Dedekind on Higher Congruences and Index Divisors, 1871 and 1878.

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Dedekind’s theorem connecting ideal theory and polynomial congruences appears in all textbooks on algebraic number theory, but few books note its connection to the problem of “common index divisors.” As part of a project to study the history of this problem, we examine in detail two of Dedekind’s papers on the subject. (See also [17] for a similar analysis of Hensel’s main work [23] on the same problem.)

We begin with a summary of the mathematical questions that Dedekind is addressing here, then consider each of the publications in turn. In each case we give a complete annotated translation. The first publication is Dedekind’s notice [4] of the second edition of Dirichlet’s Vorlesungen über Zahlentheorie. This naturally focuses on Supplement X, the main addition to the text in the new edition. Midway through, Dedekind announces some new results that were proved and further elaborated in the second paper, Über der Zusammenhang zwischen der Theorie der Ideale und der Theorie der höheren Kongruenzen [7]. Both papers are translated and annotated, both to clarify the mathematics and to highlight some historical points.

1 Mathematical Background

When Kronecker and Dedekind set out to generalize Kummer’s theory of cyclotomic integers, they quickly ran into obstacles. Finding a way around these difficulties led each of them to develop a far more complicated theory than Kummer’s. As a result, each had to justify the extra work by highlighting what made it necessary.

Suppose $n > 0$ is an integer and let $\zeta$ be a primitive $n$-th root of unity. Kummer had found an explicit description in terms of congruences of how rational primes factor in the cyclotomic integers $\mathbb{Z}[\zeta]$. It seems that both
Dedekind and Kronecker saw that Kummer’s description could be interpreted in terms of congruences between polynomials (known as “higher congruences” at the time). In modern terms, it would go something like this.

**Theorem.** Let $n > 2$ be an integer, let $ζ$ be a primitive $n$-th root of unity, and let $Φ_n(x)$ be the $n$-th cyclotomic polynomial. Fix a prime number $p ∈ ℤ$ and let

\[ Φ_n(x) ≡ F_1(x)^{e_1}F_2(x)^{e_2}...F_r(x)^{e_r} \pmod{p} \]

be the factorization of $Φ(x)$ modulo $p$, where the $F_i(x)$ are distinct irreducible polynomials in $ℤ_p[x]$. Then the factorization of $(p)$ in $ℤ[ζ]$ is

\[ (p) = p_1^{e_1}p_2^{e_2}...p_r^{e_r}, \]

with distinct prime ideals $p_i = (p, F_i(ζ))$.

Of course, Kummer did not speak of ideals; instead, he thought of $p_i$ as the “ideal prime divisor” determined by $p$ and $F_i(x)$. This amounted to an explicit method for determining the exponent of $p_i$ in a factorization. In modern terms, an “ideal prime divisor” is essentially the valuation corresponding to $p_i$.

This beautiful result seemed to suggest the possibility of a very simple theory in the general case: for a general number field $ℚ(α)$, let $Φ(x)$ be the minimal polynomial for $α$ and factor it modulo $p$. One could then use this to define “ideal primes” à la Kummer.

The choice of $α$ is crucial, of course. At least one example would have been familiar to everyone: the field $ℚ(√{-3})$ is the same as the cyclotomic field of order 3. Kummer’s approach worked if one took $α$ to be a cube root of 1 but would not work if we took $α = √{-3}$. Both Dedekind and Kronecker figured out that one needed to work with all the algebraic integers in the field $ℚ(α)$.

That highlights the first difficulty: in the case of $ℚ(ζ)$ the ring of algebraic integers is exactly $ℤ[ζ]$, but this will not be true in general. If $K$ is a number field and $𝓞 ⊂ K$ is its ring of algebraic integers there may not exist any $α ∈ ℤ[α]$ such that $K = ℚ(α)$ and $𝓞 = ℤ[α]$. In such a situation, there is no obvious $Φ(x)$ to work with.

Under certain conditions we can still make it work, however. Given a prime number $p ∈ ℤ$, suppose we can find an $α$ such that $ℤ[α] ⊂ ℚ$ has index not divisible by $p$. Then factoring the minimal polynomial for $α$ modulo $p$ gives the correct factorization of $(p)$ in $𝓞$. This theorem was announced by
Dedekind in [4] and proved in [7]. It seems clear that Kronecker was also aware of it.

This allowed one to hope, then, that an explicit factorization theory could be based on a local approach: for each prime $p$, find a generator $\alpha$ such that $p$ does not divide the index $(\mathcal{O} : \mathbb{Z}[\alpha])$. Then apply the theorem to find the factorization. As we will see below, Dedekind says that he spent a long time trying to prove that such an $\alpha$ always exists.

Alas, this is not true: there exist number fields in which all of the indices have a common prime divisor. Dedekind pointed this out (and stated the factorization theorem) in [4], probably to explain why he had needed to take a different route. Kronecker says in his Grundzüge [26, §25, p. 384] of 1882 that he had found an example in 1858.

Both Dedekind and Kronecker pointed to this essential difficulty to justify introducing a new approach: ideals in Dedekind’s case, forms in many variables in Kronecker’s. Some years later, Zolotarev tried to extend Kummer’s theory directly in this style [34], but then realized that his approach would fail for finitely many primes. (Eventually, in a second paper [33], Zolotarev found still another way to work around the difficulty.) Dedekind’s paper [7] was, as is clear from the introduction, prompted by an announcement of Zolotarev’s work.

Kronecker also mentioned Zolotarev’s attempt in [26, §25], where he stated the problem in terms of discriminants. For each choice of $\alpha$, let $d(\alpha) = \text{disc}(\Phi(x))$ be the discriminant of its minimal polynomial. Let $d_K$ be the field discriminant. Then $d(\alpha) = m^2d_K$, where $m$ is exactly the index $(\mathcal{O} : \mathbb{Z}[\alpha])$. Kronecker, who always preferred specific elements to collections, thought about this as follows. The many element discriminants $d(\alpha)$ have a common factor $d_K$ which is the essential part, attached to the “Gattung” $K$ rather than to a specific element. The other factors of $d(\alpha)$ (i.e., the factors of $m$) are therefore “inessential.” So in the “bad” examples what is happening is that some prime $p$ is an inessential divisor of every element discriminant. Such primes were the “common inessential discriminant divisors.”

The name is perhaps ill-chosen, because it is perfectly possible for a prime $p$ to divide the discriminant $d_K$ and also divide the index $(\mathcal{O} : \mathbb{Z}[\alpha])$. Such a prime divisor is then both “essential” (it divides $d_K$) and “inessential”! Dedekind’s term “index divisor” seems more appropriate.

Kronecker’s example “in the thirteenth roots of 1” is probably the simplest one. He never gave the details, but they are probably as Hensel gave them in his Ph.D. thesis [22] (see also [29, 2.2]). Let $\zeta$ be a primitive 13-th
root of unity. There is a unique subfield \( K \) of degree 4 over \( \mathbb{Q} \). Since the discriminant of \( \mathbb{Q}(\zeta) \) is a power of 13, so is the discriminant of \( K \) (in fact, \( d_K = 13^3 \)). It follows from Kummer’s work that the prime number 3 is divisible by four ideal primes in \( K \), each of which has norm 3; let \( p \) be one of these. Since \( N(p) = 3 \), the field \( \mathcal{O}/p \) has three elements. Consider some \( \alpha \in K \). Since \( K \) is a normal field, he discriminant of the minimal polynomial of an integer in \( K \) is the product of differences of four integers in \( K \). Since there are only three congruence classes modulo \( p \) at least one of these differences must be divisible by \( p \). Since \( p \) lies above 3, the discriminant \( d(\alpha) \in \mathbb{Z} \) must be divisible by 3. Since \( d_K \) is a power of 13, 3 is an inessential divisor. In Dedekind’s terms, 3 is a common inessential discriminant divisor.

This set up the problem of determining exactly when this phenomenon happens. One of the things that interests us about this problem is that it was solved twice, apparently independently. Dedekind solved in his paper [7]. It is a sign of how little Kronecker followed Dedekind’s work that he suggested the problem of common inessential discriminant divisors to Hensel for his Ph.D. in 1882. Hensel did not solve it completely in his thesis [2] but he published a solution in 1894, in [23]. While Hensel refers to Dedekind’s 1878 paper, it is not clear how carefully he had read it. The relationship between the two solutions is complex and will be explored elsewhere.

Two remarks might make it easier to read these texts. First, Dedekind does not have the quotient construction, so when we would speak of \( \mathcal{O}/p \) he must talk about congruence classes modulo \( p \). For example, the norm of an ideal is defined as the number of congruence classes modulo that ideal. Second, despite the hints in Galois, at this point there was essentially no theory of finite fields. Instead, Dedekind relies on his paper [3], in which he discusses congruences modulo polynomials.

2 Dedekind’s Anzeige

The second edition of Dirichlet’s Vorlesungen über Zahlentheorie [12] appeared in 1871. The first edition, which was in fact written by Dedekind

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1This is global number field 4.0.2197.1 in [28].
2Hensel later generalized the numerical condition in the example above to give a sufficient criterion for the existence of common inessential discriminant divisors, and even attempted to prove the condition was also necessary, which it is not. See the careful discussion in [29, 2.2].
based on his notes from Dirichlet’s lectures, had contained nine supplements, mostly taken from Dirichlet’s papers and supplementary lectures. (See [14] for a translation (mostly) of the first edition; see [16] for more information on the book and its several editions.) The second edition included a new (tenth) supplement, entitled “On the Composition of Binary Quadratic Forms” [5]. Most readers would have expected to find here a simplified account of Gauss’s theory. If they read it, however, they would have been surprised to find, in the middle of the supplement, a whole new theory of factorization in general fields of algebraic numbers.

As he had done for the first edition in 1863, Dedekind wrote an article [4] for the September 20, 1871 issue of the *Göttingische gelehrte Anzeigen*. This was a weekly journal containing mostly book reviews, but Dedekind is of course writing about his own book(s). As one would expect, the article focuses almost entirely on the new content, i.e., Supplement X. It gives a cursory summary of the parts of the supplement containing well known material, then discusses the new theory of ideals in more detail. Surprisingly, after this explanation, Dedekind decided to go well beyond what is found in Supplement X. In this digression, Dedekind announced results about the relationship of ideal theory and “higher congruences.” These theorems were not in Supplement X and Dedekind would only publish their proofs 1878 ([7], translated below). Once this was done, he went back to a section-by-section discussion of the Supplement.

Other authors had attempted to generalize Kummer’s theory of ideal factors in cyclotomic fields. Since Kummer had reduced his factorization theory to congruence conditions, those attempts (like the early attempts of Dedekind and Kronecker) had turned on “higher congruences.” It seems that the discussion of “common index divisors” (the name came later) was included in the book notice exactly to explain why the approach via congruences was bound to fail. This suggests that Dedekind was well aware that his contemporaries would wonder whether something as innovative as the theory of ideals was justified. His digression into “higher congruences” makes the point that a new method is needed.

### 2.1 Translation

*Vorlesungen über Zahlentheorie*, by P. G. Lejeune Dirichlet. Edited and with additions provided by R. Dedekind. Revised and enlarged second edition. Braunschweig, Friedr. Vieweg und Sohn. 1871.
I discussed the 1863 first edition of this work in these pages (27 January 1864), and so I can refer to that previous article with regard to the origins and content of the book. The new edition differs from the first by way of a great many additions, either in footnotes or to the text itself. Many paragraphs have been completely reworked. These changes, which do not touch the essence of Dirichlet’s lectures, mainly reflect the decision to treat, in a new appendix, the tenth supplement, the theory of the composition of binary quadratic forms, which was omitted from the first edition for reasons discussed at that time. The generality with which Gauss presented the theory in the fifth section of the Disquisitiones Arithmeticae understandably causes significant difficulties for the beginner. This difficulty motivated Dirichlet to publish De formarum binariarum secundi gradus compositione, 1851. He says in his introduction:

De formarum compositione tunc non egi, quod argumentum ab illustriissimo Gauss in “Disquisitionum Artihmeticarum” sectione quinta maxima quidem generalitate sed per calculos tam prolixos tractatum esse constat, ut perpauci compositionis naturam percipere valuerint, eo magis quod summus geometra, ut ipse monuit, brevitati consulens theorematum difficiliorum demonstrationes synthetice adornavit, suppressa analysi per quam erant eruta. Quare confidere posse mihi videor, hujus argumenti expositionem novam et plane elementarem artis analyticae cultoribus non fore ingratam.

Since in this treatise only the first main theorem of the theory in question is proved and no indication is given of how to continue, I have taken a

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3 The book, including the supplements, is divided into 170 numbered “paragraphs,” which are actually sections, often several pages long. I will generally translate “section” from now on.

4 From here on everything in the book notice focuses on the content of Supplement X.

5 This is [11].

6 An idiomatic English translation might be: “I did not then take up the composition of forms, which topic was treated by the most illustrious Gauss in the fifth and largest section of this Disquisitiones Arithmeticae. That treatment is so general and contains such long calculations that very few are able to grasp the nature of the composition. This is all the more so because that great geometer, as he himself admonished in keeping an eye for brevity, gave demonstrations of more difficult theorems by synthesis, with the analysis by which they were unearthed suppressed. For that reason it seems I may be confident that a new and undoubtedly elementary exposition of this theory will not be unwelcome to the cultivators of the analytic art.”
somewhat different route, which agrees with that of Dirichlet in that only a special case of composition is considered. Sections 145–149 contain the general theorems about the composition of forms and classes of forms. These are used in sections 150, 151 to find the ratio of the class numbers of two determinants whose ratio is a square; this is the same problem treated according to Dirichlet’s method in sections 97, 99, 100. There follow in sections 152–154 the composition of genera and Gauss’s second proof of the quadratic reciprocity theorem. Sections 155–158 contain a proof of Gauss’s theorem that the duplication of any class results in the principal genus; it is based on a theorem of Lagrange and Legendre about the rational integer solutions of indeterminate equations of degree two in two unknowns.

In the paragraphs that follow I tried to introduce the reader to a higher domain in which algebra and number theory are intimately connected. In the course of my lectures on circle division and higher algebra, held in Göttingen in winter 1856–1857 before Mr. Sommer and Mr. Bachmann and in Winter 1857-1858 before Mr. Selling and Mr. Auwers, I was convinced that the study of the algebraic properties of numbers is most appropriately based on concepts that are directly linked to the simplest arithmetic principles. I replaced the name “rational domain” with the word “field,” by which I understand a system of infinitely many numbers that has the property that the sums, differences, products, and quotients of two such numbers belong to the same system. I say a field $Q$ is a divisor of a field $M$, and the latter a multiple of the first, when all the numbers contained in $A$ are also found in $M$. Any two fields $A$, $B$ always have a least common multiple, which can be denoted by $AB$, and also a greatest common divisor. When to

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7Dirichlet, following Gauss, assumes quadratic forms look like $ax^2 + 2bxy + cy^2$. Forms like $x^2 + xy + y^2$ are replaced by $2x^2 + 2xy + 2y^2$. As a result, they must allow forms where $\gcd(a, 2b, c) \neq \gcd(a, b, c)$. This is the difficulty treated in sections 97–100.

8This is the unexpected leap. To treat binary quadratic forms Dedekind wants to consider quadratic number fields. Since it is no harder (!) to treat all number fields, he proceeds to do so.

9See [21] for a reconstruction of the development of Dedekind’s theory. See [18] for an analysis of what Dedekind means by “simplest arithmetic principles.”

10Dedekind’s work is of course Körper, which translates as “body.” The standard English term is “field,” which I will use throughout.

11By “numbers” Dedekind seems to mean complex numbers. So his fields are all subfields of $\mathbb{C}$.

12So $Q$ is a divisor of $M$ if $Q \subset M$.

13We would call the lcm the compositum and the gcd the intersection of the two fields.
each number \(a\) in a field \(A\) there corresponds a number \(b = \varphi(a)\) so that 
\[\varphi(a + a') = \varphi(a) + \varphi(a') \quad \text{and} \quad \varphi(aa') = \varphi(a)\varphi(a'),\]
the the numbers \(b\) make up a field \(B = \varphi(A)\) that is conjugate to \(A\), which arises from \(A\) by the substitution \(\varphi\). These concepts are connected, in the algebraic direction, with the ideas of Galois and, in the number-theoretical side, with Kummer’s creation of the ideal numbers.\(^1\)

In section 159 are developed the general properties of a field \(\Omega\) that has only a finite number of divisors.\(^2\) In such a field there is always a finite quantity\(^3\) of numbers \(\omega_1, \omega_2, \ldots, \omega_n\) with the property that any given number \(\omega\) from the field can always and in only one way be expressed as
\[h_1\omega_1 + h_2\omega_2 + \cdots + h_n\omega_n,
\]
where \(h_1, h_2, \ldots, h_n\) are rational numbers, which are called the coordinates of the number \(\omega\) with respect to the basis \(\omega_1, \omega_2, \ldots, \omega_n\). The number \(n\) is called the degree of the field \(\Omega\). It follows quite easily that every number in the field is an algebraic number, namely the root of an equation of degree \(n\) whose coefficients are rational numbers. There are \(n\) different substitutions linking the field \(\Omega\) to conjugate fields. The product of the \(n\) values obtained from a given number \(\omega\) via these \(n\) substitutions is called the norm of \(\omega\). It is a homogeneous function of the coordinates with rational coefficients,\(^4\) therefore a rational number, which is denoted by \(N(\omega)\). Given a system of \(n\) numbers \(\alpha_1, \alpha_2, \ldots, \alpha_n\) from the field \(\Omega\), one builds the determinant of the \(n^2\) corresponding numbers in the \(n\) conjugate fields. The square of this determinant is a rational number, which I call\(^5\) the discriminant of the

\(^{14}\)Dedekind uses “substitution” for what we would call a function, here a field homomorphism.

