A Timeon Model of Quark and Lepton Mass Matrices*

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\footnote{This research was supported in part by the U.S. Department of Energy (grant no. DE-FG02-92-
ER40699)
Abstract

It is proposed that $T$ violation in physics, as well as the masses of electron and $u$, $d$ quarks, arise from a pseudoscalar interaction with a new spin 0 field $\tau(x)$, odd in $P$ and $T$, but even in $C$. This interaction contains a factor $i\gamma_5$ in the quark and lepton Dirac algebra, so that the full Hamiltonian is $P$, $T$ conserving; but by spontaneous symmetry breaking, the new field $\tau(x)$ has a nonzero expectation value $<\tau> \neq 0$ that breaks $P$ and $T$ symmetry. Oscillations of $\tau(x)$ about its expectation value produce a new particle, the "timeon". The mass of timeon is expected to be high because of its flavor-changing properties.

The main body of the paper is on the low energy phenomenology of the timeon model. As we shall show, for the quark system the model gives a compact 3-dimensional geometric picture consisting of two elliptic plates and one needle, which embodies the ten observables: six quark masses, three Eulerian angles $\theta_{12}$, $\theta_{23}$, $\theta_{31}$ and the Jarlskog invariant of the CKM matrix.

For leptons, we assume that the neutrinos do not have a direct timeon interaction; therefore, the lowest neutrino mass is zero. The timeon interaction with charged leptons yields the observed nonzero electron mass, analogous to the up and down quark masses. Furthermore, the timeon model for leptons contains two fewer theoretical parameters than observables. Thus, there are two testable relations between the three angles $\theta_{12}$, $\theta_{23}$, $\theta_{31}$ and the Jarlskog invariant of the neutrino mapping matrix.

PACS: 12.15.Ff, 11.30.Er

Key words: timeon, $CP$ and $T$ violation, CKM matrix, neutrino mapping matrix, Jarlskog invariant
1. Introduction

We suggest that the observed $CP$ and $T$ violations are due to a new $P$ odd and $T$ odd spin zero field $\tau(x)$, called the timeon field; the same field is also responsible for the small masses of $u$, $d$ quarks, as well as that of the electron. Consider first the quark system. Let $q_i(\uparrow)$ and $q_i(\downarrow)$ be the quark states "diagonal" in $W^\pm$ transitions\[1\]:

$$q_i(\downarrow) \Leftrightarrow q_i(\uparrow) + W^-$$ (1.1)

and

$$q_i(\uparrow) \Leftrightarrow q_i(\downarrow) + W^+$$ (1.2)

with $i = 1$, 2 and 3. The electric charges in units of $e$ are $+\frac{2}{3}$ for $q_i(\uparrow)$ and $-\frac{1}{3}$ for $q_i(\downarrow)$. These quark states $q_i(\uparrow)$ and $q_i(\downarrow)$ are, however, not the mass eigenstates $d$, $s$, $b$ and $u$, $c$, $t$. We assume that the mass Hamiltonians $H_\uparrow$ for $q_i(\uparrow)$ and $H_\downarrow$ for $q_i(\downarrow)$ are given by

$$H_{\uparrow/\downarrow} = \left(q_1^\dagger, q_2^\dagger, q_3^\dagger\right)_{\uparrow/\downarrow} \left(G_{\gamma_4} + iF_{\gamma_4\gamma_5}\right)_{\uparrow/\downarrow} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}_{\uparrow/\downarrow}$$ (1.3)

with the $3 \times 3$ matrices $G_{\uparrow/\downarrow}$ and $F_{\uparrow/\downarrow}$ both real and hermitian. The mass matrix $G_{\uparrow/\downarrow}$ is the same zeroth order mass matrix as $M_0(q_{\uparrow/\downarrow})$ of Ref. 1, given by

$$G_{\uparrow/\downarrow} = \begin{pmatrix} \beta\eta^2(1 + \xi^2) & -\beta\eta & -\beta\xi\eta \\ -\beta\eta & \beta + \alpha\xi^2 & -\alpha\xi \\ -\beta\xi\eta & -\alpha\xi & \alpha + \beta \end{pmatrix}_{\uparrow/\downarrow}$$ (1.4)

in which $\alpha_\uparrow$, $\beta_\uparrow$, $\xi_\uparrow$, $\eta_\uparrow$ and $\alpha_\downarrow$, $\beta_\downarrow$, $\xi_\downarrow$, $\eta_\downarrow$ are all real parameters with $\alpha_{\uparrow/\downarrow}$ and $\beta_{\uparrow/\downarrow}$ to be positive. It can be readily verified that the determinants

$$|G_{\uparrow}| = |G_{\downarrow}| = 0.$$ (1.5)

Thus, the lowest eigenvalues of $G_{\uparrow}$ and $G_{\downarrow}$ are both zero. These two real symmetric matrices can be diagonalized by real, orthogonal matrices $(U_{\uparrow})_0$ and $(U_{\downarrow})_0$, with

$$(U_{\uparrow})_0^\dagger G_{\uparrow}(U_{\uparrow})_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_0(c) & 0 \\ 0 & 0 & m_0(t) \end{pmatrix}$$ (1.6)
and 

\[(U_\uparrow)_0^\dagger G_\uparrow(U_\uparrow)_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_0(s) & 0 \\ 0 & 0 & m_0(b) \end{pmatrix}\] (1.7)

where the nonzero eigenvalues are the zeroth order masses of c, t and s, b quarks, with

\[
m_0(c) = \beta_\uparrow[1 + \eta_\uparrow^2(1 + \xi_\uparrow^2)], \tag{1.8}\n\]

\[
m_0(t) = \alpha_\uparrow(1 + \xi_\uparrow^2) + \beta_\uparrow, \tag{1.9}\n\]

\[
m_0(s) = \beta_\downarrow[1 + \eta_\downarrow^2(1 + \xi_\downarrow^2)] \tag{1.10}\n\]

and

\[
m_0(b) = \alpha_\downarrow(1 + \xi_\downarrow^2) + \beta_\downarrow. \tag{1.11}\n\]

Thus, \(G_\uparrow\) and \(G_\downarrow\) can be each represented by an ellipse of minor and major axes given by \(m_0(c)\) and \(m_0(t)\) for \(\uparrow\) and likewise \(m_0(s)\) and \(m_0(b)\) for \(\downarrow\).

The orientations of these two elliptic plates are determined by their eigenstates. As in Ref. 1, we define four real angular variables \(\theta_\downarrow\), \(\phi_\downarrow\) and \(\theta_\uparrow\), \(\phi_\uparrow\) by

\[
\xi_\downarrow = \tan \phi_\downarrow, \quad \xi_\uparrow = \tan \phi_\uparrow \\
\eta_\downarrow = \tan \theta_\downarrow \cos \phi_\downarrow \quad \text{and} \quad \eta_\uparrow = \tan \theta_\uparrow \cos \phi_\uparrow. \tag{1.12}\n\]

The eigenstates of \(G_\uparrow\) are

\[
\epsilon_\uparrow = \begin{pmatrix} \cos \theta_\uparrow \\ \sin \theta_\uparrow \cos \phi_\uparrow \\ \sin \theta_\uparrow \sin \phi_\uparrow \end{pmatrix} \quad \text{with eigenvalue } 0, \tag{1.13}\n\]

\[
p_\uparrow = \begin{pmatrix} -\sin \theta_\uparrow \\ \cos \theta_\uparrow \cos \phi_\uparrow \\ \cos \theta_\uparrow \sin \phi_\uparrow \end{pmatrix} \quad \text{with eigenvalue } m_0(c) \tag{1.14}\n\]

and

\[
P_\uparrow = \begin{pmatrix} 0 \\ -\sin \phi_\uparrow \\ \cos \phi_\uparrow \end{pmatrix} \quad \text{with eigenvalue } m_0(t). \tag{1.15}\n\]
Correspondingly, the eigenstates of $G_{\downarrow}$ are

$$
\epsilon_{\downarrow} = \left( \begin{array}{c}
\cos \theta_{\downarrow} \\
- \sin \theta_{\downarrow} \cos \phi_{\downarrow} \\
- \sin \theta_{\downarrow} \sin \phi_{\downarrow}
\end{array} \right) \text{ with eigenvalue } 0, \quad (1.16)
$$

$$
p_{\downarrow} = \left( \begin{array}{c}
\sin \theta_{\downarrow} \\
\cos \theta_{\downarrow} \cos \phi_{\downarrow} \\
\cos \theta_{\downarrow} \sin \phi_{\downarrow}
\end{array} \right) \text{ with eigenvalue } m_0(s), \quad (1.17)
$$

and

$$
P_{\downarrow} = \left( \begin{array}{c}
0 \\
- \sin \phi_{\downarrow} \\
\cos \phi_{\downarrow}
\end{array} \right) \text{ with eigenvalue } m_0(b). \quad (1.18)
$$

We note that by changing $\theta_{\uparrow}, \phi_{\uparrow}$ to $-\theta_{\downarrow}, \phi_{\downarrow}$ the unit vectors $\epsilon_{\uparrow}, p_{\uparrow}, P_{\uparrow}$ of (1.13)-(1.15) become $\epsilon_{\downarrow}, p_{\downarrow}$ and $P_{\downarrow}$ of (1.16)-(1.18). Here the signs of $\theta_{\uparrow}$ and $\theta_{\downarrow}$ are chosen so that the sign convention of the particle data group’s CKM matrix agrees with both $\theta_{\uparrow}$ and $\theta_{\downarrow}$ being positive, as we shall see. In terms of these eigenstates, the $3 \times 3$ real unitary matrices $(U_{\uparrow})_0$ and $(U_{\downarrow})_0$ of (1.6)-(1.7) are given by

$$
(U_{\uparrow})_0 = (\epsilon_{\uparrow}, p_{\uparrow}, P_{\uparrow}). \quad (1.19)
$$

and

$$
(U_{\downarrow})_0 = (\epsilon_{\downarrow}, p_{\downarrow}, P_{\downarrow}) \quad (1.20)
$$

Thus, in the absence of the $iF_{\uparrow/\downarrow}\gamma_4\gamma_5$ term in (1.3), the corresponding CKM matrix in this approximation is given by

$$
(U_{\text{CKM}})_0 = (U_{\uparrow})_0^\dagger (U_{\downarrow})_0 =
\left( \begin{array}{ccc}
\cos \theta_{\downarrow} \cos \theta_{\uparrow} & \sin \theta_{\downarrow} \cos \theta_{\uparrow} & \sin \theta_{\uparrow} \sin \phi \\
- \sin \theta_{\downarrow} \sin \theta_{\uparrow} \cos \phi & + \cos \theta_{\downarrow} \sin \theta_{\uparrow} \cos \phi & \\
- \cos \theta_{\downarrow} \sin \theta_{\uparrow} & - \sin \theta_{\downarrow} \sin \theta_{\uparrow} & \cos \theta_{\uparrow} \sin \phi \\
- \sin \theta_{\downarrow} \cos \theta_{\uparrow} \cos \phi & + \cos \theta_{\downarrow} \cos \theta_{\uparrow} \cos \phi & \\
\sin \theta_{\downarrow} \sin \phi & - \cos \theta_{\downarrow} \sin \phi & \cos \phi
\end{array} \right) \quad , \quad (1.21)
$$
in which
\[ \phi = \phi^\uparrow - \phi^\downarrow. \] (1.22)

We assume that $T$ violation and the small masses of $u$, $d$ quarks are due to the new
\[ iF\gamma_4\gamma_5 \] (1.23)
term in (1.3), with
\[ F^\uparrow = F^\downarrow = F = \tau_q f \bar{f} \] (1.24)
in which $\tau_q$ is a real constant and $f$ a 3 dimensional unit vector represented by its $3 \times 1$ real column matrix. Graphically, we can visualize $G^\uparrow$ and $G^\downarrow$ as two elliptic plates mentioned above, and $F^\uparrow/\downarrow$ as a single needle of length $\tau_q$ and direction $f$, as shown in Figure 1.

We note that the time-reversal operation $T$ in quantum mechanics involves a complex conjugation operation changing the factor $i$ to $-i$. Since the entirety of classical mechanics can be formulated with real numbers, the presence of $i$ in quantum mechanics is necessitated by the commutation or anticommutation relation between operators, such as that between $\gamma_4$ and $\gamma_5$. This led us to postulate (1.23) as the source of $T$ violation. The specific form given by (1.23)-(1.24) can be due to the spontaneous symmetry breaking of a new $T$ odd, $P$ odd and $CP$ odd, spin 0 field $\tau(x)$, which has a vacuum expectation value given by
\[ \langle \tau(x) \rangle_{\text{vac}} = \tau_q \neq 0. \] (1.25)

While the general characteristics of spontaneous time reversal symmetry breaking models have been discussed in the literature[2], one of the new features of the present model is to connect such symmetry breaking with the smallness of the light quark and electron masses.

