RESONANCES AND SCATTERING POLES ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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ABSTRACT. On an asymptotically hyperbolic manifold \((X, g)\), we show that the poles (called resonances) of the meromorphic extension of the resolvent \((\Delta_g - \lambda(n - \lambda))^{-1}\) coincide, with multiplicities, with the poles (called scattering poles) of the renormalized scattering operator, except for the points of \(\frac{\pi}{2} - N\). At each \(\lambda_k := \frac{\pi}{2} - k\) with \(k \in \mathbb{N}\), the resonance multiplicity \(m(\lambda_k)\) and the scattering pole multiplicity \(\nu(\lambda_k)\) do not always coincide: \(\nu(\lambda_k) - m(\lambda_k)\) is the dimension of the kernel of a differential operator on the boundary \(\partial X\) introduced by Graham and Zworski; in the asymptotically Einstein case, this operator is the \(k\)-th conformal Laplacian.

1. INTRODUCTION

The purpose of this work is to give a ‘more direct’ proof of the result of Borthwick and Perry \([1]\) about the equivalence between resonant resonances and scattering poles, notably in order to analyze the special points \(\left\{ \frac{\pi}{2} - k \right\}_{k \in \mathbb{N}}\) that they did not deal with. This problem is especially interesting on convex co-compact hyperbolic quotients since these are the scattering poles (not the resonances) which appear in the divisor of Selberg’s zeta function associated to the group (cf. Patterson-Perry \([14]\)).

Let \(\bar{X} = X \cup \partial X\) a \(n + 1\)-dimensional smooth compact manifold with boundary and \(x\) a defining function for the boundary, that is a smooth function \(x\) on \(\bar{X}\) such that

\[
x \geq 0, \quad \partial \bar{X} = \{ m \in \bar{X}, x(m) = 0 \}, \quad dx|_{\partial \bar{X}} \neq 0
\]

We say that a smooth metric \(g\) on the interior \(X\) of \(\bar{X}\) is conformally compact if \(x^2 g\) extends smoothly as a metric to \(\bar{X}\). An asymptotically hyperbolic manifold is a conformally compact manifold such that for all \(y \in \partial X\), all sectional curvatures at \(m \in X\) converge to \(-1\) as \(m \to y\). Notice that convex co-compact hyperbolic quotients are included in this class of manifolds. An asymptotically hyperbolic manifold is necessarily complete and the spectrum of its Laplacian \(\Delta_g\) acting on functions consists of absolutely continuous spectrum \([\frac{\pi^2}{4}, \infty)\) and a finite set of eigenvalues \(\sigma_{pp}(\Delta_g) \subset (0, \frac{\pi^2}{4})\). The resolvent \((\Delta_g - z)^{-1}\) is a meromorphic family on \(\mathbb{C} \setminus \left(\left\{ \frac{\pi^2}{4} \right\} \cup \{ n \} \right)\) of bounded operators and the new parameter \(z = \lambda(n - \lambda)\) with \(\Re(\lambda) > \frac{n}{2}\) induces a modified resolvent

\[
R(\lambda) := (\Delta_g - \lambda(n - \lambda))^{-1}
\]

which is meromorphic on \(\{ \Re(\lambda) > \frac{n}{2} \}\), its poles being the points \(\lambda_c\) such that \(\lambda_c(n - \lambda_c) \in \sigma_{pp}(\Delta_g)\). Mazzeo and Melrose \([12]\) have constructed the finite-meromorphic extension (i.e. with poles whose residue is a finite rank operator) of \(R(\lambda)\) on \(\mathbb{C} \setminus \left(\left\{ \frac{\pi^2}{4} \right\} \cup \{ n \} \right)\). We proved in a previous work \([1]\) that this extension is finite-meromorphic on \(\mathbb{C}\) if and only if the metric is even in the sense that there exists a boundary defining function \(x\) such that the metric can be expressed by

\[
g = \frac{dx^2 + h(x^2, y, dy)}{x^2}
\]

in the collar \([0, \varepsilon] \times \partial \bar{X}\) induced by \(x\), with \(h(z, y, dy)\) smooth up to \(\{ z = 0 \}\). We will only consider these cases of even metrics to simplify the statements, but our result works as long as

\[
\begin{array}{c}
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\end{array}
\]
the studied singularity is a pole of finite multiplicity for the resolvent.

The poles of the extension $R(\lambda)$ are called resonances and the multiplicity of a resonance $\lambda_0$ is defined by

$$m(\lambda_0) := \text{rank} \int_{C(\lambda_0, \epsilon)} (n - 2\lambda)R(\lambda)d\lambda = \text{rankRes}_{\lambda_0}((n - 2\lambda)R(\lambda))$$

where $C(\lambda_0, \epsilon)$ is a circle around $\lambda_0$ with radius $\epsilon > 0$ chosen sufficiently small to avoid other resonances in $D(\lambda_0, \epsilon)$ and Res means the residue. In other words, this is the rank of the residue at $z_0 = \lambda_0(n - \lambda_0)$ of the resolvent as a function of $z = \lambda(n - \lambda)$.

The scattering operator $S(\lambda)$ is the operator on $\partial X$ defined as follows: let $\lambda \in \{\Re(\lambda) = \frac{\pi}{2}\}$ and $\lambda \neq \frac{\pi}{2}$, for all $f_0 \in C^\infty(\partial X)$ there exists a unique solution $F(\lambda)$ of the problem

$$(\Delta_g - \lambda(n - \lambda))F(\lambda) = 0, \quad F(\lambda) = x^\lambda f_- + x^{n-\lambda} f_+$$

we then set $S(\lambda) : f_0 \mapsto f_-|_{\partial X}$. In fact we should use half-densities and define $S(\lambda)$ on conormal bundles on $\partial X$ to get invariance with respect to $x$, but this is dropped here. Joshi and Sá Barreto showed \cite{JoshiS} that this family of operators extends meromorphically in $\mathbb{C} \setminus \frac{1}{2}(n - N)$ in the sense of pseudo-differential operators on $\partial X$ and that $S(\lambda)$ has the principal symbol

$$(1.2) \quad \sigma_0(S(\lambda)) = c(\lambda)\sigma_0(\Lambda^{2\lambda - n}), \quad \text{with } \Lambda := (1 + \Delta_{h_0})^{\frac{1}{2}}, \quad c(\lambda) := 2^n\lambda \Gamma(\frac{\pi}{2} - \lambda)\Gamma(\lambda - \frac{\pi}{2})$$

and $h_0 := x^2 g|_{\partial X}$, which leads to the factorization (see \cite{Guillarmou, Zworski1, Zworski2, Zworski3} for a similar approach)

$$(1.3) \quad \tilde{S}(\lambda) := c(n - \lambda)\Lambda^{-\lambda}S(\lambda)\Lambda^{-\lambda + \frac{\pi}{2}} = 1 + K(\lambda)$$

with $K(\lambda)$ compact finite-meromorphic. It is clear that the poles of $S(\lambda)$ and $\tilde{S}(\lambda)$ coincide except for the points of $\frac{\pi}{2} + \mathbb{Z}$. A pole $\lambda_0$ of $\tilde{S}(\lambda)$ is called a scattering pole and we define its multiplicity by

$$\nu(\lambda_0) := -\text{Tr} \left( \frac{1}{2\pi i} \int_{C(\lambda_0, \epsilon)} \tilde{S}'(\lambda)\tilde{S}^{-1}(\lambda)d\lambda \right) = -\text{TrRes}_{\lambda_0}(\tilde{S}'(\lambda)\tilde{S}^{-1}(\lambda))$$

Using a method close to that of Guillopé-Zworski $\cite{Guillarmou}$ and Gohberg-Sigal theory $\cite{Gohberg}$, we then obtain the

**Theorem 1.1.** Let $(X, g)$ be an asymptotically hyperbolic manifold with $g$ even in the sense of $\cite{Graham}$ and let $\lambda_0 \in \{\Re(\lambda) < \frac{\pi}{2}\}$ such that $\lambda_0 \notin \{\lambda \in \mathbb{C}: \lambda(n - \lambda) \in \sigma_{pp}(\Delta_g)\} \cap \frac{1}{2}(n - N)$. Then $\lambda_0$ is a pole of $R(\lambda)$ if and only if it is a pole of $S(\lambda)$ and we have

$$(1.4) \quad m(\lambda_0) = m(n - \lambda_0) + \nu(\lambda_0) - \|_{\frac{\pi}{2} - N}(\lambda_0) \dim \ker \text{Res}_{n - \lambda_0}S(\lambda)$$

where $\|_{\frac{\pi}{2} - N}$ is the characteristic function of $\frac{\pi}{2} - N$ and Res means the residue.

