The further chameleon groups of Richard Thompson and Graham Higman: Automorphisms via dynamics for the Higman groups $G_{n,r}$.

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Abstract

We characterise the automorphism groups of the Higman groups $G_{n,r}$ as groups of specific homeomorphisms of Cantor spaces $C_{n,r}$, through the use of Rubin’s theorem. This continues a thread of research begun by Brin, and extended later by Brin and Guzmán: to characterise the automorphism groups of the ‘Chameleon groups of Richard Thompson,’ as Brin referred to them in 1996. The work here completes the first stage of that twenty-year-old program, containing (amongst other things) a characterisation of the automorphism group of $V$, which was the ‘last chameleon.’ As it happens, the homeomorphisms which arise naturally fit into the framework of Grigorchuk, Nekrashevich, and Suschanskiǐ’s rational group of transducers, and exhibit fascinating connections with the theory of reset words for automata (arising in the Road Colouring Problem), while also appearing to offer insight into the nature of Brin and Guzmán’s exotic automorphisms.

1 Introduction

In this article, we characterize the automorphism groups $\text{Aut}(G_{n,r})$ of the important Higman groups $\{G_{n,r}\}$.

The characterisation of the automorphism group of $V = G_{2,1}$ has remained a challenge to the community of researchers of the R. Thompson groups since Brin’s 1996 article [4], which characterises the automorphism groups of $F$ and $T$, but which leaves $V$ as the last ‘chameleon.’ Brin and Guzmán’s following article [5] explores many further mysteries of the automorphism groups of the generalised Thompson groups $F_n$ and $T_n$ (the groups $F$ and $T$ correspond to $F_2$ and $T_2$ in the generalised notation, which was introduced by Brown in [6]), such as the existence of ‘Exotic automorphisms’ when $n > 2$. However, Brin and Guzmán leave the mysterious groups of automorphisms of the groups $V_n$ untouched. In [7], Burillo and Cleary further study the automorphism group of $F$, providing a presentation for this group.

In this article, we characterise the automorphism groups of the Higman groups $\{G_{n,r}\}$, by providing a characterisation of the elements of these groups. As the family of groups $\{G_{n,r}\}$ contains as a sub-family isomorphic copies of the groups $V_n$ (for a given $n$, $V_n \cong G_{n,1}$), we fill one of the main gaps mentioned above. Our characterisation is similar to Brin’s (and later Brin and Guzmán’s) in that we describe our groups as subgroups of homeomorphism groups of Cantor spaces, through the use of Rubin’s theorem (and as we use the description
of the groups $G_{n,r}$ as subgroups of the homeomorphism groups $\text{Homeo}(\mathcal{C}_{n,r})$ of the Cantor spaces $\mathcal{C}_{n,r}$.

To be more specific, for given naturals $1 \leq r < n$, Rubin’s Theorem \cite{rubin} allows us to realise the group of automorphisms $\text{Aut}(G_{n,r})$ as a group $\mathcal{S}_{n,r}$ of homeomorphisms of a specific Cantor space $\mathcal{C}_{n,r}$. We describe $\mathcal{S}_{n,r}$ as a particular subgroup of $\mathcal{R}_{n,r}$: a near relative of the Rational Group $\mathcal{R}$ of Grigorchuk, Nekrashevych and Sushanskii \cite{grigorchuk}, which we also introduce here. Elements of $\mathcal{G}_{n,r}$ fix the equivalence classes of the relation that two points in $\mathcal{C}_{n,r}$ are equivalent if they admit a common infinite suffix (in the natural labelling of the points in that space). However, $G_{n,r}$ acts highly transitively on any given equivalence class under that relation, and the points of any such equivalence class are spread densely throughout the Cantor space $\mathcal{C}_{n,r}$. These features of the natural action of $G_{n,r}$ on $\mathcal{C}_{n,r}$ force that the homeomorphisms in $\mathcal{S}_{n,r}$ have a particular interesting property we introduce here: $\mathcal{S}_{n,r}$ is precisely the subgroup of $\mathcal{R}_{n,r}$ consisting of the homeomorphisms representable by ‘bi-synchronizing’ transducers. The bi-synchronizing property relates to the ‘reset words’ of the Road Colouring Problem for automata, and expresses the necessity that elements of $\mathcal{S}_{n,r}$ represent, in some sense, ‘self-righting’ transformations.

While we have not further investigated the automorphism groups of the groups $F_n$ and $T_n$ as given by Brin and Guzmán, the connection with transducers and the rational group appears to provide a very natural framework for exploring some of the questions raised in their paper. Indeed, we hope the viewpoint taken here towards the groups of automorphisms that we study may help to further understand the ‘exotic’ automorphisms found by Brin and Guzmán.

In the next subsection, for given $1 \leq r < n$, we briefly describe the Cantor space $\mathcal{C}_{n,r}$ and the groups $G_{n,r}$, $\mathcal{R}_{n,r}$, $\mathcal{S}_{n,r}$ mentioned above (and some other groups of interest as well). After that, we state our chief results. We will also interleave some discussion of the state of current research on the questions answered here.

### 1.1 Cantor spaces and groups

For the remainder of this subsection, the symbols $r$ and $n$ will represent two natural numbers so that $1 \leq r < n$. It is fine to allow $r \geq n$ as well, but we do not as under the general form of the Higman definition of the groups $G_{n,r}$, the groups $G_{n,r}$ and $G_{n,(r+n-1)}$ are isomorphic.

Given such $r$ and $n$, the Cantor space $\mathcal{C}_{n,r}$ is the space consisting of all infinite sequences defined as follows:

$$\mathcal{C}_{n,r} := \{ca_1a_2a_3 \ldots | a_i \in \{0, 1, \ldots, n-1\}, c \in \mathbf{r}\},$$

where $\mathbf{r}$ is the set $\{0, 1, \ldots, r-1\}$ which is a set of $r$ symbols disjoint from $\{0, 1, \ldots, n-1\}$. That is, $\mathcal{C}_{n,r}$ can be thought of as a disjoint union of $r$ copies of the infinite $n$-ary Cantor space $\mathcal{C}_n := \{0, 1, \ldots, n-1\}^\omega$ (the standard topology on $\mathcal{C}_{n,r}$ is the product topology, considering $\mathbf{r}$ and $\{0, 1, 2, \ldots, n-1\}$ as finite discrete spaces).

Now, the group $G_{n,r}$ is then precisely the group generated by prefix replacement maps: one specifies two incomparable finite prefixes $c_1a_1a_2 \ldots a_j$ and $c_2b_1b_2 \ldots b_k$ (for some indices $j$ and $k$), and then ‘swaps’ these prefixes (here, two prefixes are incomparable if neither is a prefix of the other). That is, e.g., a point $c_1a_1a_2 \ldots a_ja_{j+1}a_{j+2} \ldots$ would map to $c_2b_1b_2 \ldots b_ka_{j+1}a_{j+2} \ldots$ while $c_2b_1b_2 \ldots b_kb_{k+1}b_{k+2} \ldots$ would map to $c_1a_1a_2 \ldots a_jb_{k+1}b_{k+2} \ldots$. Note that one can think of this group as a group of piecewise affine transformations of the space $\mathcal{C}_{n,r}$ which are locally orientation preserving. The book \cite{higman} introduces these groups and is still a main source of information on the Higman groups, which family of groups provides the first infinite source of infinite, finitely presented simple groups (the commutator
subgroup of \( G_{n,r} \) is always simple, and is equal to \( G_{n,r} \) when \( n \) is even, or is index two in \( G_{n,r} \) when \( n \) is odd). The Higman groups are much studied but are still of topical interest, retaining as they do some cloak of mystery. Some investigations of these groups are [13, 15, 9, 14, 18].

It is well known that in the case \( n = 2 \) and \( r = 1 \), the group \( G_{2,1} \) is isomorphic to the R. Thompson group \( V \), and in general, following the notation of Brown introduced in [6], we will denote \( G_{n,1} \) as \( V_n \). The groups \( V_n \) are called the Higman–Thompson groups, as they are natural generalizations of the Thompson group \( V \) in the Higman groups framework.

Similarly, there are subgroups \( F_n < T_n < V_n \) (again adopting the notation of Brown). The groups \( F_n \), \( T_n \) and \( V_n \) naturally generalize the R. Thompson groups \( F = F_2 \), \( T = T_2 \), and \( V = V_2 \). (See Thompson’s 1965 notes [17] or the oft-cited survey [8] for more information on the R. Thompson groups.) We will not discuss the groups \( F_n \) and \( T_n \) in any depth in this article, but we will relate the work done here to previous work carried out for those groups.

For the Cantor space \( \mathcal{C}_n \) there is the group \( \mathcal{R}_n \) of homeomorphisms of \( \mathcal{C}_n \), called the rational group (on \( n \) letters) by its discoverers Grigorchuk, Nekrashevych, and Suschanski˘ ı in [10, 11]. This is the group of homeomorphisms of \( \mathcal{C}_n \) that admit a finite (asynchronous) transducer which induces the appropriate transformations of the elements of \( \mathcal{C}_n \). The defining characteristic of a homeomorphism of \( \mathcal{C}_n \) which admits a finite asynchronous transducer to represent it is that there are only finitely many “local actions” of such a transducer on the basic open sets of the relevant Cantor space; each basic open set maps to its image using a scaled version of one of these “local actions”. Thus, if one considers the nesting properties of the basic open sets, one finds that there are some local actions which have aspects of self-similarity: they “behave the same way” on a smaller basic open set as they do on a larger basic open set which happens to contain the smaller set.

By an essentially trivial modification of the groups \( \mathcal{R}_n \), we introduce here the groups \( \mathcal{R}_{n,r} \), which are just like the groups \( \mathcal{R}_n \) except that the resulting transducers process points in the Cantor spaces \( \mathcal{C}_{n,r} \).

Our chief result is a proof that automorphisms of the Higman groups \( G_{n,r} \) are representable as the elements of a particular subgroup \( S_{n,r} \) of \( \mathcal{R}_{n,r} \). Furthermore, the transducers describing elements of \( S_{n,r} \) have a special property (bi-synchronization, defined below) which allows for an easy representation of the elements of the outer automorphism group \( \text{Out}(G_{n,r}) \) as a group \( \mathcal{O}_n \) which contains further interesting subgroups \( \mathcal{L}_n \) and \( \mathcal{P}_n \). Also, the group \( \mathcal{P}_n \) corresponds to a particularly easy-to-describe natural subgroup of the group \( \text{Aut}(\{0, 1, \ldots, n - 1\}^\mathbb{Z}, \sigma) \) of automorphisms of the full shift space \( \{0, 1, \ldots, n - 1\}^\mathbb{Z} \) (the index \( r \) here turns out to be irrelevant here).

1.2 Discussion and statements of results

As mentioned above, the automorphism groups of \( F \) and \( T \) are described in Brin’s landmark paper [4]. In the later paper [5], Brin and Guzmán go on to explore the automorphism groups of \( F_n \) and \( T_n \) for \( n > 2 \), where they make the startling discovery of ‘exotic’ automorphisms.

Perhaps surprisingly, the methods of [4, 5] fail to restrict possibilities sufficiently to create a meaningful description of the automorphisms of \( V_n \). We say just a few words on this here. The groups \( F_n \) and \( T_n \) can be thought of as homeomorphism groups of the spaces \( \mathbb{R} \) or \( S^1 \), respectively. The approaches of Brin in [4] and of Brin and Guzmán in [5] both make use of Rubin’s theorem to understand an automorphism of a group \( F_n \) or \( T_n \) as a topological conjugation by a homeomorphism of the relevant space.

For a given \( n > 1 \), the group \( V_n \) can be thought of as consisting of the piecewise affine transformations of the Cantor space \( \mathcal{C}_n := \{0, 1, \ldots, n - 1\}^N \); in some sense \( V_n \) is the group
of “PL approximation homeomorphisms” of the full group of homeomorphisms of $\mathfrak{C}_n$. As such, $V_n$ represents a very “large” group, and Rubin’s theorem again applies. However, $V_n$ takes full advantage of the totally disconnected and homogeneous nature of its relevant Cantor space. In consequence, the groupoid of local germs of elements of $V_n$ turns out to be too flexible to provide sufficiently restrictive information on its own to characterize the automorphisms of $V_n$.

Our first theorem represents a resolution of the problem mentioned above. Through a heavy use of some versions of transitivity of the action of $G_{n,r}$ on $\mathfrak{C}_{n,r}$ we are able to find a type of compactness condition: an automorphism of $G_{n,r}$ will only admit finitely many types of “local actions” on $\mathfrak{C}_{n,r}$. This allows us to show that automorphisms of $G_{n,r}$ are represented by elements in $\mathcal{R}_{n,r}$ following known arguments as given in [10]. Secondly, the “transitivity” of the action of $G_{n,r}$ also allows us to see that after finitely many steps, our transducers have to be in very specific states. We call this property synchronizing, which is the common term as established by research on the Road Colouring Problem and the Černý Conjecture (see, e.g., [19, 20]).

A transducer is synchronizing at level $n$ if there is an integer $n$ so that whenever the transducer reads an input word of length $n$, the resulting active state is then known, regardless of the initial active state. A homeomorphism is representable by a bi-synchronizing transducer if there is a natural number $n$ so that the homeomorphism and its inverse are both representable by finite transducers which are synchronizing at level $n$. Note that it is very easy to build transducers inducing homeomorphisms which are synchronizing, but where the inverse of the homeomorphism cannot be represented by a transducer with synchronization (see [2] for examples of these sorts of homeomorphisms).

Note there is an unfortunate collision in nomenclature in the literature. Transducers which transform input strings in a “one letter in, one letter out” fashion from each of their states are called synchronous transducers. A typical example of such is the transducer whose states represent the standard generators of the Grigorchuk group. Meanwhile, an automaton which has the property that after reading a specific string from any state one knows which of the states of the automaton has become the “active state” (as described above) is called a synchronizing automaton (in the literature around, e.g., the Černý Conjecture). The automata which arise in this article will have important sub-automata which will be bi-synchronizing, as described above, as well as being synchronous in many cases. We hope this collision in language will lead to no confusion.

**Theorem 1.1.** Let $\mathcal{R}_{n,r}$ represent the generalized Grigorchuk, Nekrashevych, and Sushchanskii rational group of homeomorphisms of the Cantor space $\mathfrak{C}_{n,r}$ (that is, those homeomorphisms which are representable by finite initial transducers). The subgroup $\mathcal{S}_{n,r}$ of $\mathcal{R}_{n,r}$ of homeomorphisms representable by bi-synchronizing finite transducers contains $G_{n,r}$, and is isomorphic to $\text{Aut}(G_{n,r})$.

1.2.1 On outer automorphisms

As it turns out, the second parameter $r$ for the specification of the Higman group $G_{n,r}$ is not relevant for the outer automorphism group of $G_{n,r}$. That is, we have the following.

**Theorem 1.2.** Let $1 \leq r, s < n$ be positive integers, and denote by $\text{Out}(G_{n,r})$ and $\text{Out}(G_{n,s})$ the outer automorphism groups of $G_{n,r}$ and $G_{n,s}$ respectively. Then $\text{Out}(G_{n,r}) \cong \text{Out}(G_{n,s})$.

The path to the proof of this is perhaps of interest. One can show that any given transducer $A_{q_0}$ representing an element of $\mathcal{S}_{n,r}$ has a special sub-automaton (the core of $A_{q_0}$), which precisely characterises the outer-automorphism class of the image of the homeomorphism represented by $A_{q_0}$ under the natural quotient to the outer automorphism group.
set of (equivalence classes of) such core transducers admits a nice product operation under
which it is a group, which we denote by $O_n$.

There have been those who held e.g., that large, highly transitive groups of homeomor-
phisms of homogeneous spaces should have small (finite) outer automorphism groups (in
this context, highly transitive means $k$-transitive for any $k$, but restricted over some dense
subset of the underlying space). In this case, our groups satisfy the general criteria, but for
all $n \geq 2$, it is the case that $O_n$ is actually an infinite group for all $n \geq 2$. Thus, this paper
provides another set of examples that the “meta-theorem” just proposed is indeed simply
false (this was already known to be the case, e.g., for $T_n$ and $F_n$ from the work in [4, 5]).

More formally, we have the following theorem.

**Theorem 1.3.** Let $n$ be a positive integer, the group $O_n$ is infinite.

Note further that we also provide a combinatorial characterisation of elements of the
groups $\text{Out}(G_{n,r})$.

### 1.2.2 Further related groups of interest

For given $n$ (and $r$) The group $O_n \cong \text{Out}(G_{n,r})$ has some very interesting subgroups. One
of these subgroups is $\mathcal{L}_n$, which is the image in $O_n$ of those homeomorphisms representable
by bi-synchronizing transducers that have locally constant Radon–Nikodym derivative. In
$S_{n,r}$ these elements form the subgroup of homeomorphisms with bi-Lipschitz action on the
Cantor space $C_{n,r}$, which group we denote as $\mathcal{LS}_{n,r}$. The group $\mathcal{L}_n$ contains (combinatorially
defined) further subgroup $\mathcal{P}_n$ of great interest. The elements of $\mathcal{P}_n$ correspond to non-initial
core transducers which are not only bi-synchronizing, but also synchronous (one letter read
in becomes one letter written out, on each transition). The group $\mathcal{P}_n$ embeds naturally as the
subgroup of the automorphisms of the full shift $\text{Aut}([0, 1, \ldots, n-1]^\mathbb{Z}, \sigma)$ which are given
by sliding block codes which use 0 future information. We denote the pre-image of $\mathcal{P}_n$ in
$S_{n,r}$ as $\mathcal{PS}_{n,r}$.

**Theorem 1.4.** The group $O_n$ contains as a subgroup a group $\mathcal{P}_n$, which embeds as the
subgroup of $\text{Aut}([0, 1, \ldots, n-1]^\mathbb{Z}, \sigma)$ corresponding to the elements given by sliding block
code transformations that use no future information.

Throughout the paper (but, predominantly in Subsection 8.1), we provide examples of
various group elements of the various groups, by giving representative transducers. While
the definitions are sufficient to immediately prove the first point of the following theorem,
the remaining points are proven through these demonstrations of the existence of transducers
representing various group elements with appropriate properties.
Theorem 1.5. Let $1 \leq r < n$ be integers. We have

1. $\mathcal{P}_n \leq \mathcal{L}_n \leq \mathcal{O}_n \cong \text{Out}(G_{n,r})$,
2. there are elements of $\mathcal{O}_n$ which are not in $\mathcal{L}_n$,
3. there are elements of $\mathcal{L}_n$ which are not in $\mathcal{P}_n$,
4. for $n > 2$ there are elements of $\mathcal{P}_n$ of infinite order,
5. there are non-bi-Lipschitz torsion elements of $\mathcal{O}_2$, and
6. there are elements of $\mathcal{L}_2$ of infinite order.

1.2.3 An example transducer

For those readers already comfortable with transducers, the following initial transducer (initial state $q_0$) of Figure 1.2.3 represents an element of $\mathcal{A}_{3,2}$ with natural image in $\mathcal{O}_3$ non-trivial.

One can verify, for instance, that the states $q_2$, $q_3$, and $q_4$ are the states of the core, as reading any fixed word of length two in the alphabet $\{0, 1, 2\}$ from any initial state (other than $q_0$, which only takes words beginning with words with first letter from the alphabet $\{\hat{0}, \hat{1}\}$) will result in the same fixed state.

1.3 Future directions

There are some related upcoming articles [1] and [3]. In those papers, the authors investigate many further properties of the groups $\mathcal{P}_n < \mathcal{L}_n < \mathcal{O}_n$. 
Firstly, and amongst other investigations, in [1], those authors show that the subgroup \( \mathcal{P}_n \) of the group \( \mathcal{L}_n \) is actually isomorphic to \( \text{Aut}(\{0,1,\ldots,n−1\}^\omega, \sigma) \), the automorphisms of the one sided shift on \( n \) letters (a suggestion of J. Hubbard), from which it follows that \( \mathcal{P}_2 \) is actually cyclic of order two by a classic result of Hedlund [12]. Still, Bleak and Cameron provide a new proof using a close analysis of the automorphism types of specific quotients of De Bruijn graphs.

Secondly, in [3], those authors solve the order problem for elements in \( \mathcal{P}_n \), and therefore, for elements of \( \text{Aut}(\{0,1,\ldots,n−1\}^\omega, \sigma) \).

1.4 Further thoughts and acknowledgements

It seems likely that the work in this article can be used to investigate the ‘exotic’ automorphisms which arise for the subgroups \( F_n < G_{n,1} \) for various values of \( n \). Indeed, it is relatively easy to build asynchronous finite transducers \( \mathcal{C}_n \rightarrow \mathcal{C}_n \) which represent non-PL maps \( [0,1] \rightarrow [0,1] \) which conjugate \( F_n \) to \( F_n \) in ways which cannot be realized by any inner automorphism of \( F_n \) (for \( n \geq 2 \)).

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2 Language, Notation, and Groups

In this section, we will more carefully construct the Cantor spaces \( \mathcal{C}_r \) and \( \mathcal{C}_{n,r} \) for given \( 1 \leq r < n \in \mathbb{N} \), and the groups \( G_{n,r} \). We will also develop notation and equivalence classes for sets of points in \( \mathcal{C}_{n,r} \) which will be valuable in later discussions. First, here are some general conventions we will follow.

