IDENTITIES FOR THE EULER POLYNOMIALS, $p$-ADIC INTEGRALS AND WITT’S FORMULA

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Abstract. By using Cauchy’s formula, it is known that Bernoulli numbers and Euler numbers can be represented by the contour integrals

\[ B_n = \frac{n!}{2\pi i} \int \frac{z}{e^z - 1 \cdot z^{n+1}} \, dz, \]

\[ E_n = \frac{n!}{2\pi i} \int \frac{2e^z}{e^{2z} + 1 \cdot z^{n+1}} \, dz, \]

while the following Witt’s formula represents Euler polynomials through the fermionic $p$-adic integrals

\[ E_n(a) = \int_{z_p} (x + a)^n \mu_{-1}(x). \]

Based on the above Witt’s identity and the binomial theorem, we prove some new identities for the Euler polynomials briefly. In particular, some symmetry properties of Euler polynomials have been discovered, which implies many interesting identities (known or unknown), including the Kaneko-Momiyama type identities (shown by Wu, Sun, and Pan) and the Alzer-Kwong type identity for Euler polynomials.

1. Introduction
1.1. Background. Denote by $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Bernoulli polynomials $B_n(a)$ are defined by the generating function

\[ \frac{te^{at}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(a) \frac{t^n}{n!} \]  

and $B_n = B_n(0)$ are named Bernoulli numbers. These numbers and polynomials arise from Bernoulli’s calculations of power sums in 1713, that is,

\[ \sum_{j=1}^{m} j^n = \frac{B_{n+1}(m + 1) - B_{n+1}}{n + 1} \]  

(see [23] p. 5, (2.2)).

The Euler polynomials $E_n(a)$ are defined by the generating function

\[ \frac{2e^{at}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(a) \frac{t^n}{n!}. \]  

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These polynomials were introduced by Euler who studied the alternating power sums, that is,
\[ \sum_{j=1}^{m} (-1)^j j^n = \frac{(-1)^m E_n(m+1) + E_n(0)}{2} \]
(see [23, p. 5, (2.3)]). It is easy to see that the first few Euler polynomials are
\[ E_0(a) = 1, \quad E_1(a) = a - \frac{1}{2}, \quad E_2(a) = a^2 - a, \quad E_3(a) = a^3 - \frac{3}{2}a^2 + \frac{1}{4}. \]
The Euler numbers \( E_n \) are defined by
\[ E_n = 2^n E_n \left( \frac{1}{2} \right) \]
and \( E_{2n+1} = 0 \) for \( n \in \mathbb{N} \) (see for instance [1] p. 804, 23.1.2 and [16] (0.1)).
The Euler numbers and polynomials (so called by Scherk in 1825) appear in Euler’s famous book, Institutiones Calculi Differentials (1755, pp. 487–491 and p. 522) and have been found many applications in combinatorics, number theory, classical and \( p \)-adic analysis. For example, see Abramowitz and Stegun [1], Agoh and Dilcher [3], Chen and Sun [4], Gessel [6], He and Zhang [7], Kim and Hu [12], Koblitz [13], Robert [19], Simsek [21] and Sun [22].

From the definition (1.2) we can easily deduce the following identities:
\[ E_n(1 - a) = (-1)^n E_n(a), \]
\[ (-1)^n E_n(-a) + E_n(a) = 2a^n \]
(see for instance [1] p. 804, 23.1.8 and 23.1.9).

Letting \( a = 1 \) in (1.5), we have a relation between the special values of Euler polynomials at 1 and 0:
\[ E_n(1) = (-1)^n E_n(0) = \begin{cases} 1 & \text{for } n = 0 \\ -E_n(0) & \text{for } n \neq 0, \end{cases} \]
because \( E_n(0) = 0 \) if \( n \) is even.

In 2004, Wu, Sun and Pan [24, (7) and (9)] proved the following interesting formulæ:
\[ (-1)^m \sum_{i=0}^{m} \binom{m}{i} E_{n+i}(a) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} E_{m+j}(-a) \]
and
\[ (-1)^m \sum_{i=0}^{m} \binom{m+1}{i} (n + i + 1) E_{n+i}(a) \]
\[ + (-1)^n \sum_{j=0}^{n} \binom{n+1}{j} (m + j + 1) E_{m+j}(-a) \]
\[ = (-1)^{m+1} 2(m + n + 2)(E_{m+n+1}(a) - a^{m+n+1}). \]
Here \( \binom{m}{i} \) denotes the usual binomial coefficient. The identity (1.8) can be viewed as a version of the Kaneko-Momiyama type identity for Euler polynomials. In Section 3, we shall give an alternative proof of (1.8) (see Remark 1.7 below). The identity (1.9) is an Euler polynomial version of Kaneko-Momiyama relations among Bernoulli numbers. See Kaneko [8], Momiyama [17], Gessel [6] and Wu-Sun-Pan [24] for details. In fact, we can see that Theorem 1.1 reduces to (1.9) by setting \( q = k = 1 \) (see Remark 1.2 below). More recently, by using Zeilberger’s algorithm [25], Chen and Sun [4] gave new proofs of many existing recurrence relations between Bernoulli polynomials.

