A DESCENT PRINCIPLE FOR THE
DIRAC DUAL DIRAC METHOD

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Abstract. Let $G$ be a torsion free discrete group with a finite dimensional
classifying space $BG$. We show that $G$ has a dual Dirac morphism if and only
if a certain coarse (co)-assembly map is an isomorphism. Hence the existence
of a dual Dirac morphism for such $G$ is a metric, that is, coarse, invariant
of $G$. We get similar results for groups with torsion as well. The framework
that we develop is also suitable for studying the Lipschitz and proper Lipschitz
cohomology of Connes, Gromov and Moscovici.

1. Introduction

Let $G$ be a discrete group with finite classifying space $BG$. The Descent Principle
(see [7]) asserts that the Baum-Connes assembly map $K_\ast(BG) \to \mathcal{K}_\ast(C^*_{sa}G)$ is
injective if the the coarse Baum-Connes assembly map $KX_\ast(G) \to K_\ast(C^*(|G|))$ is
an isomorphism. The latter assertion only involves the large scale geometry of $G$.
Injectivity of the Baum-Connes assembly map implies the homotopy invariance of
higher signatures for $G$. It has been expected for some time that there should be
a similar descent principle underlying the Dirac dual Dirac method. The primary
goal of this article is to specify exactly such a principle.

Let $G$ be a locally compact group. There is a canonical coarse structure on
$G$ that is invariant under right multiplication. We write $|G|$ for $G$ equipped with this
coarse structure. Let $\mathcal{E}G$ be a universal proper $G$-space. Consider the map
\begin{equation}
(1) \quad p_{\mathcal{E}G}^\ast : \mathcal{K}_\ast(C, C_0(G)) \to \mathcal{K}_\ast(\mathcal{E}G, C, C_0(G))
\end{equation}
which is induced by the constant map from $\mathcal{E}G$ to a point. This article grew out of
the observation that (1) is an invariant of the coarse space $|G|$. More precisely,
given any coarse space $X$, there is a certain $C^*$-algebra $c^{\text{red}}(X)$ called the reduced
stable Higson corona of $X$, a certain graded Abelian group $KX_\ast(X)$ called the
coarse $K$-theory of $X$, and a coarse co-assembly map
\begin{equation}
(2) \quad \mu_X^\ast : K_{\ast+1}(c^{\text{red}}(X)) \to KX_\ast(X),
\end{equation}
which is equivalent to (1) for $X = |G|$. “Equivalent” means that there are isomorphisms between the sources and targets of $\mu_{|G|}^\ast$ and $p_{\mathcal{E}G}^\ast$ that identify the two maps
in the obvious way.

This article is an expanded and completely rewritten version of the eprint [4]. In
the meantime, we have published details concerning the stable Higson corona and the
coarse co-assembly map in [6]. Thus we shall only recall these constructions
rather briefly here.

The map (1) is closely related to the Dirac dual Dirac method for proving injectivity
of the Baum-Connes assembly map. Traditionally, this method requires the existence of a proper $G$-$C^*$-algebra $A$ and $d \in \mathcal{K}(A, \mathbb{C})$, $\eta \in \mathcal{K}_G(\mathcal{C}, A)$ such that $\eta \circ d = 1_A$ or at least $p_{\mathcal{E}G}^\ast(d \circ \eta) = 1_C$ (see [12]). In [15] it is shown that there is

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a canonical choice for the Dirac morphism $D \in \text{KK}^G(P, \mathbb{C})$. It is constructed using general results on triangulated categories and can be interpreted as a projective resolution of $\mathbb{C}$ in the category $\text{KK}^G$ with respect to a certain localising subcategory. The Dirac morphism is unique up to $\text{KK}^G$-equivalence. A dual Dirac morphism is defined in [15] as an element $\eta \in \text{KK}^G(C, P)$ with $\eta \circ D = 1_P$. The dual Dirac morphism is unique if it exists. If the Dirac dual Dirac method applies to $G$ in the (traditional) sense of [12], then there is a dual Dirac morphism in the sense of [15]. The converse also holds for many groups $G$. For instance, it holds if $G$ is discrete and has a finite dimensional model for $\mathcal{E}G$.

It is further shown in [15] that the canonical maps

$$(3) \quad K_{\text{top}}^*(G, A) \overset{\sim}{\to} K_{\text{top}}^*(G, P \otimes A) \overset{\sim}{\to} K_*(\langle P \otimes A \rangle \rtimes G) \overset{\sim}{\to} K_*(\langle P \otimes A \rangle \rtimes_\tau G)$$

are all isomorphisms. Here $\rtimes$ and $\rtimes_\tau$ denote the reduced and full crossed products, respectively. Therefore, we may identify $K_{\text{top}}^*(G, A)$ with $K_*(\langle P \otimes A \rangle \rtimes G)$ and the Baum-Connes assembly map with the composition

$$K_*(\langle P \otimes A \rangle \rtimes G) \overset{\partial}{\to} K_*(A \rtimes G) \to K_*(A \times_\tau G).$$

The isomorphism appearing in equation (6) below, and many other of our results, depend on this definition of the Baum-Connes assembly map, and cannot apparently be proved with the traditional one.

Let $G$ be a torsion free discrete group with finite classifying space $BG$. Our first objective is to prove that such a group $G$ has a dual Dirac morphism if and only if (1) is an isomorphism. As a result, the Dirac dual Dirac method applies to $G$ if and only if the coarse co-assembly map $\mu^*_{|G|}$ is an isomorphism. This is our new Descent Principle. We obtain a similar criterion if $G$ is a torsion free discrete group with finite dimensional $BG$. This generalisation requires the coarse co-assembly map with coefficients.

As a result, for torsion free discrete groups with finite dimensional classifying space, the existence of a dual Dirac morphism is a coarse invariant: if two such groups $G$ and $G'$ are coarsely equivalent, then $G$ has a dual Dirac morphism if and only if $G'$ has one.

We also provide results for locally compact groups with a $G$-compact model for $\mathcal{E}G$. For such groups a dual Dirac morphism exists if and only if the maps

$$(4) \quad p^*_G : KK^*_G(C, C_0(G/H)) \to \text{RKK}^*_G(\mathcal{E}G; C, C_0(G/H))$$

are isomorphisms for all compact subgroups $H \subseteq G$. We can equip $\mathbb{C}^{{\text{top}}}(\langle G \rangle)$ with a canonical action of $G$. This gives rise to an $H$-equivariant coarse co-assembly map

$$\mu^*_{|G|, H} : K_{+1}(\mathbb{C}^{{\text{top}}}(\langle G \rangle)^H) \to KX^*_H(\langle G \rangle),$$

for each compact subgroup $H \subseteq G$, which is equivalent to (4).

From the perspective of Novikov’s original conjecture, the Dirac dual Dirac method is a method of confirming homotopy invariance of all higher signatures for a group $G$ at once. Our Descent Principle shows that the success or failure of this method depends only on the large scale geometry of $G$. Instead, one may attempt to establish the homotopy invariance of a single higher signature, should it fortuitously arise from a particularly geometric construction. The stable Higson corona construction provides a good framework for pursuing this idea, which goes back to Connes, Gromov and Moscovici in [3]. We shall use it to simplify the geometric part of the proof of homotopy invariance of Gelfand-Fuchs classes, a result of [3]. More substantially, we introduce a generalization of the notion of proper Lipschitz class, which we term boundary class, and show that such classes have a number of pleasant properties analogous to those enjoyed by Lipschitz classes, including homotopy invariance.
For the purposes of this discussion, let us agree that a higher signature for a
discrete group $G$ is a linear map $\tau: K^\top_*(G) \to \mathbb{R}$. If $BG$ is a compact space, then
a higher signature $\tau$ in our sense may be used to assign a real number to each continuous map $f: M \to BG$, where $M$ is a smooth oriented manifold, by the formula $f \mapsto \tau(f_*[D^m_\text{reg}])$. Homotopy invariance means the real numbers associated to $f: M \to BG$ and $f \circ \phi: N \to BG$ are the same, where $\phi$ is a homotopy equivalence $N \to M$ (see [19]). It is well-known that a higher signature is homotopy invariant if it factorises through the analytic assembly map $\mu_*: K^\top_*(G) \to K_*(C^*_\text{red}G)$ (see [9]).

If $A$ is any $G$-$C^*$-algebra, then $K^\top_*(G, A)$ is a graded module over the graded commutative ring $\text{RKK}^G_\ast(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. In particular, this is true of $K^\top_*(G)$. Hence any class $b \in \text{RKK}^G_\ast(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ yields a $K$-theoretic higher signature by the following recipe. We set $\tau_b(a) = \text{ind}_G(a \cdot b)$, where $\text{ind}_G: K^\top_*(G) \to \mathbb{Z}$ is the index map induced by the trivial representation of $G$. If $b$ lies in the range of the map $p_{E\mathcal{G}}: K^\top_*(\mathcal{E}G; \mathbb{C}) \to \text{RKK}^G_\ast(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ then $\tau_b$ is homotopy invariant because it factorises through $\mu_*$. If $G$ has a dual Dirac morphism, then $p_{E\mathcal{G}}$ is split surjective. This yields homotopy invariance of all higher signatures at once.

By contrast, and following the spirit of [3], we show how to construct individual homotopy invariant higher signatures directly from the stable Higson corona of $G$. The latter is in a natural way a $G$-$C^*$-algebra, so that we may consider the graded group $K^\top_*(G, c^{\mathcal{E}G}(([G]))$. Since we use $K^\top_*$, the group structure of $G$ does not add any additional analytical complications. There is a canonical map

$$K^\top_+([G]) \to \text{RKK}^G_\ast(\mathcal{E}G; \mathbb{C}, \mathbb{C}).$$

We call elements in the range of this map boundary classes for the obvious reason: they depend only on the action of $G$ on its stable Higson corona. Suppose $G$ is a discrete group with a $G$-compact model for $\mathcal{E}G$. Let $\mathcal{P}$ be the source of the Dirac morphism. Then there is an isomorphism

$$K^\top_+([G]) \cong \text{RKK}^G_\ast(\mathbb{C}, \mathbb{P}).$$

This result is fundamental for the study of boundary classes. As well, it allows us to significantly refine our Descent Principle for the Dirac dual Dirac method.

Firstly, (6) implies that any boundary class yields a homotopy invariant higher signature. In fact, any boundary class must lie in the range of the map $p_{E\mathcal{G}}$.

Secondly, it implies that every proper Lipschitz class in the sense of Connes, Gromov and Moscovici is a boundary class. Therefore the notion of boundary class generalizes that of proper Lipschitz class.

Thirdly, it shows that a dual Dirac morphism may be described directly in terms of the topological $K$-theory of the stable Higson corona, namely as an element of $K^\top_1(\mathcal{E}G, c^{\mathcal{E}G}([G]))$ whose image under the map (5) is $1_{E\mathcal{G}}$. Actually, using our Descent Principle, we conclude that (5) is an isomorphism exactly when it is surjective, and this occurs exactly when $G$ has a dual Dirac morphism.

In particular, if $G$ admits a uniform embedding in Hilbert space, every class is a boundary class, and (5) is an isomorphism. For such $G$ has a dual Dirac morphism by [6].

The isomorphism (6) also enables us to construct dual Dirac morphisms directly from contractible, admissible, $G$-equivariant compactifications $\overline{\mathcal{E}G} = \mathcal{E}G \cup \partial G$. This strengthens a result of Nigel Higson ([8]). For instance, if $G$ is Gromov hyperbolic, we may compactify $\mathcal{E}G$ using the Gromov boundary of $G$. In particular, a dual Dirac morphism can be constructed which, in the appropriate sense, extends continuously to the Gromov boundary of $G$. 

The source and target of (5) are modules over the ring $\text{RKK}^G(E G; \mathbb{C}, \mathbb{C})$. Moreover, the map (5) is a module homomorphism. Hence the boundary classes form an ideal in the ring $\text{RKK}^G(E G; \mathbb{C}, \mathbb{C})$. This shows that boundary classes are rather special, even amongst classes in the range of $p_{EG}^*$, or even more generally, amongst those classes which yield homotopy invariant higher signatures. For the unit $1_{EG} \in \text{RKK}^G(E G; \mathbb{C}, \mathbb{C})$ always yields a homotopy invariant higher signature, namely the ordinary signature, assigning to a map $f: M \to BG$ the Hirzebruch signature of $M$. On the other hand, if $1_{EG}$ is a boundary class, then, since the boundary classes constitute an ideal, all classes are boundary classes. As remarked above, this is the case if and only if $G$ has a dual Dirac class.

Finally, since the source of the Dirac morphism and consequently also the group $\text{KK}^G(\mathbb{C}, P)$ are already built into the definition of the Baum-Connes assembly map in [15], the isomorphism (6) means that the stable Higson corona construction represents an intrinsic component of the apparatus surrounding the assembly map. This provides a form of converse to the Descent Principle.

We conclude by admitting that we do not know of any geometric obstructions to (5) being an isomorphism, save the existence of a positive scalar curvature metric on a realisation of $BG$ as a compact spin$^c$-manifold. The latter already obstructs surjectivity of $p_{EG}^*$ and a fortiori the surjectivity of (5) (see [20]). One would therefore expect there to be an easier obstruction. Although the existence of an expanding sequence of graphs embedded in $|G|$ has negative consequences for the coarse assembly map for $|G|$, the arguments showing this do not appear to apply to the coarse co-assembly map.

2. COARSE GEOMETRY, THE STABLE HIGSON CORONA, AND THE COARSE CO-ASSEMBLY MAP

This article is an expanded and almost completely rewritten version of the eprint [4], where we have introduced the stable Higson corona of a coarse space and the coarse co-assembly map and related it to the Dirac dual Dirac method. In the meantime, we have discussed the basic properties of these constructions in much greater detail in [6], within the broader framework of coarse spaces which need not be groups. We shall therefore use the notations and results of [6] and concentrate on things that are special in the group case. In our analysis, we are going to study actions of locally compact groups on coarse spaces. This is necessary to prove our descent theorem for groups with torsion and to define boundary classes.

2.1. Group actions on coarse spaces.

Definition 1. Let $X$ be a coarse space and let $G$ be a locally compact group that acts continuously on $X$. For compatibility with our later arguments, we let $G$ act on the right. We call the action coarse if the set of $(xg, yg)$ with $g \in K$, $(x, y) \in E$ is an entourage for any compact subset $K \subseteq G$ and any entourage $E \subseteq X \times X$. We say that $G$ acts by translations if the set of $(xg, x)$ with $g \in K$, $x \in X$ is an entourage for any compact subset $K \subseteq G$. We say that $G$ acts isometrically if any entourage is contained in an $G$-invariant entourage.

Actions by translations and isometric actions are coarse. We usually require actions to be continuous, isometric and proper. There are a few situations where other group actions are also useful.

Let $G$ be discrete. Then an action is coarse if and only if the group acts by coarse maps. The action is by translations if and only if these maps are close to the identity. If the coarse structure on $X$ is countably generated, then the action is isometric if and only if the coarse structure comes from an $G$-invariant metric.
Now let \( \mathcal{X} = \bigcup X_n \) be a \( \sigma \)-coarse space (see [6] for the definition). We will only consider group actions that leave the subsets \( X_n \) invariant. More generally, it suffices to assume that for all \( m \in \mathbb{N} \) there is \( n \in \mathbb{N} \) with \( X_m \cdot G \subseteq X_n \); then we can rewrite \( \mathcal{X} \) as \( \bigcup X_n \cdot G \). We call the action on \( \mathcal{X} \) coarse, isometric, etc., if the restrictions to \( X_n \) have the corresponding property for all \( n \in \mathbb{N} \).

### 2.2. Coarse spaces from groups and proper group actions.

**Theorem 2.** Let \( G \) be a locally compact group and let \( X \) be a \( G \)-compact, proper \( G \)-space. There is a unique coarse structure on \( X \) that is compatible with the given topology and for which \( G \) acts isometrically. It is generated by the \( G \)-invariant entourages \( E_L := \bigcup_{g \in G} Lg \times Lg \) for compact \( L \subseteq X \).

We write \(|X|\) for \( X \) equipped with this coarse structure.

**Proof.** It is easy to see that the coarse structure generated by the entourages \( E_L \) has the required properties. Equip \( X \) with any coarse structure with the required properties. Let \( L \subseteq X \) be compact. Then \( L \) is bounded, so that \( L \times L \) is contained in a \( G \)-invariant entourage. Thus \( E_L \) is an entourage. Conversely, let \( E \subseteq X \times X \) be a \( G \)-invariant entourage. Let \( K \subseteq X \) be compact such that \( K \cdot G = X \). Then \( E \cap (K \times X) \subseteq K \times L \) for some bounded and hence relatively compact subset \( L \).

We may replace \( L \) by a compact subset that contains \( K \). Since \( K \cdot G = X \), the \( G \)-invariant entourage \( E \) is determined by \( E \cap (K \times X) \). We obtain \( E \subseteq E_L \). Hence the coarse structure is equal to the one defined by the entourages \( E_L \).

In particular, we let \(|G|\) be the group \( G \) itself equipped with the action of \( G \) by right multiplication and with the unique coarse structure for which this action is isometric.

Let \( X \) be a \( G \)-compact proper \( G \)-space. For any \( x \in X \), the map \(|G| \to |X|, g \mapsto g \cdot x\), is a coarse equivalence. These maps for different points in \( X \) are close.

Now let \( X \) be a proper \( G \)-space that is not necessarily \( G \)-compact. We only require \( X \) to be a union of an increasing sequence \((X_n)_{n \in \mathbb{N}}\) of \( G \)-compact subspaces. We implicitly require the \( X_n \) to be \( G \)-invariant and closed. Even if \( X \) is not locally compact, the spaces \( X_n \) are necessarily locally compact in the subspace topology. Thus we can turn them into coarse spaces by the above prescription. The maps \(|X_m| \to |X_n|\) for \( m \leq n \) are coarse equivalences because orbit maps \(|G| \to |X_n|\) are coarse equivalences. Hence we have turned \( X \) into a \( \sigma \)-coarse space in the sense of [6]. We write \(|X|\) for this \( \sigma \)-coarse space.

Of course, the above construction is natural, that is, a \( G \)-equivariant continuous map \( f : X \to Y \) induces a coarse continuous map \(|X| \to |Y|\).

### 2.3. The coarse category of coarse spaces.

From now on, we require locally compact groups and topological spaces to be second countable and coarse structures to be countably generated. We let \( \mathcal{C} \) be the category of coarse spaces, whose objects are the coarse spaces with second countable topology and countably generated coarse structure and whose morphisms are the continuous coarse maps. Let \( H \) be a second countable locally compact group. We let \( \mathcal{C}_H \) be the category of coarse spaces as above, equipped with a continuous, proper, and isometric action of \( H \). The morphisms in \( \mathcal{C}_H \) are the \( H \)-equivariant coarse continuous maps.