Given topological spaces \( X \) and \( Y \), we will denote by \( \text{Homeo}(X,Y) \) the full set of homeomorphisms from \( X \) to \( Y \). We set \( \text{Homeo}(X) := \text{Homeo}(X,X) \), which becomes a group under composition, the group of self-homeomorphisms of the space \( X \) (we will also use \( H(X) \) for \( \text{Homeo}(X) \) when notation is too heavy, e.g., if \( G \leq \text{Homeo}(X) \) we will write \( N_{H(X)}(G) \) for the normalizer of \( G \) in the full group \( \text{Homeo}(X) \)). If \( G \leq \text{Homeo}(X) \), we say \( \langle X,G \rangle \) is a space-group pair, and we use right actions to denote the natural action of \( G \) on \( X \). In particular, if \( x \in X, \ U \subset X \), and \( g,h \in G \), then we write \( xg \text{ or } x \cdot g \) for the image of \( x \) under the map \( g \), we write

\[ \text{Ug} := \{ ug \mid u \in U \}, \]

we write \( g^h := h^{-1}gh \) and \( [g,h] := g^{-1}h^{-1}gh \). We will extend the right action language above without too much concern when it is natural to do so, e.g., if \( h : X \rightarrow Y \) is a homeomorphism, we will write \( g^h \) to represent the self-homeomorphism \( Y \rightarrow Y \) given by the rule \( y \mapsto yh^{-1}gh \).

Throughout the remainder of this article, we will use the right-action notation outlined above. Note also that from now on, we will assume at random times that we have some given \( 1 \leq r < n \in \mathbb{N} \), so that we can refer to a space \( \mathcal{C}_{n,r} \) or a group \( G_{n,r} \) without comment. Occasionally we might still explicitly instantiate these constants.
2.1 Cantor spaces revisited

Regard the Cantor space $\mathcal{C}_n$ as the set of all infinite \{0, 1, \ldots, n-1\}-sequences. That is, give the set \{0, 1, \ldots, n-1\} the discrete topology, and denote by $\mathcal{C}_n$ the Cantor space \{0, 1, \ldots, n-1\}^{\mathbb{N}} (with the product topology). Further, set

$$W_{n,\varepsilon} := \{0, 1, \ldots, n-1\}^*$$

(the set of finite or empty words (sequences) over the alphabet \{0, 1, \ldots, n-1\}, where we will always use the symbol $\varepsilon$ to denote the empty word over any alphabet), and

$$W_n := \{0, 1, \ldots, n-1\}^+$$

(the set of finite non-trivial words in the alphabet \{0, 1, \ldots, n-1\}). If $\eta, \nu \in W_{n,\varepsilon}$ are so that $\eta$ is a prefix of $\nu$, we denote this by $\eta \leq \nu$, and we also write $\eta < \nu$ if $\eta$ is a proper prefix of $\nu$.

**Definition 2.1.** Let $\nu, \eta \in W_n$. We say that $\nu$ and $\eta$ are incomparable if $\nu \nleq \eta$ and $\eta \nleq \nu$, and we denote this as $\nu \perp \eta$.

Give the set $\hat{\mathfrak{r}} := \{0, 1, \ldots, r-1\}$ the discrete topology and set $\mathcal{C}_{n,r} := \hat{\mathfrak{r}} \times \mathcal{C}_n$ (so that $\mathcal{C}_{n,r} \cong \bigsqcup_{i \in \mathfrak{r}} \mathcal{C}_n$ and hence is a Cantor space). Further, set $W_{n,r} := \hat{\mathfrak{r}} \times W_n$, and $W_{n,\varepsilon,r} := \hat{\mathfrak{r}} \times W_{n,\varepsilon}$. By an abuse of notation, we will consider $W_{n,r} \subset W_{n,\varepsilon,r}$ as sets of nontrivial finite words with first letter from the alphabet $\hat{\mathfrak{r}}$ and any latter letters from the alphabet \{0, 1, \ldots, n-1\}, and we extend the meaning of $\leq, <, \infty$ and $\perp$ to $W_{n,\varepsilon,r}$.

For $\eta \in W_{n,\varepsilon,r} \cup W_{n,\varepsilon}$ and $\nu \in W_{n,r} \cup W_{n,\varepsilon} \cup \mathcal{C}_{n,r} \cup \mathcal{C}_n$, define $\eta \wedge \nu$ to be the concatenation of the sequences $\eta$ and $\nu$. For $\eta \in W_{n,r}$, we set $U_\eta := \{\eta \wedge x \mid x \in \mathcal{C}_n\}$, and we set $U_{\eta} := \mathcal{C}_{n,r}$. We call $U_\eta$ the cone of $\eta$. Set $B_{n,r} := \{U_\eta \mid \eta \in W_{n,r}\}$. The set $B_{n,r}$ is a clopen basis of the Cantor space $\mathcal{C}_{n,r}$.

**Notation 2.2.** For $\nu, \eta \in W_{n,\varepsilon,r}$ such that $\nu \leq \eta$, there is $\tau \in W_{n,\varepsilon,r}$ such that $\nu \wedge \tau = \eta$, and in this situation we define

$$\eta - \nu = \tau.$$

For $\eta \in W_{n,\varepsilon} \cup W_{n,r}$ and $\nu \in W_{n,\varepsilon} \cup \mathcal{C}_n \cup W_{n,\varepsilon} \cup \mathcal{C}_{n,r}$, let $\eta \leq \nu$ mean that $\eta$ is a prefix of $\nu$.

**Definition 2.3.** Let $\hat{\eta} = \{\eta_0, \ldots, \eta_{k-1}\}$ be in $W_{n,r}^k$ for some integer $k \geq 1$. We call $\hat{\eta}$ a maximal anti-chain in $W_{n,r}$ if for any distinct $i, j \in \{1, \ldots, k-1\}$, $\eta_i \nleq \eta_j$, and for every $\zeta \in W_{n,\varepsilon,r}$, there is $i < k$ such that $\eta_i \leq \zeta$ or $\zeta \leq \eta_i$. In such cases we also call the integer $k$ the length of the anti-chain $\hat{\eta}$. (Note, we will only be concerned with finite anti-chains in this article.) We may also call a finite maximal anti-chain in $W_{n,r}$ a prefix code.

2.2 The groups $G_{n,r}$ and some interesting subgroups

We are also interested in building up complex maps from simpler ones. A foundation for this is provided as below.

**Definition 2.4.** If $\eta, \zeta \in W_{n,\varepsilon,r}$ define $g_{\eta,\zeta} : U_\eta \to U_\zeta$ by

$$(\eta \wedge x)g_{\eta,\zeta} := \zeta \wedge x$$

for all $x \in \mathcal{C}_n$. For given $\eta$ and $\zeta$ as above, we will refer to the map $g_{\eta,\zeta}$ as the basic cone map from $U_\eta$ to $U_\zeta$. 

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Using the cone maps above, and maximal anti-chains in the poset $W_{n,r}$, we can define a specific class of homeomorphisms on the Cantor spaces $C_{n,r}$.

**Definition 2.5.** Let $\tilde{\eta} = \{\eta_0, \ldots, \eta_{k-1}\}$ and $\tilde{\zeta} = \{\zeta_0, \ldots, \zeta_{k-1}\}$ be maximal anti-chains in $W_{n,r}$ of length $k$ for some fixed positive integer $k$. Define

$$g_{\tilde{\eta}, \tilde{\zeta}} := \prod_{i<k} g_{\eta_i, \zeta_i}.$$

In the context above, we may call such a map $g_{\tilde{\eta}, \tilde{\zeta}}$ a *prefix code map*. It is well known (and easy to check) that compositions and inversions of prefix code maps are prefix code maps, so for fixed $r$ and $n$, the prefix code maps over $C_{n,r}$ form a group of homeomorphisms. We denote the resulting group as $G_{n,r}$.

**Notation 2.6.** Let $G_{n,r} := \{g_{\tilde{\eta}, \tilde{\zeta}} \mid \tilde{\eta}, \tilde{\zeta} \text{ are maximal anti-chains of } W_{n,r}, r \text{ of the same length}\}$.

We note in passing that in Higman’s framework, the groups $G_{n,r}$ are groups of automorphisms of free term algebras. The definition we give above is a standard translation from the original Higman definition of the groups $G_{n,r}$ to a definition of these groups as groups of homeomorphisms of spaces. For a detailed discussion of the history behind these multiple points of view, we refer the reader to section 4B of [6].

Now, from the definition given above for the elements of $G_{n,r}$, for every $g \in G_{n,r}$ there is a minimal integer $k$ and two finite sets of cones

$$\{U_{\eta_0}, \ldots, U_{\eta_{k-1}}\} \subseteq B_{n,r}$$

and

$$\{V_{\zeta_0}, \ldots, V_{\zeta_{k-1}}\} \subseteq B_{n,r},$$

each partitioning $C_{n,r}$, so that $g$ maps each $U_{\eta_i}$ to $V_{\zeta_i}$ as the basic cone map $g_{\eta_i, \zeta_i}$.

The transducer in the introduction quite clearly does not represent a map in $G_{3,2}$, so it is clear that it will project to a non-trivial outer automorphism of $G_{3,2}$. However, there are easier examples of the general fact that the group $S_{3,2}$ has homeomorphisms which naturally project to non-trivial elements of $O_{n,r}$. Here we provide a simple family of such homeomorphisms for the reader to verify the claim.

**Definition 2.7.** Let $\sigma \in S_n$ be any permutation on the set $\{0, 1, \ldots, n-1\}$. Define the map $\widehat{\sigma} : C_n \to C_n$ by the rule $(a_1 a_2 a_3 \ldots) \mapsto (a_1 \sigma a_2 \sigma a_3 \sigma \ldots)$, and call this map the $\sigma$-twist of $C_n$. Further, define the map $\widehat{\sigma}_{n,r} : C_{n,r} \to C_{n,r}$ which is obtained by applying $\widehat{\sigma}$ to each cone $U_k \cong C_n$ for $k \in \{0, 1, \ldots, r-1\}$, and call this map the $\sigma$-twist of $C_{n,r}$.

We observe in passing that the $\sigma$-twists of $C_n$ and of $C_{n,r}$ are homeomorphisms of the respective Cantor spaces. (Indeed, these homeomorphisms are obtained as induced actions on the boundary of the standard infinite trees used in the construction of the Cantor spaces under consideration, where the $\pi$-twists actually represent infinite automorphisms of these trees.) The relevance of $\sigma$-twists is shown by the next remark.

**Remark 2.8** (Existence of non-trivial $\text{Out}(G_{n,r})$). Let $1 \leq r < n \in \mathbb{Z}$, and suppose $\sigma \in S_n$. The map $\widehat{\sigma}_{n,r} : G_{n,r} \to G_{n,r}$ defined by $g \mapsto \widehat{\sigma}_{n,r}^{-1} \cdot g \cdot \widehat{\sigma}_{n,r}$, for $\widehat{\sigma}_{n,r}$ the $\sigma$-twist of $C_{n,r}$ as defined above, is an automorphism of $G_{n,r}$. Furthermore, $\widehat{\sigma}_{n,r}$ belongs to $\text{Inn}(G_{n,r})$ if and only if $\sigma$ is the trivial permutation.
The fact that \( \tilde{\sigma}_{n,r} \) is an automorphism of \( G_{n,r} \) is a direct computation that we leave to the reader. If \( \pi \) is not trivial, then the conjugacy action of \( \tilde{\sigma}_{n,r} \) cannot coincide with that of an element \( g \in G_{n,r} \) as otherwise conjugation by the product \( g^{-1}\tilde{\sigma}_{n,r} \) (which is not in \( G_{n,r} \)) would induce the trivial action on \( G_{n,r} \), while it is easy to see that for any given non-trivial homeomorphism \( \rho \) of the Cantor space \( \mathcal{C}_{n,r} \), there are elements of \( G_{n,r} \) which fail to commute with \( \rho \).

**Definition 2.9.** We say that \( g \in \text{Homeo}(\mathcal{C}_{n,r}) \) is densely canonical if for every nonempty open set \( U \subseteq \mathcal{C}_{n,r} \) there are words \( \eta, \zeta \in W_{n,r,\epsilon} \) with cones \( U_{\eta} \subseteq U \) and \( U_{\zeta} \) so that \( g|_{U_{\eta}} = g_{\eta}\zeta \).

Clearly, the set \( G_{n,r,\text{dcn}} \) of densely canonical homeomorphisms of \( \mathcal{C}_{n,r} \) is a group under composition.

**Definition 2.10.** We say that \( g \in \text{Homeo}(\mathcal{C}_{n,r}) \) is pointwise canonical if for every \( x \in \mathcal{C}_{n,r} \) there are \( \eta_1, \eta_2 \in W_{n,r,\epsilon} \) and \( y \in \mathcal{C}_n \) such that \( x = \eta_1 \cdot y \) and \( xg = \eta_2 \cdot y \).

Again, the set \( G_{n,r,\text{pcn}} \) of pointwise canonical homeomorphisms of \( \mathcal{C}_{n,r} \) is a group under composition.

**Remark 2.11.** For all integers \( 1 \leq r < n \), we have \( G_{n,r} \leq G_{n,r,\text{dcn}} \cap G_{n,r,\text{pcn}} \).

We now give another definition of \( G_{n,r,\text{pcn}} \). Define a relation on \( \mathcal{C}_{n,r} \) by:

\[
x \sim y \iff \exists \eta, \nu \in W_{n,r,\epsilon}, z \in \mathcal{C}_n \text{ so that } x = \nu \cdot z \text{ and } y = \eta \cdot z.
\]

This is an equivalence relation on \( \mathcal{C}_{n,r} \). We denote the set of equivalence classes of \( \sim \) by \( \mathcal{C}_{n,r}/ \sim \), and the equivalence class of \( x \in \mathcal{C}_{n,r} \) by \( [x]_{\sim} \). Given \( x, y \in \mathcal{C}_{n,r} \), we say \( x \) and \( y \) have an equivalent tail if and only if \( y \in [x]_{\sim} \), and we call \( [x]_{\sim} \) the tail class of \( x \). Let \( H_{n,r,\sim} \) be the group of all homeomorphisms of \( \mathcal{C}_{n,r} \) that preserve \( \sim \). That is,

\[
H_{n,r,\sim} = \{ g \in \text{Homeo}(\mathcal{C}_{n,r}) \mid y \in [x]_{\sim} \iff yg \in [xg]_{\sim} \}.
\]

Clearly, \( G_{n,r,\text{pcn}} \) is a subgroup of \( H_{n,r,\sim} \). By definition, \( H_{n,r,\sim} \) has an induced action on \( \mathcal{C}_{n,r}/ \sim \), and from this point of view it is immediate that \( G_{n,r,\text{pcn}} \) represents the kernel of the action of \( H_{n,r,\sim} \) on \( \mathcal{C}_{n,r}/ \sim \). That is, \( G_{n,r,\text{pcn}} \triangleleft H_{n,r,\sim} \), and

\[
G_{n,r,\text{pcn}} = \{ g \in H_{n,r,\sim}(\mathcal{C}_{n,r}) \mid \forall x \in \mathcal{C}_{n,r}, xg \in [x]_{\sim} \}.
\]

We say that \( G_{n,r,\text{pcn}} \) fixes \( \sim \) pointwise.

We observe the following easy lemma.

**Lemma 2.12.** For all \( x \in \mathcal{C}_{n,r} \), the group \( G_{n,r} \) acts transitively on \( [x]_{\sim} \).

**Proof:** For each \( y \in [x]_{\sim} \) there are \( \nu, \eta \in W_{n,r,\epsilon} \) and \( z \in \mathcal{C}_n \) such that \( x = \nu \cdot z \) and \( y = \eta \cdot z \). There exists two maximal anti-chains \( \vec{\eta} = \{ \eta_0, \ldots, \eta_{k-1} \} \) and \( \vec{\nu} = \{ \nu_0, \ldots, \nu_{k-1} \} \) such that \( \eta = \eta_0 \) and \( \nu = \nu_0 \), and hence \( xg_{\nu,\eta} = y \).

\( \square \)

### 3 More on \( G_{n,r,\text{dcn}}, G_{n,r,\text{pcn}}, G_{n,r} \) and \( H_{n,r,\sim} \)

If \( X \) is a topological space and \( x \in X \), then we let \( \text{Nbr}^X(x) \) denote the set of open neighbourhoods of \( x \) in \( X \). A subset \( A \subseteq X \) is somewhere dense, if for some nonempty open set \( U \subseteq X \), the intersection \( A \cap U \) is dense in \( U \). The following is a version of the main result in [10]
Theorem 3.1. (M. Rubin) Let \( \langle X, G \rangle \) and \( \langle Y, H \rangle \) be space-group pairs. Assume that \( X \) is Hausdorff, locally compact, and without isolated points, and that for every \( x \in X \) and \( U \in \text{Nbr}^X(x) \), the set \( \{yg | g \in G \text{ and } g|_{(X-U)} = \text{Id}|_{(X-U)} \} \) is somewhere dense. Assume that the same holds for \( \langle Y, H \rangle \). Suppose we have a group isomorphism \( G \cong H \). Then there is \( \varphi \in \text{Homeo}(X, Y) \) such that \( \varphi \) induces \( \phi \). That is, \( g\phi = g^\varphi \) for every \( g \in G \).

Corollary 3.2. \( \text{Aut}(G_{n,r}) \cong N_{H}(\mathfrak{c}_{n,r})(G_{n,r}) \).

Proof: An automorphism of \( G_{n,r} \) is an isomorphism from \( G_{n,r} \) to itself, so we set \( X = Y = \mathfrak{c}_{n,r} \) in the statement of Rubin’s Theorem, and we see that \( N_{H}(\mathfrak{c}_{n,r})(G_{n,r}) \) quotients onto \( \text{Aut}(G_{n,r}) \). To see that this quotient is the quotient by the trivial subgroup we recall that every self-homeomorphism of \( \mathfrak{c}_{n,r} \) is a limit of elements in \( G_{n,r} \). Thus, given any non-identity element \( h \in \text{Homeo}(\mathfrak{c}_{n,r}) \), we can easily find an element of \( G_{n,r} \) which fails to commute with \( h \).

The following lemma is standard in the literature which relies upon Rubin’s theorem.

Lemma 3.3. Let \( X \) be a topological space, \( \sim \) an equivalence relation on \( X \), \( H_\sim \) the subgroup of \( \text{Homeo}(X) \) consisting of all the homeomorphisms that preserve \( \sim \) and \( G \) a subgroup of \( H_\sim \) such that every \( g \in G \) fixes \( \sim \) pointwise and \( G \) acts transitively on each equivalence class in \( X/\sim \). Then \( \{h \in \text{Homeo}(X)|h^{-1}Gh \subseteq G \} \subseteq H_\sim \). In particular, \( N_{H(X)}(G) \leq H_\sim \).

Proof: Let \( h \in \text{Homeo}(X) \) so that \( G^h \subseteq G \) and suppose \( x \in X \) and \( y \in [x]_\sim \). Since \( G \) acts transitively on \( [x]_\sim \), there exists \( g \in G \) such that \( xg = y \). Calculating, we have \( xh^hy = xhh^{-1}gh = yh \). As \( g^h \in G \), we see that \( xh \sim yh \), and since \( x \) is an arbitrary element of \( X \), we conclude that \( h \in H_\sim \).

Corollary 3.4. We have:

1. \( \text{Aut}(G_{n,r}) \cong N_{H}(\mathfrak{c}_{n,r})(G_{n,r}) \leq H_{n,r,\sim} \)
2. \( \text{Aut}(G_{n,r,\text{pcn}}) \cong H_{n,r,\sim} \).

Proof: Point (1) follows directly from Corollary 3.2 and Lemma 3.3. For Point (2), notice that by the Lemma 3.3 we have \( N_{H}(\mathfrak{c}_{n,r})(G_{n,r,\text{pcn}}) \leq H_{n,r,\sim} \). Moreover, if \( h \in H_{n,r,\sim} \), \( g \in G_{n,r,\text{pcn}} \) and \( x \in \mathfrak{c}_{n,r} \), then \( x^{-1} \sim x^{-1}g \) so \( x^{-1}h \sim x^{-1}gh = y^h \). Thus \( x \sim y^h \), and as \( x \) is arbitrary in \( \mathfrak{c}_{n,r} \), we see that \( g^h \in G_{n,r,\text{pcn}} \), and therefore \( h \in N_{H(\mathfrak{c}_{n,r})}(G_{n,r,\text{pcn}}) \).

Claim 3.5. Let \( g \in G_{n,r,\text{pcn}} \) and \( \nu, \eta \in W_{n,r} \). Then the set

\[ A_{\nu,\eta} := \{x \in \mathfrak{c}_{n,r} | x = \nu^*y \text{ and } xg = \eta^*y \text{ for some } y \in \mathfrak{c}_{n,r} \} \]

is closed.

Proof: Suppose that \( \{x_i\}_{i=0}^\infty \subseteq A_{\nu,\eta} \) is a sequence that converges to \( x \in \mathfrak{c}_{n,r} \). For every large \( i \) there is \( y_i \in \mathfrak{c}_{n,r} \) such that \( x_i = \nu^*y_i \) and \( x_ig = \eta^*y_i \). So as \( x_i \to x \), we have \( \nu^*y_i \to \nu^*y = x \) for some \( y \in \mathfrak{c}_{n,r} \) such that \( y_i \to y \). But \( g \) is continuous, so \( x_ig \to xg \), that is \( \eta^*y_i \to xg \). Therefore, \( \eta^*y = xg \), hence \( x \in A_{\nu,\eta} \).

Claim 3.6. \( G_{n,r,\text{pcn}} = H_{n,r,\sim} \cap G_{n,r,\text{dcn}} \).

