Analysis of the SGA method for obtaining energy spectra

Patricio Cordero and Jamil Daboul*
Departamento de Física, Universidad de Chile, Santiago, Chile

We analyze and clarify how the SGA (spectrum generating algebra) method has been applied to different potentials. We emphasize that each energy level $E_\nu$ obtained originally by Morse belongs to a different $\mathfrak{so}(2,1)$ multiplet. The corresponding wavefunctions $\Psi_\nu$ are eigenfunctions of the compact generators $H_\nu^0$ with the same eigenvalue $k_0$, but with different eigenvalues $q_\nu$ of the Casimir operators $Q$. We derive a general expression for all effective potentials which have $\Psi_{k_\nu+\nu+m}(r) \propto (J^r)^m \Psi_{k_\nu}(r)$ as eigenfunctions, without using supersymmetry formalism. The different actions of SGA is further illustrated by two diagrams.

PACS numbers: 03.65.Fd, 03.65.-w, 02.20.-a

I. INTRODUCTION

Pauli [1] in 1926 was the first person who calculated the energy spectrum of a Hamiltonian algebraically. He did it for the hydrogen atom. Since then his procedure was followed by many people: It is essentially based on relating the total Hamiltonian $H$ to the Casimir operator of the symmetry or degeneracy algebra, whose generators commute with $H$. (The symmetry algebra of the N-dimensional hydrogen atom is usually identified differently for different energies $E$. By replacing the Hamiltonian $H$ by its eigenvalues, one obtains $\mathfrak{so}(N+1)$, $\mathfrak{so}(N)$, and $\mathfrak{so}(N,1)$, for $E < 0$, $E = 0$ and $E > 0$, respectively [2,3,4]. However, by keeping $H$ as operator, one obtains infinite-dimensional Kac-Moody loop algebras, of standard type for even $N$ and of twisted type for odd $N$ [5,6].) The hydrogen atom and the isotropic oscillator have infinite number of states for each eigenvalue of angular momentum $\ell$. Their symmetry algebras for the bound states are compact and therefore the raising and lowering operators of these algebras can only generate a finite number of states with different values of $\ell$, for each degenerate energy eigenvalue $E_\nu$. To relate the infinite number of states for a fixed $\ell$ it is necessary to use non-compact algebras. The smallest appropriate noncompact algebra is $\mathfrak{so}(2,1)$, because it has an infinite-dimensional representations which are bounded from below, and are denoted by $D^+(\lambda)$. The $\mathfrak{so}(2,1)$ is generated by three generators $K_i$ which commute as follows

$$
\begin{align*}
[K_0, K_1] &= iK_2, \\
[K_2, K_0] &= iK_1, \\
[K_1, K_2] &= -iK_0.
\end{align*}
$$

(1)

The $K_0$ is the compact generator, while $K_1$ and $K_2$ are the non-compact ones. Once a simultaneous eigenstate of $K_0$ and $Q$ is found, then one can generate an infinite number of states by applying the raising and lowering operators $K_\pm$ repeatedly. Because of this property $\mathfrak{so}(2,1)$ has been called a spectrum-generating algebra (SGA).

The idea of using the SGA was very popular in the 1960’s and 1970’s and numerous papers [7] investigated different approaches and potentials. In fact, Wybourne devoted a whole chapter of his popular book [8, Chap. 18] to describe how one obtains the energy levels of various systems from the $\mathfrak{so}(2,1)$ SGA.

In particular, $\mathfrak{so}(2,1)$ has been applied to obtain the S-wave bound states for a important subclass of Natanzon potentials [9], which has only a finite number of bound states, such as the Morse potential. The question arises, where are the other infinitely many energy levels that can be generated by $K_\pm$?

The purpose of the present paper is to show that in the many applications of $\mathfrak{so}(2,1)$ involving Natanzon-type potentials [10], the raising and lowering operators $K_\pm$ were actually never used to generate the finite number of bound states for $\ell = 0$. Instead, it turned out that for every energy level, a different $\mathfrak{so}(2,1)$ representation was used.

For clarity, we shall concentrate on the Morse potential,

$$
V_M(r) := V_0 \left[ e^{-2(r-r_0)/a} - 2e^{-(r-r_0)/a} \right].
$$

(2)

which is the simplest example of a Natanzon potential which has a finite number of bound states for each $\ell$. The potential (2) was introduced by Morse in 1929 to obtain the vibrational levels of diatomic molecules [11]. He obtained only the S-wave solutions, by explicitly solving the Schrödinger equation. The Morse potential has since become popular, especially among chemists, because it allows disintegration of diatomic molecules, in contrast to the shifted harmonic-oscillator potential, $(k/2)(r-r_0)^2$.

