New algebraic structures in the $C_\lambda$-extended Hamiltonian system

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Abstract.

A realization of various algebraic structures in terms of the $C_\lambda$-extended oscillator algebras is introduced. In particular, the $C_\lambda$-extended oscillator algebras realization of Fairlie-Fletcher-Zachos (FFZ) algebra is given. This latter lead easily to the realization of the quantum $U_t(sl(2))$ algebra. The new deformed Virasoro algebra is also presented.

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1 Introduction

Deformations of different groups and algebras have attracted great attention during the last few years. These new mathematical objects called quantum algebras or quantum groups have found a lot of interesting physical applications. On the other hand, recently various extensions and deformations of the oscillator algebra have indeed been applied in the description of systems with nonstandard statistics, with violation of the Pauli principle, in the construction of integrable lattice models, as well as the algebraic treatment of n-particle integrable models. Among the various deformations and extensions, we mention the following:

(i) The generalized deformed oscillator algebras (GDOAs)\(^1, 2, 3\), generated by the unit, creation, annihilation, and number operators \((I, a, a^+, N)\) satisfying the Hermiticity conditions: \((a^+) = a, N^+ = N\) and the commutation relations:

\[
[N, a^\pm] = \pm a^\pm, \quad [a, a^\pm] = aa^\pm - qa^\pm a = F(N)
\]

where \(q\) is some real number and \(F(N)\) is some Hermitian, analytic function.

(ii) The G-extended oscillator algebras, where \(G\) is some finite group, appeared in connection with n-particle integrable systems. For example, in the case of Calogero model \(^4, 5, 6\) \(G\) is the symmetric group \(S_n\). For two particles \(S_2\) is nothing but the cyclic group of order 2; \(C_2 = \{I, K, /K^2 = I\}\) and the obtaining \(S_2\)-extended oscillator algebra is generated by the operators \((I, a, a^+, N\) and \(K)\) subject to the Hermiticity conditions \((a^+) = a, N^+ = N\) and \(K^+ = K^-\) and the relations:

\[
[N, a^\pm] = a^\pm, \quad [N, K] = 0, \quad K^2 = I
\]

\[
[a, a^\pm] = I + rK, \quad (r \in R) \quad a^\pm K = -Ka^\pm
\]

together with their Hermitian conjugates.

In this situation the Abelian group \(S_2\) can be realized in terms of Klein operator \(K = (-1)^N\), where \(N\) denotes the number operators. Hence the \(S_2\)-extended oscillator algebra becomes a generalized deformed algebras, where \(F(N) = I + r(-1)^N\) and \(q = 1\) and known as the Calogero-Vasiliev or also modified oscillator algebra\(^7, 8\). By replacing \(C_2\) by the cyclic group of order \(\lambda\) i.e \(C_\lambda = \{I, K, K^2, ..., K^{\lambda-1}\}\), one get a new class of G-extended oscillator algebras, generalizing the one describing the two-particle Calogero-model.

The Letter is organized as follows: in section 2 we review some basic notions concerning the \(C_\lambda\)-extended oscillator algebras. Section 3 is devoted to the construction of the FFZ and quantum \(U_I(sl(2))\) algebra. We propose a new deformed Virasoro algebras in Section 4. Concluding remarks are given in the last section.
2 Review on the $C_\lambda$-extended oscillator algebras and its properties

In this section, we briefly review the relevant definitions and results regarding the $C_\lambda$-extended oscillator algebras (for more details see ref. 9 and the references quoted therein). The $C_\lambda$-extended oscillator algebras $A_\lambda$, where $\lambda$ take any value in the set $\{2, 3, \ldots\}$, is defined as an algebra generated by the operators ($I, a, a^+, N$ and $K$) subject to the Hermiticity conditions $(a^+)^+ = a, N^+ = N$ and $K^+ = K^-$ and the relations:

$$[N, a^+] = a^+, \quad [N, K] = 0, \quad K^\lambda = I$$ (4)

$$[a, a^+] = I + \sum_{r=1}^{\lambda-1} \gamma_r K^r, \quad a^+ K = e^{-2\pi i/\lambda} K a^+$$ (5)

together with their Hermitian conjugates, where $\gamma_r$ are some complex parameters restricted by the condition $\gamma_r^* = \gamma_{\lambda-r}$, and $K$ is the generator of cyclic group $C_\lambda$. For $\lambda = 2$ we obtain the Calogero algebra characterized by the commutation relations (2), (3).

