An $O^*(2.619^k)$ algorithm for 4-path vertex cover

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Abstract

In the 4-path vertex cover problem, the input is an undirected graph $G$ and an integer $k$. The goal is to decide whether there is a set of vertices $S$ of size at most $k$ such that every path with 4 vertices in $G$ contains at least one vertex of $S$. In this paper we give a parameterized algorithm for 4-path vertex cover whose time complexity is $O^*(2.619^k)$.

Keywords  graph algorithms, parameterized complexity.

1 Introduction

For an undirected graph $G$, an $l$-path is a path in $G$ with $l$ vertices. A $l$-path vertex cover is a set of vertices $S$ such that every $l$-path in $G$ contains at least one vertex of $S$. In the $l$-path vertex cover problem, the input is an undirected graph $G$ and an integer $k$. The goal is to decide whether there is an $l$-path vertex cover of $G$ with size at most $k$. The problem for $l = 2$ is the famous vertex cover problem. The problem is NP-hard for every constant $l \geq 2$.

For every fixed $l$, the $l$-path vertex cover problem has a trivial parameterized algorithm with running time $O^*(l^k)$. It is possible to obtain better algorithm for specific values of $l$. The first non-trivial parameterized algorithm for 3-path vertex cover was given by Tu [6], which gave an $O^*(2^k)$-time algorithm. Faster algorithms for 3-path vertex cover were given in [2, 4, 5, 8, 9]. The faster algorithm for 3-path vertex cover has $O^*(1.713^k)$ running time [5]. For the 4-path vertex cover problem, Tu et al. gave an $O^*(3^k)$-time algorithm [7]. Červený gave an $O^*(4^k)$-time algorithm for 5-path vertex cover [1].

In this paper we give an algorithm for 4-path vertex cover whose time complexity is $O^*(2.619^k)$.

2 Preliminaries

For a graph $G = (V, E)$ and a vertex $v \in V$, $N(v)$ is the set of vertices that are adjacent to $v$ and $\text{deg}(v) = |N(v)|$. For set of vertices $S$, $G[S]$ is the subgraph of $G$ induced by $S$ (namely, $G[S] = (S, E \cap (S \times S))$). We also define $G - S = G[V \setminus S]$.

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In order to give an algorithm for 4-path vertex cover, we use the iterative compression method (cf. [3]). Consider the following problem called disjoint 4-path vertex cover. The input is an undirected graph \( G = (V, E) \), an integer \( k \), and a 4-path vertex cover \( V_1 \) of \( G \). The goal is to decide whether there is a 4-path vertex cover \( S \) such that \( S \cap V_1 = \emptyset \). If there is an \( O^*(c^k) \) algorithm for the disjoint 4-path vertex cover problem then there is an \( O^*((c+1)^k) \) algorithm for 4-path vertex cover. Therefore, in the rest of the paper we will describe an algorithm for disjoint 4-path vertex cover with running time \( O^*(1.619^k) \). We will assume that for an instance \( G, k, V_1 \) of the disjoint 4-path vertex cover problem, \( G[V_1] \) does not contain a 4-path since in this case the instance is trivially a no instance.

Let \( V_2 = V \setminus V_1 \). Denote \( N_i(v) = N(v) \cap V_i \) and \( \deg_i(v) = |N_i(v)| \). A vertex \( x \in V_1 \) is called a connection vertex if \( \deg_1(x) = 0 \) and \( \deg_2(x) \geq 2 \). A vertex \( x \in V_1 \) is called a thorn if \( \deg_1(x) = 0 \) and \( \deg_2(x) = 1 \). A connected component of \( G[V_i] \) is called a \( V_i \)-component. We say that a \( V_2 \)-component \( C \) is contained in a connection vertex \( x \) if \( C \subseteq N(x) \). We say that a connection vertex \( x \) splits a \( V_2 \)-component \( C \) if \( C \cap N(x) \neq \emptyset \) and \( C \not\subseteq N(x) \).

A set of vertices \( I \) is an independent set if there is no edge between two vertices of \( I \). Note that a set \( I \) of size 1 is an independent set. A graph \( G = (V, E) \) is called a star if \( |V| \geq 3 \) and there is a vertex \( v \in V \) such that \( v \) is adjacent to all the other vertices in the graphs and \( V \setminus \{v\} \) is an independent set. The vertex \( v \) is called the center of the star, and the vertices in \( V \setminus \{v\} \) are called leaves.

3 The algorithm

In this section we describe a branching algorithm for solving the disjoint 4-path vertex cover problem. We will describe an algorithm \( \text{Alg}(G, V_1, k) \) that given a graph \( G \), a set \( V_1 \) that is a 4-path vertex cover of \( G \), and an integer \( k \), the algorithm return a 4-path vertex cover \( S \) of \( G \) such that \( S \cap V_1 = \emptyset \) and \( |S| \leq k \). If no such set exists, the algorithm returns NIL.

Let \( S_1, \ldots, S_t \) be subsets of \( V_2 \). We say that the algorithm recurses on \( S_1, \ldots, S_t \) if for each \( S_i \), the algorithm tries to find a solution \( S \) that contains \( S_i \). More precisely, the algorithm perform the following lines.

