The twisted story of world-sheet scattering in $\eta$-deformed $\text{AdS}_5 \times S^5$

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We study the world-sheet scattering theory of the $\eta$ deformation of the $\text{AdS}_5 \times S^5$ superstring corresponding to the purely fermionic Dynkin diagram. This theory is a Weyl-invariant integrable deformation of the $\text{AdS}_5 \times S^5$ superstring, with trigonometric quantum deformed symmetry. We compute the two-body world-sheet $S$ matrix of this string in the light-cone gauge at tree level to quadratic order in fermions. The result factorizes into two elementary blocks, and solves the classical Yang-Baxter equation. We also determine the corresponding exact factorized $S$ matrix, and show that its perturbative expansion matches our tree level results, once we correctly identify the deformed light-cone symmetry algebra of the string. Finally, we briefly revisit the computation of the corresponding $S$ matrix for the $\eta$ deformation based on the distinguished Dynkin diagram, finding a tree level $S$ matrix that factorizes and solves the classical Yang-Baxter equation, in contrast to previous results.
The discovery and development of integrable structures in the AdS/CFT correspondence has led to impressive insights into quantum field and string theory [1, 2]. On the string theory side the canonical model is the superstring on $\text{AdS}_5 \times S^5$, a maximally supersymmetric sigma model. In recent years integrable deformations of this theory have attracted attention, building on the development of Yang-Baxter sigma models [3–5]. There is a plethora of Yang-Baxter deformations of the $\text{AdS}_5 \times S^5$ string, with distinct algebraic characteristics and interpretations in terms of string theory and AdS/CFT.
We will consider so-called inhomogeneous Yang-Baxter deformations, which algebraically correspond to trigonometric $q$ deformations [6]. These deformations are governed by an $R$ operator solving the modified classical Yang-Baxter equation (mCYBE). In the context of the AdS$_3 \times S^5$ string they are also called $\eta$ deformations.

Studies of the original $\eta$ deformation of the AdS$_3 \times S^5$ string led to a number of open questions and interesting discoveries. Namely, while Yang-Baxter deformed superstrings have $\kappa$ symmetry [5], the background of the original $\eta$-deformed AdS$_3 \times S^5$ superstring does not satisfy the supergravity equations of motion [12]. Rather it satisfies a generalized set of equations [13], which actually derive from $\kappa$ symmetry [14]. These equations are believed to guarantee scale invariance, but not Weyl invariance [13–15]. In order for a Yang-Baxter model background to solve the more restrictive supergravity equations of motion, the $R$ operator generically needs to be unimodular [18]. This raised the question whether there is a unimodular inhomogeneous deformation of AdS$_3 \times S^5$ exists, i.e. whether there is a unimodular inhomogeneous solution of the CYBE for $\mathfrak{psu}(2,2|4)$.

The canonical solution of the inhomogeneous CYBE is the so-called Drinfel’d-Jimbo $R$ operator, which is unique for a compact Lie algebra. For noncompact algebras there is freedom corresponding to a choice of simple roots relative to the real form, see [6,21] for a discussion in the present context. For superalgebras there is further freedom in whether we choose bosonic or fermionic simple roots, mirroring the lack of uniqueness of Dynkin diagrams for superalgebras. The original $\eta$ deformation [5,22,12] is based on the Drinfel’d-Jimbo $R$ matrix for the distinguished Dynkin diagram of $\mathfrak{psu}(2,2|4)$, which is not unimodular. Building an $R$ operator relative to the fermionic Dynkin diagram instead, gives a unimodular result, and a deformation of AdS$_3 \times S^5$ that solves the supergravity equations of motion [23]. We will refer to these two distinct deformations as the distinguished and fermionic ($\eta$) deformations respectively. The classical NSNS sectors of these models are equal, while their RR sectors differ.

In this paper we will be investigating the world-sheet scattering theory for the fermionic deformation. There are concrete open questions motivating our study, in addition to broader interest in the quantum integrable structure of this Weyl invariant, integrable deformation of the AdS$_3 \times S^5$ string, with trigonometric $q$-deformed symmetry. Namely, the scattering theory of the distinguished $\eta$ deformation shows some interesting features that we would like to contrast with the corresponding fermionic ones. First, the tree level $S$ matrix for the distinguished model was found not to satisfy the classical Yang-

\footnote{As the name suggests there are also homogeneous Yang-Baxter deformations [7], a class which includes e.g. the well-known real $\beta$ deformation of the AdS$_3 \times S^5$ string [8]. Algebraically these correspond to twisted symmetry [9,10], see also [11]. This twisted symmetry can be used to conjecture field theory duals [9].}

\footnote{There have been proposals suggesting that a notion of Weyl invariance may hold for these generalized backgrounds as well [16,17]. These proposals, however, have troublesome features as discussed in [17].}

\footnote{Unimodularity is sufficient, while there are subtle counterexamples to necessity, see [19,15,20].}

\footnote{This deformation can also be used as a starting point to generate new homogeneous unimodular deformations by limiting procedures [24].}

\footnote{There are unimodular deformations that one can obtain from the one of [23] by permutations of the bosonic roots as in [6,21]. Here we focus the case which gives the “standard” NSNS sector with magnetic $H$ flux.}
Baxter equation (CYBE) \cite{12}, while the model is classically integrable. A non-local two-particle change of scattering states was required to restore this hallmark requirement of integrability, as well as to match the expansion of the exact factorized $su_q(2|2)_{c.e.}$ S matrix \cite{25,26} expected to describe this model. This unexpected friction between classical integrability and tree level factorized scattering, and the subtle redefinition of scattering states, could be related to the lack of Weyl invariance of this model, which is restored for the fermionic deformation.\footnote{In general we would expect Weyl invariance to come into play only at loop level, however.} Second, the distinguished deformed model displays so-called “mirror duality” \cite{27–30} at the bosonic level and in terms of its conjectured exact S matrix. In short, in the light-cone gauge fixed theory, inversion of the deformation parameter is equivalent to a double Wick rotation on the worldsheet, which curiously relates the thermodynamic and spectral properties of the model. Studying the S matrix for the fermionic deformation is a first step towards investigating similar properties here.

We study two aspects of the world-sheet scattering theory of the fermionic $\eta$ deformation of the $AdS_5 \times S^5$ string. First, we compute the two body S matrix perturbatively at tree level with up to two fermions. We find that the resulting T matrix solves the CYBE, in line with expected integrability. The T matrix factorizes, and we expect the factors to be related to an exact S matrix for $su_q(2|2)_{c.e.}$, analogously to the undeformed and distinguished deformed string. However, only the distinguished $su_q(2|2)_{c.e.}$ S matrix is explicitly known \cite{25}. As such, second we determine the form of the exact $su_q(2|2)_{c.e.}$ S matrix for the fermionic deformation. We do this by taking advantage of a twist relating the Hopf algebras underlying the distinguished and fermionic deformations of $sl_q(2|2)_{c.e.}$. Next, based on the embedding of the two copies of $su(2|2)$ in $psu(2,2|4)$, we conjecture that the deformation of the off-shell light-cone symmetry algebra of the string takes the form $su_{1/q}(2|2)_{c.e.} \oplus su_q(2|2)_{c.e.}$. Semi-classically $q = e^{-\kappa/\hbar}$, where $\kappa$ is the deformation parameter in the action, and $\hbar$ is the string tension. The associated exact S matrix is of the form $S_0 S(1/q) \otimes S^p(q)$, where $S_0$ is a scalar prefactor, and the $-p$ denotes a particular basis permutation. The perturbative expansion of this exact S matrix matches our tree level T matrix.

We originally benchmarked our computations of the perturbative S matrix on the undeformed AdS$_5 \times S^5$ string. After we obtained our results for the fermionic deformation we decided to also run through the distinguished background given in \cite{12}. Unexpectedly, in contrast to \cite{12} we find a perturbative S matrix that directly solves the CYBE, and factorizes in line with the distinguished $su_q(2|2)_{c.e.}$ S matrix. In this case the S matrix is such that $S(1/q) = S^p(q)$, and there is effectively no distinction between $S_0 S(1/q) \otimes S^p(q)$ and $S_0 S(1/q)^{\otimes 2}$.

This paper is organized as follows. In the next section we discuss the string Lagrangian, its gauge fixing, and its expansion in powers of fields. Then in section 3 we compute the associated tree level S matrix, and discuss its factorized structure. In section 4 we review the construction of the distinguished $su(2|2)_{c.e.}$ S matrix, and twist this construction to find the fermionic exact S matrix. We then analyze the structure of the light-cone symmetry algebra in section 5, and show that the expansion of the corresponding exact S matrix matches our tree level computation. In section 6 we discuss our results regarding
the distinguished case. Finally we conclude and list several open questions. We provide appendices on our spinor conventions, our implementation of the Feynman diagram computations, and a translation of $su(2|2)R$ operators in the sigma model and exact $S$ matrix computations.

2. Deformed Lagrangian

To compute the tree-level two-body worldsheet $S$ matrix of the fermionic $\eta$ deformed string in the light-cone gauge, we need the corresponding action in the light-cone gauge, expanded to quartic order in the fields. Rather than working directly with the Yang-Baxter sigma model action [5], we will work with the standard Green-Schwarz (GS) action and substitute the background for the fermionic deformation found in [23].

2.1. The GS string to second order in fermions

Written out, the Lagrangian for a type IIB GS superstring in a generic background, to second order in the fermions, takes the form

$$L = \sqrt{-h} \hat{g}_{MN} \partial_\alpha x^M \partial_\beta x^N - \epsilon^{\alpha\beta} \hat{B}_{MN} \partial_\alpha x^M \partial_\beta x^N + i \sqrt{-h} \hat{h}^{\alpha\beta} \partial_\alpha x^M \partial_\beta x^N \theta + i \epsilon^{\alpha\beta} \partial_\alpha x^M \bar{\theta} \Gamma_M \partial_\beta \theta, \quad (2.1)$$

where $\theta = (\theta_1, \theta_2)$ is a doublet of 10D Majorana-Weyl spinors, with the Pauli matrix $\sigma_3$ acting in this two dimensional space. The world-sheet metric $h_{\alpha\beta}$ has signature $(-1, 1)$, and $\epsilon^{\tau\sigma} = 1$. In this expression we have combined certain fermionic terms with the bosonic metric $g$ and $B$ field $B$, i.e.

$$\hat{g}_{MN} = g_{MN} - \frac{i}{4} \bar{\theta} \Gamma_{(M} \phi_{N)} \theta + \frac{i}{8} \bar{\theta} \Gamma_{(M} H_{N)} \sigma_3 \theta + \frac{i}{8} \bar{\theta} \Gamma_{(M} S \Gamma_{N)} \theta, \quad (2.2)$$

$$\hat{B}_{MN} = B_{MN} + \frac{i}{4} \bar{\theta} \Gamma_{[M} \phi_{N]} \sigma_3 \theta - \frac{i}{8} \bar{\theta} \Gamma_{[M} H_{N]} \Gamma_{[PQ]} \Gamma^{PQ} \theta - \frac{i}{8} \bar{\theta} \sigma_3 \Gamma_{[M} \Gamma_{N]} \theta,$$

with round and rectangular brackets denoting symmetrization and antisymmetrization respectively, defined with the usual factor of $1/n!$. Here $\omega$ denotes the spin connection, $H = dB$, and

$$S = - \left( \epsilon \mathcal{F}^{(1)} + \frac{1}{3!} \sigma_1 \mathcal{F}^{(3)} + \frac{1}{2! 3!} \epsilon \mathcal{F}^{(5)} \right), \quad (2.3)$$

where $\epsilon \equiv i \sigma_2$. Assuming a dilaton exists, $\mathcal{F}$ encodes the RR forms and dilaton via $\mathcal{F}^{(n)} = e^\Phi F^{(n)}$.

2.2. Light-cone gauge fixing

We assume that our general background has two isometries $t$ and $\phi$, where $t$ is timelike and $\phi$ is spacelike, and introduce the light-cone coordinates

$$x^+ = \frac{1}{2} (t + \phi), \quad x^- = \phi - t. \quad (2.4)$$

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7 See e.g. [31], but note that we use a different sign convention on the fermionic world-sheet $\epsilon$ term, in line with [12].

8 Slashes denote contraction with the appropriate set of $\Gamma$ matrices: $A \equiv A_M \ldots \Gamma^M \ldots \Gamma^L$. 

The uniform light-cone gauge then consists of fixing
\[ x^+ = \tau, \quad p_- = 1, \] (2.5)
where \( \tau \) is the world-sheet time and \( p_- \) the momentum conjugate to \( x^- \). We can shortcut gauge fixing in the Hamiltonian framework by noting that momentum and winding interchange under T duality, so that if we formally T duality our model in \( x^- \), calling the dual direction \( \psi \), our uniform light-cone gauge condition becomes
\[ x^+ = \tau, \quad \psi = \sigma. \] (2.6)
Upon integrating out the world-sheet metric the T dualized Lagrangian takes the square root form typical for a light-cone gauge. In this picture the gauge condition can be directly substituted in the Lagrangian. This light-cone gauge fixing should be accompanied by a corresponding \( \kappa \)-symmetry gauge choice for the fermions of the form
\[ \Gamma^p \theta = 0, \] (2.7)
where \( \Gamma^p \) is the tangent space counterpart of \( \Gamma^+ \), defined in (A.8) for our particular case.

