INDUCTIVE SOLUTION OF THE
TANGENTIAL CENTER PROBLEM ON ZERO-CYCLES

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Abstract. Given a polynomial \( f \in \mathbb{C}[z] \) of degree \( m \), let \( z_1(t), \ldots, z_m(t) \) denote all algebraic functions defined by \( f(z_k(t)) = t \). Given integers \( n_1, \ldots, n_m \) such that \( n_1 + \ldots + n_m = 0 \), the tangential center problem on zero-cycles asks to find all polynomials \( g \in \mathbb{C}[z] \) such that \( n_1 g(z_1(t)) + \ldots + n_m g(z_m(t)) \equiv 0 \).

The classical Center-Focus Problem, or rather its tangential version in important non-trivial planar systems lead to the above problem.

The tangential center problem on zero-cycles was recently solved in a preprint by Gavrilov and Pakovich [14].

Here we give an alternative solution based on induction on the number of composition factors of \( f \) under a generic hypothesis on \( f \). First we show the uniqueness of decompositions \( f = f_1 \circ \ldots \circ f_d \), such that every \( f_k \) is 2-transitive, monomial or a Chebyshev polynomial under the assumption that in the above composition there is no merging of critical values.

Under this assumption, we give a complete (inductive) solution of the tangential center problem on zero-cycles. The inductive solution is obtained through three mechanisms: composition, primality and vanishing of the Newton-Girard component on projected cycles.

1. Introduction

A classical problem in planar polynomial vector fields starting from Poincaré is the center-focus problem. The problem asks for the determination of mechanisms leading to the creation of a center (a singularity surrounded by a continuous family of closed orbits) rather than a focus (singularity attracting or repelling all nearby trajectories). The problem has not yet been solved in a satisfactory way except for quadratic vector fields [12, 16].

The center-focus problem has its infinitesimal version: Starting from a vector field \( X_0 \) having a center, find all perturbations \( X_\lambda \) preserving the center. More generally, given a vector field \( X_0 \) having a continuous family \( \gamma_0(t) \) of closed orbits, determine the deformations \( X_\lambda \) of \( X_0 \) preserving these closed orbits. That is, we ask the deformed family \( \gamma_\lambda(t) \) of orbits of \( X_\lambda \) to be closed, too.

Taking a parametrized transversal \( T \) to the family of closed curves \( \gamma(t) \), one defines the displacement map \( \Delta \) as the first return map minus identity along trajectories of \( X_\lambda \). Then, a continuous family of closed curves is preserved, if the displacement map of the deformation along the chosen family of closed curves is identically zero.
The most popular family of polynomial systems having a continuous family of closed orbits is the family of Hamiltonian vector fields \( X_F = -\frac{\partial F}{\partial y} \frac{\partial}{\partial x} + \frac{\partial F}{\partial x} \frac{\partial}{\partial y} \). The trajectories of \( X_F \) lie in the level curves of the Hamiltonian. Consider their deformations \( X_F + \epsilon Y \). Taking a transversal \( T \) to a family \( \gamma(t) \) of closed curves of the Hamiltonian vector field \( X_F \) parametrized by \( F \), the displacement function of the above deformation is of the form

\[
\Delta(t, \epsilon) = -\epsilon \int_{\gamma(t)} \omega_Y + o(\epsilon),
\]

where \( \omega_Y \) is the dual form to the vector field \( Y \) and \( o(\epsilon) \) is a function depending on \( t \), but tending to zero faster than \( \epsilon \), for \( \epsilon \to 0 \). The function \( t \mapsto I(t) = \int_{\gamma(t)} \omega_Y \), \( \gamma(t) \subset F^{-1}(t) \), is an abelian integral.

A necessary condition for having a solution of the infinitesimal center problem is the vanishing of its first-order term given by the abelian integral \( I(t) \).

This motivates the tangential (or first-order) center problem. We formulate it in its complex form.

**Problem 1.1. Tangential center problem.** Given a polynomial \( F \in \mathbb{C}[x, y] \) and a continuous family \( \gamma(t) \) of cycles belonging to the first homology group \( H_1(F^{-1}(t)) \), find all polynomial forms \( \omega \) such that the abelian integral

\[
I(t) = \int_{\gamma(t)} \omega
\]

vanishes identically.

The problem was solved by Ilyashenko under a genericity assumption on \( F \). He proves that under a genericity condition on \( F \), for any family of cycles \( \gamma(t) \in H_1(F^{-1}(t)) \), the abelian integral (1.1) vanishes identically if and only if the form \( \omega \) is relatively exact, i.e., of the form \( \omega = GdF + dR \), for some \( G, R \in \mathbb{C}[x, y] \).

Without the genericity hypothesis on \( F \), the claim is false and the tangential center problem is open. One important non-generic case is the hyper-elliptic case when \( F(x, y) = y^2 + f(x) \), \( f \in \mathbb{C}[x] \). For hyper-elliptic \( F \), the problem was solved by Christopher and the third author under the hypothesis that the family of cycles \( \gamma(t) \) is a family of vanishing cycles. The tangential center problem is open even in the hyper-elliptic case for general family \( \gamma(t) \) of cycles.

Let us note a related problem studied by Gavrilov and Bonnet and Dimca.

**Problem 1.2. Integrability problem.** Given a polynomial \( F \in \mathbb{C}[x, y] \), find all polynomial forms \( \omega \) such that the abelian integral \( I(t) = \int_{\gamma(t)} \omega \) vanishes identically along any cycle \( \gamma(t) \in H_1(F^{-1}(t)) \).

Note the difference: in the Integrability problem one asks for the vanishing of abelian integrals along all cycles and not just one family of cycles as in the Tangential center problem. Under generic hypothesis on \( F \), Ilyashenko proves that by monodromy one family of cycles generates all cycles. Hence the vanishing of abelian integrals along one family of cycles implies the vanishing along all cycles and the two problems coincide.

In general it is false. It is easy to see that there are solutions of the tangential center problem due to the presence of a symmetry on a family of cycles. This solution will not necessarily be a solution for another family of cycles not respecting the symmetry. The form is hence not relatively exact as Abelian integrals of relatively exact forms vanish along any cycle of \( \gamma(t) \).

This paper is dedicated to the study of the tangential center problem on zero-cycles:
Problem 1.3. Tangential center problem on zero-cycles. Given a polynomial $f \in \mathbb{C}[x]$ and a family of zero-cycles $C(t)$ of $f$, determine all functions $g \in \mathbb{C}[x]$ such that $\int_{C(t)} g$ vanishes identically.

Here, integration is just calculation of the value of a function at some points $z_i(t)$ determining the cycle $C(t) = \sum n_i z_i(t)$ (for more details see the next section). Note that in this case abelian integrals on zero-cycles are in fact algebraic functions, so the problem is certainly easier than the initial problem.

Nevertheless, in [2], we showed that the tangential center problem for the hyper-elliptic case is directly related to the above tangential center problem on zero-cycles for hyper-elliptic abelian integrals (see also [14]).

In [2] we introduced the classes of balanced and unbalanced cycles. We solved the tangential center problem on zero-cycles by induction under the hypothesis that the initial cycle is unbalanced and that in the inductive process one encounters only unbalanced cycles. We called such a cycle totally unbalanced. The main result from [2] can be resumed by saying that under the hypothesis that the cycle $C(t)$ is totally unbalanced, the only mechanism producing tangential centers is a composition (i.e. symmetry) mechanism or a sum of composition mechanisms. Our proof was based on results of Pakovich-Muzychuk [19] relative to the composition conjecture in the moment problem for the Abel equation. In particular we used Pakovich-Muzychuk’s characterization of invariant irreducible spaces. In this paper we deal with the remaining case of balanced cycles or cycles leading to balanced cycles in the induction process.

While we were preparing the present paper, the preprint [14] of Gavrilov and Pakovich appeared, solving the general tangential center problem on zero-cycles.

Their solution is based on three steps:

(i): Description of all possible irreducible subspaces of $Q^m$ invariant by the monodromy $G_f$ of $f$.

(ii): Given a cycle $\delta(t) = \sum_{i=1}^{m} n_i z_i(t)$ characterized by $(n_1, \ldots, n_m) \in \mathbb{Z}^m$, provide a method allowing to decompose the invariant space generated by the action of monodromy on $(n_1, \ldots, n_m)$ in a direct sum of irreducible $G_f$-invariant subspaces.

(iii): For a given irreducible $G_f$-invariant subspace $V$, describe the space $Z_V$ consisting of polynomials $g$ such that $\int_{\delta(t)} g \equiv 0$, for all $\delta(t) \in V$.

Our approach is different. We solve the tangential center problem on zero-cycles by induction on the number of composition factors of $f$. Theorem A together with Theorem 2.3 of [2] gives the basis of induction and Theorems B and C give the induction step. The solution is given for arbitrary cycles, but under a generic hypothesis on the polynomial $f$. We think that our inductive approach sheds new light to the complicated structure of the space of solutions of the tangential center problem on zero-cycles. In particular it isolates three mechanisms leading to a tangential centers for zero-cycles: composition, primality and vanishing of the Newton-Girard component on projected cycles. We illustrate the complex structure of the solution of the tangential center problem by some examples.

First let us explain the hypothesis under which our study is done. It follows from the Burnside-Schur theorem that primitive (i.e., undecomposable) composition factors of a polynomial $f$ are of one of the following three types: 2-transitive, linearly equivalent to a monomial $z^k$ or a Chebyshev polynomial $T_k$, with $k$ prime. Note that a composition of monomials is a monomial and similarly a composition of Chebyshev polynomials is a Chebyshev polynomial.
Given a polynomial \( f \) let \( f = f_0 \circ \cdots \circ f_d \) be a decomposition of the polynomial \( f \) in its composition factors, which are 2-transitive, Chebyshev or monomial not necessarily of prime degree.

We completely solve the tangential center problem on zero-cycles when for every \( k = 1, \ldots, d \) the critical values of \( f_0 \circ \cdots \circ f_{k-1} \) and \( f_k \) do not merge in the composition \( f_0 \circ \cdots \circ f_k \) (see Definition 2.7). We prove that under the above hypothesis, the decomposition of \( f \) as \( f = f_0 \circ \cdots \circ f_d \) is unique. The hypothesis of non-merging of critical values allows us to decompose the monodromy group of \( f_0 \circ \cdots \circ f_k \) in a semidirect product of the monodromy group of \( f_0 \circ \cdots \circ f_{k-1} \) and the monodromy group of \( f_k \). Under these hypotheses, we solve recursively the tangential center problem on zero-cycles. Using new methods, we show that in addition to the composition mechanisms, two new mechanisms leading to tangential centers exist. They both appear for balanced cycles on the level of the basis of recursion and in the recursion step.

One mechanism is related to the primality of monomial or Chebyshev polynomial factors with the characteristic polynomial \( P_C(z) = \sum_{i=1}^{m} n_i z_i^{i-1} \) associated to the cycle \( C(t) = \sum_{i=1}^{m} n_i z_i(t) \). It is described by (1) of Theorem A (basis) and Theorem B (recursive step). The other mechanism is related to the vanishing of the Newton-Girard component on projected cycles in the case of 2-transitive factors. It is described by (2) of Theorem A (basis) and Theorem B (recursive step).

The motivation of the tangential center problem for the Abel equation

\[
x' = p(t)x^2 + \lambda q(t)x^3,
\]

proposed by Briskin, Françoise and Yomdin in a series of papers [3-7], was to obtain general mechanisms for the existence of centers. For \( p, q \) polynomials, the tangential center problem has been totally solved by Pakovich and Muzychuk [10] proving that there is a tangential center if and only if \( \int_0^t q(s) \, ds \) can be written as a sum of polynomials having a common factor with \( \int_0^t p(s) \, ds \). Moreover, Briskin, Roytvarf and Yomdin [8] have proved that composition (just one summand) generates all centers of (1.2) at infinity except for a finite number of exceptional cases. Nevertheless, there are centers and tangential centers for which composition is not the generating mechanism, like (1.2) when \( p, q \) are not polynomials (see [11] [9]) or, in a more general context, (hyperelliptic) planar systems (see Example 9.2 of [2]). We hope the mechanisms obtained here will shed some light on these problems.

