Quantum Hele-Shaw flow

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Abstract. In this note, we discuss the quantum Hele-Shaw flow, a random measure process in the complex plane introduced by the physicists P. Wiegmann, A. Zabrodin, et al. This process arises in the theory of electronic droplets confined to a plane under a strong magnetic field, as well as in the theory of random normal matrices. We extend a result of Elbau and Felder to general external field potentials, and also show that if the potential is $C^2$-smooth, then the quantum Hele-Shaw flow converges, under appropriate scaling, to the classical (weighted) Hele-Shaw flow, which can be modeled in terms of an obstacle problem.

1. Introduction

A weighted distribution of $N$-tuples of complex numbers. The intention of this paper is to provide a rigorous mathematical treatment of the theoretical physics model due to Wiegmann, Zabrodin, et al., which is concerned with the behavior of a finite number of electrons (charged fermions) in a strong magnetic field in the context of Quantum Theory. In part, this was done very recently by Elbau and Felder. We will say more about this below.

Let $N$ be a positive integer, and introduce the notation

$$z_{[1,N]} = (z_1, z_2, z_3, \ldots, z_N)$$

for a point in $N$-dimensional complex Euclidean space $\mathbb{C}^N$. As a real vector space, we have the identification $\mathbb{C}^N \cong \mathbb{R}^{2N}$. We let $\lambda_{2N}$ be the standard Lebesgue measure on $\mathbb{C}^N$, so that in particular,

$$d\lambda_{2N}(z_{[1,N]}) = d\lambda_2(z_1) \cdots d\lambda_2(z_N).$$

The van der Monde determinant is the quantity

$$\Delta(z_{[1,N]}) = \prod_{j,k:j<k} (z_j - z_k),$$

where we tacitly assume that the parameters $j$ and $k$ range over the set \{1, $\ldots$, $N$\}. Let $\Phi : \mathbb{C} \to \mathbb{R}$ be a $C^2$-smooth function, which grows so quickly that

$$\Phi(z) \geq A \log |z| + O(1)$$

for some positive constant $A$. We will assume that $\Phi$ is $C^2$-smooth and grows rapidly as $|z| \to \infty$. We will also assume that $\Phi$ is real-valued for $z \in \mathbb{C}$.

(1.1) Hakan Hedenmalm and Nikolai Makarov
holds for all positive real constants $A$ (with $O(1)$ dependent on $A$). Let $\beta$ denote a positive real parameter. We let $\mu_{\beta, \Phi}$ be the finite positive Borel measure on $\mathbb{C}$ given by

$$d\mu_{\beta, \Phi}(z) = \exp\{-\beta \Phi(z)\} d\lambda_2(z), \quad z \in \mathbb{C},$$

where $\lambda_2(z)$ is the Lebesgue measure on $\mathbb{C}$. We shall consider the probability measure

$$d\Pi_{N, \beta, \Phi}(z_{[1,N]}) = \frac{1}{Z_N} |\triangle(z_{[1,N]})|^2 d\mu_{\beta, \Phi}(z_1) \cdots d\mu_{\beta, \Phi}(z_N),$$

where $Z_N$ is the normalizing constant:

$$Z_N = \int_{\mathbb{C}^N} |\triangle(z_{[1,N]})|^2 d\mu_{\beta, \Phi}(z_1) \cdots d\mu_{\beta, \Phi}(z_N).$$

We intend to study the typical behavior of (1.3) for large values of $N$. To make this more precise, let $n$ be a positive integer with $1 \leq n \leq N$, and consider the $n$-point marginal distribution

$$d\Pi_{N, \beta, \Phi}^{(n)}(z_{[1,n]}) = \frac{1}{Z_N} \left\{ \int_{\mathbb{C}^{N-n}} |\triangle(z_{[1,N]})|^2 d\mu_{\beta, \Phi}(z_{n+1}) \cdots d\mu_{\beta, \Phi}(z_N) \right\}$$

$$\times d\mu_{\beta, \Phi}(z_1) \cdots d\mu_{\beta, \Phi}(z_n).$$

The measure $\Gamma_{N, \beta, \Phi}^{(n)}$ obtained by multiplying $\Pi_{N, \beta, \Phi}^{(n)}$ by $N!/(N-n)!$ is called the $n$-point correlation measure. The distribution (1.3) appears from at least two different contexts: one is the theory of eigenvalues of random normal matrices, and the other a physical configuration of $N$ electrons localized to a two-dimensional plane, while being exposed to a strong magnetic field (the strength is regulated by the parameter $\beta$) perpendicular to the given plane. In the second instance, $\beta \Phi$ is a scalar magnetic potential, $z_1, \ldots, z_N$ are the locations of the $N$ electrons, and one looks for time-independent solutions to the Schrödinger equation. The resulting cloud of electrons is an instance of the Aharonov-Bohm effect, where the magnetic potential rather than the magnetic field manifests itself. For details, we ask the reader to consult the work of Wiegmann, Zabrodin, et al. [1], [15], [17], [16], and [21]. In the first instance (random normal matrices), we should mention first that normal $N \times N$ matrices have the decomposition $M = U^*DU$, where $D$ is diagonal (consisting of the eigenvalues), and $U$ is unitary, that is, $U^*U = UU^* = I$, where $I$ is the identity matrix. To focus on the eigenvalues, then, one should remove as much as possible of the unitary part; in other words, we should mod out with respect to $\mathcal{U}(N)/\mathcal{U}_d(N)$, where $\mathcal{U}(N)$ is the group of unitary $N \times N$ matrices, and $\mathcal{U}_d(N)$ is the subgroup of unitary diagonal matrices. If this is done carefully, and a weight is introduced to get a probability measure on the normal matrices,
we get a joint distribution of the eigenvalues of the type (1.3). For details of this, see [6] and [4]; see also Mehta’s book [18].

The standard models for the eigenvalue distribution of self-adjoint matrices are by now quite well-known (see [18], [13]). In the limit, we get such models from our probability measure (1.3) by letting $\Phi(z)$ tend to $+\infty$ on $\mathbb{C} \setminus \mathbb{R}$. For instance, the semicircle law of Wigner can be obtained by taking such a limit with the appropriate interpretation.

**Quantum Hele-Shaw flow vs Hele-Shaw flow.** We show that the $n$-point correlation measure

$$\Gamma^{(n)}_{N,\beta,\Phi} = \frac{N!}{(N-n)!} \Pi^{(n)}_{N,\beta,\Phi}$$

is given in terms of a determinantal process involving reproducing kernels of polynomial subspaces of a Bargmann-Fock type space, and that it defines a monotonic growth process in $N$ (for fixed $\beta$ and $\Phi$):

$$\Gamma^{(n)}_{N,\beta,\Phi} \leq \Gamma^{(n)}_{N+1,\beta,\Phi}, \quad N = 1, 2, 3, \ldots$$

For this reason, and for reasons which will be made clearer below, we call this Quantum Hele-Shaw flow. The analogy with classical Hele-Shaw flow was made earlier by Wiegmann and Zabrodin. The ratio $N/\beta$ seems like a natural growth parameter for the process. For mathematical reasons, however, it is more natural to use

$$\tau = \frac{N - 1}{\beta}$$

instead.

The weighted energy of a compactly supported Borel probability measure $\sigma$ is (for a given positive parameter $\tau$)

$$\mathcal{E}_{\Phi}[\sigma]\left(\frac{1}{\tau}\right) = \frac{1}{2\tau} \int_{\mathbb{C}^2} \left\{ 2\tau \log \frac{1}{|z-w|} + \Phi(z) + \Phi(w) \right\} d\sigma(z)d\sigma(w);$$

the unique compactly supported Borel probability measure which minimizes this energy is called the extremal measure, or equilibrium measure, and denoted by $\hat{\sigma}_{\Phi,\tau}$, or simply by $\hat{\sigma}$, if no misunderstanding is possible. The existence and uniqueness of the extremal measure are treated in Chapter 1 of Saff’s and Totik’s book [19].

Our first main result reads as follows.

**Theorem 1.1.** If $\tau$, $0 < \tau < +\infty$, is kept fixed, and $\beta$, $N$ are connected via (1.6), then, as $N \to +\infty$, the marginal distribution has the weak-star limit

$$d\Pi^{(n)}_{N,\beta,\Phi}(z_{[1,n]}) \to d\hat{\sigma}(z_1) \cdots d\hat{\sigma}(z_n).$$

This theorem is essentially as in Johansson’s paper [13], but with important additional technical details. Elbau and Felder [6] were the first to realize that Johansson’s argument extends to the case of normal random matrices. The present
version is more general – Elbau and Felder required that the potential be infinite in a neighborhood of infinity.