To simplify expressions we introduce the T dual metric \( \hat{g} \), B field \( \hat{B} \), and gamma matrices
\[ \hat{g}_{\psi \psi} = \frac{1}{\hat{g}_{--}}, \quad \hat{g}_{\psi \bar{M}} = -\frac{\hat{B}_{-\bar{M}}}{\hat{g}_{--}}, \quad \hat{g}_{\bar{M} \bar{N}} = \hat{g}_{\bar{M} \bar{N}} - \frac{\hat{g}_{-\bar{M}} \hat{g}_{-\bar{N}} - \hat{B}_{-\bar{M}} \hat{B}_{-\bar{N}}}{\hat{g}_{--}}, \] \[ \hat{B}_{\psi \bar{M}} = -\frac{\hat{g}_{-\bar{M}}}{\hat{g}_{--}}, \quad \hat{B}_{\bar{M} \bar{N}} = \hat{B}_{\bar{M} \bar{N}} - \frac{\hat{g}_{-\bar{M}} \hat{B}_{-\bar{N}} - \hat{B}_{-\bar{M}} \hat{g}_{-\bar{N}}}{\hat{g}_{--}}, \] (2.8)
where \( \bar{M} \) and \( \bar{N} \) run over the coordinates not involved in the T duality. For \( \hat{g} \) and \( \hat{B} \) the right hand side of these equations is implicitly expanded to second order in fermions. With these definition the general gauge fixed action to quadratic order in fermions takes the form
\[ \mathcal{L}^{\text{gf}} = 2\sqrt{-G} + E, \] (2.9)
where \( G = \det G_{\alpha \beta} \) and \( E = \epsilon^{\alpha \beta} E_{\alpha \beta} \) with
\[ G_{\alpha \beta} = \hat{g}_{MN} \partial_\alpha x^M \partial_\beta x^N + i \partial_\alpha x^M \hat{\Gamma}_M \partial_\beta \theta + i \partial_\alpha \psi \hat{\Gamma}_M \partial_\beta \sigma, \]
\[ E_{\alpha \beta} = -\hat{B}_{MN} \partial_\alpha x^M \partial_\beta x^N + i \partial_\alpha \psi \hat{\Gamma}_M \partial_\beta \theta + i \partial_\alpha x^M \hat{\Gamma}_M \sigma_3 \partial_\beta \theta, \] (2.10)
again implicitly expanded to second order in fermions, and evaluated on the gauge fixing condition \( x^+ = \tau, \psi = \sigma \). The gauge-fixed string action is
\[ S = -\frac{\hbar}{2} \int d^2 \sigma \mathcal{L}^{\text{gf}} = -\hbar \int d^2 \sigma \sqrt{-G} + \frac{1}{2} E, \] (2.11)

\[ ^9 \text{The possibility of gauge fixing via T duality was originally observed for the AdS}_5 \times \text{S}^5 \text{ string in [32].} \]
where $h$ is the string tension. When we take the string tension into account in the T duality and gauge fixing, consistency of $\psi = \sigma$ with $p_- = 1$, fixes the string length to be $P_-/h$, where $P_-$ is the integrated charge associated to $p_-$, see e.g. [33, 29] for details.\(^{10}\)

2.3. $\eta$-deformed AdS\(_5\) × S\(^5\)

The classical NSNS sector for our fermionic deformation is the same as the one for the distinguished deformation, given by [34]\(^{11}\)

\[
d s^2 = \frac{1}{1 - \kappa^2 \rho^2} \left( -(1 + \rho^2) d\tau^2 + \frac{d\rho^2}{1 + \rho^2} \right) + \frac{\rho^2}{1 + \kappa^2 \rho^4 x^2} \left( (1 - x^2) d\psi_1^2 + \frac{d x^2}{1 - x^2} \right) + \rho^2 x^2 d\psi_2^2
\]

\[
+ \frac{1}{1 + \kappa^2 \rho^2} \left( (1 - r^2) d\phi_1^2 + \frac{d r^2}{1 + \rho^2} \right) + \frac{r^2}{1 + \kappa^2 \rho^4 w^2} \left( (1 - w^2) d\phi_1^2 + \frac{d w^2}{1 - w^2} \right) + r^2 w^2 d\phi_2^2,
\]

\[
B = \frac{\kappa \rho}{1 - \kappa^2 \rho^2} d\tau \wedge d\rho + \frac{\kappa \rho^4 x}{1 + \kappa^2 \rho^4 x^2} d\psi_1 \land d\tau + \frac{\kappa \rho}{1 + \kappa^2 \rho^4} d\phi_1 \land d\tau - \frac{\kappa \rho^4 w}{1 + \kappa^2 \rho^4} d\phi_1 \land d w
\]

(2.12)

where $\kappa$ is the deformation parameter. The RR sector of the fermionic deformed model has a nonzero three form and a nonzero five form. As the expressions are large, we refer to the original paper [23] instead of reproducing the RR forms here.

Our conventions for light-cone gauge fixing and the computation of the perturbative S matrix for this background are analogous to those for the undeformed model, see e.g. the review [35]. The two coordinates labeled $r$ and $\phi$ in the background above are isometric, and the coordinates used in the light-cone gauge fixing. To get the interaction Lagrangian for the perturbative S matrix we first change to a different basis of transverse fields denoted $z_i, i = 1, \ldots, 4$ and $y_j, j = 1, \ldots, 4$. These are related to the transverse coordinates used above as

\[
\begin{align*}
\frac{z_1 + i z_2}{1 - \frac{1}{4} z^2} &= \rho \sqrt{1 - x^2} e^{i \psi_1}, & \frac{z_3 + i z_4}{1 - \frac{1}{4} z^2} &= \rho x e^{i \psi_2}, & z^2 &\equiv z_i^2, \\
\frac{y_1 + i y_2}{1 + \frac{1}{4} y^2} &= r \sqrt{1 - w^2} e^{i \phi_1}, & \frac{y_3 + i y_4}{1 + \frac{1}{4} y^2} &= r w e^{i \phi_2}, & y^2 &\equiv y_i^2.
\end{align*}
\]

(2.13)

In what follows we have (implicitly) applied this coordinate redefinition to the background, including the RR fields. We fix our spinor conventions in terms of these new coordinates directly, as discussed in appendix A.

2.4. Expansion of the action

For the computation of the tree-level two-body S matrix we need the gauge-fixed action to quartic order in the transverse fields. Since we are working to quadratic order in fermions from the start, this means quartic order in transverse bosons, or quadratic order

\(^{10}\)Before substituting the gauge condition, our Nambu-Goto type action is manifestly reparametrization invariant, so we can freely rescale $\sigma$. This rescaling remains a symmetry upon gauge fixing if we correspondingly adapt the gauge condition on $\psi$.

\(^{11}\)The authors of [34] use trigonometric coordinates $\zeta$ and $\xi$ related to our $x$ and $w$ as $x = \sin \zeta, w = \sin \xi$. 
in transverse bosons and quadratic order in gauge-fixed fermions. Physically we consider the string action (2.11), rescale the transverse fields by $1/\sqrt{\hbar}$, e.g. $z_1 \to z_1/\sqrt{\hbar}$, and keep terms up to order $1/\hbar$, i.e.\textsuperscript{12}

\[ S = \int d^2 \sigma \left( \mathcal{L}_2 + \frac{1}{\hbar} \mathcal{L}_4 + \ldots \right), \quad (2.14) \]

where by convention we have absorbed a sign in the definition of $\mathcal{L}_{2,4}$. This expansion is straightforward but computationally involved due to the complicated nature of the backgrounds.\textsuperscript{13}

At the quadratic level we find

\[ \mathcal{L}_2 = -\epsilon_{ab} \epsilon_{\dot{a}\dot{b}} \left( -\partial_\tau Y^{a\dot{a}} \partial_\tau Y^{b\dot{b}} + \partial_\sigma Y^{a\dot{a}} \partial_\sigma Y^{b\dot{b}} + (1 + \kappa^2) Y^{a\dot{a}} Y^{b\dot{b}} \right) \]

\[ -\epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \left( -\partial_\tau Z^{\alpha\dot{\alpha}} \partial_\tau Z^{\beta\dot{\beta}} + \partial_\sigma Z^{\alpha\dot{\alpha}} \partial_\sigma Z^{\beta\dot{\beta}} + (1 + \kappa^2) Z^{\alpha\dot{\alpha}} Z^{\beta\dot{\beta}} \right) \]

\[ + i \theta^{\dot{a}}_{\alpha a} \partial_\tau \theta^{\alpha a} - \frac{1}{2} \left( \epsilon_{\dot{a}a} \epsilon_{\dot{b}b} \theta^{\alpha b} \theta^{\beta b} - \epsilon_{\dot{a}a} \epsilon_{\dot{b}b} \theta^{\alpha b} \theta^{\dot{\beta}\dot{b}} \right) - \frac{1}{2} + \kappa^2 \theta^{\dot{a}}_{\dot{a}a} \theta^{\dot{a}}_{\dot{b}b} \]

\[ + i \eta^{\dot{a}}_{\dot{a}a} \partial_\tau \eta^{\dot{a}a} - \frac{1}{2} \left( \epsilon_{\dot{a}a} \epsilon_{\dot{b}b} \eta^{\alpha b} \eta^{\beta b} - \epsilon_{\dot{a}a} \epsilon_{\dot{b}b} \eta^{\alpha b} \eta^{\dot{\beta}\dot{b}} \right) - \frac{1}{2} + \kappa^2 \eta^{\dot{a}}_{\dot{a}a} \eta^{\dot{a}}_{\dot{b}b}, \quad (2.15) \]

where we have introduced the complex fields $Y$ and $Z$ via\textsuperscript{14}

\[ (Y^{a\dot{a}}) = \begin{pmatrix} Y^{1\dot{1}} & -Y^{1\dot{2}} \\ Y^{1\dot{1}} & -Y^{1\dot{2}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} y_3 - iy_4 \\ y_1 + iy_2 \end{pmatrix}, \quad (2.16) \]

\[ (Z^{\alpha\dot{\alpha}}) = \begin{pmatrix} Z^{3\dot{3}} & -Z^{3\dot{3}} \\ Z^{4\dot{4}} & -Z^{4\dot{4}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z_3 - iz_4 \\ z_1 + iz_2 \end{pmatrix}, \]

in addition to the fermions $\theta^{a\dot{a}}$ and $\eta^{\alpha\dot{\alpha}}$ parametrizing the spinors, as presented in eq. (A.9) in the appendix. The indices on these fields label their transformations with respect to the $\text{su}(2)_{\dot{a}\dot{a}}$ symmetry of the undeformed model, acting from the left and the right on the matrices. We denote the indices 1 and 2 with Latin letters $(a$ and $b$) and the indices 3 and 4 with Greek letters $(\alpha$ and $\beta$). For an index running from 1 to 4 we use capital Latin letters $M,N,...$. The Levi-Civita symbols $\epsilon_{ab}$ and $\epsilon_{\alpha\beta}$ are defined for Latin and Greek indices individually, i.e. $\epsilon_{12} = \epsilon^{12} = 1$ and $\epsilon_{34} = \epsilon^{34} = 1$.

\textsuperscript{12}In the approach of the review [35] $\sigma$ is rescaled by $h$ to remove explicit dependence on $h$ from the gauge fixed Hamiltonian. Our conventions and starting point circumvent this, but of course in both cases we end up with a string length of $P_0/h$ and only an overall factor of $h$ before expanding.

\textsuperscript{13}To give some technical details, we evaluated the gauge-fixed Lagrangian described above, formally expanding in fermions whenever possible before substituting concrete expressions. We expressed everything in terms of the bosonic coordinates, the two gauge fixed spinors $\theta_1$ and $\theta_2$, and a set of canonically ordered abstract tangent space gamma matrices. We discarded any terms that are zero due to the $\kappa$-gauge fixing, expanded the resulting expressions to appropriate order in bosons, and finally substituted concrete spinors and gamma matrices. In practice we were not able to sufficiently simplify the coordinate transformed RR forms before expanding, so we resorted to expanding the contributions of the RR forms to second order in the bosons before substituting them in the gauge-fixed Lagrangian.

\textsuperscript{14}As in [12], we interchanged our indices 1 $\leftrightarrow$ 2 and 1 $\leftrightarrow$ 2 relative to the typical conventions of the review [35], for convenient comparison to the exact $S$ matrix later.
The reality of $y_i$ and $z_i$ implies the reality condition
\[(Y^{a\ddagger}) = \epsilon_{ab} \epsilon_{\ddagger b} Y^{b\ddagger}, \quad (Z^{a\ddagger}) = \epsilon_{\alpha\beta} \epsilon_{\ddagger \beta} Z^{\beta\ddagger}.\] (2.17)
so that from the world-sheet perspective the model content is 8 real scalar bosons and 8 complex scalar fermions (Grassmann fields), all with mass $\sqrt{1 + \kappa^2}$. The interaction Lagrangian $L_4$ is too large to be meaningfully presented here, but can be found in the Mathematica notebook attached to this paper’s arXiv submission.

