METRIZABLE UNIFORM SPACES

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Abstract. Three themes of general topology: quotient spaces; absolute retracts; and inverse limits — are reapproached here in the setting of metrizable uniform spaces, with an eye to applications in geometric and algebraic topology. The results include:

1) If \( X \supset A \xrightarrow{f} Y \) is a uniformly continuous partial map of metric spaces, where \( A \) is closed in \( X \), we show that the adjunction space \( X \cup_f Y \) with the quotient uniformity (hence also with the topology thereof) is metrizable, by an explicit metric. This yields natural constructions of cone, join and mapping cylinder in the category of metrizable uniform spaces, which we show to coincide with those based on subspace (of a normed linear space); on product (with a cone); and on the isotropy of the \( l_2 \) metric.

2) We revisit Isbell’s theory of uniform ANRs, as refined by Garg and Nhu in the metrizable case. The iterated loop spaces \( \Omega^n P \) of a pointed compact polyhedron \( P \) are shown to be uniform ANRs. Four characterizations of uniform ANRs among metrizable uniform spaces \( X \) are given: (i) the completion of \( X \) is a uniform ANR, and the remainder is uniformly a Z-set in the completion; (ii) \( X \) is uniformly locally contractible and satisfies the Hahn approximation property; (iii) \( X \) is uniformly \( \varepsilon \)-homotopy dominated by a uniform ANR for each \( \varepsilon > 0 \); (iv) \( X \) is an inverse limit of uniform ANRs with “nearly splitting” bonding maps.

1. Introduction

Although topological and uniform approaches to foundations of what was then known as Analysis Situs originated in the same works by M. Fréchet and F. Riesz (cf. Remark 2.4 below), uniform spaces hopelessly lagged behind in development ever since, and were never taken seriously in algebraic and geometric topology, due in part to the lack of a coherent theory of quotient spaces, and of a reasonable notion of a polyhedron in the uniform sense. Yet the opposite side of the coin features painful side effects of the usual topological foundations, including

(i) the non-metrizability of the cone over \( \mathbb{R} \);

(ii) the non-metrizability of \( \mathbb{R}P^\infty \) with the topology of a CW-complex (or the geometric realization of a simplicial set);

(iii) the uncountability of a family of polyhedra (Mardešić’s resolution [59]) needed to capture geometrically relevant homotopy properties (the shape) of a non-compact space. This results, in particular, in the awkward situation that the shape invariant homology of a space as simple as \( \mathbb{N}^+ \times \mathbb{N} \) (where \( \mathbb{N} \) denotes the infinite countable discrete space, and + denotes the one-point compactification) cannot be calculated in ZFC, as its value depends on additional axioms [60].

Supported by Russian Foundation for Basic Research Grant No. 11-01-00822.
1.A. Quotient spaces

One of the goals of the present paper is to show that (the topology of) quotient uniformity is, after all, far nicer than quotient topology in the context of metrizable spaces. In particular, we show that finite homotopy colimits (=homotopy direct limits) of metrizable uniform spaces and uniformly continuous maps are metrizable when done uniformly (Corollary 3.24); they are certainly not metrizable with the quotient topology, as we will now discuss.

1.1. Homotopy colimits. Consider the cone over the real line \( \mathbb{R} \), with the topology of the quotient space \( \mathbb{R} \times [0,1]/\mathbb{R} \times \{1\} \). This topology is non-metrizable, because it does not even have a countable base of neighborhoods at the cone vertex. Indeed, a neighborhood of the cone vertex in the quotient corresponds to a neighborhood of \( \mathbb{R} \times \{1\} \) in \( \mathbb{R} \times [0,1] \). Among such neighborhoods is the region above the graph of an arbitrary function \( f: \mathbb{R} \rightarrow [0,1] \), i.e. \( \{(x,y) \in \mathbb{R} \times [0,1] \mid y \geq f(x)\} \). Now \( f(x) \) can approach 1 with any speed as \( x \rightarrow \pm \infty \) (exponential, doubly exponential, etc.), and it is well-known and easy to see that there are uncountably many of possible speeds (i.e. the poset of functions ordered by \( f \leq g \) if \( f(x) \leq g(x) \) for all \( x \in \mathbb{R} \) has no countable cofinal subposet).

Thus the quotient topology in the cone over \( \mathbb{R} \) is not the same as the topology of the subspace \( \mathbb{R} \times [0,1] \cup \{(0,1)\} \) of \( \mathbb{R}^2 \) or the topology of the subspace of the cone over \([-\infty, \infty] \cong [-1,1] \).

Which topology on the cone is the ‘right one’? It turns out that in many situations where some actual work is being done, the quotient topology does not do its job, and has to be replaced by something else:

(i) In his classifying space construction \( BG = (G * G * \ldots)/G \), Milnor had to use a weak (initial) topology on his joins (including finite joins) rather than the strong (final) topology of the quotient \([64]\) (with his reversed terminology he called his topology a “strong topology”).

(ii) In showing that the usual homotopy category is a closed model category in the sense of Quillen (“The homotopy category is a homotopy category” \([83]\)), A. Strøm had to modify the quotient topology of the mapping cylinder in order to show that if \( f: E \rightarrow B \) is a (Hurewicz) fibration, then so is the projection \( MC(f) \rightarrow B \). His modified mapping cylinder can be identified with a subspace of Milnor’s modified join.

There are other examples of this sort in the literature (including e.g. teardrop neighborhoods); J. Strom called attention to them in his questions on MathOverflow.net.

It turns out that the trouble disappears entirely in the setting of uniform spaces, which is the subject of §3 below. If \( X \supset A \rightarrow Y \) is a uniformly continuous partial map of metric spaces, where \( A \) is closed in \( X \), we show that the adjunction space \( X \cup_f Y \) with the quotient uniformity (hence also with the topology thereof) is metrizable, by an explicit metric (Theorem 3.8). This yields natural constructions of cone, join and mapping cylinder in the category of metrizable uniform spaces, which we show to coincide with various other natural constructions (see §3.B). In particular, we show equivalence, up to
uniform homeomorphism, of a number of definitions of join of metric spaces: one based on
quotient uniformity; another based on embedding in a normed linear space; and those
based on the amalgamated union \( X \times CY \cup CX \times Y \), where the cones are defined using
either any of the previous methods or the approach of geometric group theory, based on
the isotropy of the \( l_2 \) metric (i.e., the Law of Cosines).

1.2. Topological fundamental group. Among other serious troubles with quotient
topology is that the (usual) fundamental group of a compact metric space can fail to be
a topological group as the quotient of the space of maps [30]. The basic reason is that
the product of quotient maps is not always a quotient map.

Again, there is no such problem with (the topology of) the quotient uniformity. The
product of two quotient maps is a quotient map in the category of uniform spaces and
uniformly continuous maps (cf. [50; Exercise III.8(c)]; less trivially this is so also for
infinite products [47]). Using this it is easy to show that for a compact metric space \( X \),
the fundamental group of \( X \) is a topological group with the topology of the quotient
uniformity, as a quotient of the space of maps \((S^1, pt) \to (X, pt)\) endowed with the
metric \( d(f, g) = \sup_{x \in S^1} (f(x), g(x)) \).

Remark 1.3. We note that this topology on the fundamental group of \( X \) differs from the
more intriguing topology considered in [62]; but it is noteworthy that the topology of the
still more intriguing Steenrod fundamental group of \( X \) considered in [62] coincides with
the topology of the quotient uniformity, as a quotient of the space of base-ray preserving
maps from the 2-disk into the one-point compactification of the mapping telescope.

1.B. Absolute retracts

In working with polyhedra it is convenient to separate combinatorial issues (such as
simplicial approximation and pseudo-radial projection) from topological ones, which are
well captured by the notion of an ANR. For instance, finite homotopy limits (=homotopy
inverse limits) of PL maps between compact polyhedra are still ANRs, but no longer
polyhedra in general.

The second major goal of the present paper is to prepare for the treatment of uniform
polyhedra in the sequel to this paper by advancing a theory of uniform ANRs roughly
to the level of the classical theory of ANRs as presented in the books by Borsuk [15]
and Hu [44]. In particular, we show that finite homotopy limits and colimits of uniform
ANRs are still uniform ANRs (Theorem 4.38) — although we will see in the sequel that
those of uniform polyhedra and “uniformly PL” maps are no longer uniform polyhedra
in general.

1.4. Uniform ANRs. Our notion of a uniform ANR is not entirely standard. Two
best-known analogues of ANRs in the uniform world are the semi-uniform ANRs studied
by Michael and Torunczyk (see Remarks 4.30(b) and 4.31) and the ANRUs of Isbell,
which we revisit in §4.A. While semi-uniform ANRs are more manageable in some
respects, they are at best a useful but technical tool, involving a mix of topological and
uniform notions. On the other hand, as long as metrizable spaces are concerned, it was realized independently by Garg and Nhu that Isbell’s ANRUs are only a part of the story — namely the complete part. This understanding is, however, scarcely known, and is not well established in the literature: Garg mentioned what we now call uniform ANRs only in passing (so did not even give them any name); whereas Nhu’s metric uniform ANRs (see Remark 4.30(a)), although do coincide with our uniform ANRs, but somewhat accidentally — for his metric uniform ARs differ from our uniform ARs. Above all, there seems to have been no good intuition and no readily available technique for dealing with non-complete uniform ANRs, as compared with complete ones (i.e. ANRUs).

This is now entirely changed, for we show that a metrizable uniform space is a uniform ANR if and only if its completion is an ANRU, and the remainder can be instantaneously taken off itself by a uniform self-homotopy of the completion (Theorem 4.10). Moreover, in many ways uniform ANRs turn out to be easier, and not harder than ANRUs. In particular, we show (Theorem 4.21) that a metrizable uniform space is a uniform ANR if and only if it is uniformly locally contractible (in the sense of Isbell) and satisfies the Hahn approximation property (in the sense of Isbell). This result, proved by an infinite process, improves on the metrizable case of Isbell’s characterization of ANRUs as those uniform spaces that are uniformly locally contractible and satisfy the Hahn property and the uniform homotopy extension property. (Isbell’s uniform homotopy extension property is for possibly non-closed subsets; it follows from completeness along with the uniform homotopy extension property for closed subsets.)

The above two characterizations of uniform ANRs are at the heart of a toolkit that enables one to deal with uniform ANRs just as easily as with compact ANRs. Indeed we establish uniform analogs of what appears to be the core results of the usual theory of retracts, including Hanner’s $\varepsilon$-domination criterion (Theorem 4.22(a)) and J. H. C. Whitehead’s theorem on adjunction spaces (Theorem 4.32). We also show (see Theorem 4.35 or Corollary 4.44) that the space of uniformly continuous maps from a metrizable uniform space to a uniform ANR is a uniform ANR, and extend this to maps of pairs, which is the nontrivial part. In particular, this shows that the loop space of a compact polyhedron is a uniform ANR (Corollary 4.37(b)).

1.C. Inverse limits

The final section elaborates on a uniform theory of sequential inverse limits. The role of Mardešić resolutions is played by convergent inverse sequences (see Lemma 5.12), thereby reducing much of the hassle to the simple condition of convergence. (It can be viewed as a weakening of the surjectivity of all bonding maps, and specializes to the Mittag-Leffler condition in the case of inverse sequences of discrete spaces.) This enables us to generalize to metrizable uniform spaces virtually all known theory of inverse sequences of compacta. In particular, we establish the analogue of Milnor’s lemma on extension of a map between inverse limits to the infinite mapping telescopes (Theorem 5.12).
5.14 and Corollary 5.16), which amounts to a foundation of strong shape theory (see [62]). Another noteworthy result, whose compact case the author has not seen in the literature, is a characterization of inverse limits that are uniform ANRs in terms of properties of the bonding maps (Theorem 5.18).

1.D. Acknowledgements

I would like to thank T. Banakh, N. Brodskij, A. V. Chernavskij, A. N. Dranishnikov, J. Dydak, O. Frolkina, B. LaBuz, R. Jimenez, J. Higes, J. Krasinkiewicz, S. Nowak, E. V. Shchepin, S. Spież and H. Torunczyk for useful discussions.

2. A REVIEW OF UNIFORM SPACES

This section is intended to serve as an introduction to uniform spaces for the reader who has little to no previous acquaintance with the subject. It appears to be rather different in viewpoint and in style from the existing introductions to uniform spaces in the literature, but mathematically it does not contain anything new. The section attempts to be self-contained modulo straightforwardly verified facts, for which it gives references.

The shorter subsection on metrizable uniform spaces and the denser subsection on finiteness conditions are intended to be read carefully by the reader not familiar with these matters; whereas the more leisurely subsections on general uniform spaces are largely intended to serve as a reference.

2.1. References on uniform spaces. Unsurpassed basic references for uniform spaces are still those to the founders of the subject: Isbell’s book [50], which is well complemented by Chapters II, IX and X of Bourbaki’s General Topology [17] (recall that the original Bourbaki group included A. Weil and J. Dieudonné). See also the historic survey [12]; further surveys exist [41], [55]. Other specialized sources include books by A. Weil (1937; in French), J. W. Tukey (1940), and I. M. James (1990) on uniform spaces.

There are also books by Naimpally–Warrack (1970) and Yefremovich–Tolpygo (2007; in Russian) on the closely related subject of proximity spaces (which coincide with uniform spaces in the metrizable case) and by H. Herrlich (1987; in German) and G. Preuss (1988) on more general notions of nearness spaces. Additional information can be drawn from chapters in some books on analysis and topological algebra: Gillman–Jerison (1960), W. Page (1978), Roelke–Dierolf (1981), N. R. Howes (1995); and from chapters in some books on general topology: R. Engelking [29], Hu Sze-Tsen (1966), J. L. Kelley (1955), H. Schubert (English transl. 1968), S. Willard (1970).

Much of modern development of the theory of uniform spaces seems to occur not within topology but, in particular, in Geometric Nonlinear Functional Analysis (see [13], [54]).
2.A. Metrizable uniform spaces

2.2. Uniform continuity. We recall that a map \( f: M \to N \) between metric spaces is \textit{uniformly continuous} iff any two sequences \( x_i, y_i \in M \) with \( d(x_i, y_i) \to 0 \) as \( i \to \infty \) satisfy \( d(f(x_i), f(y_i)) \to 0 \) as \( i \to \infty \). (This sequential formulation of the familiar \( \varepsilon, \delta \) definition will often be more convenient for our purposes.) Furthermore, it is not hard to see that \( f \) is uniformly continuous iff \( d(A, B) = 0 \) implies \( d(f(A), f(B)) = 0 \) for any \( A, B \subset M \), cf. [50; II.38 and II.34]; here \( d(A, B) = \inf \{d(a, b) \mid a \in A, b \in B\} \).

2.3. Metrizable uniformity. A \textit{uniform homeomorphism} between metric spaces is a bijection that is uniformly continuous in both directions. Two metrics \( d \) and \( d' \) on a set \( S \) are \textit{uniformly equivalent} if \( \text{id}_S \) is a uniform homeomorphism between \( (S, d) \) and \( (S, d') \). In particular, every metric \( d \) is uniformly equivalent to the bounded metrics \( d'(x, y) = \min(d(x, y), 1) \) and \( d''(x, y) = \frac{d(x, y)}{1 + d(x, y)} \). A metrizable \textit{uniformity} (or uniform structure) \( u \) on \( S \) is a uniform equivalence class of metrics on \( S \); each of these metrics \textit{induces} \( u \); and a metrizable \textit{uniform space} is a set endowed with a uniformity. Clearly, the topology induced by a metric \( d \) is determined by the uniformity induced by \( d \). A uniformity \( u \) on a set \( S \) is \textit{finer} than \( u' \) if \( \text{id}_S: (S, u) \to (S, u') \) is uniformly continuous.

\textit{Remark 2.4.} Historically, the idea of a metrizable uniform space emerged together with those of a metric space and a metrizable topological space. Fréchet’s thesis (1906, based on a series of 1904-05 papers), which introduced metric spaces as well as a variant of topological spaces based on limit points of sequences, also studied an axiomatic structure midway between metric and metrizable uniform spaces (see [12; §1.1]). Sets of axioms satisfied by the relation \( d(A, B) = 0 \) between the subsets \( A, B \) of a metric space have been considered by F. Riesz in the same ICM talk (1908, based on a 1906 paper), where he suggested a modification of Fréchet’s approach based on limit points of sets as opposed to countable sequences (see [12; §1.5] and [21]).

2.5. Completeness. We recall that a sequence of points \( x_n \) of a metric space \( M \) is called a \textit{Cauchy sequence} if for every \( \varepsilon > 0 \) there exists a \( k \) such that for every \( j > k \), the \( \varepsilon \)-neighborhood of \( x_j \) in \( X \) contains \( x_k \). Clearly, this notion depends only on the underlying uniform structure of \( M \). Thus a metrizable uniform space is called \textit{complete} if every its Cauchy sequence converges. Every metrizable uniform space is a dense subset of a unique complete one, which is called its \textit{completion}; every uniformly continuous map into a complete space uniquely extends over the completion of the domain (see [50]). Every \textit{compactum}, i.e. a compact metrizable space, admits a unique uniform structure, which is complete. A metrizable uniform space is called \textit{precompact} if its completion is compact. Thus a subspace of a complete metrizable uniform space is precompact iff its closure is compact.

2.6. Covers. We recall that a \textit{cover} (or a covering) of a set \( S \) is a collection of subsets of \( S \) whose union is the whole of \( S \). A cover \( C \) of \( S \) is said to \textit{refine} a cover \( D \) of \( S \) if every \( U \in C \) is a subset of some \( V \in D \). If \( C \) is a cover of \( S \), and \( f: T \to S \) is a map, we have the covers \( f(C) := \{f(U) \mid U \in C\} \) and \( f^{-1}(C) := \{f^{-1}(U) \mid U \in C\} \) of \( T \); in
the case where \( T \subseteq S \) and \( f \) is the inclusion map, we denote \( f^{-1}(C) \) by \( C \cap T \) and call it the trace of \( C \) on \( T \). If \( C \) and \( D \) are covers of \( S \), then \( C \land D := \{ U \cap V \mid U \in C, V \in D \} \) is a cover of \( S \) refining both \( C \) and \( D \). Similarly one defines the meet \( \bigwedge C_\lambda \) of a finite family of covers \( C_\lambda \); note that if the family is empty, its meet is the singleton cover \( \{ X \} \) of \( X \).

If \( T \subseteq S \) is a subset, the star \( \text{st}(T, C) \) of \( T \) in \( C \) is the union of all elements of \( C \) that intersect \( T \). We say that \( C \) star-refines \( D \) if the cover \( \{ \text{st}(\{ x \}, C) \mid x \in S \} \) refines \( D \). Next, \( C \) is said to strongly star-refine \( D \) if \( \{ \text{st}(U, C) \mid U \in C \} \) refines \( D \). It is easy to see that every strong star-refinement is a star-refinement, and every star-refinement of a star-refinement is a strong star-refinement.

2.7. Uniform covers. A cover \( C \) of a metric space \( M \) is said to be uniform if there exists a positive number \( \lambda \) such that every subset of \( M \) of diameter \( < \lambda \) is contained in some \( U \in C \); such a \( \lambda \) is called a Lebesgue number of \( C \). Note that if a cover \( C \) of \( M \) is refined by the cover \( C_2 \) by all balls of radius \( \varepsilon \), then \( C \) is uniform (with Lebesgue number \( \varepsilon \)); and conversely, every uniform cover of \( M \) with Lebesgue number \( \lambda \) is refined by \( C_{\lambda /2} \).

It is easy to see that a map \( f : M \to N \) between metric spaces is uniformly continuous iff for every uniform cover \( D \) of \( N \), the cover \( f^{-1}(D) \) of \( M \) is uniform (equivalently, there exists a uniform cover \( C \) of \( M \) such that \( f(C) \) refines \( D \cap f(M) \)). It follows that the property of being uniform for a cover of \( M \) depends only on the underlying uniform structure of \( M \). Clearly, a cover of a compactum is uniform iff it can be refined by an open cover. A metrizable uniform space \( X \) is precompact iff every uniform cover of \( X \) has a finite uniform refinement (see [50; II.28]).

A family of disjoint subsets \( X_\alpha \subseteq M \) is called uniformly discrete if it constitutes a uniform cover of its union. (In other words, if there exists an \( \varepsilon > 0 \) such that \( d(X_\alpha, X_\beta) > \varepsilon \) whenever \( \alpha \neq \beta \), where \( d(X, Y) = \sup \{ d(x, y) \mid x \in X, y \in Y \} \).) The space \( M \) itself is called uniformly discrete if the collection of its singletons is uniformly discrete. A neighborhood \( U \) of a subset \( S \) of a metrizable uniform space \( X \) is called uniform if it contains the star of \( S \) in some uniform cover of \( X \); or equivalently if \( S \) and \( X \setminus U \) constitute a uniformly discrete collection. The space \( M \) is called uniformly connected if contains no subset that is its own uniform neighborhood.

2.8. Basis of uniformity. Uniform covers can be used to axiomatize the notion of a uniform structure. Let us call a sequence of covers \( C_1, C_2, \ldots \) of a set \( S \) fundamental if it satisfies

1. each \( C_{n+1} \) star-refines \( C_n \);
2. for any distinct points \( x, y \in S \) there exists an \( n \) such that no element of \( C_n \) contains both \( x \) and \( y \).

Two fundamental sequences of covers \( C_n \) and \( D_n \) are equivalent if for each \( n \) there exists an \( m \) such that \( C_m \) refines \( D_n \) and \( D_m \) refines \( C_n \). A basis for a metrizable uniformity \( u \) on \( S \) is a fundamental sequence of covers \( C_n \) such that each \( C_n \) is uniform with respect to \( u \), and every uniform cover \( C \) of \( (S, u) \) is refined by some \( C_n \). Clearly,
every two bases of \( u \) are equivalent; and every fundamental sequence of covers of \( S \) that is equivalent to a basis of \( u \) is itself a basis of \( u \). On the other hand, if \( d \) is a metric on \( S \) inducing \( u \), then the covers \( C_n \) of \( S \) by the balls of radius \( 3^{-n} \) about all points of \( (S,d) \) form a standard basis of \( u \).

**Theorem 2.9** (Alexandroff–Urysohn [3]). There exists a bijection between metrizable uniformities on \( S \) and equivalence classes of fundamental sequences of covers of \( S \), which assigns to a uniformity the equivalence class of any its standard basis.

Of course, the statement in the 1923 paper [3] is in different terms, even though the proof is essentially the same as that given below. It is likely, however, that the authors must have been at least partially aware of this interpretation of their result, since according to Fréchet (1928; cf. [12; p. 585]), they have been thinking of avoiding the use of metric in defining the notion of uniform continuity.

**Proof.** It remains to show that every fundamental sequence of covers \( C_n \) of \( S \) is a basis of some metrizable uniformity. To this end consider an auxiliary ‘pre-distance’ function \( f(x, y) = \inf \{2^{-n} \mid x, y \in U \text{ for some } U \in C_{2n}\} \), and define \( d(x, y) \) to be the infimum of the sums \( f(x_0, x_1) + \cdots + f(x_{n-1}, x_n) \) over all finite chains \( x = x_0, \ldots, x_n = y \) of points of \( S \). Clearly, \( d \) is a pseudo-metric, i.e. it is symmetric, satisfies the triangle axiom and is such that \( d(x, x) = 0 \) for every \( x \in S \). Let \( D_{2n-1} \) be the set of all subsets of \( S \) of \( d \)-diameter at most \( 2^{-n} \). Since \( d(x, y) \leq f(x, y) \), each \( U \in C_{2n} \) also belongs to \( D_{2n-1} \), thus \( C_{2n} \) refines \( D_{2n-1} \). To prove that \( d \) is a metric and that \( \{C_{2n}\} \) is a basis for the uniformity induced by \( d \) is suffices to show that each \( D_{2n+1} \) refines \( C_{2n} \). The latter, in turn, would follow if we prove that \( f(x, y) \leq 2d(x, y) \). Let us show that \( f(x, y) \leq 2[f(x_0, x_1) + \cdots + f(x_{n-1}, x_n)] \) for every finite chain \( x = x_0, \ldots, x_n = y \) of points of \( S \). The case \( n = 1 \) is clear. Let \( \ell_{[i,j]} = f(x_i, x_{i+1}) + \cdots + f(x_{j-1}, x_j) \) whenever \( 0 \leq i < j \leq n \). Consider the maximal \( k \) such that \( \ell_{[0,k]} \leq \frac{1}{2}\ell_{[0,n]} \). Then \( \ell_{[k+1,n]} \leq \frac{1}{2}\ell_{[0,n]} \) as well. On the other hand, by the induction hypothesis, \( f(x_0, x_k) \leq 2\ell_{[0,k]} \) and \( f(x_k, x_n) \leq 2\ell_{[k+1,n]} \). Thus each of the numbers \( f(x_0, x_k), f(x_k, x_{k+1}), f(x_{k+1}, x_n) \) does not exceed \( \ell_{[0,n]} \). If \( m \) is the least integer such that \( 2^{-m} \leq \ell_{[0,n]} \), the pairs \( \{x_0, x_k\}, \{x_k, x_{k+1}\}, \{x_{k+1}, x_n\} \) are contained in some \( U_1, U_2, U_3 \in C_{2m} \). Since \( C_{2m} \) strongly refines \( C_{2m-2} \), we obtain that \( x_0 \) and \( x_n \) belong to some \( V \in C_{2m-2} \). Hence \( f(x_0, x_n) \leq 2^{-m+1} \leq 2\ell_{[0,n]} \), as required.

2.10. Uniform spaces. In some auxiliary constructions (such as quotient uniformities and semi-uniform products of metrizable uniformities) we will have to deal with general
We call a nonempty family of covers \( \{ C_\alpha \} \) of a set \( S \) fundamental if

1. each \( C_\alpha \) and \( C_\beta \) are refined by some \( C_\gamma \);
2. each \( C_\alpha \) is star-refined by some \( C_\beta \); and
3. for any distinct points \( x, y \in S \) there exists an \( \alpha \) such that no element of \( C_\alpha \) contains both \( x \) and \( y \).

A uniformity \( u \) on \( S \) is the set of all covers that are refined by some element of a fundamental family of covers \( \{ C_\alpha \} \). These covers are called uniform, the family \( \{ C_\alpha \} \) is called a basis of \( u \), and the pair \( uS = (S, u) \) is called a uniform space. A map between two uniform spaces \( f : X \to Y \) is called uniformly continuous if for every uniform cover \( C \) of \( Y \), the cover \( f^{-1}(C) \) of \( X \) is uniform.

As far as single (countable) coverings are concerned, all uniform spaces are like (separable) metrizable uniform spaces (to rephrase an expression of Isbell), in the sense that every (countable) uniform cover \( C \) of a uniform space \( X \) is refined by \( f^{-1}(D) \) for some (countable) uniform cover \( D \) of some (separable) metrizable \( Y \) and some uniformly continuous \( f : X \to Y \) [50; I.14], [36; 3.1].

If we drop condition (2), we obtain definitions of a pre-fundamental family of covers and a pre-uniformity (some authors call this a “uniformity”, and refer to one satisfying (2) as a “separated uniformity”). In other words, a pre-uniformity on \( S \) is a family of covers of \( S \) that forms a filter with respect to star-refinement (cf. [50; I.6]). Every pseudo-metric induces a pre-uniformity. Similarly to Theorem 2.9, a pre-uniformity is pseudo-metrizable iff it has a basis that is a pre-fundamental sequence of covers \( C_i \) (i.e. satisfies condition 2.8(1)).

Also similarly to Theorem 2.9, every pre-uniformity has a basis of covers \( C_{i\alpha} \), where each \( C_{i\alpha} \) consists of all balls of radius \( 2^{-i} \) with respect to some pseudo-metric \( d_\alpha \) (cf. [50; proof of I.14], [29; proof of 8.1.10]). This yields a bijective correspondence between pre-uniformities on \( S \) and uniform equivalence classes of collections \( D \) of pseudo-metrics on \( S \) such that (i) for any \( d, d' \in D \) there exists a \( d'' \in D \) with \( d'' \geq \max(d, d') \); uniformities correspond to the equivalence classes of collections \( D \) such that (ii) for each pair of distinct points \( x, y \in S \) there exists a \( d \in D \) such that \( d(x, y) > 0 \) (cf. [29; 8.1.18]). Two such collections \( D \) and \( D' \) are uniformly equivalent if id: \( (X, D) \to (X, D') \) is uniformly continuous in both directions. A definition of uniform continuity in relevant terms is, a map \( f : (X, D) \to (Y, E) \) is uniformly continuous iff for each \( \varepsilon > 0 \) and \( e \in E \) there exists a \( \delta > 0 \) and a \( d \in D \) such that \( d(x, y) \leq \delta \) implies \( e(f(x), f(y)) \leq \varepsilon \). A related criterion is, a function between pre-uniform spaces \( f : X \to Y \) is uniformly continuous iff for each uniformly continuous pseudo-metric \( e \) on \( Y \), the pseudo-metric \( d(x, y) = e(f(x), f(y)) \) is uniformly continuous (cf. [29; 8.1.22]).

If \( X \) is a uniform space and \( S \) is a subset of \( X \), the uniform structure of subspace on \( S \) is given by the covers \( C \cap S \), where \( C \) runs over all uniform covers of \( X \).
If $X$ is a pre-uniform space, its induced topology is defined by declaring a subset $S \subset X$ open iff for each $x \in S$ there exists a uniform cover $C$ of $X$ such that $\text{st}(x, C) \subset S$. In other words, a base of neighborhoods of $x$ is given by the stars of $x$ in basic uniform covers of $X$. Every uniform cover of $X$ is refined by an open cover, since this is obviously so in the metrizable case (cf. [50; I.19]). Thus every pre-uniformity has a basis consisting of open covers.

We refer to [50] for the definition and properties of complete uniform spaces.

Let $\mathcal{U}$ (resp. $\widehat{\mathcal{U}}$) denote the category of (pre-)uniform spaces and uniformly continuous maps, and $\mathcal{T}$ the category of topological spaces and continuous maps — all viewed as concrete categories over the category of sets.

2.11. Initial uniformity. Given a set $X$ and a family $f$ of maps $f_\lambda : X \to Y_\lambda$ into pre-uniform spaces, all finite meets of the form $f_\lambda_1^{-1}(C_1) \land \cdots \land f_\lambda_k^{-1}(C_k)$, where each $C_i$ is a uniform cover of $Y_\lambda$, clearly form a basis of a pre-uniformity $u_f$. Clearly $u_f$ is the coarsest pre-uniformity on $X$ making all the $f_\lambda$ uniformly continuous (cf. [50; I.8]). Moreover, it is easy to see that $u_f$ is initial in $\mathcal{U}$ with respect to $f$; that is, a map $g : Z \to X$, where $Z$ is a pre-uniform space, is uniformly continuous if (and, obviously, only if) each composition $Z \xrightarrow{g} X \xrightarrow{f_\lambda} Y_\lambda$ is uniformly continuous (cf. [50; I.17], where the non-trivial part of the argument is redundant). Conversely, if a pre-uniformity on $X$ is initial with respect to $f$, then it has to be the coarsest pre-uniformity making all the $f_\lambda$ uniformly continuous (cf. [1; 10.43]). Thus we may call $u_f$ the initial pre-uniformity in $\mathcal{U}$ with respect to $f$. The induced topology of $u_f$ is initial in $\mathcal{T}$ with respect to $f$ [50; I.16]. Corresponding to the empty family $\emptyset$ of maps on $X$ we have the anti-discrete pre-uniformity $u_\emptyset = \{\{X\}\}$, which is not a uniformity (cf. [1; 8.3]).

The pre-uniformity $u_f$ is a uniformity (and is initial in $\mathcal{U}$ with respect to $f$), provided that each $Y_\lambda$ is a uniform space, and $f$ is point-separating, i.e. for every pair of distinct points $x, y \in X$ there exists a $\lambda$ such that $f_\lambda(x) \neq f_\lambda(y)$ (cf. [50; I.8 and I.17]).

2.12. Finest uniformity. (Pre-)uniformities on a set $X$ are ordered by inclusion, as subsets of the set of all covers of $X$. Given a family $u$ of pre-uniformities $u_\lambda$ on a set $X$, the initial pre-uniformity $u_f$ corresponding to the family $f$ of maps $f_\lambda : X \to (X, u_\lambda)$ is the least upper bound sup $u$ of the family $u$ (cf. [17; §II.1.5]). If at least one $u_\lambda$ is a uniformity, then so is sup $u$. By the above, sup $u = \{C_1 \land \cdots \land C_k \mid k \in \mathbb{N}, C_i \in \bigcup \lambda u_\lambda\}$.