1. For \( i = 1, \ldots, t \):
   
   (a) \( S \leftarrow \text{Alg}(G - S_i, V_1, k - |S_i|) \).
   
   (b) If \( S \neq \text{NIL} \) return \( S \cup S_i \).

2. Return NIL.

We now describe the reduction and branching rules of the algorithm. The algorithm applies the first applicable rule from the following rules. For most of the branching rules below, the branching vector is \((1, t)\) for \( t \geq 2 \) and therefore the branching number is at most 1.619.
(1) If $k < 0$ or $k = 0$ and $G$ contains a 4-path return NIL.

(2) If $G$ does not contain a 4-path return $\emptyset$.

(3) If $C$ is a connected component of $G$ that does not contain a 4-path return $\text{Alg}(G - C, V_1 \setminus C, k)$.

(4) If $C$ is a connected component of $G$ such that $|C \cap V_2| \leq 2$, find a minimum $V_1$-disjoint cover $S_1$ of $G[C]$ (by enumerating all subsets of $C \cap V_2$). Recurse on $S_1$.

(5) If there is a vertex $v \in V_2$ such that all the vertices in $N_1(v)$ are thorns (note that this condition holds if $N_1(v) = \emptyset$) and either $\deg_2(v) = 1$ or the $V_2$-component of $v$ is a triangle, choose such a vertex $v$ for which $\deg_1(v)$ is minimal. Return $\text{Alg}(G, V_1 \cup \{v\}, k)$. See Figure 1.

Lemma 1. Rule (5) is correct.

Proof. To prove the lemma we need to show that there is a minimum $V_1$-disjoint cover of $G$ that does not contain $v$. Let $S$ be a minimum $V_1$-disjoint cover of $G$ and suppose that $v \in S$. If $\deg_2(v) = 1$, let $u$ be the single vertex in $N_2(v)$. Then, $S' = (S \setminus \{v\}) \cup \{u\}$ is also a minimum 4-path vertex cover of $G$ (Suppose conversely that $G - S'$ contains a 4-path. This path must contain $v$. However, the connected component of $v$ in $G - S'$ consists of $v$ and its adjacent thorns, and thus this component does not contain a 4-path, a contradiction).

Next consider the case in which the $V_2$-component $C$ of $v$ is a triangle. Denote by $v, v_1, v_2$ the vertices of $S$, where $v_1 \notin S$ if $|C \cap S| \leq 2$. There is a vertex $v_1 \in C$ such that $v_1 \notin S$ (otherwise, $S \setminus \{v\}$ is a $V_1$-disjoint cover of $G$, contradicting the assumption that $S$ is a minimum $V_1$-disjoint cover of $G$). Let $v_2$ be the single vertex in $C \setminus \{v, v_1\}$. Then, $S' = (S \setminus \{v\}) \cup \{v_1\}$ is also a minimum 4-path vertex cover of $G$: Conversely, if $G - S'$ contains a 4-path, it must be of the form $u_1, u_2, v_2, v$ or $u_1, v_2, v, x$, where $x \in N_1(v)$. In the former case $u_1, u_2, v_2, v_1$ is a 4-path in $G - S$, and in the latter case $u_1, v_2, v_1, x'$ is a 4-path in $G - S$, where $x' \in N_1(v_1)$ ($x'$ exists since by the choice of $v$, $\deg_1(v_1) \geq \deg_1(v) \geq 1$). In both cases we reach a contradiction. □

(6) If there is a 4-path $P$ in $G$ such that $|P \cap V_2| = 1$, recurse on $\{v\}$, where $v$ is the single vertex in $P \cap V_2$. See Figure 2.
Figure 3: An example for Rule (7).

Figure 4: An example for Rule (8).

Note that if Rule (6) is not applicable, the \( V_1 \)-components have sizes 1 or 2.

(7) If there is a path \( P = x_1, x_2, x_3 \) in \( G \) such that \(|P \cap V_2| = 1\) and \(|(N_2(x_1) \cup N_2(x_3)) \setminus P| \geq 2\), recurse on \( \{v\} \) and \( (N_2(x_1) \cup N_2(x_3)) \setminus P \), where \( v \) is the single vertex in \( P \cap V_2 \). See Figure 3.

Lemma 2. Rule (7) is correct.

Proof. If \( S \) is a minimum \( V_1 \)-disjoint cover of \( G \) such that \( v \notin S \), then for every \( u \in N_2(x_1) \setminus P, u \in S \) (since \( S \) needs to cover the path \( u, x_1, x_2, x_3 \)). Similarly, \( N_2(x_3) \setminus P \subseteq S \).

Observation 3. If Rules (1)–(7) cannot be applied and \( x \) is a connection vertex such that there is a vertex in \( N(x) \) that is adjacent to a thorn, then \( \deg_2(x) = 2 \).

For the correctness of Observation 3, note that if \( \deg_2(x) \geq 3 \) then Rule (7) can be applied on the path \( u, v, x, y \) where \( v \) is a vertex in \( N(x) \) that is adjacent to a thorn \( x' \).

