Two Gilbert-Varshamov Type Existential Bounds for Asymmetric Quantum Error-Correcting Codes

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Abstract In this note we report two versions of Gilbert-Varshamov type existential bounds for asymmetric quantum error-correcting codes.

Keywords asymmetric error · quantum error correction

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1 Introduction

Quantum error-correcting codes (QECC) are important for construction of quantum computers, as the fault-tolerant quantum computation is based on QECC [13]. There are two kinds of errors in quantum information, one is called a bit error and the other is called a phase error. Steane [16] first studied the asymmetry between probabilities of the bit and the phase errors, and he also considered QECC for asymmetric quantum errors, which are called asymmetric quantum error-correcting codes (AQECC). Research on AQECC has become very active recently, see [7, 9, 16] and the references therein.

On the other hand, in the study of error-correcting codes, it is important to know the optimal performance of codes. For classical error-correcting codes, the Gilbert-Varshamov (GV) bound [11] is a sufficient condition for existence of codes whose parameters satisfies the GV bound. By the GV bound, one can know that the optimal performance of classical codes is at least as good as the GV bound.

For QECC, Ekert and Macchiavello obtained a GV type existential bound for general QECCs. An important subclass of general QECCs is the stabilizer codes [2,3,8], as they enable efficient encoding and decoding. Calderbank et al. [2] obtained a GV type existential bound for the stabilizer QECCs. After that, Feng and Ma [6] and Jin and Xing [10] obtained improved versions of GV type bounds for the stabilizer QECCs.

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The Calderbank-Shor-Steane (CSS) QECCs \textsuperscript{[1][2][5]} are an important subclass of the stabilizer QECCs, as the CSS codes enable more efficient implementation of the fault-tolerant quantum computation than the stabilizer codes.

Those existential bounds \textsuperscript{[2][4][6][7][10]} did not consider the asymmetric quantum errors, while the asymmetry in quantum errors is important in practice \textsuperscript{[14]}. As far as the author know, nobody has reported existential bounds for the stabilizer or the CSS QECC for asymmetric quantum errors. In this note we report such ones. Our proof arguments are similar to ones in \textsuperscript{[2][4]}.  

2 A GV type existential bound for the CSS codes

An \([n, k, d_1, d_2]_q\) QECC encodes \(q\)-ary qudits into \(n\) \(q\)-ary qudits and detects up to \(d_1\) bit errors and up to \(d_2\) phase errors. It is known \textsuperscript{[15]} that a nested classical code \(C_2 \subset C_1 \subset \mathbb{F}_q^n\) with dimensions \(k_2\) and \(k_1\) can construct an \([n, \dim C_1 - \dim C_2]_q\) CSS code, where \(\mathbb{F}_q^n\) is a finite field with \(q\) elements. A quantum error can be expressed as a pair \((e_\cdot, e_e)\), where \(e_e \in \mathbb{F}_q^n\) corresponds to the bit error component of a quantum error and \(e_\cdot \in \mathbb{F}_q^n\) does to the phase error component.

Let \(\text{GL}_n(\mathbb{F}_q)\) be the group of \(n \times n\) invertible matrices over \(\mathbb{F}_q\). Let \(B_n = \{(C_1, C_2) \mid C_2 \subset C_1 \subset \mathbb{F}_q^n, \dim C_1 = k_1, \dim C_2 = k_2\}\). For a nonzero vector \(e \in \mathbb{F}_q^n\), let \(B_{n,1}(e)\) (resp. \(B_{n,2}(e)\)) be the set of nested code pairs that cannot detect \(e\) as a bit error (resp. a phase error), that is, \(B_{n,1}(e) = \{(C_1, C_2) \in B_n \mid e \in C_1 \setminus C_2\}\) (resp. \(B_{n,2}(e) = \{(C_1, C_2) \in B_n \mid e \in C_2 \setminus C_1^\perp\}\)), where \(C_1^\perp\) is the dual code of \(C_1\) with respect to the standard inner product.

**Lemma 1** For nonzero \(e\), we have

\[
\#B_{n,1}(e) = \frac{q^{k_1} - q^{k_2}}{q^n - 1}\#B_n,
\]

\[
\#B_{n,2}(e) = \frac{q^{k_2} - q^{k_1}}{q^n - 1}\#B_n.
\]

**Proof** As each pair \(C_2 \subset C_1\) has \(\#C_1 \setminus C_2 = q^{k_2} - q^{k_1}\) undetectable errors, we have

\[
\sum_{e \in \mathbb{F}_q^n} \#B_{n,1}(e) = q^{k_2} - q^{k_1}.
\]

For nonzero \(e_1, e_2 \in \mathbb{F}_q^n\), we claim \(\#B_{n,1}(e_1) = \#B_{n,1}(e_2)\). Assuming the claim, we have

\[
\sum_{e \in \mathbb{F}_q^n} \#B_{n,1}(e) = (q^n - 1)\#B_{n,1}(e).
\]