Proof: Let \( g \in G_{n,r,\text{pcn}} \) and \( U \subseteq \mathfrak{c}_{n,r} \) be an open set. For every \( \nu, \eta \in W_{n,r} \) set \( A_{\nu,\eta} = A_{\nu,\eta} \cap U \). Since \( A_{\nu,\eta} \) is closed in \( \mathfrak{c}_{n,r} \), we have that \( A_{\nu,\eta}^U \) is closed in \( U \). Since \( g \in G_{n,r,\text{pcn}} \), we also have \( U = \bigcup_{\nu,\eta \in W_{n,r}} A_{\nu,\eta}^U \). Now, \( U \) has the Baire property and there are countably many \( A_{\nu,\eta}^U \)'s in the above union so there exists some \( \nu \) and \( \eta \) so that \( A_{\nu,\eta}^U \)
is somewhere dense. Let $U_\zeta \subseteq U$ be a basic cone so that $A^U_{g,\nu,\eta} \cap U_\zeta$ is dense in $U_\zeta$. But $A^U_{g,\nu,\eta} \cap U_\zeta$ is closed in $U_\zeta$ so $A^U_{g,\nu,\eta} \cap U_\zeta = U_\zeta$ and therefore $U_\zeta \subseteq A^U_{g,\nu,\eta}$. Let $\tau \in W_{n,\varepsilon}$ be such that $\nu^* \tau = \zeta$ and set $\omega = \eta^* \tau$. For every $\zeta^* x \in U_\zeta$ we have

$$(\zeta^* x)g = (\nu^* \tau \cdot x)g = \eta^* \tau \cdot x = \omega^* x.$$ 

Thus, we have found a cone $U_\zeta \subseteq U$ and $\omega \in W_{n,\varepsilon}$ so that $g|_{U_\zeta} = g_{\zeta,\omega}$ and hence $g \in G_{n,r,dcn}$. Therefore, $G_{n,r,dcn} \subseteq H_{n,r,\sim} \cap G_{n,r,dcn}$.

Conversely, let $g \in H_{n,r,\sim} \cap G_{n,r,dcn}$. Since $g \in G_{n,r,dcn}$, there exist $\nu, \eta \in W_{n,r}$ such that $g|_{U_\nu} = g_{\nu,\eta}$. As every tail class has a representative in $U_\nu$ and $g \in H_{n,r,\sim}$, we have that $g$ fixes $\sim$ pointwise. Therefore, $H_{n,r,\sim} \cap G_{n,r,dcn} \subseteq G_{n,r,dcn}$. \hfill \square

## 4 Local Actions

Given $h \in \text{Homeo}(\mathfrak{C}_{n,r})$ and $U_\nu \in \mathfrak{B}_{n,r}$, we would like to investigate the action of $h$ on $U_\nu$. We begin by establishing some notation for $h \in \text{Homeo}(\mathfrak{C}_n)$ and $U_\nu \in \mathfrak{B}_n$. We will then briefly discuss how to generalize what is established for homeomorphisms of $\mathfrak{C}_{n,r}$.

For $U \subseteq \mathfrak{C}_n$, define the root of $U$ to be $\nu \in W_{n,\varepsilon}$ such that $U \subseteq U_\nu$ and $U_\nu$ is the minimal element in $\mathfrak{B}_n$ (with respect to inclusion) with this property. Denote

$$\nu := \text{Root}(U).$$

**Definition 4.1.** Let $h : \mathfrak{C}_n \longrightarrow \mathfrak{C}_n$ be a continuous function. The root function of $h$ is the function $\theta_h : W_{n,\varepsilon} \longrightarrow W_{n,\varepsilon}$ defined by:

$$\nu \mapsto \nu \theta_h = \text{Root}(U_\nu h)$$

for all $\nu \in W_{n,\varepsilon}$.

**Definition 4.2.** Let $h : \mathfrak{C}_n \longrightarrow \mathfrak{C}_n$ be continuous and injective, and $\nu \in W_{n,\varepsilon}$. The local action of $h$ on $\nu$ is the injective continuous map $h_\nu : \mathfrak{C}_n \longrightarrow \mathfrak{C}_n$ defined by $x \cdot h_\nu = y$, where $(\nu^* x) \cdot h = (\nu \cdot \theta_h) \cdot y$.

**Remark 4.3.** Given any homeomorphism $h : \mathfrak{C}_n \rightarrow \mathfrak{C}_n$ and $\nu \in W_{n,\varepsilon}$, it is easy to verify that:

1. the local action of $h$ on $\nu$ is well defined,
2. $U_\nu \cdot h \subseteq U_{\nu \theta_h}$,
3. $h_\nu$ is both continuous and injective,
4. $h_\nu$ is surjective (and hence $h_\nu \in \text{Homeo}(\mathfrak{C}_n)$) if and only if $U_\nu \cdot h = U_{\nu \theta_h}$, and
5. if $\nu^* \eta \in W_{n,\varepsilon}$, then $(\nu^* \eta) \theta_h = (\nu) \theta_h \cdot (\eta) \theta_{h_\nu}$.

We now discuss how to generalize these definitions to a homeomorphism $h$ from $\mathfrak{C}_{n,r}$ to itself. Let $P_h \subseteq W_{n,r,\varepsilon}$ be the unique maximal set of strings such that:

1. if $\nu \in P_h$, then $h(U_\nu)$ is contained in a specific ball $B_\alpha \in \mathfrak{B}_{n,r}$ for some $\alpha \in \hat{\mathfrak{r}}$,
2. for any proper prefix $\mu$ of $\nu \in P_h$, there are $a_1 \neq a_2 \in \hat{\mathfrak{r}}$ and $x, y \in \mathfrak{C}_n$ so that \{a_1^* x, a_2^* y\} $\subseteq (U_\mu)h$. 




Observe by the continuity of $h$ and compactness of $C_{n,r}$ that $P_h$ is a finite set which makes a complete anti-chain for $W_{n,r,\epsilon}$. Now define, for any $\mu$ a proper prefix of an element of $P_h$, that $(\mu)\theta_h := \epsilon$, so that the local action $h_\mu : C_n \rightarrow C_{n,r}$ (or from $C_{n,r}$ to $C_{n,r}$ if $\mu = \epsilon$). For each element $\nu \in P_h$, we have that $(\nu)\theta_h \in W_{n,r,\epsilon}$ is the unique maximal common prefix of all the points in $(U_\nu)h$ which will be a non-empty string beginning with a letter in $\hat{\mathcal{R}}$, and for this $\nu$, we have a local action $h_\nu : C_n \rightarrow C_{n,r}$ (or from $C_{n,r}$ to $C_{n,r}$ if $\mu = \epsilon$).

So essentially, one needs a little care with the definition of local actions for a homeomorphism $h \in \text{Homeo}(C_{n,r})$, as finitely many associated local actions are maps from $C_n$ to $C_{n,r}$, while the rest of the associated local actions are maps from $C_n$ to $C_n$.

5 On the Rational Group $\mathcal{R}_n$ and the Related Groups $\mathcal{R}_{n,r}$

In this section we will specify the group $\mathcal{R}_{n,r}$, which is akin to the group $\mathcal{R}_n$ of Grigorchuk, Nekrashevych, and Suschanskiï in [10]. For the most part, we follow the ideas and definitions of [10].

5.1 Defining standard transducers and associated notation

**Definition 5.1.** A transducer is a tuple $A = (X_i, X_o, Q, \pi, \lambda)$, where:

1. $X_i$ is a finite alphabet, the input alphabet,
2. $X_o$ is a finite alphabet, the output alphabet,
3. $Q$ is a set, the set of states,
4. $\pi : X_i \times Q \rightarrow Q$ is a mapping, the transition function, and
5. $\lambda : X_i \times Q \rightarrow X_o^*$ is a mapping, the output function. (Recall that $X_o^*$ is the set of all finite strings in the alphabet $X_o$; that is, "*" is the "Kleene star" operator.)

The language "transducer" arises, as transducers are meant to model machines which transform inputs to new outputs in a controlled fashion. In particular, one is to imagine that there is always a specified "active state" (say $q \in Q$ for the purposes of this discussion) from which the transducer will process its input. The transducer then "processes from $q$" as follows. First, it reads a letter $a \in X_i$ from an input tape, and then it performs two actions. These actions are:

1. the active state changes to the state $\pi(a, q)$, and
2. the transducer writes the word $\lambda(a, q)$ to an output tape.

Thus, transducers transform input strings to output strings.

The functions $\lambda$ and $\pi$ can be inductively extended to the set $(X_i^* \setminus \{\epsilon\}) \times Q$ according to the following recurrence rules:

$$\pi(\mu \cdot \nu, q) := \pi(\nu, \pi(\mu, q)) \quad \text{and} \quad \lambda(\mu \cdot \nu, q) := \lambda(\mu, q) \cdot \lambda(\nu, \pi(\mu, q)),$$

where $\mu \in X_i$ and $\nu \in X_o^*$. Now further extend the definition of $\lambda$ so that it is defined on inputs from $X_i^*$ in the obvious manner. Note that an infinite string might still be transformed.
to a finite string if while processing the infinite string we at some stage visit a state \( q \) with no output on the next letter, and from then on, always visit states that give no output as we process the remaining letters of the input string.

In all that follows below, we will assume that our transducers are constructed in such a way as to never transform an infinite string to a finite string. In this way, for any state \( q \in Q \), we see that \( \lambda(\cdot, q) : X_i^\infty \to X_o^\infty \) will always represent a continuous map which map we will denote as \( h_{A_q} : X_i^\infty \to X_o^\infty \).

### 5.2 Some technicalities for transducers

Let us now consider a fixed (initial) automaton \( A_{q_0} := \langle X_i, X_o, Q, \pi_A, \lambda_A, q_0 \rangle \), where here, we are specifying \( q_0 \) as representing an initial “active state” of the general automaton \( A = \langle X_i, X_o, Q, \pi_A, \lambda_A \rangle \). We say the automaton \( A_{q_0} \) is an initial transducer. For both \( A \) and \( A_{q_0} \), we say the automaton is finite whenever \( |Q| < \infty \).

Now consider the automaton \( A_{q_0} \), and consider the continuous map \( h = h_{A_{q_0}} : X_i^\infty \to X_o^\infty \) induced by \( A_{q_0} \). Recall that for any word \( \tau \in X_i^* \) we use the notation \( h_{\tau} \) to represent the local action of \( h \) at \( \tau \), while \( \theta h_{\tau} = \text{Root}(U, h) \in X_i^* \). Let \( q \in Q \) and \( \nu \in X_i^* \) be such that \( \pi_A(\nu, q_0) = q \). If \( \lambda_A(\nu, q_0) \) is a proper prefix of \( \theta h_{\nu}(\nu) \), then \( q \) is considered a state of incomplete response. Also, if for some \( q \in Q \) and for every \( \nu \in X_i^* \) we have \( \pi(\nu, q_0) \neq q \), then we say \( q \) is an inaccessible state, noting that such a state \( q \) could never have any bearing on the definition of the map \( h \). Finally, if two accessible states \( q_1, q_2 \in Q \) are so that \( h_{A_{q_1}} = h_{A_{q_2}} \), then we say that \( q_1 \) and \( q_2 \) are equivalent states.

We say the transducer \( A_{q_0} \) is minimal if \( A_{q_0} \) has no states of incomplete response, has all states accessible, and whenever \( q_1, q_2 \in Q \) are equivalent states, we have \( q_1 = q_2 \).

We note that a general transducer is asynchronous in the sense that reading any input letter from some state, we have no guarantee that the output of the function \( \lambda \) will be a single letter. In the special case that for all states \( q \in Q \) of the transducer, and for all input letters \( x \in X_i \), we have a guarantee that \( |\lambda(x, q)| = 1 \), then we say the transducer is synchronous. Thus, if we do specify that a transducer is asynchronous, then this generally means that we are reminding the reader that (at that time) we have no guarantee that the output of the function \( \lambda \) has length 1.

One can represent transducers as directed labelled graphs as follows. Let \( A \) be an automaton as above. Our graph \( \Gamma_A \) will have vertex set the set \( Q \), and a directed edge from \( q \) to \( \pi(a, q) \) for each pair \( (a, q) \in X_i \times Q \), and where we label this edge with the string “\( a/w \)” where \( w = \lambda(a, q) \) is a word in \( X_o^\infty \).

### 5.3 Transducers acting on \( \mathcal{C}_{n,r} \)

We will use transducers to model self-homeomorphisms of the spaces \( \mathcal{C}_{n,r} \). This introduces various issues that will cause us to slightly modify our definition of automata. Above, we already extended the definition of \( \lambda \) to receive infinite inputs. We still need to discuss what needs to be done to represent the fact that the letters in \( \hat{r} \) only appear once in a string representing a point in \( \mathcal{C}_{n,r} \).

Thus, we can specify an initial transducer \( A_{q_0} \) as above with alphabets \( X_i = X_o = \hat{r} \sqcup \{0, 1, 2, \ldots, n - 1\} \), and we can ask when this transducer induces a continuous map \( h : \mathcal{C}_{n,r} \to \mathcal{C}_{n,r} \) as above (perhaps even a self-homeomorphism). Such a transducer will actually map the Cantor space \( X_i^n \) to \( X_i^* \sqcup X_o^n \), which is a much bigger space than we need if we want to restrict to a map from \( \mathcal{C}_{n,r} \) to \( \mathcal{C}_{n,r} \). We thus wish to consider a class of initial transducers which are found by restricting these larger transducers to their actions on \( \mathcal{C}_{n,r} \) (and only allow such transducers which turn such infinite inputs into infinite outputs). Of
course, we would like an internal characterisation of these restricted transducers, which we provide below.

An initial transducer for $\mathcal{C}_{n,r}$ will be a tuple $A_{q_0} = (\hat{r}, \{0, 1, \ldots, n-1\}, R, S, \pi, \lambda, q_0)$ so that:

1. $R$ is a finite set, and $S$ is a (finite or countably infinite) set disjoint from $R$; in the framework above, $Q := R \sqcup S$ is the set of states,
2. $q_0$ belongs to $R$ and is the initial state,
3. $\pi$ is piecewise defined function (the transition function) taking an input and a state, and producing a state according to rules, and
4. $\lambda$ is a piecewise defined function (the output function) taking an input letter and a state, and producing a (empty, finite, or infinite) output string.

The domain of $\pi$ and $\lambda$ is

$$\left(\hat{r} \times \{q_0\}\right) \bigcup \left(\{0, 1, \ldots, n-1\} \times (Q \setminus \{q_0\})\right),$$

the range of $\pi$ is $Q \setminus \{q_0\}$, and the range of $\lambda$ is $W_{n,r,\epsilon} \sqcup \mathcal{C}_{n,r}$. (Note here that $q_0$ is not in the range of $\pi$ since $q_0$ will be the only state from which we can write an output letter from $\hat{r}$, and by design we will only wish to write one of these letters given any long initial input.)

The functions $\pi$ and $\lambda$ (as well as the extended version of $\lambda$, mapping $\mathcal{C}_{n,r} \to \mathcal{C}_{n,r}$) will obey some rules, as follows:

1. whenever $\pi(x, q_1) = q_2 \in R$, we have $q_1 \in R$ and $\lambda(x, q_1) = \epsilon$,
2. whenever $q_1 \in R$ and $\pi(x, q_1) = q_2$, with $q_2 \in S$, we have $\lambda(x, q_1) \in W_{n,r}$,
3. if $q \in S$, then for all $x \in \{0, 1, \ldots, n-1\}$ we have $\lambda(x, q) \in W_{n,r}$ and $\pi(x, q) \in S$,
4. if $q \in Q$ and $w$ is a non-empty word so that $\pi(w, q) = q$, then $q \neq q_0$ and $\lambda(w, q) \neq \epsilon$, and
5. whenever $q \in S$ and $x \in \mathcal{C}_n$, we have $\lambda(x, q) \in \mathcal{C}_n$.

The idea of these conditions is that the letters in $\hat{r}$ are not put as output more than once, and then just at the beginning of the total output string given any initial input string. Notice the last two conditions guarantee that $A_{q_0}$ will induce a function $h_{A_{q_0}} : \mathcal{C}_{n,r} \to \mathcal{C}_{n,r}$, since no point (which by definition is given as an infinite string) will be transformed into a finite output string. Also, the penultimate condition guarantees that the graph underlying the automaton admits no directed cycles in the states in $R$. If either of the last two conditions fails, the transducer is degenerate; we will call a transducer as above satisfying all of these conditions non-degenerate.

### 5.4 Transducers, continuity, and $\omega$-equivalence

Suppose we are given a non-degenerate initial transducer $A_{q_0}$ of one of the two varieties above. In the first case, our transducer induces a map $h_{A_{q_0}} : \mathcal{C}_n \to \mathcal{C}_n$, and in the second case, a map $h_{A_{q_0}} : \mathcal{C}_{n,r} \to \mathcal{C}_{n,r}$. It is very easy, as mentioned above, to prove in either case that the map $h_{A_{q_0}}$ is continuous, and we say that $A_{q_0}$ represents the continuous map $h_{A_{q_0}}$.

Note that if a continuous function $h : \mathcal{C}_n \to \mathcal{C}_n$ or $h : \mathcal{C}_{n,r} \to \mathcal{C}_{n,r}$ can be represented by an initial automaton $A_{q_0}$ (and we will show below that it can, following a discussion of [10]),
then $A_{q_0}$ is not unique amongst all of the transducers which represent $h$. Thus, two initial
transducers are said to be $\omega$-equivalent if they represent the same continuous function.

We now trace through the basics of a construction in [10] but in our context, to show
how to build a transducer representing a homeomorphism from $\mathcal{C}_{n,r}$ to $\mathcal{C}_{n,r}$. The construction
we describe can be used to build a transducer representing any continuous map from $\mathcal{C}_{n,r}$
to itself, although in this more general case, the construction might result in a transducer
with a state from which one might read in one input letter and write as output a point in
Cantor space. The construction below, even in the case of homeomorphisms, can result in a
transducer where the set $S$ (and so, $Q$) is infinite.

Let $h \in \text{Homeo}(\mathcal{C}_{n,r})$. We inductively define a non-degenerate initial transducer
$\tilde{A}_{q_0} := \{ \hat{r}, \{0, 1, \ldots, n-1\}, R, S, \pi, \lambda, q_0 \}$
representing $h$ as follows:

1. set $Q := W_{n,r,\epsilon} \sqcup \{ \epsilon \}$,
2. set $R := \hat{r}$,
3. set $S := Q \setminus R$,
4. for $a \in \hat{r}$ define $\pi(a, \epsilon) := a$,
5. for $a \in \hat{r}$ define $\lambda(a, \epsilon) := \theta_h(a)$,
6. for $\nu \in Q \setminus \{ \epsilon \}$ and $x \in \{0, 1, \ldots, n-1\}$ define $\pi(x, \nu) := \nu \hat{x}$,
7. for $\nu \in Q \setminus \{ \epsilon \}$ and $x \in \{0, 1, \ldots, n-1\}$ define
   $$\lambda(x, \nu) := \lambda(\nu, \epsilon)^{-1}(\theta_h(\nu \hat{x}) - \lambda(\nu, \epsilon)),$$
8. set $q_0 := \epsilon \in Q$ as the initial state.

**Remark 5.2.** The following points are worth mentioning. The first is natural and important
notationally, whilst the second is a technical point relating to our transition from the general
rational group to the “rational group acting on $\mathcal{C}_{n,r}$.” The third will be discussed more fully
in the paragraphs to follow.

1. It is immediate by construction that the initial automaton $\tilde{A}_{q_0}$ represents $h$. That is,
   $h = h_{\tilde{A}_{q_0}}$.
2. In this construction, $R \subset Q$ is the set of states $\nu \in W_{n,r,\epsilon} \sqcup \{ \epsilon \}$ for which $\theta_h(\nu) = \epsilon$.
3. Even though $\tilde{A}_{q_0}$ is a non-degenerate automaton with no inaccessible states and no
   states of incomplete response, it might not be minimal, and it might not be synchronous.

We discuss the last remark above. First, note that there is no general algorithm to
transform the above transducer $\tilde{A}_{q_0}$ to a synchronous $\omega$-equivalent automaton. For instance,
many homeomorphisms of $\mathcal{C}_{n,r}$ are not induced by automorphisms of trees (or more accurately,
forests consisting of $r$ infinite $n$-ary rooted trees). In general, though, we can reduce
$\tilde{A}_{q_0}$ to a minimal automaton $A_{q_0}$ as in the subsection [5.5] below (only the third step of
the reduction process will be required, which step will identify states with equivalent local
actions).
Remark 5.3. There is a bijection between the states of \( A_{q_0} \) and the set of local actions of \( h \) (including the local action of \( h \) at the empty input). Thus, while \( A_{q_0} \) is an infinite automaton, \( A_{q_0} \) might be finite. Finally, we note that if \( A_{q_0} \) represents a homeomorphism of \( C_{n,r} \), then given a letter-state pair \((x, q)\) in the domain of \( \lambda \), we have that \( \lambda(x, q) \) is always a finite word. (Note that, for instance, for a continuous function \( f : C_n \to C_n \) which sends an entire cylinder set \( U_\mu \) to an eventually periodic point, the automaton produced by following the construction above may be finite, whilst the guaranteed output after some finite input might still be an infinite word.)

Along the lines of the last remark, the following theorem is direct consequence of Theorem 2.5 of [10].

Theorem 5.4 (Grigorchuk, et al.). Let \( 1 \leq r < n \) be integers. Any homeomorphism \( h : C_{n,r} \to C_{n,r} \) can be represented by a finite non-degenerate initial transducer if and only if the set of local actions of \( h \) is finite.

5.5 Reducing transducers

When given a non-degenerate initial transducer \( A_{q_0} = (Q_A, X, \pi_A, \lambda_A, q_0) \) recall that we say that \( A_{q_0} \) is minimal if all of its states are accessible, it has no states of incomplete response, and any two distinct states represent distinct local maps.

In this section we describe the process of reduction that turns an initial automaton into a minimal automaton, following the discussion of [10]. Note that the proofs in [10] are given of \( \pi \) avoids creating a circuit of modifications (so that the resulting algorithm might never stop). However, as our transducers act on \( C_{n,r} \), we do not need to do this: the state \( q_0 \) is the unique state that reads letters from \( \hat{r} \), and will never admit incoming transitions from elsewhere in the automaton, as all of our input strings start with such a letter but also only have one such letter. In particular, we can simply execute the algorithm of [10] using
the original $q_0$ instead of a new state $q_{-1}$, thus this step of the algorithm will result in an
omega-equivalent transducer using the same set $Q_A$ of states as for the original transducer $A_{q_0}$.