(8) If there is a \( V_1 \)-component \( \{x, y\} \) of size 2, let \( v \in N_2(x) \cup N_2(y) \) such that \( \deg_2(v) = 1 \), and let \( u \) be the single vertex in \( N_2(v) \). Recurse on \( \{u\} \). See Figure 4.

Lemma 4. Rule (8) is correct.

Proof. We first show that the vertex \( v \) exists. We claim that there is a vertex \( v \in N_2(x) \cup N_2(y) \) such that \( \deg_2(v) \geq 1 \). Suppose conversely that \( \deg_2(u) = 0 \) for every \( u \in N_2(x) \cup N_2(y) \). Since Rule (6) cannot be applied, \( N_1(u) \subseteq \{x, y\} \) for every \( u \in N_2(x) \cup N_2(y) \). Since Rule (4) cannot be applied, \(|N_2(x) \cup N_2(y)| \geq 3\). Since Rule (5) cannot be applied, the connected component of \( x \) in \( G \) contains a 4-path. Therefore, \( \deg_2(x) \geq 1 \) and \( \deg_2(y) \geq 1 \). This implies that, without loss of generality, there is a vertex \( u \in N_2(x) \) such that \(|N_2(y) \setminus \{u\}| \geq 2\). Therefore, Rule (7) can be applied on the path \( u, x, y, u \), a contradiction. Thus, there is a vertex \( v \in N_2(x) \cup N_2(y) \) such that \( \deg_2(v) \geq 1 \). Assume without loss of generality that \( v \in N_2(x) \). Since Rule (7) cannot be applied, \( \deg_2(v) = 1 \) (otherwise Rule (7) can be applied on the path \( v, x, y \)). From Rule (6) we have that \( N_1(v) \subseteq \{x, y\} \).

Let \( S \) be a minimum \( V_1 \)-disjoint cover of \( G \) and suppose that \( u \notin S \). Due to the path \( u, v, x, y \) we have that \( v \in S \). Define \( S' = (S \setminus \{v\}) \cup \{u\} \). We will show that
$S'$ is a minimum $V_1$-disjoint cover of $G$. Since $|S'| = |S|$, it suffices to show that $S'$ is a $V_1$-disjoint cover of $G$.

We consider two cases. In the first case assume that $\deg_2(y) = 0$. Therefore, $N(v) = \{x, u\}$. Suppose conversely that $G - S'$ contains a 4-path. This path must contain $v$, and since $N(v) = \{x, u\}$ and $u \in S'$, it follows that the path is of the form $a, b, x, v$. But this implies that $a, b, x, y$ is a 4-path in $G - S$, a contradiction. Therefore, $S'$ is a $V_1$-disjoint cover of $G$.

In the second case $\deg_2(y) \geq 1$. We claim that in this case $N_2(x) = N_2(y) = \{v\}$. In other words, we have $|N_2(x) \cup N_2(y)| = 1$. As shown above, $|N_2(x) \cup N_2(y)|$ cannot be 3 or more. Therefore, $|N_2(x) \cup N_2(y)| = 2$. Thus, there are $v_1 \in N_2(x)$ and $v_2 \in N_2(y)$ such that $v_1 \neq v_2$. Since $v = v_i$ for some $i$, we have that $u \in N_2(v_i)$. Therefore, Rule (7) can be applied (if for example $v = v_1$ then Rule (7) can be applied on the path $v, x, y$ and $\{u, v_2\} \subseteq (N_2(v) \cup N_2(y)) \setminus \{v\}$, a contradiction. Thus, $N_2(x) = N_2(y) = \{v\}$.

Suppose conversely that $G - S'$ contains a 4-path. This path must contain $v$. However, the connected component of $v$ in $G - S'$ is $v, x, y$ and does not contain a 4-path, a contradiction. Therefore, $S'$ is a $V_1$-disjoint cover of $G$.

\begin{observation}
If Rules (1)–(8) cannot be applied, $V_1$ is an independent set.
\end{observation}

\begin{lemma}
Rule (9) is correct.
\end{lemma}

\begin{proof}
Since Rule (7) cannot be applied, there is a vertex $u \in V_2$ such that $N_2(x_1) = N_2(x_2) = \{v, u\}$ (otherwise, $|N_2(x_1) \cup N_2(x_2)) \setminus \{v\}| \geq 2$, so Rule (7) can be applied on the path $x_1, v, x_2$). We now show that a set of vertices $S$ is a $V_1$-disjoint cover of $G$ if and only if $S$ is a $V_1$-disjoint cover of $G'$. Since $G'$ is a subgraph of $G$, if $S$ is a $V_1$-disjoint cover of $G$ then $S$ is also a $V_1$-disjoint cover of $G$. To show the second direction of the claim, let $S$ be a $V_1$-disjoint cover of $G$.

