Some comments on the bi(tri)-Hamiltonian structure
of Generalized AKNS and DNLS hierarchies

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ABSTRACT

We give the correct prescriptions for the terms involving $\partial_{x}^{-1} \delta(x - y)$, in the Hamiltonian structures of the AKNS and DNLS systems, in order for the Jacobi identities to hold. We also establish that the $sl(2)$ AKNS and DNLS systems are tri-Hamiltonians and construct two compatible Hamiltonian structures for the $sl(3)$ AKNS system. We also give a derivation of the recursion operator for the $sl(n + 1)$ DNLS system.

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\(^1\)Work supported in part by CNPq
\(^2\)Supported by FAPESP
The Hamiltonian structures of the Generalized AKNS and DNLS systems were studied in refs. \[1, 2, 3\] and \[4\] respectively. In ref. \[2\] it was obtained, in a closed form, the recurrence relation for consecutive time evolutions for the Generalized AKNS hierarchy, as well as the first and second Hamiltonian structures. Ref. \[3\] discussed the compatibility of those two structures. However, some special attention must be paid to the second bracket, since it contains terms involving \(\partial_{x}^{-1}\delta(x - y)\). In this letter, we point out that the Jacobi identities for brackets containing such terms are not guaranteed to hold. The quantity \(\partial_{x}^{-1}\delta(x - y)\) is not uniquely determined in terms of Heaviside functions, and as shown in (1.12), that undeterminacy involves just one parameter. We show that the \(sl(2)\) AKNS and DNLS systems admit three Hamiltonian structures, and we establish that the Jacobi identities for such structures are only satisfied for one particular choice of that parameter. Moreover, we also check that those three structures are compatible \[5\] and so the \(sl(2)\) AKNS and DNLS systems are tri-Hamiltonian. In the case of \(sl(3)\) AKNS system we have shown that the choice of the above mentioned parameter, in order for the Jacobi identity to hold, is not the same for all terms in the second bracket, involving \(\partial_{x}^{-1}\delta(x - y)\). We have determined in fact, that there exists two possible solutions. We also give a derivation of the recursion operator for the \(sl(n + 1)\) DNLS system.

I. AKNS

The Generalized \(sl(n + 1)\) AKNS theory is defined by the linear matrix problem \[1\]

\[
A\Psi = \partial\Psi \quad (1.1)
\]

\[
B_{m}\Psi = \partial_{t_{m}}\Psi \quad m = 2, 3, \ldots (1.2)
\]

for

\[
A = \lambda E + A^{0} \quad \text{with} \quad E = \frac{2\lambda_{n} \cdot H}{\alpha_{n}^{2}} \quad (1.3)
\]

where \(\lambda_{n}\) and \(\alpha_{n}\) are the n-th fundamental weight and simple root of \(sl(n + 1)\) respectively, and

\[
A^{0} = \sum_{a=1}^{n} \left( q_{a} E_{(\alpha_{a} + \ldots + \alpha_{n})} + r_{a} E_{-(\alpha_{a} + \ldots + \alpha_{n})} \right) \quad (1.4)
\]

and where \(q_{a}\) and \(r_{a}\) are the fields of the model, satisfying the Generalized Non-Linear Schrodinger equations

\[
\frac{\partial q_{a}}{\partial t} = \partial_{x}^{2}q_{a} - 2q_{a}\sum_{b=1}^{n} q_{b} r_{b} \quad (1.5)
\]

\[
\frac{\partial r_{a}}{\partial t} = -\partial_{x}^{2}r_{a} + 2r_{a}\sum_{b=1}^{n} q_{b} r_{b}
\]

\[3\text{Throughout this paper we will use the Chevalley basis for a Lie algebra }\mathfrak{G}, \text{ satisfying } [H_{a}, H_{b}] = 0, [H_{a}, E_{\alpha}] = K_{\alpha a} E_{\alpha} \text{ and } [E_{\alpha}, E_{\beta}] \text{ is equal to } \epsilon(\alpha, \beta)E_{\alpha + \beta} \text{ if } \alpha + \beta \text{ is a root, to } \sum_{a=1}^{n} n_{a}H_{a} \text{ if } \alpha + \beta = 0, \text{ and } 0 \text{ otherwise. We have denoted } K_{\alpha a} = \frac{2n_{a}}{2n_{a}} = \sum_{a=1}^{n} n_{b}K_{ba}, \text{ with } K_{ab} \text{ being the Cartan matrix, and a root } \alpha \text{ can be expanded in terms of simple roots as } \alpha = \sum_{a=1}^{n} n_{a} \alpha_{a}. \text{ The quantities } \epsilon(\alpha, \beta) \text{ are integers (just signs for simple laced algebras) determined by use of the Jacobi identities.} \]
The model described by the eqs. (1.5) is a representative of a hierarchy which consists of an infinite set of equations involving an infinite number of times. Successive flows are related by the recursion operator $R$, as explained in [2], by