In Section 2, we begin with a general $3 \times 3$ $T$, $P$ and $CP$ odd mass matrix of the form
\[ G\gamma_4 + iF\gamma_4\gamma_5 \] (1.26)
with $G$ and $F$ both real and hermitian, then derive some useful properties of its eigenvalues and eigenvectors. In Sections 3 and 4, we summarize the analysis of how in (1.24), the length $\tau_q$ and the direction $f$ of the needle are related to the light quark masses and the Jarlskog invariant[3] $J$ of the CKM matrix[4,5].
As we shall see, this leads to
\[ \tau_q \simeq 33MeV, \quad (1.27) \]
\[ m_u \simeq \tau_q (\tilde{f} \epsilon_\uparrow)^2 \quad (1.28) \]
and
\[ m_d \simeq \tau_q (\tilde{f} \epsilon_\downarrow)^2 \quad (1.29) \]
with \( \epsilon_\uparrow \) and \( \epsilon_\downarrow \) given by (1.13) and (1.16).

An interesting feature of the model is: the implicit assumption that the constant \( \tau_q \) might be due to the spontaneous symmetry breaking of a new type of \( T \) odd and \( CP \) odd, spin \( 0 \) field \( \tau(x) \), which has a vacuum expectation value given by (1.25). As an example, we may assume that the Lagrangian density of \( \tau(x) \) is given by
\[ -\frac{1}{2} \left( \frac{\partial \tau}{\partial x_\mu} \right)^2 - V(\tau) \quad (1.30) \]
with
\[ V(\tau) = -\frac{1}{2} \lambda \tau^2 (\tau_q^2 - \frac{1}{2} \tau^2) \quad (1.31) \]
in which the (renormalized) value of \( \lambda \) is positive. This then yields (1.25). Expanding \( V(\tau) \) around its equilibrium value \( \tau = \tau_q \), we have
\[ V(\tau) = -\frac{\lambda}{4} \tau_q^4 + \frac{1}{2} m_\tau^2 (\tau - \tau_q)^2 + O[(\tau - \tau_q)^3] \quad (1.32) \]
with
\[ m_\tau = (2\lambda)^{\frac{1}{2}} \tau_q, \quad (1.33) \]
the mass of this new \( T \) violating, \( C \) violating and \( CP \) violating quantum, called timeon.

The interaction between \( \tau(x) \) and the quark field might be obtained by replacing the \( F = \tau_q f \tilde{f} \) factor of (1.24) with
\[ F = \tau(x) f \tilde{f} \quad (1.34) \]
Because of the flavor-changing property of the timeon field[6], its mass (if it exists) must be quite high. A full analysis of this interesting possibility
lies outside the scope of this paper. Here, we concentrate on the low energy phenomenology of the timeon model.

In the application to quarks, there are ten measurable parameters in $H_{\uparrow/\downarrow}$ given by (1.3). These consist of 3 angles

$$\theta_{\uparrow}, \theta_{\downarrow} \text{ and } \phi = \phi_{\uparrow} - \phi_{\downarrow} \quad (1.35)$$

of (1.21) and 4 zeroth order masses

$$m_0(c), m_0(t), m_0(s) \text{ and } m_0(b) \quad (1.36)$$
given by (1.8)-(1.11) in the description of $G_{\uparrow}$ and $G_{\downarrow}$. In addition, the timeon term (1.24) contains 3 parameters:

$$\tau_q \text{ and two angles in } f. \quad (1.37)$$

(The angle $\phi_{\uparrow} + \phi_{\downarrow}$ is an unphysical gauge parameter.) These ten theoretical parameters account for ten observables: six quark masses, three Eulerian angles

$$\theta_{12}, \theta_{23}, \theta_{31} \quad (1.38)$$

and the $T$-violating phase factor

$$e^{i\delta} \quad (1.39)$$
in the CKM matrix. The timeon model provides a simple picture, combining these ten observables into a single compact geometric structure.

In Section 5, we extend the timeon model to leptons. Similar to quarks, there are also 10 observables: six leptonic masses and four angles (as in (1.38) and (1.39)) of the neutrino mapping matrix. However, unlike the $\uparrow$ and $\downarrow$ quarks, the mass scales of the neutrinos are much smaller than those of the charged leptons. Thus, it seems reasonable to explore the interesting possibility that the timeon term is absent in the neutrino sector. In this sense, the neutrino sector may be regarded as more "primeval", with its lowest neutrino mass zero. As we shall discuss, the corresponding timeon model for leptons gives electron a mass and the number of parameters in the theory can be reduced to only 8. Thus, there are 2 testable relations between the 10 observables.
2. Eigenvalues and Eigenvectors

2.1 General Formulation

We begin with (1.26) and write the corresponding mass Hamiltonian as

\[ H = \psi^\dagger (\mathcal{G}\gamma_4 + i\mathcal{F}\gamma_5)\psi \]  \hspace{1cm} (2.1)

where \( \mathcal{G} \) and \( \mathcal{F} \) are both 3 \times 3 real hermitian matrices. Resolve the Dirac field operator \( \psi \) into a sum of left-handed and right-handed components:

\[ \psi = L + R \]  \hspace{1cm} (2.2)

with

\[ L = \frac{1}{2}(1 + \gamma_5)\psi \quad \text{and} \quad R = \frac{1}{2}(1 - \gamma_5)\psi. \]  \hspace{1cm} (2.3)

Thus, (2.1) becomes

\[ H = L^\dagger (\mathcal{G} - i\mathcal{F})\gamma_4 R + R^\dagger (\mathcal{G} + i\mathcal{F})\gamma_4 L. \]  \hspace{1cm} (2.4)

Define

\[ M \equiv \mathcal{G} - i\mathcal{F}; \]  \hspace{1cm} (2.5)

\[ M^\dagger \equiv \mathcal{G} + i\mathcal{F} \]  \hspace{1cm} (2.6)

and their product

\[ \mathcal{M}^2 \equiv MM^\dagger = (\mathcal{G} - i\mathcal{F})(\mathcal{G} + i\mathcal{F}). \]  \hspace{1cm} (2.7)

Since \( \mathcal{M}^2 \) is hermitian, it can be diagonalized by a unitary transformation. We write

\[ V_L^\dagger MM^\dagger V_L = m_D^2 = \text{diagonal} \]  \hspace{1cm} (2.8)

with

\[ V_L^\dagger V_L = 1. \]

Multiply (2.8) on the right by \( m_D^{-1} \), we see that by defining

\[ V_R \equiv M^\dagger V_L m_D^{-1}, \]  \hspace{1cm} (2.9)

we have

\[ V_L^\dagger MV_R = m_D; \]  \hspace{1cm} (2.10)
Furthermore, $V_R$ is also the unitary matrix that can diagonalize the corresponding $M^\dagger M$; i.e.,

$$V_R^\dagger M^\dagger M V_R = m_D^2.$$  \hfill (2.11)

(Note that $M^2 = MM^\dagger$, but $M$ can be quite different from $M$ or $M^\dagger$.)

For application, the matrices $G$ and $F$ can be either $G_\uparrow$, $F_\uparrow$ or $G_\downarrow$, $F_\downarrow$ of (1.3). As in (1.13)-(1.15), we write the eigenstates of $G$ as column matrices

$$\epsilon, \ p \text{ and } P.$$  \hfill (2.12)

Likewise, write the unit column matrix $f$ in (1.24) as

$$f = \begin{pmatrix} \cos a \\ \sin a \cos b \\ \sin a \sin b \end{pmatrix}.$$  \hfill (2.13)

Thus,

$$G = \nu \epsilon \tilde{\epsilon} + \mu \tilde{p} p + m \tilde{P} \tilde{P}$$  \hfill (2.14)

and correspondingly we may set

$$F = \tau f \tilde{f},$$  \hfill (2.15)

with $\nu, \mu, m, \tau$ all real constants. When $\tau = \tau_q$ and $\nu = 0$, $F$ becomes $F$ of (1.24) and $G$ can be either $G_\uparrow$ or $G_\downarrow$ of (1.4). For the moment, we retain the eigenvalue $\nu$ in (2.14) for the formal symmetry of some of the mathematical expressions (as in (2.23) below), even though $\nu = 0$ when we discuss physical applications of our model.

As in (1.19) and (1.20), we define a real unitary matrix $U_0$ whose columns are $\epsilon, \ p$ and $P$ of (2.12); i.e.,

$$U_0 = (\epsilon \ p \ P).$$  \hfill (2.16)

The matrix $U_0$ diagonalizes $G$, with

$$G' \equiv \tilde{U}_0 G U_0 = \begin{pmatrix} \nu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & m \end{pmatrix}.$$  \hfill (2.17)
It also transforms $\mathcal{F}$ into

$$\mathcal{F}' \equiv \tilde{U}_0 \mathcal{F} U_0 = \tau f' \tilde{f}'$$  \hspace{1cm} (2.18)

where

$$f' = \begin{pmatrix} f_e \\ f_p \\ f_P \end{pmatrix}$$  \hspace{1cm} (2.19)

with

$$f_e = \tilde{e} f, \quad f_p = \tilde{p} f, \quad f_P = \tilde{P} f.$$  \hspace{1cm} (2.20)

As before,

$$f_e^2 + f_p^2 + f_P^2 = 1.$$  \hspace{1cm} (2.21)

By using (2.17)-(2.19), we find that the same $U_0$ also transforms the matrix $\mathcal{M}^2$ into

$$(\mathcal{M}')^2 = \tilde{U}_0 \mathcal{M}^2 U_0 = (\mathcal{G}')^2 + (\mathcal{F}')^2 + i[\mathcal{G}', \mathcal{F'}],$$  \hspace{1cm} (2.22)

which is given by

$$(\mathcal{M}')^2 = \begin{pmatrix} \nu^2 + \tau^2 f_e^2 & \tau [\tau - i(\mu - \nu)] f_e f_p & \tau [\tau - i(m - \nu)] f_P f_e \\ \tau [\tau + i(\mu - \nu)] f_e f_p & \mu^2 + \tau^2 f_p^2 & \tau [\tau - i(m - \mu)] f_p f_P \\ \tau [\tau + i(m - \nu)] f_P f_e & \tau [\tau + i(m - \mu)] f_p f_P & m^2 + \tau^2 f_P^2 \end{pmatrix}.$$  \hspace{1cm} (2.23)

For our applications, we are only interested in the case $\nu = 0$. Define

$$\mathcal{N} \equiv \lim_{\nu \to 0} (\mathcal{M}')^2$$  \hspace{1cm} (2.24)

and let $\lambda_1^2, \lambda_2^2, \lambda_3^2$ be the eigenvalues of $\mathcal{N}$. From (2.23) and (2.24) we have

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = m^2 + \mu^2 + \tau^2$$  \hspace{1cm} (2.25)

and

$$\lambda_1^2 \lambda_2^2 \lambda_3^2 = |\mathcal{N}| = \tau^2 f_e^4 \mu^2 m^2,$$  \hspace{1cm} (2.26)

These eigenvalues are also the solution $\lambda^2$ of the cubic equation

$$|\mathcal{N} - \lambda^2| = |\mathcal{N}| + A \lambda^2 + B \lambda^4 - \lambda^6 = 0$$  \hspace{1cm} (2.27)

where

$$A = -\mu^2 m^2 - \tau^2 [m^2(1 - f_P^2)^2 + \mu^2(1 - f_P^2)^2 + 2m \mu f_p^2 f_P^2]$$  \hspace{1cm} (2.28)
and
\[ B = m^2 + \mu^2 + \tau^2. \]  
(2.29)

In the limit \( \tau \to 0 \), we see from (2.25)-(2.29) that the two heavier masses become \( \mu \) and \( m \), while the lightest mass is proportional to \( \tau \). (Readers who are only interested in perturbative solutions are encouraged to move on to Section 3 directly.)

