A SUFFICIENT CONDITION FOR CONGRUENCY OF ORBITS
OF LIE GROUPS AND SOME APPLICATIONS

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ABSTRACT. We give a sufficient condition for isometric actions to have the
congruency of orbits, that is, all orbits are isometrically congruent to each
other. As applications, we give simple and unified proofs for some known
congruence results, and also provide new examples of isometric actions on
symmetric spaces of noncompact type which have the congruency of orbits.

1. INTRODUCTION

Isometric actions of Lie groups on Riemannian manifolds $M$ and submanifold
group of their orbits are fundamental topics in geometry. In this paper, we
consider isometric actions which have the congruency of orbits, that is, all of
whose orbits are isometrically congruent to each other. The congruency of orbits
yields good benefits for studying submanifold geometry of orbits, since it is suf-
ficient to study only one orbit. It has been known that the following isometric
actions have the congruency of orbits:

(1) the actions of $U(1)$ on spheres $\mathbb{S}^{2n+1}$ which induce the Hopf fibrations,
(2) the actions of $N$ on hyperbolic spaces which induce the horosphere folia-
tions,
(3) the actions of $S_V$ on symmetric spaces of noncompact type $M = G/K$, where $S_V$
are some codimension one subgroups of $AN$ ([2], Proposition 3.1], see Section 4 for details), and
(4) the actions of $N$ on symmetric spaces of noncompact type $M = G/K$
which induce horocycle foliations ([1], Corollary 6.5]).

Note that, for a symmetric space of noncompact type $M$, we denote by $G$
the identity component of the isometry group of $M$, and by $G = KAN$ the Iwasawa
decomposition.

In this paper, we obtain a sufficient condition for isometric actions to have the
congruency of orbits (Lemma 2.1). Indeed our sufficient condition is stated in
terms of Lie algebras, and it is very practical to apply.

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compact type, Parabolic subgroups.

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The first applications of our sufficient condition are simple and unified proofs for the congruency of orbits for all of the above mentioned actions. In Section 3 we show the congruency of orbits in the case of the Hopf fibrations (1). In Section 4 we prove the congruency of orbits for a class of actions, which contains the actions (2), (3) and (4).

As the second application of our sufficient condition, in Section 5, we provide new examples of isometric actions which have the congruency of orbits. Namely, we prove the congruency of orbits of $S\Phi$ on symmetric spaces of noncompact type $M = G/K$ (Proposition 5.1), where $S\Phi$ denotes the solvable part of the parabolic subgroup $Q\Phi$ of $G$. Recently, $S\Phi$ have played very important roles in studying submanifolds in $M$ (1 [8, 3], see also a survey [8]). Among others, it is remarkable that the orbit $(S\Phi).o$ is always minimal in $M$ and is Einstein with respect to the induced metric ([7]), where $o$ is the origin of $M = G/K$. Hence, as a corollary of the congruency of orbits of $S\Phi$, we have the following: any orbits of $S\Phi$ are minimal submanifolds in $M$, and are Einstein with respect to the induced metrics.

We are interested in studying further geometric properties of $S\Phi$-orbits. For the study, the congruency of $S\Phi$-orbits is quite useful, since we have only to consider one orbit. Furthermore, our sufficient condition would be useful for further studies on isometric actions on symmetric spaces of noncompact type, since some interesting actions do satisfy our sufficient condition, and hence applicable to study of geometry of their orbits.

Throughout this paper, we denote by Isom($M$) the isometry group of a Riemannian manifold $M$, and by Lie($G$) the Lie algebra of a Lie group $G$.

2. A key lemma

In this section, we give a sufficient condition for isometric actions to have the congruency of orbits on Riemannian manifolds.

**Lemma 2.1.** Let $M$ be a Riemannian manifold and $S$ be a connected Lie subgroup of Isom($M$) with Lie($S$) = $\mathfrak{s}$, and assume that $S$ acts transitively on $M$. If $\mathfrak{s}'$ is an ideal of $\mathfrak{s}$, then all orbits of $S'$ in $M$ are isometrically congruent to each other, where $S'$ is the connected Lie subgroup of $S$ with Lie($S'$) = $\mathfrak{s}'$.

