On A Notion of Stochastic Zeroing Barrier Function

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Abstract—This note examines the safety verification of the solution of Itô’s stochastic differential equations (SDE) using the notion of stochastic zeroing barrier function (SZBF). It is shown that an extension of the recently developed zeroing barrier function concept in deterministic systems can be derived to provide a SZBF based safety verification method for Itô’s SDE sample paths. The main tools in the proposed method include Itô’s calculus and stochastic invariance concept.

I. INTRODUCTION

The fast developments and advances in sensor, computational and communication technologies have recently stimulated a growing interests in cyber-physical systems (CPS) framework for control systems design and implementation purposes [1], [2], [3]. In such a framework, a network of computers are often used to automatically manage the plant-sensor-controller-actuator interactions and data exchange via dedicated or suitable communication networks. The resulting CPS thus often contains tight interactions between physical, computational and communication processes, resulting in complex dynamics that are often not well understood. Among others, one of the frequently encountered challenges in CPS design are concerning the safety implementation of such CPS to guarantee the fulfillment of their safety-critical operational requirements. While a large number of results have so far been established, issues related to safety verification of CPS remain open problems.

In order to examine and verify the safety property of CPS, researchers in systems and control areas have recently proposed a framework based on the use of the so-called barrier function (BF). Being essentially a certificate-based strategy akin with the well-known Lyapunov’s stability analysis method, the BF approach verifies the safety property of CPS by finding some invariant set over which the considered safety property is fulfilled [4], [5], [6]. When compared to the more conventional simulation- or reachability-based methods, the BF based methods are arguably more computationally efficient as they do not require the exhaustive enumerations or simulations of the systems trajectories/paths for deciding the safety properties in questions. These have become the motivating reasons for the currently active research for further developments on both theoretical and application aspects of the BF methods in such areas as deterministic [7], hybrid [8], and stochastic systems [9] (see also e.g. [10] and the references therein).

With regard to stochastic systems, the development of BF based safety verification method was first formulated in [9] as an exit problem. More specifically, the safety property is defined as the probability of the sample paths of Itô SDE leaving a predefined safe set when initialized from a subset of that safe set. The approach in [9] essentially search for a BF in the form of a supermartingale of the process’ sample path that can be used to upper bound such an exit probability. The supermartingale requirement in [9] for the BF was later relaxed in [11] whereby it is only required to be a c–martingale (cf. e.g. [12]). Akin to the idea for finite-time stability characterization developed in [13], such a relaxation is shown in [11] to be useful for characterizing the finite-time regional safety properties of Itô type SDE sample paths.

The aim of this note is to examine an extension of the recently developed BF based method introduced in [14] to the stochastic systems’ case. Specifically, we propose a stochastic analogue of the zeroing barrier function (ZBF) based method developed in [14] to allows for the safety verification of the solution of Itô’s SDE to be done using what we refer to in this paper as a stochastic ZBF (SZBF). We show in this note that, similar to the development of stochastic stability methods based on Lyapunov’s stability analysis in deterministic systems, the extension of ZBF based method in deterministic systems can be extended to derive a SZBF based method for safety verification of Itô’s SDE.

Section II formulates the problem setup and the considered SDE model. Section III presents the main result of the paper regarding SZBF-based invariance and stability property analyses of Itô’s SDE. Section IV concludes the note.

Notation: $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of real numbers and the $n$-dimensional Euclidean space, respectively. We use $x \in \mathbb{R}^n$ to denote a real-valued $n$-dimensional vector $x$ with an Euclidean norm $|x|$. The expected value of a random variable and the probability of a random event to occur are denoted as $\mathbb{E}[\cdot]$ and $\mathbb{P}[\cdot]$, respectively. The family of all continuous strictly increasing functions $\kappa : [0, a) \to [0, \infty)$ for some $a > 0$ is denoted as class $\mathcal{K}_a$. A continuous function $\alpha : (-b, a) \to (-\infty, \infty)$ is said to belong to the extended class $\mathcal{K}$ function $\mathcal{K}_a$ if it is strictly increasing and satisfies $\alpha(0) = 0$. The family of all continuous functions $\gamma : [0, b) \times [0, \infty) \to [0, \infty)$ is denoted as class $\mathcal{KL}$ function for some $b > 0$ if for each fixed $s$, the mapping $\gamma(r, s)$ belongs to the class $\mathcal{K}_r$ function with respect to $r$ and for each fixed $r$, the mapping $\gamma(r, s)$ is decreasing with respect to $s$ and $\gamma(r, s) \to 0$ as $s \to \infty$. $C^2_c$ denotes the family of all functions $V(x) : \mathbb{R}^n \to \mathbb{R}$ that are continuously twice differentiable in $x$ with compact support.
II. Setup & Preliminaries