The aim of this paper is to present new identities for the Euler polynomials which extend (1.8) and (1.9). Our main tool is the fermionic\(-p\)-adic integral on \( \mathbb{Z}_p \), and in the next section we shall give a brief recall of the definition and identities for this integral. Using them the detail proofs for the main theorems will be shown in the final section.

1.2. Main results. Our main results and their corollaries are as follows.

**Theorem 1.1.** Let \( k, q \in \mathbb{N} \) and let \( m, n \in \mathbb{N}_0 \) such that \( m + n > 0 \). Then for given odd integer \( k \), we have

\[
(-1)^m \sum_{i=0}^{m+q} \binom{m+q}{i} \binom{n+q+i}{k} E_{n+q+i-k}(a) + (-1)^n \sum_{j=0}^{n+q} \binom{n+q}{j} \binom{m+q+j}{k} E_{m+q+j-k}(-a) = 0.
\]

**Remark 1.2.** Considering the identity (1.6), it is easily checked that

\[
(-1)^m(m + n + 2)E_{m+n+1}(a) + (-1)^n(m + n + 2)E_{m+n+1}(-a)
\]

(1.10)

\[
= (-1)^m(m + n + 2)(E_{m+n+1}(a) + (-1)^{m+n}E_{m+n+1}(-a))
\]

\[
= (-1)^m2(m + n + 2)(E_{m+n+1}(a) - a^{m+n+1}).
\]

Thus letting \( q = k = 1 \) in Theorem 1.1 we recover Wu, Sun and Pan’s formula (1.9).

Put \( m = n, q = k \) and \( a = 0 \) in Theorem 1.1 we get an Euler polynomial version of Zékiri and Bencherif’s formula which was stated for Bernoulli numbers (see [26, Theorem 1.1]).

**Corollary 1.3.** Let \( n \in \mathbb{N}_0 \). Then for given odd integers \( q \), we have

\[
\sum_{i=0}^{m+q} \binom{m+q}{i} (n+q+i)(n+q+i-1) \cdots (n+i+1)E_{n+i}(0) = 0.
\]

Note that the above identity is also an Euler polynomial version of Kaneko’s identity [8] when \( q = 1 \), Chen and Sun’s identity [4] when \( q = 3 \).

Put \( a = 0 \) and \( q = k = 1 \) in Theorem 1.1 we have the following analogue of Momiyama’s formula [17].
Corollary 1.4. Let $m, n \in \mathbb{N}_0$ such that $m + n > 0$. Then we have
\[\sum_{i=0}^{m+1} \binom{m+1}{i}(n+i+1)E_{n+i}(0) + (-1)^{m+n}\sum_{j=0}^{n+1} \binom{n+1}{j}(m+j+1)E_{m+j}(0) = 0.\]

If we put $m = n$ in Corollary 1.4, then we get the following identity.

Corollary 1.5. For any $n \in \mathbb{N}_0$,
\[\sum_{j=0}^{n+1} \binom{n+1}{j}(n+j+1)E_{n+j}(0) = 0.\]

This can be regarded as an Euler polynomial version of Kaneko’s formulae which was stated for Bernoulli numbers (see [8]). From the above identity, we have a recurrence relationship
\[E_{2n+1}(0) = -\frac{1}{2(n+1)}\sum_{j=0}^{n} \binom{n+1}{j}(n+j+1)E_{n+j}(0),\]

It is an interest, since from this, in order to to calculate $E_{2n+1}(0)$, we only needs half of the previous terms ($E_k(0)$ with $n \leq k \leq 2n$ with $k$ odd).

We show that the $p$-adic integral is also work for proving the following Sun’s identity for Euler polynomials.

Theorem 1.6 (Sun’s identity, see [22, Theorem 1.2(iii)]). We have
\[(-1)^m \sum_{i=0}^{m} \binom{m}{i} a^{m-i}E_{n+i}(b) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} a^{n-j}E_{m+j}(c),\]
where $a + b + c = 1$.