**Remark 1:** the term $m(n - \lambda_0)$ vanishes when $\lambda_0(n - \lambda_0) \notin \sigma_{pp}(\Delta_g)$ and that $\cite{Graham}$ can be extended to the line $\{\Re(\lambda) = \frac{\pi}{2}\}$ by using that $R(\lambda)$ and $\tilde{S}(\lambda)$ are continuous on this line except possibly at $\frac{\pi}{2}$, where only $R(\lambda)$ can have a pole; in this case $\nu(\lambda_0) = 0$ and $\cite{Graham}$ is satisfied.

**Remark 2:** the additional term introduced at $\lambda_0 = \frac{\pi}{2} - k$ is exactly the dimension of the kernel of the operator $p_{2k}$ defined by Graham-Zworski in $\cite{Graham}$ Prop. 3.5]. Therefore it only depends on the $2k$ first derivatives of the metric on the boundary. When the manifold is asymptotically Einstein, this is

$$\dim \ker \text{Res}_{\frac{\pi}{2} + k}S(\lambda) = \dim \ker P_k$$

$P_k$ being the $k$-th conformally invariant power of the Laplacian (cf. $\cite{Graham}$), which depends only on the conformal class of the metric $h_0 = x^2 g|_{\partial X}$ at the boundary. If $n$ is even, it is worth noting
that \( \dim \ker p_n \geq 1 \) since \( p_n \) always annihilates constants. Moreover, if \( (\partial \bar{X}, h_0) \) is conformally flat with \((X, g)\) asymptotically Einstein, the additional term is \( \dim \ker P_k = H_0(\partial \bar{X}) \), the number of connected components of the boundary.

The recent formula obtained by Patterson-Perry \[13\] and Bunke-Olbrich \[2\] for the divisor at \( \lambda_0 \in \mathbb{C} \) of Selberg’s zeta function on a convex co-compact hyperbolic quotient always makes the ‘spectral term’ \( \nu(\lambda_0) \) appear and an additional ‘topological term’ (an integer multiple of the Euler characteristic) comes when \( \lambda_0 \in -\mathbb{N}_0 \). As a matter of fact, the ‘spectral term’ at \( \lambda_0 = \frac{\pi}{2} - k \) (with \( k \in \mathbb{N} \)) could be splitted in a ‘resonance term’ \( m(\lambda_0) \) and a ‘conformal term’ \( \dim \ker p_{2k} \) with \( p_{2k} \) the residue of \( S(\lambda) \) at \( \frac{\pi}{2} + k \). Notice also that for \( \lambda_0 \in \frac{\pi}{2} - \mathbb{N}, m(\lambda_0) \) can be 0 though \( \nu(\lambda_0) \) is not (this is the case of \( \mathbb{H}^{n+1} \) when \( n + 1 \) is odd).

Moreover the Poisson formula obtained by Perry \[17\] for convex co-compact quotients is used to give a lower bound of poles of \( \widetilde{S}(\lambda) \) (with multiplicity \( \nu(\lambda_0) \)) in a disc \( D(\frac{\pi}{2}, R) \subset \mathbb{C} \) with radius \( R \). It is clear that the number of these poles is bigger than the number of resonances, in view of Theorem \[14\]. In the trivial case of \( \mathbb{H}^{n+1} \) with \( n + 1 \) odd, we notably have no resonance though the number of poles of \( \widetilde{S}(\lambda) \) in \( D(\frac{\pi}{2}, R) \) is \( CR^{n+1} \). However, in dimension \( n + 1 = 2 \), the explicit formula of the scattering matrix for a hyperbolic funnel by Guillopé-Zworski \[9\] or the work of Bunke-Olbrich \[2\], Prop.4.3] show that the conformal term cancels, so \( \nu(\lambda_0) = m(\lambda_0) \) (modulo the discrete spectrum).

To conclude it would be interesting to study the dimension of the kernels of the conformal Laplacians on such quotients to use Perry’s results and give a lower bound of the number of resonances in a disc.

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2. Background on multiplicities

Let \( \mathcal{H}_1, \mathcal{H}_2 \) some Hilbert spaces. If \( M(\lambda) \) is meromorphic on an open set \( U \subset \mathbb{C} \) with values in the space \( \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) of bounded linear operators and if \( \lambda_0 \) is a pole of \( M(\lambda) \), there exists a neighborhood \( V_{\lambda_0} \) of \( \lambda_0 \), an integer \( p > 0 \) and some \( (M_i)_{i=1, \ldots, p} \) in \( \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) such that for \( \lambda \in V_{\lambda_0} \setminus \{\lambda_0\} \)

\[
M(\lambda) = \Xi_{\lambda_0}(M(\lambda)) + H(\lambda),
\]

\[
\Xi_{\lambda_0}(M(\lambda)) = \sum_{i=1}^{p} M_i(\lambda - \lambda_0)^{-1}, \quad H(\lambda) \in \mathcal{Kol}(V_{\lambda_0}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)).
\]

We will call \( \Xi_{\lambda_0}(M(\lambda)) \) the polar part of \( M(\lambda) \) at \( \lambda_0 \), \( p \) the order of the pole \( \lambda_0 \), \( M_1 = \text{Res}_{\lambda_0} M(\lambda) \) the residue of \( M(\lambda) \) at \( \lambda_0 \), \( m_{\lambda_0}(M(\lambda)) := \text{rank} M_1 \) the multiplicity of \( \lambda_0 \) and

\[
\text{Rank}_{\lambda_0} M(\lambda) := \dim \sum_{i=1}^{p} \text{Im}(M_i)
\]

the total polar rank of \( M(\lambda) \) at \( \lambda_0 \). Finally, a meromorphic family of operators in \( \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) whose poles have finite total polar rank will be called finite-meromorphic.

Assume now that \( \mathcal{H}_1 = \mathcal{H}_2 \); taking essentially Gohberg-Sigal notations \[4\], a root function of \( M(\lambda) \) at \( \lambda_0 \) is a function \( \varphi(\lambda) \in \mathcal{Kol}(V_{\lambda_0}, \mathcal{H}_1) \) such that \( \lim_{\lambda \to \lambda_0} M(\lambda) \varphi(\lambda) = 0 \) and \( \varphi(\lambda_0) \neq 0 \), the vanishing order of \( M(\lambda) \varphi(\lambda) \) being called the multiplicity of \( \varphi(\lambda) \). The vector \( \varphi_0 := \varphi(\lambda_0) \) is called an eigenvector of \( M(\lambda) \) at \( \lambda_0 \) and the set of eigenvectors of \( M(\lambda) \) at \( \lambda_0 \) form a vectorial subspace of \( \mathcal{H}_1 \) denoted \( \ker_{\lambda_0} M(\lambda) \). The rank of an eigenvector \( \varphi_0 \) is defined as being the supremum of the multiplicities of the root functions \( \varphi(\lambda) \) of \( M(\lambda) \) at \( \lambda_0 \) such that \( \varphi(\lambda_0) = \varphi_0 \). If \( \dim \ker_{\lambda_0} M(\lambda) = \alpha < \infty \) and the ranks of all eigenvectors are finite, a canonical system of eigenvectors is a basis \( (\varphi_0^{(i)})_{i=1, \ldots, \alpha} \) of \( \ker_{\lambda_0} M(\lambda) \) such that the ranks of \( \varphi_0^{(i)} \) have the following property: the rank of \( \varphi_0^{(1)} \) is the maximum of the ranks of all eigenvectors of \( M(\lambda) \) at \( \lambda_0 \) and
the rank of \( \varphi_0^{(i)} \) is the maximum of the ranks of all eigenvectors in a direct complement of \( \text{Vect}(\varphi_0^{(1)}, \ldots, \varphi_0^{(r-1)}) \) in \( \ker \lambda_0 M(\lambda) \). A canonical system of eigenvectors is not unique but the family of ranks of its eigenvectors does not depend on the choice of the canonical system. We then denote \( r_i = \varphi_0^{(i)} \) the partial null multiplicities of \( M(\lambda) \) at \( \lambda_0 \) and

\[
N_{\lambda_0}(M(\lambda)) = \sum_{i=1}^{r_i}
\]

the null multiplicity of \( M(\lambda) \) at \( \lambda_0 \).

