On the Generation of Long Binary Sequences with Record-Breaking PSL Values

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Abstract—Binary sequences are widely used in various practical fields, such as telecommunications, radar technology, navigation, cryptography, measurement sciences, biology or industry. In this paper, a method to generate long binary sequences (LBS) with low peak sidelobe level (PSL) value is proposed. Having an LBS with length \(n\), both the time and memory complexities of the proposed algorithm are \(O(n)\). During our experiments, we repeatedly reach better PSL values than the currently known state of art constructions, such as Legendre sequences, with or without rotations, Rudin-Shapiro sequences or m-sequences, with or without rotations, by always reaching a record-breaking PSL values strictly less than \(\sqrt{n}\). Furthermore, the efficiency and simplicity of the proposed method are particularly beneficial to the lightweightness of the implementation, which allowed us to reach record-breaking PSL values for less than a second.

Index Terms—Autocorrelation, Binary Sequences, Peak Sidelobe Level (PSL)

I. INTRODUCTION

THE practical fields in which binary sequences with low PSL values could be exploitable are manifold [1]. Despite the widely usage of binary sequences in radar and sonar pulse compression systems [2], they are further used, for example, in the estimation of the shape of hemodynamic responses [3], analysis of visual neurons [4], audio watermarking [5], orthogonal frequency division multiplexing [6], CDMA systems [7], scrambling algorithms [8], motion tracking technologies [9] and many others.

Of special interest are those binary sequences possessing low PSL value. Some strategies for construction of such sequences comprise the Barker codes [10], Rudin-Shapiro sequences [11][12], maximal length shift register sequences, or m-sequences [13], Gold codes [14], Kasami codes [15], Weil sequences [16], Legendre sets [17] and others (see [18]).

M-sequences, Gold codes and Kasami sequences have ideal periodic autocorrelation functions but have no constraints on the sidelobes of their aperiodic autocorrelation functions, i.e. their PSL value is not pre-determined. Same is true for Legendre sets and Rudin-Shapiro sequences. Furthermore, it is difficult to calculate the growth of the PSL of the aforementioned families of binary sequences. It is conjectured that the PSL values of m-sequences grow like \(O(\sqrt{n})\), making them one of the best methods to straightforwardly construct binary sequences with near-optimal PSL value. However, as stated in [19]:

The claim that the PSL of m-sequences grows like \(O(\sqrt{n})\), which appears frequently in the radar literature, is concluded to be unproven and not currently supported by data.

As summarized in [20], during the years a variety of analytical constructions and computer search methods are developed in order to construct binary sequences with relatively small or minimal PSL. By an exhaustive search the minimum values of the PSL for \(n \leq 10^{5}\) are published in [29], and for some discrete values of \(n \geq 106\) in [30][31][32][33].

It appears that the current state of art computer search methods, like CAN [34], ITROX [35], MWSIL-Diag, MM-PSL [36] or DPM [37], could yield better, or at least not worse PSL values, than the algebraic constructions. However, when the length of the generated by a given heuristic algorithm binary sequences rises, so is the overall time and memory complexity of the routine. As concluded in [32]:

As an indication of the runtime complexity of our EA the computing time is 58009 s or 16.1136 h for \(L=1019\). For lengths up to 4096, the computing time required empirically shows a seemingly quadratic growth with \(L\).

Thus, the main motivation of this work is to create an efficient and lightweight algorithm, in terms of time and memory complexity, to address the heuristic generation of very long binary sequences with near-optimal PSL values.

II. PRELIMINARIES

Let \(B = (b_0, b_1, \cdots, b_{n-1})\) be a binary sequence of length \(n > 1\), where \(b_i \in \{-1, 1\}, 0 \leq i \leq n - 1\). The aperiodic autocorrelation function of \(B\), or AACF, is given by

\[
C_u(B) = \sum_{j=0}^{n-u-1} b_jb_{j+u}, \quad \text{for } u \in \{0, 1, \cdots, n-1\}.
\]