Remark 1.7. In the case $a = 1, b = t$ and $c = -t$, Theorem 1.6 reduces to (1.8).

Putting $a = 1, b = 1/2$ and $c = -1/2$ in Theorem 1.6, we get the following consequence which can also be found in Chen and Sun’s work (see [4, Theorem 6.1]).

Corollary 1.8. We have
\[(-1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{E_{n+i}}{2n+i} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} E_{m+j}\left(-\frac{1}{2}\right),\]
where $m, n \in \mathbb{N}_0$.

Motivated by the above works, we have the following general result.
Theorem 1.9. Let \( s \in \mathbb{N} \) and let \( k, m, n \in \mathbb{N}_0 \) such that \( m + n > 0 \). Then we have

\[
\delta_{s,k} \left( \sum_{i=0}^{m} (s + 1)^{m-i+1} \binom{m+1}{i} \binom{n+i+1}{k} E_{n+i-k+1}(a) 
+ (-1)^{m-n} \sum_{j=0}^{n} (s + 1)^{n-j+1} \binom{n+1}{j} \binom{m+j+1}{k} E_{m+j-k+1}(-a) \right)
= \frac{2}{k!} \sum_{l=1}^{s} (-1)^{l} P_{m,n,s}(l; a),
\]

where \( P_{m,n,s}(l; a) \) satisfy the relation

\[
P_{m,n,s}(l; a) = \frac{d^{k}}{dx^{k}} \left( (x+a)^{m+1}(x-a-s-1)^{n+1} + (-1)^{m+n}(x-a)^{n+1}(x-a-s-1)^{m+1} \right) \bigg|_{x=l}
\]

and \( \delta_{s,k} \) can be expressed as

\[
\delta_{s,k} = (-1)^{s} - (-1)^{k} = \begin{cases} 
+2 & \text{for } s \text{ even, } k \text{ odd,} \\
-2 & \text{for } s \text{ odd, } k \text{ even,} \\
0 & \text{otherwise.} 
\end{cases}
\]

Using Theorem 1.9 we may obtain many (well-known or new) properties for Euler numbers and polynomials. Here are some few examples.

Put \( m = n, s = 1 \) and \( a = 0 \) in Theorem 1.9 we have

Corollary 1.10. (1) If \( k \) is even, then

\[
\sum_{i=0}^{n} \frac{(-1)^{i}}{2^{i}} \binom{n+1}{i} \binom{n+i+1}{k} \left((-1)^{i}E_{n+i-k+1}(0) + (-1)^{n}\right) = 0.
\]

(2) If \( k \) is odd, then

\[
\sum_{i=0}^{n} \frac{(-1)^{i}}{2^{i}} \binom{n+1}{i} \binom{n+i+1}{k} = 0.
\]

Put \( m = n, s = 2 \) and \( a = 0 \) in Theorem 1.9 we obtain

Corollary 1.11. (1) If \( k \) is odd, then

\[
\sum_{i=0}^{n} (-1)^{i} 3^{n-i+1} \binom{n+1}{i} \binom{n+i+1}{k} \times \left((-1)^{i}E_{n+i-k+1}(0) + (-1)^{n}(2^{n+i-k+1} - 1)\right) = 0.
\]

(2) If \( k \) is even, then

\[
\sum_{i=0}^{n} (-1)^{i} 3^{n-i+1} \binom{n+1}{i} \binom{n+i+1}{k} (2^{n+i-k+1} - 1) = 0.
\]
Theorem 1.12. Let $k, m \in \mathbb{N}_0$ with $0 \leq k \leq m$. Then we have

$$
\sum_{\substack{i=0 \atop m+i \text{ even}}}^{m} \binom{m}{i} \binom{m+i}{k} E_{m+i-k}(a) = \sum_{j=0}^{m} (-1)^{m+j} \binom{m}{j} \binom{m+j}{k} a^{m+j-k}.
$$

This can be regarded as an Euler polynomials version of Alzer and Kwong’s formulae which was stated for Bernoulli polynomials (see [2, Theorem 1]).