Assume that \( M(\lambda) \) is meromorphic family of Fredholm operators in \( \mathcal{L}(\mathcal{H}_1) \) and \( \lambda_0 \) a pole of finite total polar rank. If the index of \( (M(\lambda) - \Xi_{\lambda_0}(M(\lambda)))_{\lambda=\lambda_0} \) is 0, Gohberg and Sigal [4] show that there exist some holomorphically invertible operators \( U_1(\lambda) \) and \( U_2(\lambda) \) near \( \lambda_0 \), some orthogonal projections \( (P_l)_{l=0, \ldots, m} \) and some non zero integers \( (k_l)_{l=1, \ldots, m} \) such that

\[
(2.2) \quad M(\lambda) = U_1(\lambda) \left( P_0 + \sum_{l=1}^{m} (\lambda - \lambda_0)^{k_l} P_l \right) U_2(\lambda),
\]

\[
P_l P_j = \delta_{l,j} P_j, \quad \text{rank}(P_l) = 1 \text{ for } l = 1, \ldots, m, \quad \dim(1 - P_0) < \infty.
\]

If moreover \( M(\lambda) \) has a meromorphic inverse \( M^{-1}(\lambda) \) (ie. when \( P_0 + \sum_{l=1}^{m} P_l = 1 \)) then \( \lambda_0 \) is at most a pole of finite total polar rank of \( M^{-1}(\lambda) \) and

\[
(2.3) \quad M^{-1}(\lambda) = U_2^{-1}(\lambda) \left( P_0 + \sum_{l=1}^{m} (\lambda - \lambda_0)^{-k_l} P_l \right) U_1^{-1}(\lambda).
\]

It is important to notice that the set of partial null multiplicities remains invariant under multiplication by a holomorphically invertible family of operators (cf. [4]). In view of (2.2) and (2.3), it is then easy to see that

\[
\dim \ker \lambda_0 M(\lambda) = \sharp \{ l; k_l > 0 \}, \quad \dim \ker \lambda_0 M^{-1}(\lambda) = \sharp \{ l; k_l < 0 \}
\]

and that the set of partial null multiplicities of \( M(\lambda) \) (resp. \( M^{-1}(\lambda) \)) at \( \lambda_0 \) is \( \{k_l; k_l > 0\} \) (resp. \( \{k_l; k_l < 0\} \)). We deduce

\[
N_{\lambda_0}(M(\lambda)) = \sum_{k_l > 0} k_l, \quad N_{\lambda_0}(M^{-1}(\lambda)) = \sum_{k_l < 0} k_l
\]

and from the factorization [2.2] Gohberg-Sigal [1] obtain the generalized logarithmic residue theorem

\[
(2.4) \quad \text{Tr} \left( \text{Res}_{\lambda_0}(M(\lambda)M^{-1}(\lambda)) \right) = N_{\lambda_0}(M(\lambda)) - N_{\lambda_0}(M^{-1}(\lambda)).
\]

This integer is essentially the order of the zero or the pole of \( \det(M(\lambda)) \) at \( \lambda_0 \) (when \( \det(M(\lambda)) \) exists).

To conclude, let \( M(\lambda) \) be a meromorphic family of Fredholm operators with index 0 in \( \mathcal{L}(\mathcal{H}_1) \) and \( \lambda_0 \) a pole of finite total polar rank. We write \( M(\lambda) \) as in (2.2) and if \( L(\lambda) := (\lambda - \lambda_0)^{-1} M(\lambda) \), we deduce that \( \dim \ker \lambda_0 L(\lambda) = \sharp \{ l; k_l > 1 \} \), the set of partial null multiplicities of \( L(\lambda) \) at \( \lambda_0 \) is \( \{k_l - 1; k_l > 1\} \) and

\[
(2.5) \quad N_{\lambda_0}(L(\lambda)) = \sum_{k_l > 1} (k_l - 1) = \sum_{k_l > 0} (k_l - 1) = N_{\lambda_0}(M(\lambda)) - \dim \ker \lambda_0 M(\lambda).
\]

This formula will be essential for what follows since the scattering operator \( S(\lambda) \) is not finite-meromorphic near \( \frac{n}{2} + k \) (with \( k \in \mathbb{N} \)) whereas \( (\lambda - \frac{n}{2} - k)S(\lambda) \) is.
3. Resonances and scattering poles

3.1. Stretched products, half-densities. To begin, let us introduce a few notations and recall some basic things on stretched products and singular half-densities (the reader can refer to Mazzeo-Melrose [12], Melrose [13] for details). Let $X$ a smooth manifold with boundary and $x$ a boundary defining function. The manifold $X \times \bar{X}$ is a smooth manifold with corners, whose boundary hypersurfaces are diffeomorphic to $\partial X \times \bar{X}$ and $\bar{X} \times \partial X$, and defined by the functions $\pi^L_x, \pi^R_x (\pi^L_x$ and $\pi^R_x$ being the left and right projections from $X \times \bar{X}$ onto $X$).

For notational simplicity, we now write $x, x'$ instead of $\pi^L_x, \pi^R_x$ and let

$$\delta_{\partial X} := \{(m, m) \in \partial X \times \partial \bar{X}; m \in \partial X\}.$$ 

The blow-up of $X \times \bar{X}$ along the diagonal $\delta_{\partial X}$ of $\partial X \times \partial \bar{X}$ will be noted $\bar{X} \times_0 X \bar{X}$ and the blow-down map

$$\beta : \bar{X} \times_0 X \bar{X} \to X \times \bar{X}$$

This manifold with corners has three boundary hypersurfaces $\mathcal{F}, \mathcal{B}, \mathcal{I}$ defined by some functions $\rho, \rho', R$ such that $\beta^*(x) = R\rho$, $\beta^*(x') = R\rho'$. Globally, $\delta_{\partial X}$ is replaced by a larger manifold, namely by its doubly inward-pointing spherical normal bundle of $\delta_{\partial \bar{X}}$, whose each fiber is a quarter of sphere. From local coordinates $(x, y, x', y')$ on $X \times \bar{X}$, this amounts to introducing polar coordinates $(R, \rho, \rho', \omega, y)$ around $\delta_{\partial \bar{X}}$:

$$R := (x^2 + x'^2 + |y - y'|^2)^{\frac{1}{4}}, \quad (\rho, \rho', \omega) := \left(\frac{x}{R}, \frac{x'}{R}, \frac{y - y'}{R}\right)$$

with $R, \rho, \rho' \in [0, \infty)$. In these polar coordinates the Schwartz kernel of $R(\lambda)$ has a better description.

Using evident identifications induced by the inclusions

$$\delta_{\partial X} \subset \partial X \times \partial \bar{X} \subset \partial X \times X \subset \bar{X} \times X,$$

we denote by $\partial X \times_0 \bar{X}$ the blow-up of $\partial X \times X$ along $\delta_{\partial X}$ and $\partial \bar{X} \times_0 \partial \bar{X}$ the blow-up of $\partial \bar{X} \times \partial \bar{X}$ along $\delta_{\partial \bar{X}}$. $\overline{\beta}$ and $\beta_\partial$ are the associated blow-down map

$$\overline{\beta} : \partial X \times_0 \bar{X} \to \partial X \times \bar{X}, \quad \beta_\partial : \partial \bar{X} \times_0 \partial \bar{X} \to \partial \bar{X} \times \partial \bar{X}$$

with $\overline{\beta} = \beta|_{\mathcal{I}}$ and $\beta_\partial = \beta|_{\mathcal{B} \cup \mathcal{I}}$. Note that $r := R|_{\mathcal{B} \cup \mathcal{I}}$ is a defining function of the boundary of $\partial X \times_0 \partial \bar{X}$ (which is the lift of $\delta_{\partial \bar{X}}$ under $\beta_\partial$).

