AN ARZELÀ-ASCOLI THEOREM FOR THE
HAUSDORFF MEASURE OF NONCOMPACTNESS

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ABSTRACT. We generalize the Arzelà-Ascoli theorem in the space
of continuous maps on a compact interval with values in Euclidean
$N$-space by providing a quantitative link between the Hausdorff
measure of noncompactness in this space and a natural measure
of non-uniform equicontinuity. The proof hinges upon a classical
result of Jung’s on the Chebyshev radius.

1. Introduction and statement of the main result

Fix $N \in \mathbb{N}_0$ and let $C = C([a, b], \mathbb{R}^N)$ be the space of continuous
$\mathbb{R}^N$-valued maps on the compact interval $[a, b]$. Let $|\cdot|$ stand for the
Euclidean norm on $\mathbb{R}^N$ and recall that a set $F \subset C$ is said to be

1. uniformly bounded iff there exists a universal constant $M > 0$
such that $|f(x)| \leq M$ for all $f \in F$ and $x \in [a, b],$

2. uniformly relatively compact iff each sequence in $F$ contains a
subsequence converging uniformly to a map in $C,$

3. uniformly equicontinuous iff for each $\epsilon > 0$ there exists $\delta > 0$
such that $|f(x) - f(y)| < \epsilon$ for all $f \in F$ and $x, y \in [a, b]$ with
$|x - y| < \delta.$

Denote the collection of uniformly bounded sets in $C$ as $B_C$. In this
setting the following theorem is a classic ([L93]).

Theorem 1.1. (Arzelà-Ascoli) For $F \in B_C$ the following are equiva-

1. $F$ is uniformly relatively compact.

2. $F$ is uniformly equicontinuous.

Recall that $C$ is a Banach space under the supremum norm

$$\|f\|_\infty = \sup_{x \in [a, b]} |f(x)|$$

and that for a set $F \in B_C$ the Hausdorff measure of noncompactness
([BG80], [WW96]) is given by

$$\mu_H(F) = \inf \sup \inf \|f - g\|_\infty,$$

where

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the first infimum running through all finite sets $\mathcal{F}_0$ in $\mathcal{C}$. It is well known that $\mathcal{F}$ is uniformly relatively compact if and only if $\mu_H(\mathcal{F}) = 0$.

For a set $\mathcal{F} \in \mathcal{B}_C$ we define the measure of non-uniform equicontinuity as

$$\mu_{uec}(\mathcal{F}) = \inf_{\delta > 0} \sup_{f \in \mathcal{F}} \sup_{|x - y| < \delta} |f(x) - f(y)|,$$

the second supremum running through all $x, y \in [a, b]$ with $|x - y| < \delta$.

It is clear that $\mathcal{F}$ is uniformly equicontinuous if and only if $\mu_{uec}(\mathcal{F}) = 0$.

In [BG80] it was shown that $\mu_{uec}$ is a measure of noncompactness on the space $\mathcal{C}$ (Theorem 11.2).

Theorem 1.2, our main result, generalizes Theorem 1.1 by linking $\mu_H$ and $\mu_{uec}$ quantitatively. The proof is deferred to section 3.

**Theorem 1.2.** (Arzelà-Ascoli for the Hausdorff measure of noncompactness) For $\mathcal{F} \in \mathcal{B}_C$ we have

$$\frac{1}{2} \mu_{uec}(\mathcal{F}) \leq \mu_H(\mathcal{F}) \leq \left( \frac{N}{2N + 2} \right)^{1/2} \mu_{uec}(\mathcal{F})$$

In particular, if $N = 1$, then

$$\mu_H(\mathcal{F}) = \frac{1}{2} \mu_{uec}(\mathcal{F}).$$

2. A preliminary result of Jung’s

For a bounded set $A \subset \mathbb{R}^N$, the diameter is given by

$$\text{diam}(A) = \sup_{x, y \in A} |x - y|$$

and the Chebyshev radius by

$$r(A) = \inf_{x \in \mathbb{R}^N} \sup_{y \in A} |x - y|.$$

It is well known that for each bounded set $A \subset \mathbb{R}^N$ there exists a unique $x_A \in \mathbb{R}^N$ such that

$$\sup_{y \in A} |x_A - y| = r(A).$$

The point $x_A$ is called the Chebyshev center of $A$. A good exposition of the previous notions in a general normed vector space can be found in [H72], section 33.

