Nuclear matrix elements for the resonant neutrinoless double electron capture

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Abstract. The rate of the neutrinoless double electron capture ($0\nu$ECEC) decay with a resonance condition depends sensitively on the mass difference between the initial and final nuclei of decay. This is where the JYFLTRAP Penning-trap measurements at the JYFL become invaluable in estimation of the half-lives of these decays. In this work the resonant $0\nu$ECEC decay is discussed from the point of view of its theoretical aspects, in particular regarding the resonance condition and the involved nuclear matrix elements (NME). The associated decay amplitudes are derived and the calculations of the NMEs by the microscopic many-body approach of the multiple-commutator model are outlined. The resonant $0\nu$ECEC decays of $^{74}$Ge and $^{136}$Ce are discussed as applications of the theory framework.

1 Introduction

At present the neutrinoless double-beta ($0\nu\beta\beta$) decay is considered to be the most easily accessible means of extracting information on the possible Majorana mass of the neutrino. The reason for this is that the $0\nu\beta\beta$ decay occurs only if the neutrino is a massive Majorana particle. The search for the $0\nu\beta\beta$ decay is mostly concentrated on the $0\nu\beta^-\beta^-\beta^+$, $0\nu\beta^-\text{EC}$ and $0\nu$ECEC, are possible but they are hard to detect owing to their small decay $Q$ values [2]. The neutrinoless double electron capture, $0\nu$ECEC, can only be realized as a resonant decay [3] or a radiative process with or without a resonance condition [4]. The resonant $0\nu$ECEC decay has attracted a lot of experimental attention recently [5–9].

The resonance condition — close degeneracy of the initial and final (excited) atomic states — can enhance the decay rate by a factor as large as $10^6$ [3]. Possible candidates for the resonant decays are many [3, 4]. Verification of the fulfillment of the resonance condition is of utmost importance before expensive experiments are conducted in the aim to detect the resonant $0\nu$ECEC. The half-life of the resonant $0\nu$ECEC depends sensitively on the mass difference between the initial and final nuclei of decay. The verification of the resonance enhancement thus calls for accurate measurements of these mass differences. At the JYFL the way to achieve the necessary accuracy — in the 100 eV range — is to engage the Penning ion trap, JYFLTRAP, for these measurements. In fact there is a campaign at the JYFL to run the Penning trap to sort out potential candidates for measurements of resonant $0\nu$ECEC decays by dedicated underground experiments.

In this work two examples — the decay of $^{74}$Se to the second $2^+$ state in $^{74}$Ge and the decay of $^{136}$Ce to the fourth $0^+$ state in $^{136}$Ba — are discussed from the point of view of the nuclear-structure aspects of resonant $0\nu$ECEC decays.

2 Theory framework of the resonant neutrinoless double electron capture

2.1 Decay half-life

In this work we study a particular type of neutrinoless double electron capture ($0\nu$ECEC) decay, namely the resonant $0\nu$ECEC process of the form

$$e^-+e^-+(A, Z) \rightarrow (A, Z-2)^* \rightarrow (A, Z-2)+\gamma+2X,$$

(1)

where the capture of two atomic electrons leaves the final nucleus in an excited state that decays by one or more gamma-rays and the atomic vacancies are filled by outer electrons with emission of X-rays. The daughter state $(A, Z-2)^*$ is a virtual state with energy

$$E = E^* + E_H + E_H^*,$$

(2)

including the nuclear excitation energy and the binding energies of the two captured electrons. For the half-life of
the parent atom the resonance condition can be written [3, 10] in the Breit-Wigner form

\[ \ln 2 = -\frac{M^2}{(Q - E)^2 + T^2/4}, \]

where \( M \) contains the leptonic phase space and the nuclear matrix element, \( T \) denotes the combined nuclear and atomic radiative widths (few tens of electron volts [11]) and \( |Q - E| \) is the so-called degeneracy parameter containing the energy (2) of the virtual final state and the difference between the initial and final atomic masses (the \( Q \) value of resonant \( 0ν\text{ECEC} \)). The \( Q \) value needs to be measured very accurately in order to judge whether the resonant \( 0ν\text{ECEC} \) is detectable or not. For this, the JYFLTRAP Penning trap is suited in a perfect manner.

To be able to estimate the half-life we have to evaluate the quantity \( M \) in (3). In the mass mode of the resonant \( 0ν\text{ECEC} \), it can be written as

\[ M = G_{0ν\text{ECEC}} M_{0ν\text{ECEC}} \langle m_ν \rangle, \]

where \( \langle m_ν \rangle \) is the effective neutrino mass of the neutrinoless double-beta decay [2] and \( G_{0ν\text{ECEC}} \) is the leptonic phase-space factor

\[ G_{0ν\text{ECEC}} = \left( \frac{G_F \cos θ_C}{\sqrt{2}} \right)^2 \frac{g_A^2}{2π} \frac{(Zαm_e)^3}{πR_A} η_r, \quad R_A = 1.2A^{1/3}, \]

