On Goldman Bracket for $G_2$ Gauge Group

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Abstract

In this paper, we derive Goldman-type bracket for traces of monodromy matrices of flat connections on a compact Riemann surface for $G_2$ gauge group. As a by-product, we give an alternative derivation of known Goldman bracket for classical gauge groups $GL(n,R)$, $SL(n,R)$, $U(n)$, $SU(n)$, $Sp(2n,R)$ and $SO(n)$.

1 Introduction

Traces of monodromy matrices of flat connections computed along closed loops on a Riemann surface $\Sigma$ are known to satisfy the Goldman’s Poisson bracket [6] for the following list of gauge groups: $GL(n,R)$, $GL(n,C)$, $GL(n,H)$, $SL(n,R)$, $SL(n,C)$, $SL(n,H)$, $O(p,q)$, $O(n,C)$, $Sp(2n,H)$, $U(p,q)$, $Sp(2n,R)$, $Sp(p,q)$ $Sp(2n,C)$ and $SU(p,q)$. The problem of computation of the bracket between traces of monodromy matrices for exceptional gauge groups so far remains open. The goal of this paper is to fill this gap for $G_2$, the simplest exceptional Lie group.

Given the fundamental group $\pi$ of a closed oriented surface $S$ and a Lie group $G$, the space $\text{Hom}(\pi,G)/G$ is defined as the quotient of the analytic variety $\text{Hom}(\pi,G)$ by the action of $G$ by conjugation. Consider a $G$-valued ($\mathcal{G}$ is the Lie algebra of $G$) flat connection $A = A_z(z,\bar{z})dz + A_{\bar{z}}(z,\bar{z})d\bar{z}$ on a compact Riemann surface $\Sigma$ of genus $g$. The Atiyah-Bott bracket on the space of flat connections can be derived from the Chern-Simons action on the 3-dimensional manifold $\Sigma \times \mathbb{R}$. Let us represent the connection 1-forms as $A_i = \sum_{a=1}^{n} A^a_i t_a$, where $i = z, \bar{z}$. The generators $\{t_a\}$ of the the gauge group $G$ are assumed to satisfy the normalization condition

$$\frac{1}{2} \text{Tr}(t_a t_b) = f(a) \delta_{ab}, \quad (1.1)$$
with \( f(a) = \pm 1 \). Then the Atiyah-Bott bracket reads
\[
\{ A^a_z, A^b_{z'} \} = \frac{f(a)}{2} \delta^{ab} \delta^{(2)}(z - z').
\] (1.2)

The space of flat connections modulo gauge transformation is finite dimensional and traces of the monodromy matrices of flat connections can be chosen to be the underlying gauge invariant observables.

Goldman in [6] derived the Poisson bracket between traces of the monodromy matrices for classical groups already listed at the start of the introduction. For example, for any two transversally intersecting oriented closed curves \( \gamma_1 \) and \( \gamma_2 \) on \( \Sigma \), the Poisson bracket between traces of \( GL(n, \mathbb{R}) \) monodromy matrices reads
\[
\{ \text{Tr} M_{\gamma_1}, \text{Tr} M_{\gamma_2} \} = \text{Tr} M_{\gamma_1 \circ \gamma_2}.
\] (1.3)

For the case of \( Sp(2n, \mathbb{R}) \) the bracket looks as follows
\[
\{ \text{Tr} M_{\gamma_1}, \text{Tr} M_{\gamma_2} \} = \frac{1}{2} \left( \text{Tr} M_{\gamma_1 \circ \gamma_2} - \text{Tr} M_{\gamma_1 \circ \gamma_2^{-1}} \right).
\] (1.4)

Here, \( \gamma_1 \circ \gamma_2 \) and \( \gamma_1 \circ \gamma_2^{-1} \) represent loops on \( \Sigma \) which are obtained from \( \gamma_1 \) and \( \gamma_2 \) by appropriate resolution of their intersection points (see figures 1 and 2).

The extensions of Goldman’s results to exceptional Lie groups were not known before. Of all five exceptional Lie groups, \( G_2 \) is simultaneously the smallest and one of the most important ones. Recently, it played pivotal roles in exceptional geometry (see [3]) and in Lattice QCD (see [8], [10], [11] and [7], for example). Manifolds admitting \( G_2 \) holonomy are also of special interest in M-theory (see [2] and [1] for a brief review).

The goal of this paper is to generalize Goldman’s bracket to the case of \( G_2 \) gauge group. Consider the \( 7 \times 7 \) monodromy matrices \( M_{\gamma_1} \) and \( M_{\gamma_2} \) to be in the fundamental representation of \( G_2 \). We prove the following formula for the Poisson bracket between traces of these monodromy matrices
\[
\{ \text{Tr} M_{\gamma_1}, \text{Tr} M_{\gamma_2} \} = \frac{1}{2} \left[ \text{Tr} M_{\gamma_1 \circ \gamma_2} - \text{Tr} M_{\gamma_1 \circ \gamma_2^{-1}} + \frac{1}{3} \sum_{i=1}^{7} \text{Tr}(M_{\gamma_1} \mathbb{O}_i) \text{Tr}(M_{\gamma_2} \mathbb{O}_i) \right].
\] (1.5)

Here \( \mathbb{O}_1, \ldots, \mathbb{O}_7 \) are skew symmetric \( 7 \times 7 \) matrices representing the right action of purely imaginary octonions lying in 6-dimensional sphere \( S^6 \) (see [9] for a detailed discussion on left and right octonionic operators). The loops \( \gamma_1 \circ \gamma_2 \) and \( \gamma_1 \circ \gamma_2^{-1} \) are obtained from \( \gamma_1 \) and \( \gamma_2 \) by an appropriate resolution of their intersection points. Let \( \phi = \sum_{i=1}^{7} \phi_i e_i \) be a purely imaginary octonion and the matrices \( \{ \mathbb{O}_i \} \) in (1.5) represent the octonionic imaginary units \( \{ e_i \} \) in the sense of [9]. Then the action of \( \mathbb{O}_i \) on \( \phi \) is defined as
\[
\mathbb{O}_i \phi = \Im(\phi e_i), \quad i = 1, 2, \ldots, 7.
\] (1.6)
Therefore, the matrices $M_{\gamma_1}O_i$’s and $M_{\gamma_2}O_i$’s, appearing in (1.5), transform the purely imaginary octonions into themselves, i.e. they map $S^6$ to itself. A new ingredient of (1.5) in comparison with the classical Goldman bracket is the term $\sum_{i=1}^{7} \text{Tr}(M_{\gamma_1}O_i) \text{Tr}(M_{\gamma_2}O_i)$. This expression turns out to be gauge invariant although none of the terms $\{\text{Tr}(M_{\gamma_1}O_i)\}$ is individually gauge invariant.

The organization of the paper is as follows. In section 2 we recall how the Atiyah-Bott bracket originates from the Hamiltonian Chern-Simons theory and derive an auxiliary expression for Poisson bracket of traces of monodromy matrices along intersecting loops. In section 3 we show how this general expression can be used to derive this bracket for a few cases from Goldman’s list. In section 4 we derive the Poisson bracket between traces of $G_2$ monodromy matrices using the formalism developed in section 2.

2 Poisson brackets for traces of monodromy matrices from the Atiyah-Bott bracket

In this paper, space-time is modelled as a 3-manifold $\Sigma \times \mathbb{R}$ where $\Sigma$ representing “space” is a compact Riemann surface and $\mathbb{R}$ represents “time”. For an arbitrary real gauge group $G$, the Chern-Simons action functional on $\Sigma \times \mathbb{R}$ reads

$$S_{CS} = 2\int_{\Sigma \times \mathbb{R}} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

The connection 1-forms on the principal $G$-bundle, taking their values in the Lie algebra $G$ of the gauge group $G$, are given by

$$A = A_z(z, \bar{z}, t)dz + A_{\bar{z}}(z, \bar{z}, t)d\bar{z} + A_0(z, \bar{z}, t)dt.$$  

Denote the generators of the group $G$ are given by $\{t_a\}$ and write

$$A_i = \sum_{a=1}^{n} A_i^a t_a,$$  

where the space-time label $i = z, \bar{z}, 0$. Here, $\{t_a\}$ are chosen such that the following holds

$$\frac{1}{2} \text{Tr}(t_a t_b) = f(a)\delta_{ab},$$

with $f(a) = \pm 1$. The curvature form $F = dA + A \wedge A$, for the principal connections (2.2), is easily found to vanish. The time component $A_0$ of the flat connections can be gauged out. From the gauge fixed Chern-Simons action, one obtains the coordinates and momenta of the underlying phase space of flat connections. At each space-time point, there are $2n$ degrees of freedom associated with $\{A_z^a\}$ and $\{A_{\bar{z}}^a\}$. The Poisson structure
of the infinite dimensional space of flat connections is given by the famous Atiyah-Bott bracket between the phase space variables at a given time slice of space-time $\Sigma \times \mathbb{R}$:

$$\{ A^a_z, A^b_{z'} \} = \frac{f(a)}{2} \delta^{ab} \delta^{(2)}(z - z'). \quad (2.5)$$

The space of flat connections modulo gauge transformation turns out to be a finite dimensional space; traces of monodromy matrices of flat connections computed along intersecting loops on $\Sigma$ can be used as gauge invariant observables. In this section, we compute the Poisson bracket between traces of the monodromy matrices along two homotopically inequivalent loops that intersect transversally at a single point using the formalism originating from the Hamiltonian theory of Solitons. The generalization to many intersection points is straightforward.

