Tests of the naturalness of the coupling constants in ChPT at order $p^6$

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Abstract. We derive constraints on combinations of $O(p^6)$ chiral coupling constants by matching a recent two-loop calculation of the $\pi K$ scattering amplitude with a set of sum rules. We examine the validity of the natural expectation that the values of the chiral couplings can be associated with physics properties of the light resonance sector. We focus, in particular, on flavour symmetry breaking of vector resonances. A resonance chiral Lagrangian is constructed which incorporates flavour symmetry breaking more completely than was done before. We use $\pi K$ unsubtracted sum rules as tests of the modelling of the resonance contributions to the chiral couplings. In some cases the $O(p^6)$ couplings are found not to be dominated by the resonance contributions.

PACS. 12.39 Fe Chiral Lagrangians – 11.55 Hx Sum rules – 13.75 Lb Meson-meson interactions

1 Introduction

An important progress in the description of QCD via effective theories was achieved by the extension of the chiral expansion formalism$^{[12]}$ to the order $p^6$. This raises the hope of attaining high precisions in the description of low energy physics using the chiral expansion, even in the case of the three flavour expansion which is expected to converge more slowly than the two flavour one. A large number of quantities have already been computed at chiral order six starting from the work of ref.$^{[7]}$. Some representative examples concerning the two-flavour case are in refs.$^{[8]}$ and in the three-flavour case in refs.$^{[10]1112131415}$.

In practice, including the $O(p^6)$ corrections was shown to clearly bring significant improvement for the two-flavour expansion.$^{[9]}$. In this case, the corrections are dominated by the chiral logarithms, the coefficients of which are known in terms of the $O(p^4)$ and $O(p^5)$ couplings$^{[10]}$, while the corrections proportional to the $O(p^6)$ couplings are comparatively smaller. The situation for the three-flavour expansion is different, in that the role of the $O(p^6)$ couplings is much more important. As an example, in order to determine the CKM matrix element $V_{us}$ at the one percent level based on experimental data on $K \rightarrow \pi l \nu$ decays it is necessary to know the values of the two LEC’s $C_{12}^r$ and $C_{34}^r$ (see e.g.$^{[15]}$).

As far as only the order of magnitude of the chiral LEC’s is concerned, it is possible to make very simple and general statements$^{[17]13}$. The order of magnitude can be argued to depend only on $F_\pi$ and on the chiral scale $\Lambda \sim M_\rho$, such that the typical size of the $O(p^4)$ LEC’s should be $L_i \sim F_\pi^2 / M_\rho$ and that of the $O(p^6)$ LEC’s should be $C_i \sim F_\pi^2 / M_\rho^4$. The natural question which arises, then, is whether it is possible to make more quantitative estimates relating the values of the LEC’s to known properties of the light resonances in the QCD spectrum. A detailed study along this line was performed in ref.$^{[19]}$ in which it was observed that it is indeed possible to reproduce the values of the $O(p^4)$ LEC’s $L_i (\mu)$ with $\mu = M_\rho$, which had previously been determined in a model independent way$^{[19]}$, in terms of observables from the light resonance sector.

A justification for such a relationship is provided by the chiral sum rules (see e.g.$^{[2]}$ for a list). A typical example, which was analyzed in refs.$^{[20]21}$ is the LEC $L_{10}^r$ which can be expressed as a convergent integral in terms of spectral functions which can be determined experimentally from $\tau$ decays. To a good approximation, the integral is found to be saturated by the contributions from the $\rho(770)$ and $a_1(1230)$ mesons. In more complicated situations, for which the integrands cannot easily be measured, one can appeal to the large $N_c$ expansion. Indeed, at leading order in $1/N_c$ QCD can be re-expressed in terms of a Lagrangian involving an infinite set of weakly interacting mesons$^{[22]}$. The precise form of this Lagrangian is not yet known from first principles, but the weak coupling property allows one to relate the coupling constants to observables using tree level calculations, and then deduce the values of these observables from experiment.

How well does resonance saturation perform in determining the size of the $O(p^4)$ LEC’s $C_i^r$ is not known at present. The main reason is that very few of these LEC’s have been determined so far. The purpose of this paper is to derive some constraints on the LEC’s $C_i^r$ obtained by equating the $\pi K$ scattering amplitude in the subthreshold region, as constructed from experimental data in ref.$^{[23]}$, with the chiral expansion calculation up to order $p^6$ which was performed in ref.$^{[15]}$ (previous work comparing dispersive representations with the chiral
expansion up to order $p^4$ \cite{24} was performed in refs. \cite{25,26}. Some of the $\pi K$ subthreshold expansion parameters can be expressed as unsubtracted sum rules. Such expressions allow one to identify resonance contributions from experiment. We will use such results to compare with the same resonance contributions as computed starting from a large $N_c$ type resonance chiral Lagrangian. We will concentrate here on the vector resonances.

We finally identify a combination of LEC’s which should be complete contribution of order $p^6$ in terms of the basis of ref. \cite{6} isovector symmetry breaking (to first order) in a more general model for the vector meson resonances which implements a model for the vector meson resonances and from scalar mesons. Describing scalar mesons starting from a resonance chiral Lagrangian presents several difficulties, notably in identifying the properties of the nonet in the chiral limit and in the treatment of the wide resonances. For this reason, we will concentrate here on the vector resonances.

The plan of the paper is as follows. We start by recalling some aspects of the correspondence between the expansion parameters around the subthreshold point $t = 0, s - u = 0$ and the $O(p^6)$ LEC’s. Results concerning the LEC’s $C_1^+$ to $C_4^+$ (which are associated with six derivatives chiral operators) are then presented. We next consider $\pi K$ subthreshold parameters associated with chiral operators involving four derivatives plus one quark mass matrix. In this sector, serious discrepancies are observed between the chiral predictions and the sum rule results. We point out some deficiencies of the resonance model employed in ref. \cite{15} for the relevant $O(p^6)$ LEC’s and propose a model for the vector meson resonances which implements flavour symmetry breaking (to first order) in a more general way. A phenomenological determination of all the parameters entering this resonance Lagrangian is performed and the complete contribution of order $p^6$ in terms of the basis of ref. \cite{6} is worked out. Tests of this modelling are performed by comparing with resonance contributions in unsubtracted sum rules. We finally identify a combination of LEC’s which should be weakly sensitive to the scalar resonance sector and discuss the result.

2 Results on $C_1^+$ to $C_4^+$

2.1 Notation

At first, let us recall some standard results and notation concerning the $\pi K$ scattering amplitude. Assuming isospin symmetry to be exact, $\pi K$ scattering is described in terms of two independent isospin amplitudes $F^I(s, t, u)$, with $I = 1/2, 3/2$ and the Mandelstam variables, $s, t, u$, satisfy

$$s + t + u = 2\Sigma, \quad \Sigma = m_K^2 + m_\pi^2.$$ (1)

Under $s, u$ crossing the following relation holds,

$$F^\frac{1}{2}(s, t, u) = - \frac{1}{2} F^\frac{1}{2}(s, t, u) + \frac{3}{2} F^\frac{1}{2}(u, t, s).$$ (2)

It is then convenient to form the two combinations $F^+$ and $F^-$ which are respectively even and odd under $s, u$ crossing,

$$F^+(s, t, u) = \frac{1}{3} F^\frac{1}{2}(s, t, u) + \frac{2}{3} F^\frac{1}{2}(s, t, u)$$

$$F^-(s, t, u) = \frac{1}{3} F^\frac{1}{2}(s, t, u) - \frac{1}{3} F^\frac{1}{2}(s, t, u).$$ (3)

Under $s, t$ crossing $F^+$ and $F^-$ are simply proportional to the $I = 0$ and the $I = 1 \pi \pi \to K\eta$ amplitudes,

$$G^0(t, s, u) = \sqrt{6} F^+(s, t, u)$$

$$G^1(t, s, u) = 2 F^-(s, t, u).$$ (4)

A region where one expects ChPT to apply is around the subthreshold point $t = 0, s - u = m_K^2 + m_\pi^2$. The $\pi K$ amplitude can be characterized in the neighbourhood of this point by performing an expansion in powers of $t$ and $s - u$ \cite{27}. The subthreshold coefficients $C_{ij}$ are dimensionless quantities defined from this expansion

$$F^+(s, t, u) = \sum_{ij} C_{ij} \frac{t^{ij} u^{2j}}{m_\pi^2 + m_\eta^2},$$

$$F^-(s, t, u) = \sum_{ij} C_{ij} \frac{t^{ij} u^{2j}}{m_\pi^2 + m_\eta^2},$$ (5)

with

$$\nu = \frac{s - u}{4m_K}. \quad (6)$$

2.2 Chiral $O(p^6)$ tree level contributions to the sub-threshold coefficients

The contributions at tree level from the $O(p^6)$ chiral Lagrangian to the sub-threshold coefficients have been worked out in \cite{15} and can be found explicitly in this reference. We will discuss what can be learned about the $O(p^6)$ LEC’s $C_1^+$ from these expressions. Let us begin by noting some general properties of the correspondence between the sub-threshold coefficients and the LEC’s. At first, the coefficients such that

$$C_{ij}^+: i + 2j \geq 4, \quad C_{ij}^-: i + 2j \geq 3$$ (7)

get no contribution at all from the $O(p^6)$ LEC’s. This implies that the chiral expressions at order $p^6$ for these coefficients obey convergent unsubtracted dispersions relations. As a simple example $C_{ij}^{++}$ can be written as (which is easily derived from eq. \cite{19} below),

$$C_{ij}^{++} = \frac{32}{9} \frac{m_\pi^4}{\pi} \int_{m_\pi^2}^{\infty} \frac{ds}{s^2} \left[ \frac{I m F^+(s', 0)_{p^4 + p^6}}{(s' - \Sigma)^2} \right]$$ (8)

