ON THE RIEmann-Hardy Hypothesis FOR THE RAMANUJAN
Zeta function

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Abstract. The Ramanujan zeta function was in 1916 proposed by an Indian mathematician Srinivasa Ramanujan. As an analogue of the Riemann hypothesis, an English mathematician Godfrey Harold Hardy proposed in 1940 that the real part of all complex zeros of the Ramanujan zeta function is 6. This is the well-known Riemann-Hardy hypothesis for the Ramanujan zeta function. This article is devoted to the proof of this hypothesis derived from the Ramanujan-Rankin function. Owing to the integral representation of the Ramanujan-De Bruijn function, we establish its series. We also reduce its product using the Hadamard’s factorization theorem. By a class with its series and product representations, we conclude that the real part of all zeros for Ramanujan-De Bruijn function is zero. We also obtain its products of Conrey and Ghosh and Hadamard-type for the Ramanujan-Rankin function. Based on the obtained result, we prove that the Riemann-Hardy hypothesis is true.

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1. Background and results

1.1. The Ramanujan zeta function. In 1916 remarkable paper of Srinivasa Ramanujan [1], who is an Indian mathematical genius, he proposed the well-known Ramanujan zeta function \( H_\tau (x) \), defined as

\[
H_\tau (x) = \sum_{\ell=1}^{\infty} \frac{\tau (\ell)}{\ell^x},
\]

where \( x \in \mathbb{A} \), \( \Re (x) > 13/2 \) and \( \ell \in \mathbb{G} \), and \( \tau (\ell) \) is the Ramanujan’s arithmetical function [2]. Here, let \( \mathbb{A} \), \( \mathbb{B} \) and \( \mathbb{G} \) be the sets of complex, real and natural numbers, respectively.

As the first conjecture of Ramanujan [1], Mordell [3, 4] proved in 1917 that \( \tau (\ell) \) is multiplicative, i.e.,

\[
\tau (\ell_1 \ell_2) = \tau (\ell_1) \tau (\ell_2),
\]

if \( \gcd (\ell_1, \ell_2) = 1 \).

Ramanujan [1] said that \( \tau (\ell) \) can also be obtained by (see [5], p.161; also see [6, 7])

\[
L (x) = q \prod_{n=1}^{\infty} \left( 1 - q^n \right)^{24} = \sum_{m=1}^{\infty} \tau (m) q^m,
\]

which is the normalized weight 12 cusp form for \( SL_2 (Z) \) [7], where \( L (x) \) is the modular form, \( q = \exp (2\pi ix) \), \( \Im (x) > 0 \) and \( m, n \in \mathbb{G} \).

As the second conjecture of Ramanujan [1], Mordell [3] also proved that

\[
\tau (p^r) = \tau (p) \tau (p^{r-1}) - p^{11} \tau (p^{r-2}),
\]

if \( p \) is prime, and \( r > 2 \).

Deligne (see [8]; for the review of the problem, also see [9]) proved that third conjecture of Ramanujan [1], i.e.,

\[
|\tau (p)| \leq 2p^{\frac{11}{2}}.
\]

Ramanujan [1] showed that Eq. (1) has an Euler-type product of the form [10]

\[
H_\tau (x) = \prod_{p} \frac{1}{1 - \tau (\ell) \frac{1}{p^{1+p^{1-p}}}},
\]

where \( \Re (s) > 13/2 \).

1.2. The Ramanujan-Rankin function. In 1939 Rankin paper [11], he first proposed the well-known Ramanujan-Rankin function \( \xi_\tau (x) \), defined by

\[
\xi_\tau (x) = (2\pi)^{-x} \Gamma (x) H_\tau (x),
\]

where \( x \in \mathbb{A} \) and \( \Gamma (x) \) is the gamma function.

The modular form \( L (\omega) \) has the functional equation of the form [11, 12]

\[
L (\omega) = \frac{1}{\omega^{1/2}} L \left( -\frac{1}{\omega} \right).
\]
for $\omega > 0$.

Let $i = \sqrt{-1}$. There exists an integral of (7) as follows [11, 12]:

$$
\xi_r(x) = \int \omega^{x-1} L(i\omega) d\omega,
$$

which can be rewritten as (11)

$$
\xi_r(x) = \int_0^1 L(i\omega) (\omega^{x-1} + \omega^{11-x}) d\omega.
$$

Wilton [13] suggested an alternative integral representation of (7), given as [14]

$$
\xi_r(x) = \int_0^\infty \omega^{x-1} L(\exp(-2\pi\omega)) d\omega
= \int_1^\infty (\omega^{x-1} + \omega^{11-x}) L(\exp(-2\pi\omega)) d\omega,
$$

provided that

$$
L(q) = q \prod_{n=1}^\infty (1 - q^n)^{24} = \sum_{m=1}^\infty \tau(m) q^m.
$$

Rankin [11] and Wilton [13] proved that (7) has the functional equation as follows (also see [12]):

$$
\xi_r(x) = \xi_r(12 - x).
$$

1.3. The product of Conrey and Ghosh. Conrey and Ghosh [15] suggested the product of (7) as follows:

$$
\xi_r(x) = e^{b + b_0 x} \prod_{x_m} \left(1 - \frac{x}{x_m}\right) \exp \left(\frac{x}{x_m}\right),
$$

where $x_m$ take over all zeros of (7), and both $b$ and $b_0$ are two unknown constants. Here, (14) is so-called product of Conrey and Ghosh.

Wilton [13] considered $x = 6 + i\varphi$ into (14) and obtained the Ramanujan-Wilton function $\Lambda(\varphi)$, defined by

$$
\Lambda(\varphi) = \xi_r(6 + i\varphi) = \xi_r(6 - i\varphi),
$$

which satisfies the functional equation [13, 16]

$$
\Lambda(-\varphi) = \Lambda(\varphi),
$$

where $\varphi \in \mathbb{A}$.

In 1950 de Bruijn [16] presented the Ramanujan-de Bruijn function $X(z)$, defined as

$$
X(z) := \Lambda(iz) = (2\pi)^{-(6+z)} \Gamma(6+z) H_{6+z}(6+z),
$$

where $z \in \mathbb{A}$. 

1.4. **The Riemann-Hardy hypothesis.** In analogy with the Riemann hypothesis, Godfrey Harold Hardy (see [16], p.174), who is an English mathematician, proposed the well-known conjecture that states the following:

**Theorem 1.** *(Riemann-Hardy hypothesis)*

All of the nontrivial zeros of the Ramanujan zeta function \( \xi(x) \) lie on the critical line \( \Re(x) = \frac{1}{2} \).

