GLOBAL SOLUTIONS AND RANDOM DYNAMICAL SYSTEMS FOR ROUGH EVOLUTION EQUATIONS

ROBERT HESSE
Friedrich Schiller University Jena
Enst-Abbe-Platz 2, 07743
Jena, Germany

ALEXANDRA NEAMȚU*
Technical University of Munich
Boltzmannstr. 3, 85748
Garching bei München, Germany

(Communicated by María J. Garrido-Atienza)

Abstract. We consider infinite-dimensional parabolic rough evolution equations. Using regularizing properties of analytic semigroups we prove global-in-time existence of solutions and investigate random dynamical systems for such equations.

1. Introduction. In this work we analyze global-in-time existence of solutions for rough stochastic partial differential equations (SPDEs)

\[
\begin{aligned}
  dy_t &= (Ay_t + F(y_t))dt + G(y_t)d\omega_t, \quad t \in [0, T] \\
  y(0) &= \xi.
\end{aligned}
\]

Here \( T > 0 \) is a fixed time-horizon, the linear part \( A \) is the generator of an analytic \( C_0 \)-semigroup \( (S(t))_{t \in [0, T]} \) on a separable Banach space \( W \) and the initial condition \( \xi \in W \). Furthermore we assume that the nonlinearity \( F : W \to W \) is Lipshitz and \( G : W \to \mathcal{L}(V, W) \) is three times continuously Fréchet-differentiable with bounded derivatives. The precise assumptions on the coefficients will be stated in Section 2. Finally, the random input \( \omega \) is a stochastic process which can be lifted to a geometric rough path \([9, 10]\), for instance a fractional Brownian motion with Hurst parameter \( H \in (1/3, 1/2) \). We refer to \([15, 24]\) for examples of such SPDEs. In order to define the mild solution of (1), i.e.

\[
y_t = S(t)\xi + \int_0^t S(t-s)F(y_s)ds + \int_0^t S(t-s)G(y_s)d\omega_s,
\]
we rely on the pathwise construction of the rough integral

\[ \int_0^t S(t-r)G(y_r) d\omega_r, \tag{2} \]

developed in [24]. The main idea is to consider smooth approximations of the noise and justify by a limiting procedure the existence of (2) for rough inputs \( \omega \). The details on this technique are provided in Section 4. This approach implicitly provides a continuous dependence of the solution with respect to the random input, which is a major advantage of the rough paths theory [9, 10]. Similar results in this context are available in [19, 20, 21] using rough paths techniques and [17] using fractional calculus and more recently in [24] using an ansatz which combines these two approaches in a suitable way. As already announced in [24] the ultimate goal is to investigate the long-time behavior of (1) and therefore this work establishes the existence of a pathwise global solution. Consequently, we can show that the solution operator of (1) generates an infinite-dimensional random dynamical system.

Referring to the monograph of Arnold [1], it is well-known that an Itô-type stochastic differential equation generates a random dynamical system under natural assumptions on the coefficients. This fact is based on the flow property, see [28, 37], which can be obtained by Kolmogorov’s theorem about the existence of a (Hölder)-continuous random field with finite-dimensional parameter range, i.e. the parameters of this random field are the time and the non-random initial data.

The generation of a random dynamical system from an Itô-type SPDE has been a long-standing open problem, since Kolmogorov’s theorem breaks down for random fields parametrized by infinite-dimensional Hilbert spaces, see [33]. As a consequence it is not trivial how to obtain a random dynamical system from an SPDE, since its solution is defined almost surely, which contradicts the cocycle property. Particularly, this means that there are exceptional sets which depend on the initial condition and it is not clear how to define a random dynamical system if more than countably many exceptional sets occur. This problem was fully solved only under very restrictive assumptions on the structure of the noise driving the equation. For instance if one deals with purely additive noise or multiplicative Stratonovich one, there are standard transformations which reduce the SPDE in a random partial differential equation. Since this can be solved pathwise it is straightforward to obtain a random dynamical system. However, for nonlinear multiplicative noise, this technique is no longer applicable, not even if the random input is a Brownian motion. As a consequence of this issue, dynamical aspects for (1) such as asymptotic stability, Lyapunov exponents, multiplicative ergodic theorems, random attractors, random invariant manifolds have not been investigated in their full generality.

Consequently, a pathwise construction of (2) and implicitly of the solution of (1) would be the first step to overcome this obstacle. Recently, there has been a growing interest to give a pathwise meaning to the solutions of SPDEs by various techniques, see e.g. [21, 17, 22]. However there are very few results that explore the pathwise character of the solutions to analyze random dynamical systems and their long-time behavior. Progress in this sense was made for instance in [13, 17] that deal with random dynamical systems for SPDEs driven by a fractional Brownian motion with Hurst parameter \( H \in (1/2, 1) \) and \( H \in (1/3, 1/2) \). Local stability statements can be looked up in [12]. Moreover, [11] and [14] prove random attractors respectively random unstable manifolds for SPDEs driven by a fractional Brownian motion.
where the range of the Hurst index is \((1/2, 1)\). All these techniques rely on fractional calculus and require strong assumptions on the coefficients of the SPDEs.

To our best knowledge there are very few works that connect the rough paths- and random dynamical systems perspectives such as [8]. Here we contribute to this aspect and provide a general framework of random dynamical systems for rough evolution equations under natural/less restrictive assumptions on the coefficients. The crucial result that opens the door for the random dynamical systems theory is the existence of a global pathwise solution for (1). It is known that global-in-time existence of solutions is a challenging question in the context of rough paths techniques, compare [17, 21, 24]. This is due to the fact that one obtains certain quadratic estimates on the norms of the solution of (1). Hence it is not straightforward if one can extend the local solution on an arbitrary time horizon. Using additional restrictions on the coefficients and of the noisy input [17] shows global-in-time existence for (1) driven by a fractional Brownian motion with Hurst index \(H \in (1/3, 1/2]\). However, in this work using regularizing properties of analytic \(C_0\)-semigroups, a-priori estimates on certain remainder terms and a standard concatenation procedure we are able to prove using rough paths techniques the global-in-time existence of solutions. Therefore we succeed in closing the gap in [24].

This work is structured as follows. Section 2 collects important auxiliary results concerning parabolic evolution equations and rough paths theory. In Section 3 we present a very general Sewing Lemma (Theorem 6), which entails the construction of the rough integral (2). Under suitable assumptions, we are able to derive additional space-regularity of the integral operator, compare [24, 19], which will turn out to be crucial for the global-in-time existence. For the convenience of the reader Section 4 summarizes basic results regarding the construction of local-in-time solutions for rough evolution equations. We point out that in the context of rough paths the solution is given by a pair containing the path together with its Gubinelli derivative. These two components satisfy suitable algebraic and analytical properties which are precisely summarized and discussed within Section 4. These are the main necessary ingredients required in order to comprehend the techniques employed in Section 5, where we establish the central result of this paper. This opens the door to infinite-dimensional random dynamical systems using a rough path approach. Here we only show the existence of a random dynamical system in Section 6 and aim to investigate its long-time behavior in future works.

### 2. Preliminaries

We let \(T > 0, V\) stand for a Hilbert space and \(W\) denote a separable Banach space. Furthermore, for any compact interval \(J \subset \mathbb{R}\) we set \(\Delta_J := \{(t, s) \in J^2 \colon t \geq s\}\) and \(\Delta_T := \Delta_{[0, T]}\). For notational simplicity, if not further stated, we write \(|\cdot|\) for the norm of an arbitrary Banach space. Furthermore \(C\) denotes a universal constant which varies from line to line. The explicit dependence of \(C\) on certain parameters will be precisely stated, whenever required. Finally, we fix \(\alpha \in (1/3, 1/2)\). This parameter indicates the Hölder-regularity of the random input. Regarding this we recall the following essential concept in the rough path theory.

**Definition 1.** (\(\alpha\)-Hölder rough path) Let \(J \subset \mathbb{R}\) be a compact interval. We call a pair \(\omega := (\omega, \omega^{(2)})\) \(\alpha\)-Hölder rough path if \(\omega \in C^\alpha(J, V)\) and \(\omega^{(2)} \in C^{2\alpha}(\Delta_J, V \otimes V)\). Furthermore \(\omega\) and \(\omega^{(2)}\) are connected via Chen's relation, meaning that

\[
\omega^{(2)}_{ts} - \omega^{(2)}_{us} - \omega^{(2)}_{tu} = (\omega_u - \omega_s) \otimes (\omega_t - \omega_u), \quad \text{for} \ s, u, t \in J, \ s \leq u \leq t.
\]  

(3)

In the literature \(\omega^{(2)}\) is referred to as Lévy-area or second order process.
We further describe an appropriate distance between two $\alpha$-Hölder rough paths.

**Definition 2.** Let $\omega$ and $\bar{\omega}$ be two $\alpha$-Hölder rough paths. We introduce the $\alpha$-Hölder rough path (inhomogeneous) metric

\[
d_{\alpha,J}(\omega, \bar{\omega}) := \sup_{(t,s) \in \Delta_J} \frac{|\omega_t - \omega_s - \bar{\omega}_t + \bar{\omega}_s|}{|t-s|^\alpha} + \sup_{(t,s) \in \Delta_J} \frac{|\omega_t^{(2)} - \bar{\omega}_t^{(2)}|}{|t-s|^{2\alpha}}.
\]

We set $d_{\alpha,T} := d_{\alpha,[0,T]}$.

For more details on this topic consult [9, Chapter 2]. We stress that in our situation we always have that $\omega(0) = 0$ and therefore (4) is a metric. We specify concrete examples in Section 6.

Having stated the random influences that we consider, we now introduce the assumptions on the linear part and on the coefficients $F$ and $G$.

Since we are in the parabolic setting, i.e. $A$ is a sectorial operator, we can introduce its fractional powers, $(-A)^\gamma$ for $\gamma \geq 0$, see [35, Section 2.6] or [30]. We denote the domains of the fractional powers of $(-A)$ with $D_\gamma$, i.e. $D_\gamma := D((-A)^\gamma)$ ($D_0 = W$) and use the following estimates.