\(C_0(B)\) is called mainlobe and the rest \(C_u(B)\) for \(u \in \{1, \cdots, n-1\}\) are called sidelobe levels. We define the peak sidelobe level of \(B\), or PSL [38] as

\[
B_{PSL} = \max_{0 < u < n} |C_u(B)|.
\]

\(\dagger\)EA stands for Evolutionary Algorithm
III. Some observations about the PSL calculation

Let us denote \( C_{n-i-1}(B) \) by \( \hat{C}_i(B) \). Since this is just a rearrangement of the sidelobes of \( B \), it follows that:

\[
B_{PSL} = \max_{0 \leq u < n} |C_u(B)| = \max_{0 \leq u < n-1} |\hat{C}_u(B)|.
\]

We will graphically represent the calculation of values of \( \hat{C}_i(B) \) for a binary sequence of length 8 in Figure 1. The x-axis indexes represent the elements of \( B = (b_0, b_1, \cdots, b_7) \), while the y axes represents the elements of \( B \) in reverse order, i.e. \( (b_7, b_6, \cdots, b_0) \). Each cell of the graphics corresponds to the product of the provided by the \( x \) and \( y \)-axis values. To calculate \( \hat{C}_i(B) = \sum_{j=0}^{i} b_j b_{j+n-i-1} \) for some \( i \) (0 \leq i \leq 7), we start from the cell with coordinates \( (b_i, b_j) \). Then, by decreasing both indexes of the current cell by 1 we jump to the next cell \( (b_{i-1}, b_j) \) which will be added to the sum. We continue this process until we reach the cell \( (b_0, b_{7-i}) \).

![Fig. 1. A visual interpretation of the sidelobe calculation process, for a binary sequence with length 8](image)

As the value of the mainlobe \( \hat{C}_7(B) \) is always 8, we can exclude it from the PSL calculation. Having this in mind, we can define the PSL of the binary sequence \( B \) as the diagonal in Figure 1 with the highest absolute sum of its elements compared to all other diagonals, excluding the main one.

Let us denote by \( \bar{b}_i \) the flipped bit \( b_i \), i.e. \( \bar{b}_i = -b_i \) and by \( \hat{C}_i(B_j) \) the sidelobe of the binary sequence \( B_j \), obtained from \( B \) by flipping the bit on position \( j \).

We can further exploit the relations between the value of the sidelobe \( \hat{C}_i(\Psi) \) of a given binary sequence \( \Psi \) with length \( n \), and the value of the sidelobe \( \hat{C}_i(\Psi_f) \), s.t. the binary sequence \( \Psi_f \) is equal to the binary sequence \( \Psi \) with the bit on position \( f \) flipped. We denote as \( \Omega \) the array of all the consequent sidelobes of \( \Psi \), i.e.:

\[
\Omega = [\hat{C}_0(\Psi), \hat{C}_1(\Psi), \cdots, \hat{C}_{n-2}(\Psi)]
\]

We denote as \( \Omega_f \) the array of all the consequent sidelobes of \( \Psi_f \), i.e.:

\[
\Omega_f = [\hat{C}_0(\Psi_f), \hat{C}_1(\Psi_f), \cdots, \hat{C}_{n-2}(\Psi_f)]
\]

For convenience, we further denote the \( i \)-th element of a given array \( A \) as \( A[i] \). For example, \( \Omega_f[3] = \hat{C}_2(\Psi) \).

The calculation of \( \Omega_f \), corresponding to some random binary sequence \( \Psi \), is not linear. The time complexity of the trivial computational approach is \( O(n^2) \) (two nested for cycles). However, as shown in Wiener–Khinchin–Einstein theorem [39], the autocorrelation function of a wide-sense-stationary random process has a spectral decomposition given by the power spectrum of that process, we can use one regular and one inverse Fast Fourier Transform (FFT), to achieve a faster way of calculating \( \Omega_f \). Despite its time complexity of \( O(n \log n) \), its memory complexity is significantly higher than the trivial computational approach.

