McKay correspondence over non algebraically closed fields

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Abstract

The classical McKay correspondence for finite subgroups \( G \) of \( \text{SL}(2, \mathbb{C}) \) gives a bijection between isomorphism classes of nontrivial irreducible representations of \( G \) and irreducible components of the exceptional divisor in the minimal resolution of the quotient singularity \( \mathbb{A}^2_C/G \). Over non algebraically closed fields \( K \) there may exist representations irreducible over \( K \) which split over \( \overline{K} \). The same is true for irreducible components of the exceptional divisor. In this paper we show that these two phenomena are related and that there is a bijection between nontrivial irreducible representations and irreducible components of the exceptional divisor over non algebraically closed fields \( K \) of characteristic 0 as well.

1 Introduction

Let \( G \) be a finite group operating on a smooth variety \( M \) over \( \mathbb{C} \), e.g. \( M = \mathbb{A}^n_C \) and a linear operation of a finite subgroup \( G \subset \text{SL}(n, \mathbb{C}) \). Usually the quotient \( M/G \) is singular and one considers resolutions of singularities \( Y \to M/G \) with some minimality property. A method to construct resolutions of quotient singularities is the \( G \)-Hilbert scheme \( \text{G-Hilb} \) introduced in \([7]\), \([8]\). Under some conditions the \( G \)-Hilbert scheme is irreducible, nonsingular and \( \text{G-Hilb} \to M/G \) a crepant resolution \([1]\). In particular, this applies to the operation of finite subgroups \( G \subset \text{SL}(2, \mathbb{C}) \) on \( \mathbb{A}^n_C \) for \( n \leq 3 \). For \( G \subset \text{SL}(2, \mathbb{C}) \) there are also other methods to show that the \( G \)-Hilbert scheme is the minimal resolution, see \([7]\), \([8]\).

The McKay correspondence in general describes the resolution \( Y \) in terms of the representation theory of the group \( G \), see \([13]\), \([14]\) for expositions of this subject. Part of the correspondence for \( G \subset \text{SL}(2, \mathbb{C}) \) is a bijection between irreducible components of the exceptional divisor \( E \) and isomorphism classes of nontrivial irreducible representations of the group \( G \) and moreover an isomorphism of graphs between the intersection graph of components of \( E \) and the representation graph of \( G \), both being graphs of ADE type. This was the observation of McKay \([11]\).

The new contribution in this paper is to consider McKay correspondence over non algebraically closed fields. We will work over a field \( K \) that is not assumed to be algebraically closed but always of characteristic 0 and extend the McKay correspondence to this slightly more general situation. Over non algebraically closed \( K \) it is natural to consider finite group schemes instead of simply finite groups. In comparison with the situation over algebraically closed fields there may exist both representations of \( G \) and components of \( E \) that are irreducible over \( K \) but split over the algebraic closure. We will see that these two kinds of splitting that arise by extending the ground field are related by investigating the operation of the Galois group. For this we introduce Galois-conjugate representations and consider the Galois operation on the \( G \)-Hilbert scheme. The following McKay correspondence over arbitrary fields \( K \) of characteristic 0 will be consequence of more detailed theorems in section 5.

**Theorem 1.1.** Let \( K \) be any field of characteristic 0 and \( G \subset \text{SL}(2, K) \) a finite subgroup scheme. Then there is a bijection between the set of irreducible components of the exceptional divisor \( E \) and the set of isomorphism classes of nontrivial irreducible representations of \( G \) and moreover an isomorphism between the intersection graph of the irreducible components of \( E_{\text{red}} \) and the representation graph of \( G \).
Examples are discussed in subsection 5.5, the possible graphs can be found in subsection 4.4. As already observed in [10], considering the rational double points over non-algebraically closed fields one finds the remaining Dynkin diagrams of types \((B_n),(C_n),(F_4),(G_2)\). The methods of this paper should also apply to other situations, in particular to the McKay correspondence for finite small subgroups of \(GL(2,\mathbb{C})\) and give a similar generalisation as in the SL-case.

This paper is organised as follows. Section 2 shortly summarises some techniques used in this paper, namely \(G\)-sheaves for group schemes \(G\) and \(G\)-Hilbert schemes. Section 3 is concerned with the relations between Galois operations and decompositions into irreducible components both of schemes and representations. We introduce the notion of Galois-conjugate representations and \(G\)-sheaves and we describe the Galois operation on \(G\)-Hilbert schemes. In section 4 we collect some data of the finite subgroup schemes of \(SL(2,K)\) and list possible representation graphs. In addition we investigate under what conditions a finite subgroup of \(SL(2,C)\), \(C\) the algebraic closure of \(K\), is realisable as a subgroup of \(SL(2,K)\). Section 5 contains the theorems of McKay correspondence over non-algebraically closed fields. We consider two constructions, the stratification of the \(G\)-Hilbert scheme and the tautological sheaves, originating from [7] and [6] respectively, that are known to give a McKay correspondence over \(C\) and formulate them for not necessarily algebraically closed \(K\).

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Notations. In general we write a lower index for base extensions, for example if \(X,T\) are \(S\)-schemes then \(X_T\) denotes the \(T\)-scheme \(X \times_S T\) or if \(V\) is a representation over a field \(K\) then \(V_L\) denotes the representation \(V \otimes_K L\) over the extension field \(L\). Likewise, if \(\varphi : X \to Y\) is a morphism of \(S\)-schemes, we write \(\varphi_T : X_T \to Y_T\) for its base extension with respect to \(T \to S\).

## 2 Preliminaries

### 2.1 G-sheaves

Let \(K\) be a field. Let \(G\) be a group scheme over \(K\) with \(p : G \to \text{Spec} \ K\) the projection, \(e : \text{Spec} \ K \to G\) the unit, and \(m : G \times_K G \to G\) the multiplication. For affine \(G = \text{Spec} \ A\), \(A\) has the structure of a Hopf algebra over \(K\), the coalgebra structure being equivalent to the group structure of \(G\).

Let \(X\) be a \(G\)-scheme over \(K\), that is a \(K\)-scheme with an operation \(s_X : G \times_K X \to X\) of the group scheme \(G\) over \(K\). We have to use a more general notion of a \(G\)-sheaf than in [1], we adopt the definition of [12]: a (quasicoherent, coherent) \(G\)-sheaf on \(X\) is a (quasicoherent, coherent) \(\mathcal{O}_X\)-module \(\mathcal{F}\) with an isomorphism \(\lambda^\mathcal{F} : s_X^* \mathcal{F} \cong \mathcal{O}_X \otimes_{\mathcal{O}_{G \times_K X}} \mathcal{P}_{23}^* \mathcal{F}\) of \(\mathcal{O}_{G \times_K X}\)-modules satisfying the conditions (i) the restriction of \(\lambda^\mathcal{F}\) to the unit in \(G_X\) is the identity, i.e. \(e_X^* \lambda^\mathcal{F} \circ s_X^* \mathcal{F} \cong e_X^* \mathcal{P}_{23}^* \mathcal{F}\) identifies with \(id_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}\), and (ii) \((m \times id_X)^* \lambda^\mathcal{F} = \mathcal{P}_{23}^* \lambda^\mathcal{F} \circ (id_G \times s_X)^* \lambda^\mathcal{F}\), where \(p_{23} : G \times_K G \times_K X \to G \times_K X\) is the projection to the factors 2 and 3.

Remark 2.1. We summarise relevant properties of \(G\)-sheaves.

1. There is the canonical notion of \(G\)-equivariant homomorphisms between \(G\)-sheaves \(\mathcal{F}, \mathcal{G}\) on \(X\), the set of these is denoted by \(\text{Hom}_G^X(\mathcal{F}, \mathcal{G})\). Kernels and cokernels of \(G\)-equivariant homomorphisms have natural \(G\)-sheaf structures.
2. Assume \(G = \text{Spec} \ A\) affine and let \(X\) be a \(G\)-scheme with trivial \(G\)-operation, i.e. \(s_X = p_X\). Then the \(G\)-sheaf structure of a \(G\)-sheaf \(\mathcal{F}\) is equivalent to a homomorphism of \(\mathcal{O}_X\)-modules
$g: \mathcal{F} \to A \otimes_K \mathcal{F}$ satisfying the usual conditions of a comodule. This relation can be constructed using the adjunction $(p_X^*, p_X)$). Further, notions such as “subcomodule”, “homomorphism of comodules”, etc. correspond to “$G$-subsheaf”, “equivariant homomorphism”, etc. The $G$-invariant part $\mathcal{F}^G \subseteq \mathcal{F}$ is defined by $\mathcal{F}^G(U) := \{ f \in \mathcal{F}(U) \mid g(f) = 1 \otimes f \}$ for open $U \subseteq X$.

(3) For an $A$-comodule $\mathcal{F}$ on $X$ a decomposition of $A$ into a direct sum $A = \bigoplus A_i$ of subcoalgebras $A_i$ determines a direct sum decomposition $\mathcal{F} = \bigoplus \mathcal{F}_i$ into subcomodules (take preimages $g^{-1}(A_i \otimes_K \mathcal{F})$), where the comodule structure of $\mathcal{F}_i$ reduces to an $A_i$-comodule structure.

(4) A $G$-sheaf on $X = \text{Spec } K$ (or an extension field of $K$) we also call a representation. Dualisation of an $A$-comodule $\mathcal{V}$ over $K$ leads to a $K^G$-module $\mathcal{V}^\vee$, where $K^G = A^\vee = \text{Hom}_K(A, K)$ with algebra structure dual to the coalgebra structure of $A$.

(5) For quasicoherent $G$-sheaves $\mathcal{F}, \mathcal{G}$ with $\mathcal{F}$ finitely presented the sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ carries a natural $G$-sheaf structure. For locally free $\mathcal{F}$ one defines the dual $G$-sheaf by $\mathcal{F}^\vee = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. In the case of trivial $G$-operation on $X$ there is the component $\text{Hom}_{\mathcal{O}_X}^G(\mathcal{F}, \mathcal{G})$ of $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, the sheaf of equivariant homomorphisms, that can either be described as $G$-invariant part ($\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))^G \subseteq \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ or by $\text{Hom}_{\mathcal{O}_X}^G(\mathcal{F}, \mathcal{G})(U) = \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ for open $U \subseteq X$.

(6) Functors for sheaves like $\otimes, f^*, \ldots$ as well have analogues for $G$-sheaves, e.g. for equivariant $f: Y \to X$ and a $G$-sheaf $\mathcal{F}$ on $X$ the sheaf $f^* \mathcal{F}$ has a natural $G$-sheaf structure.

(7) Natural isomorphisms for sheaves lead to isomorphisms for $G$-sheaves, e.g. under some conditions there is an isomorphism $f^* \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathcal{O}_Y}(f^* \mathcal{F}, f^* \mathcal{G})$ and this isomorphism becomes an isomorphism of $G$-sheaves provided that $f$ is equivariant and $\mathcal{F}, \mathcal{G}$ are $G$-sheaves. Other examples are $f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \cong f^* \mathcal{F} \otimes_{\mathcal{O}_Y} f^* \mathcal{G}$ and $\text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \cong \text{Hom}_{\mathcal{O}_X}^G(\mathcal{F}, \mathcal{G})^\vee \otimes_{\mathcal{O}_X} \mathcal{H}$.

(8) Base extension $K \to L$ makes out of a $G$-scheme $X$ over $K$ a scheme $X_L$ with a $G$-scheme or a $G_L$-scheme structure, the operation given by $s_{X_L} = (s_X)_L$. A $G$-sheaf $\mathcal{F}$ on a $G$-scheme $X$ gives rise to a $G$-sheaf $\mathcal{F}_L = \mathcal{F} \otimes_K L = f^* \mathcal{F}$ on $X_L$, where $f: X_L \to X$. $\mathcal{F}_L$ can be considered as a $G_L$-sheaf on the $G_L$-scheme $X_L$ over $L$.

