The Relations between Volume Ratios and New Concepts of GL Constants

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Abstract

In this paper we investigate a property named GL($p, q$) which is closely related to the Gordon-Lewis property. Our results on GL($p, q$) are then used to estimate volume ratios relative to $\ell_p$, $1 < p \leq \infty$, of unconditional direct sums of Banach spaces.

Introduction

In this paper we investigate a property named GL($p, q$), $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, closely related to the Gordon-Lewis property GL, and the behavior of $p$-summing norms of operators defined on direct sums of Banach spaces in the sense of an unconditional basis. These results are then used to estimate the volume ratios $vr(X, \ell_p)$, $1 < p \leq \infty$, where $X$ is a finite direct sum of finite dimensional spaces.

A Banach space is said to have GL($p, q$), $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, if there is a constant $C$ so that $i_q(T) \leq C\pi_p(T^*)$ for every finite rank operator $T$ from an arbitrary Banach space to $X$. Here $\pi_p$ denotes the $p$-summing norm and $i_q$ the $q$-integral norm. This property was also considered by Reisner [27], note however the slight difference in the notation: Our GL($p, q$) corresponds to his $q', p'$-GL-space.

We now wish to discuss the arrangement and contents of this paper in greater detail.

In Section 1 of the paper we investigate the basic properties of GL($p, q$) and prove some inequalities for $p$-summing operators, respectively $q$-integral operators, defined on,

*Supported in part by the fund for the promotion of research in the Technion and by the VPR fund.
†Supported in part by the Danish Natural Science Research Council, grants 9503296 and 9600673.
respectively with range in, a direct sum of Banach spaces in the sense of an unconditional basis. These inequalities are then used to prove that if \((X_n)\) is a sequence of Banach spaces with uniformly bounded GL\((p, q)\)-constants and \(X\) is the direct sum of the \(X_n\)'s in the sense of a \(p\)-convex and \(q\)-concave unconditional basis, then \(X\) has GL\((p, q)\) as well. More generally we obtain that if \(Y\) is a Banach space with GL\((p, q)\) and \(L\) is a \(p\)-convex and \(q\)-concave Banach lattice, then \(L(Y)\) has GL\((p, q)\). \(K^p(L)\) and \(K^q(L)\) denote the \(p\)-convexity and \(q\)-concavity constants of \(L\) respectively.

In Section 2 we combine the results of Section 1 with those of [6] to obtain some estimates of volume ratios. One of our results, Theorem 2.5, has the following geometric consequence: Let \(L\) be a \(p\)-convex and \(q\)-concave Banach lattice having an \(n\)-dimensional Banach space \(Y = (\mathbb{R}^n, \| \cdot \|)\) as an isometric quotient. Let \(1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{p'} = 1\), then there are \(n\)-dimensional linear quotients \(V_p\) and \(V_q\) of \(B_{\ell_p}\) and \(B_{\ell_q}\) respectively, so that \(V_q \subseteq B_Y \subseteq V_p\) for which

\[
\left( \frac{|V_p|}{|V_q|} \right)^{\frac{n}{p}} \leq c \sqrt{p'} \ \text{gl}_{p,q}(L) \leq c \sqrt{p'} \ K^p(L)K^q(L).
\]

If \(X\) is a finite direct sum of \(n_k\)-dimensional Banach spaces \(X_k, 1 \leq k \leq m\) in the sense of a finite 1-unconditional basis, then we prove that

\[
\left( \prod_{k=1}^{m} \text{vt}(X_k, \ell_p)^{n_k} \right)^{1/n} \approx \text{vt}(X, \ell_p)
\]

for \(1 < p \leq \infty\), where \(n = \sum_{k=1}^{m} n_k\).

0 Notation and Preliminaries

In this paper we shall use the notation and terminology commonly used in Banach space theory as it appears in [17], [18], [26] and [31].

If \(X\) and \(Y\) are Banach spaces, \(B(X,Y)\) \((B(X) = B(X,X))\) denotes the space of bounded linear operators from \(X\) to \(Y\) and throughout the paper we shall identify the tensor product \(X \otimes Y\) with the space of \(\omega^*\)-continuous finite rank operators from \(X^*\) to \(Y\) in the canonical manner. Further if \(1 \leq p < \infty\) we let \(\Pi_p(X,Y)\) denote the space of \(p\)-summing operators from \(X\) to \(Y\) equipped with the \(p\)-summing norm \(\pi_p\), \(I_p(X,Y)\) denotes the space of all \(p\)-integral operators from \(X\) to \(Y\) equipped with the \(p\)-integral
norm $i_p$ and $N_p(X,Y)$ denotes the space of all $p$-nuclear operators from $X$ to $Y$ equipped with the $p$-nuclear norm $\nu_p$. We recall that if $1 \leq p \leq \infty$ then an operator $T$ is said to factor through $L_p$ if it admits a factorization $T = BA$, where $A \in B(X, L_p(\mu))$ and $B \in B(L_p(\mu), Y)$ for some measure $\mu$ and we denote the space of all operators from $X$ to $Y$, which factor through $L_p$ by $\Gamma_p(X,Y)$. If $T \in \Gamma_p(X,Y)$ we define

$$\gamma_p(T) = \inf \{ \|A\|\|B\| \mid T = BA, \ A \text{ and } B \text{ as above} \},$$

$\gamma_p$ is a norm on $\Gamma_p(X,Y)$ turning it into a Banach space. All these spaces are operator ideals and we refer to the above mentioned books and [13], [24] and [14] for further details. To avoid misunderstanding we stress that in this paper a $p$-integral operator $T$ from $X$ to $Y$ has a $p$-integral factorization ending in $Y$ with $i_p(T)$ defined accordingly; in some books this is referred to as a strictly $p$-integral operator.

In the formulas below we shall, as is customary, interpret $\pi_\infty$ as the operator norm and $i_\infty$ as the $\gamma_\infty$-norm.

If $n \in \mathbb{N}$ and $T \in B(\ell^n_2, X)$ then following [31] we define the $\ell$-norm of $T$ by

$$\ell(T) = \left( \int_{\mathbb{R}^n} \|Tx\|^2 d\gamma(x) \right)^{\frac{1}{2}}$$

where $\gamma$ is the canonical Gaussian probability measure on $\ell^n_2$.

A Banach space $X$ is said to have the Gordon-Lewis property (abbreviated GL) [7] if every 1-summing operator from $X$ to an arbitrary Banach space $Y$ factors through $L_1$. It is easily verified that $X$ has GL if and only if there is a constant $K$ so that $\gamma_1(T) \leq K\pi_1(T)$ for every Banach space $Y$ and every $T \in X^* \otimes Y$. In that case GL$(X)$ denotes the smallest constant $K$ with this property.

We shall say that $X$ has GL$_2$ if it has the above property with $Y = \ell_2$ and we define the constant gl$(X)$ correspondingly. An easy trace duality argument yields that GL and GL$_2$ are self dual properties and that GL$(X) = \text{GL}(X^*)$, gl$(X) = \text{gl}(X^*)$ when applicable. It is known [7] that every Banach space with local unconditional structure has GL.

If $E$ is a Banach space with a 1-unconditional basis $(e_n)$ and $(X_n)$ is a sequence of Banach spaces then we put

$$\left( \sum_{n=1}^{\infty} X_n \right)_E = \{ x \in \prod_{n=1}^{\infty} X_n \mid \sum_{n=1}^{\infty} \| x(n) \| e_n \text{ converges in } E \}$$
and if \( x \in \left( \sum_{n=1}^{\infty} X_n \right)_E \) we define
\[
\| x \| = \left\| \sum_{n=1}^{\infty} x(n) e_n \right\|
\]
thus defining a norm on \( \left( \sum_{n=1}^{\infty} X_n \right)_E \) turning it into a Banach space. If \( X_n = X \) for all \( n \in \mathbb{N} \) we put \( E(X) = \left( \sum_{n=1}^{\infty} X_n \right)_E \).