Let \( X \in \mathcal{C}_H \) and let \( Y \subseteq X \) be an \( H \)-invariant closed subset. Then we give \( Y \) the subspace coarse structure, so that \( Y \in \mathcal{C}_H \). We say that \( Y \) is coarsely dense if there is an entourage \( E \subseteq X \times X \) such that for any \( x \in X \) there is \((x, y) \in E\) with \( y \in Y \). If \( Y \) is discrete, we call \( Y \) a discretisation of \( X \). A discretisation exists if and only if the orbits of the \( H \)-action are discrete.
If \( H \) is finite, we want all finite \( H \)-spaces to be coarsely equivalent. However, there is no \( H \)-equivariant map from the one point space with trivial action to \( H \). To overcome this problem, we relax \( H \)-equivariance as follows:

**Definition 3.** Let \( X, Y \in \mathcal{C}_H \) and let \( f: X \to Y \) be a coarse Borel map. Define \( f^h(x) := f(xh)h^{-1} \) for \( x \in X, h \in H \). We call \( f \) almost \( H \)-equivariant if

\[
\{ (f^h(x), f(x)) \mid x \in X, h \in H \}
\]

is an entourage.

**Definition 4.** We let \( \mathcal{C}_H \) be the category with the same objects as \( \mathcal{C}_H \) and whose morphisms are equivalence classes of almost \( H \)-equivariant coarse Borel maps, where two maps are identified if they are close. We call \( \mathcal{C}_H \) the coarse category of coarse \( H \)-spaces. A morphism \( f: X \to Y \) is called a coarse equivalence if it is an isomorphism in \( \mathcal{C}_H \).

The following lemma is easy to check:

**Lemma 5.** The embedding of a coarsely dense \( H \)-invariant subspace is a coarse equivalence.

### 2.4. An equivariant version of the Rips complex

The Rips complex can be constructed most easily for discrete coarse spaces. In [6], we have defined it for non-discrete spaces by choosing a discretisation. When we work equivariantly, this does not always exist. To get a more natural construction of \( \mathcal{P}(X) \) for non-discrete \( X \), we adapt the locally compact model for \( \mathcal{E}G \) for a locally compact group \( G \) due to Gennadi Kasparov and Georges Skandalis ([12]).

Let \( X \in \mathcal{C}_H \). Let \( \mathcal{P}(X) \) be the set of all positive Borel measures \( \mu \) on \( X \) with total volume \( 1/2 < \| \mu \|_1 \leq 1 \). We equip (subsets of) \( \mathcal{P}(X) \) with the weak topology from the pairing with \( C_0(X) \). In this topology, \( \mathcal{P}(X) \) is locally compact and second countable. Let \( E \subseteq X \times X \) be an entourage. A measurable subset \( S \subseteq X \) is called \( E \)-bounded if \( S \times S \subseteq E \). Given a closed entourage \( E \) and \( t > 1/2 \), we let

\[
P_{E,t}(X) := \{ \mu \in \mathcal{P}(X) \mid \mu(S) \geq t \text{ for some } E \text{-bounded set } S \}.
\]

Since \( X \) is countably generated, there is an increasing sequence of closed entourages \( (E_n) \) that dominates any other entourage. Write \( P_n(X) := P_{E_n,1/2+1/n} \). Then \( \mathcal{P}(X) = \bigcup P_n(X) \).

We claim that \( P_{E,t}(X) \) is a weakly closed subset of \( \mathcal{P}(X) \). Let \( (\mu_n) \) be a weakly convergent sequence in \( P_{E,t}(X) \) that converges towards some \( \mu \in \mathcal{P}(X) \). Choose \( E \)-bounded subsets \( S_n \) for \( n \in \mathbb{N} \) such that \( \mu_n(S_n) \geq t \) and choose a compact subset \( S \) with \( \mu(S) \geq t \). Then \( \mu_n(S) > 1/2 \) and hence \( S \cap S_n \neq \emptyset \) for almost all \( n \). Since the subsets \( S_n \) are compact and \( \bigcap S_n \neq \emptyset \), the subset \( \bigcup S_n \) is relatively compact. The set of compact subsets of its closure is compact in the Hausdorff metric. Hence we can find a subsequence of \( (S_n) \) that converges towards some compact subset \( S_\infty \). We may assume that \( (S_n) \) itself converges. The convergence \( \lim \mu_n = \mu \) implies \( \mu(S_\infty) \geq t \). The convergence \( S_n \to S \) implies \( S_\infty \times S_\infty \subseteq E \) because \( E \) is closed. Thus \( \mu \in P_{E,t}(X) \), so that \( P_{E,t}(X) \) is closed as asserted. It follows that \( P_{E,t}(X) \) is locally compact in the subspace topology.

We also let

\[
P^2_{E,t}(X) := \{ (\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid \mu(S) \geq t \text{ and } \nu(S) \geq t \text{ for some } E \text{-bounded subset } S \}.
\]

These subsets of \( \mathcal{P}(X) \times \mathcal{P}(X) \) define a coarse structure on \( \mathcal{P}(X) \). One checks easily that the restriction of this coarse structure to \( P^2_{E,t}(X) \) is compatible with the topology for all closed entourages \( E' \) and all \( t' > 1/2 \). Moreover, \( P_{E,t}(X) \subseteq \)
$P_{E,t}(X)$ is coarsely dense if $E \subseteq E'$ and $t \geq t'$. Thus $\mathcal{P}(X) = \bigcup P_n(X)$ is a $\sigma$-coarse space.

The action of $H$ on $X$ induces an action on $\mathcal{P}(X)$, pushing forward measures. If $E$ is an $H$-invariant entourage, then $P_{E,t}(X)$ is $H$-invariant and the restriction of the action to $P_{E,t}(X)$ is still isometric, continuous and proper. That is, $P_{E,t}(X) \in \mathcal{E}_H$ for all $H$-invariant entourages $E$ and all $t > 1/2$.

There is a canonical map $j_X : X \to \mathcal{P}(X)$, sending $x \in X$ to the Dirac measure at $X$. The construction of $\mathcal{P}(X)$ is natural: a continuous coarse map $f : X \to X'$ induces a continuous coarse map $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(X')$ which pushes forward measures along $f$. We have $\mathcal{P}(f) \circ j_X = j_{X'} \circ f$, that is, $j_X : X \to \mathcal{P}(X)$ is a natural transformation.

The above construction has another property that is useful for technical purposes. For any pair $(E, t)$, there is $n \in \mathbb{N}$ such that $P_n(X)$ is a neighbourhood of $P_{E,t}(X)$ in $\mathcal{P}(X)$. Passing to a subsequence, we can achieve that $P_{n+1}(X)$ is a neighbourhood of $P_n(X)$ for all $n \in \mathbb{N}$. Hence there is a partition of unity $(\phi_n)$ on $\mathcal{P}(X)$ with $\text{supp}\phi_n \subseteq P_{n+1}(X) \setminus P_{n-1}(X)$. Even more, we can choose the functions $(\phi_n)$ to be $H$-equivariant. These partitions of unity are useful to formulate the universal property of $\mathcal{P}(X)$. We call a $\sigma$-coarse $H$-space $\mathcal{X} = \bigcup X_n$ partitionable if it carries such a partition of unity. This holds if and only if $X_n$ in $X_{n+2}$ for all $n \in \mathbb{N}$.

Let $\sigma \mathcal{E}_H$ be the category of partitionable $\sigma$-coarse $H$-spaces.

Let $X \in \mathcal{E}_H$. We give $X \times [0,1]$ the product topology and let $E \subseteq (X \times [0,1])^2$ be an entourage if its image in $X \times X$ is one. Then $X \times [0,1] \in \mathcal{E}_H$. The evaluation maps $X \cong X \times \{t\} \subseteq X \times [0,1]$ and the projection $X \times [0,1] \to X$ are coarse equivalences. A coarse homotopy between two morphisms $f, g : X \to Y$ is an $H$-equivariant coarse continuous map $X \times [0,1] \to Y$. This generates the equivalence relation of coarse homotopy on the space of morphisms $\mathcal{E}_H(X,Y)$, which in turn generates a notion of coarse homotopy equivalence. If two maps are coarsely homotopic, then they are both close and homotopic (as maps of topological spaces). Thus a coarse homotopy equivalence is simultaneously a coarse equivalence and a homotopy equivalence. It is evident how to extend these notions to $\sigma$-coarse spaces.

**Lemma 6.** Let $X \in \mathcal{E}_H$. Then $\mathcal{P}(X) \in \sigma \mathcal{E}_H$, and $j_X : X \to \mathcal{P}(X)$ is a coarse equivalence. Let $\mathcal{Y} \in \sigma \mathcal{E}_H$ and let $f : X \to \mathcal{Y}$ be a coarse equivalence. Then there is $h \in \sigma \mathcal{E}_H(\mathcal{Y}, \mathcal{P}(X))$ such that $h \circ f$ is coarsely homotopy equivalent to $j_X$, and the map $h$ is unique up to coarse homotopy equivalence. This universal property determines $\mathcal{P}(X)$ uniquely up to coarse homotopy equivalence.

**Proof.** It is straightforward to see that $\mathcal{P}(X)$ has the required properties. The map $j_X$ is a coarse equivalence because it is an embedding with coarsely dense range. If $f_0, f_1 : \mathcal{Y} \to \mathcal{P}(X)$ are two close maps, then we can join them by the affine homotopy $(1-t)f_0 + tf_1$. One checks easily that this homotopy is coarse. Thus close coarse maps into $\mathcal{P}(X)$ are coarsely homotopic. Therefore, the proof will be finished if we construct a coarse continuous map $h : \mathcal{Y} \to \mathcal{P}(X)$ such that $h \circ f$ is close to the identity map.

Write $\mathcal{Y} = \bigcup Y_n$ and suppose that we have found maps $h_n : Y_n \to \mathcal{P}(X)$ with the required properties. Since $\mathcal{Y}$ is partitionable, we obtain a certain partition of unity $(\phi_n)$. Since $\mathcal{P}(X)$ is convex, we can define $h := \sum \phi_n h_{n+1}$. This map has the required properties. Thus it remains to construct maps $h_n : Y_n \to \mathcal{P}(X)$. Since $f$ is a coarse equivalence, there is an almost $H$-equivariant Borel map $g : Y_n \to X$ such that $gf$ is close to the identity. Choose a uniformly bounded open covering of $Y_n$ and some subordinate partition of unity $(\psi_k)_{k \in \mathbb{N}}$, and choose $x_k \in X$ close to $g(\text{supp} \psi_k)$. Define $g'(y) := \sum_{k \in \mathbb{N}} \psi_k(y) \delta_{x_k}$, where $\delta_{x_k}$ denotes the Dirac measure at $x_k$. This is a continuous map $g' : Y_n \to \mathcal{P}(X)$ that is close to $g$. Hence $g'$
implies that $C C \to X$ and hence has an inverse up to coarse homotopy. It follows that any morphism $P$ proved in [6]), we can use $P$ equivalence, the induced map $F$ any functor $(see Lemma 20). An action of $H$ yields a $P$ and hence induces a map $P$ continuous maps $Y 2.5$. Let $G$ be a second countable locally compact group and let $E G$ be any universal proper $G$-space. Then $|E G|$ is $G$-equivariantly coarsely homotopy equivalent to $P(\{G\})$.

**Proof.** If we choose the special model for $E G$ constructed in [12], then we have $P(\{G\}) = |E G|$. Since $E G$ is determined uniquely up to $G$-equivariant homotopy equivalence and since the construction $X \mapsto |X|$ is natural, any two models for $|E G|$ are $G$-equivariantly coarsely homotopy equivalent.

**Lemma 8.** The set $\mathcal{E}C H(X,Y)$ of morphisms $X \to Y$ in the coarse category is naturally isomorphic to the set of coarse homotopy classes of $H$-equivariant coarse continuous maps $\mathcal{P}(X) \to \mathcal{P}(Y)$.

**Proof.** As in the proof of Lemma 6 one shows that any almost equivariant coarse Borel map $X \to Y$ is close to an $H$-equivariant coarse continuous map $X \to \mathcal{P}(Y)$. Moreover, close maps into $\mathcal{P}(Y)$ are coarsely homotopic. Thus $\mathcal{E}C H(X,Y)$ is in bijection with coarse homotopy classes of coarse continuous maps $X \to \mathcal{P}(Y)$. Write $\mathcal{P}(Y) = \bigcup P_n(Y)$. Any map $X \to \mathcal{P}(Y)$ is a map into $P_n(Y)$ for some $n$ and hence induces a map $\mathcal{P}(X) \to \mathcal{P}(P_n(Y))$. Since $Y \to P_n(Y)$ is a coarse equivalence, the induced map $\mathcal{P}(Y) \to \mathcal{P}(P_n(Y))$ is a coarse homotopy equivalence and hence has an inverse up to coarse homotopy. It follows that any morphism $X \to \mathcal{P}(Y)$ is close to the restriction of a morphism $\mathcal{P}(X) \to \mathcal{P}(Y)$. As above, this implies that $\mathcal{E}C H(X,Y)$ is in bijection with coarse homotopy classes of morphisms $\mathcal{P}(X) \to \mathcal{P}(Y)$.

Thus the map $X \to \mathcal{P}(X)$ plays the role of an injective resolution. If we are given any functor $F$ on $\mathcal{E}H$, we obtain a functor that descends to $\mathcal{E}C H$ by applying $F$ to $\mathcal{P}(X)$ instead of $X$.

For discrete spaces, the space $\mathcal{P}(X)$ is usually constructed using probability measures with compact support. We can also do this in general. An elementary argument shows that the two versions for $\mathcal{P}(X)$ are coarsely homotopy equivalent, so that it makes no difference which one we use. However, the model for $\mathcal{P}(X)$ with compactly supported probability measures is not partitionable. This makes it more difficult to formulate the universal property. In any case, for explicit computations, one tends to look for smaller models for $\mathcal{P}(X)$. For instance, as is well-known (and proved in [6]), we can use $X$ itself if $X$ has bounded geometry and is uniformly contractible.

### 2.5. Coarse K-theory

Let $\mathcal{Y} = \bigcup Y_n$ be any $\sigma$-coarse space. For our purposes, the appropriate algebra of functions on $Y$ is the $\sigma$-$C^*$-algebra $C_0(\mathcal{Y}) := \lim C_0(Y_n)$, which consists of all functions $f : \mathcal{Y} \to \mathbb{C}$ for which $f|_{Y_n} \in C_0(Y_n)$ for all $n \in \mathbb{N}$ (see Lemma 20). An action of $H$ on $\mathcal{Y}$ turns $C_0(\mathcal{Y})$ into a projective system of $H$-$C^*$-algebras, which we call a $\sigma$-$H$-$C^*$-algebra. We define the crossed product for a $\sigma$-$H$-$C^*$-algebra $\lim_A m$ in the evident way as $(\lim_A m) \times H := \lim (A_m) \times H$. This yields a $\sigma$-$C^*$-algebra.

We define the coarse K-theory $KX^*(X)$ of a coarse space $X$ as the K-theory of the $\sigma$-$C^*$-algebra $C_0(\mathcal{P}(X))$. We define the $H$-equivariant coarse K-theory $KX_H^*(X)$ of $X \in \mathcal{E}H$ as the K-theory of the $\sigma$-$C^*$-algebra $C_0(\mathcal{P}(X) \times H)$. We may also
Let \( G \) be a locally compact group and let \( H \subseteq G \) be a closed subgroup. Then
\[
\text{KX}_H^c([G], D) \cong K_\ast \left((C_0([\mathcal{E}G]) \otimes D) \rtimes H\right)
\]
by Lemma 7. \( \text{We warn the reader that } C_0([\mathcal{E}G], D) \neq C_0(\mathcal{E}G, D) \) unless \( \mathcal{E}G \) is \( G \)-compact; elements of \( C_0([\mathcal{E}G], D) \) are possibly unbounded continuous functions \( \mathcal{E}G \to D \), only their restrictions to \( G \)-compact subsets vanish at \( \infty \).

2.6. The stable Higson corona. Let \( X \) be a coarse space and \( D \) a \( C^* \)-algebra. Let \( \mathcal{M}(D \otimes \mathbb{K}) \) be the multiplier algebra of \( D \otimes \mathbb{K} \). We identify it with the \( C^* \)-algebra of adjointable operators on the Hilbert \( D \)-module \( D \otimes \ell^2(\mathbb{N}) \). Let \( \mathfrak{B}_{\text{red}}^\tau(X, D) \) be the \( C^* \)-algebra of norm continuous, bounded functions \( f : X \to \mathcal{M}(D \otimes \mathbb{K}) \) for which \( f(x) - f(y) \in D \otimes \mathbb{K} \) for all \( x, y \in X \). We say that a function \( f \in \mathfrak{B}_{\text{red}}^\tau(X, D) \) has vanishing variation if the function \( E \ni (x, y) \to \|f(x) - f(y)\| \) vanishes at \( \infty \) for any closed entourage \( E \subseteq X \times X \). Let \( \mathfrak{c}_{\text{red}}^\tau(X, D) \) be the subalgebra of vanishing variation functions. Let
\[
\mathfrak{c}_{\text{red}}^\tau(X, D) := \mathfrak{c}_{\text{red}}^\tau(X, D)/C_0(X, D \otimes \mathbb{K}),
\]
\[
\mathfrak{B}_{\text{red}}^\tau(X, D) := \mathfrak{B}_{\text{red}}^\tau(X, D)/C_0(X, D \otimes \mathbb{K}).
\]
It is clear that \( X \to \mathfrak{c}_{\text{red}}^\tau(X, D) \) and \( X \to \mathfrak{c}_{\text{red}}^\tau(X, D) \) are contravariant functors in \( X \) for continuous coarse maps. If \( D = \mathbb{C} \), we drop it from our notation and write \( \mathfrak{c}_{\text{red}}^\tau(X) \) and \( \mathfrak{c}_{\text{red}}^\tau(X) \). We refer to \( \mathfrak{c}_{\text{red}}^\tau(X) \) and \( \mathfrak{c}_{\text{rd}}^\tau(X) \) as the reduced stable Higson compactification and the reduced stable Higson corona of \( X \), respectively.

Proposition 9 ([6]). The functor \( X \to \mathfrak{c}_{\text{red}}^\tau(X, D) \) descends to a functor on the coarse category of coarse spaces. That is, close maps \( f, f' : X \to X' \) induce the same map \( \mathfrak{c}_{\text{red}}^\tau(X', D) \to \mathfrak{c}_{\text{red}}^\tau(X, D) \) and a coarse equivalence \( X \to X' \) induces an isomorphism \( \mathfrak{c}_{\text{red}}^\tau(X', D) \cong \mathfrak{c}_{\text{red}}^\tau(X, D) \).

