GENERICALLY FREE REPRESENTATIONS I: LARGE REPRESENTATIONS

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Abstract. For a simple linear algebraic group \( G \) acting faithfully on a vector space \( V \) with zero fixed space, we show: if \( V \) is large enough, then the Lie algebra of \( G \) acts generically freely on \( V \). That is, the stabilizer in \( \text{Lie}(G) \) of a generic vector in \( V \) is zero. The bound on \( \text{dim}\, V \) is \( \Theta((\text{rank}\, G)^2) \) and holds with only mild hypotheses on the characteristic of the underlying field. The proof relies on results on generation of Lie algebras by conjugates of an element that may be of independent interest. We use the bound in subsequent works to determine which irreducible faithful representations are generically free, with no hypothesis on the characteristic of the field. This in turn has applications to the question of which representations have a stabilizer in general position as well as the determination of the invariants of the representation.

Let \( G \) be a simple linear algebraic group over a field \( k \) acting faithfully on a vector space \( V \). In the special case \( k = \mathbb{C} \), there is a striking dichotomy between the properties of irreducible representations \( V \) whose dimension is small (say, \( \leq \text{dim}\, G \)) versus those whose dimension is large, see for example [AVE68], [Ela72], [Pop88], and [PV94, §8.7]. For example, if \( \text{dim}\, V < \text{dim}\, G \), then trivially the stabilizer \( G_v \) of a vector \( v \in V \) is nonzero. Conversely (and nontrivially) for \( V \) hardly bigger than \( \text{dim}\, G \), the stabilizer \( G_v(k) \) for generic \( v \in V \) is trivial (i.e., zero); in this case one says that \( V \) is generically free or \( G \) acts generically freely on \( V \). This property has taken on increased importance recently due to applications in Galois cohomology and essential dimension, see [Rei10] and [Mer13] for the theory and [BRV10], [GG17b], [Kar10], [LMMR13], [Lot13], etc. for specific applications.

With applications in mind, it is desirable to extend the results on generically free representations to all fields. In that setting, [GLL18] showed \( k \) algebraically closed of any characteristic and \( V \) irreducible: \( \text{dim}\, V > \text{dim}\, G \) if and only if the stabilizer \( G_v(k) \) of a generic \( v \in V \) is finite. (This was previously known when \( \text{char}\, k = 0 \) [AVE68].) Moreover, except for the cases in Table 6, when \( G_v(k) \) is finite it is 1, i.e., the group scheme \( G_v \) is infinitesimal. For applications, it is helpful to know if \( G_v \) is not just infinitesimal but is the trivial group scheme. In this paper, we provide the necessary result:

Theorem A. Let \( G \) be a simple linear algebraic group over a field \( k \) such that \( \text{char}\, k \) is not special for \( G \). If \( \rho: G \to \text{GL}(V) \) is a representation of \( G \) such that \( V \) has a \( G \)-subquotient \( X \) with \( X^g = 0 \) and \( \text{dim}\, X > b(G) \) for \( b(G) \) as in Tables 1 or 2, then for generic \( v \in V \), \( \text{Lie}(G)_v = \text{ker}\, \rho \).

Of course, \( \text{Lie}(G)_v \supseteq \text{ker}\, \rho \), so equality means that \( \text{Lie}(G)_v \) is as small as possible. In this case, we write that \( \text{Lie}(G) \) acts virtually freely on \( V \). This notion is the natural generalization of “generically freely” to allow for the possibility that
G does not act faithfully. We actually prove a somewhat stronger statement than Theorem A, see Theorem 12.2 below.

Note that ker dρ can be read off the weights of V. By hypothesis, ker dρ is a proper ideal in Lie(G) and (as char k is assumed not special) is therefore contained in the center of Lie(G), i.e., Lie(Z(G)). The restriction of ρ to Z(G) and of dρ to Lie(Z(G)) is determined by the equivalence classes of the weights of V modulo the root lattice.

If we restrict our focus to representations V that are restricted and irreducible, Theorem A quickly settles whether V is virtually free for all but finitely many types of G:

**Corollary B.** Suppose G has type $A_\ell$ for some $\ell > 19$; type $B_\ell$, $C_\ell$, or $D_\ell$ with $\ell > 11$; or exceptional type, over a field k such that char k is not special for G. For ρ: $G \to \text{GL}(V)$ a restricted irreducible representation of G, Lie(G) = ker dρ if and only if dim V > dim G.

Note that the bound $b(G)$ from Theorem A is $\Theta(\text{dim } G) = \Theta((\text{rank } G)^2)$ and holds for most k. In the special case char k = 0 one can find a similar result in [AP71] where the bound is $\Theta((\text{rank } G)^3)$. The fact that the exponent in our result is 2 (and not 3) means that the restricted irreducible representations not covered by Theorem A and Corollary B are among those enumerated in [Lüb01]. We settle these cases in a separate paper, [GG17a], because the arguments are rather different and more computational. Combining the results of these two papers with [GLL18], we get descriptions of the stabilizer $G_v$ as a group scheme when V is irreducible, which we announce in Section 15.

The case where char k is special will be treated in [GG18]. We exhibit in that paper an example to demonstrate that the conclusion of Theorem A does not hold for groups of type B with char k special (i.e., 2).

**Remarks on the proof.** Corollary B may be compared to the main result of Guerreiro’s thesis [Gue97], which classifies the irreducible $G$-modules that are also Lie(G)-irreducible such that the kernel of dρ is contained in the center of Lie(G) with somewhat weaker bounds on dim V. (See also [Aul01] and [GG17b] for other

| type of G | char k | $b(G)$ | Reference |
|-----------|--------|--------|-----------|
| $A_\ell$  | $\neq 2$ | $2.25(\ell + 1)^2$ | Cor. 5.5 |
| $A_\ell$  | $= 2$   | $2(\ell + 1)^2 - 1$ | Cor. 6.2 |
| $B_\ell$  | $\neq 2$ | $8\ell^2$ | Cor. 8.2 |
| $C_\ell$  | $\neq 2$ | $6\ell^2$ | Cor. 7.3 |
| $D_\ell$  | $\neq 2$ | $2(2\ell - 1)^2$ | Cor. 8.2 |
| $D_\ell$  | $= 2$   | $4\ell^2$ | Cor. 9.6 |

**Table 1.** Bound $b(G)$ appearing in Theorem A for types A, B, C, and D

| type of G | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-----------|-------|-------|-------|-------|-------|
| $b(G)$    | 48    | 240   | 360   | 630   | 1200  |

**Table 2.** Numbers e and $b(G)$ for Lie algebras of exceptional types
results on specific representations.) Our methods are different in the sense that Guerreiro relied on computations with the weights of $V$, whereas we largely work with the natural module. We do refer to Guerreiro’s thesis in the proof of Corollary B to handle a few specific representations.

The change in perspective that leads to our stronger results in fewer pages is the replacement of the popular inequality (1.3), which involves the specific representation $V$, with (1.4) that only involves properties of the adjoint representation Lie($G$). Thus our proof of Theorem A depends on no properties of $V$ other than its dimension, providing a dramatic simplification. Furthermore we prove new bounds on the number of conjugates of a given element $x \in$ Lie($G$) that suffice to generate a Lie subalgebra containing the derived subalgebra; these results should be of independent interest. Our bounds depend upon the conjugacy class and give upper bounds for the dimension of fixed spaces for elements in the class. As a special case, we extend the main result of [CSUW01], see Proposition 14.1. We note that some generation bounds are known in the setting of groups, see for example [GS03] or [GS02].

**Notation.** For convenience of exposition, we will assume in most of the rest of the paper that $k$ is algebraically closed of characteristic $p \neq 0$. This is only for convenience, as our results for $p$ prime immediately imply the corresponding results for characteristic zero: simply lift the representation from characteristic 0 to $Z$ and reduce modulo a sufficiently large prime.

We say that char $k$ is special for $G$ if char $k = p \neq 0$ and the Dynkin diagram of $G$ has a $p$-valent bond, i.e., if char $k = 2$ and $G$ has type $B_n$ or $C_n$ for $n \geq 2$ or type $E_4$, or if char $k = 3$ and $G$ has type $G_2$. (Equivalently, these are the cases where $G$ has a very special isogeny.) This definition is as in [Ste63, §10], [Sei87, p. 15], and [Pre97]; in an alternative history, these primes might have been called “extremely bad” because they are a subset of the very bad primes — the lone difference is that for $G$ of type $G_2$, the prime 2 is very bad but not special.

Let $G$ be an affine group scheme of finite type over $k$, which we assume is algebraically closed. If $G$ is additionally smooth, then we say that $G$ is an algebraic group. An algebraic group $G$ is simple if it is not abelian, is connected, and has no connected normal subgroups $\neq 1, G$; for example $SL_n$ is simple for every $n \geq 2$.

If $G$ acts on a variety $X$, the stabilizer $G_x$ of an element $x \in X(k)$ is a subgroup-scheme of $G$ with points

$$G_x(R) = \{ g \in G(R) \mid gx = x \}$$

for every $k$-algebra $R$. Statements “for generic $x$” means that there is a dense open subset $U$ of $X$ such that the property holds for all $x \in U$.

If Lie($G$) = 0 then $G$ is finite and étale. If additionally $G(k) = 1$, then $G$ is the trivial group scheme Spec $k$. (Note, however, that when $k$ has characteristic $p \neq 0$, the sub-group-scheme $\mu_p$ of $\mu_{p^2}$ has the same Lie algebra and $k$-points. So it is not generally possible to distinguish closed-sub-group schemes by comparing their $k$-points and Lie algebras.)

We write $g$ for Lie($G$) and similarly $\text{spin}_n$ for Lie($\text{Spin}_n$), etc. We put $\mathfrak{z}(g)$ for the center of $g$; it is the Lie algebra of the (scheme-theoretic) center of $G$. As char $k = p$, the Frobenius automorphism of $k$ induces a “$p$-mapping” $x \mapsto x^{[p]}$ on $g$. An element $x \in g$ is nilpotent if $x^{[p]^n} = 0$ for some $n > 0$, toral if $x^{[p]} = x$, and
semisimple if $x$ is contained in the Lie $p$-subalgebra of $\mathfrak{g}$ generated by $x^{[p]}$, i.e., is in the subspace spanned by $x^{[p]}, x^{[p]^2}, \ldots$.

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1. Key inequalities

Inequalities. Put $\mathfrak{g} := \text{Lie}(G)$ and choose a representation $\rho: G \to \text{GL}(V)$. For $x \in \mathfrak{g}$, put

$$V^x := \{v \in V \mid d\rho(x)v = 0\}$$

and $x^G$ for the $G$-conjugacy class $\text{Ad}(G)x$ of $x$.

Lemma 1.1. For $x \in \mathfrak{g}$,

$$x^G \cap \mathfrak{g}_v = \emptyset \quad \text{for generic } v \in V$$

is implied by:

$$\dim x^G + \dim V^x < \dim V,$$

which is implied by:

There exists $e > 0$ and $x_1, \ldots, x_e \in x^G$ such that the subalgebra $s$ of $\mathfrak{g}$ generated by $x_1, \ldots, x_e$ has $V^s = 0$ and $e \cdot \dim x^G < \dim V$.

Proof. Suppose (1.3) holds and let $v \in V$. If there is $g \in G(k)$ such that $g^{-1}xgv = 0$, i.e., $xgv = 0$. Put

$$V(x) := \{v \in V \mid \text{there is } g \in G(k) \text{ s.t. } xgv = 0\} = \bigcup_{y \in x^G} V^y.$$

Define $\alpha: G \times V^x \to V$ by $\alpha(g, w) = gw$, so the image of $\alpha$ is precisely $V(x)$. The fiber over $gw$ contains $(gc^{-1}, cw)$ for $\text{Ad}(c)$ fixing $x$, and so $\dim V(x) \leq \dim x^G + \dim V^x$. Then (1.3) implies $V(x)$ is a proper subvariety of $V$, whence (1.2). (This observation is essentially in [GG17b, Lemma 2.6], [AP71, Lemma 4], or [Gue97, §3.3], for example, but we have repackaged it here for the convenience of the reader.)

Now assume (1.4). Write $c$ for the codimension of $V^x$. Iterating the formula $\dim(U \cap U') \geq \dim U + \dim U' - \dim V$ for subspaces $U, U'$ of $V$ gives

$$\dim(\cap_i V^{x_i}) \geq \left(\sum_i \dim V^{x_i}\right) - (e - 1) \dim V.$$

The left side is zero by hypothesis, hence $\dim V - \dim V^x \geq \frac{1}{c} \dim V$ $d\rho$ is $G$-equivariant, and not just a representation of $\mathfrak{g})$ and it follows that $\dim V^x \leq (1 - 1/e) \dim V$. Now $\dim x^G < \frac{1}{c} \dim V$ implies (1.3). \hfill $\Box$

We will verify (1.3) in many cases, compare Theorem 12.2. To do so, we actually prove (1.4), which does not mention $V$. This allows us to focus on the element $x$ and its action on the natural module rather than attempting to analyze $V^x$ directly, for which it is natural to require some hypothesis on the structure of $V$ beyond simply a bound on the dimension, such as that $V$ is irreducible as is assumed in [Gue97].
Comparing subalgebras. Exploiting the fact that there are only finitely many $G$-conjugacy classes of toral and nilpotent elements of $\mathfrak{g}$ for $G$ semisimple, we obtain as in [GG17b, §1]:

**Lemma 1.5.** Suppose $G$ is semisimple over an algebraically closed field $k$ of characteristic $p > 0$, and let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$.

1. If, for every $x \in \mathfrak{g} \setminus \mathfrak{h}$ such that $x^{[p]} = x$ or $x^{[p]} = 0$ for some $n$, (1.2) holds, then for generic $v \in V$, $\mathfrak{g}_v \subseteq \mathfrak{h}$.

2. If $\mathfrak{h}$ consists of semisimple elements and equation (1.2) holds for every $x \in \mathfrak{g} \setminus \mathfrak{h}$ with $x^{[p]} \in \{0, x\}$, then for generic $v \in V$, $\mathfrak{g}_v \subseteq \mathfrak{h}$.

Taking $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$ in the preceding lemma feeds into the following.

**Lemma 1.6.** Suppose $G$ is connected reductive. If, for generic $v \in V$, $\mathfrak{g}_v \subseteq \mathfrak{z}(\mathfrak{g})$, then $\mathfrak{g}$ acts virtually freely on $V$.

**Proof.** $\mathfrak{z}(\mathfrak{g})$ is the Lie algebra of the center of $G$, so $d\rho(\mathfrak{z}(\mathfrak{g}))$ consists of semisimple linear transformations that pairwise commute and hence are simultaneously diagonalizable. It follows that $\mathfrak{z}(\mathfrak{g})_v = \ker d\rho|_{\mathfrak{z}(\mathfrak{g})} \subseteq \mathfrak{g}_v$ for all $v$, whence equality because the converse implication holds for generic $v$ by hypothesis.

**Examples.**

**Example 1.7** ($\text{SL}_2$). Recall that an irreducible representation $\rho: \text{SL}_2 \to \text{GL}(V)$ of $\text{SL}_2$ is specified by its highest weight $w$, a nonnegative integer. Let $\text{char } k =: p \neq 0$. We claim:

(i) If $\text{char } k$ divides $w$ (e.g., if $w = 0$), then $d\rho(\mathfrak{sl}_2) = 0$.

