ON THE UNIQUENESS THEOREMS FOR TRANSMISSIONS PROBLEMS RELATED TO MODELS OF ELASTICITY, DIFFUSION AND ELECTROCARDIOGRAPHY

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Abstract. We consider a generalization of the inverse problem of the electrocardiography in the framework of the theory of elliptic and parabolic differential operators. More precisely, starting with the standard bidomain mathematical model related to the problem of the reconstruction of the transmembrane potential in the myocardium from known body surface potentials we formulate a more general transmission problem for elliptic and parabolic equations in the Sobolev type spaces and describe conditions, providing uniqueness theorems for its solutions. Next, the new transmission problem is interpreted in the framework of the elasticity theory applied to composite media. Finally, we prove a uniqueness theorem for an evolutionary transmission problem that can be easily adopted to many models involving the diffusion type equations.

Introduction

Transmission problems for differential equations appear in many applications, see, for instance, [3] in relation to the elliptic theory or [5] for parabolic operators. One of the topical example is the inverse problem of electrocardiography. It is the problem of (numerical) reconstruction of cardiac electrical activity from ECG measurements on the body surface having a significant value for diagnostics and treatment of cardiac arrhythmias, see [4], [8], [32] and elsewhere. The problem involves several boundary value problems for elliptic and parabolic differential operators. First, it is Cauchy problem for elliptic operators that can be treated in the framework of the theory of the ill-posed problems, see [16], [12], [36], [37]. Second, these are the Dirichlet problem and the Neumann problem for strongly elliptic operators possessing the Fredholm property (see, for instance, [9], [19], [20], [22] and [28] for their treatment in various function spaces). The model contains also an evolutionary part, see [2], [32], involving rather general non-linear parabolic equations, that in some particular cases can be treated by the classical methods, see, for instance, [15], [17].

Recently, theoretical investigations of the steady part of the model led to interesting results about non-uniqueness and existence of its solutions in Hardy type spaces, see [11]. Paper [24] was devoted to a larger class of similar transmission problems in the Sobolev spaces in the framework of general theory of elliptic operators with constant coefficients. However the both results were obtained under the following very restrictive assumptions: all the elliptic operators involved in the model should be proportional.

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In the present work, we aim to describe conditions, providing uniqueness theorems for essentially general transmission problems of this kind involving both elliptic and parabolic differential operators. As an example, the new steady transmission problem was interpreted in the framework of the elasticity theory applied to composite media. An uniqueness theorem was also proved for an evolutionary transmission problem that can be interpreted in the framework of the theory of diffusion processes.

1. THE BIDOMAIN MODEL OF THE ELECTROCARDIOGRAPHY

Let θ be a measurable set in \( \mathbb{R}^n \), \( n \geq 2 \) (of course, for models of cardiology we may always restrict ourselves to \( n = 3 \)).

Denote by \( L^2(\theta) \) a Lebesgue space of functions on \( \theta \) with the inner product

\[
(u, v)_{L^2(\theta)} = \int_\theta v(x)u(x) \, dx.
\]

If \( D \) is a domain (an open connected set) in \( \mathbb{R}^n \) with a piecewise smooth boundary \( \partial D \), then for \( s \in \mathbb{N} \) we denote by \( H^s(D) \) the standard Sobolev space with the inner product

\[
(u, v)_{H^s(D)} = \int_D \sum_{|\alpha| \leq s} (\partial^\alpha v)(\partial^\alpha u) \, dx.
\]

As usual, denote by \( H_0^s(D) \) the closure of the subspace \( C_0^{\infty}(D) \) in \( H^s(D) \), where \( C_0^{\infty}(D) \) is the linear space of functions with compact supports in \( D \).

Next, given any non-integer \( s > 0 \), let \( H^s(D) \) stand for the so-called Sobolev-Slobodetskii space. It can be defined as the completion of \( C^{\infty}(\mathbb{D}) \) with respect to the norm

\[
\| u \|_{H^s(D)} = \left( \| u \|^2_{L^2(D)} + \int_D \sum_{|\alpha| \leq s} |(\partial^\alpha u(x) - \partial^\alpha u(y))|^2 |x - y|^{n+2(s-[s])} \, dx \, dy \right)^{1/2},
\]

where \([s]\) is the integer part of \( s \), see [29].

If the boundary \( \partial D \) of the domain \( D \) is sufficiently smooth, then, using the standard volume form \( d\sigma \) on the hypersurface \( \partial D \) induced from \( \mathbb{R}^n \), we may consider the Sobolev-Slobodeckii spaces \( H^s(\partial D) \) on \( \partial D \). Namely, let \( L^2(\partial D) \) be the Lebesgue space of functions on \( \partial D \) with the inner product

\[
(u, v)_{L^2(\partial D)} = \int_{\partial D} v(x)u(x) \, d\sigma(x).
\]

If \( 0 < s < 1 \) and \( \partial D \in C^1 \) then we define \( H^s(\partial D) \) to be the completion of \( C^1(\partial D) \) with respect to the norm

\[
\| u \|_{H^s(\partial D)} = \left( \| u \|^2_{L^2(\partial D)} + \int_{\partial D \times \partial D} \frac{|u(x) - u(y)|^2}{|x - y|^{n-1+2s}} \, d\sigma(x) \, d\sigma(y) \right)^{1/2}.
\]

For \( s \geq 1 \) we have to consider more smooth hypersurfaces. For instance, if \( \partial D \in C^{[s]+1} \) then we may define the space \( H^s(\partial D) \) using local coordinates on \( \partial D \) and a suitable partition of unity.

It is known that the functions of \( H^s(D) \), where \( s > 1/2 \), possess well-defined traces on the Lipschitz surface \( \partial D \). For \( s \in \mathbb{N} \), the trace operator

\[
t_s : H^s(D) \to H^{s-1/2}(\partial D)
\]
obtained in this way acts continuously, if $\partial D \in C^*$; moreover, in this case it possesses a bounded right inverse, see for instance [18, Ch. 1, § 8].

Let us consider now a mathematical model describing the electrical activity of the heart. Assume that the myocardial domain $\Omega_m$ is surrounded by a volume conductor $\Omega_b$. The total domain, including the myocardium and the human torso $\Omega = \Omega_b \cup \overline{\Omega_m}$, where $\overline{\Omega_m}$ is the closure of heart domain, is surrounded by a non-conductive medium (air). Assuming that intracellular, extracellular and extracardiac media are homogeneous and isotropic, denote by $M_i$, $M_e$, $M_b$ the conductivity matrices in the intra-, extracellular and extracardiac spaces, and by $\nu_i$, $\nu_e$ the outward normal vectors to the surfaces of the heart and body volume ($\Omega_m$ and $\Omega_b$), respectively.

Denote by $\nabla$ the gradient operator and by div the divergence operator in $\mathbb{R}^n$. It is convenient to set

$$\Delta_e = -\text{div } M_e \nabla, \quad \Delta_i = -\text{div } M_i \nabla, \quad \Delta_b = -\text{div } M_b \nabla.$$  

Assuming that $M_i$, $M_e$, $M_b$ are symmetric non-degenerate $(n \times n)$-matrices with real entries, satisfying

$$(1.2) \quad \zeta \cdot M \zeta = \zeta^T M \zeta > 0 \text{ for each } \zeta \in \mathbb{R}^n \setminus \{0\},$$

we obtain strongly elliptic operators $\Delta_e$, $\Delta_i$, $\Delta_b$ with constant coefficients.

If the functions $u_i, u_e$ over $\Omega_m$, and the function $u_b$ over $\Omega_b$ stand for intra-, extracellular and extracardiac (electrical) potentials, respectively, then the intra-, extracellular and extracardiac (electrical) currents are given by

$$J_i = -M_i \nabla u_i, \quad J_e = -M_e \nabla u_e, \quad J_b = -M_b \nabla u_b,$$

respectively. As the intracellular (electrical) charge $q_i$ and the extracellular charge $q_e$ should be balanced in the heart tissue, using the divergence operator, we arrive at the following equations involving the time variable $t$:

$$(1.3) \quad \frac{\partial (q_i + q_e)}{\partial t} = 0 \text{ in } \Omega_m,$$

$$(1.4) \quad -\Delta_i u_i = \frac{\partial q_i}{\partial t} + \chi I_{\text{ion}} \text{ in } \Omega_m,$$

$$(1.5) \quad -\Delta_e u_e = \frac{\partial q_e}{\partial t} - \chi I_{\text{ion}} \text{ in } \Omega_m,$$

where $I_{\text{ion}}$ is the ionic current across the cell membrane and $\chi I_{\text{ion}}$ is ionic current per unit tissue. Of course, the charge densities $q_i, q_e$ and the potentials $u_i, u_e$ are actually defined on different domains: extracellular and intracellular spaces, respectively. Thus, equations (1.3), (1.4) and (1.5) reflect the fact that a homogenization procedure is at the bottom of the considered model.

Next, combining (1.3), (1.5) (1.4) we obtain the conservation law for the total current ($J_i + J_e$):

$$\Delta_i u_i + \Delta_e u_e = 0 \text{ in } \Omega_m.$$

In the heart surrounded by a conductor, the normal component of the total current should be continuous across the boundary of the heart:

$$(1.6) \quad \nu_i \cdot (J_i + J_e) = \nu_e \cdot J_b \text{ on } \partial \Omega_m.$$
Taking in account the current behaviour at the torso, we arrive at a steady-state
version of the bidomain model of the electrocardiography [8], [4], [32]:
\begin{align}
&\Delta_i u_i + \Delta_e u_e = 0 \text{ in } \Omega_m, \\
&\Delta_b u_b = 0 \text{ in } \Omega_b, \\
&u_e = u_b \text{ on } \partial \Omega_m, \\
&\nu_i \cdot (M_e \nabla u_e) = -\nu_e \cdot (M_b \nabla u_b) \text{ on } \partial \Omega_m, \\
&\nu_i \cdot (M_i \nabla u_i) = 0 \text{ on } \partial \Omega_m, \\
&\nu_e \cdot (M_b \nabla u_b) = 0 \text{ on } \partial \Omega,
\end{align}
where (1.10), (1.11) are consequences of (1.6) and the assumption that the intra-
cellular domain is completely insulated.

However, the primary equations (1.3), (1.4), (1.5) are actually evolutionary. That
is why the model contains a large evolutionary part, too. For example, (1.4), (1.5)
imply
\begin{equation}
- \Delta_i u_i + \Delta_e u_e = \frac{\partial (q_i - q_e)}{\partial t} + 2 \chi I_{\text{ion}} \text{ in } \Omega_m \times (0, T),
\end{equation}
On the other hand, the so-called transmembrane potential
\[ v = u_i - u_e \]
satisfies
\begin{equation}
\frac{1}{2} \chi C_m (u_i - u_e) = \rho_i - q_e \text{ in } \Omega_m
\end{equation}
where $C_m$ is the capacitance of the cell membrane. Thus, using (1.13) and (1.14)
we arrive at the so-called cable equation
\begin{equation}
\frac{1}{2} \chi \left( - \Delta_i u_i + \Delta_e u_e \right) = C_m \frac{\partial (u_i - u_e)}{\partial t} + I_{\text{ion}} \text{ in } \Omega_m \times (0, T),
\end{equation}
see, for instance, [32, §2.2.2]. Of course, there are many possibilities to supplement
the model by more advanced and complicated relations.

We begin the discussion from the following problem that is known as the steady
inverse problem of the electrocardiography.