Meanwhile, Hardy (see [16], p.174) also conjectured an analogue of the Riemann-Siegel-like formula and von Mangoldt-like theorem. As an analogous work of Hardy theorem for the Riemann zeta function, it was proved by Lekkerkerker [19]. The latter was studied by generalized by Berndt [20] and further developed by Hafner [21], Ki [22] and Chirre and Castañón [23]. The former was developed by Ferguson and coauthors [24] and further reported by Keiper [25]. More importantly, the papers of Ferguson and coauthors [24] and Keiper [25] reported some nontrivial zeros for \( \xi(x) \). As one of interesting and important problems in the theory of modular form [26], the Riemann-Hardy hypothesis has not been solved for the more than nineteen-eighty years.

Wilton [13] reported (7) is an entire function, and Apostol and Sklar [27] said that (7) is an entire function of finite order. Thanks to the result of Ogg (see [28], p.21), it is proved that (7) is an entire function of order \( \lambda = 1 \). There is an unsolved problem that the product of Conrey and Ghosh [15] has the unknown parameters. The series and product of Hadamard-type representations of (7) has not considered in any paper. Theorem 1 implies that the real part of all complex zeros of (7) is 6.

In fact, de Bruijn [16] guessed that Theorem 1 is equivalent to the case that all zeros of (17) lie on the critical line \( \Re(z) = 0 \). This is the conjecture of de Bruijn. Moreover, there is no investigation for the series, product, zeros and order of (17) at present. The conjecture of de Bruijn remains an open problem.

There exists an equivalent representation of the reported works, which implies that all zeros of (17) lie on the critical line \( \Im(\phi) = 0 \). This is the conjecture of Ki [21], which is further studied by Chirre and Castañón [22]. The series, product and order of (15) have not reported. The conjecture of Ki is also an open problem.

1.5. **The main aim of the present paper.** Motivated by the ideas above, we give the outline of the present paper. In Section 2 we present the order, the series and product presentations of (7), (15) and (17). The values of the parameters in the product of Conrey and Ghosh are given in detail. The Hadamard-type product are also obtained. The product for (17) is set up based on the functional equation and Hadamard’s factorization theorem. In Section 3 we first consider the detailed proof of the conjecture of de Bruijn. We then reduce to the conjecture of Ki. Finally, we report the proof of Theorem 1.

2. **Preliminary results**

2.1. **The series and orders.** To begin with, we investigate the series representation of \( \xi(x) \) as follows.

Let \( \mathcal{H} = \mathcal{G} \cup \{0\} \).
Proposition 1. \( \text{Let } x \in \mathfrak{A}. \text{ Then there exists} \)
\[
(18) \quad \xi_\tau (x) = \sum_{s=0}^{\infty} \chi (s) (x - 6)^{2s}
\]
where
\[
(19) \quad \chi (s) = 2 \int_{0}^{1} L (i\omega) \omega^5 \left[ \frac{(\ln \omega)^{2s}}{(2s)!} \right] d\omega
\]
are the positive real coefficients and \( s \in \mathfrak{N} \).

Proof. By using (11), we present
\[
(20) \quad \xi_\tau (x) = \int_{0}^{1} L (i\omega) \left( \omega^{x-1} + \omega^{11-x} \right) d\omega
\]
\[
= \int_{0}^{1} L (i\omega) \omega^5 \left( \omega^{x-6} + \omega^{6-x} \right) d\omega
\]
\[
= \int_{0}^{1} L (i\omega) \omega^5 \{ \exp [(x - 6) \ln \omega] + \exp [(6 - x) \ln \omega] \} d\omega.
\]
To simply (20), we obtain
\[
(21) \quad \xi_\tau (x) = 2 \int_{0}^{1} L (i\omega) \omega^5 \cosh [(x - 6) \ln \omega] d\omega.
\]
Consider that
\[
(22) \quad \cosh [(x - 6) \ln \omega] = \sum_{s=0}^{\infty} \frac{[(x - 6) \ln \omega]^{2s}}{(2s)!}.
\]
By the substitution of (22) into (21),
\[
(23) \quad \xi_\tau (x) = 2 \int_{0}^{1} L (i\omega) \omega^5 \left\{ \sum_{s=0}^{\infty} \frac{[(x - 6) \ln \omega]^{2s}}{(2s)!} \right\} d\omega
\]
\[
= \sum_{s=0}^{\infty} \left\{ 2 \int_{0}^{1} L (i\omega) \omega^5 \left[ \frac{(\ln \omega)^{2s}}{(2s)!} \right] d\omega \right\} (x - 6)^{2s}.
\]
In fact,

\[(24) \quad \chi(s) = 2 \int_0^1 L(i\omega) \omega^5 \left[ \frac{(\ln \omega)^{2s}}{(2s)!} \right] d\omega > 0 \]

provided that

\[(25) \quad L(i\omega) = \exp(-2\pi\omega) \prod_{n=1}^{\infty} \left[ 1 - \exp(-2n\pi\omega) \right]^{24} > 0 \]

is true.

We thus complete the proof. \(\square\)

**Proposition 2.** If \(\chi(s)\) is defined as in Proposition 1, then there exists

\[(26) \quad \Lambda(\varphi) = \sum_{s=0}^{\infty} \chi(s) (-1)^s \varphi^{2s}, \]

where \(\varphi \in \mathfrak{A}.\)

**Proof.** Putting \(x = 6 + i\varphi\) in Proposition 1 implies the desired result. \(\square\)

Here, we need to prove the following:

**Proposition 3.** \(\Lambda(\varphi)\) is an even entire function of order \(\lambda = 1.\)

**Proof.** Owing to (26), we rewrite

\[(27) \quad \chi(s) = 2 \int_0^1 L(i\omega) \omega^5 \left[ \frac{(\ln \omega)^{2s}}{(2s)!} \right] d\omega \]

as

\[(28) \quad \chi(s) = \frac{2}{(2s)!} \int_0^1 L(i\omega) \omega^5 \exp(2s \ln \omega) d\omega > 0. \]

By virtue of

\[(29) \quad 0 < \exp(2s \log \omega) \leq 1 \]

for \(0 < \omega \leq 1,\) there exists

\[(30) \quad |\chi(s)| = \frac{2}{(2s)!} \int_0^1 L(i\omega) \omega^5 \exp(2s \ln \omega) d\omega. \]

Because of

\[(31) \quad 0 < \int_0^1 L(i\omega) \omega^5 \exp(2s \ln \omega) d\omega \leq C, \]
where

\[(32) \quad C = \int_{0}^{1} L(i\omega)\omega^5 d\omega = \int_{0}^{1} L(i\omega)\omega^5 d\omega = \xi_\tau(6) < \infty,\]

we have from (30) and (31) that

\[(33) \quad |\chi(s)| = \frac{2}{(2s)!} \int_{0}^{1} L(i\omega)\omega^5 \exp(2s \ln \omega) d\omega \leq \frac{2C}{(2s)!}.\]

By the formula (1.04) in Levin’ book ([29], p.4), we have

\[(34) \quad \lim_{s \to \infty} \sqrt{s}|\chi(s)| = 0\]

such that \(\Lambda(\varphi)\) is an entire function.

