Parametric excitation of a Bose-Einstein condensate in a 1D optical lattice

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We study the response of a Bose-Einstein condensate to a periodic modulation of the depth of an optical lattice. Using Gross-Pitaevskii theory, we show that a modulation at frequency \( \Omega \) drives the parametric excitation of Bogoliubov modes with frequency \( \Omega/2 \). The ensuing nonlinear dynamics leads to a rapid broadening of the momentum distribution and a consequent large increase of the condensate size after free expansion. We show that this process does not require the presence of a large condensate depletion. Our results reproduce the main features of the spectrum measured in the superfluid phase by Stöferle et al., Phys. Rev. Lett. 92, 130403 (2004).

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Exploring the effects of interaction and low dimensionality is one of the central themes in current research on ultracold Bose gases. Optical lattices are ideal tools in this sense. A major achievement was the demonstration that, by increasing the depth of an optical lattice, a quantum phase transition occurs from the superfluid to a Mott insulating phase \[1\]. More recently, Stöferle et al. \[2, 3\] made an effort to characterize this phenomenon by measuring the excitation spectrum. The atoms are loaded into the ground state of an harmonic trap plus an optical lattice, and they are subsequently excited by modulating the lattice depth. Then the gas is released from the trap and the width of the expanding cloud is taken as a measure of the excitation energy. The quantum phase transition shows up as the crossover from a broad continuum of excitations in the superfluid phase to a discrete spectrum in the Mott insulating phase.

The broad resonance in the superfluid regime was unexpected. In fact, the Gross-Pitaevskii (GP) theory predicts that, in the linear response regime, a modulation of the lattice depth cannot excite the gas \[4\]. For this reason, it was conjectured that the measured spectra might be the effect of strong interactions, not included in the GP theory, which yield a significant quantum depletion of the condensate \[2, 3\]. Here we show that the observed resonance is caused by the parametric amplification of Bogoliubov states and the subsequent nonlinear dynamics, which leads to the broadening of the momentum distribution. We describe both processes by using the GP theory and assuming the initial condensate to be out of equilibrium. A tiny initial occupation of Bogoliubov modes is found to be enough to produce a strong response at high frequency, while larger fluctuations are needed to explain the observed spectrum at low frequency.

We simulate the dynamics of the condensate by solving the GP equation \[4\] for an axially symmetric condensate of \(^{87}\)Rh, similar to typical “tubes” in the experiments of Ref. \[2\]. Since the transverse and axial trapping frequencies are such that \( \omega_\perp/\omega_z \gg 1 \), one can factorize the order parameter in the product of a Gaussian radial component of \( z \)-and \( t \)-dependent width, \( \sigma(z,t) \), and an axial wave function \( \Psi(z,t) \). The GP equation yields \[6\]

\[
\frac{i}{\hbar} \frac{\partial \Psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\sigma^2} + \frac{1}{a_\perp^2} \right) + V + \frac{gN|\Psi|^2}{2\pi\sigma^2} \right] \Psi,
\]

and \( \sigma = a_\perp(1 + 2aN|\Psi|^2)^{1/4} \). Here \( a_\perp = [\hbar/(m\omega_\perp)]^{1/2} \), \( N \) is the number of particles, \( g = 4\pi\hbar^2a/m \), with \( a \) the s-wave scattering length and \( m \) the atomic mass. The axial potential \( V \) is the sum of the harmonic trap and the optical lattice:

\[
V(z,t) = (m/2)\omega_z^2z^2 + sE_R[1 + A\sin(\Omega t)]\sin^2(q_Bz),
\]

where \( q_B = \pi/d \) is the Bragg wave vector, \( d \) is the lattice spacing and \( s \) is the lattice depth in units of the recoil energy \( E_R = \hbar^2q_B^2/2m \). To simulate a single tube of Ref. \[2\], we use \( d = 413 \, \text{nm}, \omega_z = 2\pi \times 84.6 \, \text{Hz} \) and \( \omega_\perp = 2\pi \times 36.5 \, \text{KHz} \), yielding \( E_R/h = 2\pi \times 3.34 \, \text{KHz} \). We first calculate the ground state for given \( N \) and \( s \), by solving the stationary GP equation with \( A = 0 \). Then we solve the time-dependent equation \[11\] with \( A \neq 0 \) for a modulation time \( t_m \). Finally, we switch off the external potential and let the condensate expand for 25 ms \[8\].

