SMOOTHNESS OF COHOMOLOGY SHEAVES OF STACKS OF SHTUKAS

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Abstract. We prove, for all reductive groups, that the cohomology sheaves with compact support of stacks of shtukas are ind-smooth over \((X \setminus N)^I\) and that their geometric generic fibers are equipped with an action of \(\text{Weil}(X \setminus N, \eta)^I\). Our method does not use any compactification of stacks of shtukas.

Introduction

Let \(X\) be a smooth projective geometrically connected curve over a finite field \(\mathbb{F}_q\). We denote by \(F\) the function field of \(X\).

Let \(G\) be a connected reductive group over \(F\). Let \(\ell\) be a prime number not dividing \(q\). Let \(E\) be a finite extension of \(\mathbb{Q}_\ell\) containing a square root of \(q\).

In the introduction, we assume that \(G\) is split to simplify the notation. Let \(\hat{G}\) be the Langlands dual group of \(G\) over \(E\). Let \(I\) be a finite set and \(W\) be a finite dimensional \(E\)-linear representation of \(\hat{G}^I\). Let \(N \subset X\) be a finite subscheme. Varshavsky \([\text{Var04}]\) and V. Lafforgue \([\text{Laf18}]\) defined the stack classifying \(G\)-shtukas with level structure on \(N\):

\[ \text{Cht}_{G,N,I,W} \text{ over } (X \setminus N)^I \]

and its degree \(j \in \mathbb{Z}\) cohomology sheaf with compact support with \(E\)-coefficients:

\[ \mathcal{H}_{G,N,I,W}^j \text{ over } (X \setminus N)^I \]

which is an inductive limit of constructible \(E\)-sheaves. The cohomology sheaf \(\mathcal{H}_{G,N,I,W}^j\) is equipped with an action of the Hecke algebra, and equipped with an action of the partial Frobenius morphisms.

Let \(\eta\) be the generic point of \(X\) and \(\overline{\eta}\) be a geometric point over \(\eta\). Let \(\eta_I\) be the generic point of \(X^I\) and \(\overline{\eta_I}\) be a geometric point over \(\eta_I\). Then \(\mathcal{H}_{G,N,I,W}^j|_{\overline{\eta_I}}\) is equipped with an action of \(\text{Weil}(\eta, \overline{\eta})^I\) and an action of the partial Frobenius morphisms.

In \([\text{Xue18b}]\), by a variant of a lemma of Drinfeld, we equipped \(\mathcal{H}_{G,N,I,W}^j|_{\overline{\eta_I}}\) with an action of \(\text{Weil}(\eta, \overline{\eta})^I\). The proof uses the fact that \(\mathcal{H}_{G,N,I,W}^j|_{\overline{\eta_I}}\) is of finite type as a module over a Hecke algebra (proved in loc.cit.). The case of non-split groups of the last assertion has not yet been written.

In this paper, in Section 1 we give a different proof of the fact that \(\mathcal{H}_{G,N,I,W}^j|_{\overline{\eta_I}}\) is equipped with an action of \(\text{Weil}(\eta, \overline{\eta})^I\) (Proposition 1.3.4). This proof is based on
the Eichler-Shimura relations in [Laf18]. It does not use the fact that $\mathcal{H}^0_{G,N,I,W}|_{(\eta)}$ is of finite type as module over a Hecke algebra. The new proof has the advantage that it is easy to generalize to the case of non-split groups.

In Section 1, we also prove that $\mathcal{H}^0_{G,N,I,W}|_{(\eta)}$, the restriction of $\mathcal{H}^0_{G,N,I,W}$ to the scheme $(\eta)^I := \eta \times_{\text{Spec} \mathbb{F}_q} \cdots \times_{\text{Spec} \mathbb{F}_q} \eta$ ($I$ times), is ind-smooth (i.e. an inductive limit of smooth $E$-sheaves). Then Proposition 1.3.4 implies that $\mathcal{H}^0_{G,N,I,W}|_{(\eta)}$ is a constant sheaf.

In Section 2, we generalize the results in Section 1. For any partition $I = I_1 \sqcup I_2$ and for any geometric point $\eta$ over a closed point of $(X \setminus N)^I$, we prove in Proposition 2.4.1 that the restriction of $\mathcal{H}^0_{G,N,I,W}$ to the scheme $(\eta)^{I_1} \times_{\text{Spec} \mathbb{F}_q} \eta$ is a constant sheaf, where $(\eta)^{I_1} = \eta \times_{\text{Spec} \mathbb{F}_q} \cdots \times_{\text{Spec} \mathbb{F}_q} \eta$ ($I_1$ times). When $I_2$ is the empty set, we recover Section 1.

In Sections 3-4, we prove that $\mathcal{H}^0_{G,N,I,W}$ is ind-smooth over $(X \setminus N)^I$ in the following sense (which is equivalent to be an inductive limit of smooth $E$-sheaves by Lemma 1.1.5):

**Theorem 0.0.1.** (Theorem 4.2.3) For any geometric point $\eta$ of $(X \setminus N)^I$ and any specialization map $\text{sp}_\eta : \eta \to \eta$, the induced morphism

$$\text{sp}_\eta^* : \mathcal{H}^0_{G,N,I,W}|_{\eta} \to \mathcal{H}^0_{G,N,I,W}|_{(\eta)}$$

is an isomorphism.

The idea is that using creation and annihilation operators in [Laf18] and Proposition 2.4.1 we construct a morphism $\Upsilon : \mathcal{H}^0_{G,N,I,W}|_{(\eta)} \to \mathcal{H}^0_{G,N,I,W}|_{(\eta)}$. Then using the "Zorro" lemma, we prove that the composition $\Upsilon \circ \text{sp}_\eta^*$ and $\text{sp}_\eta^* \circ \Upsilon$ are isomorphisms.

As an immediate consequence, in Section 5, we prove

**Proposition 0.0.2.** (Proposition 5.0.4) The action of $\text{Weil}(\eta, \eta)^I$ on $\mathcal{H}^0_{G,N,I,W}|_{(\eta)}$ factors through $\text{Weil}(X \setminus N, \eta)^I$.

In Section 6, we treat the case where $G$ is not necessarily split. The results in Sections 1-5 still hold.

In Section 7, with the help of Theorem 0.0.1 we extend for split $G$ the constant term morphism constructed in [Xue18a] from $\eta^I$ to $(X \setminus N)^I$. Then we define a smooth cuspidal cohomology subsheaf $\mathcal{H}^0_{G,N,I,W}$ of $\mathcal{H}^0_{G,N,I,W}$ over $(X \setminus N)^I$.

For the cohomology sheaves with integral coefficients, we still have the main results as in Sections 1-7. For this, we will need the context in [XZ17]. We will write these in a future version.

Proposition 0.0.2 was already proved by Xinwen Zhu in the MSRI 2019 hot topic workshop. But Theorem 0.0.1 is new.
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1. THE COHOMOLOGY SHEAVES ARE CONSTANT OVER $(\mathcal{T})^I$

In this section, we first recall some general results about ind-smooth sheaves in 1.1 and partial Frobenius morphisms in 1.2. Then we recall the cohomology sheaves of stacks of shtukas, prove Proposition 1.3.4 and Proposition 1.4.3. As a result, we deduce Proposition 1.5.1.

1.1. Reminders on ind-smooth $E$-sheaves.

1.1.1. We use the definition in [SGA5] VI 1 and [De80] 1.1 for constructible $E$-sheaves (for étale topology) and smooth constructible $E$-sheaves. In this paper, a smooth $E$-sheaf always means a smooth constructible $E$-sheaf.

We define an ind-constructible $E$-sheaf to be an inductive limit of constructible $E$-sheaves (i.e. an object in the abstract category of inductive limits of constructible $E$-sheaves). Its fiber at a geometric point is defined to be the $E$-vector space which is the inductive limit of fibers (i.e. in the category of $E$-vector spaces).

1.1.2. We use [SGA4] VIII 7 for the definition of specialization maps.

1.1.3. We denote by $\mathcal{O}_E$ the ring of integers of $E$ and $\lambda_E$ a uniformizer. Let $\Lambda = \mathcal{O}_E/\lambda_E^s \mathcal{O}_E$ for $s \in \mathbb{N}$.

Let $Y$ be a noetherian scheme over $\mathbb{F}_q$. Let $\mathcal{F}$ be a constructible $\Lambda$-sheaf over $Y$. By [SGA4] IX Proposition 2.11, the sheaf $\mathcal{F}$ is locally constant if and only if for any geometric points $\overline{x}$, $\overline{y}$ of $Y$ and any specialization map $sp : \overline{y} \to \overline{x}$, the induced morphism

$$sp^* : \mathcal{F}|_{\overline{y}} \to \mathcal{F}|_{\overline{x}}$$

is an isomorphism.

Using the definition of constructible $\mathcal{O}_E$-sheaves (resp. constructible $E$-sheaves), we deduce that a constructible $\mathcal{O}_E$-sheaf (resp. constructible $E$-sheaf) $\mathcal{F}$ over $Y$ is smooth if and only if for any geometric points $\overline{x}$, $\overline{y}$ of $Y$ and any specialization map $sp : \overline{y} \to \overline{x}$, the induced morphism

$$sp^* : \mathcal{F}|_{\overline{x}} \to \mathcal{F}|_{\overline{y}}$$

is an isomorphism.
1.1.4. Let $Y$ be a normal connected noetherian scheme over $\mathbb{F}_q$. Let $\mathcal{H} = \lim_{\lambda \in \Omega} \mathcal{F}_\lambda$ be an inductive limit of constructible $E$-sheaves over $Y$, where $\Omega$ is a numerable filtered set. We say that the ind-constructible $E$-sheaf $\mathcal{H}$ is ind-smooth if we can write $\mathcal{H}$ as an inductive limit of smooth $E$-sheaves over $Y$, i.e. there exists a numerable filtered set $\Omega'$ and smooth $E$-sheaves $G_\lambda$ for $\lambda \in \Omega'$ such that $\mathcal{H} = \lim_{\lambda \in \Omega'} G_\lambda$.

Lemma 1.1.5. For $Y$ as in 1.1.4, an ind-constructible $E$-sheaf $\mathcal{H}$ over $Y$ is ind-smooth if and only if for any geometric points $\overline{y}, \overline{y}'$ of $Y$ and any specialization map $sp : \overline{y}' \to \overline{y}$, the induced morphism $sp^* : \mathcal{H}_{\overline{y}} \to \mathcal{H}_{\overline{y}'}$ is an isomorphism.

1.1.6. To prove Lemma 1.1.5, we need some preparations. Let $\mathcal{H} = \lim_{\lambda \in \Omega} \mathcal{F}_\lambda$ as above. For any $\lambda \leq \mu$ in $\Omega$, the kernel $\text{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_\mu)$ is a constructible $E$-sheaf $\mathcal{F}_\lambda$. For $\lambda \leq \mu_1 \leq \mu_2$, we have $\text{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_{\mu_1}) \subset \text{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_{\mu_2}) \subset \text{Ker}(\mathcal{F}_\lambda \to \mathcal{H}) \subset \mathcal{F}_\lambda$.

Since $\mathcal{F}_\lambda$ is constructible and $Y$ is noetherian, we deduce that there exists $\lambda_0$, such that for all $\mu \geq \lambda_0$, we have $\text{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_{\mu_1}) = \text{Ker}(\mathcal{F}_\lambda \to \mathcal{F}_{\mu_2})$ (The argument is similar to the proof of Lemma 58.73.2 of [StacksProject]). So $\text{Im}(\mathcal{F}_\lambda \to \mathcal{F}_{\mu_1}) \to \mathcal{F}_\lambda$. We denote by $\mathcal{F}_\lambda := \text{Im}(\mathcal{F}_\lambda \to \mathcal{F}_{\lambda_0})$. We have

$$\lim_{\lambda \in \Omega} \mathcal{F}_\lambda = \lim_{\lambda \in \Omega} \mathcal{F}_\lambda.$$  

Proof of Lemma 1.1.5. One direction is obvious. Let’s prove the converse direction. Suppose that all of the specialization homomorphisms $sp^* : \mathcal{H}_{\overline{y}} \to \mathcal{H}_{\overline{y}'}$ are isomorphisms.

Since every $\mathcal{F}_\lambda$ is a constructible $E$-sheaf over $Y$, there exists an open dense subscheme $U_\lambda$ of $Y$ such that $\mathcal{F}_\lambda$ is smooth over $U_\lambda$. Let $j_\lambda : U_\lambda \hookrightarrow Y$ be the embedding. Let

$$G_\lambda := (j_\lambda)_*(\mathcal{F}_\lambda|_{U_\lambda}).$$

To prove that $\mathcal{H}$ is ind-smooth, it is enough to prove that

1. every $G_\lambda$ is a smooth $E$-sheaf over $Y$

$$\lim_{\lambda \in \Omega} \mathcal{F}_\lambda = \lim_{\lambda \in \Omega} G_\lambda$$

First we prove (1): the adjunction morphism $\text{Id} \to (j_\lambda)_*(j_\lambda)^*$ induces a morphism

$$\mathcal{F}_\lambda \to (j_\lambda)_*(j_\lambda)^*\mathcal{F}_\lambda = G_\lambda$$

Taking limit, we deduce a morphism $\lim \mathcal{F}_\lambda \to \lim G_\lambda$. Let $\eta_Y$ be the generic point of $Y$ and $\overline{y}$ a geometric point over $\eta_Y$. For any geometric point $\overline{y}$ of $Y$ and any
specialization map \( sp_\eta : \eta \to \eta \), we have a commutative diagram

\[
\begin{array}{ccc}
\lim_{\eta} \tilde{G}_\lambda |_{\eta} & \xrightarrow{sp_\eta} & \lim_{\eta} \tilde{F}_\lambda |_{\eta} \\
\downarrow & & \downarrow \sim \\
\lim_{\eta} G_\lambda |_{\eta} & \xrightarrow{sp_\eta} & \lim_{\eta} G_\lambda |_{\eta}
\end{array}
\]

By the hypothesis and (1.1), the upper line of (1.3) is an isomorphism. By the definition of \( G_\lambda \), the right vertical line of (1.3) is an isomorphism. Thus the lower line of (1.3) is surjective.

Note that for every \( \lambda \), the morphism \( sp_\eta : G_\lambda |_{\eta} \to G_\lambda |_{\eta} \) is injective. In fact, by definition we have

\[
G_\lambda |_{\eta} = \Gamma(Y(\eta), G_\lambda) = \Gamma(Y(\eta), (j_\lambda)_* (\tilde{F}_\lambda |_{U_\lambda}))) = \Gamma(Y(\eta) \times_Y U_\lambda, \tilde{F}_\lambda |_{U_\lambda})
\]

where \( Y(\eta) \) is the strict henselization of \( Y \) at \( \eta \). By [SGA1] I Proposition 10.1, since \( Y \) is normal connected, the fiber product \( Y(\eta) \times_Y U_\lambda \) is connected. Since \( \tilde{F}_\lambda |_{U_\lambda} \) is smooth, the restriction

\[
\Gamma(Y(\eta) \times_Y U_\lambda, \tilde{F}_\lambda |_{U_\lambda}) \to \tilde{F}_\lambda |_{\eta}
\]

is injective.

Now we want to prove that for any \( \eta \) and any \( sp_\eta \), the induced morphism \( sp_\eta : G_\lambda |_{\eta} \to G_\lambda |_{\eta} \) is surjective. Let \( a \in G_\lambda |_{\eta} \). Since the lower line of (1.3) is surjective, there exists \( \mu \geq \lambda \) and \( b \in G_\mu |_{\eta} \) such that the image of \( b \) by \( G_\mu |_{\eta} \) coincides with the image of \( a \) in \( \lim_{\eta} G_\lambda |_{\eta} \). Note that

\[
G_\mu |_{\eta} = \Gamma(Y(\eta), G_\mu) = \Gamma(Y(\eta) \times_Y U_\mu, \tilde{F}_\mu) = \Gamma(Y(\eta) \times_Y \eta_Y, \tilde{F}_\mu)
\]

where the last equality is because that \( \tilde{F}_\mu |_{U_\mu} \) is smooth. We have a commutative diagram

\[
\begin{array}{ccc}
\Gamma(Y(\eta) \times_Y \eta_Y, \tilde{F}_\mu) & \xrightarrow{=} & \Gamma(Y(\eta) \times_Y \eta_Y, \tilde{F}_\mu) \\
\downarrow \text{rest} & & \downarrow \text{rest} \\
G_\mu |_{\eta} & \xrightarrow{sp_\eta} & G_\mu |_{\eta}
\end{array}
\]

As before, since \( Y \) is normal connected, the fiber product \( Y(\eta) \times_Y \eta_Y \) is connected. Over \( Y(\eta) \times_Y \eta_Y \), \( \tilde{F}_\lambda \) is a smooth subsheaf of the smooth sheaf \( \tilde{F}_\mu \). For \( b \in \Gamma(Y(\eta) \times_Y \eta_Y, \tilde{F}_\mu) \), if \( \text{rest}(b) \) is in \( \tilde{F}_\lambda |_{\eta} \), then \( b \in \Gamma(Y(\eta) \times_Y \eta_Y, \tilde{F}_\lambda) = G_\lambda |_{\eta} \). We deduce that \( sp_\eta(b) = a \). Thus \( sp_\eta : G_\lambda |_{\eta} \to G_\lambda |_{\eta} \) is surjective.

