Graph Isomorphism by Conversion to Chordal (6, 3) Graphs

M. Delacorte

Abstract

Babel has shown that for an extended class of chordal (6, 3) graphs the coarsest regular simplicial partition is equivalent to the graph's automorphism partition. We give a reversible transformation for any graph to one of these graphs by using Booth's reduction of a graph to a chordal graph and elimination of Babel's forbidden subgraphs for these graphs by adding edges to them.

1 Introduction

Chordal graphs are graphs where every cycle of size 4 or greater in the graph is chorded. In a \((q, t)\) graph [2] any set of \(q\) or less vertices will induce at most \(t\) \(P_4\)s (where \(P_4\) denotes a chordless path on four vertices). Given a graph \(G = (V, E)\) let \(V\) be its set of vertices, \(E\) its edge set, \(n = |V|, m = |E|\). For \(x \in V\) let \(N(x) = \{y \in V : xy \in E\}\) be the set of neighbours of \(x\). If \(x \cup N(x)\) induces the only clique of \(G\) that contains \(x\) then \(x\) is a simplicial vertex. For chordal graphs every induced subgraph has a least one simplicial vertex. An ordered partition \(V = S_1 \cup S_2 \cup ... \cup S_q\) of \(V\) is simplicial iff for \(1 < i < q\) all the nodes in \(S_i\) are simplicial nodes of the subgraph \(G(S_i \cup S_{i+1} ... \cup S_q)\). The sets \(S_i\) are called cells. Two cells \(S_i\) and \(S_j\) are called adjacent iff at least one node of \(S_i\) is adjacent to at least one node in \(S_j\); they are called totally adjacent iff all node of \(S_i\) are adjacent to all nodes in \(S_j\). The coarsest simplicial partition consists of \(S_i\) such that \(S_i\) equals the set of all simplicial nodes in \(G(S_i \cup ... \cup S_q)\). A simplicial partition \(S\) of \(G\) is called regular iff for any cells \(S_i, S_j\) in the partition and for any \(x, x'\) in \(S_i\) we have \(|N_j(x)| = |N_j(x')|\). For extended discussions of simplicial and regular partitions see [3, 4].

Babel gives a set of five 6 vertex forbidden graphs [1] which can be used to characterizes an extended class of chordal (6, 3) graphs (see fig. 1). He also characterizes three families of graphs stars, thin leg spiders and thick leg spiders for these graphs (see fig. 2).

A star graph. The vertex set \(V\) can be partitioned into non empty sets \(K_0, K_1, ..., K_r, r \geq 1\), such that

(i) \(|K_i| = |K_j|\) for \(1 \leq i, j \leq r\).

(ii) There are no edges between different sets \(K_i\) and \(K_j\), \(1 \leq i, j \leq r\).

(iii) \(K_0 \cup K_i, 1 \leq i \leq r\), induces a clique. \(K_0\) is the center of the star and, obviously, induces a complete subgraph.

The spider graphs. The vertex set \(V\) can be partitioned into sets \(K, S\) such that

(i) \(|K| = |S| \geq 2\), \(K\) induces a complete subgraph, \(S\) is a stable set.

(ii) There exists a bijection \(f : S \rightarrow K\) such that either

(a) for all \(s \in S, k \in K : sk \in E \leftrightarrow f(s) = k\) or

(b) for all \(s \in S, k \in K : sk \notin E \leftrightarrow f(s) = k\).

If the first of the two alternatives holds then \(G\) is said to be a spider with thin legs, otherwise with thick legs. A \(P_4\) is considered to be a spider with thin legs. Obviously, the complement of a spider with thin legs is a spider with thick legs and vice versa. \(K\) is the center of the spider.
Let $S_i, S_j$ be adjacent cells with $j > i$. Then $G(S_i \cup S_j)$ is the disjoint union of isomorphic stars or isomorphic spiders. $S_j$ consists exactly of the centers (which are of equal size).

Given a spider in $G(S_i \cup S_j)$, we now denote the vertices of the center by $w_1, w_2, \ldots, w_t$ and the remaining vertices by $u_1, u_2, \ldots, u_t$ in such a way that $u_1w_1 \in E$ for a spider with thin legs resp. $u_1w_1 \notin E$ for a spider with thick legs, $l = 1, 2, \ldots, t$.