By exploiting the observations made in this section, we present an algorithm which can calculate the array \( \Omega_f \), if we hold the array \( \Omega_f \) in memory, with time and memory complexity of \( O(n) \). The pseudo-code of the algorithm is given in Algorithm 1. The following notations are used:

- \( \min(x,y) \) : returns \( x \), if \( x \leq y \); otherwise, returns \( y \).
- \( \max(x,y) \) : returns \( x \), if \( x \geq y \); otherwise, returns \( y \).
- \( x = y \) : same as \( x = x - y \)
- \( x *= y \) : same as \( x = x * y \)

**Algorithm 1 In-memory flip**

1: procedure FLIP(\( f, \Psi, \Omega_f, n \))
2: \( \delta_{min} \leftarrow \min(n-f-1, f) \)
3: \( \delta_{max} \leftarrow \max(n-f, f) \)
4: if \( f \leq \frac{n-1}{2} \) then
5: for \( q \in [0, \delta_{max} - \delta_{min} - 1] \) do
6: \( \Omega_f[\delta_{min} + q] = 2 \Psi[f] \Psi[n-q-1] \)
7: end for
8: else
9: for \( q \in [0, \delta_{max} - \delta_{min}] \) do
10: \( \Omega_f[\delta_{min} + q] = 2 \Psi[f] \Psi[q] \)
11: end for
12: end if
13: if \( f \leq \frac{n-1}{2} \) then
14: for \( q \in [0, n - \delta_{max}] \) do
15: \( \Omega_f[\delta_{max} + q - 1] = 2 \Psi[f] (\Psi[2f - q] + \Psi[q]) \)
16: end for
17: else
18: for \( q \in [0, n - \delta_{max} - 1] \) do
19: \( \Omega_f[\delta_{max} + q] = 2 \Psi[f] (\Psi[\delta_{max} - \delta_{min} + q] + \Psi[n-q-1]) \)
20: end for
21: end if
22: end if
23: \( \Psi[f] *= -1 \)
24: end procedure

The procedure introduced in Algorithm 1 performs an in-place memory update of \( \Omega_f \), when a single bit on position \( f \) of \( \Psi \) is flipped. Therefore, when the procedure ends, both \( \Psi \) and \( \Omega_f \) are transformed to \( \Psi_f \) and \( \Omega_f \). We will note that the procedure is reversible, i.e. if an in-place memory update of \( \Omega_f \) is made, when a single bit on position \( f \) of \( \Psi_f \) is flipped, both \( \Psi_f \) and \( \Omega_f \) are transformed back to \( \Psi \) and \( \Omega_f \).

IV. Algorithm for finding very long binary sequences with low PSL values

The basic ingredients of some heuristic algorithm could be summarized as:

- \( A \): Metaheuristic algorithm, like hill climbing, simulated annealing, tabu search, etc.
- \( I \): search operator, which is used to generate the candidates
- \( F \): fitness function, which is used to compare the candidates

In our previous work \cite{33}, we have used shotgun hill climbing as \( A \), a neighborhood search as \( I \), and the following fitness function as \( F \):

\[
F(B) = \sum_{u=1}^{n-1} |C_u(B)|^4 = \sum_{u=1}^{n-1} \left( \sum_{j=0}^{n-u-1} b_j b_{j+u} \right)^4 ,
\]

where \( B \) is a binary sequence with length \( n \). However, using shotgun hill climbing metaheuristic algorithm for finding very long binary sequences with low PSL is not time efficient because the number of hops required to reach some local optimum, grows exponentially when the length of the binary sequence increases.

Using a neighborhood search to consequently pick the best candidate among all neighbors could be beneficial in finding LBS with low PSL. However, in the aspect of very long binary sequences this search strategy is extremely slow. For example, in the case of a binary sequence with length \( 2^{16} \), and \( I \) equivalent to a single flip, in each optimization step we need to fitness all the \( 2^{16} \) neighbors of the current state \( S \) and to pick the one with the best score yielded by \( F \). This observation is still true, even if all the neighbors of \( S \) have better scores.

