Majorana representation for dissipative spin systems

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Abstract

The Majorana representation of spin operators allows for efficient field-theoretical description of spin-spin correlation functions. Any N-point spin correlation function is equivalent to a 2N-point correlator of Majorana fermions. For a certain class of N-point spin correlation functions (including “auto” and “pair-wise” correlations) a further simplification is possible, as they can be reduced to N-point Majorana correlators. As a specific example we study the Bose-Kondo model. We develop a path-integral technique and obtain the spin relaxation rate from a saddle point solution of the theory. Furthermore, we show that fluctuations around the saddle point do not affect the correlation functions as long as the latter involve only a single spin projection. For illustration we calculate the 4-point spin correlation function corresponding to the noise of susceptibility.

Keywords: Majorana, fermions, dissipation, spin correlators

Spin systems are notoriously difficult to describe using field-theoretic methods due to non-Abelian nature of the spin operators [1]. Often one tries to circumvent the problem by mapping the spins onto a system of either bosons or fermions, for which a standard field theory can be developed [2]. There is no unique recipe for such an approach. Several formulations have been put forth for solving specific problems, including the Jordan-Wigner [3] and Holstein-Primakoff [4] transformations, the Martin [5] Majorana-fermion and Abrikosov [6] fermion representations, as well as the Schwinger-boson [7–10] and slave-fermion [11–16] techniques.

The Jordan-Wigner transformation is the only “exact” mapping between the spin-1/2 and fermion operators as it preserves not only the operator algebra but also the dimensionality of the Hilbert space. However, it is a non-local transformation specific to one spatial dimension [2] where it is often applied to Bethe-Ansatz-solvable models or their variations (a generalization to higher dimensions does exist [17, 18], but it lacks the simplicity of the original approach). All other mappings suffer from the following two problems: (i) the Hilbert space of the fermionic or bosonic operators (the “target” Hilbert space) is enlarged as compared to the original spin Hilbert space, and (ii) the resulting theory in the fermionic or bosonic representation needs to be treated perturbatively, which often leads to complicated vertex structures (see e.g. [19]). The former issue may be resolved by additional constraints [10, 16] or by projecting out unphysical states [6] at the expense of further complications such as the appearance of non-Abelian gauge fields [9–10, 13, 16].

The Majorana-fermion representation, suggested by Martin [5], offers a possibility to avoid both types of problems mentioned above: (i) The target Hilbert space is indeed enlarged, but merely contains two (or more) copies of the original physical spin Hilbert space [20, 21]. Matrix elements of physical quantities between different copies of the original Hilbert space vanish and thus the correlation functions may be evaluated directly in the target Hilbert space (this fact is often not fully appreciated; below we justify and further illustrate this statement). (ii) The Martin transformation [5] represents the spin operators in terms of bilinear combinations of Majorana fermions. The resulting Majorana theory appears to be interacting (similarly to the situation with the Jordan-Wigner transformation [3] and other fermionic representations [11–16]) and N-point spin correlation functions are equivalent to 2N-fermion
correlators. However, a certain class of \(N\)-point spin correlation functions can be reduced to \(N\)-point Majorana correlators [21,22]. In this case, complicated vertex structures do not arise. This significant simplification applies to correlation functions which involve only a single spin (“auto-correlation” functions) or an even number of spin operators for each spin (“pair-wise” correlations). In the present paper we generalize this approach to higher-order correlation functions and develop a technique for practical calculations based on the Keldysh formalism [23].

We illustrate our general arguments by a specific example. For a model of paramagnetic spins coupled to an Ohmic bath [24] we determine the four-point correlation function describing the noise of susceptibility [25]. This quantity is closely related to the recently measured inductance fluctuations in SQUIDs [26]. In addition, it is a direct measure of non-Gaussian fluctuations that are relevant also in other physical contexts [27–29]. The noise of susceptibility is distinct from the better-known four-point correlator “second noise”, which always comprises a significant Gaussian contribution [30–32].

The paper is divided into two major parts. In Section 1 we introduce the Majorana representation of spin operators and address general issues related to this approach. In Subsection 1.2 we discuss the fact that spin-spin correlation functions can be directly calculated as correlations of the Majorana fermions despite the enlargement of the fermion Hilbert space. In Subsection 1.3 we demonstrate the simplification of the theory for the case of auto- and pair-wise-correlations. In Subsections 1.4 and 1.5 we formulate of the above correlators within the Keldysh path-integral approach.

In Section 2 we apply our general arguments to the problem of the noise of higher-order spin correlators in the Bose-Kondo model. Here we introduce the path-integral formalism in the Matsubara representation and show that certain higher-order spin correlators can be calculated in the saddle-point approximation. Further technical details are provided in the Appendices. In Appendix A we present the traditional diagrammatic perturbation theory. In Appendix B we justify the saddle-point approximation used in Section 2. Finally, in Appendix C we analyze a remarkable gauge freedom in our model.

1. Majorana Representation for Spin Operators

1.1. The Martin transformation

In this paper we focus on the following Majorana representation of the spin-1/2 operators introduced by Martin in 1959 [5] in the framework of generalized classical dynamics:

\[
\hat{S}^a = -\frac{i}{2} \epsilon_{\alpha\beta\gamma} \hat{\eta}_\alpha \hat{\eta}_\beta \hat{\eta}_\gamma, \quad \hat{S}^x = -i\hat{\eta}_y \hat{\eta}_z, \quad \hat{S}^y = -i\hat{\eta}_z \hat{\eta}_x, \quad \hat{S}^z = -i\hat{\eta}_x \hat{\eta}_y.
\]  

(1)

The Majorana operators obey the Clifford algebra

\[
\{\hat{\eta}_\alpha, \hat{\eta}_\beta\} = \delta_{\alpha\beta}, \quad \hat{\eta}^2 = \frac{1}{2}, \quad \hat{\eta}^\dagger_\alpha = \hat{\eta}_\alpha.
\]  

(2)

where \(\{,\}\) denotes the anticommutator. The Majorana representation has been used in a variety of physical contexts, cf. Refs. [2,21,22,33]. (We remark that another normalization of the Majorana operators, \(\hat{\eta}^2 = 1\), used in Ref. [21] only changes some numerical prefactors at intermediate stages of the calculation.)

The above representation with the real Majorana fermion operators,

\[
\hat{\eta}^\dagger_\alpha = \hat{\eta}_\alpha,
\]  

(3)

perfectly reproduces the SU(2) algebra of the operators \(\hat{S}^a\)

\[
[\hat{S}^a, \hat{S}^b] = i\epsilon_{a\beta\gamma} \hat{S}^\beta,
\]  

(4)

and explicitly preserves the spin-rotation symmetry.

Applying standard field-theoretical methods to fermionic systems implies the existence of a Fock space. In a faithful representation of a spin 1/2, the dimensionality of the fermionic Fock space should coincide with the dimensionality of the Hilbert space of the spin. For example, the Jordan-Wigner transformation represents a system of \(N\) spins-1/2 with the \(2^N\)-dimensional Hilbert space in terms of \(N\) fermions with the \(2^N\)-dimensional Fock space.
However, each Jordan-Wigner fermion may be expressed in terms of two Majorana fermions. In contrast, the Martin transformation \( \mathbf{1} \) represents each spin in terms of three Majorana fermions and hence does not preserve the dimensionality of the spin Hilbert space.

One method of dealing with this issue is to express the Majorana (or “real fermion”) \( \hat{\eta} \)-operators in terms of “complex” or “Dirac” fermions (this is a common slang used to distinguish usual fermions from Majoranas; these fermions do not necessarily obey the Dirac equation) and use their respective standard Fock spaces. This requires an even number of Majorana fermions. The simplest possibility is to add one auxiliary Majorana (cf. the drone-fermion fermions do not necessarily obey the Dirac equation) and use their respective standard Fock spaces. This requires an “complex” or “Dirac” fermions (this is a common slang used to distinguish usual fermions from Majoranas; these

1.2. Equivalence of spin and Majorana correlation functions

Now we show explicitly that correlation functions of spin-1/2 operators can be directly computed in the Majorana representation \( \mathbf{1} \) (regardless of the explicit construction of the Majorana Hilbert space). Indeed, the equation \( \mathbf{1} \) gives a representation of SU(2) with \( \hat{S}_z = 3/4 \). Thus, it is necessarily a direct sum of a certain number of irreducible, two-dimensional spin-1/2 representations, i.e., we obtain an integer number of copies of the spin (the actual number is determined by the particular choice of the number of auxiliary Majorana operators). The spin operators \( \mathbf{1} \) do not switch between the copies, thus any trace of an operator built out of spin operators \( \mathbf{1} \) is given by the number of copies times the trace in a two-dimensional spin space. This leads directly to the desired result as we discuss in more detail below.

Let us remind the reader that any function of the spin-1/2 operators (i.e., of Pauli matrices) is in fact a linear function. This follows from the following relation for the spin-1/2 operators

\[
\hat{S}^\alpha \hat{S}^\beta = \frac{i}{2} \epsilon_{\alpha\beta\gamma} \hat{S}^\gamma + \frac{1}{4} \delta_{\alpha\beta}.
\]

For an arbitrary number of operators describing the same spin this implies that

\[
\hat{S}^{a_1} \ldots \hat{S}^{a_n} = a_0 \hat{S}^\alpha + a_0, \quad a_0, a_0, a_i, a_i \in \mathbb{C},
\]

where the complex numbers \( a_0, a_0, a_i, a_i \) depend on the sequence \( \{a_i\} \). Since spin operators are traceless, the trace over the 2-dimensional \((d_\mathbf{S} = 2)\) spin Hilbert space \( \text{Tr}_\mathbf{S} \) of the above combination of spin operators is given by the constant term \( a_0 \):

\[
\text{Tr}_\mathbf{S} \{ \hat{S}^{a_1} \ldots \hat{S}^{a_n} \} = d_\mathbf{S} a_0(a_1, \ldots, a_n), \quad d_\mathbf{S} = 2.
\]

The Martin representation \( \mathbf{1} \) preserves the commutation relations as well as the relation \( \mathbf{5} \), and hence the equality \( \mathbf{6} \) remains valid in the Majorana representation (where the spin operators should be understood as Majorana bilinears). In particular, the coefficients \( a_i \) remain the same. The trace over the Majorana Hilbert space can be performed by noting that Majorana bilinears are traceless due to their anticommutation properties. Again [cf. Eq. \( \mathbf{7} \)], the only remaining contribution is given by the constant term \( a_0 \):

\[
\text{Tr}_\mathbf{M} \{ \hat{S}^{a_1} \ldots \hat{S}^{a_n} \} = d_\mathbf{M} a_0(a_1, \ldots, a_n),
\]

where \( d_\mathbf{M} \) is the dimension of the Majorana Hilbert space. Thus, \textit{tracing an arbitrary product of the spin-1/2 operators over the spin and Majorana Hilbert spaces yields the same result up to a numerical factor, determined by the dimensionality of the Hilbert spaces).}

Similar arguments were put forth in Ref. \( \mathbf{20} \) in the context of the drone-fermion representation. The results of this Subsection were implied in Refs. \( \mathbf{21, 22} \) and given without proof in Ref. \( \mathbf{25} \).

The above statement can be readily generalized to an arbitrary ensemble of spins. Indeed, any function of spin operators is still linear in the components of each spin. If the operators on the left-hand side of Eq. \( \mathbf{6} \) describe more than one spin, then on the right-hand side additional terms appear, which contain all possible products of operators.
related to different spins (for example, in the case of two spins the right-hand side reads \( a_0 + a_\alpha \hat{S}_0^\alpha + b_\alpha \hat{S}_0^\alpha + c_{\alpha\beta} \hat{S}_0^\alpha \hat{S}_0^\beta \)). However, all such additional terms are still traceless, and hence the only change in Eqs. (7) and (8) will be in the constants \( d_S \) and \( d_M \).