An application of Theorem 1.12 leads to

Theorem 1.13. (1) For $0 \leq k \leq m$, we have

$$
\sum_{\substack{i=0 \atop m+i \text{ even}}}^{m} \binom{m}{i} \binom{m+i}{k} \binom{m+i-k}{m-k} E_i(0) = (-1)^m \binom{m}{k}.
$$

(2) For $0 \leq k \leq m-1$, we have

$$
\sum_{\substack{i=0 \atop m+i \text{ even}}}^{m} \binom{m}{i} \binom{m+i}{k} \binom{m+i-k}{m-k-1} E_{i+1}(0) = 0.
$$

(3) For $0 \leq l \leq m-k-1$, we have

$$
\sum_{\substack{i=0 \atop m+i \text{ even}}}^{m} \binom{m}{i} \binom{m+i}{k} \binom{m+i-k}{l} E_{m+i-k-l}(0) = 0.
$$

(4) For $1 \leq j \leq m$ and $0 \leq k \leq m$, we have

$$
\sum_{\substack{i=j \atop m+i \text{ even}}}^{m} \binom{m}{i} \binom{m+i}{k} \binom{m+i-k}{m+j-k} E_{i-j}(0) = (-1)^{m+j} \binom{m}{j} \binom{m+j}{k}.
$$

Remark 1.14. In the case $k = 0$ and $l = 1$, Theorem 1.13(3) turns out to be Corollary 1.5 since $E_n(0) = 0$ if $n$ is even. When $k = 0$ and $l = 3$, Theorem 1.13(3) yield the following interesting identity:

$$
(1.12) \sum_{i=0}^{m} \binom{m}{i} (m+i)(m+i-1)(m+i-2) E_{m+i-3}(0) = 0, \quad m \geq 3,
$$

since $E_m(0) = 0$ if $m$ is even, and it remains valid if replaced $E_n(0)$ by Bernoulli numbers $B_n$ (see [4, Theorem 7.1]).
2. The fermionic $p$-adic integrals and Witt’s formula

In the next two sections, we shall use the following notations.

- $p$ – an odd rational prime number.
- $\mathbb{Z}_p$ – the ring of $p$-adic integers.
- $\mathbb{Q}_p$ – the field of fractions of $\mathbb{Z}_p$.
- $\mathbb{C}_p$ – the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

By using Cauchy’s formula, it is known that Bernoulli numbers and Euler numbers can be represented by the contour integrals

\begin{align*}
B_n &= \frac{n!}{2\pi i} \oint \frac{z}{e^z - 1} \, dz, \\
E_n &= \frac{n!}{2\pi i} \oint \frac{2e^z}{e^{2z} + 1} \, dz
\end{align*}

(see [4, (2.1) and (2.2)]).

The following Witt’s formula represents Euler polynomials through the fermionic $p$-adic integrals.

Lemma 2.1 (Witt’s formula). For any $n \in \mathbb{N}_0$ and $x \in \mathbb{Q}_p$, we have

$$\int_{\mathbb{Z}_p} (x + a)^n d\mu_{-1}(x) = E_n(a).$$

Furthermore, $\int_{\mathbb{Z}_p} d\mu_{-1}(x) = E_0(a) = 1$.

In this section, we shall give a brief overview of the definition and identities for the fermionic $p$-adic integrals, for details, we refer to [12]. The fermionic $p$-adic integral of a function $f : \mathbb{Z}_p \to \mathbb{C}_p$ is defined by

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} f(x)(-1)^x,$$

and this limits exists if $f$ is continuous on $\mathbb{Z}_p$. The fermionic $p$-adic integral (2.3) were independently found by Katz [3, p. 486] (in Katz’s notation, the $\mu^{(2)}$-measure), Shiratani and Yamamoto [20], Christol [5], Osipov [18], Lang [14] (in Lang’s notation, the $E_{1,2}$-measure), T. Kim [10] from very different viewpoints. Recently, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is used by the authors [12] to present many properties of the $p$-adic Hurwitz-type Euler zeta functions, including Laurent series expansions, functional equations, derivative formulas, and $p$-adic Raabe formulas, and it has also been used by Kim [11] and Maïga [15] to give some new identities and congruences concerning Euler numbers and polynomials.

Let $D \subset \mathbb{C}_p$ is an arbitrary subset closed under $a \to a + x$ for $x \in \mathbb{Z}_p$ and $a \in D$. That is, $D$ could be $\mathbb{C}_p \setminus \mathbb{Z}_p$, $\mathbb{Q}_p \setminus \mathbb{Z}_p$ or $\mathbb{Z}_p$.

