Gauged $\mathcal{N} = 2$ Supergravity and Partial Breaking of Extended Supersymmetry

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Abstract

We review the general gauged $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector multiplets and hypermultiplets. We consider two different models where $\mathcal{N} = 2$ supersymmetry is broken to $\mathcal{N} = 1$ spontaneously, one has a $U(1)$ vector multiplet and the other has a $U(N)$ vector multiplet. In both cases, partial breaking of $\mathcal{N} = 2$ supersymmetry is accomplished by the Higgs and the super-Higgs mechanisms. The mass spectrum can be evaluated and we conclude that the resulting models have $\mathcal{N} = 1$ supersymmetry. This is based on master thesis submitted to Graduate School of Science, Osaka City University, in March 2006.

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Chapter 1

Introduction

The standard model is a successful theory describing the high energy physics. The remarkable agreement of theory and experiment gives us confidence that it is the correct description of strong, weak and electric interactions of elementary particles. However there are several serious problems, such as the gauge hierarchy problem and the inability to include the general relativity. These facts suggest that the standard model is not final theory but rather an effective theory describing the electroweak energy scale region.

Supersymmetry has become the dominant framework of formulating physics beyond the standard model. Supersymmetric extension of the standard model can solve some problems. Since supersymmetry requires that each scalar field have fermionic partner of the same mass, the quadratic divergences of the scalar mass terms automatically vanish. Also three gauge coupling constants modify to meet accurately at very high energy. But supersymmetry is not observed in nature, so it must be broken at low energy scale. If a supersymmetry is broken spontaneously, a massless fermion called Nambu-Goldstone fermion appears. Thus, global supersymmetry should not be broken spontaneously. Though, in the local supersymmetry, the Nambu-Goldstone fermion is absorbed by gravitino through the super-Higgs mechanism. This motivates to study supergravity as a candidate beyond the standard model.

The fact that extended ($\mathcal{N} > 1$) supersymmetric theories cannot have the chiral structure of the standard model make it difficult to deal with phenomenologically. But many theorists have been researched extended supersymmetric theories using their rich geometrical structures. In $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, by using its symplectic structure, Seiberg and Witten have showed that the prepotential function can be determined exactly including the non-perturbative effects [1, 2]. Similarly, $\mathcal{N} = 2$ supergravity has symplectic structure. The most general form of Lagrangian coupled to an arbitrary number of vector multiplets and hypermultiplets in presence of a general gauging of the isometries of both vector and hypermultiplet scalar manifolds has been obtained by [3, 4]. This construction uses a coordinate independent and manifestly symplectic covariant formalism which in particular does not require the use of a prepotential function.

Partial supersymmetry breaking plays an important role of relating the extended supersymmetric field theories with the phenomenological models. It has been shown that, in $U(1)$ gauged global $\mathcal{N} = 2$ supersymmetric theory with electric and magnetic Fayet-Iliopoulos term, the $\mathcal{N} = 2$ supersymmetry is broken to $\mathcal{N} = 1$ [5, 6, 7]. It has been generalized to the case of $U(N)$ gauged theory by [8, 9, 10]. In $\mathcal{N} = 2$ supergravity, partial supersym-
metry breaking have been accomplished by simultaneous realization of the Higgs and the super-Higgs mechanisms [11, 12, 13, 14, 15, 16, 17, 37]. But it is notable that these results have been obtained in the microscopic theories. Also, it is known that \( \mathcal{N} = 2 \) supergravity relates to \( \mathcal{N} = 2 \) global supersymmetric theory [13, 19]. In this thesis, we will see that in \( U(N) \) gauged \( \mathcal{N} = 2 \) supergravity as low energy effective theory the half of supersymmetry is broken to \( \mathcal{N} = 1 \) counterpart spontaneously [20].

The Organization of This Thesis

The first half of this thesis (chapter 2 ~ chapter 4) is review of the general matter coupled \( \mathcal{N} = 2 \) supergravity in four dimensions. On the other hand, in the latter half of this thesis (chapter 5 and chapter 6), we deal with the partial supersymmetry breaking.

In chapter 2, we define special Kähler manifolds and quaternionic Kähler manifolds. These manifolds describe scalar sector of vector multiplet and hypermultiplet respectively.

In chapter 3, we review the gauging procedure of special and quaternionic Kähler manifolds. Notion of momentum map which is introduced in section 3.1 plays an important role for the gauging.

Finally, the general matter coupled Lagrangian is given in chapter 4. It is consistent with the Bianchi identities which are obtained in appendix B. Also, the supersymmetry transformation laws of all the fields are introduced at the end of this chapter.

In chapter 5, we review the simplest \( \mathcal{N} = 2 \) model where the \( \mathcal{N} = 2 \) supersymmetry is broken to \( \mathcal{N} = 1 \) [12]. This model has \( U(1) \) gauged vector multiplet and a hypermultiplet. If we choose symplectic section such as no prepotential exists, partial supersymmetry breaking occurs. It is accomplished by simultaneous realization of the Higgs and the super-Higgs mechanisms. We, also, see that the mass spectrum. In this model, the symplectic section has chosen to be the simplest function. Therefore, this model is microscopic theory such that higher order coupling terms of the scalar fields are not exist.

Chapter 6 is the main part of this thesis. We extend the model in the chapter 5 to \( U(N) \) gauged model. We give \( \mathcal{N} = 2 \) supergravity model coupled to a \( U(N) \) gauged vector multiplet and a hypermultiplet. In particular, we do not choose the symplectic section as the simplest function. It leads to the effective theory which contains higher order coupling terms of the scalar fields. We observe that the \( \mathcal{N} = 2 \) supersymmetry is broken to \( \mathcal{N} = 1 \) spontaneously. Furthermore, by redefining the fluctuations of the scalar fields from the vacuum expectation value as new \( \mathcal{N} = 1 \) scalar fields, we can write down \( \mathcal{N} = 1 \) supergravity model in terms of the superpotentials.

Field Contents

Before going to the next chapter, we present, here, the field contents of \( \mathcal{N} = 2 \) supergravity. The general matter coupled \( \mathcal{N} = 2 \) supergravity contains a gravitational multiplet, \( m \) vector multiplets and \( k \) hypermultiplets. All the component fields of these multiplets are massless and describe as follows:

- a gravitational multiplet
  This multiplet is described by the vierbein \( e^i_\mu \) \((i, \mu = 0, 1, 2, 3)\), two gravitini \( \psi^A_\mu \)
$(A = 1, 2)$ and the graviphoton $A^0_\mu$. (The upper and lower position of the index $A$ represent left and right chirality, respectively.)

- **$m$ vector multiplets**
  Each vector multiplet contain a gauge boson $A^a_\mu (a = 1, \ldots, m)$, two gauginos $\lambda^{aA}$ and a complex scalar $z^a$. (The chirality notation is opposite to that of gravitinos, that is, the upper and lower position denote right and left chirality, respectively.)

- **$k$ hypermultiplets**
  Each multiplet contain two hyperini $\zeta^\alpha (\alpha = 1, \ldots, 2k)$ and four real scalar $b^u (u = 1, \ldots, 4k)$. (The upper and lower position of the index $\alpha$ represent left and right chirality, respectively.)

The chirality notations are explained in detail in the appendix A.
Chapter 2

Special and quaternionic Kähler manifolds

The vector multiplets and the hypermultiplets contain the different kinds of the scalar fields and they are governed by the different kinds of geometries. The scalar field sector of vector multiplets is described by special Kähler manifold $SM$, while the scalar sector of hypermultiplets is described by quaternionic Kähler manifold $HM$. Special Kähler manifold is Hodge-Kähler manifold which has the particular bundle structure. Therefore, in section 2.1, we define a Hodge-Kähler manifold. The definition of the special Kähler manifold is given in section 2.2. This definition is independent of whether prepotential exists or not. We also give a different definition of the special Kähler manifold which depend on the existence of the prepotential. On the other hand, quaternionic Kähler geometry is defined in section 2.3.

There are many works on special and quaternionic Kähler geometries. Special Kähler geometry based on the special coordinates was introduced in [21, 22, 23]. The definition of the special Kähler geometry in terms of the symplectic bundles is in [25, 24, 3, 26, 4, 27, 28, 29, 30]. On the other hand, the geometric interpretation of the coupling of hypermultiplets was introduced in [31] and [32, 33, 3, 4]. Here we use the notations of [4].

2.1 Hodge-Kähler manifolds

Kähler manifolds

First of all, let us see the notations of Kähler manifold. A Kähler manifold is defined as follows. Consider a complex manifold $M$ equipped hermitian metric $g$. A Kähler manifold $M$ is a Hermitian manifold $M$ whose Kähler 2-form $K$ is closed: $dK = 0$. In this case, the metric $g$ is called Kähler metric of $M$. The Kähler 2-form can be written in terms of the Kähler metric as,

$$K = \frac{i}{2\pi} g_{ab\ast}dz^a \wedge d\bar{z}^b, \quad (2.1.1)$$

where $a = 1, \ldots, n$, and $n$ is complex dimension of the Kähler manifold. The Kähler metric is locally given by

$$g_{ab\ast} = \partial_a\partial_{\ast b}K(z, \bar{z}), \quad (2.1.2)$$
where $\mathcal{K}$ is real function and is called the Kähler potential. It is defined up to holomorphic function $f(z)$, that is, the Kähler metric is invariant under the Kähler transformation:

$$\mathcal{K} \to \mathcal{K} + f(z) + \bar{f}(\bar{z}).$$ (2.1.3)

The only non-vanishing components of the Levi-Civita connection is $\Gamma^a_{bc}$ and $\Gamma^a_{b^*c^*}$. With these components, for example, the covariant derivatives of vector $V_a$ are

$$\nabla_a V_b = \partial_a V_b + \Gamma^c_{ab} V_c,$$

$$\nabla_{b^*} V_{b^*} = \partial_{b^*} V_{b^*}. \quad (2.1.4)$$

We require the metric compatibility which can be written, in components, as

$$\nabla_a g_{bc} = 0,$$ (2.1.5)

which leads to the following relations:

$$\Gamma^a_{bc} = -g^{ad} \partial_c g_{bd},$$

$$\Gamma^a_{b^*c^*} = -g^{a^*d} \partial_{c^*} g_{b^*d^*}. \quad (2.1.6)$$

Furthermore, the non-vanishing components of curvature 2-form are written as

$$R^a_{b^*} = R^a_{b^*c^*d} dz^c \wedge dz^d, \quad R^a_{bc^*d} = \partial_c \Gamma^a_{bd},$$

$$R^a_{b^*c^*} = R^a_{b^*c^*d} dz^c \wedge dz^{d^*}, \quad R^a_{b^*c^*d^*} = \partial_{c^*} \Gamma^a_{b^*d^*}. \quad (2.1.7)$$

**Hodge-Kähler manifolds**

Let us consider a line bundle $\mathcal{L} \to \mathcal{M}$ over a Kähler manifold. This is the holomorphic vector bundle whose fibre is $\mathbb{C}^1$ and its structure group is subgroup of $GL(1, \mathbb{C})$. Thus, the Chern class is written as,

$$c(\mathcal{L}) = 1 + \frac{i}{2\pi} F,$$ (2.1.8)

where $F$ is the field strength of the $GL(1, \mathbb{C})$ connection. Since $c_j = 0$ for $j > 1$, the only available Chern class is $c_1$:

$$c_1(\mathcal{L}) = \frac{i}{2\pi} F = \frac{i}{2\pi} \partial \theta = \frac{i}{2\pi} \partial (h^{-1} \partial h) = \frac{i}{2\pi} \partial \partial \log h,$$ (2.1.9)

where $h(z, \bar{z})$ and $\theta$ are the hermitian metric and the canonical hermitian connection on $\mathcal{L}$ respectively. In (2.1.9), we have used the relation between $h(z, \bar{z})$ and $\theta$:

$$\theta = h^{-1} \partial h = h^{-1} \partial_a h dz^a,$$

$$\bar{\theta} = h^{-1} \partial h = h^{-1} \partial_{a^*} h dz^{a^*}. \quad (2.1.10)$$

Let $f(z)$ be a holomorphic section of $\mathcal{L}$. The norm of $f(z)$ is given as $||f(z)|| = h(z, \bar{z})f(\bar{z})f(z)$. Since under the action of the operator $\partial \partial$ the term $\log(f(\bar{z})f(z))$ yields a vanishing contribution, we can rewrite (2.1.9) as

$$c_1(\mathcal{L}) = \frac{i}{2\pi} \partial \partial \log || f(z) ||^2. \quad (2.1.11)$$
Definition 2.1. A Kähler manifold $\mathcal{M}$ is a Hodge-Kähler manifold $\mathcal{M}_{\text{Hodge}}$ if there exists a line bundle $\mathcal{L} \rightarrow \mathcal{M}$ whose first Chern class is equivalent to the cohomology class of the Kähler 2-form $K$, 

$$c_1(\mathcal{L}) = [K]. \quad (2.1.12)$$

Eq. (2.2.2) implies that there is a holomorphic section $W(z)$ such that:

$$K = \frac{i}{2\pi} g_{ab} \ast dz^a \wedge d\bar{z}^b = \frac{i}{2\pi} \partial \bar{\partial} \log |W(z)|^2 \equiv \frac{i}{2\pi} \partial_a \partial_b \ast \log |W(z)|^2 dz^a \wedge d\bar{z}^b. \quad (2.1.13)$$

It follows from (2.1.13) that if the manifold $\mathcal{M}$ is a Hodge-Kähler manifold, then the exponential of the Kähler potential can be interpreted as the metric on a line bundle $\mathcal{L}$:

$$h(z, \bar{z}) = \exp(K(z, \bar{z})). \quad (2.1.14)$$

By substituting (2.1.14) into (2.1.10), we obtain

$$\theta = \partial K = \partial_a K dz^a,$$

$$\bar{\theta} = \bar{\partial} K = \partial_a \ast K dz^a. \quad (2.1.15)$$

In the $\mathcal{N} = 1$ supergravity, $c_1(K) = [K]$ and this restricts $\mathcal{M}$ to be a Hodge-Kähler manifold.

**Principal $U(1)$ bundle**

It is known that there exists a correspondence between line bundles and principal $U(1)$ bundles. Thus, the covariant derivatives with respect to the canonical connection of the line bundle are related to those of the $U(1)$ bundle. We can express this correspondence, in terms of the canonical connection, as

$$Q = \text{Im} \theta = \frac{i}{2}(\theta - \bar{\theta}), \quad (2.1.16)$$

where $Q$ is the canonical connection on $U(1)$ bundle $\mathcal{U} \rightarrow \mathcal{M}$. If we apply the above formula to the case of the $U(1)$ bundle $\mathcal{U} \rightarrow \mathcal{M}$ associated with Hodge-Kähler manifold, that is, the line bundle $\mathcal{L}$ whose first Chern class equals the Kähler 2-form, we get

$$Q = -\frac{i}{2} \left( \partial_a K dz^a - \partial_a \ast K dz^a \right). \quad (2.1.17)$$

Let us consider a principal $U(1)^p$ bundle. Let $\Phi(z, \bar{z})$ be a section of $\mathcal{U}^p$. Its covariant derivative is

$$\nabla \Phi = (d + ipQ)\Phi, \quad (2.1.18)$$
or, in components,
\[ \nabla_a \Phi = (\partial_a + \frac{1}{2} p \partial_a K) \Phi, \]
\[ \nabla_a^* \Phi = (\partial_a^* - \frac{1}{2} p \partial_a^* K) \Phi. \] (2.1.19)

A covariantly holomorphic section of \( \mathcal{U} \) is defined by the equation: \( \nabla_a^* \Phi = 0 \). We can easily map each section \( \Phi(z, \bar{z}) \) of \( \mathcal{U} \) into a section \( \tilde{\Phi} \) of the line bundle \( \mathcal{L} \) by setting,
\[ \tilde{\Phi} = e^{-pK/2} \Phi \] (2.1.20)

Under the map covariantly holomorphic sections of \( \mathcal{U} \) flow into holomorphic section of \( \mathcal{L} \) and vice versa, that is,
\[ \partial_a^* \tilde{\Phi} = e^{-pK/2} \nabla_a^* \Phi \] (2.1.21)

2.2 Special Kähler manifolds

The scalar field sector of vector multiplets in \( \mathcal{N} = 2 \) supergravity is described by special Kähler geometry of local type. Suppose that there are \( m \) vector multiplets, so there exist \( m \) complex scalar fields \( z^a, \ a = 1, \ldots, m \). It is known that these scalar fields span a special Kähler manifold \( SM \). The number of the complex scalar fields, \( m \), corresponds to the complex dimension of the special Kähler manifold.

Here we only give definition of the special Kähler manifold. Thus, there is no guarantee that the vector multiplet scalar sector of \( \mathcal{N} = 2 \) supergravity is described by the special Kähler geometry. However, in the appendix B, we will confirm that it is true.

2.2.1 Definitions

We consider, in addition to the line bundle \( \mathcal{L} \) which has been introduced in the Hodge-Kähler manifold, the bundle \( SV \) as follows: \( SV \rightarrow SM \) denotes a holomorphic flat vector bundle with structure group \( Sp(2m+2, \mathbb{R}) \). Consider tensor bundle of the type \( H = SV \otimes L \).

A holomorphic section \( \Omega \) of such a bundle will have the following structure,
\[ \Omega(z) = \begin{pmatrix} X^\Lambda(z) \\ F_{\Sigma}(z) \end{pmatrix}, \quad \Lambda, \Sigma = 0, 1, \ldots, m. \] (2.2.1)

**Definition 2.2.** A special Kähler manifold \( SM \) of the local type is an \( m \)-dimensional Hodge-Kähler manifold \( M_{Hodge} \) if there exists a bundle \( H \) with the following properties.

1. For some holomorphic section \( \Omega \), the Kähler 2-form is given by
\[ K = -\frac{i}{2\pi} \partial \bar{\partial} \log (i \langle \Omega | \bar{\Omega} \rangle). \] (2.2.2)

2. On overlaps of charts \( i \) and \( j \), the symplectic section \( \Omega \) are connected by transition functions of the following form
\[ \Omega_{(i)} = \begin{pmatrix} X^\Lambda \\ F_{\Sigma} \end{pmatrix}_{(i)} = e^{f_{ij}(z)} M_{ij} \begin{pmatrix} X^\Lambda \\ F_{\Sigma} \end{pmatrix}_{(j)}, \] (2.2.3)

with \( f_{ij} \) holomorphic and \( M_{ij} \in Sp(2m+2, \mathbb{R}) \) ,

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3. The transition functions satisfy the following cocycle conditions on overlap regions of three charts:

\[ e^{f_{ij} f_{jk} f_{ki}} = 1, \]
\[ M_{ij} M_{jk} M_{ki} = 1. \] (2.2.4)

Eq. (2.2.4) implies that we have a local expression for the Kähler potential in terms of the holomorphic section Ω:

\[ K = -\log \langle \Omega | \overline{\Omega} \rangle = -\log [i(\tilde{X}^\Lambda F^\Lambda - F^\Lambda \tilde{X}^\Lambda)]. \] (2.2.5)

We introduce a non-holomorphic section V of the bundle H according to,

\[ V = \left( \frac{L^\Lambda}{M^\Sigma} \right) = e^{K/2} \Omega = e^{K/2} \left( \frac{X^\Lambda}{F^\Sigma} \right), \] (2.2.6)

so that (2.2.5) becomes

\[ 1 = i \langle V | \overline{V} \rangle = i(\overline{L}^\Lambda M^\Lambda - M^\Lambda L^\Lambda). \] (2.2.7)

It can be seen as the section on the associated U(1) bundle (2.1.18). Therefore, by using (2.1.19), its covariant derivatives are written as,

\[ U_a \equiv \nabla_a V = \left( \partial_a + \frac{1}{2} \partial_a K \right) V = \left( \frac{f^\Lambda_a}{h^\Sigma_a} \right), \] (2.2.8)

\[ U^*_a \equiv \nabla^*_a V = \left( \partial^*_a - \frac{1}{2} \partial^*_a K \right) V = 0, \] (2.2.9)

\[ \bar{U}^*_a \equiv \nabla^*_a \bar{V} = \left( \partial^*_a + \frac{1}{2} \partial^*_a K \right) \bar{V} = \left( \frac{\bar{f}^\Lambda_a}{h^{\Sigma_a}} \right), \] (2.2.10)

where \( \bar{U}^*_a \) has been defined to be complex conjugate of \( U_a \). At this stage, we can derive the following equations

\[ \langle V | U_a \rangle = \langle V | U^*_a \rangle = \langle V | \bar{U}^*_a \rangle = 0, \] (2.2.11)

\[ g_{ab^*} = -i \langle U_a | \bar{U}^*_b \rangle, \] (2.2.12)

\[ \nabla_{[a} U_b] = 0, \] (2.2.13)

where \( \nabla_a \) denotes the covariant derivative containing both the canonical connection \( \theta \) on the line bundle \( \mathcal{L} \) and the Levi-Civita connection:

\[ \nabla_a U_b = \partial_a U_b + \frac{1}{2} \partial_a K U_b + \Gamma^c_{ab} U_c. \] (2.2.14)

Defining

\[ C_{abc} = \langle \nabla_a U_b | U_c \rangle, \] (2.2.15)

we can see that it is completely symmetric in its indices. With this, (2.2.14) can be written as,

\[ \nabla_a U_b = i C_{abc} g^{cd^*} \bar{U}^*_d \] (2.2.16)
Lemma 2.1. The completely symmetric tensor $C_{abc}$ is covariantly holomorphic, namely,

$$\nabla_d^* C_{abc} = 0. \quad (2.2.17)$$

Proof. First of all, we have to observe that $U_a$ satisfies the following condition:

$$
\langle U_a | U_b \rangle = \nabla_b \langle U_a | V \rangle - \langle \nabla_b U_a | V \rangle = -i C_{abc} g^{cd*} \langle \bar{U}_d^* | V \rangle = 0, \quad (2.2.18)
$$

where we have used (2.2.11) and (2.2.16). We can also evaluate the covariant derivatives, $\nabla_a \nabla_b^* U_c$ and $\nabla_b^* \nabla_a U_c$, as follows:

$$
\nabla_a \nabla_b^* U_c = \partial_a (\nabla_b^* U_c) + \frac{1}{2} \partial_a K (\nabla_b^* U_c) + \Gamma_{ad}^c (\nabla_b^* U_d),
$$

$$
\nabla_b^* \nabla_a U_c = \partial_b^* (\nabla_a U_c) - \frac{1}{2} \partial_b^* K (\nabla_a U_c) + \frac{1}{2} \partial_a \nabla_b^* U_c + \Gamma_{ac}^d \nabla_b^* U_d + (\partial_b^* \Gamma_{ac}^d) U_d,
$$

which lead to

$$
[\nabla_a \nabla_b^* - \nabla_b^* \nabla_a ] U_c = -g_{ab^*} U_c - (\partial_b^* \Gamma_{ac}^d) U_d. \quad (2.2.19)
$$

Thus, using (2.2.18), we derive

$$
\nabla_d^* C_{abc} = \langle \nabla_d^* \nabla_a U_b | U_c \rangle + \langle \nabla_a U_b | \nabla_d^* U_c \rangle = \langle (\nabla_a \nabla_d^* U_b | U_c \rangle + i C_{abc} g^{df^*} g^{cd*} \langle \bar{U}_{f^*} | V \rangle = 0, \quad (2.2.20)
$$

in the last equality, we have used $\nabla_a \nabla_d^* U_b = \nabla_a (g_{bd^*} V) = g_{bd^*} U_a$. Therefore, the lemma 2.1 has been proved.

We have to introduce one more important quantity, the generalized gauge coupling matrix $N$ which will appear in the Lagrangian of the $N = 2$ supergravity \(^1\). It is introduced by the following relations,

$$
\bar{M}_\Lambda = \bar{N}_{\Lambda \Sigma} \bar{L}^\Sigma, \quad h_{\Lambda | \alpha} = \bar{N}_{\Lambda \Sigma} f^\Sigma_\alpha, \quad (2.2.21)
$$

which can be solved constructing the two $(N + 1) \times (N + 1)$ matrices

$$
f^\Lambda_f = \begin{pmatrix} f^\Lambda_f \\ \bar{L}^\Lambda \end{pmatrix}, \quad h_{\Lambda | f} = \begin{pmatrix} h_{\Lambda | \alpha} \\ M_\Lambda \end{pmatrix}, \quad (2.2.22)
$$

\(^1\)We will see it in the chapter 4.
and setting:
\[
\tilde{N}_{\Lambda \Sigma} = h_{\Lambda |j} \circ (f^{-1})^j_{\Sigma}.
\]

From the previous formulae it is easy to derive a set of useful relations:
\[
(\text{Im } N_{\Lambda \Sigma}) L^\Lambda L^\Sigma = -\frac{1}{2},
\]
\[
U^{\Lambda \Sigma} \equiv f_a^\Lambda \bar{f}_b^\Sigma g^{ab} = -\frac{1}{2} (\text{Im } N)^{-1|\Lambda \Sigma} - \bar{L}^\Lambda L^\Sigma,
\]
\[
C_{abc} = f_a^\Lambda \partial_b \tilde{N}_{\Lambda \Sigma} f_c^\Sigma = (N - \bar{N})_{\Lambda \Sigma} f_a^\Lambda \partial_b f_c^\Sigma,
\]

2.2.2 The holomorphic prepotential

So far we have not mentioned about the holomorphic prepotential \( F(X) \). Indeed, when the definition of special Kähler manifolds is given in intrinsic terms, as we did in the previous section, the holomorphic prepotential \( F \) can be dispense of. Actually, it appears that some physically interesting cases (for example, partial supersymmetry breaking) are precisely instances where \( F(X) \) does not exist.

However, we give the definition of special Kähler manifold, which depends on the existence of the prepotential here.

Definition 2.3. A special Kähler manifold is an \( m \)-dimensional Hodge-Kähler manifold with the following properties.

