LAGRANGIANS FOR THE W-ALGEBRA MODELS

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Abstract

The field algebra of the minimal models of W-algebras is amenable to a very simple description as a polynomial algebra generated by few elementary fields, corresponding to order parameters. Using this description, the complete Landau-Ginzburg lagrangians for these models are obtained. Perturbing these lagrangians we can explore their phase diagrams, which correspond to multicritical points with $D_n$ symmetry. In particular, it is shown that there is a perturbation for which the phase structure coincides with that of the IRF models of Jimbo et al.

1 Introduction

The classification of two-dimensional conformal field theories (2dCFT) is a flourishing branch of mathematical physics. It can be applied in essentially two different directions. One is string theory; the other is phase transitions in two dimensions and, hopefully, also higher dimensions. With regards to this second application, the information provided by 2dCFT may seem excessive. In phase transition physics, before addressing the question of critical exponents and, therefore, dimensions of fields, one is usually more interested in determining the phase diagram. For this purpose, the essential information consists of the order parameters (with their symmetry) and the relevant fields formed out of them. We are used to seeing this information coming from the Landau potential, or its field-theory version, the Landau-Ginzburg lagrangian. This

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lagrangian is not a datum of 2dCFT but it can be obtained from it, as was shown by A.B. Zamolodchikov [1] in the simple case of the unitary series of minimal models. All these models have only one order parameter and $Z_2$ symmetry. In order to describe phase transitions with higher symmetry, one is compelled to consider 2dCFT with an extended algebra. Among them, the minimal models of the W-algebras [2] are the natural generalization of the Virasoro minimal models.

The Zamolodchikov’s procedure to associate lagrangians to 2dCFT begins by giving a polynomial structure to the algebra of primary fields. He found that all the relevant primary fields of a minimal model can be expressed as composite (powers) of the most relevant one (elementary field), which plays the role of order parameter. This is a necessary condition for the existence of a lagrangian and it is actually sufficient to guess it. Nevertheless, there is a direct way to obtain it, relying on the fact that the composite field next to the most relevant field has to be identified with the descendant of the elementary field, giving therefore the equation of motion. In this paper, we will generalize this procedure to the minimal models of the W-algebras. An attempt in this direction was already made by I. Koh and S. Yang [3], but they did not obtain the complete structure of the algebra of primary fields and their lagrangians missed some important terms.

The Theory of Catastrophes is relevant in the context of Landau potentials, as a mathematical framework in which many intuitive concepts of phase transitions can be precisely formulated. Its classificatory power has already found wide application in 2d $N = 2$ superconformal theories. However, it is still a rather specialized tool and we should not assume that the reader is familiar with it. Therefore, we will put it aside, though making some remarks in footnotes.

The paper is divided in three parts. We begin with the simplest case, namely the $W_{(3)}$ algebra minimal models. In the first part, we explicitly identify all the relevant fields as powers of two elementary ones and therefore endow the algebra with a polynomial structure. The equations of motion and the lagrangian ensue from the identification of further powers of the elementary fields. In the second part, the phase structure that results from these potentials is briefly analyzed. The third part is devoted to the general W-algebra models. To deal with them one has to take into account some new features that already appear for $W_{(4)}$. The first model of the $W_{(4)}$ series has central charge $c = 1$ and belongs to the Ashkin-Teller critical line [4]. For its importance we take it as an example to study those new features. We finally discuss the phase structure of the general models and their relation with the solvable statistical models defined by the japanese group [5] (JMO models).
2 Structure of the field algebra of the $W_{(3)}$ models

The minimal models of the $W_{(3)}$ algebra are constructed in analogy with the current Virasoro minimal models, but using instead a two-component free field [6]. The procedure relies on the Dotsenko-Fateev screening charge method. There are four screening charges; the primary fields are therefore labelled by a $2 \times 2$ matrix

$$\Phi\left(\begin{array}{cc}
n & m \\
n' & m'
\end{array}\right).$$

Every unitary model is still determined by one integer $p$ and noted $W_{(3)}^p$. This model contains $p(p - 1)^2(p - 2)/12$ independent primary fields and has $D_3$ symmetry. The fusion rules are derived from the neutrality condition including definite numbers of screening charges,

$$\alpha(3) = \alpha(1) + \alpha(2) + N\alpha_1 e_1 + N'\alpha_2 e_2 + M\alpha_{-1} e_1 + M'\alpha_{-2}, \quad (1)$$

with

$$\alpha(i) \equiv \alpha\left(\begin{array}{c}
n(i) \\
n'(i) \\
m(i) \\
m'(i)
\end{array}\right) = ([1 - n(i)]\alpha_1 + [1 - m(i)]\alpha_{-1})\omega_1 +$$

$$([1 - n'(i)]\alpha_2 + [1 - m'(i)]\alpha_{-2})\omega_2. \quad (2)$$

Here $e_i$ and $\omega_i$ are the positive roots and fundamental weights of $SU(3)$ ($A_2$ algebra), respectively. Note that Eq. (1) expresses the Clebsch-Gordan decomposition of the tensor product of two $SU(3) \otimes SU(3)$ representations with highest weights $-\alpha(1)$ and $-\alpha(2)$. The solution for $n(3), n'(3), m(3), m'(3)$ is

$$n(3) = n(1) + n(2) - 1 - 2N + N'$$

$$n'(3) = n'(1) + n'(2) - 1 - 2N' + N$$

and similar equations for $m(3), m'(3)$.

