Harmonic Analysis

The John–Nirenberg inequality with sharp constants

Meilleures constantes dans l’inégalité de John–Nirenberg

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A B S T R A C T
We consider the one-dimensional John–Nirenberg inequality:

$$\left| \left\{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \right\} \right| \leq C_1 |I_0| \exp \left( -\frac{C_2}{\|f\|_*} \alpha \right).$$

A. Korenovskii found that the sharp $C_2$ here is $C_2 = 2/e$. It is shown in this paper that if $C_2 = 2/e$, then the best possible $C_1$ is $C_1 = \frac{1}{2} e^{4/e}$.

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1. Introduction

Let $I_0 \subset \mathbb{R}$ be an interval and let $f$ be an integrable function on $I_0$. Given a measurable set $E \subset \mathbb{R}$, denote by $|E|$ its Lebesgue measure. Given a subinterval $I \subset I_0$, set $f_I = \frac{1}{|I|} \int_I f$ and

$$\Omega(f; I) = \frac{1}{|I|} \int_I |f(x) - f_I| \, dx.$$

We say that $f \in BMO(I_0)$ if $\|f\|_* = \sup_{I \subset I_0} \Omega(f; I) < \infty$. The classical John–Nirenberg inequality [1] says that there are $C_1, C_2 > 0$ such that for any $f \in BMO(I_0)$,

$$\left| \left\{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \right\} \right| \leq C_1 |I_0| \exp \left( -\frac{C_2}{\|f\|_*} \alpha \right) \quad (\alpha > 0).$$

A. Korenovskii [4] (see also [5, p. 77]) found the best possible constant $C_2$ in this inequality, namely, he showed that $C_2 = 2/e$:

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\[
\left| \{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \} \right| \leq C_1 |I_0| \exp\left( -\frac{2/e}{\|f\|_*} \alpha \right) \quad (\alpha > 0),
\]

(1.1)

and in general the constant \(2/e\) here cannot be increased.

A question about the sharp \(C_1\) in (1.1) remained open. In [4], (1.1) was proved with \(C_1 = e^{1+2/e} = 5.67323 \ldots\). The method of the proof in [4] was based on the Riesz sunrise lemma and on the use of non-increasing rearrangements. In this paper, we give a different proof of (1.1), yielding the sharp constant \(C_1 = 2/e = 2.17792 \ldots\).

**Theorem 1.1.** Inequality (1.1) holds with \(C_1 = 2/e\), and this constant is the best possible.

We also use as the main tool the Riesz sunrise lemma. But instead of the rearrangement inequalities, we obtain a direct pointwise estimate for any \(BMO\)-function (see Theorem 2.2 below). The proof of this result is inspired (and close in spirit) by a recent decomposition of an arbitrary measurable function in terms of mean oscillations (see [2,6]).

We mention several recent papers [7,8] where sharp constants in some different John–Nirenberg-type estimates were found by means of the Bellman function method.

**2. Proof of Theorem 1.1**

We shall use the following version of the Riesz sunrise lemma [3].

**Lemma 2.1.** Let \(g\) be an integrable function on some interval \(I_0 \subset \mathbb{R}\), and suppose \(g_{I_0} \leq \alpha\). Then there is at most countable family of pairwise disjoint subintervals \(I_j \subset I_0\) such that \(g_{I_j} = \alpha\), and \(g(x) \leq \alpha\) for almost all \(x \in I_0 \setminus (\bigcup_j I_j)\).

Observe that the family \(\{I_j\}\) in Lemma 2.1 may be empty if \(g(x) < \alpha\) a.e. on \(I_0\).

