The spectral problem and algebras associated with extended Dynkin graphs. I.

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Abstract

There is a connection between *-representations of algebras associated with graphs and the problem about the spectrum of a sum of Hermitian operators (spectral problem). For algebras associated with extended Dynkin graphs we give an explicit description of the parameters for which there are *-representations and an algorithm for constructing these representations.

KEYWORDS: Hilbert space, irreducible representation, graph, quiver, Coxeter functor, Horn’s problem, extended Dynkin diagram, *-algebra

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Introduction.

1. Let $A_1, A_2, \ldots, A_n$ be Hermitian $m \times m$ matrices with given eigenvalues: $\tau(A_j) = \{\lambda_1(A_j) \geq \lambda_2(A_j) \geq \ldots \geq \lambda_m(A_j)\}$. The well-known classical problem about the spectrum of a sum of two Hermitian matrices (Horn’s problem) is to describe a connection between $\tau(A_1), \tau(A_2), \tau(A_3)$ such that $A_1 + A_2 = A_3$. In more symmetric setting one can seek for a connection between $\tau(A_1), \tau(A_2), \ldots, \tau(A_n)$ necessary and sufficient for the existence of Hermitian operators $A_1 + A_2 + \ldots + A_n = \gamma I$ for a fixed $\gamma \in \mathbb{R}$.

A recent solution of this problem (see [4, 5] and others) gives a complete description of possible $\tau(A_1), \tau(A_2), \ldots, \tau(A_n)$ in terms of linear inequalities. Note that the number of necessary inequalities increases with $m$.

2. In [6, 7] was considered the following modifications of the problem mentioned above, called henceforth the spectral problem (resp. strict spectral problem). We will consider bounded linear Hermitian operators $A_1, A_2, \ldots, A_n$ on a separable Hilbert space $H$. For an operator $X$ denote by $\sigma(X)$ its spectrum. Let $M_1, M_2, \ldots, M_n$ be given closed subsets of $\mathbb{R}$ and $\gamma \in \mathbb{R}$. The problem consists of the following: 1) to

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determine whether there are Hermitian operators \( A_1, A_2, \ldots, A_n \) on \( H \) such that \( \sigma(A_j) \subseteq M_j \) (respectively \( \sigma(A_j) = M_j \)) \( (1 \leq j \leq n) \), and \( A_1 + A_2 + \ldots + A_n = \gamma I \) if the answer is in the affirmative, to give a description (up to unitary equivalence) of the operators. In this work the sets \( M_1, M_2, \ldots, M_n \) will be finite. Note that even for finite \( M_k \) the second part of the problem can be very complicated if \(|M_k|\) is large enough.

The essential difference with Horn’s classical problem is that we do not fix the dimension of \( H \) (it may be finite or infinite) and the spectral multiplicities. It seems that the solution of the spectral and strict spectral problems could not be deduced directly from the Horn inequalities, since the number of necessary inequalities increases with \( m \).

3. Spectral problem could be reformulated in terms of \(*\)-representations of \(*\)-algebras. Namely, let \( \alpha(j) = (\alpha_1(j), \alpha_2(j), \ldots, \alpha_m(j)) \) \( (1 \leq j \leq n) \) be vectors with positive strictly decreasing coefficients. Put \( M_j = \alpha(j) \). Let us consider the associative algebra defined by the following generators and relations (see [10]):

\[
A_{M_1, \ldots, M_n, \gamma} = \mathbb{C} \langle p^{(1)}_1, p^{(2)}_1, \ldots, p^{(1)}_{m_1}, p^{(2)}_1, p^{(2)}_{m_2}, \ldots, p^{(n)}_1, p^{(n)}_2, \ldots, p^{(n)}_{m_n} \rangle
\]

\[
p_k^{(i)} = p_k^{(i)} \sum_{i=1}^{m_i} \alpha_k^{(i)} p_k^{(i)} = \gamma e,
\]

\[
\sum_{k=1}^{m_i} p_k^{(i)} = e, p_j^{(i)} p_k^{(i)} = 0, j, k = 1, \ldots, m_i, j \neq k, i = 1, \ldots, n).
\]

Here \( e \) is the identity of the algebra. This is a \(*\)-algebra, if we declare all generators to be self-adjoint.

A \(*\)-representation \( \pi \) of \( A_{M_1, \ldots, M_n, \gamma} \) determines an \( n \)-tuple of non-negative operators \( A^{(j)} = \sum_{k=1}^{m_k} \alpha_k^{(j)} P_k^{(j)} \), where each of the families of orthoprojections, \( \{P_i = \pi(p_i)\}, i = 1, \ldots, k \) forms a resolution of the identity and such that \( A^{(1)} + \ldots + A^{(n)} = \gamma I \). Moreover, \( \sigma(A^{(j)}) \subseteq M_j \). And vice versa, any \( n \)-tuple of Hermitian matrices \( A^{(1)}, A^{(2)}, \ldots, A^{(n)} \) such that \( A^{(1)} + \ldots + A^{(n)} = \gamma I \) and \( \sigma(A^{(j)}) \subseteq M_j \) determines a representation \( \pi \) of \( A_{M_1, \ldots, M_n, \gamma} \) by taking \( \pi(p_k^{(j)}) \) to be the spectral projections corresponding to eigenvalues \( \alpha_k^{(j)} \) respectively. So in terms of \(*\)-representations, the spectral problem is a problem consisting of the following two parts: 1) a description of the set \( \Sigma_{m_1, m_2, \ldots, m_n} \) of the parameters \( \alpha_k^{(j)}, \gamma \) for which there exist \(*\)-representations of \( A_{M_1, \ldots, M_n, \gamma} \). 2) a description of \(*\)-representations \( \pi \) of the \(*\)-algebra \( A_{M_1, M_2, \ldots, M_n, \gamma} \).

4. A natural way to try to solve the spectral problem is to describe all irreducible \(*\)-representations up to unitary equivalence and then all \(*\)-representations as sums or direct integrals of irreducible representations. Obviously if there is a \(*\)-representation of the algebra \( A_{M_1, \ldots, M_n, \gamma} \) then there is its irreducible \(*\)-representation. Hence the set \( \Sigma_{m_1, m_2, \ldots, m_n} \) coincides with the set of parameters for which algebra \( A_{M_1, \ldots, M_n, \gamma} \) has at least one irreducible \(*\)-representation; The second part of the spectral problem could be formulated in the following way: find the formulae for the irreducible \(*\)-representations of the algebra \( A_{M_1, \ldots, M_n, \gamma} \) for parameters \( (\lambda_k^{(j)}, \gamma) \in \Sigma_{m_1, \ldots, m_n} \) or at least present an algorithm to construct such representations. The strict spectral
problem could also be reformulated in terms of representation theory of these algebras but we will not discuss it here.