### 2.2 G-Hilbert schemes

Let $G = \text{Spec } A$ be a finite group scheme over a field $K$, assume that its Hopf algebra $A$ is cosemisimple (that is, $A$ is sum of its simple subcoalgebras, see [17, Ch. XIV] and subsection $\S3.1$ below).

For us the $G$-Hilbert scheme $\text{G-Hilb}_K X$ of a $G$-scheme $X$ over $K$ will be by definition the moduli space of $G$-clusters, i.e. parametrising $G$-stable finite closed subschemes $Z \subseteq X_L$, $L$ an extension field of $K$, with $H^0(Z, \mathcal{O}_Z)$ isomorphic to the regular representation of $G$ over $L$. We recall its construction (a variation of the Quot scheme construction of [3]), for a detailed discussion including the generalisation to finite group schemes with cosemisimple Hopf algebra over arbitrary base fields see [2].

Let $X$ be a $G$-scheme algebraic over $K$, assume that a geometric quotient $\pi: X \to X/G$, $\pi$ affine, exists. Then the $G$-Hilbert functor $\text{G-Hilb}_K X: (K\text{-schemes})^0 \to (\text{sets})$ given by

$$ \text{G-Hilb}_K X(T) := \left\{ \begin{array}{ll} \text{Quotient } G\text{-sheaves } [0 \to \mathcal{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \text{ on } X_T, \\
Z \text{ finite flat over } T, \text{ for } t \in T: H^0(Z_t, \mathcal{O}_{Z_t}) \text{ isomorphic to the regular representation} 
\end{array} \right\} $$

is represented by an algebraic $K$-scheme $\text{G-Hilb}_K X$. Here we write $[0 \to \mathcal{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0]$ for an exact sequence $0 \to \mathcal{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0$ of quasicoherent $G$-sheaves on $X_T$ with
\(\mathcal{F}, \mathcal{O}_Z\) specified up to isomorphism, that is either a quasicoherent \(G\)-subsheaf \(\mathcal{F} \subseteq \mathcal{O}_X\) or an equivalence class \([\mathcal{O}_X \to \mathcal{O}_Z]\) of surjective equivariant homomorphisms of quasicoherent \(G\)-sheaves with two of them equivalent if their kernels coincide.

There is the natural morphism \(\tau: G\text{-Hilb}_K X \to X/G\), which is projective and as a map of points takes \(G\)-clusters to the corresponding orbits.

In this paper we are interested in the case \(G \subset \text{SL}(2, K)\) operating on \(X = \mathbb{A}_{K}^{2}\) over fields \(K\) of characteristic 0.

**Proposition 2.2.** The \(G\)-Hilbert scheme \(G\text{-Hilb}_K \mathbb{A}_{K}^{2}\) is irreducible and nonsingular. The morphism \(\tau: G\text{-Hilb}_K \mathbb{A}_{K}^{2} \to \mathbb{A}_{K}^{2}/G\) is birational and the minimal resolution of \(\mathbb{A}_{K}^{2}/G\).

*Proof.* This is known for algebraically closed fields of characteristic 0 [7], [8], [1]. From this the statements about irreducibility and nonsingularity for not necessarily algebraically closed \(K\) follow, use that for \(C\) the algebraic closure \((G\text{-Hilb}_K \mathbb{A}_{K}^{2})_C \cong G_{C}\text{-Hilb}_C \mathbb{A}_{C}^{2}\) (see [2]). The morphism \(\tau: G\text{-Hilb}_K \mathbb{A}_{K}^{2} \to \mathbb{A}_{K}^{2}/G\) is known to be birational. The base extension \((G\text{-Hilb}_K \mathbb{A}_{K}^{2})_C \to (\mathbb{A}_{K}^{2}/G)_C\) identifies with the natural morphism \(G_{C}\text{-Hilb}_C \mathbb{A}_{C}^{2} \to \mathbb{A}_{C}^{2}/G_C\) (follows directly from the functorial definition of \(\tau\), see e.g. [2]). So the statement about minimality as well follows from the same statement for algebraically closed fields. \(\Box\)

### 3 Galois operation and irreducibility

#### 3.1 (Co)semisimple (co)algebras and Galois extensions

Let \(K\) be a field and \(K \to L\) a Galois extension, \(\Gamma := \text{Aut}_K(L)\). As reference for simple and semisimple algebras we use [3, Algèbre, Ch. VIII], for coalgebras and comodules [17]. Note that for a \(K\)-vector space \(V\) (maybe with some additional structure) \(\Gamma\) operates on the base extension \(V_L = V \otimes_K L\) via the second factor.

**Proposition 3.1.** Let \(F\) be a simple \(K\)-algebra. Assume that \(F_L\) is semisimple, let \(F_L = \bigoplus_{i=1}^{r} F_{L,i}\) be its decomposition into simple components. Then \(\Gamma\) permutes the simple summands \(F_{L,i}\) and the operation on the set \(\{F_{L,1}, \ldots, F_{L,r}\}\) is transitive.

*Proof.* The \(F_{L,i}\) are the minimal two-sided ideals of \(F_L\). Since any \(\gamma \in \Gamma\) is an automorphism of \(F_L\) as a \(K\)-algebra or ring, the \(F_{L,i}\) are permuted by \(\Gamma\).

Let \(U = \sum_{\gamma \in \Gamma} \gamma F_{L,1}\) and \(V\) the sum over the remaining \(F_{L,i}\). Then \(F_L = U \oplus V\), \(U\) and \(V\) are \(\Gamma\)-stable and thus \(U = U_L'\), \(V = V_L'\) for \(K\)-subspaces \(U', V' \subseteq F\) by [3, Algebra II, Ch. V, §10.4], since \(K \to L\) is a Galois extension. It follows that \(F = U' \oplus V'\) with \(U', V'\) two-sided ideals of \(F\). Since \(F\) is simple, \(V' = 0\), \(U = F_L\) and the operation is transitive. \(\Box\)

A coalgebra \(C \neq 0\) is called simple, if it has no subcoalgebras except \(\{0\}\) and \(C\). A coalgebra is called cosemisimple, if it is the sum of its simple subcoalgebras, in which case this sum is direct. For cosemisimple \(C\) the simple subcoalgebras are the isotypic components of \(C\) as a \(C\)-comodule (left or right), so they correspond to the isomorphism classes of simple representations of \(G\) over \(K\).

**Proposition 3.2.** Let \(C\) be a finite dimensional coalgebra over \(K\). Then \(C\) is cosemisimple if and only if \(C_L\) is cosemisimple.

*Proof.* This is equivalent to the dual statement for finite dimensional semisimple \(K\)-algebras [3, Algèbre, Ch. VIII, §7.6, Thm. 3, Cor. 4]. \(\Box\)
For simple coalgebras there is a result similar to proposition 3.1 and proven analogously, note that simple coalgebras are finite dimensional.

**Proposition 3.3.** Let $C$ be a simple coalgebra over $K$. Then $C_L$ is cosemisimple, and if $C_L = \bigoplus_i C_{L,i}$ is its decomposition into simple components, then $\Gamma$ transitively permutes the simple summands $C_{L,i}$.

**Corollary 3.4.** Let $C$ be a cosemisimple coalgebra over $K$. Then $C_L$ is cosemisimple, and if $C = \bigoplus_j C_j$ resp. $C_L = \bigoplus_j C_{L,i}$ are the decompositions into simple subcoalgebras, then:

(i) The decomposition $C_L \cong \bigoplus_i C_{L,i}$ is a refinement of the decomposition $C_L \cong \bigoplus_j (C_j)_L$.

(ii) $\Gamma$ transitively permutes the summands $C_{L,i}$ of $(C_j)_L$ for any $j$.

Therefore $(C_j)_L = \sum_{\gamma \in \Gamma} \gamma C_{L,i}$, if $C_{L,i}$ is a summand of $(C_j)_L$.

This applies to the situation considered in this paper. Assume that the field $K$ is of characteristic 0 and let $G = \text{Spec} A$ be a finite group scheme over $K$, $|G| := \dim_K A$. Define $KG$ to be the $K$-vector space $A' = \text{Hom}_K(A, K)$ with algebra structure dual to the coalgebra structure of $A$.

In this situation the algebra $A$ is always reduced and for a suitable algebraic extension field $L$ of $K$ the group scheme $G_L$ is discrete. Then $G(L)$ is a finite group of order $|G|$ and the algebra $LG = (KG)_L$ is isomorphic to the group algebra of the group $G(L)$ over $L$. By semisimplicity of group algebras for finite groups over fields of characteristic 0 and proposition 3.2 one obtains:

**Proposition 3.5.** Let $G = \text{Spec} A$ be a finite group scheme over a field $K$ of characteristic 0. Then the Hopf algebra $A$ is cosemisimple and so are its base extensions $A_L$ with respect to field extensions $K \to L$.

### 3.2 Irreducible components of schemes and Galois extensions

Let $X$ be a $K$-scheme. For an extension field $L$ of $K$ the group $\Gamma = \text{Aut}_K(L)$ operates on $X_L = X \times_K \text{Spec} L$ by automorphisms of $K$-schemes via the second factor. For simplicity we denote the morphisms $\text{Spec} L \to \text{Spec} L$, $X_L \to X_L$ coming from $\gamma : L \to L$ by $\gamma$ as well.

A point of $X$ may decompose over $L$, this way a point $x \in X$ corresponds to a set of points of $X_L$, the preimage of $x$ with respect to the projection $X_L \to X$. In particular this applies to closed points and to irreducible components. These sets are known to be exactly the $\Gamma$-orbits.

**Proposition 3.6.** Let $X$ be an algebraic $K$-scheme and $K \to L$ be a Galois extension, $\Gamma := \text{Aut}_K(L)$. Then points of $X$ correspond to $\Gamma$-orbits of points of $X_L$, the $\Gamma$-orbits are finite.

**Proof.** Taking fibers, the proposition reduces to the following statement:

Let $F$ be the quotient field of a commutative integral $K$-algebra of finite type. Then $F_L = F \otimes_K L$ has only finitely many prime ideals and they are $\Gamma$-conjugate.

**Proof.** $F_L$ is integral over $F$ because this property is stable under base extension [3, Commutative Algebra, Ch. V, §1.1, Prop. 5]. It is clear that every prime ideal of $F_L$ lies above the prime ideal $(0)$ of $F$. There are no inclusions between the prime ideals of $F_L$ [3, Commutative Algebra, Ch. V, §2.1, Proposition 1, Corollary 1]. Since every prime ideal of $F_L$ is a maximal ideal and $F_L$ is noetherian (a localisation of an $L$-algebra of finite type), $F_L$ is artinian, it has only finitely many prime ideals $Q_1, \ldots, Q_r$.

$F_L$ has trivial radical [3, Algèbre, Ch. VIII, §7.3, Thm. 1, also §7.5 and §7.6, Cor. 3]. Being an artinian ring without radical, i.e. semisimple [3, Algèbre, Ch. VIII, §6.4, Thm. 4, Cor. 2 and Prop. 9], $F_L$ decomposes as a $L$-algebra into a direct sum

$$F_L \cong \bigoplus_{i=1}^r F_{L,i}$$
of fields $F_{L,i} \cong F_L/Q_i$ (this can easily be seen directly, however, it is part of the general theory of semisimple algebras developed in [3] Algèbre, Ch. VIII that contains the representation theory of finite groups schemes with cosemisimple Hopf algebra as another special case).

$\Gamma$ operates on $F_L$, it permutes the $Q_i$ and the simple components $F_{L,i}$ of $F_L$ transitively by proposition [4,1].

\section*{3.3 Galois operation on G-Hilbert schemes}

Let $Y$ be a $K$-scheme, $L$ an extension field of $K$ and $\Gamma = \text{Aut}_K(L)$.

