Tail asymptotics for the bivariate equi-skew

Variance-Gamma distribution

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Abstract

We derive the asymptotic rate of decay to zero of the tail dependence of the bivariate skew Variance Gamma (VG) distribution under the equal-skewness condition, as an explicit regularly varying function. Our development is in terms of a slightly more general bivariate skew Generalized Hyperbolic (GH) distribution. Our initial reduction of the bivariate problem to a univariate one is motivated by our earlier study of tail dependence rate for the bivariate skew normal distribution.

Keywords: Asymptotic tail dependence coefficient; bivariate variance gamma dis-

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1 Introduction

The coefficient of lower tail dependence of a random vector \( \mathbf{X} = (X_1, X_2)^\top \) with marginal inverse distribution functions \( F_1^{-1} \) and \( F_2^{-1} \) is defined as

\[
\lambda_L = \lim_{u \to 0^+} \lambda_L(u), \quad \text{where} \quad \lambda_L(u) = P(X_1 \leq F_1^{-1}(u) | X_2 \leq F_2^{-1}(u)).
\]  

(1)

\( \mathbf{X} \) is said to have asymptotic lower tail dependence if \( \lambda_L \) exists and is positive. If \( \lambda_L = 0 \), then \( \mathbf{X} \) is said to be asymptotically independent in the lower tail.

This quantity provides insight on the tendency for the distribution to generate joint extreme event since it measures the strength of dependence (or association) in the lower tail of a bivariate distribution. If the marginal distributions of these random variables are continuous, then from (1), it follows that \( \lambda_L(u) \) can be expressed in terms of the copula of \( \mathbf{X} \), \( C(u_1, u_2) \), as

\[
\lambda_L(u) = \frac{P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u))}{P(X_2 \leq F_2^{-1}(u))} = \frac{C(u, u)}{u}.
\]

If \( \lambda_L = 0 \) in (1), that is, if asymptotic lower tail independence obtains, the asymptotic rate of convergence to zero as \( u \to 0^+ \) of the copula \( C(u, u) \) is tantamount to that of \( \lambda_L(u) \) through the relation:

\[
C(u, u) = u\lambda_L(u).
\]
The central purpose of this paper is to provide an analytic result on the asymptotic tail independence for the bivariate skew Variance-Gamma (VG) model. We will consider this problem in terms of the more general skew Generalized Hyperbolic (GH) distribution.

The (standardized) bivariate skew GH distribution, \( GH(0, R, \theta, p, a, b) \) is defined by its variance-mean mixing representation as

\[
X = \theta W + \sqrt{W} Z
\]  \hspace{1cm} (2)

where \( X = (X_1, X_2)^\top, \theta^\top = (\theta_1, \theta_2) \), and \( W \sim GIG(p, a, b) \) is independently distributed of \( Z \sim N(0, R) \). Here \( R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \), with \(-1 < \rho < 1\).

Recall that a random variable \( W \) is said to have a (univariate) Generalised Inverse Gaussian (GIG) distribution, denoted by \( GIG(p, a, b) \), if it has density

\[
f_{GIG}(w) = \frac{1}{2K_p(a, b)} w^{p-1} \exp\left(-\frac{1}{2}(a^2 w^{-1} + b^2 w)\right), \quad w > 0;
\]

\[
= 0, \quad \text{otherwise};
\]

where

\[
K_p(a, b) = \begin{cases} 
\left(\frac{a}{b}\right)^p K_p(ab), & p \in \mathbb{R}, \text{ if } a, b > 0; \\
(b^{-2p} \Gamma(p) 2^{-p-1}), & p, b > 0, \text{ if } a = 0; \\
(a^{2p} \Gamma(-p) 2^{-p-1}), & a > 0 \text{ and } p < 0, \text{ if } b = 0,
\end{cases}
\]  \hspace{1cm} (3)

Here \( K_p(\omega), \omega > 0 \), is the modified Bessel function of the second kind (Erdélyi, Magnus, Oberhettinger, and Tricomi (1954)) with index \( p \in \mathbb{R} \).