5. An key step in solving spectral problem is to describe the irreducible non-degenerate representations. Let us call a \(*\)-representation \(\pi\) of the algebra \(\mathcal{A}_{M_1, \ldots, M_n, \gamma}\) non-degenerate if \(\pi(p_{(j)}^{(k)}) \neq 0\) for all \(k, j\). Consider the following set: \(\Sigma_{m_1, \ldots, m_n} = \{\{\{\lambda_{(j)}^{(k)}\}, \gamma\} | \text{there is a non-degenerate } \ast\text{-representation of } \mathcal{A}_{M_1, \ldots, M_n, \gamma}\}\); which depends only on \((m_1, \ldots, m_n)\). Every irreducible representation of the algebra \(\mathcal{A}_{M_1, M_2, \ldots, M_n, \gamma}\) irreducible non-degenerate \(*\)-representation of an algebra \(\mathcal{A}_{\tilde{M}_1, \ldots, \tilde{M}_n, \gamma}\) for some subsets \(\tilde{M}_j \subset M_j\). Hence \((M_1, \ldots, M_n, \gamma) \in \Sigma_{m_1, \ldots, m_n}\) if there exist \((M_1, \ldots, \tilde{M}_n, \gamma) \in \Sigma_{m_1, \ldots, m_n}\). Thus the description of \(\Sigma_{m_1, \ldots, m_n}\) follows from the description of \(\Sigma_{k_1, \ldots, k_n}\) where \(k_j \leq m_j, 1 \leq j \leq n\).

6. With an integer vector \((m_1, \ldots, m_n)\) we will associate a non-oriented star-shape graph \(G\) with \(n\) branches of the lengths \(m_1, m_2, \ldots, m_n\) stemming from a single root. The graph \(G\) and vector \(\chi = (\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_{m_1}^{(1)}, \alpha_1^{(2)}, \ldots, \alpha_{m_2}^{(2)}, \ldots, \alpha_1^{(n)}, \ldots, \alpha_{m_n}^{(n)}, \gamma)\) completely determine the algebra \(\mathcal{A}_{M_1, \ldots, M_n, \gamma}\) so we will use the following notation \(\mathcal{A}_{G, \chi}\).

Henceforth we will denote \(\Sigma\) by \(\Sigma(G)\) where \(G\) is the tree mentioned above. The spectral problem for operators on a Hilbert space can be reformulated in the following way: 1) for a given graph \(G\) describe the set \(\Sigma(G)\); 2) describe non-degenerate representations \(\mathcal{A}_{M_1, \ldots, M_n, \gamma}\) up to unitary equivalence.

If the graph is a Dynkin graph (extended Dynkin graph), the problem is greatly simplified. The algebras \(\mathcal{A}_{M_1, \ldots, M_n, \gamma}\) associated with Dynkin graphs (resp. extended Dynkin graphs) have a more simple structure than in other cases. In particular, the algebras \(\mathcal{A}_{M_1, \ldots, M_n, \gamma}\) are finite dimensional (resp. have polynomial growth) if and only if the associated graph is a Dynkin graph (resp. an extended Dynkin graph) (see [12]). As shown in [9] irreducible representations of the algebras associated with Dynkin graphs exist only in certain dimensions that are bounded from above. In [6, 7, 8] we have given a complete description of \(\Sigma(G)\) for all Dynkin graphs \(G\) and an algorithm for finding all irreducible representations. In present paper we do the same for extended Dynkin graphs.

7. To solve the spectral problem for extended Dynkin graphs we will follow the following scheme:

1.) As it was mentioned above it in suffices to describe the sets \((M_1, \ldots, M_n, \gamma)\) for which there exist irreducible non-degenerate representations of the algebra \(\mathcal{A}_{M_1, \ldots, M_n, \gamma}\). Any such representation is finite dimensional (see [11]).

2.) Using the connection between representation theory of \(\mathcal{A}_{M_1, \ldots, M_n, \gamma}\) and locally-scalar representations of the associated graph (see s. 9) we can find generalized dimensions of such representations of the algebra since they correspond to the positive roots of the corresponding root system. The root can be real or imaginary.

3.) In the former case such representation can be obtained using Coxeter functors (see s. 1) starting from the simplest ones. The corresponding sets of parameters in case of graph \(\tilde{E}_6\) are described in theorems 3 and 5.

4.) In the latter case the parameters of the algebra belong to a certain hyperplane (see s. 3). In this case the dimension of irreducible representation is the minimal
imaginary root (or real root which is considered before) which is unique for each extended Dynkin graph. Since in this case the dimension is fixed the solution of the spectral problem could be obtained by direct application of Horn’s inequalities. For example for $\tilde{E}_6$ graph corresponding inequalities are written out in theorem $\text{[7]}$.

Since the description of $\ast$-representations of the $\ast$-algebras associated with extended Dynkin graphs can be reduced to a description of non-degenerate irreducible representations, in this article we essentially solve the spectral problem for the algebras associated with extended Dynkin graphs.

In the present article we develop the general setup for solving the spectral problem for extended Dynkin graphs illustration the concepts on example of $\tilde{E}_6$. In further publications we will present explicit solution for all extended Dynkin graphs.

1 Locally-scalar graph representations and representations of the algebras generated by orthoprojections.

The main tool for our classification is Coxeter functors for locally-scalar graph representations. They allow one to construct all representations starting from the simplest ones which correspond to the vertices of the graph. First we will recall a connection between category of $\ast$-representation of algebra $\mathcal{A}_{M_1, \ldots, M_n, \gamma}$ associated with the graph $G$ and locally-scalar representations of the graph $G$. For more details see $\text{[6]}$. Note that Coxeter functors could be constructed directly for the categories of representations of algebras $\mathcal{A}_{M_1, \ldots, M_n, \gamma}$ (see $\text{[10]}$) but the simplest representations of a graph does not correspond to a representation of corresponding algebra. This force us to use graph representation terminology and techniques.

Henceforth we will use definitions, notations and results about representations of graphs in the category of Hilbert spaces found in $\text{[9]}$. Let us recall some of them.

A graph $G$ consists of a set of vertices $G_v$ a set of edges $G_e$ and a map $\varepsilon$ from $G_e$ into the set of one- and two-element subsets of $G_v$ (the edge is mapped into the set of incident vertices). Henceforth we consider connected finite graphs without cycles (trees). Fix a decomposition of $G_v$ of the form $G_v = G_{v, \text{even}} \sqcup G_{v, \text{odd}}$ (unique up to permutation) such that for each $\alpha \in G_{v, \text{even}}$ one of the vertices from $\varepsilon(\alpha)$ belongs to $G_{v, \text{even}}$ and the other to $G_{v, \text{odd}}$. Vertices in $G_{v, \text{even}}$ will be called even, and those in the set $G_{v, \text{odd}}$ odd. Let us recall the definition of a representation $\Pi$ of a graph $G$ in the category of Hilbert spaces $\mathcal{H}$. Let us associate with each vertex $g \in G_v$ a Hilbert space $\Pi(g) = H_g \in \text{Ob} \mathcal{H}$, and with each edge $\gamma \in G_e$ such that $\varepsilon(\gamma) = \{g_1, g_2\}$ a pair of mutually adjoint operators $\Pi(\gamma) = \{\Gamma_{g_1, g_2}, \Gamma_{g_2, g_1}\}$, where $\Gamma_{g_1, g_2} : H_{g_2} \to H_{g_1}$. We now construct a category $\text{Rep}(G, \mathcal{H})$. Its objects are the representations of the graph $G$ in $\mathcal{H}$. A morphism $C : \Pi \to \Pi$ is a family $\{C_g\}_{g \in G_v}$ of operators $C_g : \Pi(g) \to \Pi(g)$.
such that the following diagrams commute for all edges $\gamma_{g_2,g_1} \in G_e$:

$$
\begin{array}{ccc}
H_{g_1} & \xrightarrow{\Gamma_{g_2,g_1}} & H_{g_2} \\
C_{g_1} \downarrow & & \downarrow C_{g_2} \\
H_{g_1} & \xrightarrow{\hat{\Gamma}_{g_2,g_1}} & H_{g_2}
\end{array}
$$

Let $M_g$ be the set of vertices connected with $g$ by an edge. Let us define the operators

$$A_g = \sum_{g' \in M_g} \Gamma_{gg'} \Gamma_{g'g}.$$

A representation $\Pi$ in $\text{Rep}(G, \mathcal{H})$ will be called \textit{locally-scalar} if all operators $A_g$ are scalar, $A_g = \alpha_g I_{H_g}$. The full subcategory $\text{Rep}(G, \mathcal{H})$, the objects of which are locally-scalar representations, will be denoted by $\text{Rep} G$ and called the category of locally-scalar representations of the graph $G$.