(with $m_{\pi} = m_K + m_\pi$). In this expression one can compute $I m F^+(s', 0)_{p^4 + p^6}$ by expanding over partial waves and, for each partial-wave amplitude, using the chiral expansion of the unitarity relation

$$I m f^{ij}_i(s')_{p^4 + p^6} = \sqrt{\frac{s}{s'}} f_i^j(s')_{p^2} \left[ f_i^j(s')_{p^2} + 2 \text{Re} f_i^j(s')_{p^2} \right].$$ (9)
In this manner, we could reproduce precisely the numerical result $C_{30}^{+} = 0.23$ obtained in ref.\textsuperscript{[15]}.

Next, the chiral expressions for the set of coefficients which satisfy
\begin{equation}
C_{ij}^+ : i + 2j = 3, \quad C_{ij}^- : i + 2j = 2
\end{equation}
involve the four LEC’s $C_{11}^+, C_{12}^+, C_{20}^+, C_{30}^+$\textsuperscript{[15]} which are associated with the following four chiral Lagrangian terms (the definitions of the various chiral building blocks $u_µ$, $h_µν$, etc... which appear below can be found, for instance, in ref.\textsuperscript{[6]})
\begin{align}
O_1 &= (u_µ u_ν h_µν h_λν) \\
O_2 &= (u_µ u_ν)(h_µν h_λν) \\
O_3 &= (h_µν u_ν h_µν u_µ) \\
O_4 &= (h_µν(u_µ h_µν u_ν + u_ν h_νμ u_µ)) .
\end{align}

These terms contain six derivatives and do not involve quark masses. We will discuss below the determination of these LEC’s obtained from the sub-threshold $πK$ amplitudes as well as from $ππ$ amplitudes.

We next consider the sub-threshold coefficients which satisfy
\begin{equation}
C_{ij}^+ : i + 2j = 2, \quad C_{ij}^- : i + 2j = 1 ,
\end{equation}
i.e. the three coefficients $C_{20}^+, C_{01}^+, C_{10}^-$. Their chiral expansions involve, in addition to $C_{11}^+, C_{12}^+, C_{20}^+$ the eight LEC’s $C_{5}^+ \cdots C_{8}^+$, $C_{10}^- \cdots C_{13}^-$ and the three LEC’s $C_{22}^+ , C_{23}^+ C_{25}^-$. We reproduce the corresponding Lagrangian terms below for the convenience of the reader
\begin{align}
O_5 &= \langle (u_µ u_µ)^2 \rangle \chi_+ \\
O_7 &= \langle (u_µ u_µ)\{u_λ u_ν\} \rangle \chi_+ \\
O_9 &= \langle u_µ u_µ u_ν u_ν u_λ u_λ \rangle \\
O_{11} &= \langle \chi_+ \rangle \langle (u_µ u_µ u_ν u_ν u_λ u_λ) \chi_+ \rangle \\
O_{12} &= \langle h_µν u_ν h_µν \chi_+ \rangle \\
O_{13} &= \langle h_µν u_ν h_µν \chi_+ \rangle \\
O_{22} &= \langle \chi_+ \{h_µν u_ν h_µν u_λ u_λ\} \rangle \\
O_{23} &= \langle \chi_+ h_µν u_ν h_µν \chi_+ \rangle .
\end{align}

These terms contain four derivatives and a single insertion of the quark mass matrix. The information provided by the $πK$ amplitude is not sufficient to determine separately all these LEC’s. Previously, the LEC $C_{12}$ (as well as the LEC $C_{34}$) have been determined based on the $\Delta S = 1$ scalar form-factor\textsuperscript{[28]} by combining the chiral $O(p^4)$ calculations of ref.\textsuperscript{[13]} with the dispersive construction method of ref.\textsuperscript{[29]}. Constraints on $C_{12}$ and $C_{13}$ have also been obtained from $\Delta S = 0$ scalar form-factors\textsuperscript{[14]}.

Finally, the coefficients with $i + 2j = 0, 1$: $C_{00}^+, C_{10}^- , C_{00}^-$ involve thirteen more $O(p^4)$ LEC’s among those associated with chiral Lagrangian terms containing two or three insertions of the quark mass matrix.

2.3 Determination of $C_{1}^+ \cdots C_{6}^+$

The set of sub-threshold coefficients defined in eq.\textsuperscript{[10]} constrain the values of the four LEC’s $C_{1}^+ \cdots C_{6}^+$. The relevant formulas from ref.\textsuperscript{[15]} read
\begin{align}
C_{30}^+|_{C_i} &= \frac{1}{2} (-7C_{1}^r - 32C_{2}^r + 2C_{3}^r + 10C_{4}^r) \frac{m_K^6}{F_π^4} \\
C_{11}^+|_{C_i} &= 8 (3C_{1}^r + 6C_{2}^r - 2C_{3}^r) \frac{m_K^4 m_K}{F_π^4} \\
C_{20}^-|_{C_i} &= 6 (-C_{1}^r + 2C_{3}^r + 2C_{4}^r) \frac{m_K^3 m_K}{F_π^4} .
\end{align}

The $O(p^4)$ LEC’s $L_i^r$ contribute to these coefficients only via one-loop diagrams such that one may use for the $L_i^r$ the numerical values determined at $O(p^6)$. The last two equations\textsuperscript{[13]} involve the same combination of LEC’s $C_{1}^r$. Therefore, we can determine three combinations, for instance $C_{1}^r + 4C_{5}^r$, $C_{2}^r$, $C_{3}^r + 3C_{5}^r$.

Independent informations are provided by $π π$ scattering. The $π π$ amplitude constrains the two combinations $C_{4}^r + 3C_{5}^r$ and $C_{1}^r + 4C_{3}^r + 2C_{5}^r$. The numerical values which we quote in table\textsuperscript{[1]} make use of the expressions from\textsuperscript{[30,31]}
\begin{align}
r_5^r &= F_π^2 (-8C_{1}^r - 16C_{2}^r + 10C_{3}^r + 14C_{4}^r) + 23F_π^2 \frac{1}{15360} m_K^4 + \log s \\
r_6^r &= F_π^2 (6C_{2}^r + 4C_{3}^r) + \frac{F_π^2}{15360} m_K^4 + \log s
\end{align}
and the numerical values for $r_5^r, r_6^r$ obtained in ref.\textsuperscript{[32]} from a Roy equations analysis. The right-hand sides of eqs.\textsuperscript{[15]} involve a quadratic polynomial in $\log (m_K^2/\mu^2)$ and $\log (m_K^2/\mu^2)$ which we have not determined. We have attempted to minimize its influence by performing the matching at a scale $\mu^2 = m_K m_p$ before evolving the scale to $M_p$.

It is of interest to compare these results from those of the resonance saturation model. In the case of $C_{1}^r , \ldots , C_{6}^r$ it suffices to consider resonances in the chiral limit as was the case for the $O(p^4)$ LEC’s\textsuperscript{[19]}. If one uses simply the same Lagrangian as in ref.\textsuperscript{[19]} (which was also used in the $πK$ analysis of ref.\textsuperscript{[15]}) one obtains
\begin{align}
C_{1}^{V + S} &= \frac{G_V^2}{8M_V^4} - \frac{c_4^2}{4M_S^4} , \quad C_{2}^{V + S} = 0 \\
C_{2}^{V + S} &= \frac{c_4}{12M_S^4} - \frac{c_4^2}{4M_S^4} , \quad C_{4}^{V + S} = \frac{G_V^2}{8M_V^4} .
\end{align}

Contributions from resonance Lagrangian terms like
\begin{align}
\langle \nabla ^λ V_μ [h_{νμ}, u_ν^r] \rangle , \quad \langle \nabla ^λ V_μ [h_{νμ}, u_ν^r] \rangle
\end{align}
should, in principle, also be considered but we will not do so here\textsuperscript{1}. In discussing such higher derivative terms, it is important to implement proper asymptotic conditions.

\textsuperscript{1} While this paper was being completed a preprint appeared\textsuperscript{[33]} containing a general discussion of resonance Lagrangian terms contributing at order $p^6$. 

