Anomalies in Quantum Field Theory: 
Dispersion Relations and Differential Geometry*

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Abstract

We present two different aspects of the anomalies in quantum field theory. One is the dispersion relation aspect, the other is differential geometry where we derive the Stora–Zumino chain of descent equations.

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1 INTRODUCTION

Anomalies represent the breakdown of a classical symmetry, the breakdown of the corresponding conservation law — here the conservation of the axial or chiral current — by quantum effects [1,2]. They play an important rôle in physics. Some recent work on the anomaly has been presented already on this conference, for instance, the decay $K^+ \rightarrow \pi^+ \pi^0 \gamma$ [3], the spin of the proton [4,5], or the $U(1)$–problem [6].

I want to concentrate more on the theoretical aspects. Anomalies have been first discovered in perturbation theory by UV–regularizing a divergent diagram [1,2]. But they are not just a regularization effect, they also show up in quite different procedures. For example, in the method of dispersion relations where they occur as an IR–singularity of the transition amplitude [7,8], or within sum rules [9]. Using a quite different approach, working with path integrals, the anomalies are detected by the chiral transformation of the path integral measure [10]. In the last years a development in modern mathematical techniques attracted much attention in describing the anomalies. This was differential geometry [11–17], cohomology [18,19] and topology (Atiyah–Singer index theorem) [20–29].

I would like to report on two aspects. One is the dispersion relation aspect which is a conventional and practical technique, and the other is the differential geometric approach, more precisely, the Stora–Zumino chain of descent equations which has the appeal of mathematical elegance, and which provides an interesting relation among several ‘anomalous’ terms.

2 DISPERSION RELATION APPROACH

The use of dispersion relations to calculate the axial anomaly has been proposed originally by Dolgov and Zakharov [7] (for a review see Hořejší [30] and Kerson–Huang [31]). Even before, Kummer [32] determined the anomaly in a dispersive way relying on an at that time fashionable pion–nucleon model. We, however, want to demonstrate the dispersion relation approach in a 2–dimensional example. The advantage is its simplicity and we find all features of that method. We follow here closely a work of Adam, Berthmann and Hofer [8,33].

In quantum field theory we work with Green functions. They have to satisfy the Ward identities (WI) — the field theory equivalents of the classical conservation laws — to achieve the renormalizability of the theory. In order to detect the anomaly in two dimensions we consider the 2–point function

$$\langle 0 | T_{\mu\nu}(x) j^\nu_{\gamma}(y) | 0 \rangle.$$  \hspace{1cm} (1)

Then we find for the axial Ward identity (AWI, and FT stands for Fourier
Transformation)

\[ q^\nu T_{\mu \nu}^5 = \text{FT} \partial^{\nu} \langle 0 | T j_\mu(x) j_\nu^5(y) | 0 \rangle \]
\[ = \text{FT} \langle 0 | T j_\mu(x) \partial^\nu j_\nu^5(y) | 0 \rangle \]
\[ = 2m \text{FT} i \langle 0 | T j_\mu(x) P(y) | 0 \rangle \]
\[ = 2m P^5_\mu \] \hspace{1cm} (2)

if we insert the classical result

\[ \partial^\nu j_\nu^5(y) = 2imP(y) = 2im\bar{\psi}(y)\gamma_5\psi(y). \] \hspace{1cm} (3)

Analogously when considering the 2–point function

\[ \langle 0 | T j_\mu(x) j_\nu(y) | 0 \rangle \] \hspace{1cm} (4)

we would get for the vector Ward identity (VWI)

\[ q^\mu T_{\mu \nu} = \text{FT} \langle 0 | T \partial^{\mu} j_\mu(x) j_\nu(x) | 0 \rangle = 0 \]
\[ q^\nu T_{\mu \nu} = 0 \] \hspace{1cm} (5)

if we rely on the classical conservation law

\[ \partial^\mu j_\mu(x) = 0. \] \hspace{1cm} (6)

But now we will calculate the Green functions explicitly, and we do it in a dispersive way, and we check whether these WI's are satisfied.