In Figs. 1 and 2 we report the results for \( N = 100, s = 4, A = 0.2 \) and \( \Omega = 1.33E_R/h \). The solid line in the upper panel of Fig. \[1\] shows the width of the density distribution at the end of the expansion as a function of the in-trap modulation time \( t_m \). Before about 20 ms nothing seems to occur. Then the width starts increasing and eventually reaches values of about 10 times the size of the expanding ground state. The density and momentum distributions of the in-trap condensate at the instants A, B, C, and D are plotted in Fig. 2. At the early time A, both are still indistinguishable from those of the ground state. In contrast, the momentum distribution in B clearly features new sharp peaks at \( \pm q \), with \( q = 0.56q_B \) and the density is modulated on a lengthscale of about 4 sites, accordingly. By the time C, the populations at \( \pm q \) have grown large. Besides, the lateral peaks at \( \pm 2q_B \), associated with the density modulation of the ground state in the lattice, are smaller and contributions from momenta throughout the first and second Brillouin
zones start showing up. At D, the system hardly bears any resemblance with the ground state and its broad momentum distribution explains the increase of the width of the expanding cloud shown in Fig. 1.

FIG. 1: Upper panel: the width of the density distribution \( \langle z^2 \rangle^{1/2} \) after 25 ms of free expansion is plotted as a function of the modulation time \( t_m \), as obtained with \( \Omega = 1.33E_R/\hbar \) and \( A = 0.2 \); Lower panel: population \( N_q \) of the parametrically excited mode with \( q = 0.56q_B \) and \( \omega = \Omega/2 \). The different lines refer to different initial states. Solid line: ground state; dashed: ground state plus a weak Bragg pulse; dotted: ground state plus a small white noise.

This behavior is a consequence of the parametric instability of Bogoliubov modes. A parametric instability corresponds to the exponential growth of certain modes of a system induced by the periodic variation of a parameter \([10]\). Systems governed by a nonlinear Schrödinger equation, as the GP equation, can exhibit a wide variety of parametric instabilities (see e.g., \([11]\) and references therein). In our case, the modulation of the lattice depth at frequency \( \Omega \) causes the instability of modes with frequency \( \omega(q) = \Omega/2 \). To demonstrate this, we perform simulations at various \( \Omega \) and, from each one, we extract the positions \( \pm q \) of the first extra peaks which grow in the momentum distribution. Given these values of \( q \), in Fig. 3 we plot the frequencies \( \Omega/2 \) as a function of \( q \) (points), together with the dispersion law of the lowest Bogoliubov band, \( \omega(q) \) (solid line). The latter is calculated by keeping the same optical lattice and transverse trap, but neglecting the weak axial confinement; the number of atoms per lattice site is taken to be equal to the average number of atoms per site in the actual trapped condensate \([12, 13]\). The agreement between points and solid line clearly indicates that the dynamics of the condensate is directly related to the spectrum of Bogoliubov modes.

FIG. 2: Density (left) and momentum (right) distributions at different times A, B, C and D given in Fig. 1.

In order to be parametrically amplified, the modes at \( \pm q \) must preexist at \( t = 0 \). This means that our initial state is not a pure ground state. Indeed some numerical noise is always present and yields a very small occupation of Bogoliubov modes. To understand this point, we define the fraction of atoms, \( N_q(t) \), which contributes to the peaks at \( \pm q \) \([14]\) and plot it in Fig. 1. The solid line starts at about \( 5 \times 10^{-6} \) which, for the parameters of this simulation, coincides with the \( \pm q \) components of the exact ground state. These nonvanishing components are due to the finite axial size of the condensate; they do not grow in time and must not be confused with the population of the Bogoliubov modes. The seed population is instead provided by the numerical noise and is much less than \( 10^{-6} \) at \( t = 0 \). Then it grows exponentially and it becomes visible in Fig. 1 after about 15 ms, when it exceeds the ground state level. It eventually saturates at a value of the order of \( 10^{-1} \). The time scale

FIG. 3: For each modulation of given frequency \( \Omega \), we draw a point at \((q, \omega)\), where \( q \) is the wavevector of the parametrically excited mode and \( \omega = \Omega/2 \). The solid line is the lowest Bogoliubov band.
of this growth is much longer than the lattice modulation period. The growth rate that we extract from our simulations is linear in $\Omega$ and $A$ within our accuracy. This linear dependence agrees with the results of a semi-analytic model which describes the mean-field coupling between the ground state and $\pm q$ excitations within a quasiparticle projection method \[15\].