1. On every chart there exist complex projective coordinate functions \( X^\Lambda(z) \), where \( \Lambda = 0, \ldots, m \) and a holomorphic function (prepotential) \( F(X^\Lambda) \) which is homogeneous of second degree, such that the Kähler potential is
\[
\mathcal{K} = -\log i \left[ \tilde{X}^\Lambda \frac{\partial}{\partial X^\Lambda} F(X) - X^\Lambda \frac{\partial}{\partial X^\Lambda} \tilde{F}(X) \right],
\]

2. On overlaps of charts \( i \) and \( j \), the symplectic vector \( \Omega \) which is constructed from \( X \) and \( F \) in property 1 are connected by transition functions of the following form
\[
\Omega_{(i)} = \left( \frac{X}{\partial F} \right)_{(i)} = e^{f_{ij}(z)} M_{ij} \left( \frac{X}{\partial F} \right)_{(j)},
\]

with \( f_{ij} \) holomorphic and \( M_{ij} \in Sp(2m + 2, \mathbb{R}) \).

3. The transition functions satisfy the cocycle conditions \( \text{2.2.4} \) on overlap regions of three charts.

This definition of a special Kähler manifold clearly depends on the existence of the prepotential \( F(X) \). In reference of [27], it has been proved that these definitions are equivalent each other.
2.3 Quaternionic Kähler manifolds

Let us turn to the hypermultiplet sector. There are 4 real scalar fields for each hypermultiplet and, at least locally, they can be regarded as 4 components of a quaternion. These scalar fields span a quaternionic manifold \(\mathcal{H}M\). If we have \(k\) hypermultiplets, the manifold \(\mathcal{H}M\) has dimension \(4k\).

Let us consider a \(4k\)-dimensional real manifold with a metric \(h\):
\[
 ds^2 = h_{uv}(b)db^u \otimes db^v ; \quad u,v = 1,\ldots, 4m, \tag{2.3.1}
\]
and three complex structures \(J^x\) that satisfy the quaternionic algebra
\[
 J^x J^y = -\delta^{xy}I + \epsilon^{xyz} J^z, \tag{2.3.2}
\]
and respect to which the metric is hermitian: for any tangent vector \(X,Y\) on \(\mathcal{H}M\),
\[
 h(J^x X, J^y Y) = h(X, Y). \tag{2.3.3}
\]
From (2.3.3), it follows that one can introduce a triplet of 2-forms
\[
 K^x = K^x_{uv} db^u \wedge db^v, \\
 K^x_{uv} = h_{uv}(J^x)^w_v \tag{2.3.4}
\]
The triplet of 2-forms, \(K^x\), is named the hyperKähler form. It provides the generalization of the Kähler form introduced in the complex case. It is an \(SU(2)\) Lie-algebra valued 2-form in the same way as the Kähler form is a \(U(1)\) Lie-algebra valued 2-form. In the complex case, the definition of Kähler manifold involves the statement that the Kähler 2-form is closed and, in Hodge-Kähler manifold, the Kähler 2-form can be identified with the curvature of a line bundle. Similar steps can be taken also here which lead to quaternionic manifolds.

Consider a principal \(SU(2)\) bundle \(SU \to \mathcal{H}M\) that play for hypermultiplets the same role played by the line bundle \(L \to SM\) in the case of vector multiplets. Let \(\omega^x\) denote a connection on such a bundle. To obtain a quaternionic manifold we must impose the condition that the hyperKähler 2-form is covariantly closed with respect to the connection \(\omega^x\)
\[
 \nabla K^x \equiv dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0. \tag{2.3.5}
\]
Furthermore, we define the \(SU\) curvature by
\[
 \Omega^x = d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z. \tag{2.3.6}
\]

**Definition 2.4.** A quaternionic manifold \(\mathcal{H}M\) is a \(4m\)-dimensional manifold with the structure described above and such that the curvature of the \(SU\) bundle is proportional to the HyperKähler 2-form as
\[
 \Omega^x = \lambda K^x. \tag{2.3.7}
\]
where \(\lambda\) is a non-vanishing real number.
Holonomy

As a consequence of the above structure, the holonomy group of the manifold \( \mathcal{HM} \) is a subgroup of \( Sp(2) \times Sp(2k, \mathbb{R}) \). If we introduce flat indices \( A, B = 1, 2 \); \( \alpha, \beta = 1, \ldots, 2k \) that run in the fundamental representations of \( SU(2) \) and \( Sp(2k, \mathbb{R}) \) respectively, we can find vielbein 1-form

\[
U^A = U^A_u db^u, \tag{2.3.8}
\]

such that

\[
h_{uv} = U^A_u U^B_v \epsilon_{AB}, \tag{2.3.9}
\]

where \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \) and \( \epsilon_{AB} = -\epsilon_{BA} \) are, respectively, the flat \( Sp(2m) \) and \( Sp(2) \sim SU(2) \) invariant metrics \(^1\).

The vielbein \( U^A \) is covariantly closed with respect to the \( SU(2) \) connection \( w_x \) and to some \( Sp(2k, \mathbb{R}) \) Lie algebra valued connection \( \Delta^{\alpha\beta} = \Delta^{\beta\alpha} \):

\[
\nabla U^{A} = dU^{A} + \frac{i}{2} \epsilon^{x}_{\sigma} \epsilon^{-1} A^x \wedge U^{B\alpha} + \Delta^{\alpha\beta} \wedge U^{A\gamma} \epsilon_{\beta\gamma} = 0, \tag{2.3.10}
\]

where \( (\epsilon^{x})^B_x \) are the standard Pauli matrices \(^{\ddagger}\) and \( Sp(2k, \mathbb{R}) \) Lie algebra valued connection \( \Delta^{\alpha\beta} \) satisfies:

\[
\Delta^{\alpha} \beta = \epsilon^{\beta\gamma} \Delta_{\gamma\alpha}, \tag{2.3.11}
\]

Furthermore, \( U^A \) satisfies the reality condition:

\[
U^A = (U^A)^* = \epsilon_{AB} \epsilon_{\alpha\beta} U^B. \tag{2.3.12}
\]

Eq. (2.3.12) defines the rule to lower the symplectic indices by means of the flat symplectic metrics \( \epsilon_{AB} \) and \( \epsilon_{\alpha\beta} \). More specifically we can write a stronger version of \(2.3.9\) \[\ddagger\]:

\[
(U^A_u U^B_v + U^A_v U^B_u) \epsilon_{\alpha\beta} = h_{uv} \epsilon^{AB}, \tag{2.3.13}
\]

\[
(U^A_u U^B_v + U^A_v U^B_u) \epsilon_{AB} = h_{uv} \frac{1}{m} C_{\alpha\beta}. \tag{2.3.14}
\]

We have also the inverse vierbein \( U^u_{Aa} \) defined by the following equation:

\[
U^u_{Aa} U^A = \delta^u_v, \tag{2.3.15}
\]

Flattening a pair of indices of the Riemann tensor \( R^{uv}_{ts} \) we obtain

\[
R^{uv}_{ts} U^A_u U^B = -\frac{i}{2} \epsilon^{AC} R^B_{C|ts} \epsilon^{\alpha\beta} + \epsilon^{\alpha\beta} \epsilon_{AB}, \tag{2.3.16}
\]

\(^1\)Notations are fixed in the appendix A

\(^{\ddagger}\) Also, see appendix A
where $R^B_A$ and $R^{\alpha\beta}$ are the curvature of the SU(2) connection and the Sp(2m) connection respectively:

$$
R^B_A = R^B_A |_{uv} dB^u \wedge dB^v, \\
\equiv \Omega^x_{uv} (\sigma_x)_A^B, \tag{2.3.17}
$$

$$
R^{\alpha\beta} = R^{\alpha\beta}_{ts} db^t \wedge db^s \\
\equiv d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta} C_{\gamma\delta}. \tag{2.3.18}
$$

Eq. (2.3.16) is the explicit statement that the Levi-Civita connection associated with the metric $h$ has a holonomy group contained in $SU(2) \otimes Sp(2m)$.

Let us consider (2.3.2), (2.3.4) and (2.3.7), so we easily obtain the following relation:

$$
h^{st} K^x_{us} K^y_{tw} = -\delta^{xy} h_{uw} + \epsilon^{xyz} K^z_{uw}. \tag{2.3.19}
$$

By using (2.3.7), (2.3.19) can be rewritten as follows:

$$
h^{st} \Omega^x_{us} \Omega^y_{tw} = -\lambda^2 \delta^{xy} h_{uw} + \lambda \epsilon^{xyz} \Omega^z_{uw}. \tag{2.3.20}
$$
Chapter 3

The gauging

In this chapter, we discuss the gauging procedure. The gauge group is identified with a subgroup of isometries of the product manifold $SM \times HM$. The isometries are generated by Killing vectors. Also, the Killing vectors are written by the Killing potential which is, in geometrical point of view, identical with the momentum map providing the Poisson realization of Lie algebra on the manifold.

Firstly, we review about the Killing vectors and the Killing potential. We also see the notion of the momentum map. Furthermore, the construction of the momentum map on special Kähler manifolds or quaternionic Kähler manifolds is considered respectively. Finally, we gauge all the connections which have been given in the last chapter. These are based on the works of [33, 3, 4].

3.1 The momentum map

3.1.1 The Killing vectors

Consider a Kähler manifold $M$ of complex dimension $m$. A vector field $X$ is said a Killing vector field if an infinitesimal displacement $\epsilon X$ generates an isometry, that is, under the displacement the Kähler metric is invariant:

$$ (\mathcal{L}_X g)_{\mu\nu} = 0, \quad (3.1.1) $$

where $\mathcal{L}_X$ is the Lie derivative along the vector field $X$. Let $k^a_\Lambda$ ($a = 1 \ldots m$) be a component of holomorphic Killing vectors, that is,

$$ X = a^\Lambda k_\Lambda, $$

$$ k_\Lambda = k^a_\Lambda \frac{\partial}{\partial z^a} + k^{a*}_\Lambda \frac{\partial}{\partial z^{a*}}, \quad (3.1.2) $$

where $k_\Lambda$ is a basis of the Killing vectors. The above statement can be rewritten such that, under the infinitesimal holomorphic coordinate transformations

$$ \delta z^a = \epsilon^\Lambda k^a_\Lambda(z), \quad (3.1.3) $$

the Kähler metric is invariant.
Eq. (3.1.1) implies the following Killing equations:

\[ \nabla_a k_b \Lambda + \nabla_b k_a \Lambda = 0, \]
\[ \nabla_a^* k_b \Lambda + \nabla_b^* k_a \Lambda = 0, \] (3.1.4)

where we have defined as \( k_b = g_{ba} k^a \), \( k_a^* = g_a^* k^b \). Holomorphicity of the Killing vectors means the following differential constraints:

\[ \partial_b k_a^\Lambda(z) = 0, \]
\[ \partial_b k_a^\Lambda(z) = 0. \] (3.1.5)

Let us consider a compact Lie group \( G \) acting on \( M \) by means of Killing vector fields \( X \). From (3.1.1), we obtain for the Kähler 2-form,

\[ 0 = \mathcal{L}_X K = i_X dK + d(i_X K) = d(i_X K), \] (3.1.6)

where \( i_X \) denotes the contraction with \( X \). In the third equality, we have used the fact that the Kähler 2-form is closed. In general, if \( M \) is simply connected, first de Rham cohomology group is trivial \(^\dagger\). Thus, the closed one form is also exact. In this case, the one form \( i_X K \) is exact, then we can write

\[ -\frac{1}{2\pi} d\mathcal{P}_X = i_X K, \] (3.1.7)

where a function \( \mathcal{P}_X \) is defined up to a constant. If we expand \( X = a^\Lambda k_\Lambda \) in a basis of Killing vectors \( k_\Lambda \), we can also expand \( \mathcal{P}_X \) as,

\[ \mathcal{P}_X = a^\Lambda \mathcal{P}_\Lambda. \] (3.1.8)

Using this basis, we can rewrite (3.1.7) in components

\[ k^a_\Lambda = i g^{ob} \partial_b^* \mathcal{P}_\Lambda, \]
\[ k^{a^*}_\Lambda = -i g^{a^*b} \partial_b \mathcal{P}_\Lambda. \] (3.1.9)

The function \( \mathcal{P}_\Lambda \) is called Killing potential.

Indeed the Killing vectors which are written as (3.1.9) satisfy the Killing equations (3.1.4). The first equation of (3.1.4) is automatically satisfied, because

\[ \nabla_a k_b = \partial_a k_b + \Gamma^c_{ab} k_c = (\partial_a g_{bd^*} + \Gamma^c_{ab} g_{cd^*} + \Gamma^{c^*}_{ad^*} g_{bc^*}) k^{d^*} = (\nabla_a g_{bd^*}) k^{d^*} = 0. \] (3.1.10)

where we have used \( \Gamma^c_{ad^*} = 0 \). Also, since (3.1.9) implies

\[ \nabla_a^* k_b \Lambda = -i \partial_a^* \partial_b \mathcal{P}_\Lambda, \]
\[ \nabla_b k_a^* \Lambda = i \partial_b \partial_a^* \mathcal{P}_\Lambda, \] (3.1.11)

\(^\dagger\)see, for example, chapter 6 of [34]
thus the second equation of (3.1.4) is satisfied. In other words if we can find a function $P_\Lambda$ such that the expression $i_\gamma^{ab} \partial_b P_\Lambda$ is holomorphic, then (3.1.9) defines a Killing vector.

If we substitute $g_{ab} = \partial_a \partial_b \mathcal{K}$ into (3.1.9), we obtain an expression for the Killing potential in terms of the Kähler potential,

$$i P_\Lambda = \frac{1}{2} (k^a_\Lambda \partial_a \mathcal{K} - k^a_\Lambda \partial^{a*} \mathcal{K}) = k^a_\Lambda \partial_a \mathcal{K} = -k^a_\Lambda \partial_a \mathcal{K}.$$  \hspace{1cm} (3.1.12)

Eq. (3.1.12) is true if the Kähler potential is exactly invariant under the transformations of the isometry group $G$ and not only up to a Kähler transformation as defined in (2.1.3). In other words (3.1.12) is true if

$$0 = L_\Lambda \mathcal{K} = k^a_\Lambda \partial_a \mathcal{K} + k^a_\Lambda \partial^{a*} \mathcal{K}.$$  \hspace{1cm} (3.1.13)

Note that not all the isometries of a general Kähler manifold have such a property.

### 3.1.2 The momentum map

The construction of the Killing potential can be stated in a more precise geometrical formulation which involves the notion of momentum map. Let us review this construction which reveals another deep connection between supersymmetry and geometry.

Let us firstly prove the following relation:

$$X(P_Y) = P_{[X,Y]}.$$ \hspace{1cm} (3.1.14)

We refer to this as equivariance relation.

**Proof.** Since the Kähler 2-form is closed, we get

$$0 = dK(X,Y,Z) = X(K(Y,Z)) - Y(K(X,Z)) + Z(K(X,Y))$$

$$= -K([X,Y],Z) + K([X,Z],Y) - K([Y,Z],X).$$ \hspace{1cm} (3.1.15)

On the other hand, using (3.1.6), we obtain

$$0 = d(i_X K)(Y,Z) = Y(K(X,Z)) - Z(K(X,Y)) - K(X,[Y,Z]),$$ \hspace{1cm} (3.1.16)

$$0 = d(i_Y K)(X,Z) = X(K(Y,Z)) - Z(K(Y,X)) - K(Y,[X,Z]).$$ \hspace{1cm} (3.1.17)

The above three equations imply

$$Z(K(Y,X)) = K([X,Y],Z),$$ \hspace{1cm} (3.1.18)

which leads to:

$$d \circ i_X \circ i_Y(K) = i_{[X,Y]} K,$$ \hspace{1cm} (3.1.19)

where we have used that for any function $f$, $X(f) = i_X(df)$. However, the left hand side of (3.1.19) can be rewritten as

$$-2\pi d(i_X \circ i_Y(K)) = d(i_X (dP_Y)) = d(X(P_Y)),$$ \hspace{1cm} (3.1.20)

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in the first equality we have used (3.1.7). Therefore, we have proved (3.1.14) up to constant.

Suppose that we expand \( X \) in a basis \( k_\Lambda \) such that

\[
[k_\Lambda, k_\Sigma] = f^\Gamma_{\Lambda\Sigma} k_\Gamma.
\] (3.1.21)

In the following we will use the shorthand notation \( \mathcal{L}_{k_\Lambda}, i_{k_\Lambda} \) for the Lie derivative and the contraction along the chosen basis of Killing vectors \( k_\Lambda \). The left hand side of the equivariance relation (3.1.14) can be represented in terms of the Killing vectors as follows,

\[
X(\mathcal{P}_Y) = a^\Lambda k_\Lambda (a^\Sigma \mathcal{P}_\Sigma) = ia^\Lambda a^\Sigma g_{ab^*} (k^a_\Lambda k^b_\Sigma - k^a_\Sigma k^b_\Lambda),
\] (3.1.22)

where we have used (3.1.9). On the other hand, the right hand side of (3.1.14) is written as

\[
\mathcal{P}_{[X,Y]} = a^\Lambda a^\Sigma f^\Gamma_{\Lambda\Sigma} \mathcal{P}_\Gamma.
\] (3.1.23)

Therefore, the equivariance relation implies

\[
ig_{ab^*} (k^a_\Lambda k^b_\Sigma - k^a_\Sigma k^b_\Lambda) = f^\Gamma_{\Lambda\Sigma} \mathcal{P}_\Gamma.
\] (3.1.24)

**Momentum map**

There is another way of stating the equivariance relation. It is based on the notion of the momentum map. A momentum map is constituted by \( \mathcal{P}_X \). This can be regarded as a map,

\[
\mathcal{P} : \mathcal{M} \rightarrow \mathbb{R} \otimes \mathfrak{g}^*,
\] (3.1.25)

where \( \mathfrak{g}^* \) denotes the dual of the Lie algebra \( \mathfrak{g} \) of the Lie group \( G \). Indeed let \( g \in \mathfrak{g} \) be the Lie algebra element corresponding to the Killing vector \( X \); then, for a given \( p \in \mathcal{M}, \)

\[
\mu(m) : g \rightarrow \mathcal{P}_X(p) \in \mathbb{R}
\] (3.1.26)

is a linear functional on \( \mathbb{R} \).

The momentum map is the Hamiltonian function providing the Poissonian realization of the Lie algebra on the Kähler manifold. Indeed the very existence of the closed Kähler 2-form \( K \) guarantees that every Kähler space is a symplectic manifold and that we can define a Poisson bracket.

Consider (3.1.9). To every generator of the abstract Lie algebra \( \mathfrak{g} \) we have associated a function \( \mathcal{P}_\Lambda \) on \( \mathcal{M} \). The Poisson bracket of \( \mathcal{P}_\Lambda \) with \( \mathcal{P}_\Sigma \) is defined as follows,

\[
\{ \mathcal{P}_\Lambda, \mathcal{P}_\Sigma \} = 4\pi K(\Lambda, \Sigma),
\] (3.1.27)

where \( K(\Lambda, \Sigma) = K(k\Lambda, k\Sigma) \) is the value of \( K \) along the pair of Killing vectors.

**Lemma 3.1.** The following identity is true,

\[
\{ \mathcal{P}_\Lambda, \mathcal{P}_\Sigma \} = f_{\Lambda\Sigma}^\Gamma \mathcal{P}_\Gamma + C_{\Lambda\Sigma},
\] (3.1.28)
where $C_{\Lambda\Sigma}$ is a constant tensor satisfying the cocycle condition
\begin{equation}
 f_{\Lambda\Sigma}^\Delta C_{\Delta\Gamma} + f_{\Sigma\Gamma}^\Delta C_{\Delta\Lambda} + f_{\Gamma\Lambda}^\Delta C_{\Delta\Sigma} = 0. \tag{3.1.29}
\end{equation}

**Proof.** Using (3.1.7), we have
\begin{align*}
4\pi K(\Lambda, \Sigma) &= -4\pi K(\Sigma, \Lambda) = 2\pi i_\Sigma i_\Lambda K = -2\pi i_\Lambda i_\Sigma K = -i_\Sigma dP_\Lambda = i_\Lambda dP_\Sigma \\
&= \frac{1}{2}(i_\Lambda dP_\Sigma - i_\Sigma dP_\Lambda) \\
&= \frac{1}{2}(\mathcal{L}_\Lambda P_\Sigma - \mathcal{L}_\Sigma P_\Lambda). \tag{3.1.30}
\end{align*}

Since the exterior derivative commutes with the Lie derivative, $[d, \mathcal{L}_\Lambda] = 0$, we find
\begin{align*}
4\pi dK(\Lambda, \Sigma) &= \frac{1}{2}(\mathcal{L}_\Lambda dP_\Sigma - \mathcal{L}_\Sigma dP_\Lambda) \\
&= -\pi(\mathcal{L}_\Lambda i_\Sigma K - \mathcal{L}_\Sigma i_\Lambda K) \\
&= -2\pi i_{[\Lambda, \Sigma]} K \\
&= f_{\Lambda\Sigma}^\Gamma dP_\Gamma, \tag{3.1.31}
\end{align*}
in the third equality, we have used $[i_\Lambda, \mathcal{L}_\Sigma] K = i_{[\Lambda, \Sigma]} K$ and $\mathcal{L}_\Lambda K = 0$. Using (3.1.27), we obtain
\begin{equation}
d(\{P_\Lambda, P_\Sigma\} - f_{\Lambda\Sigma}^\Gamma P_\Gamma) = 0, \tag{3.1.32}
\end{equation}
which proves (3.1.28). The cocycle condition (3.1.29) follows from the Jacobi identities satisfied by (3.1.27).

If the Lie algebra $\mathfrak{g}$ has a trivial second cohomology group $H^2(\mathfrak{g}) = 0$, then the cocycle $C_{\Lambda\Sigma}$ is a coboundary, namely $C_{\Lambda\Sigma} = f_{\Lambda\Sigma}^\Gamma C_\Gamma$ and $C_\Gamma$ are suitable constants. Therefore, if we assume $H^2(\mathfrak{g}) = 0$, we can absorb $C_\Lambda$ in the definition of $P_\Lambda$:
\begin{equation}
P_\Lambda \rightarrow P_\Lambda + C_\Lambda, \tag{3.1.33}
\end{equation}
and we obtain the stronger equation
\begin{equation}
\{P_\Lambda, P_\Sigma\} = f_{\Lambda\Sigma}^\Gamma P_\Gamma. \tag{3.1.34}
\end{equation}

Note that $H^2(\mathfrak{g}) = 0$ is true for all semi-simple Lie algebras. Using (3.1.27), (3.1.34) can be rewritten in components as follows,
\begin{equation}
\frac{i}{2} g_{ab^*} (k_a^\alpha k_{b^*}^\beta - k_a^{\alpha b^*} - k_{b^*}^{\alpha a}) = \frac{1}{2} f_{\Lambda\Sigma}^\Gamma P_\Gamma. \tag{3.1.35}
\end{equation}
This equation is identical with the equivariance relation (3.1.24).
The momentum map on special Kähler manifolds

We consider the momentum map on special Kähler manifolds. In order to distinguish the holomorphic momentum map from the triholomorphic one \( P^0_\Lambda \) which is defined on quaternionic Kähler manifolds in next subsection, we adopt the notation \( P^0_\Lambda \). The Lie derivative of the covariantly holomorphic section \( V \) defined in (2.2.7) is

\[
L_\Lambda V = k^a_\Lambda \partial_a V + k^a_\Lambda \partial_{a^*} V = T_\Lambda V + V f_\Lambda(z),
\]

where

\[
T_\Lambda = \begin{pmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{pmatrix} \in Sp(2n+2,\mathbb{R})
\]

is some element of the real symplectic Lie algebra and \( f_\Lambda(z) \) corresponds to an infinitesimal Kähler transformation.