Now let us examine the connection between the $C_\lambda$-extended oscillator algebras and the generalized deformed algebras (GDOA’s). To begin, note that the cyclic group that $C_\lambda$ has $\lambda$ inequivalent unitary irreducible matrix representation $\Gamma^\nu$, ($\nu = 0, 1, 2, \ldots, \lambda - 1$), which are one dimensional such that $\Gamma^\nu(K^r) = \exp(2\pi i\nu r/\lambda)$, for $r = 0, 1, 2, \ldots, \lambda - 1$. Hence the projection operator on the carrier space of $\Gamma^\nu$ may be written as:

$$P_\mu = \frac{1}{\lambda} \sum_{r=0}^{\lambda-1} (\Gamma^\mu(K^r))^* K^r = \frac{1}{\lambda} \sum_{r=0}^{\lambda-1} e^{-2\pi i\mu r/\lambda} K^r$$ (6)

and conversely we have :

$$K^r = \sum_{\mu=0}^{\lambda-1} e^{2\pi i\mu r/\lambda} P_\mu.$$ (7)

Then the algebra $A_\lambda$ equations (Eqs(4),(5)) can be rewritten in terms of $I, a, a^+, N$ and $P_\mu = P^\mu_+$ as follows :

$$[N, a^+] = a^+, \quad [N, P_\mu] = 0, \quad \sum_{\mu=1}^{\lambda-1} P_\mu = I$$ (8)

$$[a, a^+] = I + \sum_{\mu=1}^{\lambda-1} \alpha_\mu P_\mu, \quad a^+ P_\mu = P_{\mu+1} a^+ \text{ and } P_\mu P_\nu = \delta_{\mu\nu} P_\nu$$ (9)

with the conventions $B_\lambda \sim B_0$ and $B_{-1} \sim B_{\lambda-1}$ (where $B = P, a$). The parameters $\alpha_\mu$ are given by:
\[ \alpha_\mu = \sum_{r=1}^{\lambda-1} \exp(2i\pi \mu r / \lambda) \gamma_r, \]  
(10)

restricted by the conditions \( \sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0 \). Hence we may eliminate one of them, for instance \( \alpha_{\lambda-1} \). In this situation the cyclic group generators \( K \) and the projection operators \( P_\mu \) can be realized in terms of \( N \) as:

\[ K = e^{2\pi i N / \lambda}, \quad P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{2\pi i (N-\nu) / \lambda} \quad (\mu, 1, 2, ..., \lambda - 1) \]  
(11)

respectively. With such a choice, the algebra becomes a \((\text{GDOA}'s)\) where \( G(N) = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \), where \( P_\mu \) is given by the above equation.

In the bosonic Fock space representation [9], we may consider the bosonic oscillator Hamiltonian, defined as usual by:

\[ H_0 = \frac{1}{2} \{ a^+, a \}, \]  
(12)

which can be rewritten in terms of the projection operators as:

\[ H_0 = N + \frac{1}{2} I + \sum_{\mu=0}^{\lambda-1} j_\mu P_\mu \]  
(13)

where \( j_0 = \frac{1}{2} \alpha_0 \) and \( j_\mu = \sum_{\nu=0}^{\mu-1} \gamma_\nu + \frac{1}{2} \gamma_\mu \) for all \( \mu = 1, 2, ..., \lambda - 1 \).