The time needed to saturate depends on the amount of seed. To explore this dependence, we mimic an initial “seed” in a controllable way. For instance, just before $t = 0$ we excite the condensate with a short and weak Bragg pulse in resonance with the Bogoliubov mode at $q$. Alternatively, we simply add random noise with uniform (white noise) or thermal-like distributions. The effects of the Bragg pulse and white noise are shown in Fig. 1 where the seed is chosen to be $\sim 10^{-5}$ in both cases. The main effect of this extra seed is to anticipate the instant at which the parametric excitation becomes noticeable. Yet, the growth rate and the level at which $\langle \Delta^2 \rangle$ saturate remain the same as for the solid line, which corresponds to less than $10^{-15}$ seed excitations. This means that, for this $\Omega$, the amount of seed which is enough to trigger the parametric instability is indeed very small. In the experiments the system is certainly not in its groundstate at $t = 0$ due to thermal and quantum depletion, and to excitations produced when loading the condensate into the lattice \[16\]. Our calculations show that even a very tiny deviation from the groundstate can produce a strong response on the timescale of the experiment.

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The evolution leading up to the results in Figs. 1 and 2 not only involves the parametric excitation of the $\pm q$ modes, but also the subsequent broadening of the momentum distribution, as in C and D. This is due to the nonlinear mean-field dynamics of the condensate governed by the GP equation. This nonlinear dynamics is triggered by the growth of the parametrically unstable modes, but its outcome is little affected by changing $\Omega$ or $A$. Even more, it is hardly influenced by the external perturbation; in fact, switching off the lattice modulation at time C has very little effect on the outcome at time D or later. This implies that the response of the condensate as a function of $\Omega$ mainly reflects the $\Omega$-dependence of the growth rates of the parametrically excited modes. Moreover, the time scale for the nonlinear broadening of the momentum distribution is also relatively long, so that the increase of the width $\langle \Delta^2 \rangle^{1/2}$ occurs on a scale of several tens of ms, as in Fig. 1. This agrees with the behavior observed in the experiments (see Fig. 3 of Ref. \[17\]).

As seen in Fig. 3, the main effect of changing the lattice modulation frequency is that of picking out a different Bogoliubov mode for parametric excitation. In the limit $\Omega \rightarrow 0$ the growth rate vanishes and the width $\langle \Delta^2 \rangle^{1/2}$ at a fixed $t_m$ becomes equal to the width of a condensate which expands without being modulated. The same limiting value is obtained on the opposite side, namely at $\Omega$ close to twice the Bogoliubov bandwidth, $2\omega(q_B)$, where parametric excitations are not possible due to the presence of the gap in the Bogoliubov spectrum at the zone boundary. Therefore the response of the system takes the form of a broad peak in the range $0 < \Omega < 2\omega(q_B)$.

![Graph](image_url)

**FIG. 4:** Width $\langle \Delta^2 \rangle^{1/2}$ of the expanded array of condensates as function of $\Omega$. Top panel: experimental data of Ref. \[2\], for $s = 4$; Solid line in bottom panel: our simulations starting from a small white noise. Empty circles and solid squares: same but with a much larger white noise and thermal-like noise, respectively (see text for explanation).

In order to compare our results with the experimental observations of Ref. \[2\], a simulation with a single tube is not enough. The actual condensate is made of an array of tubes having different population, $N$, and hence different Bogoliubov bandwidths. Given the total number of atoms in the array and the trapping frequencies, one can estimate the population of each tube by calculating the “average” density distribution within a local density approximation as in Refs. \[13, 18\]. Then we perform several simulations for different values of $N$ and we sum their contributions to $\langle \Delta^2 \rangle^{1/2}$ using the “average” density distribution as a weighting function. The central condensate is taken to have $N = 90$ which is expected to be reasonably close to reality. The results are shown in the lower panel of Fig. 4. The solid line is the result of GP simulations with a white noise as a seed excitation, corresponding to $N_q \sim 10^{-3}$ at $t = 0$. This small seed is enough to produce a broad response which qualitatively agrees with the experimental one (upper panel). The height of the peak is of the same order and its slope at high frequency is very similar. Within our approach, this slope originates from the fact that the response vanishes at twice the Bogoliubov bandwidth, $2\omega(q_B)$, but this bandwidth is different for tubes with different $N$ in the array and this provides a smoothing of the high frequency tail. By repeating the calculation for a deeper lattice ($s = 6$, instead of 4), we find that this tail shifts to lower frequencies by about 10-15%. This agrees with
the experimental observations and is consistent with the calculated decrease of the bandwidth.