Now let \( \mathcal{X} = \bigcup X_n \) be a \( \sigma \)-coarse space. We have already defined \( C_0(\mathcal{X}, D) := \varinjlim C_0(X_n, D) \) above. Let
\[
\mathfrak{c}_{\text{red}}^\tau(\mathcal{X}, D) := \varprojlim \mathfrak{c}_{\text{red}}^\tau(X_n, D), \quad \mathfrak{c}_{\text{red}}^\tau(\mathcal{X}, D) := \varinjlim \mathfrak{c}_{\text{red}}^\tau(X_n, D).
\]
Equivalently, \( \mathfrak{c}_{\text{red}}^\tau(\mathcal{X}, D) \) is the \( \sigma \)-\( C^* \)-algebra of all functions \( f : \mathcal{X} \to \mathcal{M}(D \otimes \mathbb{K}) \) for which \( f|_{X_n} \in \mathfrak{c}_{\text{red}}^\tau(X_n, D) \) for all \( n \in \mathbb{N} \). Since the maps \( X_m \to X_n \) for \( m \leq n \) are coarse equivalences, Proposition 9 yields that \( \mathfrak{c}_{\text{red}}^\tau(\mathcal{X}, D) \cong \mathfrak{c}_{\text{red}}^\tau(X_n, D) \) for all \( n \in \mathbb{N} \), so that \( \mathfrak{c}_{\text{red}}^\tau(\mathcal{X}, D) \) is still a \( C^* \)-algebra. It is observed in [6] that the obvious maps give rise to an extension of \( \sigma \)-\( C^* \)-algebras
\[
0 \to C_0(\mathcal{X}, D \otimes \mathbb{K}) \to \mathfrak{c}_{\text{red}}^\tau(\mathcal{X}, D) \to \mathfrak{c}_{\text{red}}^\tau(\mathcal{X}, D) \to 0.
\]
Now let a locally compact group \( H \) act continuously and coarsely on a coarse space \( X \) and let \( D \) be an \( H \)-\( C^* \)-algebra. Let \( \mathbb{K}_H := \mathbb{K}(\ell^2 \mathbb{N} \otimes L^2 H) \) and let \( H \) act on \( D \otimes \mathbb{K}_H \) and \( \mathcal{M}(D \otimes \mathbb{K}_H) \) in the obvious way. We let \( \mathfrak{B}_{\text{red}}^{\tau H}(X, D) \) be the \( H \)-continuous subspace of the \( C^* \)-algebra of norm continuous, bounded functions \( f : X \to \mathcal{M}(D \otimes \mathbb{K}_H) \) for which \( f(x) - f(y) \in D \otimes \mathbb{K}_H \) for all \( x, y \in X \). The group \( H \) acts on \( \mathfrak{B}_{\text{red}}^{\tau H}(X, D) \) by \( (h \cdot f)(x) := h \cdot (f(xh)) \). As above, we let \( \mathfrak{c}_{\text{red}}^{\tau H}(X, D) \) be the subspace of vanishing variation functions in \( \mathfrak{B}_{\text{red}}^{\tau H}(X, D) \). This subalgebra is \( H \)-invariant. It contains \( C_0(\mathcal{X}, D \otimes \mathbb{K}_H) \) as an \( H \)-invariant ideal. Thus \( \mathfrak{c}_{\text{red}}^{\tau H}(X, D) := \mathfrak{c}_{\text{red}}^{\tau H}(X, D)/C_0(X, D \otimes \mathbb{K}_H) \) is an \( H \)-\( C^* \)-algebra. The same holds for \( \mathfrak{B}_{\text{red}}^{\tau H}(X, D) := \mathfrak{B}_{\text{red}}^{\tau H}(X, D)/C_0(X, D \otimes \mathbb{K}_H) \).
It is necessary to restrict attention to $H$-continuous elements in order to form the crossed product $\mathcal{C}^*_H(D) \rtimes H$. We warn the reader that, although a vanishing variation function $|G| \to D$ is automatically $G$-continuous for the action by left multiplication, which is by translations, this implies nothing about the action by right multiplication, which is the one we use.

The following is the central definition of this paper.

**Definition 10.** We refer to $K_{\text{top}}^0(H, \mathcal{C}^*_H(D))$ as the $H$-equivariant boundary $K$-theory of $X$ with coefficients $D$.

We are particularly interested in the case where $G$ is a discrete group, $D = \mathbb{C}$ and $X = |G|$, and where $H$ is a subgroup of $G$ acting by right multiplication on $G$. When $H$ is the trivial group, we obtain $K_*(\mathcal{C}^*_H(|G|))$, the $K$-theory of the stable Higson corona of $G$, which is used for descent. The equivariant version $K_{\text{top}}^0(G, \mathcal{C}^*_H(|G|))$ gives the boundary classes in $\text{RKK}^*_G(EG, C, C)$.

**2.7. The coarse co-assembly map.** Let $X \in \mathcal{C}_H$ and let $D$ be an $H$-$C^*$-algebra. Form the Rips complex $\mathcal{P}(X)$ as above. Let $D \in \text{KK}^G(P, \mathbb{C})$ be a Dirac morphism for $G$ as in [15]. Recall that we identify $K_{\text{top}}^0(H, D) := K_*((D \otimes P) \rtimes H)$. We also use this definition for $\sigma$-$H$-$C^*$-algebras. Since the action of $H$ on $\mathcal{P}(X)$ is proper, $C_0(\mathcal{P}(X), D)$ is an inverse system of proper $G$-$C^*$-algebras ($B_m$). These belong to $\langle \mathcal{C}_E \rangle$ by [15, Corollary 7.3]. Hence the maps $\mathcal{D}_*: (B_m \otimes P) \rtimes H \to B_m \times H$ are isomorphisms on $K$-theory. It follows from the Milnor $\lim_{\leftarrow}$-sequence of [18] that the map $K_*((\lim B_m \otimes P) \rtimes H) \to K_*((\lim B_m \times H)$ is an isomorphism as well. Thus
\[
K_*((C_0(\mathcal{P}(X), D) \otimes P) \rtimes H) \cong KX^*_H(X, P \otimes D) \cong KX^*_H(X, D).
\]

The coarse equivalence $j_X: X \to \mathcal{P}(X)$ induces an isomorphism
\[
\mathcal{C}^*_H(\mathcal{P}(X), D) \cong \mathcal{C}^*_H(X, D).
\]

We have a canonical extension of $\sigma$-$H$-$C^*$-algebras
\[
0 \to C_0(\mathcal{P}(X), D \otimes K_H) \to \mathcal{C}^*_H(\mathcal{P}(X), D) \to \mathcal{C}^*_H(\mathcal{P}(X), D) \to 0.
\]
Taking (maximal) tensor products with the source of the Dirac morphism $P$ and (full) crossed products with $H$, we obtain an extension of $\sigma$-$C^*$-algebras
\[
0 \to (C_0(\mathcal{P}(X), D \otimes K_H) \otimes P) \rtimes H \to (\mathcal{C}^*_H(\mathcal{P}(X), D) \otimes P) \rtimes H
\]
\[
\to (\mathcal{C}^*_H(\mathcal{P}(X), D) \otimes P) \rtimes H \to 0.
\]

We have canonical isomorphisms
\[
(\mathcal{C}^*_H(\mathcal{P}(X), D) \otimes P) \rtimes H) \cong K_{\text{top}}^0(H, \mathcal{C}^*_H(X, D)),
\]
\[
K_*((C_0(\mathcal{P}(X), D \otimes K_H) \otimes P) \rtimes H) \cong KX^*_H(X, D).
\]

Via these isomorphisms, the $K$-theory boundary map for (7) is equivalent to a map
\[
\mu^*_X,H,D: K_{\text{top}}^0(H, \mathcal{C}^*_H(\mathcal{P}(X), D)) \to KX^*_H(X, D).
\]

**Definition 11.** We call (9) the $H$-equivariant coarse co-assembly map for $X$ with coefficients $D$.

We obtain an equivalent map if we replace $\mathcal{P}(X)$ by a coarsely homotopy equivalent object of $\sigma \mathcal{C}_H$. The map $\mu^*_X,H,D$ is natural for almost equivariant coarse Borel maps by Lemma 8; that is, any such map gives rise to a commuting square diagram. The map $\mu^*_X,H,D$ is an isomorphism if and only if $K_{\text{top}}^0(H, \mathcal{C}^*_H(\mathcal{P}(X), D)) = 0$.

If $G$ is a locally compact group and $\mathcal{E}G$ is a universal proper $G$-space, then $|\mathcal{E}G|$ is coarsely homotopy equivalent to $\mathcal{P}(|G|)$ by Lemma 7. Hence we may use $|\mathcal{E}G|$ instead of $\mathcal{P}(|G|)$ to construct $\mu^*_G,H,D$ for any closed subgroup $H \subset G$, acting on $|G|$ by right multiplication.
For instance, let $G$ be a discrete group. Set $D = \mathbb{C}$ and $X = |G|$. We may then allow $H$ to run through the subgroups of $G$. If $H$ is the trivial subgroup we obtain the ordinary coarse co-assembly map for $|G|$:  
\[ \mu_{|G|}: K^{*}_{+1}(\mathcal{E}|G|) \to KX^{*}(|G|) \cong K_{*}(C_{0}(\mathcal{E}|G|)) \]
If $H = G$, we obtain a map  
\[ \mu_{|G|,G}^{\text{top}}: K^{*}_{+1}(G, c^{\text{red}}|G|) \to KX^{*}(|G|) \cong K_{*}(C_{0}(\mathcal{E}|G|) \times G) \]

2.8. The co-assembly map as a forgetful map. Let $X$ be a uniformly contractible coarse space with bounded geometry (without group action). It is shown in [6] that the natural map $X \to \mathcal{P}(X)$ is a coarse homotopy equivalence in this case. Hence $KX^{*}(X, D) \cong K_{*}(C_{0}(X, D))$, and the coarse co-assembly map is equivalent to the connecting map for the extension  
\[ 0 \to C_{0}(X, D \otimes \mathbb{K}) \to c^{\text{red}}(X, D) \to c^{\text{red}}(X, D) \to 0. \]
We can express the latter as a forget-control map as follows. By definition, we have $c^{\text{red}}(X, D) \subseteq \mathcal{B}^{\text{red}}(X, D)$. This yields a morphism of $C^{*}$-extensions  
\[ \begin{diagram}
0 & \rightarrow & C_{0}(X, D \otimes \mathbb{K}) & \rightarrow & c^{\text{red}}(X, D) & \rightarrow & c^{\text{red}}(X, D) & \rightarrow & 0 \\
\, & \, & \, & \, & \, & j \downarrow & \, & \, & \, \\
0 & \rightarrow & C_{0}(X, D \otimes \mathbb{K}) & \rightarrow & \mathcal{B}^{\text{red}}(X, D) & \rightarrow & \mathcal{B}^{\text{red}}(X, D) & \rightarrow & 0.
\end{diagram} \]

**Proposition 12.** If $X$ is a uniformly contractible coarse space of bounded geometry, then the $K$-theory connecting map $\partial^{*}: K^{*}_{+1}(\mathcal{B}^{\text{red}}(X, D)) \to K_{*}(C_{0}(X, D))$ is an isomorphism. Hence $\mu_{X, D}$ is equivalent to the map  
\[ j_{*}: K^{*}_{+1}(c^{\text{red}}(X, D)) \to K^{*}_{+1}(\mathcal{B}^{\text{red}}(X, D)). \]

To prove Proposition 12, it suffices to show that the $C^{*}$-algebra $\mathcal{B}^{\text{red}}(X, D)$ has vanishing $K$-theory whenever $X$ is uniformly contractible. Actually, this will be the case under the weaker assumption that $X$ is contractible.

Let $\mathcal{B}(X, D)$ denote the $C^{*}$-algebra of bounded, norm continuous maps $X \to D \otimes \mathbb{K}$. There is a canonical embedding $D \otimes \mathbb{K} \to \mathcal{B}(X, D)$. We have:

**Lemma 13.** The inclusion $\mathcal{B}(X, D) \to \mathcal{B}^{\text{red}}(X, D)$ induces a canonical isomorphism  
\[ K_{*}(\mathcal{B}(X, D))/K_{*}(D) \cong K_{*}(\mathcal{B}^{\text{red}}(X, D)). \]

**Proof.** Let $p$ be the composition $\mathcal{B}^{\text{red}}(X, D) \to \mathcal{B}(D \otimes \mathbb{K}) \to \mathcal{Q}(D \otimes \mathbb{K})$, where the first map is evaluation at some point $\ast \in X$, the second map is the quotient map, and where $\mathcal{Q}(D \otimes \mathbb{K})$ denotes $\mathcal{B}(D \otimes \mathbb{K})/D \otimes \mathbb{K}$. The induced map on $K$-theory vanishes because it factorises through the $K$-theory of $\mathcal{B}(D \otimes \mathbb{K})$, which is zero. The kernel of $p$ is evidently $\mathcal{B}(X, D)$. By considering the associated six-term exact sequence and making the identification $K_{*+1}(\mathcal{Q}(D \otimes \mathbb{K})) \cong K_{*}(D)$, one obtains short exact sequences  
\[ 0 \rightarrow K_{*}(D) \rightarrow K_{*}(\mathcal{B}(X, D)) \rightarrow K_{*}(\mathcal{B}^{\text{red}}(X, D)) \rightarrow 0. \]
It is easily checked that the first map is the canonical inclusion. \hfill \Box

**Lemma 14.** For any $D$ there is a canonical isomorphism  
\[ K_{*+1}(\mathcal{B}(X, D)) \cong \text{RK} K_{*}(X; \mathbb{C}, D). \]
Proof. We claim that $\mathcal{M}(C_0(X, D \otimes \mathbb{K})) \cong \mathcal{M}(\mathfrak{B}(X, D))$. The multipliers of $C_0(X, D \otimes \mathbb{K})$ are the bounded, strictly continuous functions $X \to \mathcal{M}(D \otimes \mathbb{K})$. These also act as multipliers on $\mathfrak{B}(X, D)$. The converse is clear because $C_0(X, D \otimes \mathbb{K})$ is an ideal in $\mathfrak{B}(X, D)$. It follows that the $K$-theory of $\mathcal{M}(\mathfrak{B}(X, D))$ vanishes. Thus the $K$-theory of $\mathfrak{B}(X, D)$ is isomorphic to the $K$-theory of $\mathcal{M}(\mathfrak{B}(X, D))/\mathfrak{B}(X, D)$. It remains to identify the $K$-theory of the latter with $\text{RKK}_0(X; \mathbb{C}, D)$.

We sketch the proof for $*=0$. Cycles for $\text{RKK}_0(X; \mathbb{C}, D)$ are $F \in \mathcal{M}(C_0(X, D \otimes \mathbb{K}))$ for which $M_\phi \cdot (FF^*-1)$ and $M_\phi \cdot (F^*F-1)$ lie in $C_0(X, D \otimes \mathbb{K})$ for all $\phi \in C_0(X)$. Equivalently, $FF^*-1$ and $F^*F-1$ belong to $\mathfrak{B}(X, D)$.

Two cycles differ by a compact perturbation if and only if their difference belongs to $\mathfrak{B}(X, D)$. Thus equivalence classes of cycles for $\text{RKK}_0(X; \mathbb{C}, D)$ up to compact perturbation are the same as unitaries in $\mathcal{M}(\mathfrak{B}(X, D))/\mathfrak{B}(X, D)$. Moreover, operator homotopy of cycles corresponds to homotopy of unitaries, and degenerate cycles correspond to unitaries in $\mathcal{M}(\mathfrak{B}^{red}(X, D))$. These observations together with the homotopy invariance of Kasparov theory yields the assertion. A similar argument also occurs in the proof of Lemma 17. \hfill \Box

Proof of Proposition 12. Since $X$ is uniformly contractible, it is contractible. This implies that $\text{RKK}_0(X; \mathbb{C}, D) \cong K_*(D)$. Using Lemmas 13 and 14, we conclude that the $K$-theory of $\mathfrak{B}^{red}(X, D)$ vanishes. Thus $\partial'$ is an isomorphism. The maps $\mu_{X, D}$ and $\partial$ are equivalent because $X \to \mathcal{P}(X)$ is a coarse homotopy equivalence. We have $\partial = \partial' \circ j_\ast$ by the naturality of the connecting map. \hfill \Box

The above description of the co-assembly map in terms of forgetting control is also available in the equivariant case. This is a pleasant and novel feature of our framework. An analogue of Proposition 12 for the coarse Baum-Connes assembly map is the de-localisation description in [21]. Let $G$ be a totally disconnected locally compact group with a $G$-compact model for $\mathcal{E}G$. The space $\mathcal{E}G$ is uniformly contractible and, moreover, $H$-equivariantly contractible for all compact subgroups $H \subseteq G$. Hence, by a small elaboration of the argument above, $K_*^H(\mathfrak{B}^{red}_G(\mathcal{E}G, D))$ vanishes for all compact subgroups $H \subseteq G$. This implies vanishing of $K_*^{top}(G, \mathfrak{B}^{red}_G(\mathcal{E}G, D))$ by [5]. Hence we have:

**Theorem 15.** Let $G$ be a totally disconnected group with a $G$-compact model for $\mathcal{E}G$. Then the $G$-equivariant coarse co-assembly map for $G$ is equivalent to the map

$$j_\ast : K_*^{top}(G, \mathcal{E}_G^{red}(\mathcal{E}G, D)) \to K_{*+1}(G, \mathfrak{B}^{red}_G(\mathcal{E}G, D))$$

induced by the inclusion $j : \mathcal{E}_G^{red}(\mathcal{E}G, D) \to \mathfrak{B}^{red}_G(\mathcal{E}G, D)$.

3. The coarse co-assembly map and equivariant Kasparov theory

We first identify the equivariant co-assembly map for $|G|$ with coefficients with a map of the form $p^G_\mathcal{E} : \text{KK}^G(\mathbb{C}, B) \to \text{RKK}_G^G(\mathcal{E}G; \mathbb{C}, B)$ for suitable $B$. Then we prove a weaker result for general $B$.

3.1. An equivalence of maps. Throughout this section, we fix a locally compact group $G$, a compact subgroup $H \subseteq G$, and an $H$-$C^*$-algebra $D$. The induced $G$-$C^*$-algebra $\text{Ind}_H^G(D)$ is defined as

$$\text{Ind}_H^G(D) := \{ f \in C_0(G, D) \mid \alpha_h(f(gh)) = f(g) \text{ for all } h \in H, g \in G \},$$

with $G$-action $(gf)(g') = f(g^{-1}g')$. If $\mathcal{E}$ is an $H$-equivariant Hilbert $D$-module, then a similar formula defines a $G$-equivariant Hilbert $\text{Ind}_H^G(D)$-module $\text{Ind}_H^G(\mathcal{E})$.

We also let $\mathcal{E}G$ be a universal proper $G$-space. Given two $G$-$C^*$-algebras $A$ and $B$, we define the bivariant Kasparov groups $\text{RKK}^G(\mathcal{E}G; A, B)$ as in [10] and let

$$p^G_\mathcal{E} : \text{Kk}^G(A, B) \to \text{RKK}^G(\mathcal{E}G; A, B)$$
be the natural map induced by the constant map from $\mathcal{E}G$ to a point.