Consider now a real-time spin auto-correlation function:

\[
\langle \hat{S}^\alpha(t_1) \ldots \hat{S}^\alpha(t_n) \rangle = \frac{\text{Tr} \left[ \hat{S}^\alpha(t_1) \ldots \hat{S}^\alpha(t_n) \hat{\rho} \right]}{\text{Tr} \left[ \hat{\rho} \right]},
\]

where \( \hat{\rho} = \exp(-\beta \hat{H}) \) is the non-normalized Gibbs density matrix. If in addition the spins are coupled to other degrees of freedom denoted collectively by \( \xi \) (this is also described by \( \hat{H} \)), then following the line of arguments presented above we can write \( \hat{\rho} = \hat{\rho}_0(\xi) + \hat{S}^\alpha \hat{\rho}_0(\xi) \), where \( \hat{\rho}_0(\xi) \) and \( \hat{\rho}_0(\xi) \) are matrices in the \( \xi \)-space. Moreover, \( \hat{\rho}_0(\xi) \) is the reduced density matrix describing the rest of the system (the \( \xi \)-degrees of freedom). In this case, the partition function may be written as

\[
\text{Tr} \left[ \hat{\rho} \right] = \text{Tr}_X \text{Tr}_\xi \left[ \hat{\rho} \right] = \text{Tr}_X \text{Tr}_\xi \left[ \hat{\rho}_0(\xi) + \hat{S}^\alpha \hat{\rho}_0(\xi) \right] = d_s \text{Tr}_X \left[ \hat{\rho}_0(\xi) \right].
\]

Now, each Heisenberg spin operator \( \hat{S}^\alpha(t) \) is related to the Schrödinger operators (i.e., the Pauli matrices) by the time evolution operator \( \hat{U}(t, \tau) = \exp[-i \int_\tau^t d\tau' \hat{H}(\tau')] \). The latter can be expanded similarly to the density matrix. Thus we can write \( \hat{S}^\alpha(t_1) \ldots \hat{S}^\alpha(t_n) \hat{\rho} = \hat{A}_0(\xi) + \hat{S}^\alpha \hat{\rho}_0(\xi) \), where \( \hat{A}_0(\xi) \) and \( \hat{\rho}_0(\xi) \) are matrices in \( \xi \)-space. As a result, we can formally perform the trace over the spin variables in the auto-correlation function (9) and find

\[
\langle \hat{S}^\alpha(t_1) \ldots \hat{S}^\alpha(t_n) \rangle_s = \frac{\text{Tr}_X \text{Tr}_\xi \left[ \hat{A}_0(\xi) + \hat{S}^\alpha \hat{\rho}_0(\xi) \right]}{d_s \text{Tr}_X \left[ \hat{\rho}_0(\xi) \right]} = \frac{\text{Tr}_X \left[ \hat{A}_0(\xi) \right]}{\text{Tr}_X \left[ \hat{\rho}_0(\xi) \right]},
\]

In the Majorana representation the structure of the above equations remains the same. The averages can be formally calculated similarly to Eq. (8). As a result, we arrive at the expression

\[
\langle \hat{S}^\alpha(t_1) \ldots \hat{S}^\alpha(t_n) \rangle_M = \frac{\text{Tr}_X \text{Tr}_M \left[ \hat{A}_0(\xi) + \hat{S}^\alpha \hat{A}_0(\xi) \right]}{d_M \text{Tr}_X \left[ \hat{\rho}_0(\xi) \right]} = \frac{\text{Tr}_X \left[ \hat{A}_0(\xi) \right]}{\text{Tr}_X \left[ \hat{\rho}_0(\xi) \right]},
\]

which is identical to Eq. (11). The generalization to the case of general (multi-spin) correlation function is straightforward (see above). Thus we have demonstrated that spin correlation functions can be calculated with the help of the Majorana representation (1) without any projection onto “physical” states:

\[
\langle \hat{S}^\alpha(t_1) \ldots \hat{S}^\alpha(t_n) \rangle_s = \langle \hat{S}^\alpha(t_1) \ldots \hat{S}^\alpha(t_n) \rangle_M.
\]

The operators on the right-hand of Eq. (13) represent the Majorana bilinears (1).

### 1.3 Simplified representation for autocorrelation functions

The arguments presented in the previous Section show that any \( N \)-point spin correlation function may be represented in terms of a \( 2N \)-point correlation function of Majorana fermions. Here we demonstrate that for a certain class of correlation functions this correspondence can be significantly simplified [21, 22, 36].

Let us rewrite Eq. (1) as follows

\[
\hat{\Theta} = \Theta \hat{n}_\alpha, \quad \Theta = -2i\hat{n}_\alpha \hat{n}_\alpha, \quad \Theta^2 = 1/2.
\]

One can easily verify that the operator \( \Theta \) commutes with all three Majorana operators \( \eta_\alpha \). Consequently, this operator also commutes with any Hamiltonian expressed in terms of \( \eta_\alpha \), and thus the corresponding Heisenberg operator is time-independent. The averaged product of a pair of spin operators can now be represented as follows

\[
\langle \hat{S}^\alpha(t) \hat{S}^\alpha(t') \rangle_M = \langle \hat{\Theta} \hat{\eta}_\alpha(t) \hat{\Theta} \hat{\eta}_\alpha(t') \rangle = \frac{1}{2} \left\langle \hat{\eta}_\alpha(t) \hat{\eta}_\alpha(t') \right\rangle.
\]
Thereby a two-point spin correlation function reduces to a two-point (rather than four-point) Majorana-fermion correlation.

Unfortunately, the above relation cannot be directly generalized to time-ordered correlators (or Green’s functions) due to the fact that spin and Majorana operators are influenced by time ordering in different ways. Explicitly, a time-ordered average of two spin operators is given by

\[ \langle \hat{S}^a(t)\hat{S}^b(t') \rangle_{\text{T}} = \begin{cases} \langle \hat{S}^a(t)\hat{S}^b(t') \rangle, & t > t' \\ \frac{1}{2} \left[ \langle \hat{\eta}_a(t)\hat{\eta}_b(t') \rangle + \langle \hat{\eta}_b(t)\hat{\eta}_a(t') \rangle \right], & t < t' \end{cases} \]

(16)

where the latter expression differs from the time-ordered average of the two Majorana-fermion operators by the absence of the minus sign in the lower line.

While this problem can be circumvented by introducing Green’s functions of the operator \( \hat{\Theta} \) [37], we consider here a simpler approach. The missing sign can be compensated for with the help of an auxiliary Majorana fermion \( \hat{m} \) that anti-commutes with the three operators \( \hat{\eta}_a \), i.e., \([\hat{m}, \hat{\eta}_a] = 0 \) and \( \hat{m}^2 = 1/2 \). Since any Hamiltonian will be expressed in terms of bilinears of \( \hat{\eta}_a \), the operator \( \hat{m} \) commutes with the Hamiltonian and, thus, is time independent. We keep, however, its formal time argument in order to be able to treat time-ordered operator products correctly. We notice that

\[ \langle \hat{m}(t)\eta_a(t)i\hat{m}(t')\eta_b(t') \rangle = \begin{cases} \langle \text{im}(t)\eta_a(t)i\text{im}(t')\eta_b(t') \rangle, & t > t' \\ \frac{1}{2} \left[ \langle \eta_a(t)\eta_b(t') \rangle + \langle \eta_b(t)\eta_a(t') \rangle \right], & t < t' \end{cases} \]

Thus we arrive at the identity

\[ \langle \hat{S}^a(t)\hat{S}^b(t') \rangle_{\text{M}} = \langle \hat{m}(t)\hat{\eta}_a(t)i\hat{m}(t')\hat{\eta}_b(t') \rangle \].

(17)

Here the 2-point spin correlator is again expressed in terms of a 4-point Majorana correlation function. However, in contrast to the direct application of the Martin transformation [1], here two of the Majorana operators \( \hat{m}(t) \) and \( \hat{m}(t') \) do not have any dynamical properties and only serve the purpose of writing the time ordering [16] in a compact form.

Similarly, we can use the auxiliary operator \( \hat{m}(t) \) to express higher-order correlation functions. For a 4-point spin correlator we find

\[ \langle \hat{S}^a(t_1)\hat{S}^b(t'_1)\hat{S}^c(t_2)\hat{S}^d(t'_2) \rangle_{\text{M}} = \langle \hat{m}(t_1)\hat{\eta}_a(t_1)i\hat{m}(t'_1)\hat{\eta}_b(t'_1)\hat{m}(t_2)\hat{\eta}_d(t_2)i\hat{m}(t'_2)\hat{\eta}_c(t'_2) \rangle \].

(18)

The correlation functions [17] and [15] are in fact auto-correlation functions in the sense that they involve operators describing the same spin. Clearly, the simplification [15] cannot be extended to different spins since the operators \( \hat{\Theta}_1 \) and \( \hat{\Theta}_2 \) anti-commute and \( \hat{\Theta}_1\hat{\Theta}_2 \neq 1 \). Therefore, at the level of 2-point correlation functions the simplification described in this section applies to auto-correlation functions only. Generalizing this technique to higher-order correlation functions, we can compute autocorrelators, such as the 4-point function [18], as well as “pair-wise” correlators comprised of pairs of operators for each spin, such as

\[ \langle \hat{S}^a(t_1)\hat{S}^b(t'_1)\hat{S}^c(t_2)\hat{S}^d(t'_2) \rangle_{\text{M}} \]

Other correlators, such as \( \langle \hat{S}^a(t_1)\hat{S}^b(t'_2) \rangle \), have to be computed by different methods.

### 1.4. Spin correlation functions in the Keldysh formalism

Real-time correlation functions at finite temperatures can be conveniently computed within the Keldysh formalism [1][23]. The calculation amounts to finding the generating functional \( Z_A \) [23] and then taking the derivative with respect to the source fields.

For a spin system, the generating functional may be defined as follows

\[ Z_A = \int D[\ldots] \exp \left\{ iS_0 + \int dt \left( \mathcal{L}^{\alpha\beta}_{\alpha\beta} \hat{S}^\alpha(t)\hat{S}^\beta(t) \right) \right\}, \]

(19)
where \( D[... \] \) denotes the appropriate measure of integration whereas \( S_0 \) represents the action of the model under consideration. In particular one could choose to integrate over the SU(2) group manifold and \( D[... \] \) would represent then the appropriate Haar measure \( [1] \). In this paper we choose a more straightforward method of integrating over real Grassmann variables representing the Majorana operators of \( [\text{1}] \). In \( (19) \), \( x_a^{\pm}\) are the source fields. The superscripts \( cl \) and \( q \) refer to the “classical” and “quantum” variables \( [23] \) that are defined as the sum and difference of the corresponding fields belonging to the upper (\( u \)) and lower (\( d \)) branch of the Keldysh contour

\[
S_a^{cl,q} = \frac{1}{\sqrt{2}} (S_a^u \pm S_a^d), \quad x_a^{cl,q} = \frac{1}{\sqrt{2}} (x_a^u \pm x_a^d).
\]

The “classical” source term defined in this way describes the physical probing field, \( x_a^{cl} \equiv \sqrt{2} B_s \), while the “quantum” term is only needed to construct the correlation function and is set to zero at the end of the calculation.

Taking the derivative of the functional \( (19) \) with respect to the source fields \( x_a^{cl,q} \), one finds the spin correlation functions. In particular, the one-point function defines the magnetization

\[
\sqrt{2} M^a = \langle S^{a,cl}(t) \rangle = -i \frac{\delta Z_A^{\alpha}}{\delta S^{a,cl}(t)} \bigg|_{t=0},
\]

(20)

The spin susceptibility is given by a 2-point function

\[
\chi_{\alpha,\beta}(t,t') = i \langle T_k S^{a,cl}(t) S^{b,cl}(t') \rangle = -i \frac{\delta^2 Z_A^{\alpha}}{\delta S^{a,cl}(t) \delta S^{b,cl}(t')} \bigg|_{t=0},
\]

(21)

while the noise spectrum \( [30] \) is determined by a different 2-point correlator, \( \langle T_k S^a(t) S^a(t') \rangle \).

