Research of Reliability Allocation Based on Gray Theory

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Abstract. A comprehensive reliability allocation method, which is applicable to the cases of lack of data in design stage, as well as the complexity of the influence factors and uncertainty of expert evaluation, is presented based on the gray system theory. The key of this method is to modify the weights of influence factors which are associated with expert scoring data by gray correlation analysis before reliability indices are assigned linearly to each subsystem. Based on gray evaluation with 4 levels, the application of this method is illustrated with the reliability allocation of a rotary machine under the consideration that complexity, technical level, importance and environment condition are chosen as influence factors. Compared with the actual faults statistics results, the validity and feasibility of the method is verified.

1. Introduction
Reliability is an important research content in the development of mechanical products. When the product is delivered to the user, whether the reliability level meets the design requirements is a major indicator for the acceptance by the ordering party. Therefore, the reliability index of the whole machine should be distributed at the beginning of the design of mechanical products, and the index requirements of each component or subsystem should be made clear.

Reliability allocation is an essential and important part of reliability design process. The engineers design and verify the reliability indicators (such as product reliability, failure efficiency or MTBF allocated by each subsystem or component, and the rationality and accuracy of the reliability allocation results are very important.

Expert scoring method is usually used for reliability allocation. Multiple experts score according to certain principles from complexity, technical difficulty, importance and work environment. Through data processing of scoring results, the scoring coefficient of each component or subsystem is obtained, and then the reliability allocation results of each component or subsystem are obtained. This approach is obviously subjective. In the early stage of product development, people's cognitive information of components is incomplete and inaccurate.

More and more scholars at home and abroad have realized the subjectivity and uncertainty of expert scoring process, so they try to use modern mathematical methods to deal with scoring data, among which fuzzy mathematics theory and gray system theory are the main ones.

In literature [1], a reliability allocation method of mechanical system, which is based on fuzzy comprehensive evaluation, is put forward based on fuzzy theory. In literature [2], the comprehensive evaluation method in the fuzzy mathematics theory is adopted to allocate the
reliability of the ship’s main power unit system, and the problem of fuzzy evaluation in the reliability allocation is solved. Yang Zhaojun et al. combine interval analysis, fuzzy comprehensive evaluation and analytic hierarchy process to establish the reliability distribution model of CNC machine tools[3]. Yang Chao et al. introduce interval analysis method to further modify the mathematical model of reliability distribution and improve the accuracy of fuzzy reliability distribution[4].

Compared with the fuzzy mathematics theory, the gray system theory[5] is more applicable to solve the problem of the uncertainty and poor information, which is effective to deal with the adverse effects caused by incomplete data and heavy subjective factors. Tang Wuxiang[6] introduces the gray evaluation method into the system reliability distribution to deal with the scoring results of experts. In literature [7], entropy weight is introduced and gray evaluation method is used to allocate the reliability index of CNC machining center, which achieves good application effect. Zhang Genbao et al. combine the gray decision making with the gray correlation analysis method to analyze the relative failure relation of each unit, and obtain the reliability distribution model [8]. Li Tong et al. apply the gray system theory to the reliability distribution of fuel cell vehicles, which expands the application field of this method [9]. In all of above methods, the gray evaluation method is used to calculate the experts’ scoring results, and the scoring coefficients of each system are obtained. However, in the actual situation, the importance of each influencing factor is different, it is necessary to analyze the weights of each influencing factor and give a reasonable weight value.

In this paper, on the basis of scoring different influencing factors with expert scoring method, weight analysis is carried out based on experts' knowledge and experience on the importance degree of influencing factors, and the weights of influencing factors are corrected with gray correlation analysis method, and then the comprehensive weights of each influencing factor are obtained.