In 1970 Cordero and Hojman [12] (hereafter will be quoted as CH) reproduced algebraically the finite S-wave energy spectrum for the Morse potential [11], by using $\mathfrak{so}(2,1)$. Therefore we shall analyze this paper in particular, and show that CH did not use a single $\mathfrak{so}(2,1)$ representation to get the energy levels, but actually a different $\mathfrak{so}(2,1)$ representation for each energy level. We explain how CH succeeded nevertheless in obtaining the correct energy spectrum. This should be worthwhile, since the CH paper succeeded for the first time to produce a finite number of bound states from a SGA.

In Sec. II we review the SGA method for the 3-D oscillator and in Sec. III we review the paper of CH on the Morse potential, and point out the basic difference in the two cases. We shall see that $K_0$ commutes with $H_1$ in the case of the oscillator, but it does not commute with $H_2$ in the Morse case; in the latter case the Casimir operator $Q$ commutes with $H_3$. In Sec. IV we derive the effective poten-

*On Sabatical leave from the Physics Department, Ben Gurion University of the Negev, 84105 Beer Sheva, Israel (e-mail: daboul@bgu.ac.il)
tials \( V_M(k_m, r_m; r) \) (see \[43\] below), which have the functions \( \Psi_{\nu, \nu + m}(r) \propto (J'_{\nu})^m \Psi_{\nu, \nu}(r) \) as eigenstates, where \( \Psi_{\nu, \nu}(r) \) are the wavefunctions of the Morse potential \[4\] and \( J'_\nu \) are raising operators. Finally, in Sec. V we give a summary.

II. ENERGY SPECTRUM OF THE 3-D HARMONIC OSCILLATOR

In this section we review the derivation of the energy spectra for partial-wave Hamiltonians \( H_\ell \) of the 'generalized' (by adding the \( \epsilon \)) harmonic oscillator

\[
H_\ell = -\hbar^2 \frac{\partial^2}{\partial r^2} + \alpha \left( \ell(\ell + 1) + \epsilon \right) \hbar^2 + \frac{k}{2} r^2,
\]

\[
= \frac{\hbar^2}{2M a^2} \left( -\alpha^2 \frac{d^2}{dr^2} + \frac{\alpha \alpha^2}{r^2} + \frac{\alpha^4 M k}{\hbar^2} r^2 \right)
\]

\[
= \delta \left( \frac{\alpha^2}{2M a^2} + \frac{\alpha \alpha^2}{r^2} + \left( \frac{\alpha^4 M k}{\hbar^2} \right)^2 y^2 \right)
\]

(3)

for all \(-\infty < k < \infty\), where we use the notation

\[
\omega \equiv \sqrt{|k|/M}, \quad y \equiv r/a,
\]

\[
\alpha \equiv \ell(\ell + 1) + \epsilon, \quad \text{and} \quad \delta \equiv \frac{\hbar^2}{2M a^2}
\]

(4)

Note that \( \alpha \ell \) are dimensionless constants. We introduced the scaling factor \( \alpha \) which has the dimension of length, and whose value will be determined below in \[5\]. Thus, \( y \) becomes a dimensionless variable, and \( \delta \) has the dimension of energy.

Note that we defined \( H_\ell \) for the attractive oscillator \( \ell > 0 \) and also for repulsive oscillator \( \ell < 0 \). Usually the oscillator is only studied for attractive case, but \( k \) was defined in \[13\] for all real values of \( k \), in order to study the contraction of the algebra \( su(2) \) to the Euclidean algebra \( e(2) \) or to the Heisenberg algebra \( h(3) \), as \( k \to 0 \).

A. Realization of \( so(2,1) \) generators for the oscillator

It is easy to check that the following three generators satisfy the commutation relations \[1\] of \( so(2,1) \)

\[
K_0(\alpha) := -\frac{d^2}{dy^2} + \frac{\alpha}{y} + \frac{y^2}{16},
\]

\[
K_1(\alpha) := -\frac{d^2}{dy^2} + \frac{\alpha}{y} - \frac{y^2}{16} = K_{-1} - \frac{y^2}{8},
\]

\[
K_2 := -\frac{i}{2} \left( \frac{y}{dy} + \frac{1}{2} \right),
\]

and thus yield different realizations of \( so(2,1) \) for every value of the constant \( \alpha \). (The generators in \[5\] follow from those in \[3\] Eq.(18.7)) by multiplying \( K_0 \) and \( K_1 \) by a minus sign, which leaves the commutation relations unchanged, and then by replacing \( \alpha \) by \(-\alpha\). The above generators are so constructed, that if they are applied to an eigenfunction of \( K_0 \), then their Casimir operator is related to \( \alpha \), as follows

\[
Q := K_0^2 - K_1^2 - K_2^2 = \left( \frac{\alpha}{4} - \frac{3}{16} \right) I =: q(\alpha) I,
\]

(6)

where \( I \) is identity operator. If we factorize \( q \) as follows

\[
q(\alpha) := \lambda (\lambda - 1) = \frac{\alpha}{4} - \frac{3}{16},
\]

(7)

and solve the quadratic equation in \( \lambda \), we obtain for \( \lambda \) the values

\[
\lambda_\pm(\alpha) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4q},
\]

\[
\lambda_\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{\alpha + 1}, \quad \text{so that} \quad \alpha \geq -1/4.
\]

It is interesting to note that the realization \[5\] for \( \alpha = 0 \) also yields the SGA of the one-dimensional harmonic oscillator. In this case, \[5\] yields the well-known values \( \lambda_- = 1/4 \) and \( \lambda_+ = 3/4 \), which define the two infinite-dimensional representations, the Fock states \( |2m\rangle \) and \( |2m + 1\rangle \), \( m = 0, 1, 2, \cdots \) respectively \[13\].

