Faddeev-Popov method for anomalous quasigroups

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Abstract
1 Introduction

The quantization of anomalous gauge theories has received much attention during the last few years. Following the work of Faddeev and Shatashvili [1], some proposals appeared [2] [3] for cancelling the anomalies through Wess-Zumino terms [4]. The main observation of these works was that, in the path integral quantization of chiral gauge theories, the integration over the elements of the gauge group must be taken into account. In this way additional fields as well as the Wess-Zumino action emerge naturally in the process of quantization. A similar observation was made earlier by Polyakov [5] in the path integral quantization of the bosonic string in the presence of the conformal anomaly.

On the other hand, a proposal to deal with anomalous gauge theories with closed, irreducible gauge algebra in the framework of the field-anti field formalism was described recently in [6]. There, by extending the original configuration space with extra degrees of freedom, a general expression for the Wess-Zumino action in terms of the anomalies of the theory was obtained and the form of the gauge transformations for the extra fields was given.

The aim of this letter is to describe a generalization of the Faddeev-Popov procedure [7] in order to treat on the same footing the path integral quantization of either anomalous and non-anomalous gauge theories with closed, irreducible gauge algebra. Our construction closely follows the results of ref. [8], where the quantization a la Faddeev-Popov of non-anomalous quasigroups was considered. As a byproduct, we obtain expressions for the Wess-Zumino action and for the gauge transformations of the additional fields which are in complete agreement with the ones previously given in [6].

Prior to that, since along our development some technical tools about the so-called quasigroup structure are needed, a summary of the results given in [8] is presented.

2 Brief overview of Quasigroups

The quasigroup structure is a generalization of the Lie group structure, introduced by Batalin [8]. The main difference between this structure and a Lie group relies on the fact that the quasigroup composition law depends not only on the parameters of the transformations, but also on the variables on which these transformations act.

To introduce the concept of quasigroup, it is useful to have in mind a manifold $\mathcal{M}$ with coordinates $\phi^i$, $i = 1, \ldots, n$. Under these conditions, consider a continuous transformation acting on the coordinates of $\mathcal{M}$ given by

$$\phi^i = P^i(\phi, \theta),$$

with $\theta^\alpha$, $\alpha = 1, \ldots, r$, a set of real parameters.\footnote{Along this paper, for simplicity, we will restrict ourselves to the bosonic case, i.e., $\epsilon(\phi^i) = \epsilon(\theta^\alpha) = 0.$}
Assume now that the transformations $F^i(\phi, \theta)$ satisfy the following properties:

1. For $\theta^\alpha = 0$, we have the identity transformation
   \[
   \phi^i = F^i(\phi, 0).
   \]

2. The composition law between two finite transformations reads
   \[
   F^i(F(\phi, \theta), \theta') = F^i(\phi, \varphi(\theta, \theta'; \phi)),
   \tag{2.1}
   \]
   where $\varphi^\alpha(\theta, \theta'; \phi)$ is the composition law of the quasigroup.

3. Left and right units coincide
   \[
   \varphi^\alpha(\theta, 0; \phi) = \varphi^\alpha(0, \theta; \phi) = \theta^\alpha.
   \tag{2.2}
   \]

4. A modified associativity law holds
   \[
   \varphi^\alpha(\varphi(\theta, \theta'; \phi), \theta''; \phi) = \varphi^\alpha(\theta, \varphi(\theta', \theta''; F(\phi, \theta)); \phi).
   \tag{2.3}
   \]

5. There exists an inverse transformation given by
   \[
   \phi^i = F^i(\tilde{\phi}, \tilde{\theta}(\theta, \tilde{\phi})),
   \]
   with the inverse $\tilde{\theta}^\alpha(\theta, \tilde{\phi})$ satisfying the relations
   \[
   \varphi^\alpha(\tilde{\theta}(\theta, \tilde{\phi}), \theta; \tilde{\phi}) = \varphi^\alpha(\theta, \tilde{\theta}(\theta, \tilde{\phi}); \phi) = 0.
   \]

The above conditions 1)-5) define the quasigroup structure. Some of the relations describing the quasigroup at the infinitesimal level, which are useful in the present study, can be obtained from these conditions in the following way.