2.1 Elementary and composite fields

The most relevant fields are

$$\sigma = \Phi\left(\begin{array}{cc}
2 & 2 \\
1 & 1
\end{array}\right), \quad \bar{\sigma} = \Phi\left(\begin{array}{cc}
1 & 1 \\
2 & 2
\end{array}\right),$$

which represent the $D_3$ spin density and its conjugate [8]. They are the obvious candidates for elementary fields. It is straightforward to prove that a certain number of the next most relevant fields are obtained from them by the Zamolodchikov’s method.
with the fusion rule (1). Besides, they arrange themselves in a triangular structure corresponding to the $SU(3)$ lattice of dominant weights. Before proceeding further, let us recall Zamolodchikov’s field identifications in more detail. The primary field content of the Virasoro minimal model $M_p$ can be visualized on a grid of dimension $p \times (p - 1)$ symmetric with respect to the center. The elementary field is placed at $(2, 2)$ and its powers are placed on the main diagonal, up to $(p - 1, p - 1)$. The next power is at $(p, p - 2)$ or equivalently at $(1, 2)$. This is the lowest end of the second diagonal, which contains the remaining powers, up to the $2p - 4$th, corresponding to the least relevant field.

In the present case, the conformal grid is four-dimensional and the equivalent of the "main diagonal" is a two-dimensional triangular section of it. To be precise, this "main diagonal" exactly corresponds to the $SU(3)$ lattice of dominant weights of level $p - 3$, as it is proved by the identifications

$$\sigma^k \bar{\sigma}^l = \Phi^{\left( k + 1 \atop l + 1 \right) \left( k + 1 \atop l + 1 \right)},$$

(4)

with $0 \leq k + l \leq p - 3$, obtained from (3) for the trivial $N = N' = M = M' = 0$ case. The next power, $k + l = p - 2$, leads to essentially three different choices. For $l = 0$, $k = p - 2$, the appropriate choice is $N = 1, N' = M = M' = 0$, and

$$\sigma^{p-2} = \Phi^{\left( p - 3 \atop 2 \right) \left( p - 1 \atop 1 \right)} = \Phi^{\left( 2 \atop 1 \atop 1 \right)}.$$

(5)

This choice holds for the subsequent powers filling the triangle subtended from it up to the $2p - 6$th power, giving

$$\sigma^{p-2+k} \bar{\sigma}^l = \Phi^{\left( k + 2 \atop l + 1 \atop l + 1 \right) \left( k + 1 \atop l \atop l \right)},$$

(6)

with $0 \leq k + l \leq p - 4$. There are similar identifications for $\bar{\sigma}^{p-2}$ and the triangle conjugate to the one mentioned above, with $N = 0, N' = 1, M = M' = 0$. We are left with the middle triangle, which requires $N = 1, N' = 1, M = M' = 0$. A convenient form of the ensuing identifications is

$$\sigma^{p-k-2} \bar{\sigma}^{k+l-1} = \Phi^{\left( k \atop l + 1 \atop l \right) \left( k + 1 \atop l \atop l \right)},$$

(7)

with $1 \leq k, l$ and $k + l \leq p - 2$. In particular, for $k = 1, l = p - 3$,

$$(\sigma \bar{\sigma})^{p-3} = \Phi^{\left( 1 \atop p - 2 \atop p - 3 \right) \left( 1 \atop 2 \atop 2 \right)} = \Phi^{\left( 1 \atop 2 \atop 2 \right)}.$$

(8)
we get the least relevant field of the thermal subalgebra (containing the fields with \(n = n'\) and \(m = m'\)). This subalgebra is actually generated by its most relevant field, the energy density

\[ \epsilon = \sigma \bar{\sigma} = \Phi \left( \begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \end{array} \right). \]

Its successive powers span the subalgebra up to \(\epsilon^{p-3}\), identified above. This field will be important in the sequel. Its conformal dimension, as obtained with the general formulas [6], is \((p - 2)/(p + 1)\).

The three triangles containing the powers of the elementary field ranging from the \(p-2nd\) to the \(2p-6th\) form a two-dimensional section of the conformal grid equivalent to the second diagonal of the Virasoro case. The field identifications above are consistent across the borders, although in a nontrivial way. The fields on the lower side of the upper triangle are

\[ \sigma^{p-2} \bar{\sigma}^l = \Phi \left( \begin{array}{c} 2 \\ l + 1 \\ 1 \\ l + 1 \end{array} \right). \]  \(9\)

They must be compared to the products of sigma with the fields on the upper side of the middle triangle

\[ \sigma^{p-3} \bar{\sigma}^l = \Phi \left( \begin{array}{c} 1 \\ l + 1 \\ 2 \\ l \end{array} \right). \]  \(10\)

There is exact coincidence if the fusion with \(\sigma\) is performed according to equations [3] with \(M = 1, N = N' = M' = 0,\)

\[ \sigma (\sigma^{p-3} \bar{\sigma}^l) = \Phi \left( \begin{array}{c} 2 \\ l + 1 \\ 1 \\ l + 1 \end{array} \right). \]  \(11\)

We know that in the Virasoro case, the main and second diagonals of the conformal grid contain all the relevant fields of the model. We have here a similar property; namely, the big triangle with all the powers up to the \(2p-6th\) contains all the relevant fields of the \(W_p^3\) model, as can be checked with the dimension formula.