Numerical values are shown in table 1, using the same values for the couplings as in ref.[19], i.e.

\[ G_V = 53 \text{ MeV}, \ c_d = 32 \text{ MeV}, \ c_m = 42 \text{ MeV}. \]  

(18)

We note that this value of \( G_V \) is somewhat smaller than the one which derives from the \( \rho \to 2\pi \) width \((G_V \sim 64.1 \text{ MeV}, \) see sec.3\) but was shown to yield good results for the \( O(p^4) \) LEC's. In the case of \( C_2 \), which is OZI suppressed we show, for illustration, the value derived from the OZI violation model A of ref.[23]. The table also shows that the results obtained using \( C_{20} \) as input and those using \( C_{01} \) are compatible for \( C_1^+ + 4C_3^+ \) and for \( C_2^+ \) but not quite so for \( C_4^+ + 3C_5^+ \). The error, however, does not take higher order chiral effects into account. The results which use \( C_{01} \) are compatible with the \( \pi\pi \) results.

The simplest resonance saturation model is seen to give correct signs and order of magnitudes for the LEC’s shown in table 1 but the agreement is certainly not as good as in the case of the \( O(p^4) \) couplings.

### 3 Symmetry breaking in the vector meson chiral Lagrangian revisited

#### 3.1 Observation of some discrepancies

Let us now turn our attention to the three coefficients \( C_{20}^+, C_{01}^+, \) and \( C_{10}^− \). As mentioned above, their chiral expansions get tree level contributions from the Lagrangian terms, \( O_5,...,O_{13} \) and \( O_{22},...,O_{25} \) which contain four derivatives and one quark mass factor. Their chiral expansions also receive \( O(p^4) \) tree level contributions involving the LEC’s \( L_1, L_2, L_3 \). In general, in such a situation, the hope is that one may use a resonance model estimate for the \( O(p^6) \) LEC’s and then derive improved determinations for the LEC’s \( L_i \). This idea was actually followed in the series of papers [13, 35] which used as experimental input the pseudoscalar meson masses, decay constants and the \( K_{13} \) decay form-factors. Using the determination of the chiral coupling constants obtained in these references from this procedure, the three \( \pi K \) sub-threshold coefficients can be predicted. The results obtained in [15] are reproduced in table 2.

Looking at table 2 it is rather striking that there is a serious discrepancy, for all these three sub-threshold coefficients, between the chiral predictions and the dispersive calculations.

### 3.2 Should one blame the dispersive representations?

A possible explanation for these discrepancies could be that the dispersive calculations are not correct. Let us argue, considering the particular example of \( C_{01}^- \) which is rather simple, that this is unlikely to be the case. One may start with a fixed-\( t \) dispersive representation, at \( t = 0 \), of the amplitude \( F^+(s, t) \) with two subtractions,

\[ F^+(s, 0) = c^+(0) + \frac{1}{\pi} \int_{m_+^2}^\infty ds' \left[ \frac{1}{s' - s} + \frac{1}{s' - u} \right. \]

\[ \left. - \frac{2(s' - \Sigma)}{(s' - m_+^2)(s' - m_-^2)} \right] \text{Im} F^+(s', 0). \]  

(19)

The validity of this kind of dispersion relation as well as that of the Froissart bound which ensures convergence can be established in a rigorous manner [36, 37]. From eq. [19] it is straightforward to derive the following sum rule for the sub-threshold parameter \( C_{01}^+ \)

\[ C_{01}^+ = \frac{8m_K^2m_\pi^2}{\pi} \int_{m_+^2}^\infty ds' \frac{\text{Im} F^+(s', 0)}{(s' - \Sigma)^2} ds'. \]  

(20)

(Note that unlike the case of \( C_{02}^+ \), this sum rule is useless for deriving the chiral result.) The integrand needed in this sum rule is displayed in fig. 1. The following remarks can be made. The contributions from the high energy region \( \sqrt{s} \geq 2 \text{ GeV} \) are negligibly small. Most of the contributions are from the \( S \) and \( P \) waves, they are concentrated in the region \( \sqrt{s} \leq 1 \) GeV and there are no numerical difficulties in computing the integral. The \( S \) wave in the lower energy range is the part affected with the largest error. In this region, one can compare with ChPT calculations (which up to order six do not depend on the LEC’s \( C_i^+ \)); the difference is of the order of 20% at most and the ChPT result for \( \text{Im} F^+(s', 0) \) tends to be smaller and not larger than the one derived from experiment. In conclusion, this sum rule seems fairly solid: any reasonable fit to the experimental data of refs. [38, 39] will give a number \( C_{01}^+ \approx 2 \).

#### 3.3 Vector resonance chiral Lagrangian

Another possible cause for the discrepancies revealed in table 2 could be that the resonance saturation model used to evaluate

| \( p^0 \) \rangle_{L_1=0} | \( p^0 \) \rangle_{L_1=L_2=0} | \( p^0 + p^0 \rangle_{\text{total}} | \text{Dispersive} |
|-----------------|-----------------|-----------------|-----------------|
| \( C_{20}^+ \) | 0.0255 | -0.0254 | 0.003 | 0.024 ± 0.006 |
| \( C_{01}^+ \) | 1.673 | 1.492 | 3.8 | 2.07 ± 0.10 |
| \( C_{10}^+ \) | -0.0253 | 0.121 | 0.09 | 0.31 ± 0.01 |

Table 2. Comparison of the dispersive results for three sub-threshold parameters (last column) with the chiral calculation of ref. [15] at order \( p^6 \). The second and third columns display results obtained when the LEC’s \( L_i^+ (\mu) \) and \( C_i^+ (\mu) \) are set equal to zero at \( \mu = 0.77 \text{ GeV} \). The fourth column displays the full chiral result from ref. [15].
In ref. [15] the vector field formalism (see [42]) was used and flavour symmetry breaking is described via a single term in connection with the finite chiral order to each resonance field. A detailed discussion such a transformation rule ensures that one can ascribe a definite chiral order to each resonance field. We will re-examine the case of the vector mesons here, and follow the approach to first construct a Lagrangian containing the resonance fields and then integrate them out. A convenient method for this construction (see e.g. [40]) is to make use of non-linear representations of the chiral group [41]. For the purpose of generating chiral Lagrangian terms it is also convenient to adopt a homogeneous chiral transformation rule for all the resonances

\[ R \rightarrow h[\phi] R h[\phi]^{\dagger} . \]  

Such a transformation rule ensures that one can ascribe a definite chiral order to each resonance field. A detailed discussion in connection with the \( O(p^4) \) LEC’s can be found in refs. [19] [42]. In ref. [15] the vector field formalism (see [42]) was used and flavour symmetry breaking is described via a single term

\[ \mathcal{L}^0_V = f_\chi \langle V_\mu [u^\mu, \chi_-] \rangle . \]  

This term is the unique one relevant to the \( O(p^6) \) LEC’s because in the vector field formalism the field \( V_\mu \) has chiral order three. The coupling constant \( f_\chi \) was determined such as to reproduce the experimental value for the ratio \( \Gamma(K^* \rightarrow K\pi)/\Gamma(\rho \rightarrow \pi\pi) \) which gave

\[ f_\chi = -0.025 . \]  

In seeking for an improvement we note that, in this formalism, the symmetry breaking effects induced from the masses of the vector mesons are absent at order \( p^6 \), which seem somewhat unnatural. This suggests to investigate different formalisms. A discussion of symmetry breaking based on the massive Yang-Mills approach was performed in ref. [43]. We will make use here of the formalism which uses anti-symmetric tensors [2] [19] instead of vector fields. The part which is relevant for the \( O(p^4) \) LEC’s was considered in ref. [19]

\[ \mathcal{L}^0_{AT} = -\frac{1}{2} \langle \nabla^\lambda V_{\mu \lambda} \nabla_\nu V^{\mu \nu} \rangle + \frac{1}{4} M_V^2 \langle V_{\mu \nu} V^{\mu \nu} \rangle \]

\[ \quad + \frac{F_V}{2 \sqrt{2}} (V_{\mu \nu} f_{\mu \nu}^{\mu \nu}) + \frac{i G_{V}}{\sqrt{2}} (V_{\mu \nu} u^{\mu} u^{\nu}) . \]  

From eq. (24) one can deduce the chiral order of the resonance field

\[ V_{\mu \nu} \sim O(p^2) . \]  

As a consequence, the kinetic energy term in eq. (24) is \( O(p^6) \) while the other terms are \( O(p^4) \). Let us now consider all possible terms which are chiral symmetry breaking corrections to the terms in eq. (24). Neglecting OZI rule violation, we find that there are six independent such terms which have chiral order six (some of these have been considered also recently in ref. [44])

\[ \mathcal{L}^m_{AT} = \]

\[ \frac{1}{2} e_V^m \langle \chi_+ V^{\mu \nu} \chi_+ \rangle + \frac{g_V^m}{M_V} \langle V^{\mu \nu} \{ \chi_+, u_{\mu \nu} \} \rangle + \frac{g_V^m}{M_V} \langle \nabla_\nu V^{\mu \nu} \chi_-, u_{\mu \nu} \rangle \]

\[ + \frac{f_V^m}{M_V} \langle [V_{\mu \nu} f^{\mu \nu}_{\chi_+}], [\chi_+, \chi_+] \rangle + \frac{f_V^m}{M_V} \langle [V_{\mu \nu} f^{\mu \nu}_{\chi_-}], [\chi_-, \chi_-] \rangle . \]  

Only the first four of these terms play a role in \( \pi K \) scattering. Instead of a single coupling constant, \( f_\chi \), in the vector formalism, one has four independent couplings here: \( e_V^0, g_V^0, g_V^2 \), \( f_V^0 \), see [42]. In ref. [42] it was shown how these two formalisms can be made to give exactly equivalent results for the \( O(p^4) \) LEC’s. This necessitates that a number of asymptotic constraints for Green’s functions, form-factors or scattering amplitudes be implemented.

