$U_q(sl(2))$ as Dynamical Symmetry Algebra of the Quantum Hall Effect

Ömer F. Dayi*
Feza Gürsey Institute, P.O.Box 6, 81220 Çengelköy–Istanbul, Turkey.

Abstract

Quantum Hall effect wavefunctions corresponding to the filling factors $1/2p+1$, $2/2p+1$, $\cdots$, $2p/2p+1$, 1, are shown to form a basis of irreducible cyclic representation of the quantum algebra $U_q(sl(2))$ at $q^{2p+1}=1$. Thus, the wavefunctions $\Psi_{P/Q}$ possessing filling factors $P/Q < 1$ where $Q$ is odd and $P$, $Q$ are relatively prime integers are classified in terms of $U_q(sl(2))$. Adopted as dynamical symmetry this leads to non–existence of a “universal microscopic theory” of the quantum Hall effect, defined as the eigenvalue problem of a differential operator $O$: $O\Psi_{\{\nu\}} = \ell_{\{\nu\}}\Psi_{\{\nu\}}$ for $\nu = 1, 1/3, 2/3, \cdots$, in the complex plane.

1. Introduction:

Microscopic theory of the fractional quantum Hall effect (QHE) is not well established. Its theoretical understanding mostly is due to trial wavefunctions [1]. For filling factors $1/m$ where $m$ is an odd integer, trial wavefunctions were given by Laughlin[2]. Trial wavefunctions for the other filling factors $\nu = P/Q < 1$, where $P$, $Q$ are relatively prime integers and $Q$ is odd, were constructed in terms of some hierarchy schemes[3]–[4] where they were obtained from a parent state which is a full filled Landau level or a Laughlin wavefunction. However, general properties of the QHE should be independent of the explicit form of the trial wavefunctions, but depend on their universal features as their orthogonality.

The integral QHE is understood in terms of non-interacting electrons which fully fill a certain amount of Landau levels. i.e. there exists a microscopic hamiltonian $h$ independent of filling factor $n = 1, 2, \cdots$, satisfying $h\Phi_n = E_n\Phi_n$, where $\Phi_n$ is the

*Talk presented in “Quantum Groups, Deformations and Contractions”, Istanbul 1997. E-mail: dayi@gursey.gov.tr.
wavefunction corresponding the filling factor \( n \), and \( E_n \) is the related eigenvalue. In fact, solving the eigenvalue problem of a given differential operator \( O \) which defines a physical system, is the usual procedure in quantum mechanics. Once the eigenfunctions of \( O \) are found, they may be classified as representations of an algebra (group) thus named dynamical symmetry algebra of the system. In the contrary, if the underlying differential operator of a physical system is not available, knowing dynamical symmetry algebra of the system can give some hints about it.

When one deals with the QHE (\( \nu \leq 1 \)) a "universal microscopic theory" given by a differential operator \( O \) which is independent of the available filling factors \( \{\nu\} = 1, 1/3, 2/3, 1/5, \cdots \), and satisfies the eigenvalue equation \( O\Psi\{\nu\} = \ell\{\nu\}\Psi\{\nu\} \), is not known. Microscopic theories which we know are given for one value of the filling factor \( \nu \), and their excitations, e.g. see \cite{5} and the references therein. i.e. they are given in terms of the eigenvalue equations as \( H_\nu \Psi_{\nu,k} = e_{\nu,k}\Psi_{\nu,k} \), where \( k \) labels the energy eigenvalues and the ground state \( \Psi_{\nu,0} \) can be a full filled Landau level or a Laughlin wavefunction. These are effective theories which depend on the number of the levels occupied by the electrons.

We utilize orthogonality of the QHE states for different filling factors, independent of their explicit form, to show that they can be classified as irreducible cyclic representations of \( U_q(sl(2)) \) at roots of unity\cite{6}. In our scheme, states corresponding to filling factors possessing a common denominator are in the same representation. Based on this classification \( U_q(sl(2)) \) at roots of unity is proposed to be dynamical symmetry algebra of the QHE, which interrelates states of different filling factors. This leads to the conclusion that a "universal microscopic theory" of the fractional QHE, in the common sense, does not exist. This would be the explanation why such a microscopic theory of the fractional QHE is not known.

2. Cyclic Representation of \( U_q(sl(2)) \):

The deformed algebra \( U_q(sl(2)) \)

\[
[E_+, E_-] = \frac{K - K^{-1}}{q - q^{-1}},
KE_\pm K^{-1} = q^{\pm 2}E_\pm.
\]  
(1)

at roots of unity i.e. \( q^{2p+1} = 1, p \) a positive integer, has a finite dimensional irreducible representation which has no classical finite dimensional analog. This is the cyclic representation whose dimension is \( 2p + 1 \)\cite{7}. Cyclic means that there are no heighest or lowest weight states in the spectrum. i.e. \( E_+|\cdots > \neq 0 \) and \( E_-|\cdots > \neq 0 \) for any state.