### 2.2 Field equations and Landau-Ginzburg lagrangians

The fields corresponding to the \(2p-6th\) power are specially important. The most relevant is \(\epsilon^{p-3}\). Pursuing the analogy with the Virasoro case, we expect that further products with \(\sigma\) or \(\bar{\sigma}\) must include the lowest dimension descendants, namely \(\partial \sigma\) or \(\partial \bar{\sigma}\), and therefore originate just field equations. However, in contrast with the Virasoro case, other fields already identified appear before these descendants in the regular part of the operator product expansion (OPE); they have to be kept in the field equations. This new feature was already remarked for the \(p = 4\) case in [4]. The OPE for \(\epsilon^{p-3}\) in
particular, can be written as

\[ \sigma(z) \epsilon^{p-3}(0) = \Phi \left( \frac{2}{1} \frac{2}{1} \right) \Phi \left( \frac{1}{1} \frac{2}{2} \right) = \frac{1}{z^{\Delta_{\epsilon^{p-3}}}} \sigma + \ldots \]

\[ + \frac{1}{z^{\Delta_{\sigma} + \Delta_{\epsilon^{p-3}} - \Delta_{\phi}}} \phi + \frac{1}{z^{\Delta_{\epsilon^{p-3}} - 1}} \partial \sigma + \ldots, \tag{12} \]

where we have shown the most singular term and the initial regular terms; numerical coefficients have been omitted. The field \( \phi \), yet to be determined, must have been already identified as a power of \( \sigma \) and \( \bar{\sigma} \), and needs to be such that the corresponding exponent of \( z \) is positive and smaller than \( 1 - \Delta_{\epsilon^{p-3}} \), that is to say

\[ \Delta_{\phi} < 1 + \Delta_{\sigma} = 1 + \frac{4}{3p(p+1)}. \]

We notice that the fusion rule Eq. 3 with \( M = 1, N = N' = M' = 0 \), gives

\[ \sigma \epsilon^{p-3} = \Phi \left( \frac{2}{1} \frac{1}{3} \right) = \Phi \left( \frac{p-3}{2} \frac{p-3}{1} \right) = \sigma^{p-4} \bar{\sigma}^{p-2} \tag{13} \]

with conformal dimension

\[ \Delta \left( \frac{2}{1} \frac{1}{3} \right) = \frac{3p(p-1) + 4}{3p(p+1)}. \tag{14} \]

This field therefore satisfies the condition and can be taken as \( \phi \).

With \( \phi \) identified as above, the OPE (12) yields the following field equation (Henceforth, the symbol \( \simeq \) will stand for equality up to numerical coefficients)

\[ \partial^2 \sigma \simeq \sigma^{p-4} \bar{\sigma}^{p-2} + \sigma (\sigma \bar{\sigma})^{p-3}. \tag{15} \]

This equation leads to the lagrangian

\[ \mathcal{L} \simeq \partial \sigma \partial \bar{\sigma} + \sigma^{p-4} \bar{\sigma}^{p-1} + (\sigma \bar{\sigma})^{p-2} + \text{c.c.} \]

\[ = \partial \sigma \partial \bar{\sigma} + (\sigma \bar{\sigma})^{p-4}(\sigma^3 + \bar{\sigma}^3) + (\sigma \bar{\sigma})^{p-2}. \tag{16} \]

The new composites appearing in it can be identified as

\[ (\sigma \bar{\sigma})^{p-2} = \Phi \left( \frac{2}{2} \frac{1}{1} \right), \tag{17} \]

\[ \sigma^{p-4} \bar{\sigma}^{p-1} = \Phi \left( \frac{1}{4} \frac{2}{2} \right), \tag{18} \]

which are of course irrelevant fields. The first one is the least irrelevant thermal field \( \epsilon^{p-2} \) of conformal dimension \( \Delta_{\epsilon^{p-2}} = 1 + (3/p) \). The other is slightly more irrelevant,
\[ \Delta = 1 + \left[ 3(p + 2)/p(p + 1) \right], \]
and is essential to endow the lagrangian with the necessary \( D_3 \) symmetry. Fields with higher dimension must also appear in the lagrangian, for there are more field equations coming from the products of \( \sigma \) (\( \bar{\sigma} \)) and other fields with the \( 2p - 6 \)th power. Those additional fields have to be neutral, and correspond to monomial terms with \( 2p - 4 \)th and \( 2p - 5 \)th powers,

\[ (\sigma \bar{\sigma})^{p-5}(\sigma^6 + \bar{\sigma}^6), (\sigma \bar{\sigma})^{p-7}(\sigma^9 + \bar{\sigma}^9), \]

etc. These terms are important for the potential to be well defined\(^1\) but, since they represent more irrelevant fields, a renormalization-group argument would tell us that their effect becomes negligible close to the multicritical point. This circumstance can be mimicked by assigning them small coefficients. Therefore, they are not expected to play any relevant role in the phase structure of the models and will be ignored henceforth. Nevertheless, the term \((\sigma \bar{\sigma})^{p-4}(\sigma^3 + \bar{\sigma}^3)\) in (16) needs to be kept, not only to enforce the \( D_3 \) symmetry but also to prevent the potential from being degenerate.

The degeneracy of the potential consisting of just the \((\sigma \bar{\sigma})^{p-2}\) term (the one found in (3)) manifests itself as the possibility of obtaining under some perturbations an infinite number of minima (one or several circumferences). Summarizing, the lagrangian (16) is the correct starting point for the study of the phase diagram in the next section.

