GEOMETRY AND REAL-ANALYTIC INTEGRABILITY

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Abstract. This note constructs a compact, real-analytic, riemannian 4-manifold \((\Sigma, g)\) with the properties that: (1) its geodesic flow is completely integrable with smooth but not real-analytic integrals; (2) \(\Sigma\) is diffeomorphic to \(\mathbb{T}^2 \times S^2\); and (3) the limit set of the geodesic flow on the universal cover is dense. This shows there are obstructions to real-analytic integrability beyond the topology of the configuration space.

1. Introduction

Taîmanov proves that if \((\Sigma, g)\) is a compact real-analytic manifold whose geodesic flow is integrable with real-analytic first integrals, then there is an invariant torus \(T\) in the unit-tangent sphere bundle \(S\Sigma\) such that \(\pi_1(T) \rightarrow \pi_1(\Sigma)\) is almost surjective [5]. The example in [2] shows that this is false in the smooth category. In that example \((\Sigma, g)\) is a compact real-analytic riemannian 3-manifold with a nilpotent \(\pi_1\) that is not almost abelian.

To state the present note’s results: Let \(E\) be the total space of a \(C^1\)-vector bundle over \(\mathbb{T}^2\) with even Euler number. The inclusion of \(C\) into \(S^2 = C \cup \{\infty\}\) induces a compactification of \(E\) into an \(S^2\)-bundle \(\Sigma\) over \(\mathbb{T}^2\) with a natural action of \(\text{SO}_2\mathbb{R}\) on its fibres. Define a metric \(g\) on \(\Sigma\) by \(\text{SO}_2\mathbb{R}\)-equivariantly identifying \(T\Sigma\) as the orthogonal direct sum of the sub-bundle \(V \simeq TS^2\) of vertical fibres and a horizontal sub-bundle \(H \simeq TT^2\) and equip each fibre of \(V\) (resp. \(H\)) with the metric of the round sphere of unit radius (resp. standard flat metric). There are natural identifications of \(H\) and \(V\) that allow \(g\) to be uniquely defined (see sections 3.3–3.4).

Theorem 1.1. The compact, real-analytic, riemannian 4-manifold \((\Sigma, g)\) has the following properties

1. its geodesic flow, \(\varphi : \mathbb{R} \times S\Sigma \rightarrow S\Sigma\), is completely (resp. non-commutatively) integrable with smooth but not real-analytic integrals;
2. \(\Sigma\) is bundle isomorphic to \(\mathbb{T}^2 \times S^2\);
3. the limit set of its geodesic flow on the universal cover is dense; and
4. its geodesic flow has zero topological entropy.

Since the geodesic flow of \((\mathbb{T}^2 \times S^2, g_{\text{flat}} \times g_{\text{round}})\) is real-analytically integrable, the topology of \(\Sigma\) does not preclude real-analytically integrable geodesic flows. How does one know that the first integrals in Theorem 1.1 cannot be real analytic? It is shown that if \(T \subset S\Sigma\) is a regular \(\varphi\)-invariant

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torus, then the subgroup \( \text{im} (\pi_1(T) \to \pi_1(\Sigma)) \) has rank at most 1. By Taïmanov’s result, mentioned above, this precludes real-analytic integrability (see sections 3.5–3.6).

The geodesic flow \( \varphi \) is more than just smoothly integrable. Let us recall some terminology to describe a type of topologically-tame integrability [4]:

A \( C^r \) \((1 \leq r \leq \infty)\) action \( \phi : \mathbb{R}^s \times M \to M \) is integrable if there is an open, dense subset \( R \) that is covered by angle-action charts \((\theta, I) : U \to T^k \times \mathbb{R}^l\) which conjugate \( \phi_t (t \in \mathbb{R}^s) \) with a translation-type map \((\theta, I) \mapsto (\theta + \omega(I)t, I)\) where \( \omega : \mathbb{R}^l \to \text{Hom}(\mathbb{R}^s, \mathbb{R}^k) \) is a smooth map. Evidently, there is an open dense subset \( L \subset R \) fibred by \( \phi \)-invariant tori \([1]\). Let \( f : L \to B \) be the \( C^r \) fibration which quotients \( L \) by these invariant tori and let \( \Gamma = M - L \) be the singular set. If \( \Gamma \) is a tamely-embedded polyhedron, then \( \phi \) is called \( k \)-semisimple with respect to \((f, L, B)\).

