Pulsating strings in Schr$_5 \times T^{1,1}$ background

A Golubtsova$^{1,2}$, H Dimov$^{1,3}$, I Iliev$^3$, M Radomirov$^3$, R C Rashkov$^{3,4}$ and T Vetsov$^{3,5}$

$^1$ The Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia
$^2$ Dubna State University Universitetskaya str., 141980 Dubna, Moscow region, Russia
$^3$ Department of Physics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria
$^4$ Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstr. 8–10, 1040 Vienna, Austria

E-mail: golubtsova@theor.jinr.ru, dimov@theor.jinr.ru, ivo.n.iliiev@abv.bg, h_dimov@phys.uni-sofia.bg, radomirov@phys.uni-sofia.bg, rash@phys.uni-sofia.bg and vetsov@phys.uni-sofia.bg

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Abstract
The quest for extension of holographic correspondence to non-relativistic sectors naturally includes Schrödinger backgrounds and their field theory duals. In this paper we study the holography by probing the correspondence with pulsating strings. The case we consider is pulsating strings in five-dimensional Schrödinger space times five-torus $T^{1,1}$, which has as field theory dual a dipole CFT. First we find particular pulsating string solutions and then semi-classically quantize the theory. We obtain the wave function of the problem and thoroughly study the corrections to the energy, which by duality are supposed to give anomalous dimensions of certain operators in the dipole CFT.

Keywords: string theory, pulsating strings, Schrödinger spacetime, holography

1. Introduction

In recent years the intensive development of the AdS/CFT correspondence has provided us with powerful tools for studying important aspects of string theory and quantum field theories. This unusual correspondence is a true duality in a sense that the strong coupling quantum regime of one of the theories is equivalent to the weakly coupled semi-classical regime of the other one. The latter enables us to establish a dictionary between objects on both sides of the duality, which opens a window to non-perturbative physics.

$^*$Author to whom any correspondence should be addressed.
Although the correspondence has been successfully applied to the description of highly supersymmetric theories, it has also been extended to less supersymmetric models, which are more interesting for phenomenology. From gravitational point of view, a straight-forward way to reduce supersymmetry in a given model can be achieved by deforming the original spacetime background such as the new background is again an Einstein manifold.

A simple procedure, proposed in [1], enables us to explicitly generate new supergravity solutions and also gives suggestions about their dual gauge theories. This approach is based on the global symmetries underlying the starting theory. Particularly, having two global $U(1)$ isometries one can interpret them geometrically as a two-torus with action of the associated $SL(2,\mathbb{R})$ symmetry on the torus parameter, which are given by $\tau \rightarrow \tilde{\tau} = \tau/(1 + \gamma \tau)$. Considering type IIB backgrounds, one already has $SL(2,\mathbb{R})$ symmetry of the ten-dimensional background. In this case, the existing two $SL(2,\mathbb{R})$ symmetries get combined to form $SL(3,\mathbb{R})$, which can be seen as a compactification of $M$-theory on $T^3$. However, from holographic point of view, it is important where in the spacetime the two-torus, associated with the global isometries, are located. Indeed, if the torus lives entirely in the asymptotically AdS part of the background geometry the dual gauge theory becomes essentially non-commutative. Here, the $U(1)$ charges are identified as momenta and the product in the field theory is replaced by the Moyal star-product. If however, the torus resides in the complementary part of the given spacetime, the product between the fields is the ordinary product, but the theory exhibits a Leigh–Strassler deformation [2]. Since the deformation is taken in a direction along the $R$-symmetry it is quite clear that the supersymmetry in the new theory will be at least partially broken. This is in agreement with the gauge theory side, as shown in [2]. This solution generating technique can be applied to wide range of cases even with or without the presence of the $SL(2,\mathbb{R})$ symmetry. A well-known example is the exactly marginal deformations of $\mathcal{N} = 4$ super Yang–Mills theory, considered in [2], whose gravity dual (for the so-called $\beta$-deformation) was derived in [1]. Other examples also include deformations of Lunin–Maldacena background, which have been thoroughly studied in the literature, for instance in [3–12].

Recently, in an attempt to generalize the AdS/CFT correspondence to strongly coupled non-relativistic field theories [13, 15], qualitatively different solutions have been obtained. A particularly interesting example includes the non-relativistic Schrödinger spacetimes, where the isometry group of the background geometry on string side is the Schrödinger group. It consists of time and space translations, space rotations, Galilean boosts and dilatations. It has also been shown that possible duals to such theories could be realized as the so called dipole gauge theories. These quantum field models are characterized with non-locality, but still living on an ordinary commutative space. Further investigations on dipole theories can be traced for instance in [16–18].

One direction, where these solutions have been used, is in the description of ordinary $\mathcal{N} = 1$ SQCD-like gauge theories, which are considered in the context of $D$-brane constructions. This type of deformations are typically used to decouple spurious effects coming from Kaluza–Klein modes on cycles wrapped by the $D$-brane [6–10]. Furthermore, models with non-relativistic symmetries are related to a number of other physically interesting studies, such as the Sachdev–Ye–Kitaev (SYK) model [19], Fermi unitary gas [20], and models with trapped supercooled atoms [13, 14], which in most cases are strongly correlated.

For this reason, we are motivated to investigate the properties of these holographic models on the string side, where calculations can be performed explicitly. An important step in this line of investigations has been done in [21] where strong arguments for integrability and quantitative check of the matching between string and gauge theory predictions have been presented. These studies triggered a new interest to holography in such backgrounds, since they open the option to gain important knowledge about dipole theories for instance. Particular string
solutions beyond the supergravity approximation has been found in [21–26]. String theory on the \( pp \)-wave geometry of the non-supersymmetric Schrödinger background has been studied in [27]. A semiclassical quantization has been also considered in [28] and very recently finite corrections has been obtained in [29]. More aspects of Schrödinger holography can be found for instance in [30–34].

In the current paper we focus on investigating the dynamics of pulsating strings on spacetime with Schrödinger symmetry. On the gauge theory side the models are known as non-relativistic duals. Gravity duals of theories with Schrödinger symmetry were obtained in [15]. The relevant for our considerations Schrödinger part of the background was obtained in [30] by TsT deformation along certain directions from \( \text{AdS}_5 \times S^5 \). As shown for instance in [32], the generated solutions are twisted and the supersymmetry is completely broken. More detailed description of this technique can be found for instance in [1]. A brief review of pulsating strings in holography has been presented in [30], where the authors also investigated the case of pulsating strings in \( \text{Sch} \times T^{1,1} \). Our considerations will be restricted to the bosonic sector of the theory with complementary manifold of type \( T^{1,1} \) conifold, which is interesting as being a part of the Calabi–Yau manifolds from the string landscape.

Our studies begin in section 2, where we consider the standard Polyakov string action in conformal gauge for pulsating strings in \( \text{Sch} \times T^{1,1} \) spacetime. Here, we impose an appropriate pulsating string ansatz and explicitly solve the corresponding classical equations of motion together with the Virasoro constraints in terms of simple trigonometric or Jacobi elliptic functions. Furthermore, in this section we identify the relevant conditions of the parameters for realizing a pulsating string configuration. In section 3 we focus on calculating the energy spectrum, the corresponding first order correction in powers of \( \lambda \) for the pulsating string configuration, and the anomalous dimensions of the operators in the dual gauge theory. In this case, the calculations are possible in perturbation theory due to the fact that the squared string Hamiltonian takes the form of a point particle one. Finally, in section 4 we make a brief conclusion of our results.

2. Pulsating string solutions in \( \text{Sch} \times T^{1,1} \)

In this section we will construct the ten-dimensional \( \text{Sch} \times T^{1,1} \) background by applying a TsT transformation on the original \( \text{AdS}_5 \times T^{1,1} \), which initially comes without a \( B \)-field. Consequently, the new background will acquire non-trivial \( B \)-field, thus essentially changing the character of the string dynamics. However, we show that the classical string equations of motion in \( \text{Sch} \times T^{1,1} \) admit explicit analytical solutions in terms of simple trigonometric and Jacobi elliptic functions.