We discuss the extremal measure in Section 3 and connect it to an obstacle problem in Section 4. This leads to our second main theorem. For the formulation, we need the following standard notation: $\Delta$ is the standard Laplacian in the plane, while given a subset $E$ of the complex plane $\mathbb{C}$, $1_E$ is the characteristic function of the set $E$.

**Theorem 1.2.** For each $\tau$, $0 < \tau < +\infty$, there exists an unbounded open set $\mathcal{D}_\Phi(\tau)$ in $\mathbb{C}$ whose complement is compact, such that the extremal measure has the form

$$d\hat{\sigma}_{\Phi, \tau}(z) = \frac{1}{4\pi \tau} 1_{\mathbb{C}\setminus\mathcal{D}_\Phi(\tau)}(z) \Delta \Phi(z) \, d\lambda_2(z), \quad z \in \mathbb{C}.$$ 

Moreover, as $\tau$ increases, the open sets $\mathcal{D}_\Phi(\tau)$ form a continuous decreasing chain, and the measures $\tau \hat{\sigma}_{\Phi, \tau}$ grow with $\tau$.

This theorem justifies the claim of Wiegmann, Zabrodin, et al., that in the limit one gets a monotone family of domains (and not only measures) in the complex plane. The proof is an application of the general theory of free boundary problems due to Caffarelli et al.

We should add that, if the weight function $\Phi$ is real analytic, then the boundary $\partial \mathcal{D}_\Phi(\tau)$ consists of real analytic curves, except for finitely many cusps (pointing outward) or so-called contact points. The evolution of $\mathcal{D}_\Phi(\tau)$ is quite similar to the Hele-Shaw flow on hyperbolic or weakly hyperbolic surfaces described in [12] and [11]. In fact, the present situation corresponds to Hele-Shaw flow on surfaces where the metric may be negative in portions of space, since $\Phi$ is a kind of metric potential, with associated metric

$$ds(z)^2 = \Delta \Phi(z) |dz|^2.$$ 

Generally speaking, Hele-Shaw flow models the expansion of viscous fluid in a surrounding medium (which we may think of as vacuum), as new fluid is injected at a constant rate at one or several source points. In the present setting, we have a single source point at infinity.

A final remark we wish to make here is that essentially all results of the paper remain valid with little change if the $C^2$-smoothness of $\Phi$ is weakened to local $W^{2,p}$-smoothness, for some $p$, $1 < p < +\infty$.

2. Quantum Hele-Shaw flow and Bargmann-Fock type spaces

**Weighted Bargmann-Fock spaces and reproducing kernels.** Let $\mu$ be a positive finite Borel measure on the complex plane $\mathbb{C}$, with

$$0 < \int_{\mathbb{C}} |z|^{2n} \, d\mu(z) < +\infty, \quad n = 0, 1, 2, 3, \ldots.$$ 

The Hilbert space $L^2(\mathbb{C}, \mu)$ consists of the square-integrable complex-valued functions with respect to $\mu$, where two functions are identified if the coincide $\mu$-almost
Quantum Hele-Shaw flow

everywhere. We shall assume that \( \mu \) is supported on an infinite set, so that \( L^2(\mathbb{C}, \mu) \) becomes infinite-dimensional. The above integrability condition ensures that all (analytic) polynomials belong to \( L^2(\mathbb{C}, \mu) \). Let \( P^2(\mathbb{C}, \mu) \) denote the closure of the linear space of polynomials in \( L^2(\mathbb{C}, \mu) \). Also, for \( N = 1, 2, 3, \ldots \), let \( P^2_N(\mathbb{C}, \mu) \) be the \( N \)-dimensional linear space of polynomials of degree \( \leq N - 1 \), supplied with the Hilbert space structure of \( L^2(\mathbb{C}, \mu) \). The point evaluations in \( \mathbb{C} \) are linear functionals on \( P^2_N(\mathbb{C}, \mu) \), and thus given by the inner product with an element of the space:

\[
p(z) = \langle p, K_{\mu,N}(\cdot, z) \rangle_{L^2(\mu)} = \int_{\mathbb{C}} K_{\mu,N}(w, z) p(w) d\mu(w), \quad z \in \mathbb{C},
\]

where, for fixed \( z \in \mathbb{C} \), the function \( K_{\mu,N}(\cdot, z) \) is an element of \( P^2_N(\mathbb{C}, \mu) \), and so is \( p \). The function \( K_{\mu,N} \) is called the reproducing kernel for the space \( P^2_N(\mathbb{C}, \mu) \).

The monomials

\[
e_j(z) = z^{j-1}, \quad j = 1, 2, \ldots, N,
\]

form a vector space basis in \( P^2_N(\mathbb{C}, \mu) \), but they need not be orthogonal to one another. To rectify this, we may apply the Gram-Schmidt process: let \( \varphi_1 = e_1 \), and for \( k = 2, 3, \ldots, N \), put

\[
\varphi_k = e_k + \sum_{j=1}^{k-1} \lambda_{j,k} e_j,
\]

where the complex scalars \( \lambda_{j,k} \) are chosen so that \( \varphi_k \) becomes perpendicular to all the vectors \( e_1, e_2, \ldots, e_{k-1} \). The functions \( \varphi_1, \varphi_2, \varphi_3, \ldots \) obtained in this fashion form an orthogonal sequence. To get an orthonormal sequence, we normalize:

\[
\phi_j(z) = \frac{\varphi_j(z)}{\| \varphi_j \|_{L^2(\mu)}}, \quad z \in \mathbb{C}.
\]

The reproducing kernel may then be conveniently expressed in terms of these orthogonal functions:

\[
K_{\mu,N}(z, w) = \sum_{j=1}^{N} \phi_j(z) \overline{\phi_j(w)}, \quad (z, w) \in \mathbb{C} \times \mathbb{C}.
\]

Matrix-valued reproducing kernels and correlation measures. For two vectors \( z_{[1,n]} = (z_1, \ldots, z_n) \) and \( w_{[1,n]} = (w_1, \ldots, w_n) \), we form the \( n \times n \) matrix

\[
K_{\mu,N}^n(z_{[1,n]}, w_{[1,n]}) = [K_{\mu,N}(z_j, w_k)]_{j,k=1}^n,
\]

which we may think of as a matrix-valued reproducing kernel.

We return to the joint distribution function \( \Pi_{\mu,N} \), in the slightly altered form

\[
d \Pi_{\mu,N}(z_{[1,N]}) = \frac{1}{Z_N} | \Delta(z_{[1,N]}) |^2 d\mu(z_1) \cdots d\mu(z_N),
\]
where \( z_{[1,N]} = (z_1, \ldots, z_N) \), and \( Z_N \) is the normalizing constant:

\[
Z_N = \int_{\mathbb{C}^N} |\Delta(z_{[1,N]})|^2 \, d\mu(z_1) \cdots d\mu(z_N).
\]

We fix an integer \( n \), \( 1 \leq n \leq N \), and split \( z_{[1,N]} = (z_{[1,n]}, z_{[n,N]}) \), with \( z_{[1,n]} = (z_1, \ldots, z_n) \) and \( z_{[n,N]} = (z_{n+1}, \ldots, z_N) \). The \( n \)-point marginal distribution measure is

\[
\begin{aligned}
d\Pi_{\mu,N}(z_{[1,n]}) &= \frac{1}{Z_N} \left\{ \int_{\mathbb{C}^{N-n}} |\Delta(z_{[1,N]})|^2 \, d\mu(z_{n+1}) \cdots d\mu(z_N) \right\} \\
&\quad \times d\mu(z_1) \cdots d\mu(z_n),
\end{aligned}
\]

while the \( n \)-point correlation measure is

\[
\begin{aligned}
d\Gamma_{\mu,N}^{(n)}(z_{[1,n]}) &= \frac{N!}{(N-n)!} \, d\Pi_{\mu,N}(z_{[1,n]}), \\
z_{[1,n]} &\in \mathbb{C}^n,
\end{aligned}
\]

where the integration takes place with respect to the variables \( z_{n+1}, \ldots, z_N \) only. For \( n = 1 \), the measure \( \Gamma_{\mu,N}^{(1)} \) describes the joint density of the eigenvalues. For measures \( \mu \) of the form \( (1.2) \), with \( \Delta \Phi \) a positive constant in a large disk centered at the origin, \( \Gamma_{\mu,N}^{(1)} \) is rather similar to the majorization function of Aleman, Richter, and Sundberg [2]; one major difference, though, is that here we consider backward shift invariant subspaces \( (P^2_N(\mathbb{C}, \mu)) \), while in [2], the object of study are the forward shift invariant subspaces.