**Theorem 1.2.** The geodesic flow \( \varphi \) is 4-semisimple with respect to an \((f, L, B)\) such that \( f \) has non-trivial monodromy and a trivial Chern class; it is also 3-semisimple with respect to an \((f', L', B')\) such that \( f' \) has trivial monodromy and a non-trivial Chern class. The fibres of \( f' \) are contractible in \( S\Sigma \).

**Remarks.** (1) The contractibility of the fibres of \( f' \) and the minimality of \( \varphi \) restricted to a generic fibre of \( f' \) implies that \( \varphi \) is not real-analytically integrable (see sections 3.5–3.6). The contractibility of the fibres of \( f' \) also implies the density of the geodesic flow’s limit set on the universal cover.

(2) If \( \Sigma \) is constructed from \( E \) with an odd Euler number, then parts 1, 3, 4 of Theorem [1] and Theorem [2] are true. The cohomology and homotopy rings of \( \Sigma \) are isomorphic to those of \( T^2 \times S^2 \), but \( \Sigma \) is not isomorphic to \( T^2 \times S^2 \). (3) The example of Theorem [1] generalizes via the non-degenerate 2-step nilmanifolds of [3]. These examples are multiply-connected. Are there simply-connected examples? (4) Theorem [1] arose from an attempt to understand whether a hyperbolic manifold admits a semisimple geodesic flow. If so, then its invariant tori are contractible and property 3 of Theorem [1] is true — morally, at least. See Theorems 6–7 and Question C of [4] for precise statements. The examples of the present note suggest that there may exist a semisimple geodesic flow on a hyperbolic manifold.

2. THE TOPOLOGY OF \( \Sigma \)

**Terminology.** Let \( G \) be a group. The anti-diagonal subgroup is the subgroup \( \{(g, g^{-1}) : g \in G\} \) of \( G \times G \). If \( X \) and \( Y \) are \( G \)-spaces, then define \( X \times_G Y \) to be the set of orbits of the anti-diagonal subgroup.

Since \( \Sigma \) is an \( S^2 \)-bundle over \( T^2 \), there is a canonical principal \( \text{SO}_3 \mathbb{R} \)-bundle \( \text{SO}_3 \mathbb{R} \leftarrow P \to T^2 \) such that \( \Sigma = P \times_{\text{SO}_3 \mathbb{R}} S^2 \) is an associated bundle.

**Lemma 2.1.** The Euler number of \( E \) is even iff \( P \) (hence \( \Sigma \)) is trivial.

**Proof.** It suffices to show that \( P \) admits a section under the stated condition. To do this, let us recall some obstruction theory. Let \( G \) be a compact, connected Lie group and let \( G \leftarrow B \to X \) be a principal \( G \)-bundle over a
CW complex $X$. Let us try to construct a section of $B$. Let $X_p$ be the $p$-skeleton of $X$ and let $\mathcal{D}_p$ be $X$’s cellular chain group in dimension $p$. Assume that $s_p : X_p \to B|X_p$ is a section of $B|X_p$. The obstruction to extending $s_p$ to a section $s_{p+1}$ over $X_{p+1}$ is a cochain $\theta \in \text{Hom}(\mathcal{D}_{p+1}; \pi_p(G))$ — the coefficients are untwisted because $B$ is a principal fibre bundle.

The cochain $\theta$ can be defined on the generators of $\mathcal{D}_{p+1}$ as follows. Let $f_i : (D^{p+1}, S^p) \to (X_{p+1}, X_p)$ be the attaching map of a $p$-cell and $\tau_i = f_i_*[D^{p+1}, S^p]$ be the induced cellular chain (orient $D^{p+1}$ arbitrarily). The pullback bundle $f_i^*B$ is trivial since $D^{p+1}$ is contractible, so let $\mu_i : f_i^*B \to G$ be a principal fibre-bundle map. Define $\langle \theta, \tau_i \rangle \in \pi_p(G)$ to be the homotopy class of the map

$$S^p \xrightarrow{f_i} X_p \xrightarrow{s_p} B|X_p \xrightarrow{\text{incl}} f_i^*B \xrightarrow{\mu_i} G.$$  

The obstruction cochain $\theta$ measures the compatibility of the trivialization of $B|X_p$ induced by $s_p$ and the trivialization of $f_i^*B$ induced by the contractibility of $D^{p+1}$.