**Problem 1.1.** Given the values of electrical potential $u_b$ on the boundary of the
body
\begin{equation}
\text{ (1.16) } \quad u_b = f_0 \text{ on } \partial \Omega,
\end{equation}
find the intracellular potential $u_i$ and extracellular potential $u_e$ in $\Omega_m$ and extrac-
cardiac potential $u_b$ in $\Omega_b$, satisfying equations (1.7), (1.8) and boundary conditions
(1.9)-(1.12).

Note that Problem 1.1 includes the Cauchy problem (1.3), (1.12), (1.16) for the
elliptic operator $\Delta_b$ that is usually ill-posed in all the standard function spaces,
see, for instance, [16], [36], or elsewhere. The uniqueness theorem for the Cauchy
problem related to elliptic equations (see, for example, [27, Theorem 2.8]) provides
the uniqueness of the potential $u_b$ in the Lebesgue and the Sobolev type spaces
if it exists. However, the uniqueness of the potentials $u_i$, $u_e$, satisfying relations
(1.7)-(1.12), (1.16) and the transmembrane potential $v$ was not mathematically
established.
The uniqueness of solutions to Problem 1.1 was investigated in [11] in Hardy type spaces over smooth domains:

$$\mathcal{H}(\Omega) = \{ u \in H^1(\Omega), \frac{\partial u}{\partial \nu} \in L^2(\partial \Omega), \Delta u \in L^2(\Omega) \}$$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the usual Laplace operator in $\mathbb{R}^n$ and $\frac{\partial}{\partial \nu}$ is the normal derivative with respect to $\partial \Omega$. However it was essential in [11] that the matrices $M_i, M_e$, were proportional:

$$M_e = \gamma M_i$$

with some positive number $\gamma$. In particular, this means that a linear change of variables reduces the consideration to the situation where

$$\Delta_i = -\sigma_i \Delta, \Delta_e = -\sigma_e \Delta, \gamma = \frac{\sigma_e}{\sigma_i},$$

and $\sigma_i, \sigma_e$, are positive numbers characterizing the electrical conductivity of the corresponding media. It was proved that under this very restrictive assumption the null-space of Problem 1.1 consists of all the triples

$$\begin{cases} u_b = 0 \quad \text{in} \quad \Omega_b, \\ u_e = u \quad \text{in} \quad \Omega_m, \\ u_i = -\frac{\sigma_e}{\sigma_i} u + c \quad \text{in} \quad \Omega_m, \end{cases}$$

where $c$ is an arbitrary constant and $u$ is an arbitrary function from $\mathcal{H}(\Omega_m)$ satisfying

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega_m,$$

cf. Theorem 2.4 below. In this way the so-called transmembrane potential $v$ can be defined on $\partial \Omega_m$ up to an arbitrary constant summand $c$: it can be uniquely defined on $\partial \Omega_m$ if we supplement the bidomain model with the following calibration assumption that always is achievable for isotropic conductivity: there is a constant $c_0$ such that

$$\int_{\partial \Omega_m} (u_i + c_0 u_e)(y) d\sigma(y) = 0.$$  

As for the Existence Theorem for Problem 1.1 the ill-posed Cauchy problem (1.8), (1.12), (1.16) can be treated with the standard regularization methods described in [12], [16], [36]. Moreover for the potentials $u_i, u_e$ in this very particular case we have

$$u_i = -\frac{\sigma_e}{\sigma_i} u_e + \frac{\sigma_e}{\sigma_i} N_i(0, \nu_i \cdot (M_b \nabla u_b)) + c$$

where $c$ is an arbitrary constant, $u_e$ is an arbitrary function from $\mathcal{H}(\Omega_m)$ satisfying (1.9), (1.10) and $N_i(g, u_0)$ is the unique solution to the Neumann problem

$$\begin{cases} \Delta N_i(g, u_0) = g \quad \text{in} \quad \Omega_m, \\ \nu_i \cdot (M_i \nabla u) N_i(g, u_0) = u_0 \quad \text{on} \quad \partial \Omega_m, \end{cases}$$

satisfying

$$\int_{\partial D} N_i(g, u_0)(x) d\sigma(x) = 0.$$
Of course, Problem 1.21 is not always solvable but it has the Fredholm property in many Sobolev type spaces. More precisely, it is solvable in the standard Sobolev spaces if and only if

\[
\int_{\partial \Omega_m} u_0 d\sigma + \int_{\Omega_m} g dx = 0,
\]

see, for instance, [28]. The last identity can be easily verified for the data chosen in (1.20) because of the relations in the bidomain model, see also Theorem 2.5 below for a more general situation. Again, if calibration assumption (1.19) holds for the pair \( u_i, u_e \) then the constant \( c \) in (1.20) may be uniquely defined by

\[
c = -c_0 \left( \int_{\partial \Omega_m} d\sigma(y) \right)^{-1} \int_{\partial \Omega_m} u_b(y) d\sigma(y).
\]

In particular, this scheme gives a possibility to find the potential \( u_b \) in \( \Omega_b \) and the potentials \( u_i, u_e \) on \( \partial \Omega_m \).

But, from mathematical point of view, this means that in the present form the steady part of the bidomain model (1.7)–(1.12), (1.16) has too many degrees of freedom. It seems, that one equation related to the potentials \( u_i, u_e \) in \( \Omega_m \) is still missing.

In the next sections we will discuss what kind of equation can be added to steady bidomain model even in a much more general situation in order to provide uniqueness of its solutions. We will also discuss the uniqueness of solutions to an evolutionary bidomain model.

2. A MORE GENERAL STEADY PROBLEM

Let us consider a more general steady problem. With this purpose, recall that a linear (matrix) differential operator

\[
A(x, \partial) = \sum_{|\alpha| \leq m} A_\alpha(x) \partial^\alpha
\]

of order \( m \) with \((l \times k)\)-matrices \( A_\alpha(x) \), having entries from \( C^\infty(X) \) on an open set \( X \subset \mathbb{R}^n \), is called an operator with injective symbol on \( X \) if \( l \geq k \) and for its principal symbol

\[
\sigma(A)(x, \zeta) = \sum_{|\alpha| = m} A_\alpha(x) \zeta^\alpha
\]

we have

\[
rang(\sigma(A)(x, \zeta)) = k \text{ for any } x \in X, \zeta \in \mathbb{R}^n \setminus \{0\}.
\]

An operator \( A \) is called (Petrovsky) elliptic, if \( l = k \) and its symbol is injective (or, the same, non-degenerate) on \( X \).

Then let \( S_A(D) \) be the space of generalized solutions to the equation \( Ah = 0 \) in a domain \( D \). If the operator \( A \) has an injective symbol and its coefficients are real analytic, then the Petrovsky theorem yields that the elements of the space \( S_A(D) \) are real analytic vector functions in \( D \).

An operator \( L(x, \partial) \) is called strongly elliptic if it is elliptic, its order is even (and equals to \( 2m \)) and there is a positive constant \( c_0 \) such that

\[
(-1)^m \Re(w^* \sigma(L)(x, \zeta) w) \geq c_0 |\zeta|^{2m} |w|^2 \text{ for any } x \in X, \zeta \in \mathbb{R}^n, w \in \mathbb{C}^k
\]

where \( w^* = w^T \) and \( w^T \) is the transposed vector for \( w \in \mathbb{C}^k \).
Actually, any operator $A^*A$ is strongly elliptic of order $2m$ if the principal symbol of $A$ is injective and

$$A^*(x, \partial) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (A^*_\alpha(x))$$

is the formal adjoint for $A$ with the adjoint matrices $A^*_\alpha(x)$. The typical operator of such type are the (minus) Laplacians

$$-\Delta = \nabla^* \nabla = -\text{div } \nabla, \quad \Delta_M = -\text{div } M \nabla,$$

where $M$ is a self-adjoint non-negative non-degenerate $(n \times n)$-matrix with constant entries.

Next, we recall that a set of linear differential operators $\{B_0, B_1, \ldots, B_{m-1}\}$ is called a $(k \times k)$-matrix Dirichlet system of order $(m-1)$ on $\partial D$ if

1) the operators are defined in a neighbourhood of $\partial D$;

2) the order of the differential operator $B_j$ equals to $j$;

3) the map $\sigma(B_j)(x, \nu(x)) : \mathbb{C}^k \rightarrow \mathbb{C}^k$ is bijective for each $x \in \partial D$, where $\nu(x)$ will denote the outward normal vector to the hypersurface $\partial D$ at the point $x \in \partial D$, see [22, 36 §9.2.2].

According to the Trace Theorem, if $\partial D \in C^s$, $s \geq m \geq 1$ then each operator $B_j$ induces a bounded linear operator

$$(2.1) \quad B_j : [H^s(D)]^k \rightarrow [H^{s-j-1/2}(\partial D)]^k.$$ Easily, if a first order operator $A$ has injective symbol in a neighbourhood of the closure $\overline{D}$ of a smooth domain $D$ then the pair $\{I_k, \sigma^*(A)(\nu)A\}$ is a Dirichlet system of the second order near $\partial D$ where $I_k$ is the unit $(k \times k)$-matrix.

Now we may proceed with the formulation of the transmission problem.

Let $\Omega_m$ and $\Omega$ be smooth bounded domains in $\mathbb{R}^n$, $n \geq 2$, such that $\Omega \supset \overline{\Omega}_m$ where $\Omega_m$ is the closure of the domain $\Omega_m$. We set $\Omega_b = \Omega \setminus \overline{\Omega}_m$. Denote by $A^{(i)}, A^{(e)}, A^{(b)}$ matrix differential operators with real analytic coefficients and injective symbols in some neighbourhoods $U_m$ and $U_b$ of the compacts $\Omega_m$ and $\Omega_b$, respectively. Then the differential operators

$$\Delta^{(e)} = (A^{(e)})^* A^{(e)}, \quad \Delta^{(b)} = (A^{(b)})^* A^{(b)}, \quad \Delta^{(b)} = (A^{(b)})^* A^{(b)}$$

are elliptic and strongly elliptic over $U_m$ and $U_b$, respectively.

We also fix the first order $(k \times k)$-matrix boundary operators $B^{(i)}_1$ near $\partial \Omega_b$ and $B^{(i)}_1, B^{(e)}_1, B^{(b)}_1$ near $\partial \Omega_m$ such that $(I_b, B^{(b)}_1), (I_b, B^{(i)}_1)$ and $(I_b, B^{(e)}_1)$ are Dirichlet pairs and

$$(2.2) \quad \int_{\partial \Omega_m} v^* B^{(i)}_1 ud\sigma = \int_{\Omega_m} ((A^{(i)} v)^* A^{(i)} u - v^* \Delta^{(i)} u) dy,$$

$$(2.3) \quad \int_{\partial \Omega_m} v^* B^{(e)}_1 ud\sigma = \int_{\Omega_m} ((A^{(e)} v)^* A^{(e)} u - v^* \Delta^{(e)} u) dy$$

for all $u \in [H^2(\Omega_m)]^k$, $v \in [H^{1}(\Omega_m)]^k$,

$$(2.4) \quad \int_{\partial \Omega_b} v^* B^{(b)}_1 ud\sigma = \int_{\Omega_b} ((A^{(b)} v)^* A^{(b)} u - v^* \Delta^{(b)} u) dy$$

for all $u \in [H^2(\Omega_b)]^k$, $v \in [H^{1}(\Omega_b)]^k$. For instance, one may take

$$(2.5) \quad B^{(b)}_1 = \sigma^*(A^{(b)})(\nu_b)A^{(b)}, \quad B^{(i)}_1 = \sigma^*(A^{(i)})(\nu_i)A^{(i)}, \quad B^{(e)}_1 = \sigma^*(A^{(e)})(\nu_e)A^{(e)},$$
where \( \nu_i \) and \( \nu_e \) are the outward normal vectors to the surfaces \( \Omega_m \) and \( \Omega_b \), respectively, because by Ostrogradsky-Gauss formula we have:

\[
(2.15) \quad \int_{\partial D} v^* \sigma^*(A)(\nu) Au \, d\sigma = \int_D \left( (Av)^* Au - v^* A^* Au \right) dx
\]

for all \( v \in [H^1(D)]^k \), \( u \in [H^2(D)]^k \) and any first order differential operator \( A \) with an injective symbol over \( \overline{D} \).