Let us denote by $V_G$ the real vector space consisting of sets $x = (x_g)$ of real numbers $x_g, g \in G_v$. Elements $x$ of $V_G$ we will call $G$-vectors. A vector $x = (x_g)$ is called positive, $x > 0$, if $x \neq 0$ and $x_g \geq 0$ for all $g \in G_v$. Denote $V_G^+ = \{x \in V_G | x > 0\}$. If $\Pi$ is a finite dimensional representation of the graph $G$ then the $G$-vector $(d(g))$, where $d(g) = \text{dim}(\Pi(g))$ is called the \textit{dimension} of $\Pi$. If $A_g = f(g) I_{H_g}$ then the $G$-vector $f = (f(g))$ is called the \textit{character} of the locally-scalar representation $\Pi$ and $\Pi$ is called the $f$-representation in this case. The \textit{support} $G_v^\Pi$ of $\Pi$ is $\{g \in G_v | \Pi(g) \neq 0\}$. A representation $\Pi$ is \textit{faithful} if $G_v^\Pi = G_v$. A character of the locally-scalar representation $\Pi$ is uniquely defined on the support $G_v^\Pi$ and non-uniquely on its complement. In the general case, denote by $\{f_\Pi\}$ the set of characters of $\Pi$. For each vertex $g \in G_v$, denote by $\sigma_g$ the linear operator on $V_G$ given by the formulae:

$$
(\sigma_g x)_{g'} = x_{g'} \text{ if } g' \neq g,
$$

$$
(\sigma_g x)_g = -x_g + \sum_{g' \in M_g} x_{g'}.
$$

The mapping $\sigma_g$ is called the \textit{reflection} at the vertex $g$. The composition of all reflections at odd vertices is denoted by $\hat{c}$ (it does not depend on the order of the factors), and at all even vertices by $\hat{c}$. A Coxeter transformation is $c = \hat{c} \hat{c}$, $c^{-1} = \hat{c} \hat{c}$.

The transformation $\hat{c}$ ($\hat{c}$) is called an odd (even) Coxeter map. Let us adopt the following notations for compositions of the Coxeter maps: $\hat{c}_k = \ldots \hat{c} \hat{c}_c (k \text{ factors})$, $\hat{c}_k = \ldots \hat{c} \hat{c}_c (k \text{ factors})$.

Any real function $f$ on $G_v$ can be identified with a $G$-vector $f = (f(g))_{g \in G_v}$. If $d(g)$ is the dimension of a locally-scalar graph representation $\Pi$, then

$$
\begin{align}
\circ c(d)(g) &= \begin{cases} 
-d(g) + \sum_{g' \in M_g} d(g'), & \text{if } g \in G_v^\circ, \\
-d(c(c) d(g)), & \text{if } g \in G_v, \\
d(g), & \text{if } g \in G_v, 
\end{cases} \\
\bullet c(d)(g) &= \begin{cases} 
-d(g) + \sum_{g' \in M_g} d(g'), & \text{if } g \in G_v^\circ, \\
-d(c(c) d(g)), & \text{if } g \in G_v, \\
d(g), & \text{if } g \in G_v.
\end{cases}
\end{align}
$$

(1.1) (1.2)
For \( d \in \mathbb{Z}_G^+ \) and \( f \in V_G^+ \), consider the full subcategory \( \text{Rep}(G, d, f) \) in \( \text{Rep} G \) (here \( \mathbb{Z}_G^+ \) is the set of positive integer \( G \)-vectors), with the set of objects \( \text{Ob} \text{Rep}(G, d, f) = \{ \Pi \mid \dim \Pi(g) = d(g), f \in \{ f_i \} \} \). All representations \( \Pi \) from \( \text{Rep}(G, d, f) \) have the same support \( X = X_d = G_v^\Pi = \{ g \in G_v \mid d(g) \neq 0 \} \). We will consider these categories only if \( (d, f) \in S = \{(d, f) \in \mathbb{Z}_G^+ \times V_G^+ \mid d(g) + f(g) > 0, g \in G_v \} \). Let \( \hat{X} = X \cap \hat{G}_v \), \( \hat{X} = X \cap \hat{G}_v \). \( \hat{X} = X \cap \hat{G}_v \). \( \hat{X} = \text{Rep}_o(G, d, f) \subset \text{Rep}(G, d, f) \) (\( \text{Rep}_o(G, d, f) \subset \text{Rep}(G, d, f) \)) is the full subcategory with objects \( (\Pi, f) \) where \( f(g) > 0 \) if \( g \in \hat{X} \) \((f(g) > 0 \) if \( g \in \hat{X} \)). Let \( S_0 = \{(d, f) \in S \mid f(g) > 0 \) if \( g \in \hat{X}_d \}, S_* = \{(d, f) \in S \mid f(g) > 0 \) if \( g \in \hat{X}_d \}\). 

Put

\[
\begin{align*}
\bullet \cdot_d(f)(g) = f_d(g) = \begin{cases} 
\circ(f)(g), & \text{if } g \in \hat{X}_d, \\
 f(g), & \text{if } g \notin \hat{X}_d, 
\end{cases} & \quad (1.3) \\
\circ \cdot_d(f)(g) = f_d(g) = \begin{cases} 
\circ(f)(g), & \text{if } g \in \hat{X}_d, \\
 f(g), & \text{if } g \notin \hat{X}_d. 
\end{cases} & \quad (1.4)
\end{align*}
\]

Let us denote \( \hat{\circ} \cdot_d^k(f) = \ldots \hat{\circ} \cdot_d^2(d) \hat{\circ} \cdot_d(d) \hat{\circ} \cdot_d(f) (k \text{ factors}), \hat{\circ} \cdot_d^k(f) = \ldots \hat{\circ} \cdot_d^2(d) \hat{\circ} \cdot_d(d) \hat{\circ} \cdot_d(f) (k \text{ factors}) \). The even and odd Coxeter reflection functors are defined in [12], \( \hat{F} : \text{Rep}_o(G, d, f) \to \text{Rep}_o(G, \hat{\circ} \cdot_d(d), \hat{f}_d) \) if \( (d, f) \in S_0 \), \( \hat{F} : \text{Rep}_o(G, d, f) \to \text{Rep}_o(G, \hat{\circ} \cdot_d(d), \hat{f}_d) \) if \( (d, f) \in S_* \); they are equivalences of the categories. Let us denote \( \hat{F}_k(\Pi) = \ldots \hat{F} \hat{F} \hat{F}(\Pi) (k \text{ factors}), \hat{F}_k(\Pi) = \ldots \hat{F} \hat{F} \hat{F}(\Pi) (k \text{ factors}) \), if the compositions exist. Using these functors, an analog of Gabriel’s theorem for graphs and their locally-scalar representations has been proven in [12]. In particular, it has been proved that any locally-scalar graph representation decomposes into a direct sum (finite or infinite) of finite dimensional indecomposable representations, and all indecomposable representations can be obtained by odd and even Coxeter reflection functors starting from the simplest representations \( \Pi_g \) of the graph \( G \) \((\Pi_g(g) = \mathbb{C}, \Pi_g(g') = 0 \) if \( g \neq g' \); \( g, g' \in G_v \)).