**Proof.** Take any $p, q \in M$. We shall show that the orbits $S'.p$ and $S'.q$ are isometrically congruent. Owing to transitivity of the action of $S$, there exists $g \in S$ such that $p = g.q$ holds. Since $S$ and $S'$ are connected and $\mathfrak{s}'$ is an ideal in $\mathfrak{s}$, one knows that $S'$ is a normal subgroup of $S$ (see for instance [9] Theorem 3.48]). Hence one has $g^{-1}S'g = S'$. Thus we obtain

$$g.(S'.q) = g.(g^{-1}S'g).q = S'.(g.q) = S'.p,$$

which implies $S'.p$ and $S'.q$ are isometrically congruent. \(\square\)
3. Hopf fibrations

In this section, by applying Lemma 2.1, we give a simple proof for the congruency of orbits of the actions $U(1)$ on spheres $S^{2n+1}$ which induce Hopf fibrations.

Let $S^{2n+1}$ be the unit sphere in $\mathbb{C}^{n+1}$. Consider the natural action of the unitary group $U(n+1)$ on $S^{2n+1}$, and let $U(1)$ be the center of $U(n+1)$. It is well-known that the orbit space of the action of $U(1)$ on $S^{2n+1}$ satisfies

$$U(1) \setminus S^{2n+1} = U(1) \setminus U(n+1)/U(n) = U(n+1)/(U(1) \times U(n)) = \mathbb{C}P^n.$$ 

The natural projection from $S^{2n+1}$ onto $\mathbb{C}P^n$ provides the Hopf fibration.

We now show the congruency of orbits of the action above, as an application of Lemma 2.1.

**Proposition 3.1.** Under the action of $U(1)$ on $S^{2n+1}$ defined above, all orbits of $U(1)$ in $S^{2n+1}$ are isometrically congruent to each other.

**Proof.** Recall that $U(n+1)$ acts transitively on $S^{2n+1}$. We know that $u(1)$ is an ideal in $u(n+1)$, or that $U(1)$ is a normal subgroup of $U(n+1)$. Hence the proof easily follows from Lemma 2.1. □

When $n = 1$, the proof of Proposition 3.1 can be found, for example, in [6, Section 2]. We note that its proof depends on the fact that $S^3$ can be identified with $\text{Sp}(1)$ equipped with the bi-invariant metric. Hence, our sufficient condition gives another proof, which can also be applied to an arbitrary $n$.

4. Horospheres and their generalizations

In this section, we give further applications of Lemma 2.1, which provide the congruency of orbits of certain isometric actions on Riemannian symmetric spaces of noncompact type and of arbitrary rank. The result of this section contains, as special cases, simple and unified proofs of the congruency of orbits of (2), (3) and (4) mentioned in Section 1.

First of all, we recall some fundamental notions of symmetric spaces of noncompact type. Refer to [4, 5]. Let $M = G/K$ be a connected Riemannian symmetric space of noncompact type, where $G$ is the identity component of $\text{Isom}(M)$, and $K$ is the isotropy subgroup of $G$ at some point $o$, called the origin. Let us denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively, and by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form $B$ of $\mathfrak{g}$. One thus obtains that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$. Denote by $\theta$ the corresponding Cartan involution. We then introduce a positive definite inner product on $\mathfrak{g}$ by $\langle X, Y \rangle := -B(X, \theta Y)$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and denote the dual space of $\mathfrak{a}$ by $\mathfrak{a}^*$. Then we define

$$\mathfrak{g}_\lambda := \{ X \in \mathfrak{g} : \text{ad}(H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a} \}$$
for each $\lambda \in \mathfrak{a}^*$, and call $\lambda \in \mathfrak{a}^* \setminus \{0\}$ the restricted root if $g_\lambda \neq 0$. Denote by $\Sigma$ the set of restricted roots. Let $\Lambda$ be a set of simple roots of $\Sigma$, and then denote by $\Sigma^+$ the set of positive roots associated with $\Lambda$.