By [1.1.3] we deduce that \( G_\lambda \) is a smooth \( E \)-sheaf over \( Y \).
Now we prove (2): by definition, for all \( \lambda_1 \leq \lambda_2 \), the morphism \( \tilde{\mathcal{F}}_{\lambda_1} \to \tilde{\mathcal{F}}_{\lambda_2} \) is injective. So the morphism \( \mathcal{G}_{\lambda_1} \mid_{\eta_Y} \to \mathcal{G}_{\lambda_2} \mid_{\eta_Y} \) is injective. Since \( \mathcal{G}_{\lambda_i} \) are smooth, we deduce that the morphism \( \mathcal{G}_{\lambda_1} \to \mathcal{G}_{\lambda_2} \) is injective.

We proved that for any \( \lambda \), the morphism \( \text{sp}^*_{\mathcal{G}_{\lambda}} : \mathcal{G}_{\lambda} \mid_{\eta_Y} \to \mathcal{G}_{\lambda} \mid_{\eta_Y} \) is injective. Since \( \mathcal{G}_{\lambda_1} \) and \( \mathcal{G}_{\lambda_2} \) are smooth, we deduce that the morphism \( \mathcal{G}_{\lambda_1} \to \mathcal{G}_{\lambda_2} \) is injective.

We deduced that the lower line of (1.3) is injective. So the lower line of (1.3), thus the left vertical line, is an isomorphism.

For any \( \lambda \), we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{F}}_{\lambda} \mid_{\eta_Y} & \xleftarrow{\text{lim}} & \tilde{\mathcal{F}}_{\lambda} \mid_{\eta_Y} \\
\downarrow & & \downarrow \\
\mathcal{G}_{\lambda} \mid_{\eta_Y} & \xleftarrow{\text{lim}} & \mathcal{G}_{\lambda} \mid_{\eta_Y}
\end{array}
\]

We deduced that \( \tilde{\mathcal{F}}_{\lambda} \mid_{\eta_Y} \to \mathcal{G}_{\lambda} \mid_{\eta_Y} \) is injective. Since this is true for any \( \eta_Y \), we deduce that (1.2) is injective.

Now fix \( \lambda \). For any \( \mu \), consider the subset of \( Y \)

\[
C_{\mu} := \{ y \in Y \text{ such that } \mathcal{G}_{\lambda} \mid_{\eta_Y} \not\subseteq \text{Im}(\tilde{\mathcal{F}}_{\mu} \mid_{\eta_Y} \to \mathcal{G}_{\mu} \mid_{\eta_Y}) \}
\]

It is constructible. For any \( \mu_1 \leq \mu_2 \), we have \( C_{\mu_1} \supset C_{\mu_2} \). We have \( \cap_{\mu} C_{\mu} = \emptyset \). We deduce that there exists \( \mu'(\lambda) \), such that for any \( \mu \geq \mu'(\lambda) \), we have \( C_{\mu} = \emptyset \). In particular, \( \mathcal{G}_{\lambda} \subseteq \text{Im}(\tilde{\mathcal{F}}_{\mu'(\lambda)} \to \mathcal{G}_{\mu'(\lambda)}) \). Thus for any \( \lambda \), we have

\[
\tilde{\mathcal{F}}_{\lambda} \subseteq \mathcal{G}_{\lambda} \subseteq \tilde{\mathcal{F}}_{\mu'(\lambda)}
\]

This implies

\[
\lim_{\lambda \in \Omega} \tilde{\mathcal{F}}_{\lambda} = \lim_{\lambda \in \Omega} \mathcal{G}_{\lambda}.
\]

1.1.7. Let \( Y \) be a normal connected noetherian scheme. Let \( \eta_Y \) be the generic point of \( Y \) and \( \eta_Y \) a geometric point over \( \eta_Y \). Then the functor \( \mathcal{F} \mapsto \mathcal{F} \mid_{\eta_Y} \) from the category of smooth \( E \)-sheaves over \( Y \) to the category of finite dimensional \( E \)-vector spaces with continuous \( \pi_1(Y, \eta_Y) \)-actions is an equivalence. (cf. [SGA5] VI 1.)

Let \( \mathcal{H} = \lim \mathcal{F}_{\lambda} \) be an inductive limit of smooth \( E \)-sheaves \( \mathcal{F}_{\lambda} \) over \( Y \). Replacing \( \mathcal{F}_{\lambda} \) by \( \tilde{\mathcal{F}}_{\lambda} \) as in 1.1.6 we can suppose that for all \( \lambda_1 \leq \lambda_2 \), the morphism \( \mathcal{F}_{\lambda_1} \mid_{\eta_Y} \to \mathcal{F}_{\lambda_2} \mid_{\eta_Y} \) is injective.

Suppose that the action of \( \pi_1(Y, \eta_Y) \) on \( \mathcal{H} \mid_{\eta_Y} \) is trivial. Then for every \( \lambda \) the action of \( \pi_1(Y, \eta_Y) \) on \( \mathcal{F}_{\lambda} \mid_{\eta_Y} \) is trivial. Since \( \mathcal{F}_{\lambda} \) is a smooth \( E \)-sheaf, by the above equivalence every \( \mathcal{F}_{\lambda} \) is a constant sheaf. Thus \( \mathcal{H} \) is a constant sheaf.

Remark 1.1.8. In this paper, we use the étale topology. One can also use the pro-étale topology instead of the étale topology. In the context of pro-étale topology, Lemma 1.1.3 would be more obvious.
1.2. Reminders on partial Frobenius morphisms.

1.2.1. In this paper, $\times_{\mathbb{F}_q}$ means $\times_{\text{Spec} \, \mathbb{F}_q}$ and $\times_{\overline{\mathbb{F}_q}}$ means $\times_{\text{Spec} \, \overline{\mathbb{F}_q}}$.

1.2.2. We denoted by $F$ the function field of $X$. Fix an algebraic closure $\overline{F}$ of $F$ and an embedding $\mathbb{F}_q \subset \overline{F}$. We denote by $\eta$ the generic point of $X$ and by $\overline{\eta}$ the geometric point over $\eta$.

Let $I$ be a finite set. We denote by $X^I := X \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X$ ($I$ copies). We denote by $F^I$ the function field of $X^I$. Fix an algebraic closure $\overline{F^I}$ of $F^I$. We denote by $\overline{\eta}$ the generic point of $X^I$ and by $\overline{\eta}^I$ the geometric point over $\overline{\eta}$.

Our notation is slightly different from [Laf18], where the function field of $X^I$ is denoted by $F^I$ and the generic point of $X^I$ is denoted by $\eta^I$. We will never use the notations $F^I$ and $\eta^I$ in this paper.

Notation 1.2.3. We denote by $(\eta)^I := \eta \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \eta$ and $(\overline{\eta})^I := \overline{\eta} \times_{\overline{\mathbb{F}_q}} \cdots \times_{\overline{\mathbb{F}_q}} \overline{\eta}$.

1.2.4. Note that $(\overline{\eta})^I$ is an integral scheme over $\text{Spec} \, \mathbb{F}_q$. In fact, we have

$$(\overline{\eta})^I = \lim_{\leftarrow} Y \otimes_{\mathbb{F}_q} \cdots \otimes_{\mathbb{F}_q} Y$$

where the projective limit is over affine étale $Y = \text{Spec} \, A$ over $X_{\overline{\mathbb{F}_q}}$. By hypothesis $X_{\overline{\mathbb{F}_q}}$ is irreducible. Every $A$ is an integral domain which is finitely generated as $\mathbb{F}_q$-algebra, thus the product $A \otimes_{\mathbb{F}_q} \cdots \otimes_{\mathbb{F}_q} A$ is still an integral domain (see for example [StacksProject] Part 2, Lemma 33.3.3). We deduce that $\overline{F} \otimes_{\overline{\mathbb{F}_q}} \cdots \otimes_{\overline{\mathbb{F}_q}} \overline{F}$ is an integral domain.

As a consequence, $\overline{\eta}^I$ is a geometric generic point of $(\overline{\eta})^I$.

1.2.5. For any scheme $Y$ over $\mathbb{F}_q$, we denote by $\text{Frob} : Y \rightarrow Y$ the Frobenius morphism over $\mathbb{F}_q$.

For any subset $J \subset I$, we denote by

$$\text{Frob}_J : X^I \rightarrow X^I$$

the morphism sending $(x_i)_{i \in I}$ to $(x_i')_{i \in I}$, with $x_i' = \text{Frob}(x_i)$ if $i \in J$ and $x_i' = x_i$ if $i \notin J$.

In particular, for $J = \{i\}$ a singleton, we have the morphism

$$\text{Frob}_{\{i\}} : X^I \rightarrow X^I.$$  

1.2.6. As in [Laf18] Remarque 8.18, we define

$$\text{FWeil}(\eta^I, \overline{\eta}^I) := \{ \varepsilon \in \text{Aut}_{\overline{\mathbb{F}_q}}(\overline{F^I}) \mid \exists (n_i)_{i \in I} \in \mathbb{Z}^I, \varepsilon \big|_{(F^I)^\text{perf}} = \prod_{i \in I} (\text{Frob}_{\{i\}})^{n_i} \}.$$  

where $(F^I)^\text{perf}$ is the perfection of $F^I$. 

1.2.7. \textbf{[Laf18 Remarque 8.18]} Let \( \Delta : X \to X^I \) be the diagonal inclusion. We fix a specialization map in \( X^I \)

\[ \text{sp : } \eta_I \to \Delta(\eta) \]

i.e. a morphism from \( \eta_I \) to the strict henselization of \( X^I \) at \( \Delta(\eta) \). In particular, for every étale neighbourhood \( U \) of \( \Delta(\eta) \) in \( X^I \), the specialization map \( \text{sp} \) induces a \( X^I \)-morphism \( \eta_I \to U \). The scheme \( (\eta)_I \) equipped with the diagonal morphism \( \Delta(\eta) \to (\eta)_I \) is a projective limit of étale neighbourhoods of \( \Delta(\eta) \) in \( X^I \) of the form \( V^I \) with \( V \) étale neighbourhood of \( \eta \) in \( X \). We deduce a morphism

\[ \eta_I \to (\eta)_I, \]

i.e. an inclusion \( (1.6) \)

\[ F \otimes_{F_q} \cdots \otimes_{F_q} F \subset F_I. \]

Let \( \varepsilon \in \text{FWeil}(\eta_I, \eta) \). For any \( i \in I \), let \( \varepsilon_i \) be the restriction of \( \varepsilon \) to the algebraic closure in \( F_I \) of \( F_q \otimes_{F_q} \cdots \otimes_{F_q} F \otimes_{F_q} \cdots \otimes_{F_q} F \), where \( F \) is the \( i \)-th factor. We have a surjective morphism

\[ \Psi : \text{FWeil}(\eta_I, \eta) \to \text{Weil}(\eta, \eta)_I \]

\[ \varepsilon \mapsto ((\text{Frob}_{(i)})^{-n_i} \circ \varepsilon_i)_{i \in I}. \]

We have exact sequences:

\[ 0 \to \pi_1^{\text{geom}}(\eta^I, \eta^I) \to \text{FWeil}(\eta^I, \eta^I) \to \mathbb{Z}^I \to 0 \]

\[ 0 \to \pi_1^{\text{geom}}(\eta, \eta)_I \to \text{Weil}(\eta, \eta)_I \to \mathbb{Z}^I \to 0 \]

1.2.8. Let \( \mathcal{G} \) be an ind-constructible \( E \)-sheaf over \( (\eta)_I \), equipped with an action of the partial Frobenius morphisms, i.e. equipped with isomorphisms \( F_{(i)} : \text{Frob}_{(i)}^* \mathcal{G} \to \mathcal{G} \) commuting to each other and whose composition is the total Frobenius isomorphism \( \text{Frob}^* \mathcal{G} \to \mathcal{G} \) over \( (\eta)_I \).

Then \( \mathcal{G}|_{\eta_I} \) is equipped with an action of \( \text{FWeil}(\eta_I, \eta_I) \) in the following way: for any \( \varepsilon \in \text{FWeil}(\eta_I, \eta_I) \) with \( \varepsilon|_{(F_I)^{\text{perf}}} = \prod_{i \in I} (\text{Frob}_{(i)})^{n_i} \), it induces a commutative diagram

\[ \begin{array}{ccc}
F_I & \xrightarrow{\varepsilon} & F_I \\
\downarrow & & \downarrow \\
(F_I)^{\text{perf}} & \xrightarrow{\prod_{i \in I} (\text{Frob}_{(i)})^{n_i}} & (F_I)^{\text{perf}}
\end{array} \]

In other words, a commutative diagram

\[ (1.7) \]

\[ \begin{array}{ccc}
\eta_I & \xrightarrow{\text{Spec } \varepsilon} & \eta_I \\
\downarrow & & \downarrow \\
\text{Spec}(F_I)^{\text{perf}} & \xrightarrow{\prod_{i \in I} (\text{Frob}_{(i)})^{n_i}} & \text{Spec}(F_I)^{\text{perf}}
\end{array} \]
Moreover, we have a commutative diagram

\[
\begin{array}{ccc}
\text{Spec } \varepsilon & \xrightarrow{\text{Spec } \varepsilon} & \eta_I \\
\downarrow & & \downarrow \\
\text{Spec } \mathbb{F}_q & \xrightarrow{\eta_I} & \eta_I
\end{array}
\]

By (1.7), we deduce a specialization map in \(X^I\):

\[\text{sp}_\varepsilon : \prod_{i \in I} \text{Frob}_{(i)}^n(\eta_I) \to \eta_I\]

which is in fact an isomorphism of schemes over \(\text{Spec}(F_I)^{\text{perf}}\) and over \(\text{Spec } \mathbb{F}_q\).

The action of \(\varepsilon\) on \(\mathcal{G}|_{\eta_I}\) is defined to be the composition:

\[
\mathcal{G}|_{\eta_I} \xrightarrow{\text{sp}_\varepsilon^*} \mathcal{G}|_{\prod_{i \in I} \text{Frob}_{(i)}^n(\eta_I)} = (\prod_{i \in I} (\text{Frob}_{(i)}^n)^* \mathcal{G})|_{\eta_I} \xrightarrow{\prod_{i \in I} F_{(i)}^{n_i}} \mathcal{G}|_{\eta_I}
\]

We deduce an action of \(\text{FWeil}(\eta_I, \eta_I)\).

1.2.9. An action of \(\text{FWeil}(\eta_I, \eta_I)\) on a finite dimensional \(E\)-vector space is said to be continuous if the action of \(\pi_1^{\text{geom}}(\eta_I, \eta_I)\) is continuous.

More generally, an action of \(\text{FWeil}(\eta_I, \eta_I)\) on an \(E\)-vector space \(M\) is said to be continuous if \(M\) is an inductive limit of finite dimensional \(E\)-vector subspaces which are stable under \(\pi_1^{\text{geom}}(\eta_I, \eta_I)\) and on which the action of \(\pi_1^{\text{geom}}(\eta_I, \eta_I)\) is continuous.

We will need the following two variants of a lemma of Drinfeld (these variants are proved by Drinfeld and recalled in [Xue18b]).

**Lemma 1.2.10.** ([Xue18b] Lemma 3.2.10) A continuous action of \(\text{FWeil}(\eta_I, \eta_I)\) on a finite dimensional \(E\)-vector space factors through \(\text{Weil}(\eta, \eta_I)^I\).

**Lemma 1.2.11.** ([Xue18b] Lemma 3.2.13) Let \(A\) be a finitely generated \(E\)-algebra. Let \(M\) be an \(A\)-module of finite type. Then a continuous \(A\)-linear action of \(\text{FWeil}(\eta_I, \eta_I)\) on \(M\) factors through \(\text{Weil}(\eta, \eta_I)^I\).

We will also need

**Lemma 1.2.12.** ([Xue18b] Lemma 3.3.4) Let \(\mathcal{F}\) be a constructible sheaf over \(X^I\), equipped with an action of the partial Frobenius morphisms. Then there exists an open dense subscheme \(U\) of \(X\) such that \(\mathcal{F}\) is smooth over \(U^I\).

1.2.13. We begin by a too simple case to illustrate the case 1.2.14. Let \(\mathcal{G}\) be a constructible \(E\)-sheaf over \((\eta)^I\), equipped with an action of the partial Frobenius morphisms.

Firstly, by Lemma 1.2.12 \(\mathcal{G}\) is smooth over \((\eta)^I\). In particular, \(\mathcal{G}\) is smooth over \((\eta_I)^I\).
Secondly, as in 1.2.8, $\mathcal{G}|_{\eta_I}$ is equipped with a continuous action of $\text{FWeil}(\eta_I, \overline{\eta_I})$. By Lemma 1.2.10, this action factors through $\text{Weil}(\eta, \overline{\eta})$. We deduce that the action of $\text{Weil}((\eta_I), \overline{\eta_I})$ on $\mathcal{G}|_{\eta_I}$ is trivial.

By the equivalence in 1.1.7, we deduce that $\mathcal{G}|_{(\eta_I)}$ is a constant sheaf over $(\eta_I)$.

1.2.14. Let $A$ be a finitely generated $E$-algebra. Let $\mathcal{G}$ be an ind-constructible $E$-sheaf over $(\eta_I)$ equipped with an action of the partial Frobenius morphisms and an action of $A$, such that

- these two actions commute with each other
- $\mathcal{G}|_{\eta_I}$ is an $A$-module of finite type

As in 1.2.8, $\mathcal{G}|_{\eta_I}$ is equipped with a continuous $A$-linear action of $\text{FWeil}(\eta_I, \overline{\eta_I})$. By Lemma 1.2.11, this action factors through $\text{Weil}(\eta, \overline{\eta})$. Thus the action of $\text{Weil}((\eta_I), \overline{\eta_I})$ on $\mathcal{G}|_{\eta_I}$ is trivial.

