Supplementary Material to: *Remnant geometric Hall response in a quantum quench*

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We provide a detailed calculation, and description of the quench and probe protocol described in the letter. The protocol described in letter consists of the following steps: (1) prepare the state in the ground state of some Hamiltonian \(|\psi_0\rangle\); (2) at \(t = 0\) quench a parameter in that Hamiltonian; (3) pulse the evolved wave function at \(t = t_1\) (called \(|\psi_1\rangle\)) by letting \(\mathbf{k} \rightarrow \mathbf{k} - e\mathbf{A}\); and (4) measure Hall current after the wave function has evolved to \(t = t_2\) (called \(|\psi_2\rangle\)).

I. THE SINGLE MASSIVE DIRAC FERMION

We first consider the specific case of the Haldane model. In this model, there are two Dirac cones in a 2D Brillouin zone where one of the cones is (near the Dirac point)

\[
h(\mathbf{k}, \Delta) = v_F \mathbf{k} \cdot \sigma + \Delta \sigma_z. \tag{1}
\]

The second cone is similar except the kinetic energy term is related by the time-reversal operator \(T = K\) (complex conjugation), but time-reversal symmetry is broken so \(\Delta \rightarrow -\Delta\) as well. As we discuss in the main text, as well as below, breaking of time-reversal symmetry is essential, for a non-zero Hall response \(T\).

We proceed in the following sections by making the approximation that \(\mathbf{A}\) is small compared to the gap \(\Delta\). In this way, we can isolate a long-lived remnant Hall response that is reminiscent of a Hall conductivity and has properties related to the underlying symmetries which we can explicitly explore.

A. State preparation

Beginning with \(h(\mathbf{k}, \Delta)\), we can prepare the state in the ground state. This state is represented by

\[
|\psi_0\rangle = \cos \frac{\theta}{2} |\uparrow\rangle - e^{i\phi} \sin \frac{\theta}{2} |\downarrow\rangle, \tag{2}
\]

where

\[
\cos \theta = -\frac{\Delta}{\sqrt{k^2 + \Delta^2}}, \tag{3}
\]

\[
ke^{i\phi} = k_x + ik_y. \tag{4}
\]

B. Quench and evolve

With this state, we quench into \(h_0(\mathbf{k}, 0)\). The time evolution operator, as can be checked, is

\[
e^{-it_1h_0} = \cos kt_1 - i \sin kt_1 \frac{k \cdot \sigma}{k} \tag{5}
\]

\[
= \begin{pmatrix}
\cos kt_1 & -i \sin kt_1 e^{-i\phi} \\
-i \sin kt_1 e^{i\phi} & \cos kt_1
\end{pmatrix}. \tag{6}
\]

Thus, our state evolves as

\[
|\psi_1\rangle = f(t_1) |\uparrow\rangle - e^{i\phi} g(t_1) |\downarrow\rangle, \tag{7}
\]

where

\[
f(t_1) = \cos \frac{\theta}{2} \cos kt_1 + i \sin \frac{\theta}{2} \sin kt_1, \tag{8}
\]

\[
g(t_1) = \sin \frac{\theta}{2} \cos kt_1 + i \cos \frac{\theta}{2} \sin kt_1. \tag{9}
\]

C. Pulse

We now consider pulsing the state. The new evolution operator is the same as the above in Eq. (5), but with the substitution \(\mathbf{k} \rightarrow \mathbf{k} - e\mathbf{A}\).

In order to obtain analytically tractable solutions, we assume that \(e|\mathbf{A}| \ll k\) which will be equivalent to \(e|\mathbf{A}| \ll \Delta / v_F\) as we will show when we consider the entire band. Without loss of generality, we take the pulse in the \(x\)-direction, and the above expansion implies

\[
|\mathbf{k} - e\mathbf{A}| = k - eA_x \cos \phi, \tag{10}
\]

\[
e^{i(\phi_A - \phi)} = 1 + \frac{eA_x}{k} \sin \phi, \tag{11}
\]

where \(e^{i\phi_A} = (k_x - eA_x) + ik_y\). The time evolution operator can then be appropriately expanded while making no assumptions regarding the time:

\[
e^{-i\delta t h_A} = \begin{pmatrix}
\cos[(k - eA_x \cos \phi)\delta t] & -i \sin[(k - eA_x \cos \phi)\delta t](1 + i\frac{eA_x}{k} \sin \phi)e^{i\phi} \\
-i \sin[(k - eA_x \cos \phi)\delta t](1 + i\frac{eA_x}{k} \sin \phi)e^{-i\phi} & \cos[(k - eA_x \cos \phi)\delta t]
\end{pmatrix}. \tag{12}
\]
Here, time $δt = t_2 - t_1$. Now, we can evolve our state with $|ψ_2⟩ = e^{-i(t_2-t_1)h_{AA}} |ψ_1⟩$. The resulting expression can be deduced by matrix multiplication of Eq. (12) and Eq. (7).

**D. Measuring the current**

We are interested in the current for the entire system where the $k$ states in the valence band of $h(k, Δ)$ are fully occupied. This analysis can also be applied for filled states up to some chemical potential $μ$. In this section, we build up to that by looking first at the current of an individual state at $k$ in the valence band, then of a ring of states with the same $|k|$, and finally at the entire band itself.

