Four-Loop QCD Corrections to the $\rho$ Parameter

K.G. Chetyrkin$^1$, M. Faisst$^1$, J.H. Kühn$^1$, P. Maierhöfer$^1$, and C. Sturm$^2$

$^1$Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany and $^2$Dipartimento di Fisica Teorica, Università di Torino, Italy & INFN, Sezione di Torino, Italy

The four-loop QCD corrections to the electroweak $\rho$ parameter arising from top and bottom quark loops are computed. Specifically we evaluate the missing “non-singlet” piece. Using algebraic methods the amplitude is reduced to a set of around 50 new master integrals which are calculated with various analytical and numerical methods. Inclusion of the newly completed term halves the final value of the four-loop correction for the minimally renormalized top-quark mass. The predictions for the shift of the weak mixing angle and the $W$-boson mass is thus stabilized.

Electroweak precision measurements and calculations provide stringent and decisive tests of the quantum fluctuations predicted from quantum field theory. As a most notable example, the indirect determination of the top quark mass, $m_t$, mainly through its contribution to the $\rho$ parameter\footnote{The $\rho$ parameter arises from top and bottom quark loops.}, coincides remarkably well with the mass measurement performed by the CDF and D0 experiments at the TEVATRON\footnote{TEVATRON is the particle accelerator at Fermilab.}. Along the same line, the bounds on the mass of the Higgs-boson depend critically on the knowledge of $m_t$ and the control of the top-mass dependent effects on precision observables.

A large group of dominant radiative corrections can be absorbed in the shift of the $\rho$ parameter from its lowest order value $\rho_{\text{Born}} = 1$. The result for the one-loop approximation

$$\delta \rho = 3 x_t = \frac{G_F m_t^2}{8 \sqrt{2} \pi^2},$$

hence quadratic in $m_t$, was first evaluated in\footnote{The term $G_F$ is the Fermi constant.} and used to establish a limit on the mass splitting within one fermion doublet. In order to make full use of the present experimental precision, this one-loop calculation has been improved by two-loop\footnote{The two-loop calculation improves the accuracy.} and even three-loop QCD corrections\footnote{Three-loop calculations provide more accurate results.}. Also important are two-loop\footnote{Two-loop corrections are less accurate than three-loop corrections.} and three-loop\footnote{Three-loop corrections are more accurate than two-loop corrections.} electroweak effects proportional to $x_t^2$ and $x_t$, respectively, and the three-loop mixed corrections of order $\alpha_s x_t^2$.

An important ingredient for the interpretation of these results in terms of top mass measurements performed at hadron colliders or at a future linear collider is the relation between the pole mass and the MS-mass definitions, the former being useful for the determination of $m_t$ at colliders, the latter being employed in actual calculations and in short-distance considerations. To match the present three-loop precision of the $\rho$ parameter, this relation must be known in two-loop approxima-



where $\Pi^{W/Z}_T (0)$ are the transversal parts of the $W$- and $Z$-boson self-energies at zero momentum transfer, respectively. The calculation is thus reduced to the evaluation of vacuum ( tadpole) diagrams.

The $W$-self-energy receives contributions from the correlator of the “non-diagonal” $t$-$b$-current only. Contributions to the $Z$ self-energy originate only from the “diagonal” axial current correlator induced by top quark loops.
The vector current part vanishes due to current conservation, the bottom quark is taken as massless. The non-vanishing parts of \( \Pi_Z^W(0) \) are conveniently decomposed into non-singlet and singlet pieces characterized by Feynman diagrams where the external current couples to the same and to two different closed fermion lines, respectively. In three-loop approximation the singlet-piece is larger than the non-singlet piece by nearly a factor twenty. This has motivated the authors of [27] to evaluate, in a first step, the four-loop singlet piece. The strategy employed in that paper was based on the algebraic reduction of the amplitudes to a small set of master-integrals with the help of the Laporta algorithm [28, 29] an approach which was recently also applied to the evaluation of the two lowest non-vanishing Taylor coefficients of the vacuum polarization [30, 31] and to the decoupling of heavy quarks in QCD [32, 33, 34], both in four-loop approximation.