A. System Description

Assume a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a standard $\mathbb{R}^m$-valued Wiener process $w_1$ defined on this space. Let $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ be the right continuous filtration generated by $w_1$, i.e., $\mathcal{F}_t := \sigma(w_s; 0 \leq s \leq t) \vee \mathcal{N}$ in which $\mathcal{N}$ denotes the class of all $\mathbb{P}$-negligible sets. We consider a stochastic process $x_t$ which evolves on this space according to the stochastic differential equation (SDE) of the form

$$dx_t = b(x_t) dt + \sum_{k=1}^{m} \sigma(x_t) dw^k_t$$

(1)

with initial value $x_0$ at time $t = 0$. In (1), both $b(\cdot)$ and $\sigma(\cdot)$ with $b(0) = 0$, $\sigma(0) = 0$ are functional mappings from $\mathbb{R}^n$ into $\mathbb{R}^m$ which satisfy Assumption (1) below.

**Assumption 1:** For $b(\cdot)$ and $\sigma_k(\cdot)$ in (1):

1) there exists a nonnegative constant $L$ such that for all $x \in \mathbb{R}^n$, the following holds.

$$|b(x)|^2 + \sum_{k=1}^{m} |\sigma(x)|^2 \leq L (1 + |x|^2)$$

(2)

2) for any $x$, $y \in \mathbb{R}^n$, the following holds.

$$|b(x) - b(y)| + \sum_{k=1}^{m} |\sigma_k(x) - \sigma_k(y)| \leq |x - y|$$

(3)

Under Assumption (1) a unique strong solution of the SDE (1) is known to exist in Itô’s sense and is given by [15], [16]

$$x_t = x_0 + \int_{0}^{t} b(x_s) ds + \sum_{k=1}^{m} \int_{0}^{t} \sigma_k(x_s) dw^k_s$$

(4)

where $x_0 \in \mathbb{R}^n$ is given. In what follows, for any $s \geq 0$ and $x \in \mathbb{R}^n$, we use $x^s$ to denote the solution of (1) of the form (4) at time $s \leq t$ when initialized from $x$.

To the solution $x_t$ in (4), we associate the infinitesimal generator which is defined by an operator $L$ acting on a function $h(x_t) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ of the form

$$Lh(x_t) := \lim_{t \searrow 0} \frac{\mathbb{E}[h(x_t)] - h(x)}{t}$$

(5)

Here, we consider $h(x)$ to be twice continuously differentiable on $x$ with compact support (denote this class of functions as $C^2_e$) such that (5) becomes [16]

$$Lh(x) = \sum_{i=1}^{m} b^i(x) \frac{\partial h(x)}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{m} \sigma^i_k(x) \sigma^j_k(x) \frac{\partial^2 h(x)}{\partial x^i \partial x^j}.$$ 

(6)

Furthermore, for any function $h(x) \in C^2_e$ and the solution (4) of the SDE (1), Itô’s lemma [17] states that the following holds for $h(x_t)$.

$$dh(x_t) = \frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} dx_t + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} dx_t(\partial h(x_t) \frac{\partial h'(x)}{\partial x} dx_t)$$

(7)

where $dx_t$ is defined as in (1), while $dt dt = 0$, $dt dw^k_t = 0$ and $dw^k_t dw^l_s = \delta_{k,l} \delta_{s,t} dt$ for any $1 \leq k \leq m$, $\delta$ being a dirac delta function, are used as a convention for (7).

