WDVV solutions from orthocentric polytopes and Veselov systems *

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Abstract

$\mathcal{N}$=4 superconformal $n$-particle quantum mechanics on the real line is governed by two prepotentials, $U$ and $F$, which obey a system of partial nonlinear differential equations generalizing the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation. For $U=0$ one remains with the WDVV equation which suggests an ansatz for $F$ in terms of a set of covectors to be found. One approach constructs such covectors from suitable polytopes, another method solves Veselov’s $\vee$-conditions in terms of deformed Coxeter root systems. I relate the two schemes for the $A_n$ example.

*Contribution to “Modern Problems in Theoretical Physics” for the 60th birthday of Joseph L. Buchbinder
1 Introduction
The issue of constructing $\mathcal{N}=4$ superconformal extensions of Calogero-type multi-particle quantum mechanics in one dimension has been attacked in several works [1]–[4]. In [1, 2] it was discovered that this task leads to the (generalized) Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation known from two-dimensional topological field theory [5, 6]. A physicist’s classification of $\mathcal{N}=4$ superconformal mechanics models based on particular WDVV solutions has been advanced in [3, 4], where new models (with a purely quantum potential based on orthocentric simplices) were found. Independently, mathematicians’ efforts revealed WDVV solutions derived from Coxeter systems and certain deformations thereof and lead to the notion of Veselov $\lor$-systems [7]–[13]. In the current paper I relate the mathematics approach with the physicist’s picture for solving the (generalized) WDVV equation. In particular, the deformed $A_n$ solutions of [9] will be mapped to the orthocentric simplices of [4].

In section 2 I recall the formulation $\mathcal{N}=4$ superconformal $n$-particle mechanics in terms of $su(1,1|2)$ generators. The closure of the superconformal algebra poses constraints on the interaction, which for an ansatz quartic in the fermionic coordinates lead to the WDVV equation plus a homogeneity condition for a quantum prepotential $F$ and to related differential equations for a classical prepotential $U$. Section 3 expresses these prepotentials in terms of a system of covectors, thereby turning the differential to nonlinear algebraic equations. Putting $U$ to zero, a family of WDVV solutions is constructed in section 4. Its covectors deform the $A_n$ root system and are parametrized by the shape moduli of orthocentric $n$-simplices. The different formulations of the WDVV equation are related in section 5, where the geometry of the deformed $A_n \lor$-systems is made explicit.

2 WDVV equations from $\mathcal{N}=4$ superconformal quantum mechanics
Let me consider a quantum mechanical system of $n$ identical particles with unit mass on the real line, described by positions $x^i$ and momenta $p_i$, and enhanced by fermionic degrees of freedom $\psi_i^\alpha$ and $\overline{\psi}_i^\alpha = (\psi_i^\alpha)^\dagger$, where $i = 1, \ldots, n$ and $\alpha = 1, 2$. Spinor indices are raised and lowered with the invariant tensor $\epsilon^{\alpha \beta}$ and its inverse $\epsilon_{\alpha \beta}$, where $\epsilon^{12} = 1$. Further, I impose the canonical quantization rules

$$[x^i, p_j] = i \delta^i_j \quad \text{and} \quad \{\psi_i^\alpha, \overline{\psi}_j^\beta\} = \delta^\beta_\alpha \delta^{ij},$$

(2.1)

with all other (anti)commutators vanishing. At this stage I have introduced a Euclidean metric ($\delta_{ij}$) in the configuration space $\mathbb{R}^{n|4n}/S_n$.

