CRITICAL VALUES AND LEVEL SETS OF DISTANCE FUNCTIONS IN RIEMANNIAN, ALEXANDROV AND MINKOWSKI SPACES

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Abstract. Let $F \subset \mathbb{R}^n$ be a closed set and $n = 2$ or $n = 3$. S. Ferry (1975) proved that then, for almost all $r > 0$, the level set (distance sphere, $r$-boundary) $S_r(F) := \{x \in \mathbb{R}^n : \text{dist}(x, F) = r\}$ is a topological $(n-1)$-dimensional manifold. This result was improved by J.H.G. Fu (1985). We show that Ferry’s result is an easy consequence of the only fact that the distance function $d(x) = \text{dist}(x, F)$ is locally DC and has no stationary point in $\mathbb{R}^n \setminus F$. Using this observation, we show that Ferry’s (and even Fu’s) result extends to sufficiently smooth normed linear spaces $X$ with $\dim X \in \{2, 3\}$ (e.g., to $\ell^p_n$, $n = 2, 3$, $p \geq 2$), which improves and generalizes a result of R. Gariepy and W.D. Pepe (1972). By the same method we also generalize Fu’s result to Riemannian manifolds and improve a result of K. Shiohama and M. Tanaka (1996) on distance spheres in Alexandrov spaces.

1. Introduction

Let $X$ be a metric space and $F \subset X$ a closed set. We will study level sets of the distance function $S_r(F) := \{x \in \mathbb{R}^n : \text{dist}(x, F) = r\}$, $r > 0$. We will call these sets (following [31]) distance spheres; they are sometimes called also $r$-boundaries of $F$ (see [10]). There is a number of articles that investigate properties of distance spheres. R. Gariepy and W.D. Pepe [14] studied distance spheres in a Minkowski space (= finite dimensional Banach space) $X$. S. Ferry [10] proved that if $X = \mathbb{R}^2$ or $X = \mathbb{R}^3$, then, for almost all $r > 0$, the distance sphere ($r$-boundary) $S_r(F)$ is a topological $(n-1)$-dimensional manifold. This result was improved by J.H.G. Fu [11], who proved that these topological manifolds are very nice: they are semiconcave surfaces. Moreover, he proved that, for $n = 2$, the above property of $S_r(F)$ is valid for all $r > 0$ except a relatively closed set $N \subset (0, \infty)$ with $\mathcal{H}^{1/2}(N) = 0$.

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We observe that Ferry’s result is an easy consequence (see Theorem 3.4) of the only fact that the distance function \( d(x) = \text{dist}(x, F) \) is locally DC (i.e., a difference of two convex functions) and has no stationary point in \( \mathbb{R}^n \setminus F \). Using this observation, we show that Ferry’s (and even Fu’s) result extends to sufficiently smooth normed linear spaces \( X \) with \( \dim X \in \{2, 3\} \) (e.g., to \( \ell^p_n, n = 2, 3, \ p \geq 2 \)), which improves and generalizes a result of R. Gariepy and W.D. Pepe [14].

If \( X \) is a Riemannian manifold, then it is well-known (see [13, p. 34] or [22]) that \( d(x) = \text{dist}(x, F) \) is locally semiconcave (and therefore also locally DC) in arbitrary local coordinates. So, we can apply Theorem 3.4 and obtain Ferry’s (and even Fu’s) results.

K. Shiohama and M. Tanaka [31] studied distance spheres of compact subsets \( F \) of a connected two-dimensional complete Alexandrov space without boundary \( X \). They proved that then, for almost all \( r > 0 \), the distance sphere \( S_r(F) \) is rectifiable and consists of a disjoint union of finitely many simply closed curves. Using our method and Perelman’s DC structure on Alexandrov spaces, we obtain a result, which improves that of [31]. Namely, we show that \( S_r(F) \) is a one-dimensional Lipschitz manifold for all \( r > 0 \) except a closed set \( N \subset [0, \infty) \) with \( \mathcal{H}^{1/2}(N) = 0 \).

If \( X \) is a three-dimensional complete Alexandrov space (possibly with boundary points), then our method gives only that, for almost every \( r > 0 \), the set \( S_r(F) \cap X^* \) is a two-dimensional Lipschitz manifold, where \( X^* \) is the set of all “Perelman regular” points (note that \( X^* \) is an open dense convex subset of \( X \), cf. Section 6). Consequently, if \( \mathcal{H}^1(X \setminus X^*) = 0 \), then Ferry’s result extends to \( X \).

In all types of spaces considered above, we obtain also weaker results on distance spheres in \( n \)-dimensional spaces with arbitrary \( n \). Namely, we prove that, except a countable set of radii \( r \), there exists an \((n-1)\)-dimensional Lipschitz manifold \( A_r \subset S_r(F) \) such that \( A_r \) is open and dense in \( S_r(F) \) and \( \mathcal{H}^{n-1}(S_r(F) \setminus A_r) = 0 \). (For the density of \( A_r \) in \( S_r(F) \) in Alexandrov spaces we need that \( X = X^* \).)

2. Preliminaries

The symbol \( B(x, r) \) will denote the open ball with center \( x \) and radius \( r \).

**Definition 2.1.** Let \( X \) be a metric space. Given a nonempty subset \( A \subset X \) and \( p \in A \), the reach of \( A \) at \( p \), \( \text{reach}(A, p) \), is defined as the supremum of all \( \varepsilon > 0 \) such that any point \( q \in X \) with \( \text{dist}(p, q) < \varepsilon \) has its unique nearest point in \( A \). We set \( \text{reach} A := \inf_{p \in A} \text{reach}(A, p) \) and say that \( A \) has positive reach if \( \text{reach} A > 0 \). The set \( A \) is said to have locally positive reach if \( \text{reach}(A, p) > 0 \) for all \( p \in A \).
Remark 2.2. Sets with positive reach were introduced by Federer [9] in the Euclidean space and by Kleinjohann [17] in Riemannian manifolds. Note that if \( A \) is compact then \( \text{reach} \ A > 0 \) whenever \( A \) has locally positive reach. It follows from the obvious fact that \( \text{reach} \ (A, p) \) depends continuously on \( p \in A \).

Definition 2.3. (cf. [12], p. 622) We say that a metric space \( X \) is an \( m \)-dimensional Lipschitz manifold if for every \( a \in X \) there exists an open neighbourhood \( U \) of \( a \) and a bilipschitz homeomorphism \( \varphi \) of \( U \) onto an open subset of \( \mathbb{R}^m \).

Let \( X \) be a normed linear space and let \( f \) be a real function defined on an open set \( G \subset X \).

The directional derivative and the one-sided directional derivative of \( f \) at \( a \in G \) in the direction \( v \in X \) are defined by

\[
\begin{align*}
\frac{d^f(a,v)}{t} & := \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} \quad \text{and} \quad \frac{d^+f(a,v)}{t} := \lim_{t \to 0^+} \frac{f(a + tv) - f(a)}{t}.
\end{align*}
\]

Now suppose that \( f \) is locally Lipschitz on \( G \). Then

\[
\frac{d_0^f(a,v)}{t} := \limsup_{z \to a, t \to 0^+} \frac{f(z + tv) - f(z)}{t}
\]

is the Clarke derivative of \( f \) at \( a \in G \) in the direction \( v \in X \) and

\[
\partial^C f(a) := \{ x^* \in X^* : \langle x^*, v \rangle \leq f_0^0(a,v) \text{ for all } v \in X \}
\]

is the Clarke subdifferential of \( f \) at \( a \). We shall use the following terminology (see [11]).

Definition 2.4. Let \( f \) be a locally Lipschitz function on an open subset \( G \) of a normed linear space. Then we say that \( a \) is a regular point of \( f \) if \( 0 \notin \partial^C f(a) \). If \( 0 \in \partial^C f(a) \), we say that \( a \) is a critical point of \( f \). The set of all critical points of \( f \) will be denoted by \( \text{Crit}(f) \). By the set of critical values of \( f \) we mean the set \( \text{cv}(f) := f(\text{Crit}(f)) \).

We will need the following easy lemma. Because of a lack of a reference we supply the obvious proof.

Lemma 2.5. Let \( f \) be a locally Lipschitz function on an open set \( G \subset \mathbb{R}^n \) and \( a \in G \). Then the following conditions are equivalent.

(i) \( a \notin \text{Crit}(f) \).