Below in Section 3 we will be interested in a specific 4-point function that determines the experimentally accessible noise of susceptibility \( [25, 26] \)

\[
C_A(t_1, t_1', t_2, t_2') = -\langle T_k S^{a,cl}(t_1) S^{a,cl}(t_1') S^{a,cl}(t_2) S^{a,cl}(t_2') \rangle = - \frac{\delta^4 Z_A^{\alpha}}{\delta S^{a,cl}(t_1) \delta S^{a,cl}(t_1') \delta S^{a,cl}(t_2) \delta S^{a,cl}(t_2')} \bigg|_{t=0},
\]

(22)

and we focus on the case of identical spin indices.

1.5. Correlation functions of Majorana fermions in the path-integral representation

Let us now reformulate the mapping between the spin and Majorana fermion operators discussed above in Sections 1.2 and 1.3 in the language of the Keldysh functional integrals. In the operator language we have established the correspondence \( [13] \) between the spin \( N \)-point functions and Majorana \( 2N \)-point functions. In special cases (namely, for autocorrelation functions and ‘pair’ correlators) we found simpler relations \( [17, 18] \) and

For usual (Dirac) fermions the path integral approach is based on the concept of coherent states \( [1, 23] \), which are eigenstates of the fermionic annihilation operators. An annihilation operator can only be constructed from Majorana fermions. In many-body problems, this is typically achieved by either “halving” \( [38, 40] \) or “doubling” \( [38, 39, 41] \) the number of Majorana operators. Keeping in mind applications to systems with a small number of degrees of freedom (such as the single-spin problem discussed below), we adopt the doubling procedure, where we add a free Majorana fermion to each of the operators introduced by the Martin transformation \( [1] \). The interaction part of the Keldysh action can then be formulated in terms of the Grassmann variables \( \eta \) corresponding to the Majorana operator \( \eta_0 \) in the previous sections. The additional Grassmann variables appear only in the “free” quadratic part of the action and are decoupled from the physical system under consideration.

The Majorana-fermion \( 2n \)-point correlation function on the right-hand side of Eq. \( [13] \), or rather its time-ordered counterpart, can be obtained either by \( 2n \) differentiations of the generating functional with respect to Grassmann source fields each coupled to a single Grassmann variable \( \eta \), or by \( n \) differentiations with respect to \( c \)-number source fields coupled to pairs of Grassmann variables \( \eta \). These pairs should then be chosen to represent the spin components according to Eq. \( [1] \) as in \( [12] \). For the autocorrelation functions or the pair correlators we would like to use the simplified correspondence, i.e., Eqs. \( [17] \) and \( [18] \). Here the auxiliary Majorana fermion \( \delta \) can be represented either by one of the free Grassmann variables used to construct the functional integral, or by one out of yet another pair of Grassmann variables that are added to the theory specifically for the purpose of computing the correlators \( [17] \) and \( [18] \). The source fields can again be chosen as either Grassmann variables or \( c \)-numbers. In the explicit calculation below we chose the latter option. Of course, physical results are independent of these technical details.
2. Spin correlators in the Bose-Kondo model

In this section we apply the general conclusion reached in the previous section to a specific example. For simplicity of the presentation we use, first, the Matsubara technique, while the final results are formulated in real time in the frame of the Keldysh formalism.

As a model we choose a zero-field spin-isotropic Bose-Kondo model (see Ref. [42, 43] and references therein). This model appears in various physical contexts, including spin glasses and liquids [44, 45]. In the context of $1/f$ noise a similar model is known as the Dutta-Horn model [24], where a large number of independent spins are coupled each to their own bath. In the Majorana representation introduced above [1], the model Hamiltonian reads

$$H = \vec{S} \vec{X} + H_B = -\frac{i}{2} X^\alpha \epsilon_{\alpha \beta \gamma} \eta_{\beta} \eta_{\gamma} + H_B,$$

where $H_B$ is the Hamiltonian of the bosonic bath controlling the free dynamics of $\vec{X}$. The latter may be characterized by a Matsubara correlation function $(TX_\alpha(\tau)X_\beta(\tau')) = \delta_{\alpha \beta} \Pi(\tau - \tau')$. The function $\Pi(\tau - \tau')$ can be written as

$$\Pi(i\omega_n) = \int_{-\Lambda}^{\Lambda} \frac{dx}{\pi} \rho(|x|) \text{sign} x \frac{1}{x - i\omega_n},$$

where $\rho(|x|)$ is the bath spectral density and $\omega_n = 2\pi n T$. In the Ohmic case considered here, $\rho(|x|) = g|x|$. For $\omega_m \ll \Lambda$, this gives $\Pi(i\omega_n) \approx (2g/\pi)\Lambda - g|\omega_n|$. One can perform an RG procedure by integrating out energies of the bath between $\Lambda/b$ and $\Lambda$. As a result the coupling constant $g$ is rescaled. The RG differential equation reads $dg/d\ln b = -2g^2/\pi$ (see, e.g., Ref. [42, 43]). One can supplement this with the scaling equation for the quasiparticle weight $d\ln Z/db = -2g/\pi$, which could be important [45] as we are interested in Green’s functions of the Majorana fermionic operators. For $g_0 = g(\ln b = 0) \ll 1$, the renormalization effects are not important as long the the temperature is high $T \gg T_K \equiv \Lambda \exp[-\pi/(2g_0)]$. Here we assume this to be the case. Thus we can safely reduce the cutoff $\Lambda(b)$ to a value of the order of temperature.

2.1. Matsubara path integral

We use the Matsubara imaginary-time technique ($t = -i\tau$, $\partial_t = -i\partial_\tau$). The partition function ($Z = \int D[\ldots] \exp[iS]$), for brevity we omit the source fields) reads

$$Z = \int D[\vec{X}] D[\eta_\alpha] \exp \left\{ iS_B + \int_0^{1/T} d\tau \left[ \frac{1}{2} \eta_\alpha(\tau) i \partial_\tau \eta_\alpha(\tau) + \frac{1}{2} X^\alpha(\tau) \epsilon_{\alpha \beta \gamma} \eta_\beta(\tau) \eta_\gamma(\tau) \right] \right\}. \quad (25)$$

Here $S_B$ is the free bosonic action. The first step is to average over the fluctuations of $\vec{X}$, yielding

$$Z = \int D[\eta_\alpha] \exp \left\{ -\frac{1}{2} \int d\tau \eta_\alpha(\tau) \partial_\tau \eta_\alpha(\tau) - \frac{1}{4} \int d\tau d\tau' M_{\alpha \beta} \Pi(\tau - \tau') \eta_\alpha(\tau) \eta_\beta(\tau') \eta_\beta(\tau') \right\}, \quad (26)$$

the matrix $M$ is of the form

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \quad (27)$$

Next we decouple the quartic Majorana-interaction in a different channel. To this end we rearrange

$$iS_{M,\text{int}}[\eta_\alpha] = -\frac{1}{4} \int d\tau d\tau' M_{\alpha \beta} \Pi(\tau - \tau') \eta_\alpha(\tau) \eta_\beta(\tau) \eta_\beta(\tau')$$

$$= \frac{1}{4} \int d\tau d\tau' \Pi(\tau - \tau') [\eta_\alpha(\tau) \eta_\alpha(\tau')] M_{\alpha \beta} [\eta_\beta(\tau) \eta_\beta(\tau')] . \quad (28)$$
We now employ the Hubbard-Stratonovich transformation by introducing fields $\Sigma_\alpha$. These fields inherit the symmetry of Majorana propagators, therefore $\Sigma_\alpha(\tau, \tau') = -\Sigma_\alpha(\tau', \tau)$. The new effective action reads

$$iS[\eta_\alpha, \Sigma_\alpha] = \int d\tau d\tau' \left( \frac{1}{2} \eta_\alpha(\tau) (G^{\alpha^\dagger}_\alpha)^{\alpha\tau} \eta_\alpha(\tau') - \frac{1}{4} \Sigma_\alpha(\tau, \tau') M^{-1}_{\alpha\beta} \Sigma_\beta(\tau, \tau') \right).$$

(29)

The Majorana Green’s function in (29) is

$$(G^{\alpha^\dagger}_\alpha)^{\alpha\tau} = -\delta(\tau - \tau') \partial_{\tau'} - \Sigma_\alpha(\tau, \tau').$$

(30)

The function $\Pi(\tau - \tau')$ is positive and non-zero. The standard form reads

$$\Pi(\tau - \tau') = \frac{gT^2}{\sin^2(\pi T(\tau - \tau'))}.$$ 

(31)

It is cut off at short times, $|\tau - \tau'| < 1/\Lambda$, leading to the maximal value of order $g\Lambda^2$. We use the divergent form keeping the regularization and renormalization in mind. Diagonalizing $M^{-1}$, we choose the eigenmodes with positive eigenvalues to be real and eigenmodes with negative eigenvalues to be imaginary, such that the overall sign of the action term in the exponent is negative, and the functional integral over $\Sigma_\alpha$ converges. In other words, we choose $\Sigma = \Sigma' + i\Sigma''$, where the real part is ‘diagonal’, $\Sigma' = \Sigma' \cdot (1, 1, 1)$, and the imaginary part $\Sigma''$ is orthogonal to it, (i.e., $\Sigma'' + \Sigma'' + \Sigma'' = 0$ and $\Sigma''$ describes two degenerate modes); with this choice the 3D integral over $\Sigma'$ and $\Sigma''$ converges. The redecoupled action (29) is again quadratic in Majorana Grassmann variables $\eta_\alpha$, which allows us to integrate them out and to obtain an effective action of $\Sigma$-fields:

$$iS[\Sigma_\alpha] = \frac{1}{2} \sum_{\alpha=\{L,R\}} \text{Tr} \log \left(G^{\alpha^\dagger}_\alpha\right) - \frac{1}{4} \int d\tau d\tau' \frac{\Sigma_\alpha(\tau, \tau') M^{-1}_{\alpha\beta} \Sigma_\beta(\tau, \tau')}{\Pi(\tau - \tau')}.$$ 

(32)

2.2. Saddle point solution

We can now identify the saddle point and fluctuations of the effective $\Sigma$-action. The saddle-point solution is found by expanding $\Sigma_\alpha = \Sigma_{\alpha 0} + \delta \Sigma_\alpha$. The linear order in $\delta \Sigma$ vanishes for

$$\Sigma_{\alpha 0}(\tau - \tau') = \Pi(\tau - \tau') M_{\alpha\beta} G_{\beta\gamma}(\tau - \tau'),$$

(33)

where $G_{\gamma\beta} = G_{\beta\gamma}[\Sigma_{\beta\gamma}]$. A straightforward calculation, in which we disregard the broadening of $G_{\beta\gamma}$ on the r.h.s. of Eq. (33), leads to

$$\Sigma_{\alpha 0}(i\epsilon_n) = -\int^\Lambda dx \frac{g x \coth(\beta x/2)}{\pi} \frac{1}{x - i\epsilon_n}.$$ 

(34)

Upon the analytic continuation $i\epsilon_n \to \epsilon + i0$ and for $\epsilon \to 0$, we obtain the retarded self-energy

$$\Sigma^R_{\alpha 0}(\epsilon \to 0) = -2igT = -i\Gamma.$$ 

(35)

Here we recognize $\Gamma = 2gT$ to be the (Korringa) relaxation rate. In terms of the eigenmodes this solution means $\Sigma''_0 = 0$, whereas $\Sigma'_0(i\epsilon_n) = \Sigma_{\alpha 0}(i\epsilon_n)$.