Let us define
\[
\mathfrak{n} := \mathop{\oplus}_{\lambda \in \Sigma^+} g_\lambda, \quad \mathfrak{s} := \mathfrak{a} \oplus \mathfrak{n},
\]
which yields that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, the Iwasawa decomposition of $\mathfrak{g}$. Note that $\mathfrak{n}$ is a nilpotent subalgebra and $\mathfrak{s}$ is a solvable subalgebra. Furthermore, one can check easily that
\[
[\mathfrak{s}, \mathfrak{s}] = \mathfrak{n}.
\]
(4.1)

Let $S$ be the connected Lie subgroup of $G$ with $\text{Lie}(S) = \mathfrak{s}$. This solvable subgroup $S$ is simply-connected and acts simply transitively on $M$.

We now give examples of isometric actions on $M$, which have the congruency of orbits. Denote by $\ominus$ the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$.

**Proposition 4.1.** Let $V$ be any linear subspace of $\mathfrak{a}$ and define $\mathfrak{s}_V := \mathfrak{s} \ominus V = (\mathfrak{a} \ominus V) \oplus \mathfrak{n}$. Then all orbits of $S_V$ in $M$ are isometrically congruent to each other, where $S_V$ is the connected Lie subgroup of $S$ with $\text{Lie}(S_V) = \mathfrak{s}_V$.

**Proof.** Recall that $S$ acts transitively on $M$. Then, by Lemma 2.1, we have only to prove that $\mathfrak{s}_V$ is an ideal in $\mathfrak{s}$. It follows easily from (4.1) and the definitions of $\mathfrak{s}$ and $\mathfrak{s}_V$ that
\[
[\mathfrak{s}, \mathfrak{s}_V] \subset [\mathfrak{s}, \mathfrak{s}] = \mathfrak{n} \subset \mathfrak{s}_V.
\]
Thus $\mathfrak{s}_V$ is an ideal in $\mathfrak{s}$. This completes the proof. □

The actions of $S_V$ on $M$ have been studied in [1], and are always hyperpolar. Furthermore, the class of $S_V$-actions contains some interesting subclasses.

**Remark 4.2.** Although the proof of Proposition 4.1 is very easy, it gives simple and unified proofs of some known results as follows.

(1) In the case when $\text{rank } M = 1$ and $\text{dim } V = 1$, Proposition 4.1 gives a proof of the congruency of horospheres in hyperbolic spaces. Indeed horospheres coincide with $N$-orbits, where $N$ stands for the connected Lie subgroup of $G$ with $\text{Lie}(N) = \mathfrak{n}$. Hence, setting $V := \mathfrak{a}$ we have $N = S_V$.

(2) In the case when $\text{rank } M > 1$ and $\text{dim } V = 1$, Proposition 4.1 gives another proof of [2, Proposition 3.1]. We note that the proof in [2] involves some geometric arguments, but our proof is purely Lie algebraic. We also note that the action of $S_V$ on $M$ is of cohomogeneity one.

(3) In the case when $\text{rank } M > 1$ and $\text{dim } V = \text{rank } M$, Proposition 4.1 also gives another proof of [1, Corollary 6.5]. Note that $S_V = N$ in this case, whose orbits form the horocycle foliation.

In the case when $\text{rank } M > 1$ and $\text{dim } V$ generic, Proposition 4.1 is a slight extension of the above mentioned results.
In this section we introduce new examples of isometric actions having the congruency of orbits on Riemannian symmetric spaces of noncompact type. They are induced by the solvable parts of parabolic subalgebras.