with \( G_F \) the Fermi coupling constant, \( θ_C \) the Cabibbo angle, \( g_A = 1.25 \) the axial-vector weak coupling constant, \( Z \) the charge of the mother nucleus, \( α \) the fine-structure constant, \( m_e \) the electron rest mass and \( R_A \) the radius of a nucleus with mass number \( A \). The quantity \( η_r \) in (5) is a suppression factor depending on the atomic orbitals where the two electrons are captured from. Solution of the Schrödinger equation for a point-charge nucleus produces the values

\[ η_{kk} = 1, \quad η_{KL} = \frac{1}{\sqrt{8}}, \quad η_{L1} = \frac{1}{8}, \]

\[ η_{L2} = η_{L2} = η_{L3} = η_{L3} = \frac{1}{96}. \]

This approximation is reasonable and compares well with the Dirac solution for a homogeneously charged spherical nucleus [12]. Expressions for the involved nuclear matrix element \( M_{0ν\text{ECEC}} \) are given in sect. 2.2 and 2.3.

### 2.2 Decays to 0+ final states

The nuclear matrix element (NME) involved in (4) for decays to the 0+ final states is defined as

\[ M_{0ν\text{ECEC}} = \frac{1}{R_A} \left[ M_{GT\text{ECEC}} - \left( \frac{g_V}{g_A} \right)^2 M_{F\text{ECEC}} \right], \]

where \( R_A = 1.2A^{1/3} \text{fm} \) is the nuclear radius, \( g_V \) (\( g_A \)) is the vector (axial-vector) weak coupling constant, \( M_{GT\text{ECEC}} \) is the Gamow-Teller and \( M_{F\text{ECEC}} \) is the Fermi NME. The two matrix elements are given by

\[ M_{F\text{ECEC}} = \sum_a \left( 0_f^T \right) \left( m_{mn} t_{mn}^a h_F(r_{mn}, E_a) \right) \left| 0_i^T \right\rangle, \]

\[ M_{GT\text{ECEC}} = \sum_a \left( 0_f^T \right) \left( m_{mn} t_{mn}^a h_{GT}(r_{mn}, E_a) \sigma_m \sigma_n \right) \left| 0_i^T \right\rangle. \]

Here the summation runs over all the states \( J^z \) of the intermediate nucleus and \( r_{mn} = |\mathbf{r}_m - \mathbf{r}_n| \) is the relative coordinate between the nucleons \( m \) and \( n \). The isospin raising operators \( r_{mn}^a \) convert the \( n \)-th proton to a neutron and \( σ \) are the usual Pauli spin matrices. Here \( 0_i^+(0^+) \) is the ground state of the even-even mother (daughter) nucleus and the neutrino potential \( h_K(r_{mn}, E_a) \), \( K = F, GT \), is defined as

\[ h_K(r_{mn}, E_a) = \frac{2}{π} R_A \int dq \frac{q h_K(q^2)}{q + E_a - (E_1 + E_t)/2} j_0(q r_{mn}), \]

where \( j_0 \) is the spherical Bessel function. The term \( h_K(q^2) \) in (10) includes the contributions arising from the short-range correlations, nucleon form factors and higher-order terms of the nucleonic weak current [13–15].

Next, we write the nuclear matrix elements explicitly. They are given by

\[ M_{K\text{ECEC}} = \sum_{J^z,k_1,k_2,J} \sum_{pp'} \sum_{n'n''} \left( -1 \right)^{J_1 + J_n + J_p + J'} \left[ \left( \begin{array}{ccc} J & J_1 & J_p \\ J & J_1 & J_p \\ J & J_1 & J_p \end{array} \right) \left( \begin{array}{ccc} n & n' & n'' \\ n & n' & n'' \end{array} \right) \right] O_K \left( \begin{array}{ccc} pp' & J' \\ pp' & J' \end{array} \right) \]

\[ \times \left( 0_i^T \left| \begin{array}{cc} c_p^T & c_p^T \\ c_p^T & c_p^T \end{array} \right| J_{K_i}^T \right) \times \left\langle J_{K_i}^T \left| \begin{array}{cc} J_{k_1}^T & J_{k_2}^T \\ J_{k_1}^T & J_{k_2}^T \end{array} \right| \left( \begin{array}{ccc} c_p^T & c_p^T \\ c_p^T & c_p^T \end{array} \right) \right\rangle \left| 0_i^T \right\rangle, \quad K = F, GT, \]

where \( k_1 \) and \( k_2 \) label the different nuclear-model solutions for a given multipole \( J^z \), the set \( k_1 \) stemming from the calculation based on the final nucleus and the set \( k_2 \) stemming from the calculation based on the initial nucleus. The operators \( O_K \) inside the two-particle matrix element derive from (8) and (9) and they can be written as

\[ O_F = h_F(r_{12}, E_k), \]

\[ O_{GT} = h_{GT}(r_{12}, E_k) \sigma_1 \cdot \sigma_2, \]

\[ r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|, \]

