Conserved system with limited number of entities follows non-Maxwellian, non-Gaussian and non-chi-square distributions obtained by Havrda-Charvát-Tsallis entropy

Jae Wan Shim

Materials and Life Science Research Division, Korea Institute of Science and Technology, and Major of Nanomaterials Science and Engineering, KIST Campus, Korea University of Science and Technology, 5 Hwarang-ro 14-gil, Seongbuk, Seoul 02792, Republic of Korea

Abstract

We demonstrate that the most probable state of a conserved system with limited number of entities is the state where non-Maxwellian, non-Gaussian and non-chi-square distributions govern. We have used the following setup of a thought experiment where a conserved system consists of limited number of entities or molecules. We have verified the mathematical derivation correctly predict the results obtained by computer simulations. The derived distributions approach the Maxwellian and the chi-square distributions as the number of entities approaches infinity. The derived distributions will have an important role in the fields of medical research where the number of entities in the system of interest is limited. Especially, the non-chi-square distribution can be interpreted by an asset distribution achieved after a repetitive game where an arbitrary portion of one’s asset is transferred to other arbitrary entity among a number of entities whose abilities are equal.

Keywords: non-Maxwellian energy distribution, non-Maxwellian velocity distribution, non-chi-square distribution
1 Introduction

By using the concept of infinity, the central limit theorem provides us the opportunity to use the Maxwellian (the Gaussian), and chi-square distributions instead of exact distributions under various circumstances. In this paper, we deal with exact distributions which we can derive by rigorous mathematical methods. The exact distributions such as non-Maxwellian distributions appear in certain environments as the followings. An evident example is space plasmas. It is widely confirmed by scientific observations; the high energy tails deviated from the Maxwellian distribution is described by the so-called Kappa distribution \[1, 2, 3\] and related researches are reported \[4, 5, 6, 7, 8, 9\]. Other examples of non-Maxwellian distributions are found in the fields including one-dimensional granular media \[10\] and an upper level of an atmosphere \[11\] and electron energy distribution in sub-quarter-micron devices \[12\] and stationary solutions of the Fokker-Planck equation \[13\]. Meanwhile, the non-Maxwellian distributions having shorter tails have not yet extensively been discussed. Moreover, the velocity or energy distribution for rarefied gas of which the Knudsen number is very large is fundamental but still uncertain. The regime of large Knudsen numbers includes molecular dynamics in micro chambers and interplanetary space as well as in ultrahigh vacuum chambers \[14, 15\]. The molecular velocity distribution function for ideal gas in an equilibrium state is commonly assumed to be the Maxwellian distribution at large Knudsen numbers even though disagreement between the Maxwellian and experimental data exists \[16, 17\]. Here, we have rigorously derived a molecular velocity distribution function and an energy distribution function with respect to a number of entities or molecules \(N\), which is applicable for a system of limited number of bodies.

2 Derivation of energy or asset distribution

To obtain the non-Maxwellian distribution for a system with limited number of entities or molecules, we begin our step with an equation which uses the Dirac delta function, that is inspired by the Cercignani’s which was used for deriving the Maxwellian velocity distribution \[18\]. He used velocities as the independent variables of his equation, however, here we use energies as the independent variables to emphasize that our distribution derived in this paper is applicable to a system conserving quantities with respect to not only a square sum of the values of entities but also a direct sum of the values of entities such as an economic system having a fixed global asset by analogy with a fixed global energy. Let us define a space \(S\) and a region \(R \in S\). The region \(R\), a conserved system with respect to the number of entities or molecules \(N\) and with respect to a given energy, is bounded. We can, then, write an energy distribution function \(f_\varepsilon\) as

\[
f_\varepsilon(\varepsilon_1, \cdots, \varepsilon_N) = c_\varepsilon \delta \left( \sum_{i=1}^{N} \varepsilon_i - E \right)
\]  

(1)
where $\epsilon_i$ is the energy of the $i$th molecule, $c_\epsilon$ is a normalizing constant, $\delta(\cdot)$ is the Dirac delta function, and $E$ is a given energy of the system in $\mathbb{R}$.