Theorem 2.1 provides a relation between the diameter and the Chebyshev radius of a bounded set in $\mathbb{R}^N$. A beautiful proof can be found in [BW41]. For extensions of the theorem we refer to [A85], [AFS00], [R02] and [NN06].

**Theorem 2.1.** (Jung) For a bounded set $A \subset \mathbb{R}^N$ we have

$$\frac{1}{2} \text{diam}(A) \leq r(A) \leq \left( \frac{N}{2N + 2} \right)^{1/2} \text{diam}(A).$$
3. Proof of Theorem 1.2

We first need two simple lemmas on linear interpolation.

For \( c_0 \in \mathbb{R}^N \) and \( r \in \mathbb{R}_0^+ \) we denote the closed ball with center \( c_0 \) and radius \( r \) as \( B^*(c_0, r) \).

**Lemma 3.1.** Consider \( c_1, c_2 \in \mathbb{R}^N \) and \( r \in \mathbb{R}_0^+ \) and assume that \( B^*(c_1, r) \cap B^*(c_2, r) \neq \emptyset \). Let \( L \) be the \( \mathbb{R}^N \)-valued map on the compact interval \([\alpha, \beta] \) defined by

\[
L(x) = \frac{\beta - x}{\beta - \alpha} c_1 + \frac{x - \alpha}{\beta - \alpha} c_2.
\]

Then, for all \( x \in [\alpha, \beta] \) and \( y \in B^*(c_1, r) \cap B^*(c_2, r) \),

\[
|L(x) - y| \leq r.
\]

**Proof.** The calculation

\[
|L(x) - y| = \left| \frac{\beta - x}{\beta - \alpha} (c_1 - y) + \frac{x - \alpha}{\beta - \alpha} (c_2 - y) \right|
\]

\[
\leq \frac{\beta - x}{\beta - \alpha} |c_1 - y| + \frac{x - \alpha}{\beta - \alpha} |c_2 - y|
\]

\[
\leq \frac{\beta - x}{\beta - \alpha} r + \frac{x - \alpha}{\beta - \alpha} r
\]

\[
= r
\]

proves the lemma. \( \square \)

**Lemma 3.2.** Consider \( c_1, c_2, y_1, y_2 \in \mathbb{R}^N \) and \( \epsilon > 0 \) and suppose that \( |c_1 - y_1| \leq \epsilon \) and \( |c_2 - y_2| \leq \epsilon \). Let \( L \) and \( M \) be the \( \mathbb{R}^N \)-valued maps on the compact interval \([\alpha, \beta] \) defined by

\[
L(x) = \frac{\beta - x}{\beta - \alpha} c_1 + \frac{x - \alpha}{\beta - \alpha} c_2
\]

and

\[
M(x) = \frac{\beta - x}{\beta - \alpha} y_1 + \frac{x - \alpha}{\beta - \alpha} y_2.
\]

Then

\[
\|L - M\|_{\infty} \leq \epsilon.
\]

**Proof.** The calculation

\[
|L(x) - M(x)| = \left| \frac{\beta - x}{\beta - \alpha} (c_1 - y_1) + \frac{x - \alpha}{\beta - \alpha} (c_2 - y_2) \right|
\]

\[
\leq \frac{\beta - x}{\beta - \alpha} |c_1 - y_1| + \frac{x - \alpha}{\beta - \alpha} |c_2 - y_2|
\]

\[
\leq \frac{\beta - x}{\beta - \alpha} \epsilon + \frac{x - \alpha}{\beta - \alpha} \epsilon
\]

\[
= \epsilon
\]

proves the lemma. \( \square \)
Proof. (of Theorem 1.2) Let $F \in B_C$.