However, in general $\mathcal{G}$ may not be ind-smooth over $(\eta_I)$. For example, let $I = \{1, 2\}$ and $\mathcal{E}_n$ be the extension by zero of the constant sheaf $\mathcal{E}$ over $\text{Frob}^i_{\{1\}}(\Delta)$, where $\Delta$ is the image of the diagonal morphism $X \hookrightarrow X^2$. Then $\mathcal{G} = \oplus_{n \in \mathbb{Z}} \mathcal{E}_n$ satisfies the above condition, but is not ind-smooth.

1.2.15. In our situation 1.3-1.5 below, we will apply 1.2.14 to the cohomology sheaves of stacks of shtukas, and prove that for other reasons the cohomology sheaves are ind-smooth over $(\eta_I)$. Then they are constant sheaves over $(\overline{\eta})$.

1.3. Action of $\text{Weil}(\eta, \overline{\eta})$ on cohomology sheaves of stacks of shtukas.

1.3.1. Let $W \in \text{Rep}_E(\hat{G}^I)$, where $\text{Rep}_E(\hat{G}^I)$ denotes the category of finite dimensional $E$-linear representations of $\hat{G}^I$.

As in [Laf18 Définition 4.7 and Xue18a Section 2.5], let $\text{Cht}_{G,N,I,W}/\Xi$ be the stack of $G$-shtukas, $\mathcal{F}_{G,N,I,W}$ be the canonical perverse sheaf over $\text{Cht}_{G,N,I,W}/\Xi$ with $E$-coefficients and $p_G : \text{Cht}_{G,N,I,W}/\Xi \to (X \setminus N)^I$ be the morphism of paws. We denote by $\Lambda^+_G$ the set of dominant coweights of $G^{ad}$. For any $\mu \in \Lambda^+_G$ and any $j \in \mathbb{Z}$, we define the sheaf of degree $j$ cohomology with compact support for the Harder-Narasimhan truncation indexed by $\mu$

$$\mathcal{F}^{\leq \mu}_{G,N,I,W} := R^j(p_G)_! \mathcal{F}_{G,N,I,W}|_{\text{Cht}_{G,N,I,W}/\Xi}$$

It is a constructible $E$-sheaf over $(X \setminus N)^I$. We define the degree $j$ cohomology sheaf

$$\mathcal{H}^{\leq \mu}_{G,N,I,W} := \lim_{\mu \in \Lambda^+_G} \mathcal{F}^{\leq \mu}_{G,N,I,W}$$

in the abstract category of inductive limits of constructible $E$-sheaves over $(X \setminus N)^I$.

1.3.2. In the following, to shorten the notation, we omit the indices $G$ and $N$, i.e. we write $\mathcal{H}^{\leq \mu}_{I,W}$ for $\mathcal{H}_{G,N,I,W}^{\leq \mu}$ and $\mathcal{H}^j_{I,W}$ for $\mathcal{H}_{G,N,I,W}^j$.
1.3.3. As in [Laf18] Section 4.3, there exists $\kappa \in \Lambda^+_G\text{-gal}$ big enough (depending on $W$) such that for any $i \in I$ and any $\mu$, we have morphisms of constructible $E$-sheaves over $(X \setminus N)^I$:

$$F_{\{i\}} : \text{Frob}_{\{i\}}^* \mathcal{H}_{I,W}^{j,\leq \mu} \to \mathcal{H}_{I,W}^{j,\leq \mu + \kappa}$$

where $\text{Frob}_{\{i\}}$ is defined in [1.2.5]. The composition for all $i \in I$ is the total Frobenius morphism followed by an increase of the Harder-Narasimhan truncation:

$$\text{Frob}_{\{i\}}^* \mathcal{H}_{I,W}^{j,\leq \mu} \to \mathcal{H}_{I,W}^{j,\leq \mu + (H)^{\kappa}}$$

Taking inductive limit, we have canonical morphisms

$$F_{\{i\}} : \text{Frob}_{\{i\}}^* \mathcal{H}_{I,W}^{j,\leq \mu} \to \mathcal{H}_{I,W}^j$$

whose composition is the total Frobenius morphism

$$\text{Frob}_{\{i\}}^* \mathcal{H}_{I,W}^{j,\leq \mu} \to \mathcal{H}_{I,W}^j.$$

Hence $\mathcal{H}_{I,W}^j$ is equipped with an action of the partial Frobenius morphisms.

As a consequence, as in [1.2.8] $\mathcal{H}_{I,W}^j \mid_{\eta}$ is equipped with a continuous action of $\text{FWeil}(\eta, \overline{\eta})$.

**Proposition 1.3.4.** The action of $\text{FWeil}(\eta, \overline{\eta})$ on $\mathcal{H}_{I,W}^j \mid_{\eta}$ factors through $\text{Weil}(\eta, \overline{\eta})^I$.

**Remark 1.3.5.** Proposition 1.3.4 is already proved in [Xue18b] by using the main result in loc.cit. that $\mathcal{H}_{I,W}^j \mid_{\eta}$ is of finite type as a module over a Hecke algebra.

In the following, we give a new proof of Proposition 1.3.4, which follows the arguments of [Laf18] Proposition 8.27. The proof uses a weaker result (Lemma 1.3.7 below) than [Xue18b]. The advantage of this proof is that it is easily generalised to not necessarily split groups in Section 6.

To prove Proposition 1.3.4, we begin by some preparations.

1.3.6. For any family $(v_i)_{i \in I}$ of closed points of $X$, we denote by $\times_{i \in I} v_i$ their product over $\text{Spec} \mathbb{F}_q$. This is a finite union of closed points of $X'$.

**Lemma 1.3.7.** ([Laf18] Proposition 7.1) Let $W = \bigotimes_{i \in I} W_i$ with $W_i \in \text{Rep}_E(\hat{G})$. Let $(v_i)_{i \in I}$ be a family of closed points of $X \setminus N$. Then there exists $\kappa$, such that for any $\mu$ and any $i \in I$, we have

$$\sum_{\alpha=0}^{\dim W_i} (-1)^\alpha S_{\Lambda_{\dim W_i},\alpha}(F_{\{i\}}^{\deg(v_i)})^\alpha = 0 \text{ in } \text{Hom}(\mathcal{H}_{I,W}^{j,\leq \mu} \mid_{\times_{i \in I} v_i}, \mathcal{H}_{I,W}^{j,\leq \mu + \kappa} \mid_{\times_{i \in I} v_i})$$

where $S_{\Lambda_{\dim W_i},\alpha}(F_{\{i\}}^{\deg(v_i)})$ is defined in [Laf18] Section 6, and we restrict it to $\mathcal{H}_{I,W}^{j,\leq \mu} \mid_{\times_{i \in I} v_i}$. □

1.3.8. For any place $v$ of $X \setminus N$, we denote by $\mathcal{O}_v$ the complete local ring at $v$ and $F_v$ its field of fractions. Let $\mathcal{H}_{G,v} := C_c(G(\mathcal{O}_v)) \setminus G(F_v)/G(\mathcal{O}_v), E)$ be the Hecke algebra of $G$ at the place $v$. 
1.3.9. ([Laf18] Section 4.4) Let
\[
T(h_{\lambda^{\dim W_i - \alpha W_i, v_i}}) : \mathcal{H}^i_{I,W}(X \smallsetminus (N \cup v_i)^I) \rightarrow \mathcal{H}^i_{I,W}(X \smallsetminus (N \cup v_i)^I)
\]
be the Hecke operator in $\mathcal{H}_{G,v_i}$ defined by Hecke correspondence.

**Lemma 1.3.10.** ([Laf18] Proposition 6.2) The operator $S_{\lambda^{\dim W_i - \alpha W_i, v_i}}$, which is a morphism of sheaves over $(X \smallsetminus N)^I$, extends the action of the Hecke operator $T(h_{\lambda^{\dim W_i - \alpha W_i, v_i}}) \in \mathcal{H}_{G,v_i}$, which is a morphism of sheaves over $(X \smallsetminus (N \cup v_i))^I$.

The combination of Lemma 1.3.7 and Lemma 1.3.10 is called the Eichler-Shimura relations.

We will use Lemma 1.3.7 and Lemma 1.3.10 to prove the following Lemma 1.3.11.

**Lemma 1.3.11.** $\mathcal{H}^j_{I,W}|_{\mathfrak{m}}$ is an increasing union of $E$-vector subspaces $\mathcal{M}$ which are stable by $\text{FWeil}(\eta_I, \mathfrak{m})$, and for which there exists a family $(\mathcal{M}_i)_{i \in I}$ of closed points in $X \smallsetminus N$ (depending on $\mathcal{M}$) such that $\mathcal{M}$ is stable under the action of $\otimes_{i \in I} \mathcal{H}_{G,v_i}$ and is of finite type as module over $\otimes_{i \in I} \mathcal{H}_{G,v_i}$.

**Proof.** Since the category $\text{Rep}_E(\hat{G}^I)$ is semisimple, it is enough to prove the lemma for $W$ irreducible, which is of the form $W = \bigotimes_{i \in I} W_i$ with $W_i \in \text{Rep}_E(\hat{G})$ (after increasing $E$).

For any $\mu \in \hat{A}_G^+$, we choose a dense open subscheme $\Omega$ of $(X \smallsetminus N)^I$ such that $\mathcal{H}^j_{I,W}|_{\Omega}$ is smooth. We choose a closed point $v$ of $\Omega$. Let $v_i$ be the image of $v$ under $(X \smallsetminus N)^I \overset{pr_i}{\to} X \smallsetminus N$, where $pr_i$ is the projection to the $i$-th factor. Then $\times_{i \in I} v_i$ is a finite union of closed points containing $v$. Let $\mathcal{M}_\mu$ be the image of

\[
\sum_{(n_i)_{i \in I} \in \mathbb{N}^I} (\otimes_{i \in I} \mathcal{H}_{G,v_i}) \cdot \left( \prod_{i \in I} F^{n_i}_{t_i} \left( \prod_{i \in I} \text{Frob}^{n_i}_{t_i} \right) \right) \mathcal{H}^j_{I,W}|_{\mathfrak{m}}
\]
in $\mathcal{H}^j_{I,W}|_{\mathfrak{m}}$. We have $\text{Im}(\mathcal{H}^j_{I,W}|_{\mathfrak{m}}) \subset \mathcal{M}_\mu$ and

\[
\mathcal{H}^j_{I,W}|_{\mathfrak{m}} = \bigcup_\mu \mathcal{M}_\mu.
\]

By definition, $\mathcal{M}_\mu$ is stable under the action of the partial Frobenius morphisms and the action of $\text{Weil}(\eta_I, \mathfrak{m})$, so is stable by $\text{FWeil}(\eta_I, \mathfrak{m})$. We only need to prove that $\mathcal{M}_\mu$ is of finite type as $\otimes_{i \in I} \mathcal{H}_{G,v_i}$-module.

We fix a geometric point $\overline{v}$ over $v$ and a specialization map $\text{sp}_v : \eta_I \to \overline{v}$. For any $n_i$, since

\[
F^{\deg(v_i)n_i}_{t_i} : (\text{Frob}^{\deg(v_i)n_i}_{t_i})^* \mathcal{H}^j_{I,W} \to \mathcal{H}^j_{I,W}
\]
is a morphism of sheaves, the specialization map $\text{sp}_v$ induces a commutative diagram

$$
\begin{array}{ccc}
(F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}} & \xrightarrow{\text{sp}_v^*} & (F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}} \\
\downarrow F_{\{i\} \deg(v_i) n_i} & & \downarrow F_{\{i\} \deg(v_i) n_i} \\
\mathcal{H}_{i,W}^{\leq \mu} & \xrightarrow{\text{sp}_v^*} & \mathcal{H}_{i,W}^{\leq \mu}
\end{array}
$$

The upper line of (1.12) is an isomorphism because $(F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}}$ is smooth over $(F_{\{i\} \deg(v_i) n_i})^{-1}\Omega$. Note that

$$(F_{\{i\} \deg(v_i) n_i}(v) = v \in \Omega
$$

thus $v \in (F_{\{i\} \deg(v_i) n_i})^{-1}\Omega$.

By Lemma 1.3.7, for each $i \in I$, we have

$$
\sum_{\alpha=0}^{\dim W_i} (-1)^\alpha S_{\lambda^{\dim W_i - \alpha}} W_i, v_i (F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_\alpha = 0 \quad \text{in } \text{Hom}(\mathcal{H}_{I,W}^{\leq \mu}, \mathcal{H}_{I,W}^{\leq \mu})
$$

We deduce that

$$
F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i} W_i \left((F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}} \right)
\subset \sum_{\alpha=0}^{\dim W_i - 1} S_{\lambda^{\dim W_i - \alpha}} W_i, v_i F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i} \left((F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}} \right)
$$

Since $S_{\lambda^{\dim W_i - \alpha}} W_i, v_i$ and $F_{\{i\}}$ are morphisms of sheaves, they commute with $\text{sp}_v^*$.

We have

$$
F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i} W_i \left(\text{sp}_v^* (F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}} \right)
\subset \sum_{\alpha=0}^{\dim W_i - 1} S_{\lambda^{\dim W_i - \alpha}} W_i, v_i F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i} \left(\text{sp}_v^* (F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}} \right)
$$

Since the upper line of (1.12) is an isomorphism, we deduce that

$$
F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i} W_i \left((F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}} \right)
\subset \sum_{\alpha=0}^{\dim W_i - 1} S_{\lambda^{\dim W_i - \alpha}} W_i, v_i F_{\{i\} \deg(v_i) n_i}^{\deg(v_i) n_i} \left((F_{\{i\} \deg(v_i) n_i})_{\mathcal{H}_{i,W}^{\leq \mu}} \right)
$$

By [Laf18], the action of the partial Frobenius morphisms commute with the action of Hecke algebras. Moreover, by Lemma 1.3.10 $S_{\lambda^{\dim W_i - \alpha}} W_i, v_i$ acts over $\mathcal{H}_{i,W}^{\leq \mu}$.
by an element of $H_{G,v_i}$. We deduce that

$$\text{RHS of } (1.13) = \sum_{\alpha=0}^{\dim W_i - 1} F^{\deg(v_i)\alpha} \left( (\text{Frob}^{\deg(v_i)\alpha})^* (S_{\alpha}^{\dim W_i - \alpha} W_i, v_i H_{I,W}) \right)$$

(1.14)

We deduce from (1.13) and (1.14) that $M_\mu$ is equal to the image of (1.15)

$$\sum_{(n_i)_{i\in I}\in \prod_{i\in I} \{0,1,\ldots,\deg(v_i),\dim W_i - 1\}} \left( \otimes_{i\in I} H_{G,v_i} \cdot \otimes_{i\in I} F^{n_i} \right) \left( \left( \otimes_{i\in I} \text{Frob}^{n_i}_{\{i\}} \right)^* \mathcal{H}_{I,W}^{j,\leq \mu} \right)$$

in $\mathcal{H}_{I,W}^{j,\mu}_{I,W}$. Thus $M_\mu$ is of finite type as $\otimes_{i\in I} H_{G,v_i}$-module. \hfill \Box

Proof of Proposition 1.3.4. For every $\mu$, applying Lemma 1.2.11 to $A = \otimes_{i\in I} H_{G,v_i}$ and $M = M_\mu$ (which is possible because of Lemma 1.3.11), we deduce that the action of $\text{Weil}(\eta_I, \eta_I)$ on $M_\mu$ factors through $\text{Weil}(\eta, \eta)$. Since $\mathcal{H}_{I,W}^{j,\mu}_{I,W}$ is an inductive limit of $M_\mu$, we deduce Proposition 1.3.4. \hfill \Box

1.4. Smoothness of cohomology sheaves over $(\eta)^I$.

1.4.1. In [Laf18] Proposition 8.32, V. Lafforgue proved that for a specialization map $\eta_I \to \Delta(\eta)$, the induced morphism

$$\mathcal{H}_{I,W}^{j,\mu}_{I,W} \mid_{\Delta(\eta)} \to \mathcal{H}_{I,W}^{j,\mu}_{I,W} \mid_{\eta_I}$$

is injective. In fact, the same argument gives a more general result: the injectivity of morphism (1.16) below.

Moreover, a similar argument as [Laf18] Proposition 8.31 gives a more general result: the surjectivity of morphism (1.16) below.

1.4.2. Let $\overline{\eta}$ be a geometric point of $(\eta)^I$. We fix a specialization map

$$\text{sp}_{\overline{\eta}} : \eta_I \to \overline{\eta}$$

It induces a morphism

(1.16)

$$\text{sp}_{\overline{\eta}} : \mathcal{H}_{I,W}^{j,\mu}_{I,W} \mid_{\overline{\eta}} \to \mathcal{H}_{I,W}^{j,\mu}_{I,W} \mid_{\eta_I}$$

Proposition 1.4.3. The morphism (1.16) is an isomorphism.

To prove Proposition 1.4.3, we need the following lemma.

Lemma 1.4.4. ([Lau04] Lemma 9.2.1) Let $x$ be a point of $(\eta)^I$. The set $\{ (\prod_{i\in I} \text{Frob}^{m_i}_{\{i\}})(x) \mid (m_i)_{i\in I} \in \mathbb{N}^I \}$ is Zariski dense in $X^I$.