2 Root systems associated with extended Dynkin diagrams.

Let us recall a few facts about root systems. Let \( G \) be a simple connected graph. Then its \textit{Tits form}

\[
q(\alpha) = \sum_{i \in G_v} \alpha_i^2 - \frac{1}{2} \sum_{\beta \in G_v, \langle \alpha, \beta \rangle = 0} \alpha_i \alpha_j (\alpha \in V_G).
\]

The \textit{symmetric bilinear form} \((\alpha, \beta) = q(\alpha + \beta) - q(\alpha) - q(\beta)\). Vector \( \alpha \in V_G \) is called sincere if each component is non-zero.
It is well known that for Dynkin graphs (and only for them) bilinear form $(\cdot,\cdot)$ is positive definite. The form is positive semi-definite for extended Dynkin graphs. And in the letter case $\text{Rad} \ q = \{v | q(v) = 0\}$ is equal to $\mathbb{Z} \delta$ where $\delta$ is a minimal imaginary root. For other graphs (which are neither Dynkin nor extended Dynkin) there are vectors $\alpha \geq 0$ such that $q(\alpha) < 0$ and $(\alpha, \epsilon_j) \leq 0$ for all $j$.

If $G$ is an extended Dynkin graph a vertex $j$ is called extending if $\delta_j = 1$. The graph obtained by deleting extending vertex is the corresponding Dynkin graph. The set of roots is $\Delta = \{\alpha \in V_G | \alpha_i \in \mathbb{Z} \forall i \in G_v, \alpha \neq 0, q(\alpha) \leq 0\}$. A root $\alpha$ is real if $q(\alpha) = 1$ and imaginary if $q(\alpha) = 0$. Every root is either positive or negative. For our classification purposes we will need the following fact (see [2]): for an extended Dynkin graph the set $\Delta \cup \{0\}/Z\delta$ is finite. Moreover, if $e$ is an extending vertex then the set $\Delta_e = \{\alpha \in \Delta \cup \{0\} | \alpha_e = 0\}$ is a complete set of representatives of the cosets from $\Delta \cup \{0\}/Z\delta$. If $\alpha$ is a root then $\alpha + \delta$ is again a root. We will call the coset $\alpha + \delta\mathbb{Z}$ a $\delta$-series. If $\alpha$ is a root then its images under the action of the group generated by $\hat{e}$ and $\check{e}$ will be called a Coxeter series or $C$-series for short. It turns out that each $C$-series decomposes into a finite number of $\delta$-series of roots. We will use this decompositions to give an explicit formula for generalized dimensions and corresponding parameters of algebras.

Let $G$ be the extended Dynkin graph $\tilde{E}_6$.

The vertices $g_0, g_1, g_2, g_3$ will be called odd and marked with $\bullet$ on the graph, the vertices $g_2, g_4, g_5, g_6$ are even and indicated with $\circ$. The parameters of the corresponding algebra $\mathcal{A}_{\alpha, \beta, \delta, \gamma}$ are enumerated according to the following picture:

We will write dimension and parameter vectors as $(v_1, v_2, v_3, v_4, v_6, v_0)$. The minimal imaginary root for $\tilde{E}_6$ is the following vector $(1, 2, 1, 2, 1, 2, 3)$

Now we describe the root system for the diagram $\tilde{E}_6$. It consists of 72 series $\{\alpha_j + k\delta\}_{k \in \mathbb{Z}}$ where $\alpha_j \in \Delta_f \cup -\Delta_f$

\[
\Delta_f = \{(0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1, 1),
(0, 0, 0, 0, 1, 1, 0), (0, 0, 0, 0, 1, 1, 1), (0, 0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1, 1),
(0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0),
(0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0),
(0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0),
(0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0),
(0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0),
\}

The root system decomposes onto 7 $C$-series. Taking into account the obvious symmetry of the graph $\tilde{E}_6$ we need to consider only three of them containing vectors
(1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1) correspondingly. These C-series are

$K_1 = \{(0, -2, -1, -2, -1, -2, -3), (0, -1, -1, -2, -1, -2, -3), (0, -1, -1, -1, -1, -1, -1), (0, -1, 0, 0, 0, 0, -1), (0, 0, -1, -1, -1, -1, -1), (0, 0, 0, -1, 0, -1, -1), (0, 0, 0, 1, 0, 1, 1), (0, 0, 1, 0, 0, 0, 0), (0, 1, 1, 1, 1, 1, 1), (0, 1, 1, 2, 1, 2, 3), (0, 2, 1, 2, 1, 2, 3)\} + \delta \mathbb{Z}$

$K_2 = \{(0, -1, -1, -2, -1, -2, -2), (0, -1, -1, -1, -1, -2), (0, -1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), (0, 1, 1, 1, 1, 1, 2), (0, 1, 1, 2, 1, 2, 2)\} + \delta \mathbb{Z}$

$K_3 = \{(0, -1, -1, 0, -1, -1), (0, 0, 0, 0, 0, -1), (0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 1, 0, 1)\} + \delta \mathbb{Z}$

### 3 Representations of algebras generated by projections.

Let us consider a tree $G$ with vertices $\{g_i, i = 0, \ldots, k + l + m\}$ and edges $\gamma_{g_i g_j}$, see the figure.

![Diagram of a tree with vertices and edges](attachment:image.png)

We will establish a connection between *-representations of the *-algebra $A_{G, \chi}$ and locally-scalar representations of the graph $G$, see [1].

Let us remark that we can assume that each set $M_1, M_2, \ldots, M_n$ contains zero (this can be achieved by a translation). Henceforth, $M_j = \alpha^{(j)} \cup \{0\}$. For three operators we will use $\alpha$, $\beta$, $\delta$ instead of $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$. Thus vector $\chi$ will be $(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_l, \delta_1, \delta_2, \ldots, \delta_m, \gamma)$.

**Definition 1** An irreducible finite dimensional *-representation $\pi$ of the algebra $A_{G, \chi}$ such that $\pi(p_i) \neq 0(1 \leq i \leq k)$, $\pi(q_j) \neq 0(1 \leq j \leq l)$, $\pi(s_d) \neq 0(1 \leq d \leq m)$, and $\sum_{i=1}^{k} \pi(p_i) \neq I, \sum_{j=1}^{l} \pi(q_j) \neq I, \sum_{d=1}^{m} \pi(s_d) \neq I$, will be called non-degenerate. By $\text{Rep}A_{G, \chi}$ we will denote the full subcategory of non-degenerate representations in the category $\text{Rep}A_{G, \chi}$ of *-representations of the *-algebra $A_{G, \chi}$ in the category $\mathcal{H}$ of Hilbert spaces.