1. The current of a single state

Having constructed the single-particle state $|ψ_2⟩$, we can calculate the expectation value of the perpendicular current $j_y = -eσ_y$ with the use of

\[
\langle ψ_2 | σ_y | ψ_2 \rangle = \text{Im} \, e^{iφ} \left\{ -i|f(t_1)|^2 - |g(t_1)|^2 \right\} (1 + \text{Re} e^{iφ} f^∗ g) \sin 2(k - eA_x \cos φ)(t_2 - t_1)
\]

and it is easily shown that

\[
|f(t_1)|^2 - |g(t_1)|^2 = \cos θ \cos 2kt_1, \quad (14)
\]

\[
f(t_1)^∗ g(t_1) = ½ (sin θ + i cos θ sin 2kt_1). \quad (15)
\]

Notice how the remnant Hall current previously alluded to is already showing up. If one drops out all oscillating currents and terms that do not depend on $A_x$, we obtain the term $2 \frac{A_x}{p} \sin φ \text{Re} e^{iφ} f$. This term will continue throughout the calculations, giving the remnant Hall effect.

We note the current is oscillatory. However, to understand how it is deviating from the non-pulsed current, we plot in Fig. 1 the difference $⟨σ_y⟩ - ⟨σ_y⟩ |A_x → 0$. In this figure, note that the pulse can change the frequency, leading to a beating effect. But for states that have momentum perpendicular to the pulse ($φ = π/2$), the center of oscillation is shifted.

2. The current of states with the same momentum

Now, we take the above and integrate it around a ring of constant of momentum. In particular, $j_y(k) = -e ∫_0^{π} \frac{dk}{2π} (ψ_2(k)|σ_y|ψ_2(k))$, and we obtain

\[
j_y(k) = -\frac{Δ}{2k^2 + Δ^2} \left\{ \frac{e^2 A_x}{2k} \sin 2kt_1 - eJ_1[2eA_x(t_2 - t_1)] \left( \cos 2kt_2 + \frac{\sin 2kt_2}{2k(t_2 - t_1)} \right) \right\}, \quad (16)
\]

with the assumption that $eA_x ≪ Δ/ν_F$. Going from the assumption that $eA_x ≪ k$ to $eA_x ≪ Δ/ν_F$ is nontrivial, but roughly: most states contributing to current have $k ∼ O(Δ/ν_F)$. We will make this precise below.

Therefore, we use Eq. (16) to evaluate the total current. Inspection of Eq. (16) reveals that there are only two integrals to consider and we consider them in turn:

\[
A_1 = ∫_0^{∞} \frac{kdk}{2π|Δ|} \frac{\cos 2kt}{\sqrt{k^2 + Δ^2}}, \quad (18)
\]

\[
A_2 = ∫_0^{∞} \frac{dk}{2π} \frac{\sin 2kt}{k^2 + Δ^2}. \quad (19)
\]

It seems as though $A_1$ is infinity, but we can evaluate it...
FIG. 1. The single-particle current response: $\langle j_y \rangle - \langle j_y \rangle |_{A_x=0}$ using state $|\psi_2\rangle$. For (a), the state’s momentum is parallel to the pulse and we see the resulting beating due to a change in frequency. On the other hand, in (b) we see no sign of beating but the center of oscillation is shifted from zero for different pulse times $t_1$.

FIG. 2. The combined current for all states at a constant momentum $p = 3\Delta/v_F$ and $eA_x = 0.3\Delta/v_F$. We illustrate the Bessel function behavior at a finite pulse-time $t_1$. While difficult to discern, this still oscillates about a finite, persistent value.

$$A_1 = \int_0^{\infty} dx \cos 2xt|\Delta| \frac{2\pi}{2\pi \sqrt{x^2 + 1}}$$

$$= \int_0^{\infty} \frac{dx}{2\pi} \cos 2xt|\Delta| + \int_0^{\infty} \frac{dx}{2\pi} \cos 2xt|\Delta|$$

The first term is purely oscillatory and can be evaluated as a $\delta$-function. The second term can be rewritten as a convergent integral using complex analysis;

$$A_1 = \frac{1}{2} \delta(2t|\Delta|) - \int_0^{\pi/2} d\phi \sin 2\phi e^{-2t|\Delta| \sin \phi}. \quad (22)$$

On the other hand, $A_2$ can be easily converted as well into

$$A_2 = \int_0^{\pi/2} d\phi e^{-2t|\Delta| \sin \phi}. \quad (23)$$

In terms of modified Bessel functions ($I_\nu(x)$) and modified Struve functions ($L_\nu(x)$), we can write these as

$$A_1 = \frac{1}{2} \delta(2t|\Delta|) + \frac{1}{2} (I_1(2t|\Delta|) - L_1(2t|\Delta|)),$$

$$A_2 = \frac{1}{2} (I_0(2t|\Delta|) - L_0(2t|\Delta|)). \quad (24)$$

The $\delta$-function will not be an issue since it has support only when $t_2 = 0$.