2.1. Generating \( \text{Sch} \times T^{1,1} \)

It is well know for nearly fifty years that the symmetry of the free Schrödinger equation

\[
\frac{\partial^2}{\partial t^2} \phi - 2 \text{Im} \frac{\partial}{\partial t} \phi = 0,
\]

is the so called Schrödinger group. In \( d \) space dimensions the group consists of spatial translations indicated by \( \vec{A} \), rotations given by the matrix \( \Omega \), and Galilean boosts with velocity \( \vec{v} \),

\[
t \to t' = \frac{at + b}{ct + d}, \quad \vec{r} \to \vec{r}' = \frac{\Omega \vec{r} + \vec{v} t + \vec{A}}{ct + d}, \quad ad - bc = 1.
\]
Furthermore, one has dilatation, where time and space dilate with different factors
\[ t \rightarrow \lambda^2 t, \quad \vec{r} \rightarrow \lambda \vec{r}, \] (2.3)
and one additional special conformal transformation
\[ t \rightarrow \frac{t}{1 + \lambda t}, \quad \vec{r} \rightarrow \frac{\vec{r}}{1 + \lambda t}. \] (2.4)

From group point of view, the Schrödinger group can be thought of as a non-relativistic analogue of the conformal group in \( d \) dimensions. In fact, as can be seen from above, the Schrödinger group in \( d \) spatial dimensions can be embedded into the relativistic conformal group in \( d + 1 \) dimensions \( SO(2, d + 2) \), or particular contraction of the conformal group [35].

For purposes of holographic correspondence it is important to consider spaces with the Schrödinger group being the maximal group of isometries. As it was mentioned in the introduction, there is a simple way to obtain the spacetime geometry possessing this symmetry. Having the comments above, it is a natural guess to assume that it can be done starting from the AdS space associated with the corresponding conformal group. Being particular case of the so called Drinfel’d–Reshetikhin twist, the TsT procedure has been used for generating many backgrounds keeping integrability (or part of it) intact. Specific point in generating Schrödinger backgrounds via TsT transformations is that one of the light-cone variables is involved. This type deformations are known also as null-Melvin twist. To generate null-Melvin twist we implement the following steps:

- write the theory in light-cone coordinates and identify a Killing direction, say \( \psi \),
- T-dualize along the Killing direction \( \psi \),
- boost the geometry in the Killing direction by an amount \( \hat{\mu} \), i.e. \( x^- \rightarrow x^- - \hat{\mu} \tilde{\psi} \), where \( \tilde{\psi} \) is the T-dualized \( \psi \),
- T-dualize the geometry back to IIA/IIB along \( \tilde{\psi} \).

In order to accomplish the desired result consider general background in the form \( \text{AdS}_5 \times X^5 \), where the metric on \( X^5 \) is \( g_{\alpha\beta} \). Now we perform a null Melvin twist along a Killing vector \( K \) on \( X^5 \). The result is
\[ ds_{10}^2 = ds_{\text{Schr}}^2 + ds_{X^5}^2, \] (2.5)
where
\[ ds_{\text{Schr}}^2 = -\Omega z^4 + \frac{1}{z^4}(-2 \, dv \, du + dx_1^2 + dx_2^2 + dz^2), \quad \Omega = ||K||^2 = g_{\alpha\beta} K^\alpha K^\beta. \] (2.6)
It is clear that \( \Omega \) is non-negative being a square length of the Killing vector. The generated \( B \)-field has the form
\[ B_{(2)} = \frac{1}{z^2} K \wedge du. \] (2.7)

An important remark is that in order to make holographic sense of these solutions one has to impose some conditions. In particular for these to be holographic duals to non-relativistic field

\footnote{For a detailed group-theoretical perspective on non-relativistic holography see [36].}
theories the light-cone coordinate $v$ should be periodic, $v \sim v + 2\pi r_v$ [13–15]. The momentum along this compact direction is quantized in units of the inverse radius $r_v^{-1}$.

Let us translate the above procedure to our case introducing the notations we will use from now on. We proceed with AdS$_5 \times T^{1,1}$ background, which arises from a stack of $N$ D-branes at the tip of the conifold, the conic Calabi–Yau three-fold whose base space is $T^{1,1}$ [37]. The manifold $T^{1,1}$ is the coset space $(SU(2) \times SU(2))/U(1)$. This supergravity solution is the dual to the $\mathcal{N} = 1$ supersymmetric Yang–Mills theory. The metric of AdS$_5 \times T^{1,1}$ is defined as

$$\text{d}x^2 = \text{d}s_{\text{AdS}}^2 + \ell^2 \! \frac{b}{4} \left[ \sum_{i=1}^2 (\text{d}x^i + \sin^2 \theta_i \text{d}\phi_i^2) + b \left( \text{d}\psi - \sum_{i=1}^2 \cos \theta_i \text{d}\phi_i \right)^2 \right], \quad (2.8)$$

where the metric on AdS$_5$ is written in light-cone coordinates by

$$\text{d}s_{\text{AdS}}^2 = \ell^2 \left( \frac{\text{d}x^+ \text{d}x^- + \text{d}x^i \text{d}x_i + \text{d}z^2}{z^2} \right). \quad (2.9)$$

The metric of $T^{1,1}$ is clearly an $S^1$ Hopf fibration over $S^2 \times S^2$ with $\psi$ parameterizing the fiber circle which winds over the two base spheres once. The isometry group of $T^{1,1}$ is $SU(2) \times SU(2)$ and in particular, it has three commuting Killing vectors: $\partial/\partial \phi_1, \partial/\partial \phi_2, \partial/\partial \psi$.

It is easily seen that the generated metric by doing $\text{T}s\text{T}$ along $\psi$ and $x^i$ will produce $5d$ Schrödinger spacetime times $T^{1,1}$. Explicitly, the procedure reads: make a $T$-duality along $\psi$, then make a shift $x^i \rightarrow x^i - \mu \psi$, where $\psi$ is $T$-dualized $\psi$ and at the end make $T$-duality back on $\psi$. The result reads

$$\text{d}x^2 = \text{d}s_{\text{Schr}}^2 + \ell^2 \! \frac{b}{4} \left[ \sum_{i=1}^2 (\text{d}x^i + \sin^2 \theta_i \text{d}\phi_i^2) + b \left( \text{d}\psi - \sum_{i=1}^2 \cos \theta_i \text{d}\phi_i \right)^2 \right], \quad (2.10)$$

where

$$\text{d}s_{\text{Schr}}^2 = \ell^2 \left( \frac{\mu^2 (\text{d}x^+)^2}{z^4} + \frac{2 \text{d}x^+ \text{d}x^- + \text{d}x^i \text{d}x_i + \text{d}z^2}{z^2} \right). \quad (2.11)$$

and there is also a generated $B$-field given as (3.5). Now we proceed with choosing an ansatz for our problem.

### 2.2. Ansatz and classical equations of motion

In the previous subsection the metric has not been written in global coordinates. The ten-dimensional line element of the background in global coordinates is given by

$$\text{d}s_{\text{Schr} \times T^{1,1}}^2 = \text{d}s_{\text{Schr}}^2 + \text{d}s_{T^{1,1}}^2, \quad (2.12)$$

where the Schrödinger part of the metric in global spacetime coordinates is written by [30]

$$\frac{\text{d}s_{\text{Schr}}^2}{\ell^2} = - \left( 1 + \frac{\mu^2}{Z^2} + \frac{\bar{X}^2}{Z^2} \right) \text{d}T^2 + \frac{2}{Z^2} \text{d}V + \frac{2 \text{d}X^2 + \text{d}Z^2}{Z^2}. \quad (2.13)$$

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6 When the Hopf numbers of the $S^1$ bundle over the base 2-spheres are $p$ and $q$ the above generalizes to the manifolds $T^{p,q}$.
and the $T^{1,1}$ part yields
\begin{equation}
\frac{dS_{T^{1,1}}}{\ell^2} = \frac{b}{4} \left[ \sum_{i=1}^{2} \left( d\theta_i^2 + \sin^2 \theta_i \, d\phi_i^2 \right) + b \left( d\psi - \sum_{i=1}^{2} \cos \theta_i \, d\phi_i \right)^2 \right]. \tag{2.14}
\end{equation}