\(^{15}\)As Dedekind said at the beginning of the paragraph, he is consciously creating a link between algebra and number theory, perhaps inspired by his teacher Dirichlet’s linking analysis and number theory.

\(^{16}\)Rather than use the dimension to define a finite extension, Dedekind focuses on the number of subfields, which allows him to stick to his “sounds like arithmetic” point of view. But he immediately points out that having finitely many subfields implies that there is a finite basis.

\(^{17}\)We struggled to translate “Anzahl von Zahlen,” which is literally the awkward “number of numbers.” We settled for “quantity of numbers.”

\(^{18}\)Dedekind remains attentive to the theory of forms, which is the ostensible subject of the supplement; the norm is a form of degree \(n\) in \(n\) variables.

\(^{19}\)This extension of the notion of discriminant seems to have appeared first here and in Supplement X.
numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and denote by $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_n)$. It is not possible and not necessary to go into the analytical developments that are linked to these concepts; they are only given in this paragraph to the extent that it seemed appropriate for a better understanding.

In the following section 160 all the algebraic numbers (which also form a field) are divided into integral and fractional numbers. An [algebraic] integer is understood to mean a root of an equation with highest coefficient $= 1$ and whose other coefficients are rational integers. From this concept simple propositions about divisibility, units, and relatively prime numbers are derived for later use.

The following section 161 contains an auxiliary theorem for our theory through which Gauss’s notion of congruence between numbers can be generalized. By a module I understand a system whose sums and differences still belong to the same system. The congruence $\omega \equiv \omega' \pmod{m}$ means that the difference $\omega - \omega'$ belongs to the system $m$. This concept has a broader scope than its extraordinary simplicity seems to promise, but we only give here what will serve to facilitate the subsequent presentation.

After these preparations, the integers of a field $\Omega$ of degree $n$ are investigated in section 162. They form a module $\mathcal{O}$, and it is shown first that one can find $n$ integers $\omega_1, \omega_2, \ldots, \omega_n$ that are basis numbers of the field, so that any integer can be represented as

$$\omega = h_1\omega_1 + h_2\omega_2 + \cdots + h_n\omega_n,$$

where all the coordinates $h_1, h_2, \ldots, h_n$ are whole numbers. The discriminant $\Delta(\omega_1, \omega_2, \ldots, \omega_n)$ of such a basis, which I call a fundamental series, has the smallest possible absolute value. This nonzero rational integer is of particular significance for the field $\Omega$, it is called the discriminant or the fundamental

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20 To a modern reader, the end of section 159 is very hard to follow; perhaps Dedekind’s readers would have agreed. He skips all of it here.

21 Dedekind consistently writes “ganze Zahl” for an algebraic integer and “ganze rational Zahl” for an element of $\mathbb{Z}$. I will translate “integer” and “rational integer” respectively.

22 This typical Dedekindian move feels perfectly comfortable for the modern reader, but it was not the way things were usually done in the 19th century. Dedekind here introduces a new algebraic idea, a “module,” and proceeds to establish the basic properties before returning to the theory of fields.

23 Dedekind’s modules are free $\mathbb{Z}$-submodules of $\mathbb{C}$.

24 The first step is to show the existence of an integral basis. A few lines later Dedekind will use the term “fundamental series” for such a basis.

25 Grundreihe.
number\textsuperscript{26} and denoted by $\Delta(\Omega)$. It divides the discriminant of any system of $n$ integers, and the quotient is a square. Furthermore, if $\mu$ is a nonzero number in $\mathcal{O}$, the number of incongruent integers with respect to $\mu$ is equal to the absolute value of the norm $N(\mu)$. We then draw attention to a strange phenomenon\textsuperscript{27} first observed in the case of cyclotomic fields. It consists in this: a integer that cannot be decomposed as a product of other integers does not always play the role of a true prime number. This was the starting point for Kummer's creation of ideal numbers.

My goal in section 163 is to propose a theory\textsuperscript{28} that applies to all [finite] fields. The fundamental idea is as follows. If $\mu$ is a nonzero number in $\mathcal{O}$, then the system $\mathfrak{m}$ of all numbers in $\mathcal{O}$ that are divisible by $\mu$ has the following two properties:

I. The sum and difference of two numbers in $\mathfrak{m}$ is a number in $\mathfrak{m}$; that is, $\mathfrak{m}$ is a module.

II. Every product of a number in $\mathfrak{m}$ with a number in $\mathcal{O}$ is also a number in $\mathfrak{m}$.

It is not true that conversely, every system $\mathfrak{m}$ of integers from a field that has these two properties, which from now on I will call an ideal, is always the set of numbers that are divisible by some fixed $\mu$. When this is the case, I say $\mathfrak{m}$ is a principal ideal and denote it by the symbol $i(\mu)\textsuperscript{29}$. We then investigate the properties of all the ideals of the field $\Omega$, and the following main result follows. Multiplying each number of an ideal $\mathfrak{a}$ by each number of an ideal $\mathfrak{b}$, these products and their sums make up an ideal, which is the product of the two factors $\mathfrak{a}$ and $\mathfrak{b}$ and is denoted by $\mathfrak{ab}\textsuperscript{30}$. It then clearly follows that $\mathfrak{a}\mathcal{O} = \mathfrak{a}$, $\mathfrak{ab} = \mathfrak{ba}$, $(\mathfrak{ab})\mathfrak{c} = \mathfrak{a}(\mathfrak{bc})$, and that from $\mathfrak{ab} = \mathfrak{ac}$ it follows that $\mathfrak{b} = \mathfrak{c}$. One says an ideal $\mathfrak{p}$ different from $\mathcal{O}$ is a prime ideal when it has no factors different from $\mathcal{O}$ and $\mathfrak{p}\textsuperscript{31}$. A composite ideal can be decomposed as a product of prime ideals and in only one way. One then defines the norm $N(\mathfrak{a})$ of an ideal $\mathfrak{a}$ to be the quantity of numbers in $\mathcal{O}$ that are incongruent

\textsuperscript{26}Grundzahl.

\textsuperscript{27}The “strange phenomenon” is the failure of unique factorization, but Dedekind describes it by saying that some irreducible elements of $\mathcal{O}$ do not behave like true primes.

\textsuperscript{28}A theory of factorization is meant. The ideal primes of Kummer will be replaced by prime ideals.

\textsuperscript{29}In [7], the notation was changed to either $\mathcal{O}\mu$ or $\mathcal{O}(\mu)$, the latter when $\mu$ is an explicit number. See below.

\textsuperscript{30}Check against Supp X!

\textsuperscript{31}Dedekind's definition of prime ideal sticks to the analogy with ordinary arithmetic. This definition is shown to be equivalent to the modern definition in [7]; see p. 27 below.
with respect to the module $a$. We have $N(ab) = N(a)N(b)$. In this way we obtain a complete analogy with the laws of divisibility in rational number theory.

This entire theory is intimately connected with the so-called theory of higher congruences, which was suggested by Gauss and developed on the work of Galois, Schönemann and others. It was first the works of Kummer on cyclotomic ideal numbers and the study of the algebraic investigations of Galois that led me to consider the theory of higher congruences, and I published a brief outline of that theory (Crelle’s Journal, Vol. 54) later sought, with its help, to create a general theory of ideal numbers, but was distracted from it by other work until the publication of this work led me back to that subject. The renewed effort led me to my new theory of ideals, which seems preferable to me because it is based on much simpler concepts. In my presentation I did not deal closely with the connection with the theory of higher congruences, because I feared that the extent of my appendix would become too large. For readers who are interested in this connection, I hope the following remarks may be useful.

Let $\omega$ be an arbitrary number in $\mathcal{O}$, and set

$$\Delta(1, \omega, \omega^2, \ldots, \omega^{n-1}) = D^2 \Delta(\Omega).$$

Then $D$ is always a rational integer, namely a homogeneous function of degree $\frac{1}{2} n(n-1)$ of the coordinates with rational integer coefficients. If then $p$ is

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32. Here Dedekind veers off the track. So far he has given a blow-by-blow account of Supplement X, but none of the material on higher congruences is found there.

33. This is [3], which discusses both congruences between polynomials and “double congruences” that amount to the theory of finite fields, hence the reference to Galois.

34. Presumably the publication of the first edition of the Vorlesungen?

35. Many of Dedekind’s contemporaries did not feel Dedekind’s approach was in any way “simpler.” In particular, there was a lot of resistance to working with infinite sets as objects. As such, ideals seemed very abstract.

36. The supplement was 118 pages long, and, split into supplements X and XI, came to be much longer in later editions.

37. By which we suspect he means those who want to know why Dedekind did not stick to the straightforward approach.

38. Here begins the digression; this material is not in Supplement X.

39. Dedekind does not mention that this is the discriminant of the polynomial of degree $n$ with $\omega$ as a root, but of course that was the original sense of “discriminant” that was generalized to $n$-tuples. He also does not mention the possibility that $D = 0$; he is more explicit about this in the second paper.

40. This homogeneous function was later called the “index form.”
a rational prime number and we are given a number $\omega$ for which $D$ is not divisible by $p$, then the decomposition of the principal ideal $i(p)$ as a product of prime ideals is easily found via the theory of higher congruences.\textsuperscript{41} The number $\omega$ satisfies an equation of degree $n F(\omega) = 0$ and if

$$F(x) \equiv P_1(x)^{e_1} P_2(x)^{e_2} \cdots P_m(x)^{e_m} \pmod{p},$$

where $P_1, P_2, \ldots, P_m$ are pairwise distinct prime functions\textsuperscript{42} of the variable $x$ of degrees $f_1, f_2, \ldots, f_m$ respectively, then we have

$$i(p) = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m},$$

where $p_1, p_2, \ldots, p_m$ are pairwise distinct prime ideals with norms $p^{f_1}, p^{f_2}, \ldots, p^{f_m}$, respectively. From this follows easily\textsuperscript{43} the following theorem, which is fruitful for both algebraic and number-theoretic investigations:

*The prime number $p$ divides the fundamental number $\Delta(\Omega)$ of the field $\Omega$ if and only if $p$ is divisible by the square of a prime ideal.*\textsuperscript{44}

At first I thought it very likely\textsuperscript{45} that for any given prime number $p$ there would exist an integer $\omega$ such that the number $D$ was not divisible by $p$. Only when all my attempts to prove the existence of such a number were unfruitful did I set myself the task of investigating whether this conjecture was incorrect.\textsuperscript{46} We conjecture that whenever $p$ is divisible by $r$ distinct prime ideals $p$ whose norms have value $p^f$, there must exist $r$ distinct prime functions $P$ of degree $f$. Conversely,\textsuperscript{47} when this last condition is always satisfied, then one can prove the existence of a number $\omega$ with the desired property. In the simplest case when $f = 1$, there are exactly $p$ distinct prime functions of degree one. The question then becomes whether there exists a field $\Omega$ in which $p$ is divisible by $(p + 1)$ distinct prime ideals, all of which of norm $p$. The degree of such a field must then be $= p + 1$. The simplest case arises when one takes $p = 2$, leading to the question: do there exist cubic

\textsuperscript{41}This is called “Dedekind’s Theorem” in many modern textbooks. The proof was first given in [7]; see below.

\textsuperscript{42}Dedekind uses “prime function” for irreducible polynomial.

\textsuperscript{43}As Dedekind will clarify, it follows easily only for primes that do not divide $D$; see pages [15] and [53] below.

\textsuperscript{44}No italics in the original, but the statement does get its own paragraph.

\textsuperscript{45}Compare the very similar comments in [7] below, page [38].

\textsuperscript{46}As indeed it is, which Dedekind will show. Kronecker claimed that he knew this in 1858.

\textsuperscript{47}This is one of the main results in [7]; it is proved again in [23].
fields in which the number 2 is divisible by three distinct prime ideals? In such a field $D$ would always be an even number. One can always assume that the fundamental series of a cubic field consists of the number 1 and two integers $\alpha, \beta$ whose product is rational. One then has

$$\alpha \alpha = a' \alpha + b \beta - bb'$$
$$\beta \beta = a \alpha + b' \beta - aa'$$
$$\alpha \beta = ab$$

where $a, b, a', b'$ are rational integers with no common divisor and we can compute

$$\Delta(\Omega) = \Delta(1, \alpha, \beta) = a'^2 b^2 + 18 a b a' b' - 4 a a'^3 - 4 b b'^3 - 27 a^2 b^2.$$ 

If we now set

$$\omega = z + x \alpha + y \beta,$$

with $z, x, y$ any rational integers, then

$$\omega^2 = z^2 = z^2 - bb' x^2 - aa' y^2 - aa' y^2 + 2abxy$$
$$+ (a' x^2 + ay^2 + 2xz) \alpha + (bx^2 + b' y^2 + 2yz) \beta,$$

48Dedekind has twice reduced to the “simplest case” in order to find his example. The task now is to find a cubic field in which 2 splits completely, which will force $D$ to be even for every choice of $\omega$. In his example, however, Dedekind proves directly that $D$ is always even by computing it explicitly, and then appears to conclude that 2 splits completely.

49Dedekind will begin by taking an integral basis and considering the corresponding multiplication table. This provides him with a number of parameters he can adjust to obtain the desired field.

50If $\alpha \beta = \ell \alpha + m \beta + n$, replace $\alpha$ by $\alpha - m$ and $\beta$ by $\beta - \ell$.

51Dedekind offers no explanation for why the formulas should look like this. To spare the reader some time, here is an explanation. Define integers $a, a', b, b', c, c', n$ by $\alpha \beta = n, a^2 = a' \alpha + b \beta - c$ and $\beta^2 = a \alpha + b' \beta - c'$. Then notice that $n \beta = a \beta^2$. Expanding the latter and equating basis coefficients gives $n = ab$, $c = bb'$, $c' = aa'$, as Dedekind says. Note, however, that $a, b, a', b'$ are not arbitrary: the minimal polynomials of $\alpha$ and $\beta$ depend on them, and bad choices will give polynomials that are not irreducible, so that the $\mathbb{Q}$-algebra defined by these equations will not be a field.

52If some prime divides all four integers, then $p^2$ would divide $\alpha^2, \alpha \beta, \text{ and } \beta^2$, and hence $p^2$ would divide $(\alpha + \beta)^2$, and so $\frac{1}{p} \alpha + \frac{1}{p} \beta \in \mathcal{O}$, contradicting the assumption that $\{1, \alpha, \beta\}$ is an integral basis.

53Given the information we have, we can compute the traces of $\alpha^2, \beta^2, \text{ and } \alpha \beta$; from that information it is easy to compute the discriminant.
and it follows\(^\text{54}\) that
\[
D = bx^3 - a'x^2y + b'xy^2 - ay^3,
\]

independent of \(z\), which is expected from the definition of \(D\). Even though \(a, b, a', b'\) have no common divisor, \(D\) will be an even number whenever \(a\) and \(b\) are even and \(a'\) and \(b'\) are odd.\(^\text{55}\) It must then be that the number \(2\) is divisible by three distinct prime ideals. This is completely confirmed by the example\(^\text{56}\)
\[
a = b = 2, \quad a' = -b' = 1, \quad \Delta(\omega) = -503;
\]

we have\(^\text{57}\)
\[
i(2) = abc, \quad i(\alpha) = a^c, \quad i(\beta) = b^c,
\]

where \(a, b, c\) are three distinct prime ideals.\(^\text{58}\)

Another example can be obtained in the following way. With respect to the modulus \(p = 2\) there exists only one prime function of degree two, namely \(x^2 + x + 1\). Therefore when in a field \(\Omega\) the integer \(2\) is divisible by at least two distinct prime ideals whose norm \(= p^2 = 4\), then \(D\) must be even. In

\(^{54}\) We have expressed the basis \(\{1, \omega, \omega^2\}\) as a linear combination of \(\{1, \alpha, \beta\}\); \(D\) is the determinant of that matrix.

\(^{55}\) If \(a, b\) are even and \(a', b'\) are odd,
\[
D \equiv x^2y + xy^2 \equiv xy(x + y) \equiv 0 \pmod{2}
\]

for all integers \(x, y\). Thus, Dedekind has shown directly that \(2\) is a common index divisor for any cubic field of this form. It remains to show that there is actually a choice of \(a, b, a', b'\) that makes it all work.

\(^{56}\) Dedekind now chooses the quadruple \((2, 2, 1, -1)\). To see that this is not a random choice, note that the minimal polynomial for \(\alpha\) is \(x^3 - a'x^2 + bb'x - ab^2\). While this is irreducible for most choices of the quadruple \((a, b, a', b')\), that is not always the case. Dedekind’s choice gives \(x^3 - x^2 - 2x - 8\), which is indeed irreducible and so we have a cubic field of discriminant \(-503\), global number field \(3.1.503.1\) in \([28]\). But if we chose \((2, 2, 1, 1)\) we would get \(x^3 - x^2 + 2x - 8 = (x - 2)(x^2 + x + 4)\). Even more dramatically, \((6, 2, 9, 13)\) would give \(\Delta = 1\), which is impossible for a number field, and indeed \(x^3 - 9x^2 + 26x - 24 = (x - 2)(x - 3)(x - 4)\).

\(^{57}\) To confirm that he has the example he wants, Dedekind writes out (without proof) the factorizations of \(2, \alpha, \beta\). Note, however, that he has already shown that \(2\) is a common index divisor; given that, his converse theorem forces the factorization to be as he wants, since there do exist irreducible polynomials of degree two and three in \(F_2[x]\).

\(^{58}\) See the more complete discussion in \([7]\), page \(48\) below, where Dedekind defines the three ideals explicitly and checks all of these statements.
this case the degree of the field must be at least \(= 4\). The phenomenon in fact occurs in the biquadratic field\(^59\) defined by the equation

\[ \alpha^4 - \alpha^3 + \alpha^2 - 2\alpha + 4 = 0. \]

The numbers \(1, \alpha, \beta = 2 : \alpha, \text{ and } \gamma = \alpha^2 - \alpha\) are a fundamental series and the fundamental number is \(13^2 \cdot 17\).