**Proposition 2.1.** We have that

\[
\det \left[ K_{\mu,N}^N(z_{[1,N]}, z_{[1,N]}) \right] = N! \frac{1}{Z_N} |\Delta(z_{[1,N]})|^2.
\]

**Proof.** We first apply simple row or column operations on the representation of the van der Monde determinant as the determinant of the \( N \times N \) matrix with entries \( e_j(z_k) \), for \( j, k = 1, \ldots, N \). The result is that the van der Monde determinant may be calculated based on the entries \( \varphi_j(z_k) \) instead, whence the result follows from the formula defining the determinant in terms of permutations, as in [18], pp. 89–95.

The following “linear algebra” description of \( \Gamma_{\mu,N}^{(n)} \) in terms of the determinant of a matrix reproducing kernel is essentially known (see [18] for the Gaussian situation).

**Theorem 2.2.** We have that

\[
d\Gamma_{\mu,N}^{(n)}(z_{[1,n]}) = \det \left[ K_{\mu,N}^n(z_{[1,n]}, z_{[1,n]}) \right] \, d\mu(z_1) \cdots d\mu(z_n).
\]
Proof. We are to show that
\[
\frac{N!}{(N-n)!} \int_{\mathbb{C}^{N-n}} |\triangle(z[1,N])|^2 \, d\mu(z_{n+1}) \cdots d\mu(z_N) = Z_N \det [K_{\mu,N}^n(z_{[1,n]},z_{[1,n]})].
\]
We follow the approach of [18], pp. 89–95, and use that by Proposition 2.1, the identity is valid for \( n = N \), which we take as our starting point. Then successive applications of Theorem 5.2.1 in [18] (suitable modified by including complex conjugates where needed) yields the result for general \( n = 1, \ldots, N \). The proof is complete.

Corollary 2.3. The \( n \)-point correlation measure \( d\Gamma^{(n)}_{\mu,N} \) increases monotonically with increasing \( N \). Its total mass equals \( \frac{N!}{(N-n)!} \).

Proof. That the total mass of \( d\Gamma^{(n)}_{\mu,N} \) is \( \frac{N!}{(N-n)!} \) follows from the definition (2.6) of the normalizing constant \( Z_N \). In view of (2.3), we have
\[
K_{\mu,N+1}(z,w) = K_{\mu,N}(z,w) + \phi_{N+1}(z) \bar{\phi}_{N+1}(w),
\]
so that
\[
K_{\mu,N+1}^n(z_{[1,n]},w_{[1,n]}) = K_{\mu,N}^n(z_{[1,n]},w_{[1,n]}) + \Phi_{\mu,N+1}(z_{[1,n]},w_{[1,n]}),
\]
where
\[
\Phi_{\mu,N+1}(z_{[1,n]},w_{[1,n]}) = (\phi_{N+1}(z_j) \bar{\phi}_{N+1}(w_k))_{j,k=1}^n.
\]
The matrix \( \Phi_{\mu,N+1}(z_{[1,n]},w_{[1,n]}) \) is positive definite, and so is \( K_{\mu,N}^n(z_{[1,n]},w_{[1,n]}) \). As the determinant operation is monotonic with respect to increasing positive definiteness (see Proposition 2.4 below), the assertion follows from Theorem 2.2. The proof is complete.

A linear algebra result. The following proposition is known; for instance, it should be possible to derive it from the Horn inequalities, which describe the possible eigenvalues of the sum of two Hermitian matrices in terms of the eigenvalues of the summands (see, e.g., the survey paper [7]). However, we find it convenient to supply instead a short direct proof.

Proposition 2.4. Let \( A \) and \( B \) be two positive semi-definite (Hermitian) \( N \times N \) matrices. Then
\[
\max \{ \det(A), \det(B) \} \leq \det(A + B).
\]

Proof. Clearly, by symmetry, it suffices to show that
\[
\det(A) \leq \det(A + B).
\]
If \( \det(A) = 0 \), the assertion follows trivially, as the determinant of a positive semi-definite matrix is always \( \geq 0 \). In the non-singular case \( \det(A) > 0 \), we note that \( A^{-1/2} \) is a well-defined positive semi-definite matrix, and that \( A^{-1/2} B A^{-1/2} \) is positive semi-definite as well. We write
\[
\det(A + B) = \det(A) \det(I + A^{-1/2} B A^{-1/2}),
\]
where \( I \) denotes the \( N \times N \) identity matrix. The assertion now follows from the fact that the eigenvalues of \( A^{-1/2} B A^{-1/2} \) are all positive, so that the eigenvalues of \( I + A^{-1/2} B A^{-1/2} \) are all greater than 1 (after all, the determinant equals the product of the eigenvalues).

\[ \]

3. Weighted Fekete points and the extremal measure

**Weighted Fekete points.** Let us analyze the probability measure (1.3), written in the form

\[ (3.1) \quad d\Pi_{N,\beta,\varphi}(\mathbf{z}_{[1,N]}) = \frac{1}{Z_N} \exp \left\{ -\beta \sum_{j=1}^{N} \varphi(z_j) \right\} |\triangle(\mathbf{z}_{[1,N]})|^2 \, d\lambda_{2N}(\mathbf{z}_{[1,N]}), \]

where \( \mathbf{z}_{[1,N]} = (z_1, \ldots, z_N) \), and \( Z_N \) is the normalization constant. The most likely configuration \( \mathbf{z}_{[1,N]} \) is the one that maximizes

\[ \exp \left\{ -\beta \sum_{j=1}^{N} \varphi(z_j) \right\} |\triangle(\mathbf{z}_{[1,N]})|^2 \]

among all vectors in \( \mathbb{C}^N \). Considering that

\[ \sum_{j,k: j<k} \left[ \varphi(z_j) + \varphi(z_k) \right] = (N-1) \sum_{j=1}^{N} \varphi(z_j), \]

we may rewrite this expression:

\[ (3.2) \quad \exp \left\{ -\beta \sum_{j=1}^{N} \varphi(z_j) \right\} |\triangle(\mathbf{z}_{[1,N]})|^2 = \prod_{j,k: j<k} \left\{ \exp \left( -\frac{\beta}{N-1} \left[ \varphi(z_j) + \varphi(z_k) \right] \right) |z_j - z_k|^2 \right\}. \]

For positive real \( \theta \), let \( E_{\Phi}(z, w; \theta) \) denote the function

\[ (3.3) \quad E_{\Phi}(z, w; \theta) = |z - w| \exp \left( -\theta \left[ \varphi(z) + \varphi(w) \right] \right); \]

we are interested in maximizing

\[ \prod_{j,k: j<k} E_{\Phi}(z_j, z_k; \theta), \]

with

\[ \theta = \frac{\beta}{2(N-1)}. \]

Let \( M_{\Phi,N}(\theta) \) be this maximum raised to the power \( 2/(N(N-1)) \):

\[ M_{\Phi,N}(\theta) = \sup_{\mathbf{z}_{[1,N]} \in \mathbb{C}^N} \left\{ \prod_{j,k: j<k} E_{\Phi}(z_j, z_k; \theta) \right\}^{2/(N(N-1))}. \]
The points \( z_{[1,N]}^* = z_{[1,N]}^* \) which achieve this maximum are called weighted Fekete points (see [19]). In order that the maximum be assumed, we should add an assumption on the weight \( \Phi \):

\[
E(z, w; \theta) \to 0 \quad \text{as} \quad \max\{|z|, |w|\} \to +\infty.
\]

An easy argument shows that (3.4) follows from (1.1) with an appropriate choice of the parameter \( A \). It follows from Theorem 1.1 [19, p. 143] that \( M_{\Phi,N}(\theta) \) is decreasing in \( N \), provided that \( \Phi \) is fixed and \( \theta \) is held constant. Moreover, under the same conditions, \( M_{\Phi,N}(\theta) \to M_{\Phi}(\theta) \) as \( N \to +\infty \), where

\[
M_{\Phi}(\theta) = \exp(-\kappa_{\Phi}(\theta)),
\]

and \( \kappa_{\Phi}(\theta) \) is results from a certain minimization problem, outlined below. The quantity \( M_{\Phi}(\theta) \) can be thought of as a weighted capacity (of the whole plane).

**The extremal measure.** Let \( P_c(\mathbb{C}) \) denote the convex “body” of all compactly supported Borel probability measure \( \sigma \) on \( \mathbb{C} \). For \( \sigma \in P_c(\mathbb{C}) \), we introduce the energy functional

\[
E_{\Phi}[\sigma](\theta) = 2\theta \int_{\mathbb{C}} \Phi(z) d\sigma(z) + \iint_{\mathbb{C}^2} \log \frac{1}{|z - w|} d\sigma(z) d\sigma(w)
\]

\[
= \iint_{\mathbb{C}^2} \left\{ \log \frac{1}{|z - w|} + \theta \Phi(z) + \theta \Phi(w) \right\} d\sigma(z) d\sigma(w)
\]

\[
= \iint_{\mathbb{C}^2} \log \frac{1}{E_{\Phi}(z, w; \theta)} d\sigma(z) d\sigma(w),
\]

and consider the problem of minimizing \( E_{\Phi}[\sigma](\theta) \) over all \( \sigma \). Define

\[
\kappa_{\Phi}(\theta) = \inf_{\sigma \in P_c(\mathbb{C})} E_{\Phi}[\sigma](\theta);
\]

this quantity is related to \( M_{\Phi}(\theta) \) via (3.5), in view of Theorem 1.3 [19, p. 145]. Any \( \sigma \in P_c(\mathbb{C}) \) which achieves the maximum in (3.7) is called an extremal (or equilibrium) measure. In view of (3.4), which follows from (1.1), it follows from Theorem 1.3 of [19, p. 27] that there exists a unique extremal measure, which we will denote by \( \tilde{\sigma} \), or, when it necessary to indicate the dependence on the parameter \( \theta \) as well as on the weight \( \Phi \), by \( \tilde{\sigma}_{\Phi,\theta} \).