$B|X_1$ admits a section $s_1$ since $\pi_1(B)$ acts trivially on $\pi_4(G)$. Assume that $X_2$ contains a single 2-cell with attaching map $f$. The obstruction cochain is the homotopy class of

$$S^1 \xrightarrow{f} X_1 \xrightarrow{s_1} B|X_1 \xrightarrow{\text{incl.}} f^*B \xrightarrow{\mu} G.$$  

Let $E_1 \subset E$ be the unit-circle bundle, which is also a principal $SO_2\mathbb{R}$-bundle. When $(G, B, X) = (SO_2\mathbb{R}, E_1, \mathbb{T}^2)$, the above discussion shows that the obstruction cochain $\theta$ lies in $\text{Hom}(\mathbb{Z}; \pi_1(SO_2\mathbb{R})) = \mathbb{Z}$ and it can be identified with the Euler class of $E$. When $(G, B, X) = (SO_3\mathbb{R}, P, \mathbb{T}^2)$, the obstruction cochain $\theta'$ lies in $\text{Hom}(\mathbb{Z}; \pi_1(SO_3\mathbb{R})) = \mathbb{Z}_2$. Moreover, it is clear that the map $\pi_1(SO_2\mathbb{R}) \to \pi_1(SO_3\mathbb{R})$ maps $\theta \to \theta'$, i.e. the reduction of $\theta$ mod 2 is $\theta'$. This proves the lemma. 

3. The Metrics

The metric $g$ is constructed from an $SO_2\mathbb{R} \times SO_2\mathbb{R}$-invariant metric on $E_1 \times S^2$ using the representation

$$\Sigma = E_1 \times_{SO_2\mathbb{R}} S^2.$$  

We construct an $SO_2\mathbb{R}$-invariant metric on $E_1$, first.

3.1. $E_1$. The unit-circle bundle $E_1$ is diffeomorphic to the nilmanifold

$$\Gamma \backslash N,$$

where $N = Nil$ is the unipotent group $3 \times 3$ upper triangular matrices and

$$\Gamma = \left\langle \alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

where $k \neq 0$ is the Euler number of $E_1$. Define a riemannian metric $g_1$ on $E_1$ by declaring that $\log \alpha$, $\log \beta$, $\log \gamma$ is an orthonormal, left-invariant frame on $N$. Let $x : \Gamma \backslash N \to S^1$ (resp. $y : \Gamma \backslash N \to S^1$) be the map induced by $g \in N \mapsto \langle \log g, \log \alpha \rangle$ (resp. by $\beta$). The map $(x, y) : E_1 \to \mathbb{T}^2$ is a non-canonical form of the fibre bundle map $E_1 \to \mathbb{T}^2.$
Lemma 3.1. trivial to verify that \( \Psi \) subset of \( g \) which has smooth first integrals \( f \) Poisson commute with \( n \) where \( \alpha \) 4 L. T. BUTLER

\[ \Psi = \Psi_1 \text{ and first integrals } f_2 \text{ of } SO_g \text{ where } \log \text{ standard orthonormal basis of } so_3. \]

3.3. Let \( S^2 = \{ \xi \in R^3 : |\xi| = 1 \} \) and let \( T^*S^2 = \{ (\xi, p) \in R^3 \times R^3 : |\xi| = 1, (\xi, p) = 0 \} \) where \( \langle \cdot, \cdot \rangle \) is the euclidean metric on \( R^3 \). Let \( g_2 \) be the round metric on \( S^2 \) induced by the euclidean metric. Let \( \mu_i \) be the standard orthonormal basis of \( so_3 \) and define the 1-form \( \alpha_i \) at \( \xi \) to be \( \alpha_i(\bullet) = \langle \cdot, \mu_i \xi \rangle \). The round metric is expressed as

\[ g_2 = \alpha^2_1 + \alpha^2_2 + \alpha^2_3 = \alpha_1^2 + 4\pi^2 \sin^2 \phi \ d\phi^2, \]

where \( (r, \phi \text{ mod } 1) \) are radial coordinates in which the vector field \( \mu_1 \xi \) equals \( \frac{1}{2\pi} \frac{\partial}{\partial \phi} \).