where \( E_k \) is the average of the \( k \)-th eigenvalues of the nuclear-model calculations based on the initial and final nuclei of the decay. Here the one-body transition densities are \( \left( 0_i^T \right) \left| \begin{array}{cc} c_p^T & c_p^T \end{array} \right| J_{K_i}^T \) and \( \left( J_{k_1}^T \right) \left| \begin{array}{cc} c_p^T & c_p^T \end{array} \right| \left( 0_i^T \right) \), and they are given separately for the different nuclear-structure formalisms in sect. 3. The quantity \( \left\langle J_{K_i}^T \left| \begin{array}{cc} J_{k_1}^T & J_{k_2}^T \\ J_{k_1}^T & J_{k_2}^T \end{array} \right| \left( \begin{array}{ccc} c_p^T & c_p^T \\ c_p^T & c_p^T \end{array} \right) \right\rangle \left| 0_i^T \right\rangle \) is the overlap factor and its purpose is to match the the two sets of \( J^z \) states that stem from the two different nuclear-model calculations—one starting from the mother nucleus (the \( k_2 \)
states) and the other starting from the daughter nucleus (the \( k_1 \) states). Its explicit form for the pnQRPA nuclear model is given in sect. 3.1.

The two-particle matrix element of (11) can be written as

\[
\langle nn' : J'|O_K||pp' : J' \rangle = \sum_{J} \sum_{J'} \langle J|O_K|J' \rangle \langle J' : n|pp|n' \rangle \langle n' : J' |J' \rangle F
\]

where the neutrino potentials are those of (10) and we have defined

\[
F_F = 1, \quad F_{GT} = 6(-1)^{S+1} \left\{ \frac{\lambda}{2} \frac{\lambda}{2} S \right\}
\]

The quantities \( M_\lambda \) are the Moshinsky brackets that mediate the transformation from the laboratory coordinates \( r_1 \) and \( r_2 \) to the center-of-mass coordinate \( R = \frac{1}{\sqrt{2}} (r_1 + r_2) \) and the relative coordinate \( r = \frac{1}{\sqrt{2}} (r_1 - r_2) \). In this way, the short-range correlations of the two decaying protons are easily incorporated in the theory. The wave functions \( \phi_{n_l}(r) \) are taken to be the eigenfunctions of the isotropic harmonic oscillator.

### 2.3 Decays to \( 2^+ \) final states

The \( 0\nu\)ECEC transition to the \( 2^+ \) final states is mediated by a spherical tensor of rank two. Denoting this tensor by \( T_2 \) one can write the transition matrix element as

\[
M_{0\nu}^{\text{ECEC}} = \sum_{a} \left( 2^+_{J_1} || t^+_m r^+_n T_2(mn; a)||0^+_1 \right)
\]

\[
= \frac{1}{\sqrt{5}} \sum_{J,J_1,J_2} \sum_{J'} \langle J' | J_2 \rangle \langle J_1 | J_2 \rangle \langle J' | J_2 \rangle \langle J_1 | J_2 \rangle \langle J_1 | J_2 \rangle
\]

where

\[
\mathcal{O}_2 = g_{GT}(r_{12}, E_k)[\sigma_1 \sigma_2], \quad r_{12} = |r_1 - r_2|
\]

where \( E_k \) is again the average of the \( k \)-th eigenvalues of the two nuclear-model calculations. The resulting two-particle matrix element is given by

\[
\langle nn' : J'|\mathcal{O}_2||pp' : J' \rangle = 6\sqrt{5} \sum_{J,J'} \langle J' | J_2 \rangle \langle J_1 | J_2 \rangle \langle J_1 | J_2 \rangle \langle J_1 | J_2 \rangle \langle J_1 | J_2 \rangle
\]

where the neutrino potential is the usual one (10) and the other quantities have been defined in sect. 2.2.

### 3 Nuclear-structure models

The starting point in the present calculations is the theoretical framework of the quasiparticle random-phase approximation (QRPA). This framework is based on the quasiparticles that are created in a BCS calculation via the Bogoliubov-Valatin transformation [16]. We use two kinds of QRPA approaches. The proton-neutron QRPA (pnQRPA) is used to produce the \( J^\pi \) states of the intermediate odd-odd nucleus and the charge-conserving QRPA (ccQRPA) is used to produce the excited states of the even-even daughter nucleus. The states of the pnQRPA and the ccQRPA are connected by decay amplitudes that are calculated in the higher-QRPA framework called the multiple-commutator model (MCM). All these approaches are discussed below.

#### 3.1 Proton-neutron quasiparticle random-phase approximation

We use the pnQRPA to form the wave functions that represent states of an odd-odd nucleus. The pnQRPA state is obtained by acting by a pnQRPA creation operator on the QRPA ground state (QRPA). This action can be expressed in the form

\[
|J^\pi E_\pm M \rangle = Q^\dagger (J^\pi E_\pm M)|\text{QRPA}\rangle
\]

\[
= \sum_{pn} \left( X_{pn}^J [a^\dagger_p a_n^J]_{JM} - Y_{pn}^J [a^\dagger_p a_n^J]^J_{JM} \right)
\]

Here \( J^\pi \) is the multipolarity of the nuclear state and \( J^\pi \) enumerates the phonons. The operator \( a^\dagger_p (a_n^J) \) creates a...
proton (neutron) quasiparticle in the orbital \( p \) \((n)\). The sum runs over all proton-neutron configurations in the chosen valence space.