2. Main results

Let \( f \in \mathbb{C}[z] \) be a polynomial of degree \( m \). Points \( z \in \mathbb{C} \) such that \( f'(z) = 0 \) are critical points of \( f \). Corresponding values \( t = f(z) \) are critical values and non-critical values are called regular values. Let \( \Sigma \) denote the set of critical values of \( f \). Then \( f : f^{-1}(\mathbb{C} \setminus \Sigma) \rightarrow \mathbb{C} \setminus \Sigma \) is a fibration with zero-dimensional fiber. For any regular value of \( t \) the fiber \( f^{-1}(t) \) consists of \( m \) distinct points. These points can be continuously transported along any path in \( \mathbb{C} \setminus \Sigma \). Replacing in the fibration \( f : f^{-1}(\mathbb{C} \setminus \Sigma) \rightarrow \mathbb{C} \setminus \Sigma \) the fibers \( f^{-1}(t) \) by their 0-th homology groups \( H_0(f^{-1}(t)) \) or their reduced 0-th homology groups \( \tilde{H}_0(f^{-1}(t)) \), one obtains the homology or reduced homology fibration.

Let \( t_0 \) be a regular value of \( f \) and \( \pi_1(\mathbb{C} \setminus \Sigma, t_0) \) the first homotopy group of the base with base point \( t_0 \). Then the continuous transport gives a morphism \( \pi_1(\mathbb{C} \setminus \Sigma, t_0) \rightarrow \text{Aut}(H_0(f^{-1}(t_0))) \) from the first homotopy group of the base to the group of automorphisms of the homology fiber. Its image is called the monodromy group \( G_f \) of \( f \). This group is isomorphic to the Galois group of \( f(z) - t \) seen as a polynomial over \( \mathbb{C}(t) \) (see [10] Th. 3.3, for example). Any section \( C \) of
the homology fibration is called a zero-chain. A section of the reduced homology fibration is called a zero-cycle.

Choosing an order \( \{z_1(t), \ldots, z_m(t)\} \) among the \( m \)-preimages of \( t \) by \( f \), a zero-chain of \( f \) is represented by

\[
C(t) = \sum_{i=1}^{m} n_i z_i(t), \quad n_i \in \mathbb{Z}
\]

and zero-cycles moreover satisfy

\[
\sum_{i=1}^{m} n_i = 0.
\]

Note that \( z_i(t) \), as well as \( C(t) \) are multivalued functions on \( \mathbb{C} \setminus \Sigma \). We also consider chains with coefficients in \( \mathbb{C} \) when necessary. Then we will specify the coefficient ring.

Let \( C \) be a chain of \( f \) and \( g \) a polynomial. The Abelian integral of \( g \) along a zero-cycle \( C \) is defined by

\[
\int_{C(t)} g := \sum_{i=1}^{m} n_i g(z_i(t)).
\]

This paper is dedicated to the solution of Problem 1.3 above.

In \cite{2} we introduced the notions of balanced and unbalanced cycles. We proved that if the cycle is unbalanced, then the tangential center problem is equivalent to solving some induced tangential center problems with \( f \) replaced by some composition factors of \( f \), hence polynomials of smaller complexity and smaller degree. Let us precise the notions. In order to be able to perform the induction we need to extend the notions from cycles to chains. Let \( \Gamma_m(f) \subset G_f \) denote the conjugacy class of \( \tau_\infty \), where \( \tau_\infty \in G_f \) is associated to a path winding once counter-clockwise around infinity. That is, \( \Gamma_m(f) \) is the set of all \( \sigma \circ \tau_\infty \circ \sigma^{-1} \), for any \( \sigma \in G_f \).

We label the roots so that \( \tau_\infty \), which is a permutation cycle of order \( m \), shifts the indices of the roots by one, i.e., \( \tau_\infty = (1, 2, \ldots, m). \)

**Definition 2.1.** We say that a chain \( C(t) \) of \( f \) is balanced if

\[
\sum_{i=1}^{m} n_{p_i} \epsilon_m^i = 0, \quad \text{for every } \tau = (p_1, p_2, \ldots, p_m) \in \Gamma_m(f),
\]

where \( \epsilon_m \) is any primitive \( m \)-th root of unity. If \( C(t) \) is not balanced, we say that \( C(t) \) is unbalanced.

The notion of balanced chain is independent on the way how permutation cycles are written. Let \( \tau \in \Gamma_m(f) \) be of the form \( \tau = \sigma \circ \tau_\infty \circ \sigma^{-1}, \ \sigma \in G_f \). Put \( (p_1, p_2, \ldots, p_m) = (\sigma(1), \sigma(2), \ldots, \sigma(m)) \) and

\[
(n_{p_1}, n_{p_2}, \ldots, n_{p_m}) = (n_{\sigma(1)}, n_{\sigma(2)}, \ldots, n_{\sigma(m)}) = (n_1, n_2, \ldots, n_m).
\]

Then, we can rephrase the definition of a chain \( C(t) \) being balanced as

\[
\sum_{i=1}^{m} n_{\sigma(i)} \epsilon_m^i = 0, \quad \text{for every } \sigma \in G_f.
\]

**Definition 2.2.** We call the polynomial

\[
P_C(z) = \sum_{n=1}^{m} n_j z^{j-1}
\]

characteristic polynomial of the chain \( C(t) \).
Note that if $C(t)$ is balanced, then $P_C(\epsilon_m) = 0$. The converse does not hold in general. The fact that the characteristic polynomial has the root $\epsilon_m$ depends on the election of the cycle of infinity, so indeed we should define the characteristic ideal as in [2], but in the special cases we use the characteristic polynomial, $C(t)$ is balanced if and only if $P_C(\epsilon_m) = 0$.

Assume that $f = \tilde{f} \circ h$, for $\tilde{f}, h \in \mathbb{C}[z]$, $\deg(h) = d$. Consider the imprimitivity system $B = \{B_k\}_{k=1,\ldots,m/d}$, where $B_k = \{i \in \{1,2,\ldots,m\} | h(z_i) = h(z_k(t))\}$, associated to $h$. (The general definition of imprimitivity system is recalled below in this section.)

**Definition 2.3.** The cycle $h(C(t))$ of $\tilde{f}$ called the projection of $C(t)$ by $h$ is defined by

\begin{equation}
h(C(t)) = \sum_{h(z_i(t))} \left( \sum_{h(z_j) = h(z_i)} n_j \right) h(z_i(t)) = \sum_{i \in B_k} \left( \sum_{j \in B_k} n_i \right) w_k(t).
\end{equation}

Here $w_1(t), \ldots, w_{m/d}(t)$ are all the different roots $h(z_i(t))$ of $\tilde{f}(z) = t$.

With the above notation, the main result of [2], Theorem 2.2 (ii), can be stated as follows.

**Theorem 2.4 ([2]).** Let $f \in \mathbb{C}[z]$, and let $C(t)$ be an unbalanced cycle. Then,

\[
\int_{C(t)} g \equiv 0, \quad \text{for } g \in \mathbb{C}[z]
\]

if and only if there exist $f_i, g_i, h_i \in \mathbb{C}[z], 1 \leq i \leq d$ such that $\deg(h_i) > 1$, $f = f_i \circ h_i$, $1 \leq i \leq d$, $g = \sum_{i=1}^d g_i \circ h_i$ and $\int_{h_i(C(t))} g_i \equiv 0$.

**Remark 2.5.** If the cycle $C(t)$ is unbalanced, then the tangential center problem reduces to solving it for the projected cycles. If some of these projected cycles are unbalanced, Theorem 2.4 applies again. In particular, if all projected cycles (in every successive step) are unbalanced, we call the initial cycle totally unbalanced and the problem is completely solved by induction using Theorem 2.4.

In this paper we study the remaining case, when $C(t)$ is balanced or in the progress of projecting we arrive at some balanced cycle. It appears that the decompositions of the polynomial $f$ play a central role in the problem.

The action of the monodromy group $G_f$ of $f$ on the set of solutions $z_i(t)$ of the equation $f(z) = t$ is closely related to the decomposability of $f$. When a group $G$ acts on a finite set $X$, it is said that the action is **imprimitive** if $X$ can be non-trivially decomposed into subsets of the same cardinality $B_i$ such that every $\sigma \in G$ sends each $B_i$ into $B_j$ for some $j$. Otherwise, the action is called **primitive**. The action of $G$ is said to be $2$-**transitive** if given any two pairs of elements of $X$, $(i,j)$ and $(k,l)$, there is an element $\sigma \in G$ such that $\sigma(i) = k$ and $\sigma(j) = l$. It is easy to prove that in these cases $X$ cannot be divided into disjoint subsets such that $G$ acts on them, which is one of the trivial cases of a primitive action. These disjoint sets $B_i$ are called **blocks** and they form a partition of $X$ called an **imprimitivity system**. This definition is consistent with the previous one (see e.g. [2] Prop. 4.1).

Precisely, the action of $G_f$ on $\{z_i(t)\}$ is imprimitive if and only if $f$ is decomposable (see [10] Prop. 3.6). Burnside-Schur Theorem (see e.g. [10] Th. 3.8) classifies primitive polynomials (i.e., a polynomial that cannot be written as a composition of two polynomials of degree greater than one):

**Burnside-Schur Theorem** Let $f$ be a primitive polynomial and $G_f$ its monodromy group. Then, one of the following holds:
(1) The action of $G_f$ on the $z_i(t)$ is 2-transitive. We call such a polynomial 2-transitive.
(2) $f$ is (linearly) equivalent to a Chebyshev polynomial $T_p$, where $p$ is prime.
(3) $f$ is (linearly) equivalent to $z^p$ where $p$ is prime.

We give a solution of the tangential center problem by induction on the number of composition factors $f$, under some additional hypotheses. First we solve the problem for basic composition factors: 2-transitive, monomial or Chebyshev of not necessarily prime degree (basis of induction). Monomial and Chebyshev cases are similar. They are hence treated together.

We introduce first some notations. Given a polynomial $f$ of degree $m$ and $z_i(t)$, $i = 1, \ldots, m$ the roots of $f(z) = t$, let $s_k$, $k = 0, \ldots, m - 1$ denote the sums of $k$-th powers of roots of $f - t$:

$$s_k = \sum_{i=1}^{m} z_i^k(t).$$

Note that the Newton-Girard formulae express $s_0, \ldots, s_{m-1}$ in function of the coefficients of $f$. In particular they are independent of $t$. Let $s(f) = (s_0, \ldots, s_{m-1}) \in \mathbb{C}^m$ denote the Newton-Girard vector of $f$.

When convenient, using the division algorithm, we will write $g(z)$ as $g(z) = \sum_{k=0}^{m-1} g_k(f)z^k$. That is, we express $g(z)$ as a polynomial in $z$ of degree $\deg g < m = \deg f$, with coefficients in $\mathbb{C}[f]$. Substituting $f = t$ for integration each coefficient $g_k$ becomes a polynomial $g_k(t)$ in $t$.

We denote $g(t) = (g_0(t), \ldots, g_{m-1}(t))$. Let $\prec, \succ$ denote the scalar product in $\mathbb{C}^m$: $\langle g(t), s(f) \rangle = \sum_{i=0}^{m-1} g_i(t)s_i$.