![Fig. 1. Integrand to be used in the sum rule, eq. (20)](image-url)
Let us now discuss the determination of these couplings from experiment.

### 3.4 Determination of the vector Lagrangian coupling constants

#### 3.4.1 Determination of $e^{\pi\rho}_{V}$

At first, it is not difficult to determine $e^{\pi\rho}_{V}$ based on the mass relations

$$
\begin{align*}
M_{\rho}^2 &= M_{\rho}^2 + 8e^{\pi\rho}_{V}B_{0}\hat{m} \\
M_{K^*}^2 &= M_{K^*}^2 + 4e^{\pi\rho}_{V}B_{0}(m_{s} + \hat{m}) .
\end{align*}
$$

(27)

Isospin breaking is neglected here and we have denoted $\hat{m} = (m_{u} + m_{d})/2$. For the numerics, we can use the results from the chiral expansion at leading order,

$$
\begin{align*}
2B_{0}\hat{m} &= M_{\rho}^2 = (134.98 \text{ MeV})^2 \\
\frac{m_{s}}{\hat{m}} &= \frac{M_{K^*}^2 + M_{K^0}^2 - M_{\pi^+}^2}{M_{\pi^0}^2} \approx 25.90 \\
B_{0}(m_{u} - m_{d}) &= M_{K^*}^2 - M_{K^0}^2 - M_{\pi^+}^2 + M_{\pi^0}^2 \\
&\approx -0.285 M_{\pi^0}^2 .
\end{align*}
$$

(28)

Using the experimental values of the $K^*(892)$ and the $\rho(770)$ masses we obtain

$$
e^{\pi\rho}_{V} \approx 0.22 .
$$

(29)

If, instead, one uses the masses of the $\phi(1020)$ and the $\rho(770)$ mesons one would obtain $e^{\pi\rho}_{V} \approx 0.24$, suggesting that the error should not dynamically emerge for this quantity.

#### 3.4.2 Determination of $f_{X}$ and $f_{V}^{m}$

As a next step we consider the coupling constant $f_{X}$. In ref. [15], $f_{X}$ was related to symmetry breaking in the decays of vectors into two pseudoscalars. Here, we will argue that these decays determine the two couplings $G'_{V}$ and $G''_{V}$. Concerning $f_{X}$, a physically plausible estimate can be obtained by relating it to the decay of the $\pi(1300)$ resonance. Let us denote the $\pi(1300)$ nonet matrix by $P$ and consider the Lagrangian,

$$
L^{\pi(1300)} = \frac{1}{2}\langle \nabla_{\mu}P\nabla_{\mu}P\rangle - \frac{1}{2}M_{\rho}^2\langle P^2 \rangle + i\hat{d}_{m}\langle P_{\pi^-} \rangle + iG'_{V}\langle \nabla_{\mu}V^{\mu
u}[P_{\rho}, P_{\pi}] \rangle + iG''_{V}\langle V_{\mu
u}[f^{\mu\nu}, P] \rangle .
$$

(30)

This extends the Lagrangian considered in ref. [19] by the last two terms proportional to $G'_{V}$ and $G''_{V}$, respectively and which have chiral order equal to six. Integrating out the $\pi(1300)$ meson, one finds that the couplings $f_{X}$ and $f_{V}^{m}$ which were appearing in eq. (26) are proportional respectively to $G'_{V}$ and $G''_{V}$

$$
f_{X} = G'_{V}\frac{d_{m}M_{V}}{M_{P}}, \quad f_{V}^{m} = G''_{V}\frac{d_{m}M_{V}}{M_{P}} .
$$

(31)

The coupling $d_{m}$ was introduced in ref. [19]. It can be estimated by appealing to a chiral super-convergence sum-rule associated with the correlator of two scalar currents minus the correlator of two pseudoscalar currents (see ref. [2]). Saturating the sum rule from the contributions of the pion, the $\pi(1300)$ and the $a_{0}(980)$ one gets the relation

$$
8d_{m}^2 + F_{0}^2 - 8c_{m}^2 = 0 .
$$

(32)

Using the value $c_{m} \simeq 42$ MeV which was obtained ref. [19] then gives

$$
d_{m} \simeq 26 \text{ MeV} .
$$

(33)

The coupling $G'_{V}$ can be related to the decay amplitude $\pi(1300) \rightarrow \rho\pi$,

$$
\Gamma_{\pi(1300)\rightarrow \rho\pi} = \frac{4(97)^2}{M_{\pi}} .
$$

(34)

The total width of the $\pi(1300)$ is known to be rather large but has not actually been very precisely determined (the PDG quotes a range of values between 200 and 600 MeV). For definiteness, let us use the result obtained in ref. [46] who also find the $\rho\pi$ decay mode to be the dominant one

$$
\Gamma_{\pi(1300)\rightarrow \rho\pi} \simeq 268 \pm 50 \text{ MeV} .
$$

(35)

This gives the estimate

$$
|G'_{V}| \simeq 0.23
$$

(36)

yielding

$$
|f_{X}| \simeq 2.8 \times 10^{-3} .
$$

(37)

This value is one order of magnitude smaller than the one obtained in ref. [15]. One consequence concerns the lifetime of the $\pi(1300)$ atom which receives a contribution (via resonance saturation of the LEC’s $G_{i}$) which is quadratic in $f_{X}$. If one uses the numerical value for $f_{X}$, the size of the $O(p^6)$ contribution to the lifetime is rather small (see the detailed discussion in ref. [47]).

A somewhat different approach is to consider the 3-point correlation function $< VAP >$, model it in terms of a finite number of resonances, and constrain the coupling constants in order to enforce the proper QCD asymptotic conditions. This was reconsidered recently by Cirigliano et al. who improved on earlier work by including the $\pi(1300)$ nonet in the construction together with the vector, axial-vector and pion multiplets. In this manner, they have obtained a determination of the $\pi(1300)$ couplings $G'_{V}, G''_{V}$ in terms of the vector and axial-vector resonance masses,

$$
G'_{V} = -\sqrt{M_{A}^2 - M_{V}^2} , \quad G''_{V} = -\frac{0.2M_{A}^2 - M_{V}^2}{8M_{A}} .
$$

(38)

Using $M_{A} = \sqrt{2}M_{V}$ this gives

$$
f_{X} \simeq -4.2 \times 10^{-3} , \quad f_{V}^{m} \simeq -1.1 \times 10^{-3} .
$$

(39)

This method provides a determination of $f_{X}$ which is in reasonably good agreement with the one based on the $\pi(1300)$ decay width and gives also the sign as well as a determination of the coupling $f_{V}^{m}$.
3.4.3 Determination of $g^m_{V1}$

The coupling $g^m_{V1}$ can be determined from the decay amplitudes of vector mesons into two pseudoscalars. The decay amplitudes have the following form

$$T(V_a \rightarrow \phi_b \phi_c) = M_{V_a} e \cdot (p_1 - p_2) T_{abc}. \quad (40)$$

Correspondingly, the decay width is given by

$$\Gamma(V_a \rightarrow \phi_b \phi_c) = |T_{abc}|^2 \frac{\rho_m^m}{6\pi}. \quad (41)$$

Using the Lagrangian [26] these amplitudes can be expressed as a function of two combinations of the couplings $g^m_{V1}$, $g^m_{V2}$ and $f_\chi$ for which we introduce the notation

$$\hat{g}^m_{V1} = g^m_{V1} + \frac{1}{2} f_\chi$$
$$\hat{g}^m_{V2} = g^m_{V2} + f_\chi. \quad (42)$$

The amplitude for $\rho^+ \rightarrow \pi^+ \pi^0$, at first, reads

$$T_{\rho^+ \rightarrow \pi^+ \pi^0} = \frac{1}{F^2} G^V_{eff},$$

$$G^V_{eff} = G_V + \frac{4\sqrt{2}\hat{\rho}_m}{M_V} (2\hat{g}^m_{V1} + \hat{g}^m_{V2}). \quad (43)$$

Using the experimental values for the mass $m_\rho = 775.5 \pm 0.5$ MeV and the width $\Gamma = 150.2 \pm 2.4$ MeV from ref. [45] gives

$$G^V_{eff} \approx 65.8 \text{ MeV.} \quad (44)$$

Next, we consider the decays $K^* \rightarrow K\pi$ and $\phi \rightarrow K\bar{K}$.