Denote two loops on $\Sigma$ by $\gamma_1$ and $\gamma_2$ that intersect transversally at a single point on $\Sigma$. Without loss of generality, let us assume that the paths $\gamma_1$ and $\gamma_2$ intersect orthogonally at $O$. These two loops are illustrated schematically by $x_1 O x_2$ and $y_1 O y_2$ in figure 1. The parts $x_1 O x_2$ and $y_1 O y_2$ are taken to lie along $X$ and $Y$ axes, respectively. The relevant transition matrices are denoted by $T(x_1, x_2)$ and $T(y_1, y_2)$. Let us denote the monodromy matrices computed along $\gamma_1$ and $\gamma_2$ by $M_{\gamma_1}$ and $M_{\gamma_2}$, respectively. They are given by

$$M_{\gamma_1} = T(x_1, x_2) \widetilde{M}_{\gamma_1}, \quad M_{\gamma_2} = T(y_1, y_2) \widetilde{M}_{\gamma_2}, \quad (2.6)$$

where $\widetilde{M}_{\gamma_1}$ and $\widetilde{M}_{\gamma_2}$ are the remaining contributions of monodromy matrices $M_{\gamma_1}$ and $M_{\gamma_2}$ due to the paths $x_2 x_1$ and $y_2 y_1$, respectively (see figure 1). The matrices $\widetilde{M}_{\gamma_1}$ and $\widetilde{M}_{\gamma_2}$ Poisson commute with each other and with other transition matrices in question since they are due to parts of the loops far away from the intersection point $O$ and hence have nothing to do with each other. There are two distinct ways of resolving the point of intersection $O$. One of them is shown in figure 1 to obtain the loop $\gamma_1 \circ \gamma_2$. Monodromy matrix around the loop $\gamma_1 \circ \gamma_2$ is denoted by $M_{\gamma_1 \circ \gamma_2}$.

Here, the matrices $M_{\gamma_1}$ and $M_{\gamma_2}$ take their values in the gauge group $G$. Let us represent the connection 1-form $A$ on $\Sigma$ as

$$A = A_1(z, \bar{z}) dz + A_2(z, \bar{z}) d\bar{z} = A_1(x, y) dx + A_2(x, y) dy. \quad (2.7)$$

In view of (2.7), the 1-forms, restricted to the real and imaginary axes, read

$$A(x, 0) = A_1(x, 0) dx, \quad \text{and} \quad A(0, y) = A_2(0, y) dy. \quad (2.8)$$

In terms of the real and imaginary parts of the connection 1-forms, i.e. $A_1$ and $A_2$, the Atiyah-Bott bracket (2.5) reduces to

$$\{ A^a_1(x, y), A^b_2(x', y') \} = \frac{1}{2} f(a) \delta^{ab} \delta(x - x') \delta(y - y'). \quad (2.9)$$
Lemma 2.1. The fundamental Poisson brackets between $\mathcal{G}$ valued 1-forms are given by

$$\{A_1(x,0), A_2(0,y)\} = \frac{1}{2} \delta(x) \delta(y) \Gamma,$$

(2.10)

where $\Gamma$ is the Casimir tensor for $\mathcal{G}$ given by

$$\Gamma = \sum_{a=1}^{n} f(a) (t_a \otimes t_a).$$

(2.11)

Proof. The above lemma is just a consequence of (2.9).

$$\{A_1(x,0), A_2(0,y)\} = \left\{ \sum_{a=1}^{n} A_1^a(x,0) t_a \otimes \sum_{b=1}^{n} A_2^b(0,y) t_b \right\}$$

$$= \sum_{a=1}^{n} \sum_{b=1}^{n} \{A_1^a(x,0), A_2^b(0,y)\} (t_a \otimes t_b)$$

$$= \frac{1}{2} \delta(x) \delta(y) \sum_{a=1}^{n} f(a) (t_a \otimes t_a).$$

Remark 2.1. We should emphasize in the context of lemma 2.1 that the basis of the underlying Lie algebra is chosen in such a way that the trace form between the group generators is diagonalised in order to comply with what was used in the derivation of the
Atiyah-Bott bracket \(2.5\). The statement of lemma 2.1 is independent of the representation of the Lie algebra, though. All it means is that the same representation has to be chosen during both the derivations of the Atiyah-Bott brackets and the fundamental Poisson brackets.

Using lemma 2.1 one obtains the Poisson bracket between transition matrices along two small paths of the given loops around the intersection point \(O\) as illustrated in figure 1.

**Lemma 2.2.** Let \(T(x_1, x_2)\) and \(T(y_1, y_2)\) be the transition matrices corresponding to paths \(x_1Ox_2\) and \(y_1Oy_2\) as indicated in figure 1. The Poisson brackets between them is given by

\[
\{T(x_1, x_2) \otimes T(y_1, y_2)\} = \frac{1}{2} [T(x_1, 0) \otimes T(y_1, 0)] \Gamma [T(0, x_2) \otimes T(0, y_2)],
\]

where \(\Gamma\) is the Casimir tensor given by (2.11). Here, \(T(x_1, y_2)\) and \(T(y_1, x_2)\), appearing in the right side of (2.12), are computed along the loop \(\gamma_1 \circ \gamma_2\) of figure 1.

**Proof.** The Poisson brackets between transition matrices in the context of Hamiltonian theory of Solitons are given in ([5], page 192). In our setting, this formula gives

\[
\frac{1}{2} [T(x_1, 0) \otimes T(y_1, 0)] \Gamma [T(0, x_2) \otimes T(0, y_2)].
\]

\(\square\)

**Lemma 2.3.** The Poisson bracket between traces of monodromy matrices is as follows

\[
\{\text{Tr } M_{\gamma_1}, \text{Tr } M_{\gamma_2}\} = \frac{1}{2} \text{Tr}_{12}[(T(0, x_2) \bar{M}_{\gamma_1} T(x_1, 0) \otimes T(0, y_2) \bar{M}_{\gamma_2} T(y_1, 0)] \Gamma],
\]

where \(M_{\gamma_1}\) and \(M_{\gamma_2}\) are given by (2.6). In (2.14), \(\text{Tr}\) and \(\text{Tr}_{12}\) denote trace in the vector spaces \(\mathbb{R}^n\) and \(\mathbb{R}^n \otimes \mathbb{R}^n\), respectively.

**Proof.** Using (2.6), one obtains

\[
\{M_{\gamma_1} \otimes M_{\gamma_2}\} = \{T(x_1, x_2) \bar{M}_{\gamma_1} \otimes T(y_1, y_2) \bar{M}_{\gamma_2}\}
\]

\[
= \{T(x_1, x_2) \otimes T(y_1, y_2) \bar{M}_{\gamma_2}\}(\bar{M}_{\gamma_1} \otimes I_2)
\]

\[
+ [T(x_1, x_2) \otimes I_2]\{\bar{M}_{\gamma_1} \otimes T(y_1, y_2) \bar{M}_{\gamma_2}\}
\]

\[
= \{T(x_1, x_2) \otimes T(y_1, y_2)\}(I_2 \otimes \bar{M}_{\gamma_2})(\bar{M}_{\gamma_1} \otimes I_2),
\]

\(\square\)
where we have exploited the fact that \( \widetilde{M}_{\gamma_1} \) and \( \widetilde{M}_{\gamma_2} \) both Poisson commute with \( T(x_1, x_2) \) and \( T(y_1, y_2) \), and amongst themselves. Using lemma 2.2 one obtains

\[
\{ M_{\gamma_1} \otimes M_{\gamma_2} \} = \{ T(x_1, x_2) \otimes T(y_1, y_2) \} (\widetilde{M}_{\gamma_1} \otimes \widetilde{M}_{\gamma_2})
\]

\[
= \frac{1}{2} [T(x_1, 0) \otimes T(y_1, 0)] [T(0, x_2) \otimes T(0, y_2)] (\widetilde{M}_{\gamma_1} \otimes \widetilde{M}_{\gamma_2})
\]

Taking trace on both sides of equation (2.16) and subsequently making use of the cyclic property of trace, one finally obtains

\[
\{ \operatorname{Tr} M_{\gamma_1}, \operatorname{Tr} M_{\gamma_2} \} = \frac{1}{2} \operatorname{Tr}_{12} [T(0, x_2) \widetilde{M}_{\gamma_1} T(x_1, 0) \otimes T(0, y_2) \widetilde{M}_{\gamma_2} T(y_1, 0)] \Gamma. \tag{2.17}
\]

\]

3 Examples of Poisson brackets between traces of monodromy matrices for some known real Lie groups

In the previous section, we obtained an auxiliary formula (2.14) for Poisson brackets between traces of monodromy matrices computed along free homotopy classes of loops on \( \Sigma \). In this section, we shall use it to reproduce Goldman’s brackets for \( GL(n, \mathbb{R}) \), \( U(n) \), \( SL(n, \mathbb{R}) \), \( SU(n) \), \( Sp(2n, \mathbb{R}) \) and \( SO(n) \) gauge groups.

We, first, note that the generalized Gell-Mann matrices in \( n \) dimensions read

\[
h_1^n = \sqrt{2} \sum_{i=1}^{n} e_{ii},
\]

\[
h_k^n = \sqrt{2} \sum_{i=1}^{k-1} e_{ii} - \sqrt{2} \sum_{k}^{n} e_{kk}, \quad \text{for} \ 1 < k \leq n, \tag{3.1}
\]

\[
f_{k,j}^n = e_{kj} + e_{jk}, \quad \text{for} \ k < j,
\]

\[
f_{k,j}^n = -i(e_{jk} - e_{kj}), \quad \text{for} \ k > j.
\]

Here, \( e_{jk} \) is an \( n \times n \) matrix with 1 in the \( (j, k) \) entry and 0 elsewhere.

A couple of preparatory lemmas are required in order to derive the Poisson bracket between traces of monodromy matrices for some known real Lie groups from (2.14).