For $\eta, \kappa \in \mathbb{R}$ we have

\[
|S(t)||_{\mathcal{L}(D_\eta, D_\kappa)} = |((-A)^\sigma S(t)||_{\mathcal{L}(D_\eta, W)} \leq Ct^{k-\eta}, \text{ for } \eta \geq \kappa
\]

\[
|S(t) - \text{Id}||_{\mathcal{L}(D_\eta, D_\lambda)} \leq C \eta^{\sigma-\lambda}, \text{ for } \sigma - \lambda \in [0,1].
\]

Furthermore, one can show that the following assertions hold true, consult [35, Chapter 3].

**Lemma 3.** For any $\nu, \eta, \mu \in [0,1]$, $\kappa, \gamma, \rho \geq 0$ such that $\kappa \leq \gamma + \mu$, there exists a constant $C > 0$ such that for $0 < q < r < s < t$ we have that

\[
|S(t) - S(t - r)|_{\mathcal{L}(D_\kappa, D_\gamma)} \leq C(t - r)^\mu(t - r)^{-\mu - \gamma + \kappa},
\]

\[
|S(t) - S(s - r) - S(t - q) + S(s - q)|_{\mathcal{L}(D_\kappa, D_\gamma)} \leq C(t - s)^\nu(r - s)^{\nu + \eta}.
\]

For our aims we introduce following function spaces. Let $\beta \in (0,1)$ be fixed and let $\mathcal{W}$ stand for a further Hilbert space. We recall that $C^\beta([0,T], \mathcal{W})$ represents the space of $W$-valued Hölder continuous functions on $[0,T]$ and denote by $C_\alpha(\Delta_T, \mathcal{W})$ the space of $\mathcal{W}$-valued functions on $\Delta_T$ with $z_{tt} = 0$ for all $t \in [0,T]$ and

\[
||z||_\alpha := \sup_{0 \leq t \leq T} |z_0| + \sup_{0 \leq s < t \leq T} \frac{|z_s - z_t|}{(t-s)^\alpha} < \infty.
\]

Furthermore, we define $C^{\beta,\beta}_\alpha([0,T], \mathcal{W})$ as the space of $W$-valued continuous functions on $[0,T]$ endowed with the norm

\[
||y||_{\beta,\beta} := ||y||_\infty + ||y||_{\beta,\beta} := \sup_{0 \leq t \leq T} |y_t| + \sup_{0 < s < t \leq T} s^{\beta} \frac{|y_t - y_s|}{(t-s)^{\beta}}.
\]

Similarly we introduce $C^{\alpha+\beta,\beta}(\Delta_T, \mathcal{W})$ with the norm

\[
||z||_{\alpha+\beta,\beta} := \sup_{0 \leq t \leq T} |z_0| + \sup_{0 < s < t \leq T} s^{\beta} \frac{|z_t - z_s|}{(t-s)^{\alpha+\beta}}.
\]

Again $z_{tt} = 0$ for all $t \in [0,T]$.

These modified Hölder spaces are well-known in the theory of maximal regularity for parabolic evolution equations, consult [30]. These were also used in [17].
In this framework we emphasize the following result which will be employed throughout this work. It is well-known that analytic $C_0$-semigroups are not Hölder continuous in 0. However, the following lemma holds true.

**Lemma 4.** Let $(S(t))_{t \geq 0}$ be an analytic $C_0$-semigroup on $W$. Then we have for all $x \in W$ and all $\beta \in [0, 1]$ that

$$\|S(\cdot)x\|_{\beta,\beta} \leq C|x|,$$

where $C$ depends only on the semigroup and on $\beta$.

**Proof.**

$$\|S(\cdot)x\|_{\beta,\beta} = \sup_{0 \leq t \leq T} |S(t)x| + \sup_{0 < s < t \leq T} s^\beta \frac{|(S(t) - S(s))x|}{(t-s)^\beta} \leq \sup_{0 \leq t \leq T} |S(t)x| + \sup_{0 < s < t \leq T} s^\beta \frac{|(S(t) - S(s))x|}{(t-s)^\beta} \leq C|x|,$$

recall (5) and (6).

This justifies our choice of working with the function space $C^{\beta,\beta}$. Note that if one lets $x \in D_\beta$ it suffices to consider only $C^\beta$. However, since we analyze random dynamical systems generated by (1) in $W$ (compare Section 6), we need to take the initial condition $\xi \in W$ instead of $D_\beta$.

On the coefficients we impose:

(F): $F: W \to W$ is Lipschitz continuous.

(G): $G: W \to L(V,D_\beta)$ is bounded and three times Fréchet differentiable with bounded derivatives. Here we demand $\alpha + 2\beta > 1$.

**Remark 5.**

1) As in [24] we set $F \equiv 0$ for simplicity, since this term does not cause additional technical difficulties.

2) To our best knowledge (F) and (G) are the most general assumptions made on the coefficients of the SPDE (1), compare [17, 21] and the references specified therein.

Finally, we fix some important notations from the rough paths theory, see also [21, 6] and random dynamical systems which will be required later on.

**Notations.** For $y \in C([0,T],W)$ and $z \in C(\Delta T, W)$ we set

$$(\delta y)_{ts} := y_t - y_s,$$

$$(\hat{\delta}y)_{ts} := y_t - S(t-s)y_s,$$

$$(\delta z)_{t\tau s} := z_{ts} - z_{t\tau} - z_{\tau s},$$

$$(\hat{\delta}z)_{t\tau s} := z_{ts} - z_{t\tau} - S(t-\tau)z_{\tau s}.$$ 

Furthermore we use the notation $\tilde{\theta}$ in order to indicate the usual shift, namely

$$\tilde{\theta}_\tau y_t := y_{t+\tau},$$

$$\tilde{\theta}_{t\tau} z_{ts} := z_{t+\tau,s+t}.$$ 

The notation $\theta$ always stands for the Wiener shift (this represents an appropriate shift with respect to the noise), more precisely

$$\theta_\tau \omega_t := \omega_{t+\tau} - \omega_\tau,$$
which is explained in detail in Section 33. This is mainly required in the random dynamical systems theory.

3. Sewing Lemma revised. In this section we collect concepts from the rough paths theory \[20, 21\] and recall some important results regarding the construction and properties of (2). For further details and complete proofs of the following statements, consult \[24, Section 4\]. A key point in this framework is given by the Sewing Lemma \[21\]. This ensures the existence of a rough integral under suitable assumptions. Here we use a special case of the Sewing Lemma proved in \[24\]. Based on this we develop a more general statement which is crucial for Section 5.

**Theorem 6** (Sewing Lemma, Theorem 4.1 \[24\]). Let $W$ be a separable Banach space and $(S(t))_{t \geq 0}$ be an analytic $C_0$-semigroup on $W$. Furthermore, let $\Xi \in C(\Delta_T, W)$ be an approximation term satisfying the following properties for all $0 \leq u \leq m \leq v \leq T$:

\[
|\Xi_{vu}| \leq c_1 (v - u)\alpha, \quad (7)
\]

\[
\left|\hat{\delta}_2 \Xi\right|_{vu} \leq c_2 (v - u)\rho. \quad (8)
\]

Here we impose $0 \leq \alpha \leq 1$ and $\rho > 1$.

Then there exists a unique $\mathcal{I}\Xi \in C([0, T], W)$, such that

\[
\mathcal{I}\Xi_0 = 0,
\]

\[
\left|\hat{\delta}_2 \mathcal{I}\Xi\right|_{ts} \leq C (c_1 + c_2) (t - s)^\alpha \quad (9)
\]

\[
|\mathcal{I}\Xi|_{ts} - \Xi|_{ts} \leq Cc_2 (t - s)^\rho. \quad (10)
\]

In order to interpret $\mathcal{I}\Xi$ as a rough integral it is crucial that this fulfills integral-like properties, namely it has to be given by a limit of finite sums and satisfy a shift property. These have been rigorously verified in \[24, Section 4\].

**Corollary 1** (Approximation by finite sums, Corollary 4.3 \[24\]). Under the assumptions of Theorem 6 it holds that

\[
\left(\hat{\delta}_2 \mathcal{I}\Xi\right)_{ts} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} S(t - v)\Xi_{vu}, \quad (12)
\]

where $|\mathcal{P}|$ stands for the mesh of the given partition $\mathcal{P} = \mathcal{P}(s,t)$.

**Remark 7.** Corollary 1 implies the additivity of the rough integral.

In order to introduce the shift property of the rough integral $\mathcal{I}\Xi$ we recall that for $\tau > 0$

\[
\tilde{\theta}_\tau \Xi|_{vu} = \Xi|_{u+\tau,u+\tau},
\]

see Section 2. Considering this, one can easily verify the shift property of $\mathcal{I}\Xi$.

**Lemma 8** (Shift property, Corollary 4.5 \[24\]). Under the assumptions of Theorem 6 we have

\[
(\hat{\delta}_2 \mathcal{I}\Xi)_{ts} = (\hat{\delta}_2 \tilde{\theta}_\tau \Xi)_{t-\tau,s-\tau}, \text{ for } \tau \leq s \leq t.
\]

In order to obtain a global solution for (1) we have to precisely analyze the spatial regularity of $\mathcal{I}\Xi$. To this aim we formulate the main result of this section.
Corollary 2. Additionally to the restrictions of Theorem 6 we further assume that
\[ |S(v - u)\Xi_{vu}|_{D_x} \leq c'_1 (v - u)^{\alpha'} \]
where \( 0 \leq \alpha' \leq 1 \) and \( 0 \leq \varepsilon < 1 \). Then we have
\[ |(\hat{\delta}I\Xi)_{ts}|_{D_x} \leq C \left( c'_1 (t - s)^{\alpha'} + c_2 (t - s)^{\rho - \varepsilon} \right). \]

Proof. The computation is similar to [24, Corollary 4.6]. We define \( P_n \) as the \( n \)-th dyadic partition of \([s,t]\) and set
\[
N^n_{ts} := \sum_{[u,v] \in P_n} S(t - v)\Xi_{vu}, \\
\hat{N}^n_{ts} := \sum_{[u,v] \in P_n} S(t - v)\Xi_{vu}, \\
\tau_n := \max \{ v < t : [u,v] \in P_n \}.
\]

By standard computation we get
\[
\hat{N}^n_{ts} - \hat{N}^{n+1}_{ts} = \sum_{[u,v] \in P_n \setminus \{ t \}} S(t - v)(\hat{\delta_2} \Xi)_{vmu} - S(t - \tau_{n+1})\Xi_{\tau_{n+1} \tau_n}.
\]