(ii) There exist \( \delta > 0 \), \( \varepsilon > 0 \), and \( v \in \mathbb{R}^n \) such that

\[
\frac{f(x + tv) - d(x)}{t} < -\varepsilon, \quad \text{whenever} \quad t > 0, x \in U_\delta(a), x + tv \in U_\delta(a).
\]

Proof. The condition (i) (i.e., \( 0 \notin \partial^C f(a) \)) holds if and only if there exists \( v \in \mathbb{R}^n \) such that \( f_0^0(a,v) = \limsup_{z \to a, t \to 0^+} \frac{f(z + tv) - f(z)}{t} < 0 \). It is easy to see that the last condition is equivalent to (ii). \( \square \)
Lemma 2.5 immediately implies the well-known fact that
(1) \( \text{Crit}(f) \) is closed in \( G \).

If \( f \) is a real function on a normed linear space \( X \), then the symbol \( f'(a) \) stands for the (Fréchet) derivative of \( f \) at \( a \in X \). If \( f'(a) \) exists and
\[
\lim_{x,y \to a, x \neq y} \frac{f(y) - f(x) - f'(a)(y - x)}{\|y - x\|} = 0,
\]
then we say that \( f \) is strictly differentiable at \( a \) (cf. [21, p. 19]).

Lemma 2.5 easily implies the well-known fact (see, e.g., [7, Proposition 2.2.4], where a weaker notion of strict differentiability is used) that
(2) If \( f'(a) \neq 0 \) and \( f \) is strictly differentiable at \( a \), then \( a \notin \text{Crit}(f) \).

**Definition 2.6.** Let \( C \) be a nonempty convex set in a real normed linear space \( X \). A function \( f : C \to \mathbb{R} \) is called DC (or d.c., or “delta-convex”) if it can be represented as a difference of two continuous convex functions on \( C \).

If \( Y \) is a finite-dimensional normed linear space, then a mapping \( F : C \to Y \) is called DC, if \( y^* \circ F \) is a DC function on \( C \) for each linear functional \( y^* \in Y^* \).

**Remark 2.7.**
(i) To prove that \( F \) is DC, it is clearly sufficient to show that \( y^* \circ F \) is DC for each \( y^* \) from a basis of \( Y^* \).

(ii) Each DC mapping is clearly locally Lipschitz.

We will need the following properties of DC functions and mappings.

**Lemma 2.8.** Let \( X, Y, Z \) be finite-dimensional normed linear spaces, let \( C \subset X \) be a nonempty convex set, and \( U \subset X \) and \( V \subset Y \) open sets.

(a) If the derivative of a function \( f \) on \( C \) is Lipschitz, then \( f \) is DC. In particular, each affine mapping is DC.

(b) Let a mapping \( F : U \to Y \) be locally DC, \( F(U) \subset V \), and let \( G : V \to Z \) be locally DC. Then \( G \circ F \) is locally DC on \( U \).

(c) If mappings \( F : U \to Y \) and \( G : U \to Y \) are locally DC, then \( F + G \) is also locally DC.

(d) Let \( n \in \{1, 2\} \), \( \dim X = n \), and let \( f \) be a locally DC function on \( U \). Let \( S := \{x \in U : f'(x) = 0\} \) be the set of all stationary points of \( f \). Then \( \mathcal{H}^{n/2}(f(S)) = 0 \).

The proofs of (a)–(c) can be found in [32]. Let us note that (a) was at first proved in [1], and (b) in [16].

The Morse-Sard theorem (d) was for \( n = 2 \) published by Landis [19] with a sketch of the proof. A detailed proof based on the modern theory of BV functions can be found in [24, Corollary 4.5]. The easier case \( n = 1 \) is proved in [23].

An important subclass of the class of DC functions is formed by semiconcave functions (cf. [6]).
Definition 2.9. Let $H$ be a unitary space. A real function $u$ on an open convex set $C \subset H$ is called \textit{semiconcave} (with a semiconcavity constant $c$) if the function 
$$ g(x) := u(x) - \left(\frac{c}{2}\right)\|x\|^2 $$

is concave on $C$.

A function $u$ on an open set $G \subset H$ is called \textit{locally semiconcave} if, for each $x \in G$, there exists $\delta > 0$ such that $u$ is semiconcave on $B(x, \delta)$. A function $g$ on $G$ is called \textit{locally semiconvex} if $-g$ is locally semiconcave.

Remark 2.10. (i) $g$ is locally semiconvex on $G$ if and only if, for each $x \in G$, there exists $\delta > 0$ and a $C^\infty$ smooth function $s$ on $B(x, \delta)$ such that the function $g + s$ is convex on $B(x, \delta)$. It follows, e.g., from [6, Proposition 1.1.3] applied to $u := -g$.

(ii) If $g$ is locally semiconvex on $G$, $a \in G$ and $v \in H$, then $g^0(a, v) = g_+(a, v)$. It follows, e.g., from [6, Theorem 3.2.1] applied to $u := -g$.

Definition 2.11. Let $H$ be an $n$-dimensional unitary space and $1 \leq k < n$.

(i) We say that a set $\emptyset \neq M \subset H$ is a \textit{k-dimensional Lipschitz surface} (resp. a \textit{k-dimensional DC surface}) in $H$, if for each $x \in M$ there exists a $k$-dimensional linear space $Q \subset H$, an open neighbourhood $W$ of $x$, a set $G \subset Q$ open in $Q$ and a Lipschitz (resp. locally DC) mapping $h : G \to Q^\perp$ such that 
$$ M \cap W = \{u + h(u) : u \in G\}. $$

(ii) We say that a set $\emptyset \neq M \subset H$ is an \textit{$(n-1)$-dimensional semiconcave surface} in $H$, if for each $x \in M$ there exists an $(n-1)$-dimensional linear space $Q \subset H$, an open neighbourhood $W$ of $x$, a set $G \subset Q$ open in $Q$, a vector $0 \neq v \in Q^\perp$, and a locally semiconcave function $s : G \to \mathbb{R}$ such that 
$$ M \cap W = \{u + s(u)v : u \in G\}. $$

By a 0-dimensional Lipschitz (resp. DC, resp. semiconcave) surface we mean a singleton.

Remark 2.12. Obviously, each $k$-dimensional DC surface in $H$ is a $k$-dimensional Lipschitz surface in $H$, and each $(n-1)$-dimensional semiconcave surface in $H$ is an $(n-1)$-dimensional DC surface in $H$.

Using the preceding definition, we can formulate some versions of known implicit function theorems for Lipschitz, DC and semiconcave functions in a concise form.

Proposition 2.13. Let $f$ be a locally Lipschitz function on an open set $G \subset \mathbb{R}^n$, and let $a \in G \setminus \text{Crit}(f)$. Denote $M := \{x \in G : f(x) = f(a)\}$. Then there exists $\delta > 0$ such that:
(i) $M \cap B(a, \delta)$ is an $(n-1)$-dimensional Lipschitz surface in $\mathbb{R}^n$.

(ii) If $f$ is locally DC, then $M \cap B(a, \delta)$ is an $(n-1)$-dimensional DC surface in $\mathbb{R}^n$.

(iii) If $f$ is locally semiconcave, then $M \cap B(a, \delta)$ is an $(n-1)$-dimensional semiconcave surface in $\mathbb{R}^n$. Moreover, $\text{reach}(\{x \in G : f(x) \geq f(a)\}, a) > 0$.

Proof. The statement (i) is an obvious reformulation of [11, Theorem 3.1], which is an easy consequence of Clarke’s implicit function theorem.

We will show that the statement (ii) is an easy consequence of [32, Proposition 5.9]. To this end, choose $\delta > 0$, $\varepsilon > 0$ and $v \in \mathbb{R}^n$ as in Lemma 2.5(ii).

Let $Y$ be the linear span of $\{v\}$ and $X := Y^\perp$. Identifying by the standard way $\mathbb{R}^n$ with $X \times Y$ and, by linear isometries, $X$ with $\mathbb{R}^{n-1}$ and $Y$ with $\mathbb{R}$, we can apply [32, Proposition 5.9] (with $G := f$) and obtain the assertion of (ii). Indeed, the fact that $\partial_2^2 f(a)$ contains surjective linear mappings $\mathbb{R} \to \mathbb{R}$ only is an easy consequence of the choice of $v$ (see Lemma 2.5(ii)). Note also that Lemma 2.5(ii) immediately implies the local validity of the inequality $\|G(x, y) - G(x, \mathbf{0})\| \geq c\|y - \mathbf{0}\|$ (with $c := \varepsilon$) which is claimed without a proof in the proof of [32, Proposition 5.9].

The first part of the assertion (iii) follows immediately from [11, Theorem 3.3]. The second part follows easily from the proof of [11, Corollary 3.4] or from [3, Theorem].