**Theorem 16.** There are isomorphisms $\text{KK}^G_*(\mathbb{C}, \text{Ind}_H^G D) \cong K^H_{*+1}(\mathfrak{c}_{H}^0([G], D))$ and $\text{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \text{Ind}_H^G D) \cong X^K_H([G], D)$ making the following diagram commute:

$$
\begin{array}{ccc}
\text{KK}^G_*(\mathbb{C}, \text{Ind}_H^G D) & \xrightarrow{\mu^*_G} & \text{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \text{Ind}_H^G D) \\
\cong & & \cong \\
K^H_{*+1}(\mathfrak{c}_{H}^0([G], D)) & \xrightarrow{\mu^*_{G|,H,D}} & X^K_H([G], D).
\end{array}
$$

Note that if $D = \mathbb{C}$ with trivial action of $H$, then $\text{Ind}_H^G D = C_0(G/H)$. Hence Theorem 16 identifies (4) with the $H$-equivariant coarse co-assembly map $\mu^*_{|G|,H}$ and (1) with the coarse co-assembly map $\mu^*_{|G|}$. We prepare the proof of Theorem 16 with several lemmas. If $H$ acts on a $C^*$-algebra $A$, let $A^H$ be the subalgebra of $H$-invariant elements of $A$.

**Lemma 17.** There are natural isomorphisms

$$
\text{KK}^G_*(\mathbb{C}, \text{Ind}_H^G D) \cong K^H_{*+1}(\mathfrak{c}_{H}^0([G], D)^H) \cong K^H_{*+1}(\mathfrak{c}_{H}^0([G], D)).
$$

**Proof.** We only treat the case $* = 0$, the case $* = 1$ is similar. To prove the first isomorphism, we describe the cycles for $\text{KK}^G_*(\mathbb{C}, \text{Ind}_H^G D)$ more concretely. Such a cycle is given by two $G$-equivariant Hilbert modules $\mathcal{E}_\pm$ over $\text{Ind}_H^G D$ and a $G$-continuous adjointable operator $F: \mathcal{E}_+ \to \mathcal{E}_-$ for which $1 - FF^*$, $1 - F^*F$ and $gF - F$ for $g \in G$ are compact. Let $D_H^\mathcal{E}_\pm := D \otimes L^2(\mathbb{H}) \otimes \ell^2(\mathbb{N})$ be the standard $H$-equivariant Hilbert module over $D$. Then $\text{Ind}_H^G(D_H^\mathcal{E}_\pm)$ is naturally isomorphic to the standard Hilbert module $\text{Ind}_H^G D \otimes L^2(G) \otimes \ell^2(\mathbb{N})$ over $\text{Ind}_H^G D$. Since $\text{Ind}_H^G D$ is a proper $G$-$C^*$-algebra, the Equivariant Stabilisation Theorem of [13] applies and yields that every countably generated Hilbert module over $\text{Ind}_H^G D$ is absorbed by the standard one. It follows that we can also define $\text{KK}^G_*(\mathbb{C}, \text{Ind}_H^G D)$ using only those “special” cycles where $\mathcal{E}_\pm = \mathcal{E}_- = \text{Ind}_H^G(D_H^\mathcal{E}_-)$.

Elements of $\text{Ind}_H^G(D_H^\mathcal{E}_-)$ are functions in $C_0(G, D_H^\mathcal{E}_-)$ that satisfy $f(g) = \alpha_h(f(gh))$ for all $g \in G$, $h \in H$. The $\text{Ind}_H^G D$-Hilbert module structure is given by pointwise multiplication and pointwise inner products. The group $G$ acts by left translation. Thus the space of adjointable operators on $\text{Ind}_H^G(D_H^\mathcal{E}_-)$ can be identified with the space of $*$-strictly continuous functions $f: G \to \mathbb{B}(D_H^\mathcal{E}_-)$ that are $H$-invariant, that is, $f(g) = \alpha_h(f(gh))$ for all $g \in G$, $h \in H$. In particular, we can view $F$ as such a function, which we still denote $F$.

The $G$-continuity of $F$ means that this function is not just strictly continuous: it is uniformly norm continuous, that is,

$$
\lim_{g \to 1} \sup_{x \in G} \|F(g^{-1}x) - F(x)\| = 0.
$$

Given (10), the condition $gF - F \in K(\text{Ind}_H^G D_H^\mathcal{E}_-)$ for $g \in G$ translates into the two conditions $F(g^{-1}x) - F(x) \in K(D_H^\mathcal{E}_-)$ for all $g, x \in G$ and

$$
\lim_{x \to \infty} \sup_{g \in K} \|F(g^{-1}x) - F(x)\| = 0.
$$

Conversely, (11) together with ordinary continuity implies (10). Thus we get exactly the condition that $F \in \mathfrak{c}_H^0([G], D)^H$. Since $\mathbb{K}(\text{Ind}_H^G D_H^\mathcal{E}_-) \cong C_0(G, D \otimes \mathbb{K}(H))^H$, the compactness of $1 - FF^*$ and $1 - F^*F$ means that the image of $F$ in $\mathfrak{c}_H^0([G], D)^H$ is unitary. Summing up, “special” cycles for $\text{KK}^G_*(\mathbb{C}, \text{Ind}_H^G D)$ are in bijection with elements of $\mathfrak{c}_H^0([G], D)^H$ whose image in $\mathfrak{c}_H^0([G], D)^H$ is unitary.

Two cycles for $\text{KK}^G_*(\mathbb{C}, \text{Ind}_H^G D)$ differ by a compact perturbation if and only if they have the same image in $\mathfrak{c}_H^0([G], D)^H$. The map $\mathfrak{c}^{\text{spec}}_H([G], D)^H \to \mathfrak{c}_H^0([G], D)^H$
is surjective because $H$ is compact. Therefore, equivalence classes of “special cycles” up to compact perturbation correspond bijectively to unitaries in $c^{\text{red}}_H([G], D)^H$. A cycle is degenerate if and only if it is a constant function on $G$. Cycles are operator homotopic if and only if the resulting unitaries in $c^{\text{red}}_H([G], D)^H$ are homotopic.

It is easy to see that $c^{\text{red}}_H([G], D)^H$ is matrix stable. Hence we do not have to adjoin matrices to compute its K-theory. The subalgebra of constant functions in $c^{\text{red}}_H([G], D)^H$ is isomorphic to $\mathcal{M}(D \otimes \mathbb{K}_H)^H$ and hence has vanishing K-theory: the same Eilenberg swindle that proves this fact for stable multiplier algebras works equivariantly. As a result, addition of degenerate cycles and operator homotopy generate the same equivalence relation on “special” cycles for KK$_0(G, Ind^G_D)$ as stable homotopy equivalence for unitaries in $c^{\text{red}}_H([G], D)^H$. Since operator homotopy and homotopy generate the same equivalence relation, we get KK$_0(G, Ind^G_D) \cong K_1(c^{\text{red}}_H([G], D)^H)$ as claimed.

To prove the second isomorphism, we show that $c^{\text{red}}_H(X, D)^H$ and $c^{\text{red}}_H(X, D) \rtimes H$ are Morita-Rieffel equivalent (notice that both algebras are $\sigma$-unital). If $H$ is finite, then $\mathbb{K}(L^2 H)$ is finite dimensional, so that

$$c^{\text{red}}_H(X, D)^H \cong (c^{\text{red}}_H(X, D) \otimes \mathbb{K}(L^2 H))^H \cong c^{\text{red}}_H(X, D) \rtimes H$$

by the proof of the Green-Julg Theorem. Thus we have an isomorphism in this case. For general compact $H$, we use that the fixed point algebra is Morita-Rieffel equivalent to a certain ideal $I$ in the crossed product (see [14]). The imprimitivity bimodule is $c^{\text{red}}_H(X, D)$ equipped with appropriate structure. We embed $C(H) \subseteq \mathbb{B}(L^2 H) \subseteq c^{\text{red}}_H(X, D)$ unitally as constant functions on $X$. Since this embedding is equivariant, the ideal $I$ contains the corresponding ideal for $C(H)$, which is all of $C(H) \rtimes H \cong \mathbb{K}(L^2 H)$. It follows that $I$ contains an approximate identity and hence must be all of $c^{\text{red}}_H(X, D) \rtimes H$.

We next recall the following well-known facts.

**Lemma 18.** Let $G$ be a locally compact group and $H$ a closed subgroup. Let $A$ be a $\sigma$-$G$-$C^*$-algebra and $B$ a $\sigma$-$H$-$C^*$-algebra. 

18.1. The $\sigma$-$G$-$C^*$-algebras $A \otimes Ind^G_H B$ and $Ind^G_H(A \otimes B)$ are isomorphic. 

18.2. The $\sigma$-$C^*$-algebras $(Ind^G_H A) \rtimes G$ and $A \rtimes H$ are Morita-Rieffel equivalent. 

**Corollary 19.** Let $G$ be a discrete group, $H$ a finite subgroup, and $D$ an $H$-$C^*$-algebra. Then there is a canonical isomorphism 

$$K_*(C_0([E_G], D) \rtimes H) \cong K_*(C_0([E_G], Ind^G_H D) \rtimes G).$$

**Proof.** Lemma 18 implies 

$$C_0([E_G], Ind^G_H D) \rtimes G \cong Ind^G_H(C_0([E_G]) \otimes D) \rtimes G \cong C_0([E_G], D) \rtimes H,$$

where $\cong$ denotes isomorphism and $\sim$ denotes Morita-Rieffel equivalence. 

**Lemma 20.** Let $G$ be a locally compact group and let $X$ be a locally compact proper $G$-space that can be written as a union of an increasing sequence $(X_n)$ of $G$-compact closed subspaces. Let $A$ be a $C^*$-algebra with trivial action of $G$ and let $B$ be a $G$-$C^*$-algebra. Then there is a natural isomorphism 

$$\text{RKK}^G(X; A, B) \cong KK_*(A, \lim_{\leftarrow} C_0(X_n, B) \rtimes G).$$

Here we use the bivariant Kasparov theory for $\sigma$-$C^*$-algebras defined by Alexander Bonkat in [2]. We will only apply this lemma for $A = \mathbb{C}$, where this reduces to K-theory for $\sigma$-$C^*$-algebras as defined in [18].
Proof. We check that both groups agree on the level of cycles after some standard simplifications. Since $C^*_0(X, B)$ is a proper $G$-$C^*$-algebra, the reduced and full crossed products for $C^*_0(X, B)$ agree. Moreover, the $C^*$-categories of $G$-equivariant Hilbert modules over $C^*_0(X, B)$ and of Hilbert modules over $C^*_0(X, B) times G$ are equivalent (see [14]). That is, any $G$-equivariant Hilbert module $\mathcal{E}$ over $C^*_0(X, B)$ corresponds to a Hilbert module $\hat{\mathcal{E}}$ over $C^*_0(X, B) \rtimes G$. The correspondence is such that $\mathbb{B}(\hat{\mathcal{E}})$ is naturally isomorphic to the $C^*$-algebra $\mathbb{B}(\mathcal{E})^G$ of $G$-equivariant adjointable operators on $\mathcal{E}$. The compact operators on $\hat{\mathcal{E}}$ correspond to the generalised fixed point algebra of $\mathbb{B}(\mathcal{E})$, which is the closed linear span of operators of the form $\int_G \alpha_g(\langle \xi | \eta \rangle) \, dg$, where $\xi, \eta \in \mathcal{E}$ are compactly supported sections. (The support of $\xi$ is the set of $x \in X$ with $\xi_x \neq 0$.) More generally, if $T \in \mathbb{K}(\mathcal{E})$ has compact support, then $\int_G \alpha_g(T) \, dg$ belongs to the generalised fixed point algebra.

To simplify our notation, we consider the $\sigma$-locally compact space $\mathcal{X} := \bigcup X_n$ and let $C^*_0(\mathcal{X}, B) \rtimes G := \lim_{\leftarrow} C^*_0(X_n, B) \rtimes G$. We have natural maps

$$C^*_0(X, B) \rtimes G \to C^*_0(\mathcal{X}, B) \rtimes G \to C^*_0(X_n, B) \rtimes G$$

for all $n \in \mathbb{N}$. If $\hat{\mathcal{E}}$ is a Hilbert module over $C^*_0(X, B) \rtimes G$, its restriction to $X_n$ is the Hilbert module $\hat{\mathcal{E}}_n := \hat{\mathcal{E}} \otimes_{C^*_0(X_n, B) \rtimes G} C^*_0(X_n, B) \rtimes G$ over $C^*_0(X_n, B) \rtimes G$. Then

$$\hat{\mathcal{E}} := \lim_{\leftarrow} \hat{\mathcal{E}}_n \cong \hat{\mathcal{E}} \otimes_{C^*_0(X_n, B) \rtimes G} C^*_0(X_n, B) \rtimes G$$

is a Hilbert module over $C^*_0(\mathcal{X}, B) \rtimes G$. Conversely, given a Hilbert module $\hat{\mathcal{E}}$ over $C^*_0(\mathcal{X}, B) \rtimes G$, we obtain a Hilbert module over $C^*_0(X, B)$ by completing the subspace of compactly supported sections $\hat{\mathcal{E}} \cdot C^*_0(X, B) \rtimes G$. It is easy to see that these two operations are inverse to each other. We have

$$\mathbb{K}(\hat{\mathcal{E}}) := \lim_{\leftarrow} \mathbb{K}(\hat{\mathcal{E}}_n), \quad \mathbb{B}(\hat{\mathcal{E}}) := \lim_{\leftarrow} \mathbb{B}(\hat{\mathcal{E}}_n).$$

We can describe $\mathbb{B}(\hat{\mathcal{E}})$ as the $C^*$-subalgebra of bounded elements in the $\sigma$-$C^*$-algebra $\mathbb{B}(\hat{\mathcal{E}})$. Thus any $\ast$-homomorphism $A \to \mathbb{B}(\hat{\mathcal{E}})$ factorises through $\mathbb{B}(\hat{\mathcal{E}})$.

Cycles for $\text{RKK}^G(X, A, B)$ are triples $\langle \mathcal{E}, \phi, F \rangle$ where $\mathcal{E}$ is a graded $G$-equivariant Hilbert module over $C^*_0(X, B)$, $\phi : A \to \mathbb{B}(\mathcal{E})$ is an equivariant $\ast$-homomorphism and $F \in \mathbb{B}(\mathcal{E})$ is a self-adjoint, $G$-equivariant, odd operator for which $M_h \cdot [\phi(a), F]$ and $M_h \phi(a)(1 - F^2)$ are compact for all $h \in C^*_0(X)$, $a \in A$. It is shown in [10, 13] that we can arrange $F$ to be strictly equivariant, using that $X$ is proper. Since $G$ acts trivially on $A$, the range of $\phi$ consists of $G$-equivariant operators on $\mathcal{E}$.

By our category equivalence, this data is equivalent to a triple $\langle \hat{\mathcal{E}}, \hat{\phi}, \hat{F} \rangle$, where $\hat{\mathcal{E}}$ is a Hilbert module over $C^*_0(\mathcal{X}, B) \rtimes G$ and $\hat{\phi}$ and $\hat{F}$ are obtained from $\phi$ and $F$ using $\mathbb{B}(\mathcal{E})^G \cong \mathbb{B}(\hat{\mathcal{E}}) \subseteq \mathbb{B}(\hat{\mathcal{E}})$. Thus $\hat{\phi}$ is a $\ast$-homomorphism and $\hat{F}$ is a bounded, odd, self-adjoint operator. We claim that this construction yields a bijection between cycles $\langle \mathcal{E}, \phi, F \rangle$ for $\text{RKK}^G(X, A, B)$ with equivariant $F$ and cycles $\langle \hat{\mathcal{E}}, \hat{\phi}, \hat{F} \rangle$ for $\text{KK}^0(A, C^*_0(\mathcal{X}, B) \rtimes G)$. Let $S$ be $\phi(a)(F^2 - 1)$ or $[F, \phi(a)]$ for some $a \in A$ and let $\hat{S}$ be the associated operator on $\hat{\mathcal{E}}$. The proof is finished if we show that $M_h S$ is compact for all $h \in C^*_0(X)$ if and only if $\hat{S}$ is compact.

Assume first that $M_h S \in \mathbb{K}(\mathcal{E})$ for all $h \in C^*_0(X)$. Choose $n \in \mathbb{N}$. By the properness of the $G$-action, there is a function $h \in C_c(X)$ with $\int_G h(xg) \, dg = 1$ for all $x \in X_n$. Then $S$ and $\int_G \alpha_g(M_h S) \, dg$ have the same restriction to $X_n$. Since $M_h S$ is compact by hypothesis and has compact support, this integral belongs to $\mathbb{K}(\mathcal{E}_n)$. This implies $\hat{S} \in \mathbb{K}(\hat{\mathcal{E}})$. Suppose conversely that $\hat{S} \in \mathbb{K}(\hat{\mathcal{E}})$ and fix $h \in C_c(X)$. Choose $n$ so that $X_n$ contains the support of $h$. Thus the product $M_h S$ only sees the restriction of $\hat{S}$ to $X_n$. The operator on $\mathcal{E}_n$ induced by $\hat{S}$ is compact. Thus $S$ belongs to the generalised fixed point algebra of $\mathcal{E}_n := \mathcal{E} \otimes_{C^*_0(X_n, B)} C^*_0(X_n, B)$. That is, it can be approximated by operators of the form $\int_G \alpha_g(T) \, dg$ for a finite rank
operator $T$ on $E_n$ with compact support. Hence the function $g \mapsto M_h \alpha_g(T)$ has compact support, so that $M_h \int G \alpha_g(T) \, dq$ is a compact operator on $E_n$. Since these operators approximate $M_h S$, we get $M_h S \in \mathbb{K}(E_n) \subseteq \mathbb{K}(E)$.

**Proof of Theorem 16.** Lemma 20 and Corollary 19 yield isomorphisms

$$\text{RKK}^G(\mathbb{C}, \text{Ind}_H^G D) \cong \text{K}_*((\mathbb{C}|G), \text{Ind}_H^G D \times D) \cong \text{K}_*((\mathbb{C}|G), D \times H) = \text{KK}_H^G(G, D).$$

The other isomorphism required for Theorem 16 is provided by Lemma 17. To check that the resulting diagram commutes, we work with (generalised) fixed point algebras instead of crossed products, as this simplifies the arguments. We have $C^\text{g}((\mathbb{C}|G), D \otimes \mathbb{K}_H)^H \cong C^\text{g}((\mathbb{C}|G), D \otimes \mathbb{K}) \times H$ because $\mathbb{K}_H$ contains a factor $\mathbb{K}(L^2 H)$.

We again define $\text{KK}_*^G(\mathbb{C}, \text{Ind}_H^G D)$ by “special” cycles and identify them with elements of $\tilde{C}_H^\text{g}((G), D)^H$ whose image $\pi(F)$ in $C_H^\text{g}((G), D)^H$ is unitary. Thus the isomorphism $\text{KK}_*^G(\mathbb{C}, \text{Ind}_H^G D) \to K_1(\tilde{C}_H^\text{g}((G), D)^H)$ maps the class represented by the cycle $F$ to the class represented by the unitary $\pi(F)$.