Hence we obtain
\[
\left| \hat{N}^n_{ts} - \hat{N}^{n+1}_{ts} \right|_{D_x} \leq \left| S(t - \tau_{n+1})\Xi_{\tau_{n+1} \tau_n} \right|_{D_x} + \sum_{[u,v] \in P_n \setminus \{ t \}} \left| S(t - v)(\hat{\delta_2} \Xi)_{vmu} \right|_{D_x}.
\]

Note that \( t - \tau_{n+1} = \tau_{n+1} - \tau_n \) since we consider a dyadic partition. Regarding our assumptions, this further results in
\[
\left| \hat{N}^n_{ts} - \hat{N}^{n+1}_{ts} \right|_{D_x} \leq c'_1 (\tau_{n+1} - \tau_n)^{\alpha'} + Cc_2 \sum_{[u,v] \in P_n \setminus \{ t \}} (t - v)^{-\varepsilon} (v - u)^\rho \\
\leq C \left( c'_1 (t - s)^{\alpha'} 2^{-n\alpha'} + c_2 (t - s)^{\rho - \varepsilon} 2^{-n(\rho - 1)} \right).
\]

Consequently, the previous expression is summable and yields that \( \hat{N}^n_{ts} \to \hat{N}_{ts} \) (in \( D_x \)) for all \( 0 \leq s < t \leq T \). Corollary 1 entails
\[
\left| (\hat{\delta}I\Xi)_{ts} - \hat{N}_{ts} \right| = \lim_{n \to \infty} \left| N^n_{ts} - \hat{N}^n_{ts} \right| = \lim_{n \to \infty} |\Xi_{\tau_n}| \leq c_1 \lim_{n \to \infty} (t - \tau_{n+1})^\alpha = 0.
\]

Hence, we know that \( \hat{N} \equiv (\hat{\delta}I\Xi) \) and finally infer that
\[
\left| (\hat{\delta}I\Xi)_{ts} \right|_{D_x} = \left| \hat{N}_{ts} \right|_{D_x} \leq C \left( c'_1 (t - s)^{\alpha'} + c_2 (t - s)^{\rho - \varepsilon} \right).
\]

This proves the statement. \( \square \)

As already emphasized the spacial regularity of \( I\Xi \) is essential for the computation in Section 5.
4. Solution theory for rough SPDEs. For a better comprehension of Section 5 we point out certain results regarding the existence and uniqueness of a local solution for (1). Here we consider another approach than in [24, 21] which finally enables us to obtain a global-in-time solution.

Remark 9. In this setting $V \otimes V$ denotes the usual tensor product of Hilbert spaces. If one wishes to work in Banach spaces, then one should consider the projective tensor product, since the property

$$\mathcal{L}(V, \mathcal{L}(V, W)) \hookrightarrow \mathcal{L}(V \otimes V, W)$$

is required. This is known to hold true, consult Theorem 2.9 in [36]. In the following, for notational simplicity we drop the tensor symbol.

We firstly indicate a heuristic computation which is required in order to construct the rough integral (2) and therefore to give a pathwise meaning to the solution of (1). These deliberations are rigorously justified in [24], which essentially combines the techniques in [21] and [17]. Here we only want to provide the general intuition of how the solution of (1) should look like and focus on its global existence.

The strategy to define (2) relies on an approximation procedure. We firstly consider a smooth path $\omega$ and a continuous trajectory $y$. The general argument eventually follows considering smooth approximations of $\omega$, as shortly indicated below. Our aim is to define (2) using Riemann-Stieltjes sums and a first-order Taylor expansion for $G$. A formal computation entails the following approximation:

$$\int_{0}^{t} S(t-r)G(y_{r}) d\omega_{r} = \sum_{[u,v] \in P} S(t-v) \int_{u}^{v} S(v-r)G(y_{r}) d\omega_{r}$$

$$\approx \sum_{[u,v] \in P} S(t-v) \left[ \int_{u}^{v} S(v-r)G(y_{u}) d\omega_{r} + \int_{u}^{v} S(v-r)DG(y_{u})(y_{r} - y_{u}) d\omega_{r} \right]$$

$$= : \sum_{[u,v] \in P} S(t-v) \left[ \omega_{vu}^{S}(G(y_{u})) + z_{vu}(DG(y_{u})) \right].$$

Here we introduced the notation

$$\omega_{vu}^{S}(G(y_{u})) := \int_{u}^{v} S(v-r)G(y_{u}) d\omega_{r}, \quad (13)$$

respectively

$$z_{vu}(DG(y_{u})) := \int_{u}^{v} S(v-r)DG(y_{u})(\delta y)_{r, u} d\omega_{r}. \quad (14)$$

By a classical integration by parts formula, see Theorem 3.5 in [35] one can argue that the term $\omega_{vu}^{S}$ can be defined for a rough input $\omega$. However, we have to continue our deliberations to obtain a meaningful definition of $z$. To this aim we let $E \in \mathcal{L}(W; \mathcal{L}(V; W))$ denote a placeholder which stands for $DG$ and consider further Riemann-Stieltjes sums for (14). Namely, for a partition $P = \mathcal{P}([s, t])$ we have

$$z_{ts}(E) = \int_{s}^{t} S(t-r)E(y_{r} - y_{s}) d\omega_{r} = \sum_{[u,v] \in P} S(t-v) \int_{u}^{v} S(v-r)E(y_{r} - y_{s}) d\omega_{r}$$
where in the second step we subtract the expression $S(r - u) y_u$. Regarding this we make the following ansatz for the first term of the previous expression. Since $y$ is supposed to solve (1), this should satisfy the variation of constants formula

$$y_r - S(r - u) y_u = \int_u^r S(r - q) G(y_q) d\omega_q.$$  

Plugging this into the expression of $z$, we immediately obtain

$$z_{ts}(E) =: \sum_{[u,v] \in P} S(t - v) \left[ \int_u^v S(v - r) E \int_u^r S(r - q) G(y_q) d\omega_q d\omega_r + a_{vu}(E, y_u) \right]$$

$$- \omega^S_{ts}(Ey_s)$$

$$\approx \sum_{[u,v] \in P} S(t - v) \left[ \int_u^v S(v - r) E \int_u^r S(r - q) G(y_u) d\omega_q d\omega_r + a_{vu}(E, y_u) \right]$$

$$- \omega^S_{ts}(Ey_s)$$

$$=: \sum_{[u,v] \in P} S(t - v) \left[ b_{vu}(E, G(y_u)) + a_{vu}(E, y_u) \right] - \omega^S_{ts}(Ey_s).$$

For simplicity we introduced the notation

$$b_{vu}(E, G(y_u)) := \int_u^v S(v - r) E \int_u^r S(r - q) G(y_u) d\omega_q d\omega_r,$$

respectively

$$a_{vu}(E, y_u) := \int_u^v S(v - r) ES(r - u) y_u d\omega_r.$$

This indicates that we have to define $a$, $b$ and $\omega^S$ in order to fully characterize $z$. At the very first sight, it is not straightforward how to introduce $b$. Therefore we continue our heuristic computation, still for a smooth path $\omega$. We let $K$ denote a placeholder which stands for $G$. Again, using Riemann-Stieltjes sums and a suitable approximation of certain terms below we infer that

$$\int_a^z S(t - r) E \int_a^v S(r - q) K d\omega_q d\omega_r = \sum_{[u,v] \in P} S(t - v) \int_u^v S(v - r) E \int_u^r S(r - q) K d\omega_q d\omega_r$$

$$+ \sum_{[u,v] \in P} S(t - v) \int_u^v S(v - r) E \int_u^r S(r - q) K d\omega_q d\omega_r$$

$$+ \sum_{[u,v] \in P} S(t - v) \int_u^v S(v - r) E \int_u^r S(r - q) K d\omega_q d\omega_r.$$
\[ \sum_{[u,v] \in P} S(t - v) \int_{u}^{v} S(v - r) E \int_{u}^{r} S(u - q) K d\omega_q d\omega_r \]

\[ + \sum_{[u,v] \in P} S(t - v) \int_{u}^{v} S(v - r) E \int_{u}^{r} K d\omega_q d\omega_r \]

\[ =: \sum_{[u,v] \in P} S(t - v) \left[ \omega^S_{uv}(E\omega^S_{uv}(K)) + c_{uv}(E,K) \right]. \]

Here

\[ c_{ts}(E,K) := \int_{s}^{t} S(t - r) EK(\omega_r - \omega_s) d\omega_r. \]

Motivated by this heuristic computation, one can introduce similar to [15, 24] these processes for smooth paths \((\omega^n, \omega^{(2),n})\) approximating \((\omega, \omega^{(2)})\) in the \(d_{\alpha,T}\)-metric. Here

\[ \omega^{(2),n}_{ts} := \int_{s}^{t} (\delta\omega^n)_{rs} \otimes d\omega^n_r. \]

Thereafter, the passage to the limit entails a suitable construction/interpretation of all these expressions according to [24, Section 5]. More precisely, the following results hold true.

**Lemma 10.** We have that

\[ \omega^{S,n}_{ts} \to \omega^S_{ts} \text{ in } C^\alpha([0,T], \mathcal{L}(\mathcal{L}(V,W), W)) \] (15)

\[ a^n \to a \text{ in } C^\alpha([0,T], \mathcal{L}(\mathcal{L}(W \otimes V,W) \times W,W)) \] (16)

\[ c^n \to c \text{ in } C^{2\alpha}([0,T], \mathcal{L}(\mathcal{L}(W \otimes V,W) \times \mathcal{L}(V,W), W)) \] (17)

\[ b^n \to b \text{ in } C^{2\alpha}([0,T], \mathcal{L}(\mathcal{L}(W \otimes V,W) \times \mathcal{L}(V,D_\beta), W)). \] (18)

We collect further results which are essential for the computation in Section 5.