\begin{observation}
If Rules (1)–(9) cannot be applied, every vertex $v \in V_2$ is adjacent to at most one connection vertex.
\end{observation}
If \( G \) in component \( C \) at least 3 is either a star or a triangle. Therefore, \( C \) show that \( N \) vertex \( u \) obtained from \( P \) is a minimum \( V \)  

\[
\begin{align*}
\text{(10)} & \quad \text{If } x \text{ is a connection vertex that splits a } V_2\text{-component } C \text{ such that there is a vertex } u \in C \cap N(x) \text{ for which } |N_2(u) \setminus N_2(x)| \geq 2, \text{ recurse on } \{u\} \text{ and } N_2(u) \setminus N_2(x). \\
\text{See Figure 6.}
\end{align*}
\]

**Lemma 8.** Rule (10) is correct.

**Proof.** Let \( S \) be a \( V_1\)-disjoint cover of \( G \). If \( u \in S \) we are done. Otherwise, we will show that \( N_2(u) \setminus N_2(x) \subseteq S \). Fix \( v \in N_2(u) \setminus N_2(x) \). Every \( V_2\)-component of size at least 3 is either a star or a triangle. Therefore, \( C \) is either a star whose center is \( u \) or a triangle. Since Rule (5) cannot be applied, \( \deg_1(v) \geq 1 \). Choose \( y \in N_1(v) \). By definition, \( v \notin N_2(x) \), hence \( x \neq y \). The set \( S \) must contain a vertex of the path \( x, u, v, y \). Due to the assumption that \( u \notin S \), we conclude that \( v \in S \). Since this is true for every \( v \in N_2(u) \setminus N_2(x) \), we obtain that \( N_2(u) \setminus N_2(x) \subseteq S \).

If Rules (1)–(10) cannot be applied, and \( x \) is a connection vertex that splits a \( V_2\)-component \( C \), there is a unique vertex \( v \in C \) such that \( v \notin N(x) \) and \( N_2(v) \cap N(x) \neq \emptyset \). This vertex will be called the boundary vertex of \( C \) with respect to \( x \).

\[
\begin{align*}
\text{(11)} & \quad \text{If } C \text{ is a } V_2\text{-component of size at least 3 that is contained in a connection vertex } x, \text{ choose } v_1, v_2, v_3 \in C \text{ such that } v_1, v_2, v_3 \text{ is a path. Recurse on } \{v_2\}. \text{ See Figure 7.}
\end{align*}
\]

**Lemma 9.** Rule (11) is correct.

**Proof.** From Observation 3 and Observation 7 \( N_1(v_i) = \{x\} \) for all \( i \). The \( V_2\)-component \( C \) is either a star whose center is \( v_2 \) or a triangle. It follows that \( N(v_1) \setminus \{v_2\} \subseteq \{x, v_3\} \subseteq N(v_2) \) and \( N(v_3) \setminus \{v_2\} \subseteq N(v_2) \). Let \( S \) be a minimum \( V_1\)-disjoint cover of \( G \) and suppose that \( v_2 \notin S \). Due to the path \( x, v_1, v_2, v_3, S \) contains either \( v_1 \) or \( v_3 \). Without loss of generality assume that \( v_1 \in S \). Then, \( S' = (S \setminus \{v_1\}) \cup \{v_2\} \) is a minimum \( V_1\)-disjoint cover of \( G \). Suppose conversely that there is a 4-path \( P \) in \( G - S' \). The path \( P \) contains \( v_1 \) and do not contain \( v_2 \). Let \( P' \) be the path obtained from \( P \) by replacing \( v_1 \) with \( v_2 \). \( P' \) is a valid path in \( G \) due to the fact that \( N(v_1) \setminus \{v_2\} \subseteq N(v_2) \). The path \( P' \) is a path in \( G - S \), contradicting the assumption that \( S \) is a minimum \( V_1\)-disjoint cover of \( G \). Therefore, \( S' \) is a minimum \( V_1\)-disjoint cover of \( G \).
Rule (12) If $C$ is a $V_2$-component which is a triangle, let $x$ be a connection vertex that splits $C$, and let $v$ be the boundary vertex of $C$ with respect to $x$. Recurse on $\{v\}$ and $C \setminus \{v\}$. See Figure 8.

Lemma 10. Rule (12) is correct.

Proof. We first show that $x$ exists. Choose $u \in C$. Since Rule (5) cannot be applied, $u$ is adjacent to a connection vertex $x$. The vertex $x$ splits $C$ (otherwise, $C$ is contained in $x$, contradicting the assumption that Rule (11) cannot be applied).

Since Rule (10) cannot be applied, $|C \cap N(x)| = 2$. Denote $C \cap N(x) = \{u, u'\}$.

Since Rule (5) cannot be applied, $v$ is adjacent to a boundary vertex $x'$. By definition, $v \notin N(x)$ and therefore $x \neq x'$. Let $S$ be a minimum $V_1$-disjoint cover of $G$ and suppose that $v \notin S$. Due to the path $x, u, v, x'$, $S$ must contain $u$. Similarly, $u' \in S$. Therefore, $C \setminus \{v\} \subseteq S$.