B. Relating \( H_\ell \) to the generators of \( so(2,1) \)

The realization \[5\] is suitable for generating the oscillator states. In fact, \( H_\ell \) becomes proportional to \( K_0 \) for \( k > 0 \) or to \( K_1 \) for \( k < 0 \), if we scale the coefficient of \( y^2 \) in \[3\] to 1/16, i.e.

\[
\sigma \equiv \frac{a^4 M^2 \omega^2}{\hbar^2} \Rightarrow \frac{1}{16}
\]

(8)

which is equivalent to choosing \( a \), as follows

\[
a = \left( \frac{\hbar^2}{16M |k|} \right)^{1/4} = \left( \frac{\hbar}{4M \omega} \right)^{1/2}.
\]

(9)

With this choice of \( a \), we obtain

\[
\delta = \frac{\hbar^2}{2M a^2} = 2\hbar \omega.
\]

(10)

Hence \[5\] becomes

\[
H_\ell = \delta \left( -\frac{d^2}{dy^2} + \frac{\alpha}{y} + \frac{\omega^2}{16} \right),
\]

\[
= \begin{cases} 2\hbar \omega K_0 & \text{for} \ k > 0 \ \text{and} \\ 2\hbar \omega K_1 & \text{for} \ k < 0 \end{cases}
\]

(11)

Hence, the eigenfunctions of \( K_0 \) will be the eigenfunctions of \( H_\ell \) for \( k > 0 \). Since the eigenvalues of \( K_0 \) are given by \[8\] \[13\] \( \nu + \lambda, \ \nu = 0, 1, 2, \cdots \), it follows that the energy spectrum for the \( \ell \)-partial wave is given, for \( \alpha = \ell(\ell + 1) + \epsilon \), by \[10\] \[36\] and \[8\]

\[
E_{\nu,\ell} = 2\hbar \omega (\nu + \lambda_\nu(\alpha_\ell)) \quad \nu = 0, 1, 2, \cdots,
\]
which tends, in the limit $\epsilon \to 0$ to

$$E_{v,\ell} = \hbar \omega (n + 3/2), \quad \text{where} \quad n := 2v + \ell$$

It can be seen that the shift in $\lambda_\ell$ in the eigenvalues of $K_0(\alpha_t)$, due to $\epsilon$, is also multiplied by the factor $2\hbar \omega$. Note that for a fixed $\ell$ the energy levels $E_{v,\ell}$ increase by $2\hbar \omega$ rather than $\hbar \omega$.

In the case of the harmonic oscillator all the energy levels for a given $\ell$ belong to a single irreducible representation of $\mathfrak{so}(2,1)$, so that all eigenfunctions $\psi_{\ell,v}$ can be obtained by applying powers of the raising operators $K_+(\alpha_t)$ on the ground state $|\Psi_{\ell,0}\rangle$, as illustrated in Fig. 1.

![FIG. 1: The eigenstates of the harmonic oscillator for a given $\ell$ belong to a single irreducible representation of $\mathfrak{so}(2,1)$ which is characterized by $\lambda_\ell$. The excited states for a given $\ell$ can be obtained by applying the raising operator $K_+(\alpha_t)$, as illustrated by the vertical arrows.](image)

### III. ALGEBRAIC DERIVATION OF THE MORSE SPECTRUM

In CH the authors reproduced the S-wave energy spectrum algebraically by using $\mathfrak{so}(2,1)$. We now show that, in contrast to the oscillator, one cannot obtain the S-wave spectrum of the Morse potential by using one representation of $\mathfrak{so}(2,1)$. In fact, it turns out, that for each energy level $E_{v}$ one needs a different realization $J_0^\alpha$ (see [5] below) of the $\mathfrak{so}(2,1)$ generators. This fact is essentially implied by the formulas they used, but it was never stated clearly neither in the above paper, nor in subsequent papers on the Natanzon potentials [6]. Before explaining their procedure, we introduce new notation and also note some changes of notation from that in CH.

#### A. New notation

We try to make the present paper self-contained. However, if the reader likes to consult the original paper of CH, he should note the following changes of notations and definitions which we have made, so that it becomes easier to check the dimensions in the formulas and to simplify some of them.

