Fluctuations in the homogenization of the Poisson and Stokes equations in perforated domains

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We study the homogenization problem of the Poisson and Stokes equations in $\mathbb{R}^3$ perforated by $m$ spherical holes, identically and independently distributed. In the critical regime when the radii of the holes are of order $m^{-1}$, we consider the fluctuations of the solutions $u_m$ around the homogenization limit $u$. In the central limit scaling, we show that these fluctuations converge to a Gaussian field, locally in $L^2(\mathbb{R}^3)$, with an explicit covariance.

1. Introduction

In a perforated domain $\Omega_m \subseteq \mathbb{R}^3$, we consider the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u_m = f & \text{in } \Omega_m, \\ u_m = 0 & \text{in } \mathbb{R}^3 \setminus \Omega_m, \end{cases}$$

and the Stokes equations

$$\begin{cases} -\Delta u_m + \nabla p_m = f, \\ \text{div } u_m = 0, \\ u_m = 0 & \text{in } \Omega_m, \\ u_m = 0 & \text{in } \mathbb{R}^3 \setminus \Omega_m. \end{cases}$$

We consider the case when $\Omega_m$ is a random set obtained by removing $m$ spherical holes from $\mathbb{R}^3$. More precisely, let $V \in W^{1,\infty}(\mathbb{R}^3)$ be a compactly supported probability measure and let $\Phi_m = \{w_1, \ldots, w_m\}$ be given as the random set of $m$ i.i.d. points distributed with density $V$. Then, we define

$$\Omega_m := \mathbb{R}^3 \setminus \bigcup_i B_i,$$

where the spherical holes are

$$B_i := B^m(w_i) := B_{R_m}(w_i).$$

We study the classical homogenization problem $m \to \infty$, $R_m \to 0$. It is well known that the critical regime for the radius is $R_m \sim m^{-1}$ such that the total capacity of the holes is of order one. For convenience, we choose

$$R_m = \frac{1}{4\pi m}.$$
for the Poisson equation and
\[ R_m = \frac{1}{6\pi m} \] (1.5)
for the Stokes equations, respectively.

Then, \( u_m \) converges weakly in the homogeneous Sobolev space \( \dot{H}^1(\mathbb{R}^3) \) to \( u \), the unique weak solution to
\[ -\Delta u + Vu = f \quad \text{in} \quad \mathbb{R}^3, \] (1.6)
respectively,
\[ -\Delta u + Vu + \nabla p = f, \quad \text{div} \ u = 0 \quad \text{in} \quad \mathbb{R}^3. \] (1.7)

Such homogenization results have been obtained under various assumptions on the distribution of the configuration of holes, see for instance [MK74; CM82; PV80; Oza83; DMG94; GHV18] for the Poisson equation, and [All90; DGR08; GH19; GV19; CH20] for the Stokes equations. For a detailed discussion of this literature, we refer the reader to [GHV18; GH19].

The main result of the present paper provides the precise rate of convergence \( u_m \to u \) in \( L^2_{\text{loc}}(\mathbb{R}^3) \) as well as a characterization of the fluctuation field. For the statement of our main result, we introduce the notation that \( A, \) depending on \( V, \) is the solution operator for this limit problem, i.e. \( u = Af. \)

**Theorem 1.1.** Let \( f \in \dot{H}^{-1}(\mathbb{R}^3) \) and let \( u_m \) and \( u \) be defined as in (1.1) and (1.6) or as in (1.2) and (1.7), respectively.

(i) For any \( \beta < 1/2, \)
\[ m^\beta \|u_m - u\|_{L^2_{\text{loc}}(\mathbb{R}^3)} \to 0 \quad \text{in probability}. \]

(ii) For every \( g \in L^2(\mathbb{R}^3) \) with compact support,
\[ \xi_m[g] := m^{1/2}(g, u_m - u) \to \xi[g] \]
in distribution, where \( \xi[g] \) is a Gaussian field with mean zero and covariance
\[ \mathbb{E}[\xi[g_1]\xi[g_2]] = (AfAg_1, AfAg_2)_{L^2_{\text{loc}}(\mathbb{R}^3)} - (Af, Ag_1)_{L^2_{\text{loc}}(\mathbb{R}^3)}(Af, Ag_2)_{L^2_{\text{loc}}(\mathbb{R}^3)} \]
for all \( g_1, g_2 \in L^2(\mathbb{R}^3) \) with compact support, where \( (\cdot, \cdot)_{L^2_{\text{loc}}(\mathbb{R}^3)} \) denotes the \( L^2 \) scalar product with weight \( V. \)

Before we comment on related results in the literature and the main ingredients of our proof, we briefly discuss two very natural questions regarding possible generalizations of this theorem. The first addresses random radii of the holes, the second space dimensions different from \( d = 3. \)

Indeed, it is not difficult to extend the above result to the case, where the radii of the holes are also random. More precisely, assume that the radius of each hole is \( R_i^n = r_i R_m \) with \( R_m \) as in (1.4) and (1.5), respectively. Assume that the \( r_i \) are independent bounded random variables, also independent of the positions, with expectation \( \mathbb{E} r = 1. \) Then, the assertions of Theorem 1.1 still hold with an additional factor \( \mathbb{E} r^2 \) in front of the first term on the right-hand side of the covariance. For the sake of simplicity of the presentation of the proof, we will only give the proof in the case of identical radii.
On the other hand, our analysis is restricted to the physically most relevant three-dimensional case. Applying the same techniques in dimension $d = 2$ seems possible with additional technicalities due to the usual issues regarding the capacity of a set in $d = 2$.

We emphasize that, for $d \geq 4$, we do not expect Theorem 1.1 to continue to hold. Roughly speaking, the problem is that the volume occupied by the holes is too big. Indeed, the critical scaling of the radius of $m$ spherical holes in dimension $d \geq 3$ is $R_m \sim m^{-1/(d-2)}$. The results cited above ensure that under this scaling, we still have $u_m \rightharpoonup u$ weakly in $\dot{H}^1(\mathbb{R}^d)$. However, we obtain as a trivial upper bound for the rate of convergence in $L^2_{\text{loc}}$.

$$
\|u_m - u\|_{L^2_{\text{loc}}(\mathbb{R}^3)} \geq \|u_m - u\|_{L^2(\bigcup_{i=1}^m B_i)} = \|u\|_{L^2(\bigcup_{i=1}^m B_i)} \sim \left(\mathcal{L}^d \left(\bigcup_{i=1}^m B_i\right)\right)^{\frac{1}{2}} \sim m^{-\frac{1}{d-2}}.
$$

This shows, that Theorem 1.1 cannot hold in this form for $d \geq 5$. Moreover, in dimension $d = 4$, this error is of critical order, which suggests that the analysis of the fluctuations is much more delicate.

Related to these considerations, the restrictions to dimension $d = 3$ is also reflected in our proof where we will use that the fundamental solution of the Poisson and Stokes equations is in $L^2_{\text{loc}}(\mathbb{R}^3)$.

In the classical theory of stochastic homogenization of elliptic equations with oscillating coefficients, the study of fluctuations has been a very active research field in recent years. Of the vast literature, one could mention for example works by Armstrong, Kuusi and Mourrat [AKM17] and by Duerinckx, Gloria and Otto [DGO20].

Regarding the homogenization of perforated domains, related results to Theorem 1.1 have been obtained in [FOT85] by Figari, Orlandi and Teta for the Poisson equation and by Rubinstein [Rub86] for the Stokes equations. In these papers, the authors considered the Poisson and the Stokes equations (1.1) and (1.2) but with an additional massive term $\lambda u_m$. Then, they obtained a result corresponding to Theorem 1.1 provided that $\lambda$ is sufficiently large (depending on $V$).

The approach in [FOT85; Rub86] follows the approximation of the solution $u_m$ by the so called method of reflections. The idea behind this method is to express the solution operator of the problem in the perforated domain in terms of the solutions operators when only one of the holes is present. More precisely, let $v_0$ be the solution of the problem in the whole space without any holes. Then, define $v_1 = v_0 + \sum_i v_{1,i}$ in such a way that $v_0 + v_{1,i}$ solves the problem if $i$ was the only hole. Since $v_{1,i}$ induces an error in $B_j$ for $j \neq i$, one adds further functions $v_{2,i}$, this time starting from $v_1$. Iterating this procedure yields a sequence $v_k$. In general, $v_k$ is not convergent. With the additional massive term though, one can show that the method of reflections does converge, provided that $\lambda$ is sufficiently large.

In [HV18], the first author and Velázquez showed how the method of reflections can be modified to ensure convergence without a massive term and how this modified method can be used to obtain convergence results for the homogenization of the Poisson and Stokes equations. In order to study the fluctuations, a high accuracy of the approximation of $u_m$ is needed. This would make it necessary to analyze many of the terms arising from the modified method of reflections which we were allowed to disregard for the qualitative convergence result of $u_m$ in [HV18]. It seems very hard to control sufficiently well these additional terms, which either do not arise or are of higher order for the (unmodified) method of reflections used in [FOT85; Rub86].
Thus, in the present paper, we do not use the method of reflections but follow an alternative approach to obtain an approximation for $u_m$. Again, we approximate $u_m$ by $\tilde{u}_m = v_0 + \sum_i v_i$, where $v_i$ solves the homogeneous Poisson (respectively Stokes) equation outside of $B_i$. However, we do not take $v_i$ as in the method of reflections, where it is expressed in terms of $v_0$. Instead $v_i$ will depend on $u$, exploiting that we already know that $u_m$ converges to $u$. In contrast to the approximation obtained from the method of reflections, we will be able to choose $v_i$ in such a way that the approximation $\tilde{u}_m = v_0 + \sum_i v_i$ is sufficient to capture the fluctuations.

A related approach has recently been used by Gérard-Varet in [GV19] to give a very short proof of the homogenization result $u_m \to u$ weakly in $\dot{H}^1$ under rather mild assumptions on the positions of the holes. However, since we study the fluctuations in this paper, we need a more refined approximation than the one used in [GV19]. More precisely, to leading order, the function $v_i$ will only depend on the value of $u$ at $B_i$. However, $v_i$ will also include a lower-order term, which is still relevant for the fluctuations. As we will see, this lower-order term will depend in some way on the fluctuations of the positions of all the other holes.

The rest of the paper is devoted to the proof of the main result, Theorem 1.1.

In Section 2, we give a precise definition of the approximation $\tilde{u}_m = v_0 + \sum_i v_i$ as well as a heuristic explanation for this choice.

In Section 3, we state three key estimates regarding this approximation and show how the proof of Theorem 1.1 follows from these estimates.

The proof of these key estimates contains a purely analytic part as well as a stochastic part, which are given in Sections 4 and 5, respectively.

For the vast part of the proof, it does not make any difference whether Poisson or Stokes equations are considered. We therefore treat these cases simultaneously and only distinguish the two cases when necessary. In particular, our notation does not distinguish between scalar functions and vector fields.

2. The approximation for the microscopic solution $u_m$

2.1. Notation

We introduce the following notation that is used throughout the paper.

We denote by $G: \dot{H}^{-1}(\mathbb{R}^3) \to \dot{H}^1(\mathbb{R}^3)$ the solution operator for Poisson and the Stokes equations, respectively. This operator is explicitly given as a convolution operator with kernel $g$, the fundamental solution of the Poisson equations and the Stokes equations, i.e.,

$$g(x) = \frac{1}{4\pi|x|}, \quad g(x) = \frac{1}{8\pi} \left( \frac{\text{Id}}{|x|} + \frac{x \otimes x}{|x|^3} \right),$$

respectively. \hfill (2.1)

We recall from Theorem 1.1 that $A: \dot{H}^{-1}(\mathbb{R}^3) \to \dot{H}^1(\mathbb{R}^3)$ is the solution operator for the limit problem (1.6) and (1.7), respectively. We observe the identities

$$(1 + GV)A = G, \quad A(1 + VG) = G. \hfill (2.2)$$

We remark that multiplication by $V$ maps from $\dot{H}^1(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$. Indeed, this follows from $V \in L^\infty(\mathbb{R}^3)$ with compact support and the fact that $\dot{H}^1(\mathbb{R}^3) \subseteq L^6(\mathbb{R}^3)$ which implies $L^{6/5}(\mathbb{R}^3) \subseteq H^{-1}(\mathbb{R}^3)$. Furthermore, observe that $A$ and $G$ are bounded operators from $L^2(\mathbb{R}^3) \cap H^{-1}(\mathbb{R}^3)$ to $C^{0,\alpha}(\mathbb{R}^3)$, $\alpha \leq 1/2$, and from $H^1 \cap H^{-1}$ to $W^{1,\infty}$. In particular, $AV$ and

\[\text{Do we need this?}\]
are all bounded operators from $L^2(\text{supp } V)$ (and in particular from $\dot{H}^1(\mathbb{R}^3)$) to $L^\infty(\mathbb{R}^3)$ and from $\dot{H}^1(\mathbb{R}^3)$ to $W^{1,\infty}(\mathbb{R}^3)$.

We denote $G^{-1} = -\Delta$. This is the inverse of $G$ for the Poisson equation. For the Stokes equations, we have $GG^{-1} = G^{-1}G = P_\sigma$, where $P_\sigma$ is the projection to the divergence free functions. In fact, we will use $G^{-1}$ in the expression $AG^{-1}$ only. We observe that $A = AP_\sigma$ and thus

$$AG^{-1}G = A.$$ 

We denote the normalized Hausdorff measure on a sphere $\partial B^m(x)$ by

$$\delta^m_x := \frac{H^2(\partial B^m(x))}{H^2(\partial B^m(x))}$$

and write $\delta^m := \delta^m_{w_i}$.

Moreover, we denote for any function $h \in L^1(B^m(x))$ the average on $B^m(x)$ by $(h)_x$, i.e.

$$(h)_x := \frac{1}{|B^m(x)|} \int_{B^m(x)} h(y) \, dy,$$

and we abbreviate $(h)_i := (h)_{w_i}$.

We will need a cut-off version of the fundamental solution. To this end, let $\eta \in C_c^\infty(B_3(0))$ with $1_{B_2(0)} \leq \eta \leq 1_{B_3(0)}$ and $\eta_m(x) := \eta(x/R_m)$. For the Poisson equation, we define $G^m$ as the convolution operator with kernel

$$g^m = (1 - \eta_m)g,$$

where $g$ is the fundamental solution of the Poisson given in equation (2.1). For the Stokes equations, we need an additional term in order to make $g^m$ divergence free. This is obtained through the classical Bogovski operator (see e.g. [Gal11, Theorem 3.1]) which provides the existence of a sequence $\psi_m \in C_c^\infty(B_3R_m \setminus B_2R_m)$ such that $\text{div } \psi_m = \text{div}(\eta_m g)$ and

$$\|\nabla^k \psi_m\|_{L^p(\mathbb{R}^3)} \leq C(p, k)\|\nabla^{k-1} \text{div}(\eta_m g)\|_{L^p(\mathbb{R}^3)}$$

for all $1 < p < \infty$ and all $k \geq 1$. By scaling considerations, the constant $C$ is independent of $m$. Then, for the Stokes equations, we define $G^m$ as the convolution operator with kernel

$$g^m = (1 - \eta_m)g + \psi_m. \tag{2.4}$$

2.2. Approximation of $u_m$ using monopoles induced by $u$

We begin by observing that for most of the configurations of holes, the holes are sufficiently separated which allows us, to leading order, to sum the contributions coming from each hole.

**Lemma 2.1.** For $\nu < \frac{1}{3}$, $L > 0$ let $W_{m,\nu, L} \subseteq \mathbb{R}^{3m}$ be the set of all configurations of holes with

$$\min_{i \neq j} |w_i - w_j| \geq Lm^\nu R_m. \tag{2.5}$$

Then, $\lim_{m \to \infty} \mathbb{P}(W_{m,\nu, L}) = 1$. 

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This is a standard result that can, for example, be found in [Oza83]. Note that (2.5) in particular implies that with probability tending to one, the balls $B_{2R_m}(w_i)$ do not overlap.

To find a good approximation for $u_m$, we observe that $u_m$ satisfies

$$-\Delta u_m = f 1_{\Omega_m} + \sum_i f_i, \quad \text{in } \mathbb{R}^3$$

for some functions $f_i \in \dot{H}^{-1}(\mathbb{R}^3)$, each supported in $B_i$, which are the charge distributions induced in the holes due to the Dirichlet boundary conditions. (We only treat here the Poisson equation, the Stokes equations are completely analogous).