For a continuous function $f$ on $\mathbb{Z}_p$, the fermionic $p$-adic integral

$$F(a) = \int_{\mathbb{Z}_p} f(x + a) d\mu_{-1}(x), \quad (a \in D)$$
is then defined and satisfies the equation
\[(2.5) \quad F(a + 1) + F(a) = 2f(a).\]
(See, e.g., [12, p. 2982, Theorem 2.2(1)]). From \((2.5)\), we have
\[(2.6) \quad F(a + q) + F(a + q - 1) = 2f(a + q - 1),\]
where \(a \in D\) and \(q \in \mathbb{N}\). Adding and subtracting this identity alternatively for \(q = 1, 2, \ldots, n\) gives the formula
\[(2.7) \quad (-1)^{q-1}F(a + q) + F(a) = 2 \sum_{i=0}^{q-1} (-1)^i f(a + i),\]
where \(a \in D\) and \(q \in \mathbb{N}\).

Note that the above identity \((2.7)\) reduces to [10, Theorem 2] by setting \(a = 0\).

In order to prove \((1.8)\), Theorem 1.1, 1.6, 1.9, 1.12 and 1.13, we need the following lemma which has been obtained by T. Kim in [10, Lemma 1], and Kim and Hu in [12, Theorem 2.2(2)].

**Lemma 2.2.** Let \(f\) be a continuous function on \(\mathbb{Z}_p\). We have the functional equation
\[(2.8) \quad I_{-1}(f_-) = I_{-1}(f_1) = -I_{-1}(f) + 2f(0),\]
where \(f_-(x) = f(-x), f_1(x) = f(x + 1)\) for all \(x \in \mathbb{Z}_p\). In particular, if \(f\) is an even function, then
\[I_{-1}(f) = f(0).\]

Using \((2.5)\) and Lemma 2.2, we get Witt’s formula for Euler polynomials (see Lemma 2.1 above).

3. **Proofs of \((1.8)\), Theorem 1.1, 1.6, 1.9, 1.12 and 1.13**

In this section, we prove six results for Euler polynomials. Many more identities can be obtained from this way easily.

**Proof of \((1.8)\).** For \(a \in D\), we consider the polynomial function
\[f(x) = (-1)^m(x + a)^m(x + a - 1)^n\]
on \(\mathbb{Z}_p\). Then by the binomial theorem, we have
\[(3.1) \quad f(x + 1) = (-1)^m(x + a + 1)^m(x + a)^n = (-1)^m \sum_{i=0}^{m} \binom{m}{i} (x + a)^{n+i}\]
and
\[(3.2) \quad f(-x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (x - a)^{m+j}.\]

Note that \(f(x)\) is continuous on \(\mathbb{Z}_p\), as a product of two such functions, so the fermionic \(p\)-adic integral for the functions in the above equalities are
well-defined. By integrating both sides of (3.1) and (3.2) respectively, we have

\[(3.3) \int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} \int_{\mathbb{Z}_p} (x+a)^{n+i}d\mu_{-1}(x),\]

and

\[(3.4) \int_{\mathbb{Z}_p} f(-x)d\mu_{-1}(x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} \int_{\mathbb{Z}_p} (x-a)^{m+j}d\mu_{-1}(x).\]

(2.8) in Lemma 2.2 leads to

\[(3.5) \int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) = \int_{\mathbb{Z}_p} f(-x)d\mu_{-1}(x).\]

Then comparing (3.3), (3.4) and (3.5), we have

\[(-1)^m \sum_{i=0}^{m} \binom{m}{i} \int_{\mathbb{Z}_p} (x+a)^{n+i}d\mu_{-1}(x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} \int_{\mathbb{Z}_p} (x-a)^{m+j}d\mu_{-1}(x).\]

Applying Lemma 2.1 the Witt’s formula, we get (1.8).

□

Proof of Theorem 1.1. Let \(a \in D, q, k \in \mathbb{N}\) and let \(m\) and \(n\) be nonnegative integers with \(m+n > 0\). If we define the polynomials function \(h(x)\) on \(\mathbb{Z}_p\) by

\[h(x) = (x+a)^{m+q}(x+a-1)^{n+q} + (-1)^{m+n}(x-a-1)^{m+q}(x-a)^{n+q},\]

then we have

\[h(x+1) = h(-x).\]

Taking \((\frac{d}{dx})^k\) on the both sides of the above identity, we have

\[h^{(k)}(x+1) = \frac{d^k h(x+1)}{dx^k} = \frac{d^k h(-x)}{dx^k} = (-1)^k h^{(k)}(-x),\]

which gives

\[h^{(k)}(x+1) = -h^{(k)}(-x) \quad \text{for } k \text{ being odd.}\]

By integrating both sides of the above equality, we get

\[(3.6) \int_{\mathbb{Z}_p} h^{(k)}(-x)d\mu_{-1}(x) = -\int_{\mathbb{Z}_p} h^{(k)}(x+1)d\mu_{-1}(x)\]

for \(k\) being odd, since \(h^{(k)}(x)\) is continuous on \(\mathbb{Z}_p\).