2. Establishment of gray evaluation matrix

Suppose there are n factors affecting reliability distribution, and there are m experts participating in the evaluation. Among them, the evaluation value of the ith influencing factor given by the K-th expert to the s component is denoted as $v_{ki}^{(s)}$, and its value range is $[1, 10]$[8], with 10 representing the maximum influence and 1 representing the minimum influence. The evaluation result of m experts on the s component is denoted as $V^{(i)}$, which is called the sample evaluation matrix

$$V^{(i)} = \begin{bmatrix} v_{11}^{(i)} & v_{12}^{(i)} & \cdots & v_{1n}^{(i)} \\ v_{21}^{(i)} & v_{22}^{(i)} & \cdots & v_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1}^{(i)} & v_{m2}^{(i)} & \cdots & v_{mn}^{(i)} \end{bmatrix}$$

When the four-level evaluation method is adopted to classify the gray number, the gray set of the influence degree of each influencing factor on the product reliability is denoted as $T = \{t_1, t_2, \cdots, t_4\}$, $t_1, t_2, \cdots, t_4$ correspond to "highest", "higher", "general" and "low" respectively. The lower limit of the first class is set as P, the lower limit of the second class is set as G, the upper limit of the third class is set as L, and the upper limit of the fourth class is set as C, then let P=9, G=7, L=5, and C=3. According to the gray statistical method [10], the weights $s_j(v_{ki}^{(i)})$ belonging to the $j$($j=1,2,3,4$) gray group of $v_{ki}^{(i)}$ can be obtained by using the whitening weight function.

The four albino weight functions are displayed in figure 1 and figure 2 respectively, and their calculation formulas are as follows.
Figure 1. Albino weight functions of the first and third types.

Figure 2. Albino weight functions of the second and forth types.

(1) gray number $\oplus \in [P, 10)$, the formula is as follows

$$f_1(v_i^{(t)}) = \begin{cases} 0 & v_i^{(t)} \leq G \\ \frac{v_i^{(t)} - G}{P - G} & G < v_i^{(t)} < P \\ 1 & v_i^{(t)} \geq P \end{cases}$$

(2) gray number $\oplus \in [L, P)$, the formula is as follows

$$f_2(v_i^{(t)}) = \begin{cases} 0 & v_i^{(t)} \leq L \\ \frac{v_i^{(t)} - L}{G - L} & L < v_i^{(t)} < G \\ 1 & v_i^{(t)} = G \\ \frac{P - v_i^{(t)}}{P - G} & G < v_i^{(t)} < P \\ 0 & v_i^{(t)} \geq P \end{cases}$$

(3) gray number $\oplus \in [C, G)$, the formula is as follows

$$f_3(v_i^{(t)}) = \begin{cases} 0 & v_i^{(t)} \leq L \\ \frac{v_i^{(t)} - L}{G - L} & L < v_i^{(t)} < G \\ 1 & v_i^{(t)} = G \\ \frac{P - v_i^{(t)}}{P - G} & G < v_i^{(t)} < P \\ 0 & v_i^{(t)} \geq P \end{cases}$$

(4) gray number $\oplus \in [1, L)$, the formula is as follows
According to the level of gray number and gray number of each gray class, the weight vector of gray class is determined as
\[ f_i(v_i^{(s)}) = \begin{cases} 1 & v_i^{(s)} \leq C \\ \frac{L-v_i^{(s)}}{L-C} & C < v_i^{(s)} < L \\ 0 & v_i^{(s)} \geq L \end{cases} \] (5)

According to the level of gray number and gray number of each gray class, the weight vector of gray class is determined as
\[ X = (x_1, x_2, x_3, x_4) = \left( \frac{P + 10}{2}, G, L, \frac{C + 1}{2} \right) = (9.5, 7, 5, 2) \] (6)
where \( x_t \) (\( t = 1, 2, 3, 4 \)) represent the weight of the t-th gray class.

Next, the gray statistics \( n_{ij}^{(s)} \) and total gray statistics \( n_i^{(s)} \) of the evaluation matrix are calculated, where
\[ n_{ij}^{(s)} = \sum_{k=1}^{m} f_i(v_i^{(s)}) \quad \text{and} \quad n_i^{(s)} = \sum_{i=1}^{4} n_{ij}^{(s)} \quad (i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, 4) . \]

For the s-th component, the gray evaluation weight of the influencing factor i belonging to the j-th gray category is as follows.
\[ r_{ij}^{(s)} = \frac{n_{ij}^{(s)}}{n_i^{(s)}} \] (7)
The gray evaluation matrix of the s-th component is obtained.
\[ R^{(s)} = \begin{bmatrix} r_{11}^{(s)} & r_{12}^{(s)} & r_{13}^{(s)} & r_{14}^{(s)} \\ r_{21}^{(s)} & r_{22}^{(s)} & r_{23}^{(s)} & r_{24}^{(s)} \\ \vdots & \vdots & \vdots & \vdots \\ r_{n1}^{(s)} & r_{n2}^{(s)} & r_{n3}^{(s)} & r_{n4}^{(s)} \end{bmatrix} \] (8)