In the VG special case \( a = 0, b = \sqrt{\frac{2}{\nu}}, p = \frac{1}{2} \). We proceed more generally by assuming \( b > 0 \) in this note in the \( GIG(p, a, b) \) setting.
It was shown in Fung and Seneta (2011) that when $X \sim GH(0, R, \theta, p, a, b)$ with $b > 0$, then $X$ is asymptotically independent in the lower tail; that is $\lambda_L = 0$. The proof in von Hammerstein (2016) for the VG can be adjusted to give this same conclusion.

Our specific focus in the sequel is to obtain a rate of convergence result of the form:

$$C(u, u) = u^\tau L(u)$$

(4)

where $L(u)$ is a slowly varying function (SVF) as $u \to 0^+$ and $\tau > 1$, when $\theta_1 = \theta_2, = \theta$, say, so $X_i \sim \theta W + \sqrt{W}Z_i, i = 1, 2$, where $Z_i \sim N(0, 1)$. That is, the distribution functions of the $X_i, i = 1, 2$ are the same: $F_1(u) = F_2(u) = F(u)$, say. We call this assumption in a bivariate setting “equi-skewness”.

Our study thus parallels that of Fung and Seneta (2016), who treat the bivariate skew normal distributed $X$, that is $X \sim SN_2(\alpha, R)$ where in $\alpha = (\alpha_1, \alpha_2)^\top$, it is assumed that $\alpha_1 = \alpha_2, = \alpha$ say, so equi-skewness obtains.

Both treatments depend on the same initial device: that

$$Z_{(2)} = \max(Z_1, Z_2) \sim SN_1(\alpha), \quad \text{where} \quad \alpha = \sqrt{\frac{1-\rho}{1+\rho}},$$

(5)

to reduce the bivariate problem to a univariate one. Our subsequent treatment is quite different, since the setting in Fung and Seneta (2016) is just mean-mixing, and with a mixing distribution not encompassed by the GIG.

Clearly, since $\lambda_L(u)$ is a probability, the index $\tau$ in (4) must satisfy $\tau \geq 1$. We note that Ledford and Tawn (1997), Ramos and Ledford (2009), Hashorva (2010) and Hua and Joe (2011) all classified the degree of tail-dependence to the value of $\tau$ in (4). Hua and Joe (2011) define $\tau$ in (4) as the (lower) tail order of a copula. The tail order case $1 < \tau < 2$
is considered as intermediate tail dependence as it corresponds to the copula having some level of positive dependence in the tail when $\lambda_L = 0$. Thus when $\lambda_L(u) = C(u, u)/u = u^{\tau - 1} L(u), 1 < \tau < 2$, there is some measure of positive association when $\lambda_L = 0$, but the association is not as strong as when $\tau = 1$, and $\lambda_L(u) = L(u) \to \lambda_L > 0, u \to 0^+$, the case of asymptotic tail dependence. In our specific setting we shall find that $1 < \tau < \infty$.

For the case of $b = 0$ in the $GIG(p, a, b)$ setting, $X$ can be asymptotically dependent in the lower tail. The limiting and rate of convergence results for this case were discussed in Fung and Seneta (2010) and Fung and Seneta (2014) respectively.

2 The Reduction

For our equi-skew setting of (2):

$$P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u))$$

$$= P(X_1 \leq y, X_2 \leq y), \text{ where } y = F_1^{-1}(u) = F_2^{-1}(u);$$

$$= E_W(P(\theta W + \sqrt{W} Z_1 \leq y, \theta W + \sqrt{W} Z_2 \leq y))$$

$$= E_W \left[ P \left( Z_1 \leq \frac{y - \theta W}{\sqrt{W}}, Z_2 \leq \frac{y - \theta W}{\sqrt{W}} \right) \right]$$

$$= E_W \left[ P(Z_{(2)} \leq \frac{y - \theta W}{\sqrt{W}}) \right],$$

where $Z_{(2)}$ has a skew normal distribution with skew parameter $\alpha$ according to (5):

$$= \int_0^\infty P \left( Z_{(2)} \leq \frac{y - \theta w}{\sqrt{w}} \right) f_W(w) \, dw$$