The eigenvectors of \( H_0 \) are the states \( |n> = |k\lambda + \mu> \) and the corresponding eigenvalues are given by:

\[ E_{k\lambda+\mu} = k\lambda + \mu + j_\mu + \frac{1}{2}, \quad k = 0, 1, ..., \mu = 0 \quad 1, ..., \lambda - 1 \]  
(14)

In the each subspace of the \( Z_\lambda \)-graded Fock space, the spectrum of \( H_0 \) is therefore harmonic, but the \( \lambda \) infinite sets of equally spaced energy levels, corresponding to \( \mu = 0, 1, 2, ..., \lambda - 1 \), may be shifted with respect to each other by some amounts depending upon the algebra parameters \( j_0, ..., j_{\lambda-2} \), through their linear combinations \( \alpha_\mu, \mu = 0, 1, ..., \lambda - 1 \).

In the case of Calogero-Vasiliev oscillator, the situation becomes very simple and coincides with that of modified harmonic oscillator.
3 $C_\lambda$-extended oscillator realization of FFZ and $U_1(sl(2))$ symmetries.

3.1 $C_\lambda$-extended oscillator algebras realization of FFZ symmetry

Before going on, we would like to give a short review concerning the $sdiff(X^{2n})$ algebra of volume preserving diffeomorphisms on smooth manifold $X^{2n}$. Let $X^{2n}$ be a $2n$-dimensional symplectic manifold with a symplectic structure $\omega_{ab}$, which can be represented in terms of the canonical constant antisymmetric $2n \times 2n$ matrix. Then $sdiff(X^{2n})$ is defined as

$$sdiff(X^{2n}) = \{ \Phi(\sigma) \in C^\infty(X)/[\Phi_1(\sigma), \Phi_2(\sigma)] = w^{ab} \partial_a \Phi_1 \partial_b \Phi_2 \}, \quad (15)$$

where $\sigma = (\sigma_1, ..., \sigma_{2n})$ denotes the corresponding local coordinates on $X^{2n}$. In the simplest case $X^{2n} = S^1 \times S^1$, the Lie algebra elements of $sdiff(T^2)$ are given by

$$\Phi_n(\sigma) = \exp(n \times \sigma) \quad (16)$$

where $n \times \sigma = n_1 \sigma_1 + n_2 \sigma_2$, then $sdiff(T^2)$ takes the following form :

$$[\Phi_m(\sigma), \Phi_n(\sigma)] = (m \Lambda n) \Phi_{m+n} \quad (17)$$

This algebra has been studied first by Arnold [10] and investigated by other authors in the theory of relativistic surfaces (see [11] for more details). The FFZ algebra or trigonometric sine algebra is defined as the quantum deformation of the Lie algebra $sdiff(S^1 \times S^1)$, which is generated by the generators $T_m$ satisfying the following commutation relations

$$[T_m, T_n] = -2i \sin\left(\frac{2\pi}{\hbar}(m \Lambda n)\right)T_{m+n} \quad h \in C^* \quad (18)$$

One notes that the limit $\hbar \to 0$ reproduces the algebra (Eq. (17)). Another approach to the definition of the above algebra is based on the ideas of noncommutative geometry [12]. Precisely, on the quantum two torus which is defined as an associative $C^*$-algebra generated by two unitary generators $U_1$ and $U_2$ satisfying the relations : $U_1U_2 = qU_2U_1$ where $(q = e^{i\hbar})$.

Now we turn to discuss the $C_\lambda$-extended oscillator realization of FFZ algebra. To begin with, let us define the following operators:

$$T_m = e^{i\pi m_1 m_2 / \lambda (a^+)^{m_1}(K)^{m_2}} \quad (19)$$

Before going on, let us discuss the problem of the negative power of the reaction operators $a^+$, which makes this construction formal. Indeed, to overcome this difficulty one consider the Bargmann representation of the $C_\lambda$-extended oscillator algebras given in [13]. With such representation the generators of $C_\lambda$-extended oscillator algebras are identified with
differential operators.
Using the relations (4), (5) one obtains:

\[ T_m T_n = e^{-i \pi m \Lambda n / \lambda} T_{m+n} \quad (20) \]

From the above equation, one easily gets the following relations

\[ [T_m, T_n] = -2i \sin(\pi \Lambda n / \lambda) T_{m+n} \quad (21) \]

So, the \( T \)'s satisfy the FFZ algebra (18), where we have used the following change

\[ 2\lambda = \hbar. \]

In what follows, we will generalize this construction for the \( U_t(sl(2)) \) algebra.