We first review parabolic subalgebras (we refer to [5]). We use the notations in Section 4. It is known that there is a one-to-one correspondence between proper subsets \( \Phi \subset \Lambda \) and the conjugacy classes of parabolic subalgebras of \( \mathfrak{g} \). The correspondence is given as follows. For each \( \Phi \subset \Lambda \), let \( \Sigma_{\Phi} \) be the root subsystem of \( \Sigma \) generated by \( \Phi \), that is, \( \Sigma_{\Phi} \) is the intersection of \( \Sigma \) and the linear span of \( \Phi \), and put \( \Sigma_{\Phi}^+ := \Sigma_{\Phi} \cap \Sigma^+ \). Then, let us define

\[
\mathfrak{q}_{\Phi} := \mathfrak{g}_0 \oplus \left( \bigoplus_{\beta \in \Sigma_{\Phi} \cup \Sigma^+} \mathfrak{g}_\beta \right),
\]

which is a parabolic subalgebra of \( \mathfrak{g} \). It then is clear but remarkable that \( \mathfrak{s} \subset \mathfrak{q}_{\Phi} \).

(5.1)

We shall give the solvable part of \( \mathfrak{q}_{\Phi} \). Consider the Langlands decomposition \( \mathfrak{q}_{\Phi} = \mathfrak{m}_{\Phi} \oplus \mathfrak{a}_{\Phi} \oplus \mathfrak{n}_{\Phi} \), where

\[
\mathfrak{a}_{\Phi} = \{ H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Phi \},
\]

\[
\mathfrak{m}_{\Phi} = (\mathfrak{g}_0 \oplus \mathfrak{a}_{\Phi}) \oplus \left( \bigoplus_{\beta \in \Sigma_{\Phi}^+} \mathfrak{g}_\beta \right),
\]

\[
\mathfrak{n}_{\Phi} = \bigoplus_{\lambda \in \Sigma^+ \setminus \Sigma_{\Phi}^+} \mathfrak{g}_\lambda.
\]

Let us define \( \mathfrak{s}_{\Phi} := \mathfrak{a}_{\Phi} \oplus \mathfrak{n}_{\Phi} \), which is called the solvable part of the parabolic subalgebra \( \mathfrak{q}_{\Phi} \). By definition, one has

\[
\mathfrak{s}_{\Phi} \subset \mathfrak{s}.
\]

Furthermore, it follows from [5, Proposition 7.78] that \( \mathfrak{s}_{\Phi} \) is an ideal in \( \mathfrak{q}_{\Phi} \), that is,

\[
[\mathfrak{s}_{\Phi}, \mathfrak{q}_{\Phi}] \subset \mathfrak{s}_{\Phi}.
\]

(5.2)

We are now in the position to state the main result of this section.

**Theorem 5.1.** Let \( \Phi \subset \Lambda \), and \( \mathfrak{s}_{\Phi} \) be the solvable part of the parabolic subalgebra \( \mathfrak{q}_{\Phi} \). Then all orbits of \( S_{\Phi} \) in \( M \) are isometrically congruent to each other, where \( S_{\Phi} \) is the connected Lie subgroup of \( S \) with \( \text{Lie}(S_{\Phi}) = \mathfrak{s}_{\Phi} \).

**Proof.** As in Proposition 4.1, we have only to check that \( \mathfrak{s}_{\Phi} \) is an ideal in \( \mathfrak{s} \). Indeed, it follows from (5.1) and (5.2) that

\[
[\mathfrak{s}, \mathfrak{s}_{\Phi}] \subset [\mathfrak{q}_{\Phi}, \mathfrak{s}_{\Phi}] \subset \mathfrak{s}_{\Phi},
\]

which completes the proof. \( \square \)

The proof of Theorem 5.1 is easy, but it provides a lot of new examples of isometric actions on \( M \) which have the congruency of orbits. Note that there are \( 2^r - 1 \) proper subsets in \( \Lambda \), where \( r \) denotes the rank of \( M \).
We also note that the orbits \((S_{\Phi}).o\) through the origin \(o\) have been studied in [7]. Indeed, it is proved that \((S_{\Phi}).o\) is always minimal in \(M\) and is Einstein with respect to the induced metric. Hence, combining these facts with Theorem 5.1, we readily obtain the following result.

**Corollary 5.2.** All orbits of \(S_{\Phi}\) are minimal in \(M\), and Einstein with respect to the induced metrics.

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