Lemma 2.14. Let $X, Y$ be finite-dimensional unitary spaces with $\dim X = n > 0$ and $\dim Y = m > 0$. Let $G \subset X$ be an open set, and $f : G \to Y$ a locally DC mapping. Then there exists a sequence $(T_i)$ of $(n-1)$-dimensional DC surfaces in $X$ such that $f$ is strictly differentiable at each point of $G \setminus \bigcup_{i=1}^\infty T_i$.

Proof. We can suppose that $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. First suppose $n > 1$. Using separability of $X$, we can clearly suppose that $G$ is convex and $f$ is DC on $G$. Let $f = (\alpha_1 - \beta_1, \ldots, \alpha_m - \beta_m)$, where all $\alpha_j$ and $\beta_j$ are convex functions. By [33], for each $j$ we can find a sequence $T^j_k, k \in \mathbb{N}$, of $(n-1)$-dimensional DC surfaces in $G$ such that both $\alpha_j$ and $\beta_j$ are differentiable at each point of $D_j := G \setminus \bigcup_{k=1}^\infty T^j_k$. Since each convex function is strictly differentiable at each point at which it is (Fréchet) differentiable (see, e.g., [32, Proposition 3.8] for a proof of this well-known fact), we conclude that each $f_j := \alpha_j - \beta_j$ is strictly differentiable at each point of $D_j$. Since strict differentiability of $f$ clearly follows from strict differentiability of all $f_j$'s, the proof is finished after ordering all the sets $T^j_k, k \in \mathbb{N}, j = 1, \ldots, m$, to a sequence $(T_i)$.

If $n = 1$, we proceed quite similarly, using the well-known fact that a convex function on an open interval is differentiable except a countable set.
3. Critical values and level sets of DC functions

Lemma 3.1. Let \( f, g \) be convex functions on an open convex set \( C \subset \mathbb{R}^n \), and let \( d := f - g \). Assume that the directional derivatives \( f'(x, v), g'(x, v) \) exist for some \( x \in C \) and \( v \in \mathbb{R}^n \), and that \( f'(x, v) \neq g'(x, v) \). Then \( x \notin \text{Crit}(d) \).

Proof. We can suppose that \( f'(x, v) < g'(x, v) \) (otherwise consider \( -v \) instead of \( v \)). Since \( f \) is convex, we have (cf. Remark 2.10(ii))

\[
f'(x, v) = f^0(x, v) = \limsup_{y \rightarrow x, t \rightarrow 0^+} \frac{f(y + tv) - f(y)}{t}
\]

and

\[
-f'(x, v) = f'(x, -v) = f^0(x, -v) = \limsup_{y \rightarrow x, t \rightarrow 0^+} \frac{f(y - tv) - f(y)}{t}
\]

\[
= - \liminf_{y \rightarrow x, t \rightarrow 0^+} \frac{f(y) - f(y - tv)}{t} = - \liminf_{z \rightarrow x, t \rightarrow 0^+} \frac{f(z + tv) - f(z)}{t}.
\]

Consequently

\[
f'(x, v) = \lim_{y \rightarrow x, t \rightarrow 0^+} \frac{f(y + tv) - f(y)}{t}.
\]

Using this also for the convex function \( g \), we obtain

\[
\lim_{y \rightarrow x, t \rightarrow 0^+} \frac{d(y + tv) - d(y)}{t} = f'(x, v) - g'(x, v) < 0.
\]

Thus there exist \( \varepsilon > 0 \) and \( \delta > 0 \) as in (ii) of Lemma 2.5 \( \square \)

Lemma 3.2. Let \( X \) be an \( n \)-dimensional unitary space and let \( k \in \{1, 2\} \) with \( k < n \). Let \( C \subset X \) be an open convex set, and let \( d \) be a DC function on \( C \). Let \( P \subset X \) be a \( k \)-dimensional DC surface. Then

\[
\mathcal{H}^{k/2}(d(P \cap \text{Crit}(d))) = 0.
\]

Proof. Let \( d = f - g \), where \( f, g \) are convex functions on \( C \).

(i) First suppose \( k = 1 \). Using separability of \( X \), we can clearly suppose that \( P = \{t + h(t) : t \in G\} \), where \( G \) is a relatively open subset of a one-dimensional linear space \( V \subset X \), and \( h : G \rightarrow V^\perp \) is a locally DC mapping. Set \( \varphi(t) := t + h(t), t \in G \). Then \( \varphi \) is locally DC on \( G \) (Lemma 2.8(a),(c)), and so also \( f \circ \varphi, g \circ \varphi \) and \( d \circ \varphi \) are locally DC on \( G \) (Lemma 2.8(b)). So, by Lemma 2.14 there exists a countable set (countable union of 0-dimensional DC surfaces) \( A \subset G \) such that \( \varphi'(t), (f \circ \varphi)'(t) \) and \( (g \circ \varphi)'(t) \) exist for each \( t \notin A \). Set \( B := \{x \in G \setminus A : (f \circ \varphi)'(t) = (g \circ \varphi)'(t)\} \). For each \( t \in B \), we have \( (d \circ \varphi)'(t) = 0 \), and consequently \( \mathcal{H}^{1/2}(d \circ \varphi(B)) = 0 \) by Lemma 2.8(d). Set

\[
N := (d \circ \varphi)(A) \cup (d \circ \varphi)(B) = d(\varphi(A) \cup \varphi(B)).
\]
Since clearly $\mathcal{H}^{1/2}(N) = 0$, it is sufficient to prove
\[ P \cap \text{Crit}(d) \subset \varphi(A) \cup \varphi(B). \]
To this end, suppose that $x \in P \setminus (\varphi(A) \cup \varphi(B))$. Then $x = \varphi(t)$ for some $t \in G \setminus (A \cup B)$. So, $\varphi'(t)$ exists and $(f \circ \varphi)'(t) \neq (g \circ \varphi)'(t)$. So we can choose $u \in V$ such that $(f \circ \varphi)'(t, u) \neq (g \circ \varphi)'(t, u)$. Set $v := \varphi'(t)(u)$. Since $f_+(x, v), f'_+(x, -v)$ exist and $f$ is locally Lipschitz, we conclude (see, e.g. Proposition 3.6(i)) that $f'_+(x, v) = (f \circ \varphi)'(t, u)$ and similarly
\[ f'_+(x, -v) = (f \circ \varphi)'(t, -u) = -(f \circ \varphi)'(t, u) = -f'_+(x, v). \]
Consequently $f'(x, v)$ exists. Similarly we obtain that $g'(x, v)$ exists. Thus $f'(x, v) = (f \circ \varphi)'(t, u) \neq (g \circ \varphi)'(t, u) = g'(x, v)$, and consequently $x \notin \text{Crit}(d)$ by Lemma 3.1. So (3) holds.

(ii) Let now $k = 2$. Using separability of $X$, we can clearly suppose that $P = \{t + h(t) : t \in G\}$, where $G$ is a relatively open subset of a two-dimensional linear space $V \subset X$, and $h : G \to V^\perp$ is a locally DC mapping. Set $\varphi(t) := t + h(t), t \in G$. Then $\varphi$ is locally DC on $G$, and so also $f \circ \varphi, g \circ \varphi$ and $d \circ \varphi$ are locally DC on $G$. So, by Lemma 2.14 there exists a sequence $(P_i)_{i=1}^\infty$ of one-dimensional DC surfaces in $V$ such that $\varphi'(t), (f \circ \varphi)'(t)$ and $(g \circ \varphi)'(t)$ exist for each $t \in G \setminus A$, where $A := \bigcup_{i=1}^\infty P_i$. Using separability of $V$, we can suppose that each $P_i$ is of the form $P_i = \{s + g_i(s) : s \in H_i\}$, where $H_i$ is a relatively open subset of a one-dimensional linear space $W_i \subset V$ and $g_i : H_i \to (W_i^\perp \cap V)$ is a locally DC mapping. Put $Q_i := \varphi(P_i) = \{s + g_i(s) + h(s + g_i(s)) : s \in H_i\}$. Since $\psi(s) := g_i(s) + h(s + g_i(s)), s \in H_i$, is a locally DC mapping $\psi : H_i \to W_i^\perp$ (Lemma 2.3(b),(c)), we obtain that each $Q_i$ is a one-dimensional DC surface in $X$.