For an $L$-scheme $f : T \to \text{Spec} L$ and $\gamma \in \Gamma$ define the $L$-scheme $\gamma_* T$ to be the scheme $T$ with structure morphism $\gamma \circ f$. For a morphism $\alpha : T' \to T$ of $L$-schemes let $\gamma_* \alpha$ be the same morphism $\alpha$ considered as an $L$-morphism $\gamma_* T' \to \gamma_* T$.

For a morphism $\alpha : Y_L \to Y'_L$ of $L$-schemes and $\gamma \in \Gamma$ define the conjugate morphism $\alpha^\gamma$ by $\alpha^\gamma := \gamma \circ (\gamma_* \alpha) \circ \gamma^{-1}$, which again is a morphism of $L$-schemes. Here $\gamma : \gamma_* Y_L \to Y_L$ is a morphism over $L$.

Let $T$ be an $L$-scheme defined over $K$, that is $T = T'_L$ for some $K$-scheme $T'$. The group $\Gamma$ operates on the set $Y_L(T)$ of morphisms $T \to Y_L$ over $L$ by

$$
\gamma : Y_L(T) \to Y_L(T) \\
\alpha \mapsto \alpha^\gamma = \gamma \circ (\gamma_* \alpha) \circ \gamma^{-1}
$$

Consider the case of $G$-Hilbert schemes. Let $G$ be a finite group scheme over $K$, $X$ be a $G$-scheme over $K$ and assume that the $G$-Hilbert functor is represented by a $K$-scheme $G \text{-Hilb}_{K} X$. There is the canonical isomorphism of $L$-schemes $G \text{-Hilb}_{K} X) \cong G_{L} \text{-Hilb}_{L} X_{L}$ (see [2]), obtained by identifying $X \times_{K} T = X_{L} \times_{L} T$ for $L$-schemes $T$.

\begin{proposition}
Let $T$ be an $L$-scheme defined over $K$. Then, for a morphism $\alpha : T \to G_{L} \text{-Hilb}_{L} X_{L}$ of $L$-schemes corresponding to a quotient $[0 \to \mathcal{I} \to \mathcal{O}_{X_{T}} \to \mathcal{O}_{Z} \to 0]$ and for $\gamma \in \Gamma$, the $\gamma$-conjugate morphism $\alpha^\gamma$ corresponds to the quotient $[0 \to \gamma_* \mathcal{I} \to \mathcal{O}_{X_{T}} \to \mathcal{O}_{\gamma Z} \to 0]$.
\end{proposition}

\begin{proof}
For a morphism of $L$-schemes $\alpha : T \to G_{L} \text{-Hilb}_{L} X_{L} \cong (G \text{-Hilb}_{K} X) \times_{K} \text{Spec} L$ consider the commutative diagram of $L$-morphisms

$$
\begin{array}{ccc}
\gamma_* T & \xrightarrow{\gamma_* \alpha} & (G \text{-Hilb}_{K} X) \times_{K} (\gamma_* \text{Spec} L) \\
\downarrow \gamma & & \downarrow \text{id} \times \gamma \\
T & \xrightarrow{\alpha^\gamma} & (G \text{-Hilb}_{K} X) \times_{K} \text{Spec} L
\end{array}
$$

The morphism $\alpha$ is given by a quotient $[0 \to \mathcal{I} \to \mathcal{O}_{X_{T}} \to \mathcal{O}_{Z} \to 0]$ on $X_{T} = X_{L} \times_{L} T$.

Under the identification $G_{L} \text{-Hilb}_{L} X_{L} = (G \text{-Hilb}_{K} X_{K})_{L}$ the $T$-valued point $\alpha$ corresponds to a morphism $T \to G \text{-Hilb}_{K} X$ of $K$-schemes, that is a quotient $[0 \to \mathcal{I} \to \mathcal{O}_{X \times_{K} T} \to \mathcal{O}_{Z} \to 0]$ on $X \times_{K} T$, and the structure morphism $f : T \to \text{Spec} L$. We have the correspondences

$$
\begin{align*}
\alpha & \iff \begin{cases} [0 \to \mathcal{I} \to \mathcal{O}_{X \times_{K} T} \to \mathcal{O}_{Z} \to 0] \\
f : T \to \text{Spec} L \\
[0 \to \mathcal{I} \to \mathcal{O}_{X \times_{K} \gamma_{T}} \to \mathcal{O}_{Z} \to 0] \\
\gamma \circ (\gamma_{*} f) : \gamma_{*} T \to \text{Spec} L \\
[0 \to \gamma^{-1} \mathcal{I} \to \mathcal{O}_{X \times_{K} T} \to \gamma^{-1} \mathcal{O}_{Z} \to 0] \\
f = \gamma \circ (\gamma_{*} f) \circ \gamma^{-1} : T \to \text{Spec} L \\
[0 \to \gamma_{*} \mathcal{I} \to \mathcal{O}_{X \times_{K} T} \to \mathcal{O}_{\gamma Z} \to 0] \\
f = \gamma \circ (\gamma_{*} f) \circ \gamma^{-1} : T \to \text{Spec} L
\end{cases} \\
\gamma \circ (\gamma_{*} \alpha) & \iff \begin{cases} [0 \to \mathcal{I} \to \mathcal{O}_{X \times_{K} T} \to \mathcal{O}_{Z} \to 0] \\
[0 \to \mathcal{I} \to \mathcal{O}_{X \times_{K} \gamma_{T}} \to \mathcal{O}_{Z} \to 0] \\
\gamma \circ (\gamma_{*} f) : \gamma_{*} T \to \text{Spec} L \\
[0 \to \gamma^{-1} \mathcal{I} \to \mathcal{O}_{X \times_{K} T} \to \gamma^{-1} \mathcal{O}_{Z} \to 0] \\
\alpha^{\gamma} = \gamma \circ (\gamma_{*} \alpha) \circ \gamma^{-1}
\end{cases}
\end{align*}
$$

\end{proof}
Under the identification \((\text{G-Hilb}_K X)_L = G_L \cdot \text{Hilb}_L X_L\) the last morphism corresponds to the quotient \([0 \to \gamma_* F \to \mathcal{O}_T \to \mathcal{O}_Z \to 0]\) on \(T = X_L \times L T\).

In particular, in the case \(X = \mathbb{A}^2_K\) the \(\gamma\)-conjugate of an \(L\)-valued point given by an ideal \(I \subseteq L[x_1, x_2]\) or a \(G_L\)-cluster \(Z \subset \mathbb{A}^2_L\) is given by the \(\gamma\)-conjugate ideal \(\gamma^{-1} I \subseteq L[x_1, x_2]\) or the \(\gamma\)-conjugate \(G_L\)-cluster \(\gamma Z \subset \mathbb{A}^2_L\).

Every point \(x\) of the \(L\)-scheme \(G_L \cdot \text{Hilb}_L \mathbb{A}^2_L\) such that \(\kappa(x) = L\) corresponds to a unique \(L\)-valued point \(\alpha: \text{Spec} L \to G_L \cdot \text{Hilb}_L \mathbb{A}^2_L\). The \(\gamma\)-conjugate point \(\gamma x\) corresponds to the \(\gamma\)-conjugate \(L\)-valued point \(\alpha^\gamma: \text{Spec} L \to G_L \cdot \text{Hilb}_L \mathbb{A}^2_L\), which is given by the ideal \(\gamma^{-1} I \subseteq L[x_1, x_2]\).

**Corollary 3.8.** Let \(x\) be a closed point of \(G_L \cdot \text{Hilb}_L \mathbb{A}^2_L\) such that \(\kappa(x) = L\), \(\alpha: \text{Spec} L \to G_L \cdot \text{Hilb}_L \mathbb{A}^2_L\) the corresponding \(L\)-valued point given by an ideal \(I \subseteq L[x_1, x_2]\). Then for \(\gamma \in \Gamma\) the conjugate point \(\gamma x\) corresponds to the \(\gamma\)-conjugate \(L\)-valued point \(\alpha^\gamma: \text{Spec} L \to G_L \cdot \text{Hilb}_L \mathbb{A}^2_L\), which is given by the ideal \(\gamma^{-1} I \subseteq L[x_1, x_2]\).}

**3.4 Conjugate \(G\)-sheaves**

Let \(G = \text{Spec} A\) be a group scheme over a field \(K\), \(X\) be a \(G\)-scheme over \(K\), let \(K \to L\) be a field extension and \(\Gamma = \text{Aut}_K(L)\). Again, \(\Gamma\) operates on \(X_L\) by automorphisms \(\gamma: X_L \to X_L\) over \(K\), these are equivariant with respect to the \(G\)-scheme structure of \(X_L\) defined in remark 2.3 (8).

**Proposition–Definition 3.9.** Let \(\mathcal{F}\) be a \(G_L\)-sheaf on \(X_L\). For \(\gamma \in \Gamma\) the \(\mathcal{O}_{X_L}\)-module \(\gamma_* \mathcal{F}\) has a natural \(G_L\)-sheaf structure given by

\[
\begin{array}{ccc}
\gamma_* s_{\gamma x}^* \mathcal{F} & \xrightarrow{\gamma_* \lambda^\mathcal{F}} & \gamma_* p_{\gamma x}^* \mathcal{F} \\
\gamma^\gamma_* s_{\gamma x}^* \mathcal{F} & \xleftarrow{\lambda^\gamma^* \mathcal{F}} & p_{\gamma x}^* \gamma^\gamma_* \mathcal{F}
\end{array}
\]

This \(G_L\)-sheaf \(\gamma_* \mathcal{F}\) is called the \(\gamma\)-conjugate \(G_L\)-sheaf of \(\mathcal{F}\). For a morphism of \(G_L\)-sheaves \(\varphi: \mathcal{F} \to \mathcal{G}\) the morphism \(\gamma_* \varphi: \gamma_* \mathcal{F} \to \gamma_* \mathcal{G}\) is a morphism of \(G_L\)-sheaves between the sheaves \(\gamma_* \mathcal{F}\) and \(\gamma_* \mathcal{G}\) with \(\gamma\)-conjugate \(G_L\)-sheaf structure.

**Remark 3.10.** This way functors \(\gamma_*\) are defined, similarly one may define functors \(\gamma^*\), then \(\gamma_*\) and \((\gamma^{-1})^*\) are isomorphic. In the case of trivial operation they preserve trivial \(G\)-sheaf structures.

The functors \(\gamma_*\) commute with functors \(f^*, f_*\) for equivariant morphisms \(f\) and with bifunctors like \(\text{Hom}\) and \(\otimes\):

**Lemma 3.11.** There are the following natural isomorphisms of \(G_L\)-sheaves:

(i) For \(G_L\)-sheaves \(\mathcal{F}, \mathcal{G}\) on \(X_L\): \(\gamma_* (\mathcal{F} \otimes \mathcal{O}_{X_L} \mathcal{G}) \cong \gamma_* \mathcal{F} \otimes \mathcal{O}_{X_L} \gamma_* \mathcal{G}\).

(ii) Let \(f: Y \to X\) be an equivariant morphism of \(G\)-schemes over \(K\) and \(\mathcal{F}\) a \(G_L\)-sheaf on \(X_L\). Then \(\gamma_* f^* \mathcal{F} \cong f^*_L \gamma_* \mathcal{F}\).

