Anistropic Invariant FRW Cosmology

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In this paper we study the effects of including anisotropic scaling invariance in the minisuperspace Lagrangian for a universe modelled by the Friedman-Robertson-Walker metric, a massless scalar field and cosmological constant. We find that canonical quantization of this system leads to a Schrödinger type equation, thus avoiding the frozen time problem of the usual Wheeler-DeWitt equation. Furthermore, we find numerical solutions for the classical equations of motion, and we also find evidence that under some conditions the big bang singularity is avoided in this model.

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I. INTRODUCTION

The most widely accepted description of the universe is given by the ΛCDM model, which makes use of the Friedman-Robertson-Walker (FRW) metric in the framework of General Relativity (GR) supplemented with a cosmological constant \( \Lambda \). However, there are theoretical problems concerning the inclusion of the cosmological constant (see [1–3] for a review). One of these problems is that GR and Quantum Field Theory — the theory that describes the other fundamental interactions — are incompatible [4]. The tools of quantum field theory give a theory with an ill ultraviolet (UV) behaviour when applied to GR. This is one of the reasons for the interest in the search for alternative descriptions of gravity at high energies. Several modifications to GR have been proposed to describe gravitational physics at (pre-)inflationary and accelerated expansion epochs [for a review of models of modified gravity see, for instance 5].

The lack of a fundamental physical principle to construct the ultraviolet theory of gravity makes the search for such a theory a complicated quest. A pragmatic approach is to consider GR as the low energy limit of a more fundamental and so far unknown theory, the quantum theory of gravity. Any formulation of quantum gravity can have new symmetries and degrees of freedom in the UV regime, but should recover or be compatible with GR in the low energy limit.

An UV completion to GR was proposed in [6] by making the theory invariant under anisotropic scaling transformations. The resulting theory is power counting renormalizable, at the expense of losing Lorentz invariance in the UV limit as a consequence of the asymmetry in the transformations that are enforced,

\[ t \rightarrow b^z t, \quad \vec{x} \rightarrow b \vec{x}. \]

The critical exponent \( z \) is adjusted to have a renormalizable theory in the UV region, but in the IR flows to \( z = 1 \) and Lorentz invariance is recovered. Although this theory has been considered as a real candidate for the UV region of GR, it is plagued by the existence of new degrees of freedom which make it incompatible with GR at low energies [7].

The cosmological implications have been studied in Hořava gravity by solving the equations of motion for the full theory (see [8, 9]) and the quantum regime in [10]. In this paper we take a different approach, we consider just one of the key ingredients of Hořava’s proposal, the anisotropic scaling invariance and apply it on a minisuperspace having only two degrees of freedom corresponding to a metric function and a scalar field. This approach is similar to that of several works about minisuperspace noncommutativity [11] or quantum cosmology, a well established line of research that deals with quantization in the minisuperspace [12] and is likely to capture qualitative features of the full, non-symmetry reduced theory [13]. The resulting theory will be an effective theory, that recovers GR results when the anisotropy is eliminated. For the minisuperspace model we choose the flat FRW metric, so we can easily contrast our results with standard cosmology and the deviations obtained will be produced only by the modification to GR due to the invariance under anisotropic scaling. We will construct an action for this model where the minisuperspace variables are compatible with the anisotropic transformations and time reparametrization invariance is preserved. We argue that the quantization of this model leads to a Schrödinger type equation for \( z \neq 1 \), as we have shown for the Kantowski-Sachs model in [14]. This quantum dynamical equation with a first order time derivative allows us to construct a conserved probability current. In the case \( z = 1 \), the system becomes singular and we return to the original WDW equation. We also obtain the classical equations of motion and find numerical solutions for different values of the anisotropy index \( z \). The analysis is done by comparing with the solution obtained in GR.

The paper is organized as follows, in section II the Lagrangian invariant under anisotropic scaling for the FRW model is presented as well as the symmetries of the theory. Then we use canonical formalism to compute the Hamiltonian of the model and to obtain the modified
dispersion relation and the Friedmann equations. In section III we present the exact solutions for the particular cases \( \Lambda = 0 \) and \( z = 1 \), and we analyze the numerical solutions for different values of the critical exponent \( z \) – also known as anisotropic index – and the cosmological constant. Section IV is devoted for conclusions and outlook.