In two dimensions we have

\[ \gamma_\nu \gamma_5 = \varepsilon_{\nu \lambda} \gamma^\lambda \] \hspace{1cm} (7)

which implies for the amplitudes

\[ T_{\mu \nu}^5 = \varepsilon_{\nu \lambda} T^\lambda_\mu \] \hspace{1cm} (8)

and in addition we introduce the tensor structure

\[ P^5_\mu(q) = \varepsilon_{\mu \nu} q^\nu P(q^2). \] \hspace{1cm} (9)

So it is enough to consider just the amplitude \( T_{\mu \nu} \). Let us investigate its structure. The general Lorentz decomposition gives

\[ T_{\mu \nu}(q) = q_\mu q_\nu T_1(q^2) - g_{\mu \nu} T_2(q^2). \] \hspace{1cm} (10)

All we require is that it vanishes at large \( q^2 \)

\[ T_{\mu \nu}(q) \xrightarrow{q^2 \to \infty} 0 \] \hspace{1cm} (11)
supplying

\[
T_1(q^2) \xrightarrow{q^2 \to \infty} 0 \\
T_2(q^2) \xrightarrow{q^2 \to \infty} \begin{cases} 
0 & \text{const.} \neq 0
\end{cases} . \tag{12}
\]

Then Cauchy's theorem provides the dispersion relations (DR) for the amplitudes \(T_1(q^2)\) and \(T_2(q^2)\) (the amplitudes are analytic except a cut starting at \(t = 4m^2\)). The unique choice for the amplitude \(T_1(q^2)\) is an unsubtracted DR (similarly for \(P(q^2)\))

\[
T_1(q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-q^2} \text{Im} \ T_1(t) \\
P(q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-q^2} \text{Im} \ P(t). \tag{13}
\]

Concerning \(T_2(q^2)\), however, we have the option not to subtract the DR or to subtract once. The natural choice would be again an unsubtracted DR

\[
T_2(q^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-q^2} \text{Im} \ T_2(t) \tag{14}
\]

since the integral exists. Then the AWI is satisfied

\[
q^\nu T_\mu^\nu(q) = q^\nu \varepsilon_{\nu\lambda}(q_\mu q^\lambda T_1(q^2) - g_\mu^\lambda T_2(q^2)) \\
= \varepsilon_{\mu\nu} q^\nu T_2(q^2) \\
= \varepsilon_{\mu\nu} q^\nu 2mP(q^2) \\
= 2m P_5^\nu(q) \tag{15}
\]

since the imaginary parts of the amplitudes do fulfil the WI's in any case

\[
\text{AWI: Im } T_2(t) = 2m \text{ Im } P(t) \\
\text{VWI: Im } T_2(t) = t \text{ Im } T_1(t). \tag{16}
\]

However, considering the VWI we find

\[
q^{\mu\nu} T_{\mu\nu} = q_\nu (q^2 T_1(q^2) - T_2(q^2)) \\
= q_\nu \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t-q^2} \text{Im} T_1(t) \\
= q_\nu \left( -\frac{1}{\pi} \right) \int_{4m^2}^{\infty} \text{Im} T_1(t) \\
= \mathcal{A} q_\nu \tag{17}
\]

with the anomaly

\[
\mathcal{A} = -\frac{1}{\pi}. \tag{18}
\]
In the last step we had to do an explicit calculation, we calculated the imaginary part of a fermion loop with help of the Cutkosky rules [8,33]

\[ t \text{ Im } T_1(t) = \frac{2m^2}{t} \left( 1 - \frac{4m^2}{t} \right)^{-1/2}. \] (19)

Now, for physical reasons, in order to restore gauge invariance, to satisfy VWI, we subtract the DR for

\[ T_2^R(q^2) = T_2(q^2) - T_2(0) = \frac{q^2}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t(t-q^2)} \text{ Im } T_2(t) \] (20)

then the subtraction constant represents the anomaly

\[ -T_2(0) = -\frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t} \text{ Im } T_2(t) = -\frac{1}{\pi} \int_{4m^2}^{\infty} dt \text{ Im } T_1(t) = -\frac{1}{\pi} = A. \] (21)

Furthermore we have

\[ T_2^R(q^2) = \frac{q^2}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t(t-q^2)} \text{ Im } T_2(t) = \frac{q^2}{\pi} \int_{4m^2}^{\infty} dt \text{ Im } T_1(t) = q^2 T_1(q^2) \] (22)

and we achieve the familiar gauge invariant tensor structure (with \( T_1 \equiv T \))

\[ T_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})T(q^2). \] (23)