Let $0 < \nu < 1/3$. Then, by the lemma above, we know that, for most of the holes, $B_m\nu R_m(w_i)$ only contains the hole $B_i$. In this case, $f_i$ is uniquely determined by the problem

$$\begin{cases} -\Delta v_i = f & \text{in } B_{m^\nu R_m}(w_i) \setminus B_i, \\ v_i = 0 & \text{in } B_i, \\ v_i(x) \to (u)_i & \text{as } |x - w_i| \to \infty. \end{cases}$$

We simplify this problem to derive an approximation for $f_i$. First, we drop the right-hand side $f$ in (2.7). Its contribution is expected to be negligible, since the volume of $B_{m^\nu R_m}(w_i) \setminus B_i$ is small compared to the difference of the boundary data at $\partial B_i$ and $\partial B_{m^\nu R_m}(w_i)$ which is typically of order 1. Next, we know that typically $\partial B_{m^\nu R_m}(w_i)$ is very far from any hole. Since $u_m \rightharpoonup u$ in $\dot{H}^1(\mathbb{R}^3)$ we therefore replace (2.7) by

$$\begin{cases} -\Delta v_i = 0 & \text{in } B_{m^\nu R_m}(w_i) \setminus B_i, \\ v_i = 0 & \text{in } B_i, \\ v_i(x) \to (u)_i & \text{as } |x - w_i| \to \infty. \end{cases}$$

Here, we could also have chosen $u(w_i)$ instead of $(u)_i$. The precise choice that we make will turn out to be convenient later. By our choice of $R_m$ in (1.4) and (1.5), respectively, the explicit solution of (2.8) is given by by $v_i$ which solves $-\Delta v_i = f_i$ in $\mathbb{R}^3$ with

$$f_i = \frac{(u)_i}{m} \delta_i^m.$$

Therefore, resorting to (2.6), we are led to approximate $u_m$ by

$$\tilde{u}_m := G \left[ f - \frac{1}{m} \sum_{i=1}^m (u)_i \delta_i^m \right].$$

We emphasize that for this approximation it is not important to know the function $u$. We only used that $u_m \rightharpoonup u$ in $\dot{H}^1(\mathbb{R}^3)$, which is always true for a subsequence by standard energy estimates. On the contrary, we can now identify the limit $u$. Indeed, if we believe that $\tilde{u}_m$ approximates $u_m$ sufficiently well,

$$u \leftarrow u_m \approx \tilde{u}_m = G \left[ f - \frac{1}{m} \sum_{i=1}^m (u)_i \delta_i^m \right] \rightharpoonup G[f - Vu],$$

which shows that $u$ indeed solves (1.6).

This approximation $\tilde{u}_m$ cannot fully capture the fluctuations, though. In the next subsection, we thus show how to refine this approximation.
We end this subsection by comparing this approximation to the one used in [FOT85; Rub86] through the method of reflections. The first order approximation of the method of reflections is given by \( \tilde{u}_m \) as defined in (2.9) but with \( Gf \) instead of \( u \) on the right-hand side. Since this is a much cruder approximation, one needs to iterate the approximation scheme. This only yields a convergent series in [FOT85; Rub86] due to the additional large massive term. On the other hand, this series then approximates \( u_m \) sufficiently well without the refinement that we introduce in the next subsection.

2.3. Refined approximation to capture the fluctuations

We make the ansatz that, macroscopically,

\[ u_m = u + m^{-\frac{1}{2}} \xi_m + o(m^{-\frac{1}{2}}), \tag{2.11} \]

where \( \xi_m \) is a random function which needs to be determined. We assume that the fluctuations \( \xi_m \) are in some sense macroscopic, just as \( u \), such that we can follow the same approximation scheme as in the previous subsection.

More precisely, we adjust the Dirichlet problem (2.8) by adding \( m^{-\frac{1}{2}}(\xi_m)_i \) on the right-hand side of the third line. This leads to the definition

\[ \tilde{u}_m := G \left[ f - \frac{1}{m} \sum_{i=1}^{m} \left( u + m^{-\frac{1}{2}} \xi_m \right)_i \delta_{ij}^m \right]. \tag{2.12} \]

We have not defined \( \xi_m \) yet. To make a good choice for \( \xi_m \), the idea is to use a similar argument as in (2.10), but only to take the limit \( m \to \infty \) in terms which are of lower order. More precisely, we observe, again taking for granted that \( \tilde{u}_m \) approximates \( u_m \) sufficiently well, and using \( u = G(f - Vu) \)

\[ u + m^{-1/2} \xi_m \approx u_m \approx \tilde{u}_m = G \left[ f - \frac{1}{m} \sum_{i=1}^{m} \left( u + m^{-\frac{1}{2}} \xi_m \right)_i \delta_{ij}^m \right] 
= u + G \left[ Vu - \frac{1}{m} \sum_{i=1}^{m} (u)_i \delta_{ij}^m \right] - G \left[ \frac{1}{m} \sum_{i=1}^{m} (m^{-\frac{1}{2}} \xi_m)_i \delta_{ij}^m \right]. \tag{2.13} \]

We expect

\[ G \left[ \sum_{j \neq i} \frac{m^{-\frac{1}{2}} (\xi_m)_j \delta_{ij}^m}{m} \right] = G(Vm^{-\frac{1}{2}} \xi_m) + O(m^{-1}). \tag{2.14} \]

Inserting this into (2.13) leads to

\[ m^{-1/2} \xi_m + G(Vm^{-\frac{1}{2}} \xi_m) \approx G \left[ Vu - \frac{1}{m} \sum_{i=1}^{m} (u)_i \delta_{ij}^m \right]. \tag{2.15} \]

This equation could be used as a definition of \( \xi_m \). Although this turns out to be a good approximation on the level of equation (2.11), we will now argue that this is not the case for the definition of \( \tilde{u}_m \) in (2.12). Indeed, the right-hand side of (2.15) is equal to \( (u)_i \) in \( B_i \) to leading order. Hence, \( (m^{-1/2} \xi_m)_i \) would be of the same order, which would yield a contribution to \( \tilde{u}_m \) through \( \xi_m \) of order 1 instead of order \( m^{-1/2} \).
Therefore, we need to be more careful and go back to microscopic considerations: Since \( u_m = 0 \) in \( B_i \) and \( \tilde{u}_m \approx u_m \), we want to define \( \xi_m \) in such a way that \( \tilde{u}_m \approx 0 \) in \( B_i \). Thus we want to compute \( \tilde{u}_m \) in \( B_i \) in order to find a good definition of \( \xi_m \). Since we expect \( \tilde{u}_m = \bar{u}_m(w_i) + O(m^{-1}) \) in \( B_i \) (at least on average), we only compute \( \bar{u}_m(w_i) \), and by the same reasoning, we replace any average \((h)_i \) by \( h(w_i) \) at will. Then, we find

\[
\bar{u}_m(w_i) \approx u(w_i) + (GVu)(w_i) - u(w_i) - m^{-\frac{1}{2}}\xi_m(w_i) - G \left[ \frac{1}{m} \sum_{j \neq i} (u + m^{-\frac{1}{2}}\xi_m)j \delta_j^m \right]
\]

Thus, we introduce, as an approximation for \( \xi_m \), we define

\[
\xi_m \approx -m^{-\frac{1}{2}}\xi_m(w_i) + G \left[ Vu - \frac{1}{m} \sum_{j \neq i} (u)j \delta_j^m \right](w_i) - G \left[ \frac{1}{m} \sum_{j \neq i} m^{-\frac{1}{2}}(\xi_m)j \delta_j^m \right](w_i).
\]

(2.16)

Requiring \( \bar{u}_m(w_i) = 0 \) yields

\[
m^{-\frac{1}{2}}\xi_m(w_i) + G \left[ \frac{1}{m} \sum_{j \neq i} m^{-\frac{1}{2}}(\xi_m)j \delta_j^m \right](w_i) = G \left[ Vu - \frac{1}{m} \sum_{j \neq i} (u)j \delta_j^m \right](w_i).
\]

(2.17)

In order to define \( \xi_m \) from this equation, we want the sum on the right-hand side to include \( i \) such that the function is the same for every \( i \). To this end, we notice that by Lemma 2.1, with high probability, we have for all \( i \)

\[
G^m \delta_i^m = 0 \quad \text{in } B_i, \quad G^m \delta_j^m = G^m \delta_j^m \quad \text{in } B_i \quad \text{for all } j \neq i.
\]

(2.18)

where \( G^m \) is the operator introduced at the end of Section 2.1. Hence, in view of (2.17), we define

\[
m^{-\frac{1}{2}}\Theta_m = GVu - \frac{1}{m} \sum_{i=1}^{m} G^m ((u)\delta_i^m).
\]

(2.19)

We expect \( \Theta_m \sim 1 \) since the right-hand side of (2.19) represents the fluctuations of \( GVu \). As before, we replace the sum on the left-hand side of (2.17) by \( V\xi_m \). Combining these approximations leads to

\[
m^{-\frac{1}{2}}(1 + GV)\xi_m = m^{-\frac{1}{2}}\Theta_m.
\]

(2.20)

In view of (2.2), we thus define \( \xi_m \) to be the solution of

\[
\xi_m = AG^{-1}\Theta_m.
\]

(2.21)

For the Stokes equations we use that \( \text{div } \Theta_m = 0 \) to see that (2.21) is equivalent to (2.20).

Note that the only difference between this definition of \( \xi_m \) and (2.15) is the replacement of \( G \) by \( G^m \). As mentioned above, we expect that, on a macroscopic scale, the operators \( G \) and \( G^m \) are almost the same (we will make this argument rigorous in Lemma 5.3). Therefore, in equation (2.11), we expect, that it does not play a role (in \( L^2_{\text{loc}}(\mathbb{R}^3) \)) whether we take \( G \) or \( G^m \). Thus, we introduce, as an approximation for \( \xi_m \),

\[
\tau_m := AG^{-1}\Theta_m,
\]

(2.22)

\[
m^{-1/2}\Theta_m := GVu - \frac{1}{m} \sum_{i=1}^{m} G(u(w_i))\delta_i(w_i).
\]
This function bears the advantage that it is the sum of i.i.d. random variables. Hence, it is straightforward to study the limit properties of \( \tau_m[g] := (g, \tau_m) \). Notice that we both replaced the average \((u_i)\) by the value in the center of the ball \( u(w_i) \) and \( \delta_m^n \) by \( \delta_m \). Since \( u \in H^1(\mathbb{R}^3) \), \( \tau_m \) is not defined for every realization of holes. However, as we will see, it is well-defined as an \( L^2 \)-function on the probability space with values in \( L^2_{\text{loc}}(\mathbb{R}^3) \).

3. Proof of the main result

The first step of the proof is to rigorously justify the approximation of \( u_m \) by \( \tilde{u}_m \), defined in (2.12) with \( \xi_m \) and \( \Theta_m \) as in (2.21) and (2.19).

**Proposition 3.1.** For all \( \varepsilon > 0 \) and all \( \beta < 1 \)

\[
\lim_{m \to \infty} \mathbb{P}_m \left[ \left\| u_m - \tilde{u}_m \right\|_{H^1(\mathbb{R}^3)} > \varepsilon \right] \to 0.
\]

The next step is to show that we actually have

\[
\tilde{u}_m = u + m^{-1/2} \xi_m + o(m^{-1/2})
\]

which was the starting point of our heuristics, i.e. \( \xi_m \) indeed describes the fluctuations of \( \tilde{u}_m \) around \( u \). In contrast to Proposition 3.1, we can only expect local \( L^2 \)-estimates since not even \( u_m - u \) is small in \( H^1(\mathbb{R}^3) \).

**Proposition 3.2.** For all \( \varepsilon > 0 \), all bounded sets \( K' \subseteq \mathbb{R}^3 \) and all \( \beta < 1 \)

\[
\lim_{m \to \infty} \mathbb{P}_m \left[ \left\| \tilde{u}_m - u - m^{-1/2} \xi_m \right\|_{L^2(K')} > \varepsilon \right] \to 0.
\]

Combining Proposition 3.1 and 3.2, we observe that we only have to prove the statements of Theorem 1.1 with \( u_m - u \) replaced by \( m^{-1/2} \xi_m \).

The next proposition shows that, instead of \( \xi_m \), we can actually consider \( \tau_m \) introduced in the previous section.

**Proposition 3.3.** For any bounded set \( K' \subseteq \mathbb{R}^3 \) there is a constant \( C(K') > 0 \) independent of \( m \) such that

\[
\mathbb{E}_m[\left\| \xi_m \right\|_{L^2(K')}^2] \leq C(K').
\]

Let \( \tau_m \) be defined by (2.22). Then,

\[
\limsup_{m \to \infty} m \mathbb{E}_m \left[ \left\| \xi_m - \tau_m \right\|_{L^2(K')}^2 \right] \leq C(K').
\]

The proof of Theorem 1.1 is a direct consequence of the above propositions together with the classical Central Limit Theorem.

**Proof of Theorem 1.1.** Due to the uniform bound on \( \mathbb{E}_m[\left\| \xi_m \right\|_{L^2(K')}^2] \) from Proposition 3.3, assertion (i) of the main theorem follows immediately from Propositions 3.1 and 3.2 since \( H^1(\mathbb{R}^3) \) embeds into \( L^2_{\text{loc}}(\mathbb{R}^3) \).
Lemma 4.1. Let \( \Xi_m \) be defined by
\[
\Lambda_m := (G^m - G) \left( \frac{1}{m} \sum_i (u)_i \delta_i^m \right) ,
\]
\[
\Gamma_m := G^m \left[ \sum_i \left( \frac{(u)_i}{m} \right) \delta_i^m \right] + G(Vm^{-\frac{1}{2}}\xi_m) ,
\]
\[
\Xi_m := G(Vm^{-\frac{1}{2}}\xi_m) - G^m \left[ \sum_i \frac{m^{-\frac{1}{2}}(\xi_m)_i}{m} \delta_i^m \right] ,
\]
\[
\tilde{\Xi}_m := G(Vm^{-\frac{1}{2}}\xi_m) - G \left[ \sum_i \frac{m^{-\frac{1}{2}}(\xi_m)_i}{m} \delta_i^m \right] .
\]

Since convergence in probability implies convergence in distribution, Propositions 3.1, 3.2 and 3.3 imply that it suffices to prove assertion (ii) of Theorem 1.1 with \( \tau_m[g] := (g, \tau_m) \), i.e. we need to prove that
\[
\tau_m[g] \to \xi[g]
\]
in distribution for any \( g \in L^2(\mathbb{R}^3) \) with compact support. Since \( \tau_m[g] \) is a sum of independent random variables, this is a direct consequence of the Central Limit Theorem and the following computation for covariances: let \( g_1, g_2 \in L^2(\mathbb{R}^3) \) with compact support, then
\[
\mathbb{E}_m [\tau_m[g_1]\tau_m[g_2]]
\]
\[
= m^{-1} \mathbb{E}_m \left[ \left( g_1, \sum_i \frac{m}{A} (V u - u(w_i)) \delta_i w_i \right)_{L^2(\mathbb{R}^3)} \left( g_2, \sum_j A \left( V u - u(w_j) \delta_j w_j \right) \right)_{L^2(\mathbb{R}^3)} \right]
\]
\[
= \int_{\mathbb{R}^3} V(y) \left( g_1, A(V u - u(y))(\delta_y)_y \right)_{L^2(\mathbb{R}^3)} \left( g_2, A(V u - u(y))(\delta_y)_y \right)_{L^2(\mathbb{R}^3)} dy
\]
\[
= \int_{\mathbb{R}^3} V(y) \left( g_1, A(u(y))(\delta_y)_y \right)_{L^2(\mathbb{R}^3)} \left( g_2, A(u(y))(\delta_y)_y \right)_{L^2(\mathbb{R}^3)} dy - \left( Ag_1, VAf \right)_{L^2(\mathbb{R}^3)} \left( Ag_2, VAf \right)_{L^2(\mathbb{R}^3)}
\]
\[
= (A g_1, A f A g_2)_{L^2_{\text{loc}}(\mathbb{R}^3)} - (A g_1, VAf)_{L^2_{\text{loc}}(\mathbb{R}^3)} (A g_2, VAf)_{L^2_{\text{loc}}(\mathbb{R}^3)} .
\]

Here we used that \( A\delta^m_y \in L^2_{\text{loc}}(\mathbb{R}^3) \) (see Lemma 5.2) for any \( m \) and that \( A \) is a symmetric operator on \( L^2(\mathbb{R}^3) \). This finishes the proof. \( \square \)

4. Proof of Propositions 3.1 and 3.2

In this section, we will reduce the proof of Propositions 3.1 and 3.2 to proving the following single probabilistic lemma. The proof of this lemma, which is given in Section 5.3, is the main technical part of this paper. It makes rigorous the heuristic equation (2.14).