(ii) If $w = 1$ or $2$, then $\mathfrak{sl}_2$ does not act virtually freely on $V$.

(iii) If $w = p^e + 1$ for some $e > 0$, then $\mathfrak{sl}_2$ acts virtually freely on $V$ but (1.3) fails for some noncentral $x \in \mathfrak{sl}_2$ with $x^{[p]} \in \{0, x\}$.

(iv) Otherwise, (1.3) holds for noncentral $x \in \mathfrak{sl}_2$ with $x^{[p]} \in \{0, x\}$, and in particular $\mathfrak{sl}_2$ acts virtually freely on $V$.

To see this, write $w = \sum_{i \geq 0} w_i p^i$ where $0 \leq w_i < p$. By Steinberg, $V$ is isomorphic (as an $\text{SL}_2$-module) to $\otimes_i L(\omega_i)^{[p]^i}$, where the exponent $[p]^i$ denotes the $i$-th Frobenius twist, and the irreducible module $L(w_i)$ is also the Weyl module with highest weight $w_i$ by [Win77], of dimension $w_i + 1$. Thus, as a representation of $\mathfrak{sl}_2$, $V$ is isomorphic to a direct sum of $c := \prod_{i > 0} (w_i + 1)$ copies of $L(w_0)$. This proves (i), so we suppose for the remainder of the proof that $w_0 > 0$.

As in the previous paragraph, $L(1)$ is the natural representation (with generic stabilizer a parabolic subalgebra) and $L(2)$ (when $p \neq 2$) is the adjoint action on $\mathfrak{sl}_2$ (with generic stabilizer of dimension 1). This verifies (ii).

We investigate now (1.3). For $x$ nonzero nilpotent or noncentral toral, we have $\dim(x^{\text{SL}_2}) = 2$. For $x$ nonzero nilpotent, $L(w_0)^x$ is the highest weight line. If $x^{[p]} = x$, then up to conjugacy $x$ is diagonal with entries $(a, -a)$ for some $a \in \mathbb{F}_p$; as $x$ is non-central, $p \neq 2$ and $\dim L(w_0)^x = 0$ or 1 depending on whether $w_0$ is odd or even. Assembling these, we find $\dim(x^{\text{SL}_2}) = \dim L(w) \leq 2 + c$ with equality for $x$ nonzero nilpotent, whereas $\dim L(w) = cw_0 + c$. We divide the remaining cases via the product $cw_0$, where we have already treated the case (ii) where $c = 1$ and $w_0 = 1$ or 2.

Suppose $c = 2$ and $w_0 = 1$, so we are in case (iii). The action of $\mathfrak{sl}_2$ on $V$ via $d\rho$ is the same as the action of $\mathfrak{sl}_2$ on two copies of the natural module, equivalently, on
2-by-2 matrices by left multiplication. A generic matrix $v$ is invertible, so $(\mathfrak{sl}_2)_v = 0$. Yet we have verified in the previous paragraph that (1.3) fails for $x$ nonzero nilpotent, proving (iii).

The case (iv) is where $cw_0 > 2$, where we have verified (1.3), completing the proof of the claim.

Example 1.8. Let $x \in \mathfrak{g}$. If $\dim x^G + \dim (V^*)^x < \dim (V^*)$, then (1.3) holds for $x$. This is obvious, because $d\rho(x)$ and $-d\rho(x)^T$ have the same rank.

2. INTERLUDE: SEMISIMPLIFICATION

For Theorem A, we consider representations $V$ of $G$ that need not be semisimple. For each chain of submodules $0 =: V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n := V$ of $G$, we can construct the $G$-module $V' := \oplus_{i=1}^n V_i/V_{i-1}$. For example, if each $V_i/V_{i-1}$ is an irreducible (a.k.a. simple) $G$-module then $V'$ is the semisimplification of $V$. In this section, we discuss to what extent results for $V$ correspond to results for $V_i/V_{i-1}$ and for $V'$, using the notation of this paragraph and writing $\rho: G \to \text{GL}(V)$ and $\rho': G \to \text{GL}(V')$ for the actions.

From the subquotient to $V$.

Example 2.1. Suppose that for some $x \in \mathfrak{g}$ and some $1 \leq i \leq n$, we have

$$\dim x^G + \dim (V_i/V_{i-1})^x < \dim (V_i/V_{i-1}).$$

We claim that (1.3) holds for $x$. By induction it suffices to consider the case $i = 2$ and a chain $V_1 \subseteq V_2 \subseteq V$.

Suppose first that $V_1 = 0$. Then $\dim x^G + \dim V^x \leq \dim x^G + \dim V^x + \dim V/V_2$, whence the claim. Now suppose that $V_2 = V$, so $(V_2/V_1)^x$ is a submodule of $V^x$; the claim follows by Example 1.8. Combining these two cases gives the full claim.

There is an analogous statement about the dimension of generic stabilizers.

Example 2.2. For each $1 \leq i \leq n$ and generic $v \in V$ and generic $w \in V_i/V_{i-1}$, we claim that $\dim g_v \leq \dim g_w$. If $V_{i-1} = 0$, then $w$ is in $V$ and the claim follows by the fiber dimension theorem. If $V_i = V$, then any inverse image $\tilde{w} \in V$ of $w$ has $g_{\tilde{w}} \subseteq g_w$, and again the claim follows. Combining these two cases suffices to give the full claim, as in the previous example.

From $V'$ to $V$.

Example 2.3. When checking the inequality (1.3), it suffices to do it for $V'$. More precisely, for $x \in \mathfrak{g}$, we have: If $\dim x^G + \dim (V')^x < \dim V'$, then $\dim x^G + \dim V^x < \dim V$. This is obvious because $\dim V^x \leq \sum \dim (V_i/V_{i-1})^x$.

Proposition 2.4. For generic $v \in V$ and $v' \in V'$, we have $\dim g_v \leq \dim g_{v'}$.

Proof. By induction on the number $n$ of summands in $V'$, we may assume that $V' = W \oplus V/W$ for some $g$-submodule $W$ of $V$.

Suppose first that $\dim V/W = 1$. Pick $v \in V$ with nonzero image $\bar{v} \in V/W$. ut $t := \{x \in \mathfrak{g} \mid d\rho(x)v \in W\}$, a subalgebra of $\mathfrak{g}$ sometimes called the transporter of $v$ in $W$. A generic vector $v' \in V'$ is of the form $w + c\bar{v}$ for generic $w \in W$ and $c \in k^\times$. Evidently, $g_{v'} = t_w$. By upper semicontinuity of dimension, $\dim t_{v_0} \leq \dim t_v$ for generic $v_0 \in V$. On the other hand, writing $v_0 = w_0 + \lambda v$ for $\lambda \in k^\times$, for $x \in g_{v_0}$ we find $d\rho(x)v = -\frac{1}{\lambda}d\rho(x)w_0 \in W$, so $g_{v_0} = t_{v_0}$, proving the claim.
In the general case, pick a splitting \( \phi: V/W \hookrightarrow V \) and so identify \( V \) with \( V' \) as vector spaces. We may intersect open sets defining generic elements in \( V \) and \( V' \) and so assume the two notions agree under this identification. Let \( v := w + \phi(\bar{v}) \) be a generic vector in \( V \), where \( w \in W \) and \( \bar{v} \in V/W \) is the image of \( v \); \( v' := w \oplus \bar{v} \) is a generic vector in \( V' \). Defining \( t \) as in the previous paragraph, we have \( g_v, g_{v'} \subseteq t \). Replacing \( g, V, V' \) with \( t, W + kv, W + k\bar{v} \) and referring to the previous paragraph gives the claim. \( \square \)

If \( g \) acts generically freely on \( V' \) (i.e., \( g_v = 0 \)), then the proposition says that \( g \) acts generically freely on \( V \). This immediately gives the following statement about group schemes:

**Corollary 2.5.** If \( G_{v'} \) is finite étale for generic \( v' \in V' \), then \( G_v \) is finite étale for generic \( v \in V \). \( \square \)

While generic freeness of \( V' \) implies generic freeness of \( V \) for the action by the Lie algebra \( g \), it does not do so for the action by the algebraic group \( G \), as the following example shows.

**Example 2.6.** Take \( G = \mathbb{G}_a \) acting on \( V = k^3 \) via

\[
\rho(r) := \begin{pmatrix} 1 & r^p \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Let \( V_2 \subset V \) be the subspace of vectors whose bottom entry is zero. Then \( G \) acts on \( V_2 \) via \( r \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \) and in particular a generic \( v_2 \in V_2 \) has \( G_{v_2} = 1 \). On the other hand, a generic vector \( v := \begin{pmatrix} z \\ \bar{z} \\ \bar{z} \end{pmatrix} \) in \( V \) has \( G_v \) the étale subgroup with points \( \{ r \mid ry + r^p z = 0 \} \), i.e., the kernel of the homomorphism \( zF + yId: \mathbb{G}_a \to \mathbb{G}_a \) for \( F \) the Frobenius map.

Direct sums have better properties with respect to calculating generic stabilizers, see for example [Pop89, Prop. 8] and [Löt15, Lemma 2.15].

3. Deforming semisimple elements to nilpotent elements

**Fixed subspaces.** For \( x \in g \), we use the shorthand \( x^{G_{x}G} \) for the orbit of \( x \) under the subgroup of \( \text{GL}(g) \) generated by \( G_{x}m \) and \( \text{Ad}(G) \). For \( y \) in the closure of \( x^{G_{x}G} \), \( \dim V^x \leq \dim V^y \) by upper semicontinuity of dimension.

**Example 3.1.** Suppose that \( x \in g \) is non-central semisimple and let \( b \) be a Borel subalgebra containing \( x \). Because \( x \) is not central, there is a root subgroup \( U_{\alpha} \) in the corresponding Borel subgroup that does not commute with \( x \). This implies that \( x + \lambda y \) is in the same \( G \)-orbit as \( x \) for all \( \lambda \in k \) and \( y \) in the corresponding root subalgebra, and similarly \( \lambda x + y \) is in the same \( G \)-orbit as \( \lambda x \) and in particular \( y \) is in the closure of \( x^{G_{x}G} \).

**Lemma 3.2.** Suppose \( k \) is algebraically closed and let \( G = \text{GL}_n \) or \( \text{SL}_n \). Then for every non-central toral element \( x \in g \), the nilpotent \( y \) from the preceding paragraph may be chosen so that the \( \text{Ad}(G) \)-orbits of \( x \) and \( y \) have the same dimension. Furthermore, if \( x \) has exactly two eigenvalues (e.g., if \( \text{char} k = 2 \)), then \( y \) can be chosen so that additionally \( y^2 = 0 \).

**Proof.** Suppose first that \( G = \text{GL}_n \) and the transcendence degree of \( k \) over \( \mathbb{F}_p \) is at least \( n^2 \). We may assume (after conjugation) that the given toral element \( x \) is diagonal with entries in \( \mathbb{F}_p \). Permuting the basis so that vectors with the
same eigenvalue are adjacent, we may assume that \( x \) has \( 0, 1, 2, \ldots, p - 1 \) down the diagonal, with \( i \) appearing \( n_i \) times. The centralizer of \( x \) in \( \text{GL}_n \) is \( \prod_i \text{GL}_{n_i} \), of dimension \( \sum n_i^2 \).

Take now \( y \) to be a “generic” strictly upper triangular matrix block according to the eigenvalues of \( x \), i.e., such that the \( n_i \)-by-\( n_i \) blocks on the diagonal are 0 and the entries northeast of these blocks are algebraically independent over \( \mathbb{F}_p \). As \( x \) is not a scalar matrix, \( y \) is not zero. By writing out block matrices, one sees that \( x + ty \) is in \( x^G \), as desired.

Using the independence of the entries in \( y \), we deduce that the size of the Jordan blocks in the Jordan form of \( y \) are given by the partition of \( n \) into \( n_i \)'s, \( n_i \leq n/2 \), and the entries northeast of these blocks are algebraically independent over \( \mathbb{F}_p \). As \( x \) is not a scalar matrix, \( y \) is not zero. By writing out block matrices, one sees that \( x + ty \) is in \( x^G \), as desired.

A specialization argument now shows that the result holds for all \( k \) algebraically closed of characteristic \( p \) and \( G = \text{GL}_n \).

For \( G = \text{SL}_n \), each toral element is also toral in \( \text{GL}_n \) and one takes \( y \) as in the \( \text{GL}_n \) case. \( \square \)

**Generation.** Suppose now that \( X \) is an irreducible and \( G \)-invariant subset of \( \mathfrak{g} \) such that \( X \) is open in \( \overline{X} \). If, for some \( Y \subseteq \overline{X} \), \( e \) elements of \( Y(k) \) generate a \( G \)-invariant subalgebra \( M \) of \( \mathfrak{g} \), then \( e \) generic elements of \( X \) generate \( M \). (For example, one can take \( X = x^G \) and \( Y = y^G \) for some \( y \in \overline{X} \).)

To see this, let \( S \) denote the set of \( (y_1, \ldots, y_e) \in \overline{X}^\times \) such that \( (y_1, \ldots, y_e) \) generating \( M \). It is an open subset of \( \overline{X}^\times \) that is nonempty by hypothesis, so \( S \) meets the open set \( X^\times \).

Here is an application of this observation. Take \( x \) and \( y \) as in Example 3.1 and set, \( X = x^G \) and \( y^G \). As \( y \) belongs to \( \overline{X} \), if \( e \) conjugates of \( y \) generate a \( G \)-invariant subalgebra \( M \) of \( \mathfrak{g} \), then so do \( e \) conjugates of \( x \). Moreover, upper semicontinuity of dimension gives that \( \dim y^G \leq \dim x^G \).

### 4. Quasi-regular subalgebras

For this section, let \( T \) be a maximal torus in a reductive algebraic group \( G \) over an algebraically closed field \( k \). Writing \( \mathfrak{t} := \text{Lie}(T) \) and \( \mathfrak{g} := \text{Lie}(G) \), the action of \( T \) on \( \mathfrak{g} \) gives the Cartan decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \) where \( \Phi \) is the set of roots of \( G \) with respect to \( T \) and \( \mathfrak{g}_\alpha \) is the 1-dimensional root subalgebra for the root \( \alpha \). (Note that the action by \( t \) induces a direct sum decomposition on \( \mathfrak{g} \) that need not be as fine when \( \text{char} k = 2 \), for in that case \( \alpha \) and \( -\alpha \) agree on \( \mathfrak{t} \), and if furthermore \( G = \text{Sp}_{2n} \) for \( n \geq 1 \), then the centralizer of \( t \) in \( \mathfrak{g} \), the Cartan subalgebra, properly contains \( \mathfrak{t} \).) We say that a subalgebra \( L \) of \( \mathfrak{g} \) is **quasi-regular with respect to** \( T \) if

\[
L = (L \cap \mathfrak{t}) \oplus \begin{cases} \bigoplus_{\alpha \in \Phi}(L \cap \mathfrak{g}_\alpha) & \text{if } \text{char } k \neq 2 \\ \bigoplus_{\alpha \in \Phi^+}(L \cap \mathfrak{g}_{\pm \alpha}) & \text{if } \text{char } k = 2 \end{cases}
\]

as a vector space, where \( \mathfrak{g}_{\pm \alpha} := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \) and \( \Phi^+ \) denotes the set of positive roots relative to some fixed ordering. We say simply that \( L \) is quasi-regular if it is quasi-regular with respect to some torus \( T \).