$$\partial_{tn}(\begin{array}{c} r_i \\ q_l \end{array}) = R_{(i,l),(j,m)}\partial_{tn-1}(\begin{array}{c} r_j \\ q_m \end{array}) =$$

$$\begin{pmatrix} (-\partial + r_k \partial^{-1} q_k) \delta_{ij} + r_i \partial^{-1} q_j & r_i \partial^{-1} r_m + r_m \partial^{-1} r_i \\ -q_l \partial^{-1} q_j - q_j \partial^{-1} q_l & (\partial - q_k \partial^{-1} r_k) \delta_{lm} - q_l \partial^{-1} r_m \end{pmatrix}\partial_{tn-1}(\begin{array}{c} r_j \\ q_m \end{array})$$

A first Hamiltonian structure is introduced from a Poisson bracket

$$P_1(x, y) = \begin{pmatrix} \{ r_i, r_j \} & \{ r_i, q_m \} \\ \{ q_l, r_j \} & \{ q_l, q_m \} \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \delta(x - y)$$

(1.7)

A second Poisson bracket structure can be obtained from the recursion operator

$$R = P_2 P_1^{-1}$$

leading to

$$P_2(x, y) = \begin{pmatrix} 0 & \delta_{im} \\ \delta_{ij} & 0 \end{pmatrix} \partial_x \delta(x - y) +$$

$$\begin{pmatrix} r_i(x) r_j(y) + r_j(x) r_i(y) & -\delta_{im} \sum_k r_k(x) q_k(y) - r_i(x) q_m(y) \\ -\delta_{ij} \sum_k q_k(x) r_k(y) - q_i(x) r_j(y) & q_i(x) q_m(y) + q_m(x) q_l(y) \end{pmatrix} \partial^{-1}_x \delta(x - y)$$

(1.9)

We say two brackets $\{ \cdot, \cdot \}_i$ and $\{ \cdot, \cdot \}_j$ are compatible if

$$K_{ij}(A, B, C) \equiv J_{ij}(A, B, C) + J_{ji}(A, B, C) = 0$$

(1.10)

where

$$J_{ij}(A, B, C) \equiv \{ \{ A, B \}_i, C \}_j + \{ \{ C, A \}_i, B \}_j + \{ \{ B, C \}_i, A \}_j$$

(1.11)

As we have mentioned in the introduction the bracket $P_2$ does not necessarily satisfy the Jacobi identities. That can be easily seen by considering the Jacobi identity for the three fields $r_1(x)$, $r_1(y)$ and $r_2(z)$. The source of the problem lies in the way one expresses the quantity $\partial^{-1}_x \delta(x - y)$ in terms of Heaviside’s $\Theta$ functions. The generic form of such relation is

$$\partial^{-1}_x \delta(x - y) = \gamma \Theta(x - y) - (1 - \gamma) \Theta(y - x)$$

(1.12)

since (see (3.3))

$$\partial_x \partial^{-1}_x \delta(x - y) = \delta(x - y)$$

for any $\gamma$

(1.13)
I.I. The \(sl(2)\) case

As a first example consider the \(sl(2)\) case, where the first and second brackets are given respectively by

\[
\{ r(x), r(y) \}_1 = \{ q(x), q(y) \}_1 = 0 \quad \{ r(x), q(y) \}_1 = -\delta(x - y) \tag{1.14}
\]

and (from now on we will use primes to denote derivatives w.r.t. the argument of the function, e.g. \(\partial_x \delta(x - y) \equiv \delta'(x - y)\))

\[
\left\{ r(x), r(y) \right\}_2 = 2r(x) r(y) \partial_x^{-1} \delta(x - y) \tag{1.15}
\]

\[
\left\{ q(x), q(y) \right\}_2 = 2q(x) q(y) \partial_x^{-1} \delta(x - y) \tag{1.16}
\]

\[
\left\{ r(x), q(y) \right\}_2 = \delta'(x - y) - 2r(x) q(y) \partial_x^{-1} \delta(x - y) \tag{1.17}
\]

Since the bracket has to be antisymmetric under the exchange of its entries, we choose, on (1.15) and (1.16), the constant \(\gamma\) introduced in (1.12) to be \(\frac{1}{2}\). Then the Jacobi identity implies that we have to make the same value for \(\gamma\) in (1.17).