**Lemma 3.1.** Given the matrices \( h_1^n \) and \( h_k^n \) as in (3.1), we have

\[
h_1^n \otimes h_1^n + \sum_{k=2}^{n} h_k^n \otimes h_k^n = 2e_{11} \otimes e_{11} + 2 \sum_{k=2}^{n} e_{kk} \otimes e_{kk}. \tag{3.2}
\]
Proof.

\[
\sum_{k=2}^{n} h_k^n \otimes h_k^n
\]

\[
= \sum_{k=2}^{n} \left[ \left( \sqrt{\frac{2}{k(k-1)}} \sum_{i=1}^{k-1} e_{ii} - \sqrt{\frac{2}{k}} e_{kk} \right) \otimes \left( \sqrt{\frac{2}{k(k-1)}} \sum_{i=1}^{k-1} e_{ii} - \sqrt{\frac{2}{k}} e_{kk} \right) \right]
\]

\[
= \sum_{k=2}^{n} \left[ \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \frac{2}{k} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \frac{2}{k} \sum_{i=1}^{k-1} e_{ii} \otimes e_{kk} - \frac{2}{k} \sum_{i=1}^{k-1} e_{ii} \otimes e_{kk} \right]
\]

Consider the right side of (3.3) and compute

\[
\sum_{k=2}^{n} \left[ \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \frac{2}{k} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) \right]
\]

\[
= \sum_{k=2}^{n} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right)
\]

\[
= \sum_{k=2}^{n} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right)
\]

\[
+ \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes e_{k-1,k-1} \right) + \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{k-1,k-1} \otimes e_{ii} \right)
\]

\[
- \sum_{k=3}^{n+1} \frac{2}{k-1} \left( e_{k-1,k-1} \otimes e_{k-1,k-1} \right)
\]

\[
= 2e_{11} \otimes e_{11} - \frac{2}{n} \left( \sum_{i=1}^{n} e_{ii} \otimes \sum_{i=1}^{n} e_{ii} \right) + \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes e_{k-1,k-1} \right)
\]

\[
+ \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{k-1,k-1} \otimes e_{ii} \right) - \sum_{k=3}^{n+1} \frac{2}{k-1} \left( e_{k-1,k-1} \otimes e_{k-1,k-1} \right)
\]

\[
= 2e_{11} \otimes e_{11} - \frac{2}{n} \left( \sum_{i=1}^{n} e_{ii} \otimes \sum_{i=1}^{n} e_{ii} \right) + \sum_{k=2}^{n} \frac{2}{k} \left( \sum_{i=1}^{k} e_{kk} \otimes e_{ii} \right)
\]

\[
+ \sum_{k=2}^{n} \frac{2}{k} \left( \sum_{i=1}^{k} e_{kk} \otimes e_{ii} \right) - \sum_{k=2}^{n} \frac{2}{k} \left( e_{kk} \otimes e_{kk} \right)
\]

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\[ = 2e_{11} \otimes e_{11} - \frac{2}{n} \left( \sum_{i=1}^{n} e_{ii} \otimes \sum_{i=1}^{n} e_{ii} \right) + \frac{2}{k} \sum_{k=2}^{n} \left( \sum_{i=1}^{k-1} e_{ii} \otimes e_{kk} \right) + \frac{2}{k} (e_{kk} \otimes e_{kk}) \]
\[ + \frac{2}{k} \sum_{k=2}^{n} \left( \sum_{i=1}^{k-1} e_{kk} \otimes e_{ii} \right) - \frac{2}{k} (e_{kk} \otimes e_{kk}) \]
\[ = 2e_{11} \otimes e_{11} - \frac{2}{n} \left( \sum_{i=1}^{n} e_{ii} \otimes \sum_{i=1}^{n} e_{ii} \right) + \frac{2}{k} \sum_{k=2}^{n} \left( \sum_{i=1}^{k-1} e_{ii} \otimes e_{kk} \right) + \frac{2}{k} (e_{kk} \otimes e_{kk}) \]
\[ + \frac{2}{k} \sum_{k=2}^{n} \left( \sum_{i=1}^{k-1} e_{kk} \otimes e_{ii} \right). \]  

(3.4)

Plugging (3.4) into (3.3), one obtains
\[ \sum_{k=2}^{n} h_{k}^{n} \otimes h_{k}^{n} = 2e_{11} \otimes e_{11} - \frac{2}{n} \left( \sum_{i=1}^{n} e_{ii} \otimes \sum_{i=1}^{n} e_{ii} \right) + \frac{2}{k} \sum_{k=2}^{n} e_{kk} \otimes e_{kk}, \]  

(3.5)

which leads to
\[ h_{1}^{n} \otimes h_{1}^{n} + \sum_{k=2}^{n} h_{k}^{n} \otimes h_{k}^{n} = 2e_{11} \otimes e_{11} + 2 \sum_{k=2}^{n} e_{kk} \otimes e_{kk}. \]

Lemma 3.2. Given \( f_{k,j}^{n} \) for \( k < j \) and \( k > j \) as in (3.1), the following holds
\[ \sum_{k \neq j} f_{k,j}^{n} \otimes f_{k,j}^{n} = 2 \sum_{k \neq j} e_{jk} \otimes e_{kj}. \]  

(3.6)

Proof.
\[ \sum_{k < j} f_{k,j}^{n} \otimes f_{k,j}^{n} = \sum_{k < j} (e_{kj} + e_{jk}) \otimes (e_{kj} + e_{jk}) \]
\[ = \sum_{k < j} e_{kj} \otimes e_{kj} + \sum_{k < j} e_{kj} \otimes e_{jk} + \sum_{k < j} e_{jk} \otimes e_{kj} + \sum_{k < j} e_{jk} \otimes e_{jk}. \]  

(3.7)

We also have,
\[ \sum_{k > j} f_{k,j}^{n} \otimes f_{k,j}^{n} = \sum_{k > j} (-ie_{jk} + ie_{kj}) \otimes (-ie_{jk} + ie_{kj}) \]
\[ = - \sum_{k > j} e_{jk} \otimes e_{jk} + \sum_{k > j} e_{jk} \otimes e_{kj} + \sum_{k > j} e_{kj} \otimes e_{kj} - \sum_{k > j} e_{kj} \otimes e_{kj}. \]  

(3.8)

Therefore, (3.7) together with (3.8) imply
\[ \sum_{k \neq j} f_{k,j}^{n} \otimes f_{k,j}^{n} = 2 \sum_{k > j} e_{jk} \otimes e_{kj} + 2 \sum_{k < j} e_{jk} \otimes e_{kj} \]
\[ = 2 \sum_{k \neq j} e_{jk} \otimes e_{kj}. \]  

(3.9)
Let us now compute the Casimir operator $\Gamma$ appearing in (2.14) for the specific cases of $GL(n, \mathbb{R})$, $U(n)$, $SL(n, \mathbb{R})$, $SU(n)$ and $SO(n)$. In what follows, the $n^2 \times n^2$ permutation matrix is denoted by $P$. Given two $n \times n$ matrices $A$ and $B$, $P$ enjoys the following properties:

$$P(A \otimes B) = (B \otimes A)P$$

$$\text{Tr}_{12}[(A \otimes B)P] = \text{Tr}(AB).$$

(3.10)

**Proposition 1.** For $GL(n, \mathbb{R})$ and $U(n)$ gauge groups, the Casimir tensor in the auxiliary expression (2.14) of Poisson bracket between traces of the monodromy matrices reads

$$\Gamma = 2P,$$

(3.11)

where $P = \sum_{k,j=1}^{n} e_{jk} \otimes e_{kj}$ is the Permutation matrix.

**Proof.**

**Case 1: GL(n, R)**

The Lie algebra associated with $GL(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{R})$, the vector space of all real $n \times n$ matrices. The dimension of this vector space is $n^2$. We choose the matrix $h^n_1$, $(n-1)$ matrices $h^n_k$ with $1 < k \leq n$, $\frac{(n^2-n)}{2}$ matrices $f^n_{k,j}$ with $k < j$, and another $\frac{(n^2-n)}{2}$ matrices $if^n_{k,j}$ with $k > j$ from (3.1) to form a basis of $\mathfrak{gl}(n, \mathbb{R})$. Here, in (2.4), associated with the preceding choice of generators for $GL(n, \mathbb{R})$, $f(a) = -1$ for the $\frac{(n^2-n)}{2}$ basis elements $if^n_{k,j}$ with $k > j$. For the rest of the $n^2$ basis elements, we have $f(a) = 1$.

With the above choice of the basis of $\mathfrak{gl}(n, \mathbb{R})$, the Casimir tensor $\Gamma$ reads,

$$\Gamma = h^n_1 \otimes h^n_1 + \sum_{k=2}^{n} h^n_k \otimes h^n_k + \sum_{k<j} f^n_{k,j} \otimes f^n_{k,j} + \sum_{k>j} -(if^n_{k,j} \otimes if^n_{k,j})$$

$$= h^n_1 \otimes h^n_1 + \sum_{k=2}^{n} h^n_k \otimes h^n_k + \sum_{k<j} f^n_{k,j} \otimes f^n_{k,j} + \sum_{k>j} (f^n_{k,j} \otimes f^n_{k,j}).$$

(3.12)

Using lemma [3.1] together with lemma [3.2] in (3.12), one obtains the Casimir tensor for the case of $GL(n, \mathbb{R})$ gauge group:

$$\Gamma = 2 \sum_{k,j=1}^{n} e_{jk} \otimes e_{kj}.$$  

(3.13)

**Case 2: U(n)**

An appropriate choice of basis for the Lie algebra $\mathfrak{u}(n)$, in the context of (2.4), will be the $n^2$ skew-Hermitian matrices (see (3.1)) $ih^n_1$, $ih^n_k$ for $1 < k \leq n$ and $if^n_{k,j}$ for $k \neq j$. 

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In accordance with the choice of these generators of unitary group $U(n)$, $f(a) = -1$ in (2.4) for $a = 1, 2, \ldots, n^2$. The corresponding Casimir tensor $\Gamma$ reads off immediately

$$
\Gamma = -(ih_1^n \otimes ih_1^n) + \sum_{k=2}^{n} -(ih_k^n \otimes ih_k^n) + \sum_{k<j} -(if_{k,j}^n \otimes if_{k,j}^n) + \sum_{k>j} -((if_{k,j}^n \otimes if_{k,j}^n)
$$

$$
= h_1^n \otimes h_1^n + \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} f_{k,j}^n \otimes f_{k,j}^n
$$

$$
= 2 \sum_{k,j=1}^{n} e_{jk} \otimes e_{kj}. \quad (3.14)
$$

Here, again, we use lemma 3.1 and lemma 3.2 to arrive at (3.14).