2.3. Fluctuations

To study the fluctuations we expand the trace-log-term in (32) to second order in $\delta \Sigma$. The action reads

$$iS_{\delta \Sigma} = \frac{1}{4} \sum_{\alpha} \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 G_{\beta\gamma}(\tau_1 - \tau_2) \delta \Sigma_\alpha(\tau_2, \tau_3) G_{\beta\gamma}(\tau_3 - \tau_4) \delta \Sigma_\alpha(\tau_4, \tau_1) \frac{\delta \Sigma_\alpha(\tau, \tau') M^{-1}_{\alpha\beta} \delta \Sigma_\beta(\tau, \tau')}{\Pi(\tau - \tau')}.$$ 

(36)
The Fourier transform of $\delta \Sigma$ is introduced via
\[
\delta \Sigma_{\alpha}(\epsilon_n, \nu_m) = \int d\tau d\tau' e^{i\nu_m(\tau-\tau')} e^{i\epsilon_n(\tau'-\tau')} \delta \Sigma_{\alpha}(\tau, \tau').
\] (37)

The direct analysis shows that one of $\epsilon_n$ and $\nu_m$ must be fermionic and the other bosonic, so that both $\epsilon_n + \nu_m$ and $\epsilon_n - \nu_m$ are fermionic. Then we obtain
\[
iS_{\delta \Sigma} = -\frac{T^2}{4} \sum_\alpha \sum_{\epsilon_n, \nu_m} G_{0,\alpha}(\epsilon_n + \nu_m) G_{0,\alpha}(\epsilon_n - \nu_m) \partial \Sigma_{\alpha}(\epsilon_n, \nu_m) \partial \Sigma_{\alpha}(\epsilon_n, -\nu_m)
- \frac{T^3}{4} \sum_{\alpha \beta \nu_1 \nu_2} A(\epsilon_1 + \epsilon_2) \partial \Sigma_{\alpha}(\epsilon_1, \nu) M_{\alpha\beta}^{\nu_1\nu_2} \delta \Sigma_{\beta}(\epsilon_2, -\nu).
\] (38)

Here
\[
A(i\omega_m) = \frac{1}{g^2 T^2} \int_0^{1/T} \int_0^{1/T} d\tau d\tau' \sin^2(\pi \tau T) = 1 \frac{1}{2g^2 T^2} \left( \delta_{m,0} - \frac{1}{2} \delta_{m,1} - \frac{1}{2} \delta_{m,-1} \right).
\] (39)

The mean-field Green function reads
\[
G_{0,\alpha}(i\epsilon_n) = \frac{1}{i\epsilon_n + \Gamma \text{sign}(\epsilon_n)}.
\] (40)

Since $\epsilon_n$ is necessarily fermionic, we have $|G_{0,\alpha}| < 1/(\pi T)$. Thus, the first term of (38) cannot compete with the second one which is proportional to $1/g$. This important observation allows us to disregard the first term of (38) and essentially all the contributions of the second and higher orders, originating from the trace-log term of (32). This in turn simplifies calculations of the higher-order spin correlators in the next section.

The argument above for the smallness of the first term of (38) is based on the discreteness of the Matsubara fermionic frequencies. Ultimately, we are interested in real times and the behavior in various frequency ranges, including low frequencies $\omega \ll T$; hence one should be careful with the estimates. The expressions above indicate that for $\epsilon \to 0$ the Green functions $G_{0,\alpha}$ in the first term of (38) might become of order $1/T = 1/(2gT)$. This, in turn, might imply that the (prefactor of $\delta \Sigma^2$) in the first term of (38) scales with $g^{-2}$ and dominates over the second term $\propto g^{-1}$. To clarify the situation in the low-frequency range we perform a direct Keldysh calculation in Appendix B.

We conclude that the first term of (38) does not become large in the domain of low real frequencies. Thus, the second term of (38) dominates.

2.4. Averaging over fluctuations

The knowledge of the propagator of the $\delta \Sigma$-fluctuations allows one to construct a new perturbative series, starting at the fixed-point solution given in Eq. (35). The new perturbative expansion for the Green function is based on the following series:
\[
\hat{G}_\alpha(\tau, \tau') = \hat{G}_{0,\alpha}(\tau, \tau') + \int d\tau_1 d\tau_2 \hat{G}_{0,\alpha}(\tau, \tau_1) \partial \hat{\Sigma}_\alpha(\tau_1, \tau_2) \hat{G}_{0,\alpha}(\tau_2, \tau')
+ \int d\tau_1 d\tau_2 d\tau_3 \hat{G}_{0,\alpha}(\tau, \tau_1) \partial \hat{\Sigma}_\alpha(\tau_1, \tau_2) \hat{G}_{0,\alpha}(\tau_2, \tau_3) \partial \hat{\Sigma}_\alpha(\tau_3, \tau_4) \hat{G}_{0,\alpha}(\tau_4, \tau') + \ldots
\] (41)

This series is to be averaged over the fluctuations to obtain $\langle \hat{\hat{G}}_\alpha(\tau, \tau') \rangle_{\delta \Sigma}$. However, on the right-hand side the spin index $\alpha$ is the same in all the terms, essentially because the saddle-point Green function is diagonal in spin space. Taking into account the dominance of the second term in the action (38), we conclude that
\[
\langle \delta \hat{\Sigma}_\alpha(\ldots) \delta \hat{\Sigma}_\alpha(\ldots) \rangle_{\delta \Sigma} \propto M_{\alpha\alpha} = 0,
\] (42)

because the diagonal entries of the matrix $\tilde{M}$ are zeros (note that in Eq. (42) the contributions of the real and imaginary $\Sigma$-components cancel each other, cf. the discussion above Eq. (32)). Thus, the fluctuation-averaged Green function coincides with the saddle-point solution
\[
\left\langle \hat{G}_\alpha(\tau, \tau') \right\rangle_{\delta \Sigma} = \hat{G}_{0,\alpha}(\tau, \tau').
\] (43)
Moreover, for averages of Green-function products with the same spin index $\alpha$ we can substitute all $\hat{G}_\alpha$ with their saddle-point values due to the relation \eqref{eq:SaddlePoint1}, this simplifies calculations considerably. In the next section we employ this convenient property.

2.5. Correlation functions

As a first example, we calculate the diagonal spin susceptibility

$$\chi(t,t') = i \langle T_K S^{\alpha cl}(t) S^{\alpha \bar{d}}(t') \rangle = -i \langle T_K \left[ \hat{m}(t) \tilde{\eta}_\alpha(t) \right]^\dagger \left[ \hat{m}(t') \tilde{\eta}_\alpha(t') \right] \rangle .$$  \tag{44}$$

The only contribution to $\chi(t,t')$ is given by the diagram in Fig. 1. We immediately arrive at the standard result

$$\chi''(\omega) = \frac{1}{\pi} \frac{\Gamma}{\omega^2 + \Gamma^2} \approx \frac{1}{4\tau} \frac{\omega\Gamma}{\omega^2 + \Gamma^2} .$$  \tag{45}$$

Notice that no vertex corrections appear in our case, and the precision of this calculation rests solely on the precision, with which the self-energy is evaluated.

We further demonstrate the power of our approach by calculating one of the higher-order spin correlators. We evaluate the connected part of the 4-th order correlator \eqref{eq:FourthOrderCorrelator}, which is related to noise of spin susceptibility \cite{25, 26}. Using \eqref{eq:FourthOrderCorrelator} we obtain

$$C_\chi(t_1,t_2,t_3,t_4) = \langle T_K G^{\alpha cl}(t_1) G^{\alpha \bar{d}}(t_2) G^{\alpha cl}(t_3) G^{\alpha \bar{d}}(t_4) \rangle = \langle T_K \left[ \hat{m}(t_1) \tilde{\eta}_\alpha(t_1) \right]^\dagger \left[ \hat{m}(t_2) \tilde{\eta}_\alpha(t_2) \right] \left[ \hat{m}(t_3) \tilde{\eta}_\alpha(t_3) \right] \left[ \hat{m}(t_4) \tilde{\eta}_\alpha(t_4) \right] \rangle .$$  \tag{46}$$

The discussion in the previous section implies that the connected part of this correlator is given by the six diagrams depicted in Fig. 2. Here, the double solid lines stand for the saddle-point Green functions $\hat{G}_{0\alpha}$, whereas the dashed lines represent the trivial correlators of the conserved quantity $\hat{m}$. As shown in the previous section, we may use the saddle-point Green functions, since the contribution of the $\delta\Sigma_{\alpha\beta}$-fluctuations vanishes, cf. \eqref{eq:SaddlePoint1}. For completeness, we show the result, which is put into context and interpreted in detail elsewhere \cite{25}.

$$C^{\text{conv}}_\chi(\nu_1,\omega_1,\omega_2) = \frac{\Gamma^2}{8\tau^2} \frac{\omega_1 + \omega_2 + 2\Gamma}{(\omega_1 + \omega_2 + \Gamma)(\omega_1 + \nu + \Gamma)(\omega_2 + \nu + \Gamma)(\omega_2 - \nu + \Gamma)} .$$

If a spin correlation function involves different spin components, one can no longer rely on the saddle-point contributions. One example is the correlator

$$\langle T_K S^{\alpha cl}(t_1) S^{\alpha \bar{d}}(t_2) S^{\beta cl}(t_3) S^{\beta \bar{d}}(t_4) \rangle .$$  \tag{47}$$

related to the correlations of susceptibilities in different directions. The non-zero off-diagonal fluctuations $\langle \delta\Sigma_{\alpha\beta} \rangle$ contribute to this correlator, and thus additional diagrams have to be considered. This is, however, beyond the scope of the present paper.
3. Conclusion

In this paper we have demonstrated the efficiency of the Majorana representation of the spin-1/2 operators (1). We have shown that the representation (1) allows for a direct calculation of spin correlation functions in terms of Majorana fermions despite the fact that their Hilbert space is enlarged as compared to that of the spins. The precise construction of the Majorana Hilbert space was shown to be irrelevant as far as the correlation functions are concerned.

The Majorana representation is particularly efficient in the case of auto-correlation functions (or “pair-wise” correlation functions). Such N-spin functions can be represented in terms of N-point Majorana correlators, which significantly simplifies calculations. In particular complicated vertex structures do not appear.

As an example we have revisited the well known Bose-Kondo model. We have developed the Keldysh path-integral approach and have shown that the spin relaxation and susceptibility are efficiently described within the saddle-point approximation. Moreover we have shown that correlation functions containing a single spin projection can also be efficiently calculated at the saddle-point. In particular we have evaluated a 4-spin correlation function corresponding to the noise of susceptibility.

It would be interesting to apply our approach to a wider range of physical problems, for example, to a sub-Ohmic Bose-Kondo model describing physics of spin glasses [42–45].

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Appendix A. Perturbation theory

Given the Majorana representation for spins (1), the identities (17), (18), and the Hamiltonian (23), one could calculate correlation functions within perturbation theory by diagrammatic expansion. Here the zeroth-order Hamiltonian consists of the bath part $H_B$ only. The dressed Majorana Green function is self-consistently constructed, using the lowest-order self-energy. The lowest-order self-energy diagram is depicted in Fig. A.3, it contains the “free” Majorana Green function $\hat{G}_{f,a}(\omega)$,

$$G_{f,a}^{R/A}(\omega) = (\omega \mp i0)^{-1},$$

$$\Sigma^R_a(\omega) = -\int \frac{d\Omega}{2\pi} \left( \Pi^K(\omega + \Omega)G_{f,a}^{K}(\Omega) + \Pi^K(\omega + \Omega)G_{f,a}^{R}(\Omega) \right) = -2igT \left( 1 + O\left( \frac{\omega}{T} \right) \right),$$

$$G^{R}_a(\omega) = (\omega - \Sigma^R_a(\omega))^{-1} = (\omega + i\Gamma)^{-1}, \quad \Gamma = 2gT.$$

Here the Keldysh components of the bath correlation functions read

$$\hat{\Pi}^{ab}_a(t,t') = \delta_{ab} \left\langle T_{K} \hat{X}^{a}(t)\hat{X}^{b}(t') \right\rangle = \hat{\Pi}^{ab}(t,t'),$$

$$\Pi^{R/A}(\omega) = \pm g\omega - iD, \quad \Pi^K(\omega) = \coth\left( \frac{\omega}{2T} \right) \left( \Pi^K(\omega) - \Pi^K(\omega) \right),$$

$$\Pi^R(\omega) = \coth\left( \frac{\omega}{2T} \right) \left( \Pi^K(\omega) - \Pi^K(\omega) \right).$$
where \( D \equiv (2g/\pi)A \), the spin indices \( \alpha, \beta = \{ x, y, z \} \), the Keldysh indices \( a, b = \{ cl, q \} \), and the "classical" and "quantum" operators \( \hat{X}^{\alpha \beta}(q) = (X^{\alpha \beta} \pm \text{e}^{i \omega d})/\sqrt{2} \).

