Failure of Kohn’s Theorem and $f$-sum-rule in intrinsic Dirac-Weyl materials in the presence of a filled Fermi sea

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(Dated: May 11, 2018)

Kohn’s Theorem and the $f$-sum rule are powerful theorems for translationally invariant single-band electronic systems with parabolic electronic dispersion relations that impose restrictions on the effects of electron-electron interactions on electrical conductivity and on dielectric response, respectively. We show rigorously that similar theorems do not exist for intrinsic Dirac-Weyl materials with filled Fermi seas where the chemical potential is pinned at the band touching points.

I. INTRODUCTION

Kohn’s Theorem$^1$ and the $f$-sum rule$^2$ are very powerful theorems regarding the effects of electron-electron interactions in single-band metals with parabolic band dispersion relations strictly obeying translational invariance. Kohn’s Theorem states that interactions do not change the cyclotron resonance frequencies in such a system, or, if applied in the absence of a magnetic field, that electron-electron interactions cannot degrade the current in the system and therefore cannot affect the DC long-wavelength electrical conductivity. As a result, in a clean material with a parabolic dispersion relation, the conductivity cannot be changed except by Umklapp scattering (which, arising from an underlying lattice, explicitly breaks translational invariance and momentum conservation) or Baber scattering (which involves multiband systems also manifesting a breaking of translational invariance). The $f$-sum rule is a restriction on the dielectric response of such a material, and is as follows. If we let $\epsilon(q, \omega)$ be the dielectric function, then it must satisfy the identity,

$$\int_0^\infty d\omega \omega \text{Im} \left[ \frac{1}{\epsilon(q, \omega)} \right] = \frac{2\pi^2 e^2 n}{m} q^2,$$

where $e$ is the (absolute value of) the electron charge, $m$ is the noninteracting mass of the charge carriers, and $n$ is their number density. This exact formula applies specifically to single-band three-dimensional metals, but the invariance of this $f$ sum when interactions are introduced will extend to other dimensions as well as long as translational invariance and the consequent parabolic band dispersion applies. The right hand side is also equal to $-\frac{2\pi^2 \omega_p^2}{3}$, where $\omega_p$ is the plasma frequency. We note that it depends only on the density of carriers, and thus must remain constant, even in the presence of interactions. Thus, both Kohn’s Theorem and $f$-sum rule are statements about the inability of electron-electron interactions to affect specific properties of parabolically dispersing electron systems—the specific properties being long-wavelength DC conductivity and the long-wavelength plasma frequency, respectively. We note that the effective metal here has a single band with a partially filled Fermi surface containing $n$ carriers (electrons or holes) per unit volume. The current work by contrast focuses on systems with a filled and an empty Fermi sea with the chemical potential pinned at the band touching point in between, as for example, in Dirac-Weyl materials.

We note two facts that allow us to see where these theorems originate. In the case of Kohn’s Theorem, it is related to the fact that, for particles with a parabolic dispersion relation in translationally invariant systems, momentum and velocity are proportional to each other, so that the total current, which is proportional to the sum of the velocities of the electrons, is necessarily proportional to the total momentum. As a result, we would expect that electron-electron interactions, by virtue of momentum conservation, must not alter the current, i.e., momentum conservation automatically implies velocity or current conservation. As for the $f$ sum rule, it emerges from evaluating the double commutator, $\langle [H, \rho(q)], \rho(-q) \rangle$, where $H$ is the Hamiltonian of the system and $\rho(q)$ is the density operator in momentum space. The integral on the left hand side of the $f$-sum rule is proportional to the expectation value of this double commutator. It turns out that, in systems with parabolic electronic dispersions, it is proportional only to $q^2$ and to the total electron number operator. This means that the expectation value, and thus the $f$ sum, is invariant under the introduction of anything that only changes the energy distribution of the electrons, such as finite temperature or interactions.