Direct application of proposition 1 in (2.14) and subsequent use of the properties of $P$, enumerated in (3.10), yield the formula of Poisson bracket for traces of $GL(n, \mathbb{R})$ and $U(n)$ monodromy matrices as given by the following theorem:

**Theorem 2.** The Poisson bracket between traces of $GL(n, \mathbb{R})$ or $U(n)$ monodromy matrices reads

$$
\{\text{Tr } M_{\gamma_1}, \text{Tr } M_{\gamma_2}\} = \text{Tr } M_{\gamma_1 \circ \gamma_2}, \quad (3.15)
$$

where $M_{\gamma_1 \circ \gamma_2}$ is a $GL(n, \mathbb{R})$ or $U(n)$ monodromy matrix computed along the loop $\gamma_1 \circ \gamma_2$ of figure 7.

**Proposition 3.** The Casimir tensor in (2.14) for $SL(n, \mathbb{R})$ or $SU(n)$ gauge group reads

$$
\Gamma = 2P - \frac{2}{n} \mathbb{I}, \quad (3.16)
$$

with $P = \sum_{k,j=1}^{n} e_{jk} \otimes e_{kj}$ being the Permutation matrix and $\mathbb{I}$ being the $n^2 \times n^2$ identity matrix.

**Proof.**

**Case 1: SL$(n, \mathbb{R})$**

The Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ consists of traceless $n \times n$ real matrices. We, therefore, choose $(n-1)$ matrices $h_k^n$ with $1 < k \leq n$, $(\frac{n^2-n}{2})$ matrices $f_{k,j}^n$ for $k \leq j$ and another $(\frac{n^2-n}{2})$ real matrices $if_{k,j}^n$ with $k > j$ from the ones enumerated in (3.1). As was in the case of $\mathfrak{gl}(n, \mathbb{R})$, $f(a) = -1$ in (2.4) holds only for the $SL(n, \mathbb{R})$ group generators given by $if_{k,j}^n$. Therefore, the associated Casimir tensor reads

$$
\Gamma = \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} -(if_{k,j}^n \otimes if_{k,j}^n)
$$

$$
= \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} f_{k,j}^n \otimes f_{k,j}^n
$$

$$
= 2P - h_1^n \otimes h_1^n
$$

$$
= 2P - \frac{2}{n}. \quad (3.17)
$$
Case 2: $SU(n)$

The real Lie algebra $\mathfrak{su}(n)$ consists of $n \times n$ traceless skew-Hermitian matrices. As a basis of $\mathfrak{su}(n)$, we choose $(n - 1)$ traceless skew-Hermitian matrices $i\hbar^n_k$ with $1 < k \leq n$ and another $(n^2 - n)$ such matrices $i\hbar^n_{kj}$ for $k \neq j$ from the matrices enumerated in (3.1). Here, we only have $f(a) = -1$ in (2.4) for all such $(n^2 - 1)$ group generators of $SU(n)$. The corresponding Casimir tensor then reads

$$\Gamma = \sum_{k=2}^{n} -(i\hbar^n_k \otimes i\hbar^n_k) + \sum_{k<j} -(i\hbar^n_{kj} \otimes i\hbar^n_{kj}) + \sum_{k>j} -(i\hbar^n_{kj} \otimes i\hbar^n_{kj})$$

$$= \sum_{k=2}^{n} \hbar^n_k \otimes \hbar^n_k + \sum_{k<j} \hbar^n_{kj} \otimes \hbar^n_{kj} + \sum_{k>j} \hbar^n_{kj} \otimes \hbar^n_{kj}$$

$$= 2P - \hbar^n_1 \otimes \hbar^n_1$$

$$= 2P - \frac{2}{n} \mathbb{I}.$$  \hspace{1cm} (3.18)

We have repeatedly used lemma 3.1 and lemma 3.2 in establishing (3.17) and (3.18).

Following the use of proposition 3 in (2.14) and subsequent use of the properties of $P$ as given by (3.10), one obtains the Poisson bracket for $SL(n, \mathbb{R})$ and $SU(n)$ monodromy matrices.

**Theorem 4.** The Poisson bracket between traces of monodromy matrices for $SL(n, \mathbb{R})$ and $SU(n)$ gauge groups is given by

$$\{\text{Tr} \ M_{\gamma_1}, \text{Tr} \ M_{\gamma_2}\} = \text{Tr} \ M_{\gamma_1 \gamma_2} - \frac{1}{n} \text{Tr} \ M_{\gamma_1} \text{Tr} \ M_{\gamma_2}. \hspace{1cm} (3.19)$$

In course of proving theorem 4 one also makes use of the identity $\text{Tr}_{12}(A \otimes B) = \text{Tr} A \text{Tr} B$ for any two $n \times n$ matrices $A$ and $B$.

We shall now consider the case when the gauge group is $Sp(2n, \mathbb{R})$. It is being dealt separately since an appropriate choice of basis for the associated Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$, in view of (2.4), is unrelated with the generalized Gell-Mann matrices enumerated in (3.1).

The Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ is an $n(2n+1)$ dimensional real vector space. An appropriate choice of basis, along with respective $f(a) = \pm 1$ for $a = 1, 2, \ldots, n(2n+1)$ in (2.4), is outlined in the following table:
| i, j, k | Basis elements                                                                                                                                 | f(a) | No. of elements |
|---------|---------------------------------------------------------------------------------------------------------------------------------------------|------|-----------------|
| 1 ≤ i < j ≤ n | \( \frac{1}{\sqrt{2}} (e_{i,j+n} + e_{j,i+n} + e_{j+n,i} + e_{i+n,j}) \)                                                                 | 1    | \( \frac{n^2-n}{2} \) |
| 1 ≤ i < j ≤ n | \( \frac{1}{\sqrt{2}} (e_{i,j+n} + e_{j,i+n} - e_{j+n,i} - e_{i+n,j}) \)                                                                 | -1   | \( \frac{n^2-n}{2} \) |
| 1 ≤ k ≤ n   | \( e_{k,n+k} + e_{n+k,k} \)                                                                                                                | 1    | n               |
| 1 ≤ k ≤ n   | \( e_{k,n+k} - e_{n+k,k} \)                                                                                                                | -1   | n               |
| 1 ≤ i < j ≤ n | \( \frac{1}{\sqrt{2}} (e_{ij} + e_{ji} - e_{i+n,j+n} - e_{j+i+n} - e_{i+n,j+n} - e_{j+i+n}) \)                                       | 1    | \( \frac{n^2-n}{2} \) |
| 1 ≤ i < j ≤ n | \( \frac{1}{\sqrt{2}} (e_{ij} - e_{ji} + e_{i+n,j+n} - e_{j+i+n}) \)                                                                   | -1   | \( \frac{n^2-n}{2} \) |
| 1 ≤ k ≤ n   | \( e_{kk} - e_{k+n,k+n} \)                                                                                                                | 1    | n               |

Table 1: Appropriate choice of basis for \( \mathfrak{sp}(2n, \mathbb{R}) \)

Now, the Casimir tensor for the structure Lie group \( Sp(2n, \mathbb{R}) \) is provided by the following proposition

**Proposition 5.** The Casimir tensor \( \Gamma \) in (2.14), for \( Sp(2n, \mathbb{R}) \) gauge group, reads

\[
\Gamma = P + \chi,
\]

(3.20)

with \( \chi \) given as

\[
\chi = \sum_{1 \leq i < j \leq n} (e_{i,j+n} \otimes e_{i+n,j} + e_{j,i+n} \otimes e_{j+n,i} + e_{j+n,i} \otimes e_{j,i+n} + e_{i+n,j} \otimes e_{i,j+n} \\
- e_{ij} \otimes e_{i+n,j+n} - e_{i+n,j+n} \otimes e_{ij} - e_{ji} \otimes e_{j+n,i+n} - e_{i+n,j+n} \otimes e_{ij}) \\
+ \sum_{1 \leq k \leq n} (e_{k,n+k} \otimes e_{n+k,k} + e_{n+k,k} \otimes e_{k,n+k} - e_{kk} \otimes e_{k+n,k+n} - e_{k+n,k+n} \otimes e_{kk}).
\]

(3.21)

We shall be calling \( \chi \) as the defect matrix henceforth.

**Proof.** In order to prove proposition 5 we first note that, for any two \( n \times n \) matrices \( a \) and \( b \), the following holds

\[
(a + b) \otimes (a + b) - (a - b) \otimes (a - b) = 2(a \otimes b + b \otimes a).
\]

(3.22)

Using the above fact, we have the following for \( 1 \leq i < j \leq n \):

\[
\frac{1}{\sqrt{2}} (e_{i,j+n} + e_{j,i+n} + e_{j+n,i} + e_{i+n,j}) \otimes \frac{1}{\sqrt{2}} (e_{i,j+n} + e_{j,i+n} + e_{j+n,i} + e_{i+n,j}) \\
- \frac{1}{\sqrt{2}} (e_{i,j+n} + e_{j,i+n} - e_{j+n,i} - e_{i+n,j}) \otimes \frac{1}{\sqrt{2}} (e_{i,j+n} + e_{j,i+n} - e_{j+n,i} - e_{i+n,j}) \\
= e_{i,j+n} \otimes e_{i+n,j} + e_{i+n,j} \otimes e_{i,j+n} + e_{j,i+n} \otimes e_{j+n,i} + e_{j+n,i} \otimes e_{j,i+n} \\
+ e_{j+n,i} \otimes e_{i,j+n} + e_{j+n,i} \otimes e_{i,j+n} + e_{i+n,j} \otimes e_{i,j+n} + e_{i+n,j} \otimes e_{i,j+n}.
\]

(3.23)

We also compute for \( 1 \leq k \leq n \),

\[
(e_{k,n+k} + e_{n+k,k}) \otimes (e_{k,n+k} + e_{n+k,k}) - (e_{k,n+k} - e_{n+k,k}) \otimes (e_{k,n+k} - e_{n+k,k}) \\
= 2(e_{k,n+k} \otimes e_{n+k,k} + e_{n+k,k} \otimes e_{k,n+k}).
\]