Making use of the ansatz (19) one can derive (see, e.g., [16]) the pnQRPA equations of motion in the quasiboson approximation. The equations can be cast in the matrix form

\[
\begin{pmatrix}
A(J) & B(J) \\
B(J) & A(J)
\end{pmatrix}
\begin{pmatrix}
X^J \\ Y^J
\end{pmatrix} = E_J \begin{pmatrix}
X^J \\ -Y^J
\end{pmatrix},
\]

(20)

where \( J \) is the angular momentum and the matrix elements of the \( A(J) \) and \( B(J) \) matrices read

\[
A_{pn,p'n'}(J) = \delta_{pp'}\delta_{nn'}(E_p + E_n) + g_{pp}(u_p^\dagger u_n u_{p'n'} + v_p v_n u_{p'n'})
\times \langle p;n \rangle |V|^p'n' \langle J \rangle
+ g_{ph}(u_p^\dagger v_n u_{p'n'} + v_p u_n v_{p'n'})
\times \langle p;n-1 \rangle |V|_{RES}|p'n'^{-1} \langle J \rangle,
\]

(21)

and

\[
B_{pn,p'n'}(J) = -g_{pp}(u_p^\dagger u_n u_{p'n'} + v_p v_n u_{p'n'})
\times \langle p;n \rangle |V|^p'n' \langle J \rangle
+ g_{ph}(u_p^\dagger v_n u_{p'n'} + v_p u_n v_{p'n'})
\times \langle p;n-1 \rangle |V|_{RES}|p'n'^{-1} \langle J \rangle,
\]

(22)

with \( v \) and \( u \) corresponding to the occupation and unoccupation amplitudes stemming from the BCS calculation.

Above \( V \) is the normalized, \( J \)-coupled and anti-symmetrized two-body interaction matrix element (see eq. (8.16) of [16]) and the particle-hole matrix element \( |V|_{RES} \) of the residual interaction is obtained from \( V \) by the Pandya transformation

\[
\langle p;n \rangle |V|^p'n' \langle J \rangle = -\sum_{j'} j_{j'}^2 \left\{ \begin{array}{ccc} j_p & j_n & J \\ j'_{p'} & j'_{n'} & J' \end{array} \right\} \langle p'n' \rangle |V|^p'n' \langle J' \rangle.
\]

(23)

The quantities \( E_p \) and \( E_n \) in (21) are the quasiparticle energies for the proton orbital \( p \) and neutron orbital \( n \), respectively. The \( j_{p,n} \) in (23) are the total angular momenta of the proton or neutron single-particle orbitals.

In (21) and (22) the particle-hole and particle-particle parts of the pnQRPA matrices are separately scaled by the particle-hole parameter \( g_{ph} \) and particle-particle parameter \( g_{pp} \) [17–19]. The particle-hole parameter affects the position of the Gamow-Teller giant resonance and its value was fixed by the available systematics [16] on the location of the giant state. The parameter \( g_{pp} \) affects only weakly the neutrinoless double electron capture rates and it value was set to the default \( g_{pp} = 1.0 \).

The pnQRPA transition densities are given by

\[
\langle 0^+ | c_{n'}^\dagger \bar{c}_p | J^+ \rangle = \hat{J}(-1)j_{p'}^{j_{n'}} j_{n'}^{-1} j_{p'}^{-1} \times \bar{u}_{p'} \bar{v}_{n'} Y_{p'n'}^{J_{p'n'}} + u_{p'} v_{n'} Y_{p'n'}^{J_{p'n'}} \],
\]

(24)

\[
\langle J_{p'n'}^+ | c_{n'}^\dagger \bar{c}_p | 0^+ \rangle = \hat{J}(-1)j_{p'}^{j_{n'}} j_{n'}+1 j_{p'}+1 \times v_{p'n'} X_{p'n'}^{J_{p'n'}} + u_{p'n'} Y_{p'n'}^{J_{p'n'}} \],
\]

(25)

where \( v \) (\( \bar{v} \)) and \( u \) (\( \bar{u} \)) correspond to the BCS occupation and unoccupation amplitudes of the initial (final) even-even nucleus. The amplitudes \( X \) and \( Y \) come from the pnQRPA calculation starting from the initial (final) nucleus of the \( 0vEC \) decay. The overlap factor in (11) and (15) is given by

\[
\langle J_{p'n'}^- | J^+_k \rangle = \sum_{pn} \left[ X_{pn}^{J_{p'n'}} X_{pn}^{J_{k}} - Y_{pn}^{J_{p'n'}} Y_{pn}^{J_{k}} \right] \]

(26)

and it takes care of the matching of the corresponding states in the two sets of states based on the initial and final even-even reference nuclei.