2.2 Some Useful Expressions

It is convenient to arrange the three eigenvalues \( \lambda_1^2, \lambda_2^2, \lambda_3^2 \) in an ascending order, each with a new subscript:
\[ \lambda_s^2 < \lambda_l^2 < \lambda_L^2, \]  
(2.30)

Write
\[ E_i = \lambda_i^2 \]  
(2.31)

with
\[ i = s, \ l \ \text{and} \ L. \]  
(2.32)

(Here, the letters are \( s \) for small, \( l \) for large and \( L \) for very large.) Let \( \psi_i \) be the corresponding eigenstate defined by
\[ \mathcal{N}\psi_i = E_i\psi_i. \]  
(2.33)

Using (2.23)-(2.24), we can express \( \mathcal{N} \) in the form
\[ \mathcal{N} = \begin{pmatrix} h & n\chi \\ n\chi^\dagger & n^2 \end{pmatrix} \]  
(2.34)

where
\[ h = \begin{pmatrix} \tau^2 f_\epsilon^2 & \tau(\tau - i\mu)f_\epsilon f_p \\ \tau(\tau + i\mu)f_\epsilon f_p & \mu^2 + \tau^2 f_p^2 \end{pmatrix}, \]  
(2.35)

\[ n\chi = \begin{pmatrix} (\tau - im)f_\epsilon \\ (\tau - i(m - \mu))f_p \end{pmatrix}\tau f_P \]  
(2.36)

and
\[ n^2 = m^2 + \tau^2 f_P^2, \]  
(2.37)
Correspondingly, each eigenvector $\psi_i$ of $\mathcal{N}$ can be written as

$$\psi_i = \begin{pmatrix} \phi_i \\ c_i \end{pmatrix}$$  \hspace{1cm} (2.38)

with $\phi_i$ a $2 \times 1$ column matrix and $c_i$ a constant. From (2.33), (2.34) and (2.38), we have

$$h\phi_i + (nc_i)\chi = E_i \phi_i$$  \hspace{1cm} (2.39)

and

$$n\chi^\dagger \phi_i + n^2 c_i = E_i c_i.$$  \hspace{1cm} (2.40)

From (2.39), it follows that

$$\phi_i = (E_i - h)^{-1} nc_i \chi,$$  \hspace{1cm} (2.41)

and likewise from (2.40),

$$c_i = (E_i - n^2)^{-1} n\chi^\dagger \phi_i.$$  \hspace{1cm} (2.42)

Substituting (2.41) to (2.40) we find

$$m_i^2 \equiv E_i = n^2 [1 + \chi^\dagger (m_i^2 - h)^{-1} \chi].$$  \hspace{1cm} (2.43)

Likewise, (2.39) and (2.42) lead to

$$(h - \frac{n^2}{n^2 - E_i} \chi \chi^\dagger) \phi_i = E_i \phi_i.$$  \hspace{1cm} (2.44)

Both (2.43) and (2.44) are valid for all three solutions $i = s, l$ and $L$. From (2.44), we see that once the eigenvalue $E_i$ is known, the determination of the corresponding three-dimensional eigenvector $\psi_i$ reduces to the much simpler two-dimensional spinor equation (2.44). This fact will be of use in the later sections on leptons.

**Remarks** It is well known that a mass matrix of the form (2.4) can also be written as a single term

$$\psi^\dagger M \gamma_4 \psi,$$  \hspace{1cm} (2.45)

but with the hermitian matrix $M$ complex. However, the reality conditions of $G$ and $\mathcal{F}$ lead to a specific form of $M$, and that specific form may appear
"unnatural" without the "timeon" picture. Further discussions will be given in Appendix A.

### 2.3 Determination of Eigenvalues

For quarks, the top mass $m_t$ is much larger than $m_c$ and $m_u$, so is the bottom mass $m_b >> m_s$ and $m_d$. Likewise for charged leptons, $m_\tau$ is $>> m_\mu$ and $m_e$. Thus, in the notations of (2.32), for $i = L$ the largest mass $m_L$ satisfies

$$m_L >> m_l > m_s.$$ \hspace{1cm} (2.46)

Eq.(2.43) gives a convenient route to express $m_L$ in terms of the parameters in $\mathcal{N}$, given by (2.34)-(2.37). In this case, we can regard the parameter $m$ in (2.23) as satisfying

$$m = O(m_L) >> \mu \text{ and } \tau.$$ \hspace{1cm} (2.47)

Define the parameter $\epsilon$ through

$$m_L^2 = n^2 + \epsilon = m^2 + \tau^2 f_P^2 + \epsilon.$$ \hspace{1cm} (2.48)

Eq.(2.43) in the case $i = L$ gives

$$m_L^2 = n^2 + \chi^\dagger \frac{n^2}{n^2 + \epsilon - h} \chi$$ \hspace{1cm} (2.49)

which leads to

$$\epsilon = \chi^\dagger \frac{n^2}{n^2 + \epsilon - h} \chi = \chi^\dagger \left(1 - \frac{h - \epsilon}{n^2}\right)^{-1} \chi$$

$$= \chi^\dagger \chi + \chi^\dagger \frac{h - \epsilon}{n^2} \chi + \chi^\dagger \left(\frac{h - \epsilon}{n^2}\right)^2 \chi + \cdots.$$

Thus, we have the expansion

$$m_L^2 = m^2 + \tau^2 f_P^2 + \chi^\dagger \chi + \frac{1}{m^2} [\chi^\dagger h \chi - (\chi^\dagger \chi)^2] + O(m^{-4});$$ \hspace{1cm} (2.50)

i.e., on account of (2.35)-(2.37)

$$m_L^2 = m^2 + \tau^2 f_P^2 [1 + f_\epsilon^2 + \left(1 - \frac{\mu}{m}\right)^2 f_P^2] + \frac{\tau^2 \mu^2}{m^2}.$$ \hspace{1cm} (2.51)
To derive the corresponding expansions for the smaller masses $m_s^2$ and $m_l^2$, define

$$ S \equiv m_s^2 + m_l^2 $$

and

$$ P \equiv m_s^2 m_l^2. $$

Thus,

$$ m_l^2 = \frac{1}{2} [S + (S^2 - 4P)^{\frac{1}{2}}] $$

and

$$ m_s^2 = \frac{1}{2} [S - (S^2 - 4P)^{\frac{1}{2}}]. $$

From (2.25)-(2.26), we have

$$ S = m^2 + \mu^2 + \tau^2 - m_L^2 $$

and

$$ P = \tau^2 f_e^4 \mu^2 m^2 / m_L^2. $$

Combining these with (2.51), we have the expressions for $m_l^2$ and $m_s^2$. In the limit $m \to \infty$, but $\tau$ comparable to $\mu$, (2.54) becomes

$$ S \to \mu^2 + \tau^2 (f_e^2 + f_p^2)^2 $$

and

$$ P \to \tau^2 f_e^4 \mu^2. $$

If in addition $\tau \to 0$, then we have

$$ m_L^2 \to m^2, \quad m_l^2 \to \mu^2 \quad \text{and} \quad m_s^2 \to \tau^2 f_e^4. $$
3. Perturbative Solution and Jarlskog Invariant

3.1 Perturbation Series

In this and the following sections we return to the mass Hamiltonian (1.3) for quarks and calculate its eigenstates by using $G_{\gamma 4}$ as the zeroth order Hamiltonian and $i F_{\gamma 4 \gamma 5}$ as the perturbation. From the discussions given in the last section, we see that this is identical to the problem of finding the eigenstates of $\mathcal{N}$ regarding $\tau$ as the small parameter. Using (2.23)-(2.24), we may write

$$\mathcal{N} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \mu^2 & 0 \\
0 & 0 & m^2
\end{pmatrix} + \mathcal{N}_1 + O(\tau^2) \quad (3.1)$$

with

$$\mathcal{N}_1 = \tau \begin{pmatrix}
0 & -i \mu f_{\epsilon} f_p & -i m_P f_{\epsilon} \\
i \mu f_{\epsilon} f_p & 0 & -i (m - \mu) f_p f_P \\
 i m f_P f_{\epsilon} & i (m - \mu) f_p f_P & 0
\end{pmatrix} . \quad (3.2)$$

To first order in $\mathcal{N}_1$, the eigenstates of $\mathcal{N}$ can be readily obtained. For applications to physical quarks, we need only to identify that $f_{\epsilon}$, $f_p$ and $f_P$ are replaced by

$$(f_{\epsilon})_{\uparrow \downarrow} = \tilde{f}_{\epsilon_{\uparrow \downarrow}},$$

$$(f_p)_{\uparrow \downarrow} = \tilde{f}_{p_{\uparrow \downarrow}}$$

and

$$(f_P)_{\uparrow \downarrow} = \tilde{f}_{P_{\uparrow \downarrow}} .$$

Likewise, neglecting $O(\tau^2)$ corrections, we can relate $\mu^2$ and $m^2$ to (quark mass)$^2$ by

$$\mu_{\uparrow}^2 = m_{\epsilon}^2, \quad m_{\uparrow}^2 = m_{\epsilon}^2,$$

$$\mu_{\downarrow}^2 = m_{\epsilon}^2, \quad m_{\downarrow}^2 = m_{\epsilon}^2$$

and set

$$\tau = \tau_q . \quad (3.5)$$
Thus, to $O(\tau_q)$, in the $\uparrow$ sector the state vectors of $u$, $c$, $t$ are related to those of $\epsilon_\uparrow$, $p_\uparrow$ and $P_\uparrow$ by

$$
\begin{pmatrix}
  u \\
c \\
t
\end{pmatrix} =
\begin{pmatrix}
  \epsilon_\uparrow \\
  \epsilon_\uparrow \\
  \epsilon_\uparrow
\end{pmatrix}
\begin{pmatrix}
  (\epsilon_\uparrow | u) \\
  (\epsilon_\uparrow | c) \\
  (\epsilon_\uparrow | t)
\end{pmatrix}

\begin{pmatrix}
  (p_\uparrow | u) \\
  (p_\uparrow | c) \\
  (p_\uparrow | t)
\end{pmatrix}

\begin{pmatrix}
  (P_\uparrow | u) \\
  (P_\uparrow | c) \\
  (P_\uparrow | t)
\end{pmatrix}

\begin{pmatrix}
  \epsilon_\uparrow \\
p_\uparrow \\
P_\uparrow
\end{pmatrix}
$$

with

$$
(\epsilon_\uparrow | u) = 1 + O(\tau_q^2),
$$

$$
(\epsilon_\uparrow | c) = -i \frac{\tau_q}{mc} (f_c f_p)_{\uparrow} + O(\tau_q^2),
$$

$$
(\epsilon_\uparrow | t) = -i \frac{\tau_q}{mt} (f_c f_P)_{\uparrow} + O(\tau_q^2),
$$

$$
(p_\uparrow | u) = -i \frac{\tau_q}{mc} (f_c f_p)_{\uparrow} + O(\tau_q^2),
$$

$$
(p_\uparrow | c) = 1 + O(\tau_q^2),
$$

$$
(p_\uparrow | t) = -i \frac{\tau_q}{mt + mc} (f_p f_P)_{\uparrow} + O(\tau_q^2),
$$

$$
(P_\uparrow | u) = -i \frac{\tau_q}{mt} (f_c f_P)_{\uparrow} + O(\tau_q^2),
$$

$$
(P_\uparrow | c) = -i \frac{\tau_q}{mt + mc} (f_p f_P)_{\uparrow} + O(\tau_q^2)
$$

and

$$
(P_\uparrow | t) = 1 + O(\tau_q^2).
$$

Likewise, for the $\downarrow$ sector, we may write

$$
\begin{pmatrix}
  d \\
s \\
b
\end{pmatrix} =
\begin{pmatrix}
  \epsilon_\downarrow \\
  \epsilon_\downarrow \\
  \epsilon_\downarrow
\end{pmatrix}
\begin{pmatrix}
  (\epsilon_\downarrow | d) \\
  (\epsilon_\downarrow | s) \\
  (\epsilon_\downarrow | b)
\end{pmatrix}

\begin{pmatrix}
  (p_\downarrow | d) \\
  (p_\downarrow | s) \\
  (p_\downarrow | b)
\end{pmatrix}

\begin{pmatrix}
  (P_\downarrow | d) \\
  (P_\downarrow | s) \\
  (P_\downarrow | b)
\end{pmatrix}

\begin{pmatrix}
  \epsilon_\downarrow \\
p_\downarrow \\
P_\downarrow
\end{pmatrix}
$$

with

$$
(\epsilon_\downarrow | d) = 1 + O(\tau_q^2),
$$

$$
(\epsilon_\downarrow | s) = -i \frac{\tau_q}{ms} (f_c f_p)_{\downarrow} + O(\tau_q^2),
$$

$$
(\epsilon_\downarrow | b) = -i \frac{\tau_q}{mt + ms} (f_p f_P)_{\downarrow} + O(\tau_q^2).$$
\[ (e_\downarrow | b) = -i \frac{\tau_q}{m_b} (f_e f_P)_\downarrow + O(\tau_q^2), \quad (3.19) \]

\[ (p_\downarrow | d) = -i \frac{\tau_q}{m_s} (f_e f_P)_\downarrow + O(\tau_q^2), \quad (3.20) \]

\[ (p_\downarrow | s) = 1 + O(\tau_q^2), \quad (3.21) \]

\[ (p_\downarrow | b) = -i \frac{\tau_q}{m_b + m_s} (f_p f_P)_\downarrow + O(\tau_q^2), \quad (3.22) \]

\[ (P_\downarrow | d) = -i \frac{\tau_q}{m_b} (f_e f_P)_\downarrow + O(\tau_q^2), \quad (3.23) \]

\[ (P_\downarrow | s) = -i \frac{\tau_q}{m_b + m_s} (f_p f_P)_\downarrow + O(\tau_q^2) \quad (3.24) \]

and

\[ (P_\downarrow | b) = 1 + O(\tau_q^2). \quad (3.25) \]

### 3.2 Jarlskog Invariant

Write the CKM matrix as

\[ U_{CKM} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \quad (3.26) \]

Following Jarlskog[3], we introduce

\[ S_1 = U_{11}^{*} U_{12}, \quad S_2 = U_{21}^{*} U_{22}, \quad S_3 = U_{31}^{*} U_{32} \quad (3.27) \]

and define

\[ \mathcal{J} = \text{Im} S_1^{*} S_2. \quad (3.28) \]

By using (3.27) we see that

\[ \mathcal{J} = \text{Im} \left[ (U_{11} U_{22})(U_{12}^{*} U_{21}^{*}) \right], \quad (3.29) \]

Because of unitarity of the CKM matrix,

\[ S_1 + S_2 + S_3 = 0. \quad (3.30) \]
Therefore, \( J \) is equal to twice the area of the triangle whose sides are \( S_1, S_2 \) and \( S_3 \). Furthermore, from the explicit form of \( J \) given by (3.29), we see that \( J \) is symmetric with respect to the interchange between the row and column indices of the CKM matrix. It follows then in deriving \( J \), we may use the elements of any two columns and of any two rows of the CKM matrix.