Set $N_i := d (\bigcup_{i=1}^\infty Q_i \cap \text{Crit}(d))$. By part (i) of the proof, $\mathcal{H}^{1/2}(N_i) = 0$. Set $B := \{t \in G \setminus \bigcup_{i=1}^\infty P_i : (f \circ \varphi)'(t) = (g \circ \varphi)'(t)\}$. For each $t \in B$, we have $(d \circ \varphi)'(t) = 0$, and consequently $\mathcal{H}^1(d \circ \varphi(B)) = 0$ by the Morse-Sard theorem for DC functions (Lemma 2.8(d)) on $\mathbb{R}^2$. Set
\[ N := N_1 \cup (d \circ \varphi)(B) = d ((\varphi(A) \cap \text{Crit}(d)) \cup \varphi(B)). \]
Since clearly $\mathcal{H}^1(N) = 0$, it is sufficient to prove
\[ P \cap \text{Crit}(d) \subset \varphi(A) \cup \varphi(B). \]
The proof of (4) can be done literally as the proof of (3). \hfill \Box

**Proposition 3.3.** Let $n \in \{2, 3\}$ and let $d$ be a locally DC function on an open set $G \subset \mathbb{R}^n$. Suppose that $d$ has no stationary point. Let $\text{cv}(d) = d(\text{Crit}(d))$ be the set of critical values of $d$. Then $\mathcal{H}^{(n-1)/2}(\text{cv}(d)) = 0$.

**Proof.** We can and will assume that $G$ is convex. By Lemma 2.14 there exists a sequence $(P_i)_{i=1}^\infty$ of $(n-1)$-dimensional DC surfaces in $\mathbb{R}^n$ such that $d$ is strictly differentiable (and $d'(x) \neq 0$ by the assumptions) at each $x \in G \setminus \bigcup_{i=1}^\infty P_i$.
So, using (2), we obtain \( \text{Crit}(d) \subset \bigcup_{i=1}^{\infty} P_i \). Applying Lemma 3.2 for each \( i \in \mathbb{N} \), we obtain that \( \mathcal{H}^{(n-1)/2}(\text{Crit}(d) \cap P_i) = 0 \) for each \( i \), and therefore \( \mathcal{H}^{(n-1)/2}(\text{cv}(d)) = 0 \).

**Theorem 3.4.** Let \( n \in \{2,3\} \) and let \( d \) be a locally DC function on an open set \( G \subset \mathbb{R}^n \). Suppose that \( d \) has no stationary point. Then there exists a set \( N \subset \mathbb{R} \) with \( \mathcal{H}^{(n-1)/2}(N) = 0 \) such that, for every \( r \in d(G) \setminus N \), the set \( d^{-1}(r) \) is an \((n-1)\)-dimensional DC surface. If \( d \) is even locally semiconcave, we can also assert that \( d^{-1}(r) \) is an \((n-1)\)-dimensional semiconcave surface and the set \( \{ x \in G : d(x) \geq r \} \) has locally positive reach.

Moreover, \( N \) can be chosen so that \( N = d(C) \), where \( C \) is a closed set in \( G \). Namely, we can put \( N := \text{cv}(d) = d(\text{Crit}(d)) \).

**Proof.** Set \( C := \text{Crit}(d) \) and \( N := d(C) \). Then \( C \) is closed in \( G \) by \( (\text{i}) \). Proposition 3.3 yields \( \mathcal{H}^{(n-1)/2}(N) = 0 \). Let \( r \in d(G) \setminus N \). Applying Proposition 2.13 (with \( f := d \)) to each point \( a \in d^{-1}(r) \), we easily obtain that the sets \( d^{-1}(r) \) and \( \{ x \in G : d(x) \geq r \} \) have the desired properties.

The following weaker result holds for all dimensions \( n \).

**Theorem 3.5.** Let \( d \) be a locally DC (resp. locally semiconcave) function on an open set \( G \subset \mathbb{R}^n \) and assume that \( d \) has no stationary point. Then, for all \( r \in d(G) \), except a countable set, the set \( A_r := d^{-1}(r) \setminus \text{Crit}(d) \) is an \((n-1)\)-dimensional DC surface (resp. semiconcave surface) which is open and dense in \( d^{-1}(r) \) and \( \mathcal{H}^{n-1}(d^{-1}(r) \setminus A_r) = 0 \).

**Proof.** If \( r \in \mathbb{R} \) and \( A_r := d^{-1}(r) \setminus \text{Crit}(d) \) is nonempty, then it is an \((n-1)\)-dimensional DC surface (resp. semiconcave surface) by Proposition 2.13. Set

\[
N_1 := \{ r \in \mathbb{R} : \mathcal{H}^{n-1}(d^{-1}(r) \cap \text{Crit}(d)) > 0 \},
\]

\[
N_2 := \{ r \in \mathbb{R} : d \text{ has a local extreme at a point of } d^{-1}(r) \}
\]

and \( N := N_1 \cup N_2 \). By Lemma 2.14 and (2), \( \text{Crit}(d) \) can be covered by countably many \((n-1)\)-dimensional DC surfaces and therefore \( \mathcal{H}^{n-1} \) is \( \sigma \)-finite on \( \text{Crit}(d) \). Thus \( N_1 \) is countable. It is well-known (and easy to prove) that the set of (possibly non-strict) extremal values of a real function on a separable metric space \( Y \) is countable. (The proof for \( Y = \mathbb{R} \) [28, p. 43] easily generalizes to general \( Y \).) Thus \( N_2 \), and so also \( N \), is countable. Note that \( A_r \) is open in \( d^{-1}(r) \) by \( (\text{i}) \). To conclude the proof, it is sufficient to prove that \( A_r = d^{-1}(r) \setminus \text{Crit}(f) \) is dense in \( d^{-1}(r) \) for each \( r \in (0, \infty) \setminus N \).

To this end, suppose on the contrary that there exist \( r \in \mathbb{R} \setminus N \) and a point \( a \in d^{-1}(r) \setminus A_r \). Choose a convex open neighborhood \( U \subset G \) of \( a \) such that \( U \cap A_r = \emptyset \). Since \( r \notin N_2 \), we can choose points \( b, c \in U \) such that \( d(b) > r \) and \( d(c) < r \). Set \( W := (c-b)^\perp \) and \( B_\delta := \{ w \in W : ||w|| < \delta \} \).

Choose \( \delta > 0 \) so small that \( d(x) > r \) for each \( x \in b + B_\delta \) and \( d(y) < r \) for
Remark 4.1. Fu did not consider distance spheres is not essential, since the set \( \{ b + w : t \in [0, 1] \} \) with \( d(z_w) = r \). Denote \( Z := \{ z_w : w \in B_\delta \} \). Since the mapping \( z_w \mapsto w \) is Lipschitz with constant 1 on \( Z \) and \( H^{n-1}(B_\delta) > 0 \), we obtain \( H^{n-1}(U \cap d^{-1}(r)) \geq H^{n-1}(Z) > 0 \). Since \( U \cap d^{-1}(r) \subset \text{Crit}(d) \), we obtain a contradiction with \( r \notin N_1 \). \( \square \)

4. MINKOWSKI SPACES

Let \( X \) be a Minkowski space (= finite dimensional Banach space). R. Gariepy and W.D. Pepe [14] proved the following results.

(GP1) If \( \dim X = n \), the norm of \( X \) is strictly convex or differentiable and \( F \subset X \) is a closed set, then, for almost every \( r > 0 \), the distance sphere \( S_r(F) \) is either empty, or there there exists an \( (n-1) \)-dimensional Lipschitz manifold \( A_r \subset S_r(F) \) such that \( A_r \) is open in \( S_r(F) \) and \( H^{n-1}(S_r(F) \setminus A_r) = 0 \).

(GP2) If \( \dim X = 2 \), the norm of \( X \) is twice differentiable with bounded second derivative on the unit sphere and \( F \subset X \) is a closed set, then, for almost every \( r > 0 \), the distance sphere \( S_r(F) \) is either empty, or a one-dimensional Lipschitz manifold.

S. Ferry [10] proved that if \( X = \mathbb{R}^n \) with \( n \in \{2, 3\} \) then, for almost all \( r > 0 \), the distance sphere \( S_r(F) \) is either empty or a topological \( (n-1) \)-dimensional manifold. He also showed that this result does not hold in \( \mathbb{R}^n \) for \( n \geq 4 \).

J.H.G. Fu [11] essentially (cf. Remark 4.1) proved the following stronger result.

(Fu) Let \( X = \mathbb{R}^n \), \( n \in \{2, 3\} \), and \( F \subset X \) be a nonempty compact set. Then there exists a compact set \( N \subset [0, \infty) \) with \( H^{(n-1)/2}(N) = 0 \) such that, for every \( r \in (0, \infty) \setminus N \), the distance sphere \( S_r(F) \) is an \( (n-1) \)-dimensional semiconcave surface and \( \{ x : \text{dist}(x, F) > r \} \) has positive reach.