If \( (Y_n) \) is another sequence of Banach spaces and \( T_n \in B(X_n, Y_n) \) for all \( n \in \mathbb{N} \) with \( \sup_n \| T_n \| < \infty \) then we define the operator \( \oplus_{n=1}^{\infty} T_n : \left( \sum_{n=1}^{\infty} X_n \right)_E \rightarrow \left( \sum_{n=1}^{\infty} Y_n \right)_E \) by \( \left( \oplus_{n=1}^{\infty} T_n \right)(x) = (T_n x(n)) \). Clearly \( \| \oplus_{n=1}^{\infty} T_n \| \leq \sup_n \| T_n \| \).

We shall need a “continuous” version of the above direct sums so hence let \( X \) be a Banach space and \( L \) a Banach lattice. If \( \sum_{j=1}^{n} x_j \otimes y_j \in X \otimes L \) then it follows from \([18, \text{Section I d}]\) that \( \sup_{\| x^* \| \leq 1} \left| \sum_{j=1}^{n} x^*(x_j) y_j \right| \) exists in \( L \) and we put
\[
\left\| \sum_{j=1}^{n} x_j \otimes y_j \right\|_m = \sup_{\| x^* \| \leq 1} \left\| \sum_{j=1}^{n} x^*(x_j) y_j \right\|
\]
and define \( L(X) \) to be the completion of \( X \otimes L \) equipped with the norm \( \| \cdot \|_m \). Spaces of that type was originally defined and investigated by Schaefer \([29]\); we refer to \([10]\) for the properties of \( L(X) \) needed in this paper.

If \( n \in \mathbb{N} \) and \( X \) is an \( n \)-dimensional Banach space then we shall identify \( X \) with \( (\mathbb{R}^n, \| \cdot \|_X) \) by choosing a fixed basis of \( X \) and identifying it with the unit vector basis of \( \mathbb{R}^n \), and \( B_X \) will denote the unit ball of \( X \). Hence if \( B \subseteq X \) is a Borel set we can define the volume \( |B| \) of \( B \) as the Lebesgue measure of \( B \) considered as a subset of \( \mathbb{R}^n \). The volume function thus defined is uniquely determined up to a constant only depending on the chosen basis.

Let \( X \) and \( Y \) be \( n \)-dimensional Banach spaces and let \( (x_j)_{j=1}^{n} \), respectively \( (y_j^*) \) be fixed bases of \( X \), respectively \( Y^* \). If \( T \in B(X, Y) \) then we define the determinant of \( T \) by
\[
\det T = \det \{ y_j^*(Tx_i) \}.
\]

Up to a constant depending only on the chosen bases \( \det T \) is uniquely determined.

In the sequel, if \( X_k \) \( 1 \leq k \leq m \) are \( n_k \)-dimensional spaces with fixed chosen bases and \( n = \sum_{n=1}^{m} n_k \) then we shall always identify \( \prod_{n=1}^{m} X_k \) with \( \mathbb{R}^n \) via the canonical basis of the product.
If $X$ is a Banach space and $E$ is an $n$-dimensional Banach space then we define the volume ratio $\text{vr}(E, X)$, [3], [8] by

$$\text{vr}(E, X) = \inf \left\{ \left( \frac{|B_E|}{|T(B_X)|} \right)^{\frac{1}{n}} \mid T \in B(X, E), \|T\| \leq 1 \right\}.$$

When $X = \ell_\infty$, $\text{vr}(E, \ell_\infty)$ is called the "zonoid" ratio of $E$, and when $X = \ell_2$, $\text{vr}(E, \ell_2)$ is the well known classical volume ratio of $E$, see e.g. [20], [31] and the references therein.

Similarly,

$$\text{vr}(E, S(X)) = \inf \left\{ \left( \frac{|B_E|}{|T(B_F)|} \right)^{\frac{1}{n}} \mid F \subseteq X, \dim F = n, T(B_F) \subseteq B_E \right\}.$$

When $X = \ell_p$ we set $S_p = S(\ell_p)$.

Finally, if $X$ and $Y$ are Banach spaces and $T \in B(X, Y)$ then we define the $n$-th volume number $v_n(T)$ by

$$v_n(T) = \sup \left\{ \left( \frac{|T(B_E)|}{|B_F|} \right)^{\frac{1}{n}} \mid E \subseteq X, T(E) \subseteq F \subseteq Y, \dim E = \dim F = n \right\}.$$

If rank($T$) $< n$ we put $v_n(T) = 0$. Volume numbers or similar notions were discussed by [4], [20], [22], [26] and [31]. The main results on volume numbers we are going to use here can be found in [3].

\section{The GL Property and Related Invariances}

We start with the following definition

\textbf{Definition 1.1} If $1 \leq p, q \leq \infty$ then a Banach space $X$ is said to have GL($p, q$) if there exists a constant $C$ so that for all Banach spaces $Z$ and every $T \in Z^* \otimes X$ we have

$$i_q(T) \leq C\pi_p(T^*). \quad (1.1)$$

If $X$ has GL($p, q$) then the smallest constant $C$ which can be used in (1.1) is denoted by GL$_{p,q}(X)$. If $X$ satisfies the condition of Definition 1.1 for $Z = \ell_2$ then we shall say that $X$ has gl($p, q$) and define the constant gl$_{p,q}(X)$ correspondingly. It was proved in Corollary (3.12) (I) [3] that if $X$ is a finite-dimensional Banach space, $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$
then
\[ \text{vr}(X, \ell_q) \text{vr}(X^*, \ell_{p'}) \leq \frac{\pi e}{2} \text{gl}_{p,q}(X). \]

By factoring a given finite rank operator with range in \( X \) through its kernel it is readily seen that it is enough to consider finite dimensional spaces \( Z \) in Definition 1.1.

By trace duality arguments it is readily verified that if a Banach space \( X \) has \( \text{GL}(p, q) \) then \( X^* \) has \( \text{GL}(q', p') \), and hence \( X^{**} \) has \( \text{GL}(p, q) \) as well. The other direction is part of the next lemma.

**Lemma 1.2** Let \( 1 \leq p, q \leq \infty \) and let \( X \) be a Banach space. If a subspace \( Y \) of \( X^{**} \) containing \( X \) has \( \text{GL}(p, q) \) then \( X \) has it as well with
\[ \text{GL}_{p,q}(X) \leq \text{GL}_{p,q}(Y). \]

**Proof:** Let \( Z \) be a finite dimensional Banach space, \( T \in Z^* \otimes X \) and \( \varepsilon > 0 \) arbitrary. Let \( I \) denote the identity operator of \( X \) into \( Y \). Choose a finite dimensional subspace \( F \), \( IT(Z) \subseteq F \subseteq Y \), so that \( \varepsilon + i_q(IT) \geq i_q(IT : Z \to F) \). The principle of local reflexivity [11], [16], gives an isomorphism \( V : F \to X \) with \( \|V\| \leq 1 + \varepsilon \) and \( Vx = x \) for all \( x \in F \cap X \).