We would like to point out a peculiarity of the lowest-\( p \) models: Counting their primary fields one realizes that there are not enough to support the identifications above. For instance, \( W_4^{(3)} \) and \( W_4^{(3)} \) have 6 and 20 primary fields, respectively, which cannot account for all the fields mentioned above. The six fields of the former are all relevant and identifiable with the composite fields up to second power of the elementary ones. However, we know that this model coincides with the nondiagonal modular invariant \((A_4, D_4)\) of the Virasoro series [17, 3], which has additional primary fields for higher powers of the elementary fields, namely,

\[ \Phi_{(12,13)} = \sigma^3, \quad \Phi_{(13,12)} = \bar{\sigma}^3 \]

\[ \Phi_{(13)} = \epsilon^2 = (\sigma \bar{\sigma})^2. \]  

Therefore, these fields have to be \( W \)-secondaries. It is easy to find that indeed

\[ \sigma^3 = \bar{W}_{-1} \epsilon, \quad \bar{\sigma}^3 = W_{-1} \epsilon \]

\[ \epsilon^2 = \bar{W}_{-1} W_{-1} \epsilon. \]

In conclusion, some of the composite fields of the low-\( p \) models turn out to be \( W \)-secondaries with the necessary symmetry properties.

\(^1\)This fact is specially clear in the language of Catastrophe Theory, where the absence of those terms produces an indeterminate potential [3].
3 Phase structure of the $W_{(3)}$ models

The mean-field phase diagram is obtained analyzing the various configurations of minima produced upon perturbation of the potential part of the lagrangian by relevant fields. The topology of the mean-field phase diagram is identical to the actual phase diagram (this is the reason why the Landau potential suffices to study the phase structure). A complete representation of the phase diagrams is however extremely complicated, as can be imagined from the difficulties already encountered for the simplest case, the three-state Potts model $W^4_{(3)}$, worked out in [7]. Nevertheless, one is mostly interested in certain properties of the phase diagram, namely, the number of possible phases (stable states) and their topological interrelation, which only demand a limited knowledge of it. These properties are conveniently expressed by the state diagram, which we define as a set of points corresponding to the possible states (either stable or unstable) in a definite region of the phase diagram, and being linked whenever they can merge under a suitable perturbation. This diagram provides a direct connection with statistical mechanics: A perturbed conformal model can be interpreted as the renormalization-group universality class of lattice models with a state variable living on that diagram. Now we recall that the $W^p_{(n)}$ are known to describe the critical behavior of the JMO lattice models [5] (see also [10]). These are interaction round a face (IRF) models with state variables defined on the dominant-weight lattice of some Lie algebra, as a generalization of the restricted solid on solid (RSOS) models of Andrews, Baxter and Forrester, the critical behavior of which is described by the Virasoro minimal models. A motivation for the construction of these IRF models actually arose from the representation of the minimal W models as cosets $(A_n \oplus A_n, A_n)$ with level $(l - 1, 1)$, in which the $A_n$ dominant weights of level $l$ play a natural role (recall the role of $(1)$ as the Clebsch-Gordan decomposition rule).

The relation between the state diagram of the $W^4_{(3)}$ model and the diagram of dominant fundamental weights of $A_2$ is apparent in [4]. It is to be expected that the state diagram of the $W^p_{(3)}$ model coincides with the $A_2$ dominant-weight diagram of level $p - 3$. Therefore, we would be interested in obtaining that state diagram by perturbing the potential in a suitable way. Finding this perturbation in the general case turns out to be quite a nontrivial problem: Contrary to the Virasoro case, with one order parameter, for which is rather easy to discern the effect of each individual perturbation, in the case with several order parameters it is almost impossible to predict how the potential will deform under a particular perturbation. However, there are some rules, as we shall see. First of all, let us introduce real variables, by $\sigma = x + iy$ ($\bar{\sigma} = x - iy$),

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2 Hence, for phase diagrams derived from a potential the state diagram coincides with the Dynkin diagram of a level curve, as defined in [4] and used in this context in [3]. However, we prefer here the name state diagram since we will be using Lie-algebra terms, among which Dynkin diagram already has a very standard meaning, not identical to that in [3]. In fact, the state diagrams for the $W_{(n)}$ models are related with weight diagrams of the underlying Lie algebra.
more suitable for representation of the potential, and simplified notation for the two independent $D_3$ invariants

\[ I_0 = \sigma \bar{\sigma} = x^2 + y^2, \]
\[ I_1 = \sigma^3 + \bar{\sigma}^3 = x^3 - 3xy^2. \]

The potential takes the form of a polynomial in $I_0$ and $I_1$,

\[ V(x, y) = I_0^{p-2} + I_0^{p-4}I_1 + P(I_0, I_1), \tag{23} \]

where the symmetric perturbation $P(I_0, I_1)$ has degree $2p - 6$ as a polynomial in $x, y$. If the potential contained only $I_0$, it would have full $O(2)$ symmetry and depend on just the radial variable $r = \sqrt{x^2 + y^2}$. Although this possibility is not admissible in the present context, it is convenient to compare with the case with one order parameter. Here the maximum unfolding deformation is achieved for non-null values of all the coupling constants in the symmetric perturbation. The modification introduced by $I_1$ amounts to reduce the symmetry of this rotational invariant potential to $D_3$. The structure is still the same along the three directions in the $(x, y)$ plane for which $I_1 = 0$, and must be qualitative similar all over. Therefore, one has to tune carefully all the coupling constants to produce minima configurations corresponding to the desired state diagram. This problem, topological in essence, cannot be formulated in any simple algebraic form. The simplest models can be successfully handled by computer programs capable of representing algebraic surfaces, such as Mathematica. In this way, one tests different values of coupling constants to observe how the potential deforms. With some insight and a good deal of luck, one may arrive to a configuration recognizable as described by the appropriate $A_2$ dominant-weight diagram. Needless to say, the number of coupling constants increases rapidly with $p$ and beyond $p = 6$ this trial-and-error method ceases to be feasible.