Using Ogg’s result (see [28], p.21) that \(\xi_\tau(x)\) is an entire function of order \(\lambda = 1\), Theorem 2 in Levin’ book ([29], p.4) said that the order \(\lambda\) of \(\Lambda(\varphi)\) can be also given as

\[(35) \quad \lambda = \lim_{s \to \infty} \frac{s \ln s}{\ln |\chi(s)|} = 1.\]

By (35), \(\Lambda(\varphi)\) is also an entire function of order \(\lambda = 1\) because \(|\chi(s)(-1)^s| = \chi(s) > 0\).

We thus finish the proof. □

Putting \(x = 6 + z\) in Proposition 1 for \(z \in \mathcal{A}\) and using the formulae (9) and (11) of Rankin [11], we obtain

\[(36) \quad X(z) = (2\pi)^-(6+z) \Gamma(6+z) \Pi_\tau(6+z) = \int_{0}^{\infty} \omega^{5+z} L(i\omega) d\omega \]

\[= \int_{0}^{1} L(i\omega)\omega^5 (\omega^z + \omega^{-z}) d\omega,\]

which is the result of De Bruijn [16].

Based on this, we get the followings:

**Corollary 1.** If \(\chi(s)\) is defined as in Proposition 1 then we have

\[(37) \quad X(z) = \sum_{s=0}^{\infty} \chi(s) z^{2s},\]

where \(z \in \mathcal{A}\).

**Proof.** As a similar manner of Proposition 1 we have from (36) the required result. □

Moreover, we have the followings:

**Corollary 2.** The function \(X(z)\) is an even entire function of order \(\lambda = 1\).
Proof. With (37), we know
\[(38) \quad X(z) = X(-z),\]
where \(z \in \mathbb{A}\), and by using (34) and (35), we see that \(X(z)\) is an entire function of order \(\lambda = 1\).
Hence, we complete the proof. \(\square\)

2.2. The products.

Theorem 2. There exists
\[(39) \quad X(z) = X(0) \prod_{\Im(z_m) > 0} \left(1 - \frac{z^2}{z_m^2}\right),\]
where the product run over all of the zeros of \(z_m\) of \(X(z)\).

Proof. Taking \(z = 0\) in (36) implies
\[(40) \quad X(0) = \int_{0}^{\infty} \omega^5 L(i\omega) d\omega > 0\]
provided that (25) holds.
In view of Corollary 2, the Hadamard’s factorization theorem (see [30], p.250) said that
\[(41) \quad X(z) = e^{M + Lz} \prod_{z_m} \left(1 - \frac{z}{z_m}\right) \exp \left(\frac{z}{z_m}\right),\]
where the product run over all of the zeros \(z_m\) of \(X(z)\) and both \(M\) and \(L\) are two constants.
Clearly, the sequence \(\{z_m\}\) are the set of non-zero zeros of \(X(z)\) with \(|z_{m+1}| > |z_m|\).
Since \(X(z)\) is an even entire function obtained in Corollary 2,
\[(42) \quad X(z) = e^{M + Lz} \prod_{z_m} \left(1 - \frac{z}{z_m}\right) \exp \left(\frac{z}{z_m}\right)
\quad = e^{M + Lz} \prod_{\Im(z_m) > 0} \left(1 - \frac{z}{z_m}\right) \left(1 + \frac{z}{z_m}\right) \exp \left(\frac{z}{z_m} - \frac{z}{z_m}\right)
\quad = e^{M + Lz} \prod_{\Im(z_m) > 0} \left(1 - \frac{z^2}{z_m^2}\right),\]
which yields that
\[(43) \quad X(-z) = e^{M - Lz} \prod_{\Im(z_m) > 0} \left(1 - \frac{z^2}{z_m^2}\right).\]
Combining (42) and (43) and using (38), we suggest
\[(44) \quad e^{M + Lz} \prod_{\Im(z_m) > 0} \left(1 - \frac{z^2}{z_m^2}\right) = e^{M - Lz} \prod_{\Im(z_m) > 0} \left(1 - \frac{z^2}{z_m^2}\right).\]
By (40) and (44), we have 

\[ M = \ln X(0) \quad \text{and} \quad L = 0 \quad \text{such that} \quad (42) \quad \text{can be rewritten as} \]

\[ (45) \quad X(z) = X(0) \prod_{\Im(m) > 0} \left( 1 - \frac{z^2}{z_m^2} \right). \]

Thus, we obtain the required result. \(\Box\)

As a direct result, we obtain:

**Corollary 3.** There exists any positive number \(\varepsilon > 0\) such that

\[ (46) \quad \frac{1}{|z_m|^{1+\varepsilon}} \]

is convergent.

**Proof.** Since \(X(z)\) is an even entire function of order \(\lambda = 1\) and (39) is true, as an analogous result in Levin' book ([29], p.8), we have (46) such that \(\sum_{m=1}^{\infty} |z/z_m|^{1+\varepsilon}\) converges uniformly in each bounded domain.

This is the required result. \(\Box\)

2.2.1. *The product of Conrey and Ghosh.* As a direct result of Theorem 2, we see the following:

**Theorem 3.** Assume the above denotations. Then we have:

- **(B1) There is**

\[ (47) \quad \xi_r(x) = \xi_r(6) \prod_{\Im(m) > 0} \left[ 1 - \frac{(x-6)^2}{2} \right]. \]

- **(B2) (The product of Conrey and Ghosh) There is**

\[ (48) \quad \xi_r(x) = e^{b_0 x} \prod_{x_m} \left( 1 - \frac{x}{x_m} \right) \exp \left( \frac{x}{x_m} \right), \]

where the product takes over all of the zeros \(x_m\) of \(\xi_r(x)\), \(b_0 = \ln \xi_r(0) - 6 \sum x_m (1/x_m)\) and \(b = \ln \xi_r(0)\).