**Lemma 11.** Let \(K \in \mathcal{L}(V,W), E \in \mathcal{L}(W \otimes V,W)\) and \((\omega, \omega^{(2)})\) be an \(\alpha\)-Hölder rough path. Then \(\omega^S, a\) and \(c\) can be defined using integration by parts as

\[ \omega^S_{ts}(K) = S(t - s)K(\delta\omega)_{ts} - A \int_{s}^{t} S(t - r)K(\delta\omega)_{tr} dr, \] (19)

\[ a_{ts}(E,x) = \omega^S_{ts}(Ex) + \int_{s}^{t} \omega^S_{tr}(EAS(r - s)x)dr, \] (20)

\[ c_{ts}(E,K) = \omega^S_{ts}(EK(\delta\omega)_{ts}) - S(t - s)EK \omega^{(2)}_{ts} - \int_{s}^{t} AS(t - r)EK \omega^{(2)}_{tr} dr. \] (21)

As precisely stated in [24, Section 5] these processes satisfy important analytic and algebraic properties which perfectly fit in the rough path framework. We shortly indicate them together with a generalization which will be required in Section 5.
Lemma 12 (Analytic properties, Lemma 5.4 and Lemma 5.11 [24]). The following analytic estimates hold true:

\[ |\omega^S_t(K)| \leq C \|\omega\|_{\alpha} |K|(t-s)^\alpha, \]

(22)

\[ |a_{ts}(E,x)| \leq C \|\omega\|_{\alpha} |E| |x| W(t-s)^\alpha, \text{ for } x \in W, \]

(23)

\[ |a_{ts}(E,x) - \omega^S_t(Ex)| \leq C \|\omega\|_{\alpha} |E| |x|_{D_\gamma} (t-s)^{\alpha+\gamma}, \text{ for } x \in D_\gamma \text{ and } 0 < \gamma \leq 1, \]

(24)

\[ |c_{ts}(E,K)| \leq C \left( \|\omega\|_{\alpha} + \|\omega^{(2)}\|_{2\alpha} \right) |K| (t-s)^{2\alpha}, \]

(25)

\[ |b_{ts}(E,K)| \leq C |E| |K|_{\mathcal{L}(V,D_\gamma)} \left( \|\omega\|_{\alpha}^2 + \|\omega^{(2)}\|_{2\alpha}^2 \right) (t-s)^{2\alpha}. \]

(26)

The next statement gives an extension of (22).

**Lemma 13.** Let \( K \in \mathcal{L}(V,D_\gamma) \) for \( 0 \leq \gamma \leq 1 \). Then

\[ |\omega^S_t(K)|_{D_\gamma} \leq C \|\omega\|_{\alpha} |K|_{\mathcal{L}(V,D_\gamma)} (t-s)^\alpha. \]

(27)

**Proof.** The proof can immediately be derived using (19), see Lemma 5.4 in [24] for a detailed computation.

\[ \square \]

**Lemma 14** (Algebraic properties, Lemma 5.3 and Lemma 5.11 [24]). The algebraic relations are satisfied:

\[ \langle \delta_2 \omega^S \rangle_{tr}(E,K) = 0, \]

(28)

\[ \langle \delta_2 a \rangle_{tr}(E,x) = a_{tr}(E,(S(\tau-s)-Id)x), \]

(29)

\[ \langle \delta_2 \lambda \rangle_{tr}(E,K) = \omega^S_t(EK(\delta\omega)_{tr}), \]

(30)

\[ \langle \delta_2 b \rangle_{tr}(E,K) = a_{tr}(E,\omega^S_t(K)). \]

(31)

Putting all arguments from our heuristic computation together, we immediately observe that in order to define (2), the abstract approximation term \( \Xi \) in Theorem 6 is given here by

\[ \Xi_{vu}^{(y)} = \Xi_{vu}^{(y)}(y,z) = \omega^S_{vu}(G(y_u)) + z_{vu}(DG(y_u)). \]

Verifying (7) and (8) the existence of the rough integral \( \mathcal{I}\Xi^{(y)} \) is obtained. The same holds true for

\[ \Xi^{(z)}(y,y_{vu}(E) = b_{vu}(E,G(y_u)) + a_{vu}(E,y_u). \]

Regarding this one concludes that the solution of (1) has the structure

\[ y_t = S(t)x + \mathcal{I}\Xi^{(y)}(y,z)_t \]

\[ = S(t)x + \lim_{|\mathcal{P}(0,t)| \to 0} \sum_{[u,v] \in \mathcal{P}(0,t)} S(t-v)[\omega^S_{vu}(G(y_u)) + z_{vu}(DG(y_u))] \]

\[ z_{ts}(E) = (\delta\mathcal{I}\Xi^{(z)}(y,y))_{ts}(E) - \omega^S_{ts}(Ey_v) \]

\[ = \lim_{|\mathcal{P}(s,t)| \to 0} \sum_{[u,v] \in \mathcal{P}(s,t)} S(t-v)[b_{vu}(E,G(y_u)) + a_{vu}(E,y_u)] - \omega^S_{ts}(Ey_v). \]

(33)

One can show that (1) has a unique local-in-time solution in a suitable function space which incorporates the algebraic and analytic properties of the pair \((y,z)\). For
satisfies the estimates that the mapping \( r > 1 \) further.

For a pair \((y, z)\) where \((y_t)_{t \in [0, T]}\) is a \(W\)-valued path and the area term \((z_{ts})_{(t,s) \in \Delta_T}, z_{ts} \in L(\mathcal{L}(W \otimes V, W), W)\), we consider the function space \( X_{\omega, T} := \{(y, z) : y \in C^{\beta, \beta}([0, T], W), z \in C^\alpha(\Delta_T, \mathcal{L}(W \otimes V, W)), \right\} \), where \( z \in C^\alpha(\Delta_T, \mathcal{L}(W \otimes V, W)) \cap C^{\alpha+\beta, \beta}(\Delta_T, \mathcal{L}(W \otimes V, W))\). Moreover, this space is endowed with norm
\[
\|(y, z)\|_X := \|y\|_{\infty} + \|y\|_{\beta, \beta} + \sup_{0 \leq s, t \leq T} \frac{|z_{ts}|}{(t-s)^{\alpha}} + \sup_{0 < s, t \leq T} s^\beta \frac{|z_{ts}|}{(t-s)^{\alpha+\beta}}.
\] (34)

Since we also have to incorporate the initial condition \(\xi\) we introduce the mapping
\[
\mathcal{M}_{T, \omega, \xi} : X_{\omega, T} \to X_{\omega, T}, \quad \mathcal{M}_{T, \omega, \xi}(y, z) = (\tilde{y}, \tilde{z}),
\]
with
\[
\tilde{y}_t := S(t)\xi + \int\Xi(y, z)_{ts}, \quad \tilde{z}_{ts}(E) := (\delta I \Xi^{(z)}(y, \tilde{y}))_{ts}(E) - \omega^S_{ts}(E\tilde{y}_s),
\]
where
\[
\Xi(y)_{vu} = \Xi(y, z)_{vu} := \omega^S_{vu}(G(y_u)) + z_{vu}(DG(y_u)), \quad \Xi^{(z)}(y, \tilde{y})_{vu}(E) := b_{vu}(E, G(y_u)) + a_{vu}(E, \tilde{y}_u).
\]

The reason why we choose to work with these modified Hölder spaces is given by the fact that the analytic \(C_0\)-semigroup \((S(t))_{t \in [0, T]}\) is not Hölder-continuous in 0 but belongs to \(C^{\beta, \beta}\), recall Lemma 4.

In this setting one has the following existence result for (1). For further details and a complete proof, consult [24, Section 6].

**Theorem 15.** (Theorem 6.9 [24]) Let \(\frac{1}{3} < \alpha \leq \frac{1}{2}\) and \(0 < \beta < \alpha\) with \(\alpha + 2\beta > 1\). Furthermore, choose \(r > 1\) with \(|\xi| \leq r\). Then there exist a \(T = T(\omega, r) > 0\) such that the mapping \(\mathcal{M}_{T, \omega, \xi}\) has a unique fixed-point \((y, z) \in X_{\omega, T}\). Moreover, this satisfies the estimates
\[
\|y_T\|_{D_\alpha} \leq C \left( r T^{-\beta} + \|(y, z)\|_X^2 T^{\alpha-\beta} \right), \quad (35)
\]
\[
\|\mathcal{M}_T(y, z)\|_X \leq C \left( r + 1 + \|(y, z)\|_X^2 \right) T^{\alpha}.
\] (36)

If \(\xi \in D_\beta\), then \(y \in C^{\beta}([0, T], W)\) and \(z \in C^{\alpha+\beta}(\Delta_T, \mathcal{L}(W \otimes V, W), W)\).

Furthermore we provide some results regarding the fixed-points of \(\mathcal{M}_{T, \omega, \xi}\) which can be obtained by straightforward computations.

**Remark 16.** If \((y, z)\) is a fixed-point of \(\mathcal{M}_{T, \omega, \xi}\) than for any \(\tilde{T} < T\) the restriction of \((y, z)\) on \([0, \tilde{T}] \times \Delta_{\tilde{T}}\) is a fixed-point of \(\mathcal{M}_{\tilde{T}, \omega, \xi}\).

Regarding Lemma 10 we immediately obtain.

**Lemma 17.** The fixed-point of \(\mathcal{M}_{T, \omega, \xi}\) continuously depends on the initial data and on the random input.
We point out the following essential facts for the random dynamical systems theory, see Section 6.

**Remark 18.** Note that if \((\omega, \omega^{(2)})\) is an \(\alpha\)-Hölder rough path, then the shift \((\theta_\tau \omega, \tilde{\theta}_\tau \omega^{(2)})\) is again an \(\alpha\)-Hölder rough path. The proof is conducted in Lemma 32. From this fact one can infer that the \(\tilde{\theta}_\tau\)-shift of all the supporting processes depends on the shifts of the random input, i.e. \(\theta_\tau \omega\) and \(\tilde{\theta}_\tau \omega^{(2)}\).

Keeping this in mind we state:

**Lemma 19.** Let \(T > 0\) and \((y, z) \in X_{\omega,T}\) be a fixed-point of \(\mathcal{M}_{T, \omega, \xi}\). Then for any \(\tau \in [0, T)\) there exists a fixed-point of \(\mathcal{M}_{T-\tau, \theta_\tau \omega, y}\) given by \((\tilde{\theta}_\tau y, \tilde{\theta}_\tau z)\).