Fortunately, there are more powerful topological arguments that can be directly applied to solve the inverse problem; namely, finding from the state diagram a potential with a minima configuration that fits it. Afterwards, this potential can be compared with the original one. In order to expose the method, let us introduce the concept of level curves for a potential; in our case, $V(x, y) = c$ defines an algebraic curve at level $c$. This curve is singular if there are extremal points of the potential (minima, maxima or saddle points) at that level. Now we consider the $A_2$ dominant-weight diagram of level $l$; we mark the middle points on each link and construct a singular curve joining these points, as in Fig. 1. We would like to identify this topological curve with a level curve of the potential \(23\) for $l = p - 3$. The first thing we observe is that the curve is the product of a definite number of components (two in Fig. 1). We further

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3Throughout the paper, we use the term level with two different meanings, namely, as the value of $z = V(x, y)$ in the $(x, y, z)$ coordinate representation providing the graphical notion of level curves or as the integer associated to a representation of a Lie algebra.

4This process is the inverse of obtaining the Dynkin diagram of a curve according to [9].
notice that there are precisely \((l + 1)/2\) trefoil shaped curves when \(l\) is odd and \(l/2\) trefoil curves plus one of circular type when \(l\) is even. The level curve of the potential for the \(W^4_{(3)}\) model,

\[ V(x, y) = I_0^2 + I_1 + wI_0 = 0, \quad (24) \]

is the simplest algebraic curve with the trefoil shape\(^5\) (observe that there are two different topological types depending on the sign of \(w\)). The simplest circular curve is, of course, the circle \(I_0 - c = 0\). Therefore, the simplest algebraic curve which represents the whole topological curve is given by the product

\[
\prod_{i=1}^{(l+1)/2} (I_0^2 + I_1 + w_iI_0) = I_0^{l+1} + I_0^{l-1}I_1 + \cdots = 0, \quad (25)
\]

for \(l\) odd, and by

\[
\prod_{i=1}^{l/2} (I_0^2 + I_1 + w_iI_0)(I_0 - c) = I_0^{l+1} + I_0^{l-1}I_1 + \cdots = 0, \quad (26)
\]

for \(l\) even. We see that both reproduce the potential \((23)\) when \(l = p - 3\).

We should remark that the construction above yields values of the coupling constants (in fact, a range) for which the state diagram is the desired \(A_2\) dominant-weight diagram. However, there are certainly other values that produce the same diagram: All the level curves obtained by this method are singular and correspond to very special potentials with the saddle points at the same level; we can easily imagine deformations of the potential that move some saddle points off that level without altering its topology. There can actually be a simpler set of values, with some of them null. Moreover, those potentials are special in yet another sense: We have to contemplate the presence of terms with higher powers of \(I_1\) in the non-perturbed potential, as we noted at the end of the first section. These terms, though, have small coefficients and will not change the topological type of the potential. In any case, that construction provides a privileged starting point to describe the phase diagram and can be used to probe the effect of other terms. We have done so with Mathematica, drawing various contour plots for the \(p = 5\) and \(p = 6\) models. A characteristic plot clearly exhibiting the state diagram appears in Fig. 2.

4 The general \(W_{(n)}\) case

The generalization of the Dotsenko-Fateev construction to \(W_{(n)}\) demands a free massless scalar field with \(n - 1\) components. Therefore, the number of screening charges is

\(^5\)This potential has been described in a previous paper. That \((24)\) represents a trefoil can be understood without calculation when \(w = 0\): \(I_1 = 0\) gives just three straight lines intersecting at the origin and producing six sectors where \(V(x, y)\) takes negative and positive values alternatively. The effect of adding \(I_0^2\) is raising the potential for \(r\) large enough, hence closing the three negative sectors.
2(n − 1) and the primary fields are labelled by an equal number of integers. The unitary models $W_{(n)}^p$ contain $[p!(p − 1)!/[n!(n − 1)!(p − n)!(p − n − 1)!]$ spinless primary fields and they have $D_n$ symmetry [2]. The neutrality condition to derive the fusion rules is the straightforward generalization of that in (1)

$$\alpha(3) = \alpha(1) + \alpha(2) + \alpha_+ \sum_{i=1}^{n-1} N_i e_i + \alpha_- \sum_{i=1}^{n-1} M_i e_i,$$

with

$$\alpha(j) = \alpha(\{n_i(j)\} | \{m_i(j)\}) = \sum_{i=1}^{n-1} ([1 - n_i(j)] \alpha_+ + [1 - m_i(j)] \alpha_-) \omega_i. \quad (28)$$

Here $e_i$ and $\omega_i$ are the positive roots and fundamental weights of $SU(n)$ (algebra $A_{n-1}$, respectively. The fusion rule (27) represents the Clebsch-Gordan decomposition for $SU(n) \otimes SU(n)$. To obtain the numerical fusion rules one is to substitute for the roots in terms of the weights,

$$e_1 = 2 \omega_1 - \omega_2,$$
$$e_i = 2 \omega_i - \omega_{i-1} - \omega_{i+1}, \quad i = 2, \ldots, n-2,$$
$$e_{n-1} = 2 \omega_{n-1} - \omega_{n-2},$$

and solve for $n_i(3)$ and $m_i(3)$,

$$n_1(3) = n_1(1) + n_1(2) - 1 - 2N_1 + N_2,$$
$$n_i(3) = n_i(1) + n_i(2) - 1 - 2N_i + N_{i-1} + N_{i+1}, \quad i = 2, \ldots, n-2,$$
$$n_{n-1}(3) = n_{n-1}(1) + n_{n-1}(2) - 1 - 2N_{n-1} + N_{n-2}.$$

(There are similar equations for $m_i$).