(3.24)
Again, considering another set of \( n^2 - n \) generators and applying (3.22), one obtains for \( 1 \leq i < j \leq n, 
\)
\[
\frac{1}{\sqrt{2}}(e_{ij} + e_{ji} - e_{i+n,j+n} - e_{j+n,i+n}) \otimes \frac{1}{\sqrt{2}}(e_{ij} + e_{ji} - e_{i+n,j+n} - e_{j+n,i+n}) 
\]
\[
- \frac{1}{\sqrt{2}}(e_{ij} - e_{ji} + e_{i+n,j+n} - e_{j+n,i+n}) \otimes \frac{1}{\sqrt{2}}(e_{ij} - e_{ji} + e_{i+n,j+n} - e_{j+n,i+n}) 
\]
\[
= e_{ij} \otimes e_{ji} - e_{ij} \otimes e_{i+n,j+n} - e_{j+n,i+n} \otimes e_{ji} + e_{j+n,i+n} \otimes e_{i+n,j+n} + e_{ji} \otimes e_{ij} - e_{ji} \otimes e_{j+n,i+n} - e_{i+n,j+n} \otimes e_{ji} + e_{i+n,j+n} \otimes e_{j+n,i+n}.
\] (3.25)

Finally, for \( n \) diagonal generators of \( Sp(2n, \mathbb{R}) \), we obtain with \( 1 \leq k \leq n, 
\)
\[
(e_{kk} - e_{k+n,k+n}) \otimes (e_{kk} - e_{k+n,k+n}) 
\]
\[
= e_{kk} \otimes e_{kk} - e_{kk} \otimes e_{k+n,k+n} - e_{k+n,k+n} \otimes e_{kk} + e_{k+n,k+n} \otimes e_{k+n,k+n}.
\] (3.26)

Adding (3.23) to (3.25) and (3.24) to (3.26) followed by summing over \( 1 \leq i < j \leq n \) and \( 1 \leq k \leq n \), respectively and finally adding up the two summands, we obtain,

\[
\Gamma = \left[ \sum_{1 \leq i < j \leq n} \left( e_{i,j+n} \otimes e_{j+n,i} + e_{j,i+n} \otimes e_{i+n,j} + e_{i,j+n} \otimes e_{i+n,j} + e_{i+n,j} \otimes e_{j,i+n} 
\right. 
\right. 
\left. + e_{ij} \otimes e_{ji} + e_{i+n,j+n} \otimes e_{i+n,j+n} + e_{ji} \otimes e_{ij} + e_{i+n,j+n} \otimes e_{j+n,i+n} 
\right. 
\left. + \sum_{1 \leq k \leq n} \left( e_{k,n+k} \otimes e_{n+k,k} + e_{n+k,k} \otimes e_{k,n+k} + e_{kk} \otimes e_{kk} + e_{k+n,k+n} \otimes e_{k+n,k+n} \right) 
\right. 
\left. + \sum_{1 \leq i < j \leq n} \left( e_{i,j+n} \otimes e_{i+n,j} + e_{j,i+n} \otimes e_{j+n,i} + e_{j,i+n} \otimes e_{j,i+n} + e_{i+n,j} \otimes e_{i,j+n} 
\right. 
\right. 
\left. - e_{ji} \otimes e_{ij} - e_{j+n,i+n} \otimes e_{j-i+n} - e_{j+i+n} \otimes e_{j+n,i+n} - e_{i+n,j+n} \otimes e_{ij} 
\right. 
\left. + \sum_{1 \leq k \leq n} \left( e_{k,n+k} \otimes e_{n+k,k} + e_{n+k,k} \otimes e_{k,n+k} - e_{kk} \otimes e_{kk} + e_{k+n,k+n} \otimes e_{k+n,k+n} \right) 
\right) 
\right]
\]
\[
= P + \chi.
\] (3.27)

We require the following lemma to prove the main result regarding the Poisson bracket for \( Sp(2n, \mathbb{R}) \) monodromy matrices.

Lemma 3.3. For \( A, B \in Sp(2n, \mathbb{R}) \), \( \chi \) being the defect matrix as in proposition \( \Box \) and \( P \) being the permutation matrix, we have the following identity

\[
\text{Tr}_{12}[(A \otimes B)\chi] = -\text{Tr}(AB^{-1}).
\] (3.28)
**Proof.** Given the $2n \times 2n$ symplectic matrix $B$, its inverse is given by the following sets of equations:

For the matrix entries with $1 \leq i < j \leq n$,

$$(B^{-1})_{ij} = B_{j+n,i+n}, \quad (B^{-1})_{ji} = B_{i+n,j+n}, \quad (B^{-1})_{i,j+n} = -B_{j,i+n}$$

$$(B^{-1})_{i,n+j} = -B_{i,j+n}, \quad (B^{-1})_{n+i,j} = -B_{j+n,i}, \quad (B^{-1})_{j+n,i} = -B_{n+i,j} \quad (3.29)$$

Whereas, for the matrix entries with $1 \leq k \leq n$, one obtains

$$(B^{-1})_{kk} = B_{k+n,k+n}, \quad (B^{-1})_{k,n+k} = -B_{k,n+k}$$

$$(B^{-1})_{n+k,k} = -B_{n+k,k}, \quad (B^{-1})_{k+n,k+n} = B_{kk} \quad (3.30)$$

Using the explicit expression of the defect matrix $\chi$ given in (3.21) and that of the symplectic matrix $B^{-1}$ in (3.29) and (3.30), one obtains

\[
\text{Tr}_{12}[(A \otimes B)\chi] = \sum_{1 \leq i < j \leq n} (A_{j+n,i} B_{j,i+n} + A_{i+n,j} B_{i,j+n} + A_{i,j+n} B_{i+n,j} + A_{j,i+n} B_{j+n,i})
\]

\[
+ \sum_{1 \leq k \leq n} (A_{n+k,k} B_{k+n,k} + A_{k,n+k} B_{n+k,k}) - \sum_{1 \leq i < j \leq n} (A_{ji} B_{j+n,i+n} + A_{i+n,j+n} B_{ij})
\]

\[
- \sum_{1 \leq j < n} (A_{ij} B_{i+n,j+n} + A_{j+n,i+n} B_{ji}) - \sum_{1 \leq k \leq n} (A_{kk} B_{k+n,k+n} + A_{k+n,k+n} B_{kk})
\]

\[
= - \sum_{1 \leq i < j \leq n} [A_{j+n,i} (B^{-1})_{i,j+n} + A_{i+n,j} (B^{-1})_{j,i+n} + A_{i,j+n} (B^{-1})_{j+n,i} + A_{j,i+n} (B^{-1})_{i+n,j}]
\]

\[
- \sum_{1 \leq k \leq n} [A_{n+k,k} (B^{-1})_{k,n+k} + A_{k,n+k} (B^{-1})_{n+k,k}] - \sum_{1 \leq i < j \leq n} [A_{ji} (B^{-1})_{ij} + A_{i+n,j+n} (B^{-1})_{j+n,i+n}]
\]

\[
- \sum_{1 \leq i < j \leq n} [A_{ij} (B^{-1})_{ji} + A_{j+n,i+n} (B^{-1})_{i+n,j+n}] - \sum_{1 \leq k \leq n} [A_{kk} (B^{-1})_{kk} + A_{k+n,k+n} (B^{-1})_{k+n,k+n}]
\]

\[
= - \text{Tr}(AB^{-1}). \quad (3.31)
\]

We now prove the main theorem concerning the Poisson bracket between traces of $Sp(2n, \mathbb{R})$ monodromy matrices.

**Theorem 6.** The Poisson bracket between traces of $Sp(2n, \mathbb{R})$ monodromy matrices $M_{\gamma_1}$ and $M_{\gamma_2}$ is given by

\[
\{\text{Tr} M_{\gamma_1}, \text{Tr} M_{\gamma_2}\} = \frac{1}{2} \left( \text{Tr} M_{\gamma_1 \circ \gamma_2} - \text{Tr} M_{\gamma_1 \circ \gamma_2^{-1}} \right), \quad (3.32)
\]
where $M_{\gamma_1 \circ \gamma_2}$ is an $Sp(2n, \mathbb{R})$ monodromy matrix computed along the loop $\gamma_1 \circ \gamma_2$ as shown in figure [4] while the monodromy matrix $M_{\gamma_1 \circ \gamma_2^{-1}}$ is computed along the other loop $\gamma_1 \circ \gamma_2^{-1}$ as given by figure [2].

Proof. Plugging the Casimir tensor $\Gamma$ (see (3.20)) back in (2.14) and using the identity from lemma 3.3, one obtains

$$\{\text{Tr } M_{\gamma_1}, \text{Tr } M_{\gamma_2}\}$$

$$= \frac{1}{2} \text{Tr}_{12}[(T(0, x_2)\tilde{M}_{\gamma_1}T(x_1, 0) \otimes T(0, y_2)\tilde{M}_{\gamma_2}T(y_1, 0))(P + \chi)]$$

$$= \frac{1}{2} \text{Tr } M_{\gamma_1 \circ \gamma_2} + \frac{1}{2} \text{Tr}_{12}[(T(0, x_2)\tilde{M}_{\gamma_1}T(x_1, 0) \otimes T(0, y_2)\tilde{M}_{\gamma_2}T(y_1, 0))\chi]$$

$$= \frac{1}{2} \text{Tr } M_{\gamma_1 \circ \gamma_2} - \frac{1}{2} \text{Tr}[T(0, x_2)\tilde{M}_{\gamma_1}T(x_1, 0)T(0, y_1)\tilde{M}_{\gamma_2}^{-1}T(y_2, 0)]$$

$$= \frac{1}{2} \left(\text{Tr } M_{\gamma_1 \circ \gamma_2} - \text{Tr } M_{\gamma_1 \circ \gamma_2^{-1}}\right). \quad (3.33)$$

Figure 2: (a) Two loops transversally intersecting at point $O$ of $\Sigma$, (b) Superposition $\gamma_1 \circ \gamma_2^{-1}$ obtained by an appropriate resolution of the intersection point $O$.