The generators of infinitesimal transformations for the variables $\phi^i$ are
\[
R^i_\alpha(\phi) \equiv \left. \frac{\partial F^i(\phi, \theta)}{\partial \theta^\alpha} \right|_{\theta=0}.
\]

Antisymmetrizing the second derivatives of the modified composition law \(2.1\) with respect to $\theta^\alpha$, $\theta'^\beta$ in $\theta = \theta' = 0$, one obtains the algebra
\[
R^i_{\alpha,j}R^j_\beta - R^i_{\beta,j}R^j_\alpha = R^i_\gamma T^\gamma_{\alpha\beta},
\tag{2.4}
\]
with the structure functions $T^\gamma_{\alpha\beta}(\tilde{\phi})$ defined as
\[
T^\gamma_{\alpha\beta}(\tilde{\phi}) = \left. \left[ \frac{\partial^2 \varphi^\gamma(\theta, \theta'; \phi)}{\partial \theta^\beta \partial \theta'^\alpha} - (\alpha, \beta) \right] \right|_{\theta=\theta'=0}.
\]
From the associativity law (2.3), one gets the generalized Jacobi identity for the structure functions
\[ T_{\alpha \beta}^\gamma T_{\gamma \delta}^\mu - T_{\alpha \beta}^\mu R_\delta^i + \text{(cyclic perm. of } (\alpha, \beta, \delta)) = 0. \] (2.5)

Let us introduce now the matrices \( \mu, \tilde{\mu} \) defined by
\[ \mu_{\alpha \beta}(\theta, \phi) = \frac{\partial \varphi^\alpha(\theta, \theta' \phi)}{\partial \theta' \beta} \bigg|_{\theta' = 0}, \quad \tilde{\mu}_{\beta \alpha}(\theta, \phi) = \frac{\partial \varphi^\alpha(\theta' \phi, \theta)}{\partial \theta' \beta} \bigg|_{\theta' = 0}, \] (2.6)
while we denote their inverses as \( \lambda, \tilde{\lambda} \)
\[ \lambda^\alpha_{\beta \gamma} \mu_{\gamma \beta} = \delta^\alpha_{\beta}, \quad \tilde{\lambda}_\beta^\alpha \tilde{\mu}_\alpha^\beta = \delta^\beta_{\beta}. \]
These matrices will always exist, at least locally, because the property
\[ \mu_{\beta \alpha}(\theta = 0) = \tilde{\mu}^\alpha_{\beta}(\theta = 0) = \delta^\alpha_{\beta}, \]
holds by virtue of eqs. (2.2) and (2.4).

With the aid of the associativity law (2.3) one obtains an analog to the Lie equation for the transformations of the fields \( F_i \)
\[ \frac{\partial F^i}{\partial \theta^\alpha} = R_\beta^i(F) \lambda^\alpha_{\beta \gamma}(\theta, \phi), \] (2.7)
and the useful transformation rule for the generators
\[ \frac{\partial F^i}{\partial \phi^j} R^j_\beta(\phi) = R^i_\beta(F) \lambda^\alpha_{\gamma \beta}(\theta, \phi). \] (2.8)

On the other hand, differentiating the same equation (2.3) with respect to the parameters of the quasigroup, an analog to the Lie equation for the composition functions \( \varphi^\alpha \) is found
\[ \frac{\partial \varphi^\alpha(\theta, \theta' \phi)}{\partial \theta' \gamma} = \mu^\alpha_{\beta}(\varphi(\theta, \theta' \phi, \phi) \lambda^\beta_{\gamma}(\theta' \phi, F(\phi, \theta)), \] (2.9)
as well as the following commutation relation for the elements of the matrix \( \lambda \)
\[ \frac{\partial \lambda^\alpha_{\beta \gamma}}{\partial \theta^\beta} - \frac{\partial \lambda^\beta_{\gamma \delta}}{\partial \theta^\gamma} - T^\alpha_{\mu \nu}(\phi) \lambda^\mu_{\beta \gamma} \lambda^\nu_{\delta \gamma} = 0, \] (2.10)
which is the analog to the Maurer-Cartan equation for a Lie group.