It is convenient to denote \((U_{CKM})_0\) of (1.21) simply as \( V \), with

\[
V \equiv (U_{CKM})_0 = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}.
\] (3.31)

In terms of the state vectors \( \epsilon_\uparrow, p_\uparrow, P_\uparrow \) and \( \epsilon_\downarrow, p_\downarrow, P_\downarrow \) of (1.13)-(1.18), we can also write \( V \) as

\[
V = \begin{pmatrix} \langle \epsilon_\uparrow | \epsilon_\downarrow \rangle & \langle \epsilon_\uparrow | p_\downarrow \rangle & \langle \epsilon_\uparrow | P_\downarrow \rangle \\ \langle p_\uparrow | \epsilon_\downarrow \rangle & \langle p_\uparrow | p_\downarrow \rangle & \langle p_\uparrow | P_\downarrow \rangle \\ \langle P_\uparrow | \epsilon_\downarrow \rangle & \langle P_\uparrow | p_\downarrow \rangle & \langle P_\uparrow | P_\downarrow \rangle \end{pmatrix}.
\] (3.32)

Likewise, the CKM matrix is given by

\[
U_{CKM} = \begin{pmatrix} (u|d) & (u|s) & (u|b) \\ (c|d) & (c|s) & (c|b) \\ (t|d) & (t|s) & (t|b) \end{pmatrix} = V + i\tau_q W + O(\tau_q^2)
\] (3.33)

where the matrix elements of \( W \) are derived from (3.7)-(3.15) and (3.17)-(3.25). Using the perturbative solution of Sec. 3.1, we can readily express the matrix elements of \( U_{CKM} \) in terms of the corresponding ones of \( V \). The Jarlskog invariant can then be evaluated by using (3.29).

As will be shown in Appendix B, the result, accurate to the first power of \( \tau_q \), is

\[
J = \tau_q \left[ \left( \frac{f_e f_P}{m_s} \right) A_s + \left( \frac{f_e f_P}{m_b} \right) A_b + \left( \frac{f_P f_P}{m_s + m_b} \right) B_1 \right. \\
+ \left. \left( \frac{f_e f_P}{m_c} \right) A_c + \left( \frac{f_e f_P}{m_t} \right) A_t + \left( \frac{f_P f_P}{m_c + m_t} \right) B_1 \right]
\] (3.34)

where

\[
A_s = -V_{13} V_{23} V_{33} \equiv -2 \cdot 10^{-4},
\] (3.35)
\[ A_b = -V_{12}V_{22}V_{32} \cong 8.8 \cdot 10^{-3}, \quad (3.36) \]
\[ B_{\dagger} = -V_{11}V_{21}V_{31} \cong 1.10 \cdot 10^{-3}, \quad (3.37) \]
\[ A_c = V_{31}V_{32}V_{33} \cong -2 \cdot 10^{-4}, \quad (3.38) \]
\[ A_t = V_{21}V_{22}V_{23} \cong -8.8 \cdot 10^{-3} \quad (3.39) \]

and
\[ B_{\dagger} = V_{11}V_{12}V_{13} \cong 1.10 \cdot 10^{-3}. \quad (3.40) \]

From the definition (3.29) and (3.34), these coefficients \( A_s, \ldots, B_{\dagger} \) are all products of four factors of \( V_{ij} \). As will be shown in Appendix B, because \( V \) is a real orthogonal matrix, these quartic products can all be reduced to triple products given by (3.35)-(3.40). Since \( m_c \gg m_s \) and \( m_t \gg m_b \), we can, as an approximation, neglect the terms related to the up sector in (3.34).

4. Determination of \( \tau_q \) and \( f \)

4.1 A Special coordinate system

For the \( \uparrow \) quarks, the parameters \( \lambda_1, \lambda_2, \lambda_3 \) in (2.25)-(2.29) are related to the quark masses by
\[ \lambda_1 = m_u, \quad \lambda_2 = m_c \quad \text{and} \quad \lambda_3 = m_t. \quad (4.1) \]

Likewise, \( f_\epsilon \) is
\[ (f_\epsilon)_{\dagger} = \tilde{f}\epsilon_{\dagger} \quad (4.2) \]
with \( \epsilon_{\dagger} \) given by (1.13) and \( f \) the unit directional vector of (1.24). We work to the lowest order in \( \tau_q \). From (2.26), setting \( \lambda_1\lambda_2\lambda_3 = m_u\mu m \) we have
\[ m_u = \tau_q(\tilde{f}\epsilon_{\dagger})^2. \quad (4.3) \]

Likewise, for the \( \downarrow \) quarks
\[ m_d = \tau_q(\tilde{f}\epsilon_{\downarrow})^2. \quad (4.4) \]

It is convenient to introduce a special coordinate system in which
\[ \epsilon_{\downarrow} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\dagger} = \begin{pmatrix} \cos \theta_c \\ -\sin \theta_c \\ 0 \end{pmatrix}. \quad (4.5) \]
Since \( p_\perp \) and \( P_\perp \) are both \( \perp \epsilon_\perp \), we may write

\[
p_\perp = \begin{pmatrix} 0 & -\cos \gamma \\ -\sin \gamma & \sin \gamma \end{pmatrix} \quad \text{and} \quad P_\perp = \begin{pmatrix} 0 & -\sin \gamma \\ -\cos \gamma & -\cos \gamma \end{pmatrix}.
\]  

(4.6)

Furthermore, we shall set the zeroth order CKM matrix \((U_{CKM})_0\) of (1.21) to be

\[
(U_{CKM})_0 = \begin{pmatrix} (\epsilon_\uparrow|\epsilon_\downarrow) & (\epsilon_\uparrow|p_\downarrow) & (\epsilon_\uparrow|P_\downarrow) \\ (p_\uparrow|\epsilon_\downarrow) & (p_\uparrow|p_\downarrow) & (p_\uparrow|P_\downarrow) \\ (P_\uparrow|\epsilon_\downarrow) & (P_\uparrow|p_\downarrow) & (P_\uparrow|P_\downarrow) \end{pmatrix} = \begin{pmatrix} .974 & .227 & 5 \cdot 10^{-3} \\ -2.27 & .973 & .04 \\ 5 \cdot 10^{-3} & -.04 & .999 \end{pmatrix} + O(1 \cdot 10^{-3}).
\]  

(4.7)

Thus, with the same accuracy of \(O(10^{-3})\), the Cabibbo angle \( \theta_c \) is given by

\[
\cos \theta_c = .974 \quad \text{and} \quad \sin \theta_c = .227.
\]  

(4.8)

Likewise, from \((\epsilon_\uparrow|P_\downarrow) = 5 \cdot 10^{-3}\) in (4.7), in accordance with (4.5), (4.6) and

\[
(\epsilon_\uparrow|P_\downarrow) = \sin \theta_c \sin \gamma,
\]  

(4.9)

we find

\[
\sin \gamma = 2.2 \cdot 10^{-2},
\]  

(4.10)

which together with (4.5) and (4.6) give the coordinate system defined by \((\epsilon_\perp, p_\perp, P_\perp)\). Eq.(4.7) then, in turn, determines the corresponding coordinate system \((\epsilon_\uparrow, p_\uparrow, P_\uparrow)\).

Next, we shall determine the parameters \(\tau_q\) and the directional angles \(\alpha\) and \(\beta\) of the unit vector

\[
f = \begin{pmatrix} \sin \alpha \cos \beta \\ \sin \alpha \sin \beta \\ \cos \alpha \end{pmatrix}
\]  

(4.11)

in the coordinate system defined by (4.5)-(4.6).
4.2 Determination of $\beta$

From (4.5) and (4.11), we have

$$\tilde{\epsilon}_\perp = \sin \alpha \cos \beta$$ \hfill (4.12)

and

$$\tilde{\epsilon}_\parallel = \sin \alpha \cos(\beta + \theta_c).$$ \hfill (4.13)

Thus, on account of (4.3) and (4.4),

$$\frac{\cos^2(\beta + \theta_c)}{\cos^2 \beta} = \frac{m_u}{m_d}$$ \hfill (4.14)

and therefore

$$\frac{\cos(\beta + \theta_c)}{\cos \beta} = \pm \left(\frac{m_u}{m_d}\right)^{\frac{1}{2}}.$$ \hfill (4.15)

assuming

$$\frac{m_u}{m_d} \approx \frac{1}{2},$$ \hfill (4.16)

we find two solutions for $\beta$:

$$\beta \approx 48^0 \ 50'$$ \hfill (4.17)

or

$$\beta \approx 82^0 \ 20'.$$ \hfill (4.18)

4.3 Determination of $\alpha$ and $\tau_q$

We shall first determine the parameter $\alpha$ by using the Jarlskog invariant

$$J = 3.08 \cdot 10^{-5}.$$ \hfill (4.19)

Define

$$F = 10^2 J m_b / \tau_q.$$ \hfill (4.20)

From (4.4), (4.5) and (4.11), we have

$$m_d = \tau_q \sin^2 \alpha \cos^2 \beta$$ \hfill (4.21)

and therefore

$$F = 10^2 J (m_b / m_d) \sin^2 \alpha \cos^2 \beta.$$ \hfill (4.22)
For definiteness, we shall set the various quark masses as
\[ m_d \approx 5\text{MeV}, \quad m_u \approx 2.5\text{MeV} \]
\[ m_s \approx 95\text{MeV}, \quad m_c \approx 1.25\text{GeV} \]
\[ m_b \approx 4.2\text{GeV} \quad \text{and} \quad m_t \approx 175\text{GeV}, \] (4.23)
consistent with the Particle Data Group values\[7\]. Thus, (4.22) becomes
\[ F(\alpha, \beta) \approx 2.6 \sin^2 \alpha \cos^2 \beta, \] (4.24)

On the other hand, from (3.34) and by using the numerical values for \( A_s, A_b, \cdots, B_\uparrow \) of (3.35)-(3.40) together with the various quark masses given above, the same function \( F(\alpha, \beta) \) is also
\[
F(\alpha, \beta) \approx -0.88(f_\epsilon f_P)_\downarrow - 0.067(f_\epsilon f_P)_\uparrow \\
+ 0.88(f_\epsilon f_P)_\downarrow - 0.021(f_\epsilon f_P)_\uparrow \\
+ 0.11(f_p f_P)_\downarrow + 0.0026(f_p f_P)_\uparrow. \] (4.25)
As an approximation, we may neglect the contributions of the \( \uparrow \) sector. Combining (4.24) with (4.25), we find
\[ 2.6 \sin^2 \alpha \cos^2 \beta \approx 0.88 f_{\epsilon_1}[f_{P_\downarrow} - f_{P_\uparrow}] + 0.11(f_p f_P)_\downarrow \] (4.26)
with
\[ f_{\epsilon_1} = \tilde{f}_{\epsilon_1}, \] (4.27)
given by (4.12),
\[ f_{P_\downarrow} = \tilde{f}_{P_\downarrow} = - \sin \alpha \sin \beta \cos \gamma + \cos \alpha \sin \gamma \] (4.28)
and
\[ f_{P_\uparrow} = \tilde{f}_{P_\uparrow} = - \sin \alpha \sin \beta \sin \gamma - \cos \alpha \cos \gamma. \] (4.29)
By using \( \gamma \) from (4.10),
\[ \beta \approx 48^0 50' \]
from (4.17), we find
\[ \alpha \approx -36^0 10'. \] (4.30)
From (4.4) and (4.12), we have
\[
\tau_q = \frac{m_d}{(\sin^2 \alpha \cos^2 \beta)}.
\]  
(4.31)

The above values of \(\alpha\), \(\beta\) and \(m_d \approx 5\,\text{MeV}\) give
\[
\tau_q \approx 33\,\text{MeV}.
\]  
(4.32)

On the other hand, the alternative solution \(\beta \approx 82^\circ 20'\) of (4.18) leads to a much larger value \(\tau_q \sim 5.5\,\text{GeV}\). Such a large value invalidates the small \(\tau_q\) approximation. Thus, we focus only on the solution (4.32) in this paper.