In CH the Morse potential was written as $V_{old}(r) := D \left[ e^{-2a_{old}(r-r_0)} - 2he^{-a_{old}r-r_0} \right]$. In this paper we define $a := 1/a_{old}$, so that $a$ has the dimension of length. We also replace the unnecessary parameter $b$ by 1, by defining $V_0 = Db^2$ and adjusting the value of $r_0$. With the new definition the potential $V(r)$ has its minimum at $r = r_0$.

We find it quite useful to introduce the following dimensionless constant

$$k_0 \equiv \frac{1}{\hbar} \sqrt{2Ma^2V_0} = \sqrt{V_0/\mathcal{E}}, \quad \text{so that} \quad V_0 = \mathcal{E}k_0^2., \quad (14)$$

where $\mathcal{E}$ is defined by the expression (4). We shall see that $k_0$ is equal to the eigenvalue of $K_0$, when it acts on the $S$–wave solutions.

The following dimensionless exponential function is also very useful

$$h(r) := k_0e^{-(r-r_0)/a} \quad (15)$$

Note that $h(r_0) := k_0$ and that our $h(r)$ in (15) is equal to twice the $h_{old}(r)$ in CH; this redefinition simplifies many formulas, by making factors of 1/2 and 1/4 unnecessary.

We can now write the Morse potential in terms of $h(r)$ simply, as follows

$$V_M(r) = V_0 \frac{h(r)}{k_0} \left( \frac{h(r)}{k_0} - 2 \right) = \mathcal{E} \left( h(r)^2 - 2k_0h(r) \right) \quad (16)$$

#### B. Relating the partial Hamiltonian $H_\ell$ to $J_0$

The following three operators were defined in CH, which depend on a parameter $\beta$, which the authors called $-E/\mathcal{E}$,

$$J_0(\beta) := \frac{1}{2h(r)} \left[ -a^2\Delta_r + \beta + h^2(r) \right],$$

$$J_1(\beta) := J_0 - \hbar,$$

$$J_2 := i \left( a \frac{d}{dr} + \frac{a}{r} - \frac{1}{2} \right), \quad (17)$$

where $h(r)$ is the exponential function (15) and $\Delta_r$ is the radial part of the Laplacian $\Delta$, i.e.

$$\Delta_r := \frac{d^2}{dr^2} + 2 \frac{d}{dr} + \left( \frac{d}{dr} + \frac{1}{r} \right)^2 \quad (18)$$
These $J_i$ commute exactly as the $K_i$ in (1) and hence they also yield a realization of the non-compact $\mathfrak{so}(2,1)$ algebra. The realization $J_i$ in (17) is defined such that it acts on the radial part $\Psi_\nu(r) = R_\nu / r$ of the wave function $\Psi_\nu(r) = R_\nu / Y_\nu(\theta, \phi)$. It can be transformed into the realization $K_i$ of $\mathfrak{so}(2,1)$ which acts on $R(r)$, as follows

$$K_i := r J_i \frac{1}{r} \tag{19}$$

The Casimir operator $Q$ of the $J_i$ is related to the parameter $\beta$ by

$$Q := J_0^2 - J_1^2 - J_2^2 = (\beta - \frac{1}{4}) I =: q(\beta) I \tag{20}$$

so that we may replace $\beta$ in (17) by $q + 1/4$. Note that the above expression for $q(\beta)$ for the $J_i$ in (17) is different from that for $q(\alpha)$ in (6). The corresponding eigenvalue $\lambda(\beta)$ is defined such that it acts on the wave function $\Psi_\nu(r)$, and yield a realization of the non-compact $\mathfrak{so}(2,1)$ algebra.

The main observation in this section is that all eigenfunctions of $H_0$ must be eigenfunctions of different $J_0^0$, but with the same eigenvalue $\lambda_0$. In contrast, all eigenfunctions of $H_0^0$ of the oscillator are eigenfunctions of Casimir operator $Q$ for a fixed $q = \lambda_0 - 1$ and different eigenvalues of $K_0(\alpha_0)$, namely $k_0 = \nu + \lambda_0$, $\nu = 0, 1, 2, \cdots$.

D. The traditional derivation of the bound state solutions

For completeness and for comparison, we review the traditional derivation of the bound state solutions (11-16). Making the change of variables

$$\xi = 2h(r) = 2k_0 e^{-r(x-o)/a} \tag{28}$$

in Schrödinger’s equation (24) and using $\Psi(r) = R(\xi)/r$, we obtain

$$R''(\xi) + \frac{1}{\xi} R'(\xi) + \left(\frac{1}{4} - \frac{k_0}{\xi} + \frac{E}{\xi^2} \right) R(\xi) = 0 \tag{29}$$