Typically, this method is used to calculate 1- or 2-point correlation functions, e.g., magnetization or susceptibility. In lower orders Majorana propagators are never connected by a bosonic line. Such an element only appears in higher orders or in higher correlation functions as discussed in [25]. An example is shown in Fig. A.4. Here it is in principle necessary to consider more complex diagrams including, e.g., ladders of bosonic propagators. In the perturbation theory with a vanishing unperturbed spin Hamiltonian it is by no means justified to simply neglect these diagrams. As an illustration let us consider renormalization of the interaction line. The dressed interaction carries four times (or frequencies), the Keldysh indices \( a, b, c, d = \{ cl, q \} \), and the spin indices \( \alpha, \beta, \gamma, \delta = \{ x, y, z \} \); diagrammatically it is depicted by

\[
\hat{\Gamma}^{abc, cd}_{ab, cd} = \begin{array}{ccc}
\omega_1 + \nu & \leftarrow & \omega_1 \\
\omega_2 - \nu & \leftarrow & \omega_2 \\
\end{array}
\begin{array}{c}
\hat{\Gamma}^{ab}_{ab} \\
\hat{\Gamma}^{cd}_{cd} \\
\end{array}
\begin{array}{c}
\omega_1 \\
\omega_2 \\
\end{array}
\begin{array}{ccc}
1, \beta & \leftarrow & 2, \alpha \\
1, \beta & \leftarrow & 2, \alpha \\
\end{array}
\]

For demonstration, we pick one of several possible components of \( \hat{\Gamma} \) that carries identical spin indices on the left (as well as on the right) and the Keldysh indices \((21, 21)\), which correspond to the retarded-retarded component, that is \( \Gamma^{RR}_{\alpha \beta} \). The partially dressed interaction is obtained from the Dyson-type equation, depicted in Fig. A.5, by summation of the contributing diagrams in the small-\( g \) expansion. These are constructed out of the bosonic correlator \( \Pi^K \sim 2gT = \Gamma \) combined with Green’s functions \( G^R G^A \) such that the upper and lower halves of the complex plane each contain one of the Green-functions poles. These leading contributions are of the same order in \( g \) as the bare bosonic line, thus we might suspect a strong renormalization of the interaction line. This is, however, not the case due to a cancellation. Taking into account both contributions depicted in Fig. A.5, we obtain

\[
\Gamma^{RR}_{\alpha \beta}(\omega_1, \omega_2, \nu) = \Pi^K(\nu)M_{\alpha \beta} - \int \frac{d\Omega}{2\pi} \Pi^K(\nu - \Omega)M_{\alpha \gamma} G^R_\nu(\omega_1 + \Omega)G^R_\nu(\omega_2 - \Omega) + \int \frac{d\Omega}{2\pi} \Pi^K(\nu - \Omega)M_{\alpha \gamma} G^R_\nu(\omega_1 + \Omega)G^R_\nu(\omega_2 - \Omega) \Gamma^{RR}_{\alpha \beta}(\omega_1, \omega_2, \Omega) (A.7)
\]

The Green functions are calculated within the approximation \( \Sigma^K(\omega) = -2igT = -iT \), which is justified and consistent in the high-temperature regime \( T \gg T_K, \nu, \omega_1, \omega_2, \Gamma \). Let us assume that the leading term in \( \Gamma^{RR} \) does not depend on the third frequency and splits \( \Gamma^{RR} = \Gamma^{RR}_{\alpha \beta} + \Gamma^{RR}_{\gamma \delta} \). If this is the case, the integrals in the second and third term become equal, and we find that the renormalized interaction coincides with the bare one:

\[
\Gamma^{RR}_{\alpha \beta}(\omega_1, \omega_2) = \Pi^K(0)M_{\alpha \beta}. (A.8)
\]
The problem in this perturbative approach is that the non-perturbed spin Hamiltonian is zero (the non-perturbed Hamiltonian consists of the bath Hamiltonian $H_\text{b}$ only). In other words, the spin-bath interaction is not weak as compared to the energy scale of the unperturbed spin dynamics. One might therefore expect that higher-order diagrams are important. Having observed a particular cancellation we cannot in principle exclude other important higher-order contributions. Instead of evaluating multiple higher-order diagrams we choose the path-integral approach in the following section allowing for a more straightforward analysis.

Appendix B. Majorana path integral in the Keldysh representation

In Section 2 we have developed a path-integral technique, which allowed us to obtain the self-energy (Korringa relaxation rate) as a saddle-point solution. We have argued that the fluctuations around this saddle point can sometimes be disregarded. Namely, this is the case for a correlation function involving only one spin component. This conclusion was based on the smallness of the higher-than-linear contributions to the trace-log term of the action (32). We have shown this using the small parameter $\Gamma/\epsilon_0$ for the fermionic Matsubara frequencies. To be able to treat also low real frequencies, we provide here the Keldysh version of the path-integral calculation.

For the model considered here, (23), the partition function reads

$$Z = \int \mathcal{D}[\tilde{X}] \mathcal{D}[\eta_\alpha] \exp \left\{ iS_B + \frac{i}{2} \int dt \left( \eta_\alpha(t) i\partial_t \eta_\alpha(t) + iX^a(t) e_{a\beta} \eta_\beta(t) \eta_\beta(t) \right) \right\}$$

$$= \int \mathcal{D}[\tilde{X}] \mathcal{D}[\eta_\alpha] \exp \left\{ iS_B + \sum_{a=\alpha,\beta} \int \right. \left. dt \left( \eta_\alpha(t) i\partial_t \eta_\alpha(t) + i\tau^a_{\alpha\beta} X^{a\beta}(t) e_{a\beta} \eta_\beta(t) \eta_\beta(t) \right) \right\},$$

where the Keldysh indices $a$ takes the value $u$ at the forward part of the contour and $d$ on the backward part of the contour. At the first step we average over the fluctuations of $\tilde{X}$, which yields

$$Z = \int \mathcal{D}[\eta_\alpha] \exp \left\{ -\frac{1}{2} \int dt \eta^a_{\alpha}(t) \tau^a_{\alpha\beta} \eta^b_{\beta}(t) + \frac{1}{4} \int dtdt' M_{ab} \tau^a_{\alpha\beta} \Pi^{ab}(t, t') \tau^b_{\beta\gamma} \eta^c_{\gamma}(t) \eta^d_{\gamma}(t') \right\}.$$  \hfill (B.1)

Here $\langle T X^a(t)X^b(t') \rangle = \delta_{ab} \Pi^{ab}(t - t')$, and $a, b$ are the Keldysh indices over which summation is implied. As in Section 2 we decouple the quartic term in a different channel

$$iS_{\alpha\beta\alpha\beta}[\eta_\alpha] = \frac{1}{4} \int dtdt' M_{ab} \tau^a_{\alpha\beta} \Pi^{ab}(t, t') \tau^b_{\beta\gamma} \eta^c_{\gamma}(t) \eta^d_{\gamma}(t) = -\frac{1}{4} \int dtdt' \tau^a_{\alpha\beta} \Pi^{ab}(t, t') \tau^b_{\beta\gamma} \eta^c_{\gamma}(t) \eta^d_{\gamma}(t') M_{ab}(\eta^c_{\gamma}(t) \eta^d_{\gamma}(t')).$$ \hfill (B.2)

Appendix B.1. Qualitative considerations

Prior to performing the full fledged Keldysh analysis, we provide here a qualitative argument based on the locality of the bath correlation function $\Pi^{ab}(t - t')$ on the relevant time scale of order $1/\Gamma$. On this time-scale we can safely replace all components of $\Pi^{ab}(t - t')$ by its classical (Keldysh) part, i.e., $\Pi^{ab}(t - t') \approx \Pi(t_1 - t_2) = (1/2)\Pi(t - t')$. This allows us to proceed similarly to the treatment in the Matsubara case in the main text (cf. Eq. [32]), and we obtain

$$iS[\Sigma] = \frac{1}{2} \sum_{a=\alpha,\beta} \text{Tr} \log \left( G^{-1}_a \right) - \frac{1}{4} \int dtdt' \frac{\Sigma^{ab}(t, t') \tau^a_{\alpha\beta} \Sigma^{ab}(t, t')}{\Pi(t - t')} \tau^b_{\beta\gamma} M_{ab}(\eta^c_{\gamma}(t) \eta^d_{\gamma}(t')).$$ \hfill (B.4)

where

$$(G^{-1}_a)^{ab}_{\gamma\delta} = i\tau^a_{\alpha\beta} \delta(t - t') \partial_{\gamma} - \Sigma^{ab}_a (t, t').$$

One can find the saddle point and again obtain the relaxation rate $\Gamma = 2gT$. This is done below in the full Keldysh calculation. Here we concentrate on the fluctuations $\Sigma^{ab}$. On the relevant time scales ($\sim 1/\Gamma$) the function $\Pi$ is local, $\Pi(t - t') \sim 2gT \delta(t - t')$. (Note that the delta-function should be understood as such only at relatively long time scales. For instance, it does not force us to take the Grassmann variables in (B.3) at coinciding times.) This locality means, in
The Majorana Green function in (B.5) reads
\[ G(t,t') = -\Sigma_{\delta}(t-t') \].
This strong constraint implies that only off-diagonal elements of \( \Sigma_{\delta} \) fluctuate: \( \Sigma_{\delta}^{aa}(t,t') = -\Sigma_{\delta}^{da}(t,t') \sim \delta(t-t') \), whereas \( \Sigma_{\delta}^{dd} = \delta_{dd} = 0 \). Upon the Keldysh rotation (see below), this means that only the retarded and advanced components of \( \Sigma_{\delta} \) fluctuate. In addition the Keldysh component of the Majorana Green function \( G^K_{ab} \) can be neglected, since it scales as \( \propto \tanh(\omega/2T)\delta(\omega) \). As a result, upon expansion of the trace-log term of (B.4), the only terms that can appear are of the type \( \text{Tr}[G^K_{ab}\delta_\Sigma G^K_{ab}\delta_\Sigma G^K_{ab}\delta_\Sigma \ldots] \) and \( \text{Tr}[G^K_{ab}\delta_\Sigma G^K_{ab}\delta_\Sigma G^K_{ab}\delta_\Sigma \ldots] \).

Since \( \delta_\Sigma^{R/A} \) are local in time, these terms vanish. Thus we are allowed to disregard the trace-log term of (B.4). Notice, that this argument is not valid at finite magnetic fields \( B \geq \Gamma \), see below.

Appendix B.2. Full Keldysh calculation

We now go back to the full Keldysh version of (B.4) keeping all Keldysh components of \( \Pi^{ab} \). One can decouple the quartic Majorana interaction with the help of complex bosonic fields \( Q_\alpha \) via a Hubbard-Stratonovich transformation. The fields \( Q_\alpha \) inherit the symmetry of the Majorana propagators, therefore \( Q^{ab}_\alpha(t,t') = -Q^{ba}_\alpha(t,t') \). Since the correlator \( \Pi^{ab}(t-t') \) may have a complicated time-dependent structure, we choose to keep it in the numerator:
\[
iS[\{\eta_\alpha, Q_\alpha\}] = \int dt dt' \left( \frac{i}{2} \langle t \rangle_{a}^{ab}(t) (G_{a}^{ab}_{\delta}(t)) \delta(t-t') - \frac{1}{4} \partial^2_{t} \frac{1}{2} \Pi^{ab}(t,t') Q^{ab}_\alpha(t,t') M^{-1}_{ab} Q^{ab}_\alpha(t,t') \right) \]  
(B.5)
The Majorana Green function in (B.5) reads
\[
(G_{a}^{ab}_{\delta}(t)) = i\tau^2_{a} \delta(t-t') \delta^a \sigma^b - \Sigma_{\sigma}^{ab}(t,t') \]  
(B.6)
\[
\Sigma_{\sigma}^{ab}(t,t') = \tau^2_{a} \Sigma_{\sigma}^{bb}(t,t') \]  
(B.7)

After the standard Keldysh rotation,
\[
\hat{G} = LGL = \begin{pmatrix} G^K & G^K \\ G^A & 0 \end{pmatrix}, \quad \hat{\Pi} = L\Pi L = \begin{pmatrix} \Pi^K & \Pi^K \\ \Pi^K & 0 \end{pmatrix}, \quad \hat{\Sigma} = L\Sigma L = \begin{pmatrix} 0 & \Sigma^A \\ \Sigma^A & \Sigma^K \end{pmatrix}, \quad L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \]  
(B.8)
we find
\[
\langle \hat{G}^{-1}_{a}\rangle_{\nu} = i\tau^2_{a} \delta(t-t') \delta^a \sigma^b - \hat{\Sigma}_{a}(t,t') \]  
(B.9)
\[
\hat{\Sigma}_{ab}(t,t') = \frac{1}{2} \text{Tr} \left[ \hat{\gamma}^a \hat{\Pi}(t,t') \hat{\gamma}^b \hat{\Pi}_a(t,t') \right] \]  
(B.10)
where \( \hat{\gamma}^a \equiv \tau^a \) and \( \hat{\gamma}^\nu \equiv \tau_0 \).