### 3.2 \( C_\lambda \)-extended oscillator realization of \( U_t(sl(2)) \) symmetry

First let us recall that the \( U_t(sl(2)) \) algebra emerges in several contexts, e.g. in
sine-Gordon theory \[14\] and in Chern-Simon theory \[15\] and recently it is uncovered in Landau problem which is intimately connected to problem of fractional quantum Hall effect \[16\]. It is well-known \[15, 17\] that the FFZ algebra induces the quantum \( U_t(sl(2)) \) algebra. Relying on this fact, we present it \( C_\lambda \)-extended oscillator realization. To start, let's recall that the \( U_t(sl(2)) \) is defined as a complex unital associative algebra over \( \mathbb{C}(t) \), the field of fraction for the ring of formal power series in the indeterminate \( t \ (t \neq 0, 1) \), generated by the generators \( X^\pm, H \) and \( H^{-1} \) satisfying :

\[ H^{-1} = HH^{-1} = 1, \quad HX^\pm H^{-1} = t^{\pm 2} X^\pm \]

\[ [X^+, X^-] = \frac{H - H^{-1}}{t - t^{-1}} \quad (22) \]

Let us present the following construction depending on the pair \( (m, n) \) and the gen-
erators \( T, X_\pm, H \) and \( H^{-1} \)

\[ X^+ = \frac{1}{t - t^{-1}}(T_m + T_n) \]

\[ X^- = \frac{1}{t - t^{-1}}(T_{-m} + T_{-n}) \]

\[ H = T_{m-n} \quad , \quad H^{-1} = T_{n-m} \quad (23) \]

where the the deformation parameter \( t = \exp(-i \pi m \Lambda n / \lambda) \). Calculating the commu-
tation relations for \( X^\pm \) and \( H^\pm \) using Eqs (19-21), one get easily the commutation relation for \( U_t(sl(2)) \).

6
4 $C_\lambda$-deformed Virasoro algebra.

In this section we introduce the new deformed Virasoro algebra using the $C_\lambda$-extended oscillator algebras. To begin with, recall that the Virasoro algebra termed witt algebra or also conformal algebra was first introduced in the context of string theories. Its is relevant to any theory in 2-dimensional space-times which possesses conformal invariance. The Witt algebra is the complexification of the Lie algebra $Vect(S^1)$. An element of $W$ is a linear combination of the elements of the form $e^{i \theta} \frac{d}{d \theta}$ where $\theta$ is a real parameter and the Lie bracket on $W$ is given by:

$$[e^{i \theta} \frac{d}{d \theta}, e^{i \theta'} \frac{d}{d \theta'}] = i(m-n)e^{i(n+m)\theta} \frac{d}{d \theta}$$ \hspace{1cm} (24)

It is rather convenient to consider an embedding of the circle into complex plane $C$ with the coordinates $z$, so that $z = e^{i \theta}$ and the element of the basis $e_m (m \in \mathbb{Z})$ are expressed as $e_m = -z^{k+1} \partial_z$. In this basis the commutation relations have the following form:

$$[e_m, e_n] = (m-n)e_{m+n}$$ \hspace{1cm} (25)

On the other hand the deformation (q-deformation) of this algebra was first introduced by Curtright and Zachos [18] and investigated in many occasion by many authors [19, 20, 21], and defined by the following q-commutation relations:

$$[e_m, e_n]_q = q^{m-n}e_m e_n - q^{n-m}e_n e_m = \frac{(q^{m-n} - q^{n-m})}{q - q^{-1}} e_{m+n}$$ \hspace{1cm} (26)