(iii) For quasicoherent \(G_L\)-sheaves \(\mathcal{F}, \mathcal{G}\) on \(X_L\) with \(\mathcal{F}\) finitely presented: \(\gamma_* \text{Hom}_{\mathcal{O}_{X_L}}(\mathcal{F}, \mathcal{G}) \cong \gamma_* \text{Hom}_{\mathcal{O}_{X_L}}(\gamma_* \mathcal{F}, \gamma_* \mathcal{G})\). If the \(G\)-operation on \(X\) is trivial, it follows that \(\gamma_* (\text{Hom}_{\mathcal{O}_{X_L}}(\mathcal{F}, \mathcal{G})) \cong \text{Hom}_{\mathcal{O}_{X_L}}(\gamma_* \mathcal{F}, \gamma_* \mathcal{G})\).}

**Remark 3.12.** If \(\mathcal{F} \cong \mathcal{F}'_L\) for some \(G\)-sheaf \(\mathcal{F}'\) on \(X\), then there are maps (not \(L\)-linear) \(\gamma: \mathcal{F} \to \mathcal{F}\) resp. isomorphisms of \(G_L\)-sheaves \(\gamma: \mathcal{F} \to \gamma_* \mathcal{F}\) on \(X_L\). For a subsheaf \(\mathcal{G} \subseteq \mathcal{F}\) the above isomorphisms of \(G_L\)-sheaves restrict to isomorphisms of \(G_L\)-sheaves \(\gamma: \gamma^{-1} \mathcal{G} \to \gamma_* \mathcal{G}\).
3.5 Conjugate comodules and representations

Let $G = \text{Spec} \, A$ be an affine group scheme over a field $K$, $X$ be a $G$-scheme over $K$, let $K \to L$ be a field extension and $\Gamma = \text{Aut}_K(L)$.

**Remark 3.13.** For $\gamma \in \Gamma$ there are maps $\gamma : A_L \to A_L$. Taking the canonically defined conjugate Hopf algebra structure on the target, these maps become isomorphisms $\gamma : A_L \to \gamma_* A_L$ of Hopf algebras over $L$. They correspond to isomorphisms $\gamma : \gamma_* G_L \to G_L$ of group schemes over $L$.

**Proposition 3.14.** Let $\mathcal{F}$ be a $G_L$-sheaf on $X_L$, $X$ with trivial $G$-operation, the $G_L$-sheaf structure equivalent to an $A_L$-comodule structure $\gamma^\ast \mathcal{F} : \mathcal{F} \to A_L \otimes_L \mathcal{F}$. Then for $\gamma \in \Gamma$ the $G_L$-sheaf structure of the $\gamma$-conjugate $G_L$-sheaf $\gamma_* \mathcal{F}$ is equivalent to the comodule structure $\rho^\ast \gamma_* \mathcal{F} : \gamma_* \mathcal{F} \to A_L \otimes_L \gamma_* \mathcal{F}$ determined by commutativity of the diagram

$$
\begin{array}{ccc}
\gamma_* \mathcal{F} & \xrightarrow{\gamma \otimes \mathcal{F}} & \gamma_* A_L \otimes_L \gamma_* \mathcal{F} \\
id & & \gamma \otimes id \\
\gamma_* \mathcal{F} & \xrightarrow{\rho^\ast \gamma_* \mathcal{F}} & A_L \otimes_L \gamma_* \mathcal{F}
\end{array}
$$

**Proof.** Apply the construction mentioned in remark 2.1 (2) to diagram 11.

In the special case of representations the definition of conjugate $G$-sheaves leads to the notion of a conjugate representation: Instead of a sheaf $\gamma_* \mathcal{F}$ one has an $L$-vector space $\gamma_* V$, the vector space structure given by $(l,v) \mapsto \gamma(l)v$ using the original structure. The choice of a $K$-structure $V = V_L'$ gives an isomorphism $\gamma : V \to \gamma_* V$ of $L$-vector spaces and leads to the diagram

$$
\begin{array}{ccc}
\gamma_* V & \xrightarrow{\gamma \otimes V} & \gamma_* A_L \otimes_L \gamma_* V \\
id & & \gamma \otimes id \\
\gamma_* V & \xrightarrow{\rho^\ast \gamma_* V} & A_L \otimes_L \gamma_* V
\end{array}
$$

for definition of the $\gamma$-conjugate $A_L$-comodule structure $(\rho^\ast V)^\gamma$ on $V$ — this definition is made, such that $\gamma : (V, (\rho^\ast V)^\gamma) \to (\gamma_* V, \rho^\ast \gamma_* V)$ is an isomorphism of $A_L$-comodules. We write $V^\gamma$ for $V$ with the conjugate $A_L$-comodule structure.

**Remark 3.15.** Let $V'$ be an $A$-comodule over $K$ and $V = V_L'$. Then as a special case of remark 3.12 there are maps $\gamma : V \to V$ resp. isomorphisms of $A_L$-comodules $\gamma : V \to \gamma_* V$. For any $A_L$-subcomodule $U \subseteq V$ these restrict to isomorphisms of $A_L$-comodules $\gamma^{-1} U \xrightarrow{\sim} \gamma_* U \cong U^\gamma$.

3.6 Decomposition into isotypic components and Galois extensions

Let $G = \text{Spec} \, A$ be an affine group scheme over a field $K$, let $K \to L$ be a Galois extension, $\Gamma = \text{Aut}_K(L)$. Assume that $A, A_L$ are cosemisimple.

Recall the relations between the Galois operation on $A_L$ given by maps $\gamma : A_L \to A_L$ resp. isomorphisms $\gamma : A_L \to \gamma_* A_L$ of Hopf algebras or of $A_L$-comodules (see remark 3.13 or 3.15) and the decompositions $A = \bigoplus_{j \in J} A_j$ and $A_L = \bigoplus_{I \in I} A_{L,i}$ into simple subcoalgebras described in corollary 3.3. We relate this to conjugation of representations. The subcoalgebras $A_{L,i}$ are the isotypic components of $A_L$ as a left- (or right-) comodule, let $V_i$ be the isomorphism class of simple $A_{L,i}$-comodules corresponding to $A_{L,i}$. Define an operation of $\Gamma$ on the index set $I$ by $V_{\gamma(i)} = V_i^\gamma$. Using remark 3.15 one obtains:
Lemma 3.16. \( \gamma^{-1} A_{L,i} = A_{L,\gamma(i)} \).

The decomposition of \( A \) into simple subcoalgebras \( A = \bigoplus_j A_j \) gives decompositions of representations and more generally of \( G \)-sheaves on \( G \)-schemes with trivial \( G \)-operation into isotypic components corresponding to the \( A_j \) (see remark 2.1 (3)). After base extension one has decompositions of \( G_L \)-sheaves, we compare it with the decompositions coming from the decomposition of \( A_L \) into simple subcoalgebras.

Proposition 3.17. Let \( X \) be a \( G \)-scheme with trivial operation, \( \mathcal{F} \) a \( G \)-sheaf on \( X \) and let

\[ \mathcal{F} = \bigoplus_j \mathcal{F}_j, \quad \mathcal{F}_L = \bigoplus_i \mathcal{F}_{L,i} \]

be the decompositions into isotypic components as a \( G \)-sheaf resp. \( G_L \)-sheaf. Then:

(i) \( \mathcal{F}_L = \bigoplus_i \mathcal{F}_{L,i} \) is a refinement of \( \mathcal{F}_L = \bigoplus_j \mathcal{(F}_j)_L \).

(ii) The operation of \( \Gamma \) on \( \mathcal{F}_L \) (see remark 3.12) permutes the isotypic components \( \mathcal{F}_{L,i} \) of \( \mathcal{F}_L \). It is \( \gamma^{-1} \mathcal{F}_{L,i} = \mathcal{F}_{L,\gamma(i)} \), if \( V_{\gamma(i)} = V_i \).

(iii) \( \mathcal{(F}_j)_L = \sum_{\gamma \in \Gamma} \gamma \mathcal{F}_{L,i} \), if \( \mathcal{F}_{L,i} \) is a summand of \( \mathcal{(F}_j)_L \).

Sketch of proof. Combine remark 3.12, proposition 3.14 and lemma 3.16 with corollary 3.4.

Corollary 3.18. \( \Gamma \) operates by \( V_i \mapsto V_i^\gamma \) on the set \( \{V_i \mid i \in I\} \) of isomorphism classes of irreducible representations of \( G_L \). The subsets of \( \{V_i \mid i \in I\} \), which occur by decomposing irreducible representations of \( G \) over \( K \) as representations over \( L \), are exactly the \( \Gamma \)-orbits.

For similar results in the representation theory of finite groups see e.g. [4, Vol. I, §7B].

4 The finite subgroup schemes of \( SL(2, K) \): representations and graphs

In this section \( K \) denotes a field of characteristic 0.

4.1 The finite subgroups of \( SL(2, C) \)

By the well known classification any finite subgroup \( G \subset SL(2, C) \), \( C \) an algebraically closed field of characteristic 0, is isomorphic to one of the following groups (presentations and character tables are listed in appendix A).

- \( \mathbb{Z}/n\mathbb{Z} \) (cyclic group of order \( n \), \( n \geq 1 \))
- \( BD_n \) (binary dihedral group of order \( 4n \), \( n \geq 2 \))
- \( BT \) (binary tetrahedral group)
- \( BO \) (binary octahedral group)
- \( BI \) (binary icosahedral group).

4.2 Representation graphs

In the following definition we will introduce the (extended) representation graph as an in general directed graph. A loop is defined to be an edge emanating from and terminating at the same vertex. In addition we will attach a natural number called multiplicity to any vertex, and for homomorphisms of graphs in addition we will require, that for any vertex of the target its multiplicity is the sum of the multiplicities of its preimages.
Definition 4.1. The extended representation graph $\text{Graph}(G, V)$ associated to a finite subgroup scheme $G$ of $\text{GL}(n, K)$, $V$ the given $n$-dimensional representation, is defined as the following directed graph:

- vertices. A vertex of multiplicity $n$ for each irreducible representation of $G$ over $K$ which decomposes over the algebraic closure of $K$ into $n$ irreducible representations.
- edges. Vertices $V_i$ and $V_j$ are connected by $\dim_K \text{Hom}_K^G(V_i, V \otimes_K V_j)$ directed edges from $V_i$ to $V_j$. In particular any vertex $V_i$ has $\dim_K \text{Hom}_K^G(V_i, V \otimes_K V_i)$ directed loops.

Define the representation graph to be the graph, which arises by leaving out the trivial representation and all edges emanating from or terminating at the trivial representation.

We say that a graph is undirected, if between any two different vertices the numbers of directed edges of both directions coincide and for any vertex the number of directed loops is even.

Then one can form a graph having only undirected edges by defining (number of undirected edges between $V_i$ and $V_j$) := (number of directed edges from $V_i$ to $V_j$) = (number of directed edges from $V_j$ to $V_i$) for different vertices $V_i, V_j$ and (number of undirected loops of $V_i$) := $\frac{1}{2}$ (number of directed loops of $V_i$) for any vertex $V_i$.

Remark 4.2.
(1) For $G \subset \text{SL}(2, K)$ the (extended) representation graph is undirected. There is the isomorphism $\text{Hom}_K^G(V_i \otimes_K V, V_j) \cong \text{Hom}_K^G(V_i, V \otimes_K V_j)$, which follows from the isomorphism $\text{Hom}_K^G(V_i \otimes_K V, V_j) \cong \text{Hom}_K^G(V_i, V^\vee \otimes_K V_j)$ and the fact that the 2-dimensional representation $V$ given by inclusion $G \to \text{SL}(2, K)$ is self-dual. Further, that the number of directed loops of any vertex is even, follows from the fact that over the algebraic closure $C$ one has $\dim_C \text{Hom}_C^G(U_i, V_C \otimes_C U_i) = 0$ for irreducible $U_i$ over $C$.
(2) There is a definition of (extended) representation graph with another description of the edges: vertices $V_i$ and $V_j$ are connected by $a_{ij}$ edges from $V_i$ to $V_j$, where $V \otimes_K V_j = a_{ij} V_i \oplus$ other summands. The two definitions coincide over algebraically closed fields, always one has $a_{ij} \leq \dim_K \text{Hom}_K^G(V_i, V \otimes_K V_j)$, inequality comes from the presence of nontrivial automorphisms.