I want the dynamics to be invariant under $\mathcal{N}=4$ superconformal transformations. Their generators $\{H, Q_\alpha, \tilde{Q}^\alpha, D, J_a, S_\alpha, \tilde{S}^\alpha, K\}$, with $a = 1, 2, 3$ and $Q_\alpha = Q^\alpha$ as well as $(S_\alpha)^\dagger = S^\alpha$, form a (centrally extended) $su(1,1|2)$ algebra defined by the following non-
vanishing (anti)commutation relations,

\[
\begin{align*}
[D, H] &= -iH, \\
[D, K] &= +iK, \\
\{Q_\alpha, \bar{Q}^\beta\} &= 2H\delta_\alpha^\beta, \\
\{Q_\alpha, S^\beta\} &= +2i(\sigma_\alpha)_\beta^\gamma J_\alpha - 2D\delta_\alpha^\beta - iC\delta_\alpha^\beta, \\
\{S_\alpha, \bar{S}^\beta\} &= 2K\delta_\alpha^\beta, \\
\{\bar{Q}^\alpha, S_\beta\} &= -2i(\sigma_\alpha)_\beta^\alpha J_\alpha - 2D\delta_\beta^\alpha + iC\delta_\beta^\alpha, \\
\{D, Q_\alpha\} &= -\frac{i}{2}iQ_\alpha, \\
[K, Q_\alpha] &= +iS_\alpha, \\
\{Q_\alpha, K\} &= +i\bar{S}^\alpha, \\
\{D, S_\alpha\} &= +\frac{i}{2}iS_\alpha, \\
[H, S_\alpha] &= -iQ_\alpha,
\end{align*}
\]

Here, \(\epsilon_{123} = 1\), \(C\) stands for the central charge, and \(\{\sigma_1, \sigma_2, \sigma_3\}\) denote the Pauli matrices.

For a realization of the generators I try (repeated indices are summed over) \([1]-[4]\)

\[
K = \frac{1}{2}x^i x^i, \quad S_\alpha = x^i \psi_\alpha^i, \quad \bar{S}^\alpha = x^i \bar{\psi}^{i\alpha}, \\
D = -\frac{1}{2}(x^i p_i + p_i x^i), \quad J_\alpha = \frac{1}{2}\bar{\psi}^{i\alpha}(\sigma_\alpha)_\beta^{i\beta} \psi_\beta^i, \\
Q_\alpha = (p_j - i x^i U_{ij}(x)) \psi_\alpha^j - \frac{1}{2} x^i F_{ijkl}(x) \langle \psi_\beta^{i\beta} \bar{\psi}^{k\beta} \psi_\alpha^j \psi_\beta^l \rangle, \\
\bar{Q}^\alpha = (p_j + i x^i U_{ij}(x)) \bar{\psi}^{i\alpha} - \frac{1}{2} x^i F_{ijkl}(x) \bar{\psi}^{j\beta} \bar{\psi}^{k\beta} \psi_\alpha^i \psi_\beta^l, \\
H = \frac{1}{2}p_i p_i + V_B(x) - U_{ij}(x) \langle \psi_\alpha^i \bar{\psi}^{j\alpha} \rangle + \frac{1}{4} F_{ijkl}(x) \langle \psi_\alpha^i \bar{\psi}^{j\alpha} \bar{\psi}^{k\beta} \psi_\beta^l \rangle,
\]

with completely symmetric unknown functions \(V_B, U_{ij}\) and \(F_{ijkl}\) homogeneous of degree -2 in \(x \equiv \{x^1, \ldots, x^n\}\). Here, the symbol \(\langle \ldots \rangle\) stands for symmetric (or Weyl) ordering. The ordering ambiguity present in the fermionic sector affects the bosonic potential \(V_B\). In contrast to the \(\mathcal{N}=2\) superconformal extensions \([14, 15]\), the closure of the algebra demands the quartic term, and a nonzero central charge requires the quadratic term. Hence, there does not exist a free mechanical representation of the algebra \((2.2)\). A prototypical model is of the Calogero type,

\[
V_B = \sum_{i<j} \frac{g^2}{(x_i - x_j)^2}, \quad U_{ij} = ? , \quad F_{ijkl} = ? .
\]