Let $\Gamma^\beta_0(X)$ the line bundle of singular half-densities on $X$, trivialized by $\nu := |dvol|_0^\frac{1}{4}$, and $\Gamma^\{\mathcal{F}\}(\partial X)$ the bundle of half-densities on $\partial X$, trivialized by $\nu_0 := |dvol|_0^\frac{1}{4}$ (where $h_0 = x^2|_{\mathcal{I}}$).

From these bundles, one can construct the bundles $\Gamma^\beta_0(X \times \bar{X}), \Gamma^\{\mathcal{I}\}(\partial X \times X)$ and $\Gamma^\{\mathcal{F}\}(\partial \bar{X} \times \partial X)$ by tensor products and the bundles $\Gamma^\beta_0(\bar{X} \times_0 X), \Gamma^{\beta}_{00}(\partial \bar{X} \times_0 X)$ and $\Gamma^{\{\mathcal{F}\}}(\partial \bar{X} \times_0 \partial \bar{X})$ by lifting under $\beta, \overline{\beta}$ and $\beta_\partial$ the three previous bundles. If $M$ denotes $\bar{X}, \bar{X} \times \bar{X}$ or $\partial \bar{X} \times \bar{X}$, we write $\hat{C}^\infty(M, \Gamma^\beta_0)$ the space of smooth sections of $\Gamma^\beta_0(M)$ that vanish to all order at all the boundary hypersurfaces of $M$, and $C^{\infty}(M, \Gamma^\beta_0)$ is its topological dual. The Hilbert space $L^2(\bar{X}, \Gamma^\beta_0)$ and $L^2(\partial \bar{X}, \Gamma^\{\mathcal{F}\})$ are isomorphic to $L^2(X, dvol)$ and $L^2(\partial X, dvol)$, they will be denoted $L^2(X)$, $L^2(X)$.

For $\alpha \in \mathbb{R}$, let $x^\alpha L^2(X) := \{f \in C^{\infty}(\bar{X}, \Gamma^\beta_0); x^{-\alpha} f \in L^2(X)\}$ and we set $\langle.,.\rangle$ the symmetric non-degenerate products

$$\langle u, v \rangle := \int_X uv \text{ on } L^2(X), \quad \langle u, v \rangle := \int_{\partial X} uv \text{ on } L^2(\partial X).$$

We can check by using the first pairing that the dual space of $x^\alpha L^2(X)$ is isomorphic to $x^{-\alpha}L^2(X)$. We shall also use the following tensorial notation for $E = x^\alpha L^2(X)$ (resp. $E = \partial X \times_0 \partial \bar{X}$).
\[ L^2(\partial \bar{X}), \psi, \phi \in E' \]
\[ \phi \otimes \psi : \begin{cases} E &\rightarrow E' \\ f &\rightarrow \phi(\psi, f) \end{cases} \]

3.2. Resolvent. From [12] [6], we know that on an asymptotically hyperbolic manifold \((X, g)\) with \(g\) even, the modified resolvent
\[ R(\lambda) := (\Delta_g - \lambda(n - \lambda))^{-1} \]
extends for all \(N > 0\) to a finite-meromorphic family of operators in \(\{\Re(\lambda) > \frac{n}{2} - N\}\) with values in \(\mathcal{L}(x^N L^2(X), x^{-N} L^2(X))\), whose poles, the resonances, form a discrete set \(\Re\) in \(\mathbb{C}\). Moreover \(R(\lambda)\) is a continuous operator from \(C^\infty(\bar{X}, \Gamma_{\tilde{0}})\) to \(C^{-\infty}(\bar{X}, \Gamma_{\tilde{0}})\), its associated Schwartz kernel being
\[ r(\lambda) = r_0(\lambda) + r_1(\lambda) + r_2(\lambda) \in C^{-\infty}(\bar{X} \times \bar{X}, \Gamma_{\tilde{0}}) \]
with (see [12] or [11] Th. 2.1):
\[ \beta^*(r_0(\lambda)) \in I^{-2}(\bar{X} \times_0 \bar{X}, \Gamma_{\tilde{0}}), \]
\[ \beta^*(r_1(\lambda)) \in \rho^3 \rho^\lambda C^\infty(\bar{X} \times_0 \bar{X}, \Gamma_{\tilde{0}}), \]
\[ r_2(\lambda) \in x^\lambda x^{-\lambda} C^\infty(\bar{X} \times \bar{X}, \Gamma_{\tilde{0}}), \]
\[ \beta^{-1}(\{(m, m) \in X \times X; m \in X\}) \]
and vanishing to infinite order at \(\mathcal{B} \cup \mathcal{F}\) (note that the lifted interior diagonal only intersects the topological boundary of \(\bar{X} \times_0 \bar{X}\) at \(\mathcal{F}\), and it does transversally). Moreover, \((\rho \rho')^{-\lambda} \beta^*(r_1(\lambda))\) and \((x x')^{-\lambda} r_2(\lambda)\) are meromorphic in \(\lambda \in \mathbb{C}\) and \(r_0(\lambda)\) is the kernel of a holomorphic family of operators
\[ R_0(\lambda) \in \mathcal{H}ol(\mathcal{C}, \mathcal{L}(x^\alpha L^2(X), x^{-\alpha} L^2(X))), \quad \forall \alpha \geq 0. \]

Note also that Patterson-Perry arguments [14] Lem.4.9] prove that \(R(\lambda)\) does not have poles on the line \(\{\Re(\lambda) = \frac{n}{2}\}\), except maybe \(\lambda = \frac{n}{2}\). The set of poles of \(R(\lambda)\) in the half plane \(\{\Re(\lambda) > \frac{n}{2}\}\) is \(\{\lambda_c; \Re(\lambda_c) > \frac{n}{2}, \lambda_c(n - \lambda_c) \in \sigma_{pp}(\Delta_g)\}\), they are first order poles and their residue is
\[ \text{Res}_{\lambda_c} R(\lambda) = (2\lambda_c - n)^{-1} \sum_{k=1}^{n} \phi_k \otimes \phi_k, \quad \phi_k \in x^{\lambda_c} C^\infty(\bar{X}, \Gamma_{\tilde{0}}), \]
where \((\phi_k)_{k=1,...,p}\) are the normalized eigenfunctions of \(\Delta_g\) for the eigenvalue \(\lambda_c(n - \lambda_c)\). One can see by a Taylor expansion at \(x = 0\) of the eigenvector equation that if \(x^{-\lambda_c + \frac{n}{2}} \phi_k |_{\partial X} = 0\) then \(\phi_k \in \mathcal{O}^\infty(\bar{X}, \Gamma_{\tilde{0}})\), which is excluded according to Mazzeo’s results [11].

To simplify the notations, we shall set \(z(\lambda) := \lambda(n - \lambda)\) the holomorphically invertible function from \(\Re(\lambda) < \frac{n}{2}\) to \(\mathbb{C} \setminus [\frac{n^2}{4}, \infty)\).