In order to complete the evaluation of the four-loop QCD corrections to the \( \rho \) parameter, \( \Pi_Z^W(0) \) and the non-singlet parts of \( \Pi_Z^W(0) \) are required. The evaluation of \( \Pi_Z^W(0) \) is fairly straightforward: the input diagrams have been generated with QGRAF [35]; for the algebraic reduction to master integrals an efficient program has been constructed [36] which relies on FORM3 and FERMAT [37, 38, 39]. Furthermore, the full set of the corresponding master integrals is available with high precision [40, 41].

The evaluation of \( \Pi_Z^W(0) \), however, requires the knowledge of a sizeable number of new master-integrals, a major part of them (around 40) nontrivial to evaluate precisely. The master integrals can be chosen in many different ways. As discussed in [41] the choice of a so called “finite basis” leads to integrals particularly suited for the evaluation through Padé approximations. On the other hand, topologies with eight lines or less are conveniently calculated through difference equations. In the present paper we therefore employ a combined approach, which makes use of difference equations [29, 42] to evaluate the simpler topologies, i.e. those with up to eight lines, and a semi-numerical method based on Padé approximations [41, 43, 44, 45, 46]. There, a suitably chosen line of the four-loop vacuum diagram is cut, the large- and the small \( q^2 \) behaviour of the resulting three-loop propagator are calculated analytically [47, 48, 49, 50], the function in the whole region is represented by Padé approximations and the remaining \( q^2 \) integration is performed numerically (see Fig. 1).

\[
\text{4 loop} = \left[ \begin{array}{c} q \\ 3l \end{array} \right] \propto \int dq \left[ \begin{array}{c} q \\ 3l \end{array} \right] = \int dq F(q^2)
\]

Fig 1. Symbolic description of the Padé method: One line of the vacuum integral is cut, the resulting propagator is represented by a Padé approximation and integrated numerically.

An estimate of the numerical uncertainty is obtained by comparing different Padé approximations based on the same input information from the large and small \( q^2 \) region, or by increasing the input information through inclusion of more terms from the high and the low \( q^2 \) region. Furthermore, in all cases at least two different lines were cut to check the consistency of the results. A detailed discussion of the various applications, characteristic examples and comparisons with analytic results, e.g. for the lowest moment of the polarization function can be found in [46].

In contrast to the applications discussed in earlier publications [41, 43, 46], massless cuts unavoidably arise in some of the relevant diagrams and a generalization of the method is required: In addition to the introduction of a suitably chosen function needed for the subtraction of the high energy logarithms, another function is employed for the subtraction of the logarithms arising from the massless cut in the low energy limit.

The result for the shift in the \( \rho \) parameter can be cast into the following form:

\[
\delta \rho^\text{MS} = 3x_1 \sum_{i=0}^{3} \left( \frac{\alpha_s}{\pi} \right)^i \delta \rho^\text{MS}_i \tag{3}\]

Here \( x_1 \) is expressed in terms of the \( \overline{\text{MS}} \) quark mass \( m_t = m_t(\mu) \) at scale \( \mu = m_t \), and \( \alpha_s \), defined in the \( \overline{\text{MS}} \)-scheme for six flavors, is chosen at the same scale. The normalization factors are such that \( \delta \rho^\text{MS}_0 = 1 \).

For the four-loop non-singlet result, decomposed according to the various color structures and the \( n_f \)-dependence, we find:

\[
\delta \rho^\text{MS}_3 \text{ (non-singlet) } = \frac{1}{2} \left[ 1.5211 C_F^3 + 1.2363 C_F^2 C_A + 2.3132 C_F^2 T n_l - 4.5962 C_F^2 T n_h + 0.7438 C_F C_A^2 - 1.3705 C_F C_A T n_l + 2.5037 C_F C_A T n_h + 0.4681 C_F T^2 n_l^2 + 0.6880 C_F T^2 n_h^2 + 0.8495 C_F T^2 n_h n_l, \right] \tag{4}
\]
with $C_F = (N_c^2 - 1)/(2N_c)$, $C_A = N_c$, and $T = 1/2$, where $N_c = 3$ is the number of colors. $n_f$ denotes the number of active (light plus heavy) quark fields, with $n_f = n_l + n_h$. This result has been also independently obtained with the help of direct application of the Padé approximation method like it was described in \[15\] for the lowest Taylor coefficients of the vacuum polarization. We find agreement for all color structures with the relative accuracy varying between 0.4% and 4%, and confirm the result for the singlet contribution.