4. The extremal measure and Hele-Shaw flow

A change of parameters. For reasons of convenience, we will now use \( \tau = 1/(2\theta) \) in place of \( \theta \) as a positive real parameter. As before, \( \Phi : \mathbb{C} \to \mathbb{R} \) is \( C^2 \)-smooth, and subject to the growth requirement (1.1) for every choice of the real parameter \( A \). In particular, (3.4) holds with \( \theta = 1/(2\tau) \). As we mentioned in the previous section, by Theorem 1.3 [19, p. 27], there exists a unique extremal measure \( \tilde{\sigma} \in P_c(\mathbb{C}) \) which achieves the infimum in (3.7). For \( \tilde{\sigma} \), the associated logarithmic potential

\[
L[\tilde{\sigma}](z) = \int_{\mathbb{C}} \log \frac{1}{|z - w|} d\tilde{\sigma}(w), \quad z \in \mathbb{C},
\]
is locally bounded in \( \mathbb{C} \) (Theorem 4.3 \cite{19} p. 51). So, for instance, the extremal measure \( \hat{\sigma} \) has no point masses. To get an understanding of this extremal measure, we make a detour to obstacle problems and Hele-Shaw flow. A general reference for Hele-Shaw flow is the recent book of Gustafsson and Vasiliev \cite{9}.

**An obstacle problem.** Consider the following function:

\[
V(z) = -\Phi(z), \quad z \in \mathbb{C}.
\]

For \( 0 < \tau < +\infty \), let \( \text{SH}_\tau(\mathbb{C}) \) denote the collection of (extended real-valued) functions that are superharmonic in the whole plane \( \mathbb{C} \), and which are of the form

\[
-2\tau \log |z| + R(z),
\]

where \( R(z) \) is superharmonic in a punctured neighborhood of infinity, and bounded from below there. If we have two functions \( f_1, f_2 \in \text{SH}_\tau(\mathbb{C}) \), then the minimum of the two \( \min\{f_1, f_2\} \) is in \( \text{SH}_\tau(\mathbb{C}) \) as well. In addition, the decreasing limit of functions in \( \text{SH}_\tau(\mathbb{C}) \) remains in \( \text{SH}_\tau(\mathbb{C}) \) unless it degenerates to \(-\infty\) identically.

We define \( \hat{V}_\tau \) to be the least majorant (or lower envelope) of \( V \) in the class \( \text{SH}_\tau(\mathbb{C}) \). In other words, \( V \) is an obstacle, and \( \hat{V}_\tau \) solves the obstacle problem.

**The extremal measure and the obstacle problem.** The next proposition explains the relationship between the obstacle problem and the solution \( \hat{\sigma} \) to the optimization problem (3.7) with \( \theta = 1/(2\tau) \). For convenience of notation, we shall write \( \hat{\sigma} \Phi,\tau \) in place of \( \hat{\sigma} \Phi,1/(2\tau) \) when we need to indicate the dependence on the parameter \( \tau \).

**Proposition 4.1.** The logarithmic potential for the extremal measure \( \hat{\sigma} = \hat{\sigma} \Phi,\tau \) is

\[
L[\hat{\sigma}](z) = \frac{1}{2\tau} \hat{V}_\tau(z) + C_\tau,
\]

where \( C_\tau \) is the constant

\[
C_\tau = \frac{1}{2\tau} \left\{ \int_{\mathbb{C}} \Phi(z) d\hat{\sigma}(z) + \int_{\mathbb{C}^2} \log \frac{1}{|z-w|} d\hat{\sigma}(z)d\hat{\sigma}(w) \right\}.
\]

**Proof.** This follows from Theorem 4.1 \cite{19} p. 49].

We recall that the function \( V = -\Phi \) is assumed to be of class \( C^2 \) throughout \( \mathbb{C} \). We shall need the class \( C^{1,1} \), which consists of continuously differentiable functions whose gradient is locally Lipschitz. In other words, all second order partial derivatives of the function are locally bounded.

**Proposition 4.2.** The envelope \( \hat{V}_\tau \) is of class \( C^{1,1} \) throughout \( \mathbb{C} \).

**Proof.** In view of Proposition 4.1, the envelope function \( \hat{V}_\tau \) is harmonic off \( \text{supp} \hat{\sigma} \). Since the extremal measure \( \hat{\sigma} \) has compact support, it follows that we can find an open circular disk \( \mathbb{D}(0,R) \) of radius \( R, 0 < R < +\infty \), around the origin, such that \( \text{supp} \hat{\sigma} \) is compactly contained in \( \mathbb{D}(0,R) \). Along the boundary \( \mathbb{T}(0,R) \) of the
disk, $\hat{V}_\tau$ is harmonic, and in particular smooth. We consider the following obstacle problem. Suppose $v$ is superharmonic on $\mathbb{D}(0, R)$ and $C^1$-smooth on $\bar{\mathbb{D}}(0, R)$, and
\begin{equation}
\begin{cases}
v(z) \geq V(z), & z \in \mathbb{D}(0, R), \\
v(z) = \hat{V}_\tau(z), & z \in \mathbb{D}(0, R).
\end{cases}
\end{equation}

The lower envelope of all such $v$ is denoted by $\tilde{V}_\tau$. A simple argument shows that $\tilde{V}_\tau$ coincides with $\hat{V}_\tau$. Indeed, we easily see that $\tilde{V}_\tau \leq \hat{V}_\tau$ on $\mathbb{D}(0, R)$, with equality on $\mathbb{T}(0, R)$; moreover, if we extend $\tilde{V}_\tau$ to all of $\mathbb{C}$ by declaring that it should equal $\hat{V}_\tau$ off $\mathbb{D}(0, R)$, we get a superharmonic function, which belongs to $\text{SH}_\tau(\mathbb{C})$. But – by definition – $\hat{V}_\tau$ is the smallest such majorant, and we must have equality: $\tilde{V}_\tau = \hat{V}_\tau$.

In view of the known smoothness properties of solutions of obstacle problems of the type (4.1), it follows from the $C^2$-smoothness of $V$ that $\hat{V}_\tau$ is of class $C^{1,1}$ on $\mathbb{D}(0, R)$. In the rest of the plane, it is harmonic, and therefore automatically of class $C^{1,1}$. The basic references on obstacle problems are Chapter 1 in [8], or the paper [3] by Caffarelli and Kinderlehrer.

The proof is complete.

We introduce the notation
\begin{equation}
D_\Phi(\tau) = \left\{ z \in \mathbb{C} : V(z) < \hat{V}_\tau(z) \right\};
\end{equation}
this is an open set which contains a punctured neighborhood of the point at infinity.

**Proposition 4.3.** We have
\[ \{ z \in \mathbb{C} : \Delta \Phi(z) < 0 \} \subset D_\Phi(\tau). \]

**Proof.** Let $z_0 \in \mathbb{C}$ be a point with $\Delta \Phi(z_0) < 0$, and suppose that $z_0 \notin D_\Phi(\tau)$. Then
\[ \hat{V}(z_0) = V(z_0) = -\Phi(z_0), \]
so that if we put $U = \hat{V} - V$, we get $U(z_0) = 0$, while $U \geq 0$ everywhere, and $U$ is strictly superharmonic in a neighborhood of $z_0$. By the strong maximum principle, then, this is not possible. The result is immediate.

We also need the “harmonicity” set of $\Phi$:
\[ \mathcal{H}_\Phi = \{ z \in \mathbb{C} : \Delta \Phi(z) = 0 \}, \]
which is a closed set in $\mathbb{C}$. If $X$ is a Borel measurable subset of the plane $\mathbb{C}$, let us agree to say that $z_0 \in \mathbb{C}$ is a pseudo-interior point for $X$ if there exists a small open disk $D$ around $z_0$ such that
\[ |D \cap X|_2 = |D|_2 \]
holds, where $| \cdot |_2$ is the operation of taking the Lebesgue measure of the given set. For instance, all interior points of the set $X$ are pseudo-interior points of $X$.