In order to compute the image of $[\pi(F)]$ under the coarse co-assembly map, we have to describe the connecting map in $K$-theory. Let $0 \to I \to E \to Q \to 0$ be an extension of $\sigma$-$C^*$-algebras and suppose that $E$ and $Q$ are unital. Let $u \in Q$ be unitary and lift it to $F \in E \subseteq M(I)$. Since $u$ is unitary, $FF^* - 1$ and $F^*F - 1$ belong to $I$. Thus $F \in M(I)$ is a cycle for $\text{KK}_0(\mathbb{C}, I) \cong K_0(I)$. This element of $K_0(I)$ is the image of $[u]$ under the connecting map. Thus we get $\mu_{|G|, H, D}[\pi(F)]$ if we lift $\pi(F) \in C^\text{g}_H((G), D)^H$ to $C^\text{g}_H((\mathbb{C}|G), D)^H$ to an element of $C^\text{g}_H((\mathbb{C}|G), D)^H$ and then view this as a Fredholm multiplier of $C^\text{g}_0((\mathbb{C}|G), D \otimes \mathbb{K}_H)^H$.

Choose an $H$-invariant continuous function $c: EG \to \mathbb{R}_+$ with $\int_G c(xg) \, dg = 1$ for all $x \in EG$ such that $W_Y := \text{supp} c \cap Y$ is compact for all $G$-compact subsets $Y \subseteq EG$. Let $L_Y \subseteq G$ be the set of $g \in G$ with $W_Y \cap W_Y \neq \emptyset$. We let

$$\tilde{F}(x) := \int_G c(xg) F(g^{-1}) \, dg.$$ 

If $x \in W_Y$ for some $g \in G$, then $\tilde{F}(x)$ is an average of $F(h)$ with $xh^{-1} \in W_Y$, so that $h \in L_Y^{-1}$. It follows that $\tilde{F}|Y$ belongs to $C^g_H((\mathbb{C}|G), D)^H$ for all $G$-compact $Y$, that is, $\tilde{F} \in C^g_H((\mathbb{C}|G), D)^H$. The quotient map $C^g_H((\mathbb{C}|G), D)^H \to \tilde{C}^g_H((\mathbb{C}|G), D)^H$ simply restricts a function on $EG$ to any $G$-orbit in $|EG|$. Hence $\pi(\tilde{F}) = \pi(F)$ in $\tilde{C}^g_H((\mathbb{C}|G), D)^H$. Thus $\mu_{|G|, H, D}$ maps $[\pi(F)]$ to the class represented by the Fredholm multiplier $\tilde{F}$ of $C^g_0((\mathbb{C}|G), D \otimes \mathbb{K}_H)^H$.

Now go around the diagram the other way. By definition, $p^g_\text{ind}[F]$ is represented by the multiplication operator $F' f(x, g) := F(g) f(x, g)$ on $C_0(\mathbb{C}|EG, \text{Ind}_H^G D_H^\infty)$, with action of $G$ coming from the action on $\mathbb{C}|EG \times G$ by $h \cdot (x, g) = (xh^{-1}, h g)$. The same formulas work if we replace $C^g_0(\mathbb{C}|EG)$ by $C^g_0(|EG|)$. Let $c$ be as above. It is easy to check that the multiplication operator $F''$ defined by

$$(F'' f)(x, g) := \int_G c(xh) F(h^{-1} g) \, dh \cdot f(x, g)$$

is a $G$-equivariant compact perturbation of $F'$. That is, $F''$ and $F'$ have the same class in $\text{RKK}_0^G(\mathbb{C}|EG, \text{Ind}_H^G D)$ and $F''$ is a multiplier of the generalised fixed point algebra of $C^g_0(|EG|, \text{Ind}_H^G D_H^\infty)$. Restriction to $\mathbb{C}|EG \times \{1\} \subseteq \mathbb{C}|EG \times G$ identifies this generalised fixed point algebra with $C_0(|EG|, D \otimes \mathbb{K}_H)^H$. The isomorphisms

$$\text{RKK}_0^G(\mathbb{C}, \text{Ind}_H^G D) \cong K_0(C_0(|EG|, \text{Ind}_H^G D \times 1)) \cong K_0(C_0(|EG|, D \otimes \mathbb{K}) \times H) \cong K_0(C_0(|EG|, D \otimes \mathbb{K}_H)^H).$$
constructed above send \( F' \) to the class of the Fredholm multiplier \( F''|_{EG \times \{1\}} \) of \( C_0(|EG|, D \otimes \mathbb{K}_H)^H \). The reason for this is that Lemma 18.2 is proved using the same manipulations of generalised fixed point algebras that we used above to view \( F'' \) as a multiplier of \( C_0(|EG|, D \otimes \mathbb{K}_H)^H \). By construction, \( F''|_{EG \times \{1\}} = F \). Thus the diagram commutes as desired. \( \square \)

3.2. Constructing Kasparov cycles from the stable Higson corona.

**Theorem 21.** Let \( G \) and \( H \) be locally compact groups and let \( X \) be a coarse space equipped with commuting actions of \( G \) and \( H \). Suppose that \( G \) acts by translations and that \( H \) acts properly and by isometries. Let \( D \) be an \( H-C^* \)-algebra. Then there is a natural commuting diagram

\[
\begin{array}{ccc}
K_{s+1}(\mathcal{C}_H^G(X, D) \rtimes H) & \xrightarrow{\mu_{X,H,D}} & KX_H^*(X, D) \\
\psi_{G,X,H,D} & & \psi_{G,X,H,D} \\
\text{KK}_G^G(\mathbb{C}, C_0(X, D) \rtimes H) & \xrightarrow{\rho_{G}} & \text{RKK}_G^G(\mathbb{E}; \mathbb{C}, C_0(X, D) \rtimes H).
\end{array}
\]

If \( H \subseteq G \) is a compact subgroup and \( X = |G| \) with actions of \( G \) and \( H \) by multiplication, then this diagram is equivalent to the one constructed in Theorem 16 via the Morita-Rieffel equivalence \( C_0(G, D) \rtimes H \sim C_0(G, D)^H = \text{Ind}_{H}^{G}(D) \). Hence the vertical maps are isomorphisms in this special case.

We will apply this theorem to construct elements of \( \text{KK}^G(\mathbb{C}, B) \) in Section 8. It is also used to construct dual Dirac morphisms from compactifications (Theorem 49). Of course, the natural transformations \( \phi \) and \( \psi \) cannot be isomorphisms in general.

**Proof.** We only construct the diagram in the case \( * = 0 \), the case \( * = 1 \) is similar. To simplify notation, let \( A := C_0(X, D) \rtimes H \) and assume \( D \otimes \mathbb{K}_H \cong D \), so that we can omit the stabilisations in the definition of \( \mathcal{C}_H^G(X, D) \) and \( \mathcal{C}_H^G(X, D) \). Recall that we have a canonical \( C^* \)-extension

\[
0 \to A \to \mathcal{C}_H^G(X, D) \rtimes H \to \mathcal{C}_H^G(X, D) \rtimes H \to 0.
\]

Let \( u \in \mathcal{C}_H^G(X, D) \rtimes H \) be unitary and lift it to \( F \in \mathcal{C}_H^G(X, D) \rtimes H \). Then we may view \( F \) as a Fredholm multiplier \( F : A \to A \). Since \( F \in \mathcal{C}_H^G(X, D) \) and \( G \) acts on \( X \) by translations, \( g \mapsto gF - F \) is a norm continuous function from \( G \) to \( A \). Thus \( F \) defines a cycle for \( \text{KK}_G^G(\mathbb{C}, A) \). As in the proof of Lemma 17, one checks that this construction defines a map \( K_1(\mathcal{C}_H^G(X, D) \rtimes H) \to \text{KK}_G^G(\mathbb{C}, A) \).

There is a function \( c : \mathcal{E}G \to \mathbb{R}_+ \) for which \( \int_{\mathcal{E}G} c(\mu) \, dg = 1 \) for all \( \mu \in \mathcal{E}G \) and \( \supp \mu \cap Y \) is compact for \( G \)-compact \( Y \subseteq \mathcal{E}G \). Recall that \( \mathcal{P}(X) \) contains all compactly supported probability measures on \( X \). Hence we can define a map \( c_* : \mathcal{E}G \times X \to \mathcal{P}(X) \) by

\[
(c_*(\mu, x), \alpha) := \int_{\mathcal{G}} c(\mu) \alpha(g^{-1}x) \, dg
\]

for all \( \alpha \in C_0(X) \). One checks easily that this map is continuous and satisfies \( c_*(\mu g, g^{-1}x h) = c_*(\mu, x) h \) for all \( g \in G, \mu \in \mathcal{E}G, x \in X, h \in H \). If \( K \subseteq \mathcal{E}G \) is compact, then there is a compact subset \( L \subseteq G \) such that \( c(\mu) = 0 \) for \( \mu \in K \) and \( g \notin L \). Hence \( c_*(\mu, x) \) is supported in \( L^{-1}x \) for \( \mu \in S \). Since \( G \) acts on \( X \) by translations, this is contained in \( P_{E,1}(X) \) for a suitable entourage \( E \). The restriction \( c_* : K \times X \to P_{E,1}(X) \) is proper because it is close to the map

\[
K \times X \xrightarrow{c_*} X \xrightarrow{\Delta_X} P_{E,1}(X).
\]

Recall that \( KX_H^0(X, D) := K_0(C_0(\mathcal{P}(X), D) \rtimes H) \). We briefly write \( B \) for the \( \sigma-C^* \)-algebra \( C_0(\mathcal{P}(X), D) \rtimes H \). Elements of \( K_0(B) \) are represented by (bounded)
Fredholm multipliers $F \in \mathcal{M}(B)$ (that is, $1 - F^*F \in B$ and $1 - FF^* \in B$). We do not have to stabilise because $B$ is already stable. View such a Fredholm multiplier as a function $H \times \mathcal{P}(X) \to \mathcal{M}(D)$ and pull it back with $c_* : E\!G \times X \to \mathcal{P}(X)$ to a function $c^*(F) : H \times E\!G \times X \to \mathcal{M}(D)$. The equivariance of $c_*$ implies that $c^*(F)$ is a $G$-invariant multiplier of $C_0(E\!G, A)$. The restriction of $c^*(F)$ to $C(K, A)$ is Fredholm for all compact subsets $K \subseteq E\!G$ because $c_*$ restricts to a proper map $K \times X \to P_{E,1}(X)$ for some entourage $E$. Therefore, $c^*(F)$ is a cycle for $RKK^G_0(E\!G; \mathbb{C}, A)$. This yields a natural map $KX_0^H(X, D) \to RKK^G_0(E\!G; \mathbb{C}, A)$.

Finally, routine computations which we leave to the reader show that the diagram in the statement of the theorem commutes and agrees with the one in Theorem 16 for $X = |G|$. \hfill \Box

For any pair of $C^*$-algebras $B, P$ we have a canonical exact sequence
\[ 0 \to (C_0(|G|, B) \otimes P) \times G \to (\mathcal{C}^{\tau\theta}(|G|, B) \otimes P) \times G \to (\mathcal{C}^{\tau\theta}(|G|, B) \otimes P) \times G \to 0. \]

Here $\otimes$ denotes maximal tensor products. Let $\partial_{|G|, B, P}$ denote the corresponding boundary map.

**Corollary 22.** There is a natural transformation
\[ \Psi^B_{*, P} : K_*(C_0(|G|, B) \otimes P) \times G \to \text{K\kern-.1667em K}_*^G(\mathbb{C}, B \otimes P) \]

for pairs of $G$-$C^*$-algebras $B, P$, which makes the following diagram commute:

\[
\begin{array}{ccc}
K_*(C_0(|G|, B) \otimes P) \times G & \xrightarrow{\partial_{|G|, B, P}} & \text{K\kern-.1667em K}_*^G(\mathbb{C}, B \otimes P) \\
\downarrow{\Psi^B_{*, P}} & & \downarrow{\sim} \\
\text{K\kern-.1667em K}_*^G(\mathbb{C}, B \otimes P) & \xrightarrow{\mathbf{r}_G} & RKK^G_*(E\!G; \mathbb{C}, B \otimes P).
\end{array}
\]

The map $\Psi^B_{*, P}$ is an isomorphism if $P = C_0(G/H)$ for a compact open subgroup $H \subseteq G$.

We shall see later that $\Psi^B_{*, P}$ is an isomorphism for the source of the Dirac morphism $P$ provided there is a $G$-compact model for $E\!G$ and $G$ is almost totally disconnected (Theorem 47).

**Proof.** Suppose first that $P = \mathbb{C}$. Let $G$ act on $|G|$ on both sides by multiplication and identify $C_0(|G|, B) \rtimes G \cong B \otimes \mathbb{K}(L^2G) \sim B$ as usual. The map $\phi^G_{*, |G|, G, B}$ of Theorem 21 is the required map for $P = \mathbb{C}$. One checks easily that the map
\[ \phi^G_{*, |G|, G, B} : K_*(C_0(|G|, B) \rtimes G) \to RKK^G_*(E\!G; \mathbb{C}, B) \]

is the canonical isomorphism that we have already used above. To get the map for arbitrary $P$, simply use the embedding $\mathcal{C}^{\tau\theta}_G(|G|, B) \otimes P \to \mathcal{C}^{\tau\theta}_G(|G|, B \otimes P)$. If $P = C_0(G/H)$ for a compact subgroup, then $A \otimes P \rtimes G \sim A \rtimes H$ for all $G$-$C^*$-algebras $A$. If $H \subseteq G$ is open, then there is no difference between $H$-continuity and $G$-continuity. Hence the second assertion follows from the last assertion of Theorem 21. \hfill \Box

### 4. Projective resolutions, Dirac and dual Dirac morphisms

We recall some results from [15] concerning Dirac and dual Dirac morphisms and the Baum-Connes assembly map.

Let $G$ be a locally compact group and $H$ a compact subgroup of $G$. We have the restriction functor $\text{Res}^H_G : \text{K\kern-.1667em K}^G \to \text{K\kern-.1667em K}^H$, whose definition is obvious, and the induction functor $\text{Ind}^G_H : \text{K\kern-.1667em K}^H \to \text{K\kern-.1667em K}^G$, which we have already used above. We call $G$-$C^*$-algebras of the form $\text{Ind}^G_H D$ for compact $H$ compactly induced. Let
be the class of compactly induced $G$-$C^*$-algebras and let $\langle CI \rangle \subseteq KK^G$ be the localising subcategory generated by $CI$. This is the smallest full subcategory of $KK^G$ containing $CI$ that satisfies

1. $\langle CI \rangle$ is triangulated, that is, closed under suspensions and under extensions with an equivariant, completely positive, contractive section;
2. $\langle CI \rangle$ is closed under countable direct sums.

All proper $G$-$C^*$-algebras belong to $\langle CI \rangle$ by [15, Corollary 7.3].

An element $f \in KK^G(A, B)$ is called a weak equivalence if $\text{Res}_H^G(f)$ is invertible in $KK^H(A, B)$ for all compact subgroups $H \subseteq G$. An object $A \in KK^G$ is called weakly contractible if $\text{Res}_H^G(A) \cong 0$ in $KK^H$.

**Definition 23** (see [15, Definition 4.5]). A Dirac morphism for a locally compact group $G$ is a weak equivalence $D \in KK^G(P, \mathbb{C})$ with $P \in \langle CI \rangle$.

Any group $G$ has a Dirac morphism ([15, Proposition 4.6]). It is unique in the sense that if $D \in KK^G(P, \mathbb{C})$ and $D' \in KK^G(P', \mathbb{C})$ are Dirac morphisms, then there is an isomorphism $i: P \to P'$ with $D' \circ i = D$. From now on we fix a Dirac morphism $D \in KK^G(P, \mathbb{C})$. For any $A \in KK^G$, we have $A \otimes P \in \langle CI \rangle$, and $\text{id}_A \otimes D \in KK^G(A \otimes P, A)$ is a weak equivalence. Thus $A \otimes P$ is a $\langle CI \rangle$-simplicial approximation of $A$. The morphism $\text{id}_A \otimes D$ is invertible if and only if $A \in \langle CI \rangle$.

**Theorem 24** (see [15, Theorem 5.2, Proposition 10.2]). The Baum-Connes assembly map with coefficients $A$ is equivalent to the map

$$D_*: K_*(P \otimes A \rtimes_r G) \to K_*(A \rtimes_r G),$$

induced by a Dirac morphism $D \in KK^G(P, \mathbb{C})$.

Moreover, the natural projection induces an isomorphism

$$K_*(P \otimes A \rtimes_r G) \cong K_*(P \otimes A \rtimes_r G).$$

As a consequence, the functor $A \mapsto K^\text{top}(G, A)$ is the localisation (or left derived functor) of both $A \mapsto K(A \rtimes_r G)$ and $A \mapsto K(A \rtimes G)$. This justifies calling the map $D_*: F(A \otimes P) \to F(A)$ for a covariant functor $F$ the assembly map for $F$. For a contravariant functor $F$, we obtain a co-assembly map $F(A) \to F(A \otimes P)$.

We shall be particularly interested in the contravariant functor $A \mapsto KK^G(A, B)$ for $B \in KK^G$. Its localisation $A \mapsto KK^G(A \otimes P, B)$ also gives the morphisms $A \to B$ in the localisation of the category $KK^G$ at the weak equivalences. The assembly map for this functor can be described in more classical terms as follows:

**Theorem 25** (see [15, Theorem 7.1]). Let $EG$ be a locally compact model for the universal proper $G$-space. There is a natural isomorphism $KK^G(P \otimes A, B) \cong RKK^G(EG; A, B)$ making the following diagram commute:

$$\begin{array}{ccc}
KK^G(A \otimes P, B) & \cong & RKK^G(EG; A, B) \\
\downarrow D^* & & \downarrow p^*_G \\
KK^G(A, B). & & \\
\end{array}$$
It is useful to examine this in greater detail, considering the diagrams

\[
\begin{align*}
&\text{Diagram (12)} \\
\begin{array}{ccc}
\text{KK}^G(A, B) & \xrightarrow{D^*} & \text{KK}^G(A \otimes P, B) \\
D_2 & & D_2 \cong \\
\text{KK}^G(A, B \otimes P) & \xrightarrow{D^*} & \text{KK}^G(A \otimes P, B \otimes P),
\end{array}
\end{align*}
\]

\[
\begin{align*}
&\text{Diagram (13)} \\
\begin{array}{ccc}
\text{RKK}^G(\mathcal{E}G; A, B) & \xrightarrow{D^*} & \text{RKK}^G(\mathcal{E}G; A \otimes P, B) \\
\tau_p & & \tau_p \cong \\
\text{RKK}^G(\mathcal{E}G; A, B \otimes P) & \xrightarrow{D^*} & \text{RKK}^G(\mathcal{E}G; A \otimes P, B \otimes P).
\end{array}
\end{align*}
\]

Here \(\tau_p\) denotes the exterior product with \(P\). In addition, the maps \(p^*_\mathcal{E}G\) give a natural transformation between the two diagrams. That is, we get a commuting diagram in the form of a cube, which we do not draw.