As we discussed in the heuristic arguments, we will in the following exploit that the probability for very close holes is vanishing as stated in Lemma 2.1. In the notation of this lemma, we abbreviate
\[
W_m = W_{m, 0.5} .
\]

Lemma 4.1. Let \( \Gamma_m \) and \( \Xi_m \) be defined by
\[
\Lambda_m := (G^m - G) \left( \frac{1}{m} \sum_i (u)_i \delta_i^m \right) ,
\]
\[
\Gamma_m := G^m \left[ \sum_i \left( \frac{(u)_i}{m} \right) \delta_i^m \right] + G(Vm^{-\frac{1}{2}}\xi_m) ,
\]
\[
\Xi_m := G(Vm^{-\frac{1}{2}}\xi_m) - G^m \left[ \sum_i \frac{m^{-\frac{1}{2}}(\xi_m)_i}{m} \delta_i^m \right] ,
\]
\[
\tilde{\Xi}_m := G(Vm^{-\frac{1}{2}}\xi_m) - G \left[ \sum_i \frac{m^{-\frac{1}{2}}(\xi_m)_i}{m} \delta_i^m \right] .
\]
Then,

\[
\limsup_{m \to \infty} m^2 \mathbb{E}_m \left[ 1_{W_m} \left\| \nabla (u + G(Vu) + \Gamma_m + \Xi_m) \right\|^2_{L^2(\cup_i B_i)} \right] < \infty,
\]

\[
\limsup_{m \to \infty} m^4 \mathbb{E}_m \left[ 1_{W_m} \left\| \Xi_m \right\|^2_{L^2(\cup_i B_i)} \right] < \infty,
\]

\[
\limsup_{m \to \infty} m^2 \mathbb{E}_m \left[ 1_{W_m} \left\| \tilde{\Xi}_m + \Lambda_m \right\|^2_{L^2_{\text{loc}}(\mathbb{R}^3)} \right] < \infty.
\]

**Proof of Proposition 3.2.** We compute using \( u = Af = G(f - Vu) \) and \( \xi_m = AG^{-1} \Theta_m = \Theta_m - GV\xi_m \) and the definition of \( \Theta_m \)

\[
\tilde{u}_m - u - m^{-1/2} \xi_m = G \left( f - \frac{1}{m} \sum_i (u + m^{-1/2} \xi_m) \delta_i^m \right) - u - m^{-1/2} \xi_m
\]

\[
= G \left( Vu - \frac{1}{m} \sum_i (u + m^{-1/2} \xi_m) \delta_i^m \right) - m^{-1/2} \Theta_m + m^{-1/2} GV\xi_m
\]

\[
= m^{-1/2} G \left( V\xi_m - \frac{1}{m} \sum_i (\xi_m) \delta_i^m \right) + (G^m - G) \left( \frac{1}{m} \sum_i (u) \delta_i^m \right)
\]

\[
= \tilde{\Xi}_m + \Lambda_m.
\]

Hence,

\[
\mathbb{P}_m \left[ m^\beta \left\| \tilde{u}_m - u - m^{-1/2} \xi_m \right\|_{L^2(\mathbb{R}^3)} > \varepsilon \right]
\]

\[
\leq \mathbb{P}_m \left[ W^c_m \right] + C \varepsilon^{-2} m^{2\beta} \mathbb{E}_m \left[ 1_{W_m} \left( \left\| \tilde{\Xi}_m + \Lambda_m \right\|^2_{L^2_{\text{loc}}(\mathbb{R}^3)} \right) \right]
\]

and we now conclude by Lemmas 2.1 and 4.1. \(
\square
\)

**Proof of Proposition 3.1.** We observe that the assertion follows from the following claim: There exists a constant \( C \) which depends only on \( f \) and \( V \) such that for all \( (w_1, \ldots, w_m) \in W_m \) and all \( m \) sufficiently large

\[
\left\| \tilde{u}_m - u_m \right\|^2_{H^1(\mathbb{R}^3)} \leq C \left( \left\| \nabla (u + G(Vu)) \right\|^2_{L^2(\cup_i B_i)} + \left\| \nabla \Gamma_m \right\|^2_{L^2(\cup_i B_i)} + \left\| \nabla \Xi_m \right\|^2_{L^2(\cup_i B_i)} + \left\| \Theta_m \right\|^2_{L^2(\cup_i B_i)} \right).
\]

\[
(4.1)
\]

Indeed, accepting the claim for the moment, let \( \beta < 1 \) and \( \varepsilon > 0 \). Then,

\[
\mathbb{P}_m \left[ m^\beta \left\| \tilde{u}_m - u_m \right\|_{H^1(\mathbb{R}^3)} > \varepsilon \right]
\]

\[
\leq \mathbb{P}_m \left[ W^c_m \right] + C \varepsilon^{-2} m^{2\beta} \mathbb{E}_m \left[ 1_{W_m} \left( \left\| \nabla (u + G(Vu) + \Gamma_m + \Xi_m) \right\|^2_{L^2(\cup_i B_i)} + m^2 \left\| \Xi_m \right\|^2_{L^2_{\text{loc}}(\mathbb{R}^3)} \right) \right].
\]

Thus, the assertion follows again from Lemmas 2.1 and 4.1.

It remains to prove the claim above. It follows from the fact that \( u_m - \tilde{u}_m \) solves the homogeneous Poisson or Stokes equations outside of the holes. We only give the proof in the case of the Stokes equations. For the Poisson equation, the proof is slightly simpler.

Let \( (w_1, \ldots, w_m) \in W_m \). Then, by definition of this set, the balls \( B_{2R_m}(w_i) \) are disjoint for \( m \) sufficiently large and we may assume in the following that this is satisfied.
By definition of \( u_m \) and \( \tilde{u}_m \), we have \(-\Delta (\tilde{u}_m - u_m) + \nabla p = 0 \) in \( \mathbb{R}^3 \setminus \bigcup_i B_i \). By classical arguments which we include for convenience, this implies

\[
\|\tilde{u}_m - u_m\|_{H^1(\mathbb{R}^3)}^2 \leq C \left( \|\nabla \tilde{u}_m\|_{L^2(\bigcup_i B_i)}^2 + \frac{1}{m} \sum_i (\tilde{u}_m)_i^2 \right). \tag{4.2}
\]

Indeed, \( \tilde{u}_m - u_m \) minimizes the \( \dot{H}^1(\mathbb{R}^3) \)-norm among all divergence free functions \( v \) with \( v = \tilde{u}_m - u_m = \tilde{u}_m \) in \( \bigcup_i B_i \). Thus, to show (4.2), it suffices to construct a divergence free function \( v \) with \( v = \tilde{u}_m - u_m = \tilde{u}_m \) in \( \bigcup_i B_i \) such that \( \|v\|_{\dot{H}^1(\mathbb{R}^3)} \) is bounded by the right-hand side of (4.2). Since the balls \( B_{2R_m}(w_i) \) are disjoint as \( (w_1, \ldots, w_m) \in W_m \), we only need to construct functions \( v_i \) such that \( v_i \in H^1_0(B_{2R_m}(w_i)) \), \( v_i = \tilde{u}_m \) in \( B_i \) and

\[
\|v_i\|_{H^1(\mathbb{R}^3)}^2 \leq C \left( \|\nabla \tilde{u}_m\|_{L^2(B_i)}^2 + \frac{1}{m} (\tilde{u}_m)_i^2 \right).
\]

It is not difficult to see that such functions \( v_i \) exist. For the convenience of the reader, we state this result in Lemma 4.2 below. Thus, the estimate (4.2) holds.

It remains to prove that the right-hand side of (4.2) is bounded by the right-hand side of (4.1). To this end, let \( x \in B_i \) for some \( 1 \leq i \leq m \). We resort to the definition of \( \tilde{u}_m \) in (2.12) to deduce, analogously as in (2.16), that

\[
\tilde{u}_m(x) = u(x) - (u)_i - m^{-\frac{1}{2}}(\xi_m)_i + G(Vu)(x) - G \left[ \sum_{j \neq i} \frac{(u)_j}{m} \delta_j^m \right] (x) \\
- G \left[ \sum_{j \neq i} m^{-\frac{1}{2}} \frac{(\xi_m)_j}{m} \delta_j^m \right] (x).
\]

The definitions of \( \xi_m \) and \( \Theta_m \) from (2.21) and (2.19) the identity \( \xi_m = \Theta_m - GV\xi_m \) imply that for all \( y \in B_i \)

\[
m^{-\frac{1}{2}} \xi_m(y) = G(Vu)(y) - G \left[ \sum_{j \neq i} \frac{u(w_j)_m}{m} \delta_j^m \right] (y) - G(Vm^{-\frac{1}{2}}\xi_m)(y),
\]

where we used that \( (w_1, \ldots, w_m) \in W_m \) to replace \( C^m \) by \( G \). Thus,

\[
\tilde{u}_m(x) = u(x) - (u)_i + G(Vu)(x) - (G(Vu))_i + G \left[ \sum_{j \neq i} \frac{(u)_j}{m} \delta_j^m \right] _i - G \left[ \sum_{j \neq i} \frac{(u)_j}{m} \delta_j^m \right] (x) \\
+ (G(Vm^{-\frac{1}{2}}\xi_m))_i - G \left[ \sum_{j \neq i} m^{-\frac{1}{2}} \frac{(\xi_m)_j}{m} \delta_j^m \right] _i (x) \\
= (u + G(Vu))(x) - (u - G(Vu))_i + \Gamma_m(x) - (\Gamma_m)_i + \Xi_m(x).
\]

To conclude the proof, we again use \( (w_1, \ldots, w_m) \in W_m \) to replace \( G \) by \( G^m \) appropriately, combine this estimate with (4.2) and the estimate \( (\Xi_m)_i^2 \leq Cm^3 \||\Xi_m\||_{L^2(B_i)}^2 \). \( \square \)

**Lemma 4.2.** Let \( x \in \mathbb{R}^3 \), \( R > 0 \) and \( v \in H^1(B_R(x)) \) be divergence free. Then, there exists a divergence free function \( \varphi \in H^1_0(B_{2R}(x)) \) with \( \varphi = v \) in \( B_R(x) \) and

\[
\|\varphi\|_{H^1(\mathbb{R}^3)}^2 \leq C \left( \|\nabla v\|_{L^2(B_R(x))}^2 + R(v)_{x,R}^2 \right),
\]

where \( (v)_{x,R} = \int_{B_R(x)} v \) and \( C \) is a universal constant.
Proof. We write $v = v - (v)_{x,R} + (v)_{x,R}$. By a classical extension result for Sobolev function, there exists $\varphi_1 \in H^1_0(B_{2R}(x))$ such that $\varphi_1 = v - (v)_{x,R}$ in $B_R(x)$ and
\[ \| \nabla \varphi_1 \|_{L^2(\mathbb{R}^3)} \leq C \| \nabla v \|_{L^2(B_R(x))}. \]

By scaling, the constant $C$ does not depend on $R$.

Furthermore, we take $\varphi_2 = (v)_{x,R} \theta_R$ where $\theta_R \in C^\infty_c(B_{2R}(x))$ is a cut-off function with $\theta_R = 1$ in $B_R(x)$ and $\| \nabla \theta_R \|_\infty \leq CR^{-1}$. Then,
\[ \| \nabla \varphi_2 \|_{L^2(\mathbb{R}^3)}^2 \leq CR(v)_{x,R}^2. \]

Finally, applying a standard Bogovski operator, there exists a function $\varphi_3 \in H^1_0(B_{2r}(x) \setminus B_R(x))$ such that $\text{div} \varphi_3 = -\text{div}(\varphi_1 + \varphi_2)$ and
\[ \| \nabla \varphi_3 \|_{L^2(\mathbb{R}^3)} \leq C \| \text{div}(\varphi_1 + \varphi_2) \|_{L^2(\mathbb{R}^3)}. \]

Again, the constant $C$ is independent of $R$ by scaling considerations.

Choosing $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ finishes the proof.

5. Proof of probabilistic statements

This section contains the technical part of the proof, the probabilistic estimates stated in Proposition 3.3 and Lemma 4.1. The strategy that we will use to estimate all these terms is to expand the square of sums over the holes and then to use independence of the positions of the holes to calculate the expectations, distinguishing between terms where different holes appear and where one or more holes appear more than once. Then, it will remain to observe that combinatorially relevant terms cancel and that the remaining terms can be bounded sufficiently well, uniformly in $m$. This proof is quite lengthy. Indeed, expanding the square will lead to terms with up to 5 indices, thus giving rise to a huge number of cases that need to be distinguished.

However, there are only relatively few analytic tools that we will rely on to obtain these cancellations and estimates. These are collected in the following subsection. Their proofs are postponed to the appendix.

Some of those estimates concern expressions that will recurrently appear when we take expectations. Indeed, since many of the terms in Lemma 4.1 contain $L^2$-norms in the holes $B_i$, we will often deal with terms of the form
\[ \mathbb{E}_m \left[ \int 1_{B_i^m}(x) \right] = \int_{\mathbb{R}^3} 1_{B_i^m}(y) V(y) \, dy = m^{-3} (V)_x. \]

Another term that recurrently appears due to the definitions of $\tilde{u}_m$ and $\xi_m$ is
\[ (V v)(x) := \mathbb{E}_m [(v)_1 \delta_{m}^{i_1}] (x) = \int_{\mathbb{R}^3} V(y) (v)_y \delta_{y}^{m} (x) \, dy = \int_{\partial B_{m}^y} V(y) (v)_y \, dy. \]

To justify this formal computation one tests the expression with a function $\varphi \in C^\infty_c(\mathbb{R}^3)$ and performs some changes of variables.
5.1. Some analytic estimates

In this subsection, we collect some auxiliary observations and estimates for future reference.

In the following, we denote by $K$ the bounded set defined by

$$K := \{ x \in \mathbb{R}^3 : \text{dist}(x, \text{supp } V) \leq 1 \}. \quad (5.2)$$

Note that $B_i \subseteq K$ almost surely for all $1 \leq i \leq m$ and all $m \geq 1$.

**Lemma 5.1.** For all $\alpha > 0$, all $1 \leq p \leq \infty$, and all $v \in L^p(K)$, we have

$$\| V^\alpha(v) \|_{L^p(\mathbb{R}^3)} \leq C\| v \|_{L^p(K)}, \quad (5.3)$$

where the constant $C$ depends only on $V$, $p$ and $\alpha$.

Moreover, for all $1 \leq p \leq \infty$ and all $\phi \in L^p(\mathbb{R}^3)$

$$\| (\phi) \|_{L^p(\mathbb{R}^3)} \leq \| \phi \|_{L^p(\mathbb{R}^3)}. \quad (5.4)$$

Furthermore, for all $v \in \dot{H}^1(\mathbb{R}^3)$

$$\| v - (v) \|_{L^2(\mathbb{R}^3)} \leq m^{-1}\| v \|_{\dot{H}^1(\mathbb{R}^3)}. \quad (5.5)$$

The operator $V$ defined in (5.1) is a bounded operator from $L^2(K)$ to $L^2(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$ and from $H^1(K)$ to $H^1(\mathbb{R}^3)$. Moreover, there is a constant $C$ depending only on $V$ such that

$$\| (V - V)v \|_{L^2(\mathbb{R}^3)} \leq C m^{-1}\| v \|_{\dot{H}^1(\mathbb{R}^3)}, \quad (5.6)$$

$$\| (V - V)v \|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq C m^{-1}\| v \|_{L^2(K)}. \quad (5.7)$$

**Lemma 5.2.** There exists a constant $C$ such that for all $x, y \in \mathbb{R}^3$ and all $m \geq 1$, we have

$$|G_{y}^{m}(x)| \leq C \frac{1}{|x - y| + m^{-1}}, \quad (5.8)$$

$$|A_{y}^{m}(x)| \leq C \left(1 + \frac{1}{|x - y| + m^{-1}}\right), \quad (5.9)$$

$$|\nabla G_{y}^{m}(x)| \leq C \frac{1}{|x - y|^2 + m^{-2}}. \quad (5.10)$$

In particular, for any bounded set $K'$

$$\sup_{y \in \mathbb{R}^3} \left(\| G_{y}^{m} \|_{L^2(K')} + \| A_{y}^{m} \|_{L^2(K')} \right) \leq C(K'). \quad (5.11)$$

Moreover, for all $m \geq 1$ and $y \in \mathbb{R}^3$, it holds

$$\| \delta_{y}^{m} \|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq C m^{1/2}, \quad (5.12)$$

with a constant independent of $y$ and $m$.

**Lemma 5.3.** For any $k \in \mathbb{N}$, $G^{m}$ is a bounded operator from $\dot{H}^{k}(\mathbb{R}^3)$ to $\dot{H}^{k+2}(\mathbb{R}^3)$. Moreover, there is a constant $C$ that depends only on $k$ such that

$$\| G - G^{m} \|_{\dot{H}^{k}(\mathbb{R}^3) \to \dot{H}^{k}(\mathbb{R}^3)} \leq C m^{-2}, \quad (5.13)$$

$$\| G - G^{m} \|_{\dot{H}^{k}(\mathbb{R}^3) \to \dot{H}^{k+1}(\mathbb{R}^3)} \leq C m^{-1}. \quad (5.14)$$
5.2. Proof of Proposition 3.3

For the proof of Lemma 3.3, we first introduce another function, $\rho_m$, intermediate between $\tau_m$ and $\xi_m$. We first show that $\xi_m$ is close to $\rho_m$ in the following lemma, which we will also use in the proof of Lemma 4.1. In the following, we will use the notation $A \lesssim B$ for scalar quantities $A$ and $B$ whenever there is a constant $C > 0$ such that $A \leq CB$ and where $C$ depends neither directly nor indirectly on $m$.