To overcome the disadvantages mentioned above, we choose the following strategy:

- \( A \): stochastic hill climbing metaheuristic algorithm. We visit a random neighbor of the current state \( S \) and accept it, if it is a better candidate than \( S \). Otherwise, we pick another neighbor of \( S \) and repeat the process.
- \( I \): we choose a single flip as the search operator, so we can exploit memory and time efficiency of Algorithm \cite{1}.
- \( F \): since \( C(B) \)'s are rearrangements of the sidetolobes of \( B \), we can use the same fitness function \( F(B) \) as in \cite{33}, i.e.:

\[
F(B) = \sum_{u=0}^{n-2} |\hat{C}_u(B)|^4 = \sum_{u=0}^{n-2} \hat{C}_u(B)^4
\]

We need to further address the strategy described in \( A \) of picking the next candidate, or neighbor, of \( S \). Let us consider an approach of consistently probing \( x \) pseudo-randomly chosen neighbors. In case a better candidate is found, we accept it; otherwise, we try again, until we have accumulated a total number of \( t \) consequent fails. Then, we announce that we have reached a local optimum. This model can be described by the Bernoulli distribution. The probability to achieve exactly \( r \) successes in \( N \) trials is equal to:

\[
P(X = r) = \binom{N}{r} p^r q^{N-r} ,
\]

where \( p \) and \( q \) are the probabilities of success and failure respectively, i.e. \( q = 1 - p \). We can easily calculate \( P(X = 0) \):

\[
P(X = 0) = \binom{N}{0} p^0 q^{N-0} = q^N = (1 - p)^N
\]

We further calculate \( P(X \geq 1) \):

\[
P(X \geq 1) = 1 - P(X = 0) = 1 - (1 - p)^N
\]

Thus, relaying solely on pseudorandom choices of neighbors is not efficient and there is always a chance to miss the better candidate. We can increase the probability of finding the eventual better candidate, but that significantly overhead the optimization process. Missing a better candidate is undesirable behavior of the optimization process, specially when we are dealing with very long binary sequences.

The number of neighbors of a binary sequence \( B \) with length \( n \) is \( n \). Let us denote those neighbors as \( i_1, i_2, \ldots, i_n \), where the \( j \)-th neighbor \( i_j \) is equal to \( B \) with flipped bit on position \( j \). We suggest the following simple search strategy:

1) we pick a pseudorandomly generated neighbor \( i_r \)
2) we consequently try, for all \( x \in [1, n - 1] \), the neighbors \( i_r \mod n \).

We want to emphasize on the extreme situation when the local optimum is already reached, i.e. \( k = 0 \). The suggested search strategy will detect that in exactly \( n \) steps, which is an optimal scenario. Furthermore and more importantly, we never miss a better candidate, if any, and we keep the nondeterministic nature of the search routine at the same time.

We suggest Algorithm \cite{2} for finding very long binary sequences with low PSL which is based on the above described \((A, I, F)\). The following notations and functions are used in the pseudo-code:

- \( \hat{\Psi} \): a random (initial) binary sequence.
- \( x, y \leftarrow a, b \): is equivalent to the statements \( x = a \) and \( y = b \).
- \( R(n) \): a function, which generates a pseudo-random integer number \( \in [0, n) \).
- \( Q(x, B, \Omega_B) \): a function, which makes \( x \) flips at random bit positions in \( B \). We pass \( \Omega_B \) as argument, so we can use the in-place memory function \( \text{Flip} \). We apply this function to escape the local minimum, when we are stuck in such.
- \( \text{beacon} \): we further implant a beacon in the cost function \( F \), so we can simultaneously calculate the PSL of the given binary sequence. Such approach adds a negligible overhead, if any, to the cost function routine.

V. ON THE COMPLEXITY OF ALGORITHM \cite{2}

We emphasize that the complexity of Algorithm \cite{2} mainly depends on the complexity of Algorithm \cite{1} because in each iteration during the optimization process, Algorithm \cite{1} is called twice, in case the new candidate is worse than the current one, and once, if the new candidate is better. The in-memory flip function applied in Algorithm \cite{1} passes only once through \( \Omega_B \) array, without creating any memory overloads, to reach time and memory complexities of \( O(n) \). The same observation is true for the simple cost function \( F \) - it passes only once through \( \Omega_B \) to sum all quadrupled values of its elements. The function \( Q \) is a random number of calls of \( F \) (between 1 and 4). The remaining part of Algorithm \cite{2} consists of a simple automata, which rules the continuous optimization process.
Algorithm 2 An algorithm for binary sequences PSL optimization