Definition 4.3. For a finite subgroup scheme $G \subset \text{SL}(2, K)$, $V$ the given 2-dimensional representation, define a $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ on the representation ring of $G$ by

$$\langle V_i, V_j \rangle := \dim_K \text{Hom}_K^G(V_i, V \otimes_K V_j) - 2 \dim_K \text{Hom}_K^G(V_i, V_j)$$

Remark 4.4. The form $\langle \cdot, \cdot \rangle$ determines and is determined by the extended representation graph (the second equation follows from the fact, that $\dim_K \text{Hom}_K^G(V_i, V_i) =$ multiplicity of $V_i$):

$$\langle V_i, V_j \rangle = \text{number of undirected edges between } V_i \text{ and } V_j, \text{ if } V_i \not\cong V_j$$

$$\frac{1}{2} \langle V_i, V_i \rangle = \text{number of undirected loops of } V_i - \text{multiplicity of } V_i$$

4.3 Representation graphs and field extensions

Let $K \to L$ be a Galois extension, $\Gamma = \text{Aut}_K(L)$ and let $G$ be a finite subgroup scheme of $\text{SL}(2, K)$.

An irreducible representation $W$ of $G$ over $K$ decomposes as a representation of $G_L$ over $L$ into isotypic components $W = \bigoplus_i U_i$ which are $\Gamma$-conjugate by proposition 3.17. Every $U_i$ decomposes into irreducible components $U_i = V_i^{\oplus m}$ (the same $m$ for all $i$ because of $\Gamma$-conjugacy). In the following we will write $m(W, L/K)$ for this number. It is related to the Schur index in the representation theory of finite groups (see e.g. [1] Vol. II, §74)).
Proposition 4.5. For finite subgroup schemes $G$ of $\text{SL}(2, K)$ it is $m(W_j, L/K) = 1$ for every irreducible representation $W_j$ of $G$. It follows that $W_j$ decomposes over $L$ into a direct sum $(W_j)_L \cong \bigoplus_i V_i$ of $\gamma$-conjugate irreducible representations $V_i$ of $G_L$ nonisomorphic to each other.

Proof. We may assume $L$ algebraically closed. Further we may assume that $G$ is not cyclic. The natural 2-dimensional representation $W$ given by inclusion $G \subset \text{SL}(2, K)$ does satisfy $m(W, L/K) = 1$, because it is irreducible over $L$.

Following the discussion below without using this proposition one obtains the graphs in section 4.4 without multiplicities of vertices and edges but one knows which vertices over the algebraic closure may form a vertex over $K$ and which vertices are connected. Argue that if an irreducible representation $W_i$ satisfies $m(W_i, L/K) = 1$, then any irreducible $W_j$ connected to $W_i$ in the representation graph has to satisfy this property as well. \hfill \square

There is a morphism of graphs $\text{Graph}(G_L, W_L) \rightarrow \text{Graph}(G, W)$ (resp. of the nonextended graphs, the following applies to them as well). For $W_j$ an irreducible representation of $G$ the base extension $(W_j)_L$ is a sum $(W_j)_L = \bigoplus_i V_i$ of irreducible representations of $G_L$ nonisomorphic to each other by proposition 4.3. The morphism $\text{Graph}(G_L, W_L) \rightarrow \text{Graph}(G, W)$ maps components of $(W_j)_L$ to $W_j$, thereby their multiplicities are added. Further, for irreducible representations $W_j, W_j'$ of $G$ there is a bijection between the set of edges between $W_j$ and $W_j'$ and the union of the sets of edges between the irreducible components of $(W_j)_L$ and $(W_j')_L$, again using proposition 4.3. $(W_j)_L$ and $(W_j')_L$ are sums $(W_j)_L = \bigoplus_i V_i$, $(W_j')_L = \bigoplus_i' V_i'$ of irreducible representations of $G_L$ nonisomorphic to each other and one has

$$\dim_K \text{Hom}^G_K(W_j \otimes_K W, W_j') = \dim_L(\text{Hom}^G_K(W_j \otimes_K W_j') \otimes_K L) = \dim_L \text{Hom}^G_L((W_j)_L \otimes_L W_L, (W_j')_L) = \dim_L \text{Hom}^G_L(\bigoplus_i V_i \otimes_L W_L, \bigoplus_i' V_i') = \sum_{i,i'} \dim_L \text{Hom}^G_L(V_i \otimes_L W_L, V_i')$$

The Galois group $\Gamma$ operates on $\text{Graph}(G_L, W_L)$ by graph automorphisms. Irreducible representations are mapped to conjugate representations and equivariant homomorphisms to the conjugate homomorphisms. The vertices of $\text{Graph}(G, W)$ correspond to $\Gamma$-orbits of vertices of $\text{Graph}(G_L, W_L)$ by corollary 3.18.

Proposition 4.6. The (extended) representation graph of $G$ arises by identifying the elements of $\Gamma$-orbits of vertices of the (extended) representation graph of $G_L$, adding multiplicities. The edges between vertices $W_j$ and $W_j'$ are in bijection with the edges between the isomorphism classes of irreducible components of $(W_j)_L$ and $(W_j')_L$. \hfill \square

4.4 The representation graphs of the finite subgroup schemes of $\text{SL}(2, K)$

As extended representation graph of a finite subgroup scheme of $\text{SL}(2, K)$ with respect to the natural 2-dimensional representation the following graphs can occur. We list the extended representation graphs $\text{Graph}(G, V)$ of the finite subgroups of $\text{SL}(2, C)$ for $C$ algebraically closed, their groups of automorphisms leaving the trivial representation fixed and the possible extended representation graphs for finite subgroup schemes over non algebraically closed $K$, which after suitable base extension become the graph $\text{Graph}(G, V)$. We use the symbol $\circ$ for the trivial representation.
- Cyclic groups

\[(A_{2n}), n \geq 1\]

\[\mathbb{Z}/2\mathbb{Z}\]

\[(A_{2n+1}), n \geq 1\]

\[\mathbb{Z}/2\mathbb{Z}\]

\[(A_1)\]

\[\{id\}\]

- Binary dihedral groups

\[(D_n), n \geq 5\]

\[\mathbb{Z}/2\mathbb{Z}\]

\[(D_4)\]

\[S_3\]

\[\{id\}\]

- Binary tetrahedral group

\[(E_6)\]

\[\mathbb{Z}/2\mathbb{Z}\]

\[\{id\}\]

- Binary octahedral group

\[(E_7)\]

\[\{id\}\]

- Binary icosahedral group

\[(E_8)\]

\[\{id\}\]
Remark 4.7. Taking \( \frac{-2}{(2,2)} \) \( V \) for the isomorphism classes of irreducible representations \( V \) as simple roots one can form the Dynkin diagram with respect to the form \(-\langle \cdot, \cdot \rangle \) (see e.g. [3] Groupes et algèbres de Lie]). Between (extended) representation graphs and (extended) Dynkin diagrams there is the correspondence

\[
\begin{align*}
(A_n) & \quad (A_2) \quad (A_{2n+1}) \quad (A_{2n+2}) \quad (D_n) \quad (D_4) \quad (E_6) \quad (E_7) \quad (E_8) \\
(A_n) & \quad (C_1) = (A_1) \quad (C_{n+1}) \quad (C_{n+1}) \quad (D_n) \quad (B_{n-1}) \quad (G_2) \quad (E_6) \quad (F_4) \quad (E_7) \quad (E_8)
\end{align*}
\]

A long time ago, the occurrence of the remaining Dynkin diagrams of types \((B_n), (C_n), (F_4), (G_2)\) as resolution graphs had been observed in [10] with a slightly different assignment of the non extended diagrams to the resolutions of these singularities, see also [16].

### 4.5 Finite subgroups of SL(2, K)

Given a field \( K \) of characteristic 0, it is a natural question, which of the finite subgroups \( G \subset \text{SL}(2, C) \), \( C \) the algebraic closure of \( K \), are realisable over the subfield \( K \) as subgroups (not just as subgroup schemes), that is, there is an injective representation of the group \( G \) in \( \text{SL}(2, K) \).

For a finite subgroup \( G \) of \( \text{SL}(2, C) \) to occur as a subgroup of \( \text{SL}(2, K) \) it is necessary and sufficient that the given 2-dimensional representation in \( \text{SL}(2, C) \) is realisable over \( K \). This is easy to show using the classification and the irreducible representations of the individual groups. If a representation of a group \( G \) over \( C \) is realisable over \( K \), necessarily its character has values in \( K \). For the finite subgroups of \( \text{SL}(2, C) \) and the natural representation given by inclusion this means:

- \( \mathbb{Z}/n\mathbb{Z}: \xi + \xi^{-1} \in K \), \( \xi \in C \) a primitive \( n \)-th root of unity.
- \( \text{BD}_n: \xi + \xi^{-1} \in K \), \( \xi \in C \) a primitive \( 2n \)-th root of unity.
- \( \text{BT}: \) no condition.
- \( \text{BO}: \sqrt{2} \in K \).
- \( \text{BL}: \sqrt{5} \in K \).

To formulate sufficient conditions, we introduce the following notation:

**Definition 4.8.** ([15] Part I, Chapter III, §1). For a field \( K \) the Hilbert symbol \( (\langle \cdot, \cdot \rangle)_K \) is the map \( K^* \times K^* \to \{-1, 1\} \) defined by \( \langle a, b \rangle)_K = 1 \) if the equation \( z^2 - ax^2 - by^2 = 0 \) has a solution \( (x, y, z) \in K^3 \setminus \{(0, 0, 0)\} \), and \( \langle a, b \rangle)_K = -1 \) otherwise.

**Remark 4.9.** It is \( \langle -1, b \rangle)_K = 1 \) if and only if \( x^2 - by^2 = -1 \) has a solution \( (x, y) \in K^2 \).

**Theorem 4.10.** Let \( G \) be a finite subgroup of \( \text{SL}(2, C) \) such that the values of the character of the natural representation given by inclusion are contained in \( K \). Then:

1. If \( G \cong \mathbb{Z}/n\mathbb{Z} \), then \( G \) is realisable over \( K \).
2. If \( G \cong \text{BD}_n \), let \( \xi \in C \) be a primitive \( 2n \)-th root of unity and \( c := \frac{1}{2}(\xi + \xi^{-1}) \). Then \( G \) is isomorphic to a subgroup of \( \text{SL}(2, K) \) if and only if \( \langle -1, c^2 - 1 \rangle)_K = 1 \).
3. If \( G \cong \text{BT}, \text{BO} \) or \( \text{BI} \), then \( G \) is isomorphic to a subgroup of \( \text{SL}(2, K) \) if and only if \( \langle -1, -1 \rangle)_K = 1 \).

**Proof.** (i) For \( n \geq 3 \) let \( \xi \) be a primitive \( n \)-th root of unity and \( c := \frac{1}{2}(\xi + \xi^{-1}) \). By assumption \( c \in K \). Then \( \mathbb{Z}/n\mathbb{Z} \) is realisable over \( K \), there is the representation

\[
\mathbb{Z}/n\mathbb{Z} \to \text{SL}(2, K), \quad T \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 2c \end{pmatrix}
\]
For the individual groups we obtain:

The representation is realisable over $K$ and only if the representation given by $x$ having the properties \((\alpha \delta x, y, z)\) as well. After diagonalisation: \(\xi\) and let \(ξ \) be a primitive 2n-th root of unity. Then $G$ is realisable as a subgroup of $\text{SL}(2, K)$ if and only if the representation given by

\[
\sigma \mapsto \left( \begin{array}{cc} ξ & 0 \\ 0 & ξ^{-1} \end{array} \right), \quad \tau \mapsto \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)
\]  \(\text{(2)}\)

is realisable over $K$.