**Lemma 5.4.** Let $\rho_m$ be defined by

$$
\rho_m := AG^{-1}\hat{\Theta}_m, \\
m^{-1/2}\hat{\Theta}_m := GVu - \frac{1}{m} \sum_{i=1}^{m} G((u)_i\delta^m_i).
$$

Then, for every bounded $K' \subseteq \mathbb{R}^3$

$$
\mathbb{E}_m \left[ \| \xi_m - \rho_m \|_{L^2(K')}^2 \right] \leq Cm^{-1}
$$

and

$$
\mathbb{E}_m \left[ \| \nabla \xi_m - \nabla \rho_m \|_{L^2(\mathbb{R}^3)}^2 \right] \leq Cm.
$$

**Proof.** Let $K$ be the set defined in (5.2).

We argue that $AG^{-1}$ satisfies

$$
\|AG^{-1}v\|_{L^2(K')} \lesssim \|v\|_{L^2(K')}
$$

for any $K' \supset K$ and any (divergence free) $v \in L^2(K')$. Indeed, by (2.2), we observe that

$$
AG^{-1} = (1 - AV)P_{\sigma},
$$

where the projection $P_{\sigma}$ to the divergence free functions is only present for the Stokes equations.

We observe that both $\Theta$ and $\hat{\Theta}$ are divergence free in the case of the Stokes equations. Thus, by (5.13), we have for any bounded set $K' \supset K$

$$
\mathbb{E}_m \left[ \| \xi_m - \rho_m \|_{L^2(K')}^2 \right] = \frac{1}{m} \mathbb{E}_m \left[ \left\| \sum_i AG^{-1}(G - G^m)((u)_i\delta^m_i) \right\|_{L^2(K')}^2 \right] \lesssim \frac{1}{m} \mathbb{E}_m \left[ \sum_i \sum_{j \neq i} \int_{K'} ((G - G^m)((u)_i\delta^m_i))((G - G^m)((u)_j\delta^m_j)) \right] + \frac{1}{m} \mathbb{E}_m \left[ \sum_i \int_{K'} |(G - G^m)((u)_i\delta^m_i)|^2 \right] =: I_1 + I_2.
$$

Using (5.13), we deduce

$$
I_1 = (m - 1)\| (G - G^m)Vu \|_{L^2(K')}^2 \leq Cm^{-3} \| Vu \|_{L^2(\mathbb{R}^3)}^2 \leq m^{-3} \| u \|_{\dot{H}^1(\mathbb{R}^3)}^2.
$$
It remains to bound $I_2$. By combining (5.14) with (5.12), we obtain

$$
\| (G - G^m)(\delta_y^m) \|_{L^2(\mathbb{R}^3)}^2 \lesssim m^{-2} \| \delta_y^m \|_{H^{-1}(\mathbb{R}^3)}^2 \lesssim m^{-1}.
$$

Using this estimate and (5.3) yields

$$
I_2 \lesssim m^{-1} \int_{\mathbb{R}^3} V(y)(u)^2_y \, dy \lesssim m^{-1} \| u \|_{L^2(K)}^2.
$$

For the gradient estimate, we can argue similarly: Since $AG^{-1}$ is bounded from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$

$$
\mathbb{E}_m \left[ \| \nabla(\xi_m - \rho_m) \|_{L^2(\mathbb{R}^3)}^2 \right] = \frac{1}{m} \mathbb{E}_m \left[ \left\| \sum_{i=1}^m \nabla AG^{-1}(G - G^m)((u)_i\delta_i^m) \right\|_{L^2(\mathbb{R}^3)}^2 \right] \lesssim \frac{1}{m} \mathbb{E}_m \left[ \sum_{i=1}^m \sum_{j \neq i} \int_{\mathbb{R}^3} \left( \nabla(G - G^m)((u)_i\delta_i^m) \right) \left( \nabla(G - G^m)((u)_j\delta_j^m) \right) \right]
$$

$$
+ \frac{1}{m} \mathbb{E}_m \left[ \sum_{i=1}^m \int_{\mathbb{R}^3} |\nabla(G - G^m)((u)_i\delta_i^m)|^2 \right] =: I_1 + I_2.
$$

Using (5.14), we deduce

$$
I_1 = (m - 1)\| \nabla(G - G^m)V u \|_{L^2(\mathbb{R}^3)}^2 \leq Cm^{-1}\| V u \|_{L^2(\mathbb{R}^3)}^2 \leq m^{-1}\| u \|_{\dot{H}^1(\mathbb{R}^3)}^2.
$$

It remains to bound $I_2$. Using that both $G^m$ and $G$ are bounded operators from $H^{-1}$ to $\dot{H}^1$, we find with (5.12)

$$
\| \nabla(G - G^m)(\delta_y^m) \|_{L^2(\mathbb{R}^3)}^2 \lesssim \| \delta_y^m \|_{H^{-1}(\mathbb{R}^3)}^2 \lesssim m^{1}.
$$

Using this estimate and (5.3) yields

$$
I_2 \lesssim m \int_{\mathbb{R}^3} V(y)(u)^2_y \, dy \lesssim m^1 \| u \|_{L^2(K)}^2.
$$

This finishes the proof. \hfill \Box

**Proof of Proposition 3.3.** By Lemma 5.4, it remains to prove

$$
\mathbb{E}_m \left[ \| \rho_m - \tau_m \|_{L^2(K')}^2 \right] \lesssim Cm^{-1}.
$$

Following the same reasoning as in the proof of Lemma 5.4, we find

$$
\mathbb{E}_m \left[ \| \rho_m - \tau_m \|_{L^2(K')}^2 \right] \lesssim \frac{1}{m} \mathbb{E}_m \left[ \sum_{i=1}^m \sum_{j \neq i} \int_{K'} (G(u(w_i))\delta_{w_i} - (u)_i\delta_i^m) (G(u(w_j))\delta_{w_j} - (u)_j\delta_j^m) \right]
$$

$$
+ \frac{1}{m} \mathbb{E}_m \left[ \sum_{i=1}^m \int_{K'} |G(u(w_i))\delta_{w_i} - (u)_i\delta_i^m|^2 \right] =: I_1 + I_2.
$$
Using (5.7), we deduce
\[ I_1 = (m-1)\|G(V-V)u\|^2_{L^2(K')} \lesssim Cm^{-1}\|u\|^2_{L^2(K)}. \]

It remains to bound \( I_2 \)
\[ I_2 \lesssim \int_{K'} \int_{\mathbb{R}^3} V(y) \left| G \left( u(y) - (u)_y \delta_y^m \right) \right|^2 (x) \, dy \, dx \]
\[ \int_{K'} \int_{\mathbb{R}^3} V(y) \left| G \left( u(y)(\delta_y - \delta_y^m) \right) \right|^2 (x) \, dy \, dx \]
\[ =: I_{2,1} + I_{2,2}. \]

Using (5.11) and (5.5) yields
\[ I_{2,1} \lesssim C(K) \int_{\mathbb{R}^3} V(y)||(u(y) - (u)_y)||^2 \lesssim m^{-2}\|u\|^2_{H^1(\mathbb{R}^3)}. \]

Finally,
\[ I_{2,2} \lesssim m^{-1}\|u\|^2_{H^1(\mathbb{R}^3)}, \]

since
\[ \|G(\delta_y - \delta_y^m)\|^2_{L^2(\mathbb{R}^3)} \lesssim m^{-1}. \]

This estimate can either be obtained from a direct computation using the explicit expression of \( G\delta_y^m \) provided in (A.2) and (A.3), respectively. Alternatively, it is obtained from the observation that \( \|\delta_y - \delta_y^m\|_{H^{-2}(\mathbb{R}^3)} \lesssim m^{-1/2} \) due to the Sobolev embedding \([\cdot]_{C^{0,1/2}} \lesssim \|\|_{H^2(\mathbb{R}^3)}\).

Combining these estimates shows \( I_2 = O(m^{-1}) \) which proves the first claim.

For the uniform bound on \( E_m[\|\xi_m\|^2_{L^2(K')}], \) observe that by Lemma 5.4, it is enough to bound \( E_m[\|p_m\|^2_{L^2(K')}] \). By (5.15), it holds
\[ E_m[\|p_m\|^2_{L^2(K')}] = \frac{1}{m} \sum_{i=1}^m \sum_{j=1}^m (AVu - A((u)_j \delta_j^m)) \left( AVu - A((u)_j \delta_j^m) \right) \]
\[ = \frac{1}{m} \sum_{i=1}^m \left[ AVu - A((u)_i \delta_i^m) \right]^2 \]
\[ =: \tilde{I}_1 + \tilde{I}_2. \]

As before, using (5.7), we deduce
\[ \tilde{I}_1 \lesssim (m-1)\|A(V-V)u\|^2_{L^2(K')} \lesssim (m-1)\|(V-V)u\|_{H^{-1}(\mathbb{R}^3)} \lesssim \|u\|^2_{L^2(K)}. \]

For the cross terms, we do not need to use cancellations but rather estimate brutally using (5.11) and (5.3)
\[ \tilde{I}_2 \lesssim \|AVu\|^2_{L^2(K')} + \int V(y)||(u)_y||^2 \|A\delta_y^m\|^2_{L^2(K')} \, dy \lesssim \|u\|^2_{L^2(K)}. \]

This finishes the proof. \( \square \)
5.3. Proof of Lemma 4.1

We begin the proof of Lemma 4.1 by observing that we have actually already proved the required estimate for $\Lambda^m$. Indeed, $\Lambda^m = m^{-1/2}(\Theta^m - \hat{\Theta}^m)$ with $\hat{\Theta}^m$ as in Lemma 5.4. Moreover, in the proof of Lemma 5.4, we showed $\|\Theta^m - \hat{\Theta}^m\|_{L^2_{loc}(\mathbb{R}^3)}^2 \lesssim m^{-1}$.

We divide the rest of proof of Lemma 4.1 into three steps corresponding to the three terms

\[
I_1 := \mathbb{E}_m \left[ 1_{W_m} \| \nabla(u + G(Vu)) \|_{L^2(\cup_i B_i)}^2 \right], \\
I_2 := \mathbb{E}_m \left[ 1_{W_m} \| \nabla \Gamma_m \|_{L^2(\cup_i B_i)}^2 \right], \\
I_3 := m^2 \mathbb{E}_m \left[ 1_{W_m} \| \Xi_m \|_{L^2(\cup_i B_i)}^2 \right] + \mathbb{E}_m \left[ 1_{W_m} \| \hat{\Xi}_m \|_{L^2(\Gamma')}^2 \right] + \mathbb{E}_m \left[ 1_{W_m} \| \nabla \Xi_m \|_{L^2(\cup_i B_i)}^2 \right],
\]

where $\Gamma'$ is a bounded set. We need to prove $I_k \leq Cm^{-2}$ for $k = 1, 2, 3$, uniformly in $m$ with a constant depending only on $f$, $V$ and $\Gamma'$.

**Step 1: Estimate of $I_1$.**

Let $v := \nabla(u + G(Vu)) \in L^2(\mathbb{R}^3)$. Since $v$ is deterministic, and the positions of the holes $B_i$ are independent, we estimate

\[
I_1 = \mathbb{E}_m \left[ 1_{W_m} \| v \|_{L^2(\cup_i B_i)}^2 \right] \leq \mathbb{E}_m \left[ \| v \|_{L^2(\cup_i B_i)}^2 \right] = m^{-2} \int_{\mathbb{R}^3} |W^\perp v|^2 \, dx \lesssim m^{-2} \| v \|_{L^2(\mathbb{R}^3)}^2.
\]

Here we used (5.4) together with $V \in L^\infty(\mathbb{R}^3)$. To conclude, we recall that $GV$ is a bounded operator on $H^1(\mathbb{R}^3)$.

**Step 2: Estimate of $I_2$.**

Since $\Gamma_m$ depends on $m$, the computation is more involved. According to the definition of $\Gamma$, we split $I_2$ again. More precisely, it suffices to estimate

\[
I_{2,1} := \mathbb{E}_m \left[ \| \nabla G \left( \sum_{j \neq i} \frac{u_j \delta_j^m}{m} \right) \|_{L^2(\cup_i B_i)}^2 \right], \\
I_{2,2} := \mathbb{E}_m \left[ \| \nabla G(Vm^{-\frac{1}{2}} \xi_m) \|_{L^2(\cup_i B_i)}^2 \right].
\]

In the first term, we used that for $(w_1, \ldots, w_m) \in W_m$ we can replace $G^m$ by $G$ according to (2.18).

We first consider $I_{2,1}$. We expand the square to obtain

\[
I_{2,1} = \mathbb{E}_m \left[ \int_{\cup_i B_i} \left( \nabla G \left( \frac{1}{m} \sum_{j \neq i} \frac{u_j \delta_j^m}{m} \right) \right)(x) \left( \nabla G \left( \frac{1}{m} \sum_{k \neq i} \frac{u_k \delta_k^m}{m} \right) \right)(x) \right].
\]

We distinguish the cases $j \neq k$ and $j = k$ and denote the corresponding terms by $I_{2,1}^{jk}$ and $I_{2,1}^{jj}$.

In the case $j \neq k$, we apply a similar reasoning as for $I_1$: due to the independence of $w_i$, $w_j$,
where we used again (5.4). Since by Lemma 5.1 \( V \) is a bounded operator, we therefore conclude that

\[
I_{2,1}^{jk} \lesssim m^{-2} \|u\|_{L^2(K)}^2.
\]

It remains to estimate \( I_{2,1}^{ij} \). We compute

\[
I_{2,1}^{ij} = m^{-3} \int_{\mathbb{R}^3} (V) x \int_{\mathbb{R}^3} V(y) \left( \nabla G \left[ (u)_y \delta^m \right] \right)^2 \, dy
\]

\[
\lesssim m^{-3} \int_{\mathbb{R}^3} V(y)(u)_y^2 \| \nabla G \delta^m \|_{L^2(\mathbb{R}^3)}^2 \, dy.
\]

By (5.12)

\[
\| \nabla G \delta^m \|_{L^2(\mathbb{R}^3)}^2 \lesssim m.
\]

Combining this with (5.3), we conclude

\[
I_{2,1}^{ij} \lesssim m^{-2} \| V^{1/2}(u)_y \|_{L^2(\mathbb{R}^3)}^2 \lesssim m^{-2} \| u \|_{L^2(K)}^2.
\]

We now turn to \( I_{2,2} \). We estimate

\[
I_{2,2} \leq E_m \left[ \| \nabla G(Vm^{\frac{1}{2}} \rho_m) \|_{L^2(\mathbb{R}^3)}^2 \right] + E_m \left[ \| \nabla G(Vm^{\frac{1}{2}} (\xi_m - \rho_m)) \|_{L^2(\mathbb{R}^3)}^2 \right],
\]

with \( \rho_m \) from Lemma 5.4. Using this Lemma and the fact that \( GV \) is a bounded operator from \( H^1(\mathbb{R}^3) \) to \( W^{1,\infty}(\mathbb{R}^3) \), we find

\[
E_m \left[ \| \nabla G(Vm^{\frac{1}{2}} (\xi_m - \rho_m)) \|_{L^2(\mathbb{R}^3)}^2 \right] \lesssim m^{-2} \| \xi_m - \rho_m \|_{H^1(\mathbb{R}^3)}^2 \lesssim m^{-2}.
\]

Recalling the definition of \( \rho_m \) from Lemma 5.4, we have

\[
E_m \left[ \| \nabla G(Vm^{\frac{1}{2}} \rho_m) \|_{L^2(\mathbb{R}^3)}^2 \right] \leq \sum_{i=1}^m E_m \left[ \left\| \nabla G \left( VA \left[ \frac{1}{m} VU - \sum_{j \neq i} (u)_j \delta^m \right] \right) \right\|_{L^2(B_i)}^2 \right]
\]

\[
\leq \sum_{i=1}^m E_m \left[ \| \nabla G(VA(Vu)) \|_{L^2(B_i)}^2 \right]
\]

\[
+ \sum_{i=1}^m E_m \left[ \left\| \nabla G \left( VA \left[ \frac{1}{m} \sum_{j \neq i} (u)_j \delta^m \right] \right) \right\|_{L^2(B_i)}^2 \right] =: I_{2,2,1} + I_{2,2,2}.
\]
We recall from (5.17) that this is a very rough estimate, since we actually expect cancellations from the difference. Indeed, since $GVA$ is a bounded operators on $\dot{H}^1(\mathbb{R}^3)$, $I_{2,2,1}$ is controlled analogously as $I_1$. Moreover, $J$ should therefore be better by a factor $J^2$ between $J$ and $J^3$. We will focus on the proof on $J$ as this is the most difficult term. We will comment on the adjustments needed to treat $J_2$ and $J_3$ along the estimates for $J_1$. Roughly speaking, the main difference between $J_1$ and $J_2$ is that one considers $L^2(U_1 B_1)$ for $J_1$ and $L^2_{\text{loc}}(\mathbb{R}^3)$ for $J_2$. Naively, $J_1$ should therefore be better by a factor $|U_1 B_1| \sim m^{-2}$, which is exactly the estimate we obtain. Moreover, $J_3$ concerns the gradient of the terms in $J_1$. Since we may loose a factor $m^{-2}$ going from $J_1$ to $J_3$, it will not be difficult to adapt the estimates for $J_1$ to $J_3$ using the gradient estimates in Section 5.1.