- **(B3) (The product of Hadamard-type) There is**

\[ (49) \quad \xi_r(x) = \xi_r(0) \prod_{x_m} \left( 1 - \frac{x}{x_m} \right). \]

Moreover, there exists any positive number \(\delta > 0\) such that \(\sum_{m=1}^{\infty} |x_m|^{-(1+\delta)}\) is convergent.

**Proof.** Step one is to prove (B1). Taking \(z = x - 6\) in Theorem [2] and Corollary [1] we have

\[ (50) \quad X(x - 6) = X(0) \prod_{\Im(m) > 0} \left[ 1 - \frac{(x-6)^2}{2} \right]. \]
and

\[(51) \quad X (x - 6) = \sum_{s=0}^{\infty} \chi (s) (x - 6)^{2s}.\]

By Proposition [1] we see that (51) is equal to

\[(52) \quad \xi_{\tau} (x) = X (x - 6) = \sum_{s=0}^{\infty} \chi (s) (x - 6)^{2s} = X (0) \prod_{\Im (z_m) > 0} \left[ 1 - \frac{(x-6)^2}{z_m^2} \right].\]

Because (52) gives

\[(53) \quad \xi_{\tau} (6) = X (0),\]

(52) becomes

\[(53) \quad \xi_{\tau} (x) = \xi_{\tau} (6) \prod_{\Im (z_m) > 0} \left[ 1 - \frac{(x-6)^2}{z_m^2} \right].\]

Thus, we finish the proof of (B1).

Step two is to prove (B2).

By Proposition [3] we adopt the Hadamard’s factorization theorem in Titchmarsh’s book [30], p.250 to obtain the conjecture of Conrey and Ghosh [15], that is,

\[(54) \quad \xi_{\tau} (x) = e^{b+b_0 x} \prod_{x_m} \left( 1 - \frac{x}{x_m} \right) \exp \left( \frac{x}{x_m} \right),\]

where the product run over all of the zeros \(x_m\) of \(\xi_{\tau} (x)\) and both \(b\) and \(b_0\) are two constants. Obviously, the sequence \(\{x_m\}\) is the set of non-zero zeros of \(\xi_{\tau} (x)\) with \(|x_{m+1}| > |x_m|\).

Combining (21), (53) and (54), we have

\[(55) \quad \xi_{\tau} (x) = e^{b+b_0 x} \prod_{x_m} \left( 1 - \frac{x}{x_m} \right) \exp \left( \frac{x}{x_m} \right) \]

\[= \xi_{\tau} (6) \prod_{\Im (z_m) > 0} \left[ 1 - \frac{(x-6)^2}{z_m^2} \right] \]

\[= 2 \int_{0}^{1} L (i \omega) \omega^5 \cosh [(x - 6) \ln \omega] d\omega\]

such that

\[(56) \quad (x_m - 6)^2 = z_m^2.\]

From (55) and (56) we may get

\[(57) \quad \xi_{\tau} (x) = \xi_{\tau} (6) \prod_{\Im (z_m) > 0} \left[ 1 - \frac{(x-6)^2}{(x_m-6)^2} \right].\]

In view of the fact \(x_m = 6 \pm z_m\) obtained by (56), we have

\[(58) \quad \Im (z_m) = \Im (x_m - 6) = \Im (x_m)\]
such that (55) becomes

\[ \xi_\tau (x) = \xi_\tau (6) \prod_{\Im (x_m) > 0} \left[ 1 - \left( \frac{x - 6}{x_m - 6} \right)^2 \right] \]

(59)

\[ = \xi_\tau (6) \prod_{\Im (x_m - 6) > 0} \left[ 1 - \left( \frac{x - 6}{x_m - 6} \right)^2 \right] \]

\[ = \xi_\tau (6) \prod_{\Im (x_m) > 0} \left[ 1 - \left( \frac{x - 6}{x_m - 6} \right)^2 \right]. \]

Let us write

\[ \xi_\tau (x) = \xi_\tau (6) \prod_{\Im (x_m) > 0} \left[ 1 - \left( \frac{x - 6}{x_m - 6} \right)^2 \right] \]

(60)

\[ = \xi_\tau (6) \prod_{\Im (x_m) > 0} \left\{ \left( 1 - \frac{x - 6}{x_m - 6} \right) \left( 1 + \frac{x - 6}{x_m - 6} \right) \right\} \]

\[ = \xi_\tau (6) \prod_{\Im (x_m) > 0} \left\{ \left( 1 - \frac{x - 6}{x_m - 6} \right) \left( 1 - \frac{x - 6}{6 - x_m} \right) \right\} \]

\[ = \xi_\tau (6) \prod_{\Im (x_m) > 0} \left\{ \left( 1 - \frac{x - 6}{x_m - 6} \right) \left( 1 - \frac{x - 6}{12 - x_m} \right) \right\}. \]

By using the functional equation (13), the identity (60) is equal to

\[ \xi_\tau (x) = \xi_\tau (6) \prod_{\Im (x_m) > 0} \left\{ \left( 1 - \frac{x - 6}{x_m - 6} \right) \left[ 1 - \frac{x - 6}{12 - x_m} \right] \right\}. \]

(61)

From (55) we take into account

\[ \xi_\tau (x) = e^{b + b_0 x} \prod_{x_m} \left( 1 - \frac{x - 6}{x_m - 6} \right) \exp \left( \frac{x}{x_m} \right) = 2 \int_0^1 L (i \omega) \omega^5 \cosh \left[ (x - 6) \ln \omega \right] d\omega \]

such that

\[ \xi_\tau (0) = e^b = 2 \int_0^1 L (i \omega) \omega^5 \cosh (6 \ln \omega) d\omega = 2 \int_0^1 L (i \omega) \omega^{11} d\omega > 0, \]

(63)
\( \xi_\tau (6) = e^{b + 6b_0} \prod_{x_m} \left(1 - \frac{6}{x_m}\right) \exp \left(\frac{6}{x_m}\right) = 2 \int_0^1 L (i \omega) \omega^5 d\omega > 0 \)

and

\[ \xi_\tau (12) = e^{b + 12b_0} \prod_{x_m} \left(1 - \frac{12}{x_m}\right) \exp \left(\frac{12}{x_m}\right) \]

\( = 2 \int_0^1 L (i \omega) \omega^5 \cosh (6 \ln \omega) d\omega \]

\( = 2 \int_0^1 L (i \omega) \omega^{11} d\omega > 0. \)