For ease of notation, we define

$$K_\nu(x) = \int_0^{\pi/2} \frac{dz}{2\pi} (-1)^\nu \sin^\nu z e^{-2z \sin z}. \quad (26)$$

Thus, the overall current can be written as (putting in all the appropriate units)
\[ J_y = -\frac{\pi e^2}{h} \left\{ \frac{A_x \Delta}{h} K_0(t_1 |\Delta|/h) - J_1[2v_F eA_x(t_2 - t_1)/\hbar] \right\} \left[ \frac{\Delta}{ev_F(t_2 - t_1)} K_0(t_2 |\Delta|/h) + \text{sgn}(\Delta) \frac{\Delta^2}{ev_F} K_1(t_2 |\Delta|/h) \right]. \] \tag{27}

This form of \( J_y \) allows us to investigate some of the asymptotic properties. Importantly, the first term is clearly what we get as a remnant current for \( t_2 \to \infty \). Additionally, in that same limit we can use the asymptotic form of \( J_1 \) yielding the next order term that dies off as \( t_2^{-5/2} \).

The current is zero when \( t_1 = t_2 \). However, at long times, we can see the remnant effect \( J_y^\infty \) which we explicitly define

\[ J_y^\infty = -\frac{\pi e^2}{h} \left( \frac{A_x \Delta}{h} \right) K_0(t_1 |\Delta|/h). \] \tag{28}

As \( t_1 \to 0 \), we get

\[ J_y^\infty \sim -\frac{\pi e^2}{4h} \left( \frac{A_x \Delta}{h} \right) \left( 1 - \frac{t_1 |\Delta|}{\pi \hbar} \right). \] \tag{29}

On the other hand, as \( t_1 \to \infty \) (technically, \( t_1 \gg \hbar/|\Delta| \) which is allowed in our approximation scheme)

\[ J_y^\infty \sim -\text{sgn}(\Delta) \frac{e^2}{4h} \frac{A_x}{t_1}. \] \tag{30}

This die off with \( t_1 \) is due to dephasing in the system.

The current itself can be seen for various values of \( t_1 \) in Fig. 3. Note that the current starts at zero and grows to saturate at its long-time value. We note that the order of limits matters here: If we let \( t_1 = 0 \) before we integrate, we would not get any long-time value; however, if we wait any amount of time, we immediately get the large value here. This can be understood by how the states are evolving on the Bloch sphere as we will discuss in a later section and in the main text.

With all of this, we now look to analyze explicitly the remnant Hall current. In so doing, we will use to quantum geometry as a tool. We have previously assumed \( eA_x \ll k \) which is equivalent to assuming \( eA_x \ll \Delta/v_F \) since we will be integrating over all \( k \).

4. Relation between \( eA_x \ll k \) and \( eA_x \ll \Delta/v_F \)

In the following section, we crucially assume only \( eA_x \ll \Delta/v_F \), and we say \( j_y^\text{approx}(k) \) is given by Eq. (16) while \( j_y(k) \) is the exact expression.

Therefore, \( j_y(k) = j_y^\text{approx}(k) + O(eA_x/k)^2 \).

First, pick a value \( K \) that is small and for which we want to cut-off the integral

\[ J_y = \int_0^\infty \frac{k dk}{2\pi} j_y(k) \]

\[ = \int_K^\infty \frac{k dk}{2\pi} j_y(k) + \frac{1}{2} K^2 j_y(0) + \cdots. \] \tag{32}

The value of the current \( j_y(0) \) can be determined by considering the evolution of the zero-momentum state. The state begins as as \( |\downarrow\rangle \), and is quenched to \( H = 0 \) where it does not evolve. Upon application of the pulse the Hamiltonian is suddenly \( H = -v_F eA_x \sigma_x \), and the evolution of \( j_y(0) \) is given by \( \langle \sigma_y \rangle = -\sin 2v_F eA_x(t_2 - t_1) \).

Returning to Eq. (32), we can replace \( j_y(k) \) by the approximate version by assuming \( eA_x \ll K \) and the first order correction will be

\[ J_y = \int_K^\infty \frac{k dk}{2\pi} j_y^\text{approx}(k) - \frac{1}{2} K^2 \sin(2v_F eA_x t) + O((eA_x)^2 \log[v_F K/\Delta]). \] \tag{33}

We can then expand the integral assuming \( K \ll \Delta/v_F \)

\[ J_y = \int_0^\infty \frac{k dk}{2\pi} j_y^\text{approx}(k) \]

\[ -\frac{1}{2} K^2 [j_y^\text{approx}(0) - \epsilon \sin(2v_F eA_x (t_2 - t_1))] \]

\[ + O((eA_x)^2 \log[v_F K/\Delta]). \] \tag{34}

Now, we just need to choose the right \( K \); we want \( eA_x \ll K \ll \Delta/v_F \) which is satisfied if we choose a small \( \epsilon > 0 \) such that

\[ K \approx \frac{\Delta}{v_F} \left( \frac{eA_x}{\Delta} \right)^{1-\epsilon/2} \] \tag{35}

or even by \( K \approx eA_x \log[\Delta/v_F eA_x] \). The \( K^2 \) term will...