$$= \int_0^\infty P(\theta w + \sqrt{w} Z_{(2)} \leq y) f_W(w) \, dw$$

$$= P(X^* \leq y),$$
where \( X^* = \theta W + \sqrt{W} Z_{(2)} \) and \( X^* \) is defined by a variance-mean mixing of a skew normal representation. This type of distribution was considered in Arslan (2014), according to whose Proposition 1 the probability density of \( X^* \) is given by:

\[
f_{X^*}(x) = \frac{2e^{\theta x}}{\sqrt{2\pi K_p(a,b)K_{p-\frac{1}{2}}}} \left( (a^2 + x^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}} \right) P(Y \leq \alpha x)
\]

where \( Y \sim \text{GH}(0, 1, \alpha \theta, p-\frac{1}{2}, (a^2+x^2)^{\frac{1}{2}}, (\theta^2+b^2)^{\frac{1}{2}}) \) i.e. \( Y \) has a univariate GH distribution.

Now, setting: \( \beta = (\alpha^2 \theta^2 + \theta^2 + b^2)^{\frac{1}{2}}, = (\theta^2(1 + \alpha^2) + b^2)^{\frac{1}{2}} \):

\[
P(Y \leq \alpha x) = \int_{-\infty}^{\alpha x} \frac{e^{\theta z}}{\sqrt{2\pi K_p^{-\frac{1}{2}}((a^2 + x^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}})}} K_{p-1} \left( (a^2 + x^2 + z^2)^{\frac{1}{2}}, \beta \right) dz,
\]

\[
= \int_{-\infty}^{\alpha x} e^{\theta z} \left( (a^2 + x^2 + z^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}} \right) (\beta) \left( (a^2 + x^2 + z^2)^{\frac{1}{2}}, \beta \right) \nu \left( \beta(a^2 + x^2 + z^2)^{\frac{1}{2}} \right) dz
\]

from \( \text{eq}. \) As \( z \leq x \leq y \), so when \( y \to -\infty \), \( (a^2 + x^2 + z^2)^{\frac{1}{2}} \to \infty \), we can use the asymptotic behaviour of the Bessel function (see Jørgensen (1982)):

\[
K_{\nu}(y) = \sqrt{\frac{\pi}{2y}} e^{-y} \left( 1 + O \left( \frac{1}{y} \right) \right), \quad \text{as } y \to \infty
\]

and \( P(Y \leq \alpha x) \) becomes

\[
= \int_{-\infty}^{\alpha x} e^{\theta z} \left( (a^2 + x^2 + z^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}} \right) (\beta) \left( (a^2 + x^2 + z^2)^{\frac{1}{2}}, \beta \right) \nu \left( \beta(a^2 + x^2 + z^2)^{\frac{1}{2}} \right) dz
\]

\[
\times \sqrt{2\pi K_p^{-\frac{1}{2}}((a^2 + x^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}})} e^{-\beta(a^2+x^2+z^2)^{\frac{1}{2}}} \left( 1 + O \left( \frac{1}{\sqrt{a^2 + x^2 + z^2}} \right) \right) \)
\]

\[
= \int_{0|x|}^{\alpha x} e^{-\alpha \theta z} \left( (a^2 + x^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}} \right) (\beta) \left( (a^2 + x^2 + z^2)^{\frac{1}{2}}, \beta \right) \nu \left( \beta(a^2 + x^2 + z^2)^{\frac{1}{2}} \right) dz
\]

\[
\times \sqrt{2\pi K_p^{-\frac{1}{2}}((a^2 + x^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}})} e^{-\beta(a^2+x^2+z^2)^{\frac{1}{2}}} \left( 1 + O \left( \frac{1}{|x|} \right) \right) \)
\[
= \int_\alpha^\infty \frac{e^{-\alpha \theta |x|s}}{\sqrt{2\pi K_p^{-\frac{1}{2}}((a^2 + x^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}})}} \left( \frac{|x|}{\beta} \left(1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}} \right)^{p-1} \\
\times \sqrt{\frac{\pi}{2\beta|x| \left(1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}}}} e^{-\beta |x| \left(1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}}} |x| \left(1 + O \left(\frac{1}{|x|}\right)\right) ds,
\]