Remark 4.1. Fu did not consider distance spheres \( S_r(F) \) but the sets \( S^*_r(F) := \partial B_r(F) \), where \( B_r(F) := \{ x \in X : \text{dist}(x, F) \leq r \} \). However, this difference is not essential, since the set \( \{ r > 0 : S_r(F) \neq S^*_r(F) \} \) is countable (even for any \( n \in \mathbb{N} \) and any nonempty closed \( F \subset \mathbb{R}^n \)).

Fu formulated his result in a formally different way: he asserted that, for every \( r \in (0, \infty) \setminus N \), \( S^*_r(F) \) is a Lipschitz \( (n-1) \)-dimensional manifold and \( X \setminus B_r \) is a set of positive reach. However, the proofs of [11] give that, for every \( r \in (0, \infty) \setminus N \), the set \( S^*_r(F) \) is an \( (n-1) \)-dimensional semiconcave surface and \( S^*_r(F) = S_r(F) \).

Our first result generalizes in a sense (Fu) to sufficiently smooth normed linear spaces and generalizes and improves (GP2).
Theorem 4.2. Let $n \in \{2,3\}$ and let $(X, \| \cdot \|)$ be an $n$-dimensional normed linear space such that the derivative of the norm $\| \cdot \|$ is Lipschitz on the unit sphere (e.g., $X = \ell^n_2$, $p \geq 2$). Let $F \subset X$ be a nonempty closed set and denote $S_r(F) := \{ x \in X : \text{dist}_{\| \cdot \|}(x, F) = r \}$, $F_{\geq r} := \{ x \in X : \text{dist}_{\| \cdot \|}(x, F) \geq r \}$.

Then there exists a set $N \subset (0, \infty)$ with $\mathcal{H}^{(n-1)/2}(N) = 0$ such that, for every $r \in (0, \infty) \setminus N$:

(i) The distance sphere $S_r(F)$ is either empty or an $(n-1)$-dimensional Lipschitz manifold in $(X, \| \cdot \|)$.

(ii) If $\| \cdot \|_H$ is an arbitrary (equivalent) Hilbert norm on $X$, then $S_r(F)$ is either empty or an $(n-1)$-dimensional semiconcave surface in $(X, \| \cdot \|_H)$ and $F_{\geq r}$ has locally positive reach in $(X, \| \cdot \|_H)$.

(iii) If $F$ is compact then $N$ is closed in $(0, \infty)$ and $F_{\geq r}$ has positive reach in $(X, \| \cdot \|_H)$.

Proof. First observe that the norm of $X = \ell^n_2$ ($p \geq 2$) has the assumed property; see e.g. [8, proof of Corollary 1.2, p. 187]. Further observe that (ii) immediately implies (i).

To prove (ii), we will need that the distance function $g(x) := \text{dist}_{\| \cdot \|}(x, F)$ is semiconcave on $G := X \setminus F \subset (X, \| \cdot \|_H)$. It follows from the proof of [34, Theorem 5], where it is shown that, for each $x_0 \in G$, the function $g$ is semiconcave (in $(X, \| \cdot \|_H)$) on the ball $\{ x \in X : \|x-x_0\| < g(x_0)/2 \}$ (although [34, Theorem 5] only asserts that $g$ is locally DC). Further, no point $x_0 \in G$ is a stationary point of $g$. Indeed, let $y \in F$ be a point with $\|y-x_0\| = g(x_0)$. Since clearly $g(x_0 + t(y-x_0)) - g(x_0) = -t\|y-x_0\|$ if $0 < t < 1$, we see that $x_0$ is not a stationary point of $g$. Now choose an arbitrary linear isometry $L : (X, \| \cdot \|_H) \rightarrow \mathbb{R}^p$. Applying Theorem [34] to the function $d := g \circ L^{-1}$, we obtain a set $N \subset (0, \infty)$ with the desired properties.

Now suppose that $F$ is compact. Then we will use the fact that, by Theorem [34], $N$ can be chosen so that $N = g(C)$, where $C$ is a closed set in $G$. For each $0 < a < b < \infty$, we have that $N \cap [a,b] = g(C \cap \{ x \in X : \text{dist}_{\| \cdot \|}(x, F) \in [a,b] \})$ is compact, since $\{ x \in X : \text{dist}_{\| \cdot \|}(x, F) \in [a,b] \} \subset G$ is compact and $g$ is continuous. Therefore $N$ is closed in $(0, \infty)$. Finally, choose $\rho > 0$ such that $\|x\|_H \leq \rho$ for each $x \in X \setminus F_{\geq r}$, and observe that

$$\text{reach } F_{\geq r} = \min \left\{ \inf_{p \in F_{\geq r}, \|p\|_H \leq \rho + 1} \text{reach } (F_{\geq r}, p), \inf_{\|p\|_H > \rho + 1} \text{reach } (F_{\geq r}, p) \right\}. $$

The first infimum is positive since reach $(F_{\geq r}, \cdot)$ is continuous and positive, and $\{ p \in F_{\geq r} : \|p\|_H \leq \rho + 1 \}$ is compact. The second infimum is clearly greater or equal to 1. Thus, reach $F_{\geq r} > 0$ and the proof of (iii) is over. \qed

Remark 4.3. The property (ii) immediately implies that, if $r \in (0, \infty) \setminus N$, then $S_r(F)$ is either empty or an $(n-1)$-dimensional DC surface in $(X, \| \cdot \|)$, if we define this notion in normed spaces in a natural way (as in [35]).
The following result considerably improves (GP1) (since the exceptional set is countable and $A_r$ is dense in $S_r(F)$), but only in sufficiently smooth normed linear spaces. It seems to be new also in Euclidean spaces.

**Theorem 4.4.** Let $X$ be an $n$-dimensional normed linear space ($n \geq 2$) such that the derivative of the norm is Lipschitz on the unit sphere (e.g., $X = \ell^p_n$, $p \geq 2$). Consider on $(X, \| \cdot \|)$ an arbitrary equivalent Hilbert norm $\| \cdot \|_H$. Let $F \subset X$ be a nonempty closed set. Then, for all $r > 0$, except a countable set, the distance sphere $S_r(F)$ (considered in $(X, \| \cdot \|)$) is either empty, or there exists an $(n-1)$-dimensional semiconcave surface $A_r$ in $(X, \| \cdot \|_H)$ such that $A_r \subset S_r(F)$, $A_r$ is open and dense in $S_r(F)$ and $H^{n-1}(S_r(F) \setminus A_r) = 0$.

**Proof.** It was shown in the proof of Theorem 4.2 that $g(x) := \text{dist} \| \cdot \|(x,F)$ is locally semiconcave on $G := X \setminus F \subset (X, \| \cdot \|_H)$ and has no stationary points in $G$. (Indeed, the proof worked for arbitrary $n \in \mathbb{N}$.) Thus it is sufficient to apply Theorem 3.5. □

**Remark 4.5.** Obviously, $A_r$ is an $(n-1)$-dimensional Lipschitz manifold in $(X, \| \cdot \|)$.

5. Riemannian manifolds

Let $M$ be a smooth, complete and connected Riemannian manifold. By dist we denote the induced inner distance on $M$. Let $F$ be a nonempty closed subset of $M$, and denote by $d_F := \text{dist}(\cdot,F)$ the distance function from $F$. An $F$-segment is a unit speed geodesic path $\gamma : [0, a] \to M$ such that $\gamma(a) \in F$ and $a - t = d_F(\gamma(t))$, $t \in [0, a]$. Notice that if $p \in M \setminus F$ then there always exists at least one $F$-segment emanating from $p$. The following definition is commonly used in Riemannian geometry, see e.g. [26, §11.1] or [15]:

**Definition 5.1.** A point $p \in M \setminus F$ is a critical point of $d_F$ if for any tangent vector $v \in T_pM$ there exists an $F$-segment $\gamma$ emanating from $p$ and such that the angle formed by $v$ and $\dot{\gamma}(0)$ is not greater than $\frac{\pi}{2}$. Let $\text{Crit}(d_F)$ denote the set of all critical points of $d_F$ in $M$. A point $p \in M \setminus F$ is a regular point of $d_F$ if $p \notin \text{Crit}(d_F)$.