![FIG. 3. The current of the whole valence band of the Haldane model following the quench and subsequent pulse with size \( eA_x = 0.1 \Delta/v_F \). The current starts at zero until it is pulsed at \( t_1 \) at which point it moves towards its long-time value \( J_y^\infty \).](image-url)
then be the lowest order in the expression and we have
\[
J_y = \int_0^\infty \frac{k}{2\pi} j_y^{\text{approx}}(k) \, dk
- \frac{1}{2} \Delta^2 \left( \frac{\nu_F e A_x}{\Delta} \right)^{2-\epsilon} \left[ j_y^{\text{approx}}(0) - e \sin(2\nu_F e A_x t) \right].
\] (36)

Lastly, we need to know what \( j_y^{\text{approx}}(0) \) is. In fact,
\[
j_y^{\text{approx}}(0) = -e^2 A_x t_1 - eJ_1(2eA_x(t_2-t_1)) \frac{t_2}{t_2-t_1}.
\] (37)

Nearly all is well-behaved and bounded except the term linear in \( t_1 \). In order to make this a proper asymptotic expansion, we need to further assume \( \nu_F e A_x t_1 / h \lesssim O(1) \), or
\[
t_1 \lesssim O(1) \frac{h}{\nu_F e A_x} \gg \frac{h}{\Delta}.
\] (38)

In the last line we have just used that \( e A_x \ll \Delta / \nu_F \). This shows us that while we need to keep \( t_1 \) small, it can actually be much larger than the times in the system \( h/\Delta \). The time scale we need to keep \( t_1 \) under is determined by assuming the above remains a good asymptotic expansion:
\[
\frac{\Delta t_1}{h} \ll \left( \frac{\Delta}{\nu_F e A_x} \right)^{2-\epsilon}.
\] (39)

Therefore, we can safely use the approximate version of \( j_y(k) \) given in Eq. (16). In the exact expression Eq. (49), this will be explicitly seen, and the next order term will clearly be shown to be \((\nu_F e A_x / \Delta)^2\).

**II. BLOCH GEOMETRY AND THE REMNANT HALL CURRENT**

Now we consider the Bloch sphere and what it means to have a remnant Hall current. In our example, the current is given exclusively by \( \langle \sigma_y \rangle \) which is just the projection of our Bloch vector onto the \( y \)-axis.

The idea is captured in Figures 4 and 5. The key feature being: the pulse shifts the center of rotation, in part, perpendicular to the pulse. Therefore, we get an average current in that direction that is seen once the system dephases.

This remnant Hall current can be calculated exactly using these ideas without the approximation \( e A_x \ll k \) that we made in previous sections.

### A. Bloch geometry

Say we have a path on the Bloch sphere \( \mathbf{r}(t) \) which rotates around some vector \( \mathbf{B} \), then the average of this path around one cycle can be easily given by a projection \( (\mathbf{r}(t) \cdot \mathbf{B}) \mathbf{B} \) where \( \mathbf{B} = \mathbf{B}/B \) is a unit vector. If we further want to know how much of this is along the \( y \)-direction, we can project it along \( y \), giving us simply
\[
Y(\mathbf{B} ; \mathbf{r}(t)) = (\mathbf{r}(t) \cdot \mathbf{B})(\mathbf{B} \cdot \hat{y}).
\] (40)

This quantity represents the remnant Hall current for a particular state and is independent of \( t \) (given \( \mathbf{r}(t) \) circulating around \( \mathbf{B} \)), and when we integrate it over all states we will obtain the remnant Hall current as previously discussed.

After our mapping to the Bloch sphere, we begin in a state on the sphere \( \mathbf{r}_0 \), rotate into \( \mathbf{r}_1 \) at time \( t_1 \) by precession around the vector \( 2\mathbf{p} \). The pulse then causes precession about the vector \( \mathbf{B} = 2(\mathbf{p} - e \mathbf{A}) \), but averaging the result over a period of precession gives exactly Eq. (40) with \( \mathbf{r}(t) = \mathbf{r}_1 \). Therefore, to find \( \mathbf{r}_1 \), we look at the precession of our initial state \( \mathbf{r}_0 \) around the vector \( 2\mathbf{p} \) (\( |\psi_1\rangle \) is represented by Bloch vector \( \mathbf{r}_1 \) here). Mapping our vector appropriately,
\[
\mathbf{r}_0 = -\mathbf{p} \sin \theta + \mathbf{z} \cos \theta.
\] (41)

As we rotate around \( \mathbf{p} \) at a rate of \( 2\mathbf{p} \), we just need to let \( \mathbf{z} \rightarrow (\cos 2\mathbf{p} t \mathbf{z} + \sin 2\mathbf{p} t \mathbf{z} \times \mathbf{y}) \). Therefore,
\[
\mathbf{r}_1 = -\mathbf{p} \sin \theta + (\cos 2\mathbf{p} t_1 \mathbf{z} - \sin 2\mathbf{p} t_1 \mathbf{z} \times \mathbf{p}) \cos \theta.
\] (42)

The quantity we are interested in for this problem is
\[
Y(p, \phi) = \frac{|\mathbf{r}_1 \cdot (\mathbf{p} - e \mathbf{A})|| (\mathbf{p} - e \mathbf{A}) \cdot \hat{y}|}{|\mathbf{p} - e \mathbf{A}|^2}.
\] (43)

Imposing \( \mathbf{A} = A_x \mathbf{x} \), we can begin evaluating
\[
(\mathbf{p} - e \mathbf{A}) \cdot \hat{y} = p \sin \phi.
\] (44)

Then, in order to evaluate \( \mathbf{r}_1 \cdot (\mathbf{p} - e \mathbf{A}) \) we find
\[
\mathbf{p} \cdot (\mathbf{p} - e \mathbf{A}) = p - e A_x \cos \phi,
\] (45)
\[
(\mathbf{z} \times \mathbf{p}) \cdot (\mathbf{p} - e \mathbf{A}) = e A_x \sin \phi.
\] (46)