Thus, there exist fields \(\Omega\) in which the number \(D\) above is always divisible by certain singular prime numbers \(p\). Of course there are only finitely many such primes. I remark, however, that the theorem above, characterizing of the rational primes that divide the fundamental number \(\Delta(\Omega)\) of a field, remains valid in general, but it would take us too far afield were I to give a proof of this theorem or to explore its significance for the theory of fields.

After this digression, I continue to summarize the contents of the sections that follow. In section 164 the ideals of the field \(\Omega\) are divided into a finite number of classes. Two ideals are called equivalent when their product by some fixed ideal is a principal ideal. An ideal class consists of all ideals that are equivalent to a given ideal. The principal class consists of the principal ideals. These ideal classes allow a composition that has the same properties of the composition of classes of quadratic forms.

In section 165 I show the relationship between the composition of ideal classes and the decomposable homogeneous forms that arise from the same field \(\Omega\)\(^60\).

Section 166 gives Dirichlet’s theory of units in a generalized form. The presentation is completely independent of the previous one. In section 167 this theory is used to obtain an expression for the number of ideal classes by way of an infinite series, much like the determination of the class number of quadratic forms. At this point I break away from the study of the general problem, since my investigations of this topic have not yet been crowned with sufficient success to be published. The sections that follow, 168–170, illustrate the general theory by applying it to the example of quadratic fields.

So far it appears that the theory of ideal numbers has been the subject of serious research by only four or five mathematicians\(^61\). My heartfelt wish is that the new edition of Dirichlet’s Vorlesungen über Zahlentheorie may facil-

\(^59\)This is global number field 4.0.2873.1 in [28].

\(^60\)These are the forms given by the norm function; they are “decomposable” because the norm is a product by definition.

\(^61\)Who were they?
itate access to this large subject and perhaps to motivate a larger number of mathematics to apply their powers so that, amidst the tremendous recent progress made in geometry and in the theory of functions, number theory may not be left behind.

July 22 1871
R. Dedekind

3 The 1878 Paper

The title of [7] translates as “On the Relationship between the Theory of Ideals and the Theory of Higher Congruences.” It tells us that the paper will discuss the connections between two subjects about which Dedekind had already written: the theory of “higher congruences” and the theory of ideals. By “higher congruences” Dedekind means not only congruences modulo a prime between polynomials of higher degree but also the kind of congruence he will write as “\text{modd } p, P,\text{” }$ where $p$ is a prime and $P$ is a polynomial. The notation “\text{modd}” is intended to call attention that there are two moduli in play. Higher congruences had been discussed by Dedekind in his Abriß of 1857, which includes much of what we would now describe as the theory of finite fields. That paper is one of the main references used; Dedekind denotes it as “C” for short.

At this point, Dedekind had given two accounts of his theory of ideals: first in [5], Supplement X of the second edition of Dirichlet’s Vorlesungen über Zahlentheorie and then in an article [9] published in French. Dedekind refers to these as “D” and “B”; for an English reader B is the preferred reference, since it was translated by John Stillwell and published by Cambridge University Press [9].

In what follows we give a loose annotated translation of [7]. The translation is “loose” in the sense that we have not tried to preserve the exact syntactic structure of Dedekind’s long sentences nor always attempted (and certainly not always succeeded) to capture every nuance of meaning. We have, however, tried to translate the mathematical content precisely, mostly preserving Dedekind’s terminology. Our annotations are given as numbered annotations.

\footnote{Famously, this did not happen. See, for example, the correspondence with Lipschitz translated in [9, Section 0.7].}

\footnote{Today we would describe this as working in the quotient ring $\mathbb{F}_p[x]/(P)$.}

\footnote{We will nevertheless write $\text{mod } p, P$.}
footnotes; Dedekind’s own footnotes are marked with asterisks. Page numbers in [8] are indicated in the margin. Dedekind numbers his main results as I, II, III, etc.; we have labeled those theorems accordingly, but have highlighted other results (usually given by Dedekind in italics) as theorems as well.

In [8], Öystein Ore added several endnotes, which we give in summary form at the end. The editors also added a few footnotes that we have translated in annotations, distinguishing them from Dedekind’s original footnotes. There are several spelling changes made in [8]; for example, “Discriminante” becomes “Diskriminante.” When we quote the German, we have tried to stick to the original spelling.

3.1 Translation

On the Relationship between the Theory of Ideals and the Theory of Higher Congruences

by R. Dedekind

The new principles by which I arrived at a theory of ideals that is rigorous and without exceptions were first explained seven years ago in the second edition of the Lectures on Number Theory by Dirichlet (§ 159–170) and more recently given, in greater detail and in slightly modified form, in the Bulletin des sciences mathématiques et astronomiques (t. XI. p. 278; t. I (2e. serie), p. 17, 69, 144, 207). Stimulated by the great discovery of Kummer, I had been concerned with this subject for many years, starting from a completely different basis, namely the theory of higher congruences. Although these investigations brought me very close to the desired goal, I decided not to publish them, because the theory that emerges suffers from two imperfections. The first is that the investigation of a domain of integral algebraic numbers begins first with the consideration of a certain number and the equation corresponding to it, which is then interpreted as a congruence. The definitions of ideal numbers (or rather of divisibility by ideal

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numbers) are obtained in this way. Since everything depends on a specific representation, it follows that the invariant character of the definition cannot be recognized from the start. The second imperfection of this approach is that there are peculiar exceptional cases that require special treatment.

My more recent theory, on the other hand, is based exclusively on notions such as fields, [algebraic] integers, and ideals, whose definition does not require any particular form of representation of the numbers, removing the first defect. The power of these extremely simple concepts is shown by the fact that, in proving the general laws of divisibility, a distinction between several cases never occurs again. I have made some remarks about the connection between the two types of justification and stated some theorems without proof in the *Göttingischen gelehrten Anzeigen* of September 20, 1871 (pp. 1488–1492). In particular I have discovered the reason for the existence of the peculiar exceptional cases mentioned above. Since then, a theory of ideal numbers by Zolotareff appeared in 1874, in a paper in Russian with the title *Théories des nombres entiers complexes, avec une application au calcul integral*. This was announced and abstracted in the *Jahrbuch über die Fortschritte der Mathematik* (Vol. 6, p 117). From the abstract it is clear that the theory of Zolotareff is also based on the theory of higher congruences, but that the treatment of the aforementioned exceptional cases is temporarily excluded and is reserved for a later presentation. I do not know if this prospective completion has since been published. Since, however, the connection between the two types of justification of general ideal theory is of sufficient interest in itself, I allow myself to provide here the proofs of the remarks given in the *Göttingischen gelehrten Anzeigen*.

I will assume as known both my theory of ideals and the theory of higher congruences, one of the main topics of this paper.

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66 This is one of Dedekind’s fundamental methodological principles: one should always try to define things in a way that is independent of specific choices, rather than making such choices and then proving invariance. He wanted his mathematics “coordinate-free.”

67 Those special cases are the index divisors, one of the main topics of this paper.

68 Perhaps Hensel wrote because he did not think Dedekind’s “reason” was a sufficient answer to the question; see 17.

69 This is 34; the “completion” mentioned below eventually appeared as 33.

70 It seems, then, that Zolotarev’s paper was the main stimulus for writing this paper.

*I* can only refer to the abstract. After several unsuccessful attempts to get it in the bookstore, I have recently obtained the original through the kindness of Professor Wangerin, but given my ignorance of the Russian language, to my great regret I was able to understand very little, only what is clear from looking at the formulas.
congruences, of which I gave a short description earlier in Borchardt’s *Journal* (Vol. 54, p. 1). For brevity, I will cite this paper on congruences as C, the second edition of Dirichlet’s number theory as D, and the paper in the *Bulletin des sciences mathématiques* as B.

§ 1

Let $\Omega$ be a finite field of degree $n$, and let $\mathcal{O}$ be the domain of all [algebraic] integers contained in it. There always exist $n$ independent integers $\omega_1, \omega_2, \ldots, \omega_n$ which are a basis for the domain $\mathcal{O}$, that is, the system $\mathcal{O}$ is identical with the collection $[\omega_1, \omega_2, \ldots, \omega_n]$ of all numbers $\omega$ of the form

$$\omega = h_1\omega_1 + h_2\omega_2 + \cdots + h_n\omega_n,$$

where $h_1, h_2, \ldots, h_n$ are arbitrary rational integers. The discriminant

$$\Delta(\omega_1, \omega_2, \ldots, \omega_n) = \Delta(\Omega) = D,$$

71 This is [3]; Borchardt was then the editor of the *Journal für die Reine und Angewandte Mathematik*.

72 This is [12], but more specifically [5].

73 This is [6], but we cite the English translation [9].

74 This section introduces the key objects in play: the ring of integers $\mathcal{O}$ of a number field $\Omega$, the order $\mathcal{O}' = \mathbb{Z}[\theta]$, and the index $k$.

75 Dedekind says “finite field” for what we would call “a finite extension of $\mathbb{Q}$.” He never considers fields with finitely many elements.

76 Dedekind uses the lowercase fraktur $\mathfrak{o}$.

77 Dedekind uses “ganzen Zahlen,” literally “whole numbers,” for algebraic integers. I will typically translate “integers.” The elements of $\mathbb{Z}$ are “rational integers.”

78 The notion of the discriminant of a set of algebraic numbers seems to have been created by Dedekind by analogy to the older notion of the discriminant (or determinant) of a polynomial. In this paper Dedekind typically uses “Discriminante” for the general construct, reserving “Grundzahl,” which we translate as “fundamental number,” for this particular discriminant.

19
which is independent of the choice of the basis numbers, \( \omega_1, \omega_2, \ldots, \omega_n \), is called the fundamental number or the discriminant of the field \( \Omega \). (D. § 159, 160, 162; B. § 12–18).

Now if \( \theta \) is a specific algebraic integer in the field, we can set

\[
1 = c_1^{(0)} \omega_1 + c_2^{(0)} \omega_2 + \cdots + c_n^{(0)} \omega_n \\
\theta = c_1^{(1)} \omega_1 + c_2^{(1)} \omega_2 + \cdots + c_n^{(1)} \omega_n \\
\theta^2 = c_1^{(2)} \omega_1 + c_2^{(2)} \omega_2 + \cdots + c_n^{(2)} \omega_n \\
\vdots \\
\theta^{n-1} = c_1^{(n-1)} \omega_1 + c_2^{(n-1)} \omega_2 + \cdots + c_n^{(n-1)} \omega_n
\]

where all the \( n^2 \) coefficients or coordinates \( c \) are rational integers, and we will have

\[
\Delta(1, \theta, \theta^2, \ldots, \theta^{n-1}) = Dk^2,
\]

where

\[
k = \sum \pm c_1^{(0)} c_2^{(1)} \cdots c_n^{(n-1)}
\]

is a rational integer. The absolute value of this number \( k \), which is independent of the choice of integral basis, will for brevity from now on be called the index of the integer \( \theta \). If \( k \) is not 0, as we will always assume, the numbers

\[
1, \theta, \theta^2, \ldots, \theta^{n-1}
\]

will be independent of each other (D. § 159; B. § 4, 15, 17) and \( \theta \) will be the root of an irreducible equation of degree \( n \)

\[
F(\theta) = \theta^n + a_1 \theta^{n-1} + \cdots + a_n = 0,
\]

where the coefficients \( 1, a_1, a_2, \ldots, a_n \) are all rational integers.

If we let \( \varphi(t) \) be any function of the variable \( t \), — and I remark that always, by this name [function] and by an expression of the form \( \varphi(t), f(t), \ldots \)

---

79 Dedekind uses “Basiszahlen.” From here on I will use the modern term “integral basis.”

80 Dedekind writes \( c'_i \) where I have \( c_i^{(1)}, c_i^{(2)} \), etc.

81 The meaning of this notation, standard at the time, is \( k = \text{det}[c_i^{(j)}] \). Dedekind does not use matrices, which had not yet been invented, nor does he represent the determinant as an array.

82 This running assumption is crucial, but it is not mentioned every again. It is equivalent to assuming that \( \theta \) is a generator.
in this treatise one should always understand an entire function of \( t \) whose coefficients are rational integers — the set \( \mathcal{O}' \) of all other numbers of the form

\[
\omega' = \varphi(\theta)
\]

is called an order (D. § 165, 166; B. § 23); all such numbers are integers of the field \( \Omega \) and therefore are contained also in \( \mathcal{O} \). Clearly it suffices to take only the functions

\[
\varphi(t) = x_0 + x_1 t + x_2 t^2 + \cdots + x_{n-1} t^{n-1}
\]

whose degree is smaller than \( n \), since if \( \varphi_1(t) \) has degree larger than \( n \) we can divide it by

\[
F(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_{n-1} t + a_n.
\]

The remainder \( \varphi(t) \) will have degree less than \( n \) and at the same time \( \varphi_1(\theta) = \varphi(\theta) \). In the notation used above (B. § 3) we can set

\[
\mathcal{O}' = [1, \theta, \theta^2, \ldots, \theta^{n-1}].
\]

It also follows from the irreducibility of the equation \( F(\theta) = 0 \) that each number \( \omega' \) can be represented in the form \( \varphi(\theta) \) in only one way; nevertheless, in what follows we will not always restrict ourselves to that form of representation, but rather allow functions of any degree.

Prime numbers \( p \) — by which name we mean a rational positive prime number — fall in two cases once the fixed number \( \theta \) is chosen: the first case, which applies to infinitely many prime numbers, is when the index \( k \) of the number \( \theta \) is not divisible by \( p \). If \( k = \pm 1 \), then all primes are in this first case, and in fact \( \mathcal{O}' \) is identical to \( \mathcal{O} \). When however \( k^2 > 1 \), a finite number of primes will fall into the second case, namely the prime divisors of \( k \). The paragraphs that follow will show that the decomposition of the prime numbers \( p \) of the first kind (or rather the decomposition of the corresponding

---

83 So “function” always means a polynomial with integer coefficients.
84 Dedekind means \( \varphi(t) \) with \( \varphi \) of degree less than \( n \).
85 The idea, then, is to choose and fix \( \theta \in \mathcal{O} \) such that \( \Omega = \mathbb{Q}(\theta) \). Then \( \mathcal{O}' = \mathbb{Z}[\theta] \subset \mathcal{O} \) and \( k = (\mathcal{O} : \mathcal{O}') \) is the index. The rational primes \( p \) that divide \( k \) are those in the second case; the (infinitely many) other primes are in the first case.
principal ideals $\mathcal{O}_p$) as a product of prime ideals can be completely reduced to the decomposition of the function $F(t)$ as a product of functions that are prime with respect to the modulus $p$ (C. 6). On the other hand, it is not possible to do this in the same simple way for prime numbers of the second kind. The following remarks should be made before this investigation.

Let $p$ be a fixed prime of the first kind, so that $k$ is not divisible by $p$. In this case an element of $\mathcal{O}' = x_0 + x_1\theta + x_2\theta^2 + \cdots + x_{n-1}\theta^{n-1}$ is divisible by $p$ (which means it is equal to $p\omega$, with $\omega$ an integer, i.e., an element of $\mathcal{O}$) if and only if all the coefficients $x_i$ are divisible by $p$. To see that, notice that

$$\omega' = h_1\omega_1 + h_2\omega_2 + \cdots + h_n\omega_n$$

where

$$h_1 = c_1^{(0)} x_0 + c_1^{(1)} c_1 + c_1^{(2)} x_2 + \cdots + c_1^{(n-1)} x_{n-1}$$
$$h_2 = c_2^{(0)} x_0 + c_2^{(1)} c_1 + c_2^{(2)} x_2 + \cdots + c_2^{(n-1)} x_{n-1}$$
$$\ldots$$
$$h_n = c_n^{(0)} x_0 + c_n^{(1)} c_1 + c_n^{(2)} x_2 + \cdots + c_n^{(n-1)} x_{n-1}$$

It follows from the independence of $\omega_1, \omega_2, \ldots, \omega_n$ that $\omega'$ is divisible by $p$ if and only if each of the coordinates $h_1, h_2, \ldots, h_n$ is divisible by $p$. If so, each of the products $kx_0, kx_1, kx_2, \ldots, kx_{n-1}$ is also divisible by $p$, and therefore so are the coefficients $x_0, x_1, x_2, \ldots, x_{n-1}$. The same theorem can clearly also be stated as: a number $\omega'$ of the order $\mathcal{O}'$ is divisible by a prime number

---

86 Here and elsewhere Dedekind writes “Produkte aus lauter Primidealen,” literally “product of nothing but prime ideals,” to emphasize that it is a complete factorization into primes.

87 We would say irreducible modulo $p$.

88 This is the theorem announced in the Anzeige [4].

89 This paragraph is simple linear algebra. Dedekind uses the expression of the powers of $\theta$ in terms of the integral basis to rewrite $\omega'$ in terms of the basis.