Another useful relation involving the matrices \( \lambda, \tilde{\lambda}, \mu, \tilde{\mu} \) is
\[ \lambda^\delta_{\gamma}(D_\beta \mu^\alpha_{\beta}) - \tilde{\lambda}^\delta_{\gamma} \frac{\partial \tilde{\mu}^\alpha_{\beta}}{\partial \phi^i} = 0, \] (2.11)
with the operator \( D_\beta \) defined as
\[ D_\beta = \left( \frac{\partial}{\partial \theta^\beta} - R^i_\alpha(\phi) \lambda^\alpha_{\gamma \beta}(\theta, \phi) \frac{\partial}{\partial \phi^i} \right). \]
In our construction some functional integrals over the elements of the gauge quasigroup should be considered. For this reason it is convenient to introduce the concept of right and left invariant measures on the quasigroup. The right invariant measure, defined by

\[ D\theta \det \tilde{\lambda}^\alpha_\alpha(\theta, F(\theta, \tilde{\theta})) , \]  

\[ \text{(2.12)} \]

is invariant under the simultaneous change of classical fields and parameters

\[ \phi^i \rightarrow \tilde{\phi}^i = F^i(\phi, \varepsilon), \]

\[ \theta^\alpha \rightarrow \tilde{\theta}^\alpha = \varphi^\alpha(\theta, \varepsilon; F(\phi, \tilde{\theta})), \]

where \( \tilde{\theta}^\alpha = \tilde{\varphi}^\alpha(\theta, \tilde{\phi}) \) is the inverse of the parameter \( \theta^\alpha \).

On the other hand, the left invariant measure

\[ D\theta \det \lambda^\alpha_\alpha(\theta, \phi) , \]  

\[ \text{(2.13)} \]

can be shown to be invariant under the simultaneous change of classical fields and parameters

\[ \phi^i \rightarrow \tilde{\phi}^i = F^i(\phi, \varepsilon), \]

\[ \theta^\alpha \rightarrow \tilde{\theta}^\alpha = \varphi^\alpha(\varepsilon, \tilde{\phi}; \theta, \tilde{\phi}). \]

\[ \text{(2.14)} \]

In particular, the invariance property of the left invariant measure turns out to be very important in our construction. Indeed, as we will see, for the case of anomalous gauge theories, this invariance suggests considering the parameters of the quasigroup as new dynamical variables transforming under the action of the gauge quasigroup as in (2.14).

For a more exhaustive study of the quasigroup structure, we refer the reader to the original reference [8].

3 Generalized Faddeev-Popov method for quasigroups

Let us now describe a generalization of the Faddeev-Popov procedure [7] to treat on the same footing the path integral quantization of either anomalous and non-anomalous gauge theories.

We restrict ourselves to irreducible theories with closed gauge algebras. The classical action for these systems, \( S_0(\phi^i), i = 1, \ldots, n, \) is invariant under the (infinitesimal) gauge transformations

\[ \delta \phi^i = R^i_\alpha(\phi) \varepsilon^\alpha, \quad \alpha = 1, \ldots, r. \]

\[ \text{(3.1)} \]