The same intuition, however, does not apply to Dirac electrons with linear dispersion. For one, there is no direct proportionality between momentum and velocity—all electrons travel at the same speed, regardless of momentum. This opens up the possibility that electron-electron interactions can in fact degrade the current in such a system since momentum conservation does not automatically imply velocity conservation. This line of argument, of course, is purely classical in nature; we will put it on a more rigorous and quantum mechanical footing in this work. We will also find that the double commutator that yields the $f$-sum rule cannot be expressed in terms of the total electron number operator; we obtain an operator whose expectation value depends on the details of the electrons’ energy distribution, and thus can be affected by interactions.
We mention that the fact that in Dirac systems electron-electron interactions could affect the electrical conductivity has earlier been discussed in the literature in the context of quantum criticality associated with the Dirac point, and as such, the violation of Kohn’s Theorem in the Dirac system is implicitly known in the literature. Our work, however, puts the violation of Kohn’s Theorem on firm formal footing by considering the Dirac system, which, to the best of our knowledge has not been done before. Our formal proof follows Kohn’s derivation and shows that the theorem is violated in Dirac systems without any reference to quantum criticality. Also, it is well-known that in the presence of the breaking of translational invariance by some mechanism, electron-electron interactions can in fact affect the electrical conductivity even in a system with parabolic energy dispersion (but the interaction effect must disappear as the translational invariance is restored). Some well-known examples of such translational invariance breaking mechanisms leading to the violation of Kohn’s Theorem in its pristine form even for nominally parabolic band systems are Umklapp scattering in lattice systems, Baber scattering in transition metals (or generally in two-band systems), electron-hole scattering in semiconductors (or generally in any multicomponent system with more than one carrier species differing in charge or mass), electronic screening of impurity disorder, hydrodynamic effects associated with strong interactions, and Altshuler-Aronov type interaction corrections in the presence of diffusive carriers. The interesting physics in Dirac materials is that interaction affects the conductivity intrinsically by virtue of the fundamental breaking of Kohn’s Theorem as we show in this paper—no explicit mechanism breaking translational invariance (such as Umklapp or Baber or disorder, etc.) is necessary. Of course, one could argue that in a solid state system a linear band dispersion cannot arise unless the translational invariance is somehow broken (i.e., a lattice must somehow be present), but the standard continuum theory we use in the current work does not make any explicit reference to an explicit momentum conservation breaking mechanism from outside. The same applies to the breakdown of the $f$-sum rule which in ordinary materials arises from number conservation, and thus appears to be sacrosanct. But in Dirac materials the gapless nature of the electron and hole bands touching at the Dirac point leads to an apparent breakdown of number conservation since the infinitely filled valence band provides a mechanism for zero energy excitations violating the $f$-sum rule in its pristine form as we show in our work.

While all of our discussion so far has been about Dirac materials with linear dispersion, we should emphasize that the basic physics described above is by no means limited to just Dirac materials with linear dispersion—any material featuring two touching bands will display these effects as well as long as the chemical potential is pinned at the band touching point (i.e., the system is undoped). In particular, Kohn’s Theorem and the $f$-sum rule break down in materials with quadratic band touching points (e.g., bilayer graphene) as well, and we also explicitly and formally show it in this work. The key for the breakdown is not the dispersion linearity, but the simultaneous presence of a filled and an empty Fermi sea in the system with the chemical potential being precisely at the band touching point. Thus, all intrinsic Dirac-Weyl type materials in all dimensions violate Kohn’s Theorem and $f$-sum rule if the chemical potential is at the Dirac point.

The rest of this work is organized as follows. We dedicate Secs. II and III to reviewing the proofs of Kohn’s Theorem and the $f$-sum rule, respectively, for partially filled single-band systems with parabolic dispersions, and then showing that these theorems break down for systems with massless Dirac dispersions or quadratic band touching points. We then provide conclusions in Sec. IV.

## II. KOHN’S THEOREM

### A. Dirac electrons

We begin by addressing whether or not a theorem similar to Kohn’s Theorem holds for Dirac electrons. We will show that, in fact, this theorem is dependent on the parabolic dispersion and a finite Fermi surface by showing that it fails for massless Dirac electrons. The essential difference is as follows. For electrons with a parabolic dispersion, the total momentum, the sum of the velocities of the electrons, and the current are all proportional to each other, and thus conservation of momentum implies conservation of the other two quantities. This is not true for massless Dirac electrons, however—the speed of the electrons is always the same, no matter what their momenta may be. Thus, the velocity conservation fails explicitly even when momentum is conserved in the continuum model.

It might be helpful to first review Kohn’s original proof, as we will model our calculation on it. We start with the Hamiltonian,

$$
H = \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m} + \sum_{1 \leq j < k \leq N} u(\mathbf{r}_j - \mathbf{r}_k),
$$

where $u(\mathbf{r}_j - \mathbf{r}_k)$ is a two-body interaction between electrons $j$ and $k$ and $\mathbf{P}_j$ is the kinetic momentum of electron $j$. We assume that $u(-\mathbf{r}) = u(\mathbf{r})$ and work in the Landau gauge, so that $\mathbf{P}_j = \mathbf{p}_j + \frac{e}{c} \mathbf{A} \times \mathbf{p}_j$.