Here, $0 \leq \psi \leq 4\pi, 0 \leq \theta_i \leq \pi, 0 \leq \phi_i < 2\pi$, and $b = 2/3$, although we will keep the notation $b \equiv 1$. Also one has $\mu$ is a deformation parameter coming from the TsT transformation. Furthermore, the application of TsT transformations also generates a non-zero $B$-field, namely
\begin{equation}
B_{\phi i} = \frac{\ell^2 b \mu}{2Z^2} \, d\tau \wedge \left( d\psi - \sum_{i=1}^{2} \cos \theta_i \, d\phi_i \right). \tag{2.15}
\end{equation}

The Polyakov string action in conformal gauge, with Kalb–Ramond $B$-field is given by
\begin{equation}
S = -\frac{1}{4\pi\alpha'} \int d\tau \, d\sigma \left( \sqrt{-h} \alpha^\alpha \partial_\alpha X^\beta \partial_\beta X^\gamma G_{MN} - \epsilon^{\alpha\beta} \partial_\alpha X^\gamma \partial_\beta X^\gamma B_{MN} \right), \tag{2.16}
\end{equation}

where $\alpha, \beta = 0, 1, h_{\alpha\beta} = \text{diag}(-1, 1)$, $M, N = 0, \ldots, 9$ and $\epsilon^{01} = +1$. Therefore, the Lagrangian in the considered background takes the following explicit form
\begin{align}
-4\pi\alpha' L &= G_{TT}(T'^2 - \hat{T}^2) + G_{\bar{X}X}(\bar{X}'^2 - \hat{X}^2) + G_{ZZ}(Z'^2 - \hat{Z}^2) \\
&\quad + 2G_{TV}(T'V' - \hat{T}\hat{V}) + G_{\theta\theta}(\theta'^2 - \hat{\theta}^2) + G_{\theta\phi}(\phi'^2 - \hat{\phi}^2) \\
&\quad + G_{\theta\psi}(\psi'^2 - \hat{\psi}^2) + G_{\phi\phi}(\phi'^2 - \hat{\phi}^2) + G_{\phi\psi}(\phi\psi' - \hat{\phi}\hat{\psi}) + 2G_{\phi\theta} \\
&\quad \times (\phi_1'\phi_2 - \hat{\phi}_1\hat{\phi}_2) + 2G_{\phi\psi}(\phi_1'\psi' - \hat{\phi}_1\hat{\psi}) + 2G_{\phi\theta}(\phi_2'\theta' - \hat{\phi}_2\hat{\theta}) \\
&\quad + 2B_{T\phi}(T'\phi - \hat{T}\phi) + 2B_{T\psi}(T'\psi - \hat{T}\psi) + 2B_{T\theta}(T'\theta - \hat{T}\theta). \tag{2.17}
\end{align}

where we have used $X = \partial_\tau X$ and $X' = \partial_\sigma X$. In order to obtain pulsating string solutions we consider the following string ansatz ($\kappa > 0$):
\begin{align}
T &= \kappa \tau, \quad V = 0, \quad \bar{X} = \bar{0}, \quad Z = \text{const} \neq 0, \\
\theta_1 &= \theta_1(\tau), \quad \theta_2 = \theta_2(\tau), \quad \phi_1 = \phi_1(\tau), \quad \phi_2 = \phi_2(\tau), \quad \psi = \psi(\tau). \tag{2.18}
\end{align}

In this case, the equations of motion for $V, \bar{X}, \phi_1, \phi_2$ and $\psi$, are trivially satisfied, while the equations for $T, Z, \theta_1, \theta_2$ stay relevant. Let us begin with the equation along $T$,
\begin{equation}
m_1 \sin \theta_1(\tau) \dot{\theta}_1(\tau) + m_2 \sin \theta_2(\tau) \dot{\theta}_2(\tau) = 0. \tag{2.19}
\end{equation}

It can be written in the following useful form
\begin{equation}
m_1 \cos \theta_1(\tau) + m_2 \cos \theta_2(\tau) = A = \text{const}, \quad m_{1,2} \neq 0. \tag{2.20}
\end{equation}

The equation along $Z$ is given by
\begin{equation}
Z^2 = \frac{2\mu c}{b(m_3 - A)} = \text{const}, \tag{2.21}
\end{equation}

One notes that for $b = 1$ the $S^1$ metric is recovered.
while the equations along $\theta_i$, $i = 1, 2$, yield

$$\ddot{\theta}_i(\tau) + m_i \sin \theta_i(\tau) \left[ m_i \cos \theta_i(\tau) + b(m_3 - A) - \frac{2\mu_k}{Z^2} \right] = 0. \quad (2.22)$$

Using equation (2.21) one can write the previous expression in the form

$$\ddot{\theta}_i(\tau) + m_i^2 \sin \theta_i(\tau) \cos \theta_i(\tau) = 0. \quad (2.23)$$

We should also supplement the equations of motion with the Virasoro constraints:

$$\text{Vir}_1: G_{MN} (\dot{X}^M \dot{X}^N + X^M \dot{X}^N) = 0, \quad (2.24)$$
$$\text{Vir}_2: G_{MN} \dot{X}^M \dot{X}^N = 0. \quad (2.25)$$

The first equation (2.24) explicitly reads

$$\dot{\theta}_1^2(\tau) + \dot{\theta}_2^2(\tau) + m_1^2 \sin^2 \theta_1(\tau) + m_2^2 \sin^2 \theta_2(\tau) - \frac{4\kappa^2}{b} = 0, \quad (2.26)$$

and the second Virasoro constraint (2.25) is trivially satisfied. Multiplying (2.23) by $\dot{\theta}_i$ we obtain

$$\dot{\theta}_i(\tau) \dot{\theta}_i(\tau) + m_i^2 \sin \theta_i(\tau) \cos \theta_i(\tau) = 0, \quad (2.27)$$

thus one finds

$$\frac{d}{d\tau} \left\{ \dot{\theta}_i^2(\tau) + m_i^2 \sin^2 \theta_i(\tau) \right\} = 0, \quad (2.28)$$

or

$$\dot{\theta}_i^2(\tau) + m_i^2 \sin^2 \theta_i(\tau) = K_i > 0. \quad (2.29)$$

Combining equations (2.26) and (2.29) we get a relation between the constants $K_1$ and $K_2$, namely

$$K_1 + K_2 = \frac{4\kappa^2}{b}. \quad (2.30)$$

Now we are in a position to integrate equation (2.29), namely

$$\int_0^{\theta_i(\tau)} \frac{d\theta_i}{\sqrt{1 - m_i^2 \sin^2 \theta_i}} = F \left( \theta_i, \sqrt{\frac{m_i^2}{K_i}} \right) = \sqrt{K_i} \int_0^\tau d\tau. \quad (2.31)$$

Finally, the solution is given in terms of the Jacobi elliptic cosine

$$\cos \theta_1(\tau) = \text{cn} \left( \sqrt{K_1} \tau, \sqrt{\frac{m_1^2}{K_1}} \right), \quad \cos \theta_2(\tau) = \text{cn} \left( \sqrt{K_2} \tau, \sqrt{\frac{m_2^2}{K_2}} \right). \quad (2.32)$$

Now the pulsating string condition (2.20) acquires the form

$$m_1 \text{cn} \left( \sqrt{K_1} \tau, \sqrt{\frac{m_1^2}{K_1}} \right) + m_2 \text{cn} \left( \sqrt{K_2} \tau, \sqrt{\frac{m_2^2}{K_2}} \right) = A. \quad (2.33)$$
It must be satisfied for any value of \( \tau \). Setting \( \tau = 0 \) we get

\[
m_1 + m_2 = A. \tag{2.34}
\]

Finally we can find an explicit expression for the classical energy of the considered string configuration, namely

\[
E = -\int_{l/2}^{l/2} d\sigma \frac{\partial L}{\partial (\partial_\tau T)} = \frac{\ell}{\alpha'} \left[ \left( 1 + \frac{\hat{\mu}^2}{Z^2} \right) \kappa + \frac{b\hat{\mu}}{2Z^2} (m_1 + m_2 - m_3) \right], \tag{2.35}
\]

where \( l = 2\pi \ell \) is the string length and we have also used (2.33) and (2.34).