The map \(\text{KK}^G(A \otimes P, B \otimes P) \to \text{KK}^G(A \otimes P, B)\) in (12) is an isomorphism by [15, Proposition 4.4]. The composition \(D_2 \circ \tau_p\) in (12) agrees with \(D^*\) by well-known properties of the exterior product. Since \(D_2\) is an isomorphism and the square in (12) evidently commutes, the lower triangle also commutes. The same argument shows that (13) commutes. All the maps in (13) are isomorphisms because \(p^*_\mathcal{E}G(D)\) is invertible by [15, Corollary 7.3]. Theorem 25 together with invertibility of \(p^*_\mathcal{E}G(D)\) implies that the map \(p^*_\mathcal{E}G\) is an isomorphism from \(\text{KK}^G(A \otimes P, B)\) to \(\text{RKK}^G(\mathcal{E}G; A \otimes P, B)\) and, similarly, from \(\text{KK}^G(A \otimes P, B \otimes P)\) to \(\text{RKK}^G(\mathcal{E}G; A \otimes P, B \otimes P)\).

Finally, we observe that the maps in (12) are all isomorphisms for \(A \in \langle CT \rangle\) because then \(\text{id} \otimes D \in \text{KK}^G(A \otimes P, A)\) is invertible. For the same reason, the vertical maps in (12) are isomorphisms if \(B \in \langle CT \rangle\).

**Definition 26** (see [15, Definition 8.1]). If \(\eta \in \text{KK}^G(C, P)\) satisfies \(\eta \circ D = 1_p\), then we call \(\eta\) a dual Dirac morphism and \(\gamma = D \circ \eta\) a \(\gamma\)-element for \(G\).

If a dual Dirac morphism exists, then it is unique. It is shown in [15, Theorem 8.2] that a dual Dirac morphism exists whenever the group has a \(\gamma\)-element according to one of the more traditional definitions (see [12]). Conversely, if \(P\) is a proper \(G\)-\(C^*\)-algebra, then Definition 26 is equivalent to the traditional ones. We show in Section 5.1 that \(P\) can be taken to be a proper \(G\)-\(C^*\)-algebra for many groups; in particular, this holds for discrete groups with finite dimensional \(\mathcal{E}G\).

It follows from [15, Theorem 8.3.5] and the above discussion that the following assertions are all equivalent to the existence of a dual Dirac morphism:

1. the map \(p^*_\mathcal{E}G: \text{KK}^G(A, B) \to \text{RKK}^G(\mathcal{E}G; A, B)\) is an isomorphism for all \(A \in \text{KK}^G, B \in \langle CT \rangle\);
2. the map \(D^*: \text{KK}^G(A, B) \to \text{KK}^G(A \otimes P, B)\) in (12) is an isomorphism for all \(A \in \text{KK}^G, B \in \langle CT \rangle\);
3. the map \(D^*: \text{KK}^G(A, B \otimes P) \to \text{KK}^G(A \otimes P, B \otimes P)\) in (12) is an isomorphism for all \(A, B \in \text{KK}^G\).

Using the characterisation (1) and the identification of the coarse co-assembly map in Theorem 16, we obtain the following theorem:

**Theorem 27.** If \(G\) has a dual Dirac morphism (or merely an approximate dual Dirac morphism), then the equivariant coarse co-assembly map with coefficients

\[
\mu^*_G: K^H_{i+1}(\mathcal{E}G; \langle G, D \rangle) \to KX_H(\langle G, D \rangle)
\]
Lemma 28. A dual Dirac morphism for $G$ exists if and only if the map
\begin{equation}
\rho^*_G: KK^G(\mathbb{C}, P) \to RKK^G(EG; \mathbb{C}, P)
\end{equation}
is surjective. If this is the case, then (14) is an isomorphism.

Proof. Theorem 25 identifies (14) with $D^*: KK^G(\mathbb{C}, P) \to KK^G(P, P)$. This yields the first assertion because any pre-image of $1_P$ is a dual Dirac morphism. Conversely, if $G$ has a dual Dirac morphism then (14) is an isomorphism by characterisation (1) for existence of a dual Dirac morphism above. \qed

Since the category $\langle C \rangle$ to which $P$ belongs is generated by $C$, one might hope that isomorphism of $\mu_{[G], H, D}$ for all $H$, $D$ implies that (14) is an isomorphism, so that $G$ has a dual Dirac morphism by Lemma 28. Unfortunately, we know nothing about the behaviour of $B \mapsto KK^G(\mathbb{C}, B)$ for infinite direct sums, and these are needed to construct general objects of $\langle C \rangle$. In order to construct $P$ without using infinite direct sums, we need finiteness hypotheses on $EG$.

5. More about the Dirac dual Dirac method

This section contains some new results about the Dirac dual Dirac method. We first explain how to construct concrete models for Dirac morphisms. Then we prove that the existence of a dual Dirac morphism is hereditary for extensions. It follows that a locally compact group contains a dual Dirac morphism if and only if its group of connected components has one.

5.1. Detecting Dirac morphisms.

Lemma 29. Let $G$ be a locally compact group, let $A$ be a $G$-$C^*$-algebra and let $d \in KK^G(A, \mathbb{C})$. Then $d$ is a Dirac morphism for $G$ if and only if there are natural isomorphisms $KK^G(A, B) \cong RKK^G(EG; \mathbb{C}, B)$ for all $G$-$C^*$-algebra $B$ that make the following diagram commute:

\begin{equation}
\begin{array}{ccc}
& & \cong \\
KK^G(A, B) & \xrightarrow{\cong} & RKK^G(EG; \mathbb{C}, B) \\
\downarrow{a^*} & & \downarrow{\rho^*_G} \\
KK^G(\mathbb{C}, B). & & \\
\end{array}
\end{equation}

Proof. Theorem 25 shows that a Dirac morphism $D \in KK^G(P, \mathbb{C})$ has these properties. Conversely, the hypotheses on $d$ determine the functor $B \mapsto KK^G(A, B)$ and the natural transformation $a^*: KK^G(\mathbb{C}, B) \to KK^G(A, B)$ uniquely. By the Yoneda Lemma, this implies that $d$ and $D$ are equivalent. \qed

Corollary 30. Suppose that $EG$ can be realised by a proper isometric action of $G$ on a complete Riemannian manifold $M$. Then the class $[D_M] \in KK^G(C_r(M), \mathbb{C})$ constructed by Gennadi Kasparov in [10, Definition 4.2] is a Dirac morphism for $G$. 

is an isomorphism for all compact subgroups $H \subseteq G$ and all $H$-$C^*$-algebras $D$. In particular, $\mu^*_G: K_{*+1}(\mathbb{C}[G]) \to KK^G([G])$ is an isomorphism.

We do not define approximate dual Dirac morphisms here; see [15, Section 8.1] for a discussion. A group that acts properly by isometries on a weakly bolic, weakly geodesic space has an approximate dual Dirac morphism by a result of Gennadi Kasparov and Georges Skandalis ([12]). Thus Theorem 27 implies that the coarse co-assembly map is an isomorphism for such groups.

In the following, we investigate whether the converse of Theorem 27 holds, that is, whether isomorphism of $\mu_{[G], H, D}$ for all $H$, $D$ implies the existence of a dual Dirac morphism for $G$. We will use the following lemma.
Corollary 31. Suppose that $EG$ can be realised by a finite dimensional simplicial complex $X$ on which $G$ acts simplicially. Then the class $[DX] \in KK^G(A_X, C)$ constructed by Gennadi Kasparov and Georges Skandalis in [11, Definition 1.3] is a Dirac morphism for $G$.

Proof. The sufficient condition of Lemma 29 is verified in [10, Theorem 4.9] and [11, Theorem 6.5].

Remark 32. Formally, the source $P$ of the Dirac morphism cannot be graded because the Kasparov category of graded $C^*$-algebras is not triangulated. However, it is permissible to use a graded $G$-$C^*$-algebra that is $KK^G$-equivalent to an ungraded one. It is well-known that $C_r(M)$ in Corollary 30 is $KK^G$-equivalent to $C_0(T^*M)$. A similar ungraded model for $A_X$ is constructed in [11]. Therefore, we may ignore this technical issue.

We call a locally compact group $G$ almost totally disconnected if the connected component of the identity element in $G$ is compact. Of course, totally disconnected groups have this property. A group is almost totally disconnected if and only if there exists a proper simplicial action of $G$ on a simplicial complex. In this case, we can always realise $EG$ by a simplicial action on a simplicial complex. However, $EG$ need not be finite dimensional. This is the only obstruction to applying Corollary 31.

5.2. Dual Dirac morphisms for group extensions. Let $N \to E \to G$ be an extension of locally compact groups. A subgroup $U \subseteq E$ is called $N$-compact if its image in $G$ is compact. Then $U$ is an extension of $N$ by a compact group.

Theorem 33. Suppose that $G$ and all $N$-compact subgroups of $E$ have dual Dirac morphisms. Then $E$ has a dual Dirac morphism as well.

Proof. We assume that $G$ is almost totally disconnected for simplicity. This special case implies the general assertion using Corollary 34 (which only requires the special case). Let $D_G \in KK^G(P_G, C)$ and $D_E \in KK^E(P_E, C)$ be Dirac morphisms for $G$ and $E$ and let $\eta_G \in KK^G(C, P_G)$ be a dual Dirac morphism for $G$.

The homomorphism $\pi : E \to G$ induces a functor $\pi^* : KK^G \to KK^E$ satisfying $\pi^*(C) = C$. The functor $\pi$ maps weak equivalences to weak equivalences, since it maps compact subgroups to compact subgroups. Since weak equivalences between objects of $(C(I))$ are invertible, $\text{id} \otimes \pi^*(D_G) \in KK^E(P_E \otimes \pi^*(P_G), P_E)$ is invertible. We claim that

$$(15) \quad (D_E \otimes \text{id})^* : KK^E(\pi^*(A), P_E) \to KK^E(P_E \otimes \pi^* (A), P_E)$$

is an isomorphism for $A = P_G$. Before we prove this claim, we show how it yields a dual Dirac morphism for $E$. Let $\beta \in KK^E(\pi^*(P_G), P_E)$ be the pre-image of $\text{id} \otimes \pi^*(D_G)$ under the isomorphism (15) and let $\eta_E := \beta \circ \pi^*(\eta_G) \in KK^G(C, P_E)$. Then $\eta_E \circ D_E = \text{id}_{P_E} \otimes \pi^*(D_G)$. Hence $\eta_E \circ D_E$ is an idempotent weak equivalence $P_E \to P_E$ and so must be equal to $1$. Consequently, $\eta_E$ is a dual Dirac morphism for $E$.

The class of $A$ for which (15) is an isomorphism is localising (see page 19). Therefore, it suffices to prove that (15) is an isomorphism for compactly induced $A$. Equivalently, we need only consider the case where $\pi^*(A) = \text{Ind}_G^E(D)$ for some $N$-compact subgroup $U \subseteq E$. Any compact subgroup of $G$ is contained in a compact open one because $G$ is almost totally disconnected. Making $U$ larger, we may assume that $\pi(U) \subseteq G$ is open and compact.

We identify $P_E \otimes \text{Ind}_U^E(D) \cong \text{Ind}_U^E(P_E \otimes D) \cong \text{Ind}_U^E(P_U \otimes D)$ because the restriction of a Dirac morphism for $E$ is a Dirac morphism for $U$ by [15, Proposition 10.1]. We rewrite $KK^E(\text{Ind}_U^E A, B) \cong KK^U(A, \text{Res}_E^U B)$ as in [15, Proposition 3.1].
We have $\text{Res}^U_P E \in \langle CT \rangle$ by [15, Proposition 10.1]. Since $U$ has a dual Dirac morphism by hypothesis, the map

$$D_U^*: \text{KK}^U(D, \text{Res}^U_P E) \to \text{KK}^U(P_U \otimes D, \text{Res}^U_P E)$$

is an isomorphism by [15, Theorem 8.3]. This means that (15) is an isomorphism for compactly induced $A$ and hence for $A = P_G$. □

Corollary 34. Let $G$ be a locally compact group, let $G_0 \subseteq G$ be the connected component of the identity, and let $G/G_0$ be its group of connected components. Then $G$ has a dual Dirac morphism if and only if $G/G_0$ has one.

Thus when constructing dual Dirac morphisms we may always restrict attention to totally disconnected groups.

Proof. All $G_0$-compact subgroups of $G$ are almost connected and hence have a dual Dirac morphism by [10]. The assertion now follows from Theorem 33. □

6. Descent

In this section, we state and prove our Descent Principle. Fix a locally compact group $G$ and a Dirac morphism $D \in \text{KK}^G(P, \mathbb{C})$ for $G$. We first treat the case where $G$ has a $G$-compact model for $EG$. Then we treat the case where $G$ is discrete, does not have too many finite subgroups, and has a finite dimensional model for $EG$.

In both cases, we need some information on $P$, the domain of the Dirac morphism. In the case of $G$-compact $EG$, we obtain this information using generalities on compactly generated triangulated categories. For discrete $G$ with finite dimensional $EG$, we use instead the concrete description of $P$ in Corollary 31. The idea is that we want to use bootstrapping arguments to obtain assertions about $P$ from assertions about much simpler coefficient algebras.

Let $\mathcal{G}$ be some class of objects in $\text{KK}^G$. Write $\langle \mathcal{G} \rangle$ for the localising subcategory and $\langle \mathcal{G} \rangle_{\text{fin}}$ for the thick triangulated subcategory generated by it. The latter is the smallest subcategory of $\text{KK}^G$ containing $\mathcal{G}$ that is closed under suspensions, admissible extensions and retracts. The larger category $\langle \mathcal{G} \rangle$ has the same properties and is also closed under countable direct sums. In our applications, we know that something desirable happens for objects of $\mathcal{G}$, and therefore that this also happens for objects of $\langle \mathcal{G} \rangle_{\text{fin}}$ for purely formal reasons. We would therefore like $\mathcal{G}$, which is our target, but which $a$ priori only lies in $\langle \mathcal{G} \rangle$, to actually lie in $\langle \mathcal{G} \rangle_{\text{fin}}$. For this we need a hypothesis on $E_G$.

6.1. The case of finite classifying space. The following is our Descent Principle for groups $G$ admitting a $G$-compact model for $EG$.

Theorem 35. Let $G$ be a locally compact group with $G$-compact $EG$. Then $G$ has a dual Dirac morphism if and only if the $H$-equivariant coarse co-assembly map

$$\mu_{G,H}^*: \text{K}_{*+1}(\xi^G_H \otimes (|G|) \rtimes H) \to \text{KX}_{H}^*(|G|)$$

is an isomorphism for all smooth compact subgroups $H \subseteq G$.

Corollary 36. If $G$ is a torsion free discrete group with finite classifying space $BG$, then $G$ has a dual Dirac morphism if and only if the coarse co-assembly map

$$\mu_{G}^*: \text{K}_{*+1}(\xi^{\text{co}}^*(|G|)) \to \text{KX}^*(|G|)$$

is an isomorphism. In particular, the existence of a dual Dirac morphism for $G$ is a coarse property, that is, it only depends on the coarse space $|G|$. 
The proof of Theorem 35 requires some preparation. Assume first that $G$ is almost totally disconnected. In this case, any compact subgroup is contained in a compact open subgroup. We let

$$\mathcal{C}I_0 := \{ C_0(G/H) \mid H \subseteq G \text{ compact open subgroup} \}. $$

If $G$ is arbitrary, we use instead the larger class of smooth compact subgroups. The following definition is equivalent to the one in [15]. We call a compact subgroup $H \subseteq G$ smooth if there are an open almost connected subgroup $U \subseteq G$ and a compact normal subgroup $N$ in $U$ such that $U/N$ is a Lie group and $N \subseteq H \subseteq U$. Then $G/H$ is a smooth manifold, being a disjoint union of copies of the homogeneous space $(U/N)/(H/N)$. We let

$$\mathcal{C}I_1 := \{ C_0(G/H) \mid H \subseteq G \text{ smooth compact subgroup} \}. $$

If $G$ is almost totally disconnected, then $\mathcal{C}I_0 \subseteq \mathcal{C}I_1$. If $G$ is totally disconnected, then $\mathcal{C}I_0 = \mathcal{C}I_1$.

We have $P \in (\mathcal{C}I_1)$ for all $G$ by [15, Proposition 9.2]. If $G$ is almost totally disconnected, then $P \in (\mathcal{C}I_0)$. We are going to find a criterion for $P$ to belong to $(\mathcal{C}I_1)_{\text{fin}}$ or $(\mathcal{C}I_0)_{\text{fin}}$ that uses abstract results on triangulated categories. These are related to the Brown Representability Theorem with cardinality restrictions [15, Theorem 6.1].

We call $A \in KK^G$ compact if $KK^G(A, B)$ is countable for all $B \in KK^G$ and, in addition, $B \mapsto KK^G(A, B)$ commutes with countable direct sums.

**Lemma 37.** Let $\mathcal{G}$ be a countable set of compact objects of a triangulated category $\mathcal{T}$ that has countable direct sums. Let $A \in (\mathcal{G})$. Then $A$ belongs to $(\mathcal{G})_{\text{fin}}$ if and only if $A$ is compact.

**Proof.** Objects of $(\mathcal{G})_{\text{fin}}$ are compact because the compact objects in a triangulated category form a thick triangulated subcategory. Up to the fact that we only have countable direct sums, the converse is a result of Amnon Neeman [16, Lemma 2.2]. We have to check that his proof only uses countable direct sums under our cardinality hypotheses. This is routine, so that we omit the verification. We mention that the critical points of the argument are explained in greater detail in [17].  

**Lemma 38.** Let $G$ be a second countable locally compact group. If $H \subseteq G$ is smooth, then $C_0(G/H)$ is compact. Up to conjugacy there are only countably many smooth compact subgroups.

**Proof.** Let $N \subseteq H \subseteq U \subseteq G$ be such that $U$ is open and almost connected and $U/N$ is a Lie group. Let $K \subseteq U$ be a maximal compact subgroup containing $H$. [15, Corollary 3.2] implies $KK^G_K(C_0(G/H), B) \cong KK^G_K(C_0(U/H), \text{Res}^G_K B)$ because $C_0(G/H) \cong \text{Ind}^G_K C_0(U/H) \cong \text{Ind}^G_K C_0(K/H)$. We want to obtain a $K$-equivariant isomorphism $C_0(U/H) \cong C_0(U/K) \times C(K/H)$. Then the compactness of $C_0(U/H)$ follows immediately from the Poincaré duality isomorphism (see [10])

$$KK^G_K(C_0(U/H), B) \cong KK^G_K(\mathbb{C}, C_0(U/K) \otimes C_r(K/H) \otimes B). $$

We prove $U/H \cong U/K \times K/H$ following Herbert Abels ([1]). The Lie algebra $\mathfrak{k}$ of $K/N$ acts on the Lie algebra of $U/N$ by conjugation. We split the latter into invariant subspaces $\mathfrak{k} \oplus \bigoplus_{i=1}^n \mathfrak{p}_i$. Abels observes that the map

$$\prod \mathfrak{p}_i \times K \to U, \quad ((x_i), k) \mapsto \prod \exp(x_i) \cdot k$$

is a $K$-equivariant diffeomorphism. This yields the desired $K$-equivariant diffeomorphism $U/H \cong \prod \mathfrak{p}_i \times K/H \cong U/K \times K/H$.  

\[\square\]
Proposition 39. Let \( G \) be a second countable locally compact group and let \( P \) be the source of the Dirac morphism. Then \( P \in \langle CI_1 \rangle_{\text{fin}} \) if and only if the functor \( B \mapsto \text{RKK}^G_0(\mathcal{E}G; \mathbb{C}, B) \) commutes with countable direct sums and only produces countable groups for separable \( B \). If \( G \) is almost totally disconnected, this is also equivalent to \( P \in \langle CI_0 \rangle_{\text{fin}} \).