As we see in the next chapter, the Lagrangian of gauged \( \mathcal{N} = 2 \) supergravity is not necessarily invariant under the Kähler transformation (2.1.3). In order for the Lagrangian to be invariant, we should impose the following restriction:

\[
f_\Lambda(z) = 0.
\]

Under the above restriction, recalling (2.2.5) and (2.2.6), for the sections \( V \) and \( \Omega \) we have,

\[
L_\Lambda V = (k^a_\Lambda \partial_a \mathcal{K} + k^a_{\Lambda^*} \partial_{a^*} \mathcal{K}) V + e^{\mathcal{K}/2} L_\Lambda \mathcal{O}
\]

\[
= T_\Lambda e^{\mathcal{K}/2} \mathcal{O},
\]

\[
L_\Lambda \mathcal{O} = T_\Lambda \mathcal{O}.
\]

Thus, we obtain

\[
L_\Lambda \mathcal{K} = k^a_\Lambda \partial_a \mathcal{K} + k^a_{\Lambda^*} \partial_{a^*} \mathcal{K} = 0
\]

that is identical with (3.1.13). Therefore, as discussed in the last subsection, the following equation is true

\[
i P^0_\Lambda = k^a_\Lambda \partial_a \mathcal{K} = -k^a_{\Lambda^*} \partial_{a^*} \mathcal{K}.
\]

Utilizing the definition in (2.2.8) we easily obtain,

\[
k^a_\Lambda U_a = T_\Lambda V + i P^0_\Lambda V.
\]

Taking the symplectic scalar product of (3.1.42) with \( \tilde{V} \) and recalling (2.2.4) we finally get:

\[
\mathcal{P}^0_\Lambda = \langle \tilde{V} | T_\Lambda V \rangle = \langle V | T_\Lambda \tilde{V} \rangle
\]

\[
= e^\mathcal{K} \langle \Omega | T_\Lambda \Omega \rangle.
\]

In the gauging procedure, we are interested in the symplectic image of whose generators is block-diagonal and coincides with adjoint representation in each block. Namely,

\[
T_\Lambda = \begin{pmatrix} f^\Sigma_{\Lambda\Delta} & 0 \\
0 & -f^\Sigma_{\Lambda\Delta} \end{pmatrix}
\]

Then (3.1.43) becomes

\[
\mathcal{P}^0_\Lambda = e^\mathcal{K} (F_{\Delta} f^\Lambda_{\Lambda\Sigma} \tilde{X}^\Sigma + \tilde{F}_{\Delta} f^\Lambda_{\Lambda\Sigma} X^\Sigma).
\]
3.1.3 The triholomorphic momentum map

Let us turn to a discussion of isometries of the quaternionic manifold \( \mathcal{H} \mathcal{M} \) associated with hypermultiplets. The triholomorphicity of the momentum map \( \mathcal{P}^x (x = 1, 2, 3) \) comes from the quaternionic algebra \( 2.3.2 \) of \( \mathcal{H} \mathcal{M} \). We must assume that on \( \mathcal{H} \mathcal{M} \) we have an action by triholomorphic isometries of the same Lie group \( \mathcal{G} \) that acts on the special Kähler manifold \( \mathcal{S} \mathcal{M} \). This means that on product manifold \( \mathcal{S} \mathcal{M} \otimes \mathcal{H} \mathcal{M} \) we have Killing vectors

\[
\hat{k}_\Lambda = k^a_\Lambda \frac{\partial}{\partial z^a} + k^a_\Lambda \frac{\partial}{\partial z^a} + k^u_\Lambda \frac{\partial}{\partial b^u}
\]

which generate the transformation keeping invariant the metric,

\[
\hat{g} = \begin{pmatrix} g_{ab} & 0 \\ 0 & h_{uv} \end{pmatrix}
\]

and satisfy the same Lie algebra as the corresponding Killing vectors on special Kähler manifolds \( \mathcal{S} \mathcal{M} \):

\[
[\hat{k}_\Lambda, \hat{k}_\Sigma] = f^\Gamma_{\Lambda\Sigma} \hat{k}_\Gamma.
\]

In the previous section, we have obtained the Kähler 2-form is invariant under the action of Lie derivative. Similarly, the Killing vector fields leave the hyperKähler form invariant. The only difference is the freedom to do the \( SU(2) \) rotations in the \( SU(2) \) bundle \( SU \), that is,

\[
\mathcal{L}_\Lambda K^x = \epsilon^{xyz} K^y W^z_\Lambda,
\]

\[
\mathcal{L}_\Lambda \omega^x = \nabla W^x_\Lambda,
\]

where \( W^x_\Lambda \) is an \( SU(2) \) compensator associated with the Killing vector \( k^u_\Lambda \). This can be rewritten, by using the identification between hyperKähler forms and \( SU(2) \) curvatures \( 2.3.7 \), as

\[
\mathcal{L}_\Lambda \Omega^x = \epsilon^{xyz} \Omega^y W^z_\Lambda.
\]

The compensator \( W^x_\Lambda \) necessarily fulfills the cocycle condition:

\[
\mathcal{L}_\Lambda W^x_\Sigma - \mathcal{L}_\Sigma W^x_\Lambda + \epsilon^{xyz} W^y_\Lambda W^z_\Sigma = f^\Gamma_{\Lambda\Sigma} W^x_\Gamma.
\]

In full analogy with the case of Kähler manifolds, to each Killing vector we can associate a triplet \( \mathcal{P}^x_\Lambda(b) \) of Killing potentials. Indeed,

\[
\nabla W^x_\Lambda = \mathcal{L}_\Lambda \omega^x = d(i_\Lambda \omega^x) + i_\Lambda (d\omega^x) = \lambda(i_\Lambda K^x) + \nabla (i_\Lambda \omega^x),
\]

where \( \nabla \) is defined such that, for \( SU(2) \) vector \( V^x \), \( \nabla V^x = dV^x + \epsilon^{xyz} \omega^y V^z \). Therefore, if we set

\[
\mathcal{P}^x_\Lambda \equiv \lambda^{-1}(i_\Lambda \omega^x - W^x_\Lambda),
\]
we obtain
\[ i_{\Lambda} K^{x} = -\nabla P^{x}_{\Lambda} \equiv -(dP^{x}_{\Lambda} + \epsilon^{xy} \omega^{y} P^{x}_{\Lambda}). \tag{3.1.54} \]

If we expand \( X = a^{A} k_{A} \) on a basis of Killing vectors \( k_{A} \) satisfying the commutation relation \( \tag{3.1.48} \), we can set
\[ P^{x}_{X} = a^{A} P^{x}_{A}. \tag{3.1.55} \]

With this basis, we can also rewrite \( \tag{3.1.54} \) as,
\[ i_{X} K^{x} = -\nabla P^{x}_{X}. \tag{3.1.56} \]

This is a generalization of the relation \( \tag{3.1.7} \).

Furthermore, we need a generalization of the equivariance relation obtained by \( \tag{3.1.14} \).

It should be written in terms of \( P^{x}_{X} \) as follows:
\[ X(\mathcal{P}^{x}_{Y}) = \mathcal{P}^{x}_{[X,Y]}, \tag{3.1.57} \]

where the left-hand side of \( \tag{3.1.57} \) is interpreted as
\[ X(\mathcal{P}^{x}_{Y}) = i_{X} \nabla P^{x}_{Y} = X^{u} \nabla_{u} P^{x}_{Y}. \tag{3.1.58} \]

The triholomorphic momentum map

As in the Kähler manifold case, \( \tag{3.1.54} \) defines a momentum map:
\[ \mathcal{P}^{x} : \mathcal{M} \rightarrow \mathbb{R}^{3} \otimes \mathfrak{g}^{*}, \tag{3.1.59} \]

where \( \mathfrak{g}^{*} \) denotes the dual of the Lie algebra \( \mathfrak{g} \) of the group \( G \). Indeed let \( g \in \mathfrak{g} \) be the Lie algebra element corresponding to the Killing vector \( X \), then, for a given \( p \in \mathcal{M} \)
\[ \mu(p) : g \rightarrow \mathcal{P}^{x}_{X}(p) \in \mathbb{R}^{3} \tag{3.1.60} \]

is a linear functional on \( \mathfrak{g} \).

Correspondingly, the triholomorphic Poisson bracket is defined as follows:
\[ \{\mathcal{P}_{\Lambda}, \mathcal{P}_{\Sigma}\}^{x}_{\Lambda} = 2K^{x}(\Lambda, \Sigma) - \lambda \epsilon^{xy} \mathcal{P}^{y}_{\Lambda} \mathcal{P}^{x}_{\Sigma} \tag{3.1.61} \]

Lemma 3.2. The following identity is true,
\[ \{\mathcal{P}_{\Lambda}, \mathcal{P}_{\Sigma}\}^{x} = f_{\Lambda\Sigma}^{\Gamma} \mathcal{P}^{x}_{\Gamma} + C^{x}_{\Lambda\Sigma}, \tag{3.1.62} \]

where \( C^{x}_{\Lambda\Sigma} \) is covariantly constant, namely, \( \nabla C^{x}_{\Lambda\Sigma} = 0 \) and fulfills the cocycle condition
\[ f_{\Lambda\Sigma}^{\Delta} C^{x}_{\Delta\Gamma} + f_{\Sigma\Gamma}^{\Delta} C^{x}_{\Delta\Lambda} + f_{\Gamma\Lambda}^{\Delta} C^{x}_{\Delta\Sigma} = 0. \tag{3.1.63} \]

Proof. It is analogous to the proof of lemma 3.1. The difference is that we have to introduce
covariant exterior derivative and covariant Lie derivative $\tilde{\mathcal{L}}_\Lambda$ instead of the ordinary ones. $\tilde{\mathcal{L}}_\Lambda$ is defined such as for any $SU(2)$ vector $V^x$,

$$\tilde{\mathcal{L}}_\Lambda V^x = \mathcal{L}_\Lambda V^x + \epsilon^{xyz} W^y V^z.$$  \hfill (3.1.64)

With this definition, (3.1.50) is rewritten as

$$\tilde{\mathcal{L}}_\Lambda \Omega^x = 0.$$  \hfill (3.1.65)

Using (3.1.54), we have

$$2 K^x (\Lambda, \Sigma) = i_\Sigma i_\Lambda K^x = \frac{1}{2} (i_\Lambda \nabla \mathcal{P}_\Sigma - i_\Sigma \nabla \mathcal{P}_\Lambda)$$

$$= \frac{1}{2} (\mathcal{L}_\Lambda \mathcal{P}_\Sigma - \mathcal{L}_\Sigma \mathcal{P}_\Lambda + \epsilon^{xyz} (i_\Lambda \omega)^y \mathcal{P}_z^x - \epsilon^{xyz} (i_\Sigma \omega)^y \mathcal{P}_z^x).$$  \hfill (3.1.66)

Thus, we obtain

$$2 K^x (\Lambda, \Sigma) - \lambda \epsilon^{xyz} \mathcal{P}_\Lambda^y \mathcal{P}_\Sigma^z = \frac{1}{2} (\tilde{\mathcal{L}}_\Lambda \mathcal{P}_\Sigma^x - \tilde{\mathcal{L}}_\Sigma \mathcal{P}_\Lambda^x).$$  \hfill (3.1.67)

Since the covariant exterior derivative commutes with the covariant Lie derivative for any $SU(2)$ vector $V^x$, $[\nabla, \tilde{\mathcal{L}}_\Lambda] V^x = 0$, we find

$$\nabla (2 K^x (\Lambda, \Sigma) - \lambda \epsilon^{xyz} \mathcal{P}_\Lambda^y \mathcal{P}_\Sigma^z) = \frac{1}{2} (\tilde{\mathcal{L}}_\Lambda \nabla \mathcal{P}_\Sigma^x - \tilde{\mathcal{L}}_\Sigma \nabla \mathcal{P}_\Lambda^x)$$

$$= \frac{1}{2} (\tilde{\mathcal{L}}_\Lambda i_\Sigma K^x - \tilde{\mathcal{L}}_\Sigma i_\Lambda K^x)$$

$$= -[i_\lambda, \tilde{\mathcal{L}}_\Sigma] K^x$$

$$= f_{\Lambda \Sigma}^\Gamma \nabla \mathcal{P}_\Gamma^x,$$  \hfill (3.1.68)

in the third equality, we have used the fact that $[i_\Lambda, \tilde{\mathcal{L}}_\Sigma] K^x = i_{[\Lambda, \Sigma]} K^x$. Using (3.1.61), we get

$$\nabla (\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x - f_{\lambda \Sigma}^\Gamma \mathcal{P}_\Gamma^x) = 0.$$  \hfill (3.1.69)

Thus, the lemma 3.2 is proved.

If we assume that the second cohomology group is trivial, then we have

$$C_{\Lambda \Sigma}^x = f_{\Lambda \Sigma}^\Gamma C_\Gamma^x,$$  \hfill (3.1.70)

and the constant $C_\Lambda^x$ is absorbed by $\mathcal{P}_\Lambda^x$. Therefore, we obtain the Poissonian realization of the Lie algebra

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x = f_{\lambda \Sigma}^\Gamma \mathcal{P}_\Gamma^x,$$  \hfill (3.1.71)

which, in components, leads to

$$K_{uv}^x, K_{\lambda \Sigma}^x - \frac{1}{2} \epsilon^{xyz} \mathcal{P}_\Lambda^y \mathcal{P}_\Sigma^z = \frac{1}{2} f_{\lambda \Sigma}^\Gamma \mathcal{P}_\Gamma^x.$$  \hfill (3.1.72)

Eq. (3.1.72), which is the most convenient way of expressing equivariance relation in a coordinate basis, plays a fundamental role in the construction of the supersymmetric action, supersymmetry transformation rules and of the superpotential for $N = 2$ supergravity on a general quaternionic manifold.
3.2 Gauging of the composite connections

On the special and quaternionic manifold, we have introduced several connection 1-forms related with different bundles. Gauging the corresponding supergravity theory is done by gauging these composite connections of the underlying $\sigma$-model.

For the Levi-Civita connection the gauging is standard on Kähler manifold $\mathcal{M}$. Let $k^a_\Lambda(z)$ be the Killing vectors which are defined in the last section. The ordinary differential $dz^a$ is replaced by the covariant differential defined as

$$dz^a \rightarrow \nabla z^a = dz^a + gA^\Lambda k^a_\Lambda(z), \quad (3.2.1)$$

$$dz^{a*} \rightarrow \nabla z^{a*} = dz^{a*} + gA^\Lambda k^{a*}_\Lambda(z), \quad (3.2.2)$$

where $g$ is the gauge coupling constant and $A^\Lambda$ is the gauge 1-form. The Levi-Civita connection $\Gamma^a_b = \Gamma^a_{bc} dz^c$ is replaced by its gauged counterpart as

$$\Gamma^a_b \rightarrow \hat{\Gamma}^a_b = \Gamma^a_{bc} \nabla z^c + gA^\Lambda \partial_b k^a_\Lambda, \quad (3.2.3)$$

$$\Gamma^{a*}_b \rightarrow \hat{\Gamma}^{a*}_b = \Gamma^{a*}_{b,c} \nabla z^{c*} + gA^\Lambda \partial_b k^{a*}_\Lambda. \quad (3.2.4)$$

The gauged curvature 2-form is

$$\hat{\mathcal{R}}^a_b = d\hat{\Gamma}^a_b + \hat{\Gamma}^a_c \wedge \hat{\Gamma}^c_b = R^a_{bc} \nabla z^c \wedge \nabla z^d + gF^\Lambda \partial_b k^a_\Lambda(z), \quad (3.2.5)$$

where $F^\Lambda$ is the gauged field strength:

$$F^\Lambda = dA^\Lambda + \frac{1}{2} g f^\Lambda_{\Sigma} A^\Sigma \wedge A^\Gamma. \quad (3.2.6)$$

In an analogous way, the gauging of the $Sp(2k)$ connection gives

$$\Delta^\alpha\beta \rightarrow \hat{\Delta}^\alpha\beta = \Delta^\alpha\beta + gA^\Lambda \partial_u k^u_\Lambda U^\alpha U^\beta_{|v}, \quad (3.2.7)$$

and the associated gauged curvature $\hat{\mathbb{R}}$ becomes

$$\hat{\mathbb{R}}^{\alpha\beta} = \mathbb{R}^{\alpha\beta} + \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \hat{\Delta}^{\gamma\delta} = \mathbb{R}^{\alpha\beta} + db^u + gF^\Lambda \partial_u k^u_\Lambda U^\alpha U^\beta_{|v}. \quad (3.2.8)$$

The existence of the Killing potentials allow the following definitions for the $U(1)$ connection and the $SU(2)$ connection:

$$\mathcal{Q} \rightarrow \hat{\mathcal{Q}} = \mathcal{Q} + gA^\Lambda \mathcal{P}^0_\Lambda, \quad (3.2.9)$$

$$\omega^x \rightarrow \hat{\omega}^x = \omega^x + gA^\Lambda \mathcal{P}^x_\Lambda. \quad (3.2.10)$$

By computing the associated gauged curvatures, one finds the gauge covariant expressions:

$$\hat{\mathcal{K}} = d\hat{\mathcal{Q}} = ig_{ab} \nabla z^a \wedge \nabla z^b + gF^\Lambda \mathcal{P}^0_\Lambda, \quad (3.2.11)$$

$$\hat{\Omega}^x = d\hat{\omega}^x + \frac{1}{2} \epsilon^{xuyz} \hat{\omega}^y \wedge \hat{\omega}^z = \Omega^x \nabla b^u \wedge \nabla b^v + gF^\Lambda \mathcal{P}^x_\Lambda. \quad (3.2.12)$$
The Lagrangian of gauged $\mathcal{N} = 2$ supergravity

In this chapter we introduce the gauged $\mathcal{N} = 2$ supergravity Lagrangian and the supersymmetry transformation laws in terms of the geometric quantities which have been given in the last two chapters.

In [35, 31, 21, 22, 32, 23, 36], the Lagrangian of $\mathcal{N} = 2$ supergravity was constructed. In particular, the Lagrangian of gauged $\mathcal{N} = 2$ supergravity was introduced in [3, 37, 4].

4.1 The Lagrangian of gauged $\mathcal{N} = 2$ supergravity

The complete Lagrangian can be found in [4]. It is constructed such that its equations of motion are consistent to the solutions of the Bianchi identities. We will discuss with the Bianchi identities and its solutions in the appendix B. The construction of the Lagrangian is very complicated and tedious. Here we do not see the derivation of the complete Lagrangian. The general derivation of the supergravity Lagrangian is given in [38]. Also, [24, 3] will help you deriving the complete Lagrangian.

The gauged $\mathcal{N} = 2$ supergravity action is given by

$$S = \int d^4x \sqrt{-g} \left( \mathcal{L}_\text{kin} + \mathcal{L}_\text{Pauli} + \mathcal{L}_\text{non-\text{\textit{inv}}} + \mathcal{L}_\text{\textit{inv}} + \mathcal{L}_\text{Yukawa} - V(z, \bar{z}, b) \right).$$

(4.1.1)

$\mathcal{L}_\text{kin}$ consists of the kinetic terms of the component fields and is written as,

$$\mathcal{L}_\text{kin} = R + g_{ab} \nabla_\mu z^a \nabla_\mu \bar{z}^b + h_{uv} \nabla_\mu b^u \nabla_\mu b^v + \frac{\epsilon^{\mu \nu \lambda \sigma}}{\sqrt{-g}} (\bar{\psi}_A^\mu \gamma_\nu \nabla_\lambda \psi_{A\sigma} - \bar{\psi}_{A\mu} \gamma_\nu \nabla_\lambda \psi_A^\sigma + \frac{1}{4} (\text{Im} \mathcal{N})_{\lambda \Sigma} F_{\mu \nu}^A F_{\Sigma}^{\mu \nu} + \frac{1}{4} (\text{Re} \mathcal{N})_{\lambda \Sigma} F_{\mu \nu}^A \tilde{F}_{\Sigma}^{\mu \nu}$$

$$- ig_{ab} \left( \bar{\lambda}^a A^\mu \gamma_\mu \lambda_{A}^b + \bar{\lambda}^a A^\mu \gamma_\mu \lambda_{A}^a \right) - 2i (\bar{\zeta}^\alpha \gamma_\mu \nabla_\mu \zeta_\alpha + \bar{\zeta}^\alpha \gamma_\mu \nabla_\mu \zeta_\alpha) + i g_{ab} \nabla_\mu \bar{z}^b \bar{\lambda}^a A^\mu \psi_{A\nu} - g_{ab} \nabla_\mu \bar{z}^b \bar{\psi}_A^\mu \lambda_{A}^a + 2U_{u}^A \nabla_\mu b^u \zeta_{A\nu} \psi_{A\nu} - 2U_{u}^A \nabla_\mu b^u \bar{\psi}_A^\mu \zeta_{A} + h.c.) \right].$$

(4.1.2)
and $\lambda^{aA}$ are, respectively, scalar fields and $SU(2)$ doublet gaugini of the vector multiplets. On the other hand, $b^\mu$ and $c^a$ are scalar fields and $SU(2)$ doublet hyperinui of the hypermultiplets. The $SU(2)$ doublet gravitini are represented by $\psi^A_\mu$. The field strengths $F_{\mu\nu}^A$ of the gauge fields $A_\mu^a$ and the graviphoton field $A_\mu^0$ are

$$ F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f_{\Sigma\Gamma}^A A_\mu^\Sigma A_\nu^\Gamma, \quad (4.1.3) $$

and $\tilde{F}^A_{\mu\nu}$ are their Hodge duals:

$$ \tilde{F}_{\mu\nu}^A = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} F^{A\lambda\sigma}. \quad (4.1.4) $$

The normalization of the kinetic terms for the hypermultiplet scalar fields $b^\mu$ depend on the constant $\lambda$ of the quaternionic Kähler manifold. We have chosen that $\lambda = 1$ because of the positivity of the kinetic terms.

Although we need not the explicit forms of $\mathcal{L}_{\text{Pauli}}$, $\mathcal{L}^\text{inv}_{4\text{Fermi}}$ and $\mathcal{L}^\text{non--inv}_{4\text{Fermi}}$, we give them for completeness:

$$ \mathcal{L}_{\text{Pauli}} = F_{\mu\nu}^{-A}(\text{Im} \mathcal{N})_{\Lambda\Sigma} [2 L^A \psi^A_\mu \psi^B_\nu \epsilon_{AB} - 2 i f_{A}^{\Sigma} \bar{\chi}^a_\mu \gamma^\nu \psi^B_\mu \epsilon_{AB} + 1 4 \nabla_\beta \bar{\psi}^A_\mu \gamma_{\beta}\psi^B_{\nu} \epsilon_{AB} - \frac{1}{2} L^\Sigma \bar{\zeta}_\alpha \gamma^\mu \zeta_\beta C^{\alpha\beta} ] + h.c., \quad (4.1.5) $$

$$ \mathcal{L}^\text{inv}_{4\text{Fermi}} = \frac{i}{2} \left( g_{ab} \bar{\chi}^a_\alpha \gamma_\sigma B^b_\Lambda - 2 A^A_B \bar{\chi}^a_\alpha \gamma_\sigma D^b_\Lambda \epsilon_{AC} \epsilon_{BD} + h.c. \right) \bar{\psi}^A_\mu \gamma_\alpha \psi^B_\nu \epsilon_{AB} \bar{\psi}^A_\mu \gamma_\rho \psi^B_\nu \epsilon_{AB}, $$

$$ \mathcal{L}^\text{non--inv}_{4\text{Fermi}} = (\text{Im} \mathcal{N})_{\Lambda\Sigma} [2 L^A \epsilon_{AB} \bar{\psi}^A_\mu \psi^B_\nu - (\bar{\psi}^A_\mu \psi^B_\nu)^{-}\epsilon_{AB} \epsilon_{CD}] $$

$$ - 8 i L^A f_{AB}^\Sigma (\bar{\psi}^A_\mu \psi^B_\nu)^{-}(\bar{\chi}^a_\alpha \gamma_\mu \psi^B_\nu)^{-} $$

$$ - 2 f_{AB}^{1 4} (\bar{\chi}^a_\alpha \gamma_\mu \psi^B_\nu)^{-}(\bar{\chi}^b_\alpha \gamma_\rho \psi^B_\nu)^{-} \epsilon_{AB} \epsilon_{CD} $$

$$ + \frac{i}{2} L^A f_{AB}^\Sigma \epsilon^{\alpha\beta} C_{abc} (\bar{\psi}^A_\mu \psi^B_\nu) - \bar{\chi}^a_\alpha \gamma_\mu \lambda_\beta \epsilon_{AB} \epsilon_{CD} \quad (4.1.6) $$

\footnote{We have redefined the field strengths $F^\lambda_{\mu\nu}$ in \cite{BZ2} in the appendix B as $F^\lambda_{\mu\nu} \rightarrow \frac{1}{2} F^\lambda_{\mu\nu}$.}
where we have used, \( F_{\mu\nu}^{\pm A} = \frac{1}{2}(F_{\mu\nu}^{A} \pm \frac{i}{2} \epsilon^{\mu\nu\lambda\sigma} F_{\lambda \sigma}^{A}) \). Also, \((\ldots)^{-}\) denotes the self dual part of the fermion bilinears.

Because of the gauging, we obtain the following Yukawa coupling terms which include the mass terms of the fermions and scalar potential terms:

\[
L_{\text{Yukawa}} = g^2 [S_{AB} \bar{a}_{\mu B} \bar{\psi}^\mu B + ig_{ab} W^{aAB} \bar{\psi}^\mu \lambda^\mu B + 2i N_{A}^{\lambda} \bar{\psi}^\lambda \gamma_{\mu} \psi^\mu_{A} + M^{\alpha \beta} \bar{\psi}^\lambda \gamma_{\mu} \psi^\mu_{A} + M_{aB} \bar{\lambda}^a \lambda^B + M_{aA[B} \bar{\lambda}^{aA} \Lambda^{B]} + h.c.,
\]

\[
V(z, \bar{z}, b) = g^2 [g_{ab} k^a \bar{k}^b \bar{L}^A \bar{L}^\Sigma + \bar{g}_{ab}^{\mu} f_{a}^{\Sigma} \bar{f}_{b}^{\Lambda} \bar{P}_{\Lambda} \bar{P}_{\Sigma} + 4 h_{uv} k^u \bar{k}^v \bar{L}^A \bar{L}^\Sigma - 3 \bar{L}^{A} \bar{L}^{\Sigma} \bar{P}^{\rho}_{\Lambda} \bar{P}^{\rho}_{\Sigma}].
\]

The coupling constant \( g \) in \( L_{\text{Yukawa}} \), \( V \) is a symbolic notation to remind that these terms are produced by the gauging. Therefore, there is no Yukawa terms and the scalar potential term in the ungauged theory \( (g = 0) \).

\( L_{\text{Yukawa}} \) are written by the following matrices,

\[
S_{AB} = \frac{i}{2} (\sigma_{x})_{AB} \bar{P}^{\Lambda}_{\Lambda} \bar{L}^{\Sigma},
\]

\[
W^{aAB} = \epsilon^{AB} \bar{k}^{a} \bar{L}^{\Lambda} + i (\sigma_{x})^{AB} \bar{P}^{\rho}_{\rho} \bar{g}^{ab} \bar{f}^{b}_{b}^{\Lambda},
\]

\[
N_{\alpha}^{A} = 2 \bar{U}_{\alpha u} \bar{k}^{u} \bar{L}^{\Lambda},
\]

\[
M^{\alpha \beta} = - \bar{U}_{\alpha u} \bar{U}_{\beta v} \bar{\epsilon}_{AB} \bar{\nabla}^{[u} \bar{k}^{v]} \bar{L}^{\Lambda},
\]

\[
M_{bB}^{a} = - 4 \bar{U}_{B b} \bar{k}^{u} \bar{f}^{a}_{\alpha},
\]

\[
M_{aA[B} = \frac{1}{2} \left( \epsilon_{AB} \bar{g}_{ac} \bar{k}^{c} \bar{f}^{a}_{a}^{\Lambda} + i (\sigma_{x})_{AB} \bar{P}^{\rho}_{\rho} \bar{\nabla}_{b} \bar{f}^{b}_{a}^{\Lambda} \right).
\]

In the subsequent chapter, it is convenient to divide \( W^{aAB} \) and \( M_{aA[B} \) into two parts
respectively such that

\[ W_1^{aAB} = \epsilon^{AB} k^a_\alpha \bar{L}_\alpha, \]  
(4.1.16) 
\[ W_2^{aAB} = ((\sigma_x)^{AB} P^x g^{\alpha\beta} f^\beta_\mu), \]  
(4.1.17) 
\[ \mathcal{M}_{1;\alpha A|bB} = \frac{1}{2} \epsilon_{AB g\alpha c} k^c_\alpha f^\beta_\mu, \]  
(4.1.18) 
\[ \mathcal{M}_{2;\alpha A|bB} = \frac{i}{2} (\sigma_x)^{AB} P^x \nabla_b f^\alpha_a, \]  
(4.1.19) 

In the following, we refer to these matrices as mass matrices.