The elementary fields are the $n - 1$ spin fields corresponding to the fundamental representations of $SU(n)$,

$$\sigma_k = \Phi(1, \ldots, 1, \underbrace{2}_{k}, 1, \ldots, 1 | 1, \ldots, 1, \underbrace{2}_{k}, 1, \ldots, 1).$$

They support the $D_n$ representations

$$\sigma_k \rightarrow e^{2\pi ki/n}\sigma_k$$
$$\overline{\sigma}_k \rightarrow \overline{\sigma}_k;$$

namely, the most relevant fields, $\sigma_1$ and $\overline{\sigma}_1 \equiv \sigma_{n-1}$, support the standard representation and the others the remaining two-dimensional representations; (if $n$ is even, the central field forms a one-dimensional representation). The crucial fact is that all the other
relevant fields can be formed as powers of those \( n - 1 \) elementary ones and arranged on the \( SU(n) \) lattice of dominant weights of level \( 2(p-n) \). Showing this in the analogous way to the \( W(3) \) case, demands thinking of a \( 2(n-1) \)-dimensional conformal grid and various \( (n-1) \)-dimensional sections. We will restrict ourselves to \( n = 4 \) for simplicity, given that no essentially new properties arise for higher \( n \). In this case we have three elementary fields,

\[
\begin{align*}
\sigma_1 &= \Phi(1,0,0 \mid 1,0,0), \\
\sigma_2 &= \Phi(0,1,0 \mid 0,1,0), \\
\bar{\sigma}_1 &= \Phi(0,0,1 \mid 0,0,1),
\end{align*}
\]

the second one being real. The lattice of dominant weights is an isosceles pyramid (Fig. 3). The “main diagonal” contains fields up to level \( p-4 \), which are identified in a straightforward way, from (30) when all \( N_i = M_i = 0 \ (i = 1,2,3) \), with the products of the elementary fields up to maximum total power \( p-4 \),

\[
\sigma_1^k \sigma_2^l \bar{\sigma}_1^m = \Phi(k+1,l+1,m+1 \mid k+1,l+1,m+1), \quad k+l+m \leq p-4. \tag{31}
\]

The identification of the next power, \( k + l + m = p - 3 \), demands that some of the \( N_i \) be non-null. For \( \sigma_1 \) and \( \bar{\sigma}_1 \) we have choices analogous to those in the \( W(3) \) case:

- \( N_1 = 1 \) and the others null for

\[
\sigma_1^{p-3} = \Phi(p-4,2,1 \mid p-2,1,1) = \Phi(2,1,1 \mid 1,1,1). \tag{32}
\]

- The conjugate equation when \( N_3 = 1 \) and the others null.

- The analogous middle triangle is also obtained by \( N_1 = N_3 = 1 \) (the remaining null).

However, the new component \( \sigma_2 \) introduces further possibilities, the simplest of which are

\[
\begin{align*}
\sigma_1^{p-4} \sigma_2 &= \Phi(p-3,1,1 \mid p-3,1,1) \Phi(1,2,1 \mid 1,2,1) \\
&= \Phi(p-4,1,1 \mid p-3,2,1) = \Phi(1,1,2 \mid 2,1,1), \tag{33}
\end{align*}
\]

with \( N_1 = N_2 = 1 \) (remaining null), and its conjugate. Another interesting one is

\[
\begin{align*}
\sigma_2^{p-3} &= \Phi(1,p-3,1 \mid 1,p-3,1) \Phi(1,2,1 \mid 1,2,1) \\
&= \Phi(1,p-4,1 \mid 1,p-2,1) = \Phi(1,2,1 \mid 1,1,1), \tag{34}
\end{align*}
\]

with \( N_1 = N_3 = 1, N_2 = 2 \). All these choices hold for the respective domains of pyramidal shape that are subtended from the position of these fields in the big pyramid up to level \( 2p-8 \) containing all the relevant fields.
An exhaustive description of the field identification domains and their matching conditions is cumbersome but not necessary to find the Landau-Ginzburg lagrangian: Like in the $W(3)$ case, within the fields of highest level only the most relevant is needed. This field also belongs to the thermal subalgebra, now generated by two fields,

\[ \epsilon_1 = \Phi(1, 0, 1 | 1, 0, 1) = \sigma_1 \bar{\sigma}_1 \] (35)

and

\[ \epsilon_2 = \Phi(0, 2, 0 | 0, 2, 0) \simeq \sigma_1 \bar{\sigma}_1 + \sigma_2^2 \] (36)

(Recall that the symbol \( \simeq \) means equal up to numerical coefficients). The sought field is

\[ \epsilon_{p-4} = (\sigma_1 \bar{\sigma}_1)^{p-4} = \Phi(1, 1, 1 | 2, 1, 2). \] (37)

Upon product with the elementary field $\sigma_1$, we obtain the field equation

\[ \partial^2 \sigma_1 \simeq \sigma_1^{p-5} \bar{\sigma}_1^{p-4} \sigma_2 + \sigma_1(\sigma_1 \bar{\sigma}_1)^{p-4} \] (38)

(Note the similarity with (13), although the monomial for the field $\phi$ is now different, consistently with the present $D_4$ symmetry). Hence the lagrangian

\[ \mathcal{L} \simeq \partial \sigma_1 \partial \bar{\sigma}_1 + (\sigma_1 \bar{\sigma}_1)^{p-5}(\sigma_1^2 + \bar{\sigma}_1^2)\sigma_2 + (\sigma_1 \bar{\sigma}_1)^{p-3}. \] (39)