B. Problem Formulation

Given the SDE in (1), the objective of this paper is to examine the safety verification of the strong solution of (1) with respect to (wrt.) the SDE (1). In this regard, we consider by construction the following closed set $C$ (which is a subset of $\mathbb{R}^n$) defined by a function $h(x)$.

$$C = \{ x \in \mathbb{R}^n : h(x) \geq 0 \}$$

(8a)

$$C^o = \{ x \in \mathbb{R}^n : h(x) > 0 \}$$

(8b)

$$\partial C = \{ x \in \mathbb{R}^n : h(x) = 0 \}$$

(8c)

in which $\partial C$ and $C^o$ denote the boundary and interior of $C$, respectively.

Given the SDE in (1) and the closed set $C$ in (8), the main objective of this paper is to establish conditions on the defining function $h(x)$ that will guarantee the set $C$ to be invariant under the evolution of the SDE (1)’s strong solution in (4). Following the framework developed in [14] for characterizing the invariant set of deterministic systems, our approach is based on the stochastic version of the notion of zeroing barrier function. To this end, we recall in Definition (1) below the corresponding notion of stochastic invariant set to be used in this paper.

**Definition 1:** A closed subset $C \subset \mathbb{R}^n$ is said to be *stochastically invariant* wrt. the SDE (1) if for every $\mathcal{F}_0$-measurable random variable $x_0$ such that $x_0 \in C$ almost surely (a.s.), the strong solution $x_t$ in (4) satisfies $x_t \in C$ for all $t \geq 0$ a.s.

III. Main Results

This section proposes the notion of stochastic zeroing barrier function that can be used to establish the invariance of a closed set wrt. the strong solution of SDE (1) in (4).

A. Stochastic Invariance Under SZBF Existence

In this subsection, we show that the existence of a SZBF ensure the stochastic invariance of a certain subset of $\mathbb{R}^n$ wrt. the SDE in (1). To begin with, we first state the following notion of stochastic zeroing barrier function (SZBF).

**Definition 2 (SZBF):** Consider the SDE (1) and the set $C$ defined in (8) by a function $h(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ with $h(x) \in C^2_e$. If there exists a class $K_c$ function $\alpha$ and a set $D$ with $C \subseteq D \subseteq \mathbb{R}^n$ such that for all $x \in D$ the following hold:

(i) $Lh(x) \geq -\alpha(h(x))$, and (ii) $\sum_{k=1}^{m} \frac{\partial h(x)}{\partial x^i} \sigma_k(x) = 0$, then $h(x)$ is a SZBF.

We now present the main result of this paper which essentially states that the existence of a SZBF $h(x)$ as per Definition (2) implies the invariance of a closed set $C$ defined in (8) wrt. the solution of the SDE (1) in (4).

**Proposition 1:** Consider the SDE (1) and a closed set $C$ in (8) defined by some function $h(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ with $h(x) \in C^2_e$. If $h(x)$ is a SZBF defined on the set $D$ with $C \subseteq D \subseteq \mathbb{R}^n$, then $C$ is stochastically invariant wrt. the solution of (1).
Expanding the right-hand side of (10) and using the convention as stated following (7), this implies that (12) may be rewritten as

\[
dh(x) = \frac{\partial h(x)}{\partial x} dx_t + \sum_{i=1}^{m} \sigma(x_i) dw^i_t.
\]  

Since \( h(x) \) is a ZBF for the diffusion-free part of (1), we have by [14, Proposition 1] that \( h(x) \geq -\alpha(h(x)) \) in which \( \alpha \in \mathcal{K}_c \). Combining this and conditions (i)-(ii) in Lemma 1 we have \( \mathcal{L}h(x) \geq -\alpha(h(x)) \) which by Definition 2 implies \( h(x) \) is a SZBF for SDE (1) as claimed in statement 1) in the lemma. The statement 2) in the lemma follows from the fact that \( h(x) \) is a ZBF of the diffusion-free part of (1) (cf. [14, Proposition 1]) as well as a SZBF for the SDE (1) (cf. Proposition 1).

**Remark 1:** Intuitively, Lemma 1 states that the invariance property of the differential equation of the form \( \dot{x}(t) = b(x) \) may be preserved in its stochastic counterpart of the form (1) only if the diffusion part of the SDE (1) satisfies conditions (i)-(ii) in the lemma.

**B. SZBF-induced Stochastic Stability**

In this section, we show that the existence of a SZBF for the SDE in (1) induces a stochastic Lyapunov function which guarantees the stochastic stability of the corresponding invariance set. This thus essentially establishes the stochastic counterpart of [14, Proposition 2] that was developed for the deterministic systems case.