For the poles of \(R(\lambda)\) in \(\{\Re(\lambda) < \frac{n}{2}\}\), we use Lemma 2.4 and 2.11 of [9] to show the

**Lemma 3.1.** Let \(\lambda_0 \in \mathbb{R}\) and \(N\) such that \(\frac{n}{2} > \Re(\lambda_0) > \frac{n}{2} - N\), then in a neighbourhood \(V_{\lambda_0}\) of \(\lambda_0\) we have the decomposition
\[ R(\lambda) = \Phi F_1(\lambda) \left( \sum_{j=1}^{m} (z(\lambda) - z(\lambda_0))^{k_j} P_j \right) F_2(\lambda) \Phi + H(\lambda), \]
with \(m \in \mathbb{N}, k_1, \ldots, k_m \in -\mathbb{N}\),
\[ H(\lambda) \in \mathcal{H}ol(V_{\lambda_0}, \mathcal{L}(x^N L^2(X), x^{-N} L^2(X))), \quad F_i(\lambda) \in \mathcal{H}ol(V_{\lambda_0}, \mathcal{L}(\mathbb{C}^q)), \]
where \( q = -\sum_{j=1}^m \epsilon_j = m_{\lambda_0}(z'(\lambda)R(\lambda)) \) is the multiplicity of the resonance \( \lambda_0 \), \( (P_j)_{j=1,\ldots,m} \) are some orthogonal projections on \( \mathbb{C}^q \) such that \( P_j P_j = \delta_{ij} P_j \) and \( \text{rank}(P_j) = 1 \), \( \Phi \) is defined by

\[
\Phi : \begin{cases} 
  x^N L^2(X) & \rightarrow \mathbb{C}^q \\
  f & \rightarrow ((\psi_l,f))_{l=1,\ldots,q}
\end{cases}
\]

(3.4) \( \Psi \) being a basis of \( \text{Im}(A) \) with \( A := \text{Res}_{\lambda_0}(z'(\lambda)R(\lambda)) \). Moreover we have

(3.4) \( \text{Im}(A) \subset \sum_{j=0}^{p-1} x^{\lambda_0} \log^j(x)C^\infty(\mathring{X}, \Gamma_{\emptyset}^\frac{1}{2}) \)

with \( p \) the order of the pole \( \lambda_0 \) of \( R(\lambda) \).

**Proof:** it suffices to use Lemmas 2.4 and 2.11 of [1] but we factorize the resolvent and not the scattering operator. The arguments used in these lemmas are essentially that the polar part of \( R(\lambda) \) be expressed by

\[
\Xi_{\lambda_0}(R(\lambda)) = \Xi_{\lambda_0} \left( \sum_{i=1}^p \frac{(\Delta_y - z(\lambda_0))^{i-1} A}{(z(\lambda) - z(\lambda_0))^i} \right)
\]

and the factorization into its Jordan form of the nilpotent matrix of \( \Delta_y - z(\lambda_0) \) acting on \( \text{Im}(A) \). Observe that the elliptic regularity implies that the elements of \( \text{Im}(A) \) are smooth in \( X \).

To study the structure of the Schwartz kernel \( a_j \) of \( A_j \), we first consider the following operator

(3.5) \( \overline{R}(\lambda) := x^{-\lambda + \frac{1}{2}} R(\lambda) x^{-\lambda + \frac{1}{2}} \)

in a disc \( D(\lambda_0, \epsilon) \) around \( \lambda_0 \) with radius \( \epsilon \). If \( \epsilon \) is taken sufficiently small, \( \overline{R}(\lambda) \) is meromorphic in this disc with values in \( \mathcal{L}(x^{2\epsilon} L^2(X), x^{-2\epsilon} L^2(X)) \), \( \lambda_0 \) is the only pole and its order is \( p \). The Schwartz kernel \( (xx')^{-\lambda + \frac{1}{2}} R(\lambda) \) of \( R(\lambda) \) is meromorphic and its polar part at \( \lambda_0 \) is the same as the one of \( (xx')^{-\lambda + \frac{1}{2}} (r_1(\lambda) + r_2(\lambda)) \) since \( r_0(\lambda) \) is holomorphic in \( \mathbb{C} \). We then can easily check [1] Prop. 3.3] that we have in \( V_{\lambda_0} \)

(3.6) \( \Xi_{\lambda_0}(\overline{R}(\lambda)) = \sum_{j=-p}^{-1} B_j (\lambda - \lambda_0)^j \)

where \( B_j \in \mathcal{L}(x^{2\epsilon} L^2(X), x^{-2\epsilon} L^2(X)) \) has a Schwartz kernel of the form

(3.7) \( b_j(x, y, x', y') = \sum_{i=1}^{r_j} \psi_{ij}(x, y) \varphi_{ij}(x', y') \frac{dx dy dx' dy'}{|x_{n+1} x_{n+1}|^\frac{1}{2}}, \quad \psi_{ij}, \varphi_{ij} \in x^{\frac{1}{2}} C^\infty(\mathring{X}) \).

Observe now that \( x^{\lambda - \frac{1}{2}} \) has the following Taylor expansion at \( \lambda_0 \)

\[
x^{\lambda - \frac{1}{2}} = x^{\lambda_0 - \frac{1}{2}} \sum_{j=0}^{p-1} \log^j(x) \frac{(\lambda - \lambda_0)^j}{j!} + O((\lambda - \lambda_0)^p)
\]

in the sense of operators of \( \mathcal{L}(x^{N} L^2(X), x^{2\epsilon} L^2(X)) \) and \( \mathcal{L}(x^{-2\epsilon} L^2(X), x^{-N} L^2(X)) \). We deduce that \( z'(\lambda)R(\lambda) \) has a residue \( A \) satisfying

\[
\text{Im}(A) \subset \sum_{j=0}^{p-1} x^{\lambda_0} \log^j(x) C^\infty(\mathring{X}, \Gamma_{\emptyset}^\frac{1}{2})
\]

and we are done.  \( \square \)
3.3. Scattering matrix. Joshi and Sá Barreto have shown that the scattering matrix $S(\lambda)$ (defined in the introduction) has the following Schwartz kernel

$$s(\lambda) := (2\lambda - n)(\beta_0)^* \left( (x^{-\lambda+\frac{d}{2}} \partial_x^{-\lambda+\frac{d}{2}} r(\lambda)) \big|_{x=\Psi} \right)$$

Following (3.1) and (3.8) we have in $\mathbb{C} \setminus (\mathbb{R} \cup \{\frac{n}{2} + 
\}$

$$s(\lambda) = (\beta_0)^* \left( r^{-2\lambda} k_1(\lambda) \right) + k_2(\lambda),$$

where $k_1(\lambda)$ and $k_2(\lambda)$ are meromorphic in $\lambda \in \mathbb{C}$. Outside its poles, $s(\lambda)$ is a conormal distribution of order $-2\lambda$ associated to $\partial_x$ and $S(\lambda)$ is a pseudo-differential operator of order $2\lambda - n$ on $\partial_x$. In the sense of Shubin [18, Def. 11.2], $S(\lambda)$ is a holomorphic family in $\{\Re(\lambda) < \frac{n}{2}\} \setminus \mathbb{R}$ of zeroth order pseudo-differential operators. We then deduce that $S(\lambda)$ is holomorphic in the same open set, with values in $\mathcal{L}(L^2(\partial_x))$. Recall the functional equation satisfied by $S(\lambda)$ (cf. [1])

$$S(\lambda)^{-1} = S(n - \lambda) = S(\lambda)^*,$$

which also proves that $S(\lambda)$ is regular on the line $\{\Re(\lambda) = \frac{n}{2}\}$. Furthermore, (3.10) holds also for $\tilde{S}(\lambda)$ and by analytic extension we have on $\mathbb{C} \setminus \mathbb{R}$

$$\tilde{S}^{-1}(\lambda) = \tilde{S}(n - \lambda).$$

The principal symbol of $S(\lambda)$ is given in (1.2) and the renormalization $\tilde{S}(\lambda)$ of $S(\lambda)$ defined in (1.6) is Fredholm with index 0, consequently we are in the framework of Section 2.