Since \( VIT = T \) we obtain
\[ i_q(T) \leq \|V\|i_q(IT : Z \to F) \]
\[ \leq (1 + \varepsilon)i_q(IT) + (1 + \varepsilon)\varepsilon \]
\[ \leq (1 + \varepsilon)\text{GL}_{p,q}(Y)\pi_p(T^*I^*) + (1 + \varepsilon)\varepsilon \]
\[ \leq (1 + \varepsilon)\pi_p(T^*) + (1 + \varepsilon)\varepsilon. \]

Since \( \varepsilon \) was arbitrary this shows that \( X \) has \( \text{GL}(p, q) \) with \( \text{GL}_{p,q}(X) \leq \text{GL}_{p,q}(Y) \). \( \square \)

It follows immediately that \( X \) has \( \text{GL}(\infty, q) \) for some \( q, 1 \leq q < \infty \) (or dually has \( \text{GL}(p, 1) \) for some \( p, 1 \leq p < \infty \)) if and only if it is finite dimensional. Obviously, since the GL-property is self-dual, \( X \) has GL if and only if it has GL(1, \( \infty \)).

The next theorem which is the result of the work of several authors, [11], [14], [16] and [21], describes the situation for the remaining values of \( p \) and \( q \).

**Theorem 1.3** If \( X \) is a Banach space, \( 1 \leq p, q \leq \infty \), then the following statements hold:
(i) If $X$ has $GL(p, q)$ then $X$ has GL, $X$ is of cotype $\max(q, 2)$ and $X^*$ is of cotype $\max(p', 2)$. If $q < \infty$ and $1 < p < \infty$, then $X$ is of type $\min(2, p)$.

(ii) If $X$ has GL, $2 \leq q < \infty$ and $B(L_\infty, X) = \Pi_q(L_\infty, X)$ then $X$ has GL(1, q).

(iii) If $X$ has GL and is of type $p$-stable for some $p$, $1 < p \leq 2$, then there is a $q$, $1 \leq q < \infty$ so that $X$ has GL($p, q$).

(iv) If $1 < p < \infty$ then $X$ has $GL(p, p)$ if and only if $X$ is either a $L_p$-space or isomorphic to a Hilbert space.

(v) If $1 < q < p < \infty$ then $X$ has $GL(p, q)$ if and only if $X$ is isomorphic to a Hilbert space.

(vi) $X$ has $GL(\infty, \infty)$ (respectively $X$ has GL(1, 1)) if and only if it is a $L_\infty$-space (respectively a $L_1$-space).

(vii) If $X$ is a $p$-convex and $q$-concave Banach lattice then $X$ has $GL(p, q)$ with $GL_{p,q}(X) \leq K^p(X)K_q(X)$.

**Proof:**

(i) Assume that $X$ has $GL(p, q)$. If $T \in Y^* \otimes X$ then

$$\gamma_1(T^*) = \gamma_\infty(T) \leq i_q(T) \leq GL_{p,q}(X)\pi_p(T^*) \leq GL_{p,q}(X)\pi_1(T^*)$$

and hence $X^*$ and therefore also $X$ has GL.

If $q$ is finite then $X$ has property $(S_q)$ of [3] and is therefore of cotype $\max(q, 2)$ by Theorem 1.3 there. Since $X^*$ has GL($q', p'$) it is of cotype $\max(p', 2)$.

If $q < \infty$ and $1 < p < \infty$, then both $X$ and $X^*$ are of finite cotype and since in addition $X$ has GL it follows e.g. from [3, Theorem 1.9] that $X$ is $K$-convex. Therefore $X$ has type $\min(p, 2)$.

(ii) Assume that $X$ has GL and $\Pi_q(L_\infty, X) = B(L_\infty, X)$ for some $q$, $2 \leq q < \infty$ with $K$-equivalence between the norms and let $C$ be the GL-constant of $X$.

Let $Z$ be an arbitrary finite dimensional Banach space and $T \in Z^* \otimes X$. Since $X$ and hence also $X^*$ has GL with constant $C$, there exists a measure $\mu$ and operators $A \in B(X^*, L_1(\mu))$, $B \in B(L_1(\mu), Z^*)$ with $T^* = BA$ and $\|A\|\|B\| \leq C\pi_1(T^*)$. 

7
Let $\varepsilon > 0$ be arbitrary. Using the local properties of $L_\infty(\mu)$ we can find a finite dimensional subspace $E \subseteq L_\infty(\mu)$ with $d(E, \ell_\infty^{\dim E}) \leq 1 + \varepsilon$ and $B^*(Z) \subseteq E$, and hence $i_q(A^*_E) \leq (1 + \varepsilon)\pi_q(A^*_E)$. By the principle of local reflexivity there is an isomorphism $V : A^*(E) \to X$ so that $\|V\| \leq 1 + \varepsilon$ and $V x = x$ for all $x \in A^*(E) \cap X$. Since clearly $T = VA^*_EB^*$ we obtain

$$i_q(T) \leq (1 + \varepsilon)^2\|B\|\pi_q(A^*_E) \leq K(1 + \varepsilon)^2\|A\|\|B\| \leq KC(1 + \varepsilon)^2\|T^*\|$$

and hence $X$ has GL$(1, q)$.

(iii) Let $X$ have GL and be of type $p$-stable for some $p$, $1 < p \leq 2$. By $[\underline{21}]$ $\pi_p'(L_\infty, X^*) = B(L_\infty, X^*)$ and $X$ is of finite cotype so that there is a finite $q$ with $B(L_\infty, X) = \pi_q(L_\infty, X)$. The first statement implies that $\Pi_1(X^*, Z) = \Pi_p(X^*, Z)$ for any Banach space $Z$ and therefore $X$ has GL$(p, q)$ by (ii).

(iv) Let $1 < p < \infty$. By $[\underline{14}]$ $X$ has GL$(p, p)$ if and only if $X$ is isomorphic to a complemented subspace of an $L_p$-space, or equivalently $[\underline{16}]$ if and only if either $X$ is a $L_p$-space or isomorphic to a Hilbert space.

(v) Let $1 < q < p < \infty$. Since $q > 1$ it follows from $[\underline{3}]$ Proposition 0.3 and its proof that there is a universal constant $c$ so that if $Z$ is a Banach space and $T \in Z^* \otimes \ell_2$ then $\frac{1}{c\sqrt{q}}i_q(T) \leq \ell(T^*) \leq c\sqrt{p}\ \pi_p(T^*)$. This gives that $\ell_2$ has GL$(p, q)$.

If $X$ has GL$(p, q)$ then it has both GL$(p, p)$ and GL$(q, q)$ and hence it follows from (iv) that $X$ is isomorphic to a Hilbert space.

(vi) Assume that $X$ is a $L_{\infty, \lambda}$-space and let $T \in Z^* \otimes X$, where $Z$ is an arbitrary Banach space. By definition, there is a finite dimensional subspace $E \subseteq X$ with $d(E, \ell_\infty^{\dim E}) \leq \lambda$ and $T(Z) \subseteq E$. Hence $\gamma_\infty(T) \leq \lambda\|T\|$ and $X$ has GL$(\infty, \infty)$.

Assume next that $X$ has GL$(\infty, \infty)$. By using trace duality twice, we obtain that the identity operator of $X^{**}$ factors through an $L_\infty$-space and is therefore isomorphic to a complemented subspace of an $L_\infty$-space. It follows from $[\underline{16}]$ that $X^{**}$ and hence $X$ is $L_\infty$-space.

By Lemma $[\underline{1.2}]$ $X$ has GL$(1, 1)$ if and only if $X^*$ has GL$(\infty, \infty)$ and the result follows by noting that $X$ is a $L_1$-space if and only if $X^*$ is a $L_\infty$-space.
(vii) Let $Y$ be an arbitrary Banach space and $T \in Y^* \otimes X$. From [10, Theorem 1.3] or [12] it follows that there exists a factorization $T = BA$, where $A \in B(Y, L_\infty)$, $B \in B(L_\infty, X)$, $B \geq 0$ and $\|A\|\|B\| \leq K^p(X)\pi_p(T^*)$. By a theorem of Maurey, see e.g. [18, Theorem 1.d.10], $B$ is $q$-summing, with $i_q(B) = \pi_q(B) \leq K_q(X)\|B\|$. Hence, $i_q(T) \leq \|A\|i_q(B) \leq K^p(X)K_q(X)\pi_p(T^*)$. The case $p = 1$, $q = \infty$ was first proved in [7], the general formula is contained in [12] up to trace duality.