Working in the Heisenberg picture, the equation of motion for the total momentum of the system, $\mathbf{B} = \sum_j \mathbf{P}_j$, is

$$
\frac{d\mathbf{P}}{dt} = \frac{i}{\hbar} [H, \mathbf{P}] = -\frac{e}{mc} \mathbf{P} \times \mathbf{B}.
$$

At this point, we can simply set $\mathbf{B} = 0$ to obtain the familiar statement of conservation of total momentum,
and thus, as stated earlier, current. If, however, we retain the magnetic field, then we find that

\[ [H, P_\pm] = \pm \hbar \omega_c P_\pm, \]

(4)

where \( \omega_c = \frac{eB}{mc} \) is the cyclotron frequency. This means that the \( P_\pm \) operators act like the raising and lowering operators for the harmonic oscillator—applied to an eigenstate of \( H \) with energy \( E \), they produce another eigenstate with an energy \( E \pm \hbar \omega_c \). Therefore, the cyclotron resonance frequencies are unchanged.

Although Kohn’s original proof is only for explicitly translationally invariant systems as in Eq. (2) above where there is no external one-particle potential destroying translational invariance (in fact, the theorem fails generically in the presence of such a one-electron spatially varying potential since the center of mass and relative coordinates are no longer separable, and hence interaction effects can then modify the center of mass motion), it is possible to generalize Kohn’s Theorem to a situation where an explicit parabolic external potential is applied to the system. This is simply because the parabolic potential allows the separation of center of mass and relative coordinates, thus preserving Kohn’s Theorem in spite of an apparent breaking of the translational invariance.

Now we turn our attention to the case of a massless Dirac fermion, and attempt to replicate the above proof for this case. The Hamiltonian is

\[ H = v_F \sum_{j=1}^N \vec{\sigma}_j \cdot \vec{P}_j + \sum_{1 \leq j < k \leq N} u(\vec{r}_j - \vec{r}_k), \]

(5)

where \( \vec{\sigma}_j \) is the vector of Pauli matrices acting on electron \( j \). Here, we are considering a system with just one flavor of Dirac fermion for simplicity; additional flavors will not affect our conclusions here. The velocity operator is given by

\[ \vec{v}_j = v_F \vec{\sigma}_j. \]

(6)

Note that this is not proportional to the momentum operator; as implied earlier, this is the key fact that will lead to the failure of Kohn’s theorem for this case.

We begin by determining the equation of motion for the kinetic momentum. The commutator of a component of the kinetic momentum \( P_{k,\alpha} \) of electron \( k \) is

\[ [H, P_{k,\alpha}] = -i \frac{\hbar eB}{c} v_{k,x} \delta_{\alpha,y} + i \frac{\hbar eB}{c} v_{k,y} \delta_{\alpha,x} \]

\[ - i \hbar \sum_{j=k+1}^N \frac{\partial u(\vec{r}_j - \vec{r}_k)}{\partial x_{k,\alpha}} + i \hbar \sum_{j=1}^{k-1} \frac{\partial u(\vec{r}_j - \vec{r}_k)}{\partial x_{k,\alpha}}. \]

(7)

The commutator with the total momentum \( P_\alpha = \sum_{k=1}^N P_{k,\alpha} \) is then

\[ [H, P_\alpha] = -i \frac{\hbar eB}{c} (v_x \delta_{\alpha,y} - v_y \delta_{\alpha,x}). \]

(8)

We may write this equation for the \( x \) and \( y \) components in a different way; defining \( P_\pm = P_x \pm iP_y \), we find that

\[ [H, P_\pm] = \pm \frac{\hbar eB}{c} v_\pm, \]

(9)

where \( v_\pm = \sum_{j=1}^N v_{k,\alpha} \) is the sum of the electrons’ velocities, which is proportional to the total current. Therefore,

\[ \frac{dP_\pm}{dt} = \pm i \frac{eB}{c} v_\pm. \]

(10)

Because the velocity operator is not proportional to the momentum operator, we cannot necessarily conclude from this result that, in the absence of a magnetic field, the sum of the electrons’ velocities, and thus the total current, will be conserved. In fact, the above result also demonstrates explicitly that the cyclotron resonance frequency in a Dirac material will indeed be renormalized by electron-electron interaction in direct violation of Kohn’s Theorem. Indeed, the dependence of the cyclotron resonance on interaction effects has been experimentally observed in graphene, which is the prototypical Dirac material.