Turn now to the construction of the $C_\lambda$-deformed Virasoro algebra. To do this we will adopt the approach for undeformed case [22] where the generators $e_m$ are constructed from one classical oscillator pair $a^+, a$ as the infinite-dimensional extension of the following realization of $sp(2) \sim o(2, 1)$

$$e_1 = a, \hspace{0.5cm} e_0 = (a^+) a, \hspace{0.5cm} e_1 = (a^+)^2 a$$ \hspace{1cm} (27)

The extension to positive indices $m$ is straightforward

$$e_m = (a^+)^{m+1} a$$ \hspace{1cm} (28)

and the negative values ($m < -1$) are described by nonanalytic dependence (monomials of $a^+$ with negatives powers). which acts as the differential operators in Bargmann representation [13]. From the following commutation relations between the generators $a^+, a$ and $K$

$$[a, a^+] = I + \sum_{r=1}^{\lambda - 1} \gamma^r K^r, \hspace{0.5cm} a^+ K = e^{-2\pi i / \lambda} K a^+$$ \hspace{1cm} (29)

one obtain after algebraic manipulation the following relations:
\[ [a, (a^+)^{m+1}] = (m + \sum_{r=1}^{\lambda-1} f_r \gamma_r K^r)(a^+)^{m-1} \quad (30) \]

where the \( f_r \) is given by
\[
f_r = (1 + e^{r(2\pi i/\lambda)} + e^{2r(2\pi i/\lambda)} + \ldots + e^{(m-1)r(2\pi i/\lambda)}) \quad f_1 = 1. \quad (31)\]

Then from the previous equations, one get the following commutation relation for generators \( e_m(\gamma) \):
\[
[e_m(\gamma), e_n(\gamma)] = (m - n)e_{m+n}(\gamma) + \sum_{r=1}^{\lambda-1} (e^{(n+1)r(2\pi i/\lambda)} - e^{(m+1)r(2\pi i/\lambda)}) \gamma_r K^r e_{m+n}(\gamma). \quad (32)\]

Which goes to the ordinary Virasoro algebra for \( \gamma_r \to 0 \). However, when \( \gamma_r \neq 0 \) it is something new, one ask about the commutation relation between \( K \) and the generators \( e_m \), thanks to Eqs.(5),(30) one easily finds :
\[
[e_m(\gamma), K] = g(\gamma)e_m(\gamma)K \quad (33)\]

where \( g(\gamma) = (1 - \exp 2\pi i(m+1)/\lambda) \). In the case of Calogero-Vasiliev, \( \lambda = 2 \), in this situation the relation (30) becomes :
\[
[a, (a^+)^m] = (m + \frac{1}{2}(1 - (-1)^m))\gamma_1 K)(a^+)^{m-1}, \quad (34)\]

and the commutation relations between the generators \( e_m(\gamma) \) are :
\[
[e_m(\gamma), e_n(\gamma)] = (m - n)e_{m+n} + \frac{1}{2}((-1)^n - (-1)^m)\gamma_1 Ke_{m+n}(\gamma). \quad (35)\]

which is the \( K \)-deformed Virasoro algebra introduced in \cite{23}. In this case, we have \( g(2) = (1 + (-1)^m) \), hence for \( m \) odd, \( e_m \) commutes with the operators \( K \), for \( m \) even we have \( g(2) = 2 \).

5 Conclusion.

In this letter, we have presented the realization of FFZ algebra in terms \( C_\lambda \)-extended oscillator algebras, we have shown how this realization lead to obtain the realization pf the quantum quantum \( U_t(sl(2)) \). Otherwise, we have presented the new deformed Virasoro which we call the \( C_\lambda \)-deformed Virasoro algebra. It is interesting to investigated these new algebraic structures in the Calogero-Vasiliev model. Finally, note that in same way one can constructed \( C_\lambda \)-deformed \( W \)-algebras.

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