Minimal volume entropy of RAAG’s

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Abstract
Bregman and Clay recently characterized which right-angled Artin groups with geometric dimension 2 have vanishing minimal volume entropy. In this note, we extend this characterization to higher dimensions.

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1 INTRODUCTION

Let \( X \) be a finite complex with a piecewise Riemannian metric \( g \) (i.e., a collection of Riemannian metrics on cells which agree on intersections). Fix a basepoint \( x_0 \) in the universal cover \( \tilde{X} \), and let \( \tilde{g} \) be the pulled-back metric on \( \tilde{X} \). The associated volume entropy of \((X, g)\) is the exponential growth rate of the balls \( B_{x_0}(t) \) in the universal cover:

\[
\text{ent}(X, g) = \lim_{t \to \infty} \frac{1}{t} \log \text{Vol}(B_{x_0}(t), \tilde{g})
\]

This limit does not depend on the choice of basepoint. We now define the minimal volume entropy \( \omega(X) \) to be

\[
\omega(X) = \inf_{g} \text{ent}(X, g) \frac{\text{Vol}(X, g)^{1/\dim X}}{\text{Vol}(X, g)}
\]

where we minimize over all piecewise Riemannian metrics \( g \). Normalizing by the volume guarantees this does not change under scaling \( g \). This invariant was initially defined for Riemannian manifolds in [8].

Now, suppose \( G \) is a group with a finite classifying space \( BG \). Let \( \text{gdim}(G) \) be the geometric dimension of \( G \), that is, \( \text{gdim}(G) \) is the minimal dimension of such a \( BG \). We define the minimal
volume entropy of $G$, denoted $\omega(G)$, to be the infimum of $\omega(BG)$ over all finite classifying spaces $BG$ of dimension $= \text{gdim}(G)$. We say a classifying space $BG$ is minimal dimensional if $\text{dim} BG = \text{gdim}(G)$.

In this note, we study this invariant for right-angled Artin groups (from now on RAAG’s). If $L$ is a flag simplicial complex, recall that the associated RAAG $A_L$ has a presentation with generators corresponding to vertices, and where two generators commute if and only if the two vertices span an edge in $L$. We give an almost complete characterization of the (non)vanishing of $\omega(A_L)$ based on the topology of the defining flag complex $L$. The geometric dimension of $A_L$ is equal to $\text{dim} L + 1$ (an $n$-simplex in $L$ corresponds to an $\mathbb{Z}^{n+1}$ subgroup of $A_L$, so this is an obvious lower bound). We prove the following theorem:

**Theorem 1.1.** Let $L$ be a $d$-dimensional flag complex and $A_L$ be the corresponding RAAG. Then

1. If $H^d(L, \mathbb{Z}) \neq 0$, then $\omega(A_L) > 0$.
2. If $L$ embeds into a $d$-dimensional contractible complex, then $\omega(A_L) = 0$.

Bregman and Clay had previously proved this theorem when $L$ is 1-dimensional; in this case $\omega(A_L)$ vanishes if and only if $L$ is a forest [Theorem 1.2, [4]]. If $d \neq 2$, then $H^d(L, \mathbb{Z}) = 0$ is equivalent to $L$ embedding into a $d$-dimensional contractible complex, hence these conditions are complementary [Remark 1.26, [6]]. By the universal coefficient theorem, $H^d(L, \mathbb{Z}) = 0$ is equivalent to $H_d(L, \mathbb{Z}) = 0$ and $H_{d-1}(L, \mathbb{Z})$ being free abelian. Also, again by universal coefficients, it’s equivalent to $H_d(L, \mathbb{F}_p) = 0$ for all primes $p$.

**Remark.** This embedding condition was used in [6] to construct manifold models for $BA_L$ of dimension $2 \text{gdim}(A_L) - 1$ (classical arguments for any type $F$ group $G$ guarantee manifold models of dimension $2 \text{gdim}(G)$). It would be interesting to further relate nonvanishing of minimal entropy with “low”-dimensional manifold models of $BG$.

**Remark.** Kevin Li has recently shown that if $L$ is a 2-complex with $H^2(A_L, \mathbb{Z}) = 0$, then $\omega(A_L) = 0$ [9, Theorem 3.9]. This completes the characterization of nonvanishing $\omega(A_L)$. The method in [9] works in all dimensions as well, so provides an alternative proof of Theorem 1.1.