The representation \((\text{2})\) is realisable over $K$ if and only if there is a $2 \times 2$-matrix $Mτ$ over $K$ having the properties

\[
\det(Mτ) = 1, \quad \text{ord}(Mτ) = 4, \quad (MτMσ)^2 = -I, \quad \text{where } Mσ = \left( \begin{array}{cc} 0 & -1 \\ 1 & 2c \end{array} \right), \quad c = \frac{1}{2}(ξ + ξ^{-1}). \quad \text{(3)}
\]

If the representation \((\text{2})\) is realisable over $K$, then with respect to a suitable basis it maps $σ \mapsto Mσ$ and the image of $τ$ is a matrix satisfying the properties \((\text{3})\).

On the other hand, if $Mτ$ is a matrix having these properties, then $σ \mapsto Mσ, τ \mapsto Mτ$ is a representation of $G$ in $\text{SL}(2, K)$, which is easily seen to be isomorphic to the representation \((\text{2})\).

There is a $2 \times 2$-matrix $Mτ$ over $K$ having the properties \((\text{3})\) if and only if the equation

\[
x^2 + y^2 - 2cxy + 1 = 0 \quad \text{(4)}
\]

has a solution \((x, y) ∈ K^2\).

A matrix $Mτ = \left( \begin{array}{cc} α & β \\ γ & δ \end{array} \right)$ satisfies the conditions \((\text{3})\) if and only if \((α, β, γ, δ) ∈ K^4\) is a solution of $αδ - βγ - 1 = 0, α + δ = 0, β + 2cδ - γ = 0$. Such an element of $K^4$ exists if and only if there exists a solution \((x, y) ∈ K^2\) of equation \((\text{4})\).

The equation \((\text{4})\) has a solution \((x, y) ∈ K^2\) if and only if \((\langle -1, c^2 - 1 \rangle)_K = 1\).

We write the equation $x^2 + y^2 - 2cxy + 1 = 0$ as $(x, y) \left( \begin{array}{cc} 1 & -c \\ c & 1 \end{array} \right) (\frac{x}{y}) = -1$. After diagonalisation $(x, y) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) (\frac{x}{y}) = -1$ or $x^2 + (1 - c^2)y^2 + 1 = 0$. This equation has a solution $(x, y) ∈ K^2$ if and only if $(\langle -1, c^2 - 1 \rangle)_K = 1$.

(iii) Let $G = BD_n = \langle σ, τ | τ^2 = σ^n = (τσ)^2 \rangle$ (then the element $τ^2 = σ^n = (τσ)^2$ has order 2) and let $ξ$ be a primitive 2n-th root of unity. Then $G$ is isomorphic to a subgroup of $\text{SL}(2, K)$. Let $G$ be a $2$-th root of unity and let $ξ$ be a primitive 2n-th root of unity. Then $G$ is isomorphic to a subgroup of $\text{SL}(2, K)$.

Next we show:

Equation \((\text{4})\) has a solution \((x, y) ∈ K^2\) if and only if \((\langle -1, (2c)^2 - 3 \rangle)_K = 1\).

Equation \((\text{5})\) has a solution if and only if \((x, y, z) \left( \begin{array}{cc} 1 & -c \\ c & 1 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) = 0\) has a solution \((x, y, z) ∈ K^3\) with $z ≠ 0$. The existence of a solution with $z ≠ 0$ is equivalent to the existence of a solution \((x, y, z) ∈ K^3 \setminus \{(0, 0, 0)\}\) if \((x, y, 0)\) is a solution, then \((x, y, x - 2cy)\) as well. After diagonalisation: \((x, y, z) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3-(2c)^2 \end{array} \right) \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = 0\). The existence of a solution \((x, y, z) ∈ K^3 \setminus \{(0, 0, 0)\}\) for this equation is equivalent to \((\langle -1, (2c)^2 - 3 \rangle)_K = 1\).

For the individual groups we obtain:

\begin{align*}
BT: & \quad c = \frac{1}{\sqrt{2}} \left(1 - \sqrt{5}\right), \quad \langle -1, -2 \rangle_K = 1. \\
BO: & \quad c = \frac{1}{\sqrt{2}} \left(1 + \sqrt{5}\right), \quad \langle -1, 1 \rangle_K = 1. \\
BI: & \quad c = \frac{1}{\sqrt{2}} \left(1 ± \sqrt{5}\right), \quad \langle -1, (\langle 1 ± \sqrt{5} \rangle)_K = 1. \\
\end{align*}

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Each of these conditions is equivalent to \((-1, -1)_{K} = 1\). For \(BI\): \(\frac{1}{2}(3 \pm \sqrt{5}) = (\frac{1}{2}(1 \pm \sqrt{5}))^2\).

For \(BT\) one has maps between solutions \((x, y)\) for \(x^2 + y^2 = -1\) corresponding to \((-1, -1)_{K}\) and \((x', y')\) for \(x'^2 + 2y'^2 = -1\) corresponding to \((-1, -2)_{K}\) given by \(x = \frac{x'^2 + 1}{2y'} \leftrightarrow x' = \frac{x + y}{x - y}\), \(y = \frac{x' - 1}{2y'} \leftrightarrow y' = \frac{1}{x - y}\) for \(x \neq y\) resp. \(y' \neq 0\) and by \((x, x) \mapsto (0, x), (x', 0) \mapsto (x', 0)\).

5 McKay correspondence for \(G \subset SL(2, K)\)

Let \(G\) be a finite subgroup scheme of \(SL(2, K)\), \(K\) a field of characteristic 0, and \(C\) the algebraic closure of \(K\). There is the geometric quotient \(\pi: A^2_K \rightarrow A^2_K/G\) and the natural morphism \(\tau: G\text{-Hilb}_K A^2_K \rightarrow A^2_K/G\), which is the minimal resolution of this quotient singularity.

5.1 The exceptional divisor and the intersection graph

Define the exceptional divisor \(E\) by

\[ E := \tau^{-1}(\mathcal{O}) \]

where \(\mathcal{O} = \pi(O)\), \(O\) the origin of \(A^2_K\). In general \(E\) is not reduced, denote by \(E_{\text{red}}\) the underlying reduced subscheme.

**Definition 5.1.** The intersection graph of \(E_{\text{red}}\) is defined as the following undirected graph:
- **vertices.** A vertex of multiplicity \(n\) for each irreducible component \((E_{\text{red}})_{i}\) of \(E_{\text{red}}\) which decomposes over the algebraic closure of \(K\) into \(n\) irreducible components.
- **edges.** Different \((E_{\text{red}})_{i}\) and \((E_{\text{red}})_{j}\) are connected by \((E_{\text{red}})_{i} \approx (E_{\text{red}})_{j}\) undirected edges. \((E_{\text{red}})_{i}\) has \(\frac{1}{2}((E_{\text{red}})_{i} \approx (E_{\text{red}})_{i}\) loops.

If \(K\) is algebraically closed, then the \((E_{\text{red}})_{i}\) are isomorphic to \(P^1_K\) and the self-intersection of each \((E_{\text{red}})_{i}\) is \(-2\), because the resolution is crepant.

Let \(K \rightarrow L\) be a Galois extension, \(\Gamma = \text{Aut}_K(L)\). \(\Gamma\) operates on the intersection graph of \((E_{\text{red}})_{L}\) by graph automorphisms. The irreducible components \((E_{\text{red}})_{i}\) of \(E_{\text{red}}\) correspond to \(\Gamma\)-orbits of irreducible components \((E_{\text{red}})_{L, k}\) of \((E_{\text{red}})_{L}\) by proposition 3.6. For the intersection form one has

\[(E_{\text{red}})_{i} \approx (E_{\text{red}})_{j} = ((E_{\text{red}})_{i})_{L} \approx ((E_{\text{red}})_{j})_{L} = \sum_{k, l}((E_{\text{red}})_{L, k} \approx (E_{\text{red}})_{L, l})\]

where indices \(k\) and \(l\) run through the irreducible components of \(((E_{\text{red}})_{i})_{L}\) and \(((E_{\text{red}})_{j})_{L}\) respectively. Thus for the intersection graph there is a proposition similar to proposition 4.10 for representation graphs.

**Proposition 5.2.** The intersection graph of \(E_{\text{red}}\) arises by identifying the elements of \(\Gamma\)-orbits of vertices of the intersection graph of \((E_{\text{red}})_{L}\), adding multiplicities. The edges between vertices \((E_{\text{red}})_{i}\) and \((E_{\text{red}})_{j}\) are in bijection with the edges between the irreducible components of \(((E_{\text{red}})_{i})_{L}\) and \(((E_{\text{red}})_{j})_{L}\).

5.2 Irreducible components of \(E\) and irreducible representations of \(G\)

The basic statement of McKay correspondence is a bijection between the set of irreducible components of the exceptional divisor \(E\) and the set of isomorphism classes of nontrivial irreducible representations of the group scheme \(G\).
**Theorem 5.3.** There are bijections for intermediate fields $K \subseteq L \subseteq C$ between the set $\text{Irr}(E_L)$ of irreducible components of $E_L$ and the set $\text{Irr}(G_L)$ of isomorphism classes of nontrivial irreducible representations of $G_L$ having the property that for $K \subseteq L \subseteq L' \subseteq C$, if the bijection $\text{Irr}(E_L) \rightarrow \text{Irr}(G_L)$ for $L$ maps $E_i \leftrightarrow V_i$, then the bijection $\text{Irr}(E_{L'}) \rightarrow \text{Irr}(G_{L'})$ for $L'$ maps irreducible components of $(E_i)_{L'}$ to irreducible components of $(V_i)_{L'}$.

**Proof.** As described earlier, the Galois group $\Gamma = \text{Aut}_L(C)$ of the Galois extension $L \rightarrow C$, operates on the sets $\text{Irr}(G_C)$ and $\text{Irr}(E_C)$. In both cases elements of $\text{Irr}(G_L)$ and $\text{Irr}(E_L)$ correspond to $\Gamma$-orbits of elements of $\text{Irr}(G_C)$ and $\text{Irr}(E_C)$ by corollary 3.18 and proposition 3.6 respectively. This way a given bijection between the sets $\text{Irr}(G_C)$ and $\text{Irr}(E_C)$ defines a bijection between $\text{Irr}(G_L)$ and $\text{Irr}(E_L)$ on condition that the bijection is equivariant with respect to the operations of $\Gamma$. Checking this for the bijection of McKay correspondence over the algebraically closed field $C$ constructed via stratification or via tautological sheaves will give bijections over intermediate fields $L$ having the property of the theorem. This will be done in the process of proving theorem 5.6 or theorem 5.9.

Moreover, in the situation of the theorem the Galois group $\Gamma = \text{Aut}_L(C)$ operates on the representation graph of $G_C$ and on the intersection graph of $(E_{\text{red}})_C$. Then in both cases the graphs over $L$ arise by identifying the elements of $\Gamma$-orbits of vertices of the graphs over $C$ by proposition 4.6 and 5.2. Therefore an isomorphism of the graphs over $C$, the bijection between the sets of vertices being $\Gamma$-equivariant, defines an isomorphism of the graphs over $L$.

For the algebraically closed field $C$ this is the classical McKay correspondence for subgroups of $\text{SL}(2, C)$ ([11], [6], [8]). The statement, that there is a bijection of edges between given vertices $(E_{\text{red}})_{L,i} \leftrightarrow V_i$ and $(E_{\text{red}})_{L,j} \leftrightarrow V_j$, can be formulated equivalently in terms of the intersection form as $(E_{\text{red}})_{L,i}.(E_{\text{red}})_{L,j} = \langle V_i, V_j \rangle$.

**Theorem 5.4.** The bijections $E_i \leftrightarrow V_i$ of theorem 5.3 between irreducible components of $E_L$ and isomorphism classes of nontrivial irreducible representations of $G_L$ can be constructed such that $(E_{\text{red}})_{i.(E_{\text{red}})} = \langle V_i, V_j \rangle$ or equivalently that these bijections define isomorphisms of graphs between the intersection graph of $(E_{\text{red}})_L$ and the representation graph of $G_L$.