Observation 11. If Rules (1)–(12) cannot be applied and $x$ is a connection vertex that splits a $V_2$-component $C$ such that $C \cap N(x)$ is not an independent set, then there is a unique vertex in $C \cap N(x)$ that is adjacent to the boundary vertex of $C$ with respect to $x$.

Rule (13) If $x$ is a connection vertex that splits exactly one $V_2$-component $C$ and at least one of the vertices in $N(x)$ is adjacent to a thorn, recurse on $\{u\}$, where $u \in N(x)$ is a vertex that is adjacent to the boundary vertex of $C$ with respect to $x$. See Figure 9.

Lemma 12. Rule (13) is correct.

Proof. By Observation 3, $\deg(x) = 2$. Denote $N(x) = \{u, u'\}$. From Observation 7 $u$ and $u'$ are not adjacent to connection vertices other than $x$. Let $S$ be a minimum $V_1$-disjoint cover of $G$ and suppose that $u \notin S$. Due to the assumption that at least one of $u, u'$ is adjacent to a thorn, $u' \in S$. We claim that the set $S' = (S \setminus \{u'\}) \cup \{u\}$ is a minimum $V_1$-disjoint cover of $G$. Assume conversely that $G – S'$ contains a 4-path. This path must contain $u'$, so it is of the form $a, b, v, u'$ of $a, v, u', y$, where $y$ is either $x$ or a thorn adjacent to $u'$. In the former case $a, b, v, u$ is a 4-path in $G – S$, and in the latter case $a, v, u, x$ is a 4-path in $G – S$. This is a contradiction to the assumption that $S$ is a $V_1$-disjoint cover. Therefore, $S'$ is a minimum $V_1$-disjoint cover of $G$.

Figure 8: An example for Rule [12].

Figure 9: An example for Rule [13].
For the following rules, suppose that \( x \) is a connection vertex. We denote by \( S_b \) the set containing the boundary vertex of \( C \) with respect to \( x \) for every \( V_2 \)-component \( C \) that \( x \) splits and \( C \cap N(x) \) is an independent set. Additionally, \( S_n \) is a set containing the unique vertex in \( C \cap N(x) \) that is adjacent to the boundary vertex of \( C \) with respect to \( x \) for every \( V_2 \)-component \( C \) that are split by \( x \) and \( C \cap N(x) \) is not an independent set (see Observation 11).

\[(14)\] If \( x \) is a connection vertex such that \( |C \cap N(x)| = 1 \) for every \( V_2 \)-component that \( x \) splits, \( x \) contains exactly one \( V_2 \)-component \( C' \), and \( |C'| = 2 \), recurse on \( N(x) \setminus C' \) and \( S_b \cup \{u\} \), where \( u \) is a vertex in \( C' \). See Figure 10.

Lemma 13. Rule (14) is correct.

Proof. Let \( S \) be a minimum \( V_1 \)-disjoint cover of \( G \). If \( S \cap C' = \emptyset \) then \( N(x) \setminus C' \subseteq S \) (for every \( v \in N(x) \setminus C' \) there is a 4-path \( v, x, u, u' \) in \( G \), where \( u' \) is the single vertex in \( C' \), and therefore \( v \in S \)) and we are done.

Now assume that \( S \cap C' \neq \emptyset \). Let \( S_0 = N(x) \cup S_b \) and \( S' = (S \setminus S_0) \cup S_b \cup \{u\} \). The set \( S' \) is a \( V_1 \)-disjoint cover of \( G \): Assume conversely that \( G - S' \) contains a 4-path. This path must contain a vertex \( v \in N(x) \). However, the connected component of \( v \) in \( G - S' \) is a star (by Observation 3 and Observation 7), a contradiction. We will show that \( |S \cap S_0| \geq |S_b \cup \{u\}| \) and therefore \( S' \) is a minimum \( V_1 \)-disjoint cover of \( G \).

Let \( t \) be the number of \( V_2 \)-component that \( x \) splits. Since Rule (4) cannot be applied, \( t \geq 1 \). By definition, \( |S_b \cup \{u\}| = t + 1 \). If \( |S \cap C'| = 1 \) then for every \( V_2 \)-component \( C \) that \( x \) splits, \( S \) contains either the single vertex in \( C \cap N(x) \) or the boundary vertex of \( C \) with respect to \( x \). Therefore, \( |S \cap S_0| \geq t + 1 \). Otherwise (if \( |S \cap C'| = 2 \)), for every \( V_2 \)-component \( C \) that is split by \( x \) except at most one, \( S \) contains either the single vertex in \( C \cap N(x) \) or the boundary vertex of \( C \) with respect to \( x \). Therefore, \( |S \cap S_0| \geq (t - 1) + 2 = t + 1 \).

The branching vector of Rule (14) is \((t, t + 1)\), where \( t \geq 1 \). Therefore, the branching number is at most 1.619.

\[(15)\] If \( x \) is a connection vertex that contains at least one \( V_2 \)-component, recurse on \( S_c \cup S_b \cup S_n \), where \( S_c \) is a set containing one vertex from each \( V_2 \)-component of size 2 that is contained in \( x \). See Figure 11.