We now proceed to compute the Poisson bracket between traces of $SO(n)$ monodromy matrices. Let $e_{ij}$ denote an $n \times n$ matrix with 1 in $(i, j)$ entry and 0 elsewhere. There are $\frac{(n^2-n)}{2}$ basis elements for the corresponding Lie algebra $\mathfrak{so}(n)$ given by

$$t_a = e_{ij} - e_{ji} \quad \text{with} \quad 1 \leq i < j \leq n. \quad (3.34)$$
It can immediately be seen that the index $a$ runs from 1 to \( \frac{n(n-1)}{2} \). In this case, $f(a)$, appearing in (2.4) is $-1$ for all $a$. The Casimir tensor $\Gamma$ for the Lie algebra $\mathfrak{so}(n)$ now reads

\[
\begin{align*}
\Gamma &= \sum_{i<j} -[(e_{ij} - e_{ji}) \otimes (e_{ij} - e_{ji})] \\
&= -\sum_{i \neq j} (e_{ij} \otimes e_{ij}) + \sum_{i \neq j} (e_{ij} \otimes e_{ji}) \\
&= \sum_{i,j=1}^{n} (e_{ij} \otimes e_{ji}) - \sum_{i,j=1}^{n} (e_{ij} \otimes e_{ij}) \\
&= P + \chi, \quad (3.35)
\end{align*}
\]

where $P$ is the so-called Permutation matrix and $\chi$, which we refer to as the defect matrix for the Lie algebra $\mathfrak{so}(n)$, is given by

\[
\chi = -\sum_{i,j=1}^{n} (e_{ij} \otimes e_{ij}). \quad (3.36)
\]

We state the following lemma before deriving the Poisson bracket between traces of $SO(n)$ monodromy matrices.

**Lemma 3.4.** Let $A, B \in SO(n)$ and $\chi$ be as given in (3.36). Then the following holds

\[
\text{Tr}_{12}[(A \otimes B)\chi] = -\text{Tr}(AB^{-1}). \quad (3.37)
\]

**Proof.** For any two $SO(n)$ matrices $A$ and $B$, we note that

\[
\begin{align*}
\text{Tr}_{12}[(A \otimes B)\chi] &= -\text{Tr}_{12}[(A \otimes B)\left(\sum_{i,j=1}^{n} e_{ij} \otimes e_{ij}\right)] \\
&= -\text{Tr}_{12} \left( \sum_{i,j=1}^{n} Ae_{ij} \otimes Be_{ij} \right) \\
&= -\sum_{i,j=1}^{n} \text{Tr}_{12}(Ae_{ij} \otimes Be_{ij}) \\
&= -\sum_{i,j=1}^{n} \text{Tr}(Ae_{ij}) \text{Tr}(Be_{ij}) \\
&= -\sum_{i,j=1}^{n} A_{ji}B_{ji} \\
&= -\sum_{i,j=1}^{n} A_{ji}(B^T)_{ij} \\
&= -\sum_{i,j=1}^{n} A_{ji}(B^{-1})_{ij} \\
&= -\text{Tr}(AB^{-1}). \quad (3.38)
\end{align*}
\]
Using the expression (3.35) for the Casimir tensor $\Gamma$ of the Lie algebra $\mathfrak{so}(\mathfrak{n})$ in the general formula (2.14) and repeating the same computations as in the proof of theorem [6], one obtains the Poisson brackets between the traces of $SO(n)$ monodromy matrices.

We state this main result for the case of rotation group $SO(n)$ by means of the following theorem.

**Theorem 7.** The Poisson bracket between traces of $SO(n)$ monodromy matrices $M_{\gamma_1}$ and $M_{\gamma_2}$ is given by

$$\{\text{Tr} M_{\gamma_1}, \text{Tr} M_{\gamma_2}\} = \frac{1}{2} \left( \text{Tr} M_{\gamma_1 \circ \gamma_2} - \text{Tr} M_{\gamma_1 \circ \gamma_2^{-1}} \right),$$

where $M_{\gamma_1 \circ \gamma_2}$ and $M_{\gamma_1 \circ \gamma_2^{-1}}$ are $SO(n)$ monodromy matrices computed along the loops $\gamma_1 \circ \gamma_2$ and $\gamma_1 \circ \gamma_2^{-1}$, respectively. These loops are obtained by an appropriate resolution of intersection points and are illustrated by figures [7] and [2].

We note that (3.15), (3.19), (3.32) and (3.39) coincide with Goldman’s formulae in ([6], page 266). Here, we computed the Poisson bracket between traces of the monodromy matrices for a single point of transversal intersection. The proof for many intersection points follow similarly.

### 4 Poisson bracket between traces of $G_2$ monodromy matrices

In this section, we compute the Poisson bracket between traces of $G_2$ monodromy matrices which has not been considered in the literature so far. The exceptional real Lie group $G_2$ is 14-dimensional. Below is a list of the appropriately normalized (in view of (2.4)) 14 basis elements of the corresponding exceptional real simple Lie algebra $\mathfrak{g}_2$ as given in [4].

\[
\begin{align*}
C_1 &= \frac{1}{\sqrt{2}} (-e_{47} - e_{56} + e_{65} + e_{74}) \\
C_2 &= \frac{1}{\sqrt{2}} (e_{46} - e_{57} - e_{64} + e_{75}) \\
C_3 &= \frac{1}{\sqrt{2}} (-e_{45} + e_{54} - e_{67} + e_{76}) \\
C_4 &= \frac{1}{\sqrt{2}} (e_{27} + e_{36} - e_{63} - e_{72}) \\
C_5 &= \frac{1}{\sqrt{2}} (-e_{26} + e_{37} + e_{62} - e_{73}) \\
C_6 &= \frac{1}{\sqrt{2}} (e_{25} - e_{34} + e_{43} - e_{52}) \\
C_7 &= \frac{1}{\sqrt{2}} (-e_{24} - e_{35} + e_{42} + e_{53}) \\
C_8 &= \frac{1}{\sqrt{6}} (-2e_{23} + 2e_{32} + e_{45} - e_{54} - e_{67} + e_{76})
\end{align*}
\]
The Casimir tensor $\Gamma$ corresponding to the fundamental representation of the Lie algebra $\mathfrak{g}_2$ is provided by the following proposition.

**Proposition 8.** The Casimir tensor $\Gamma$ for the fundamental representation of $\mathfrak{g}_2$ reads

$$\Gamma = \sum_{i,j=1}^{7} e_{ij} \otimes e_{ij} - \sum_{i,j=1}^{7} e_{ij} \otimes e_{ij} + \frac{1}{3} \sum_{i=1}^{7} \mathcal{O}_i \otimes \mathcal{O}_i, \quad (4.2)$$

where the matrices $\{\mathcal{O}_i\}$ are given by

$$\begin{align*}
\mathcal{O}_1 &= e_{23} - e_{32} + e_{45} - e_{54} - e_{67} + e_{76} \\
\mathcal{O}_2 &= e_{31} - e_{13} + e_{46} - e_{64} + e_{57} - e_{75} \\
\mathcal{O}_3 &= e_{12} - e_{21} + e_{47} - e_{74} - e_{56} + e_{65} \\
\mathcal{O}_4 &= e_{51} + e_{62} + e_{73} - e_{15} - e_{26} - e_{37} \\
\mathcal{O}_5 &= e_{14} - e_{27} + e_{36} - e_{41} - e_{63} + e_{72} \\
\mathcal{O}_6 &= e_{17} + e_{24} - e_{35} - e_{42} + e_{53} - e_{71} \\
\mathcal{O}_7 &= e_{25} - e_{16} + e_{34} - e_{43} - e_{52} + e_{61}. \quad (4.3)
\end{align*}$$

The proof is given in the Appendix 7.

The skew-symmetric $7 \times 7$ matrices $\{\mathcal{O}_i\}$ are reminiscent of the imaginary units of the normed division algebra of octonions, the multiplication table of which can be constructed out of the following relations (see [9]):

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \quad (i,j,k = 1,\ldots,7), \quad (4.4)$$

where $\epsilon_{ijk}$ is totally antisymmetric and is unity for the following set of combinations:

$$\{123, 145, 176, 246, 257, 347, 365\}.$$

Note that $\{\mathcal{O}_i\}$ can be obtained from the $8 \times 8$ matrix representations of $\{e_i\}$ (see [9]) by deleting the first row and the first column of the respective matrices.
Use of proposition 8 in (2.14) yields the Poisson bracket between traces of $G_2$ monodromy matrices computed along homotopically inequivalent loops intersecting transversally at a point on $\Sigma$. The Poisson bracket between these gauge invariant observables is provided by the following theorem.

**Theorem 9.** The Poisson bracket between traces of $G_2$ monodromy matrices $M_{\gamma_1}$ and $M_{\gamma_2}$, computed along homotopically inequivalent and transversally intersecting loops $\gamma_1$ and $\gamma_2$, respectively, is given by

$$\{\text{Tr } M_{\gamma_1}, \text{Tr } M_{\gamma_2}\} = \frac{1}{2} \left[ \text{Tr } M_{\gamma_1 \circ \gamma_2} - \text{Tr } M_{\gamma_1 \circ \gamma_2^{-1}} + \frac{7}{3} \sum_{i=1}^{7} \text{Tr}(M_{\gamma_1 \bigcirc_i}) \text{Tr}(M_{\gamma_2 \bigcirc_i}) \right],$$

(4.5)

where $M_{\gamma_1 \circ \gamma_2}$ and $M_{\gamma_1 \circ \gamma_2^{-1}}$ are $G_2$ monodromy matrices computed along the loop $\gamma_1 \circ \gamma_2$ and $\gamma_1 \circ \gamma_2^{-1}$, respectively. These loops are illustrated in figures 7 and 8. The skew-symmetric matrices $\{\bigcirc_i\}$ are as given in (4.3).