To facilitate the integral calculations of the Dirac delta function, which will be done later, let us define variables $q_i$ by

$$q_i = \pm \sqrt{\epsilon_i}$$

which is purely a mathematical change of variables at this stage and alternatively we can define $q_i = \sqrt{\epsilon_i}$. The choice does not affect the final result of one-entity energy distribution. Let us define a distribution function with respect to $q_i$ as

$$f_q(q_1, \cdots, q_N) = c_q \delta \left( \sum_{i=1}^{N} q_i^2 - E \right)$$

where $c_q$ is another normalizing constant. Then, it is convenient to deal with the summation in the delta function by adopting the polar coordinate system as the followings,

$$q_i = r \cos \phi_i \prod_{j=1}^{i-1} \sin \phi_j$$

for $i = 1, \cdots, N - 1$ and

$$q_N = r \prod_{j=1}^{N-1} \sin \phi_j$$

where the ranges of the variables are defined by $0 \leq r \leq \infty$ and $0 \leq \phi_j \leq \pi$ for $j = 1, \cdots, N - 2$ and $0 \leq \phi_{N-1} \leq 2\pi$. Now, we have a compact expression

$$\sum_{i=1}^{N} q_i^2 = r^2$$

and the infinitesimal volume $dq_1 \cdots dq_N$ can be written as

$$dq_1 \cdots dq_N = r^{N-1} dr \, dA_N$$

where $A_N$ is the surface area of the $N$-dimensional unit ball and its value is given by

$$A_N = \frac{2 \pi^{N/2}}{\Gamma \left( \frac{N}{2} \right)}$$

where $\Gamma(\cdot)$ is the gamma function defined by using the double factorial as

$$\Gamma \left( \frac{N}{2} \right) = 2^{-\frac{N-1}{2}} (N - 2)!! \sqrt{\pi}.$$ 

Then, the normalizing constant $c_q$ is obtained by using the following equation,

$$c_q \int \delta \left( r^2 - E \right) r^{N-1} dr \int dA_N = \frac{c_q}{2} A_N E^{-\frac{N-2}{2}} = 1,$$
which is

\[ c_q = \frac{2}{A_N E^{\frac{N-2}{2}}} \]

When we integrate Eq. (2) with respect to \( q_i \) for \( i = 1, \cdots, N-1 \) with excluding \( q_N \), the result of the integration becomes as the following:

\[
F_q(q_N) = \int \cdots \int f_q(q_1, \cdots, q_N) dq_1 \cdots dq_{N-1}
\]

\[
= c_q A_{N-1} \int \delta \left( x^2 - (E - q_N^2) \right) r^{N-2} dr
\]

\[
= \begin{cases} 
  A_{N-1} \left( A_N E^{\frac{N-2}{2}} \right)^{-1} (E - q_N^2)^{\frac{N-3}{2}} & \text{if } q_N^2 \leq E \\
  0 & \text{if } q_N^2 > E.
\end{cases}
\]

Let us assume that the energy has no preference to be associated with a specific molecule, then, the distribution function must be symmetric with respect to the index changes of the molecules so that we can omit the subscript \( N \) for simplicity. By substituting \( A_{N-1} \) and \( A_N \) with the expressions of the Gamma function, we have

\[ F_q(q) = \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-1}{2} \right) \sqrt{\pi E}} \left( 1 - \frac{q^2}{E} \right)^{\frac{N-3}{2}} \]

for \( q^2 \leq E \) and otherwise \( F_q(q) = 0 \).