Making use of (63), (64) and (65), we have

\( b = \ln \xi_\tau (0) \)

and

\( \xi_\tau (0) = \xi_\tau (12) \)

such that

\( \xi_\tau (0) = e^b = e^{b + 12b_0} \prod_{x_m} \left(1 - \frac{12}{x_m}\right) \exp \left(\frac{12}{x_m}\right) = 2 \int_0^1 L (i \omega) \omega^{11} d\omega > 0. \)

By using (61), we obtain

\( \xi_\tau (x) = \xi_\tau (6) \prod_{x_m} \left(1 - \frac{x - 6}{x_m - 6}\right), \)

which leads to

\( \xi_\tau (0) = \xi_\tau (6) \prod_{x_m} \left(1 - \frac{6}{x_m - 6}\right) = \xi_\tau (6) \prod_{x_m} \left(1 + \frac{6}{x_m - 6}\right) > 0. \)

From (70) we have

\( \xi_\tau (0) = \xi_\tau (6) \prod_{x_m} \left(\frac{x_m}{x_m - 6}\right) \)

such that

\( \xi_\tau (6) = \xi_\tau (0) \prod_{x_m} \left(\frac{x_m - 6}{x_m}\right) = \xi_\tau (0) \prod_{x_m} \left(1 - \frac{6}{x_m}\right). \)

From (61) and (69) we obtain

\( \xi_\tau (6) = \xi_\tau (0) e^{6b_0} \prod_{x_m} \left(1 - \frac{6}{x_m}\right) \exp \left(\frac{6}{x_m}\right). \)

Combining (72) and (73) gives

\( \xi_\tau (6) = \xi_\tau (0) \prod_{x_m} \left(1 - \frac{6}{x_m}\right) = \left[ e^{6b_0} \prod_{x_m} \exp \left(\frac{6}{x_m}\right) \right] \prod_{x_m} \left(1 - \frac{6}{x_m}\right). \)
which leads to
\[(75)\quad \xi_\tau(0) = e^{6b_0} \prod_{x_m} \exp \left( \frac{6}{x_m} \right).\]

It follows from (75) that
\[(76)\quad e^{6b_0} = \xi_\tau(0) \prod_{x_m} \exp \left( -\frac{6}{x_m} \right),\]

which yields that
\[(77)\quad b_0 = \ln \xi_\tau(0) - 6 \sum \frac{1}{x_m}.\]

Thus, (54) is true under the conditions (66) and (77).

Step three is to prove (B3).

By using (72), we reconsider
\[(78)\quad \xi_\tau(x) = \xi_\tau(6) \prod \left( 1 - \frac{x-6}{x_m-6} \right) = \xi_\tau(0) \prod \left( 1 - \frac{6}{x_m} \right) \prod \left( 1 - \frac{x-6}{x_m-6} \right).\]

Further, (78) becomes
\[
(79) \quad \xi_\tau(x) = \xi_\tau(0) \prod \left( 1 - \frac{6}{x_m} \right) \prod \left( 1 - \frac{x-6}{x_m-6} \right) \\
= \xi_\tau(0) \prod \left( 1 - \frac{6}{x_m} \right) \cdot \prod \frac{x_m - 6 - (x - 6)}{x_m - 6} \\
= \xi_\tau(0) \prod \frac{x_m - 6}{x_m} \cdot \prod \frac{x_m - s}{x_m - 6} \cdot \prod \frac{x_m - s}{x_m - 6} \\
= \xi_\tau(0) \prod \frac{x_m - 6}{x_m} \cdot \prod \frac{x_m - s}{x_m - 6} \cdot \prod \frac{x_m - s}{x_m - 6}. \\

It follows from (79) that
\[(80)\quad \xi_\tau(x) = \xi_\tau(0) \prod \frac{x_m - x}{x_m} = \xi_\tau(0) \prod \left( 1 - \frac{x}{x_m} \right).\]

Clearly, (80) is the same as (49). Thus, this proof is finished. □

By using Proposition 3, the order \( \lambda = 1 \) of \( \xi_\tau(x) \) implies that there is any positive number \( \delta > 0 \) such that \( |x_m|^{-(1+\delta)} < \infty \). Thus, the series \( \sum_{m=1}^{\infty} |x_m|^{-(1+\delta)} \) is convergent.
Remark. With (21) and (49), we establish the relation

\[(81) \quad \xi_\tau(x) = 2 \int_0^1 L(i\omega) \omega^5 \cosh [(x - 6) \ln \omega] d\omega = \xi_\tau(0) \prod_{x_m} \left(1 - \frac{x}{x_m}\right).\]

And,

\[(82) \quad \xi_\tau(1) = 2 \int_0^1 L(i\omega) \omega^5 \cosh [(1 - 6) \ln \omega] d\omega = 2 \int_0^1 L(i\omega) \omega^{10} d\omega = \xi_\tau(0) \prod_{x_m} \left(1 - \frac{5}{x_m}\right) > 0,\]

which implies that

\[(83) \quad \aleph = \sum_{x_m} \frac{1}{x_m} < \infty.\]

Thus,

\[(84) \quad b_0 = \ln \xi_\tau(0) - 6\aleph\]

is a constant.

2.2.2. Two lines of symmetry.

Corollary 4. Suppose \(\varsigma \in \mathbb{A}\) and \(\varsigma \neq x_m\). Then we have the following equivalent representations:

- \(\text{(C1)}\) There is

\[(85) \quad \xi_\tau(x) = \xi_\tau(\varsigma) \prod_{\Im(x_m) > 0} \left[1 - \frac{(x - 6)^2 - (\varsigma - 6)^2}{(x_m - 6)^2 - (\varsigma - 6)^2}\right].\]

- \(\text{(C2)}\) There is

\[(86) \quad \xi_\tau(x) = \xi_\tau(6) \prod_{\Im(x_m) > 0} \left[1 - \left(\frac{x - 6}{x_m - 6}\right)^2\right].\]
Proof. By (49) in Theorem 3, we obtain

\[ \xi_\tau (x) = \xi_\tau (0) \prod_{x_m} \left( 1 - \frac{x}{x_m} \right) \]

\[ = \xi_\tau (0) \prod_{x_m} \left( \frac{x_m - x}{x_m} \right) \]

\[ = \xi_\tau (0) \prod_{x_m} \left( \frac{x_m - \varsigma}{x_m} \cdot \frac{x_m - x}{x_m - \varsigma} \right) \]

\[ (87) \]

\[ = \xi_\tau (0) \prod_{x_m} \frac{x_m - \varsigma}{x_m} \prod_{x_m} \frac{x_m - x}{x_m - \varsigma} \]

\[ = \xi_\tau (0) \prod_{x_m} \left( 1 - \frac{\varsigma}{x_m} \right) \prod_{x_m} \frac{x_m - \varsigma - (x - \varsigma)}{x_m - \varsigma} \]

\[ = \xi_\tau (0) \prod_{x_m} \left( 1 - \frac{\varsigma}{x_m} \right) \prod_{x_m} \left( 1 - \frac{x - \varsigma}{x_m - \varsigma} \right) \]