**Problem 2.1.** Let \( s \geq 2 \) and \( \alpha_i, \alpha_e, \beta_e, \beta_i \in \mathbb{R} \), \( \alpha_e^2 + \alpha_i^2 \neq 0 \), \( \beta_e^2 + \beta_i^2 \neq 0 \). Given vector functions

\[
f \in [H^{s-2}(\partial\Omega)]^k, \quad f_0 \in [H^{s-1/2}(\partial\Omega)]^k, \quad f_1 \in [H^{s-3/2}(\partial\Omega)]^k,
\]

find, if possible, vector functions \( u_i, u_e \in [H^s(\Omega_m)]^k, \quad u_b \in [H^s(\Omega_b)]^k \) satisfying

\[
(2.7) \quad \alpha_i \Delta^{(i)} u_i + \alpha_e \Delta^{(e)} u_e = 0 \quad \text{in} \quad \Omega_m,
\]

\[
(2.8) \quad \Delta^{(b)} u_b = f \quad \text{in} \quad \Omega_b,
\]

\[
(2.9) \quad u_e = u_b \quad \text{on} \quad \partial\Omega_m,
\]

\[
(2.10) \quad B_1^{(e)} u_e = \beta_e B_1^{(b)} u_b \quad \text{on} \quad \partial\Omega_m,
\]

\[
(2.11) \quad B_1^{(i)} u_i = \beta_i B_1^{(b)} u_b \quad \text{on} \quad \partial\Omega_m,
\]

\[
(2.12) \quad B_1^{(b)} u_b = f_1 \quad \text{on} \quad \partial\Omega,
\]

\[
(2.13) \quad u_b = f_0 \quad \text{on} \quad \partial\Omega.
\]

Besides we will use below an assumption that is similar to calibration relation (1.10): there is a constant \( c_0 \) such that

\[
(2.14) \quad \int_{\partial\Omega_m} h^*(y)(u_i + c_0 u_e)(y) d\sigma(y) = 0 \quad \text{for all} \quad h \in S_{A^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k.
\]

Problem 2.1 was considered in [24] in the Sobolev spaces over smooth domains under the following rather restrictive but partially natural assumptions:

1) the coefficients of the operators \( A^{(i)}, A^{(e)}, A^{(b)} \) are constants;
2) there is a constant \( \gamma > 0 \) such that \( A^{(e)} = \sqrt{\gamma} A^{(i)} \);
3) the spaces \( S_{A^{(i)}}(\Omega_m) \cap [H^2(\Omega_m)]^k \) and \( S_{A^{(b)}}(\Omega) \cap [H^2(\Omega)]^k \) coincide;
4) boundary operators \( B_1^{(b)}, B_1^{(i)}, B_1^{(b)} \) are given by (2.5);
5) \( \alpha_i = 1, \alpha_e = 1, \beta_e = -1, \beta_i = 0, f_1 \equiv 0, f \equiv 0; \)
6) the Shapiro-Lopatinsky conditions are fulfilled for the pair of differential operators \( (\Delta^{(i)}, B_1^{(i)}) \) over \( \overline{\Omega}_m \), see [1] Chapter 1, §3, condition II for \( q = 0 \), [23] or elsewhere.

Only assumption 6) from the list is essential for our considerations because relations (2.7), (2.11) lead us to the following Neumann problem: given pair \( g \in [H^{s-2}(\Omega_m)]^k \) and \( u_1 \in [H^{s-3/2}(\partial\Omega_m)]^k \), find, if possible, a function \( u \in [H^s(\Omega_m)]^k \) such that

\[
(2.15) \quad \left\{ \begin{array}{l}
\Delta^{(i)} u = g \quad \text{in} \quad \Omega_m, \\
B_1^{(i)} u = u_1 \quad \text{on} \quad \partial\Omega_m,
\end{array} \right.
\]

see, for instance, [23]. More precisely, the Shapiro-Lopatinsky conditions provide that problem (2.15) has the Fredholm property. Practically, under (2.2), they are equivalent to the following bound: there is a positive constant \( c_i \) such that

\[
\| u \|_{H^1(\Omega_m)} \leq c_i \| A^{(i)} u \|_{L^2(\Omega_m)} \quad \text{for all} \quad u \in (S_{A^{(i)}}(\Omega_m) \cap [H^1(\Omega_m)]^k)^\perp
\]
where \( (S_{A(0)}(\Omega_m))\cap [H^1(\Omega_m)]^k \) stands for the orthogonal complement of the subspace \( S_{A(0)}(\Omega_m) \cap [H^1(\Omega_m)]^k \) in the Hilbert space \( [H^1(\Omega_m)]^k \). In particular, the Shapiro-Lopatinsky conditions guarantee that the space \( S_{A(0)}(\Omega_m) \cap [H^s(\Omega_m)]^k \) is finite dimensional. Actually, the following theorem holds true.

**Theorem 2.2.** Let \( s \in \mathbb{N}, s \geq 2, \) and \( \partial \Omega_m \subset C^s \). If Shapiro-Lopatinsky conditions hold true the pair \( (\Delta^{(i)}, B^{(i)}_1) \) and \( (\Omega, s) \) is solvable if and only if there is a function \( x \) for all \( (\Omega, s) \).

Theorem 2.3. Results from \([27, 6, 36]\). With this purpose, we set \( (\Delta^{(i)}, B^{(i)}_1) \) for all \( i \) above.

First, we note, that similarly to Problem 1.1, Problem 2.1 includes the Cauchy problem \( (2.8), (2.12), (2.13) \) for the elliptic operator \( \Delta \) with \( (2.17) \) that is usually ill-posed in all the standard function spaces, see, for instance, \([10, 36]\), or elsewhere. However, the uniqueness theorem for the Cauchy problem related to elliptic equations (see, for example, \([27, Theorems 2.8]\)) provides the uniqueness of the vector function \( u_b \) in the Lebesgue and the Sobolev type spaces if it exists. Moreover, since the operator \( \Delta^{(b)} \) is elliptic and its coefficients are real analytic, it admits a bilateral fundamental solutions to the operator, say \( \varphi_b \), in a neighbourhood of the compact \( \Omega_b \). Let us indicate a solvability criterion and formulas for its solutions based on results from \([27, 6]\) and \([36]\). With this purpose, we set

\[
F(x) = \int_{\Omega_b} \left( \varphi_b(x, y)^* f(y) dy + \int_{\partial \Omega} (B^{(b)}_1(y, \partial_y) \varphi_b(x, y))^* f_0(y) - (\varphi_b(x, y))^* f_1(y) \right) d\sigma(y), \quad x \notin \partial \Omega.
\]

**Theorem 2.3.** Let \( B^{(b)}_1 \) satisfy \( (2.4) \). Then Cauchy problem \((2.8), (2.12), (2.13)\) with data in \( [H^{s-2}(\Omega_b)]^k \times [H^{s-1/2}(\partial \Omega)]^k \times [H^{s-3/2}(\partial \Omega)]^k \) is densely solvable in the space \( [H^s(\Omega_b)]^k \). Moreover, it has no more that one solution in \( [H^s(\Omega_b)]^k \). It is solvable for a triple \( f \in [H^{s-2}(\partial \Omega)]^k \), \( f_0 \in [H^{s-1/2}(\partial \Omega)]^k \), \( f_1 \in [H^{s-3/2}(\partial \Omega)]^k \) if and only if there is a function \( \mathcal{F} \in [H^s(X \setminus \Omega_m)]^k \) satisfying \( \Delta_b \mathcal{F} = 0 \) in \( X \setminus \Omega_m \) and such that

\[
\mathcal{F}(x) = F(x)
\]

for all \( x \in X \setminus \Omega \). Besides, the solution \( u \), if exists, is given by the following formula

\[
u_b(x) = F(x) - \mathcal{F}(x), \quad x \in \Omega_b.
\]

**Proof.** Follows from \([27, Theorems 2.8 and 5.2]\) for the case \( f = 0 \) and \([6]\) for \( f \neq 0 \) because both the set \( S = \partial \Omega \) where the boundary Cauchy data are defined and its complement \( \partial \Omega_b \setminus S = \partial \Omega_m \) are non empty and open in the relative topology.
The ill-posed Cauchy problem (2.1), (2.12), (2.13) can be also treated with the standard regularization methods described in [12], [16], [36], providing formulas for exact and approximate solutions.

Now we are ready to describe the null-space of Problem 2.1. We slightly differ the approach of [11] and [24] in this more general situation.

**Theorem 2.4.** Let \( s \geq 2, (2.2), (2.3), (2.4) \) be fulfilled and the pair \((\Delta^{(i)}, B_1^{(i)})\) satisfy Shapiro-Lopatinsky conditions in \( \Omega_m \). If \( \alpha_i \neq 0 \) and
\[
S_{A^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \subset S_{\Delta^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k
\]
then the null-space of Problem 2.1 consists of all the triples \( u_i, u_e, u_b \) from \([H^s(\Omega_m)]^k \times [H^s(\Omega_m)]^k \times [H^s(\Omega_m)]^k \) satisfying the following conditions:
\[
\begin{align*}
  u_b &= 0 & \text{in } \Omega_b, \\
  u_e &= u & \text{in } \Omega_m, \\
  u_i &= (\alpha_e/\alpha_i)\mathcal{N}^{(i)}(\Delta^{(e)}u, 0) + h_0 & \text{in } \Omega_m,
\end{align*}
\]
where \( h_0 \) is an arbitrary element of the finite dimensional space \( S_{A^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \) and \( u \) is an arbitrary vector function from \([H^s(\Omega_m)]^k \times [H^s(\Omega_m)]^k \times [H^s(\Omega_m)]^k \). Moreover, if calibration assumption (2.14) holds for a pair \( u_i, u_e \) from the null-space then the element \( h_0 \) in (2.20) equals to zero.

**Proof.** Indeed, let the triple \((u_b, u_e, u_e) \in [H^s(\Omega_b)]^k \times [H^s(\Omega_m)]^k \times [H^s(\Omega_m)]^k \) belong to the null-space of Problem 2.1. Hence \( f \equiv 0 \) in \( \Omega_b \), \( f_0 = f_1 \equiv 0 \) on \( \partial\Omega \) and then \( u_b \equiv 0 \) in \( \Omega_b \) because of Theorem 2.3. Of course, using (2.19), (2.11), we obtain
\[
u_e = u_b = B_1^{(e)}u_e = \beta_iB_1^{(b)}u_b = 0 \quad \text{on } \partial\Omega_m
\]
for \( u_e \in [H^s(\Omega_m)]^k \). Since the pair \((1, B_1^{(e)})\) is a Dirichlet system on \( \partial\Omega_m \), then, according to [10], \( u_e \in [H^2_0(\Omega_m) \cap H^s(\Omega_m)]^k \). The function \( u_i \) satisfies (2.7) and (2.11) and hence
\[
B_1^{(i)}u_i = \beta_iB_1^{(b)}u_b = 0
\]
Then, according to Theorem 2.2, this means precisely
\[
u_i = (\alpha_e/\alpha_i)\mathcal{N}^{(i)}(\Delta^{(e)}u, 0) + h_0
\]
with an arbitrary element \( h_0 \in S_{A^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \).