The AWI, on the other hand, contains the anomaly

\[ q^\nu T^5_{\mu\nu} = q^\nu \varepsilon_{\nu\lambda} (q_\mu q^\lambda T_1(q^2) - g_\mu^\lambda T_2^R(q^2)) = \varepsilon_{\mu\nu} q^\nu (T_2(q^2) - T_2(0)) = \varepsilon_{\mu\nu} q^\nu (2mP(q^2) + A). \] (24)

So the axial current is not conserved but provides the well known anomaly result (we take \( m = 0 \))

\[ \partial^\nu f_5^\nu = \frac{e}{\pi} \varepsilon_{\nu\mu} \partial^\nu A^\mu = \frac{e}{2\pi} \varepsilon_{\nu\mu} F^\nu\mu. \] (25)
If we subtract the amplitude $T_2(q^2)$ at some arbitrary point $q_0^2$ then the anomaly is distributed on both WI’s, the AWI and the VWI. There is no choice to get rid of the anomaly at all.

The source of the anomaly in this dispersive procedure is the existence of the superconvergence sum rule

$$\int_{4m^2}^{\infty} dt \, \text{Im} \, T_1(t) = 1. \quad (26)$$

The anomaly corresponds to a threshold singularity of $\text{Im} \, T_1(t)$ at $t = 4m^2$ approaching a $\delta$–function for $m \to 0$

$$\lim_{m \to 0} \text{Im} \, T_1(t) =$$

$$= \lim_{m \to 0} \frac{2m^2}{t^2} \left( 1 - \frac{4m^2}{t} \right)^{-1/2} \theta(t - 4m^2)$$

$$= \delta(t). \quad (27)$$

3 STORA–ZUMINO CHAIN OF DESCENT EQUATIONS

Now we return to four dimensions and to nonabelian gauge fields. There exist two types of anomalies (for a review see Jackiw [34]), the singlet anomaly and the nonabelian anomaly (Bardeen’s result [35]). In terms of differential forms the explicit expressions are:

singlet anomaly

$$\mathcal{A} = d \star j^5 = \frac{1}{4\pi^2} \text{tr} \,FF$$

$$= \frac{1}{4\pi^2} \text{tr} \,d \left( AdA + \frac{2}{3} A^3 \right) \quad (28)$$

nonabelain anomaly

$$-G^a [A] = (D \ast j)^a$$

$$= \frac{1}{24\pi^2} \text{tr} \, T^a d \left( AdA + \frac{1}{2} A^3 \right). \quad (29)$$

(Note that in the second case the field and the current are of positive chirality, for negative chirality we get a sign change, the chirality index is suppressed.) Although the two expressions resemble each other very closely they are of different origin. But there exists a very peculiar relation between these two anomalies. The nonabelian anomaly in $2n$ dimensions is related to the singlet anomaly in $(2n + 2)$ dimensions via some chain of differential geometric equations. This
is the Stora–Zumino (SZ) chain of descent equations we want to present now [11–16].

We carry out pure mathematics, the construction is actually algebraic. Instead of the trace we use the more general, symmetric invariant polynomial $P(F^n)$. Since $P(F^n)$ is closed (due to the Bianchi identity $DF = 0$) it is locally exact (Poincaré lemma)

$$P(F^n) = dQ_{2n-1}(A, F). \tag{30}$$

The polynomial $Q_{2n-1}$ of form degree $(2n - 1)$ — called Chern–Simons form — can be calculated explicitly

$$Q_{2n-1}(A, F) = n \int_0^1 dt P(A, F_t^{n-1}) \tag{31}$$

where

$$F_t = tF + (t^2 - t)A^2 \tag{32}$$

(geometrically, we consider a trivial fibre bundle). Equation (30) – (32), also called transgression formula, is the starting point of the following mathematical operations.

Stora [12] noticed that a shift in both, in the gauge potential — geometrically the connection — and in the derivative

$$A \rightarrow \hat{A} = A + v \quad d \rightarrow \Delta = d + s \tag{33}$$

where $v$ represents the Faddeev–Popov ghost and $s$ the BRS operator, leaves the field strength — geometrically the curvature — invariant

$$dA + A^2 = F(A) \equiv \hat{F} (\hat{A}) = \Delta \hat{A} + \hat{A}^2. \tag{34}$$

Identity (34) holds by virtue of the familiar BRS equations and is named by Stora “Russian formula”.