Besides, the generators \( R^i_\alpha \) are assumed to be independent and the relations (2.4) and (2.5) verified at any space-time point. Therefore, the gauge structure is in general that of the quasigroup defined in the preceding section at every space-time point.
In order to generalize the Faddeev-Popov procedure for a generic quasigroup let us consider, first of all, admissible gauge fixing conditions $\chi_\alpha(\phi) = 0$. After that, a representation of the unity for quasigroups is introduced \cite{8}

$$1_L = \int D\theta \delta[\chi_\alpha(\phi, \theta)] \Delta_\chi[F(\phi, \theta)]. \quad (3.2)$$

The determinant $\Delta_\chi[F(\phi, \theta)]$ in (3.2), given by

$$\Delta_\chi[F(\phi, \theta)] = \det \left( \frac{\partial \chi_\alpha(F(\phi, \theta))}{\partial \theta^\beta} \right),$$

can be written, using relation (2.7), as

$$\Delta_\chi[F(\phi, \theta)] = \det D_{\alpha\beta}(F(\phi, \theta)) \det \lambda^\alpha_\beta(\theta, \phi),$$

where the matrix $D_{\alpha\beta}$ is defined through the relation

$$D_{\alpha\beta}(\phi) = \left( \frac{\partial \chi_\alpha}{\partial \theta^\beta} R^\beta_\alpha(\phi) \right).$$

Hence, the above expressions yield the following representation of the unity $1_L$ \cite{32}

$$1_L = \int \left[ D\theta \det \lambda^\alpha_\beta(\theta, \phi) \delta[\chi_\alpha(F(\phi, \theta))] \det D_{\alpha\beta}(F(\phi, \theta)) \right],$$

where $\det D_{\alpha\beta}(\phi)$ is the usual Faddeev-Popov determinant and

$$\left[ D\theta \det \lambda^\alpha_\beta(\theta, \phi) \right] \equiv DG_L(\theta, \phi),$$

is the left invariant measure for the elements of the quasigroup (2.13).

An equivalent representation of the unity, $1_R$, can be obtained as well from \cite{32} through the change of parameters $\theta^\alpha \rightarrow \tilde{\theta}^\alpha(\theta, \phi)$,

$$1_R = \int \left[ D\theta \det \tilde{\lambda}^\alpha_\beta(\theta, F(\phi, \tilde{\theta})) \delta[\chi_\alpha(F(\phi, \tilde{\theta}))] \det D_{\alpha\beta}(F(\phi, \tilde{\theta})) \right], \quad (3.3)$$

where $\tilde{\theta}^\alpha = \tilde{\theta}^\alpha(\theta, \phi)$ is the inverse element associated with $\theta^\alpha$ and

$$\left[ D\theta \det \tilde{\lambda}^\alpha_\beta(\theta, F(\phi, \tilde{\theta})) \right] \equiv DG_R(\theta, \phi),$$

is now the right invariant measure for the elements of the quasigroup \cite{2}.12.

Consider now the vacuum-to-vacuum transition amplitude or S-matrix of the theory

$$Z = \int D\phi \exp \left\{ \frac{i}{\hbar} S_0(\phi) \right\}, \quad (3.4)$$
where $D\phi$ is the naive integration measure. After inserting in it the unity $1_R$ (3.3) in the usual way, it reads

$$Z = \int D\phi [D\theta \det \tilde{\lambda}_\beta^\alpha(\theta, F(\phi, \tilde{\theta}))] [\det D_{\alpha\beta}(F(\phi, \tilde{\theta}))] \exp \left\{ \frac{i}{\hbar} S_0(\phi) \right\}.$$  

The dependence of $D_{\alpha\beta}$ and $\chi_\alpha$ on $\theta^\alpha$ can be dropped out by performing the change of variables $\phi^i \to F^i(\phi, \theta)$. This operation yields

$$Z = \int DF(\phi, \theta) [D\theta \det \tilde{\lambda}_\beta^\alpha(\theta, \phi)] [\det D_{\alpha\beta}(\phi)] \exp \left\{ \frac{i}{\hbar} S_0(\phi) \right\}. \quad (3.5)$$

At this point, in the standard Faddeev-Popov procedure it is assumed that the functional measure is gauge invariant. However, as is well known after Fujikawa’s works [9], this is not true in general, this non-invariance being the source of potential anomalies. Therefore, a careful analysis of the Jacobian of the above change of variables

$$DF(\phi, \theta) = \det \left( \frac{\partial F^i}{\partial \phi^j} \right) D\phi = \det S^i_j D\phi, \quad (3.6)$$

should be considered.