3.2 The charge-conserving QRPA and the multiple-commutator model

The multiple-commutator model (MCM) [19,20] is designed to connect excited states of an even-even reference nucleus to states of the neighboring odd-odd nucleus. Earlier the MCM has been used extensively in the calculations of double-\( \beta \)-decay rates [21,22]. The states of the odd-odd nucleus are given by the pnQRPA in the form (19). The excited states of the even-even nucleus are generated by the (charge conserving) quasiparticle random-phase approximation (ccQRPA) described in detail in [16]. Here the symmetrized form of the phonon amplitudes is adopted contrary to ref. [16] so that the-\( k \)-th \( J^+ \) state can be written as a QRPA phonon in the form

\[
|J^+_k M \rangle = Q^t (J^+_k M) |QRPA\rangle = \sum_{ab} \left( Z_{ab} [a^+_a a^+_b]_{JM} - W_{ab} [a^+_a a^+_b]^\dagger_{JM} \right) |QRPA\rangle.
\]

(27)

The symmetrized amplitudes \( Z \) and \( W \) are obtained from the usual ccQRPA amplitudes \( X \) and \( Y \) [16] through the following transformation:

\[
Z_{ab} = \begin{cases}
X^J_{ab}, & \text{if } a = b \\
\frac{1}{2} X^J_{ab}, & \text{if } a < b, \\
\frac{1}{2} X^J_{ba}, & \text{if } a > b
\end{cases}
\]

(28)

and similarly for \( W \) in terms of \( Y \). The amplitudes \( X \) and \( Y \) are obtained by solving the QRPA equations of motion that formally look like the matrix equation (20).

In the case of the ccQRPA the matrix elements of the \( A(J) \) and \( B(J) \) matrices read

\[
A_{ab,cd} = \delta_{ac}\delta_{bd} (E_a + E_b) + g_{pp} (u_a u_d v_c v_d + v_a v_d u_c u_d)
\times \langle a; b | J | c; d \rangle \]

(29)
An ideal two-phonon state consists of partner states $V$ and $U$ that are degenerate in energy, and exactly at an energy twice the excitation energy of the $2^+_1$ state. In practice, this degeneracy is always lifted by the residual interaction between the one- and two-phonon states [23]. The related transition density of the MCM, that can be inserted in (11) or (15), attains the form

$$
(J^+_2 || [c^+_i c^+_j|^] || J^+_1) = \frac{40}{\sqrt{2}} J_L\bar{J}(1)^{J^+_2 J^+_1 J^+_1} \times \sum_{p_1 n_1} \left[ \bar{V}_{p_1}' u_{n_1}' \bar{Z}_{p_1}'^{n_1} - V_{p_1} u_{n_1} Z_{p_1}^{n_1} \right] \left\{ J J L \bar{J}, J, \bar{J}_{p_1} p_1 n_1 \right\},
$$

where, as usual, the barred quantities denote amplitudes obtained for the $0\nu\text{ECEC}$ daughter nucleus.

### 4 Nuclear-structure calculations

In this section we discuss as examples the nuclear-structure calculations related to the $0\nu\text{ECEC}$ decays of two nuclei, $^{74}\text{Se}$ and $^{136}\text{Ce}$, that are located in quite different mass regions. Since $0\nu\text{ECEC}$ decays involve charge-changing decay transitions it is advantageous to perform also an auxiliary analysis by engaging theoretical description of the lateral beta-decay feeding of the low-lying states of the final nuclei of the resonant $0\nu\text{ECEC}$ decays.

#### 4.1 Determination of the model parameters

In the case of the decay of $^{74}\text{Se}$ the nuclear-structure calculation was performed in the $1p$-$0f$-$2s$-$1d$-$0g$ (9 orbitals) single-particle valence space for both protons and neutrons. For the decay of $^{136}\text{Ce}$ the proton single-particle space was chosen as $1p$-$0f$-$2s$-$1d$-$0g$-$2p$-$1f$-$0h$ (10 orbitals) and that of the neutrons consisted of $2s$-$1d$-$0g$-$2p$-$1f$-$0h$ (11 orbitals). The single-particle energies of all these orbitals were generated by the use of a spherical Coulomb-corrected Woods-Saxon (WS) potential with a standard parametrization [24], optimized for nuclei near the line of beta stability.