We have to distinguish the cases where all $i, j, k$ are distinct and the case where $j = k$. In the first case, we can proceed analogously as for $I_{2,1}^j$. In particular, we use the definition of $V u$ to deduce

$$I_{2,2,2}^{j,k} = m^{-2}(m-1)(m-2) \int_{\mathbb{R}^3} (V)^{\frac{1}{2}} (\nabla G V A u)^2 \, dx \lesssim m^{-2} \| \nabla G V A u \|_{L^2(\mathbb{R}^3)}^2 \lesssim m^{-2} \| u \|_{H^{-1}(\mathbb{R}^3)}^2$$

since $G V A$ is also bounded from $\dot{H}^{-1}(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$.

We conclude

$$I_{2,2,2}^{j,k} \lesssim m^{-2} \| u \|_{L^2(K)}.$$

It remains to estimate $I_{2,2,2}^{j,j}$. Analogously as for $I_{2,1}^{j,j}$, we obtain

$$I_{2,2,2}^{j,j} = m^{-3} \frac{m-1}{m} \int_{\mathbb{R}^3} (V)^{\frac{1}{2}} \int_{\mathbb{R}^3} V(y) \left( \nabla G V A (u)_y \delta^m_x (x) \right)^2 \, dy \, dx \lesssim m^{-3} \int_{\mathbb{R}^3} V(y) (u)_y^2 \| \nabla G V A \delta^m_x \|_{L^2(\mathbb{R}^3)} \, dy.$$

Since $\nabla G V$ is a bounded operator from $\dot{H}^1(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$, we obtain by (5.12) combined with (5.3)

$$I_{2,2,2}^{j,j} \lesssim m^{-2} \| V^{1/2} (u) \|_{L^2(\mathbb{R}^3)}^2 \lesssim m^{-2} \| u \|_{L^2(K)}.$$

This finishes the estimate of $I_{2,2,2}$. Therefore, the estimate of $I_2$ is complete, which also finishes the estimate of $I_2$.

**Step 3: Estimate of $I_3$.**

We recall from (5.17) that $I_3$ consists of three terms, which we denote by $J_1, J_2$ and $J_3$. We will focus on the proof on $J_1$ as this is the most difficult term. We will comment on the adjustments needed to treat $J_2$ and $J_3$ along the estimates for $J_1$. Roughly speaking, the main difference between $J_1$ and $J_2$ is that one considers $L^2(U_1 B_1)$ for $J_1$ and $L^2_{\text{loc}}(\mathbb{R}^3)$ for $J_2$. Naively, $J_1$ should therefore be better by a factor $|U_1 B_1| \sim m^{-2}$, which is exactly the estimate we obtain. Moreover, $J_3$ concerns the gradient of the terms in $J_1$. Since we may loose a factor $m^{-2}$ going from $J_1$ to $J_3$, it will not be difficult to adapt the estimates for $J_1$ to $J_3$ using the gradient estimates in Section 5.1.
Step 3.1: Expansion of the terms

As is the previous step, we first want to replace all occurrences of $G^m$ by $G$. Note that $G^m$ is present both explicitly in the definition of $\Xi^m$ and also implicitly through $\xi_m$. By (2.18) and independence of the position of the holes, it holds

$$m^2 E_m \left[ 1_{W_m} \| \Xi_m \|^2_{L^2(\cup_i B_i)} \right]$$

$$\leq m^2 E_m \left[ 1_{W_m} \sum_{i=1}^m \int_{B_i} |G(V m^{-\frac{1}{2}} \xi_m) - G \left( \sum_{j \neq i} \frac{m^{-\frac{1}{2}} (\xi_m)_j \delta_j}{m} \right) |^2 \, dx \right]$$

$$= m^3 E_m \left[ 1_{W_m} \int_{B_i} |G(V m^{-\frac{1}{2}} \xi_m) - G \left( \sum_{j \neq i} \frac{m^{-\frac{1}{2}} (\xi_m)_j \delta_j}{m} \right) |^2 \, dx \right]$$

$$\lesssim m^3 \left( E_m \left[ \int_{B_i} |G(V m^{-1/2} (\xi_m - \rho_m)) |^2 \right] \right)$$

$$+ E_m \left[ 1_{W_m} \int_{B_i} |G(V m^{-1/2} \rho_m) - G \left( \sum_{j \neq i} \frac{m^{-\frac{1}{2}} (\xi_m)_j \delta_j}{m} \right) |^2 \, dx \right].$$

We use that $GV$ is a bounded operator from $L^2(K)$ to $L^\infty(B_i)$ and Lemma 5.4 to deduce

$$E_m \left[ \int_{B_i} |G(V m^{-1/2} (\xi_m - \rho_m)) |^2 \right] \lesssim m^{-3} E_m \left[ \| G(V m^{-1/2} (\xi_m - \rho_m)) \|_{L^\infty(B_i)}^2 \right]$$

$$\lesssim m^{-3} E_m \left[ \| m^{-1/2} (\xi_m - \rho_m) \|_{L^2(K)}^2 \right]$$

$$\lesssim m^{-5}.$$

This implies, that we only have to estimate the second summand which we will call $\mathfrak{J}_1$:

$$\mathfrak{J}_1 = E_m \left[ 1_{W_m} \int_{B_i} |G(V m^{-1/2} \rho_m) - G \left( \sum_{j \neq i} \frac{m^{-\frac{1}{2}} (\xi_m)_j \delta_j}{m} \right) |^2 \, dx \right].$$

Now we can insert the explicit formula for $m^{-\frac{1}{2}} \xi_m$

$$m^{-\frac{1}{2}} \xi_m = m^{-\frac{1}{2}} AG^{-1} \Theta_m = AG^{-1} \left[ \frac{1}{m} \sum_{j=1}^m \left( G(V u) - G^m[(u)_j \delta_j^m] \right) \right],$$

and $m^{-\frac{1}{2}} \rho_m$

$$m^{-\frac{1}{2}} \rho_m = \frac{1}{m} m^{-\frac{1}{2}} \sum_{j=1}^m \left( AV u - A[(u)_j \delta_j^m] \right).$$
We use again (2.18) to replace $G^m$ by $G$, and obtain

$$\mathcal{J}_1 \leq m^3 \mathbb{E}_m \left[ \int_{B_i} \left( \frac{1}{m^2} \sum_{j=1}^{m} \sum_{k=1}^{m} G \left[ V A (V u - (u)_k \delta_k^m) \right] 
\right.
- (1 - \delta_{ij}) G \left[ (A (V u - (1 - \delta_{jk})(u)_k \delta_k^m)) j \delta_j^m \right] \left( \frac{1}{m^2} \sum_{n=1}^{m} \sum_{\ell=1}^{m} G \left[ V A (V u - (u)_\ell \delta_\ell^m) \right] 
\right.
- (1 - \delta_{in}) G \left[ (A (V u - (1 - \delta_{n\ell})(u)_\ell \delta_\ell^m)) n \delta_n^m \right] \right)$$

$$= m^{-1} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{n=1}^{m} \sum_{\ell=1}^{m} I_{3j,k,n,\ell}^i.$$ 

Here $i$ is any of the identically distributed holes.

We denote

$$I_{3j,k,n,\ell}^i = \mathbb{E}_m \left[ \int_{B_i} \nabla \Psi_{j,k}(x) \Psi_{n,\ell}(x) \, dx \right]$$

with

$$\Psi_{j,k}(x) = G \left[ V A (V u - (u)_k \delta_k^m) \right] - (1 - \delta_{ij}) G \left[ (A (V u - (1 - \delta_{jk})(u)_k \delta_k^m)) j \delta_j^m \right]. \quad (5.18)$$

(Strictly speaking $\Psi_{j,k}$ depends on $i$, but we omit this dependence for the ease of notation.)

Similarly, we have the estimate

$$\mathcal{J}_3 \leq m^{-3} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{n=1}^{m} \sum_{\ell=1}^{m} \mathbb{E}_m \left[ \int_{B_i} \nabla \nabla \Psi_{j,k}(x) \Psi_{n,\ell}(x) \, dx \right],$$

where, again, we estimate

$$\mathcal{J}_3 \lesssim \mathbb{E}_m \left[ \int_{\bigcup_i B_i} \left| \nabla G(V m^{-\frac{1}{2}}(\xi - \rho_m)) \right|^2 \right] + \mathcal{J}_3 \lesssim m^{-2} + \mathcal{J}_3,$$

with the same proof as before using that $\nabla GV$ is a bounded operator from $\dot{H}^1(\mathbb{R}^3)$ to $W^{1,\infty}(\mathbb{R}^3)$ and the second part of Lemma 5.4. Observe that we can loose exactly $O(m^2)$ in the bounds for the gradient.

Furthermore,

$$\mathcal{J}_2 \lesssim \mathbb{E}_m \left[ \left\| G(V m^{-\frac{1}{2}}(\xi - \rho_m)) \right\|_{L^2(K')}^2 \right] + \mathcal{J}_2 \lesssim m^{-2} + \mathcal{J}_2.$$ 

Here,

$$\mathcal{J}_2 \leq m^{-1} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{n=1}^{m} \sum_{\ell=1}^{m} \int_{K'} \mathbb{E}_m \left[ \tilde{\Psi}_{j,k}(x) \tilde{\Psi}_{n,\ell}(x) \right] \, dx,$$

where $\tilde{\Psi}_{j,k}$ denotes the function that is obtained by omitting the factor $(1 - \delta_{ij})$ in (5.18).
Relying on this structure enables us to make more precise the argument why the estimate for $J_1$ is most difficult compared to $J_2$ and $J_3$. Indeed, for the estimate for $J_3$, one just follows the same argument as for $J_1$. The relevant estimates in Section 5.1 show that whenever $\nabla G$ instead of $G$ appears, we lose (at most) a factor $m^{-1}$. For completeness, we provide the proof of the estimates regarding $J_3$ in the appendix.

On the other hand, for $J_2$, we can use the estimates that we will prove for the terms of $J_1$ in the case when the index $i$ is different from all the other indices. Indeed, in those cases, $\Psi_{j,k} = \tilde{\Psi}_{j,k}$, and we will always estimate

$$|I_{3}^{i,j,k,n,\ell}| = \left| m^{-3} \int_{\mathbb{R}^3} (V)_x E_m [\Psi_{j,k} \Psi_{n,\ell}] \, dx \right| \lesssim m^{-3} \| E_m [\Psi_{j,k}(x) \Psi_{n,\ell}(x)] \|_{L^1_{\text{loc}}(\mathbb{R}^3)}.$$

Thus, the bound for $J_2$ is a direct consequence of the bound for $J_1$.

Recall that we need to prove $|J_1| \lesssim m^{-2}$. We will split the sum into the cases $\# \{i, j, k, n, \ell\} = \alpha$, $\alpha = 1, \ldots, 5$. Then, since $i$ is fixed, there will be $m^{\alpha-1}$ summands for the case $\# \{i, j, k, n, \ell\} = \alpha$. Thus, it is enough to show that in each of these cases

$$|I_{3}^{i,j,k,n,\ell}| \lesssim m^{-\alpha}.$$

To prove this estimate, we have to rely on cancellations between the terms that $\Psi_{j,k}$ is composed of. To this end, we denote the first part of $\Psi_{j,k}$ by

$$\Psi_{k}^{(1)} := \Psi_{k}^{(1,1)} + \Psi_{k}^{(1,2)} := G [V A V u - V A [(u)_{k} \delta_{k}^{m}]],$$

and the second part by

$$\Psi_{j,k}^{(2)} := \Psi_{j}^{(2,1)} + \Psi_{j,k}^{(2,2)} := (1 - \delta_{ij})G \left[ (A (V u - (1 - \delta_{jk})(u)_{k} \delta_{k}^{m}))_{j} \delta_{j}^{m} \right].$$

We observe that

$$E_m[\Psi_{k}^{(1,1)}] = GVAV u, \quad E_m[\Psi_{k}^{(1,2)}] = GVAV u, \quad E_m[\Psi_{j}^{(2,1)}] = (1 - \delta_{ij})GVAV u, \quad E_m[\Psi_{j,k}^{(2,2)}] = (1 - \delta_{ij})(1 - \delta_{jk})GVAV u.$$

(5.19)

Step 3.2: The cases in which at most 2 indices are equal

In many cases, we can rely on cancellations within $\Psi_{k}^{(1)}$ and $\Psi_{j,k}^{(2)}$. Indeed, we will prove the following lemma:

**Lemma 5.5.** Let $K' \subseteq \mathbb{R}^3$ be bounded. Then,

$$\left\| E_m[\Psi_{k}^{(1)}] \right\|_{L^2(K')} \lesssim m^{-1}, \quad (5.20)$$

$$\left\| E_m[\Psi_{j,k}^{(2)}] \right\|_{L^2(K')} \lesssim m^{-1} \quad \text{if} \ j \neq k. \quad (5.21)$$

There are only three cases (up to symmetry), where we have to rely on cancellations between $\Psi_{k}^{(1)}$ and $\Psi_{j,k}^{(2)}$ to estimate $I_{3}^{i,j,k,n,\ell}$. These are the cross terms, when either $j = n$, or $k = \ell$, or $j = \ell$, and all the other indices are different. In these cases, we will rely on the following lemma:
Lemma 5.5. \(K' \subseteq \mathbb{R}^3\) be bounded. Then,
\[
\|E_m[\Psi_{j,k}]\|_{L^1(K')} \lesssim m^{-2} \quad \text{if} \ \#\{i,j,k,\ell\} = 4,
\]
\[
\|E_m[\Psi_{j,k}\Psi_{n,k}]\|_{L^1(K')} \lesssim m^{-2} \quad \text{if} \ \#\{i,j,k,n\} = 4,
\]
\[
\|E_m[\Psi_{j,k}\Psi_{n,j}]\|_{L^1(K')} \lesssim m^{-2} \quad \text{if} \ \#\{i,j,k,n\} = 4.
\]

Finally, we obtain the following estimates, useful in particular for the cases in which \(i = k\).

Lemma 5.7. \(K' \subseteq \mathbb{R}^3\) be bounded. Then, for any \(i,j,k\),
\[
\|E_m[\Psi^{(1,1)}]\|_{L^2(K')} + \|E_m[\Psi^{(1,2)}]\|_{L^2(K')}
+ \|E_m[\Psi^{(2,1)}]\|_{L^2(K')} + \|E_m[\Psi^{(2,2)}]\|_{L^2(K')} \lesssim 1.
\]

Combining these lemmas allows us to estimate \(I^{i,j,k,n,\ell}_3\) in any of the cases when \(\alpha = \#\{i,j,k,n,\ell\} \geq 4\).

Corollary 5.8. The following estimates hold true where the implicit constants are independent of \(m\):

1. If \(\#\{i,j,k,n,\ell\} = 5\), it holds
\[
|I^{i,j,k,n,\ell}_3| \lesssim m^{-5}.
\]

2. Let \(\#\{i,j,k,n,\ell\} = 4\), it holds
\[
|I^{i,j,k,n,\ell}_3| \lesssim m^{-4}.
\]

Proof. If \(\#\{i,j,k,n,\ell\} = 5\), then by independence, the Hölder inequality and Lemma 5.5
\[
|I^{i,j,k,n,\ell}_3| \leq \|E_m[1_{B_m^\infty}(w_i)]\|_{L^\infty(\mathbb{R}^3)} \|E_m[\Psi_{j,k}]\|_{L^2(K)} \|E_m[\Psi_{n,\ell}]\|_{L^2(K)}
\lesssim m^{-3}m^{-1}m^{-1} = m^{-5}.
\]

If \(\#\{i,j,k,n,\ell\} = 4\), we need to distinguish all the possible combinations of two indices being equal. Depending on which indices coincide, we split the product by independence of the other indices. If \(j = n, k = \ell\) or \(j = \ell\) (or \(k = n\) which is the same), we rely on Lemma 5.6 and gain an additional factor \(m^{-3}\) from the expectation of \(1_{B_m^\infty}\).

If \(j = k\) (or \(n = \ell\)), the expectation completely factorizes into \(E_m[1_{B_m^\infty}]E_m[\Psi_{j,k}]E_m[\Psi_{n,\ell}]\) and we can apply (5.25) for the second factor and Lemma 5.5 for the third factor.

Finally, in all the other cases we can, without loss of generality, split the expectation into \(E_m[1_{B_m^\infty}\Psi_{j,k}]E_m[\Psi_{n,\ell}]\) and apply (5.26) for the first factor and Lemma 5.5 for the second factor.