$$T_{K^{*+} \rightarrow K^0\pi^+} = \frac{\sqrt{2}}{2F_{K^0}\phi_{K^0}} \left\{ G^V_{eff} + \frac{4\sqrt{2}\hat{g}^m_{V1}}{M_V} B_0(m_s - \hat{m}) \right\}$$
$$T_{K^{*+} \rightarrow K^0\pi^0} = \frac{1}{2F_{K^+}\phi_{K^0}} \left\{ G^V_{eff} + \frac{4\sqrt{2}\hat{g}^m_{V1}}{M_V} B_0(m_s - \hat{m}) \right\}$$
$$T_{\phi \rightarrow K^0\pi^-} = -\frac{\sqrt{2}e^2 F^2}{6M^2} \left\{ G^V_{eff} + \frac{8\sqrt{2}\hat{g}^m_{V1}}{M_V} B_0(m_s - \hat{m}) \right\}$$
$$T_{\phi \rightarrow K^0\eta} = \frac{\sqrt{2}}{2F_{K^*\eta}} \left\{ G^V_{eff} + \frac{4\sqrt{2}\hat{g}^m_{V1}}{M_V} B_0(m_s - \hat{m}) \right\}$$
$$T_{\phi \rightarrow K^0\eta} = -\frac{\sqrt{2}}{2F_{K^*\eta}} \left\{ G^V_{eff} + \frac{4\sqrt{2}\hat{g}^m_{V1}}{M_V} B_0(m_s - \hat{m}) \right\}.$$  

These expressions include isospin breaking contributions proportional to $m_u - m_d$ and those proportional to $e^2 F^2$ induced by the coupling of the neutral vector mesons to the photon. We have also taken into account the influence of wave-function renormalization of the pseudoscalar mesons. If we ignore isospin breaking, i.e. set $m_u = m_d$, then the decay amplitudes no longer depend on $\hat{g}^m_{V2}$, which allows us to determine $\hat{g}^m_{V1}$. Combining the experimental values for the $K^*$ and $K^0\phi$ decay widths into $K\pi$ we obtain

$$\hat{g}^m_{V1} \approx 6.0 \times 10^{-3}. \quad (46)$$

If one uses the $\phi$ decay widths into $K^+ K^-$ and $K^0\bar{K}^0$ instead, one obtains a smaller but not very different value,

$$\hat{g}^m_{V1} \approx 4.3 \times 10^{-3}. \quad (47)$$

From these two results one can infer $\hat{g}^m_{V1} = (5.2 \pm 1.5) \times 10^{-3}$.

3.4.4 Determination of $g^m_{V2}$

Finally, we have to determine $g^m_{V2}$. The results of the previous subsection shows that if one forms isospin breaking combinations

$$T_{K^{*+} \rightarrow K^0\pi^+} - \sqrt{2} T_{K^{*+} \rightarrow K^+\pi^0},$$

$$T_{\phi \rightarrow K^0\pi^-} - T_{\phi \rightarrow K^0\eta},$$

the coupling $g^m_{V2}$ is the only one which contributes. In practice, however, it turns out not to be possible to determine $g^m_{V2}$ in this way. Precise experimental information exists for isospin violation in $\phi$ decays but, in this case, there are significant electromagnetic contributions as well, which are difficult to evaluate. Further amplitudes which vanish in the isospin limit are $\omega \rightarrow \pi^+ \pi^-$ and $\rho^+ \rightarrow \pi^+ \eta$. These amplitudes have the following expressions,

$$T_{\omega \rightarrow \pi^+ \pi^-} = \frac{G_V}{F^2 (M^2_{\pi^+} - M^2_{\pi^0})} \times$$

$$\left\{ \frac{m_u - m_d}{m_s - \hat{m}} (M_{K^0} - M_{\rho}) + \frac{2}{3} \right\}$$
$$T_{\rho^+ \rightarrow \pi^+ \eta} = \frac{\sqrt{2}G_{eff}}{4 F_{\pi} F_{\eta}} \left( m_u - m_d \right)$$
$$T_{\rho^+ \rightarrow \pi^+ \eta} = \frac{\sqrt{2}G_{eff}}{4 F_{\pi} F_{\eta}} \left( m_u - m_d \right). \quad (49)$$

In these cases the contribution proportional to $g^m_{V2}$ can be estimated to be relatively small such that it is again difficult to precisely extract its value.

The coupling $g^m_{V2}$ appears in the amplitude $\rho \rightarrow K\bar{K}$, as one can see from the expression,

$$T(\rho^+ \rightarrow K^+\bar{K}^-) =$$

$$\frac{1}{\sqrt{2}F_{K^0}} \left\{ G^V_{eff} + \frac{4\sqrt{2}\hat{g}^m_{V2}}{M_V} B_0(m_s - \hat{m}) \right\}. \quad (50)$$
From an experimental point of view, one can hope to determine this amplitude from the $\tau$ decay process $\tau \to K\bar{K}\nu\tau$. It is customary to approximate the dynamics of $\tau$ hadronic decays as proceeding via a few resonances. In the case of the $K\bar{K}$ channel, the $p(770)$ and the $p(1450)$ resonances can contribute. The resonance $p(1450)$ has a rather small coupling to $K\bar{K}$ and its contribution is also suppressed by phase-space such that it seems a plausible approximation to saturate the integrated $\tau \to K\bar{K}\nu\tau$ decay width from just the $\rho$ contribution. In order to compute this decay width from our resonance model we first introduce the charged vector current matrix element which, in the isospin limit, involves a single form-factor

$$\langle K^- (p_1)K^0 (p_2)|\bar{d}\gamma^\mu u|0 \rangle = (p_1 - p_2)^\mu F^K_V (s),$$

$$s = (p_1 + p_2)^2.$$  

(51)

Computing the form factor from our effective Lagrangian, we obtain

$$F^K_V (s) = 1 + \frac{F_V}{F^K} \left( G^{ff}_V + \frac{4\sqrt{2}}{M_V} \hat{g}^m_{\nu_2} B_0 (m_s - \hat{m}) \right) \frac{s}{M_V^2 - s}. $$

(52)

The $\tau$ decay rate into $K\bar{K}\nu\tau$ has the following expression

$$\Gamma_{K\bar{K}} = V^{\alpha \beta}_{ud} \frac{G^2_F M_\tau^5}{16\sqrt{2}\pi} \int_{4m_\pi^2}^{M_\tau^2} \frac{ds}{M_\tau^2} \left( 1 - \frac{4m_\pi^2}{s} \right)^{\frac{3}{2}} \times \left( \frac{1 - \frac{s}{M_\tau^2}}{1 + \frac{2s}{M_\tau^2}} \right) |F^K_V (s)|^2. $$

(53)

In practice, the formula (52), which is obtained from a tree level calculation, does not account for the $\rho$ meson width. On may account for this effect in a phenomenological way by replacing $M_\rho^2 - iM_\rho^\prime$ in the propagator in eq. (52) by $M_\rho^2 - iM_\rho^\prime \Gamma(s)$. In the energy range relevant for $\tau$ decay we retain the contributions to the $\rho$ width arising from the $\pi\pi$ and the $K\bar{K}$ channels as well as the $4\pi$ channel simply approximated as $\omega\pi$ which gives, in the region $s \geq 4m_K^2$

$$M_\Gamma (s) = \frac{M_\rho^2 M_\rho^\prime}{\sqrt{s}} \left[ \left( \frac{s - 4m_\pi^2}{M_\rho^2 - 4m_\pi^2} \right)^{\frac{3}{2}} + \frac{1}{2} \left( \frac{s - 4m_\pi^2}{M_\rho^2 - 4m_\pi^2} \right) \right] + G^2_{\omega\rho\pi} \frac{[s - (M_\omega - 4m_\pi)^2]^{\frac{3}{2}}}{24s}. $$

(54)

The coupling constant $G_{\omega\rho\pi}$ may be estimated using vector meson dominance and the experimental value of the $\omega \to \gamma\pi$ width

$$G_{\omega\rho\pi}^2 = 24 \text{ GeV}^{-2}. $$

(55)

Using the expression (55) for the imaginary part of the $\rho$ meson propagator and the experimental value (55) of the $\tau \to K\bar{K}\nu\tau$ decay rate $R = (15.4 \pm 1.6) \times 10^{-3}$ we obtain

$$\hat{g}^m_{\nu_2} \simeq 0.015. $$

(56)

Ignoring completely the $\rho$ width gives a larger value $\hat{g}^m_{\nu_2} \simeq 0.022$. Alternatively, one may estimate $\hat{g}^m_{\nu_2}$ by making use of an asymptotic constraint, namely imposing that the form-factor $F^V (s)$ goes as $1/s^{\frac{3}{2}}$ asymptotically. This yields a somewhat smaller value $\hat{g}^m_{\nu_2} \simeq 0.011$. This discussion allows us to estimate that the error on the estimate (56) should be of the order of 50% i.e. $\hat{g}^m_{\nu_2} \simeq 0.015 \pm 0.0007$.