by letting \( z = |x|s; \)

\[
\left| \frac{\int_\alpha^\infty e^{-\alpha \theta |x|s}}{\sqrt{2\pi K_p^{-\frac{1}{2}}((a^2 + x^2)^{\frac{1}{2}}, (\theta^2 + b^2)^{\frac{1}{2}})}} \left( \frac{|x|}{\beta} \left(1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}} \right)^{p-1} \\
\times \sqrt{\frac{\pi}{2\beta|x| \left(1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}}}} e^{-\beta |x| \left(1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}}} |x| \left(1 + O \left(\frac{1}{|x|}\right)\right) ds\right|.
\]

Hence, from (6) and (8), as \( x \to -\infty \):

\[
f_{X*}(x) = \frac{e^{-\theta |x|}}{\sqrt{2\pi K_p(a, b)\beta^{p-\frac{1}{2}}}} \left(1 + O \left(\frac{1}{|x|}\right)\right) \\
\times \int_\alpha^\infty \left(1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}(p-\frac{3}{2})} e^{-|x|\left[1 + \frac{a^2}{x^2} + s^2 \right]^{\frac{1}{2}} + \alpha \theta s} ds
\]

(9)

### 3 Asymptotic bivariate equi-skew form

We next need to investigate the asymptotic behaviour of the integral in (9) as \( x \to -\infty \).

To this end define

\[
\phi(s) = \beta \left(1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}} + \alpha \theta s
\]

(10)

where as before \( \beta = (\alpha^2 \theta^2 + \theta^2 + b^2)^{\frac{1}{2}}, \alpha > 0, \theta \in \mathbb{R} \).

We shall need in the sequel

\[
\phi(\alpha) + \theta > 0, \; \phi(\alpha) > 0,
\]

(11)
To see this

\[
\phi(\alpha) - |\theta| = \beta \left( 1 + \frac{\alpha^2}{x^2} + \beta^2 \right)^{\frac{1}{2}} + \alpha^2 \theta - |\theta|
\]

\[
\geq \beta (1 + \alpha^2)^{\frac{1}{2}} + \alpha^2 \theta - |\theta|
\]

\[
= (1 + \alpha^2)^{\frac{1}{2}}((\alpha^2 \theta^2 + \theta^2)^{\frac{1}{2}} + \alpha^2 \theta - |\theta|)
\]

\[
> (1 + \alpha^2)^{\frac{1}{2}}((\alpha^2 \theta^2 + \theta^2)^{\frac{1}{2}} + \alpha^2 \theta - |\theta|), \text{ since } b > 0,
\]

\[
= (1 + \alpha^2)|\theta| + \alpha^2 \theta - |\theta|
\]

\[
= \alpha^2(|\theta| + \theta) \geq 0.
\]

Hence \(\phi(\alpha) > |\theta| > 0\), and (11) follows. Next

\[
\phi'(s) = \frac{\beta s}{(1 + \frac{\alpha^2}{x^2} + s^2)^{\frac{1}{2}}} + \alpha \theta > 0
\]

for all \(s \geq \alpha\) with \(\alpha, \beta > 0\) and \(\theta \in \mathbb{R}\) providing \(|x| \geq \frac{|\theta a|}{b}\). To see this we can show, similarly to the above, that

\[
\left( \beta s - \alpha |\theta| \left( 1 + \frac{\alpha^2}{x^2} + s^2 \right)^{\frac{1}{2}} \right) > 0
\]

providing \(s^2 > (\frac{\alpha a}{b})^2\), so that, if we take \(s \geq \alpha\), providing \(|x| > \frac{|\theta a|}{b}\). Given that we shall need \(|x| \to \infty\), the inequality (12) will hold for all \(s \geq \alpha\) for any fixed \(\alpha > 0, b > 0, \theta \in \mathbb{R}\).