We can also introduce a third bracket by the relation

\[
P_3 \equiv P_2 P_1^{-1} P_2 \tag{1.18}
\]

giving

\[
\left\{ r(x), r(y) \right\}_3 = - (r(x) r'(y) + r'(x) r(y)) \epsilon(x - y) \tag{1.19}
\]

\[
\left\{ q(x), q(y) \right\}_3 = (q(x) q'(y) + q'(x) q(y)) \epsilon(x - y) \tag{1.20}
\]

\[
\left\{ r(x), q(y) \right\}_3 = -\delta''(x - y) - (r(x) q'(y) - r'(x) q(y)) \epsilon(x - y) + 4r(x) q(y) \delta(x - y) \tag{1.21}
\]

We have checked, using the Mathematica program, that the three brackets defined above satisfy the Jacobi identities, and in addition, that they are all compatible with each other, in the sense of (1.10). The above results establish that the \(sl(2)\)-AKNS system is tri-Hamiltonian.

I.II. The \(sl(3)\) case

We have checked that in order for the Jacobi identity to be satisfied, the second bracket for the \(sl(3)\)-AKNS system given in (1.10), one can not choose the value of \(\gamma\) introduced in (1.12), to be the same for all brackets. Because of antisymmetry, one has to take \(\gamma = \frac{1}{2}\) for brackets involving the same type of fields, and for the other brackets we determine the values of \(\gamma\) imposing the Jacobi identity. The result we got, using Mathematica, is that there are two possibilities for the choices of \(\gamma\), specified below by the parameter \(\eta\) which can take the values \(\pm 1\).

\[
\left\{ r_i(x), r_i(y) \right\}_2 = r_i(x) r_i(y) \epsilon(x - y) \quad \text{for } i = 1, 2 \tag{1.22}
\]

\[
\left\{ q_i(x), q_i(y) \right\}_2 = q_i(x) q_i(y) \epsilon(x - y) \quad \text{for } i = 1, 2 \tag{1.23}
\]
\[ \{ r_i(x), q_i(y) \}_2 = \delta'(x-y) - r_i(x) q_i(y) \epsilon(x-y) \]
\[ + \eta \sum_{j=1}^2 \varepsilon_{ij} r_j(x) q_j(y) \Theta(\eta \varepsilon_{ij}(y-x)) \quad \text{for } i = 1, 2 \]  
\[ (1.24) \]
\[ \{ r_1(x), r_2(y) \}_2 = r_1(x) r_2(y) \Theta(x-y) - \eta r_1(y) r_2(x) \Theta(\eta(y-x)) \]
\[ (1.25) \]
\[ \{ q_1(x), q_2(y) \}_2 = q_1(x) q_2(y) \Theta(x-y) + \eta q_1(y) q_2(x) \Theta(\eta(x-y)) \]
\[ (1.26) \]
\[ \{ r_1(x), q_2(y) \}_2 = -r_1(x) q_2(y) \Theta(x-y) \]
\[ (1.27) \]
\[ \{ r_2(x), q_1(y) \}_2 = r_2(x) q_1(y) \Theta(y-x) \]
\[ (1.28) \]

where \( \varepsilon_{ij} \) is antisymmetric and \( \varepsilon_{12} = 1 \).

It is now straightforward to show that the first bracket defined in (1.7) is compatible with the above second brackets for the two choices of \( \eta \), namely \( \eta = \pm 1 \). Therefore the \( sl(3) \)-AKNS system is bi-Hamiltonian. Recently, Liu \[3\] has argued that the \( sl(3) \)-AKNS system is bi-Hamiltonian. However, it was not taken into account the fact that the Jacobi identity may not be satisfied, if a careful prescription for \( \partial^{-1} \delta(x-y) \) (see (1.12)) is not made.

Let us mention that through (1.18) we can calculate the third bracket for this case. However, we need to check the Jacobi identity and the compatibility condition (1.10) with the other two brackets in order to establish that the system is indeed tri-Hamiltonian.