**Remark 5.2.** Other definitions of critical and regular points of distance functions on Riemannian manifolds appear in the literature (see, e.g., [13, p. 34] or [3, p. 55]); fortunately, they are all known to be equivalent. This will be shown for completeness in Lemma 5.5 and follows essentially from the following observation: For a point $p \in M \setminus F$, $p \notin \text{Crit}(d_F)$ if and only if there exists a tangent vector $v \in T_pM$ and $\varepsilon > 0$ such that

$$d_F(c_v(t)) \geq d_F(p) + \varepsilon t$$

for all sufficiently small $t > 0$, where $c_v$ is the geodesic curve defined on a neighbourhood of 0 such that $c_v(0) = p$ and $\dot{c}_v(0) = v$, see [15, p. 360].
Theorem 3.4 yields the following extension of Fu’s result to Riemannian manifolds.

**Theorem 5.3.** Let \( n \in \{2, 3\} \) and let \( F \) be a nonempty closed subset of a connected complete smooth \( n \)-dimensional Riemannian manifold \( M \). Then, setting \( N := d_F(\text{Crit}(d_F)) \subset (0, \infty) \), we have \( \mathcal{H}^{(n-1)/2}(N) = 0 \) and for all \( r \in d_F(M \setminus F) \setminus N \),

(i) \( S_r(F) \) is an \((n - 1)\)-dimensional Lipschitz manifold,
(ii) \( \{ p \in M : d_F(p) \geq r \} \) has locally positive reach.

If, moreover, \( F \) is compact then \( N \) is relatively closed in \((0, \infty)\), and \( \{ p \in M : d_F(r) \geq r \} \) has positive reach for all \( r \in d_F(M \setminus F) \setminus N \).

**Definition 5.4.** A function \( f \) on \( M \) is said to be **locally semiconvex** (resp. **locally semiconcave**) on an open subset \( G \subset M \) if for any chart \((U, \varphi)\) with \( U \subset G \), \( f \circ \varphi^{-1} \) is locally semiconvex (resp. locally semiconcave).

It is well known that the distance function \( d_F \) to a closed subset \( F \subset M \) is locally semiconcave on \( M \setminus F \), see [22] (cf. also [13, p. 34]).

Bangert [2, 3] studied a system \( F(M) \) of functions on \( M \) which turns out to be just the system of locally semiconvex functions (by Remark 2.10(ii) and the proofs in [22]). He showed [2] that the directional derivative \( \partial_p f(v) \) of \( f \in F(M) \) at \( p \in M \) exists in any direction \( v \in T_p M \), and defined [3] regular points of \( f \) as those points \( p \in M \) for which

\[
\exists v \in T_p M : \quad \partial_p f(v) < 0.
\]

This definition (which has in [3] a formally different, but clearly equivalent form) can be, of course, extended to functions that are locally semiconvex on an open subset of \( M \) only.

The following lemma shows that Bangert’s terminology is consistent with Definitions 2.4 and 5.1.

**Lemma 5.5.** Let \( f \) be a locally semiconvex function on an open set \( G \subset M \), \( p \in G \), and let \( \varphi : U \to \mathbb{R}^n \) be a chart about \( p \) with \( U \subset G \). Then

(i) **Condition** (5) holds if and only if \( p \notin \text{Crit}(f \circ \varphi^{-1}) \).
(ii) The set of points \( p \in G \) with property (5) (regular points of \( f \) in the sense of Bangert) is open.
(iii) If, in particular, \( f = -d_F \) for some closed subset \( F \subset M \), then a point \( p \in M \setminus F \) satisfies (5) if and only if \( p \notin \text{Crit}(d_F) \). Moreover, \( \text{Crit}(d_F) \) is a closed subset of \( M \setminus F \).

**Proof.** From the proof of [2] (3.1)Satz], it follows that

\[
\partial_p f = (f \circ \varphi^{-1})'_+(\varphi(p), \cdot) \circ (d\varphi(p)).
\]
Since $f \circ \varphi^{-1}$ is locally semiconvex, we have
\[(f \circ \varphi^{-1})'_+(\varphi(p), \cdot) = (f \circ \varphi^{-1})^0(\varphi(p), \cdot)\]
by Remark 2.10 (ii), hence,
\[\partial_p f = (f \circ \varphi^{-1})^0(\varphi(p), \cdot) \circ (d\varphi(p)).\]
Assertion (i) follows then directly from the definitions. Statement (ii) follows from the fact that $\varphi(U) \setminus \text{Crit}(f \circ \varphi^{-1})$ is open for any chart $\varphi$, and each $\varphi$ is a homeomorphism. Statement (iii) follows from Remark 5.2 and (ii).

In the proof of Theorem 5.3 we use the following result due to Bangert ([3, Theorem]).

*(Ban)* Let $f$ be locally semiconvex on $M$ and $r \in \mathbb{R}$ be such that every point $p \in f^{-1}(r)$ is a regular point of $f$. Then \( \{ p \in M : f(p) \leq r \} \) has locally positive reach.

(In fact, Bangert showed a stronger result in [3], namely that a weaker regularity condition is equivalent to the property of locally positive reach.)

We shall also use the fact that each chart $\varphi : U \rightarrow \mathbb{R}^n$ of a smooth Riemannian manifold is locally bilipschitz (with respect to the induced inner metric on $M$). (See, e.g., the proof of [26, Theorem 3.4].)

**Proof of Theorem 5.3.** Recall that $dF$ is locally semiconcave on $M \setminus F$. Take a countable atlas $(U_i, \varphi_i)$ of $M \setminus F$ and notice that $N = \bigcup_i N_i$ with
\[N_i := \text{cv}(dF \circ \varphi_i^{-1}) = dF \circ \varphi_i^{-1}(\text{Crit}(dF \circ \varphi_i^{-1})), \quad i \in \mathbb{N},\]
by Lemma 5.5 (i) and (iii). Further, $dF \circ \varphi_i^{-1}$ has no stationary point. Indeed, for any $p \in U_i$ there exists a unit direction $v \in T_pM$ with directional derivative $\partial_p dF(v) = -1$ (take $v = \dot{\gamma}(0)$ for an $F$-segment $\gamma$ emanating from $p$), and notice that $(dF \circ \varphi_i^{-1})'(\varphi_i(p), d\varphi_i(p)v) = -1$, hence, $\varphi_i(p)$ cannot be a stationary point of $dF \circ \varphi_i^{-1}$. Hence, $\mathcal{H}^{(n-1)/2}(N) = 0$ by Proposition 3.3. Since $M$ is complete, it is boundedly compact by the Hopf-Rinow theorem [26, Theorem 7.1] and, hence, if $F$ is compact then, by the continuity of $dF$ and Lemma 5.5 (iii), $\text{Crit}(dF) \cap (dF)^{-1}[a, b]$ is compact for any $0 < a < b < \infty$. Hence, using the continuity of $dF$ again,
\[N \cap [a, b] = dF(\text{Crit}(dF) \cap (dF)^{-1}[a, b])\]
is compact for any $0 < a < b < \infty$. Thus, $N$ is closed in $(0, \infty)$.

We shall verify now (i) and (ii) for $r \in dF(M \setminus F) \setminus N$. By Theorem 3.3, for $r \in dF(M \setminus F) \setminus N$ and for each $i$,
\[\varphi_i(S_r(F) \cap U_i) = (dF \circ \varphi_i^{-1})^{-1}(r)\]
is either empty or an $(n - 1)$-dimensional semiconcave manifold. As $\varphi_i$ is locally bilipschitz, (i) follows.
To show (ii), note that if \( r \in d_F(M \setminus F) \setminus N \) then all points of \( S_r(F) \) are regular points of \(-d_F\) (in the sense of Bangert). Consider any connected component \( M' \) of \( M \setminus F \) and let \( f \) be the restriction of \(-d_F\) to \( M' \). As \( f \) is locally semiconvex, Bangert’s result (Ban) cited above implies that \( \{ p \in M' : f(p) \leq r \} = \{ p \in M' : d_F(p) \geq r \} \) has locally positive reach in \( M' \). It follows easily that \( \{ p \in M : d_F(p) \geq r \} \) has locally positive reach in \( M \) as well.

Let \( F \subset M \) be compact. Denoting \( F_{\geq r} := \{ p \in M : d_F(p) \geq r \} \) for brevity, we have

\[
\text{reach } F_{\geq r} = \min \left\{ \inf_{r \leq d_F(p) \leq 2r} \text{reach}(F_{\geq r}, p), \inf_{d_F(p) > 2r} \text{reach}(F_{\geq r}, p) \right\}.
\]

The first infimum is positive since \( \text{reach}(F_{\geq r}, \cdot) \) is continuous and positive, and \( \{ p : r \leq d_F(p) \leq 2r \} \) is compact. The second infimum is clearly greater or equal to \( r \). Thus, \( \text{reach } F_{\geq r} > 0 \).