**Proof.** The proof follows the lines of Lemma 6.11 in [24]. We use the notation \(\Xi_{(y/z)} \omega\) and \(\Xi_{(y/z)} \theta_\tau \omega\) in order to emphasize the shifts with respect to \(\omega\). By standard computations we get

\[
\begin{align*}
\tilde{\theta}_\tau y_t &= y_{t+\tau} = S(t+\tau) \xi + \tilde{\Theta}_t(y, z)_{t+\tau} \\
&= S(t) y_t + (\delta \tilde{\Theta}_t(y, z))_{t+\tau} \\
&= S(t) y_t + \tilde{\Theta}_t(y, \tilde{\theta}_\tau z)_t.
\end{align*}
\]

Furthermore,

\[
\begin{align*}
\tilde{\theta}_\tau z_{ts}(E) &= z_{t+s, t+s+\tau}(E) = (\delta \tilde{\Theta}_t(y, y))_{t+s, t+s+\tau}(E) - \omega_S^{s+t+s+\tau}(E_{y_{t+s}}) \\
&= (\delta \tilde{\Theta}_t(y, \tilde{\theta}_\tau y))_{t+s}(E) - \tilde{\theta}_\tau \omega_S^{s+t+s+\tau}(E_{\tilde{\theta}_\tau y}).
\end{align*}
\]

The deliberations conducted in Section 5 improve these results.

5. **Construction of the global-in-time solution.** As recalled in the previous section, working with (34) leads to quadratic estimates for the norm of \((y, z)\) in \(X_{\omega,T}\). From this approach it is not clear how/if one can extend the unique local solution on an arbitrary time horizon. Therefore we need different arguments for the global-in-time existence. To this aim, similar to the finite-dimensional case, see [9, Section 8.5], it is convenient to work with the norm of certain remainder terms, which is common in the rough paths theory.

Keeping Theorem 15 in mind, we recall that \(0 < \beta < \alpha\) and \(\alpha + 2\beta > 1\).

**Definition 20.** Let \((y, z) \in X_{\omega,T}\). Then we define the remainders

\[
\begin{align*}
R^Y_{ts} := (\delta y)_{ts} - \omega_S^y(G(y_s)), \\
R^Z_{ts}(E) := z_{ts}(E) - b_{ts}(E, G(y_s)).
\end{align*}
\]

**Remark 21.** If \(S = \text{Id}\) and \((y, z)\) is a fixed-point of \(\mathcal{M}\), then the previous terms read as

\[
R^Y_{ts} = (\delta y)_{ts} - G(y_s)(\delta \omega)_{ts},
\]

respectively

\[
R^Z_{ts}(E) = E \int_s^t R^Y_{rs} d\omega_r.
\]

The deliberations conducted in Section 5 improve these results.
The expression for the remainder $R^Y$ is the same as the one in the finite-dimensional case, compare [9, Section 8.5]. In contrast to the finite-dimensional setting, $R^Z$ is required here to estimate the quadratic terms appearing in (36).

**Definition 22.** Let $(y, z) \in X_{\omega, T}$ such that
\[
\Phi_T(y, z) := \|y\|_{\infty, D_{2\beta}, T} + \|R^Y\|_{2\beta, T} + \|R^Z\|_{\alpha+2\beta, T} < \infty, \quad (37)
\]
where
\[
\|y\|_{\infty, D_{2\beta}, T} := \sup_{0 \leq t \leq T} \|y_t\|_{D_{2\beta}}, \\
\|R^Y\|_{2\beta, T} := \sup_{0 \leq s < t \leq T} \frac{|R^Y_{ts}|}{(t-s)^{2\beta}}, \\
\|R^Z\|_{\alpha+2\beta, T} := \sup_{0 \leq s < t \leq T} \sup_{|E| \leq 1} \frac{|R^Z_{ts}(E)|}{(t-s)^{\alpha+2\beta}}.
\]

The space of all pairs $(y, z)$ satisfying (37) is denoted by $\tilde{X}_{\omega, T}$.

In the following computations, we emphasize that $C > 0$ is an arbitrary constant which varies from line to line. It also depends on the random input $\omega$, more precisely $C = C (\|\omega\|_{\alpha} \cdot \|\omega^{(2)}\|_{2\alpha})$, as justified below. For notational simplicity, we drop the $\omega$-dependence on $C$, since this does not influence our deliberations.

**Lemma 23.** Let $(y, z) \in \tilde{X}_{\omega, T}$. Then we obtain the following estimates
\[
\|y\|_{\beta, T} \leq C \left(T^{\alpha-\beta} + T^\beta \Phi_T(y, z)\right), \quad (38)
\]
\[
\|z\|_{\alpha+\beta, T} \leq C \left(T^{\alpha-\beta} + T^\beta \Phi_T(y, z)\right). \quad (39)
\]

**Proof.** Regarding the definition of $R^Y$, $\delta y$ and Lemma 13, we immediately obtain
\[
\|y\|_{\beta, T} = \sup_{0 \leq s < t \leq T} \frac{|(\delta y)_{ts}|}{(t-s)^{2\beta}} \\
\leq \sup_{0 \leq s < t \leq T} \frac{|R^Y_{ts}|}{(t-s)^{2\beta}} + \sup_{0 \leq s < t \leq T} \frac{|(S(t-s) - \text{Id}) y_s|}{(t-s)^{2\beta}} + \sup_{0 \leq s < t \leq T} \frac{\|\omega^2_{ts}(G(y_s))\|}{(t-s)^{2\beta}} \\
\leq \|R^Y\|_{2\beta, T} T^\beta + C \|y\|_{\infty, D_{2\beta}, T} T^\beta + C \|\omega\|_{\alpha, T} T^{\alpha-\beta} \\
\leq C \left(T^{\alpha-\beta} + T^\beta \Phi_T(y, z)\right),
\]
which proves the first statement.

Furthermore, due to (26), the estimates for $z$ result in
\[
\|z\|_{\alpha+\beta, T} = \sup_{0 \leq s < t \leq T} \sup_{|E| \leq 1} \frac{|z_{ts}(E)|}{(t-s)^{\alpha+\beta}} \\
\leq \sup_{0 \leq s < t \leq T} \sup_{|E| \leq 1} \frac{|R^Z_{ts}(E)|}{(t-s)^{\alpha+\beta}} + \sup_{0 \leq s < t \leq T} \sup_{|E| \leq 1} \frac{|b_{ts}(E, G(y_s))|}{(t-s)^{\alpha+\beta}} \\
\leq \|R^Z\|_{\alpha+2\beta, T} T^\beta + C \left(\|\omega\|_{\alpha}^2 + \|\omega^{(2)}\|_{2\alpha}^2\right) T^{\alpha-\beta} \\
\leq C \left(T^{\alpha-\beta} + T^\beta \Phi_T(y, z)\right).
\]
\[\square\]
The next result indicates the connection between the space-regularity of $y$ and of the initial data $\xi$.

**Lemma 24.** Let $\xi \in D_\beta$ and $(y, z)$ be a fixed-point of $M_{T, \omega, \xi}$. Then for $t \in (0, T]$ we have that $y_t \in D_{2\beta}$.

**Proof.** By Theorem 15 we know that $y \in C^\beta$ and $z \in C^{\alpha+\beta}$. In order to apply Corollary 2 we have to compute

$$
(\delta_2 \Xi^{(y)})_{\nu u} = \Xi^{(y)}_{\nu u} - \Xi^{(y)}_{\nu m} - S(v-m)\Xi^{(y)}_{\nu u}.
$$

Due to (28) we immediately obtain that

$$
(\delta_2 \Xi^{(y)})_{\nu u} = \omega^{S}_{\nu u}(G(y_u) - G(y_m)) + (\delta_2 z)_{\nu u}(DG(y_u)) + z_{\nu m}(DG(y_u) - DG(y_m)).
$$

Therefore, we estimate

$$
\left| (\delta_2 \Xi^{(y)})_{\nu u} \right| \leq \omega^{S}_{\nu u}(G(y_u) - G(y_m) + DG(y_u)(\delta y)_{\nu u}) + |z_{\nu m}(DG(y_u) - DG(y_m))|.
$$

(40)

For the first term we have applying (22) that

$$
|\omega^{S}_{\nu u}(G(y_u) - G(y_m) + DG(y_u)(\delta y)_{\nu u})| 
\leq C \|\omega\|_{\alpha} (v-m)^{\alpha} |G(y_u) - G(y_m) + DG(y_u)(\delta y)_{\nu u}|.
$$

Furthermore,

$$
|G(y_u) - G(y_m) + DG(y_u)(\delta y)_{\nu u}| \leq C \|y\|_{2, \beta, T}^{2} (m-u)^{2\beta},
$$

and

$$
|z_{\nu m}(DG(y_u) - DG(y_m))| \leq C \|z\|_{\alpha+\beta} \|y\|_{\beta, T} (v-u)^{\alpha+2\beta}.
$$

Summarizing, we obtain

$$
\left| (\delta_2 \Xi^{(y)})_{\nu u} \right| \leq C \left(1 + \|y\|_{2, \beta, T}^{2} + \|z\|_{\alpha+\beta, T}^{2}\right) (v-u)^{\alpha+2\beta}.
$$

(41)

On the other hand

$$
\left| S(v-u)\Xi^{(y)}_{\nu u} \right|_{D_{2\beta}} \leq \left| S(v-u)\omega^{S}_{\nu u}(G(y_u)) \right|_{D_{2\beta}} + \left| S(v-u)z_{\nu u}(DG(y_u)) \right|_{D_{2\beta}} 
\leq C (v-u)^{-\beta} \|\omega^{S}_{\nu u}(G(y_u))\|_{D_{2\beta}} + C(v-u)^{-2\beta} |z_{\nu u}(DG(y_u))|.
$$

Using (27) we get

$$
\left| S(v-u)\Xi^{(y)}_{\nu u} \right|_{D_{2\beta}} \leq C(1 + \|z\|_{\alpha+\beta, T}) (v-u)^{-\alpha-\beta}.
$$

Hence, Corollary 2 entails

$$
\left| (\delta_2 \Xi^{(y)})_{ts} \right|_{D_{2\beta}} = \left| (\delta y)_{ts} \right|_{D_{2\beta}} \leq C \left(1 + \|y\|_{2, \beta, T}^{2} + \|z\|_{\alpha+\beta, T}^{2}\right) (t-s)^{\alpha-\beta},
$$

which simply yields

$$
|y_t|_{D_{2\beta}} \leq |S(t)\xi|_{D_{2\beta}} + \left| (\delta y)_{ts} \right|_{D_{2\beta}} 
\leq C t^{-\beta} \|\xi\|_{D_{2\beta}} + C \left(1 + \|y\|_{2, \beta, T}^{2} + \|z\|_{\alpha+\beta, T}^{2}\right) t^{\alpha-\beta}.
$$

(43)

This proves the statement.