For \( L \) quasi-regular, \( \mathfrak{t} \) evidently normalizes \( L \), i.e., \( L + \mathfrak{t} \) is also a quasi-regular subalgebra.
Example 4.1. Suppose there is a $t \in \mathfrak{t} \cap L$ such that
\begin{equation}
\pm \alpha(t) \neq \pm \beta(t) \quad \text{for all } \alpha \neq \beta \in \Phi^+ \cup \{0\}.
\end{equation}
Put $m(x)$ for the minimal polynomial of $\text{ad}(t)$. For each $\alpha \in \Phi \cup \{0\}$, evaluating $m(x)/(x - \alpha(t))$ at $\text{ad}(t)$ gives a linear map $\mathfrak{g} \to \mathfrak{g}$ with image $\mathfrak{g}_\alpha$ (if $\text{char } k \neq 2$) or $\mathfrak{g}_{\pm \alpha}$ (if $\text{char } k = 2$). Restricting $t$ to $L$ shows that $L \cap \mathfrak{g}_\alpha$ or $L \cap \mathfrak{g}_{\pm \alpha}$ is contained in $L$, i.e., $L$ is quasi-regular.

Example 4.3. Suppose $G = \text{SL}_n$ or $\text{GL}_n$ for $n \geq 4$. If $L$ contains a copy of $\mathfrak{sl}_{n-1}$ (say, the matrices with zeros along the rightmost column and bottom row), then $L$ is quasi-regular. Indeed, taking $T$ to be the diagonal matrices in $G$ and $t \in \mathfrak{t}$ to have distinct indeterminates in the first $n - 2$ diagonal entries and a zero in the last diagonal entry, we find that $t$ satisfies (4.2). This $L$ is quasi-regular, but need not be regular, in the sense that it need not contain a maximal toral subalgebra of $\mathfrak{g}$.

Remark 4.4. Suppose $\text{char } k \neq 2$ and $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$, or $\mathfrak{sp}_{2n}$. If $\mathfrak{h}$ is a Lie subalgebra that contains a maximal toral subalgebra $\mathfrak{t}$ (so $\mathfrak{h}$ is quasi-regular) and acts irreducibly on the natural module, then $\mathfrak{h} = \mathfrak{g}$. To see this, note that $\mathfrak{h}$ is a sum of $\mathfrak{t}$ and the root spaces it contains (using the char $k \neq 2$), and so is determined by $\mathfrak{t}$ and a closed subset of the root system of $\mathfrak{g}$, whose classification over $k$ is the same as the Borel-de Siebenthal classification over $\mathbb{C}$.

The claim is clear if $\mathfrak{h}$ is contained in a maximal parabolic subalgebra, for such subalgebras act reducibly (even stabilizing a totally singular subspace for $\mathfrak{g} = \mathfrak{so}_n$, or $\mathfrak{sp}_{2n}$), see for example [CG06, §3]. Otherwise, $\mathfrak{h}$ stabilizes a nondegenerate subspace (compare for example [Dyn57, Table 9]) and again the claim follows. See [BGGT12, Lemma 3.6] for a similar statement on the level of groups.

The subsystem subalgebra. Suppose $L$ is a quasi-regular subalgebra of $\mathfrak{g}$ with respect to $T$. Define $L_0$ to be the subalgebra of $L$ generated by the $L \cap \mathfrak{g}_\alpha$, for $\alpha \in \Phi$.

Lemma 4.5. If
\begin{enumerate}
  \item char $k \neq 2$ or
  \item char $k = 2$, $\Phi$ is irreducible, and all roots have the same length,
\end{enumerate}
then $L_0$ is an ideal in $L + \mathfrak{t}$.

Proof. If char $k \neq 2$, then $L_0 \cap \mathfrak{g}_\alpha = L \cap \mathfrak{g}_\alpha$ for all $\alpha$ and the claim is trivial, so assume (2) holds. As $L_0$ is evidently stable under $\text{ad } \mathfrak{t}$, it suffices to check that, for $x_\beta \in \mathfrak{g}_\beta$, $x_{-\beta} \in \mathfrak{g}_{-\beta}$, and $c \in k$ such that $x_\beta + cx_{-\beta} \in L$, we have
\[
[x_\beta + cx_{-\beta}, x_\alpha] = [x_\beta, x_\alpha] + c[x_{-\beta}, x_\alpha] \in L_0.
\]
However, by hypothesis $\alpha + \beta$ and $\alpha - \beta$ cannot both be roots, so at least one of the two terms in the displayed sum is zero and the expression belongs to $L \cap \mathfrak{g}_{\alpha + \beta}$ or $L \cap \mathfrak{g}_{\alpha - \beta}$, hence to $L_0$.

Example 4.6. Let $L$ be the space of symmetric $n$-by-$n$ matrices in $\mathfrak{sl}_n$. It is a Lie subalgebra when char $k = 2$, and, in that case, it is quasi-regular with respect to the maximal torus $T$ of diagonal matrices in $\text{GL}_n$ and $L_0 = 0$.

Lemma 4.7. Suppose $L$ is a quasi-regular subalgebra of $\mathfrak{gl}(V)$ with respect to a maximal torus $T$. Then $L_0$ is irreducible on $V$ if and only if $L_0 + \mathfrak{t}$ is irreducible on $V$ if and only if $L_0 = \mathfrak{sl}(V)$, if and only if $L_0 + \mathfrak{t} = \mathfrak{gl}(V)$. 

Proof. The algebra $L_0$ is $(L_0 \cap t) \oplus \bigoplus_{\alpha \in S} \alpha$, where $S$ is a closed subsystem of a root system of type $A$. Therefore $S = \Phi$ (in which case $L_0$ acts irreducibly and $L_0 = \mathfrak{sl}(V)$) or $S$ is contained in a proper subsystem (which normalizes a proper $T$-invariant subspace of $V$).

Application to type $A$.

Theorem 4.8. Suppose $L$ is a subalgebra of $\mathfrak{sl}_n$ for some $n \geq 2$ that is quasi-regular and acts irreducibly on the natural representation of $\mathfrak{sl}_n$. Then

1. $L$ contains $\mathfrak{sl}_n$, or
2. $\text{char } k = 2$ and $L$ is $\text{GL}_n$-conjugate to a subalgebra of symmetric $n$-by-$n$ matrices.

Proof. Let $T$ be the maximal torus with respect to which $L$ is quasi-regular. After conjugation by an element of $\text{GL}_n(k)$, we may assume that $T$ is the diagonal matrices. If $L_0 = L$ or even $L_0 + t = \mathfrak{gl}(V)$, we conclude that $L$ contains $\mathfrak{sl}_n$.

Case: $L_0 \neq 0$. Suppose $L_0 \neq 0$. We claim that (1) holds. Replacing $L$ with $L + t$, we may assume that $L_1 := L_0 + t$ acts irreducibly on $V$.

Suppose $W \subset V$ is a subspace on which $L_1$ acts nontrivially and irreducibly. Conjugating by a monomial matrix, we may assume that $W$ is the subspace consisting of vectors whose nonzero entries are in the first $w := \dim W$ coordinates. Now $L_1 \cap \mathfrak{gl}(W)$ is a quasi-regular subalgebra of $\mathfrak{gl}(W)$ acting irreducibly on $W$ and it is generated by $t \cap \mathfrak{gl}(W)$ and those $\mathfrak{g}_\alpha$ contained in $L$, so by Lemma 4.7 it equals $\mathfrak{gl}(W)$.

If $W \not\subset V$, then there is a $\beta \in \Phi$ such that $\mathfrak{g}_{\pm \beta} \cap L_0 = 0$ yet $(\mathfrak{g}_{\pm \beta} \cap L)W \not\subset W$. That is, there exists $i > w$ and $j \leq w$ such that $E_{ij} - cE_{ji} \in L$ for some $c \in k^\times$, where $E_{ij}$ denotes the matrix whose unique nonzero entry is a 1 in the $(i,j)$-entry. As $\dim W \geq 2$, there is $\ell \leq w$, $\ell \neq j$ and $E_{ij} - E_{j\ell} \in \mathfrak{sl}(W) \subset L_0$. So $[E_{ij}, E_{ij} - cE_{j\ell}] = -cE_{\ell i}$ is in $L$, hence in $L_0$, yet $E_{\ell i}W \not\subset W$, a contradiction. Thus $W = V$, i.e., $L_1$ acts irreducibly on $V$ and $L_0 = \mathfrak{sl}_n$.

Case: $L_0 = 0$. Suppose $L_0 = 0$. If $\text{char } k \neq 2$, then $L \subset t$ cannot be irreducible, so assume $\text{char } k = 2$. We prove (2). We may replace $L$ with $L + t$ and so assume that $L$ contains $t$.

Define $\hat{L}$ to be the subspace generated by $t$ and those $\mathfrak{g}_{\pm \alpha}$ with nonzero intersection with $L$. It is closed under the bracket. Indeed, fixing nonzero elements $x_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in \Phi$, those $\mathfrak{g}_{\pm \alpha}$ that meet $L$ are spanned by an element $x_\alpha + c_\alpha x_{-\alpha}$ for some $c_\alpha \in k^\times$. If $\mathfrak{g}_{\pm \beta}$ also meets $L$, then

$$[x_\alpha + c_\alpha x_{-\alpha}, x_\beta + c_\beta x_{-\beta}] \in \mathfrak{g}_{\pm (\alpha + \beta)} + \mathfrak{g}_{\pm (\alpha - \beta)}.$$ 

As $L$ acts irreducibly on $V$, so does $\hat{L}$, and Lemma 4.7 gives that $\hat{L} = \mathfrak{sl}_n$ and in particular $\mathfrak{g}_{\pm \alpha}$ meets $L$ for every root $\alpha$.

For each simple root $\alpha_i$, set $h_i : \mathbb{G}_m \to \text{GL}_n$ to be a cocharacter such that $\alpha_j \circ h_i : \mathbb{G}_m \to \mathbb{G}_m$ is $t \mapsto 1$ if $i \neq j$ and $t \mapsto t^{r_i}$ for some $r_i \neq 0$ if $i = j$. As

$$\text{Ad}(h_i(t))(x_\alpha, c_\alpha x_{-\alpha}) = t^{r_i} x_\alpha + \frac{c_\alpha}{t^{r_i}} x_{-\alpha},$$

there is a $t_i \in k^\times$ for each $i$ so that $\text{Ad}(h_i(t_i))(\mathfrak{g}_{\pm \alpha}) \cap L$ is generated by $E_{i,i+1} + E_{i+1,i}$. Conjugating $L$ by $\prod h_i(t_i)$ arranges this for all simple roots $\alpha_i$ at once, and it follows that the resulting conjugate of $L$ is the algebra of symmetric matrices. □
5. Type $A$ and $\text{char } k \neq 2$

Recall that $\mathfrak{sl}_n$ is either simple (char $k$ does not divide $n$) or has a unique nontrivial ideal, the center (consisting of the scalar matrices, in case char $k$ does divide $n$).

Example 5.1. Suppose that char $k \neq 2$ and $x$ is regular nilpotent; we claim that $e(x) = 2$. We may assume $k$ is algebraically closed because the set of $t$-tuples of conjugates of $x$ that generate $\mathfrak{sl}_n$, for any particular $t$, is open. Up to conjugacy, $x$ has 1’s on the superdiagonal and 0’s in all other entries. Among the conjugates of $x$ one finds an element $x_2$ with algebraically independent entries on the subdiagonal and 0’s elsewhere; put $\mathfrak{h}$ for the subalgebra generated by $x, x_2$. Then $t = [x, x_2]$ is diagonal and satisfies (4.2), so $\mathfrak{h} = \mathfrak{sl}_n$ (Theorem 4.8).

The following lemma has no restrictions on the characteristic of $k$. For $x \in \mathfrak{sl}_n$, put $\alpha(x)$ for the dimension of the largest eigenspace.

Lemma 5.2. For non-central $x \in \mathfrak{sl}_n$, if $e > \frac{n-1}{n-\alpha(x)}$, then the subalgebra of $\mathfrak{sl}_n$ generated by $x$ generic conjugates of $x$ fixes no 1-dimensional subspace nor codimension-1 subspace of the natural module.

The hypothesis that $x$ is non-central serves to ensure that the denominator $n - \alpha(x)$ is not zero.

Proof. Suppose the subalgebra generated by $e$ generic conjugates of $x$ fixes a line. Then, arguing as in [BGGT12, Lemma 3.4(ii)], every subalgebra generated by $e$ conjugates fixes a line. Putting $X := x^{\mathfrak{sl}_n}$, there is a map $G \times (x^eX) \to x^eX$ via $(g, x_1, \ldots, x_e) \mapsto (\text{Ad}(g)x_1, \ldots, \text{Ad}(g)x_e)$, and by hypothesis $x^eX$ belongs to the image of $G \times (x^e(X \cap \mathfrak{p}))$ where $\mathfrak{p}$ is the stabilizer of the first basis vector in the natural module, the Lie algebra of a parabolic subgroup $P$ of $\mathfrak{sl}_n$. Thus

$$e \cdot \dim X \leq \dim P^1 + e \cdot \dim (X \cap \mathfrak{p}),$$

and consequently

$$e(\dim X - \dim (X \cap \mathfrak{p})) \leq \dim(G/P) = n - 1. \tag{5.3}$$

Now consider the variety $Y \subset X \times \mathbb{P}^{n-1}$ with $k$-points

$$Y(k) = \{ (y, \omega) \in X(k) \times \mathbb{P}(k^n) \mid y\omega = \omega \}.$$

The projection of $Y$ on the first factor maps $Y$ onto $X$ with fibers of dimension $\alpha(x) - 1$. The projection of $Y$ on the second factor maps $Y$ onto $\mathbb{P}^{n-1}$ with fibers of dimension $\dim(X \cap \mathfrak{p})$. Consequently,

$$\dim X + \alpha(x) - 1 = \dim Y = (n - 1) + \dim(X \cap \mathfrak{p}).$$

Combining this with (5.3) gives $e \leq \frac{n-1}{n-\alpha(x)}$.

Now suppose each subalgebra $\mathfrak{g}$ generated by $e$ generic conjugates of $x$ fixes a codimension-1 subspace $V$ of the natural module. Using the dot product we may identify the natural module $k^n$ with its contragradient $(k^n)^*$, and it follows that the subalgebra $\{ y^\top \mid y \in \mathfrak{g} \}$ fixes the line in $(k^n)^*$ of elements vanishing on $V$. Consequently $e \leq \frac{n-1}{n-\alpha(x^\top)}$. As $\alpha(x^\top) = \alpha(x)$, the claim is proved.

Proposition 5.4. Assume char $k \neq 2$. For each nonzero nilpotent $x \in \mathfrak{sl}_n$, $e$ generic conjugates of $x$ generate $\mathfrak{sl}_n$, where:

1. $e = 3$ if $x$ has Jordan canonical form with partition $(2, 2, \ldots, 2)$ or $(2, 2, \ldots, 2, 1)$. 

(2) \( e = 2 \) if \( \alpha(x) \leq \lceil n/2 \rceil \) but we are not in case (1).
(3) \( e = \lceil \frac{n}{n-\alpha(x)} \rceil \) if \( \alpha(x) > \lceil n/2 \rceil \).