**Remark 5.6.** The sets \( S_r(F) \), for \( r \in d_F(M \setminus F) \setminus N \), are rather regular Lipschitz manifolds. Indeed, our proof gives that they are “semiconcave surfaces” in the sense that, for each chart \((U, \varphi)\) on \( M \), the image of \( S_r(F) \cap U \) under \( \varphi \) is either empty, or a semiconcave surface in \( \mathbb{R}^n \).

Finally, we apply Theorem 3.5 to Riemannian manifolds (of arbitrary dimension).

**Theorem 5.7.** Let \( F \) be a nonempty closed subset of a connected complete smooth \( n \)-dimensional Riemannian manifold \( M \) with \( n \geq 2 \). Then, for all \( r \in d_F(M \setminus F) \), up to a countable set, the set \( A_r := S_r(F) \setminus \text{Crit}(d_F) \) is an \((n - 1)\)-dimensional Lipschitz manifold which is open and dense in \( S_r(F) \) and \( \mathcal{H}^{n-1}(S_r(F) \setminus A_r) = 0 \).

**Proof.** Let \((U, \varphi)\) be any chart in \( M \setminus F \). Applying Theorem 3.5 to the locally semiconcave function \( d_F \circ \varphi^{-1} \), we obtain a countable set \( N^\varphi \subset d_F(U) \) such that whenever \( r \in d_F(U) \setminus N^\varphi \), then

\[
B_r^\varphi := (d_F \circ \varphi^{-1})^{-1}(r) \setminus \text{Crit}(d_F \circ \varphi^{-1})
\]

is an \((n - 1)\)-dimensional semiconcave surface which is open dense in \((d_F \circ \varphi^{-1})^{-1}(r) = \varphi(S_r(F) \cap U) \) and fulfills \( \mathcal{H}^{n-1}(\varphi(S_r(F) \cap U) \setminus B_r^\varphi) = 0 \). Since \( \varphi \) is locally bilipschitz, \( A_r^\varphi := \varphi^{-1}(B_r^\varphi) = A_r \cap U \) is an \((n - 1)\)-dimensional Lipschitz manifold, it is open dense in \( S_r(F) \cap U \) and \( \mathcal{H}^{n-1}(S_r(F) \cap U \setminus A_r^\varphi) = 0 \). Considering a countable atlas of \( M \setminus F \), the proof is finished in a standard way. \(\square\)

6. **Alexandrov spaces**

Let \( M \) be an \( n \)-dimensional Alexandrov space \((n \geq 2)\) with lower curvature bound (i.e., \( M \) is a complete, locally compact length space with lower curvature
bound in the sense of Alexandrov, and with finite Hausdorff dimension $n$, see [4, Chapter 10]).

A point $p \in M$ is called regular if the space of directions at $p$, $\Sigma_p(M)$, is isometric to the unit sphere $S^{n-1}$. Otherwise, $p \in M$ is called singular; we denote by $S_M$ the set of all singular points of $M$. The set of singular points has Hausdorff dimension at most $n-1$ and if $X$ has no boundary, then $\dim_H S_M \leq n-2$ (see [5, §10.6]). If $n = 2$ and $M$ has no boundary then $S_M$ is even countable (see [20, Lemma 1.3]).

Perelman [25] introduced the set $M^* \subset M$ of all points $p \in M$ such that there exist $\xi_1, \ldots, \xi_{n+1} \in \Sigma_p(M)$ with $\angle(\xi_i, \xi_j) > \pi/2$ for any $1 \leq i < j \leq n+1$. We shall call the points of $M^*$ “Perelman regular”, and the remaining points in $M$ “Perelman singular”. It is well-known (and easy to see) that any regular point is Perelman regular as well. Thus, $M \setminus M^*$ is countable if $n = 2$ and $M$ has no boundary. Further, $M^*$ is a dense, open and convex subset of $M$ (see [25, the end of §3]). Perelman introduced and applied a “DC structure” on $M^*$. We will need only the following fact about it (see [25, p. 3, l. 14-15, and Proposition (C)]).

(Per) For any $p \in M^*$ there exists an open neighbourhood $U$ of $p$ in $M$ and a bilipschitz mapping $\varphi : U \to \mathbb{R}^n$ such that $\varphi(U)$ is open and, if $f$ is a function on $U$ that is semiconcave in the intrinsic sense, then $f \circ \varphi^{-1}$ is locally DC on $\varphi(U)$.

Following [18, §2.7], we shall call the pair $(U, \varphi)$ from (Per) a DC local chart. Note that semiconcavity in the intrinsic sense is defined by means of semiconcavity along geodesic paths, see [27, Definition 124] for a precise definition.

The proof of (Per) is contained only in the unpublished preprint [25], but its validity is adopted and used by experts in the theory of Alexandrov spaces (cf., e.g., [18]).

As in the previous chapters, we shall use the notation $d_F$ for the distance function from a closed set $F \subset M$.

**Theorem 6.1.** Let $n \in \{2, 3\}$ and let $M$ be an $n$-dimensional Alexandrov space with lower curvature bound and $F$ a closed subset of $M$. Then, the following hold.

(i) There exists a set $N \subset (0, \infty)$ with $\mathcal{H}^{(n-1)/2}(N) = 0$ such that for all $r \in d_F(M^* \setminus F) \setminus N$, $S_r(F) \cap M^*$ is an $(n-1)$-dimensional Lipschitz manifold.

(ii) If, moreover, $\mathcal{H}^{(n-1)/2}(M \setminus M^*) = 0$ then there exists a set $N' \subset (0, \infty)$ with $\mathcal{H}^{(n-1)/2}(N') = 0$ such that for all $r \in d_F(M \setminus F) \setminus N'$, $S_r(F)$ is an $(n-1)$-dimensional Lipschitz manifold. If, in addition, $F$ is compact then $N'$ can be chosen to be relatively closed in $(0, \infty)$. 
Corollary 6.2. Let $M$ be a two-dimensional Alexandrov space with lower curvature bound and without boundary, and let $F$ be a compact subset of $M$. Then there exists a relatively closed subset $N$ of $(0, \infty)$ with $\mathcal{H}^{1/2}(N) = 0$ such that for all $r \in d_F(M \setminus F) \setminus N$, $S_r(F)$ is a one-dimensional Lipschitz manifold.

Remark 6.3. Corollary 6.2 improves partially Shiohama’s and Tanaka’s result [31, Theorem B], where the exceptional set is of one-dimensional measure zero and need not be closed.

Proof. (i) Let $(U, \varphi)$ be a DC local chart in $M^*$. Since the distance function $d_F$ is semiconcave on $M \setminus F$ in the intrinsic sense (see [27, Proposition 125]), the composed mapping $d_F \circ \varphi^{-1}$ is locally DC on $\varphi(U) \subset \mathbb{R}^n$ by (Per).

We shall show that $d_F \circ \varphi^{-1}$ has no stationary point. Take a point $p \in U$ and notice that, since $M$ is complete and boundedly compact, there exists at least one $F$-segment emanating from $p$, i.e., a unit-speed geodesic path $\gamma : [0, a] \to M$ such that $\gamma(0) = p, \gamma(a) \in F$ and $a - t = d_F(\gamma(t)), t \in [0, a]$. Then, denoting $x_t := \varphi(\gamma(t))$, we have

$$-t = d_F(\gamma(t)) - a = d_F \circ \varphi^{-1}(x_t) - d_F \circ \varphi^{-1}(x_0).$$

Since $|x_t - x_0| \leq ct$, where $c > 0$ is the Lipschitz constant of $\varphi$, we get

$$|d_F \circ \varphi^{-1}(x_t) - d_F \circ \varphi^{-1}(x_0)| \geq c^{-1}|x_t - x_0|,$$

which shows that $x_0$ cannot be a stationary point of $d_F \circ \varphi^{-1}$, as $x_t \to x_0$ ($t \to 0+$).

Thus, we can apply Theorem 3.4 and find a set $N \subset (0, \infty)$ with $\mathcal{H}^{(n-1)/2}(N) = 0$ and such that for all $r \in d_F(U) \setminus N, \varphi(d_F^{-1}(r))$ is an $(n-1)$-dimensional DC surface. Since $\varphi$ is bilipschitz, $S_r(F) \cap U = \varphi^{-1}(\varphi(d_F^{-1}(r)))$ is an $(n-1)$-dimensional Lipschitz manifold. Using the separability of $M^*$, we can find a countable family $(U_i, \varphi_i)$ of DC local charts such that $\bigcup_i U_i = M^* \setminus F$, apply the above procedure to each of these charts and find a common exceptional set $N \subset (0, \infty)$ with $\mathcal{H}^{(n-1)/2}(N) = 0$ and such that for all $r \in d_F(M^* \setminus F) \setminus N$, $S_r(F) \cap M^*$ is an $(n-1)$-dimensional Lipschitz manifold.