Because of

\[ \xi_\tau (\varsigma) = \xi_\tau (0) \prod_{x_m} \left( 1 - \frac{\varsigma}{x_m} \right), \]

we obtain from (87) that

\[ (88) \]

\[ \xi_\tau (x) = \xi_\tau (\varsigma) \prod_{x_m} \left( 1 - \frac{x - \varsigma}{x_m - \varsigma} \right). \]

Using the functional equation (13), we reconsider (88) as

\[ (89) \]

\[ \xi_\tau (x) = \xi_\tau (\varsigma) \prod_{x_m > 0} \left( 1 - \frac{x - \varsigma}{x_m - \varsigma} \right) \]

\[ = \xi_\tau (\varsigma) \prod_{\exists (x_m) > 0} \left( 1 - \frac{x - \varsigma}{x_m - \varsigma} \right) \left( 1 - \frac{x - \varsigma}{12 - x_m - \varsigma} \right) \]

\[ = \xi_\tau (\varsigma) \prod_{\exists (x_m) > 0} \frac{(x_m - 6) - (x - 6)}{(x_m - 6) - (\varsigma - 6)} \prod_{\exists (x_m) > 0} \frac{(6 - x_m) - (x - 6)}{(6 - x_m) + (6 - \varsigma)} \]
where
\[
\prod_{\Im(x_m) > 0} \left( 1 - \frac{x - \varsigma}{x_m - \varsigma} \right) = \prod_{\Im(x_m) > 0} \left( \frac{(x - 6) - (\varsigma - 6)}{(x_m - 6) - (\varsigma - 6)} \right) \\
= \prod_{\Im(x_m) > 0} \frac{\Im(x_m - 6) - (\varsigma - 6) - (x - 6) - (\varsigma - 6)}{(x_m - 6) - (\varsigma - 6)} \\
= \prod_{\Im(x_m) > 0} \frac{(x_m - 6) - (x - 6)}{(x_m - 6) - (\varsigma - 6)}
\]

and
\[
\prod_{\Im(x_m) > 0} \left( 1 - \frac{x - \varsigma}{12 - x_m - \varsigma} \right) = \prod_{\Im(x_m) > 0} \left( 1 - \frac{(x - 6) - (\varsigma - 6)}{(6 - x_m) + (6 - \varsigma)} \right) \\
= \prod_{\Im(x_m) > 0} \frac{[6 - x_m] + (6 - \varsigma) - (x - 6) - (\varsigma - 6)}{(6 - x_m) + (6 - \varsigma)} \\
= \prod_{\Im(x_m) > 0} \frac{(6 - x_m) - (x - 6)}{(6 - x_m) + (6 - \varsigma)}.
\]

Further,
\[
\xi_r (x) = \xi_r (\varsigma) \prod_{\Im(x_m) > 0} \left[ \frac{(x_m - 6) - (x - 6)}{(x_m - 6) - (\varsigma - 6)} \cdot \frac{(6 - x_m) - (x - 6)}{(6 - x_m) + (6 - \varsigma)} \right] \\
= \xi_r (\varsigma) \prod_{\Im(x_m) > 0} \left[ \frac{(x_m - 6) - (x - 6)}{(x_m - 6) - (\varsigma - 6)} \cdot \frac{(x_m - 6) + (x - 6)}{(x_m - 6) + (\varsigma - 6)} \right] \\
= \xi_r (\varsigma) \prod_{\Im(x_m) > 0} \frac{(x_m - 6)^2 - (x - 6)^2}{(x_m - 6)^2 - (\varsigma - 6)^2}.
\]

Finally,
\[
\xi_r (x) = \xi_r (\varsigma) \prod_{\Im(x_m) > 0} \frac{(x_m - 6)^2 - (x - 6)^2}{(x_m - 6)^2 - (\varsigma - 6)^2} \\
= \xi_r (\varsigma) \prod_{\Im(x_m) > 0} \left[ \frac{(x_m - 6)^2 - (\varsigma - 6)^2}{(x_m - 6)^2 - (\varsigma - 6)^2} - \frac{(x - 6)^2 - (\varsigma - 6)^2}{(x - 6)^2 - (\varsigma - 6)^2} \right] \\
= \xi_r (\varsigma) \prod_{\Im(x_m) > 0} \left[ 1 - \frac{(x - 6)^2 - (\varsigma - 6)^2}{(x_m - 6)^2 - (\varsigma - 6)^2} \right],
\]
which is the required result.
On the substituting \( \zeta = 6 \) into (93), we obtain
\[
\xi_r (x) = \xi_r (\zeta) \prod_{\Im(x_m) > 0} \left[ 1 - \frac{(x-6)^2}{(x_m-6)^2} \right].
\]
Thus, the proof is finished. \( \square \)

**Remark.** From (C1) in Corollary 4 we discover that \( \xi_r (x) \) has two lines of symmetry as follows: \( x = 6 \) and \( \zeta = 6 \). With (C2) in Corollary 4 we also discover that we can remove the line \( \zeta = 6 \) of symmetry of \( \xi_r (x) \) to obtain the equation (94) by putting \( \zeta = 6 \) into (93).

As a direct result of Corollary 4, we also obtain the following:

**Corollary 5.** There is
\[
\Lambda (\varphi) = \Lambda (0) \prod_{\Re(\varphi_m) > 0} \left( 1 - \frac{\varphi^2}{\varphi_m^2} \right),
\]
where the product runs over all zeros \( \varphi_m \) of \( \Lambda (\varphi) \).