Proof. In fact, the Zariski closure of this set is a closed subscheme $Z$ of $X^I$, invariant by the partial Frobenius morphisms. If $Z$ is not equal to $X^I$, by Lemma 9.2.1 of [Lau04] (recalled in the proof of Lemme 8.12 of [Laf18]), $Z$ is included in a finite union of vertical divisors (i.e. the inverse image of a closed point by one
of the projections $X^l \to X$). However, the image of $x$ in $X^l$ is not included in any vertical divisor. This is a contradiction. We deduce that $Z = X^l$. \hfill \square

**Proof of Proposition 1.4.3** Since the category $\text{Rep}_E(\widehat{G}^l)$ is semisimple, it is enough to prove the proposition for $W$ irreducible, which is of the form $W = \coprod_{i \in I} W_i$ with $W_i \in \text{Rep}_E(\widehat{G})$ (after increasing $E$).

**Injectivity:** the proof is the same as Proposition 8.32 of [Laf18], except that we replace everywhere $\Delta(\overline{y})$ by $\overline{x}$ and replace everywhere $\Delta(\overline{\nu})$ by $\overline{\nu}$ (defined below). For the convenience of the reader, we briefly recall the proof. Let $a \in \text{Ker}(\text{sp}_{\overline{y}})$. We want to prove that $a = 0$.

There exists $\mu_0$ large enough and $\overline{a} \in \mathcal{H}_{I,W}^{j,\leq \mu_0}|_{\overline{x}}$, such that $a$ is the image of $\overline{a}$ in $\mathcal{H}_{I,W}^j|_{\overline{x}}$. We denote by $x$ the image of $\overline{x}$ in $(X \setminus N)^l$ and $\overline{\{x\}}$ the Zariski closure of $x$. Let $\Omega_0$ be a dense open subscheme of $\overline{\{x\}}$ such that $\mathcal{H}_{I,W}^{j,\leq \mu_0}|_{\Omega_0}$ is smooth.

Let $y$ be a closed point in $\Omega_0$. Let $\overline{y}$ be a geometric point over $y$ and $\text{sp}_y : \overline{x} \to \overline{y}$ a specialization map over $\Omega_0$. We have a commutative diagram

\begin{equation}
\begin{aligned}
\mathcal{H}_{I,W}^{j,\leq \mu_0}|_{\overline{y}} & \xrightarrow{\text{sp}_y^*} \mathcal{H}_{I,W}^{j,\leq \mu_0}|_{\overline{x}} \\
\mathcal{H}_{I,W}^j|_{\overline{y}} & \xrightarrow{\text{sp}_y^*} \mathcal{H}_{I,W}^j|_{\overline{x}}
\end{aligned}
\end{equation}

The upper horizontal morphism is an isomorphism because $\mathcal{H}_{I,W}^{j,\leq \mu_0}|_{\Omega_0}$ is smooth. Thus there exists $\overline{b} \in \mathcal{H}_{I,W}^{j,\leq \mu_0}|_{\overline{y}}$ such that $\overline{a} = \text{sp}_y^*(\overline{b})$. Let $b$ be the image of $\overline{b}$ in $\mathcal{H}_{I,W}^j|_{\overline{x}}$. We have $a = \text{sp}_y^*(b)$.

Let $y_i$ be the image of $y$ by $(X \setminus N)^l \xrightarrow{\nu_{\nu}} X \setminus N$. Then $\cap_{i \in I} y_i$ is a finite union of closed points containing $y$. Let $d_i = \text{deg}(y_i)$. For any $(n_i)_{i \in I} \in \mathbb{N}^l$, we have $\prod_{i \in I} \text{Frob}_{n_i}^{d_i}(\overline{y}) = \overline{y}$. (Note that in general $\prod_{i \in I} \text{Frob}_{n_i}^{d_i}(\overline{x}) \neq \overline{x}$.) We have the partial Frobenius morphism

$$
\prod_{i \in I} F_{d_i,n_i}^{n_i} : \mathcal{H}_{I,W}^j|_{\overline{y}} = (\prod_{i \in I} \text{Frob}_{n_i}^{d_i})^* \mathcal{H}_{I,W}^j|_{\overline{x}} \to \mathcal{H}_{I,W}^j|_{\overline{y}}
$$

Let

$$
b_{(n_i)_{i \in I}} = \prod_{i \in I} F_{d_i,n_i}^{n_i}(b) \in \mathcal{H}_{I,W}^j|_{\overline{x}} \quad \text{and} \quad a_{(n_i)_{i \in I}} = \text{sp}_y^*(b_{(n_i)_{i \in I}}) \in \mathcal{H}_{I,W}^j|_{\overline{x}}
$$

In particular, $b_{(0)_{i \in I}} = b$ and $a_{(0)_{i \in I}} = a$. 


Let \( d = \deg(y) = \text{ppcm}\{d_i\}_{i \in I} \). Note that \( \prod_{i \in I} \text{Frob}_{(i)} \) is the total Frobenius morphism, thus the morphism

\[
\prod_{i \in I} F_{(i)}^{d_n} : \mathcal{H}_{I,W}^j \rightarrow \mathcal{H}_{I,W}^j
\]

is bijective. We have

\[
a_{(n_i \cdot \text{nd/d}_i)}_{i \in I} = \prod_{i \in I} F_{(i)}^{d_n} (a_{(n_i)})_{i \in I}.
\]  

(Laf18) Lemma 8.33 is still true if we replace everywhere \( \Delta(\overline{\eta}) \) by \( \overline{\eta} \) and replace everywhere \( \Delta(\overline{\eta}) \) by \( \overline{\eta} \). Thus we have:

(1) for all \( k \in I \) and for all \( (n_i)_{i \in I} \in \mathbb{N}^I \),

\[
\sum_{\alpha = 0}^{\dim W_k} (-1)^{\alpha} \lambda^{\dim W_k - \alpha} W_k \cdot y_k (a_{(n_i + \alpha \delta_i,k)})_{i \in I} = 0 \quad \text{in} \quad \mathcal{H}_{I,W}^j \bigg|_{\overline{\eta}}.
\]

(2) Let \( \mu_1 \geq \mu_0 \) such that \( \text{sp}^*_\overline{\eta}(\overline{a}) \in \mathcal{H}_{I,W}^{j,\leq \mu_0} \bigg|_{\overline{\eta}} \) has zero image in \( \mathcal{H}_{I,W}^{j,\leq \mu_1} \bigg|_{\overline{\eta}} \). Let \( \Omega_1 \) be a dense open subscheme of \((X \setminus N)^I \) such that \( \mathcal{H}_{I,W}^{j,\leq \mu_1} \bigg|_{\Omega_1} \) is smooth. Then for every \( (m_i)_{i \in I} \in \mathbb{N}^I \) such that \( \prod_{i \in I} \text{Frob}_{(i)}^{d_i m_i} (x) \in \Omega_1 \), we have \( a_{(m_i)}_{i \in I} = 0 \) in \( \mathcal{H}_{I,W}^j \bigg|_{\overline{\eta}} \).

Note that the open subscheme

\[
\bigcap_{(\alpha_i)_{i \in I} \in \mathbb{N}^I \setminus \{0, \ldots, \dim W_i - 1\}} (\prod_{i \in I} \text{Frob}_{(i)}^{d_i \alpha_i})^{-1}(\Omega_1)
\]

is also dense in \( X^I \). By Lemma 1.1.4 there exists \( (N_i)_{i \in I} \in \mathbb{N}^I \) such that \( \prod_{i \in I} \text{Frob}_{(i)}^{d_i N_i} (x) \) is in (1.20). We deduce

\[
\prod_{i \in I} \text{Frob}_{(i)}^{d_i (N_i + \alpha_i)} (x) \in \Omega_1 \quad \text{for all} \quad (\alpha_i)_{i \in I} \in \bigcup_{i \in I} \{0, \ldots, \dim W_i - 1\}.
\]

By (2), we deduce that

\[
a_{(N_i + \alpha_i)}_{i \in I} = 0 \quad \text{for all} \quad (\alpha_i)_{i \in I} \in \prod_{i \in I} \{0, \ldots, \dim W_i - 1\}.
\]

By (1), for every \( k \in I \) and \( (n_i)_{i \in I} \in \mathbb{N}^I \),

\[
a_{(n_i + \dim W_k \delta_i,k)}_{i \in I} = \sum_{\alpha = 0}^{\dim W_k - 1} (-1)^{\alpha + \dim W_k} \lambda^{\dim W_k - \alpha} W_k \cdot y_k (a_{(n_i + \alpha \delta_i,k)})_{i \in I}
\]

Using (1.21) and (1.22), by induction we deduce that

\[
a_{(n_i)}_{i \in I} = 0 \quad \text{for all} \quad (n_i)_{i \in I} \in \mathbb{N}^I \quad \text{such that} \quad n_i \geq N_i, \forall i \in I.
\]

Thus for \( n \geq N_i \) for all \( i \in I \), we have \( a_{(n \cdot \text{nd/d}_i)}_{i \in I} = 0 \). Then (1.18) implies \( a_{(0)}_{i \in I} = 0 \). This proves the injectivity of \( \text{sp}^*_\overline{\eta} \).
**Surjectivity:** the proof is similar as [Laf18] Proposition 8.31. By (1.11), we have \( \mathcal{H}_{I,W}^j|_{\overline{\eta}} = \bigcup \mathcal{M}_\mu \), where \( \mathcal{M}_\mu \) is defined in (1.10). To prove that \( \text{sp}_x^* \) is surjective, it is enough to prove that for every \( \mu \), we have \( \mathcal{M}_\mu \subset \text{Im}(\text{sp}_x^*) \).

There exist \( \mu_2 \) large enough and \( \omega_1, \ldots, \omega_m \in \mathcal{H}_{I,W}^j|_{\overline{\eta}} \) such that \( \omega_1, \ldots, \omega_m \) is a family of generators of \( \mathcal{M}_\mu \) as \( \otimes_{i \in I} \mathcal{H}_{G,v_i} \)-module. Let \( \Omega_2 \) be a dense open subscheme of \( (X \smallsetminus N)^I \) such that \( \mathcal{H}_{I,W}^j|_{\Omega_2} \) is smooth. By Lemma 1.4.4 the set \( \{ (\prod_{i \in I} \text{Frob}_{(i)}^{ni})(x), (m_i)_{i \in I} \in \mathbb{N}^I \} \) is Zariski dense in \( X^I \). We deduce that there exists \( (n_i)_{i \in I} \in \mathbb{N}^I \), such that \( (\prod_{i \in I} \text{Frob}_{(i)}^{ni})(x) \in \Omega_2 \).

Let the specialization map

\[
(\prod_{i \in I} \text{Frob}_{(i)}^{ni}) \text{sp}_x : (\prod_{i \in I} \text{Frob}_{(i)}^{ni})(\overline{\eta}) \to (\prod_{i \in I} \text{Frob}_{(i)}^{ni})(\overline{x})
\]

be the image of \( \text{sp}_x \) by \( \prod_{i \in I} \text{Frob}_{(i)}^{ni} \). It induces a morphism

\[
((\prod_{i \in I} \text{Frob}_{(i)}^{ni}) \text{sp}_x)^* : \mathcal{H}_{I,W}^j|_{\overline{\eta}} \to \mathcal{H}_{I,W}^j|_{(\prod_{i \in I} \text{Frob}_{(i)}^{ni})(\overline{\eta})}
\]

This is an isomorphism because \( \mathcal{H}_{I,W}^j|_{\Omega_2} \) is smooth. In particular,

\[
\mathcal{H}_{I,W}^j|_{(\prod_{i \in I} \text{Frob}_{(i)}^{ni})(\overline{\eta})} \subset \text{Im}((\prod_{i \in I} \text{Frob}_{(i)}^{ni}) \text{sp}_x)^*.
\]

Since the action of the Hecke algebra is given by morphisms of sheaves, it commutes with \((\prod_{i \in I} \text{Frob}_{(i)}^{ni}) \text{sp}_x)^* \). We deduce that

(1.23) \( \otimes_{i \in I} \mathcal{H}_{G,v_i} \cdot \mathcal{H}_{I,W}^j|_{(\prod_{i \in I} \text{Frob}_{(i)}^{ni})(\overline{\eta})} \subset \text{Im}((\prod_{i \in I} \text{Frob}_{(i)}^{ni}) \text{sp}_x)^* \).

As in the proof of [Laf18] Proposition 8.31. we have a commutative diagram

(1.24)

\[
\begin{array}{ccc}
\mathcal{H}_{I,W}^j|_{(\prod_{i \in I} \text{Frob}_{(i)}^{ni})(\overline{\eta})} & \xrightarrow{((\prod_{i \in I} \text{Frob}_{(i)}^{ni}) \text{sp}_x)^*} & \mathcal{H}_{I,W}^j|_{(\prod_{i \in I} \text{Frob}_{(i)}^{ni})(\overline{\eta})} \\
\cong \prod_{i \in I} \mathcal{F}_{(i)}^{n_i} & \cong \prod_{i \in I} \mathcal{F}_{(i)}^{n_i} & \cong \prod_{i \in I} \mathcal{F}_{(i)}^{n_i} \\
\mathcal{H}_{I,W}^j|_{\overline{x}} & \xrightarrow{\text{sp}_x^*} & \mathcal{H}_{I,W}^j|_{\overline{\eta}} \\
\end{array}
\]

where \( \mathcal{M}_\mu' := (\prod_{i \in I} \mathcal{F}_{(i)}^{n_i})^{-1} \mathcal{M}_\mu \) is the inverse image of \( \mathcal{M}_\mu \). Note that \( \mathcal{M}_\mu \) is stable under the action of \( \text{FWeil}(\overline{\eta}, \overline{\eta}) \) (recall that the action is defined in (1.28)). Thus any choice of isomorphism \( \beta \) between \((\prod_{i \in I} \text{Frob}_{(i)}^{ni})(\overline{\eta})\) and \( \overline{\eta} \) identifies \( \mathcal{M}_\mu' \) and...
Let $\bar{\omega}_1, \ldots, \bar{\omega}_m \in \mathcal{H}_{I, W}^{\leq \mu_2}|_{(\prod_{i \in I} \text{Frob}^n_{(i)})((\bar{\eta})^I)}$ denote the images of $\omega_1, \ldots, \omega_m$ by $\beta^*$.

Then $\bar{\omega}_1, \ldots, \bar{\omega}_m$ is a family of generators of $\mathcal{M}_\mu'$ as $\otimes_{i \in I} H_{G, v_i}$-module. We deduce that

$$\mathcal{M}_\mu' \subset (\otimes_{i \in I} H_{G, v_i}) \cdot \mathcal{H}_{I, W}^{\leq \mu_2}|_{(\prod_{i \in I} \text{Frob}^n_{(i)})((\bar{\eta})^I)}.$$

By (1.23), $\mathcal{M}_\mu' \subset \text{Im}((\prod_{i \in I} \text{Frob}^n_{(i)})\text{sp}_\tau)^*$. We deduce from (1.24) that $\mathcal{M}_\mu \subset \text{Im}((\text{sp}_\tau)^*)$.

\[ \Box \]

1.5. The cohomology sheaves are constant over $(\eta)^I$.

**Proposition 1.5.1.** $\mathcal{H}_{I, W}^{\leq \mu_2}|_{(\eta)^I}$ is a constant sheaf over $(\eta)^I$.

**Proof.** By Proposition 1.4.3, the sheaf $\mathcal{H}_{I, W}^{\leq \mu_2}|_{(\eta)^I}$ is ind-smooth over $(\eta)^I$. By Proposition 1.3.4, the action of Weil($(\eta)^I, (\eta)^I$) on $\mathcal{H}_{I, W}^{\leq \mu_2}|_{(\eta)^I}$ is trivial.

By Lemma 1.1.5 and 1.1.7, we deduce Proposition 1.5.1.

\[ \Box \]

2. The cohomology sheaves are constant over $(\eta)^I \times \overline{\mathbb{F}_q}$

2.0.1. Let $I$ be a disjoint union $I_1 \sqcup I_2$. Let $u$ be a closed point of $(X \setminus N)^I_2$. Let $\overline{u} = \text{Spec} \overline{\mathbb{F}_q}$ be a geometric point over $u$.

In this section, we will generalize the results in Section 1 from $(\eta)^I$ to $(\eta)^I \times \overline{\mathbb{F}_q}$. When $I_1 = I$ and $I_2 = \emptyset$, the results in Section 2 recover the results in Section 1.

2.1. Reminders on partial Frobenius morphisms.

2.1.1. Let $\eta_{I_1}$ be the generic point of $X^{I_1}$. Fix a geometric point $\overline{\eta_{I_1}}$ over $\eta_{I_1}$. By (1.2.4), the scheme $(\eta)^I_1 \times \overline{\mathbb{F}_q}$ is an integral scheme. Note that $\overline{\eta_{I_1}} \times \overline{\mathbb{F}_q}$ is a geometric generic point of $(\eta)^I_1 \times \overline{\mathbb{F}_q}$. 
2.1.2. Let $\mathcal{F}$ be an ind-constructor $E$-sheaf over $(\eta)^{I_1} \times_{\bar{\eta}_q} u$, equipped with an action of the partial Frobenius morphisms, i.e. for every $i \in I_1$, an isomorphism $F_{i}^*: \text{Frob}_{i}^{*} \mathcal{F} \simeq \mathcal{F}$ and an isomorphism $F_{I_1}^* : \text{Frob}_{I_1}^{*} \mathcal{F} \simeq \mathcal{F}$, commuting to each other and whose composition is the total Frobenius isomorphism $\text{Frob}^* \mathcal{F} \simeq \mathcal{F}$ over $(\eta)^{I_1} \times_{\bar{\eta}_q} u$.