Let $G(S_i \cup S_j)$, $j > i$, be the disjoint union of spiders and let $S_k$ be adjacent to $S_i$, $k > i$, $k \neq j$. Then $N_k(u_l) = N_k(w_1)$ holds for all $l$.

Let $G(S_i \cup S_k)$, $k > i$, be the disjoint union of stars for all cells $S_k$ which are adjacent to $S_i$. Further let $j$ be minimal, $j > i$, such that $S_j$ is adjacent to $S_i$ and $k \neq j$. Then $N_k(u) = N_k(w)$ holds for all $u \in S_i$, $w \in S_j$ with $uw \in E$.

Finally using the above lemmas it is proved that the coarsest regular simplicial partition of an extended class of chordal (6, 3) graph is the same as its automorphism partition.

2 Preliminaries

Before giving the algorithm to convert a graph to an extended class of chordal (6, 3) graph we give a few preliminaries. Graph isomorphism is polynomial time reducible to chordal graph isomorphism [4] as follows map given graph $G$ to its subdivision graph, and then connect all the vertices which were vertices in $G$ to each other. To transform a chordal graph to an extended class of chordal (6, 3) graph edges are added to each of the forbidden subgraphs in the graph (see fig. 3).
A system for marking the edges added to Babel's forbidden subgraphs to convert them to the extended chordal (6, 3) graphs is needed. To each node that is part of an added edge a tree graph is attached. This tree can encode information specific to the node i.e., which type of forbidden subgraph it is part of what position it holds in the subgraph (see fig. 4, 5). After a graph's forbidden subgraphs have been found and the edges added to them and marked new forbidden subgraphs may be created. So a second round of finding forbidden subgraphs adding edges and marking them will be necessary. To mark the second set of added edges the height of the marking trees has to be increased, to create a second level. There may be forbidden subgraphs that contain one or two root nodes of marking trees. Edges will not be added to these nodes (see fig. 6). After the second round of finding and marking a third round may be necessary. Since the graph can have at most $n(n - 1)/2$ edges this processes has to stop.

Figure 3: Edges added to forbidden graphs, in red to convert them to chordal (6, 3).
Figure 4: Tree graph for marking end nodes of edges added to eliminate forbidden graphs.
Figure 5: Examples of round 1 marked forbidden graphs
3 Isomorphism Testing

To test whether two graphs are isomorphic first test if graphs are chordal. If the graphs are not chordal apply Booth’s reduction to them. If the graphs are chordal test whether they contain any of the forbidden subgraphs. If the graphs contain forbidden subgraphs eliminate them by adding edges to them and marking them repeat this elimination process till the graphs are free of forbidden subgraphs. There are now two extended chordal (6, 3) graphs which may be tested as in Babel.

An alternate procedure that does not use marking trees can be done as follows. Create the two extended chordal (6, 3) graphs as above using booth reduction and adding edges to eliminate forbidden graphs as needed.
Test the graphs as in Babel if the automorphism partition is trivial the graphs are not isomorphic. If the partition is not trivial find the corresponding vertices in the two graphs from the simplicial partition align the graphs and see if the original edges in the two graphs match. If the edges do not match the graphs are not isomorphic.

References

[1] L. Babel. Isomorphism of chordal (6, 3) graphs Computing 54(4): 303-316 (1995)

[2] L. Babel, S. Olariu. On the isomorphism of graphs with few P4S. Workshop on Graph-Theoretic Concepts in Comp. Sci. WG ’95, M. Nagl, ed., Lecture Notes in Comput. Sci, 1017 (1995), 24-36

[3] L. Babel, I.N.Ponomarenko, and G. Tinhofer. The Isomorphism Problem For Directed Path Graphs and For Rooted Directed Path Graphs J. Algorithms 21 (1996) 542-564.

[4] Booth, Kellogg S.; Colbourn, C. J. Problems polynomially equivalent to graph isomorphism. Technical Report, No. CS-77-04, Computer Science Department, University of Waterloo 1979.