3. Energy corrections and anomalous dimensions

In this section we will quantize the pulsating string semi-classically. We will calculate the corresponding energy spectra and their perturbative quantum corrections in powers of a proper small parameter \( \epsilon \). As in the other cases of pulsating strings, it appears that the small parameter \( \epsilon \) is proportional to \( \lambda/n^2 \) as we show in subsection 3.3. According to the AdS/CFT dictionary, the anomalous dimensions of the corresponding SYM operators are directly related to the corrections of the string energy. The analysis below will follow a strategy along the lines in [38, 39, 44, 45].

3.1. Derivation of the Schrödinger equation

In order to semi-classically quantize the bosonic sector of the pulsating string configuration one invokes the Nambu–Goto string action, namely

\[
S_{NG} = -\frac{1}{2\pi \alpha'} \int d\tau \, d\sigma \sqrt{-\det (G_{MN} \partial_\alpha X^M \partial_\beta X^N - B_{MN} \partial_\alpha X^M \partial_\beta X^N)}. \tag{3.1}
\]

The first ingredient toward finding the spectrum is to make a pullback of the line element of the metric of \( \text{Schr}_3 \times T^{1,1} \) to the subspace, where string dynamics takes place. The result for the metric is

\[
ds^2 = \ell^2 \left( -|G_{00}| \, dT^2 + \sum_{i,j=1}^{2} G_{ij} \, d\theta^i \, d\theta^j + \sum_{k,l=1}^{3} \tilde{G}_{kl}(\theta_1, \theta_2) \, d\phi^k \, d\phi^l \right), \tag{3.2}
\]

where the following quantities have been defined\(^8\)

\[
|G_{00}| = 1 + \frac{\hat{\mu}^2}{Z^2}, \quad (G_{ij}) = \frac{b}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.3}
\]

and

\[
(\tilde{G}_{kl}) = \frac{b}{4} \begin{pmatrix} b \cos^2 \theta_1 + \sin^2 \theta_2 & b \cos \theta_1 \cos \theta_2 & -b \cos \theta_1 \\ b \cos \theta_1 \cos \theta_2 & b \cos^2 \theta_2 + \sin^2 \theta_2 & -b \cos \theta_2 \\ -b \cos \theta_1 & -b \cos \theta_2 & b \end{pmatrix}. \tag{3.4}
\]

\(^8\) From now on we will use the notation \( |G_{00}| = |G_{TT}| \).
The $B$-field also changes to
\[ B_{(2)} = \ell^2 \frac{b \mu}{2Z^2} dT \wedge (d\psi - \cos \theta_1 \, d\phi_1 - \cos \theta_2 \, d\phi_2) = \ell^2 \sum_{k=1}^{3} b_{0k}(\theta_1, \theta_2) dT \wedge d\phi^k, \] (3.5)
where
\[ b_{01} = -\frac{b \mu}{2Z^2} \cos \theta_1, \quad b_{02} = -\frac{b \mu}{2Z^2} \cos \theta_2, \quad b_{03} = \frac{b \mu}{2Z^2}. \] (3.6)

Based on [40], in this section we can consider a more extended ansatz than (2.18), such as
\[ T = k\tau, \quad V = 0, \quad \vec{X} = \vec{0}, \quad Z = \text{const} \neq 0, \]
\[ \theta_1 = \theta_1(\tau), \quad \theta_2 = \theta_2(\tau), \] (3.7)
\[ \phi_1 = m_1 \sigma + h_1(\tau), \quad \phi_2 = m_2 \sigma + h_2(\tau), \quad \psi = \phi_3 = m_3 \sigma + h_3(\tau). \] (3.8)

Taking into account the above ansatz one can also calculate the components of the induced metric and the $B$-field on the worldsheet:
\[ \frac{d^2 s^\nu}{d^2 s} = \left( -|G_{00}| \kappa^2 + G_{ij} \dot{\phi}^i \dot{\phi}^j + \hat{G}_{pq} h^p h^q \right) d\sigma^2 + \hat{G}_{pq} m_p m_q \, d\sigma^2 \]
\[ + 2\hat{G}_{pq} m_p \dot{h}^q \, d\tau \, d\sigma, \]
(3.9)

\[ B_{(2)}^{\nu} = \ell^2 B_{\nu\sigma} d\tau \wedge d\sigma, \quad B_{\nu\sigma} = -B_{\sigma\nu} = \sum_{i=1}^{3} b_{0i}(\theta_1, \theta_2) \kappa m_i. \] (3.10)

By taking into account (3.6) one finds that $B_{(2)}^{\nu} \equiv B^{(1)}(\theta_1, \theta_2)$, i.e.
\[ B_{\nu\sigma} \equiv B^{(1)}(\theta_1, \theta_2) = \frac{b^2 \mu^2 \kappa^2}{4Z^4}(m_3 - m_1 \cos \theta_1 - m_2 \cos \theta_2)^2 \geq 0. \] (3.11)

Now, we can write the Nambu–Goto action (3.1) in terms of the notations we have introduced above,
\[ S_{NG} = -\frac{\ell^2}{\alpha'} \int d\tau \sqrt{||\vec{m}||^2 \left( |G_{00}| \kappa^2 - G_{ij} \dot{\phi}^i \dot{\phi}^j - \hat{G}_{pq} h^p h^q \right) + \left( \hat{G}_{pq} m_p m_q \right)^2 - B^2}, \] (3.12)
where $\ell^2 / \alpha' = \sqrt{\lambda}$ is the 't Hooft coupling constant and
\[ ||\vec{m}||^2 = \sum_{k=1}^{3} \hat{G}_{ik}(\theta_1, \theta_2) m_k m_k \]
\[ = \frac{b}{4} \left( \sum_{i=1}^{2} m_i^2 \sin^2 \theta_i + b \left( m_3 - \sum_{i=1}^{2} m_i \cos \theta_i \right)^2 \right) > 0, \] (3.13)

or equivalently
\[ ||\vec{m}||^2 = \frac{b}{4} \left\{ (b \cos^2 \theta_1 + \sin^2 \theta_1) m_1^2 + (b \cos^2 \theta_2 + \sin^2 \theta_2) m_2^2 + bm_3^2 \right. \]
\[ + 2b \cos \theta_1 \cos \theta_2 m_1 m_2 - 2b \cos \theta_1 m_1 m_3 + 2b \cos \theta_2 m_2 m_3 \}. \] (3.14)
It is useful to consider the Hamiltonian formulation of the problem. In our case, the canonical momenta are given by

\[ \Pi_i = \frac{\partial L}{\partial \dot{q}^i} = \sqrt{\lambda} \frac{\| \vec{m} \|^2 G_{ij} \dot{\theta}^j}{\sqrt{\| \vec{m} \|^2 \left( |G_{00}| \kappa^2 - G_{ij} \dot{\theta}^i \dot{\theta}^j - G_{pq} \dot{\theta}^p \dot{\theta}^q \right) + \left( \hat{G}_{pq} \theta^p \right) \theta^q} - B^2}, \]  

(3.15)

\[ \Pi_p = \frac{\partial L}{\partial \dot{p}^p} = \sqrt{\lambda} \frac{\| \vec{m} \|^2 \hat{G}_{pq} \dot{\theta}^q G_{pq} m_q}{\sqrt{\| \vec{m} \|^2 \left( |G_{00}| \kappa^2 - G_{ij} \dot{\theta}^i \dot{\theta}^j - G_{pq} \dot{\theta}^p \dot{\theta}^q \right) + \left( \hat{G}_{pq} \theta^p \right) \theta^q} - B^2}, \]  

(3.16)

One also notes that there is a relation between the momenta \( \hat{\Pi}_p \), namely

\[ m_1 \hat{\Pi}_1 + m_2 \hat{\Pi}_2 + m_3 \hat{\Pi}_3 = 0, \]  