A sufficient condition for this is the existence of a \( G \)-compact model for \( \mathcal{E}G \).

Proof. Lemma 38 shows that Lemma 37 applies to \( CI_1 \), and to \( CI_0 \) if \( G \) is almost totally disconnected. We have \( P \in \langle CI_1 \rangle \) by [15, Proposition 9.2], and \( P \in \langle CI_0 \rangle \) if \( G \) is almost totally disconnected. Theorem 25 and Lemma 20 yield

\[
\text{KH}^G_0(P, B) \cong \text{RKK}^G_0(\mathcal{E}G; \mathbb{C}, B) \cong K_0(C_0(\mathcal{E}G); B \times G).
\]

Thus \( P \) is a compact object of \( \text{KK}^G \) if and only if \( \text{RKK}^G_0(\mathcal{E}G; \mathbb{C}, B) \) has the properties required in the statement of the theorem. Moreover, if we can find a \( G \)-compact model for \( \mathcal{E}G \), then \( C_0(\mathcal{E}G) \) is a \( C^* \)-algebra. In this case, the isomorphism (16) implies immediately that \( P \) is compact by well-known properties of K-theory. \( \square \)

We can now prove Theorem 35. We remark that the argument only uses the potentially weaker hypothesis \( P \in \langle CI_1 \rangle_{\text{fin}} \). However, we know no example of a group satisfying the latter condition but without a \( G \)-compact model for \( \mathcal{E}G \).

Proof of Theorem 35. By Theorem 16, the assumption on \( \mu^*_{\langle GI \rangle, H} \) implies that

\[
p^*_\mathcal{E}G : \text{KR}^G(\mathbb{C}, B) \to \text{RKK}^G(\mathcal{E}G; \mathbb{C}, B)
\]

is an isomorphism for all \( B \in CI_1 \). The class of objects \( B \) for which (17) is an isomorphism is a thick triangulated subcategory of \( \text{KK}^G \). Therefore, it contains \( \langle CI_1 \rangle_{\text{fin}} \). By Proposition 39, it contains \( P \). Thus a dual Dirac morphism exists by Lemma 28. The converse assertion is Theorem 27. \( \square \)

6.2. Discrete groups with finite dimensional classifying space. We now pass to the case when \( G \) has a finite dimensional model for \( \mathcal{E}G \). We also assume \( G \) to be discrete. Therefore, \( \mathcal{E}G \) can be realised by a simplicial complex. We assume that (1) we can choose \( \mathcal{E}G \) finite dimensional; (2) there are only finitely many conjugacy classes of finite subgroups in \( G \).

Of course, the second condition is trivially satisfied for torsion free groups. It is known that there exist groups with finite dimensional \( \mathcal{E}G \) that violate (2).

As in Section 6.1, what we really need is a condition on the domain of the Dirac morphism \( P \). Let \( CI_2 \) be the set of \( C^* \)-algebras of the form \( C_0(\mathbb{N} \times G/H) \) as \( H \) ranges over the finite subgroups of \( G \). We let \( \langle CI_2 \rangle_{\text{fin}} \) be the thick triangulated subcategory generated by \( CI_2 \). Since \( C_0(G/H) \) is a retract of \( C_0(\mathbb{N} \times G/H) \), this category contains \( \langle CI_0 \rangle_{\text{fin}} \). The same argument as in the proof of Theorem 35 yields the following lemma:

Lemma 40. Suppose that \( P \in \langle CI_2 \rangle_{\text{fin}} \). Then \( G \) has a dual Dirac morphism if and only if the \( H \)-equivariant coarse co-assembly map with coefficients \( C_0(\mathbb{N}) \) is an isomorphism for all finite subgroups \( H \leq G \).

Theorem 41. Let \( G \) be a discrete group that satisfies the conditions (1) and (2) above. Then \( P \in \langle CI_2 \rangle_{\text{fin}} \). Hence \( G \) has a dual Dirac morphism if and only if the \( H \)-equivariant coarse co-assembly map with coefficients \( C_0(\mathbb{N}) \)

\[
\mu^*_{\langle GI \rangle, H, C_0(\mathbb{N})} : K_{*+1}(C_{\text{red}}^*(\mathcal{E}G; C_0(\mathbb{N}) \times H)) \to \text{KX}_H(\mathcal{E}G; C_0(\mathbb{N}))
\]

is an isomorphism for all finite subgroups \( H \leq G \).
We use constructions of Gennadi Kasparov and Georges Skandalis in [11]. Choose a finite dimensional simplicial model for $E_G$ as above. By a barycentric subdivision, we can arrange that the action of $G$ is “type preserving”. Hence $G$ acts on the $C^*$-algebra $A_{E_G}$ defined in [11]. As we observed in Corollary 31, $A_{E_G}$ is a model for $P$. Thus we have to prove that $A_{E_G} \in \langle CI_2 \rangle_{\text{fin}}$. In different notation, this is already shown in [11]. We sketch the argument. The skeletal filtration of $E_G$ gives rise to a filtration of $A_{E_G}$ by ideals
\[0 = A_{E_G}^{(-1)} \subseteq A_{E_G}^{(0)} \subseteq A_{E_G}^{(1)} \subseteq \cdots \subseteq A_{E_G}^{(n)} = A_{E_G},\]
where $n$ is the dimension of $E_G$. The resulting extensions
\[0 \to A_{E_G}^{(k-1)} \to A_{E_G}^{(k)} \to A_{E_G}^{(k)}/A_{E_G}^{(k-1)} \to 0\]
have $G$-equivariant, completely positive, contractive sections because all occurring $G$-$C^*$-algebras are proper and nuclear. Hence these extensions are admissible. By induction on $k$, it follows that $A_{E_G}^{(k)}$ belongs to the triangulated subcategory of $KK^G$ generated by the subquotients $A_{E_G}^{(j)}/A_{E_G}^{(j-1)}$ for all $j$. Thus it remains to prove that these subquotients belong to $\langle CI_2 \rangle_{\text{fin}}$. They are $KK^G$-equivalent to $C_0(E_G^{(j)})$, where $E_G^{(j)}$ denotes the set of $j$-cells of $E_G$, viewed as a discrete $G$-space. Thus $E_G^{(j)}$ is a disjoint union of homogeneous spaces $G/H$ for finite subgroups $H \subseteq G$. By assumption, there are at most finitely many non-isomorphic proper homogeneous spaces $G/H$. Hence we can write $E_G^{(j)}$ as a finite disjoint union of spaces of the form $G/H \times I$, where $I$ is some countable set. This implies $C_0(E_G^{(j)}) \in \langle CI_2 \rangle_{\text{fin}}$ as desired.

**Corollary 42.** If $G$ is a torsion free discrete group with finite dimensional classifying space $E_G$, then $G$ has a dual Dirac morphism if and only if

$$\mu^*_{[G],C_0(\mathbb{N})}: K_{*+1}(\mathcal{C}^0([G],C_0(\mathbb{N}))) \to KK^*(|G|,C_0(\mathbb{N}))$$

is an isomorphism. In particular, the existence of a dual Dirac morphism for $G$ is a coarse property, that is, it only depends on the coarse space $[G]$.

7. **Geometric K-theory**

In this section we define the closely related notions of boundary class and regularising class in $KK^G(E_G;\mathbb{C},\mathbb{C})$ and observe that such classes give rise to homotopy invariant higher signatures. The boundary classes arise from the equivariant topological K-theory of the stable Higson corona. The regularising classes arise from the Dirac dual Dirac method. For totally disconnected groups with $G$-compact $E_G$, boundary classes and regularising classes coincide. We will also use these ideas to construct dual Dirac morphisms and boundary classes from certain geometric situations.

7.1. **Boundary classes and regularising classes.**
Proposition 43. Let $D \in \text{KK}^G(P, \mathbb{C})$ be a Dirac morphism for $G$. Then the following diagram commutes

\[
\begin{array}{cccc}
\text{KK}^G(\mathbb{C}, P) & \xrightarrow{p_{EG}^*} & \text{RKK}^G(\mathbb{E}G; \mathbb{C}, P) & \xrightarrow{D_*} & \text{RKK}^G(\mathbb{E}G; \mathbb{C}, \mathbb{C}) \\
\Psi_* & \xrightarrow{\cong} & & & \xrightarrow{\cong} \\
K^\text{top}_{*+1}(G, \xi^*G(|G|)) & \xrightarrow{\partial_{|G|, P}} & K_*(C_0(|\mathbb{E}G|, P) \times G) & \xrightarrow{D_*} & K_*(C_0(|\mathbb{E}G|) \times G) \\
\end{array}
\]

where $\Psi_*$ is as in Corollary 22 and $\mu^*_G$ is the $G$-equivariant coarse co-assembly map for $|G|$. 

Proof. The left square diagram is the special case of Corollary 22 with $B = \mathbb{C}$ and $P = P$, together with our identification $K^\text{top}_*(G, A) = K_*(A \otimes P) \times G$). The rest of the diagram evidently commutes, given our definition of the coarse co-assembly map. The vertical isomorphisms in the right square are contained in Lemma 20. The other isomorphisms follow from the invertibility of $p_{EG}^*$, see Section 4. □

The above diagram points to a particular subgroup of $\text{RKK}^G(\mathbb{E}G; \mathbb{C}, \mathbb{C})$, namely the range of the composition $D_* \circ p_{EG}^* = p_{EG}^* \circ D_* : \text{KK}^G(\mathbb{C}, P) \to \text{RKK}^G(\mathbb{E}G; \mathbb{C}, \mathbb{C})$. It contains the classes that come from $K^\text{top}_{*+1}(G, \xi^*G(|G|))$ via $\mu^*_G$. Accordingly we make the following definition.

Definition 44. Let $\alpha \in \text{RKK}^G(\mathbb{E}G; \mathbb{C}, \mathbb{C})$. We call $\alpha$ a **boundary class** if it belongs to the range of the map $K^\text{top}_{*+1}(G, \xi^*G(|G|)) \to \text{RKK}^G(\mathbb{E}G; \mathbb{C}, \mathbb{C})$ in Proposition 43. We call $\alpha$ **regularising** if it belongs to the range of the map $\text{KK}^G_*(\mathbb{C}, P) \to \text{RKK}^G(\mathbb{E}G; \mathbb{C}, \mathbb{C})$ of the same Proposition.

It follows immediately from Proposition 43 that boundary classes are regularising and that regularising classes belong to the range of $p_{EG}^* : \text{KK}^G_*(\mathbb{C}, \mathbb{C}) \to \text{RKK}^G(\mathbb{E}G; \mathbb{C}, \mathbb{C})$ and hence yield homotopy invariant higher signatures. The group $K^\text{top}_{*+1}(G, \xi^*G(|G|))$ should be thought of as a sort of geometric $K$-theory group for $G$. Given its importance, we state this as a proposition.

Proposition 45. Regularising classes, whence also boundary classes, are in the range of $p_{EG}^*$ and thus yield homotopy invariant higher signatures.

We also have the following.

Proposition 46. The boundary classes and the regularising classes form graded ideals in $\text{RKK}^G_*(\mathbb{E}G; \mathbb{C}, \mathbb{C})$.

Thus, if $\alpha, \beta \in \text{RKK}^G_*(\mathbb{E}G; \mathbb{C}, \mathbb{C})$ and $\alpha$ is regularising, then $\alpha \beta$ is also regularising and hence homotopy invariant. This is the reason for calling these classes regularising.

Proof. We return to our discussion of the diagrams (12) and (13). We have already observed that the maps

\[
\begin{align*}
\text{KK}^G(P, P) & \xrightarrow{p_{EG}} \text{RKK}^G(\mathbb{E}G; P, P) \\
\end{align*}
\]

are isomorphisms. Since they are graded ring homomorphisms, $\text{KK}^G(P, P)$ and $\text{RKK}^G(\mathbb{E}G; \mathbb{C}, \mathbb{C})$ are isomorphic as graded rings. It is well-known that the graded

...
Then for every $B$ of Corollary 22 is an isomorphism for which the diagram of Theorem 35. It implies in particular that for groups $I$ algebra, so that elements of Theorem 47.

Thus the RKK for $\Psi^*$, the other three maps are evidently module homomorphisms. This implies that the images of $K^{\text{top}}(G, \mathcal{C}^{\text{red}}(G))$ and $K^{G}(\mathbb{C}, P)$ in $RKK^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ are ideals.

We are compelled to conclude on purely formal grounds that regularising classes and boundary classes are rather special even among classes in the range of $p^*_G$, and in particular amongst those classes yielding homotopy invariant higher signatures. The unit is always among the latter. On the other hand, if it is regularising or a boundary class, then every class is such because these form ideals. In this case $G$ actually has a dual Dirac morphism by Lemma 28.

We also note that if we use the traditional definition of $K^{\text{top}}(G)$, then the action of $RKK^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ on topological K-theory discussed above may be described rather easily. Thus the $RKK^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$-module structure of $K^{\text{top}}(G, \mathcal{C}^{\text{red}}(G))$ is not special to our setup. However, the construction of the isomorphism $\Psi_*$ identifying regularising and boundary classes is difficult with the traditional definition.

If $X \in (\mathcal{C}I)$, in particular if $X$ is proper, then $1_X \otimes 0$ is invertible in $K^{G}(X \otimes P, X)$. Given $A, B \in KK^G$, $X \in (\mathcal{C}I)$, and $\alpha \in KK^G(A, X)$, $\beta \in KK^G(X, B)$, we define $\beta \bullet \alpha \in KK^G(A, B \otimes P)$ as the composition $(1_P \otimes \beta \circ (1_X \otimes D)^{-1} \circ \alpha \in KK^G(A, P \otimes B)$. Naturality of exterior products implies that $D_\ast (\beta \bullet \alpha) = \beta \circ \alpha$. Hence $p^*_G(\beta \circ \alpha) = p^*_G \circ D_\ast (\beta \bullet \alpha) = D_\ast \circ p^*_G (\beta \bullet \alpha)$ is regularising. Conversely, by the definition of regularising, any regularising class has the form $D_\ast (\alpha)$ for some $\alpha \in RKK^G(\mathcal{C}, P)$.

To summarize, an element of $RKK^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ is regularising if and only if it can be factorised through an object of $(\mathcal{C}I)$. For many groups, $P$ is a proper $G$-$C^\ast$-algebra, so that elements of $I$ even factor through a proper $G$-$C^\ast$-algebra.

The following is a central result of this paper. It may be regarded as a refinement of Theorem 35. It implies in particular that for groups $G$ with $G$-compact $\mathcal{E}G$, regularising classes and boundary classes are the same

**Theorem 47.** Let $G$ be an almost totally disconnected group with $G$-compact $\mathcal{E}G$.

Then for every $B \in KK^G$, the map

$$\Psi^B_\ast : K^{\text{top}}_{i+1}(G, \mathcal{C}^{\text{red}}(|G|), B) \to KK^G_{i}(\mathbb{C}, B \otimes P)$$

of Corollary 22 is an isomorphism for which the diagram

$$\begin{array}{ccc}
K^{\text{top}}_{i+1}(G, \mathcal{C}^{\text{red}}(|G|), B) & \xrightarrow{\mu^{|G|, G, B}} & KK^G_{i}(|G|, B) \\
\downarrow{\Psi^B_\ast} & & \downarrow{\approx} \\
KK^G_{i}(G, B \otimes P) & \xrightarrow{p^*_G} & RKK^G_{i}(\mathcal{E}G; \mathbb{C}, B \otimes P)
\end{array}$$

commutes. In particular, $K^{\text{top}}_{i+1}(G, \mathcal{C}^{\text{red}}(|G|))$ is naturally isomorphic to $KK^G_{i}(\mathbb{C}, P)$.

Moreover, $KK^G_{i}(\mathbb{C}, B \otimes P) \cong KK^G_{i}(\mathbb{C}, B)$ if $B$ is a proper $G$-$C^\ast$-algebra or, more generally, if $B \in (\mathcal{C}I)$.

**Proof.** Corollary 22 with $P = P$ yields the desired diagram. The map $\Psi^B_\ast$ is an isomorphism for $P \in \mathcal{C}I_0$. The class of $P$ for which this is the case is thick and triangulated. Hence it contains $P$ by Proposition 39. The isomorphism statement follows. Proper $G$-$C^\ast$-algebras belong to $(\mathcal{C}I)$ by [15, Corollary 7.3]. We have $B \otimes P \cong B$ if and only if $B \in (\mathcal{C}I)$ by [15, Theorem 4.7].
We summarize our main result as follows.

**Corollary 48.** Let $G$ be an almost totally disconnected group with $G$-compact $\mathcal{E}G$ and let $a \in \text{RKK}^G_0(\mathbb{E}G; \mathbb{C}, \mathbb{C})$. Then the following are equivalent:

1. $a$ can be factorised through an object of $\langle \mathcal{C} \mathcal{T} \rangle$;
2. $a$ is regularising;
3. $a$ is a boundary class.

### 7.2. Dual Dirac morphisms from compactifications

We are going to construct dual Dirac morphisms from contractible admissible compactifications, thereby strengthening a result of Nigel Higson ([8]). We assume throughout that there is a $G$-compact model for $\mathcal{E}G$. Hence there is no difference between $C_0(\mathcal{E}G)$ and $C_0([\mathcal{E}G])$.

Recall that a metrisable compactification $Z \supseteq [\mathcal{E}G]$ is called admissible if all (scalar valued) continuous functions on $Z$ have vanishing variation. If this is the case for scalar valued functions, it automatically holds for operator valued functions because $C(Z, D) \cong C(Z) \otimes D$. An equivariant compactification of $[\mathcal{E}G]$ is a compactification together with a $G$-action that extends the given action on $[\mathcal{E}G]$.

**Theorem 49.** Let $G$ be a locally compact group with a $G$-compact model for $\mathcal{E}G$ and let $[\mathcal{E}G] \subseteq Z$ be an admissible equivariant compactification. If $Z$ is $H$-equivariantly contractible for all compact subgroups $H \subseteq G$, then $G$ has a dual Dirac morphism.