3. Determination of weight set of influencing factors

The main factors affecting product reliability are complexity, technical difficulty, importance, working time, environmental conditions and economy, etc. Obviously, the relative importance of various influencing factors varies. In this paper, on the basis of the importance score of influencing factors, the gray correlation method is used to analyze the score result of influencing factor set, and the weight of the importance degree of influencing factors is corrected, so as to obtain the comprehensive weight set of influencing factors.

The relative importance of the s-th component shall be scored according to the influencing factors, and the expert shall mark "\(^{\checkmark}\)" in the table 1. Count the times of marking "\(^{\checkmark}\)" in each lattice, and the matrix of the times of influencing factor combination weight \( Y^{(s)} \) is constructed. The importance of influencing factors is divided into 5 levels, and the weight vector of the levels formed by the corresponding values in the order of high to low is denoted as \( M = (9, 7, 5, 3, 1) \). Then the combination weight vector of the influencing factors is as follows.
\[ Y^{(s)} = (y_1^{(s)}, \ldots, y_n^{(s)}) = \frac{1}{m} MN^{(s)} \] (9)
it reflects the tendency of experts to evaluate the importance of influencing factors.

**Table 1. A valuation of combination weight of influence factors**

| Relative importance | Influencing factors | \( a_1 \) | \( a_2 \) | \ldots | \( a_n \) |
|---------------------|---------------------|-----------|-----------|-----|-----------|
| Most important      |                     |           |           |     |           |
| Very important      |                     |           |           |     |           |
In the gray correlation theory \cite{11}, the group influencing factors are taken as the characteristic sequence of the system, and the individual influencing factors are taken as the behavior sequence, and then the correlation degree between them is obtained by comparing the behavior sequence with the characteristic sequence. Obviously, individual factors with high correlation degree have the greatest influence on the system and their weight is also the largest. By analyzing the gray relational degree of gray assessment matrix and combining the combined weight of influencing factors obtained in the table 1, the comprehensive weight of influencing factors is obtained. The characteristic sequence of the s-th component is as follows.

\[ U_o(s) = (u_{01}(s), u_{02}(s), \ldots, u_{0m}(s))^T \]  

(10)

where \( u_{0i}(s) = \max \{v_{i1}(s), v_{i2}(s), \ldots, v_{in}(s)\} \) \( i = 1, 2, \ldots, m \).

The score of the influence factors of each component by experts are taken as its behavior sequence \( U_k = \{v_{ik1}, v_{ik2}, \ldots, v_{ikm}\}^T \), \( k = 1, 2, \ldots, n \), that is the k-th column in formula (1).

The behavior sequence is compared with the characteristic sequence to obtain the corresponding residual sequence, which is listed in the order of each residual and forms the absolute residual sequence \( o_{ik} \).

The correlation degree reflects the dominant relation of the behavior sequence to the feature sequence. The correlation degree between the behavior sequence of the s-th component and the characteristic sequence is as follows.

\[ \Delta_i(s) = (\delta_{1i}(s), \delta_{2i}(s), \ldots, \delta_{ni}(s)) \]  

(11)

where \( \delta_{ji}(s) = \left\{ \left| v_{ij1}(s) - u_{j1}(s) \right|, \left| v_{ij2}(s) - u_{j2}(s) \right|, \ldots, \left| v_{ijn}(s) - u_{jn}(s) \right| \right\}^T \) \( (i = 1, 2, \ldots, n) \).

The element of its corresponding correlation coefficient matrix can be obtained by using formula (12)

\[ \xi_{ij}(s) = \frac{\min_{k} \Delta_i(s)(k) + \rho \max_{k} \Delta_j(s)(k)}{\Delta_i(s)(k) + \rho \max_{k} \Delta_j(s)(k)} \]  

(12)

where \( \rho \) is the distinguishing coefficient.