Setting $n_h = 1$, $n_l = 5$ and the color coefficients to their natural values, we find

$$\delta\rho^{\overline{MS}}_3 = \delta\rho_3^{\overline{MS}}(\text{singlet}) + \delta\rho_3^{\overline{MS}}(\text{non-singlet}) = -3.2866 + 1.6067 = -1.6799,$$  \hspace{1cm} (5)

where we have also displayed the result of \[27\] for the singlet piece. The singlet piece is still larger than the non-singlet piece by a factor two. Nevertheless, the hierarchy is less pronounced than in the three-loop case. Numerically, the overall correction looks small, just as in the two- and three-loop case. However, if the result is expressed in terms of the pole mass, a major shift originates from the large correction in the pole-$\overline{MS}$ relation:

$$\delta\rho_3^{\text{pole}} = -93.1501.$$  \hspace{1cm} (6)

For fixed top mass, this corresponds to a shift of around 2 MeV in the $W$-boson mass, well below the precision anticipated for future experiments.

In conclusion, the full $\mathcal{O}(X_t^2\alpha_s^2)$ contribution to the $\rho$ parameter proves to be small and the result based on the three-loop calculation is stabilized.

Acknowledgments: We would like to thank P. Baikov and V. Smirnov for useful cross checks of some diagrams and M. Steinhauser for support in the usage of MATAD. This work has been supported by DFG through SFB/TR 9 and by BMBF, Grant No. 05HT4VKA/3. The work of C.S. was also partially supported by MIUR under contract 2001023713-J006.

---

Note added.
The results of our calculations have been recently confirmed in the independent work \[51\].

*Permanent address: Institute for Nuclear Research, Russian Academy of Sciences, Moscow 117312, Russia.

---

[1] LEP Electroweak Working Group hep-ex/0511027.
[2] Tevatron Electroweak Working Group (2006), hep-ex/0603039.
[3] M. J. G. Veltman, Nucl. Phys. B123, 89 (1977).
[4] A. Djouadi and C. Verzegnassi, Phys. Lett. B195, 265 (1987).
[5] A. Djouadi, Nuovo Cim. A100, 357 (1988).
[6] B. A. Kniehl, J. H. Kühn, and R. G. Stuart, Phys. Lett. B214, 621 (1988).
[7] L. Avdeev, J. Fleischer, S. Mikhailov, and O. Tarasov, Phys. Lett. B336, 560 (1994), hep-ph/9406363.
[8] K. G. Chetyrkin, J. H. Kühn, and M. Steinhauser, Phys. Lett. B351, 331 (1995), hep-ph/9502291.
[9] J. J. van der Bij and F. Hoogeveen, Nucl. Phys. B283, 477 (1987).
[10] R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci, and A. Vicere, Phys. Lett. B288, 95 (1992), hep-ph/9205238.
[11] R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci, and A. Vicere, Nucl. Phys. B409, 105 (1993).
[12] J. Fleischer, O. V. Tarasov, and F. Jegerlehner, Phys. Lett. B319, 249 (1993).
[13] J. Fleischer, O. V. Tarasov, and F. Jegerlehner, Phys. Rev. D51, 3820 (1995).
[14] J. J. van der Bij, K. G. Chetyrkin, M. Faisst, G. Jikia, and T. Seidensticker, Phys. Lett. B498, 156 (2001), hep-ph/0011373.
[15] M. Faisst, J. H. Kühn, T. Seidensticker, and O. Veretin, Nucl. Phys. B665, 649 (2000), hep-ph/0002275.
[16] N. Gray, D. J. Broadhurst, W. Grafe, and K. Schilcher, Z. Phys. C48, 673 (1990).
[17] F. Jegerlehner and M. Y. Kalmykov, Acta Phys. Polon. B34, 5335 (2003), hep-ph/0310361.
[18] M. Faisst, J. H. Kühn, and O. Veretin, Phys. Lett. B589, 35 (2004), hep-ph/0403026.
[19] D. Eiras and M. Steinhauser, JHEP 02, 010 (2006), hep-ph/0512099.
[20] K. G. Chetyrkin and M. Steinhauser, Phys. Rev. Lett. 83, 4001 (1999), hep-ph/9907509.
[21] K. G. Chetyrkin and M. Steinhauser, Nucl. Phys. B573, 617 (2000), hep-ph/9911434.
[22] K. Melnikov and T. v. Ritbergen, Phys. Lett. B482, 99 (2000), hep-ph/9912391.
[23] K. G. Chetyrkin, J. H. Kühn, and M. Steinhauser, Phys. Rev. Lett. 75, 3394 (1995), hep-ph/9504413.
[24] J. A. Aguilar-Saavedra et al. (ECFA/DESY LC Physics Working Group) (2001), hep-ph/0106315.
[25] G. P. Zeller et al. (NuTeV), Phys. Rev. Lett. 88, 091802 (2002), hep-ex/0110059.
[26] C. Rosenbleck (2005), hep-ex/0505033.
[27] Y. Schröder and M. Steinhauser, Phys. Lett. B622, 124 (2005), hep-ph/0504055.
[28] S. Laporta and E. Remiddi, Phys. Lett. B379, 283 (1996), hep-ph/9602417.
[29] S. Laporta, Int. J. Mod. Phys. A15, 5087 (2000), hep-ph/0102033.
[30] K. G. Chetyrkin, J. H. Kühn, and C. Sturm (2006), hep-ph/0604234.
[31] R. Boughenaz, M. Czakon, and T. Schutzmeier (2006), hep-ph/0605023.
[32] K. G. Chetyrkin, J. H. Kühn, and C. Sturm (2006), hep-ph/0604187.
[33] Y. Schröder and M. Steinhauser, JHEP 01, 051 (2006), hep-ph/0512058.
[34] K. G. Chetyrkin, J. H. Kühn, and C. Sturm, Nucl. Phys. B744, 121 (2006), hep-ph/0512060.
[35] P. Nogueira, J. Comput. Phys. 105, 279 (1993).
[36] C. Sturm, PhD thesis, Cuvillier Verlag, Goettingen.
ISBN 3-86537-569-X (2005).