5. Applications to Leptons

The mapping matrix for leptons presents us with a different quantitative picture from the CKM matrix for quarks.

In the CKM matrix, the only off-diagonal elements of a relatively significant size are those associated with the Cabibbo angle \(\theta_c\), which is also not large. In the lepton mapping matrix \(U_l\), except for its \(T\)-violating element ((\(U_l\))\(_{13}\) in the standard form), all matrix elements are not small. Furthermore, within the present \(\sim 1\sigma\) accuracy, \(U_l\) is given by the Harrison-Perkins-Scott (HPS) form[8]
\[
(U_l)_{0} = \begin{pmatrix} \sqrt{\frac{T}{4}} & \sqrt{\frac{T}{2}} & 0 \\ -\sqrt{\frac{T}{4}} & \sqrt{\frac{T}{2}} & \sqrt{\frac{T}{2}} \\ \sqrt{\frac{T}{6}} & -\sqrt{\frac{T}{3}} & \sqrt{\frac{T}{2}} \end{pmatrix}.
\]
(5.1)

This leads us to propose the following simple Timeon model for leptons.

5.1 Mass Matrices for Leptons

As in (1.1)-(1.2), we define \(l_i(\uparrow)\) and \(l_i(\downarrow)\) to be the lepton states "diagonal" in \(W^\pm\) transitions, so that
\[
l_i(\uparrow) \doteq l_i(\downarrow) + W^+ 
\]  
and
\[
l_i(\downarrow) \doteq l_i(\uparrow) + W^-.
\]
(5.2)
with \( i = 1, 2 \) and 3. Their electric charges in units of \( e \) are 0 for \( l_i(\uparrow) \) and \(-1\) for \( l_i(\downarrow) \). However, these \( l_i(\uparrow) \) and \( l_i(\downarrow) \) are not the mass eigenstates \( \nu_1, \nu_2, \nu_3 \) and \( e, \mu, \tau \). Their mass Hamiltonians \( H_\uparrow \) and \( H_\downarrow \) are given by

\[
H_{\uparrow/\downarrow} = (\begin{pmatrix} l_1^\dagger, l_2^\dagger, l_3^\dagger \end{pmatrix})_{\uparrow/\downarrow} (G(l) \gamma_4 + iF(l) \gamma_4 \gamma_5)_{\uparrow/\downarrow} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}_{\uparrow/\downarrow} .
\] (5.3)

We choose \( G_\uparrow \) and \( G_\downarrow \) so that if \( F_{\uparrow/\downarrow} \) were to vanish, the lepton mapping matrix would be (5.1) exactly and, in addition, the lowest lepton mass in either \( \uparrow \) or \( \downarrow \) would be zero. We further simplify our model by assuming that the timeon field does not couple to neutrinos; i.e.,

\[
F(l)_{\uparrow} = 0.
\] (5.4)

Thus, the lightest neutrino is massless. All departures of the lepton mapping matrix from the HPS form (5.1) stem from the single timeon term

\[
F(l)_{\downarrow} = \tau_\nu \bar{\nu}
\] (5.5)

for the charged leptons, with \( \nu \) a real 3 dimensional unit vector like \( f \) of (1.24) for quarks. The same timeon term (5.5) also gives the electron mass. Thus, all departures from (5.1) would come from the charged leptons. A consequence of the assumption \( F(l)_{\uparrow} = 0 \) is that the predictions of the model will now depend on the choice (formerly arbitrary) of the "hidden" bases \( (l_1, l_2, l_3)_{\uparrow/\downarrow} \) of (5.2). As illustrated in Figure 2, we have chosen this basis to recognize the factorization of (5.1) with

\[
(U_\nu)_0 \equiv (V_\downarrow)_0 V_\uparrow_0
\] (5.6)

in which

\[
(V_\downarrow)_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}
\] (5.7)

and

\[
(V_\uparrow)_0 = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\] (5.8)
It appears natural to associate the left-hand factor in the product (5.6) with the charged leptons and the right-hand factor with the neutrinos; i.e., through 

\((V_\downarrow)_0\) of (5.7), we identify

\[ (l_1)_\downarrow = e_0 \]

and through \((V_\uparrow)_0\) of (5.8),

\[ (l_3)_\uparrow = \nu_3. \]

In (5.9), \(e_0\) denotes the zeroth order electron state (i.e., without its timeon correction). It is convenient to consider (5.6) as a product of two consecutive rigid body rotations with \(e_0\) and \(\nu_3\) as their respective fixed axes of rotations of the same rigid body. In this picture, we may identify

\[ l_i(\uparrow) = l_i(\downarrow). \]

A comparison between (5.9) and (5.10) suggests that like \(e_0\), \(\nu_3\) is also massless. (There is an alternative possibility with \(\nu_1\) massless, as will also be discussed below.)

### 5.2 Analysis of Mass Matrices

We begin with \(H_\downarrow\) of (5.3) for the charged leptons by assuming

\[
G(l)_\downarrow = \begin{pmatrix}
0 & 0 & 0 \\
0 & a_\downarrow + b_\downarrow & -a_\downarrow \\
0 & -a_\downarrow & a_\downarrow + b_\downarrow
\end{pmatrix},
\]  

(5.12)

with \(a_\downarrow, b_\downarrow\) both positive. Using (5.7) we can diagonalize \(G(l)_\downarrow\) through

\[
(V_\downarrow)_0^{-\dagger}G(l)_\downarrow(V_\downarrow)_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & b_\downarrow & 0 \\
0 & 0 & 2a_\downarrow + b_\downarrow
\end{pmatrix}
\]

(5.13)

and leads to the zeroth order mass of \(e\) to be 0, those of mu and tau to be

\[ \mu = b_\downarrow \]

and

\[ m = 2a_\downarrow + b_\downarrow. \]

(5.14)
The physical masses of $e$, $\mu$ and $\tau$ can then be obtained by using (2.23)-(2.29), with the parameters $\tau$ and $f$ replaced by $\tau_l$ and $v$ of the leptonic timeon term (5.5).

In the \uparrow sector, in accordance with (5.4) the neutrino masses are only due to the $G(l)_{\uparrow}$ term, with the lowest neutrino mass zero. There are two such possibilities, of which the first is

(i)

$$G(l)_{\uparrow} = \begin{pmatrix} \frac{1}{2}a_{\uparrow} + b_{\uparrow} & \sqrt{\frac{2}{3}}a_{\uparrow} & 0 \\ \sqrt{\frac{2}{3}}a_{\uparrow} & a_{\uparrow} + b_{\uparrow} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (5.15)$$

where $a_{\uparrow}$ and $b_{\uparrow}$ are both positive. The masses of $\nu_1$, $\nu_2$ and $\nu_3$ are given by

$$m_1 = b_{\uparrow}, \quad m_2 = \frac{3}{2}a_{\uparrow} + b_{\uparrow} \quad (5.16)$$

and

$$m_3 = 0. \quad (5.17)$$

In this case, we would have from the present experimental data[9,10]

$$\frac{1}{2}(m_1^2 + m_2^2) \approx 2.39 \times 10^{-3}eV^2 \quad (5.18)$$

and

$$m_2^2 - m_1^2 \approx 7.67 \times 10^{-5}eV^2. \quad (5.19)$$

(ii) The second possibility is,

$$G(l)_{\uparrow} = \begin{pmatrix} \frac{1}{2}a_{\uparrow} & \sqrt{\frac{2}{3}}a_{\uparrow} & 0 \\ \sqrt{\frac{2}{3}}a_{\uparrow} & a_{\uparrow} & 0 \\ 0 & 0 & b_{\uparrow} \end{pmatrix}, \quad (5.19)$$

which leads to

$$m_1 = 0, \quad (5.20)$$

and

$$m_2 = \frac{3}{2}a_{\uparrow}, \quad m_3 = b_{\uparrow}. \quad (5.21)$$

In this case, instead of (5.18), we have

$$m_2^2 \approx 7.67 \times 10^{-5}eV^2 \quad (5.22)$$

and

$$m_3^2 \approx 2.43 \times 10^{-3}eV^2. \quad (5.23)$$
In either case, (i) or (ii) the mass matrix (5.15) or (5.19) can be diagonalized with the same real unitary matrix \((V \uparrow)_0\) of (5.8) so that
\[
(V \uparrow)_0^\dagger G(l) \uparrow (V \uparrow)_0 = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},
\]
and the zeroth order neutrino mapping matrix (5.6) is the HPS form (5.1).

In Fig. 2, we give a graphic illustration of these two rotations \((V \downarrow)_0\) and \((V \uparrow)_0\). In either (i) or (ii), the parameters in the neutrino sector are determined. The remaining parameters are
\[
\mu, \ m
\]
the two mass parameters of (5.14) for the charged leptons, and the three parameters
\[
\tau_l \text{ and two angular variables}
\]
that characterize the unit vector \(v\) of the timeon term (5.5). On the other hand, these five parameters in (5.23)-(5.24) should account for seven observables: the three charged lepton masses
\[
m_e, \ m_\mu, \ m_\tau
\]
and the four parameters
\[
\theta_{12}, \ \theta_{23}, \ \theta_{31}
\]
and
\[
\text{the Jarlskog invariant } J_l
\]
of the lepton mapping matrix. Hence, the model defined in this section predicts two relations between these seven observables (5.25)-(5.27), as we shall discuss. At present, the existing knowledge of the neutrino mapping matrix agrees with the HPS form (5.1) to \(\approx 10\%\). Anticipating that future experiments may improve the accuracy to 1% level, we shall calculate the elements of the lepton mapping matrix to all orders in \(\tau_l/\mu\), first order in \(\tau_l/m\), but neglecting all \(1/m^2\) corrections.

5.3 Statevectors \(e, \mu\) and \(\tau\)
Consider the $l_i(\downarrow)$ sector. From (5.13)-(5.14), we have
\[
(V_\downarrow)_0^\dagger G(l)_1(V_\downarrow)_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & m \end{pmatrix}. \tag{5.28}
\]

As in (2.12) and (2.14), we can express $G(l)_\downarrow$ in terms of its eigenvalues $\mu$, $m$ and their corresponding eigenvectors $\tilde{\epsilon}$, $\tilde{p}$, $\tilde{P}$ as
\[
G(l)_\downarrow = \mu \tilde{p} \tilde{\epsilon} + m \tilde{P} \tilde{\epsilon}. \tag{5.29}
\]

Under the same transformation $(V_\downarrow)_0$, the $T$ odd timeon term $F(l)_\downarrow$ of (5.5) becomes
\[
(V_\downarrow)_0^\dagger F(l)_\downarrow(V_\downarrow)_0 = \tau_l f' \tilde{f}' \tag{5.30}
\]
with $f'$ related to the unit vector $v$ of (5.5) by
\[
f' = (V_\downarrow)_0^\dagger v. \tag{5.31}
\]
As in (2.19), write
\[
f' = \begin{pmatrix} f_\epsilon \\ f_p \\ f_P \end{pmatrix}, \tag{5.32}
\]
where in place of (2.20) we have
\[
f_\epsilon = \tilde{\epsilon} v, \quad f_p = \tilde{p} v, \quad f_P = \tilde{P} v. \tag{5.33}
\]

Equate the matrices $G$ and $F$ of (2.1) with $G(l)_\downarrow$ and $F(l)_\downarrow$, the matrix $N$ of (2.24) becomes
\[
N = (V_\downarrow)_0^\dagger M^2(V_\downarrow)_0 \tag{5.34}
\]
with $M^2$ of (2.7) given now by
\[
M^2 = [G(l)_\downarrow - iF(l)_\downarrow][G(l)_\downarrow + iF(l)_\downarrow]. \tag{5.35}
\]

For applications to charged leptons, we identify the subscripts $\epsilon$, $\mu$ and $\tau$ of (2.32) in Section 2.2 with $e$, $\mu$ and $\tau$. Thus, (2.31) and (2.32) become
\[
E_i = \lambda_i^2 = m_i^2 \quad \tag{5.36}
\]
and

\[ i = e, \mu \text{ and } \tau. \]  \hspace{1cm} (5.37)

These lepton masses and their corresponding eigenvectors \( \psi_e, \psi_\mu \) and \( \psi_\tau \) can be readily obtained by using results derived in Sections 2.2 and 2.3, as we shall see.