Using Lemmas 3.1 and (3.9), we then obtain the

**Lemma 3.2.** Let $\lambda_0 \in \{\Re(\lambda) < \frac{n}{2}\}$ a pole of $S(\lambda)$. Then $\lambda_0 \in \mathbb{R}$ and, following the notations of Lemma 3.1, we have near $\lambda_0$

$$S(\lambda) = (2\lambda - n)^{\Phi(\lambda)} F_1(\lambda) \left( \sum_{j=1}^{m} (z(\lambda) - z(\lambda_0))^{k_j} P_j \right) F_2(\lambda)^{-\Phi(\lambda)} + H^\delta(\lambda)$$

with $H^\delta(\lambda) \in \mathcal{H}\text{ol}(V_{\lambda_0}, \mathcal{L}(L^2(\partial_x)))$ and $\Phi(\lambda) \in \mathcal{H}\text{ol}(V_{\lambda_0}, \mathcal{L}(L^2(\partial_x), C^q))$.

**Proof:** the fact that $\lambda_0 \in \mathbb{R}$ is straightforward since if $r(\lambda)$ was holomorphic one would have $s(\lambda)$ holomorphic in view of (3.8). Now, $\tilde{R}(\lambda)$ being defined in (3.10), we saw in Lemma 3.1 that the polar part of $\tilde{R}(\lambda)$ at $\lambda_0$ has a Schwartz kernel $\Xi_{\lambda_0}(\tilde{r}(\lambda))$ satisfying

$$\Xi_{\lambda_0}(\tilde{r}(\lambda)) \in (x')^\frac{d}{2} C^\infty(\tilde{X} \times \tilde{X}, \Gamma_0^{\frac{d}{2}}).$$

Let $\Phi(\lambda) := \sum_{i=0}^{p-1} \frac{(\lambda - \lambda_0)^i}{i!} \frac{d^i}{d\lambda^i} (\Phi(x^{-\lambda+\frac{d}{2}}))_{\lambda=\lambda_0}$ in the sense of operators of $\mathcal{L}(x^{2\lambda} L^2(X), C^q)$:

$$\Phi(\lambda) : \left\{ \begin{array}{l}
x^{2\lambda} L^2(X) \rightarrow C^q \\
f \mapsto \left( \sum_{j=0}^{p-1} \frac{(\lambda - \lambda_0)^j}{j!} \left( \log^j(x) x^{-\lambda_0 + \frac{d}{2}} \psi_1, f \right) \right)_{l=1, \ldots, q} \end{array} \right..$$

Lemma 3.1 implies that

$$\Xi_{\lambda_0}(\tilde{R}(\lambda)) = \Xi_{\lambda_0} \left( \Phi(\lambda) F_1(\lambda) \left( \sum_{j=1}^{m} (z(\lambda) - z(\lambda_0))^{k_j} P_j \right) F_2(\lambda)^{-\Phi(\lambda)} \right).$$

Let $C := \sum_{j=0}^{-p} \text{Im}(B_j)$ with $B_j$ the operators defined in (3.8) and let $\Pi_C$ be the orthogonal projection of $x^{-2\lambda} L^2(X)$ onto $C$. We multiply (3.13) on the left by $\Pi_C$ and on the right by $\Pi_C$,
and using that $\Xi_{\lambda_0}(\tilde{R}(\lambda))$ is symmetric (since $\Re(\lambda) = \Re(\lambda)$) we deduce that \( \Xi_{\lambda_0}(\tilde{R}(\lambda)) \) remains true if $\Phi(\lambda)$ is replaced by

$$
x^{2\varepsilon}L^2(X) \to \mathbb{C}^q,
$$

$$
f \to \left( \sum_{j=0}^{n-1} \left( \frac{\lambda_0 - \lambda}{j} \right)^l \langle \Pi_C(\log^j(x)x^{-\lambda_0 + \frac{\pi}{2}\psi_1}, f) \rangle \right)_{l=1,\ldots,q}
$$

so that the logarithmic terms disappear. Finally, we can use the representation of $S(\lambda)$ by its Schwartz kernel \( \Xi_{\lambda_0}(\tilde{R}(\lambda)) \) and we obtain

$$
\Xi_{\lambda_0}(S(\lambda)) = \Xi_{\lambda_0} \left( (2\lambda - n)\Phi(\lambda)F_1(\lambda) \left( \sum_{j=1}^{m} (z(\lambda) - z(\lambda_0))^k P_j \right) F_2(\lambda) \right),
$$

with

$$
\Phi(\lambda) : \left\{ \begin{array}{l}
L^2(\partial \tilde{X}) \to \mathbb{C}^q \\
f \to \left( \sum_{j=0}^{n-1} \left( \frac{\lambda_0 - \lambda}{j} \right)^l \langle \Pi_C(\log^j(x)x^{-\lambda_0 + \frac{\pi}{2}\psi_1})|_{\partial \tilde{X}}, f \rangle \right)_{l=1,\ldots,q},
\end{array} \right.
$$

the proof is achieved.

From this lemma, we deduce the

**Corollary 3.3.** If $\lambda_0 \in \{ \Re(\lambda) < n \}$ is a pole of $S(\lambda)$, it is a pole of $R(\lambda)$ such that

$$
m_{\lambda_0}(z(\lambda)R(\lambda)) \geq N_{\lambda_0} \left( c(n - \lambda)\tilde{S}(n - \lambda) \right).
$$

**Proof:** firstly, \( \Xi_{\lambda_0}(\tilde{R}(\lambda)) \) can be expressed by

$$
c(\lambda)\tilde{S}(n - \lambda) = F_3(\lambda) \left( \sum_{j=1}^{m} (z(\lambda) - z(\lambda_0))^k P_j \right) F_4(\lambda) + \tilde{H}^2(\lambda),
$$

$$
F_3(\lambda) := (2\lambda - n)\Lambda^{-\lambda + \frac{\pi}{2}i} \Phi(\lambda)F_1(\lambda), \quad F_4(\lambda) := F_2(\lambda)\Phi(\lambda)\Lambda^{-\lambda + \frac{\pi}{2}i},
$$

$$
\tilde{H}^2(\lambda) := (2\lambda - n)\Lambda^{-\lambda + \frac{\pi}{2}i} H^2(\lambda)\Lambda^{-\lambda + \frac{\pi}{2}i}.
$$

Note that we can take $k_1 \leq \cdots \leq k_m < 0$ and set $(\varphi_0^{(j)})_{j=1,\ldots,M}$ a canonical system of eigenvectors of $c(n - \lambda)\tilde{S}(n - \lambda)$ at $\lambda_0$ with $r_1 \geq \cdots \geq r_M$ the associated partial null multiplicities (this canonical system exists and is deduced from the one of $\tilde{S}(n - \lambda)$). Let us show that $M \leq m$ and, by induction, that $r_j \leq -k_j$ for all $j = 1,\ldots,M$.

If $\varphi_0^{(j)}(\lambda)$ is a root function of $c(n - \lambda)\tilde{S}(n - \lambda)$ at $\lambda_0$ corresponding to $\varphi_0^{(j)}$, there exists a holomorphic function $\varphi^{(j)}(\lambda)$ such that

$$
c(n - \lambda)\tilde{S}(n - \lambda)\varphi^{(j)}(\lambda) = (z(\lambda) - z(\lambda_0))^r \varphi^{(j)}(\lambda)
$$

with $\varphi^{(j)}(\lambda_0) \neq 0$, hence when $\lambda$ approaches $\lambda_0$ in the following identity

$$
\varphi^{(j)}(\lambda) = \sum_{i=1}^{m} (z(\lambda) - z(\lambda_0))^{r_i + k_i} F_3(\lambda)P_i F_4(\lambda) \varphi^{(j)}(\lambda) + (z(\lambda) - z(\lambda_0))^{r_j} \tilde{H}^2(\lambda) \varphi^{(j)}(\lambda),
$$

we find that $r_1 \leq -k_1$ and $\varphi_0^{(j)}$ is in the vectorial space

$$
E_j := \text{Vect}\{F_3(\lambda_0)P_i F_4(\lambda_0)L^2(\partial \tilde{X}); r_j \leq -k_i\}.
$$