The covariant derivatives are defined as follows:

\[ \nabla_\mu \psi_{\lambda\nu} = \partial_\mu \psi_{\lambda\nu} - \frac{1}{4} \gamma_{ij} \omega_{\mu}^{ij} \psi_{\lambda\nu} + \frac{i}{2} \hat{Q}_\mu \psi_{\lambda\nu} + \hat{\omega}_{A\mu} \psi_{B\nu}, \]  
(4.1.20a) 
\[ \nabla_{\mu} \psi^A = \partial_{\mu} \psi^A - \frac{1}{4} \gamma_{ij} \omega_{\mu}^{ij} \psi^A - \frac{i}{2} \hat{Q}_\mu \psi^A + \hat{\omega}_B \psi^B, \]  
(4.1.20b) 
\[ \nabla_{\mu} z^a = \partial_{\mu} z^a + g A^A_{\mu} k^a_\alpha, \]  
(4.1.20c) 
\[ \nabla_{\mu} \lambda^a A = \partial_{\mu} \lambda^a A - \frac{1}{4} \gamma_{ij} \omega_{\mu}^{ij} \lambda^a A - \frac{i}{2} \hat{Q}_\mu \lambda^a A + \hat{\Gamma}_B \lambda^B + \hat{\omega}_B \lambda^B, \]  
(4.1.20d) 
\[ \nabla_{\mu} \lambda^a A = \partial_{\mu} \lambda^a A - \frac{1}{4} \gamma_{ij} \omega_{\mu}^{ij} \lambda^a A^a + \frac{i}{2} \hat{Q}_\mu \lambda^a A + \hat{\Gamma}_B \lambda^B + \hat{\omega}_B \lambda^B, \]  
(4.1.20e) 
\[ \nabla_{\mu} b^a = \partial_{\mu} b^a + g A^A_{\mu} k^a_\alpha, \]  
(4.1.20f) 
\[ \nabla_{\mu} \xi_\alpha = \partial_{\mu} \xi_\alpha - \frac{1}{4} \gamma_{ij} \omega_{\mu}^{ij} \xi_\alpha - \frac{i}{2} \hat{Q}_\mu \xi_\alpha + \hat{\Gamma}^\beta_{\alpha} \xi_\beta, \]  
(4.1.20g) 
\[ \nabla_{\mu} \xi_\alpha = \partial_{\mu} \xi_\alpha - \frac{1}{4} \gamma_{ij} \omega_{\mu}^{ij} \xi_\alpha + \frac{i}{2} \hat{Q}_\mu \xi_\alpha + \hat{\Gamma}^\beta_{\alpha} \xi_\beta. \]  
(4.1.20h)

In the above equations, \( \hat{Q}_\mu, \hat{\Gamma}^a_{\beta\mu}, \hat{\omega}_A^B \) and \( \hat{\Gamma}_B \) are the space-time components of the gauged connections which have introduced at the end of the last chapter:

\[ \hat{Q}_\mu = -\frac{i}{2} (\partial_{\mu} K \partial_{\mu} z^a - \partial_{\alpha} K \partial_{\mu} z^a), \]  
(4.1.21a) 
\[ \hat{\Gamma}^a_{\beta\mu} = \Gamma^a_{bc} \partial_{\mu} z^c + g A^A_{\mu} \partial_{\mu} k^a_\alpha, \]  
(4.1.21b) 
\[ \hat{\omega}_A^B = \frac{i}{2} \sigma_x \hat{F}_A^B = \frac{i}{2} (\omega_{\mu} \partial_{\mu} b^a + g A^A_{\mu} P^x) (\sigma_x)^B_A, \]  
(4.1.21c) 
\[ \hat{\Gamma}_B = \Delta_\alpha^a \partial_{\mu} b^a + g A^A_{\mu} \partial_{\mu} k^a_\alpha U^a|A\alpha U^\beta_A. \]  
(4.1.21d)

where \( \omega_{\mu}^{ij} \) is the spin-connection. Also, the coupling constant \( g \) in the covariant derivatives is a symbolic notation which comes from the gauging. If we consider the ungauged theory, these terms are vanish and all the gauged covariant derivatives reduce to ordinary ones.

### 4.2 The supersymmetry transformation laws

In order to discuss whether supersymmetry are broken or not in the subsequent chapter, we need the supersymmetry transformation laws of all the fermions. Let \( \epsilon^A \) be the infinitesimal
parameter of the supersymmetry transformations. The supersymmetry transformation laws of the fermion fields are given by

\[ \delta \psi_{\alpha A} = \mathcal{D}_\mu \epsilon_A - \frac{1}{4} \left( \partial_\mu \kappa \bar{\chi}^a_{\alpha B} \epsilon_B - \partial_\mu \bar{\lambda}^a_{\alpha B} \epsilon_B \right) \psi_{\alpha A} + \omega_{AB} \epsilon_C \left( \epsilon^{CD} \bar{\psi}_{C_{\beta D}} + \bar{\zeta}_{\alpha} \epsilon^C \right) \psi_{B \mu} + (A^\mu_b \eta_m + A^\mu_\nu \frac{\gamma_{\mu \nu}}{2}) \epsilon_B + \epsilon_{AB} (T_{\mu \nu}^{-} + U_{\mu \nu}^{+}) \epsilon^B + i g S_{AB} \eta_{\mu \nu} \epsilon^B \]

\[ \delta \lambda^a = \frac{1}{4} \left( \partial_\mu \kappa \bar{\chi}^a_{\alpha B} \epsilon_B - \partial_\mu \bar{\lambda}^a_{\alpha B} \epsilon_B \right) \lambda^a A 
- \omega^A_{\alpha B} \epsilon_C \left( \epsilon^{CD} \bar{\psi}_{C_{\beta D}} + \bar{\zeta}_{\alpha} \epsilon^C \right) \lambda^a B 
- \Gamma^a_{bc} \bar{\chi}^a_{\beta B} (\epsilon^b \epsilon^c \gamma^B + \bar{\zeta}^a \gamma^B) \right) \lambda^a B 
+ i \left( U^B_{\beta \alpha} \epsilon_B - \epsilon^{BC} \bar{\psi}_{C_{\gamma B}} \bar{\zeta} + \bar{\zeta}^a \psi^B \right) \gamma^B \epsilon^A \epsilon_{AB} \epsilon_C \alpha \beta 
+ g \lambda^a \epsilon_A. \]

The supersymmetry transformation laws of the bosons are

\[ \delta e^i = -i \bar{\psi}_{\alpha A} \gamma^i \epsilon_A - i \bar{\psi}_A \gamma^i \epsilon_A, \]

\[ \delta A^\mu = 2 L A \bar{\psi}_{\alpha A} \epsilon_B \epsilon_{AB} + 2 L A \bar{\psi}_{\alpha A} \epsilon_B \epsilon_{AB} + i (f^A_{\alpha \beta} \epsilon_B \epsilon_{AB} + f^A_{\alpha \beta} \epsilon_B \epsilon_{AB}), \]

\[ \delta \alpha^a = \bar{\lambda}^a \epsilon_A, \]

\[ \delta b^a = U_{\alpha A} (\bar{\zeta}^a \epsilon^A + \epsilon^{A B} \bar{\zeta} B \epsilon_B), \]

where

\[ A^\mu_A = -i g_a^b \left( \bar{\lambda}^a_{\beta A} \gamma^\mu \lambda^b_{\beta} - \delta^a_{\beta A} \gamma^\mu \lambda^b_{\beta} \right), \]

\[ A^\mu_A = i g_a^b \left( \bar{\lambda}^a_{\beta A} \gamma^\mu \lambda^b_{\beta} - \delta^a_{\beta A} \gamma^\mu \lambda^b_{\beta} \right) - i \frac{a}{4} \delta^B \bar{\zeta}^a \epsilon^\beta \epsilon_B, \]

\[ T_{\mu \nu}^- = i (\text{Im} \mathcal{N}) \lambda \Sigma L^A \left( \bar{F}_{\mu \nu}^\Sigma - \frac{1}{4} \nabla_a f^\Sigma_{\alpha A} \bar{\lambda}^a_{\alpha B} \epsilon_B \epsilon_{AB} \right) \]

\[ T_{\mu \nu}^+ = i (\text{Im} \mathcal{N}) \lambda \Sigma L^A \left( \bar{F}_{\mu \nu}^\Sigma + \frac{1}{4} \nabla_a f^\Sigma_{\alpha A} \bar{\lambda}^a_{\alpha B} \epsilon_B \epsilon_{AB} \right). \]
\[ U_{\mu
u}^- = -\frac{i}{4} C^{\alpha\beta} \bar{\zeta}_\alpha \gamma_{\mu\nu} \zeta_\beta, \quad (4.2.10a) \]
\[ U_{\mu
u}^+ = -\frac{i}{4} C_{\alpha\beta} \bar{\zeta}^\alpha \gamma_{\mu\nu} \zeta^\beta, \quad (4.2.10b) \]
\[ G_{\mu
u}^- = -\frac{1}{2} g^{ab} \tilde{f}_{\mu
u} (\text{Im} \mathcal{N}) \Sigma A \left( \tilde{F}_{\mu
u}^\Lambda + \frac{1}{4} \nabla_a \bar{f}_b \tilde{\lambda}^a \bar{\gamma}_{\mu\nu} \lambda^b \epsilon_{AB} - \frac{1}{2} C^{\alpha\beta} \bar{\zeta}_\alpha \gamma_{\mu\nu} \zeta_\beta L^A \right), \quad (4.2.11a) \]
\[ G_{\mu
u}^+ = -\frac{1}{2} g^{ab} \tilde{f}_{\mu
u} (\text{Im} \mathcal{N}) \Sigma A \left( \tilde{F}_{\mu
u}^{\Lambda^+} + \frac{1}{4} \nabla^c \tilde{f}_b \tilde{\lambda}^c \bar{\gamma}_{\mu\nu} \lambda^d \epsilon^{AB} - \frac{1}{2} C_{\alpha\beta} \bar{\zeta}^\alpha \gamma_{\mu\nu} \zeta^\beta L^A \right), \quad (4.2.11b) \]
\[ Y_{\mu
u}^{AB} = i \frac{1}{2} g^{ab} C_{b^c d^e} \bar{\lambda}^c \bar{\lambda}^d \epsilon^{AC} \epsilon^{BD}. \quad (4.2.12) \]

In the above equations, \( \tilde{F}_{\mu
u}^\Lambda \) are the supercovariant field strengths defined by:
\[ \tilde{F}_{\mu\nu}^\Lambda = F_{\mu\nu}^\Lambda + \bar{L}^A \bar{\psi}_\mu \psi_\nu \epsilon_{AB} + \bar{L}^\Lambda \bar{\psi}_{A\mu} \psi_{B\nu} \epsilon^{AB} \]
\[ -i \bar{f}_a \tilde{\lambda}^a \bar{\gamma}_{\nu \rho} \epsilon_{AB} - i \bar{f}_a \tilde{\lambda}^a \bar{\gamma}_{\nu \rho} \epsilon^{AB}. \quad (4.2.13) \]
Chapter 5

Partial supersymmetry breaking in the $\mathcal{N} = 2$ $U(1)$ gauged model

The simplest realization of the partial breaking of $\mathcal{N} = 2$ local supersymmetry has been discussed in reference [12]. The model includes a $U(1)$ vector multiplet and a hypermultiplet. In this chapter, we review this model and it will be generalized to $U(N)$ gauged model in next chapter.

First of all, the parametrizations of the vector multiplet and the hypermultiplet are given in section 5.1. In section 5.2, we see that the $\mathcal{N} = 2$ local supersymmetry is broken to $\mathcal{N} = 1$. This is confirmed by the appearance of a Nambu-Goldstone fermion and the mass spectrum which is computed in section 5.3. Finally, in section 5.4, we summarize the results and discuss its applications.

5.1 $\mathcal{N} = 2$ $U(1)$ gauged supergravity model

5.1.1 Vector multiplet

The $U(1)$ vector multiplet contains a complex scalar field $z$ which spans the special Kähler manifold of complex dimension 1. We start from the case in which the holomorphic prepotential $F(X^0, X^1)$ exists. Since the prepotential $F(X)$ is a homogeneous of second degree of the coordinates $X^\Lambda(z)$ ($\Lambda = 0, 1$), we can write $F$ as follows,

$$F(X^0, X^1) = (X^0)^2 F(X^1/X^0). \tag{5.1.1}$$

We can evaluate $F_\Lambda$ by taking the derivative of $F$ with respect to $X^\Lambda$,

$$F_0 = 2X^0 F(X^1/X^0) - X^1 \frac{\partial}{\partial(X^1/X^0)} F(X^1/X^0),$$

$$F_1 = X^0 \frac{\partial}{\partial(X^1/X^0)} F(X^1/X^0). \tag{5.1.2}$$

It is natural to choose the coordinate $X^\Lambda(z)$ linearly independent:

$$X^0(z) = \frac{1}{\sqrt{2}}, \quad X^1(z) = \frac{1}{\sqrt{2}} z. \tag{5.1.3}$$

To be precise, the gauge symmetry is $U(1) \times U(1)_{\text{graviphoton}}$. 

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Substituting (5.1.3) into (5.1.2), we have
\[ F_0(z) = \frac{1}{\sqrt{2}} \left( 2F(z) - z \frac{\partial F(z)}{\partial z} \right), \]
\[ F_1(z) = \frac{1}{\sqrt{2}} \frac{\partial F(z)}{\partial z}. \] (5.1.4)

To specify the model, we have to choose a particular form of the holomorphic function \( F(z) \). Our choice, here, is
\[ F(z) = -iz. \] (5.1.5)

This is the simplest choice of the function \( F(z) \): it corresponds to consider the microscopic (or bare) theory. The holomorphic section \( \Omega(z) \) can be written as,
\[ \Omega(z) = \left( \begin{array}{c} X^\Lambda \\ F_\Lambda \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ z \\ -iz \\ -i \end{array} \right). \] (5.1.6)

We can easily compute the Kähler potential (2.2.5) and its derivative:
\[ K = -\log(z + \bar{z}), \]
\[ \partial_z K = -(z + \bar{z})^{-1}. \] (5.1.7) (5.1.8)

By using above equations, the Kähler metric and the Levi-Civita connection (2.1.6) are
\[ g_{zz} = \partial_z \partial_{\bar{z}} K = (z + \bar{z})^{-2}, \]
\[ \Gamma_{zz}^x = -g^{zz'} \partial_z g_{zz'} = 2(z + \bar{z})^{-1}, \] (5.1.9) (5.1.10)
where we have used the shorthand notation: \( \partial_z = \partial/\partial z \) and \( \partial_{\bar{z}} = \partial/\partial \bar{z} \).

Now perform the symplectic transformation,
\[ \Omega \rightarrow \hat{\Omega} = S \Omega = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ z \\ -iz \\ -i \end{array} \right), \] (5.1.11)
where \( S \in Sp(4, \mathbb{R}) \). After this mapping, the transformed section \( \hat{\Omega} \) can no longer be written in the standard form with the prepotential \( \hat{F} \), because the last two components clearly can not be written as functions of the first two. Thus no prepotential \( \hat{F}(\hat{X}^0, \hat{X}^1) \) exists. We still have the same Kähler manifold (5.1.7)-(5.1.10) ††, but with different couplings of the scalar fields to the vectors and to the fermions.

In the following, we use the transformed section (5.1.11) and omit tilde: (5.1.11)
\[ X^\Lambda = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ i \end{array} \right), \quad F_\Lambda = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -iz \\ z \end{array} \right). \] (5.1.12)

††The Kähler potential \( K \) has been defined by the symplectic invariant way. Therefore, the Kähler metric and the Levi-Civita connection are same.
For future reference, let us evaluate $f_\Lambda^z$ and $h_{\Lambda|z}$.
Substituting (5.1.7)-(5.1.11) into (2.2.8), we obtain

\begin{align}
\label{5.1.13}
f_\Lambda^z &= e^{K/2}(\partial_z + \partial_z K)X^\Lambda \\
&= -\frac{1}{\sqrt{2}}(z + \bar{z})^{-3/2}\begin{pmatrix} 1 \\ i \end{pmatrix},
\end{align}

\begin{align}
\label{5.1.14}
h_{\Lambda|z} &= e^{K/2}(\partial_z + \partial_z K)F_\Lambda \\
&= \frac{1}{\sqrt{2}}(z + \bar{z})^{-3/2}\begin{pmatrix} -i \\ 1 \end{pmatrix},
\end{align}

Furthermore, the covariant derivative of $f_\Lambda^z$ can be obtained by substituting (5.1.7)-(5.1.11) into (2.2.14),

\begin{align}
\nabla_z f_\Lambda^z &= \partial_z f_\Lambda^z + \frac{1}{2}\partial_z K f_\Lambda^z + \Gamma^z_{zz} f_\Lambda^z \\
&= 0.
\end{align}

If we compare (5.1.15) with (2.2.16), we obtain

\begin{equation}
C_{zzz} = 0.
\end{equation}

This is because $F$ has been chosen as the simplest function.

### 5.1.2 Hypermultiplet

Before going to details, we must explain why one hypermultiplet is needed to break half supersymmetry. This discussion is same for the $U(N)$ gauged case in the next chapter. If $\mathcal{N} = 2$ local supersymmetry is broken to $\mathcal{N} = 1$ spontaneously, one gravitino remains massless but the other becomes massive by super-Higgs mechanism. So, it is important to note that in addition to super-Higgs mechanism Higgs mechanism must occur. Since $\mathcal{N} = 1$ supersymmetry is manifest, the massive gravitino forms $\mathcal{N} = 1$ massive multiplet of spin $(3/2, 1, 1, 1/2)$. Thus two gauge bosons, that is $U(1)$ gauge boson and graviphoton, must become massive by absorbing the scalar fields, in other words, Higgs mechanism must happen so as to keep $\mathcal{N} = 1$ supersymmetry. Therefore we need at least one hypermultiplet and at least two $U(1)$ translational isometries which provide Higgs mechanism.

In this chapter (and in the next chapter), we take the same parametrizations as those of [12, 13, 19, 20]. The $SU(2)$ connection $\omega^x$ and the $SU$ curvature (2.3.7) are parametrized as,

\begin{align}
\omega^x_y &= \frac{1}{b^1}\delta^x_u, \\
\Omega^x_u &= -\frac{1}{2(b^0)^2}\delta^x_u, \\
\Omega^x_{yz} &= \frac{1}{2(b^0)^2}\epsilon^{xyz}.
\end{align}

Furthermore, by using (2.3.20), the metric $h_{uv}$ of this manifold is given by

\begin{equation}
h_{uv} = \frac{1}{2(b^0)^2}\delta_{uv}.
\end{equation}
In order to write down the fermion mass matrices and the supersymmetry transformation laws, we also need the symplectic vielbein $U_{u}^{\alpha A} db^u$ ($\alpha, A = 1, 2$). It leads:

$$U_{u}^{\alpha A} = \frac{1}{2 \theta^b} \epsilon^{\alpha \beta} (db^0 - i \sigma^x db^x)^A_{\beta}.$$  (5.1.19)

where $\sigma^x$ are the standard Pauli matrices **.

The metric (5.1.18) is invariant under arbitrary constant translations of the coordinates $b^1, b^2, b^3$ because it does not depend on $b^1, b^2, b^3$. As we have seen above, in order for the $\mathcal{N} = 2$ supersymmetry to be broken partially, we have to gauge two of the isometries. Here we choose the following translational isometries of two coordinates $b^2, b^3$ as the gauged isometries:

$$b^2 \rightarrow b^2 + \epsilon^2 (g_2 + g_3),$$  (5.1.20)

$$b^3 \rightarrow b^3 + \epsilon^3 g_1,$$  (5.1.21)

where $g_1, g_2, g_3, \in \mathbb{R}$. The Killing vectors $k^u_{\Lambda}$ which generate these isometries can be written as follows:

$$k^u_0 = g_1 \delta^u_3 + g_2 \delta^u_2,$$  (5.1.22)

$$k^u_1 = g_3 \delta^u_2,$$  (5.1.23)

and the associated Killing potentials $P^x_{\Lambda}$ for these vectors are

$$P^x_0 = \frac{1}{b^0} (g_1 \delta^x_3 + g_2 \delta^x_2),$$  (5.1.24)

$$P^x_1 = \frac{1}{b^0} g_3 \delta^x_2.$$  (5.1.25)

### 5.2 Partial supersymmetry breaking

In this section, we see that the local $\mathcal{N} = 2$ supersymmetry is broken to $\mathcal{N} = 1$ in the simple model which has been given in the last section. This will be confirmed by the appearance of a Nambu-Goldstone fermion: the fermion whose supersymmetry transformation on the vacuum is non-zero. Since the supersymmetry transformation laws of the fermions are mainly written by the mass matrices, let us evaluate it here. By substituting the parametrizations in the last section into (4.1.10)-(4.1.19), we have

$$S_{AB} = -\frac{ie^{K/2}}{2\sqrt{2b^0}} \begin{pmatrix} ig_2 + g_3 & g_1 \\ g_1 & ig_2 + g_3 \end{pmatrix},$$  (5.2.1)

$$W^z_{1AB} = 0,$$  (5.2.2)

$$W^z_{2AB} = -\frac{e^{K/2}}{\sqrt{2b^0}} (z + \bar{z}) \begin{pmatrix} g_2 + ig_3 & ig_1 \\ ig_1 & g_2 + ig_3 \end{pmatrix},$$  (5.2.3)

**The notations are explained in appendix A.
\[
N^{A}_{\alpha} = \frac{ie^{K/2}}{\sqrt{2}b^0} \begin{pmatrix} g_1 & -ig_2 + g_3 \\ ig_2 - g_3 & -g_1 \end{pmatrix},
\]
(5.2.4)

\[
\mathcal{M}^{\alpha\beta} = \frac{ie^{K/2}}{\sqrt{2}b^0} \begin{pmatrix} -ig_2 - g_3 & g_1 \\ g_1 & -ig_2 - g_3 \end{pmatrix},
\]
(5.2.5)

\[
\mathcal{M}^{\alpha}_{zB} = \sqrt{2}ie^{K/2}b^0(z + \bar{z})^{-1} \begin{pmatrix} g_1 & ig_2 + g_3 \\ -ig_2 - g_3 & -g_1 \end{pmatrix},
\]
(5.2.6)

\[
\mathcal{M}_{1;A|zB} = 0,
\]
(5.2.7)

\[
\mathcal{M}_{2;A|zB} = 0.
\]
(5.2.8)

where (5.2.8) comes from (5.1.15)

The scalar potential

The scalar potential \( V(z, \bar{z}, b) \) is given by (4.1.9). If we substitute the parametrizations in the last section into \( V \), we have

\[
V(z, \bar{z}, b) = g^{zz'}P_{\Lambda}^{\mu}P_{\Sigma}^{\nu}f_{z}^\Lambda f_{\bar{z}}^\Sigma + 4h_{uv}k_{\Lambda}^{u}k_{\Sigma}^{v}\bar{L}^\Lambda L^\Sigma - 3P_{\Lambda}^{\mu}P_{\Sigma}^{\nu}\bar{L}^\Lambda L^\Sigma = 0,
\]
(5.2.9)

identically, for any value of \( g_1, g_2, g_3 \) and also of \( z \) and \( b^u \). In the following, suppose that we choose one of the vacua, that is, we know the value of \( z \) at the vacuum and write the vacuum expectation value of \( \ldots \) as \( \langle \ldots \rangle \). In this model, we cannot determine the vacuum expectation value of \( z \). In the next chapter, the scalar potential takes the non-trivial form and we can determine the value of \( z \) at the vacuum.

Supersymmetry transformations of the fermions

Using (5.2.1)-(5.2.4), the vacuum expectation values of the supersymmetry transformation laws of the fermions (4.2.1)-(4.2.3) are

\[
\langle \delta \psi_{A\mu} \rangle = i\langle S_{AB} \rangle \gamma_\mu \epsilon^B = \langle e^{K/2} \rangle \begin{pmatrix} ig_2 + g_3 \\ g_1 \end{pmatrix} \gamma_\mu \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix},
\]
(5.2.10)

\[
\langle \delta \lambda^z A \rangle = Wz_{AB} \epsilon_B = -\langle e^{K/2} (z + \bar{z}) \rangle \begin{pmatrix} g_2 + ig_3 \\ ig_1 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix},
\]
(5.2.11)

\[
\langle \delta \zeta_\alpha \rangle = N^{A}_{\alpha} \epsilon_A = \langle e^{iK/2} \rangle \begin{pmatrix} g_1 & -ig_2 + g_3 \\ ig_2 - g_3 & -g_1 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}.
\]
(5.2.12)

If each matrix has one zero-eigenvalue, the supersymmetry transformation corresponding to one direction is zero and that corresponding to another direction is non-zero, so we can
see the $\mathcal{N} = 2$ local supersymmetry is broken to $\mathcal{N} = 1$. In order for each matrix to have one zero-eigenvalue, we have to impose the following conditions on the coupling constants:

$$g_2 = 0, \quad g_1 = \pm g_3. \quad (5.2.13)$$

Notice that $g_1, g_2, g_3 \in \mathbb{R}$. In other words, when the conditions (5.2.13) are satisfied, $\mathcal{N} = 2$ local supersymmetry is broken to $\mathcal{N} = 1$.