Let us define the one-entity energy distribution \( F_\epsilon(\epsilon_N) \) by using Eq. (1) as

\[ F_\epsilon(\epsilon_N) = \int \cdots \int f_\epsilon(\epsilon_1, \cdots, \epsilon_N) d\epsilon_1 \cdots d\epsilon_{N-1} \]

and with considering the following identity

\[ \int_0^\infty F_\epsilon(\epsilon) d\epsilon = 2 \int_0^\infty F_q(q) dq \]

with \( dq = (2\sqrt{\pi})^{-1} d\epsilon \), the energy distribution \( F_\epsilon(\epsilon_N) \) or simply \( F_\epsilon(\epsilon) \) is, then, obtained by

\[ F_\epsilon(\epsilon) = \frac{1}{\sqrt{\epsilon_N}} F_q(q) \]

\[ = \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-1}{2} \right) \sqrt{\pi \epsilon E}} \left( 1 - \frac{\epsilon}{E} \right)^{\frac{N-3}{2}} \]

for \( \epsilon \leq E \) and otherwise \( F_\epsilon(\epsilon) = 0 \).

When \( N \) approaches infinity, Eq. (7) approaches the Maxwellian energy distribution which is also a chi-square distribution with one degree of freedom as

\[ \chi^2_\epsilon(\epsilon) = \frac{1}{\sqrt{2\pi \epsilon E}} \exp \left( -\frac{\epsilon}{2E} \right) \]
where the total energy $E$ is assumed to be linearly dependent on $N$ as $E = N \bar{E}$. One can use $\bar{E} = (kT)/2$ from the following relation of the kinetic theory of gases that the sum of the kinetic energies of molecules in one-dimensional space can be expressed by

$$E = \frac{m}{2} \sum_i v_i^2 = \frac{1}{2} N kT$$

where $v_i$ is the velocity of the $i$th molecule, $k$ is the Boltzmann constant, and $T$ is a temperature. Then, Eq. (8) becomes

$$M_\epsilon(\epsilon) = \frac{1}{\sqrt{\pi kT}} \exp \left( -\frac{\epsilon}{kT} \right).$$

By defining $\epsilon = \bar{E} \bar{\epsilon}$, we respectively obtain dimensionless versions of Eqs. (7) and (8) as

$$F_{\bar{\epsilon}}(\bar{\epsilon}) = \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-1}{2} \right) \sqrt{\pi \bar{\epsilon} N kT}} \left( 1 - \frac{\bar{\epsilon}}{N} \right)^{\frac{N-3}{2}}$$

and

$$\chi_\epsilon^2(\bar{\epsilon}) = \frac{1}{\sqrt{2 \pi \bar{\epsilon}}} \exp \left( -\frac{\bar{\epsilon}}{2} \right).$$

The graphs of Eqs. (10) and (11) are shown in Fig. (1). The solid (black) lines correspond to the graphs of Eq. (10) for $N = 4$, 5, and 10; and the dashed (red) line corresponds to the graph of Eq. (11). Note that the mean of $\bar{\epsilon}$ with respect to $F_{\bar{\epsilon}}(\bar{\epsilon})$ is 1 and the variance is $2(N-1)/(N+2)$.

### 3 Derivation of velocity distribution

It is convenient to start from Eq. (5) to show a velocity distribution. By letting

$$q_i = \sqrt{\frac{m}{2}} v_i,$$

we have

$$F_q(v) = \frac{\Gamma \left( \frac{N}{2} \right) \sqrt{m}}{\Gamma \left( \frac{N-1}{2} \right) \sqrt{\pi N kT}} \left( 1 - \frac{mv^2}{NkT} \right)^{\frac{N-3}{2}}$$

and it is straightforward to show that Eq. (12) approaches the Maxwellian velocity distribution as $N$ increases. By defining

$$g(v) = \left( 1 - \frac{mv^2}{NkT} \right)^{\frac{N-3}{2}},$$

we have

$$\lim_{N \to \infty} \ln g(v) = -\frac{mv^2}{2kT}.$$
And also we have
\[ \lim_{N \to \infty} \frac{\Gamma \left( \frac{N}{2} \right) \sqrt{m}}{\Gamma \left( \frac{N-1}{2} \right) \sqrt{\pi N kT}} = \frac{\sqrt{m}}{\sqrt{2\pi kT}} \]
so that we can obtain the molecular velocity distribution when \( N \) approaches infinity as
\[ M(v) = \lim_{N \to \infty} F_v(v) = \sqrt{\frac{m}{2\pi kT}} \exp \left( -\frac{mv^2}{2kT} \right) \]
which is the Maxwellian velocity distribution in one-dimensional space.