When \( q^{2p+1} = 1 \) irreducible cyclic representation of \( U_q(sl(2)) \) can be written in some basis \( \{v_0, v_1, \cdots, v_{2p}\} \) as \( Kv_m = \lambda q^{-2m}v_m \).
where \( m = 0, \cdots, 2p \), and we defined \( v_0 \equiv v_{2p+1}, \ v_{-1} \equiv v_{2p} \). \( \lambda, \ g_m \), and \( f_m \) are some complex constants which are nonzero and in the case of requesting that the representation in unitary, we should restrict their values such that

\[
K^\dagger = K^{-1}; \quad E_+ = E_-. \tag{3}
\]

3. Classification:

QHE trial wavefunctions in the standard hierarchy scheme are given by\[^3\],\[^8\]

\[
\psi_\nu(z_1, \cdots, z_{N_0}) = \int \prod_{\alpha=1}^{r} \prod_{i_\alpha=1}^{N_\alpha} \left[ d^2 z^{(\alpha)}_{i_\alpha} \right] e^{-\frac{i}{2} \sum_{i=1}^{N_0} |z_i|^2} \prod_{\beta=0}^{r} \prod_{i_\beta<j_\beta}^{N_\beta} (z^{(\beta)}_{i_\beta} - z^{(\beta)}_{j_\beta}) a_{\beta}
\]

\[
\times \prod_{i_\beta+1<j_\beta=1}^{N_{\beta+1},N_{\beta}} (z^{(\beta+1)}_{i_\beta+1} - z^{(\beta)}_{j_\beta}) b_{\beta,_{\beta+1}}, \tag{4}
\]

where \( z^{(0)}_{i_\alpha} \equiv z_i \). The measure \( \prod [d^2 z^{(\alpha)}_{i_\alpha}] \) depends on \( a_{\beta} \) and \( |z^{(\beta)}_{i_\beta} - z^{(\beta)}_{j_\beta}| \), however the detailed form of it does not affect the filling factor \( \nu = \frac{P}{Q} \). \( a_0 \) is an odd positive integer, \( a_\alpha \) for \( \alpha \neq 0 \) are even integers which can be positive or negative and \( b_{\beta+1,\beta} = \pm 1 \), except \( b_{r,r+1} = 0 \). By placing the \( N_0 \) electrons on a spherical surface in a monopole magnetic field, one can find that filling factor of (4) is given by

\[
\nu = \frac{1}{a_0 - \frac{1}{a_1 - \cdots - \frac{1}{a_r}}}. \tag{5}
\]

Factors with negative powers may be replaced by complex-conjugate factors with positive powers multiplied by some exponential factors. Hence, (4) can equivalently be given as\[^8\]

\[
\psi_\nu(z_1, \cdots, z_{N_0}) = \int \prod_{\alpha=1}^{r} \prod_{i_\alpha=1}^{N_\alpha} \left[ d^2 z^{(\alpha)}_{i_\alpha} \prod_{i_\alpha<j_\alpha}^{N_\alpha} |z^{(\alpha)}_{i_\alpha} - z^{(\alpha)}_{j_\alpha}|^2 \right] e^{-\frac{1}{2} \sum_{i=1}^{N_0} |z_i|^2} \prod_{\beta=0}^{r} \prod_{i_\beta<j_\beta}^{N_\beta} (z^{(\beta)}_{i_\beta} - z^{(\beta)}_{j_\beta})^{p_\beta} \prod_{i_\beta+1<j_\beta=1}^{N_{\beta+1},N_{\beta}} (z^{(\beta+1)}_{i_\beta+1} - z^{(\beta)}_{j_\beta})^{q_{\beta}}, \tag{6}
\]

where \( z^{(\beta)}_{i_\beta} = z^{(\beta)}_{i_\beta} \) for \( \beta = \text{even} \) and \( z^{(\beta)}_{i_\beta} = \bar{z}^{(\beta)}_{i_\beta} \) for \( \beta = \text{odd} \) and

\[
\begin{align*}
\theta_0 &= 0, & \theta_r &= \frac{(-1)^r}{p_r - (-1)^r \theta_{r-1}}, \\
q_0 &= -1, & q_r &= (-1)^{r+1} q_{r-1} \theta_r.
\end{align*}
\]
Now, filling factor is
\[ \nu = \frac{1}{p_0 + \frac{1}{p_1 + \frac{1}{p_2 + \cdots + \frac{1}{p_r}}}}, \]
where \( p_0 \) is odd and the other \( p_i \) are even integers.