**Step 2:** (Removing inaccessible states) Remove from $Q$ every state that can not be reached
from the initial state $q_0$. That is, if for some $q \in Q$ and for every $\nu \in W_{n,r,\epsilon}$ we have
$\pi(\nu, q_0) \neq q$, then remove $q$ from $Q$. As $Q$ is finite, we can detect in finite time if at some
length $k$, the words in $W_{n,r,\epsilon}$ of length $k$ only cause the transitions of $\pi$ from $q_0$ to hit states
already seen by following transitions from $q_0$ for shorter words, at which point any unvisited
states will never be visited, and can be removed. Thus, our new automaton may have fewer
states than the original.

**Step 3:** (Identifying states representing the same local actions) If for two accessible states
$q, p \in Q \setminus \{q_0\}$ (in the case of a map on $C_n$, do this across all of $Q$) we have that for every
$\nu \in W_n$ the equation $\lambda(\nu, q) = \lambda(\nu, p)$ holds, then we can identify $p$ and $q$ as a single state.
Note that as in Step 2, we can detect our condition after processing finitely many words from
$W_n$. (Note that the initial transducers $A_p$ and $A_q$ are omega-equivalent, so there is no reason to
treat $p$ and $q$ as separate states in $A_{q_0}$.) After these potential identifications, we see that our
final state sets will be a quotient set of a subset of the initial set of states $Q_A$, so again our
new transducer can have no more states than the original transducer $A$.

Grigorchuk et al. prove (in their context, but the proof is essentially the same in ours)
that these steps produce the unique (up to isomorphism) reduced, non-degenerate initial
transducer in any omega-equivalency class (Proposition 2.8 of [10]).

### 5.6 The groups $R_n$ and $R_{n,r}$

We are now in position to define the rational groups $R_n$ and $R_{n,r}$.

**Definition 5.5.** Define $R_n \subseteq \text{Homeo}(C_n)$ and $R_{n,r} \subseteq \text{Homeo}(C_{n,r})$ to be the sets of homeo-
morphisms which can be represented by minimal transducers which are finite.

With definitions in place, we can now state the following lemma.

**Lemma 5.6.** For integers $1 \leq r < n$, each of the sets $R_n$ and $R_{n,r}$ forms a group under
composition.

Note that this is proven for $R_n$ in [10], and the details of that proof work just as well for
$R_{n,r}$.

**Remark 5.7.** The group $G_{n,r}$ coincides with a subgroup of $R_{n,r}$. In particular, for $g \in \text{Homeo}(C_{n,r})$, we will have $g \in G_{n,r}$ if and only if $g$ is in $R_{n,r}$ and if, when $g$ is represented by
a minimal non-degenerate automaton $A_{q_0} = (\hat{r}, R, S, \pi, \lambda, q_0)$, then there is some constant $m$ and a state $q \in S$ so that for any word $w \in W_{n,r}$ with $|w| \geq m$, we have $\pi(w, q_0) = q$, and for all $x \in \{0, 1, \ldots, n-1\}$, we have $\lambda(x, q) = x$.

The following is a general proposition for transducers transforming a Cantor space into
itself. For instance, it will apply to elements of the various types of rational groups we are
discussing here. We give the statement for the Grigorchuk, Nekrashevych, Suschanskii form
of the rational group.

**Proposition 5.8.** Suppose $A = (\{0, 1, \ldots, n-1\}, Q, \pi_A, \lambda_A)$ is a finite transducer. If, for
each state $q \in Q$, the induced map $h_A : C_n \rightarrow C_n$ is a self-homeomorphism of $C_n$, then $A$
is synchronous, has no states with incomplete response, and for each state $q$, the restricted
map $\lambda_A(\cdot, q) : \{0, 1, \ldots, n-1\} \rightarrow \{0, 1, \ldots, n-1\}$ is a permutation.
Proof: Recall that the map \( A_q : C_n \to C_n \) is defined as the map on Cantor space induced as the infinite extension of the map \( \lambda_A(\cdot, q) : \{0, 1, \ldots, n-1\}^* \to \{0, 1, \ldots, n-1\}^* \). Now, for each letter \( a \in \{0, 1, \ldots, n-1\} \), we have a word \( w_a := \lambda_A(a, q) \) and a state \( q_a := \pi(a, q) \in Q \).

By assumption, the image of \( A_q \) is the whole of the Cantor space, and \( A_q \) is also injective. Therefore, we must have

\[
C_n = \text{Image}(h_{A_q}) = \bigcup_{a \in \{0, 1, \ldots, n-1\}} w_a \cdot \text{Image}(h_{A_q}) = \bigcup_{a \in \{0, 1, \ldots, n-1\}} w_a \cdot C_n.
\]

We know that for any \( a \in \{0, 1, \ldots, n-1\} \), we have \( \text{Image}(h_{A_q}) \) is all of Cantor space because the map \( h_{A_q} \) is a homeomorphism for each state \( q_a \) (and so, \( w_a \) must represent a complete response for the state \( q \) with input \( a \)). Moreover, and we know the union above is also a disjoint union of these cylinder sets: if there are \( a, b \in \{0, 1, \ldots, n-1\} \) with \( a \neq b \) but \( w_a \cdot C_n \cap w_b \cdot C_n \neq \emptyset \), then the map \( h_{A_q} \) would not be injective.

But this means that the set of words \( \{w_a | a \in \{0, 1, \ldots, n-1\}\} \) is a complete antichain (with cardinality \( n \)) for the poset of words \( \{0, 1, \ldots, n-1\}^* \), and the only such antichain is the set of \( n \) distinct words of length \( 1 \). In particular, for all \( a \in \{0, 1, \ldots, n-1\} \), we have \( |\lambda(a, q)| = 1 \), and the restricted map \( \lambda_A(\cdot, q) : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\} \) is actually a permutation.

\[ \square \]

6 Automorphisms Admit Few Local Actions

This section is devoted to the proof of the following result.

Theorem 6.1. Let \( \phi \in \text{Aut}(G_{n,r}) \), and let \( \varphi : C_{n,r} \to C_{n,r} \) be the Rubin conjugator representing \( \phi \). Then the set

\[ \mathcal{L} \varphi = \{ \varphi : C_n \to C_n \mid \nu \in W_{n,r} \} \]

is finite.

To show this theorem, we start with a series of definitions and lemmas that will be quite useful.

Definition 6.2. For any clopen set \( U \subseteq C_{n,r} \) there is a minimal finite anti-chain \( A \subseteq B_{n,r} \) such that \( U = \bigcup A \). We denote this anti-chain by \( \text{Dec}(U) \), and we call it the decomposition of \( U \).

Note that if \( A \) is an anti-chain in \( B_{n,r} \) such that \( U = \bigcup A \), then for every \( U_n \in A \), the set \( (A \{U_n\}) \cup \{U_{n-l} \mid 0 \leq l \leq n-1\} \) is also an anti-chain in \( B_{n,r} \) that satisfies \( U = \bigcup A \). Thus, by starting from \( \text{Dec}(U) \) and proceeding this way, the next definition makes sense.

Definition 6.3. The anti-chain \( \text{Dec}_{\text{bal}}(U) \) is the minimal anti-chain \( A \) in \( B_{n,r} \) satisfying \( U = \bigcup A \) and such that \( |\eta_1| = |\eta_2| \) holds for every \( U_m, U_m \in A \).

Although the next two lemmas are somewhat obvious, we provide complete proofs.

Lemma 6.4. Let \( g, h : C_{n,r} \to C_{n,r} \) be two different continuous functions. Then there exists \( U_\nu \in B_{n,r} \) such that \( g(U_\nu) \cap h(U_\nu) = \emptyset \). In particular for every \( U_{\tau} \subseteq g(U_\nu) \) and \( U_{\tau'} \subseteq h(U_\nu) \), we have \( \tau \perp \tau' \).

Proof: Let \( \zeta \in C_{n,r} \) such that \( g(\zeta) \neq h(\zeta) \). Then there exist \( U_\mu, U_{\mu'} \in B_{n,r} \) such that \( U_\mu \cap U_{\mu'} = \emptyset \), \( g(\zeta) \in U_\mu \) and \( h(\zeta) \in U_{\mu'} \). Thus, \( g^{-1}(U_\mu) \cap h^{-1}(U_{\mu'}) \neq \emptyset \) is an open set, hence it contains a basic set \( U_\nu \) which satisfies \( g(U_\nu) \cap h(U_\nu) \subseteq U_\mu \cap U_{\mu'} = \emptyset \). \( \square \)
Lemma 6.5. Let $h \in \text{Homeo}(\mathcal{C}_{n,r})$ and $U_\tau, U_\eta \in \mathcal{B}_{n,r}$. Then for every $U_\tau' \subseteq h(U_\tau)$ there exist $U_\eta' \subseteq h(U_\eta)$ and $\chi \in W_n$ such that $h(U_\tau^- \chi) \subseteq U_\tau'$ and $h(U_\eta^- \chi) \subseteq U_\eta'$.

Proof: Set $\chi_\tau := \text{Root}(h(U_\tau))$ and $\chi_\eta := \text{Root}(h(U_\eta))$. Also set $\alpha' := \tau' - \chi_\tau$. Then there exists $U_{\alpha'} \subseteq h_\eta(\mathcal{C}_{n,r})$ such that $(h_\eta)^{-1}(U_{\alpha'}) \cap (h_\eta)^{-1}(U_{\beta'})$ is a nonempty clopen set. So there exists $U_\chi \subseteq (h_\eta)^{-1}(U_{\alpha'}) \cap (h_\eta)^{-1}(U_{\beta'})$. Setting $\eta' := \chi_\eta \sim \beta'$ we get that $h(U_\tau^- \chi) \subseteq U_\tau'$ and $h(U_\eta^- \chi) \subseteq U_\eta'$.

The following represents a key step in the proof of Theorem 6.1.

Proposition 6.6. Let $h \in \text{Homeo}(\mathcal{C}_{n,r})$ and let $U_\tau, U_\eta \in \mathcal{B}_{n,r}$. Suppose that $h_{\tau} \sim \chi \neq h_{\eta} \sim \chi$ holds for every $\chi \in W_{n,r}$. Then $h \notin H_{n,r,\infty}$.

Proof: Let $\{(\zeta_i, \rho_i)\}_{i=0}^{\infty} \subset W_{n,r} \times W_{n,r}$ be such that for every $\zeta, \rho \in W_{n,r}$ we have

$$\left| \{ i \mid (\zeta, \rho) = (\zeta_i, \rho_i) \} \right| = \aleph_0.$$

We will next construct by induction four convergent sequences of members of $W_{n,r,\tau}$, namely $\{\zeta_i\}_{i=0}^{\infty} \subseteq \eta_i$, $\{\tau_i\}_{i=0}^{\infty} \subseteq \eta_i$, $\{\zeta_i\}_{i=0}^{\infty} \subseteq \eta_i$, and $\{\zeta_i\}_{i=0}^{\infty} \subseteq \eta_i$, such that $\tau_i \sim \zeta_i$, $\eta_i \sim \rho_i$, $\tau_i \sim \zeta_i$, and $\eta_i \sim \rho_i$ for some $\zeta, \rho, \zeta', \rho'$ in $\mathcal{C}_{n,r}$. Furthermore, we will have by construction that $\tau_0 = \tau$ and $\eta_0 = \eta$, and for every $i \in \mathbb{N}$, the following properties will hold:

1. $\tau_i \sim \zeta_i$, $\eta_i \sim \eta_i$, $\tau_i \sim \tau_i$, and $\eta_i \sim \eta_i$.
2. $\tau_i - \tau_i \sim \eta_i - \eta_i$.
3. $h(U_{\tau_i}) \subseteq U_{\tau_i}$ and $h(U_{\eta_i}) \subseteq U_{\eta_i}$.
4. $\zeta_i - \tau_0 \sim \rho_i - \eta_0$.

Assume for a while that we have constructed these sequences. Then we have

1. $\bigcap_{i=0}^{\infty} U_{\tau_i} = \{ \zeta \}$ and $\bigcap_{i=0}^{\infty} U_{\eta_i} = \{ \rho \}$,
2. $\bigcap_{i=0}^{\infty} U_{\tau_i} = \{ \zeta' \}$ and $\bigcap_{i=0}^{\infty} U_{\eta_i} = \{ \rho' \}$,
3. $\zeta - \tau_0 = \rho - \eta_0$.

Property 3. above yields $\zeta \sim \rho$, and by Properties 1. and 2.,

$$\{ h(\zeta) \} = h\left( \bigcap_{i=0}^{\infty} U_{\tau_i} \right) = \bigcap_{i=0}^{\infty} h(U_{\tau_i}) \supseteq \bigcap_{i=0}^{\infty} U_{\tau_i} = \{ \zeta' \},$$

$$\{ h(\rho) \} = h\left( \bigcap_{i=0}^{\infty} U_{\eta_i} \right) = \bigcap_{i=0}^{\infty} h(U_{\eta_i}) \supseteq \bigcap_{i=0}^{\infty} U_{\eta_i} = \{ \rho' \}.$$ 

Thus, $h(\zeta) = \zeta'$ and $h(\rho) = \rho'$. However, we claim that $\zeta' \sim \rho'$, and hence, $h \notin H_{n,r,\infty}$. Indeed, if there are $\alpha, \beta \in W_{n,r}$ and $\delta \in \mathcal{C}_{n,r}$ such that $\zeta' = \alpha \delta$ and $\rho' = \beta \delta$, then choosing $i \in \mathbb{N}$ satisfying $(\zeta_i, \rho_i) = (\alpha, \beta)$ and large-enough so that $\alpha \leq \tau_i$ and $\beta \leq \eta_i$, we have that both $\tau_i - \zeta_i$ and $\eta_i - \rho_i$ are initial segments of $\delta$, so they are compatible. But by property (P3) above, we have $(\tau_i - \zeta_i) \|^\perp (\eta_i - \rho_i)$, a contradiction.

To begin with the construction of the sequences satisfying (P3), (P4), (P5) and (Pd) above, carry out the following steps:

1. Set $\tau_0 := \tau$, $\eta_0 := \eta$, $\chi_{\tau_0} := \text{Root}(h(U_{\tau_0}))$ and $\chi_{\eta_0} := \text{Root}(h(U_{\eta_0})).$
2. Compare $(\zeta_0, \rho_0)$ with $(x_{n_0}, x_{n_0})$.

(a) If $\zeta_0 \not\leq x_{n_0}$ or $\rho_0 \not\leq x_{n_0}$, then:

i. If $\zeta_0 \not\leq x_{n_0}$, then choose any $U_{\alpha'} \subseteq h_{n_0}(E_{n,r})$ such that $\zeta_0 \not\leq x_{n_0} \cap \alpha'$.

ii. Else $\rho_0 \not\leq x_{n_0}$, then choose any $U_{\beta'} \subseteq h_{n_0}(E_{n,r})$ such that $\rho_0 \not\leq x_{n_0} \cap \beta'$.

(b) If $\zeta_0 \leq x_{n_0}$ and $\rho_0 \leq x_{n_0}$, then:

i. If $\zeta_0 = x_{n_0}$ and $\rho_0 = x_{n_0}$, then $h_{n_0} \neq h_{n_0}$, by Lemma 6.4 there exists $U_{\nu} \in B_{n,r}$ such that $h_{n_0}(U_{\nu}) \cap h_{n_0}(U_{\nu}) = \emptyset$, so choose any $U_{\alpha'} \subseteq h_{n_0}(U_{\nu})$.

ii. Else $x_{n_0} - \zeta_0 \leq x_{n_0} - \rho_0$, and then $\delta := (x_{n_0} - \rho_0) - (x_{n_0} - \zeta_0)$ and choose any $U_{\alpha'} \subseteq h_{n_0}(E_{n,r})$ such that $\delta \cap \alpha'$.

iii. Else $x_{n_0} - \zeta_0 \leq x_{n_0} - \rho_0$, and then $\delta := (x_{n_0} - \zeta_0) - (x_{n_0} - \rho_0)$ and choose any $U_{\beta'} \subseteq h_{n_0}(E_{n,r})$ such that $\delta \cap \beta'$.

iv. Else $(x_{n_0} - \zeta_0) \perp (x_{n_0} - \rho_0)$, and then choose any $U_{\alpha'} \subseteq h_{n_0}(E_{n,r})$.

3. If we have chosen $U_{\alpha'}$, then set $\tau_0 := x_{n_0} \cap \alpha'$. By Lemma 6.5 there exists $\eta'$ and $\chi \in W_{n,r}$ such that $h(U_{\alpha'} \cap \chi) \subseteq U_{\tau_0} \subseteq h(U_{\eta'})$ and $h(U_{\eta'}) \subseteq U_{\eta'} \subseteq h(U_{\eta})$. Then set $\eta_0 := \eta'$, $\tau_1 := \tau_0 \cap \chi$ and $\eta_1 := \eta_0 \cap \chi$.

4. If we have chosen $U_{\beta'}$, then set $\eta_0 := x_{n_0} \cap \beta'$. By Lemma 6.5 there exists $\tau'$ and $\chi \in W_{n,r}$ such that $h(U_{\beta'} \cap \chi) \subseteq U_{\tau'} \subseteq h(U_{\eta})$ and $h(U_{\eta}) \subseteq U_{\eta} \subseteq h(U_{\eta})$. Then set $\tau_0 := \tau'$, $\tau_1 := \tau_0 \cap \chi$ and $\eta_1 := \eta_0 \cap \chi$.

We have

(a) $\tau_0 \leq \tau_1$, $\eta_0 \leq \eta_1$ and $\tau_1 - \tau_0 = \eta_1 - \eta_0$,

(b) $h(U_{\tau_1}) \subseteq U_{\tau_0} \subseteq h(U_{\eta_1})$ and $h(U_{\eta_1}) \subseteq U_{\eta_0} \subseteq h(U_{\eta_0})$,

(c) if $\zeta_0 \leq \tau_0$ and $\rho_0 \leq \eta_0$, then $(\tau_0 - \zeta_0) \perp (\eta_0 - \rho_0)$,

(d) $h_{\tau_1} \neq h_{\eta_1}$.

Note that (c) above is due to step 2., and (d) is guaranteed by the hypothesis of the proposition.

Next, assume that we have already defined $\{\tau_0, \ldots, \tau_k\}, \{\eta_0, \ldots, \eta_k\}, \{\tau_{0}', \ldots, \tau_{k-1}'\}$ and $\{\eta_{0}', \ldots, \eta_{k-1}'\}$, in such a way the following holds:

(a) for every $0 \leq i < k$, we have $\tau_i \leq \tau_{i+1}$, $\eta_i \leq \eta_{i+1}$, $\tau_{i+1} - \tau_i = \eta_{i+1} - \eta_i$, and for $0 \leq i < k - 1$, we have $\tau_i \leq \tau_{i+1}$ and $\eta_i \leq \eta_{i+1}$.

(b) $h(U_{\tau_{i+1}}) \subseteq U_{\tau_i} \subseteq h(U_{\eta_{i+1}})$ and $h(U_{\eta_{i+1}}) \subseteq U_{\eta_i} \subseteq h(U_{\eta_i})$ for every $0 \leq i < k$,

(c) if $\zeta_i \leq \tau_i'$ and $\rho_i \leq \eta_i'$, then $(\tau_i' - \zeta_i) \perp (\eta_i' - \rho_i)$,

(d) $h_{\tau_i} \neq h_{\eta_i}$.

We want to define $\tau_{k+1}, \eta_{k+1}, \tau_{k}'$ and $\eta_{k}'$. To do this, we carry out the following steps:

1. Set $\chi_{k+1} := \text{Root}(h(U_{\tau_k}))$ and $\chi_{\eta_k} := \text{Root}(h(U_{\eta_k}))$, and note that $\tau_{k-1} \leq \chi_{\tau_k}$ and $\eta_{k-1} \leq \chi_{\eta_k}$.

2. Compare $(\zeta_k, \rho_k)$ with $(\chi_{\tau_k}, \chi_{\eta_k})$.

(a) If $\zeta_k \not\leq \chi_{\tau_k}$ or $\rho_k \not\leq \chi_{\tau_k}$ then:

i. If $\zeta_k \not\leq \chi_{\tau_k}$, then choose any $U_{\alpha'} \subseteq h_{\tau_k}(E_{n,r})$ such that $\zeta_k \not\leq \chi_{\tau_k} \cap \alpha'$.

ii. Else $\rho_k \not\leq \chi_{\tau_k}$, then choose any $U_{\beta'} \subseteq h_{\eta_k}(E_{n,r})$ such that $\rho_k \not\leq \chi_{\eta_k} \cap \beta'$.

(b) If $\zeta_k \leq \chi_{\tau_k}$ and $\rho_k \leq \chi_{\eta_k}$, then:

i. If $\zeta_k = \chi_{\tau_k}$ and $\rho_k = \chi_{\eta_k}$, then as $h_{\tau_k} \neq h_{\eta_k}$, by Lemma 6.4 there exists $U_{\nu} \in B_{n,r}$ such that $h_{\tau_k}(U_{\nu}) \cap h_{\eta_k}(U_{\nu}) = \emptyset$, so choose any $U_{\alpha'} \subseteq h_{\tau_k}(U_{\nu})$. 


ii. Else if $\chi_{\tau_k} - \zeta_k \leq \chi_{\eta_k} - \rho_k$, and then set $\delta := (\chi_{\eta_k} - \rho_k) - (\chi_{\tau_k} - \zeta_k)$ and choose any $U_{\alpha'} \subseteq h_{\tau_k}(\mathcal{C}_{n,r})$ such that $\delta \perp \alpha'$.

iii. Else if $\chi_{\tau_k} - \zeta_k \geq \chi_{\eta_k} - \rho_k$, and then set $\delta := (\chi_{\eta_k} - \rho_k) - (\chi_{\tau_k} - \zeta_k)$ and choose any $U_{\beta'} \subseteq h_{\eta_k}(\mathcal{C}_{n,r})$ such that $\delta \perp \beta'$.

iv. Else $(\chi_{\tau_k} - \zeta_k) \perp (\chi_{\eta_k} - \rho_k)$. Here, choose any $U_{\alpha'} \subseteq h_{\tau_k}(\mathcal{C}_{n,r})$.