Inserting the representation \((2.3)\) into the algebra \((2.2)\), one produces a fairly long list of constraints on \(V_B, U_{ij}\) and \(F_{ijkl}\). One of the consequences is that \([1, 2, 3]\)

\[
\begin{align*}
U_{ij} &= \partial_i \partial_j U \quad \text{and} \quad F_{ijkl} = \partial_i \partial_j \partial_k \partial_l F, \\
V_B &= \frac{1}{2} (\partial_i U)(\partial_i U) + \frac{g^2}{8} (\partial_i \partial_j \partial_k F)(\partial_i \partial_j \partial_k F),
\end{align*}
\]

which introduces two scalar prepotentials. Note that a quadratic polynomial in \(F\) or a constant in \(U\) are irrelevant. The constraints then turn into the following system of
nonlinear partial differential equations [2, 3] (see also [1]),
\[
(\partial_i \partial_k \partial_p F)(\partial_j \partial_l \partial_p F) = (\partial_j \partial_k \partial_p F)(\partial_i \partial_l \partial_p F) , \quad x^i \partial_i \partial_j \partial_k F = -\delta_{jk} , \quad \tag{2.6}
\]
\[
\partial_i \partial_j U - (\partial_i \partial_j \partial_k F) \partial_k U = 0 , \quad x^i \partial_i U = -C , \quad \tag{2.7}
\]
which I refer to as the “structure equations”. Notice that these equations are quadratic in \( F \) but only linear in \( U \). The first of (2.6) is a kind of zero-curvature condition for a connection \( \partial^3 F \). It coincides with the (generalized) WDVV equation known from topological field theory [5, 6]. The first of (2.7) is a kind of covariant constancy for \( \partial U \) in the \( \partial^3 F \) background. Since its integrability implies the WDVV equation projected onto \( \partial U \), I call it the “flatness condition”.

The right equations in (2.6) and (2.7) represent homogeneity conditions for \( U \) and \( F \). They are are inhomogeneous with constants \( \delta_{jk} \) and \( C \) (the central charge) on the right-hand side and display an explicit coordinate dependence. Furthermore, the second equation in (2.6) can be integrated twice, arriving at
\[
(x^i \partial_i - 2) F = -\frac{1}{2} x^i x^i \quad \text{and} \quad x^i \partial_i U = -C . \quad \tag{2.8}
\]
where I used the freedom in the definition of \( F \) to put the integration constants – a linear function on the right-hand side – to zero.

There are some dependencies among the equations (2.6) and (2.7). The contraction of two left equations with \( x^i \) is a consequence of the two right equations, and therefore only the components orthogonal to \( x \) are independent, effectively reducing the dimension to \( n-1 \). This means that only \( \frac{1}{12} n(n-1)^2(n-2) \) WDVV equations need to be solved and only \( \frac{1}{2} n(n-1) \) flatness conditions have to be checked. For \( n=2 \) in particular, the single WDVV equation follow from the homogeneity condition in (2.6), and the three flatness conditions are all equivalent. Hence, the nonlinearity of the structure equations becomes only relevant for \( n \geq 3 \).