We finish this step by giving the proofs of Lemmas 5.5, 5.6 and 5.7.
Proof of Lemma 5.5. By (5.19), we have
\[ \mathbb{E}_m[\Psi^{(1)}_k] = GVA (V - \mathcal{V}) u, \]
and using (5.6) yields (5.20).
Similarly, for \( j \neq k, i \neq j \),
\[ \mathbb{E}_m[\Psi^{(2)}_{j,k}] = GVA (V - \mathcal{V}) u. \]
Using again (5.6) yields (5.21). For \( i = j \), \( \Psi^{(2)}_{j,k} = 0 \) and there is nothing to prove. \( \square \)

Proof of Lemma 5.6. Regarding (5.22), we have
\[
\mathbb{E}_m [\Psi_{j,k} \Psi_{j,\ell}] = \int V(y) V(z) V(w) \left( G [VA (V u - (u)_z \delta^m_z)] - G \left( A (V u - (u)_z \delta^m_z) \right) y_g \right) dy \, dz \, dw \\
\quad \cdot \left( G [VA (V u - (u)_w \delta^m_w)] - G \left( A (V u - (u)_w \delta^m_w) \right) y_g \right) dy \, dz \, dw \\
= \int V(y) \left( GVA (V - \mathcal{V}) u - (A (V - \mathcal{V}) u)_g \delta^m_y \right)^2 dy.
\]
We obtain
\[
\| \mathbb{E}_m [\Psi_{j,k} \Psi_{j,\ell}] \|_{L^1(K')} \lesssim \| GVA (V - \mathcal{V}) u \|^2_{L^2(K')} + \int V(y) (A (V - \mathcal{V}) u)_g \| G \delta^m_y \|^2_{L^2(K')} dy \\
\lesssim m^{-2} + \| A (V - \mathcal{V}) u \|^2_{L^2(K')} \\
\lesssim m^{-2},
\]
where we used (5.11) for both terms and (5.11) and (5.3) for the second term.

Regarding (5.23), we compute
\[
\mathbb{E}_m [\Psi_{j,k} \Psi_{n,k}] = \int V(y) V(z) V(w) \left( G [VA (V u - (u)_z \delta^m_z)] - G \left( A (V u - (u)_z \delta^m_z) \right) y_g \right) dy \, dz \, dw \\
\quad \cdot \left( G [VA (V u - (u)_z \delta^m_z)] - G \left( A (V u - (u)_z \delta^m_z) \right) y_g \right) dy \, dz \, dw \\
= \int V(z) (G (V - \mathcal{V}) AV u - (u)_z G (V - \mathcal{V}) A \delta^m_z)^2 dz.
\]
Thus, we obtain
\[
\| \mathbb{E}_m [\Psi_{j,k} \Psi_{n,k}] \|_{L^1(K')} \lesssim \| G (V - \mathcal{V}) AV u \|^2_{L^2(K')} + \sup_z \| G (V - \mathcal{V}) A \delta^m_z \|^2_{L^2(K')} \int V(z) (u)_z^2 dz \\
\lesssim m^{-2},
\]
where we used (5.11) for both terms and (5.11) for the second term.
Finally, to prove (5.24), we just apply Young’s inequality to reduce to the previous two estimates. Indeed,

\[
\mathbb{E}_m [\Psi_{j,k} \Psi_{n,j}] = \left\| \iint V(y)V(z)V(w) \left( G \left[ V_A(Vu - (u)\delta^m_y) \right] - G \left[ A \left( Vu - (u)\delta^m_y \right) \right] \delta^m_y \right) \right\|_{L^2(\mathbb{R}^3)} \\
\leq \sup_{y \in \mathbb{R}^3} \left\| GVA \delta^m_y \right\|_{L^\infty(\mathbb{R}^3)} \left\| \int V(y)(u)_y \iint V(z)GVA(Vu - (u)\delta^m_y) dy \right\|_{L^2(\mathbb{R}^3)} \\
\lesssim m^{-3} \left\| (Vu)_y \right\|_{L^2(\mathbb{R}^3)} \\
\lesssim m^{-3},
\]

where we used (5.11), (5.4) and (5.3). Since for \( j \neq i \),

\[
\mathbb{E}_m [1_{B}\Psi_{j,i}^{(2,2)}] = \int V(y)1_{B}(y)GVA \left[ (u)_y \delta^m_y \right] dy,
\]

the estimate of this term is analogous.  

**Step 3.3: The cases in which the number of different indices is 3 or less.**

It remains to estimate \( |I_3^{i,j,k,n,\ell}| \), when \#\{i, j, k, n, \ell\} \leq 3. We will show that \( |I_3^{i,j,k,n,\ell}| \lesssim m^{-3} \) for \#\{i, j, k, n, \ell\} = 3, and \( |I_3^{i,j,k,n,\ell}| \lesssim m^{-2} \) for \#\{i, j, k, n, \ell\} \leq 2. Formally, a factor \( m^{-3} \) can be expected to come from the term \( 1_{B_3} \), so that cancellations are not needed for the estimates.
of those term. We will see that this strategy works for all the terms except for $I^{i,j,i,j,\ell}$ with $i, j, \ell$ mutually distinct.

Thus, in all cases except $I^{i,j,i,j,\ell}$ with $i, j, \ell$ mutually distinct, we just brutally estimate the product $\Psi_{j,k}\Psi_{n,\ell}$ via the triangle inequality

$$
\left| I^{i,j,k,n,\ell}_3 \right| \leq \sum_{\alpha,\beta,\gamma,\delta=1}^2 \left| \mathbb{E}_m \left[ 1_{B_i^m} \Psi^{(\alpha,\beta)}_{j,k} \Psi^{(\gamma,\delta)}_{n,\ell} \right] \right|
$$

with the convention that $\Psi^{(1,1)}_{j,k} = \Psi^{(1,1)}_{j,k}$, and similarly for $\Psi^{(1,2)}_{j,k}$ and $\Psi^{(2,1)}_{j,k}$.

We now consider all possible cases of $(\alpha,\beta,\gamma,\delta) \in \{1,2\}^4$ and $\#\{i,j,k,n,\ell\} \leq 3$. Since $\Psi^{(1,1)}_{j,k}$ does not depend on any index and both $\Psi^{(1,2)}_{j,k}$ and $\Psi^{(2,1)}_{j,k}$ only depend on one index (not taking into account the dependence of $i$ since $\Psi^{(2,1)}_{i,j} = 0$ anyway), the number of cases to be considered considerably reduces for these terms.

In order to exploit this in the sequel, we introduce the following slightly abusive notation. When considering the term $\mathbb{E}_m[1_{B_i^m} \Psi^{(\alpha,\beta)}_{j,k} \Psi^{(\gamma,\delta)}_{n,\ell}]$ for fixed $\alpha,\beta,\gamma,\delta$, we define the notion of relevant indices to be the subset of indices $\{i,j,k,n,\ell\}$ appearing in this product after replacing $\Psi^{(1,1)}_{j,k}$ by $\Psi^{(1,1)}_{j,k}$ and similarly for $\Psi^{(1,2)}_{j,k}$, $\Psi^{(2,1)}_{j,k}$ and for the indices $n,\ell$.

To further reduce the number of cases that we have to consider, we next argue that we do not have to consider the cases $\{j,k,n,\ell\}$ with $J \cap \{j,k\} \cap \{n,\ell\} = \emptyset$, where $J$ is the set of relevant indices. Indeed, in all these cases, the expectation factorizes, and we conclude by the bounds provided by Lemma 5.7. In particular, we do not have to consider any case where $\Psi^{(1,1)}_{j,k}$ appears.

Moreover, if $j$ is a relevant index and $i = j$, then $\Psi^{(2,2)}_{j,k} = \Psi^{(2,1)}_{j} = 0$, and therefore, there is nothing to estimate. If $j$ and $k$ are both relevant indices and $j = k$, then $\Psi^{(2,2)}_{j,k} = 0$, and therefore, there is nothing to estimate either. The same reasoning applies to the cases where $i = n$ and $n = \ell$, respectively.

We now list all the cases that are left to consider. Cases that are equivalent by symmetry we list only once. We use the convention here, that we only specify which relevant indices coincide; relevant indices which are not explicitly denoted as equal are assumed to be different. The indices which are not relevant may take any number, in particular coinciding with each other or with relevant indices.

1. $(\alpha,\beta,\gamma,\delta) = (2,2,2,2)$: Relevant indices: $\{i,j,k,n,\ell\}$. Since all the indices are relevant, we only have to consider cases where at least two pairs or three indices coincide. All the other cases are already covered when we have estimated $I^{i,j,k,n,\ell}$ with $\#\{i,j,k,n,\ell\} \geq 4$. The cases left to consider are
   a) $i = k$, $j = n$,
   b) $i = k$, $j = \ell$,
   c) $i = k$, $\ell$,
   d) $j = n$, $k = \ell$,
   e) $j = \ell$, $k = n$,
   f) $i = k = \ell$, $j = n$.

2. $(\alpha,\beta,\gamma,\delta) = (2,1,2,2)$: Relevant indices: $\{i,j,n,\ell\}$. Cases to consider:
Combining this with the pointwise estimate (5.9) yields cancellations with listed above. Hence, since (5.8) the case (1a). As mentioned at the beginning of Step 3.3, this is the case, where we rely on cancellations with \( \Psi^{(2,1)} \). We thus estimate

\[
\mathbb{E}_m \left[ 1_{B^m}(x) \Psi_j^{(2,2)}(x)(\Psi_j^{(2,1)} - \Psi_j^{(2,2)})(x) \right] = \int V(z)V(y)1_{B^m(y)}(x)G \left[ A \left[ (u)_y \delta_z^{m} \right] \right] (x) G \left[ (A(V - V)u)_z \delta_z^{m} \right] (x) \, dz \, dy
\]

\[
= \int V(z)V(y)1_{B^m(y)}(x) \left[ A \left[ (u)_y \delta_z^{m} \right] \right] (x) \left( G\delta_z^{m} \right)^2(x) (A(V - V)u)_z \, dz \, dy.
\]

Hence, since \( A \) maps \( L^2(\mathbb{R}^3) \cap H^{-1}(\mathbb{R}^3) \) to \( L^\infty(\mathbb{R}^3) \) and by (5.6) and (5.7)

\[
\int \left| \mathbb{E}_m \left[ 1_{B^m}(x) \Psi_j^{(2,2)}(x)(\Psi_j^{(2,1)} - \Psi_j^{(2,2)}) \right] \right| \, dx \lesssim m^{-1} \iint V(z)V(y)1_{B^m(y)}(x) \left| A \left[ (u)_y \delta_z^{m} \right] \right| (G\delta_z^{m})^2(x) \, dz \, dy \, dx.
\]

By (5.8)

\[
\int 1_{B^m(y)}(x)(G\delta_z^{m})^2(x) \, dx \lesssim m^{-3} \frac{1}{|z - y|^2 + m^{-2}}.
\] (5.27)

Combining this with the pointwise estimate (5.9) yields

\[
\int \left| \mathbb{E}_m \left[ 1_{B^m}(x) \Psi_j^{(2,2)}(x)(\Psi_j^{(2,1)} - \Psi_j^{(2,2)}) \right] \right| \, dx \lesssim m^{-4} \int \int V(z)V(y) |(u)_y| \frac{1}{|z - y|^2 + m^{-2}} \left( 1 + \frac{1}{|z - y| + m^{-1}} \right) \, dz \, dy
\]

\[
\lesssim m^{-4} \log m \int V(y) |(u)_y| \, dy
\]

\[
\lesssim m^{-4} \log m,
\]

In order to conclude the proof of the lemma, it now remains to give estimates for the cases listed above.

The case (1a). As mentioned at the beginning of Step 3.3, this is the case, where we rely on cancellations with \( \Psi^{(2,1)} \). We thus estimate

\[
\mathbb{E}_m \left[ 1_{B^m}(x) \Psi_j^{(2,2)}(x)(\Psi_j^{(2,1)} - \Psi_j^{(2,2)})(x) \right] = \int V(z)V(y)1_{B^m(y)}(x)G \left[ A \left[ (u)_y \delta_z^{m} \right] \right] (x) G \left[ (A(V - V)u)_z \delta_z^{m} \right] (x) \, dz \, dy
\]

\[
= \int V(z)V(y)1_{B^m(y)}(x) \left[ A \left[ (u)_y \delta_z^{m} \right] \right] (x) \left( G\delta_z^{m} \right)^2(x) (A(V - V)u)_z \, dz \, dy.
\]

Hence, since \( A \) maps \( L^2(\mathbb{R}^3) \cap H^{-1}(\mathbb{R}^3) \) to \( L^\infty(\mathbb{R}^3) \) and by (5.6) and (5.7)

\[
\int \left| \mathbb{E}_m \left[ 1_{B^m}(x) \Psi_j^{(2,2)}(x)(\Psi_j^{(2,1)} - \Psi_j^{(2,2)}) \right] \right| \, dx \lesssim m^{-1} \iint V(z)V(y)1_{B^m(y)}(x) \left| A \left[ (u)_y \delta_z^{m} \right] \right| (G\delta_z^{m})^2(x) \, dz \, dy \, dx.
\]

By (5.8)

\[
\int 1_{B^m(y)}(x)(G\delta_z^{m})^2(x) \, dx \lesssim m^{-3} \frac{1}{|z - y|^2 + m^{-2}}.
\] (5.27)

Combining this with the pointwise estimate (5.9) yields

\[
\int \left| \mathbb{E}_m \left[ 1_{B^m}(x) \Psi_j^{(2,2)}(x)(\Psi_j^{(2,1)} - \Psi_j^{(2,2)}) \right] \right| \, dx \lesssim m^{-4} \int \int V(z)V(y) |(u)_y| \frac{1}{|z - y|^2 + m^{-2}} \left( 1 + \frac{1}{|z - y| + m^{-1}} \right) \, dz \, dy
\]

\[
\lesssim m^{-4} \log m \int V(y) |(u)_y| \, dy
\]

\[
\lesssim m^{-4} \log m,
\]
where we used (5.3).

The case (1b) is similar. However, it turns out to be easier, since the singularity is subcritical, so we do not need to take into account cancellations. Indeed,

\[
\mathbb{E}_m \left[ 1_{B_i^m}(x) \Psi^{(2,2)}_{ji}(x) \Psi^{(2,2)}_{nj}(x) \right] = \int V(z) V(y) 1_{B^m(y)}(x) G \left[ (A |(u)_y \delta^m_{y}) \right] \delta^m_z (x) \int (A |(u)_z \delta^m_z) w V(w) \delta^m_w dw \right] (x) dz dy
\]

\[
= \int V(z) V(y)(u)_y(u)_z 1_{B^m(y)}(x) \left( A \delta^m_y \right)_z (G \delta^m_z)(x) (GV A \delta^m_z) (x) dz dy.
\]

Thus, since \( GV \) maps \( L^2(K) \) to \( L^\infty(\mathbb{R}^3) \) and by (5.11)

\[
\int \left| \mathbb{E}_m \left[ 1_{B_i^m} \Psi^{(2,2)}_{ji} \Psi^{(2,2)}_{nj} \right] \right| dx \lesssim \int \int V(z) V(y) |(u)_y|(u)_z 1_{B^m(y)}(x) \left( A \delta^m_y \right)_z |(G \delta^m_z)| (x).
\]

Now we proceed as in the previous case to estimate

\[
\int \left| \mathbb{E}_m \left[ 1_{B_i^m} \Psi^{(2,2)}_{ji} \Psi^{(2,2)}_{nj} \right] \right| dx \lesssim m^{-3} \int V(z) V(y) \left( (u)_z^2 + (u)_y^2 \right) \frac{1}{|z - y| + m^{-1}} \left( 1 + \frac{1}{|z - y| + m^{-1}} \right) dz dy \lesssim m^{-3}.
\]

The case (1c): We have

\[
\mathbb{E}_m \left[ 1_{B_i^m}(x) \Psi^{(2,2)}_{ji}(x) \Psi^{(2,2)}_{ni}(x) \right] = \int V(y) 1_{B^m(y)}(x) G \left[ \int V(z) \left( A |(u)_y \delta^m_{y}\right) \delta^m_z dz \right] \right]^2 dy
\]

\[
= \int V(y) (u)_y^2 1_{B^m(y)}(x) (GV \delta^m_y)^2 dy.
\]

Thus, using first that \( \|GV \delta^m_y\|_{L^\infty(\mathbb{R}^3)} \lesssim 1 \) as above, and then (5.4) together with (5.3).

\[
\int \left| \mathbb{E}_m \left[ 1_{B_i^m} \Psi^{(2,2)}_{ji} \Psi^{(2,2)}_{ni} \right] \right| dx \lesssim m^{-3} \int (V(u)_y^2) dx \lesssim m^{-3}.
\]

The case (1d): We compute

\[
\mathbb{E}_m \left[ 1_{B_i^m}(x) \Psi^{(2,2)}_{jk}(x) \Psi^{(2,2)}_{jk}(x) \right] = m^{-3} \int (V)_y V(y) V(z) \left( A \left[ (u)_y \delta^m_{y}\right] \delta^m_z \right) (x) \right]^2 dy dz
\]

\[
= m^{-3} \int (V)_y V(y) V(z)(u)_y^2 \left( A \delta^m_y \right)_z (G \delta^m_z)^2 (x) dy dz.
\]
Using (5.11) twice and (5.3), we can successively estimate the integral in $x$, $z$ and $y$ to deduce
\[
\left| \mathbb{E}_m \left[ 1_{B^m_y} \phi_{jk}^{(2,2)}(x) \phi_{jk}^{(2,2)}(x) \right] \right| \ dx \\
\lesssim m^{-3} \int V(y) V(z) \left( u_y^2 \left( A \left[ \phi_{jk}^{(2,2)}(x) \phi_{jk}^{(2,2)}(x) \right] \right) \right) \ dy \ dz \\
\lesssim m^{-3} \int V(y) \ dy \\
\lesssim m^{-3}.
\]

The case (1e): We just observe that
\[
\left| \mathbb{E}_m \left[ 1_{B^m_y} \phi_{jk}^{(2,2)}(x) \phi_{jk}^{(2,2)}(x) \right] \right| \ dx \leq \int \mathbb{E}_m \left[ 1_{B^m_y} \left( \phi_{jk}^{(2,2)}(x) \phi_{jk}^{(2,2)}(x) \right) \right] \ dx.
\]
Thus, this case is reduced to case (1d).