Let us define a dimensionless velocity \( \bar{v} = v\sqrt{m/(kT)} \). Then, we have
\[ F_v(\bar{v}) = \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-1}{2} \right) \sqrt{\pi N}} \left( 1 - \frac{\bar{v}^2}{N} \right)^{\frac{N-3}{2}} \]
and
\[ M(\bar{v}) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\bar{v}^2}{2} \right). \]

In Fig. (2), we present the graphs of \( F_v(\bar{v}) \) for \( N \in \{3, 4, 5, 10\} \) and \( M(\bar{v}) \). As \( N \) increases, \( F_v(\bar{v}) \) approaches the Maxwellian distribution \( M(\bar{v}) \).
Figure 2: Graphs of $F_{\bar{v}}(\bar{v})$ for $N \in \{3, 4, 5, 10\}$ and $\bar{M}(\bar{v})$ defined by Eqs. (13) and (14). The straight line (black) is the graph of $F_{\bar{v}}(\bar{v})$ when $N = 3$ and the curved lines (black) are the graphs when $N \in \{4, 5, 10\}$. The dashed line (red) is the graph of $\bar{M}(\bar{v})$. Note that $F_{\bar{v}}(\bar{v})$ approaches $\bar{M}(\bar{v})$ as $N$ increases.

For the case of the Maxwellian distribution, the expectation values of the square of velocity and the speed are respectively obtained by

$$E_M(v^2) = \int v^2 M(v) dv = \frac{kT}{m}$$

and

$$E_M(|v|) = \int |v| M(v) dv = \sqrt{\frac{2kT}{\pi m}}.$$  

Meanwhile, for the case of the non-Maxwellian distribution $F_v(v)$, the expectation values are respectively obtained by

$$E(v^2) = \int v^2 F_v(v) dv = E_M(v^2)$$

and

$$E(|v|) = \int |v| F_v(v) dv = \sqrt{\frac{N}{2}} \frac{\Gamma(N/2)}{\Gamma((N+1)/2)} E_M(|v|).$$

We define

$$\gamma(N) = \frac{E(|v|)}{E_M(|v|)} = \sqrt{\frac{N}{2}} \frac{\Gamma(N/2)}{\Gamma((N+1)/2)}$$

and plot $\gamma(N)$ with respect to $N$ in Fig. 3. The $\gamma(N)$ that is the ratio between $E(|v|)$ and $E_M(|v|)$ rapidly approaches 1 as $N$ increases.
4 Verification

We have performed computer simulations where molecules are exchanging energies with random amounts and the results are shown in Fig. (4). The upper left figure shows a reference that is calculated by using Eq. (13) for \( N = 5 \) and 10 to be compared with the results obtained by the computer simulations. Note that the thick (red) line is for \( N = 5 \) and the dashed (black) line is for \( N = 10 \). The upper right, the lower left, and the lower right figures show the simulation results obtained by analyzing ten thousand, one hundred thousand, and one million samples, respectively. Note that the results of \( N = 5 \) (red) and 10 (gray) are overlapped. The curves obtained by using Eq. (13) are drawn in the lower right figure. We observe that the theoretical result is a very good agreement to the simulation result.