Proof. The conjugacy class of \( x \) is determined by its Jordan form, which corresponds to a partition \((p_1, \ldots, p_\alpha)\) of \( n \), i.e., a list of numbers \( p_1 \geq p_2 \geq \cdots \geq p_\alpha > 0 \) such that \( p_1 + \cdots + p_\alpha = n \). If \( x \) has partition \((n)\), then \( e(x) = 2 \) by Example 5.1.

If \( x \) has partition \((2, 1, \ldots, 1)\), i.e., the Jordan form of \( x \) has a unique nonzero entry, then \( x \) generates a root subalgebra, and we may assume it corresponds to a simple root. The other root subalgebras for simple roots and for the lowest root suffice to generate \( \mathfrak{sl}_n \), so in this case \( e = \lfloor n/(n - (n - 1)) \rfloor \) conjugates suffice to generate.

Thus we may assume that \( n \geq 4 \).

Suppose first that \( x \) has partition \((2, 2, \ldots, 2)\) and view \( x \) as the image of a regular nilpotent in \( \mathfrak{sl}_2 \) under the diagonal embedding in \( \mathfrak{sl}_2^{\times n/2} \subset \mathfrak{sl}_n \). As in Example 5.1, two \( \mathfrak{sl}_2^{\times n/2} \)-conjugates suffice to generate \( \mathfrak{sl}_2^{\times n/2} \). As the adjoint representation of \( \mathfrak{sl}_n \) restricts to a multiplicity-free representation of \( \mathfrak{sl}_2^{\times n/2} \), there are only a finite number of Lie algebras lying between \( \mathfrak{sl}_2^{\times n/2} \) and \( \mathfrak{sl}_n \). Now \( x^{\mathfrak{sl}_n} \) generates \( \mathfrak{sl}_n \) as a Lie algebra, so it is not contained in any of these proper subalgebras and the irreducible variety \( x^{\mathfrak{sl}_n} \) is not contained in the union of the proper subalgebras. This proves the claim that \( 3 \) conjugates suffice to generate \( \mathfrak{sl}_n \).

If \( x \) has partition \((2, 2, \ldots, 2, 1)\), then we view it as the image of \( x' \in \mathfrak{sl}_{n-1} \) where \( x' \) has partition \((2, 2, \ldots, 2, 2)\), for which three \( \mathfrak{sl}_{n-1} \)-conjugates generate \( \mathfrak{sl}_{n-1} \). That is, three generic \( \mathfrak{sl}_n \)-conjugates of \( x \) generate a subalgebra \( \mathfrak{h} \) that is quasi-regular (Example 4.3). Moreover, as \( n = 2\alpha - 1 \), \( \mathfrak{h} \) does not fix a 1-dimensional or codimension-1 subspace of the natural module (Lemma 5.2), and therefore \( \mathfrak{h} \) acts irreducibly and \( \mathfrak{h} \) is the whole algebra \( \mathfrak{sl}_n \) (Remark 4.4).

Now suppose \( \alpha(x) \leq n/2 \) and we are not in case (1). Then \( p_1 \geq 3 \) and by passing to an element in the closure of \( x^{\mathfrak{sl}_n} \), we can reduce to the cases

(a) \( n \) is even and \( x \) has partition \((3, 2, \ldots, 2, 1)\); or
(b) \( n \) is odd and \( x \) has partition \((3, 2, \ldots, 2)\).

In case (a), we see by induction that we can generate \( \mathfrak{sl}_{n-1} \) with two \( \mathfrak{sl}_n \)-conjugates and we argue as in the preceding case.

In case (b), deform to \( y \in x^{\mathfrak{sl}_n} \) with partition \((3, 2, \ldots, 2, 1, 1)\). It is the image of \( y' \in \mathfrak{sl}_{n-1} \) with partition \((3, 2, \ldots, 2, 1)\). By induction on \( n \), two \( \mathfrak{sl}_{n-1} \)-conjugates of \( y' \) generate a copy of \( \mathfrak{sl}_{n-1} \). Arguing as in the preceding cases concludes the proof of (2).

Finally, suppose \( \alpha(x) > \lceil n/2 \rceil \), so in particular \( p_\alpha = 1 \). Put \( x' \in \mathfrak{sl}_{n-1} \) for a nilpotent with partition \((p_1, \ldots, p_{\alpha-1})\). By induction, we find that \( \lceil n/(n - \alpha) \rceil \) \( \mathfrak{sl}_{n-1} \)-conjugates suffice to generate a copy of \( \mathfrak{sl}_{n-1} \), and we complete the proof as before.

Corollary 5.5. For noncentral \( x \in \mathfrak{gl}_n \) such that \( x^{[p]} \in \{0, x\} \), there exist \( x_1, \ldots, x_{\epsilon} \in x^{\mathfrak{sl}_n} \) such that \( x_1, \ldots, x_{\epsilon} \) generate a subalgebra containing \( \mathfrak{sl}_n \) and \( \epsilon \cdot \dim x^{\mathfrak{sl}_n} \leq \frac{9}{4} n^2 \).

Proof. Suppose first that \( x^{[p]} = 0 \). We consider the three cases in Proposition 5.4.
In case (1), we have \( \dim x^{\mathfrak{sl}_n} \leq n^2/2 \) and \( e(x) = 3 \), so the claim is clear. In case (2), \( e = 2 \) and \( \dim x^{\mathfrak{sl}_n} < n^2 \). In case (3), among those nilpotent \( y \) with rank \( n - \alpha(x) \), the one with the largest \( \mathfrak{sl}_n \)-orbit has partition \((n - \alpha(x) + 1, 1, \ldots, 1)\),
whose orbit has dimension $n^2 - n - \alpha(x)^2 + \alpha(x)$. Consequently,
\[ e(x) \cdot \dim x^{SL_n} < (n + \alpha(x) - 1)(2n - \alpha(x)). \]
This is a quadratic polynomial in $\alpha(x)$ opening downwards with maximum at $(n + 1)/2$, so the right side has maximum $\frac{4}{9}n^2 - 3n/2 - 3/4$, verifying the claim for $x$ nilpotent.

For $x \in sl_n$ noncentral toral, let $y$ be the nilpotent element provided by Lemma 3.2. Then $\dim x^G = \dim y^G$ and the same number of conjugates suffice to generate a subalgebra containing $sl_n$. \hfill $\square$

6. Type $A$ and $\text{char } k = 2$

Proposition 6.1. Suppose $\text{char } k = 2$ and let $x \in sl_n$ be a nilpotent element of square 0 and rank $r$. Then $sl_n$ can be generated by $e := \max\{3, \left\lceil \frac{n}{r} \right\rceil \}$ conjugates of $x$.

Proof. Note the result is clear if $x$ is a root element by taking root elements in each of the simple positive root subalgebras and in the root subalgebra corresponding to the negative of the highest root. This gives the result for $n = 2, 3$ and shows that for $n = 4$, it suffices to consider $r = 2$. Choose two conjugates of $x$ and $y$ generating $sl_2 \times sl_2$. It is straightforward to see for a generic conjugate $z$ of $x$, the elements $x, y$ and $z$ generate $sl_4$. So assume $n > 4$.

If $n$ is odd, it follows by induction on $n$ that $e$ conjugates of $x$ can generate an $sl_{n-1}$. On the other hand, the condition on the rank implies by Lemma 5.2 that $e$ generic conjugates of $x$ do not fix a 1-space or a hyperplane. Thus, generically $e$ conjugates of $x$ generate a subalgebra that acts irreducibly and is quasi-regular. Also, we see that generically the dimension of the Lie algebra generated by $e$ conjugates has dimension at least $(e - 1)^2 - 1$. Since $n > 4$, this is larger than the dimension of the space of symmetric matrices, whence by Theorem 4.8, we see that $e$ generic conjugates generate $sl_n$.

Now assume that $n$ is even. By passing to closures we may assume that $r < n/2$ (since $n > 4$, $e = 3$ for both elements of rank $n/2$ and rank $n/2 - 1$). Now argue just as for the case that $n$ is odd. \hfill $\square$

Remark. The result also holds for idempotents of rank $e \leq n/2$ by a closure argument.

Corollary 6.2. Suppose $\text{char } k = 2$. For noncentral $x \in gl_n$ with $n \geq 2$ such that $x^{[2]} \in \{0, x\}$, there exist $x_1, \ldots, x_e \in x^{SL_n}$ such that $x_1, \ldots, x_e$ generate a subalgebra containing $sl_n$ and $e \cdot \dim x^{SL_n} \leq 2n^2 - 2$.

Proof. Let $x \in gl_n \setminus \mathfrak{l}(sl_n)$ satisfy $x^{[2]} = 0$ and put $r$ for the rank of $x$. Then $
 \dim x^{SL_n} = n^2 - (r^2 + (n - r)^2) = 2r(n - r). \n$ If 3 conjugates of $x$ generate $sl_n$, then $3 \cdot \dim x^{SL_n} = 6r(n - r)$. This has a maximum at $r = n/2$, where it is $\frac{1}{2}n^2 < 2n^2 - 2$. Otherwise $(n + r)/r$ conjugates suffice to generate, and we have $e \dim x^{SL_n} \leq 2(n^2 - r^2) \leq 2n^2 - 2$.

Now suppose that $x \in sl_n$ is noncentral toral. Taking $y$ such that $y^{[2]} = 0$ as in Lemma 3.2, we find that (1.3) holds also for $x$ noncentral toral. \hfill $\square$
7. Type C and char \(k\) Odd

**Proposition 7.1.** Assume \(\text{char } k \neq 2\). For every nonzero nilpotent \(x \in \mathfrak{sp}_{2n}\) for \(n \geq 1\) of rank \(r\), \(e\) generic conjugates of \(x\) generate \(\mathfrak{sp}_{2n}\), where:

1. \(e = 3\) if \(x\) has Jordan canonical form with partition \((2, 2, \ldots, 2)\).
2. \(e = 2\) if \(r \geq n\) but we are not in case (1).
3. \(e = 2\lceil n/r \rceil\) if \(r < n\).

**Proof.** The conjugacy class of \(x\) is determined by its Jordan form, which corresponds to a partition \((p_1, \ldots, p_n)\) of \(2n\) with \(p_1 \geq p_2 \geq \cdots \geq p_n\) such that odd numbers appear with even multiplicity. Note that \(\mathfrak{sp}_2 = \mathfrak{sl}_2\), so the \(n = 1\) case holds by Example 5.1.

By specialization (replacing \(x\) with an element of \(x^{\mathfrak{sp}_{2n}}\) as in §3), we may replace in the partition of \(x\)

\[(7.2) \ (2s + 2, 1, 1) \mapsto (s + 1, s + 1, 2) \text{ or } (2s + 1, 2s + 1, 1, 1) \mapsto (2s, 2s, 2, 2) \text{ for } s \geq 2\]

without changing the rank \(r\) of \(x\) nor whether the partition is \((2, \ldots, 2)\). In this way, we may assume that \(p_α \geq 2\) or \(p_1 \leq 4\).

**Case (1).** Suppose that \(x\) has partition \((2, 2, \ldots, 2)\). Two conjugates of \(x\) suffice to generate a copy of \(\mathfrak{sl}_2^{\times n} \subset \mathfrak{sp}_{2n}\), and this contains a regular semisimple element of \(\mathfrak{sp}_{2n}\). Furthermore, the natural representation of \(\mathfrak{sp}_{2n}\) is multiplicity-free for \(\mathfrak{sl}_2^{\times n}\), so one further conjugate suffices to produce a subalgebra that is irreducible on the natural module. Appealing to Remark 4.4, the claim follows in this case.

**Case \(\mathfrak{sp}_4\).** For the case \(n = 2\), it remains to consider \(x\) with partition \((4)\), i.e., a regular nilpotent. A pair of generic conjugates generates an irreducible subalgebra. By passing to \((2, 2)\), we see it also generically contains an element as in \((4, 2)\), whence the result.

**Case \(\mathfrak{sp}_6\).** Suppose \(x \in \mathfrak{sp}_6\) has partition \((4, 1, 1)\). By induction, two conjugates suffice to generate an \(\mathfrak{sp}_4\). Moreover, by counting dimensions, we see in fact we can generate [\(\mathfrak{p}, \mathfrak{p}\)] where \(\mathfrak{p}\) is the stabilizer of a 1-space (take \(x, y \in \mathfrak{sp}_4\) in the Levi subalgebra and consider \(x^q, y^{q'}\) where \(q, q'\) are in the unipotent radical of \(P\), the parabolic with Lie algebra \(\mathfrak{p}\). If none generate [\(\mathfrak{p}, \mathfrak{p}\)], then they would have to generate (modulo the root algebra) a Levi subalgebra, but since all Levi’s are conjugate (because they are the centralizer of a \(l\)-dimensional torus in the group), the space of Levi’s is a 4-dimensional variety while the different \(x^q, y^{q'}\) form a 6-dimensional variety. Thus, if the result fails, we have \(\langle x, y \rangle = \langle x^q, y^{q'} \rangle\) with at least one of \(q, q'\) not centralizing \(x, y\) respectively. This is implies that say \(x - x^q\) is in the algebra, a contradiction. Thus, we see that generically, a pair does not stabilize a 4-space but has a 4-dimensional composition factor — whence it has a composition series \(1, 4, 1\) — but we also know that generically it does not stabilize a 1-space.

Alternatively, we see that by passing to closures (to the class \((2, 2, 2)\)) we can generate something containing \(\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2\) and so the smallest composition factor of the natural module for the subalgebra generated by a generic pair has dimension at most 2 and the subalgebra also contains a strongly regular semisimple element. So if two conjugates do not suffice to generate \(\mathfrak{sp}_6\), we see that generically there must be a 4-dimensional composition factor and the other 2-dimensional, whence
it must be contained in $\mathfrak{sp}_4 \times \mathfrak{sp}_2$ and so indeed in $\mathfrak{sp}_4$ which contradicts that the smallest composition factor is generically at least 2-dimensional.

For $x \in \mathfrak{sp}_6$, it remains to treat the case where $x$ has partition $(3,3)$. By induction, a pair can generate $\mathfrak{sl}_3$ and so generically it contains a strongly regular semisimple element. We see that generically all composition factors are at least 3-dimensional. By taking closures, we can pass to $(2,2,2)$ and we see that not every pair stabilizes a 3-dimensional space, whence a generic pair is irreducible and the proposition is proved for $\mathfrak{sp}_6$.

**Case** $2n \geq 8$ and $x$ has partition $(3,3,2,\ldots,2,1,1)$. Suppose now that $x$ has partition $(3,3,2,\ldots,2,1,1)$ so $r = n$. By induction, two conjugates can generate $\mathfrak{sp}_{2n-2} \times \mathfrak{sp}_2$ and in particular generically the algebra contains an element as in (4.2). If $n$ is even, we can also generate $\mathfrak{sl}_n$ and so the smallest composition factor has dimension $n$, whence the algebra is generically irreducible and the result follows. In either case ($n$ even or odd), we can also argue as follows:

(a) Show that the variety

$$\{ (W, x_1, x_2) \mid W \text{ is a nondegenerate } 2\text{-space}, x_1, x_2 \in x^{\mathfrak{sp}_{2n}} \text{ and } x_iW = W \}$$

has dimension less than $2 \dim x^{\mathfrak{sp}_{2n}}$, or

(b) Show that we can generate a subalgebra that acts uniserially with series $9, 1, 2n - l, 2n$.

For (a), one can compute $\dim x^{\mathfrak{sp}_{2n}}$ by [LS12] or by noting that it is just the 0 eigenspace on the symmetric square of the natural module. For (b), we can argue as for $\mathfrak{sp}_6$.