The momentum map of the \( SO_2 \) action on \( T^*S^2 \) is

\[ \Psi_2(\xi, p) = \langle p, \mu_1 \xi \rangle = \frac{1}{2\pi} p_\phi, \]

while the hamiltonian of the metric \( g_2 \) is

\[ H_2 = \frac{1}{2} \left( \langle p, \mu_1 x \rangle^2 + \langle p, \mu_2 x \rangle^2 + \langle p, \mu_3 x \rangle^2 \right) = \frac{1}{2} \left( p_1^2 + (2\pi \sin r)^{-2} p_\phi^2 \right). \]

3.3. \( E_1 \times S^2 \) and \( \Sigma \). The riemannian metric \( g_1 \times g_2 \) has hamiltonian \( H_1 + H_2 \) and first integrals \( f_1, f_2, f_3 \& \Psi_1, \Psi_2 \) [N.B. \( f_1 = \Psi_1 \)]. The metric is also \( SO_2 \times SO_2 \)-invariant, hence it is invariant under the anti-diagonal action of \( SO_2 \). Therefore, there is a well-defined submersion metric \( g \) on \( \Sigma = E_1 \times SO_2 S^2 \). The momentum map of the anti-diagonal \( SO_2 \) action is \( \Psi = \Psi_1 - \Psi_2 \).

It is well-known that \( T^*\Sigma \) is symplectomorphic to \( \Psi^{-1}(0)/SO_2 \). It is trivial to verify that

**Lemma 3.1.** \( H_1, H_2, f_1, f_2, f_3 \) are functionally independent on an open dense subset of \( \Psi^{-1}(0) \). Moreover, \( H_1, H_2, f_1, f_2 \) Poisson commute while \( H_1, H_2, f_1 \) Poisson commute with \( f_2, f_3 \).
This Lemma proves that the geodesic flow of $(\Sigma, g)$ is completely integrable with the integrals induced by $H_1, H_2, f_1, f_2$; and it is non-commutatively integrable with the integrals induced by $H_1, H_2, f_1, f_2, f_3$. Note that $f_2$ and $f_3$ are only smooth.

3.4. An explicit expression for $g$. Let $\hat{E}_1 = N/Z(\Gamma)$, the universal abelian covering space of $E_1$, and let $\hat{\Sigma} = \hat{E}_1 \times_{SO_2 \mathbb{R}} S^2$ be the universal covering space of $\Sigma$. There is the commutative diagram of riemannian im/submersions

$$(R^2 \times S^1 \times S^2, \hat{K}) \xrightarrow{w} (\hat{E}_1 \times S^2, \hat{g}_1 + g_2) \xrightarrow{v} (\hat{E}_1 \times S^2, \hat{g}_1 + g_2),$$

where dotted arrows indicate maps that remain to be defined. An explicit formula for $g$ in local coordinates is provided by $\hat{k}$.

Let $(x, y, z \mod 1)$ be coordinates on $\hat{E}_1 = N/Z(\Gamma)$ and let $(r, \phi \mod 1)$ be polar coordinates on $S^2$ (section 3.1[3.2]). Identify $SO_2 \mathbb{R}$ with $R/Z$. The anti-diagonal action of $SO_2 \mathbb{R}$ on $\hat{E}_1 \times S^2$ is

$$\theta * (x, y, z \mod 1, r, \phi \mod 1) = (x, y, z + \theta \mod 1, r, \phi - \theta \mod 1),$$

for all $\theta$ in $SO_2 \mathbb{R}$. Let $s = z + \phi$ and $t = z$, so that $(x, y, r, s, t)$ is a system of local coordinates on $\hat{E}_1 \times S^2$. This defines $W$; since the $SO_2 \mathbb{R}$-orbits are the circles $(x, y, r, s) = constant$, this also defines $\hat{\rho}$ and $w$. Since $\hat{K} = W^*(\hat{g}_1 + g_2)$,

$$\hat{K} = dx^2 + dy^2 + (dt - \frac{1}{2}(xy - yx))^2 + dx^2 + 4\pi^2 \sin^2 x (ds - dt)^2.$$