We first show that

$$\mu_H(F) \leq \left( \frac{N}{2N+2} \right)^{1/2} \mu_{\text{uec}}(F).$$

Fix $\epsilon > 0$. Then, $F$ being uniformly bounded, we can take a constant $M > 0$ such that

$$\forall f \in F, \forall x \in [a, b] : |f(x)| \leq M. \quad (1)$$

Pick a finite set $Y \subset \mathbb{R}^N$ for which

$$\forall z \in B^*(0, 3M), \exists y \in Y : |y - z| \leq \epsilon. \quad (2)$$

Now let $0 < \alpha \leq 2M$ be so that $\mu_{\text{uec}}(F) < \alpha$, i.e. there exists $\delta > 0$ for which

$$\forall f \in F, \forall x, y \in [a, b] : |x - y| < \delta \Rightarrow |f(x) - f(y)| \leq \alpha. \quad (3)$$

Then choose points

$$a = x_0 < x_1 < \ldots < x_{2n} < x_{2n+1} = b,$$

put

$$I_0 = [0, x_2[,$$

$$I_k = ]x_{2k-1}, x_{2k+2}[ \text{ if } k \in \{1, \ldots, n-1\},$$

$$I_n = ]x_{2n-1}, x_{2n+1}]$$

and assume that we have made this choice such that

$$\forall k \in \{0, \ldots, n\} : \text{diam}(I_k) < \delta. \quad (4)$$

Furthermore, for each $(y_0, \ldots, y_{2n+1}) \in Y^{2n+2}$, let $L_{(y_0, \ldots, y_{2n+1})}$ be the $\mathbb{R}^N$-valued map on $[a, b]$ defined by

$$L_{(y_0, \ldots, y_{2n+1})}(x) = \begin{cases} 
\frac{x_{1}-x}{x_{1}-x_{0}} y_{0} + \frac{x-x_{0}}{x_{1}-x_{0}} y_{1} & \text{if } x \in [x_{0}, x_{1}] \\
\vdots \\
\frac{x_{k+1}-x}{x_{k+1}-x_{k}} y_{k} + \frac{x-x_{k}}{x_{k+1}-x_{k}} y_{k+1} & \text{if } x \in [x_{k}, x_{k+1}] \\
\vdots \\
\frac{x_{2n+1}-x}{x_{2n+1}-x_{2n}} y_{2n} + \frac{x-x_{2n}}{x_{2n+1}-x_{2n}} y_{2n+1} & \text{if } x \in [x_{2n}, x_{2n+1}] 
\end{cases}$$

and put

$$F_0 = \{ L_{(y_0, \ldots, y_{2n+1})} \mid (y_0, \ldots, y_{2n+1}) \in Y^{2n+2} \}.$$

Then $F_0$ is a finite subset of $C$. Now fix $f \in F$ and let $c_{f,k}$ stand for the Chebyshev center of $f(I_k)$ for each $k \in \{0, \ldots, n\}$. It follows from $(3)$ and $(4)$ that $\text{diam} f(I_k) \leq \alpha$ and thus, by Theorem 2.1,

$$\forall k \in \{0, \ldots, n\} : \sup_{x \in I_k} |c_{f,k} - f(x)| \leq \left( \frac{N}{2N+2} \right)^{1/2} \alpha. \quad (5)$$
Let \( \tilde{f} \) be the \( \mathbb{R}^N \)-valued map on \([a, b]\) defined by