Due to the “Russian formula” we can equate the transgression formula (30) with its shifted version

$$\Delta Q_{2n-1}(A + v, F) =$$

$$\quad = P(\hat{F}^n) \equiv P(F^n) = dQ_{2n-1}(A, F). \tag{35}$$

Next we expand the Chern–Simons form in powers of the Faddeev–Popov ghost $v$

$$Q_{2n-1}(A + v, F) =$$

$$\quad = Q_{2n-1}^0(A, F) + Q_{2n-2}^1(v, A, F)$$

$$\quad + Q_{2n-3}(v, A, F) + \ldots + Q_{2n-1}^0(v) \tag{36}$$
where the upper index denotes the powers of $v$ and the lower the form degree. We insert this expansion into equation (35)

\[
(d + s)Q_{2n-1}^0 + (d + s)Q_{2n-2}^1 + (d + s)Q_{2n-3}^2 \\
+ \ldots + (d + s)Q_{2n-1}^{2n-1} = dQ_{2n-1}^0, \tag{37}
\]

we compare the terms of same form degree and same power in $v$ then we obtain a chain of descent equations (we have followed here Stora’s approach [12], the view of Zumino [14] is more geometric but equivalent)

\[
P(F^n) - dQ_{2n-1}^0 = 0 \\
sQ_{2n-1}^0 + dQ_{2n-2}^1 = 0 \\
sQ_{2n-2}^1 + dQ_{2n-3}^2 = 0 \\
\ldots \\
sQ_{2n-2}^{2n-2} + dQ_{2n-1}^{2n-1} = 0 \\
sQ_{2n-1}^{2n-1} = 0. \tag{38}
\]

The chain we can solve now term by term. Starting with the Chern–Simons form we apply the BRS operator $s$, respect the BRS rules and we find $Q_{2n-2}^1$, and so forth. The explicit solutions in case of $n = 3$ are (they are not unique, general formulae are given by Zumino [16])

\[
Q_0^0 = \text{tr} \left[ A(dA)^2 + \frac{3}{2} A^3 dA + \frac{3}{5} A^5 \right] \\
Q_1^1 = \text{tr} v d \left( AdA + \frac{1}{2} A^3 \right) \\
Q_2^2 = -\frac{1}{2} \text{tr} \left[ (v^2 A + v A v + A v^2) dA + v^2 A^3 \right] \\
Q_3^3 = \frac{1}{2} \text{tr} \left[ -v^3 dA + A v A v^2 \right] \\
Q_4^4 = \frac{1}{2} \text{tr} v^4 A \\
Q_5^5 = \frac{1}{10} \text{tr} v^5. \tag{39}
\]

What we observe now is that $Q_1^1$ provides precisely the nonabelian anomaly (29) besides the normalization. The reason is that equation

\[
sQ_{2n-2}^1 + dQ_{2n-3}^2 = 0 \tag{40}
\]

represents a local version of the Wess–Zumino consistency condition — the condition which defines the anomaly. So we can identify the chain term $Q_{2n-2}^1$ (mathematics) with the anomaly $G(v, A)$ (physics)

\[
G(v, A) = \int_M v^a G^a[A] = N \int_M Q_{2n-2}^1. \tag{41}
\]
The normalization $N$ we must take from somewhere else, from perturbation theory or from path integral methods or from topological methods.

Let us summarize. On pure mathematical grounds we can derive a system of equations — the SZ chain of descent equations — which we can solve. Several terms have a correspondence in physics what we list at the end.

Physical meaning of the chain terms:

- $P(F^n) = \text{tr } F^n$
  singlet anomaly in $2n$ dimensions
- $Q_{2n-1}^0$
  Chern–Simons form, ingredient for topological field theories [36–39]
- $Q_{2n-2}^1$
  nonabelian anomaly, solution of the Wess–Zumino consistency condition [12,14]
- $Q_{2n-3}^2$
  Schwinger term in an equal time commutator of Gauss–law operators [40]
- $Q_{2n-4}^3$
  represents the violation of the Jacobi identity for velocity operators in the presence of a magnetic monopole [41,42]

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