In the following, we choose the coupling constants as follows,

$$g_2 = 0, \quad g_1 = g_3. \quad (5.2.14)$$

Substituting (5.2.14) into (5.2.10)-(5.2.12),

$$\langle \delta \psi_A \mu \rangle = \left( \frac{e^{K/2}}{2\sqrt{2\theta}} \right) g_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \gamma_\mu \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad (5.2.15)$$

$$\langle \delta \lambda z \rangle = -\left( \frac{ie^{K/2}(z + \bar{z})}{\sqrt{2\theta}} \right) g_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad (5.2.16)$$

$$\langle \delta \zeta_\alpha \rangle = \left( \frac{ie^{K/2}}{\sqrt{2\theta}} \right) g_1 \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}. \quad (5.2.17)$$

If we define $\phi_{\pm} = \pm \sqrt{2}(\phi \pm \phi_2)$ where $\phi \in \psi, \zeta, \lambda$ (the left or right chirality are denoted by the upper or lower position of the index $\pm$), their supersymmetry transformations on the vacuum are

$$\langle \delta \psi_{+\mu} \rangle = \left( \frac{e^{K/2}}{\sqrt{2\theta}} \right) g_1 \gamma_\mu (\epsilon_1 + \epsilon_2),$$

$$\langle \delta \lambda_{z+} \rangle = -\left( \frac{2ie^{K/2}(z + \bar{z})}{\theta} \right) g_1 (\epsilon_1 + \epsilon_2),$$

$$\langle \delta \zeta_- \rangle = \left( \frac{3ie^{K/2}}{\theta} \right) g_1 (\epsilon_1 + \epsilon_2),$$

$$\langle \delta \psi_{-\mu} \rangle = \langle \lambda_{z-} \rangle = \langle \zeta_+ \rangle = 0. \quad (5.2.18)$$

As we will see in the next section, the gravitino $\psi^+_{\mu}$ becomes massive by the super-Higgs mechanism, while $\psi^-_{\mu}$ remains massless. We define linear combinations of the fermions $\lambda^{z+}$ and $\zeta_-$ such that

$$\chi_\bullet \equiv -(z + \bar{z})^{-1}\lambda^{z+} + 2\zeta_-, \quad \eta_\bullet \equiv (z + \bar{z})^{-1}\lambda^{z+} + \zeta_-, \quad (5.2.19)$$

whose supersymmetry transformations on the vacuum are

$$\langle \delta \chi_\bullet \rangle = \left( \frac{3\sqrt{2ie^{K/2}}}{\theta} \right) g_1 (\epsilon_1 + \epsilon_2),$$

$$\langle \delta \eta_\bullet \rangle = 0. \quad (5.2.20)$$
where the upper or lower position of dot represents left or right chirality respectively. Note that we cannot make the expectation values of the supersymmetry transformations of all the (spin-1/2) fermions zero and only the supersymmetry transformations of the fermion \( \chi_* \) takes the non-zero value. Thus, \( \chi_* \) is the Nambu-Goldstone fermion and the \( \mathcal{N} = 2 \) supersymmetry is broken to \( \mathcal{N} = 1 \).

5.3 The mass spectrum

5.3.1 Fermion mass

Let us discuss the occurrence of the super-Higgs mechanism. To see this, we have to evaluate the fermion mass terms of the Lagrangian \( \mathcal{L}_{\text{Yukawa}} \).

\[
\mathcal{L}_{\text{Yukawa}} = -\left( \frac{\sqrt{2} i g_1 e^{K/2}}{b_0} \right) \left( \bar{\psi}^+ \gamma^\mu \psi^+ - i \chi^* \gamma_\mu \psi^+ + \frac{1}{3} \bar{\chi} \chi^* - \frac{1}{3} \bar{\eta} \eta^* \right) + \ldots + \text{h.c.} \tag{5.3.1}
\]

We can, also, confirm \( \chi_* \) is the Nambu-Goldstone fermion from the fact that it couples to the gravitino \( \psi^+ \) in the second term. Such a field can be gauged away by a suitable gauge transformation of the gravitino \( \psi^+ \):

\[
\psi^+_\mu \rightarrow \psi^+_\mu + \frac{i}{6} \gamma_\mu \chi^*, \tag{5.3.2}
\]

which results in

\[
\mathcal{L}_{\text{Yukawa}} = -i\left( \frac{\sqrt{2} g_1 e^{K/2}}{b_0} \right) \left( \bar{\psi}^+ \gamma^\mu \psi^+ - \frac{1}{3} \bar{\eta} \eta^* \right) + \ldots + \text{h.c.} \tag{5.3.3}
\]

The super-Higgs mechanism has occurred, that is, by absorbing the Nambu-Goldstone fermion, the gravitino \( \psi^+_\mu \) has acquired a mass.

The kinetic terms of the massive fermions in the \( \mathcal{L}_{\text{kin}} \) are

\[
\mathcal{L}_{\text{kin}} = \frac{\epsilon^{\mu \nu \lambda \sigma}}{\sqrt{-g}} \bar{\psi}^A \gamma_\mu \partial_\lambda \psi^A - \frac{i}{3} \eta^* \gamma_\mu \partial_\lambda \eta^* + \ldots + \text{h.c.} \tag{5.3.4}
\]

It is easy to find the mass of the fermions by evaluating the equations of motion of them. As a result, the gravitino mass \( m \) is

\[
m = \left| \left( \frac{\sqrt{2} g_1 e^{K/2}}{b_0} \right) \right|. \tag{5.3.5}
\]

Notice that the mass of the physical fermion \( \eta_* \) is the same as the gravitino, that is, \( m \).

Therefore, we can anticipate that \( \psi^+_\mu \) and \( \eta_* \) form a \( \mathcal{N} = 1 \) massive multiplet of spin (3/2,1,1,1/2). On the other hand, \( \lambda^* \), together with the scalar fields, are expected to form \( \mathcal{N} = 1 \) massless chiral multiplet. To make sure, we must consider the masses of the gauge bosons and the scalar fields.
5.3.2 Boson mass

Let us consider the masses of the gauge bosons. They appear in the kinetic term of the hypermultiplet scalar:

\[ h_{uv} \nabla_\mu b^u \nabla^\mu b^v = \frac{1}{2(\partial^\mu)^2} \delta_{uv} (\partial_\mu b^u + A_\mu^k k^k_u)(\partial^\mu b^v + A^\lambda k^\lambda_v) \]

\[ = \frac{1}{2(\partial^\mu)^2} (g_1^2 A_\mu^0 A_\mu^0 + g_1^2 A_\mu^1 A_\mu^1 + 2g_1 (A_\mu^0 \partial^\mu b^3 + A_\mu^1 \partial^\mu b^2)) + \ldots \]

(5.3.6)

Furthermore the massless scalar fields \( b^2, b^3 \) can be eliminated from (5.3.6) by employing the gauge transformations of \( A_\mu^0, A_\mu^1 \):

\[ A_\mu^0 \rightarrow A_\mu^0 - \frac{1}{g_1} \partial_\mu b^3, \quad A_\mu^1 \rightarrow A_\mu^1 - \frac{1}{g_1} \partial_\mu b^2. \]

(5.3.7)

This is the ordinary Higgs mechanism and the scalar fields \( b^2, b^3 \) are the Nambu-Goldstone bosons.

The gauge kinetic terms are \( \frac{1}{4} (\text{Im} \langle N \rangle)_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^{\Sigma \mu \nu} \), so we must compute the generalized coupling matrix \( N \) (2.2.23):

\[ \tilde{N}_{\Lambda \Sigma} = h_{\Lambda|I} (f^{-1})^I_{\Sigma}, \]

(5.3.8)

with

\[ h_{\Lambda|I} = \begin{pmatrix} M_\Lambda & h_{\Lambda|z} \end{pmatrix} = \frac{e^{K/2} \bar{z}}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & \bar{z} + z \end{pmatrix}, \]

(5.3.9)

\[ f^I_{\Lambda} = \begin{pmatrix} \bar{L}^\Lambda & f^I_{\Sigma} \end{pmatrix} = \frac{e^{K/2}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & \bar{z} + z \end{pmatrix}. \]

(5.3.10)

Therefore, the matrix \( N_{\Lambda \Sigma} \) is

\[ N_{\Lambda \Sigma} = -iz \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

(5.3.11)

The kinetic terms of the gauge bosons are written as follows,

\[ \frac{1}{4} \text{Im} \langle N_{\Lambda \Sigma} \rangle F^\Lambda_{\mu \nu} F^{\Sigma \mu \nu} = -\frac{1}{4} \frac{(e^{-K})}{2} F^0_{\mu \nu} F^{0\mu \nu} - \frac{1}{4} \frac{(e^{-K})}{2} F^1_{\mu \nu} F^{1\mu \nu}. \]

(5.3.12)

From (5.3.6) and (5.3.12), we can read off the masses of the gauge bosons. As a result, both of them agree with (5.3.5), that is,

\[ m = \sqrt{\frac{2g_1 e^{K/2}}{\partial^\mu}}. \]

(5.3.13)

The \( U(1) \) gauge boson and the graviphoton have acquired the mass.

Since the scalar potential \( V \) is identically zero, there is no mass term of the vector multiplet scalar \( z \). Thus, all the scalar fields are massless.

We summarize the mass spectrum of our model in the table 5.1:

\( \mathcal{N} = 1 \) gravity multiplet contains the vierbein \( e^I_\mu \) and the gravitino \( \bar{\psi}_-^I \). On the other hand,
the massive gravitino $\psi_+^{\mu}$, the $U(1)$ gauge boson $A_1^{\mu}$, the graviphoton $A_0^{\mu}$ and the fermion $\eta_*$ form a massive spin-$3/2$ multiplet. The gaugino $\lambda_{z}^-$ and the complex scalar field $z$ form the massless chiral multiplet. The hyperino $\zeta_+$ and the scalar $b_0, b_1$ form $N = 1$ chiral multiplet. Note that the $U(1) \times U(1)$ graviphoton gauge symmetry is completely broken and the vacuum lie in the Higgs phase.

| $\mathcal{N} = 1$ multiplet      | field  | mass |
|----------------------------------|--------|------|
| gravity multiplet                | $e_{\mu}^i, \psi_{\mu}^-$ | 0    |
| spin-3/2 multiplet               | $\psi_+^{\mu}, A_0^{\mu}, A_1^{\nu}, \eta_*$ | $m$  |
| chiral multiplet                 | $\lambda_{z}^-, z$       | 0    |
| chiral multiplet                 | $\zeta_+, b_0, b_1$      | 0    |

Table 5.1: the mass spectrum

5.4 Conclusion and discussion

We have seen that, in the $\mathcal{N} = 2$ $U(1)$ gauged supergravity model, the $\mathcal{N} = 2$ supersymmetry has been broken to $\mathcal{N} = 1$ counterpart. The super-Higgs and the Higgs mechanisms are observed: The Nambu-Goldstone fermion and the Nambu-Goldstone bosons are absorbed by the gravitino and the gauge bosons respectively. Also the gauge symmetry is broken by the Higgs mechanisms.

To make more realistic model, we have to consider non-Abelian gauge group. But, if $\mathcal{N} = 2$ local supersymmetry is broken to $\mathcal{N} = 1$, by the Higgs mechanisms, the gauge symmetry corresponding to the graviphoton and to one of the gauge bosons of the vector multiplet sector have to be broken. Thus, it is desirable that the gauge symmetry of the vector multiplet sector is $U(1) \times$ (non-Abelian). Indeed it has been showed that, in the case of the $U(1) \times$ (compact group), partial supersymmetry breaking has occurred [14].

The model which we have used in this section is the simplest one because we have chosen the holomorphic function $F$ as the simple function. The resulting theory is the microscopic (or bare) theory. This leads to the following question; Does partial supersymmetry breaking occur in $\mathcal{N} = 2$ effective theory ? We will answer the question in the next chapter: we will use $U(N)$ gauged effective model by keeping $F$ general holomorphic function.
Chapter 6

Partial supersymmetry breaking in the $\mathcal{N} = 2$ $U(N)$ gauged model

In this chapter, we will see that, in the $\mathcal{N} = 2$ supergravity model which includes a $U(N)$† vector multiplet and a hypermultiplet, the $\mathcal{N} = 2$ supersymmetry is broken to $\mathcal{N} = 1$ spontaneously with keeping the $SU(N)$ gauge symmetry manifest [20]. This is a generalization of the model which we have reviewed in the last chapter. We do not set the section as the simplest function here. It leads to the effective theory which contains higher order coupling terms of the scalar fields. In this sense, it is not simple generalization.

The organization is parallel to that of the chapter 5. First of all, the parametrizations of the vector multiplet and hypermultiplet are given in section 6.1. In section 6.2, we observe that the $\mathcal{N} = 2$ local supersymmetry is broken to $\mathcal{N} = 1$ spontaneously. The mass spectrum is computed in section 6.3 and the resulting $\mathcal{N} = 1$ Lagrangian is obtained in section 6.4. Finally, in section 6.5, we summarize the results.

6.1 $\mathcal{N} = 2$ $U(N)$ gauged supergravity model

6.1.1 $U(N)$ vector multiplet

The vector multiplets contain the complex scalar fields $z^a$. The index $a = 1, \ldots, N^2$ label the $U(N)$ gauge group and $N^2$ refers to the overall $U(1)$. For simplicity, we write $N^2 = n$ below. We start from the case where the holomorphic prepotential $F(X^0, X^a)$ exists. The prepotential $F(X^0, X^a)$ can be written,

$$F(X^0, X^a) = (X^0)^2 F(X^a/X^0).$$  \hfill (6.1.1)

By taking the $X^\Lambda$ ($\Lambda = 0, 1 \ldots, n$) derivative of $F$, we obtain

$$F_0 = 2X^0 F(X^b/X^0) - X^a \frac{\partial}{\partial(X^a/X^0)} F(X^b/X^0),$$

$$F_a = X^0 \frac{\partial}{\partial(X^a/X^0)} F(X^b/X^0).$$  \hfill (6.1.2)

†To be precise, the gauge symmetry is $U(N) \times U(1)$. The $U(1)$ gauge group comes from the graviphoton.
It is natural to choose the upper part of the holomorphic section, \( X^\Lambda(z) \), as
\[
X^0(z) = \frac{1}{\sqrt{2}}, \quad X^a(z) = \frac{1}{\sqrt{2}} z^a, \quad (6.1.3)
\]
which leads to
\[
F_0(z) = \frac{1}{\sqrt{2}} \left( 2F(z) - z^a \frac{\partial F(z)}{\partial z^a} \right), \\
F^a(z) = \frac{1}{\sqrt{2}} \frac{\partial F(z)}{\partial z^a}. \quad (6.1.4)
\]
where we have defined \( \partial_a = \partial/\partial z^a \). We keep \( F(z) \) general function for a moment.

Let us perform the symplectic rotation which is similar to that in the last chapter. We rotate the overall \( U(1) \) part, that is, \( X^n \to -F_n \) and \( F_n \to X^n \). As a result, the holomorphic sections are
\[
X^0(z) = \frac{1}{\sqrt{2}}, \quad F_0(z) = \frac{1}{\sqrt{2}} \left( 2F(z) - z^a \frac{\partial F(z)}{\partial z^a} \right), \\
X^\hat{a}(z) = \frac{1}{\sqrt{2}} \hat{z}^\hat{a}, \quad F^\hat{a}(z) = \frac{1}{\sqrt{2}} \frac{\partial F(z)}{\partial \hat{z}^\hat{a}}, \quad (6.1.5)
\]
\[
X^n(z) = \frac{1}{\sqrt{2}} \frac{\partial F(z)}{\partial z^n}, \quad F_n(z) = -\frac{1}{\sqrt{2}} \hat{z}^n,
\]
where the index \( \hat{a} = 1, \ldots, n-1 \), label the \( SU(N) \) subgroup. Our choice of sections is such that no holomorphic prepotential exists.

With this choice, Kähler potential and its derivatives are
\[
\mathcal{K} = -\log \mathcal{K}_0, \quad (6.1.6)
\]
\[
\mathcal{K}_0 = i \left( F - \bar{F} - \frac{1}{2} (z^a - \bar{z}^a)(F_a + \bar{F}_a) \right), \quad (6.1.7)
\]
\[
\partial_a \mathcal{K} = -\frac{i}{2 \mathcal{K}_0} (F_a - \bar{F}_a - (z^c - \bar{z}^c)F_{ac}), \quad (6.1.8)
\]
where \( F_a = \partial F/\partial z^a \). The Kähler metric and the Levi-Civita connection are
\[
g_{ab} = \partial_a \partial_b \mathcal{K} = \partial_a \mathcal{K} \partial_b \mathcal{K} - \frac{i}{2 \mathcal{K}_0} (F_{ab} - \bar{F}_{ab}), \quad (6.1.9)
\]
\[
\Gamma_{bc}^{\alpha} = -g^{\alpha \beta} \partial_{\beta} g_{\alpha \gamma} = -\delta^\alpha_b \partial_c \mathcal{K} - \delta^\alpha_c \partial_b \mathcal{K} + \frac{1}{\mathcal{K}_0} g^{\alpha \beta} (\partial_b \partial_c \partial_\alpha \mathcal{K} + \partial_b \partial_\alpha \partial_c \mathcal{K} + \partial_\alpha \partial_c \partial_b \mathcal{K}). \quad (6.1.10)
\]
Furthermore, by using (2.2.16) and (6.1.10), \( \nabla_a f^0_b \) and \( \nabla_a f^n_b \) are evaluated as follows,

\[
\nabla_a f^0_b \equiv \frac{ie^{\mathcal{K}/2}}{\sqrt{2}} C_{abc} g^{cd\ast} \partial_d \mathcal{K}
\equiv \partial_a f^0_b + \Gamma^c_{ab} f^0_c + \frac{1}{2} \partial_a \mathcal{K} f^0_b
= \frac{e^{\mathcal{K}/2}}{\sqrt{2}} \left( \partial_a \partial_b \mathcal{K} - \partial_a \mathcal{K} \partial_b \mathcal{K} + \frac{1}{K_0} g^{cd\ast} (\partial_a \partial_b K_0 \partial_d \mathcal{K} + \partial_a \partial_b \partial_d K_0) \partial_c \mathcal{K} \right), \tag{6.1.11}
\]

\[
\nabla_a f^n_b \equiv \frac{ie^{\mathcal{K}/2}}{\sqrt{2}} C_{abc} g^{cd\ast} (\bar{F}_{nd} + \partial_d \mathcal{K} \bar{F}_n)
\equiv \partial_a f^n_b + \Gamma^c_{ab} f^n_c + \frac{1}{2} \partial_a \mathcal{K} f^n_b
= \frac{e^{\mathcal{K}/2}}{\sqrt{2}} \left( F_{nab} - \partial_a \mathcal{K} \partial_b \mathcal{K} F_n + \partial_a \partial_b \mathcal{K} F_n \right)
+ \frac{e^{\mathcal{K}/2}}{\sqrt{2} K_0} g^{cd\ast} (\partial_a \partial_b K_0 \partial_d \mathcal{K} + \partial_a \partial_b \partial_d K_0) (\mathcal{F}_{nc} + \partial_c \mathcal{K} \mathcal{F}_n). \tag{6.1.12}
\]

These equations will be used in the analysis of the scalar potential.

**U(N) gauging**

In order to gauge the vector multiplets, we define the Killing vectors as follows,

\[
k^c_a \partial_c = f^c_{ab} z^b \partial_a, \quad k^{\ast c}_a \partial^{\ast c} = f^{\ast c}_{ab} \bar{z}^b \partial^a, \tag{6.1.13}
\]

where \( f^a_{bc} \) is the structure constant of the \( U(N) \) gauge group satisfying

\[
[t_a, t_b] = i f^c_{ab} t_c. \tag{6.1.14}
\]

In this way, for example, the covariant derivative of the scalar fields can take the standard form;

\[
\nabla_\mu z^a = \partial_\mu z^a + A^A_{\mu} k^A_a
= \partial_\mu z^a + f^a_{bc} A^b_{\mu} \bar{z}^c. \tag{6.1.15}
\]

Note that these Killing vectors satisfy

\[
0 = \mathcal{L}_A \mathcal{K} = k^A_a \partial_b \mathcal{K} + k^{A^a}_a \partial_b \mathcal{K}. \tag{6.1.16}
\]

### 6.1.2 Hypermultiplet

We take the same parametrizations as those of the section 5.1.2. Since in order for \( \mathcal{N} = 2 \) supersymmetry to be broken partially, as we have seen in the last chapter, two gauge boson have to be massive by the Higgs mechanism, let the overall \( U(1) \) gauge boson and the
graviphoton be the gauge bosons which become massive. For this purpose, we generalize as follows,

\[ k_0^0 = g_1 \delta^u + g_2 \delta^u, \quad k_0^u = 0, \quad k_n^u = g_3 \delta^u, \]

\[ \mathcal{P}_0^x = \frac{1}{b_0} (g_1 \delta^x + g_2 \delta^x^2), \quad \mathcal{P}_x^0 = 0, \quad \mathcal{P}_x^x = \frac{1}{b_0} g_3 \delta^x^2, \quad (6.1.17) \]

### 6.2 Partial supersymmetry breaking

In order to write down the supersymmetry transformations of the fermions, let us evaluate the mass matrices here. Substituting (6.1.6)-(6.1.9) and the hypermultiplet parametrizations into (6.1.6)-(6.1.9), we have

\[
S_{AB} = -\frac{ie^{K/2}}{2\sqrt{b_0}} \begin{pmatrix} i(g_2 + g_3 \bar{F}_n) & g_1 \\ g_1 & i(g_2 + g_3 \bar{F}_n) \end{pmatrix},
\]

\[
W_1^{\alpha AB} = -ie^{K/2} D^a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
W_2^{\alpha AB} = \frac{e^{K/2}}{\sqrt{2b_0}} g^{a b r} \begin{pmatrix} g_2 \partial_0 \mathcal{K} + g_3 (\mathcal{F}_{n b} + \partial_0 \mathcal{K} \mathcal{F}_n) & ig_1 \partial_0 \mathcal{K} \\ ig_1 \partial_0 \mathcal{K} & g_2 \partial_0 \mathcal{K} + g_3 (\mathcal{F}_{n b} + \partial_0 \mathcal{K} \mathcal{F}_n) \end{pmatrix},
\]

\[
N_\alpha^A = \frac{ie^{K/2}}{\sqrt{2b_0}} \begin{pmatrix} g_1 & -i(g_2 + g_3 \bar{F}_n) \\ i(g_2 + g_3 \bar{F}_n) & -g_1 \end{pmatrix},
\]

\[
M_{\alpha \beta} = \frac{ie^{K/2}}{\sqrt{2b_0}} \begin{pmatrix} -i(g_2 + g_3 \bar{F}_n) & g_1 \\ g_1 & -i(g_2 + g_3 \bar{F}_n) \end{pmatrix},
\]

\[
M_{\alpha bb} = -\frac{\sqrt{2}ie^{K/2}}{b_0} \begin{pmatrix} g_1 \partial_0 \mathcal{K} & i(g_2 \partial_0 \mathcal{K} + g_3 (\mathcal{F}_{n b} + \partial_0 \mathcal{K} \mathcal{F}_n)) \\ -i(g_2 \partial_0 \mathcal{K} + g_3 (\mathcal{F}_{n b} + \partial_0 \mathcal{K} \mathcal{F}_n)) & -g_1 \partial_0 \mathcal{K} \end{pmatrix},
\]

\[
M_{1,\alpha | A | b B} = -\frac{ie^{K/2}}{2} g_{ac} (\partial_b + \partial_0 \mathcal{K}) D^c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
M_{2,\alpha | A | b B} = -\frac{1}{2b_0} \begin{pmatrix} g_2 \nabla_b f_{a}^0 + g_3 \nabla_b f_{a}^0 & -ig_1 \nabla_b f_{a}^0 \\ -ig_1 \nabla_b f_{a}^0 & g_2 \nabla_b f_{a}^0 + g_3 \nabla_b f_{a}^0 \end{pmatrix},
\]

\[
= \frac{ie^{K/2}}{2\sqrt{2}} C_{a b c} \gamma^0 \begin{pmatrix} g_2 \partial_0 \mathcal{K} + g_3 (\mathcal{F}_{n b} + \partial_0 \mathcal{K} \mathcal{F}_n) & -ig_1 \partial_0 \mathcal{K} \\ -ig_1 \partial_0 \mathcal{K} & g_2 \partial_0 \mathcal{K} + g_3 (\mathcal{F}_{n b} + \partial_0 \mathcal{K} \mathcal{F}_n) \end{pmatrix},
\]

where we have introduced,

\[
D^a = \frac{i}{\sqrt{2}} f^a_{b c} z^b \bar{z}^c.
\]

(6.2.2)
6.2.1 The scalar potential

By the gauging of hypermultiplet, the scalar potential $V$ takes a nontrivial form:

$$V(z, \bar{z}, b) = e^K g_{ab} D^a \bar{D}^b + \frac{e^K}{(b^0)^2} g^{ab} D_b \bar{D}_a^e$$

$$- \frac{e^K}{2(b^0)^2} (\mathcal{E}^x + \mathcal{M}^x \mathcal{F}_n)(\mathcal{E}^x + \mathcal{M}^x \bar{\mathcal{F}}_n),$$  \hspace{1cm} (6.2.3)

with

$$D_a^e = \frac{1}{\sqrt{2}} (\mathcal{E}^x \partial_a \mathcal{K} + \mathcal{M}^x (\mathcal{F}_{na} + \partial_a \mathcal{K} \mathcal{F}_n)), \hspace{1cm} (6.2.4)$$

$$\mathcal{E}^x = (0, g_2, g_1),$$

$$\mathcal{M}^x = (0, g_3, 0).$$

The first term comes from $U(N)$ gauging of the vector multiplet, while the second and the last terms correspond to gauging of the hypermultiplet. Note that this potential is quite different from what we have seen in the last chapter. If we choose $\mathcal{F}$ as the simplest function of the holomorphic sections $X^\Lambda$, we obtain flat potential and cannot determine the expectation values of the scalar fields. On the other hand, in our model, we can obtain it by determining the minimum of the potential.