□

In the rest of this section we let $E$ denote a Banach space with a normalized 1-unconditional basis $(e_j)$ and biorthogonal system $(e^*_j)$. Our first result is a generalization of [3, Theorem 1.18].

**Theorem 1.4** Let $(X_n)$ be a sequence of Banach spaces, $Z$ a Banach space, $1 \leq p \leq \infty$ and $T : (\sum_{n=1}^\infty X_n)_E \to Z$ a $p$-summing operator. Put $T_n = T|_{X_n}$ for all $n \in \mathbb{N}$.

If $E^*$ is $p$-convex then for all $m \in \mathbb{N}$

$$\left\| \sum_{j=1}^m \pi_p(T_j)e^*_j \right\| \leq K^p(E^*)\pi_p(T).$$

(1.2)

If, in addition, $(e^*_j)$ is boundedly complete or $\sum_{n=1}^\infty T_n$ converges to $T$ in the $p$-summing norm, then $\sum_{j=1}^\infty \pi_p(T_j)e^*_j$ converges in $E^*$ and

$$\left\| \sum_{j=1}^\infty \pi_p(T_j)e^*_j \right\| \leq K'_{p'}(E)\pi_p(T).$$

(1.3)

**Proof:** It is obvious that (1.2) implies (1.3) under the additional assumptions, so let us concentrate on (1.2).

Let $\varepsilon > 0$ be given arbitrarily. For every $n \in \mathbb{N}$ we can choose a finite set $\sigma_n \subseteq \mathbb{N}$ and \{\{x_i(n) \mid i \in \sigma_n\} \subseteq X_n\} so that

$$\pi_p(T_n)^p \leq \sum_{i \in \sigma_n} \|Tx_i(n)\|^p + \varepsilon \cdot 2^{-n},$$

(1.4)

$$\sup \left\{ \sum_{i \in \sigma_n} |x^*(x_i(n))|^p \mid x^* \in X_n^*, \|x^*\| \leq 1 \right\} \leq 1.$$  

(1.5)
For every sequence \((\alpha_n) \subseteq \mathbb{R}_+ \cup \{0\}\) and every \(m \in \mathbb{N}\) we obtain:

\[
\sum_{n=1}^{m} \alpha_n \pi_p(T_n)^p \leq \sum_{n=1}^{m} \sum_{i \in \sigma_n} \|T(\alpha_n^{1/p} x_i(n))\|^p + \varepsilon
\]

\[
\leq \pi_p(T)^p \sup \left\{ \sum_{n=1}^{m} \alpha_n \sum_{i \in \sigma_n} |\langle x^*(n), x_i(n) \rangle|^p \mid x^* \in \left( \sum_{n=1}^{\infty} X_n^{*} \right)_{E^{*}}, \|x^*\| \leq 1 \right\} + \varepsilon
\]

\[
\leq \pi_p(T)^p \sup \left\{ \sum_{n=1}^{m} \|x^*(n)\|^p \alpha_n \sum_{i \in \sigma_n} \left| \langle x^*(n), x_i(n) \rangle \right|^p \mid x^* \in \left( \sum_{n=1}^{\infty} X_n^{*} \right)_{E^{*}}, \|x^*\| \leq 1 \right\} + \varepsilon
\]

\[
\leq \pi_p(T)^p \sup \left\{ \sum_{n=1}^{m} \|x^*(n)\|^p \alpha_n \mid x^* \in \left( \sum_{n=1}^{\infty} X_n^{*} \right)_{E^{*}}, \|x^*\| \leq 1 \right\} + \varepsilon \quad \text{ (1.6)}
\]

Let \(E_{(p)}^*\) denote the \(p\)-concavification of \(E^*\) (see [18] for details). If in (1.6) we take the supremum over all sequences \((\alpha_n) \in (E_{(p)}^*)^*\) with \(\|(\alpha_n)\| \leq 1\) and let \(\varepsilon \to 0\) we get

\[
\| \sum_{n=1}^{m} \pi_p(T_n) e_n^* \|^p \leq (K^p(E^*))^p \pi_p(T)^p
\]

which is (1.7).

By the trace duality between the \(p'\)-integral and the \(p\)-summing norms we obtain with the same notation as in Theorem 1.4.

**Corollary 1.5** Let \(E\) be \(q\)-concave for some \(q, 1 \leq q \leq \infty\), \(S \in B(Z, (\sum_{j=1}^{\infty} X_j)_{E})\) and denote by \(P_n : (\sum_{j=1}^{\infty} X_j)_{E} \to X_n\) the canonical projection for all \(n \in \mathbb{N}\).

If \(\sum_{n=1}^{\infty} i_q(P_n S) e_n\) converges in \(E\) then

\[
i_q(S) \leq K_q(E) \left\| \sum_{n=1}^{\infty} i_q(P_n S) e_n \right\|_E.
\]

(1.8)

The analogous result holds for \(q\)-nuclear operators.

**Proof:** Let \(\varepsilon > 0\) be arbitrary. For every \(n \in \mathbb{N}\) we can find a measure \(\mu_n, A_n \in B(Z, L_q(\mu_n))\) and \(B_n \in B(L_q(\mu_n), X_n)\) so that \(i_q(A_n) \leq i_q(P_n S) + \varepsilon : 2^{-n}, \|B_n\| \leq 1\) and \(P_n S = B_n A_n\).

Let \(m \in \mathbb{N}\) and consider the \(q\)-integral operator \(\sum_{n=1}^{m} A_n : Z \to (\sum_{n=1}^{m} L_q(\mu_n))_{E}\). By trace duality [23] we can find a \(q'\)-summing operator \(W : (\sum_{n=1}^{m} L_q(\mu_n))_{E} \to Z\) with
\[ \pi_{q'}(W) = 1 \text{ so that with } W_n = W_{L_q(\mu_n)} \text{ we have} \]

\[
\begin{align*}
    i_q(\sum_{n=1}^{m} A_n) & \leq \sum_{n=1}^{m} \text{tr}(WA_n) + \varepsilon \\
    & \leq \sum_{n=1}^{m} \pi_{q'}(W_n)i_q(A_n) + \varepsilon \\
    & \leq \| \sum_{n=1}^{m} \pi_{q'}(W_n)e_n^* \| \left( \| \sum_{n=1}^{m} i_q(P_nS)e_n \| + \varepsilon \right) + \varepsilon \\
    & \leq K_q(E)\| \sum_{n=1}^{m} i_q(P_nS)e_n \| + \varepsilon(K_q(E) + 1) \tag{1.9}
\end{align*}
\]

where we have used Theorem 1.4 to get the last inequality.

Formula (1.9) shows that \( \sum_{n=1}^{\infty} A_n \) converges in the \( q \)-integral norm to an operator \( A : Z \to (\sum_{n=1}^{\infty} L_q(\mu))_E \) with

\[
    i_q(A) \leq K_q(E)\| \sum_{n=1}^{\infty} i_q(P_nS)e_n \| + \varepsilon(1 + K_q(E)). \tag{1.10}
\]

The operator \( B = \bigoplus_{n=1}^{\infty} B_n : (\sum_{n=1}^{\infty} L_q(\mu_n))_E \to (\sum_{n=1}^{\infty} X_n)_E \) is clearly bounded with \( \| B \| \leq 1 \) and \( S = BA \). Hence \( S \) is \( q \)-integral with

\[
    i_q(S) \leq K_q(E)\| \sum_{n=1}^{\infty} i_q(P_nS)e_n \|. \tag{1.11}
\]

The statement on \( q \)-nuclear operators can be proved in a similar manner or by duality. □

The next theorem was originally proved by Reisner [27] using other methods; we can also obtain it directly from Theorem 1.3 and Corollary 1.5.