We are now in a position to reformulate in precise terms our second theorem from the introduction (Theorem 1.2).
**Theorem 4.4.** In the sense of measures, we have
\[ d\hat{\sigma}_{\Phi, \tau}(z) = \frac{1}{4\pi\tau} 1_{C \setminus \mathcal{D}_\Phi(\tau)}(z) \Delta \Phi(z) \, d\lambda_2(z), \quad z \in \mathbb{C}. \]
In particular, the support of the measure \( \hat{\sigma}_{\Phi, \tau} \) consists of all points of \( \mathbb{C} \) which are not pseudo-interior for the set \( \mathcal{D}_\Phi(\tau) \cup \mathcal{H}_\Phi \).

**Proof.** We get from Proposition 4.1 that
\[ (4.3) \quad -2\pi d\hat{\sigma}(z) = \Delta L[\hat{\sigma}](z) \, d\lambda_2(z) = \frac{1}{2\tau} \Delta \hat{V}_\tau(z) \, d\lambda_2(z). \]
According to [14, p. 53], we have
\[ (4.4) \quad \Delta \hat{V}_\tau(z) = \Delta V(z) = -\Delta \Phi(z), \quad z \in \mathbb{C} \setminus \mathcal{D}_\Phi(\tau), \]
in the almost-everywhere sense. On the non-coincidence set, however, \( \hat{V}_\tau \) is harmonic:
\[ (4.5) \quad \Delta \hat{V}_\tau(z) = 0, \quad z \in \mathcal{D}_\Phi(\tau). \]
This follows by a standard Perron process argument (see, for instance, Proposition 2.2 in [12]). By Proposition 4.2, the function \( \Delta \hat{V}_\tau \) is in \( L^\infty_{\text{loc}}(\mathbb{C}) \), so that we see from (4.3), (4.4), and (4.5) that
\[ d\hat{\sigma}_{\Phi, \tau}(z) = \frac{1}{4\pi\tau} 1_{C \setminus \mathcal{D}_\Phi(\tau)}(z) \Delta \Phi(z), \quad z \in \mathbb{C}, \]
as asserted. The statement regarding the support of \( \hat{\sigma}_{\Phi, \tau} \) is an easy consequence of this identity. The proof is complete.

The obstacle problem considered here share many features in common with the Hele-Shaw flow domains considered by Hedenmalm and Shimorin in [12]. However, the time parameter \( \tau \) flows in the opposite direction as compared with the time parameter \( t \) used in [12]. This means that any cusps of the compact set \( \mathbb{C} \setminus \mathcal{D}_\Phi(\tau) \) should point outward. Also, if \( \Phi \) is real-analytic, the smoothness analysis of based on Sakai’s work [20] done in [12] carries over to the situation treated here.

The domains \( \mathcal{D}(\tau) \) grow as \( \tau \) decreases, and the “harmonic moments” are preserved, with the exception of the first; this is the content of the following proposition.

**Proposition 4.5.** Let \( 0 < \tau < \tau' < +\infty \). Suppose \( h \) is harmonic and bounded in \( \mathcal{D}_\Phi(\tau) \), with an extension to \( \mathbb{C} \) that is locally of Sobolev class \( W^{2,2} \). We then have the equality
\[ 4\pi (\tau' - \tau) h(\infty) = \int_{\mathcal{D}_\Phi(\tau) \setminus \mathcal{D}_\Phi(\tau')} h(z) \Delta \Phi(z) \, d\lambda_2(z). \]

**Proof.** This follows from a suitable application of Green’s formula, analogous to what is done in [12]. \[\square\]
If the domains $D_\Phi(\tau)$ are smooth and vary smoothly with $\tau$, then, by arguing as in [12], pp. 188-189, we see that the assertion of Proposition 4.5 has the interpretation that the boundary $\partial D_\Phi(\tau)$ propagates with velocity proportional to a weight times the gradient of the Green function with singularity at infinity.

The growth of the complementary sets $\mathbb{C} \setminus D_\Phi(\tau)$ as $\tau$ increases is quite an interesting process. For $\tau$ close to 0, we should expect each such complementary set to be localized close to the points where the minimum of the potential function $\Phi$ is attained. If there is only one minimum point $z_0$, and the function $\Phi$ is convex in a neighborhood of $z_0$, then we can show that a small neighborhood of $z_0$ is contained in $\mathbb{C} \setminus D_\Phi(\tau)$, for each fixed $\tau$, by comparing with a concave majorant.

An interesting question seems to be whether

$$\bigcap_{0<\tau<+\infty} D_\Phi(\tau) = \emptyset.$$ 

It is easy to see that a necessary condition for this to happen is that $\Phi$ be subharmonic everywhere. We do not know to what extent this condition is sufficient.

5. Condensation of quantum Hele-Shaw flow

The continuous limit of eigenvalues of normal matrices. For positive real $\beta$, let $\mu_{\beta, \Phi}$ denote the measure

$$d\mu_{\beta, \Phi}(z) = \exp\{-\beta \Phi(z)\} d\lambda_2(z), \quad z \in \mathbb{C},$$

where we recall that $d\lambda_2$ is area measure in the plane. Throughout this section, we shall use – as before – the following scaling choice of $\beta$ as we increase $N$:

$$\beta = 2\theta (N - 1) = \frac{N - 1}{\tau},$$

where $\tau$ is a fixed positive real parameter. Also, we recall the previously used notation

$$z_{[1,n]} = (z_1, \ldots, z_n), \quad z_{[1,N]} = (z_1, \ldots, z_N).$$

Given this setup, we consider the $n$-point correlation measure $\Gamma^{(n)}_{\mu,N}$ with $\mu = \mu_{\beta, \Phi}$ and $\beta$ given by (5.2). We want to understand the asymptotic behavior of this measure as $N \to +\infty$. This means that we should understand the impact of working with the $L^2$ norm in place of the $L^\infty$ norm in the setting of (5.1).

We recall that $\Phi$ is a $C^2$-smooth real-valued function, which tends to infinity at infinity at a pace prescribed by (1.1). It follows from this that there exists a compact subset $K$ of $\mathbb{C}$ such that

$$\kappa_{\Phi}(\theta) + 1 < \log \frac{1}{E_{\Phi}(z, w; \theta)}$$

$$= \theta \left[ \Phi(z) + \Phi(w) \right] + \log \frac{1}{|z - w|}, \quad (z, w) \in \mathbb{C}^2 \setminus (K \times K),$$
where we recall the definition \((3.7)\) of \(\kappa_\Phi(\theta)\). In analogy with the energy function \((3.6)\), we introduce the function \(\mathcal{E}_\Phi^\sharp [z_{[1,N]}](\theta)\),

\[(5.4) \quad \mathcal{E}_\Phi^\sharp [z_{[1,N]}](\theta) = \frac{2}{N(N-1)} \log \prod_{j,k; j < k} \frac{1}{E_\Phi(z_j, z_k; \theta)} = \frac{1}{N(N-1)} \sum_{j,k; j \neq k} \log \frac{1}{E_\Phi(z_j, z_k; \theta)},\]

where \(E_\Phi(z, w; \theta)\) is given by \((3.3)\), and \(\theta = 1/(2\tau)\). We recall that

\[(5.5) \quad \log \frac{1}{M_{\Phi,N}(\theta)} \leq \mathcal{E}_\Phi^\sharp [z_{[1,N]}](\theta), \quad z_{[1,N]} \in \mathbb{C}^N,\]

with equality only at the weighted Fekete points. For positive real \(\epsilon\), let

\[\mathfrak{A}_{\Phi,N}(\epsilon, \theta) = \left\{ z_{[1,N]} \in \mathbb{C}^N : \mathcal{E}_\Phi^\sharp [z_{[1,N]}](\theta) \leq \log \frac{1}{M_{\Phi,N}(\theta)} + \epsilon \right\}.\]

We need to know that the proportion of points in \(\mathfrak{A}_{\Phi,N}(\epsilon, \theta)\) which stays in the compact set \(K\) converges to 1 as \(N \to +\infty\) and \(\epsilon \to 0\).