Let $\pi$ be a *-representation of $A_{G, \chi}$ on a Hilbert space $H_0$. Put $P_i = \pi(p_i)$, $1 \leq i \leq k$, $Q_j = \pi(q_j)$, $1 \leq j \leq l$, $S_t = \pi(s_t)$, $1 \leq t \leq m$. Let $H_{p_i} = 3m P_i$, $H_{q_j} = 3m Q_j$, $H_{s_t} = 3m S_t$. Denote by $\Gamma_{p_i}$, $\Gamma_{q_j}$, $\Gamma_{s_t}$ the corresponding natural isometries $H_{p_i} \rightarrow H_0$, $H_{q_j} \rightarrow H_0$, $H_{s_t} \rightarrow H_0$. Then, in particular, $\Gamma_{p_i}^* \Gamma_{p_i}$ is
the identity operator on $H_{p_i}$ and $\Gamma_{p_i} \Gamma_{p_i}^* = P_i$. Similar equalities hold for the operators $\Gamma_{q_i}$ and $\Gamma_{s_i}$. Using $\pi$ we construct a locally-scalar representation $\Pi$ of the graph $G$.

Let $\Gamma_{ij} : H_j \to H_i$ denote the operator adjoint to $\Gamma_{ji} : H_i \to H_j$, i.e. $\Gamma_{ij} = \Gamma_{ji}^*$. Put

$$
\Pi(g_0) = H^{g_0} = H_0,
\Pi(g_k) = H^{g_k} = H_{p_1} \oplus H_{p_2} \oplus \ldots \oplus H_{p_k},
\Pi(g_{k-1}) = H^{g_{k-1}} = H_{p_2} \oplus \ldots \oplus H_{p_{k-1}} \oplus H_{p_k},
\Pi(g_{k-2}) = H^{g_{k-2}} = H_{p_2} \oplus H_{p_3} \oplus \ldots \oplus H_{p_{k-1}},
\ldots
$$

In these equalities the summands are omitted from the left and the right in turns. Analogously, we define subspaces $\Pi(g_i)$ for $i = k+1, \ldots, k+l$ and $i = k+l+1, \ldots, k+l+m$. Define the operators $\Gamma_{g_0,g_i} : H^{g_i} \to H^{g_0}$, where $i \in \{k, k+l, k+l+m\}$, by the block-diagonal matrices

$$
\Gamma_{g_0,g_k} = [\sqrt{\alpha_1} \Gamma_{p_1} | \sqrt{\alpha_2} \Gamma_{p_2} | \ldots | \sqrt{\alpha_k} \Gamma_{p_k}],
\Gamma_{g_0,g_{k+1}} = [\sqrt{\beta_1} \Gamma_{q_1} | \sqrt{\beta_2} \Gamma_{q_2} | \ldots | \sqrt{\beta_l} \Gamma_{q_l}],
\Gamma_{g_0,g_{k+l+m}} = [\sqrt{\delta_1} \Gamma_{s_1} | \sqrt{\delta_2} \Gamma_{s_2} | \ldots | \sqrt{\delta_m} \Gamma_{s_m}].
$$

Now we define the representation $\Pi$ on the edges $\gamma_{g_0,g_k}$, $\gamma_{g_0,g_{k+1}}$, $\gamma_{g_0,g_{k+l+m}}$ by the rule

$$
\Pi(\gamma_{g_0,g_k}) = \{\Gamma_{g_0,g_k}, \Gamma_{g_k,g_0}\},
\Pi(\gamma_{g_0,g_{k+1}}) = \{\Gamma_{g_0,g_{k+1}}, \Gamma_{g_{k+1},g_0}\},
\Pi(\gamma_{g_0,g_{k+l+m}}) = \{\Gamma_{g_0,g_{k+l+m}}, \Gamma_{g_{k+l+m},g_0}\}.
$$

It is easy to see that

$$\Gamma_{g_0,g_k} \Gamma_{g_k,g_0} = \Gamma_{g_0,g_{k+1}} \Gamma_{g_{k+1},g_0} = \Gamma_{g_0,g_{k+l+m}} \Gamma_{g_{k+l+m},g_0} = \gamma_{I_{H^{g_0}}}. $$

Let $O_{H,0}$ denote the operators from the zero space to $H$, and $O_{0,H}$ denote the zero operator from $H$ into the zero subspace. For the operators $\Gamma_{g_j,g_i} : H^{g_i} \to H^{g_j}$ with $i, j \neq 0$, put

$$
\Gamma_{g_{k-1},g_k} = O_{0,H_{p_1}} \oplus \sqrt{\alpha_1} - \alpha_2 I_{H_{p_2}} \oplus \sqrt{\alpha_1} - \alpha_3 I_{H_{p_3}} \oplus \ldots \oplus \sqrt{\alpha_1} - \alpha_k I_{H_{p_k}},
\Gamma_{g_{k-1},g_{k-2}} = \sqrt{\alpha_2 - \alpha_k I_{H_{p_2}}} \oplus \sqrt{\alpha_3 - \alpha_k I_{H_{p_3}}} \oplus \ldots \oplus \sqrt{\alpha_{k-1} - \alpha_k I_{H_{p_{k-1}}} \oplus O_{H_{p_k},0}},
\Gamma_{g_{k-3},g_{k-2}} = O_{0,H_{p_2}} \oplus \sqrt{\alpha_2 - \alpha_3 I_{H_{p_3}}} \oplus \sqrt{\alpha_2 - \alpha_4 I_{H_{p_4}}} \oplus \ldots \oplus \sqrt{\alpha_2 - \alpha_{k-1} I_{H_{p_{k-1}}}},
\ldots
$$

The corresponding operators for the rest of the edges of $G$ can be constructed analogously. One can check that the operators $\Gamma_{g_j,g_i}$, where $\Gamma_{g_j,g_i} = \Gamma_{g_i,g_j}^*$, define a locally-scalar representation of the graph $G$ with the following character $f$:
\[ f(g_k) = \alpha_1, \quad f(g_{k+i}) = \beta_1, \quad f(g_{k+l+m}) = \delta_1, \]
\[ f(g_{k-1}) = \alpha_1 - \alpha_k, \quad f(g_{k+l-1}) = \beta_1 - \beta_l, \quad f(g_{k+l+m-1}) = \delta_1 - \delta_m, \]
\[ f(g_{k-2}) = \alpha_2 - \alpha_k, \quad f(g_{k+l-2}) = \beta_2 - \beta_l, \quad f(g_{k+l+m-2}) = \delta_2 - \delta_m, \]
\[ f(g_{k-3}) = \alpha_2 - \alpha_{k-1}, \quad f(g_{k+l-3}) = \beta_2 - \beta_{l-1}, \quad f(g_{k+l+m-3}) = \delta_2 - \delta_{m-1}, \]
\[ f(g_{k-4}) = \alpha_3 - \alpha_{k-1}, \quad f(g_{k+l-4}) = \beta_3 - \beta_{l-1}, \quad f(g_{k+l+m-4}) = \delta_3 - \delta_{m-1}, \]
\[ \ldots \]

\[ f(g_0) = \gamma. \]

And vice versa, if a locally-scalar representation of the graph \( G \) with the character \( f(g_i) = x_i \in \mathbb{R}^* \) corresponds to a non-degenerate representation of \( \mathcal{A}_{G,X} \), then one can check that
\[ \alpha_1 = x_k, \]
\[ \alpha_k = x_k - x_{k-1}, \alpha_2 = x_k - x_{k-1} + x_{k-2}, \]
\[ \alpha_{k-1} = x_k - x_{k-1} + x_{k-2} - x_{k-3}, \]
\[ \alpha_3 = x_k - x_{k-1} + x_{k-2} - x_{k-3} + x_{k-4}, \]
\[ \ldots \]

Here \( x_j = 0 \) if \( j \leq 0 \). Analogously one can find \( \beta_j \) and \( \delta_l \). We will denote \( \Pi \) by \( \Phi(\pi) \).