**Lemma 25.** If $\xi \in D_{2\beta}$ and $(y, z)$ is a fixed-point of $M_{T, \omega, \xi}$ then $\Phi_{T}(y, z) < \infty$. 


Then it holds
\[ \Phi_T(y, z) \leq C \left( ||\xi||_{D_{2\beta}} + T^{\alpha-\beta} + T^{\alpha} \Phi_T(y, z) \right). \]
Proof. Recall that \( \Phi_T(y, z) = \|y\|_{\infty, D_{2\beta}, T} + \|R^Y\|_{2\beta, T} + \|R^Z\|_{\alpha+2\beta, T} \). We begin with \( \|R^Y\|_{2\beta, T} \) and further use that

\[
|G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu}| = \int_0^1 |DG(y_u + q(\delta y)_{mu}) - DG(y_u)| dq \quad (\delta y)_{mu} \\
\leq \int_0^1 |DG(y_u + q(\delta y)_{mu}) - DG(y_u)| dq \\
\quad \cdot \left[ \left| R^Y_{mu} \right| + \left| (S(m - u) - 1) y_u \right| + \left| \omega^S_{mu}(G(y_u)) \right| \right] \\
\leq C \left[ \|R^Y\|_{2\beta, T} (m - u)^{2\beta} + \|y\|_{\infty, D_{2\beta}, T} (m - u)^{2\beta} + \|y\|_{\beta, T} \|\omega\|_{\alpha} (m - u)^{\alpha+\beta} \right].
\]

Applying (38) results in

\[
|G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu}| \\
\leq C \left[ \Phi_T(y, z)(m - u)^{2\beta} + (T^{\alpha-\beta} + T^\beta \Phi_T(y, z))(m - u)^{\alpha+\beta} \right].
\]

All in all we obtain for the first term in (41)

\[
|\omega^S_{vm}(G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu})| \leq C(1 + \Phi_T(y, z))(v - u)^{\alpha+2\beta}.
\]

For the second term in (41) we have

\[
|z_{vm}(DG(y_u) - DG(y_m))| \\
\leq \|z_{vm}(DG(y_u) - DG(y_m))\| + \|b_{vm}(DG(y_u) - DG(y_m), G(y_m))\| \\
\leq C \|z_{vm}(DG(y_u) - DG(y_m))\| + \left( \|\omega\|_{2\alpha} + \|\omega^{(2)}(\omega)\|_{2\alpha} \right) (m - u)^{2\alpha} \|y\|_{\beta, T} (m - u)^{\beta}.
\]

Again, we apply (38) and derive

\[
|z_{vm}(DG(y_u) - DG(y_m))| \leq C(1 + \Phi_T(y, z))(v - u)^{\alpha+2\beta}.
\]

Summarizing, we obtain

\[
\left| \delta \Xi^{(y)}_{vm} \right| \leq C(1 + \Phi_T(y, z))(v - u)^{\alpha+2\beta}.
\]

Then (11) yields

\[
\left| \delta \Xi^{(y)}(s)_{ts} - \Xi^{(y)}_{ts} \right| \leq C(1 + \Phi_T(y, z))(t - s)^{\alpha+2\beta}.
\]

Consequently,

\[
|R^Y_{ts}| = \left| (\delta y)_{ts} - \omega^S_{ts}(G(y_s)) \right| \\
\leq \left| (\delta \Xi^{(y)}_{ts} - \Xi^{(y)}_{ts}) \right| + \left| z_{ts}(DG(y_s)) \right| \\
\leq C(1 + \Phi_T(y, z))(t - s)^{\alpha+2\beta} + \|z\|_{\alpha+\beta, T} (t - s)^{\alpha+\beta}.
\]

Now (39) entails the first important estimate on the \(2\beta\)-norm of \(R^Y\), namely

\[
\|R^Y\|_{2\beta, T} \leq C (T^{\alpha-\beta} + T^\alpha \Phi_T(y, z)) . \quad (46)
\]
We now continue investigating \( \|y\|_{\infty,D_{2\beta},T} \). In order to apply Corollary 2 we firstly consider

\[
|S(v-u)\Xi_{vu}|_{D_{2\beta}} \leq |S(v-u)\omega^S_v(G(y_u))|_{D_{2\beta}} + |S(v-u)\omega_{vu}(DG(y_u))|_{D_{2\beta}} \\
\leq C (v-u)^{-\beta}\omega^S_v(G(y_u))|_{D_{2\beta}} + C(v-u)^{-2\beta}|\omega_{vu}(DG(y_u))|.
\]

Using (27) and (39) we get

\[
|S(v-u)\Xi_{vu}|_{D_{2\beta}} \leq (1 + \|z\|_{\alpha+\beta,T}) (v-u)^{\alpha-\beta} \\
\leq C (1 + T^\beta \Phi_T(y,z))(v-u)^{\alpha-\beta}.
\]

Hence, by Corollary 2 we obtain

\[
|\hat{\delta}\Xi_{(y)}|_{D_{2\beta}} \leq C (1 + T^\beta \Phi_T(y,z))(t-s)^{\alpha-\beta} + C (1 + \Phi_T(y,z))(t-s)^{\alpha} \\
\leq C (T^{\alpha-\beta} + T^\alpha \Phi_T(y,z)).
\]

Regarding this we immediately obtain

\[
|y_t|_{D_{2\beta}} \leq \left| \hat{\delta}\Xi_{(y)} \right|_{D_{2\beta}} + |S(t)\xi|_{D_{2\beta}} \\
\leq C (|\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^\alpha \Phi_T(y,z)).
\]

This obviously implies the second important estimate, namely

\[
\|y\|_{\infty,D_{2\beta},T} \leq C (|\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^\alpha \Phi_T(y,z)).
\]

Finally, we only have to compute \( \|R^Z\|_{\alpha+2\beta,T} \) analogously to the proof of Lemma 25.

Applying (46) and (38) to (44) entails

\[
|\hat{\delta}\Xi_{(z)}|_{a_t}(E) - \xi_{a_t}(E) | \leq C (T^{\alpha-\beta} + T^\alpha \Phi_T(y,z)) |E| (t-s)^{\alpha+2\beta}.
\]

Consequently, (24) further leads to

\[
|k_{a_t}(E)| \leq |\hat{\delta}\Xi_{(z)}|_{a_t}(E) - \xi_{a_t}(E) | + |a_{a_t}(E, y_s) - \omega_{a_t}(E, y_s) | \\
\leq C (T^{\alpha-\beta} + T^\alpha \Phi_T(y,z)) |E| (t-s)^{\alpha+2\beta} + C \|\omega\|_{a_t} \|y\|_{\infty,D_{2\beta},T} |E| (t-s)^{\alpha+2\beta}.
\]

Regarding this and plugging in (47), we derive the third and final important estimate for the terms defining \( \Phi_T \), namely

\[
\|R^Z\|_{\alpha+2\beta,T} \leq C \left( |\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^\alpha \Phi_T(y,z) \right).
\]

This proves the statement, i.e.

\[
\Phi_T(y,z) \leq C \left( |\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^\alpha \Phi_T(y,z) \right).
\]

\( \square \)

We now derive a crucial estimate which will be required for the concatenation procedure.

**Lemma 27.** Let \( T > 0, r \geq 1 \vee |\xi|_{D_{2\beta}} \) and let \((y,z)\) be a fixed-point of \( M_{T,\omega,\xi} \). Then there exists a constant \( M > 0 \) independent of \( r \), such that

\[
\|y\|_{\infty,D_{2\beta},T} \leq r M e^{MT}.
\]
Lemma 28. Let \( y^0 \) be a fixed-point of \( \mathcal{M}_{T_1, \omega, \xi} \) and \( (y^1, z^1) \) be a fixed-point of \( \mathcal{M}_{T_2, \omega, y^0 T_1} \). Then we obtain a fixed-point \((y, z)\) of \( \mathcal{M}_{T_1 + T_2, \omega, \xi} \) via

\[
y_t := \begin{cases} y^1_t, & 0 \leq t \leq T_1 \\
y^2_{t-T_1}, & T_1 \leq t \leq T_1 + T_2,
\end{cases}
\]
Proof. The statement follows by a standard computation. We only focus on certain cases, since the rest are straightforward. For the beginning we consider the appropriate shifts with respect to $\mathcal{T}_1$. Consequently, where we use in the last step that $\hat{T}_1 = \xi_t$. Since

$$
\Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1) = b_{y^2}(E, G(y^2 - T_1)) + a_{y^2}(E, y^2 - T_1),
$$

further entails

$$
\bar{\theta}_1 \Xi(y^2, y^2 - T_1) = \bar{\theta}_1 b_{y^2}(E, G(y^2 - T_1)) + \bar{\theta}_1 a_{y^2}(E, y^2 - T_1) = \Xi(y^2, y^2 - T_1).
$$

Hence, we infer using Lemma 8 that

$$
\Xi(y^2, y^2 - T_1) = \Xi(y^2, y^2 - T_1).
$$

and

$$
z_t(s) = \begin{cases} \tilde{z}_t(s), & 0 \leq s \leq t \leq T_1 \\ \omega^s_{T_1}(E (\delta y^1)_{T_1}) + \tilde{z}_t - T_1, o(E) + S(t - T_1)z^{s}_{T_1, s}(E), & 0 \leq s \leq T_1 \leq t \leq T_1 + T_2 \\ \tilde{z}_t - T_1, s - T_1(E), & t \leq s \leq t \leq T_1 + T_2. \end{cases}
$$

$$
\Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1) = \Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1).
$$

Recall that

$$
\Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1) = \Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1).
$$

Now, Lemma 8 entails

$$
\Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1) = \Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1).
$$

Consequently,

$$
\Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1) = \Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1).
$$

Now, let $0 \leq s \leq T_1 < t \leq T_1 + T_2$. Then we have

$$
\Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1) = \Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1).
$$

Further, we infer using Lemma 8 that

$$
\Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1) = \Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1).
$$

Hence, we infer using Lemma 8 that

$$
\Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1) = \Xi(y^2, y^2 - T_1, y^2 - T_1, y^2 - T_1).
$$
Regarding all the previous deliberations we can now state the main results of this section.