These equations imply
\[
Y(p, \phi) = \frac{p[- \sin \theta (e A_x \cos \phi - e A_x \sin 2pt_1 \cos \theta \sin \phi \sin \phi)]}{p^2 + e^2 A_x^2 - 2eA_x p \cos \phi}.
\] (47)

Just as before, we integrate first around \( d\phi \). With some complex integration, we can write the final result as
\[
\int \frac{d\phi}{2\pi} Y(p, \phi) = \frac{1}{2} \cos \theta \sin 2pt_1 \left\{ \frac{e A_x}{p} \right\} \begin{cases} \frac{|e A_x|}{p} & |e A_x| < p, \\ \frac{|e A_x|}{p} & |e A_x| > p. \end{cases}
\] (48)

And finally, we can integrate this expression over momentum to obtain \( J_y \). Inserting all physical constants:
At lowest order Eq. (49) agrees with Eq. (28), and we see that the higher order terms do in fact die off as \((a.A_.2/\Delta)^2\).

**B. General two-band theory**

We now consider the general two-band theory for this Hall effect. In general, we define the remnant Hall response as the antisymmetric part of the tensor \(\Xi_{\nu}^\mu\) that is obtained from

\[
J^\infty_\mu = \Xi_{\nu}^\mu A_\nu + O(A^2).
\]

Since we are working in two-dimensions, this is just a single value we call \(\Xi_{\nu}^\mu\) Hall = \(\Xi_{\nu}^z - \Xi_{\nu}^y\).

The general two-band model we have in mind is

\[
h(p, \Delta) = \epsilon_0(p)I + d(p, \Delta) \cdot \sigma.
\]

FIG. 4. The green arrows represent the states being considered: Both have constant \(p\) and are at angles \(\phi = \pm \pi/2\). The pulse causes the states to rotate clockwise around a point north of it on the equator (it rotates around the momentum vector \(p\)). At time \(t_1 = \frac{\pi}{8kvp}\), both states are on the equator and we apply a pulse \(eA_\nu > 0\). This shifts the center of rotation making the circle representing Rabi oscillation smaller for \(\phi = \pi/2\) and larger for \(\phi = -\pi/2\). In Fig. 5 we show how this leads to a shift in the average \(<\sigma_y>\).

FIG. 5. From a bird’s eye-view, we can see that the Rabi oscillations represented in Fig. 4 cause a movement in the center of rotation along the \(y\)-direction (represented by \(\Delta y\) in the figure). Just as in Fig. 4, we have taken \(p\) the same for both and \(t_1 = \frac{\pi}{8kvp}\).

\[
J^\infty_y = \frac{e^2}{2h} \left( \frac{A_\Delta}{h} \right) \left[ \int_0^{\pi/2} dz \, e^{-2t_1|\Delta| \sin z} + \frac{|eA_x|}{\Delta} \int_0^1 dx \, \frac{x^2 - 1}{\sqrt{1 + (\frac{2A_x}{\Delta})^2}} \sin(2|eA_x|xt_1) \right]. \tag{49}
\]

Without loss of generality, we quench \(\Delta \to 0\) at \(t = 0\). The current operator for such a theory is \(\partial_\mu \epsilon_0(p, \Delta) = \partial_\mu \epsilon_0(p)I + \partial_\mu d(p, \Delta)\).

First, let us show that \(\partial_\mu \epsilon_0(p)I\) does not contribute to the Hall conductivity. Considering the state \(|\psi_2>\), the first term in the expression for current is simply

\[
<\psi_2|\partial_\mu \epsilon_0(p - eA)|\psi_2> = \partial_\mu \epsilon_0(p - eA) = \partial_\mu \epsilon_0(p) - eA_\nu \partial_\nu \partial_\mu \epsilon_0(p). \tag{53}
\]

The last term is symmetric in \(\mu\) and \(\nu\) and therefore cannot contribute to a Hall response. Therefore, we only need to consider \(\partial_\mu d(p, \Delta) \cdot \sigma\) for the Hall current.

Taking the view from before, we have simply that the current at infinite time should be represented by

\[
J^\infty_\mu(p, t_1) = -e \partial_\mu d(p - eA, \Delta) \cdot <\psi_2|\sigma|\psi_2>, \tag{54}
\]

where \(<\psi_2|\sigma|\psi_2>\) represents the time-average of the state over a period of Larmor precession.
The average is purely determined by the state right after the pulse \( |\psi_1\rangle \) which is a point on the Bloch sphere, and this is determined by the evolution of \( \hat{n} = \langle \psi(t) | \sigma | \psi(t) \rangle \) in the time frame \( t \in [0, t_1] \). The equation of motion is

\[
\partial_t \hat{n} = 2d(p, 0) \times \hat{n}, \quad \hat{n}(t_1) = -\hat{d}(p, \Delta) .
\] (55)

After the pulse, the new center of rotation is defined by the vector \( \hat{d}(p - eA) \), and the average is simply

\[
\langle \psi_2 | \sigma | \psi_2 \rangle = \hat{d}(p - eA, 0) \cdot [\hat{d}(p - eA, 0) \cdot \hat{n}(t_1)] .
\] (56)

It is then a simple matter to show

\[
j^\mu_\infty(p, t_1) = -e[\hat{d}(p - eA, 0) \cdot \hat{n}(t_1)] \partial_\mu d(p - eA, 0) .
\] (57)