Thus in view of (10), (12), \(\phi(s), s \geq \alpha\), has an inverse function \(\phi^{-1}(s), s \geq \phi(\alpha)\). Next we consider, with reference to (9),

\[
|x|^{p-\frac{1}{2}} \int_{\alpha}^{\infty} \left( 1 + \frac{\alpha^2}{x^2} + s^2 \right)^{\frac{1}{2}(p-\frac{3}{2})} e^{-|x|\phi(s)} ds
\]

and change variable of integration to \(w = \phi(s) - \phi(\alpha)\), so the expression becomes:

\[
= |x|^{p-\frac{1}{2}} e^{-|x|\phi(\alpha)} \int_{w=0}^{\infty} \frac{(1 + \frac{\alpha^2}{x^2} + (\phi^{-1}(w + \phi(\alpha)))^2)^{\frac{1}{2}(p-\frac{3}{2})}}{\phi'(\phi^{-1}(w + \phi(\alpha)))} e^{-|x|w} dw
\]

8
\[ |x|^{p-\frac{3}{2}} e^{-|x|\phi(\alpha)} \{ |x| \int_{w=0}^{w=\infty} \tilde{\theta}(w) e^{-|x|w} \, dw \} \]  

(14)

where

\[ \tilde{\theta}(w) = \frac{\left(1 + \frac{\alpha^2}{x^2} + (\phi^{-1}(w + \phi(\alpha)))^2 \right)^{\frac{1}{2}(p-\frac{3}{2})}}{\phi'(\phi^{-1}(w + \phi(\alpha)))}. \]  

(15)

We now consider each of the multiplicands in (14) separately. First, using integration by parts, and putting for convenience \( v = |x| \) we have

\[
v \int_{w=0}^{w=\infty} \tilde{\theta}(w)e^{-vw} \, dw \\
= v \left[ \tilde{\theta}(w)e^{-vw} \right]_{w=0}^{w=\infty} - v \int_{w=0}^{w=\infty} e^{-vw} \tilde{\theta}'(w) \, dw \\
= \tilde{\theta}(0) + \int_{w=0}^{w=\infty} \tilde{\theta}'(w)e^{-vw} \, dw.
\]

We now assume for the moment that \( a = 0 \), so that \( \tilde{\theta}(w) \) does not involve \( v = |x| \); and note in passing that the case of the GH of special interest to us, the VG, would still be encompassed by an initial assumption that \( a = 0 \).

From the fact that \( \tilde{\theta}'(w) \) is bounded near \( w = 0^+ \) and \( \tilde{\theta}'(w) \) for large positive \( w \) is of asymptotic growth: \( \text{Const.} \times w^\kappa \), for some fixed \( \kappa \) as \( w \to \infty \), we obtain as \( v \to \infty \):

\[
\int_{w=0}^{w=\infty} \tilde{\theta}'(w)e^{-vw} \, dw = O \left( \frac{1}{v} \right)
\]

so that

\[
v \int_{w=0}^{w=\infty} \tilde{\theta}(w)e^{-vw} \, dw = \tilde{\theta}(0) \left( 1 + O \left( \frac{1}{v} \right) \right), \quad v \to \infty,
\]

where from (15) and (12) with \( a = 0 \)

\[
\tilde{\theta}(0) = \frac{(1 + \alpha^2)^{\frac{1}{2}(p-\frac{3}{2})}}{\phi'(\alpha)} = \frac{(1 + \alpha^2)^{\frac{1}{2}(p-\frac{3}{2})}}{\frac{\beta\alpha}{(1+\alpha^2)^{\frac{3}{2}}} + \alpha\beta} = \frac{(1 + \alpha^2)^{\frac{1}{2}(p-\frac{3}{2})}}{\alpha(\beta + \theta(1 + \alpha^2)^{\frac{1}{2}})}.
\]

(16)
Thus from (13)
\[ |x|^{p-\frac{3}{2}} \int_{\alpha}^{\infty} \left( 1 + \frac{a^2}{x^2} + s^2 \right)^{\frac{1}{2}(p-\frac{3}{2})} e^{-|x|\phi(s)} \, ds = \tilde{\theta}(0) |x|^{p-\frac{3}{2}} e^{-|x|\phi(\alpha)} \left( 1 + O \left( \frac{1}{|x|} \right) \right) \] (17)

when \( a = 0 \) as \( x \to -\infty \), where \( \tilde{\theta}(0) \) is given by (16), and
\[ \phi(\alpha) = \beta \left( 1 + \alpha^2 \right)^{\frac{1}{2}} + \alpha^2 \theta \] (18)