The decoupled action (B.5) is again quadratic in Majorana Grassmann variables \( \eta_\alpha \), enabling us to integrate them out and to obtain an effective action for the \( Q \)-fields. Thereafter, we can identify the saddle point and fluctuations of the effective \( Q \)-action.
\[
iS[Q_\alpha] = \frac{1}{2} \text{Tr} \sum_{a} \log \langle \hat{G}_{a}(Q_{\alpha}) \rangle^{-1} + \frac{1}{8} \int dt dt' \hat{\Pi}^{ab}(t,t') M^{-1}_{ab} \text{Tr} \left[ \hat{\gamma}^a \hat{\Pi}(t,t') \hat{\gamma}^b \hat{\Pi}_a(t,t') \right]. \]  
(B.12)
Here \( \text{Tr} \) denotes the trace in the Keldysh and time space and \( \text{Tr} \) is the trace in the Keldysh space. The saddle-point solution is found by expansion taking the linear order in \( \delta \hat{Q} \). The solution must be stationary, depending only on the time difference: \( \hat{G}_{0\beta}(t,t') = \hat{G}_{0\beta}(t-t') \). We obtain the self-consistency equation
\[
\hat{Q}_{a\beta}(t-t') = M_{\alpha\beta} \hat{G}_{0\beta}(t-t'). \]  
(B.13)

In the high-temperature regime, \( T \gg T_K \), it is easy to obtain the self-consistent solution for the self-energy. In frequency space, one finds (summation over double indices assumed)
\[
\Sigma^{R}_{0\alpha}(\omega) = -\frac{1}{2} M_{\alpha\beta} \int \frac{d\Omega}{2\pi} \left( \Pi^{K}(\omega + \Omega) G^{A}_{0\beta}(\Omega) + \Pi^{R}(\omega + \Omega) G^{K}_{0\beta}(\Omega) \right) = -2it \left( 1 + O \left( \frac{\omega}{T} \right) \right) \]  
(B.14)
\[
G^{R}_{0\alpha}(\omega) = \left( \omega - \Sigma^{R}_{0\alpha}(\omega) \right)^{-1} \approx \left( \omega + 2it \right)^{-1}, \]  
(B.15)
coinciding with the saddle-point result \((35)\) and the results of the perturbation theory \((A.2)\) and \((A.3)\).

To analyze the fluctuations \(\delta \hat{Q}\), we expand the trace-log-term in \(B.12\) up to the second order in \(\delta \hat{Q}\). With the help of the Fourier transform of \(\delta \hat{Q}\), introduced as

\[
\delta \hat{Q}_\alpha(\omega, \nu) = \int dt dt' e^{-i\omega(t-t')} e^{-i\nu(t'-t)} \delta \hat{Q}_\alpha(t, t'),
\]

we rewrite the action in the form \(iS_{\delta \hat{Q}} = iS_{\delta \hat{Q}}^{(1)} + iS_{\delta \hat{Q}}^{(2)}\). The first term originates in the expansion of the trace-log term of \(B.12\):

\[
iS_{\delta \hat{Q}}^{(1)} = \frac{1}{2} \int dt dt' dt'' dt''' \Tr \left[ \hat{G}_{0,\alpha}(t, t') \delta \hat{\Sigma}_\alpha(t', t) \hat{G}_{0,\alpha}(t_1, t') \delta \hat{\Sigma}_\alpha(t', t) \right] \]

\[
= \frac{1}{2} \int d\omega d\Omega \left( \frac{2\pi}{2}\right)^3 \Tr \left[ \hat{G}_{0,\alpha}(\Omega - \nu/2) \delta \hat{\Sigma}_\alpha(\Omega) \hat{G}_{0,\alpha}(\Omega + \nu/2) \delta \hat{\Sigma}_\alpha(-\nu) \right].
\]  

(B.17)

Here, for brevity, we expressed \(iS_{\delta \hat{Q}}^{(1)}\) via the self-energy fluctuations

\[
\delta \hat{\Sigma}_\alpha^{(\delta)}(\Omega, \nu) = -\frac{1}{2} \int \frac{d\omega}{2\pi} \Tr \left[ \gamma^a \hat{\Pi}(\Omega + \omega) \gamma^b \delta \hat{Q}_\alpha(\omega, -\nu) \right].
\]  

(B.18)

The second term appeared after the Hubbard-Stratonovich transformation:

\[
iS_{\delta \hat{Q}}^{(2)} = \frac{M^{-1}_{adj}}{8} \int dt dt' \hat{\Pi}^{(\delta)}(t, t') \Tr \left[ \gamma^a \delta \hat{Q}_\alpha(t, t') \gamma^b \delta \hat{Q}_\beta(t', t) \right]
\]

\[
= \frac{M^{-1}_{adj}}{8} \int d\omega d\alpha d\omega' d\nu \Tr \left[ \gamma^a \delta \hat{Q}_\alpha(\omega, \nu) \gamma^b \delta \hat{Q}_\beta(\omega', -\nu) \right].
\]  

(B.19)

In this Appendix we focus on the first term \(B.17\), which emerged from the expansion of the trace-log term. We give a detailed discussion of this term and conclude that it may be neglected as compared to the second term \(B.19\).

This is done on the basis of a small-\(g\)-expansion, justifying our course of action in Section 2.3 of the main text. In order to symmetrize and simplify the problem we parametrize fluctuations around the saddle point in terms of the mode \(R^{(1)}\), symmetric with respect to time, and three antisymmetric modes \(R^{(2)}\), \(R^{(3)}\) and \(R^{(4)}\):

\[
\delta \hat{Q}_\alpha(\omega, \nu) = \left( R^{(3)}(\omega, \nu) + iR^{(4)}(\omega, \nu) \right) \frac{R^{(3)}(\omega, \nu) - iR^{(4)}(\omega, \nu)}{\left( R^{(3)}(\omega, v) + iR^{(4)}(\omega, v) \right)^2} \frac{R^{(3)}(\omega, \nu) + iR^{(4)}(\omega, \nu)}{\left( R^{(3)}(\omega, v) - iR^{(4)}(\omega, v) \right)^2} \delta \hat{Q}_\alpha^{(1)}(\omega, \nu).
\]  

(B.20)

\[
R^{(1)}(t, t') = R^{(1)}(t', t), \quad R^{(2)}(t, t') = -R^{(2)}(t', t), \quad R^{(3)}(t, t') = -R^{(3)}(t', t), \quad R^{(4)}(t, t') = -R^{(4)}(t', t).
\]  

(B.21)

Clearly, \(B.19\) is proportional to \(g\) essentially originating from the bath correlation function \(\hat{\Pi}\). In \(B.17\), each \(\delta \hat{\Sigma}\) contains a factor of \(\hat{\Pi}\), therefore the whole term appears to be of at least second order in \(g\) unless the Green’s functions yield an inverse factor \(g\). The only combination of Green’s functions yielding \(1/g\) is \(G^4(\Omega + \nu/2)G^8(\Omega - \nu/2)\) (or vice versa, \(R/\Lambda \to \Lambda/R\)). For example, one of the contributions to \(B.17\) has the form

\[
\int \frac{d\omega d\nu}{(2\pi)^2} G^4(\Omega + \nu/2)G^8(\Omega - \nu/2)\hat{\Pi}^{(3)}(\Omega + \omega)\hat{\Pi}^{(3)}(\Omega + \omega') R^{(3)}(\omega, v) R^{(3)}(\omega', -\nu), \quad i, j = 3, 4.
\]  

(B.22)

In this term, assuming \(\hat{\Pi}^k \approx 4gT\) to be constant (at low frequencies), the \(\Omega\) integration of \(G^8G^4\) yields an inverse factor of \(g\) since \(1/G = (2gT)^{-1}\). The structure of this term resembles the structure of the diagrammatic elements \(A.7\) discussed Appendix A. However, in writing \(B.22\) we did not take into account the antisymmetry of \(R^{(3)}\) and \(R^{(4)}\) as explained in \(B.21\). Due to the antisymmetry terms of the kind \(B.22\) cancel out. This is the same cancellation which we encountered in perturbation theory in Eq. \(A.7\), thus we conclude that the above mentioned divergent terms also cancel out in perturbation theory if symmetries are respected during the re-summation.
To substantiate our claim we provide here a rigorous analysis of (B.17). For this purpose we decompose $\delta\hat{\Sigma}_\omega(\Omega, \nu)$, use the explicit form of $\Pi^{R/A}$ and take advantage of the symmetry relations (B.21) of $R^{(\omega)}$.

$$\delta\Sigma_{\omega}^{11}(\Omega, -\nu) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ 2g\Omega R_{\omega}^{(1)}(\omega, \nu) + \frac{\Pi^K(\omega + \Omega) - \Pi^K(-\omega + \Omega)}{2} \left( R_{\omega}^{(2)}(\omega, \nu) - R_{\omega}^{(3)}(\omega, \nu) \right) \right]$$

$$\delta\Sigma_{\omega}^{12}(\Omega, -\nu) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[ \frac{\Pi^K(\omega + \Omega) + \Pi^K(-\omega + \Omega)}{2} iR_{\omega}^{(1)}(\omega, \nu) + \frac{\Pi^K(\omega + \Omega) - \Pi^K(-\omega + \Omega)}{2} R_{\omega}^{(2)}(\omega, \nu) - 2g\omega R_{\omega}^{(3)}(\omega, \nu) \right]$$

$$\delta\Sigma_{\omega}^{21}(\Omega, -\nu) = \delta\Sigma_{\omega}^{12}(\Omega, -\nu; R_{\omega}^{(1)} \rightarrow -R_{\omega}^{(1)}; R_{\omega}^{(4)} \rightarrow -R_{\omega}^{(4)})$$

$$\delta\Sigma_{\omega}^{22}(\Omega, -\nu) = \delta\Sigma_{\omega}^{11}(\Omega, -\nu; R_{\omega}^{(1)} \rightarrow -R_{\omega}^{(1)}; R_{\omega}^{(4)} \rightarrow -R_{\omega}^{(4)})$$

In addition

$$\Pi^{R/A}(\Omega + \omega) = \pm g(\Omega + \omega) - iD, \quad \Pi^K(\Omega + \omega) = 2g(\Omega + \omega) \coth \left( \frac{\Omega + \omega}{2T} \right).$$

At low frequencies $\Omega \ll T$ the terms containing the ‘classical’ contribution $\Pi^K \approx 4gT$ dominate over those containing $\Pi^{R/A}$. Indeed $\Pi^{R/A}$ are important only in the ‘quantum’ region $\Omega \gtrsim T$, then $\Pi^{R/A} \approx g\Omega$. Here we recall the discussion about the RG in Section 3. We can disregard most of the ‘quantum’ domain $\Omega \gtrsim T$ because these frequencies could be integrated out in the initial RG procedure.