3. If we have chosen $U_{\alpha'}$, then set $\tau'_k := \chi_{\tau_k} \setminus \alpha'$. By Lemma 6.12 there exist $\eta'$ and $\chi \in W_{n,r}$ such that $h(U_{\tau_k} \setminus \chi) \subseteq U_{\tau'_k} \subseteq h(U_{\tau_k})$ and $h(U_{\eta_k} \setminus \chi) \subseteq U_{\eta'_k} \subseteq h(U_{\eta_k})$. Then set $\eta'_k := \eta'$, $\tau_{k+1} := \tau_k \setminus \chi$ and $\eta_{k+1} := \eta_k \setminus \chi$.

4. If we have chosen $U_{\eta'}$, then set $\eta'_k := \chi_{\eta_k} \setminus \beta'$. By Lemma 6.13 there exist $\tau'$ and $\chi \in W_{n,r}$ such that $h(U_{\eta_k} \setminus \chi) \subseteq U_{\tau'_k} \subseteq h(U_{\eta_k})$ and $h(U_{\eta_k} \setminus \chi) \subseteq U_{\eta'_k} \subseteq h(U_{\eta_k})$. Then set $\tau'_k := \tau'$, $\tau_{k+1} := \tau_k \setminus \chi$ and $\eta_{k+1} := \eta_k \setminus \chi$.

The verification of the desired properties works as in the case $k = 0$. This then completes the construction of the sequences, and thus the proof of the proposition.

Corollary 6.7. If $h \in H_{n,r,\sim}$ then there exist $U_{\nu}, U_{\eta} \in B_{n,r}$ such that $\nu$ and $\eta$ are incomparable, $U_{\nu} \cup U_{\eta} \neq \mathcal{C}_{n,r}$ and $h_{\nu} = h_{\eta}$.

Definition 6.8. For $h \in \text{Homeo}(\mathcal{C}_{n,r})$ and $U_{\nu}, U_{\eta} \in B_{n,r}$, we say that $h$ acts on $U_{\nu}$ and $U_{\eta}$ in the same fashion provided that $h_{\nu} = h_{\eta}$.

For two clopen sets $U, W \subseteq \mathcal{C}_{n,r}$ we say that $h$ acts on $U$ and $W$ in the same fashion everywhere provided that $h$ acts on $U_{\nu}$ and $U_{\eta}$ in the same fashion for all $U_{\nu} \in \text{Dec}(U)$ and $U_{\eta} \in \text{Dec}(W)$.

Definition 6.9. Let $h \in \text{Homeo}(\mathcal{C}_{n,r})$ and $U$ and $W$ clopen sets in $\mathcal{C}_{n,r}$. We say that $h$ acts on $U$ and $W$ almost in the same fashion provided that $h_{\nu} \cdot \chi = h_{\eta} \cdot \chi$ holds for all $\chi \in W_n$ with $|\chi| \geq k$ and all $U_{\nu} \in \text{Dec}(U)$, $U_{\eta} \in \text{Dec}(W)$.

When $h$ acts on $U$ and $W$ almost in the same fashion and $k$ is the minimal natural number which satisfies the condition in the above definition, we define $\text{crit}_h(U, W) := k$, which we refer to as the critical level of $U$ and $W$ with respect to $h$.

Definition 6.10. We say that $h$ acts on $U$ and $W$ in the same fashion uniformly provided that for every $U_{\nu} \in \text{Dec}(U)$ and $U_{\eta} \in \text{Dec}(W)$, we have $h_{\nu} = h_{\eta} \cdot \zeta$ for all $\zeta \in W_{n,\zeta}$.

We say that $h$ acts on $U$ and $W$ almost in the same fashion uniformly provided that there exists $k \in \mathbb{N}$ such that for every $U_{\nu} \in \text{Dec}(U)$, $U_{\eta} \in \text{Dec}(W)$ and $\chi, \zeta \in W_n$ with $|\zeta| \geq k$, we have $h_{\nu} \cdot \chi = h_{\eta} \cdot \zeta$.

Remark 6.11. Let $g \in G_{n,r}$ and let $U$ and $W$ be any two clopen sets. Then $g$ acts on $U$ and $W$ almost in the same fashion uniformly. Indeed, local actions associated to small-enough cones are all the identity map.

The following lemma should be obvious but the two that follow after will require some explanation.

Lemma 6.12. Let $g, h \in \text{Homeo}(\mathcal{C}_{n,r})$ and let $U_{\nu}, U_{\eta} \in B_{n,r}$. Suppose that $U_{\nu}g = U_{\nu'}, U_{\eta}g = U_{\eta'} \in B_{n,r}$, $g$ acts on $U_{\nu}$ and $U_{\eta}$ in the same fashion and $h$ acts on $U_{\nu'}$ and $U_{\eta'}$ in the same fashion. Then $gh$ acts on $U_{\nu}$ and $U_{\eta}$ in the same fashion.

Lemma 6.13. Let $g, h \in \text{Homeo}(\mathcal{C}_{n,r})$ and let $U_{\nu}, U_{\eta} \in B_{n,r}$. Suppose that $g$ acts on $U_{\nu}$ and $U_{\eta}$ in the same fashion and $h$ acts on $U_{\nu}g$ and $U_{\eta}g$ almost in the same fashion uniformly. Then $gh$ acts on $U_{\nu}$ and $U_{\eta}$ almost in the same fashion.
Proof: Suppose for a contradiction that $gh$ does not act almost in the same fashion on $U_\nu$ and $U_\eta$. In this case, for all $k \in \mathbb{N}$ there is $\chi \in W_n$ of length $k$ such that $h_\nu \cdot \chi \neq h_\eta \cdot \chi$. Using the finite intersection property, we conclude that there is some point $y \in \mathcal{C}_n$ so that if we set $x_\nu := \nu \cdot y$ and $x_\eta := \eta \cdot y$, then for all $\gamma$ a prefix of $\delta$, we have

$$(y - \gamma) \cdot (gh)_\nu \cdot \gamma \neq (y - \gamma) \cdot (gh)_\eta \cdot \gamma.$$  

Let $\alpha, \beta \in W_{n,r}$ be so that $x_\nu g \in U_\alpha$ and $x_\eta g \in U_\beta$, where $U_\alpha \in \text{Dec}(U_\nu g)$ and $U_\beta \in \text{Dec}(U_\eta g)$. Since $g$ acts on $U_\nu$ and $U_\eta$ in the same fashion, there is $y' \in \mathcal{C}_n$ so that $x_\nu g = \alpha \cdot y'$ whilst $x_\eta g = \beta \cdot y'$. Since $h$ acts on $U_\nu g$ and $U_\eta g$ almost in the same fashion uniformly, there is $m \in \mathbb{N}$ such that if $\chi \in W_n$ has length $m$, then for all $U_\rho \in \text{Dec}(U_\nu g)$ and all $U_\tau \in \text{Dec}(U_\eta g)$, we have $h_\rho \cdot \chi = h_\tau \cdot \chi$. Thus, if we let $\chi$ be any prefix of $\delta'$ of length larger than or equal to $m$, then

$$(y' - \gamma) h_\alpha \cdot \chi = (y' - \gamma) h_\beta \cdot \chi.$$  

Now, take $\gamma \in W_n$ a large-enough prefix of $\delta$ so that there is $\chi \in W_n$ of length at least $m$ such that $U_\nu \cdot \gamma g \subset U_\alpha \cdot \chi$ and $U_\eta \cdot \gamma g \subset U_\beta \cdot \chi$. Then we see immediately that

$$(x_\nu - (\nu \cdot \gamma)) \cdot (gh)_\nu \cdot \gamma = (x_\eta - (\eta \cdot \gamma)) \cdot (gh)_\eta \cdot \gamma,$$

that is,

$$(x - \gamma) \cdot (gh)_\nu \cdot \gamma \neq (x - \gamma) \cdot (gh)_\eta \cdot \gamma,$$

which is contrary to our assumption. \(\square\)

Lemma 6.14. Let $\nu_1, \nu_2, \eta_1, \eta_2 \in W_{n,r}$ such that $\nu_1 \perp \nu_2$, $\eta_1 \perp \eta_2$, $U_{\nu_1} \cup U_{\nu_2} \neq \mathcal{C}_{n,r}$ and $U_{\eta_1} \cup U_{\eta_2} \neq \mathcal{C}_{n,r}$. Then there exists $g \in G_{n,r}$ such that $g \mid U_{\nu_1} = g_{\nu_1, \nu_2}$ and $g \mid U_{\nu_2} = g_{\nu_2, \nu_1}$.

Proof: It is enough to show that there exists two maximal ordered anti-chains $\vec{\nu} = \{\nu_1, \ldots, \nu_t\}$ and $\vec{\eta} = \{\eta_1, \ldots, \eta_t\}$ in $W_{n,r}$ such that $t = l$. For that, it is enough to find such two maximal ordered anti-chains such that $t$ and $l$ are congruent modulo $n - 1$ since by assumption we have $t > 2$ and $l > 2$, so we can enlarge the smaller anti-chain by taking the $n$ immediate successors of the third member instead of that member and by that means enlarge the cardinality of the anti-chain by $n - 1$ (and so on).

Assume without meaningful loss of generality that $|\nu_1| \leq |\nu_2|$ and $|\eta_1| \leq |\eta_2|$. Let $\chi_\nu \in W_{n,r}$ such that $|\chi_\nu| = |\nu_1|$ and $\chi_\nu \leq \nu_2$, and let $\chi_\eta \in W_{n,r}$ such that $|\chi_\eta| = |\eta_1|$ and $\chi_\eta \leq \eta_2$.

Let $A_\nu := \{\zeta \in W_{n,r} \mid |\zeta| = |\nu_1|, \zeta \neq \nu_1, \zeta \neq \nu_2\}$ and $B_\nu := \{\zeta \in W_{n,r} \mid |\zeta| = |\nu_2|, \chi_\nu \leq \zeta, \zeta \neq \nu_2\}$. Then $D_\nu := A_\nu \cup B_\nu \cup \{\nu_1, \nu_2\}$ is a maximal anti-chain which contains $\nu_1$ and $\nu_2$, and

$$|D_\nu| = (rn|\nu_1| - 1) + (rn|\nu_2| - |\nu_1| - 1) + 2 = rn|\nu_2| + rn|\nu_1| - 1.$$  

Let $A_\eta := \{\zeta \in W_{n,r} \mid |\zeta| = |\eta_1|, \zeta \neq \eta_1, \zeta \neq \eta_2\}$ and $B_\eta := \{\zeta \in W_{n,r} \mid |\zeta| = |\eta_2|, \chi_\eta \leq \zeta, \zeta \neq \eta_2\}$. Then $D_\eta := A_\eta \cup B_\eta \cup \{\eta_1, \eta_2\}$ is a maximal anti-chain which contains $\eta_1$ and $\eta_2$, and

$$|D_\eta| = (rn|\eta_1| - 1) + (rn|\eta_2| - |\eta_1| - 1) + 2 = rn|\eta_1| + rn|\eta_2| - 1.$$  

Thus, $|D_\nu| \equiv |D_\eta|(mod n - 1)$. Finally, order $D_\nu$ so that $\nu_1$ and $\nu_2$ are the first elements and order $D_\eta$ so that $\eta_1$ and $\eta_2$ are the first elements. \(\square\)
Corollary 6.15. Let $h \in \text{Homeo}(\mathfrak{C}_{n,r})$ such that $h^{-1}G_{n,r}h \subseteq G_{n,r}$. Then for every $U_\nu, U_\eta \in \mathcal{B}_{n,r}$ such that $U_\nu \cap U_\eta \neq \mathfrak{C}_{n,r}$, the map $h$ acts on $U_\nu$ and $U_\eta$ almost in the same fashion.

Proof: By Lemma 6.3, we have $h \in H_{n,r,\infty}$, so by Corollary 6.7 there exist $U_\nu, U_\eta \in \mathcal{B}_{n,r}$ such that $h$ acts on $U_\nu$ and $U_\eta$ in the same fashion. Moreover, we can choose these $U_\nu$ and $U_\eta$ such that $\nu' \perp \eta'$ and $U_\nu \cup U_\eta \neq \mathfrak{C}_{n,r}$.

Let $U_\nu, U_\eta \in \mathcal{B}_{n,r}$ be such that $U_\nu \cup U_\eta \neq \mathfrak{C}_{n,r}$. Assume first that $\nu \perp \eta$. By Lemma 6.14 there exists $g \in G_{n,r}$ such that $g|U_\nu = g_\nu'$ and $g|U_\eta = g_\eta'$. So, by Lemma 6.12 $gh$ acts on $U_\nu$ and $U_\eta$ in the same fashion. Letting $f := h^{-1}gh \in G_{n,r}$, we have $hf = gh$. By Remark 6.11 and Lemma 6.13 $h = ghf^{-1}$ acts on $U_\nu$ and $U_\eta$ almost in the same fashion.

Next, assume that $\nu \leq \eta$ or $\eta \leq \nu$. Then there exists $\beta \in W_{n,r}$ such that $\nu \perp \beta$, $\eta \perp \beta$ and $U_\nu \cup U_\eta \neq \mathfrak{C}_{n,r}$. By the choice of $k$, $h$ acts on $U_\nu$ and $U_\beta$ almost in the same fashion and $h$ acts on $U_\eta$ and $U_\beta$ almost in the same fashion. That is, there exists $k_1 \in \mathbb{N}$ such that for every $\zeta \in W_n$ with $|\zeta| \geq k_1$, we have that $h$ acts on $U_\nu \cdot \zeta$ and $U_\beta \cdot \zeta$ in the same fashion, and there exists $k_2 \in \mathbb{N}$ such that for every $\chi \in W_n$ with $|\chi| \geq k_2$, we have that $h$ acts on $U_\eta \cdot \chi$ and $U_\beta \cdot \chi$ in the same fashion. Taking $k := \max\{k_1, k_2\}$, we get that for every $\mu \in W_n$ with $|\mu| \geq k$, the map $h$ acts on $U_\nu \cdot \mu$, $U_\beta \cdot \mu$ and $U_\eta \cdot \mu$ in the same fashion. Hence, $h$ acts on $U_\nu$ and $U_\eta$ almost in the same fashion.

We are finally in position to complete the proof of Theorem 6.1 that is, if $h \in N_{\mathcal{H}(\mathfrak{C}_{n,r})}(G_{n,r})$, then $h$ admits only finitely many types of local actions (recall this implies that $h$ can be represented by a transducer with only finitely many states). Notice that the statement below is slightly more general in that we also allow homeomorphisms $h$ which conjugate $G_{n,r}$ into $G_{n,r}$, even if the image of $G_{n,r}$ under this general conjugation is a proper subset of $G_{n,r}$. (Such homeomorphisms play a role in [2], for instance.)

Corollary 6.16. Let $h \in \text{Homeo}(\mathfrak{C}_{n,r})$ be such that $h^{-1}G_{n,r}h \subseteq G_{n,r}$. Then $h$ uses only finitely many types of action.

Proof: Let us fix $A$ a complete antichain having at least three elements. For instance, if $n = 2$, we can take $A := \{000, 001, 010, 011\}$, and if $n > 2$, we can take either $A := \{0, 1, \ldots, n-1\}$ or $A := \{0, 1, \ldots, n-1\}^2$. In all cases, $A$ can be taken so that all of its words are length three.

By Corollary 6.15 for every $\nu, \eta \in A$ and $a \in \{0, 1, \ldots, n-1\}$, the map $h$ acts on $U_\nu$ and $U_\eta$ almost in the same fashion, and it also acts on $U_\nu$ and $U_\nu \cdot a$ almost in the same fashion. Set

$$k := \max \{ \text{crit}(h_\nu)|\nu, \eta \in A \text{ or } \nu \in A \text{ and } \eta = \nu \cdot a \text{ for some } a \in \{0, 1, \ldots, n-1\} \}.$$ 

Now, let $B := \{\nu_0, \ldots, \nu_{k-1}\}$ be the set of all the $\nu \in W_n$ with $|\nu| = k$ ordered by the lexicographic order of $W_{n,r}$.

We claim that for every $\eta \in W_{n,r}$ with $|\eta| \geq k+3$, there exists $\nu \in B$ such that $h_\eta = h_{\nu \cdot \tau}$ holds for every $\tau \in A$.

To show the claim above, we proceed by induction on $m := |\eta|$. If $m = k+3$, then $\eta = \chi \cdot \nu$ for some $\chi \in A$ and $\nu \in B$. Now, by the choice of $k$, we have $h_\chi \cdot \nu = h_{\nu \cdot \tau}$ for every $\tau \in A$, as desired.

Assume now that for every $\eta \in W_{n,r}$ with $|\eta| = m$ the claim is true. Let $\eta \in W_{n,r}$ be such that $|\eta| = m+1$. Write $\eta = \chi \cdot b \cdot \eta'$ for $\chi \in A$ and $b \in \{0, 1, \ldots, n-1\}$. Then $|\eta'| \geq k$, so by the choice of $k$, we have that $h_\chi \cdot b \cdot \eta' = h_{\eta \cdot \tau}$ holds for every $\tau \in A$. By the induction hypothesis, for $\chi \cdot \eta'$ there exists $\eta \in B$ such that $h_{\chi \cdot \eta'} = h_{\eta \cdot \tau}$ holds for every $\tau \in A$. Thus, for every $\tau \in A$, we have $h_\eta = h_{\chi \cdot b \cdot \eta'} = h_{\chi \cdot \eta'} = h_{\eta \cdot \tau}$, as desired.
Thus, for \( h \in N_{H(\mathfrak{c}_{n,r})}(G_{n,r}) \), we now have that \( h \) admits only finitely many local actions, and so \( h \in \mathcal{R}_{n,r} \) and can be represented by a minimal initial automaton

\[
A_{q_0} := \{ \hat{r}, \{ 0, 1, \ldots, n - 1 \}, R, S, \pi_A, \lambda_A, q_0 \}.
\]

Now, for \( \nu \in W_{n,r} \) minimality ensures us that the local map \( h_\nu : \mathfrak{c}_n \to \mathfrak{c}_n \) is representable by the initial automaton \( A_{\pi_A(\nu, q_0)} \) (since \( \pi_A(\nu, q_0) \) is not a state of incomplete response). That is, we have \( h_\nu = h_{A_{\pi_A(\nu, q_0)}} \). We now introduce simplified notation, reflecting the perspective that the local maps of \( h \) are determined by the states of \( A_{q_0} \).

**Notation 6.17.** Suppose \( h \in \mathcal{R}_{n,r} \) is represented by the minimal initial automaton

\[
A_{q_0} := \{ \hat{r}, \{ 0, 1, \ldots, n - 1 \}, R, S, \pi_A, \lambda_A, q_0 \}
\]

and that \( q \in Q = R \sqcup S \). By the notation \( h_q \) we will mean the local map \( h_\nu \) where \( \nu \in W_{n,r,e} \) is such that \( \pi_A(\nu, q_0) = q \).

### 7 Finding our Place in the Rational Group \( \mathcal{R}_{n,r} \)

Corollary 3.2 shows that \( \text{Aut}(G_{n,r}) \cong N_{H(\mathfrak{c}_{n,r})}(G_{n,r}) \) where \( \text{Homeo}(\mathfrak{c}_{n,r}) \) is the full group of homeomorphisms \( \text{Homeo}(\mathfrak{c}_{n,r}) \). Meanwhile, Theorem 6.1 shows that any element \( \phi \in N_{H(\mathfrak{c}_{n,r})}(G_{n,r}) \) is a homeomorphism of \( \mathfrak{c}_{n,r} \) that admits only finitely many types of local actions. Thus, Theorem 5.3 then implies that any such homeomorphism \( \phi \) is actually an element of \( \mathcal{R}_{n,r} \), since it is a homeomorphism that can be represented by a (non-degenerate) finite transducer.

In this section, we isolate exactly which elements of \( \mathcal{R}_{n,r} \) represent the Rubin conjugators of \( G_{n,r} \), corresponding to its automorphisms.

#### 7.1 Getting in synth

Here we define a property of transducers that will be of great value to us. The central idea of the following definition is that the automata which satisfy this property are ‘synchronizing’ in a sense similar to the way that this word occurs with relation to the Road Colouring Problem, and indeed, in a particularly strong form: given a large enough input word, and starting with any general state in the automaton as the initial active state, then the active state resulting from reading the input word is fully determined only by that initial word, and not in any way by whichever state was the original initial active state.

**Definition 7.1.** Let \( A = (X_i, X_o, Q_A, \pi_A, \lambda_A) \) be a transducer, \( m \) be a natural number, and \( s : X^m_i \to Q_A \) be a function so that if \( w \in X^m_i \) and \( q \in Q_A \), then \( \pi_A(w, q) = s(w) \). In this case we call \( s \) a synchronizing map for \( A \) and we say that \( A \) is synchronizing at level \( m \). If \( A \) is initial and \( A \) represents a homeomorphism \( h \) on \( \mathfrak{c}_n \) and \( h^{-1} \) also admits a representative initial transducer \( B_{m_0} = (X_o, X_i, Q_B, \pi_B, \lambda_B, q_0) \) which is synchronizing at level \( m \), then we say \( A \) (or \( h \)) is bi-synchronizing (at level \( m \)).

In a slightly more general way, we say that a word \( \omega \in X^m_i \) is a synchronizing word or is synchronizing, if \( \pi(\omega, q) \) does not depend on the state \( q \). Thus, \( A \) is synchronizing at level \( m \) whenever all words of length \( m \) are synchronizing.

The above definitions extend to transducers acting on \( \mathfrak{c}_{n,r} \) in the obvious way, using all valid strings of length \( m \), and applying them only to the states where these strings can be applied (so if a string begins with a letter from \( r \), then it must be processed from the initial state).
Remark 7.2. The transducer in Figure 1.2.3 of the Introduction is bi-synchronizing at level 2 and represents a homeomorphism of $C_{3,2}$.

The reader can also easily verify the points of the following remark for general transducers and also our more specific transducers which take inputs from $C_{n,r}$.