On the other side of the peak, at low frequency, the response turns out to be more sensitive to the seed. In order to reproduce the observed response around 2000 Hz, we have to start with a condensate in which a significant number of atoms populate the lowest Bogoliubov modes. This also yields an increase of the axial width of the expanded condensate without modulation. The effect of a larger seed is represented in Fig. 4 by empty circles and solid squares, which are the results of calculations with a white noise and a thermal-like seed, respectively. The thermal seed is simulated by random fluctuations, where the average occupation number of Bogoliubov excitations follows a Bose distribution at a given temperature. The dispersion $\omega(q)$ of a uniform condensate, calculated as for Fig. 3, and the corresponding quasiparticle amplitudes are used to relate the occupation of the quasiparticle states to the momentum distribution of the atoms. Both the temperature in the thermal-like seed and the level of the white noise are chosen in such a way that the width without modulation is close to the experimental value, $(\langle z^2 \rangle)^{1/2} \approx 25 \mu m$. This implies a total depletion of the order of 20% in the case of white noise and the temperature $k_B T \sim 2 \times 10^{-3} \hbar \omega_{\perp}$ for thermal noise. The latter value is compatible with the experimental estimate $k_B T < 6 \times 10^{-3} \hbar \omega_{\perp}$ given in Ref. [16]. As one can see, in both cases the overall response is higher and broader, and the qualitative agreement with the experiments is improved. This result is also compatible with the observation made in Ref. [2] that the excitation efficiency decreases when the transverse confinement is reduced and, hence, the initial depletion of the condensate is reduced as well. A quantitative comparison is however difficult at this level. In the experiment a thermalization process is performed after the modulation and before the expansion, by lowering the depth of the transverse optical lattice and letting the tubes interact for a while. This is not included in our simulations, since it would require a full 3D calculation. Also the procedure of averaging over many tubes, some of them with only a few atoms, limits the accuracy of the comparison. Nevertheless, our analysis suggests that a characterization of classical and/or quantum fluctuations can indeed be possible by appropriately choosing the trapping geometry and the experimental procedure and using the parametric resonances for a selective amplification of initial fluctuations [14].

In conclusion we have shown that the broad resonance observed in condensates subject to a periodic modulation of the optical lattice can be interpreted as the effect of a parametric instability of Bogoliubov modes. The occurrence of this type of instability in Bose-Einstein condensates is an interesting property of these systems, which originates from their superfluid nature and the existence of undamped collective excitations. The parametric instability and the subsequent nonlinear dynamics are nicely accounted for by the GP equation. Since our calculations are purely mean-field in nature, they are not conclusive in excluding possible many-body processes in the explanation of the experimental observations. Exploiting all predictions of GP theory is however essential in view of investigations on beyond mean-field effects due to strong interactions and low dimensionality.

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[1] M. Greiner et al., Nature (London) 415, 39 (2002).
[2] T. Stöferle et al., Phys. Rev. Lett. 92, 130403 (2004).
[3] C. Schori et al., Phys. Rev. Lett. 93, 240402 (2004).
[4] C. Menotti et al., Phys. Rev. A 67, 053609 (2003).
[5] H.P. Büchler and G. Blatter, e-print cond-mat/0512526.
[6] F. Dalfovo et al., Rev. Mod. Phys. 71, 463 (1999).
[7] L. Salasnich et al., Phys. Rev. A 65, 043614 (2002).
[8] We treat the expansion as in P. Massignan and M. Modugno, Phys. Rev. A 67, 023614 (2003). For these condensates the effects of the mean-field interaction on the axial expansion are negligible and the final density distribution directly reflects the in situ momentum distribution.
[9] As in the experiments, when calculating the width $\langle z^2 \rangle^{1/2}$ we ignore the lateral peaks due to the $\pm 2q_B$ components of the momentum distribution.
[10] L.D. Landau and E.M. Lifshitz, Mechanics (Oxford, Pergamon, 1973).
[11] J.J. García-Ripoll et al, Phys. Rev. Lett. 83, 1715 (1999); P.G. kevrekidis et al., J. Low Temp. Phys. 120, 205 (2000); L. Salasnich et al., J. Phys. B: At. Mol. Opt. Phys. 35, 3205 (2002); Z. Rapti et al., J. Phys. B: At. Mol. Opt. Phys. 37, S257 (2004).
[12] As far as longitudinal excitations are concerned, the effects of the axial harmonic confinement on $\omega(q)$ are small and the condensate can be safely assumed to be infinite, thus allowing one to exploit the Bloch symmetry of Bogoliubov excitations.
[13] M. Krämer et al., Eur. Phys. J. D 27, 247 (2003).
[14] $N_q$ is the integral of the momentum distribution in an interval $\Delta q \sim 2\pi/\hbar Z$ around $\pm q$, divided by the total number of particles, where $Z$ is the axial size of the condensate.
[15] C. Tozzo, M. Krämer, and F. Dalfovo, in preparation.
[16] A process in which $\pm q$ quasiparticles are created via stimulated emission in the linear response regime and in the presence of quantum depletion is suggested in Ref. [2]. This mechanism can also provide a seed for the subsequent parametric amplification.
[17] M. Köhl et al., J. Low. Temp. Phys. 138, 635 (2005).
[18] M. Krämer, L. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 88, 180404 (2002).
[19] H. Moritz et al., Phys. Rev. Lett. 91, 250402 (2003).