An expression for this Jacobian can be obtained, as described in [8], by differentiating its logarithm with respect to the parameters of the quasigroup. Proceeding in this way and taking into account eqs. (2.7) and (2.8) we find

$$\frac{\partial}{\partial \theta^\alpha} \left\{ \ln[\det S^i_j] \right\} = R^i_{\beta,i}(F)\lambda^\beta_\alpha - \mu^\sigma_{\beta,i}R^i_{\gamma}(\phi)\tilde{\lambda}^\sigma_\gamma \lambda^\beta_\alpha.$$ 

The last term of the above expression can be written, using (2.10) and (2.11), as

$$\mu^\sigma_{\beta,i}R^i_{\gamma}(\phi)\tilde{\lambda}^\sigma_\gamma \lambda^\beta_\alpha = -T^\gamma_{\gamma\beta}(F)\lambda^\beta_\alpha - \frac{\partial}{\partial \theta^\alpha} \left\{ \ln \left( \frac{\det \lambda}{\det \tilde{\lambda}} \right) \right\},$$

from which we obtain

$$\frac{\partial}{\partial \theta^\alpha} \left\{ \ln[\det S^i_j] \right\} = A^\beta_\alpha(\phi)\lambda^\beta_\alpha + \frac{\partial}{\partial \theta^\alpha} \left\{ \ln \left( \frac{\det \lambda}{\det \tilde{\lambda}} \right) \right\}, \quad (3.7)$$

with the quantities $A^\alpha_\alpha(\phi)$ given by

$$A^\alpha_\alpha(\phi) = (R^i_{\alpha,i} + T^\beta_{\beta\alpha})(\phi). \quad (3.8)$$

It should be noted that for a local gauge theory the quantities $A^\alpha_\alpha$ are proportional to $\delta(0)$. Therefore, in order to make sense out of this construction, some scheme to regularize them should be considered.

\footnote{From now on, $A^\alpha_\alpha$ will stand for the regularized expression of (3.8).}
Introduce now an object $M_1(\phi, \theta)$ verifying the differential equation
\[ \frac{\partial M_1}{\partial \theta^\alpha} = -iA_\beta(F(\phi, \theta))\lambda^\beta_\alpha(\theta, \phi), \quad (3.9) \]
with the boundary condition $M_1(\phi, \theta = 0) = 0 \mod 2\pi$. This definition allows to solve immediately eq.(3.7) and to write the jacobian (3.6) as
\[ [\det S^i_j](\phi, \theta) = \exp\left\{iM_1(\phi, \theta)\right\} \left(\frac{\det \lambda(\theta, \phi)}{\det \lambda(\theta, \phi)}\right)^{-1}, \]
from which we obtain, substituting it into (3.5), the following expression for the S-matrix
\[ Z = \int D\phi D\lambda D\tilde{\lambda} D\chi \left[\delta[S_\alpha(\phi)] \det D_{\alpha\beta}(\phi) \right] \exp\left\{\frac{i}{\hbar}[S_\alpha(\phi) + \hbar M_1(\phi, \theta)]\right\}. \]