**Proposition 2:** Let \( h(x) : \mathcal{D} \to \mathbb{R} \) with \( h \in C^2_b \) be defined on an open subset \( \mathcal{D} \subseteq \mathbb{R}^n \) such that \( \mathcal{C} \subseteq \mathcal{D} \subseteq C^2_b \) where \( C \) is defined as in (8). If \( h(x) \) is a SZBF wrt. the SDE in (1), then the set \( \mathcal{C} \) is stochastically stable.

**Proof:** Note that \( h(x) \) being a SZBF on \( \mathcal{D} \) induces a stochastic Lyapunov function \( V_C(x) : \mathcal{D} \to \mathbb{R} \) of the form

\[
V_C(x) = \begin{cases} 
0, & \text{if } x \in \mathcal{C}, \\
-h(x), & \text{if } x \in \mathcal{D} \setminus \mathcal{C}.
\end{cases}
\]  

With such a choice, one may notice that \( V_C(x) \) is continuous on its domain and \( V_C(x) \in C^2_b \) at every point \( x \in \mathcal{D} \setminus \mathcal{C} \). Furthermore: i) \( V_C(x) = 0 \) for \( x \in \mathcal{C} \); ii) \( V_C(x) > 0 \) for \( x \in \mathcal{D} \setminus \mathcal{C} \); and iii) for \( x \in \mathcal{D} \setminus \mathcal{C} \), then \( \mathcal{L}V_C(x) \) satisfies

\[
\mathcal{L}V_C(x) = -\mathcal{L}h(x) \leq \alpha \circ h(x) = \alpha(-V_C(x)) \leq 0,
\]  

where \( \alpha \in \mathcal{K}_c \).

Let us consider a solution \( x_t^{0,x_0} := x_0 \in I_t \subseteq \mathcal{C} \) of (1) in which \( I_t \) denotes the zero set of the function \( V_C(x_t) \) at time \( t \) (i.e., \( x_0 \) belongs to the zero set of \( V_C(x) \) at \( t = 0 \)). Assume that \( I_t \) is closed in \( \mathcal{C} \) such that \( I_t \) is also closed in \( \mathcal{C} \), then by the continuous everywhere property of \( \mathcal{D} \) in (1), there exists a time \( t_c \) such that \( x_t^{0,x_0} \in \mathcal{C} \) for a certain
time interval $[0, t_C)$ and that $x_{t_C}^{0,x_0}$ is also a strong solution of (1). In this regard, $t_C > 0$ with probability 1 is the first instance when $x_{t_C}^{0,x_0}$ leaves the set $\mathcal{C}$. Using (11) to compute Itô formula for $V_C(x)$ on the interval $[0, t_C)$, we have that

$$V_C(x(t_C \wedge t)) - V_C(x_0) = \int_0^{t_C \wedge t} \mathcal{L}V_C(x_s) \, ds - \sum_{k=1}^m \int_0^{t_C \wedge t} \frac{\partial V_C(x_s)}{\partial x} \sigma_k(x_s) \, d{w^k}_s \, ds$$

Taking the expectation of the above equation gives

$$\mathbb{E}[V_C(x(t_C \wedge t))] - V_C(x_0) = \mathbb{E} \left[ \int_0^{t_C \wedge t} \mathcal{L}V_C(x_s) \, ds \right]$$

which by (16) implies that

$$\mathbb{E}[V_C(x(t_C \wedge t))] \leq 0.$$ (18)

Combining (19) with the property that $V_C = 0$ for $x \in \mathcal{C}$ and positive for $x \in \mathcal{D} \setminus \mathcal{C}$, we then have that

$$V_C(x(t_C \wedge t)) = 0$$ (19)

with probability 1 which thus implies that $(t_C \wedge t) := t$ holds with probability 1. Then by [16, Lemma 7.4], we conclude that the set $\mathcal{C}$ is stable in probability.

IV. REMARK AND DISCUSSION

This paper has presented an approach for the safety verification of the solution of Itô’s stochastic differential equations using the notion of stochastic zero barrier function. It is shown that the extension of ZBF based method in deterministic systems can be extended to provide SZBF based method for safety verification of Itô’s stochastic differential equation.

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