Where do large contributions to the $\Omega$-integral in (B.17) come from? In the region $\Omega \sim T$ Green’s functions generate a factor of $1/T_2$, the expression as a whole scales as $g^2$ and can therefore be neglected as compared with (B.19), which scales as $g$. The remaining region to consider is $\Omega \sim \Gamma \ll T$. There, terms containing the Keldysh Green function

$$G^R(\Omega) = -\frac{2T \tanh \frac{\Omega}{2T}}{\Omega^2 + \Gamma^2}$$

get another order of $g$ due to the hyperbolic tangent in the small $g$ expansion: $\tanh \Omega/(2T) \sim \Gamma/T = 2g$. Neglecting $G^K$-terms we write the remaining terms of (B.17) as

$$G_{\alpha\beta}^R \partial^2 G_{\alpha\beta}^R G_{\alpha\beta}^A \partial^2 G_{\alpha\beta}^A + G_{\alpha\beta}^R \partial^2 G_{\alpha\beta}^A \partial^2 G_{\alpha\beta}^A + G_{\alpha\beta}^R \partial^2 G_{\alpha\beta}^A \partial^2 G_{\alpha\beta}^A + G_{\alpha\beta}^R \partial^2 G_{\alpha\beta}^A \partial^2 G_{\alpha\beta}^A.$$

In the limit $\Omega \ll T$ most prefactors in $\delta\hat{\Sigma}$ are linear in $\Omega$. Considering the region $\Omega \sim \Gamma = 2gT$ the linear prefactor $\Omega$ yields another order of $g$, and therefore the corresponding terms can also be neglected. Finally, the $\Omega$-independent term $a_\omega \partial R^{(4)}$ in $\delta\Sigma^{(12)}$ only appears combined with $G^R G^R$ and $G^A G^A$, having both poles on the same side of the real axis. The integration by residue theorem yields zero. We conclude that all terms of (B.17) are of higher order in $g$ than those of (B.19). This line of reasoning still holds if a magnetic field $B$ is included in the problem provided that $B \ll \Gamma$. For larger fields, which are however still smaller than the temperature, the frequency in the hyperbolic tangent would essentially be replaced by $B$ and thus yield a factor $\tanh B/(2T) \approx B/T > g$.

As a result we have confirmed the conclusion of the main text: it is justified to neglect the first term in the action (B.17), which was obtained from the expansion of the trace-log term around the saddle point. The action of $\delta\hat{Q}$-fluctuations around the saddle point is governed by second term, (B.19), generated in the Hubbard-Stratonovich decoupling of the quartic Majorana term with $Q$-fields.

Appendix C. Gauge freedom

In this Appendix we explore the curious gauge freedom present in our problem. Interestingly, we find that the saddle-point solution (B.13) and particularly the imaginary part of the self-energy acquire a gauge-field dependence. As a result, the physical Green function no longer coincides with the saddle point Green function. We find that fluctuations in turn become important, and their role is to compensate for the effect of the introduced gauge fields.

We recapitulate the quartic term in the Majorana action (B.3), before the $Q$-fields were introduced.

$$iS_{M,\text{intr}}[\eta_\alpha] = -\frac{1}{4} \int dt dt' \gamma_3^\alpha \Pi^{\beta\gamma}(t, t') \eta_\alpha^{\beta\gamma}(\eta_\alpha^{\alpha\beta} M_{\alpha\beta} \eta_\alpha^{\delta\gamma} \gamma_3^\delta)$$

(C.1)

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Due to the property $\eta^2 = 0$ of Grassmann variables, adding real/complex finite entries $A_x, A_y$, and $A_z$ on the diagonal of the matrix $M$ does not change the action. The range of possible values of $A_\alpha$ is limited by the constraints that $M$ is invertible (or equivalently, $\det M \neq 0$) and its eigenvalues have to be real. We interpret the $A_\alpha$ as gauge fields, which may be fixed by some condition.

$$M = \begin{pmatrix} A_x & 1 & 1 \\ 1 & A_y & 1 \\ 1 & 1 & A_z \end{pmatrix}. \quad \text{(C.2)}$$

The redecoupling of the quartic term with the help of complex bosonic fields $Q_\alpha$ via a Hubbard-Stratonovich transformation is not modified, the action is still of the form given in the main text in Eq. (B.5). The only new feature consists in the diagonal non-zero entries of the matrix $M$.

$$iS[\eta_\alpha, Q_\alpha] = \int dt \left( \frac{i}{2} \eta_\alpha(t) G^{-1}_\alpha(t) \eta_\alpha(t') - \frac{1}{4} \tau^a_1 \tau^b_2 \Pi^{ab}(t, t') Q^{ab}_\alpha(t, t') M^{-1}_{ab} Q^{ab}_\beta(t, t') \right). \quad \text{(C.3)}$$

Integrating out the Majorana fermions one again obtains (32)

$$iS[Q_\alpha] = \frac{1}{2} \text{Tr}_a \sum_{\nu} \log (G_\nu(Q_\nu))^{-1} + \frac{1}{8} \int dt dt' \tilde{\Pi}^{ab}(t, t') M^{-1}_{ab} \text{Tr} \left[ \tilde{\gamma}^a \tilde{Q}_\alpha(t, t') \tilde{\gamma}^b \tilde{Q}_\beta(t', t) \right], \quad \text{(C.4)}$$

and the saddle point equations

$$\dot{Q}_{0,\alpha}(t, t') = M_{ab} \dot{G}_{0}(t, t'), \quad \left(\dot{G}_{ab}(t, t') = i\tau_1 \delta(t - t') \partial_t - \Sigma_{ab}(t, t') \right), \quad \text{(C.5)}$$

We emphasize that this set of equations now depends on the gauge fields $A_\alpha$, introduced above, suggesting that there is not just one but instead a set of saddle points, characterized by the values of $\{A_x, A_y, A_z\}$. This fact becomes obvious if we write down the self-consistent solution in the high-temperature regime explicitly:

$$\Sigma_{ab}^R(\omega) = -\frac{1}{2} M_{ab} \int d\Omega 2\pi \left( \Pi^K(\omega + \Omega) G^a_\nu(\Omega) + \Pi^K(\omega + \Omega) G^b_\nu(\Omega) \right)$$

$$\left(1 + A_\alpha \right) + O(\Omega^2) \right). \quad \text{(C.6)}$$

As the imaginary part of $\Sigma^K$ now depends on the gauge fields $A_\alpha$, we can no longer identify it as a physically observable rate. For any finite $\{A_x, A_y, A_z\}$ equation (C.4) fails and therefore the saddle point Green function and the physical Green function do not coincide, in contrast to the result (32) in the main text. In order to find the physical Green function, we have to reconsider fluctuations around the saddle points for finite gauge fields.

To describe the fluctuations we evaluate the action (B.19), that is the leading second term of $iS[\delta Q]$. To simplify the discussion we define

$$R^{(i)}_a(\omega, \nu) = \int_0^\infty \frac{d\omega}{2\pi} R^{(i)}_a(\omega, \nu), \quad R^{(i)}_a(\nu) = \int_0^\infty \frac{d\omega}{2\pi T} R^{(i)}_a(\omega, \nu) \quad \text{for} \ i = 2, 3, 4. \quad \text{(C.7)}$$

Written in the matrix form in terms of $(R^{(i)}_a, R^{(2)}_a, R^{(3)}_a, R^{(4)}_a)$, the leading terms in the high-temperature regime are $(i, j = 1, 2, 3, 4)$

$$iS^{(ij)}_{\delta Q} = 2gM_\nu^{-1} \int \frac{dv}{2\pi} \int_0^\infty \frac{d\omega d\nu'}{(2\pi)^2} R^{(i)}_a(\omega, \nu) \left[ \begin{array}{cccc} 2T & 0 & 0 & i\omega' \\ 0 & 0 & \frac{\omega'}{T} & 0 \\ 0 & \frac{\omega'}{T} & 0 & 0 \\ i\omega & 0 & 0 & -\frac{\omega'}{T} \end{array} \right] R^{(j)}_b(\omega', -\nu), \quad \text{(C.8)}$$

$$= -\frac{1}{2} \int \frac{dv}{2\pi} R^{(i)}_a(\nu) \left( D^{-1} \right)_{ab}^{(ij)} R^{(j)}_b(-\nu), \quad \text{(C.9)}$$

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where

\[
P_{ij}^{(i)} = \frac{M_{ij}}{4gT} \begin{pmatrix} 1 & 0 & 0 & 3i \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 3i & 0 & 0 & -6 \end{pmatrix}
\]

and

\[
(D^{-1})^{ij}_{ab} = -4gT M^{-1}_{ij} \begin{pmatrix} 2 & 0 & 0 & i \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ i & 0 & 0 & -\frac{1}{3} \end{pmatrix}.
\]

(C.10)

Thus, we obtain for the correlator of the fluctuations \( \{ R_\alpha^{(i)}(v_1) R_\alpha^{(j)}(v_2) \} = D_{ij}^{(i)} \frac{2\pi\delta(v_1 + v_2)}{} \).

Knowing the propagator of fluctuations, we can compute the fluctuation-averaged self-energy. As a starting point we use the self-consistent Dyson equation, which has to be satisfied by the correct Green function.

\[
\hat{G}_{t,f}^{-1} \circ \hat{G}_a = \mathbb{1}.
\]

(C.11)

Before the average is actually performed causality does not necessarily apply, and we have to allow for a finite ‘anti-Keldysh’ 22 component of the fluctuating self-energy. For the 12-component of the Green function, which will become retarded after the averaging, we obtain the following equation

\[
i\partial_t G_{\alpha}^{12}(t,t') = \delta(t-t') + \int dt_1 \Sigma_{\alpha}^{21}(t,t_1)G_{\alpha}^{12}(t_1,t') + \int dt_1 \Sigma_{\alpha}^{22}(t,t_1)G_{\alpha}^{22}(t_1,t').
\]

(C.12)

We have found that the propagator (C.10) does not depend on the frequency \( \omega \) corresponding to the time difference \( t - t_1 \). Therefore, we assume that \( \Sigma_{\alpha}^{21}(t,t_1) = \delta(t-t_1) \Sigma_{\alpha}^{21}(t) \) is local in time, but keep the dependence on total time for the fluctuation average later on. We also neglect \( \Sigma_{\alpha}^{22} \Sigma_{\alpha}^{22} \), which is of higher order because \( \Sigma_{\alpha}^{22} \sim \Pi^B G^{R/A} + \Pi^E G^K \) is smaller than the ‘big’ combination \( \Pi^E G^{R/A} \).

\[
i\partial_t G_{\alpha}^{12}(t,t') = \delta(t-t') + \Sigma_{\alpha}^{21}(t)G_{\alpha}^{12}(t,t').
\]

(C.13)

The above equation is solved by the following ansatz. The self-energy \( \Sigma_{\alpha}^{21} \) includes the constant saddle-point contribution and fluctuations. The saddle-point part is \( \Sigma_{\alpha}^{21} \) and can be separated. The resulting prefactor is identified with the saddle-point Green’s function.

\[
G_{\alpha}^{12}(t,t') = -i\Theta(t-t') \exp \left\{ -i \int_{t'}^t dt_1 \Sigma_{\alpha}^{21}(t_1) \right\} = -i\Theta(t-t') \exp \left\{ -i \int_{t'}^t dt_1 \delta \Sigma_{\alpha}^{21}(t_1) \right\}
\]

(C.14)

We treat the \( \delta \hat{Q} \) fluctuations using the usual Gaussian averaging procedure and use a ‘self-energy correlator’ for simplification.

\[
G_{\alpha}^{12}(t,t') \equiv \left\langle G_{\alpha}^{12}(t,t') \right\rangle_{\delta \hat{Q}} = G_{\alpha,0}^{12}(t,t') \left\langle \exp \left\{ -i \int_{t'}^t dt_1 \delta \Sigma_{\alpha}^{21}(t_1) \right\} \right\rangle_{\delta \hat{Q}}
\]

(C.15)

The 21-component of the ‘self-energy correlator’ is decomposed into a bath correlator and fluctuating modes according to Eq. (B.18).