II. DNLS

The Generalized \( sl(n+1) \)-DNLS system is defined by (1.1)-(1.2) for \( A \) given by \[4\]
\[ A = \lambda^2 E + \lambda A^0 \]
with \( E \) given by (1.3), and
\[ A^0 = \sum_{a=1}^n \left( -q_a E_{(a_a+...+a_n)} + r_a E_{-(a_a+...+a_n)} \right) \]
\[ (2.1) \]
\[ (2.2) \]
The corresponding Generalized DNLS equations of motion are
\[ \frac{\partial q_a}{\partial t} = \partial^2_x q_a + 2 \partial_x \left( q_a \sum_{b=1}^n q_b r_b \right) \]
\[ \frac{\partial r_a}{\partial t} = -\partial^2_x r_a + 2 \partial_x \left( r_a \sum_{b=1}^n q_b r_b \right) \]
\[ (2.3) \]

Let us now discuss the recursion operator for such system. Consider the Zakharov-Shabat equations for consecutive times \( t_m \) and \( t_{m-1} \)
\[ \partial_{t_m} A - \partial_x B_m + [ A, B_m ] = 0 \]
\[ (2.4) \]
\[ \partial_{t_{m-1}} A - \partial_x B_{m-1} + [ A, B_{m-1} ] = 0 \]
\[ (2.5) \]

In order to determine the general solution of these equations we take the ansatz
\[ B_m = \lambda^2 B_{m-1} + \lambda^2 C_m + \lambda D_m + Y_m \]
\[ (2.6) \]
Multiplying (2.4) by \(\lambda\), (2.3) by \(\lambda^3\), taking the difference and using the ansatz (2.6), one gets

\[
\lambda \partial_{t_m} A^0 - \lambda^3 \partial_{t_{m-1}} A^0 - \partial_x \left( \lambda^2 C_m + \lambda D_m + Y_m \right) - \lambda^2 \left[ E, \lambda^2 C_m + \lambda D_m + Y_m \right] \\
+ \lambda \left[ A^0, \lambda^2 C_m + \lambda D_m + Y_m \right] = 0
\]  

(2.7)

The \(\lambda\) independent components yields

\[
\partial_x Y_m = 0
\]  

(2.8)

so that we can choose \(Y_m = 0\). The other components yields the following equations

\[
\partial_{t_m} A^0 - \partial_x D_m = 0
\]  

(2.9)

\[
\partial_x C_m - [A^0, D_m] = 0
\]  

(2.10)

\[
\partial_{t_{m-1}} A^0 - [E, D_m] - [A^0, C_m] = 0
\]  

(2.11)

\[
[E, C_m] = 0
\]  

(2.12)

From the last equation we conclude that \(C_m \in \text{Ker (ad } E\)\). Since \(A^0 \in \text{Im (ad } E\)\), the first equation implies that also \(D_m \in \text{Im (ad } E\)\) (except for a \(x\) independent component in \(\text{Ker (ad } E\) which we do not consider).

II.I. The \(sl(2)\) case

In this case one has \(E = \frac{1}{2} \sigma_3\), \(A^0 = -q \sigma_+ + r \sigma_-\)

(2.13)

and so, since \(C_m \in \text{Ker (ad } E\) and \(D_m \in \text{Im (ad } E\) one has

\[
C_m = c_m \sigma_3, \quad D_m = d^+_m \sigma_+ + d^-_m \sigma_-\]

(2.14)

Replacing (2.13) into (2.10) and (2.11) one gets

\[
\partial_x c_m + q d^-_m + r d^+_m = 0
\]  

(2.15)

\[
\partial_{t_{m-1}} q - d^-_m - 2q c_m = 0
\]  

(2.16)

\[
\partial_{t_{m-1}} r + d^-_m - 2rc_m = 0
\]  

(2.17)

From these equations one finds

\[
c_m = \partial^{-1}_x \partial_{t_{m-1}} (r q)
\]  

(2.18)

Replacing into (2.16) and (2.17) leads to

\[
d^+_m = -\partial_{t_{m-1}} \partial^{-1}_x \partial_{t_{m-1}} (r q)
\]  

(2.19)

\[
d^-_m = -\partial_{t_{m-1}} \partial^{-1}_x \partial_{t_{m-1}} (r q)
\]  

(2.20)

\(\sigma_{3,\pm}\) stands for the combinations of the Pauli matrices such that, \([\sigma_3, \sigma_{\pm}] = \pm 2\sigma_{\pm}\) and \([\sigma_+, \sigma_-] = \sigma_3\).
Substituting into (2.9), one can write it in the form of (1.6) with the recursion operator given by

\[
R(x, y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} \delta(x - y) + 2 \begin{pmatrix} q(x) r(x) & r(x)^2 \\ q(x)^2 & q(x) r(x) \end{pmatrix} \delta(x - y) \\
+ 2 \begin{pmatrix} r'(x) q(y) & r'(x) r(y) \\ q'(x) q(y) & q'(x) r(y) \end{pmatrix} \partial_x^{-1} \delta(x - y)
\]