A cover $C_i$ of a set $X$ is called normal with respect to a family $F$ of covers of $X$, if it can be included in an infinite sequence $C_1; C_2, C_3, \ldots$ of covers of $X$ such that each $C_{i+1}$ star-refines $C_i$, and each $C_i$ is refined by some element of $F$. If $F$ is nonempty and every two elements of $F$ have a common refinement in $F$, then it is easy to see that the family of covers of $X$, normal with respect to $F$, constitutes a pre-uniformity $u_F$ on $X$. Clearly, $u_F$ is the finest among those pre-uniformities $u_\lambda$ that have a basis contained in $F$; that is, $u_F = \text{sup } u$ and $u_F \in u$, where $u$ is the family of all the $u_\lambda$ (cf. [50; I.10]).
2.13. Fine uniformity. The induced topology of every uniformity is Tychonoff (=completely regular Hausdorff=\(T_{3\frac{1}{2}}\)) [50; I.11], and every pre-uniformity whose induced topology is \(T_1\) is clearly a uniformity. Given a Tychonoff topological space \(X\), its topology is initial with respect to the family \(f\) of all continuous maps \(f_\lambda : X \to \mathbb{R}\), and therefore is induced by the uniformity \(u_f\), with respect to the usual uniformity on \(\mathbb{R}\) (cf. [50; I.15]). Once the set of uniformities inducing the given topology on \(X\) is non-empty, there exists a finest such uniformity \(u_X\), consisting of all covers of \(X\) that are normal with respect to the family of all open covers of \(X\) (cf. [50; I.20]). This is the fine uniformity of the Tychonoff topological space \(X\). A map from \((X, u_X)\) into a pre-uniform space is uniformly continuous iff it is continuous, and \(u_X\) is characterized by this property [17; Exer. IX.1.5]; at the same time, a map \((X, u_f) \to \mathbb{R}\) is uniformly continuous iff it is continuous. By [17; Exer. IX.1.5], \(u_X\) corresponds to the family of all pseudo-metrics on \(X\) that are uniformly continuous as functions \(X \times X \to \mathbb{R}\); whereas by [17; Example at the end of §IX.1.2], \(u_f\) corresponds to the family of all pseudo-metrics \(d_\lambda(x,y) = |f_\lambda(x) - f_\lambda(y)|\).

It is well-known that every open cover of a paracompact topological space has an open star-refinement (cf. [29; 5.1.12]) and that every metrizable topological space is paracompact (see [14] or [29]). It follows that the fine uniformity of a metrizable topological space \(X\) consists of all covers that can be refined by open covers. This uniformity is itself almost never metrizable — specifically, it is metrizable if and only if the set \(K\) of non-isolated points of \(X\) is compact, and the complement to any uniform neighborhood of \(K\) is uniformly discrete (see [4], [56], [75]).

2.14. Final pre-uniformity. Given a set \(X\) and a family \(f\) of maps \(f_\lambda : Y_\lambda \to X\) from pre-uniform spaces, the collection \(u_F\) of all covers of \(X\), normal with respect to the family \(F\) of all covers \(C\) of \(X\) such that the cover \(f_\lambda^{-1}(C)\) of \(Y_\lambda\) is uniform for each \(\lambda\), is a pre-uniformity \(u^f\) on \(X\). Clearly \(u^f\) is the finest pre-uniformity on \(X\) making all the \(f_\lambda\) uniformly continuous (cf. [50; Exer. I.7]). Moreover, it is easy to see that \(u^f\) is final in \(\mathcal{U}\) with respect to \(f\); that is, a map \(g : (X, u^f) \to Z\), where \(Z\) is a pre-uniform space, is uniformly continuous if (and, obviously, only if) each composition \(Y_\lambda \xrightarrow{f_\lambda} (X, u^f) \xrightarrow{g} Z\) is uniformly continuous. (To see this, note that if \(D'\) star-refines \(D\), then \(g^{-1}(D')\) star-refines \(g^{-1}(D)\).) Conversely, if a pre-uniformity on \(X\) is final with respect to \(f\), then it is the finest pre-uniformity making each \(f_\lambda\) uniformly continuous (this is the dualization of [1; 10.43]). Thus we may call \(u^f\) the final pre-uniformity in \(\mathcal{U}\) with respect to \(f\). Corresponding to the empty family \(\emptyset\) of maps into \(X\) we have the discrete pre-uniformity \(u^0\), which is a uniformity (cf. [1; 8.1]). We note that the fine uniformity of a topological space \((S, t)\) is nothing but the final uniformity corresponding to the family of inclusions in \(S\) of all compacta in \((S, t)\) considered as uniform spaces (with their unique uniformity).

In general, the question when \(u^f\) is a uniformity is not easy (see [50; Exer. I.7], [36; Theorem 2.2] for partial results). Instead, one has the following construction.

2.15. Uniform space associated to a pre-uniform space. If \(X\) is a pre-uniform space, let \(\sim\) be the separating equivalence relation on \(X\) defined by \(x \sim y\) iff every
uniform cover of \( X \) contains an element \( U \) such that \( x, y \in U \). Equivalently, \( x \sim y \) iff \( d(x, y) = 0 \) for each uniformly continuous pseudo-metric on \( X \). Let \( f: X \to X/\sim \) assign to each point its separating equivalence class. If \( C \) is a uniform cover of \( X \), then \( f(C) \) is normal with respect to \( f \); indeed, \( f^{-1}(f(U)) \subset \text{st}(U, C') \) for every \( U \in C \) and every uniform cover \( C' \) of \( X \) (in particular, this holds with \( C' = C \)), and therefore \( f^{-1}(f(C)) \) refines \( D \) whenever \( C \) strongly star-refines \( D \). It follows that the covers \( f(C) \), where \( C \) runs over all uniform covers of \( X \), form the pre-uniformity \( u^f \) on \( X/\sim \); by construction it is a uniformity. (Cf. \([17; \S II.3.8]\).

2.16. Coarsest pre-uniformity. Given a family \( u \) of pre-uniformities \( u_\lambda \) on a set \( X \), the final pre-uniformity \( u_f \) corresponding to the family \( f \) of maps \( f_\lambda: (X, u_\lambda) \to X \) is the greatest lower bound \( \inf u \) of the family \( u \). Alternatively, \( \inf u = \sup u_* \), where the set \( u_* \) of lower bounds of \( u \) among all pre-uniformities on \( X \) is non-empty as it contains the anti-discrete pre-uniformity \( u_0 \) (cf. \([17; \S II.1.5]\)). Similarly \( \sup u = \inf u^* \), where the set \( u^* \) of upper bounds of \( u \) among all pre-uniformities on \( X \) is non-empty as it contains the discrete uniformity \( u^0 \). By the above, \( \inf u \) consists of all covers, normal with respect to the family \( \bigcap \lambda u_\lambda \).

If \( F \) is a family of covers of a set \( X \) such that every \( C \in F \) is normal with respect to \( F \), then it is easy to see that the family of covers \( C_1 \land \cdots \land C_k \), where \( k \in \mathbb{N} \) and each \( C_i \in F \), is a base of a pre-uniformity \( u^F \) on \( X \). Clearly, \( u^F \) is the coarsest among those pre-uniformities \( u_\lambda \) that contain \( F \); that is, \( u^F = \inf u \) and \( u^F \in u \), where \( u \) is the family of all the \( u_\lambda \) (cf. \([50; 1.9]\)).

2.17. Coarse uniformity. A Tychonoff space \( X \) admits a coarsest uniformity \( u^X \) inducing its topology iff \( X \) is locally compact; when \( X \) is locally compact, \( u^X \) coincides with the uniformity of the subspace of the one-point compactification of \( X \), as well as with the initial uniformity with respect to the family of all continuous maps \( X \to \mathbb{R} \) that vanish on the complement to a compact set \([80; \text{Theorem XIV}] \) (see also \([50; \text{Exer. II.10}] \), \([17; \text{Exer. IX.1.15}] \)). Furthermore, a metrizable space \( X \) admits a coarsest metrizable uniformity inducing its topology iff \( X \) is locally compact and separable \([81; \text{Corollary to \textbf{Theorem 1}}]\).

2.18. Product. The product \( \prod X_\lambda \) of uniform spaces \( X_\lambda \) is their set-theoretic product \( X \) endowed with the initial uniformity with respect to the family of projections \( \pi_\lambda: X \to X_\lambda \) (cf. \([1; 10.53]\)). Thus the induced topology of \( \prod X_\lambda \) is the product topology, and a cover of \( \prod X_\lambda \) is uniform iff it is refined by \( \pi_X^{-1}(C_1) \land \cdots \land \pi_X^{-1}(C_k) \) for some uniform covers \( C_1, \ldots, C_k \) of some finite subcollection \( X_{\lambda_1}, \ldots, X_{\lambda_k} \). It is easy to check that \( (X, \pi_X) \) is also the product of \( X_\lambda \)'s in \( \mathcal{U} \) in the sense of abstract category theory (see \([50; \text{p. 14}]\)).

If \( X \) and \( Y \) are metrizable uniform spaces and \( C_n, D_n, n = 1, 2, \ldots \) are bases of their uniform covers, then clearly \( E_n := \pi_X^{-1}(C_n) \land \pi_X^{-1}(D_n) \) form a basis of uniform covers of \( X \times Y \), where \( \pi_X \) and \( \pi_Y \) denote the projections. It follows that given metrics, denoted \( d \), on \( X \) and \( Y \), then a metric on \( X \times Y \) is given by \( d_{E_n}(x, y, (x', y')) = \max\{d(x, x'), d(y, y')\} \). Since \( a + b \geq \max\{a, b\} \geq \frac{1}{2}(a + b) \) whenever \( a, b \geq 0 \), it is uniformly equivalent to \( d_{t_1} \), where \( d_{t_1}((x, y), (x', y')) = d(x, x') + d(y, y') \). Since
2.19. Disjoint union. The disjoint union \( \bigsqcup X_\lambda \) of uniform spaces \( X_\lambda \) is their set-theoretic disjoint union \( X \) endowed with the final pre-uniformity with respect to the injections \( \iota_\lambda: X_\lambda \to X \) (cf. [1; 10.67(2)]). This pre-uniformity is obviously a uniformity (cf. [50; Exer. I.7(i)]). A cover \( C \) of \( \bigsqcup X_\lambda \) is uniform iff \( \iota_\lambda^{-1}(C) \) is uniform for each \( \lambda \); indeed, every cover \( C \) satisfying the latter condition is star-refined by another such cover \( \bigsqcup \iota_\lambda(C_\lambda) \), where each \( C_\lambda \) star-refines \( \iota_\lambda^{-1}(C) \). It is easy to check that \( (X, \iota_\lambda) \) is the coproduct of \( X_\lambda \)’s in \( \mathcal{U} \) in the sense of abstract category theory (see [50; p. 14]), and that its underlying topology is the topology of disjoint union (see [50; II.8]).

We note that infinite disjoint unions of metrizable uniform spaces normally fail to be metrizable. Finite disjoint unions preserve metrizability. Indeed, if \( X \) and \( Y \) are metric spaces of diameter \( \leq 1 \), we can extend their metrics to a metric on the set-theoretic disjoint union of \( X \) and \( Y \) by \( d(x, y) = 1 \) whenever \( x \in X \), \( y \in Y \); clearly, it induces the uniformity of the disjoint union.

A metrizable replacement of the countable disjoint union \( \bigsqcup X_i \) is the inverse limit of the finite disjoint unions \( Y_i := (X_1 \sqcup \cdots \sqcup X_i) \sqcup \mathbb{N} \) and the maps \( f_i: Y_{i+1} \to Y_i \) defined by \( f_i|_{X_j} = \text{id} \) for \( j \leq i \), \( f_i(X_{i+1}) = 0 \) and \( f_i(i) = i - 1 \) for \( i > 0 \).

2.20. Mono- and epimorphisms. Monomorphisms in \( \mathcal{U} \) are injective \( \mathcal{U} \)-morphisms [50; II.4]; in other words, a \( \mathcal{U} \)-morphism \( f: X \to Y \) is injective iff \( \mathcal{U} \)-morphisms \( g, h: Z \to X \) coincide whenever their compositions with \( f \) coincide. Epimorphisms in \( \mathcal{U} \) are \( \mathcal{U} \)-morphisms with dense image (cf. [50; p. 15 and I.13]); in other words, a \( \mathcal{U} \)-morphism \( f: X \to Y \) has dense image iff \( \mathcal{U} \)-morphisms \( g, h: Y \to Z \) coincide whenever their pre-compositions with \( f \) coincide.

It is not hard to see that monomorphisms in \( \mathcal{U} \) are again injective \( \mathcal{U} \)-morphisms (cf. [1; 7.38]); whereas epimorphisms in \( \mathcal{U} \) are surjective \( \mathcal{U} \)-morphisms (cf. [1; 7.45 for \( \subseteq \); 21.13(1) and 21.8(1) for \( \subset \)).

More generally, a family \( f \) of uniformly continuous maps \( f_\lambda: X \to Y_\lambda \) is a mono-source in \( \mathcal{U} \) iff it is point-separating; the former means that two maps \( g, h: Z \to X \) coincide whenever the compositions \( Z \xrightarrow{g} X \xrightarrow{f_\lambda} Y_\lambda \) and \( Z \xrightarrow{h} Y \xrightarrow{f_\lambda} Y_\lambda \) coincide for each \( \lambda \) [1; 10.8]. Dually, a family \( f \) of uniformly continuous maps \( f_\lambda: Y_\lambda \to X \) is an epi-sink in \( \mathcal{U} \) iff it is jointly surjective (i.e. \( \bigcup \lambda f(Y_\lambda) = X \)); the former means that two maps

\[ \max\{a, b\}^2 \leq a^2 + b^2 \leq (a + b)^2, \] these metrics are also uniformly equivalent to \( d_{ij} \), where \( d_{ij}((x, y), (x', y'))^2 = d(x, x')^2 + d(y, y')^2 \).

Now let \( X_1, X_2, \ldots \) be metrizable uniform spaces and for each \( i \) let \( C_n^{(i)} \), \( n = 1, 2, \ldots \) be a basis of uniform covers of \( X_i \). Then a basis of uniform covers of \( \prod X_i \) is given by

\[ D_n := \pi_1^{-1}(C_n^{(1)}) \land \pi_2^{-1}(C_n^{(2)}) \land \cdots \land \pi_n^{-1}(C_n^{(n)}) \],

which is indeed a fundamental sequence. It follows that if \( d_i \) is a metric on \( X_i \) such that \( X_i \) is of diameter at most 1, the uniformity on \( \prod X_i \) is induced by the \( l_1 \) metric \( d((x_i), (y_i)) = \sum_i 2^{-i}d_i(x_i, y_i) \) and by the \( l_\infty \) metric \( d((x_i), (y_i)) = \sup\{2^{-i}d_i(x_i, y_i)\} \).
Proposition 2.21. (a) Given a family of uniformly continuous maps \( f_\lambda : X \to Y_\lambda \) between pre-uniform spaces, there exist a uniformly continuous surjection \( h : Z \to Y \) and a point-separating family \( g \) of uniformly continuous maps \( g_\lambda : Z \to Y_\lambda \), where \( Z \) has initial uniformity with respect to \( g \), such that each \( f_\lambda = g_\lambda h \). Such a factorization is unique up to uniform homeomorphism.

(b) Given a family of uniformly continuous maps \( f_\lambda : Y_\lambda \to X \) between pre-uniform spaces, there exist a uniformly continuous injection \( h : Z \to Y \) and a jointly surjective family \( g \) of uniformly continuous maps \( g_\lambda : Y_\lambda \to Z \), where \( Z \) has final uniformity with respect to \( g \), such that each \( f_\lambda = h g_\lambda \). Such a factorization is unique up to uniform homeomorphism.

2.22. Embedding. A uniformly continuous map \( f : A \to X \) of uniform spaces is called an embedding if it is injective, and the uniformity on \( A \) is initial with respect to \( f \) (cf. [1, 8.6]). Thus if \( f : A \to X \) is an embedding, a basis of the uniformity of \( A \) is given by the covers \( f^{-1}(C) \), where \( C \) runs over all uniform covers of \( X \). In fact, all uniform covers of \( A \) are of this form; for if \( D \) is refined by \( f^{-1}(C) \), then the cover \( E := \{U \cup f(V) \mid V \in D, U \in C, f^{-1}(U) \subset V\} \) is refined by \( C \) and satisfies \( f^{-1}(E) = D \) since \( f^{-1}(f(V)) = V \) due to the injectivity of \( f \). Thus an injective map between uniform spaces is an embedding iff it is a uniform homeomorphism onto its image with the subspace uniformity.

Composition of embeddings is an embedding; and if the composition \( X \xrightarrow{f} Y \to Z \) is an embedding, then so is \( f \) [1, 8.9]. By Proposition 2.21(a), every uniformly continuous map between pre-uniform spaces is a composition of a uniformly continuous surjection and an embedding; this factorization is unique up to uniform homeomorphism (cf. [50, II.5]).

2.23. Extremal and regular monomorphisms. Embeddings coincide with extremal monomorphisms in \( \mathcal{U} \) and also with regular monomorphisms in \( \mathcal{U} \) [1, 21.13(4) and 21.8(1)]. A \( \mathcal{U} \)-monomorphism (i.e. a uniformly continuous injection) \( f : A \to X \) is extremal in \( \mathcal{U} \) if, once \( f \) factors in \( \mathcal{U} \) through a \( \mathcal{U} \)-epimorphism (i.e. a uniformly continuous surjection) \( g : A \to B \), this \( g \) must be a uniform homeomorphism. A uniformly continuous map \( f : A \to X \) is a regular monomorphism in \( \mathcal{U} \) if there exist \( \mathcal{U} \)-morphisms \( g, h : X \to Y \) such that \( f \) is their equalizer; that is, \( gf = hf \), and any \( \mathcal{U} \)-morphism \( f' : B \to X \) satisfying \( gf' = hf' \) uniquely factors through \( f \) in \( \mathcal{U} \).

It is easy to see that extremal monomorphisms in \( \mathcal{U} \) coincide with embeddings onto closed subspaces. To see that regular monomorphisms in \( \mathcal{U} \) coincide with embeddings onto closed subspaces, note that if \( A \) is a closed subspace of a uniform space \( X \), then for each \( x \in X \setminus A \) there exists a uniformly continuous map \( g_x : X \to [0, \infty) \) such that
\(g_x(A) = \{0\}\) and \(g_x(x) \neq 0\) [50; I.13]; consequently, the uniformly continuous map
\(\prod g_x: X \to \prod_{x \notin A}[0, \infty)\) satisfies \((\prod g_x)^{-1}(0) = A\).

It follows that the pullback of an embedding is an embedding [1; II.18].

2.24. Quotient. A uniformly continuous map \(f: X \to Q\) of pre-uniform spaces is called a quotient map if it is surjective, and the pre-uniformity on \(Q\) is final with respect to \(f\) (cf. [1; 8.10]). This quotient pre-uniformity therefore consists of all covers \(C_1\) of \(Y\) such that \(C_1\) admits a star-refinement \(C_2\), which in turn admits a star-refinement \(C_3\), etc., so that \(f^{-1}(C_i)\) is uniform for each \(i\). Note that the uniform space associated to a pre-uniform space (see 2.15 above) is its quotient. On the other hand, if \(X\) is a uniform space, then the quotient pre-uniformity does not need to be a uniformity in general. For instance, if \(f\) has a non-closed point-inverse, then no uniformity on \(Q\) can make \(f\) uniformly continuous. See [36; Theorem 2.2] for a characterization of quotients of uniform spaces whose pre-uniformity is a uniformity. Given a uniform space \(X\) and an equivalence relation \(R\) on the underlying set of \(X\), the quotient \(X/R\) is the set of equivalence classes of \(R\) endowed with the quotient pre-uniformity.

Composition of quotient maps is a quotient map; and if a composition \(X \to Y \xrightarrow{f} Z\) is a quotient map, then so is \(f\) [1; 8.13]. Every uniformly continuous retraction is a quotient map [1; 8.12(2)]. By Proposition 2.21(b), every uniformly continuous map between pre-uniform spaces is a composition of a quotient map and a uniformly continuous injection; by the above, the latter is in turn a composition of a uniformly continuous bijection and an embedding; this factorization into three maps is unique up to uniform homeomorphism (cf. [50; II.5]).

2.25. Extremal and regular epimorphisms. Quotient maps coincide with extremal epimorphisms in \(\bar{U}\) and also with regular epimorphisms in \(\bar{U}\) [1; 21.13(5) and 21.8(1)]. A \(\bar{U}\)-epimorphism (i.e. a uniformly continuous surjection) \(f: X \to Q\) is extremal in \(\bar{U}\) if, once \(f\) factors in \(\bar{U}\) through a \(\bar{U}\)-monomorphism (i.e. an uniformly continuous injection) \(p: R \to Q\), this \(p\) must be a uniform homeomorphism. A uniformly continuous map \(f: X \to Q\) is a regular epimorphism in \(\bar{U}\) if there exist \(\bar{U}\)-morphisms \(g, h: Y \to X\) such that \(f\) is their coequalizer; that is, \(fg = fh\), and any \(\bar{U}\)-morphism \(f': B \to X\) satisfying \(f'g = f'h\) uniquely factors through \(f\) in \(\bar{U}\).

Extremal epimorphisms in \(\bar{U}\) coincide again with quotient maps, for they coincide with extremal epimorphisms in \(\bar{U}\) as long as they are surjective — which they have be due to the second condition in the definition of an extremal epimorphism. To see that regular epimorphisms in \(\bar{U}\) coincide with quotient maps, note that a quotient map \(q: X \to Q\) is the coequalizer of the projections of the subspace \(\{(x, y) \mid q(x) = q(y)\}\) of \(X \times X\) onto the factors; and the coequalizer of a pair of maps from a pre-uniform space \(Y\) into \(X\) equals the coequalizer of the resulting maps from the uniform space associated to \(X\).

It follows that the pushout of a quotient map is a quotient map (dually to [1; II.18]).

2.26. Extremal mono-sources and epi-sinks. The uniqueness part of Proposition 2.21 implies the following. A family \(f\) of \(\bar{U}\)-morphisms \(f_\lambda: X \to Y_\lambda\) is an extremal
mono-source in $\mathcal{U}$ iff it is point-separating, and $X$ has the initial uniformity with respect to $f$. The extremality means that once $f$ factors in $\mathcal{U}$ through a $\mathcal{U}$-epimorphism (i.e. a uniformly continuous surjection) $g: X \to Z$, this $g$ must be a uniform homeomorphism. Dually, a family $f$ of $\mathcal{U}$-morphisms $f_\lambda: Y_\lambda \to X$ is an extremal epi-sink in $\mathcal{U}$ iff it is jointly surjective, and $X$ has the final uniformity with respect to $f$. Here the extremality means that once $f$ factors in $\mathcal{U}$ through a $\mathcal{U}$-monomorphism (i.e. an uniformly continuous injection) $p: R \to Q$, this $p$ must be a uniform homeomorphism.

Combining the above with [1; 10.26(2)] and its dual, we obtain

**Corollary 2.27.** (a) Let $f$ be a family of $\mathcal{U}$-morphisms $f_\lambda: X \to Y_\lambda$. Then $\prod f: X \to \prod Y_\lambda$ is an embedding if and only if $f$ is point-separating and $X$ has the initial uniformity with respect to $f$.

(b) Let $f$ be a family of $\mathcal{U}$-morphisms $f_\lambda: Y_\lambda \to X$. Then $\bigsqcup f: \bigsqcup Y_\lambda \to X$ is a quotient map if and only if $f$ is jointly surjective and $X$ has the final uniformity with respect to $f$.

From [1; 10.35(5)] we deduce

**Proposition 2.28.** $f_\lambda: X_\lambda \to Y_\lambda$ be a family of uniformly continuous maps between pre-uniform spaces.

(a) If each $f_\lambda$ is an embedding, then so is $\prod f_\lambda: \prod Y_\lambda \to \prod Y_\lambda$.

(b) If each $f_\lambda$ is a quotient map, then so is $\bigsqcup f_\lambda: \bigsqcup X_\lambda \to \bigsqcup X_\lambda$.

From [1; 27.15] (see also [1; 28.14]) and its dual we deduce

**Proposition 2.29.** (b) If $g: A \to X$ is an embedding and $Y$ is a uniform space, then $g \sqcup \text{id}_Y: A \sqcup Y \to X \sqcup Y$ is an embedding. In particular, $\iota_2: Y \to X \sqcup Y$ is an embedding.

(b) If $q: X \to Q$ is a quotient map and $Y$ is a uniform space, then $q \times \text{id}_Y: X \times Y \to Q \times Y$ is a quotient map. In particular, $\pi_2: X \times Y \to Y$ is a quotient map.

In fact, a product of quotient maps is a quotient map [47; Theorem 1] (for the case of a finite product see also [50; Exer. III.8(c)]). A product of two sequential direct limits is the sequential direct limit of the products [8] (which is not the case for topological spaces).

**2.30. Graph.** If $f: X \to Y$ is a possibly discontinuous map between uniform spaces, its graph $\Gamma_f$ is the subspace $\{(x, f(x)) \mid x \in X\}$ of $X \times Y$. If $f$ is (uniformly) continuous, $\Gamma_f$ is (uniformly) homeomorphic to $X$, via the composition $\Gamma_f \to X \times Y \to X$ of the inclusion and the projection, whose inverse is given by $X \xrightarrow{\text{id}_X \times f} X \times Y$.

In particular, every continuous map $f: X \to Y$ between uniform spaces is the composition of the homeomorphism $X \to \Gamma_f$ and the uniformly continuous map $\Gamma_f \to X \times Y \to X$. Similarly, every uniformly continuous map between metric spaces is a composition of a uniform homeomorphism and a 1-Lipschitz map.
2.D. Function spaces

If $X$ and $Y$ are uniform spaces, and $C$ is a uniform cover of $Y$, let $U(X,C)$ be the cover of the set $U(X,Y)$ of uniformly continuous maps $X \to Y$ by the sets $O_f := \{ g \mid \forall x \in X \exists U \subseteq C f(x), g(x) \in U \}$ for all $f \in U(X,Y)$. One can check that the covers $U(X,C)$ form a base of a uniform structure on $U(X,Y)$, which therefore becomes a uniform space.

If $Y$ is metrizable, then so is $U(X,Y)$; specifically, if $d$ is a bounded metric on $Y$, then $d(f,g) = \sup_{x \in X} d(f(x),g(x))$ is a bounded metric on $U(X,Y)$.

If $Y$ is complete, then so is $U(X,Y)$ (cf. [50; III.31]). If $X$ is compact and $Y$ is separable metrizable, then $U(X,Y)$ is separable (see [29; 4.2.18]).

2.31. Absolute extensors. We call a uniform space $Y$ an A[N]EU if for every (not necessarily metrizable) uniform space $X$ and every (not necessarily closed) subspace $A \subseteq X$, every uniformly continuous map $A \to X$ extends to a uniformly continuous map $X \to Y$ [respectively, $N \to Y$, where $N$ is a uniform neighborhood of $A$ in $X$]. AEU's are also known as “injective spaces” [50] since they are the injective objects of the category of uniform spaces and uniformly continuous maps.

By Katetov's theorem (see [50; III.9] or [34] or [53] or [6]; see also [89], [66]), the unit interval $I$ is an AEU. Isbell’s finite-dimensional uniform polyhedra are known to be AEU's [48; 1.9], [50; V.15]. (Infinite-dimensional uniform polyhedra will be treated in the sequel to this paper.)

Every AEU is complete, using, inter alia, that the only uniform neighborhood of $X$ in its completion $\bar{X}$ is the entire $X$ (see [50; V.14], [48; I.7], [50; III.8]).

**Theorem 2.32.** [50; III.14] If $D$ is a discrete uniform space and $Y$ is an A[N]EU, then $U(D,Y)$ is an A[N]EU.

In particular, $U(D,I)$ is an AEU.

**Proof.** If $Z$ a uniform space, it is easy to see (cf. [50; III.13]) that a map $f: Z \to U(D,Y)$ is uniformly continuous if and only if $F: Z \times D \to Y$, defined by $F(z,m) = f(z)(m)$, is uniformly continuous. But since $X$ is an A[N]EU, every uniformly continuous map $A \times D \to Y$ extends to a uniformly continuous map $Z \times D \to Y$ [respectively, $N \times D \to Y$, where $N$ is a uniform neighborhood of $A$ in $Z$, for every $A \subseteq Z$].

We note that $U(\mathbb{N},I)$ is inseparable, where $\mathbb{N}$ denotes the infinite countable discrete space and $I = [0,1]$. On the other hand, $q_0 := U((\mathbb{N}^+,\infty),(I,0))$ is separable, where $+$ stands for the one-point compactification. (We recall that the functional space $U((\mathbb{N}^+,\infty),(\mathbb{R},0))$ is known as $c_0$.)

**Corollary 2.33.** $q_0$ is an AEU.

**Proof.** (Compare [13; Example 1.5].) We define $r: U(\mathbb{N},I) \to q_0$ by $(r(x))_n = 0$, if $x_n < d(x,q_0)$, and $(r(x))_n = x_n - d(x,q_0)$ if $x_n \geq d(x,q_0)$, where $d(x,q_0) = \limsup x_n$. Clearly, $r$ is a uniformly continuous retraction. Since $U(\mathbb{N},I)$ is an AEU, we infer that so is $q_0$. □
2.34. Semi-uniform product. A subset \( S \subset U(X,Y) \) is called \textit{uniformly equicontiuous} (or equiuniformly continuous) if for each uniform cover \( D \) of \( Y \) there exists a uniform cover \( C \) of \( X \) such that for each \( f \in S \), the cover \( f^{-1}(D) \) is refined by \( C \) (equivalently, \( f(C) \) refines \( D \cap f(X) \)).

Given a map \( f : Z \to U(X,Y) \), consider \( F : Z \times X \to Y \), defined by \( F(z,x) = f(z)(x) \). The following are equivalent \([50; \text{III.21, III.22, III.26}]:\)

- \( f \) is uniformly continuous and its image is uniformly equicontinuous;
- \( F \) is uniformly continuous;
- \( F(z,*) : X \to Y \) is a uniformly equicontinuous family and \( F(*,x) : Z \to Y \) is a uniformly equicontinuous family.

The \textit{semi-uniform product} \( Z \times X \) of a uniform space \( Z \) and a metrizable uniform space \( X \) is the set \( Z \times X \) endowed with the uniformity with basis consisting of covers \( \{ U_i \times V_j \} \), where \( \{ U_i \} \) is a uniform cover of \( Z \) and for each \( i \), \( \{ V_j \} \) is a uniform cover of \( X \) (see \([49; \text{2.2 and 2.5}], [50; \text{III.23 and III.28}]\)). Note that \( Z \times X \) does not have to be metrizable if \( Z \) and \( X \) are. \( Z \times X \) is uniformly homeomorphic to \( Z \times X \) when \( Z \) is compact or \( X \) is discrete \([50; \text{III.24}]\) and is always homeomorphic to \( Z \times X \) \([50; \text{III.22}]\).

Given a map \( f : Z \to U(X,Y) \), consider \( \Phi : Z \times X \to Y \), defined by \( \Phi(z,x) = f(z)(x) \). The following are equivalent \([50; \text{III.22, III.26}]:\)

- \( f \) is uniformly continuous;
- \( \Phi \) is uniformly continuous;
- \( \Phi(z,*): X \to Y \) is uniformly continuous for each \( z \in Z \), and \( \Phi(*,x): Z \to Y \) is a uniformly equicontinuous family.

\textit{Remark 2.35.} Note that the uniform continuity of \( \Phi(z,*) : X \to Y \) for each \( z \in Z \) and of \( \Phi(*,x) : Z \to Y \) for each \( x \in X \) is equivalent to the uniform continuity of \( f : Z \to U_{pw}(X,Y) \), where \( U_{pw} \) is the space of uniformly continuous maps with the uniformity of pointwise convergence; the corresponding uniformity \( X \bowtie Y \) on the product \( X \times Y \) need not be pre-compact even if both \( X \) and \( Y \) are compact \([50; \text{Exer. III.7}]\). Another symmetric version \( \sup(u_{XY}, u_{YX}) \) of the uniformity \( u_{XY} \) of \( X \bowtie Y \) is studied in \([10]\).

\textit{Theorem 2.36.} \([49; \text{2.6}], [50; \text{III.25}]\) \textit{If \( X \) is a metrizable uniform space and \( Y \) is an A[N]/EU, then \( U(X,Y) \) is an A[N]/EU.}\n
We note that the proof essentially involves non-metrizable spaces when \( Y \) is metrizable. A rather technical proof, not involving non-metrizable spaces, of the case \( Y = I \) is found in \([50; \text{III.18}]\).

\textit{Proof.} If \( X \) is an AEU, every uniformly continuous map \( A \times X \to Y \) extends to a uniformly continuous map \( Z \times X \to Y \), for every \( A \subset Z \).