(ii) If $\mathcal{H}^{(n-1)/2}(M \setminus M^*) = 0$ then, since $d_F$ is Lipschitz, we have

$$\mathcal{H}^{(n-1)/2}(d_F(M \setminus M^*)) = 0$$

as well. Hence, enlarging the exceptional set $N$ to $N' := N \cup d_F(M \setminus M^*)$, we obtain that $\mathcal{H}^{(n-1)/2}(N') = 0$ and the level set $S_r(F)$ itself is an $(n-1)$-dimensional Lipschitz manifold for $r \in d_F(M \setminus F) \setminus N'$.

Let now $F$ be compact, in addition, and let $(U_i, \varphi_i)$ be the countable atlas of DC local charts covering $M^* \setminus F$, as above. It follows from Theorem 3.4 that we can take for the exceptional set $N' := d_F(Q)$, where

$$Q = (M \setminus M^*) \cup \{p \in M^* \setminus F : \forall i, p \in U_i \implies \varphi_i(p) \in \text{Crit}(d_F \circ \varphi_i^{-1})\}.$$
Using that $M^*$ is open, each $\text{Crit}(d_F \circ \phi_i^{-1})$ is closed in $\phi_i(U_i)$ and $\phi_i$ are homeomorphisms, we get that the set $Q$ is closed in $M \setminus F$. If $0 < a < b < \infty$, the set $d_F^{-1}[a,b]$ is compact (since $d_F$ is continuous and $M$ is boundedly compact, see [4, §10.8]) and, hence,

$$N' \cap [a,b] = d_F(Q \cap d_F^{-1}[a,b])$$

is compact as well. Consequently, $N'$ is closed in $(0, \infty)$. □

Remark 6.4. It is easy to see that in a general (possibly with boundary) three-dimensional (and even two-dimensional) Alexandrov space $M$ with lower curvature bound, Ferry’s result (almost all distance spheres are topological manifolds) does not hold (for $M$ we can take a closed ball in $\mathbb{R}^2$ or $\mathbb{R}^3$). However, we do not know whether Ferry’s result holds in each three-dimensional Alexandrov space with lower curvature bound and without boundary. In this case our method cannot be used since there exists a three-dimensional convex surface $X$ in $\mathbb{R}^4$ for which $\mathcal{H}^4(X \setminus X^*) > 0$ (see Example 6.5).

However, Ferry’s result holds in every three-dimensional complete convex surface $X$ in $\mathbb{R}^4$; it is proved in [29] without using Perelman’s DC structure.

Example 6.5. We shall demonstrate on a particular example that all points on one-dimensional “sufficiently sharp” edges of three-dimensional convex surfaces are Perelman singular. Let $A = \{(x, y, z) \in \mathbb{R}^3 : \alpha^2(x^2 + y^2) = z^2, z \geq 0\}$ with $\alpha \geq \sqrt{2\pi^2 - 1}$ and consider the convex cone $C = A \times \mathbb{R}$ in $\mathbb{R}^4$. Then, for the convex surface $X = \partial C$, any point of the edge $\{(0, 0, 0)\} \times \mathbb{R}$ is Perelman singular. Of course, it suffices to show that the origin $0$ is Perelman singular. There even do not exist three directions of $X$ at $0$ forming obtuse angles each with other. To see this, let $\xi_1, \xi_2, \xi_3$ be three non-zero vectors from $X$ determining three directions of $X$ at $0$. Multiplying by positive factors, we can suppose that these vectors can be written as

$$\xi_i = (r \cos \vartheta_i, r \sin \vartheta_i, \alpha r, s_i), \quad i = 1, 2, 3,$$

where $r \geq 0$, $\vartheta_i \in [0, 2\pi)$ and $s_i \in \mathbb{R}$, $i = 1, 2, 3$. At least two of the three numbers $s_i$ must have nonnegative product, so assume without loss of generality that $s_1, s_2 \geq 0$. We shall show that the angle formed by the directions of $\xi_1$ and $\xi_2$ on $X$ is not obtuse. Since $C$ is a cone, the angle can be obtained as

$$\angle(\xi_1, \xi_2) = \arccos \frac{||\xi_1||^2 + ||\xi_2||^2 - \text{dist}^2(\xi_1, \xi_2)}{2||\xi_1||||\xi_2||},$$

cf. [4, §3.6.5] (dist is the intrinsic distance in $X$). The points $\xi_1$ and $\xi_2$ can be connected by the following path on $X$

$$\gamma : t \mapsto \left( r \cos(\vartheta_1 + t(\vartheta_2 - \vartheta_1)), r \sin(\vartheta_1 + t(\vartheta_2 - \vartheta_1)), \alpha r, s_1 + t(s_2 - s_1) \right), \quad t \in [0, 1]$$
of length
\[ \text{length} \gamma = \int_0^1 \| \gamma'(t) \| \, dt = \sqrt{(\vartheta_2 - \vartheta_1)^2 r^2 + (s_2 - s_1)^2}. \]

Hence,
\[ \text{dist}^2(\xi_1, \xi_2) \leq (\text{length} \gamma)^2 \leq 4\pi^2 r^2 + s_1^2 + s_2^2 \]
which is less or equal to
\[ \| \xi_1 \|^2 + \| \xi_2 \|^2 = 2(1 + \alpha^2)r^2 + s_1^2 + s_2^2 \]
since \( \alpha^2 \geq 2\pi^2 - 1 \). Hence, the angle formed by \( \xi_1 \) and \( \xi_2 \) is not obtuse.

We finish this section by an application of Theorem 3.5 to Alexandrov spaces of any dimension.

**Theorem 6.6.** Let \( M \) be an \( n \)-dimensional \((n \geq 2)\) Alexandrov space with lower curvature bound and without boundary, and let \( F \subset M \) be closed. Then, for all \( r \in d_F(M \setminus F) \) except a countable set, either \( \mathcal{H}^{n-1}(S_r(F)) = 0 \), or there exists an \((n-1)\)-dimensional Lipschitz manifold \( A_r \subset S_r(F) \) which is open in \( S_r(F) \) and \( \mathcal{H}^{n-1}(S_r(F) \setminus A_r) = 0 \) holds.

**Remark 6.7.** If, in addition, \( M = M^* \) then the manifolds \( A_r \) in Theorem 6.6 can be found so that there are moreover dense in \( S_r(F) \).

**Proof.** Let \((U_i, \varphi_i)\) be a countable atlas of DC local charts covering \( M^* \setminus F \), as in the proof of Theorem 6.1. Applying Theorem 3.5 to \( d_F \circ \varphi_i^{-1} \) (the validity of assumptions was shown in the proof of Theorem 6.1) and the bilipschitz property of \( \varphi_i \), we obtain countable sets \( N_i \subset d_F(U_i) \) such that for all \( i \) and all \( r \in d_F(U_i) \setminus N_i \),
\[ P_i := (S_r(F) \cap U_i) \setminus \varphi_i^{-1}(\text{Crit}(d_F \circ \varphi_i^{-1})) \]
is an \((n-1)\)-dimensional Lipschitz manifold with \( \mathcal{H}^{n-1}((S_r(F) \cap U_i) \setminus P_i) = 0 \). Set \( N := \bigcup_i N_i \) and for \( r \in d_F(M^* \setminus F) \setminus N \),
\[ A_r := \{ p \in S_r(F) \cap M^* : \exists \delta > 0, S_r(F) \cap B(p, \delta) \text{ is an } (n-1)\text{-dimensional Lipschitz manifold} \}. \]

Clearly, \( A_r \) is an \((n-1)\)-dimensional Lipschitz manifold open in \( S_r(F) \). Note that \( \bigcup_i P_i \subset A_r \) and recall that \( \dim_H(M \setminus M^*) \leq \dim_H(S_M) \leq n - 2 \). Hence,
\[ \mathcal{H}^{n-1}(S_r(F) \setminus A_r) \leq \mathcal{H}^{n-1}(M \setminus M^*) + \mathcal{H}^{n-1}((S_r(F) \cap M^*) \setminus \bigcup_i P_i) = 0 \]
for all \( r \in d_F(M^* \setminus F) \setminus N \), as required. If \( r \in d_F(M \setminus F) \setminus d_F(M^* \setminus F) \) then \( \mathcal{H}^{n-1}(S_r(F)) \leq \mathcal{H}^{n-1}(M \setminus M^*) = 0. \) \( \square \)
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