Fractional charge of quasi-particles on the horizon of black holes

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In the previous works, it was claimed that black holes can be considered as topological insulators. In this paper, we will show that they are actually fractional topological insulators. That is, the quasi-particles and quasi-holes can have fractional charges and statistics (spins). For BTZ black hole, the filling fraction is $v = \frac{1}{2}$. For Kerr black hole, the filling fraction is $v = \frac{S}{S}$, where $S$ is the entropy of black hole.
I. INTRODUCTION

Thanks to the works of Event Horizon Telescope Collaboration, now we have the first picture of black hole. It will accelerate the study of black hole and the gravity theory. For black hole, there are still some difficult problems, such as information loss paradox. An approach to solving this paradox is to find black hole-like objects, that is, artificial black holes in laboratory, such as acoustic black hole and so on. One can study the Hawking radiation and the correlations in those systems, and recently the thermal Hawking radiation was found in an analogue black hole.

In the previous works, the author claim that the black hole can be considered as topological insulator. Based on this claim, we study the symmetry group of both sides and find that they are closed related. We also give the microscopic states of BTZ black hole and Kerr black holes based on the boundary scalar field. Those microscopic states account for the entropy of black hole.

An important question is: are those topological insulators integral or fractional? That is, can the quasi-particles have fractional charges and statistics or not. In this paper, we will show that they are fractional topological insulators, and the quasi-particles can have fractional spin (statistics). More precisely, BTZ black hole can be considered as fractional topological insulator with filling fraction \( v = \frac{1}{2} \). For Kerr black hole, the filling fraction is \( v = \frac{S}{2\pi} \), where \( S \) is the entropy of black hole.

The paper is organized as follows. In section II, we analyse the BTZ black hole. In section III, the Kerr black hole is studied. Section IV is the conclusion.

II. BTZ BLACK HOLE AS FRACTIONAL TOPOLOGICAL INSULATOR

First let us review the Laughlin states in fractional quantum Hall states (FQHE). The low energy edge excitations can be described by compact chiral boson theory. On a circle, the chiral boson can be expanded as

\[
\phi_L(t, \sigma) = \phi_0 + p_{\phi}(t + \sigma) + i \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(t+\sigma)}. \tag{1}
\]

The Hilbert space of chiral boson can be written as

\[
\mathcal{H} = \mathcal{H}_{KM} \otimes \mathcal{H}_p, \tag{2}
\]

where \( \mathcal{H}_{KM} \) is generated by the oscillator part \( \alpha_n \), and represent the zero charge states, that is the phonon. The \( \mathcal{H}_p \) is generated by the zero mode part \( e^{i\phi_0} \), and represent the charged states, that is quasi-particles and quasi-holes.

The quasi-holes and quasi-particles have the spectrum

\[
v = \frac{1}{m}, \quad Q = \pm \frac{n}{m} e, \quad J = \frac{n^2}{2m}, \quad m, n \in \mathbb{Z}, \tag{3}
\]

where \( v \) is the filling fraction, \( Q \) the electric charge, \( e \) the unit electric charge, \( J \) the spin and \( n \) the number of the quasi-particle.

Now let us consider the BTZ black hole case. The boundary degrees of freedom can be described two chiral massless scalar fields \( \Psi_L, \Psi_R \) with opposite chirality, or equivalently a non-chiral boson field \( \phi \). Those fields have mode...
informations about the fractional charges of the quasi-particles. The zero modes contribution of Hamiltonian and angular momentum of BTZ black hole are given by
\[
\psi_L(\phi - \frac{v}{l}) = \psi_{L0} - \alpha_n\tan(\phi + \frac{v}{l}) + i \sum_{n \neq 0} \frac{\alpha_n}{n} e^{in(\phi + \frac{v}{l})}, \quad \alpha_n = \alpha_{-n},
\]
\[
\psi_R(\phi - \frac{v}{l}) = \psi_{R0} - \alpha_\phi(\varphi - \frac{v}{l}) + i \sum_{n \neq 0} \frac{\alpha_n}{n} e^{in(\varphi - \frac{v}{l})}, \quad (\alpha_n)^* = \alpha_{-n},
\]
where \(v' = \frac{r_+}{L}v, k = \frac{L}{2\alpha}, \omega_n = |\frac{n}{r_+}|, k_n = n \) and \(A = 2\pi r_+ \) is the length of the circle. The relation between those components are
\[
\phi_0 = -(\psi_{L0} + \psi_{R0}), \quad p_\phi = -\frac{\alpha_0 - \alpha_{-\phi}}{r_+}, \quad p_\phi = \alpha_0 + \alpha_{-\phi},
\]
and
\[
a_n = \begin{cases} 
-\frac{i\alpha_0^{+}\sqrt{n+2k}}{n}, & n > 0 \\
\frac{i\alpha_0^{-}\sqrt{n-2k}}{n}, & n < 0 
\end{cases}, \quad (\alpha_n)^* = \begin{cases} 
\frac{i\alpha_0^{+}\sqrt{n+2k}}{n}, & n > 0 \\
\frac{i\alpha_0^{-}\sqrt{n-2k}}{n}, & n < 0 
\end{cases}.
\]