The integral of this over the occupied momenta \( p \) will give the total current at infinite-time \( J^\mu_\infty \). However, we are interested in picking out the Hall contribution. We find that if we define \( j^\mu_\infty(p, t_1) \approx \chi_{\mu\nu}^N(p, t_1)A_\nu \) and \( \chi_{\text{Hall}}^\infty(p, t_1) = \frac{1}{2} \chi_{xy}^\infty(p, t_1) \), then \( \Xi_{\text{Hall}} = \int_p \chi_{\text{Hall}}^\infty \), \( \hat{d}_0 = \hat{d}(p, 0) \), and \( \hat{d} = \hat{d}(p, \Delta) \), and

\[
\chi_{\text{Hall}}^\infty(p, t_1) = e^2[\partial_\mu d_0 \partial_\mu d_0 - \partial_\mu d_0 \partial_\mu d_0] \cdot \hat{n}(t_1) .
\] (58)

1. **Condition for no remnant Hall current**

We can solve Eq. (55) and obtain

\[
\hat{n}(t_1) = \hat{d}_0 \cdot \hat{d}_0 - \hat{d}_0 \times (\hat{d}_0 \times \hat{d}) \cos 2d_0 t_1 - \hat{d}_0 \times \hat{d} \sin 2d_0 t_1 .
\] (59)

This can be substituted into Eq. (58), and we get

\[
\chi_{\text{Hall}}^\infty(p, t_1) = e^2[\partial_\mu d_0 \partial_\mu d_0 - \partial_\mu d_0 \partial_\mu d_0] \cdot [\hat{d} \cos 2d_0 t_1 - \hat{d}_0 \times \hat{d} \sin 2d_0 t_1] .
\] (60)

These terms will integrate to zero given some special symmetries.

The natural symmetry to consider is time-reversal symmetry. Indeed, this helps, but only partially, since for a time-reversal preserving system, we have

\[
T^{-1}h(-p)T = h(p)
\] (61)

\[
\hat{d}(-p) \cdot T^{-1} \sigma T = \hat{d}(p) \cdot \sigma .
\] (62)

As an anti-unitary operator, \( T = UK \) where \( U \) is a unitary and \( K \) is complex conjugation. Complex conjugation will just let \( \sigma_y \rightarrow -\sigma_y \), and \( U \) will rotate along the Bloch sphere. Thus, any sort of triple product will change sign, and we have

\[
\partial_\mu d_0[\partial_\mu d \cdot (\hat{d}_0 \times \hat{d})] \rightarrow -\partial_\mu d_0[\partial_\mu d \cdot (\hat{d}_0 \times \hat{d})] .
\] (63)

Naturally \( d_0(-p) = d_0(p) \) as well. Thus, with both momenta connected by time-reversal symmetry, they lie at the same energy and any integral over a finite chemical potential will cancel their contributions.

However, there is still the possibility that the term that goes as \( \cos 2d_0 t_1 \) in Eq. (58) will be finite.

To address this term, if both systems respect a mirror symmetry (unitary or anti-unitary) along one plane, this term will also vanish. In particular, let us say the “mirror plane” is along the \( x \)-direction without loss of generality. Then, define the mirror symmetry operator \( M_y = UK \) (where \( a = 0 \) if unitary and \( a = 1 \) if anti-unitary) as acting on \( H \) such that

\[
M_y^{-1}h(p_x, -p_y)M_y = h(p_x, p_y)
\] (64)

\[
\hat{I} \hat{d}(p_x, -p_y) \cdot \sigma = \hat{d}(p_x, p_y) \cdot \sigma ,
\] (65)

where \( \hat{I} \) is either the identity or an inversion operator on \( \hat{d} \). In fact, \( \hat{I} \) should act as the identity on \( d(p_x, 0) \), and so all \( d(p_x, 0) \) span an invariant subspace for \( \hat{I} \). Generally this space will be more than one dimensional, and so if it is two-dimensional (if it is three-dimensional, \( \hat{I} \) will just be the identity), we have

\[
h(p_x, 0) = d_x(p_x, 0)\sigma_x + d_z(p_x, 0)\sigma_z ,
\] (66)

where without loss of generality we chose the \( x \)- and \( z \)-directions to be the invariant directions. Thus, we have \( d_y(p_x, p_y) \rightarrow 0 \) as \( p_y \rightarrow 0 \). Furthermore, inversion can only occur in that direction, so we have

\[
d(p_x, -p_y) = (d_x(p_x, p_y), d_z(p_x, p_y), d_z(p_x, p_y)).
\] (67)

This allows us to say

\[
\partial_\mu d_0 \partial_\mu d_0 \cdot d(p_x, -p_y) \rightarrow -\partial_\mu d_0 \partial_\mu d_0 \cdot \hat{d} .
\] (68)

And by similar reasoning as before, once we integrate over all momenta, there will be no contribution to the Hall current.

Thus, we have proven the following statement: \( h(p, \Delta) \) and \( h(p, 0) \) both have time-reversal symmetry and mirror symmetry along the same plane, then \( \Xi_{\text{Hall}} = 0 \).

Another way to phrase this is: if \( h(p, \Delta) \) has mirror symmetry along some axis for all \( \Delta \), then time-reversal symmetry before and after the quench implies \( \Xi_{\text{Hall}} = 0 \). The models we study are just that: Models that have such a mirror symmetry.