By generalizing the calculations of Laughlin given in Ref. [1] and making use of the scalar product
\[ (\psi_\nu, \psi_{\nu'}) \equiv \int d^2 z_1 \cdots d^2 z_N \bar{\psi}_\nu(z_1, \cdots, z_N) \psi_{\nu'}(z_1, \cdots, z_N). \]
one can show that \( \psi_\nu \) states are orthogonal.[8]

To emphasize the second quantized character of our construction let us introduce the states
\[ |i, p>^T = \int d^2 z_1 \cdots d^2 z_{N_0} e^{-\frac{1}{2} \sum_{k=1}^{N_0} |z_k|^2} \psi_{\nu + \frac{1}{2p+1}}(z_1, \cdots, z_{N_0})|z_1, \cdots, z_{N_0}>, \]
where \( i = 1, \cdots, 2p+1; \ p = 1, 2, \cdots, \) so that any filling factor \( \nu = P/Q \) is considered.

We used the vectors
\[ |z_1, \cdots, z_{N_0} > = \frac{1}{\sqrt{N_0!}} \varphi^\dagger(z_1) \cdots \varphi^\dagger(z_{N_0})|0>. \]

The fermionic operators \( \varphi(z), \varphi^\dagger(z) \) satisfy the anticommutation relation
\[ \{\varphi^\dagger(z), \varphi(z')\} = e^{z\bar{z}}. \]

The subscript \( T \) denotes the fact that trial wave functions are used to give an explicit realization. The states [4] are orthonormal:
\[ <i, p|j, p'>^T = \delta_{i,j}\delta_{p,p'}. \]

We have shown that the states \( |i, p>, \ p >^T \) are orthonormal by using the explicit form of trial wavefunctions. However, this should be a universal feature of QHE wavefunctions. Then, even if we do not know the explicit form, we can say that exact states of the QHE which we indicate with \( |i, p>, \ p > \), should be orthonormal:
\[ <i, p|j, p'> = \delta_{i,j}\delta_{p,p'}. \]

Indeed, in the following we will use this universal property of QHE states without referring to any trial wavefunction.

Let us deal with the states
\[ |1, p>, \ |2, p>, \cdots, \ |2p, p>, \ |2p + 1, p>, \]
 corresponding to the filling factors
\[ \nu = \frac{1}{2p + 1}, \frac{2}{2p + 1}, \cdots, \frac{2p}{2p + 1}. \]
Define the following second quantized operators acting in the space spanned by the states (13),

\[ \tilde{K} = \sum_{i=1}^{2p+1} q^{i} |i, p><i, p|, \]  
\[ \tilde{E}_{+} = \sum_{i=1}^{2p+1} a_{i} |i, p><i+1, p|, \]  
\[ \tilde{E}_{-} = \sum_{i=1}^{2p+1} \bar{a}_{i} |i+2, p><i, p|, \]

where

\[ q^{2p+1} = 1. \]

To obtain the compact forms we adopted the definitions

\[ |2p + 2, p><1, p|, |2p + 3, p><2, p|. \]

By using the orthonormality condition (11) one observes that inverse of \(\tilde{K}\) is

\[ \tilde{K}^{-1} = \sum_{i=1}^{2p+1} q^{-i} |i, p><i, p| = \tilde{K}^\dagger. \]

Let the coefficients \(a_{i}\) are nonzero and satisfy

\[ |a_{2p+1}|^2 - |a_{2p-1}|^2 = 0, \]
\[ |a_{2p}|^2 - |a_{2p-2}|^2 = -1, \]
\[ |a_{l+2}|^2 - |a_{l}|^2 = \frac{q^{l+2} - q^{-l-2}}{q - q^{-1}}, \]

where \(l = -1, 0, \ldots (2p - 3)\); \(a_{-1} \equiv a_{2p}, a_{0} \equiv a_{2p+1}\). Then, in terms of the basis (\(|1, p>, \cdots, |2p + 1, p>|\)) the operators (15)–(17) lead to a \((2p + 1)\) dimensional unitary irreducible cyclic representation of \(U_{q}(sl(2))\) at \(q\) satisfying (18).

