A 2-D meshless time-domain algorithm for solving Maxwell’s equations based on Steger-Warming flux vector splitting approach

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Abstract. A 2-D meshless time-domain algorithm is proposed for solving Maxwell’s equations. The spatial derivatives related to the algorithm are approximated by using an expanded Taylor series and the least square technique in each cloud of points, and then a Steger-Warming flux vector splitting approach is introduced for computing the physical flux of the governing equations. After that, an explicit four-stage Runge-Kutta scheme is employed in time-marching. Combined with solving Maxwell’s equations, the implementations of the present algorithm are described in details. Based on the proposed algorithm, bistatic radar cross sections for typical 2-D objects in different frequency regions are given, which show a good agreement with those of the series solutions. The paper ends with the presentation of the electromagnetic scattering fields for a 2-D aircraft model, which display the characteristics of the algorithm presented in dealing with time-domain problems to a certain extent.

1. Introduction

Recently, both academic and engineering circles have focused more and more attention on stealth characteristics of military aircrafts. Theoretically, the stealth characteristics such as radar cross section (RCS) for a given aerodynamic body can be obtained by solving the fundamental time-domain Maxwell’s equations. With the development of computing facilities, more and more researchers begin to numerically direct solve the Maxwell’s equations so as to compute the RCS of the target [1-3].

In spite of the great success of many numerical methods as effective tools for solving the Maxwell’s equations, there is still a growing interest in the development of new advanced methods. Meshless methods, which only make use of clouds of points and do not require the points be connected to form a grid, have been proposed and become one of the hottest research areas in the last decade. These new kinds of numerical methods distribute a set of points in the computational domain interested, which make them flexible for treating complex geometries [4]. Representative meshless methods for analyzing electromagnetic fields contain the Element Free Galerkin method and meshless methods based on radial basis functions [5-7]. However, such meshless methods usually need to choose shape functions or basis functions, which can affect the quality of the companion matrix and the implementation of the boundary condition [8]. We notice that another type of meshless method has been applied to solve Euler equations in Computational Fluid Dynamics (CFD), which is different...
from those in Computational Electromagnetics (CEM) in computation of spatial derivatives and physical flux related to the governing equations [4].

Taking into account that Maxwell’s equations and Euler equations have the similar mathematical properties such as the hyperbolic nature, we aim to propose a CFD based meshless time-domain algorithm for solving Maxwell’s equations in CEM. Based on the clouds of points related to the meshless time-domain algorithm, the spatial derivatives are approximated by using an expanded Taylor series and the least square technique. Different from other published papers for the study of meshless methods [9], the physical flux of the Maxwell’s equations is computed by a Steger-Warming flux vector splitting approach. The physical quantities at the midpoint between the central and satellite points related to the flux computation are approximated by using the linear function reconstruction. After that, an explicit four-stage Runge-Kutta scheme is employed in time-marching. Then, typical 2-D examples are given to validate the algorithm, and a study on electromagnetic scattering characteristics of a 2-D aircraft model is also presented at the end of this paper.

2. Meshless time-domain algorithm

2.1. Governing equations

In 2-D Cartesian coordinate system, the non-dimensional form of Maxwell’s equations for Transverse Magnetic (TM) polarization may be written as:

\[
\frac{\partial \mathbf{W}}{\partial t} + \left( \frac{\partial \mathbf{F}_1}{\partial x} + \frac{\partial \mathbf{F}_2}{\partial y} \right) = \mathbf{S} \tag{1}
\]

Where:

\[
\mathbf{W} = \left[ \varepsilon E_z \quad \mu H_x \quad \mu H_y \right]^T, \quad \mathbf{F}_1 = \left[ -H_y \quad 0 \quad -E_z \right]^T, \quad \mathbf{F}_2 = \left[ H_x \quad E_z \quad 0 \right]^T, \quad \mathbf{S} = \left[ -\varepsilon E_z \quad -\sigma_m H_x \quad -\sigma_m H_y \right]^T.
\]

\(E_z\) indicates the scalar component of the electric field vector \(E\) in the Cartesian \(z\) direction, \(H_x\) and \(H_y\) indicate the scalar components of the magnetic field vector \(H\) in the Cartesian \(x\) and \(y\) directions respectively, \(\varepsilon\) is the electric permittivity, \(\mu\) is the magnetic permeability, \(\sigma\) is the conductivity, \(\sigma_m\) is the magnetic resistivity.