The frame $\left\{ \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial t}, \frac{\partial}{\partial y} + \frac{\partial}{\partial \rho} \right\}$ on $R^2 \times S^2$ horizontally lifts to the frame $\left\{ X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial t}, Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial \rho}, R = \frac{\partial}{\partial t}, S = \frac{\partial}{\partial \theta} + \frac{4\pi^2 \sin^2 \theta}{1 + 4\pi^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \right\}$ on $R^2 \times S^1 \times S^2$. Along with $T = \frac{\partial}{\partial \phi}$, this forms a $\hat{K}$-orthogonal frame. A simple calculation shows that $X, Y$ and $R$ have unit norm and the norm of $S$ is $\sqrt{\frac{4\pi^2 \sin^2 x}{1 + 4\pi^2 \sin^2 x}}.$ Therefore

$$\hat{k} = dx^2 + dy^2 + dx^2 + \frac{4\pi^2 \sin^2 \theta}{1 + 4\pi^2 \sin^2 \theta} (ds - \frac{1}{2}(xy - yx))^2.$$

Remarks. (1) Clearly $\hat{k}_{\{x=const., y=const.\}}$ equals $dx^2 + \frac{4\pi^2 \sin^2 \theta}{1 + 4\pi^2 \sin^2 \theta} \, ds^2$, which is a non-round $SO_2 \mathbb{R}$-invariant metric on $S^2$. In addition, it is clear that these fibres are totally geodesic. (2) If $u$ is a smooth function which vanishes at $r = 0, \pi$, has $u'(0) = 2\pi, u'(\pi) = -2\pi$ and the even derivatives of $u$ vanish at $r = 0, \pi$, then

$$\hat{k}_u = dx^2 + dy^2 + dx^2 + u(r)^2 (ds - \frac{1}{2}(xy - yx))^2.$$
defines a smooth metric on $\mathbb{R}^2 \times S^2$. A simple computation shows that $k_u$ is invariant under the deck-transformation group and so induces a metric $g_u$ on $\Sigma$. Theorem 1.1 is true for $(\Sigma, g_u)$, and if $u$ is analytic, then Theorem 1.2 is true also. If $u(r) = 2\pi \sin r$, then $\{x = \text{const.}, y = \text{const.}\}$ is a totally geodesic round $S^2$ but $(\Sigma, g_u)$ is not isometric to $(T^2 \times S^2, g_{\text{flat}} + g_{\text{round}})$.

### 3.5. The limit set

Say that a point is recurrent for a flow if its orbit visits arbitrarily small neighbourhoods of itself in both forward and backward time. The limit set is the closure of the set of recurrent points.

Let $(\Sigma, g)$ be the universal riemannian cover of $(\Sigma, g)$ and let $\tilde{\varphi}$ be the geodesic flow of $(\Sigma, \tilde{g})$. The functions $x, y, a, b, c$ induce well-defined smooth functions on $T^*\Sigma$ and hence on $T^*\tilde{\Sigma}$. Let $\tilde{x}(\tilde{y})$ be the single-valued function on $T^*\tilde{\Sigma}$ induced by $x (y)$. The map

$$\nu = (a + c\tilde{y}, b - c\tilde{x}, c), \quad \nu : T^*\tilde{\Sigma} \to \mathbb{R}^3$$

is a first integral of $\tilde{\varphi}$ (it is a coordinatized incarnation of the momentum map of $\tilde{N}$’s left action on $T^*\Sigma$). For $\nu_0 = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ such that $\nu_3 \neq 0$, it is apparent that if $\nu(P) = \nu_0$, then

$$|\tilde{x}(P)| \leq \left| \frac{b(P) + \nu_2}{\nu_3} \right|, \quad |\tilde{y}(P)| \leq \left| \frac{a(P) + \nu_1}{\nu_3} \right|.$$ (★)

On $S\Sigma$ the functions $a$ and $b$ are bounded above by unity, so

**Lemma 3.2.** The map $\nu|\{c \neq 0\} \cap S\tilde{\Sigma}$ is proper.

Let $S \subset S\Sigma$ be a regular invariant 3-torus and let $\tilde{S} \subset S\tilde{\Sigma}$ be a lift of this torus. Since $\tilde{S}$ is regular it lies in $\{c \neq 0\}$, and therefore it is a closed subset of a fibre of $\nu$. Hence $\tilde{S}$ is compact. But $\tilde{\varphi}|\tilde{S}$ is a translation-type flow, so its limit set is $\tilde{S}$. Since the union of regular tori is dense, this proves that the limit set is dense. Since the limit set is closed, it is $S\Sigma$. This proves part 3 of Theorem 1.1.