II. ANISOTROPIC SCALING INVARIANT FRW COSMOLOGY

The flat Friedman-Robertson-Walker (FRW) cosmological model is described, in the ADM decomposition, by the line element

\[
ds^2 = -N^2 dt^2 + e^{2\alpha(t)} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)],
\]

where \( N(t) \) is the lapse function and \( e^{\alpha(t)} = a(t) \) is the scale factor. The Einstein-Hilbert action minimally coupled to a massless scalar field \( \phi \) and a cosmological constant \( \Lambda \) for this flat FRW metric is

\[
S(\alpha, \phi) = \int dt \left[ -\frac{3a^2}{k^2 N} + a^3 \left( \frac{\dot{\phi}^2}{2N} + \Lambda N \right) \right],
\]

where the dot represents derivatives with respect to \( t \). The first goal in this work is to look for generalizations of this action incorporating the invariance under anisotropic scale transformations, \( \vec{x} \rightarrow \vec{b} \vec{x}, t \rightarrow b^2 t \), where \( z \) characterizes the anisotropy of the system. Besides this invariance, we also demand the action to be invariant under time reparametrizations, which means that if we see the time coordinate as a function of a parameter \( \tau \), then the action has to be invariant under any choice of \( \tau \). A final requirement that we impose on the action is that in the limit \( z = 1 \) it has to reduce to the usual Lorentz invariant action for the FRW model in GR. The only action that we could find satisfying the above conditions is

\[
S(a, \phi, t) = \int d\tau \left\{ -\frac{3a^2(\dot{a})^2}{k^2 N \tau^2} + a^3 \frac{\dot{\phi}^2}{2N \tau^2} + \frac{a^3 \Lambda}{\tau} \right\},
\]

where now dots denote derivatives with respect to \( \tau \). The appearance of \( \tau \) in the action is necessary to achieve explicit invariance under time reparametrizations. It is straightforward to see that when \( z = 1 \) the standard FRW action is recovered. The invariances that we are requiring are satisfied if under time reparametrization \( \tau = \tau(f) \) we have \( N \rightarrow \dot{\phi} N \) and under anisotropic scaling

\[
N \rightarrow b^{\frac{2z-2z}{z^2}} N, \quad \Lambda \rightarrow b^{\frac{2z-2z}{z^2}} \Lambda, \quad k^2 \rightarrow b^{-2} k^2.
\]

These transformations for \( N \) might seem to restrictive, however we will see that this is not a problem when trying to solve the equations of motion resulting from (3). Indeed, even the gauge \( N = 1 \) that is usually chosen in order to simplify the equations in the \( z = 1 \) case is compatible with (4).

Now we proceed to apply the canonical formalism to the anisotropic scaling invariant action (3) in order to obtain the respective Hamiltonian and the equations of motion.

A. Canonical formalism for the anisotropic scaling invariant FRW action

The dynamical variables that appear in action (3) are the scale factor \( a \), the scalar field \( \phi \) and an explicit time parametrization \( t \). Their corresponding canonical momenta are given, as usual, by

\[
\Pi_a = \frac{\partial L}{\partial \dot{a}} = \frac{6az(\dot{a})^{2z-1}\dot{a}}{k^2 N^2(t)^z},
\]

\[
\Pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{a^3 z(\dot{\phi})^{2z-1}\phi}{N^2(t)^z},
\]

\[
\Pi_t = \frac{\partial L}{\partial \dot{t}} = \frac{3a^2(\dot{a})^2(1-z)}{k^2 N^2(t)^z} + \frac{a^3(\dot{\phi})^2(1-z)}{2N^2(t)^z} + \frac{1-z}{2z} a^{2z+2 - 2z(N\Lambda)^{-\frac{1}{2z}}} t^{\frac{1}{2z}} \approx 0.
\]

One can verify that by choosing \( \dot{t} = 1 \) then \( H_c \) can be written as a true Hamiltonian and it takes the form \( N^{\frac{1}{2z}} H_c(a, \phi, \Pi_a, \Pi_\phi) \), making evident the roles of \( N \) as a Lagrange multiplier and of \( H_c \) as a constraint. The weak equality \( \approx \) means that strictly \( H_c \) is equal to zero only on a constraint hypersurface in the phase space. Comparing the last result with the expression for \( \Pi_t \) we can obtain the formal relation

\[
H_c = \frac{z}{1-z} \Pi_t.
\]

Unlike the weak constraint (6), this last relation is valid in the entire phase space. This is a result with remarkable consequences at the quantum level, since the canonical
quantization of this relation leads to a Schrödinger-type equation, where a first order derivative in time appears [14], offering a possibility to solve the frozen time problem of the Wheeler-DeWitt equation that appears when canonical quantization is applied to GR. For a review, see for example [15]. However, our focus in this work is on the classical theory. We must note that the relation (7) between \(H_c\) and \(H_t\) does not hold for the FRW model in GR since it is not well defined for \(z = 1\). This should not be surprising as for \(z = 1\) the action (3) does not contain any factors of \(t\), so a canonical momenta \(H_t\) cannot be defined.