It can be shown that (17) holds with these same values of \( \tilde{\theta}(0) \) and \( \phi(\alpha) \) for general \( a \). Hence, for GH,
\[ f_{X^*}(x) = \frac{\tilde{\theta}(0) e^{-\theta|x|} |x|^{p-\frac{3}{2}} e^{-|x|\phi(\alpha)}}{\sqrt{2\pi} \, K_p(a,b) \beta^{p-\frac{1}{2}}} \left( 1 + O \left( \frac{1}{|x|} \right) \right) \]
as \( x \to -\infty \).

Thus for large negative \( y \), noting from (11) that \( \phi(\alpha) + \theta > 0 \), so the integral is well-defined:
\[ P(X^* \leq y) = \frac{\tilde{\theta}(0)}{\sqrt{2\pi} \, K_p(a,b) \beta^{p-\frac{1}{2}}} \int_{-\infty}^{y} |x|^{p-\frac{3}{2}} e^{-|x|\phi(\alpha) + \theta} \left( 1 + O \left( \frac{1}{|x|} \right) \right) \, dx = \frac{\tilde{\theta}(0)}{\sqrt{2\pi} \, K_p(a,b) \beta^{p-\frac{1}{2}}} \int_{|y|}^{\infty} v^{p-\frac{3}{2}} e^{-v(\phi(\alpha) + \theta)} \left( 1 + O \left( \frac{1}{v} \right) \right) \, dv \]
where \( v = -x = |x| \).

Now, from L’Hôpital’s rule, for \( \alpha \in \mathbb{R}, \beta > 0 \)
\[ \int_{s}^{\infty} y^{\alpha-1} e^{-\beta y} \, dy = \frac{1}{\beta} s^{\alpha-1} e^{-\beta s} \left( 1 + o(1) \right), \ s \to \infty, \]
\[ = \frac{1}{\beta} s^{\alpha-1} e^{-\beta s} + \frac{\alpha - 1}{\beta} \int_{s}^{\infty} y^{\alpha-2} e^{-\beta y} \, dy \] (19)
by integration by parts. Thus
\[ \left| \int_{s}^{\infty} y^{\alpha-1} e^{-\beta y} \, dy - \frac{1}{\beta} s^{\alpha-1} e^{-\beta s} \right| \leq \left| \frac{\alpha - 1}{\beta s} \right| \int_{s}^{\infty} y^{\alpha-2} e^{-\beta y} \, dy \]
\[ = \left| \frac{\alpha - 1}{\beta s} \right| \frac{1}{\beta} s^{\alpha-1} e^{-\beta s} \left( 1 + o(1) \right), \ \text{from (19)}, \]
\[ = \frac{1}{\beta} s^{\alpha-1} e^{-\beta s} \left( O \left( \frac{1}{s} \right) \right) \]
so that
\[ \int_{s}^{\infty} y^{\alpha-1} e^{-\beta y} dy = \frac{1}{\beta} s^{\alpha-1} e^{-\beta s} \left( 1 + O \left( \frac{1}{s} \right) \right), \quad s \to \infty, \]

whence, as \( y \to -\infty \):

\[ P(X^* \leq y) = \frac{\tilde{\theta}(0)}{\sqrt{2\pi} K_p(a, b) \beta^{p-\frac{1}{2}} (\phi(\alpha) + \theta)} |y|^{p-\frac{n}{2}} e^{-\frac{1}{2} (\phi(\alpha) + \theta) |y|} \left( 1 + O \left( \frac{1}{|y|} \right) \right) \] (20)

where \( \tilde{\theta}(0) \) is given by (16), and \( \phi(\alpha) + \theta = (1 + \alpha^2)^{\frac{1}{2}} (\beta + \theta (1 + \alpha^2)^{\frac{1}{2}}) \), and recall:

\[ P(X_1 \leq y, X_2 \leq y) = P(X^* \leq y). \] (21)