Thus, any triple \((u_b, u_e, u_e) \in H^2(\Omega_b) \times H^2(\Omega_m) \times H^2(\Omega_m) \), belonging to the null-space of Problem 2.1, has the form as in (2.14) with an arbitrary element \( h_0 \) of the finite dimensional space \( S_{A^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \) and an arbitrary vector function \( u = u_e \in [H^s(\Omega_m) \cap H^2_0(\Omega_m)]^k \).

Let a triple \((u_b, u_e, u_e) \in [H^s(\Omega_b)]^k \times [H^s(\Omega_m)]^k \times [H^s(\Omega_m)]^k \) have the form as in (2.14) with an arbitrary element \( h_0 \in S_{A^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \) and an arbitrary vector function \( u \in [H^s(\Omega_m) \cap H^2_0(\Omega_m)]^k \). Then, obviously, \( f \equiv 0 \) in \( \Omega \), \( f_0 = f_1 \equiv 0 \) on \( \partial\Omega \). Moreover, integrating by parts with the use of (2.3) we easily obtain
\[
\int_{\Omega_m} h^{(e)}Du = \int_{\Omega_m} (\Delta^{(e)}h)^{-1}Du = 0 \quad \text{for all } h \in S_{\Delta^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k
\]
because \( u \in H^2(\Omega_m) \) and embedding (2.19) is fulfilled. Hence, as (2.2) holds true, Theorem 2.2 implies that there is a solution \( w \) to Neumann problem (2.15) for the operator \( \Delta^{(i)} \):
\[
\begin{align*}
  \Delta^{(i)}w &= (\alpha_e/\alpha_i)\Delta^{(e)}u & \text{in } \Omega_m, \\
  B_1^{(i)}w &= 0 & \text{on } \partial\Omega_m.
\end{align*}
\]
According to Theorem 2.2, the general form of such a solution is precisely

\[ w = \left( -\frac{\alpha_e}{\alpha_i} \right) \mathcal{N}^{(i)}(\Delta_e u, 0) + h_0 \]

with an arbitrary element \( h_0 \in S_{A^{(i)}}(\Omega_m) \cap \left[ H^s(\Omega_m) \right]^k \). If we take \( u_i = w \) then

\[
\begin{align*}
    u_b &= 0 \quad \text{in} \quad \Omega_b, \\
    u_e &= u \quad \text{in} \quad \Omega_m, \\
    B_1^{(i)} u_e &= 0 \quad \text{on} \quad \partial \Omega_m, \\
    u_c &= 0 \quad \text{on} \quad \partial \Omega_m, \\
    \Delta^{(i)} u_i &= \left( -\frac{\alpha_e}{\alpha_i} \right) \Delta^{(e)} u \quad \text{in} \quad \Omega_m, \\
    B_2^{(i)} u_i &= 0 \quad \text{on} \quad \partial \Omega_m
\end{align*}
\]

(2.21)

with any \( u \in \left[ H^2_0(\Omega_m) \cap H^s(\Omega_m) \right]^k \).

Thus, any triple \((u_b, u_e, u_c) \in [H^s(\Omega_b)]^k \times [H^2(\Omega_m)]^k \times [H^2(\Omega_m)]^k\) having the form as in (2.20) with a arbitrary element \( h_0 \in S_{A^{(i)}}(\Omega_m) \cap \left[ H^s(\Omega_m) \right]^k \) and an arbitrary vector function \( u \in \left[ H^2_0(\Omega_m) \cap H^s(\Omega_m) \right]^k \) belongs to the null-space of Problem 2.1.

Finally, if calibration assumption (2.14) is fulfilled then, as \( u_e = u \in [H^2_0(\Omega_m)]^k \), condition (2.17) yields

\[ 0 = \int_{\partial \Omega_m} h^* (u_i + c_0 u_e)(y) d\sigma(y) = \int_{\partial \Omega_m} h^* h_0 d\sigma \]

for all \( S_{A^{(i)}}(\Omega_m) \cap \left[ H^s(\Omega_m) \right]^k \). In particular, for \( h = h_0 \) we obtain

\[ \int_{\partial \Omega_m} |h_0|^2 d\sigma = 0 \]

i.e. \( h_0 = 0 \) on \( \partial \Omega_m \). Finally, as \( h_0 \in S_{A^{(i)}}(\Omega_m) \cap \left[ H^s(\Omega_m) \right]^k \) we may use the Uniqueness Theorem [27, Theorem 2.8] applying it to the Cauchy problem

\[
\begin{align*}
    A^{(i)} h_0 &= 0 \quad \text{in} \quad \Omega_m, \\
    h_0 &= 0 \quad \text{on} \quad \partial \Omega_m,
\end{align*}
\]

and concluding that \( h_0 \equiv 0 \) in \( \Omega_m \). \( \square \)

Let us formulate an existence theorem for the Problem 2.1. With this purpose we note that the scale of Sobolev spaces can be extended for negative smoothness indexes, too. Namely, for \( s > 0 \), denote by \( H^{-s}(D) \) the completion of \( C^\infty(\overline{D}) \) with respect to the norm

\[ \| u \|_{H^{-s}(D)} = \sup_{v \in C^\infty_0(D) \backslash \{0\}} \frac{|(v, u)_{L^2(D)}|}{\|v\|_{H^s(D)}}. \]

Actually, \( H^{-s}(D) \) can be identified with the dual of \( H^s_0(D) \) with respect to the pairing induced by \( (\cdot, \cdot)_{L^2(D)} \).

We denote also by \( P_0 \) the \([L^2(\partial \Omega_m)]^k\)-orthogonal projection onto the closed finite dimensional subspace \( S_{A^{(i)}}(\Omega_m) \cap \left[ H^s(\Omega_m) \right]^k \subset [L^2(\partial \Omega_m)]^k \). Also, we assume that the following relations are fulfilled:

\[ S_{A^{(i)}}(\Omega_m) \cap \left[ H^s(\Omega_m) \right]^k \subset S_{A^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k, \]

(2.22)

\[ S_{A^{(i)}}(\Omega_m) \cap \left[ H^s(\Omega_m) \right]^k = S_{A^{(i)}}(\Omega) \cap \left[ H^s(\Omega) \right]^k \subset S_{A^{(i)}}(\Omega_b) \cap [H^s(\Omega_b)]^k. \]

(2.23)
Theorem 2.5. Let $s \geq 2$, (2.22), (2.3), (2.4) be fulfilled and the pair $(\Delta^{(i)}, B_1^{(i)})$ satisfy Shapiro-Lopatinsky conditions in $\Omega_m$ and let embeddings (2.22), (2.23) hold true. If $\alpha_i \neq 0$ then, given $f \in [H^{s-2}((\partial \Omega_b)]^k$, $f_0 \in [H^{s-1/2}((\partial \Omega_b)]^k$, $f_1 \in [H^{s-3/2}((\partial \Omega_b)]^k$ admitting the solution $u_0 \in [H^s(\Omega_b)]^k$ to (2.3), (2.10) and (2.13), there are functions $u_e, u_i \in [H^s(\Omega_m)]^k$ satisfying (2.7), (2.9), (2.10), (2.11) if and only if

$$
(2.24) \quad (\beta_e \alpha_e + \alpha_i \beta_i) \left( \int_{\Omega_b} h^*(y) f(y) \, dy + \int_{\partial \Omega_b} h^*(y) f_1(y) \, d\sigma(y) \right) = 0
$$

for all $h \in S_{A(i)}(\Omega_b) \cap [H^s(\Omega_b)]^k$.

Proof. Indeed, as $u_b \in [H^s(\Omega_b)]^k$ we see that

$$
Q u = g \quad \text{in} \quad \Omega_m,
$$

$$
u = u_b \quad \text{on} \quad \partial \Omega_m,
$$

$$
B_1^{(e)} u = \beta_e B_1^{(i)} u_b \quad \text{on} \quad \partial \Omega_m
$$

with an arbitrary function $g \in [H^{s-1}((\Omega_m)]^k$ and an arbitrary strongly elliptic formally non-negative operator $Q$ of the fourth order and with real analytic coefficients in a neighbourhood of $\Omega_m$. Indeed, under these assumptions the Dirichlet problem (2.25) admits one and only one solution in $[H^s(\Omega_m)]^k$, for instance, [20], [22] Ch. 5 or elsewhere. For instance, one may take $Q = (\Delta^{(e)})^2$ because $\Delta = (\Delta^{(e)})^*$ and hence the operator

$$(\Delta^{(e)})^2 = (\Delta^{(e)})^* \Delta^{(e)}$$

is strongly elliptic formally non-negative and of fourth order; in particular, the linear space $S_Q(\Omega_m) \cap [H^2(\Omega_m)]^k$ is trivial.

If $\alpha_e \neq 0$ then, integrating by parts with the use of (2.3), (2.4), (2.8), (2.10), (2.22), (2.24), we obtain

$$
(2.26) \quad -\frac{\beta_i - \beta_e}{\alpha} \int_{\Omega_m} h^*(y) \Delta^{(e)} u_e(y) \, dy = \frac{\alpha_e}{\alpha_i} \int_{\Omega_m} h^* B_1^{(e)} u_e \, d\sigma - \frac{\alpha_e}{\alpha_i} \int_{\Omega_m} (A^{(e)})^* A^{(e)} u_e(y) \, dy =
$$

$$(\beta_e + \frac{\beta_i - \beta_e}{\alpha} \frac{\alpha_e}{\alpha_i}) \int_{\Omega_m} h^* B_1^{(i)} u_b \, d\sigma - (\beta_e \frac{\alpha_e}{\alpha_i} + \beta_i) \int_{\Omega_m} h^*(B_1^{(e)} u_b - B_1^{(i)} u_b) \, d\sigma =
$$

$$
-(\beta_e \alpha_e / \alpha_i + \beta_i) \left( \int_{\Omega_b} (A^{(e)})^* A^{(f)} u_b \, dy - \int_{\Omega_b} h^* \Delta^{(e)} u_b \, dy \right)
$$

$$
-\beta_i \int_{\partial \Omega_m} h^* B_1^{(i)} u_b \, d\sigma + (\beta_e \alpha_e / \alpha_i + \beta_i) \int_{\partial \Omega} h^* B_1^{(i)} u_b \, d\sigma =
$$

$$
-\beta_i \int_{\partial \Omega_m} h^* B_1^{(i)} u_b \, d\sigma + \frac{\alpha_e \beta_e + \alpha_i \beta_i}{\alpha_i} \left( \int_{\Omega_b} h^* f \, dy + \int_{\partial \Omega} h^* f_1 \, d\sigma \right)
$$

for all $h \in S_{A(i)}(\Omega_m) \cap [H^s(\Omega_m)]^k$. 
If \( \alpha_e = 0 \) then, similarly,

\[
(2.27) \quad \beta_i \int_{\partial \Omega_m} h^* B_1^{(b)} u_b d\sigma = -\beta_i \int_{\partial \Omega_b} h^* B_1^{(b)} u_b d\sigma + \beta_i \int_{\partial \Omega} h^* B_1^{(b)} u_b d\sigma = \beta_i \int_{\partial \Omega} h^* f_1 d\sigma + \int_{\Omega_b} h^* f d\gamma
\]

for all \( h \in S_{A^{(1)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \).