90 If the matrix $[c_j^i]$ is invertible mod $p$, then $\vec{h} \equiv \vec{0}$ if and only if $\vec{x} \equiv \vec{0}$.

*This notation for principal ideals is more appropriate than $i(p)$, which I used earlier (D. § 163).
$p$ of the first kind if the quotient $\frac{\omega'}{p}$ is itself in the order $O'$. Conversely, when all the coefficients $x_0, x_1, x_2, \ldots, x_{n-1}$ are all divisible by $p$, then obviously $\omega'$ is divisible by $p$. Therefore two numbers $\varphi_1(\theta)$ and $\varphi_2(\theta)$ of the order $O'$ are congruent modulo $p$ (i.e., their difference $\varphi_1(\theta) - \varphi_2(\theta)$ is divisible by $p$) if and only if the coefficients of the two functions $\varphi_1(t)$ and $\varphi_2(t)$ are all congruent modulo $p$, i.e., in the sense of the theory of higher congruences, when we have

$$\varphi_1(t) \equiv \varphi_2(t) \pmod{p}$$

(C. 1). For this conclusion, however, we need to assume that the degrees of the functions $\varphi_1(t)$ and $\varphi_2(t)$ are less than $n$. If that is not the case, after dividing by $F(t)$ we obtain an identity of the form

$$\varphi_1(t) - \varphi_2(t) = F(t)\psi(t) + \psi_1(t),$$

where $\psi_1(t)$ has degree less than $n$, and then $\varphi_1(\theta) - \varphi_2(\theta) = \psi_1(\theta)$. We will have

$$\varphi_1(\theta) \equiv \varphi_2(\theta) \pmod{p}$$

when $\psi_1(t) = p\psi_2(t)$, that is, when

$$\varphi_1(t) - \varphi_2(t) = F(t)\psi(t) + p\psi_2(t).$$

The existence of such an identity is described in the theory of higher congruences as

$$\varphi_1(t) - \varphi_2(t) \equiv F(t)\psi(t) \pmod{p}$$

or simply as (C. 7) as

$$\varphi_1(t) \equiv \varphi_2(t) \pmod{p, F(t)}.$$

Conversely, it is clear that from that function congruence the number congruence

$$\varphi_1(\theta) \equiv \varphi_2(\theta)$$

\[91\text{Added a paragraph break here.}\]

\[92\text{Dedekind will now translate congruences between elements } \varphi(\theta) \text{ of } \mathbb{Z}[\theta] \text{ into congruences between the polynomials } \varphi(t). \text{ He does so first under the assumption that } \deg(\varphi) < n, \text{ and then in the general case. For the latter he uses the “double modulus” } p, F(t).\]
also follows; the two congruences are therefore equivalent. Thus in $\mathcal{O}'$ there are as many numbers $\varphi(\theta)$ that are incongruent modulo $p$ as there are functions $\varphi(t)$ incongruent with respect to the double modulus $p, F(t)$; there are $p^n$ of the latter (C. 8), which is also the number \((\mathcal{O} : \mathcal{O}p) = N(p)\) of numbers in $\mathcal{O}$ that are incongruent modulo $p$ (B. § 18; D. § 162), which implies the following result: each number $\omega$ of the domain $\mathcal{O}$ is congruent modulo $p$ to a number $\omega'$ of the order $\mathcal{O}'$.  

The same conclusion can be reached directly by the following simple argument. From the $n$ relations between the numbers $1, \theta, \theta^2, \ldots, \theta^{n-1}$, on the one hand, and the numbers $\omega_1, \omega_2, \ldots, \omega_n$, on the other, it follows that the products $k\omega_1, k\omega_2, \ldots, k\omega_n$ are contained in the order $\mathcal{O}'$, and therefore so are all the products $k\omega$ for any $\omega$ belonging to $\mathcal{O}$. Therefore we have $k\omega = \varphi(\theta)$. Now since $k$ is not divisible by $p$, we can choose a rational integer $l$ such that $kl \equiv 1 \pmod{p}$, and then it follows that $\omega \equiv kl\omega \equiv l\varphi(\theta) \pmod{p}$, so that $\omega$ is congruent modulo $p$ to a number $l\varphi(\theta)$ that is in the order $\mathcal{O}'$.  

Things are completely different when $p$ is a prime the second kind. In that case the determinant $k$ is divisible by $p$, and it is easy to prove that there exist $n$ rational integers $x_0, x_1, \ldots, x_{n-1}$, not all divisible by $p$, such that the corresponding numbers $h_1, h_2, \ldots, h_n$ are all divisible by $p$. Then the corresponding number

$$\omega' = x_0 + x_1\theta + x_2\theta^2 + \cdots + x_{n-1}\theta^{n-1}$$

is in fact divisible by $p$ even though the coefficients $x_0, x_1, \ldots, x_{n-1}$ are not all divisible by $p$. It follows that the number $(\mathcal{O}' : \mathcal{O}p)$ of incongruent elements in $\mathcal{O}'$ is smaller than $p^n$. It follows that there are numbers $\omega$ in $\mathcal{O}$ that are not congruent modulo $p$ to any element $\varphi(\theta)$, i.e., there exist congruence classes (mod $p$) in $\mathcal{O}$ for which there is no representative in $\mathcal{O}'$. The precise determination of the number $(\mathcal{O}' : \mathcal{O}p)$ is not necessary for our purposes.

---

93Dedekind writes $(\mathcal{O}, \mathcal{O}p)$ for the index.

94What Dedekind has shown is that when $p$ does not divide the index the quotient $\mathcal{O}/p\mathcal{O}$ is isomorphic to $\mathcal{O}'/p\mathcal{O}'$. As he shows next, if $k$ is the index $(\mathcal{O} : \mathcal{O}')$ the isomorphism is given by multiplication by $lk$, where $l$ is any rational integer such that $lk \equiv 1 \pmod{p}$.

95For index divisors, $\mathcal{O}'/p\mathcal{O}' \rightarrow \mathcal{O}/p\mathcal{O}$ is not onto.

96We would write $(\mathcal{O}' : \mathcal{O}p)$, but Dedekind does not.

97The editors of [8] add a footnote here: “In Zolotareff one also finds the theorem that the exceptional prime numbers are precisely those for which there is a number $\omega'$ in the order $\mathcal{O}'$ that are divisible by $p$ but whose coefficients are not all divisible by $p$. Zolotareff does not say, however, that these prime numbers are the index divisors.”
In this paragraph we consistently make the assumption that \( p \) is a prime number of the first kind. We want to prove that in this case the theory of higher congruences gives an easy way to decompose a principal ideal \( \mathcal{O}_p \) into its prime factors. This happens because the function \( F(t) \), which we will denote \( F \) for brevity, factors modulo \( p \) as a product of prime functions \( P(t) \) (C. 6). If we assume, for convenience, that each prime function \( P \) has highest coefficient \( = 1 \), it follows that two incongruent prime functions are always relatively prime (C. 5). Combining all the congruent factors into powers we get

\[ F \equiv P_1^{e_1}P_2^{e_2}\cdots P_m^{e_m} \pmod{p} \]

where the \( P_i \) are all the incongruent prime functions contained in \( F \).

Let \( P \) be any one of these \( m \) prime functions, and let \( \rho = P(\theta) \). Then there is an ideal \( \mathfrak{p} \) that is the greatest common divisor of \( \mathcal{O}_p \) and \( \mathcal{O}_\rho \). To study the properties of this ideal \( \mathfrak{p} \), we first determine all the elements \( \varphi(\theta) \) contained in the order \( \mathcal{O}' \) that are divisible by \( p \) (i.e., are contained in \( \mathfrak{p} \)).

We want to prove that the congruence

\[ \psi(\theta) \equiv 0 \pmod{p} \]  \hspace{1cm} (1)

is completely equivalent to the function congruence

\[ \psi(t) \equiv 0 \pmod{p, P} \]  \hspace{1cm} (2)

Indeed by definition (D. § 163; B. § 19) the ideal \( \mathfrak{p} \) is the collection of all numbers of the form

\[ \rho \alpha + p \beta, \]

---

98 This section states and proves “Dedekind’s Theorem,” describing the factorization of primes of the first kind in terms of higher congruences. Dedekind will use the phrase “prime function” to mean a monic polynomial that is irreducible modulo \( p \). Everything in this section assumes that \( p \) does not divide \( k \).

99 Dedekind knows that there is unique factorization in \( \mathbb{F}_p[x] \). This is one of the many results in [3].

100 Dedekind doesn’t state this as a separate Lemma but he uses it over and over in the sequel.

101 Here begins the proof. Recall that \( \rho = P(\theta) \) where \( P \) is an irreducible factor of \( F \).
where $\alpha, \beta$ are arbitrary numbers from the domain $\mathcal{O}$. By § 1, each number $\alpha$ is congruent modulo $p$ to some number $\varphi(\theta)$ in the order $\mathcal{O}'$, so from (1) we get a congruence of the form
\[
\psi(\theta) \equiv P(\theta)\varphi(\theta) \pmod{p};
\]
this is equivalent (as in § 1) to the function congruence
\[
\psi(t) \equiv P(t)\varphi(t) \pmod{p, F},
\]
and therefore also equivalent to congruence (2), since $F$ is divisible by $P$. Conversely, it follows immediately\footnote{From (2) we get $\psi(t) = P(t)\varphi(t) + pG(t)$; plugging in $\theta$ gives $\psi(\theta) = \rho\varphi(\theta) + pG(\theta) \in \mathfrak{p}$, since $\varphi(\theta) \equiv \alpha \pmod{p}$.} from (2) that any $\psi(\theta)$ is of the form $\rho\alpha + p\beta$, and so is $\equiv 0 \pmod{p}$ as well. This proves our claim above.\footnote{The lemma is now proved.}

With the help of these results we can easily\footnote{The argument is to pass from $\mathcal{O}/\mathfrak{p}$ to $\mathcal{O}'/\mathfrak{p}$ and then to translate congruences between elements of $\mathcal{O}'$ into congruences of polynomials using the Lemma above. This reduces the problem to counting incongruent polynomials modulo $p, P$, which Dedekind had already done in \[3\].} compute the norm of the ideal $\mathfrak{p}$, i.e., the number $(\mathcal{O} : \mathfrak{p}) = N(\mathfrak{p})$ of elements of $\mathcal{O}$ that are incongruent modulo $\mathfrak{p}$. So let $\alpha_1, \alpha_2$ be any two numbers in $\mathcal{O}$. From § 1 we now know that there exist two numbers $\varphi_1(\theta), \varphi_2(\theta)$ in $\mathcal{O}'$ that are congruent modulo $p$ to $\alpha_1, \alpha_2$. Since $\mathfrak{p}$ divides $p$, we also have
\[
\alpha_1 \equiv \varphi_1(\theta), \; \alpha_2 \equiv \varphi_2(\theta) \pmod{\mathfrak{p}}.
\]
So the two numbers $\alpha_1, \alpha_2$ are congruent modulo $\mathfrak{p}$ if and only if
\[
\varphi_1(\theta) \equiv \varphi_2(\theta) \pmod{p}.
\]
This congruence is equivalent, as above, to the congruence
\[
\varphi_1(t) \equiv \varphi_2(t) \pmod{p, P}.
\]
Therefore there are as many numbers $\alpha$ that are incongruent modulo $\mathfrak{p}$ as there are functions $\varphi(t)$ incongruent with respect to the double modulus $p, P$;\footnote{This quantity is $= p^f$, where $f$ is the degree of the function $P$ (C. 8), so we have $N(\mathfrak{p}) = p^f$.} this quantity is $= p^f$, where $f$ is the degree of the function $P$ (C. 8), so we have
\[
N(\mathfrak{p}) = p^f.
\]

26
With that it is easy to prove \( p \) is a prime ideal. First, we know \( f \geq 1 \), so \( N(p) \neq 1 \), so \( p \) cannot be equal to \( \mathcal{O} \). It suffices then to show\(^{105}\) that \( p \) is not a decomposable ideal, i.e., that it is not a product of the form \( a_1a_2 \), where \( a_1, a_2 \) are ideals and neither is equal to \( \mathcal{O} \). Such a decomposable\(^{106}\) ideal \( m = a_1a_2 \) has the characteristic property that there are always two numbers \( \alpha_1, \alpha_2 \), neither divisible by \( m \), whose product \( \alpha_1\alpha_2 \) is divisible by \( m \). This is because both the ideals \( a_1, a_2 \) are different from \( \mathcal{O} \), so neither of them can be divisible by their product \( m = a_1a_2 \). So there must exist a number \( \alpha_1 \) that is divisible by \( a_1 \) but not by \( m \), and similarly an \( \alpha_2 \) that is divisible by \( a_2 \) but not by \( m \). So \( p \) will be a prime ideal if we can show\(^{107}\) that a product \( \alpha_1\alpha_2 \) cannot be divisible by \( p \) unless at least one of the factors \( \alpha_1, \alpha_2 \) is divisible by \( p \). For this\(^{108}\) we set, as above,

\[
\alpha_1 \equiv \varphi_1(\theta), \alpha_2 \equiv \varphi_2(\theta) \pmod{p},
\]

so that

\[
\alpha_1\alpha_2 \equiv \varphi_1(\theta)\varphi_2(\theta) \pmod{p},
\]

and since \( \alpha_1\alpha_2 \equiv 0 \pmod{p} \), we must have

\[
\varphi_1(\theta)\varphi_2(\theta) \equiv 0 \pmod{p}
\]

and so

\[
\varphi_1(t)\varphi_2(t) \equiv 0 \pmod{p,P}.
\]

Since \( P \) is a prime function it follows\(^{109}\) that one of the two congruences

\[
\varphi_1(t) \equiv 0 \pmod{p,P} \quad \text{or} \quad \varphi_2(t) \equiv 0 \pmod{p,P}
\]

must hold (C. 6). So at least one of the congruences

\[
\varphi_1(\theta) \equiv 0 \pmod{p} \quad \text{or} \quad \varphi_2(\theta) \equiv 0 \pmod{p}
\]

\(^{105}\)Here we see that (at this time) Dedekind’s working definition of “prime ideal” is not the same as the one we learn today. The next several lines explain why it is enough to prove that \( \alpha_1\alpha_2 \in p \) implies that either \( \alpha_1 \) or \( \alpha_2 \) is in \( p \). The argument is straightforward; note that Dedekind consistently writes “\( m \) divides \( \alpha \)” instead of “\( \alpha \) belongs to \( m \).”

\(^{106}\)The proof starts here.

\(^{107}\)We have shown that an ideal \( I \) is indecomposable if and only if \( ab \in I \) implies either \( a \in I \) or \( b \in I \).

\(^{108}\)Now we will prove \( p \) is prime; as usual, we reduce to elements of \( \mathcal{O}' \) and then to polynomial congruences.

\(^{109}\)Irreducibles in \( \mathbb{F}_p[t] \) are prime, which Dedekind had proved in \([3]\).
must be true, that is, one of the two numbers $\alpha_1, \alpha_2$ must $\equiv 0 \pmod{p}$.

Therefore $p$ is a prime ideal, and we know (B. § 21) that $p$ is a prime ideal of degree $f$, since $N(p) = p^f$.

Now we would like to prove that the highest exponent $e$ of $P$ in the factorization of $F$ and the highest exponent of $p$ in the factorization of $p$ are equal. Indeed, if $F$ is divisible modulo $p$ by $P^e$ but not by $P^{e+1}$, we have

$$F \equiv SP^e \pmod{p},$$

where $S$ is not divisible by $P$. It follows as above that

$$\sigma = S(\theta)$$

is not divisible by $p$. Since $p$ is the greatest common divisor of $Op$ and $O\rho$, we know that

$$Op = pa, \quad O\rho = pb$$

with $a$ and $b$ relatively prime. So what we need to prove is that the highest power of $p$ contained in $a$ is $p^{e-1}$. For this, consider the number

$$\eta = \sigma \rho^{e-1} = S(\theta)P(\theta)^{e-1},$$

which cannot be divisible by $p$, since the degree of the polynomial $SP^{e-1}$ is less than $n$ and its highest coefficient is $= 1$. On the other hand, $\eta$ is divisible by $p^{e-1}$, since $\rho$ is divisible by $p$. From the congruence $F \equiv SP^e \pmod{p}$, we see that $\eta \rho = \sigma \rho^e$ is divisible by $p$. So the ideal $\eta p \rho$ is divisible by $pa$, and therefore $\eta p \rho$ is divisible by $a$; since $a$ and $b$ are relatively prime, we see that $\eta$ is divisible by $a$. So let

$$O\eta = ac,$$

where $c$ is an ideal not divisible by $p$ because otherwise $\eta$ would be divisible by $ap = Op$, which we know is not the case. Since $\eta$ is divisible by $p^{e-1}$, so is $a$.

---

110 We have a prime ideal dividing $p$, so it remains to determine the valuation, i.e., the highest power of $p$ dividing $p$.

111 Dedekind first proves that $a$ is divisible by $p^{e-1}$, and then proves it cannot be divisible by $p^e$.

112 Added a paragraph break here. The first part of the proof is finished: $a$ is divisible by $p^{e-1}$ and so $Op = pa$ is divisible by $p^e$; in valuation terms, $v_p(a) \geq e - 1$.

*It follows that $a$ is the greatest common divisor of the ideals $Op$ and $O\eta$, and so $\eta \rho$ is the least common multiple of $Op$ and $O\eta$, i.e., $p$ is the collection of all roots $\pi$ of the congruence $\eta \pi \equiv 0 \pmod{p}$. This could also have been used to define the ideal $p$.  

28
We now only need to show that \(a\) is not divisible by \(p^e\). Since \(e \geq 1\), if \(a\) is divisible by \(p^e\), then it is certainly divisible by \(p\) itself. Now if \(a\) is divisible by \(p\), \(b\) cannot be divisible by \(p\), and therefore \(\rho\) is not divisible by \(p^2\). From that it follows that \(\sigma\) is not divisible by \(p\), so in this case \(p^{e-1}\) is the highest power of \(p\) contained in the number \(\eta = \sigma p^{e-1}\). So \(\eta\), and therefore the ideal \(a\) contained in it, cannot be divisible by \(p^e\), which was to be proved.

After this the investigation of a specific prime function \(P\) contained in \(F\) and its corresponding prime ideal \(p\) is complete. We now apply the results to all the functions contained in \(F\),

\[
F \equiv P_1^{e_1} P_2^{e_2} \cdots P_m^{e_m} \pmod{p},
\]

with incongruent prime functions \(P_1, P_2, \ldots, P_m\) of degrees, respectively, \(f_1, f_2, \ldots, f_m\).