### 3 Covector ansatz for the prepotentials

For a particular solution to (2.8), I make the ansatz [1, 3, 4]
\[
F = -\frac{1}{2} \sum_{\alpha} f_{\alpha} \alpha(x)^2 \ln |\alpha(x)| \quad \text{and} \quad U = -\sum_{\alpha} g_{\alpha} \ln |\alpha(x)| \quad \tag{3.1}
\]
with real coefficients \( f_{\alpha} \) and \( g_{\alpha} \), where \( \alpha \) runs over a finite set of (unlabelled) noncollinear covectors in \( \mathbb{R}^n \), i.e.
\[
\alpha(x) = \alpha_i x^i \quad \text{for each covector} \ \alpha . \quad \tag{3.2}
\]
The center-of-mass degree of freedom corresponds to \( \alpha(x) = \rho(x) \equiv \sum_i x^i \), and the relative particle motion is translation invariant only if \( \alpha_i \rho_i = 0 \ \forall \alpha \neq \rho \), meaning that the other covectors span only the hyperplane perpendicular to \( \rho \) and \( \{ \alpha \} \) decomposes orthogonally. Identical particles require the set \( \{ \alpha \} \) to be invariant (up to sign) under permutations of the components \( \alpha_i \) and enforce equality of the \( f_{\alpha} \) (and \( g_{\alpha} \)) coefficients for permutation-related covectors. Relative translation invariance and permutation symmetry are coordinate-dependent properties; they are not preserved by a generic SO(\( n \)) coordinate transformation. Therefore, demanding either will severely restrict the coordinate choice.
Finally, a rescaling of $\alpha$ may be absorbed into a renormalization of $f_\alpha$. Therefore, only the rays $\mathbb{R}_+\alpha$ are invariant data. I cannot, however, change the sign of $f_\alpha$ in this manner.

Compatibility of (3.1) with the conditions (2.8) directly yields

$$\sum_\alpha f_\alpha \alpha_i \alpha_j = \delta_{ij} \quad \text{and} \quad \sum_\alpha g_\alpha = C . \quad (3.3)$$

The second relation fixes the central charge, and the $g_\alpha$ are independent free couplings if not forced to zero. The first relation amounts to a decomposition of the identity $(\delta_{ij})$ into (usually non-orthogonal) rank-one projectors and imposes $\frac{1}{2} n(n+1)$ relations on the coefficients $f_\alpha$ for a given set $\{\alpha\}$.

From (3.1) one derives

$$\partial_i \partial_j \partial_k F = - \sum_\alpha f_\alpha \frac{\alpha_i \alpha_j \alpha_k}{\alpha(x)} \quad \text{and} \quad \partial_i U = - \sum_\alpha g_\alpha \frac{\alpha_i}{\alpha(x)} , \quad (3.4)$$

and so the bosonic part of the potential takes the form

$$V_B = \frac{1}{2} \sum_{\alpha,\beta} \frac{\alpha \cdot \beta}{\alpha(x) \beta(x)} \left( g_{\alpha \beta} + \frac{\hbar^2}{4} f_\alpha f_\beta (\alpha \cdot \beta)^2 \right) \quad (3.5)$$

with the covector scalar product

$$\alpha \cdot \beta = \alpha_i \delta^{ij} \beta_j = \alpha_i \beta_i . \quad (3.6)$$

The remaining structure equations in (2.6) and (2.7) become

$$\sum_{\alpha,\beta} f_\alpha f_\beta \frac{\alpha \cdot \beta}{\alpha(x) \beta(x)} (\alpha \wedge \beta)^{\otimes 2} = 0 \quad \text{and} \quad (3.7)$$

$$\sum_\beta \left( g_\beta \frac{1}{\beta(x)} - f_\beta \sum_\alpha g_\alpha \frac{\alpha \cdot \beta}{\alpha(x)} \right) \frac{1}{\beta(x)} \beta \otimes \beta = 0 \quad (3.8)$$

with

$$(\alpha \wedge \beta)^{\otimes 2}_{ijkl} = (\alpha_i \beta_j - \alpha_j \beta_i)(\alpha_k \beta_l - \alpha_l \beta_k) \quad \text{and} \quad (\beta \otimes \beta)_{ij} = \beta_i \beta_j . \quad (3.9)$$

The task is to first solve (3.7) and (3.3), i.e. find sets $\{\alpha, f_\alpha\}$, and then to determine $\{g_\alpha\}$ from (3.8), subject to (3.3). Many $F$ backgrounds do not admit a $C \neq 0$ solution, but a homogeneous $U$ can always be found [4]. I close the section with a simplifying observation. If a set of covectors decomposes into mutually orthogonal subsets, (3.7) and (3.8) hold for each subset individually, and their prepotentials just add up to the total $F$ or $U$. Therefore, one may restrict the analysis to indecomposable covector sets.