Lemma 14. Rule (15) is correct.
Rule (16) is correct.

Lemma 15.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure12.png}
\caption{An example for Rule (16).}
\end{figure}

Proof. Since Rules (4), (7) and (13) cannot be applied, there is no vertex in $N(x)$ that is adjacent to a thorn. From Observation 7 we obtain that $N_1(u) = \{x\}$ for every $u \in N(x)$.

Let $S$ be a minimum $V_1$-disjoint cover of $G$. Let $S_0 = N(x) \cup S_b$. The set $S' = (S \setminus S_0) \cup (S_c \cup S_b \cup S_n)$ is a $V_1$-disjoint cover of $G$ (if $G - S'$ contains a 4-path then this path must contain a vertex $u \in N(x)$). However, the connected component of $u$ in $G - S'$ is a star, a contradiction. We will show that $|S \cap S_0| \geq |S_c \cup S_b \cup S_n|$ and therefore $S'$ is a minimum $V_1$-disjoint cover of $G$.

Let $s_1$ (resp., $s_2$) be the number of $V_2$-components of size 1 (resp., size 2) that are contained in $x$. Let $t_1$ (resp., $t_2$) be the number of $V_2$-components $C$ that $x$ splits and $|C \cap N(x)| = 1$ (resp., $|C \cap N(x)| \geq 2$). By definition, $|S_c \cup S_b \cup S_n| = s_2 + t_1 + t_2$.

Suppose that $s_2 = 0$. Then, from the assumption that $x$ contains at least one $V_2$-component, $s_1 \geq 1$. If for every $V_2$-component $C$ that $x$ splits, $S$ contains either a vertex from $C \cap N(x)$ or the boundary vertex of $C$ with respect to $x$, then $|S \cap S_0| \geq t_1 + t_2 = s_2 + t_1 + t_2$. Otherwise, let $C$ be a $V_2$-component that $x$ splits and $S$ does not contain a vertex from $C \cap N(x)$ and does not contain the boundary vertex of $C$ with respect to $x$. It follows that $N(x) \setminus C \subseteq S$. Therefore,

$$|S \cap S_0| \geq |N(x) \cap C| \geq s_1 + (t_1 + t_2 - 1) \geq t_1 + t_2 = s_2 + t_1 + t_2.$$ \[16\]

Now suppose that $s_2 \geq 1$. Since Rule (14) cannot be applied, either $s_1 \geq 1$ or $t_2 \geq 1$. If there is a $V_2$-component $C$ of size 2 that is contained in $x$ and $C \cap S = \emptyset$, then $N(x) \setminus C \subseteq S$ and thus $|S \cap S_0| \geq s_1 + 2(s_2 - 1) + t_1 + 2t_2 \geq s_2 + t_1 + t_2$. Otherwise, $S$ contains at least one vertex from each $V_2$-component of size 2 that is contained in $x$. If $S$ contains at least one vertex from each $V_2$-component of size 2 that is contained in $x$, and there is at least one such component $C$ such that $|C \cap S| = 1$, then for every $V_2$-component $C'$ that $x$ splits, $S$ contains either a vertex from $C' \cap N(x)$ or the boundary vertex of $C'$ with respect to $x$. Therefore, $|S \cap S_0| \geq s_2 + t_1 + t_2$. Otherwise (if $S$ contains the vertices of every $V_2$-component of size 2 that is contained in $x$), for every $V_2$-component $C'$ that $x$ splits except at most one, $S$ contains either a vertex from $C' \cap N(x)$ or the boundary vertex of $C'$ with respect to $x$. Therefore, $|S \cap S_0| \geq 2s_2 + (t_1 + t_2 - 1) \geq s_2 + t_1 + t_2$.

If $x$ is a connection vertex that splits exactly one $V_2$-component $C$, recurse on $\{v\}$, where $v$ is defined as follows. If $N(x)$ is not an independent set and $\deg(x) \geq 3$, $v$ is the unique vertex in $C \cap N(x)$ that is adjacent to the boundary vertex of $C$ with respect to $x$. Otherwise, $v$ is the boundary vertex of $C$ with respect to $x$. See Figure 12.

Lemma 15. Rule (16) is correct.
Proof. By Observation 7 and since Rule (13) cannot be applied, $N_1(u) = \{x\}$ for every $u \in N(x)$. Additionally, $x$ does not contain $V_2$-components (due to Rule (15)). Let $S$ be a minimum $V_1$-disjoint cover of $G$ such that $v \notin S$. $S$ must contain at least one vertex from $N(x) \cup \{v\}$. Therefore, the set $S' = (S \setminus N(x)) \cup \{v\}$ is a minimum $V_1$-disjoint cover of $G$ (note that $G - S'$ is either a triangle or a star).

**Observation 16.** If Rules (1)–(16) cannot be applied, every connection vertex $x$ splits at least two $V_2$-components.

(17) If $x$ is a connection vertex that splits a $V_2$-component $C$ such that the boundary vertex $v$ of $C$ with respect to $x$ satisfies $\deg_1(v) \geq 1$ and at least one of the vertices in $N(x)$ is adjacent to a thorn, recurse on $\{u\}$ and $\{v\} \cup (N(x) \setminus \{u\})$ where $u$ is the single vertex in $C \cap N(x)$. See Figure 13.