**Theorem 1.6** Let \( 1 \leq p \leq q \leq \infty \) and let \( (X_n) \) be a sequence of Banach spaces all having \( \text{GL}(p, q) \) with \( M = \sup \{ \text{GL}_{p, q}(X_n) \mid n \in \mathbb{N} \} < \infty \). If \( E \) is \( p \)-convex and \( q \)-concave then \( (\sum_{n=1}^{\infty} X_n)_E \) has \( \text{GL}(p, q) \) with

\[
    \text{GL}_{p, q}((\sum_{n=1}^{\infty} X_n)_E) \leq MK^p(E)K_q(E). \tag{1.12}
\]
Proof: Let $Z$ be an arbitrary Banach space and $T \in Z^* \otimes (\sum_{n=1}^\infty X_n)_E$. Put $S = T^*|_{(\sum_{n=1}^\infty X_n)_E^*}$. From Theorem 1.4 and Corollary 1.5 we obtain

\[
i_q(T) \leq K_q(E)\|\sum_{n=1}^\infty i_q(P_n T)e_n\|
\leq MK_q(E)\|\sum_{n=1}^\infty \pi_p(T^* P_n^*)e_n\| \leq MK^p(E)K_q(E)\pi_p(S)
\leq MK^p(E)K_q(E)\pi_p(T^*),
\]

from which the statement directly follows. □

The case $p = 1$ and $q = \infty$ gives:

**Corollary 1.7** If $(X_n)$ is a sequence of Banach spaces all having GL with $M = \sup\{GL(X_n) \mid n \in \mathbb{N}\} < \infty$ then $(\sum_{n=1}^\infty X_n)_E$ has GL with

\[
GL((\sum_{n=1}^\infty X_n)_E) \leq M.
\]

(1.14)

We now wish to generalize Theorem 1.6 and its corollary to the space $L(X)$, where $L$ is a Banach lattice and $X$ a Banach space. For this we need the following lemma on $p$-summing norms which might also be useful in other situations. Before we state it we need a little notation: If $X$ and $Y$ are Banach spaces, $T \in X^* \otimes Y$ and $F$ is a finite dimensional subspace of $Y$ with $T(X) \subseteq F$ then $T_F$ denote the operator $T$ considered as an operator from $X$ to $F$.

**Lemma 1.8** Let $X$ and $Y$ be Banach spaces and $T \in X^* \otimes Y$. If $1 \leq p \leq \infty$ and $\mathcal{F}$ is an upwards directed set of finite dimensional subspaces all containing $T(X)$ with $\cup\{F \mid F \in \mathcal{F}\} = Y$, then

\[
\lim_{F \in \mathcal{F}} \pi_p(T^*_F) = \pi_p(T^*).
\]

(1.15)

**Proof:** We can without loss of generality assume that $X$ is finite dimensional and let us also assume that $1 \leq p < \infty$; the case $p = \infty$ is easier and left to the reader.

The net $(\pi_p(T^*_F))$ is non-negative and decreasing and hence convergent to $\alpha$, say; clearly $\pi_p(T^*) \leq \alpha$.
Let now \( \varepsilon > 0 \) be arbitrary. Since \( \text{rank}(T_F^*) = \text{rank}(T^*) \) for all \( F \) it follows from [3, Theorem 5 and its proof] that we can find an \( m \) independent of \( F \), so that \( \pi_p(T_F^*) \) can be computed up to \( \varepsilon \) using \( m \) vectors from \( F^* \). Hence for every \( F \in \mathcal{F} \) we can find \( \{x_{j,F}^* \mid 1 \leq j \leq m \} \subseteq X^* \) so that

\[
\|x_{j,F}^*\| < 1 \quad \text{for all} \quad 1 \leq j \leq m
\]

(1.16)

\[
\sum_{j=1}^{m} |x_{j,F}^*(x)|^p < 1 \quad \text{for all} \quad x \in F, \ |x| \leq 1
\]

(1.17)

\[
\left(\sum_{j=1}^{m} \|T^*x_{j,F}^*\|^p\right)^{1/p} \geq \pi_p(T_F^*) - \varepsilon.
\]

(1.18)

(1.16) gives that there is a subnet \((x_{j,F'}^*)\) and an \( x_j^* \in X^* \) so that \((x_{j,F'}^*)\) converges \( w^* \) to \( x_j^* \) for all \( j, 1 \leq j \leq m \). Since \( \bigcup_{F \in \mathcal{F}} F = Y \), (1.17) gives that \( \sum_{j=1}^{m} |x_j^*(x)|^p \leq 1 \) for all \( x \in Y, \ |x| \leq 1 \).

From the \( w^* \)-continuity of \( T^* \) it follows that \((T^*x_{j,F'}^*)\) converges \( w^* \) to \( T^*x_j^* \) and therefore also in norm, since \( X \) is finite dimensional. Hence going to the limit in (1.18) we get

\[
\pi_p(T^*) \geq \left(\sum_{j=1}^{m} \|T^*x_j^*\|^p\right)^{1/p} \geq \alpha - \varepsilon
\]

which implies that \( \pi_p(T^*) \geq \alpha \), since \( \varepsilon \) was arbitrary. \( \square \)

We are now able to prove:

**Theorem 1.9** Let \( 1 \leq p \leq q \leq \infty \) and let \( X \) be a Banach space with \( \text{GL}(p,q) \). If \( L \) is a \( p \)-convex and \( q \)-concave Banach lattice then \( L(X) \) has \( \text{GL}(p,q) \) with

\[
\text{GL}_{p,q}(L(X)) \leq \text{GL}_{p,q}(X)K^p(L)K_q(L)
\]

(1.19)

**Proof:** If the statement of the theorem has been proved for order complete Banach lattices, then since \( L^{**} \) is order complete and \( L(X) \subseteq L^{**}(X) \subseteq L(X)^{**} \) it follows from Lemma 1.2 that \( L(X) \) has \( \text{GL}(p,q) \). It is therefore no restriction to assume that \( L \) is order complete.

Let \( Z \) be a Banach space, \( T \in Z^* \otimes L(X) \) with \( \|T\| \leq 1 \) and \( \varepsilon > 0 \) arbitrary. From Lemma 1.8 it follows that there is an \( n \in \mathbb{N} \) and an \( n \) dimensional subspace \( F \subseteq L(X) \) so
that $T(Z) \subseteq F$ and

$$\pi_p(T_F^*) \leq \pi_p(T^*) + \varepsilon. \quad (1.20)$$

Let $(u_j^n)_{j=1}^n$ be an Auerbach basis, \[7\], of $F$ with biorthogonal system $(u_j^*)_n \subseteq F^*$. By \[10, Lemma 2.15\], and the order completeness of $L$ there is an $m \in \mathbb{N}$, a set $\{e_i \mid 1 \leq i \leq m\} \subseteq L$ consisting of mutually disjoint positive vectors of norm 1 and $\{v_j \mid 1 \leq j \leq n\} \subseteq [e_i](X) = Y$ (naturally considered as a subspace of $L(X)$) so that

$$\|u_j - v_j\| \leq \frac{\varepsilon}{n} \text{ for all } 1 \leq j \leq n. \quad (1.21)$$