**Proof.**

$$\{\text{Tr } M_{\gamma_1}, \text{Tr } M_{\gamma_2}\}$$

$$= \frac{1}{2} \text{Tr}_{12}[(T(0, x_2) \tilde{M}_{\gamma_1} T(x_1, 0) \otimes T(0, y_2) \tilde{M}_{\gamma_2} T(y_1, 0)) \Gamma]$$

$$= \frac{1}{2} \text{Tr}_{12} \left[ (T(0, x_2) \tilde{M}_{\gamma_1} T(x_1, 0) \otimes T(0, y_2) \tilde{M}_{\gamma_2} T(y_1, 0)) \times \left( \sum_{i,j=1}^{7} e_{ij} \otimes e_{ji} - \sum_{i,j=1}^{7} e_{ij} \otimes e_{ij} + \frac{7}{3} \sum_{i=1}^{7} \bigcirc_i \otimes \bigcirc_i \right) \right]$$

$$= \frac{1}{2} \text{Tr}_{12}[(T(0, x_2) \tilde{M}_{\gamma_1} T(x_1, 0) \otimes T(0, y_2) \tilde{M}_{\gamma_2} T(y_1, 0)) P]$$

$$- \frac{1}{2} \text{Tr}_{12} \left[ \sum_{i,j=1}^{7} (T(0, x_2) \tilde{M}_{\gamma_1} T(x_1, 0) e_{ij} \otimes T(0, y_2) \tilde{M}_{\gamma_2} T(y_1, 0) e_{ij}) \right]$$

$$+ \frac{1}{6} \text{Tr}_{12} \left[ \sum_{i=1}^{7} (T(0, x_2) \tilde{M}_{\gamma_1} T(x_1, 0) \bigcirc_i \otimes T(0, y_2) \tilde{M}_{\gamma_2} T(y_1, 0) \bigcirc_i) \right]$$

$$= \frac{1}{2} \text{Tr } M_{\gamma_1 \circ \gamma_2} - \frac{1}{2} \sum_{i,j=1}^{7} (T(0, x_2) \tilde{M}_{\gamma_1} T(x_1, 0))_{ji} (T(0, y_2) \tilde{M}_{\gamma_2} T(y_1, 0))_{ji}$$

$$+ \frac{1}{6} \sum_{i=1}^{7} \text{Tr}(M_{\gamma_1 \bigcirc_i}) \text{Tr}(M_{\gamma_2 \bigcirc_i})$$

$$= \frac{1}{2} \text{Tr } M_{\gamma_1 \circ \gamma_2} - \frac{1}{2} \sum_{i,j=1}^{7} (T(0, x_2) \tilde{M}_{\gamma_1} T(x_1, 0))_{ji} (T(0, y_1) \tilde{M}_{\gamma_2}^{-1} T(y_2, 0))_{ij}$$

$$+ \frac{1}{6} \sum_{i=1}^{7} \text{Tr}(M_{\gamma_1 \bigcirc_i}) \text{Tr}(M_{\gamma_2 \bigcirc_i})$$

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\[ 
\begin{align*}
&= \frac{1}{2} \text{Tr} M_{\gamma_1 \circ \gamma_2} - \frac{1}{2} \text{Tr} (T(0, x_2) \tilde{M}_{\gamma_1} T(x_1, 0) T(0, y_1) \tilde{M}_{\gamma_2}^{-1} T(y_2, 0)) \\
&\quad + \frac{1}{6} \sum_{i=1}^{7} \text{Tr}(M_{\gamma_1} \circ_1) \text{Tr}(M_{\gamma_2} \circ_2) \\
&= \frac{1}{2} \left[ \text{Tr} M_{\gamma_1 \circ \gamma_2} - \text{Tr} M_{\gamma_1 \circ \gamma_2}^{-1} \right] + \frac{1}{3} \sum_{i=1}^{7} \text{Tr}(M_{\gamma_1} \circ_1) \text{Tr}(M_{\gamma_2} \circ_2). \quad (4.6)
\end{align*}
\]

Remark 4.1. We remark here that the term \( \sum_{i=1}^{7} \text{Tr}(M_{\gamma_1} \circ_1) \text{Tr}(M_{\gamma_2} \circ_2) \) in (4.5) is \( G_2 \)-gauge invariant while none of the terms \( \{ \text{Tr}(M_{\gamma_1} \circ_1) \} \) is, as can easily be verified with the help of different 1-parametric subgroups of \( G_2 \).

5 Conclusion and outlook

In this paper, we have generalized the Goldman bracket to the case of \( G_2 \) gauge group. The expression for the Poisson bracket between traces of \( G_2 \)-valued monodromy matrices contains a gauge invariant term of new type which were not present in the cases of classical gauge groups obtained by Goldman in [6]. As a by-product, we present an alternative derivation of the well-known Goldman’s bracket for the following gauge groups: \( GL(n, \mathbb{R}) \), \( U(n) \), \( SL(n, \mathbb{R}) \), \( SU(n) \), \( Sp(2n, \mathbb{R}) \) and \( SO(n) \). In future, we plan to extend our formalisms to find Goldman-type brackets for the other exceptional gauge groups: \( F_4 \), \( E_6 \), \( E_7 \) and \( E_8 \).

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7 Appendix

This Appendix is devoted to the lengthy proof of proposition \( \S \) which enumerates the Casimir tensor of the Lie algebra \( g_2 \) for its fundamental representation.

Proof of proposition \( \S \)

Let us, first, concentrate on one of the commuting pairs \( \{C_1, C_9\} \) of the \( g_2 \) basis
elements (see (4.1)). We have,

\[- (C_1 \otimes C_1) = - \frac{1}{2} \left[ (-e_{47} - e_{56} + e_{65} + e_{74}) \otimes (-e_{47} - e_{56} + e_{65} + e_{74}) \right] \]

\[= -\frac{1}{2} \left( e_{47} \otimes e_{47} + e_{56} \otimes e_{56} + e_{65} \otimes e_{65} + e_{74} \otimes e_{74} \right) + \frac{1}{2} \left( e_{47} \otimes e_{74} + e_{56} \otimes e_{65} + e_{65} \otimes e_{56} + e_{74} \otimes e_{47} \right) - \frac{1}{2} \left( e_{47} \otimes e_{56} - e_{47} \otimes e_{65} + e_{56} \otimes e_{47} - e_{56} \otimes e_{74} \right) - e_{65} \otimes e_{47} + e_{65} \otimes e_{74} - e_{74} \otimes e_{56} + e_{74} \otimes e_{65}. \]  

(7.1)

We also have,

\[- (C_9 \otimes C_9) = \frac{1}{6} \left[ (-2e_{12} + 2e_{21} + e_{47} - e_{56} + e_{65} - e_{74}) \otimes (-2e_{12} + 2e_{21} + e_{47} - e_{56} + e_{65} - e_{74}) \right] \]

\[= -\frac{1}{6} \left( e_{12} \otimes e_{12} + e_{21} \otimes e_{21} + \frac{2}{3} (e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) \right) - \frac{1}{6} \left( e_{47} \otimes e_{74} + e_{56} \otimes e_{65} + e_{65} \otimes e_{56} + e_{74} \otimes e_{47} \right) - \frac{1}{3} \left( e_{12} \otimes e_{47} - e_{12} \otimes e_{56} + e_{12} \otimes e_{65} - e_{12} \otimes e_{74} - e_{21} \otimes e_{47} + e_{21} \otimes e_{56} - e_{21} \otimes e_{65} + e_{21} \otimes e_{74} + e_{47} \otimes e_{12} - e_{47} \otimes e_{21} + e_{56} \otimes e_{21} + e_{65} \otimes e_{12} - e_{65} \otimes e_{21} - e_{74} \otimes e_{12} + e_{74} \otimes e_{21} - e_{56} \otimes e_{12} \right) - \frac{1}{6} \left( e_{47} \otimes e_{56} - e_{47} \otimes e_{65} + e_{56} \otimes e_{47} - e_{56} \otimes e_{74} - e_{74} \otimes e_{56} + e_{74} \otimes e_{65} + e_{12} \otimes e_{47} + e_{12} \otimes e_{56} - e_{12} \otimes e_{65} + e_{12} \otimes e_{74} + e_{21} \otimes e_{47} - e_{21} \otimes e_{56} + e_{21} \otimes e_{65} - e_{21} \otimes e_{74} + e_{47} \otimes e_{12} - e_{47} \otimes e_{21} + e_{56} \otimes e_{21} + e_{65} \otimes e_{12} - e_{65} \otimes e_{21} - e_{74} \otimes e_{12} + e_{74} \otimes e_{21} - e_{56} \otimes e_{12} \right) - \frac{1}{6} \left( e_{47} \otimes e_{56} - e_{47} \otimes e_{65} + e_{56} \otimes e_{47} - e_{56} \otimes e_{74} - e_{74} \otimes e_{56} + e_{74} \otimes e_{65} + e_{12} \otimes e_{47} + e_{12} \otimes e_{56} - e_{12} \otimes e_{65} + e_{12} \otimes e_{74} + e_{21} \otimes e_{47} - e_{21} \otimes e_{56} + e_{21} \otimes e_{65} - e_{21} \otimes e_{74} + e_{47} \otimes e_{12} - e_{47} \otimes e_{21} + e_{56} \otimes e_{21} + e_{65} \otimes e_{12} - e_{65} \otimes e_{21} - e_{74} \otimes e_{12} + e_{74} \otimes e_{21} - e_{56} \otimes e_{12} \right) \]

(7.2)