2.2. Points distribution and generation of clouds of points

Different from traditional mesh-based methods, the meshless time-domain algorithm only requires clouds of points distributed in the computational domain. Sometimes points based on grids can be used for convenience, and sometimes techniques of distributing points directly can be employed if needed. After points being distributed in the computational domain, clouds of points related to the meshless time-domain algorithm should be constructed. For 2-D situations, how the cloud of points \(C_i\) constructed can refer to reference [9]. In figure 1, point \(i\) is the central point and points 1, 2, 3, 4, 5, 6 are satellite points on the cloud of points \(C_i\).

**Figure 1.** Cloud of points \(C_i\).
2.3. Approximation of spatial derivatives

Based on the cloud of points $C_i$ above, considering any sufficiently differentiable function $f = f(x, y)$ near the central point $i$, the Taylor series about the point $i$ can be expressed in the following form:

$$f = f_i + a_i h + a_2 l + \cdots + l h f_i \frac{\partial f}{\partial x} + l l h f_i \frac{\partial^2 f}{\partial y^2} + O(h^{m\epsilon}, l^{m\epsilon})$$

(2)

Where $f_i = f(x_i, y_i)$, $h = x - x_i$, $l = y - y_i$, $a_i(i = 1, 2)$ represent the spatial derivatives of the function at the central point $i$. For the linear approximation, equation (2) can be written as a linear function $\tilde{f} = f_i + a_i h + a_2 l$. Based on the meshless time-domain algorithm, the function values at all satellite points $k$ ($k = 1, \cdots, M$) are available, denoted by $f_k$. Then, in order to minimize the total error of $f_k$, the spatial derivatives can be solved by the following equations [9]:

$$\begin{bmatrix} \sum h_i^2 \\ \sum h_i l_k \\ \sum l_i^2 \end{bmatrix} \begin{bmatrix} a_i \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum h_k (f_k - f_i) \\ \sum l_k (f_k - f_i) \end{bmatrix}$$

(3)

Then the linear approximated function can be written as $f(x, y) = f_i + \sum a_k (f_k - f_i) h + \sum \beta_k (f_k - f_i) l$, where coefficients $a_k$ and $\beta_k$ can be obtained before time-marching, because they are only related to the coordinates of discrete points on $C_i$. The spatial derivatives of the function at the central point $i$ then can be approximated as:

$$a_i = \sum \alpha_k (f_k - f_i), a_2 = \sum \beta_k (f_k - f_i)$$

(4)

The spatial derivatives can also be estimated by using the following formulas:

$$a_i = \sum \alpha_k (f_k - f_i), a_2 = \sum \beta_k (f_k - f_i)$$

(5)

Where coefficients $\alpha_k$ and $\beta_k$ can also be obtained before time-marching.

2.4. Computation of physical flux

Based on the cloud of points $C_i$, if equations (5) are applied to the convective flux of the Maxwell’s equations, the following expression is obtained:

$$\left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) = \sum \alpha_{ik} (F_{1ik} - F_{1i}) + \sum \beta_{ik} (F_{2ik} - F_{2i}) = \sum (\alpha_{ik} F_{1i} + \beta_{ik} F_{2i}) - \sum (\alpha_{ik} F_{1i} + \beta_{ik} F_{2i})$$

(6)

![Figure 2. The virtual interface at the midpoint on the cloud of points $C_i$.](image)

For the coefficients $\alpha_{ik}$ and $\beta_{ik}$ as well as the function value $f_i$ at the central point $i$ being available, then the flux term $\sum (\alpha_{ik} F_{1i} + \beta_{ik} F_{2i})$ is available. In the following text, how to accommodate the other flux term $\sum (\alpha_{ik} F_{1ik} + \beta_{ik} F_{2ik})$ will be introduced. Create a virtual interface
at the midpoint between the central and each satellite point of the gridless cloud (see figure 2), where ‘+’ corresponds to the side of the central point \( i \) and ‘-’ corresponds to the side of the satellite point \( k \).