4 WDVV solutions from orthocentric simplices

For the rest of the paper I put $U$ to zero and investigate solutions to the WDVV equations (3.7), subject to the homogeneity condition

$$\sum_\alpha f_\alpha \alpha \otimes \alpha = 1 . \quad (4.1)$$
Let me look for indecomposable sets of covectors obeying the WDVV equation (3.7). In one dimension, the equation is trivial. For \( n=2 \), it follows from the homogeneity condition (4.1), which can actually be satisfied for any set \( \{ \alpha \} \) of coplanar covectors [4]. Nevertheless, it is instructive to outline the simplest examples. For the case of two covectors \( \{ \alpha, \beta \} \) one is forced to \( \alpha \cdot \beta = 0 \). For three coplanar covectors \( \{ \alpha, \beta, \gamma \} \), the homogeneity condition (4.1) uniquely fixes the \( f \) coefficients to

\[
 f_{\alpha} = -\frac{\beta \cdot \gamma}{\alpha \wedge \beta \wedge \gamma \wedge \alpha} \quad \text{and cyclic},
\]

(4.2)
due to the identity

\[
 \beta \wedge \gamma \cdot \alpha^i \alpha^j + \text{cyclic} = -\alpha \wedge \beta \wedge \gamma \wedge \alpha \delta^{ij}.
\]

(4.3)
The traceless part of the homogeneity condition should imply the single WDVV equation (3.7) in two dimensions. Indeed, the choice (4.2) turns the latter into

\[
 \alpha \wedge \beta \gamma(x) + \beta \wedge \gamma \alpha(x) + \gamma \wedge \alpha \beta(x) = 0
\]

(4.4)
which is identically true. Without loss of generality I may assume that \( \alpha + \beta + \gamma = 0 \), i.e. the three covectors form a triangle. In this case I have \( \alpha \wedge \beta = \beta \wedge \gamma = \gamma \wedge \alpha = 2A \), where the area \( A \) of the triangle may still be scaled to \( \frac{1}{2} \), and (4.2) simplifies to

\[
 f_{\alpha} = -\frac{\beta \cdot \gamma}{4 A^2} \quad \text{and cyclic}.
\]

(4.5)

Figure 1: Triangular configuration of covectors

In dimension \( n=3 \), the minimal set of three covectors must form an orthogonal basis, with \( f^{-1}_{\alpha} = \alpha \cdot \alpha \). Let me skip the cases of four and five covectors and go to the situation of six covectors because the homogeneity condition (4.1) then precisely determines all \( f \) coefficients. However, it is not true that six generic covectors can be scaled to form the edges of a polytope. The space of six rays in \( \mathbb{R}^3 \) modulo rigid SO(3) is nine dimensional, while the space of tetrahedral shapes (modulo size) has only five dimensions. In order to generalize the \( n=2 \) solution above, let me assume that my six covectors can be scaled to form a tetrahedron, with volume \( V \) and edges \( \{ \alpha, \beta, \gamma, \alpha', \beta', \gamma' \} \) where \( \alpha' \) is skew to \( \alpha \) and so on. Any such tetrahedron is determined by giving three nonplanar covectors, say
\{\alpha, \beta, \gamma\}', which up to rigid rotation are fixed by six parameters, corresponding to the shape and size of the tetrahedron.