Let \( \pi \) and \( \tilde{\pi} \) be non-degenerate representations of the algebra \( \mathcal{P}_{\alpha,\beta,\delta,\gamma} \) and \( C_0 \) an intertwining operator for these representations; this is a morphism from \( \pi \) to \( \tilde{\pi} \) in the category \( \text{Rep} G \), \( C_0 : H_0 \to H_0, C_0 \pi = \tilde{\pi} C_0 \). Put
\[ C_{p_i} = \Gamma_{p_i}^* C_0 \Gamma_{p_i}, C_{p_i} : H_{p_i} \to \tilde{H}_{p_i}, 1 \leq i \leq k, \]
\[ C_{q_j} = \Gamma_{q_j}^* C_0 \Gamma_{q_j}, C_{q_j} : H_{q_j} \to \tilde{H}_{q_j}, k + 1 \leq j \leq k + l, \]
\[ C_{s_t} = \Gamma_{s_t}^* C_0 \Gamma_{s_t}, C_{s_t} : H_{s_t} \to \tilde{H}_{s_t}, k + l + 1 \leq t \leq k + l + m, \]
\[ \ldots \]

Put
\[ C^{(g_0)} = C_0 : H^{(g_0)} \to \tilde{H}^{(g_0)}, \]
\[ C^{(g_k)} = C_{p_1} \oplus C_{p_2} \oplus \ldots \oplus C_{p_k} : H^{(g_k)} \to \tilde{H}^{(g_k)}, \]
\[ C^{(g_{k-1})} = C_{p_2} \oplus \ldots \oplus C_{p_{k-1}} \oplus C_{p_k} : H^{(g_{k-1})} \to \tilde{H}^{(g_{k-1})}, \]
\[ C^{(g_{k-2})} = C_{p_2} \oplus C_{p_3} \oplus \ldots \oplus C_{p_{k-1}} \oplus C_{p_k} : H^{(g_{k-2})} \to \tilde{H}^{(g_{k-2})}, \]
\[ \ldots \]

Analogously one can construct the operators \( C^{(g_i)} \) for \( i \in \{k + l, \ldots, k + l + m\} \). It is routine to check that the operators \( \{C^{(g_i)}\}_{0 \leq i \leq k + l + m} \) intertwine the representations \( \Pi = \Phi(\pi) \) and \( \tilde{\Pi} = \Phi(\tilde{\pi}) \). Put \( \Phi(C_0) = \{C^{(g_i)}_0\}_{0 \leq i \leq k + l + m} \). Thus we have defined a functor \( \Phi : \text{Rep} \mathcal{A}_{G,X} \to \text{Rep} G \), see [4]. Moreover, the functor \( \Phi \) is univalent and full.
Let $\tilde{\text{Rep}}(G, d, f)$ be the full subcategory of irreducible representations of $\text{Rep}(G, d, f)$. $\Pi \in \text{Ob} \tilde{\text{Rep}}(G, d, f)$, $f(g_i) = x_i \in \mathbb{R}^+$, $d(g_i) = d_i \in \mathbb{N}_0$, where $f$ is the character of $\Pi$, $d$ its dimension. It easy to verify that the representation $\Pi$ is isomorphic (unitary equivalent) to an irreducible representation from the image of the functor $\Phi$ if and only if

1. $0 < x_1 < x_2 < \ldots < x_k; 0 < x_{k+1} < x_{k+2} < \ldots < x_{k+l}$; \hfill (3.2)  
2. $0 < x_{k+l+1} < x_{k+l+2} < \ldots < x_{k+l+m}$; \hfill (3.3)  
3. $0 < d_1 < d_2 < \ldots < d_k < d_0; 0 < d_{k+1} < d_{k+2} < \ldots < d_{k+l}$; \hfill (3.4)  
4. $d_{k+l} < d_0; 0 < d_{k+l+1} < d_{k+l+2} < \ldots < d_{k+l+m} < d_0$. \hfill (3.5)

(All matrices of the representation of the graph $G$, except for $\Gamma_{g_0,g_k}, \Gamma_{g_k,g_0}, \Gamma_{g_0,g_{k+l}}, \Gamma_{g_{k+l},g_0}, \Gamma_{g_0,g_{k+l+m}}, \Gamma_{g_{k+l+m},g_0}$, can be brought to the "canonical" form (3.1) by admissible transformations. Then the rest of the matrices will naturally be partitioned into blocks, which gives the matrices $\Gamma_{p_i}, \Gamma_{q_i}, \Gamma_{s_i}$, and thus the projections $P_i, Q_i, R_i$.)

An irreducible representation $\Pi$ of the graph $G$ satisfying conditions (3.2)–(3.5) will be called non-degenerate. Let

$$\dim H_{p_i} = n_i, 1 \leq i \leq k;$$
$$\dim H_{q_j} = n_{k+j}, 1 \leq j \leq l;$$
$$\dim H_{s_t} = n_{k+l+t}, 1 \leq t \leq m;$$
$$\dim H_0 = n_0.$$

The vector $n = (n_0, n_1, \ldots, n_{k+l+m})$ is called the generalized dimension of the representation $\pi$ of the algebra $\mathcal{A}_{G,\chi}$. Let $\Pi = \Phi(\pi)$ for a non-degenerate representation of the algebra $\mathcal{A}_{G,\chi}$, $d = (d_0, d_1, \ldots, d_{k+l+m})$ be the dimension of $\Pi$. It is easy to see that

$$n_1 + n_2 + \ldots + n_k = d_k,$$
$$n_2 + \ldots + n_{k-1} + n_k = d_{k-1},$$
$$n_2 + \ldots + n_{k-1} = d_{k-2},$$
$$n_3 + \ldots + n_{k-2} + n_{k-1} = d_{k-3},$$
$$\ldots$$

Thus

$$n_1 = d_k - d_{k-1},$$
$$n_k = d_{k-1} - d_{k-2},$$
$$n_2 = d_{k-2} - d_{k-3},$$
$$\ldots$$

(3.6)

Analogously one can find $n_{k+1}, \ldots, n_{k+l}$ from $d_{k+1}, \ldots, d_{k+l}$ and $n_{k+l+1}, \ldots, n_{k+l+m}$ from $d_{k+l+1}, \ldots, d_{k+l+m}$.
Denote by $\overrightarrow{\text{Rep}} G$ the full subcategory in $\text{Rep} G$ of non-degenerate locally-scalar representations of the graph $G$. As a corollary of the previous arguments we obtain the following theorem.