Define a numerical flux term \( \mathbf{Q}_{ik} = \xi_{ik} F(W_{ik}) + \eta_{ik} F_s(W_{ik}) \), where \( \xi_{ik} = \alpha_{ik} / \sqrt{\alpha_{ik}^2 + \beta_{ik}^2} \), \( \eta_{ik} = \beta_{ik} / \sqrt{\alpha_{ik}^2 + \beta_{ik}^2} \). Define a vector \( \mathbf{d}_{ik} = (\alpha_{ik}, \beta_{ik}) \), then equation (6) can be written as:

\[
\left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right)_{ik} = \sum_{k=1}^{M} \mathbf{Q}_{ik} |d_{ik} - \sum_{k=1}^{M} (\alpha_{ik} F_{1i} + \beta_{ik} F_{2i})
\]

In order to compute the numerical flux term \( \mathbf{Q}_{ik} \), apply Steger-Warming flux vector splitting approach [10] to split \( T_{ik} = \partial \mathbf{Q}_{ik} / \partial W_{ik} \) (the Jacobian matrix of \( \mathbf{Q}_{ik} \), and then \( \mathbf{Q}_{ik} \) can be written in a splitting form as \( \mathbf{Q}_{ik} = \mathbf{Q}_{ik}^+ + \mathbf{Q}_{ik}^- = M_{ik} W_{ik}^+ + M_{ik} W_{ik}^- \), where \( M_{ik}^+ \) and \( M_{ik}^- \) are the split matrices of \( T_{ik} \), \( W_{ik}^+ \) and \( W_{ik}^- \) can be determined by the linear function reconstruction \( U^+ = U_i + 0.5\nabla U_i \cdot \mathbf{r}_k \), \( U^- = U_k - 0.5\nabla U_k \cdot \mathbf{r}_k \), where \( \mathbf{r}_k = (x_k - x_i, y_k - y_i) \). It is important to illustrate that \( U \) represents any component of the vector \( \mathbf{W} \), and the gradient terms (spatial derivatives) \( \nabla U_i \) and \( \nabla U_k \) can be computed by equations (4).

2.5. Time-marching and boundary conditions

After the spatial discretization, the semi-discretization form of the Maxwell’s equations on the cloud of points \( C_i \) can be expressed as:

\[
\frac{\partial \mathbf{W}}{\partial t} = - \sum_{k=1}^{M} \mathbf{Q}_{ik} |d_{ik} - \sum_{k=1}^{M} (\alpha_{ik} F_{1i} + \beta_{ik} F_{2i}) + S_i | = - \mathbf{R}_i
\]

Where \( \mathbf{R}_i \) represents the residual error at the point \( i \). Then, the four-stage Runge-Kutta scheme \((m = 4)\) is used to solve equation (8):

\[
\mathbf{W}^{n+k/m} = \mathbf{W}^n - \Delta t \gamma_k \mathbf{R}(\mathbf{W}^{n+(k-1)/m}) \quad k = 1, \ldots, m
\]

Where \( \gamma_k = 1/(m - k + 1) \).

For the solution of the Maxwell’s equations in this paper, the perfect conductor boundary condition is used on the wall (solid body) and the perfectly matched layer boundary condition is imposed in the far field, where the values of coefficients related to the formulas can refer to references [10].