The triangle result (4.5) can be employed to patch together the unique solution to the homogeneity condition (4.1) for the tetrahedron, but only if the geometric constraints

$$\alpha \cdot \alpha' = 0, \quad \beta \cdot \beta' = 0, \quad \gamma \cdot \gamma' = 0$$

(4.6)

are obeyed for the pairs of skew edges. In this situation, the identity

$$\beta \cdot \gamma \beta' \cdot \gamma' \alpha^i \alpha^j + \beta \cdot \gamma' \beta' \cdot \gamma \alpha'^i \alpha'^j + \text{cyclic} = -36V^2 \delta^{ij},$$

(4.7)

guarantees the homogeneity condition (4.1) for

$$f_\alpha = -\frac{\beta \cdot \gamma \beta' \cdot \gamma'}{36V^2}$$

and

$$f_{\alpha'} = -\frac{\beta' \cdot \gamma \beta \cdot \gamma'}{36V^2}$$

(4.8)

plus their cyclic images. Tetrahedra subject to (4.6) are called “orthocentric” [16]. They are characterized by the fact that all four altitudes are concurrent (in the orthocenter) and their feet are the orthocenters of the faces. The space of orthocentric tetrahedra is of codimension two inside the space of all tetrahedra and represents a three-parameter deformation of the $A_3$ root system (ignoring the overall scale).

What about the WDVV equation in this case? The 15 pairs of edges in the double sum of (3.7) group into four triples corresponding to the tetrahedron’s faces plus the three skew pairs. It is not hard to see that for each face the contributions add to zero, and so the concurrent edge pairs do not contribute to the double sum in (3.7). This leaves the three skew pairs, but their contribution is killed by the orthocentricity constraint (4.6), and the WDVV equation is indeed obeyed.

Although I do not know the $f$ coefficients for a general tetrahedron, I can offer the following proof that the WDVV equation already enforces the orthocentricity. Consider the limit $\hat{n}(x) \to \infty$ for some fixed covector $\hat{n}$ of unit length. Decomposing

$$\alpha = \alpha \cdot \hat{n} \hat{n} + \alpha_{\perp} \quad \longrightarrow \quad \alpha(x) = \alpha \cdot \hat{n}(x) + \alpha_{\perp}(x)$$

(4.9)

we see that any factor $\frac{1}{\alpha(x)}$ vanishes in this limit unless $\alpha \cdot \hat{n} = 0$. Thus, only covectors perpendicular to $\hat{n}$ survive in (3.7) and (3.8), reducing the system to the hyperplane...
orthogonal to \( \hat{n} \). On the other hand, any solution to these equations, being an identity in \( x \), must carry over to a solution of the limiting equations, which correspond to the dimensionally reduced system. In a general tetrahedron, take \( \hat{n} \propto \alpha \wedge \alpha' \). Then, the limit \( \hat{n}(x) \to \infty \) in (3.7) retains only the covectors \( \alpha \) and \( \alpha' \), and the WDVV equation reduces to a single term, which vanishes only for \( \alpha \cdot \alpha' = 0 \). Equivalently, the plane spanned by \( \alpha \) and \( \alpha' \) contains no further covector, and two covectors in two dimensions must be orthogonal. The same argument applies to \( \beta \cdot \beta' \) and \( \gamma \cdot \gamma' \), completing the proof.

\[ \beta^{(n-2)} \gamma^{(n-2)} \]

... 

\[ \alpha \]

**Figure 3:** Faces sharing an edge of an \( n \)-simplex

This scheme may be taken to any dimension \( n \). A simplicial configuration of \( \frac{1}{2}n(n+1) \) covectors is already determined by \( n \) independent covectors, which modulo \( \text{SO}(n) \) are given by \( \frac{1}{2}n(n+1) \) parameters. The homogeneity condition (4.1) uniquely fixes the \( f \) coefficients. Employing an iterated dimensional reduction to any plane spanned by a skew pair of edges and realizing that no other edge lies in such a plane, one sees that the WDVV equation always demands such an edge pair to be orthogonal. This condition renders the \( n \)-simplex orthocentric and reduces the number of degrees of freedom to \( n+1 \) (now including the overall scale given by the \( n \)-volume \( V \)). In this situation I can write down the unique solution to both the homogeneity condition and the WDVV equation,

\[ f_{\alpha} = \frac{\beta \cdot \gamma \, \beta' \cdot \gamma' \, \beta'' \cdot \gamma'' \, \ldots \, \beta^{(n-2)} \cdot \gamma^{(n-2)}}{(n! \, V)^2} \]