**Theorem A.** Assume that $C(t) = \sum n_j z_j(t)$ is a balanced chain of a polynomial $f$.

(1) If the action of $G_f$ is 2-transitive, then there exists $n$ such that $n_j = n$ for every $1 \leq j \leq m$. Moreover, for every $g(z) = \sum_{k=0}^{m-1} g_k(f)z^k$, $\int_{C(t)} g \equiv 0$ if and only if $g(t) \in s(f)^\perp$.

(2) If $f(z) = z^m$ (resp. $f(z) = T_m(z)$), then in the solution $g = \sum_{i=0}^{m-1} g_j h_j(z)$, with $h_j(z) = z^j$, (resp. $h_j(z) = T_j(z)$) of the tangential center problem $\int_{C(t)} g \equiv 0$ the only terms $h_j$ present are for $j$ such that $k = \gcd(j, m)$ such that $P_C(\epsilon_m^k) = 0$, where $\epsilon_m$ is a primitive $m$-th root of unity.

Condition (2) can also be written as: If $f(z) = z^m$ (resp. $f(z) = T_m(z)$), then $\int_{C(t)} g \equiv 0$ if and only if $g(z) = \sum_{j=0}^{m-1} g_j h_j(z)$, where $g_j \in \mathbb{C}[f]$, $h_j(z) = z^j$ (resp. $h_j(z) = T_j(z)$), for any $j$ is such that $\Phi_{m/k}(z)|P_C(z)$, where $k = \gcd(m, j)$, $P_C(z) = \sum_{i=1}^{m} n_i z_i^{j-1}$ is the characteristic polynomial of the chain $C$ and $\Phi_n(z)$ is the $n$-th cyclotomic polynomial, that is,

$$\Phi_n(z) = \prod_{1 \leq j \leq n, \gcd(j, n) = 1} (z - \epsilon_n^{\frac{jm}{n}}).$$

**Remark 2.6.** (1) We explain here the geometric idea behind the proof of Theorem A. Let $C(t)$ be a balanced chain. In order to prove (1), one considers the case when the monodromy group $G_f$ of $f$ is 2-transitive and one fixes any root $z_i(t)$ and considers its stabilizer $H_i$. The assumption that $G_f$ is 2-transitive means that the stabilizer $H_i$ of the root acts transitively on the other roots. Averaging by the action of the stabilizer $H_i$, the condition that the chain $C(t)$ is balanced, gives a relation independent of $i$. From this relation it follows that in the chain $C(t) = \sum_i n_i z_i(t)$, all $n_i$ coincide. If the chain is a cycle, the cycle is trivial.
Once we know that all $a_i$ in a chain coincide, the characterization of the vanishing of the integral $\int_{C(t)} g$ by the orthogonality of the vector $g(t)$ to the Newton-Girard vector $s(f)$ is just rewriting the vanishing of the integral.

(2) If $f(z) = z^m$, then the roots $z_i$ are simply given by the roots of unity $e^{i\pi/m}$.

By explicit calculation $\int_{C(t)} g = \sum_{j=0}^{m-1} g_j s_j P_C(\epsilon_m)$ and the theorem follows. The orbit of $C(t)$ by monodromy is a sum of irreducible invariant spaces. In this case the irreducible invariant spaces are easily calculated and are related to the divisors of $m$. The result follows from explicit calculations.

If $f(z) = T_m(z)$, then $\int_{C(t)} g = \alpha_j(t) P_C(\epsilon_m) + \overline{\alpha_j(t)} P_C(\bar{\epsilon}_m)$, with $\alpha_j(t) = \frac{1}{2} e^{i\xi(t)}$ and $\xi(t) = \arccos(t)$. This case is next treated similarly as the monomial case.

Next we show how to reduce the tangential center problem on a cycle $C(t)$ of $f = f \circ h$, where $h$ is 2-transitive or linearly equivalent to a monomial or a Chebyshev polynomial of not necessarily prime degree. In the step of induction, we reduce the original tangential center problem of a cycle of $f$ to the tangential center problem of the projected cycle $h(C(t))$, which is a cycle of the function $\tilde{f}$, simpler than the original function $f$. We do the induction step under the generic hypothesis that the critical values of $\tilde{f}$ and $h$ do not merge:

**Definition 2.7.** Let $\tilde{f}$ and $h$ be two nonlinear polynomials and let $f = \tilde{f} \circ h$. We say that the critical values of $f$ and $h$ do not merge if

1. \( \{ f(z); \tilde{f}'(h(z)) = 0 \} \cap \{ f(z); h'(z) = 0 \} = \emptyset \),
2. $\tilde{f}$ is injective on the set of critical values of $h$.

As we prove in Lemma 5.1, condition (1) of Definition 2.7 assures that the monodromy group $G_\tilde{f}$ is a semidirect product of the subgroups $N_h$ and $G_f$, $G_\tilde{f} = N_h \rtimes G_f$, where $N_h$ is the normal closure of the monodromy group $G_h \subset G_\tilde{f}$ with the natural injection (which in general is not a group morphism). Condition (2) of Definition 2.7 assures that there exists a group morphisms $\phi : G_h \to G_\tilde{f}$.

We divide the induction step in two cases, according to the nature of $h$. We study first the case when $h$ is 2-transitive.

**Theorem B.** Let $f(z)$ be a polynomial such that $f = \tilde{f} \circ h$ with $\tilde{f}, h \in \mathbb{C}[z]$, $1 < \deg(h) = d < m = \deg f$. Let $C(t) \in \widehat{H}_0(f^{-1}(t))$ be a balanced cycle of $f$.

Assume that the critical values of $\tilde{f}$ and $h$ do not merge and that $h$ is 2-transitive.

Then $g(z) = \sum_{i=0}^{d-1} z^i g_i(h(z))$, verifies

\[
\int_{C(t)} g \equiv 0
\]

if and only if

\[
\int_{h(C(t))} \tilde{g} \equiv 0,
\]

and $g_i \in \mathbb{C}[w]$, $i = 0, \ldots, d - 1$, are solutions of the linear system

\[
\langle g, s \rangle \equiv \tilde{g}(w).
\]

**Remark 2.8.** Thus, the solution $g$ and the inductive solution $\tilde{g}(w)$ are related by the linear system of equations $\tilde{g}(w) = \sum_{i=0}^{d-1} s_i g_i(w)$. Here $s_i$, $i = 0, \ldots, d - 1$ are expressed through the coefficients of $h$ by the Newton-Girard formulae for $h(z) - w$ and they are independent of $w$. We call $s(h) = (s_0, \ldots, s_{d-1}) \in \mathbb{C}^d$ the Newton-Girard vector of $h$. Let $\pi_h(g) = \frac{\langle g(w), s(h) \rangle}{|s(h)|}$ be the component of the vector $g(w) = (g_0(w), \ldots, g_{d-1}(w)) \in \mathbb{C}[w]^d$ representing $g$ in the direction of the Newton-Girard
vector $s(h)$. Theorem 2.1 can be resumed by saying that under above conditions, (2.7) is equivalent to the vanishing of $\int_{h(C(t))} \pi_h(g)$, where $\pi_h(g)$ is the component of the vector representing $g$ in the direction of the Newton-Girard vector of $h$. See Example 2.13.

Remark 2.9. The geometric idea of the proof of Theorem 2.1 is similar to the one of Theorem A in the case of $h$ 2-transitive. One applies averaging by the stabilizer $H_{i_0,j_0}$ of a root $z_{i_0,j_0}$ to the identity satisfied by balanced cycles. The roots are regrouped in groups defined by the imprimitivity system of $h$. One does not prove now that all coefficients $n_{ij}$ of the cycle are the same, but that all coefficients $n_{i,j_0}$ corresponding to roots in the same block $B_{i_0}$, are the same (Proposition 5.1).

Next, given $g(z) = \sum_{i=0}^{n-1} z^i g_i(h(z))$, in the integral $\int_{C(t)} g$ one regroups all terms according to the block $B_{i_0}$ to which they belong. This allows to express $\int_{C(t)} g$ through an integral on the projected cycle $h(C(t))$ and the condition $\int_{C(t)} g = 0$ becomes equivalent to the vanishing of $\int_{h(C(t))} \pi_h(g)$.

Assume that $f = \tilde{f} \circ h$, for $\tilde{f}, h \in \mathbb{C}[z]$, $\deg(h) = d$. In order to formulate the induction step in the case of monomial or Chebyshev $h$ (Theorem 3.2), in addition to the projection of a cycle $C(t)$ by $h$ we need the notion of $h$-invariant parts of $C(t)$.

Definition 2.10. For each cycle $C(t)$ of $f$ and the decomposition $f = \tilde{f} \circ h$ of $f$, consider the imprimitivity system $B = \{B_1, \ldots, B_{m/d}\}$ associated to $h$. We define the $h$-invariant parts of $C(t)$ as the chains $C_k(t) = \sum_{i \in B_k} n_i z_i(t)$ of $f$, or as chains $\check{C}_k(w) = \sum_{i \in B_k} n_i z_i(w)$, $h(z_i) = w$, of $h$.

The $h$-invariant part $C_k(t)$ corresponds to the part of the cycle presented by the roots in the $k$-th line in Figure 4. Relations $C(t) = \sum_{k=1}^{m/d} C_k(t)$ and $\check{C}_k(w) = C_k(\check{f}(w))$ hold. Note that even if $C(t)$ is a cycle, the $h$-invariant parts $C_k(t)$ are only chains in general.

Theorem C. Let $f(z)$ be a polynomial such that $f = \tilde{f} \circ h$ with $\tilde{f}, h \in \mathbb{C}[z]$, $1 < \deg(h) = d < m = \deg f$. Let $C(t) \in \tilde{H}_0(f^{-1}(t))$ be a balanced cycle of $f$.

Assume that the critical values of $\tilde{f}$ and $h$ do not merge and let $h$ be a monomial or a Chebyshev polynomial. Let $\check{z}_k(w)$, $k = 1, \ldots, d$, denote the zeros of $h(z) = w$. Then the $h$-invariant parts $\check{C}_k(w)$, $k = 1, \ldots, m/d$, of $C(t)$ are balanced.

Moreover,

$$\int_{C(t)} g \equiv 0$$

if and only if $g$ is of the form

$$g(z) = \tilde{g}(h(z))/(d-1) + u(z),$$

where $\tilde{g}(w)$ is a polynomial such that

$$\int_{h(C(t))} \tilde{g} \equiv 0,$$

and

$$\int_{\check{C}_k(w)} u \equiv 0, \quad 1 \leq k \leq m/d.$$  

Remark 2.11. The geometric idea of the proof of Theorem C is the following. One considers the imprimitivity decomposition $\{B_1, \ldots, B_m\}$ of the roots associated to $h$. Fix $1 \leq i_0 \leq m$. Let $H_{i_0,j}$, $j = 1, \ldots, d$, be the stabilizers of the root $z_{i_0,j} \in B_{i_0}$ and $H_{i_0} = \bigcap_{j=1, \ldots, d} H_{i_0,j}$ their intersection. Averaging with respect to $H_{i_0}$ the identity
Moreover, averaging with respect to $H_{10}$ in the zero-dimensional abelian integral, one obtains that there exists a polynomial $p_{10}$ such that the vanishing of the abelian integral is equivalent to the integral on the $h$-invariant parts being equal to $p_{10}$. For these systems, we prove that the general solution is equal to the sum of a particular solution $\tilde{y}(h(z))/(d-1)$ of the non-homogeneous system with the general solution $u$ of the homogeneous system.

Note that as $\tilde{C}_k(w)$ are balanced chains of $h$, (2.11) is solved in Theorem $A$ and the solution of (2.10) is given by the induction hypothesis.

We prove Theorems $B$ and $C$ in Section 6.

If $\tilde{f}$ and $h$ merge, then Theorems $B$ and $C$ also gives tangential centers. However, there can be other solutions not covered by them. Indeed, by perturbing $f = \tilde{f} \circ h$ we can assure that $\tilde{f}$ and $h$ do not merge and Theorems $B$ and $C$ gives solutions of the deformed system, which in the limit give solutions of the original system.