### 3.6. Real-analytic non-integrability

Let’s complete the proof of part 1 of Theorem 1.1. For the next seven paragraphs inclusive $(\Sigma, g)$ is a compact real-analytic riemannian manifold with geodesic flow $\varphi$ and first Betti number $q$. Assume that $\varphi$ is real-analytically integrable (or more generally, geometrically simple [5]) and let $\mathcal{T}$ be the induced singular fibration of $S\Sigma$. A regular fibre of $\mathcal{T}$ is an isotropic torus. Let the lift of $•$ on $S\Sigma$ to the universal cover $S\tilde{\Sigma}$ be denoted by $\tilde{•}$.

**Lemma 3.3.** There is an open set $\mathcal{U}$ of fibres of $\mathcal{T}$ such that for each $T \in \mathcal{U}$, $T$ is diffeomorphic to a cylinder $T^r \times \mathbb{R}^q$ where $r + q \leq \dim \Sigma$.

**Proof.** Taïmanov’s Theorem says that there is an open set $\mathcal{U} \subset \mathcal{T}$ of regular invariant tori such that for each $T \in \mathcal{U}$ the $\pi_1$-image of $T \to \Sigma$ has finite index. Therefore, there is a splitting $T = T^r \times T^q$ where $\pi_1(T) \to \pi_1(\Sigma)$ factors through an injection $\pi_1(T^r) \to \pi_1(\Sigma)$.

**Remark.** It is clear from the construction of action-angle variables that the splitting of $T$ can be made compatible with its tautological affine structure.

Let us continue with the notation and hypotheses of Lemma 3.3. In addition,
Lemma 3.4. Let $S$ be a singular fibration of $S\Sigma$ whose regular fibres are $s$-dimensional $\varphi$-invariant tori. If there is a dense set of fibres of $\tilde{S}$ on which $\tilde{\varphi}$ is minimal, then $s \leq r$.

Proof. Recall that a flow is minimal if every orbit is dense. Let $v \in S\Sigma$ be a regular point of each singular fibration and let $\tilde{S} \in \tilde{S}$ (resp. $\tilde{T} \in \tilde{T}$) be the regular $S$-fibre (resp. $\tilde{T}$-fibre) through $v$. It can be assumed that $v$ is chosen so that $\tilde{\varphi}|S$ is minimal and $\tilde{T} \in \tilde{U}$. Since $\tilde{\varphi}|S$ is minimal, $v$ is a recurrent point for $\tilde{\varphi}|\tilde{T}$. Therefore, all points of $\tilde{T}$ are recurrent. Let $\tilde{T} = T^r \times R^q$ be a splitting from Lemma 3.3 which is compatible with the tautological affine structure on $\tilde{T}$ and let $v = (v', v'')$ relative to this splitting. Since $v$ is a recurrent point for $\tilde{\varphi}|\tilde{T}$, the orbit closure $\overline{\tilde{\varphi}_R(v)}$ must be contained in $T^r \times \{v''\}$. Since $\tilde{\varphi}|S$ is minimal, $\overline{\tilde{\varphi}_R(v)} = S$. Therefore $S \subset T^r \times \{v''\}$. Thus $s \leq r$.

Proof of Part 1 of Theorem 1.1 Let us return to the $(\Sigma, g)$ constructed in sections 3.1–3.4. In this case $q = 2$ and $S$ is the singular fibration of $S\Sigma$ by the connected components of the common level sets of $H_1, H_2, f_1, f_2$ and $f_3$, so $s = 3$. The inequalities (*) and the subsequent argument shows that every regular fibre of $S$ is a 3-torus. The geodesic flow $\varphi$ is minimal on a dense set of regular fibres of $S$, hence $\tilde{S}$. This follows from equation (9) of [2], which constructs action-angle variables for the geodesic flow of $(E_1, g_1)$. By Lemma 3.3 if $\varphi$ is real-analytically integrable, then $r \geq s = 3$. Since $r + q \leq 4$, $q \leq 1$. As $q = 2$, this is absurd. Therefore, $\varphi$ is not real-analytically integrable in either the commutative or non-commutative sense. This proves part 1 of Theorem 1.1.