Again, substituting $R(\xi) = e^{-\xi/2} \xi^s F(\xi)$ into (31), where $s = \sqrt{-E/\xi}$, yields a differential equation for $F(s + 1 - \xi) F'(\xi) + (k_0 - 1/2 - s) F(\xi) = 0$,

whose solutions are the confluent hypergeometric functions $\_1F_1(s + 1/2 - k_0, 2s + 1; \xi)$. These functions become polynomials and yield normalizable wavefunctions for $s = k_0 -...
$1/2 - \nu \geq 0$ where $\nu$ is nonnegative integer. This condition yields the energy levels

$$-E_\nu = \varepsilon s^2 = \varepsilon (k_0 - 1/2 - \nu)^2 > 0, \quad \nu = 0, 1, 2, \ldots, \nu_{\text{max}}$$

and the corresponding wave functions

$$\Psi_\nu(r) = \frac{1}{r} R(2h(r)) \approx e^{-h (k_0 - 1/2 - \nu)} f_1(-\nu, 2k_0 - 2\nu; 2h). \quad (32)$$

It is interesting to note that if $k_0 = n + 1/2$ we obtain $E_n = 0$, but this solution does not correspond to a bound state, since the solution is not normalizable for $s = 0 [11]$. This result can be understood intuitively, because we are dealing with S-wave solutions, so that there is no potential barrier which can prevent the particle from escaping to infinity. In contrast, an $E = 0$ solution would probably be normalizable for $\ell \geq 1$, since in the latter case the effective potential $U(r) = \ell(\ell + 1)/(2M^2r) + V_M(r)$ approaches $r \to \infty$ from above, and thus provides a potential barrier of infinite range. This was illustrated by Daboul and Nieto [17], who studied $E = 0$ solutions for a class of potentials.

IV. EFFECTIVE MORSE POTENTIALS GENERATED BY $J^\nu_\pm = J_\pm(\varepsilon E_\nu/\varepsilon')$

Eqs. [17] define a realization $J_1(\beta)$ of the so(2,1) algebra for every value of the parameter $\beta \geq 0$. However, since we are interested in eigenfunctions of $J_0$ which are normalizable, we restrict the values of $\beta$ to the discrete set $\beta_n = -E_n/\varepsilon'$, where the $E_n$ are the discrete eigenvalues of $H_S$. We denote the corresponding generators by

$$J^\nu_\pm := J_1(\beta_\nu) = J_\pm(\varepsilon E_\nu/\varepsilon'), \quad \nu = 0, 1, 2, \ldots \quad (33)$$

In [35] we found that

$$J^0_0 \Psi_{\lambda_\nu, \nu_\nu}(r) = k_0 \Psi_{\lambda_\nu, \nu_\nu}(r) \quad (34)$$

Using the raising and lowering operators

$$J_\pm := J_1 \pm iJ_2, \quad (35)$$

which obey

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0, \quad (36)$$

the following functions can be defined, by acting with $(J^\nu_\pm)^m$ onto $\Psi_{\lambda_\nu, \nu_\nu}$

$$\Psi_{\lambda_\nu, \nu_\pm \nu \nu}(r) := (J^\nu_\nu)^m \Psi_{\lambda_\nu, \nu_\nu}(r). \quad (37)$$

If these states exist and are normalizable, then they must be eigenfunctions of $J_0$ with eigenvalues $k_0 \pm m$, since

$$J^0_0 \Psi_{\lambda_\nu, \nu_\nu \nu \nu}(r) = J^\nu_\nu (J^\nu_\pm)^m \Psi_{\lambda_\nu, \nu_\nu}(r) = (k_0 \pm m) (J^\nu_\nu)^m \Psi_{\lambda_\nu, \nu_\nu}(r) = (k_0 \pm m) \Psi_{\lambda_\nu, \nu_\nu \nu \nu}(r) \quad (38)$$

and we used the following general relations

$$J^\nu_0 (J^\nu_\pm)^m = (J^\nu_\pm)^m J^\nu_0 + [J^\nu_\nu, (J^\nu_\pm)^m] = (J^\nu_\pm)^m J^\nu_0 \pm m(J^\nu_\perp)^m$$

Multiplying [39] by the factor $\varepsilon' 2h(r)$ and substituting the expression [25] for the $J^\nu_0$ operator, yields the following differential equations

$$0 = \varepsilon' 2h(r) [J^\nu_0 \nu_\nu - (k_0 \pm m)] \Psi_{\lambda_\nu, \nu_\pm \nu \nu}(r) = \varepsilon' -a^2 \Delta r + \varepsilon' h^2(r) - (k_0 \pm m)2h(r) \Psi_{\lambda_\nu, \nu_\pm \nu \nu}(r) \quad (39)$$

The radial functions $\Psi_{\lambda_\nu, \nu_\pm \nu \nu}(r)$ can therefore be interpreted as eigenstates of the S-wave Schrödinger equation for the following potentials

$$V_{\text{eff}}(m, r) = \frac{\hbar^2}{2ma^2} [h^2(r) - (k_0 + m)2h(r)] = V_0 \left[ e^{-2(r-r_0)/a} - 2 \frac{k_0}{k_0 - (k_0 - k_0/r_0)}/a \right] \quad (40)$$