Let us find the conditions which minimize the potential. Firstly, let us consider the variation of $V$ with respect to the scalar field $z^a$. The derivative of the second and the last terms of $V$ is

$$\partial_a \left( \frac{e^K}{(b^0)^2} g^{ab} D_b \bar{D}_a^e - \frac{e^K}{2(b^0)^2} (\mathcal{E}^x + \mathcal{M}^x \mathcal{F}_n)(\mathcal{E}^x + \mathcal{M}^x \bar{\mathcal{F}}_n) \right)$$

$$= \frac{e^K}{(b^0)^2} \left( (\partial_a \mathcal{K}) g^{bc} D_b \bar{D}_c^e + (\partial_a g^{bc}) D_b^e \bar{D}_c^e + g^{bc} (\partial_a D_b \mathcal{E}^x) \bar{D}_c^e \right)$$

$$= \frac{e^K}{(b^0)^2} g^{bc} D_c^e \left( \partial_a D_b^e - (\partial_b \mathcal{K}) D_a^e + \frac{1}{k_0} g^{cd} (\partial_a \partial_b \mathcal{K} \partial_d \mathcal{K} \partial_0) D_c^e \right)$$

$$= \frac{ie^K}{(b^0)^2} C_{abc} g^{bd} \tilde{D}_d^e \bar{D}_c^e, \hspace{1cm} (6.2.5)$$

where we have used (6.1.12) and (6.1.12) in the last equality. Thus, the condition which minimizes the potential is

$$0 = (\partial_c V) = (\partial_c \left( e^K g_{ab} D^a \bar{D}^b \right)) + \frac{e^K}{(b^0)^2} i \left( C_{abc} g^{bd} \bar{D}_d^e \bar{D}_c^e \right).$$  \hspace{1cm} (6.2.6)

Of course, we must consider the variation with respect to the hypermultiplet scalar fields $b^a$. Since $V$ contains only $b^0$, it is straightforward to compute the derivative in terms of $b^0$. The condition which determines the vacuum is

$$0 = \frac{(\partial V)}{(\partial b^0)} = - \frac{e^K}{(b^0)^3} (2g^{ab} D_a^e \bar{D}_b^e \mathcal{E}^x \bar{D}_a^e (\mathcal{E}^x + \mathcal{M}^x \mathcal{F}_n)(\mathcal{E}^x + \mathcal{M}^x \bar{\mathcal{F}}_n)) \delta_{ab}.$$

$$\hspace{1cm} (6.2.7)$$

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The holomorphic function $\mathcal{F}$

Since we would like to keep $SU(N)$ gauge symmetry manifest, we will work on the condition $\langle z^a \rangle = \delta^{an} \lambda$. Then $\langle \mathcal{D}^a \rangle = \langle i \sqrt{2} f^a_{\beta \gamma} z^\beta \bar{z}^\gamma \rangle = 0$ holds. Thus we have $\langle \partial_a (e^k g_{ab} \mathcal{D}^a \mathcal{D}^b) \rangle = 0$. Moreover, we assume a form of the gauge invariant function $\mathcal{F}(z)$, which parallels that of [9], as

\begin{equation}
\mathcal{F}(z) = \frac{iC}{2} (z^n)^2 + \mathcal{G}(z),
\end{equation}

\begin{equation}
\mathcal{G}(z) = \sum_{l=0}^{k} \frac{C_l}{l!} tr(z^a t_a)^l,
\end{equation}

where $C \in \mathbb{R}$ is constant. We will see that $C$ must be non-zero in order for the inverse of the Kähler metric to exist.

Let us compute the expectation values of some geometrical quantities. Firstly, the expectation value of the derivative of $\mathcal{F}$ is

\begin{align*}
\langle \mathcal{F}_a \rangle &= \delta_{an} \langle \mathcal{F}_n \rangle, \\
\langle \mathcal{F}_{na} \rangle &= \delta_{an} \langle \mathcal{F}_{nn} \rangle, \\
\langle \mathcal{F}_{\hat{a} \hat{b}} \rangle &= \delta_{\hat{a} \hat{b}} \langle \mathcal{F}_{nn} - iC \rangle, \\
\langle \mathcal{F}_{nab} \rangle &= \delta_{ab} \langle \mathcal{F}_{nnn} \rangle,
\end{align*}

(6.2.10)

where the explicit forms of $\langle \mathcal{F}_n \rangle$ and $\langle \mathcal{F}_{nn} \rangle$ are

\begin{align*}
\langle \mathcal{F}_n \rangle &= \sum_{l} \frac{C_l}{(l-1)!} \lambda^{l-1} + iC \lambda, \\
\langle \mathcal{F}_{nn} \rangle &= \sum_{l} \frac{C_l}{(l-2)!} \lambda^{l-2}.
\end{align*}

(6.2.11)

It is easy to compute the expectation value of $\partial_a \mathcal{K}$ by using (6.1.8):

\begin{align*}
\langle \partial_a \mathcal{K} \rangle &= -\frac{i}{2} \langle e^\mathcal{K} \rangle (\mathcal{F}_a - \bar{\mathcal{F}}_a - (\lambda + \bar{\lambda}) \mathcal{F}_a) \\
&= \delta_{an} \langle \partial_n \mathcal{K} \rangle.
\end{align*}

(6.2.12)

Furthermore, the vacuum expectation value of the Kähler metric $g_{ab}$ (6.1.9) can be evaluated as follows:

\begin{align*}
\langle g_{ab} \rangle &= \begin{pmatrix}
\langle g_{11} \rangle & \langle g_{11} \rangle & \cdots & \langle g_{1n} \rangle \\
\langle g_{11} \rangle & \langle g_{11} \rangle & \cdots & \langle g_{2n} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \langle g_{nn} \rangle & \langle g_{nn} \rangle
\end{pmatrix},
\end{align*}

(6.2.13)
with

\[ \langle g_{11\ast} \rangle = -\frac{i\langle e^K \rangle}{2} (\mathcal{F}_{nn} - \bar{\mathcal{F}}_{nn} - 2iC), \]

\[ \langle g_{nn} \rangle = \langle \partial_n K \rangle^2 - \frac{i\langle e^K \rangle}{2} (\mathcal{F}_{nn} - \bar{\mathcal{F}}_{nn}). \quad (6.2.14) \]

Note that only the diagonal components are non-zero and these take the same value except for \( \langle g_{nn} \rangle \). Finally, we have to compute \( \langle D^x_a \rangle \) and \( \langle C_{abc} \rangle \) by using the above equations;

\[ \langle D^x_a \rangle = \delta_{an} \frac{1}{\sqrt{2}} (\mathcal{E}^x_1 + \mathcal{M}^x (\mathcal{F}_{na} + \partial_a K_1)); \]

\[ \langle C_{abc} \rangle = \frac{\langle e^K \rangle}{2} \langle \mathcal{F}_{abc} \rangle. \quad (6.2.15) \]

### 6.2.2 Vacuum conditions

Now we are ready to analyze (6.2.6) and (6.2.7). Substituting (6.2.10)-(6.2.15) into (6.2.6), we obtain

\[ 0 = \langle ie^{2K} \rangle \langle \mathcal{F}_{nn} g^{nn}_+ \bar{D}^x_{a\ast} g^{nn}_- \bar{D}^x_{b\ast} \rangle. \quad (6.2.16) \]

The points \( \langle \mathcal{F}_{nn} \rangle = 0 \) are unstable vacua because \( \langle \partial_a \partial_b V \rangle = 0 \). Also, the points which satisfy \( \langle g^{nn}_+ \rangle = 0 \) and \( \langle \partial_n K \rangle = 0 \) are not stable. Thus, the vacuum condition reduces to

\[ \langle \bar{D}^x_{a\ast} \bar{D}^x_{b\ast} \rangle = 0, \quad (6.2.17) \]

which implies

\[ \langle \mathcal{F}_{nn} + \mathcal{F}_{nn} \partial_n K \rangle = -\left( \frac{g_2}{g_3} \pm i \frac{g_1}{g_3} \right). \quad (6.2.18) \]

where we use \( \langle ... \rangle \) for those vacuum expectation value which are determined as the solutions to (6.2.17). We have also assumed \( g_3 \neq 0 \). Note that if \( g_3 = 0 \), (6.2.17) leads to \( g_1 = g_2 = 0 \) and the supersymmetry is unbroken. We are not interested in such a case.

By using (6.2.18), the second condition (6.2.7) leads to

\[ 0 = \left\langle \left( \mathcal{E}^x + \mathcal{M}^x \mathcal{F}_{nn} \right) \left( \mathcal{E}^x + \mathcal{M}^x \bar{\mathcal{F}}_{nn} \right) - 2g^{ab\ast} D^x_a D^x_{b\ast} \right\rangle 
\]

\[ = g_1 + ig_3 \left\langle \frac{F_{nn}}{\partial_n K} \right\rangle^2 + g_1^2 - \left\langle g^{nn}_+ \partial_n K \right\rangle^2 \right\rangle 2g_1^2. \quad (6.2.19) \]

It can be shown that this condition (6.2.19) leads to \( \langle \mathcal{F}_{nn} \rangle = 0 \): If \( \langle \mathcal{F}_{nn} \rangle = 0 \), (6.2.19) is automatically satisfied. Thus, let us consider the case \( \langle \mathcal{F}_{nn} \rangle \neq 0 \) and prove that it conflicts with the assumption, \( g_3 \neq 0 \). We write \( \mathcal{F}_{nn} \) as

\[ \mathcal{F}_{nn} = F_1 + iF_2, \quad (6.2.20) \]
where $F_1, F_2 \in \mathbb{R}$. From (6.2.12), (6.2.14) and (6.2.18), by using $F_1$ and $F_2$, we obtain

\[
\langle \langle g^{nn^*} \rangle \rangle = \langle \langle |\partial_n K|^2 \rangle \rangle + \frac{F_2}{\|K_0\|}, \quad (6.2.21)
\]

\[
\langle \partial_n K + \partial_n K^* \rangle = \frac{i}{\|K_0\|} \left( \left\langle \frac{F_{nn}}{\partial_n K} - \bar{F}_{nn}^{\dagger} \partial_n K^* \right\rangle \pm 2i \frac{g_1}{g_3} + (\lambda - \bar{\lambda}) F_1 \right), \quad (6.2.22)
\]

\[
\langle \partial_n K - \partial_n K^* \rangle = \frac{1}{\|K_0\|} (\lambda - \bar{\lambda}) F_2. \quad (6.2.23)
\]

Then the condition (6.2.19) can be written as

\[
0 = 2g_1^2 - 2\Delta g_1^2 \left\langle |g^{nn^*}|^2 \right\rangle + iYg_1^2 + g_3^2 \left\langle \left| \frac{F_{nn}}{\partial_n K} \right|^2 \right\rangle, \quad (6.2.24)
\]

where we have defined $Y$ as

\[
Y \equiv \left\langle \frac{F_{nn}}{\partial_n K} - \frac{\bar{F}_{nn}^{\dagger}}{\partial_n K^*} \right\rangle
\]

\[
= \frac{1}{\|\partial_n K\|^2} \left[ F_1 \langle \partial_n K - \partial_n K^* \rangle - \frac{F_2}{\|K_0\|} \left( Y + 2i \frac{g_1}{g_3} + (\lambda - \bar{\lambda}) F_1 \right) \right]. \quad (6.2.25)
\]

In the second equality, we have used (6.2.22). Using (6.2.21), we can solve the above equation for $Y$:

\[
Y \langle g^{nn^*} \rangle = \pm \frac{F_2}{\|K_0\|} \frac{2i g_1}{g_3}. \quad (6.2.26)
\]

Substituting (6.2.26) into (6.2.24), we get

\[
0 = 2g_1^2 - 2\Delta g_1^2 \left\langle |g^{nn^*}|^2 \right\rangle - 2\Delta g_1^2 \left\langle g^{nn^*} \frac{F_2}{\|K_0\|} \right\rangle + g_3^2 \left\langle \left| \frac{F_{nn}}{\partial_n K} \right|^2 \right\rangle
\]

\[
= g_3^2 \left\langle \left| \frac{F_{nn}}{\partial_n K} \right|^2 \right\rangle, \quad (6.2.27)
\]

where we have used (6.2.21). Therefore, we conclude that when $\langle F_{nn} \rangle \neq 0$, the vacuum condition leads to $g_3 = 0$. This conflicts with the assumption which is written in below (6.2.18). Thus, we can say that the second vacuum condition implies

\[
\langle F_{nn} \rangle = 0. \quad (6.2.28)
\]

In the rigid theory [8, 9], the second vacuum condition is not needed, because there is no hypermultiplet in the model. In fact, we will see later that $\mathcal{N} = 2$ local supersymmetry is not broken partially without the second vacuum condition.

In the following, we choose the vacuum condition as

\[
\langle F_n \rangle = - \left( \frac{g_2}{g_3} + \frac{ig_1}{g_3} \right). \quad (6.2.29)
\]
With this, we can evaluate the expectation value of $\partial_a K$ and $g_{ab}^*$ such that

$$
\langle \langle \partial_a K \rangle \rangle = -\delta_{an} \langle \langle e^K \rangle \rangle \frac{g_1}{g_3},
$$

$$
\langle \langle g_{11}^* \rangle \rangle = -\langle \langle e^K \rangle \rangle C,
$$

$$
\langle \langle g_{nn}^* \rangle \rangle = |\langle \langle \partial_n K \rangle \rangle|^2 = \langle \langle e^{2K} \rangle \rangle \left( \frac{g_1}{g_3} \right)^2.
$$

(6.2.30)

Note that $C = 0$ is necessary for the Kähler metric to be invertible.

### 6.2.3 Supersymmetry transformations of the fermions

Let us compute the vacuum expectation values of the mass matrices. Substituting (6.2.29)–(6.2.30) into the mass matrices (6.2.1a)–(6.2.1i),

$$
\langle \langle S_{AB} \rangle \rangle = -\left\langle \frac{ie^{K/2}}{2\sqrt{2b_0}}g_1 \right\rangle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
$$

(6.2.31)

$$
\langle \langle W^a_{2AB} \rangle \rangle = \delta_{an} \left\langle \frac{ie^{K/2}}{\sqrt{2b_0}}(\partial_n K)^{-1}g_1 \right\rangle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
$$

(6.2.32)

$$
\langle \langle N^A_\alpha \rangle \rangle = \left\langle \frac{ie^{K/2}}{\sqrt{2b_0}}g_1 \right\rangle \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},
$$

(6.2.33)

$$
\langle \langle M^{\alpha\beta} \rangle \rangle = \left\langle \frac{ie^{K/2}}{\sqrt{2b_0}}g_1 \right\rangle \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},
$$

(6.2.34)

$$
\langle \langle M^a_{AB} \rangle \rangle = -\left\langle \frac{\sqrt{2}ie^{K/2}}{b_0}g_1 \delta_a K \right\rangle \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},
$$

(6.2.35)

$$
\langle \langle M_{2,aAB} \rangle \rangle = \left\langle \frac{e^{K/2}}{2\sqrt{2b_0}}g_1 C_{abc}g^{cd^*} \partial_{d^*} K \right\rangle \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.
$$

(6.2.36)

Notice that each matrix has a zero eigenvalue. The expectation values of the supersymmetry transformations of the fermions are

$$
\langle \langle \delta \psi_{+\mu} \rangle \rangle = \left\langle \frac{ie^{K/2}}{2b_0}g_1 \right\rangle \gamma_\mu (\epsilon_1 + \epsilon_2),
$$

$$
\langle \langle \delta \lambda^{a+} \rangle \rangle = \delta_{an} \left\langle \frac{ie^{K/2}}{b_0}g_1(\partial_n K)^{-1} \right\rangle (\epsilon_1 + \epsilon_2),
$$

$$
\langle \langle \delta \zeta_- \rangle \rangle = \left\langle \frac{ie^{K/2}}{b_0}g_1 \right\rangle (\epsilon_1 + \epsilon_2),
$$

$$
\langle \langle \delta \psi_{-\mu} \rangle \rangle = \langle \langle \lambda^{a-} \rangle \rangle = \langle \langle \zeta_+ \rangle \rangle = 0.
$$

(6.2.37)
Note that only the supersymmetry transformations of $\psi^+_\mu$, $\lambda^{a+}$ and $\zeta_-$ at the vacuum are non-zero. Furthermore, quite similar to the last chapter, we define linear combination of the fermions, $\lambda^{a+}$ and $\zeta_-$ as

$$
\chi^\bullet = \langle \partial_\mu K \rangle \lambda^{a+} + 2\zeta_-,
\eta^\bullet = -\langle \partial_\mu K \rangle \lambda^{a+} + \zeta_-,
$$

(6.2.38)

whose supersymmetry transformations are

$$
\langle \delta \chi^\bullet \rangle = \langle \delta \eta^\bullet \rangle = 0.
$$

(6.2.39)

Therefore, the fermion $\chi^\bullet$ is the Nambu-Goldstone fermion.

### 6.3 The mass spectrum

#### 6.3.1 Fermion mass

Let us consider the fermion mass spectrum. By substituting (6.2.31)-(6.2.36) into $L_{\text{Yukawa}}$, we obtain

$$
L_{\text{Yukawa}} = -i \left\langle \frac{\sqrt{2}e^{K/2}}{b^0} g_1 \right\rangle \left( \bar{\psi}^+_\mu \gamma^{\mu \nu} \psi^+_\nu - i \bar{\chi} \gamma_\mu \psi^+_\mu + \frac{1}{3} \bar{\chi} \chi^\bullet - \frac{1}{3} \bar{\eta} \eta^\bullet \right)
$$

$$
+ \frac{1}{2\sqrt{2}} \left\langle \frac{e^{K/2}}{b^0} g_3 F_{aan} \right\rangle \bar{\lambda}^a - \lambda^{a-} + \ldots + \text{h.c.},
$$

(6.3.1)

The Nambu-Goldstone fermion $\chi^\bullet$ couples to the gravitino $\psi^+_\mu$ in the second term and we can remove it from the action by redefining the gravitino such that,

$$
\psi^+_\mu \rightarrow \psi^+_\mu + i \frac{6}{\gamma} \gamma^\mu \chi^\bullet,
$$

(6.3.2)

which results in

$$
L_{\text{Yukawa}} = -i \left\langle \frac{\sqrt{2}e^{K/2}}{b^0} g_1 \right\rangle \left( \bar{\psi}^+_\mu \gamma^{\mu \nu} \psi^+_\nu - \frac{1}{3} \bar{\eta} \eta^\bullet \right)
$$

$$
+ \frac{1}{2\sqrt{2}} \sum_{a=1}^n \left\langle \frac{e^{K/2}}{b^0} g_3 F_{aan} \right\rangle \bar{\lambda}^a - \lambda^{a-}
$$

$$
+ \ldots + \text{h.c.}.
$$

(6.3.3)

The gravitino $\psi^+_\mu$ has acquired a mass by the super-Higgs mechanism.

The kinetic terms of the massive fermions are

$$
L_{\text{kin}} = \frac{\epsilon^{\mu \nu \lambda \sigma}}{\sqrt{-g}} \bar{\psi}^+_\mu \gamma_\nu \partial_\lambda \psi_{\lambda a} - i \frac{5}{3} \bar{\eta} \gamma_\mu \partial^\mu \eta^\bullet - i \sum_{a=1}^n \left\langle g_{aa*} \right\rangle \bar{\lambda}^{a-} \gamma_\lambda \chi^{a+} + \ldots + \text{h.c.}
$$

(6.3.4)
It is easy to find the mass of the fermions by evaluating the equations of motion. The gravitino mass $m$ and the masses of the gauginos $m_a$ are

$$m = \left| \left\langle \frac{\sqrt{2eK}}{b^0} g_1 \right\rangle \right|,$$

$$m_a = \left| \left\langle \frac{eK}{2b^0} g_3 F_{aan} g^{aa^*} \right\rangle \right|. \quad (6.3.5)$$

Notice that the mass of the physical fermion $\eta$ is the same as the gravitino, that is, $m$. We can expect that $\psi^+ + \mu$ and $\eta$ form a $\mathcal{N} = 1$ massive multiplet of spin $(3/2,1,1,1/2)$. On the other hand, $\lambda^a -$, together with the scalar fields, form $\mathcal{N} = 1$ massive chiral multiplet. To make sure, we must consider the masses of the gauge bosons and the scalar fields.

### 6.3.2 Boson mass

Let us compute the masses of the scalar fields. If we define the fluctuations from the expectation value of the scalar fields as the new scalar fields $\tilde{z}^a$, that is, $\tilde{z}^a = z^a - \langle \langle z^a \rangle \rangle$. The scalar potential can be expanded as,

$$V = \langle \langle V \rangle \rangle + \frac{1}{2} \langle \langle \partial_a \partial_b V \rangle \rangle \bar{z}^a \bar{z}^b + \frac{1}{2} \langle \langle \partial_\mu \partial_\mu V \rangle \rangle \bar{z}^a \bar{z}^a + \ldots \quad (6.3.6)$$

where we use the fact that first derivative of $V$ is zero by the vacuum condition. The second derivative can be easily evaluated,

$$\langle \langle \partial_a \partial_b V \rangle \rangle = \delta_{ab} \left\langle \frac{eK}{2(b^0)^2} |g_3 F_{aan}|^2 g^{aa^*} \right\rangle,$$

$$\langle \langle \partial_\mu \partial_\mu V \rangle \rangle = 0 \quad (6.3.7).$$

Thus, the kinetic terms and the mass terms of the scalar fields $\tilde{z}^a$ are

$$\sum_a \left( \langle g_{aa^*} \rangle \partial_\mu \tilde{z}^a \partial^\mu \tilde{z}^{a^*} - \left\langle \frac{eK}{2(b^0)^2} |g_3 F_{aan}|^2 g^{aa^*} \right\rangle \tilde{z}^a \bar{z}^{a^*} \right). \quad (6.3.8)$$

We can see the mass of $\tilde{z}^a$ is the same as (6.3.5), namely, the mass of gaugino $\lambda^a -$. Therefore, as we have anticipated, they form $\mathcal{N}^2$ massive chiral multiplets.

The gauge boson masses appear in the kinetic terms of the hypermultiplet scalars:

$$h_{uv} \nabla_\mu b^u \nabla_\mu b^v = \frac{1}{2(b^0)^2} \delta_{uv} (\partial_\mu b^u + A^A_\mu k^A_\mu)(\partial^\mu b^v + A^A_\mu k^A_\mu)$$

$$= \frac{1}{2(b^0)^2} (g_1^2 A^0_\mu A^{0\mu} + g_3^2 A^m_\mu A^{m\mu}) + \ldots, \quad (6.3.9)$$

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where we have defined the gauge boson $A^n_\mu$ as

$$A^n_\mu = A^n_\mu + \left(\frac{g_2}{g_3}\right) A^0_\mu.$$  \hspace{1cm} (6.3.10)

Since the kinetic terms of the gauge bosons are $\frac{1}{4} (\text{Im} \mathcal{N})_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^{\Sigma \mu \nu}$, we must compute the generalized coupling matrix $\langle \mathcal{N} \rangle$ on the vacuum:

$$\langle \mathcal{N} \rangle = \begin{pmatrix}
\langle \mathcal{N}_{00} \rangle & 0 & \cdots & \cdots & 0 & \langle \mathcal{N}_{0n} \rangle \\
0 & \langle \mathcal{G}_{11} \rangle & 0 & \cdots & 0 & 0 \\
\vdots & 0 & \langle \mathcal{G}_{22} \rangle & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots \\
0 & 0 & \cdots & 0 & \langle \mathcal{G}_{n-1,n-1} \rangle & 0 \\
\langle \mathcal{N}_{n0} \rangle & 0 & \cdots & \cdots & 0 & \langle \mathcal{N}_{nn} \rangle
\end{pmatrix}, \hspace{1cm} (6.3.11)

with

$$\text{Im} \langle \mathcal{N}_{00} \rangle = \left\langle \frac{e^{-K}}{2} \right\rangle \frac{g_2^2 + g_3^2}{g_1^2},$$
$$\text{Im} \langle \mathcal{N}_{0n} \rangle = \text{Im} \langle \mathcal{N}_{n0} \rangle = \left\langle \frac{e^{-K}}{2} \right\rangle \frac{g_2 g_3}{g_1^2},$$
$$\text{Im} \langle \mathcal{N}_{nn} \rangle = \left\langle \frac{e^{-K}}{2} \right\rangle \left(\frac{g_3}{g_1}\right)^2. \hspace{1cm} (6.3.12)$$

Therefore the gauge boson kinetic terms are

$$\frac{1}{4} \text{Im} \langle \mathcal{N} \rangle F^\Lambda_{\mu \nu} F^{\Sigma \mu \nu} = - \left\langle \frac{e^{-K}}{8} \right\rangle F^0_{\mu \nu} F^{0 \mu \nu} - \left\langle \frac{e^{-K}}{8} \right\rangle \left(\frac{g_3}{g_1}\right)^2 F^{\mu n} F^{n \mu \nu} + \frac{1}{4} \sum_a \text{Im} \langle \mathcal{G}_{a a} \rangle F_{\mu \nu}^{a} F^{a \mu \nu}, \hspace{1cm} (6.3.13)$$

where we have defined $F^\Lambda_{\mu \nu} = \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu$. We can read off the masses of gauge boson $A^0_\mu$ and $A^n_\mu$ from (6.3.9) and (6.3.13). As a result, both of them agree with (6.3.5). Thus the $U(N) \times U(1)$ graviphoton gauge symmetry is broken to $SU(N)$ and the vacuum lies in the Higgs phase.

We summarize the mass spectrum of our model in the table 6.1:

$\mathcal{N} = 1$ gravity multiplet contains the vierbein and the gravitino $\psi^-_\mu$, and the massive gravitino $\psi^+_\mu$, U(1) gauge boson $A^n_\mu$, the graviphoton $A^0_\mu$ and the fermion $\eta_\mu$ form a massive spin-3/2 multiplet. The $\mathcal{N} = 2$ vector multiplets have been divided into $\mathcal{N} = 1$ vector multiplets and chiral multiplets. The $\mathcal{N} = 1$ vector multiplets described by massless gauge bosons $A^a_\mu$ and gauginos $\lambda^{a+}$ which become to $SU(N)$ vector multiplet. On the other hand, the gaugino $\lambda^{a-}$ and the scalar field $z^a$ form chiral multiplets which belong to the $SU(N)$ adjoint representation. The gaugino $\lambda^{a-}$ and the complex scalar $z^a$ form a $\mathcal{N} = 1$ massive chiral multiplet. Also the hyperino $\zeta_+$ and the scalar $b^0, b^1$ form a chiral multiplet. Note that although this mass spectrum is analogous to the rigid counterpart [10], the phase of the theory is different: our vacuum lies in the Higgs phase, while the rigid one in the Coulomb phase.
### Table 6.1: the mass spectrum

| Multiplet                         | Field          | Mass |
|-----------------------------------|----------------|------|
| gravity multiplet                 | \( e_i^\mu, \psi^-_\mu \) | 0    |
| spin-3/2 multiplet                | \( \psi^+_\mu, A^0_\mu, A^m_\mu, \eta \) | \( m \) |
| SU(\( N \)) vector multiplet     | \( A^{a\mu}, \lambda^{a-} \) | 0    |
| SU(\( N \)) adjoint chiral multiplet | \( \lambda^{a+}, z^a \) | \( m^a \) |
| chiral multiplet                  | \( \lambda^{n-}, z^n \) | \( m^n \) |
| chiral multiplet                  | \( \zeta^+, b^+, b^- \) | 0    |

6.4 \( \mathcal{N} = 1 \) Lagrangian

In the last section, we have considered the lowest order terms with respect to the fermion fields and the shifted scalar fields \( \tilde{z}^a \). Now, we would like to know all of the terms in \( \mathcal{L}_{\text{Yukawa}} \) and \( V \). This can be done exactly and, at the end of this section, we will see that \( \mathcal{N} = 1 \) Lagrangian can be written by the superpotentials.