We note that our conventions at this point differ from those of [34,12]. Namely, their authors parametrized the string tension as $h = g\sqrt{1 + \kappa^2}$, and rescaled the fields by $1/\sqrt{g}$ rather than $1/\sqrt{\hbar}$. Hence our interaction terms, had [34,12] worked in a Lagrangian framework, are related as
\[L_4(\varphi) = (1 + \kappa^2) \tilde{L}_4(\tilde{\varphi}).\] (2.18)
where we denote quantities from [34,12] with bars, with $\varphi$ collectively denoting the rescaled transverse fields. Moreover, in light-cone gauge fixing we implicitly rescale $\sigma$ by $1/h$ compared to the implicit rescaling by $1/g$ of [34,12]. As a result
\[\sigma = \frac{1}{\sqrt{1 + \kappa^2}} \tilde{\sigma} \quad \Rightarrow \quad p = \sqrt{1 + \kappa^2} \tilde{p},\] (2.19)
where $p$ is the spatial world-sheet momentum used in the S matrix below. Under these identifications, our quadratic world-sheet momentum matches the one of [12]. Our bosonic interaction Lagrangian should correspond to the bosonic interactions of [34,12] in the Hamiltonian setting, while the fermionic interaction terms are inherently different.

3. Perturbative S matrix

With our kinetic and interaction Lagrangians we are ready to compute the tree level S matrix.
3.1. On-shell mode expansion

For the in- and out-states of the Feynman amplitudes we need the classical solutions of $\mathcal{L}_2$. Its equations of motion are solved by the on-shell mode expansions\(^{15}\)

\[
Y^{a\dot{a}}(\tau, \sigma) = \frac{1}{\sqrt{2\pi}} \int dp \frac{1}{2\sqrt{\omega_p}} \left(e^{i(p\sigma - \omega_p \tau)} a^{\dot{a}} a(p) + e^{-i(p\sigma - \omega_p \tau)} e^{\dot{a} b} a^b(p)\right),
\]

\[
Z^{a\dot{a}}(\tau, \sigma) = \frac{1}{\sqrt{2\pi}} \int dp \frac{1}{2\sqrt{\omega_p}} \left(e^{i(p\sigma - \omega_p \tau)} a^{\dot{a}} a(p) + e^{-i(p\sigma - \omega_p \tau)} e^{\dot{a} \dot{b}} a^{\dot{b}}(p)\right),
\]

\[
\theta^{a\dot{a}}(\tau, \sigma) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int dp \frac{1}{\sqrt{\omega_p}} \left(-ie^{i(p\sigma - \omega_p \tau)} f^a_p a^{\dot{a}}(p) + ie^{-i(p\sigma - \omega_p \tau)} h^a_p a^{\dot{a}}(p)\right),
\]

\[
\eta^{a\dot{a}}(\tau, \sigma) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int dp \frac{1}{\sqrt{\omega_p}} \left(+ie^{i(p\sigma - \omega_p \tau)} f^a_p a^{\dot{a}}(p) - ie^{-i(p\sigma - \omega_p \tau)} h^a_p a^{\dot{a}}(p)\right),
\]

where in comparison to the distinguished case of [12] it is more convenient to use $f^a_p$ and $h^a_p$ for $\theta^{a\dot{a}}$, as this enables a direct comparison with the exact result of section 4. The dispersion relation is

\[
\omega_p = \sqrt{1 + \kappa^2 + p^2},
\]

and the wave functions for the fermions are given by

\[
f_p = \sqrt{\frac{p + i\kappa}{p - i\kappa}} \sqrt{\frac{\omega_p + \sqrt{1 + \kappa^2}}{2}}, \quad h_p = \frac{p}{2f_p},
\]

\[
|f_p|^2 - |h_p|^2 = \sqrt{1 + \kappa^2}, \quad |f_p|^2 + |h_p|^2 = \omega_p. \tag{3.4}
\]

We reformulated $f_p$ from [12] to obtain manifestly continuous amplitudes for all $p > p'$ when choosing the standard branch for the square root function.

Upon quantization we have $(a^{MN})^\dagger = a^{\dagger MN}$ for all operators. For the bosons this stems from the reality condition (2.17), for the fermions it is a result of the equations of motion. It reduces the number of degrees of freedom on-shell effectively from 8 complex to 8 real scalar fermions.

3.2. T matrix

We are going to calculate the $2 \rightarrow 2$ scattering matrix $\mathcal{S}$ from the gauge fixed, deformed and expanded Lagrangian. For this, we expand $\mathcal{S}$ in terms of the tree-level $T$ matrix as

\[
\mathcal{S} = \mathbb{1} + \frac{i}{\hbar} T + \ldots \tag{3.5}
\]

and follow the standard Feynman diagram procedure, adapted to some of the intricacies of our model – details are presented in appendix B. The scattering process includes

\(^{15}\)Note that in the limit $\kappa \to 0$ the review [35] gives an expansion that differs by factors of $\pm i$ for the fermions. This would give a $T$ matrix that differs by some (physically inconsequential) signs from the $T$ matrix of [36], which is the one reproduced in [35].
two momenta, \( p \) and \( p' \) with \( p > p' \). The scattering states are \( |a_{M\beta}^\dagger(p)\alpha^P_{\alpha\beta}(p')\rangle = a_{M\beta}^\dagger(p)\alpha^P_{\alpha\beta}(p')\rangle \). Staying in line with the existing literature, we label these states by their particle content and add a prime to an operator if it depends on \( p' \) and leave it without if it depends on \( p \). This gives for example

\[
|Y_{aa}\theta'_{b\beta}\rangle \equiv |a_{aa}^\dagger(p)\alpha^P_{b\beta}(p')\rangle, \quad |Z_{aa}\eta'_{b\beta}\rangle \equiv |a_{aa}^\dagger(p)\alpha^P_{b\beta}(p')\rangle.
\]

The \( T \) matrix is given in the following by its action on the two particle states.

**Boson-Boson**

\[
T|Y_{aa}Y'_{bb}\rangle = +2A|Y_{aa}Y'_{bb}\rangle + (B - W\epsilon_{ab})|Y_{ab}Y'_{ba}\rangle + (B - W\epsilon_{ab})|Y_{ba}Y'_{ab}\rangle \\
+ \tilde{C}^\alpha_{ab}\epsilon^\beta_{a\beta} |\theta_{aa}\theta'_{b\beta}\rangle + C^\alpha_{ab}\epsilon^\alpha_{a\beta} |\eta_{aa}\eta'_{b\beta}\rangle
\]

\[
T|Z_{aa}Z'_{bb}\rangle = -2A|Z_{aa}Z'_{bb}\rangle + (-B - W\epsilon_{\alpha\beta})|Z_{\alpha\beta}Z'_{\beta\alpha}\rangle + (-B - W\epsilon_{\alpha\beta})|Z_{\beta\alpha}Z'_{\alpha\beta}\rangle \\
- \tilde{C}^\beta_{ab}\epsilon^\alpha_{b\alpha} |\eta_{aa}\eta'_{b\beta}\rangle - C^\beta_{ab}\epsilon^\alpha_{b\alpha} |\eta_{aa}\eta'_{b\beta}\rangle
\]

**Fermion-Fermion**

\[
T|\theta_{aa}\theta'_{b\beta}\rangle = + \tilde{\tilde{C}}^\alpha_{ab}\epsilon^\beta_{a\beta} |Y_{aa}Y'_{bb}\rangle - C^\alpha_{ab}\epsilon^\alpha_{a\beta} |Z_{aa}Z'_{\beta\beta}\rangle
\]

\[
T|\eta_{aa}\eta'_{b\beta}\rangle = - \tilde{\tilde{C}}^\alpha_{ab}\epsilon^\beta_{a\beta} |Z_{aa}Z'_{bb}\rangle + C^\alpha_{ab}\epsilon^\alpha_{a\beta} |Y_{aa}Y'_{bb}\rangle
\]

\[
T|\theta_{aa}\eta'_{b\beta}\rangle = - \tilde{\tilde{H}}^\beta_{a\beta}|Y_{ab}Z'_{ba}\rangle - H^\beta_{a\beta}|Z_{\beta\alpha}Y'_{ab}\rangle
\]

\[
T|\eta_{aa}\theta'_{b\beta}\rangle = + \tilde{\tilde{H}}^\beta_{a\beta}|Z_{\beta\alpha}Y'_{ba}\rangle + \tilde{H}^\beta_{a\beta}|Y_{ba}Z'_{a\beta}\rangle
\]
Boson-Fermion

\[ T |Y_{ab}\theta'_{b\beta}\rangle = (A + G) |Y_{ab}\theta'_{b\beta}\rangle + (B - W\epsilon_{ab}) |Y_{ba}\theta'_{a\beta}\rangle \]
\[ + H^{3a}_{a\beta} |\theta_{a\beta} Y'_{a\beta}\rangle + C_{ab}^{\alpha\beta} \epsilon_{ab}\epsilon^{\alpha\beta} |\eta_{aa} Z'_{\beta\beta}\rangle \]
\[ T |Y_{aa}\eta'_{\beta\beta}\rangle = (A + G) |Y_{aa}\eta'_{\beta\beta}\rangle + (B - W\epsilon_{ai}) |Y_{ai}\eta'_{\beta\beta}\rangle \]
\[ + H^{3a}_{a\beta} |\eta_{aa} Y'_{a\beta}\rangle - \dot{C}_{ab}^{\alpha\beta} \epsilon_{ab}\epsilon^{\alpha\beta} |\theta_{aa} Z'_{\beta\beta}\rangle \]
\[ T |\theta_{aa} Y'_{bb}\rangle = (A - G) |\theta_{aa} Y'_{bb}\rangle + (B - W\epsilon_{ab}) |\theta_{ba} Y'_{ab}\rangle \]
\[ + \dot{H}^{\beta\lambda}_{ab} |Y_{ab}\theta'_{a\lambda}\rangle - C_{ab}^{\alpha\beta} \epsilon_{ab}\epsilon^{\alpha\beta} |Z_{ad}\theta'_{d\beta}\rangle \]
\[ T |Z_{aa}\theta'_{b\beta}\rangle = - (A + G) |Z_{aa}\theta'_{b\beta}\rangle + (-B - W\epsilon_{a\beta}) |Z_{a\beta}\theta'_{a\lambda}\rangle \]
\[ - \dot{H}^{\beta\lambda}_{ab} |\theta_{ba} Z'_{a\lambda}\rangle + \dot{C}_{ab}^{\alpha\beta} \epsilon_{ab}\epsilon^{\alpha\beta} |\eta_{aa} Y'_{bb}\rangle \]
\[ T |Z_{aa}\eta'_{\beta\beta}\rangle = - (A + G) |Z_{aa}\eta'_{\beta\beta}\rangle + (-B - W\epsilon_{a\beta}) |Z_{a\beta}\eta'_{a\lambda}\rangle \]
\[ - \dot{H}^{\beta\lambda}_{ab} |\eta_{aa} Z'_{a\lambda}\rangle - C_{ab}^{\alpha\beta} \epsilon_{ab}\epsilon^{\alpha\beta} |\theta_{aa} Y'_{bb}\rangle \]
\[ T |\theta_{aa} Z'_{b\beta}\rangle = - (A - G) |\theta_{aa} Z'_{b\beta}\rangle + (-B - W\epsilon_{a\beta}) |\theta_{a\beta} Z'_{a\lambda}\rangle \]
\[ - \dot{H}^{\beta\lambda}_{ab} |Z_{a\lambda}\theta'_{a\beta}\rangle - \dot{C}_{ab}^{\alpha\beta} \epsilon_{ab}\epsilon^{\alpha\beta} |Y_{aa}\eta'_{\beta\beta}\rangle \]
\[ T |\eta_{aa} Z'_{\beta\beta}\rangle = - (A - G) |\eta_{aa} Z'_{\beta\beta}\rangle + (-B - W\epsilon_{a\beta}) |\eta_{a\beta} Z'_{a\lambda}\rangle \]
\[ - \dot{H}^{\beta\lambda}_{ab} |Z_{a\beta}\eta'_{a\lambda}\rangle + C_{ab}^{\alpha\beta} \epsilon_{ab}\epsilon^{\alpha\beta} |Y_{aa}\theta'_{b\beta}\rangle \]
Because we work only up to quadratic order in fermions, we were not able to determine the expressions for four-fermion processes. The coefficients used above are defined as

\[
A = \frac{1}{4} \frac{(p - p')^2 + \kappa^2(\omega - \omega')^2}{p\omega' - p'\omega}, \\
B = \frac{pp' + \kappa^2\omega\omega'}{p\omega' - p'\omega}, \\
G = -\left(1 + \kappa^2\right) \frac{\omega^2 - \omega'^2}{4 p\omega' - p'\omega}, \\
W = i\kappa, \\
C_0 = -\left(1 + \kappa^2\right) \frac{1}{\sqrt{p^2 + \kappa^4}} \frac{\sinh\left(\frac{1}{2} (\text{arsinh} \frac{p}{\sqrt{1 + \kappa^2}} - \text{arsinh} \frac{p'}{\sqrt{1 + \kappa^2}})\right)}{p\omega' - p'\omega}, \\
G = -\left(1 + \kappa^2\right) \frac{\omega^2 - \omega'^2}{4 p\omega' - p'\omega}, \\
C_0 = -\left(1 + \kappa^2\right) \frac{1}{\sqrt{p^2 + \kappa^4}} \frac{\sinh\left(\frac{1}{2} (\text{arsinh} \frac{p}{\sqrt{1 + \kappa^2}} - \text{arsinh} \frac{p'}{\sqrt{1 + \kappa^2}})\right)}{p\omega' - p'\omega}.
\]

This permutation appears only on terms with dotted indices, and is just a basis transformation.\(^{16}\) This can be avoided by choosing a different basis to label our fields from the start. We fixed our current basis for comparison to established literature, in particular regarding the distinguished deformation.\(^{13}\)
3.3. Factorization