**Proposition 5.1.** Suppose \(z_{[1,N]} \in \mathfrak{A}_{\Phi,N}(\epsilon, \theta)\), and that \(0 < \epsilon < \frac{1}{2}\). Let \(N_K\) denote the number of indices \(j\) for which \(z_j \in K\). Then, for sufficiently large \(N\), we have

\[\frac{N_K}{N} \geq 1 - 2\epsilon.\]

**Proof.** Let \(X \subset \{1, \ldots, N\}\) be the subset of all indices \(j\) for which \(z_j \in K\), and let \(Y\) be the complement in \(\{1, \ldots, N\}\). We split the sum defining \(\mathcal{E}_\Phi^\sharp [z_{[1,N]}](\theta)\) accordingly:

\[\mathcal{E}_\Phi^\sharp [z_{[1,N]}](\theta) = \frac{1}{N(N-1)} \sum_{j,k \in X; j \neq k} \log \frac{1}{E_\Phi(z_j, z_k; \theta)}
+ \frac{2}{N(N-1)} \sum_{j \in X, k \in Y; j \neq k} \log \frac{1}{E_\Phi(z_j, z_k; \theta)}
+ \frac{1}{N(N-1)} \sum_{j \in Y; j < k} \log \frac{1}{E_\Phi(z_j, z_k; \theta)},\]

The sum of the last two terms is estimated from below by

\[\left(1 - \frac{N_K(N_K-1)}{N(N-1)}\right)(\kappa_\Phi(\theta) + 1),\]
while the first term of the right hand side is estimated in the following manner:

\[
\log \frac{1}{M_{\Phi,N_K}(\theta)} \leq E_\Phi[z_{[1,N]}](\theta) = \sum_{j,k \in X \setminus \{j\}} \log \frac{1}{E_\Phi(z_j,z_k;\theta)}, \quad z_{[1,N]} \in \mathbb{C}^N.
\]

We now see that for \(z_{[1,N]} \in \mathcal{A}_N(\epsilon,\theta)\),

\[
\frac{N_K(N_K - 1)}{N(N - 1)} \log \frac{1}{M_{\Phi,N_K}(\theta)} + \left(1 - \frac{N_K(N_K - 1)}{N(N - 1)}\right)(\kappa_\Phi(\theta) + 1)
\]

\[
\leq E_\Phi[z_{[1,N]}](\theta) \leq \log \frac{1}{M_{\Phi,N}(\theta)} + \epsilon.
\]

It follows that

\[
(5.6) \quad 1 + \kappa_\Phi(\theta) - \log \frac{1}{M_{\Phi,N}(\theta)} - \epsilon \leq \frac{N_K(N_K - 1)}{N(N - 1)} \left(1 + \kappa_\Phi(\theta) - \log \frac{1}{M_{\Phi,N_K}(\theta)}\right).\]

For large values of \(N\), the left hand side is positive, and hence the right hand side is, too. We quickly rule out the possibility that the first on the right hand side is negative (which corresponds to \(N_K = 0\)), in which case we are left with both factors on the right hand side being positive. We get from (5.6) that

\[
(5.7) \quad \frac{N_K(N_K - 1)}{N(N - 1)} \geq 1 - \frac{\epsilon}{1 + \kappa_\Phi(\theta) - \log \frac{1}{M_{\Phi,N_K}(\theta)}}.
\]

if we recall that \(M_{\Phi,N}(\theta) \leq M_{\Phi,N_K}(\theta)\). Next, we note that \(N_K \to +\infty\) as \(N\) tends to infinity; therefore, for sufficiently large \(N\),

\[
0 \leq \log \frac{1}{M_{\Phi,N_K}(\theta)} - \kappa_\Phi(\theta) \leq \frac{1}{2}.
\]

The assertion of the proposition now follows from (5.7). \(\square\)

**Lemma 5.2.** Let \(\sigma \in \mathcal{P}_c(\mathbb{C})\) be absolutely continuous with respect to two-dimensional Lebesgue measure,

\[
d\sigma(z) = S(z) \, d\lambda_2(z),
\]

where \(S \geq 0\) is area-summable and \(S \log^+ S\) is area-summable as well. The normalization constant \(Z_N\) in the definition of \(d\Pi_{N,\beta,\Phi}(z_{[1,N]})\) then has the following bound:

\[
Z_N \geq \exp \left\{-N(N - 1) \mathcal{E}_\Phi[\sigma](\theta) - N \int_{\mathbb{C}} \log S(z) \, d\sigma(z)\right\},
\]

with the understanding that \(\log S \, d\sigma\) vanishes off the support of \(\sigma\).

**Proof.** We recall first the definition of \(Z_N\):

\[
(5.8) \quad Z_N = \int_{\mathbb{C}^N} \exp \left\{-\beta \sum_{j=1}^N \Phi(z_j) \right\} \left|\Delta(z_{[1,N]})\right|^2 \, d\lambda_2 N(z_{[1,N]}).
\]
Let $\Sigma$ denote the set where $S(z) > 0$; as $\sigma$ has compact support, we may assume that the set $\Sigma$ is bounded. We then have

$$Z_N \geq \int_{\Sigma^N} \exp \left\{ -\beta \sum_{j=1}^{N} \Phi(z_j) \right\} \left| \Delta(z_{[1,N]}) \right|^2 \frac{1}{S(z_1) \ldots S(z_N)} \exp \left\{ -\beta \sum_{j=1}^{N} \Phi(z_j) + 2 \log \left| \Delta(z_{[1,N]}) \right| - \sum_{j=1}^{N} \log S(z_j) \right\} \sigma(z_1) \ldots \sigma(z_N),$$

where the second estimate is due to Jensen’s inequality. In view of (5.2) and the definition of the energy functional, we have

$$Z_N \geq \exp \left\{ -N(N-1) E_{\Phi}(\sigma)(\theta) - N \int_{\Sigma} \log S(z) \, d\sigma(z) \right\},$$

as claimed.

Remark 5.3. We note that in view of Theorem 4.4, the extremal measure $\sigma = \hat{\sigma}$ enjoys the regularity property of Lemma 5.2.

The next proposition is central to the argument; its proofs mimics Johansson’s method for eigenvalues of self-adjoint matrices [13].

**Proposition 5.4.** For sufficiently large $N$, we have

$$\Pi_{N,\beta,\Phi}(\mathbb{C}^N \setminus \mathfrak{A}_N(\epsilon, \theta)) \leq e^{-\epsilon N(N-1)/2}.$$

**Proof.** For $z_{[1,N]} \in \mathbb{C}^N \setminus \mathfrak{A}_N(\epsilon, \theta)$, we have

$$E_{\Phi}[z_{[1,N]}](\theta) = \frac{1}{N(N-1)} \sum_{j,k : j \neq k} \log \frac{1}{E_{\Phi}(z_j, z_k; \theta)} \geq \log \frac{1}{M_{\Phi,N}(\theta)} + \epsilon \geq \kappa_{\Phi}(\theta) + \epsilon.$$

We recall the estimate (1.1) from below of the weight $\Phi$, which for an appropriate choice of the parameter $A$ gives

$$E_{\Phi}[z_{[1,N]}](\theta) \geq -C + \frac{1}{N} \sum_{j=1}^{N} \log (1 + |z_j|^2), \quad z_{[1,N]} \in \mathbb{C}^N,$$
for a positive constant $C$ that only depends on $\theta$ and $\Phi$. We form a convex combination of these two estimates ($0 \leq \gamma \leq 1$):

\begin{equation}
(5.9) \quad E^\#_\Phi(z_{[1,N]})(\theta) \geq (1 - \gamma) \left[ \kappa_\Phi(\theta) + \epsilon \right] - C\gamma
+ \frac{\gamma}{N} \sum_{j=1}^{N} \log \left(1 + |z_j|^2\right), \quad z_{[1,N]} \in \mathbb{C}^N \setminus \mathcal{A}_N(\epsilon, \theta).
\end{equation}

We recall the formula defining $\Pi_{N,\beta,\Phi}$, while keeping in mind the identity \((3.2)\),

\begin{equation}
(5.10) \quad d\Pi_{N,\beta,\Phi}(z_{[1,N]}) = \frac{1}{Z_N} \prod_{j,k: j \neq k} E_\Phi(z_j, z_k; \theta) \, d\lambda_{2N}(z_{[1,N]})
= \frac{1}{Z_N} \exp \left\{ - N(N-1) E^\#_\Phi(z_{[1,N]})(\theta) \right\} d\lambda_{2N}(z_{[1,N]}).
\end{equation}

So, on the set $\mathbb{C}^N \setminus \mathcal{A}_N(\epsilon, \theta)$, we get, in view of \((5.9)\),

\begin{equation*}
\begin{aligned}
d\Pi_{N,\beta,\Phi}(z_{[1,N]}) &\leq \frac{1}{Z_N} \exp \left\{ - N(N-1)(1 - \gamma) \left[ \kappa_\Phi(\theta) + \epsilon \right] 
+ CN(N-1)\gamma - \gamma(N-1) \sum_{j=1}^{N} \log \left(1 + |z_j|^2\right) \right\} \, d\lambda_{2N}(z_{[1,N]}).
\end{aligned}
\end{equation*}

By Lemma \ref{lem:5.2}, we have

\begin{equation*}
Z_N \geq \exp \left\{ - N(N-1) \kappa_\Phi(\theta) - BN \right\},
\end{equation*}

where $B$ is the real number

\begin{equation*}
B = \int_\mathbb{C} \log \Delta_\Phi(z) \, d\tilde{\sigma}(z) + \log \frac{\theta}{\pi}.
\end{equation*}