(3.17)

which will lead to a corresponding relation (3.35) between the quantum numbers as well. Using Legendre transformation, \( L = \Pi_k \dot{\theta}^k - H \), we find the (square of) the pulsating string Hamiltonian

\[ H^2 = \frac{\| \vec{m} \|^2 |G_{00}| \kappa^2 - B^2}{\| \vec{m} \|^2} \left( \sum_{i,j=1}^{2} \hat{G}^{ij} \hat{\Pi}_i \hat{\Pi}_j + \sum_{p,q=1}^{3} \hat{G}^{pq} \hat{\Pi}_p \hat{\Pi}_q \right) + \lambda \left( \| \vec{m} \|^2 |G_{00}| \kappa^2 - B^2 \right). \]  

(3.18)

We observe that \( H^2 \) looks like a point–particle Hamiltonian, which seems to be characteristic feature of pulsating strings in holography. The last term,

\[ U(\theta_1, \theta_2) = \| \vec{m} \|^2 |G_{00}| \kappa^2 - B^2 \]

\[ = \frac{b \kappa^2}{4} \left\{ b \left( m_1^2 + m_2^2 + m_3^2 \right) + \left( |G_{00}| - b \right) \left[ m_1^2 \sin^2 \theta_1 + m_2^2 \sin^2 \theta_2 \right] + 2b m_1 m_2 \cos \theta_1 \cos \theta_2 - 2b m_3 \left( m_1 \cos \theta_1 + m_2 \cos \theta_2 \right) \right\}, \]  

(3.19)

serves as an effective potential, which encodes the relevant dynamics of the strings. The \( H^2 \) is an effective point–particle Hamiltonian, which can write in the following form

\[ H^2 = \]  

\[ \frac{\| \vec{m} \|^2 |G_{00}| \kappa^2 - B^2}{\| \vec{m} \|^2} \bar{P}^2 + \lambda U(\theta_1, \theta_2). \]  

(3.20)

Since our considerations are valid only at high energies, this suggests that one can think of the potential term \( \lambda U \) as a perturbation to the large kinetic part of \( H^2 \). It is also clear that

\[ \frac{\| \vec{m} \|^2 |G_{00}| \kappa^2 - B^2}{\| \vec{m} \|^2} > 0. \]  

(3.21)

The kinetic term of the Hamiltonian (3.18) is considered as a five dimensional Laplace–Beltrami operator of \( T^{1,1} \)

\[ \bar{P}^2 = \left( \sum_{i,j=1}^{2} \hat{G}^{ij} \hat{\Pi}_i \hat{\Pi}_j + \sum_{p,q=1}^{3} \hat{G}^{pq} \hat{\Pi}_p \hat{\Pi}_q \right) \rightarrow \Delta_{T^{1,1}}, \]  

(3.22)
which defines the eigenfunctions of the Hamiltonian, satisfying the following Schrödinger equation

\[
\frac{\|\hat{m}\|^2 |G_{00}| \kappa^2 - B^2}{\|\hat{m}\|^2} \Delta_{T^{1,1}} \Psi = -E^2 \Psi, \tag{3.23}
\]

or

\[
\left[ 1 - \frac{B^2(\theta_1, \theta_2)}{|G_{00}| \kappa^2 \|\hat{m}\|^2(\theta_1, \theta_2)} \right] \Delta_{T^{1,1}} \Psi = -\frac{E^2}{|G_{00}| \kappa^2} \Psi. \tag{3.24}
\]

Here we have

\[
0 \leq \frac{B^2(\theta_1, \theta_2)}{|G_{00}| \kappa^2 \|\hat{m}\|^2(\theta_1, \theta_2)} = \frac{b \mu^2}{Z^4 + \mu^2} \left( \frac{m_3 - m_1 \cos \theta_1 - m_2 \cos \theta_2}{\sum_{i=1}^2 m_i^2 \sin^2 \theta_i + \frac{3}{2} m_i^2} \right) < 1. \tag{3.25}
\]

Moreover, at high energies, the inequality is satisfied

\[
0 \leq \frac{B^2(\theta_1, \theta_2)}{|G_{00}| \kappa^2 \|\hat{m}\|^2(\theta_1, \theta_2)} \ll 1. \tag{3.26}
\]

At quantum level the wave function accounts for all quantum fluctuations. As far as we are interested in the quasi-classical quantum behavior it is convenient to split the fluctuations into fast and slow variables. At the relevant accuracy level the wave function ‘feels’ only the slow fluctuations while the fast ones enter the wave equation with their average over the fluctuation period. In our case fast variables are the angles \( \theta_1 \) and \( \theta_2 \). Indeed, it is easy to see from the explicit solutions (2.32) and (2.30) that the periods \( m_i / \sqrt{\kappa} \) of \( \theta_i(\sqrt{\kappa} = r) \) are very small provided \( \kappa^2 \) is very large. In this sense, the wave function will ‘feel’ the average values of \( \cos \theta_i \) and \( \cos^2 \theta_i \). Moreover, their oscillation amplitudes are in terms with a weight much smaller than one (3.26). Therefore, we can approximate the small term (3.26) as follows

\[
0 \leq \frac{B^2(\theta_1, \theta_2)}{|G_{00}| \kappa^2 \|\hat{m}\|^2(\theta_1, \theta_2)} \approx \frac{b \mu^2}{(Z^4 + \mu^2) \left( \frac{m_3^2 + \frac{3}{2} m_1^2 m_2^2}{m_1^2} \right)} \ll 1. \tag{3.27}
\]

Then, we have to solve the following equations

\[
\Delta_{T^{1,1}} \Psi = -\frac{E^2}{\kappa^2 Z^4} \left[ \frac{2b m_1^2 + (1 + b)(m_1^2 + m_2^2)}{\sin \theta_1 \partial \theta_1} \left( \sin \theta_1 \frac{\partial}{\partial \theta_1} \right) + \frac{b \cos \theta_1}{\sin^2 \theta_1} \left( \frac{\partial^2}{\partial \phi_1^2} + \frac{2b \cos \theta_1}{\sin^2 \theta_1} \frac{\partial^2}{\partial \phi_1 \partial \phi_1} \right) \right] \Psi. \tag{3.28}
\]

3.2. Laplace–Beltrami operator and wave function

3.2.1. Laplace–Beltrami operator on \( T^{1,1} \). Using the line element of \( T^{1,1} \) given by (2.14) and the standard definition of the Laplace–Beltrami operator we find (see section 3 of [41] for the general case of \( T^{p,q} \))

\[
\Delta_{T^{1,1}} = \frac{4}{b^2} \left[ \frac{b}{\sin \theta_1 \partial \theta_1} \left( \sin \theta_1 \frac{\partial}{\partial \theta_1} \right) + \frac{b \cos \theta_1}{\sin^2 \theta_1} \left( \frac{\partial^2}{\partial \phi_1^2} + \frac{2b \cos \theta_1}{\sin^2 \theta_1} \frac{\partial^2}{\partial \phi_1 \partial \phi_1} \right) \right].
\]
The Schrödinger equation for the wave function is

\[ J. \text{ Phys. A: Math. Theor. Z} \gg \text{directions} \]

\( (3.34) \) in (3.32), together with (3.30), we arrive at

\[ \text{We can rewrite this expression in a form, which is more useful for our analysis} \]

\[ \Delta_{T^{1,1}} = 4 \frac{b^3}{Z^2} \left[ b \left( \frac{1}{\sin \theta_1} \frac{\partial}{\partial \theta_1} \left( \sin \theta_1 \frac{\partial}{\partial \theta_1} \right) + \frac{1}{\sin^2 \theta_1} \left( \frac{\partial}{\partial \phi_1} + \cos \theta_1 \frac{\partial}{\partial \psi} \right)^2 \right) \right. \]

\[ + 4 \left( \frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left( \sin \theta_2 \frac{\partial}{\partial \theta_2} \right) + \frac{1}{\sin^2 \theta_2} \left( \frac{\partial}{\partial \phi_2} + \cos \theta_2 \frac{\partial}{\partial \psi} \right)^2 \right) \left. \right] \frac{\partial^2}{\partial \theta^2} \phi. \] (3.30)