**Theorem 1** Let $\mathcal{A}_{G, \chi}$ be associated with a graph $G$. The functor $\Phi$ is an equivalence of the categories $\overrightarrow{\text{Rep}} \mathcal{A}_{G, \chi}$ of non-degenerate $*$-representations of the algebra $\mathcal{A}_{G, \chi}$ and the category $\text{Rep} G$ of non-degenerate locally-scalar representations of the graph $G$.

Let us define the Coxeter functors for the $*$-algebras $\mathcal{A}_{G, \chi}$, by putting $\Psi = \Phi^{-1} \Phi \Phi$ and $\mathcal{A}_{G, \chi} = \Phi^{-1} \Phi \Phi$. Now we can use the results of \cite{1} to give a description of representation of the $*$-algebra $\mathcal{A}_{G, \chi}$.

Note that to find formulae of the locally-scalar representations of a given extended Dynkin graph we need to consider two principally different cases: the case when the vector of generalized dimension is a real root and the case when it is an imaginary root. In the latter case the vector of parameters $\chi$ must satisfy (in order for the representations to exist) a certain linear relation ("traces equality"). Hence $\chi$ must belong to a certain hyperplane $h_G$ which depends only on the graph $G$. A simple calculation yields that for extended Dynkin graphs $\tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ these hyperplanes are the following:

| $D_4$ | $E_6$ | $E_7$ | $E_8$ |
|-------|-------|-------|-------|
| $\alpha_1 + \beta_1 + \delta_1 + \eta_1 = 2\gamma$ | $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \delta_1 + \delta_2 = 3\gamma$ | $\alpha_1 + \alpha_2 + \beta_3 + \beta_4 + 2\delta_1 = 4\gamma$ | $2(\alpha_1 + \alpha_2) + \beta_1 + \beta_2 + \beta_3 + \beta_4 + 3\delta_1 = 6\gamma$ |

It is known (see \cite{12}) that in case $\chi \in h_G$ the dimension of any irreducible representation is bounded (by 2 for $\tilde{D}_4$, by 3 for $\tilde{E}_6$, by 4 for $\tilde{E}_7$ and by 6 for $\tilde{E}_8$). Thus in case of the hyperplane we can describe the set of admissible parameters $\chi$ using Horn's inequalities. In case $\chi \notin h_G$ the dimension of any irreducible locally-scalar representation is a real root. In what follows we will relay on the following result due to V.Ostrovskij \cite{11}.

**Theorem 2** Let $\pi$ be an irreducible $*$-representation of the algebra $\mathcal{A}_{G, \chi}$ associated with Extended Dynkin graph $G$ and $\tilde{\pi}$ corresponding representation of the graph $G$. Then either generalized dimension $d$ of $\tilde{\pi}$ is a singular root or vector-parameter $\chi \in h_G$.

Hence we will solve the spectral problem if we describe the parameters $\chi$ for which there are locally-scalar representations with vector of generalized dimension being real singular roots and parameters belonging to hyperplane $h_G$ for which there exist representations of the algebra. Firstly we will consider the case $\chi \notin h_G$.

In the next section we will do the following: we know how to construct all irreducible locally-scalar representations of Dynkin graphs with the aid of Coxeter reflection functors starting from the simplest ones. In particular, we can find their dimensions and characters \cite{9}. Next we single out non-generate representations and apply the equivalence functor $\Phi$, see Theorem 1.
4 Algebras associated with extended Dynkin graphs.

Thus in "generic" situation the powers of the Coxeter map $C^k = (CC)^k$ acts on vectors (in quiver notations) by the following formulas:

if $k \equiv 0 \pmod{3}$

$$
\begin{pmatrix}
11 + 3(-1)^k & 4 & -1 + 3(-1)^k & 4 & -1 + 3(-1)^k & 4 & 3(3 - (-1)^k) \\
-4 & 4 & -4 & -8 & -4 & -8 & -12 \\
-1 + 3(-1)^k & 4 & 11 + 3(-1)^k & 4 & -1 + 3(-1)^k & 4 & 3(3 - (-1)^k) \\
-4 & -8 & -4 & 4 & -4 & -8 & -12 \\
-1 + 3(-1)^k & 4 & -1 + 3(-1)^k & 4 & 11 + 3(-1)^k & 4 & 3(3 - (-1)^k) \\
-4 & -8 & -4 & -8 & -4 & 4 & -12 \\
3(3 - (-1)^k) & 12 & 3(3 - (-1)^k) & 12 & 3(3 - (-1)^k) & 12 & 3(9 + (-1)^k)
\end{pmatrix}
$$

if $k \equiv 1 \pmod{3}$

$$
\begin{pmatrix}
1 + (-1)^k & 4 & 1 + (-1)^k & 0 & 1 + (-1)^k & 0 & 3 - (-1)^k \\
-4 & -4 & 0 & 0 & 0 & 0 & -4 \\
1 + (-1)^k & 0 & 1 + (-1)^k & 4 & 1 + (-1)^k & 0 & 3 - (-1)^k \\
0 & 0 & -4 & -4 & 0 & 0 & -4 \\
1 + (-1)^k & 0 & 1 + (-1)^k & 0 & 1 + (-1)^k & 4 & 3 - (-1)^k \\
0 & 0 & 0 & 0 & -4 & -4 & -4 \\
3 - (-1)^k & 4 & 3 - (-1)^k & 4 & 3 - (-1)^k & 4 & 9 + (-1)^k
\end{pmatrix}
$$

if $k \equiv 2 \pmod{3}$

$$
\begin{pmatrix}
-5 + 3(-1)^k & -4 & 7 + 3(-1)^k & 8 & 7 + 3(-1)^k & 8 & 3(3 - (-1)^k) \\
4 & -4 & -8 & -4 & -8 & -4 & -12 \\
7 + 3(-1)^k & 8 & -5 + 3(-1)^k & -4 & 7 + 3(-1)^k & 8 & 3(3 - (-1)^k) \\
-8 & -4 & 4 & -4 & -8 & -4 & -12 \\
7 + 3(-1)^k & 8 & 7 + 3(-1)^k & 8 & -5 + 3(-1)^k & -4 & 3(3 - (-1)^k) \\
-8 & -4 & -8 & -4 & 4 & -4 & -12 \\
3(3 - (-1)^k) & 12 & 3(3 - (-1)^k) & 12 & 3(3 - (-1)^k) & 12 & 3(9 + (-1)^k)
\end{pmatrix}
$$

Let us consider the first C-series which contains simple root $(1,0,0,0,0,0,0)$. It decomposes onto $12\delta$-series. The map $C^6$ takes simple representation of the quiver to the non-degenerate one. To obtain further representations of this series we need to apply $C^k$ written explicitly above.

Let $M_d$ denote transition matrix from generalized dimension of the quiver to the generalized dimension of the algebra and $M_f$ denote the transition matrix from algebra parameters to quiver parameters.

We will write $v \geq 0$ meaning that $v_j > 0$ for $1 \leq j \leq 6$ and $v_7 = 0$.

**Theorem 3** Consider the following vectors $v_1 = (1,0,0,0,0,0,0), v_2 = (1,1,0,0,0,0,0), v_3 = (0,1,0,0,0,0,1), v_4 = (0,0,0,1,0,1,1), v_5 = (0,0,1,1,1,1,1), v_6 = (0,1,1,1,1,1,1), v_7 = (1,1,0,1,0,1,2), v_8 = (1,2,0,1,0,1,2), v_9 = (1,2,1,1,1,1,2), v_{10} = (1,1,1,2,1,2,2), v_{11} = (0,1,1,2,1,2,3), v_{12} = (0,2,1,2,1,2,3)$. Put $d_k = v_k \mod 12 + \left\lfloor \frac{k}{12} \right\rfloor \delta$. 