1: BestCost, Cost ← $F(\Omega_{\Psi})$, 0
2: isGImpr, isLImpr ← True, False
3: while true do
4:   if isGImpr then
5:     r ← R(n)
6:   for i ∈ [0, n) do
7:     $\text{Flip}((r + i) \% n, \Psi, \Omega_{\Psi})$
8:     Cost ← $F(\Omega_{\Psi})$  \> * the beacon is here *
9:   if BestCost > Cost then
10:      BestCost ← Cost
11:      isLImpr ← True
12:      break
13:   else
14:      $\text{Flip}((r + i) \% n, \Psi, \Omega_{\Psi})$
15:   end if
16: end for
17: if isLImpr then
18:     isGImpr, isLImpr ← True, False
19:     continue
20: else
21:     isGImpr ← False
22: end if
23: else
24:   r ← R(4)
25:   Q(1+r, \Psi, \Omega_{\Psi})
26:   isGImpr, isLImpr ← True, False
27: end if
28: end while

Therefore, both time and memory complexities of Algorithm 2 are $O(n)$.

VI. RESULTS

We have implemented Algorithm 2 by using the C language and a mid-range computer station. Given the linear time and memory complexity of the algorithm, we were able to repeatedly generate binary sequences with record-breaking PSL values for less than a second. As stated in [32], the time required to reach a PSL value 26, for a binary sequence with length 1019, is 58009 seconds or 16.1136 hours. For comparison, by using Algorithm 2 we reach this value for less than a second.

We present the results achieved by Algorithm 2 for binary sequences with lengths $x^2$ for $x \in [18,44]$, compared with the currently known state of art algorithms found in the literature, like CAN [34], ITROX [35], MWISL-Diag, MM-PSL [36], DPM [37], 1bCAN [40]. We will refer to this collection of algorithms as collection A. We want to emphasize, that the differences between the proposed algorithm with algorithms from collection A are manifold. For example, we do not use converging functions, mini regular or quadratic optimization problems, floating-based arithmetic. Furthermore, the provided algorithm does not suffer from an unique navigation trace through the sequence search space. The experiments were based on 12 instances of each algorithm (each ran to a distinct thread of the processor). Furthermore, the lifetime of our algorithm is restricted to 1 minute. As shown in Figure 2 we significantly outperform the best results achieved by state of art algorithms. In fact, for some of the lengths, less than a second was needed to reach a record-breaking PSL.

![Fig. 2: Comparison to other state of the art algorithms known in literature](image)

In contrast to some other state of the art algorithms, the computing complexity of the algorithm presented in this work does not grow quadratically. Maybe this is the reason for the lack of published results for binary sequences of lengths greater than $2^{12}$. Nevertheless, the results with which we can further compare are $m$-sequences. However, such sequences exists only for lengths $2^n - 1$, $n \geq 1$, $n \in N$.

In Table I we present the best PSL values of binary $m$-sequences with length $n$ (with or without rotation), yielded by some primitive polynomial of degree $n$ over $GF(2)$ from [41] denoted by $M_n^p$ and the binary sequences generated by Algorithm 2 denoted by $A_n$ for lengths $2^n - 1$ and $13 \leq n \leq 17$. As it can be seen from Table I our results significantly outperform the best results achieved by $m$-sequences.

VII. CONCLUSIONS

In this paper we present an efficient heuristic algorithm for finding very long binary sequences with record-breaking PSL values. Since the time and memory complexities of the suggested algorithm are both $O(n)$, we were able to construct binary sequences with record-breaking PSL values for less than a second.

| $n$ | $2^n - 1$ | $M_n^p$ | $A_n$ |
|-----|----------|--------|------|
| 13  | 8191     | 85     | 77   |
| 14  | 16383    | 125    | 115  |
| 15  | 32767    | 175    | 171  |
| 16  | 65535    | 258    | 254  |
| 17  | 131071   | 363    | 360  |
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