These are the crucial terms that determine the phase structure. However, like in the $W(3)$ case, other terms corresponding to more irrelevant fields are also required for the potential to be well defined. In addition, we will consider again a symmetric perturbation, formed by lower degree terms corresponding to relevant fields. To simplify the aspect of all these terms, it is convenient again to regard the potential as a polynomial in the basic invariants. The invariants for the standard representation of $D_4$ and their expression in terms of real variables ($\sigma_1 = x + iy$) are

\[ I_0 = \sigma_1 \bar{\sigma}_1 = x^2 + y^2, \]
\[ I_1 = \sigma_1^4 + \bar{\sigma}_1^4 = 2(x^4 + y^4 - 6x^2y^2). \]

Now the presence of $\sigma_2$, forming a one-dimensional $D_4$ representation\(^6\), produces new invariants

\[ I_2 = \sigma_2^2 = z^2 \quad (z \equiv \sigma_2) \]
\[ I_{12} = (\sigma_1^2 + \bar{\sigma}_1^2)\sigma_2 = 2(x^2 - y^2)z \]

That is to say, four basic invariants altogether. However, they are not independent, for they satisfy the relation

\[ I_{12}^2 = (I_1 + 2I_0)I_2. \] (40)

\(^6\)The complete representation of $D_4$ for the $W_{(4)}$ models is the direct sum $A_2 \oplus E$, in the notation of Landau-Lifshitz’s textbooks.
This fact introduces some complications but it will suffice here to say that it amounts to the possibility of substituting for any power of $I_{12}$ (higher than the first), which in consequence will not appear in the potential. Therefore, the potential takes the form

$$V(x, y, z) = I_{0}^{p-3} + I_{0}^{p-5}I_{12} + \mathcal{P}(I_{0}, I_{1}, I_{2}, I_{12}),$$

(41)

where the symmetric perturbation

$$\mathcal{P}(I_{0}, I_{1}, I_{2}, I_{12}) = \mathcal{P}_{0}(I_{0}, I_{1}, I_{2}) + I_{12}\mathcal{P}_{1}(I_{0}, I_{1}, I_{2})$$

has degree $2p - 8$ as a polynomial in $x, y, z$. The irrelevant additional terms are $I_{0}^{p-5}I_{1}, I_{0}^{p-4}I_{2}, I_{0}^{p-5}I_{2}^{2}, I_{0}^{p-7}I_{12}I_{1}, I_{0}^{p-6}I_{1}I_{2},$ etc., and will be ignored henceforth.

### 4.1 Phase structure and its relation with IRF models

In the study of phase structure we would like to generalize the method applied to the $W_{(3)}$ models. If it applies to the $W_{(4)}$ ones as well, we should expect to have configurations of minima that reproduce the successive sets of level-$l$ dominant integral weights of $A_{3}$. The simplest model, $W_{(4)}^{5}$, is again very illustrative. We will take advantage of the fact that there is an alternative description of this model: It is a special point on the line of 2dCFT defined on the orbifolds of a circle with variable radius. These 2dCFT, with $c = 1$ and $D_{4}$ symmetry, correspond to critical Ashkin-Teller models with variable 4-spin coupling ([4, 12]). They have two twist fields with conformal dimension $(1/16, 1/16)$, realizing the standard $D_{4}$ representation (our $x, y$), and one field with dimension depending on the coupling constant $K$ as $\frac{1}{8\pi K}$, which realizes the non-trivial one-dimensional representation (in our model this field is $z$, with $\Delta_{z} = 1/12$).

Specifying in (41) and after performing a $\pi/4$ rotation on the $xy$-plane to write the cubic invariant in a more suitable form, the perturbed potential for the $W_{(4)}^{5}$ model can be written as

$$V(x, y, z) = I_{0}^{2} + I_{12} + wI_{0} + w'I_{2} = (x^{2} + y^{2})^{2} + xyz + w(x^{2} + y^{2}) + w'z^{2},$$

(42)

This potential is analogous to (24) except for the presence of two independent energy perturbations; the cubic term determines likewise its symmetry. The equation $I_{12} = 0$ represents the three coordinate planes dividing the space in eight sectors for which the potential takes minus and plus signs alternatively. Now the effect of adding $I_{0}^{2}$ does not suffice to close a surface and produce minima, for there is no term $z^{4}$, and the potential remains unbounded in this direction. Nevertheless, the quadratic perturbation $z^{2}$ bounds the potential, giving minima in pyramidal configuration.

---

7For a short account on Invariant Theory bearing on this problem, see [11].

8Note that the algebraic surface $(x^{2} + y^{2} + z^{2})^{2} + xyz = 0$ has a four-lobed tetrahedral shape and therefore possesses higher symmetry, namely, the full group of the tetrahedron $T_{d}$ (Landau-Lifshitz notation), isomorphic to the symmetric group $S_{4}$. 

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Let us attempt to calculate the position of the minima of the full potential (42) from the equations

\[
\begin{align*}
\frac{\partial V}{\partial x} &= 4x(x^2 + y^2) + yz + 2wx = 0, \\
\frac{\partial V}{\partial y} &= 4y(x^2 + y^2) + xz + 2wy = 0, \\
\frac{\partial V}{\partial z} &= xy + 2w'z = 0.
\end{align*}
\] (43, 44, 45)

The last equation is particularly simple to solve, giving

\[z = -xy/2w',\]

which substituted back into the others yields the same equations as those coming from the two-variable potential with \(D_4\) symmetry

\[V(x, y) = (x^2 + y^2)^2 + (2w')^{-1}x^2y^2 + w(x^2 + y^2) = x^4 + y^4 + (2 - \frac{1}{2w'})x^2y^2 + w(x^2 + y^2).\] (47)