Then $\mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u}$ is equipped with an action of $\text{FWeil}(\eta_{I_1}, \bar{\eta}_{I_1})$ in the following way: let $\varepsilon \in \text{FWeil}(\eta_{I_1}, \bar{\eta}_{I_1})$ with $\varepsilon|_{(F_{I_1})^{\text{post}}} = \prod_{i \in I_1} (\text{Frob}_{i})^{n_i}$. By Lemma 1.2.8, we have a morphism $\text{sp}_\varepsilon : \prod_{i \in I_1} (\text{Frob}_{i})^{n_i}(\eta_{I_1}) \to \eta_{I_1}$ over $\text{Spec} \bar{\eta}_q$. Thus, tensoring with $\text{Id}_\mathcal{F}$, we deduce a specialization map in $X^I$ (which is in fact an isomorphism of schemes):

$$\text{sp}_\varepsilon \otimes \text{Id}_\mathcal{F} : \prod_{i \in I_1} (\text{Frob}_{i})^{n_i}(\eta_{I_1}) \times_{\bar{\eta}_q} \bar{u} \to \eta_{I_1} \times_{\bar{\eta}_q} \bar{u}$$

The action of $\varepsilon$ is defined to be the composition:

$$\mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u} \xrightarrow{(\text{sp}_\varepsilon \otimes \text{Id}_\mathcal{F})^*} \mathcal{F}|_{\prod_{i \in I_1} (\text{Frob}_{i})^{n_i}(\eta_{I_1}) \times_{\bar{\eta}_q} u} \xrightarrow{\prod_{i \in I_1} (\text{Frob}_{i})^{n_i}} \mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u}$$

We deduce an action of $\text{FWeil}(\eta_{I_1}, \bar{\eta}_{I_1})$.

2.1.3. As in 2.1.2, we begin by considering a too simple case to illustrate the case 2.1.4. Let $\mathcal{F}$ be a constructible $E$-sheaf over $(\eta)^{I_1} \times_{\bar{\eta}_q} u$, equipped with an action of the partial Frobenius morphisms.

By Lemma 1.2.12 applied to $I_1$, we deduce that $\mathcal{F}$ is smooth over $(\eta)^{I_1} \times_{\bar{\eta}_q} u$.

As in 2.1.2, $\mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u}$ is equipped with a continuous action of $\text{FWeil}(\eta_{I_1}, \bar{\eta}_{I_1})$.

By Lemma 1.2.10, this action factors through $\text{Weil}(\eta, \bar{\eta})^{I_1}$. We deduce that the action of $\text{Weil}(\eta^{I_1}, \bar{\eta}_{I_1})$ on $\mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u}$ is trivial. Note that

$$\text{Weil}(\eta^{I_1} \times_{\bar{\eta}_q} u, \eta_{I_1} \times_{\bar{\eta}_q} \bar{u}) \simeq \text{Weil}(\eta^{I_1}, \bar{\eta}_{I_1})$$

Thus the action of $\text{Weil}(\eta^{I_1} \times_{\bar{\eta}_q} u, \eta_{I_1} \times_{\bar{\eta}_q} \bar{u})$ on $\mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u}$ is also trivial.

By 1.1.7, $\mathcal{F}$ is constant over $(\eta)^{I_1} \times_{\bar{\eta}_q} u$.

2.1.4. As in 2.1.4, let $A$ be a finitely generated $E$-algebra. Let $\mathcal{F}$ be an ind-constructor $E$-sheaf over $(\eta)^{I_1} \times_{\bar{\eta}_q} u$, equipped with an action of the partial Frobenius morphisms $A$, and an action of $A$, such that

- these two actions commute with each other
- $\mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u}$ is an $A$-module of finite type

Then as in 2.1.2, $\mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u}$ is equipped with a continuous $A$-linear action of $\text{FWeil}(\eta_{I_1}, \bar{\eta}_{I_1})$. By Lemma 1.2.11, this action factors through $\text{Weil}(\eta, \bar{\eta})^{I_1}$. Thus the action of $\text{Weil}(\eta^{I_1} \times_{\bar{\eta}_q} u, \eta_{I_1} \times_{\bar{\eta}_q} \bar{u})$ on $\mathcal{F}|_{\eta_{I_1} \times_{\bar{\eta}_q} u}$ is trivial.

However, in general $\mathcal{F}$ is not ind-smooth over $(\eta)^{I_1} \times_{\bar{\eta}_q} u$. 
2.1.5. In our situation below, we will apply 2.1.4 to the cohomology sheaves of stacks of shtukas, and prove that for other reasons the cohomology sheaves are ind-smooth over $(\eta)_1^I \times \mathbb{F}_q \overline{u}$. Then they are constant over $(\eta)_1^I \times \mathbb{F}_q \overline{u}$.

2.2. Action of Weil($\eta, \overline{\eta}$) on cohomology of stacks of shtukas.

2.2.1. Consider the sheaf $\mathcal{H}_{1,W}^j \mid_{(\eta)_1^I \times \mathbb{F}_q u}$ over $(\eta)_1^I \times \mathbb{F}_q u$. By 1.3.3 it is equipped with an action of the partial Frobenius morphisms. As in 2.1.2 the fiber $\mathcal{H}_{1,W}^j \mid_{\eta \times \mathbb{F}_q \overline{u}}$ is equipped with an action of FWeil($\eta_1, \overline{\eta}_1$).

Proposition 2.2.2. The action of FWeil($\eta_1, \overline{\eta}_1$) on $\mathcal{H}_{1,W}^j \mid_{\eta_1 \times \mathbb{F}_q \overline{u}}$ factors through Weil($\eta, \overline{\eta}$)$^I_1$.

Proof. The proof is similar to the proof of Proposition 1.3.4. We use Lemma 2.2.3 below. Then to each $\mathcal{M}$, we apply 2.1.4 to $A = \otimes_{i \in I_1} \mathcal{H}_{G,v_i}$ and $\mathcal{F} = \mathcal{M}$. □

Lemma 2.2.3. $\mathcal{H}_{1,W}^j \mid_{\eta_1 \times \mathbb{F}_q \overline{u}}$ is an inductive limit of sub-$E$-modules $\mathcal{M}$ which are stable under FWeil($\eta_1, \overline{\eta}_1$), and for which there exists a family $(v_i)_{i \in I_1}$ of closed points in $X \setminus N$ (depending on $\mathcal{M}$) such that $\mathcal{M}$ is stable under the action of $\otimes_{i \in I_1} \mathcal{H}_{G,v_i}$ and is of finite type as module over $\otimes_{i \in I_1} \mathcal{H}_{G,v_i}$.

Proof. The proof is similar to the proof of Lemma 1.3.11.

For any $\mu$, we choose a dense open subscheme $\Omega$ of $(X \setminus N)^I_1$ such that $\mathcal{H}_{1,W}^{j, \leq \mu} \mid_{\Omega \times \mathbb{F}_q u}$ is smooth. We choose a closed point of $X \setminus N$. Let $v_i$ be the image of $v$ under $(X \setminus N)^I_1 \xrightarrow{pr_i} X \setminus N$. Let $\mathcal{M}_\mu$ be the image of

$$\sum_{(n_i)_{i \in I_1} \in \mathbb{N}^{I_1}} (\otimes_{i \in I_1} \mathcal{H}_{G,v_i}) \cdot \left( \prod_{i \in I_1} F_{\{i\}}^n i \left( \prod_{i \in I_1} \text{Frob}_{\{i\}}(i)^* \mathcal{H}_{1,W}^{j, \leq \mu} \mid_{\eta_1 \times \mathbb{F}_q \overline{u}} \right) \right)$$

in $\mathcal{H}_{1,W}^j \mid_{\eta_1 \times \mathbb{F}_q \overline{u}}$.

Then the proof of Lemma 1.3.11 works if we replace everywhere $\mathfrak{v}$ by $\mathfrak{v} \times \mathbb{F}_q \overline{u}$ and replace $\eta\mathfrak{v}$ by $\eta_1 \times \mathbb{F}_q \overline{u}$. □

2.3. Smoothness of cohomology sheaves over $(\eta)_1^I \times \mathbb{F}_q \overline{u}$.

2.3.1. Let $\mathfrak{v}$ be a geometric point of $(\eta)_1^I$. Then $\eta_1 \times \mathbb{F}_q \overline{u}$ and $\mathfrak{v} \times \mathbb{F}_q \overline{u}$ are geometric points of $(X \setminus N)^I$. Let

$$\text{sp}_\mathfrak{v} : \eta_1 \times \mathbb{F}_q \overline{u} \to \mathfrak{v} \times \mathbb{F}_q \overline{u}$$

be a specialization map in $(X \setminus N)^I$. It induces a morphism

$$\text{sp}_\mathfrak{v}^* : \mathcal{H}_{1,W}^j \mid_{\mathfrak{v} \times \mathbb{F}_q \overline{u}} \to \mathcal{H}_{1,W}^j \mid_{\eta_1 \times \mathbb{F}_q \overline{u}}$$

Proposition 2.3.2. The morphism (2.1) is an isomorphism.

Proof. The proof is similar to the proof of Proposition 1.3.3. We replace everywhere $\mathfrak{v}$ by $\mathfrak{v} \times \mathbb{F}_q \overline{u}$ and replace everywhere $\eta\mathfrak{v}$ by $\eta_1 \times \mathbb{F}_q \overline{u}$. □
2.4. The cohomology sheaves are constant over \((\overline{\eta})^I_1 \times \overline{\pi}\).

**Proposition 2.4.1.** \(\mathcal{H}_{j, W}^j |_{(\overline{\eta})^I_1 \times \overline{\pi}}\) is a constant sheaf over \((\overline{\eta})^I_1 \times \overline{\pi}\).

**Proof.** By Proposition 2.3.2, the sheaf \(\mathcal{H}_{j, W}^j |_{(\overline{\eta})^I_1 \times \overline{\pi}}\) is ind-smooth over \((\overline{\eta})^I_1 \times \overline{\pi}\). By Proposition 2.2.2, the action of Weil\((\overline{\eta})^I_1 \times \overline{\pi}, (\overline{\eta})^I_2 \times \overline{\pi})\) on \(\mathcal{H}_{j, W}^j |_{(\overline{\eta})^I_1 \times \overline{\pi}}\) is trivial. By Lemma 1.1.5 and 1.1.7, we deduce Proposition 2.4.1. \(\square\)

2.4.2. Let \(s\) be a closed point of \(X \setminus N\) and \(s = \text{Spec } \overline{\mathbb{F}_q}\) a geometric point over \(s\). Let \((\overline{s})^I_2 := s \times \overline{\mathbb{F}_q} \cdots \overline{\mathbb{F}_q} \overline{s}\).

Then \(\overline{\pi} = (\overline{s})^I_2\) is a geometric point of \((X \setminus N)^I_2\) as in 2.0.1. By Proposition 2.4.1, \(\mathcal{H}_{j, W}^j |_{(\overline{\eta})^I_1 \times \overline{\pi} (\overline{s})^I_2}\) is a constant sheaf over \((\overline{\eta})^I_1 \times \overline{\pi} (\overline{s})^I_2\).

3. A CONDITION OF SMOOTHNESS

The goal of this section is to prove Proposition 3.3.4 which will be used in the next section. To illustrate, we begin with a simple case, that is Proposition 3.2.1.

3.1. pseudo-product sheaves.

3.1.1. As in [SGA7 XIII], let \(S\) be a trait (i.e. spectrum of a DVR) which is henselian. Let \(s = \text{Spec } k\) be the closed point and \(\delta = \text{Spec } K\) the generic point. Fix an algebraic closure \(K\) of \(K\). We denote by \(\delta = \text{Spec } K\). It will be enough for us to consider only the case where \(k\) is separately closed, i.e. we assume \(\overline{s} = s\).

In this section, we write \(\times\) for \(\times \text{Spec } k\).

3.1.2. Let \(I\) be a finite set. Let \((\overline{\pi})_{i \in I}\) be a family of geometric points of \(S\) such that \(\overline{\pi}_i \in \{\overline{s}, \overline{\delta}\}\). We denote by \(\times_{i \in I} \overline{\pi}_i\) the fiber product over \(\text{Spec } k\). As in 1.2.4, \(\times_{i \in I} \overline{\pi}_i\) is an integral scheme over \(\text{Spec } k\).

3.1.3. Let \(\Lambda = \mathcal{O}_E/\lambda_F \mathcal{O}_E\) or \(\mathcal{O}_E\) or \(E\).

**Definition 3.1.4.** Let \(\mathcal{G}\) be an ind-constructible \(\Lambda\)-sheaf over \(S^I\). We say that \(\mathcal{G}\) is a pseudo-product if for any family \((\overline{\pi})_{i \in I}\) of geometric points of \(S\) such that \(\overline{\pi}_i \in \{\overline{s}, \overline{\delta}\}\), the restriction \(\mathcal{G}|_{\times_{i \in I} \overline{\pi}_i}\) is a constant sheaf over \(\times_{i \in I} \overline{\pi}_i\).

**Notation 3.1.5.** For pseudo-product sheaf \(\mathcal{G}\), we denote \(\mathcal{G}|_{\times_{i \in I} \overline{\pi}_i} := \Gamma(\times_{i \in I} \overline{\pi}_i, \mathcal{G})\).

**Example 3.1.6.** If \(\mathcal{G} = \bigotimes_{i \in I} \mathcal{F}_i\), where every \(\mathcal{F}_i\) is an ind-constructible \(\Lambda\)-sheaf over \(S\), then \(\mathcal{G}\) is a pseudo-product.
Lemma 3.1.7. ([SGA7 XIII 1.2.2]) We have a functor $\Theta_S$ from
\[ \mathcal{A} := \{ \text{ind-constructible $\Lambda$-sheaf $G$ over $S$} \} \]
to
\[ \mathcal{B} := \{ (\mathcal{G}|_{\overline{\mathcal{S}}}, \mathcal{G}|_{\overline{\eta}}, \phi), \mathcal{G}|_{\overline{\mathcal{S}}} \text{ is a trivial $\text{Gal}(\overline{\delta}/\delta)$-module,} \]
\[ \mathcal{G}|_{\overline{\eta}} \text{ is a $\text{Gal}(\overline{\delta}/\delta)$-module,} \]
\[ \phi : \mathcal{G}|_{\overline{\mathcal{S}}} \to \mathcal{G}|_{\overline{\eta}} \text{ is a $\text{Gal}(\overline{\delta}/\delta)$-equivariant morphism} \} . \]
It is an equivalence of categories.

Remark 3.1.8. In general, we have an exact sequence
\[ 1 \to \mathcal{I} \to \mathcal{G}(\overline{\eta}/\overline{\eta}) \to \mathcal{G}(\overline{\delta}/\overline{\delta}) \to 1 \]
where $\mathcal{I}$ is the inertia group. The morphism $\phi$ factors through
\[ \phi : \mathcal{G}|_{\overline{\mathcal{S}}} \to \mathcal{G}|_{\overline{\eta}} \to \mathcal{G}|_{\overline{\delta}}. \]
Thus if $\phi$ is an isomorphism, then the action of $\mathcal{I}$ on $\mathcal{G}|_{\overline{\eta}}$ is trivial.

Here, under our hypothesis that $\overline{\mathcal{S}} = s$, we have $\mathcal{I} = \text{Gal}(\overline{\eta}/\eta)$. If $\phi$ is an isomorphism, then the action of $\text{Gal}(\overline{\eta}/\eta)$ on $\mathcal{G}|_{\overline{\eta}}$ is trivial. We deduce that an object $(\mathcal{G}|_{\overline{\mathcal{S}}}, \mathcal{G}|_{\overline{\eta}}, \phi : \mathcal{G}|_{\overline{\mathcal{S}}} \to \mathcal{G}|_{\overline{\eta}})$ in the category $\mathcal{B}$ corresponds to a constant sheaf over $S$. Thus a smooth sheaf over $S$ is the same thing as a constant sheaf over $S$.

Another way to see this fact is that $S$ does not have non trivial etale covering.

3.1.9. Let $Y$ be a scheme over $s$. We have a commutative diagram
\[
\begin{array}{ccc}
Y \times s & \xrightarrow{i} & Y \times S & \xrightarrow{j} & Y \times \delta \\
\downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{i} & S & \xrightarrow{j} & \delta
\end{array}
\]

3.1.10. Applying [SGA7 XIII 1.2.4] to $Y = S$ and using Lemma 3.1.7, we deduce

Lemma 3.1.11. We have a functor $\Theta_{S \times S}$ from
\[ \mathcal{A} := \{ \text{pseudo-product ind-constructible $\Lambda$-sheaf $G$ over $S \times S$} \} \]
to
\[ \mathcal{B} := \{ (\mathcal{G}|_{\overline{\mathcal{S} \times \mathcal{S}}}, \mathcal{G}|_{\overline{\eta \times \eta}}, \mathcal{G}|_{\overline{\delta \times \delta}}, \phi_{00,10}, \phi_{00,01}, \phi_{10,11}, \phi_{01,11}) \}
where $\mathcal{G}|_{\overline{\mathcal{S} \times \mathcal{S}}}, \mathcal{G}|_{\overline{\eta \times \eta}}, \mathcal{G}|_{\overline{\delta \times \delta}}$ are $\text{Gal}(\overline{\delta}/\delta)^2$-modules ,
the action of $\text{Gal}(\overline{\delta}/\delta)^2$ is trivial on $\mathcal{G}|_{\overline{\delta \times \delta}}$,
factors through the first factor on $\mathcal{G}|_{\overline{\eta \times \eta}}$,
factors through the second factor on $\mathcal{G}|_{\overline{\delta \times \delta}}$;
$\phi_{00,10}, \phi_{00,01}, \phi_{10,11}, \phi_{01,11}$ are $\text{Gal}(\overline{\delta}/\delta)^2$-equivariant morphisms
such that the following diagram is commutative \}
This functor is an equivalence of categories.