Remark. The fact that $\varphi$ is non-commutatively integrable appears to be important for the above proof of real-analytic non-integrability. This is mistaken. The hamiltonian $H_1 = (2 + \sin 2\pi y)H_1$ on $T^r E_1$ plus $H_2$ on $T^s S^2$ induces a real-analytic metric $g'$ on $\Sigma$ whose geodesic flow $\varphi'$ is completely integrable with integrals $H_1', H_2, f_1$ and $f_2$. Construction of action-angle variables for the induced singular toral fibration $\tilde{S}$ shows that $\varphi'$ is minimal on a generic fibre of $S$. The inequality (*) for $\tilde{y}$ shows that the regular fibres of $\tilde{S}$ can only go to infinity in the $\tilde{x}$-direction, so they are either compact $T^4$ or cylinders $T^3 \times R^1$.

On the other hand, assume that $\varphi'$ is real-analytically integrable with induced singular fibration $\tilde{T}$. Let $v \in S\Sigma$ be a regular point of both singular fibrations. Let $S \in S$ and $\tilde{T} \in \tilde{T}$ be the respective fibres through $v$. By the density of minimal fibres, it can be assumed that $\varphi'|S$ is minimal and that $\tilde{T} \in \tilde{U}$. Since $\tilde{T}$ is a $\varphi'$-invariant torus, minimality implies $S \subset T$. Since $T$ is isotropic, it is a lagrangian torus of the same dimension of $S$. Therefore, $S$ is open and closed in $T$, hence $S = T$. By Lemma 3.3 $\tilde{T}$ splits as $T^2 \times R^2$. By the above comments $\tilde{S} = T^3 \times R^1$ or $T^4$. But $T$ is homeomorphic to $S$. Absurd.
4. Semisimplicity

Let $\Gamma'_1 = \{ p \in \mathcal{T}^* \Sigma : c(2H_1 - c^2)(2H_2 - c^2) = 0 \}$, $L'_1 = \mathcal{T}^* \Sigma - \Gamma'_1$ and $B'_1 = \mathcal{T}^2 \times \mathbb{R}^* \times \mathbb{R}^+ \times \mathbb{R}^+$. Define the map $f'_1 : L'_1 \to B'_1$ by

$$f'_1 = \left( \frac{a}{c} + y \mod 1, \frac{b}{c} - x \mod 1, c, 2H_2 - c^2, 2H_1 - c^2 \right).$$

This map is a proper submersion whose fibres are isotropic, $\varphi$-invariant 3-tori. The singular set $\Gamma'_1$ is a real-analytic set, hence a tamely-embedded polyhedron.

Similarly, let $\Gamma_1 = \Gamma'_1$, $L_1 = L'_1$ and $B_1 = \mathcal{T}^1 \times \mathbb{R}^* \times \mathbb{R}^+ \times \mathbb{R}^+$. Define the map $f_1 : L_1 \to B_1$ by

$$f_1 = \left( \frac{a}{c} + y \mod 1, c, 2H_2 - c^2, 2H_1 - c^2 \right).$$

This map is a proper submersion whose fibres are isotropic, $\varphi$-invariant 4-tori. These two constructions, along with the arguments in [4], imply all but the final sentence of Theorem 1.2.

The map $f'_1$ extends to a map $f'_2 : L_2' = \{ c \neq 0 \} \to B_2' = \mathcal{T}^2 \times \mathbb{R}^* \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$. By the homotopy-lifting theorem, the inclusion of a fibre of $f'_1$ is homotopic to the inclusion of the fibre $T_{s,t}$ over $(0 \mod \mathbb{Z}^2, 1, s, t)$ $(s, t > 0)$. As $T_{0,0}$ is an elliptic critical fibre for $f'_2$, and an $SO_2 \mathbb{R}$-orbit, it follows that $T_{s,t}$ is contractible in $L_2'$. This suffices to complete the proof of Theorem 1.2.

References

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