For \( i = e \) and \( \mu \), the eigenfunction \( \psi_i \) can be written as, in accordance with (2.38),

\[ \psi_e = \begin{pmatrix} \phi_e \\ c_e \end{pmatrix} \quad \text{and} \quad \psi_\mu = \begin{pmatrix} \phi_\mu \\ c_\mu \end{pmatrix}. \]  \hspace{1cm} (5.38)

By neglecting \( n^{-2} = O(m^{-2}) \), we can approximate (2.44) as

\[ (h - \chi \chi^\dagger) \phi_i = E_i \phi_i \]  \hspace{1cm} (5.39)

with

\[ E_i = m_{ei}^2 \text{ or } m_{\mu i}^2. \]  \hspace{1cm} (5.40)

To the same approximation, (2.40) gives

\[ c_i = -\chi^\dagger \phi_i / n. \]  \hspace{1cm} (5.41)

In terms of the Pauli spin matrices \( \sigma_i \), write

\[ h - \chi \chi^\dagger = a + b_1 \sigma_1 + b_2 \sigma_2 + b_3 \sigma_3 \]  \hspace{1cm} (5.42)

where, neglecting \( O(m^{-2}) \),

\[ a = \frac{1}{2} \left[ \mu^2 + \tau_i^2 (f_c^2 + f_p^2)^2 + 2 \tau_i \tau_p f_c^2 f_p^2 \right] \]  \hspace{1cm} (5.43)

and

\[ b_1 = \tau_i^2 (f_c^2 + f_p^2 + \frac{\mu}{m} f_p^2) f_c f_p, \]
\[ b_2 = \tau_i \mu f_c f_p, \]
\[ b_3 = -\frac{1}{2} \mu^2 + \frac{1}{2} \tau_i^2 [(f_c^4 - f_p^4) - 2 \frac{\mu}{m} f_p^2 f_p^2]. \]  \hspace{1cm} (5.44)

The eigenvalues of (5.42) are

\[ m_{\mu i}^2 = a + b \]  \hspace{1cm} (5.45)

and

\[ m_{ei}^2 = a - b \]
with

\[ b = (b_1^2 + b_2^2 + b_3^2)^{\frac{1}{2}}. \]  

(5.46)

In the charged lepton sector, it is convenient to introduce the polar coordinates \( \alpha_l \) and \( \beta_l \) through

\[ b_1 = b \sin \alpha_l \cos \beta_l, \quad b_2 = b \sin \alpha_l \sin \beta_l \quad \text{and} \quad b_3 = b \cos \alpha_l. \]  

(5.47)

Neglecting \( O(m^{-2}) \), we find the components of the eigenfunctions \( \psi_e \) and \( \psi_\mu \) in (5.38) to be

\[ \phi_e = N_e \left( \begin{array}{c} \sin \frac{1}{2} \alpha_l \\ -e^{i\beta_l} \cos \frac{1}{2} \alpha_l \end{array} \right) \]

and

\[ \phi_\mu = N_\mu \left( \begin{array}{c} e^{-i\beta_l} \cos \frac{1}{2} \alpha_l \\ \sin \frac{1}{2} \alpha_l \end{array} \right). \]

(5.48)

There are arbitrary phase factors in \( N_e \) and \( N_\mu \), which will be discussed in Appendix C. Here we simply set these normalization factors to be

\[ N_e = N_\mu = 1 + O(m^{-2}). \]

(5.49)

Combining (5.38) with (5.48), we have

\[ \psi_e = \begin{pmatrix} \langle \epsilon | e \rangle \\ \langle p | e \rangle \\ \langle P | e \rangle \end{pmatrix} = \begin{pmatrix} \sin \frac{1}{2} \alpha_l \\ -e^{i\beta_l} \cos \frac{1}{2} \alpha_l \\ c_e \end{pmatrix} \]

(5.50)

and

\[ \psi_\mu = \begin{pmatrix} \langle \epsilon | \mu \rangle \\ \langle p | \mu \rangle \\ \langle P | \mu \rangle \end{pmatrix} = \begin{pmatrix} e^{-i\beta_l} \cos \frac{1}{2} \alpha_l \\ \sin \frac{1}{2} \alpha_l \\ c_\mu \end{pmatrix} \]

(5.51)

in which

\[ c_e = -m^{-1} \chi^\dagger \begin{pmatrix} \sin \frac{1}{2} \alpha_l \\ -e^{i\beta_l} \cos \frac{1}{2} \alpha_l \end{pmatrix}, \]

(5.52)

\[ c_\mu = -m^{-1} \chi^\dagger \begin{pmatrix} e^{-i\beta_l} \cos \frac{1}{2} \alpha_l \\ \sin \frac{1}{2} \alpha_l \end{pmatrix}. \]
and

\[ \chi \equiv \begin{pmatrix} \chi_e \\ \chi_p \end{pmatrix} \]  

(5.53)

given by (2.36)-(2.37). Correspondingly, neglecting \( O(m^{-2}) \) we find

\[ \psi_\tau = \begin{pmatrix} <\epsilon|\tau> \\ <p|\tau> \\ <P|\tau> \end{pmatrix} = \begin{pmatrix} m^{-1}\chi_e \\ m^{-1}\chi_p \\ 1 \end{pmatrix}. \]  

(5.54)

Define the transformation matrix \( W \) to be

\[ W = \begin{pmatrix} <e|\epsilon> & <\mu|\epsilon> & <\tau|\epsilon> \\ <e|p> & <\mu|p> & <\tau|p> \\ <e|P> & <\mu|P> & <\tau|P> \end{pmatrix} \]  

(5.55)

with \( \epsilon, p, P \) being the same eigenvectors \( \epsilon, p, P \) of (5.29). Thus,

\[ \begin{pmatrix} e \\ p \\ P \end{pmatrix}_\downarrow = W \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}\]

(5.56)

\[ \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} = W^\dagger \begin{pmatrix} e \\ p \\ P \end{pmatrix}_\downarrow \]

and the lepton mapping matrix is given by

\[ V_{l-map} = W^\dagger \cdot (U_{l})_0 \]  

(5.57)

with \((U_{l})_0\) given by the HPS form (5.1).

Combining these matrices, we derive the lepton mapping matrix \( V_{l-map} \) given in Table 1. To compare with the experimental neutrino mapping matrix, there is still a phase convention which will be discussed in Appendix C.

5.4 Jarlskog Invariant
The Jarlskog invariant \( J_l \) can be calculated by using (3.29) and replacing \( U_{CKM} \) by \( V_{l-map} \) of Table 1. The result is

\[
J_l = \frac{N}{6(m_\mu^2 - m_\tau^2)[1 + O(m^{-2})]} [1 + O(m^{-2})]
\]  

(5.58)

with

\[
N = \mu \tau_l f_\epsilon f_p + (\tau_l/m)f_\epsilon f_p \left[ \mu^2 - \tau_l^2 (f_\epsilon^2 + f_p^2)(f_\epsilon - f_p)(f_\epsilon + 2f_p) \right]
\]  

(5.59)

In the limit \( m \to \infty \), \( J_l \) becomes

\[
J_l = [6(m_\mu^2 - m_\tau^2)]^{-1} m_\mu \tau_l f_\epsilon f_p [1 + O(m^{-1})].
\]  

(5.60)

In the limit \( \tau_l \to 0 \), we have

\[
J_l = \frac{1}{6} \tau_l (m_\mu^{-1} f_\epsilon f_p + m_\tau^{-1} f_\epsilon f_p) + O(\tau_l^3).
\]  

(5.61)

The same result can also be derived by using the perturbative expression (3.34) and replacing \( V \) in (3.35)-(3.40) by \( V_{l-map} \).

Remarks

As mentioned before, the leptonic timeon model predicts two relations between the seven observables (5.25)-(5.27). To see how a test can be made, we may take the five parameters in the theory to be

\[
m, \mu, \tau_l, f_\epsilon \text{ and } f_p
\]  

(5.62)

of (5.23)-(5.24), or the equivalent set

\[
m, a, b, \alpha_l \text{ and } \beta_l
\]  

(5.63)

with \( a, b \) given by (5.43), (5.46) and \( \alpha_l, \beta_l \) by (5.47). Using (2.50) and setting \( m_L = m_\tau \), the mass of the heavy lepton \( \tau \), we have for the first parameter in (5.63)

\[
m^2 = m_\tau^2 + O(\tau_l^2).
\]  

(5.64)

Likewise, from (5.45)

\[
a = \frac{1}{2} (m_p^2 + m_\epsilon^2)
\]  

(5.65)
and

\[ b = \frac{1}{2}(m_\mu^2 - m_e^2). \]  

(5.66)

The two remaining angular parameters \( \alpha_l \) and \( \beta_l \) in (5.63), or the equivalent parameters \( f_\epsilon \) and \( f_p \) can then be determined by using

\[ J_l \text{ and any } |V_{ij}|; \]  

(5.67)

i.e., the absolute value of any element of the experimentally measured neutrino mapping matrix. All other remaining elements of the neutrino mapping matrix may serve as part of the test of the timeon model. Further discussions are given in Appendix D.

Acknowledgement

We wish to thank N. Samios for discussions.

Table 1

\[
V_{l\text{-map}} = \begin{pmatrix}
\sqrt{\frac{2}{3}} \sin \frac{\alpha}{2} + \sqrt{\frac{1}{6}} \cos \frac{\alpha}{2} e^{i\beta} & \sqrt{\frac{1}{6}} c_e & \sqrt{\frac{1}{3}} (\sin \frac{\alpha}{2} - \cos \frac{\alpha}{2} e^{i\beta} - c_e) & \sqrt{\frac{1}{2}} (\cos \frac{\alpha}{2} e^{i\beta} + \sin \frac{\alpha}{2} - c_e) \\
\sqrt{\frac{2}{3}} \cos \frac{\alpha}{2} e^{-i\beta} - \sqrt{\frac{1}{6}} \sin \frac{\alpha}{2} + \sqrt{\frac{1}{6}} c_\mu & \sqrt{\frac{1}{3}} (\cos \frac{\alpha}{2} e^{-i\beta} + \sin \frac{\alpha}{2} - c_\mu) & \sqrt{\frac{1}{2}} (\sin \frac{\alpha}{2} + c_\mu) \\
\frac{1}{m} (\sqrt{\frac{2}{3}} \chi_\epsilon - \sqrt{\frac{1}{6}} \chi_\mu) + \sqrt{\frac{1}{6}} & \frac{1}{m} \sqrt{\frac{1}{3}} (\chi_\epsilon + \chi_\mu) - \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} (\frac{1}{m} \chi_\epsilon + 1)
\end{pmatrix}
\]

Table 1. Lepton Mapping Matrix (neglecting \( O(m^{-2}) \)), with the parameters \( \alpha = \alpha_l \) and \( \beta = \beta_l \) given by (5.47); \( c_e, c_\mu \) by (5.52), and \( \chi_\epsilon, \chi_\mu \) by (5.53) and (2.36)-(2.37). In the limit \( m \to \infty \), \( c_e, c_\mu, m^{-1} \chi_\epsilon \) and \( m^{-1} \chi_p \) all become 0.
Appendix A  Two Forms of Mass Matrix

A.1 General Formulation

As in (2.45), the mass matrix \( M \) and its related Hamiltonian \( H \) of a Dirac field operator \( \Psi \) with \( n \)-generation components can be written as

\[
H = \Psi^\dagger M \gamma_4 \Psi \quad \text{(A.1)}
\]

in which

\[
M = M^\dagger, \quad \text{(A.2)}
\]
denoting a hermitian matrix. Decompose \( \Psi \) into a sum of left-handed and right-handed parts:

\[
\Psi = \mathcal{L} + \mathcal{R} \quad \text{(A.3)}
\]

with

\[
\mathcal{L} = \frac{1}{2}(1 + \gamma_5)\Psi \quad \text{and} \quad \mathcal{R} = \frac{1}{2}(1 - \gamma_5)\Psi. \quad \text{(A.4)}
\]

Correspondingly, (A.1) becomes

\[
H = \mathcal{L}^\dagger M \gamma_4 \mathcal{R} + \mathcal{R}^\dagger M \gamma_4 \mathcal{L}. \quad \text{(A.5)}
\]

Assume \( n \geq 3 \) and \( M \) to have an imaginary part so that \( H \) is \( T, C \) and \( CP \) violating.