If $S = \sum_{j=1}^nu_j^* \otimes v_j : F \to Y$, then for all $u \in F$ we have

$$\|u - Su\| \leq \sum_{j=1}^n \|u_j^*\| \|u_j - v_j\| \leq \varepsilon. \quad (1.22)$$

Considering $T - ST$ as an operator from $Z$ to $L(X)$ it has the representation

$$T - ST = \sum_{j=1}^n T_F^* u_j^* \otimes (u_j - v_j) \quad (1.23)$$

and therefore

$$\nu_1(T - ST) \leq \sum_{j=1}^n \|T_F^* u_j^*\| \|u_j - v_j\| \leq \varepsilon. \quad (1.24)$$

Applying Theorem \[1.6\] and the previous inequalities we obtain

$$i_q(T) \leq i_q(ST) + i_q(T - ST) \leq i_q((ST)_Y) + \nu_1(T - ST) \leq GL_{p,q}(X)K^p(L)K_q(L)\pi_p((ST)_Y^*) + \varepsilon$$

$$\leq GL_{p,q}(X)K^p(L)K_q(L)\|S\| \pi_p(T_F^*) + \varepsilon \leq GL_{p,q}(X)K^p(L)K_q(L)(1 + \varepsilon)(\pi_p(T^*) + \varepsilon) + \varepsilon. \quad (1.25)$$

From (1.25) we conclude that $L(X)$ has $GL(p, q)$ and since $\varepsilon$ was arbitrary (1.19) follows. \[ \square \]

As a corollary we obtain
Corollary 1.10 Let $X$ be a Banach space with GL and $L$ a Banach lattice. Then $L(X)$ has GL with

$$GL(L(X)) = GL(X).$$

2 Volume Ratios of Direct Sums of Banach Spaces

In this section we shall use the results of Section 1 to compute volume ratios of certain direct sums of finite dimensional Banach spaces. Throughout the section we let $m \in \mathbb{N}$, $n_k \in \mathbb{N}$ for all $1 \leq k \leq m$, let $E$ denote an $m$ dimensional Banach space with a normalized 1-unconditional basis $(e_k)_k^{m-1}$ and biorthogonal system $(e_k^*)_k^{m-1}$ and let $X_k$ and $Y_k$ be Banach spaces with dim $X_k = \dim Y_k = n_k$ for all $1 \leq k \leq m$; put $n = \sum_{k=1}^m n_k$, $X = \left( \sum_{k=1}^m X_k \right)_E$ and $Y = \left( \sum_{k=1}^m Y_k \right)_E$. We wish to compute $\text{vr}(X, \ell_p)$ for $1 < p \leq \infty$, but before we can do that we need a few lemmas.

Lemma 2.1

$$\frac{|B_X|}{|B_Y|} = \prod_{k=1}^m \frac{|B_{X_k}|}{|B_{Y_k}|}$$

Proof: We will iteratively interchange the $X_k$’s by the $Y_k$’s. So, we define $Z_{m-1} = (\sum_{k=1}^{m-1} X_k \oplus Y_m)_E$. By choosing a basis in each of the involved spaces we may identify $X_k, Y_k$ respectively $X, Y, Z_{m-1}$ with $\mathbb{R}^{n_k}$ respectively $\mathbb{R}^n$ in a canonical manner. Then we define $r : \mathbb{R}^{m-1} \to \mathbb{R}$ by

$$r(t_1, t_2, \ldots, t_{m-1}) = \inf \{ t \in \mathbb{R} \mid \| \sum_{j=1}^{m-1} t_j e_j + t e_m \|_X = 1 \} \text{ for all } (t_1, t_2, \ldots, t_{m-1}) \in \mathbb{R}^{m-1}. \quad (2.1)$$

For every $x_k \in X_k$, $1 \leq k \leq m - 1$ we put

$$A(x_1, x_2, \ldots, x_{m-1}) = r(\| x_1 \|_{X_1}, \| x_2 \|_{X_2}, \ldots, \| x_{m-1} \|_{X_{m-1}}) B_{X_m} \quad (2.2)$$
and consider $\mathbb{R}^n = \prod_{k=1}^{m} \mathbb{R}^{n_k}$. With this notation we obtain

$$|B_X| = \int_{\mathbb{R}^{nm}} \cdots \int_{\mathbb{R}^{n_1}} 1_{B_X}(x_1, \ldots, x_m) dx_1 \cdots dx_m = \int_{\mathbb{R}^{n_{m-1}}} \cdots \int_{\mathbb{R}^{n_1}} \left[ \int_{\mathbb{R}^{nm}} 1_{A(x_1, \ldots, x_{m-1})}(x_m) dx_m \right] dx_1 \cdots dx_{m-1} = |B_{X_m}| \int_{\mathbb{R}^{n_{m-1}}} \cdots \int_{\mathbb{R}^{n_1}} r(||x_1||_X, \ldots, ||x_{m-1}||_X) dx_1 \cdots dx_{m-1} \quad (2.3)$$

Using the same calculation for $Z_{m-1}$ in (2.3) we get

$$|B_{Z_{m-1}}| = |B_{Y_m}| \int_{\mathbb{R}^{n_{m-1}}} \cdots \int_{\mathbb{R}^{n_1}} r(||x_1||_X, \ldots, ||x_{m-1}||_X) dx_1 \cdots dx_{m-1}$$

and hence:

$$|B_X| = |B_{Z_{m-1}}| \frac{|B_{X_m}|}{|B_{Y_m}|}. \quad (2.4)$$

The result now follows by iterating (2.4). □

**Lemma 2.2** Let $1 \leq p < \infty$ and for every $1 \leq k \leq m$ let $T_k \in X_k^* \otimes \ell_2$. If $T = \bigoplus_{k=1}^{m} T_k \in X^* \otimes \ell_2^m(\ell_2)$ then

$$\pi_p(T) \leq c\sqrt{p} \left\| \sum_{k=1}^{m} \pi_p(T_k)e_k^* \right\| \leq c\sqrt{p} K_{p'}(E) \pi_p(T) \quad (2.5)$$

where $c$ is a universal constant.

**Proof:** The right hand side inequality (2.5) is formula (1.2). If $2 < p$ then it follows from Maurey’s extension theorem [19] that every $p$-summing operator from an arbitrary Banach space to $\ell_2$ is already 2-summing and $\pi_2(T) \leq c\sqrt{p} \pi_p(T)$ (see e.g. [25]). Therefore it suffices to prove (2.5) for $1 \leq p \leq 2$.

By the factorization theorem of Pietsch (see e.g. [17]) we can for every $1 \leq k \leq m$ find a Radon probability measure $\mu_k$ on $B_{X_k}$ so that for every $x_k \in X_k$,

$$\|T_k x_k\| \leq \pi_p(T_k) \left( \int_{B_{X_k}} |x_k^*(x_k)|^p d\mu_k(x_k) \right)^{\frac{1}{p}}.$$
Put $B = \prod_{k=1}^m B_{X_k^*}$, $\mu = \prod_{k=1}^m \mu_k$ and $\tau = \left\| \sum_{k=1}^m \pi_p(T_k) e_k^* \right\|$ and let $(r_k)$ denote the sequence of Rademacher functions on $[0,1]$. Since $(e_k^*)$ is 1-unconditional the function $f$ defined by $f(t,x^*) = \tau^{-1}(r_k(t)\pi_p(T_k)x_k^*)$ for all $t \in [0,1]$ and all $x^* = (x_k^*) \in X^*$ maps $[0,1] \times B$ into $B_{X^*}$ and hence $d\nu = [dt \times d\mu] \circ f^{-1}$ is a probability measure on $B_{X^*}$ (actually concentrated on the sphere).