From now, let us consider a system having \( D \) degrees of freedom such as the Maxwellian distribution with respect to speed in three-dimensional space which is expressed, in a neat form with using dimensionless variables as in Eq. (14), by

\[
M_{\tilde{v}}(|\tilde{v}|) = \left( \frac{2}{\pi} \right)^{1/2} |\tilde{v}|^2 \exp \left( -\frac{1}{2} |\tilde{v}|^2 \right) \tag{16}
\]

where \( \tilde{v} = (\tilde{v}_x, \tilde{v}_y, \tilde{v}_z) \). Previously, we have obtained the energy distribution Eq. (17) that approaches the Maxwellian energy distribution with one degree of freedom as \( N \) increases. Indeed, it is straightforward to obtain the energy

![Figure 3: Log-linear plot of \( \gamma(N) \) with respect to \( N \) in Eq. (15). The ratio \( \gamma(N) \) approaches 1 as \( N \) increases.](image)
Figure 4: Speed distributions obtained by theoretical methods and computer simulations. The upper left figure shows a reference that is calculated by using Eq. (13) for $N = 5$ (thick red) and 10 (dashed black). The upper right, the lower left, and the lower right figures show the simulation results, where molecules are exchanging energies with random amounts, obtained by analyzing ten thousand, one hundred thousand, and one million samples, respectively. Note that the results of $N = 5$ (red) and 10 (gray) are overlapped in the figures of the simulation results.
distribution of $D$ degrees of freedom. The result is expressed by

$$F_\epsilon(\epsilon) = \frac{\Gamma\left(\frac{DN}{2}\right) E^{-D/2} \epsilon^{(D-2)/2}}{\Gamma\left(\frac{E}{2}\right) \Gamma\left(\frac{D}{2}\right)} \left(1 - \frac{\epsilon}{E}\right)^{\frac{D(N-1)-2}{2}}$$  \hspace{1cm} (17)$$

where $0 \leq \epsilon \leq E$. One way to obtain Eq. (17) is to define

$$\epsilon_i = q_i^2 = \sum_{p=1}^{D} q_{i,p}$$

where $q_{i,p}$ is defined by the polar coordinate system in the same manner to that of $q_i$ but with another set of variables $\varphi_p$ instead of $\phi_i$. Now if we substitute

$$E = \frac{DNkT}{2}$$

and applying a change of variables in Eq. (17) as

$$\epsilon = \frac{m|\vec{v}|^2}{2} = \frac{kT}{2}|\vec{v}|^2,$$

then we have

$$F_{|\vec{v}|}^D(|\vec{v}|) = \Gamma\left(\frac{DN}{2}\right) |\vec{v}|^{(D-1)} \left(\frac{DN}{2}\right)^{D/2} \left(1 - \frac{|\vec{v}|^2}{DN}\right)^{\frac{D(N-1)-2}{2}}$$  \hspace{1cm} (18)$$

and when $D = 3$, we have compared $F_{|\vec{v}|}^3$ to the corresponding Maxwellian distribution Eq. (16) in Fig. (5). The dashed line (red) corresponds to the result of $N$ being infinity or the Maxwellian distribution and the solid line (black) and the dot-dashed line (black) do to the results of $N = 5$ and 10, respectively. Note that the result of short tails with respect to the Maxwellian in Fig. (5) is in accordance to the experimental results of molecular evaporation of acid dimers reported in [19] and hydrogen molecules desorbing from a copper crystal [16] where the figures of the references use flight time instead of velocity so that the positions of peaks are inversed, and normalize the heights of peaks.