The ANEU case is similar, using additionally that every uniform neighborhood of \( C \times X \) in \( Z \times X \) contains \( U \times X \) for some uniform neighborhood \( U \) of \( C \) in \( Z \). \( \square \)

2.37. Hyperspace. If \( M \) is a metric space, the space \( H(M) \) of all nonempty closed subsets of \( M \) is endowed with the Hausdorff metric \( d'_H = \min(d_H,1) \), where \( d_H(A,B) = \)}
max\{\sup\{d(a, B) \mid a \in A\}, \sup\{d(A, b) \mid b \in B\}\}. If \( X \) is a metrizable uniform space, the induced uniform structure of its hyperspace \( H(X) \) is well-defined, and if \( Y \) is complete or compact, so is \( H(X) \) \cite[II.48, II.49]. One can show that \( d_H(A, B) \) equals the supremum of \( |d(x, A) - d(x, B)| \) over all \( x \in X \); this identifies \( H(X) \) with a subspace of \( U(X, [0,1]) \) (see \cite{11}).

2.E. Finiteness properties of coverings

It is not hard to see that every uniform cover of a separable uniform space has a countable uniform refinement (cf. \cite[2.3]).

2.38. Dimension. A cover \( C \) of a set \( S \) is said to be of multiplicity \( \leq m \) if every \( x \in X \) belongs to at most \( m \) elements of \( C \). We say that a uniform space \( X \) is of dimension \( \leq d \) if every uniform cover of \( X \) is refined by a uniform cover of multiplicity \( \leq d + 1 \). (Isbell calls this the “large dimension” of \( X \); however, when finite, it coincides with what he calls the “uniform dimension” \cite[Theorem V.5].) We call \( X \) residually finite-dimensional if every uniform cover of \( X \) is refined by a uniform cover of finite multiplicity (in the literature these are also known as “finistic” or “distal” uniform spaces). Note that every compactum is residually finite-dimensional.

Lemma 2.39. The completion of a residually finite dimensional space is residually finite dimensional.

Proof. This is similar to \cite[IV.23] for the reader’s convenience we include a direct proof. Let \( C \) be a uniform cover of \( X \) with a Lebesgue number \( 3\lambda \). Then every subset of \( X \) of diameter \( \lambda \) has its \( \lambda \)-neighborhood contained in some \( U \in C \), and therefore is itself contained in \( U' := \{x \in X \mid d(x, X \setminus U) > \lambda\} \). Thus \( D := \{U' \mid U \in C\} \) is a uniform cover of \( X \). The cover \( \bar{D} \) of \( \bar{X} \) by the closures of the elements of \( D \) is uniform \cite[II.9], and obviously its multiplicity does not exceed that of \( C \). \( \square \)

2.40. Point-finite, star-finite and Noetherian spaces. We recall that a cover \( \{U_\alpha\} \) of a set \( X \) is called point-finite if each \( x \in X \) belongs to only finitely many \( U_\alpha \)’s. Next, \( \{U_\alpha\} \) is called star-finite if each \( U_\beta \) meets only finitely many \( U_\alpha \)’s. Following \cite[§7]{42}, we call \( \{U_\alpha\} \) Noetherian if there exists no infinite sequence \( U_{\alpha_1}, U_{\alpha_2}, \ldots \) such that \( U_{\alpha_1} \cap \cdots \cap U_{\alpha_n} \) is nonempty for each \( n \in \mathbb{N} \). The following implications are straightforward.

\[
\begin{align*}
\text{star-finite} & \quad \text{finite multiplicity} \\
\Downarrow & \Downarrow \\
\text{Noetherian} & \Downarrow \\
\text{point-finite}
\end{align*}
\]

It is well-known that every metrizable space is paracompact so in particular weakly paracompact, i.e. every its open cover admits an open point-finite refinement; and that every separable metrizable space is strongly paracompact, i.e. every its open cover admits
an open star-finite refinement (see [14] or [29]). It is asserted in [42; §7] that every open cover of every paracompact space admits an open Noetherian refinement.

A uniform space \( X \) is called point-finite (star-finite; Noetherian) if every uniform cover of \( X \) has a point-finite (star-finite; Noetherian) uniform refinement. We caution the reader that in the literature, “uniform paracompactness” and its variations refer to a completely different menagerie of properties, involving arbitrary open coverings besides uniform coverings.

**Proposition 2.41.** A uniform cover of every \( d \)-dimensional (resp. of every residually finite-dimensional) star-finite uniform space has a star-finite uniform refinement of multiplicity \( \leq d + 1 \) (resp. of finite multiplicity).

For a deeper study of compatibility of properties of covers see [36; §1].

**Proof.** Let \( C \) be the given uniform cover of the given space \( X \). Then \( C \) has a star-finite uniform refinement \( D \), which in turn has a uniform refinement \( E \) of multiplicity \( \leq d + 1 \) (finite multiplicity). Thus each \( U \in E \) lies in some \( V = f(U) \in D \). Then \( \bigcup f^{-1}(V) \mid V \in D \) is a cover of \( X \) with the desired properties. \( \square \)

It has been a long-standing open problem of Stone (1960), reiterated in [50; Research Problem B3], whether every metrizable uniform space (or every uniform space indeed) is point-finite. It has been resolved in the negative by J. Pelant and (independently) E. V. Shchepin (1975; see references in [42]); only recently it has been shown that \( U(N, I) \) fails to be point-finite [42].

In contrast, the separable space \( c_0 := U((N^+, \infty), (\mathbb{R}, 0)) \) is point-finite, as observed in [70; 2.3]; here \( N^+ = N \cup \{\infty\} \) is the one-point compactification of the countable discrete uniform space \( N \). Indeed, for each \( \varepsilon > 0 \), the covering of \( c_0 \) by all balls of radius \( \frac{2}{3}\varepsilon \) centered at points of the lattice \( U((N^+, \infty), (\varepsilon\mathbb{Z}, 0)) \) is uniform (with Lebesgue number \( \frac{1}{3}\varepsilon \)) and point-finite.

**Theorem 2.42 ([88]).** Every separable metrizable uniform space is point-finite.

**Proof.** Let \( D \) be the given uniform cover of the given space \( X \), and let \( C = \{U_i\} \) be a countable strong star-refinement of a strong star-refinement of \( D \). Let us consider \( W_n = \text{st}(U_n, C) \setminus \text{st}(U_1, C) \cup \cdots \cup \text{st}(U_{n-1}, C) \) and let \( V_n = \text{st}(W_n, C) \). Since \( \{W_i\} \) is a cover of \( X \), \( \{V_i\} \) is a uniform cover of \( X \). Since \( V_i \subset \text{st}(U_i, C) \), \( \{V_i\} \) refines \( D \). Let us prove that each \( U_i \) meets only finitely many \( V_n \)'s. If \( U_i \cap \text{st}(W_n, C) \neq \emptyset \), then \( \text{st}(U_i, C) \cap W_n \neq \emptyset \), hence \( i \leq n \) by the construction of \( W_n \). \( \square \)

Theorem 2.42 also follows from Aharoni’s theorem: every separable metric space admits a Lipschitz (hence uniform) embedding in \( c_0 \) (see [70; Theorem 3.1]). We find it easier, however, to prove 2.42 directly and then use it to give a short proof (following [70; 2.1]) of the relevant part of Aharoni’s theorem.

**Theorem 2.43** (Aharoni; see [70]). Each separable metrizable uniform space uniformly embeds into the function space \( q_0 = U((N^+, \infty), ([0, 1], 0)) \), where \( N^+ \) is the one-point compactification of the countable discrete uniform space \( N \).
Proof. Let $M$ be the given space with some fixed metric. Let $C_n$ be the covering of $M$ by balls of radius $2^{-n}$. By 2.42 it has a countable uniform point-finite refinement $D_n$. Let $\lambda_n$ be a Lebesgue number of $D_n$, and suppose that $D_n = \{V_{n1}, V_{n2}, \ldots \}$. Define $f: M \times \mathbb{N} \times \mathbb{N} \to I$ by $f(x, n, i) = \min\{d(x, M \setminus V_{ni}), \lambda_n\}$, and extend $f$ over $M \times (\mathbb{N} \times \mathbb{N})^+$ by $f(x, \infty) = 0$.

Since $D_n$ is point-finite, each restriction $f|_{M \times \{n\} \times \mathbb{N}}$ has support in $M \times \{n\} \times S_n$ for some finite set $S_n \subset \mathbb{N}$. Furthermore, $f(M \times \{n\} \times S_n) \subset [0, 2^{-n}]$ since $D_n$ refines $C_n$. Hence $f(x, *, *) : (\mathbb{N} \times \mathbb{N})^+ \to I$ is uniformly continuous for each $x \in M$, and $f(\ast, n, i) : M \to I$ plus $f(\ast, \infty)$ is a uniformly equicontinuous family. Thus $f$ determines a uniformly continuous map $F: M \to U((\mathbb{N} \times \mathbb{N})^+, I, 0)$.

If $d(x, y) > 2^{-n+1}$, then no element of $D_n$ contains both $x$ and $y$. Let $V_{ni}$ be an element of $D_n$ containing the $\frac{1}{2}\lambda_n$-neighborhood of $x$. Then $f(x, n, i) > \frac{1}{2}\lambda_n$ and $f(y, n, i) = 0$. Hence $d(F(x), F(y)) \leq \frac{1}{2}\lambda_n$ implies $d(x, y) \leq 2^{-n+1}$. Thus $F$ is injective and $F^{-1}$ is uniformly continuous. \qed

Remark 2.44. Theorem 2.43 should be compared with the following results.

(i) — easy (see [50; II.19, II.21], [13; 1.1(i)]) Every metric space $M$ of diameter $\leq 1$ isometrically embeds in $U(dM, I)$, where $dM$ is the set $M$ endowed with discrete uniformity. Consequently, every uniform space uniformly embeds in a product of spaces of the form $U(D, I)$, where $D$ is discrete.

(ii) — hard (Banach–Mazur; see [29; 4.5.21], [14; Prop. II.1.5]) Every separable metric space of diameter $\leq 1$ isometrically embeds in $U(I, I)$.

(iii) — (Menger–Nöbeling–Pontryagin) Every compactum topologically (hence, uniformly) embeds in the Hilbert cube. (Note that the Hilbert cube is an AEU, being a product of AEU’s.)

It is not hard to see that the separable space $C\mathbb{N}$, the cone over the countable uniformly discrete space, is not star-finite. (See §3.B for the general definition of a cone; for the time being, we may define $C\mathbb{N}$ as the subspace of $U(\mathbb{N}, I)$ consisting of all functions with support in at most one point.)

Every subspace of the countable product $\mathbb{R}^\infty$ of lines is star-finite: a fundamental sequence of star-finite covers of $\mathbb{R}^\infty$ is given by $f_n^{-1}(C_n)$, where $f_n : \mathbb{R}^\infty \to \mathbb{R}^n$ is the projection and $C_n$ is the set of the open (cubical) stars of vertices of the standard cubulation of $\mathbb{R}^n$ by cubes with edge length $2^{-n}$.

Proposition 2.45. A [d-dimensional] (completely) metrizable topological space is homeomorphic to a [d-dimensional] star-finite metrizable (complete) uniform space.

See [50; VI.19, VI.24] for related assertions.

Proof. Let $C_1, C_2, \ldots$ be a basis for a (completely) metrizable uniformity $uX$ on the given topological space $X$. Without loss of generality, each $C_i$ is an open cover. Set $D_1 = C_1$, and suppose that an open cover $D_i$ of $X$ has been constructed. Since $X$ is paracompact, there exists an open star-refinement $D'_i$ of $D_i$ (see [29]). Then $D''_i := D'_i \cap C_{i+1}$ is an open
cover of $X$ refining $C_{i+1}$ and star-refining $D_i$. [In the case where $X$ is $d$-dimensional, $D_i''$ has an open refinement of multiplicity $\leq d+1$, and so itself will be assumed without loss of generality to be of multiplicity $\leq d+1$.] Since $X$ is strongly paracompact, there exists an open star-finite refinement $D_{i+1}$ of $D_i''$ (see [14] or [29]). Thus we obtain a sequence of star-finite open covers $D_1, D_2, \ldots$ such that each $D_{i+1}$ refines $C_{i+1}$ and $\text{refines a cover of multiplicity } \leq d+1$ that in turn star-refines $D_i$. The former implies that the sequence $D_1, D_2, \ldots$ separates points of $X$, and thus is a basis for some $[d$-dimensional$]$ star-finite metrizable uniformity $u'X$ on the underlying set of $X$.

If a subset $T \subset S$ is open in the induced topology of $u'X$ (that is, for every $x \in T$ there exists an $n$ such that $\text{st}(x, D_n) \subset T$), then $T$ is open in $X$ due to the openness of the covers $D_i$. Since each $D_i$ refines $C_i$, the identity map $u'X \to uX$ is uniformly continuous. Hence $u'X$ induces the original topology of $X$. It remains to observe that a uniformity finer than a complete one and inducing the same topology is complete [17]; for (in the metrizable case) every Cauchy sequence in $u'X$ is Cauchy in $uX$ and hence convergent in $X$. □

It is proved in [42; Theorem 7.1] that the unit ball $Q_0 = U((N^+, \infty), ([−1, 1], 0))$ of the separable space $c_0$ fails to be Noetherian.

2.46. Uniform local compacta. By a local compactum we mean a locally compact separable metrizable space\(^1\); or equivalently a metrizable topological space that is a countable union of compacta $X_i$ such that each $X_i \subset \text{Int} X_{i+1}$; or equivalently the complement to a point in a compactum; or equivalently the complement to a compactum in a compactum (see e.g. [29; 3.3.2, 3.8.C, 3.5.11]).

By a uniform local compactum we mean a metrizable uniform space that has a countable uniform cover by compacta. A map $f$ from a uniform local compactum $X$ into a metric space is uniformly continuous iff it is continuous and every two proper maps $\varphi, \psi : N \to X$ such that $d(\varphi(n), \psi(n)) \to 0$ as $n \to \infty$ satisfy $d(f\varphi(n), f\psi(n)) \to 0$ as $n \to \infty$. Every closed subset of a finite-dimensional Euclidean space is a uniform local compactum.

Clearly, every uniform local compactum is complete and its underlying topological space is a local compactum. The converse to the latter is false: $N \times N$ with the metric $d((m, n), (m', n')) = 1$ if $m \neq m'$ and $d((m, n), (m, n')) = \frac{1}{m}$ is not a uniform local compactum, although it is topologically discrete. However, an ANRU (see §4.A) whose underlying topological space is a local compactum is a uniform local compactum [49; 5.4].

Proposition 2.47. A local compactum is homeomorphic to a uniform local compactum.

A version for not necessarily metrizable spaces goes back to A. Weil (cf. [12; p. 590], see also [17; Exer. IX.1.15(d)]).

\(^1\)These are also sometimes called “$\sigma$-compacta”, compare [27]; we refrain from this terminology since $\sigma$-compacta are more commonly not required to be locally compact.
Proof. Let $d$ be some metric on the one-point compactification of the given local compactum $X$. Then $d'(x, y) = d(x, y) + \frac{1}{|d(x, \infty)|} - \frac{1}{|d(y, \infty)|}$ is a complete metric on $X$, inducing the same topology as $d$, moreover every ball (of finite radius) in $(X, d')$ is compact. \hfill \square

The constructed uniform structure is not canonical: it depends in general on the choice of $d$ in its uniform equivalence class.

**Theorem 2.48 ([48; 1.15], [76; Prop. 1]).** Uniform local compacta are star-finite.

Proof. Let $Q = \{q_1, q_2, \ldots\}$ be a countable dense subset of $X$, and suppose that closed $3\varepsilon$-balls in $X$ are compact. Define $Z \subset Q$ by $q_1 \in Z$, and $q_{i+1} \in Z$ unless $q_{i+1}$ is contained in the union of the open $\varepsilon$-balls about the points of $\{q_1, \ldots, q_i\} \cap Z$. On the other hand, let $\{V_\alpha\}$ be the cover of $X$ by the open $\varepsilon/2$-balls about all $x \in X$. If $K$ is the closed $3\varepsilon$-ball about some $z_0 \in Z$, its cover $\{V_\alpha \cap K\}$ has a finite subcover $\{W_j\}$. Since $Z$ contains no pair of points at distance $<\varepsilon$ from each other, each $W_j$ contains at most one $z \in Z$. So $K \cap Z$ is finite. Then the cover of $X$ by the $\frac{3}{2}\varepsilon$-balls about the points of $Z$ is star-finite; clearly, it is also uniform. \hfill \square

### 2.49. Bornological conditions

Let $X$ be a uniform space.

We recall that $X$ is precompact if and only if for each uniform cover $C$ of $X$ there exists a finite set $F \subset X$ such that $st(F, C) = X$. Every separable metrizable space admits a uniformly continuous homeomorphism onto a (non-canonical) precompact metrizable space, cf. [40].

A subset $S$ of $X$ is called $\mathbb{R}$-bounded if for each uniform cover $C$ of $X$ there exists a finite set $F \subset X$ and a positive integer $n$ such that $S \subset st^n(F, C)$; here $st^0(F, C) = F$, and $st^{n+1}(F, C) = st(st^n(F, C), C)$ (see [45], [17; Exer. II.4.7]; an anticipatory definition is found in [46] and in an older edition of [17]). It is not hard to see that $\mathbb{R}$-bounded subsets of a uniformly locally precompact space are precompact (cf. [38; 1.18]). On the other hand, every uniformly contractible uniform space (for instance, $U(X, I)$ for every uniform space $X$) is $\mathbb{R}$-bounded (as a subset of itself).

It turns out that $S \subset X$ is $\mathbb{R}$-bounded if and only if every uniformly continuous map of $X$ into the real line $\mathbb{R}$ with the Euclidean uniformity is bounded on $S$ [38; 1.14] (see also [4; Theorem 2], [5; Theorem 7]). In particular, $\mathbb{R}$-boundedness is preserved under uniformly continuous surjections. In fact, $S$ is $\mathbb{R}$-bounded if and only if it has finite diameter with respect to every uniformly continuously pseudo-metric on $X$; when $X$ is metrizable, $S$ is $\mathbb{R}$-bounded if and only if it has finite diameter with respect to every metric on $X$ [38; 1.12, 1.13]. Further references on $\mathbb{R}$-bounded spaces include [39], [33] and those therein.

A uniform space $X$ is called fine-bounded if every uniformly continuous map from $X$ to a fine uniform space $F$ has precompact image. Equivalently (see [85]), if every uniformly continuous map from $X$ to $\mathbb{R}$ with the fine uniformity is bounded.

A uniform space $X$ is called $\mathbb{N}$-bounded if it admits no countable cover by uniformly disjoint sets (see [85]). Equivalently, if every uniformly continuous map of $X$ into the countable uniformly discrete space $\mathbb{N}$ is bounded. Clearly,
compact ⇒ precompact ⇒ $\mathbb{R}$-bounded ⇒ fine-bounded ⇒ $\mathbb{N}$-bounded.

The fine uniformity on $\mathbb{R}$ is $\mathbb{N}$-bounded but not fine-bounded, and the Euclidean uniformity on $\mathbb{R}$ is fine-bounded but not $\mathbb{R}$-bounded. However, metrizable $\mathbb{N}$-bounded spaces are fine-bounded [84] (see also [77; proof of Theorem 3]).

3. Metrization of quotient spaces

The following proposition does not seem to be particularly helpful, but may be relevant for motivation of subsequent considerations.

**Proposition 3.1.** Let $X$ be a metrizable uniform space with a basis $C_1, C_2, \ldots$ of uniformity, and let $Y$ be a pre-uniform space and $f : X \to Y$ a quotient map. Consider the covers $D_{i,0} = f(C_i)$ and $D_{i,j+1} = \{\operatorname{st}(\{y\}, D_{ij}) \mid y \in Y\}$ of $Y$. Given an increasing $\varphi : \mathbb{N} \to \mathbb{N}$, write $D_{\varphi} = D_{\varphi(0),0} \cup D_{\varphi(1),1} \cup \ldots$.

(a) A basis of the pre-uniformity of $Y$ consists of the $D_{\varphi}$ for all increasing $\varphi : \mathbb{N} \to \mathbb{N}$.

(b) $Y$ is pseudo-metrizable iff there exist increasing $\varphi_1, \varphi_2, \ldots : \mathbb{N} \to \mathbb{N}$ such that for every increasing $\varphi : \mathbb{N} \to \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $D_{\varphi_n}$ refines $D_{\varphi}$.

An analogue of (a) in the language of pseudo-metrics is established in [71].

*Proof.* (a). Each $D_{\varphi}$ is star-refined by $D_{\varphi,\sigma}$, where $\sigma(k) = k + 1$. Each $D_{\varphi}$ and $D_{\psi}$ are refined by $D_{\chi}$, where $\chi(n) = \max(\varphi(n), \psi(n))$. Thus the $D_{\varphi}$ for increasing $\varphi$ form a pre-fundamental family of covers of $Y$. Finally, suppose that $E_0, E_1, \ldots$ is a sequence of covers of $Y$ such that each $E_{i+1}$ star-refines $E_i$ and each $f^{-1}(E_i)$ is uniform. Then $f^{-1}(E_i)$ is refined by $C_{\varphi(i)}$ for some $\varphi : \mathbb{N} \to \mathbb{N}$, or equivalently $E_i$ is refined by $f(C_{\varphi(i)}) = D_{\varphi(i),0}$. Then $E_0$ is refined by $D_{\varphi}$.

(b). Suppose that $Y$ is metrizable, and let $E_1, E_2, \ldots$ be a basis of its uniformity. Then by (a) each $E_k$ is refined by $D_{\varphi_k}$ for some increasing $\varphi_k : \mathbb{N} \to \mathbb{N}$, and each $D_{\varphi}$ is refined by some $E_n$.

Conversely, suppose that there exist $\varphi_i$ as specified. Then by (a), every uniform cover of $Y$ is refined by some $D_{\varphi}$, and by the hypothesis, $D_{\varphi}$ is refined by $D_{\varphi_n}$ for some $n$. Also for each $m$ there exists an $n$ such that $D_{\varphi_m,\sigma}$ is refined by $D_{\varphi_n}$, where $\sigma(k) = k + 1$ so $D_{\varphi_m}$ is star-refined by $D_{\varphi_n,\sigma}$. By a renumbering we may assume that $n = m + 1$ without loss of generality. Then $D_{\varphi_1}, D_{\varphi_2}, \ldots$ is a basis of the pre-uniformity of $Y$. □

3.A. Quotient maps of finite type

We say that a map $q : X \to Y$ between two uniform spaces is a *quotient map of finite type* if it is uniformly continuous and surjective, and there exists an $n$ such that for every uniform cover $C$ of $X$, the cover $f_n(C) = f(\operatorname{st}(f^{-1}(f_{n-1}(C)), C)$ is uniform, where $f_0(C)$ is the cover of $Y$ by singletons. Specifically we will say that $q$ is of *type $n$*. Most important are type 1 quotient maps, which were introduced in [90].

**Lemma 3.2.** Let $X$ be a metrizable uniform space, $Q$ a set, $q : X \to Q$ a surjection, and $u_q$ the quotient pre-uniformity on $Q$. If $d$ is a metric on $X$, let $d_n(x,y) =$
\[ \inf_{x=x_0,\ldots,x_n=y} d(q^{-1}(x_i), q^{-1}(x_{i+1})) \text{ and } d_\infty(x,y) = \inf_{n \in \mathbb{N}} d_n(x,y). \] Clearly, \( d_\infty \) is a pseudo-metric on \( Q \).

(a) [61] If \( u_q \) is metrizable, then it is induced by \( d_\infty \) for some metric \( d \) on \( X \).

(a') If \( u_q \) is metrizable, and \((Q,d_\infty) \xrightarrow{id} (Q,d_n)\) is uniformly continuous for some \( n \) (regardless of whether \( d_n \) is a pseudo-metric), then \( u_q \) is induced by \( d_\infty \).

(b) If \( q: X \to (Q,u) \) is a quotient map of finite type for some uniformity \( u \) on \( Q \), then it is a quotient map (i.e. \( u = u_q \)), and \( u \) is pseudo-metrizable.

(c) If \( d_{2n} \) is a metric, then \( q: X \to (Q,d_{2n}) \) is a type \( n \) quotient map.

(c') If \( (Q,d_\infty) \xrightarrow{id} (Q,d_{2n}) \) is uniformly continuous (regardless of whether \( d_{2n} \) is a pseudo-metric), then \( q: X \to (Q,d_\infty) \) is a type \( n \) quotient map.

(d) [61] \( d_\infty = d_n \) iff \( d_n \) is a pseudo-metric. If \( d_n \) is a metric, then it induces \( u_q \).

(d') If \( (Q,d_\infty) \xrightarrow{id} (Q,d_n) \) is uniformly continuous for some \( n \) (regardless of whether \( d_n \) is a pseudo-metric), then \( u_q \) is induced by \( d_\infty \).

(e) The composition of a type \( m \) quotient map and a type \( n \) quotient map is a type \( mn \) quotient map.

To this lemma one should add that if \( X \) is a compactum and \( u_q \) is a uniformity, then \( q: X \to (Q,u_q) \) is a type 1 quotient map [90].

The proof of (a) is easy, and we repeat it as it is used in the proof of (a'). Our proof of (d) is much simpler than the original one.

Part (e) is surprising, because the composition of two quotient maps between uniform spaces for which \( d_m \) and \( (d_m)_n \) are metrics is obviously a quotient map for which \( d_{mn} \) is a metric, and easy examples show that this estimate is sharp. Thus a composition of \( k \) of quotient maps for which \( d_2 \) is a metric is still a type 1 quotient map, but with only \( d_{qk} \) being a metric!

**Proof.** (a). Suppose that \( u_q \) is induced by a metric \( d_Q \). Then \( d'(x,y) := d_Q(q(x),q(y)) \) is a uniformly continuous pseudo-metric on \( X \). The pseudo-metric \( d''_\infty(x,y) \) on \( Q \) is defined as above in the case of a metric. Since \( d_Q \) satisfies the triangle axiom, \( d''_\infty = d_Q \).

If \( d \) is a metric on \( X \), then \( d'' := d' + d \) is a metric, which is uniformly equivalent to \( d \), and \( d''_\infty \geq d_\infty = d_Q \). Since \( d_Q \) is a metric (rather than just a pseudo-metric), so is \( d''_\infty \). Finally, \( d''_\infty \) is uniformly continuous on \( (Q,d_Q) = (Q,u_q) \), and therefore is uniformly equivalent to \( d_Q \). \( \Box \)

(a'). In the notation of the proof of (a), we note that \( d''_\infty \geq d_\infty \). Then to prove that \( d_\infty \) is uniformly equivalent to \( d''_\infty \) (and hence induces \( u_q \)), it suffices to show that \( d''_\infty \) is uniformly continuous on \( (Q,d_\infty) \). Now \( D(x,y) := d''_\infty(q(x),q(y)) \) is a uniformly continuous pseudo-metric on \( X \). So for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( D(x,y) < \delta \) implies \( D(x,y) < \frac{\varepsilon}{n} \). By the hypothesis there exists a \( \gamma > 0 \) such that \( d_\infty(v,w) < \gamma \) implies \( d_n(v,w) < \frac{1}{2}\gamma \). Then there exist \( x_0, y_1, x_1, \ldots, y_n \in X \) such that \( q(x_0) = v \), \( q(x_i) = q(y_i) \) for \( i = 1, \ldots, n \), \( q(y_n) = w \) and \( d(x_0,y_1) + \cdots + d(x_{n-1},y_n) < \delta \). In particular, each \( d(x_i,y_{i+1}) < \delta \). Then \( D(x_i,y_{i+1}) < \frac{\delta}{n} \); whereas \( D(y_i,x_i) = 0 \) for each \( i \). Hence \( d''_\infty(v,w) \leq D(x_1,y_n) \leq \varepsilon \). \( \Box \)
(b). To prove that \( q \) is a quotient map, we need to show that \( u \) includes every sequence of covers \( D_i \) of \((Q, u)\) such that \( f^{-1}(D_i) \) is uniform and \( D_{i+1} \) strongly star-refines \( D_i \) for each \( i \). Indeed, if \( C = f^{-1}(D_n) \), then \( f_1(C) = D_n \), \( f_i+1(C) = st(f_i(C), D_n) \), and so \( f_n(C) \) refines \( D_i \).

To prove that \( u \) is pseudo-metrizable, we use Theorem 2.9. If \( E_1, E_2, \ldots \) is a fundamental sequence of covers for \( X_i \), and \( D_i \) is a uniform cover of \((Q, u)\), let \( D_{i+1} \) strongly star-refine \( D_i \), and let \( C \) be an \( E_i \) that refines \( f^{-1}(D_n) \). Then \( f_n(C) \) refines \( D_i \), so \( u \) has a countable basis.

(c). Suppose that \( d_{2n} \) is a metric. From its definition, \( q: X \to (Q, d_{2n}) \) is uniformly continuous. Let \( C_\varepsilon \) be the cover of \( X \) by closed balls of radius \( \varepsilon \). It suffices to show that \( f_n(C_\varepsilon) \) is uniform for every \( \varepsilon > 0 \). If \( x \in X \), the element \( f_n(x) \) of \( f_n(C_\varepsilon) \) consists of all \( y \in Q \) such that there exists a chain \( f(x) = x_0, \ldots, x_n = y \) satisfying \( d(q^{-1}(x_i), q^{-1}(x_{i+1})) < \varepsilon \) for each \( i \). Then every pair of points \( y, y' \in Q \) at \( d_{2n} \)-distance \( < \varepsilon \) is contained in some element of \( f_n(C_\varepsilon) \).

(c'). \( q: X \to (Q, d_{2n}) \) is uniformly continuous by the definition of \( d_\infty \). By the hypothesis, for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that every pair of points \( y, y' \in Q \) at \( d_\infty \)-distance \( < \delta \) satisfies \( d_{2n}(y, y') < \varepsilon \). Now the proof of (c) applies.

(d). If \( d_n \) satisfies the triangle axiom, then \( d_n = d_k \) for all \( k > n \). Conversely, \( d_n = d_{2n} \) implies the triangle axiom for \( d_n \). The final assertion follows from (b) and (c).

(d'). This follows from (b) and (c).

(e). Let \( h: X \xrightarrow{f} Y \xrightarrow{g} Z \) be the given composition. Given a uniform cover \( C \) and a cover \( D \) of \( X \), let \( f_1(D, C) = st(f^{-1}(f(D)), C) \) and \( f_{i+1}(D, C) = f_i(f_i(D, C), C) \). Then \( h_1(D, C) \) is refined by \( f_1(D, C) \), hence \( h_i(D, C) \) is refined by \( f_i(D, C) \). Writing \( X \) for the cover of \( X \) by singletons, we obtain that \( f(h_m(X, C)) \) is refined by \( f(f_m(X, C)) = f_m(C) \) and therefore is uniform. Let \( C' \) be a star-refinement of \( f(h_m(X, C)) \).

Given a cover \( D' \) of \( Y \), let \( f_1(D', C') = f(st(f^{-1}(D'), C')) \) and let \( f_{i+1}(D', C') = f_i(f_i(D', C'), C') \). Then \( f_m(D', C') \) is refined by \( st(D', C') \). If \( D'' \) is a cover of \( Z \), then \( h_m(D'', C') \) is refined by \( g(f_m(g^{-1}(D''), C')) \). The latter is in turn refined by the cover \( g(st(g^{-1}(D''), C')) = g_1(D'', C') \).

Finally, let \( h_1(D'', C) = h_m(D'', C) \) and \( h_{i+1}(D'', C) = h_i(h_i(D'', C), C) \). Then \( h_n(D'', C) \) is refined by \( g_n(D'', C') \). Hence \( h_m(C) = h_m(Z, C) = h_n(Z, C) \) is refined by \( g_n(Z, C') \) and therefore is uniform.

**Corollary 3.3 (Marxen [61]).** Let \( X \) be a metrizable uniform space and \( S_\alpha \) a uniformly discrete family of closed subsets of \( X \). Then the quotient \( Q \) of \( X \) by the equivalence relation whose only non-singleton equivalence classes are the \( S_\alpha \) is metrizable.

Moreover, if \( d \) is a metric on \( X \) then \( d_2 \) is a metric on \( Q \).

**Proof.** It is easy to see that \( d_\infty = d_2 \) (cf. [61]; see also [90]) and that \( d_2(x, y) \neq 0 \) whenever \( x \neq y \).
Lemma 3.4 (Nhu [68; 2.5]). If \( X \) is a metrizable uniform space and \( A \subset X \) is a closed subset, then every bounded metric on \( A \) extends to a bounded metric on \( X \).

The proof given below depends on 3.3, in contrast to Nhu’s paper, which the author became aware of long after writing up the proof below. The similar assertion for pseudometrics is well-known [48; Lemma 1.4] (a direct proof), [50; III.16]. An extension theorem for a (possibly unbounded) metric on a closed subset of a metrizable topological space was proved by F. Hausdorff (1930).

We note that the use of functional spaces in the following proof may be replaced by a reference to [48; Lemma 1.4], whose proof is elementary.