Each of these effective potentials has its minimum at $r_m$, where

$$\frac{r_m}{a} = \frac{m_0}{a} - \ln \left( \frac{k_0 + m_0}{k_0} \right) = \frac{m_0}{a} - \ln \left( \frac{k_0}{k_0} \right), \quad (41)$$

so that

$$e^{-(r-r_0)/a} = \frac{k_0}{k_0} e^{-(r-r_0)/a}. \quad (42)$$

By substituting this expression into [39], we obtain

$$V_M(k_m, r_m) = V_m \left[ e^{-2(r-r_0)/a} - 2 e^{-(r-r_0)/a} \right] = \varepsilon' k_m^2 \left[ e^{-2(r-r_0)/a} - 2 e^{-(r-r_0)/a} \right]. \quad (43)$$

The effective potentials [43] look exactly as the original Morse potential [24], except that the parameters $(k_0, r_0)$ get changed into $(k_m, r_m)$. As $m$ increases the effective potentials [43] will have shorter range and their minima

$$V_{\text{eff}}(r_m) = -V_0 \left( \frac{k_0 + m}{k_0} \right)^2 = -\varepsilon' k_m^2 \quad (44)$$

become deeper and deeper, decreasing almost quadratically with $m$. The associated energy eigenvalues are

$$E_\nu^{(m)} = -\varepsilon' \left( k_m - \frac{1}{2} - \nu \right)^2, \quad \nu = 0, 1, \ldots, \lfloor k_m - 1/2 \rfloor, \quad (45)$$

$$= -\varepsilon' \left( k_0 + m - \frac{1}{2} - \nu \right)^2 \quad (46)$$

$$= E_{\nu+n}^{(m+n)} \quad \text{for } n \geq -\nu. \quad (47)$$
A. Connection to SUSY-QM

Using the relation (45) we obtain immediately

\[ E_{v} \equiv E_{v}^{(0)} = -\varepsilon \left( k_{0} - \frac{1}{2} - v \right)^{2} = -\varepsilon \left( k_{-v} - \frac{1}{2} \right)^{2} \]

\[ = E_{0}^{(-v)} \quad \text{for} \quad v = 1, 2, \cdots, v_{\text{max}}. \]  (46)

Hence the energies \( E_{v}, \ v = 1, 2, \cdots, v_{\text{max}} \) of the excited states of the original Morse potential (2) are equal to ground state energies \( E_{0}^{(-v)} \) of the effective potentials \( V_{\text{eff}}(-v, r) \), \( v = 1, 2, \cdots, v_{\text{max}} \). This is one of the interesting results of the quantum-mechanical supersymmetry (SUSY-QM) formalism [18]. We derived it here without using the latter formalism and without the need of finding out the relevant supersymmetric potential \( W(x,a) \).

To understand the above result more thoroughly we give a second proof that the \( \Psi_{\lambda_{v}, v}(r) \equiv (J_{v}^{
u})^{m} \Psi_{\lambda_{v}, v}(r) \) are eigenfunctions of \( H_{5} \) with the potential \( V_{\text{eff}}(m, r) \).

For this, we first note the important relation (42); it tells us that the \( h(r) \) defined in (39) is invariant under the transformation \((k_{0}, r_{0}) \) to \((k_{m}, r_{m})\), i.e.

\[ h(r) \equiv h(k_{0}, r_{0}) = k_{0} e^{-(r-r_{0})/\alpha} = k_{m} e^{-(r-r_{m})/\alpha} \]

\[ = h(k_{m}, r_{m}). \]  (47)

Hence, also the generators

\[ J_{v}^{
u} \equiv J_{1}(k_{0}, r_{0}, -E_{v}/\varepsilon; r) = J_{m}(k_{m}, r_{m}, -E_{v}/\varepsilon; r), \]

as defined in (17), do not depend on \( m \), but differ for different \( E_{v}/\varepsilon \).

Now we compare the following two equations:

\[ J_{v}^{
u}(k_{0}, r_{0}; r) \Psi_{k_{0}-v, v+m}(k_{0}, r_{0}; r) = (k_{0} + m) \Psi_{k_{0}-v, v+m}(k_{0}, r_{0}; r) \]

\[ J_{v}^{
u}(k_{m}, r_{m}; r) \Psi_{k_{m}-v, v+m}(k_{m}, r_{m}; r) = k_{m} \Psi_{k_{m}-v, v+m}(k_{m}, r_{m}; r), \]  (49)

\[ (50) \]

where the first follows from (38) and the second follows from the condition (42) on the Morse eigenfunctions for the potential \( V_{\text{eff}}(m, r) \). The two \( \Psi \) are solutions of the same differential operator \( J_{v}^{
u} \) with the same eigenvalue. Since the eigenvalues of \( J_{v}^{
u} \) are not degenerate, the two functions must be the same, except for a constant factor. Hence, \( \Psi_{\lambda_{v}, v}(r) \) is the \((v + m)\)th excited state of the Morse potential \( V_{\text{eff}}(m, r) \).