We will consider two ways to construct bijections between nontrivial irreducible representations and irreducible components with the properties of theorem 5.3 and 5.5. A stratification of $G\text{Hilb}_K \mathbb{A}^2_K$ ([7], [8]) and the tautological sheaves on $G\text{Hilb}_K \mathbb{A}^2_K$ ([6], [9]).

### 5.3 Stratification of $G\text{Hilb}_K \mathbb{A}^2_K$

Let $S := K[x_1, x_2]$, let $O \subset \mathbb{A}^2_K$ be the origin, $S \subset C$ the corresponding maximal ideal, $\mathcal{O} := \pi(O) \subset \mathbb{A}^2_K/G$ with corresponding maximal ideal $n \subset S^G$, let $\mathcal{S} := S/nS$ with maximal ideal $\mathfrak{m}$. An $L$-valued point of the fiber $E = r^{-1}(\mathcal{O})$ corresponds to a $G$-cluster defined by an ideal $I \subset S_L$ such that $n_L \subset I$ or equivalently an ideal $\overline{T} \subset \overline{S}_L = S_L/n_LS_L$. For such an ideal $I$ define the representation $V(I)$ over $L$ by $V(I) := \overline{T}/\mathfrak{m}_L\overline{T}$.

**Lemma 5.5.** For $\gamma \in \text{Aut}_K(L)$: $V(\gamma^{-1}I) \cong V(I)^\gamma$.

**Proof.** As an $A_L$-comodule $\overline{T} = \overline{T} \oplus L\overline{T}$, where $\overline{T}_0 \cong \overline{T}/\mathfrak{m}_L\overline{T}$. Then $\gamma^{-1}T = \gamma^{-1}\overline{T}_0 \oplus \mathfrak{m}_L(\gamma^{-1}\overline{T})$ and $V(\gamma^{-1}I) = \gamma^{-1}(\overline{T}/\mathfrak{m}_L(\gamma^{-1}\overline{T}) \cong \gamma^{-1}\overline{T}_0 \cong V(I)^\gamma$ by remark 3.15 applied to $\overline{T}_0 \subseteq \mathcal{S}_L$.

**Theorem 5.6.** There is a bijection $E_j \leftrightarrow V_j$ between the set $\text{Irr}(E)$ of irreducible components of $E$ and the set $\text{Irr}(G)$ of isomorphism classes of nontrivial irreducible representations of $G$. 

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such that for any closed point \( y \in E \): If \( I \subset S_{\kappa(y)} \) is an ideal defining a \( \kappa(y) \)-valued point of the
scheme \( \{ y \} \subset E \), then

\[
\text{Hom}^G_{\kappa(y)}(V(I), (V_j)_{\kappa(y)}) \neq 0 \iff y \in E_j
\]

and \( V(I) \) is either irreducible or consists of two irreducible representations not isomorphic to
each other. Applied to the situation after base extension \( K \to L \), \( L \) an algebraic extension of
\( K \), one obtains bijections \( \text{Irr}(E_L) \leftrightarrow \text{Irr}(G_L) \) having the properties of theorems 5.3 and 5.4.

**Proof.** In the case of algebraically closed \( K \) the theorem follows from [8].

In the general case denote by \( U_i \) the isomorphism classes of nontrivial irreducible representations of
\( G_C \) over the algebraic closure \( C \). Over \( C \) the theorem is valid, let \( E_{C,i} \) be the component
corresponding to \( U_i \).

We show that this bijection is equivariant with respect to the operations of \( \Gamma = \text{Aut}_K(C) \).

Let \( x \in E_{C,i} \) be a closed point such that \( x \notin E_{C,i'} \) for \( i' \neq i \). Then for the corresponding
\( C \)-valued point \( \alpha: \text{Spec} C \to E_{C,i} \) given by an ideal \( I \subset S_C \) one has \( V(I) \cong U_i \). By corollary 3.3 the \( C \)-valued point corresponding to \( \gamma x \) is \( \alpha \gamma \) given by the ideal \( \gamma^{-1} I \subset S_C \). By lemma 5.5
\( V(\gamma^{-1} I) \cong U_{\gamma(i)} \), where \( U_{\gamma(i)} = U_i^{\gamma} \). Therefore \( \gamma x \in E_{\gamma(i)} \) and \( \gamma E_i = E_{\gamma(i)} \).

For an irreducible representation \( V_j \) of \( G \) define \( E_j \) to be the component of \( E \) which decomposes
over \( C \) into the irreducible components \( E_{C,j} \) satisfying \( U_i \subseteq (V_j)_C \). This method, applied to the
situation after base extension \( K \to L \), leads to bijections having the properties of theorems 5.3 and 5.4.

We show that this bijection is given by the condition in the theorem. Let \( y \) be a closed point of
\( E \) and \( \alpha \) a \( \kappa(y) \)-valued point of the scheme \( \{ y \} \subset E \) given by an ideal \( I \subset S_{\kappa(y)} \). \( K \to \kappa(y) \) is an
algebraic extension, embed \( \kappa(y) \) into \( C \). After base extension \( \kappa(y) \to C \) one has the \( C \)-valued
point \( \alpha_C: \text{Spec} C \to \{ y \}_C \) given by \( I_C \subset S_C \). Then \( V(I)_C \cong V(I_C) \) and \( I_C \) corresponds to a
closed point \( z \in \{ y \}_C \subset E_C \). Therefore

\[
y \in E_j \iff z \in E_{C,i} \text{ for some } i \text{ satisfying } U_i \subseteq (V_j)_C \notag
\]

\[
\iff \text{Hom}^G_C(V(I_C), U_i) \neq 0 \text{ for some } i \text{ satisfying } U_i \subseteq (V_j)_C \notag
\]

\[
\iff \text{Hom}^G_{\kappa(y)}(V(I), (V_j)_{\kappa(y)}) \neq 0 \notag
\]

\[\Box\]

### 5.4 Tautological sheaves

Let \( 0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{A}^2_K} \to \mathcal{O}_Z \to 0 \) be the universal quotient of \( Y := \text{G-Hilb}_K \mathbb{A}^2_K \). The projection
\( p: Z \to Y \) is a finite flat morphism, \( p_* \mathcal{O}_Z \) is a locally free \( G \)-sheaf on \( Y \) with fibers \( p_* \mathcal{O}_Z \otimes_{\mathcal{O}_Y} \kappa(y) \)
isomorphic to the regular representation over \( \kappa(y) \).

Let \( V_0, \ldots, V_n \) the isomorphism classes of irreducible representations of \( G \), \( V_0 \) the trivial representa-
tion. The \( G \)-sheaf \( \mathcal{I} := p_* \mathcal{O}_Z \) on \( Y \) decomposes into isotypic components (see remark 2.1(3) and subsection 3.6)

\[
\mathcal{I} \cong \bigoplus_{j=0}^n \mathcal{I}_j
\]

where \( \mathcal{I}_j \) is the component for \( V_j \).

**Definition 5.7.** For any isomorphism class \( V_j \) of irreducible representations of \( G \) over \( K \) define the sheaf \( \mathcal{F}_j \) on \( Y = \text{G-Hilb}_K \mathbb{A}^2_K \) by

\[
\mathcal{F}_j := \text{Hom}_{\mathcal{O}_Y}(V_j \otimes_K \mathcal{O}_Y, \mathcal{I}_j) = \text{Hom}_{\mathcal{O}_Y}(V_j \otimes_K \mathcal{O}_Y, \mathcal{I})
\]

For a field extension \( K \to L \) denote by \( \mathcal{F}_{L,i} \) the sheaf \( \text{Hom}_{\mathcal{O}_{Y_L}}(U_i \otimes_L \mathcal{O}_{Y_L}, \mathcal{I}_L) \) on \( Y_L \), \( U_i \) an
irreducible representation of \( G_L \) over \( L \).
Remark 5.8.

(1) For $K = C$ the sheaves $F_j$ were studied in [6, 9], they may be defined as well as $F_j = \tau^* \mathcal{H}om^G_{\mathbb{A}^2_K/(\mathcal{O}_{\mathbb{A}^2_K} \otimes \mathcal{O}_K)}(V_j \otimes \mathcal{O}_K, \mathcal{O}_{\mathbb{A}^2_K})$ using the canonical morphisms in the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{p} & Z \\
\downarrow{\pi} & & \downarrow{q} \\
\mathbb{A}^2_K/\mathcal{G} & \xrightarrow{\pi} & 
\end{array}
\]

(2) $F_j$ is a locally free sheaf of rank $\dim_K V_j$.

(3) For each $j$ there is the natural isomorphism of $G$-sheaves $F_j \otimes_{\End_K(V_j)} V_j \xrightarrow{\sim} \mathcal{G}_j$.

Let $K \rightarrow L$ be a Galois extension and $U_0, \ldots, U_r$ be the isomorphism classes of irreducible representations of $G_L$ over $L$. Then a decomposition $(V_j)_L = \bigoplus_{i \in I_j} U_i$ over $L$ of an irreducible representation $V_j$ of $G$ over $K$ gives a decomposition of the corresponding tautological sheaf

$$
(V_j)_L = \mathcal{H}om^G_{\mathcal{O}_Y}(V_j \otimes \mathcal{O}_Y, F)_L \cong \mathcal{H}om^G_{\mathcal{O}_Y}(\bigoplus_{i \in I_j} U_i \otimes \mathcal{O}_Y, F)_L \cong \bigoplus_{i \in I_j} \mathcal{H}om^G_{\mathcal{O}_Y}(U_i \otimes \mathcal{O}_Y, F)_L = \bigoplus_{i \in I_j} F_{L,i}
$$

We have used the fact that the $U_i$ occur with multiplicity 1 as it is the case for finite subgroup schemes of $SL(2, K)$, see proposition 3.3.

The tautological sheaves $F_j$ can be used to establish a bijection between the set of irreducible components of $E_{\text{red}}$ and the set of isomorphism classes of nontrivial irreducible representations of $G$ by considering intersections $L_j.(E_{\text{red}})_{j'}$, i.e. the degrees of restrictions of the line bundles $L_j := \bigwedge^r F_j, F_j$ to the curves $(E_{\text{red}})_{j'}$.

**Theorem 5.9.** There is a bijection $E_j \leftrightarrow V_j$ between the set $\text{Irr}(E)$ of irreducible components of $E$ and the set $\text{Irr}(G)$ of isomorphism classes of nontrivial irreducible representations of $G$ such that

$$
L_j.(E_{\text{red}})_{j'} = \dim_K \text{Hom}^G_K(V_j, V_{j'})
$$

where $L_j = \bigwedge^r F_j, F_j$.

Applied to the situation after base extension $K \rightarrow L$, $L$ an algebraic extension field of $K$, one obtains bijections $\text{Irr}(E_L) \leftrightarrow \text{Irr}(G_L)$ having the properties of theorems 5.3 and 5.4.