**Theorem 2.2.** Let \(f \in BMO(I_0)\), and let \(0 < \gamma < 1\). Then there is at most countable decreasing sequence of measurable sets \(G_k \subset I_0\) such that \(|G_k| \leq \min(2\gamma^k, 1)|I_0|\) and for a.e. \(x \in I_0\),

\[
|f(x) - f_{I_0}| \leq \frac{\|f\|_*}{2\gamma} \sum_{k=0}^\infty x_{G_k}(x).
\]

(2.1)

**Proof.** Given an interval \(I \subset I_0\), set \(E(I) = \{x \in I : f(x) > f_I\}\). Let us show that there is at most a countable family of pairwise disjoint subintervals \(I_j \subset I_0\) such that \(\sum_j |I_j| \leq \gamma |I_0|\) and for a.e. \(x \in I_0\),

\[
(f - f_{I_0})x_{E(I_0)} \leq \frac{\|f\|_*}{2\gamma} x_{E(I_0)} + \sum_j (f - f_{I_j})x_{E(I_j)}.
\]

(2.2)

We apply Lemma 2.1 with \(g = f - f_{I_0}\) and \(\alpha = \frac{|I_0|}{2\gamma}\). One can assume that \(\alpha > 0\) and the family of intervals \(\{I_j\}\) from Lemma 2.1 is non-empty (since otherwise (2.2) holds trivially only with the first term on the right-hand side). Since \(g_{I_j} = \alpha\), we obtain:

\[
\sum_j |I_j| = \frac{1}{\alpha} \int_{\bigcup_j I_j} (f - f_{I_0}) \, dx \leq \frac{1}{\alpha} \int_{\{x \in I_0 : f(x) > f_{I_0}\}} (f - f_{I_0}) \, dx
\]

\[
= \frac{1}{2\alpha} \gamma (f; I_0)|I_0| \leq \gamma |I_0|.
\]

Since \(g_{I_j} = \alpha\), we have \(f_{I_j} = f_{I_0} + \alpha\), and hence:

\[
f - f_{I_0} = (f - f_{I_0})x_{I_0 \setminus \bigcup_j I_j} + \alpha x_{\bigcup_j I_j} + \sum_j (f - f_{I_j})x_{I_j}.
\]

This proves (2.2) since \(f - f_{I_0} \leq \alpha\) a.e. on \(I_0 \setminus \bigcup_j I_j\).

The sum on the right-hand side of (2.2) consists of the terms of the same form as the left-hand side. Therefore, one can proceed iterating (2.2). Denote \(I_j^k = I_{j^k}\), and let \(I_j^k\) be the intervals obtained after the \(k\)-th step of the process. Iterating (2.2) \(m\) times yields:

\[
(f - f_{I_0})x_{E(I_0)} \leq \frac{\|f\|_*}{2\gamma} \sum_{k=0}^{m} \sum_j x_{E(I_j^k)}(x) + \sum_i (f - f_{I_{j^i}^{m+1}})x_{E(I_{j^i}^{m+1})}.
\]
(where \( I_j^0 = I_0 \)). If there is \( m \) such that for any \( i \) each term of the second sum is bounded trivially by \( \frac{\|f\|_*}{2^i} \chi_{E(I_j^m)} \), we stop the process, and we would obtain the finite sum with respect to \( k \). Otherwise, let \( m \to \infty \). Using that

\[
\left| \bigcup_i I_j^{m+1} \right| \leq \gamma \left| \bigcup_i I_j^m \right| \leq \cdots \leq \gamma^{m+1} |I_0|,
\]

we get that the support of the second term will tend to a null set. Hence, setting \( E_k = \bigcup_j E(I_j^k) \), for a.e. \( x \in E(I_0) \) we obtain:

\[
(f - f_{I_0}) \chi_{E(I_0)} \leq \frac{\|f\|_*}{2^i} \left( \chi_{E(I_0)}(x) + \sum_{k=1}^\infty \chi_{E_k}(x) \right).
\]

Observe that \( E(I_j) = \{ x \in I_j : f(x) > f_{I_0} + \alpha \} \subset E(I_0). \) From this and from the above process we easily get that \( E_{k+1} \subset E_k. \) Also, \( E_k \subset \bigcup_j I_j^k, \) and hence \( |E_k| \leq \gamma^k |I_0| \).