**Proof.** Since $Z$ is admissible, we have an embedding $\mathfrak{B}_G^{\mathcal{E}G}(Z) \subseteq \mathfrak{c}_G^{\mathcal{E}G}(\mathcal{E}G)$. Let $\partial Z := Z \setminus \mathcal{E}G$ be the boundary of the compactification. Identifying $\mathfrak{B}_G^{\mathcal{E}G}(\partial Z) \cong \mathfrak{B}_G^{\mathcal{E}G}(Z)/C_0(\mathcal{E}G, \mathbb{K}_G)$, we obtain a morphism of extensions

$$
\begin{array}{ccc}
0 & \longrightarrow & C_0([\mathcal{E}G], \mathbb{K}_G) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathfrak{B}_G^{\mathcal{E}G}(Z) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathfrak{B}_G^{\mathcal{E}G}(\partial Z) & \longrightarrow 0
\end{array}
$$

Let $H$ be a compact subgroup. Since $Z$ is compact, $\mathfrak{B}_G(Z)$ is identical with $C(Z, \mathbb{K})$. In particular, since $Z$ is $H$-equivariantly contractible by hypothesis, $\mathfrak{B}_G(Z)$ is $H$-equivariantly homotopy equivalent to $\mathcal{C}$. Hence $\mathfrak{B}_G^{\mathcal{E}G}(Z)$ has vanishing $H$-equivariant $K$-theory. This implies $K_0^\text{top}(G, \mathfrak{B}_G^{\mathcal{E}G}(Z)) = 0$ by [5], so that the connecting map

$$K_0^\text{top}(G, \mathfrak{B}_G^{\mathcal{E}G}(\partial Z)) \rightarrow K_0^\text{top}(G, C_0(\mathcal{E}G)) \cong K_*^0(C_0(\mathcal{E}G) \times G)$$

is an isomorphism. This in turn implies that the connecting map

$$K_0^\text{top}(G, e_G^{\mathcal{E}G}(\mathcal{E}G)) \rightarrow K_0^\text{top}(G, C_0(\mathcal{E}G))$$

is surjective. Thus we can lift $1 \in \text{RKK}_0^G(\mathbb{E}G; \mathbb{C}, \mathbb{C}) \cong K_0(\mathcal{E}G) \times G)$ to

$$\alpha \in K_0^\text{top}(G, e_G^{\mathcal{E}G}(\mathcal{E}G)) \cong K_0^\text{top}(G, e_G^{\mathcal{E}G}(|\mathcal{E}G|)) = K_0^\text{top}(G, e_G^{\mathcal{E}G}(|\mathcal{E}G|)).$$

Then $\Psi_* (\alpha) \in \text{KK}_0^G(\mathbb{C}, P)$ is the desired dual Dirac morphism. \(\square\)

Theorem 49 applies to the Gromov boundary for a hyperbolic group. We conclude that a dual Dirac morphism must come from the topological $K$-theory of the Gromov boundary. In fact, the above argument shows that all boundary classes for a hyperbolic group come from the Gromov boundary.
8. LIPSCHITZ AND PROPER LIPSCHITZ K-THEORY CLASSES

The idea of [3] is to prove homotopy invariance for a particular higher signature by showing that it arises from a specific geometric construction. This motivated us to formulate the notion of boundary classes and prove Theorem 21, which is a direct analogue of some constructions in [3]. In this section, we show how Lipschitz cohomology classes and boundary classes are related. In the case of proper Lipschitz classes, the relationship is exact: every proper Lipschitz class is a boundary class. The proof is a consequence of Corollary 48: proper Lipschitz classes are regularising.

We also use the stable Higson corona construction to approach non proper Lipschitz classes, simplifying the geometric part of the proof of homotopy invariance of Gelfand-Fuchs cohomology classes, as well as to describe a more general geometric situation in which the essential idea of [3] can be made to work.

8.1. Construction of higher signatures. Let $X$ be a locally compact $G$-space, and suppose that $G$ acts by translations with respect to some coarse structure on $X$. From Theorem 21 we have a map

$$K_{*+1}(c^{*\mathcal{O}}(X)) \xrightarrow{\psi^{G,X}} KK^{G}_{*}(\mathbb{C}, C_{0}(X)).$$

Hence given any nonzero class in the K-theory of the stable Higson corona of $X$, we may push it forward and obtain a class $\alpha \in KK^{G}_{*}(\mathbb{C}, C_{0}(X))$. This class in turn induces a map $K_{*}(\text{max} C_{*}(G)) \to K_{*+N}(\text{max} C_{0}(X) \rtimes G)$. If $b: K_{*+N}(\text{max} C_{0}(X) \rtimes G) \to \mathbb{R}$ is a linear functional, then we may then construct a higher signature for $G$ (see the Introduction for a brief discussion of higher signatures), by the composition

$$K^{\text{top}}_{*}(G) \to K_{*}(\text{max} C_{*}G) \xrightarrow{\alpha} K_{*+N}(\text{max} C_{0}(X) \rtimes G) \xrightarrow{b} \mathbb{R}, \tag{18}$$

where the first map is the analytic assembly map. This higher signature is homotopy invariant by construction since it factorises through the analytic assembly map.

Such a ‘geometric’ higher signature therefore has two components: the construction of a nonzero class in the K-theory of the Higson corona of a $G$-space $X$, and the construction of a linear functional $b$ as above. The former is obviously related to coarse geometry; the latter is, however, not. If $X$ is a proper $G$-space, then topology provides us with many classes in $KK^{G}_{*}(\text{max} C_{0}(X), \mathbb{C})$, whence linear functionals $b$ above, and the nontrivial part of the above procedure is finding nonzero classes in the K-theory of the stable Higson corona of $X$. It is possible to find such classes if the coarse co-assembly map for $X$ is an isomorphism, for then the K-theory of the stable Higson corona is the same as the coarse K-theory of $X$.

If $X$ is proper, then the higher signature constructed above corresponds to a regularising class in $\text{RK}^{G}(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. Indeed, assuming that our linear functional $b$ comes from a class $\beta \in KK^{G}_{*}(\text{max} C_{0}(X), \mathbb{C})$, we can form the class $\beta \bullet \alpha \in KK^{G}_{*}(\mathbb{C}, \mathbb{C})$.

We may then apply the isomorphism $D_{*} \circ p^{G}_{\mathcal{E}G}: KK^{G}_{*}(\mathbb{C}, \mathbb{P}) \xrightarrow{\cong} \text{RK}^{G}_{*}(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ to obtain a class in $\text{RK}^{G}_{*}(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. The latter class is by construction regularising.

The higher signature corresponding to it (see Introduction) is the same as that constructed by the recipe (18). If $G$ has a $G$-compact model for $\mathcal{E}G$, we conclude that the higher signatures constructed by (18) where $X$ is a proper $G$-space correspond in fact to boundary classes.

In this paper we discuss three natural examples where the above situation arises for an action of $G$ on a locally compact space $X$: when $X$ has the coarse structure coming from an admissible compactification; when $X$ admits commuting actions of $G$ and another group $H$, whence admits a coarse structure inherited from the action of $H$; and when $X$ admits a proper map to Euclidean space satisfying a certain displacement condition, whence admits a pulled-back coarse structure.
The first situation we have already implicitly discussed whilst proving Theorem 49. The second will appear in the context of Gelfand-Fuchs classes below. The third amounts to a reformulation of the ideas of Connes, Gromov and Moscovici in [3], and we begin with it.

8.2. Pulled-back coarse structures and Lipschitz classes. Let $X$ be a $G$-space, let $Y$ be a coarse space and let $\alpha: X \to Y$ be a proper continuous map. We pull back the coarse structure on $Y$ to a coarse structure on $X$, so that $E \subseteq X \times X$ is an entourage if and only if $\alpha_*(E) \subseteq Y \times Y$ is one. Since $\alpha$ is proper and continuous, this coarse structure is compatible with the topology on $X$. The group $G$ acts by translations with respect to this coarse structure if and only if $\alpha$ satisfies the following displacement condition: for any compact subset $K \subseteq G$,

$$\{(\alpha(gx), \alpha(x)) \mid x \in X, \ g \in K\} \subseteq Y \times Y$$

is an entourage. The map $\alpha$ becomes a coarse map. Hence we obtain canonical maps

$$K_{*+1}(c^{e\theta}(Y)) \xrightarrow{\alpha^*} K_{*+1}(c^{e\theta}(X)) \xrightarrow{\psi^{X,G}_{*}} KK^G_*(\mathbb{C}, C_0(X)).$$

where $\psi^{G,X}_{*}$ is as in Theorem 21.

In particular, we can push forward any class in the group $K_{*+1}(c^{e\theta}(Y))$ to a class $\alpha \in KK^G_*(\mathbb{C}, C_0(X))$ and proceed as in Section 8.1.

The constructions of [3, Section I.10] involve the specific choice of $Y = \mathbb{R}^N$ with the coarse structure from the Euclidean metric. The coarse co-assembly map is an isomorphism for $\mathbb{R}^N$ because $\mathbb{R}^N$ is scalable. Moreover, $\mathbb{R}^N$ is uniformly contractible and has bounded geometry. Hence the map $j: \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ is a coarse homotopy equivalence. We have, therefore canonical isomorphisms

$$K_{*+1}(c^{e\theta}(\mathbb{R}^N)) \cong K_*(C_0(\mathbb{R}^N)) \cong K_{*+N}(\mathbb{C})$$

and thus the $K$-theory of $c^{e\theta}(\mathbb{R}^N)$ is infinite cyclic. We denote the generator by $[\partial\mathbb{R}^N] \in K_{1-N}(c^{e\theta}(\mathbb{R}^N))$. Of course, this is nothing but the usual dual Dirac morphism for the locally compact group $\mathbb{R}^N$. Thus we obtain a class

$$[\alpha] := \psi(\alpha^*[\partial\mathbb{R}^N]) \in KK^G_{*N}(\mathbb{C}, C_0(X))$$

for any map $\alpha: X \to \mathbb{R}^N$ that satisfies the displacement condition above.

There is a slightly more general setup, also contained in [3], where we replace a map to $\mathbb{R}^N$ by a section of an $N$-dimensional vector bundle, as follows. Let $P$ be a $G$-space and let $\pi: X \to P$ be a $G$-equivariant Spin($N$)-principal bundle. That is, the action of $G$ on $X$ commutes with the action of Spin($N$) and $G$ is $G$-equivariant.

Let $T := X \times_{\text{Spin}(N)} \mathbb{R}^N$ be the associated vector bundle over $P$. It carries a $G$-invariant Euclidean metric and spin structure. If $\alpha: P \to T$ is a section, then we can define a map $\alpha': X \to \mathbb{R}^N$ by sending $x \in X$ to the coordinates of $\alpha\pi(x)$ in the orthogonal frame described by $x$. This map is Spin($N$)-equivariant with respect to the standard action of Spin($N$) on $\mathbb{R}^N$. Conversely, any Spin($N$)-equivariant map $\alpha': X \to \mathbb{R}^N$ arises in this fashion. Since Spin($N$) is compact, the map $\alpha'$ is proper if and only if $p \mapsto \|\alpha(p)\|$ is a proper function on $P$.

As above, we can use a Spin($N$)-equivariant proper continuous map $\alpha': X \to \mathbb{R}^N$ to pull back the coarse structure of $\mathbb{R}^N$ to $X$. Then Spin($N$) acts by isometries. The group $G$ acts by translations if and only if the section $\alpha: P \to T$ associated to $\alpha'$ satisfies the displacement condition that

$$\sup\{\|g\alpha(g^{-1}x) - \alpha(x)\| \mid x \in X, \ g \in K\}$$

be bounded for all compact subsets $K \subseteq G$. Suppose that $\alpha$ satisfies this. Then we are in the situation of Theorem 21 with $H = \text{Spin}(N)$. Since $H$ acts freely,
\[ C_0(X) \times \text{Spin}(N) \text{ is Morita-Rieffel equivalent to } C_0(P), \text{ and this Morita-Rieffel equivalence is } G\text{-equivariant. We obtain canonical maps} \]

\[
K^\text{Spin}(N)_{s+1}(\mathbb{R}^N) \xrightarrow{\psi} K^\text{Spin}(N)_{s+1}(\mathbb{R}^N) \rightarrow KK^G_*(\mathbb{C}, C_0(X) \times \text{Spin}(N)) \cong KK^G_*(\mathbb{C}, C_0(P)).
\]

The space \( \mathbb{R}^N \) is Spin(\(N\))-equivariantly scalable, so that the Spin(\(N\))-equivariant coarse co-assembly map for \( \mathbb{R}^N \) is an isomorphism. Moreover, the action of Spin(\(N\)) on \( \mathbb{R}^N \) is spin by definition. Hence

\[
K^\text{Spin}(N)_{s+1}(\mathbb{R}^N) \cong K^\text{Spin}(N)_{s}(C_0(\mathbb{R}^N)) \cong K^\text{Spin}(N)_{s}(\mathbb{C}).
\]

The usual dual Dirac morphism \([\partial \mathbb{R}^N] \in K^\text{Spin}(N)_{1}(\mathbb{R}^N)\) for \( \mathbb{R}^N \) is the image of the trivial representation of Spin(\(N\)) in \( K^\text{Spin}(N)_{0}(\mathbb{C}) \). As above, we obtain a class \([\alpha] := \psi(\alpha^*[\partial \mathbb{R}^N]) \in KK^G_{s,N}(\mathbb{C}, C_0(X))\) for any proper section \(\alpha : P \rightarrow T\) satisfying the displacement condition.

The classes constructed in the above manner are called \textit{Lipschitz classes}; if the space \( \mathcal{X} \) is proper, they are called proper Lipschitz classes. In view of our discussion in Section 8.1, we have:

\textbf{Corollary 50.} If \( G \) is a discrete group with a \( G\)-compact model for \( \mathcal{E} \mathcal{G} \), then every proper Lipschitz \( K \)-theory class in \( RKK^G(\mathcal{E} \mathcal{G}; \mathbb{C}, \mathbb{C}) \) is regularising, whence a boundary class.

\[ \text{8.3. Coarse structures on jet bundles.} \text{ We recall the setup for the proof of homotopy invariance of Gelfand-Fuchs cohomology classes in [3]. Let } M \text{ be an oriented compact manifold and let } \text{Diff}^+(M) \text{ be the infinite dimensional Lie group of orientation preserving diffeomorphisms on } M. \text{ Let the locally compact group } G \text{ act on } M \text{ by orientation preserving diffeomorphisms, that is, by a continuous group homomorphism } G \rightarrow \text{Diff}^+(M). \text{ We are interested in the classes in the cohomology of } G \text{ that we obtain by pulling back the Gelfand-Fuchs cohomology for } M \text{ (the latter being part of the group cohomology of } \text{Diff}^+(M)\text{; see [3]).}
\]

Let \( \pi^k : J^k_+(M) \rightarrow M \) be the \textit{oriented }\( k\)-jet bundle over \( M \). That is, a point in \( J^k_+(M) \) is the \( k \)-th order Taylor series at 0 of a germ of an orientation preserving diffeomorphism of a neighbourhood of \( 0 \in \mathbb{R}^n \) into \( M \). Germs of diffeomorphisms of neighbourhoods of \( 0 \in \mathbb{R}^n \) form a connected Lie group \( H \). Its Lie algebra \( \mathfrak{h} \) is the space of polynomial maps \( p : \mathbb{R}^n \rightarrow \mathbb{R}^n \) of order \( k \) with \( p(0) = 0 \), with an appropriate Lie algebra structure. The maximal compact subgroup \( K \subseteq H \) is isomorphic to \( \text{SO}(n) \), acting by isometries on \( \mathbb{R}^n \). It acts on \( \mathfrak{h} \) by conjugation.

The bundle \( J^k_+(M) \) is an \( H \)-principal bundle over \( M \). Since the action of \( H \) on \( J^k_+(M) \) is natural, it commutes with the action of \( G \). We let \( H \) act on the right and \( G \) on the left. Define \( X_k := J^k_+(M)/K \). This is the bundle space of a fibration over \( M \) with fibres \( H/K \).

The complex that computes Gelfand-Fuchs cohomology can be represented in a canonical way as a complex of \( \text{Diff}^+(M) \)-invariant differential forms on \( X_k \). These differential forms generate cyclic cocycles on \( C^\infty_*(X_k) \rtimes_{\text{alg}} G \). Using the formalism of \( n \)-traces, these cyclic cocycles can be extended to linear functionals \( K_*(C^\infty_*(X_k) \rtimes_{\text{alg}} G) \rightarrow \mathbb{C} \). Once we have an appropriate element \( \alpha \in KK^G_*(\mathbb{C}, C_0(X_k)) \), we can pull back these classes to linear functionals on \( K_*(C^\text{alg}_* G) \) and thus prove the homotopy invariance of Gelfand-Fuchs cohomology classes. We can construct \( \alpha \) in the following fashion.

We equip \( J^k_+(M) \) with the unique coarse structure for which \( H \) acts isometrically defined in Theorem 2. The compactness of \( J^k_+(M)/H \equiv M \) implies easily
that $G$ acts by translations on $J^k_+(M)$. Moreover, any orbit map $|H| \to J^k_+(M)$ is a coarse equivalence. We have a Morita-Rieffel equivalence $C_0(X_k) \sim C_0(J^k_+(M) \rtimes K)$ because $K$ acts freely on $J^k_+(M)$. Thus we want to look at the map

$$K_{n+1}(\tau_K^G(J^k_+(M) \rtimes K) \xrightarrow{\psi} KK^G(C, C_0(J^k_+(M) \rtimes K) \cong KK^G(C, C_0(X_k))$$

produced by Theorem 21. Since $H$ is almost connected, it has a dual Dirac morphism by [10]. Moreover, $H/K$ is a model for $\mathcal{E}G$ by [1]. Together with Theorem 27 this implies

$$K_{n+1}(\tau_K^G(J^k_+(M) \rtimes K) \cong K_{n+1}(\tau_K^G(|H|) \rtimes K) \cong KK^*_K(|H|) \cong K^*_K(H/K).$$

Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of $H$ and $K$. There is a $K$-equivariant homeomorphism $\mathfrak{h}/\mathfrak{k} \cong H/K$, where $K$ acts on $\mathfrak{h}/\mathfrak{k}$ by conjugation. Now we need to know whether there is a $K$-equivariant spin structure on $\mathfrak{h}/\mathfrak{k}$. One can check that this is the case if $n$ is even or if $k \equiv 0, 1 \bmod 4$. Since we can choose $k$ as large as we like, we can always assume that this is the case. Hence we may identify $K^*_K(H/K)$ with the representation ring of $K$ in degree $-N$ where $N = \dim \mathfrak{h}/\mathfrak{k}$. The trivial representation of $K$ yields a canonical element in $KK^G_{N-N}(C, C_0(X_k))$.

The above construction is essentially the same as in [3]. We get some additional trouble with spin structures because we want to use only the case of “fixed target” in the notation of [3]. Our framework allows us to use the existence of a dual Dirac morphism for $H$. In contrast, this fact is reproved in different notation in [3].

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