Proof. We may assume that the diameter of \( A \) does not exceed 1. Then there exists an isometric embedding of \( A \) into \( U(dA, I) \), where \( dA \) is \( A \) endowed with discrete uniformity (see Remark 2.44(i)), which in turn extends to a uniformly continuous map \( f: X \to U(dA, I) \) (see Theorem 2.32).\(^2\) Pick some metric on \( X/A \), which exists by Corollary 7.4, and consider the \( l_\infty \) product metric (or the \( l_1 \) product metric) on \( U(dA, I) \times (X/A) \). Then \( g = f \times q: X \to U(dA, I) \times (X/A) \) is injective, uniformly continuous, and isometric on \( A \). It suffices to prove that its inverse is uniformly continuous. Suppose that \( x_i, y_i \in X \) are such that \( d(g(x_i), g(y_i)) \to 0 \) as \( i \to \infty \) but \( d(x_i, y_i) \) is bounded below by some \( \varepsilon > 0 \).

By passing to a subsequence and interchanging \( x_i \) with \( y_i \), we may assume that either

1. \( d(x_i, A) \to 0 \) and \( d(y_i, A) \to 0 \) as \( i \to \infty \), or
2. \( d(x_i, A) \to 0 \), while \( d(y_i, A) \) is bounded below by some \( \delta > 0 \), or
3. \( d(x_i, A) \) and \( d(y_i, A) \) are bounded below by some \( \delta > 0 \).

In the second case, \( d(q(x_i), q(y_i)) \) is bounded below by \( \delta/2 \) for sufficiently large \( i \), hence \( d(g(x_i), g(y_i)) \) is bounded below. In the third case, let \( Z \) be the complement in \( X \) to the open \( \delta \)-neighborhood of \( A \). Then \( q|Z \) is a uniform embedding, so \( d(q(x_i), q(y_i)) \) is bounded below since \( d(x_i, y_i) \) is. In the first case, there exist \( x'_i, y'_i \in A \) such that \( d(x_i, x'_i) \to 0 \) and \( d(y_i, y'_i) \to 0 \) as \( i \to \infty \). Then \( d(x'_i, y'_i) \) is bounded below by \( \varepsilon/2 \) for sufficiently large \( i \). Hence \( d(f(x'_i), f(y'_i)) \) is bounded below. By the triangle axiom, \( d(f(x_i), f(y_i)) \) is bounded below. Thus \( d(g(x_i), g(y_i)) \) is bounded below. \( \square \)

3.5. Amalgamated union. Let \( f: A \to X \) and \( g: A \to Y \) be embeddings between uniform spaces with closed images. The pushout of the diagram \( X \xleftarrow{f} A \xrightarrow{g} Y \) is called the amalgamated union of \( X \) and \( Y \) along the copies of \( A \), and is denoted \( X \cup_A Y \); a more detailed notation is \( X \cup_{h} Y \), where \( h = g f^{-1} \) is the uniform homeomorphism between \( f(A) \) and \( g(A) \). Thus \( X \cup_A Y \) is the quotient of \( X \sqcup Y \) by the equivalence relation \( x \sim y \) if \( \{x, y\} = \{f(a), g(a)\} \) for some \( a \in A \). It is not hard to see that \( X \) and \( Y \) (and consequently also \( A \)) can be identified with subspaces of \( X \cup_A Y \).

\(^2\)Note that \( U(dA, I) \) is inseparable (unless \( A \) is finite). One can remain within the realm of separable spaces here (as long as \( X \) itself is separable) by the price of using a more subtle uniform embedding (see Theorem 2.43) along with a more subtle extension (see Corollary 2.33).
Corollary 3.6. Let $X$ and $Y$ be metrizable uniform spaces and $h$ is a uniform homeomorphism between closed subsets $A \subset Y$ and $B \subset Z$, then the amalgamated union $X \cup_h Y$ is metrizable.

Moreover, there exists a metric on $X$ such that $h$ is an isometry; and if $d$ is any such metric, then $d_2$ is a metric on $X \cup_h Y$.

The metric $d_2$ on the amalgamated union (not identified as a metric of the uniform quotient) has been considered by Nhu [66], [67] and elsewhere [18], [20], [87].

On the other hand, in Corollary 4.41 below we give an alternative proof of the metrizability of the amalgamated union, which is more direct (based on Theorem 2.9) but does not produce any nice explicit metric. It is, however, the explicit $d_2$ metric that will be crucial for applications of Corollary 3.6.

Proof. The existence of a metric such that $h$ is an isometry follows from Lemma 3.4. If $h$ is an isometry, then it is easy to see that $d_\infty = d_2$ and that $d_2(x, y) \neq 0$ whenever $x \neq y$ (cf. [18; I.5.24]). □

3.7. Adjunction space. Let $X$ and $Y$ be uniform spaces, $A \subset X$ and $f: A \to X$ a uniformly continuous map. The adjunction space $X \cup_f Y$ is the pushout (in the category of pre-uniform spaces) of the diagram $X \supset A \xrightarrow{f} Y$. In other words, $X \cup_f Y$ is the quotient of $X \cup Y$ by the equivalence relation $x \sim y$ if $x \in A$ and $f(x) = y$. Its equivalence classes are $[y] = \{y\} \cup f^{-1}(y)$ for $y \in Y$ and $[x] = \{x\}$ for $x \in X \setminus A$.

We are now ready to prove the main result of this subsection.

Theorem 3.8. Let $X$ and $Y$ be metrizable (complete) uniform spaces and $A$ a closed subset of $X$. If $f: A \to Y$ is a uniformly continuous map, then $X \cup_f Y$ is metrizable (and complete).

Moreover, there exists a metric $d$ on $X \cup Y$ such that $f$ is 1-Lipschitz; for any such metric, $d_3$ is a metric on $X \cup_f Y$.

We note that this includes 3.3 and 3.6 as special cases, apart from the explicit metrics.

It might be possible to prove the metrizability of the adjunction space, without producing any nice explicit metric, by a more direct method, akin to the proof of Corollary 4.41 below. In fact, G. L. Garg gave a (rather complicated) construction of a metrizable uniform space with properties resembling those of the adjunction space [35]; in his Zentralblatt review of Garg’s paper, J. R. Isbell claims, without giving any justification, that “the author constructs the pushout of a closed embedding and a surjective morphism in the category of metrizable uniform spaces”. We note that due to the nature of Garg’s construction, a proof of Isbell’s claim would be unlikely to produce any nice explicit metric on the adjunction space. It is, however, the explicit $d_3$ metric that will be crucial for applications of Theorem 3.8.

Proof. Let $d_X$ and $d_Y$ be some bounded metrics on $X$ and $Y$. Let $D(a, b) = d_X(a, b) + d_Y(f(a), f(b))$ for all $a, b \in A$. Then $D$ is a bounded metric on $A$; clearly it is uniformly equivalent to the restriction of $d_X$ over $A$; and $d_Y(f(a), f(b)) \leq D(a, b)$ for all $a, b \in A$. 


(Note that $D$ is nothing but the restriction of the $l_1$ product metric to the graph of $f$.) By Lemma 3.4 $D$ extends to a metric, also denoted $D$, on the uniform space $X$. Define a metric $d$ on $X \sqcup Y$ by $d(x, x') = D(x, x')$ if $p, q \in X$, by $d(y, y') = d_Y(y, y')$ if $y, y' \in Y$ and by $d(x, y) = 1$ whenever $x \in X$ and $y \in Y$. Then $d(f(a), f(b)) \leq d(a, b)$ for $a, b \in A$. Since $A$ is closed, $d(x, A) > 0$ for every $x \in X \setminus A$. We have

$$d([y], [z]) = \min(d(y, z), d(f^{-1}(y), f^{-1}(z))) = d(y, z)$$

for all $y, z \in f(X)$, and it follows that

$$d_\infty([y], [z]) = d_1([y], [z]) = d(y, z)$$

for all $y, z \in Y$. Therefore $Y$ is identified with a subspace of $X \sqcup f Y$, and it is easy to see that

$$d_\infty(x, [y]) = d_2(x, [y]) = \inf_{z \in f(X)} d(x, f^{-1}(z)) + d(z, y) \geq d(x, A) > 0$$

for all $x \in X \setminus A$ and $y \in Y$ and

$$d_\infty([x], [x']) = d_3([x], [x']) = \min(d(x, x'), \inf_{y,z \in f(X)} d(x, f^{-1}(y)) + d(y, z) + d(f^{-1}(z), x'))$$

$$\geq \min(d(x, x'), d(x, A) + d(x', A)) > 0$$

for any pair of distinct $x, x' \in X \setminus A$. Thus $d_3$ is a metric (see the first assertion of Lemma 3.2(d)). Hence by the second assertion of Lemma 3.2(d), $X \sqcup f Y$ is metrizable.

Now suppose that $X$ and $Y$ are complete. Let $q_1, q_2, \ldots$ be a Cauchy sequence in $Q$. Suppose that $d_3(q_n, Y)$ is bounded below. Then each $q_n = \{x_n\}$, where $x_n \in X \setminus A$ and $d(x_n, A)$ is bounded below. Then $x_n$ is a Cauchy sequence in $X$ and so converges to an $x \in X \setminus A$. Then $q_n$ converges to $\{x\}$ in $Q$. If $d_3(q_n, Y)$ is not bounded below, then by passing to a subsequence we may assume that $d_3(q_n, Y) \to 0$ as $n \to \infty$. Then there exist $y_1, y_2, \cdots \in Y$ such that $d_3(q_n, [y_n]) \to 0$ as $n \to \infty$. Then $[y_n]$ is a Cauchy sequence in $Q$, whence $y_n$ is a Cauchy sequence in $Y$ and so has a limit $y \in Y$. Then $[y_n]$ converges to $[y]$ in $Q$. Since $q_1, [y_1], q_2, [y_2], \ldots$ is a Cauchy sequence and it has a subsequence converging to $[y]$, it converges to $[y]$ itself. □

3.8. Uniform Join. Let $X$ and $Y$ be uniform spaces. We define their join to be the quotient $X \times Y \times [-1, 1]/\sim$, where $(x, y, t) \sim (x', y', t')$ if either $t = 1$ and $x = x'$ or $t = -1$ and $y = y'$. Thus we have the pushout diagram

$$
\begin{array}{ccc}
X \times Y \times \{1, -1\} & \subset & X \times Y \times [-1, 1] \\
\pi \downarrow & & \downarrow \\
X \sqcup Y & \to & X \ast Y,
\end{array}
$$

where $\pi(x, y, 1) = x$ and $\pi(x, y, -1) = y$. By Lemma 2.29(b) and Lemma 2.28(b), $\pi$ is a quotient map. So if $X$ and $Y$ are metrizable or complete, then by Theorem 3.8 so is $X \ast Y$. More specifically, given metrics, denoted $d$, on $X$ and $Y$, then a metric on $X \ast Y$
Let us fix some metrics on $\Pi$.

Lemma 3.11. is given by

$$d_3([(x, y, t)], [(x', y', t')]) = \min \left\{ \begin{array}{l} d(x, x') + d(y, y') + |t - t'|, \\
      d(x, x') + (t + 1) + (t' + 1), \\
      d(y, y') + (1 - t) + (1 - t'), \\
      (2 - |t - t'|) + 2 \end{array} \right\}.$$ 

(In the case $t, t' \in (-1, 1)$, the four options correspond respectively to the chains

$$(x, y, t) \rightarrow (x', y', t');$$
$$(x, y, t) \rightarrow (x, *, -1) \rightarrow (x', *, -1) \rightarrow (x', y', t');$$
$$(x, y, t) \rightarrow (*, y, 1) \rightarrow (y', 1) \rightarrow (x', y', t');$$
$$(x, y, t) \rightarrow (*, y, 1) \rightarrow (x', *, -1) \rightarrow (x', y', t');$$

other cases involve subchains of these chains.) The formula for $d_3$ implies that if $X$ and $Y$ are metrizable uniform spaces, and $A \subset X$ and $B \subset Y$ are their subspaces, then $A * B$ is a subspace of $X * Y$. In particular, $X * \emptyset$ and $\emptyset * Y$ are subspaces of $X * Y$. Since they are uniformly disjoint, their union may be identified with $X \sqcup Y$.

3.10. Uniform cone. The uniform join $X * pt$ is denoted $CX$ and called the cone over the uniform space $X$. Thus $CX$ may be viewed as the quotient space $X \times [0, 1]/X \times \{1\}$. Given a metric $d$ on $X$, by Lemma 3.3, a metric on $CX$ is given by

$$d_2([(x, t)], [(x', t')]) = \min\{d(x, x') + |t - t'|, (1 - t) + (1 - t')\}.$$

If the diameter of $X$ is $\leq 2$, then $(X, d)$ is isometric to the subset $X \times \{0\}$ of $(CX, d_2)$.

Lemma 3.11. Let $X$ and $Y$ be metrizable uniform spaces. Then $X * Y$ is uniformly homeomorphic to $CX \times Y \cup_{X \times Y} X \times CY$.

Proof. Let us fix some metrics on $X$ and $Y$ such that $X$ and $Y$ have diameters $\leq 2$. Viewing $CX$ as $X \times [0, 1]/X \times \{1\}$ and $CY$ as $Y \times [-1, 0]/Y \times \{-1\}$, a metric on $CX \times CY$ is given by

$$d([(x, y, t)], [(x', y', t')]) = \min\{d(x, x') + d(y, y') + |t - t'|, d(y, y') + (1 - t) + (1 - t')\},$$

and a metric on $X \times CY$ is given by

$$d([(x, y, t)], [(x', y', t')]) = \min\{d(x, x') + d(y, y') + |t - t'|, d(x, x') + (t + 1) + (t' + 1)\}.$$

Since the diameters of $X$ and $Y$ are $\leq 2$, the metrics on $CX \times Y$ and $X \times CY$ both induce the same metric $d(x, x') + d(y, y')$ on $X \times Y$. Then it is easy to see that the metric on $CX \times Y \cup_{X \times Y} X \times CY$ given by 3.6 coincides (exactly, not just up to uniform equivalence) with the above metric $d_3$ of $X * Y$.

3.12. Euclidean cone. Let $d$ be a metric on a set $X$ such that the diameter of $X$ does not exceed $\pi$. Then $X \times [0, 1]/X \times \{0\}$ can be endowed with a metric $d_E$ modelled on the Law of Cosines that holds in Euclidean spaces: $d_E([(x, t)], [(y, s)])^2 = t^2 + s^2 - 2ts \cos d(x, y)$. (Think of $X$ as a subset of the Euclidean unit sphere in some Euclidean space.) It is well-known $d_E$ is indeed a metric, and that it is complete if $d$ is (see e.g.
It is not hard to see that the uniform equivalence class of $d_E$ does not depend on the choice of $d$ in its uniform equivalence class.

Note that $d_E([(x, t)], [(y, t)]) = 2t \sin \frac{d(x, y)}{2}$, which is bounded below by $\frac{1}{2}td(x, y)$ and above by $td(x, y)$. This is somewhat “smoother” than with our standard cone metric: $d_2([(x, 1 - t)], [(y, 1 - t)]) = \min\{2t, d(x, y)\}$.

**Lemma 3.13.** If $X$ is a metrizable uniform space, the Euclidean cone over $X$ is uniformly homeomorphic to $CX$ keeping $X$ fixed.

**Proof.** Given a metric $d$ on $X$, we have the Euclidean metric $d_E$ on $X \times [0, 1]/X \times \{0\}$ and the metric $d_2$ on $CX = X \times [0, 1]/X \times \{1\}$. Define a metric $d_e$ on $CX$ by $d_e([(x, t)], [(y, s)]) = d_E([(x, 1 - t)], [(y, 1 - s)])$. Note that if $v$ is the cone vertex, $d_2(v, [(x, t)]) = 1 - t = d_e(v, [(x, t)])$.

Let $x_i, y_i \in CX$ be two sequences. By passing to subsequences and interchanging $x_i$ with $y_i$, we may assume that either

(i) $d(x_i, v) \to 0$ and $d(y_i, v) \to 0$ as $i \to \infty$, or
(ii) $d(x_i, v) \to 0$, while $d(y_i, v)$ is bounded below by some $\delta > 0$, or
(iii) $d(x_i, v)$ and $d(y_i, v)$ are bounded below by some $\delta > 0$,

where $d(*, v) = d_2(*, v) = d_e(*, v)$. We claim that $d_2(x_i, y_i) \to 0$ as $i \to \infty$ if and only if $d_e(x_i, y_i) \to 0$ as $i \to \infty$. Indeed, both hold in the case (i); none holds in the case (ii); and to do with the case (iii) in suffices to show that the restrictions of $d_2$ and $d_e$ over $X \times [0, 1 - \varepsilon]$ are uniformly equivalent for each $\varepsilon > 0$.

It is clear that the restriction of $d_2$ over $X \times [0, 1 - \varepsilon]$ is uniformly equivalent to the $l_1$ product metric for each $\varepsilon > 0$. The restriction of $d_e$ over $X \times [0, 1 - \varepsilon]$ is uniformly equivalent to the $l_2$ product metric for each $\varepsilon > 0$, using that

$$d_E([(x, t)], [(y, s)])^2 = (t - s)^2 + 2ts(1 - \cos d(x, y)) = (t - s)^2 + 4ts \sin^2 \frac{d(x, y)}{2},$$

where $ts \in [\varepsilon^2, 1]$, and $\frac{\varepsilon}{2} \leq \sin z \leq z$ as long as $z \in [0, \frac{\pi}{2}]$.

**3.14. Rectilinear cone and join.** Let $V$ be a vector space. Given two subsets $S, T \subset V$, their rectilinear join $S \cdot T$ is the union of $S, T$, and all straight line segments with one endpoint in $S$ and another in $T$. Obviously, rectilinear join is associative.

We define the rectilinear cone $cS = (S \times \{0\}) \cdot \{(0, 1)\} \subset V \times \mathbb{R}$. We identify $S$ with $S \times \{0\} \subset cS$.

Given two vector spaces $V$ and $W$ and subsets $S \subset V$ and $T \subset W$, their independent rectilinear join $ST = (S \times \{0\} \times \{-1\}) \cdot (\{(0, 1) \times \{0\} \times \{1\}) \subset V \times W \times \mathbb{R}$. We identify $S, T$ with $S0, \emptyset T \subset ST$.

**Lemma 3.15.** Let $V$ and $W$ be normed vector spaces, and let $X \subset V$ and $Y \subset W$. Then there exists a uniform homeomorphism between $c(XY)$ and $cX \times cY$, taking $XY$ onto $cX \times Y \cup X \times cY$. 


Lemma 3.17. Let $V$ be a normed vector space and $X$ a bounded subset of $V$. Then the rectilinear cone $cX$ is uniformly homeomorphic to $cX$ keeping $X$ fixed.

The proof is based on Lemma 3.13.

Proof. Let us consider the $l_\infty$ norm $||(v, t)|| = \max\{||v||, t\}$ on $V \times \mathbb{R}$. By scaling the norm of $V$ we may assume that $X$ lies in the unit ball $Q$ of $V$. Without loss of generality $X = Q$. Let $f : Q \times I \to cQ$ send $Q \times \{0\}$ onto the cone vertex $\{(0, 1)\}$, and each cylinder generator $\{x\} \times I$ linearly onto the cone generator $c\{x\}$.

If $x \in Q$, the cone generator $c\{x\}$ has $\mathbb{R}$-length 1 (i.e. its projection onto $\mathbb{R}$ has length 1) and $V$-length $\leq 1$. Therefore if $y \in Q$ and $t, s \in I$, the straight line segment $[f(x, t), f(y, s)]$ with endpoints $f(x, t)$ and $f(y, s)$ has $\mathbb{R}$-length $|s - t|$ and the segment $[f(z, \max(t, s)), f(z, \min(t, s))]$, where $z = x$ if $t > s$ and $z = y$ if $t < s$, has $V$-length $\leq |s - t|$. Since $[f(x, \max(t, s)), f(y, \max(t, s))]$ is parallel to $V$, it has $V$-length $\max(t, s)D$, where $D = ||x - y||$. Hence by the triangle inequality, $[f(x, t), f(y, s)]$ has $V$-length between $\max(t, s)D - |s - t|$ and $\max(t, s)D + |s - t|$. Thus $S := ||f(x, t) - f(y, s)||$ is bounded below by $\max(|s - t|, \max(t, s)D - |s - t|)$ and above by $\max(t, s)D + |s - t|$.

On the other hand, note that $1 - \frac{D^2}{8} \leq \cos D \leq 1 - \frac{D^2}{8}$ due to $\frac{\pi}{2} \leq \sin \frac{D}{2} \leq \frac{\pi}{2}$ for $D \in [0, \frac{\pi}{2}]$. If we take $d(x, y) = ||x - y||$ in the definition of $d_E$, these inequalities imply that $\bar{E}^2 := d_E([x, t], [(y, s)])^2$ is bounded above by $tsD^2 + (s - t)^2$ and below by $\frac{1}{4}tsD^2 + (s - t)^2$.

We consider two cases. When $|s - t| \leq \frac{1}{2} \max(t, s)D$, we have $S \geq \max(t, s)D - |s - t| \geq \frac{1}{2} \max(t, s)D$. When $|s - t| \geq \frac{1}{2} \max(t, s)D$, we have $S \geq |s - t| \geq \frac{1}{2} \max(t, s)D$. Thus in either case $S \geq \frac{1}{2} \max(t, s)D \geq \frac{1}{2} \sqrt{ts}D$ and $S \geq |s - t|$. Hence $4S^2 + S^2 \geq tsD^2 + (s - t)^2 \geq E^2$ and so $3S \geq E$.

We consider two cases. When $|s - t| \leq \frac{1}{2} \max(t, s)$, we get $ts \geq \max(t, s)^2 - \max(t, s)|s - t| \geq \frac{1}{2} \max(t, s)^2$ and therefore $\bar{E}^2 \geq \frac{1}{8}tsD^2 \geq \frac{1}{8} \max(t, s)^2D^2$. When $|s - t| \geq \frac{1}{2} \max(t, s)$, we get $\bar{E}^2 \geq (s - t)^2 \geq \frac{1}{4} \max(t, s)^2 \geq \frac{1}{8} \max(t, s)^2D^2$. In either case, $E \geq \frac{1}{4} \max(t, s)D$ and $\bar{E}^2 \geq (s - t)^2$. Hence $4E + E \geq \max(t, s)D + |s - t| \geq S$.  

Lemma 3.17. If $X, Y$ and $Z$ are uniform spaces, $x \in X$ and $y \in Y$, then the subspace $X \times \{y\} \times Z \cup \{x\} \times Y \times Z$ of $X \times Y \times Z$ is uniformly homeomorphic to the amalgamated union $X \times Z \cup_{\{x\} \times Z = \{y\} \times Z} Y \times Z$.  

Proof. We have $cX = (X \times \{0\}) \cdot \{(0, 1)\} \subset V \times I$ and $cY = (Y \times \{0\}) \cdot \{(0, 1)\} \subset W \times I$, where $I = [0, 1] \subset \mathbb{R}$, and

$$c(XY) = (X \times \{0\}) \cdot \{(0, 1)\} \cdot \{(0) \times X \times \{(1, 0)\}\} \cdot \{(0, 0, 0, 1)\} \subset V \times W \times \Delta,$$

where $\Delta = c(\{-1\} \cdot \{1\})$ is the convex hull of $(-1, 0), (1, 0)$ and $(0, 1)$ in $\mathbb{R}^2$. Define $\varphi : I \times I \to \Delta$ by $\varphi(0, 0) \to (0, 0), (1, 0) \to (1, 0), (0, 1) \to (-1, 0), (1, 1) \to (0, 1)$ and by extending linearly to the convex hull of $(0, 0), (1, 0), (1, 1)$ and to the convex hull of $(0, 0), (0, 1), (1, 1)$. Clearly $\varphi$ is a homeomorphism, and hence (by compactness) a uniform homeomorphism. Therefore $f : V \times I \times W \times I \to V \times W \times \Delta$, defined by $f(x, y, z) = (x, y, \varphi(t, z))$ is a uniform homeomorphism. It is easy to see that $f^{-1}(c(XY)) = cX \times cY$ and $f^{-1}(XY) = cX \times Y \cup X \times cY$.  \(\square\)
This can be seen as a purely category-theoretic fact.

**Proof.** The amalgamated union maps onto the subspace in the obvious way. Given maps $f: X \times \{y\} \times Z \to W$ and $g: \{x\} \times Y \times Z \to W$ agreeing on $\{x\} \times \{y\} \times Z$, one defines $f \cup g$ using the projections of $X \times Y \times Z$ onto $X \times Z$ and onto $Y \times Z$. □

**Lemma 3.18.** Let $V$ and $W$ be normed vector spaces, and let $X \subseteq V$ and $Y \subseteq W$ be bounded subsets. Then the subspace $cX \times Y \cup X \times cY$ of $cX \times cY$ is uniformly homeomorphic to the amalgam $cX \times Y \cup_{X \times Y} X \times cY$.

**Proof.** Lemma 3.16 implies that a uniform neighborhood of $X$ in $cX$ is uniformly homeomorphic to $X \times I$. Denote this uniform neighborhood by $U_X$. Then the uniform neighborhood $U_X \times U_Y$ of $X \times Y$ in $cX \times cY$ is uniformly homeomorphic to $X \times Y \times I \times I$. Hence by Lemma 3.17, the subspace $U_X \times Y \cup X \times U_Y$ of $U_X \times U_Y$ is uniformly homeomorphic to the amalgam $U_X \times Y \cup_{X \times Y} X \times U_Y$. The definition of the amalgam as a pushout yields a uniformly continuous bijection $f: cX \times Y \cup_{X \times Y} X \times cY \to cX \times Y \cup X \times cY$ such that $f^{-1}$ is uniformly continuous on $cX$ and on $X \times cY$. By the above, $f^{-1}$ is also uniformly continuous on $U_X \times Y \cup X \times U_Y$. Since these three subsets of $cX \times Y \cup X \times cY$ form its uniform cover, $f^{-1}$ is uniformly continuous. □

**Theorem 3.19.** Let $V$ and $W$ be normed vector spaces, and let $X \subseteq V$ and $Y \subseteq W$ be bounded subsets. Then the independent rectilinear join $XY$ is uniformly homeomorphic to $X \ast Y$.

**Proof.** By Lemma 3.15, $XY$ is uniformly homeomorphic to the subspace $cX \times Y \cup X \times cY$ of $cX \times cY$. By Lemma 3.18, the latter is uniformly homeomorphic to the amalgam $cX \times Y \cup_{X \times Y} X \times cY$. By Lemma 3.16 the latter is in turn uniformly homeomorphic to the amalgam $cX \times Y \cup_{X \times Y} X \times CY$. Finally, by Lemma 3.11, the latter is uniformly homeomorphic to $X \ast Y$. □

**Corollary 3.20.** Let $V$ and $W$ be normed vector spaces, $X$ and $Y$ metrizable uniform spaces, and $f: X \hookrightarrow V$ and $g: Y \hookrightarrow W$ embeddings onto bounded subsets. Then the uniform homeomorphism type of the independent rectilinear join $f(X)g(Y)$ does not depend on the choices of $V$, $W$, $f$ and $g$.

**Corollary 3.21.** Given metrizable uniform spaces $X$ and $Y$, there exists a uniform homeomorphism between $C(X \ast Y)$ and $CX \times CY$ taking $X \ast Y$ onto $CX \times Y \cup X \times CY$.

**Proof.** Pick bounded metrics on $X$ and $Y$. Then there exists an isometric embedding of $X$ onto a bounded subset of the normed vector space $U(dX, \mathbb{R})$, where $dX$ denotes $X$ endowed with discrete uniformity (see Remark 2.44(i)), and similarly for $Y$. (If $X$ is separable, it uniformly embeds onto a bounded subset of a separable normed vector space, see Theorem 2.43 or Remark 2.44(ii).) Then Theorem 3.19 and Lemmas 3.16, 3.15 yield uniform homeomorphisms $C(X \ast Y) \to C(XY) \to c(XY) \to cX \times cY \to CX \times CY$. □
3.22. Mapping cylinder. The mapping cylinder $MC(f)$ is the adjunction space $X \times I \cup_f Y$, where $f'$ is the partial map $X \times I \supset X \times \{1\} = X \overset{f}{\rightarrow} Y$.

Lemma 3.23. Let $f: X \rightarrow Y$ be a uniformly continuous map between metrizable uniform spaces.
(a) If $A$ is a subspace of $X$, then $MC(f|_A)$ is a subspace of $MC(f)$.
(b) $MC(f)$ is uniformly homeomorphic to the image of $\Gamma_f \times I \cup X \times Y \times \{1\}$ under the map $X \times Y \times I \rightarrow CX \times Y$.

Proof. (a). Given bounded metrics $d_X$ on $X$ and $d$ on $Y$, define a new metric $d$ on $X$ by $d(x, x') = d_X(x, x') + d(f(x), f(x'))$. Clearly, $d$ is bounded and uniformly equivalent to $d_X$, and $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. Using $d$ we define the $l_1$ product metric on $X \times I$, which is bounded; and hence the metric of disjoint union, also denoted $d$, on $X \times I \cup Y$. Then $d(f(p), f(q)) \leq d(p, q)$ for all $p, q \in X \times \{1\}$, and thus from Theorem 3.8 we have the following $d_3$ metric on $MC(f)$: $d_3([y], [y']) = d(y, y')$, $d_3([(x, t)], [y]) = (1 - t) + d(f(x), y)$ and $d_3([(x, t)], [(x', t')]) = \min\{d(x, x') + |t - t'|, (1 - t) + (1 - t') + d(f(x), f(x'))\}$ for all $y, y' \in Y$, $x, x' \in X$ and $t, t' \in I$. Clearly, this coincides with the similar $d_3$ metric on $MC(f|_A)$. □

(b). The uniform homeomorphism between $X$ and $\Gamma_f$ yields a uniform homeomorphism between $MC(f)$ and $MC(\pi|_{\Gamma_f})$, where $\pi: X \times Y \rightarrow Y$ is the projection. On the other hand, by the proof of Lemma 3.11, $CX \times Y$ is uniformly homeomorphic to $MC(\pi)$. Hence the assertion follows from (a). □

Corollary 3.24. If $\Delta$ is a finite diagram of metrizable uniform spaces and uniformly continuous maps, its homotopy colimit is a metrizable uniform space.

4. Theory of retracts

4.1. Absolute retracts and extensors. As long as we have finite coproducts, as well as embeddings and quotient maps in a concrete category $\mathcal{C}$ over the category of sets, we also have in $\mathcal{C}$ the notions of map extension and neighborhood. A diagram $X \leftrightarrow A \overset{f}{\rightarrow} Y$, where $f$ is a $\mathcal{C}$-morphism and $A$ is a $\mathcal{C}$-embedding is called a partial $\mathcal{C}$-morphism, and is said extend to a $\mathcal{C}$-morphism $\tilde{f}: X \rightarrow Y$ if the composition $A \overset{a}{\leftarrow} X \overset{f}{\rightarrow} Y$ equals $f$. Given a $\mathcal{C}$-embedding $g: X \leftarrow Y$, its decomposition into a pair of $\mathcal{C}$-embeddings $X \overset{a}{\leftarrow} N \overset{n'}{\rightarrow} Y$ is a neighborhood of $g$ if there exists a $\mathcal{C}$-embedding $g': X' \rightarrow Y$ such that $g \sqcup g': X \sqcup X' \rightarrow Y$ is a $\mathcal{C}$-embedding and $n \sqcup g': N \sqcup X' \rightarrow Y$ is a $\mathcal{C}$-quotient map.

An AE($\mathcal{C}$) is any $\mathcal{C}$-object $Y$ such that every partial $\mathcal{C}$-morphism $X \overset{a}{\leftarrow} A \overset{f}{\rightarrow} Y$ extends to a $\mathcal{C}$-morphism $X \rightarrow Y$. If $Y$ satisfies this for the special case of $f = \text{id}_Y$, then $Y$ is called an AR($\mathcal{C}$). Absolute extensors are also known as “injective objects”, cf. [50; p. 39], [1; 9.1; see also 9.6]
An ANE(C) is any C-object Y such that for every partial C-morphism X $\xymatrix{a \ar[r]^f & Y}$ there exists a C-neighborhood $A \xymatrix{\hookleftarrow & N \ar[r]^{f^{-1}} & X}$ of a such that the partial C-morphism $N \xymatrix{\hookleftarrow & A \ar[r]^f & Y}$ extends to a C-morphism $N \ar[r] & Y$. If Y satisfies the above for the special case of $f = \text{id}_Y$, then Y is called an ANR(C). Note the obvious implications
\[
\text{AE}(C) \Rightarrow \text{ANE}(C) \\
\Rightarrow \Rightarrow \\
\text{AR}(C) \Rightarrow \text{ANR}(C).
\]

4.2. A[N]Rs and A[N]Es for various subcategories of $\mathcal{U}$. Among full subcategories of the category $\mathcal{U}$ of uniform spaces and uniformly continuous maps are the categories $\mathcal{CM}$ of compact metrizable spaces (topological, or equivalently, uniform); $\mathcal{CU}$ of complete uniform spaces; $\mathcal{MU}$ of metrizable uniform spaces; $\mathcal{SU}$ of uniform spaces whose topology is separable; their pairwise intersections $\mathcal{MCU}$, $\mathcal{SMU}$, $\mathcal{SCU}$; and the category $\mathcal{SMCU} = \mathcal{SU} \cap \mathcal{MU} \cap \mathcal{CU}$ of Polish uniform spaces.