### 3.4.5 Determination of $f^m_{V_1}$

Finally, let us consider $f^m_{V_1}$. This parameter controls flavour symmetry breaking in the matrix elements of the vector current between a vector meson and the vacuum,

$$F_{K^*} - F_\rho = \frac{8\sqrt{2}}{M_V} f^m_{V_1} B_0 (m_s - \hat{m}).$$

(57)

We can extract the relevant information from the $\tau$ decay processes $\tau \to \rho^-\nu\tau$ and $\tau \to K^*^-\nu\tau$. Using the experimental results from (55) we obtain

$$F_\rho = 146.3 \pm 1.2 \text{ MeV}, \quad F_{K^*} = 155.1 \pm 4.0 \text{ MeV}, $$

(58)

from which we finally deduce

$$f^m_{V_1} = 0.0027 \pm 0.0013. $$

(59)

### 3.5 Vector meson contributions to the LEC’s

Let us now integrate out the vector meson from the Lagrangian (24), (26) and consider the $O(p^6)$ chiral Lagrangian terms which are generated. One finds

$$L^{(6)}_{\text{AT}} = \frac{G^2_V}{4M_V} \langle \nabla \lambda [u^\mu, u^\nu] \nabla \nu [u^\nu, u^\mu] \rangle - \left( \frac{e_V G_V}{2M_V^3} - \frac{\sqrt{2} G_V g^m_{V_1}}{M_V^3} \right) \langle [u^\mu, u^\nu] u^\mu u^\nu \chi_+ \rangle + \frac{G_V g^m_{V_1}}{\sqrt{2}M_V^3} \langle [u^\mu, u^\nu] u^\mu \chi^\nu \rangle + \frac{G_V F_V}{2M_V^4} i \langle \nabla_\mu [u^\nu, u^\mu] u^\nu \rangle - \frac{F_V}{4M_V^4} \langle \nabla^\lambda f_{\lambda\mu} \nabla_\nu f_{\nu\mu} \rangle + \frac{F_V f_\chi}{\sqrt{2}M_V^8} \langle f_{\mu\nu} \nabla^\lambda [\chi_-, u^\nu] \rangle + \left( \frac{F_V G_V e^m_{V_1}}{2M_V^3} - \frac{2G_V f^m_{V_1}}{\sqrt{2}M_V^3} - \frac{F_V g^m_{V_1}}{\sqrt{2}M_V^3} \right) i \langle f_{\mu\nu} \chi^\nu [\chi_-, u^\nu] \rangle + \left( \frac{e_V F_V^2}{4M_V^4} - \frac{2F_V f^m_{V_1}}{\sqrt{2}M_V^3} \right) \langle \chi_+ f_{\mu\nu} f^\nu_+ \rangle $$
\(- \frac{F_V f_{\mu \nu}}{\sqrt{2} M_V} \langle f_{\mu \nu} | f_{\rho \sigma} \rangle \),
\[- \frac{2 G_V f_{\mu \nu}}{\sqrt{2} M_V} \langle f_{\mu \nu} | \chi \rangle \langle \chi | u^\mu u^\nu \rangle \). \hspace{1cm} (60)

In the vector field formalism one term, proportional to \( f_X^2 \), is generated which does not appear in eq. (60). In the spirit of ref. [15] we may simply add this term here:\footnote{Alternatively, one may describe spin one resonances in terms of a pair of fields \( V_{\mu \nu} \) and \( V_\chi \). A more detailed discussion of this framework will be presented elsewhere [54].}

\[ L^{(6)} = - \frac{f_X^2}{2 M_V^2} \langle [u_\mu, \chi | [u^\mu, \chi] \rangle . \hspace{1cm} (61) \]

In this way, we recover exactly the results of ref. [15], if we set the extra coupling constants in our vector Lagrangian equal to zero. Next, we can expand the chiral Lagrangian terms over the canonical \( O(p^3) \) basis established in ref. [15]. After some calculation, we obtain contributions to 45 different LEC's

\[
C^V_{14} = \frac{G^2_V}{4 M_V^3}, \\
C^V_{48} = - \frac{G^2_V}{8 M_V^4}, \\
C^V_{50} = \frac{G_V F_V}{4 M_V^3} + \frac{f_X F_V}{\sqrt{2} M_V^3}, \\
C^V_{51} = - \frac{G^2_V}{4 M_V^3} + \frac{G_V F_V}{4 M_V^3} + \frac{f_X F_V}{\sqrt{2} M_V^3}, \\
C^V_{52} = - \frac{G_V F_V}{4 M_V^3} - \frac{f_X F_V}{\sqrt{2} M_V^3}, \\
C^V_{53} = \frac{G_V F_V}{8 M_V^4} - \frac{3 f_X^2}{16 M_V^3} - \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{55} = \frac{G_V F_V}{8 M_V^4} + \frac{3 f_X^2}{16 M_V^3} + \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{56} = \frac{G_V F_V}{4 M_V^3} + \frac{f_X^2}{8 M_V^4} - \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{57} = \frac{G_V F_V}{2 M_V^3} + \frac{F^2_V}{8 M_V^4} + \frac{\sqrt{2} f_X F_V}{M_V^3}, \\
C^V_{59} = \frac{G_V F_V}{8 M_V^4} - \frac{F^2_V}{4 M_V^3} - \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{61} = \frac{e^{-} G^2_V}{4 M_V^3} - \frac{\sqrt{2} F_V f_{\mu \nu}}{M_V^3}, \\
C^V_{63} = - \frac{\sqrt{2} f_X G_V}{M_V^3} + \frac{e^{-} F_V G_V}{2 M_V^3} - \frac{F_V f_{\mu \nu}}{\sqrt{2} M_V^3}, \\
C^V_{65} = \frac{F_V f_{\mu \nu}}{\sqrt{2} M_V^3}, \\
C^V_{66} = \frac{G^2_V}{8 M_V^4}, \\
C^V_{69} = - \frac{G^2_V}{8 M_V^4}, \\
C^V_{70} = \frac{g_{\mu \nu}}{M_V^3} - \frac{G_V F_V}{8 M_V^4} + \frac{F^2_V}{8 M_V^4} - \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{72} = \frac{G_V F_V}{8 M_V^4} - \frac{F^2_V}{8 M_V^4} + \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{73} = \frac{G_V F_V}{4 M_V^3} - \frac{F^2_V}{8 M_V^4} - \frac{f_X F_V}{\sqrt{2} M_V^3}, \\
C^V_{74} = - \frac{G^2_V}{8 M_V^4}, \\
C^V_{76} = - \frac{G_V F_V}{8 M_V^4} + \frac{F^2_V}{16 M_V^3} - \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{78} = \frac{G_V F_V}{8 M_V^4} + \frac{F^2_V}{4 M_V^3} + \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{79} = \frac{G_V F_V}{8 M_V^4} + \frac{F^2_V}{8 M_V^4} - \frac{f_X F_V}{2 \sqrt{2} M_V^3}, \\
C^V_{82} = - \frac{G_V F_V}{16 M_V^3} - \frac{F^2_V}{16 M_V^3} - \frac{f_X F_V}{4 \sqrt{2} M_V^3} - \frac{f_{\mu \nu} F_V}{\sqrt{2} M_V^3}, \]
\[ C^{V}_{85} = \frac{3G^2}{16M_V^2} + \frac{f_s G_V}{2\sqrt{2}M_V^3} - \frac{\sqrt{2} f_{\rho}^2 G_V}{M_V^3}, \]
\[ C^{V}_{87} = \frac{F^2}{8M_V^4}, \]
\[ C^{V}_{88} = \frac{G_F V}{4M^3_V} - \frac{f_s F_V}{\sqrt{2}M_V^3}, \]
\[ C^{V}_{89} = \frac{F^2}{2M_V^4} + \frac{G_F V}{4M^3_V}, \]
\[ C^{V}_{90} = -\frac{f_s F_V}{\sqrt{2}M_V^3}, \]
\[ C^{V}_{92} = \frac{F^2}{M_V^4}, \]
\[ C^{V}_{93} = -\frac{F^2}{4M_V^4}. \] (62)

In these formulas \( n \) stands for the number of flavours and should be set to \( n = 3 \).

These results can be verified to agree with the ones obtained in ref. [55] when retaining the same resonance coupling constants as in our Lagrangian. The correspondence in the notation between the coupling constants appearing in our eqs. [26] and those in ref. [33] is as follows:

\[ e^m_{\nu} \frac{F^2}{2M^2_V} = \lambda^S_{\nu \nu} - \lambda^V \]
\[ g^m_{\nu 1} \frac{V}{M^2_V} = \lambda^S_{\nu \nu} \]
\[ g^m_{\nu 2} \frac{V}{M^3_V} = \lambda^V \]
\[ f^m_{\nu} \frac{V}{M^2_V} = -\lambda^S_{\nu \nu} + \frac{1}{2} \lambda^V \]
\[ f^m_{\nu 1} \frac{V}{M^3_V} = \lambda^S_{\nu \nu} - \lambda^V. \] (63)

These relationships may be derived by making a field redefinition on the scalar and pseudoscalar resonance fields used in ref. [33]

\[ S \to \tilde{S} + e_m \frac{V}{M_S}, \quad P \to \tilde{P} + id_m \frac{V}{M_P}. \] (64)

We note that the terms proportional to \( f^2 \) in \( C^{V}_{25} \) and \( C^{V}_{26} \) which are generated in the \( V \)-formalism but not directly in the \( AT \)-formalism have not been considered in ref. [33].