Setting now \( F(I) = \{ x \in I : f(x) \leq f_I \} \), and applying the same argument to \( (f_{I_0} - f) \chi_{F(I)} \), we obtain:

\[
(f_{I_0} - f) \chi_{F(I_0)} \leq \frac{\|f\|_*}{2^i} \left( \chi_{F(I_0)}(x) + \sum_{k=1}^\infty \chi_{F_k}(x) \right),
\]

where \( F_{k+1} \subset F_k \) and \( |F_k| \leq \gamma^k |I_0| \). Also, \( F_k \cap E_k = \emptyset. \) Therefore, summing \((2.3)\) and \((2.4)\) and setting \( G_0 = I_0 \) and \( G_k = E_k \cup F_k, k \geq 1 \), we get \((2.1)\). \( \square \)

**Proof of Theorem 1.1.** Let us show first that the best possible \( C_1 \) in \((1.1)\) satisfies \( C_1 \geq \frac{1}{4} e^{4/\varepsilon}. \) It suffices to give an example of \( f \) on \( I_0 \) such that for any \( \varepsilon > 0, \)

\[
\left| \{ x \in I_0 : |f(x) - f_{I_0}| > 2(1 - \varepsilon)\|f\|_* \} \right| = |I_0|/2.
\]

Let \( I_0 = [0, 1] \) and take \( f = \chi_{[0,1/4]} - \chi_{[3/4, 1]} \). Then \( f_{I_0} = 0. \) Hence, \((2.5)\) would follow from \( \|f\|_* = 1/2. \) To show the latter fact, take an arbitrary \( I \subset I_0. \) It is easy to see that computations reduce to the following cases: \( I \) contains only \( 1/4 \) and \( I \) contains both \( 1/4 \) and \( 3/4. \)

Assume that \( I = (a, b), 1/4 \in I \) and \( b < 3/4 \). Let \( \alpha = \frac{1}{4} - a \) and \( \beta = b - \frac{1}{4} \). Then \( f_I = \alpha / (\alpha + \beta) \) and:

\[
\Omega(f; I) = \frac{2}{\alpha + \beta} \int_{\{x \in I : f > f_I\}} (f - f_I) = \frac{2\alpha \beta}{(\alpha + \beta)^2} \leq 1/2
\]

with \( \Omega(f; I) = 1/2 \) if \( \alpha = \beta. \)

Consider the second case. Let \( I = (a, b), a < 1/4 \) and \( b > 3/4. \) Let \( \alpha \) be as above and \( \beta = b - \frac{3}{4} \). Then:

\[
\Omega(f; I) = \frac{2}{\alpha + \beta + 1/2} \int_{\{x \in I : f > f_I\}} (f - f_I) = \frac{4\alpha(4\beta + 1)}{(2\alpha + 2\beta + 1)^2}.
\]

Since

\[
\sup_{0 \leq \alpha, \beta \leq 1/4} \frac{4\alpha(4\beta + 1)}{(2\alpha + 2\beta + 1)^2} = 1/2,
\]

this proves that \( \|f\|_* = 1/2. \) Therefore, \( C_1 \geq \frac{1}{4} e^{4/\varepsilon}. \) Let us show now the converse inequality.

Let \( f \in \text{BMO}(I_0) \). Setting \( \psi(x) = \sum_{k=0}^\infty \chi_{G_k}(x) \), where \( G_k \) are from \( \text{Theorem 2.2} \), we have:

\[
\left| \{ x \in I_0 : \psi(x) > \alpha \} \right| = \sum_{k=0}^\infty |G_k| \chi_{[k,k+1)}(\alpha)
\]

\[
\leq |I_0| \sum_{k=0}^\infty \min(1, 2\gamma^k) \chi_{[k,k+1)}(\alpha).
\]