\[
\delta \Sigma_{\alpha}^{21}(t,t') = -\frac{1}{2} \text{Tr} \left[ \tilde{\Pi}(t,t') \tau_i \delta \hat{Q}_a(t',t) \right]
\]

\[
= \frac{i}{2} \Pi^k(t-t') R_{\alpha}^{(1)}(t',t) - \frac{1}{2} \Pi^k(t-t') R_{\alpha}^{(2)}(t',t) - \frac{1}{2} \left( \Pi^k(t-t') + \Pi^k(t-t') \right) R_{\alpha}^{(3)}(t',t)
\]

\[
- \frac{1}{2} \left( \Pi^k(t-t') \right) R_{\alpha}^{(4)}(t',t)
\]

(C.18)
\[\delta \Sigma^{21}_{\alpha}(\Omega, \nu) = \frac{1}{2} \int \frac{d\omega}{2\pi} \left( \Pi^{K}(\omega + \Omega) + \Pi^{K}(-\omega + \Omega) \right) i R^{(1)}_{\alpha}(\omega, \nu) - \frac{\Pi^{K}(\omega + \Omega) - \Pi^{K}(-\omega + \Omega)}{2} R^{(2)}_{\alpha}(\omega, \nu) - \alpha \omega R^{(3)}_{\alpha}(\omega, \nu) \]

\[\approx \int_{0}^{\infty} \frac{d\omega}{2\pi} \left( 4igTR^{(1)}_{\alpha}(\omega, \nu) - 2g\omega R_{\alpha}^{(4)}(\omega, \nu) \right) = 4igTR^{(1)}_{\alpha}(\nu) - 2gTR^{(4)}_{\alpha}(\nu) \]

\[\Rightarrow \delta \Sigma^{21}_{\alpha}(\Omega, -\nu) = 2gT \begin{pmatrix} R^{(1)}_{\nu} & R^{(2)}_{\nu} & R^{(3)}_{\nu} & R^{(4)}_{\nu} \end{pmatrix} \]

(C.16)

Considering the exponent of (C.15) we find

\[-\frac{1}{2} \int_{r} dt_{2} dt_{3} \left( \delta \Sigma^{21}_{\alpha}(t_{2}) \delta \Sigma^{21}_{\alpha}(t_{3}) \right)_{\delta Q} = -\frac{1}{2} \int_{r} dt_{2} dt_{3} \int \frac{d\nu_{2}}{2\pi} \int \frac{d\nu_{3}}{2\pi} \left( \delta \Sigma^{21}_{\alpha}(\nu_{2}) \delta \Sigma^{21}_{\alpha}(\nu_{3}) \right)_{\delta Q} e^{-i\nu_{2}+\nu_{3},t_{2}+t_{3}} \]

\[= -\frac{(2gT)^2}{2} (t - t') \begin{pmatrix} 2i & 0 & 0 \ 0 & 0 & -1 \end{pmatrix} D^{(ij)}_{\alpha} = -gT (t - t') A_{\alpha}. \quad \text{(C.17)} \]

At the saddle point the imaginary part of the self-energy (C.6) was found to depend on the arbitrary constants \( A_{\alpha} \). Hence it cannot correspond to the physical decay rate. We define the physical decay rates \( \Gamma_{\alpha} \) using the physical, fluctuation-averaged 12-component of the Green’s function

\[iG^{R}_{\alpha}(t, t') \equiv i \left( G^{(12)}_{\alpha}(t, t') \right)_{\delta Q} = \Theta(t - t') e^{-\Gamma_{\alpha}(t - t')} \quad \text{(C.18)} \]

and find

\[\Gamma_{\alpha} = -\text{Im} \Sigma^{R}_{0,\alpha} = gTA_{\alpha} = 2gT. \quad \text{(C.19)} \]

In conclusion, we found that for arbitrary \( A_{\alpha} \) the physical decay rate is not given by the self-energy at the saddle point but rather by the decay rate \( \Gamma_{\alpha} \) of the fluctuation-averaged Green function, which is independent of \( A_{\alpha} \).

After having identified \( A_{\alpha} \) as kind of an arbitrary gauge, we can now choose \( A_{\alpha} = 0 \). Then, the saddle-point solution coincides with the correct decay rate, and corrections due to fluctuations cancel. In other words, the saddle point Majorana Green function coincides with the physical Green function. Analogously this applies to the self-energy.

References

[1] A. Altland, B. Simons, Condensed Matter Field Theory, 2nd Edition, Cambridge University Press, 2010.
[2] A. M. Tsvelik, Quantum field theory in condensed matter physics, Cambridge University Press, Cambridge, 1996.
[3] P. Jordan, E. Wigner, Zeitschrift für Physik 47 (1928) 631–651.
[4] T. Holstein, H. Primakoff, Phys. Rev. 58 (1940) 1098–1113.
[5] J. L. Martin, Generalized classical dynamics, and the ‘classical analogue’ of a Fermi oscillator, Proc. R. Soc. London A 251 (1267) (1959) 536–542.
[6] R. Fréssard, J. Kroha, P. Wölfle, Ch. 3 in Strongly correlated systems, Vol. 1: Theoretical methods of Springer series in solid-state sciences; 171, Springer, Berlin, 2012.
[7] J. Schwinger, in: L. Biedenharn, H. van Dam (Eds.), Quantum Theory of Angular Momentum, Academic Press, New York, 1965, p. 229.
[8] D. P. Arovas, A. Auerbach, Functional integral theories of low-dimensional quantum Heisenberg models, Phys. Rev. B 38 (1988) 316–332.
[9] N. Read, S. Sachdev, Large-\( n \) expansion for frustrated quantum antiferromagnets, Phys. Rev. Lett. 66 (1991) 1773–1776.
[10] F. Wang, Schwinger boson mean field theories of spin liquid states on a honeycomb lattice: Projective symmetry group analysis and critical field theory, Phys. Rev. B 82 (2010) 024419.
[11] I. Affleck, Z. Zou, T. Hsu, P. W. Anderson, SU(2) gauge symmetry of the large-\( n \) limit of the Hubbard model, Phys. Rev. B 38 (1988) 745–747.
[12] J. B. Marston, I. Affleck, Large-\( n \) limit of the Hubbard-Heisenberg model, Phys. Rev. B 39 (1989) 11538–11558.
[13] N. Andrei, P. Coleman, Cooper instability in the presence of a spin liquid, Phys. Rev. Lett. 62 (1989) 595–598.
E. Dagotto, E. Fradkin, A. Moreo, SU(2) gauge invariance and order parameters in strongly coupled electronic systems, Phys. Rev. B 38 (1988) 2926–2929.

X. G. Wen, Mean-field theory of spin-liquid states with finite energy gap and topological orders, Phys. Rev. B 44 (1991) 2664–2672.

F. J. Burnell, C. Nayak, SU(2) slave fermion solution of the Kitaev honeycomb lattice model, Phys. Rev. B 84 (2011) 125125.

E. Fradkin, Jordan-Wigner transformation for quantum-spin systems in two dimensions and fractional statistics, Phys. Rev. Lett. 63 (1989) 322–325.

L. Huerta, J. Zanelli, Bose-Fermi transformation in three-dimensional space, Phys. Rev. Lett. 71 (1993) 3622–3624.

A. Zawadowski, P. Fazekas, Dynamics of impurity spin above the Kondo temperature, Zeitschrift für Physik A, Hadrons and Nuclei 226 (1969) 235–265.

H. J. Spencer, Theory of s – d scattering in dilute magnetic alloys with spin-1/2 impurities, Phys. Rev. 171 (1968) 515–530.

A. Shnirman, Y. Makhlkin, Spin-spin correlators in the Majorana representation, Phys. Rev. Lett. 91 (2003) 207204.

W. Mao, P. Coleman, C. Hooley, D. Langreth, Spin dynamics from Majorana Fermions, Phys. Rev. Lett. 91 (2003) 207203.

A. Kamenev, Field theory of non-equilibrium systems, 1st Edition, Cambridge Univ. Press, Cambridge [u.a.], 2011.

P. Dutta, P. M. Horn, Low-frequency fluctuations in solids: 1/f noise, Rev. Mod. Phys. 53 (1981) 497–516.

P. Schad, B. N. Narozhny, G. Schönh, A. Shnirman, Nonequilibrium spin noise and noise of susceptibility, Phys. Rev. B 90 (2014) 205419.

S. Sendelbach, D. Hover, M. Mück, R. McDermott, Complex Inductance, Excess Noise, and Surface Magnetism in dc SQUIDs, Phys. Rev. Lett. 103 (11) (2009) 117001.

M. B. Weissman, 1/f noise and other slow, nonexponential kinetics in condensed matter, Rev. Mod. Phys. 60 (2) (1988) 537–571.

M. B. Weissman, What is a spin glass? A glimpse via mesoscopic noise, Rev. Mod. Phys. 65 (3) (1993) 829–839.

Burnett J., Faoro L., Wisby I., Gurtovoi V. L., Chernykh A. V., Mikhailov G. M., Tulin V. A., Shaikhaidarov R., Antonov V., Meeson P. J., Tzalenchuk A. Ya., Lindström T., Evidence for interacting two-level systems from the 1/f noise of a superconducting resonator, Nature Communications 5.

S. M. Kogan, Electronic noise and fluctuations in solids, Cambridge University Press, Cambridge, 1996.

H. G. E. Beck, W. P. Spruit, 1/f noise in the variance of Johnson noise, J. Appl. Phys. 49 (6) (1978) 3384–3385.

G. T. Seidler, S. A. Solin, Non-Gaussian 1/f noise: Experimental optimization and separation of high-order amplitude and phase correlations, Phys. Rev. B 53 (15) (1996) 9753–9759.

F. Berezin, M. Marinov, Particle spin dynamics as the Grassmann variant of classical mechanics, Annals of Physics 104 (2) (1977) 336 – 362.

R. P. Kenan, Perturbation expansion for the magnetization of the spin antiferromagnet, Journal of Applied Physics 37 (3) (1966) 1453–1454.

H. J. Spencer, S. Doniach, Low-temperature anomaly of electron-spin resonance in dilute alloys, Phys. Rev. Lett. 18 (1967) 994–997.

P. Coleman, L. B. Ioffe, A. M. Tsvelik, Simple formulation of the two-channel Kondo model, Phys. Rev. B 52 (1995) 6611–6627.

M. Cabrera Cano, S. Flores, Magnetotransport in the Kondo model with ferromagnetic exchange interaction, Phys. Rev. B 88 (2013) 035104.

J. de Boer, B. Peeters, K. Skenderis, P. van Nieuwenhuizen, Loop calculations in quantum mechanical non-linear sigma models with fermions and applications to anomalies, Nuclear Physics B 459 (3) (1996) 631 – 692.

F. Bastianelli, P. v. Nieuwenhuizen, Path integrals and anomalies in curved space, Cambridge monographs on mathematical physics, Cambridge University Press, Cambridge, 2009.

R. Shankar, A. Vishwanath, Equality of bulk wave functions and edge correlations in some topological superconductors: A spacetime derivation, Phys. Rev. Lett. 107 (2011) 106803.

J. Nilsson, M. Bazzanella, Majorana fermion description of the Kondo lattice: Variational and path integral approach, Phys. Rev. B 88 (2013) 045112.

G. Zaránd, E. Demler, Quantum phase transitions in the Bose-Fermi Kondo model, Phys. Rev. B 66 (2002) 024427.

L. Zhu, Q. Si, Critical local-moment fluctuations in the Bose-Fermi Kondo model, Phys. Rev. B 66 (2002) 024426.

A. Georges, O. Parcollet, S. Sachdev, Mean field theory of a quantum Heisenberg spin glass, Phys. Rev. Lett. 85 (2000) 840–843.

A. Georges, O. Parcollet, S. Sachdev, Quantum fluctuations of a nearly critical Heisenberg spin glass, Phys. Rev. B 63 (2001) 134406.

M. Vogt, C. Buragohain, S. Sachdev, Quantum impurity dynamics in two-dimensional antiferromagnets and superconductors, Phys. Rev. B 61 (2000) 15152–15184.