It follows from the Burnside-Schur Theorem that any polynomial $f$ can be decomposed as $f = f_0 \circ \cdots \circ f_d$, where each factor $f_k$ is either 2-transitive, linearly equivalent to a monomial or linearly equivalent to a Chebyshev polynomial. Note that a composition of monomials is a monomial and a composition of Chebyshev polynomials is a Chebyshev polynomial. We do not assume that the degree of these polynomials $f_k$ linearly equivalent to a monomial or to a Chebyshev polynomial is prime. Putting together Theorem 2.4 and Theorems $A$, $B$ and $C$, we solve inductively the tangential center problem under the following hypothesis on $f$:

Hypothesis 2.12. Let $f \in \mathbb{C}[z]$ have a decomposition $f = f_0 \circ \cdots \circ f_d$, with $f_k$ 2-transitive, linearly equivalent to a monomial or to a Chebyshev polynomial. We assume that the critical values of $f_0 \circ \cdots \circ f_{k-1}$ and $f_d$ do not merge for any $1 \leq k \leq d$.

We show that under Hypothesis 2.12, there exists a unique decomposition $f = f_0 \circ \cdots \circ f_d$, with $f_k$ 2-transitive, linearly equivalent to a monomial or to a Chebyshev polynomial, satisfying Hypothesis 2.12. More precisely:

Proposition 2.13. Assume that $f = f_0 \circ \cdots \circ f_d$ for some polynomials $f_0, \ldots, f_d$ such that every $f_k$ is either 2-transitive, linearly equivalent to a monomial or linearly equivalent to a Chebyshev polynomial, and that Hypothesis 2.12 holds. If $f = f_0 \circ \cdots \circ f_{d}$ is another decomposition of $f$ for some polynomials $f_0, \ldots, f_{d}$ such that every $f_k$ is either 2-transitive, linearly equivalent to a monomial or linearly equivalent to a Chebyshev polynomial, then there exist $\tilde{f}_{k,1}, \ldots, \tilde{f}_{k,d(k)}$, $\tilde{f}_{k,1} = j_{k-1,d(k)-1} + 1$ ($k > 0$), such that (up to linear transformations)

$$
\tilde{f}_k = \tilde{f}_{k,1} \circ \cdots \circ \tilde{f}_{k,d(k)}, \quad \text{for every } k = 0, 1, \ldots, d.
$$

Moreover, if the critical values of $f_0 \circ \cdots \circ \tilde{f}_{k-1}$ and $\tilde{f}_k$ do not merge for any $1 \leq k \leq d$, then $d = \tilde{d}$, and (up to linear transformations)

$$
\tilde{f}_k = \tilde{f}_k, \quad \text{for every } k = 0, 1, \ldots, d.
$$

A decomposition of $f$ in primitive polynomials is not unique in general. To get uniqueness, we have to regroup successive monomials or Chebyshev factors (or factors linearly equivalent to one of them) that commute (see [13]). Ritt’s Theorem states that if a polynomial $f$ admits two different decompositions, then there exist monomials or Chebyshev factors (or a factor linearly equivalent to one of the previous types) in a decomposition of $f$ into primitive polynomials that commute. Note for instance that $z^5 = (z^2)^3 = (z^3)^2$. Section 5, where we prove Proposition 2.13, deals with the converse problem, that is, given a decomposition of $f = f \circ h$, Hypothesis 2.12 ensures that any other decomposition of $f$ is a further
decomposition of $\tilde{f}$ and $h$, or, in other words, there exists no commuting factors between $\tilde{f}$ and $h$.

To conclude this section, we give two examples of the application of Theorems A, B and C.

**Example 2.14.** First, let us consider a polynomial composition of two 2-transitive polynomials $f = \tilde{f} \circ h$, with $\tilde{f}(z) = z^3 - z^2 + z$ and $h(z) = z^3 + 2z^2 - 1$, and the cycle

$$C(t) = z_1(t) - z_2(t) + z_4(t) - z_5(t) + z_7(t) - z_8(t),$$

where we are assuming that $(1, 2, 3, 4, 5, 6, 7, 8, 9)$ is the permutation associated to a loop around infinity. Then, the imprimitivity systems are the equivalence classes modulo divisors of $m$, and it can be easily checked that Hypothesis 2.12 holds. Moreover, $C(t)$ is balanced.

![Figure 1. Cycles $C(t)$ and $h(C(t))$ in Example 2.14](image)

Write $g(z) = g_0(h(z)) + zg_1(h(z)) + z^2g_2(h(z))$, $g_i \in \mathbb{C}[w]$. By Theorem B, $g$ satisfies $\int_{C(t)} g \equiv 0$ if and only if

$$\int_{h(C(t))} \tilde{g} \equiv 0,$$

where $h(C(t)) = w_1(t) - w_2(t)$ is unbalanced, and $\tilde{g}(w) = 3g_0(w) - 2g_1(w) + 4g_2(w)$.

Now, the solutions of $\int_{h(C(t))} \tilde{g} \equiv 0$ are given by Theorem 2.4. As $\tilde{f}$ is primitive, the solutions are of the form $\tilde{g}(w) = k_0(\tilde{f}(w))$, for any polynomial $k_0 \in \mathbb{C}[w]$.

Then $\int_{C(t)} g \equiv 0$ if and only if $k_0(\tilde{f}(w)) = 3g_0(w) - 2g_1(w) + 4g_2(w)$. For instance, if $k_0(w) = 1$, and $g_0, g_1, g_2$ are quadratic polynomials: $g_i(z) = a_{i2}z^2 + a_{i1}z + a_{i0}$, then the space of solutions is 6-dimensional given by

$$g(z) = \frac{1}{3} + \frac{2a_{10} - 2a_{11} + 2a_{12} - 4a_{20} + 4a_{21} - 4a_{22} + (a_{10} - a_{11} + a_{12}) z}{3}$$

$$+ \frac{1}{3} \left( 4a_{11} - 8a_{12} + 3a_{20} - 11a_{21} + 19a_{22} \right) z^2 + \frac{4}{3} \left( 2a_{11} - 4a_{12} - a_{21} + 2a_{22} \right) z^3$$

$$+ \left( a_{11} + \frac{2}{3} a_{12} + 3a_{21} - 14a_{22} \right) z^4 + \frac{1}{3} \left( 20a_{12} + 3a_{21} - 22a_{22} \right) z^5$$

$$+ \frac{2}{3} (7a_{12} + 4a_{22}) z^6 + (a_{12} + 4a_{22}) z^7 + a_{22} z^8$$

for any $a_{10}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22} \in \mathbb{C}$.
Example 2.15. Let us consider a polynomial composition of a 2-transitive polynomial and a monomial, \( f = f \circ h \), with \( f(w) = w^3 - w^2 + w \) and \( h(z) = z^6 \), and the cycle

\[
C(t) = z_1(t) - z_2(t) + z_7(t) - z_6(t) + z_{13}(t) - z_{14}(t),
\]

where we are assuming that \( (1, 2, \ldots, 17, 18) \) is the permutation associated to a loop around infinity. It can be checked that Hypothesis 2.12 holds. Moreover, \( C(t) \) is balanced.

It is easy to see that \( h(C(t)) = 3w_1(t) - 3w_2(t) \) is an unbalanced cycle of \( \tilde{f} \). Then, the solutions of

\[
\int_{h(C(t))} \tilde{g} \equiv 0
\]

are given by Theorem 2.4. More precisely, as \( \tilde{f} \) is primitive, it follows that \( \tilde{g}(w) \) is a function of \( \tilde{f} \). Let \( \tilde{g}(w) = g_0(f(w)) \). On the other hand, the \( h \)-invariant parts of the cycle \( C(t) \) are given by \( \tilde{C}_1(t) = w_1(t) + w_3(t) + w_5(t), \tilde{C}_2(t) = -w_1(t) - w_3(t) - w_5(t) \) and \( \tilde{C}_3(t) = 0 \). Hence \( P_{\tilde{C}_1}(w) = 1 + w^2 + w^4, P_{\tilde{C}_2} = -P_{\tilde{C}_1} \) and \( P_{\tilde{C}_3} = 0 \).

If \( u = \sum_{i=1}^5 u_j z^j \), where \( u_j \in \mathbb{C}[z^6] \), by Theorem A (2), the solutions of

\[
\int_{\tilde{C}_k(t)} u \equiv 0 \quad \text{for } k = 0, 1, 2,
\]

are given by \( u_0 = u_3 = 0 \).

Finally, by Theorem C, the solutions of

\[
\int_{C(t)} g \equiv 0
\]

are

\[
g(z) = g_0(f(z)) + u(z),
\]

for any polynomials \( g_0 \) and \( u = \sum_{i=1}^5 u_j z^j \) such that \( u_0 = u_3 = 0 \).

3. Structure of the space of solutions

In this section we give the general structure of the space of solutions. It will be used in the next section for solving the tangential center problem in two basic cases: the monomial and the Chebyshev case.

Choosing a basis \( z_1(t), \ldots, z_m(t) \) in \( H_0(f^{-1}(t)) \), one can identify each chain \( C(t) = \sum_{i=1}^m n_i z_i(t) \) with \( n(C) = (n_1, \ldots, n_m) \in \mathbb{C}^m \). Similarly, to each vector \( v = (v_1, \ldots, v_m) \in \mathbb{C}^m \) we associate the chain \( C_v := \sum_{i=1}^m v_i z_i(t) \). Then,

\[
\int_{C_v(t)} g = < (g(z_1(t)), \ldots, g(z_m(t))), \bar{n}(C(t)) >,
\]

where \( < -, - > \) is the usual scalar product in \( \mathbb{C}^m \) and \( \bar{n} \) denotes the complex conjugate of \( n \).

Given \( f \) and \( C \) we search for all \( g \) such that \( \int_{C(t)} g \equiv 0 \). Let \( G_f \) be the monodromy group of \( f \) and \( V(t) \subset H_0(f^{-1}(t)) \) the vector space generated by the orbit of a chain \( C(t) \) of \( f \) by \( G_f \). By analytic continuation, the vanishing of \( \int_{C(t)} g \) is equivalent to the vanishing of \( \int_{\sigma(C(t))} g \), for any \( \sigma(C(t)) \in V(t) \). Let \( H^0(f^{-1}(t)) \) be the dual space to \( H_0(f^{-1}(t)) \). Hence, from abstract point of view, the tangential center problem is simply the problem of determining \( (V(t)^{\perp})^* \): the dual space to the orthogonal complement of \( V \). The space \( H^0(f^{-1}(t)) \) is organized as an \( m \)-dimensional \( \mathbb{C}[t] \)-module, with multiplication defined by

\[
P(t)g(z) = P(f(z))g(z).
\]

Let \( V_r \) denote \( r \)-periodic (mod \( m \)) vectors in \( \mathbb{Q}^m \) and let \( D(f) \) be the set of positive divisors \( d \) of \( m = deg(f) \) such that there exists a decomposition \( f = f \circ h \).
where \( \text{deg}(h) = d \). The structure of the \( G_f \)-invariant subspaces of \( \mathbb{Q}^m \) is determined by Lemma 5.1 of \cite{19}.

**Lemma 3.1** (\cite{19}). Any \( G_f \)-irreducible invariant subspace of \( V(t) \) is of the form

\[
U_r(t) = V_r(t) \cap (V_{r_1}(t)^\perp \cap \cdots V_{r_j}(t)^\perp),
\]

where \( r \in D(f) \) and \( \{r_1, \ldots, r_j\} \) is a complete set of divisors of \( m \) covered by \( r \), that is, they are all the maximal divisors of \( r \) in \( D(f) \). The subspaces \( U_r(t) \) are mutually orthogonal and any \( G_f \)-invariant subspace of \( \mathbb{Q}^m \) is a direct sum of some \( U_r(t) \) as above.