**Case** $r \geq n$. We now consider the case where $r \geq n$ (and $2n \geq 8$).

If $p_\alpha \geq 2$, then, as $o = 2n - r \leq n$ and we are not in case (1), we may replace $2s \rightsquigarrow (s, s)$ for $s \geq 3$, $(s, s) \rightsquigarrow (s - 1, s - 1, 1, 1)$ for $s \geq 4$, or $(4, 2) \rightsquigarrow (3, 3)$ as long as we retain the property that rank $x \geq n$. In this way, we may assume that $p_\alpha \leq 1$ or $p_1 \leq 3$.

So suppose $p_\alpha = 1$, in which case we may assume that $p_1 \leq 4$. We may replace $(4, 4, 1, 1) \rightsquigarrow (3, 3, 2, 2)$, $(4, 2) \rightsquigarrow (3, 3)$, or $(4, 3, 3, 1, 1) \rightsquigarrow (3, 3, 2, 2, 2)$ without changing the rank of $x$. Repeating these reductions and those in the previous paragraph, we are reduced to considering partitions $(4, 1, \ldots, 1)$ of rank 3 (excluded because $r \geq n \geq 4$) or $p_1 = 3$.

If there are at least four 3's, we substitute $(3^3, 1^2) \rightsquigarrow (3^2, 1^4)$ if $p_\alpha = 1$ or $(3^4) \rightsquigarrow (3^2, 2^3)$ if $p_\alpha > 1$. Thus we may assume that $x$ has partition $(3^2, 2r - 4, 1^4)$. As $2r \geq 2n = 2r - 2 + t$, we find that $x$ has partition $(3^2, 2r - 4, 1^2)$ with $r = n$ (in which case the proposition has already been proved) or partition $(3^2, 2r - 4)$ with $r = n + 1$, which specializes to the previous case.

**Case** $r < n$. Now suppose that $x$ has rank $r < n$, so in particular $p_\alpha = 1$ and we may assume that $p_1 \leq 4$. Specializing as in (7.2) also with $s = 1$, we may assume that $x$ has partition $(2^r, 1^{2n - 2r})$. If $r = 1$, then $2n$ conjugates suffice to generate $\mathfrak{sp}_{2n}$ by, for example, [CSUW01]. So assume $r \geq 2$.

Clearly, $n/r \leq n/2 < n - r$, so there are at least $2v + 2$ 1-by-1 Jordan blocks in $x$ for $e := 2[n/r] = 2v + 2$. We then subdivide $x$ into two blocks on the diagonal, with partitions $(2, 1^{2v})$ and $(2^{r-1}, 1^{2n-2r-2v})$. By the $r = 1$ case, $e$ generic conjugates of the first generate an $\mathfrak{sp}_e$ subalgebra and by induction $\max \{ 3, 2 [(n - v - 1)/(r - 1)] \}$ conjugates of the second generate an $\mathfrak{sp}_{2n-e}$ subalgebra. As $2n \leq re$, we have
(n - v - 1)/(r - 1) \leq n/r$, and the max in the preceding sentence is at most $e$. Note that $\mathfrak{sp}_v \times \mathfrak{sp}_{2n - v}$ contains a regular semisimple element of $\mathfrak{sp}_{2n}$ and the natural module has composition factors of size $e$, $2n - e$.

Alternatively, we may subdivide $x$ into blocks with partitions $(2^r, 1^{2n-2r-2})$ and $(1^2)$. By induction, $e$ generic conjugates of this element give an $\mathfrak{sp}_{2n-2}$ subalgebra, with composition factors of size $1$, $1$, $2n - 2$. As this list does not meet the list of composition factors from the previous paragraph, the generic subalgebra generated by $e$ conjugates acts irreducibly on the natural module, and we are done via an application of Remark 4.4.

$\Box$

**Corollary 7.3.** Assume $\text{char } k \neq 2$. For $x \in \mathfrak{sp}_{2n}$ nonzero nilpotent or noncentral semisimple, there exist $x_1, \ldots, x_r \in x^{\mathfrak{sp}_{2n}}$ such that $x_1, \ldots, x_n$ generate $\mathfrak{sp}_{2n}$ and $e \cdot \dim x^{\mathfrak{sp}_{2n}} \leq 6n^2$.

**Proof.** Note that we are done if 3 conjugates of $x$ suffice to generate $\mathfrak{sp}_{2n}$, as $\dim x^G \leq 2n^2$.

**Nilpotent case.** Suppose that $x$ is nonzero nilpotent and put $e(x)$ for the minimal number of conjugates of $x$ needed to generate $\mathfrak{sp}_{2n}$. We have $e(x) \leq 2\lfloor n/(n - \alpha(x)) \rfloor$ by Proposition 7.1 and in particular we may assume that $\alpha(x) > n$. To bound $\dim x^G$, we replace $x$ with $y$ such that $\alpha(y) = \alpha(x)$ and $y$ specializes to $x$, i.e., $x$ belongs to the closure of $y^G$. Then $\mathfrak{sp}_{2n}$ is also generated by $2\lfloor n/(n - \alpha(x)) \rfloor$ conjugates of $y$ and $\dim x^G \leq \dim y^G$. The element $x$ is given by a partition $(p_1, \ldots, p_n)$ as in the proof of Proposition 7.1.

We claim that $y$ can be taken to have partition $(2s, 2, 1^\alpha(x)-2)$ or $(2s, 1^\alpha(x)-1)$. Indeed, let $I := \{i \mid i > 1$ and $p_i > 2\}$. Then the element $y$ with partition $(p'_1, p'_2, \ldots, p'_n)$ where

$$ p'_i = \begin{cases} 
2 & \text{if } i \in I \\
p_i & \text{if } i > 1 \text{ and } i \notin I \\
p_1 + \sum_{i \in I} (p_i - 2) & \text{if } i = 1 
\end{cases} $$

specializes to $x$, compare [CM93, 6.2.5]. Replacing $x$ with $y$ we find an element with partition $(2s, 2^r, 1^\alpha(x) - r - 1)$ for some $s \geq 1$ and some $r$. If $r > 2$ and $s > 1$, then we may replace $x$ with an element with partition $(2s + 2, 2^r - 2, 1^\alpha(x) - r + 1)$ and repeating this procedure gives the claim.

The formula for $\dim C_{\mathfrak{sp}_{2n}(t)}(y)$ in [LS12, p. 39] gives that it is at least $n + (\alpha(x)^2 - 1)/2$. Applying \(\lfloor n/(2n - \alpha(x)) \rfloor \leq (3n - \alpha)/(2n - \alpha)\), we find that $e(x) \cdot \dim x^G < 6n^2 + (n - \alpha) + 1/(2n - \alpha)$. As $n - \alpha$ is negative, we have verified the required inequality for $x$ nilpotent.

**Semisimple case.** We may assume $x$ is diagonal. Put $\alpha_0$ for the number of nonzero entries in $x$; we will construct a nilpotent $y$ in the closure of $x^{G_{\mathfrak{sp}_{2n}}}$ so that $\alpha(y) = \alpha_0$. Recall that the diagonal of $x$ consists of pairs $(t, -t)$ with $t \in k$.

Suppose first that $\alpha_0 \geq n$. We pick $y$ to be block diagonal as follows. For a 4-by-4 block with entries $(0, 0, t, -t)$ for some $t \in k^\times$, we make a 4-by-4 block in $y$ in the same location, where the 2-by-2 block in the upper right corner is generic for $\mathfrak{sp}_4$. As $\alpha_0 \geq n$, by permuting the entries in $x$ we may assume that all pairs $(0, 0)$ on the diagonal of $x$ are immediately followed by a $(t, -t)$ with $t \neq 0$. Thus, it remains to specify the diagonal blocks in $y$ at the locations corresponding to the remaining 2-by-2 blocks $(t, -t)$ for $t \neq 0$ in $x$, for which we take $y$ to have a 1 in
upper right corner. Thus we have constructed a nilpotent \( y \) with \( \alpha(y) = n \), and by hypothesis \( e(y) \leq 3 \). Therefore \( e(x) \cdot \dim x^{i \infty} \leq 3 \dim x^{Sp_{2n}} \leq 6n^2 \).

Now suppose \( \alpha_0 < n \). Let \( x_0 \) be a \( 2\alpha_0 \times 2\alpha_0 \) submatrix consisting of all the nonzero diagonal entries in \( x \) together with \( \alpha_0 \) zero entries. Take \( y_0 \) to be the nilpotent element constructed from \( x_0 \) as in the preceding paragraph, and extend it by zeros to obtain a nilpotent \( y \) with \( \alpha(y) = 2n - \alpha_0 \). Then \( y \) is in the closure of \( x^{G_m Sp_{2n}} \) and \( e(y) \leq 2|n/\alpha_0| < 2(n + \alpha_0)/\alpha_0 \). On the other hand, the centralizer of \( x \) has dimension at least \( \dim Sp_{2n-\alpha_0} + \alpha_0/2 = 2n^2 - 2n\alpha_0 + \alpha_0^2/2 + n \). Thus \( \dim x^{Sp_{2n}} \leq 2n\alpha_0 - \alpha_0^2/2 \). In summary, \( e(x) \cdot \dim x^{Sp_{2n}} < (n + \alpha_0)(4n - \alpha_0) = 4n^2 + 3\alpha_0 n - \alpha_0^2/2 \). As a function of \( \alpha_0 \), it is a parabola opening down with max at \( \alpha_0 = 1.5n \), so its maximum for \( \alpha_0 < n \) is where \( \alpha_0 = n - 1 \), i.e., the max is at most \( 6n^2 - n - 1 \).

\( \square \)

8. Types \( B \) and \( D \) with \( \text{char } k \) odd

Proposition 8.1. Assume \( \text{char } k \neq 2 \). For every nonzero nilpotent \( x \in \mathfrak{so}_n \) for \( n \geq 5 \), \( \max \{4, \lceil n/\alpha(x) \rceil \} \) conjugates of \( x \) generate \( \mathfrak{so}_n \).

Proof. The conjugacy class of \( x \) is determined by its Jordan form, which is given by a partition \((p_1, \ldots, p_a)\) of \( n \) where even values occur with even multiplicity. We go by induction on \( n \). As \( \mathfrak{so}_5 \cong \mathfrak{sp}_4 \), the \( n = 5 \) case is covered by Proposition 7.1, which gives 4 as the largest number of conjugates needed to generate. For \( n = 6, \mathfrak{so}_6 \cong \mathfrak{sl}_4 \), and this case is handled by Proposition 5.4. So assume \( n \geq 7 \).

Suppose first that the number \( \delta \) of 1’s in the partition for \( x \) is at most 1. Then we can find an element \( y \) in the closure of \( x^{SO_n} \) with partition

\[
\begin{align*}
&\text{(i) } (2^{n/2}) \text{ if } n \equiv 0 \text{ mod } 4; \\
&\text{(ii) } (2^{(n-1)/2}, 1) \text{ if } n \equiv 1 \text{ mod } 4; \\
&\text{(iii) } (2^2, 2^{n-6}/2) \text{ if } n \equiv 2 \text{ mod } 4; \text{ or} \\
&\text{(iv) } (3, 2^{(n-3)/2}) \text{ if } n \equiv 3 \text{ mod } 4.
\end{align*}
\]

To see this, we specialize \((2s, 2s) \leadsto (s^4)\) for \( s \geq 2; s \leadsto (s-4, 2, 2)\) for odd \( s \geq 7; \) or \((s, 1) \leadsto ((s+1)/2, (s+1)/2))\) for odd \( s \geq 3 \) and \( \delta = 1 \). Together with trivial reductions such as \((5^2) \leadsto (3^2, 2^2)\) brings us to a partition of the form \((3^b, 2^c, 1^d)\) for some \( b \leq 3 \) and some \( c \) from which the claim quickly follows. For such a \( y, 2 \) conjugates suffice to generate a copy of \( \mathfrak{sl}_2^{\times n/2}, \mathfrak{sl}_2^{\times (n-1)/2}, \mathfrak{so}_3 \times \mathfrak{so}_3 \times \mathfrak{sl}_2^{\times (n-6)/2}, \) or \( \mathfrak{so}_3 \times \mathfrak{sl}_2^{\times (n-3)/2} \) respectively. As in the proof of Proposition (7.1), it follows that 3 conjugates are enough to generate \( \mathfrak{so}_n \).

Now suppose there are more 1’s in the partition for \( x \). We specialize using

\[
(2s+1, 1) \leadsto (s+1, s+1) \text{ for } s \geq 1 \quad \text{and} \quad (s, s, 1, 1) \leadsto (s-1, s-1, 2, 2) \text{ for } s \geq 4.
\]

If, after a step in this specialization process, we find that only 0 or 1 1-by-1 blocks remain, we are done by the preceding paragraph. Therefore, we may assume that \( x \) has partition \((2^2, 1^u)\) for \( u \geq 2 \).

Write out \( t = 2t_0 + \delta \) for \( \delta = 0 \) or 1, and set \( v = 2t_0 \lceil \frac{u}{2t} \rceil \). We can view \( x \) as block diagonal where the first block has partition \((2^{2t_0}, 1^v)\) and the second has partition \((2^{2t_0+2\delta}, 1^{u-v})\). For the first block,

\[
e := 2 + \left\lceil \frac{v}{2t_0} \right\rceil = 2 + \left\lceil \frac{u}{2t} \right\rceil
\]
conjugates suffice to generate an $\mathfrak{so}_{2t_0e}$ subalgebra by induction on $n$. For the second block, we note that
\[ \frac{u - v}{2t_0 + 2\delta} \leq \frac{u}{2t} \]
so, by induction, $e$ conjugates suffice to generate an $\mathfrak{so}_{n-2t_0e}$ subalgebra. Because $\mathfrak{so}_{2t_0e} \times \mathfrak{so}_{n-2t_0e}$ contains a regular semisimple element and the natural module has composition factors of size $2t_0e$ and $n - 2t_0e$, we conclude as in the proof of Proposition 7.1 that $e$ conjugates of $x$ suffice to generate $\mathfrak{so}_n$. \hfill \Box

**Corollary 8.2.** Assume $p := \text{char } k \neq 2$. For noncentral $x \in \mathfrak{so}_n$ such that $x^{[p]} \in \{0, x\}$, there exist $x_1, \ldots, x_e \in x^{\mathfrak{so}_n}$ that generate $\mathfrak{so}_n$ and such that $e \cdot \dim x^{\mathfrak{so}_n} \leq 2(n - 1)^2$.

**Proof.** As $\text{char } k \neq 2$, we identify $\mathfrak{spin}_n$ with $\mathfrak{so}_n$ via the differential of the covering map $\mathfrak{spin}_n \to \mathfrak{so}_n$. We argue as in the proof of Corollary 7.3, replacing $\mathfrak{sp}_{2n}$ with $\mathfrak{so}_n$ and references to Proposition 7.1 with references to 8.1. We may assume that $e(x) > 4$, for in that case $e(x) \cdot x^{\mathfrak{so}_n} \leq 4 \cdot \left(\binom{n}{2} - \lfloor n/2 \rfloor \right) \leq 2(n - 1)^2$.