The full measure on \( T^{1,1} \) is

\[ d\Omega = \sqrt{\det(G_{\mu \nu})} d\theta_1 d\theta_2 d\phi_1 d\phi_2 d\psi = \frac{b^3}{32} \sin \theta_1 \sin \theta_2 d\theta_1 d\theta_2 \sin \phi_1 d\phi_2 d\psi. \] (3.31)

### 3.2.2. Wave function

The Schrödinger equation for the wave function is

\[ \Delta_{T^{1,1}} \Psi(\theta_1, \theta_2, \phi_1, \phi_2, \psi) = -M^2 \frac{E^2}{\kappa^2} \Psi(\theta_1, \theta_2, \phi_1, \phi_2, \psi), \] (3.32)

where

\[ M^2 = \frac{Z^4 \left[ 2bm_3 + (1 + b)(m_1^3 + m_2^3) \right]}{Z^4 \left[ 2bm_3 + (1 + b)(m_1^3 + m_2^3) \right] + \mu^2 (m_1^3 + m_2^3)}. \] (3.33)

One notes that the order of magnitude for \( M^2 \sim 1 \), when we are situated far from the boundary, \( Z \gg 1 \). To separate the variables, we define \( \Psi \) as

\[ \Psi(\theta_1, \theta_2, \phi_1, \phi_2, \psi) = e^{i l_1 \theta_1} e^{i l_2 \theta_2} e^{i l_3 \psi} f_1(\theta_1) f_2(\theta_2), \quad l_1, l_2, l_3 \in \mathbb{Z}. \] (3.34)

With this choice we can solve for the eigenfunctions, replacing the derivatives along Killing directions \((\partial_{\theta_1}, \partial_{\theta_2}, \partial_{\psi})\) by \((il_1, il_2, il_3)\) correspondingly. Note that the constraint (3.17) implies the following relation among the quantum numbers \( l_{1,2,3} \)

\[ m_1 l_1 + m_2 l_2 + m_3 l_3 = 0, \] (3.35)

hence one of the \( l_{1,2,3} \) quantum numbers can be expressed in terms of the other two. Substituting (3.34) in (3.32), together with (3.30), we arrive at

\[ E^2 = \frac{4}{b^2} \left( bE_1^2 + bE_2^2 + l_3^2 \right), \] (3.36)

where \( E_1 \) and \( E_2 \) are determined by the ordinary differential equations

\[ \left[ \frac{1}{\sin \theta_i} \frac{\partial}{\partial \theta_i} \left( \sin \theta_i \frac{\partial}{\partial \theta_i} \right) - \frac{1}{\sin^2 \theta_i} (l_i + \cos \theta_i l_3)^2 \right] f_i(\theta_i) = -M^2 \frac{E_i^2}{\kappa^2} f_i(\theta_i), \quad i = 1, 2. \] (3.37)
It is convenient to define new variables \( z_i = \cos \theta_i \). Then the equations can be written as
\[
(1 - z_i^2) \frac{d^2}{dz_i^2} - 2z_i \frac{d}{dz_i} - \frac{1}{1 - z_i^2} (l_i + z_i J_3) ^2 + \frac{M^2 E_i ^2}{\kappa^2} f_i (z_i) = 0. \tag{3.38}
\]
The solutions to these equations are given by the standard hypergeometric functions. However, we choose the regular solution
\[
f_i (z_i) = (1 - z_i)^{\frac{b_i}{2}} (1 + z_i)^{-\frac{a_i}{2}} F_1 \left[ \frac{1}{2} \left( \alpha_i + \beta_i + 1 - \sqrt{1 + 4 \left( \frac{l_i ^2 + M^2 E_i ^2}{\kappa^2} \right)} \right), 1 + \alpha_i, \frac{1}{2} \beta_i \right]. \tag{3.39}
\]
where \( \alpha_i \equiv |l_i - l_j| \) and \( \beta_i \equiv |l_i + l_j| \). In addition, we have to ensure that the solutions \( f_i (\theta_i) \) are square integrable with respect to the measure for \( \theta_i \), which leads to the following restrictions on the parameters
\[
\sqrt{1 + 4 \left( \frac{l_i ^2 + M^2 E_i ^2}{\kappa^2} \right)} - \alpha_i - \beta_i - 1 = 2n_i, \quad n_i \in \mathbb{N}. \tag{3.40}
\]
From (3.36) and (3.40) one finds the squared semi-classically quantized energy of the pulsating string configuration
\[
E^2 = \frac{4 \kappa^2}{b_i^2 M^2} \left( \frac{b}{4} \sum_{i=1}^{2} (2n_i + \alpha_i + \beta_i + 1)^2 - \left( 2b - \frac{M^2}{\kappa^2} \right) l_i ^2 - \frac{b}{2} \right). \tag{3.41}
\]
In this case, the large quantum numbers \( n_{1,2} \gg l_{1,2,3} \gg 1 \) assure \( E^2 > 0 \). Now, the solution can be written in terms of Jacobi polynomials [42]
\[
f_i (z_i) = (1 - z_i)^{\alpha_i/2} (1 + z_i)^{-\beta_i/2} \frac{n_i ! \Gamma (\alpha_i + 1)}{\Gamma (\alpha_i + 1 + n_i)} p_{n_i}^{(\alpha_i, \beta_i)} (z_i), \tag{3.42}
\]
thus one can write the normalized wave functions such as
\[
\Psi_{n_i}^{\alpha_i, \beta_i} (z_i) = \sqrt{\frac{(\alpha_i + \beta_i + 1 + 2n_i) n_i ! \Gamma (\alpha_i + \beta_i + 1 + n_i)}{2^{\alpha_i + \beta_i + 1 + 2n_i} n_i ! \Gamma (\alpha_i + 1 + n_i) \Gamma (\beta_i + 1 + n_i)}} \times (1 - z_i)^{\alpha_i/2} (1 + z_i)^{-\beta_i/2} p_{n_i}^{(\alpha_i, \beta_i)} (z_i), \quad i = 1, 2. \tag{3.43}
\]
Therefore, the full wave function in \( T^{1,1} \) (3.34) looks like
\[
\Psi = \frac{1}{\sqrt{16 \pi ^3}} e^{il_1 \phi_1} e^{il_2 \phi_2} e^{i \theta_1 \psi} \Psi_{n_1}^{\alpha_1, \beta_1} (z_1) \Psi_{n_2}^{\alpha_2, \beta_2} (z_2). \tag{3.44}
\]
Taking the limit \( \tilde{\mu} \to 0 \) (Schrödinger \( \times T^{1,1} \to \text{AdS}_4 \times T^{1,1} \)) we find perfect agreement with the results for the energy and the wave function given in [40].
3.3. Leading correction to the energy

In terms of the new variables $z_i$ the potential (3.19) becomes

$$U(z_1, z_2) = |\tilde{m}|^2 |G_{00}| \kappa^2 - B^2$$

$$= \frac{bc^2}{4} \left\{ b \left( m_1^2 + m_2^2 + m_3^2 \right) + (|G_{00}| - b) \left[ m_1^2 \left( 1 - z_1^2 \right) + m_2^2 \left( 1 - z_2^2 \right) \right] + 2bm_1m_2 z_1z_2 - 2bm_3 (m_1z_1 + m_2z_2) \right\}.$$  \hspace{1cm} (3.45)

Consequently the measure (3.31) also changes to

$$d\Omega = \frac{b^3}{32} dz_1 \, dz_2 \, d\phi_1 \, d\phi_2 \, d\psi, \quad -1 \leq z_1, z_2 \leq 1.$$  \hspace{1cm} (3.46)

Hence, considering the large energies (large quantum numbers $n_{1,2}$ and $l_{1,2,3}$), one can find the first correction to the energy

$$\delta E^2 = \lambda \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{2\pi} |\Psi(z_1, z_2, \phi_1, \phi_2, \psi)|^2 U(z_1, z_2) d\Omega(z_1, z_2, \phi_1, \phi_2, \psi)$$

$$= \frac{b^3 \kappa^2}{32} \int_{-1}^{1} \int_{-1}^{1} \left\{ |\psi_{n_1, \beta_1}^{\alpha}(z_1)|^2 |\psi_{n_2, \beta_2}^{\alpha}(z_2)|^2 U(z_1, z_2) dz_1 \, dz_2. \right\}.$$  \hspace{1cm} (3.47)