To these functions correspond prime ideals \(p_1, p_2, \ldots, p_m\) with the corresponding degrees, so that

\[
N(p_1) = p^{f_1}, N(p_2) = p^{f_2}, \ldots, N(p_m) = p^{f_m}
\]

and

\[
p_1^{e_1}, p_2^{e_2}, \ldots, p_m^{e_m}
\]

are the highest powers of these ideals contained \(p\). These \(m\) ideals are all distinct, for example, \(p_2\) is not divisible by \(P_1 \pmod{p}\), so the number \(P_2(\theta)\)

---

113 The argument opens with “if \(a\) is divisible by \(p^e\),” which seems like setting up a proof by contradiction. But that is not where the argument goes. I think it is better expressed as two cases: if \(a\) is not divisible by \(p\), then since \(0 = v_p(a) \geq e - 1\) we must have \(e = 1\) and we are done. If \(a\) is divisible by \(p\), then Dedekind shows that \(v_p(\rho) = 1\) and so the number \(\eta = \sigma p^{e-1}\) is divisible by \(a\) but not by \(p^e\). From that it follows that \(v_p(a) \leq e - 1\), hence must be exactly \(e - 1\).

114 For each different irreducible factor \(P\) we have found a prime ideal \(p\) dividing \(p\) and shown that the multiplicity of \(P\) as a factor of \(F\) is the same as the multiplicity of \(p\) as a factor of \(p\). To complete the proof we put these all together and then show that these are no other prime ideals dividing \(p\).

115 This is a key fact later on: distinct factors correspond to distinct ideals and vice versa. Dedekind shows only that different \(P\) give different ideals.
is divisible by \( p_2 \) but not by \( p_1 \), and it follows that \( p_1 \) and \( p_2 \) are different ideals. Finally, we know that \( p \) cannot be divisible by any other prime ideal, since

\[
P_1(\theta)^{e_1} P_2(\theta)^{e_2} \ldots P_m(\theta)^{e_m} \equiv 0 \pmod{p}.
\]

If \( p \) is divisible by a prime ideal, that ideal has to divide one of the \( m \) numbers \( \rho = P(\theta) \); but then that ideal must be identical to the prime ideal \( p \), which is the greatest common divisor of \( \mathcal{O}p \) and \( \mathcal{O}\rho \).

From all this it follows (D. § 163, B. § 25) that

\[
\mathcal{O}p = p_1^{e_1} p_2^{e_2} \ldots p_m^{e_m}.
\]

A consequence of this, found by taking norms, is

\[
n = e_1 f_1 + e_2 f_2 + \cdots + e_m f_m.
\]

Thus we have proved the following theorem, which I announced in the *Göttingischen gelehrten Anzeigen* in September 20, 1871.

**Theorem (I).** Let \( k \) be the index of the number \( \theta \) that satisfies the irreducible equation of degree \( N F(\theta) = 0 \). If \( k \) is not divisible by \( p \) and if

\[
F \equiv P_1^{e_1} P_2^{e_2} \ldots P_m^{e_m} \pmod{p}
\]

where the \( P_1, P_2, \ldots, P_m \) are incongruent prime functions of degree \( f_1, f_2, \ldots, f_m \), respectively, then we have

\[
\mathcal{O}p = p_1^{e_1} p_2^{e_2} \ldots p_m^{e_m},
\]

where \( p_1, p_2, \ldots, p_m \) are pairwise distinct prime ideals whose degrees are, respectively, \( f_1, f_2, \ldots, f_m \), and for each distinct prime function \( P \) the corresponding prime ideal \( p \) is the greatest common divisor of the ideals \( \mathcal{O}p \) and \( \mathcal{O}P(\theta) \).

---

116The last thing to note is that we have the complete factorization: no other prime ideals divide \( p \). This is easy to see.

117The product of all the \( P_i(\theta)^{e_i} \) is congruent mod \( p \) to \( F(\theta) = 0 \).

118Surprisingly, this famous formula does not appear in [5] or [9].

119Usually known today as “Dedekind’s theorem.”

120Recall the running assumption that \( k \neq 0 \), so that \( \Omega = \mathbb{Q}(\theta) \) and the minimal polynomial is of degree \( n \).
§ 3

From this theorem it follows that on the basis of a specific integer $\theta$ from the field $\Omega$, which allows one to represent as $\varphi(\theta)$ infinitely many integers, one can find the factorization of all the prime numbers $p$ that do not divide the index corresponding to a the chosen $\theta$. It is therefore very important to know whether a prime number $p$ is a divisor of the index $k$ or not. If we have a basis $\omega_1, \omega_2, \ldots, \omega_n$ of the domain $\mathcal{O}$, or even just know the fundamental number $D$ of the field $\Omega$, it is easy to answer the question, since in that case we can find $k$ directly. From the coefficients of the equation $F(\theta) = 0$ we can compute its discriminant

$$\Delta(1, \theta, \theta^2, \ldots, \theta^{n-1}) = (-1)^{\frac{n(n-1)}{2}} N(F'(\theta)) = Dk^2,$$

and from that we can find the square of the index $k$ by dividing by $D$. In most investigations, however, things are very different, since only the equation $F(\theta) = 0$ is known, and not the fundamental number $D$ of the corresponding field $\Omega$. We would like to decide on that basis whether or not a specific prime number $p$ divides the unknown index of the number $\theta$. This is in fact possible, as we will now show, with the help of the theory of higher congruences. Using our previous notation, the answer turns out to depend on the nature of the function $M$ that appears in the identity

$$F = P_{e_1}^1 P_{e_2}^2 \ldots P_{e_m}^m - pM.$$

This will be the content of the next two theorems.

**Theorem (II).** If the index of the number $\theta$ is not divisible by $p$, then $M$ cannot be divisible mod $p$ by any prime function $P$ whose square divides $F$ mod $p$.

---

121 We are not sure what Dedekind means here; perhaps it is this. Given $\theta$, the infinitely many elements of $\mathbb{Z}[\theta]$ are algebraic integers in $\Omega$. The running assumption that $k \neq 0$ means that $\mathbb{Z}[\theta]$ has finite index in $\mathcal{O}$.

122 Dedekind points out that this is easy if we have the discriminant $D$ but his goal is to answer the question solely in terms of the minimal polynomial $F$.

123 That is, solely on the basis of the equation.

124 This equation defines the polynomial $M$: it is the result of dividing the difference by $p$. Since $F$ is congruent mod $p$ to the product, $M$ is a polynomial with rational integer coefficients.
To prove this, we can use the results in the previous paragraph, which were all obtained under the assumption that \(p\) does not divide \(k\). Retaining the same notation we used there, write \(F \equiv SP^e \pmod{p}\), or

\[ F = SP^e - pM, \]

and suppose \(e \geq 2\). Then \(p\) is divisible by \(p^2\), so \(a\) is divisible by \(p\) and \(b\) is not. Therefore \(p^e\) is the highest power of \(p\) dividing \(S(\theta)P(\theta)^e = pM(\theta)\). Since \(p\) is divisible by \(p^e\), it follows that \(M(\theta)\) cannot be divisible by \(p\) and so \(M \not\equiv 0 \pmod{p,P}\), as claimed.

It is also possible to prove the theorem without using the results in the previous paragraph, in following indirect but equivalent form:

**Lemma:** Suppose \(P,R,S,T \in \mathbb{Z}[t]\), \(P \equiv R \pmod{p}\), \(S \equiv T \pmod{p}\), \(P\) is irreducible mod \(p\), and that

\[ F = P^e S - pM = R^e T - pN \]

with \(e \geq 2\). Then \(M - N\) is divisible by \(P\).

**Proof of Lemma:** The equation

\[ P^e S - pM = R^e T - pN \]

gives

\[ M - N = \frac{1}{p}(P^e S - R^e T), \]

so we need to know \(P^e S - R^e T \pmod{p^2}\). Writing \(R = P + pX\), \(T = S + pY\) we get

\[ P^e S - R^e T \equiv P^e S - (P^e + epP^{e-1}X)(S + pY) \equiv p(eP^{e-1}S + P^e Y) \pmod{p^2}. \]

Dividing by \(p\) gives

\[ M - N \equiv eP^{e-1}S + P^e Y \pmod{p}, \]

and since \(e \geq 2\) we are done. □

Dedekind does not prove this lemma; rather, he *deduces* it from the fact that the question of whether the index is divisible by \(p\) is independent of the choice of lifts. But he does say it can be checked directly, and it seems better to do that.

So \(S\) is the product of all the irreducible factors different from \(P\); in particular, \(S\) is not divisible by \(P\).

Since \(a\) and \(b\) are relatively prime.

Since the principal ideal \(P\rho\) is equal to \(p\beta\), we know that \(\rho = P(\theta)\) is divisible by \(p\) only once, and that \(\sigma = S(\theta)\) is not divisible by \(p\).

Dedekind means \(p^e\) is the highest power of \(p\) that divides \(p\), of course.

This concludes the proof of Theorem II.
Theorem. If $F$ is divisible mod $p$ by the square of an irreducible polynomial $P$, so that $F = SP^e - pM$ with $e \geq 2$, and $M$ is divisible by $P$, then the index $k$ of the number $\theta$ will be divisible by $p$.

Let the letters $\rho$, $\sigma$, $\eta$ have the same meanings as in the previous paragraph, so that we set

$$\rho = P(\theta), \quad \sigma = S(\theta), \quad \eta = \sigma \rho^{e-1}.$$  

Using the results of § 1\(^{131}\) the proof of our theorem will be complete if we can show that the number $\eta = S(\theta)P(\theta)^{e-1}$ must be divisible by $p$, since the function $SP^{e-1}$ is of degree lower than $n$ and not $\equiv 0 \pmod{p}$. To prove that $\eta$ is divisible by $p$, it suffices to show that each power of a prime ideal dividing $p$ also divides $\eta$ (D. § 163, B. § 25). To this end set

$$\mu = M(\theta);$$

consider the equation

$$\sigma \rho^e = \eta \rho = p \mu.$$  

First, if $p$ is a prime ideal dividing $p$ but not dividing $\rho$, then from $\eta \rho = p \mu$ it follows at once that $\eta$ is divisible by the highest power of $p$ dividing $p$.

Next, suppose $p$ divides both $p$ and $\rho$. Since $S$ and $P$ are relatively prime functions\(^{132}\) there exist (C. 4) two functions $U$, $V$ such that the congruence \(^{215}\)

$$SU + PV \equiv 1 \pmod{p}$$

holds. From that we get the numerical congruences\(^{133}\)

$$\sigma U(\theta) + \rho V(\theta) \equiv 1 \pmod{p}$$

$$\sigma U(\theta) \equiv 1 \pmod{p}.$$  

\(^{131}\)If $p$ does not divide the index $k$, then two elements of $O'$ are congruent mod $p$ if and only if the corresponding polynomials are congruent mod $p, F$. In our case the polynomial will have degree less than $n$, so being congruent mod $p, F$ is equivalent to being congruent mod $p$.

\(^{132}\)Dedekind doesn’t say so, but he means relatively prime mod $p$.

\(^{133}\)For the first one, we just plug in $\theta$; for the second, remember that $p$ divides both $p$ and $\rho$.  

33
and it follows that \( \sigma \) is not divisible by \( p \). Let \( p^h, p^r, p^m \) be the highest powers of \( p \) dividing \( p, \rho, \mu \), respectively. Since \( \sigma \rho^e = p\mu \) and \( \eta = \sigma \rho^{e-1} \), we see that 

\[
er = h + m,
\]

and also that the highest power of \( p \) appearing in \( \eta \) is equal to 

\[(e - 1)r = h + m - r.\]

Since we want to show that \( \eta \) is divisible by \( p^h \), it remains to prove that 

\[m \geq r.\]

Now we have to consider two cases. 135 In the first, \( r \geq h \), we use the first assumption of our theorem, namely that \( e \geq 2 \). Then \( h + m = er \geq 2r \), and so \( m - r \geq r - h \geq 0 \), as claimed. In the second case, \( r \leq h \), we use the second assumption in our theorem, namely that \( M \equiv 0 \pmod{p^r} \), i.e., \( M \equiv PT \pmod{p} \). Therefore \( \mu \equiv \rho T(\theta) \pmod{p} \). Since \( \rho \) is divisible by \( p^r \), it follows from this congruence that \( \mu \) is also divisible by \( p^r \), and so that \( m \geq r \), as we wanted to prove.

Now that we have proved Theorem II in two different ways, we will also show the correctness of the converse.

**Theorem (III).** If \( M \) is not divisible mod \( p \) by any prime function \( P \) whose square divides \( F \) mod \( p \), the index \( k \) of the number \( \theta \) is not divisible by \( p \).

The same theorem clearly can also be stated in the following form:

**Theorem.** If the index \( k \) of a number \( \theta \) is divisible by \( p \), there exists a prime function \( P \) dividing \( M \) whose square divides \( F \) modulo \( p \).

We present the proof 136 of the latter form [of the theorem], because the assumption that \( k \) is divisible by \( p \) is easier to use, insofar as (according to

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134 From here on we are basically computing \( p \)-adic valuations.
135 The two cases are \( r \geq h \) and \( r \leq h \). Dedekind will use the assumption that \( e \geq 2 \) to handle the first case and the assumption that \( M \) is divisible by \( P \) to handle the second.
136 The structure of the proof is as follows. If \( p|k \) then there exists a polynomial \( \varphi(t) \in \mathbb{Z}[t] \) such that \( \varphi(t) \equiv 0 \pmod{p} \) but \( \varphi(\theta) \) is divisible by \( p \) in \( \mathcal{O} \). We look at \( A = \gcd(F, \varphi) \) (over \( \mathbb{F}_p \), but choose a monic lift of degree \( < n \)) and set \( F = AB - pM \). Then we show that any prime divisor \( P \) of \( B \) in \( \mathbb{F}_p[t] \) also divides \( M \), therefore divides \( F \), and we can then show it divides \( A \) as well, so that \( P^2|F \). Factoring out the largest power of \( P \) gives \( F = P^{e-1}A'B' - pM \) with \( P \nmid A'B' \), \( e \geq 2 \), \( P|M \), which is what we want. Notice that the polynomial denoted by \( M \) might change in the course of the argument.
§ 1) it implies the existence of a number

$$\varphi(\theta) = x_0 + x_1\theta + x_2\theta^2 + \cdots + x_{n-1}\theta^{n-1}$$

which is divisible by $p$ but whose coefficients $x_0, x_1, x_2, \ldots, x_{n-1}$ are not all divisible by $p$. Let us first denote by $A$ the greatest common divisor of $\varphi(t)$ and $F$ modulo $p$. The degree of $A$ is smaller than $n$, since $\varphi$ has degree smaller than $n$, and it is also not $\equiv 0 \pmod{p}$. Write

$$F = AB - pM,$$

so that $B$ is not a constant. There exist (C. 4) two functions $\varphi_1, \varphi_2$ such that

$$\varphi(t)\varphi_1(t) + F(t)\varphi_2(t) \equiv A(t) \pmod{p}.$$ 

From this it follows that the number $A(\theta)$ is also divisible by $p$. From that we get an equation of the form

$$A(\theta)^s + ph_1A(\theta)^{s-1} + \cdots + p^sh^s = 0,$$

where $h_1, h_2, \ldots, h_s$ are rational integers (D. § 160; B. § 13). Since the equation $F(\theta) = 0$ is irreducible, this results in an equation that holds identically in the variable $t$ of the form

$$A^s + ph_1A^{s-1} + \cdots + p^sh^s = FG,$$

which implies also the congruence

$$A^s \equiv 0 \pmod{p, F}.$$ 

137 We can say something a little stronger that will help below: if we had all but $x_0$ divisible by $p$, then $\varphi(\theta) \equiv x_0 \pmod{p}$ and so $x_0$ is also divisible by $p$. This means that $\varphi(t)$ is not a constant mod $p$. 

138 $\deg(B) = \deg(F) - \deg(A) = n - \deg(A) \neq 0$. 

139 Since $F(\theta) = 0$ and $\varphi(\theta)$ is divisible by $p$, it follows that $A(\theta)$ is divisible by $p$. In particular, $A$ cannot be a constant mod $p$. This transfers the assumption that $\varphi(\theta)$ is divisible by $p$ to $A(\theta)$ where $A|F$ in $\mathbb{F}_p[t]$.

140 The assumption that $\varphi(\theta)$ is divisible by $p$ leads to the conclusion that $\frac{1}{p}A(\theta)$ is an algebraic integer, so it satisfies a monic equation with integer coefficients. Multiplying by a power of $p$ gives the equation below.

141 I.e., an equation in $\mathbb{Z}[t]$.

*In a similar way one can easily show that the criterion for the divisibility by $p$ of a number $\varphi(\theta)$ consists in the congruence $\varphi(t) \equiv 0 \pmod{p, K}$, where $K$ is a completely determined divisor of the function $F$ modulo $p$. 
Therefore the function $A$ must be divisible modulo $p$ by every prime function that divides $F$ modulo $p$ (C. 5 and 6). Now taking the equation above that is satisfied by the number $A(\theta)$ and multiplying it by $B(\theta)^s$, and recalling that $A(\theta)B(\theta) = pM(\theta)$ we get

$$M(\theta)^s + h_1M(\theta)^{s-1}B(\theta) + \cdots + h_sB(\theta)^s = 0,$$

and therefore an identity of the form

$$M^s + h_1M^{s-1}B + h_2M^{s-2}B^2 + \cdots + h_sB^s = FH.$$

so $M^s \equiv 0 \pmod{p,B}$, which again implies that any prime function dividing $B$ modulo $p$ must also divide $M$. But we proved above that $B$ is not a constant, so it has at least one prime divisor $P$, which must then also divide $M$. Since $F$ is a multiple of $B$ mod $p$, it must also divide $F$. But every prime function dividing $F$ must divide $A$, as we showed above, so $P$ must divide both $A$ and $B$, which shows $P^2$ must divide $F$, since $F \equiv AB \pmod{p}$. So we have shown that there is a prime function contained in $M$ whose square is contained in $F$, which is what we wanted to prove.