**Nilpotent case.** Suppose that $x$ is nonzero nilpotent. We have $e(x) \leq \lfloor n/(n - \alpha) \rfloor$ and in particular we may assume that $\alpha > \frac{4}{3}n$. Recall that $x$ is determined by a partition $(p_1, \ldots, p_n)$ of $n$, where even numbers appear with even multiplicity. As in the proof of Corollary 7.3, we may replace $x$ with $y$ with partition $(p_1 + \sum_{i=2}^n (p_i - 1), 1^{n-1})$. This element has $\alpha(y) = \alpha(x)$ and orbit of size $\binom{n}{2} - \binom{n}{2}$. As $e(x) < (2n - \alpha)/(n - \alpha)$, it follows that $e(x) \cdot x^{\mathfrak{so}_n} < \frac{1}{2}(2n - \alpha)(n + \alpha - 1)$. The upper bound is maximized for $\frac{1}{2}n \leq \alpha < n$ at the lower bound, where it is $\frac{3}{4}n^2 - \frac{3}{4}n < 2(n - 1)^2$.

**Semisimple case.** Suppose that $x$ is noncentral diagonal in $\mathfrak{so}_n$.

Suppose first that $n$ is even. If $\alpha_0 \geq n/2$, then pick $y$ as is done in Corollary 7.3, so $\alpha(y) = n/2$, $e(y) \leq 4$, and we are done. If $\alpha_0 < n/2$, we perform the same construction as in the last paragraph of the proof of 7.3 to obtain $y$ with $\alpha(y) = n - \alpha_0$, so $e(y) \leq \max\{4, \lfloor n/\alpha_0 \rfloor\}$; suppose $\lfloor n/\alpha_0 \rfloor > 4$, i.e., $n/\alpha_0 > 4$, i.e., $\alpha_0 < n/4$. The orbit of $x$ has dimension at least $\dim \mathfrak{so}_n - \dim \mathfrak{so}_{n-\alpha_0} - \alpha_0/2$, whence $e(x) \cdot \dim x^{\mathfrak{so}_n} < (n + \alpha_0)(n - \alpha_0)/2$, where the right side is maximized at $\alpha_0 = n/4$ and again we verify that the upper bound is at most $2(n - 1)^2$.

When $n$ is odd, we view $x$ as lying in the image of $\mathfrak{so}_{n-1} \hookrightarrow \mathfrak{so}_n$ and take $y$ in this same image as constructed by the method in the previous paragraph. Computations identical to the ones just performed again verify $e(x) \cdot \dim x^{\mathfrak{so}_n} < 2(n - 1)^2$. \hfill \Box

9. **Type $D$ with $\text{char } k = 2$**

**Concrete descriptions.** For sake of precision, we first give concrete descriptions of the groups and Lie algebras associated with a nondegenerate quadratic from $q$ on a vector space $V$ of even dimension $2n$ over a field $k$ (of any characteristic). The orthogonal group $O(q)$ is the sub-group-scheme of $\text{GL}(V)$ consisting of elements that preserve $q$, i.e., such that $q(gv) = q(v)$ for all $v \in V \otimes R$ for every commutative $k$-algebra $R$, the special orthogonal group $SO(q)$ for the kernel of the Dickson invariant $O(q) \to Z/2$, and the group of norm similarities $GO(q)$ and proper norm similarities $SGO(q)$ are the sub-group-schemes of $\text{GL}(V)$ generated by the scalar transformations and $O(q)$ or $SO(q)$ respectively, see for example [KMRT98, §12 and
p. 348] or [Km91, Ch. IV]. The group SO(q) is semisimple of type $D_n$, but neither simply connected nor adjoint.

The statement that $q$ is nodegenerate means that the bilinear form $b$ on $V$ defined by $b(v, v') := q(v + v') - q(v) - q(v')$ is nondegenerate. Viewing the Lie algebra of a group $G$ over $k$ as the kernel of the homomorphism $G(k) \rightarrow G(k)$ induced by the map $\varepsilon \mapsto 0$ from the dual numbers $k[\varepsilon]$ to $k$, one finds that $\mathfrak{o}(q)$ is the set of $x \in \mathfrak{gl}(V)$ such that $b(xv, v) = 0$ for all $v \in V$. Since $O(q)/SO(q) \cong \mathbb{Z}/2$, $\mathfrak{o}(q) = \mathfrak{o}(q)$. As $b$ is nodegenerate, the equation $b(Tv, v') = b(v, \sigma(T)v')$ defines an involution $\sigma$ on $\text{End}(V)$, and the set of alternating elements $\{T - \sigma(T) \mid T \in \text{End}(V)\}$ is contained in $\mathfrak{o}(q)$ and also has dimension $2n^2 - n$ [KMRT98, 2.6], therefore the two subspaces are the same. The Lie algebra $\mathfrak{o}(q)$ of $O(q)$ and $\text{SO}(q)$ is the set of elements $x \in \mathfrak{gl}(V)$ such that there exists a $\mu_x \in k$ so that $b(xv, v) = \mu_x q(v)$ for all $v$. It has dimension one larger than $\mathfrak{o}(q)$.

**Example 9.1.** $V = k^{2n}$ and $q$ is defined by $q(v) = \sum^n_{i=1} v_i v_{i+n}$, we write $\mathfrak{so}_{2n}$ instead of $\mathfrak{o}(q)$, etc. The linear transformation $x$ obtained by projecting on the first $n$ coordinates and then multiplying by $-1$ satisfies $b(xv, v) = -q(v)$ for all $v \in V$, so it and $\mathfrak{so}_{2n}$ span $\mathfrak{g}l_{2n}$.

Suppose $x \in \mathfrak{so}(q)$ is a projection, i.e., $x^2 = x$, so $x$ gives a decomposition $k^{2n} = \ker x \oplus \text{im } x$ as vector spaces. If $x$ belongs to $\mathfrak{so}_{2n}$, then this is an orthogonal decomposition and $b$ is nodegenerate on $\ker x$ and $\text{im } x$. Otherwise, $\text{im } x$ and $\ker x$ are maximal totally isotropic subspaces. To see this, note that if $q(v) \neq 0$, then $b(xv, v) = \mu_x q(v) \neq 0$, which is impossible if $xv \in \{0, v\}$.

We assume for the remainder of the section that $\text{char } k = 2$. As was done for other types, we consider how many conjugates of an $x \in \mathfrak{so}_{2n}$ with $x^{[2]} \in \{0, x\}$ suffice to generate a subalgebra of $\mathfrak{so}_{2n}$ containing the derived subalgebra $[\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]$.

**Example 9.2.** One can verify by computing with an example that for $x \in \mathfrak{so}_{2n}$ with $x^{[2]} = 0$, $e$ conjugates suffice to generate $[\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]$ in the cases (a) $n = 4$, $e = 4$, and $x$ is a root element; (b) $n = 5$, $e = 5$, and $x$ is a root element; (c) $n = 7$ or $8$, $e = 4$, and $x$ has rank 4. (In the last case, note that $x$ can be taken to have Jordan form with partition $(2^4, 1^{2n-4})$.) Magma code is provided with the arxiv version of this paper.

**Lemma 9.3.** Let $g = \mathfrak{so}_{2n}$ with $n > 4$. If $x$ is a root element, and $m > 4$, then $m$ generic conjugates of $x$ generate the derived subalgebra of a natural $\mathfrak{so}_{2m}$.

**Proof.** The case $n = 4$ is from Example 9.2.

Now assume that $n > 4$ and $4 \leq m < n$. By induction, we know that $m$ conjugates can generate the derived subalgebra of a copy of $\mathfrak{so}_{2m}$. Clearly any $m$ conjugates have a fixed space of dimension at least $2n - 2m$ and generically this space will be nondegenerate, whence this $\mathfrak{so}_{2m}$ is naturally embedded in $\mathfrak{so}_{2n}$.

Now assume that $m = n$; by Example 9.2 we may assume that $n \geq 6$. So now take $n - 2$ generic root elements, $x_1, \ldots, x_{n-2}$; they generate the derived subalgebra of a natural $\mathfrak{so}_{2n-4}$ by induction. Let us take a basis of $k^{2n}$ as in Example 9.1. We identify our $\mathfrak{so}_{2n-4}$ as the one acting trivially on the subspace spanned by $v_1, v_{n+1}, v_2, v_{n+2}$.

Then consider two copies of the derived subalgebra of $\mathfrak{so}_{2n-4}$ acting on the spaces spanned by $v_i$ and $v_{n+i}$ for $1 \leq i < n$ and for $1 < i \leq n$. These both contain our $\mathfrak{so}_{2n-4}$ and by induction we can choose $x, y$ respectively so that $x, x_1, \ldots, x_{n-2}$
generate the first copy of the derived subalgebra of \( \mathfrak{so}_{2n-2} \) and \( x_1, \ldots, x_{n-2}, y \) generate the second copy. These two copies generate the derived subalgebra of \( \mathfrak{so}_{2n} \), as can be seen by considering the root elements in each one. \( \square \)

**Proposition 9.4.** Let \( \mathfrak{g} = \mathfrak{so}_{2n} \) with \( n \geq 4 \) over an algebraically closed field \( k \) of characteristic 2. Let \( x \in \mathfrak{g} \) satisfy \( x^2 \in \{0, x\} \) have rank 2r (with \( r \leq n \)). Then \( \max\{4, [n/r]\} \) conjugates of \( x \) generate a Lie subalgebra containing \( [\mathfrak{so}_{2n}, \mathfrak{so}_{2n}] \).

**Proof.** By passing to closures, we may assume that \( x \) is a root element or \( W \)-dimensional indecomposable denoted by \( X \).

Let \( r \) be the highest root, there exist \( n/r \) conjugates having composition factors of dimensions \( \leq (n+1)/2, n(n-1)/2 \) and also one where there is a composition factor of dimension at least \((n-1)(2n-3) - 1\). Thus, generically there is a composition factor of dimension at most \( 2n^2 - 5n + 2 \) and the smallest composition factor is at least \( n(n-1)/2 \). Since the sum of these two numbers (for \( n \geq 9 \)) is greater than \( \dim X = 2n^2 - n - 2 \), we see that generically \( e \)-conjugates acts irreducibly on \( X \), whence they generate \( \mathfrak{g} \).

Suppose that \( n \) is even. The same argument shows that \( e \)-conjugates can generate a subalgebra having composition factors on \( [\mathfrak{g}, \mathfrak{g}] \) of dimensions \( 2, n^2 - 2, n(n-1)/2 \) and another \( e \)-conjugates having composition factors of dimensions \( 1, 2n^2 - 5n + 2, 2n^2 - 2n - 2 \). This implies that generically \( e \)-conjugates act irreducibly on \( [\mathfrak{g}, \mathfrak{g}]/\mathfrak{z}(\mathfrak{g}) \) and this implies they generate \( [\mathfrak{g}, \mathfrak{g}] \).

**Example 9.5.** Suppose \( x \in \mathfrak{so}_{2n} \) satisfies \( x^2 = 0 \), so the Jordan form of \( x \) has 2r 2-by-2 blocks and \( 2n - 4r \) 1-by-1 blocks for some \( r \leq n \). There are two possibilities for the conjugacy class of \( x \), see [Hes79, 4.4] or [LS12, p. 70]. We focus on the larger class, the one where the restriction of the natural module to \( x \) includes a 4-dimensional indecomposable denoted by \( W_2(2) \) in [LS12]. The centralizer of \( x \) in \( \text{SO}_{2n} \) has dimension

\[
\sum_{i=1}^{2r} 2(i-1) + \sum_{i=2r+1}^{2n-2r} (i-1) = \binom{2n-2r}{2} + \binom{2r}{2},
\]

and therefore \( \dim x^{\text{SO}_{2n}} = 4r(n-r) \). (The other class has dimension \( 2r(2n-2r-1) \).)

**Corollary 9.6.** Suppose \( \text{char } k = 2 \). For every noncentral \( x \in \mathfrak{so}_{2n} \) such that \( x^2 \in \{0, x\} \), there exist \( x_1, \ldots, x_r \in x^{\text{SO}_{2n}} \) that generate a subalgebra containing \( [\mathfrak{so}_{2n}, \mathfrak{so}_{2n}] \) and such that \( e \cdot \dim x^{\text{GO}_{2n}} \leq 4n^2 \).

**Proof.** Suppose \( x \) has rank \( 2r \leq n \) and \( x^2 = 0 \) as in Example 9.5. The condition we need is that \( 4n^2 \geq e4r(n-r) \). If the maximum in Prop. 9.4 is 4, i.e., if \( r \geq n/4 \), then as a function of \( r \), \( 16r(n-r) \) has a maximum of \( 4n^2 \) at \( r = n/2 \). Otherwise,
the maximum is \( e = \lceil n/r \rceil < (n + r)/r \), so \( e \dim x^G < 4(n^2 - r^2) \). The right side has a maximum of \( 4n^2 - 4 \) at \( r = 1 \).

If \( x^{[2]} = x \) and \( x \in \mathfrak{so}_{2n} \), the centralizer of \( x \) in \( GO_{2n} \) has codimension 1 in \( GO_{2r} \times GO_{2(n-r)} \) when \( x \) has rank 2r. Thus, \( \dim x^G = 4r(n-r) \) as for \( x \) nilpotent. Restricting now \( x \) to \( im x \), we can view \( x \) as stabilizing two maximal isotropic subspaces \( V_1, V_2 \) such that \( \mathrm{d}\pi(x)|_{V_1} \) has rank \( r \), that is, \( \mathrm{d}\pi(x) \) is the image of some rank \( r \) toral element \( \hat{x} \) under a map \( \mathfrak{gl}_n \to \mathfrak{so}_{2n} \). Let \( \hat{y} \) denote the nilpotent provided by Lemma 3.2. Inspecting the proof shows that this can be done so that \( \hat{y} \) has rank \( r \). The image \( y \in \mathfrak{so}_{2n} \) of \( \hat{y} \) then has rank 2r and we are done by the previous paragraph.

If \( x^{[2]} = x \) and \( x \notin \mathfrak{so}_{2n} \), then \( x \) is determined by choosing an ordered pair of “parallel” maximal isotropic subspaces and so the dimension of \( x^{GO_{2n}} \) agrees with the dimension of the flag variety of \( D_n \) of parabolics with Levi subgroups of type \( A_{n-2} \), which has dimension \( (n^2 + n - 2)/2 \). Up to conjugacy, we may assume \( x \) is the element from Example 9.1. Let \( y_0 \) be an \( n \)-by-\( n \) nilpotent matrix of with \( [n/2] \) 2-by-2 rank 1 Jordan blocks down the diagonal. Then \( y = (y_0 w_0) \) is in \( \mathfrak{so}_{2n} \), is nilpotent, and 4 conjugates of \( y \) suffice to generate a subalgebra containing \( \mathfrak{so}_{2n} \) (Prop. 9.4), so 4 conjugates of \( x \) suffice as well. As \( 2n^2 + 2n - 4 < 4n^2 \), the claim is proved in this case.

\[ \square \]

10. Lemmas on the structure of \( \mathfrak{g} \)

**Lemma 10.1.** Let \( G \) be a simple algebraic group over \( k \) and assume that \( \text{char} \, k \neq (SL_2, 2) \), then:

1. \( \mathrm{d}\pi(\hat{\mathfrak{g}}) = \langle \mathfrak{g}_\alpha \mid \alpha \in \Phi \rangle \supseteq [\mathfrak{g}, \mathfrak{g}] \).

If furthermore \( (G, \text{char} \, k) \neq (SL_2, 2) \), then:

2. Equality holds in (1).

3. If \( \mathfrak{s} \) is a subalgebra such that \( \mathfrak{s} + \mathfrak{j}(\mathfrak{g}) \supseteq [\mathfrak{g}, \mathfrak{g}] \), then \( \mathfrak{s} \supseteq [\mathfrak{g}, \mathfrak{g}] \).