In this context, the explicit form of the correction is obtained by plugging the various wave functions and the potential in (3.47), namely

$$\delta E^2 = \lambda \frac{b^3 \kappa^2}{128} \int_{-1}^{1} \left( \sum_{i=1}^{3} m_i^2 \right) + \frac{(|G_{00}| - b - 1) \sum_{i=1}^{2} m_i^2 \int_{-1}^{1} (1 - z_i^2) |\psi_{n_i, \beta_i}^{\alpha}(z_i)|^2 dz_i}{ \right\}.$$  \hspace{1cm} (3.48)

In short notations it looks like

$$\delta E^2 = \lambda \frac{b^3 \kappa^2}{128} \left( \frac{|G_{00}| - b - 1}{b} \right) \left( m_1^2 l_1 + m_2^2 l_2 \right) + 2m_1m_2 l_1l_2 - 2m_3 (m_1l_1 + m_2l_2) + \sum_{j=1}^{3} m_j^2 \right\},$$  \hspace{1cm} (3.49)

where the introduced integrals are explicitly calculated as follows

$$I_1 = \int_{-1}^{1} \left( 1 - z_i^2 \right) |\psi_{n_i, \beta_i}^{\alpha}(z_i)|^2 dz_i$$

$$= \frac{(m_i + \alpha_i + \beta_i + 1)(m_i + \alpha_i + \beta_i + 2)(m_i + \alpha_i + 1)(m_i + \beta_i + 1)}{(2m_i + \alpha_i + \beta_i + 1)(2m_i + \alpha_i + \beta_i + 2)(2m_i + \alpha_i + \beta_i + 3)} \right\}.$$  \hspace{1cm} (3.50)
\[ \tilde{I}_i = \int_{-1}^{1} z_i |\Psi_{n_i}^{(\alpha_i, \beta_i)}(z)|^2 dz_i \]
\[ \quad = \frac{2(n_i + \beta_i)(n_i + \alpha_i + \beta_i)}{2(n_i + \alpha_i + \beta_i + 1)(2n_i + \alpha_i + \beta_i)} + \frac{2(n_i + 1)(n_i + \alpha_i + 1)}{(2n_i + \alpha_i + \beta_i + 1)(2n_i + \alpha_i + \beta_i + 2)}. \] (3.51)

The expression for the correction to the energy looks very complicated. Therefore, we use the fact that the approximation we work in is for large quantum numbers, say \( n_{1,2} \gg l_{1,2,3} \gg 1 \).

Within this approximation the integrals behave like
\[ \tilde{I}_i = \frac{1}{8} + \frac{1}{32} (2\alpha_i^2 + 2\beta_i^2 - 1) \frac{1}{n_i^2} + O\left( \frac{1}{n_i^3} \right), \] (3.52)
\[ \tilde{I}_i = 1 + \frac{1}{4} (\beta_i^2 - \alpha_i^2) \frac{1}{n_i^2} + O\left( \frac{1}{n_i^3} \right). \] (3.53)

Since \( \alpha_i \equiv |l_i - l_3| \) and \( \beta_i \equiv |l_i + l_3| \), the above integrals look like
\[ I_i = \frac{1}{8} + \frac{1}{8} \left( \frac{l_i^2 + l_3^2 - \frac{1}{4}}{n_i^2} \right) + O\left( \frac{1}{n_i^3} \right), \] (3.54)
\[ \tilde{I}_i = 1 + \frac{l_i l_3}{n_i^2} + O\left( \frac{1}{n_i^3} \right). \] (3.55)

As expected, the energy correction \( \delta E^2 \) reduces to the result\(^9\) of AdS\(_5\) \( \times \) \( T^{1,1} \) obtained in [45], when \( \mu \to 0 \).

In order to extract the anomalous dimension of the operators in the dual field theory we have to define an appropriate expansion of the energy \( \tilde{E} \). For this purpose, let us represent the square root \( \sqrt{E^2 + \delta E^2} \) in the following way
\[ \tilde{E} = E \sqrt{1 + \frac{\delta E^2}{E^2}} = E \sqrt{1 + \epsilon}. \] (3.56)

The squared quasi-classical energy \( E^2 \) is large, because it is proportional to the sum of large quantum numbers \( n_{1,2} \) and \( l_{1,2,3} \). Therefore, one requires \( \delta E^2 < E^2 \), in which case \( \epsilon < 1 \). One can see this from the asymptotic form of \( \epsilon \) for large \( n_{i} \), namely
\[ \epsilon = \frac{b_0 M^2 \left( \left( \frac{\epsilon_0}{\lambda} \right)^2 + 7 \right) (m_1^2 + m_2^2) + 8m_3^2 - 16m_3(m_1 + m_2) + 16m_1m_2}{4096 (n_1^2 + n_2^2)} \lambda \]
\[ + O\left( \frac{1}{n_i^3} \right), \] (3.57)
assuming that the ratio \( \lambda/(n_1^2 + n_2^2) \) is kept small. Expanding the energy up to the first power of \( \epsilon \) and using (3.41) and (3.49) one finds

\(^9\)In [45] the authors consider \( \kappa = 1 \).
\[
\tilde{E} = \frac{2\kappa}{bM} \sqrt{\frac{b}{2} \sum_{i=1}^{2} (2n_i + \alpha_i + \beta_i + 1)^2 - \left(2b - \frac{M^2}{\kappa^2}\right) I_2^2 - \frac{b}{2}} \\
+ \frac{b^6 \kappa MY}{512 \sqrt{\frac{b}{2} \sum_{i=1}^{2} (2n_i + \alpha_i + \beta_i + 1)^2 - \left(2b - \frac{M^2}{\kappa^2}\right) I_2^2 - \frac{b}{2}}} \lambda + \mathcal{O}(c^2),
\]

where

\[
Y = \left(\frac{(G_{0i})}{b} - 1\right) (m_i^2 I_1 + m_i^2 I_2) + 2m_1 m_2 I_1 I_2 - 2m_3 (m_1 I_1 + m_2 I_2) + \sum_{j=1}^{3} m_j^2.
\]

Hence, by the dictionary of the correspondence the anomalous dimension \(\Delta\) of the operators in the dual gauge theory is given by

\[
\Delta = \frac{b^6 \kappa MY}{512 \sqrt{\frac{b}{2} \sum_{i=1}^{2} (2n_i + \alpha_i + \beta_i + 1)^2 - \left(2b - \frac{M^2}{\kappa^2}\right) I_2^2 - \frac{b}{2}}} \lambda.
\]

Since \(n_i\) are large we can further write

\[
\Delta \approx \frac{sMB^2 \frac{1}{4} \left(\frac{(G_{0i})}{b} + 7\right) (m_1^2 + m_2^2) + 8m_3^2 - 16m_3(m_1 + m_2) + 16m_1m_2}{4096 \sqrt{n_1^2 + n_2^2}} \lambda.
\]

### 3.4. Short comments on field theory side

The dual theory is conjectured to be a specific field theory, namely dipole field theory. The TsT transformation which produces bulk theory corresponds to a particular deformation (Drinfel’d–Reshetikhin twist) on field theory side so it is worth to give very brief comments about field operators. To this end let us say a few words about gauge theory side. The symmetries of gauge theory originate from bulk geometry. For instance, in the case of AdS5 geometry the string states on \(S^5\) are described by conserved momenta \((J_1, J_2, J_3)\) which are the three \(R\)-charges of the theory. On the other the scalar operators in the SYM theory can be characterized by the highest weight \((J_1, J_2, J_3)\) of an SO(6) representation with Dynkin indices \([J_3 + J_1 - J_2, J_2 - J_1]\). Explicitly, on Yang–Mills side the simplest operators are of the form \(\text{Tr}(X^a Y^b Z^c)\), where \(X, Y, Z\) are the standard complex scalars of the \(\mathcal{N} = 4\) supermultiplet, \(X = 1/\sqrt{2} (\varphi_1 + i\varphi_2), Y = 1/\sqrt{2} (\varphi_3 + i\varphi_4), Z = 1/\sqrt{2} (\varphi_5 + i\varphi_6)\). As it is well known, the \(R\)-charge are ordering independent under the trace however, operators having different ordering mix under renormalization. It is this mixing which makes computation of one-loop anomalous dimensions a nontrivial issue. The particular class of pulsating string solutions in \(S^5\) and AdS5 part of the geometry correspond to operators of the form \(\text{Tr}(ZZ)^{\Delta-J_1/2} X^{J_1}\) and \(\text{Tr}(DD)^{\Delta-J_2/2} Z^{J_2}\), respectively.