2. **Relation to Berry curvature**

We can now make a more precise claim about the Hall current’s relation to Berry curvature.

First, we take the equation for Larmor precession Eq. (55) and rewrite it as

\[
\hat{d}_0 = \frac{\hat{n} \times \partial_\mu \hat{n}}{2d_0} - \hat{n}(\hat{d}_0 \cdot \hat{d}) .
\] (69)
The term that is important for Eq. (58) is

$$\hat{n} \cdot \partial_{\mu} \hat{d}_0 = \hat{n} \cdot \left[ \frac{\partial_{\mu} \hat{n} \times \partial_{\nu} \hat{n}}{2d_0} \right] - \partial_{\mu}(\hat{d}_0 \cdot \hat{d}).$$

(70)

Now, we can write the Berry curvature as $\Omega_{\mu\nu} = \frac{1}{2} \hat{n} \cdot (\partial_{\mu} \hat{n} \times \partial_{\nu} \hat{n})$, and thus, we have

$$\chi_H(\mathbf{p}, t_1) = e^2 \{ \partial_{\nu} \log d_0[\Omega_{\nu\kappa 1} - \partial_\kappa (\hat{d}_0 \cdot \hat{d})] - \partial_\kappa \log d_0[\Omega_{\nu\kappa t_1} - \partial_\kappa (\hat{d}_0 \cdot \hat{d})] \}. \quad (71)$$

If we assume the system has the mirror-symmetry described in the preceding section, then

$$\chi_H^\infty(\mathbf{p}, t_1) = e^2 \int_p \partial_{\nu} \log d_0 - \Omega_{\nu\kappa t_1} \partial_\kappa \log d_0. \quad (72)$$

This simplifies one step further if we began in a filled band, then we can integrate by parts without picking up boundary terms and we get

$$\chi_H^\infty = e^2 \int_p \partial_{\nu} \Omega_{\nu\kappa \nu} \log d_0. \quad (73)$$

The response of the system is purely described in terms of a weighted integral over the derivative of the Berry curvature. This Berry curvature is evaluated for the state at the time of the pulse.

3. Full time current response

For completeness, we just mention that if we define $\hat{n}_2 = \langle \psi_2 | \sigma | \psi_2 \rangle$, we can easily write the expression for the single particle current response:

$$j_{\mu}(\mathbf{p}; t_2, t_1) = j_{\mu}^\infty(\mathbf{p}; t_1) \hat{d} \cdot (\hat{d} \times \partial_{\kappa \nu} \hat{n}_2), \quad (74)$$

where we just defined $\hat{d} \cdot \hat{d} = \hat{d}(\mathbf{p} - e\mathbf{A}, 0)$.

III. THE TIME-REVERSAL BROKEN BHZ MODEL

Lastly, we consider the time-reversal broken BHZ model with

$$h(\mathbf{k}, M) = \mathbf{d}(\mathbf{k}, M) \cdot \sigma, \quad (75)$$

where

$$\mathbf{d}(\mathbf{k}, M) = (\sin k_x, \sin k_y, M + 2 - \cos k_x - \cos k_y). \quad (76)$$

This model admits 4 gapped (or ungapped at the phase transitions) Dirac cones. In equilibrium, depending on the sign of the gap at each point (controlled by $M$), we can have a positive, negative or zero Hall conductance overall.

Generically, we will quench from $M$ to $M'$, then measure current to determine the long-time remnant Hall response. The ingredients we need are (1) the ground state of $h(\mathbf{k}, M)$, (2) the time evolution with the quenched Hamiltonian $h(\mathbf{k}, M')$, and (3) the eigenstates and energies of $h(\mathbf{k} - e\mathbf{A}, M')$.

We shall denote our resulting $d$-vectors by $h(\mathbf{k}, M) = \mathbf{d} \cdot \sigma$, $h(\mathbf{k}, M') = \hat{d}_0 \cdot \sigma$, and $h(\mathbf{k} - e\mathbf{A}, M') = \hat{d} \cdot \sigma$.

We can then write the ground state of $h(\mathbf{k}, M)$ as

$$|\psi_0\rangle = \frac{(d - d_z)|↑\rangle - (d_x + id_y)|↓\rangle}{\sqrt{2d(d - d_z)}}, \quad (77)$$

the time evolution of $h_0 = h(\mathbf{k}, M')$ as

$$e^{-i\hbar h_0} = \begin{pmatrix}
\cos td_0 - i\frac{d_{0z}}{d_0} \sin td_0 & -i\frac{d_{0x} - id_{0y}}{d_0} \sin td_0 \\
-i\frac{d_{0x} + id_{0y}}{d_0} \sin td_0 & \cos td_0 + i\frac{d_{0x} - id_{0y}}{d_0} \sin td_0
\end{pmatrix}, \quad (78)$$

and the energies and eigenstates of $h_A = h(\mathbf{k} - e\mathbf{A}, M')$ as

$$\epsilon_{A, \pm} = \pm d_A, \quad (79)$$

$$|\epsilon_{A, \pm}\rangle = \frac{(d_A \pm d_{A, \pm}) |↑\rangle \pm (d_{A, x} + id_{A, y}) |↓\rangle}{\sqrt{2d_A(d_A \pm d_{A, \pm})}}. \quad (80)$$

With these ingredients we can solve for the remnant Hall current for a single momentum $\mathbf{k}$, and then integrate over the Brillouin zone to obtain the total remnant Hall current.