In Fig. 2 we display the states \( \Psi_{\lambda_{v}, v}(r) \), defined in (37), in the \((\lambda_{v}, n)\) plane, where \( \lambda_{v} = k_{0} - v = \sqrt{|E_{v}|/\varepsilon} - 1/2 \) and \( n \equiv v \pm m \pm 1 \). All the eigenstates for a fixed effective potential \( V_{\text{eff}}(k_{m}, r_{m}; r) \) lie on a single diagonal line. But, the states along a horizontal line have the energy eigenvalue, but belong to different potentials \( V_{\text{eff}}(m, r) \). The states \( \Psi_{\lambda_{v}, v}(r) \) for the original Morse potential (2) can be obtained from the ground states of \( V_{\text{eff}}(-v, r) \) by applying \((J_{v}^{
u})^{\nu}\), as illustrated in the figure by horizontal arrows.

V. SUMMARY AND CONCLUSIONS

The S-wave energy levels of the Morse potential have been known since 1929. Thus, the algebraic derivation of these levels has not brought anything new, as far as applications are concerned. What is interesting in the algebraic treatment is how the mathematical formalism works. In the present paper we did clarify when it works and how it works.

We showed in particular that \( \mathfrak{so}(2,1) \) is applied in a completely different manner to the oscillator and the Morse potentials: for the harmonic oscillator a single raising operator \( K_{e}^{r}(\ell) \) yields all the eigenstates for the given \( \ell \). In contrast, for the Morse potential the raising operators \( J_{v}^{r}(\ell = 0) \) map the wave functions of the original Morse potential onto wavefunctions of other Morse potentials. This contrast is illustrated by figures 1 and 2. Thus, the message of the present paper is that one should be more critical and check carefully how the SGA are applied.

For example, with our present insight we wanted to check how Wybourne obtained the spectrum of the Morse potential in [8, §8.1]. It turned out that Wybourne did not even give the \( \mathfrak{so}(2,1) \) generators for every potential, as was done, for exam-
Instead, he used a slightly different version of the realization of \( \mathfrak{so}(2,1) \) for the harmonic oscillator, and showed that the algebra describes the energy spectrum of the following differential equation \( L \) (Eq. (18.19))

\[
\left( \frac{d^2}{dy^2} + \frac{a}{y^2} + by^2 + c \right) \Psi(y) = 0.
\]

Then he simply transformed the Schrödinger equation of different potentials to the above differential equation. But this is essentially what Morse did, already in 1929, by transforming the S-wave Schrödinger equation to a differential equation, which was solved by Schrödinger. It is also what Landau and Lifschitz \( \S 23 \) did for many potentials, by transforming their Schrödinger equations to the confluent hypergeometric equation.

There is no doubt that group theory helps us understand many results in physics, such as the degeneracies of the eigenstates of the hydrogen atom. It has many useful applications in elementary particles such as flavor and color \( \text{SU}(3) \). However, it seems to us that many of the papers on the SGA can even mislead non-experts, as we demonstrated in the present paper. They might believe, for example, that one could obtain a finite number of states by just using the raising operators of a single realization of the \( \mathfrak{so}(2,1) \) algebra, as is the case for the oscillator.

Apparently, some experts have noticed that. In a well-written and easy to read article \( \text{[13]} \), the authors give a review of the super symmetry (SUSY-QM) formalism and of shape invariant potentials and mention that the Morse potential is of the invariant type. This means that one can obtain the excited states of the Morse potential from its ground state, not by applying the supersymmetric raising and lowering operators \( A^\dagger(a_i) \) and \( A(a_i) \) operators, which have the same structure, but which depend on different parameters \( a_i \). We write this statement, using their notation, as follows

\[ \Psi^{(\nu)}_{n+1}(x,a_0) \sim A^\dagger(a_1)A^\dagger(a_2) \cdots A^\dagger(a_n) \Psi^{(\nu)}_0(x,a_0) \]

We showed in Sec. \( \text{[IV.A]} \) that for the Morse case the following equivalent statement holds

\[ \Psi_{0,-\nu,V}(r,k_0) \sim (J_\nu^V)^{V} \Psi_{k+\nu,0}(r;k_0 V) , \]

where all the raising operators are equal, as we showed in \( \text{[48]} \). Thus, we gave an explanation of why the excited states \( \Psi_{0,-\nu,V}(r,k_0) \) of the original Morse potential \( \text{[2]} \) are related to the ground states \( \Psi_{k+\nu,0}(r;k_0 V) \) of related effective potentials \( V \), without using the SUSY formalism. We illustrate this action in Fig. \( \text{[1]} \) by applying \( J_\nu^V \) three times on the ground state, \( \Psi_{k,0}(r;k=-3) \), of \( V(-3) \) and obtain the third excited state of \( V(0) \). A more recent and detailed review of the SUSY formalism can be found in \( \text{[19]} \).