As the two-body interaction we have used the Bonn-A $G$-matrix and we have renormalized it in the standard way [17–19]: The quasiparticles are treated in the BCS formalism and the pairing matrix elements are scaled by a common factor, separately for protons and neutrons. In practice these factors are fitted such that the lowest quasiparticle energies obtained from the BCS match the experimental pairing gaps for protons and neutrons, respectively. The average behavior of both the proton and neutron pairing gaps is given by the three-point formulae (see, e.g., [16])

$$
\Delta_p = \frac{1}{4} \left( -1 \right)^{Z_1+1} \left[ S_p (A + 1, Z + 1) - 2 S_p (A, Z) + S_p (A - 1, Z - 1) \right],
$$

$$
\Delta_n = \frac{1}{4} \left( -1 \right)^{N_1+1} \left[ S_n (A + 1, Z) - 2 S_n (A, Z) + S_n (A - 1, Z) \right],
$$

An ideal two-phonon state consists of partner states $\pi = 0^+, 2^+, 4^+$ that are degenerate in energy, and exactly at an energy twice the excitation energy of the $2^+_1$ state. In practice, this degeneracy is always lifted by the residual interaction between the one- and two-phonon states [23]. The related transition density of the MCM, that can be
where \( S_p \) and \( S_n \) are the experimental separation energies obtained from the measured nuclear masses (see, e.g., [25]) for protons and neutrons, respectively, and \( Z \) and \( N \) are the proton and neutron numbers.

The intermediate \( J^+ \) states were generated both from the mother and daughter ground states by the use of the pnQRPA, discussed in sect. 3.1. The two obtained sets of \( J^+ \) states were matched by using the overlap (26). The particle-hole strength parameter \( g_{ph} \) of the pnQRPA, shown in (21) and (22), determines the energy of the Gamow-Teller giant resonance (GTGR) and its value was fixed by fitting the empirical location of the GTGR [16]. The particle-particle parameter \( g_{pp} \) has little effect on the \( \beta^+ / EC \) type of transitions and we have used the default value \( g_{pp} = 1.0 \).

The value of the NME in (4) is affected by both the value of the axial-vector coupling constant \( g_A \) and the nucleon-nucleon short-range correlations (SRC). In this work the quenched value \( g_A = 1.0 \) is assumed and we use the unitary correlation operator method (UCOM) to account for the SRC. The UCOM has recently been applied successfully in calculations of the neutrinoless double-beta decay [13–15,26]. The effect of the UCOM SRC is to slightly reduce the magnitude of the NME by gently shifting the wave function of the relative motion away from the touching point of the two decaying nucleons. Also the dipole form factors of the nucleons and higher-order nucleonic currents have been included in the present calculations in the way described in [13–15].

The final states of the presently discussed decays are the second \( 2^+ \) state, \( 2_2^+ \), in \(^{74}\text{Ge}\) and the fourth \( 0^+ \) state, \( 0_4^+ \), in \(^{136}\text{Ba}\) (the \( 0^+ \) ground state is counted as the first \( 0^+ \) state). The \( 2_2^+ \) state in \(^{74}\text{Ge}\) is assumed to be a member of a two-phonon triplet with a wave function (33). The experimental energy of the involved first \( 2^+ \) state was reproduced in the ccQRP A calculation by varying the value of the particle-hole strength \( g_{ph} \) that scales the particle-hole matrix elements in the way presented in eqs. (29) and (30). The value of the particle-particle strength can safely be taken to be \( g_{pp} = 1.0 \) since the properties of the ccQRPA states depend only very weakly on the value of this parameter. The \( 0^+ \) states are a special case in a ccQRPA calculation: The first ccQRPA root is spurious and has to be removed by setting its value to zero. It has to be noted that in the ccQRPA the other \( 0^+ \) states are not contaminated by this spuriousity [27]. To bring the first ccQRPA root to zero one has to change the values of both \( g_{ph} \) and \( g_{pp} \). The values of these parameters can be fixed uniquely by reproducing at the same time the low-energy spectrum of the \( 0^+ \) states. The low-energy spectra of \(^{74}\text{Ge}\) and \(^{136}\text{Ba}\) are discussed in sect. 4.2.

Finally, it should be stressed that the present calculations assume spherical shape of the involved nuclei. In fact many of the nuclei under study here possess a non-zero static oblate or prolate deformation in their ground state. The corresponding deformation parameter is, however, not very large, of the order of \( |\beta| \leq 0.25 \). Thus one should expect to encounter effects arising from the nuclear deformation itself or the possible deformation differences between different nuclei participating the weak decay processes. These effects should show up already at the level of single beta decays and that is why it pays to study the lateral beta-decay feeding of the nuclei involved in the double-beta transitions. This aspect, among others, is addressed in sect. 4.2.

4.2 Auxiliary analysis through lateral beta decays

The available data on beta-decay rates allow for studies of the lateral beta-decay feeding of the low-energy states in the \( 0\nu\overline{\text{EC}} \) daughter nuclei \(^{74}\text{Ge}\) and \(^{136}\text{Ba}\). The transitions are first-forbidden for \(^{74}\text{As}(2_1^+ \rightarrow 74\text{Ge}(0^+_1, 2^+_1, 4^+_1)) \) and allowed Gamow-Teller for \(^{136}\text{La}(1^+_1 \rightarrow 136\text{Ba}(0^+_1, 2^+_1)). \)