Our tree level result matches a T matrix written in the factorized form
\[
T = T(-\kappa) \otimes 1 + 1 \otimes T^p(\kappa),
\]
\[
T_{MNPQ}^{\text{PQ\hat{Q}}} = (-1)^{\epsilon_M(\epsilon_N + \epsilon_Q)} T_{MN}^{\text{PQ\hat{Q}}}(-\kappa) \delta^\text{P}_M \delta^\text{\hat{Q}}_N + (-1)^{\epsilon_Q(\epsilon_M + \epsilon_P)} \delta^\text{P}_M \delta^\text{Q}_N T_{MPQ}^{\text{\hat{P}Q\hat{Q}}}(\kappa),
\]
up to four fermion amplitudes that we did not compute. Here the first and second factor of the tensor product acts respectively on the undotted or dotted indices. \(\epsilon_M\) describes the statistics of the index, i.e. it is zero for Latin indices (1 and 2) and one for Greek indices (3 and 4). The matrix \(T\) is the tree-level expansion of the exact fermionic \(\mathfrak{su}_q(2|2)\) c.e. \(S'\) matrix that will be derived in the next section. The entries of \(T(\kappa)\) are
\[
\begin{align*}
\mathcal{T}_{ab}^{cd} &= A \delta^c_a \delta^d_b + (B + W\epsilon_{ab}) \delta^d_a \delta^c_b, \\
\mathcal{T}_{ab}^{\gamma\delta} &= -A \delta^\gamma_a \delta^\delta_b + (-B + W\epsilon_{ab}) \delta^\delta_a \delta^\gamma_b, \\
\mathcal{T}_{a\beta}^{cd} &= G \delta^c_a \delta^d_\beta, \\
\mathcal{T}_{a\beta}^{\gamma\delta} &= -G \delta^\gamma_a \delta^\delta_\beta, \\
\mathcal{T}_{ab}^{\gamma\delta} &= C_{ab}^{\gamma\delta} \epsilon_{ab}, \\
\mathcal{T}_{a\beta}^{\gamma\delta} &= \mathcal{C}_{a\beta}^{\gamma\delta} \epsilon_{ab}, \\
\mathcal{T}_{ab}^{\gamma\delta} &= \mathcal{H}_{a\beta}^{\gamma\delta} \epsilon_{ab}, \\
\mathcal{T}_{a\beta}^{\gamma\delta} &= \mathcal{H}_{a\beta}^{\gamma\delta} \epsilon_{ab},
\end{align*}
\]
with the coefficients from eq. (3.7). The sign flip of \(\kappa\) and the permutation of indices in eq. (3.9) both respectively leave the terms involving \(A, B\) and \(G\) invariant and change the \(W\) term by a sign. The \(C\) and \(H\) terms in turn transform in a non-trivial way.

4. Exact \(S\) matrix

In this section we derive the exact \(\mathfrak{su}_q(2|2)\) c.e. \(S\) matrix for the fermionic deformation. We exploit the fact that at the level of the complexified superalgebra \(\mathfrak{sl}_q(2|2)\) c.e. the Hopf algebras constructed using respectively the distinguished and fully fermionic Dynkin diagram of \(\mathfrak{sl}(2|2)\) have coproducts related by a twist. The \(S\) matrix associated to the fully fermionic Dynkin diagram can thus be obtained from the \(\mathfrak{sl}_q(2|2)\) c.e. \(S\) matrix associated to the distinguished Dynkin diagram through twisting and upon imposing appropriate reality conditions.

4.1. \(\mathfrak{su}_q(2|2)\) c.e. Hopf algebra

Let us first recall the defining relations of the \(\mathfrak{su}_q(2|2)\) superalgebra. For this we start by considering a Cartan-Weyl basis of the complexified \(\mathfrak{sl}(2|2)\) superalgebra, formed by Cartan elements \(H_j\), positive roots \(E_j\) and negative roots \(F_j\), where the index \(j = 1, 2, 3\). The \(q\)-deformation is defined through the relations
\[
q^{H_j} E_k = q^{A_{jk}} E_k q^{H_j}, \quad q^{E_j} F_k = q^{-A_{jk}} F_k q^{E_j}, \quad [E_j, F_k] = d_j \delta^{jk} [H_j]_q,
\]
with \([x]_q = (q^x - q^{-x})/(q - q^{-1})\) and \(A\) a symmetric Cartan matrix associated to \(\mathfrak{sl}(2|2)\), obtained from the original unsymmetrized Cartan matrix \(\hat{A}\) through \(A = DA\) with
$D = \text{diag}(d_1, d_2, d_3)$. A particularity of Lie superalgebras that sets them apart from ordinary Lie algebras is that they admit inequivalent Dynkin diagrams, depending on the number of bosonic simple roots in the chosen root system. Each Dynkin diagram is associated to a different Cartan matrix. $\mathfrak{sl}(2|2)$ admits three inequivalent Dynkin diagrams. We will focus on two of them, the distinguished Dynkin diagram, which has the maximum number of bosonic simple roots (two), and the fully fermionic Dynkin diagram, where all the three simple roots are fermionic.

The relations (4.1) are not enough to completely fix the $\mathfrak{sl}(2|2)$ superalgebra but need to be supplemented with standard and higher-order Serre relations. We do not write these conditions in their most general form here (independent on the choice of Dynkin diagram), but rather later when considering the distinguished and fully fermionic Dynkin diagram.

There are several coproducts under which the quantum deformed algebra becomes a Hopf algebra. Here we choose the one whose action on the Cartan elements, positive and negative simple roots is given by

$$\Delta(H_j) = H_j \otimes 1 + 1 \otimes H_j ,$$

$$\Delta(E_j) = E_j \otimes 1 + q^{-H_j} \otimes E_j ,$$

$$\Delta(F_j) = F_j \otimes q^{H_j} + 1 \otimes F_j .$$

In order to obtain a non-trivial S matrix one needs to introduce the braiding. The coproduct of Cartan elements and bosonic simple roots remains unchanged, but the coproduct of fermionic simple roots needs to be adapted. Again, we postpone the explicit expression of the coproduct with braiding to when we consider specific Dynkin diagrams.

**Distinguished Dynkin diagram.** The distinguished Cartan matrix corresponds to choosing a root system with the maximum number of bosonic simple roots. In the case of $\mathfrak{sl}(2|2)$, this corresponds to two bosonic simple roots and one fermionic simple root. This is the Dynkin diagram chosen in [25], and we review the main characteristics of the corresponding Hopf algebra here.

The unsymmetrised and symmetrised distinguished Cartan matrices are

$$\hat{A} = \begin{pmatrix} +2 & -1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +2 \end{pmatrix}, \quad A = \begin{pmatrix} +2 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -2 \end{pmatrix}, \quad D = \text{diag}(+1, -1, -1) . \quad (4.5)$$

The standard Serre relations are

$0 = [E_1, E_3] = [F_1, F_3] = E_2E_2 = F_2F_2$

$= E_1E_1E_2 - (q + q^{-1})E_1E_2E_1 + E_2E_1E_1 = E_3E_3E_2 - (q + q^{-1})E_3E_2E_3 + E_2E_3E_2$

$= F_1F_1F_2 - (q + q^{-1})F_1F_2F_1 + F_2F_1F_2 = F_3F_3F_2 - (q + q^{-1})F_3F_2F_3 + F_2F_3F_2 , \quad (4.6)$

and the higher order Serre relations take the form $P = 0$ and $K = 0$ with

$$P = E_1E_2E_3E_2 + E_2E_3E_2E_1 + E_3E_2E_2E_1 + E_2E_1E_2E_3 - (q + q^{-1})E_2E_1E_3E_2 ,$$

$$K = F_1F_2F_3F_2 + F_2F_3F_2F_1 + F_3F_2F_1F_2 + F_2F_1F_2F_3 - (q + q^{-1})F_2F_1F_3F_2 . \quad (4.7)$$
The Cartan matrix has non-maximal rank 2 and there is thus a central element, given by
\[ C = -H_2 - \frac{1}{2}(H_1 + H_3). \]
In fact, it can be shown that the higher-order Serre relations (4.7) can be consistently dropped, in which case also P and K become central elements and one obtains the triply centrally extended algebra \( \mathfrak{sl}_q(2|2) \ltimes \mathbb{R}^2 \).

The coproduct (including the braiding factor \( U \)) of the Cartan elements and simple roots is
\[
\Delta(H_j) = H_j \otimes 1 + 1 \otimes H_j, \quad (4.8)
\]
\[
\Delta(E_j) = \begin{cases} E_j \otimes 1 + q^{-H_j} \otimes E_j, & j = 1, 3 \\ E_j \otimes U^{-1/2} + q^{-H_j} U^{1/2} \otimes E_j, & j = 2 \end{cases}, \quad (4.9)
\]
\[
\Delta(F_j) = \begin{cases} F_j \otimes q^{H_j} + 1 \otimes F_j, & j = 1, 3 \\ F_j \otimes q^{H_j} U^{1/2} + U^{-1/2} \otimes F_j, & j = 2 \end{cases}. \quad (4.10)
\]
This in turn fixes the coproduct of the three central elements to be
\[
\Delta(C) = C \otimes 1 + 1 \otimes C, \quad (4.11)
\]
\[
\Delta(P) = P \otimes U^{-1} + q^{2C} U \otimes P, \quad (4.12)
\]
\[
\Delta(K) = K \otimes q^{-2C} U + U^{-1} \otimes K. \quad (4.13)
\]
The S matrix should satisfy
\[
\Delta^{\text{op}}(X) S = S \Delta(X), \quad \forall X \in \mathfrak{sl}(2|2). \quad (4.14)
\]
In particular if \( X \) is central this implies \( \Delta^{\text{op}}(X) = \Delta(X) \). While this is immediately satisfied for C, imposing it for P and K partially fixes them to be
\[
P = \alpha \beta U^{-1} \left( 1 - q^{2C} U^2 \right), \quad K = \alpha^{-1} \beta U \left( q^{-2C} - U^{-2} \right), \quad (4.15)
\]
where \( \alpha \) and \( \beta \) are yet undetermined complex numbers.

**Fermionic Dynkin diagram.** For the fully fermionic Dynkin diagram on the other hand all the three simple roots are fermionic. The non-symmetrised and symmetrised Cartan matrices are (we use primes to denote quantities related to the fully fermionic Dynkin diagram)
\[
A' = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & -1 \\ 0 & +1 & -1 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad D' = \text{diag}(-1, +1, -1). \quad (4.16)
\]
The standard Serre relations are
\[
0 = E'_1E'_1 = F'_1F'_1 = E'_2E'_2 = F'_2F'_2 = E'_3E'_3 = F'_3F'_3, \quad (4.17)
\]
togther with \( P' = 0 \) and \( K' = 0 \), where
\[
P' = [E'_1, E'_3], \quad K' = [F'_1, F'_3]. \quad (4.18)
\]
The higher-order Serre relations are automatically satisfied. The reason why we have separated the standard Serre relations into (4.17) and (4.18) is that the second constraints can be consistently dropped, leading to a triply centrally extended $\mathfrak{sl}_2$ superalgebra with central elements $P'$, $K'$ and

$$C' = \frac{1}{2}(H'_1 + H'_3).$$

(4.19)

The coproduct (including the braiding) is

$$\Delta'(H'_j) = H'_j \otimes 1 + 1 \otimes H'_j,$$

(4.20)

$$\Delta'(E'_j) = \begin{cases} E'_j \otimes U^{1/2} + q^{-H'_j} U^{1/2} \otimes E'_j, & j = 1, 3, \\ E'_j \otimes U^{1/2} + q^{-H'_j} U^{-1/2} \otimes E'_j, & j = 2, \end{cases}$$

(4.21)

$$\Delta'(F'_j) = \begin{cases} F'_j \otimes q^{H'_j} U^{1/2} + U^{-1/2} \otimes F'_j, & j = 1, 3, \\ F'_j \otimes q^{H'_j} U^{-1/2} + U^{1/2} \otimes F'_j, & j = 2. \end{cases}$$

(4.22)

For the three central elements we obtain

$$\Delta'(C') = C' \otimes 1 + 1 \otimes C',$$

(4.23)

$$\Delta'(P') = P' \otimes U^{-1} + q^{2C'} U \otimes P',$$

(4.24)

$$\Delta'(K') = K' \otimes q^{-2C'} U + U^{-1} \otimes K',$$

(4.25)

and $P'$, $K'$ obey analogous relations to (4.15).