Moreover, the order on $(r_j)_{j=1,\ldots,M}$ implies that $E_j \subset E_M$ for $j = 1,\ldots,M$ but dim $E_M \leq m$ since rank$(P_i) = 1$, thus we necessarily have $M \leq m$, $(\varphi_0^{(j)})_{j=1,\ldots,M}$ being independent by assumption. Now let $j \leq M$ and suppose that $r_i \leq -k_i$ for all $i \leq j$. We first note that $E_j \subset E_{j+1}$ since $r_{j+1} \leq -k_{j+1}$. If $r_{j+1} > -k_{j+1}$, we would have dim $E_{j+1} \leq j$ but $E_{j+1}$ contains the linearly independent vectors $\varphi_0^{(j_1)}, \ldots, \varphi_0^{(j_{j+1})}$, a contradiction. One concludes that $r_{j+1} \leq -k_{j+1}$ and

$$
N_{\lambda_0} \left( c(n - \lambda)\tilde{S}(n - \lambda) \right) = \sum_{j=1}^{M} r_j \leq - \sum_{i=1}^{m} k_i = q = m_{\lambda_0}(z(\lambda)R(\lambda)),
$$

and using that $\Xi_{\lambda_0}(\tilde{R}(\lambda))$ is symmetric (since $\Re(\lambda) = \Re(\lambda)$) we deduce that \( \Xi_{\lambda_0}(\tilde{R}(\lambda)) \) remains true if $\Phi(\lambda)$ is replaced by

$$
x^{2\varepsilon}L^2(X) \to \mathbb{C}^q,
$$

$$
f \to \left( \sum_{j=0}^{n-1} \left( \frac{\lambda_0 - \lambda}{j} \right)^l \langle \Pi_C(\log^j(x)x^{-\lambda_0 + \frac{\pi}{2}\psi_1}, f) \rangle \right)_{l=1,\ldots,q}
$$

so that the logarithmic terms disappear. Finally, we can use the representation of $S(\lambda)$ by its Schwartz kernel \( \Xi_{\lambda_0}(\tilde{R}(\lambda)) \) and we obtain

$$
\Xi_{\lambda_0}(S(\lambda)) = \Xi_{\lambda_0} \left( (2\lambda - n)\Phi(\lambda)F_1(\lambda) \left( \sum_{j=1}^{m} (z(\lambda) - z(\lambda_0))^k P_j \right) F_2(\lambda) \right),
$$

with

$$
\Phi(\lambda) : \left\{ \begin{array}{l}
L^2(\partial \tilde{X}) \to \mathbb{C}^q \\
f \to \left( \sum_{j=0}^{n-1} \left( \frac{\lambda_0 - \lambda}{j} \right)^l \langle \Pi_C(\log^j(x)x^{-\lambda_0 + \frac{\pi}{2}\psi_1})|_{\partial \tilde{X}}, f \rangle \right)_{l=1,\ldots,q},
\end{array} \right.
$$

the proof is achieved. □
the corollary is proved. □

Lemma 3.4. Let $\lambda_0 \in \{ \Re(\lambda) < \frac{c}{2} \}$ be a pole of $R(\lambda)$ of finite multiplicity. If $\lambda_0(n - \lambda) \notin \sigma_{pp}(\Delta_g)$ or $\lambda_0 \notin \frac{1}{2}(n - N)$, then $\lambda_0$ is a pole of $S(\lambda)$ such that

$$m_{\lambda_0}(z'(\lambda)R(\lambda)) \leq N_{\lambda_0}(c(n - \lambda)S(n - \lambda)).$$

Proof: we first suppose that $\lambda_0$ is not a pole of $c(\lambda)$ (i.e. $\lambda_0 \notin \frac{c}{2} - N$). From Gohberg-Sigal theory, one can factorize $\tilde{S}(\lambda)$ near $\lambda_0$ as in (2.2)

$$c(\lambda)\tilde{S}(\lambda) = U_1(\lambda) \left( P_0 + \sum_{l=1}^m (\lambda - \lambda_0)^{k_l} P_l \right) U_2(\lambda)$$

with $U_1(\lambda)$, $U_2(\lambda)$ some holomorphically invertible operators near $\lambda_0$ and

$$P_l P_j = \delta_{ij} P_j, \quad \text{rank}(P_l) = 1 \text{ for } l = 1, \ldots, m, \quad 1 = \sum_{j=0}^m P_j, \quad k_l \in \mathbb{Z}^*.$$

Take the Green equation between the resolvent and the scattering operator (see [15, 16, 7, 9, 6])

$$R(\lambda) - R(n - \lambda) = (2\lambda - n)E(n - \lambda)\lambda^{\frac{c}{2}}c(\lambda)\tilde{S}(\lambda)\lambda^{\frac{c}{2}}E(n - \lambda)$$

on $\mathcal{L}(x^N L^2(X), x^{-N} L^2(X))$ with $\frac{c}{2} - N < |\Re(\lambda)| < \frac{c}{2}$ and $E(\lambda)$ the transpose of the Eisenstein operator, its Schwartz kernel being

$$e(\lambda) := \overline{\beta}_c (\beta^* (x^-\lambda^{\frac{c}{2}} \tau(\lambda))|_T).$$

We can suppose that $k_1 \leq \cdots \leq k_m$ and set $p := \max(0, -k_1)$. We consider the following Laurent expansions at $\lambda_0$

$$(n - 2\lambda)R(\lambda - \lambda_0) = \sum_{i=-1}^{p-1} R_i(\lambda - \lambda_0)^i + O((\lambda - \lambda_0)^p),$$

$$(2\lambda - n)U_2(\lambda)\lambda^{\frac{c}{2}}E(n - \lambda) = \sum_{i=-1}^{p-1} E_1^{(2)}(\lambda - \lambda_0)^i + O((\lambda - \lambda_0)^p),$$

$$(n - 2\lambda)E(n - \lambda)\lambda^{\frac{c}{2}}U_1(\lambda) = \sum_{i=-1}^{p-1} E_2^{(1)}(\lambda - \lambda_0)^i + O((\lambda - \lambda_0)^p),$$

where $R_{-1}$ and $E_{-1}^{(j)}$ are not 0 if and only if $\lambda_0(n - \lambda_0) \in \sigma_{pp}(\Delta_g)$, and in this case

$$R_{-1} = - \sum_{i=1}^{k} \phi_i \otimes \phi_i,$$

$$E_{-1}^{(2)} = \sum_{i=1}^{k} U_2(\lambda_0)\lambda^{\frac{c}{2}}(x^{\lambda_0 - \frac{c}{2}} \phi_i)|_{\partial X} \otimes \phi_i,$$

$$E_{-1}^{(1)} = - \sum_{i=1}^{k} \phi_i \otimes U_1(\lambda_0)\lambda^{\frac{c}{2}}(x^{\lambda_0 - \frac{c}{2}} \phi_i)|_{\partial X},$$

with $\phi_i \in x^{n - \lambda_0} C^\infty(\overline{\mathcal{X}}, \Gamma_{\mathcal{O}}^{-\frac{c}{2}})$ the normalized eigenfunctions of $\Delta_g$ for the eigenvalue $\lambda_0(n - \lambda_0)$.