**Remark 3.1.12.** The morphism $\phi_{0,10}$ (resp. $\phi_{0,01}$) factors through

\[ g|_{\mathcal{X} \times S} \to g|_{\mathcal{X} \times S} \quad (\text{resp. } \mathcal{I} \to \mathcal{I}) \]

where $\mathcal{I}$ is the inertia group. Thus $\phi_{0,11} := \phi_{10,11} \circ \phi_{0,10} = \phi_{0,11} \circ \phi_{0,01}$ factors through

\[ g|_{\mathcal{X} \times S} \to g|_{\mathcal{X} \times S} \quad (\text{resp. } \mathcal{I} \to \mathcal{I}) \]

When they are all isomorphism, the action of $\mathcal{I}$ on $g|_{\mathcal{X} \times S}$ (resp. $\mathcal{I}$) is trivial and the action of $\mathcal{I}^2$ on $\mathcal{I}$ is trivial. In our case $\mathcal{S} = \mathcal{S}$, we have $\mathcal{I} = \text{Gal}(\overline{\mathcal{S}}/\mathcal{S})$. Via the equivalence of categories in Lemma 3.1.11, we deduce that a smooth pseudo-product sheaf over $S \times S$ is the same as a constant sheaf.

**Example 3.1.13.** In Lemma 3.1.11 when $\mathcal{G} = \mathcal{F}_1 \boxtimes \mathcal{F}_2$, (3.1) coincides with

\[ \mathcal{F}_1|_{\mathcal{S} \times \mathcal{X}} \otimes \mathcal{F}_2|_{\mathcal{S} \times \mathcal{X}} \xrightarrow{\phi_{0,1} \otimes \text{Id}} \mathcal{F}_1|_{\mathcal{S} \times \mathcal{X}} \otimes \mathcal{F}_2|_{\mathcal{S} \times \mathcal{X}} \]

3.1.14. Applying [SGA7 XIII 1.2.4] to $Y = S \times S$ and using Lemma 3.1.11 we deduce

**Lemma 3.1.15.** We have a functor $\Theta_{S \times S \times S}$ from

\[ \mathcal{A} := \{ \text{pseudo-product ind-constructible } \Lambda\text{-sheaf } \mathcal{G} \text{ over } S \times S \times S \} \]

to

\[ \mathcal{B} := \{ (\text{Gal}(\overline{\mathcal{S}}/\mathcal{S})^3)^3\text{-modules } : \mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}} \to \mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}}, \mathcal{G}|_{\mathcal{X} \times \mathcal{S}}, \mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}} : \text{Gal}(\overline{\mathcal{S}}/\mathcal{S})^3\text{-equivariant morphisms } : \phi_{0,0,0,0,1,0,1,0,1,1,1,1,1,1,1} \} \]

where the action of $\text{Gal}(\overline{\mathcal{S}}/\mathcal{S})^3$ is trivial on $\mathcal{G}|_{\mathcal{X} \times \mathcal{S}}$, factors through the first (resp. second, third) factor on $\mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}}$ (resp. $\mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}}$, $\mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}}$), factors through the $\{1,2\}$ (resp. $\{1,3\}, \{2,3\}$)-factors on $\mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}}$ (resp. $\mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}}$, $\mathcal{G}|_{\mathcal{X} \times \mathcal{S}, \mathcal{X} \times \mathcal{S}}$); morphisms are such that the following diagram is commutative
This functor is an equivalence of categories.

**Remark 3.1.16.** Similarly to Remark 3.1.12, a smooth pseudo-product sheaf over $S \times S \times S$ is the same as a constant sheaf.

3.1.17. Let $G$ be a pseudo-product sheaf over $S \times S \times S$. We have a partial diagonal morphism

$$S \times S \xrightarrow{(\Delta^{(1,2)}, \text{Id})} S \times S \times S. \tag{3.3}$$

Then $G_{|S \times \Delta^{(1,2)}(S) \times S}$ is a pseudo-product sheaf over $S \times S$. Applying [SGA7 XIII 1.2.7 (a)] to $Y = S$, $Y' = S \times S$ and the diagonal morphism $\Delta^{(1,2)} : S \to S \times S$, we deduce that $\Theta_{S \times S}(G_{|S \times \Delta^{(1,2)}(S) \times S})$ coincides with the commutative sub-diagram

$$G_{|S \times \Delta^{(1,2)}(S) \times S} \xrightarrow{\phi_{000,000}} \cdots \xrightarrow{\phi_{000,011}} \cdots \xrightarrow{\phi_{001,101}} G_{|S \times \Delta^{(1,2)}(S) \times S}$$

of (3.2).

Similarly, we have a partial diagonal morphism

$$S \times S \xrightarrow{(\text{Id}, \Delta^{(2,3)})} S \times S \times S. \tag{3.4}$$

$G_{|S \times S \times S}$ is a pseudo-product sheaf over $S \times S$ and $\Theta_{S \times S}(G_{|S \times S \times S})$ coincides with the commutative sub-diagram

$$G_{|S \times S \times S} \xrightarrow{\phi_{100,100}} \cdots \xrightarrow{\phi_{101,111}} \cdots \xrightarrow{\phi_{100,110}} G_{|S \times S \times S}$$

of (3.2).
3.2. A condition of smoothness.

**Proposition 3.2.1.** Let $S$ be a strictly henselian trait as in [3.1.1]. Let $\mathcal{F}$ be an ind-constructible $E$-sheaf over $S$. Suppose that there exists a pseudo-product ind-constructible $E$-sheaf $\mathcal{G}$ over $S^{(1,2,3)}$, and morphisms over $S \times S$

$$\mathcal{C}^f : \mathcal{F} \boxtimes E \to \mathcal{G}_{|S \times \Delta^{(2,3)}(S)} \text{ and } \mathcal{C}^g : \mathcal{G}_{|\Delta^{(1,2)}(S) \times S} \to E \boxtimes \mathcal{F}$$

where $E$ is the constant sheaf over $S$, such that the composition of the restriction over $S$

$$\mathcal{F} \simeq \mathcal{F} \otimes E \xrightarrow{\mathcal{C}^f} \mathcal{G}_{|\Delta^{(1,2,3)}(S)} \xrightarrow{\mathcal{C}^g} E \otimes \mathcal{F} \simeq \mathcal{F}$$

is an isomorphism. Then $\mathcal{F}$ is constant over $S$, i.e. the canonical morphism

$$\mathcal{F}|_\pi \to \mathcal{F}|_\pi$$

is an isomorphism.

**Proof.** First we construct a morphism $\mathcal{F}|_\pi \to \mathcal{F}|_\pi$. Applying the functor $\Theta_{S \times S \times S}$ in Lemma 3.1.11 to $\mathcal{G}$, we obtain a commutative diagram

$$\mathcal{G}_{|S \times S \times S} \xrightarrow{\phi_{00,100}} \mathcal{G}_{|S \times S \times S} \xrightarrow{\phi_{00,110}} \mathcal{G}_{|S \times S \times S} \xrightarrow{\phi_{100,110}} \mathcal{G}_{|S \times S \times S}$$

In particular, we obtain a morphism $\phi_{100,110} : \mathcal{G}_{|S \times \Delta^{(2,3)}(\pi)} \to \mathcal{G}_{|\Delta^{(1,2)}(S) \times \pi}$. We construct the composition of morphisms

$$\mathcal{F}|_\pi \simeq \mathcal{F}|_\pi \otimes E|_\pi \xrightarrow{\mathcal{C}^f} \mathcal{G}_{|S \times \Delta^{(2,3)}(\pi)} \xrightarrow{\phi_{100,110}} \mathcal{G}_{|\Delta^{(1,2)}(S) \times \pi} \xrightarrow{\mathcal{C}^g} E|_\pi \otimes \mathcal{F}|_\pi \simeq \mathcal{F}|_\pi$$

We want to prove that (3.7) is the inverse of (3.5), up to isomorphism.

Now we prove that (3.7) $\circ$ (3.5) is an isomorphism. This will imply that (3.5) is injective.

Applying the functor $\Theta_{S \times S}$ in Lemma 3.1.11 to $\mathcal{C}^f : \mathcal{F} \boxtimes E \to \mathcal{G}_{|S \times \Delta^{(2,3)}(S)}$, we obtain a commutative diagram

$$\mathcal{F}|_\pi \otimes E|_\pi \xrightarrow{\mathcal{C}^f} \mathcal{F}|_\pi \otimes E|_\pi$$

where the horizontal morphisms are the canonical morphisms $\phi_{00,10}$ in Lemma 3.1.11.
Applying the functor $\Theta_{S \times S}$ in Lemma 3.1.11 to $G: S^{(1,2)} \times S \to E \boxtimes F$, we obtain a commutative diagram
\[
G|_{\Delta^{(1,2)}(\overline{S}) \times \overline{S}} \ar[r]^{\Theta} \ar[d]^{\Theta} & G|_{\Delta^{(1,2)}(\overline{S}) \times \overline{S}} \ar[d]^{\Theta} \\
E|_{\overline{S}} \otimes F|_{\overline{S}} \ar[r]^{\Theta} & E|_{\overline{S}} \otimes F|_{\overline{S}}
\]
where the horizontal morphisms are the canonical morphisms $\phi_{00,10}$ in Lemma 3.1.11.

By 3.1.17, the lower line of (3.8) and the upper line of (3.6) coincide. The left lower line of (3.6) and the upper line of (3.9) coincide. Combining (3.8), (3.6) and (3.9), we deduce that the following diagram is commutative:
\[
F|_{\overline{S}} \otimes E|_{\overline{S}} \ar[r]^{\Theta} \ar[d]^{\Theta} & F|_{\overline{S}} \otimes E|_{\overline{S}} \ar[d]^{\Theta} \\
G|_{\Delta^{(1,2)}(\overline{S}) \times \overline{S}} \ar[r]^{\Theta} \ar[d]^{\Theta} & G|_{\overline{S} \otimes \Delta^{(2,3)}(\overline{S})} \ar[d]^{\Theta} \\
E|_{\overline{S}} \otimes F|_{\overline{S}} \ar[r]^{\Theta} & E|_{\overline{S}} \otimes F|_{\overline{S}}
\]

The right vertical line is (3.7). Taking into account $F|_{\overline{S}} \otimes E|_{\overline{S}} \simeq F|_{\overline{S}}$ and $F|_{\overline{S}} \otimes E|_{\overline{S}} \simeq F|_{\overline{S}}$, by Example 3.1.13, we deduce that the upper line of (3.10) is nothing but the canonical morphism (3.5).

By the hypothesis, the composition of the left vertical morphisms is an isomorphism. Since $E$ is a constant sheaf over $S$, the lower line is identity. These imply that (3.5) is injective.

Now we prove that (3.5) $\circ$ (3.7) is an isomorphism. This will imply that (3.5) is surjective.

Applying the functor $\Theta_{S \times S}$ in Lemma 3.1.11 to $G: F \boxtimes E \to G|_{S \times \Delta^{(2,3)}(S)}$, we obtain a commutative diagram:
\[
F|_{\overline{S}} \otimes E|_{\overline{S}} \ar[r]^{\Theta} \ar[d]^{\Theta} & F|_{\overline{S}} \otimes E|_{\overline{S}} \ar[d]^{\Theta} \\
G|_{\overline{S} \times \Delta^{(2,3)}(\overline{S})} \ar[r]^{\Theta} \ar[d]^{\Theta} & G|_{\overline{S} \times \Delta^{(2,3)}(\overline{S})} \ar[d]^{\Theta} \\
E|_{\overline{S}} \otimes F|_{\overline{S}} \ar[r]^{\Theta} & E|_{\overline{S}} \otimes F|_{\overline{S}}
\]

where the horizontal morphisms are the canonical morphisms $\phi_{10,11}$ in Lemma 3.1.11.
Applying the functor \( \Theta_{S \times S \times S} \) in Lemma 3.1.15 to \( \mathcal{G} \), we obtain a commutative diagram

\[
\begin{array}{ccc}
\mathcal{G} |_{\bar{S} \times \bar{S} \times \bar{S}} & \xrightarrow{\phi_{100,111}} & \mathcal{G} |_{\bar{S} \times \bar{S} \times \bar{S}} \\
\phi_{100,110} & & \phi_{110,111} \\
\mathcal{G} |_{\bar{S} \times \bar{S} \times \bar{S}} & & \\
\end{array}
\]

Applying the functor \( \Theta_{S \times S} \) in Lemma 3.1.11 to \( \mathcal{C} \): \( G \)\( \xrightarrow{\Delta} \{1,2\} (S) \times S \to E \square \mathcal{F} \), we obtain a commutative diagram

\[
\begin{array}{ccc}
\mathcal{G} |_{\Delta(1,2)(\bar{S}) \times \bar{S}} & \xrightarrow{\phi} & \mathcal{G} |_{\Delta(1,2)(\bar{S}) \times \bar{S}} \\
E |_{\bar{S}} \otimes \mathcal{F} |_{\bar{S}} & \xrightarrow{\phi} & E |_{\bar{S}} \otimes \mathcal{F} |_{\bar{S}} \\
\end{array}
\]

where the horizontal morphisms are the canonical morphisms \( \phi_{10,11} \) in Lemma 3.1.11.

By 3.1.17, the lower line of (3.11) and the upper line of (3.12) coincide. The right lower line of (3.12) and the upper line of (3.13) coincide. Note that \( \mathcal{G} |_{\bar{S} \times \bar{S} \times \bar{S}} \) is constant, thus \( \Gamma(\bar{S} \times \bar{S} \times \bar{S}, \mathcal{G}) = \mathcal{G} |_{\Delta(1,2,3)(\bar{S})} \). Combining (3.11), (3.12) and (3.13), we obtain a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F} |_{\bar{S}} \otimes E |_{\bar{S}} & \xrightarrow{\phi^2} & \mathcal{F} |_{\bar{S}} \otimes E |_{\bar{S}} \\
\mathcal{G} |_{\bar{S} \times \Delta(2,3)(\bar{S})} & \xrightarrow{\phi^3} & \mathcal{G} |_{\Delta(1,2,3)(\bar{S})} \\
\mathcal{G} |_{\Delta(1,2)(\bar{S}) \times \bar{S}} & \xrightarrow{\phi^3} & \mathcal{G} |_{\Delta(1,2,3)(\bar{S})} \\
E |_{\bar{S}} \otimes \mathcal{F} |_{\bar{S}} & \xrightarrow{\phi^2} & E |_{\bar{S}} \otimes \mathcal{F} |_{\bar{S}} \\
\end{array}
\]

The left vertical line is (3.7). By Example 3.1.13, we deduce that the lower line of (3.14) is nothing but the canonical morphism (3.5).

By hypothesis, the composition of right vertical morphisms is an isomorphism. Since \( E \) is a constant sheaf over \( S \), the upper line is identity. These imply that (3.5) is surjective. \( \square \)

3.3. A generalization.

3.3.1. Let \( I = \{1, 2, \cdots, n\} \) be a finite set. Let \( (S_i)_{i \in I} \) be a family of strictly henselian traits with

\[
\overline{s_i} = s_i \to S_i \leftarrow \delta_i \leftarrow \overline{\delta_i}
\]
Lemma 3.3.2. We have a functor \( \Theta_{x_i \in S_i} \) from
\[
\mathcal{A} := \{ \text{pseudo-product ind-constructible } \Lambda\text{-sheaf } \mathcal{F} \text{ over } \times_{i \in I} S_i \}
\]
to
\[
\mathcal{B} := \{ (x_i \in I \text{ Gal}(\delta_i/\delta_i)\text{-modules } \mathcal{F}|_{x_i \in I \bar{\pi}_i} \text{ for all families } (\bar{\pi}_i)_{i \in I} \text{ with } \bar{\pi}_i \in \{ \bar{s}_i, \delta_i \}, \\
\times_{i \in I} \text{ Gal}(\delta_i/\delta_i)\text{-equivariant morphisms } \phi_{(\bar{\pi}_i), (\bar{\delta}_i)} : \mathcal{F}|_{x_i \in I \bar{\pi}_i} \to \mathcal{F}|_{x_i \in I \bar{\pi}_i}
\]
for all families \((\bar{\pi}_i)_{i \in I}, (\bar{\delta}_i)_{i \in I}\) with \bar{\pi}_i, \bar{\delta}_i \in \{ \bar{s}_i, \delta_i \} \text{ and } \bar{\pi}_i \text{ a specialization of } \bar{\delta}_i)

where the action of \times_{i \in I} \text{ Gal}(\delta_i/\delta_i) \text{ on } \mathcal{F}|_{x_i \in I \bar{\pi}_i} \text{ factors through the factors } i
\]
where \( \bar{\pi}_i = \delta_i \), the morphisms \( \phi_{(\bar{\pi}_i), (\bar{\delta}_i)} \) are such that the corresponding

diagram is commutative \}

This functor is an equivalence of categories.

Proof. We argument by induction on \( I \). When \( I \) is a singleton, this is Lemma
\[ 3.1.7 \] In general, suppose that we have the equivalence \( \Theta_{S_1 \times \cdots \times S_m} \) for some \( m \geq 1 \).