**The twist.** As already mentioned, the $q$-deformed superalgebra generated by the fully fermionic Cartan matrix is isomorphic to the $q$-deformed superalgebra generated by the distinguished Cartan matrix, while the coproducts are related by a twist. This remains true even after introducing the braiding. Indeed, the Lusztig transformation $\omega : ([., .], A') \rightarrow ([., .], A)$ defined by [37]

$$\omega(H'_1) = H_1 + H_2,$$  

$$\omega(E'_1) = E_1 E_2 - q E_2 E_1,$$  

$$\omega(F'_1) = F_2 F_1 - q^{-1} F_1 F_2,$$

$$\omega(H'_2) = -H_2,$$  

$$\omega(E'_2) = -F_2 q H_2,$$  

$$\omega(F'_2) = -q^{-H_2} E_2,$$

$$\omega(H'_3) = H_2 + H_3,$$  

$$\omega(E'_3) = E_3 E_2 - q^{-1} E_2 E_3,$$  

$$\omega(F'_3) = F_2 F_3 - q F_3 F_2,$$

(4.26)

is such that

$$[\omega(X'), \omega(Y')] = \omega([X', Y']) , \quad \forall X', Y' \in \mathfrak{sl}_2(2|2).$$

(4.27)

Moreover, under this map the central elements are transformed into one another

$$\omega(C') = C,$$  

$$\omega(P') = P,$$  

$$\omega(K') = K,$$

(4.28)

The co-products on the other hand are related by a Drinfel’’d twist

$$(\omega \otimes \omega)\Delta'(X') = F^{-1} \Delta'(\omega(X')) F,$$  

$$F = 1 \otimes 1 - (q - q^{-1}) U^{1/2} F_2 \otimes U^{1/2} E_2,$$

(4.29)
with \( F \) satisfying the cocycle condition
\[
(F^{-1} \otimes 1)(\Delta \otimes 1)F^{-1} = (1 \otimes F^{-1})(1 \otimes \Delta)F^{-1}.
\] (4.30)

Therefore, the S matrix for the fully fermionic Dynkin diagram is
\[
S' = F^{-\text{op}}SF.
\] (4.31)

**Reality conditions.** Until now we have worked with the complexified algebra \( \mathfrak{sl}_q(2|2) \) and have not imposed any reality conditions to obtain \( \mathfrak{su}_q(2|2) \). The S matrix \( S' \) of (4.31) is thus not a priori unitary. Imposing the reality conditions
\[
H_j^\dagger = H_j, \quad E_j^\dagger = q^{-H_j}F_j, \quad U^\dagger = U^{-1},
\] (4.32)
produces a unitary S matrix \( S \) associated to the distinguished Dynkin diagram, but the sought after S matrix \( S' \), associated to the fully fermionic Dynkin diagram, is not unitary due to the twist. Therefore, we need to adapt the reality conditions so that \( S \) is not unitary but \( S' \) is. In other words, instead of using the reality conditions (4.32) that are compatible with the coproduct (4.8), we choose reality conditions that are compatible with the coproduct (4.20). These are
\[
H_j^\dagger = H_j, \quad E_j^\dagger = q^{-H_j}F_j, \quad U^\dagger = U^{-1}.
\] (4.33)

Imposing
\[
\omega^\dagger(X') = \omega(X'),
\] (4.34)
then gives rise to
\[
E_1^\dagger = q^{-1}q^{-2H_2-H_1}F_1, \quad E_2^\dagger = q^{H_2}F_2, \quad E_3^\dagger = q q^{-2H_2-H_3}F_3.
\] (4.35)

It thus follows that a way to obtain the exact \( q \)-deformed S matrix for the fully fermionic Dynkin diagram is to twist the \( \mathfrak{su}_q(2|2)_{c.e.} \) S matrix associated to the distinguished Dynkin diagram and impose the reality conditions (4.35).

### 4.2. Fundamental S matrix

The \( q \)-deformed S matrix based on the distinguished Dynkin diagram of \( \mathfrak{sl}_q(2|2) \) has been derived by Beisert and Koroteev in [25]. For completeness we review the construction here, with some slight changes. In particular, we use a symmetric braiding and work with the shifted and rescaled variables of [38].

**Fundamental representation.** The fundamental representation of the centrally extended \( \mathfrak{sl}_q(2|2) \) superalgebra is spanned by four states \( |\phi_1\rangle, |\phi_2\rangle, |\psi_1\rangle, |\psi_2\rangle \), with \( |\phi_j\rangle \) bosonic and \( |\psi_j\rangle \) fermionic obeying
\[
\begin{align*}
H_1 |\phi_1\rangle &= - |\phi_1\rangle, & H_2 |\phi_1\rangle &= -(C - 1/2) |\phi_1\rangle, & H_3 |\phi_1\rangle &= 0 \\
H_1 |\phi_2\rangle &= + |\phi_2\rangle, & H_2 |\phi_2\rangle &= -(C + 1/2) |\phi_2\rangle, & H_3 |\phi_2\rangle &= 0 \\
H_1 |\psi_2\rangle &= 0 & H_2 |\psi_2\rangle &= -(C + 1/2) |\psi_2\rangle & H_3 |\psi_2\rangle &= + |\psi_2\rangle \quad (4.36) \\
H_1 |\psi_1\rangle &= 0 & H_2 |\psi_1\rangle &= -(C - 1/2) |\psi_1\rangle & H_3 |\psi_1\rangle &= - |\psi_1\rangle.
\end{align*}
\]
where $C$ is the central charge for the fundamental representation. The action of the simple roots is given by

\begin{align}
E_1 |\phi_1\rangle &= \bar{a} |\phi_2\rangle, \\
E_2 |\phi_2\rangle &= a |\psi_2\rangle, \\
E_3 |\psi_2\rangle &= \bar{b} |\psi_1\rangle, \\
E_2 |\psi_1\rangle &= b |\phi_1\rangle, \\
F_1 |\phi_1\rangle &= \bar{c} |\phi_2\rangle, \\
F_2 |\phi_2\rangle &= c |\psi_1\rangle, \\
F_2 |\psi_1\rangle &= d |\phi_1\rangle, \\
F_1 |\psi_1\rangle &= \bar{d} |\psi_2\rangle,
\end{align}

where $\bar{a}, a, \bar{b}, b, c, \bar{c}, d, \bar{d}$ are coefficients constrained by the commutation relations of the $q$-deformed algebra. By renormalising the states one could in principle eliminate two barred coefficients, setting for instance $\bar{a} = \bar{b} = 1$, but here we prefer to keep the coefficients free, while ensuring the normalisation $\langle \phi_a | \phi_a \rangle = \langle \psi_\alpha | \psi_\alpha \rangle = 1$. Choosing the basis of states

\begin{align}
|\phi_1\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
|\phi_2\rangle &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \\
|\psi_2\rangle &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \\
|\psi_1\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\end{align}

the positive and negative simple roots have matrix realisations

\begin{align}
E_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \bar{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_2 &= \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{b} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
F_1 &= \begin{pmatrix} 0 & \bar{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
F_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}, \\
F_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align}

The matrix realisation of the other generators can easily be deduced from their expressions in terms of $E_j$ and $F_j$. Taking the commutator between a positive and a negative simple root one obtains the relations

\begin{align}
\bar{a} \bar{c} &= 1, \\
\bar{b} \bar{d} &= 1,
\end{align}

and

\begin{align}
ad &= \left[ C + \frac{1}{2} \right]_q, \\
bc &= \left[ C - \frac{1}{2} \right]_q.
\end{align}

The commutation relations involving a Cartan element and a positive or negative simple root are automatically satisfied. Furthermore, the central charges $P$ and $K$, expectation values of the central elements $P$ and $K$ respectively, are given by

\begin{align}
P &= \bar{a} \bar{b} \bar{a}, \\
K &= \bar{c} \bar{d} \bar{d}.
\end{align}

This in turns implies the closure condition

\begin{align}
\left[ C + \frac{1}{2} \right]_q \left[ C - \frac{1}{2} \right]_q = PK &= \beta^2 (1 - q^{2C} U^2)(q^{-2C} - U^{-2}),
\end{align}
where for the last equality we plugged in the explicit expressions for $P$ and $K$ derived in (4.15). This can be recast into

$$ (V - V^{-1})^2 = \xi^2 (U - U^{-1})^2 + (1 - \xi^2)(q^{1/2} - q^{-1/2})^2 , $$

(4.44)

where we introduced

$$ V = q^C , \quad \xi = -i \frac{\beta (q - q^{-1})}{\sqrt{1 - \beta^2 (q - q^{-1})^2}} . \quad (4.45) $$

The labeling of states by $\phi$ and $\psi$ as used in this section and in [25], in our conventions corresponds to the sigma model indices 1, 2, 3, 4 as

$$ \psi_1 \leftrightarrow 1 , \quad \psi_2 \leftrightarrow 2 , \quad \phi_1 \leftrightarrow 3 , \quad \phi_2 \leftrightarrow 4 \quad (4.46) $$

with a second copy of these for the dotted indices.

**Deformation of the Zhukovsky variables.** As customary in the context of $q$-deformations we introduce deformations of the Zhukovsky variables,

$$ U^2 = q^{-1} \frac{x^+ + \xi}{x^- + \xi} = q^{1/2} \frac{x^- + \xi}{1 + \xi} , \quad V^2 = q^{-1} \frac{1 + x^+ \xi}{1 + x^- \xi} = q^{1/2} \frac{\xi / x^- + 1}{\xi / x^+ + 1} , \quad (4.47) $$

and hence in the $x^{\pm}$ variables the closure condition (4.44) becomes

$$ q^{-1} \left( x^+ + \frac{1}{x^+} \right) - q \left( x^- + \frac{1}{x^-} \right) - (q - q^{-1}) \left( \xi + \frac{1}{\xi} \right) = 0 . \quad (4.48) $$

The variables $\bar{a}$ and $\bar{b}$ remain free, while the others are

$$ \bar{c} = \frac{1}{\bar{a}} , \quad \bar{d} = \frac{1}{\bar{b}} , \\ a = \sqrt{\beta} \gamma U^{-1/2} \frac{1}{\sqrt{\bar{ab}}} , \\ b = \sqrt{\beta} \alpha \gamma^{-1} U^{-1/2} \left( 1 - \frac{x^+}{x^-} \right) \frac{1}{\sqrt{\bar{ab}}} , \\ c = i \sqrt{\beta} \alpha^{-1} \gamma \sqrt{1 - \xi^2 q^{1/2} U^{1/2} V^{-1}} \frac{1}{x^+ + \xi} \sqrt{\bar{ab}} , \\ d = i \sqrt{\beta} \gamma^{-1} \sqrt{1 - \xi^2 q^{1/2} U^{1/2} V} \frac{x^- - x^+}{1 + x^+ \xi} \sqrt{\bar{ab}} . \quad (4.49) $$
The $s_{1/2}(2|2)$ c.e. S matrix. The S matrix satisfying the defining equality (4.14) is [25]\(^{17}\)

\[
S\left|\phi_0\phi_0\right> = A_{12}\left|\phi_0\phi_0\right>, \quad S\left|\psi_0\psi_0\right> = -D_{12}\left|\psi_0\psi_0\right>,
\]

\[
S\left|\phi_0\phi_2\right> = \frac{\bar{a}_1}{a_2}\left(1 - \frac{q + q^{-1}}{2}\right)\left|\phi_2\phi_1\right> + \frac{q(A_{12} - B_{12})}{2}\left|\phi_1\phi_2\right> - \frac{\bar{b}_1}{a_2}\frac{C_{12}}{a_2 q^2 + 1}\left|\psi_2\psi_1\right> + \frac{\bar{b}_1}{a_2}\frac{q C_{12}}{a_2 q^2 + 1}\left|\psi_1\psi_2\right>,
\]

\[
S\left|\phi_2\phi_1\right> = \frac{A_{12} - B_{12}}{q + q^{-1}}\left|\phi_2\phi_1\right> + \frac{\bar{a}_2}{a_1}\frac{q^{-1}A_{12} + B_{12}}{q + q^{-1}}\left|\phi_1\phi_2\right> + \frac{\bar{b}_2}{a_1}\frac{C_{12}}{a_1 q + q^{-1}}\left|\psi_2\psi_1\right> - \frac{\bar{b}_2}{a_1}\frac{q C_{12}}{a_1 q + q^{-1}}\left|\psi_1\psi_2\right>,
\]

\[
S\left|\psi_2\psi_1\right> = \frac{\bar{b}_1 q^2 D_{12} + E_{12}}{q + q^{-1}}\left|\psi_2\psi_1\right> - \frac{\bar{b}_1 q^{-1}D_{12} + q E_{12}}{q + q^{-1}}\left|\psi_1\psi_2\right> - \frac{\bar{a}_1}{b_2}\frac{F_{12}}{b_2 q + q^{-1}}\left|\phi_2\phi_1\right> - \frac{\bar{a}_2}{b_2}\frac{q F_{12}}{b_2 q + q^{-1}}\left|\phi_1\phi_2\right>,
\]

\[
S\left|\psi_1\psi_1\right> = -\frac{D_{12} - E_{12}}{q + q^{-1}}\left|\psi_1\psi_1\right> - \frac{\bar{b}_1 q^{-1}D_{12} + q E_{12}}{q + q^{-1}}\left|\psi_1\psi_2\right> - \frac{\bar{a}_1}{b_2}\frac{F_{12}}{b_2 q + q^{-1}}\left|\phi_2\phi_1\right> - \frac{\bar{a}_2}{b_2}\frac{q F_{12}}{b_2 q + q^{-1}}\left|\phi_1\phi_2\right>,
\]

\[
S\left|\phi_1\psi_1\right> = G_{12}\left|\phi_1\psi_1\right> + \frac{\bar{b}_1}{b_1}\frac{q H_{12}}{b_1}\left|\psi_1\psi_1\right>, \quad S\left|\psi_1\psi_1\right> = L_{12}\left|\psi_1\psi_1\right> + \frac{\bar{b}_2}{b_1}\frac{K_{12}}{b_1}\left|\phi_1\psi_1\right>,
\]

\[
S\left|\phi_1\psi_2\right> = G_{12}\left|\phi_1\psi_2\right> + H_{12}\left|\psi_1\psi_1\right>, \quad S\left|\psi_1\psi_2\right> = L_{12}\left|\psi_1\psi_2\right> + K_{12}\left|\phi_1\psi_2\right>,
\]

\[
S\left|\phi_2\psi_1\right> = G_{12}\left|\phi_2\psi_1\right> + \frac{\bar{a}_2}{a_1}\frac{H_{12}}{a_1}\left|\psi_1\psi_1\right>, \quad S\left|\psi_2\psi_1\right> = L_{12}\left|\psi_2\psi_1\right> + \frac{\bar{a}_1}{a_2}\frac{K_{12}}{a_2}\left|\phi_2\psi_1\right>,
\]

\[
S\left|\phi_2\psi_2\right> = G_{12}\left|\phi_2\psi_2\right> + \frac{\bar{a}_2}{a_1}\frac{H_{12}}{a_1}\left|\psi_1\psi_1\right>, \quad S\left|\psi_2\psi_2\right> = L_{12}\left|\psi_2\psi_2\right> + \frac{\bar{a}_1}{a_2}\frac{K_{12}}{a_2}\left|\phi_2\psi_2\right>. \quad (4.50)
\]