\[
\tilde{f}(x) = \begin{cases} 
  \frac{x^2 - x}{x_2 - x_1} c_{f,0} & \text{if } x \in [x_0, x_1] \\
  \frac{x^2 - x}{x_2 - x_1} c_{f,0} + \frac{x - x_1}{x_2 - x_1} c_{f,1} & \text{if } x \in [x_1, x_2] \\
  c_{f,1} & \text{if } x \in [x_2, x_3] \\
  \frac{x - x_1}{x_4 - x_3} c_{f,1} + \frac{x - x_2}{x_4 - x_3} c_{f,2} & \text{if } x \in [x_3, x_4] \\
  \vdots & \\
  \frac{x_{2k} - x}{x_{2k} - x_{2k-1}} c_{f,k-1} + \frac{x - x_{2k-1}}{x_{2k} - x_{2k-1}} c_{f,k} & \text{if } x \in [x_{2k-1}, x_{2k}] \\
  c_{f,k} & \text{if } x \in [x_{2k}, x_{2k+1}] \\
  \frac{x_{2k+1} - x}{x_{2k+2} - x_{2k+1}} c_{f,k+1} & \text{if } x \in [x_{2k+1}, x_{2k+2}] \\
  \vdots & \\
  \frac{x_{2n} - x}{x_{2n} - x_{2n-1}} c_{f,n-2} + \frac{x - x_{2n-3}}{x_{2n} - x_{2n-3}} c_{f,n-1} & \text{if } x \in [x_{2n-3}, x_{2n-2}] \\
  c_{f,n-1} & \text{if } x \in [x_{2n-2}, x_{2n-1}] \\
  \frac{x_{2n+1} - x}{x_{2n+2} - x_{2n+1}} c_{f,n} & \text{if } x \in [x_{2n-1}, x_{2n}] \\
  c_{f,n} & \text{if } x \in [x_{2n}, x_{2n+1}] 
\end{cases}
\]

Then (\ref{3.1}) and Lemma \ref{3.1} learn that

\[
\|\tilde{f} - f\|_{\infty} \leq \left( \frac{N}{2N + 2} \right)^{1/2} \alpha. 
\] (6)

Also, it easily follows from (1) and (5) that \( \|\tilde{f}\|_{\infty} \leq 3M \) and thus (2) allows us to choose \((y_0, \ldots, y_{2n+1}) \in \mathbb{R}^{2n+2}\) such that

\[
\forall k \in \{0, \ldots, 2n + 1\} : |y_k - \tilde{f}(x_k)| \leq \epsilon. 
\] (7)

Combining (7) and Lemma \ref{3.1} reveals that

\[
\|L_{(y_0, \ldots, y_{2n+1})} - \tilde{f}\|_{\infty} \leq \epsilon. 
\] (8)

But then we have found \( L_{(y_0, \ldots, y_{2n+1})} \) in \( \mathcal{F}_0 \) for which, by (6) and (8),

\[
\|L_{(y_0, \ldots, y_{2n+1})} - f\|_{\infty} \leq \left( \frac{N}{2N + 2} \right)^{1/2} \alpha + \epsilon
\]

which, by the arbitrariness of \( \epsilon \), entails that \( \mu_H(\mathcal{F}) \leq \left( \frac{N}{2N + 2} \right)^{1/2} \alpha \) and thus, by the arbitrariness of \( \alpha \), the inequality

\[
\mu_H(\mathcal{F}) \leq \left( \frac{N}{2N + 2} \right)^{1/2} \mu_{uec}(\mathcal{F})
\]

is established.

We now prove that

\[
\frac{1}{2} \mu_{uec}(\mathcal{F}) \leq \mu_H(\mathcal{F})
\]

Let \( \alpha > 0 \) be so that \( \mu_H(\mathcal{F}) < \alpha \). Then there exists a finite set \( \mathcal{F}_0 \subset \mathcal{C} \) such that for all \( f \in \mathcal{F} \) there exists \( g \in \mathcal{F}_0 \) for which \( \|g - f\|_{\infty} \leq \alpha \).
Take $\epsilon > 0$. Since $\mathcal{F}_0$ is uniformly equicontinuous there exists $\delta > 0$ so that

\[
\forall g \in \mathcal{F}_0, \forall x, y \in [a, b] : |x - y| < \delta \Rightarrow |g(x) - g(y)| \leq \epsilon.
\]  

(9)

Now, for $f \in \mathcal{F}$, choose $g \in \mathcal{F}_0$ such that

\[
\|g - f\|_\infty \leq \alpha.
\]  

(10)

Then, for $x, y \in [a, b]$ with $|x - y| < \delta$, we have, by (9) and (10),

\[
|f(x) - f(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| \leq 2\alpha + \epsilon
\]

which, by the arbitrariness of $\epsilon$ reveals that $\mu_{uec}(\mathcal{F}) \leq 2\alpha$ and thus, by the arbitrariness of $\alpha$, the inequality

\[
\frac{1}{2}\mu_{uec}(\mathcal{F}) \leq \mu_H(\mathcal{F})
\]

holds.

\[\square\]

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