From II and III, the question of whether $p$ divides $k$ reduces to looking at the factorization

$$F = P_1^{e_1}P_2^{e_2} \cdots P_m^{e_m} - pM$$

of any function $F$ into prime functions modulo $p$. In particular, if $F$ is not divisible by the square of any prime function so that all the exponents $e_1, e_2, \ldots, e_m$ are equal to 1 or when it happens that none of the prime

142 This is the key conclusion. Since $A^s$ is divisible by $F$ in $\mathbb{F}_p[t]$, every irreducible factor of $F$ must also divide $A$.

143 After dividing by $p^s$.

144 $B$ is a divisor of $F$ in $\mathbb{F}_p[t]$, so every term but the first is divisible by $B$ in $\mathbb{F}_p[t]$.

145 Factoring out the highest powers of $P$ dividing $A$ and $B$ we get $F \equiv P^eA'B' \pmod{p}$, with $P \nmid A'B'$. By unique factorization in $\mathbb{F}_p[t]$, $A'B'$ is the rest of the factorization of $F$ and $M = \frac{1}{p}(F - P^eA'B')$ is (a possible choice for) the polynomial we are studying. We know that $e \geq 2$ and $P$ divides $M$. As we observed above, this property is independent of the choice of $M$, so it follows that no matter how we factor $F$ we will have $F = P^eS - pM$ with $P$ dividing $M$.

146 All we need to check is there exists an $i$ such that $e_i \geq 2$ and $P_i$ divides $M$.

147 This is not the interesting case, since, as Dedekind’s footnote points out, if $F$ is not divisible by the square of a polynomial mod $p$ it follows that $p$ does not divide the polynomial discriminant $\Delta = Dk^2$, so of course it does not divide $k$.

*This will be the case if and only if the discriminant $\Delta(1, \theta, \theta^2, \ldots, \theta^n)$ of the equation $F(\theta) = 0$ is not divisible by $p$. 

36
functions whose squares divide \( F \) are contained in \( M \), then \( k \) is not divisible by \( p \), and Theorem I from § 2 applies. But if there is a prime function dividing \( M \) whose square also divides \( F \), then \( k \) is divisible by \( p \) and the second proof of Theorem II shows that the factorization of the ideal \( \mathcal{O} \) into prime factors is different from the one determined in Theorem I.

To this result we add the following remark. If the functions \( R_1, R_2, \ldots, R_m \) are congruent to the functions \( P_1, P_2, \ldots, P_m \), then we have

\[
F = R_1^{e_1} R_2^{e_2} \ldots R_m^{e_m} - pN
\]

and \( N \) certainly does not need to be congruent mod \( p \) to \( M \). On the other hand, the divisibility of the index \( k \) by \( p \) is independent of the choice of (lifts of) the divisors mod \( p \), so we must have that the property of \( M \) that is key for this result will also hold for \( N \). This can easily be confirmed directly by calculation. If we denote by \( Q \) the product of all the prime functions contained in \( F \) whose squares are not contained in \( F \), one can, by a suitable choice of the functions \( R_1, R_2, \ldots, R_m \), always arrive at a function \( N \) which is relatively prime to \( Q \), but if there is a prime function \( P \) that divides \( M \) such that \( P^2 \) divides \( F \), a calculation shows that then \( P \) divides \( N \) as well.

---

148 Dedekind doesn’t explain why, but we think it might be this. If \( F \equiv P^e S \pmod{p} \), \( P \) does not divide \( S \), and \( p = \gcd(p, P(\theta)) \), we expect that \( p^e \) is a divisor of \( p \). But in the second proof of Theorem II we showed that if \( p \mid k \) the number \( \eta = S(\theta)P(\theta)^{e-1} \) is divisible by \( p \), but it is not divisible by \( p^e \). Hence \( p^e \) does not in fact divide \( p \).

149 We did this in the footnote on page 62. It is not quite clear that Dedekind’s argument works without proving this first, but here he notes that it can easily be checked by a direct calculation.

150 If \( F = PB \pmod{p} \) with \( P \nmid B \) in \( \mathbb{F}_p[t] \), we can always replace \( P \) with \( P + pC \) where \( P \nmid C \) in \( \mathbb{F}_p[t] \). That replaces \( M \) by \( M + CB \), which is not divisible by \( P \) in \( \mathbb{F}_p[t] \).

151 The reference in Dedekind’s footnote, in which “12” should read “17,” is 30.

*It follows from this that the ideal theory of Zolotareff is limited to the case in which the index \( k \) is not divisible by \( p \). At least this seems to follow from the following words, which we can find in the abstract mentioned above (Jahrbuch über die Fortschritte der Mathematik, Vol. 6): “To present the theory in its simplest form, the author assumes that \( F_1(x) \) is not divisible by any of the functions \( V, V_1, V_2, \ldots \). If this condition does not hold, one can transform the equation \( F(x) = 0 \) modulo \( p \) so that it does hold. The author reserves the discussion of this transformation for another opportunity.” — Since according to my investigations (see § 5 of this paper) there exist fields in which the indices of all integers \( \theta \) are divisible by a certain prime number \( p \), it follows that all equations \( F(\theta) = 0 \) have the unfortunate property that impedes the application of Zolotareff’s theory. Hence I suppose that there is a misunderstanding in the quoted words from the abstract. It is possible that the author’s completion of the theory will be based on considerations similar to those in Seling’s theory of ideal numbers (Schlömich’s Zeitschrift, Vol. 10, p. 12ff.)

37
In the number domains $\mathcal{O}$ first considered by Kummer, which come from a primitive root of the equation $\theta^m = 1$, the happy circumstance occurs that the powers $1, \theta, \theta^2, \ldots, \theta^{n-1}$, with $n = \varphi(m)$, form a basis of the domain $\mathcal{O}$. It follows that the index $k$ of the number $\theta$, on which the entire investigation is based, is $= 1$. I soon realized, however, that in the general investigation of any finite field $\Omega$ and domain $\mathcal{O}$ containing all the integers in $\Omega$ this simple case rarely occurs. I thought for a long time that it was likely that for every given prime number $p$ it might be possible to find an integer $\theta$ in the field $\Omega$ whose index is not divisible by $p$. If so, with the help of that $\theta$ we could succeed in determining the ideal factors of $p$. Since all my attempts to prove the existence of such a number $\theta$ were unsuccessful, I finally decided, if possible, to prove that this assumption was false. I achieved this goal, as I have already indicated in the *Göttingischen gelehrten Anzeigen* of September 20, 1871, through the considerations that form the content of this and the following paragraph.

Let $p$ be a fixed prime number and let $p_1, p_2, \ldots, p_m$ be all the distinct prime ideals dividing $p$; we will denote their degrees by $f_1, f_2, \ldots, f_m$, so that, for example, $N(p_1) = p^{f_1}$. If there exists an integer $\theta$ whose index $k$ is not divisible by $p$, it follows from Theorem I that there exist $m$ polynomials $P_1, P_2, \ldots, P_m$ of degrees $f_1, f_2, \ldots, f_m$, pairwise incongruent modulo $p^{f_1}$. It is now of the greatest importance for our investigation that this conclusion holds.

---

152 If $\theta$ is an $m$-th root of unity, we have $\mathcal{O} = \mathbb{Z}[\theta]$, which is what allows Kummer’s general approach to work.

153 Dedekind says “rarely” or “exceptionally.” It is unclear how many examples he knew at this point, so this is an impressive insight. One expects that in fact the set of number fields with monogenic rings of integers has density zero. See, for example, [1].

154 This is one of Dedekind’s tantalizing accounts of why he ended up creating ideal theory: the “local” approach that might allow one to reduce everything to “higher congruences” is defeated by the existence of common index divisors. Kronecker [26, § 25] makes a similar argument.

155 Dedekind assumes he knows the factorization of $p$.

156 Since different prime ideals dividing $p$ correspond to distinct irreducible polynomials in $\mathbb{F}_p[t]$, such polynomials must exist. Conversely, if we cannot find enough irreducible polynomials of the required degrees, there cannot be any polynomial $F$ whose factorization matches the factorization of $p$, and so $k$ must be divisible by $p$ no matter which $\theta$ is chosen.
may be reversed so that the following theorem holds.

**Theorem (IV).** Let $f_1, f_2, \ldots, f_m$ be the degrees of the distinct prime ideals $p_1, p_2, \ldots, p_m$ contained in $p$. Suppose that modulo $p$ there exist $m$ incongruent prime functions $P_1, P_2, \ldots, P_m$ of degrees $f_1, f_2, \ldots, f_m$ respectively. Then there exists an integer $\theta$ in $\Omega$ whose index $k$ is not divisible by $p$.

Before giving a proof of this theorem, we will make a few general observations that do not depend on all of its hypotheses.

Let $p$ be a prime ideal dividing $p$, of degree $f$ at least one. Then all the integers $\omega$ of the field $\Omega$ satisfy the congruence $\omega^{pf} - \omega \equiv 0 \pmod{p}$ (D. § 163; B. § 26, 3°). Now if $t$ is a variable, the function $t^{pf} - t$ is congruent mod $p$ to the product of all the incongruent-mod-$p$ prime functions whose degree is a divisor of the number $f$ (C. 19). Among them we can choose at will a prime function $P$ whose degree $f = f$; this is always possible because there always exists at least one such function (C. 20). Then

$$t^{pf} - t \equiv P(t)H(t) \pmod{p},$$

Footnotes:

157 So the existence of sufficiently many irreducible polynomials is enough to guarantee the existence of the appropriate $\theta$.

158 On page 456 of [20], Hasse says that “In deriving this criterion, Hensel gave the first demonstration of the power of his new foundation of algebraic number theory.” The criterion there comes with an explicit formula for the number of irreducible monic polynomials of degree $f$ in $\mathbb{F}_p[t]$, but this formula was certainly known to Dedekind as well. Hensel proves the same theorem in [23]; see [17]. Note, however, that to use this criterion one needs to know the factorization of $p$. That motivated Hensel to look for another criterion in [23], but the real solution came from the theory of $p$-adic numbers, which is what Hasse refers to as the “new foundation of algebraic number theory.”

159 Making “some observations” is Dedekind’s way to prove some lemmas. We will indicate each observation with a footnote.

160 The first lemma says that if $p$ divides $p$ and has degree $f$, then there exists an irreducible polynomial $P \in \mathbb{F}_p[t]$ of degree $f$ and an element $\alpha \in O$ such that $P(\alpha) \equiv 0 \pmod{p}$.

161 Since $O/p$ is a field with $p^f$ elements, every element has order $p^f - 1$.

162 The (unique up to isomorphism) field with $p^f$ elements contains all the fields with $p^d$ elements with $d | f$. Again we notice that [3] is basically a theory of finite fields.

163 So $P$ is monic, irreducible mod $p$, and of degree $f$, and therefore a divisor of $t^{pf} - t$.
and therefore
\[ \omega^{p^f} - \omega \equiv P(\omega)H(\omega) \pmod{p}. \]
Since \( p \) divides \( p \), we see that for every number \( \omega \) contained in \( \mathcal{O} \) we have the congruence
\[ P(\omega)H(\omega) \equiv 0 \pmod{p}. \]
Therefore the number of roots \( \omega \) that are incongruent modulo \( p \) is exactly
\[ \left[ \mathcal{O} : p \right] = N(p) = p^f, \]
therefore equal to the degree of the congruence. Using the same simple arguments as in rational number theory (D. § 26), one can easily prove that a congruence of degree \( r \) modulo a prime ideal \( p \) can have no more incongruent roots in the number domain \( \mathcal{O} \) than the degree \( r \). I will omit the proof for brevity.\[ ^{164} \]
Therefore in our case the congruence \( H(\omega) \equiv 0 \pmod{p} \) can have at most \((p^f - f)\) incongruent roots, and it follows that the representatives \( \omega \) of the \( f \) other number classes must satisfy the congruence \( P(\omega) \equiv 0 \pmod{p} \). For our purposes, however, it is sufficient to know that this congruence has at least one root. Let \( \alpha \) be one such root, so that
\[ P(\alpha) \equiv 0 \pmod{p}. \]
We now consider all the number of the form \( \varphi(\alpha) \) and we want to prove\[ ^{165} \]
that the congruence
\[ \varphi(\alpha) \equiv 0 \pmod{p} \]
is equivalent to the function congruence
\[ \varphi(t) \equiv 0 \pmod{p, P}. \]
Indeed, if the latter congruence holds, then also
\[ \varphi(t) \equiv P(t)\psi(t) \pmod{p}, \]
and so
\[ \varphi(\alpha) \equiv P(\alpha)\psi(\alpha) \pmod{p}, \]
and since both of the numbers \( p \) and \( P(\alpha) \) are divisible by \( p \), we get \( \varphi(\alpha) \equiv 0 \pmod{p} \). Conversely, if \( \varphi(t) \) is not divisible by the prime function \( P(t) \) then
\[ ^{164} \text{A polynomial over a field cannot have more roots than its degree.} \]
\[ ^{165} \text{The second lemma says that an element of } \mathbb{Z}[\alpha] \text{ is divisible by } p \text{ if and only if } \alpha = \varphi(t) \text{ with } \varphi(t) \text{ divisible by } P(t) \text{ in } \mathbb{F}_p[t]. \text{ Essentially, } \alpha \text{ is a generator of the residue field } \mathcal{O}/p \text{ and its minimal polynomial over } \mathbb{F}_p \text{ is } P. \]
\( \varphi(t) \) and \( P(t) \) will be relatively prime functions, and if follows that there exist two functions \( \varphi_1(t), \varphi_2(t) \) such that the congruence
\[
\varphi(t)\varphi_1(t) + P(t)\varphi_1(t) \equiv 1 \pmod{p}
\]
holds (C. 5). Then we have
\[
\varphi(\alpha)\varphi_1(\alpha) + P(\alpha)\varphi_1(\alpha) \equiv 1 \pmod{p},
\]
and if follows that \( \varphi(\alpha) \) is \( \not\equiv 0 \pmod{p} \). So we have proved the claim above.

In the case that \( p \) is divisible by \( p^2 \), we also want to choose the root \( \alpha \) of the congruence \( P(\alpha) \equiv 0 \pmod{p} \) so that the number \( P(\alpha) \) is \( \not\equiv 0 \pmod{p^2} \). This is always possible: if \( \alpha \) is a root of the congruence \( P(\alpha) \equiv 0 \pmod{p^2} \), then one can choose a number \( \lambda \) that is divisible by \( p \) but not by \( p^2 \) and set \( \alpha' = \alpha + \lambda \). Then
\[
P(\alpha') = P(\alpha) + \lambda P'(\alpha) + \lambda^2 P''(\alpha) + \ldots \\
\equiv \lambda P'(\alpha) \pmod{p^2}
\]
and since the derivative function \( P'(t) \) has degree \( f - 1 \) and is not \( \equiv 0 \pmod{p} \), it cannot be \( \equiv 0 \pmod{p, P} \), and therefore the number \( P'(\alpha) \) above is not divisible by \( p \). So the number \( \lambda P'(\alpha) \), and therefore also the number \( P(\alpha') \), is divisible by \( p \) but not by \( p^2 \). So we have proved the existence of a number \( \alpha' \). Let’s remove the accent and thus assume that \( P(\alpha) \) is divisible by \( p \) but not by \( p^2 \).

Let \( p^e \) be the highest power of \( p \) contained in \( p \); we want to prove that the numerical congruence
\[
\varphi(\alpha) \equiv 0 \pmod{p^e}
\]
\[166\] The third lemma is that if necessary we can change \( \alpha \) to make sure \( P(\alpha) \) is not divisible by \( p^2 \).
\[167\] The editors of [8] note here that here \( P''(\alpha) \) really should be \( \frac{P''(\alpha)}{2} \) and similarly for higher terms. They do not discuss whether the denominators will create trouble for the argument.
\[168\] Dedekind knows that “finite fields are perfect” but only in the language of \([4]\).
\[169\] So now we have an irreducible polynomial \( P \) of degree \( f \), an element \( \alpha \in \mathcal{O} \) such that \( P(\alpha) \equiv 0 \pmod{p} \); if \( p^2 \) divides \( p \), we can assume that \( P(\alpha) \not\equiv 0 \pmod{p^2} \).
\[170\] The fourth lemma is another translation into polynomial congruences, this time for divisibility by a power of \( p \).
\[171\] The proof is identical to the previous one. The key result from \([4]\) is that the gcd of two polynomials is a linear combination.
is equivalent to the function congruence
\[ \varphi(t) \equiv 0 \pmod{p, P^e}. \]

If the latter holds, then
\[ \varphi(t) \equiv P(t)^e \psi(t) \pmod{p}, \]
so also
\[ \varphi(\alpha) \equiv P(\alpha)^e \psi(\alpha) \pmod{p}. \]

Since both \( p \) and \( P(\alpha)^e \) are divisible by \( p^e \), it follows that \( \varphi(\alpha) \equiv 0 \pmod{p^e} \).
Conversely, if the function congruence does not hold, the greatest common divisor mod \( p \) between \( \varphi(t) \) and \( P(t)^e \) must be of the form \( P(t)^s \) for some \( s < e \). So (C. 4) we have polynomials \( \varphi_1(t), \varphi_2(t) \) such that
\[ \varphi(t) \varphi_1(t) + P(t)^e \varphi_2(t) \equiv P(t)^s \pmod{p}. \]
Since both \( p \) and \( P(\alpha)^e \) are divisible by \( p^e \), we get
\[ \varphi(\alpha) \varphi_1(\alpha) \equiv P(\alpha)^s \pmod{p^e}. \]
Since \( s < e \) and \( P(\alpha) \) is not divisible by \( p^2 \), it follows that \( \varphi(\alpha) \) is not \( \equiv 0 \pmod{p^2} \), and our claim is proved.