The ten coefficients are given by

\[
A_{12} = S_0\frac{U_1 V_1}{U_2 V_2}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+},
\]

\[
B_{12} = S_0\frac{U_1 V_2}{U_2 V_1}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\left(1 - \frac{q + q^{-1}}{2}\right)\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\frac{x_1^- - x_2^+}{x_2^- - x_1^+}\frac{x_1^+ - x_2^-}{x_2^- - x_1^+},
\]

\[
C_{12} = -S_0\left(1 - \frac{q + q^{-1}}{2}\right)\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\frac{x_1^- - x_2^+}{x_2^- - x_1^+}\frac{x_1^+ - x_2^-}{x_2^- - x_1^+}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+},
\]

\[
D_{12} = -S_0, \quad E_{12} = -S_0\left(1 - \frac{q + q^{-1}}{2}\right)\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\frac{x_1^- - x_2^+}{x_2^- - x_1^+}\frac{x_1^+ - x_2^-}{x_2^- - x_1^+}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+},
\]

\[
F_{12} = -S_0\left(1 - \frac{q + q^{-1}}{2}\right)\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\frac{x_1^- - x_2^+}{x_2^- - x_1^+}\frac{x_1^+ - x_2^-}{x_2^- - x_1^+}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+},
\]

\[
G_{12} = S_0\frac{1}{q^{1/2}U_2V_2}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+},
\]

\[
H_{12} = S_0\frac{\gamma_1 x_2^+ - x_2^-}{\gamma_2 x_2^- - x_2^+},
\]

\[
K_{12} = S_0\frac{U_1 V_1}{U_2 V_2}\frac{\gamma_2 x_2^+ - x_2^-}{\gamma_1 x_2^- - x_2^+},
\]

\[
L_{12} = S_0 U_1 V_1 q^{1/2}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}. \quad (4.51)
\]

\(^{17}\)Due to our choice of normalisation of the fields, the S matrix is obtained from [25] by sending $|\phi_2\rangle \rightarrow \bar{a}|\phi_2\rangle$ and $|\psi_1\rangle \rightarrow \bar{b}|\psi_1\rangle$.  

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This S matrix satisfies the quantum Yang-Baxter equation
\[ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} . \]

Here we view S as an operator acting in the tensor product of two spaces, each spanned by the \(|\phi_a\rangle\) and \(|\psi_b\rangle\), cf. equation (4.50). Then the quantum Yang-Baxter equation is an equation in the triple tensor product space, where here and below the indices on S indicate the factors of the tensor product in which S acts nontrivially.

4.2.1. Distinguished Dynkin diagram

Let us briefly discuss the exact \(su_q(2|2)\) c.e. S matrix associated with the distinguished Dynkin diagram.

**Reality conditions.** In order to obtain the exact S matrix for the deformed model based on the distinguished Dynkin diagram we need to impose the reality conditions (4.32). This in turn implies \(\xi \in i\mathbb{R}\) together with the constraints
\[
|\bar{a}|^2 = q , \quad |\bar{b}|^2 = q^{-1} , \\
|\gamma|^2 = i(1 - qV^2)\frac{\sqrt{1 - \xi^2}}{\xi} , \quad |\alpha|^2 = 1 .
\]

A solution to these equations is given by
\[
\bar{a} = \sqrt{q} , \quad \bar{b} = \frac{1}{\sqrt{q}} , \quad \alpha = 1 , \quad \gamma = \sqrt{-q^{1/2}UV(x^+ - x^-)} .
\]

With this choice of coefficients, the S matrix precisely matches the one of [25].

**A symmetry of the exact S matrix.** An interesting property of the exact S matrix for the distinguished Dynkin diagram with (4.53) is its invariance under the map
\[
q \rightarrow q^{-1} , \quad |\phi_1\rangle \leftrightarrow |\phi_2\rangle , \quad |\psi_1\rangle \leftrightarrow |\psi_2\rangle .
\]

To show this, let us analyse the consequences of sending \(q \rightarrow q^{-1}\) on the coefficients of the S matrix. First of all, the braiding factor \(U\) is independent on \(q\) and thus remains unchanged. By the definitions (4.45) we have \(V \rightarrow V^{-1}\) and \(\xi \rightarrow -\xi\). The variables \(x^\pm\) also need to be modified as they are subject to the conditions (4.47) and (4.48), which depend on \(q\), \(V\) and \(\xi\). The solutions to the modified constraints are
\[
x^- \rightarrow \frac{x^- + \xi}{1 + x^-\xi} , \quad x^+ \rightarrow \frac{x^+ + \xi}{1 + x^+\xi} .
\]

This in turns implies
\[
\gamma \rightarrow \gamma \frac{\sqrt{1 - \xi^2}}{1 + x^+\xi} .
\]
Under these transformations the ten coefficients $A_{12}, B_{12}, \ldots, L_{12}$ remain invariant. The latter however enter the $S$ matrix with factors of the deformation parameter $q$. Therefore the transformation $q \to q^{-1}$ is not itself a symmetry of the exact $S$ matrix associated to the distinguished Dynkin diagram, but it is easy to see that it becomes one when supplemented with the exchange of states as in (4.54).

### 4.2.2. Fully fermionic Dynkin diagram

We now consider the $su_q(2|2)_{c.e.}$ $S$ matrix associated to the fermionic Dynkin diagram.

**Implementing the twist.** In order to obtain the exact $S$ matrix $S'$ associated to the fully fermionic Dynkin diagram we implement the twist (4.31), $S' = F^{-op} SF$. In the fundamental representation, $F$ only differs from the identity for the following matrix elements:

\[
(F - 1 \otimes 1) |\phi_1 \phi_2\rangle = -(q - q^{-1})U_1^{1/2}U_2^{1/2}c_1 a_2 |\psi_1 \psi_2\rangle ,
\]

\[
(F - 1 \otimes 1) |\phi_1 \psi_1\rangle = -(q - q^{-1})U_1^{1/2}U_2^{1/2}c_1 b_2 |\psi_1 \phi_1\rangle ,
\]

\[
(F - 1 \otimes 1) |\psi_2 \phi_2\rangle = -(q - q^{-1})U_1^{1/2}U_2^{1/2}d_1 a_2 |\phi_2 \psi_2\rangle ,
\]

\[
(F - 1 \otimes 1) |\psi_2 \psi_1\rangle = -(q - q^{-1})U_1^{1/2}U_2^{1/2}d_1 b_2 |\phi_2 \phi_1\rangle .
\]

**Reality conditions.** Finally, to obtain a unitary $S$ matrix we impose the reality conditions (4.35), leading to

\[
|\bar{a}|^2 = V^2 q^{-1} , \quad |\bar{b}|^2 = V^2 q , \quad |\bar{\gamma}|^2 = -\alpha q^{3/2} U V (x^+ - x^-) , \quad |\alpha|^2 = 1 .
\]

With this choice of coefficients, the $S$ matrix is unitary: $(S'_{12})^\dagger S'_{12} = 1 \otimes 1$.

### 4.2.3. Expansion of the exact $S$ matrix

In order to obtain the tree-level expansion of the fermionic $su_q(2|2)_{c.e.}$ $S$ matrix, we need to provide a physical meaning to the purely algebraic quantities used to construct the exact fermionic $S$ matrix. We assume that the energy and the momentum are defined as in the undeformed case through

\[
\mathbb{C} |\Phi\rangle = C |\Phi\rangle = \frac{\omega}{2} |\Phi\rangle , \quad \mathbb{U} |\Phi\rangle = U |\Phi\rangle = e^{ip} |\Phi\rangle ,
\]

with $\Phi$ standing for an element of $\{\phi_1, \phi_2, \psi_1, \psi_2\}$. The exact $S$ matrix has two free parameters $q$ and $\xi$. Based on experience with the distinguished case [34], we take these to be related to the deformation parameter $\kappa$ and the string tension $h$ entering the string sigma model through\(^{18}\)

\[
q = e^{-\kappa/h} , \quad \xi = i \kappa .
\]

\(^{18}\)The expression for $\xi$ is equivalent to taking $\beta = h/(2\sqrt{1 + \kappa^2})$. Taking $h = g\sqrt{1 + \kappa^2}$, $\beta = g/2$ gives the expressions of [34].
Rescaling the momentum $p \rightarrow p/T$ we find, to linear order in $h$,

\[ U = 1 + \frac{ip}{2h} + \ldots , \quad V = 1 - \frac{\kappa \omega}{2h} + \ldots . \]  
(4.60)

Solving the variables $x^\pm$ as functions of $U,V$, and using the closure condition (4.44) yields the dispersion relation (3.2). Finally, taking the scalar factor $S_0$ equal to the one of the distinguished case [26,34,27],19 expanding the exact S-matrix gives

\[ S(q) = 1 + \frac{i}{\hbar} T(\kappa) + \ldots , \]  
(4.61)

matching indices as in eqs. (4.46). We precisely recover the coefficients (3.10) for the matrix $T(\kappa)$.

5. Comparison of the perturbative and exact S matrix

In section 3.3 we found a tree level $T$ matrix of the form

\[ T = T(-\kappa) \otimes 1 + 1 \otimes T^p(\kappa). \]  
(5.1)

This structure is a deformation of the one for the undeformed string, which has two identical $T$ factors in its $T$ matrix. It suggests that the light-cone symmetry algebra is not a straightforward identical deformation of both factors of the $su(2|2)^\vee \oplus 2_{c.e.}$ light-cone symmetry of the undeformed string. We can determine its precise form by considering the embedding of the relevant $su(2|2)$ algebras in $su(2,2|4)$, and the action of the $R$ operator at both levels.

5.1. Light-cone symmetry algebra

In the matrix conventions of [35] (see e.g. equation (2.123) there), the two copies of $su(2|2)$ are embedded in $su(2,2|4)$ as

\[
\begin{pmatrix}
\mathbb{R} & 0 & -Q^\dagger & 0 \\
0 & \mathbb{R} & 0 & Q \\
Q & 0 & \mathbb{L} & 0 \\
0 & \tilde{Q}^\dagger & 0 & \tilde{\mathbb{L}}
\end{pmatrix},
\]  
(5.2)

in $2 \times 2$ block notation, with one copy of $su(2|2)$ generated by $R,L,Q,Q^\dagger$, and the other by their dotted counterparts. Note the different relative placement of $Q$ and $\tilde{Q}$.20

This structure needs to be contrasted with the action of the fermionic $R$ operator defining the action, and the fermionic $R$ operator $R_{su(2|2)}$ corresponding to the $q$ deformation of

19It is consistent to use the scalar factor of the distinguished case (translated to our conventions), because the distinguished and fermionic models and S matrices are identical at the purely bosonic level. The scalar factor only affects diagonal entries at leading order in $1/h$.

20While $Q \leftrightarrow Q^\dagger$ is an automorphism of $su(2|2)$, the central extensions $\{Q,Q\} \sim \mathbb{C}$ and $\{\tilde{Q},\tilde{Q}\} \sim \mathbb{C}$ that appear off shell meaningfully fix this embedding.
the exact $\mathfrak{su}(2|2)$ $S$ matrix. The $R$ operator defining the action, acts on elements $M$ of $\mathfrak{su}(2,2|4)$ as \cite{23}

$$R(M)_{ij} = -i\epsilon_{ij}M_{ij}, \quad \epsilon = \begin{pmatrix}
0 & +1 & +1 & +1 & +1 & +1 \\
-1 & 0 & +1 & +1 & +1 & +1 \\
-1 & -1 & 0 & +1 & +1 & +1 \\
-1 & +1 & +1 & 0 & +1 & +1 \\
-1 & -1 & +1 & +1 & 0 & +1 \\
-1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}, \quad (5.3)$$

where we have highlighted the blocks corresponding to the undotted and dotted copies of $\mathfrak{su}(2|2)$ in green and yellow respectively. In appendix C we translate between conventions to determine the $R$ operator corresponding to the exact $S$ matrix of section 4, acting on a copy of $\mathfrak{su}(2|2)$ of the form

$$\begin{pmatrix}
R & -Q^\dagger \\
Q & L
\end{pmatrix}. \quad (5.4)$$

This $R$ operator is

$$R_{\mathfrak{su}(2|2)}(M)_{ij} = -i\epsilon_{ij}M_{ij}, \quad \epsilon = \begin{pmatrix}
0 & -1 & -1 & -1 \\
+1 & 0 & -1 & -1 \\
+1 & -1 & 0 & -1 \\
+1 & +1 & +1 & 0
\end{pmatrix}. \quad (5.5)$$

Comparing the action of $R_{\mathfrak{su}(2|2)}$ to the action of $R$ on the two $\mathfrak{su}(2|2)$s as in equation (5.3), we see that $R$ acts like $-R_{\mathfrak{su}(2|2)}$ on the undotted copy of $\mathfrak{su}(2|2)$. For the dotted copy, the difference between $R$ and $R_{\mathfrak{su}(2|2)}$ is more involved, but can be matched by an index permutation. Namely, working at the level of the indices of the corresponding generators (see e.g. section 2.4.2 of \cite{35}),\footnote{Recall that we permute indices 1 and 2, and 1 and 2, relative to \cite{35}.} under the permutation $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, (5.2) transforms as

$$\begin{pmatrix}
34 & 33 & 24 & 14 \\
44 & 43 & 23 & 13 \\
42 & 32 & 21 & 22 \\
41 & 31 & 11 & 12
\end{pmatrix} \rightarrow \begin{pmatrix}
34 & 33 & 24 & 14 \\
44 & 43 & 23 & 13 \\
42 & 32 & 21 & 22 \\
41 & 31 & 11 & 12
\end{pmatrix}$$

where we have indicated the action of the $R$ operator by color-coding the entries in red ($+i$), blue ($-i$) and white (0). We see that after the permutation $R$ acts precisely
 oppositely on the dotted and undotted (indexed) generators of the two copies of $\mathfrak{su}(2|2)$. Since $R$ acted like $-R_{\mathfrak{su}(2|2)}$ on the undotted $\mathfrak{su}(2|2)$, it acts like $+R_{\mathfrak{su}(2|2)}$ on the dotted $\mathfrak{su}(2|2)$ with permuted indices.