**Proof.** The map \( \mathrm{d}\pi \) restricts to an isomorphism \( \mathfrak{g}_\alpha \cong \mathfrak{g} \) for each root \( \alpha \), and in particular \( \mathrm{d}\pi(\hat{\mathfrak{g}}) \supseteq \langle \mathfrak{g}_\alpha \rangle \). The subalgebra \( \mathfrak{j}(\mathfrak{g}) \) is an ideal in \( \mathfrak{g} \) and \( \mathfrak{g}/\mathfrak{j}(\mathfrak{g}) \) is abelian, so \( \langle \mathfrak{g}_\alpha \rangle \supseteq [\mathfrak{g}, \mathfrak{g}] \), proving (1). The hypotheses on \( (G, k) \) give \( [\mathfrak{g}, \mathfrak{j}(\mathfrak{g})] = \hat{\mathfrak{g}} \) [Pre97, Lemma 2.3(ii)], proving (2).

For (3), we may assume that \( \mathfrak{j}(\mathfrak{g}) \neq 0 \), and in particular the center of \( G \) is not étale and \( G \) does not have type \( A_1 \).

If \( G = \tilde{G} \), then for each \( g \in G(k) \), there is \( z_g \in \mathfrak{j}(\mathfrak{g}) \) such that \( z_g + gx_{\tilde{\alpha}} \in \mathfrak{s} \), where \( \tilde{\alpha} \) denotes the highest root. Thus, \( \mathfrak{s} \) contains \( [z_g + gx_{\tilde{\alpha}}, z_{g'} + gx_{\tilde{\alpha}'}] = [gx_{\tilde{\alpha}}, g'x_{\tilde{\alpha}'}] \) for all \( g, g' \in G(k) \), hence \( \mathfrak{s} = \mathfrak{g} \) by [Pre97, Lemma 2.3(ii)].

Dropping now the assumption that \( G = \tilde{G} \), we put \( q : G \to \tilde{G} \) for the natural map to the adjoint group. The claim is that \( dq(\hat{\mathfrak{s}}) \supseteq dq([\mathfrak{g}, \mathfrak{g}]) \), which by (1) equals \( dq \mathrm{d}\pi(\hat{\mathfrak{g}}) \). For a basis \( s_1, \ldots, s_r \) of \( \mathfrak{s} \), pick \( \hat{s}_i \in \hat{\mathfrak{g}} \) mapping to \( s_i \) and set \( \hat{\mathfrak{s}} = (\hat{s}_1, \ldots, \hat{s}_r) \). Then the claim is \( dq \mathrm{d}\pi(\hat{\mathfrak{s}}) = dq \mathrm{d}\pi(\hat{\mathfrak{g}}) \), and we are done by the case where \( G \) is simply connected.

\[ \square \]

When \( G = \text{SL}_2 \) and char \( k = 2 \), \( \mathfrak{j}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \) is a/the maximal toral subalgebra of \( \mathfrak{g} \), so (3) fails.

**Example 10.2.** Let \( G = \text{Sp}_{2n} \) over a field \( k \) with char \( k = 2 \), so char \( k \) is special for \( G \). Fix a nondegenerate quadratic form \( q \) on \( k^{2n} \) whose bilinearization is the
symplectic form stabilized by $G$; then, as a subspace of $\mathfrak{sp}_{2n}$, $\mathfrak{so}(q) = [\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n}]$
does not depend on $q$ and we denote it by $\mathfrak{so}_{2n}$. Moreover, it has codimension $2n$ in
$\mathfrak{sp}_{2n}$, which shows that some hypothesis on $G$ or $k$ is necessary in Lemma 10.1(1).

Consider the irreducible “spin” representation $\rho: \text{Sp}_{2n} \to \text{GL}(V)$ for some $n \geq 2$
over a field of characteristic 2. It has dimension $2^n$ and factors through the special
isogeny $\pi: \text{Sp}_{2n} \to \text{Spin}_{2n+1}$. The image $d\pi(\mathfrak{sp}_{2n})$ in $\mathfrak{spin}_{2n+1}$ has dimension $2n + 1$
with socle $d\pi(\mathfrak{so}_{2n})$, the 1-dimensional center of $\mathfrak{spin}_{2n+1}$, compare [His84], [Hog82],
or [CGP15, §7.1]. Now $\rho$ is the composition of $\pi$ with the spin representation of
$\mathfrak{spin}_{2n+1}$, which is minuscule, so the center of $\mathfrak{spin}_{2n+1}$ acts by scalars on $V$. It
follows that $\ker d\rho$ is $\ker d\pi$, the subalgebra of $\mathfrak{sp}_{2n}$ generated by the short root
subalgebras, which equals the derived subalgebra $[\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]$.

The following result does not have any hypothesis on char $k$.

**Lemma 10.3.** Let $G$ be a simple algebraic group over $k$. Then for every $G$-module $V$:

1. $V^{[\mathfrak{g}, \mathfrak{g}]} = V^\theta$.
2. $(V/V^\theta)^\theta = 0$.

**Proof.** As the action of $\mathfrak{g}$ on $V$ is $G$-equivariant, $V^{[\mathfrak{g}, \mathfrak{g}]} \supseteq V^\theta$, are $G$-submodules, so
we may assume that $V^\theta = 0$ and $V$ is annihilated by $[\mathfrak{g}, \mathfrak{g}]$. Replacing $V$ with a
$G$-submodule, we may further assume that $V$ is irreducible; we aim to show that $V = 0$.

By Steinberg’s Tensor Product Theorem [Ste63], $V = V_0 \otimes V_1^{[\mathfrak{p}, \mathfrak{p}]}$ for irreducible
representations $V_0, V_1$ with $V_0$ restricted. As a representation of $\mathfrak{g}$, this is the same
as a direct sum of copies of $V_0$, so we may replace $V$ with $V_0$ and assume that
the highest weight $\lambda$ of $V$ is restricted. If $\lambda = 0$, we are done, so assume not. If $G = \text{SL}_2$ and char $k = 2$, then $V$ is the tautological representation, $[\mathfrak{g}, \mathfrak{g}]$ are the
scalars acting on $X$, and $V^{[\mathfrak{g}, \mathfrak{g}]} = 0$, a contradiction.

As $\lambda \neq 0$, so there is a simple root $\alpha$ with $\langle \lambda, \alpha^\vee \rangle$ not zero in $k$. Put $x_\alpha, x_{-\alpha}$
for basis elements of the root subalgebras for $\pm \alpha$. Then $x_\alpha x_{-\alpha} v = (\lambda, \alpha^\vee) v \neq 0$ as
in the proof of [Ste63, Lemma 4.3(a)], hence $x_{-\alpha} v \neq 0$ and $\mathfrak{g}_{\pm \alpha}$ acts nontrivially
on $V$. If $\mathfrak{g}_{\pm \alpha} \subseteq [\mathfrak{g}, \mathfrak{g}]$, we obtain a contradiction that proves (1). Note that this
happens if char $k$ is not special (Lemma 10.1(2)) or $\alpha$ is short [Hog82] as well as in
all the cases with char $k$ special except when $G = \text{Sp}_{2n}$ and char $k = 2$.

So suppose we are in that case. We may assume by the previous paragraph that $\langle \lambda, \alpha^\vee \rangle = 0$
for all short simple roots $\alpha$. As $\lambda$ is restricted, it is the fundamental
dominant weight dual to the unique short simple coroot, i.e., $X$ is the spin
representation. In that case, (1) is proved in Example 10.2.

Suppose $V^\theta \subseteq W \subseteq V$ for some $\mathfrak{g}$-submodule $W$ such that $\mathfrak{g}W \subseteq V^\theta$. Then
for $x, y \in \mathfrak{g}$ and $w \in W$, $[x, y]w = x(yw) - y(xw) \in xV^\theta - yV^\theta = 0$. That is,$W \subseteq V^{[\mathfrak{g}, \mathfrak{g}]} = V^\theta$, proving (2).

□

11. Exceptional types

The aim of this section is to provide the necessary material to prove Theorem A
for exceptional groups, but we begin with some general-purpose observations. Recall
that a root element of a Lie algebra $\mathfrak{g}$ of $G$ is a generator for a one-dimensional
root subalgebra $\mathfrak{g}_\alpha$ of $\mathfrak{g}$.

**Lemma 11.1.** Let $G$ be a reductive algebraic group. For each nonzero nilpotent
$x \in \mathfrak{g}$, there is a root element in the closure of $x^G$. 
Proof. Write $x = \sum_{\alpha \in S} X_\alpha$ where $S$ is a nonempty set of positive roots (relative to some torus $T$) and $X_\alpha$ is a generator for $g_\alpha$. Pick a subtorus $T'$ of $T$ that centralizes some $X_\alpha$ but not some $X_{\alpha'}$ for some $\alpha \neq \alpha' \in S$. Now in the closure of $x^{T'}$ we find a nonzero nilpotent supported on $S \setminus \{\alpha'\}$, and by induction we are done. \hfill \Box

We say that a root element in $g_\alpha$ is long (resp. short) if $\alpha$ is long (resp. short).

**Lemma 11.2.** Let $G$ be a simple linear algebraic group over a field $k$ such that $\text{char } k$ is not special for $G$. For every nonzero nilpotent $x \in g$, there is a long root element in the closure of $x^G$.

**Proof.** By Lemma 11.1, we may assume that $G$ has two root lengths and that $x$ is a root element for a short root $\alpha$.

Suppose first that $G$ has rank 2, so $G$ has type $G_2$ and $\text{char } k \neq 3$ or $G$ has type $C_2$ and $\text{char } k \neq 2$. Let $\alpha$ be the short simple root, $\gamma$ be the highest root (a long root), and take $\beta := \gamma - \alpha$. Let $x_\alpha, x_\beta : G_\alpha \to G$ be the corresponding root subgroups. These pick a generator $X_\alpha := \text{ad}(x_\alpha(1))$ of $g_\alpha$ such that

$$\text{ad}(x_\beta(t))X_\alpha = X_\alpha + N_{\beta,\alpha}X_\gamma,$$

where $X_\gamma$ generates $g_\gamma$, cf. [Ste16, Ch. 3]. As $\text{char } k$ is not special for $G$, $N_{\beta,\alpha}$ is not zero in $k$, and arguing as in the proof of Lemma 11.1 we conclude that $k^\times X_\gamma$ meets the closure of $(X_\alpha)^G$, proving the claim in case $G$ has rank 2.

If $G$ has rank at least 3, pick a long root $\beta$ that is not orthogonal to $\alpha$ and let $G'$ be the subgroup of $G$ corresponding to the rank 2 sub-root-system generated by $\alpha, \beta$. The ratio of the square-lengths of $\alpha, \beta$ is 2 so $G'$ has type $C_2$ and $\text{char } k \neq 2$. Then the closure of $x^{G'}$ contains a long root element in $G'$, hence in $G$. \hfill \Box

Now we focus on exceptional groups. Recall Table 2 from the introduction.

**Proposition 11.3.** Suppose $G$ is simple of exceptional type over a field $k$ such that $\text{char } k$ is not special for $G$. For $e$ and $b(G)$ as in Table 2 and $x \in g$ nonzero nilpotent or noncentral semisimple, we have:

1. there are $x_1, \ldots, x_e \in x^G$ generating a Lie subalgebra of $g$ containing $[g, g]$, and
2. $e \cdot \dim x^G \leq b(G)$.

**Proof.** The crux is to prove (1). By taking closures as in §3, we may assume that the orbit $x^G$ of $x$ consists of root elements. Moreover, as $k$ is not special, by Lemma 11.2 we may assume that $x^G$ consists of long root elements. In view of Lemma 10.1(1), we may assume $g$ is simply connected.

If $p \neq 2$, we can apply the result of [CSUW01] to obtain (1). We now prove the result for $p = 2$; in most cases, the argument also gives another proof for all $p$.

If $G = G_2$, we consider the $A_2$ subalgebra $\mathfrak{h}$ generated by the long roots so $g/\mathfrak{h}$ has the weights of $k^3 \oplus (k^3)^*$ as a representation of $\mathfrak{h}$, so it is multiplicity free. As $\mathfrak{h}$ can be generated with 3 root elements (Prop. 6.1), the claim follows.

If $G = E_8$, one uses that 4 root elements generate the $D_4$ inside $E_8$ (Example 9.2) and argue as for $G_2$, or one computes directly that five random root elements generate $g$. This completes the proof of (1).

Claim (2) follows because

$$b(G) = e \cdot (\dim G - \text{rank } G) \geq e \cdot \dim x^G.$$

\hfill \Box
12. Proof of Theorem A

Lemma 12.1. Let \( G \) be a simple algebraic group over a field \( k \) such that \( p := \text{char } k \) is not special. Then for all noncentral \( x \in \mathfrak{g} \) such that \( x^{[p]} \in \{0, x\} \), there exists \( x_1, \ldots, x_e \in x^G \) generating a subalgebra \( \mathfrak{s} \) of \( \mathfrak{g} \) containing \( [\mathfrak{g}, \mathfrak{g}] \) and \( e \cdot \text{dim } x^G \leq b(G) \).

Proof. Put \( \pi : \tilde{G} \to G \) for the simply connected cover of \( G \). If \( d\pi : \tilde{\mathfrak{g}} \to \mathfrak{g} \) is an isomorphism, then we apply 5.5 or 6.2 for type A, 8.2 for types B or D if \( p \neq 2, 7.3 \) for type C, and 11.3(2) for the exceptional types. If \( G \) is adjoint of type \( E_6 \) and \( \text{char } k = 3 \), we are done by Prop. 11.3.

Therefore, we may assume that \( G = \text{SL}_n / \mu_n \) and \( p \mid m \), or \( G \) has type \( D_n \) and \( p = 2 \). In these cases, 5.5, 6.2, and 9.6 concern not \( G \) but a group \( G'' := (G' \times \mathbb{G}_m)/Z(G') \) for some \( G' \) isogenous to \( G \). In particular, putting \( q : G'' \to G \) for the natural surjection onto the adjoint group, the induced map \( dq : \text{Lie}(G'') \to \text{Lie}(G) \) is also a surjection.

Consider now the case \( G = \tilde{G} \). Pick \( y \in \text{Lie}(G'') \) such that \( dq(y) = x \). The results cited in the previous paragraph provide elements \( y_1, \ldots, y_e \in y^{G''} \) such that \( \mathfrak{s}'' := \langle y_1, \ldots, y_e \rangle \) contains \( [\mathfrak{s}'', \mathfrak{g}''] \), and \( e \cdot \text{dim } y^{G''} \leq b(G) \). Taking \( x_i := dq(y_i) \), we obtain the desired result.

In the general case, write now \( q \) for the natural map \( G \to \tilde{G} \). For \( z := dq(x) \), let \( z_1, \ldots, z_e \in z^{G} \) by the elements provided by the adjoint case of the lemma. Pick \( g_i \in G(k) \) such that \( z_i = \text{Ad}(g_i)z \) and set \( x_i := \text{Ad}(g_i)x \). Then \( x_1, \ldots, x_e \) generate a subalgebra \( \mathfrak{s} \) such that \( dq(s) \supseteq [\mathfrak{g}, \mathfrak{g}] \). Lemma 10.1 completes the proof. □

We now prove the following result, which has the same hypotheses as Theorem A and a stronger conclusion.