**Proof.** Taking \( x = 6 + i\varphi \) into (86), we have
\[
\Lambda (\varphi) = \xi_r (6 + i\varphi) = \xi_r (6) \prod_{\Im(x_m) > 0} \left[ 1 + \frac{\varphi^2}{(x_m-6)^2} \right].
\]

From (96) all zeros \( \varphi_m \) of \( \Lambda (\varphi) \) are given as follows:
\[
x_m - 6 = \pm i\varphi_m.
\]

Substituting (97) into (96), we show
\[
\Lambda (\varphi) = \xi_r (6) \prod_{\Im(x_m) > 0} \left[ 1 + \frac{\varphi^2}{(x_m-6)^2} \right] = \xi_r (6) \prod_{\Re(\varphi_m) > 0} \left( 1 - \frac{\varphi^2}{\varphi_m^2} \right).
\]

We easily see that the sequence \( \{ \varphi_m \} \) is the set of non-zero zeros of \( \Lambda (\varphi) \) with \( |\varphi_{m+1}| > |\varphi_m| \).

By (95), we obtain the relation
\[
\Lambda (\varphi) = \xi_r (6 + i\varphi),
\]
which leads to
\[
\Lambda (0) = \xi_r (6).
\]
Putting (100) into (98) implies the required result. \( \square \)

3. The Riemann-Hardy hypothesis

3.1. The conjecture of de Bruijn. At first, we have the following:

**Theorem 4.** *(The conjecture of de Bruijn)* All of the zeros of \( X (z) \) lie on the critical line \( \Re (z) = 0 \).
Proof. By Corollary 1 and Theorem 2, we set up a class of \( X(z) \) as follows:

\[
X(z) = \sum_{s=0}^{\infty} \chi(s) z^{2s} = \chi(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{z^2}{z_m^2} \right).
\]

From (28) and (101) we have

\[
X(z) = \sum_{s=0}^{\infty} \chi(s) z^{2s}
\]

such that

\[
\overline{X(z)} = \sum_{s=0}^{\infty} \chi(s) z^{2s} = \chi(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{z^2}{z_m^2} \right).
\]

In which \( \overline{X(z)} \) and \( \overline{z} \) are the complex conjugates of \( X(z) \) and \( z \), respectively.

It follows from (103) that

\[
\overline{X(z)} = X(\overline{z}).
\]

On adopting (101) and (104), we obtain the first product of \( \overline{X(z)} \) as follows:

\[
\overline{X(z)} = X(\overline{z}) = \chi(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{z^2}{z_m^2} \right).
\]

Similarly, by finding the complex conjugate of \( X(z) \) in (101) and using (40), we get

\[
\overline{X(z)} = \chi(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{z_m^2}{z^2} \right).
\]

Combining (103) and (106), we present

\[
\overline{X(z)} = X(\overline{z}) = \chi(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{z^2}{z_m^2} \right) = \chi(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{z_m^2}{z^2} \right).
\]

Taking the complex conjugate of (107), we may get

\[
X(z) = X(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{z^2}{z_m^2} \right) = X(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{z_m^2}{z^2} \right).
\]

By Corollary 1, we know that there exists any positive number \( \varepsilon > 0 \) such that the series

\[
\sum_{m=1}^{\infty} \left| z_m \right|^{-(1+\varepsilon)}
\]

is convergent.
This implies that

\[ \sum_{m=1}^{\infty} |z_m|^{-1+\varepsilon} \]

is convergent.

From (109) and (110),

\[ \sum_{m=1}^{\infty} |z_m|^{-2} \]

and

\[ \sum_{m=1}^{\infty} |\overline{z}_m|^{-2} \]

are convergent.

Putting \( z = 1 \) into (108), we present

\[ X(1) = X(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{1}{z_m} \right) = X(0) \prod_{\Im(z_m) > 0} \left( 1 - \frac{1}{\overline{z}_m} \right). \]

By using the fact (111) and (112) are convergent, Theorem 5 in Knopp’s monograph (see [31], p.10) said that (113) is absolutely convergent. Following Theorem 3 in Knopp’s monograph (see [31], p.10), we see that (113) converges.

Making use of Theorem 3 in Knopp’s monograph (see [31], p.10) and (113), this implies that

\[ \sum_{m=1}^{\infty} \frac{1}{z_m^2} \]

and

\[ \sum_{m=1}^{\infty} \frac{1}{\overline{z}_m^2} \]

are convergent, and

\[ \sum_{m=1}^{\infty} \frac{1}{z_m^2} = \sum_{m=1}^{\infty} \frac{1}{\overline{z}_m^2}. \]

From (116) we have

\[ z_m^2 - \overline{z}_m^2 = 0 \]

such that

\[ (z_m - \overline{z}_m)(z_m + \overline{z}_m) = 0. \]

By using the fact \( z_m - \overline{z}_m = 2\Im(z_m) \neq 0 \), it follows from (118) that

\[ z_m + \overline{z}_m = 2\Re(z_m) = 0. \]
Thus,
\[(120)\] 
\[\Re (z_m) = 0\]
implies that
\[(121)\] 
\[z_m = i \Im (z_m) .\]
Substituting \((121)\) back into \((116)\) leads to the series
\[\sum_{m=1}^{\infty} \frac{1}{z_m^2} = \sum_{m=1}^{\infty} \frac{1}{\Im (z_m)^2}\]
converges.

Applying Theorem 4 in Knopp’s monograph (see [31], p.10), this implies that \((113)\) and \((121)\) converge. Once again, it implies that we go back to verify the truth of \((113)\) and \((121)\).

Let \(|\Im (z_m)| = \gamma_m > 0\). Then we have from Theorem 2 that
\[(122)\] 
\[X (z) = X (0) \prod_{\Im (z_m) > 0} \left( 1 - \frac{z^2}{z_m^2} \right) = X (0) \prod_{m=1}^{\infty} \left( 1 + \frac{z^2}{\gamma_m^2} \right).\]
Thus, we complete the proof. \(\square\)

3.2. The conjecture of Ki. We now present the following result:

**Theorem 5. (The conjecture of Ki)** All of the zeros of \(\Lambda (\varphi)\) lie on the critical line \(\Im (\varphi) = 0\).

**Proof.** By using \((122)\), we have
\[(123)\] 
\[\Lambda (\varphi) = X (i\varphi)\]
such that
\[(124)\] 
\[\Lambda (\varphi) = X (i\varphi) = X (0) \prod_{m=1}^{\infty} \left( 1 - \frac{\varphi^2}{\gamma_m^2} \right).\]
By using \(\Lambda (0) = X (0)\), \((124)\) can be rewritten as
\[(125)\] 
\[\Lambda (\varphi) = \Lambda (0) \prod_{m=1}^{\infty} \left( 1 - \frac{\varphi^2}{\gamma_m^2} \right).\]
From \((124)\) we obtain the required result.