The case (1f). Note that $\# \{i, j, k, n, \ell \} = 2$. Hence, we only need a bound $m^{-2}$. We have
\[
\mathbb{E}_m \left[ 1_{B^m_y} \phi_{ijk}^{(2,2)}(x) \phi_{ijk}^{(2,2)}(x) \right] \\
= \int V(y) V(z) 1_{B^m_y} \left( A \left[ \phi_{ijk}^{(2,2)}(x) \phi_{ijk}^{(2,2)}(x) \right] \right) \ dy \ dz \\
= \int V(y) V(z) \left( u_y^2 1_{B^m_y} \left( A \delta_{y}^m \right) \right) \ dy \ dz.
\]
We can estimate the integral in $x$ using again (5.27)
\[
\left| \mathbb{E}_m \left[ 1_{B^m_y} \phi_{ijk}^{(2,2)}(x) \phi_{ijk}^{(2,2)}(x) \right] \right| \ dx \\
\lesssim \int V(y) V(z) \left( u_y^2 1_{B^m_y} \left( A \delta_{y}^m \right) \right) \ dy \ dz \ dx \\
\lesssim m^{-3} \int V(y) V(z) \left( u_y^2 \left( A \delta_{y}^m \right) \right) \ dy \ dz \\
\lesssim m^{-3} \int V(y) V(z) \left( u_y^2 \left( A \delta_{y}^m \right) \right) \ dy \ dz.
\]
Moreover, using (5.9), we find
\[
\left| \mathbb{E}_m \left[ 1_{B^m_y} \phi_{ijk}^{(2,2)}(x) \phi_{ijk}^{(2,2)}(x) \right] \right| \ dx \\
\lesssim m^{-3} \int V(y) V(z) \left( u_y^2 \left( A \delta_{y}^m \right) \right) \ dy \ dz \\
\lesssim m^{-2} \int V(y) \ dy \\
\lesssim m^{-2},
\]
where we used (5.3) in the last estimate.

The case (2a) is reduced to the cases (3) and (1d) by Young’s inequality, analogously as in the case (1e).

The case (2b) was estimated together with the case (1a) if $k$ is different from the other indices.

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If $k$ coincides with one of the other indices, the number of different indices is $2$ and we can reduce the case to the cases (3) and (1f) by Young’s inequality.

The case (3): In this case we get a factor $m^{-3}$ from $1_{B_i^m}$ and thus the desired estimate follows from

$$\|E_m[|\Psi_j^{(2,1)}|^2]\|_{L^1(K)} \lesssim \int V(y)(AVu_y)^2\|G\delta_y^m\|_{L^2(K)}^2 \lesssim 1,$$

where we used (5.11) and (5.3).

The case (4a) is estimated by an analogous computation as the one at the end of the proof of Lemma 5.7, relying on the fact that $\|\Psi_k^{(1,2)}\|_{L^\infty(\mathbb{R}^3)} \lesssim |(u)_k|$, (5.29) which is a direct consequence of (5.11) and the fact that $GV$ is bounded from $L^2(K)$ to $L^\infty(\mathbb{R}^3)$. Since the index $n$ is free, a similar bound can be used for $\Psi_{n,\ell}^{(2,2)}$. More precisely,

$$|E_m[1_{B_i^m}\Psi_k^{(1,2)}\Psi_{n,\ell}^{(2,2)}]| \lesssim \int V(y)1_{B_m(y)}GV A[(u)_y\delta_y^m]GV A[(u)_y\delta_y^m] dy \lesssim \int V(y)1_{B_m(y)}|(u)_y|^2 dy,$$

since $GV$ maps $L^2(K)$ to $L^\infty(\mathbb{R}^3)$ and using again (5.11). As before, integrating in $x$ yields a factor $m^{-3}$.

The case (4b): Using (5.29) yields

$$|E_m[1_{B_i^m}\Psi_k^{(1,2)}\Psi_{k,i}^{(2,2)}]| \lesssim \int V(y)1_{B_m(y)}V(z)|(u)_y||(u)_z|G[\delta_z^m](A\delta_y^m)d\gamma d\delta,$$

which is the same as (5.28) which we have already estimated.

The case (4c) is reduced to the cases (6a) and (1d) by Young’s inequality.

The case (5) is reduced to the cases (6a) and (3) by Young’s inequality.

The cases (6a) and (6b) are estimated by an analogous computation as the one at the end of the proof of Lemma 5.7, relying on (5.29) again.

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A. Appendix

A.1. Proofs of the auxiliary estimates from Section 5.1

Proof of Lemma 5.1. Define

\[ [h](x) = \int_{\partial B^m} h(y) \, d\mathcal{H}^2(y). \]

We observe that for \( h \in W^{1,p}(\mathbb{R}^3) \), \( 1 \leq p < \infty \)

\[ \| h \|_{L^p(\mathbb{R}^3)}^p = \int_{\mathbb{R}^3} \left| \int_{\partial B^m(x)} h(y) \, d\mathcal{H}^2(y) \right|^p \, dx \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} 1_{|x-y| = m^{-1}} |h(y)|^p \, d\mathcal{H}^2(y) \, dx \]

\[ = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} 1_{|y'| = m^{-1}} |h(y' + x)|^p \, d\mathcal{H}^2(y') \, dx \]

\[ = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} 1_{|y'| = m^{-1}} |h(x')|^p \, d\mathcal{H}^2(y') \, dx' \]

\[ = \| h \|_{L^p(\mathbb{R}^3)}^p. \]

By density, the operator \([\cdot]\) is defined on \( L^p(\mathbb{R}^3) \).

Using an analogous argument also for the average \((\cdot)\) over the full ball yields (5.4). Moreover, if \( v \in L^p(K) \), the fact that \( V \in L^\infty \) has compact support in \( K \) implies (5.3).

We note that \( \mathcal{V}v = [V(v)] \). Thus, \( \mathcal{V} \) is a bounded operator from \( L^2(K) \) to \( L^2(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3) \) and from \( H^1(K) \) to \( H^1(\mathbb{R}^3) \) by the previous estimates, together with the assumption that \( V \in W^{1,\infty} \) with compact support and \( L^{3/2}(\mathbb{R}^3) \subseteq \dot{H}^{-1}(\mathbb{R}^3) \).

To prove (5.5), we first establish the following inequality:

Let \( R > 0 \) and \( \varphi \in L^1(\mathbb{R}^3) \) with \( \varphi \geq 0 \), \( \text{supp} \varphi \subseteq B_R(0) \) and \( \| \varphi \|_{L^1} = 1 \). Let \( v \in \dot{H}^1(\mathbb{R}^3) \), then

\[ \| \varphi * v - v \|_{L^2(\mathbb{R}^3)} \leq R \| \nabla v \|_{L^2(\mathbb{R}^3)}. \]  

(A.1)

There are several ways to prove this. By scaling, it is enough to consider the case \( R = 1 \). We can use the Fourier transform: observe that \( \hat{\varphi} \in C^\infty(\mathbb{R}^3) \) with

\[ |\nabla \hat{\varphi}| = |\mathcal{F}(x \varphi)| \in L^\infty(\mathbb{R}^3). \]

Since \( \hat{\varphi}(0) = 1 \), this shows that there is a constant \( C > 0 \) such that \( |(1 - \hat{\varphi})(k)| \leq C|k| \). Hence, \n
\[ \| \varphi * v - v \|_{L^2(\mathbb{R}^3)}^2 = \| (1 - \hat{\varphi}) \hat{v} \|_{L^2(\mathbb{R}^3)}^2 \leq \| k \hat{v} \|_{L^2(\mathbb{R}^3)}^2 \leq C \| \nabla v \|_{L^2(\mathbb{R}^3)}^2. \]

Now, (5.5) follows by choosing \( \varphi(x) = 1_{B^m(0)}(x) \).

For the estimate (5.6), we compute

\[ \| \mathcal{V}v - Vv \|_{L^2(\mathbb{R}^3)} \]

\[ = \left\| \int_{\partial B^m(x)} V(y)(v)_y \, d\mathcal{H}^2(y) - V(x)v(x) \right\|_{L^2(\mathbb{R}^3)} \]

\[ \leq \left\| \int_{\partial B^m(x)} (V(y) - V(x))(v)_y \, d\mathcal{H}^2(y) \right\|_{L^2(\mathbb{R}^3)} + \left\| \int_{\partial B^m(x)} V(x)((v)_y - v(x)) \, d\mathcal{H}^2(y) \right\|_{L^2(\mathbb{R}^3)} \]

\[ = J_1 + J_2. \]
Further, it is by Jensen’s inequality
\[
J_1^2 = \int_{\mathbb{R}^3} \left( \int_{\partial B^m(x)} (V(y) - V(x)) (v)_y \, d\mathcal{H}^2(y) \right)^2 \, dx \\
\leq \int_{\mathbb{R}^3} \left( \int_{\partial B^m(x)} |V(y) - V(x)|^2 |(v)_y|^2 \, d\mathcal{H}^2(y) \right) \, dx \\
\leq m^{-2} \|\nabla V\|^2_{L^\infty(\mathbb{R}^3)} \|v\|^2_{L^2(\mathbb{R}^3)},
\]
where we used (5.4). Moreover,
\[
J_2^2 = \int_{\mathbb{R}^3} \left( \int_{\partial B^m(x)} V(x) \int_{B^m(y)} v(z) - v(x) \, dz \, dy \right)^2 \, dx \\
\leq \|V\|^2_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \left( \int_{\partial B^m(x)} \int_{B^m(y)} v(z) \, dz \, dy - v(x) \right)^2 \, dx \\
= \|V\|^2_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \int_{\partial B^m(x)} |B^m|^{-1} 1_{|y-z| \leq R_m} \, dy \right) (v(z)) \, dz - v(x)^2 \, dx \\
= \|V\|^2_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(x-z) v(z) \, dz - v(x)^2 \, dx,
\]
with the choice
\[
\varphi(x) = \int_{\partial B^m(x)} |B^m|^{-1} 1_{|y| \leq R_m} \, dy.
\]
Using Fubini, we easily see that \(\varphi\) satisfies the assumptions to apply (A.1). Hence
\[
J_2^2 \leq C m^{-2} \|V\|^2_{L^\infty(\mathbb{R}^3)} \|\nabla u\|^2_{L^2(\mathbb{R}^3)}.
\]
Finally, estimate (5.7) follows from testing with \(\psi \in H^1(\mathbb{R}^3)\)
\[
\langle Vv - \mathcal{V}v, \psi \rangle = \langle v, V\psi - \mathcal{V}\psi \rangle \leq m^{-1} \|v\|_{L^2(\mathbb{R}^3)} \|V\|_{W^{1,\infty}(\mathbb{R}^3)} \|\psi\|_{H^1(\mathbb{R}^3)}.
\]
To justify the first line, observe that
\[
\int_{\mathbb{R}^3} (Vv)(x) \psi(x) \, dx = \int_{\mathbb{R}^3} V(x)(v)_x \int_{\partial B^m(x)} \psi(y) \, d\mathcal{H}^2(y) \, dx \\
= \int_{\mathbb{R}^3} V(x) \left( \int_{\mathbb{R}^3} 1_{|x-z| \leq 1/m} v(z) \, dz \right) \int_{\partial B^m(x)} \psi(y) \, d\mathcal{H}^2(y) \, dx \\
= \int_{\mathbb{R}^3} v(z) \left( \int_{\mathbb{R}^3} 1_{|x-z| \leq 1/m} V(x) \int_{\partial B^m(x)} \psi(y) \, d\mathcal{H}^2(y) \, dx \right) \, dz \\
= \int_{\mathbb{R}^3} v(z) (\mathcal{V}\psi)(z) \, dz.
\]
This finishes the proof.

\textbf{Proof of Lemma 5.2.} We first give the proof in the case of the Poisson equation. Recalling the definition of \(B^m(y)\) from (1.3) and (1.4), we have
\[
G_{\delta^m_y}(x) = \begin{cases} 
  m & x \in B^m(y) \\
  \frac{1}{4\pi |x-y|} & x \in \mathbb{R}^3 \setminus B^m(y), \quad (A.2)
\end{cases}
\]

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so that (5.8), (5.10) and (5.11) follow immediately. (5.10) implies that \( \| G \delta \|_{\dot{H}^1(\mathbb{R}^3)} \lesssim m^{1/2} \) and, since \( G \) is an isometry from \( \dot{H}^{-1}(\mathbb{R}^3) \) to \( \dot{H}^1(\mathbb{R}^3) \), this proves (5.12).

For the Stokes equations (recall from (1.4) and (1.5) the different definition of the radius \( R_m \) of the ball \( B_m \) for the Stokes equations), it is well-known that

\[
G_\delta^m(x) = \begin{cases} 
  \text{Id} & x \in B_m(y) \\
  g(x - y) - \frac{R_m^2}{6} \Delta g(x - y) & x \in \mathbb{R}^3 \setminus B_m(y),
\end{cases}
\]

with \( g \) as in (2.1). The desired properties for \( G_\delta^m(x) \) thus follow similarly as for the Poisson equation. The bounds for \( A \) follow by using the identity \( A = G - AVG \) and that \( AV \) maps \( L^2_{\text{loc}}(\mathbb{R}^3) \) to \( L^\infty(\mathbb{R}^3) \).

**Proof of Lemma 5.3.** We only give the proof in the case of the Stokes equations. The Poisson equation is easier. By (2.4), \( G - G^m \) is a convolution operator with convolution kernel

\[
\tilde{g}_m := \eta_m g - \psi_m.
\]

Thus, to prove (5.13) and (5.14) it suffices to show

\[
\| \nabla^l \tilde{g}_m \|_{L^1(\mathbb{R}^3)} \lesssim m^{-2+l}
\]

for \( l = 0, 1 \). Moreover, (A.4) for \( l = 2 \) implies that \( G^m \) is a bounded operator from \( \dot{H}^l(\mathbb{R}^3) \) to \( \dot{H}^{l+2}(\mathbb{R}^3) \) since we know that \( G \) is a bounded operator from \( \dot{H}^l(\mathbb{R}^3) \) to \( \dot{H}^{l+2}(\mathbb{R}^3) \).

By definition of \( \eta_m \), we have for all \( l \in \mathbb{N} \)

\[
| \nabla^l (\eta_m g) | \lesssim m^{1+l} \mathbf{1}_{B_{3R_m}(0) \setminus B_{2R_m}(0)}.
\]

In particular, for all \( 1 \leq p \leq \infty \) and all \( l \in \mathbb{N} \)

\[
\| \nabla^l (\eta_m g) \|_{L^p(\mathbb{R}^3)} \lesssim m^{1+l-3/p}.
\]

In view of (2.3), this implies

\[
\| \nabla^l (\eta_m g) \|_{L^p(\mathbb{R}^3)} \lesssim m^{1+l-3/p},
\]

for all \( l \geq 1 \) and all \( 1 < p < \infty \). By the Hölder inequality, this bound also holds for \( p = 1 \) and by the Poicaré inequality also for \( l = 0 \). Combining (A.5) and (A.6) yields (A.4).

**A.2. Estimates for \( J_3 \)**

We follow the same strategy as for \( J_1 \) described in Steps 3.2 and 3.3 of the proof of Lemma 4.1. Therefore, we just name and prove the relevant lemmas. Observe that we need weaker bounds. If we want to show \( |J_3| \lesssim m^{-2} \), this requires

\[
I_{3,\n}^{l,j,k,l} = E_m \left[ \int_{B_i} \nabla \Psi_{j,k}(x) \nabla \Psi_{n,\ell}(x) \, dx \right] \lesssim m^{-\alpha+2}, \quad \alpha = \# \{ i, j, k, n, \ell \}.
\]

As before, we write \( \nabla \Psi_{j,l} = \nabla \Psi_k^{(1)} + \nabla \Psi_k^{(2)} \), where

\[
\nabla \Psi_k^{(1)} := \nabla \Psi_k^{(1,1)} + \nabla \Psi_k^{(1,2)} := \nabla G[V A (V u - (u)_k \delta_k^m)],
\]

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Lemma A.1. \[
\n\nabla \Psi_{j,k}^{(2)} := \nabla \Psi_j^{(2,1)} + \nabla \Psi_{j,k}^{(2,2)} := (1 - \delta_{ij}) \nabla G \left[ A (V u - (u_k \delta_j^m) \right] \delta_j^m \right] .
\]

We observe that
\[
\begin{align*}
\mathbb{E}_m[\nabla \Psi_{1,1}^{(1,1)}] &= \nabla G V A V u, \\
\mathbb{E}_m[\nabla \Psi_{1,2}^{(1,2)}] &= \nabla G V A V u, \\
\mathbb{E}_m[\nabla \Psi_{1,1}^{(2,1)}] &= (1 - \delta_{ij}) \nabla G V A V u, \\
\mathbb{E}_m[\nabla \Psi_{1,2}^{(2,2)}] &= (1 - \delta_{ij})(1 - \delta_{jk}) \nabla G V A V u.
\end{align*}
\]

Furthermore, we observe that the only difference is that the outmost $G$ is replaced by $\nabla G$. Hence, we will apply the same strategy as before using the analogous auxiliary estimates for the gradient.