For any natural \( k \), \( w_k \) denotes the complex vector \( (1, e_m^{k}, e_m^{2k}, \ldots, e_m^{(m-1)k}) \in \mathbb{C}^m \), where \( e_m = \exp(2\pi i/m) \) is a primitive \( m \)-th root of unity. For \( k = 1, \ldots, m \) these vectors are orthogonal and form a basis of \( \mathbb{C}^m \).

A choice of the chain \( C(t) \) determines the corresponding invariant space \( V(t) \). Note, moreover, that if a chain \( C(t) \) has real coefficients \( r_j \), then for any \( k \) the vectors \( w_k \) and \( w_{m-k} \) simultaneously belong to \( V(t)^G \) or \( (V(t)^\perp)^G \).

Let us recall that \( (V_r)^G \) is generated by the vectors \( w_k \) such that \( m/r \) divides \( k \). From now on, when necessary we will assume that all \( \mathbb{Q} \)-vector spaces \( (V(t), V(t)^\perp, V_d, U_d, \text{etc}) \) are complexified.

Let \( C_k := w_k \) and let \( \{C_k : k \in S \subset \{1, \ldots, m\}\} \) be a basis of the space \( V(t) \). Then the solution of the tangential center problem is given by the space generated by the dual basis \( C_k^* \), \( k \in \{1, \ldots, m\} \setminus S \).

Let \( P_C \) denote the characteristic polynomial \( P_C(z) = \sum_{j=1}^m n_j z^{j-1} \) and let

\[
\Phi_j(z) = \prod_{1 \leq k < j, \gcd(k,j) = 1} (z - e^{2\pi i j/k})
\]

be the \( j \)-th cyclotomic polynomial. Note that \( P_C(e_m^k) = \langle C, w_k \rangle = \langle w_k, C \rangle \).

**Lemma 3.2.** Let \( f(z) \) be a polynomial of degree \( m \) and \( C(t) \) a chain of \( f \). Then:

1. The \( \mathbb{C}[t] \)-module of solutions of the tangential center problem is given by \( U^* = U_{d_1}^* \oplus \cdots \oplus U_{d_t}^* \). Where a basis of \( U_{d_i}^* \) is given by \( C_{m}^* \) and \( U_m^* \) contains functions \( C_{m}^* \), with \( k \) relatively prime with \( m \). For any \( d_i \) divisor of \( m \), \( U_{d_i}^* \) contains functions of the form \( C_{k}^* \), with \( k \) a multiple of \( m/d_i \), but not of any prime factors of \( d_i \), i.e., \( \gcd(m,k) = m/d_i \).
2. The subspace \( U_{d_i}^* \) is a subspace of \( U^* \) if and only if \( C(t) \) is a cycle. The subspace \( U_m^* \) is a subspace of \( U^* \) if and only if \( C(t) \) is balanced.
3. The above spaces \( U_{d_i}^* \) are mutually orthogonal. Their dimensions satisfy \( \dim_C(U_{d_i}^*) = 1 \), \( \dim_C(U_m^*) \geq \phi(m) \), \( \dim_C(U_k^*) \geq \phi(d_k) \).

**Proof.** Assume first that \( C(t) \) is a cycle. By definition of a cycle, this means that the chain \( C_m \) associated to \( w_m = (1, \ldots, 1) \) belongs to \( V(t)^\perp \). That is, \( P_C(e_m^k) = P_C(1) = 0 \). This is equivalent to \( \Phi_1(z) \) being a factor of \( P_C(z) \) and \( C_m^* \) belonging to the space of solutions.

A chain \( C(t) \) is unbalanced if for any permutation cycle at infinity one has \( \sum_j n_j e^j \neq 0 \), i.e., the chain \( C_1 \) does not belong to \( V(t)^\perp \). Assume now that \( C(t) \) is balanced. This means that the chain \( C_1 \) also belongs to \( V(t)^\perp \), that is, \( P_C(e_m^k) = 0 \). By Lemma 3.1 \( U_m = V_{r_1}^\perp \cap \cdots \cap V_{r_j}^\perp \), where \( r_i \) are all maximal divisors of \( m \) in \( D(f) \), but strictly smaller than \( m \). This means that \( m/r_i \) are prime factors of \( m \) in \( D(f) \). Then \( U_m \) contains \( C_k \), where \( k \) is not a multiple of any prime factor of \( m \) in \( D(f) \). In particular, \( U_m \) contains \( C_k \) when \( k \) is coprime with \( m \). Note then that the vector \( w_1 \) belongs to \( U_m \), and
therefore \( U_m \subset V(t) \). That gives that the functions \( C_k^* \) for \( k \) coprime with \( m \) are in the space of solutions of the center problem and \( \Phi_m(z) \) divides \( P_{C^*}(z) \).

Finally, we calculate the remaining elements of the basis of the solution vector space. They correspond to some irreducible components of \((V^+)^*\). By Lemma 3.1 each one of them is of the form \( U_d = V_d \cap (V_{r_1}^+ \cap \cdots \cap V_{r_l}^+) \), for some divisor \( d \) of \( m \) different from 1 and \( m \) and a complete set \( \{r_1, \ldots, r_l\} \) of elements of \( D(f) \) covered by \( d \). Similarly as for \( U_m \), now \( U_d \) contains \( C_k \), where \( k \) is a multiple of \( m/d \), but not of any prime factors of \( d \). By duality, for the same \( k \) the dual functions \( C_k^* \) belong to a basis of solutions of \( U_d^* \). Besides, that also means that \( \Phi_d(z) \) divides \( P_{C^*}(z) \).

For some intermediate steps in the solution of the Tangential center problem we shall need to solve not the Tangential center problem but the more general equation

\[
\int_{C(t)} g = p(t),
\]

where the righthand side is a polynomial \( p(t) \in \mathbb{C}[t] \).

**Proposition 3.3.** Let \( p(t) \in \mathbb{C}[t] \), \( p(t) \neq 0 \). There exists a solution of \( \int_{C(t)} g = p(t) \) if and only if \( C(t) \) is not a cycle.

Moreover, if \( C(t) \) is not a cycle, then \( \int_{C(t)} g = p(t) \) if and only if

\[
g = \frac{p \circ f}{\sum n_i} + u,
\]

where \( u \) is a solution of \( \int_{C(t)} u = 0 \).

**Proof.** Assume that \( \int_{C(t)} g = p(t) \). Since \( p(t) \) is invariant by the action of \( G_f \), then

\[
|G_f| p(t) = \sum_{\sigma \in G_f} \sigma(p(t)) = \sum_{\sigma \in G_f} \int_{\sigma(C(t))} g = \frac{|G_f|}{m} \sum_{k=1}^{m} \left( \sum_{i=1}^{m} n_i \right) g(z_k(t)).
\]

If \( C(t) \) is a cycle, it follows \( p(t) = 0 \), contrary to the assumption.

If \( C(t) \) is not a cycle, then \( \int_{C(t)} g = p(t) \) is equivalent to

\[
0 \equiv \int_{C(t)} g - p(t) = \sum n_i \left( g(z_i(t)) - \frac{p(f(z_i(t)))}{\sum n_i} \right) = \int_{C(t)} u.
\]

\( \square \)

4. Solution for 2-transitive, monomials and Chebyshev polynomials

In this section we prove Theorem A which we have divided into Propositions 4.1 and 4.3. That is, we shall solve the zero-dimensional tangential center problem for the basis of induction, this is, when \( f \) is a 2-transitive polynomial, a monomial or a Chebyshev polynomial. We need to consider chains instead of cycles for they appear in the induction process.

First assume that \( f(z) = z^m \) or \( f(z) = T_m(z) \), where \( T_m = \cos(m \arccos(z)) \) is the \( m \)-th Chebyshev polynomial, with \( m \) not necessarily prime. Let \( C(t) \) be a balanced cycle of \( f \) (with real coefficients). Under the above assumptions we calculate \( (V(t)^+)^* \) explicitly. The key point of the calculation resides in the fact that dual vectors \( C_k^* \) of the chains \( C_k \), are easily calculated in the monomial case. Similarly, in the Chebyshev case, the dual space \( \text{Vect}(C_k, C_{m-k})^* \) of the inseparable space \( \text{Vect}(C_k, C_{m-k}) \) is easily calculated.

**Proposition 4.1.** Let \( f(z) = z^m \) (resp. \( f(z) = T_m(z) \)) be a monomial function (resp. Chebyshev polynomial) and \( C(t) \) a balanced chain of \( f \). Then:
(1) The $\mathbb{C}_[t]$-module of solutions of the tangential center problem is given by
\[ U^* = U^*_m \oplus U^*_{d_1} \oplus \cdots \oplus U^*_{d_j}. \]
where a basis of $U^*_1$ is given by $g_d(z) = 1$ and a basis of $U^*_m$ is given by the functions $g_k(z) = z^k$ (resp. $g_k(z) = T_k(z)$) for $k$ relatively prime with $m$. For any $d_j$ divisor of $m$, a basis of $U^*_{d_j}$ is given by functions of the form $g_k(z) = z^k$ (resp. $g_k(z) = T_k(z)$) for $k$ a multiple of $m/d_j$, but not of any prime factors of $d_j$, i.e., $\operatorname{g.c.d.}(m,k) = m/d_j$.

(2) The above spaces $U^*_j$ are mutually orthogonal. Their dimensions are given by
\[ \dim_{\mathbb{C}}(U^*_1) = 1, \quad \dim_{\mathbb{C}}(U^*_m) = \psi(m), \quad \dim_{\mathbb{C}}(U^*_d) = \psi(d). \]

(3) The space of solutions of the tangential center problem is generated by
\[ \{g_j(z) = z^j \ (\text{resp. } g_j(z) = T_j(z)) : \Phi_{m/\operatorname{g.c.d.}(m,j)}(z)|P_{G}(z)\}. \]

Proof. Let $f(z) = z^n$ or $f(z) = T_m(z)$, $C(t)$ a balanced chain of $f$ and $V(t)$ the orbit of $C(t)$ defined as above. Let $D(m)$ be the set of divisors of $m$. Note that for any divisor $k \in D(m)$ one has $f = g_{m/k} \circ g_k$, where $g(z) = z^l$ (resp. $g_l(z) = T_l(z)$). This shows that in the two particular cases for $f$ a monomial or a Chebyshev polynomial, the set $D(f)$ coincides with $D(m)$, since the decompositions of $f$ are given by the divisors of $m$. By Lemma 3.2 we now know the complete decomposition of the space of solutions in $G_f$-invariant spaces, which is the same in both cases. We have to calculate the dual spaces of each of the direct summands of $V(t)^\perp$.

Let $g_j(z) = z^j$ and $z_j(t) = t^{l/j} \epsilon_m^{-1}$, $j = 1, \ldots, m$, for $\epsilon_m$ a primitive $m$-th root of unity. Then
\[
\int_{C_h} g_l = \sum_{j=1}^{m} \frac{w_{k,j}}{w_k} g_l(z_j(t)) = \sum_{j=1}^{m} \frac{\epsilon_m^{k(j-1)} t^{l/j} \epsilon_m^{(j-1)}}{w_k} = t^{l/m} \sum_{j=1}^{m} \epsilon_m^{k(j-1)} \epsilon_m^{(j-1)} = t^{l/m} < w_l, w_k > \]
\[ = mt^{l/m} \delta_{lk}. \]
This shows that
\[ g_l = \epsilon_l C_l^*, \]
where $\epsilon_l$ is a nonzero constant. The claim in the monomial case follows now from Lemma 3.2 (note that the inequalities of the dimensions must be equalities in this case).