We know that its extrema are placed on the vertices of a square. For the full 3-variable potential, the respective \(z\) obtained from (46) are identical in absolute value but negative or positive according to whether \(x\) and \(y\) have the same sign or not. Therefore, they form the vertices of an isosceles pyramid. However, it is not possible to check if the dimensions of this pyramid coincide with those of the \(A_3\) dominant-weight lattice of level one (as drawn in Fig. 3). In fact, this question is nearly meaningless, given that there is no relation between \(z\) units and \(x\) or \(y\) units. Since the potential is at most quadratic in \(z\), implying that this variable cannot affect critical behavior, a healthy point of view is to disregard it altogether and consider just (47) as the whole potential. This potential has already been associated to the critical line of the Ashkin-Teller model [14]. This line arises as a result of the existence of a marginal field, identified with the term \(x^2y^2\), in that model [15]. Its coefficient \(\alpha = 2 - (1/2w')\) is directly related to the coupling constant \(\pi K = (\alpha/2) + 1\), that is to say to the dimension of the field \(z\). For \(\alpha = 0\) we have two decoupled Ising models. When \(\alpha = 2\) the symmetry augments to \(O(2)\), representing the Kosterlitz-Thouless point of the XY model. In principle, the parameter \(\alpha\) can also take negative values as long as \(\alpha > -2\). For instance, the value

\[\alpha = 0\] is essential in Catastrophe theory, where a quadratic potential is the simplest Morse function, which can be added to any catastrophe to give an equivalent one. The particular equivalence to which we are led in our case discarding \(z\) is \(T_{244} \simeq X_9\) of Arnold’s classification [13].

\[\alpha = 2\] is called double-cusp catastrophe (\(X_9\)) and has aroused interest as the simplest catastrophe with a modal deformation [13], which is just the one caused by the \(x^2y^2\) term. The physical consequences of modality were studied in [15].
that gives the dimension of the field \( z \) in the \( W_{5(4)} \) model is \(-2/3\). However, at \( \alpha = -1 \) something singular happens: The conformal dimension of \( z \) decreases to 1/16, the same as that of \( x \) or \( y \), and the symmetry augments to \( S_4 \), representing the 4-state Potts model [4]. The potential has to include now the \( z^4 \) term, becoming

\[
V(x, y) = x^4 + y^4 + z^4 + xyz + w(x^2 + y^2 + z^2),
\]

which was considered years ago for the 4-state Potts model in any dimension [10]. For smaller values of \( \alpha \), it is the term \( x^4 + y^4 \) that has to be dropped; hence, the symmetry reduces to \( D_4 \) again.

The reader may be wondering about the bearing of the previous arguments on the models with \( p > 5 \). Here the field \( z \) is certainly not to be disregarded but the symmetry considerations still stand. In any case, the argument used for the \( W_{(3)} \) potentials applies: If the term with \( I_{12} \) in (11) is omitted, the maximum unfolding consists of minima on spherical surfaces and the perturbation that produces it is easily found. The effect of \( I_{12} \) is enforcing the symmetry to give the pyramidal shape. A detailed proof would involve a construction with algebraic surfaces on the selected \( A_3 \) diagram of dominant weights. It is straightforward and will not be expounded here. Finally, it is clear as well how to generalize these methods to higher values of \( n \).

5 Conclusions

There are two main utilities of the lagrangian approach to 2dCFT: First, it endows the field algebra with further structure that constitutes a much simpler picture of it. Second, it allows a direct study of the phase diagram that does not need to deal with the complications inherent to other methods of treating perturbed 2dCFT (Bethe ansatz, etc). It also provides with simple objects, such as the state diagram, that can be directly related to statistical models known to have critical behavior described by those 2dCFT. It is therefore a sort of link between statistical models and 2dCFT. In this context, it may be of some help to people concerned with subjects that manifest themselves in both areas, like quantum groups.

In this paper we have developed the lagrangian approach for the minimal models of \( W \)-algebras, relying on the free field construction, which directly relates them to Lie algebras (to be precise, Kac-Moody algebras). We have seen how all the relevant fields of \( W_{(n)} \) can be identified with monomials of the elementary fields that fit on the \( A_{n-1} \) dominant-weight lattice of level \( 2(p-n) \). The main field equation follows from an OPE, in the Zamolodchikov’s way. A new feature, worth emphasizing, is that one has to save in that OPE two fields instead of one to get the correct field equation. Furthermore, this seems to be the only requirement for consistency of the lagrangian description. We have obtained afterwards the Landau-Ginzburg lagrangians, noting that they posses the \( D_n \) symmetry of these models. The potential contains the necessary information to
study the whole phase diagram. We have however limited ourselves to the phase structure produced by symmetric perturbations. In particular, we have shown that there exists a perturbation that originates a state diagram identifiable with that defining the statistical models of Jimbo et al. [8], namely, the $A_{n-1}$ dominant-weight diagram of level $p-n$. On the other hand, the phase transition between definite regimes of these models has already been shown to be described by the W-models. In consequence, our result closes the loop, showing a two-way equivalence.

Many possibilities remain to be explored. For instance, we have confined ourselves to the W models that are diagonal modular invariants. Extending the methods in this paper to the non-diagonal ones seems quite straightforward. It would be interesting to see whether the resulting potentials fit in a classification (ADE or similar) agreeing with that known for the non-diagonal modular invariants of W models.

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Figure Captions

1. Level curve for a minima configuration with the shape of the $A_2$ weight diagram of level 3.

2. Contour Plot of the potential $V(x, y) = 0.49I_1^2 + (I_0^2 + I_1)(I_0 - 0.033)$, corresponding to $W^5_{(3)}$.

3. $A_3$ Lattice of dominant weights of level 3 with some field identifications.