From (3.14), (3.15) and (3.16) we obtain

$$A := \text{Res}_{\lambda_0}((n - 2\lambda)R(\lambda)) = R_{-1} + \sum_{j+i+k_l=0} E_1^{(j)} P_l E_1^{(2)} + \sum_{j+i+k_l=1} E_1^{(1)} P_l E_2^{(2)}$$

where by convention $k_l = 0 \iff l = 0$. We set $V := \text{Im}(A_1) + \text{Im}(A_2)$ with

$$A_1 := R_{-1} + E_{-1}^{(2)} P_0 E_0^{(2)} + E_{-1}^{(1)} \left( \sum_{k_l=1}^m P_l \right) E_{-1}^{(2)},$$

$$A_2 := E_0^{(1)} P_0 E^{-2}_{-1} + \sum_{j+i+k_l=1} E_1^{(1)} P_l E_2^{(j)}.$$

Remark from (3.17) that

$$\text{Im}(A_1) \subset x^{n - \lambda_0} C^\infty(\overline{\mathcal{X}}, \Gamma_{\mathcal{O}}^{-\frac{c}{2}}), \quad (\Delta_g - \lambda_0(n - \lambda_0))A_1 = 0$$
and in view of Lemma 8.1, we know that there exists $p \in \mathbb{N}$ such that
\[
\text{Im}(A) \subset \sum_{j=0}^{p-1} x^{\lambda_j} \log^j(x) C^\infty(X, \Gamma_0^+), \quad (\Delta_g - \lambda_0(n - \lambda_0))^p A = 0
\]
thus we can argue that
\[
\forall u \in V, \quad (\Delta_g - \lambda_0(n - \lambda_0))^p u = 0.
\]
Note that if $\lambda_0 \notin \frac{1}{2}(n - N)$, we clearly have
\[
x^{n-\lambda_0} C^\infty(X, \Gamma_0^+) \cap \sum_{j=0}^{p-1} x^{\lambda_j} \log^j(x) C^\infty(X, \Gamma_0^+) \subset C^\infty(X, \Gamma_0^+),
\]
therefore, if $V_1, V_2$ are defined by
\[
V_1 = V \cap x^{n-\lambda_0} C^\infty(X, \Gamma_0^+), \quad V_2 = V \cap \sum_{j=0}^{p-1} x^{\lambda_j} \log^j(x) C^\infty(X, \Gamma_0^+),
\]
we deduce from the unique continuation principle proved by Mazzeo [11] that
\[
V_1 \cap V_2 \subset C^\infty(X, \Gamma_0^+) \cap \ker L^2(\Delta_g - \lambda_0(n - \lambda_0))^p = 0.
\]
Hence, we can split $V = V_1 \oplus V_2 \oplus V_3$ with $V_3$ a direct complement of $V_1 \oplus V_2$ in $V$. Let $\Pi_{V_3}$ be the projection of $V$ onto $V_2$ parallel to $V_1 \oplus V_3$, $\Pi_V$ the orthogonal projection of $x^{-N} L^2(X)$ onto $V$ and $\iota_V$ the inclusion of $V$ into $x^{-N} L^2(X)$. We multiply \eqref{3.15} on the left by $\Pi_V^\prime := \iota_V \Pi_V \Pi_V^\prime$ and on the right by $\Pi_{V_2}^\prime$ to obtain
\[
A = \sum_{j+i+k_1=-1}^{p-1} \Pi_{V_2}^\prime E_i^{(1)} P_i E_j^{(2)} \Pi_{V_2}^\prime
\]
by construction of $V_2$ and using the symmetry $\iota^\prime A = A$ (since $\iota^\prime R(\lambda) = R(\lambda)$). Now remark that
\[
\sum_{j+i+k_1=-1}^{p-1} \Pi_{V_2}^\prime E_i^{(1)} P_i E_j^{(2)} \Pi_{V_2}^\prime = \sum_{k_i<0}^{k_1} \sum_{i=0}^{-k_i-1} \Pi_{V_2}^\prime E_i^{(1)} P_i E_j^{(2)} \Pi_{V_2}^\prime
\]
and the rank of this operator is bounded by $-\sum_{k_i<0} k_i = N_{\lambda_0}(c(n - \lambda) \tilde{S}(n - \lambda))$ since rank($P_i$) = 1. The lemma is proved when $\lambda_0 \notin \frac{3}{2} - N$.

On the other hand if $\lambda_0 \in \frac{3}{2} - N$ and $\lambda_0(n - \lambda_0) \notin \sigma_{pp}(\Delta_g)$, we have $R_{-1} = 0$, $E_{-1}^{(1)} = 0$ and $E_{-1}^{(2)} = 0$ in \eqref{3.15} and \eqref{3.15} by
\[
c(\lambda) \tilde{S}(\lambda) = U_1(\lambda) \left( \lambda - \lambda_0 \right) P_0 + \sum_{i=1}^{m} \left( \lambda - \lambda_0 \right)^{k_i+1} P_i U_2(\lambda),
\]
\[
\text{Res}_{\lambda_0}(n - 2\lambda) R(\lambda) = \sum_{j+i+k_1=-2}^{j+i+k_1} \sum_{k_i<1} E_i^{(1)} P_i E_j^{(2)}
\]
the first one being obtained from Gohberg-Sigal factorization \eqref{2.2} of $\tilde{S}(\lambda)$ at $\lambda_0$. Now observe that the rank of
\[
\sum_{j+i+k_1=-2}^{j+i+k_1} \sum_{k_i<1} \Pi_{V_2}^\prime E_i^{(1)} P_i E_j^{(2)} \Pi_{V_2}^\prime = \sum_{k_i<0}^{k_1} \sum_{i=0}^{-k_i-2} \Pi_{V_2}^\prime E_i^{(1)} P_i E_j^{(2)} \Pi_{V_2}^\prime
\]
is bounded by
\[
- \sum_{k_i<0} (k_i + 1) = - \sum_{k_i<0} (k_i + 1) = N_{\lambda_0}(\tilde{S}(n - \lambda)) - \dim \ker_{\lambda_0} \tilde{S}(n - \lambda) = N_{\lambda_0}(c(n - \lambda) \tilde{S}(n - \lambda))
\]
in view of (2.3), the proof is complete. \qed

**Proof of Theorem 1.1** we combine the results of Corollary 3.3 and Lemma 3.4, and observe that

$$\ker_{\lambda_0} \tilde{S}(n - \lambda) = \ker \tilde{S}(n - \lambda_0) = \ker \text{Res}_{n - \lambda_0} S(\lambda),$$

then it remains to show that

(3.19) $$N_{\lambda_0}(\tilde{S}(\lambda)) = m_{n - \lambda_0}.$$ 

Whereas the case $$\lambda_0(n - \lambda_0) \notin \sigma_{pp}(\Delta_g)$$ is clear since $$\tilde{S}(\lambda)^{-1} = \tilde{S}(n - \lambda)$$ is holomorphic near $$\lambda_0$$ and $$m_{n - \lambda_0} = 0$$, the case $$\lambda_0(n - \lambda_0) \in \sigma_{pp}(\Delta_g)$$ needs a little more care. In view of (3.2), $$\tilde{S}(\lambda)$$ has the following polar part at $$n - \lambda_0$$

$$C(\lambda_0)(\lambda - n + \lambda_0)^{-1} \sum_{j=1}^{k} \Lambda^{\lambda_0 - \frac{2}{k}} \phi_j^2 \otimes \Lambda^{\lambda_0 - \frac{2}{k}} \phi_j^2$$

with $$C(\lambda_0) \neq 0$$ if $$\lambda_0 \notin \frac{2}{k} - \mathbb{N}$$, $$k = m_{n - \lambda_0}$$ and $$\phi_j^2 := x^{\lambda_0 - \frac{2}{k}} \phi_j|_{\tilde{X}}$$ (where $$(\phi_j)_j$$ is an orthonormal basis of $$\ker_{x^2}(\Delta_g - \lambda_0(n - \lambda_0))$$ as in (3.2)). It is not difficult to see that $$(\phi_j)_j$$ are independent, otherwise there would exist a non zero solution $$u \in x^{n - \lambda_0 + 1} C^\infty(\tilde{X}, \Gamma_0^2)$$ of $$(\Delta_g - \lambda_0(n - \lambda_0))u = 0$$ and a Taylor expansion of this equation at $$x = 0$$ proves that $$u \in C^\infty(\tilde{X}, \Gamma_0^2)$$, which is excluded according to Mazzeo’s result [11]. Since the pole is a first order pole, the factorization of $$\tilde{S}(\lambda)$$ as in (2.2) near $$n - \lambda_0$$ is clear for the $$k_l < 0$$: we have $$m = k$$ and $$k_l = -1$$ for $$l = 1, \ldots, k$$. Using (3.2) and $$\tilde{S}(\lambda)^{-1} = \tilde{S}(n - \lambda)$$, one then obtain that the partial null multiplicities of $$\tilde{S}(\lambda)$$ at $$\lambda_0$$ are $$\{-k_1, \ldots, -k_k\}$$ which gives (3.19) and the theorem. \qed

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