Consider now the Chebyshev case. Let $T_m(z) = \cos(m \arccos(z))$ be the $m$-th Chebyshev polynomial, which is a polynomial of degree $m$. If we take $f(z) = T_m(z)$, then for any $t \in \mathbb{C}$, $f(z) = t$ gives $m$ preimages
\[ z_k(t) = (T_m^{-1}(t))_k = \cos \left( \frac{1}{m} \arccos_k(t) \right), \]
where we choose the range of $\arccos_k$ in $[0, 2\pi) + 2(k-1)\pi$ (indeed $\arccos_k(t) = \arccos_1(k) + 2(k-1)\pi$, see Figure 2). Note that a loop of $t$ around infinity transforms $\arccos_k(t)$ into $\arccos_{k+1}(t)$ (see Figure 3).

The Chebyshev polynomials $T_0, \ldots, T_{m-1}$ form a basis of the space of polynomials as a $\mathbb{C}_[t]$-module.
If we take $t \in \mathbb{R}$ and denote $\xi(t) = \arccos_1(t) \in \mathbb{R}$, $\epsilon_m = e^{i2\pi/m}$, then for any chain $C(t)$ we obtain

$$ \int_{C(t)} T_j(z) = \sum_{k=1}^{m} n_k T_j(z_k) = \sum_{k=1}^{m} n_k \cos \left( \frac{j}{m} \arccos_k(t) \right) $$

$$ = \sum_{k=1}^{m} n_k \cos \left( \xi(t) \frac{j}{m} + 2(k-1)\pi \frac{j}{m} \right) $$

$$ = \sum_{k=1}^{m} n_k \left( \frac{e^{i\xi(t)} + e^{i2(k-1)\pi}}{2} \frac{e^{-i\xi(t)} - e^{-i2(k-1)\pi}}{2} \right) $$

$$ = \frac{e^{i\xi(t)}}{2} \left( \sum_{k=1}^{m} n_k \epsilon_m^{j(k-1)} \right) + \frac{e^{-i\xi(t)}}{2} \left( \sum_{k=1}^{m} n_k \epsilon_m^{-j(k-1)} \right) $$

$$ = \frac{e^{i\xi(t)}}{2} P_C(\epsilon_m^j) + \frac{e^{-i\xi(t)}}{2} P_C(\overline{\epsilon_m^j}) $$

$$ = \alpha_j(t) P_C(\epsilon_m^j) + \alpha_j(t) P_C(\overline{\epsilon_m^j}), $$

where $\alpha_j(t) = \frac{e^{i\xi(t)}}{2}$. If the chain $C(t)$ has real coefficients, this gives

$$ \int_{C(t)} T_j(z) = 2 \text{Re}(\alpha_j(t) P_C(\epsilon_m^j)), $$
due to the fact that $P_C(z) \in \mathbb{R}[z]$.

Consider in particular the chain $C_k$. Note that $P_{C_k}(e_m^j) = \langle w_j, w_k \rangle$. Hence (4.12) gives

$$\int_{C_k(t)} T_j = \alpha_j(t) < w_j, w_k > + \alpha_j(t) < w_{m-j}, w_k >.$$

It follows that $\int_{C_k(t)} T_j = 0$ if $k \not\in \{j, m-j\}$.

On the other hand, as our balanced cycle $C(t)$ is real, it follows that the polynomial $P_C(z)$ has real coefficients and hence $< w_{m-j}, C > = P_C(e_m^{m-j}) = P_C(e_m^{j}) = P_C(C_3) = P_C(C_4) = \langle w_j, C \rangle$. This means that $w_j \in V(t)^\perp$, if and only if $w_{m-j} \in V(t)^\perp$. It follows that the two-dimensional dual space of the space generated by $w_j$ and $w_{m-j}$ is the space generated by $T_j$ and $T_{m-j}$. The claim now follows as in the monomial case from Lemma 3.2.

Example 4.2. Let $f(z) = z^6$. Then $D(f) = \{1, 2, 3, 6\}$ and using Lemma 3.4 we get $H_0(f^{-1}(t)) \equiv \mathbb{C}^6 = U_1 \oplus U_2 \oplus U_3 \oplus U_6$, where $U_1 = \langle C_0 \rangle$, $U_2 = \langle C_4 \rangle$, $U_3 = \langle C_2, C_4 \rangle$, $U_6 = \langle C_1, C_5 \rangle$. Let $C$ be a cycle of $f$, $V$ its orbit and $V^\perp$ the orthogonal complement. By the cycle condition $U_1 \subset V^\perp$.

Assume that the cycle $C(t)$ is balanced. This is equivalent to assuming $U_6 \subset V^\perp$.

Now various balanced cycles can be considered. For instance, if $C$ is such that $V = U_2$, then the space of solutions $g$ as a $\mathbb{C}[t]$-module is generated by $\{1, z, z^2, z^4, z^5\}$. If $V = U_3$, then the solution is generated by $\{1, z, z^3, z^5\}$.

By the Burnside-Schur Theorem, if $f$ is primitive and it is not in the cases above, then it is 2-transitive. The unbalanced case is solved in Theorem 2.4, therefore only the balanced case remains. The following result solves the tangential center problem for a balanced cycle and a 2-transitive $f$.

Proposition 4.3. Let $f \in \mathbb{C}[z]$ be a polynomial with a 2-transitive monodromy group and $C(t) = \sum_{j=1}^{m} n_j z_j(t)$ a chain of $f$.

1. If $C(t)$ is a balanced chain, then there exists $n$ such that $n_j = n$ for every $1 \leq j \leq m$. In particular if $C(t)$ is a balanced cycle, then it is trivial.

2. If $C(t)$ is a balanced chain, then $\int_{C(t)} g \equiv 0$ if and only if

$$\sum_{k=1}^{\deg(g)} s_k g_k = 0,$$

where $g(z) = \sum g_k z^k$ and $s_k$ are given by the Newton-Girard formulae, as

$$s_k = \sum_{i=1}^{m} z_i^k(t).$$

Proof. (1) Let us assume that $C(t)$ is balanced. It means that $\sum_{j=1}^{m} n_{\sigma(j)} e_m^j = 0$, for any $\sigma \in G_f$. Let $H_1 = \{ \sigma \in G_f | \sigma(1) = 1 \}$ be the stabilizer of $z_1(t)$. Then

$$\sum_{\sigma \in H_1} \sum_{j=1}^{m} n_{\sigma(j)} e_m^j = 0.$$

That is,

$$|H_1| n_1 e_m + \sum_{j=2}^{m} \sum_{\sigma \in H_1} n_{\sigma(j)} e_m^j = 0.$$

(4.13) Now the assumption that $G_f$ is 2-transitive and $H_1$ is the stabilizer of $z_1(t)$ implies that $H_1$ acts transitively on $z_2(t), \ldots, z_m(t)$. Hence, for each $j \in \{2, \ldots, m\}$ and
each \( k \in \{2, \ldots, m\} \) there is the same number of occurrences of \( n_k \) in the sum \( \sum_{\sigma \in H_i} n_{\sigma(j)} e_j^m \). This number is \( \frac{|H_i|}{m-1} \) and

\[
\sum_{j=2}^{m} \sum_{\sigma \in H_i} n_{\sigma(j)} e_j^m = \frac{|H_i|}{m-1} \sum_{k=2}^{m} n_k \sum_{j=2}^{m} e_j^m = \frac{|H_i|}{m-1} \left( -n_1 + \sum_{k=1}^{m} n_k \right) \sum_{j=2}^{m} e_j^m.
\]

Observing that \( \sum_{j=2}^{m} e_j^m = -e_m \), by Equation (4.13) we get

\[
0 = |H_i| n_1 e_m - |H_i| \frac{n_1 + m}{m-1} \epsilon_m \left( -n_1 + \sum_{k=1}^{m} n_k \right) = |H_i| \epsilon_m \left( n_1 + \frac{n_1 + m}{m-1} \right) - \sum_{k=1}^{m} n_k \sum_{j=2}^{m} e_j^m.
\]

Since \( |H_i| \neq 0 \) and \( \epsilon_m \neq 0 \), then

\[
n_1 = \frac{\sum_{k=1}^{m} n_k}{m}.
\]

The choice of index 1 was arbitrary. Hence \( n_j \) is a constant not depending on \( j \).

If \( C(t) \) is a cycle, then this constant must be zero.

(2) Let us compute the solutions of \( \int_{C(t)} g = 0 \), with \( g(z) = \sum g_k z^k \), \( g_k \in \mathbb{C} \). By (1), we can assume that \( C(t) = n \sum z_i(t) \). Then

\[
\int_{C(t)} g = n \sum_{i=1}^{m} g(z_i(t)) = n \sum_{i=1}^{\deg(g)} \sum_{k=1}^{\deg(g)} g_k z_i^k(t) = n \sum_{k=1}^{\deg(g)} g_k \sum_{i=1}^{m} z_i^k(t) \equiv 0,
\]

and the solution follows from the Newton-Girard formulae. Recall that Newton-Girard formulae express explicitly \( s_k(t) = \sum z_i^k(t) \) in function of the coefficients of the polynomial \( f(z) - t \).

\[\square\]

5. Monodromy group of imprimitive polynomials

In the next section we will prove Theorems [3] and [4]. We shall assume that Hypothesis [2.12] holds, which, as we prove in this section, will allow us to write the monodromy group of \( f \) as a semidirect product defined by the monodromy groups of \( \tilde{f} \) and \( h \).

First, we introduce a new numbering in the preimages of \( t \) by \( f \). Fix a regular value \( t \) of \( f \) and take the preimage by \( \tilde{f} \) of \( t \). We obtain \( m/d \) points, \( w_1(t), \ldots, w_{m/d}(t) \).

For each \( w_i(t) \), let \( z_{i,j}(t) \), \( j = 1, \ldots, d \) denote each of the preimages of \( w_i(t) \) by \( h \).

Then, according to what we saw in Section [1], the blocks of the imprimitivity system associated to \( f = \tilde{f} \circ h \) are

\( B_i = \{ z_{i,j}(t) : j = 1, \ldots, d \} \), \( i = 1, \ldots, m/d \).

They correspond to rows of circles in Figure [3].

Differentiating \( f = \tilde{f} \circ h \), we obtain \( f'(z) = \tilde{f}'(h(z)) h'(z) \). Thus, critical points of \( f \) correspond to either the preimage by \( h \) of critical points of \( \tilde{f} \) or critical points of \( h \). Let us denote

\[
\{ z : f'(z) = 0 \} = \{ a_1, \ldots, a_{d(m/d - 1)}, b_1, \ldots, b_{d - 1} \},
\]

where \( \tilde{f}'(h(a_i)) = 0, h'(b_i) = 0 \).

Let \( \alpha_i \) denote the permutation associated to \( f(a_i) \) and \( \beta_i \) the permutation associated to \( f(b_i) \). Each permutation \( \alpha_i \) (resp. \( \beta_i \)) corresponds to winding counterclockwise around only one critical value \( f(a_i) \) (resp. \( f(b_i) \)) along a closed path.
Note that by assumption the paths giving $\beta_i$ lift to closed paths (loops) based at whatever $w_i(t)$ we take as starting point as they encircle no critical value of $\tilde{f}$. Similar claim is valid for $\alpha_i$. This gives

$$\alpha_k(z_{i,j}) = z_{\alpha_k(i),j}, \quad \beta_k(z_{i,j}) = z_{i,\beta_k(i,j)},$$

so that $\alpha_k$ exchange whole blocks, while $\beta_k$ only moves elements inside every block.

Thanks to Hypothesis 2.12, we can split the monodromy group $G_f$ of $f$ in terms of the monodromy group of $\tilde{f}$ and a normal subgroup of $G_f$ which contains (a subgroup isomorphic to) the monodromy group of $h$.