Finally, the introduction of ghosts $C^\alpha$, antighosts $\tilde{C}^\alpha$ and auxiliary fields $B^\alpha$ enables to exponentiate the Faddeev-Popov determinant and the gauge fixing conditions in the usual way, yielding
\[ Z = \int D\phi D\lambda D\tilde{\lambda} D\chi D\tilde{\chi} \left[\delta[S_\alpha(\phi; C, \tilde{C}, B)] \det D_{\alpha\beta}(\phi) \right] \exp\left\{\frac{i}{\hbar}[S_\alpha(\phi; C, \tilde{C}, B) + \hbar M_1(\phi, \theta)]\right\}, \quad (3.10) \]
where $S_\chi$ is the quantum gauge fixed action
\[ S_\chi = S_0 + \tilde{C}^\alpha D_{\alpha\beta}C^\beta + B^\alpha \chi_\alpha, \]
and $DG_L(\theta, \phi)$ is the left invariant measure for the elements of the gauge quasigroup (2.13). Therefore, apart from the explicit integration over the elements of the gauge quasigroup and the presence of the $M_1$ term in (3.10), the result we arrive for the S-matrix is the same as the one obtained in the standard Faddeev-Popov method.

Let us now consider in more detail the $M_1$ term. As is well known, the integrability conditions for an equation of the type (3.9) are
\[ \frac{\partial^2 M_1}{\partial \theta^\alpha \partial \theta^\beta} - \frac{\partial^2 M_1}{\partial \theta^\beta \partial \theta^\alpha} = 0. \]
In the present case, after using eqs.(2.7) and (2.10), these conditions yield
\[ \left(\frac{\partial A_\alpha}{\partial \phi^i} R^i_\beta - \frac{\partial A_\beta}{\partial \phi^i} R^i_\alpha - A_\gamma T^\gamma_{\alpha\beta}\right) (F) \lambda^\alpha_\mu \lambda^\beta_\nu = 0, \]
which, taking into account the invertibility of the matrix $\lambda^\beta_\alpha$, are equivalent to the Wess-Zumino consistency conditions [4]
\[ \left(\frac{\partial A_\alpha}{\partial \phi^i} R^i_\beta - \frac{\partial A_\beta}{\partial \phi^i} R^i_\alpha - A_\gamma T^\gamma_{\alpha\beta}\right) (\phi) = 0. \quad (3.11) \]
Hence, from this result we conclude that the above construction makes sense if the scheme introduced to regularize the quantities \( A_\alpha \) is "consistent", i.e., if the regularized expression of \( A_\alpha \) verifies the Wess-Zumino consistency conditions (3.11).

Under such assumptions, the solution of equation (3.9) with the appropriate boundary condition is given by the integral

\[
M_1(\phi, \theta) = -i \int_0^\theta A_\beta(F(\phi, \theta')) \lambda_\alpha^\beta(\theta', \phi) d\theta'^\alpha, \tag{3.12}
\]

which, in view of the above discussion, does not depend upon the form of the integration path. Taking this fact into account and using the analog to the Lie equation for the composition functions (2.9), as well as (2.1) and (2.2), it is possible to verify that expression (3.12) for the \( M_1 \) term fulfills the so-called 1-cocycle condition

\[
M_1(\phi, \theta) + M_1(F(\phi, \theta), \theta') - M_1(\phi, \varphi(\theta, \theta'; \phi)) = 0 \pmod{2\pi}.
\]

Therefore, we conclude that the quantities \( A_\alpha \) and the \( M_1 \) term are the candidates to be the anomalies and the Wess-Zumino term for the case of an anomalous gauge theory. Note also that their expressions as well as the expression (3.10) for the S-matrix coincide with the ones obtained in [6] in the framework of the field-antifield formalism.

4 Anomalous gauge theories

The analysis performed so far has been given in full generality. Let us now analyze in which conditions the theory can be considered to be anomalous or not. To this end consider expression (3.10) for the S-matrix and perform the integration over the elements of the quasigroup. Once this is done, we have in general

\[
\exp\{i\tilde{M}_1(\phi)\} = \int \mathcal{D}G_L(\theta, \phi) \exp\{iM_1(\phi, \theta)\}.
\]

After that, with the expression for the \( \tilde{M}_1 \) term at hand, we face two possibilities:

- a) \( \tilde{M}_1(\phi) \) is a local functional. In this case \( \exp\{i\tilde{M}_1(\phi)\} \) can be considered as part of the measure of the fields of the theory and absorbed in it through a suitable redefinition of them. This is the situation which corresponds to a non-anomalous gauge theory.