4 Quantile function and asymptotic copula

The marginal density of each of \( X_1, X_2 \) is given by

\[ f_{X_1}(x) = \frac{e^{\theta x}}{\sqrt{2\pi} K_p(a,b)} K_{p-1/2}((x^2 + a^2)^{1/2}, (\theta^2 + b^2)^{1/2}), \quad x \in \mathbb{R}, \]

as expressed in Fung and Seneta (2011), equation (20), following Blæsild (1981). Hence after some algebra using (7)

\[ F_1(x) = P(X_1 \leq x) = A |x|^{p-1} e^{-\left( (\theta^2 + b^2)^{\frac{n}{2}} + \theta \right) |x|} \left( 1 + O \left( \frac{1}{|x|} \right) \right) \] (22)

where \( A^{-1} = 2K_p(a,b) (\theta^2 + b^2)^{\frac{n}{2}} (\theta^2 + b^2)^{\frac{n}{2}} + \theta \), as \( x \to -\infty \). In the VG special case where \( a = 0, b = \sqrt{\frac{2}{\nu}}, p = 1/2, \mu = 0, \sigma^2 = 1 \), this is equation (18) of Fung and Seneta (2011).

The expression (22) itself is a special case of distribution functions with a generalized gamma-type tail considered by Fung and Seneta (2018), equation (6), where in the
notation of that paper on the left-hand side of the following: \( a = A, b = p - 1, c = (\theta^2 + b^2)^{\frac{1}{2}} + \theta, d = 1, e = 1 \). So from (9) of that paper, in our notation, we have

\[
F_1^{-1}(u) = \frac{\log u}{(\theta^2 + b^2)^{\frac{1}{2}} + \theta} - \frac{(p - 1) \log |\log u|}{(\theta^2 + b^2)^{\frac{1}{2}} + \theta} - \frac{(p - 1) \log \left( \frac{A_1}{(\theta^2 + b^2)^{\frac{1}{2}} + \theta} \right)}{(\theta^2 + b^2)^{\frac{1}{2}} + \theta} + O \left( \frac{\log |\log u|}{|\log u|} \right)
\]
as \( u \to 0^+ \).

We finally address the rate of convergence. To simplify notation put

\[
\gamma = \left( \beta + (1 + \alpha^2)^{\frac{1}{2}} \theta \right) = (\phi(\alpha) + \theta) / (1 + \alpha^2)^{\frac{1}{2}}
\]

\[
\delta = (\theta^2 + b^2)^{\frac{1}{2}} + \theta
\]

\[
\tau = \frac{(1 + \alpha^2)^{\frac{1}{2}} \gamma}{\delta}
\]

\[
C_1 = \frac{(1 + \alpha^2)^{\frac{1}{2}} (p - \frac{3}{2})}{\sqrt{2\pi K_\rho(a, b)\beta^{p - \frac{1}{2}} \alpha(1 + \alpha^2)^{\frac{1}{2}} \gamma^2 \delta^{p - \frac{1}{2}}}}
\]

\[
C_2 = A^{-\tau} \delta^{(p - 1)\tau}
\]

Then from (20), (21) we have

\[
P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u))
\]

\[
\sim C_1 \left( |\log u + O(\log |\log u|)| \right)^{p - \frac{3}{2}} e^{-\frac{\log u}{3} - \frac{(p - 1) \log |\log u|}{3} - \frac{(p - 1) \log C_2}{3} + O \left( \frac{\log |\log u|}{|\log u|} \right) \left( 1 + \alpha^2 \right)^{\frac{1}{2}} \gamma}
\]

\[
\sim C_1 \times u^\tau \times \left( |\log u| \right)^{p - \frac{3}{2} - (p - 1)\tau} \times C_2.
\]

As a result,

\[
\frac{P(X_1 \leq F_1^{-1}(u), X_2 \leq F_2^{-1}(u))}{u} = u^{\tau - 1} L(u)
\]

where

\[
L(u) \sim C_1 C_2 \left( |\log u| \right)^{(p - 1)(1 - \tau) - \frac{3}{2}}
\]

12
is a slowly varying function.

Obviously, the rate for the VG is obtained explicitly by letting $a = 0, b = \sqrt{\frac{2}{\nu}}, p = \frac{1}{2}$.

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