Adding (7.1) to (7.2), one obtains the following

\[-(C_1 \otimes C_1) - (C_9 \otimes C_9) \]

\[= -\frac{2}{3} \left( e_{12} \otimes e_{12} + e_{21} \otimes e_{21} + e_{47} \otimes e_{47} + e_{56} \otimes e_{56} + e_{65} \otimes e_{65} + e_{74} \otimes e_{74} \right) + \frac{2}{3} \left( e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{47} \otimes e_{74} + e_{56} \otimes e_{65} + e_{65} \otimes e_{56} + e_{74} \otimes e_{47} \right) - \frac{1}{3} \left( e_{47} \otimes e_{56} - e_{47} \otimes e_{65} + e_{56} \otimes e_{47} - e_{56} \otimes e_{74} - e_{74} \otimes e_{56} + e_{74} \otimes e_{65} + e_{12} \otimes e_{47} + e_{12} \otimes e_{56} - e_{12} \otimes e_{65} + e_{12} \otimes e_{74} + e_{21} \otimes e_{47} - e_{21} \otimes e_{56} + e_{21} \otimes e_{65} - e_{21} \otimes e_{74} + e_{47} \otimes e_{12} - e_{47} \otimes e_{21} + e_{56} \otimes e_{21} + e_{65} \otimes e_{12} - e_{65} \otimes e_{21} - e_{74} \otimes e_{12} + e_{74} \otimes e_{21} - e_{56} \otimes e_{12} \right) - \frac{2}{3} \left( e_{12} \otimes e_{12} + e_{21} \otimes e_{21} + e_{47} \otimes e_{47} + e_{56} \otimes e_{56} + e_{65} \otimes e_{65} + e_{74} \otimes e_{74} \right) + \frac{2}{3} \left( e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{47} \otimes e_{74} + e_{56} \otimes e_{65} + e_{65} \otimes e_{56} + e_{74} \otimes e_{47} \right) \]

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\[-\frac{1}{3}[e_{47} \otimes (e_{56} - e_{65} + e_{21} - e_{12}) + e_{74} \otimes (e_{65} - e_{56} + e_{12} - e_{21}) + e_{56} \otimes (e_{47} - e_{74} + e_{12} - e_{21}) + e_{65} \otimes (e_{74} - e_{47} + e_{21} - e_{12}) + e_{12} \otimes (e_{56} - e_{47} - e_{65} + e_{74}) + e_{21} \otimes (e_{47} - e_{56} + e_{65} - e_{74})] \]

\[= -\frac{2}{3}(e_{12} \otimes e_{12} + e_{21} \otimes e_{21} + e_{47} \otimes e_{47} + e_{56} \otimes e_{56} + e_{65} \otimes e_{65} + e_{74} \otimes e_{74}) \]

\[+ \frac{1}{3}[(e_{12} - e_{21}) \otimes (e_{12} - e_{21} - e_{56} - e_{74} + e_{65} + e_{47}) + (e_{47} - e_{74}) \otimes (e_{47} - e_{74} - e_{56} - e_{21} + e_{65} + e_{12}) + (-e_{56} + e_{65}) \otimes (e_{65} - e_{56} + e_{47} + e_{12} - e_{74} - e_{21})] \]

\[-\frac{1}{3}(e_{12} - e_{21}) \otimes (e_{12} - e_{21}) - \frac{1}{3}(e_{47} - e_{74}) \otimes (e_{47} - e_{74}) \]

\[-\frac{1}{3}(e_{65} - e_{56}) \otimes (e_{65} - e_{56}) \]

\[= -\frac{2}{3}(e_{12} \otimes e_{12} + e_{21} \otimes e_{21} + e_{47} \otimes e_{47} + e_{56} \otimes e_{56} + e_{65} \otimes e_{65} + e_{74} \otimes e_{74}) \]

\[+ \frac{1}{3}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{47} \otimes e_{74} + e_{56} \otimes e_{65} + e_{65} \otimes e_{56} + e_{74} \otimes e_{47}) \]

\[+ \frac{1}{3}(e_{12} - e_{21} + e_{47} - e_{74} + e_{65} - e_{56}) \otimes (e_{12} - e_{21} + e_{47} - e_{74} + e_{65} - e_{56}) \]

\[-\frac{1}{3}(e_{12} \otimes e_{12} + e_{21} \otimes e_{21}) + \frac{1}{3}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) \]

\[-\frac{1}{3}(e_{47} \otimes e_{47} + e_{74} \otimes e_{74}) + \frac{1}{3}(e_{47} \otimes e_{47} + e_{74} \otimes e_{47}) \]

\[-\frac{1}{3}(e_{65} \otimes e_{65} + e_{56} \otimes e_{56}) + \frac{1}{3}(e_{65} \otimes e_{56} + e_{56} \otimes e_{65}) \]

\[= -(e_{12} \otimes e_{12} + e_{21} \otimes e_{21} + e_{47} \otimes e_{47} + e_{74} \otimes e_{74} + e_{56} \otimes e_{56} + e_{65} \otimes e_{65}) \]

\[+ (e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{47} \otimes e_{74} + e_{74} \otimes e_{47} + e_{56} \otimes e_{65} + e_{65} \otimes e_{56}) \]

\[+ \frac{1}{3}(e_{12} - e_{21} + e_{47} - e_{74} + e_{65} - e_{56}) \otimes (e_{12} - e_{21} + e_{47} - e_{74} + e_{65} - e_{56}). \]

(7.3)

Similarly, for the commuting pairs of the basis elements \{C_2, C_{10}\}, \{C_3, C_8\}, \{C_4, C_{11}\}, \{C_5, C_{12}\}, \{C_6, C_{13}\} and \{C_7, C_{14}\}, one obtains

\[-(C_2 \otimes C_2) - (C_{10} \otimes C_{10}) \]

\[= -(e_{13} \otimes e_{13} + e_{31} \otimes e_{31} + e_{46} \otimes e_{46} + e_{64} \otimes e_{64} + e_{57} \otimes e_{57} + e_{75} \otimes e_{75}) \]

\[+ (e_{13} \otimes e_{31} + e_{31} \otimes e_{13} + e_{46} \otimes e_{64} + e_{64} \otimes e_{46} + e_{57} \otimes e_{75} + e_{75} \otimes e_{57}) \]

\[+ \frac{1}{3}(e_{31} - e_{13} + e_{46} - e_{64} + e_{57} - e_{75}) \otimes (e_{31} - e_{13} + e_{46} - e_{64} + e_{57} - e_{75}). \]

(7.4)

\[-(C_3 \otimes C_3) - (C_8 \otimes C_8) \]

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\[-(C_4 \otimes C_4) - (C_{11} \otimes C_{11})
\quad = -(e_{14} \otimes e_{14} + e_{41} \otimes e_{41} + e_{27} \otimes e_{27} + e_{72} \otimes e_{72} + e_{36} \otimes e_{36} + e_{63} \otimes e_{63})
\quad + (e_{14} \otimes e_{41} + e_{41} \otimes e_{14} + e_{27} \otimes e_{72} + e_{72} \otimes e_{27} + e_{36} \otimes e_{63} + e_{63} \otimes e_{36})
\quad + \frac{1}{3} (e_{14} - e_{41} + e_{36} - e_{63} + e_{72} - e_{27}) \otimes (e_{14} - e_{41} + e_{36} - e_{63} + e_{72} - e_{27}).
\] (7.5)

\[-(C_5 \otimes C_5) - (C_{12} \otimes C_{12})
\quad = -(e_{15} \otimes e_{15} + e_{51} \otimes e_{51} + e_{26} \otimes e_{26} + e_{62} \otimes e_{62} + e_{37} \otimes e_{37} + e_{73} \otimes e_{73})
\quad + (e_{15} \otimes e_{51} + e_{51} \otimes e_{15} + e_{26} \otimes e_{62} + e_{62} \otimes e_{26} + e_{37} \otimes e_{73} + e_{73} \otimes e_{37})
\quad + \frac{1}{3} (e_{51} - e_{15} + e_{62} - e_{26} + e_{73} - e_{37}) \otimes (e_{51} - e_{15} + e_{62} - e_{26} + e_{73} - e_{37}).
\] (7.6)

\[-(C_6 \otimes C_6) - (C_{13} \otimes C_{13})
\quad = -(e_{16} \otimes e_{16} + e_{61} \otimes e_{61} + e_{25} \otimes e_{25} + e_{52} \otimes e_{52} + e_{34} \otimes e_{34} + e_{43} \otimes e_{43})
\quad + (e_{16} \otimes e_{61} + e_{61} \otimes e_{16} + e_{25} \otimes e_{52} + e_{52} \otimes e_{25} + e_{34} \otimes e_{43} + e_{43} \otimes e_{34})
\quad + \frac{1}{3} (e_{61} - e_{16} + e_{25} - e_{52} + e_{34} - e_{43}) \otimes (e_{61} - e_{16} + e_{25} - e_{52} + e_{34} - e_{43}).
\] (7.7)

\[-(C_7 \otimes C_7) - (C_{14} \otimes C_{14})
\quad = -(e_{17} \otimes e_{17} + e_{71} \otimes e_{71} + e_{24} \otimes e_{24} + e_{42} \otimes e_{42} + e_{35} \otimes e_{35} + e_{53} \otimes e_{53})
\quad + (e_{17} \otimes e_{71} + e_{71} \otimes e_{17} + e_{24} \otimes e_{42} + e_{42} \otimes e_{24} + e_{35} \otimes e_{53} + e_{53} \otimes e_{35})
\quad + \frac{1}{3} (e_{17} - e_{71} + e_{24} - e_{42} + e_{35} - e_{53}) \otimes (e_{17} - e_{71} + e_{24} - e_{42} + e_{35} - e_{53}).
\] (7.8)
Adding (7.3), (7.4), (7.5), (7.6), (7.7) and (7.8) to (7.9), one obtains

\[ \Gamma = \sum_{i=1}^{7} -(C_i \otimes C_i) \]

\[ = \sum_{i \neq j} e_{ij} \otimes e_{ji} - \sum_{i \neq j} e_{ij} \otimes e_{ij} + \frac{1}{3} \sum_{i=1}^{7} \mathbb{O}_i \otimes \mathbb{O}_i \]

\[ = \left( \sum_{i \neq j} e_{ij} \otimes e_{ji} + \sum_{i=1}^{7} e_{ii} \otimes e_{ii} \right) - \left( \sum_{i \neq j} e_{ij} \otimes e_{ij} + \sum_{i=1}^{7} e_{ii} \otimes e_{ii} \right) + \frac{1}{3} \sum_{i=1}^{7} \mathbb{O}_i \otimes \mathbb{O}_i \]

\[ = \sum_{i,j=1}^{7} e_{ij} \otimes e_{ji} - \sum_{i,j=1}^{7} e_{ij} \otimes e_{ij} + \frac{1}{3} \sum_{i=1}^{7} \mathbb{O}_i \otimes \mathbb{O}_i. \quad (7.10) \]

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