The allowed Gamow-Teller beta-decay transitions of interest in this work are of the type \( 1^+ \rightarrow 0^+, 2^+ \). For them the \( log ft \) value is defined as [16]

\[
\log ft = \log(f_{0\nu}t_{1/2}) = \log \left( \frac{6147}{B_{\text{GT}}} \right),
\]

\[
B_{\text{GT}} = \frac{g_A^2}{2J_1 + 1} (J^+ |\sigma|1^+)^2,
\]

(37)

for the initial \( 1^+ \) and final \( J^+ = 0^+, 2^+ \) states. Here \( f_{0\nu} = f_{1u}^f + f_{1u}^{EC} \) is the the leptonic phase-space factor for the allowed \( \beta^+ / EC \) decays defined in [16]. The first-forbidden decay transitions can be divided into two categories: those with an angular-momentum change of two units (first-forbidden unique, FFU) and those with an angular-momentum change of at most one unit (first-forbidden non-unique, FFNU). For the first-forbidden unique transitions \( 2^+ \rightarrow 0^+, 4^+ \) we can define [16]

\[
\log ft = \log(f_{1u}t_{1/2}) = \log \left( \frac{6147}{J_2 B_{1u}} \right),
\]

\[
B_{1u} = \frac{g_A^2}{2J_1 + 1} M_{1u}^2,
\]

(38)

where

\[
M_{1u} = \frac{m_e e^2}{\sqrt{\pi}} \langle J^+ |[\sigma t]|2^- \rangle.
\]

(39)

for the initial \( 2^- \) and final \( J^+ = 0^+, 4^+ \) states. Here \( f_{1u} = f_{1u}^f + f_{1u}^{EC} \) is the the leptonic phase-space factor for the FFU \( \beta^+ / EC \) decays as given in [16]. For the first-forbidden non-unique transitions \( 2^+ \rightarrow 2^+ \) one defines [16,28,29]

\[
\log ft = \log(f_{0\nu}t_{1/2}) = \log \left( \frac{6147}{S_1} \right),
\]

\[
S_1 = S_{1^+}^0 + S_{1^+}^{EC},
\]

(40)

where \( S_1 \) is the integrated shape factor that is a complex combination of the various nuclear matrix elements and phase-space factors, see ref. [29].

Results of the beta-decay calculations for \(^{74}\text{Ge}\) are shown in table 1. There the experimental and computed energies of the \( 2_1^+ \) state and the two-phonon triplet in
Table 1. Experimental and theoretical level energies in $^{74}$Ge and log $ft$ values for the feeding of these levels through the first-forbidden $\beta^+/\text{EC}$ decay of the $2^-$ ground state of $^{74}$As. The experimental data is taken from [30].

| State | Experiment | Theory |
|-------|------------|--------|
|       | $E$(MeV) | log $ft$ | $E$(MeV) | log $ft$ |
| $0^+_t$ | 0.000 | 9.7 | 0.000 | 8.95 |
| $2^+_t$ | 0.596 | 6.96 | 0.596 | 6.77 |
| $2^+_f$ | 1.204 | 8.25 | 1.195 | 7.43 |
| $4^+_f$ | 1.464 | 11.28 | 1.195 | 10.45 |
| $0^+_f$ | 1.483 | 10.34 | 1.195 | 9.03 |

Table 2. Experimental and theoretical level energies in $^{136}$Ba and log $ft$ values for the feeding of these levels through the allowed $\beta^+/\text{EC}$ decay of the $1^+$ ground state of $^{136}$La. The experimental data is taken from [31].

| State | Experiment | Theory |
|-------|------------|--------|
|       | $E$(MeV) | log $ft$ | $E$(MeV) | log $ft$ |
| $0^+_t$ | 0.000 | 4.56 | 0.000 | 4.02 |
| $2^+_t$ | 0.818 | 5.90 | 0.818 | 4.93 |
| $2^+_f$ | 1.551 | 7.39 | 1.636 | 8.12 |
| $0^+_f$ | 1.579 | 6.18 | 1.636 | 7.54 |
| $2^+_f$ | 2.080 | 6.51 | 1.971 | 4.29 |
| $2^+_g$ | 2.129 | 5.97 | 2.256 | 4.41 |
| $0^+_g$ | 2.141 | 5.72 | 1.946 | 4.62 |
| $0^+_h$ | 2.315 | 6.25 | 2.358 | 5.11 |
| $2^+_h$ | 2.486 | 6.28 | 2.458 | 5.44 |

$^{74}$Ge are shown in columns two and four. In this table also the experimental and computed log $ft$ values corresponding to the $\beta^+/\text{EC}$-decay feeding of these states by the $2^-$ ground state of $^{74}$As are shown. Decays to the $0^+$ and $4^+$ states are first-forbidden unique and the decays to the $2^+$ states are non-unique. The correspondence of the computed log $ft$ values with the experimental ones is reasonable, i.e. what could be expected from a simple MCM description of $\beta^+/\text{EC}$-decay transitions. In particular, the theoretical decay rate to the two-phonon $2^+$ state, $2^+_t$, is too high pointing to a slight overestimation of the magnitude of the corresponding $0^+\text{EC}$ NME.