Hence, by \((2.1)\),

\[
\left| \{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \} \right| \leq \left| \{ x \in I_0 : \psi(x) > 2\gamma \alpha / \|f\|_* \} \right|
\]

\[
\leq |I_0| \sum_{k=0}^\infty \min(2\gamma^k, 1) \chi_{[k,k+1)}(2\gamma \alpha / \|f\|_*).
\]
This estimate holds for any $0 < \gamma < 1$. Therefore, taking here the infimum over $0 < \gamma < 1$, we obtain:

$$\left| \{ x \in I_0 : |f(x) - f_{I_0}| > \alpha \} \right| \leq \varrho \left( \frac{2e}{\|f\|_W} \alpha \right) |I_0|,$$

where

$$\varrho(\xi) = \inf_{0 < \gamma < 1} \sum_{k=0}^{\infty} 2\gamma^k, 1, X_{(k,k+1)}(\gamma^\xi).$$

Thus, the theorem would follow from the following estimate:

$$\varrho(\xi) \leq \frac{1}{2} e^{4 - \xi} \quad (\xi > 0). \quad (2.6)$$

It is easy to see that $\varrho(\xi) = 1$ for $0 < \xi < 2/e$, and in this case (2.6) holds trivially. Next, $\varrho(\xi) = \frac{2}{e^\xi}$ for $2/e \leq \xi \leq 4/e$. Using that the function $e^{\xi}/\xi$ is increasing on $(1, \infty)$ and decreasing on $(0, 1)$, we get:

$$\max_{\xi \in [2/e, 4/e]} 2e^{\xi}/\xi = \frac{1}{2} e^{4/e},$$

verifying (2.6) for $2/e \leq \xi \leq 4/e$.

For $\xi \geq 1$ we estimate $\varrho(\xi)$ as follows. Let $\xi \in [m, m+1)$, $m \in \mathbb{N}$. Taking $\gamma_i = i/e^\xi$ for $i = m$ and $i = m + 1$, we get:

$$\varrho(\xi) \leq 2 \min \left( \frac{m}{e^\xi}, \left( \frac{m+1}{e^\xi} \right)^{m+1} \right) = 2 \left( \frac{m}{e^\xi} \right)^{m} X_{(m,m)}(\xi) + \left( \frac{m+1}{e^\xi} \right)^{m+1} X_{(m+1,m+1)}(\xi), \quad (2.7)$$

where $\xi_m = \frac{1}{e^{(m+1)/m}}$. Using the fact that the function $e^{\xi}/\xi^m$ is increasing on $(m, \infty)$ and decreasing on $(0, m)$, by (2.7) we obtain that for $\xi \in [m, m+1)$,

$$\varrho(\xi) e^{\xi} \leq 2 \left( \frac{m}{e^\xi m} \right)^{m} e^{\xi_m} = 2 \left( \frac{1}{(1 + 1/m)^m} \right)^{m+1} = c_m.$$

Let us show now that the sequence $\{c_m\}$ is decreasing. This would finish the proof since $c_1 = \frac{1}{2} e^{4/e}$. Let $\eta(x) = (1 + 1/x)^x$ for $x > 0$, and

$$v(x) = \left( e^{\eta(x)/e} / \eta(x) \right)^{x+1}.$$

Then $c_m = 2v(m)$ and hence it suffices to show that $v'(x) < 0$ for $x > 1$. We have:

$$v'(x) = v(x) \left( \log \frac{e}{\eta(x)} \frac{1}{1 - \eta(x)/e} \right).$$

Since $\eta(x)(1 + 1/x) > e$, we get $\mu(x) = \frac{\eta(x)}{\eta(x) - 1} > x$. From this and from the fact that the function $(1 + 1/x)^{1+x}$ is decreasing, we obtain:

$$\left( \frac{1}{x^{1+\mu(x)}} \right) \left( \frac{1}{x^{1+\mu(x)}} \right) = (1 + 1/x)^{1+\mu(x)}(1 + 1/x)^{1+x},$$

which is equivalent to that $v'(x) < 0$. \ □

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