Although we cannot write \(H_c\) without any explicit reference to the velocities of the dynamical variables, we can at least remove the dependence on \(\dot{a}\) and \(\dot{\phi}\). It is straightforward to find

\[
\frac{z}{1 - z} \dot{t} H_t = \left[ \frac{3^2 - 1}{2} \right] \frac{N^2}{a} \left[ \frac{z^2}{1 - 12z} \right] \frac{1}{a^3} + \frac{1}{2} \left[ \frac{N^2}{a} \right] \frac{1}{a^3 t^2} + \frac{z}{1 - 2z} \left[ \frac{a^{4 - z} N \Lambda t}{t} \right] \frac{1}{a^3}. \tag{8}
\]

By construction, this model is invariant under time reparametrizations, thus we can choose \(t = \tau\). This choice is convenient to cast the right hand side of (8) as a true Hamiltonian, which on-shell satisfies the weak constraint \(H_c \approx 0\). Let us now derive the Friedmann equations for this modified FRW cosmological model.

### B. Friedmann equations

In the framework of general relativity it is customary to obtain the Friedmann equations through the Einstein equations. Nevertheless, they can also be obtained from the Hamiltonian formalism as this is just another way to describe the same physics. Specifically, they arise from the constraint \(H = 0\) and from the Hamilton equations for the momenta of the scale factor and of the scalar field. Using this identification, we find that the analogues of the Friedmann equations for arbitrary values of \(z\) are

\[
3z(\dot{a}^2) = \frac{a^3}{2} \left( \frac{z^2}{1 - 12z} \right) \frac{1}{a^3} + \frac{z}{1 - 2z} \left[ \frac{a^{4 - z} N \Lambda t}{t} \right] \frac{1}{a^3} = 0, \tag{9}
\]

\[
(6z + A(6z) \frac{4z^2}{12z} + 6z(2z - 1) a(\dot{a})^2) = 0,
\]

\[
-3Bz^2 \frac{a}{a^3} \dot{a}^2 = 0,
\]

\[
3a^2 \dot{a} z (\dot{\phi}^2)^2 - 3z(2z - 1) \dot{\phi}^2 = 0,
\]

where we have defined the constants

\[
A = \frac{1}{1 - 2z} \left( \frac{3^2 - 1}{12z} \right) \frac{1}{a^3}, \quad B = \frac{3}{2(1 - 2z)} \left( \frac{1}{z} \right) \frac{1}{a^3},
\]

\[
C = \frac{z^2(4 - z)}{(1 - 2z)(2z - 1)}.
\]

We will refer to these generalized Friedmann equations simply as the Friedman equations. For \(\Lambda \neq 0\), analytical solutions to (9) cannot be found, so we need to resort to numerical methods. Before showing numerical results we present the exact solution for \(\Lambda = 0\) and we compare the respective numerical result with it, this is done in the following section. When \(z = 1\) the Friedmann equations (9) reduce to the ones obtained in GR for a flat FRW metric, a cosmological constant and a massless scalar field:

\[
3a^2 \dot{a}^2 + a^2 \dot{\phi}^2 = \frac{a^3}{2} \phi^2 \tag{10}
\]

\[
\dot{a}^2 + 3a \dot{a} \dot{\phi} + a^2 \dot{\phi}^2 = 0.
\]

In the next section we explore the properties of the model described by (9).

### III. NUMERICAL AND ANALYTICAL SOLUTIONS

We can find analytical solutions for \(\Lambda = 0\) and for \(\Lambda \neq 0\) with \(z = 1\). Although some properties of the solutions for different \(z\) can be studied in the first case, the main use of these exact solutions is that we can compare them to the numerical solutions in order to check the consistency of our numerical results.