Apply [SGA7 XIII 1.2.4] to \( S = S_{m+1} \) and \( Y = S_1 \times \cdots \times S_m \). Using the induction hypothesis and the hypothesis that \( \mathcal{G} \) is pseudo-product, we deduce the equivalence \( \Theta_{S_1 \times \cdots \times S_{m+1}} \).

Remark 3.3.3. Similarly to Remark 3.1.12, a smooth pseudo-product sheaf over \( \times_{i \in I} S_i \) corresponds to an object in \( \mathcal{B} \) where all \( \phi_{(\bar{\pi}_i), (\bar{\delta}_i)} \) are isomorphism, thus the \( \times_{i \in I} \text{ Gal}(\delta_i/\delta_i)\text{-modules are trivial. As a consequence, a smooth pseudo-product sheaf over } \times_{i \in I} S_i \text{ is the same as a constant sheaf.}

Proposition 3.3.4. Let \( \mathcal{F} \) be a pseudo-product ind-constructible \( \Lambda\text{-sheaf over } \times_{i \in I} S_i \). Suppose that there exists a pseudo-product ind-constructible \( \Lambda\text{-sheaf } \mathcal{G} \) over \( (\times_{i \in I} S_i)^{(1,2,3)} \), and morphisms over \( (\times_{i \in I} S_i)^{\Delta(1,2,3)} \times (\times_{i \in I} S_i) \)

\[
C^\sharp : \mathcal{F} \boxtimes E \to \mathcal{G}|_{(\times_{i \in I} S_i)^{\Delta(2,3)}(\times_{i \in I} S_i)} \quad \text{and} \quad C^\flat : \mathcal{G}|_{\Delta(1,2)(\times_{i \in I} S_i) \times (\times_{i \in I} S_i)} \to E \boxtimes \mathcal{F}
\]

where \( E \) is the constant sheaf over \( (\times_{i \in I} S_i) \), such that the composition of the restriction over \( \times_{i \in I} S_i \)

\[
\mathcal{F} = \mathcal{F} \boxtimes E \xrightarrow{C^\sharp} \mathcal{G}|_{\Delta(1,2,3)(\times_{i \in I} S_i)} \xrightarrow{C^\flat} E \boxtimes \mathcal{F} = \mathcal{F}
\]
is an isomorphism.

Then for any families of \((\bar{\pi}_i)_{i \in I}\) and \((\bar{\delta}_i)_{i \in I}\) with \( \bar{\pi}_i, \bar{\delta}_i \in \{ \bar{s}_i, \delta_i \} \) and \( \bar{\pi}_i \) a specialization of \( \bar{\delta}_i \) as in Lemma 3.3.2, the canonical morphism

\[
(3.15) \quad \phi_{(\bar{\pi}_i), (\bar{\delta}_i)} : \mathcal{F}|_{\times_{i \in I} \bar{\pi}_i} \to \mathcal{F}|_{\times_{i \in I} \bar{\delta}_i}
\]
is an isomorphism. In particular, \( \mathcal{F} \) is a constant sheaf.
**Proof.** The proof is similar to the proof of Proposition 3.2.1. First we construct a morphism of the inverse direction

\[
\begin{align*}
\mathcal{F}|_{x_i \in \mathcal{Y}} \oplus \mathcal{E}|_{x_i \in \mathcal{Y}} & \overset{\epsilon\eta}{\rightarrow} \mathcal{G}|_{\Delta^{(1,2)}(x_i \in \mathcal{Y}) \times \Delta^{(2,3)}(x_i \in \mathcal{Y})} \\
\mathcal{G}|_{\Delta^{(1,2)}(x_i \in \mathcal{Y}) \times \Delta^{(2,3)}(x_i \in \mathcal{Y})} & \overset{\epsilon\eta}{\rightarrow} \mathcal{E}|_{x_i \in \mathcal{Y}} \oplus \mathcal{F}|_{x_i \in \mathcal{Y}} \\
\mathcal{F}|_{x_i \in \mathcal{Y}} & \cong \mathcal{F}|_{x_i \in \mathcal{Y}} \oplus \mathcal{E}|_{x_i \in \mathcal{Y}} \overset{\epsilon\eta}{\rightarrow} \mathcal{G}|_{\Delta^{(1,2)}(x_i \in \mathcal{Y}) \times \Delta^{(2,3)}(x_i \in \mathcal{Y})} \\
\mathcal{G}|_{\Delta^{(1,2)}(x_i \in \mathcal{Y}) \times \Delta^{(2,3)}(x_i \in \mathcal{Y})} & \overset{\epsilon\eta}{\rightarrow} \mathcal{E}|_{x_i \in \mathcal{Y}} \oplus \mathcal{F}|_{x_i \in \mathcal{Y}} \cong \mathcal{F}|_{x_i \in \mathcal{Y}}.
\end{align*}
\]

Injectivity of (3.15): using Lemma 3.3.2, we construct the following commutative diagram

\[
\begin{align*}
\mathcal{F}|_{x_i \in \mathcal{Y}} \oplus \mathcal{E}|_{x_i \in \mathcal{Y}} & \overset{\epsilon\eta}{\rightarrow} \mathcal{F}|_{x_i \in \mathcal{Y}} \oplus \mathcal{E}|_{x_i \in \mathcal{Y}} \\
\mathcal{G}|_{\Delta^{(1,2)}(x_i \in \mathcal{Y}) \times \Delta^{(2,3)}(x_i \in \mathcal{Y})} & \overset{\epsilon\eta}{\rightarrow} \mathcal{E}|_{x_i \in \mathcal{Y}} \oplus \mathcal{F}|_{x_i \in \mathcal{Y}} \cong \mathcal{F}|_{x_i \in \mathcal{Y}}.
\end{align*}
\]

The upper line identifies with (3.15). The composition of the left vertical morphisms is an isomorphism. The lower line is identity. Thus (3.15) is injective.

Surjectivity of (3.15): using Lemma 3.3.2, we construct the following commutative diagram

\[
\begin{align*}
\mathcal{F}|_{x_i \in \mathcal{Y}} \oplus \mathcal{E}|_{x_i \in \mathcal{Y}} & \overset{\epsilon\eta}{\rightarrow} \mathcal{F}|_{x_i \in \mathcal{Y}} \oplus \mathcal{E}|_{x_i \in \mathcal{Y}} \\
\mathcal{G}|_{\Delta^{(1,2)}(x_i \in \mathcal{Y}) \times \Delta^{(2,3)}(x_i \in \mathcal{Y})} & \overset{\epsilon\eta}{\rightarrow} \mathcal{E}|_{x_i \in \mathcal{Y}} \oplus \mathcal{F}|_{x_i \in \mathcal{Y}} \cong \mathcal{F}|_{x_i \in \mathcal{Y}}.
\end{align*}
\]

The lower line identifies with (3.15). The composition of the right vertical morphisms is an isomorphism. The upper line is identity. Thus (3.15) is surjective.

□
4. Smoothness of the cohomology of stacks of shtukas

The goal of this section is to prove Theorem 4.2.3. We illustrate the proof by a simple case in Section 4.1.

4.1. Example: when \( I \) is a singleton.

4.1.1. Let \( I = \{1\} \) and \( W \in \text{Rep}_E(\hat{G}) \). For any \( j \in \mathbb{Z} \), we have the degree \( j \) cohomology sheaf \( \mathcal{H}^j_{\{1\},W} \) over \( X \smallsetminus N \).

4.1.2. We recall the definition of the creation operator and the annihilation operator in \[\text{Laf18}\] Section 5.

Let \( W^* \) be the dual representation of \( W \). Let \( \{e_k\} \) be a basis of \( W \) and \( \{e_k^*\} \) be the dual basis. We denote by \( 1 \) the trivial representation of \( \hat{G} \).

Let \( \delta : 1 \to W^* \otimes W \) be the morphism sending 1 to \( \sum_k e_k^* \otimes e_k \).

The creation operator \( \phi^{(2,3)}_\delta \) is defined to be the composition of morphisms of sheaves over \( (X \smallsetminus N) \times (X \smallsetminus N) \):

\[
\mathcal{H}^j\{1\},W \boxtimes E_{(X \smallsetminus N)} \xrightarrow{\phi^{(2,3)}_\delta} \mathcal{H}^j\{1\},W \boxtimes (W^* \otimes W)
\]

(4.1)

Let \( \chi_{(2,3)} \) be the degree \( 2 \) cohomology sheaf associated to \( (X \smallsetminus N) \times (X \smallsetminus N) \):

\[
\mathcal{H}^j\{1\},W \boxtimes E_{(X \smallsetminus N)} \xrightarrow{\chi_{(2,3)}} \mathcal{H}^j\{1\},W \boxtimes (W^* \otimes W)
\]

(4.2)

where \( \mathcal{H}(\text{Id}_W \boxtimes \delta) \) follows from the functoriality and \( \chi_{(2,3)} \) is the fusion isomorphism \((\text{Laf18})\) Proposition 4.12) associated to the map

\[
\{1, 2, 3\} \rightarrow \{1, 0\}, \; 1 \mapsto 1, \; 2 \mapsto 0, \; 3 \mapsto 0
\]

Let \( \text{ev} : W \otimes W^* \rightarrow 1 \) be the evaluation map. The annihilation operator \( \phi^{(1,2)}_{\text{ev}} \) is defined to be the composition of morphisms of sheaves over \( (X \smallsetminus N) \times (X \smallsetminus N) \):

\[
\mathcal{H}^j\{1, 2, 3\},W \boxtimes (W^* \otimes W) \xrightarrow{\chi_{(1,2)}} \mathcal{H}^j\{0, 3\},(W \otimes W^*) \boxtimes W
\]

(4.3)

where \( \chi_{(1,2)} \) is the fusion isomorphism associated to the map

\[
\{1, 2, 3\} \rightarrow \{0, 3\}, \; 1 \mapsto 0, \; 2 \mapsto 0, \; 3 \mapsto 3
\]

Lemma 4.1.3. ("Zorro" lemma, \[\text{Laf18}\] (6.18)) The composition of morphisms of sheaves over \( X \smallsetminus N \):

\[
\mathcal{H}^j\{1\},W = \mathcal{H}^j\{1\},W \boxtimes E_{(X \smallsetminus N)} \xrightarrow{\phi^{(2,3)}_\delta} \mathcal{H}^j\{1, 2, 3\},W \boxtimes (W^* \otimes W) \xrightarrow{\phi^{(1,2)}_{\text{ev}}} E \boxtimes \mathcal{H}^j\{3\},W = \mathcal{H}^j\{3\},W
\]

is the identity.

\[\square\]

Proposition 4.1.4. \( \mathcal{H}^j\{1\},W \) is ind-smooth over \( X \smallsetminus N \).
**Proof.** It is enough to prove that for any geometric point \( \overline{\tau} \) of \( X \setminus N \) and any specialization map \( sp : \overline{\tau} \rightarrow \tau \), the induced morphism

\[
s^{*} : \mathcal{H}^{I_{1}}_{\{1\},W} \big|_{\tau} \rightarrow \mathcal{H}^{I_{1}}_{\{1\},W} \big|_{\overline{\tau}}
\]

is an isomorphism.

Since \( X \setminus N \) is smooth of dimension 1, the strict henselization \( (X \setminus N)_{\overline{\tau}} \) of \( X \setminus N \) at \( \overline{\tau} \) is a trait. Apply Proposition \(3.2.1\) to

- \( S = (X \setminus N)_{\overline{\tau}} \), \( s = \overline{\tau} \), \( \delta = \overline{\tau} \), \( \delta \) the image of \( \tau \) in \( S \),
- \( \mathcal{F} = \mathcal{H}^{I}_{\{1\},W} \),
- \( \mathcal{G} = \mathcal{H}^{I}_{\{1,2,3\},W \otimes W} \). Note that by Proposition \(2.4.1\), \( \mathcal{H}^{I}_{\{1,2,3\},W \otimes W} \) is a pseudo-product sheaf over \( S_{\{1,2,3\}} \).
- \( \mathcal{C}^{\flat} \) the creation morphism \( \mathcal{C}^{\flat}_{\delta} \{2,3\} \)
- \( \mathcal{C}^{\sharp} \) the annihilation morphism \( \mathcal{C}^{\sharp}_{\delta} \{1,2\} \).

By Lemma \(4.1.3\) the hypothesis of Proposition \(3.2.1\) is satisfied. We deduce that \( sp^{*} \) is an isomorphism. \( \square \)

### 4.2. General case.

Let \( I \) be a finite set and \( W \in \text{Rep}_{E}(\hat{G}^{I}) \).

#### 4.2.1. Let \( I_0 = I_1 = I_2 = I_3 = I \). We denote by

\[
\Delta_{I_0 \sqcup I_2 \sqcup I_3} : (X \setminus N)^{I} \rightarrow (X \setminus N)^{I_1} \times (X \setminus N)^{I_2} \times (X \setminus N)^{I_3},
\]

\[
(x_{i})_{i \in I} \mapsto ((x_{i})_{i \in I_1}, (x_{i})_{i \in I_2}, (x_{i})_{i \in I_3})
\]

\[
\Delta_{I_1 \sqcup I_2} : (X \setminus N)^{I} \rightarrow (X \setminus N)^{I_1} \times (X \setminus N)^{I_2}, \quad (x_{i})_{i \in I} \mapsto ((x_{i})_{i \in I_1}, (x_{i})_{i \in I_2})
\]

\[
\Delta_{I_2 \sqcup I_3} : (X \setminus N)^{I} \rightarrow (X \setminus N)^{I_2} \times (X \setminus N)^{I_3}, \quad (x_{i})_{i \in I} \mapsto ((x_{i})_{i \in I_2}, (x_{i})_{i \in I_3})
\]

We denote by \( \mathbf{1} \) the trivial representation of \( \hat{G}^{I} \). As in [Laf18] Section 5, we define the creation operator \( \mathcal{C}^{\sharp}_{\delta} \{I_0 \sqcup I_2 \sqcup I_3\} \) (creating paw \( I_2 \sqcup I_3 \)) to be the composition of morphisms of complexes over \( (X \setminus N)^{I} \times (X \setminus N)^{I} \):

\[
\mathcal{H}_{I_0,W} \boxtimes E_{(X \setminus N)^{I}} \xrightarrow{\mathcal{H}_{I_0 \sqcup I_2 \sqcup I_3,W \otimes W}} \mathcal{H}_{I_0 \sqcup I_2 \sqcup I_3,W \otimes (W^* \otimes W)}
\]

(4.4)

where \( \chi_{I_2 \sqcup I_3} \) is the fusion isomorphism ([Laf18] Proposition 4.12) associated to the map

\[
I_1 \sqcup I_2 \sqcup I_3 \rightarrow I_1 \sqcup I_0
\]

sending \( I_1 \) to \( I_1 \) by identity, \( I_2 \) to \( I_0 \) by identity and \( I_3 \) to \( I_0 \) by identity.

We define the annihilation operator \( \mathcal{C}^{\sharp}_{ev,I_0 \sqcup I_2} \) (annihilating paws \( I_1 \sqcup I_2 \)) to be the composition of morphisms of complexes over \( (X \setminus N)^{I} \times (X \setminus N)^{I} \):

\[
\mathcal{H}_{I_0 \sqcup I_2,W \otimes (W^* \otimes W)} \xrightarrow{\mathcal{H}_{I_0 \sqcup I_2,W \otimes W} \otimes \mathcal{H}_{I_0 \sqcup I_2,W \otimes W}} \mathcal{H}_{I_0 \sqcup I_2,W \otimes (W^* \otimes W)}
\]

(4.5)

\[
\mathcal{H}_{I_0 \sqcup I_2 \sqcup I_3,W \otimes (W^* \otimes W)} \xrightarrow{\mathcal{H}_{I_0 \sqcup I_2 \sqcup I_3,W \otimes (W^* \otimes W)}} \mathcal{H}_{I_0 \sqcup I_2 \sqcup I_3,W \otimes (W^* \otimes W)} \xrightarrow{\mathcal{H}_{I_0 \sqcup I_2 \sqcup I_3,W \otimes (W^* \otimes W)}} \mathcal{H}_{I_0 \sqcup I_2 \sqcup I_3,W \otimes (W^* \otimes W)}
\]

(4.5)
where $\chi_{I_1 \sqcup I_2}$ is the fusion isomorphism associated to the map

$$I_1 \sqcup I_2 \sqcup I_3 \to I_0 \sqcup I_3$$

sending $I_1$ to $I_0$ by identity, $I_2$ to $I_0$ by identity and $I_3$ to $I_3$ by identity.

**Lemma 4.2.2.** The composition of morphisms of complexes over $(X \setminus N)^I$:

$$\mathcal{H}_{I_1, W}^{e_d} \xrightarrow{\mathcal{F}_{I_1, I_2 \sqcup I_3, W}^{e_d}} \mathcal{H}_{I_1 \sqcup I_2 \sqcup I_3, W}^{e_d} \xrightarrow{\mathcal{F}_{I_1 \sqcup I_2 \sqcup I_3, W}^{e_d}} \mathcal{H}_{I_0 \sqcup I_3, W}^{e_d}$$

is the identity.