As changing the sign of the $R$ operator is equivalent to changing the sign of $\kappa$ or inverting $q$, the two copies of $\mathfrak{su}(2|2)$ effectively have opposite deformation parameters, upon implementing this permutation. Putting everything together, our light-cone symmetry algebra is expected to be $\mathfrak{su}_{1/2}(2|2)_{c.e.} \oplus \mathfrak{su}_{q}(2|2)_{c.e.}$.

### 5.2. Expanded exact S matrix

The above structure for the light-cone symmetry algebra is compatible with our tree level $T$ matrix, and suggest that the exact $S$ matrix is of the form $S(1/q) \otimes S^p(q)$, where the factors correspond to the exact $\mathfrak{su}_q(2|2)_{c.e.}$ $S$ matrix $S(q)$. Using the tree level expansion of $S(q)$ given in equation (4.61), we find

$$S(1/q) \otimes S^p(q) = 1 + \frac{i}{\hbar} (T(-\kappa) \otimes 1 + 1 \otimes T^p(\kappa)) + \ldots = 1 + \frac{i}{\hbar} T + \ldots .$$

(5.7)

In other words we find perfect agreement between the exact $\mathfrak{su}_{1/2}(2|2)_{c.e.} \oplus \mathfrak{su}_{q}(2|2)_{c.e.}$ $S$ matrix and our tree level $T$ matrix, provided $q = e^{-\kappa/\hbar}$, at least semiclassically. Taking into account the relative parametrizations, this identification of $q$ matches the one of [6].

### 6. Distinguished deformation

With the framework set up, we can quickly revisit the computation of the tree level $T$ matrix for the distinguished deformation. Taking the distinguished background presented in [12] as input, we proceed exactly as described before, except that we use the unconjugated $f_p$ and $h_p$ in the mode expansion of $\theta$

$$\vartheta^{a\dot{a}}(\tau, \sigma) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int dp \frac{1}{\sqrt{\omega_p}} \left( -ie^{i(p\sigma - \omega_p \tau)} f_p a^{\alpha \dot{a}}(p) + ie^{-i(p\sigma - \omega_p \tau)} h_p e^{ab} \epsilon^{\dot{\alpha} \dot{\beta}} a^\dagger_{b\dot{\beta}}(p) \right),$$

(6.1)

for a convenient direct comparison. The resulting $T$ matrix factorizes, and solves the CYBE. As for the fermionic case, it is of the form

$$T = T(-\kappa) \otimes 1 + 1 \otimes T^p(\kappa)$$

(6.2)

where $T$ is now given by eq. (3.10) with $A$, $B$, $G$ and $W$ as in the fermionic case, see eq. (3.7), and

$$C^{\alpha \beta}_{ab}(\kappa) = \bar{C}^{\dot{a} \dot{b}}_{\alpha \beta}(\kappa) = C_0, \quad \mathcal{H}^{ab}_{\alpha \beta}(\kappa) = \bar{\mathcal{H}}^{\dot{a} \dot{b}}_{\alpha \beta}(\kappa) = \mathcal{H}_0 .$$

(6.3)

Although the $R$ operator is different, the light-cone symmetry algebra has the same overall structure as in the fermionic case. As discussed in appendix C, we now have

$$R_{\mathfrak{su}(2|2)}(M)_{ij} = -i\epsilon_{ij} M_{ij}, \quad \epsilon = \begin{pmatrix} 0 & -1 & -1 & -1 \\ +1 & 0 & -1 & -1 \\ +1 & +1 & 0 & -1 \\ +1 & +1 & +1 & 0 \end{pmatrix},$$

(6.4)
and the analogue of (5.6) becomes

\[
\begin{pmatrix}
34 & 33 & 24 & 14 \\
44 & 43 & 23 & 13 \\
34 & 33 & 21 & 22 \\
44 & 43 & 14 & 22
\end{pmatrix} \rightarrow
\begin{pmatrix}
34 & 33 & 24 & 14 \\
44 & 43 & 23 & 13 \\
42 & 32 & 11 & 12 \\
41 & 31 & 14 & 22
\end{pmatrix}
\]

(6.5)

upon the same permutation \(1 \leftrightarrow 2 \) and \(3 \leftrightarrow 4 \). We see that also in the distinguished case we expect \(\mathfrak{su}_{1/q}(2|2)\text{c.e.} \oplus \mathfrak{su}_q(2|2)\text{c.e.} \) symmetry. Expanding the corresponding exact S matrix reproduces our tree level result here as well.

The inversion of the deformation parameter is less significant here than it was in the fermionic case. As discussed around equations (4.54), for the distinguished deformation the permutation and inversion of deformation parameter correspond to a symmetry of the exact S matrix

\[
S^p(q) = S(1/q), \quad \Rightarrow \quad T^p(\kappa) = T(-\kappa),
\]

so that there is no real distinction between \(\mathfrak{su}_{1/q}(2|2)\text{c.e.} \oplus \mathfrak{su}_q(2|2)\text{c.e.} \) and \(\mathfrak{su}_q(2|2)\oplus^2\text{c.e.} \) symmetry in this case. In our current conventions, if we strip off the permutation, the T matrix is

\[
T = T(-\kappa) \otimes 1 + 1 \otimes T(-\kappa)
\]

(6.7)

and hence manifestly compatible with \(\mathfrak{su}_{1/q}(2|2)\oplus^2\text{c.e.} \) symmetry.

Our present results conflict with those of [12], whose T matrix only factorizes and satisfies the CYBE after a nonlocal redefinition of the scattering states. Our results agree in the purely bosonic sector, but differ for the fermions.\(^{22}\) As our setups differ throughout the various stages of the computation, it is not straightforward to conclusively determine the origin for this mismatch. It is likely due to a subtle difference in the gauge fixing of \(\kappa \) symmetry.\(^{23}\) Our results show that there exists a gauge choice for which this classically integrable field theory admits a tree level S matrix that solves the CYBE, which seems like a natural consistency requirement.

7. Conclusions

In this paper we studied the world-sheet scattering theory of the light-cone gauge-fixed fermionic \(\eta \) deformation of the AdS\(_5 \times S^5\) string. We started by computing the tree-level

\(^{22}\)To directly compare, note that [12] defines epsilon with lower indices with the opposite sign from us. Moreover, at the exact S matrix level, at face value we have an inverted deformation parameter compared to [34,12]. However, for the distinguished deformation this can be undone by a basis change and is thus inconsequential. In fact, the identification of \(q \) depends on mapping from the exact and tree level S matrix basis in the first place.

\(^{23}\)We thank G. Arutyunov, R. Borsato and S. Frolov for discussions on this point.
world-sheet S matrix, showing that it satisfies the classical Yang-Baxter equation and has a factorized structure. Based on expectations regarding the light-cone symmetry algebra, we then determined the exact S matrix factor compatible with $\mathfrak{su}_q(2|2)_{\text{c.e.}}$ symmetry for the fermionic Dynkin diagram. By considering the embedding of the light-cone symmetry algebra in the full symmetry algebra relative to the action of the $R$ operator governing the deformation, we found that the two factors of the light-cone symmetry algebra are in fact deformed oppositely, resulting in $\mathfrak{su}_{1/q}(2|2)_{\text{c.e.}} \oplus \mathfrak{su}_q(2|2)_{\text{c.e.}}$ symmetry. The corresponding full exact S matrix is compatible with our tree-level world-sheet computation.

We also revisited the distinguished deformation of AdS$_5 \times S^5$ in our setup, similarly finding a factorized tree level $T$ matrix that solves the classical Yang-Baxter equation. In this case the light-cone symmetry algebra is based on the distinguished Dynkin diagram, and it turns out that inversion of the deformation parameter is effectively a symmetry of the exact S matrix factor, so that pragmatically there is no distinction between $\mathfrak{su}_{1/q}(2|2)_{\text{c.e.}} \oplus \mathfrak{su}_q(2|2)_{\text{c.e.}}$ and $\mathfrak{su}_q(2|2)_{\text{c.e.}}$ symmetry.

There are a number of questions we did not address in this paper. First, it would be interesting to understand what effect the change from distinguished to fermionic deformation has on the exact spectrum of the model, or whether perhaps the models should ultimately be considered equivalent, and if so, how this relates to Weyl invariance. This first of all requires analysis of the corresponding Bethe equations. Second, coming back to the mirror duality mentioned in the introduction, it would be interesting to see whether this feature of the distinguished deformation is also present for our fermionic one. It would not only be interesting to answer this question for the current exact S matrix, but also to investigate the tree level and exact S matrices for deformations of AdS$_3 \times S^3$ where it is possible to realize mirror duality explicitly also in the fermionic sector of the sigma model [39]. Next, for the exact S matrix and Bethe ansatz description of these models it is important to understand the precise identification of the exact parameters $q$ and $\xi$ and the Lagrangian parameters $\kappa$ and $h$. This is related to questions surrounding quantum corrections to Yang-Baxter deformed backgrounds, where initial studies have thus far focused on $\alpha'$ corrections for homogeneous deformations [40,41], and corrections to (not Weyl invariant) deformed backgrounds to maintain compatibility with RG flow [42,43], see also the very recent [44,45]. At the level of quantum corrections, it would also be great to investigate the one-loop S matrices for both the distinguished and fermionic deformations along the lines of [40], as at loop level we are generically sensitive to Weyl invariance.\footnote{For the distinguished case the one-loop S matrix has been studied using unitarity techniques in [47].}

We could also consider further unimodular inhomogeneous deformations with differing bosonic sectors as in [6,21]. In [21] it was shown that the bosonic S matrices of these models are related to the one of the standard inhomogeneous deformation by one-particle momentum-dependent changes of basis. It would be interesting to understand whether this picture continues to hold when including fermions, and if so, to find a matching algebraic picture at the level of the exact S matrix. Finally, it would be very interesting to understand whether the fermionic deformation of the AdS$_5 \times S^5$ string that we considered here, can be given an interpretation in terms of AdS/CFT.
Acknowledgements

We would like to thank Gleb Arutyunov, Riccardo Borsato, Sergey Frolov, Ben Hoare, Alessandro Sfondrini and Linus Wulff for discussions, and Gleb Arutyunov, Riccardo Borsato, Sergey Frolov, Ben Hoare and Arkady Tseytlin for comments on the draft of this paper. The work of FS is supported by the Swiss National Science Foundation through the NCCR SwissMAP. The work of ST and YZ is supported by the German Research Foundation (DFG) via the Emmy Noether program “Exact Results in Extended Holography”. ST is supported by LT.

A. Spinor conventions

Here we briefly set out our conventions regarding the spinors of the Green-Schwarz action and associated objects such as gamma matrices, the vielbein and spin connection. Our conventions are close to those of [12] but differ in the labeling of coordinates and coset model gamma matrices.