**Lemma 17.** Rule (17) is correct.

**Proof.** By Observation 3, $\deg(x) = 2$. Using Observation 16 we obtain that $|C \cap N(x)| = 1$. Let $x'$ be a vertex in $N_1(v)$. By definition, $v$ is not adjacent to $x$ and therefore $x \neq x'$. Let $S$ be a minimum $V_1$-disjoint cover of $G$ such that $u \notin S$. Due to the path $x', v, u, x$ in $G$, $S$ must contain $v$. Since at least one of the two vertices in $N(x)$ is adjacent to a thorn, $S$ must contain the single vertex in $N(x) \setminus \{u\}$.

(18) If $x$ is a connection vertex that splits a $V_2$-component $C$ such that the boundary vertex $v$ of $C$ with respect to $x$ satisfies $\deg_1(v) \geq 1$, recurse on $\{u\}$ and $\{v\} \cup S_b \cup S_n$, where $u$ is a vertex in $C \cap N(x)$, See Figure 14.

**Lemma 18.** Rule (18) is correct.

**Proof.** Let $x'$ be a vertex in $N_1(v)$. Let $S$ be a minimum $V_1$-disjoint cover of $G$ and assume that $u \notin S$. Due to the path $x', v, u, x$, $S$ must contain $v$. Let $S_0 = N(x) \cup S_b$ and $S' = (S \setminus S_0) \cup (S_b \cup S_n)$. Using similar argument used in the proofs above, $S'$ is a $V_1$-disjoint cover of $G$ and $|S \cap S_0| \geq |S_b \cup S_n|$. Therefore, $S'$ is a minimum $V_1$-disjoint cover of $G$.

From Rule (5) and Rule (18) we obtain the following observation.
Observation 19. If Rules (1)–(18) cannot be applied and C is a V₂-component that
is split by a connection vertex x, then C is a star whose center v is the boundary
vertex of C with respect to x. Additionally, \( \deg_1(v) = 0 \).

(19) If \( x \) is a connection vertex that splits a V₂-component C such that \( |C \cap
N(x)| \geq 2 \), recurse on \( S_b \). See Figure 15.

Lemma 20. Rule (19) is correct.

Proof. Let \( S \) be a minimum \( V_1 \)-disjoint cover of \( G \). Define \( S_0 = N(x) \cup S_b \) and
\( S' = (S \setminus S_0) \cup S_b \). The connected component of \( x \) in \( G - S' \) is a star. Therefore, \( S' \)
is a \( V_1 \)-disjoint cover of \( G \). We will show that \( |S \cap S_0| \geq |S_b| \) and therefore \( S' \) is a
minimum \( V_1 \)-disjoint cover of \( G \).

Let \( t_1 \) (resp., \( t_2 \)) be the number of V₂-components \( C \) that are split by \( x \) and
\( |C \cap N(x)| = 1 \) (resp., \( |C \cap N(x)| \geq 2 \)). By definition, \( |S_b| = t_1 + t_2 \).

Suppose first that for every V₂-component \( C' \) such that \( x \) splits \( C' \) and \( |C' \cap
N(x)| = 1 \) we have that \( S \) contains at least one vertex from \( C' \cap S_0 \). For every
V₂-component \( C' \) such that \( x \) splits \( C' \) and \( |C' \cap N(x)| \geq 2 \), \( S \) must contain at least
one vertex from \( C' \cap S_0 \). Therefore, \( |S \cap S_0| \geq t_1 + t_2 \).

Now, suppose that there is a V₂-component \( C'' \) such that \( x \) splits \( C'' \), \( |C'' \cap N(x)| = 1 \),
and \( S \) does not contain a vertex from \( C'' \cap S_0 \). In this case we have \( N(x) \setminus C'' \subseteq S \).
Therefore, \( |S \cap S_0| \geq t_1 - 1 + 2t_2 \geq t_1 + t_2 \).  

(20) If \( x \) is a connection vertex that splits at least three V₂-components, recurse
on \( S_b \) and on \( S_i = (S \setminus (N(x) \cup S_b)) \cup (N(x) \setminus C_i) \) for every \( i \leq t \), where \( C_1, \ldots, C_t \)
are the V₂-components that \( x \) splits. See Figure 16.

Lemma 21. Rule (20) is correct.

Proof. Let \( S \) be a minimum \( V_1 \)-disjoint cover of \( G \). Let \( S_0 = N(x) \cup S_b \) and
\( S' = (S \setminus S_0) \cup S_b \). By Observation 3 the vertices of \( N(x) \) are not adjacent to
thorns, and it follows that \( S' \) is a \( V_1 \)-disjoint cover of \( G \). If \( S \) contains at least one
vertex from \( C_i \cap S_0 \) for all \( i \leq t \) then \( |S \cap S_0| \geq t = |S_b| \). It follows that \( S' \) is a
minimum \( V_1 \)-disjoint cover of \( G \).