With a change of integration variable, it is straightforward to obtain the non-Maxwellian distribution with respect to velocity that is

$$F_{\vec{v}}^D(\vec{v}) = \Gamma\left(\frac{DN}{2}\right) (\pi DN)^{-D/2} \left(1 - \frac{|\vec{v}|^2}{DN}\right)^{\frac{D(N-1)-2}{2}}$$  \hspace{1cm} (19)$$

where the support is defined by $|\vec{v}|^2 \leq DN$. One may obtain Eq. (19) equation by using Havrda-Charvát-Tsallis entropy [20]. Note that the Maxwellian distribution in three-dimensional space $M_\vec{v}(\vec{v})$ is obtained by just multiplying the Maxwellian distributions in one-dimensional space $M(\vec{v}_x), M(\vec{v}_y), \text{ and } M(\vec{v}_z)$ as

$$M_\vec{v}(\vec{v}) = M(\vec{v}_x)M(\vec{v}_y)M(\vec{v}_z),$$
Figure 5: Speed distributions $F_{|v|}$ are shown to compare them to the Maxwellian. The dashed line (red) corresponds to the result of $N$ being infinity or the Maxwellian distribution and the solid line (black) and the dot-dashed line (black) do to the results of $N = 5$ and $10$, respectively.

Figure 6: Speed distributions calculated from $\tilde{F}_\psi(\bar{v})$. The line (blue) corresponds to the result of $N$ being infinity or the Maxwellian distribution and the cross and the circular symbols do to the results of $N = 5$ and $10$, respectively.
however, \( F_D^3(\vec{v}) \) of Eq. (19) with \( D = 3 \) is different from the tensor product of the distributions of one degree of freedom as

\[
\tilde{F}_\vec{v}(\vec{v}) = \left( \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-1}{2} \right) \sqrt{\pi N}} \right)^3 \left( 1 - \frac{\vec{v}_x^2}{N} \right) \left( 1 - \frac{\vec{v}_y^2}{N} \right) \left( 1 - \frac{\vec{v}_z^2}{N} \right) \frac{(1 - \vec{v}^2)}{N-3} \cdot (20)
\]

The result of a numerical calculation of the speed distribution obtained by using Eq. (20) is shown in Fig. (6). The line (blue) corresponds to the result of \( N \) being infinity or the Maxwellian distribution and the cross and the circular symbols do to the results of \( N = 5 \) and 10, respectively. We have obtained the results of the numerical calculations by generating quite evenly distributed points on the surface of spheres having different radii that are equivalent to speeds and calculating probabilities by summing up the values of Eq. (20) for each set of points having the same radius. We emphasize that the directional independence of the velocity distribution is only valid in the Maxwellian distribution where \( N \) is assumed to be infinity. Otherwise, the directional probability is dependent. Note that when \( \vec{v} = (0, 0, 0) \) and \( N = 6.02 \times 10^{23} \) which is the Avogadro’s number, \( F_D^3(\vec{v}) - \tilde{F}_\vec{v}(\vec{v}) = -1.05 \times 10^{-25} \) and \( F_D^3(\vec{v}) - M_\vec{v}(\vec{v}) = 3.59 \times 10^{-8} \) which means the infinity assumption generates more error than the directional independence assumption even in a scale of the Avogadro’s.

5 Conclusion

In conclusion, we have obtained distribution functions for a system with a limited number of entities, which are non-Maxwellian (non-Gaussian) and non-chi-squared, though, they approach the corresponding Maxwellian distributions as the number of entities approaches infinity. If we use a terminology of a system of ideal gas, we confirm that a molecular velocity distribution for a system with even a limited number of molecules can be approximated by the Maxwellian molecular velocity distribution when the number of molecules of the system is sufficiently enough. Otherwise, a system having a small number of molecules should be considered by using the non-Maxwellian molecular velocity distribution derived in this paper. When comparing the non-Maxwellian molecular velocity distribution to the Maxwellian molecular velocity distribution, the former has shorter tails than the latter and the expectation value of the square of velocity is identical; however, the expectation value of the speed is not. We have demonstrated that the directional independence of the Maxwellian distribution is a special character achieved from the number of molecules being infinity. A number of entities whose abilities are equal will have an asset distribution according to the derived non-chi-square distribution after participating in a game where an arbitrary portion of one’s asset is transferred to other arbitrary entity in a repetitive way.
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References

[1] I. Zouganelis. Measuring suprathermal electron parameters in space plasmas: Implementation of the quasi-thermal noise spectroscopy with kappa distributions using in situ Ulysses/URAP radio measurements in the solar wind. *Journal of Geophysical Research: Space Physics*, 113(A8), 2008.