**Theorem 4.2.3.** $\mathcal{H}_{I, W}^I$ is ind-smooth over $(X \setminus N)^I$.

**Proof.** Step (1): For every $i \in I$, for any geometric point $\overline{\eta}_i = \text{Spec} \overline{\mathbb{F}}_q$ over a closed point of $X \setminus N$ and any specialization map $\text{sp}_i : \overline{\eta} \to \overline{\eta}_i$, we denote by $(X \setminus N)(\overline{\eta}_i)$ the strict henselisation of $X \setminus N$ at $\overline{\eta}_i$. Since $X \setminus N$ is smooth of dimension 1, $(X \setminus N)(\overline{\eta}_i)$ is a trait.

Applying Proposition 3.3.1 to:

- $S_i = (X \setminus N)(\overline{\eta}_i)$, $\overline{s}_i = \overline{\eta}_i$, $\overline{\delta}_i$ the image of $\overline{\eta}$ in $S_i$,
- $\mathcal{F} = \mathcal{H}_{I, W}^{(1, 2, 3)}$, which is a pseudo-product sheaf over $\times_{i \in I} S_i$ by Proposition 2.4.1,
- $\mathcal{G}_i = \mathcal{H}_{I, W}^{(1, 2, 3)}$, which is a pseudo-product sheaf over $\times_{i \in I} S_i$ by Proposition 2.4.1,
- $\mathcal{C}_1$ the creation morphism $\delta_i^{(1, 2, 3)}$,
- $\mathcal{C}_2$ the annihilation morphism $\delta_i^{(1, 2, 3)}$.

By Lemma 4.2.2 the hypothesis of Proposition 3.3.1 is satisfied. We deduce that $\mathcal{H}_{I, W}^{I}$ is constant over $\times_{i \in I} S_i$.

Step (2): We denote by $(X \setminus N)^I(\times_{i \in I} \overline{\eta}_i)$ the strictly henselisation of $(X \setminus N)^I$ at $\times_{i \in I} \overline{\eta}_i$. For every $j \in I$, the projections to $j$-th factor $(X \setminus N)^I \to X \setminus N$ and $\times_{i \in I} \overline{\eta}_i \to \overline{\eta}_j$ induce a morphism

$$f_j : (X \setminus N)^I(\times_{i \in I} \overline{\eta}_i) \to (X \setminus N)(\overline{\eta}_j) = S_j.$$

We deduce a morphism

$$(f_j)_{j \in I} : (X \setminus N)^I(\times_{i \in I} \overline{\eta}_i) \to \times_{j \in I} S_j.$$

Note that

$$(X \setminus N)^I(\times_{i \in I} \overline{\eta}_i) = (\times_{i \in I} S_i)(\times_{i \in I} \overline{\eta}_i)$$

Thus any specialization map $\text{sp} : \overline{\eta} \to \times_{i \in I} \overline{\eta}_i$ in $(X \setminus N)^I$ is also a specialization map in $\times_{i \in I} S_i$. By Step (1), $\mathcal{H}_{I, W}^{I}$ is constant over $\times_{i \in I} S_i$. Thus the induced morphism

$$\mathcal{H}_{I, W}^{I} \xrightarrow{\text{sp}^*} \mathcal{H}_{I, W}^{I}$$

is an isomorphism.
5. Action of Weil($X \smallsetminus N, \overline{\eta}$) on cohomology of stacks of shtukas

5.0.1. Let $v = \text{Spec } k(v)$ be a place of $X \smallsetminus N$. Fix an embedding $\mathcal{F} \subset \mathcal{F}_v$. Thus we have an inclusion $\text{Weil}(\mathcal{F}_v/F_v) \subset \text{Weil}(\mathcal{F}/F)$.

Let $\overline{k(v)}$ be the algebraic closure of $k(v)$ in $\mathcal{F}_v$. Let $\mathcal{F} = \text{Spec } \overline{k(v)}$ be the geometric point over $v$. Let $\mathcal{I}_v = \text{Ker}(\text{Weil}(\mathcal{F}_v/F_v) \rightarrow \text{Weil}(\overline{k(v)}/k(v)))$ be the inertia group at $v$.

5.0.2. Let $S = (X \smallsetminus N)_{\mathcal{F}}$ be the strict henselisation of $X \smallsetminus N$ at $\mathcal{F}$. By step (1) of the proof of Theorem 12.3, the restriction $\mathcal{H}^j_{I,W} |_{S^I}$ is a constant sheaf over $S^I$. As a consequence, the action of $(\mathcal{I}_v)^I$ on $\mathcal{H}^j_{I,W} |_{\mathcal{F}}$ is trivial.

5.0.3. The group $\text{Weil}(X \smallsetminus N, \overline{\eta})$ is the quotient of $\text{Weil}(\eta, \overline{\eta})$ by the subgroup generated by the $\mathcal{I}_v$ for all places $v \subset X \smallsetminus N$ and their conjugates.

For any $v$ and any embedding $\mathcal{F} \subset \mathcal{F}_v$, the action of $(\mathcal{I}_v)^I$ on $\mathcal{H}^j_{I,W} |_{\mathcal{F}}$ is trivial. We deduce
Proposition 5.0.4. The action of Weil(η, π) on $H^j_{I,W}|_{\mathfrak{m}^T}$ (defined in Proposition 1.3.4) factors through Weil($X \setminus N, \pi$).

6. THE CASE OF NON NECESSARILY SPLIT GROUPS

In this section, we use the context of [Laf18] Section 12.

6.0.1. Let $G$ be a geometrically connected smooth reductive group over $F$. As in [Laf18] Section 12.1, let $U$ be the maximal open subscheme of $X$ such that $G$ extends to a smooth reductive group scheme over $U$. We choose a parahoric integral model of $G$ on every point of $X \setminus U$. Then by gluing we obtain a smooth group scheme over $X$ that we still denote by $G$. Now $G$ is a smooth group scheme over $X$, reductive over $U$, of parahoric type at every point of $X \setminus U$. All fibers of $G$ are geometrically connected.

We denote by $\tilde{N} := |N| \cup (X \setminus U)$.

6.0.2. We use the Harder-Narasimhan truncation for $GL_r$ as in [Laf18] Section 12.1. We choose a vector bundle $V$ of rank $r$ over $X$ equipped with a trivialization of $\det(V)$ and an embedding $\rho : G_{ad} \to SL(V)$. We have $\rho^* : Bun_{G_{ad}} \to Bun_{0, GL_r}$. For any dominant coweight $\mu$ of $GL_r$, we define $Bun_{\leq \mu} G_{ad}$ to be $(\rho^*)^{-1}(Bun_{0, GL_r}^{\leq \mu})$. We define $Bun_{\leq \mu} G$ as inverse image of $Bun_{\leq \mu} G_{ad}$. The inductive systems induced by different choices of $V$ and $\rho$ are compatible.

6.0.3. As in [Laf18] Section 12.1, let $\tilde{F}$ be the finite Galois extension of $F$ such that $\text{Gal}(\tilde{F}/F)$ is the image of $\text{Gal}(F/F)$ in the group of automorphisms of the Dynkin diagram of $G$. Let the $L$-group $^L G$ be the semi-direct product $\hat{G} \rtimes \text{Gal}(\tilde{F}/F)$, where the semi-direct product is for the action of $\text{Gal}(\tilde{F}/F)$ on $\hat{G}$ preserving a pinning.

6.0.4. As in [Laf18] Section 12.3, we suppose $E$ large enough such that all irreducible representations of $^L G$ are defined over $E$.

We denote by $\text{Rep}_E((^L G)^I)$ the category of finite dimensional $E$-linear representation of $(^L G)^I$. Let $I$ be a finite set and $W \in \text{Rep}_E((^L G)^I)$. As in [Laf18] Section 12.3, we have a variant of the geometric Satake equivalence (loc.cit. Théorème 12.16). We define the stack of $G$-shtukas $\text{Ch}_{G,N,I,W}/\Xi$ over $(X \setminus \tilde{N})^I$, the canonical perverse sheaf $\mathcal{F}_{G,N,I,W}^\Xi$ over $\text{Ch}_{G,N,I,W}/\Xi$ and the morphism of paws $p_G : \text{Ch}_{G,N,I,W}/\Xi \to (X \setminus \tilde{N})^I$. For any dominant coweight $\mu$ of $GL_r$, we define the truncated stack of shtukas $\text{Ch}_{G,N,I,W}^{\leq \mu}/\Xi$ where the truncation follows from 6.0.2. For any $j \in \mathbb{Z}$, we define the sheaf of degree $j$ cohomology with compact support

$$\mathcal{H}^j_{G,N,I,W}^{\leq \mu} := R^j(p_G)_!(\mathcal{F}_{G,N,I,W}^\Xi)|_{\text{Ch}_{G,N,I,W}^{\leq \mu}/\Xi}$$
It is a constructible $E$-sheaf over $(X \smallsetminus \hat{N})^I$. We define
\[ \mathcal{H}_{G, N, I, W}^j := \lim_{\mu} \mathcal{H}_{G, N, I, W}^{j, \leq \mu} \]
in the abstract category of inductive limits of constructible $E$-sheaves over $(X \smallsetminus \hat{N})^I$.

As in [Laf18] Section 12.3, the sheaf $\mathcal{H}_{G, N, I, W}^j$ is equipped with an action of the partial Frobenius morphisms and an action of the Hecke algebra.

6.0.5. The results in Sections 1-5 still hold if we replace everywhere $X \smallsetminus N$ by $X \smallsetminus \hat{N}$ and replace everywhere $\hat{G}$ by $L^G$. We also replace the Eichler-Shimura relations in [Laf18] Sections 6-7 by the more general Eichler-Shimura relations in [Laf18] Section 12.3.3. Here are the slight modifications.

**Lemma 6.0.6.** ([Laf18] Section 12.3.3) Let $(v_i)_{i \in I}$ be a family of closed points of $X \smallsetminus \hat{N}$. Let $W = \bigotimes_{i \in I} W_i$ with $W_i \in \text{Rep}_E(L^G)$. Then there exists $\kappa$, such that for any $\mu$ and any $i \in I$, we have
\[ \sum_{\alpha=0}^{\dim W_i} (-1)^\alpha S_{\lambda_{\text{dim} W_i - \alpha W_i, v_i}}(F_{\deg(v_i)})^{\alpha} = 0 \text{ in } \text{Hom}(\mathcal{H}_{I, W}^{\leq \mu, x_i}, \mathcal{H}_{I, W}^{\leq \mu + \kappa, x_i}) \]
where $S_{\lambda_{\text{dim} W_i - \alpha W_i, v_i}} : \mathcal{H}_{I, W}^{\leq \mu, x_i} \to \mathcal{H}_{I, W}^{\leq \mu + \kappa, x_i}$ is defined in [Laf18] Section 12.3.3. \hfill $\square$

**Lemma 6.0.7.** ([Laf18] Section 12.3.3) The operator $S_{\lambda_{\text{dim} W_i - \alpha W_i, v_i}}$, which is a morphism of sheaves over $(X \smallsetminus \hat{N})^I$, extends the action of the Hecke operator $T(h_{\lambda_{\text{dim} W_i - \alpha W_i, v_i}}) \in \mathcal{H}_{G, v_i}$, which is a morphism of sheaves over $(X \smallsetminus (\hat{N} \cup v_i))^I$ defined by Hecke correspondence. \hfill $\square$

**Remark 6.0.8.** In [Laf18], there are more general statements which use the local $L$-group. But Lemma 6.0.6 and Lemma 6.0.7 are enough for us.

**Lemma 6.0.9.** $\mathcal{H}_{G, N, I, W}^j|_{\mathcal{M}}$ is an increasing union of $E$-vector subspaces $\mathcal{M}$ which are stable by $\text{FWeil}(\eta_1, \eta_1)$, and for which there exists a family $(v_i)_{i \in I}$ of closed points in $X \smallsetminus \hat{N}$ (depending on $\mathcal{M}$) such that $\mathcal{M}$ is stable under the action of $\otimes_{i \in I} \mathcal{H}_{G, v_i}$ and is of finite type as module over $\otimes_{i \in I} \mathcal{H}_{G, v_i}$.

**Proof.** Note that the category $\text{Rep}_E(L^G)^I$ is semisimple, it is enough to prove the lemma for $W$ irreducible, which is of the form $W = \bigotimes_{i \in I} W_i$ with $W_i \in \text{Rep}_E(L^G)$. Then the proof is the same as the proof of Lemma 13.4. \hfill $\square$

**Proposition 6.0.10.** The action of $\text{FWeil}(\eta_1, \eta_1)$ on $\mathcal{H}_{G, N, I, W}^j|_{\mathcal{M}}$ factors through $\text{Weil}(\eta_1, \eta_1)^I$. \hfill $\square$

**Proposition 6.0.11.** $\mathcal{H}_{I, W}^j|_{\mathcal{M}}$ is smooth over $(\eta_1)^I$. \hfill $\square$
Proof. The proof is the same as the proof of Proposition 4.3.3 \[\square\]

The same argument as in Section 4 proves

**Theorem 6.0.12.** \(\mathcal{H}^{j}_{G,N,I,W}\) is ind-smooth over \((X \smallsetminus \hat{N})^I\). In other words, for any geometric point \(\overline{x}\) of \((X \smallsetminus \hat{N})^I\) and any specialization map \(\text{sp}_{\overline{x}} : \overline{\eta} \to \overline{x}\), the induced morphism

\[
(6.1) \quad \text{sp}_{\overline{x}}^* : \mathcal{H}^{j}_{G,N,I,W} \mid_{\overline{x}} \to \mathcal{H}^{j}_{G,N,I,W} \mid_{\overline{\eta}}
\]

is an isomorphism. \[\square\]

The same argument as in Section 5 proves

**Proposition 6.0.13.** The action of \(\text{Weil}(\eta, \overline{\eta})^I\) on \(\mathcal{H}^{j}_{G,N,I,W} \mid_{\overline{\eta}}\) (defined in Proposition 6.0.10) factors through \(\text{Weil}(X \smallsetminus \hat{N}, \overline{\eta})^I\). \[\square\]

7. Smooth cuspidal cohomology sheaf

In this section, we assume that \(G\) is split because the constant term morphisms are written only in the split case in [Xue18a].

7.0.1. Let \(P\) be a parabolic subgroup and \(M\) be its Levi quotient. In [Xue18a] Section 3.4, we defined the cohomology sheaf of stack of \(M\)-shtukas: \(\mathcal{H}^{j}_{M,N,I,W}\) over \((X \smallsetminus N)^I\). In loc.cit. Section 3.5, we constructed the constant term morphism of sheaves over \(\eta_I\)

\[
(7.1) \quad \mathcal{C}^{P,j}_{G,N} : \mathcal{H}^{j}_{G,N,I,W} \mid_{\eta_I} \to \mathcal{H}^{j}_{M,N,I,W} \mid_{\eta_I}
\]

Similarly to Theorem 4.2.3, we prove

**Proposition 7.0.2.** For any geometric point \(\overline{x}\) of \((X \smallsetminus N)^I\) and any specialization map \(\text{sp}_{\overline{x}} : \overline{\eta} \to \overline{x}\), the induced morphism

\[
(7.2) \quad \text{sp}_{\overline{x}}^* : \mathcal{H}^{j}_{M,N,I,W} \mid_{\overline{x}} \to \mathcal{H}^{j}_{M,N,I,W} \mid_{\overline{\eta}}
\]

is an isomorphism.

By Theorem 4.2.3 and Proposition 7.0.2, \(\mathcal{H}^{j}_{G,N,I,W}\) and \(\mathcal{H}^{j}_{M,N,I,W}\) are ind-smooth. We extend morphism (7.1) to a morphism over \((X \smallsetminus N)^I\)

\[
(7.3) \quad \mathcal{C}^{P,j}_{G,N} : \mathcal{H}^{j}_{G,N,I,W} \to \mathcal{H}^{j}_{M,N,I,W}
\]

**Definition 7.0.3.** We define the cuspidal cohomology sheaf over \((X \smallsetminus N)^I\) to be

\[
\mathcal{H}^{j,\text{cusp}}_{G,N,I,W} := \bigcap_{P < G} \text{Ker} \mathcal{C}^{P,j}_{G,N}
\]
7.0.4. By definition, $\mathcal{H}^{j, \text{cusp}}_{G,N,I,W}|_{\mathcal{X}}$ is the cuspidal cohomology group $H^{j, \text{cusp}}_{G,N,I,W}$ defined in Xue18a Definition 3.5.13.

**Proposition 7.0.5.** $\mathcal{H}^{j, \text{cusp}}_{G,N,I,W}$ is a smooth E-sheaf over $(X \setminus N)^{\mathfrak{f}}$.

**Proof.** We deduce from Theorem 4.2.3 and Proposition 7.0.2 that for any geometric point $\mathfrak{f}$ of $(X \setminus N)^{\mathfrak{f}}$ and any specialization map $sp_{\mathfrak{f}}: \mathcal{X} \to \mathfrak{f}$, the induced morphism

\[ sp_{\mathfrak{f}}^*: \mathcal{H}^{j, \text{cusp}}_{G,N,I,W}|_{\mathfrak{f}} \to \mathcal{H}^{j, \text{cusp}}_{G,N,I,W}|_{\mathcal{X}} \]

is an isomorphism. So $\mathcal{H}^{j, \text{cusp}}_{G,N,I,W}$ is ind-smooth.

Moreover, by Xue18a Theorem 0.0.1, $\mathcal{H}^{j, \text{cusp}}_{G,N,I,W}|_{\mathcal{X}}$ has finite dimension. Thus $\mathcal{H}^{j, \text{cusp}}_{G,N,I,W}$ is a constructible sheaf.

We deduce that $\mathcal{H}^{j, \text{cusp}}_{G,N,I,W}$ is smooth. \(\square\)

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