A different form of an \( n \)-generation \( T \) and \( CP \) violating mass Hamiltonian can be written in the form similar to (1.3), also with a Dirac operator \( \psi \) of \( n \) components:

\[
H = \psi^\dagger (G \gamma_4 + iF \gamma_4 \gamma_5) \psi, \quad \text{(A.6)}
\]

where \( G \) and \( F \) are both \( n \)-dimensional hermitian matrices,

\[
G = G^\dagger \quad \text{and} \quad F = F^\dagger. \quad \text{(A.7)}
\]

For \( n \geq 3 \), \( G \) and \( F \) both nonzero, the Hamiltonian \( H \) is \( T, P \) and \( CP \) violating. As in (A.3)-(A.4), we resolve \( \psi \) in a similar form:

\[
\psi = L + R \quad \text{(A.8)}
\]
with
\[ L = \frac{1}{2}(1 + \gamma_5)\psi \quad \text{and} \quad R = \frac{1}{2}(1 - \gamma_5)\psi. \]  
(A.9)

Thus, (A.6) becomes
\[ H = L^\dagger(G - iF)\gamma_4R + R^\dagger(G + iF)\gamma_4L, \]  
(A.10)
different from (A.5).

In the standard model, excluding the mass Hamiltonian, only the left hand components of $\uparrow$ and $\downarrow$ quarks are linked by their $W$-interaction. Hence, the right-hand component $R$ or $R^\dagger$ can undergo an independent arbitrary unitary transformation. Because of this freedom, we can bring (A.10) into the form (A.5), or vice versa, as is well known. We shall review this equivalence, and then discuss how this equivalence can be altered by imposing new restrictions on these matrices.

To show this, we begin with the form (A.10). Define
\[ M = G - iF \]  
(A.11)
and assume it to be nonsingular (i.e., the eigenvalues of $M^\dagger M$ are all nonzero.) On account of (A.7), the hermitian conjugate of $M$ is
\[ M^\dagger = G + iF. \]  
(A.12)

Since $MM^\dagger$ is hermitian, there exists a unitary matrix $V_L$ that can diagonalize $MM^\dagger$, with
\[ V_L^\dagger MM^\dagger V_L = m_D^2 = \text{Diagonal}. \]  
(A.13)

For every eigenvector $\phi$ of $MM^\dagger$ with eigenvalue $\lambda$, the corresponding vector $M^\dagger\phi$ is an eigenvector of $M^\dagger M$ with the same eigenvalue $\lambda$. Thus, $M^\dagger M$ can also be diagonalized by another unitary matrix $V_R$ as
\[ V_R^\dagger M^\dagger MV_R = m_D^2, \]  
(A.14)
with $m_D^2$ the same diagonal matrix of (A.13).

Multiply (A.13) on the right by $m_D^{-1}$, it follows that
\[ V_L^\dagger MV_R = m_D, \]  
(A.15)
provided that we define

\[ V_R = M^\dagger V_L m_D^{-1}. \]  \hspace{1cm} (A.16)

One can readily see that \( V_L \) and \( V_R \) thus defined satisfies \( V_L^\dagger V_L = 1 \), \( V_R^\dagger V_R = 1 \) as well as (A.13) and (A.14). Since \( R \) can be transformed independently from \( L \), we can transform the \( \psi \) field by

\[ L \to V_L L \]  \hspace{1cm} (A.17)

and

\[ R \to V_R R. \]  \hspace{1cm} (A.18)

Next let us examine the mass matrix \( \mathcal{M} \) of (A.1)-(A.2). Because \( \mathcal{M} \) is hermitian, it can be diagonalized by a single unitary transformation \( V \), with the left-handed and right-handed components of the field operator \( \Psi \) undergoing the same transformation; i.e., in contrast to (A.17)-(A.18), we have

\[ \mathcal{L} \to V \mathcal{L}, \]  \hspace{1cm} (A.19)

\[ \mathcal{R} \to V \mathcal{R} \]  \hspace{1cm} (A.20)

and correspondingly

\[ \mathcal{H} \to \Psi^\dagger m_D \gamma_4 \Psi \]  \hspace{1cm} (A.21)

with \( m_D \) being the corresponding diagonal matrix. So far as the mass matrices are concerned, we regard these two mass Hamiltonians \( \mathcal{H} \) and \( H \) as equivalent, if the diagonal matrix \( m_D \) of (A.21) has the same set of eigenvalues as those in (A.15). In this case, we can without loss of generality set

\[ \mathcal{M}^2 = (\mathcal{G} - i\mathcal{F})(\mathcal{G} + i\mathcal{F}) \]  \hspace{1cm} (A.22)

and

\[ V = V_L; \]  \hspace{1cm} (A.23)

hence (A.13) becomes

\[ V^\dagger \mathcal{M}^2 V = m_D^2 \]  \hspace{1cm} (A.24)

and therefore

\[ V^\dagger \mathcal{M} V = m_D. \]  \hspace{1cm} (A.25)

(Note that \( \mathcal{M} \neq \mathcal{M}^\dagger \), even though \( \mathcal{M}^2 = \mathcal{M} \mathcal{M}^\dagger \).)
A.2 Restricted Class with $G$ and $F$ Real

We will now discuss theories in which both matrices $G$ and $F$ are real; i.e.,

$$G = G^* \quad \text{and} \quad F = F^*. \quad (A.26)$$

Since $G$ and $F$ are also hermitian; they must both be symmetric matrices. In $n$-dimension, each of these matrices can carry $\frac{1}{2}n(n+1)$ independent real parameters, giving a total of $n(n+1)$ real parameters. On the other hand, $M$ being a single hermitian matrix consists of only $n^2$ real parameters. Thus, knowing $G$ and $F$, by using (A.22), we can always determine uniquely the corresponding $M$, but not the converse, by expanding in power series as follows.

Decompose the hermitian $M$ into its real and imaginary parts:

$$M = R + iI. \quad (A.27)$$

with $R$ and $I$ both real; hence, $R$ is symmetric and $I$ antisymmetric. On account of (A.26), the real part of (A.22) is

$$R^2 - I^2 = G^2 + F^2, \quad (A.28)$$

and the imaginary part is

$$\{R, I\} = [G, F]. \quad (A.29)$$

In what follows, we assume that $G$ and $F$ are both known, as in the case when the mass matrix is given by (1.3). In addition, $F$ can be regarded as small compared to $G$. Hence, we can expend $R$ and $I$ in powers of $F$. Write

$$R = G + R_2 + R_4 + R_6 + \cdots \quad (A.30)$$

and

$$I = I_1 + I_3 + I_5 + \cdots, \quad (A.31)$$

with $R_n$ and $I_m$ to be of the order of $F^n$ and $F^m$ respectively. Eqs.(A.28) and (A.29) give

$$\{G, I_1\} = [G, F],$$

$$\{G, R_2\} = F^2 + I_1^2,$$
\{G, I_3\} = -\{R_2, I_1\}, \quad (A.32)
\{G, R_4\} = \{I_1, I_3\} - R_2^2, \text{ etc.}

As noted before, so far as these mass matrices are concerned, the two formalisms (A.1) and (A.6) are regarded as equivalent to each other, provided that (A.28) and (A.29) hold. Then (A.32) gives the conditions determining the series expansions (A.30)-(A.31) of \(R\) and \(I\) in terms of \(G\) and \(F\).

It is convenient to write \(G\) in terms of its eigenvalues \(\nu, \mu, m\) and their corresponding eigenvectors \(\epsilon, p, P\) (as in (2.14)):

\[ G = \nu \hat{\epsilon} \hat{\epsilon} + \mu \hat{p} \hat{p} + m \hat{P} \hat{P}. \quad (A.33) \]

Let \(\vec{A}\) be a vector whose \(k^{th}\) component is given by the \((i, j)^{th}\) component of the commutator between \(G\) and \(F\):

\[ [G, F]_{ij} = \epsilon_{ijk} A_k \quad (A.34) \]

with \(\epsilon_{ijk} = \pm 1\) depending on \((ijk)\) being an even or odd permutation of \((1, 2, 3)\), and 0 otherwise. From (A.33) and (2.18)-(2.19), we can readily verify that

\[ A_k = \tau \left( \nu (\hat{\epsilon} \cdot \hat{f})(\hat{\epsilon} \times \hat{f}) + \mu (\hat{p} \cdot \hat{f})(\hat{p} \times \hat{f}) + m (\hat{P} \cdot \hat{f})(\hat{P} \times \hat{f}) \right)_k. \quad (A.35) \]

Since \(I\) is antisymmetric, so is \(I_1\). Write its \((ij)\)th component as

\[ (I_1)_{ij} = \epsilon_{ijk} J_k. \quad (A.36) \]

By using the first equation of (A.32) with (A.33)-(A.36), it can be readily verified that \(\vec{J}\) is related to \(\vec{A}\) by

\[ \vec{J} \cdot \hat{\epsilon} = (\mu + m)^{-1} \vec{A} \cdot \hat{\epsilon} \]
\[ \vec{J} \cdot \hat{p} = (m + \nu)^{-1} \vec{A} \cdot \hat{p} \quad (A.37) \]

and

\[ \vec{J} \cdot \hat{P} = (\nu + \mu)^{-1} \vec{A} \cdot \hat{P}. \]

In the same way, we can solve for \(R_2, I_3, \cdots\).
Appendix B  Proof of Eq.(3.34) for Jarlskog Invariant

B.1 Definitions, Corollaries and Conventions

Let \( V = (V_{i\alpha}) \) be a 3 \times 3 real orthogonal matrix with positive determinant, and indices \( i \) and \( \alpha = 1, 2 \) and \( 3 \); hence,
\[
V = V^*, \quad V^{-1} = \tilde{V} \quad \text{and} \quad |V| = 1. \tag{B.1}
\]
Given any value of \( i \), define \( i', i'' \) by
\[
\epsilon_{ii'i''} = 1, \tag{B.2}
\]
so that \((i, i', i'')\) is a cyclic permutation of \((1, 2, 3)\). We shall use the same definition for each of such similar indices \( j, k, l, \alpha, \beta, \gamma \). Thus, for a given \( j \), the corresponding \( j' \) and \( j'' \) satisfy
\[
\epsilon_{jj'j''} = 1, \tag{B.3}
\]
and likewise
\[
\epsilon_{kk'k''} = \epsilon_{ll'l''} = \epsilon_{\alpha\alpha'\alpha''} = \epsilon_{\beta\beta'\beta''} = \epsilon_{\gamma\gamma'\gamma''} = 1.
\]
Furthermore, for any pair \( i, \alpha \), we have by the expression for \( V^{-1} \)
\[
\begin{vmatrix}
V_{i'\alpha'} & V_{i'\alpha''} \\
V_{i''\alpha'} & V_{i''\alpha''}
\end{vmatrix} = |V|(V^{-1})_{i\alpha} = V_{i\alpha} \tag{B.4}
\]
on account of (B.1). This identity will enable us to reduce certain quartic products of \( V_{i\alpha} \) to triple products of \( V_{i\alpha} \), as we shall see.

B.2 Jarlskog Invariant

Similar to the relation between \( V \) of (3.32) and
\[
U = U_{CKM} \tag{B.5}
\]
of (3.33), we define \( W = (W_{i\alpha}) \) through
\[
U = V + i\tau_q W \tag{B.6}
\]
where \( W \) has the form
\[
W_{i\alpha} = \sum_{j \neq i} F_{ij} V_{j\alpha} - \sum_{\beta \neq \alpha} \mathcal{F}_{\alpha\beta} V_{i\beta}.
\]
(Eq. B.7)

For our purpose here, it is necessary to specify only that \( F \) and \( \mathcal{F} \) are real and symmetric; i.e.,
\[
F_{ij} = F_{ji} = F_{ij}^* \quad \text{and} \quad \mathcal{F}_{\alpha\beta} = \mathcal{F}_{\beta\alpha} = \mathcal{F}_{\alpha\beta}^*.
\]
(Eq. B.8)

We also define for any particular pair of indices \( k \) and \( \gamma \),
\[
J = U_{k\gamma} U_{k'\gamma'} U_{k\gamma'}^* U_{k'\gamma}^* \quad \text{and} \quad J_0 = V_{k\gamma} V_{k'\gamma'} V_{k\gamma'}^* V_{k'\gamma}^*.
\]
(Eq. B.10)

Thus, substituting (B.6) into (B.10), we find, to first order in \( \tau_q \),
\[
J - J_0 = i\tau_q \Delta_{k\gamma}
\]
(Eq. B.12)

where
\[
\Delta_{k\gamma} = J_0 \left( \frac{W_{k\gamma}}{V_{k\gamma}} + \frac{W_{k'\gamma'}}{V_{k'\gamma'}} - \frac{W_{k\gamma'}}{V_{k\gamma'}} - \frac{W_{k'\gamma}}{V_{k'\gamma}} \right).
\]
(Eq. B.13)

Note that \( k, \gamma \) are subject to the cyclic convention typified by (B.2). [It will turn out that \( \Delta_{k\gamma} \) is independent of the choice of \( k \) and \( \gamma \), even though this is not assumed.]