### 3.5.1 Resonance saturation versus experiment for \( C_{61} \)

Only one of the LEC’s which appear in eqs. [62] (except for \( C_{10} \) and \( C_{14} \)) has actually been determined from experiment. Let us consider the two-point correlator of two vector currents

\[ i \int d^4 x e^{ipx} \langle 0 | T(V^{ij}_\mu(x)V^{ij}_\nu(0)) | 0 \rangle \]
\[ = (p_\mu p_\nu - p^2 g_{\mu \nu}) \Pi^{ij}(p^2) + g_{\mu \nu} p^2 \Pi_0^{ij}(p^2) \]

with

\[ V^{ij}_\mu(x) = \tilde{\psi}^i(x) \gamma_\mu \psi^j(x), \] (66)

and then consider the difference

\[ \Delta \Pi = \Pi^{ud}(0) - \Pi^{us}(0). \] (67)

The chiral computation of this quantity at order \( p^6 \) was first performed in ref. [55] and the result was confirmed and expressed in terms of the canonical set of \( O(p^6) \) LEC’s in ref. [11]. The chiral expansion involves no LEC at all at chiral order \( p^6 \) and a single LEC at chiral order \( p^8 \), which is \( C_{61}^{\prime} \). Using finite-energy sum rule techniques, the value of \( \Delta \Pi \) can be determined from experiment [55] (earlier related calculations were performed in refs. [56, 57])

\[ \Delta \Pi_{\text{exp}} = 0.0203 \pm 0.0032. \] (68)

This result translates into the following value for the \( O(p^6) \) LEC

\[ C_{61}^{\prime}(m_\rho) = (1.24 \pm 0.44) \times 10^{-3} \text{ GeV}^{-2}. \] (69)

On the other hand, our resonance saturation model, using the results from eqs. [62] and the determination of the resonance parameters discussed above, yields

\[ C_{61}^{V} = 2.10 \times 10^{-3} \text{ GeV}^{-2} \] (70)

(using \( F_V = F_p \), see eq. [55]) which is in qualitative agreement with the experimental determination.

### 3.6 Comparison between resonance saturation and the dispersive representations

We can now return to the \( \pi K \) scattering amplitude and compute the vector meson contributions generated from the saturation of LEC’s \( C_i \) as shown above [62]. We quote the result for the three sub-threshold coefficients under consideration in this section,

\[ C_{20}^{+ \mid C_V^{\prime}} = \left[ -\frac{7}{8} G^2 \frac{m_K^2 + m_\rho^2}{M_V^4} + 3 \frac{G^2}{2} e_m \frac{m_K^2}{M_V^4} \right] M^4_{\pi} \]
\[ - \frac{3}{2} G^2 \frac{5 m_K^2 m_\rho^2 + m_\rho^2 m_\pi^2}{M_V^4} \]
\[ C_{01}^{+ \mid C_V^{\prime}} = \left[ 2 G^2 \frac{m_K^2 + m_\rho^2}{M_V^4} - 8 G^2 \frac{m_K^2}{M_V^4} \right] \frac{m_K^2 m_\rho^2}{M_V^4} \]
\[ + 8 \sqrt{2} G^2 \frac{5 m_K^2 m_\rho^2 + m_\rho^2 m_\pi^2}{M_V^4} \frac{m_K^2 m_\rho^2}{M_V^4} \]
\[ C_{10}^{+ \mid C_V^{\prime}} = \left[ 3 G^2 \frac{m_K^2 + m_\rho^2}{M_V^4} - 4 G^2 e_m \frac{m_K^2}{M_V^4} + 2 m_\pi^2 \right] \]
\[ + 4 \sqrt{2} G^2 \frac{5 m_K^2 m_\rho^2 + m_\rho^2 m_\pi^2}{M_V^4} \frac{m_K^2 m_\rho^2}{M_V^4} \]
\[ \times \frac{m_K m_\rho^2}{F^4_{\pi}}. \] (71)

A comparison of the numerical results for the resonance saturated part of these sub-threshold parameters between the vector field model and the antisymmetric tensor model is performed in table 3. One can see that the differences are substantial. In two cases even the sign of the result is different.

One can perform a check of the resonance saturation model in the following way. Consider the set of sub-threshold coefficients which can be written as sum rules with no subtractions.

At this level, it is easy to identify a particular resonance \( R \) contribution: it suffices to restrict the integration region to the
neighbourhood of the resonance mass and to restrict the sum over partial-waves to the one which corresponds to the spin of the resonance. This is illustrated in fig. 2 which shows the integrands (in both the s and the t channel) associated with the coefficient $C_{ij}^{20}$. In this case, the contribution from the vector resonance can be isolated in the s channel and the contributions from the scalar resonances can be identified in both the s and the t channel. Fig. 3 illustrates the situation for the coefficient $C_{ij}^{10}$: in this case the vector contribution appears in both the s and the t channels.

From the point of view of ChPT now, we can split the contributions to a given sub-threshold coefficient $C_{ij}$ into one part, $C_{ij}^{\text{loop}}$, which arises from loop diagrams and one part $C_{ij}^{\text{tree}}$ which arises from tree level diagrams. The latter piece, up to chiral order $p^2$, involves terms linear in the $O(p^0)$ LEC’s $L_i^\mu$, terms which are quadratic in the $O(p^0)$ LEC’s and, finally, those which are linear in the LEC’s $C_i^\mu$. Both $C_{ij}^{\text{loop}}$ and $C_{ij}^{\text{tree}}$ depend on the chiral renormalization scale $\mu$. Let us assume that a proper scale $\mu$ exists such that $C_{ij}^{\text{loop}}$ corresponds to the low energy integration part of the coefficient $C_{ij}$ and $C_{ij}^{\text{tree}}$ to the higher energy part. We can then make a check of our resonance saturation model by computing $C_{ij}^{\text{tree}}$ using the resonance-saturated values of the LEC’s $L_i$ and $C_i$ and comparing the result with the dispersive integral calculation in which the integral is computed over an energy range $E > E_0$. The lower boundary of the integration range should be somewhat below the resonance mass. We will only consider the role of the vector mesons here. In the resonance saturation model, we correspondingly keep the terms proportional to the coupling $G_V$. The terms arising from the LEC’s $C_i$ were shown in eq. (71). Upon using the resonance model of ref. [19] and retaining the contributions proportional to $G_V$ the terms which are linear or quadratic in the LEC’s $L_i$ yield

$$C_{ij}^{+\mu}_{L+LL} = \left[ 3 \frac{G_i^2}{4 m_V^2} \left[ 1 - \frac{8 c d c m (m_k^2 - m^2)}{F^2 m_s^2} \right] m_i^2 F^2 \right]$$

$$C_{10}^{+\mu}_{L+LL} = \left[ \frac{2 G_i^2}{m_V^2} \left[ 1 - \frac{8 c d c m (m_k^2 - m^2)}{F^2 m_s^2} \right] m_i^2 m_k^2 \right]$$

$$C_{10}^{+\mu}_{L+LL} = \left[ \frac{G_i^2}{m_V^2} \left[ 1 - \frac{8 c d c m (m_k^2 - m^2)}{F^2 m_s^2} \right] m_i^2 m_k^2 \right]$$

(72)

The comparison, as discussed above, of the resonance saturation result with the dispersive resonance calculation is performed in table 4. The table shows that the results from the antisymmetric tensor model for the relevant $C_i$’s when added to the contributions linear and quadratic in the $L_i$’s compares rather well with the resonance contributions as computed from the sum rules.

### Table 3. Results on the $O(p^0)$ part involving the $C_{ij}$ LEC’s of some sub-threshold coefficients, using two different vector resonance saturation models of these.

| L+LL | (L+LL+C)$_V$ | (L+LL+C)$_{AT}$ |
|-------|--------------|-----------------|
| $C_{20}^{+\mu}$ | -0.0065 | -0.012 |
| $C_{01}^{+\mu}$ | 0.439 | 0.17 |
| $C_{10}^{+\mu}$ | 0.185 | 0.08 |

### Table 4. Comparison between vector resonance contributions to three subthreshold coefficients as computed from sum rules (last column) and as computed from resonance saturation models of the LEC’s. The second column shows the contributions which are linear and quadratic in the LEC’s $L_i$ while the third and fourth column show the additional effect of the LEC’s $C_i$ using the vector or the antisymmetric tensor model respectively.