The A[N]R($\mathcal{CM}$) coincide with the A[N]E($\mathcal{CM}$) (see \cite[III.3.2(k)]{44}); and also with A[N]Rs in the notation of Borsuk \cite[15]{15}, which are the same as compact A[N]Rs in the notation of Hu \cite[15; V.1.2]{44} or \cite[III.5.3]{44}).

By definition, the A[N]E($\mathcal{U}$) are the same as the A[N]EU of §2.D. Following Isbell \cite[48, 49]{48}, \cite[49]{49}, we also abbreviate A[N]R($\mathcal{U}$) to A[N]RU. Using Remark 2.44(i), it is easy to see that these are in fact the same as the A[N]EU, cf. \cite[p. 111]{48}, \cite[V.14]{50}. We further note that restricting the class of $\mathcal{U}$-embeddings to all extremal/regular monomorphisms of $\mathcal{U}$, that is, embeddings onto closed subsets (in the spirit of the non-uniform A[N]R theory \cite[15]{15}, \cite[44]{44}) does not affect the definitions of an A[N]RU and A[N]EU (see \cite[III.8]{50}, \cite[proof of I.7]{48}). Every A[N]RU is an A[N]R($\mathcal{MU}$), and more generally an A[N]R for the category of collectionwise normal spaces \cite[4.4]{49}.

Every A[N]RU is an A[N]R($\mathcal{CU}$) since it is complete. Conversely, every A[N]R($\mathcal{CU}$) is an A[N]RU by considering the completion of the domain (see \cite[II.10]{50}). Similar (and simpler) arguments show that the A[N]R($\mathcal{MCU}$) coincide with the A[N]R($\mathcal{MU}$); the A[N]R($\mathcal{SMCU}$) coincide with the A[N]R($\mathcal{SMU}$); etc.

It is known that the following properties are equivalent: being an A[N]R($\mathcal{MU}$); being an A[N]E($\mathcal{MU}$); being a metrizable A[N]RU (see \cite[p. 111]{48}).

Lemma 4.3. (a) The following properties are equivalent: being an A[N]R($\mathcal{SMCU}$); being an A[N]E($\mathcal{SMCU}$); being a Polish A[N]RU.

(b) The following properties are equivalent: being an A[N]R($\mathcal{CM}$); being a compact metrizable A[N]RU.

Proof. (a). The implications
\[
\text{Polish A[N]EU} \Rightarrow \text{A[N]E}(\mathcal{SMCU}) \\
\Rightarrow \Rightarrow \\
\text{Polish A[N]RU} \Rightarrow \text{A[N]R}(\mathcal{SMCU})
\]
are obvious. Thus it suffices to show that if $Y$ is an $A[N]R(SMCU)$, then it is an $A[N]EU$. By the hypothesis $Y$ is a Polish uniform space. Then by Aharoni’s Theorem 2.43 it uniformly embeds onto a closed subset of $q_0$. By Corollary 2.33, $q_0$ is an AEU. Hence $f: A \to Y \subset q_0$ extends to a uniformly continuous map $\tilde{f}: X \to q_0$. Then $\tilde{f}$ composed with a uniformly continuous retraction of $Y$ in $q_0$ onto $Y$ is the required extension.

(b). Obviously, a product of AEU’s is an AEU; in particular, the Hilbert cube $I^\infty$ is an AEU. On the other hand, it is well-known that every compactum embeds in $I^\infty$. Hence every $A[N]R(CM)$ is a [neighborhood] retract of $I^\infty$, and therefore (similarly to the proof of (a)) it is an $A[N]EU$. □

Remark 4.4. The implication $A[N]R(SMCU) \Rightarrow A[N]E(SMCU)$ in (a) could be alternatively proved using an embedding into the nonseparable space $U(dY, I)$, where $dY$ is the discrete uniform space of the cardinality of $Y$ (see Remark 2.44(i)). The full assertion of (a) could be alternatively proved using the Banach–Mazur Theorem (see Remark 2.44(ii).

Remark 4.5. Let us mention some results on ARUs and ANRUs arising from Banach (=complete normed vector) spaces and Frechét (=complete metrizable locally convex vector) spaces. These will not be used in the sequel.

(a) If a Banach space $V$ is an ANRU, then every (bounded) closed convex body $B$ in $V$ is an ANRU (ARU), cf. [49; comments preceding 3.1].

Indeed, by translating we may assume without loss of generality that 0 is an interior point of $B$. Then the Minkowski functional $\|v\|_B = \inf\{r > 0 \mid v/r \in B\}$ yields a uniformly continuous retraction $v \mapsto v/\max\{1, \|v\|_B\}$. (Note that if $B$ is the unit ball, $\|v\|_B = \|v\|$. Hence $B$ is an ARU.

If in addition $B$ is bounded, then $v \mapsto (1 - t)v, t \in [0, 1]$, is a uniformly continuous null-homotopy of $B$ in itself. So $B$ is a uniformly contractible ANRU, hence an ARU [48; 1.11] (see also a somewhat different proof in Theorem 4.27 below). In fact, $v \mapsto \frac{\|v\|}{\|v\|_B}v$ is a uniform homeomorphism of $B$ onto the unit ball.

(b) Injective objects in the category of Banach spaces are ANRUs [49; second assertion of 3.1]. The space $L_\infty(\mu)$ for every measure $\mu$, in particular, the space $l_\infty$ of bounded real sequences, is injective as a Banach space [13; beginning of §1.5], and hence is an ANRU. In particular, its unit ball is an ARU by (a).

On the other hand, $c_0$ is not injective as a Banach space, for it is not a bounded linear retract of $l_\infty$ (R. S. Philips, 1940; see [58; proof of I.2.f.3]). However, $c_0$ is injective as a separable Banach space (see [58; I.2.f.4]) and is a uniform retract of $l_\infty$ [13; Example 1.5], [57; Theorem 6(a)]. Hence it is an ANRU, and its unit ball $Q_0 = U((N^+, \infty), ((-1, 1], 0))$ is an ARU. Note that $(x_1, x_2, \ldots) \mapsto (|x_1|, |x_2|, \ldots)$ is a uniformly continuous retraction of $Q_0$ onto $q_0$.

(c) A Banach space $V$ is an ANRU if it has uniformly normal structure [13; 1.26], that is, if there exists a $\gamma < 1$ such that every convex subset of $E$ of diameter 1 is contained in a ball of radius $\gamma$. The Banach spaces $L_p(\mu)$ with $1 < p < \infty$ and every measure $\mu$,
including the sequence spaces $l_p$ with $1 < p < \infty$, have uniformly normal structure (see [13; A.9] and [58; II.1.f.1, “in addition” part]). In particular, they are ANRUs, though this has more direct proofs [57; Theorem 8] (correcting a mistake in [49; 3.1, proof of (c)]), [13; 1.29]. In particular, the unit balls of these spaces are ARUs by (a).

The unit ball of $L_1(\mu)$ for every measure $\mu$ is uniformly homeomorphic to those of $L_p(\mu)$ for $1 < p < \infty$ (see [13; Remark to 1.29]), hence is also an ARU. In particular, the unit ball of $l_1$ is an ARU; in fact, according to [50; Remarks after the proof of 3.2], $l_1$ itself is an ANRU. A necessary condition for a Banach space $V$ to have unit ball uniformly homeomorphic to that of an $L_p(\mu)$, $1 \leq p < \infty$, is that the $n$-dimensional vector space with the $l_\infty$-norm) do not uniformly embed in $V$ for each $n$ [13; 9.21]. This condition is not sufficient [13; 9.23], but can be made sufficient by further hypotheses [13; 9.4, 9.7].

(d) Given a normed vector space $V$ identified with a subspace of the countably dimensional vector space $\bigoplus_{i=1}^{\infty} \mathbb{R}$ and given normed vector spaces $V_1, V_2, \ldots$, let us write $\bigoplus_i V_i$ for the normed vector space $f^{-1}(V)$, $||x|| = ||f(x)||_V$, where $f(x_1, x_2, \ldots) = (||x_1||_{V_1}, ||x_2||_{V_2}, \ldots)$. Each $\bigoplus_i V_i$ is isomorphic to a subspace of $c_0$ (see [57; (5.2)]).

The Banach space $(l_1^p \oplus l_2^p \oplus \ldots)_{c_0}$, where $1 \leq p \leq \infty$ and $l_1^p$ is the $n$-dimensional vector space with the $l_1$-norm, is an ARU [57; proof of Theorem 13].

On the other hand, none of the following Banach spaces, nor even their unit balls, is an ANRU: $(l_1^p \oplus l_2^p \oplus \ldots)_{c_0}$ and $(l_1^p \oplus l_2^p \oplus \ldots)_p$, as well as $(l_1^{n_1} \oplus l_2^{n_2} \oplus \ldots)_{c_0}$ and $(l_1^{n_1} \oplus l_2^{n_2} \oplus \ldots)_p$, where $1 < p_k < \infty$ for each $k$ and $p_k \to \infty$ as $k \to \infty$; in addition $1 \leq p \leq \infty$ and $(\log n_k)/p_k \to \infty$ as $k \to \infty$ [57; Corollary 1 of Theorem 10 and subsequent Remark 2] (see also [13; Example 1.30]).

(e) If $V$ is a Banach space, the space $H_{cb}(V)$ of all closed bounded nonempty convex subsets of $V$ with the Hausdorff metric is an ARU [49; 3.2].

(f) If $B$ is the unit ball of a Banach space, and $X$ is a residually finite-dimensional uniform space, then every uniformly continuous partial map $X \supset A \to Y$ extends to a uniformly continuous map $X \to Y$ [89].

(g) Frechét spaces are characterized as limits of inverse sequences of Banach spaces (in the category of topological vector spaces), see [14; Proposition I.6.4]. If a Frechét space $V$ is an ANRU, then every closed (bounded) convex body in $V$ is an ANRU (ARU). Indeed, if $V$ is the limit of an inverse sequence $\ldots \xrightarrow{f_1} V_1 \xrightarrow{f_0} V_0$ of Banach spaces $V_i$ and continuous linear maps, then each $f_i^n : V \to V_i$ is continuous, so its kernel $K_i$ is closed. Without loss of generality 0 is an interior point of $B$. Then $B_i := B + K_i$ is a closed convex body in $V$ that is absorbing (i.e. for every $v \in V$ there exists an $r$ such that $v/r \in B_i$), and then similarly to (a), $B_i$ is a uniform retract of $V$, and in particular of $B_{i-1}$. Since $B$ is the inverse limit of the $B_i$, it follows (see the proof of Theorem 5.18 below) that $B$ is a uniform retract of $B_{[0, \infty]}$, and therefore (see the proof of Theorem 4.21 below) also of $V$. The remainder of the proof is similar to that in (a).

(h) If $V$ is an injective object in the category of Frechét spaces, then every closed bounded convex body $B$ in $V$ is an ARU. Indeed, $V$ is injective as a locally convex space
(see [26; Lemma 0]), and $B$ is a uniform retract of $V$ by the argument in (g). Now the assertion follows from [66; Theorem 1.6].

Injective Frechet spaces include injective Banach spaces (see [26; Lemma 0]) and hence their countable products, and these appear to be all known examples (see [22] and [25]).

4.B. Uniform ANRs

Isbell’s proof that ANRUs are complete uses non-metrizable test spaces such as the ordinal $\omega_1$ in the order topology in an essential way, even when the given ANRU is itself metrizable. Relevance of this argument for the purposes of geometric topology is questionable; one would not be comfortable using this construction as a basis for geometrically substantial results. An alternative, more transparent approach (implicit in the papers of Garg [35] and Nhu [66], [67]; see also [91]) results from replacing embeddings with extremal (or regular) epimorphisms (i.e. embeddings onto closed subsets) in the definitions of $A[N]R(M\mu)$ and $A[N]E(M\mu)$.

Thus by a uniform $A[N]E$ we mean a metrizable uniform space $Y$ such that given a closed subset $A$ of a metrizable uniform space $X$, every uniformly continuous $f: A \to Y$ extends to a uniformly continuous map defined on [a uniform neighborhood of $A$ in] $X$. Similarly, by a uniform $A[N]R$ we mean a metrizable uniform space $Y$ such that for every uniform embedding $i$ of $Y$ onto a closed subspace of a metrizable uniform space $Z$ there exists a uniformly continuous retraction of [a uniform neighborhood of $Y$ in] $Z$ onto $i(Y)$. Obviously uniform $A[N]Rs$ include all metrizable $A[N]RUs$ and no other complete spaces.

Proposition 4.6 (Garg [35]). A metrizable uniform space $Y$ is a uniform $A[N]R$ if and only if it is a uniform $A[N]E$.

Proof. This follows from Theorem 3.8 by the usual adjunction space argument (see [44; §III.3]).

Remark 4.7. (i) The above proof is much easier than the original one in [35]; a proof that is arguably still easier (as it does not use metrizability of the adjunction space) is presented in [91; Appendix] modelled on an argument in [66].

(ii) Proposition 4.6 holds, with the same proof, if “metrizable” is replaced by “separable metrizable” throughout, including the definitions of uniform $A[N]R$ and $A[N]E$.

4.8. Homotopy completeness. We say that a uniform space $X$ is homotopy complete if there exists a uniform homotopy $H: \tilde{X} \times I \to \tilde{X}$, where $\tilde{X}$ is the completion of $X$, such that $H(x,0) = x$ and $H(\tilde{X} \times (0,1]) \subset X$. Note that if $X$ is homotopy complete, then it is uniformly $\varepsilon$-homotopy equivalent to its completion, for each $\varepsilon > 0$.

Remark 4.9. A subspace $A$ of a separable metrizable topological space $Y$ such that there exists a homotopy $h_t: Y \to Y$ satisfying $h_0 = \text{id}$ and $h_t(Y) \cap A = \emptyset$ for $t > 0$ is called a ‘$Z$-set’ by some authors including S. Ferry [31] and a ‘homotopy negligible set’ by some authors including T. Banakh (see [7]). It is well-known that under this assumption,
Y is an ANR if and only if \( Y \setminus A \) is an ANR (cf. [31; 1.1(iv)], [7; Proposition 1.2.1, Exercise 1.2.16]). Beware that Z-sets in a more usual sense (of Toruńczyk) are something else in general [7; Exercise 1.2.11] but the same in ANRs [7; Theorem 1.4.4, Exercise 1.2.10(a)\( \Leftrightarrow \) (c)]; homotopy negligible sets in a more usual sense (of Eels and Kuiper) omit the control (see [7; Exercises 1.2.10(a)\( \Leftrightarrow \) (b) and 1.2.12(a)\( \Leftrightarrow \) (d))].

**Theorem 4.10.** Suppose that \( X \) is a metrizable uniform space. Then \( X \) is a uniform \( A[N]R \) if and only if it is homotopy complete and its completion is an \( A[N]RU \).

A few month after having written up the proof of Theorem 4.10, the author learned that its analogue for semi-uniform ANRs (see Remark 4.30(b)) had been known [79].

**Proof.** For the “if” direction, we consider the case of ARUs; the case of ANRUs is similar. Let \( Y \) be a metrizable uniform space and \( A \) a closed subset of \( Y \). Then \( D: Y \to I \) defined by \( D(y) = \min\{d(y, A), 1\} \) satisfies \( D^{-1}(0) = A \). Given a uniformly continuous \( f: A \to X \), the hypothesis yields a uniformly continuous extension \( \bar{f}: Y \to \bar{X} \). Let \( F \) be the composition \( \bar{Y} \xrightarrow{f \times D} \bar{X} \times I \xrightarrow{H} \bar{X} \). Then \( F|_A = f \), and \( F(Y \setminus A) \subset H(\bar{X} \times (0, 1)) \subset X \). Thus \( F \) is a uniformly continuous extension of \( f \) with values in \( X \).

Conversely, suppose \( X \) is a uniform \( A[N]R \). Let \( Z_\varepsilon \) denote the subspace \( X \times (0, 1] \cup \bar{X} \times (0, 1) \) of \( \bar{X} \times [0, 1) \). Then \( X \times \{0\} \) is a closed subset of \( Z_1 \). Since \( X \) is a uniform ANR, \( \text{id}_X \) extends to a uniformly continuous map \( Z_\varepsilon \to X \) for some \( \varepsilon > 0 \). The latter has a unique extension over the completions, which yields the required homotopy.

It remains to show that \( \bar{X} \) is an \( A[N]RU \), or equivalently an \( A[N]RU(\mathcal{MU}) \); here we consider only the case of ARUs, the case of ANRUs being similar. Let \( Y \) be a metrizable uniform space and \( A \) a subset of \( Y \). Given a uniformly continuous \( f: A \to \bar{X} \), we uniquely extend it to a uniformly continuous \( \bar{f}: \bar{A} \to \bar{X} \). Recall that the homotopy \( H \) has been constructed; consider the composition \( F: \bar{A} \times I \xrightarrow{\bar{f} \times \text{id}_I} \bar{X} \times I \xrightarrow{H} \bar{X} \). Then \( F \) sends \( \bar{A} \times (0, 1] \) into \( X \), and so \( F|_{\bar{A} \times (0, 1]} \) extends to a uniformly continuous map \( \bar{F}: \bar{Y} \times (0, 1] \to X \). If a point \( (a, t) \) of \( \bar{A} \times I \) is close to a point \( (y, s) \) of \( \bar{Y} \times (0, 1] \), then they are both close to the point \( (a, s) \) of the intersection (by considering the \( l_\infty \) metric on the product); thus \( F \cup \bar{F}: \bar{A} \times I \cup \bar{Y} \times (0, 1] \to \bar{X} \) is uniformly continuous. Define \( D: \bar{Y} \to I \) by \( D(y) = \min\{d(y, \bar{A}), 1\} \); since \( \bar{A} \) is closed in \( \bar{Y} \), we have \( D^{-1}(0) = \bar{A} \).

Then the composition of the graph \( \Gamma_D: \bar{Y} \to \bar{A} \times \{0\} \cup \bar{Y} \times (0, 1] \) with \( F \cup \bar{F} \) provides the required extension of \( f \).

\( \square \)

**Corollary 4.11.** (a) [35; Example 2], [66; Remark 2.8] The half-open interval \( (0, 1] \) and the open interval \( (0, 1) \) are uniform ARs.

(b) \([-1, 0) \cup (0, 1) \) is not a uniform ANR.

**Corollary 4.12.** A separable metrizable uniform space is a uniform \( A[N]R \) if and only if it is a [neighborhood] retract of every separable metrizable uniform space containing it as a closed subset.

This follows from Lemma 4.3(a) and the proof of Theorem 4.10, which also works in the separable case.
4.13. Uniform local contractibility. We call a metric space $X$ uniformly locally contractible if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every uniform space $Y$, every two $\delta$-close uniformly continuous maps $f, g: Y \to X$ are uniformly $\varepsilon$-homotopic (that is, are joined by a uniformly continuous homotopy $Y \times I \to X$ that is $\varepsilon$-close to the composition $Y \times I \rightharpoonup Y \to X$). Obviously uniform local contractibility does not depend on the choice of the metric on $X$ in its uniform equivalence class.

Remark 4.14. If $X$ is uniformly locally contractible, then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\delta$-balls in $X$ contract within their concentric $\varepsilon$-balls, by a uniformly equicontinuous family of null-homotopies (see [49; 4.2]).

Remark 4.15. If $X$ is uniformly locally contractible, then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $n$ and every uniform space $Y$, every $(n-1)$-sphere $F: Y \times S^{n-1} \to X$ of maps $Y \to X$ that are within $\delta$ of each other bounds an $n$-ball $\bar{F}: Y \times B^n \to X$ of maps $Y \to X$ that are within $\varepsilon$ of each other. Indeed, the definition of uniform local contractibility yields an $\varepsilon$-homotopy $h_i$ between $F$ and the composition of the projection $Y \times \partial I^n \to Y$ and $F|_{Y \times pt}: Y \to X$; then $\bar{F}$ can be defined by $\bar{F}(y, [(r, \varphi)]) = h_{1-r}(y, \varphi)$, where $[(r, \varphi)]$ is the image of $(r, \varphi) \in [0, 1] \times S^{n-1}$ in $B^n = [0, 1] \times S^{n-1}/\{(0) \times S^n\}$.

Lemma 4.16. Let $X$ be a (separable; compact) metric space. Then $X$ is uniformly locally contractible if either

(a) $X$ is a uniform ANR, or

(b) for each (separable; compact) metrizable uniform space $Y$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that every two $\delta$-close uniformly continuous maps $f, g: Y \to X$ are uniformly $\varepsilon$-homotopic.

We note that the hypothesis of (b) is a weakening of the condition of uniform local contractibility, with restrictions imposed on $Y$ and with $\delta$ allowed to depend on $Y$. It follows from (b) that $X$ is uniformly locally contractible if and only if it is LCU in the sense of Isbell (whose $\delta$ is allowed to depend on $Y$) [49]. That ANRUs are LCUs was known to Isbell [49; 4.2]; his proof of this fact is rather different (via functional spaces).

Proof. Let $U_i$ be the $\frac{1}{m}$-neighborhood of the diagonal in $X \times X$ in the $l_\infty$ product metric on $X \times X$, and let $f_i, g_i: U_i \subset X \times X \to X$ be the two projections. We shall show that, under the hypothesis of either (a) or (b), for each $\varepsilon > 0$ there exists an $i$ such that $f_i$ and $g_i$ are uniformly $\varepsilon$-homotopic.

(a) Let $Y_n = \bigsqcup_{i \geq n} U_i \times [0, \frac{1}{i}]$, and let $A_n = \bigsqcup_{i \geq n} U_i \times \{0, \frac{1}{i}\}$. Define $f: A_1 \to X$ via $f_i$ on $U_i \times \{0\}$ and via $g_i$ on $U_i \times \{\frac{1}{i}\}$. Since $X$ is a uniform ANR, $f$ has a uniformly continuous extension $\bar{f}$ over a uniform neighborhood $U$ of $A_1$ in $Y_1$. This $U$ contains the $\frac{1}{m}$-neighborhood of $A$ for some $n$, in the $l_\infty$ product metric on $Y \subset X \times X \times N \times [0, 1]$, which in turn contains $Y_n$. Moreover, for each $\varepsilon > 0$ there exists an $m \geq n$ such that $\bar{f}$ takes $\frac{1}{m}$-close points into $\varepsilon$-close points. Then $\bar{f}$ restricted to $U_m \times (0, \frac{1}{m})$ is a uniform $\varepsilon$-homotopy between $f_m$ and $g_m$.

(b) Consider $F = \bigcup_{i \in \mathbb{N}} f_i$ and $G_n := \bigcup_{i \in [n]} f_i \cup \bigcup_{i \in \mathbb{N} \setminus [n]} g_i$, both mapping $Y := \bigcup U_i$ into $X$. Note that $d(F, G_n) \to 0$ as $n \to \infty$. Then by the hypothesis of (b), for each
there exists an \( n \) such that \( F \) and \( G_n \) are uniformly \( \varepsilon \)-homotopic. Then also \( f_{n+1} \) and \( g_{n+1} \) are uniformly \( \varepsilon \)-homotopic.

We have thus shown, in both (a) and (b), that for each \( \varepsilon > 0 \) there exists a uniform \( \varepsilon \)-homotopy \( h_i: U_i \to X \) between \( f_i \) and \( g_i \). Now given \( \frac{1}{n} \)-close maps \( f, g: Y \to X \), the image of \( f \times g: Y \to X \times X \) lies in \( U_i \). Then the composite homotopy \( Y \xrightarrow{f \times g} U_n \xrightarrow{h_i} X \) is a uniform \( \varepsilon \)-homotopy between \( f \) and \( g \).

\[ \square \]

**Remark 4.17.** Let us call a metric space \( X \) *uniformly locally equiconnected* if for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every uniform space \( Y \), and a subset \( A \) of \( Y \), every two \( \delta \)-close uniformly continuous maps \( f, g: Y \to X \) that agree on \( A \) are uniformly \( \varepsilon \)-homotopic keeping \( A \) fixed. Then Lemma 4.16 has a relative version: (a) uniform ANRs are uniformly locally equiconnected; (b) in the definition of uniform local equiconnectedness, \( Y \) may be assumed to be metrizable and \( A \) to be closed; and if \( X \) is separable (compact), \( Y \) may be additionally assumed to be separable (compact). The proof is similar, and shows that, under the hypothesis of either (a) or (b), the uniform \( \varepsilon \)-homotopy between \( f_i \) and \( g_i \) can be chosen so as to keep the diagonal \( \Delta_X \subset U_i \) fixed.

Uniformly locally equiconnected compacta coincide with locally equiconnected compacta [28; Theorem 2.5]. Whether locally equiconnected compacta are ANRs is a long standing open problem going back to Fox [32]. Its special case is the so-called “compact AR problem” of whether a compact convex subset of a metrizable topological vector space is an AR.

Uniformly locally contractible compacta were considered in [69].

**4.18. Hahn property.** Following Isbell [49], we say that a metric space satisfies the *Hahn property* if for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every uniform space \( Y \) and every \( \delta \)-continuous map \( \Xi: Y \to X \) (that is, a possibly discontinuous map such that there exists a uniform cover \( C \) of \( Y \) such that \( \Xi \) sends every element of \( C \) into a set of diameter at most \( \delta \)), there exists a uniformly continuous map \( f: Y \to X \) that is \( \varepsilon \)-close to \( \Xi \). Obviously the satisfaction of the Hahn property does not depend on the choice of the metric on \( X \) in its uniform equivalence class.

**Lemma 4.19.** Let \( X \) be a (separable; compact) metric space. Then \( X \) satisfies the Hahn property if either

(a) \( X \) is a uniform ANR, or

(b) for each (separable; compact) metrizable uniform space \( Y \) and every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that every \( \delta \)-continuous map \( Y \to X \) is \( \varepsilon \)-close to a uniformly continuous map \( Y \to X \).

We note that the hypothesis of (b) is a weakening of the Hahn condition, with restrictions imposed on \( Y \) and with \( \delta \) allowed to depend on \( Y \). That ANRUs satisfy the Hahn property was shown by Isbell [49; 4.2]; this along with Theorem 4.10 implies (a).

**Proof of (b).** Let \( F \) be a (separable; compact) metric ARU containing \( X \) (see Theorem 2.43 and Remark 2.44). Let \( U_i \) be the \( \frac{1}{i} \)-neighborhood of \( X \) in \( F \), and let \( U = \bigcup U_i \).
Pick a $\frac{1}{3n}$-continuous retraction $\xi_i: U_i \to X$ (cf. the proof of Theorem 4.21), and define $\Xi_n: U \to X$ via $\xi_i$ on each $U_i$ for $i \geq n$, and by a constant map on $U_1 \sqcup \cdots \sqcup U_{n-1}$. Thus $\Xi_n$ is $\frac{1}{3n}$-continuous. By the hypothesis, for each $\varepsilon > 0$ there exists an $n \geq \frac{1}{\varepsilon}$ such that $\Xi_n$ is $\frac{\varepsilon}{2}$-close to a uniformly continuous map $U \to X$. Then $\xi_n$ is $\frac{\varepsilon}{2}$-close to a uniformly continuous map $r_n: U_n \to X$.

By (a), $F$ satisfies the Hahn property. So for each $n$ there exists a $\delta > 0$ such that every $\delta$-continuous map $\Xi: Y \to X$ is $1_n$-close to a uniformly continuous map $f: Y \to U_n \subset F$. Then $\Xi$ is $\varepsilon$-close to the composition $Y \overset{f}{\to} U_n \overset{r_n}{\to} X$. $\square$

**Theorem 4.20.** The following are equivalent for a compactum $X$:

(i) $X$ satisfies the Hahn property;

(ii) for each $\varepsilon > 0$ there exists a compact polyhedron $P$ and continuous maps $X \overset{f}{\to} P \overset{g}{\to} X$ whose composition is $\varepsilon$-close to $id_X$;

(iii) for each $\varepsilon > 0$ there exists a compactum $Y$ satisfying the Hahn property and continuous maps $X \overset{f}{\to} Y \overset{g}{\to} X$ whose composition is $\varepsilon$-close to $id_X$.

This means that compacta satisfying the Hahn property coincide with the “approximate ANRs” of Noguchi and Clapp [23], which are also known as “NE-sets” after Borsuk [16] and as compacta that are “approximate polyhedra” in the sense of Mardešić [59].

We shall generalize Corollary 4.20 to all separable metrizable uniform spaces in the sequel to this paper which deals with uniform polyhedra.

**Proof.** (i) implies (ii) using that $X$ is the inverse limit of an inverse sequence of compact polyhedra (see also Lemma A.12(a) below). Since compact polyhedra are ANRs, hence uniform ANRs, and in particular satisfy the Hahn property, (ii) implies (iii). The implication (iii)$\Rightarrow$(i) is an easy exercise. $\square$

**Theorem 4.21.** Let $X$ be a metrizable uniform space. Then $X$ is a uniform ANR if and only if $X$ is uniformly locally contractible and satisfies the Hahn property.

This improves on the metrizable case of Isbell’s characterization of ANRUs [49; 4.3], which additionally involved the homotopy extension property (see Lemma 4.26 below).

A few months after having written up the proof of Theorem 4.21, the author discovered that a rather similar characterization of non-uniform ANRs is stated without proof in [2; Theorem 3.7] (some relevant ideas can be found also in Theorems 3.2 and 4.8 in [2]).

**Proof.** If $X$ is a uniform ANR, it is uniformly locally contractible by Lemma 4.16(a) and satisfies the Hahn property by Lemma 4.19(a). Conversely, suppose $Y$ is a metric space, $A$ is a closed subset of $Y$, and $f: A \to X$ is a uniformly continuous map. Fix some metric on $X$. Let $\gamma_i = \delta_{LCU}(2^{-n})$ be given by the uniform local contractibility of $X$ with $\varepsilon = 2^{-n}$; let $\beta_i = \delta_{Hahn}(\gamma_i)$ be given by the Hahn property of $X$ with $\varepsilon = \gamma_i$; and let $\alpha_i = \delta_{f}(\beta_i)$ be such that $f$ sends $3\alpha_i$-close points into $\beta_i$-close points. We may assume that $\alpha_{i+1} \leq \alpha_i$, $\gamma_{i+1} \leq \gamma_i$ and $\beta_i \leq \gamma_i$ for each $i$. Let $U_i$ be the $\alpha_i$-neighborhood of $A$ in $Y$; thus $A = \bigcap U_i$. 
Define $\Xi_i: U_i \to X$ by sending every $x \in U_i$ into $f(x'_i)$, where $x'_i \in A$ is $\alpha_i$-close to $x$. (When $Y$ is separable, it is easy to find such an $x'_i$ using only the countable axiom of choice.) Then $\Xi_i$ sends $\alpha_i$-close points $x, y$ into $\beta_i$-close points $f(x'_i), f(y'_i)$ (using that $x'_i$ is $3\alpha_i$-close to $y'_i$). By the Hahn property, there exists a uniformly continuous map $f_i: U_i \to X$ such that $f_i$ is $\gamma_i$-close to $\Xi_i$; in particular, $f_i|_A$ is $\gamma_i$-close to $f$. Given an $x \in U_{i+1}$, it is $\alpha_i$-close to both $x'_i$ and $x'_{i+1}$, which are therefore $2\alpha_i$-close to each other. Hence $\Xi_i(x) = f(x'_i)$ is $\beta_i$-close to $f(x'_{i+1}) = \Xi_{i+1}(x)$. Thus $f_i|_{U_{i+1}}$ is $3\gamma_i$-close to $f_{i+1}$. Then by the uniform local contractibility, $f_i|_{U_{i+1}}$ is uniformly $2^{-i}$-homotopic to $f_{i+1}$. These homotopies combine into a uniformly continuous map $H$ on the extended mapping telescope $U_{[1,\infty]} := A \cup \bigcup_{i \in \mathbb{N}} U_i \times [2^{-i-1}, 2^{-i}] \subset U_1 \times [0, 1]$ of the inclusions $\cdots \subset U_2 \subset U_1$, which restricts to the identity on $A$.