where the edge \( \alpha \) is shared by the \( n-1 \) faces \( \langle \alpha \beta \gamma \rangle, \langle \alpha \beta' \gamma' \rangle, \ldots, \langle \alpha \beta^{(n-2)} \gamma^{(n-2)} \rangle \), and I have oriented all edges as pointing away from \( \alpha \). This formula works because any sub-simplex, in particular any tetrahedral building block, is itself orthocentric. To summarize, the WDVV solutions for simplicial covector configurations in any dimension are exhausted by an \( n \)-parameter deformation of the \( A_n \) root system. The \( n \) moduli are relative angles and do not include the \( \frac{1}{2}n(n+1) \) trivial covector rescalings, which, apart from the common scale, destroy the tetrahedron.

These findings suggest that covector configurations corresponding to deformations of other roots systems may solve the WDVV equations as well. For verification, I propose to consider the polytopes associated with the weight systems of a given Lie algebra, since their edge sets are built from the root covectors. The idea is then to relax the angles of such
polytopes and analyze the constraints from the homogeneity and WDVV equations. The above $n$-dimensional orthocentric hypertetrahedra emerge simply from the fundamental representations of $A_n$. Extending this strategy to other representations and Lie algebras could lead to many more solutions.

5 WDVV solutions from Veselov systems

In the mathematical literature, the (generalized) WDVV equation is usually formulated as

$$W_i W_k^{-1} W_j = W_j W_k^{-1} W_i \quad \text{for} \quad i, j, k = 1, \ldots, n ,$$  \hfill (5.1)

where $W_i$ is an $n \times n$ matrix with entries

$$(W_i)_{lm} = \partial_i \partial_l \partial_m W \quad \text{for} \quad W = W(y^1, \ldots, y^n) ,$$  \hfill (5.2)

and $\partial_i \equiv \frac{\partial}{\partial y^i}$. It is easy to show [7] that (5.1) is equivalent to

$$W_i G^{-1} W_j = W_j G^{-1} W_i \quad \text{with} \quad G = -y^k W_k ,$$  \hfill (5.3)

which in components reads

$$\left(\partial_i \partial_l \partial_p W\right) G^{pq} \left(\partial_q \partial_m \partial_j W\right) = \left(\partial_j \partial_l \partial_p W\right) G^{pq} \left(\partial_q \partial_m \partial_i W\right) ,$$  \hfill (5.4)

where the index position distinguishes between the metric $G$ and its inverse $G^{-1}$. For the covector ansatz (3.1)

$$W = -\frac{1}{2} \sum_{\beta} f_{\beta} \beta(y)^{\beta} \ln |\beta(y)|$$  \hfill (5.5)

it follows that

$$W_i = - \sum_{\beta} f_{\beta} \frac{\beta_i}{\beta(y)} \beta \otimes \beta \quad \rightarrow \quad G = \sum_{\beta} f_{\beta} \beta \otimes \beta .$$  \hfill (5.6)

How is this related to the material of the previous sections? Comparing with (4.1), it seems that one must impose the additional condition of $G = -\mathbb{1}$. However, this is not so, because such a choice may be achieved by a linear coordinate change

$$x^i = y^j M_j^i \quad \rightarrow \quad \beta_i = M_j^i \alpha_j$$  \hfill (5.7)

so that for $F(x) = W(y)$ one gets

$$W_i = M_j^i F_j \quad \text{and} \quad G_{lm} = -y^k W_{klm} = -M_t^i M_m^j x^k F_{kij} = M_t^i \delta_{ij} M_j^m ,$$  \hfill (5.8)

where the right equation in (2.6) was used in the last step. This converts the metric $(G_{ij})$ of the $y$-frame to the Euclidean metric $(\delta_{ij})$ in the $x$-frame,\(^2\) and changes the covector scalar product accordingly,

$$\beta \cdot \beta' = \beta_i G^{ij} \beta_j = \alpha_k M_t^k \left(G^{ij} M_j^i \alpha_i'\right) = \alpha_k \delta^{kl} \alpha_l' = \alpha \cdot \alpha' ,$$  \hfill (5.9)