Applying (2.8) in Lemma 2.2 to the left hand side of (3.6), we have

\[(3.7) \int_{\mathbb{Z}_p} h^{(k)}(-x)d\mu_{-1}(x) = \int_{\mathbb{Z}_p} h^{(k)}(x+1)d\mu_{-1}(x).\]

Combining (3.6) with (3.7), we get the following formula

\[\int_{\mathbb{Z}_p} h^{(k)}(x+1)d\mu_{-1}(x) = -\int_{\mathbb{Z}_p} h^{(k)}(x+1)d\mu_{-1}(x),\]
or equivalently,
\[ (3.8) \quad \int_{\mathbb{Z}_p} h^{(k)}(x+1)d\mu_1(x) = 0 \]
for \( k \) being odd. To calculate the left-hand side of the above equation, we rewrite \( h(x+1) \) in the form
\[ h(x+1) = \sum_{i=0}^{m+q} \binom{m+q}{i} (x+a)^{n+q+i} + (-1)^{m+n} \sum_{j=0}^{n+q} \binom{n+q}{j} (x-a)^{m+q+j}. \]

Then
\[ h^{(k)}(x+1) = k! \sum_{i=0}^{m+q} \binom{m+q}{i} \binom{n+q+i}{k} (x+a)^{n+q+i-k} \]
\[ + (-1)^{m+n} k! \sum_{j=0}^{n+q} \binom{n+q}{j} \binom{m+q+j}{k} (x-a)^{m+q+j-k}. \]

Applying (3.8) to the above \( k \)-th derivative \( h^{(k)}(x+1) \), and use Lemma 2.1, the Witt's formula, we conclude our assertion for \( k \) being odd. \( \square \)

Proof of Theorem 1.6. For \( a+b+c=1 \), we consider the polynomial function
\[ g(x) = (-1)^m(x+a+b-1)^m(x+b-1)^n \]
on \( \mathbb{Z}_p \).

(1) We have
\[ (3.9) \quad g(x+1) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} a^{m-i}(x+b)^{n+i}. \]

(2) Since \( a+b+c=1 \), we have
\[ g(-x) = (-1)^m(-x+a+b-1)^m(-x+b-1)^n \]
\[ = (-1)^n (x-a-b+1)^m (x+b+1)^n \]
\[ = (-1)^n (x+c)^m (x+a+c)^n \]
\[ = (-1)^n \sum_{j=0}^{n} \binom{n}{j} a^{n-j}(x+c)^{m+j}. \]

By integrating both sides of (3.9) and (3.10) respectively, we have
\[ (3.11) \quad \int_{\mathbb{Z}_p} g(x+1)d\mu_1(x) = (-1)^m \sum_{i=0}^{m} \binom{m}{i} a^{m-i} \int_{\mathbb{Z}_p} (x+b)^{n+i}d\mu_1(x) \]
and
\[ (3.12) \quad \int_{\mathbb{Z}_p} g(-x)d\mu_1(x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} a^{n-j} \int_{\mathbb{Z}_p} (x+c)^{m+j}d\mu_1(x). \]

(2.8) in Lemma 2.2 leads to
\[ (3.13) \quad \int_{\mathbb{Z}_p} g(x+1)d\mu_1(x) = \int_{\mathbb{Z}_p} g(-x)d\mu_1(x). \]
Then comparing (3.11) (3.12) and (3.13), we get

\[ (-1)^m \sum_{i=0}^{m} \binom{m}{i} a^{m-i} \int_{\mathbb{Z}_p} (x + b)^{n+i} d\mu_{-1}(x) \]

\[ = (-1)^n \sum_{j=0}^{n} \binom{n}{j} a^{n-j} \int_{\mathbb{Z}_p} (x + c)^{m+j} d\mu_{-1}(x). \]