In any case, (2.24), (2.26), (2.27) and Theorem 2.2 yield the existence of a vector function \( u_i \in [H^s(\Omega_m)]^k \) satisfying

\[
(2.28) \quad \begin{cases} 
\Delta^{(i)} u_i = (-\alpha_e/\alpha_i) \Delta^{(e)} u_e & \text{in } \Omega_m, \\
B_1^{(i)} u_i = \beta_i B_1^{(b)} u_b & \text{on } \partial \Omega_m.
\end{cases}
\]

More precisely, Theorem 2.2 states that \( u_i \) is given by

\[
(2.29) \quad u_i = N^{(i)}((-\alpha_e/\alpha_i) \Delta^{(e)} u_e, \beta_i B_1^{(b)} u_b) + h_0
\]

with an arbitrary element \( h_0 \in S_{A^{(1)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \).

Again, if calibration assumption (2.14) holds for a pair \( u_i, u_e \) then

\[
0 = \int_{\partial \Omega_m} h^* \left( h_0 + N^{(i)}((-\alpha_e/\alpha_i) \Delta^{(e)} u_e, \beta_i B_1^{(b)} u_b) + c_0 u_e \right) d\sigma(y) = \int_{\partial \Omega_m} (\Pi_0 h)^* \left( h_0 + c_0 u_e \right) d\sigma = \int_{\partial \Omega_m} h^* \left( h_0 + c_0 \Pi_0 u_b \right) d\sigma
\]

for any \( h \in S_{A^{(1)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \) because of normalising condition (2.12). Thus, the element \( h_0 \) in (2.27) may be uniquely defined by

\[
(2.30) \quad h_0 = -c_0 \Pi_0 u_b.
\]

Finally, chain of equalities (2.26) tell us that condition (2.21) is necessary for the solvability of problem (2.7), (2.9), (2.10), (2.11), that was to be proved. \( \square \)

As we have mentioned above, in the very particular case where, additionally, the operators \( A^{(e)}, A^{(i)}, A^{(b)} \) are proportional, similar theorems were proved in [24].

**Remark 2.6.** Assumptions (2.19), (2.22), (2.23) are fulfilled in the following reasonable situation:

\[
(2.31) \quad A^{(i)} = M^{(i)} A, \quad A^{(e)} = M^{(e)} A, \quad A^{(b)} = M^{(b)} A,
\]

where \( M^{(i)}, M^{(e)}, M^{(b)} \) are \((l \times l)\)-matrices with real analytic coefficients over \( U_m \) and \( U_b \), satisfying (1.2) and \( A \) is an \((l \times k)\) holonomic differential operator, i.e. it is an operator with constant coefficients such that for any domain \( D \subset \mathbb{R}^n \) we have

\[
S_A(D) = S_A(\mathbb{R}^n), \quad \dim(S_A(\mathbb{R}^n)) < \infty,
\]

and there is a positive constant \( c_A \) such that

\[
\|u\|_{H^1(\Omega)} \leq c_A \|Au\|_{L^2(\Omega)} \quad \text{for all } u \in (S_A(\Omega) \cap [H^1(\Omega)]^k)^\perp
\]

for any bounded domain \( \Omega \) with smooth boundary.

For instance, in the models of the electrocardiography we have \( A = \nabla \),

\[
S_A(D) = S_V(D) = S_V(\mathbb{R}^n) = S_A(\mathbb{R}^n) = \mathbb{R}, \quad \dim(S_A(\mathbb{R}^n)) = 1,
\]

for any domain \( D \subset \mathbb{R}^n \) and then

\[
S_{A^{(i)}}(\Omega_m) = S_{A^{(1)}}(\Omega) = S_{A^{(e)}}(\Omega_m) = S_{A^{(0)}}(\Omega_b) = \mathbb{R},
\]
Example 2.7. Consider the situation where the case \( A^{(i)} \), \( A^{(i)} \), and \( A^{(b)} \) satisfy (2.31) and the operators \( A^{(i)} \), \( A^{(i)} \) are proportional (cf. 11, 24). Then, in particular, the pair \((\Delta^{(i)} \sigma^{*}(A^{(i)})(u_{b})A^{(i)})\) satisfies Shapiro-Lopatinsky conditions in \( \Omega_{m} \), embeddings (2.19), (2.22), (2.23) hold true and

\[
\Delta^{(i)} = \gamma \Delta^{(i)} \quad \text{with some } \gamma > 0.
\]

Then, with any pair \((\Delta^{(i)}, B_{1}^{(i)})\) satisfying Shapiro-Lopatinsky conditions and (2.22), we have

\[
N^{(i)}(\Delta^{(i)}u_{b}, 0) = \gamma N^{(i)}(\Delta^{(i)}u, 0) = \gamma u
\]

for each \( u \in [H^{s}(\Omega_{m}) \cap H_{0}^{2}(\Omega_{m})]^{k} \). Thus, according to Theorem 2.2

\[
\begin{cases}
  u_{e} = u & \text{in } \Omega_{m},
  u_{i} = (-\alpha_{c}/\alpha_{i}) \gamma u + h_{0} & \text{in } \Omega_{m},
\end{cases}
\]

for each pair \( u_{i}, u_{e} \) from the null-space of Problem 1.1 where \( h_{0} \) is an arbitrary element of \( S_{A^{(i)}(\Omega_{m})} \cap [H^{s}(\Omega_{m})]^{k} \) and \( u \) is an arbitrary function from \([H^{s}(\Omega_{m}) \cap H_{0}^{2}(\Omega_{m})]^{k}\). Again, if calibration assumption (2.14) holds for the pair \( u_{i}, u_{e} \) from the null-space then the element \( h_{0} \) in (2.33) equals to zero.

As for the Existence Theorem, in this case

\[
N^{(i)}((-\alpha_{c}/\alpha_{i}) \Delta^{(i)}u_{e}, \beta_{i} B_{1}^{(b)}u_{b}) =
N^{(i)}((-\alpha_{c}/\alpha_{i}) \gamma \Delta^{(i)}u_{e}, \left((-\alpha_{c}/\alpha_{i}) \gamma \beta_{e} + \beta_{i} - (-\alpha_{c}/\alpha_{i}) \gamma \beta_{e}\right) B_{1}^{(b)}u_{b}) =
-(-\alpha_{c}/\alpha_{i}) \gamma u_{e} + (\beta_{i} + (\alpha_{c}/\alpha_{i}) \gamma \beta_{e}) N^{(i)}(0, B_{1}^{(b)}u_{b}).
\]

Thus, formula (2.29) has the form

\[
(3.4) \quad u_{i} = -(-\alpha_{c}/\alpha_{i}) \gamma u_{e} + (\beta_{i} + (\alpha_{c}/\alpha_{i}) \gamma \beta_{e}) N^{(i)}(0, B_{1}^{(b)}u_{b}) + h_{0}
\]

where \( h_{0} \) is an arbitrary element of \( S_{A^{(i)}(\Omega_{m})} \cap [H^{s}(\Omega_{m})]^{k} \); if (2.24) is fulfilled. Again, if calibration assumption (2.14) holds for the pair \( u_{i}, u_{e} \) then the element \( h_{0} \) in the last formula may be uniquely defined by (2.30).

Remark 2.8. We note that for \( \alpha_{i} \neq 0 \), similarly to the results of 11 Theorem 2.3 means that Problem 2.4 has too many degrees of freedom. Actually, for \( \alpha_{i} = 0 \) the problem is even more unbalanced. Indeed, using (2.5), (2.7), (2.10) and the Uniqueness Theorem 27 Theorem 2.8 for the Cauchy problem, it is easy to check that in this case the null-space of Problem 2.4 consists of all the triples \( u_{i}, u_{e}, u_{b} \) from \([H^{s}(\Omega_{m})]^{k} \times [H^{s}(\Omega_{m})]^{k} \times [H^{s}(\Omega_{m})]^{k}\) satisfying the following conditions:

\[
\begin{cases}
  u_{b} = 0 & \text{in } \Omega_{b},
  u_{e} = 0 & \text{in } \Omega_{m},
  u_{i} = u & \text{in } \Omega_{m},
\end{cases}
\]

where \( u \in [H^{s}(\Omega_{m})]^{k} \) is an arbitrary function satisfying

\[
B_{1}^{(i)} u = 0 \quad \text{on } \partial \Omega_{m}.
\]
On the other hand, as \( \alpha_e \neq 0 \), the existence of a solution to Problem 2.1 depends upon the solvability of the following Cauchy problem

\[
\begin{align*}
\Delta^{(e)} u_e &= 0 \quad \text{in } \Omega_m, \\
u_e &= u_b \quad \text{on } \partial \Omega_m, \\
u_e &= \beta_e B^{(e)}_1 u_b \quad \text{on } \partial \Omega_m,
\end{align*}
\]

(2.36)

corresponding to (2.8), (2.9), (2.10). Since the boundary data are given on all the surface \( \partial \Omega_m \), this Cauchy problem is normally solvable in the declared spaces, see [27, Theorems 2.8 and 5.2]. However it has a large co-kernel: it is solvable if and only if

\[
\int_{\partial \Omega_m} (B^{(e)}_1(y, \partial_y) \varphi_e(x, y))^* u_b(y) - \beta_e (\varphi_e(x, y))^* B^{(e)}_1 u_b(y) \, d\sigma(y) = 0
\]

for all \( x \in \Omega \). In particular, this implies that the data \( f_0 \in [H^{s-1/2}((\partial \Omega))], \, f_1 \in [H^{s-3/2}((\partial \Omega))] \), \( f \in [H^{s-2}(\Omega)] \), defining the function \( u_b \), can not be arbitrary. For this reason, we will concentrate our efforts on the case where \( \alpha_i \neq 0 \).

As we have noted in Remark 2.8 the null-space of Problem 2.1 is too large. Practically, this means that at least one equation related to the unknown vector functions in \( \Omega_m \) is still missing. On the other hand, the proof of Theorem 2.5 suggests us to supplement Problem 2.1 with a fourth order strongly elliptic equation

(2.37)

\[ Q u_e = g \quad \text{in } \Omega_m \]

with a given function \( g \) in \( \Omega_m \).

**Corollary 2.9.** Let \( s \geq 2 \), \( \alpha_i \neq 0 \), \( 2.22, \, \, 2.23, \, \, 2.24 \) be fulfilled and the pair \( (\Delta^{(e)}, B^{(e)}_1) \) satisfy Shapiro-Lopatinsky conditions in \( \Omega_m \), let embeddings \( 2.22, \, \, 2.23 \) hold true and the triple

\[ f \in [H^{s-2}(\Omega)] \], \( f_0 \in [H^{s-1/2}((\partial \Omega))] \), \( f_1 \in [H^{s-3/2}((\partial \Omega))] \)

admit the solution \( u_0 \in [H^s(\Omega)] \) to (2.8), (2.12), (2.13) and satisfy (2.24). If \( Q \) is a fourth order strongly elliptic operator over \( \Omega_m \), then, given vector \( g \in [H^{s-4}(\Omega_m)] \), problem (2.7), (2.9), (2.10), (2.11), (2.37) has the Fredholm property. If \( Q \) is a fourth order formally non-negative strongly elliptic operator with real analytic coefficients over \( \Omega_m \), then, given vector \( g \in [H^{s-4}(\Omega_m)] \), problem (2.7), (2.9), (2.10), (2.11), (2.37) has one and only one solution \( (u_i, u_e) \in [H^s(\Omega_m)]^{k} \times [H^s(\Omega_m)]^{k} \).