Since the cotype 2 constant of $L_p(\mu)$ is majorized by $\sqrt{2}$ for $1 \leq p \leq 2$ (the constant in Khintchine’s inequality for $p = 1$, see [9] and [30]), the following inequalities hold for every $x = (x_k) \in X$ (putting $x^* = (x_k^*) \in X^*$):

$$\|Tx\| = \left( \sum_{k=1}^m \|T_kx_k\|^2 \right)^{\frac{1}{2}} \leq \left[ \sum_{k=1}^m \pi_p(T_k)^2 \left( \int_B |x_k^*(x_k)|^p d\mu(x^*) \right)^{\frac{1}{p}} \right]^{\frac{1}{2}}$$

$$\leq \sqrt{2} \left( \int_0^1 \int_B \left| \sum_{k=1}^m r_k(t)\pi_k(T_k)x_k^*(x_k) \right|^p d\mu(x^*) dt \right)^{\frac{1}{p}}$$

$$= \sqrt{2} \tau \left( \int_{B_{X^*}} |x^*(x)|^p d\nu(x^*) \right)^{\frac{1}{p}}. \quad (2.6)$$

This shows that $T$ is $p$-summing with $\pi_p(T) \leq c\sqrt{p} \tau$ for all $1 \leq p < \infty$. \hfill \Box

**Lemma 2.3** For every $1 \leq k \leq m$ there exist $\alpha_k \geq 0$, $\beta_k \geq 0$ so that

$$\left\| \sum_{k=1}^m \alpha_k e_k^* \right\| = 1 = \left\| \sum_{k=1}^m \beta_k e_k \right\|$$

and

$$\prod_{k=1}^m (\alpha_k^{n_k} \beta_k^{n_k} n_k^{-n_k}) \geq n^{-n}.$$ 

**Proof:** Consider the space $Z = \left( \sum_{k=1}^m \ell_1^{n_k} \right)_{E^*}$, with its canonical 1-unconditional basis. Applying a result of Lozanovskii [13] on 1-unconditional bases (see Corollary 3.4 in [20]) to $Z$ we see that there exist numbers $\tau_{jk} \geq 0$ and $\sigma_{jk} \geq 0$ for $1 \leq k \leq m$ and $1 \leq j \leq n_k$ so that if $D_\tau : \ell_\infty^m \to Z$, respectively $D_\sigma : Z \to \ell_1^n$ are the diagonal operators determined by $(\tau_{jk})$, respectively $(\sigma_{jk})$, then:

$$\|D_\tau\| = 1 = \|D_\sigma\|; \quad D_\tau D_\sigma = \frac{1}{n} \text{id}_Z. \quad (2.7)$$
If we for every $1 \leq k \leq m$ define

$$\alpha_k = \sum_{j=1}^{n_k} \tau_{jk}, \quad \beta_k = \max_{1 \leq j \leq n_k} \sigma_{jk},$$

then

$$\| \sum_{k=1}^{m} \alpha_k e_k^* \| = \| D_\tau \| = 1 = \| D_\sigma \| = \| \sum_{k=1}^{m} \beta_k e_k \|.$$  \hspace{1cm} (2.9)

From (2.7) we obtain for all $1 \leq k \leq m$:

$$\frac{1}{n} = \left( \prod_{j=1}^{k} \sigma_{jk} \tau_{jk} \right)^{\frac{1}{n_k}} \leq \frac{1}{n_k} \left( \sum_{j=1}^{n_k} \tau_{jk} \right) \sup_{1 \leq j \leq n_k} \sigma_{jk} = \frac{1}{n_k} \alpha_k \beta_k,$$

and hence

$$n^{-n} \leq \prod_{k=1}^{m} (\alpha_k^{n_k} \beta_k^{n_k} n_k^{-n_k}).$$  \hspace{1cm} (2.10)

Our next lemma follows from \[2\] and Lemma 2.3.

**Lemma 2.4** There is a universal constant $c > 0$ so that if $T_k \in X_k^* \otimes \ell_2^{n_k}$, $\alpha_k$ is as in Lemma 2.3 for $1 \leq k \leq m$ and $T = \oplus_{k=1}^{m} T_k : X \to \ell_2^n$, then

$$c^n \prod_{k=1}^{m} n_k^{n_k} |T_k(B_{X_k})| \leq n^n |T(B_X)|.$$  \hspace{1cm} (2.12)

**Proof:** Let $\{ \sigma_{jk} \mid 1 \leq j \leq n_k, 1 \leq k \leq m \}$, $\beta_k$ for $1 \leq k \leq m$ and $Z$ be defined as in the proof of Lemma 2.3 and let $\varepsilon_{jk} = \pm 1$ for $1 \leq j \leq n_k, 1 \leq k \leq m$. If $\{ f_{jk} \mid 1 \leq j \leq n_k, 1 \leq k \leq m \}$ denotes the canonical basis of $Z^*$ and $B : \ell_2^n \to Z^*$ is the diagonal operator defined by $(\sigma_{jk})$, then it follows from \[2\], Corollary 1.4(e)] that there is a universal constant $d > 0$.
so that
\[
\frac{1}{n} \prod_{k=1}^{m} \beta_{n_k}^{n_k} \leq |B_{Z^*}|.
\]
(2.14)

and hence
\[
d^n \prod_{k=1}^{m} \beta_{n_k}^{n_k} \leq |B_{Z^*}|.
\]
(2.14)

From Lemma 2.4, Lemma 2.3 and (2.14) we now obtain:
\[
|T(B_X)| = |B_X| |\det(T)| = |B_X| \prod_{k=1}^{m} \alpha_{n_k}^{n_k} |\det(T_k)|
\]
\[
\geq 2^{-m} \prod_{k=1}^{m} n_{n_k}^{n_k} |T_k(B_{X_k})| \cdot \prod_{k=1}^{m} \alpha_{n_k}^{n_k} n_{n_k}^{n_k} \beta_{n_k}^{n_k}
\]
\[
\geq 2^{-m} d^n n^{-n} \prod_{k=1}^{m} n_{n_k}^{n_k} |T_k(B_{X_k})|
\]
(2.15)

which is (2.12) with \(c = \frac{d}{2}\).

\[\square\]

**Theorem 2.5** Let \(1 \leq r, p \leq \infty\). There is a universal constant \(C\) so that if \(Y\) is a finite dimensional quotient of a Banach space \(Z\) with \(\text{gl}(r, p)\) then
\[
\max\{\text{vr}(Y, \ell_p), \text{vr}(Y^*, S_r')\} \leq \text{vr}(Y, \ell_p)\text{vr}(Y^*, S_{r'}) \leq C\sqrt{r'} \text{gl}_{r, p}(Z).
\]
(2.16)

In particular, if \(Y\) is a finite-dimensional quotient of a \(r\)-convex \((1 < r)\) and \(p\)-concave Banach lattice \(Z\) then
\[
\text{vr}(Y, \ell_p)\text{vr}(Y^*, S_{r'}) \leq C\sqrt{r'} K^r(Z) K_p(Z).
\]
**Proof:** Let $Q : Z \to Y$ be a quotient map, put $n = \dim Y$ and let $S \in Y^* \otimes \ell_2^n$ be arbitrary. It follows from [3, Theorem 3.10 i)] that

\[ n^{1/2} v_n(S) v_{Y^*}(S^*) \leq c \nu_{\ell' \to \ell'_p}(Q^* S^*) \leq c \nu_{\ell' \to \ell'_p}(S) \leq c \sqrt{\nu_{\ell' \to \ell'_p}(Z)} \pi_{p'}(S). \]  

(2.17)