**Theorem 29.** Let $\xi$ in $D_{2\beta}$. Then for any $T > 0$ there exists a unique global solution, i.e. there exists a unique fixed-point of $M_{T,\omega,\xi}$.

**Proof.** Let $r = 1 \lor |\xi|_{D_{2\beta}}$. By Lemma 27 we know that every fixed-point of $M_{T,\omega,\xi}$ must satisfy the estimate

$$
\|y\|_{\infty,D_{2\beta},T} \leq rM e^{MT} =: \tilde{r}.
$$

Particularly, this means that $|y_t|_{D_{2\beta}} \leq \tilde{r}$, for all $t \leq T$. Applying Theorem 15 with $|\xi|_{D_{2\beta}} \leq \tilde{r}$ entails the existence of a unique local solution on a time interval $[0,T^*]$, where $T^* = T^*(\omega,\tilde{r})$, i.e. there is a unique fixed-point $(y,z)$ of $M_{T^*,\omega,\xi}$. Note that $T^*$ can be chosen to depend on the behavior of $\omega$ on the whole time interval $[0,T]$, not only on $[0,T^*]$. For simplicity, since we can choose $T^*$ arbitrary small, we set $N := \frac{T}{T^*} \in \mathbb{N}$ for $N \geq 2$.

Note that $|y_{T^*}| \leq \tilde{r}$. Hence, we can derive by using again Theorem 15 the existence of a unique fixed-point of $M_{T^*,\theta_{T^*}\omega,y_{T^*}}$. Furthermore, Lemma 28 shows that we can concatenate them and obtain a fixed-point $(y,z)$ of $M_{2T^*,\omega,\xi}$. Again we have $|y_{2T^*}| \leq \tilde{r}$.

Iterating this argument entails the existence of a unique fixed-point $(y,z)$ of $M_{T,\omega,\xi}$ for any $T > 0$.

**Corollary 3.** Let $\xi$ in $W$. Then for any $T > 0$ there exists a unique fixed-point of $M_{T,\omega,\xi}$.

**Proof.** Theorem 15 gives us the existence of a unique fixed-point $(y,z)$ of $M_{T_t,\omega,\xi}$, where $T_t = T_t(\omega,\xi)$. Furthermore due to (35) we obtain $y_{T_t} \in D_{2\beta}$. Then by Theorem 15 we know that there exists a unique-fixed point of $M_{T_t,\theta_{T_t}\omega,y_{T_t}}$, which according to Lemma 28 can be concatenated with the previous one to a fixed-point $(y,z)$ of $M_{2T_t,\omega,\xi}$. Again we have $|y_{2T_t}| \leq \tilde{r}$.

Iterating this argument entails the existence of a unique fixed-point $(y,z)$ of $M_{T_t,\omega,\xi}$ for any $T > 0$.

6. **Random dynamical systems.** Based on the results derived in the previous section we investigate random dynamical systems for (1). There are very few works that deal with random dynamical systems for SPDEs driven by nonlinear multiplicative rough noise, see for instance [8]. In the finite-dimensional setting this topic was considered in [2, 18].

We start by introducing the next fundamental concept in the theory of random dynamical systems, which describes a model of the driving noise.

**Definition 30.** Let $(\Omega, F, \mathbb{P})$ stand for a probability space and $\theta : \mathbb{R} \times \Omega \to \Omega$ be a family of $\mathbb{P}$-preserving transformations (i.e., $\theta_t \mathbb{P} = \mathbb{P}$ for $t \in \mathbb{R}$) having following properties:

(i): the mapping $(t, \omega) \mapsto \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \otimes F, F)$-measurable;

(ii): $\theta_0 = \text{Id}_\Omega$;

(iii): $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$. 

\[\square\]
Then the quadrupel \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) is called a metric dynamical system.

Motivated by this we precisely describe the random input driving \((1)\). Therefore, our aim is introduce the (canonical) probability space associated to a Hilbert space-valued \(\alpha\)-Hölder rough path. We recall that \(\alpha \in (\frac{1}{4}, \frac{1}{2})\) was fixed at the beginning of this work. An example is constituted by a trace-class \(V\)-valued fractional Brownian motion with Hurst index \(H \in (1/3, 1/2]\). In order to construct it, we recall that a two-sided real-valued fractional Brownian motion \(\tilde{\beta}^H(\cdot)\) with a Hurst index \(H \in (0, 1)\) is a centered Gaussian process with covariance function

\[
E(\tilde{\beta}^H(t)\tilde{\beta}^H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad \text{for } s, t \in \mathbb{R}.
\]

In order to introduce a \(V\)-valued process, we let \(Q\) stand for a positive symmetric operator of trace-class on \(V\), i.e. \(\text{tr}_V Q < \infty\). This has a discrete spectrum which will be denoted by \((\lambda_n)_{n \in \mathbb{N}}\). It is well-known that the eigenvectors \((e_n)_{n \in \mathbb{N}}\) build an orthonormal basis in \(V\). Then a \(V\)-valued two-sided \(Q\)-fractional Brownian motion \(\omega(\cdot)\) is represented by

\[
\omega(t) = \sum_{n=1}^{\infty} \sqrt{n^{\beta n}(t)} e_n, \quad t \in \mathbb{R}, \quad (49)
\]

where \((\tilde{\beta}^H_n(\cdot))_{n \in \mathbb{N}}\) is a sequence of one-dimensional independent standard two-sided fractional Brownian motions with the same Hurst parameter \(H\) and \(\text{tr}_V Q = \sum_{n=1}^{\infty} \lambda_n < \infty\). In the following sequel we further fix \(H \in (\frac{1}{3}, \frac{1}{2}]\).

Keeping \((49)\) in mind we are justified to introduce the canonical probability space \((C_0(\mathbb{R}, V), \mathcal{B}(C_0(\mathbb{R}, V)), \mathbb{P}, \theta)\). Here \(C_0(\mathbb{R}, V)\) denotes the set of all \(V\)-valued continuous functions which are zero in zero endowed with the compact open topology and \(\mathbb{P}\) is the fractional Gauß-measure which is uniquely determined by \(Q\). As already introduced in Section 2, we take for \(\theta\) the usual Wiener-shift, namely

\[
\theta_t \omega_t = \omega_{t+\tau} - \omega_\tau, \quad \text{for } \omega \in C_0(\mathbb{R}, V).
\]

It is well-known that the above introduced quadruple is a metric dynamical system. For our aims we restrict it to the set \(\Omega := C_0^\alpha(\mathbb{R}, V)\) of all \(\alpha'\)-Hölder-continuous paths on any compact interval, where \(\frac{1}{4} < \alpha < \alpha' < H \leq \frac{1}{2}\). We equip this set with the trace \(\sigma\)-algebra \(\mathcal{F} := \Omega \cap \mathcal{B}(C_0(\mathbb{R}, V))\) and take the restriction of \(\mathbb{P}\) as well. Then \(\Omega \subset C_0(\mathbb{R}, V)\) has full measure and is \(\theta\)-invariant. Moreover, the new quadrupel \((\Omega, \mathcal{F}, \mathbb{P}, \theta)\) as introduced above forms again a metric dynamical system which we will further be restricted later on.

We point out the following result regarding the existence/construction of the Lévy-area \(\omega^{(2)}\) for an element \(\omega \in \Omega\). We stress the fact that it is necessary to let \(\omega\) be \(\alpha'\)-Hölder continuous for \(\frac{1}{4} < \alpha < \alpha' < H \leq \frac{1}{2}\). This is required in order to lift \(\omega\) to a rough path \(\omega = (\omega, \omega^{(2)})\). To this aim we furthermore have to consider the restriction of \(\omega\) on compact intervals. The precise setting is stated below.

Lemma 31. Let \(\frac{1}{4} < \alpha < \alpha' < H \leq \frac{1}{2}\) and \(\omega \in \Omega\) be a \(Q\)-fractional Brownian motion with Hurst index \(H\). Then there is a \(\theta\)-invariant subset \(\Omega' \subset \Omega\) of full measure such that for any \(\omega \in \Omega'\) and for any compact interval \(J \subset \mathbb{R}\) there exists a Lévy-area \(\omega^{(2)} \in C^{2\alpha}(\Delta_J, V \otimes V)\) such that \(\omega = (\omega, \omega^{(2)})\) defines an \(\alpha\)-Hölder rough path. This can further be approximated by a sequence \(\omega^n := ((\omega^n, \omega^{(2),n}))_{n \in \mathbb{N}}\) in
the corresponding $d_{\alpha,J}$-metric. Here $(\omega^n)_{n\in\mathbb{N}}$ are piecewise dyadic linear functions and
\[ \omega^{(2),n}_{ts} = \int_s^t (\delta \omega^n)_{rs} \otimes d\omega^r. \]

**Proof.** Let $j, k \in \mathbb{N}$ and $T \in \mathbb{N}$ be such that $J \subseteq [-T, T]$. We introduce
\[ \omega^{(2)}_{ts}(j, k) := \int_s^t (\delta \beta^H_j(r) - \beta^H_j(s)) \, d\beta^H_k(r), \text{ for } -T \leq s \leq t \leq T. \] (50)

This process exists almost surely according to Theorem 2 in [5], see also Corollary 10 in [9]. Regarding (49) we can represent the infinite-dimensional Lévy-area component-wise as
\[ \omega^{(2)}_{ts} = \sum_{j,k=1}^{\infty} \sqrt{\lambda_j} \sqrt{\lambda_k} \omega^{(2)}_{ts}(j, k) \, e_j \otimes e_k. \] (51)

This is well-defined almost surely due to the fact that $\text{tr}_Y Q < \infty$. Moreover one has that $\omega^{(2),n} \to \omega^{(2)}$ in $C^{2\alpha}(\Delta_{[-T,T]}, \mathbb{V} \otimes \mathbb{V})$ almost surely. The proof of these assertions relies on a standard Borel-Cantelli argument combined with the Garsia-Rodemich-Rumsey inequality and follows the lines of Lemma 2 in [16]. Since $J \subseteq [-T, T]$, one clearly concludes that $\omega^n$ converges to $\omega$ with respect to the $d_{\alpha,J}$-metric. This immediately yields that $\Omega'$ has full measure and is $\theta$-invariant. \qed

From now on we work with the metric dynamical system $(\Omega', \mathcal{F}', \mathbb{P}', \theta)$ corresponding to $\Omega'$ constructed in Lemma 31. As above we set $\mathcal{F}' := \Omega' \cap \mathcal{F}$ and take $\mathbb{P}'$ as the restriction of $\mathbb{P}$.