Next, we determine the equation of motion for the sum of the electron velocity operators. The commutator of \( v_\pm \) with the Hamiltonian is

\[ [H, v_\pm] = \pm v_F \sum_{k=1}^N (v_{k,\pm} P_{k,z} - v_{k,z} P_{k,\pm}), \]

(11)

so that the corresponding equation of motion is

\[ \frac{dv_\pm}{dt} = \pm i \frac{eB}{\hbar} \sum_{k=1}^N (v_{k,\pm} P_{k,z} - v_{k,z} P_{k,\pm}). \]

(12)

We thus have obtained an expression that cannot be written entirely in terms of sums of the electrons’ momenta and velocities. We now consider the equation of motion for \( \sum_{k=1}^N (v_{k,\pm} P_{k,z} - v_{k,z} P_{k,\pm}) \). To this end, we will need the identity, \([A, BC] = [A, B]C + B[A, C]\), along with the following commutators:

\[ [H, P_{k,z}] = -i \hbar \sum_{j=k+1}^N \frac{\partial u(\vec{r}_j - \vec{r}_k)}{\partial z_k} \]

\[ + i \hbar \sum_{j=1}^{k-1} \frac{\partial u(\vec{r}_j - \vec{r}_k)}{\partial z_k}, \]

(13)

\[ [H, v_{k,z}] = \frac{1}{2} v_F (v_{k,-} P_{k,+} - v_{k,+} P_{k,-}). \]

(14)

Combining these results, we get

\[ \sum_{j=1}^N [H, v_{j,\pm} P_{j,z} - v_{j,z} P_{j,\pm}] \]

\[ = \sum_{j=1}^N \{ v_F (v_{j,\pm} P_{j,z} - v_{j,z} P_{j,\pm}) P_{j,z} \]
\[
\frac{dv_j}{dt} = \frac{iv_F}{2\hbar} \sum_{k=1}^{N} \left( v_{k,+} P_{j,+} - v_{j,+} P_{k,+} - v_{k,-} P_{j,-} + v_{j,-} P_{k,-} \right)
\]
and the commutator of the operator on the right-hand side of it is
\[
\sum_{j=1}^{N} [H, v_{j,+} P_{j,+} - v_{j,-} P_{j,-}] = \sum_{j=1}^{N} \{-v_F(v_{j,+} P_{j,+} - v_{j,-} P_{j,-})P_{j,-} - v_F(v_{j,-} P_{j,-} - v_{j,+} P_{j,+})P_{j,+} + v_{j,+} \left[ -\frac{eB}{c} v_{j,+} + i\hbar \sum_{k=1}^{j-1} \left( \frac{\partial u(\mathbf{r}_k - \mathbf{r}_j)}{\partial x_k} + i \frac{\partial u(\mathbf{r}_k - \mathbf{r}_j)}{\partial y_k} \right) \right] - v_{j,-} \left[ -\frac{eB}{c} v_{j,-} + i\hbar \sum_{k=1}^{j-1} \left( \frac{\partial u(\mathbf{r}_k - \mathbf{r}_j)}{\partial x_k} - i \frac{\partial u(\mathbf{r}_k - \mathbf{r}_j)}{\partial y_k} \right) \right] \}
\]

The terms that depend on the electron-electron interaction do not cancel out of these expressions, and thus we conclude that the time evolution of the sum of the electrons’ velocities, and thus the total current in the system, depends on the interaction, in contrast with the case with a parabolic dispersion.

We emphasize that our formal results arise from the fact that the relative and center of mass coordinates are hopelessly intermixed in the dynamics of Dirac systems, and thus, electron-electron interaction, in spite of being dependent only on relative coordinates, affects the total momentum, and hence the conductivity. The fact that the correct calculation of the dielectric function for the Dirac problem must account for the intraband and interband processes on an equal footing was already pointed out in Ref. 13.