Within this situation, it is of interest to consider the particular case when \( M_1 \) is zero and the gauge group is a Lie group. Under such conditions the volume of the gauge group, \( \int \mathcal{D}G_L(\theta) = V_{\text{gauge}} \), can be factorized from the functional integration, yielding as a physically relevant quantity

\[
Z_\chi = \int \mathcal{D}\phi \mathcal{D}\bar{C} \mathcal{D}C \mathcal{D}B \exp \left\{ \frac{i}{\hbar} S_\chi(\phi; C, \bar{C}, B) \right\},
\]
obtaining in this way the standard Faddeev-Popov result for non-anomalous gauge groups.

• b) $\tilde{M}_1(\phi)$ is a non-local functional. This is the way in which anomalous gauge theories manifest themselves in this formulation. In this case in order to use standard techniques of quantum field theory is better to take expression (3.10) as a starting point. $\theta^\alpha$ are then interpreted as the additional fields which appear in anomalous gauge theories due to the loss of the gauge symmetry at the quantum level and the $M_1$ term as the corresponding Wess-Zumino action.

In the case of an anomalous gauge theory, the invariance property of the left measure (2.13) for the elements of the gauge quasigroup suggests taking (2.14) as the gauge transformation for the extra fields $\theta^\alpha$. It is also of interest to consider the infinitesimal form of this transformation which reads

$$\delta \theta^\alpha = -\bar{\mu}_\beta^\alpha(\theta, \phi)\varepsilon^\beta.$$  (4.1)

To complete the construction, it is instructive to verify that with the above choice for the gauge transformations of the additional fields $\theta^\alpha$ the (infinitesimal) gauge variation of the Wess-Zumino action (3.12) reproduces the anomalies $A_\alpha$. Indeed, using the infinitesimal transformations (3.1) and (4.1), it is

$$\delta M_1(\phi, \theta) = -i \int_0^\theta \delta[\lambda^\alpha_\beta(\theta')\lambda^\beta_\alpha(\theta', \phi)]d\theta'^\alpha + i A_\sigma(F(\phi, \theta))\lambda^\alpha_\beta \bar{\mu}_\beta^\alpha(\theta, \phi)\varepsilon^\gamma,$$  (4.2)

where the second term comes from the variation of the upper integration limit. On the other hand, the gauge variation of the integrand yields a total derivative

$$\delta[\lambda^\alpha_\beta(\theta')\lambda^\beta_\alpha(\theta', \phi)] = \frac{\partial}{\partial \theta'^\alpha} \left[ A_\sigma(F(\phi, \theta'))\lambda^\sigma_\beta \bar{\mu}_\beta^\alpha(\theta', \phi)\varepsilon^\gamma \right],$$  (4.3)

where in evaluating this expression use has been made of the consistency conditions for $A_\alpha$ (3.11) and the commutation relation for $\lambda^\sigma_\beta$ (2.10). Finally, the value of the boundary term (1.3) in the upper limit of the integral exactly cancels the second term in (4.2), obtaining in the end the expected result

$$\delta M_1(\phi, \theta) = i A_\alpha(\phi)\varepsilon^\alpha.$$  

In summary, we have described a generalization of the Faddeev-Popov method to treat either anomalous and non-anomalous gauge theories in an unified way. As a byproduct, explicit expressions for the Wess-Zumino action in terms of the anomalies of the theory $A_\alpha$ as well as for the gauge transformations of the extra fields have been obtained, in complete agreement with the results presented in [6].
Finally, we would like to comment about the relation of our results with the ones presented in [10], where a derivation of the Faddeev-Popov formula for a generic non-linear sigma model was performed. Although both results are equivalent in the non-anomalous case, the procedure followed in this paper is different from ours. Indeed, while we start from the naive S-matrix (3.4), in [10] an S-matrix with a reparametrization invariant measure, including an extra factor, is considered. On the other hand, a major difference comes from the fact that the representation of the unity considered there is constructed in terms of a path integral along the gauge orbits, without any reference to the gauge group. Clearly a further study is needed to elucidate the relationship between both procedures.