Now suppose that there is an index \( i \) such that \( S \) does not contain a vertex from
\( C_i \cap S_0 \). Then, \( S \) must contain every vertex in \( C_i \setminus S_0 \). Additionally, \( S \) must contain
every vertex in \( N(x) \setminus C_i \). Therefore, \( S_i \subseteq S \).
The branching vector of Rule (20) is at least $(t, t, \ldots, t)$, where the value $t$ is repeated $t$ time. The worst case is when $t = 3$, and the branching number of $(3, 3, 3)$ is at most $1.588$.

(21) If $x$ is a connection vertex that splits the $V_2$-components $C$ and $C'$ such that $|C'| \geq 4$ and the vertices in $N(x)$ are not adjacent to thorns, recurse on $S_1 = \{v, v'\}$, $S_2 = (C \setminus \{u, v\}) \cup \{u'\}$, and $S_3 = \{u\} \cup (C' \setminus \{u', v'\})$, where $u$ (resp., $u'$) is the single vertex in $C \cap N(x)$ (resp., $C' \cap N(x)$), and $v$ (resp., $v'$) is the boundary vertex of $C$ (resp., $C'$) with respect to $x$. See Figure 17.

Lemma 22. Rule (21) is correct.

Proof. Let $S$ be a minimum $V_1$-disjoint cover of $G$. Due to the paths $x, u, v, v'$ for every $v' \in C \setminus \{u, v\}$, there are three possible cases: (1) $u \in S$, (2) $u \notin S$ and $v \in S$ (3) $u, v \notin S$ and $C \setminus \{u, v\} \subseteq S$.

If Case (3) occurs then $u' \in S$ (due to the path $v, u, x, u'$). Thus, $S_2 \subseteq S$. If Case (2) occurs then $S$ contains either $u'$ or $v'$. Therefore, $S' = (S \setminus \{u'\}) \cup \{v'\}$ is a minimum $V_1$-disjoint cover of $G$ and $S_1 \subseteq S'$.

Now suppose that Case (1) occurs. If $u' \in S$ or $v' \in S$, the set $S'' = (S \setminus \{u, u'\}) \cup \{v, v'\}$ is a minimum $V_1$-disjoint cover of $G$ and $S_1 \subseteq S''$. Otherwise ($u', v' \notin S$) we have $C' \setminus \{u', v'\} \subseteq S$. Therefore, $S_3 \subseteq S$.

The branching vector of Rule (21) is at least $(2, 2, 3)$, and the branching number is at most $1.619$.

(22) If $C$ is a $V_2$-component of size at least 4, recurse on $S_i = \{u_i\} \cup \{v_1, \ldots, v_s\} \setminus \{v_i\}$ for $i = 1, \ldots, s$, where $v_1, \ldots, v_s$ are the leaves of $C$, and $u_i$ is the unique vertex in $V_2$ such that is adjacent to the unique connection vertex that is adjacent to $v_i$. See Figure 18.

Lemma 23. Rule (22) is correct.

Proof. Due to Rule (5) and Rule (9), $v_i$ is adjacent to a unique connection vertex $x_i$. Since Rule (20) cannot be applied, there is a unique vertex $u_i \neq v_i$ that is adjacent to $x_i$. Therefore, the definition of Rule (22) is valid.
Let $S$ be a minimum $V_1$-disjoint cover of $G$. If there is an index $i$ such that $v_i \notin S$, then $u_i \in S$ (since at least one of the vertices $u_i$ and $v_i$ is adjacent to a thorn). Additionally, $(\{v_1, \ldots, v_s\} \setminus \{v_i\}) \subseteq S$ (for every $j \neq i$, $S$ must contain $v_j$ due to the path $x_i, v_i, v, v_j$, where $v$ is the center of $S$). Therefore, $S_i \subseteq S$. If $v_i \in S$ for all $i$, define $S' = (S \setminus \{v_1\}) \cup \{u_1\}$. The set $S'$ is a minimum $V_1$-disjoint cover of $G$ and $S_1 \subseteq S'$.

The branching vector of Rule (22) is $(s, s, \ldots, s)$, where the value $s$ is repeated $s$ times. Since $s \geq 3$, the branching number of this rule is at most 1.588.

(23) If none of the previous rules is applicable, create a set $S_1$ as follows. Go over all the connected components of $G$. Let $C$ be a connected component. $C$ of $G$ has the form $C = C_1 \cup \cdots \cup C_s \cup \{x_1, \ldots, x_s\} \cup T$ where $C_1, \ldots, C_s$ are $V_2$-components that are stars of size 3, $x_1, \ldots, x_s$ are connection vertices, $x_i$ splits $C_i$ and $C_{i+1}$ for $i = 1, \ldots, s - 1$, $x_s$ splits $C_1$ and $C_s$, and $T$ is a set of thorns that are adjacent to vertices in $N(x_1) \cup \cdots \cup N(x_s)$. Add to $S$ the set $\bigcup_{i=1}^{s}(C_i \cap N(x_i))$. Recurse on $S_1$. See Figure 19.

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