To help explicitly illustrate the basic physics behind Kohn’s Theorem for electrons with parabolic dispersions and its failure for Dirac electrons, it might be helpful to consider collisions between two electrons in each case. This will also help us to quantify how severe the effects of this breakdown will be for Dirac electrons. In both cases, the conservation of momentum is very well known:
\[
\mathbf{p}_{1,i} + \mathbf{p}_{2,i} = \mathbf{p}_{1,f} + \mathbf{p}_{2,f}.
\]

In the case of a parabolic dispersion, the momentum is just \( \mathbf{p} = m \mathbf{v} \), so that, dividing by \( m \),
\[
\mathbf{v}_{1,i} + \mathbf{v}_{2,i} = \mathbf{v}_{1,f} + \mathbf{v}_{2,f}.
\]
Therefore, the sum of the velocities of the two electrons is also conserved, and thus this collision will not degrade the total current carried by them. On the other hand, if the electrons have a Dirac dispersion, then \( \mathbf{v} = \frac{\mathbf{p}}{m \gamma} \). In this case, the conservation of momentum becomes
\[
\mathbf{v}_{1,i}\mathbf{p}_{1,i} + \mathbf{v}_{2,i}\mathbf{p}_{2,i} = \mathbf{v}_{1,f}\mathbf{p}_{1,f} + \mathbf{v}_{2,f}\mathbf{p}_{2,f}.
\]

In contrast to the parabolic case, this equation does not necessarily imply conservation of the total velocity. Thus, electron-electron interaction, in spite of being momentum conserving, can indeed relax the charge current flow and hence lead to a finite electrical conductivity. However, total velocity conservation is approximately true in the case of low-energy excitations, i.e., all electronic wave vectors are near the Fermi surface. In this case, \( n_i/f \approx \hbar k_F \). This approximation should hold very well in cases where the temperature is far below the chemical potential, i.e., \( k_B T \ll \mu \), since the Pauli exclusion principle will help to “freeze out” scattering into or from states with momenta other than \( \hbar k_F \). Note, however, that at the Dirac point, where \( \mu = 0 \) in our notation, the system is always susceptible to interaction effects, and is thus a non-Fermi-liquid, which has led people to dub the Dirac point a quantum critical point lying between an electron metal and a hole metal. Thus, for an undoped intrinsic system with the chemical potential at the Dirac point, the system violates current conservation in the presence of electron-electron scattering.

B. Quadratic band touching

We now turn our attention to the case of a quadratic band touching, e.g., in bilayer graphene. We will find that similar effects occur here as well provided that the Fermi level is pinned at the band touching point (i.e., no partial band filling). The Hamiltonian for this case is
\[
H = \frac{1}{2m} \sum_{j=1}^{N} \left( (P_{j,x}^2 - P_{j,y}^2)\sigma_{j,x} + 2P_{j,x}P_{j,y}\sigma_{j,y} \right) + \sum_{1 \leq j < k \leq N} u(\mathbf{r}_j - \mathbf{r}_k),
\]
where all symbols have the same basic meaning as before. If we determine the velocity operators for this system, we get
\[
v_{j,x} = \frac{P_{j,x}}{m} \sigma_{j,x} + \frac{P_{j,y}}{m} \sigma_{j,y},
\]
\[ v_{j,y} = \frac{P_{x,j}}{m} \sigma_{j,y} - \frac{P_{y,j}}{m} \sigma_{j,x} \]  \hspace{1cm} (23)

We note that the direct proportionality between momentum and velocity is broken in a different way here—the velocity component operators depend on both components of the (kinetic) momentum operator. We should emphasize here, however, that this lack of direct proportionality between the velocity and momentum operators is just a symptom of the underlying physics at work here, namely, the presence of a filled negative-energy Fermi sea.

If we now determine the equations of motion for the kinetic momentum, we obtain similar results as before. Letting \( P_{\alpha} = \sum_{j} P_{j,\alpha} \) be the total momentum, we find that
\[ \frac{dP_{x}}{dt} = -\frac{e B}{c} v_{y}, \]  \hspace{1cm} (24)
\[ \frac{dP_{y}}{dt} = \frac{e B}{c} v_{x}. \]  \hspace{1cm} (25)

We now want to determine the equations of motion for the velocity. Following a similar procedure as for the Dirac case, we find that
\[ \frac{dv_{k,x}}{dt} = \frac{e B}{m c} v_{k,y} \sigma_{k,x} + \frac{e B}{m c} v_{k,x} \sigma_{k,y} \]
\[ - \frac{1}{m} \sum_{j=1}^{N} \frac{\partial u(\vec{r}_{j} - \vec{r}_{k})}{\partial x_{k}} - \frac{\partial u(\vec{r}_{k} - \vec{r}_{j})}{\partial x_{k}} \]
\[ - \frac{1}{m} \sum_{j=1}^{N} \frac{\partial u(\vec{r}_{j} - \vec{r}_{k})}{\partial y_{k}} - \frac{\partial u(\vec{r}_{k} - \vec{r}_{j})}{\partial y_{k}} \]
\[ + \frac{1}{h m} \frac{P_{k,x}}{P_{k,x} - P_{k,y}} \sigma_{k,z} \]
\[ - \frac{1}{h m} \frac{P_{k,y}}{P_{k,x} - P_{k,y}} \sigma_{k,z} \]  \hspace{1cm} (26)