The correlation degree reflects the dominant relation of the behavior sequence to the feature sequence, and the formula is as follows.

\[ \gamma_{s}(i) = \frac{1}{m} \sum_{i=1}^{m} \xi_{ij}(s), \quad i = 1, 2, \ldots, n \]  

(13)

Finally, the modified comprehensive weight vector of the influencing factors is obtained.

\[ W(s) = (\tilde{w}_1(s), \tilde{w}_2(s), \ldots, \tilde{w}_n(s)) \]  

(14)

where \( \tilde{w}_i(s) = \frac{w_i(s) \gamma_{s}(i)}{n \sum_{i=1}^{n} w_i(s) \gamma_{s}(i)} \), \( i = 1, 2, \ldots, n \).

According to the expert evaluation allocation method, the comprehensive gray evaluation value of the s-th component is obtained.

\[ Z(s) = W(s) \cdot R(s) \cdot X^T \]  

(15)

The final failure rate assigned to the s-th component is as follows.

\[ \lambda(s) = \frac{Z(s)}{\sum_{s=1}^{l} Z(s)}, \quad s = 1, 2, \ldots, l \]  

(16)
where \( l \) is the number of components, and \( \lambda_T \) is the failure rate of the product.

4. Case study

A rotating machinery product is composed of six parts: main body, left support system, right support system, damping system, power device and protection device, and the reliability model of these parts is a series model. The top design requires that the annual failure efficiency of the mechanical product shall not be higher than 0.04, that is \( \lambda_T \leq 0.04 \).

In the early stage of development, five experienced experts evaluate the six major parts of the mechanical product from four aspects: complexity, technical difficulty, importance and environmental conditions. Take the main body as an example to introduce the reliability allocation process. Evaluation matrix of main body is as follows.

\[
V^{(i)} = \begin{bmatrix}
9 & 8 & 7 & 8 \\
8 & 8 & 8 & 7 \\
8 & 9 & 8 & 9 \\
9 & 8 & 7 & 8 \\
8 & 9 & 8 & 7 \\
\end{bmatrix}
\]

For the main body, the statistics of influencing factors "complexity" belonging to each gray group are as follows

\[
j = 1: n_{i_1}^{(i)} = f_1(9) + f_1(8) + f_1(8) + f_1(9) + f_1(8) \\
\quad = 1 + 0.5 + 0.5 + 1 = 3.5 \\
j = 2: n_{i_2}^{(i)} = f_2(9) + f_2(8) + f_2(9) + f_2(8) \\
\quad = 0.5 + 0.5 + 0.5 + 1 = 1.5 \\
j = 3: n_{i_3}^{(i)} = f_3(9) + f_3(8) + f_3(9) + f_3(8) \\
\quad = 0 \\
j = 4: n_{i_4}^{(i)} = f_4(9) + f_4(8) + f_4(8) + f_4(9) + f_4(8) \\
\quad = 0
\]

The total gray statistic of the "complexity degree" of the influencing factors is as follows.

\[
n_i^{(i)} = n_{i_1}^{(i)} + n_{i_2}^{(i)} + n_{i_3}^{(i)} + n_{i_4}^{(i)} = 5
\]

According to formula (7), the gray evaluation weights of the influencing factors "complexity" belonging to each gray category can be obtained.

\[
r_{1_1}^{(i)} = \frac{n_{i_1}^{(i)}}{n_i^{(i)}} = 0.7 \quad r_{1_2}^{(i)} = \frac{n_{i_2}^{(i)}}{n_i^{(i)}} = 0.3 \quad r_{1_3}^{(i)} = \frac{n_{i_3}^{(i)}}{n_i^{(i)}} = 0 \quad r_{1_4}^{(i)} = \frac{n_{i_4}^{(i)}}{n_i^{(i)}} = 0
\]

In the same way, the gray statistics and the total gray statistics of other influencing factors can be obtained, and then the corresponding gray evaluation weights can be calculated. Finally, the gray evaluation matrix of the main body is obtained.

\[
R^{(i)} = \begin{bmatrix}
0.7 & 0.3 & 0 & 0 \\
0.6 & 0.4 & 0 & 0 \\
0.2 & 0.7 & 0.1 & 0 \\
0.4 & 0.6 & 0 & 0 \\
\end{bmatrix}
\]