**Example.** Here is a curious example. Suppose $L_1$ is a flag triangulation of $\mathbb{R}P^2$, and suppose that $L_2$ is a flag triangulation of a two-dimensional $\mathbb{Z}/3$-Moore space (for instance, $L_2$ is obtained by attaching a disc to a circle by a degree 3 map). By Theorem 1.1, both $A_{L_1}$ and $A_{L_2}$ have nonvanishing minimal volume entropy. On the other hand, again by Theorem 1.1, their product has vanishing minimal volume entropy (as the join $L_1 \ast L_2$ has $H_5(L, \mathbb{F}_p) = 0$ for all $p$). This is in contrast with the simplicial volume $||M||$ of a closed manifold $M$; Gromov proved in [8] the inequality

$$||M|| ||N|| \leq ||M \times N|| \leq \left(\frac{\dim M + \dim N}{\dim M}\right)||M|| ||N||.$$

For a closed $m$-manifold $M$, Gromov also showed the inequality

$$\omega(M)^m \geq C_m ||M||$$

for a constant $C_m$ only dependent on $m$. In particular, the product of any two manifolds $M \times N$ with $||M||, ||N|| > 0$ has $\omega(M \times N) > 0$. 
To prove Theorem 1.1, we use the following fibering criteria of Babenko-Sabourau [2]. Bregman and Clay used the same criteria in [4] to compute $\omega(A_L)$ in the one-dimensional case.

**Definition 1.2.** Let $X$ be a simplicial complex. We say $X$ has FCA (short for fiber collapsing assumption) if there is a simplicial complex $P$ with $\dim P < \dim X$ and a simplicial map $f : X \to P$ so that for all $p \in P$, if $F_p$ is a component of $f^{-1}(p)$ then the image subgroup $i_*(\pi_1(F_p))$ in $\pi_1(X)$ is subexponentially growing with subexponential growth rate $\leq 1 - \frac{\dim P}{\dim X}$.

Babenko and Sabourau show that if $X$ has FCA then $\omega(X) = 0$ [Theorem 1.3, [2]]. If $X$ is a minimal dimensional classifying space for $G$, this of course implies that $\omega(G) = 0$. In our setting, all subexponentially growing subgroups of RAAG’s are free abelian (this follows from a theorem of Baudisch described below). The subexponential growth rate of a finitely generated free abelian group is 0, so we do not require going into the details of this rate.

**Definition 1.3.** A group $G$ has uniform exponential growth if there is a $\delta > 0$ so that the growth of $G$ with respect to any finite generating set $S$ is exponential with growth rate $> \delta$. We say $X$ has FNCA (fiber non-collapsing assumption) if there is a $\delta > 0$ so that every map $f : X \to P$ with $\dim P < \dim X$ has a connected component $F_p$ of a point preimage $f^{-1}(p)$ with the image subgroup $i_*(\pi_1(F_p))$ in $\pi_1(X)$ having uniform exponential growth $> \delta$.

Babenko and Sabourau also show that if $X$ has FNCA, then $\omega(X) > \epsilon_m$, where $\epsilon_m$ only depends on the dimension $m$ of $X$ and $\delta$ [Theorem 1.5, [2]]. The conditions FCA and FNCA are almost, but not quite, complementary. On the other hand, Bregman and Clay [4] show that they are for classifying spaces of RAAG’s. This follows from the following facts: RAAG’s have so-called uniform uniform exponential growth, in the sense that there is a $\delta > 0$ so that every non-abelian finitely generated subgroup of a given RAAG has uniform exponential growth $> \delta$ (every non-abelian subgroup on two generators is free by a theorem of Baudisch [3]). Now, if a classifying space for a RAAG does not have the FCA, then any map to a smaller dimensional complex has a point preimage whose image in $\pi_1$ is non-abelian. Hence, this subgroup has uniform exponential growth $> \delta$, and in particular has the FNCA. It follows that for a RAAG $A_L$, if we can show that every minimal dimensional classifying space $BA_L$ has FNCA, this will imply that $\omega(A_L) > 0$ (in fact, in this case it suffices to exhibit FNCA for one model of $BA_L$, see Proposition 3.9 of [4]).