Let us pay attention to the zero mode part \(p_\phi, p_\phi(\alpha_0^{\pm})\), since as the quantum Hall states, they will give important informations about the fractional charges of the quasi-particles. The zero modes contribution of Hamiltonian and angular momentum of BTZ black hole are given by
\[
H_0 = \pi m_0 r_+ (p_\phi^2 + p_\phi^2),
\]
\[
J_0 = 2\pi m_0 r_+ p_\phi p_\phi.
\]
With the zero mode of the Kac-Moody algebra,
\[
\alpha_0^+ = \frac{p_\phi - p_\phi r_+}{2}, \quad \alpha_0^- = \frac{p_\phi + p_\phi r_+}{2},
\]
they can be written as
\[
H_0 r_+ = ML = k((\alpha_0^+)^2 + (\alpha_0^-)^2), \quad J_0 = J = k((\alpha_0^+)^2 - (\alpha_0^-)^2).
\]
The zero modes satisfy the quantization condition
\[
\alpha_0^\pm = \pm \frac{n_\pm}{2k}.
\]
The entropy and the angular momentum satisfy
\[
S = 2\pi k(\alpha_0^+ - \alpha_0^-) = \pi(n_+ + n_-), \quad J = \frac{1}{4k}(n_+^2 - n_-^2) = J_+ - J_-.
\]
Compared with the spectrum of FQHE, we can make the following correspondence
\[
m \leftrightarrow 2k, \quad Q \leftrightarrow \alpha_0^\pm, \quad J \leftrightarrow J_\pm.
\]
We can also interpret the \(n_\pm\) as numbers of quasi-holes and quasi-particles on the right- and left-sector respectively. It is easy to show that they satisfy
\[
n_+ + n_- = \frac{S}{\pi}, \quad n_+ - n_- = \frac{S_-}{\pi},
\]
where \(S_-\) is the entropy of the inner horizon.
For Kerr black hole, we also consider the zero mode part. The massless scalar field has the following mode expansion

\[
\phi(v', \theta, \varphi) = \phi_0 + p_v v' \ln(\cot \frac{\theta}{2}) + p_\varphi \varphi + \sqrt{\frac{1}{m_0 A}} \sum_{l \neq 0} \sqrt{\frac{1}{2\omega_l}} [a_{l,m} e^{-i\omega_l v'} Y_l^m(\theta, \varphi) + a_{l,m}^+ e^{i\omega_l v'} (Y_l^m)^*(\theta, \varphi)],
\]

where \(\omega_l^2 = \frac{l(l+1)}{r^2} + \frac{a^2}{r^2}\), \(Y_l^m(\theta, \varphi)\) are spherical harmonics and \(A = 4\pi r_+^2\).

The energy and angular momentum can be written as

\[
H_0 = \frac{m_0 A}{2}(p_v^2 + |p_\varphi|^2),
\]

\[
J_0 = m_0 A p_v |p_\varphi|,
\]

where \(m_0 = \frac{M^2}{2\pi r_+}\).

Define

\[
\alpha_0^+ = \frac{|p_\varphi| - p_v r_+}{2} = \pi T_R, \quad \alpha_0^- = \frac{|p_\varphi| + p_v r_+}{2} = -\pi T_L.
\]

Assume that the zero mode part of the energy and angular momentum can be rewritten as

\[
H_0 r_+ = \frac{M r_+}{2} = k((\alpha_0^+)^2 + (\alpha_0^-)^2), \quad J_0 = \frac{J}{\gamma} = k((\alpha_0^+)^2 - (\alpha_0^-)^2), \quad \gamma \equiv \frac{r_+^2 + a^2}{r_+^2}.
\]

From the above equations, one can get

\[
k = 2M^2, \quad S = 2\pi k (\alpha_0^+ - \alpha_0^-).
\]

In the previous work [12], it is shown that due to the compact of the horizon, the entropy and the angular momentum are quantized to

\[
S = 2\pi n_1, \quad J = n_2, \quad n_1, n_2 \in \mathbb{N}.
\]

Then it is to show that the zero modes of the Kac-Moody algebra are

\[
\alpha_0^\pm = \pm \frac{n_\pm}{2k}, \quad n_\pm = n_1 \pm n_2.
\]

Then

\[
S = 2\pi k (\alpha_0^+ - \alpha_0^-) = \pi (n_+ + n_-), \quad J = \frac{\gamma}{4k} (n_+^2 - n_-^2) = J_+ - J_-.
\]

So we can make the correspondence

\[
m \leftrightarrow \frac{2k}{\gamma} = n_+ + n_- = \frac{S}{\pi}, \quad Q \leftrightarrow \gamma \alpha_0^\pm, \quad J \leftrightarrow J_\pm.
\]

We can also interpret the \(n_\pm\) as numbers of quasi-particles on the left- and right-sector respectively. It is easy to show that they satisfy

\[
n_+ + n_- = \frac{S}{\pi}, \quad n_+ - n_- = 2J.
\]
IV. CONCLUSION

In this paper, we claim that the black holes can be considered as fractional topological insulators. For BTZ black hole in three dimensional spacetime, the filling fraction is \( v = \frac{1}{2k} \). For Kerr black hole in four dimension, the filling fraction is \( v = \frac{\pi}{S} \), where \( S \) is the entropy of black hole. The quasi-particles (quasi-holes) can have fractional spin \( J = \frac{v}{2} \), thus fractional statistics. They also have fractional charge \( \alpha^+ = \pm \frac{\pi}{v} \), which can be considered as re-scale of the supertranslation \( T_\theta \), and related to the entropy of the black hole \( S \).

It is well known that the fractional charge and statistics are closely related with the topological order. Thus the black holes can also have topological order. It is interesting to investigate if they can support non-abelian anyons which can be used to built topological quantum computer.

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