One can now apply the results described above to each of the prime ideals \( p_1, p_2, \ldots, p_m \). One chooses arbitrary prime functions \( P_1, P_2, \ldots, P_m \) whose degrees \( f_1, f_2, \ldots, f_m \) are the same as the degree of the corresponding prime ideal. As above, one determines as many numbers \( \alpha_1, \alpha_2, \ldots, \alpha_m \) such that \( P(\alpha_1), P(\alpha_2), \ldots, P(\alpha_m) \) are respectively divisible by \( p_1, p_2, \ldots, p_m \) and such that in the case that \( p \) is divisible by \( p_r^2 \) the corresponding \( P_r(\alpha_r) \) is not divisible by \( p_r^2 \).

Since the ideals \( p_1, p_2, \ldots, p_m \) are distinct, their squares are [pairwise] relatively prime, so one can find (D. § 163; B. § 26) a number \( \theta \) such that
\[
\begin{align*}
\theta &\equiv \alpha_1 \pmod{p_1^2} \\
\theta &\equiv \alpha_2 \pmod{p_2^2} \\
& \quad \quad \vdots \\
\theta &\equiv \alpha_m \pmod{p_m^2}
\end{align*}
\]
\footnote{The final lemma shows, using the Chinese Remainder Theorem, that we can find a single \( \theta \) satisfying the same properties as the \( \alpha_i \). This will, of course, eventually be the \( \theta \) whose existence is claimed in Theorem IV.}
\footnote{This is the Chinese Remainder Theorem in \( \mathcal{O} \); Dedekind does not use that name.}
Then we have

\[
P_1(\theta) \equiv P_1(\alpha_1) \pmod{p_1^2}
\]
\[
P_2(\theta) \equiv P_2(\alpha_2) \pmod{p_2^2}
\]
\[
\cdots\cdots
\]
\[
P_m(\theta) \equiv P_m(\alpha_m) \pmod{p_m^2}
\]

It follows that the numbers \(P_1(\theta), P_2(\theta), \ldots, P_m(\theta)\) are divisible respectively by \(p_1, p_2, \ldots, p_m\), but, in the case when \(p\) is divisible by \(p_r^2\), the number \(P_r(\theta)\) is not divisible by \(p_r^2\). The number \(\theta\) therefore unites in itself all the properties that each of the numbers \(\alpha_r\) has with respect to the corresponding prime ideal \(p_r\). Now let

\[
\mathcal{O}p = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m},
\]

so that we have, by taking the norm,

\[
n = e_1 f_1 + e_2 f_2 + \cdots + e_m f_m.
\]

A number of the form \(\varphi(\theta)\) is divisible by one of the powers \(p_1^{e_1}, p_2^{e_2}, \ldots, p_m^{e_m}\) if and only if the corresponding function congruence

\[
\varphi(t) \equiv 0 \pmod{p_1^{e_1}}
\]
\[
\varphi(t) \equiv 0 \pmod{p_2^{e_2}}
\]
\[
\cdots\cdots
\]
\[
\varphi(t) \equiv 0 \pmod{p_m^{e_m}}
\]

holds. An integer from the field is divisible by \(p\) if and only if it is divisible by each of the \(m\) powers \(p_1^{e_1}, p_2^{e_2}, \ldots, p_m^{e_m}\), and therefore a numerical congruence

\[
\varphi(\theta) \equiv 0 \pmod{p}
\]

is equivalent to the system of \(m\) function congruences above.

So far we have intentionally put no restriction on the choice of the prime \(\text{[223]}\)

\[174\text{So now we have a single number } \theta \text{ that "works" for all } i \text{ simultaneously, rather than individual } \alpha_i.\]

\[175\text{I think Dedekind has finished the "general observations" mentioned above, so that the proof of Theorem IV now begins. One could, however, argue that this is one more lemma, and that the real proof begins when he invokes the key assumption in the next paragraph.}\]
functions $P_1, P_2, \ldots, F_m$ except that their degrees are respectively those of the prime ideals $p_1, p_2, \ldots, p_m$, so that, for example, if $f_1 = f_2$ nothing stops us from choosing $P_1 = P_2$. We now want to introduce the main assumption of the theorem, namely that we can find $m$ pairwise incongruent prime functions of the desired degree, and we will assume that $P_1, P_2, \ldots, P_m$ are such pairwise incongruent prime functions. Then the powers $P_1^{e_1}, P_2^{e_2}, \ldots, P_m^{e_m}$ will be pairwise relatively prime; if we let

$$R = P_1^{e_1} P_2^{e_2} \ldots P_m^{e_m}$$

be their product, then the numerical congruence

$$\varphi(\theta) \equiv 0 \pmod{p}$$

is equivalent the system of $m$ function congruences given above and so (C. 5) is equivalent to the [single] function congruence

$$\varphi(t) \equiv 0 \pmod{p, R}.$$

Notice that the degree of the product $R$ is

$$e_1 f_1 + e_2 f_2 + \cdots + e_m f_m,$$

and so is $= n$. Therefore a number

$$\varphi(\theta) = x_0 + x_1 \theta + x_2 \theta^2 + \cdots + x_{n-1} \theta^{n-1}$$

can only be divisible by $p$ if $\varphi(t) \equiv 0 \pmod{p}$, i.e., only if all the $x_j$ are divisible by $p$. It follows (from § 1) that the index of $\theta$ is not divisible by $p$. So we have proved the Theorem stated above, and we now want to add the following remark.

---

176 The key assumption is used here.
177 First use the equivalence we just proved.
178 This is the Chinese Remainder Theorem for polynomial congruences.
179 Since $\varphi(t)$ is a polynomial of degree $n - 1$ it can only be divisible by $R$ modulo $p$ if it is zero modulo $p$.
180 If there are enough irreducible polynomials modulo $p$, we can find a number $\theta$ whose index is not divisible by $p$.
181 The remark is just that the polynomials we have found are exactly the factors (modulo $p$) of the irreducible polynomial of $\theta$. 

44
Since \( k \) is not divisible by \( p \), it is also not equal to 0. Thus, the number \( \theta \) we obtained is the root of an irreducible equation \( F(\theta) = 0 \) of degree \( n \). Then \( F(\theta) \equiv 0 \pmod{p} \), so the function \( F \) must be divisible by \( R \pmod{p} \). Since both functions have degree \( n \) and have highest coefficient 1, we must have \( F \equiv R \pmod{p} \), i.e.

\[
F \equiv P_1^{e_1} P_2^{e_2} \ldots P_m^{e_m} \pmod{p},
\]

and we have now returned to the starting point of our investigation in § 2.

§ 5

Our last investigation has yielded a criterion that answers the question of whether \( \Omega \) contains an integer \( \theta \) whose index is not divisible by \( p \). When we have

\[
\mathcal{O} p = p_1^{e_1} p_2^{e_2} \ldots p_m^{e_m},
\]

where \( p_1, p_2, \ldots, p_m \) are distinct prime ideals whose degrees are, respectively, \( f_1, f_2, \ldots, f_m \), then the singular case in which the indices of all the integers in \( \Omega \) are divisible by \( p \) happens when and only when it is not possible to find \( m \) prime functions of degree \( f_1, f_2, \ldots, f_m \) that are pairwise incongruent mod \( p \). Now we must ask whether the case when there are not enough

\[\text{To summarize, Dedekind has proved the following: if we know the factorization of } p \text{ in } \Omega \text{ and we choose any list of polynomials of the appropriate degrees, then we can find an element } \theta \text{ that generates } \Omega \text{ over } \mathbb{Q} \text{ and whose index is not divisible by } p. \text{ It will not necessarily be a nice generator.}
\]

For example, suppose \( \Omega = \mathbb{Q}(\sqrt{2}) \) and \( p = 7 \). Then in fact \( \mathcal{O} = \mathbb{Z}[\sqrt{2}] \). The principal ideal \( \mathcal{O} 7 \) factors as \( \mathcal{O} 7 = \mathcal{O}(3 + \sqrt{2}) \cdot \mathcal{O}(3 - \sqrt{2}) \). The factors \( p_1 = \mathcal{O}(3 + \sqrt{2}) \) and \( p_2 = \mathcal{O}(3 - \sqrt{2}) \) are both prime ideals of degree one. Let’s deliberately make the “wrong” choice of two distinct polynomials of degree one in \( F_7[t] \): \( P_1 = t, \ P_2 = t - 1 \). We now need to find \( \alpha_1, \alpha_2 \) such that \( P_i(\alpha_i) \equiv 0 \pmod{p_i} \). Clearly we can take \( \alpha_1 = 3 + \sqrt{2} \) and \( \alpha_2 = 4 - \sqrt{2} \). Solving \( \theta \equiv \alpha_1 \pmod{p_1^2} \), we get \( \theta \equiv 25 + 27\sqrt{2} \pmod{49} \), so let \( \theta = 25 + 27\sqrt{2} \). The minimal polynomial for \( \theta \) is

\[
\theta^2 - 50\theta - 833 \equiv \theta^2 - \theta \pmod{7}
\]

as expected, and the discriminant of \( \theta \) is 5832, so the index of \( \mathbb{Z}[\theta] \) is 729 = 3\(^6\), which is not divisible by 7.

More specifically, it gives an answer to the question of whether \( \theta \) exists if we know the factorization of the principal ideal \( \mathcal{O} p \).
such polynomials ever does occur. To answer this, we will take the simplest possible approach. The incongruent prime functions of degree one are the following:

\[ t, t+1, t+2, \ldots, t+(p-1). \]

Their number is \( p \). So the singular case above will occur in a field \( \Omega \) whenever a prime number \( p \) factors as the product of \( p+1 \) distinct prime ideals of degree 1. By the norm computation above, the degree \( n \) of such a field must be \( n = p+1 \). If, in order to obtain the simplest case, one takes the smallest prime number \( p = 2 \), the question arises whether there are cubic fields \( \Omega \) in which the number 2 is divisible by three distinct prime ideals of degree one. In such a field, the indices of all algebraic integers will be even. This investigation was carried out in full generality the Göttingischen gelehrten Anzeigen of September 20, 1871, and led to an affirmative answer; here I will be content to give a single example that has already been mentioned there.

Let \( \alpha \) be a root of the irreducible polynomial of degree 3

\[ F(\alpha) = \alpha^3 - \alpha^2 - 2\alpha - 8 = 0. \]

To find the discriminant we find the number

\[ F'(\alpha) = \delta = -2 - 2\alpha + 3\alpha^2. \]

---

184 More generally, Dedekind’s argument shows that if \( p \) splits completely in a field of degree \( n > p \), then it will be a common index divisor.

185 So we can find fields with common index divisors if we can solve the following problem: given a prime \( p \) and an integer \( n > p \), find a field in which \( p \) splits completely. Dedekind chooses \( p = 2 \) and \( n = 3 \).

186 Dedekind does much more than just give the example. Starting from an irreducible polynomial of degree three, Dedekind finds an integral basis, computes the discriminant, finds an explicit factorization of 2, and computes explicitly the products of all the ideals that divide 2. One gets the impression that he wants to show that everything in his theory can be computed explicitly, given enough time and patience.

187 The editors of [8] add a footnote: “The notice [Anzeige] just mentioned contains an explanation of the method by which Dedekind came up with the example discussed here. It also contains another example of a field with a common index divisor, namely a quartic field in which the prime number 2 decomposes into two prime ideals of degree two.”

188 Dedekind uses the theorem that the discriminant of the minimal polynomial for \( \alpha \) is equal to \((-1)^{n(n-1)/2} N(F'(\alpha)) \).
Then, repeatedly using $F(\alpha) = 0$, we compute the products

\[ \delta \alpha = 24 + 4\alpha + \alpha^2 \]
\[ \delta \alpha^2 = 8 + 26\alpha + 5\alpha^2 \]

and via linear elimination of $1, \alpha, \alpha^2$ we find that $\delta$ is a root of the equation

\[
\begin{vmatrix}
-2 - \delta & -2 & 3 \\
24 & 4 - \delta & 1 \\
8 & 26 & 5 - \delta \\
\end{vmatrix} = 0
\]

that is,

\[ \delta^3 - 7\delta^2 - 2012 = 0. \]

Therefore we have

\[ \Delta(1, \alpha, \alpha^2) = -N(\delta) = -2012 = -2^2 \cdot 503. \]

Since 503 is a prime number, the only two square divisors of this discriminant are 1 and 4, so the index $k$ of the number $\alpha$ is either 1 or 2. It is therefore the function

\[ F(t) = t^3 - t^2 - 2t - 8 \]

that we need to investigate modulo $p = 2$. Clearly

\[ F = P_1^2 P_2 - 2M \equiv P_1^2 P_2 \pmod{2}, \]

where

\[ P_1 = t, \quad P_2 = t - 1, \quad M = t + 4. \]

Since $P_1$ is a factor of $M$ and $P_1^2$ is a factor of $F$ modulo 2, it follows (from the second proof of Theorem II in § 3) that the number

\[ P_1(\alpha)P_2(\alpha) = \alpha(\alpha + 1) \]

is divisible by 2, and therefore $k = 2$. That is immediately confirmed by the fact that the number

\[ \beta = \frac{1}{2} \alpha(\alpha - 1) - 1 \]

189 We would say “using the Cayley-Hamilton theorem.”

190 If we believe the theorems we are done: this is the condition for 2 to be an index divisor for this polynomial, and so $k = 2$. But Dedekind will check that his theorems are true each time.
turns out to be an algebraic integer. In fact, using the equation $F(\alpha) = 0$ we find that

$$\alpha^2 = 2 + \alpha + 2\beta$$
$$\beta^2 = -2 + 2\alpha - \beta$$
$$\alpha\beta = 4$$

and so

$$\beta^3 + \beta^2 + 2\beta - 8 = 0.$$  

It follows that

$$1 = 1 \cdot 1 + 0 \cdot \alpha + 0 \cdot \beta$$
$$\alpha = 0 \cdot 1 + 1 \cdot \alpha + 0 \cdot \beta$$
$$\alpha^2 = 2 \cdot 1 + 1 \cdot \alpha + 2 \cdot \beta$$

and so

$$\Delta(1, \alpha, \alpha^2) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{vmatrix}^2 \Delta(1, \alpha, \beta) = 2^2 \Delta(1, \alpha, \beta),$$

from which we see that

$$\Delta(1, \alpha, \beta) = -503.$$  

Since this number is not divisible by any square (except 1), it is the fundamental number $D$ of our cubic field $\Omega$, and the numbers $1, \alpha, \beta$ are a basis for all the integers $\omega$ belonging to the domain $\mathcal{O}$, i.e.,

$$\mathcal{O} = [1, \alpha, \beta]$$

in the notation we have used before. Any algebraic integer in $\mathcal{O}$ can be written in the form

$$\omega = z + x\alpha + y\beta,$$

where $z, x, y$ are arbitrary rational integers.

We now want to use these results to determine the factorization of the number 2. Since

$$\alpha^2 = 2 + \alpha + 2\beta \equiv \alpha \pmod{2}$$
$$\beta^2 = -2 + 2\alpha + \beta \equiv \beta \pmod{2}$$

We have computed the discriminant $D$ and an integral basis of $\mathcal{O}$.  

This is the key: he wants to check that 2 splits completely, but he cannot use Theorem I, so he goes for a direct computation. The first step is to show 2 is unramified.
we get
\[(z + x\alpha + y\beta)^2 \equiv z^2 + x^2\alpha^2 + y^2\beta^2 \equiv z + x\alpha + y\beta \pmod{2},\]
so that every \(\omega \in \mathcal{O}\) satisfies \(\omega^2 - \omega \equiv 0 \pmod{2}\). If follows, first, that 2 cannot be divisible by the square of a prime ideal\(^{193}\). Indeed, if \(\mathcal{O}(2) = p^2q\), with \(p\) a prime ideal in \(\mathcal{O}\) or even any ideal different from \(\mathcal{O}\), then \(pq\) is not divisible by \(\mathcal{O}(2)\), so there exists an element \(\omega\) such that \(pq\) divides \(\omega\) but 2 does not divide \(\omega\). Then \(\omega^2\) is divisible by \(p^2q^2\) and therefore by 2, and this contradicts\(^{193}\) the congruence \(\omega^2 \equiv \omega \pmod{2}\) above. So \(\mathcal{O}(2)\) is either a prime ideal or a product of distinct prime ideals\(^{193}\). Let \(p\) be a prime ideal dividing 2, then we must have \(\omega^2 \equiv \omega \pmod{p}\) for every \(\omega\) in \(\mathcal{O}\). The number of incongruent roots of this congruence is \((\mathcal{O}; p) = N(p)\), but the number of roots cannot be larger than the degree of the congruence, so we get \(N(p) \leq 2\) and hence \(N(p) = 2\). So \(p\) is a prime ideal and is not all of \(\mathcal{O}\), since \(N(p) > 1\). Therefore every prime ideal contained in 2 is of degree one, and it follows, since \(N(2) = 2^3 = 8\), that
\[\mathcal{O}(2) = abc,\]
where \(a, b, c\) are three distinct prime ideals of degree one\(^{196}\). This shows that the singular case described above does occur, and we will check\(^{197}\) that indeed the indices of all the number \(\omega\) will be divisible by 2. In fact, if we set\(^{198}\)
\[z' = z^2 + 2x^2 - 2y^2 + 8xy\]
\[x' = x^2 + 2y^2 + 2xz\]
\[y' = 2x^2 - y^2 + 2yz\]
we have
\[\omega^2 = z' + x'\alpha + y'\beta,\]
from which it follows that the index of \(\omega\) is equal to the determinant
\[
\begin{vmatrix}
1 & 0 & 0 \\
z & x & y \\
z' & x' & y'
\end{vmatrix}
= xy' - x'y = 2x^3 - x^2y - xy^2 - 2y^3,
\]
\(^{193}\)The next two sentences give a proof.\(^{194}\)Indeed, \(0 \equiv \omega^2 \not\equiv \omega \pmod{2}\).\(^{195}\)Now we need to prove 2 is not a prime in \(\mathcal{O}\).\(^{196}\)We are done. But Dedekind will prove it again.\(^{197}\)Again, we already know this must happen, but Dedekind will check it explicitly.\(^{198}\)Dedekind is just computing \(\omega^2\).
which is always even.  