This procedure can be done easily numerically, and we obtain what is plotted in Fig. 6 (we restore physical units in the plot). Notice that we get an effect independent of what phase we are quenching from or to. For reference, the phases are trivial for $M > 0$, $-2 < M < 0$ is a topological insulator with $\sigma_{xy} = -1$, $-4 < M < -2$ is a topological insulator with $\sigma_{xy} = +1$, and $M < -4$ is back to a trivial insulator.

IV. DISSIPATION

With Eq. (55), we can add in dissipation in analogy with nuclear magnetic resonance techniques. The simplest way of including it is in terms of $T_1$ and $T_2$ times (not to be confused with $t_1$ and $t_2$ from before). For simplicity, say that $\mathbf{d} = d\hat{z}$, then our equations are modified such that

$$\partial_t n_x = 2|d \times n|_x = \frac{1}{T_2} n_x, \quad (81)$$

$$\partial_t n_y = 2|d \times n|_y = \frac{1}{T_2} n_y, \quad (82)$$

$$\partial_t n_z = 2|d \times n|_z = \frac{1}{T_1} (n_z + 1). \quad (83)$$

Note that we have relaxed the condition that $\mathbf{n}$ be a unit vector.
Curiously, if we just consider the $T_2$ time: the oscillations are damped out and the remnant Hall response easier to see in a shorter amount of time. The remnant Hall response truly becomes a dephasing phenomenon. We can understand this by returning to the model discussed in Sec. I D 2. Assuming $T_1 \to \infty$ and $T_2$ is finite and only depends on the magnitude of the momentum $k$, we can rewrite the current along a ring of constant momentum as

$$j_y(k) = -\frac{\Delta}{\sqrt{k^2 + \Delta^2}} \left\{ \frac{e^2 A_x}{2k} \sin 2kt_1 - eJ_1[2eA_x(t_2 - t_1)]e^{-(t_2 - t_1)/T_2(k)} \times \left( \cos 2kt_2 + \frac{\sin 2kt_2}{2k(t_2 - t_1)} \right) \right\}. \quad (84)$$

Notice how the oscillatory part of the current is damped by the appearance of a $T_2(k)$. We can easily plot the effects of this, and Fig. 2 is modified so that we obtain Fig. 7

It only is the time $T_1$ that hurts the remnant Hall response. In fact, if we neglect the oscillatory parts of the current and focus on the long-time response $j_{\text{Hall}}(t_1)$, then we find that

$$j_{\text{Hall}}(t_1, t_2) = j_{\text{Hall}}(t_1)e^{-t_2/T_1} + \text{Oscillatory terms.} \quad (85)$$

To estimate the $T_1$, we use Fermi’s golden rule: the transition rate is

$$\Gamma_k = \frac{2\pi}{\hbar} \left[ \int dS_{k'} |\langle k' | V | k \rangle|^2 \right] \rho(\epsilon_k), \quad (86)$$

where $\rho(\epsilon_k)$ is the density of states (if $\epsilon_0(k) = 0$, then $\rho(\epsilon) = \int \frac{d^2 k}{(2\pi)^2} [\delta(\epsilon - |d|) + \delta(\epsilon - |d|)]$, the integral over $dS_{k'}$ is the surface in $k'$ space such that $\rho(\epsilon_k) = \rho(\epsilon_{k'})$, and $\langle k' | V | k \rangle$ is the perturbing matrix element (interaction or disorder) that connects the two $k$ states. The relaxation time is then given by $T_1(\epsilon_k) = 1/\Gamma_k$, so that $1/T_1(\epsilon_k) = \gamma_k \rho(\epsilon_k)$. For simplicity, we assume further that $\gamma_k$ is independent of $k$.

For the single Dirac cone model, $d(k, 0) \sim k$, we have $\rho(\epsilon) \sim |\epsilon|$. Therefore, we have $j_{\text{Hall}}(t_1, t_2) = j_{\text{Hall}}e^{-\gamma|k|t_2}$.

In this particular case (where we have quenched to a “critical” point; i.e. to graphene), we can evaluate long-
time behavior (dropping the oscillatory term)

\[ J_{\text{Hall}}(t_2, t_1) = -\int \frac{k \, dk}{(2\pi)^2} \frac{\Delta}{\sqrt{k^2 + \Delta^2}} \frac{e^2 A_x}{2k} \sin 2kt_1 \, e^{-\gamma kt_2} \]

\[ = -\int \frac{dx/\gamma t_2 \Delta \sin(2xt_1/\gamma t_2)}{(2\pi)^2} \frac{e^2 A_x}{2} \sin 2x \, e^{-x} \tag{87} \]

\[ \sim \frac{1}{\gamma t_2} \int \frac{dx}{(2\pi)^2} \frac{(2xt_1/\gamma t_2)}{\sqrt{x^2/\gamma^2 t_2^2 + \Delta^2}} \frac{e^2 A_x}{2} \, e^{-x} \tag{88} \]

\[ \sim \frac{e^2 t_1 A_x}{\hbar t_2^2}, \quad (90) \]

where we have dropped constants in the last line. We see that we have a power law of \( t_1/t_2^2 \) at long times.