**Proof.** In the case of algebraically closed $K$ the theorem follows from [6].

In the general case denote by $U_0, \ldots, U_r$ the isomorphism classes of irreducible representations of $G_C$ over the algebraic closure $C, U_0$ the trivial one. Over $C$ the theorem is valid, let $E_{C,i}$ be the component corresponding to $U_i$, what means that $L_{C,i}.(E_{\text{red}})_{C,i'} = \delta_{i,i'}$, where $L_{C,i} = \bigwedge^r F_{C,i}, F_{C,i}$. To show that the bijection over $C$ is equivariant with respect to the operations of $\Gamma = \text{Aut}_K(C)$, one has to show that $\gamma_* L_{C,i} \cong L_{C,\gamma(i)}$, where $U_{\gamma(i)} = U^\gamma_i$. Then $L_{C,i}.E_{C,i'} = \gamma_* L_{C,i}.\gamma E_{C,i'} = L_{C,\gamma(i)}.\gamma E_{C,i'}$ and therefore $\gamma E_{C,i'} = E_{C,\gamma(i')}$. It is $\gamma_* L_{C,i} \cong L_{C,\gamma(i)}$, because using lemma 3.11 and remark 3.12

$$
\gamma_* F_{C,i} \cong \mathcal{H}om^G_{\mathcal{O}_Y}(\bigoplus_{i \in I_j} U_i \otimes \mathcal{O}_Y, \mathcal{G}_C) \cong \mathcal{H}om^G_{\mathcal{O}_Y}(U^\gamma_i \otimes \mathcal{O}_Y, \mathcal{G}_C) = F_{C,\gamma(i)}
$$

Since the bijection over $C$ is equivariant with respect to the $\Gamma$-operations on $\text{Irr}(G_C)$ and $\text{Irr}(E_C)$, one can define a bijection $\text{Irr}(G) \leftrightarrow \text{Irr}(E)$: For $V_j \in \text{Irr}(G)$ let $E_j$ be the element of $\text{Irr}(E)$ such that $(V_j)_C = \bigoplus_{i \in I_j} U_i$ and $(E_j)_C = \bigcup_{i \in I_j} E_{C,i}$ for the same subset $I_j \subseteq \{1, \ldots, r\}$. This method
applied to the situation after base extension $K \to L$ leads to bijections having the properties of theorems 5.3 and 5.4. We show that this bijection is given by the construction of the theorem. It is $(\mathcal{F}_j)_C = \bigoplus_{i \in I_j} \mathcal{F}_{C,i}$ and therefore
\[
\mathcal{L}_j((E_{\text{red}})') = (\mathcal{L}_j)_C((E_{\text{red}})')C = (\bigotimes_{i \in I_j} \mathcal{L}_{C,i}, (\sum_{i' \in I_j} (E_{\text{red}})_{C,i'}) = \sum_{i,i'} \mathcal{L}_{C,i}(E_{\text{red}})_{C,i'} = \sum_{i,i'} \dim_C \Hom_{C,i}(U_i, U_{i'}) = \dim_C \Hom_{C,i}(V_j, V_j') = \dim_K \Hom_{K}(V_j, V_j') \]

### 5.5 Examples

**Finite subgroups of $\text{SL}(2, K)$.** In the case of subgroups $G \subset \text{SL}(2, K)$ the representation graph can be read off from the table of characters of the group $G$ over an algebraically closed field, since in this case representations are conjugate if and only if the values of their characters are.

We have the following graphs for the finite subgroups of $\text{SL}(2, K)$:
- cyclic group $\mathbb{Z}/n\mathbb{Z}$, $n \geq 1$. It is $\xi + \xi^{-1} \in K$, $\xi$ a primitive $n$-th root of unity. Diagram $(A_{n-1})$ if $\xi \in K$, otherwise $(A_{n-1})'$.
- binary dihedral group $BD_n$, $n \geq 2$. It is $c = \frac{1}{2}(\xi + \xi^{-1}) \in K$, $\xi$ a primitive $2n$-th root of unity, and $(\langle -1, c^2 - 1 \rangle)_K = 1$. Diagram $(D_{n+2})$ if $n$ even or $\sqrt{-1} \in K$, otherwise $(D_{n+2})'$.
- binary tetrahedral group $BT$. It is $(\langle -1, 1 \rangle)_K = 1$. Diagram $(E_6)$ if $K$ contains a primitive 3rd root of unity, otherwise $(E_6)'$.
- binary octahedral group $BO$. It is $(\langle -1, 1 \rangle)_K = 1$ and $\sqrt{2} \in K$. Diagram $(E_7)$.
- binary icosahedral group $BI$. It is $(\langle -1, 1 \rangle)_K = 1$ and $\sqrt{3} \in K$. Diagram $(E_8)$.

Examples for the graphs $(A_0)'$, $(D_{2m+1})'$, $(E_6)'$:
- $(A_0)'$: $\mathbb{Z}/(n+1)\mathbb{Z}$ over $\mathbb{Q}(\xi + \xi^{-1})$, $\xi$ a primitive $(n + 1)$-th root of unity.
- $(D_{2m+1})'$: $BD_{2m-1}$ over $\mathbb{Q}(\xi)$, $\xi$ a primitive $2(2m - 1)$-th root of unity.
- $(E_6)'$: $BT$ over $\mathbb{Q}(\sqrt{-1})$.

**Abelian subgroup schemes.** In the case of abelian subgroup schemes of $\text{SL}(2, K)$ the graphs $(A_n)$ and $(A_n)'$ occur.
- the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ is realisable as the subgroup of $\text{SL}(2, K)$ generated by $g := \begin{pmatrix} 0 & 1 \\ 1 & \xi + \xi^{-1} \end{pmatrix}$, if the field $K$ contains $\xi + \xi^{-1}$ for $\xi$ a primitive $n$-th root of unity. If $K$ does not contain $\xi$, then there are 1-dimensional representations over the algebraic closure that are not realisable over $K$, one has diagram $(A_{n-1})'$.
- for the subgroup scheme $G = \mu_n \subset \text{SL}(2, K)$ the Hopf algebra $K[y]/(y^n)$ decomposes into a direct sum of simple subcoalgebras $\langle y \rangle_K$ corresponding to 1-dimensional representations of $G$. Thus one has diagram $(A_{n-1})$.

**The graph $(D_{2m})'$.** Let $n \geq 2$, $\varepsilon$ a primitive $4n$-th root of unity and $\xi = \varepsilon^2$. Put $K = \mathbb{Q}(\varepsilon + \varepsilon^{-1})$, $C = \mathbb{Q}(\varepsilon)$ and $\Gamma = \text{Aut}_K(C) = \{id, \gamma\}$. One has the injective representation of $BD_n = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = \sigma^m = (\tau \sigma)^2 \rangle$ in $\text{SL}(2, C)$ given by
\[
\sigma \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & -\xi \\ \xi^{-1} & 0 \end{pmatrix}
\]
We will identify $BD_n$ with its image in $\text{SL}(2, C)$ and regard it as a subgroup scheme of $\text{SL}(2, C)$. $\Gamma$ operates on $\text{SL}(2, C)$, the $K$-automorphism $\gamma \in \Gamma$, $\varepsilon \mapsto \varepsilon^{-1}$ of order 2 operates nontrivially on the closed points of $BD_n$ by $\sigma \mapsto \sigma^{-1}$, $\tau \mapsto \tau \sigma$. The subgroup scheme $BD_n \subset \text{SL}(2, C)$ is defined over $K$, let $G \subset \text{SL}(2, K)$ such that $G_C = BD_n$. The closed points of $G$ correspond to $\Gamma$-orbits of closed points of $BD_n$, they have the form $\{id\}$, $\{-id\}$, $\{\sigma^k, \sigma^{-k}\}$, $\{\tau \sigma^k, \tau \sigma^{-k+1}\}$. The automorphism $\gamma$ operates on the characters of $BD_n$ trivially except that for even $n$ it permutes two of the irreducible 1-dimensional representations. One has the graph $(D_{n+2})'$ for $n$ even and the graph $(D_{n+2})$ for $n$ odd.
A  Finite subgroups of $\text{SL}(2, C)$: Presentations and character tables

- Cyclic groups

The irreducible representations are $\chi_j : \mathbb{Z}/n\mathbb{Z} \rightarrow C^*$, $i \mapsto \xi^{ij}$ for $j \in \{0, \ldots, n-1\}$, where $\xi$ is a primitive $n$-th root of unity.

- Binary dihedral groups: $BD_n = \langle \sigma, \tau \mid \tau^2 = \sigma^n = (\tau \sigma)^2 \rangle$, $-id := (\tau \sigma)^2$.

| $id$ | $-id$ | $\sigma^k$ | $\tau$ | $\tau \sigma$ |
|------|-------|-----------|--------|-------------|
| 1    | 1     | 1         | 1      | 1           |
| $1'$ | 1     | 1         | 1      | $-1$        |
| $1''$| 1     | $-1$     | $(-1)^k$ | $i$         |
| $1'''$| 1 | $-1$ | $(-1)^k$ | $-i$ |
| $2\mathbb{Z}$ | 2 | $(-1)^j/2$ | $\xi^{b} + \xi^{-b}$ | 0 |

$\xi$ a primitive $3n$-th root of unity and $j = 1, \ldots, n - 1$.

- Binary tetrahedral group: $BT = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle$, $-id := (ab)^2$.

| $id$ | $-id$ | $a$ | $-a$ | $b$ | $-b$ |
|------|-------|-----|------|-----|------|
| 1    | 1     | 1   | 1    | 1   | 1    |
| $1'$ | 1     | $\omega$ | $\omega^2$ | $\omega^2$ | 1 |
| $1''$| 1     | $\omega^2$ | $\omega$ | $\omega$ | 1 |
| $2'$ | 2     | $-2$ | 1    | $-1$ | 1    |
| $2''$| 2     | $-2$ | $\omega$ | $-\omega$ | $\omega^2$ |

$\omega$ a primitive 3rd root of unity.

- Binary octahedral group: $BO = \langle a, b \mid a^3 = b^4 = (ab)^2 \rangle$, $-id := (ab)^2$.

| $id$ | $-id$ | $ab$ | $a$ | $-a$ | $b$ | $-b$ |
|------|-------|-----|-----|------|-----|------|
| 1    | 1     | 1   | 1   | 1    | 1   | 1    |
| $1'$ | 1     | $-1$ | $1$ | $-1$ | $-1$ | 1    |
| $2''$| 2     | 0   | $-1$ | $-1$ | 0   | 2    |
| $3'$ | 3     | 1   | 0   | $-1$ | $-1$ | $-1$ |
| $3''$| 3     | $-1$ | 0   | 1   | $-1$ | $-1$ |
| $2'$ | 2     | $-2$ | 0   | $1$ | $-\sqrt{2}$ | $-\sqrt{2}$ |
| $2'''$| 2 | $-2$ | 0 | $1$ | $-\sqrt{2}$ | $\sqrt{2}$ |
| 4    | 4     | $-4$ | 0   | $1$ | 1   | 0    |

- Binary icosahedral group: $BI = \langle a, b \mid a^3 = b^5 = (ab)^2 \rangle$, $-id := (ab)^2$.

| $id$ | $-id$ | $a$ | $-a$ | $b$ | $-b$ | $b^2$ | $-b^2$ | $ab$ |
|------|-------|-----|------|-----|------|-------|--------|------|
| 1    | 1     | 1   | 1    | 1   | 1    | 1     | 1      | 1    |
| 3    | 3     | 0   | 0    | $\mu^+$ | $\mu^+$ | $\mu^-$ | $\mu^-$ | $-1$ |
| $3'$ | 3     | 0   | 0    | $\mu^-$ | $\mu^-$ | $\mu^+$ | $\mu^+$ | $-1$ |
| $4'$ | 4     | 4   | 1    | $-1$ | $-1$ | $-1$ | $-1$ | $0$ |
| 5    | 5     | $-1$ | $-1$ | 0    | 0    | 0    | 0    | 0 |
| 2    | 2     | $-2$ | 1    | $-1$ | $-1$ | $-1$ | $-1$ | 0 |
| $2'$ | 2     | $-2$ | 1    | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $4'$ | 4     | $-4$ | 1    | 1    | $-1$ | $-1$ | $1$   | $0$ |
| 6    | 6     | $-6$ | 0    | 0    | $-1$ | 1    | $-1$ | $0$ |

$\mu^* := \frac{1}{2}(1 + \sqrt{5})$, $\mu^- := \frac{1}{2}(1 - \sqrt{5})$. 

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