**Proof.** Recall that a problem related to an operator equation

\[ Ru = f \]

with a linear bounded operator \( R : X_1 \to X_2 \) in Banach spaces \( X_1, X_2 \) has the Fredholm property, if the kernel \( \ker(R) \) of the operator \( R \) and the cokernel \( \coker(R) \) (i.e. the kernel \( \ker(R^*) \) of its adjoint operator \( R^* : X^*_2 \to X^*_1 \)) are finite-dimensional vector spaces and the range of the operator \( R \) is closed in \( X_2 \).

Under the hypothesis of this corollary both Dirichlet problem (2.9), (2.10), (2.37), see, for instance, [22] and Neumann problem (2.7), (2.11), see, for instance, [28], have Fredholm property in the relevant Sobolev spaces. Hence the first part of the statement of the corollary is proved.

If we additionally assume that \( Q \) is a fourth order formally non-negative strongly elliptic operator with real analytic coefficients over \( \Omega_m \), then, given vector \( g \in [H^{s-4}(\Omega_m)] \),
problem Dirichlet problem (2.9), (2.10), (2.37) has one and only one solution \( u_e \in [H^s(\Omega_m)]^k \). Moreover, as we have seen in the proof of Theorem 2.5, Neumann problem (2.7), (2.11) is uniquely solvable in the space \([H^s(\Omega_m)]^k \). Thus, problem (2.7), (2.9), (2.10), (2.11), (2.14), (2.37) has one and only one solution \((u_i, u_e)\in [H^s(\Omega_m)]^k \times [H^s(\Omega_m)]^k \).

This finishes the proof of the corollary.\(\square\)

We emphasize that the Fredholm property for a problem is not always the desirable result in applications because of the possible lack of the uniqueness and possible absence of solutions. As the index (the difference between the dimensions of its kernel and co-kernel) of the Dirichlet problem in the standard setting equals to zero, the lack of uniqueness immediately implies some necessary solvability conditions applied to the given vector \( g \) in the Corollary 2.9.

Example 2.10. First of all, we note that Problem 1.1 is perfectly fit for the new more general model with \( \alpha_i = 1, \alpha_e = 1, \beta_e = -1, \beta_i = 0, f = 0, f_1 = 0 \) and \( A^{(i)} = M^{(i)} \nabla, A^{(e)} = M^{(e)} \nabla \ A^{(b)} = M^{(b)} \nabla \)
and \( M^{(i)}, M^{(e)}, M^{(b)} \) are \((l \times l)\)-matrices with real analytic coefficients over \( U_m \) and \( U_b \) satisfying (1.2). Comparing with the results of §1, we have
\[
M_i = (M^{(i)}(x))^2, M_e = (M^{(i)}(x))^2, M_b = (M^{(i)}(x))^2.
\]
i.e. we may consider matrices with real analytic entries.

Unfortunately, we do not know any published modifications of the standard bidomain model of the electrocardiography involving higher order strongly elliptic equation (2.37). The following example has been reported to us by Vitaly Kalinin\(^{1}\).

Namely, consider Problem 1.1 in the situation where assumption (1.17) is fulfilled. Next we assume that the function \( f_0 \) in (1.16) does not depend on the time variable \( t \), calibration condition (1.19) is fulfilled and that the following electrodynamic relation holds true for the steady current \( u_e \):
\[
(2.38) \quad \Delta u_e = -\frac{q_e}{\varepsilon \varepsilon_0}.
\]
Hence, substituting (2.38) into (1.4) and (1.5) we obtain formulas that can be useful if we need to transform evolutionary equations to stationary ones:
\[
(2.39) \quad \frac{\partial \Delta u_e}{\partial t} = -\frac{1}{\varepsilon \varepsilon_0} \left( \sigma_e \Delta u_e + \chi I_{ion} \right).
\]
Now, taking in account (1.7), cable equation (1.15) and (1.20) we obtain the following equation in the sense of distributions in \( \Omega_m \times (0, T) \):
\[
(2.40) \quad \frac{\sigma_l \sigma_e \varepsilon \varepsilon_0}{\sigma_e + \sigma_l} \Delta^2 u_e = -\chi C_m \sigma_e \Delta u_e - C_m \chi I_{ion} - \frac{\chi \sigma_i \sigma_e \varepsilon \varepsilon_0}{\sigma_e + \sigma_i} \Delta I_{ion}.
\]
If we are to stay within the framework of linear theory we may assume that the ionic current is given by
\[
(2.41) \quad I_{ion}(v) = \sum_{j=1}^{n} a_j \partial_j v + a_0 v + b
\]

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with some function $b \in L^2(\Omega)$, and some constants $a_j$, $0 \leq j \leq n$. Then, as the operator $\Delta^2$ is strongly elliptic, using (1.20) and (2.40) we arrive at the fourth order strongly elliptic equation

$$
(2.42) \quad \frac{\sigma \sigma_0 e \Delta u_c}{\sigma + \sigma_0} + \chi C_m \sigma_0 e \Delta u_c + \left( \frac{C_m (\sigma e + \sigma_i)}{\sigma_i} \right) \chi I_{ion}(u_c) + 
\chi \sigma_0 e \epsilon_0 \Delta I_{ion}(v_c)
$$

There is little hope that Dirichlet problem (2.42), (1.9), (1.10) is uniquely solvable, taking into account that the coefficient $\epsilon_0$ is practically very small. Hence we may grant the Fredholm property only for problem (1.7), (1.9), (1.10), (1.11), (2.42) even under calibration assumption (1.19).

Also we note that in the practical models of the electrocardiography the term $I_{ion}(v, x, t)$ is usually non-linear with respect to $v$. For general non-linear Fredholm problems one may provide under reasonable assumptions a discrete set of solutions only, see [30] for the second order elliptic operators in Hölder spaces. Thus one should specify the type of the non-linearities under the consideration. For example, in the models of the cardiology the non-linear term is often taken as a polynomial of second or third order with respect to $v$, see, for instance, [2], [32], though these choices do not fully correspond to the real processes in the myocardium.

Example 2.11. Consider the following $((n^2 + 1) \times n)$-matrix differential operator

$$
(2.43) \quad \mathcal{A} = \begin{pmatrix}
\nabla & 0 & 0 & \ldots & 0 \\
0 & \nabla & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & \nabla \\
\partial_1 & \partial_2 & \partial_3 & \ldots & \partial_n
\end{pmatrix}.
$$

Its symbol

$$
\sigma(\mathcal{A})(\zeta) = \begin{pmatrix}
\zeta & 0 & 0 & \ldots & 0 \\
0 & \zeta & 0 & \ldots & 0 \\
0 & 0 & \zeta & \ldots & 0 \\
0 & 0 & 0 & \zeta & 0 \\
\zeta_1 & \zeta_2 & \zeta_3 & \ldots & \zeta_n
\end{pmatrix}
$$

is injective for any $\zeta \in \mathbb{R}^n \setminus \{0\}$ because it contains submatrices of the type $\zeta_j I_n$, $1 \leq j \leq n$.

Obviously, the space of its solutions $S_{\mathcal{A}}(D)$ coincide with $\mathbb{R}^n$ and the operator is holonomic. Taking a diagonal $((n^2 + 1) \times (n^2 + 1))$-matrix

$$
(2.44) \quad \mathcal{M}(x) = \begin{pmatrix}
\mu(x) & 0 & 0 & \ldots & 0 \\
0 & \mu(x) & 0 & \ldots & 0 \\
\cdots & \cdots & \cdots & \ldots & \cdots \\
0 & 0 & 0 & \mu(x) & 0 \\
0 & 0 & \ldots & 0 & \lambda(x) + \mu(x)
\end{pmatrix}
$$

with real analytic entries $\lambda(x)$, $\mu(x)$ over a domain $X \subset \mathbb{R}^n$ we obtain $(n \times n)$-differential operator

$$
\mathcal{A}^* \mathcal{M} \mathcal{A} = -\left( \text{div}(\mu(x) I_n) \nabla + \nabla (\lambda(x) + \mu(x)) \text{div} \right).
$$
In many applications it is known as the Lamé operator; it is elliptic, strongly elliptic and formally non-negative if

\[
\begin{cases}
\mu(x) \geq m_0 \text{ for all } x \in X, \\
\lambda(x) + 2\mu(x) \geq m_0 \text{ for all } x \in X, \\
\lambda(x) + \mu(x) \geq 0 \text{ for all } x \in X,
\end{cases}
\]

where \( m_0 \) is a positive number, because with \( x \in X, \zeta \in \mathbb{R}^n, w \in \mathbb{C}^n \) we have:

\[
\sigma(\mathcal{A}^* \mathcal{M} \mathcal{A})(\zeta) = -\left(\mu(x)|\zeta|^2 I_n + (\lambda(x) + \mu(x))\zeta^T\right),
\]

\[
\det \sigma(\mathcal{A}^* \mathcal{M} \mathcal{A})(\zeta) = |\zeta|^{2n} \mu^{n-1}(x)(\lambda(x) + 2\mu(x)),
\]

\[-\Re\left(w^* \sigma(\mathcal{A}^* \mathcal{M} \mathcal{A})(\zeta) w\right) = \left(\mu(x)|\zeta|^2 |w|^2 + (\lambda(x) + \mu(x))|\zeta^T w|^2\right).
\]

If the functions \( \mu \) and \( \lambda \) are constant then its bilateral fundamental solution of convolution type is given by the Kelvin-Somigliana matrix \( \Phi(x) = (\Phi_{mj}(x)) \) with the components

\[
\Phi_{mj}(x) = \frac{1}{2\mu(\lambda + 2\mu)} \left(\delta_{mj} (\lambda + 3\mu)\varphi_n(x) - (\lambda + \mu) x_j \frac{\partial}{\partial x_m} \varphi_n(x)\right)
\]

where \( \delta_{mj} \) is the Kronecker delta, and \( \varphi_n(x) \) is the standard fundamental solution to the Laplace operator in \( \mathbb{R}^n \) (see, for example, [13, Part II, §2, (1.7)]).

As it is known from the linear Elasticity Theory (for \( n = 2 \) and \( n = 3 \)), the system of equations

\[\mathcal{A}^* \mathcal{M} \mathcal{A} u = f \text{ in } D\]

describes the displacement vector \( u(x) \) of points \( x \) of an elastic body \( D \) under the action of the force \( f(x) \); in these case \( \mu \) and \( \lambda \) are the so-called Lamé constants characterizing elastic properties of body’s material, see, for instance, [13, Ch. 1, §11, formula (11.7)].