Tangent space

We introduce tangent space gamma matrices as follows

\[
\Gamma_a = \begin{cases} 
\sigma_1 \otimes \gamma_0 \otimes \mathbb{1}_4 & a = 0, \\
-\sigma_2 \otimes \mathbb{1}_4 \otimes \gamma_5 & a = 1, \\
\sigma_1 \otimes \gamma_{a-1} \otimes \mathbb{1}_4 & a = 2, 3, 4, 5, \\
-\sigma_2 \otimes \mathbb{1}_4 \otimes \gamma_{a-5} & a = 6, 7, 8, 9, 
\end{cases} \tag{A.1}
\]

where

\[
\gamma_0 = i\sigma_3 \otimes \sigma_0 = i\gamma_5, \\
\gamma_1 = \sigma_2 \otimes \sigma_2, \\
\gamma_2 = -\sigma_2 \otimes \sigma_1, \\
\gamma_3 = \sigma_1 \otimes \sigma_0, \\
\gamma_4 = \sigma_2 \otimes \sigma_3, \tag{A.2}
\]

are the coset model \(\gamma\) matrices of [35], and the \(\sigma_i\) denote the Pauli matrices, with \(\sigma_0 \equiv \mathbb{1}_2\). The associated charge conjugation matrix

\[
C = i\sigma_2 \otimes K \otimes K, \quad K = -i\sigma_0 \otimes \sigma_2, \tag{A.3}
\]

satisfies

\[
\Gamma_a^t = -C \Gamma_a C^{-1}, \quad C^t C = \mathbb{1}, \quad C^t = -C. \tag{A.4}
\]

In these conventions

\[
\Gamma_{11} \equiv \Gamma_0 \Gamma_1 \ldots \Gamma_9 = \sigma_3 \otimes \mathbb{1}_{16}. \tag{A.5}
\]
A Majorana-Weyl spinor satisfies
\[ \theta' C = \bar{\theta} \equiv \theta^\dagger \Gamma^0, \quad \text{and} \quad \Gamma_{11} \theta = \theta. \quad (A.6) \]

In the light-cone gauge we fix kappa symmetry as
\[ \Gamma^p \theta = 0, \quad (A.7) \]

where we introduce tangent space light-cone coordinates similarly to the curved ones
\[ \Gamma^p = \frac{1}{2} (\Gamma^0 + \Gamma^1), \quad \Gamma^m = \Gamma^1 - \Gamma^0, \quad (A.8) \]

with labels \( p \) and \( m \) to distinguish them from the curved space + and –.\(^{25}\) We parametrize the components of our two kappa-gauge-fixed Majorana-Weyl spinors as
\[
\begin{pmatrix}
0 \\
0 \\
-\text{i}(\eta^{41})^* + \eta^{32} \\
\text{i}(\eta^{42})^* + \eta^{21} \\
0 \\
0 \\
\text{i}(\eta^{31})^* + \eta^{42} \\
-\text{i}(\eta^{32})^* + \eta^{41} \\
-(\theta^{14})^* - \text{i}\theta^{23} \\
(\theta^{24})^* - \text{i}\theta^{13} \\
0 \\
0 \\
(\theta^{13})^* - \text{i}\theta^{24} \\
-(\theta^{23})^* - \text{i}\theta^{14} \\
0 \\
\vdots \\
0
\end{pmatrix},
\]
\[ \theta_1 = \frac{1}{2} \begin{vmatrix} \eta \rightarrow \text{i}\eta \end{vmatrix}, \quad \theta_2 = \begin{vmatrix} \theta \rightarrow -\text{i}\theta \end{vmatrix} \quad (A.9) \]

This index assignment matches the behavior of the components under the \( \mathfrak{su}(2) \) transformations of the \( Z \) and \( Y \) fields of the main text, see equation (2.16), here in the spinorial representation. This parametrization can be read off by translating the spinor \( \theta^a_2 \) of [12] to an 8 \times 8 matrix using the matrix generators of \( \mathfrak{su}(2,2|4) \) used there, and comparing this to the standard parametrization of the fermions in the coset formulation, see e.g. equation (1.139) of [35].\(^{26}\) In matrix form the kappa gauge \( \Gamma^p \theta = 0 \) becomes the one used in the coset model formulation, see e.g. equation (1.87) of [35]. Our spinors contain eight complex Grassmann-valued fields: four \( \eta \)s and four \( \theta \)s.

\(^{25}\)In this gauge, any fermion bilinear \( \bar{\theta}_i \Gamma^a \ldots \Gamma^n \theta_j \) involving purely transverse tangent space gamma matrices – those with indices other than \( p \) or \( m \) (0 or 1) – is zero.

\(^{26}\)In line with appendix C.2 of [12], relative to [35] we permute indices 1 and 2, and replace \( \theta \rightarrow \text{i}\theta \) and \( \eta \rightarrow -\text{i}\eta \).
Spacetime

Spacetime $\Gamma$ matrices $\Gamma_M$ are defined as

$$\Gamma_M = e^a_M \Gamma_a,$$  \hspace{1cm} (A.10)

where $e$ is the vielbein. In our case the vielbein needs to be chosen appropriately to maintain a straightforward link to coset sigma model objects and associated conventions. Our vielbein is determined by the deformed current $A$ of the sigma model, see eqs. (2.8) and (2.19) of [18]. Taking a coset element appropriate for our $z$ and $y$ variables as in eqn. (1.147) of [35], and evaluating the deformed current, we find

$$e^{aM} = \begin{pmatrix}
-\frac{1-z^2}{4} & \kappa_{z_1} & \kappa_{z_2} & \kappa_{z_3} & \kappa_{z_4} \\
-\frac{\kappa_{z_1}}{4} & 1 - \frac{z^2}{4} & -\frac{\kappa(z^2 + z_1^2)}{4} & \frac{\kappa_{z_2}}{4} & \frac{\kappa_{z_3}}{4} \\
-\frac{\kappa_{z_2}}{4} & \frac{\kappa(z_1^2 + z_2^2)}{4} & 1 - \frac{z^2}{4} & -\frac{\kappa_{z_3}}{4} & -\frac{\kappa_{z_4}}{4} \\
-\frac{\kappa_{z_3}}{4} & -\frac{\kappa_{z_2}z_3}{4} & \kappa_{z_3} & 1 - \frac{z^2}{4} & 0 \\
-\frac{\kappa_{z_4}}{4} & -\frac{\kappa_{z_2}z_4}{4} & -\frac{\kappa_{z_3}z_4}{4} & 0 & 1 - \frac{z^2}{4}
\end{pmatrix},$$

for the deformed AdS factor with $a = 0, 2, 3, 4, 5$ and $M = t, z_1, z_2, z_3, z_4$, and

$$e^{aM} = \begin{pmatrix}
\frac{1+y^2}{4} & -\kappa y_1 & -\kappa y_2 & -\kappa y_3 & -\kappa y_4 \\
-\frac{\kappa y_1}{4} & 1 + \frac{y^2}{4} & \frac{\kappa(y^2 + y_1^2)}{4} & -\frac{\kappa y_2}{4} & -\frac{\kappa y_3}{4} \\
-\frac{\kappa y_2}{4} & -\frac{\kappa(y_1^2 + y_2^2)}{4} & 1 + \frac{y^2}{4} & \frac{\kappa y_3}{4} & \frac{\kappa y_4}{4} \\
-\frac{\kappa y_3}{4} & -\frac{\kappa y_2y_3}{4} & -\frac{\kappa y_3}{4} & 1 + \frac{y^2}{4} & 0 \\
-\frac{\kappa y_4}{4} & -\frac{\kappa y_2y_4}{4} & -\frac{\kappa y_3y_4}{4} & 0 & 1 + \frac{y^2}{4}
\end{pmatrix},$$

for the deformed sphere factor with $a = 1, 6, 7, 8, 9$ and $M = \phi, y_1, y_2, y_3, y_4$. Other components of the vielbein vanish. We raised the curved index to get more compact expressions. At $\kappa = 0$ the vielbein is diagonal and associates tangent indices to coordinates as

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{pmatrix}$$

(A.11)

The spin connection can be similarly extracted, however it is not independent and can also be found via

$$\omega^a_{\cdot} = -2e^{b|N} \partial_M e^{[b]} - e^{aP} e^{bQ} \partial_Q e^{cP} e^{cM},$$  \hspace{1cm} (A.12)
B. Feynman diagrammatics

We used standard Feynman diagram methods to determine the perturbative T matrix – with the two major steps being the calculation of the Feynman rules and Feynman amplitudes. We performed these two procedures in Mathematica, using the packages FeynRules [48] and FeynArts [49] respectively.

This section states the implementation details and highlights certain issues arising from the intricacies of the model at hand. In particular it has scalar valued fermions, which further are complex off shell, but become real on shell (see eq. (3.1)). The bosons in turn are always constrained to be real by the reality condition eq. (2.17).

Feynman rules  We describe our model as 8 real bosons and 8 complex fermions and will take the (on-shell) reality conditions into account only when calculating the amplitudes in the second step.

For the scalar fermions we follow the algorithm for Feynman diagrams with general fermionic fields [50]. In particular this requires us to keep track of a fermion flow direction for each vertex involving fermions. It is not sufficient to simply look at the particle/anti-particle flow (like can be done for Dirac fermions), because certain interaction terms with fermionic parts break this flow. (For example terms proportional to $\theta \theta$ or $\theta^\dagger \theta^\dagger$.) FeynRules is interfering with the proper tracking of the fermion flow by not respecting the ordering of fermionic fields in the input and bringing them into alphabetic order internally. We were able to resolve this issue by ordering the input already before giving it to the package, adding extra signs from anticommuting fermions if necessary.

Our light cone gauge explicitly breaks the Lorentz invariance of the interaction terms. To support this, we had to perform minor modifications to the code of FeynRules.

Lastly we uncovered a bug in version 2.3.36 of the package, which caused certain vertices with momentum dependence to be dropped from the output. We reported this to the developers and proposed a fix.

Finally we obtain all the 4-point vertices coming from the interaction Lagrangian and can use them to calculate the Feynman amplitudes.

Feynman amplitudes  With the 4-point vertices at hand we can determine the amplitudes for the $2 \rightarrow 2$ scattering described in section 3.2. For the in- and out-states we take the data from the on-shell mode expansion eq. (3.1), in particular the dispersion relation for $\omega_p$, various prefactors and the fermionic wave functions $f_p$ and $h_p$. The in- and out-states are assigned according to the occurrence of the operators $a_{MN}$ and $a^\dagger_{MN}$ in the mode expansion eq. (3.1). (Note the change of indices of $a^\dagger$ caused by the Levi-Civita symbols.) To account for the on-shell reality of the fermions, we sum the contributions from their fields and anti-fields.

For Feynman diagrams containing fermionic fields FeynArts already implements the algorithm described in [50] and only requires the flipping rules for reversed vertices and wave functions as input. Due to the fermions being scalars these rules simply become the addition of an extra minus sign.
Besides this, FeynArts orders the particles in the in-state opposite to how the existing literature does and how we present them. To account for this we have to add an extra minus sign for amplitudes with two fermionic in-states. Finally, our sought for $T$ is related to the amplitudes $M$ calculated by FeynArts by

$$
T(p, p') = \int dk \, dk' \, \delta(p + p' - k - k') \, \delta(\omega_p + \omega_{p'} - \omega_k - \omega_{k'}) \, M(p, p', k, k')
$$

$$
= \frac{\omega_p^2 \omega_{p'}}{|p \omega_{p'} - p' \omega_p|} (M(p, p', p, p') + M(p, p', p', p)).
$$

(B.1)

Here $k$ and $k'$ denote the momenta of the outgoing particles. Due to energy and momentum conservation these are restricted to take on the same values as the incoming momenta $p$ and $p'$. To follow the existing literature we assume that $p > p'$.

C. $su(2|2)$ $R$ operators

Here we derive the precise form of the $R$ operators corresponding to the fermionic and distinguished $q$ deformations of $su(2|2)$ used in section 4 and [25], and express them in a basis of the form

$$
\begin{pmatrix}
R & -Q^\dagger \\
Q & L
\end{pmatrix},
$$

(C.1)

referred to in sections 5 and 6. As the fermionic case is built on the distinguished case, we first consider the latter.

Distinguished deformation. We start with eqs. (4.37-4.39), taking $\bar{a} = \bar{b} = \bar{c} = \bar{d} = 1$ for unitarity in the undeformed limit, and $a = d = 1$ and $b = c = 0$ for the standard fundamental representation of $su(2|2)$ with $C = +1/2$. Note the anti-canonical ordering of $|\psi_1\rangle$ and $|\psi_2\rangle$ with respect to the basis vectors in eqs. (4.38). We have

$$
E_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
E_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
E_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
$$

(C.2)

while the remaining non-simple positive roots can be obtained by repeated commutators of these. In this case all positive roots are lower diagonal, and all negative roots are upper diagonal. The $R$ operator that acts as multiplication by $-i$ on the positive roots, $+i$ on the negative roots, and 0 on the Cartan generators, is then given by

$$
R_{su(2|2)}(M)_{ij} = -i \epsilon_{ij} M_{ij}, \quad \epsilon = \begin{pmatrix}
0 & -1 & -1 & -1 \\
+1 & 0 & -1 & -1 \\
+1 & +1 & 0 & -1 \\
+1 & +1 & +1 & 0
\end{pmatrix}.
$$

(C.3)

The overall sign of the $R$ operator acting on (C.1) can also be directly confirmed by considering eqs. (2.7) of [25], where the fermionic positive simple root is taken from $Q$. 

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In sections 5 and 6 we use a canonically ordered matrix basis labeled by indices ordered as 3, 4, 2, 1. The present conventions match those of sections 5 and 6, and correspond to a copy of $\mathfrak{su}(2|2)$ of the form (C.1) if we identify our index labels as in equation (4.46), i.e. $\psi_i \leftrightarrow i$ and $\phi_i \leftrightarrow i + 2$, for $i = 1, 2$.  

Fermionic deformation. We can use the Lusztig transformation of eqs. (4.26) to determine our new simple roots, and commutators for the remainder. Demanding the usual action of $R$ on these roots then gives

$$R_{\mathfrak{su}(2|2)}(M)_{ij} = -i\epsilon_{ij}M_{ij}, \quad \epsilon = \begin{pmatrix} 0 & -1 & -1 & -1 \\ +1 & 0 & +1 & -1 \\ +1 & -1 & 0 & -1 \\ +1 & +1 & +1 & 0 \end{pmatrix}. \quad (C.4)$$

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\[\text{Footnote 14} \] The review [35] uses a canonically ordered matrix basis labeled by indices ordered as 3, 4, 1, 2. Our permutation of indices 1 and 2 with respect to [35] described in footnote 14 is then the counterpart to the anti-canonical ordering of the $\psi$ indices with respect to the matrix basis used here.
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