[37] J. A. M. Vermaseren (2000), math-ph/0010025.
[38] R. H. Lewis, Fermat’s User Guide, http://www.bway.net/~lewis/.
[39] M. Tentyukov and J. A. M. Vermaseren (2006), cs.sc/0604052.
[40] Y. Schröder and A. Vuorinen (2005), hep-ph/0503209.
[41] K. G. Chetyrkin, M. Faisst, C. Sturm, and M. Tentyukov, Nucl. Phys. B742, 208 (2006), hep-ph/0601165.
[42] S. Laporta, Phys. Lett. B549, 115 (2002), hep-ph/0210336.
[43] J. Fleischer and O. V. Tarasov, Z. Phys. C64, 413 (1994), hep-ph/9403230.
[44] P. A. Baikov and D. J. Broadhurst (1995), hep-ph/9504398.
[45] M. Faisst, K. G. Chetyrkin, and J. H. Kühn, Nucl. Phys. Proc. Suppl. 135, 307 (2004).
[46] M. Faisst, PhD thesis, Cuvillier Verlag, Goettingen ISBN 3-86537-506-5 (2005).
[47] S. G. Gorishnii, S. A. Larin, L. R. Surguladze, and F. V. Tkachov, Comput. Phys. Commun. 55, 381 (1989).
[48] M. Steinhauser, Comput. Phys. Commun. 134, 335 (2001), hep-ph/0009029.
[49] R. Harlander, T. Seidensticker, and M. Steinhauser, Phys. Lett. B426, 125 (1998), hep-ph/9712228.
[50] T. Seidensticker (1999), hep-ph/9905298.
[51] R. Boughezal and M. Czakon (2006), hep-ph/0606232.
[52] A. L. Kataev and S. Kumano, J. Phys. G29, 1925 (2003), hep-ph/0211052.
[53] S. A. Kulagin, Phys. Rev. D67, 091301 (2003), hep-ph/0301045.
[54] K. S. McFarland and S.-O. Moch (2003), hep-ph/0306052.
[55] K. P. O. Diener, S. Dittmaier, and W. Hollik, Phys. Rev. D72, 093002 (2005), hep-ph/0509084.
[56] J. Erler (2006), hep-ph/0604035.
[57] For more detailed discussion of various aspects of the issue see e.g. works 52, 53, 54, 55, 56 and references therein.