Next, the matrix \( \mathcal{T} = (\mathcal{T}_{mj}(x)) \) with the entries

\[
\mathcal{T}_{mj}(x) = \mu \delta_{mj} \frac{\partial}{\partial x_j} + \lambda \nu_m(x) \frac{\partial}{\partial x_j} + \mu \nu_j(x) \frac{\partial}{\partial x_m} \quad (m, j = 1, ..., n),
\]

is known as the boundary stress operator near \( \partial D \) if \( \nu_i(x) \) are the components of the outward unit normal vector to \( \partial D \) at the point \( x \). Applying Ostrogradsky-Gauss formula we see that

\[
(2.45) \quad \int_D v^* \mathcal{T} u d\sigma = \int_D \left((\mathcal{A} v)^* \mathcal{M} \mathcal{A} u - v^* \mathcal{A}^* \mathcal{M} \mathcal{A} u\right) dx
\]

for all \( v \in [H^1(D)]^n, u \in [H^2(D)]^n \), i.e. we may consider Problem 2.1 for operators

\[A^{(i)} = M^{(i)} \mathcal{A}, \quad A^{(e)} = M^{(e)} \mathcal{A}, \quad A^{(b)} = M^{(b)} \mathcal{A}\]

related to operator (2.43) and square roots \( M^{(i)}, M^{(e)}, M^{(b)} \) of \((n \times n)\)-matrices \( \mathcal{M}^{(i)}, \mathcal{M}^{(e)}, \mathcal{M}^{(b)} \) given by (2.44) with Lamé constants \( \mu^{(i)}, \lambda^{(i)}, \mu^{(e)}, \lambda^{(e)}, \mu^{(b)}, \lambda^{(b)} \), respectively, and boundary first order operators

\[B_{1}^{(i)} = \mathcal{F}^{(i)}, \quad B_{1}^{(e)} = \mathcal{F}^{(e)}, \quad B_{1}^{(b)} = \mathcal{F}^{(b)}.
\]

The problem for an elastic composite body \( \Omega = \Omega_b \cup \overline{\Omega}_m \) then consists in the following:

1) the description of the displacement vector \( u_b \) of the ‘exterior’ elastic body \( \Omega_b \) by the known force \( f \) in \( \Omega_b \), the displacement \( f_0 = u_b \) and the stress \( \mathcal{T}^{(b)} u_b = f_1 \) on the surface \( \partial \Omega \), see (2.3), (2.12), (2.13).
2) the description of the displacement vectors \( u_i, u_e \) of the ‘interior’ composite body \( \Omega_m \subset \Omega \), where two more elastic materials are mixed in such a way that
a) the displacement vectors \( u_i, u_e \) inside \( \Omega_m \) are linked via some homogenization procedure with the use of equation (2.7);

b) relations \((2.9), (2.10), (2.11)\) connect the displacement \( u_b \) and the stress \( \mathbf{T}^{(b)}\) with the displacement \( u_e \), the stress \( \mathbf{T}^{(c)}\) and the stress \( \mathbf{T}^{(i)}\) on the surface \( \partial\Omega_m \).

The situation become sufficiently realistic if we assume \( \alpha_i = \alpha_e, \beta_i = 0, \beta_e = 1 \), i.e. the loads applied to different materials inside \( \Omega_m \) are the same, the stress \( \mathbf{T}^{(i)} \) equals to zero on the surface \( \partial\Omega_m \) (for instance, because of the corresponding material never contact with the surface) and \( \mathbf{T}^{(c)}\) on \( \partial\Omega_m \).

3. An evolutionary problem

We recall that the primary equations \( (1.3), (1.4), (1.5) \), leading to the classical steady bidomain model of the electrocardiography are actually evolutionary. That is why, let us obtain a uniqueness theorem for a generalized evolutionary problem, too.

With this purpose, we introduce the suitable spaces for investigation of parabolic equations, see, for instance, [15] Ch. 1.

For \( T > 0 \) we set \( \Omega_T = \Omega \times (0, T) \). Let \( C^{2s,s}(\Omega_T) \) be the set of all continuous functions \( u \) on \( \Omega_T \), having on \( \Omega_T \) continuous partial derivatives \( \partial_t^j \partial_x^s u \) for all multi-indices \( (\alpha, j) \in \mathbb{Z}^n_+ \times \mathbb{Z}_+ \), satisfying \( |\alpha| + 2j \leq 2s \). Clearly, for the cylinder domain \( \Omega_T \) we have \( \Omega_T = \Omega \times [0, T] \). Then \( C^{2s,s}(\Omega_T) \) denotes the subset in \( C^{2s,s}(\Omega_T) \), such that for any function \( u \in C^{2s,s}(\Omega_T) \) and any multi-index \( (\alpha, j) \in \mathbb{Z}^n_+ \times \mathbb{Z}_+ \), there is a function \( u_{\alpha,j} \), continuous on \( \Omega_T \) and such that \( \partial_t^j \partial_x^s u = u_{\alpha,j} \) in \( \Omega_T \).

Let us denote by \( H^{2s,s}(\Omega_T), s \in \mathbb{Z}_+ \), anisotropic (parabolic) Sobolev spaces, see, for instance, [15], i.e. the set of such measurable functions \( u \) on \( \Omega_T \) that the partial derivatives \( \partial_t^j \partial_x^s u \) belong to the Lebesgue space \( L^2(\Omega_T) \) for all multi-indices \( (\alpha, j) \in \mathbb{Z}^n_+ \times \mathbb{Z}_+ \) satisfying \( |\alpha| + 2j \leq 2s \). This is a Hilbert space with the inner product

\[(3.1) \quad (u, v)_{H^{2s,s}(\Omega_T)} = \sum_{|\alpha| + 2j \leq 2s} \int_{\Omega_T} \partial_t^j \partial_x^s v(x, t) \partial_t^j \partial_x^s u(x, t) dx dt.\]

We may also define \( H^{2s,s}(\Omega_T) \) as the completion of the linear space \( C^{2s,s}(\Omega_T) \) with respect to the norm \( \| \cdot \|_{H^{2s,s}(\Omega_T)} \) induced by the inner product \( (3.1) \). In particular, if \( s = 0 \) then \( H^{0,0}(\Omega_T) = L^2(\Omega_T) \).

We will also use the so-called Bochner spaces of functions depending on \( (x, t) \) over \( \Omega_T \). Namely, if \( \mathcal{B} \) is a Banach space (possibly, a space of functions over \( \Omega \)) and \( p \geq 1 \), we denote by \( L^p([0, T], \mathcal{B}) \) the Banach space of measurable maps \( u : [0, T] \to \mathcal{B} \) with the norm

\[ \|u\|_{L^p([0, T], \mathcal{B})} := \|u(\cdot, t)\|_{\mathcal{B}} \|L^p([0, T]), \text{see, for instance, [17] Ch. §1.2}.\]

Now, taking in account cable equation \( (1.16) \), we consider a modified Problem \( 2.1 \) adding the time variable \( t \in [0, T] \).

**Problem 3.1.** Let \( \alpha_i, \alpha_e, \beta_e, \beta_i, \mu_i, \mu_e \in \mathbb{R}, \alpha_i^2 + \alpha_e^2 \neq 0, \beta_i^2 + \beta_e^2 \neq 0, \mu_i^2 + \mu_e^2 \neq 0 \). Given vector functions

\[ f \in L^2(\Omega_m \times (0, T))^k, f_0 \in L^2([0, T], H^{3/2}(\partial\Omega_m))^k, f_1 \in L^2([0, T], H^{1/2}(\partial\Omega_m))^k, \]


and mapping \( I : [H^{2,1}(\Omega_m \times (0, T))]^k \rightarrow [L^2(\Omega_m \times (0, T))]^k \), find unknown vector functions \( u_b \in [H^{2,1}(\Omega_b \times (0, T))]^k \), \( u_i, u_e \in [H^{2,1}(\Omega_m \times (0, T))]^k \) satisfying

\[
(3.2) \quad \alpha_i \Delta^{(i)} u_i + \alpha_e \Delta^{(e)} u_e = 0 \text{ in } \Omega_m \times [0, T],
\]
\[
(3.3) \quad \Delta^{(b)} u_b = f \text{ in } \Omega_b \times [0, T],
\]
\[
(3.4) \quad u_e = u_b \text{ on } \partial \Omega_m \times [0, T],
\]
\[
(3.5) \quad B_1^{(e)} u_e = \beta_e B_1^{(b)} u_b \text{ on } \partial \Omega_m \times [0, T],
\]
\[
(3.6) \quad B_1^{(i)} u_i = \beta_i B_1^{(b)} u_b \text{ on } \partial \Omega_m \times [0, T],
\]
\[
(3.7) \quad B_2^{(b)} u_b = f_1 \text{ on } \partial \Omega \times [0, T],
\]
\[
(3.8) \quad u_b = f_0 \text{ on } \partial \Omega \times [0, T],
\]
\[
(3.9) \quad -\mu_i \Delta^{(i)} u_i + \mu_e \Delta^{(e)} u_e = \frac{\partial (u_i - u_e)}{\partial t} + I(u_i - u_e) \text{ in } \Omega \times (0, T).
\]

As in \( \text{[1, 2]} \), it is reasonable to supplement the problem with calibration assumption: there is a function \( c_0(t) \in C[0, T] \) such that

\[
(3.10) \quad \int_{\partial \Omega_m} h^*(y)(u_i(y, t) + c_0(t)u_e(y, t))d\sigma(y) = 0
\]

for all \( h \in S_{A^{(i)}}(\Omega_m) \cap [H^s(\Omega_m)]^k \) and for almost all \( t \in [0, T] \).

The further developments depend essentially on the structure of the mapping \( I \). We continue the discussion with the situation considered in Example \( \text{[2.7]} \).

**Theorem 3.2.** Let \( s \geq 1 \), \( \alpha_i \neq 0 \), \( \mu_i \alpha_e + \mu_e \alpha_i \neq 0 \), \( \text{[2.2], [2.3], [2.4]} \) hold true and the pair \((\Delta_i, B_1^{(i)})\) satisfy Shapiro-Lopatinsky conditions in \( \Omega_m \). Let also the coefficients of the operator \( \Delta_i \) constants. If \( \text{[2.32] and [3.10]} \) are fulfilled and

\[
(3.11) \quad I(v) = \sum_{j=1}^n a_j \partial_j v + a_0 v + g
\]

with some function vector \( g \in [L^2(\Omega_m \times (0, T))]^k \), and some \((k \times k)\)-matrices \( a_j \), \( 0 \leq j \leq n \), then Problem \( \text{[3.1]} \) has no more than one solution \((u_i, u_e, u_b)\) in the space \([H^{2,1}(\Omega_m \times (0, T))]^k \times [H^{2,1}(\Omega_m \times (0, T))]^k \times [H^{2,1}(\Omega_b \times (0, T))]^k \).