and
\[ \frac{dv_{k,y}}{dt} = \frac{e B}{m c} v_{k,x} \sigma_{k,y} - \frac{e B}{m c} v_{k,y} \sigma_{k,x} \]
\[ - \frac{1}{m} \sum_{j=1}^{N} \frac{\partial u(\vec{r}_{j} - \vec{r}_{k})}{\partial x_{k}} - \frac{\partial u(\vec{r}_{k} - \vec{r}_{j})}{\partial x_{k}} \]
\[ + \frac{1}{m} \sum_{j=1}^{N} \frac{\partial u(\vec{r}_{j} - \vec{r}_{k})}{\partial y_{k}} - \frac{\partial u(\vec{r}_{k} - \vec{r}_{j})}{\partial y_{k}} \]
\[ - \frac{1}{h m} \frac{P_{k,x}}{P_{k,x} - P_{k,y}} \sigma_{k,z} \]
\[ - \frac{1}{h m} \frac{P_{k,y}}{P_{k,x} - P_{k,y}} \sigma_{k,z} \]  \hspace{1cm} (27)

We see that, if we were to sum the velocities of all of the particles, then the terms involving the interaction will not cancel out. As a result, we find that Kohn’s Theorem is also broken for a system with quadratic band touching points. As noted before, the fact that the exact nature of the dispersion does not matter points to the underlying reason for this failure of Kohn’s Theorem being the presence of an empty electron band and a filled hole band with the Fermi level exactly at the band touching point, rather than just a partially filled electron band as in ordinary metals.

### III. f-SUM RULE

We now turn our attention to the f-sum rule, also referred to as the Thomas-Reiche-Kuhn sum rule for metals. In systems with parabolic electronic dispersion relations, it has been shown that what is known as the f-sum rule, Eq. (1), holds. None of the quantities on the right hand side of this relation are dependent on such things as temperature or the presence of interactions. Therefore, this relation provides a powerful constraint on models of the dielectric function for a system with a parabolic dispersion, as it can be calculated for noninteracting electrons, but must be followed even in the presence of interactions. In fact, this sum rule necessarily restricts the high-frequency long wavelength dielectric function of a simple metal to have the form,
\[ \epsilon(\vec{q}, \omega) = 1 - \frac{\omega_{p}^{2}}{\omega^{2}}, \]  \hspace{1cm} (28)
where \( \omega_{p} \) is the standard electronic plasma frequency. We will illustrate that, in contrast, there is no simple f-sum rule for systems with a massless Dirac dispersion—the right-hand side of the analogous relation will actually depend on the details of the energy distribution of the electrons, and thus interactions or even temperature can change it. This failure of the simple f-sum rule arises from the presence of the filled negative energy hole Fermi sea in the system.

We will begin by stating the most general form of the above relation. The right-hand side is proportional to the expectation value of the double commutator, \([[H, \rho(\vec{q})], \rho(-\vec{q})]]\), where \( H \) is the Hamiltonian and \( \rho(\vec{q}) \) is the density operator in momentum space. Following a derivation similar to that of Ref. [3], we find that
\[
\int_{0}^{\infty} d\omega \omega \text{Im} \left[ \frac{1}{\epsilon(\vec{q}, \omega)} \right] = \frac{\pi e^{2} V C(\vec{q})}{2 \hbar^{2} V} \langle [[H, \rho(\vec{q})], \rho(-\vec{q})]] \rangle,
\]  \hspace{1cm} (29)
where \( V \) is the volume of the system and \( V C(\vec{q}) \) is the Coulomb interaction in momentum space. In the case of a parabolic dispersion, one finds that the double commutator is
\[ [[H, \rho(\vec{q})], \rho(-\vec{q})]] = -\frac{\hbar^{2} q^{2}}{m} N, \]  \hspace{1cm} (30)
where \( N = \sum_{\vec{k}} n(\vec{k}) \) is the total electron number operator. If we substitute this into the previous equation and use the fact that, in three dimensions, \( V C(\vec{q}) = \frac{4 \pi e^{2} q^{2}}{q^{2}} \), we will recover Eq. (1). The fact that this expression for the
double commutator only depends on the total electron number operator is what leads to the invariance of the \( f \) sum under the introduction of interactions.