In order to complete our example and to confirm the predictions derived from general theory by calculation, we want finally to represent the ideals that appear here in the form of finite modules of rank three (D. § 161; B.§ 3), i.e., to determine these ideals by finding their bases. These representations are as follows:

\[
\begin{align*}
\mathbf{a} &= [2, \alpha, 1 + \beta] \\
\mathbf{b} &= [2, 1 + \alpha, \beta] \\
\mathbf{c} &= [2, \alpha, \beta].
\end{align*}
\]

The system \( \mathbf{a} \) of all numbers of the form

\[
\alpha' = 2z + \alpha z + (1 + \beta)y,
\]

where \( z, x, y \) are arbitrary rational integers, indeed has the fundamental properties of an ideal, namely:

I. The sums and differences of two numbers \( \alpha' \) in the system \( \mathbf{a} \) belong to the same system \( \mathbf{a} \).

II. Each product of a number \( \alpha' \) from the system \( \mathbf{a} \) and a number \( \omega \) from the domain \( \mathcal{O} \) is still a number from the system \( \mathbf{a} \). \[228\]

The first property is clear. To prove the second it suffices to check that the product of each of the basis numbers 2, \( \alpha \), \( 1 + \beta \) of \( \mathbf{a} \) by each of the basis numbers 1, \( \alpha \), \( \beta \) of \( \mathcal{O} \) belongs to \( \mathbf{a} \). This is clear right away for the five products

\[
2.1, \alpha.1, (1 + \beta).1, 2.\alpha, 2.\beta = -2 + 2(1 + \beta).
\]

For the remaining four the same follows from the equations

\[
\alpha.\alpha = \alpha + 2(1 + \beta), \quad \alpha.\beta = 2.2,
\]

\[199\]This will later be called computing the “index form,” especially in the Kronecker school. Hensel showed in [23] that the index form has content 1, i.e., it does not have an integer factor bigger than 1. On the other hand, it may be that the values of the index form are always divisible by some prime. Here the form is congruent mod 2 to \( (x^2 - x)y - (y^2 - y)x \) and hence is always 0 (mod 2). In the same paper Hensel proved that \( p \) is a common divisor of the values if and only if the form involves \( u^p - u \), as here.

\[200\]So we are going to check everything explicitly.

\[201\]Dedekind say “dreigliedrigen Moduln,” which means something like “triple modules” or “trinomial modules.”

\[202\]Dedekind doesn’t explain how he computed these (it’s easy enough), but he will check that these modules are indeed ideals and that their product is 2.
\[(1 + \beta)\alpha = 2.2 + \alpha, \quad (1 + \beta)\beta = -2 + 2\alpha.\]

In the same way one can check that the systems \(b\) and \(c\) are ideals.

The Norm \(N(m)\) of an ideal \(m\) is the number \((\mathcal{O} : m)\) of numbers that are incongruent mod \(m\) (D. § 163; B. § 20), which is equal to the determinant of the expressions that give the basis numbers of \(m\) as linear combinations of the basis numbers of \(\mathcal{O}\) (D. § 161; B. § 4, 4°). So, for example,

\[
N(a) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2,
\]

and in the same way

\[N(b) = N(c) = 2.\]

When, however, the norm of an ideal is a prime number, that ideal must necessarily be a prime ideal\(^{203}\) since in general we have \(N(a_1a_2) = N(a_1)N(a_2)\).

Therefore \(a\), \(b\), \(c\) are prime ideals. Further, they are pairwise distinct, since the number \(\beta\) belongs to both \(b\) and \(c\) but not to \(a\), and the number \(\alpha\), which is contained in \(c\), is not contained in \(b\). The number 2 is contained in all three ideals and so must also be contained in the product \(abc\), so \(\mathcal{O}(2) = mabc\), where \(m\) is some ideal. But computing the norm we get

\[N(2) = 8 = N(m)N(a)N(b)N(c) = 8N(m),\]

therefore \(N(m) = 1\), so \(m = \mathcal{O}\) and \(\mathcal{O}(2) = abc\).\(^{204}\) But we also want to check this result, which follows from general theorems, by a direct computation, i.e., through the actual multiplication of the ideals (D. § 165; B. § 12).\(^{205}\)

By the product \(ab\) of two ideals we understand the system of all products \(\alpha'\beta'\) and all sums of such products \(\alpha'\beta'\), where \(\alpha', \beta'\) are any numbers belonging respectively to the ideals \(a, b\) (D. §163; B. § 22). Such a product is therefore first a finite module whose basis numbers are all the products of

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\(^{203}\)This checks that the three factors are prime ideals.

\(^{204}\)We have proved the factorization a second time.

\(^{205}\)Dedekind’s approach initially distinguished between two notions of divisibility. On the one hand, an ideal \(a\) divides \(b\) if \(b \subset a\). On the other, \(a\) divides \(b\) if there exists an ideal \(c\) such that \(b = ac\). In [9] he says that proving these two notions are equivalent is “the main difficulty of the theory,” which he has overcome. He has established the factorization of 2 using the first point of view. Now he will check it from the second point of view.
each basis number of \( a \) by each basis number of \( b \). In our case, then, \( ab \) is the finite module whose basis numbers are the nine products

\[
2 \cdot 2 = 4, \quad 2(1 + \alpha) = 2 + 2\alpha, \quad 2\beta = 2\beta,
\]

\[
\alpha \cdot 3 = 2\alpha, \quad \alpha(1 + \alpha) = 2 + 2\alpha + 2\beta, \quad \alpha\beta = 4,
\]

\[
(1 + \beta)2 = 2 + 2\beta, \quad (1 + \beta)(1 + \alpha) = 5 + \alpha + \beta, \quad (1 + \beta)\beta = -2 + 2\alpha.
\]

Of those nine number only three are mutually independent (D. § 159; B. § 4), so by the method I have described in detail (B. § 4, we can reduce this module with nine generators to one with three generators. Doing this very simple and easy calculation one gets the six following equations:

\[
\begin{align*}
\mathbf{a}^2 &= [4, \alpha, 3 + \beta] ; \\
\mathbf{b}^2 &= [4, 1 + \alpha, \beta] ; \\
\mathbf{c}^2 &= [4, 2 + \alpha, 2 + \beta] ; \\
\mathbf{a}\mathbf{b} &= [2, 2\alpha, 1 + \alpha + \beta] .
\end{align*}
\]

We now proceed in the same way. Multiplying each of those by \( a \), \( b \), \( c \) using the same method, we obtain the following ten principal ideals:

\[
\begin{align*}
\mathbf{a}\mathbf{b}\mathbf{c} &= [2, 2\alpha, 2\beta] = \mathcal{O}(2) \\
\mathbf{a}^2 \mathbf{c} &= [4, \alpha, 2 + 2\beta] = \mathcal{O}(2) \\
\mathbf{b}^2 \mathbf{c} &= [4, 2 + 2\alpha, \beta] = \mathcal{O}(2) \\
\mathbf{a} \mathbf{c}^2 &= [4, 2 + \alpha, 2\beta] = \mathcal{O}(2 - \beta) \\
\mathbf{b} \mathbf{c}^2 &= [4, 2\alpha, 2 + \beta] = \mathcal{O}(2 - \beta) \\
\mathbf{a}^2 \mathbf{b} &= [4, 2\alpha, 3 + \alpha + \beta] = \mathcal{O}(3 + \alpha + \beta) \\
\mathbf{a}\mathbf{b}^2 &= [4, 2 + 2\alpha, 1 + \alpha + \beta] = \mathcal{O}(1 + \alpha + \beta) \\
\mathbf{a}^3 &= [8, 4 + \alpha, 3 + \beta] = \mathcal{O}(3 + 2\alpha + \beta) \\
\mathbf{b}^3 &= [8, 1 + \alpha, 4 + \beta] = \mathcal{O}(1 + \alpha) \\
\mathbf{c}^3 &= [8, 2 + \alpha, 2 + \beta] = \mathcal{O}(\alpha + \beta - 4).
\end{align*}
\]

The ten numbers \( \mu \) to which these principal ideals \( \mathcal{O}\mu = [\mu, \alpha\mu, \beta\mu] \) correspond are connected to each other by the following easily checked relations:

\[
\alpha(\alpha - 2)(1 + \alpha) = 2^3; \quad \alpha\beta = (\alpha - 1)(1 + \alpha + \beta^2)
\]

---

He says “neungliedrigen Modul.”

Dreigliedrigen.”
\[(\alpha - 2)(3 + \alpha + \beta) = 2\alpha; \quad \alpha(2 - \beta) = 2(\alpha - 2)\]
\[(\alpha - 2)(3 + 2\alpha + \beta) = \alpha^2; \quad \alpha(\alpha + \beta - 4) = (\alpha - 2)^2\]

This example, to which one add many others, makes it clear that there are fields \(\Omega\) in which the indices of all integers are divisible by the same prime number \(p\). This result is not a welcome one in some respects. Indeed, there are many theorems in the theory of ideals that would be easy to prove via the theory of higher congruences were it not for the fact that Theorem I in § 2 requires the assumption that the index \(k\) of the integer \(\theta\) not be divisible by \(p\). We have now seen, however, that in many cases this hypothesis cannot be satisfied no matter which number \(\theta\) we choose, and it follows that the approach suggested by that Theorem will not work in full generality. For example, I mention the following important theorem which I also used in the Göttingischen gelehrten Anzeigen of 20 September 1971:

*The fundamental number \(D\) of a field \(\Omega\) is the product of those and only those rational prime numbers \(p\) that are divisible by the square of a prime ideal in that field.*

If there is an integer in \(\Omega\) whose index is not divisible by the prime number \(p\), the truth of this result clearly follows very easily\(^{208}\) from § 2. But this obviously does not lead to a proof of the general theorem, and it was only after several unsuccessful attempts that I succeeded in finding a general proof. I must, however, reserve the detailed development of this subject, in which the theorem itself will be considerably generalized, for another occasion.

### 3.2 Notes by Öystein Ore

In Dedekind’s collected works, this paper is followed by some two pages of “Erläuterungen zur vorstehenden Abhandlung” signed by Öysten Ore, one of the three editors. We present only some highlights of what Ore has to say. Quotations are from [3] pp. 230–232.

“...The problem of generalizing the Kummer theory of ideals in cyclotomic fields to general fields leads naturally to a definition of ideals by means of

\(^{208}\)If \(p\) divides \(D\), then it divides \(d(\theta)\) for any \(\theta\), which means that the corresponding polynomial \(F(x)\) has a double root modulo \(p\) and hence is divisible by the square of an irreducible polynomial. If we assume there is a \(\theta\) whose index is not divisible by \(p\), we can use Theorem I, which tells us that \(p\) is divisible by the square of a prime ideal. Conversely, if \(p\) is divisible by the square of a prime ideal, \(F(\theta)\) is divisible by the square of an irreducible polynomial mod \(p\), and so \(p|d(\theta)\). Since \(d(\theta) = k^2D\) and \(p \not| k\) then \(p|D\).
higher congruences. Selling (Zeitschr. f. Math n. Phys., vol 10, pp. 17–47 (1865)) already takes this path, and it is possible, using Galoisian imaginaries and other auxiliary fields, to obtain a general theory of ideals in Galois fields. The prime ideal decomposition of a prime number \( p \) is obtained from the factorization mod \( p^n \) of the defining equation in these auxiliary fields. A proof of the invariance of these ideals, i.e., of their independence of the chosen defining equation, is not clear.” The Selling article is the same one cited by Dedekind, [30]. See Ore’s comments below on factoring modulo powers of \( p \).

“As can be seen from the introduction,” Dedekind had tried this method as well[209] but then abandoned it in favor of an abstract theory of ideals as presented in the second edition of Dirichlet’s Zahlentheorie. This form of the theory does not give us, however, an explicit way to determine the factorization of given numbers in the field. Theorem I solves that problem for primes that do not divide the index, but the existence of common index divisors (or common inessential discriminant divisors) blocks that path in general.

The next few paragraphs focus on Zolotarev. In Zolotarev’s first paper [34], something like Theorem I is used as the definition of prime divisors, but of course this means it does not work for all primes. Ore says that Zolotarev’s second paper [33] solved the problem, but to do that had to abandon the approach based on higher congruences. Ore explains Zolotarev’s “semi-local” approach; there are good expositions in [31] and [15]. Ore says he will not get into all the alternative ways to lay the foundations.

Kronecker’s approach based on forms gives a theoretically very simple determination (“eine theoretisch besonders einfache Bestimmung”) of the prime divisors of a rational prime[210] “As was first shown in full generality by Hensel. . . there is a complete analogue to Dedekind’s theorem for all prime numbers in this theory.” The reference is to [24], where Hensel shows that one can overcome the existence of common index divisors by studying the ”Fundamentalgleichung“ [211]

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209 This is even clearer from the discussion in the beginning of § 4, where Dedekind says he thought for a long time that this would be possible.

210 Does Ore mean that Kronecker’s theory yields an algorithm? Or that it is computable in theory but not in practice?

211 This is the polynomial in \( t \) with coefficients in \( \mathbb{Z}[x_0, x_1, \ldots, x_{n-1}] \) that has the generic algebraic integer \( \omega = x_0 + x_1 \omega + \cdots + x_{n-1} \omega^{n-1} \) and its conjugates as roots. Hensel proved that the discriminant of this equation is a homogeneous form in \( n \) variables with content \( D \), and also that the factorization modulo \( p \) of the fundamental equation always gives the
“However, this solution to the problem does not provide any information about the relationship between the properties of the field equation and prime ideal decompositions, as is the case with Dedekind’s theorem. In Hensel’s $p$-adic theory of algebraic numbers, this gap is partially filled by showing that the decomposition of the defining equation into irreducible $p$-adic factors corresponds to the decomposition of $p$ into prime ideal powers. For the complete determination of the prime ideal decomposition, one must also use Kronecker’s theory here. (See K. Hensel: *Theorie der algebraischen Zahlen I*, Leipzig 1908.)” So even in 1930 Ore does not see the $p$-adic solution as a complete answer. The reference is to Hensel’s [25]; there was never a volume II.

“It can be shown, however, that the difficulties of Dedekind’s theory can be completely eliminated if, instead of congruences (mod $p$) one always considers congruences (mod $p^\alpha$) where $\alpha > \delta$ if the discriminant of the corresponding equation is exactly divisible by $p^\delta$. The corresponding irreducible factors are then not determined (mod $p^\alpha$), but rather (mod $p^{\alpha-\delta}$). The common index divisors then completely lose their exceptional character and one obtains a clear correspondence between prime ideal decomposition and factors of the equation (O. Ore, *Math. Ann.*, Vol. 96, pp. 315–352 (1926) and Vol. 97, pp. 569–598 (1927)). Furthermore, Dedekind’s representation of the prime ideals in the form $\beta = (p, \varphi(\theta))$ can be recovered by a method that shows great similarity to the determination of the series development of algebraic functions (O. Ore, *Math. Ann.*, Vol. 99, pp. 84–117 (1928)).” It is surprising that Ore does not recognize this as equivalent to Hensel’s $p$-adic approach.

The theorem in section 4 gives a criterion for the existence of common index divisors. Hensel gave a different one in [23] (see [17] for a translation); Hensel’s criterion is in terms of the index form. In the same paper Hensel also proved Kronecker’s conjecture that if a number field $K$ has common index divisors then there exists an extension field for which the values of the index form do not have a common divisor.

Hensel’s criterion implies that if $K$ is a number field of degree $n$ for which $p$ is a common index divisor then $p < 1/2n(n-1)$. Ore notes several improvements on this estimate: E. v. Zylinsky proved [32], using Dedekind’s criterion, that in fact $p < n$. M. Bauer showed [2] that if $p < n$ there always exists a field of degree $n$ for which $p$ is a common index divisor. Bauer’s result also follows from general theorems of Hasse [19] showing the existence correct factorization of $p$ in the corresponding number field.
of fields in which \( p \) has prescribed factorization.

## 4 Conclusion

Richard Dedekind was the first to give an example of a field in which there is a common index divisor, in his 1871 *Anzeige*. (Kronecker says he knew an example in 1858, but he did not mention it in print until 1882.) In his 1878 paper, Dedekind showed that such common index divisors were entirely a “small primes” effect. Specifically, \( p \) is a common index divisor for \( K \) if and only if translating its factorization in \( \mathcal{O}_K \) into a polynomial factorization modulo \( p \) requires too many irreducible polynomials. This is exactly the criterion that Hasse gives in [20, p. 456] and attributes to Hensel.

As Ore pointed out in his notes, the usefulness of the criterion is limited in that it requires knowledge of the factorization of \( p \). This is frustrating, since starting point of the investigation was exactly the use of Dedekind’s theorem to determine the factorization of \( p \). The status of the problem in 1878 was this:

- Given a generator \( \alpha \) and a prime \( p \), one can tell, looking at the factorization of the minimal polynomial modulo \( p \), whether \( p \) divides the index \( (\mathcal{O} : \mathbb{Z}[\alpha]) \). This is one of the main results in the paper translated above.

- If \( p \) did not divide the index, one could determine its factorization in \( \mathcal{O} \) from the factorization modulo \( p \) of the irreducible polynomial.

- In some cases, however, it is impossible to find such a generator \( \alpha \). This happens when there are not enough irreducible polynomials modulo \( p \) to reflect the correct factorization of \( p \) in \( \mathcal{O} \).

- For such common index divisors, no algorithm was available to determine the factorization.

Kronecker assigned the problem to Hensel sometime in the early 1880s. It was the topic of Hensel’s dissertation and of several papers until the culminating paper of 1894, which we translate in [17]. This suggests that neither Kronecker nor Hensel had fully absorbed Dedekind’s 1878 paper, though both cite it.
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