As we combine this with \((5.9)\), we arrive at

\begin{equation*}
\begin{aligned}
d\Pi_{N,\beta,\Phi}(z_{[1,N]}) &\leq \exp \left\{ - N(N-1) \left[ \gamma \kappa_\Phi(\theta) + (1 - \gamma)\epsilon \right] + B N 
+ C\gamma N(N-1) - \gamma(N-1) \sum_{j=1}^{N} \log \left(1 + |z_j|^2\right) \right\} \, d\lambda_{2N}(z_{[1,N]})
= \exp \left\{ - N(N-1) \left[ \gamma \kappa_\Phi(\theta) + (1 - \gamma)\epsilon \right] + B N + C\gamma N(N-1) \right\}
\times \prod_{j=1}^{N} \left(1 + |z_j|^2\right)^{-\gamma(N-1)} \, d\lambda_{2N}(z_{[1,N]})
\end{aligned}
\end{equation*}

on the set $\mathbb{C}^N \setminus \mathcal{A}_N(\epsilon, \theta)$. Considering that

\begin{equation*}
\int_\mathbb{C} \left(1 + |z|^2\right)^{-\gamma(N-1)} \, d\lambda_2(z) = \frac{\Pi}{\gamma(N-1) - 1},
\end{equation*}
provided that \(1/(N - 1) < \gamma \leq 1\), we find that

\[
\Pi_{N, \beta, \Phi}(C^N \setminus A_N(\epsilon, \theta)) \leq \exp \left\{ -N(N - 1)[\gamma \kappa_{\Phi}(\theta) + (1 - \gamma)\epsilon - C\gamma] + BN \right\} \times \left( \frac{\Pi}{\gamma(N - 1)} \right)^N.
\]

The dominant contribution in the expression which is exponentiated is

\[-N(N - 1)[\gamma \kappa_{\Phi}(\theta) + (1 - \gamma)\epsilon - C\gamma].\]

We would like to pick \(\gamma\), \(1/(N - 1) < \gamma < 1\), such that

\[0 < \gamma \kappa_{\Phi}(\theta) + (1 - \gamma)\epsilon - C\gamma.\]

This is possible; indeed, without loss of generality, we may assume that \(C\) is greater than \(\kappa_{\Phi}(\theta)\), in which case one such choice is

\[\gamma = \frac{\epsilon}{2(C - \kappa_{\Phi}(\theta) + \epsilon)}.\]

This value of \(\gamma\) yields the assertion of the proposition. The proof is complete. \(\square\)

For a point \(z_{[1,N]} \in C^N\), we define the associated weighted sum of point masses \(\sigma[z_{[1,N]}] \in \mathcal{P}_c(C)\) by the formula

\[
(5.11) \quad d\sigma[z_{[1,N]}](z) = \frac{1}{N} \sum_{j=1}^{N} d\delta_{z_j}(z), \quad z \in C,
\]

where \(\delta_w\) means the Dirac point mass at \(w \in C\). Also, let \(C_b(C) = C(C) \cap L^\infty(C)\) denote the space bounded complex-valued continuous functions on \(C\).

**Proposition 5.5.** Suppose \(\sigma_N = \sigma[z_{[1,N]}]\) is as above, with \(z_{[1,N]} = (z_1, \ldots, z_N) \in C^N\).

Suppose, moreover, that

\[\mathcal{E}_{\Phi}^+ [z_{[1,N]}](\theta) \to \kappa_{\Phi}(\theta)\]

as \(N \to +\infty\). Then \(\sigma_N\) converges to \(\widehat{\sigma}\), the extremal measure, in the weak-star topology, as \(N \to +\infty\). In other words, for each \(f \in C_b(C)\), we have

\[\int_C f(z) \, d\sigma_N(z) \to \int_C f(z) \, d\widehat{\sigma}(z) \quad \text{as} \quad N \to +\infty.\]

**Proof.** First, fix a small but positive \(\epsilon\). By assumption, for sufficiently big \(N\),

\[z_{[1,N]} \in A_N(\epsilon, \theta),\]

so that by Proposition 5.1 \(\sigma_N(K) \geq 1 - 2\epsilon\). As \(\epsilon\) can be made as small as we like, and each \(\sigma_N\) is a probability measure, we see that

\[\sigma_N(K) \to 1 \quad \text{as} \quad N \to +\infty.\]
The space of all finite complex Borel measures on $K$ is a Banach space, with weak-star compact unit ball. This means that each subsequence of the sequence $\sigma_1|_K, \sigma_2|_K, \sigma_3|_K, \ldots$ has a weak-star convergent subsequence. We shall show that any such limit coincides with $\hat{\sigma}$, from which the assertion follows.

Without loss of generality, then, by passing to a subsequence, we may assume that the sequence $\sigma_N|_K, N = 1, 2, 3, \ldots$, converges weak-star itself to a limit, which we call $\bar{\sigma}$. The weak-star limit of $\sigma_N$ is then also $\bar{\sigma}$, given that $\sigma_N(\mathbb{C} \setminus K) \to 0$ as $N \to +\infty$. By testing with the function $f = 1$, we see that $\tilde{\sigma}$ is a probability measure supported on $K$, so that $\tilde{\sigma} \in P_c(\mathbb{C})$. We claim that $E \Phi[\tilde{\sigma}](\theta) \leq \kappa \Phi(\theta)$. Once this is established, the equality $\tilde{\sigma} = \hat{\sigma}$ follows from the uniqueness of the extremal measure. We use a cut-off argument: we note that

$$E^\tau_\Phi [z[1,N]](\theta) = \frac{2\theta}{N} \frac{\sum_{j=1}^N \Phi(z_j)}{N(N-1)} + \frac{1}{N(N-1)} \sum_{j,k: j \neq k} \log \frac{1}{|z_j - z_k|},$$

so that if $L$ is a real parameter,

$$E^\tau_\Phi [z[1,N]](\theta) \geq \frac{2\theta}{N} \frac{\sum_{j=1}^N \Phi(z_j)}{N(N-1)} + \frac{1}{N(N-1)} \sum_{j,k=1}^N \min \left\{ \log \frac{1}{|z_j - z_k|}, L \right\} - \frac{L}{N-1}.$$

Now, let $N \to +\infty$, so that we get, in view of our assumptions,

$$\kappa \Phi(\theta) \geq 2\theta \int_{\mathbb{C}} \Phi(z) d\bar{\sigma}(z) + \int_{\mathbb{C}^2} \min \left\{ \log \frac{1}{|z - w|}, L \right\} d\bar{\sigma}(z) d\bar{\sigma}(w).$$

As we let $L$ tend to $+\infty$, the right hand side approaches $E_\Phi[\tilde{\sigma}](\theta)$, whence the claim is immediate.

As in the introduction, we let $\Pi^{(n)}_{N,\beta,\Phi}$ be the $n$-point marginal distribution measure and $\Gamma^{(n)}_{N,\beta,\Phi}$ the $n$-point correlation measure for the probability distribution measure $\Pi_{N,\beta,\Phi}$. We keep $\beta$ connected with $\tau$, $N,$ and $\theta$ via (5.2) and fix $\theta$ (or, if you like, $\tau$) while $N$ grows.

Let $C_b(\mathbb{C}^n) = C(\mathbb{C}^n) \cap L^\infty(\mathbb{C}^n)$ denote the Banach space of bounded complex-valued continuous functions on $\mathbb{C}^n$. We arrive at a precise reformulation of our first main result (Theorem 1.1).

**Theorem 5.6.** We have

$$\Pi^{(n)}_{N,\beta,\Phi}(\mathbb{C}^n) = 1,$$

while, as $N \to +\infty$,

$$\Pi^{(n)}_{N,\beta,\Phi}(K^n) \to 1.$$

Moreover, for each $f \in C_b(\mathbb{C}^n)$, we have, as $N \to +\infty$,

$$\int_{\mathbb{C}^n} f(z[1,n]) d\Pi^{(n)}_{N,\beta,\Phi}(z[1,n]) \to \int_{\mathbb{C}^n} f(z[1,n]) d\tilde{\sigma}(z_1) \cdots d\tilde{\sigma}(z_n).$$
In other words, in the weak-star topology of measures, we have, as $N \to +\infty$,
\[ d\Pi_{N,\beta,\Phi}^{(n)}(z_{[1,n]}) \to d\hat{\sigma}(z_1) \cdots d\hat{\sigma}(z_n). \]

Here, $\hat{\sigma} = \hat{\sigma}_{\Phi,\theta}$ is the extremal measure.

Proof. We note that the total mass of the measure $\Pi_{N,\beta,\Phi}^{(n)}$ is 1.