3. Numerical results and discussion

![Figure 3. Incident plane wave.](image)

In this section, typical 2-D numerical examples are given to validate the algorithm presented. Then, a study on the electromagnetic scattering characteristics of a 2-D aircraft model is carried out. Figure 3 shows an incident plane wave [11] propagating along the \( k \) axis in \( xoy \) plane. Define \( \varphi \) as the angle between the direction of the \( k \) axis and the \( x \) coordinate axis. The following \( x \) and \( y \) coordinate
values are all nondimensionalized by the wavelength $\lambda$. For the incident TM wave situation, the normalized incident wave component can be represented as follows:

$$E_i = \cos[2\pi(k - t)]; \quad H_i^x = \sin\phi \times \cos[2\pi(k - t)]; \quad H_i^y = -\cos\phi \times \cos[2\pi(k - t)] \quad (10)$$

Where $k = x\cos\phi + y\sin\phi$.

3.1. 2-D circular cylinders in different frequency regions

In order to validate the algorithm presented and show its ability to accommodate configurations in different frequency regions, a set of 2-D circular cylinder cases are simulated first. The circular cylinder are assumed to be perfectly electric conductor with the radius $r = 0.1\lambda$ (low frequency), $1.0\lambda$ (middle frequency), and $5.0\lambda$ (high frequency) respectively. The computational domains are set as $6\lambda \times 6\lambda$, $10\lambda \times 10\lambda$, and $30\lambda \times 30\lambda$ respectively with 19339, 16654 and 65733 points distributed. The incident harmonic TM wave is propagating along the $x$ coordinate axis with $\phi = 0^\circ$.

Contours of scattered fields are presented in figure 4, which are in good agreement with the FVTD results [10] (Due to limited space, the FVTD results are not presented). The calculated bistatic RCS are presented in figure 5, which are in good agreement with the series solutions [12]. The convergence history of the present algorithm is also given in figure 6. It could be observed that the present algorithm has a fast convergence speed in middle and high frequency regions. When the iteration cycle achieves 35 and 68 respectively, the calculated logarithmic residual errors have already converged below $1.0 \times 10^{-7}$ (see figure 6 (b), (c)), which is much faster than the experiential conclusion of reference [11] (The iteration cycle should achieve 60 and 180 respectively when the numerical results have converged to the stable solutions for the time-harmonic field).

![Figure 4. Contours of scattered fields for the 2-D circular cylinders.](Image)

![Figure 5. Bistatic RCS for the 2-D circular cylinders.](Image)
3.2. 2-D aircraft model
In order to display the characteristics of the algorithm presented in dealing with time-domain problems, a 2-D aircraft model [13] is simulated. The length of the aircraft model is set as $8\lambda$ and the computational domain is set as $20\lambda \times 20\lambda$ with 25889 points distributed. The incident harmonic TM wave is propagating with the incident angle $\varphi = 45^\circ$.

Contours of scattered fields in different times are presented in figure 7. Assume that the incident wave just hits the plane when $t = 0$, then the scattering wave generates (see figure 7(a)). As time goes on, the scattered field is gradually expanding (see figure 7(b), (c)). The scattered field in the computational domain reaches periodic steady state before $t = 120$ (see figure 7(d)). It could be observed from figure 7(d) that scattered fields are strong in the direction of the bistatic angle of $\varphi = 45^\circ$ and $\varphi = 315^\circ$, which is in agreement with the physical theory.

4. Conclusions
A 2-D meshless time-domain algorithm for solving Maxwell’s equations is proposed. Based on the clouds of points related to the algorithm, a least square technique is introduced in dealing with approximating the spatial derivatives and a Steger-Warming flux vector splitting approach is employed for computing the physical flux of the governing equations. The calculated bistatic RCS for typical verification examples in different frequency regions are obtained, which are in good agreement with the series solutions. Numerical examples also show that the present algorithm has a fast convergence speed in middle and high frequency regions. The paper ends with the analysis of the electromagnetic scattering characteristics for a 2-D aircraft model, which display the characteristics of the algorithm presented in dealing with time-domain problems to a certain extent.
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