We now calculate the double commutator for Dirac electrons. The density operators commute with the interaction term, so we just need to find it for the noninteracting case. The noninteracting part of the Hamiltonian \( H_0 \) is, in second-quantized form,

\[
H_0 = \hbar v_F \sum_k \Psi^\dagger(k) \sigma \cdot k \Psi(k),
\]

where \( \Psi^T(k) = [a(k), b(k)] \) is the vector of annihilation operators for pseudospins \( a \) and \( b \) (e.g., sublattice), and

\[
\rho(q) = \sum_k \Psi^\dagger(k) \Psi(k + q).
\]

Applying the usual anticommutation relations for fermionic operators, we find that

\[
[[H, \rho(q)], \rho(-q)] = \hbar v_F \sum_k [2 \Psi^\dagger(k) \sigma \cdot k \Psi(k) - \Psi^\dagger(k + q) \sigma \cdot k \Psi(k + q) - \Psi^\dagger(k - q) \sigma \cdot k \Psi(k - q)].
\]

If we now perform the unitary transformation that diagonalizes \( H_0 \), we may split this expression into two sets of terms, which we will denote \( C_1 \) and \( C_2 \), i.e., \([ [H, \rho(q)], \rho(-q)] = C_1 + C_2 \), where

\[
C_1 = \hbar v_F \sum_k [2k \Psi^\dagger(k) \Psi_+ - 2k \Psi^\dagger(k) \Psi_- - \Psi^\dagger(k + q) \Psi_+ - \Psi^\dagger(k - q) \Psi_-]
\]

and

\[
C_2 = \hbar v_F \sum_k [\Psi^\dagger(k + q) \sigma \cdot q \Psi(k + q) - \Psi^\dagger(k - q) \sigma \cdot q \Psi(k - q)].
\]

In these expressions, \( \Psi_\pm(k) \) is the annihilation operator for electrons in positive- \( (+) \) and negative- \( (-) \) energy single-particle eigenstates. We note that a similar expression to ours has been derived for the case of monolayer graphene in Ref. [3] there, only the terms corresponding to our \( C_2 \) are obtained. It turns out that the \( C_1 \) terms have a nonzero expectation value; we believe that this arises from the same phenomenon mentioned therein (the “anomalous commutator” problem).

We note that, in contrast to the case of an electron-band-only parabolic dispersion, the expression that we obtain for \([ [H, \rho(q)], \rho(-q)] \) cannot be expressed in terms of the total number of particles; the expectation value of this expression will necessarily depend on the details of the electron energy distribution. Therefore, there is no \( f \)-sum rule for Dirac electrons, and thus the \( f \) sum can in fact be altered by the presence of interactions.

We find similar results for the intrinsic quadratic band touching case as well. The noninteracting Hamiltonian, again in second-quantized form, is

\[
H_0 = \frac{\hbar^2}{2m} \sum_k \Psi^\dagger(k) [(k_x^2 - k_y^2) \sigma_x + 2k_x k_y \sigma_y] \Psi(k),
\]

where all symbols have similar meanings as in the Dirac case. If we now calculate \([ [H, \rho(q)], \rho(-q)] \), we find that it is given by \( D_1 + D_2 \), where

\[
D_1 = \frac{\hbar^2}{2m} \sum_k [2k^2 \Psi^\dagger_+(k) \Psi_+ - 2k^2 \Psi^\dagger_-(k) \Psi_- - \Psi^\dagger_+(k + q) \Psi_+ - \Psi^\dagger_-(k - q) \Psi_-] \]

and

\[
D_2 = \frac{\hbar^2}{2m} \sum_k [\Psi^\dagger(k + q) M_+ \Psi_+ - \Psi^\dagger(k - q) M_- \Psi_-] \]

As in the Dirac case, the expectation value of the double commutator will depend on the details of the electronic energy distribution, and thus will be changed by the presence of electron-electron interactions. Thus, the \( f \)-sum rule is violated in the intrinsic undoped Dirac-Weyl system independent of energy dispersion.

As we already stated in the Introduction, this failure of \( f \)-sum rule arises from the presence of the infinite filled valence band in the continuum Dirac-Weyl system—one can of course impose a sum rule by imposing a physical
energy or momentum cut off on the spectrum, but then the result becomes explicitly dependent on this cut off. Thus, the invariable presence of both intraband and interband processes destroy the simplicity of an \( f \)-sum rule for these systems.