\(^2\)Note that for the $y$-frame one must replace $\delta^{ij}$ with $G^{ij}$ in the quantization rule (2.1).
in short:
\[
G = M \delta M^\top \quad \text{and} \quad \delta = M^\top G^{-1} M .
\] (5.10)
Thus, solutions to (5.4) of the form (5.5) can be translated to solutions to (2.6) of the form (3.1) by a linear transformation.

For a prominent example, I turn to the \(n\)-parameter deformation of the \(A_n\) root system first proposed in [9],
\[
\{\beta\} = \left\{ \sqrt{c_i c_j} \left( e^i - e^j \right), \sqrt{c_i} e^i \mid 1 \leq i < j \leq n \right\} \quad \text{(no sums)} ,
\] (5.11)
where \(e^i(y) = y^i\) and the \(c_i\) are arbitrary (positive) parameters. It was shown that this covector set satisfies the so-called \(\lor\)-conditions, which implies that (with \(f_\beta = 1\)) it provides an \(n\)-parameter family of solutions (5.5) to the WDVV equation. For this case, the metric and its inverse are quickly evaluated,
\[
G_{ij} = (1 + \sum_k c_k) c_i \delta_{ij} - c_i c_j \quad \text{and} \quad G^{ij} = (1 + \sum_k c_k)^{-1} \left( c_i^{-1} \delta^{ij} + 1 \right) ,
\] (5.12)
but in order to compute the corresponding transformation matrix \(M\) (or its inverse \(M^{-1}\)) via (5.10) one has to diagonalize \(G\) (or \(G^{-1}\)), which is not an easy task.

However, in order to interpret the solution (5.11) in the \(x\)-frame, it suffices to study its geometric (frame-independent) properties. First, I rescale each \(\beta\) by shifting the square roots into \(f_\beta\) coefficients,
\[
\{\gamma\} = \{ e^i - e^j, e^i \} \quad \text{and} \quad \{ f_\gamma \} = \{ c_i c_j , c_i \} \quad \text{for} \quad 1 \leq i < j \leq n ,
\] (5.13)
and observe that the new covectors fulfil the incidence relations of an \(n\)-simplex. Second, I must figure out the angles formed by its edges,
\[
\cos \angle(\gamma, \gamma') = \frac{\gamma \cdot \gamma'}{\sqrt{\gamma \cdot \gamma} \sqrt{\gamma' \cdot \gamma'}} \quad \text{with} \quad \gamma \cdot \gamma' = \gamma_i G^{ij} \gamma'_j ,
\] (5.14)
These angles depend on the deformation parameters \(c_i\), except for
\[
e^i \cdot (e^j - e^k) = 0 \quad \text{for} \quad i, j, k \text{ mutually distinct} ,
\] (5.15)
which means that non-concurrent edges are orthogonal to one another! This is a frame-independent statement and qualifies the polytope based on (5.11) as an orthocentric one.

Clearly, I have rediscovered the solution family of section 4. As a side result, one obtains an explicit parametrization of orthocentric \(n\)-simplices,
\[
\{ \alpha \} (c) = \left\{ M^{-1} (e^i - e^j) , M^{-1} e^i \right\} ,
\] (5.16)
where the \(c_i\)-dependence enters via the matrix \(M^{-1}\). The (physical) geometries corresponding to the other known \(\lor\)-systems remain to be worked out.

Acknowledgments
I thank A. Galajinsky and K. Polovnikov for pleasant collaborations on which this contribution is based. Furthermore, I am grateful to Misha Feigin for enlightening discussions. The research was supported by DFG grant 436 RUS 113/669/0-3.
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