(2.21)

One can verifies that the recursion operator can be decomposed in to

\[
R = P_3 P_2^{-1}
\]

(2.22)

where

\[
P_3(x, y) = \begin{pmatrix} 0 & \delta'(x - y) \\ \delta'(x - y) & 0 \end{pmatrix}
\]

(2.23)

and

\[
P_2(x, y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \delta(x - y) + 2 \begin{pmatrix} r(x) r(y) & -r(x) q(y) \\ -q(x) r(y) & q(x) q(y) \end{pmatrix} \partial_x^{-1} \delta(x - y)
\]

(2.24)

The brackets associated to the above \( P_2 \) and \( P_3 \) were proposed in [6], by choosing the prescription \( \gamma = \frac{1}{2} \) in (1.12), i.e.

\[
2 \partial_x^{-1} \delta(x - y) = \epsilon(x - y)
\]

(2.25)

in all entries of (2.24).

We can obtain a further bracket, denoted \( P_4 \), by

\[
R = P_4 P_3^{-1}
\]

(2.26)

It gives

\[
\{ r(x), r(y) \}_4 = -r'(x) r'(y) \epsilon(x - y) + 2r'(x) r(y) \delta(x - y) + 2r(x)^2 \delta'(x - y)
\]
\[
\{ q(x), q(y) \}_4 = -q'(x) q'(y) \epsilon(x - y) + 2q'(x) q(y) \delta(x - y) + 2q(x)^2 \delta'(x - y)
\]
\[
\{ r(x), q(y) \}_4 = -\delta''(x - y) - r'(x) q'(y) \epsilon(x - y) + 2r'(x) q(x) \delta(x - y)
\]
\[
+ 2r(x) q(x) \delta'(x - y)
\]

(2.27) \hspace{1cm} (2.28) \hspace{1cm} (2.29)

The third and fourth brackets, \( P_3 \) and \( P_4 \), were introduced in ref. [7], from which they constructed the recursion operator \( R \) through (2.26).

We have verified, using the Mathematica program, that the three brackets \( P_2, P_3 \) and \( P_4 \) satify the Jacobi identities and are compatible with each other in the sense of (1.10). Therefore, we have established that such system is tri-Hamiltonian.
II.II. The $sl(3)$ case

Let us parametrize, in this case, the quantities introduced in (2.6) as

\[ C_m = c_m^{(a_1)} E_{a_1} + c_m^{(-a_1)} E_{-a_1} + c_m^{(1)} H_1 + c_m^{(2)} H_2 \]  
\[ D_m = d_m^{(a_1+a_2)} E_{a_1+a_2} + d_m^{(a_2)} E_{a_2} + d_m^{(-a_1-a_2)} E_{-a_1-a_2} + d_m^{(-a_2)} E_{-a_2} \]  
\[ A^0 = -q_1 E_{a_1+a_2} - q_2 E_{a_2} + r_1 E_{-a_1-a_2} + r_2 E_{-a_2} \]  

Substituting in (2.10) one gets

\[ \partial_x c_m^{(a_1)} = -q_1 d_m^{(-a_2)} - r_2 d_m^{(a_1+a_2)} \]  
\[ \partial_x c_m^{(-a_1)} = -q_2 d_m^{(-a_1-a_2)} - r_1 d_m^{(a_1+a_2)} \]  
\[ \partial_x c_m^{(1)} = -q_1 d_m^{(-a_1-a_2)} - r_1 d_m^{(a_1+a_2)} \]  
\[ \partial_x c_m^{(2)} = -q_1 d_m^{(-a_1-a_2)} - r_1 d_m^{(a_1+a_2)} - q_2 d_m^{(-a_2)} - r_2 d_m^{(a_2)} \]  

whilst (2.11) leads to

\[ d_m^{(a_1+a_2)} = -\partial_{t_{m-1}} q_1 - q_1 \left( c_m^{(1)} + c_m^{(2)} \right) - q_2 c_m^{(a_1)} \]  
\[ d_m^{(a_2)} = -\partial_{t_{m-1}} q_2 - q_2 \left( -c_m^{(1)} + 2c_m^{(2)} \right) - q_1 c_m^{(-a_1)} \]  
\[ d_m^{(-a_1-a_2)} = -\partial_{t_{m-1}} r_1 + r_1 \left( c_m^{(1)} + c_m^{(2)} \right) + r_2 c_m^{(-a_1)} \]  
\[ d_m^{(-a_2)} = -\partial_{t_{m-1}} r_2 + r_2 \left( -c_m^{(1)} + 2c_m^{(2)} \right) + r_1 c_m^{(a_1)} \]  