In terms of \( \tilde{z}^a \), the holomorphic function \( \mathcal{F}(z) \) is expanded as,

\[
\mathcal{F}(z) = \mathcal{F}(\langle \langle z \rangle \rangle + \tilde{z})
\]

\[= \langle \langle \mathcal{F} \rangle \rangle + \tilde{\mathcal{F}}, \tag{6.4.1} \]

where

\[
\tilde{\mathcal{F}} = \langle \langle \mathcal{F}_a \rangle \rangle \tilde{z}^a + \frac{1}{2!} \langle \langle \mathcal{F}_{ab} \rangle \rangle \tilde{z}^a \tilde{z}^b + \frac{1}{3!} \langle \langle \mathcal{F}_{abc} \rangle \rangle \tilde{z}^a \tilde{z}^b \tilde{z}^c + \ldots. \tag{6.4.2} \]

Similarly, \( \mathcal{F}_a \) and \( \mathcal{F}_{ab} \) are

\[\mathcal{F}_a = \langle \langle \mathcal{F}_a \rangle \rangle + \langle \langle \mathcal{F}_{ab} \rangle \rangle \tilde{z}^b + \frac{1}{2!} \langle \langle \mathcal{F}_{abc} \rangle \rangle \tilde{z}^b \tilde{z}^c + \ldots. \tag{6.4.3} \]

\[= \tilde{\mathcal{F}}_a, \]

\[\mathcal{F}_{ab} = \langle \langle \mathcal{F}_{ab} \rangle \rangle + \langle \langle \mathcal{F}_{abc} \rangle \rangle \tilde{z}^c + \ldots. \tag{6.4.4} \]

\[= \tilde{\mathcal{F}}_{ab}.\]

We have redefined \( \tilde{\mathcal{F}}_a \) and \( \tilde{\mathcal{F}}_{ab} \) as the derivative of \( \tilde{\mathcal{F}} \) with respect to \( \tilde{z}^a \). The Kähler potential, its derivative and the Kähler metric are, respectively,

\[
\mathcal{K} = -\log i \left[ \langle \langle \mathcal{F} - \tilde{\mathcal{F}} \rangle \rangle + \tilde{\mathcal{F}} - \tilde{\mathcal{F}} - \frac{1}{2}(\langle \langle z^a - \tilde{z}^a \rangle \rangle + z^a - \tilde{z}^a)(\tilde{\mathcal{F}}_a + \tilde{\mathcal{F}}_a) \right], \tag{6.4.5} \]

\[
\partial_a \mathcal{K} = -\frac{i}{2\mathcal{K}_0}(\tilde{\mathcal{F}}_a - \tilde{\mathcal{F}}_a - (\langle \langle z^a - \tilde{z}^a \rangle \rangle + z^a - \tilde{z}^a)\tilde{\mathcal{F}}_{ab}) \tag{6.4.6} \]

\[= \tilde{\partial}_a \mathcal{K},\]

\[g_{ab^*} = \tilde{\partial}_a \mathcal{K} \tilde{\partial}_{b^*} \mathcal{K} - \frac{i}{2\mathcal{K}_0}(\tilde{\mathcal{F}}_{ab} - \tilde{\mathcal{F}}_{ab}) \tag{6.4.7} \]

\[= \tilde{g}_{ab^*}.\]
where \( \tilde{\partial}_a = \partial/\partial \tilde{z}^a \). In this way, we can write down all the terms in the Lagrangian in terms of the new scalar field \( \tilde{z}^a \). In the following, let us see several terms of the Lagrangian.

**The Yukawa interaction terms**

The Yukawa interaction terms \( \mathcal{L}_{\text{Yukawa}} \) can be easily rewritten with these notations. First of all, we define the following quantities,

\[
\mathcal{W}(\tilde{z}, \bar{\tilde{z}}) \equiv e^{K/2} \mathcal{W}(\tilde{z}) \equiv 2(S_{11} - S_{12}) = \frac{e^{K/2}}{\sqrt{2b_0}} g_3 (\bar{\mathcal{F}}_n - \langle \mathcal{F}_n \rangle),
\]

\[
\mathcal{S}(\tilde{z}, \bar{\tilde{z}}) \equiv e^{K/2} \mathcal{S}(\tilde{z}) \equiv 2(S_{11} + S_{12}) = \frac{e^{K/2}}{\sqrt{2b_0}} (2g_2 + g_3 (\bar{\mathcal{F}}_n + \langle \mathcal{F}_n \rangle)),
\]

where \( S_{AB} \) is the gravitino mass matrix. Note that \( \mathcal{W} \) and \( \mathcal{S} \) are related as follows:

\[
\mathcal{W} = \mathcal{S} + \frac{i\sqrt{2} g_1 e^{K/2} b_0}{b_0}. \quad (6.4.10)
\]

Thus they are not independent. However, in the following, for convenience, we will write down the resulting Lagrangian, using both \( \mathcal{W}, \mathcal{S} \).

The first term of \( \mathcal{L}_{\text{Yukawa}} \) is rewritten in terms of \( \mathcal{W} \) and \( \mathcal{S} \) as

\[
2S_{AB} \psi^A_{\mu} \gamma^{\mu\nu} \psi^B_{\nu} = \mathcal{W} \bar{\psi}^A_{\mu} \gamma^{\mu\nu} \psi^B_{\nu} + \mathcal{S} \bar{\psi}^A_{\mu} \gamma^{\mu\nu} \psi^B_{\nu}. \quad (6.4.11)
\]

Thus, we refer to \( \mathcal{W} \) and \( \mathcal{S} \) as superpotentials. Since the usual \( \mathcal{N} = 1 \) supergravity models coupled to chiral multiplets contain one gravitino, there is only one superpotential. But, in this \( \mathcal{N} = 1 \) model obtained through the partial supersymmetry breaking, there are two gravitini. Therefore there exist two superpotentials which are related by eq. (6.4.10).

The covariant derivatives of \( \mathcal{W} \) and that of \( \mathcal{S} \) are respectively

\[
\tilde{\nabla}_a \mathcal{W} = \frac{e^{K/2}}{\sqrt{2b_0}} g_3 (\bar{\mathcal{F}}_n a + \bar{\tilde{\partial}}_a K \bar{\mathcal{F}}_n - \bar{\tilde{\partial}}_b K \langle \mathcal{F}_n \rangle)
\]

\[
\tilde{\nabla}_a \mathcal{S} = \frac{e^{K/2}}{\sqrt{2b_0}} (2g_2 \bar{\tilde{\partial}}_a K + g_3 (\bar{\mathcal{F}}_n a + \bar{\tilde{\partial}}_a K \bar{\mathcal{F}}_n + \bar{\tilde{\partial}}_a K \langle \mathcal{F}_n \rangle))
\]

and the second derivatives of them are evaluated as,

\[
\tilde{\nabla}_a \tilde{\nabla}_b \mathcal{W} = \sqrt{2} (\mathcal{M}_{2:1|1} - \mathcal{M}_{2:1|2}), \quad (6.4.14)
\]

\[
\tilde{\nabla}_a \tilde{\nabla}_b \mathcal{S} = \sqrt{2} (\mathcal{M}_{2:1|1} + \mathcal{M}_{2:1|2}), \quad (6.4.15)
\]

\(^1\)For example, [39] or [40, 41].
where $\tilde{W}_{ab} = (W^{ab})^*$. This can be used to rewrite the second term and the last term of $\mathcal{L}_{\text{Yukawa}}$:

$$i g_{ab} W^{aAB} \bar{\chi}_A^b \gamma_\mu \psi_B^\mu = e^{K/2} \bar{g}_{ab} \ast \mathcal{D}^a (\bar{\chi}_A^b \gamma_\mu \psi_B^\mu - \bar{\lambda}_A^b \gamma_\mu \psi_B^\mu)$$

$$+ i \tilde{\nabla}_a \bar{W} \bar{\lambda}_A^b \gamma_\mu \psi_B^\mu + i \tilde{\nabla}_a \bar{S} \bar{\lambda}_A^b \gamma_\mu \psi_B^\mu, \quad (6.4.16)$$

$$\mathcal{M}_{aA|bB} \bar{\chi}_A^b \bar{\lambda}_B^a = \mathcal{M}_{1;1|2}(\bar{\chi}_A^a - \lambda_A^+ + \bar{\lambda}_A^a + \lambda_B^b)$$

$$+ \frac{1}{\sqrt{2}} \tilde{\nabla}_a \bar{\nabla}_b \bar{W} \bar{\lambda}_A^a - \lambda_B^b + \frac{1}{\sqrt{2}} \tilde{\nabla}_a \bar{\nabla}_b \bar{S} \bar{\lambda}_A^a + \lambda_B^b. \quad (6.4.17)$$

In this way, we can evaluate all the terms of $\mathcal{L}_{\text{Yukawa}}$. As a result, we obtain

$$\mathcal{L}_{\text{Yukawa}} = \bar{W} \bar{\psi}_- \gamma^\mu \psi_- + i(\tilde{\nabla}_a \bar{W} \bar{\lambda}_A^a - 2\bar{V} \bar{\zeta}^+) \gamma_\mu \psi_- - e^{K/2} \bar{g}_{ab} \ast \mathcal{D}^a \bar{\lambda}_A^b \gamma_\mu \psi_-$$

$$+ S \bar{\psi}_+ \gamma^\mu \psi_+ + i(\tilde{\nabla}_a S \bar{\lambda}_A^a - 2S \bar{\zeta}^-) \gamma_\mu \psi_+ + e^{K/2} \bar{g}_{ab} \ast \mathcal{D}^a \bar{\lambda}_A^b \gamma_\mu \psi_+$$

$$+ \bar{W} \bar{\zeta}_+ \zeta_+ + S \bar{\zeta}_+ \zeta_- - 2 \tilde{\nabla}_a \bar{W} \bar{\zeta}_+ \lambda_- + 2 \tilde{\nabla}_a S \bar{\zeta}_- \lambda_+ + \mathcal{M}_{1;1|2}(\bar{\chi}_A^a - \lambda_A^+ - \bar{\lambda}_A^a + \lambda_B^b)$$

$$+ \frac{1}{\sqrt{2}} \tilde{\nabla}_a \bar{\nabla}_b \bar{W} \bar{\lambda}_A^a - \lambda_B^b + \frac{1}{\sqrt{2}} \tilde{\nabla}_a \bar{\nabla}_b \bar{S} \bar{\lambda}_A^a + \lambda_B^b. \quad (6.4.18)$$

Note that if we substitute $\bar{z}^a = 0$ into $\mathcal{L}_{\text{Yukawa}}$, it reduces to the fermion mass terms which have been obtained in the last section.

**The scalar potential**

We turn to the scalar potential $V$. It is useful to rewrite it in terms of the mass matrices as

$$V = -12 S_1^A S_{A1} + \bar{g}_{ab} \ast \bar{W}_{1A}^b W^{a1A} + 2 N_1^a N_1^a. \quad (6.4.19)$$

which is also obtained from the supergravity Ward identities in reference [12]. Note that $S_1^{AB} = (S_{AB})^*$ and $N_1^a = (N_A^a)^*$.

Let us rewrite (6.4.19) in terms of the superpotentials. The first term is

$$-12 S_1^A S_{A1} = -6((S_{11} - S_{12})(S_{11} - S_{12}) + (S_{11} + S_{12})(S_{11} + S_{12}))$$

$$= \frac{3}{2} (|W|^2 + |S|^2), \quad (6.4.20)$$

and the second term is

$$\bar{g}_{ab} \ast \bar{W}_{1A}^b W^{a1A} = \bar{g}_{ab} \ast \left( \bar{W}_{11}^b W_{1}^{a11} + \bar{W}_{21}^b W_{2}^{a11} + \bar{W}_{12}^b W_{2}^{a12} \right)$$

$$= e^{K/2} \bar{g}_{ab} \ast D^a D^b + \frac{1}{2} g^{ab} \tilde{\nabla}_a \bar{W} \tilde{\nabla}_a W + \frac{1}{2} g^{ab} \tilde{\nabla}_a \bar{S} \tilde{\nabla}_a S. \quad (6.4.21)$$

In the first equality, we have used (6.1.16). The last term is

$$2 N_1^a N_1^a = |W|^2 + |S|^2$$

$$= \frac{1}{2} h^{uv} \nabla_u W \nabla_v \bar{W} + \frac{1}{2} h^{uv} \nabla_u S \nabla_v \bar{S}, \quad (6.4.22)$$

55
where \(u, v = 0, 1\) and \(h_{uv} \equiv \delta_{uv}/(b^0)^2\). Note that \(\nabla_u W = \partial_u W\). Substituting (6.4.20) - (6.4.22) into (6.4.19), we have

\[
V = e^{K/2}g_{ab}D^aD^b + \frac{1}{2}g^{ab}\tilde{\nabla}_a W\tilde{\nabla}_b W + \frac{1}{2}g^{ab}\tilde{\nabla}_a S\tilde{\nabla}_b S

- \frac{3}{2}|W|^2 - \frac{3}{2}|S|^2 + \frac{1}{2}h_{uv}\nabla_u W\nabla_v \bar{W} + \frac{1}{2}h_{uv}\nabla_u S\nabla_v \bar{S}.
\]

(6.4.23)

This is the final form of the scalar potential. We can see that \(L_{Yukawa}\) and the scalar potential \(V\) takes essentially the same form as the usual \(\mathcal{N} = 1\) supergravity model (such as [39] or [40, 41]). Note that the superpotentials \(W\) and \(S\) are related by (6.4.10).

### 6.5 Conclusion

We have seen that, in the \(\mathcal{N} = 2\) \(U(N)\) gauged supergravity model, the \(\mathcal{N} = 2\) supersymmetry has been broken to \(\mathcal{N} = 1\) counterpart spontaneously. In particular, we have not chosen the symplectic section as the simplest function. This leads to the effective theory which contains higher order coupling terms of the scalar fields. As we have seen in the last section, the Nambu-Goldstone fermion and the Nambu-Goldstone bosons are absorbed by the gravitino and the gauge bosons respectively, through the super-Higgs and the Higgs mechanisms. All the masses of the fermions and the bosons have been evaluated. The gauge symmetry \(U(N) \times U(1)_{\text{graviphoton}}\) is broken to \(SU(N)\) and the resulting model lies in the Higgs phase. Finally, we have considered the \(\mathcal{N} = 1\) Lagrangian. The \(\mathcal{N} = 1\) Yukawa interaction terms and the \(\mathcal{N} = 1\) scalar potential can be written in terms of the superpotentials.

As is pointed out in [13], if we force the gravity and the hypermultiplet to decouple, the gravitino mass (6.3.5) becomes zero. Thus, the gauge boson corresponding to the overall \(U(1)\) and the graviphoton become massless in this limit. The Higgs phase of \(U(1) \times U(1)_{\text{graviphoton}}\) approaches the Coulomb phase.

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Appendix A

Conventions and notations

Minkowski metric is defined as $\eta_{ij} \equiv (1, -1, -1, -1)$. The Riemann Tensor is

$$R^\mu{}_{\nu} = d\Gamma^\mu{}_{\nu} + \Gamma^\mu{}_{\rho} \wedge \Gamma^\rho{}_{\nu} \equiv -\frac{1}{2}R^\mu{}_{\nu\rho\sigma}dx^\rho \wedge dx^\sigma.$$

Decomposition of tensors in self-dual and antiself-dual parts is

$$T^\pm{}_{\mu\nu} = \frac{1}{2} \left( T_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} \right).$$

A.1 Notations in quaternionic Kähler manifolds

SU(2) and Sp(2k) metrics

The flat SU(2) and Sp(2k) metrics satisfy:

$$\epsilon^{AB}\epsilon_{BC} = -\delta^A_C, \quad \epsilon^{AB} = -\epsilon^{BA}, \quad \epsilon^{12} = \epsilon_{12} = +1,$$

$$C^{\alpha\beta}C_{\beta\gamma} = -\delta^\alpha_\gamma, \quad C^{\alpha\beta} = -C^{\beta\alpha}, \quad C^{12} = C_{12} = +1.$$

For any SU(2) vector $P_A$ we have:

$$\epsilon_{AB}P^B = P_A,$$

$$\epsilon^{AB}P_B = -P^A,$$

and for any Sp(2k) vectors $P_\alpha$ we have:

$$C_{\alpha\beta}P^\beta = P_\alpha,$$

$$C^{\alpha\beta}P_\beta = -P^\alpha.$$

Pauli matrices

The standard Pauli matrices $(\sigma^x)^B_A$ ($x = 1, 2, 3$) are

$$(\sigma^1)^B_A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^2)^B_A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\sigma^3)^B_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
The Pauli matrices with two lower indices are defined as follows:

\[(\sigma^x)_{AB} \equiv (\sigma^x)_A^C \epsilon_{BC}, \tag{A.1.8}\]
\[(\sigma^x)^{AB} \equiv -(\sigma^x)_B^C \epsilon^{AC}. \tag{A.1.9}\]

The above equations can be written as

\[(\sigma^1)_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (\sigma^2)_{AB} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, (\sigma^3)_{AB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A.1.10}\]
\[(\sigma^1)^{AB} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, (\sigma^2)^{AB} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, (\sigma^3)^{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.1.11}\]

These imply the following equation:

\[(\sigma^x_{AB})^* = -\sigma^x_{AB}. \tag{A.1.12}\]

If we define \((\sigma^x)^A_B\) as

\[(\sigma^x)^A_B = -\epsilon^{AC}(\sigma^T)_C^B, \tag{A.1.13}\]

we obtain

\[(\sigma^1)^A_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^2)^A_B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (\sigma^3)^A_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1.14}\]

### A.2 Spinor conventions

**Clifford algebra**

The gamma matrices in 4-dimensions are given as follows:

\[\{\gamma_i, \gamma_j\} = 2\eta_{ij},\]
\[[\gamma_i, \gamma_j] = 2\gamma_{ij},\]
\[\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5^\dagger = \gamma_5, \quad \gamma_5^2 = 1,\]
\[\{\gamma_5, \gamma_i\} = 0,\]
\[\gamma_0^\dagger = \gamma_0,\]
\[\epsilon_{ijkl}\gamma^{kl}2i\gamma_{ij}\gamma_5, \quad \epsilon^{ijkl}\gamma_{kl} = 2i\gamma_{ij}\gamma_5.\]
Chirality

The upper or lower position of the indices of the spinors fix their chirality as follows:

\[
\gamma_5 \begin{pmatrix}
\psi_A \\
\chi^a A \\
\zeta_\alpha \\
\chi^* \\
\eta^* 
\end{pmatrix} = \begin{pmatrix}
\psi_A \\
\chi^a A \\
\zeta_\alpha \\
\chi^* \\
\eta^* 
\end{pmatrix} : \text{right handed}, \quad (A.2.2)
\]

\[
\gamma_5 \begin{pmatrix}
\psi^A \\
\lambda^a_A \\
\zeta^\alpha_a \\
\lambda^a_A \\
\chi^* \\
\eta^* 
\end{pmatrix} = - \begin{pmatrix}
\psi^A \\
\lambda^a_A \\
\zeta^\alpha_a \\
\lambda^a_A \\
\chi^* \\
\eta^* 
\end{pmatrix} : \text{left handed}. \quad (A.2.3)
\]

Majorana conditions

For any fermion \( \phi \), the Majorana condition is

\[
\bar{\phi} \equiv \phi^\dagger \gamma_0 = \phi^T C, \quad (A.2.4)
\]

where the charge conjugation matrix has the following properties:

\[
C^2 = -1, \quad C^T = -C, \quad (C\gamma^i)^T = C\gamma^i, \quad (C\gamma^{ij})^T = C\gamma^{ij}, \quad (C\gamma_5)^T = -C\gamma_5, \quad (C\gamma_5\gamma^i)^T = -C\gamma_5\gamma^i,
\]

\[
(C\gamma_5\gamma^{ij})^T = -C\gamma_5\gamma^j\gamma^i, \quad (C\gamma_5\gamma^{ij})^T = C\gamma_5\gamma^{ij}. \quad (A.2.5)
\]

Hermiticity of currents

For 0-form spinors:

\[
(\bar{\chi} \eta^* \eta^*)^\dagger = \bar{\eta}^* \chi^* = \bar{\chi}^* \eta^*, \quad (A.2.6)
\]

\[
(\bar{\chi} \gamma^i \eta^* \eta^*)^\dagger = \bar{\eta}^* \gamma^i \chi^* = -\bar{\chi}^* \gamma^i \eta^*, \quad (A.2.7)
\]

\[
(\bar{\chi} \gamma^{ij} \eta^* \eta^*)^\dagger = -\bar{\eta}^* \gamma^{ij} \chi^* = \bar{\chi}^* \gamma^{ij} \eta^*. \quad (A.2.8)
\]

For 1-form spinors

\[
(\bar{\psi}_A \psi_B)^\dagger = -\bar{\psi}_B \psi_A = \bar{\psi}_A \psi_B, \quad (A.2.9)
\]

\[
(\bar{\psi}^A \gamma^i \psi_B)^\dagger = -\bar{\psi}^B \gamma^i \psi_A = -\bar{\psi}^A \gamma^i \psi_B, \quad (A.2.10)
\]

\[
(\bar{\psi}^A \gamma^{ij} \psi_B)^\dagger = \bar{\psi}_B \gamma^{ij} \psi_A = \bar{\psi}_A \gamma^{ij} \psi_B. \quad (A.2.11)
\]
Appendix B

The Bianchi identities and its solutions

B.1 Ungauged case

Let us start from ungauged case. We introduce the basic fields of $\mathcal{N} = 2$ supergravity and their associated curvatures as differential forms in superspace. We take the one-forms $e^i$ and $\psi_A$, as a basis for $\mathcal{N} = 2$ superspace, whose space-time components $e^i_\mu$ and $\psi_{A\mu}$ are the ordinary vierbein and gravitino fields. The curvatures of $\mathcal{N} = 2$ ungauged supergravity coupled to the vector multiplets and hypermultiplet are as follows:

The superspace curvatures in gravitational sector are defined as,

\begin{align}
T^i &\equiv D e^i - i \bar{\psi}_A &\wedge & \gamma^i \psi^A, \\
\rho_A &\equiv \nabla \psi_A \equiv d \psi_A - \frac{1}{4} \gamma_{ij} \omega^{ij} \wedge \psi_A + \frac{i}{2} Q \wedge \psi_A + \omega^B_A \wedge \psi_B, \\
\rho^A &\equiv \nabla \psi^A \equiv d \psi^A - \frac{1}{4} \gamma_{ij} \omega^{ij} \wedge \psi^A - \frac{i}{2} Q \wedge \psi^A + \omega^B_A \wedge \psi_B, \\
R^{ij} &\equiv d \omega^{ij} - \omega^i_k \wedge \omega^{kj},
\end{align}

where \(T^i\) is the torsion 2-form, \(R^{ij}\) is the space-time Ricci 2-form, \(\omega^{ij}\) is the spin-connection 1-form, \(Q\) is $U(1)$ bundle connection which has been defined in (2.1.17), \(\omega^{B}_A \equiv \frac{1}{2} (\sigma_x)^B_A \omega^x\) (\(\omega^A_B = \epsilon^{AC} \omega^D_C \epsilon_{DB}\)) are $SU(2)$ bundle connection which has been defined in the section 2.3.

The curvatures and covariant derivatives in the vector multiplet sector are

\begin{align}
\nabla z^a &= dz^a, \\
\nabla \bar{z}^a &= d\bar{z}^a, \\
\nabla \lambda^{aA} &\equiv d \lambda^{aA} - \frac{1}{4} \gamma_{ij} \omega^{ij} \lambda^{aA} - \frac{i}{2} Q \lambda^{aA} + \Gamma^a_{bA} \lambda^b + \omega^B_A \lambda^{aB}, \\
\nabla \bar{\lambda}^{a*} &\equiv d \bar{\lambda}^{a*} - \frac{1}{4} \gamma_{ij} \omega^{ij} \bar{\lambda}^{a*} + \frac{i}{2} Q \bar{\lambda}^{a*} + \Gamma^a_{bA} \bar{\lambda}^{b*} + \omega^B_A \bar{\lambda}^{aB*}, \\
F^A &\equiv d A^A + \tilde{L} \bar{\psi}_A \wedge \psi_B \epsilon^{AB} + L^A \bar{\psi} \wedge \psi_B \epsilon_{AB},
\end{align}
where $A^\Lambda (\Lambda = 0, \ldots, m)$ is the gauge connection 1-form: the value $\Lambda = 0$ corresponds to the graviphoton and $\Lambda = 1, \ldots, m$ corresponds to the gauge bosons of $m$ vector multiplets. The quantities $L^\Lambda$ and $\bar{L}^\Lambda$ are arbitrary functions of $z^a$ and $\bar{z}^a$ on Kähler manifold, but, as we will see later, the Bianchi identities constrain in such a way that they coincide with the objects defined in (2.2.6). Note that we do not assume that the vector multiplet scalar sector is described by the special Kähler geometry rather we only assume it is described by Hodge-Kähler geometry.