For the case \(\Lambda = 0\), we combine the first two equations in (9) obtaining

\[
2z(\dot{a}^2) = 2z(2z - 1)^2 a(\dot{a}^2) - 1 \dot{a} = 0. \tag{11}
\]

The general solution to this differential equation is

\[
a(t) = \left[ \frac{1}{2} \frac{(2z - 1)^2}{(4z^2 + 1 - 2z)(c_2 + c_1 t)} \right] \frac{1}{(z - 1)^{1/3}} \tag{12}
\]

Once we have \(a(t)\), the solution for \(\dot{a}\) is obtained directly from the third equation in (9), i.e. the momentum conservation for the scalar field, but that is not relevant at this moment. The quotient of \(c_2\) and \(c_1\) is related to the time \(t_0\) at which \(a = 0\), whereas \(c_2\) and \(z\) determine the value of \(a\) at \(t = 0\). As expected, this solution is well behaved in the limit \(z \rightarrow 1\) and it recovers the usual solutions \(a(t) \sim t^{1/3}\) corresponding to the massless scalar field that we are considering. Nevertheless, it is straightforward to see that (12) does not reflect the desirable characteristics that we would have expected in order to make the addition of anisotropic invariance relevant at a classical level, as would be the existence of an accelerating scale factor in the absence of the cosmological constant or the removal of the singularity \(a = 0\).

In Fig(1), we plot the analytical solutions for some values of \(z\) as well as the corresponding numerical solutions. The constants in (12) are chosen for each \(z\) in order to make \(\dot{a}\) coincide for the numerical and exact solutions. The thick lines correspond to the analytical solutions and the thin lines to the numerical solutions for
FIG. 1. Solution to the Friedmann equations with $\Lambda = 0$, the same type of line is used both for the numerical and exact solutions that correspond to the same $z$, but the thicker lines are for the numerical results.

We can see that numerical and exact solutions are in good agreement. Also from this plot, we can verify that for $z > \frac{1}{2}$ the exponent in $a(t)$ is negative and therefore there is no acceleration. This can be verified explicitly from the analytical solution.

For the general case, $\Lambda \neq 0$ and $z \neq 1$ we will rely on the numerical results to extract conclusions on the model. We take the system of Hamilton equations derived from (6), which has the advantage of being a first order system, and solve it numerically. The initial conditions are chosen such that $H = 0$. Before describing the results for $z \neq 1$, we check the consistency of the numerical results for $z = 1$ and $\Lambda$ positive or negative. We do this by comparing the numerical results to the exact solution of the equation

$$4a\ddot{a} + 2a^2 \dot{a} + 2\Lambda a^3 = 0,$$

which is obtained by taking $z = 1$ in the Friedman equations (9) and making some algebraic manipulations with them. A solution (not the most general one) is given by

$$a(t) = \left[ -C \sin(\sqrt{3}\sqrt{\Lambda}t + D)\sqrt{\Lambda} \right]^{\frac{1}{3}}.$$

The constants $C$ and $D$ are related to the initial conditions for $a(t)$ and its velocity. Figure (2) shows a comparison of (14) with the numerical solutions of the Hamilton’s equations for $\Lambda = 0.1$ and $\Lambda = -0.1$. In both cases there exist a $t_0$ such that $a(t_0) = 0$, thus these solutions always have an initial big bang singularity.

Now we turn our attention to the solutions for $z \neq 1$. As already mentioned, here all the results are numerical. Figure (3) shows the results for $\Lambda < 0$. We find that as $z$ becomes larger than 1 the accelerated growth of $a(t)$ is rapidly suppressed, in fact the scale factor grows almost linearly with time for large $t$. In contrast, when $z < 1$ the accelerated growth of the scale factor is reinforced, so that the universe expands faster than in the solution for $z = 1$. There are no clear indications that the initial singularity can be avoided in this scenario.

For $\Lambda > 0$ the results are shown in Fig. 4. The value of $z$ has remarkable consequences on the behaviour of the scale factor. For example, for $z = 5$ we see that there is not a $t_0$ such that $a(t_0) = 0$. This characteristic is also present and more noticeable for larger values of $z$, as shown in Fig.(5), where we can see in more detail the region where $a(t)$ reaches its minimum value, $a_{\text{min}}$. The precise value of $a_{\text{min}}$ depends on $z$ and $\Lambda$. For $z < 5$, all the solutions describe qualitatively the same type of universes, and we cannot say with confidence if all of them show a big bang singularity or not. Unlike the

FIG. 2. Comparison of the numerical and analytical solutions to the Friedmann equations for $\Lambda = -0.1$ and $\Lambda = 0.1$. In both cases $z = 1$.