Remark 7.3. 1. Suppose $A$ is a transducer and $m$ is a natural number so that $A$ is synchronizing at level $m$. Then for all natural $n \geq m$, we have that $A$ is synchronizing at level $n$.

2. Suppose $A = (\hat{r}, \{0,1, \ldots, n-1\}, R, S, \pi, \lambda, q_0)$ represents a homeomorphism $h \in \text{Homeo}(C_{n,r})$ and is synchronizing at level $m$ for some positive integer $m$ with synchronizing map $s$ taking all inputs of length $m$ to states in $Q := R \cup S$. Then we have the following:

- for all $q_1 \in Q$ and $q_2 \in \text{Image}(s)$, there is a nontrivial word $w$ so that $\lambda(w, q_1) = q_2$,
- for all $q \in \text{Image}(s)$ and all $w \in W_n$, we have $\pi(w, q) \in \text{Image}(s)$, and
- $\text{Image}(s) \subseteq S$.

The image of the synchronizing map is therefore an inescapable set of states in its automaton. In the remainder of this section, our initial transducers will all be given as acting on some space $C_{n,r}$, but all of the definitions apply similarly in the context of the original rational groups $\{R_n\}_{n \geq 1}$ of Grigorchuk, Nekrashevych, and Suschanskii.

Again, let $A_{q_0} = (\hat{r}, \{0,1, \ldots, n-1\}, R, S, \pi, \lambda, q_0)$ be an automaton that is synchronizing at level $m$. We call the maximal sub-automaton of $A_{q_0}$ which uses as its set of states the set $\text{Image}(s)$ the core of $A$, which we denote as Core$(A)$ (note that we drop the reference to the initial state $q_0$ in this notation, as the core is independent of initial state if $A_{q_0}$ is synchronizing). Observe that Core$(A)$ is a non-initial automaton in its own right with transition function $\tilde{\pi}$ and output function $\tilde{\lambda}$ where these functions are defined as the restrictions of the functions $\pi$ and $\lambda$ to the domain $\{0,1, \ldots, n-1\} \times \text{Image}(s)$. That is, we have the induced automaton

$$\text{Core}(A) := (\{0,1, \ldots, n-1\}, \text{Image}(s), \tilde{\pi}, \tilde{\lambda}).$$

Note further that if $A_{q_0}$ is synchronizing at level $m$ for some integer $m$, then there is a least integer $k$ so that $A_{q_0}$ is synchronizing at level $k$, so we say $A_{q_0}$ synchronizes at level $k$. Observe also that if $A_{q_0}$ synchronizes at level $n$ for some integer $n$, then there is some $m \leq n$ so that Core$(A)$ synchronizes at level $m$.

Theorem 7.4. Let $M_{n,r}$ be the set of all endomorphisms $\tau : C_{n,r} \to C_{n,r}$ which are representable by synchronizing transducers with all outputs finite. The set $M_{n,r}$ forms a monoid under composition of maps.

Proof. First observe that if $\phi, \rho$ are two endomorphisms of $C_{n,r}$ that are representable by synchronizing transducers then these transducers are $\omega$-equivalent to finite transducers, since the core of each of these transducers is finite, and one can only visit finitely many states on directed paths in these transducers from the initial states to the cores (else these automata would admit accessible cycles containing states outside of their core states, which would violate the synchronizing condition). Therefore, we will assume in the remainder of this argument that we have two initial transducers $A$ and $B$ representing the maps $\phi$ and $\rho$ respectively, which are minimal and synchronizing, and so finite. We will denote the output and transition functions of these transducers as $\lambda_A, \lambda_B, \pi_A$, and $\pi_B$ in the obvious fashion.
It follows similarly to the arguments given in [10] that the composition of two endomor-
phisms \( \phi \) and \( \rho \) in \( \mathcal{M}_{n,r} \) can be represented by a minimal finite transducer with set of states
given as a subset of the product of the states of \( A \) and \( B \), and with all outputs finite, and
we follow the construction there to form this product and pass to its minimised equivalent
transducer (this will require us only to remove some states, and possibly to finitely modify
some outputs to remove states of incomplete response, for the remaining transitions).

We claim that the resulting transducer \( C \) is also synchronizing.

To this end observe firstly that the resulting transducer (and as well the transducers \( A 
\) and \( B \)) has no cycles of states with \( \varepsilon \) outputs, lest the transducer map a point in the Cantor
space \( \mathcal{C}_{n,r} \) to a finite string. In particular, their is a number \( N \) so that from any state \( q \) in
\( A \), if we read an input of length \( N \), the output will be at least length one.

Next, suppose that \( A \) synchronizes at level \( k \) while \( B \) synchronizes at level \( m \). Increase
\( N \) if necessary, so that \( mN > k \), notice that \( A \) still has the property that upon reading an
input of length \( N \) from any state \( q \), the output will be a non-trivial word. We claim that \( C 
\) synchronizes at level \( mN \).

The reader can see this by carefully considering how the output and transition func-
tions of \( C \) are defined. If one considers some state \( (a,b) \) as the active state for \( C \), where
\( a \) is a state in \( A \) and \( b \) is a state in \( B \), then for each letter \( j \in \{0,1,\ldots,n-1\} \) we have
\( \lambda_C(j,(a,b)) = \lambda_B(\lambda_A(j,a),b) \) while \( \pi_C(j,(a,b)) = (\pi_A(j,a),\pi_B(\lambda_A(j,a),b)) \). As the transitions in \( A \) synchronise on inputs of length \( k < mN \), then as the first coordinate of the
transitions of \( C \) mirrors the transitions of \( A \), we see that the first coordinate is completely
determined by an input of length \( mN \). Similarly, as \( A \) must produce a word of length at
least \( m \) on reading any word of length \( mN \) from any state, and as the second coordinate of
the transitions of \( C \) mirrors the transitions of \( B \) over the outputs of \( A \), we see that the
second coordinate must be synchronised as well, and our claim is supported.

Therefore, it is the case that \( \mathcal{M}_{n,r} \) is at least a semigroup, however, the set \( \mathcal{M}_{n,r} \) contains
an endomorphism representing the identity map on \( \mathcal{C}_{n,r} \) (since this can be represented by a
two-state, synchronous and synchronizing transducer), and thus \( \mathcal{M}_{n,r} \) is a monoid.

Remark 7.5. Observe that for two endomorphisms of \( \mathcal{M}_{n,r} \), represented by synchronizing
transducers \( A \) and \( B \) with all outputs finite, the transducer representing the product of these
endomorphisms has as set of states a subset of the product of the states of \( A \) and of \( B \), and
has its core occurring over a subset of the states arising in the product of the states in the
cores of \( A \) and \( B \).

Definition 7.6. We denote by \( \mathcal{S}_{n,r} \) the set of all homeomorphisms of \( \mathcal{C}_{n,r} \) which are repre-
sentable by bi-synchronizing transducers.

We observe in passing that of course \( \mathcal{S}_{n,r} \) is a subset of \( \mathcal{R}_{n,r} \), and it is very easy to see
that this is a proper containment.

Theorem 7.4 has the following corollary.

Corollary 7.7. The subset \( \mathcal{S}_{n,r} \) of \( \mathcal{R}_{n,r} \) forms a subgroup of \( \mathcal{R}_{n,r} \) under composition of
homeomorphisms.

We call the group \( \mathcal{S}_{n,r} \) the group of bi-synchronizing homeomorphisms of \( \mathcal{C}_{n,r} \), or corre-
spondingly the bi-synchronizing group when the Cantor space being acted upon is clear.

We point out, from the proof of Theorem 7.4, that if one simply takes the full transducer
product of two minimal transducers representing elements of \( \mathcal{S}_{n,r} \), then the result will be a
connected transducer. However, it can happen that some of the states of that product will
be inaccessible. Indeed, it can happen that a state corresponding to the product of two core
states also might not be accessible (although, one can always read some input starting from
this state to get into the core of the product automaton). Thus, after computing any such general product, it is practically of great value to minimize the resulting automaton.

7.2 Properties of synchronizing and bi-synchronizing transducers

In this subsection, we take a detour to unearth a more detailed structure theory for minimal, synchronizing (or bi-synchronizing) transducers.

We begin our investigations by looking into how synchronization combines with the basic methods of minimization.

Lemma 7.8. Suppose $0 < m \in \mathbb{N}$ and that $A$ is an initial transducer that synchronizes at level $m$ and which represents a self-homeomorphism $\phi$ of $\mathcal{C}_{n,r}$. Let $B$ be the minimal transducer $\omega$-equivalent to $A$ produced by the minimisation algorithm. Then $B$ has finitely many states and synchronizes at level $m$ as well.

Proof: We suppose that $0 < m \in \mathbb{N}$ and that $A_{q_0} = (\mathfrak{r}, \{0, 1, \ldots, n-1\}, R_1, S_1, \pi_1, \lambda_1, q_0)$ is an initial transducer that synchronizes at level $m$ and which represents a self-homeomorphism $\phi$ of $\mathcal{C}_{n,r}$ for some $0 < r < n \in \mathbb{N}$.

We consider the minimisation algorithm as discussed in Section 5.3.

The first operation is to remove states of incomplete response. Recall that a state $q$ has incomplete response to an input letter $x$ if the word $z = \lambda(x, q)$ is smaller than the guaranteed eventual output of $q$ on reading long enough inputs with initial letter $x$. In the process as given in [10] one starts by adding a new initial state $q_{-1}$, so as to remove the possibility of an infinite loop of adjustments. In our context of removing states of incomplete response for transducers representing homeomorphisms of $\mathcal{C}_{n,r}$, we do not need to add this new state $q_{-1}$, since the initial state $q_0$ cannot be in any cycle in the graph underlying the automaton (since the prefixes of outputs eventually created from the state $q_0$ always begin with a single letter in $\mathfrak{r}$). Thus, the process of removing states of incomplete response in our context is simply to inductively adjust all outputs from accessible states by “back-propagating” guaranteed responses as far as possible. The resulting transducer is $\omega$-equivalent to the original, and it has the further property that for all accessible states, there is no guaranteed response on an empty input. In any case, the transitions in the new automaton mirror the transitions in the old automaton, so the synchronization level is preserved.

The next operation is to remove inaccessible states. It is immediate that this does not increase the synchronization level (it could decrease the synchronization level, if we remove an inaccessible state $q$ from which the transitions into Core($A$) require long inputs). Also, we observe that the number of states has not increased, and that $q_0$ is still the initial state of the resulting automaton, which we will denote by $A_1$.

We can observe that the automaton $A_1$ must now have only finitely many states: the states in the core, (which is a finite set), and all the states one can visit on the way to the core from the initial state (again, a finite set, since after we have read an input of length $m$, we are in the core).

The third operation is to identify each set of equivalent states to a single state. The point to understand in this case is that if two states $q_1$ and $q_2$ are to be identified, then the resulting state will be the image under the new synchronizing map of any word that would have resulted under the old map in either of the two initial states. To understand that this still makes sense, we observe that if an input letter $x$ is processed by $A_1$ at the two states $q_1$ and $q_2$, then we will have that $\lambda_{A_1}(x, q_1) = \lambda_{A_1}(x, q_2)$ and while the two states $\pi_{A_1}(x, q_1)$ and $\pi_{A_1}(x, q_2)$ might be different in $A_1$, they will induce identical local maps (or $q_1$ and $q_2$ would not be able to be identified), so that we see that $\pi_{A_1}(x, q_1)$ and $\pi_{A_1}(x, q_2)$ will also
have been identified. Therefore, the synchronization level of the resulting automaton might become shorter, but it certainly cannot be increased by this process.

We call a word \( w \in W_n \) a prime word if there is no prefix \( z < w \) and \( k \in \mathbb{N} \), with \( k > 1 \), so that \( z^k = w \). For all \( w \in W_n \), there is a shortest prefix \( z \) of \( w \) so that there is a positive integer \( k \) so that \( w = z^k \), we call \( z \) the prime root of \( w \), and note that the prime root of \( w \) is in fact a prime word. If a word \( w \in W_n \) admits two non-trivial subwords \( g \) and \( h \) so that \( w = gh \), then we say \( w \) is a cyclic rotation of the word \( h \circ g \). Also, suppose \( A_{p_0} = (\hat{r}, \{0, 1, \ldots, n-1\}, R_A, S_A, \pi_A, \lambda_A, p_0) \) is a synchronizing finite transducer representing a continuous map \( \mathcal{C}_{n,r} \rightarrow \mathcal{C}_{n,r} \). We say a word \( w \in W_n \) represents a basic circuit of \( A_{p_0} \) if there is state \( p \) so that \( \pi_A(w, p) = p \) and if \( w_1 \) and \( w_2 \) are non-trivial prefixes of \( w \) so that \( \pi_A(w_1, p) = \pi_A(w_2, p) \) then \( w_1 = w_2 \). In this case we say \( w \) represents a basic circuit of \( A \) based at \( p \).

The following lemma consists of various points which are all easy exercises. We will prove points \( \text{[1]} \) \( \text{[2]} \) and \( \text{[7]} \) leaving the other points to the reader.

**Lemma 7.9.** Suppose \( A_{p_0} = (\hat{r}, \{0, 1, \ldots, n-1\}, R_A, S_A, \pi_A, \lambda_A, p_0) \) is a synchronizing finite transducer, with all outputs on finite inputs finite.

1. Given a core state \( p \) and a word \( w \) so that \( \pi(w, p) = p \), then if \( z \) is the prime root of \( w \), then \( \pi(z, p) = p \).

2. Given any word \( w \in W_n \), there is a unique state \( p \) so that \( \pi_A(w, p) = p \).

3. Suppose \( w \) represents a basic circuit of \( A_{p_0} \), then the unique state \( p \) so that \( \pi_A(w, p) = p \) has the property that \( p \) is in \( \text{Core}(A) \), and furthermore, the word \( w \) is prime.

4. Let \( p \in \text{Core}(A) \), and let \( W_p \) represent the set of all words representing basic circuits of \( A \) (at \( p \)). Then, \( W_p \) is finite.

5. The sets \( W_p \), for \( p \) running over the states in \( \text{Core}(A) \), partition the set of all words representing basic circuits of \( A \).

6. Given a core state \( p \), given any letter \( x \in \{0, 1, \ldots, n-1\} \), there is a word in \( W_p \) beginning with the letter \( x \).

7. For any state \( q \) the set of minimal elements of the set \( \{ \lambda_q(w) | w \in X^* \} \setminus \{ \varepsilon \} \) has no common initial prefix.

**Proof.** Suppose \( m \in \mathbb{N}, w \in W_n, \) and \( A_{p_0} = (\hat{r}, \{0, 1, \ldots, n-1\}, R_A, S_A, \pi_A, \lambda_A, p_0) \) a synchronizing finite transducer, with all outputs on finite inputs finite, and where \( A_{p_0} \) synchronizes at level \( m \) for some non-negative integer \( m \).

First suppose \( p \) is a state of \( \text{Core}(A) \) and \( \pi(w, p) = p \). Let \( z \) be the prime root of \( w \), and \( k \) be a positive integer so that \( z^k = w \). Consider the word \( z^m \). It is immediate that \( z^m \) is at least \( m \) in length, and so it synchronizes \( A_{p_0} \) to some state. Meanwhile, \( \pi(w^m, p) = p \) and \( w^m \) has suffix \( z^m \), so \( z^m \) must synchronize \( A_{p_0} \) to \( p \). But now, \( z^{m+1} \) also synchronizes \( A_{p_0} \) to \( p \), since \( z^{m+1} \) has \( z^m \) as a suffix. Therefore, \( \pi(z, p) = p \). This shows point \( \text{[1]} \) Release the variable \( p \) now.

Since \( |w| > 0 \) it is immediately clear that there is a length \( m \) suffix of \( w^m \) so in particular \( w^m \) synchronizes \( A_{p_0} \) to some state \( p \in \text{Core}(A) \). It is then the case that \( \pi(w^m, p) = p \). Note that the prime root \( z \) of \( w \) is also the prime root of \( w^m \), so by point \( \text{[1]} \) the prime root \( z \) of \( w \) has \( \pi(z, p) = p \). It is then immediate that \( \pi(w, p) = \pi(z^k, p) = p \), which shows point \( \text{[2]} \)

Note that point \( \text{[7]} \) follows from the fact that our automaton is minimal. If there were a non-trivial common prefix \( w \) for the output words from a state \( q \), then the minimalisation
procedure would have the transitions to $q$ having all their outputs ending with $w$ as a suffix, while $w$ would be removed from the prefixes of all of the outputs of $q$. □

The following lemma is effectively a continuation of Lemma 7.9, now under the context that our automaton is bi-synchronizing (and hence, acts bijectively on its target Cantor space).

**Lemma 7.10.** Let $1 \leq r < n \in \mathbb{N}$ and $h \in \text{Homeo}(C_{n,r})$ be represented by a finite minimal bi-synchronizing transducer $A$ with set of states $Q$. We have the following.

1. Every finite word $w$ is obtained as a prefix of some output from a state in $Q$.

2. For any periodic equivalence class of $\sim$ with (minimal) period $w$ there exists a unique minimal $t \in \mathbb{N}$ and a unique circuit in $A$ with output $w^t$.

3. If $q \in Q$ so that the local map $h_q : C_n \to C_n$ is bijective, then for all $p \in Q$ such that $\pi(l, p) \neq q$ for some $l \in \{0, 1, \ldots, n-1\}$, we have $\lambda(l, p) \neq \varepsilon$.

4. If $p \in Q$ so that the local map $h_p : C_n \to C_n$ is a homeomorphism, and so that its outputs $\{\lambda(l, p) \mid l \in \{0, 1, \ldots, n-1\}\}$ are all different from the empty word, then each such output word has length one, and the induced map $\lambda(\cdot, p) : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$ is a permutation.

5. If $p \in Q$ so that the local map $h_p : C_n \to C_n$ is a homeomorphism, and so that its outputs $\{\lambda(l, p) \mid l \in \{0, 1, \ldots, n-1\}\}$ are all of length one, then for all $q$ so that $\pi(l, p) = q$ for some $l \in \{0, 1, \ldots, n-1\}$, the local map $h_q$ is also a homeomorphism.

**Proof:** To see point 1, simply observe that is some finite word $w$ is not obtained as a segment of output of some state in $Q$, then the image of $h$ cannot be clopen, since some points will be missing from the image of $h$ in each basic cone of $C_{n,r}$.

Point 2 is essentially the dual point to Lemma 8.2’s point 2. One can work in a minimal transducer $B$ representing $h^{-1}$, to detect the word $v$ of output produced by reading $w$ on the cycle that accepts $w$ in $B$. Since $w$ is prime, the word $v$ is also prime (otherwise, if $v = r^t$ for some $t > 1$, we have that $B$ has a cycle that accepts $r$, and the output of $r$) Then, there is a unique cycle $C$ of $A$ labelled by $v$, and one can detect the value $t$ by seeing how many times $w$ is produced upon reading $v$ from the state that begins the cycle $C$.

Point 3 follows from the fact that as $q$ can produce a full copy of $C_n$ as output. Let $l$ be a letter in $\{0, 1, \ldots, n-1\}$ so that $\pi(l, p) = q$, and $x \in \{0, 1, \ldots, n-1\} \setminus \{l\}$. Then, there is an output from $p$ obtained by first reading $x$, and then reading any infinite word. This output can also be obtained from $p$ by first reading $l$. In particular, $h$ would fail to be injective.

Point 4 follows as we must be able, for any given word $w \in W_n$, to find some word $z \in W_n$ so that $\lambda(z, p)$ has $w$ as a prefix. Therefore, we need to be able to write words with the $n$ distinct one letter prefixes provided by our alphabet $\{0, 1, \ldots, n-1\}$, so, the $n$ outputs of $p$ must begin with the $n$ distinct letters in $\{0, 1, \ldots, n-1\}$. Furthermore, if for some $j \in \{0, 1, \ldots, n-1\}$, the output $w_j = \lambda(j, p)$ has length greater than one and initial letter $k$, then from the state $p$ there will be at least $n-1$ outputs beginning with the letter $k$ that cannot occur as prefixes of any output written from the state $p$. In particular, the length of all outputs from state $p$ on one-letter inputs is one, and these outputs form a function $\lambda(\cdot, p) : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$ which must be a permutation.

Finally, point 5 follows as each of the states in the image of the transition function $\pi(\cdot, p) : \{0, 1, \ldots, n-1\} \to Q$ is responsible for a full Cantor set of outputs. (I.e., the state
\(\pi(j, p)\) must produce all prefixes of all words in the Cantor space \(\mathcal{C}_n\) so that \(p\) itself will admit all outputs with prefix the letter \(\lambda(j, p)\). \(\square\)

**Lemma 7.11.** Let \(1 \leq r < n \in \mathbb{N}\) and \(h \in \mathcal{S}_{n, r}\) be represented by a finite minimal bi-synchronizing transducer \(A\) with set of states \(Q\). If for some state \(q\) in \(Q\), in the initial automaton \(A_q\) any finite word in \(W_n\) can be obtained as an initial segment of some output, then the local map \(h_q\) is a self-homeomorphism of \(\mathcal{C}_n\).

**Proof.** By definition, \(\text{Range}(\lambda_q)\) is a clopen set in \(\mathcal{C}_n\) and by our assumption, every basic cone in \(\mathcal{C}_n\) intersects \(\text{Range}(\lambda_q)\), thus \(\text{Range}(\lambda_q) = \mathcal{C}_n\). \(\square\)

**Definition 7.12.** Let \(h \in \mathcal{S}_{n, r}\) be represented by a minimal non-degenerate automaton \(A_{q_0} = (\hat{r}, R, S, \pi_A, \lambda_A, q_0)\). We call any state \(q \in (R\setminus\{q_0\} \sqcup S\) where the map \(h_q : \mathcal{C}_n \to \mathcal{C}_n\) is a homeomorphism a homeomorphism state.