**Proof.** Fix vector functions \( f \in [L^2(\Omega_m \times (0, T))]^k \), \( f_0 \in L^2([0, T], H^{3/2}(\partial \Omega)) \), \( f_1 \in L^2([0, T], H^{1/2}(\partial \Omega)) \), admitting a solution \( u_b \in H^{2,1}(\Omega_b \times (0, T)) \) to \( \text{[3.3]}, \text{[3.7]}, \text{[3.8]} \). Let \((\tilde{u}_i, \tilde{u}_e, \tilde{u}_b)\) and \((\hat{u}_i, \hat{u}_e, \hat{u}_b)\) be two solutions to Problem \( \text{[3.1]} \). Then the
vector \((w_i, w_e, w_b) = (\hat{u}_i, \hat{u}_e, \hat{u}_b)\) satisfies
\begin{align}
(3.12) & \quad \alpha_i \Delta^{(i)} w_i + \alpha_e \Delta^{(e)} w_e = 0 \text{ in } \Omega_m \times [0, T], \\
(3.13) & \quad \Delta^{(b)} w_b = 0 \text{ in } \Omega_b \times [0, T], \\
(3.14) & \quad w_e = w_b \text{ on } \partial \Omega_m \times [0, T], \\
(3.15) & \quad B^{(e)} w_e = \beta_e B^{(b)} w_b \text{ on } \partial \Omega_m \times [0, T], \\
(3.16) & \quad B_1^{(i)} w_i = \beta_i B_1^{(b)} w_b \text{ on } \partial \Omega_m \times [0, T], \\
(3.17) & \quad B_1^{(b)} w_b = 0 \text{ on } \partial \Omega \times [0, T], \\
(3.18) & \quad w_b = 0 \text{ on } \partial \Omega \times [0, T], \\
(3.19) & \quad -\mu_i \Delta^{(i)} w_i + \mu_e \Delta^{(e)} w_e = \frac{\partial(w_i - w_e)}{\partial t} + I(w_i - w_e) \text{ in } \Omega \times (0, T).
\end{align}

the last equation being satisfied in \(\Omega \times (0, T)\). Then by Theorem 2.4 we have
\begin{align}
(3.20) & \quad \left\{ \begin{array}{ll}
w_b(x, t) = 0 & \text{if } (x, t) \in \Omega_b \times [0, T], \\
w_e(x, t) = w & \text{if } (x, t) \in \Omega_m \times [0, T], \\
w_i(x, t) = \mathcal{N}^{(i)}((\alpha_e/\alpha_i)^2 \Delta^{(e)} w(-t, 0))(x) & \text{if } (x, t) \in \Omega_m \times [0, T],
\end{array} \right.
\end{align}

where, as before, \(\mathcal{N}^{(i)}\) is the Neumann operator related to \(\Delta^{(i)}\) and \(w\) is a function from the space \(L^2([0, T], [H^2_0(\Omega_m)]^k) \cap [H^{2s, \gamma}(\Omega_m \times (0, T))]^k\) providing that parabolic equation (3.19) is fulfilled and calibration assumption (3.10) holds true.

Since \(\Delta^{(i)} = \gamma \Delta^{(i)}\) with some \(\gamma > 0\), then, according to (2.33) and (3.20), we have
\begin{align}
(3.21) & \quad \left\{ \begin{array}{ll}
w_b(x, t) = 0 & \text{if } (x, t) \in \Omega_b \times [0, T], \\
w_e(x, t) = w & \text{if } (x, t) \in \Omega_m \times [0, T], \\
w_i(x, t) = (-\alpha_e/\alpha_i)^2 \gamma w & \text{if } (x, t) \in \Omega_m \times [0, T],
\end{array} \right.
\end{align}

where \(w\) is a function from \(L^2([0, T], [H^2_0(\Omega_m)]^k) \cap [H^{2s, \gamma}(\Omega_m \times (0, T))]^k\) satisfying the following reduced version of equation (3.19):
\begin{align}
(3.22) & \quad (1 + (\alpha_e/\alpha_i)^2) \frac{\partial w}{\partial t} + (\mu_i (\alpha_e/\alpha_i) + \mu_e) \Delta^{(e)} w = \left(I(\hat{u}_i - \hat{u}_e) - I(\hat{u}_i - \hat{u}_e)\right) \text{ in } \Omega_m \times (0, T).
\end{align}

Clearly, (3.23)
\begin{align}
(3.23) & \quad \hat{v} - \tilde{v} = (\hat{u}_i - \hat{u}_e) - (\tilde{u}_i - \tilde{u}_e) = w_i - w_e = -(\alpha_e/\alpha_i)^2 \gamma + 1)w,
\end{align}

and then (3.11), (3.12), (3.22) imply the following relation in \(\Omega_m \times (0, T)\):
\begin{align}
(3.24) & \quad (\alpha_i + \alpha_e \gamma) \frac{\partial w}{\partial t} + (\mu_i \alpha_i + \mu_e \alpha_e) \Delta^{(e)} w + (\alpha_i + \alpha_e \gamma) \left(\sum_{j=1}^{n} \alpha_j \partial_j w + a_0 w\right) = 0.
\end{align}

If \((\alpha_i + \alpha_e \gamma) = 0\) then \(w(t) = 0\) for each \(t \in [0, T]\) because the Uniqueness Theorem for the Cauchy problem for the second order elliptic operator \(\Delta^{(e)}\), see [27] Theorem 2.8), for \(w \in [H^2_0(\Omega_m)]^k\) and \((\mu_i \alpha_e - \mu_e \alpha_i) \neq 0\).

If \((\alpha_i + \alpha_e \gamma) \neq 0\) then, similarly to elliptic theory, we may use integral representations in parabolic (backward parabolic) theory. Namely, consider the following differential operator
\begin{align}
(3.25) & \quad \mathcal{L} = \frac{\partial}{\partial t} + (\mu_i \alpha_i + \mu_e \alpha_e)(\alpha_i + \alpha_e \gamma)^{-1} \Delta^{(e)} + \sum_{j=1}^{n} a_j \partial_j + a_0
\end{align}
constant coefficients. It is parabolic if \( (\mu_c \alpha_i + \mu_s \alpha_e) (\alpha_i + \alpha_e \gamma)^{-1} > 0 \) and it is backward parabolic if \( (\mu_c \alpha_i + \mu_s \alpha_e) (\alpha_i + \alpha_e \gamma)^{-1} < 0 \).

We proceed with the parabolic case because the arguments for the backward parabolic are similar. Indeed, under the assumptions above \( L \) admits a fundamental solution, say, \( \Psi \).

Proof. See, for instance, [33, Ch. 6, §1.5, Theorem 2.8], [34, Theorem 2.4.8], and hence it admits a suitable integral formula. Namely, denote by \( S \) a relatively open subset of \( \partial \Omega \) and set \( S_T = S \times (0, T) \). For functions \( g \in [L^2(\Omega_T)]^k, v \in L^2([0, T], [H^{1/2}(\partial \Omega)]^k), w \in L^2([0, T], [H^{3/2}(\partial \Omega)]^k) \), \( h \in [H^{1/2}(\Omega)]^k \) we introduce the following potentials:

\[
I_\Omega(h)(x, t) = \int_{\Omega} \Psi_L(x, y, t) h(y, 0) dy,
\]

\[
G_\Omega(f)(x, t) = \int_{\Omega}^{t} \int_{\partial \Omega} \Psi_L(x, y, t, \tau) f(y, \tau) dy d\tau,
\]

\[
V_S(v)(x, t) = \int_{S_T} \tilde{B}_0(y) \Psi_L(x, y, t, \tau) v(y, \tau) dy d\tau,
\]

\[
W_S(w)(x, t) = -\int_{S_T} \tilde{B}_1(y) \Psi_L(x, y, t, \tau) w(y, \tau) dy d\tau,
\]

(see, for instance, [7, Ch. 1, §3 and Ch. 5, §2]), where \( \tilde{B} = (\tilde{B}_0, \tilde{B}_1) \) is the dual Dirichlet pair for the elliptic operator

\[
D = (\mu_c \alpha_i + \mu_s \alpha_e)(\alpha_i + \alpha_e \gamma)^{-1} \Delta^{(e)} + \sum_{j=1}^{n} a_j \partial_j + a_0
\]

and the Dirichlet pair \( B = (I_k, B^{(e)}_i) \) over \( \partial \Omega \), i.e.

\[
\int_{\partial \Omega} ((\tilde{B}_1 v)^* u + (\tilde{B}_0 v)^* B^{(e)}_i u) d\sigma = \int_D (v^* Du - (D^* v)^* u) dx.
\]

for all \( u, v \in [C^\infty(\Omega)]^k \), see, for instance, [33, Lemma 8.3.3], [36, Lemma 9.27].

By the construction, all these potentials are (improper) integral depending on the parameters \((x, t)\).

Next, we formulate the so-called Green formula for the parabolic operator \( L \).

**Lemma 3.3.** Assume that the coefficients of the operator \( \Delta_i \) and \( a_j, 0 \leq j \leq n \), are constant. Then for all \( T > 0 \) and all \( u \in [H^{2,1}(\Omega_T)]^k \) the following formula holds:

\[
(3.26) \quad u(x, t), (x, t) \in \Omega_T \setminus 0, (x, t) \notin \Omega_T \to \left( I_{\Omega}(u) + G_{\Omega}(Lu) + V_{\partial \Omega}(\partial_{e,m} u) + W_{\Omega}(u) \right)(x, t).
\]

**Proof.** See, for instance, [33, Ch. 6, §12] or [34, Theorem 2.4.8] even for more general linear operators admitting fundamental solutions or parametrix.

Taking into account Green formula [33, 20], we obtain

\[
(3.27) \quad w(x, t), (x, t) \in \Omega_m \times (0, T) \setminus 0, (x, t) \notin \Omega_m \times [0, T) \to I_{\Omega_m}(w)(x, t).
\]
It is well known that the fundamental solution $\Psi_L(x,t)$ is real analytic with respect to the space variable $x$ for each $t > 0$, see [5], [31]. In particular, this means that the potential $I(\Omega)(u)(x,t)$ is real analytic with respect to $x$ for each $t > 0$, too. However, according to (3.27), it equals to zero outside $\Omega_T$. Therefore it is identically zero for each $t > 0$ and then $w \equiv 0$ in $\Omega_T$, cf. [14], [21] for the similar uniqueness theorem related to the heat equation and the parabolic Lamé type systems.

Finally, we see that $(w_i, w_e, w_b) = 0$ because of (3.21). $\square$

Again, we note that Problem 1.1 (i.e. the inverse problem of the electrocardiography) supplemented with cable equation (1.13) is perfectly corresponds to Problem 3.1 with $\alpha_i = 1$, $\alpha_e = 1$, $\beta_e = -1$, $\beta_i = 0$, $\mu_i = 1$, $\mu_e = 1$, $f = 0$, $f_1 = 0$ and the specific choice of the Laplacians $\Delta^{(i)}$, $\Delta^{(c)}$, $\Delta^{(b)}$ as in Example 2.10.

On the other hand, we see that Problem 3.1 can be easily adopted to many models involving diffusion equations. Again, in the practical models of such kind the term $I(v, x, t)$ is usually non-linear with respect to $v$. Thus, the uniqueness and the existence theorems to Problem 3.1 under assumptions (2.32) and (3.10) are closely related to these type of theorems for the following non-standard Cauchy problem for a quasilinear parabolic equation:

\begin{equation}
\left\{
\begin{array}{ll}
\mathcal{L}v = F(v) & \text{in } \Omega_m \times (0,T), \\
v = g_0 & \text{on } \partial\Omega_m \times [0,T], \\
B^{(c)}_1 v = g_1 & \text{on } \partial\Omega_m \times [0,T],
\end{array}
\right.
\end{equation}

with some data $g_0, g_1$ and (possibly, non-linear) term $F$. Even in the case where $F$ is linear with respect to $u$, problem (3.28) might be ill-posed in some cases, cf., [14], [21]. Thus, for both linear and the non-linear case, a thorough investigation of (3.28) is necessary. It looks that the problem can be treated with the use of Cauchy-Kowalevskaya theorem. But in a matter of facts, the question is much more delicate because the Cauchy-Kowalevskaya theorem is related to real analytic solutions for real analytic data in a small neighborhood of a real analytic surface. Even if we assume that the surface $\partial\Omega_m$ and the data are real analytic, the structure of the fundamental solutions to parabolic equations with constant coefficients makes us admit that solutions of such equations are often real analytic with respect to the space variables $(x_1, \ldots, x_n)$ but unlikely to be analytic with respect the time variable $t$.

However, as the primary goals of the paper were uniqueness theorems, it is worth to say that the real analyticity of solutions to (3.28) with respect to the space variables would leave us a good hope for a uniqueness theorem for Problem 3.1 in a non-linear situation, too.

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