**Lemma 5.1.** Assume that Hypothesis 2.12 holds. Let us denote

$$G_f = \langle \alpha_k \rangle, \quad N_h = \langle \alpha_i \beta_k \alpha_i^{-1} : i, k \rangle.$$

Then $G_f$ is the semidirect product $N_h \rtimes G_{\tilde{f}}$ of $N_h$ and $G_{\tilde{f}}$ with respect to $\phi : G_f \to \text{Aut}(N_h)$, $\phi(\alpha) = \phi_{\alpha}$, where $\phi_{\alpha}(\sigma) = \alpha \sigma \alpha^{-1}$.

**Proof.** First, we define the group semidirect product $N_h \rtimes G_{\tilde{f}}$ as the cartesian product set $N_h \times G_{\tilde{f}}$ with the following operation defined by $\phi$:

$$(\sigma, \alpha)(\tilde{\sigma}, \tilde{\alpha}) = (\sigma \phi_{\alpha}(\tilde{\sigma}), \alpha \tilde{\alpha}) = (\sigma \tilde{\alpha} \alpha^{-1}, \alpha \tilde{\alpha}).$$

Obviously $N_h \cap G_{\tilde{f}} = \{1d\}$. Hence, if we prove that any element $\sigma \in G_f$ is written in the form $\sigma = \tau \alpha$, where $\tau \in N_h$ and $\alpha \in G_{\tilde{f}}$, then this decomposition is unique (if $\tau \alpha = \tilde{\tau} \tilde{\alpha}$, then $\tilde{\tau}^{-1} \tau = \tilde{\alpha} \alpha^{-1} \in N_h \cap G_{\tilde{f}} = \{1d\}$ and $\tau = \tilde{\tau}$, $\alpha = \tilde{\alpha}$) and we will have the group isomorphism

$$G_f \quad \longrightarrow \quad N_h \rtimes G_{\tilde{f}}$$

$$\sigma = \tau \alpha \quad \mapsto \quad (\tau, \alpha)$$

Since $G_f$ is generated by the permutations $\{\alpha_i, \beta_j : i = 1, \ldots, d(m/d - 1), j = 1, \ldots, d - 1\}$, every element $\sigma$ in $G_f$ is a finite product of $\alpha$’s and $\beta$’s. In order to write $\sigma$ as the product of a permutation $\tau$ in $N_h$ and a permutation $\alpha$ in $G_{\tilde{f}}$, we group together the $\alpha$’s and $\beta$’s that appear in the expression of $\sigma$ in the following way:

$$\sigma = \prod_{i \in I_2^\alpha} \beta_i \prod_{j \in I_2^\beta} \alpha_j \prod_{k \in I_2^\gamma} \beta_k \cdots \prod_{l \in I_2^\delta} \alpha_l,$$

where the set of indices $I_2^\beta$ are permutations with repetitions of $\{1, \ldots, d - 1\}$ and the set of indices $I_2^\alpha$ are permutations with repetitions of $\{1, \ldots, d(m/d - 1)\}$. The first product belongs to $N_h$, but the second product $\prod_{j \in I_2^\beta} \alpha_j$ is in $G_f$, so we
multiply the third product $\prod_{k \in I_2^0} \beta_k$ on the right by $(\prod_{j \in I_2} \alpha_j)^{-1} \prod_{j \in I_2^0} \alpha_j = Id$.

Now we rewrite $\sigma$ as follows

$$\sigma = \prod_{i \in I_1^0} \beta_i \left( \prod_{j \in I_2} \alpha_j \prod_{k \in I_2^0} \beta_k \left( \prod_{j \in I_2} \alpha_j \right)^{-1} \right) \prod_{j \in I_2^0} \alpha_j \ldots \prod_{i \in I_1^0} \alpha_i,$$

where the first and the second products $\prod_{i \in I_1^0} \beta_i, \left( \prod_{j \in I_2} \alpha_j \prod_{k \in I_2^0} \beta_k \left( \prod_{j \in I_2} \alpha_j \right)^{-1} \right)$ are in $N_h$.

We follow this procedure with the new third product until there is no product of $\beta$’s left, and in a finite number of times we have $\sigma$ rewritten as a product of several permutations in $N_h$ and a final one in $G_f$, which can be renamed as $\tau \in N_h$ and $\alpha \in G_f$. \hfill \square

**Lemma 5.2.** Assume that $f = \tilde{f} \circ h$ satisfies Hypothesis 2.12. Let $H_{\alpha, \beta}$ denote the stabilizer of $(i_0, j_0)$ in $N_h$. Then:

1. $G_f$ is isomorphic to the monodromy group of $\tilde{f}$.
2. $N_h$ is a normal subgroup of $G_f$ such that for every block $B_i$ of $f$ there exists a subgroup $G_i$ of $N_h$ such that $G_i$ leaves fixed all the elements $(j, k) \notin B_i$.
   Moreover, $G_i$ is isomorphic to the monodromy group $G_h$ of $h$.
3. The subgroup $\prod_{j=1, \ldots, d} H_{\alpha, j} \subseteq N_h$ acts transitively on the elements of $B_i$ for every $i \neq i_0$.
4. If $h$ is 2-transitive, then $H_{\alpha, \beta}$ acts transitively on the other elements of the block $B_{i_0}$, thus, on $\{(i_0, j) : j \neq j_0\}$.

**Proof.** (1) and (4) follow easily and (3) is a consequence of (2), since $\prod_{i=1, \ldots, d} H_{i_0, j}$ contains $G_i$ for every $i \neq i_0$.

To conclude, we prove (2). The group $N_h$ is generated by elements of the form $\tau = \alpha \beta \alpha^{-1}$, with $\alpha \in G_f$ and $\beta \in G_h$. By Hypothesis 2.12, $\beta$ moves elements of at most one block $B_i$. Since the group $G_f$ acts transitively on each column, then for every $B_i$ there exists a subgroup of $N_h$, which we will call $G_i$, isomorphic to $G_h$ that moves only the elements of $B_i$. \hfill \square

Now, we shall prove that the non-merging hypothesis on critical values implies that for any two decompositions of a polynomial, one of the inner factors factorizes by the other one. We recall that $\tau_{\infty} = (1, 2, \ldots, m) \in G_f$.

**Proposition 5.3.** Assume that $f = f_0 \circ h_0$ satisfies Hypothesis 2.12, $f, f_0, h_0 \in \mathbb{C}[z]$. If there exist $f_1, h_1 \in \mathbb{C}[z]$ such that $f = f_1 \circ h_1$, then there exists a polynomial $w$ such that either $h_1 = w \circ h_0$ or $h_0 = w \circ h_1$.

**Proof.** Let $B_0$ be a block associated to the decomposition $f = f_0 \circ h_0$ such that $1 \in B_0$, and let $B_1$ a block associated to $f = f_1 \circ h_1$ such that $1 \in B_1$. Then either $B_1 \subset B_0$ or there exists $i \in B_1 \setminus B_0$.

If $B_1 \subset B_0$, then there exists $w \in \mathbb{C}[z]$ such that $w(h_1) = h_0$; we know that $B_0 = \{j \in \{1, \ldots, m\} : h_0(z_j) = h_0(z_j)\}$ and $B_1 = \{k \in \{1, \ldots, m\} : h_1(z_k) = h_1(z_k)\}$. Their stabilizers $G_{B_0} = \{\sigma \in G_f : \sigma(B_0) = B_0\}$ and $G_{B_1} = \{\sigma \in G_f : \sigma(B_1) = B_1\}$ verify that $G_{B_1} \subseteq G_{B_0}$ and, therefore, $L^{G_{B_0}} = \mathbb{C}(h_0(z_1)) \subseteq L^{G_{B_1}} = \mathbb{C}(h_1(z_1))$, where $L = \mathbb{C}(z_1(t), \ldots, z_m(t))$ and $L^{G_k}$ denotes the elements of $L$ invariants by the action of $G_k$. As a consequence, there exists a rational function $w$, which we can assume polynomial, such that $h_0 = w \circ h_1$.

If there exists $i \in B_1 \setminus B_0$, then $\tau_{\infty}^{-1}(B_0) \subset B_1$, and there exists $w \in \mathbb{C}[z]$ such that $w(h_0) = h_1$; since $\tau_{\infty}^{-1}(1) = i \in \tau_{\infty}^{-1}(B_0)$, this one is a block of the imprimitivity system associated to the decomposition $f = f_0 \circ h_0$ different from $B_0$, then $1 \notin \tau_{\infty}^{-1}(B_0)$. By Lemma 5.2 (3), we can assure that the stabilizer $H_1$ (in $G_f$)
of 1 is transitive on every other block different from \(B_0\). Therefore, \(H_1\) is transitive on \(\tau_{\infty}^{-1}(B_0)\) and the orbit of \(i\) by \(H_1\) contains \(\tau_{\infty}^{-1}(B_0)\). Since every permutation in \(H_1\) leaves \(B_1\) fixed and \(i \in B_1\), \(\tau_{\infty}^{-1}(B_0) \subset B_1\) and following a similar argument as in the previous case, there exist \(w \in \mathbb{C}[z]\) such that \(h_1 = w \circ h_0\). □

Proposition 5.3 implies that when we have chains of decompositions satisfying Hypothesis 2.12 then we have uniqueness.

Proof of Proposition 2.13. We shall prove it by induction on the number \(\bar{d}\) of factors in the decomposition of \(f\).

First, assume that \(d = 0\) and \(f = \tilde{f}_0\) is 2-transitive, Chebyshev or a monomial, and let \(f = f_0 \circ \ldots \circ f_d\) be a decomposition of \(f\) for some polynomials \(f_0, \ldots, f_d\) such that every \(f_k\) is either 2-transitive, Chebyshev or a monomial, and that Hypothesis 2.12 holds for every \(f_0 \circ \ldots \circ f_k = \left( f_0 \circ \ldots \circ f_{k-1} \right) \circ f_k\). Since 2-transitive polynomials are primitive and monomials and Chebyshev polynomials cannot decompose satisfying Hypothesis 2.12 then \(d = 0\).

Now, assume that \(f = f_0 \circ \ldots \circ f_d\) for some polynomials \(f_0, \ldots, f_d\) such that every \(f_k\) is either 2-transitive, Chebyshev or a monomial, and that Hypothesis 2.12 holds for every \(f_0 \circ \ldots \circ f_k = \left( f_0 \circ \ldots \circ f_{k-1} \right) \circ f_k\).

If \(f = \tilde{f}_0 \circ \ldots \circ \tilde{f}_d\) is another decomposition of \(f\) for some polynomials \(\tilde{f}_0, \ldots, \tilde{f}_d\) such that every \(\tilde{f}_k\) is either 2-transitive, Chebyshev or a monomial, then Proposition 5.3 there exists \(w \in \mathbb{C}[z]\) such that either \(f_d = w \circ \tilde{f}_d\) or \(f_d = w \circ \tilde{f}_d\).

Assume that \(f_d = w \circ \tilde{f}_d\) for some non-linear \(w\). In particular this implies that \(w, \tilde{f}_d\) are both monomials or both Chebyshev polynomials. Then

\[
f_0 \circ \ldots \circ f_{d-1} = \tilde{f}_0 \circ \ldots \circ \tilde{f}_{d-1} \circ w.
\]

Thus, Hypothesis 2.12 does not hold, since \(\tilde{f}_0 \circ \ldots \circ \tilde{f}_{d-1} \circ w\) and \(f_d\) share a critical value.

Therefore, we may assume that \(f_d = w \circ \tilde{f}_d\). But in this case

\[
f_0 \circ \ldots \circ f_{d-1} \circ w = \tilde{f}_0 \circ \ldots \circ \tilde{f}_{d-1}.
\]

Note that \(f_0 \circ \ldots \circ f_{d-1} \circ w\) still satisfies Hypothesis 2.12 since the critical values of \(w\) are a subset of those of \(f_d\). We conclude by induction.