#### 3.7 A LEC combination with dominant vector contributions

In general, the low-energy couplings get important contributions from the light vector mesons and also from the light scalar resonances [19]. Accounting for the scalar contributions is made difficult by several features. Firstly, the OZI rule is rather strongly violated in the scalar meson sector. This induces a large number of parameters in the resonance Lagrangian which cannot be determined unless some assumptions are made: see e.g. ref. [53] for a recent discussion and some examples of such assumptions. A second difficulty is caused by the presence of the wide scalars (the $\sigma$ or $\kappa$ mesons). Interferences between the contributions from the wide scalars and the narrow ones lead to structures in the partial wave amplitudes (see e.g. fig. 2 right) which are not well approximated by computing tree level diagrams from a resonance Lagrangian. For these reasons, it is useful to try to identify specific combinations of LEC’s which receive small contributions from the scalar mesons. We can generate one such combination by starting from $\pi K$ sub-threshold coefficients and forming the following combination

$$C_{NS} = C_{01}^+ + \frac{2 m_K}{m_\pi} C_{10}^- .$$

(73)

Indeed, this quantity satisfies an unsubtracted dispersion relation and, by construction, it receives no resonant S-wave contributions from either the $t$– or the $s$–channels. The only resonant contributions are from the $l \geq 1$ partial waves. The s-channel integrand is shown in fig. 4 while the t-channel integrand is the same, up to a scale factor, as that shown in fig. 3. Computing the integrals we find the experimental value of this quantity

$$C_{NS} = 4.27 \pm 0.17 .$$

(74)

Using the chiral expansion for $C_{NS}$ one finds that the following combination of $O(p^4)$ and $O(p^6)$ LEC’s is involved,

$$L_2^{eff} (\mu) = L_2^{\mu} + (m_K^2 + m_\pi^2) (-2 C_{4}^{+} + C_{10}^{-} - 2 C_{12}^{+} + 2 C_{22}^{+} + 2 C_{23}^{+} - C_{25}^{-}) + (4 m_\pi^2 + 2 m_\pi^2) (C_{14}^{-} - 2 C_{13}^{-}) .$$

(75)

According to the remarks made above, this combination of LEC’s receives no contributions from the scalar mesons corresponding to virtual exchanges in the $\pi K$ scattering amplitude. It does, however, pick up contributions from the scalars via tadpole-type diagrams$^4$. Such contributions have been accounted for in our approach via flavour symmetry breaking effects with the exception, however, of the LEC $C_{12}^+$ and for the

$^4$ We thank Roland Kaiser for pointing this out to us.
1/$N_c$ suppressed LEC’s). This LEC receives no contribution from the resonance Lagrangian terms we have considered. Fortunately, direct determinations exist for $C_{12}^r$ based on the scalar form factors with either $\Delta S = 0$ (ref.\cite{14}) or $\Delta S = 1$ (ref.\cite{28}). The latter determination seems more precise and gives a value in the range $-0.6 \leq 10^4 C_{12}^r(m_\rho) \leq 0.6 \text{ GeV}^{-2}$ which implies that the corresponding contribution in eq.\eqref{eq:75} is negligibly small.

We can determine the experimental value of $L_{2}^{\text{eff}}$ from the experimental value of the combination $C_{NS}^{+}$ (\cite{24}) and its chiral expansion. If we use the expansion up to order $p^4$ we find

$$L_{2}^{\text{eff}}(m_\rho)\big|_{p^4} \simeq 1.32 \times 10^{-3} .$$

This value agrees rather well with that found from the resonance saturation model $L_{2}^{V} = 1.2 \times 10^{-3}$ \cite{19}. If we now include the $O(p^6)$ correction in the chiral expansion we find

$$L_{2}^{\text{eff}}(m_\rho)\big|_{p^4+p^6} = (0.16 \pm 0.08) \times 10^{-3} . \quad (77)$$

We would like now to compare with the result from the resonance saturation model also including $O(p^6)$ corrections which has the following expression,

$$L_{2}^{\text{eff}}\big|_{V} = \frac{G_{V}^{2}}{4M_{V}^{2}} \left\{ 1 + \frac{m_{K}^{2} + m_{\pi}^{2}}{M_{V}^{2}} \left[ 1 - 2e_{V}^{m} \right] + 2\sqrt{2} \frac{M_{V}^{2}}{G_{V}} \left( \hat{g}_{V1}^{m} + \hat{g}_{V2}^{m} \right) \right\} . \quad (78)$$

Numerically, using the results from sec. 3.4, one obtains

$$L_{2}^{\text{eff}}\big|_{V} \simeq 2.04 \times 10^{-3} . \quad (79)$$
Keeping in mind that the determination of the resonance Lagrangian couplings is approximate (due, in particular, to the use of large $N_c$ type approximations), it is nevertheless clear that the value of $L_2^{eff}$ obtained above \((79)\) from our resonance saturation model differs quite substantially (by about a factor of ten) from the experimental determination of $L_2^{eff}(\mu)$ when $\mu = m_\rho$.

\subsection*{3.8 Discussion}

This problem cannot be attributed to the resonance model itself since we have checked that the results do correspond, at least approximately, to the contribution from the resonance region in the sum rule expression of $C_{NS}$ (the integrand is shown in fig. 3). It must therefore be concluded that the values of the LEC’s can fail to be dominated by the resonance contributions at $O(p^6)$ with $\mu = m_\rho$.

One obvious possible reason for the failure of resonance saturation is that the variation of the LEC’s as a function of $\mu$ at $O(p^6)$ can be much faster than it is for the $O(p^4)$ LEC’s. This is illustrated in fig. 3 which shows the behaviour of $L_2^{eff}$ as a function of the scale. The figure shows that, in fact, a scale $\mu_0$ does exist such that resonance saturation of $L_2^{eff}$ is exact, but its value, $\mu_0 \simeq 0.45$ GeV is significantly smaller than $m_\rho$.

One must also keep in mind that the renormalized coupling constants are obtained from the bare ones by a minimal subtraction procedure. Their values thus depend both on the regularization scheme and on the subtraction convention. The procedure adopted in ChPT (based on dimensional regularization and modified minimal subtraction) was shown to lead to natural values for the couplings at order $p^4$. This, however, is not guaranteed to remain true at arbitrary higher orders. A remark is in order, finally, concerning the chiral expansion of the quantity $L_2^{eff}$. In the resonance saturation model, the contribution of order $p^6$ is rather large, amounting to a 50% correction as compared to the $O(p^4)$ one. At first sight, the situation seems to be worse for $L_2^{eff}(\mu)$: if we set $\mu = m_\rho$, the contribution of order $p^6$ practically cancels that of order $p^4$. In this case, however, the relative contributions strongly depend on the scale: if we take $\mu \simeq 0.55$ GeV the $O(p^6)$ contribution will be much smaller than the $O(p^4)$ one, while if we take $\mu \simeq 0.45$ GeV the relative contributions become similar to those in the resonance saturation model.

\section*{4 Summary}

Our goal was to extract some model independent informations about the $O(p^6)$ chiral coupling constants, about which little is known at present, and probe the validity, in this sector, of the idea of resonance dominance. We used input from the $\pi K$ scattering amplitude in the subthreshold region derived from experimental data using dispersion relations. In this way, we generated three constraints on the four LEC’s $C_1$ to $C_4$ and three constraints on eleven LEC’s among $C_5$ to $C_{22}$. These are associated with chiral operators which involve one insertion of the quark mass matrix. In line with the earlier work of ref.\([15]\) it appears natural, assuming resonance dominance, to associate the values of these LEC’s with flavour symmetry breaking in the light resonance sector. In order to implement this, we have considered a (vector) resonance Lagrangian which is more general than the one used in ref.\([15]\). We determined all the coupling constants in this Lagrangian from experiment, in a large $N_c$ spirit. In principle, a more consistent approach to the determination of such couplings is to appeal to asymptotic constraints\([12]\). In practice, the two approaches...
usually give similar results and, furthermore, it is often not possible to satisfy all the relevant asymptotic constraints using a minimal number of resonances (e.g. [49,58]). Here, in order to test some of our estimates for the resonance content of the LEC’s, we have used unsubtracted sum rules in which one restricts the integration range to the resonance region.

One of our initial motivations was to try to understand the reason for a number of significant discrepancies between the chiral $O(p^6)$ predictions of ref. [15] for certain subthreshold expansion parameters of the πK amplitude and the dispersive results. We found that improving the vector resonance Lagrangian does not help in resolving these discrepancies. We made no attempt to improve the scalar resonance Lagrangian but we identified a specific combination of chiral LEC’s which should be insensitive to that sector (beyond the effect of generating flavour symmetry breaking). A clear outcome of our analysis is that, if one sets the value of the chiral scale $\mu$ equal to the $\rho$-meson mass, then the value of this combination of LEC’s is not dominated by the resonance contribution. We have also encountered examples for which resonance dominance was reasonable, see sec.3.5.1. This suggests that in parallel to the efforts which are pursued in order to develop consistent resonance models (e.g. [33]) one should also try to obtain further direct determinations of the LEC’s $C_i$.

This result may be compared with the observation made in the baryon sector of ChPT [59] already at one loop. In dimensional regularization, the one-loop corrections to the baryon masses were found to be rather large requiring, in order to compensate for that, that the low-energy couplings be set to values which are unnaturally large. The origin of the problem was traced to the regularization procedure and the physical interpretation of the chiral scale $\mu$. One expects $\mu$ to correspond approximately to a momentum cutoff in the loop integrals. The authors of ref. [59] show that this expectation can break down when unequal mass particles propagate inside the loops. As a possible cure to this problem they proposed to use a regularization method different from dimensional regularization.

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