This implies Theorem 1.6 by using Lemma 2.1. \hfill □

Proof of Theorem 1.9. Let \( a \in D, s \in \mathbb{N} \) and let \( k, m, n \in \mathbb{N}_0 \) such that \( m + n > 0 \). Let us define the polynomials \( P_{m,n,s}(x; a) \) by

\[ P_{m,n,s}(x; a) = (x + a)^{m+1} + (-1)^{m+n}(x - a)^{n+1} \]

on \( \mathbb{Z}_p \). Then we have

\[ P_{m,n,s}(x + s + 1; a) = P_{m,n,s}(-x; a), \]

which gives

\[ P^{(k)}_{m,n,s}(x + s + 1; a) = (-1)^{k} P^{(k)}_{m,n,s}(-x; a). \]

It is easily seen that

\[ \sum_{l=1}^{s} (-1)^l \left( P^{(k)}_{m,n,s}(x + j + 1; a) + P^{(k)}_{m,n,s}(x + j; a) \right) \]

\[ = -P^{(k)}_{m,n,s}(x + 1; a) + (-1)^{s} P^{(k)}_{m,n,s}(x + s + 1; a). \]

Therefore we have

\[ (-1)^{s} \int_{\mathbb{Z}_p} P^{(k)}_{m,n,s}(x + s + 1; a) d\mu_{-1}(x) \]

\[ = \sum_{l=1}^{s} (-1)^l \int_{\mathbb{Z}_p} \left( P^{(k)}_{m,n,s}(x + l + 1; a) + P^{(k)}_{m,n,s}(x + l; a) \right) d\mu_{-1}(x) \]

\[ + \int_{\mathbb{Z}_p} P^{(k)}_{m,n,s}(x + 1; a) d\mu_{-1}(x) \]

\[ = 2 \sum_{l=1}^{s} (-1)^l P^{(k)}_{m,n,s}(l; a) + \int_{\mathbb{Z}_p} P^{(k)}_{m,n,s}(-x; a) d\mu_{-1}(x) \]

(using (2.5) and (2.8) in Lemma 2.2)

\[ = 2 \sum_{l=1}^{s} (-1)^l P^{(k)}_{m,n,s}(l; a) + (-1)^{k} \int_{\mathbb{Z}_p} P^{(k)}_{m,n,s}(x + s + 1; a) d\mu_{-1}(x) \]

(using (3.15))

yields that

\[ \delta_{s,k} \int_{\mathbb{Z}_p} P^{(k)}_{m,n,s}(x + s + 1; a) d\mu_{-1}(x) = 2 \sum_{l=1}^{s} (-1)^l P^{(k)}_{m,n,s}(l; a), \]
where $\delta_{s,k} = ((-1)^s - (-1)^k)$. From (3.14) and the binomial theorem, we have

$$P_{m,n,s}(x + s + 1; a) = \sum_{i=0}^{m+1} \binom{m+1}{i} (s+1)^{m-i+1} (x+a)^{n+i+1}$$

$$+ (-1)^{m+n} \sum_{j=0}^{n+1} \binom{n+1}{j} (s+1)^{n-j+1} (x-a)^{m+j+1},$$

which implies

(3.17)

$$P_{m,n,s}^{(k)}(x + s + 1; a) = k! \sum_{i=0}^{m+1} \binom{m+1}{i} \binom{n+i+1}{k} (s+1)^{m-i+1}$$

$$\times (x+a)^{n+i-k+1}$$

$$+ (-1)^{m+n} k! \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{m+j+1}{k} (s+1)^{n-j+1}$$

$$\times (x-a)^{m+j-k+1}.$$

Since $\int_{\mathbb{F}_p} d\mu_{-1}(x) = E_0(a) = 1$, substituting (3.17) into the left hand side of (3.16), then using Lemma 2.1, we get

$$\delta_{s,k} \left( \sum_{i=0}^{m+1} (s+1)^{m-i+1} \binom{m+1}{i} \binom{n+i+1}{k} E_{n+i-k+1}(a) \right)$$

$$+ (-1)^{m+n} \sum_{j=0}^{n+1} (s+1)^{n-j+1} \binom{n+1}{j} \binom{m+j+1}{k} E_{m+j-k+1}(-a)$$

$$= \frac{2}{k!} \sum_{l=1}^{s} (-1)^l P_{m,n,s}^{(l)}(l; a).$$

This completes our proof. \qed

Proof of Theorem 1.12. For $a \in D$, we consider the polynomial function

$$q(x) = (x + a)^m (x + a - 1)^m$$

on $\mathbb{F}_p$. By the binomial theorem, the formula $q(x)$ and $q(x+1)$ can be rewritten as