Given \(h \in \mathcal{S}_{n, r}\) for some \(1 \leq r < n \in \mathbb{N}\), represented by a minimal transducer \(A\), we have seen above that \(A\) has, for each letter \(l \in \{0, 1, \ldots, n - 1\}\), a unique state \(q_l\) so that \(\pi(l, q_l) = q_l\). We call the state \(q_l\) the \(l\)-loop state. We define the set

\[
\text{LoopStates}(A) = \{q \in Q | q \text{ is the } l\text{-loop state for some } l \in \{0, 1, \ldots, n - 1\}\}.
\]

**Lemma 7.13.** Let \(1 \leq r < n \in \mathbb{N}\), and \(h \in \mathcal{S}_{n, r}\) be represented by a minimal transducer \(A\). For each \(l \in \{0, 1, \ldots, n - 1\}\) let \(q_l \in \text{LoopStates}(A)\) be the \(l\)-loop. Suppose that for some \(l \in \{0, 1, \ldots, n - 1\}\) we have that for all \(j \in \{0, 1, \ldots, n - 1\} \setminus \{l\}\) the word \(\lambda(j, q_l)\) is not trivial and has no non-trivial common prefix with \(\lambda(l, q_l)\). Then, the local map \(h_{q_l} : \mathcal{C}_n \to \mathcal{C}_n\) is a homeomorphism, and furthermore in this case we have the following:

1. \(|\lambda(j, q_l)| = 1\), for each \(j \in \{0, 1, \ldots, n - 1\}\),
2. the map \(\lambda(\cdot, q_l) : \{0, 1, \ldots, n - 1\} \to \{0, 1, \ldots, n - 1\}\) is a permutation, and
3. for each state \(p_j := \pi(j, q_l)\), the state \(p_j\) is a homeomorphism state.

**Proof.** Assume \(q_l\) is the loop state for some \(l \in \{0, 1, \ldots, n - 1\}\) and for all \(j \in \{0, 1, \ldots, n - 1\} \setminus \{l\}\)

the word \(\lambda(j, q_l)\) is not trivial and has no non-trivial common prefix with \(\lambda(l, q_l)\). For all \(j \in \{0, 1, \ldots, n - 1\}\), set \(w_j := \lambda(j, q_l)\). Observe that \(\{q_l\}\) represents the only circuit with output \(w_l\). Further observe that as the outputs from the state \(q_l\) on reading one-letter inputs are never trivial, that given any letter \(k \in \{0, 1, \ldots, n - 1\}\), there is some \(j \in \{0, 1, \ldots, n - 1\}\) so that \(\lambda(j, q_l)\) begins with the letter \(k\). This follows, as otherwise the homeomorphism \(h\) would not be surjective, as it would be impossible, for large enough values of \(t\), to produce any infinite sequence of letters representing a point in Cantor space with a contiguous substring of the form \((\lambda(l, q_l)^t \cdot k\).

Therefore, we see that from the state \(q_l\) we can produce all strings of length 1 as prefixes of outputs. However, the argument above is generic, since given any word \(w \in W_n\), the only way for \(h\) to produce infinite sequences representing points of Cantor space, with contiguous substrings of the form \((\lambda(l, q_l)^t \cdot w\) for arbitrarily large \(t\), is to have an infinite string of input with a substring of the form \(l^m\) for some \(s\) near \(t\) (that is, for values of \(t\) much larger than the synchronizing length \(m\)), followed by some word \(z\) so that \(\lambda(z, q_l)\) has prefix \(w\).
Therefore, \( q_i \) is a homeomorphism state by Lemma 7.11.

Furthermore, as \( q_i \) is a homeomorphism state with all transitions with non-empty outputs, we have by Lemma 7.10 Point 4 that all the outputs on those transitions are length one and that the map \( \lambda(\cdot, q_i) : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\} \) is a permutation. We also have that for all states \( q \) in the set \( \{\pi(j, q_i) | j \in \{0, 1, \ldots, n-1\}\} \) that \( A_q \) induces a homeomorphism on \( C_n \) by Lemma 7.10 Point 5.

### 7.3 The isomorphism \( \text{Aut}(G_{n,r}) \cong S_{n,r} \)

Recall that the purpose of this section is to determine the subgroup of \( R_{n,r} \) corresponding to the automorphisms of the \( G_{n,r} \).

Note that in the proof below, we are not using the notation \( q_i \) to represent a loop state, but rather, the initial state of one of the four essential transducers that arise in the proof.

**Theorem 7.14.** Let \( r < n \) be positive integers, and suppose that \( \varphi : C_{n,r} \to C_{n,r} \) is a homeomorphism represented by a minimal initial automaton

\[
A_{q_1} = (\hat{r}, \{0, 1, \ldots, n-1\}, R_1, S_1, \pi_1, \lambda_1, q_1)
\]

which happens to be finite. The automaton \( A_q \) is bi-synchronizing at level \( m \) for some \( m \) if and only if \( \varphi \in N_{H(C_{n,r})}(G_{n,r}) \).

**Proof.** Throughout this proof, we will suppose \( \varphi \) is a homeomorphism of \( C_{n,r} \) that is representable by a finite minimal initial automaton

\[
A_{q_1} = (\hat{r}, \{0, 1, \ldots, n-1\}, R_1, S_1, \pi_1, \lambda_1, q_1)
\]

and that \( \varphi^{-1} \) is representable by a finite minimal automaton

\[
B_{q_2} = (\hat{r}, \{0, 1, \ldots, n-1\}, R_2, S_2, \pi_2, \lambda_2, q_2)
\]

For the first half of our proof we will further suppose that \( \varphi \in S_{n,r} \) so that there is a positive integer \( m \) so that \( A_{q_1} \) and \( B_{q_2} \) are both synchronizing at level \( m \). We will also suppose that \( g \in G_{n,r} \), and argue that \( g^g \in G_{n,r} \).

One path to see this is to note that \( g \) is representable by a bi-synchronizing transducer \( C_{q_1} \), and that the product \( \varphi^{-1}g\varphi \) is therefore a homeomorphism in \( S_{n,r} \), representable by some minimal bi-synchronizing transducer \( D_{q_4} \). Further, we will explain why the core of \( D_{q_4} \) contains only a single state which acts as the identity.

Firstly, as \( g \in G_{n,r} \), we already understand that \( g \) admits finitely many types of action. Represent \( g \) by a prefix exchange map

\[
\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle \mapsto \langle \beta_1, \beta_2, \ldots, \beta_k \rangle,
\]

with the condition that every prefix that appears in this map is at least length two (this will simplify some constructions appearing below, and we can always achieve it by splitting a prefix and its corresponding prefix into their \( n \) extensions, representing their \( n \) child nodes, and assigning the corresponding extended mapping from the original map). We now build a transducer \( C_{q_1} = (\hat{r}, \{0, 1, \ldots, n-1\}, R_3, S_3, \pi_3, \lambda_3, q_3) \) that represents \( g \) as follows.

For every word \( w \) that represents a prefix of a word in the set \( \chi := \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \), we determine a state. We also will have one further state which we will denote by \( 1_g \). The set \( S_3 \) will contain only the state \( 1_g \), and all remaining states will be in the set \( R_3 \).
The empty word is in the list of prefixes of the words $\alpha_i$, and this prefix corresponds to the initial state $q_0$.

For any proper non-empty prefix $w$ of a word in $\chi$, denote its corresponding state as $q_w$. Then, for all words in $R_3$, we can determine the transitions and outputs as follows.

- if $r \in \hat{r}$, set $\pi_3(r, q_3) = q_r \in R_3$, and $\lambda_3(r, q_3) = \varepsilon$;
- if $u < v$ are two proper prefixes of some word $\alpha_i$, with $|u| \geq 1$, and with $l \in \{0, 1, \ldots, n - 1\}$ so that $v = u \cdot l$, set $\pi_3(l, u) = v$ and $\lambda_3(l, u) = \varepsilon$; and finally,
- if $u$ is a proper prefix of some word $\alpha_i$, so that $\alpha_i = u \cdot l$ for some $l \in \{0, 1, \ldots, n - 1\}$, then set $\pi_3(l, u) = 1_g$, and $\lambda_3(l, u) = \beta_i$.

Naturally, the state $1_g$ acts as the identity, so that for all $l \in \{0, 1, \ldots, n - 1\}$, we determine $\pi_3(l, 1_g) = 1_g$ and $\lambda_3(l, 1_g) = l$.

It is immediate that $g$ is synchronizing, and in particular, by following a similar process for $g^{-1}$, we see that $g \in S_{n,r}$. Indeed, it is clear that $G_{n,r}$ embeds nicely in $S_{n,r}$ as the set of all homeomorphisms representable by bi-synchronizing transducers with trivial (acting) cores (that is, with cores of one state, simply carrying out the iterated identity permutation).

It now follows that $g^2 \in S_{n,r}$. It remains to observe that if $D_{q_i} = (\hat{r}, \{0, 1, \ldots, n - 1\}, R_4, S_4, \pi_4, \lambda_4, q_4)$ is a minimal bi-synchronizing transducer representing $g^2$ then the core of $D_{q_i}$ is a single state transducer acting as the identity map on $C_n$. However, this follows by construction, since the core of a product of transducers in $S_{n,r}$ is determined by taking the product of the cores, and determining which states are in the core of the result. In our case, as the core of $C_{q_i}$ is the single state identity transducer, it will not interfere with the structure or the transitions of the resulting core of the three-fold product. (Removing the middle coordinate (which is always $1_g$) from the product structure representing all states in the three-fold product of the cores produces a bijection between that product’s states and the states of the product of the cores of the transducers $A_{q_1}$ and $B_{q_2}$ that commutes with the transition and output functions. That is, these automata are strongly isomorphic.)

In particular, the resulting core of the three-fold product transforms the Cantor space $C_n$ identically to the core of the product of the transducers $A_{q_1}$ and $B_{q_2}$. However, this last core represents the identity transformation, since $A_{q_1}$ and $B_{q_2}$ are transducers representing inverse elements. Therefore, $g^2$ is representable by a bi-synchronizing transducer with core a single state acting as the identity map, and in particular, $g^2 \in G_{n,r}$.

Now to finish the proof of Theorem 7.14, we need to show that as $\varphi \in N_H(\epsilon_{n,r})(G_{n,r})$ is representable by the finite transducer $A_{q_1}$ (by Theorem 6.1), then there is a natural number $m$ so that $A_{q_1}$ is actually bi-synchronizing at level $m$. We may assume that $A_{q_1}$ is reduced and that

$$A_{q_1} = (\hat{r}, \{0, 1, \ldots, n - 1\}, R_1, S_1, \pi_1, \lambda_1, q_1).$$

Furthermore, let $B_{q_2}$ represent $\varphi^{-1}$, and assume that $B_{q_2}$ is reduced as well (and so, finite) and that

$$B_{q_2} = (\hat{r}, \{0, 1, \ldots, n - 1\}, R_2, S_2, \pi_2, \lambda_2, q_2).$$

We will proceed by contradiction: assume that for any natural number $m$, one of $\varphi$ or $\varphi^{-1}$ cannot be synchronizing at level $m$. Since $\varphi$ and $\varphi^{-1}$ are both in $N_H(\epsilon_{n,r})(G_{n,r})$, we can assume without meaningful loss of generality that $A_{q_1}$ cannot be synchronizing at level $m$ for any natural numbers $m$. In particular, for each such a positive integer $m$, define $w_m \in W_{n,e}$ to be a word of length $m$ so that the set

$$\{ \pi_1(w_m, q) \mid q \in (R_1 \backslash \{q_1\}) \cup S_1 \}$$

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has cardinality greater than one (note that any two distinct states in such a set actually represent distinct local maps for \( \varphi \), so that there will be some finite word that will produce incomparable outputs from the pair of states).

By the definition of the words \( w_m \), for each natural number \( m \), there are states \( r_{1,m} \) and \( r_{2,m} \) so that \( \pi_1(w_{m, r_{1,m}}) \neq \pi_1(w_{m, r_{2,m}}) \) holds. As \( A_q \) is finite, there is some pair \((r_1, r_2)\) from \( S_1^2 \) so that for infinitely many values of \( m \), we have that \( \pi_1(w_{m, r_1}) \neq \pi_1(w_{m, r_2}) \). Indeed, as \( A_q \) is finite, there is a particular \( m \) and a pair \((s_1, s_2)\) with \( s_1 \neq s_2 \) and two distinct prefixes \( w_{i,m} \) of \( w_m \) so that \( \pi_1(w_{i,m, r_1}) = s_1 = \pi_1(w_{i,m, r_2}) \) and \( \pi_1(w_{i,m, r_2}) = s_2 = \pi_1(w_{j,m, r_2}) \). Indeed, if \( w \) is the suffix of \( w_{i,m} \) so that \( w_{i,m} \vdash w = w_{j,m} \), then we have \( \pi_1(w, s_1) = s_1 \) and \( \pi_1(w, s_2) = s_2 \) while \( s_1 \neq s_2 \). Define an infinite sequence of words \( (x_k)_{k<\infty} \) as \( x_k := w^k \) so that for all \( k \) we have \( \pi_1(x_k, s_1) = s_1 \neq s_2 = \pi_1(x_k, s_2) \).

Define \( f_1 := \lambda_1(w, s_1) = f_{1,1}f_{1,2} \ldots f_{1,t_1} \in W_n \) and \( f_2 := \lambda_1(w, s_2) = f_{2,1}f_{2,2} \ldots f_{2,t_2} \in W_n \), to be the words produced by the transducer \( A_q \) from the states \( s_1 \) and \( s_2 \) respectively when the cycle label \( w \) is read in.

Note that as \( A_q \) is reduced, there are points \( u, v_1, v_2 \in \mathcal{C}_n \) so that \( \lambda_1(u, s_1) \perp \lambda_1(u, s_2) \) and also so that the first letter of output of \( \lambda_1(u, s_1) \) does not agree with the first letter of output of \( \lambda_1(u, s_1) \), and the first letter of the output of \( \lambda_1(v_2, s_2) \) does not agree with the first letter of the output of \( \lambda_1(u, s_2) \). Set for all positive integers \( k \), \( y_k := x_k \vdash u \) and \( z_k := x_k \vdash v \).

Now let \( \gamma_1, \gamma_2 \in W_{n,r,e} \) be two minimal length words so that \( \pi_1(\gamma_1, q_1) = s_1 \) and \( \pi_1(\gamma_2, q_1) = s_2 \). Suppose that \( \mu_1, \mu_2 \in W_{n,r} \), so that \( \lambda_1(\gamma_1, q_1) = \mu_1 \) and \( \lambda_1(\gamma_2, q_1) = \mu_2 \). Pick \( g \in G_{n,r} \) so that \( \gamma_1 \vdash a_1a_2 \ldots g = \gamma_2 \vdash a_1a_2 \ldots \) for all \( a_1a_2 \ldots \in \mathcal{C}_n \).

**Claim 7.15.** The homeomorphism \( h := \varphi^{-1}g\varphi \) does not belong to \( G_{n,r} \).

**Proof of claim:** We prove this by contradiction. So suppose \( h \) lies in \( G_{n,r} \).

We consider the action of \( h \) on the family of elements \( b_k := (\gamma_1 \vdash y_k) \cdot \varphi \). We observe that for these elements, \( b_k \cdot h = \lambda_1(\lambda_2(b_k, q_2) \cdot g, q_1) \).

\[
\lambda_1(\lambda_2(b_k, q_2) \cdot g, q_1) = \lambda_1(\gamma_1 \vdash x_k \vdash u \cdot g, q_1) = \lambda_1(\gamma_2 \vdash x_k \vdash u, q_1) = \lambda_1(\gamma_2 \vdash x_k, q_1) \lambda_1(u, s_2)) = \mu_2 ^{(f_2)^k} \lambda_1(u, s_2).
\]

Now there are some points to be made.

Firstly, as we assume that \( h \) is in \( G_{n,r} \), it must be the case that for \( k \) very large, the point \( b_k \) will have a suffix of the form \((f_2)^k \lambda_1(u, s_2)\) as well. Meanwhile, if we replaced \( g \) by some other element \( g' \in G_{n,r} \) which carried out, as part of its prefix code, the identity substitution for the prefix \( \gamma_1 \), then we can form \( h' := (g')^g \), and the calculation of \( b_k \cdot (g')^g \) must result in a point of the form \( \mu_1 ^{(f_1)^k} \lambda_1(u, s_1) \), so in particular, there are long identical suffixes of these two resulting points, one beginning somewhere (early, given large enough \( k \)) in the string \((f_2)^k \) and the other beginning somewhere (early, given large \( k \)) in the string \((f_1)^k \).

This has consequent consequences. Firstly, it must be the case that \( f_1 \) or \( f_2 \) are powers of, respectively, a word \( f_3 \) and a cyclic rotation of \( f_3 \). Similarly, by substituting in \( v_1 \) or \( v_2 \) for \( u \), we can see that \( f_1 \) and \( f_2 \) must actually have the same length, or else these substitutions will show that one of the elements \( h \) or \( h' \) is scaling input strings in a non-one-to-one fashion at arbitrary depths, for some of their input strings, so that \( h \) or \( h' \) will not be Lipschitz and thus not in \( G_{n,r} \). Hence these two words are in fact cyclic rotations of each other, and that the distinction between \( \lambda_1(u, s_1) \) and \( \lambda_1(u, s_2) \) is that one is an infinite suffix of the other. We can now assume without meaningful loss of generality that indeed \( \lambda_1(u, s_2) = P \vdash \lambda_1(u, s_2) \) for \( P \) a nontrivial prefix of \( f_1 \) for some rotation \( f \) of \( f_1 \) and some large enough integer \( l \).
At this stage it is possible for us to measure precisely what the offset is for the size of the prefix substitution carried out by $h$ on the words $b_k$ for large enough $k$ (assuming $h$ were actually an element of $G_{n,r}$). First replace $u$ by $v_2$ in the whole calculation process above for the input to $h$ (so, we apply $\varphi$ to get $b_{k,v_2}$, and plug $b_{k,v_2}$ into $h$). The smallest index where the result changes is the index the output $\lambda_1(u, s_2)$ starts being written. Meanwhile, by substituting $v_1$ for $u$, and applying $\varphi$ to the result, we get a string which when plugged into $h'$ will in fact be unchanged (as would $b_k$), and which will be different from the string $b_k$ at some specific index; this shows at what index the output $\lambda_1(u, s_1)$ is written in order to create the identity action of $h'$ on $b_k$. But then, as we can determine what is the offset $h$ is to have on $b_k$ as a prefix code map, and we can now detect that $h$ is inserting (or deleting) a non-trivial prefix $P$ at arbitrary large depths in the indices, and so cannot actually be in $G_{n,r}$.

In particular, our assumption that $A_{q_1}$ is not synchronising is false.

By similar logic, as the normaliser of $G_{n,r}$ is a subgroup of $R_{n,r}$, it is closed under inversion, so it is also the case that $B_{q_2}$ must be synchronising as well, so therefore $A_{q_1}$ is bi-synchronising and $\varphi \in S_{n,r}$.

8 On the Natural Quotient $S_{n,r} \rightarrow \text{Out}(G_{n,r})$

Now that we understand that $S_{n,r}$ represents precisely the group of automorphisms of $G_{n,r}$, we can of course take the natural quotient, to examine the outer automorphisms of $G_{n,r}$.

We have seen that the result of multiplying an element of $S_{n,r}$ by an element of $G_{n,r}$ is an element of $S_{n,r}$ with a core that is strongly isomorphic to the original core (here, by strongly isomorphic automata, we mean that there is a bijection between the states of those automata which commutes with the transition and output functions). Thus, the minimal core of a reduced automaton representing an element in $S_{n,r}$ naturally represents the corresponding outer automorphism class, and we expand on this here.

Grigorchuk, Nekraschevych, and Suschanski˘ı’s argue the uniqueness (up to what we call strong isomorphism in this article) of any two minimal finite transducers representing the same homeomorphism of Cantor space. Therefore, we introduce the following notation.

**Notation 8.1.** Let $h \in R_{n,r}$. By $A_h$ we will mean a minimal initial finite automaton representing $h$.

We can now proceed to develop our understand of Out($G_{n,r}$).

**Lemma 8.2.** For any $g, h \in S_{n,r}$, we have:

1. $\text{Core}(A_{gh}) \subset \text{Core}(A_g) \text{Core}(A_h)$, where the product of these transducers is computed as in [10].

2. if $\text{Core}(A_g)$ is strongly isomorphic to $\text{Core}(A_h)$, then there is an element $v \in G_{n,r}$ so that $gv = h$.

3. the set \{Core($g$) | $g \in S_{n,r}$\} is a set of non-initial transducers representing a group $\mathcal{O}_n$, and

4. $\mathcal{O}_n \cong \text{Out}(G_{n,r})$.

**Proof:** We proceed point by point.
1. The first point is by direct computation. One computes the automata product for $A_g$ and $A_h$. The states of the product are a subset of the direct product of the states of the original transducers. After reading a long enough string of input, both coordinates must be processing from inside the core of their respective transducers.

2. This follows from the inversion algorithm of Grigorchuk, Nekrashevych, and Suschanskiĭ (see the proof of their Proposition 2.21) which applies to any finite non-generator automaton $A_\phi$ inducing a homeomorphism $\phi$ of Cantor space. That algorithm builds, for each state $q$ of the states of $A_\phi$, the (finite) set of incomparable finite words that are not prefixes of outputs from $q$, and for a word $w$ in this set of words, the pair $(w, q)$ becomes a state of the inverse transducer. Similarly, the transitions and outputs for the transducer representing the inverse function depend precisely on the transitions and outputs of the original automaton. In particular, since $g$ and $h$, in our context, have strongly isomorphic cores, then the inversion algorithm just described will build automata with a strong isomorphism between the sub-automata created by running this inversion algorithm upon the cores of these automata, and as a consequence the resulting automata (and their core sub-automata) will also be strongly isomorphic. Thus, the resulting automata $A_{g^{-1}}$ and $A_{h^{-1}}$ will also have strongly isomorphic cores. It is now the case that the product automata $A_{g^{-1}} \times A_h$ must have its core as being strongly isomorphic to the core of the product $A_{h^{-1}} \times A_h$, which core is a sub-automata acting as the identity map. From this it follows that $A_{g^{-1}h}$ has core acting as the identity, so that $v := g^{-1}h$ must be in $G_{n,r}$.