Before going to the main case in our study let us say a few words about Schrödinger holographic model from field theory perspective. The TsT transformations on string theory side translates to the dual theory as a star product. When the directions involved in TsT are trans-
verse to the stack of branes from which geometry descends, it produces a twist in the field theory having ordinary product. If however, one of the directions is along the branes the star product is non-trivial and the theory becomes dipole one

\[(\Phi_1 \star \Phi_2)(x) = \Phi_1(x + L_1)\Phi_2(x - L_2),\]  

(3.62)

where \(L = L_1 + L_2\) is the dipole length associate with \(R\)-charges. The big advantage of \(\text{Schr}_5 \times S^7\) holographic model is that it is integrable. Proving that, the authors of [21] were able to map the composite operators of monomial form to a spin chain. To study field theory side one must just to replace the ordinary product with the star one.

\[O = \text{tr}(\Phi_1 \star \Phi_2 \star \cdots).\]  

(3.63)

One can choose to work either using Seiberg–Witten map or working directly with the star product. A nice analysis has been presented in [21].

Let us turn now to the case of \(\text{AdS}_5 \times T^{1,1}\) where the supersymmetry gets reduced from \(\mathcal{N} = 4\) to \(\mathcal{N} = 1\). As we discuss in this paper, the parent, after certain TsT-transformation, produces \(\text{Schr}_5 \times T^{1,1}\) holographic model. This geometry appears as the large \(N\) limit of a stack of \(D3\) branes with attached six-dimensional conifold. This six-dimensional cone \(M^6\) comes with metric

\[ds^2_{M^6} = dr^2 + r^2 d\Omega_{M^5}^2,\]  

(3.64)

where \(M^5_{S^E}\) is Sasaki–Einstein five-manifold, is Kähler and Ricci-flat and thus, a Calabi–Yau manifold. Actually the Sasaki–Einstein five-manifold \(M^5_{S^E}\) is a \(U(1)\) fibration over a four-dimensional Kähler–Einstein manifold \(M^4_{K^E}\). The dual to \(\text{AdS}_5 \times T^{1,1}\) field theory is \(\mathcal{N} = 1\) superconformal gauge theory known as the Klebanov–Witten theory and was originally described in [37]. The geometry contains \(p^1 \times p^1\) which suggests that field theory has flavor symmetry \(SU(2) \times SU(2)\). The elementary degrees of freedom are denoted by the fields \(A\) and \(B\), each a doublet of the factor groups \(SU(2) \times SU(2)\) and with conformal anomalous dimension \(\Delta_{AB} = 3/4\). The gauge group is \(SU(N) \times SU(N)\), and the two chiral multiplets \(A\) and \(B\) are correspondingly in the \((N, \bar{N})\) and \((\bar{N}, N)\) representations. The field theory superpotential is easy to find

\[W = \frac{\lambda}{2} \epsilon^{ijk} \text{tr}[AB_iA,B_j],\]  

(3.65)

with \(i = 1, 2\). The chiral operators analogue of the \((X, Y, Z)\) operators in \(\mathcal{N} = 4\) SYM are given by \(\text{tr}(AB)^k\) with \(R\)-charge \(k\) and in the \((\frac{1}{2}, \frac{1}{2})\) representation of the flavor group \(SU(2) \times SU(2)\). Unfortunately, unlike the case of \(\text{AdS}_5 \times S^5\), it has been shown that this theory possess chaotic behavior [46]. Although there are island of integrability, the theory is not integrable in general. This point toward the conclusion that powerful Bethe ansatz technique is not applicable for further analysis.

Turning to the case of \(\text{Schr}_5 \times T^{1,1}\) background, one can repeat the arguments from [21] and replace the operators product with the star one. The big disadvantage however is that the theory is not integrable and one cannot expect that it can be mapped to a spin chain. Thus, we cannot implement the sophisticated analysis of [21] to our case. At this point we can only say that a study of particular subsectors is in progress.
4. Conclusion

Our study is focused on the dynamics of pulsating strings in Schrödinger background times $T^{1,1}$ conifold with non-zero $B$-field. The Schrödinger part of the spacetime was obtained by TsT deformations of the original AdS$_5 \times T^{1,1}$ space involving the time direction and one spatial dimension in the internal space. This particular TsT technique is also known as null Melvin twist transformation, which is a special case of the general Drinfel’d–Reshetikhin twist.

Previous studies of pulsating strings in particular backgrounds with external fluxes have been conducted for instance in [43]. In order to find pulsating string solutions in Schr$_5 \times T^{1,1}$ background, we have employed an appropriate ansatz (2.18) for the string configuration. Considering the bosonic part of the Polyakov string action in conformal gauge we have found the relevant classical equations of motion (2.19)–(2.23) together with the Virasoro constraints (2.26). The solutions are in terms of Jacobi elliptic functions (2.32). Furthermore, we managed to explicitly calculate also the classical energy of the pulsating string configuration in terms of the relevant parameters of the problem.

A great simplification in the semi-classical treatment of the problem came from the fact that the squared Hamiltonian (3.18) of the pulsating strings has the form of a point–particle Hamiltonian. This allowed us to recognize an effective string potential, which was later used to obtain the corrections to the energy via perturbation theory in powers of a properly defined small parameter $\epsilon$, which is proportional to a ratio of the ’t Hooft coupling $\lambda$ and the eigenvalues of the semi-classical energy $E$. The holographic AdS/CFT dictionary relates the calculated energy corrections to the anomalous dimensions (3.60) of the operators in the dual gauge theory. Thus, the next issue we focused on has been the calculation of corrections to the energy of the bulk theory. The resulting equation (3.24) for the wave function is a second order Fuchsian differential equation, which is hard to solve analytically. However, since we consider correspondence at quasi-classical level the wave equation can be reduced to simpler expressions applying physically relevant approximation.

We solved the approximated wave equation and obtained the wave function. Analogously to the case of AdS$_5 \times S^5$, the semi-classical quantized energy is given in terms of principal quantum numbers $(l_1, l_2, l_3)$ associated with the variables $\phi_1, \phi_2$ and $\psi$ as well as the integer numbers $(n_1, n_2)$ dubbed as quantization conditions. The first correction to the energy is obtained by making use of the wave function and potential to the first order in the coupling $\lambda$ (3.49). According to the holographic correspondence corrections to the bulk energy determine anomalous dimensions of the dual field theory operators (3.60). As expected, the energy corrections depend on the parameter $\hat{\mu}$ coming from generating background TsT procedure. It measured the deviation from the most studied case of AdS$_5 \times T^{1,1}$ [45]. Indeed, the potential for the effective Hamiltonian (3.19), compared to [45], contains deformations not only in the first term but receives a new contribution from the $B$-field. However, if we take a limit $\hat{\mu} \to 0$ the theory should be reduced to holography in AdS$_5 \times T^{1,1}$. Indeed, conducting this limit the effective theory in terms of energy correction reduces to the one obtained in [45].

This study complements the results for non-relativistic holographic correspondence, which is much less investigated compared to the relativistic AdS/CFT correspondence. In this context there are many ways to proceed. One aspect is to extend this kind of analysis to other backgrounds with non-relativistic field theory duals. Another issue is the identification of (at least of particular class of) field theory operators and thorough investigation of their correlation functions. We hope to address these issues in the near future.
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ORCID iDs

M Radomirov https://orcid.org/0000-0001-6180-9215
T Vetsov https://orcid.org/0000-0003-2912-7964

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