### IV. CONCLUSION

We have demonstrated that two well-known theorems, Kohn’s Theorem and \( f \)-sum rule, that have been derived for single-band electrons with parabolic dispersions and partially filled Fermi surfaces (i.e., metals) do not apply to intrinsic Dirac-Weyl electron materials. These theorems for electrons with parabolic dispersions are very powerful—Kohn’s Theorem places restrictions on the ability of electron-electron interactions to affect cyclotron resonance frequency and DC conductivity, while the \( f \)-sum rule imposes a restriction on the correct mathematical models of dielectric responses. While the charge carriers in many semiconductors and metals possess parabolic dispersions, there are a number of materials, including 2D graphene and 3D Dirac and Weyl materials, that have massless Dirac dispersions instead. Our results imply, for one, that electron-electron interactions can in fact change the electrical conductivity of these materials, even in the absence of Umklapp scattering or other explicit momentum-conservation-breaking mechanisms.

We expect that significant interaction and temperature effects on conductivity will occur at temperatures comparable to or higher than the chemical potential, i.e., \( k_B T \gtrsim \mu \). The reason for this is that, if the temperature is much lower than the chemical potential, then scattering processes that can alter the sum of the velocities of the electrons are “frozen out” due to the Pauli exclusion principle (a similar argument forms the basis for Landau’s Fermi liquid theory). We note that, in Dirac systems, therefore, a fundamental difference exists between \( \mu < k_B T \) and \( \mu > k_B T \) (where \( \mu = 0 \) is the Dirac point or the band touching point in our notation) with the higher-temperature regime corresponding to an “intrinsic” non-Fermi-liquid type system where interactions matter nontrivially whereas the lower-temperature regime being more like a standard Fermi liquid system. It follows that the pure semimetal with the chemical potential precisely at the Dirac point is always a non-Fermi-liquid and explicitly violates Kohn’s Theorem and \( f \)-sum rule always. As for the failure of the \( f \) sum rule, it can be traced back in part to the presence of the negative-energy Dirac sea of electrons. The usual electronic systems with parabolic dispersions possess no similar “infinite” sea of electrons. Due to this, there is only a finite amount of “weight” that interactions can redistribute in the \( f \) sum. This is not true for massless Dirac electrons; the presence of the infinite Dirac sea means that there is an infinite amount of “weight” present. This is, of course, in addition to the fact that the double commutator, \([ [H, \rho(\vec{q})], \rho(-\vec{q})] \), yields an operator with an expectation value that will depend on the details of the energy distribution of the electrons.

We should emphasize that the linear Dirac cone nature of the dispersion is not the ultimate source of this breakdown of Kohn’s Theorem and the \( f \)-sum rule, but rather the presence of a filled Fermi sea of negative-energy electrons. To help illustrate this, we also investigated the case of materials with quadratic band touchings in their spectra, such as bilayer graphene, and showed that these two theorems break down in them as well. The common theme with these is the presence of an “infinite” sea of electrons and the lack of a band gap. If a band gap were to be opened, then this will “freeze out” scattering processes that involve the Fermi sea, thus restoring Kohn’s Theorem and the \( f \)-sum rule as long as the temperature is much less than the band gap. This other method for “rescuing” these theorems, of course, is only relevant when the chemical potential is close to the band minimum; otherwise, there is no real difference with the case of a large chemical potential discussed above.

Thus a gapless semimetal with the chemical potential pinned at the band touching point with a completely filled and a completely empty Fermi sea (i.e., an intrinsic undoped Dirac-Weyl system) is always a non-Fermi-liquid (independent of the energy band dispersion) in the sense that interactions affect its conductivity even in the absence of disorder. The result arises simply from the presence of zero energy interband excitations which make the system fundamentally different from a single-band metal with a partially filled Fermi sea. If the chemical potential \( \mu \) is finite (i.e., away from the Dirac point), the system still behaves as an intrinsic material as long as the temperature is high enough: \( k_B T \gg \mu \). It is interesting to note that \( k_B T > \mu \) is in some sense the classical limit of the system, and the classical limit manifests a strong quantum critical effect of the underlying Dirac point whereas the quantum limit, \( \mu \gg k_B T \), is benign and behaves as an ordinary metal. The reason for this apparently puzzling behavior is physically obvious: Only in the high-temperature limit, the effective low-energy interband excitations proliferate leading to the strange quantum critical behavior involving the violation of Kohn’s Theorem and \( f \)-sum rule discussed in the current work.

### Acknowledgments

This work is supported by the Laboratory for Physical Sciences.

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