FIG. 3. Numerical and exact solutions to the Friedmann equations for $\Lambda = -0.1$ and different values of $z$. 

FIG. 4. The value of $z$ has remarkable consequences on the behaviour of the scale factor. For example, for $z = 5$ we see that there is not a $t_0$ such that $a(t_0) = 0$. This characteristic is also present and more noticeable for larger values of $z$, as shown in Fig.(5), where we can see in more detail the region where $a(t)$ reaches its minimum value, $a_{\text{min}}$. The precise value of $a_{\text{min}}$ depends on $z$ and $\Lambda$. For $z < 5$, all the solutions describe qualitatively the same type of universes, and we cannot say with confidence if all of them show a big bang singularity or not. Unlike the
FIG. 4. Numerical solutions to the Friedmann equations for $\Lambda = 0.1$ and different values of $z$.

FIG. 5. Numerical solutions to the Friedmann equations for $\Lambda = 0.1$. We see a non zero minimal value for the scale factor.

case $\Lambda < 0$, here there is not a relevant difference when $z$ changes from $z > 1$ to $z < 1$, epochs of accelerated expansion do not exist for any choice of $z$.

IV. CONCLUSIONS

In this work we have analysed the effects of including invariance under anisotropic scaling in the minisuperspace as an explicit symmetry of the action. We study a simple cosmological model consisting of the flat FRW metric plus a massless scalar field and a cosmological constant. A first relevant result of our analysis is that the Hamiltonian formulation of the model reveals that the inclusion of anisotropic invariance is promising at the quantum level, specifically with respect to the so called problem of frozen time, since the canonical formalism of the model leads to a dispersion relation that includes a linear momenta associated to the time coordinate. After canonical quantization, this relation leads to a dynamical quantum equation that includes a first derivative in time, providing a way out to the frozen time problem.

In the main part of this work, we studied the classical solutions of the model for various values of $z > 1/2$. The reason to study only these values is the following: for $z = 1/2$ the last term in the action (3) becomes singular, thus this value of $z$ can be thought as separating the anisotropic index in the ranges $z > 1/2$, which includes the usual Lorentz invariant solutions when $z = 1$, and $z < 1/2$. Although we can suppose that the anisotropic invariance was present at very early stages of the universe and therefore $z$ could take any value, we want to be able to recover the Lorentz invariant ($z = 1$) solutions at late times. Assuming that the transition from anisotropic to Lorentz invariance is continuous, this requirement imposes that we have to stay in the $z > 1/2$ branch.

With respect to the solutions that we found to the anisotropic scaling invariant model, some remarks are in order:

- The limit $z \to 1$ is well defined for all of them.
- Without a cosmological constant, the inclusion of anisotropic invariance makes the volume of the universe to grow almost linearly with time. When this linear regime is reached the velocity $\dot{a}$ is larger for larger $z$.
- With a negative cosmological constant, the accelerated growth of $a(t)$ is suppressed for $z > 1$, whereas for $z < 1$ it is reinforced.
- With a positive cosmological constant we have the most remarkable modification introduced by the invariance under anisotropic scaling, namely, we can remove the Big Bang singularity, obtaining an $a_{\text{min}} \neq 0$. The minimum $z$ for which this is attainable depends on $\Lambda$ and on the initial conditions.

As an overall conclusion, the inclusion of anisotropic invariance results in desirable characteristics both at the quantum and classical levels. It gives a possible solution to the problem of time in quantum cosmology as well as a resolution of the Big Bang singularity. However, we need to seek further for explicit exact solutions that allow us to establish firmly the existence of a non-vanishing minimum value for the scale factor, and to find analytical conditions to determine the relation between the values of $z$, $a_{\text{min}}$ and $\Lambda$. The existence of a mechanism to change dynamically the anisotropic index in such a way that the model can flow from an anisotropic invariant epoch at high energies – close to the Planck scale – to a Lorentz invariant epoch at lower energies is a non-trivial open question that would be interesting to study, since such a mechanism would allow us to have relevant deviations from classical solutions at high energies while recovering standard general relativity results in the infrared regime.
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