[2] M. Maksimovic, V. Pierrard, and P. Riley. Ulysses electron distributions fitted with Kappa functions. *Geophysical Research Letters*, 24(9):1151–1154, 1997.

[3] W. G. Pilipp, H. Miggenrieder, M. D. Montgomery, K. H. Mühlhäuser, H. Rosenbauer, and R. Schwenn. Characteristics of electron velocity distribution functions in the solar wind derived from the Helios Plasma Experiment. *Journal of Geophysical Research: Space Physics*, 92(A2):1075–1092, 1987.

[4] M. Lockwood, B. J. I. Bromage, R. B. Horne, J.-P. St-Maurice, D. M. Willis, and S. W. H. Cowley. Non-Maxwellian ion velocity distributions observed using EISCAT. *Geophysical Research Letters*, 14(2):111–114, 1987.

[5] N. Rubab and G. Murtaza. Debye length in non-Maxwellian plasmas. *Physica Scripta*, 74(2):145–148, jul 2006.

[6] N. Rubab and G. Murtaza. Dust-charge fluctuations with non-Maxwellian distribution functions. *Physica Scripta*, 73(2):178–183, Jan 2006.

[7] D. Hubert. Non-Maxwellian velocity distribution functions and incoherent scattering of radar waves in the auroral ionosphere. *Journal of Atmospheric and Terrestrial Physics*, 46(6):601 – 611, 1984.

[8] S. Zaheer and G. Murtaza. Weibel instability with non-Maxwellian distribution functions. *Physics of Plasmas*, 14(2):022108, 2007.

[9] T. Hori, M. D. Bowden, K. Uchino, and K. Muraoka. Measurement of non-Maxwellian electron energy distributions in an inductively coupled plasma. *Applied Physics Letters*, 69(24):3683–3685, 1996.

[10] D. Benedetto, E. Caglioti, J. A. Carrillo, and M. Pulvirenti. A non-Maxwellian steady distribution for one-dimensional granular media. *Journal of Statistical Physics*, 91(5):979–990, Jun 1998.

[11] J. W. Chamberlain and F. J. Campbell. Rate of Evaporation of a Non-Maxwellian Atmosphere. *Astrophysical Journal*, 149:687, Sep 1967.
[12] A. Gehring, T. Grasser, H. Kosina, and S. Selberherr. Simulation of hot-electron oxide tunneling current based on a non-Maxwellian electron energy distribution function. *Journal of Applied Physics*, 92(10):6019–6027, 2002.

[13] G. Kaniadakis and P. Quarati. A set of stationary non-Maxwellian distributions. *Physica A: Statistical Mechanics and its Applications*, 192(4):677 – 690, 1993.

[14] M. Knudsen. *The Kinetic Theory of Gases: Some Modern Aspects*. Methuen, 1950.

[15] A. Chambers. *Modern Vacuum Physics*. Masters series in physics and astronomy. CRC Press, 2004.

[16] W. Jitschin and G. Reich. Molecular velocity distribution at large knudsen numbers. *Journal of Vacuum Science & Technology A*, 9(5):2752–2756, 1991.

[17] J. Denur. Speed-dependent weighting of the Maxwellian distribution in rarefied gases: A second-law paradox? *Foundations of Physics*, 37:1685–1706, 2007.

[18] C. Cercignani. *Mathematical methods in kinetic theory*. Plenum Press, 1969.

[19] M. Faubel and Th. Kisters. Non-equilibrium molecular evaporation of carboxylic acid dimers. *Nature*, 339(6225):527–529, 1989.

[20] J. W. Shim. Entropy formula of N-body system. *Scientific Reports*, 10(1):14029, 2020.