Since (2.17) holds for all $S$ it follows from [3, Theorem 3.7 (ii)] that there is a universal constant $C$ so that

\[ v_{Y^*}(S) \leq C \sqrt{\nu_{\ell' \to \ell'_p}(Z)} \]

and apply now Theorem 1.3 (vii). \[ \square \]

**Remark:** We now note that

\[ v_{Y}(\ell) v_{Y^*}(S) \sim \inf \left( \frac{v_{Y}(\ell)}{v_{Y^*}(S)} \right), \]

where the infimum is taken over all $n$-dimensional linear quotients $V_r$ of $\ell$ and $V_p$ of $\ell'$ so that $V_p \subseteq B_Y \subseteq V_r$. Indeed, by definition

\[ v_{Y}(\ell) = \inf \left\{ \left( \frac{v_{Y}(\ell)}{v_{Y^*}(S)} \right) \mid V_p \subseteq B_Y \right\}, \]

\[ v_{Y^*}(S) = \inf \left\{ \left( \frac{v_{Y^*}(S)}{v_{T(B_{S^*})}} \right) \mid S_r \subseteq \ell_r, T(B_{S^*}) \subseteq B_{Y^*} \right\}. \]

Now, $[T(B_{S^*})] = W(B_{\ell_r}) = V_r$ for some linear $W : \ell_r \to \mathbb{R}^n$. By [28] and [1] for any $n$-dimensional space $X$, $\left( |B_X|/|B_{X^*}| \right)^{1/n} \sim 1$, and therefore

\[ v_{Y^*}(S) \sim \left( \frac{v_{Y}(\ell)}{v_{Y^*}(S)} \right)^{1/n} \]

where $B_Y \subseteq V_r$, hence the result follows.

We are now able to prove:

**Theorem 2.6** Let $E$ be a $m$ dimensional Banach space with a normalized 1-unconditional
basis, let $X_k$, $k = 1, \ldots, m$, be $n_k$-dimensional Banach spaces and $X = \bigoplus_{k=1}^m X_k$. Let $1 < r \leq p < \infty$ or $r = 1$ and $p = \infty$. There is a universal constant $c_0 > 0$ and a constant $C(r, p)$ so that for $n = \sum_{k=1}^m n_k$

$$\frac{1}{c_0} p^r \left( \prod_{k=1}^m \nu r(X_k, \ell_p)^{n_k} \right)^{1/n} \leq \nu r(X, \ell_p) \leq c_0 C(r, p) \left( \prod_{k=1}^m \nu r(X_k, \ell_p)^{n_k} \right)^{1/n}$$

(2.18)

where $C(r, p) = \sqrt{r'}$ for $1 < r \leq p < \infty$ and $C(1, \infty) = 1$.

**Proof:** Let us first prove the left inequality of (2.18). By [6, Theorem 3.7(ii)] in the case $1 < p \leq \infty$ there is a universal constant $A$ and operators $T_k \in X_k^* \otimes \ell_2^{n_k}$ with $\pi p'(T_k) = 1$ for $1 \leq k \leq m$ so that

$$A^{-1} \nu r(X_k, \ell_p) \leq n_k |T_k(B_{X_k})|^{1/nk} \leq A \sqrt{p'} \nu r(X_k, \ell_p).$$

(2.19)

Let $\alpha_k$, $1 \leq k \leq m$ be chosen as in Lemma 2.3 and put $T = \bigoplus_{k=1}^m \alpha_k T_k$. Lemma 2.2 and Lemma 2.3 now give

$$\pi p'(T) \leq c \sqrt{p'} \left\| \sum_{k=1}^m \alpha_k \pi p'(T_k) e_k^* \right\| = c \sqrt{p'} \left\| \sum_{k=1}^m \alpha_k e_k^* \right\| = c \sqrt{p'}$$

(2.20)

and hence by (2.19), (2.20) and Lemma 2.4, there is a $c' > 0$ so that

$$c' \left( \prod_{k=1}^m \nu r(X_k, \ell_p)^{n_k} \right)^{1/n} \leq c' A \left[ \prod_{k=1}^m (n_k^{n_k} |T_k(B_{X_k})|) \right]^{1/n}$$

$$\leq A n |T(B_X)|^{1/n} \leq c A^2 p' \nu r(X, \ell_p)$$

(2.21)

which shows the left inequality in (2.18).

To prove the right inequality we can by definition find operators $W_k \in B(\ell_p, X_k)$ for $1 \leq k \leq m$ so that $\|W_k\| = 1$ and

$$\nu r(X_k, \ell_p) = \left( \frac{|B_{X_k}|}{|W_k(B_{\ell_p})|} \right)^{n_k}. \quad (2.22)$$

Put for every $1 \leq k \leq m$ $Y_k = \ell_p/W_k^{-1}(0)$, let $Q_k$ denote the quotient map of $\ell_p$ onto $Y_k$, define $V_k \in B(Y_k, X_k)$ so that $W_k = V_k Q_k$, let $Y = (\sum_{k=1}^m Y_k)_E$ and put $V = \bigoplus_{k=1}^m V_k \in B(Y, X)$. 21
If \( S \in B(\ell_p, Y) \) so that \( \|S\| = 1 \) and \( \frac{|B_Y|}{|\frac{B_Y}{S(B_{\ell_p})}|} = \text{vr}(Y, \ell_p) \) then since \( \|V\| = 1 \) we get using Lemma 2.1:

\[
\text{vr}(X, \ell_p)^n \leq \frac{|B_X|}{|V S(B_{\ell_p})|} = (\text{vr}(Y, \ell_p))^n \frac{|B_X|}{|B_Y|} \frac{|S(B_{\ell_p})|}{|V S(B_{\ell_p})|} \\
= (\text{vr}(Y, \ell_p))^n \det V^{-1} \prod_{k=1}^m \frac{|B_{X_k}|}{|B_{Y_k}|} \\
= (\text{vr}(Y, \ell_p))^n \det V^{-1} \prod_{k=1}^m \frac{|B_{X_k}|}{|W_k(B_{\ell_p})|} \prod_{k=1}^m \frac{|V_k(B_{Y_k})|}{|B_{Y_k}|} \\
= (\text{vr}(Y, \ell_p))^n \prod_{k=1}^m \text{vr}(X_k, \ell_p)^n_k. \tag{2.23}
\]

If \( p = \infty \) then \( Y^*_k \) is a subspace of an \( L_1 \)-space hence \( \text{GL}(Y^*_k) = \text{GL}(Y^*_k) = 1 \). It now follows from Corollary 1.7 that \( Y \) has \( \text{GL} \) as well with \( \text{GL}(Y) = 1 \) and hence by the result of \( [8] \) there is a universal constant \( C \) so that

\[
1 \leq \text{vr}(Y, \ell_\infty) \text{vr}(Y^*_\infty, \ell_\infty) \leq C \text{gl}(Y) = C. \tag{2.24}
\]

If \( 1 < r \leq p < \infty \) then it follows from Theorem 1.6 that \( E(\ell_p) \) has \( \text{GL}(r, p) \) with \( \text{GL}_{r,p}(E(\ell_p)) \leq K^r(E) K_p(E) \) (this can also easily be obtained from the fact that \( E(\ell_p) \) is an \( r \)-convex and \( p \)-concave Banach lattice) and the operator \( Q = \oplus_{k=1}^m Q_k \) is readily seen to be a quotient map of \( E(\ell_p) \) onto \( Y \). Hence Theorem 2.5 assures the existence of a universal constant \( C \) so that

\[
\text{vr}(Y, \ell_p) \leq C \sqrt{r' K^r(E) K_p(E)}. \tag{2.25}
\]

Combining (2.23) with (2.24) and (2.25) we obtain the right inequality of (2.18). \( \square \)

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