It is interesting to note that by using a ‘direct approach’, it was possible to obtain many results \( \text{[20]} \), among them a construction of new quasi-exactly solvable deformation of the Morse potential, which the authors have not been able to obtain by Lie-algebraic methods.

The Morse potential is one of the simplest examples of the general class of Natanzon’s potentials \( \text{[10]} \) which have been studied by using SGA algebraic methods \( \text{[3]} \). Therefore a deeper understanding of the SGA method in the Morse’s case should help us understand the more general cases as well.

Acknowledgments

One of us (P.C.) acknowledges partial financial support from Fondecyt grant 1030993 and Fonlap grant 1198002. J. Daboul would like to thank Prof. Victor Fuenzalida, Chairman of the Physics Department, for his hospitality and Dr. Juan Paulo Wiff for his great help with the computer.

---

[1] W. Pauli, Z. Phys. 36, 336 (1926); English translation in: Sources of Quantum Mechanics, (North-Holland, Amsterdam, 1967), B.L. van der Waerden (Ed.).
[2] L. I. Schiff, Quantum mechanics, 3rd Ed., McGraw-Hill, New York, 1968.
[3] A. Sudbery, Quantum mechanics and the particles of nature, Cambridge University Press, Cambridge, 1986.
[4] M. M. Nieto, Am. J. Phys. 47, 1067, 1979.
[5] J. Daboul, P. Slodowy and C. Daboul, Phys. Lett. B 317, 321 (1993); C. Daboul, J. Daboul, and P. Slodowy, The Dynamical Algebra of the Hydrogen Atom as a Twisted Loop Algebra, in " Group Theoretical Methods in Physics", A. Arima, T. Eguchi and N. Nakanishi (Eds.), (World Scientific, Singapore, 1995) p. 175-178. [hep-th/9408080]. These papers contain a review of the basic concepts of the Kac-Moody algebras.
[6] C. Daboul and J. Daboul, Phys. Lett. B 425, 135 (1998).
[7] Y. Dothan, M. Gell-Mann and Y. Ne’eman, Phys. Lett. 17 148 (1965); A.O. Barut, P. Budini and C. Fronsdal, Proc. Roy. Soc. A 291, 106 (1966); R.H. Pratt and T.F. Jordan, Phys. Rev 148, 1276 (1966); R. Musto, Phys. Rev. 148, 1247 (1966); M. Bandier and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966); R.C. Hwa and J. Nuyts, 145, 1188 (1966); C. Fronsdal, Phys. Rev. 156, 1665 (1967); P. Budini, Acta Phys. Austria, Suppl. 4, 118 (1967); J. Lanik, Nucl. Phys. 2B, 263 (1967); ibid 5B, 523 (1968); P. Aldrovandi and P. Leal Ferreira, Lett. Nuovo Cimento 1, 317 (1969); J. Lanik, Czech. Journ. Phys. 19B, 1540 (1969); P. Cordero, Lett. Nuovo Cimento 4, 164 (1970); P. Cordero and G.C. Ghirardi, Nuovo Cimento 2A, 217 (1971).
[8] B. G. Wybourne, Classical Groups for Physicists, John Wiley, New York, 1974.
[9] P. Cordero and S. Salamó, J. Phys. A: Math. Gen. 24 5299, (1991); P. Cordero, S. Salamó, Journal of Math. Phys. 35 3301-3307 (1994).
[10] G. A. Natanzon, Teor. Mat. Fiz. 38, 146 (1979).
[11] P. M. Morse, Phys. Rev. 34, 57 (1929)
[12] P. Cordero and S. Hojman, Lettere al Nuovo Cimento 4, 1123 (1970)
[13] J. Daboul, Contractions via Kac-Moody formalism, in Proceedings of the 25th International Colloquium on Group Theoretical Methods in Physics, (Cocoyoc, Mexico, 2-6 August 2004), George S. Pogosyan, Luis Edgar Vicent and Kurt Bernardo Wolf (Eds.), IOP Publishing Ltd, to appear in 2005.
[14] A. Vourdas and A. Wünsche, J. of Phys. A: Math. Gen. 31, 9341 (1998).
[15] A. O. Barut and C. Fronsdal, Proc. Roy. Soc. A 287, 532 (1965)
[16] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, 2. Ed., Pergamon Press, 1965.
[17] J. Daboul and M. M. Nieto, Phys. Lett. A 317, 321 (1993); J. Daboul and M. M. Nieto, Int. J. Mod. Phys. A 11, 3801 (1996).
[18] A. Gangopadhyaya, J. V. Mallow and U. P. Sukhatme, Phys. Rev. 58, 4287 (1998).
[19] M. R. Kibler and M. Daoud, On supersymmetric quantum mechanics, quant-ph 0409169
[20] D. Gómez-Ullate, N. Kamran and R. Milson, Quasi-Exact Solvability and the direct approach to invariant subspaces, to be published in J. Math. Phys.