Pick a uniformly continuous function $\varphi: U_1 \to [0, 1]$ such that $\varphi^{-1}(0) = A$ and $\varphi^{-1}([0, 2^{-i}]) \subset U_i$. For instance, $\varphi(x) = \alpha d(x, A)$ works, where $\alpha: [0, \alpha_1] \to [0, 1]$ is a monotone homeomorphism such that $\alpha^{-1}(2^{-i}) \leq \alpha_i$ for each $i$. Then $\text{id} \times \varphi: U_1 \to U_1 \times [0, 1]$ embeds $U_i$ into $U_{[1,\infty]}$ and restricts to the identity on $A$. Hence the composition $U_1 \xrightarrow{\text{id} \times \varphi} U_{[1,\infty]} \xrightarrow{H} X$ is a uniformly continuous extension of $f$. \hfill \square

As a consequence we get the following uniform analogues of Hanner’s two characterizations of ANRs ([37; Theorems 7.1 and 7.2], [44; Theorems IV.5.3 and IV.6.3]).

**Corollary 4.22.** (a) A metric space is a uniform ANR if it is uniformly $\varepsilon$-dominated by a uniform ANR for each $\varepsilon > 0$.

(b) Suppose that a metric space $X$ is uniformly embedded in a uniform ANR $Z$. Then $X$ is a uniform ANR if and only if for each $\varepsilon > 0$ there exists a uniformly continuous map $\varphi$ of a uniform neighborhood of $X$ in $Z$ into $X$ such that $\varphi|_X$ is uniformly $\varepsilon$-homotopic to $\text{id}_X$ with values in $X$.

**Proof.** (b). The “only if” assertion follows using that $X \times \{0\}$ is closed in $X \times \{0\} \cup Z \times (0, 1]$ (compare with Theorem 4.10).

To prove the “if” assertion, fix an $\varepsilon > 0$ and let $\delta$ be such that that the uniform neighborhood $U$ provided by the hypothesis contains the $\delta$-neighborhood of $X$ in $Z$, and the map $\varphi$ provided by the hypothesis is $(\delta, \varepsilon)$-continuous.

By Lemma 4.16(a), there exists a $\gamma > 0$ such that for every uniform space $Y$, every two $\gamma$-close uniformly continuous maps $f, g: Y \to X$ are uniformly $\delta$-homotopic with values in $Z$. Then the homotopy actually has values in $U$, and therefore $\varphi f$ and $\varphi g$ are uniformly $\varepsilon$-homotopic with values in $X$. On the other hand, by the hypothesis they are uniformly $\varepsilon$-homotopic to $f$ and $g$, respectively, with values in $X$. Thus $X$ is uniformly locally contractible.

By Lemma 4.19(a), there exists a $\beta > 0$ such that for every uniform space $Y$, every $\beta$-continuous map $\Xi: Y \to X$ is $\varepsilon$-close to a uniformly continuous map $h: Y \to Z$. Then $h$ actually has values in $U$, and then $\varphi h: Y \to X$ is $\varepsilon$-close to $\varphi \Xi$. By the hypothesis, the latter is in turn $\varepsilon$-close to $\Xi$. Thus $X$ satisfies the Hahn property.

So we infer from Theorem 4.21 that $X$ is a uniform ANR. \hfill \square
(a). Let $X$ be the given space. We may assume that it is embedded in a uniform ANR (for instance, in some ARU) $Z$. Given an $\varepsilon > 0$, the hypothesis provides a uniform ANR $Y$ that uniformly $\varepsilon$-dominates $X$. Thus we are given uniformly continuous maps $u: X \to Y$ and $d: Y \to X$ such that the composition $X \xrightarrow{u} Y \xrightarrow{d} X$ is uniformly $\varepsilon$-homotopic to the identity. Let $\delta$ be such that $d$ is $(\delta, \varepsilon)$-continuous. Since $Y$ is a uniform ANR, and $X \times \{0\}$ is closed in $X \times \{0\} \cup Z \times (0, 1]$, $u$ is uniformly $\delta$-homotopic to the restriction of a uniformly continuous map $\bar{u}: U \to Y$ for some uniform neighborhood $U$ of $X$ in $Z$. Then $du$ is uniformly $\varepsilon$-homotopic to $d\bar{u}|_X$. On the other hand, $du$ is uniformly $\varepsilon$-homotopic to the identity. So we infer from (b) that $X$ is a uniform ANR.

Another consequence of Theorem 4.21 is the following

**Theorem 4.23.** A finite dimensional metrizable uniform space is a uniform ANR if and only if it is uniformly locally contractible.

**Proof.** Let $X$ be a uniformly locally contractible $n$-dimensional space. Then its completion $\bar{X}$ is still $n$-dimensional (see Lemma 2.39). By [50; Theorem V.34], $\bar{X}$ is the limit of an inverse sequence of $n$-dimensional uniform polyhedra $P_i$. Since $X$ is uniformly locally contractible, every collection of small spheroids $S^{k+1} \to X$ bounds a uniformly equicontinuous family of small singular disks $D^{k+1} \to X$. Using this, it is easy to construct maps $r_i: P_i \to X$ by induction on skeleta, so that for each $\varepsilon > 0$ there is an $i$ such that the composition $X \subset \bar{X} \xrightarrow{\pi_i} P_i \xrightarrow{r_i} \to X$ is $\varepsilon$-close to the identity. By [48; 1.9], each $P_i$ is a uniform ANR and so (see Lemma 4.19(a)) satisfies the Hahn property. Since each $p_i^\infty: \bar{X} \to P_i$ is uniformly continuous, we infer that $X$ also satisfies the Hahn property. By Theorem 4.21, $X$ is a uniform ANR.

**Remark 4.24.** Borsuk’s example of a locally contractible compactum that is not an ANR (see [15]) shows that uniform local contractibility of a residually finite dimensional metrizable uniform space $X$ does not follow from uniform local connectedness of each of the functional spaces $F_n = U(S^n \times \mathbb{N}, X)$. By the proof of Theorem 4.23, it does follow when $X$ is finite-dimensional.

**Remark 4.25.** By the proof of Theorem 4.23 (see also §5), if $X$ is a uniformly locally contractible metrizable uniform space, then for every finite-dimensional uniform space $Y$, every uniformly continuous partial map $Y \supset A \to X$ extends to a uniformly continuous map $Y \to X$.

The following uniform Borsuk homotopy extension lemma is proved in a straightforward way; similarly to [48; 1.10], [50; VI.17] (see also [15; IV.8.1] for the non-uniform version).

**Lemma 4.26.** Suppose that $Y$ is a uniform ANR, $X$ is a metrizable uniform space and $A$ is a closed subset of $X$. If a homotopy $h: A \times I \to Y$ extends over $X \times \{0\}$, then it extends over $X \times I$.

**Theorem 4.27.** A metrizable uniform space is a uniform AR if and only if it is a uniform ANR and is uniformly contractible.
This is parallel to [48; 1.11], apart from using the metrizability of the cone.

Proof. If \( Y \) is a uniform AR, by definition it is a uniform ANR. The cone \( CY \) is metrizable, so \( \text{id}_Y \) extends to a uniformly continuous map \( CY \to Y \); the composition \( Y \times I \to CY \to Y \) is then a uniform null-homotopy of \( \text{id}_Y \).

Conversely, let \( A \) be a closed subset of a metrizable uniform space \( X \). If \( Y \) is uniformly contractible, every uniformly continuous map \( f : A \to Y \) is homotopic to a constant map, which extends over \( X \). If additionally \( Y \) is a uniform ANR, then by Lemma 4.26 the homotopy extends over \( X \times I \); in particular, \( f \) extends over \( X \).

\[\square\]

Lemma 4.28. Let \( X \) be a metrizable uniform space and \( A \) a closed subset of \( X \).

(a) Suppose that \( A \) is a uniform ANR. Then \( A \) is a uniform deformation retract of \( X \) if and only if the inclusion \( A \hookrightarrow X \) is a uniform homotopy equivalence.

(b) Suppose that \( X \) is a uniform ANR. Then \( A \) is a uniform deformation retract of \( X \) if and only if \( A \) is a uniform strong deformation retract of \( X \).

This is proved by usual arguments [82; 1.4.10 and 1.4.11] making use of the uniform Borsuk lemma (Lemma 4.26).

Lemma 4.29. If \( f : X \to Y \) is a uniform homotopy equivalence of uniform ANRs, then \( f \) is a pointed uniform homotopy equivalence of \( (X, x) \) and \( (Y, f(x)) \) for each \( x \in X \).

This is proved by usual arguments (see e.g. [74; Lecture 4, Proposition 1]) making use of the uniform Borsuk lemma (Lemma 4.26).

Remark 4.30. Several variations of the notion of a uniform A[N]R exist in the literature.

(a) Nhu [66], [67] uses closed isometric, rather than uniform, embeddings to define what may be termed metric uniform A[N]Rs. For bounded metrics the distinction is vacuous by Lemma 3.4, but in general it is not: the real line with its usual metric is a metric uniform AR; whereas every uniform AR is uniformly contractible by Theorem 4.27 and therefore is \( \mathbb{R} \)-bounded (cf. §2.49). Nevertheless, metric uniform ARs of finite diameter are precisely uniform ARs with a choice of a metric [66]; and (arbitrary) metric uniform ANRs are precisely uniform ANRs with a choice of a metric [91; Appendix].

(b) Michael calls a map \( f : X \to Y \) of metric spaces uniformly continuous at a closed subset \( A \subseteq X \) if for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such \( f \) is \((\delta, \varepsilon)\)-continuous on the \( \delta \)-neighborhood of \( A \) [63]. Michael [63] and Torunczyk [86] (see also [67; p. 193]) use continuous retractions of uniform neighborhoods that are uniformly continuous at their target to define what may be termed semi-uniform A[N]Rs. These lie between usual (metrizable) A[N]Rs and our uniform A[N]Rs; in fact every A[N]R is the underlying space of a semi-uniform A[N]R [63], [86]. Sakai established the analogue of Theorem 4.10 for semi-uniform ANRs [79].

(c) Lipschitz, and 1-Lipschitz ANRs and ARs have been studied. According to [52; p. 65], 1-dimensional topologically complete ARs are metrizable as 1-Lipschitz ARs (see [72] for a proof in the compact case). On the other hand, Isbell showed that 2-dimensional non-collapsible compact polyhedra are not metrizable as 1-Lipschitz ARs.
[52]. It appears to be unknown whether every ANR is homeomorphic to a Lipschitz ANR (cf. [43]). However, Hohti showed that every LC$_n$ compactum can be remetrized so as to be Lipschitz $n$-LC for all $n > 0$ [43].

**Remark 4.31.** We mention some facts relating to semi-uniform ANRs.

(a) Similarly to the proof of Theorem 4.21, a metrizable uniform space is a semi-uniform ANR if and only if it is semi-uniformly locally contractible and satisfies the weak Hahn property. We call a metric space $M$ semi-uniformly locally contractible if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every two $\delta$-close continuous maps of a metrizable topological space into $M$ are $\varepsilon$-homotopic. (That semi-uniform ANRs satisfy this property but not conversely was observed by Michael [63].) We say that $M$ satisfies the weak Hahn property if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\gamma > 0$, every $(\gamma, \delta)$-continuous map $f$ of a metric space $N$ into $M$ is $\varepsilon$-close to a continuous map. (Note that the metric on $N$ is irrelevant, and the hypothesis on $f$ is equivalent to saying that for every convergent sequence $(x_n)$ in $N$, the set of limit points of the sequence $(f(x_n))$ has diameter $< \delta$.)

(b) Similarly to the proof of Theorem 4.23, the following are equivalent for a finite dimensional metrizable uniform space $X$: (i) $X$ is a semi-uniform ANR; (ii) $X$ is semi-uniformly locally contractible; (iii) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every continuous map $S^n \to X$ with image of diameter $\leq \delta$ bounds a continuous map $B^{n+1} \to X$ with image of diameter $\leq \varepsilon$.

4.C. Homotopy limits and colimits of uniform ANRs

**Theorem 4.32.** Let $X$, $Y$ and $A$ be uniform ANRs, where $A$ is a closed subset of $X$. If $f: A \to Y$ is a uniformly continuous map, then the adjunction space $X \cup_f Y$ is a uniform ANR.

This is a uniform analog of J. H. C. Whitehead’s theorem (see [15; Theorem V.9.1]). Our proof is a modification of Hanner’s proof of Whitehead’s theorem [37; Theorem 8.2], [44; Theorem VI.1.2].

**Lemma 4.33.** Assume the hypothesis of Theorem 4.32 and fix a metric on $X \cup_f Y$. Then for each $\varepsilon > 0$ there exists a uniform $\varepsilon$-homotopy $H: X \cup_f Y \to X \cup_f Y$ keeping $Y$ fixed and such that $H_0$ is the identity, and $H_1$ retracts a uniform neighborhood of $Y$ in $X \cup_f Y$ onto $Y$.

Moreover, $H_t$ lifts to a uniform homotopy $h: X \to X$ keeping $A$ fixed and such that $h_0$ is the identity and $h_1$ retracts a uniform neighborhood of $A$ in $X$ onto $A$.

**Proof.** Since $A$ is a uniform ANR, there is a uniform retraction $r$ of a uniform neighborhood $U$ of $A$ in $X$ onto $A$. The projection $X \times \{0\} \cup A \times I \to X$ combines with $r$ into a uniformly continuous map $R: X \times \{0\} \cup A \times I \cup U \times \{1\} \to X$. Since $X$ is a uniform ANR, the latter extends to a uniformly continuous map $\tilde{R}$ on a uniform neighborhood $W$ of $X \times \{0\} \cup A \times I \cup U \times \{1\}$ in $X \times I$. This $W$ contains the region $\Phi$ below the graph of a function $\varphi: (X, V) \to ([\delta, 1], \{1\})$, where $V$ is a uniform neighborhood of $A$ in $U$.
and $\delta > 0$. Then $h_t: X \to X$, defined by $h_t(x) = \bar{R}(x, t\varphi(x))$, is a uniform homotopy, keeping $A$ fixed, between $id_X$ and an extension $\bar{r}$ of $r$ over $X$. Since $\Phi$ may be assumed to be contained in any uniform neighborhood of $X \times \{0\} \cup A \times I$ given in advance, $h_t$ may be assumed to be an $\varepsilon$-homotopy.

Consider the self-homotopy $H_t$ of $X \cup_f Y$ defined by $H_t(y) = y$ for each $y \in Y$ and all $t \in I$ and by $H_t([x]) = [h_t(x)]$ for $x \in X$ and all $t \in I$. It is well-defined since $h_t$ fixes $A$. To prove that $H_t$ is uniform, we consider the $d_3$ metric as in Theorem 3.8. Given $x, x' \in X$, we have $d_3([x], [x']) = \min\{d(x, x'), \inf_{a, a' \in A} d(x, a) + d(f(a), f(a')) + d(a', x')\}$. Hence if $([x], t)$ is $\alpha$-close to $([x'], t')$, then $|t - t'| \leq \alpha$, and either $d(x, x') \leq \alpha$, or there exist $a, a' \in A$ with $d(x, a) \leq \alpha$, $d(x', a') \leq \alpha$ and $d(f(a), f(a')) \leq \alpha$. Suppose $h_t$ is $(\alpha, \beta)$-continuous, viewed as a map $X \times I \to X$, where $\beta \geq \alpha$. Then either $h_t(x)$ is $\beta$-close to $h_t(x')$, or $|t - t'| \leq \beta$ and there exist $a, a' \in A$ such that $h_t(x)$ is $\beta$-close to $h_t(a) = a$, $h_t(x')$ is $\beta$-close to $h_t(a') = a'$, and $d(f(a), f(a')) \leq \beta$. Therefore $d_3([h_t(x)], [h_t(x')]) \leq 3\beta$. Thus $H_t$ is a uniform homotopy.

It remains to observe that the image of $V \cup Y$ in $X \cup_f Y$ is a uniform neighborhood of $Y$ in $X \cup_f Y$, by considering the $d_3$ metric. 

\[\square\]

Proof of Theorem 4.32. We only consider the case of ANRs; the case of ARs is similar, and alternatively it can be deduced from the case of ANRs using Theorem 4.27.

Suppose $X \cup_f Y$ is a closed subset of a metrizable uniform space $Z$. We are going to apply Theorem 4.22(b); to this end, fix an $\varepsilon > 0$, and feed it into the preceding lemma. Let $U_Y$ be the uniform neighborhood of $A$ in $X$ provided by the preceding lemma, and let $h_t$ and $H_t$ be the homotopies provided by the preceding lemma. Define $\bar{f}: X \to X \cup_f Y$ by $\bar{f}(x) = [x]$. Let $U_X$ be a uniform neighborhood of $X \setminus U_Y$, uniformly disjoint from $A$; then $\bar{f}$ uniformly embeds $U_X$. Write $V = X \cup_f Y$, $V_A = \bar{f}(U_X \cap U_Y)$, $V_X = \bar{f}(U_X)$ and $V_Y = Y \cup \bar{f}(U_Y)$. Let $Z_X$ and $Z_Y$ be uniformly disjoint open uniform neighborhoods respectively of $V_X \setminus V_A$ and of $V_Y \setminus V_A$ in $Z$, and set $Z_A = Z \setminus (Z_X \cup Z_Y)$.

Since $A$ is a uniform ANR, $h_1\bar{f}^{-1}|_{V_A}$ extends to a uniformly continuous map $\varphi_A: W_A \to A$, where $W_A$ is a uniform neighborhood of $V_A$ in $V_A \cup Z_A$. Since $V_A$ is a uniform neighborhood of $Z_A \cap V$ in $V$, there exists a uniform neighborhood $N_A$ of $V$ in $Z$ such that $N_A \cap Z_A \subset W_A$. Since $X$ is a uniform ANR, $(h_1\bar{f}^{-1}|_{V_X}) \cup \varphi_A: V_X \cup W_A \to X$ extends to a uniformly continuous map $\varphi_X: W_X \cup W_A \to X$, where $W_X$ is a uniform neighborhood of $V_X$ in $Z_X \cup V_X$. Since $V_X$ is a uniform neighborhood of $Z_X \cap V$ in $V$, there exists a uniform neighborhood $N_X$ of $V$ in $Z$ such that $N_X \cap Z_X \subset W_X$. Since $Y$ is a uniform ANR, $H_1|_{V_X} \cup \bar{f}\varphi_A: V_Y \cup W_A \to Y$ extends to a uniformly continuous map $\varphi_Y: W_Y \cup W_A \to Y$, where $W_Y$ is a uniform neighborhood of $V_Y$ in $Z_Y \cup V_Y$. Since $V_Y$ is a uniform neighborhood of $Z_Y \cap V$ in $V$, there exists a uniform neighborhood $N_Y$ of $V$ in $Z$ such that $N_Y \cap Z_Y \subset W_Y$.

Since $Z_X$ and $Z_Y$ are uniformly disjoint, so are $W_X$ and $W_Y$, and therefore the map $(f\varphi_X) \cup \varphi_Y: W_X \cup W_A \cup W_Y \to V$ is well-defined and uniformly continuous. By construction it restricts to $H_1$ on $V$. On the other hand, $W_X \cup W_A \cup W_Y$ contains the
uniform neighborhood $N_X \cap N_A \cap N_Y$ of $V$ in $Z$. So we infer from Theorem 4.22(b) that $V$ is a uniform ANR.

**Corollary 4.34.** If $X$ and $Y$ are uniform $A[N]Rs$ each containing a closed copy of a uniform $A[N]R$ $A$, then the amalgamated union $X \cup_A Y$ is a uniform $A[N]R$.

Modulo Corollary 3.6, this also follows from the results of Nhu [66, 67], who used the $d_2$ metric on the underlying set of $X \cup_A Y$ but did not identify it as a metric of the quotient uniformity.

**Theorem 4.35.** If $Y$ is a metrizable uniform space, $B \subset Y$, and $X$ and $A \subset X$ are uniform $A[N]Rs$, then $U((Y, B), (X, A))$ is a uniform $A[N]R$.

For an alternative proof see Corollary 4.44 below.

**Proof.** We consider the case of ANRs; the case of ARUs follows using Theorem 4.27.

By Theorem 4.10, the completion $\bar{X}$ of $X$ is an ANRU. Hence by Theorem 2.36, $U(Y, \bar{X})$ is an ANRU. Again by Theorem 4.10, there exists a uniform homotopy $h_t$ of $\bar{X}$ such that $h_0 = \text{id}$ and $h_t(\bar{X}) \subset X$ for $t > 0$. Now $H_t: f \mapsto h_t f$ is a uniform homotopy of $U(Y, \bar{X})$ such that $H_0 = \text{id}$ and $H_t(U(Y, \bar{X})) \subset U(Y, X)$ for $t > 0$. Hence by Theorem 4.10, $U(Y, X)$ is a uniform ANR.

Since $A$ is a uniform ANR, by Theorem 4.22(b) for each $\varepsilon > 0$ there exists uniformly continuous map $g: U \to A$, where $U$ is a uniform neighborhood of $A$ in $X$ such that $g|_A$ is uniformly $\varepsilon$-homotopic to $\text{id}_A$ with values in $A$. Then $G: f \mapsto gf$ is a uniformly continuous map $U((Y, B), (X, U)) \to U((Y, B), (X, A))$ such that $G|_{U((Y, B), (X, A))}$ is uniformly $\varepsilon$-homotopic to $\text{id}|_{U((Y, B), (X, A))}$ with values in $U((Y, B), (X, A))$. Since $U((Y, B), (X, U))$ is a uniform neighborhood of $U((Y, B), (X, A))$ in $U(Y, X)$, again by Theorem 4.22(b), $U((Y, B), (X, A))$ is a uniform ANR.

**Remark 4.36.** Under the hypothesis of 4.35, the path components of $U((X, A), (Y, B))$ form a uniformly disjoint collection (see [50; V.16] and its proof). In particular, each path component of $U((X, A), (Y, B))$ is a uniform ANR. It follows, for instance, that the subspace of self-homotopy equivalences in $U((X, A), (X, A))$ is a uniform ANR.

Note that Theorem 4.35 implies Corollary 2.33. Here is another consequence:

**Corollary 4.37.** If $P$ is a uniform ANR, then the iterated loop space $\Omega^n(P, pt) = U(S^n, pt), (P, pt))$ is a uniform ANR.

**Theorem 4.38.** Let $\Delta$ be a finite diagram of uniform ANRs and uniformly continuous maps. Then the homotopy limit and the homotopy colimit of $\Delta$ are uniform ANRs.

**Proof.** We explicitly consider only the mapping cylinder and the mapping cocylinder; the general case is established similarly. Let $f: X \to Y$ be a uniformly continuous map between uniform ANRs. Then $MC(f)$ is the adjunction space $X \times I \cup_{f'} Y$, where $f'$ is the map $X \times \{0\} \to Y$, and therefore a uniform ANR by Theorem 4.32.
We recall the mapping cocylinder \( MCC(f) = \{(x, p) \in X \times U(I, Y) \mid f(x) = p(0)\} \). For each \( \varepsilon > 0 \), we define a uniform embedding \( j_\varepsilon \) of \( MCC(f) \) into \( MCC'(f) := U((I, \{0\}, \{1\}, (MC(f), X, Y)) \) by

\[
j_\varepsilon(x, p)(t) = \begin{cases} 
[(x, t/\delta)], & t \leq \delta, \\
[p(t-\delta)], & t \geq \delta,
\end{cases}
\]

where \( \delta = \delta(p, \varepsilon) \) is given by Lemma 4.39 below, and the square brackets denote the equivalence class in \( MC(f) = X \times I \sqcup Y/\sim \). By Theorem 4.44, either generalized from pairs to triples or coupled with Remark 4.36, \( MCC'(f) \) is a uniform ANR. On the other hand, we have a uniformly continuous map \( r : MCC'(f) \to MCC(f) \) given by \( r(p) = (p(0), \pi p) \), where \( \pi : MC(f) \to Y \) is the projection. Clearly, \( rj_\varepsilon \) is uniformly \( \varepsilon \)-homotopic to the identity. Hence \( MCC(f) \) is a uniform ANR by Corollary 4.22(a). □

**Lemma 4.39.** Let \( X \) and \( Y \) be metric spaces. Then for each \( \varepsilon > 0 \) there exists a uniformly continuous function \( \delta : U(X, Y) \to (0, 1] \) such that each \( p \in U(X, Y) \) is \((\delta(p), \varepsilon)\)-continuous.

**Proof.** Let \( Z_n \) be the set of all \((2^{-n}, \frac{\varepsilon}{2})\)-continuous maps in \( Z := U(X, Y) \). Clearly, \( Z_0 \subset Z_1 \subset \ldots \), and \( \bigcup Z_n = Z \). Let \( U_n \) be the \((1 - 2^{-n-1})\frac{\varepsilon}{3}\)-neighborhood of \( Z_n \). Thus each \( p \in U_n \) is \((2^{-n}, \varepsilon)\)-continuous. Since \( Z_n \subset Z_m \) for \( m > n \), the \((2^{-n-1} - 2^{-m-1})\frac{\varepsilon}{3}\)-neighborhood of \( U_n \) lies in \( U_m \). For each \( p \in U_{m+1} \setminus U_n \) let \( r_n(p) = \max(\frac{2}{3}d(p, U_n), 2^{-n-2}) \), and let \( \delta(p) = 2^{-n-1} - r_n(p) \). Since \( p \in U_{m+1} \) and \( \delta(p) \leq 2^{-n-1} \), we infer that \( p \) is \((\delta(p), \varepsilon)\)-continuous.

By construction \( \delta \) is \( \frac{2}{3} \)-Lipschitz on all pairs \((p, p')\) with \( p, p' \in U_{m+1} \setminus U_n \). On the other hand, since \( U_m \) contains the \((2^{-n-1} - 2^{-m-1})\frac{\varepsilon}{3}\)-neighborhood of \( U_n \), we have \( d(p, U_n) + d(p, Z \setminus U_m) \geq (2^{-n-1} - 2^{-m-1})\frac{\varepsilon}{3} \) for all \( p \in U_m \setminus U_n \). Hence \( r_n'(p) + q_m(p) \geq 2^{-n-1} - 2^{-m-1} \), where \( r_n'(p) = \frac{2}{3}d(p, U_n) \) and \( q_m(p) = \frac{2}{3}d(p, Z \setminus U_m) \). If \( p \notin U_{m+1} \), we additionally have \( q_m(p) \geq 2^{-n-2} - 2^{-m-1} \), and it follows that \( r_n(p) + q_m(p) \geq 2^{-n-1} - 2^{-m-1} \). Therefore \( \delta(p) = 2^{-n-1} - r_n(p) \leq 2^{-m-1} + q_m(p) \) for all \( p \in U_{m+1} \setminus U_n \). Given a \( p' \in U_{m+1} \setminus U_m \), we have \( \delta(p') = 2^{-m-1} - r_m(p') \), and therefore

\[
0 \leq \delta(p) - \delta(p') \leq q_m(p) + r_m(p') \leq \frac{2}{3}[d(p, Z \setminus U_m) + d(p', U_m)] \leq \frac{2}{3}[d(p, p') + d(p', p)].
\]

Thus \( \delta \) is \( \frac{6}{\varepsilon} \)-Lipschitz on all pairs \((p, p')\) with \( p' \in U_{m+1} \setminus U_m \) and \( p \in U_{m+1} \setminus U_n \) with \( m > n \). But all pairs are either of this type or of the similar type with \( m = n \), which has already been treated. Thus \( \delta \) is \( \frac{6}{\varepsilon} \)-Lipschitz.

### 4.D. Relative functional spaces via non-metrizable amalgam

This subsection is devoted to an alternative proof of Theorem 4.35 (with a similar result for non-metrizable ARUs and ANRUs as a byproduct), avoiding the infinite construction in the proof of Theorem 4.21 but involving a study of amalgamated unions of non-metrizable uniform spaces.
Lemma 4.40. A cover of $X \cup_A Y$ is uniform if and only if it is refined by a cover of the form $C + D := \{\text{st}(z, C \cup D) \mid z \in X \cup_A Y\}$, where $C$ is a uniform cover of $X$ and $D$ a uniform cover of $Y$.

Proof. Let $E$ be a uniform cover of $X \cup_A Y$. Then $E$ is star-refined by a cover $E_s$ such that $C := E_s \cap X$ is a uniform cover of $X$ and $D := E_s \cap Y$ is a uniform cover of $Y$. Then $C \cup D$ refines $E_s$, hence $C + D$ refines $E$.

Conversely, let $C$ be a uniform cover of $X$ and $D$ a uniform cover of $Y$. We need to construct a sequence of covers $E_0, E_1, \ldots$ of $X \cup Y$ such that $E_0 = C + D$, each $E_{i+1}$ star-refines $E_i$, and each $E_i \cap X$ is a uniform cover of $X$ and each $E_i \cap Y$ is a uniform cover of $Y$. First note that $(C + D) \cap X$ is uniform, for it is refined by $\{\text{st}(x, C) \mid x \in X\}$, which in turn is refined by $C$. Similarly $(C + D) \cap Y$ is uniform.

Let $C_s$ be a uniform cover of $X$ star-refining $C$ and let $D_s$ be a uniform cover of $Y$ star-refining $D$. Then $C_A := C_s \cap A$ and $D_A := D_s \cap A$ are uniform covers of $A$, hence so is $F_s := C_s \cap D_A = C_A \cap D_A$. Since $C_A$ is a uniform cover of $A$, it is of the form $C_Y \cap A$ for some uniform cover $C_Y$ of $Y$. Similarly $D_A$ is of the form $D_X \cap A$ for some uniform cover $D_X$ of $X$. Then $C_s' := C_s \cap D_X$ is a uniform cover of $X$ star-refining $C$ and $D_s' := D_s \cap C_Y$ is a uniform cover of $Y$ star-refining $D$. In addition, $C_s' \cap A = F_s = D_s' \cap A$.

Next, let $C_{ss}$ be a uniform cover of $X$ star-refining $C_s'$ and let $D_{ss}$ be a uniform cover of $Y$ star-refining $D_s'$. Let $F_{ss} = C_{ss} \cap D_{ss}$, and define $C_{ss}'$ and $D_{ss}'$ similarly to the above. Then $C_{ss}'$ is a uniform cover of $X$ star-refining $C_s'$ and $D_{ss}'$ is a uniform cover of $Y$ star-refining $D_s'$; in addition, $C_{ss}' \cap A = F_{ss} = D_{ss}' \cap A$.

We claim that $C_{ss}' + D_{ss}'$ star-refines $C + D$; iterating the construction of $C_{ss}'$ and $D_{ss}'$ would then yield the required sequence $E_1, E_2, \ldots$ (with $E_1 = C_{ss} + D_{ss}'$). Given a $z \in X \cup_A Y$, we will show that $\text{st}(z, C_{ss}' + D_{ss}')$ lies in $\text{st}(z', C) \cup \text{st}(z', D) = \text{st}(z', C \cup D)$ for some $z' \in X \cup_A Y$. By symmetry we may assume that $z \in X$. Let $U$ be an element of $C_{ss}' + D_{ss}'$ containing $z$. Then $U = \text{st}(w, C_{ss}') \cup \text{st}(w, D_{ss}')$ for some $w \in X \cup_A Y$. We consider two cases.

I. First suppose that $z \notin \text{st}(A, C_{ss}')$. Then $w \in X \setminus A$, hence $U = \text{st}(w, C_{ss}')$. Then $U$ is contained in an element of $C_{ss}'$, which is in turn contained in an element of $C$. Thus $U \subseteq \text{st}(z, C)$ and we may set $z' = z$.

II. It remains to consider the case where $z \in \text{st}(A, C_{ss}')$. Then $z \in \text{st}(z', C_{ss}')$ for some $z' \in A$. Also since $z \in U$, either $z \in \text{st}(w, C_{ss}')$ or $z \in \text{st}(w, D_{ss}')$. We consider these two cases.

1. If $z \in \text{st}(w, D_{ss}')$ then $z \in Y$, whence $z \in A$. Since $z$ and $z'$ are contained in one element of $C_{ss}'$ and also in $A$, they are contained in one element of $F_{ss}$, hence in one element of $D_{ss}'$. Thus $z \in \text{st}(z', D_{ss}')$ and $w \in \text{st}(z, D_{ss}')$, whence $w \in \text{st}(z', D_{ss}')$. In particular, $w \in Y$. We consider two cases.
   a) If $w \notin A$, then $U = \text{st}(w, D_{ss}')$. Then $U \subseteq \text{st}(z', D)$ and we are done.
   b) If $w \in A$, then $w \in \text{st}(z', F_s)$, and therefore $w \in \text{st}(z', C_s)$. Then $U = \text{st}(w, C_{ss}') \cup \text{st}(w, D_{ss}')$ is contained in $\text{st}(z', C) \cup \text{st}(z', D)$ and we are done.
2. If \( z \in \text{st}(w, C'_s) \) then also \( w \in \text{st}(z, C''_s) \), and therefore \( w \in \text{st}(z', C''_t) \). In particular, \( w \in X \). We consider two cases.

   a) If \( w \notin A \), then \( U = \text{st}(w, C'_s) \). Then \( U \subset \text{st}(z', C) \) and we are done.

   b) If \( w \in A \), then \( w \in \text{st}(z', F_s) \), and therefore \( w \in \text{st}(z', D_t) \). Then \( U = \text{st}(w, C'_s) \cup \text{st}(w, D'_s) \) is contained in \( \text{st}(z', C) \cup \text{st}(z', D) \) and we are done.