As a second step, we prove (5.12) under the slightly more restrictive assumption $f \in C_c(\mathbb{C})$, which means that the test function $f$ has compact support.

Let $\varsigma$ be a permutation of $\{1, 2, 3, \ldots, N\}$. Then, due to the symmetry properties of $\Pi_{N,\beta,\Phi}$,
\[ \int_{\mathbb{C}^n} f(z_{[1,n]}) d\Pi_{N,\beta,\Phi}^{(n)}(z_{[1,n]}) = \int_{\mathbb{C}^N} f(z_1, \ldots, z_n) d\Pi_{N,\beta,\Phi}(z_{[1,N]}) \]
\[ = \int_{\mathbb{C}^N} f(z_{\varsigma(1)}, \ldots, z_{\varsigma(n)}) d\Pi_{N,\beta,\Phi}(z_{[1,N]}), \]
from which we quickly deduce that
\[ \int_{\mathbb{C}^n} f(z_{[1,n]}) d\Pi_{N,\beta,\Phi}^{(n)}(z_{[1,n]}) = \frac{1}{N!} \sum_{\varsigma} \int_{\mathbb{C}^N} f(z_{\varsigma(1)}, \ldots, z_{\varsigma(n)}) d\Pi_{N,\beta,\Phi}(z_{[1,N]}), \]
where the sum runs over all permutations of $\{1, 2, 3, \ldots, N\}$. We fix a small positive $\epsilon$, and split the integral:
\[ \int_{\mathbb{C}^n} f(z_{[1,n]}) d\Pi_{N,\beta,\Phi}^{(n)}(z_{[1,n]}) = \frac{1}{N!} \sum_{\varsigma} \int_{\mathbb{C}^N} f(z_{\varsigma(1)}, \ldots, z_{\varsigma(n)}) d\Pi_{N,\beta,\Phi}(z_{[1,N]}), \]
\[ + \frac{1}{N!} \sum_{\varsigma} \int_{\mathbb{C}^N \setminus \mathcal{A}(\epsilon, \theta)} f(z_{\varsigma(1)}, \ldots, z_{\varsigma(n)}) d\Pi_{N,\beta,\Phi}(z_{[1,N]}). \]

By Proposition 5.3, the last term is bounded in modulus by
\[ e^{-\epsilon N(N-1)/2} \|f\|_{L^\infty(\mathbb{C})}, \]
for large $N$. To deal with the first term on the right hand side, we should understand the behavior of
\[ \frac{1}{N!} \sum_{\varsigma} f(z_{\varsigma(1)}, \ldots, z_{\varsigma(n)}), \quad z_{[1,N]} \in \mathcal{A}(\epsilon, \theta). \]

Let us consider the simplest case $n = 1$ first. Then (5.15) amounts to
\[ \frac{1}{N} \sum_{j=1}^N f(z_j), \quad z_{[1,N]} \in \mathcal{A}(\epsilon, \theta). \]
By letting $\epsilon$ approach 0 slowly as $N \to +\infty$, we may ensure that (5.14) tends to 0 as $N \to +\infty$, while (5.16) approaches
$$\int_{\mathbb{C}} f(z) \, d\hat{\sigma}(z).$$

It is easy to check that the latter statement entails
$$\frac{1}{N!} \int_{\mathbb{A}_{N}(\epsilon, \theta)} \sum_{j=1}^{N} f(z_j) \, d\Pi_{N, \beta, \Phi}(z_{[1,N]}) \rightarrow \int_{\mathbb{C}} f(z) \, d\hat{\sigma}(z)$$
as $N \to +\infty$, if we allow $\epsilon$ to approach 0 slowly.

The remaining case $n > 1$ is handled in an analogous manner: Proposition 5.5 should be replaced by a multidimensional analogue, which we get by iterated integration. It is useful to keep in mind that for large $N$ and fixed $n$, the following collections of sequences are asymptotically (as $N \to +\infty$) the same:
$$(\varsigma(1), \ldots, \varsigma(n)) \quad \text{and} \quad (j(1), \ldots, j(n)),$$
where $\varsigma$ runs over all permutations of $\{1, \ldots, N\}$, and $j$ runs over all functions $j : \{1, \ldots, n\} \to \{1, \ldots, N\}$. For instance,
$$\frac{N^n(N - n)!}{N!} \rightarrow 1 \quad \text{as} \quad N \to +\infty.$$

There are a couple of assertion that remain to check. If we notice that the support of $\hat{\sigma}$ is actually contained in the interior of the compact set $K$, then, by choosing a smooth cut-off function $\chi \in \mathcal{C}_{c}(\mathbb{C}^n)$ with $0 \leq \chi(z_{[1,n]}) \leq 1$ everywhere, $\chi(z_{[1,n]}) = 1$ on $(\text{supp} \, \hat{\sigma})^n$, and $\chi(z) = 0$ off $K^n$, we find from (5.12) that
$$\liminf_{N \to +\infty} \Pi_{N, \beta, \Phi}^{(n)}(K^n) \geq 1.$$Since we are dealing with probability measures, it follows that
$$\lim_{N \to +\infty} \Pi_{N, \beta, \Phi}^{(n)}(K^n) = 1,$$as claimed.

It remains to verify that (5.12) holds for all $f \in \mathcal{C}_b(\mathbb{C}^n)$. To this end, we use a smooth cut-off function $\chi$ similar to what we defined above, and write $f = \chi f + (1 - \chi) f$. Then $\chi f \in \mathcal{C}_c(\mathbb{C})$, while the integral of the remainder $(1 - \chi) f$ tends to 0 as $N \to +\infty$, by a simple estimate.

The proof is complete.

\[\square\]

Remark 5.7. Let us think of the $N$-tuple $z_{[1,N]} = (z_1, \ldots, z_N)$ as a random variable taking values in $\mathbb{C}^N$, with distribution measure $\Pi_{N, \beta, \Phi}$. The marginal distribution of the first $n$ ($1 \leq n \leq N$) coordinates $z_{[1,n]} = (z_1, \ldots, z_n)$ is then given by the measure $\Pi_{N, \beta, \Phi}^{(n)}$; to stress the dependence on $N$, let us write
$$z_{[1,n]|N} = (z_1|N, \ldots, z_n|N)$$
for the marginal random variable. Now, if \( \beta \) grows with \( N \) according to (5.2), for some fixed positive \( \theta \), Theorem 5.6 says that as \( N \to +\infty \), \( z_1|_N, \ldots, z_n|_N \) become identically distributed independent random variables, each having \( \hat{\sigma}_{\Phi, \theta} \) as distribution measure. In particular, the covariance type matrix (with \( \mu = \mu_{\beta, \Phi} \))

\[
K_{\mu, N}^y(z_{[1,n]}, z_{[1,n]}) = \left( K_{\mu, N}(z_j, z_k) \right)_{j,k=1}^n
\]

becomes asymptotically diagonal as \( N \to +\infty \).

**Example.** We need to give an example to illustrate the result. We consider the rather trivial case of \( \Phi(z) = |z|^2 \), and put \( n = 1 \). Then \( \Delta \Phi(z) = 4 \) is constant. A computation shows that with \( \mu = \mu_{\beta, \Phi} \) given by (1.2), the reproducing kernel function for the polynomial subspace is

\[
K_{\mu, N}(z, w) = \frac{\beta}{\pi} \sum_{j=0}^{N-1} \left( \frac{\beta z \bar{w}}{j!} \right)^j.
\]

We then have

\[
d\Pi_{N, \beta, \Phi}^{(n)}(z) = \frac{1}{N} K_{\mu}(z, z) e^{-\beta |z|^2} d\lambda_2(z) = \frac{\beta}{N \pi} e^{-\beta |z|^2} \sum_{j=0}^{N-1} \left( \frac{\beta |z|^2}{j!} \right)^j d\lambda_2(z),
\]

while it is easy to check that

\[
D_{\Phi}(\tau) = \{ z \in \mathbb{C} : |z| > \sqrt{\tau} \}.
\]

In view of this, the content of Theorem 5.6 in this simple case is that, for fixed positive \( \tau \), the function

\[
e^{-(N-1)|z|^2/\tau} \sum_{j=0}^{N-1} \frac{1}{j!} \left( \frac{(N-1)|z|^2}{\tau} \right)^j,
\]

which is real-valued, with values between 0 and 1, tends to 1 as \( N \to +\infty \) for \( |z| < \sqrt{\tau} \), and to 0 for \( |z| > \sqrt{\tau} \). This fact is of course well-known. In [6], the domain \( D_{\Phi}(\tau) \) is computed explicitly and shown to be an ellipse in the more general case (still with constant \( \Delta \Phi(z) = 4 \))

\[
\Phi(z) = |z|^2 + a \text{ Re } (z^2), \quad -1 < a < 1.
\]

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