From the eqs. (2.33)-(2.40) one gets

\[ c_m^{(2)} = \partial_x^{-1} \partial_{t_{m-1}} (r_1 q_1 + r_2 q_2) \]  

and

\[ \partial_x c_m^{(a_1)} - q_1 r_2 \left( 2c_m^{(1)} - c_m^{(2)} \right) - (r_2 q_2 - r_1 q_1) c_m^{(a_1)} = \partial_{t_{m-1}} (q_1 r_2) \]  
\[ \partial_x c_m^{(-a_1)} + q_2 r_1 \left( 2c_m^{(1)} - c_m^{(2)} \right) + (r_2 q_2 - r_1 q_1) c_m^{(-a_1)} = \partial_{t_{m-1}} (q_2 r_1) \]  
\[ \partial_x c_m^{(1)} - r_1 q_2 c_m^{(a_1)} + r_2 q_1 c_m^{(-a_1)} = \partial_{t_{m-1}} (q_1 r_1) \]  

One can write the above system of differential equations in a more elegant way, introducing

\[ c_m^\pm = c_m^{a_1} \pm c_m^{-a_1}, \quad c_m^0 = 2c_m^1 - c_m^2 \]  

and also

\[ \gamma_\pm \equiv q_1 r_2 \pm q_2 r_1, \quad \gamma_0 \equiv q_1 r_1 - q_2 r_2 \]  

Now, introduce the matrices

\[ c \equiv \begin{pmatrix} c_+ \\ c_0 \\ c_- \end{pmatrix}, \quad \gamma \equiv \begin{pmatrix} \gamma_+ \\ \gamma_0 \\ \gamma_- \end{pmatrix} \]
and

\[ W \equiv \begin{pmatrix} 0 & -\gamma & \gamma_0 \\ -\gamma & 0 & -\gamma_+ \\ \gamma_0 & -\gamma_+ & 0 \end{pmatrix} \quad (2.48) \]

Then, the solution for the system (2.42)-(2.44) can be written formally as

\[ c = D^{-1}_{x} \partial_{t_{m-1}} \gamma \quad D_x \equiv \partial_x + W \quad (2.49) \]

Notice that \( W \) is a matrix in the adjoint of \( sl(2) \), and therefore in the actual integration of the above equations such algebraic structure should play an important role.

III. Appendix

We give here some definitions and relations involving delta and Heaviside functions.

The Heaviside function is defined by

\[ \Theta(x - y) \equiv \begin{cases} 1 & \text{for } x > y \\ 0 & \text{for } x < y \\ \frac{1}{2} & \text{for } x = y \end{cases} \quad (3.1) \]

We also introduce the sign function as

\[ \epsilon(x - y) \equiv \Theta(x - y) - \Theta(y - x) \quad (3.2) \]

One also has

\[ \partial_x \Theta(x - y) = -\partial_y \Theta(x - y) = \delta(x - y) \quad (3.3) \]

and so

\[ \partial_x \epsilon(x - y) = -\partial_y \epsilon(x - y) = 2\delta(x - y) \quad (3.4) \]

In order to verify Jacobi identities and other relations involving fields at different points we used the strategy of using delta functions and its derivatives to try to write the fields at the same point. The relations used can be derived from the identity

\[ f(y) \delta(x - y) = f(x) \delta(x - y) \quad (3.5) \]

Indeed, differentiating it one gets

\[ f(y) \frac{d^n \delta(x - y)}{dx^n} = \sum_{l=0}^{n} \binom{n}{l} \frac{d^{n-l} \delta(x - y)}{dx^{n-l}} \frac{d^l f(x)}{dx^l} \quad (3.6) \]

We also have used the identities

\[ \Theta(y - x) \Theta(z - x) - \Theta(y - z) \Theta(z - x) - \Theta(y - x) \Theta(z - y) = 0 \]

\[ \Theta(x - y) \Theta(z - x) + \Theta(y - x) \Theta(z - y) - \Theta(z - x) \Theta(z - y) = 0 \quad (3.7) \]

Acknowledgements

We are grateful to H. Aratyn for the correspondence on the subject.
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