We note that Lemma 4.40 yields an alternative proof of Corollary 4.41, apart from the explicit metric:

**Corollary 4.41.** If \( X \) and \( Y \) are metrizable uniform spaces, every amalgamated union \( X \cup_A Y \) is metrizable.

*Proof.* Let \( C_1, C_2, \ldots \) be a basis of the uniformity of \( X \) and \( D_1, D_2, \ldots \) be a basis of the uniformity of \( Y \). If \( C \) is a uniform cover of \( X \) and \( D \) is a uniform cover of \( Y \), then there exists an \( i \) such that \( C_i \) refines \( C \) and \( D_i \) refines \( D \). Then \( C_i + D_i \) refines \( C + D \). Hence every uniform cover of \( X \cup_A Y \) is refined by one of \( C_1 + D_1, C_2 + D_2, \ldots \). By Theorem 2.9, \( X \cup_A Y \) is metrizable.

**Corollary 4.42.** Let \( X \) and \( Y \) be uniform spaces and \( A \subset X \) and \( B \subset Y \) closed subspaces.

   (a) The subspace \( X \times B \cup A \times Y \) of \( X \times Y \) is uniformly homeomorphic to the amalgamated union \( X \times B \cup_{A \times B} A \times Y \).

   (b) If \( X \) is metrizable, then the natural maps \( B \times A \to B \times X, B \times A \to Y \times A \) and \( Y \times A \cup_{B \times A} B \times X \to Y \times X \) are uniform embeddings.

Part (a), whose proof is similar to (b) (but easier), will not be used below; the metrizable case of (a) can also be deduced from Corollary 3.6.

The first two assertions of (b) are proved in [50].

*Proof of (b).* Since \( X \) is metrizable, a base of uniform covers of \( B \times X \) is given by covers of the form \( \{ U_a \times V^i_a \} \), where \( \{ U_a \} \) is a uniform cover of \( B \) and for each \( i \), \( \{ V^i_a \} \) is a uniform cover of \( X \). It follows that \( B \times A \to B \times X \) is a uniform embedding (cf. [50; III.29]). Since \( A \) is metrizable, a base of uniform covers of \( Y \times A \) is given by covers of the form \( \{ U_a \times V^i_a \} \), where \( \{ U_a \} \) is a uniform cover of \( Y \) and for each \( i \), \( \{ V^i_a \} \) is a uniform cover of \( A \). It follows that \( B \times A \to Y \times A \) is a uniform embedding (cf. [50; III.25]). Since \( X \) is metrizable, a base of uniform covers of \( Y \times X \) is \( \{ U_a \times V^i_a \} \), where \( \{ U_a \} \) is a uniform cover of \( Y \) and for each \( i \), \( \{ V^i_a \} \) is a uniform cover of \( X \). Then another such base is given by covers of the form \( W^i_a := \text{st}(p, \{ U_a \times V^i_a \}) \mid p \in Y \times X \).

On the other hand, by Lemma 4.40, a base of uniform covers of \( Y \times A \cup_{B \times A} B \times X \) is given by \( W^i_a := \{ U_a \times V^i_a \} \cup \{ U_a \times V^i_a \} \) in the same notation as above. Since each \( W^i_a = W^i_a \cap (Y \times A \cup B \times X) \), we obtain that the injective map \( Y \times A \cup_{B \times A} B \times X \to Y \times X \) (furnished by the categorical definition of a pushout) is a uniform embedding. 

**Theorem 4.43.** If \( X \) is a metrizable uniform space, \( A \) is a subset of \( X \), and \( Y \) and \( B \subset Y \) are \( A[N]RU \)s, then \( U((X, A), (Y, B)) \) is an \( A[N]RU \).
The case \( A = \emptyset \) is known (see Theorem 2.36); the proof of the general case is based on the same idea but additionally employs Corollary 4.42(b).

**Proof.** We consider the ARU case; the ANRU case is similar (cf. the proof of Theorem 2.36). Since \( B \) is an ARU, it is complete, and therefore closed in \( Y \). Then without loss of generality \( A \) is closed in \( X \) (else it can be replaced by its closure). Pick a pair \((Z, C)\) of uniform spaces with \( C \) closed in \( Z \) and a uniformly continuous \( f : C \to U((X, A), (Y, B)) \). We now apply Corollary 4.42(b). Since \( C \times A \to C \times X \) is a uniform embedding, \( f \) determines a uniformly continuous map \( \Phi : (C \times X, C \times A) \to (Y, B) \). Since \( C \times A \to Z \times A \) is a uniform embedding and \( B \) is an ARU, the restriction \( \psi : C \times A \to B \) of \( \Phi \) extends to a uniformly continuous \( \overline{\psi} : Z \times A \to B \). Since \( Z \times A \cup_{C \times A} C \times X \to Z \times X \) is a uniform embedding and \( Y \) is an ARU, \( \Phi \cup_{\psi} \overline{\psi} : Z \times A \cup_{C \times A} C \times X \to Y \) extends to a uniformly continuous \( \Phi : Z \times X \to Y \). This map \( \Phi : (Z \times X, Z \times A) \to (Y, B) \) determines a uniformly continuous \( \bar{f} : Z \to U((X, A), (Y, B)) \) extending \( f \). \( \square \)

**Corollary 4.44.** If \( Y \) is a metrizable uniform space, \( B \subset Y \), and \( X \) and \( A \subset X \) are uniform \( A[N]R \)s, then \( U((Y, B), (X, A)) \) is a uniform \( A[N]R \).

For an alternative proof, see Theorem 4.35.

**Proof.** We consider the case of uniform ANRs. The case of uniform ARs follows e.g. using Theorem 4.27.

We first construct a uniformly continuous homotopy \( h_\varepsilon \) of \((\bar{X}, \bar{A})\) such that \( h_0 = \text{id} \) and \( h_\varepsilon(\bar{X}, \bar{A}) \subset (X, A) \) for \( t > 0 \). Let \( h_\varepsilon^X \) and \( h_\varepsilon^X \) be uniformly continuous homotopies of \( \bar{A} \) and \( \bar{X} \) such that \( h_0^X = \text{id} \), \( h_\varepsilon^X \) is uniformly continuous, \( h_\varepsilon^X(\bar{A}) \subset A \) and \( h_\varepsilon^X(\bar{X}) \subset X \) for \( t > 0 \) (see Lemma 4.10). Define \( h_\varepsilon^X(\bar{X}) \to \bar{X} \) by \( h_\varepsilon^X(x) = h_\varepsilon^X(d(x, A)) \), where \( d \) is a metric on \( \bar{X} \) bounded by \( 1 \). Then \( h_\varepsilon^X(t) : \bar{X} \to \bar{X} \) is uniformly continuous, \( h_0^X = \text{id} \), \( h_\varepsilon^X|_{\bar{A}} = \text{id} \), and \( h_\varepsilon^X(\bar{X} \setminus \bar{A}) \subset X \) for \( t > 0 \).

On the other hand, since \( X \) is a uniform ANR and \( \bar{A} \cap X \) is closed in \( X \), the homotopy \( h_\varepsilon^X|_{\bar{A} \cap X} \) can be extended to a uniformly continuous homotopy \( h_\varepsilon : \bar{U} \to \bar{X} \), where \( U \) is a closed uniform neighborhood of \( A \) in \( X \), so that \( h_0 = \text{id} \). The latter in turn extends by continuity to a uniformly continuous homotopy \( h_\varepsilon : \bar{U} \times I \to \bar{X} \), which necessarily restricts to \( h_\varepsilon^X \) on \( \bar{A} \). Define \( r : \bar{X} \times I \to \bar{U} \times I \cup \bar{X} \times \{0\} \) by \( r(x, t) = (x, t(1 - ud(x, A))) \), where \( u = 1/d(\bar{U}, \bar{X} \setminus \bar{U}) \). Then \( r \) is the identity on \( \bar{A} \times I \), and \( \bar{U} \times I \) into itself, and \( (\bar{X} \setminus \bar{U}) \times I \) into \( (\bar{X} \setminus \bar{U}) \times \{0\} \). Define \( h_\varepsilon^X(\bar{X}) \to \bar{X} \) by \( h_\varepsilon^X(x) = h_\varepsilon^X(r(x, t)) \) for \( x \in \bar{U} \) and by \( h_\varepsilon^X(x) = x \) for \( x \notin \bar{U} \). Then \( h_0^X \): \( \bar{X} \times I \to \bar{X} \) is uniformly continuous, \( h_0^X = \text{id} \), \( h_\varepsilon^X(\bar{X}) = h_\varepsilon^X(\bar{U}) \cup h_\varepsilon^X(\bar{X} \setminus \bar{U}) \subset X \) for all \( t \), and \( h_\varepsilon^X(\bar{A}) \subset A \) for \( t > 0 \).

We set \( h_t = h_\varepsilon|_{\bar{A} \setminus X} \). Then \( h_0 = \text{id} \), \( h_t(\bar{X}) = h_\varepsilon^X(h_\varepsilon(\bar{A}) \cup h_\varepsilon^X(\bar{X} \setminus \bar{A})) \subset h_\varepsilon^X(\bar{A} \cup X) \subset X \) for \( t > 0 \), and \( h_t(\bar{A}) = h_\varepsilon^X(h_\varepsilon(\bar{A})) = h_\varepsilon^X(\bar{A}) \subset A \) for \( t > 0 \).

Now define a homotopy \( \varphi_t \) of \( U((Y, B), (\bar{X}, \bar{A})) \) by \( \varphi_t(f) = h_t f \). Then \( \varphi_t \) is uniformly continuous, \( \varphi_0 = \text{id} \) and \( \varphi_t \) sends \( U((Y, B), (\bar{X}, \bar{A})) \) into \( U((Y, B), (X, A)) \) for \( t > 0 \). It follows that \( U((Y, B), (\bar{X}, \bar{A})) \) is a completion of \( U((Y, B), (X, A)) \). Now the assertion follows from Theorems 4.10 and 4.43. \( \square \)
5. Inverse limits

5.A. Convergence and stability

5.1. Inverse limits. Let \( X_1, X_2, \ldots \) be metrizable uniform spaces. Given uniformly continuous maps \( f_i : X_{i+1} \rightarrow X_i \) for each \( i \), the inverse limit \( L := \lim \left( f_1, X_1, f_0, X_0 \right) \) is defined to be the subset of \( \prod X_i \) consisting of \emph{threads}, i.e. sequences \((x_1, x_2, \ldots)\) such that each \( f_i(x_{i+1}) = x_i \). The map \( f_i^\infty : L \rightarrow X_i \) is defined by restricting the projection \( \pi_i : \prod X_j \rightarrow X_i \). The bonding maps \( f_i \) have compositions \( X_j \xrightarrow{f_{j-1}} \ldots \xrightarrow{f_1} X_i \) denoted by \( f_i^j \). Since every two uniform covers of each \( X_{i+1} \) can be refined by a single uniform cover, we conclude that a cover of \( L \) is uniform iff it can be refined by the preimage of a single uniform cover of some \( X_i \). It is easy to check that \((L, f_i^\infty)\) is the category-theoretic inverse limit, i.e. every family of uniformly continuous maps \( \varphi_i : L' \rightarrow X_i \) commuting with the bonding maps \( f_i \) factors through a unique map \( \varphi : L' \rightarrow L \) (so that each \( \varphi_i = f_i^\infty \varphi \)).

It is easy to check that if each \( X_i \) is complete; separable; point-finite; star-finite; or Noetherian, then so is \( L \). And if each \( X_i \) is a uniform local compactum and each \( f_i \) is proper (i.e. the preimage of every compactum is a compactum), then \( L \) is a uniform local compactum.

5.2. \( \varepsilon \)-Separating maps. Let \( f : X \rightarrow Y \) be a map between metric spaces. We recall that \( f \) is uniformly continuous iff for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( f \) is \((\delta, \varepsilon)\)-continuous, that is, sends \( \delta \)-close (=at most \( \delta \)-close) points into \( \varepsilon \)-close points. Note that the Hahn property (see §4.B) involved \( \varepsilon \)-continuous maps, i.e. maps that are \((\delta, \varepsilon)\)-continuous for some \( \delta > 0 \). Dually, we say that \( f \) is \((\varepsilon, \delta)\)-separating if \( \delta \)-close points have \( \varepsilon \)-close point-inverses; and \( \varepsilon \)-separating if it is \((\varepsilon, \delta)\)-separating for some \( \delta > 0 \). Note that when \( X \) is compact, the latter is equivalent to the more familiar notion of an “\( \varepsilon \)-map”, which is that \( f^{-1}(x) \) is of diameter \( < \varepsilon \) for every \( x \in X \). \( \varepsilon \)-Separating maps were known to Isbell [51], who called them simply “\( \varepsilon \)-mappings”.

Lemma 5.3. Given an inverse sequence of metric spaces \( X_i \) and uniformly continuous maps \( p_i \), for each \( \varepsilon > 0 \) there exists an \( i \) such that \( p_i^\infty : \lim X_j \rightarrow X_i \) is \( \varepsilon \)-separating.

Proof. Let \( C \) be the cover of \( \lim X_j \) by all sets of diameter \( \varepsilon \). Since \( C \) is uniform, it is refined by \( (p_i^\infty)^{-1}(C_i) \) for some uniform cover \( C_i \) of \( X_i \). If \( \lambda \) is the Lebesgue number of \( C_i \), we obtain that \( p_i^\infty \) is \((\varepsilon, \lambda)\)-separating. \( \square \)

5.4. Freudenthal’s space and mapping telescope. Consider an inverse sequence \( X = (\ldots \xrightarrow{f_3} X_1 \xrightarrow{f_0} X_0) \) of uniformly continuous maps between metrizable uniform spaces. Let \( X_{\mathbb{N}\cup\infty} = \lim (\ldots \xrightarrow{f_3} X_2 \cup X_1 \cup X_0 \xrightarrow{id \cup id \cup f_1} X_1 \cup X_0 \xrightarrow{id \cup f_0} X_0) \). Given any subset \( J \subset \mathbb{N} \cup \{\infty\} \), we denote by \( X_J \) the preimage of \( J \) under the projection \( X_{\mathbb{N}\cup\infty} \rightarrow \mathbb{N} \cup \{\infty\} \). Note that \( X_\infty \) is identified with \( \lim X \), and its complement \( X_N \) is homeomorphic to the topological space \( \bigsqcup_{i \in \mathbb{N}} X_i \). If each \( X_i \) is complete, \( X_{\mathbb{N}\cup\infty} \) is the completion of \( X_\mathbb{N} \).
5.5. Convergent and Cauchy inverse sequences. Let us call the inverse sequence $X$ convergent if every uniform neighborhood of $\lim X$ in $X_{\mathbb{N} \cup \infty}$ (or equivalently in $X_{[0, \infty)}$) contains all but finitely many of $X_i$'s. We say that $X$ is Cauchy if for every $\varepsilon > 0$ there exists a $k$ such that for every $j > k$, the $\varepsilon$-neighborhood of $X_j$ in $X_N$ (or equivalently in $X_{[0, \infty)}$) contains $X_k$. Clearly, these notions depend only on the underlying uniform structures. Here is an example of a Cauchy inverse sequence that is divergent: $\ldots \subset (0, \frac{1}{2}] \subset (0, \frac{1}{3}] \subset (0, 1]$. The inverse sequence $\ldots \subset [2, \infty) \subset [1, \infty) \subset [0, \infty)$ fails to be Cauchy.

The close analogy with the definition of a convergent/Cauchy sequence can be formalized. It is easy to see that the inverse sequence $X$ is convergent (Cauchy) iff the sequence of the closed subsets $X_i$ of $X_{\mathbb{N} \cup \infty}$ is convergent (Cauchy) in the hyperspace $H(X_{\mathbb{N} \cup \infty})$, or equivalently in $H(X_{[0, \infty)})$. This implies parts (a) and (b) of the following lemma.

Lemma 5.6. [62] Let $X = (\ldots \overset{f_i}{\rightarrow} X_1 \overset{f_0}{\rightarrow} X_0)$ be an inverse sequence of uniformly continuous maps between metric spaces.

(a) If each $X_i$ is compact, $X$ is convergent.
(b) If $X$ is convergent, it is Cauchy; the converse holds when each $X_i$ is complete.
(c) $X$ is convergent if and only if for each $i$, each uniform neighborhood of $f_i^\infty(\lim X)$ in $X_i$ contains all but finitely many of $f_i^j(X_j)$'s.
(d) $X$ is Cauchy if and only if for each $i$ and every $\varepsilon > 0$ there exists a $k$ such that for every $j > k$, the $\varepsilon$-neighborhood of $f_i^j(X_j)$ in $X_i$ contains $f_i^k(X_k)$.
(e) If each $X_i$ is uniformly discrete, $X$ is convergent if and only if it satisfies the Mittag-Leffler condition: for each $i$, the images of $X_k$ in $X_i$ stabilize, i.e. there exists a $j > i$ such that $p_k^j(X_k) = p_i^j(X_j)$ for each $k > j$.
(f) If $X$ is convergent and each $X_i$ is non-empty (resp. uniformly connected), then $\lim X$ is non-empty (resp. uniformly connected).
(g) If $Y = (\ldots \overset{g_i}{\rightarrow} Y_1 \overset{g_0}{\rightarrow} Y_0)$ is another inverse sequence of uniformly continuous maps between metric spaces, $h_i: X_i \rightarrow Y_i$ are surjections commuting with the bonding maps and $X$ is convergent (Cauchy), then $Y$ is convergent (resp. Cauchy).

It should be noted that the map $\lim X \rightarrow \lim Y$ in (g) is not necessarily a surjection, in contrast to the compact case. For instance, take $X_i = \mathbb{N} := \{0, 1, \ldots \}$ and $Y_i = [i] := \{0, \ldots, i - 1\}$ (discrete uniform spaces), and let each $f_i$ be the identity map, and $g_i: [i + 1] \rightarrow [i]$ and $h_i: \mathbb{N} \rightarrow [i]$ the retractions with the only non-degenerate point-inverses being those of $i - 1$. Then $\lim Y$ is homeomorphic to the one-point compactification of $\mathbb{N}$, where the remainder point is not in the image of $\lim X$.

See [62] for an alternative proof of (a) and a more detailed proof of (c).
Proof. Parts (a) and (b) have been proved above. Parts (c) and (d) follow using that each \( p^\infty_i : X_{\{i,i+1,...,\infty\}} \to X_i \) (which is the restriction of \( p^\infty_i : X_{N\cup\infty} \to X_1 \sqcup \cdots \sqcup X_i \)) is uniformly continuous and \( \varepsilon_i \)-separating, where \( \varepsilon_i \to 0 \) as \( i \to \infty \), by Lemma 5.3. Parts (e) and (f) follow from (c); part (g) follows from (c) and (d).

From (b) and (d) we immediately obtain (compare [62])

**Corollary 5.7** (Bourbaki’s Mittag-Leffler Theorem). Let \( L \) be the limit of an inverse sequence \( X = (\ldots \overset{f_i}{\to} X_1 \overset{f_0}{\to} X_0) \) of uniformly continuous maps between complete metrizable uniform spaces. If each \( f_i(X_{i+1}) \) is dense in \( X_i \), \( f_0^\infty(L) \) is dense in \( X_0 \).

As observed by V. Runde, a special case of 5.7 is (compare [62])

**Corollary 5.8** (Baire Category Theorem). The intersection of a countable collection of dense open sets in a complete metrizable uniform space is dense.

The substantial result of this subsection is that inverse limits are stable under sufficiently small perturbations of inverse sequences.

**Proposition 5.9.** Let \( \ldots \overset{p_i^2}{\to} X_1 \overset{p_0^1}{\to} X_0 \) and \( \ldots \overset{q_i^2}{\to} Y_1 \overset{q_0^1}{\to} Y_0 \) be inverse sequences of uniformly continuous maps between metric spaces, where the \( Y_i \) are complete, and let \( X \) and \( Y \) be their inverse limits. Let \( f_i : X_i \to Y_i \), \( i = 0,1,... \), be uniformly continuous maps. A uniformly continuous map \( f : X \to Y \) such that each \( q_i^\infty f \) is \( 2\beta_i \)-close to \( f_i p_i^\infty \)

(a) exists, provided that

(i) \( f_i p_i^{i+1} \) and \( q_i^{i+1} f_{i+1} \) are \( \alpha_i \)-close for each \( i \), where

(ii) \( \alpha_i > 0 \) is such that \( q_j^i \) is \( (\alpha_i,2^{j-i} \beta_j) \)-continuous for each \( j \leq i \);

(b) is unique, if in addition to (i) and (ii) the following holds:

(iii) each \( \beta_i > 0 \) is such that \( q_i^\infty \) is \( (\delta_i,9\beta_i) \)-separating, where \( \delta_i > 0 \)

is such that \( q_i^\infty \) is \( \delta_i \)-separating and \( \delta_i \to 0 \) as \( i \to \infty \);

(c) is a uniform homeomorphism onto its image, if

(iv) each \( f_i \) is \( (\gamma_i,5\beta_i) \)-separating, where

(v) each \( \gamma_i > 0 \) is such that \( p_i^\infty \) is \( (\varepsilon_i,\gamma_i) \)-separating, where \( \varepsilon_i > 0 \) is such that \( p_i^\infty \) is \( \varepsilon_i \)-separating and \( \varepsilon_i \to 0 \) as \( i \to \infty \);

(d) is surjective, if in addition to (i)–(v) the following holds:

(vi) every \( y_i \in Y_i \) is \( \alpha_i \)-close to some \( y_i \in f_i(X_i) \);

(vii) \( X \) is complete and \( \ldots \overset{p_i^2}{\to} X_1 \overset{p_0^1}{\to} X_0 \) is convergent.

Admittedly the statement of Proposition 5.9 is rather cumbersome. For some purposes it becomes more revealing if simplified in one or both of the following ways.

**Corollary 5.10.** Proposition 5.9(a,b,c,d) holds

(e) if each \( X_i \) has diameter \( \leq 1 \), and condition (v) is replaced by

(v') each \( \gamma_i > 0 \) is such that \( p_j^i \) is \( (\gamma_i,2^{j-i}) \)-continuous for all \( j \leq i \);

(f) if each \( Y_i \) has diameter \( \leq 1 \), and condition (iii) is replaced by

(iii') each \( \beta_i > 0 \) is such that \( q_j^i \) is \( (9\beta_i,2^{j-i}) \)-continuous for \( j \leq i \);
(g) when (e) and (f) are combined.

Proof. If each $X_i$ has diameter $\leq 1$, we may endow $X$ with the metric $d((x_i), (x_i')) = \sup \{2^{-i}d(x_i, x_i') \mid i \in \mathbb{N}\}$ and take $\varepsilon_i = 2^{-i}$. Then (v) follows from (v’).

Similarly, if each $Y_i$ has diameter $\leq 1$, we may endow $Y$ with the metric $d((y_i), (y_i')) = \sup \{2^{-i}d(y_i, y_i') \mid i \in \mathbb{N}\}$ and take $\delta_i = 2^{-i}$. Then (iii) follows from (iii’).

In the compact case, a version of Corollary 5.10(g) was obtained by Rogers [78]. His proof is by a different method, reducing the general case to the case where $p_j^i$ and $q_j^i$ are embeddings, and involving what appears to be a substantial use of compactness.

We note the following regarding the proof of Proposition 5.9. The proof of (d) will not be simplified if the $f_i$ are assumed to be surjective. If the $f_i$ are only assumed to be continuous, rather than uniformly continuous, then the hypotheses of (a) and (c) still imply that $F$ is a homeomorphism onto its image and $f^{-1}$ is uniformly continuous. The constant 9 in (iii) and (iii’) is relevant for (d), but can be replaced by 5 for the purposes of (b).

Proof. (a). Let us consider the compositions $F_j^{(i)} : X \xrightarrow{p_j^{\infty}} X_i \xrightarrow{f_i} Y_i \xrightarrow{q_j^i} Y_j$. By (i) and (ii), $F_j^{(i+1)}$ and $F_j^{(i)}$ are $2^{-i}\beta_j$-close for every $i \geq j$. Since $Y_j$ is complete, $F_j^{(j+k)}$ uniformly converge to a map $F_j : X \to Y_j$. Since $2^0 + 2^{-1} + \cdots = 2$, it is $2\beta_j$-close to $F_j^{(j)} = f_j p_j^{\infty}$.

Each $f_i$ is uniformly continuous, hence so is each $F_j^{(j+k)}$ and consequently their uniform limit $F_j$. Since each $F_j^{(i)} = q_j^{i+1}F_j^{(i+1)}$, we obtain $F_j = q_j^{i+1}F_{j+1}$. Then by the universal property of inverse limits there exists a uniformly continuous map $f : X \to Y$ such that $q_j^{\infty}f = F_j$ for each $j$. In particular, $q_j^{\infty}f$ is $2\beta_j$-close to $f_j p_j^{\infty}$ for each $j$. □

(b). Given another map $f' : X \to Y$ such that $q_j^{\infty}f'$ is $2\beta_j$-close to $f_j p_j^{\infty}$ for each $j$, we have that $q_j^{\infty}f'$ is $4\beta_j$-close to $q_j^{\infty}f$ for each $j$. Hence by (iii), $f'$ is $\delta_j$-close to $f$ for each $j$, i.e. $f' = f$. □

(c). By (iv) and (v), each $f_j p_j^{\infty}$ is $(\varepsilon_j, 5\beta_j)$-separating. By the hypothesis it is $2\beta_j$-close to $q_j^{\infty}f$, so the latter must be $(\varepsilon_j, 5\beta_j - 2\beta_j - 2\beta_j)$-separating. In particular, $q_j^{\infty}f$ is $\varepsilon_j$-separating; hence so is $f$. Since this holds for every $j$, and $\varepsilon_j \to 0$ as $j \to \infty$, $f$ must be injective. On the other hand, since $q_j^{\infty}f$ is $(\varepsilon_j, \beta_j)$-separating, and the uniformly continuous $q_j^{\infty}$ is $(\lambda_j, \beta_j)$-continuous for some $\lambda_j > 0$, $f$ is $(\varepsilon_j, \lambda_j)$-separating. Then $f^{-1} : f(X) \to X$ is $(\lambda_j, \varepsilon_j)$-continuous for every $j$. Since $\varepsilon_j \to 0$ as $j \to \infty$, we conclude that $f^{-1}$ is uniformly continuous. □

(d). Pick some $y \in Y$. By (vi), each $q_k^{\infty}(y)$ is $\alpha_i$-close to some $y_i = f_i(x_i)$ for some $x_i \in X_i$. Let $\mu_j$ be such that $f_j$ is $(\mu_j, \alpha_j)$-continuous. By the convergence hypothesis and Lemma 5.6(c), there exists an $i = \varphi(j)$ such that $p_j^i(X_i)$ is contained in the $\mu_j$-neighborhood of $p_j^{\infty}(X)$. Then $p_j^i(x_i)$ is $\mu_j$-close to $p_j^{\infty}(x_{(j)})$ for some $x_{(j)} \in X$. Hence $y_j^i := f_j p_j^i(x_i)$ is $\alpha_j$-close to $y_j^{(j)} := f_j p_j^{\infty}(x_{(j)})$. Now let us consider an arbitrary $k < j$.

By (ii), $q_k^j(y_j^i)$ is $\frac{1}{2}\beta_k$-close to $q_k^j(y_j^{(j)})$. On the other hand, by (i) and (ii), $q_k^j f_j$ is $2\beta_k$-close to $f_k p_k^j$ (and even $(2\beta_k - \nu)$-close for some $\nu > 0$, using that $1 + \frac{1}{2} + \cdots + 2^{k-1} < 2$); hence
\( q^j(y_j) \) is \( 2\beta_k \)-close to \( y_k \), and \( q^j_i(y_j) \) is \( 2\beta_k \)-close to \( y_k := f_k p_k^\infty(x_j) \). Summing up, \( y_k \) is \( 2\beta_k \)-close to \( y_k \). By the above, the latter is in turn \( \alpha_k \)-close, hence by (ii) \( \beta_k \)-close to \( y_k \). Thus \( y_k \) is \( \frac{1}{2} \beta_k \)-close to \( y_k \). Since by (iv) and (v), \( f_k p_k^\infty \) is \( (\varepsilon_k, 5 \beta_k) \)-separating, we conclude that \( x_j \) is \( \varepsilon_k \)-close to \( x_k \). Consequently \( x_{(1)} \), \( x_{(2)} \), \ldots is a Cauchy sequence, and since \( X \) is complete, it converges to some \( x \in X \).

By (b) we have \( q^\infty_j f = F_j \) in the notation in the proof of (a), where \( F_j(x) \) is the limit of \( F_j^{(i)}(x) \)'s as \( i \to \infty \), each \( F_j^{(i)}(x) \) in turn being the limit of \( F_j^{(i)}(x_{(l)}) \)'s as \( l \to \infty \). We have \( F_j^{(i)}(x_{(l)}) = q_j^i f_j p_j^\infty(x_{(l)}) = q_j^i(y_i^{(l)}) \). From (i) and (ii), \( q_j^i(y_i^{(l)}) \) is \( 2\beta_j \)-close to \( y_j^{(l)} \). Without loss of generality, \( l > j \); then by the above, \( y_j^{(l)} \) is \( \frac{1}{2} \beta_j \)-close to \( y_j^{m} \), where \( m = \varphi(l) \). From (i) and (ii), \( y_j^{m} \) is \( 2\beta_j \)-close to \( q_j^{m}(y_m) \). By our choice of \( y_m \), this \( y_m \) is \( \alpha_m \)-close to \( q_m^{\infty}(y) \), hence by (ii), \( q_j^{m}(y_m) \) is \( \frac{1}{2} \beta_j \)-close to \( q_j^{\infty}(y) \) (using that \( m \geq l < j + 1 \)). To summarize, \( F_j^{(i)}(x_{(l)}) = q_j^i(y_i^{(l)}) \) is \( 9\beta_j \)-close to \( q_j^{\infty}(y) \); in fact, they are even \( (9\beta_j - i) \)-close for some \( i > 0 \). Hence \( F_j^{(i)}(x) \) is \( (9\beta_j - \frac{i}{2}) \)-close to \( q_j^{\infty}(y) \) for each \( i \). Then \( F_j(x) \) is \( 9\beta_j \)-close to \( q_j^{\infty}(y) \). Since \( F_j = q_j^{\infty} f \), by (iii), \( f(x) \) is \( \delta_j \)-close to \( y \) for each \( j \). Since \( \delta_j \to 0 \) as \( j \to \infty \), we obtain that \( f(x) = y \). 

\[ \Box \]

The following is a direct consequence of Corollary 5.10(a,b,f); the compact case (apart from the uniqueness) is due to Mioduszewski \[65] \].

**Corollary 5.11.** Let \( \ldots \xrightarrow{p_{n-1}} X_1 \xrightarrow{p_n} X_0 \) be an inverse sequence of uniformly continuous maps between complete metric spaces, and let \( X \) be its inverse limit. Then there exists a sequence of \( \beta_i^* > 0 \) such that for each sequence of \( \beta_i \in (0, \beta_i^*) \) there exists a sequence of \( \alpha_i > 0 \) such that the following holds. Suppose \( \ldots \xrightarrow{p_{n-1}} Y_1 \xrightarrow{p_n} Y_0 \) is an inverse sequence of uniformly continuous maps between metrizable uniform spaces, and \( Y \) is its inverse limit. If \( n_i \) is a non-decreasing unbounded sequence of natural numbers, and \( f_i : X_{n_i} \to Y_i \) are uniformly continuous maps such that the diagram

\[
\begin{array}{ccc}
X_{n_{i+1}} & \xrightarrow{f_{i+1}} & Y_{i+1} \\
\downarrow{p_{n_{i+1}}} & & \downarrow{q_{i+1}} \\
X_{n_i} & \xrightarrow{f_i} & Y_i
\end{array}
\]

\( \alpha_i \)-commutes for each \( i \), then there exists a unique uniformly continuous map \( f : X \to Y \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p_{n_i}} & & \downarrow{q_i} \\
X_{n_i} & \xrightarrow{f_i} & Y_i
\end{array}
\]

\( \beta_i \)-commutes for each \( i \).
5.B. Inverse sequences of uniform ANRs

**Lemma 5.12.** Let $X$ be the limit of a convergent inverse sequence of metrizable uniform spaces $X_i$ and uniformly continuous maps $p_i$, and let $Y$ be a metric space.

(a) Suppose that $Y$ satisfies the Hahn property. Then for every uniformly continuous map $f: X \to Y$ and each $\varepsilon > 0$ there exists a $j$ and a uniformly continuous map $g: X_j \to Y$ such that $f$ is $\varepsilon$-close to the composition $X \xrightarrow{p_j} X_j \xrightarrow{g} Y$.