$$q(x) = \sum_{i=0}^{m} (-1)^{m+i} \binom{m}{i} (x+a)^{m+i},$$

$$q(x+1) = \sum_{i=0}^{m} \binom{m}{i} (x+a)^{m+i},$$
respectively. Hence, we have

\[ q^{(k)}(x + 1) + q^{(k)}(x) = k! \sum_{i=0}^{m} \binom{m}{i} \binom{m + i}{k} (x + a)^{m+i-k} (1 + (-1)^{m+i}) \]

\[ = \begin{cases} 
2k! \sum_{i=0}^{m} \binom{m}{i} (m+i) (x + a)^{m+i-k} & \text{if } m + i \text{ even} \\
0 & \text{if } m + i \text{ odd} 
\end{cases} \]

and

\[ q^{(k)}(0) = k! \sum_{j=0}^{m} (-1)^{m+j} \binom{m}{j} \binom{m + j}{k} a^{m+j-k}. \]

The second equality in (2.8) of Lemma 2.2 implies

\[ \int_{\mathbb{Z}_p} (q^{(k)}(x + 1) + q^{(k)}(x))d\mu_{-1}(x) = 2q^{(k)}(0). \]

On expanding (3.20) by (3.18) and (3.19), we obtain

\[ \sum_{i=m+i \text{ even}}^{m} \binom{m}{i} \binom{m + i}{k} \int_{\mathbb{Z}_p} (x + a)^{m+i-k} d\mu_{-1}(x) \]

\[ = \sum_{j=0}^{m} (-1)^{m+j} \binom{m}{j} \binom{m + j}{k} a^{m+j-k}, \]

and the result follows from Lemma 2.1.

**Proof of Theorem 1.13.** We consider the polynomial function

\[ r(x) = x^m(x - 1)^m \]

on \( \mathbb{Z}_p \). By the binomial theorem, the formula \( r(x) \) and \( r(x + 1) \) can be rewritten as

\[ r(x) = \sum_{i=0}^{m} (-1)^{m+i} \binom{m}{i} x^{m+i} \quad \text{and} \quad r(x + 1) = \sum_{i=0}^{m} \binom{m}{i} x^{m+i}, \]

respectively. Hence, we have

\[ r^{(k)}(x + 1) + r^{(k)}(x) = k! \sum_{i=0}^{m} \binom{m}{i} \binom{m + i}{k} x^{m+i-k} (1 + (-1)^{m+i}) \]

\[ = \begin{cases} 
2k! \sum_{i=0}^{m} \binom{m}{i} (m+i) x^{m+i-k} & \text{if } m + i \text{ even} \\
0 & \text{if } m + i \text{ odd} 
\end{cases} \]

and

\[ r^{(k)}(x) = k! \sum_{j=0}^{m} (-1)^{m+j} \binom{m}{j} \binom{m + j}{k} x^{m+j-k}. \]
To see Part (1), note that by (3.21),
\[(3.23)\]
\[
\left(\frac{d}{dx}\right)^{m-k} \left[ r^{(k)}(x+1) + r^{(k)}(x) \right]
= 2k!(m-k)! \sum_{\substack{i=0 \atop m+i \text{ even}}}^{m} \binom{m + i}{k} \binom{m + i - k}{m - k} x^i.
\]
Similarly, by (3.22) we obtain
\[(3.24)\]
\[
\left. \left(\frac{d}{dx}\right)^{m-k} \left( r^{(k)}(x) \right) \right|_{x=0}
= k!(m-k)! \sum_{j=0}^{m} (-1)^{m+j} \binom{m+j}{j} \binom{m + j - k}{m - k} x^j \bigg|_{x=0}
= k!(m-k)!(-1)^m \binom{m}{k},
\]
since $0^j = 1$ if $j = 0$ and $0^j = 0$ if $j \in \mathbb{N}$.

The second equality in (2.8) of Lemma 2.2 implies
\[(3.25)\]
\[
\int_{\mathbb{Z}_p} (r^{(m)}(x+1) + r^{(m)}(x)) d\mu_{-1}(x) = 2r^{(m)}(0).
\]
Substituting (3.23) and (3.24) into (3.25), we have
\[(3.26)\]
\[
\sum_{\substack{i=0 \atop m+i \text{ even}}}^{m} \binom{m + i}{k} \binom{m + i - k}{m - k} \int_{\mathbb{Z}_p} x^i d\mu_{-1}(x) = (-1)^m \binom{m}{k},
\]
which leads to Part (1) by using Lemma 2.1 with $a = 0$.

The Parts (2), (3) and (4) can be shown in a similar way with Part (1). \qed

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