Combining these two equalities, we have

\[
\#B_{n,1}(e) = \frac{q^{k_1} - q^{k_2}}{q^n - 1}\#B_n.
\]

We finish the proof by proving the claim. Let \(e_1, e_2\) be nonzero vectors. We have

\[
\#B_{n,1}(e_1) = 2\#\{C_1, C_2 \in B_n \mid e_1 \in C_1 \setminus C_2\}
\]

\[
= 2\#\{(\tau C_1, \tau C_2) \mid \tau \in \text{GL}_n(\mathbb{F}_q), e_1 \in C_1 \setminus C_2\}
\]

\[
= 2\#\{(\tau C_1, \tau C_2) \mid \tau \in \text{GL}_n(\mathbb{F}_q), \tau' e_1 \in C_1 \setminus C_2\}
\]

\[
= \#B_{n,1}(\tau' e_1),
\]
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where \( \tau' \in \text{GL}_n(\mathbb{F}_q) \) such that \( \tau' e_i = e_2 \).

For phase errors, we can make a similar argument with \( C_2^+ \supset C_1^+ \). □

**Theorem 2** Let \( n, k_1, k_2, d_s, d_e, d_z \) be positive integers such that

\[
\frac{q^{k_2} - q^{k_1}}{q^n - 1} \sum_{i=1}^{d_s-1} \binom{n}{i} (q - 1)^i \left( \frac{q^{n-k_2} - q^{n-k_1}}{q^n - 1} \right) n \binom{n}{i} (q - 1)^i < 1, \tag{1}
\]

then an \( [[n, k_1 - k_2, d_s, d_e, d_z]]_q \) CSS QECC exists.

**Proof** Recall that each quantum error can be expressed by its bit error component \( e_s \in \mathbb{F}_q^n \) and its phase error component \( e_e \in \mathbb{F}_q^n \). The bit error component \( e_s \) cannot be detected by codes in \( B_{n,q}(e_s) \) and the phase error component \( e_e \) cannot be detected by codes in \( B_{n,q}(e_e) \). The detectabilities of the bit errors and the phase errors are independent of each other. Therefore, if Eq. (1) holds then there exists at least one (\( C_1, C_2 \) \( \in B_n \) that can detect all the bit errors with weight up to \( d_s - 1 \) and all the phase errors with weight up to \( d_z - 1 \), which implies it is an \( [[n, k_1 - k_2, d_s, d_e, d_z]]_q \) quantum code. □

Classical coding theorists often have interest in asymptotic versions of GV type existential bounds [11]. They are stated in terms of information rate and relative distance of classical error-correcting codes. In the classical error correction, information rate is the ratio of the number of information symbols to the code length, and relative distance is the ratio of the minimum distance to the code length.

We can also derive an asymptotic version of Theorem 2. For an \( [[n, k, d_s, d_e]]_q \) QECC, we may define the relative distance \( \delta_s \) for bit errors as \( d_s/n \), and the relative distance \( \delta_e \) for bit errors as \( d_e/n \). The information rate of an \( [[n, k]]_q \) QECC is defined as \( k/n \) [13].

Recall [11] that for \( 0 \leq \delta \leq 1 - 1/q \) we have

\[
\sum_{i=1}^{\lceil \log_q n \rceil} \binom{n}{i} (q - 1)^i \leq q^{\log_q (1 - \delta)}, \tag{2}
\]

where \( h_q(\delta) = \delta \log_q (q - 1) - \delta \log_q \delta - (1 - \delta) \log_q (1 - \delta) \).

**Corollary 3** Let \( \delta_s \) and \( \delta_e \) be real numbers such that \( 0 \leq \delta_s \leq 1 - 1/q \) and \( 0 \leq \delta_e \leq 1 - 1/q \). If

\[
h_q(\delta_s) < 1 - R_1, \tag{3}
\]

\[
h_q(\delta_e) < R_2, \text{ and} \tag{4}
\]

\[
0 \leq R_1 - R_2,
\]

then, for sufficiently large \( n \), there exists an \( [[n, [n R_1], [n R_2], [n \delta_s], [n \delta_e]]]_q \) CSS QECC exists.

In Corollary 3, \( R_1 \) is the information rate of classical ECC \( C_1 \), and \( R_2 \) is the information rate of classical ECC \( C_2 \). The corresponding quantum CSS code has information rate \( R_1 - R_2 \), relative distance \( \delta_s \) for bit errors, and relative distance \( \delta_e \) for phase errors.
Proof Assume that Eq. (3) holds. Then for sufficiently large $n$ we have

$$nh_3(\delta_x) < n - n R_1$$

$$\Rightarrow q^{nh_3(\delta_x)} < (1/2)q^{n - n R_1}$$

$$\Rightarrow q^{n R_1} - q^{nh_3(\delta_x)} < 1/2$$

$$\Rightarrow q^{n R_1 - n} - q^{nh_3(\delta_x)} < (1/2)q^{n - 1} \sum_{i=1}^{[n]} \binom{n}{i} (q - 1)^i < 1/2.$$  \hspace{1cm} (5)