According to the above scoring method, five experts scored the weights of the four influencing factors of the subject, and the combined weight matrix of the influencing factors \( N^{(i)} \) is obtained.
According to the formula (7), the weight vector of the combination of influencing factors is calculated.

\[ Y^{(i)} = (6, 6, 7, 8, 7, 4) \]

The characteristic sequence of the main body is \( U^{(i)}_0 = (9, 8, 9, 9, 9) \). Absolute residual sequence is shown in Table 2.

| Expert | \( \Delta_1 \) | \( \Delta_2 \) | \( \Delta_3 \) | \( \Delta_4 \) |
|--------|----------------|----------------|----------------|----------------|
| 1      | 0              | 1              | 2              | 1              |
| 2      | 0              | 0              | 0              | 1              |
| 3      | 1              | 0              | 1              | 0              |
| 4      | 0              | 1              | 2              | 1              |
| 5      | 1              | 0              | 1              | 2              |

\( \min \min \Delta^{(i)} = 0, \max \max \Delta^{(i)} = 2, i = 1, \ldots, 4, k = 1, \ldots, 5 \). The distinguishing coefficient \( \rho = 0.5 \).

From the equation (12) and (13), we can get the correlation between the influencing factors.

\[ r^{(i)} = (0.8, 0.8, 0.5333, 0.5666) \]

The correlation degree is normalized to get the weight set of each influencing factor.

\[ w^{(i)} = (w_1, w_2, w_3, w_4) = (8/27, 8/27, 16/81, 1/5) \]

The comprehensive weight of influencing factors for reliability allocation of the main body is determined after modification.

\[ W^{(i)} = (0.2745, 0.2912, 0.2163, 0.2180) \]

The comprehensive gray evaluation coefficient of the main body is \( Z^{(i)} = W^{(i)} \cdot R^{(i)} \cdot X^T = 8.2001 \).

In the same way, the comprehensive evaluation coefficients of the other five components can be obtained, \( Z^{(2)} = 5.1894 \), \( Z^{(3)} = 7.5408 \), \( Z^{(4)} = 5.2547 \), \( Z^{(5)} = 7.0836 \), \( Z^{(6)} = 2.1873 \).

The total failure rate of the mechanical product is \( \lambda_T = 0.04/a \). The results allocated by this method are shown in Table 3.

| Component name     | Comprehensive evaluation coefficient | Failure rate/a |
|--------------------|--------------------------------------|----------------|
| Main body          | 8.2001                               | 0.0093         |
| Left support system| 5.1894                               | 0.0058         |
| Right support system| 7.5408                              | 0.0085         |
| Damping system     | 5.2547                               | 0.0059         |
| Power device       | 7.0836                               | 0.0080         |
Protection device & 2.1873 & 0.0025 \\

The basis of reliability analysis is the statistical analysis of product failure data. Before the reliability distribution, the field data of the old models that are already in use are counted to obtain the fault statistics of the old models, as shown in the table 4. The components of new and old models have little change in structural form, material and function, and have inheritance in design. It is of reference value to consider the field use of old models from similar product design.

| Component name          | Failure time | Failure frequency |
|-------------------------|--------------|-------------------|
| Main body               | 45           | 36.89%            |
| Left support system     | 10           | 8.20%             |
| Right support system    | 32           | 26.23%            |
| Damping system          | 11           | 9.02%             |
| Power device            | 23           | 18.85%            |
| Protection device       | 1            | 0.82%             |

From table 4, it can be seen that the failure rate of each part of the old model from high to low is as follows: main body, right support part, power part, damping part, left support part and protection device. According to the principle of reliability allocation: for complex systems and units with high technical difficulty, on the premise of meeting the overall reliability index of the system, in order to reduce the development cost and shorten the development cycle, lower reliability index can be allocated. Compared with other components, the main body, right support components and power components of the old model show a higher failure rate. Therefore, in the reliability allocation of the new model, these components can be assigned a higher failure rate, which is consistent with the results in this paper, so the allocation method is rational.

5. Conclusion
The subjective opinions of experts on influencing factors are considered, and the gray correlation analysis method is used to correct the weight of the importance of influencing factors. The subjective problems of expert scoring and the complexity of influencing factors in the process of reliability allocation are effectively solved.

The calculation process of this method is simple and convenient for engineering application, which can be extended to other industries.

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