3. This follows from the previous points. The process of taking products of core transducers and reducing the result to its core is well defined, and by the discussion in the previous point the core of the inverse automaton $A_{g^{-1}}$ for an element $g \in S_{n,r}$ is actually a sub-automaton of the result of the inversion process applied to the core of $A_g$, so the set of cores has a well defined product and inverse operation, and this product directly inherits associativity from the associativity of composition of the original functions in $S_{n,r}$.

4. This follows from point (2), and from the fact that $G_{n,r}$ is the precisely the subgroup of $S_{n,r}$ consisting of the homeomorphisms that are representable by transducers with representative cores which act as the identity.

\[ \square \]

8.1 On Lipschitz conditions, the groups $O_n$, and subgroups of interest

Recall our notation that $\mathcal{LS}_{n,r}$ represents the subgroup of $S_{n,r}$ corresponding to the homeomorphisms of $\mathcal{C}_{n,r}$ that are bi-lipschitz. Also recall that we call $O_n$ the image of $S_{n,r}$ under the quotient by its normal subgroup $G_{n,r}$, and the image of $\mathcal{LS}_{n,r}$ in $O_n$ is the group $\mathcal{L}_n$. We further have the group $\mathcal{P}_{S_{n,r}}$, which is the subgroup of $\mathcal{LS}_{n,r}$ consisting of homeomorphisms which can be represented by $c$ with cores that are in fact synchronous, as well as bi-synchronizing, and we call the image of this group in $O_n$ the subgroup $\mathcal{P}_n$. In this sub-section we explore some properties of these various groups. The examples and discussion provided here imply all of the statements of Theorem 1.5.

For given $1 \leq r < n \in \mathbb{N}$, recall that elements of $G_{n,r}$ are (bi-)Lipschitz. Also, due to the strong bi-synchronizing condition on elements of $S_{n,r}$, one might be inclined to believe that any general element of $S_{n,r}$ must also be bi-Lipschitz. However, this first impression turns out to be false.
Theorem 8.3. For $1 \leq r < n$, for $n > 2$, the group $S_{n,r}$ contains elements which are not bi-Lipschitz.

Proof. The transducer $B$ depicted in Figure 2 is minimal and bi-synchronizing (at level two), and is its own core. However, it is not synchronous. In this case we are demonstrating for $n = 3$.

![Figure 2: A non-bi-Lipschitz core transducer of infinite order.](image)

Although $B_a$ is bi-synchronizing (at level two) and will induce a self-homeomorphism of $C_3$ (that is, if we take in particular $a$ as its initial state), it does not produce a Lipschitz map on Cantor space. Consider the cycle of edges labelled by inputs 2 and 0 respectively (connecting $a$ to $b$ and then going back to $a$). This cycle has length two in the directed graph underlying the transducer and so input word of length two. However, the output word written on reading this input while traversing this cycle has length three. In particular, the point $p = 202\overline{0}$ maps to $q = 100\overline{00}$ in a non-Lipschitz fashion as points from ever-smaller basic cones about $202\overline{0}$ face an ever-increasing general contraction factor (a witness to this is the sequence of points $((20)^k \overline{0})_{k \in \mathbb{N}}$, where the $k^{th}$ such point has distance $1/2^{(2k+1)}$ from $p$, but lands under $\phi$ at the point $(100)^{k+1}_2$ at distance $1/2^{3k+1}$ from $q$.

It is not hard to build similar examples for any alphabet of size $n$, for $n > 2$, based on this example. In particular, we can build elements of $S_{n,r}$ for any $n > 2$ which are not bi-Lipschitz.

To see that we can build a bi-synchronizing automaton $B_{n,r}$ representing an element $\tilde{B}_{n,r}$ of $S_{n,r}$ which uses $B$ as part of its core, we will simply add a state to the transducer $B$ depicted above, and many transition edges, as described below, to produce the transducer $B_{n,r}$ which induces the map $\tilde{B}_{n,r} \in S_{n,r}$.

We include a figure below depicting the resulting transducer $B_{4,2}$ (so, for $n = 4$ and $r = 2$). Details of the construction follow.
The added state will be an initial state \( q_0 \). For each symbol \( x \in \mathcal{r} \), the transitions will be given by the rules \( \pi(x, q_0) = a \) and \( \lambda(x, q_0) = x \). That is, \( q_0 \) admits \( r \) edges from \( q_0 \) to \( a \), one for each input from \( \mathcal{r} \), and each transition acts as the identity transformation on its letter of \( \mathcal{r} \).

The remaining transitions are as follows. For each symbol \( x \) in \( \{0, 1, \ldots, n-1\} \setminus \{0, 1, 2\} \), set \( \pi(x, a) = a \) and \( \lambda(x, a) = x \). Furthermore, set \( \pi(x, b) = a \) and \( \lambda(x, b) = x \). Finally, set \( \pi(x, c) = a \) and \( \lambda(x, c) = 0x \).

The reader can verify that the transducers so constructed are bi-synchronizing and represent self-homeomorphisms for the spaces \( \mathfrak{C}_{n,r} \).

While it is perhaps surprising that for \( n > 1 \), the group \( S_{n,r} \) contains elements that are not bi-Lipschitz, one wonders how complicated the subgroup of \( S_{n,r} \) will be, which consists of the bi-Lipschitz homeomorphisms. It seems to be a very strong further condition on the maps, and we can hope that the automata involved will all be strongly restricted.

The example depicted in Figure 4 is of an element in \( L_2 \) that is not in the subgroup \( \mathcal{P}_2 \), and which is of infinite order.

One way to verify that the transducer in Figure 4 is actually of infinite order is to find a witness point in \( \{0, 1\}^\mathbb{Z} \) that is on an infinite orbit under the action of this transducer. For this purpose, the point \( \ldots 111(001)^\omega \) serves nicely (say, the point with coordinate entries of value 1 at all negative indices, and with its first entry of value 0 occurring at index 0), which the reader can verify by a simple induction tracing the orbit as given below.

\[
\ldots 111(001)^\omega \mapsto \ldots 111 \cdot 01 \cdot (001)^\omega \\
\ldots 111 \cdot 0101 \cdot (001)^\omega \mapsto \ldots 111 \cdot (01)^3(001)^\omega \mapsto^k \\
\ldots 111 \cdot (01)^{k+3} \cdot (001)^\omega
\]
It is not hard, following the method of the construction following the example in Figure 3, to increase the alphabet size to $n$ for any given integer $n > 2$.

We give two further example transducers, exploring the boundary cases when $n$ is small.

The first such example, depicted in Figure 5, is of a transducer representing an element of $P_3$ which is of infinite order. One can prove by induction that the point $\overline{p} := (\overline{p}_i) \in \{0, 1, 2\}^\mathbb{Z}$ given by the rules

$$p_i = \begin{cases} 
2 & \text{if } i < 0 \\
1 & \text{if } i \geq 0.
\end{cases}$$

witnesses an infinite orbit under the iterated action of the transducer of Figure 5.

Our final example, depicted in Figure 6, is of a non-bi-Lipschitz torsion element of $O_{2,1}$, given as below. One can check that the homeomorphism represented by this transducer has order two.
9 Root functions, and detecting flavours of synchronicity.

The goal of this section is to explore further properties of the root function, and to determine when a specific homeomorphism of $C_{n,r}$ begins to act (locally) as a synchronous transducer. While not central to the main results of the paper, we find these results to be of interest when working with the group $S_{n,r}$, and in particular when one is trying to discern conditions that force a general element in $S_{n,r}$ to actually reside in one of the subgroups $L S_{n,r}$ or $P S_{n,r}$.

We start this discussion with the next lemma.

**Lemma 9.1.** Let $g, h \in \text{Homeo}(C_{n,r})$, and let $\nu, \eta \in W_{n,r,\epsilon}$. Then:

1. $\theta_h$ is monotonic.
2. $\theta_h(\nu) = \theta_h(\eta)$ and $h_\nu = h_\eta \implies \nu = \eta$.
3. $\theta_h$ is injective $\iff$ $\theta_h$ is an automorphism of $W_{n,r,\epsilon}$.
4. $\theta_g(\theta_h(\nu)) \leq \theta_{gh}(\nu)$.
5. $\theta_g(\theta_h(\nu)) = \theta_{gh}(\nu) \iff g_{\theta_h(\nu)} \circ h_\nu = (gh)_\nu$.
6. $\theta_{h^{-1}}(\theta_h(\nu)) = \nu \iff (h^{-1})_{\theta_h(\nu)} = (h_\nu)^{-1} \iff h(U_\nu) = U_{\theta_h(\nu)}$.
7. If $\theta_h(\nu) \leq \theta_h(\nu')$ implies $\nu \leq \nu'$ for every $\nu' \in W_{n,r,\epsilon}$, then $h(U_\nu) = U_{\theta_h(\nu)}$.
8. Assume that $h(U_\nu) = U_{\theta_h(\nu)}$ and $\theta_h(\nu) < \theta_h(\nu' \cdot a)$ for all $a \in \{0, 1, \ldots, n - 1\}$. Then there is a permutation $\pi \in S_n$ so that $h(U_\nu \cdot a) = U_{\theta_h(\nu) \cdot \pi(a)}$ for all $a \in \{0, 1, \ldots, n - 1\}$.
9. $h(U_\nu) = U_{\theta_h(\nu')} \text{ holds for every } \nu' \in W_{n,r,\epsilon} \text{ if and only if } \theta_h \in \text{Aut}(W_{n,r,\epsilon})$.
10. There is no common nontrivial initial segment for all minimal elements of the set $\{\theta_h(\tau) | \tau \in W_n\} \setminus \{\epsilon\}$.
Proof: (1) For \( \nu, \eta \in W_{n,r} \), we have
\[
\nu \leq \eta \iff U_\nu \supseteq U_\eta \iff h(U_\nu) \supseteq h(U_\eta),
\]
and the last inclusion implies that \( \text{Root}(h(U_\nu)) \subseteq \text{Root}(h(U_\eta)) \), thus \( \theta_h(\nu) \leq \theta_h(\eta) \).

(2) \( \theta_h(\nu) = \theta_h(\eta) \) and \( h_\nu = h_\eta \) imply that \( h(U_\nu) = h(U_\eta) \), which happens if and only if \( U_\nu = U_\eta \), thus \( \nu = \eta \).

(3) One direction is trivial. In the other direction, assume that \( \theta_h \) is injective. We prove by induction on \( l = |\nu| \) that \( |\nu| = |\theta_h(\nu)| \). That will imply that \( \theta_h \) is a bijection, and since it is also monotonic, the assertion follows.

The claim is obvious for \( l = 0 \). Assume that for all \( \eta \in W_{n,r,e} \) with \( |\eta| < l \), \( |\eta| = |\theta_h(\eta)| \) and \( h(U_\eta) = U_{\theta_h(\eta)} \). Let \( \eta, \nu \in W_{n,r,e} \) and \( j \in \{0, 1, \ldots, n - 1\} \) such that \( \nu = \eta \setminus j \) and \( |\eta| = l - 1 \). Then by the induction hypothesis,
\[
h(U_\eta) = U_{\theta_h(\eta)} = h(U_{\eta^{-1}}) \cup h(U_{\eta^{-2}}) \cup \ldots \cup h(U_{\eta^{-n-1}}).
\]
So as \( \theta_h \) is injective we must have that
\[
\{h(U_{\eta^{-0}}), h(U_{\eta^{-1}}), \ldots, h(U_{\eta^{-n-1}})\} = \{U_{\theta_h(\eta)^{-0}}, U_{\theta_h(\eta)^{-1}}, \ldots, U_{\theta_h(\eta)^{-n-1}}\}.
\]
Thus \( h(U_\nu^{-j}) = U_{\theta_h(\nu)^{-i}} \) for some \( i \in \{0, 1\} \) which also implies that \( \theta_h(\nu) = \theta_h(\eta) \sim i \). Hence \( h(U_\nu) = U_{\theta_h(\nu)} \) and \( |\nu| = |\theta_h(\nu)| \).

(4) Since \( h(U_\nu) \subseteq U_{\theta_h(\nu)} \) we have that \( gh(U_\nu) = g(h(U_\nu)) \subseteq g(U_{\theta_h(\nu)}) \). Thus \( \theta_g(\theta_h(\nu)) \leq \theta_g(\nu) \).

(5) For every \( y \in \mathfrak{C}_{n,r} \) we have on one hand that \( gh(\nu \setminus y) = \theta_{gh}(\nu \setminus (gh)_\nu(y)) \) and in the other that \( g(h(\nu \setminus y)) = \theta_g(\theta_h(\nu \setminus h_\nu(y))) = \theta_g(\theta_h(\nu)) \setminus g_{\theta_h(\nu)}(h_\nu(y)) \). Thus \( \theta_g(\theta_h(\nu)) = \theta_g(\nu) \iff (gh)_\nu(y) = g_{\theta_h(\nu)}(h_\nu(y)) \) for every \( y \in \mathfrak{C}_{n,r} \).

(6) The first equivalent is a special case of (5) with \( g = h^{-1} \). For the second equivalent note that \( \theta_{h^{-1}}(\theta_h(\nu)) = \nu \implies h^{-1}(U_{\theta_h(\nu)}) \subseteq U_\nu \iff U_{\theta_h(\nu)} \subseteq h(U_\nu) \). But by definition \( U_{\theta_h(\nu)} \supseteq h(U_\nu) \), thus \( U_{\theta_h(\nu)} = h(U_\nu) \). The other direction is clear.

(7) If \( h(U_\nu) \not\subseteq U_{\theta_h(\nu)} \), then there exists \( \nu \not\in W_{n,r,e} \) such that \( h(U_\eta) \not\subseteq U_{\theta_h(\nu)} \), hence \( \theta_h(\nu) \not\leq \theta_h(\eta) \).

(8) Assume that \( h(U_\nu) = U_{\theta_h(\nu)} \) and \( \theta_h(\nu) < \theta_h(\nu \setminus a) \) for all \( a \in \{0, 1, \ldots, n - 1\} \). By the first condition we get that
\[
U_{\theta_h(\nu)} = \bigcap_{a \in \{0, 1, \ldots, n-1\}} h(U_\nu \setminus a).
\]
So by the second condition, we must necessarily have \( \theta_h(\nu \setminus a) = \theta_h(\nu) \setminus \pi(a) \) for all \( a \in \{0, 1, \ldots, n-1\} \) for some permutation \( \pi \) of \( \{0, 1, \ldots, n-1\} \). Thus, for all \( a \in \{0, 1, \ldots, n-1\} \), we have \( h(U_\nu \setminus a) = U_{\theta_h(\nu) \setminus \pi(a)} \).

(9) If \( \theta_h \in \text{Aut}(W_{n,r,e}) \) then \( \theta_{h^{-1}} = \theta_{h^{-1}} \). Therefore, by (6), for every \( \nu' \in W_{n,r,e} \) we have \( h(U_{\nu'}) = U_{\theta_h(\nu')} \). Conversely, assume that for every \( \nu' \in W_{n,r,e} \) we have \( h(U_{\nu'}) = U_{\theta_h(\nu')} \). Then by (6), we have \( \theta_{h^{-1} \circ \theta_h} = Id \), which implies that \( \theta_h \) is injective, and therefore by (3), \( \theta_h \in \text{Aut}(W_{n,r,e}) \).

(10) This is because \( \theta_{h_\nu}(\tau) = \text{Root}(h_\nu(U_\tau)) \).

\[\square\]

Lemma 9.2. Let \( h \in \text{Homeo}(\mathfrak{C}_{n,r}) \) and \( \nu, \eta, \tau \in W_{n,r} \) be such that \( U_\eta \in \text{Dec}(h(U_\nu)) \). Then:

1. If \( \eta \leq \tau \), then \( \nu \leq \theta_{h^{-1}}(\tau) \).
2. If \( \tau < \eta \), then \( \theta_{h^{-1}}(\tau) < \nu \).
Proof: (1) Since \( U_\eta \in \text{Dec}(h(U_\nu)) \), we have \( U_\eta \subset h(U_\nu) \), hence \( h^{-1}(U_\eta) \subset U_\nu \). If \( \eta \leq \tau \) then this implies \( h^{-1}(U_\tau) \subset U_\nu \), thus \( \nu \leq \theta_{h^{-1}}(\tau) \).
(2) If \( \tau < \eta \) then \( U_\tau \not\subset h(U_\nu) \), hence \( h^{-1}(U_\nu) \not\subset U_\nu \). But \( \theta_{h^{-1}}(\tau) \leq \theta_{h^{-1}}(\eta) \) and \( \nu \leq \theta_{h^{-1}}(\eta) \) so we must have \( \theta_{h^{-1}}(\tau) < \nu \). □

When we are given a homeomorphism \( h \in \text{Homeo}(\mathcal{C}_{n,r}) \), starting with the root function \( \theta_h \) of \( h \), we can build another function \( \tilde{\theta}_h : W_{n,r,\varepsilon} \rightarrow W_{n,r,\varepsilon} \), which we call the local root function of \( h \).

**Definition 9.3.** Define \( \tilde{\theta}_h(\varepsilon) := \varepsilon \), and for every \( \nu \in W_{n,r,\varepsilon} \) and \( l \in \{0, \ldots, n-1\} \), define \( \tilde{\theta}_h(\nu \cdot l) := \theta_h(\nu \cdot l) - \theta_h(\nu) \).

Note that this function essentially detects the suffix which is added to \( \theta_h(\nu) \) when constructing \( \theta_h(\nu \cdot l) \).

**Lemma 9.4.** For all \( h \in \text{Homeo}(\mathcal{C}_{n,r}) \), the following are equivalent:

1. \( \theta_h \in \text{Aut}(W_{n,r,\varepsilon}) \),
2. \( \tilde{\theta}_h(\nu) \in \{0, \ldots, n-1\} \) holds for every \( \nu \in W_{n,r} \),
3. \( \tilde{\theta}_h(\nu) \neq \varepsilon \) holds for every \( \nu \in W_{n,r} \),
4. \( \theta_h \) is injective.

**Proof:** (1) \( \Rightarrow \) (2): This follows from the monotonicity of \( \theta_h \).
(2) \( \Rightarrow \) (3): This is trivial.
(3) \( \Rightarrow \) (1) Assume that for every \( \nu \in W_{n,r} \), we have \( \tilde{\theta}_h(\nu) \neq \varepsilon \). Then as \( h(U_\varepsilon) = h(\mathcal{C}_{n,r}) = \mathcal{C}_{n,r} = U_{\theta_h(\varepsilon)} \), by Lemma 9.1 (8), we get that for every \( j \in \{0, \ldots, n-1\} \) there exists \( l \in \{0, \ldots, n-1\} \) such that \( \tilde{\theta}_h(j) = l \) and \( h(U_j) = U_l \). Now the claim follows by induction.
(1) \( \iff \) (4) By Lemma 9.1 (3).

Recall that for a permutation \( \pi \in \text{Sym}(\{0, \ldots, n-1\}) \), we defined (c.f., Definition 2.7) the maps \( \pi_n : \mathcal{C}_n \rightarrow \mathcal{C}_n \) and \( \pi_{n,r} : \mathcal{C}_{n,r} \rightarrow \mathcal{C}_{n,r} \) by twisting each entry according to \( \pi \) (except for the first one -which remains unchanged- in the case of \( \mathcal{C}_{n,r} \)). The next lemma should be obvious.

**Lemma 9.5.** An element \( h \in \text{Homeo}(\mathcal{C}_{n,r}) \) coincides with \( \hat{\pi} \) up to a permutation of the first coordinate if and only if for every \( \nu \in W_{n,r} \) we have that \( h(U_\eta) \) is a ball \( U_\eta \in B_{n,r} \) and \( h(U_\nu \cdot l) = U_\eta \cdot \pi(l) \) for all \( 0 \leq l \leq n-1 \).

The following proposition detects when a homeomorphism of \( \mathcal{C}_{n,r} \) acts, on some cone, as an iterated permutation.

**Proposition 9.6.** Let \( h \in \text{Homeo}(\mathcal{C}_{n,r}) \) and \( \nu \in W_{n,r} \). If \( h_\nu = h_{\nu \cdot l} \) holds for every \( 0 \leq l \leq n-1 \), then \( h_\nu = \hat{\pi}_n \) for some \( \pi \in \text{Sym}(\{0, \ldots, n-1\}) \).
closed, and hence the complement of this set is both open and closed. In particular, since \( h_\nu \)
is not surjective, there is a finite word \( \Gamma \in W_n \) so that the image of \( h_\nu \) is disjoint from
the ball \( U_\Gamma \) in \( \mathcal{C}_\nu \). That is, for no input to \( A \) will \( A \) produce an output with prefix \( \Gamma \). Now, let
\( w \) be the output of \( A \) from \( q \) on reading the input ‘0’. Clearly the point \( \overline{w} \) is in the image
of \( h_\nu \), and there is a minimal length prefix \( x \) so that \( \nu \cdot \overline{0} \cdot h = x \cdot \overline{w} \). However, there is a
sequence of points \( (p_i) \in \mathcal{C}_{n,r} \setminus U_\nu \) so that \( z_i = p_i \cdot h = x \cdot w_{\theta^i} \Gamma \overline{w} \) for \( \theta \) some increasing
function, since \( h \) is surjective, but the points \( w_{\theta^i} \Gamma \overline{w} \) are not in the image of \( h_\nu \). Thus,
there is some point \( p \) (a limit of a subsequence of \( (p_i) \)) in \( \mathcal{C}_{n,r} \setminus U_\nu \) so that \( p \cdot h = x \cdot \overline{w} \) as well,
which contradicts the fact that \( h \) is a homeomorphism. Therefore, we may indeed assume that \( h_\nu \) is surjective.

Our argument is actually complete: the automaton \( A \) has all states (there is only one)
simultaneously injective and surjective, and so by Proposition 5.8, the state \( q \) also must act
locally as a permutation.

\[
\square
\]

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