Similarly, for sufficiently large $n$ Eq. (4) implies

$$nh_2(\delta_x) < n R_2$$

$$\Rightarrow q^{nh_2(\delta_x)} < (1/2)q^{n - n R_2}$$

$$\Rightarrow q^{n(1 - R_2)} - q^{nh_2(\delta_x)} < 1/2$$

$$\Rightarrow q^{n R_1 - n} - q^{nh_2(\delta_x)} < (1/2)q^{n - 1} \sum_{i=1}^{[n]} \binom{n}{i} (q - 1)^i < 1/2.$$  \hspace{1cm} (6)

Equations (5) and (6) imply that the assumption of Theorem 2 becomes true for sufficiently large $n$, which shows Corollary 3. \hspace{1cm} \square

3 A GV type existential bound for the stabilizer codes

Let $C \subset \mathbb{F}_q^{2n}$ be a $\mathbb{F}_q$-linear space of dimension $n - k$ self-orthogonal with respect to the standard symplectic inner product in $\mathbb{F}_q^{2n}$. $C$ can be viewed as an $[[n, k]]_q$ stabilizer QECC. Let $A_n$ be the set of all such $C$’s. A nonzero $e \in \mathbb{F}_q^{2n}$ can be viewed as a quantum error on $n$ qudits. Let $A_n(e)$ be the set of stabilizer codes that cannot detect $e$ as an error, that is, $A_n(e) = \{C \in A_n \mid e \in C^\perp \setminus C\}$, where $C^\perp$ is the dual of $C$ with respect to the symplectic inner product. Then $\#A_n(e) \leq \frac{1}{q^{2n}} \cdot \frac{1}{q^{n-2k}} \#A_n$ [12 Lemma 9].

Recall that, for $C$ to be $[[n, k, d_x, d_z]]_q$, $C$ must be able to detect all $d_x$ or less bit errors and all $d_z$ or less phase errors. The number of such errors is

$$\sum_{i=1}^{d_x-1} \binom{n}{i} (q - 1)^i \times \sum_{i=1}^{d_z-1} \binom{n}{i} (q - 1)^i.$$  

By the same argument as [12][Remark 10] (or as the last section), we have the following theorem:

Theorem 4 Let $n, k_1, k_2, d_x$ and $d_z$ be positive integers such that

$$1 - q^{-2k} \leq q^{-2k} \frac{1}{q^{2n}} \sum_{i=1}^{d_x-1} \binom{n}{i} (q - 1)^i \times \sum_{i=1}^{d_z-1} \binom{n}{i} (q - 1)^i < 1$$

then there exists an $[[n, k, d_x, d_z]]_q$ stabilizer QECC. \hspace{1cm} \square
By almost the same argument as Corollary 3 we can derive the following asymptotic version of Theorem 4.

**Corollary 5** Let \( \delta_x \) and \( \delta_z \) be real numbers such that \( 0 \leq \delta_x \leq 1 - \frac{1}{q} \) and \( 0 \leq \delta_z \leq 1 - \frac{1}{q} \). If

\[
h_q(\delta_x) + h_q(\delta_z) < 1 - R \leq 1,
\]

then, for sufficiently large \( n \), there exists an \([n, [nR], [n\delta_x], [n\delta_z]]_q\) stabilizer QECC. □

The quantum stabilizer code in Corollary 5 has information rate \( R \), relative distance \( \delta_x \) for bit errors, and relative distance \( \delta_z \) for phase errors.

By the relation between the CSS and the stabilizer QECCs 3, we see that the assumption in Corollary 3 is less demanding than that in Corollary 5 for the same \( n, R = R_1 - R_2, \delta_x \) and \( \delta_z \), which means that Corollary 5 is a stronger existential bound than Corollary 3.

**Remark 6** Theorems 2 and 4, and Corollaries 3 and 5 do not admit direct comparisons against previously known GV type bounds even when \( d_x = d_z \). The reason is as follows: For a binary QECC to be \([n,k,2,2]_2\), it must detect at least \( n^2 \) different errors. On the other hand, for a binary \([n,k]_2\) QECC to detect all single symmetric errors, it only has to detect \( 3n \) errors, which is generally much fewer than \( n^2 \). The above example shows that the number of asymmetric quantum errors is much different from that of corresponding symmetric quantum errors, even if we assume the same number of bit errors and phase errors in asymmetric quantum errors.

In addition, the famous \([5,1,3]_2\) binary stabilizer code in 3 can detect up to four bit errors if there is no phase error, and can detect up to four phase errors if there is no bit error. Thus it is simultaneously both \([5,1,1,5]_2\) AQECC and \([5,1,5,1]_2\) AQECC. This phenomenon makes the direct comparison even more difficult.

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