The $0^+_t$ state in $^{136}$Ba is presumed to be a ccQRPA phonon of the form (27). The experimental and computed energies of this state and a number of other low-energy $0^+$ and $2^+$ states in $^{136}$Ba are shown in table 1 in columns two and four. As can be seen the computed energies agree very well with the experimental ones. In the same table we show also the log $ft$ values of the $\beta^+/\text{EC}$-decay feeding of these states via allowed Gamow-Teller transitions from the $1^+$ ground state of $^{136}$La. As can be seen the computed log $ft$ values are usually a bit too small indicating that the associated transitions are predicted to be too fast by the MCM formalism. In particular, the theoretical decay rate to the $0^+_t$ state is too high pointing to a possible overestimation of the magnitude of the corresponding $0^+\text{EC}$ NME.

One possible source of the noticed differences between the computed and measured log $ft$ values could be the omission of the deformation degree of freedom in the present calculations. In particular the deformation differences between the mother and daughter nuclei could drive the suppression of beta-decay transition rates. Such tendencies are visible in tables 1 and 2, especially for the decay of $^{136}$La in table 2. However, it seems that the deformation effects are not so strong as to ruin the overall compatibility of the trends visible in the computed and measured log $ft$ values.

4.3 Results for the NMEs of the resonant $0^+\text{ECEC}$ decays

We can use relations (3) and (4) to write the half-life of the resonant $0^+\text{ECEC}$ in the form

$$T_{1/2} = \frac{C_{\text{ECEC}}}{(\langle n_\nu \rangle [\text{eV}])^2} \text{years},$$

where the effective neutrino mass has to be inserted in units of eV. The $0^+\text{ECEC}$-decay half-life of $^{74}$Se can now be obtained directly by the use of eqs. (41), (4), (15), (25) and (34), where the second leg of the decay is evaluated by the use of the MCM-calculated transition density (34). The corresponding value for the matrix element in (15) is $M_{\text{ECEC}}(0^+ \rightarrow 2^+_t) = 0.624$ MeV. However, additional considerations, discussed extensively in [32], reduce this value by some two orders of magnitude. Such a small value of $M_{\text{ECEC}}$ results in an extremely long $0^+\text{ECEC}$ half-life [32] irrespective of the fulfillment of the resonance condition (3). Also the uncertainty in the value of the final NME should be accounted for. In this case, the uncertainty is considerable due to the recoil structure of the NME. The decay rate is further suppressed by the JYFLTRAP-measured $Q$ value that yields a rather large uncertainty $|Q - E_{\text{min}}| = 2.4$ keV for the degeneracy parameter. An additional suppression is imposed by the necessary capture from the $L_2$ (2p$_{1/2}$) and $L_3$ (2p$_{3/2}$) atomic orbitals (see (6)). Summing up all these effects produces a half-life value (41) with $C_{\text{ECEC}} \approx 2 \times 10^{-43}$ indicating of the large uncertainty in the recoil structure of the NME, not so much of the uncertainties in the nuclear wave functions.

The $0^+\text{ECEC}$-decay half-life of $^{136}$Ce can be obtained by the use of (41), (4), (7), (11), (25) and (32), where the second leg of the decay uses the MCM-calculated transition density (32). The corresponding value for the matrix element (7) is computed to be $M_{\text{ECEC}}(0^+ \rightarrow 0^+_f) = 8.0 - 22$ MeV depending on which of the ccQRPA roots would correspond to the high-lying experimental $0^+_f$ state. The JYFLTRAP-measured $Q$ value is $|Q - E_{\text{min}}| = 11.7$ keV [33] for the atomic $KK$ capture. In this case then $C_{\text{ECEC}} \approx (3 - 23) \times 10^{42}$ in (41).
5 Conclusions

A general theoretical framework of the resonant neutrinoless double electron capture is presented from the point of view of the involved nuclear matrix elements. Use of this theory framework together with the \( Q \) value measurements by the JYFLTRAP produces information about the nuclei whose \( 0\nu \)ECEC decays could be detected in dedicated underground experiments. This, in turn, is important in pinning down the value of the possible Majorana mass of the neutrino.

In this work explicit expressions for the amplitudes of the resonant \( 0\nu \)ECEC decays have been derived. The nuclear matrix elements have been evaluated within the higher-QRPA framework of the multiple commutator model. The associated transition densities have been derived and presented in the article. An auxiliary analysis has been performed by the use of the lateral beta-decay feeding of the final nuclei of the double electron capture decays. The related allowed and first-forbidden transitions have been defined and computed by the use of the multiple-commutator model. As an example, the developed formalism is applied to the decays of \( {}^{74}\text{Se} \) and \( {}^{136}\text{Ce} \). The values of the computed matrix elements and the \( Q \) values measured by the JYFLTRAP suggest that the resonant double electron captures in \( {}^{74}\text{Se} \) and \( {}^{136}\text{Ce} \) are impossible to detect in foreseeable future.

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