We start by giving the corresponding lemmas in the case $\#\{i, j, k, n, \ell\} \geq 4$.

Lemma A.1. \[
\begin{align*}
\|\mathbb{E}_m[\nabla \Psi_k^{(1)}]\|_{L^2(\mathbb{R}^3)} &\lesssim m^{-1}, \\
\|\mathbb{E}_m[\nabla \Psi_j^{(2)}]\|_{L^2(\mathbb{R}^3)} &\lesssim m^{-1} \text{ if } j \neq k.
\end{align*}
\]

Lemma A.2. \[
\begin{align*}
\|\mathbb{E}_m[\nabla \Psi_{j,k} \nabla \Psi_{j,\ell}]\|_{L^1(\mathbb{R}^3)} &\lesssim m^{-1} \text{ if } \#\{i, j, k, \ell\} = 4, \\
\|\mathbb{E}_m[\nabla \Psi_{j,k} \nabla \Psi_{n,k}]\|_{L^1(\mathbb{R}^3)} &\lesssim m^{-1} \text{ if } \#\{i, j, k, n\} = 4, \\
\|\mathbb{E}_m[\nabla \Psi_{j,k} \nabla \Psi_{n,j}]\|_{L^1(\mathbb{R}^3)} &\lesssim m^{-1} \text{ if } \#\{i, j, k, n\} = 4.
\end{align*}
\]

Lemma A.3. We have for any $i, j, k$
\[
\begin{align*}
\|\mathbb{E}_m[\nabla \Psi_{1,1}^{(1,1)}]\|_{L^2(\mathbb{R}^3)} + \|\mathbb{E}_m[\nabla \Psi_{1,2}^{(1,2)}]\|_{L^2(\mathbb{R}^3)} \\
+ \|\mathbb{E}_m[\nabla \Psi_{1,1}^{(2,1)}]\|_{L^2(\mathbb{R}^3)} + \|\mathbb{E}_m[\nabla \Psi_{1,2}^{(2,2)}]\|_{L^2(\mathbb{R}^3)} &\lesssim m.
\end{align*}
\]

Proof of Lemma A.1. By (A.7), we have
\[
\mathbb{E}_m[\Psi_k^{(1)}] = \nabla G V A (V - V) u.
\]

Using (5.7) yields (A.8).

Similarly, for $j \neq k, i \neq j$,
\[
\mathbb{E}_m[\Psi_{j,k}^{(2)}] = \nabla G V A (V - V) u.
\]

Using again (5.7) yields (A.9).
Proof of Lemma A.2. Regarding (A.10), we have
\[ E_m [\nabla \Psi_{j,k} \nabla \Psi_{j,l}] = \int V(y) \left( \nabla GVA(V - \mathcal{V})u - (A(V - \mathcal{V})u)_y \nabla G\delta^m_y \right)^2 dy, \]
and hence
\[ \|E_m [\nabla \Psi_{j,k} \nabla \Psi_{j,l}]\|_{L^1(\mathbb{R}^3)} \lesssim \|\nabla GVA(V - \mathcal{V})u\|_{L^2(\mathbb{R}^3)}^2 + \int V(y) (A(V - \mathcal{V})u)_y \|\nabla G\delta^m_y\|_{L^2(K)}^2 dy \lesssim m^{-2} + m^{-1} \lesssim m^{-1}, \]
where we used (5.7) for both terms and (5.12) for the second term.

Regarding (A.11), we compute
\[ E_m [\nabla \Psi_{j,k} \nabla \Psi_{n,k}] = \int \hat{V}(y) (\nabla G(V - \mathcal{V})AVu - (A(V - \mathcal{V})u)_y \nabla G\delta^m_y)_2 dy. \]
Hence, we obtain
\[ \|E_m [\nabla \Psi_{j,k} \nabla \Psi_{n,k}]\|_{L^1(\mathbb{R}^3)} \lesssim \|\nabla G(V - \mathcal{V})AVu\|_{L^2(\mathbb{R}^3)}^2 + \sup_y \|\nabla G(V - \mathcal{V})A\delta^m_y\|_{L^2(\mathbb{R}^3)}^2 \int V(z)(u)_z^2 dz \lesssim m^{-1}, \]
where we used (5.7) for both terms and (5.11) for the second term. Finally, (A.12) follows from (A.10) and (A.11) via Young’s inequality. \( \square \)

Proof of Lemma A.3. The first estimate, (A.13), follows directly from (A.7) together with the fact that the operators \( \nabla GVA, \nabla GVA^2, \nabla G^2AV, \nabla GVA^2 \) are all bounded operators from \( \dot{H}^1(\mathbb{R}^3) \) to \( \dot{H}^1(\mathbb{R}^3) \).

Regarding (A.14), these estimates follow from (A.13) if \( i \neq k \). If \( i = k \), we only need to consider those terms, in which \( k \) appears, i.e. \( \nabla \Psi_{k}^{(1,2)} \) and \( \nabla \Psi_{j,k}^{(2,2)} \). Again, we only need to consider the case \( j \neq k = i \).

Then
\[ \|E_m \left[ 1_{B^m_i} \nabla \Psi_{j,i}^{(1,2)} \right] \| = \left\| \int V(y) 1_{B^m(y)} \nabla GVA[(u)_y \delta^m_y] dy \right\| \leq \sup_{y \in \mathbb{R}^3} \|\nabla GVA\delta^m_y\|_{L^\infty(\mathbb{R}^3)} \left\| \int V(y)(u)_y 1_{B^m_y} dy \right\|_{L^2(\mathbb{R}^3)} \lesssim m^{-5/2}. \]
Here, we used (5.12) and that \( GV \) maps \( \dot{H}^1(\mathbb{R}^3) \) to \( W^{1,\infty}(\mathbb{R}^3) \) for the first term, and (5.4) followed by (5.3) for the second. Since for \( j \neq i \),
\[ E_m[1_{B^m_j} \nabla \Psi_{j,i}^{(2,2)}] = \int V(y) 1_{B^m(y)} \nabla GVA \left[ (u)_y \delta^m_y \right] dy, \]
the estimate of this term is analogous. \( \square \)
This finishes the cases in which at most 2 indices are equal. For the remaining cases, we can again follow the same strategy as for $J_1$. We provide here only the necessary estimates. All the other estimates follow by applying Young’s inequality and reducing the proofs to the estimates given here, just as in the proof for $J_1$.

**Lemma A.4.** The corresponding estimates in the case $(\alpha, \beta, \gamma, \delta) = (2, 2, 2, 2)$ are:

\[
i = k, j = n: \quad \int |E_m \left[ B^n_i \nabla \Psi_{j,i}^{(2,2)}(\nabla \Psi_j^{(2,1)} - \nabla \Psi_j^{(2,2)}) \right] | \, dx \lesssim m^{-2}. \tag{A.15}
\]

\[
i = k, j = \ell: \quad \int |E_m \left[ B^n_i \nabla \Psi_{j,i}^{(2,2)} \nabla \Psi_{n,j}^{(2,2)} \right] | \, dx \lesssim m^{-2}. \tag{A.16}
\]

\[
i = k = \ell: \quad \int |E_m \left[ B^n_i \nabla \Psi_{j,i}^{(2,2)} \nabla \Psi_{n,i}^{(2,2)} \right] | \, dx \lesssim m^{-2}. \tag{A.17}
\]

\[
j = n, k = \ell: \quad \int |E_m \left[ B^n_i \nabla \Psi_{j,k}^{(2,2)} \right] | \, dx \lesssim m^{-2}. \tag{A.18}
\]

\[
i = k = \ell, j = n: \quad \int |E_m \left[ B^n_i \nabla \Psi_{j,i}^{(2,2)} \nabla \Psi_{n,i}^{(2,2)} \right] | \, dx \lesssim 1. \tag{A.19}
\]

The corresponding estimate in the case $(\alpha, \beta, \gamma, \delta) = (2, 1, 2, 1)$ is:

\[
j = n: \quad \int |E_m \left[ B^n_i \nabla \Psi_j^{(2,1)} \nabla \Psi_j^{(2,1)} \right] | \, dx \lesssim m^{-2}. \tag{A.20}
\]

The corresponding estimates in the case $(\alpha, \beta, \gamma, \delta) = (1, 2, 2, 2)$ are:

\[
i = k = \ell: \quad \int |E_m \left[ B^n_i \nabla \Psi_i^{(1,2)} \nabla \Psi_{n,i}^{(2,2)} \right] | \, dx \lesssim m^{-2}. \tag{A.21}
\]

\[
i = \ell, k = n: \quad \int |E_m \left[ B^n_i \nabla \Psi_k^{(1,2)} \nabla \Psi_{k,i}^{(2,2)} \right] | \, dx \lesssim m^{-1}. \tag{A.22}
\]

**Proof of Lemma A.4.** For (A.15), it is

\[
E_m \left[ B^n_i \nabla \Psi_{j,i}^{(2,2)}(\nabla \Psi_j^{(2,1)} - \nabla \Psi_j^{(2,2)}) \right] = \iint V(z)V(y)B^n_y(x) \left( A \left[ (u) \delta^m_y \right] \right)_z (\nabla G \delta^m_z)^2(x) (A(V - V)u)_z \, dz \, dy.
\]

By (5.10), it holds

\[
\int B^n_y(x)(\nabla G \delta^m_z)^2(x) \, dx \lesssim m^{-3} \frac{1}{|z - y|^4 + m^{-4}}, \tag{A.23}
\]

and thus analogously as in the corresponding term for $J_1$

\[
\int E_m \left[ B^n_i \nabla \Psi_{j,i}^{(2,2)}(\nabla \Psi_j^{(2,1)} - \nabla \Psi_j^{(2,2)}) \right] \, dx \\
\lesssim m^{-4} \int V(z)V(y)(u)_y \frac{1}{|z - y|^4 + m^{-4}} \left( 1 + \frac{1}{|z - y| + m^{-1}} \right) \, dz \, dy \\
\lesssim m^{-2}.
\]
Regarding (A.16), we compute
\[
\mathbb{E}_m \left[ \mathbf{1}_{B^m}(x) \nabla \Psi^{(2,2)}_{j_1} (x) \nabla \Psi^{(2,2)}_{n_1} (x) \right] \\
= \int \int V(z) V(y) (u)_y (u)_y \mathbf{1}_{B^m}(y) (A \delta^m_y) (\nabla G \delta^m_z) (x) (\nabla G \mathcal{A} \delta^m_z) (x) \, dz \, dy.
\]

Now we use that \( \mathcal{G} \mathcal{V} \) maps \( \dot{H}^1(\mathbb{R}^3) \) to \( W^{1,\infty}(\mathbb{R}^3) \) to deduce as in the previous case
\[
\int \left| \mathbb{E}_m \left[ \mathbf{1}_{B^m}(x) \nabla \Psi^{(2,2)}_{j_1} (x) \nabla \Psi^{(2,2)}_{n_1} (x) \right] \right| \, dx \\
\lesssim m^{1/2} m^{-3} \int V(z) V(y) (u)_y^2 (u)_y^2 \left( 1 + \frac{1}{|z - y| + m^{-1}} \right) \, dz \, dy \\
\lesssim m^{-5/2} \log m.
\]

For (A.17), we get
\[
\mathbb{E}_m \left[ \mathbf{1}_{B^m}(x) \nabla \Psi^{(2,2)}_{j_1} (x) \nabla \Psi^{(2,2)}_{n_1} (x) \right] = \int V(y) (u)_y^2 \mathbf{1}_{B^m}(y) (\nabla G \mathcal{A} \delta^m_y) (x)^2 \, dy.
\]
Thus by (5.11) and (5.3), it is
\[
\int \left| \mathbb{E}_m \left[ \mathbf{1}_{B^m}(x) \nabla \Psi^{(2,2)}_{j_1} (x) \nabla \Psi^{(2,2)}_{n_1} (x) \right] \right| \, dx \lesssim m^{-2}.
\]

The case (A.18):
\[
\mathbb{E}_m \left[ \mathbf{1}_{B^m}(x) \nabla \Psi^{(2,2)}_{j_1} (x) \nabla \Psi^{(2,2)}_{j_2} (x) \right] \\
= m^{-3} \int \int (V)_x V(y) V(z) (u)_y^2 (A \delta^m_y)_z^2 (\nabla G \delta^m_z)^2 (x) \, dy \, dz.
\]

Using (5.12), (5.11) and (5.3), we get
\[
\int \left| \mathbb{E}_m \left[ \mathbf{1}_{B^m}(x) \nabla \Psi^{(2,2)}_{j_1} (x) \nabla \Psi^{(2,2)}_{j_2} (x) \right] \right| \, dx \lesssim m^{-2} \int V(y) V(z) (u)_y^2 (A \delta^m_y)_z^2 \, dy \, dz \lesssim m^{-2}.
\]

For the next estimate (A.19), we get
\[
\mathbb{E}_m \left[ \mathbf{1}_{B^m}(x) \nabla \Psi^{(2,2)}_{j_1} (x) \nabla \Psi^{(2,2)}_{j_1} (x) \right] \\
= \int \int V(y) V(z) (u)_y^2 \mathbf{1}_{B^m}(y) (A \delta^m_y)_z^2 (\nabla G \delta^m_z)^2 (x) \, dy \, dz.
\]

By using again (A.23) and (5.9), we get
\[
\int \left| \mathbb{E}_m \left[ \mathbf{1}_{B^m} \nabla \Psi^{(2,2)}_{j_1} \nabla \Psi^{(2,2)}_{j_1} \right] \right| \, dx \\
\lesssim m^{-3} \int V(y) V(z) (u)_y^2 \left( \frac{1}{|y - z|^4 + m^{-4}} + \frac{1}{|y - z|^6 + m^{-6}} \right) \, dz \, dy \\
\lesssim \int V(y) (u)_y^2 \, dy \\
\lesssim 1.
\]
To estimate (A.20), observe

$$
\mathbb{E}_m \left[ 1_{B^m_i} |\nabla \Psi^{(2,1)}_j|^2 \right] \lesssim m^{-3} \int (AV u)^2_y |\nabla G \delta^m_y| (x)^2 \, dy,
$$

and hence by (5.12), it holds

$$
\int \mathbb{E}_m \left[ 1_{B^m_i} |\nabla \Psi^{(2,1)}_j|^2 \right] \, dx \lesssim m^{-2} \int V(y) (AV u)^2_y \, dy \lesssim m^{-2}.
$$

For (A.21), it holds

$$
|\mathbb{E}_m \left[ 1_{B^m_i} \nabla \Psi^{(1,2)}_i \nabla \Psi^{(2,2)}_{n,i} \right] | \\
\leq \int V(y) 1_{B^m_i} |\nabla G V A \left[ (u)_y \delta^m_y \right] | |\nabla G V A \left[ (u)_y \delta^m_y \right] | \, dy \\
\lesssim m \int V(y) 1_{B^m_y} (u)_y^2 \, dy,
$$

where we used (5.12). Thus

$$
\int |\mathbb{E}_m \left[ 1_{B^m_i} \nabla \Psi^{(1,2)}_i \nabla \Psi^{(2,2)}_{n,i} \right] | \, dx \lesssim m^{-2}.
$$

Finally for (A.22), it is

$$
|\mathbb{E}_m \left[ 1_{B^m_i} \nabla \Psi^{(1,2)}_k \nabla \Psi^{(2,2)}_{k,i} \right] | \\
\leq \int 1_{B^m_y} V(y) V(z)(u)_z \left| \nabla G V A \delta^m_z \right| \left| A \delta^m_z \right| \, dy \, dz \\
\lesssim m^{1/2} \int 1_{B^m_y} V(y) V(z)(u)_z \left| \nabla G \delta^m_z \right| \, dy \, dz,
$$

where we used (5.11). This is estimated as in (A.16) to get

$$
\int |\mathbb{E}_m \left[ 1_{B^m_i} \nabla \Psi^{(1,2)}_k \nabla \Psi^{(2,2)}_{k,i} \right] | \, dx \lesssim m^{-1}.
$$

\[ \square \]

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