13
The algebra \( A_{\tilde{E}_{6,\chi}} \) associated with the Dynkin graph \( \tilde{E}_6 \) has an irreducible non-degenerate representation in generalized dimensions \( M_d k \) for all \( k \geq \frac{15}{2} \) iff the parameters \( \chi \) satisfy the following condition: \( D_k \chi \geq 0 \) where matrix

\[
D_k = D_1 \begin{cases} 
C^{6-s} M_f & \text{if } k = 2s \\
C^{6-s} C M_f & \text{if } k = 2s + 1 
\end{cases}
\]

and

\[
D_1 = \begin{pmatrix}
3 & -3 & 1 & -3 & 1 & -3 & 5 \\
1 & -1 & 1 & -2 & 1 & -1 & 2 \\
3 & -3 & 2 & -3 & 1 & -2 & 4 \\
1 & -1 & 1 & -1 & 1 & -2 & 2 \\
3 & -3 & 1 & -2 & 2 & -3 & 4 \\
5 & -4 & 2 & -4 & 2 & -4 & 6 \\
2 & -2 & 1 & -2 & 1 & -2 & 3 
\end{pmatrix}
\]

**Theorem 4** Consider the following vectors \( v_1 = (0, 1, 0, 0, 0, 0, 0), v_2 = (1, 1, 0, 0, 0, 0, 1), v_3 = (1, 1, 0, 1, 0, 1, 1), v_4 = (0, 1, 1, 1, 1, 1, 2), v_5 = (0, 1, 1, 1, 1, 1, 2), v_6 = (1, 1, 1, 2, 1, 2, 3). Put \( d_k = v_k \mod 6 + \lfloor \frac{k}{4} \rfloor \delta \).

The algebra \( A_{\tilde{E}_{6,\chi}} \) associated with the Dynkin graph \( \tilde{E}_6 \) has an irreducible non-degenerate representation in generalized dimensions \( M_d d_k \) for all \( k \geq 8 \) iff the parameters \( \chi \) satisfy the following condition: \( D_k \chi \geq 0 \) where matrix

\[
D_k = D_2 \begin{cases} 
C^{3-s} M_f & \text{if } k = 2s \\
C^{3-s} C M_f & \text{if } k = 2s + 1 
\end{cases}
\]

where

\[
D_2 = \begin{pmatrix}
0 & 0 & 1 & -1 & 1 & -1 & 1 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & -1 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
1 & -1 & 1 & -1 & 0 & -1 & 2 \\
2 & -1 & 1 & -2 & 1 & -2 & 3 \\
1 & -1 & 1 & -2 & 1 & -2 & 3 
\end{pmatrix}
\]

**Theorem 5** Consider the following vectors \( v_1 = (0, 0, 0, 0, 0, 0, 1), v_2 = (0, 1, 0, 1, 0, 1, 1), v_3 = (1, 1, 1, 1, 1, 1, 2), v_4 = (1, 2, 1, 2, 1, 2, 2). Put \( d_k = v_k \mod 4 + \lfloor \frac{k}{4} \rfloor \delta \).

The algebra \( A_{\tilde{E}_{6,\chi}} \) associated with the Dynkin graph \( \tilde{E}_6 \) has an irreducible non-degenerate representation in generalized dimensions \( M_d d_k \) for all \( k \geq 5 \) iff the parameters \( \chi \) satisfy the following condition: \( D_k \chi \geq 0 \) where matrix

\[
D_k = D_3 \begin{cases} 
C^{s-2} M_f & \text{if } k = 2s \\
C^{s-2} C M_f & \text{if } k = 2s + 1 
\end{cases}
\]

14
where

\[
D_3 = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 1 & -1 & 1 \\
-1 & 2 & -1 & 2 & -1 & 2 \\
\end{pmatrix}
\]

**Theorem 6** Up to symmetry all irreducible non-degenerate *-representations of the algebra \(A_{\tilde{E}_6, \chi}\) with vector of parameters not on a hyperplane are described in theorem \(\Delta\).

5 The hyperplane case.

Applying Horn's inequalities we get the following theorem:

**Theorem 7** The algebra \(A_{\tilde{E}_6, \chi}\) associated with the Dynkin graph \(\tilde{E}_6\) has *-representation in generalized dimensions \((1, 1; 1, 1; 1, 1; 3)\) iff the parameters \(\chi\) satisfy the following conditions:

\[
2(\alpha_1 + \beta_1) > \alpha_2 + \beta_2 + \delta_1 + \delta_2, 2(\alpha_1 + \delta_1) > \alpha_2 + \beta_1 + \beta_2 + \delta_2, \\
2(\beta_1 + \delta_1) > \alpha_1 + \alpha_2 + \beta_2 + \delta_2, \alpha_1 + \alpha_2 + \beta_1 + \delta_1 > 2(\beta_2 + \delta_2), \\
2(\alpha_2 + \beta_2 + \delta_1) > \alpha_1 + \beta_1 + \delta_2, \alpha_1 + \beta_1 + \beta_2 + \delta_1 > 2(\alpha_2 + \delta_2), \\
2(\alpha_2 + \beta_1 + \delta_2) > \alpha_1 + \beta_2 + \delta_1, 2(\alpha_1 + \beta_2 + \delta_2) > \alpha_2 + \beta_1 + \delta_1, \\
\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \delta_2 > 2\delta_1, \alpha_1 + \beta_1 + \delta_1 + \delta_2 > 2(\alpha_2 + \beta_2), \\
\alpha_1 + \alpha_2 + \beta_2 + \delta_1 + \delta_2 > 2\beta_1, \alpha_2 + \beta_1 + \beta_2 + \delta_1 + \delta_2 > 2\alpha_1.
\]

The following theorem gives a complete solution to the spectral problem in case of Dynkin graph \(\tilde{E}_6\).

**Theorem 8** A non-degenerate *-representation of the algebra \(A_{\tilde{E}_6, \chi}\) exists iff \(\chi\) satisfies at the conditions of at least one of the theorems \(\Delta\).

**Proof:** The case \(\chi \notin h_{\tilde{E}_6}\) is covered by theorems \(\Delta\).

If \(\chi \in h_{\tilde{E}_6}\) then for any irreducible *-representation \(\pi\) of the algebra \(A_{\tilde{E}_6, \chi}\) the generalized dimension of the corresponding representation \(\hat{\pi}\) of the graph \(\tilde{E}_6\) is the minimal imaginary root \(\delta = (1, 2; 1, 2; 1, 2; 3)\) (then dimension of \(\pi\) is \((1, 1; 1, 1; 1, 1; 3)\)) or is a real root \(d\).

We have already described the case of singular roots in theorems \(\Delta\). But \(d\) cannot be regular. Because otherwise application of the Coxeter functors would produce irreducible representations in all dimensions of the form \(d + k\delta\) where \(k\) is a positive integer. This would be a contradiction with the fact that \(A_{\tilde{E}_6, \chi}\) is PI-algebra for \(\chi \in h_{\tilde{E}_6}\). Thus we have exhausted all the possibilities.
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