By substituting (B.7) into (B.13), we must obtain an expression of the form
\[
\Delta_{k\gamma} = \sum_l A_{ll'} F_{ll'} + \sum_\lambda A_{\lambda\nu} \mathcal{F}_{\lambda\nu}
\]
(Eq. B.14)

where the \( A \)'s and \( \lambda \)'s are to be determined. From (B.7), (B.13) and (B.14), we see that each \( A \) is made of terms having the form \( J_0 V_{j\alpha}/V_{i\alpha} \). Consider \( A_k \): in (B.14) we must put \( l'' = k \); hence \( l, l' \) are \( k', k'' \) in some order. Thus in (B.7), \( i \) is either \( k' \) or \( k'' \). But the index \( k'' \) does not occur in (B.13). Therefore, we have \( i = k', j = k'' \) and \( \alpha = \gamma \) or \( \gamma' \). Therefore,
\[
A_k = J_0 \left( 0 + \frac{V_{k''\gamma'}}{V_{k'\gamma'}} - 0 - \frac{V_{k''\gamma}}{V_{k'\gamma}} \right)
\]

41
\[ V_{k\gamma} V_{k'\gamma} \begin{vmatrix} V_{k''\gamma} & V_{k''\gamma'} \\ V_{k''\gamma'} & V_{k''\gamma''} \end{vmatrix} = V_{k\gamma} V_{k'\gamma'} V_{k''\gamma''} \]  \hspace{1cm} (B.15)\\

on account of (B.4). Likewise,

\[ A_{k'} = V_{k'\gamma} V_{k'\gamma'} V_{k''\gamma''}. \]  \hspace{1cm} (B.16)\\

For \( k'' \), the calculation is different, even though the result will be similar. Note that in (B.14) \( l, l' \) can be \( k, k' \) in either order, so that in (B.7) \( i \) and \( j \) can also be \( k, k' \) in either order. Thus, we now have four terms instead of two:

\[
A_{k''} = J_0 \left( \frac{V_{k'\gamma}}{V_{k\gamma}} + \frac{V_{k\gamma'}}{V_{k'\gamma'}} - \frac{V_{k'\gamma'}}{V_{k\gamma}} - \frac{V_{k\gamma'}}{V_{k'\gamma}} \right)
\]

\[
= V_{k'\gamma} V_{k'\gamma'} \begin{vmatrix} V_{k''\gamma} & V_{k''\gamma'} \\ V_{k''\gamma'} & V_{k''\gamma''} \end{vmatrix} + V_{k\gamma} V_{k'\gamma'} \begin{vmatrix} V_{k'\gamma'} & V_{k'\gamma} \\ V_{k''\gamma'} & V_{k''\gamma''} \end{vmatrix}
\]

\[
= -V_{k'\gamma} V_{k'\gamma'} V_{k''\gamma''} - V_{k\gamma} V_{k'\gamma'} V_{k''\gamma''} = +V_{k''\gamma} V_{k'\gamma'} V_{k''\gamma''} \]  \hspace{1cm} (B.17)\\
since \( V \) is orthogonal.

Similar formulas are obtained for \( A_\gamma, A'_\gamma, A''_\gamma \) with a change of sign. Since (B.15)-(B.17) all have the same form, the choice of \( k, \gamma \) in (B.10) and (B.11) is immaterial and we have

\[ J = J_0 + i\tau_q \Delta \]  \hspace{1cm} (B.18)\\

where

\[
\Delta = \sum_l F_{ll'} \prod_\alpha V_{l''\alpha} - \sum_\chi \mathcal{F}_{\chi\chi'} \prod_i V_{i\chi''}.
\]  \hspace{1cm} (B.19)\\

B.3 Applications to quarks

By consulting (3.7)-(3.15) and (3.17)-(3.25), we find that

\[ F_{ij} = \frac{f_i f_j}{m_i + m_j}, \quad \mathcal{F}_{\alpha\beta} = \frac{\bar{f}_\alpha \bar{f}_\beta}{\mu_\alpha + \mu_\beta} \]  \hspace{1cm} (B.20)\\

where

\[
f_1 = (f_\epsilon)_1, \quad f_2 = (f_p)_1, \quad f_3 = (f_p)_1,
\]

\[
\bar{f}_1 = (f_\epsilon)_1, \quad \bar{f}_2 = (f_p)_1, \quad \bar{f}_3 = (f_p)_1
\]  \hspace{1cm} (B.21)\\

with

\[ m_1 = m_u = 0, \quad m_2 = m_c, \quad m_3 = m_t, \]
\[ \mu_1 = m_d = 0, \quad \mu_2 = m_s, \quad \mu_3 = m_b. \]  
\[ (B.22) \]

Then (3.34) is seen to be (B.14) if we identify
\[ A_c = A_3, \quad A_t = A_2, \quad B_\uparrow = A_1, \]
\[ A_s = A_3, \quad A_b = A_2 \quad \text{and} \quad B_\downarrow = A_1. \]
\[ (B.23) \]

With these identifications, (3.38)-(3.40) are (B.15)-(B.17) and (3.35)-(3.37) are the corresponding formulas for \( A_1 \), \( A_2 \) and \( A_3 \).

**Appendix C Phase Convention in \( V_{\text{l-map}} \)**

In order to compare the lepton mapping matrix \( V_{\text{l-map}} \) given in Table 1 and the experimentally measured neutrino mapping matrix
\[ U_\nu = (U_{ij}), \]  
\[ (C.1) \]

there are certain phase conventions. In definition of
\[ V_{\text{l-map}} = \begin{pmatrix} <\nu_1|e> & <\nu_2|e> & <\nu_3|e> \\ <\nu_1|\mu> & <\nu_2|\mu> & <\nu_3|\mu> \\ <\nu_1|\tau> & <\nu_2|\tau> & <\nu_3|\tau> \end{pmatrix} \]  
\[ (C.2) \]
given by Table 1, we made arbitrary phase choices of these statevectors. This allows us to consider the following transformations:
\[ \begin{pmatrix} |e> \\ |\mu> \\ |\tau> \end{pmatrix} \rightarrow \Omega_l \begin{pmatrix} |e> \\ |\mu> \\ |\tau> \end{pmatrix} \]  
\[ (C.3) \]
and
\[ \begin{pmatrix} |\nu_1> \\ |\nu_2> \\ |\nu_3> \end{pmatrix} \rightarrow \Omega_\nu \begin{pmatrix} |\nu_1> \\ |\nu_2> \\ |\nu_3> \end{pmatrix}, \]  
\[ (C.4) \]
where
\[ \Omega_l = \begin{pmatrix} e^{i\xi_e} & 0 & 0 \\ 0 & e^{i\xi_\mu} & 0 \\ 0 & 0 & e^{i\xi_\tau} \end{pmatrix} \]  
\[ (C.5) \]
and
\[
\Omega_\nu = \begin{pmatrix}
e^{i\eta_1} & 0 & 0 \\
0 & e^{i\eta_2} & 0 \\
0 & 0 & e^{i\eta_3}
\end{pmatrix}.
\] (C.6)

The experimentally measured neutrino mapping matrix \(U_\nu\) is related to \(V_{l-map}\) by
\[
U_\nu = \Omega_l V_{l-map} \Omega^{-1}_\nu,
\] (C.7)

with the conditions
\[
|U_\nu| = 1
\] (C.8)

and the following four matrix elements of \(U_\nu\),
\[
U_{11}, U_{12}, U_{23} \text{ and } U_{33}
\] (C.9)

all real and positive. Thus, (C.8) and (C.9) determine five of the six phase factors in (C.5)-(C.6). The remaining one
\[
\xi_e + \xi_\mu + \xi_\tau + \eta_1 + \eta_2 + \eta_3
\] (C.10)

does not appear in \(U_\nu\).

**Appendix D  Test for Leptonic System**

Following the discussion given in Remarks at the end of Section 5, we may consider
\[
J_l \text{ and } |V_{11}|
\] (D.1)

as an example of (5.67), and two other members, say
\[
|V_{12}| \text{ and } |V_{23}|.
\] (D.2)

From Table 1 and noting that \(c_e\) and \(c_\mu\) are \(O(m^{-1})\), we have
\[
|V_{11}|^2 = \frac{2}{3} \sin^2 \frac{\alpha}{2} + \frac{1}{6} \cos^2 \frac{\alpha}{2} + \frac{2}{3} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \cos \beta \\
+ \frac{1}{3} \text{Re} \left[ c_e^*(2 \sin \frac{\alpha}{2} + e^{i\beta} \cos \frac{\alpha}{2}) \right],
\]

44
\[ |V_{12}|^2 = \frac{1}{3} \sin^2 \frac{\alpha}{2} + \frac{1}{3} \cos^2 \frac{\alpha}{2} - \frac{2}{3} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \cos \beta \]

\[ - \frac{2}{3} \text{Re} \left[ c^*_e \left( \sin \frac{\alpha}{2} - e^{i\beta} \cos \frac{\alpha}{2} \right) \right] \]  

(D3)

and

\[ |V_{23}|^2 = \frac{1}{2} \sin^2 \frac{\alpha}{2} + \text{Re}[c^*_\mu \sin \frac{\alpha}{2}], \]

in which \( \alpha = \alpha_t, \beta = \beta_t \) and all \( O(m^{-2}) \) terms are not included. Using (5.52)-(5.53) and noting that, to our approximation, (2.36)-(2.37) yield

\[ \chi = \begin{pmatrix} \chi_e \\ \chi_p \end{pmatrix} = -i \tau f_P \begin{pmatrix} f_e \\ f_p \end{pmatrix} ; \]

(D.4)

therefore, in accordance with (5.52)-(5.53)

\[ c^*_e = i \frac{\tau}{m} f_P (f_e \sin \frac{\alpha}{2} - f_p e^{-i\beta} \cos \frac{\alpha}{2}) \]  

(D.5)

and

\[ c^*_\mu = i \frac{\tau}{m} f_P (f_e e^{i\beta} \cos \frac{\alpha}{2} + f_p \sin \frac{\alpha}{2}) \]

where

\[ \tau = \tau_t . \]  

(D.6)

From (D.3) and (D.5) and eliminating \( f_P \) by (2.21), we find

\[ |V_{11}|^2 = \frac{5}{12} - \frac{1}{4} \cos \alpha + \frac{1}{3} \sin \alpha \cos \beta \]

\[- \frac{1}{6} \tau \left( 1 - f_e^2 - f_p^2 \right)^{\frac{1}{2}} (f_e + 2 f_p) \sin \alpha \sin \beta, \]

\[ |V_{12}|^2 = \frac{1}{3} - \frac{1}{3} \sin \alpha \cos \beta \]

\[- \frac{1}{3} \tau \left( 1 - f_e^2 - f_p^2 \right)^{\frac{1}{2}} (f_e - f_p) \sin \alpha \sin \beta \]  

(D.7)

and

\[ |V_{23}|^2 = \frac{1}{4} - \frac{1}{4} \cos \alpha \]

\[- \frac{1}{2} \tau \left( 1 - f_e^2 - f_p^2 \right)^{\frac{1}{2}} f_e \sin \alpha \sin \beta, \]
in which $\alpha = \alpha_l$ and $\beta = \beta_l$ are defined by (5.47). Thus, $|V_{11}|^2$, $|V_{12}|^2$ and $|V_{23}|^2$ are all functions of $m$, $\mu$, $\tau$, $f_\epsilon$ and $f_p$.

Finally, the Jarlskog invariant is given by (5.58)-(5.59). These expressions provide the explicit forms for the four observables in (D.1)-(D.2). Together with (5.45) and (5.64) for the masses of $e$, $\mu$ and $\tau$, we have seven observables in terms of five parameters.

Figure 1. A schematic drawing of the quark mass matrix $M_{l/l} = G_{\gamma_4} + iF\gamma_4\gamma_5$. The vibration of $\tau(x)$ is timeon.
Figure 2. Geometric Representation of Harrison, Perkins, Scott Transformation (5.1) and (5.6)-(5.8)

The axes $OX$, $OY$ and $OZ = -OD$ represent $e$, $\mu$ and $\tau$. An $\alpha = 45^0$ left-hand rotation $(V_1)_0$ along $e$ takes $\tau$ to $\nu_3$ and $\mu$ to $OA$. A second $\beta = \sin^{-1} \frac{1}{\sqrt{3}}$ left-hand rotation $(V_1)_0$ along $\nu_3$ takes $e$ to $\nu_1$ and $OA$ to $\nu_2$ which is along $OB$. The hidden bases are $l_1 = e$, $l_2 \parallel OA$ and $l_3 = \nu_3$. 
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