4. Discussions:

It is shown that QHE wavefunctions can be classified as irreducible cyclic representations of \(U_{q}(sl(2))\) at roots of unity in a very natural way. This naturalness follows from the fact that the most significant physical quantity of the QHE \(\nu = P/Q\) fits very well with the integer \((m\) in (2)) characterizing irreducible cyclic representations of \(U_{q}(sl(2))\).
Because of this classification we can propose $U_q(sl(2))$ at roots of unity as dynamical symmetry algebra of the QHE. Obviously, any set of orthogonal states possessing a quantum number which permits a partition of unity like $\nu$,

$$\sum_{i=1}^{2p+1} \nu(|i, 2p + 1>) = 1,$$

can be classified as irreducible cyclic representation of $U_q(sl(2))$ at a root of unity. Hence, our observation is not enough to prove that $U_q(sl(2))$ at roots of unity is the real dynamical symmetry algebra of the QHE. However, there exists another evidence to believe that the proposed dynamical symmetry is the one chosen by nature. If there exists a “universal microscopic theory” of the QHE given in terms of a differential operator depending on $z_k, \bar{z}_k$ and their derivatives and moreover, possessing this dynamical symmetry, it should be in the form

$$O_q \Phi_i = \ell_i \Phi_i; \ i = 1, \ldots, 2p + 1; \ q^{2p+1} = 1.$$

Here, $O_q$ denotes a differential operator which is a function of the generators of the dynamical symmetry algebra $U_q(sl(2))$ and $\Phi_i$ are its eigenfunctions corresponding to the eigenvalues $\ell_i$ and possessing the filling factors $\nu(\Phi_i) = i/2p + 1$. Thus, one should find a differential realization of the generators of $U_q(sl(2))$ which determine the cyclic representations. These differential operators should act on polynomials in $z_k$. Because, the $\nu = 1$ wavefunction which is an element of the basis of irreducible cyclic representation is exact and it is a polynomial in $z_k$. There are some differential realizations of these generators leading to the cyclic representations of $U_q(sl(2))$ given in a space of polynomials if the following equivalence relation is satisfied

$$z^{j+2p+1} \sim z^j.$$

But, these constraints are not permitted in the complex plane where QHE wavefunctions should be constructed. This leads to the conclusion that a “universal microscopic theory” for the fractional QHE (in the common sense), does not exist if its dynamical algebra is $U_q(sl(2))$ at roots of unity. This may explain why one could not find a universal microscopic theory of the fractional QHE.

How one can utilize the proposed dynamical symmetry for the QHE to calculate some physical quantities? Here, one of the most significant physical quantities is the partition function which may be obtained if the Green function on the space defined by $U_q(sl(2))$ at $q$ roots of unity with cyclic representation is available. In Ref. [10], Green function on the space defined by the $q$-deformed group $SU_q(2)/U(1)$ for $q$ not a root of unity is obtained without referring to explicit form of the representations but depending only on their general features. We hope that a similar calculation can be used in our case. Then, we can obtain Green function and the partition function. This may lead to a decisive answer if the proposed dynamical symmetry is the real symmetry of the QHE and moreover, it may give some hints about its physical interpretation.
References

[1] *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).

[2] R. B. Laughlin, Phys. Rev. Lett. **50** (1983) 1395.

[3] F. D. M. Haldane, Phys. Rev. Lett. **50** (1983) 605; B. I. Halperin, Phys. Rev. Lett. **52** (1984) 1583.

[4] J. K. Jain, Phys. Rev. Lett. **63** (1989) 199; X. G. Wen and A. Zee, Phys. Rev. B **46** (1992) 2290.

[5] S. C. Zhang, Int. J. Mod. Phys. B **6** (1992) 25; R. K. Ghosh and S. Rao, “Fractional QHE States as Exact Ground States”, cond-mat/9703217.

[6] Ö. F. Dayi, “QHE Wavefunctions as Cyclic Representations of $U_q(sl(2))$”, J. Phys. A in press, q-alg/9704010.

[7] G. Lusztig, Adv. Math. **70** (1988) 237; M. Rosso, Commun. Math. Phys. **117** (1988) 581; P. Roche and D. Arnaudon, Lett. Math. Phys. **17** (1989) 295; C. de Concini and V. G. Kac, *Representations of Quantum Groups at Root of Unity*, Progress in Mathematics Vol. 92 (Birkhäuser, Boston, 1990); L. C. Biedenharn and M. A. Lohe, *Quantum Group Symmetry and q–Tensor Algebras*, (World Scientific, Singapore, 1995).

[8] N. Read, Phys. Rev. Lett. **65** (1990) 1502.

[9] B. Blok and X. G. Wen, Phys. Rev. B **43** (1991) 8337.

[10] H. Ahmedov and I.H. Duru, “Green Function on the q–Symmetric Space $SU_q(2)/U(1)$”, RIBS-PH-4/97 and q-alg/9703032.