(b) Suppose that $f,g: X_i \to Y$ are uniformly continuous maps such that the two compositions $X \xrightarrow{p_i} X_i \xrightarrow{f} Y$ are $\varepsilon$-close. Then for each $\delta > 0$ there exists a $k$ such that the two compositions $X_k \xrightarrow{p_k} X_i \xrightarrow{g} Y$ are $(\varepsilon + \delta)$-close.

The conclusion resembles Mardesˇi´c’s definition of “resolution” which is the basis of his approach to shape theory [59].

A special case of (a), with the convergence hypothesis strengthened to surjectivity of all bonding maps, was known to Isbell [49; 7.4].

(a.). Let $\delta = \delta_{\text{Hahn}}(\varepsilon/2)$ be given by the Hahn property for $\varepsilon_{\text{Hahn}} = \varepsilon/2$. Let us fix some metrics on $X$ and the $X_i$’s. There exists a $\gamma > 0$ such that $f$ is $(\gamma, \delta)$-continuous. Then by Lemma 5.3 there exists an $i$ such that $p_i^\infty: X \to X_i$ is $\gamma$-separating. Hence $p_i^\infty$ is $(\gamma, 3\beta)$-separating for some $\beta > 0$. Then by Lemma 5.6(c) there exists a $j$ such that the $\beta$-neighborhood of $p_i^\infty(X)$ contains $p_j(X_j)$.

Now every $x \in U_\beta$ is $\beta$-close to $p_i^\infty(\varphi(x))$ for some $\varphi(x) \in X$. (If $X_i$ is separable, the definition of $\varphi: U_\beta \to X$ requires only the countable axiom of choice. We do not require that $\varphi p_i^\infty = \text{id}_X$.) Moreover, if $y$ is $\beta$-close to $x$, then $p_i^\infty(\varphi(y))$ is $3\beta$-close to $p_i^\infty(\varphi(x))$. Hence $\varphi(y)$ is $\gamma$-close to $\varphi(x)$. Thus $\varphi$ is $(\beta, \gamma)$-continuous.

The composition $U_\beta \xrightarrow{\varphi} X \xrightarrow{f} Y$ is $(\beta, \delta)$-continuous, and so is $(\varepsilon/2)$-close to a uniformly continuous map $\psi: U_\beta \to Y$. Hence also the composition $X \xrightarrow{p_i^\infty} U_\beta \xrightarrow{\varphi} X \xrightarrow{f} Y$ is $(\varepsilon/2)$-close to the composition $X \xrightarrow{p_i^\infty} U_\beta \xrightarrow{\psi} Y$. Since $f$ is $(\gamma, \beta)$-separating, the composition $X \xrightarrow{\varphi} U_\beta \xrightarrow{\varphi} X$ is $\gamma$-close to $\text{id}_X$, and therefore the composition $X \xrightarrow{p_i^\infty} U_\beta \xrightarrow{\varphi} X \xrightarrow{f} Y$ is $\delta$-close to $f$. We may assume that $\delta \leq \varepsilon/2$. Then we conclude that $f$ is $\varepsilon$-close to the composition $X \xrightarrow{p_i^\infty} X_j \xrightarrow{p_j} U_\beta \xrightarrow{\psi} Y$.  

(b). Let us fix some metric on the $X_i$, and pick a $\gamma > 0$ such that $f$ and $g$ are $(\gamma, \delta/2)$-continuous. By Lemma 5.6(c) there exists a $k$ such that the $\gamma$-neighborhood of $p_i^\infty(X)$ contains $p_k^\infty(X_k)$. Given an $x \in X_k$, pick a $z \in X$ such that $p_i^\infty(z)$ is $\gamma$-close to $p_k^\infty(x)$. Then the $f$- and $g$-images of $p_k^\infty(x)$ are $(\delta/2)$-close to those of $p_i^\infty(z)$, which are in turn $\varepsilon$-close to each other.  

**Theorem 5.13.** Let $\ldots \xrightarrow{q_2} Y_1 \xrightarrow{q_1} Y_0$ be an inverse sequence of uniformly continuous maps between metric spaces satisfying the Hahn property, and let $Y$ be its inverse limit. Suppose $f: X \to Y$ is a uniformly continuous map, where $X$ is the limit of a convergent
inverse sequence \( \ldots \xrightarrow{p_1} X_1 \xrightarrow{p_0} X_0 \) of uniformly continuous maps between metrizable uniform spaces. Then for each sequence of \( \alpha_i > 0 \) there exist an increasing sequence of natural numbers \( n_i \), and uniformly continuous maps \( f_i : X_{n_i} \to Y_i \) such that the diagrams

\[
\begin{array}{ccc}
X_{n_{i+1}} & \xrightarrow{f_{i+1}} & Y_{i+1} \\
\downarrow p_{n_{i+1}} & & \downarrow q_i \\
X_{n_i} & \xrightarrow{f_i} & Y_i
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow p_n & & \downarrow q_n \\
X_n & \xrightarrow{f_n} & Y_n
\end{array}
\]

\( \alpha_i \)-commute for each \( i \).

The compact case is due essentially to Mioduszewski [65].

Proof. By Lemma 5.12(a), there exist an \( n_0 \) and a uniformly continuous \( f_0 : X_{n_0} \to Y_0 \) such that the composition \( X \xrightarrow{p_{n_0}} X_{n_0} \xrightarrow{f_0} Y_0 \) is \((\alpha_0/3)\)-close to \( X \xrightarrow{f} Y \xrightarrow{q_0} Y_0 \). Similarly for each \( \alpha'_i > 0 \) there exist an \( n'_1 > n_0 \) and an \( f'_i : X_{n'_1} \to Y_1 \) such that the composition \( X \xrightarrow{p_{n'_1}} X_{n'_1} \xrightarrow{f'_i} Y_1 \) is \( \alpha'_i \)-close to \( X \xrightarrow{f} Y \xrightarrow{q'_1} Y_1 \). Let \( \alpha'_1 < \alpha_1/3 \) be such that \( q_1 \) is \((\alpha'_1, \alpha_0/3)\)-continuous. Then the compositions \( X \xrightarrow{p_{n_0}} X_{n_0} \xrightarrow{f_0} Y_0 \) and \( X \xrightarrow{p_{n'_1}} X_{n'_1} \xrightarrow{f'_i} Y_1 \) are \((2\alpha_0/3)\)-close. Hence by Lemma 5.12(b), there exists an \( n_1 > n'_1 \) such that the compositions \( X_{n_1} \xrightarrow{p_{n_1}} X_{n_0} \xrightarrow{f_0} Y_0 \) and \( X_{n_1} \xrightarrow{p_{n'_1}} X_{n'_1} \xrightarrow{f'_i} Y_1 \xrightarrow{q'_1} Y_0 \) are \( \alpha_0 \)-close. We define \( f_1 \) to be the composition \( X_{n_1} \xrightarrow{p_{n_1}} X_{n'_1} \xrightarrow{f'_i} Y_1 \), and proceed similarly.

Theorem 5.14. Let \( \ldots \xrightarrow{q_i} Y_1 \xrightarrow{q_0} Y_0 \) be an inverse sequence of uniformly continuous maps between uniform ANRs, and let \( Y \) be its inverse limit. Suppose \( f : X \to Y \) is a uniformly continuous map, where \( X \) is the limit of a convergent inverse sequence \( \ldots \xrightarrow{p_i} X_1 \xrightarrow{p_0} X_0 \) of uniformly continuous maps between metrizable uniform spaces.

Then there exists an increasing sequence \( n_i \) and a level-preserving uniformly continuous extension \( f_n : X_{n(0, \infty)} \to Y_{[0, \infty]} \) of \( f \).

Moreover, given another such extension \( f'_n \), there exists an increasing subsequence \( l_i \) of \( n_i \) such that the compositions \( X_{l(0, \infty)} \xrightarrow{p_{l_i}} X_{n(0, \infty)} \xrightarrow{f_n} Y_{[0, \infty]} \) and \( f'_{l_i} \) are uniformly homotopic through level-preserving extensions \( X_{l(0, \infty)} \to Y_{[0, \infty]} \) of \( f \).

Proof. By Lemmas 4.19 and 4.16, each \( Y_i \) satisfies the Hahn property and is uniformly locally contractible. Let \( \alpha_i = \delta_{LCU}(2^{-i}) \) be given by the uniform local contractibility of \( Y_i \) corresponding to \( \varepsilon = 2^{-i} \). The first assertion now follows from Theorem 5.13.

The moreover part is established by similar arguments, but replacing
- Lemma 4.19(a) with Lemma 4.16(a);
- Lemma 4.16(a) with its 1-parameter version (see Remark 4.15);
- Lemma 5.12(b) with Lemma 5.12(a);
- Lemma 5.12(b) with itself applied to \( X \times I \).

We now sketch a slightly different approach.
Theorem 5.15. Let \( \{ q_i \} \) \( Y_1 \to Y_0 \) be an inverse sequence of uniformly continuous maps between uniform ANRs, and let \( Y \) be its inverse limit. Then the mapping telescope \( Y_{[0,\infty)} \) and the extended mapping telescope \( Y_{[0,\infty]} \) are uniform ANRs.

Moreover, if \( Y_0 \) is a uniform AR, then \( Y_{[0,\infty)} \) and \( Y_{[0,\infty]} \) are uniform ARs.

Proof. Each \( Y_{[i,i+1]} \) is a uniform ANR by Theorem 4.32, so each \( Y_{[0,i]} \) is a uniform ANR by Corollary 4.34. Now \( Y_{[0,\infty)} \) and \( Y_{[0,\infty]} \) are uniformly \( 2^{-i} \)-dominated by \( Y_{[0,i]} \) for each \( i \), whence they are uniform ANRs by Corollary 4.22(a). The moreover part follows from Theorem 4.27, since \( Y_{[0,\infty)} \) and \( Y_{[0,\infty]} \) uniformly deformation retract onto \( Y_0 \).

Theorem 5.15 immediately implies the following result, which is easily seen to be equivalent (cf. [62; Lemma 2.5]) to Theorem 5.14.

Corollary 5.16. Let \( \{ q_i \} \) \( Y_1 \to Y_0 \) be an inverse sequence of uniformly continuous maps between uniform ANRs, where \( Y_0 \) is a uniform AR, and let \( Y \) be its inverse limit. Suppose \( f : X \to Y \) is a uniformly continuous map, where \( X \) is the limit of a convergent inverse sequence \( \{ p_i \} \) \( X_1 \to X_0 \) of uniformly continuous maps between metrizable uniform spaces.

Then there exists a uniformly continuous extension \( f_{[0,\infty)} : X_{[0,\infty)} \to Y_{[0,\infty)} \) of \( f \) sending \( X_{[0,\infty)} \) into \( Y_{[0,\infty)} \). Moreover, every two such extensions are uniformly homotopic through such extensions.

Most of the compact case of Theorem 5.15 and Corollary 5.16 was proved by J. Milnor (1961; published 1995) and rediscovered in mid-70s independently by J. Krasinkiewicz; Y. Kodama; Chapman–Siebenmann; and Dydak–Segal (see references in [62; §2]).

Theorem 5.17. Let \( \{ p_i \} \) \( X_1 \to X_0 \) be a convergent inverse sequence of uniformly continuous maps between uniform ANRs, where \( X_0 \) is a uniform AR [resp. no condition on \( X_0 \)], and let \( X \) be its inverse limit.

Then \( X \) is a uniform ANR if and only if it is a uniform retract of \( X_{[0,\infty]} \) [resp. of \( X_{[n,\infty]} \) for some \( n \)].

A similar characterization of non-uniform ANRs is found in [79; Theorem 1].

Proof. The ‘if’ direction follows from Theorem 5.15. But let us sketch an alternative proof, avoiding the use of Theorem 4.32. Given a metrizable uniform space \( Y \) and a closed subset \( A \subset Y \), we may embed \( Y \) in the mapping telescope \( U_{[0,\infty)} \) of appropriate uniform neighborhoods \( U_i \) of \( A \) in \( Y \) (with \( U_0 = Y \)) like in the end of the proof of Theorem 4.21. Given a uniformly continuous map \( A \to X \), by Theorem 5.14 it extends to a uniformly continuous map \( U_{[0,\infty)} \to X_{[0,\infty]} \). The required extension is now given by the composition \( Y \subset U_{[0,\infty)} \to X_{[0,\infty]} \to X \) [resp. by its restriction of the form \( U_m \subset U_{[m,\infty)} \to X_{[n,\infty]} \to X \)].

The “only if” direction follows by the definition of a convergent inverse sequence. □

Theorem 5.18. Let \( \{ p_i \} \) \( X_1 \to X_0 \) be a convergent inverse sequence of uniformly continuous maps between uniform ANRs, and let \( X \) be its inverse limit. Then there
exists a sequence of \( \alpha_i^* > 0 \) such that for each sequence of \( \alpha_i \in (0, \alpha_i^*] \) the following are equivalent:

(a) \( X \) is a uniform ANR;
(b) there exists an \( i_0 \) such that for each \( i \geq i_0 \) there exists an \( n_i \geq i \) and a uniformly continuous map \( s_i : X_{n_i} \to X_{n_{i+1}} \) such that the composition \( X_{n_i} \xrightarrow{s_i} X_{n_{i+1}} \xrightarrow{p_{n_{i+1}}^i} X \) is \( \alpha_i \)-close to \( p_{n_{i+1}}^i \); and an \( l_{i+1} \geq n_{i+1} \) such that \( p_{n_{i+1}}^{l_{i+1}} \) is uniformly homotopic to the composition \( X_{l_{i+1}} \xrightarrow{p_{l_{i+1}}^{n_{i+1}}} X_{n_{i+1}} \xrightarrow{s_i} X_{n_{i+1}} \).

Proof. Let \( \alpha_i^* \) be bounded above by the \( \alpha_i \) in Corollary 5.11 (upon setting \( \beta_i = \beta_i^* \) in there). Since each \( X_i \) is uniformly locally contractible (see Lemma 4.16(a)), we may assume that every two \( \alpha_i \)-close maps into \( X_i \) are uniformly homotopic.

Suppose that \( X \) is a uniform ANR. Since \( X \) is uniformly locally contractible, there exists an \( \varepsilon > 0 \) such that every two \( \varepsilon \)-close maps into \( X \) are uniformly homotopic. Given an \( i \), let \( \delta_{i} > 0 \) be such that \( \delta_{i} < \varepsilon \) and \( p_i^\alpha \) is \((\delta_{i}, \alpha_i/2)\)-continuous. Since the inverse sequence is convergent and \( X \) satisfies the Hahn property (see Lemma 4.19(a)), there exists an \( m_i \geq i \) and a uniformly continuous map \( r_i : X_{m_i} \to X \) such that the composition \( X \xrightarrow{p_i^m} X_{m_i} \xrightarrow{r_i} X \) is \( \delta_i \)-close to the identity. Then \( p_i^\alpha \) is \( \frac{\alpha_i}{2} \)-close to the composition \( X \xrightarrow{p_i^m} X_{m_i} \xrightarrow{r_i} X \xrightarrow{p_i^\alpha} X_i \). Since the inverse sequence is convergent, by Lemma 5.12(b) there exists an \( n_i \geq m_i \) such that \( p_i^{n_i} \) is \( \alpha_i \)-close to the composition \( X_{n_i} \xrightarrow{p_i^{n_i}} X_{m_i} \xrightarrow{r_i} X \xrightarrow{p_i^{\infty}} X_i \). In particular, they are joined by a uniform homotopy \( \nu_i : X_{n_i} \times I \to X_i \). Set \( s_i \) to be the composition \( X_{n_i} \xrightarrow{p_i^{n_i}} X_{m_i} \xrightarrow{r_i} X \xrightarrow{p_i^{\infty}} X_j \), where \( j = n_{i+1} \). Thus the composition \( X_{n_i} \xrightarrow{s_i} X_j \xrightarrow{p_j^j} X_i \) is \( \alpha_i \)-close to \( p_i^{n_i} \).

Now the composition \( X \xrightarrow{p_i^m} X_{m_i} \xrightarrow{r_i} X \) is \( \delta_i \)-close to the identity, which is in turn \( \delta_i \)-close to the composition \( X \xrightarrow{p_i^m} X_{m_i} \xrightarrow{r_i} X \). Since \( m_j \geq m_i \) and the inverse sequence is convergent, by Lemma 5.12(b) there exists an \( l = l_{i+1} \geq m_j \) such that the compositions \( X_l \xrightarrow{p_l^m} X_{m_i} \xrightarrow{r_i} X \) and \( X_l \xrightarrow{p_l^m} X_{m_j} \xrightarrow{r_j} X \) are \( 2(\delta_i + \delta_j) \)-close, hence \( \varepsilon \)-close. Therefore they are joined by a uniform homotopy \( \lambda_i : X_l \times I \to X \). Then the composition \( X_l \xrightarrow{\lambda_i} X \xrightarrow{p_j^{\infty}} X_j \) along with the composition \( X_l \xrightarrow{\lambda_i \times \text{id}_I} X_{n_j} \times I \xrightarrow{\nu_j} X_j \) combine to form a uniform homotopy \( X_l \times I \to X_j \) between \( p_j^j \) and the composition \( X_l \xrightarrow{p_l^{m_i}} X_{m_i} \xrightarrow{r_i} X \xrightarrow{p_j^{\infty}} X_j \), which is the same as \( X_l \xrightarrow{p_l^{m_i}} X_{n_i} \xrightarrow{s_i} X_j \).

Conversely, suppose that (b) holds. We may assume that \( i_0 = 0 \) and that each \( l_{i+1} \geq l_i \). Then the uniform homotopy between \( p_{l_{i+1}}^{n_{i+1}} \) and the composition \( X_{l_{i+1}} \xrightarrow{p_{l_{i+1}}^{l_{i+1}}} X_{n_{i+1}} \xrightarrow{s_i} X_{n_{i+1}} \) yields a uniformly continuous map \( \varphi_i : MC(p_{l_{i+1}}^{l_{i+1}}) \to X_{n_{i+1}} \) that restricts to \( p_{l_{i+1}}^{l_{i+1}} \) on \( X_{l_{i+1}} \) and to the composition \( X_{l_i} \xrightarrow{p_{l_i}^{l_i}} X_{n_i} \xrightarrow{s_i} X_{n_{i+1}} \) on \( X_{l_i} \). Let us write \( Y_i = X_{l_i} \) so that \( MC(p_{l_{i+1}}^{l_{i+1}}) \) becomes \( Y_{[i,i+1]} \). For each \( j \geq i + 1 \) let \( \varphi_j^i \) denote the composition
We obtain a uniformly continuous map $\Psi: X_{n+1} \to X_n$. Then $\varphi_j^0, \ldots, \varphi_j^{j-1}$ combine into a uniformly continuous map $\Phi: X \to X_j$. By the hypothesis each composition $X_{[i,j+1]} \to X_j$ is $\alpha_j$-close to the composition $\Phi$. Hence the composition $\Psi_j: Y[0,j] \to X_j$ is $\alpha_j$-close to the composition $\Psi_j: Y[0,j] \to X_j$. Thus by Corollary 5.11 we obtain a uniformly continuous map $\Psi: Y[0,\infty] \to X$, which by construction (or alternatively by the uniqueness in Corollary 5.11) restricts to the identity on $X$. The assertion now follows from Corollary 5.17.

Theorem 5.19. Let $\ldots \to X_1 \to X_0$ be a convergent inverse sequence of uniformly continuous maps between uniform ANRs, and let $X$ be its inverse limit. Then there exists a sequence of $\alpha^*_i > 0$ such that for each sequence of $\alpha_i \in (0, \alpha^*_i]$ the following are equivalent:

(a) $X$ satisfies the Hahn property;

(b) there exists an $i_0$ such that for each $i \geq i_0$ there exists an $n_i \geq i$ and a uniformly continuous map $s_i: X_{n_i} \to X_{n+1}$ such that the composition $X_{n_i} \to X_{n+1} \to X_i$ is $\alpha_i$-close to $p_i^{n_i}$.

The proof is similar to (and easier than) that of Theorem 5.18.

5.C. Uniform homeomorphisms between inverse limits

Theorem 5.20. Let $\ldots \to X_1 \to X_0$ and $\ldots \to Y_1 \to Y_0$ be inverse sequences of uniformly continuous maps between complete metric spaces, and let $X$ and $Y$ be their inverse limits. Then there exists a sequence of $\alpha_i > 0$ such that the following holds. Suppose that there exist non-decreasing unbounded sequences of natural numbers $n_i$ and $m_i$, and uniformly continuous maps $f_i: X_{m_i} \to Y_{m_i}$ and $g_i: Y_{m_i} \to X_{n_i}$ such that the diagrams

$$
\begin{align*}
X_{n_{i+1}} & \xrightarrow{f_{i+1}} Y_{m_{i+1}} \\
X_{n_i} & \xrightarrow{f_i} Y_{m_i}
\end{align*}
and
\begin{align*}
X_{n_{i+1}} & \xleftarrow{g_{i+1}} Y_{m_{i+1}} \\
X_{n_{i-1}} & \xleftarrow{g_i} Y_{m_i}
\end{align*}
$$

Diagram 5.20.
respective $\alpha_{m_i}$- and $\alpha_{n_{i-1}}$-commute, and the compositions $Y_{m_{i+1}} \xrightarrow{g_{i+1}} X_{n_i} \xrightarrow{f_i} Y_{m_i}$ and $X_{n_i} \xrightarrow{f_i} Y_{m_i} \xrightarrow{f_{i-1}} X_{n_{i-1}}$ are respectively $\alpha_{m_i}$- and $\alpha_{n_{i-1}}$-close to the bonding maps, for each $i$. Then $X$ and $Y$ are uniformly homeomorphic.

Moreover, there exists a sequence of $\beta^* > 0$ such that for each sequence of $\beta_i \in (0, \beta^*]$, the $\alpha_i$ can be chosen so that there exists a unique uniform homeomorphism $h : X \to Y$ such that the diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
p^\infty_{n_i} & & q^\infty_{m_i} \\
X_{n_i} & \xrightarrow{f_i} & Y_{m_i}
\end{array}
\quad \quad \quad
\begin{array}{ccc}
X & \xleftarrow{h^{-1}} & Y \\
p^\infty_{m_{i-1}} & & q^\infty_{m_i} \\
X_{n_{i-1}} & \xleftarrow{g_i} & Y_{m_i}
\end{array}
\]

respectively $\beta_{m_i}$- and $\beta_{n_{i-1}}$-commute for each $i$.

The compact case (apart from the uniqueness) is due to Mioduszewski [65].

**Proof.** It suffices to prove the moreover assertion. We may assume that the $\alpha_i$ are such that $p^i_k$ is $(\alpha_i, 2^{k-l}\beta_k)$-continuous for each $k$ and each $l > k$. It follows that every diagram of the form

\[
\begin{array}{ccc}
X_{n_j} & \xleftarrow{g_{j+1}} & Y_{m_{j+1}} \\
p^\infty_{n_j} & & q^\infty_{m_{j+1}} \\
X_{n_i} & \xrightarrow{f_i} & Y_{m_i}
\end{array}
\]

$\alpha_{m_i} + \beta_{m_i}$-commutes, since it splits into $j - i$ square diagrams and one triangular diagram as in the hypothesis.

On the other hand, Corollary 5.11 yields uniformly continuous maps $f : X \to Y$ and $g : Y \to X$ satisfying the desired conditions in place of $h$ and $h^{-1}$. It remains to show that $fg = \text{id}_Y$ and $gf = \text{id}_X$. Let $\gamma_i$ be such that $f_i$ is $(\gamma_i, \beta_{m_i})$-continuous. We may assume that the $\beta_i$ are such that each $p^i_k$ is $(\beta_i, \frac{1}{k})$-continuous. Then for each $i$ there exists a $j$ such that $p^i_{m_j}$ is $(\beta_{m_j}, \gamma_i)$-continuous. Then the two compositions

\[
\begin{array}{ccc}
Y & \xrightarrow{g^\infty_{m_{j+1}}} & X \\
X_{n_j} & \xleftarrow{g_{j+1}} & Y_{m_{j+1}} \\
p^\infty_{n_j} & & q^\infty_{m_{j+1}} \\
X_{n_i} & \xrightarrow{f_i} & Y_{m_i}
\end{array}
\quad \quad \quad
\begin{array}{ccc}
Y & \xrightarrow{q^\infty_{m_i}} & X \\
X_{n_i} & \xrightarrow{f_i} & Y_{m_i}
\end{array}
\]

are $\beta_{m_i}$-close. Since $f$ satisfies the desired condition on $h$, the left-hand composition is in turn $\beta_{m_i}$-close to the composition $Y \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{q^\infty_{m_i}} Y_{m_i}$. On the other hand, by the above the right-hand composition is $(\alpha_{m_i} + \beta_{m_i})$-close to $Y \xrightarrow{q^\infty_{m_i}} Y_{m_i}$.

To summarize, $Y \xrightarrow{fg} Y \xrightarrow{q^\infty_{m_i}} Y_{m_i}$ is $(3\beta_{m_i} + 2\alpha_{m_i})$-close to $Y \xrightarrow{q^\infty_{m_i}} Y_{m_i}$. We may assume that each $\alpha_i \leq \beta_i$, and that each the $\beta_i$ are such that each $q^\infty_{m_i}$ is $(\varepsilon_i, 5\beta_i)$-separating for
some zero-convergent sequence of \( \varepsilon_i > 0 \). Thus \( fg \) is \( \varepsilon_i \)-close to the identity for each \( i \), whence it is the identity. Similarly \( gf \) is the identity. 

Theorem 5.20 immediately implies the following well-known result, whose compact case is due to M. Brown [19].

**Corollary 5.21** ([51; remark to Lemma B], [50; Exer. IV.7(a)], [24]). Let \( X \) be the limit of an inverse sequence \( \ldots \xrightarrow{p_i} X_1 \xrightarrow{p_0} X_0 \) of uniformly continuous maps between complete metric spaces. Suppose that for each \( i \) we are given a sequence of uniformly continuous maps \( q_1, q_2, \ldots \) uniformly convergent to \( p_i \). Then there exists a sequence of \( n_i \in \mathbb{N} \) such that for each sequence of \( m_i \geq n_i \), the limit \( Y_{(m_i)} \) of the inverse sequence \( \lim(\ldots q_{i,m_i} \xrightarrow{m_i} X_1 \xrightarrow{q_0} X_0) \) is uniformly homeomorphic to \( X \).

Note that under the additional hypothesis that \( \ldots \xrightarrow{p_i} X_1 \xrightarrow{p_0} X_0 \) is convergent (which holds if the \( X_i \) are compact), Corollary 5.21 follows directly from Corollary 5.10(a,c,d,g).

**Theorem 5.22.** Let \( \ldots \xrightarrow{p_i} X_1 \xrightarrow{p_0} X_0 \) and \( \ldots \xrightarrow{q_i} Y_1 \xrightarrow{q_0} Y_0 \) be convergent inverse sequences of uniformly continuous maps between metric spaces satisfying the Hahn property, and suppose that their inverse limits \( X \) and \( Y \) are uniformly homeomorphic by a homeomorphism \( h \). Then for each sequence of \( \alpha_i > 0 \) there exist increasing sequences of natural numbers \( n_i \) and \( m_i \), and uniformly continuous maps \( f_i: X_{n_i} \rightarrow Y_{m_i} \) and \( g_i: Y_{m_i} \rightarrow X_{n_{i-1}} \) such that the diagrams

\[
\begin{align*}
X_{n_{i+1}} & \xrightarrow{f_{i+1}} Y_{m_{i+1}} & X & \xrightarrow{h} Y \\
p_{n_{i+1}} & \downarrow & p_{m_{i+1}} & \downarrow q_{m_{i+1}} \\
X_{n_i} & \xrightarrow{f_i} Y_{m_i} & X_{n_i} & \xrightarrow{f_i} Y_{m_i}
\end{align*}
\]

\( \alpha_{m_i} \)-commute, and the diagrams

\[
\begin{align*}
X_{n_{i}} & \xleftarrow{g_{i+1}} Y_{m_{i+1}} & X & \xleftarrow{h^{-1}} Y \\
p_{n_{i-1}} & \downarrow & p_{m_{i-1}} & \downarrow q_{m_{i-1}} \\
X_{n_{i-1}} & \xleftarrow{g_i} Y_{m_i} & X_{n_{i-1}} & \xleftarrow{g_i} Y_{m_i}
\end{align*}
\]

\( \alpha_{n_{i-1}} \)-commute, and the compositions \( Y_{m_{i+1}} \xrightarrow{g_{i+1}} X_{n_i} \xrightarrow{f_i} Y_{m_i} \) and \( X_{n_i} \xrightarrow{f_i} Y_{m_i} \xrightarrow{f_{i-1}} X_{n_{i-1}} \) are respectively \( \alpha_{m_i} \)- and \( \alpha_{n_{i-1}} \)-close to the bonding maps, for each \( i \).

A version of Theorem 5.22 is found in [51; Theorem 1]; a closer version of the compact case is also found in [65].

The proof of Theorem 5.22 employs the same ideas as that of Theorem 5.13, and we leave the details to the reader.

**Theorem 5.23.** Let \( \ldots \xrightarrow{p_i} X_1 \xrightarrow{p_0} X_0 \) and \( \ldots \xrightarrow{q_i} Y_1 \xrightarrow{q_0} Y_0 \) be convergent inverse sequences of uniformly continuous maps between complete metric spaces satisfying the Hahn property, and let \( X \) and \( Y \) be their inverse limits. Then there exists a sequence of \( \alpha_i^* > 0 \) such that for each sequence of \( \alpha_i \in (0, \alpha_i^* \] the following are equivalent:
(i) $X$ and $Y$ are uniformly homeomorphic;
(ii) there exist non-decreasing unbounded sequences of natural numbers $n_i$ and $m_i$, and uniformly continuous maps $f_i: X_{n_i} \to Y_{m_i}$ and $g_i: Y_{m_i} \to X_{n_{i-1}}$ such that the diagrams

\[
\begin{array}{ccc}
X_{n_j} & \overset{g_{j+1}}{\leftarrow} & Y_{m_{j+1}} \\
\downarrow{p_{n_j}} & & \downarrow{q_{m_{j+1}}} \\
X_{n_i} & \overset{f_{j+1}}{\rightarrow} & Y_{m_i}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_{n_j} & \overset{f_j}{\rightarrow} & Y_{m_j} \\
\downarrow{p_{n_{i-1}}} & & \downarrow{q_{m_j}} \\
X_{n_{i-1}} & \overset{g_i}{\leftarrow} & Y_{m_i}
\end{array}
\]

respectively $\alpha_{m_i}$- and $\alpha_{n_{i-1}}$-commute, for each $i$ and each $j \geq i$.

$(i) \Rightarrow (ii)$. Let $n_i$, $m_i$, $f_i$ and $g_i$ be given by Corollary 5.22. We may assume that the $\alpha_i$ are such that $p_k^n$ is $(\alpha_i, 2^{i-k}\alpha_k)$-continuous for each $k$ and each $l > k$. Each diagram in condition (ii) splits into $j-i$ square diagrams and one triangular diagram which approximately commute by Corollary 5.22, and the assertion follows. \[\square\]

$(ii) \Rightarrow (i)$. Let $\gamma_i$ be such that $f_i$ is $(\gamma_i, \alpha_{m_i})$-continuous. We may assume that the $\alpha_i$ are such that each $p_k^n$ is $(\alpha_i, \frac{1}{k} \alpha_k)$-continuous. Then for each $i$ there exists a $j$ such that $p_{n_j}$ is $(\alpha_{n_j}, \gamma_i)$-continuous. The composition $X_{n_{j+1}} \xrightarrow{f_{j+1}} Y_{m_{j+1}} \xrightarrow{g_{j+1}} X_{n_j}$ is $\alpha_{n_j}$-close to the bonding map, whereas the composition $Y_{m_{j+1}} \xrightarrow{g_{j+1}} X_{n_j} \xrightarrow{p_{n_j}} X_{n_i} \xrightarrow{f_i} Y_{m_i}$ is $\alpha_{m_i}$-close to the bonding map. It follows that the diagram

\[
\begin{array}{ccc}
X_{n_{j+1}} & \overset{f_{j+1}}{\rightarrow} & Y_{m_{j+1}} \\
\downarrow{p_{n_j}} & & \downarrow{q_{m_{j+1}}} \\
X_{n_i} & \overset{f_i}{\rightarrow} & Y_{m_i}
\end{array}
\]

$2\alpha_{m_i}$-commutes. Similarly, for each $i$ there exists a $j$ so that the diagram

\[
\begin{array}{ccc}
X_{n_j} & \overset{g_{j+1}}{\leftarrow} & Y_{m_{j+1}} \\
\downarrow{p_{n_{i-1}}} & & \downarrow{q_{m_j}} \\
X_{n_{i-1}} & \overset{g_i}{\leftarrow} & Y_{m_i}
\end{array}
\]

$2\alpha_{n_{i-1}}$-commutes. Hence after an appropriate thinning out of indices, Corollary 5.20 applies to produce a uniform homeomorphism between $X$ and $Y$. \[\square\]

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