If Hypothesis 2.12 holds for every \(f_0 \circ \ldots \circ f_k = \left( f_0 \circ \ldots \circ f_{k-1} \right) \circ f_k\) interchanging \(\{f_k\}\) and \(\{\tilde{f}_k\}\) in the previous arguments we conclude. □

6. Proof of Theorems [3] and [C]

Let \(C(t)\) be a balanced cycle of an imprimitive polynomial \(f = \tilde{f} \circ h\), with \(h\) 2-transitive, Chebyshev or a monomial. According to the type of \(h\), we deduce first some information on how the cycle \(C(t)\) is positioned with respect to the imprimitivity system defined by \(h\).

Proposition 6.1. Assume that \(f = \tilde{f} \circ h\) satisfies Hypothesis 2.12. Let \(C(t)\) be a balanced cycle of \(f\) and let \(B_h = \{B_1, \ldots, B_{m/d}\}\) denote the imprimitivity system corresponding to \(h\).

1. If \(h\) is 2-transitive, then \(\tau_k\) is constant for \(k \in B_j\), for every \(B_j \subset B_h\).
2. If \(h(z) = z^d\) or \(h(z) = T_d(z)\), then the restriction of \(C(t)\) to each block of \(B_h\) is balanced.

Proof. (1) Let us fix \(i_0, j_0\) and let us denote \(H_0 = H_{i_0,j_0} \subset G_f\) the stabilizer of \(z_{i_0,j_0}\). For every preimage \(z_{i,j}(t)\) of \(t\) by \(f\), let us denote by \(k(i,j)\) its position when we enumerate them so that \(\tau_{\infty} = (z_1(t), z_2(t), \ldots, z_m(t))\). Besides, \(z_{i,j}(t) = z_{k(i,j)}(t)\).
Therefore, since $C(t)$ is balanced,
\[
0 = \sum_{\tau \in H_0} \sum_{i,j} n_{\tau(i,j)k_m(i,j)}
\]
\[
= |H_0| n_{i_0,j_0} \epsilon_m k_m(i_0,j_0) + |H_0| \left( \sum_{j=1,j\neq j_0}^{d} \frac{\sum_{j=1,j\neq j_0}^{d} n_{i_0,j} k_m(i_0,j)}{d-1} \sum_{j=1,j\neq j_0}^{d} \epsilon_m k_m(i_0,j) \right) + |H_0| \left( \sum_{i=1,i\neq i_0}^{m/d} \sum_{j=1}^{d} n_{i,j} \sum_{j=1}^{d} \epsilon_m k_m(i,j) \right),
\]
where the second summand is due to Lemma 5.2 (4), since $h$ is 2-transitive, and the last summand is due to Lemma 5.2 (3). Now, observe that for every $i$, since it is the sum of all the powers of $\epsilon_m$ corresponding to a block $B_i$, whose elements are the residue class mod $m/d$ for some rational function $\tilde{g}$ (see, for instance, [19 Lemma 3.1]), that is, \[\{i,j\} \in \{i_0,j_0\} \Rightarrow \tilde{g}(\tilde{t}) \text{ is constant along the blocks of the subgroup } G, \]\[\text{for some rational function } \tilde{g}(z) \text{ restricted to the block } B_{i_0}.\]

Thus, $n_{i_0,j_0}$ does not depend on $j_0$. Therefore $n_{i_0,j_0}$ is constant in the block $B_{i_0}$.

(2) Assume now that $h(z) = z^d$, or $h(z) = T_d(z)$. Let $H_0 = \bigcap_{j=1,...,d} H_{i_0,j}$. The elements \(\{(i,j)\} \in C^d(f)\) with $i \neq i_0$ fixed are moved transitively by Lemma 5.2 (3). Therefore, using similar arguments as in (1),
\[
0 = \sum_{\tau \in H_0} \sum_{i,j} n_{\tau(i,j)k_m(i,j)} = |H_0| \left( \sum_{i=1}^{d} n_{i_0,j_0} k_m(i_0,j_0) \right).
\]

Then, $C(t)$ restricted to the block $B_{i_0}$ (the $i_0$-th $h$-invariant part of $C(t)$) is balanced. \(\square\)

**Proof of Theorem 3.** By Proposition 6.1 $C(t)$ is constant along the blocks of the imprimitivity system corresponding to $h$, $B_h = \{B_1,...,B_{m/d}\}$. Therefore,
\[
\int_{C(t)} g = \sum_i n_i g(z_i(t)) = \sum_{i=1}^{m/d} n_i \sum_{k \in B_i} g(z_k(t)).
\]

Let us observe that $\sum_{k \in B_i} g(z_k(t))$ is invariant by $G_{B_i} = \{\sigma \in G_f : \sigma(B_i) = B_i\}$. Then, if we denote $L = \mathbb{C}(z_1(t),...,z_m(t))$, $\sum_{k \in B_i} g(z_k(t)) \in L^{G_{B_i}} = \mathbb{C}(h(z_k(t)))$ for every $k \in B_i$. Let us denote $w_i(t) = h(z_i(t))$, which is constant in $k \in B_i$; then $w_i(t)$ is one of the $m/d$ preimages of $t$ by $f$. Therefore, $\sum_{k \in B_i} g(z_k(t)) = \tilde{g}_i(w_i(t))$ for some rational function $\tilde{g}_i(z)$, which we can assume is a polynomial. Since every subgroup $G_{B_i}$ of $G_f$ is conjugated to another $G_{B_j}$, it can easily be proved that $\tilde{g}_i$ does not depend on the block $B_i$, that is, there is only one polynomial $\tilde{g}(z)$ such that
\[
\sum_{k \in B_i} g(z_k(t)) = \tilde{g}(w_i(t)), \quad i = 1,...,m/d.
\]
Let us observe that 

\[ \int_{C(t)} g = \sum_{i=1}^{m/d} n_i \tilde{g}(w_i(t)) = \frac{\int h(C(t)) \tilde{g}}{d}. \]

Since \( z_k(t) \) for \( k \in B_i \) are the preimages of \( w_i \) by \( h \), Equation (6.15) can be rewritten as

\[ \sum_{k \in B} g(z_k(w)) = \tilde{g}(w). \]

Replacing \( g \) by its linear expansion in terms of the basis \( \{1, z, \ldots, z^{d-1}, h(z), zh(z), \ldots, z^{d-1}h(z), h^2(z), \ldots\} \) of \( C[z] \), namely \( g(z) = \sum_{i=0}^{d-1} z^i g_i(h(z)) \), we get

\[
\tilde{g}(w) = \sum_{k \in B} \sum_{i=0}^{d-1} z^i_k(w) g_i(h(z_k(w))) = \sum_{k \in B} \sum_{i=0}^{d-1} z^i_k(w) g_i(w)
= \sum_{i=0}^{d-1} \left( \sum_{k \in B} z^i_k(w) \right) g_i(w) = \sum_{i=0}^{d-1} s_i g_i(w),
\]

where \( s_i \) can be obtained by the Newton-Girard formulae applied to \( h \).

**Proof of Theorem 6.4**. Assume that \( f = \tilde{f} \circ h \), \( h(z) = z^d \) or \( h(z) = T_d(z) \) and Hypothesis 6.12 is verified. Let \( C(t) \) be a balanced cycle of \( f \). Assume that

\[ \int_{C(t)} g \equiv 0, \]

that is, \( \sum_{i,j} n_{i,j} g(z_{i,j}(t)) = 0 \). Fix \( i_0 \in \{1, \ldots, m/d\} \). Let \( H_{i_0} \) be the stabilizer of \( (i_0, j), j = 1, \ldots, d \) in \( N_h \). By Lemma 6.2 (3), the elements \( (i, j), j = 1, \ldots, d \) with \( i \neq i_0 \) are moved transitively by \( H_{i_0} \). Therefore,

\[
0 = \sum_{\sigma \in H_{i_0}} \sum_{i,j} n_{\sigma(i,j)} g(z_{i,j}(t)) = |H_{i_0}| \sum_{j=1}^{d} n_{i_0,j} g(z_{i_0,j}(t)) + \frac{|H_{i_0}|}{d-1} \sum_{i=1,i \neq i_0}^{m/d} \left( \sum_{j=1}^{d} n_{i,j} \right) \left( \sum_{j=1}^{d} g(z_{i,j}(t)) \right).
\]

Let us observe that \( \sum_{j=1}^{d} g(z_{i,j}(t)) \) is invariant by \( G_{B_i} = \{ \sigma \in G_f : \sigma(B_i) = B_i \} \), for \( i \neq i_0 \). As a consequence, if we denote \( L = \mathbb{C}(z_1(t), \ldots, z_m(t)) \), we obtain that \( \sum_{j=1}^{d} g(z_{i,j}(t)) \) belongs to \( L^{G_{B_i}} = \mathbb{C}(h(z_{i,j}(t))) = \mathbb{C}(w_i(t)) \), where \( w_i(t) = h(z_{i,j}(t)) \) for \( j = 1, \ldots, d \). By Lüroth Theorem, there exists a polynomial \( \tilde{g}(w) \) (which arguing as in the previous proof does not depend on \( i \)) such that

\[ \sum_{j=1}^{d} g(z_{i,j}(t)) = \tilde{g}(w_i(t)) \]

and, consequently,

\[
\sum_{j=1}^{d} n_{i_0,j} g(z_{i_0,j}(t)) = \frac{-1}{d-1} \sum_{i=1,i \neq i_0}^{m/d} \left( \sum_{j=1}^{d} n_{i,j} \right) \tilde{g}(w_i(t)).
\]
Replacing in the equation \( \int_{C(t)} g = 0 \),

\[
0 = \sum_{i=1}^{m/d} \sum_{j=1}^{d} n_{i,j} g(z_{i,j}(t)) = \frac{-1}{d-1} \sum_{i_0=1}^{m/d} \sum_{i \neq i_0}^{m/d} \left( \sum_{j=1}^{d} n_{i,j} \right) \tilde{g}(w_i(t))
\]

\[
= - \frac{m}{d} \sum_{i=1}^{m/d} \left( \sum_{j=1}^{d} n_{i,j} \right) \tilde{g}(w_i(t)).
\]

Thus,

(6.16)

\[
\int_{h(C(t))} \tilde{g} = 0.
\]

On the other hand,

\[
\sum_{j=1}^{d} n_{i_0,j} g(z_{i_0,j}(t)) = \frac{-1}{d-1} \sum_{i=1}^{m/d} \left( \sum_{j=1}^{d} n_{i,j} \right) \tilde{g}(w_i(t))
\]

\[
= \frac{1}{d-1} \left( \sum_{j=1}^{d} n_{i_0,j} \right) \tilde{g}(w_{i_0}(t)).
\]

In consequence, putting

(6.17)

\[
p_{i_0}(w) := \frac{\sum_{j \in B_{i_0}} n_j}{d-1} \tilde{g}(w) \in \mathbb{C}[w]
\]

for every \( i_0 \), \( g \) is a solution of

(6.18)

\[
\int_{\tilde{C}_{i_0}(w)} g = p_{i_0}(w), \quad 1 \leq i_0 \leq m/d.
\]

Recall that \( \tilde{C}_k(w) \) is a balanced chain of \( h \) and the right-hand part of (6.18) is a polynomial. Then, by Proposition 3.3 a particular solution of (6.18) is given by

\[
\left( p_{i_0} \circ h \right)(z) = \tilde{g}(h(z)) \right) \sum_{i \in B_{i_0}} n_i \right) = \frac{\tilde{g}(h(z))}{d-1},
\]

and the general solution is given by adding general solutions of \( \int_{\tilde{C}_{i_0}(w)} u = 0 \).

\[ \square \]

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