For the sake of completeness we indicate the following result regarding the shift-property of an $\alpha$-Hölder rough path.

**Lemma 32.** For an $\alpha$-Hölder rough path $(\omega, \omega^{(2)})$ and $\tau \in \mathbb{R}$, the time-shift $(\theta_{\tau}\omega, \tilde{\theta}_{\tau}\omega^{(2)})$
\[ \theta_{\tau}\omega_t = \omega_{t+\tau} - \omega_t \]
\[ \tilde{\theta}_{\tau}\omega^{(2)}_{ts} = \omega^{(2)}_{t+\tau,s+\tau} \]
is again an $\alpha$-Hölder rough path.

**Proof.** The time-regularity is straightforward and one can easily verify Chen’s relation (3). This reads as
\[ \tilde{\theta}_{\tau}\omega^{(2)}_{ts} - \tilde{\theta}_{\tau}\omega^{(2)}_{ts} - \tilde{\theta}_{\tau}\omega^{(2)}_{tu} = \omega^{(2)}_{t+\tau,s+\tau} + \omega^{(2)}_{u+\tau,s+\tau} + \omega^{(2)}_{t+\tau,u+\tau} \]
\[ = \omega_{u+\tau,s+\tau} + \delta_{t+\tau} \omega - \omega_{t+\tau} + \omega_{u+\tau} + \omega_{u+\tau} \]
\[ = \delta_{t+\tau} \omega + \delta_{u+\tau} \omega. \] (52)

where in (52) we use Chen’s relation (3). \qed

**Definition 33.** A random dynamical system on $W$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping
\[ \varphi : \mathbb{R}_+ \times \Omega \times W \to W, \ (t, \omega, x) \mapsto \varphi(t, \omega, x), \]
which is $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F} \times \mathcal{B}(W), \mathcal{B}(W))$-measurable and satisfies:
(i): \( \varphi(0, \omega, \cdot) = \text{Id}_W \) for all \( \omega \in \Omega \);

(ii): \( \varphi(t + \tau, \omega, x) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, x)) \), for all \( x \in W, \ t, \tau \in \mathbb{R}_+, \ \omega \in \Omega \).

If one additionally assumes that

(iii): \( \varphi(t, \omega, \cdot) : W \to W \) is continuous for all \( t \in \mathbb{R}_+, \omega \in \Omega \),

then \( \varphi \) is called a continuous random dynamical system.

The second property in Definition 33 is referred to as the cocycle property. One can now expect that the solution operator of (1) generates a random dynamical system. Indeed, working with a pathwise interpretation of the stochastic integral, no exceptional sets can occur.

We can now state the main result of this work. Recall that \( \Omega' \) was constructed in Lemma 31.

**Theorem 34.** The solution operator of (1) generates a random dynamical system \( \varphi : \mathbb{R}_+ \times \Omega' \times W \to W \) given by

\[
\varphi(t, \omega, \xi) := y_t,
\]

where \( y \) is the first component of the fixed-point operator \( M_{t, \omega, \xi} \).

**Proof.** Due to Theorem 29 we know that we can define the solution \( (y, z) \) of (1) on any time-interval \([0, T]\) for \( T > 0 \). The cocycle property was proved in Lemma 19.

The continuity of \( \varphi \) with respect to time and initial condition is clear (recall Lemma 17), we only have to show the joint measurability. Therefore we consider a sequence of solutions \( ((y^n, z^n))_{n \in \mathbb{N}} \) corresponding to the smooth approximations \( ((\omega^n, \omega^{(2, n)}))_{n \in \mathbb{N}} \), recall Lemma 31. Note that the mapping \( \omega \mapsto (\omega^n, \omega^{(2, n)}) \) is measurable. Due to the fact that \( \omega^n \) is smooth \( y^n \) is a classical solution of (1).

Hence the mapping

\[
[0, T] \times \Omega' \times W \ni (t, \omega, \xi) \mapsto y^n_t \in W
\]

is \( (B([0, T]) \otimes \mathcal{F}' \otimes B(W), B(W)) \)-measurable. Regarding Lemma 17 one can immediately infer that the solution \( (y, z) \) continuously depends on \( (\omega^n, \omega^{(2, n)}) \). This leads to

\[
\lim_{n \to \infty} y^n_t = y_t,
\]

which gives us the measurability of \( y_t \) with respect to \( \mathcal{F}' \otimes B(W) \). Since \( y \) is continuous with respect to \( t \), we obtain by Lemma 3 in [3] the joint measurability, i.e. the \( (B([0, T]) \otimes \mathcal{F}' \otimes B(W), B(W)) \) measurability of the mapping

\[
[0, T] \times \Omega' \times W \ni (t, \omega, \xi) \mapsto y_t \in W.
\]

Since (55) holds true for any \( T > 0 \), one obviously concludes that \( \varphi \) is \( (B(\mathbb{R}_+) \otimes \mathcal{F}' \otimes B(W), B(W)) \)-measurable. \( \square \)

**Remark 35.** This work is complementary to [17], where similar statements are obtained using fractional calculus. We emphasize that rough paths techniques are not restricted to the case of the fractional Brownian motion, but apply to a wider class of Gaussian processes, their covariance functions satisfy certain criteria, as specified in [9, Chapter 10] and [10, Chapter 15]. Therefore, the results presented in this work can be applied to Gaussian processes with stationary increments that can be lifted to \( \alpha \)-Hölder rough paths. The stationarity assumption is required here for the random dynamical systems approach.
REFERENCES

[1] L. Arnold, Random Dynamical Systems, Springer, Berlin Heidelberg, Germany, 2003.
[2] I. Bailleul, S. Riedel and M. Scheutzow, Random dynamical system, rough paths and rough flows, J. Differential Equat., 262 (2017), 5792–5823.
[3] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, Vol. 580. Springer-Verlag, Berlin-New York, 1977.
[4] L. Coutin and A. Lejay, Sensitivity of rough differential equations: An approach through the Omega lemma, J. Differential Equat., 264 (2018), 3899–3917.
[5] L. Coutin and Z. Qian, Stochastic analysis, rough path analysis and fractional Brownian motions, Probab. Theory Related Fields, 122 (2002), 108–140.
[6] A. Deya, M. Gubinelli and S. Tindel, Non-linear rough heat equations, Probab. Theory Related Fields, 153 (2012), 97–147.
[7] A. Deya, A. Neuenkirch and S. Tindel, A Milstein-type scheme without Lévy area terms for SDES driven by fractional Brownian motion, Ann. Inst. H. Poincaré Probab. Statist., 48 (2012), 518–550.
[8] B. Fehrmann and B. Gess, Well-posedness of stochastic porous media equations with non-linear, conservative noise, Arch. Ration. Mech. Anal., 233 (2019), 249–322.
[9] P. K. Friz and M. Hairer, A Theory of Regularity Structures, Invent. Math., 198 (2014), 269–504.
[10] M. Gubinelli, Controlling rough paths, J. Func. Anal., 216 (2004), 86–140.
[11] M. Gubinelli, Young integrals and SPDEs, Potential Anal., 25 (2006), 307–326.
[12] M. Gubinelli and S. Tindel, Rough evolution equations, Ann. Probab., 38 (2010), 1–75.
[13] M. Hairer, A theory of regularity structures, Invent. Math., 198 (2014), 269–504.
[14] M. Hairer, Ergodicity of stochastic differential equations driven by fractional Brownian motion, Ann. Probab., 33 (2005), 703–758.
[15] R. Hesse and A. Neamtu, Local mild solutions for rough stochastic partial differential equations, J. Differential Equat., 267 (2019), 6480–6538.
[16] Y. Hu and D. Nualart, Rough path analysis via fractional calculus, Trans. Amer. Math. Soc., 361 (2009), 2689–2718.
[17] P. Imkeller and C. Lederer, The cohomology of stochastic and random differential equations and local linearization of stochastic flows, Stoch. Dyn., 2 (2002), 131–159.
[18] L. W. Kantorowitsch and G. P. Akilow, Funktionalanalysis in Normierten Räumen, Verlag Harri Deutsch, 1978.
[19] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge University Press, 1990.
[29] C. Lederer, *Kojugation Stochastischer Und Zufälliger Stationärer Differentialgleichungen Und Eine Version Des Lokalen Satzes von Hartman–Grobman Für Stochastische Differentialgleichungen*, PhD Thesis. HU Berlin, 2001.
[30] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, 1995.
[31] B. Maslowski and D. Nualart, *Evolution equations driven by a fractional Brownian motion*, *J. Funct. Anal.*, 202 (2003), 277–305.
[32] B. Maslowski and B. Schmalfuß, *Random dynamical systems and stationary solutions of differential equations driven by the fractional Brownian motion*, *Stochastic Anal. Appl.*, 22 (2004), 1577–1607.
[33] S. Mohammed, T. Zhang and H. Zhao, *The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations*, *Memoirs of the AIMS*, 196 (2008), vi+105 pp.
[34] D. Nualart and A. Răşcanu, *Differential equations driven by fractional Brownian motion*, *Collect. Math.*, 53 (2002), 55–81.
[35] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Applied Mathematical Series, Springer–Verlag, Berlin, 1983.
[36] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer, 2002.
[37] M. Scheutzow, On the perfection of crude cocycles, *Random Comput. Dynam.*, 4 (1996), 235–255.
[38] A. Yagi, *Abstract Parabolic Evolution Equations and their Applications*, Springer, 2010.
[39] L. C. Young, *An integration of Hölder type, connected with Stieltjes integration*, *Acta Math.*, 67 (1936), 251–282.
[40] M. Zähle, *Integration with respect to fractal functions and stochastic calculus I*, *Probab. Theory Related Fields*, 111 (1998), 333–374.

Received for publication April 2019.

E-mail address: robert.hesse@uni-jena.de
E-mail address: alexandra.neamtu@tum.de