Theorem 12.2. Let \( G \) be a simple linear algebraic group over a field \( k \) such that \( p := \text{char } k \) is not special for \( G \). If \( \rho : G \to \text{GL}(V) \) is a representation of \( G \) such that \( V \) has a \( G \)-subquotient \( X \) with \( X^\rho = 0 \) and \( \dim X > b(G) \) for \( b(G) \) as in Tables 1 or 2, then \( \dim x^G + \dim V^x < \dim V \) for all noncentral \( x \in \mathfrak{g} \) with \( x^{[p]} \in \{0, x\} \).

Proof. Assume for the moment that \( V = X \). We verify the inequality (1.3) for the set \( X \) of noncentral \( x \in \mathfrak{g} \) such that \( x^{[p]} \in \{0, x\} \). As \( V^\rho = 0 \), Lemma 10.3 gives that \( V^{[s, s]} = 0 \). Combining Lemma 12.1 with §1 shows that (1.3) holds for \( x \in X \).

For general \( V \), it follows then that (1.3) holds for \( x \in X \) as in Example 2.1. □

Proof of Theorem A. Combine Theorem 12.2 with Lemmas 1.1 and 1.6. □

13. Small examples; proof of Corollary B

Before proving Corollary B, we provide some examples that we treat in greater generality than is required for proving the corollary.

Lemma 13.1. Let \( p \) be an odd prime. Let \( G = \text{SO}(V) \) with \( \dim V = n \). Let \( W \) be the irreducible composition factor of \( S^2V \) of dimension \( n(n+1)/2 - 1 \) if \( \text{char } k \) does not divide \( n \), or \( n(n+1)/2 - 2 \) if \( \text{char } k \) divides \( n \). The annihilator in \( \text{Lie}(G) \) of a generic \( v \in V \) is zero.

Proof. Let \( S = \text{Sym}^2(V) \) which we identify with \( n \)-by-\( n \) symmetric matrices and we identify \( L \) with skew symmetric matrices. Then we see \( W \) inside \( S \) (with \( L \) acting via Lie bracket in \( \mathfrak{g}l_n \)).
If \( p \) does not divide \( n \), \( W \) is just the trace zero matrices in \( S \). If \( p \) divides \( n \), then \( W \) is the set of trace zero matrices modulo scalars.

If we take an element of trace zero that is diagonal and generic, then its centralizer in \( \mathfrak{gl}_n \) is just diagonal matrices (and even so for commuting modulo scalars). Thus, its centralizer in \( L \) is 0, whence the generic stabilizer in \( L \) is 0. \( \square \)

**Example 13.2.** Let \( G \) be a simple algebraic group and put \( L(\tilde{\alpha}) \) for the irreducible representation with highest weight the highest root \( \tilde{\alpha} \). It is a composition factor of the adjoint module.

If \( G \) has type \( C_n \) for \( n \geq 1 \) (including \( A_1 = C_1 \) and \( B_2 = C_2 \)) and \( \text{char} \ k = 2 \), then \( L(\tilde{\alpha}) \) is a Frobenius twist of the natural representation of dimension \( 2n \), so \( \mathfrak{g} \) acts as zero on \( L(\tilde{\alpha}) \).

Suppose now that we are not in the case of the preceding paragraph and \( \text{char} \ k \) is not special for \( G \). Put \( \pi : \tilde{G} \to G \) for the simply connected cover of \( G \). The hypotheses give that \( L(\tilde{\alpha}) \cong \tilde{\mathfrak{g}}/3(\tilde{\mathfrak{g}}) \) as \( G \)-modules and that Cartan subalgebras of \( \tilde{\mathfrak{g}} \) and \( \mathfrak{g} \) are Lie algebras of maximal tori. It follows, then, that there is an open subset \( U \) of \( \mathfrak{g} \) that meets \( \text{Lie}(T) \) for every maximal torus \( T \) of \( \tilde{G} \) such that for \( a \in U \) the subalgebra \( \text{Nil}(a, \tilde{\mathfrak{g}}) := \cup_{m > 0} \ker(\text{ad}a)^m \) has minimal dimension (i.e., \( a \) is regular in the sense of \([DG70, \S XIII.4]\)). Pick \( a \in U \cap T \) and put \( \tilde{a} \in L(\tilde{\alpha}) \) for the image of \( a \). Then

\[
\text{Lie}(\tilde{T}) \subseteq \tilde{\mathfrak{g}}_3 = \{ x \in \tilde{\mathfrak{g}} | \text{ad}(x)\tilde{a} \in 3(\tilde{\mathfrak{g}}) \} \subseteq \text{Nil}(a, \tilde{\mathfrak{g}}) = \text{Lie}(\tilde{T}),
\]

where the last equality is by, for example, \([DG70, \text{Cor. XIII.5.4}]\). The image \( T \) of \( \tilde{T} \) in \( G \) is a maximal torus that fixes \( \tilde{a} \), so \( \mathfrak{g}_3 \) is generated by \( \text{Lie}(T) \) and the root subgroups it contains. But any such root subgroup would be the image of the corresponding root subgroup of \( \tilde{\mathfrak{g}} \), which does not stabilize \( \tilde{a} \), and therefore \( \mathfrak{g}_3 = \text{Lie}(T) \). In particular, \( \mathfrak{g} \) does not act virtually freely on \( L(\tilde{\alpha}) \).

**Proof of Corollary B.** Suppose first that \( G \) has classical type. Set \( M := \ell^3 \) if \( G \) has type \( B_t, C_t \) or \( D_t \), and \( M := \ell^3/8 \) in case \( G \) has type \( A_t \). If \( \dim V > M \), then examining the bounds in Table 1 shows that \( \dim V > b(G) \) and by Theorem A \( \mathfrak{g} \) acts virtually freely on \( V \). So assume \( \dim V \leq M \); by \([\text{Libb01, Th. 5.1}]\), \((G, V)\) appears in Table 2 of ibid. Those representations of dimension less than \( \dim G - \dim 3(\mathfrak{g}) \) cannot be virtually free. The remaining possibilities are settled in Lemma 13.1 and Example 13.2.

Suppose now that \( G \) has exceptional type. The case \( V = L(\tilde{\alpha}) \) has been treated in Example 13.2. Otherwise, Tables A.49–A.53 in Lübeck provide the following list of possibilities for \( V \) with \( b(G) \geq \dim V \geq \dim G - \dim 3(\mathfrak{g}) \), up to graph automorphism and assuming \( \text{char} \ k \) is not special, where we denote the highest weights as in \([\text{Libb01}]\): \( G_2 \) with highest weight 02 and dimension 26 or 27 (which factors through \( \text{SO}_7 \) and so is virtually free by Lemma 13.1); \( G_2 \) with highest weight 11 and dimension 38 and \( \text{char} \ k = 7 \); \( F_4 \) with highest weight 0010 and dimension 196 and \( \text{char} \ k = 3 \); \( E_6 \), with highest weight 000002 or 000010 and dimension 324 or 351. These representations have \( \dim V > \dim G \) and are virtually free by \([\text{Gue97, Th. 4.3.1}]\). Note that for any particular \( V \) and \( \text{char} \ k \), one can verify that the representation is virtually free using a computer, as described in \([\text{GG17a}]\). \( \square \)
14. How many conjugates are needed to generate Lie(G)?

The results in the previous section suffice to prove the following, which generalizes the main result (Th. 8.2) of [CSUW01].

**Proposition 14.1.** Let $G$ be a simple linear algebraic group over an algebraically closed field $k$ such that char $k$ is not special for $G$, and let $e$ be as in Table 3.

1. If $G$ is simply connected and $x \in g$ is noncentral, then there exists $e$ $G$-conjugates of $x$ that generate $g$.
2. If $x \in g$ is noncentral, then there exists $e$ $G$-conjugates of $x$ that generate a subalgebra containing $[g, g]$.

| type of $G$ | $e$ |
|-------------|-----|
| $A_n$ ($n \geq 1$) or $B_n$ ($n \geq 3$) | $n + 1$ |
| $C_n$ ($n \geq 2$) | $2n$ |
| $D_n$ ($n \geq 4$) | $n$ |

Table 3. Number of conjugates $e$ needed to generate, as in Proposition 14.1.

The new results here are types $A$, $D$, $E$, and $G_2$ when char $k = 2$. The related result in [CSUW01] is stated for long root elements only, but the proof below shows that the long root elements are the main case.

**Proof.** We first assume that $x$ is a root element and $G$ is simply connected. If $G$ is of exceptional type, we apply Proposition 11.3, so assume that $G$ has type $A$, $B$, $C$, or $D$. For type $A_n$, i.e., $G = SL_{n+1}$, $n + 1$ conjugates suffice by Proposition 5.4(3) if char $k \neq 2$ and Proposition 6.1 if char $k = 2$. For type $C_n$ ($Sp_{2n}$) with $n \geq 2$, $2n$ conjugates suffice by Proposition 7.1(3). For types $B$ and $D$, long root elements have rank 2 so Proposition 8.1 gives the claim. If char $k = 2$ and $G$ has type $D_n$, then the claim follows for $so_{2n}$ by Lemma 9.3 and the claim follows for groups isogenous to $G$ by Lemma 10.1.

If $x$ is nonzero nilpotent, then by Lemma 11.2 and deforming as in §3 we are reduced to the previous case.

Generally, $x$ has a Jordan decomposition $x = x_s + x_n$ where $x_s$ is semisimple and $x_n$ is nilpotent and we may assume $x_s \neq 0$. If $x_s$ is noncentral, then we replace $x$ with $x_s$ (whose orbit is closed in the closure of $xG$) and then replace $x_s$ with a root element as in Example 3.1.

Therefore, we may assume that $x_s, x_n \neq 0$ and $x_s$ is central. Deforming, it suffices to treat the case where $x_n$ is a root element. The line $tx_s + x_n$ for $t \in k$ has an open subset consisting of elements such that $e$ conjugates suffice to generate $g$ (in case (1)) or containing $[g, g]$ (in case (2)), and this set is nonempty because it contains $x_n$, so it contains $t_0x_s + x_n$ for some $t_0 \in k^\times$. The element $x_n$ and $t_0^{-1}x_n$ are in the same $G$-orbit, so the same is true of $x$ and $x_s + t_0^{-1}x_n$; this proves the claim.

In the proof, the final paragraph could have been replaced by an argument by mapping $x$ into the Lie algebra of the adjoint group of $G$ and applying the result for nilpotent elements there together with Lemma 10.1.
15. The generic stabilizer in $G$ as a group scheme

Let $G$ be an algebraic group over a field $k$ and $\rho: G \to \text{GL}(V)$ a representation. We say that $G$ acts generically freely on $V$ if there is a dense open subset $U$ of $V$ such that, for every extension $K$ of $k$ and every $u \in U(K)$, the stabilizer $G_u$ (a closed sub-group-scheme of $G \times K$) is the trivial group scheme. Of course, the kernel $\ker \rho \subseteq G_u$ for all $u$, so it is natural to replace $G$ with $\rho(G)$ and assume that $G$ acts faithfully on $V$, i.e., $\ker \rho$ is the trivial group scheme.

In this section, we announce results on determining the generic stabilizer as a group scheme when $V$ is faithful and irreducible. The proofs are combinations of the main results in this paper, the sequels $[GG17a]$ and $[GG18]$ (which build on this paper), and $[GLL18]$.

**Corollary C.** Suppose $G$ acts faithfully and irreducibly on $V$. Then $G_v$ is finite étale for generic $v \in V$ if and only if $\dim V > \dim G$ and $(G, \text{char } k, V)$ does not appear in Table 5.

**Proof.** Write the highest weight $\lambda$ as $\lambda = \lambda_0 + p\lambda_1$ where $\lambda_0$ is restricted and $p := \text{char } k$. As $V$ is faithful, $\lambda_0 \neq 0$. If $\lambda_1 = 0$, then $V$ is restricted and the claim is Corollary B in this paper and $[GG17a, \text{Th. A}]$ when $\text{char } k$ is not special and $[GG18]$ when $\text{char } k$ is special.

If $\lambda_1 \neq 0$, then $\dim V \geq m(G)^2 > \dim G$, where $m(G)$ is the dimension of the smallest nontrivial irreducible representation of $G$. Then $\mathfrak{g}$ acts generically freely by $[GG17a, \text{Th. B}]$ and $[GG18, \text{Th. C}]$. □

The previous result addressed when $G_v$ is finite étale. We now give a result concerning when it is the trivial group scheme, which requires reference also to $[GLL18]$.

**Corollary D.** Suppose $G$ acts faithfully and irreducibly on $V$. Then $G_v$ acts generically freely on $V$ if and only if $\dim V > \dim G$ and $(G, \text{char } k, V)$ appears in neither Table 5 nor Table 6.

**Proof.** By Corollary C, we may assume that $\dim V > \dim G$ and our task is to determine such $V$ where $G_v(k) \neq 1$ $[GLL18, \text{Table 1}]$ or $\mathfrak{g}_v \neq 0$ (Table 5, by Cor. C) for generic $v \in V$. □

The results above settle completely the question of determining which faithful irreducible representations of simple $G$ are generically free. It is natural to ask which of these hypotheses are necessary. For example, if $\text{char } k$ is special for $G$, there are irreducible but non-faithful representations that factor through the very special isogeny; whether or not these are virtually free is settled in $[GG18]$. Another way that $G$ may fail to act faithfully is if $V$ is the Frobenius twist of a representation $V_0$; in that case $\mathfrak{g}$ acts trivially on $V$, so $G$ acts virtually freely if and only if the group $G(k)$ of $k$-points acts virtually freely on $V_0$. One could ask: What about analogues of the main results for $G$ semisimple?

One could also ask for a stronger bound in Theorem A. What is the smallest constant $c$ such that the conclusion holds when we set $b(G) = c \dim G$? What about to guarantee $G_v$ étale? Or $G_v = 1$? Table 5 shows that $c$ must be greater than 1. Does $c = 2$ suffice?
Table 4. Dynkin diagrams of simple root systems of classical type, with simple roots numbered as in [Lüb01].

| $G$ | char $k$ | weight | dim $V$ | dim $\mathfrak{g}_v$ |
|-----|----------|--------|--------|-----------------|
| $A_7$ | 2 | $\wedge^4$ | 70 | 4 |
| $A_8$ | 3 | $\wedge^3$ or $\wedge^6$ | 84 | 3 |
| $D_8$ | 2 | half-spin | 128 | 4 |

Table 5. Irreducible and restricted representations $V$ of simple $G$ with $\dim V > \dim G$ that are not virtually free for $\mathfrak{g}$. For each, the stabilizer $\mathfrak{g}_v$ of a generic $v \in V$ is a toral subalgebra, and $\dim \mathfrak{g}_v$ is given for the case where $G$ is simply connected.

| $G$ | char $k$ | weight | dim $V$ | dim $\mathfrak{g}_v$ |
|-----|----------|--------|--------|-----------------|
| $A_7$ | 2 | $\wedge^4$ | 70 | 4 |
| $A_8$ | 3 | $\wedge^3$ or $\wedge^6$ | 84 | 3 |
| $D_8$ | 2 | half-spin | 128 | 4 |

Table 6. Irreducible faithful representations $V$ of simple $G$ with $\dim V > \dim G$ such that $G_v$ is finite étale and $\neq 1$ for generic $v \in V$, up to graph automorphism.
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