5 Example: The chiral Schwinger model

To conclude, let us apply the previous results to a typical example of an anomalous gauge theory: the chiral Schwinger model. The classical action for this system

$$S_0(A_\mu; \psi, \bar{\psi}) = \int d^2x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i \bar{\psi} D^{(1-\gamma_5)/2} \psi \right],$$

is invariant under the infinitesimal gauge transformations

$$\delta A_\mu = \partial_\mu \epsilon, \quad \delta \psi = i \psi \epsilon, \quad \delta \bar{\psi} = -i \bar{\psi} \epsilon,$$

where \(\epsilon\) is the infinitesimal parameter of the abelian gauge group.

After introducing a consistent regularization scheme (e.g. Pauli-Villars regularization, . . . ), we obtain for the consistent anomaly \(\mathcal{A}\)

$$\mathcal{A}(A_\mu) = \frac{i}{4\pi} [(1-a) \partial_\mu A^\mu - \epsilon^{\mu\nu} \partial_\mu A_\nu], \quad (5.1)$$

where \(a\) is an arbitrary regularization parameter [11].

Let us now concentrate on the \(M_1\) term. As a path of integration in the general expression (3.12) we can take

$$\theta' = \theta \cdot t, \quad t \in [0, 1],$$

from which we obtain, taking into account that for the abelian gauge group in consideration \(\lambda_3^a = 1\),

$$M_1(A_\mu, \theta) = \frac{1}{4\pi} \int d^2x \int_0^1 dt \left[ (1-a) \partial_\mu A^\mu - \epsilon^{\mu\nu} \partial_\mu A_\nu \right] \theta,$$

where \(A'_\mu = A'_\mu(t)\) is the gauge transformed of \(A_\mu\) with parameter \(\theta \cdot t\), i.e., \(A'_\mu(t) = (A_\mu + \partial_\mu \theta \cdot t)\). The substitution of this expression in (5.2) and the integration over the parameter \(t\) yield (modulo total derivatives)

$$M_1(A_\mu, \theta) = \frac{1}{4\pi} \int d^2x \left\{ \frac{(a-1)}{2} \partial_\mu \theta \partial^\mu \theta - \theta \left[ (a-1) \partial_\mu A^\mu + \epsilon^{\mu\nu} \partial_\mu A_\nu \right] \right\},$$
which is exactly the Wess-Zumino action for this model.

It is easy to verify that the gauge variation of $M_1$ reproduces the anomaly $A$ \[ (5.1) \]

\[ \delta M_1(A_\mu, \theta) = \int d^2x \left[ iA \cdot \varepsilon \right], \]

with the gauge transformation for the extra field $\theta$ given by

\[ \delta \theta = -\varepsilon. \]

Finally, the functional integration

\[ \int \mathcal{D} \theta \exp \{ iM_1(A_\mu, \theta) \} \sim \exp \{ i\tilde{M}_1(A_\mu) \}, \]

yields the effective action

\[ \tilde{M}_1(A_\mu) = \frac{1}{8\pi} \int d^2x \left\{ A_\mu \left[ ag^{\mu\nu} - (g^{\mu\alpha} + \varepsilon^{\mu\alpha}) \left( \frac{\partial\alpha}{\Box} \right) (g^{\nu\beta} + \varepsilon^{\nu\beta}) \right] A_\nu \right\}, \]

which is non-local as corresponds to an anomalous gauge theory.

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