This proof of Theorem 5 is now finished. \(\square\)

3.3. The proof of Theorem 11. We now return to its proof.

Applying \((17)\) and \((121)\), we obtain
\[(126)\] 
\[\xi_{\tau} (x) = \xi_{\tau} (6) \prod_{\Im (z_m) > 0} \left[ 1 - \frac{(x-6)^2}{z_m^2} \right] = \xi_{\tau} (6) \prod_{\Im (z_m) > 0} \left[ 1 + \frac{(x-6)^2}{\Im (z_m)^2} \right].\]
On substituting \(|\Im (z_m)| = \gamma_m > 0\) into \((126)\), this gives
\[(127)\] 
\[\xi_{\tau} (x) = \xi_{\tau} (6) \prod_{\Im (z_m) > 0} \left[ 1 + \frac{(x-6)^2}{\gamma_m^2} \right] = \xi_{\tau} (6) \sum_{m=1}^{\infty} \left[ 1 + \frac{(x-6)^2}{\gamma_m^2} \right].\]
Adopting Theorem 1 in Knopp’s book (see [31], p.9) and taking $\xi_\tau(x) = 0$ in (127),

$$1 + \frac{(x - 6)^2}{\gamma_m^2} = 0,$$

is the root of $\xi_\tau(x) = 0$. Thus,

$$x_m = 6 \pm i\gamma_m,$$

where $\gamma_m > 0$.

From (7) and (129), we deduce that all zeros of $\xi_\tau(x)$ lies on $\Re(x_m) = 6$.

Thus, we complete the proof of Theorem 1.

References

[1] S. Ramanujan, On certain arithmetical functions, Transactions of the Cambridge Philosophical Society, 22 (1916) (9), 159-184.
[2] G. H. Hardy, Note on Ramanujan’s arithmetical function $\tau(n)$, Mathematical Proceedings of the Cambridge Philosophical Society, 23 (1927) (06), 675.
[3] L. J. Mordell, On Mr Ramanujan’s empirical expansions of modular functions, Proceedings of the Cambridge Philosophical Society, 19(1917), 117-124.
[4] W. C. Winnie Li, The Ramanujan conjecture and its applications, Philosophical Transactions of the Royal Society A, 378 (2020)(2163), 20180414.
[5] R. A. Rankin, Contributions to the theory of Ramanujan’s function $\tau(n)$ and similar arithmetical functions: I, the order of the Fourier coefficients of integral modular forms, Mathematical Proceedings of the Cambridge Philosophical Society, 35(1939) (3), pp. 357-372.
[6] M. Rogers, Identities for the Ramanujan zeta function, Advances in Applied Mathematics, 51(2013) (2), 266-275.
[7] B. Heim, M. Neuhauser and F. Rupp, Fourier coefficients of powers of the Dedekind eta function, The Ramanujan Journal, 48 (2019) (1), 1-11.
[8] P. Deligne, La conjecture de Weil. I., Publications Mathématiques de l’Institut des Hautes Études Scientifiques, 43 (1974)(1), 273-307.
[9] J. S. Balakrishnan, W. Craig and K. Ono, Variations of Lehmer’s conjecture for Ramanujan’s tau-function, Journal of Number Theory, 237 (2022), 3-14.
[10] C. J. Moreno, Prime number theorems for the coefficients of modular forms, Bulletin of the American Mathematical Society, 78 (1972) (5), 796-798.
[11] R. A. Rankin, Contributions to the theory of Ramanujan’s function $\tau(n)$ and similar arithmetical functions: I. The zeros of the function on the line, Mathematical Proceedings of the Cambridge Philosophical Society, 35 (1939) (3), 351-356.
[12] J. B. Conrey and A. Ghosh, Turán inequalities and zeros of Dirichlet series associated with certain cusp forms, Transactions of the American Mathematical Society, 342 (1994) (1), 407-419.
[13] J. R. Wilton, A note on Ramanujan’s arithmetical function $\tau(n)$, Mathematical Proceedings of the Cambridge Philosophical Society, 25(1929) (2), 121-129.
[14] B. C. Berndt and M. I. Knopp, Hecke’s theory of modular forms and Dirichlet series, Vol. 5, World Scientific, 2008.
[15] J. B. Conrey and A. Ghosh, Simple zeros of the Ramanujan $\tau$-Dirichlet series, Inventiones mathematicae, 94 (1988) (2), 403-419.
[16] N. G. De Bruijn, The roots of trigonometric integrals, Duke Mathematical Journal, 17 (1950) (3), 197-226.
[17] G. H. Hardy, Ramanujan twelve lectures on subjects suggested by his life and work, Cambridge University Press, Cambridge, 1940.
[18] R. Courant and F. John, Introduction to Calculus and Analysis I, Springer, 2012.
[19] C. G. Lekkerkerker, On the zeros of a class of Dirichlet series, van Gorcum, 1955.
[20] B. C. Berndt, On the zeros of a class of Dirichlet series I, Illinois Journal of Mathematics, 14 (1970) (2), 244-258.
[21] J. L. Hafner, On the zeros of Dirichlet series associated with certain cusp forms, Bulletin of the American Mathematical Society, 8 (1983) (2), 340-342.
[22] H. Ki, On the zeros of approximations of the Ramanujan $\Xi$-function, The Ramanujan Journal, 17 (2008) (1), 123-143.
[23] A. Chirre and O.V. Castañón, A note on the zeros of approximations of the Ramanujan $\Xi$-function, The Ramanujan Journal, 57 (2020), 389-400.
[24] H. R. P. Ferguson, R. D. Major, K. E. Powell and H. G. Throolin, On zeros of mellin transforms of $SL_2(\mathbb{Z})$ cusp forms, Mathematics of Computation, 42 (1984) (165), 241.
[25] J. B. Keiper, On the zeros of the Ramanujan $\tau$-Dirichlet series in the critical strip, Mathematics of Computation, 65 (1996) (216), 1613-1620.
[26] P. Sarnak, Some applications of modular forms, Vol. 99, Cambridge University Press, 1990.
[27] T. M. Apostol and A. Sklar, The approximate functional equation of Hecke’s Dirichlet series, Transactions of the American Mathematical Society, 86 (1957) (2), 446-462.
[28] A. Ogg, Modular forms and Dirichlet series, Vol. 39, WA Benjamin, New York, 1969.
[29] B. Y. Levin, Distribution of zeros of entire functions, Vol. 150, American Mathematical Society, 1980.
[30] E. C. Titchmarsh, The theory of functions, Oxford University Press, 1939.
[31] K. Knopp, Theory of functions, Parts II, Dover Publications, New York, 1947.

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