The covariant derivatives in the hypermultiplet sector are

$$\nabla \zeta_\alpha \equiv d \zeta_\alpha - \frac{1}{4} \gamma_{ij} \omega^{ij} \zeta_\alpha - \frac{i}{2} Q^\alpha_\beta \zeta_\beta + \Delta_\beta^\alpha \zeta_\beta,$$

$$\nabla \bar{\zeta}_\bar{\alpha} \equiv d \bar{\zeta}_\bar{\alpha} - \frac{1}{4} \gamma_{ij} \bar{\omega}^{ij} \bar{\zeta}_\bar{\alpha} + \frac{i}{2} \bar{Q}^\alpha_\beta \bar{\zeta}_\beta + \Delta_\beta^\alpha \bar{\zeta}_\beta,$$

where $\Delta_\beta^\alpha$ is the gauged Levi-Civita connection on $\mathcal{HM}$ defined in the section 2.3, satisfying the conditions (2.3.11). It is convenient to convert the world index of the curvature $db^u$ into a flat index $A, \alpha$ by means of the quaternionic vielbein such that,

$$U_A^\alpha \equiv \mathcal{U}_i^A db^u.$$

The Bianchi identities in the ungauged case

We can derive the following Bianchi identities: in the gravitational sector,

$$\mathcal{D}T^i + R^{ij} \wedge e_j - i \bar{\psi}^A \wedge \gamma^i \rho_A + i \tilde{\psi}^A \wedge \gamma^i \psi_A = 0,$$

$$\nabla \rho_A + \frac{1}{4} \gamma_{ij} R^{ij} \wedge \psi_A - \frac{i}{2} K \wedge \psi_A - R^A_B \wedge \psi_B = 0,$$

$$\nabla \rho^A + \frac{1}{4} \gamma_{ij} \bar{R}^{ij} \wedge \bar{\psi}_A + \frac{i}{2} K \wedge \bar{\psi}_A - R^A_B \wedge \bar{\psi}_B = 0,$$

$$\mathcal{D}R^{ij} = 0,$$

where $R^A_B$ is the $SU(2)$ curvature defined in (2.3.17) and $K$ is the Kähler 2-form, $K = dQ$. The covariant derivative $\mathcal{D}$ have defined as, for vector $V^i$ and tensor $V^{ij}$,

$$\mathcal{D}V^i = dV^i - \omega^i_j \wedge V^j,$$

$$\mathcal{D}V^{ij} = dV^{ij} - \omega^i_k \wedge V^{kj} + \omega^j_k \wedge V^{ik}.$$

In the vector multiplet sector, we obtain

$$d^2 z^a = d^2 \bar{z}^a = 0,$$

$$\nabla^2 \lambda^a = \frac{1}{4} \gamma_{ij} R^{ij} \lambda^a + \frac{i}{2} K \lambda^a + R^a_b \lambda^b - \frac{i}{2} R^A_B \lambda^a = 0,$$

$$\nabla^2 \lambda^a = \frac{1}{4} \gamma_{ij} \bar{R}^{ij} \lambda^a - \frac{i}{2} K \lambda^a + R^a_b \lambda^b - \frac{i}{2} R^A_B \lambda^a = 0,$$

$$dF^A = \epsilon_{AB} \nabla L^A \wedge \bar{\psi}^A \wedge \psi^B + 2 \epsilon_{AB} L^A \bar{\psi}^A \wedge \rho^B - \epsilon_{AB} \bar{L}^A \wedge \psi_A \wedge \psi_B + 2 \epsilon_{AB} \bar{L}^A \bar{\psi}_A \wedge \rho_B = 0,$$
where $R^\alpha_b$ is the Ricci 2-form of the Hodge-Kähler manifold.

In the hypermultiplet sector, we have

\[
\nabla^2 \zeta_\alpha + \frac{1}{4} \gamma_{ij} R^{ij}_\alpha \zeta_\alpha + R^\alpha_\beta \zeta_\beta + \frac{i}{2} K \zeta_\alpha = 0,
\]

(B.1.9a)

\[
\nabla^2 \zeta^\alpha + \frac{1}{4} \gamma_{ij} R^{ij\alpha} + R^\alpha_\beta \zeta^\beta - \frac{i}{2} K \zeta^\alpha = 0,
\]

(B.1.9b)

\[
\nabla U^{\alpha} = 0,
\]

(B.1.9c)

where $\mathbb{R}^\alpha_\beta$ is the $Sp(2m)$ curvature which has been defined in (2.3.18).

**The solutions of the Bianchi identities in the ungauged case**

The solutions of the Bianchi identities (B.1.5a)–(B.1.9c) in the ungauged case are written as follows:

\[
T^i = 0,
\]

(B.1.10)

\[
\rho_A = \rho_{A|ij} e^i \wedge e^j + (A^B_{A|j} \eta^{ij} + A^{IB}_{A|j} \gamma^j) \psi_B \wedge e^i + \epsilon_{AB} (T^i + U^i) \gamma^j \psi^B \wedge e^i,
\]

(B.1.11)

\[
\rho^A = \rho_{ij}^A e^i \wedge e^j + (A^B_{B|j} \eta^{ij} + A^{IB}_{B|j} \gamma^j) \psi_B \wedge e^i + \epsilon^{AB} (T^i + U^i) \gamma^j \psi^B \wedge e^i,
\]

(B.1.12)

\[
R^{ij} = R^{ij}_{kl} e^k \wedge e^j - i(\overline{\psi}_A \theta^A_{ij} + \overline{\psi}_B \theta^{ij}) \wedge e^k + \epsilon^{ijkl} \overline{\psi}_A \wedge \gamma_k \psi_B (A^B_{A|k} - A^A_{B|k}) + i \epsilon^{AB} \overline{\psi}_A \wedge \psi_B (T^{+ij} + U^{-ij}) - i \epsilon_{AB} \overline{\psi}_A \wedge \psi^B (T^{-ij} + U^{+ij}),
\]

(B.1.13)

\[
d z^a = Z^a_i e^i + \overline{\psi}_A \lambda^A_i,
\]

(B.1.14)

\[
d z^{\alpha^*} = Z^{\alpha^*}_i \bar{e}^i + \overline{\psi}_A \lambda^A_{\alpha^*},
\]

(B.1.15)

\[
\nabla \lambda^{\alpha A} = \nabla_i \lambda^{\alpha A} e^i + i Z^{\alpha A}_i \gamma^i \psi_A + G_{ij}^{-a} \gamma^i \psi^B e^{AB} + Y^{aAB} \psi_B,
\]

(B.1.16)

\[
\nabla \lambda^A_{\alpha^*} = \nabla_i \lambda^A_{\alpha^*} e^i + i Z^A_{\alpha^*} \gamma^i \psi_A + G_{ij}^{+a} \gamma^i \psi^B e^{AB} + Y_{AB}^a \psi_B,
\]

(B.1.17)

\[
F^A = F^A_i e^i \wedge e^j + i (j^A_{\alpha^*} \lambda^{A}_{\alpha^*} \gamma_i \psi^B e^{AB} + j^A_{\alpha} \bar{\lambda}^{A}_{\alpha} \gamma_i \psi_B e^{AB}) \wedge e^i,
\]

(B.1.18)

\[
\nabla \zeta_\alpha = \nabla_i \zeta_\alpha e^i + i U^{B\beta}_i \gamma^i \psi^A e^{AB} C_{\alpha\beta},
\]

(B.1.19)

\[
\nabla \zeta^\alpha = \nabla_i \zeta^\alpha e^i + i U^{A\alpha}_i \gamma^i \psi_A,
\]

(B.1.20)

\[
U^{\alpha A} = U^{\alpha A}_i e^i + \epsilon^{AB} C_{\alpha\beta} \psi_B \zeta_\beta + \overline{\psi}_A \zeta^\alpha,
\]

(B.1.21)
where
\[ A_A^{iB} = -\frac{i}{4} g_{aB} (\bar{\lambda}_A^a \gamma_i \lambda^B) - \delta^n_A \bar{\lambda}_A^a \gamma_i \lambda^B, \quad (B.1.22a) \]
\[ A_A^{iB} = \frac{i}{4} g_{aB} (\bar{\lambda}_A^a \gamma_i \lambda^B) - \frac{1}{2} \delta^n_A \bar{\lambda}_A^a \gamma_i \lambda^B + \frac{i}{4} \lambda \delta^n_A \zeta_\alpha \gamma_i \zeta^\alpha, \quad (B.1.22b) \]
\[ \theta^{ij;k}_{\Lambda} = 2 \gamma^{[i} P_{\Lambda}^{ij} + \gamma^j P_{\Lambda}^{ij}; \quad \theta^{ij;k|\Lambda} = 2 \gamma^{[i} P_{\Lambda}^{ijk] + \gamma^k P_{\Lambda}^{ijk|\Lambda}, \quad (B.1.23) \]

\[ T_{ij}^- = 2i (\text{Im} N)_{\Lambda \Sigma} L^\Sigma \left( F_{ij}^- + \frac{1}{8} \nabla_a f_b \bar{\lambda}_A^a \gamma_{ij} \lambda^B \epsilon_{AB} + \frac{\lambda}{4} C^{\alpha \beta \gamma_{ij} \zeta_{\beta} L^\Lambda} \right), \quad (B.1.24a) \]

\[ T_{ij}^+ = 2i (\text{Im} N)_{\Lambda \Sigma} L^\Sigma \left( F_{ij}^- + \frac{1}{8} \nabla_a f_b \bar{\lambda}_A^a \gamma_{ij} \lambda^B \epsilon_{AB} + \frac{\lambda}{4} C^{\alpha \beta \gamma_{ij} \zeta_{\beta} L^\Lambda} \right), \quad (B.1.24b) \]

\[ U_{ij}^- = \frac{i}{4} \lambda C^{\alpha \beta \gamma_{ij} \zeta_{\beta}}, \quad (B.1.25a) \]

\[ U_{ij}^+ = \frac{i}{4} \lambda C^{\alpha \beta \gamma_{ij} \zeta_{\beta}}, \quad (B.1.25b) \]

\[ C_{ij}^- = -g^{ab} f_b \Sigma (\text{Im} N)_{\Lambda \Sigma} \left( F_{ij}^- + \frac{1}{8} \nabla_a f_b \bar{\lambda}_A^a \gamma_{ij} \lambda^B \epsilon_{AB} + \frac{\lambda}{4} C^{\alpha \beta \gamma_{ij} \zeta_{\beta} L^\Lambda} \right), \quad (B.1.26a) \]

\[ G_{ij}^+ = -g^{ab} f_b \Sigma (\text{Im} N)_{\Lambda \Sigma} \left( F_{ij}^+ + \frac{1}{8} \nabla_a f_b \bar{\lambda}_A^a \gamma_{ij} \lambda^B \epsilon_{AB} + \frac{\lambda}{4} C^{\alpha \beta \gamma_{ij} \zeta_{\beta} L^\Lambda} \right), \quad (B.1.26b) \]

\[ Y_{A^a B^b} = \frac{i}{2} g^{ab} C_{b^c c^d} \bar{\lambda}_A^c \lambda^D \epsilon^{AC} \epsilon^{BD}, \quad (B.1.27a) \]

\[ Y_{A^a B^b} = -\frac{i}{2} g^{ab} C_{b^c c^d} \bar{\lambda}_A^c \lambda^D \epsilon^{AC} \epsilon^{BD}. \quad (B.1.27b) \]

Furthermore, from the closure of the Bianchi identities, we obtain the constraints on \( L^\Lambda, \quad \bar{L}^\Lambda, \quad f_a^A, \quad f_a^A, \) and \( C_{abc} \) which are the geometrical object of the Hodge-Kähler manifold. This constraints restrict the Hodge-Kähler manifold to be a special Kähler manifold. We will discuss this at the end of the next section.

**B.2 Gauged case**

Let us consider the modifications to the previous results when the theory is gauged. The discussion in this section is parallel to that in the previous section. In this case, we have to redefine the curvature \((B.1.1a) - (B.1.3b)\) according to the discussions in the section 3.2. In particular, for the scalar fields \( z^a, \bar{z}^a, \) and \( b^a, \) we replace as follows:

\[ dz^a \rightarrow \nabla z^a = dz^a + g A^A k^a_A(z), \quad (B.2.1a) \]

\[ d\bar{z}^a \rightarrow \nabla \bar{z}^a = d\bar{z}^a + g A^A k^a_A(\bar{z}), \quad (B.2.1b) \]

\[ db^a \rightarrow \nabla b^a = db^a + g A^A k^a_A(b). \quad (B.2.1c) \]
Eq. (B.2.1c) implies that the gauged quaternionic vielbein $\hat{U}^{\dot{A}\alpha}$ is given by
\[\hat{U}^{\dot{A}\alpha} = U_{uv}^{\dot{A}\alpha} \nabla b^u \wedge \nabla b^v. \tag{B.2.2}\]

We also have to replace the curvatures of the connections with their gauged expressions:
\[R^a_b \rightarrow \hat{R}^a_b, \tag{B.2.3a}\]
\[K \rightarrow \hat{K}, \tag{B.2.3b}\]
\[R^{\alpha\beta} \rightarrow \hat{R}^{\alpha\beta}, \tag{B.2.3c}\]
\[R_B^A \rightarrow \hat{R}_B^A \equiv i \frac{2}{2}(\sigma_x)_B^A \hat{\Omega}^x. \tag{B.2.3d}\]

where $\hat{R}^a_b, \hat{K}, \hat{R}^{\alpha\beta}$ and $\hat{R}_B^A$ are given in (3.2.3), (3.2.11), (3.2.8) and (3.2.12). Furthermore, we have to replace $dA^\Lambda$ in (B.1.2e) with the complete gauge curvature (3.2.6):
\[dA^\Lambda \rightarrow F^\Lambda \equiv dA^\Lambda + \frac{1}{2}gf_{\Sigma}^\Lambda A^\Sigma \wedge A^\Gamma. \tag{B.2.4}\]

For completeness, let us collect the curvatures of all the sectors. The curvatures in gravitational sector are
\[T^i \equiv \mathcal{D}^i - i\bar{\psi}_A \gamma_i \psi^A, \tag{B.2.5a}\]
\[\rho_A \equiv \nabla \psi_A \equiv d\psi_A - \frac{1}{2} \gamma_{ij} \omega^{ij} \wedge \psi_A + \frac{i}{2} \hat{Q} \wedge \psi_A + \hat{\omega}_B \wedge \psi_B, \tag{B.2.5b}\]
\[\rho^A \equiv \nabla \psi^A \equiv d\psi^A - \frac{1}{2} \gamma_{ij} \omega^{ij} \wedge \psi^A - \frac{i}{2} \hat{Q} \wedge \psi^A + \hat{\omega}_B \wedge \psi^B, \tag{B.2.5c}\]
\[R^{ij} \equiv d\omega^{ij} - \omega^k \wedge \omega^{kj}. \tag{B.2.5d}\]

The curvatures and covariant derivatives in the vector multiplet sector are
\[\nabla z^a = dz^a + gA^\Lambda k_A^a (z), \tag{B.2.6a}\]
\[\nabla z^a^* = dz^a^* + gA^\Lambda k_A^a^* (\bar{z}), \tag{B.2.6b}\]
\[\nabla \lambda^{aA} \equiv d\lambda^{aA} - \frac{1}{4} \gamma_{ij} \omega^{ij} \lambda_A^a - \frac{i}{2} \hat{Q} \lambda^a A + \hat{\Gamma}^a_b \lambda^b A + \hat{\omega}_B \lambda_A^a B, \tag{B.2.6c}\]
\[\nabla \lambda^{aA} \equiv d\lambda^{aA} - \frac{1}{4} \gamma_{ij} \omega^{ij} \lambda^a A - \frac{i}{2} \hat{Q} \lambda^a A + \hat{\Gamma}^a_b \lambda^b A + \hat{\omega}_B \lambda^a B, \tag{B.2.6d}\]
\[\hat{F}^A \equiv F^A + \bar{L}^A \psi_A \wedge \psi_B^{AB} + \bar{L} \bar{\psi}^A \wedge \psi^{AB}. \tag{B.2.6e}\]

We have used the notation $\hat{F}$ in order to distinguish from $F^\Lambda = dA^\Lambda + \frac{1}{2}gf_{\Sigma}^\Lambda A^\Sigma \wedge A^\Gamma$. In the hypermultiplet sector the covariant derivatives are
\[U^{\dot{A}\alpha} \equiv U_{uv}^{\dot{A}\alpha} \nabla b^u \equiv U_{uB}^{\dot{A}\alpha} (db^u + gA^\Lambda k_A^u (b)), \tag{B.2.7a}\]
\[\nabla \zeta_\alpha \equiv d\zeta_\alpha - \frac{1}{2} \gamma_{ij} \omega^{ij} \zeta_\alpha - \frac{i}{2} \hat{Q} \zeta_\alpha + \hat{\Delta}_\alpha^\beta \zeta_\beta, \tag{B.2.7b}\]
\[\nabla \zeta_\alpha \equiv d\zeta_\alpha - \frac{1}{2} \gamma_{ij} \omega^{ij} \zeta_\alpha + \frac{i}{2} \hat{Q} \zeta_\alpha + \hat{\Delta}_\alpha^\beta \zeta_\beta. \tag{B.2.7c}\]
The Bianchi identities in the gauge case

The Bianchi identities in the gauged case is the same as the ungauged case \([B.1.5a] - [B.1.9c]\) except that the curvatures are replaced as \([B.2.3a] - [B.2.3d]\). In the gravitational sector, we obtain the following Bianchi identities:

\[
D T^i + R^i_j \wedge e_j - i \bar{\psi}^A \wedge \gamma^i \rho_A + i \bar{\rho}^A \wedge \gamma^i \psi_A = 0,
\]  
\text{(B.2.8a)}

\[
\nabla \rho_A + \frac{1}{4} \gamma_{ij} R^{ij}_A \wedge \psi_A - \frac{i}{2} K \wedge \psi_A - \hat{R}^B_A \wedge \psi_B = 0,
\]  
\text{(B.2.8b)}

\[
\nabla \rho^A + \frac{1}{4} \gamma_{ij} R^{ij}_A \wedge \psi^A + \frac{i}{2} K \wedge \psi^A - \hat{R}^A_B \wedge \psi^B = 0,
\]  
\text{(B.2.8c)}

\[
D R^{ij} = 0.
\]  
\text{(B.2.8d)}

In the vector multiplet sector, we get

\[
\nabla^2 z^a - g(\hat{F}^A - \bar{L}^A \bar{\bar{\psi}}_A \wedge \psi_B \epsilon^{AB} - L^A \wedge \bar{\psi}^A \wedge \psi^B \epsilon_{AB}) k^a_A = 0,
\]  
\text{(B.2.9a)}

\[
\nabla^2 \bar{z}^a - g(\hat{F}^A - \bar{L}^A \bar{\bar{\psi}}_A \wedge \psi_B \epsilon^{AB} - L^A \wedge \bar{\psi}^A \wedge \psi^B \epsilon_{AB}) k^a_A = 0,
\]  
\text{(B.2.9b)}

\[
\nabla^2 \lambda^a = \frac{1}{4} \gamma_{ij} R^{ij}_A \lambda^a_A + \frac{i}{2} K \lambda^a_A + \hat{R}^a_b \lambda^b_A - \frac{i}{2} \hat{R}^A_B \lambda^a_B = 0,
\]  
\text{(B.2.9c)}

\[
\nabla^2 \lambda^a = \frac{1}{4} \gamma_{ij} R^{ij}_A \lambda^a_A - \frac{i}{2} K \lambda^a_A + \hat{R}^a_b \lambda^b_A - \frac{i}{2} \hat{R}^A_B \lambda^a_B = 0,
\]  
\text{(B.2.9d)}

\[
\nabla^2 \lambda^a - \epsilon_{AB} \nabla \bar{L}^A \wedge \bar{\bar{\psi}}_A \wedge \psi_B + 2 \epsilon_{AB} \bar{L}^A \bar{\bar{\psi}}_A \wedge \bar{\bar{\psi}}_B
\]  
\[+ \epsilon^{AB} \nabla \bar{L}^A \wedge \bar{\bar{\psi}}_A \wedge \psi_B + 2 \epsilon^{AB} \bar{L}^A \bar{\bar{\psi}}_A \wedge \bar{\bar{\psi}}_B = 0.
\]  
\text{(B.2.9e)}

In the hypermultiplet sector, we derive

\[
\nabla^2 \zeta_\alpha - \frac{1}{4} \gamma_{ij} R^{ij}_A \zeta_\alpha + \hat{R}^A_B \zeta_\beta - \frac{i}{2} K \zeta_\alpha = 0,
\]  
\text{(B.2.10a)}

\[
\nabla^2 \zeta_\alpha + \frac{1}{4} \gamma_{ij} R^{ij}_A \zeta_\alpha + \hat{R}^A_B \zeta_\beta - \frac{i}{2} K \zeta_\alpha = 0,
\]  
\text{(B.2.10b)}

\[
\nabla U^A_\alpha - g(\hat{F}^A - \bar{L}^A \bar{\bar{\psi}}_A \wedge \psi_B \epsilon^{AB} - L^A \wedge \bar{\psi}^A \wedge \psi^B \epsilon_{AB}) k^A_A(b) U^A_\alpha = 0.
\]  
\text{(B.2.10c)}
The solutions of the Bianchi identities in the gauged case

The solutions of the Bianchi identities (B.2.8a)-(B.2.10c) in the gauged case are written as follows:

\[ T^i = 0, \]
\[ \rho_A = \rho_{A[ij]} \epsilon^i \wedge \epsilon^j + (A^B_{A[ij]} \eta^{ij} + A'^{B}_{A[ij]} \gamma^{ij}) \psi_B \wedge \epsilon^i \]
\[ + \epsilon_{AB}(T_{ij}^- + U_{ij}^+) \gamma^j \psi^B \wedge \epsilon^i + igS_{AB} \eta^{ij} \psi_B \wedge \epsilon^i, \]
\[ (B.12) \]
\[ \rho^A = \rho^{A[ij]} \wedge \epsilon^j + (A'^{A}_{B[ij]} \eta_{ij} + A'^{A}_{A[ij]} \gamma_{ij}) \psi_B \wedge \epsilon^i \]
\[ + \epsilon^{AB}(T_{ij}^+ + U_{ij}^-) \gamma^j \psi^B \wedge \epsilon^i + igS^{AB} \eta^{ij} \gamma^j \psi_B \wedge \epsilon^i, \]
\[ (B.13) \]
\[ R^{ij} = R_{klij} \epsilon^k \wedge \epsilon^l - i(\overline{\psi}_A \theta^A_{k[ij]} + \overline{\psi}_A \theta^A_{A[k]} \gamma_{ij}) \wedge \epsilon^k \]
\[ + \epsilon^{ijkl} \overline{\psi}_A \wedge \gamma_{kl} \psi_B (A'_{A[k]} - A'_{A[k]}) \]
\[ + \epsilon^{AB} \overline{\psi}_A \wedge \psi_B (T_{ij}^- + U_{ij}^+) - i \epsilon^{AB} \overline{\psi}_A \wedge \psi_B (T_{ij}^+ + U_{ij}^-) \]
\[ - gS_{AB} \overline{\psi}_A \wedge \gamma_{ij} \psi_B - gS^{AB} \overline{\psi}_A \wedge \gamma_{ij} \psi_B, \]
\[ (B.14) \]
\[ \hat{F}^A = F_{ij} \epsilon^i \wedge \epsilon^j + i(f^A_{\alpha} \lambda_{\alpha}^A \gamma^i \psi^B \epsilon^{AB} + \tilde{f}^A_{\alpha} \lambda_{\alpha}^A \gamma_i \psi_B \epsilon^{AB}) \wedge \epsilon^i, \]
\[ (B.15) \]
\[ \nabla \lambda^A = \nabla_i \lambda^A \epsilon^i + iZ_i^A \gamma^i \psi^A + G_{ij} \gamma^j \psi_B \epsilon^{AB} + (Y^A \epsilon^{AB} + gw^A \psi_B), \]
\[ (B.16) \]
\[ \nabla \lambda^A = \nabla_i \lambda^A \epsilon^i + iZ_i^A \gamma^i \psi^A + G_{ij} \gamma^j \psi_B \epsilon^{AB} + (Y^A \epsilon^{AB} + gw^A \psi_B), \]
\[ (B.17) \]
\[ \nabla z^A = Z_i^A \epsilon^i + \overline{\psi}_A \lambda^A, \]
\[ (B.18) \]
\[ \nabla z^A = Z_i^A \epsilon^i + \overline{\psi}_A \lambda^A, \]
\[ (B.19) \]
\[ \nabla \zeta_A = \nabla_i \zeta_A \epsilon^i + iU_i \gamma^i \psi^A \epsilon^{AB} \gamma^B \psi^A + gN_A \psi_A, \]
\[ (B.20) \]
\[ \nabla \zeta_A = \nabla_i \zeta_A \epsilon^i + iU_i \gamma^i \psi^A \epsilon^{AB} \gamma^B \psi^A + gN_A \psi_A, \]
\[ (B.21) \]
\[ U^{\alpha A} = U_{\alpha A} \epsilon^i + e^{AB} \gamma_{AB} \gamma^B \psi^A \zeta_A, \]
\[ (B.22) \]

where \( A^A_{[ij]}, A'^A_{A[ij]}, \theta^A_{ij}, T_{ij}, U_{ij}, G^A_{ij} \) and \( Y^{A AB} \) are identical with the ungauged case (B.1.22a)-B.1.27b. The constant \( g \) is just a symbolic notation to remind that these terms are produced by the gauging and the new terms \( S_{AB}, W^{A AB} \) and \( N^A_A \) are

\[ S_{AB} = \frac{i}{2}(\sigma_x)^{C}_{A} \epsilon^{BC} \gamma^{A} \psi^{C}, \]
\[ S^{AB} = \frac{i}{2}(\sigma_x)^{A}_{C} \epsilon^{BC} \gamma^{C} \psi^{A}, \]
\[ (B.23) \]
\[ W^{A AB} = e^{AB} \gamma^{A} \psi^{C} \epsilon^{BC} \gamma^{A} \psi^{C} + g \overline{\psi}_A \gamma^B \psi^A, \]
\[ (B.24) \]
\[ N^A_\alpha = 2U^A_{\alpha |u} \Lambda^\alpha \Lambda^A, \]
\[ N^\alpha_A = -2U^\alpha_A |u \Lambda^\alpha \Lambda^A. \]  
(B.2.25)

The constraints following from the closure of the Bianchi identities are a set of differential constraints on the upper parts \( L^A, \bar{L}^A, f^A_a \) and \( \bar{f}^A_a \) of the symplectic sections \( V \) and \( U^a \) and on \( C_{abc} \):

\[ \nabla_a^* L^A = \nabla_a \bar{L}^A = 0, \] (B.2.26)
\[ f^A_a = \nabla_a L^A, \] (B.2.27)
\[ \bar{f}^A_a = \nabla_a^* \bar{L}^A, \] (B.2.28)
\[ \nabla_d^* C_{abc} = \nabla_d C_{a^*b^*c^*} = 0, \] (B.2.29)
\[ \nabla_a f^A_b = ig^{cd} \bar{f}^A_d C_{abc}. \] (B.2.30)

These equations are the same for that of the section 2.2. Therefore the constraints \( \text{[B.2.26] - [B.2.30]} \) restrict the Hodge-Kähler manifold which we start from to be a special Kähler manifold.
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