Part (2) of Theorem 1.1 follows from an explicit construction of $BA_L$. This model for $BA_L$ is built by gluing together tori of various dimensions corresponding to the simplices of $L$. If $L$ embeds into a contractible complex $L'$ of the same dimension, then this model for $BA_L$ naturally maps to $L'$, and the preimages of points are homotopic to tori or points. Therefore, this model for $BA_L$ has FCA. To prove part (1), we consider the mod $p$ homology growth in residual chains of finite index subgroups. For RAAG’s, this was recently computed by Avramidi, Okun, and the second author [1]. If there is a minimal dimensional model for $BA_L$ with FCA, it will follow from a result of Sauer’s that this growth vanishes in the top dimension for all $p$ [13]. The computation in [1] then shows that the top $F_p$-homology of $L$ vanishes for all $p$.

## 2 Classifying Spaces of RAAG’s

Our calculation of $\omega(A_L)$ relies on the following construction of models for $BA_L$. Let $L$ be the defining flag complex of the RAAG, and let $K$ be the geometric realization of the poset of simplices
of $L$. Then $K$ is isomorphic to the cone on the barycentric subdivision of $L$, where the cone point corresponds to the empty simplex. Given a simplex $\sigma \in L$, let $A_{\sigma}$ be the corresponding free abelian subgroup of $A_L$. A point $x \in K$ is contained in some minimal simplex, which corresponds to a chain of simplices of $L$. Let $\sigma(x)$ be the smallest element in this chain. The basic construction $\mathcal{U}(A_L,K)$ is defined to be

$$\mathcal{U}(A_L,K) := A_L \times K / \sim$$

where $(g, x) \sim (g', x')$ if and only if $x = x'$ and $gA_{\sigma(x)} = g'A_{\sigma(x)}$.

Then $G$ acts on $\mathcal{U}(A_L,K)$ with strict fundamental domain $1 \times K$, which we identify with $K$. † For RAAG’s, it is known that $\mathcal{U}(A_L,K)$ is contractible, in fact it admits a natural CAT(0) cubical structure.

The stabilizer of a simplex $\tau \subset K$ is the free abelian special subgroup $A_{\min \tau}$, where $\min \tau$ is the smallest element in the corresponding chain of simplices of $L$. In particular, the stabilizer is trivial if and only if $\tau$ contains the cone point corresponding to the empty simplex.

Let $\mathcal{U} = \mathcal{U}(A_L,K)$. To build a classifying space $BA_L$, we use the Borel Construction $EA_L \times_{A_L} \mathcal{U}$ (if $X$ and $Y$ are two $G$-spaces, then $X \times_G Y$ is the direct product $X \times Y$ quotiented by the diagonal $G$-action). Since $\mathcal{U}$ is contractible, this produces a (noncompact) model of $BA_L$.

The action of the stabilizer of any cell in $\mathcal{U}$ fixes the cell. It follows that $EA_L \times_{A_L} \mathcal{U}$ naturally maps to $\mathcal{U}/A_L = K$. The preimage of an open simplex $\tau$ in $K$ is homeomorphic to $\tau \times EA_L/A_{\min \tau}$, hence homotopy equivalent to a torus $T^{\min \tau}$. A rebuilding procedure of Geoghegan lets us build a compact model for $BA_L$, denoted $X_L$, which maps to $K$ and the preimage of a simplex $\tau$ is homeomorphic to $\tau \times T^{\min \tau}$ [Chapter 6, [7]]. Note that $X_L$ is $(\dim L + 1)$-dimensional; it follows that the geometric dimension of $A_L$ is $\dim L + 1$.

In particular, we can assume that if $\tau$ is a simplex in $K$ which contains the cone point, then the preimage in $X_L$ is a copy of $\tau$. Therefore, $X_L = \text{Cone}(L) \cup Y_L$, where $Y_L$ is the preimage of $L$ in $BA_L$. We say $Y_L$ is the toral subcomplex of $X_L$. The following lemma is immediate from the discussion.

**Lemma 2.1.** If $L$ is contractible, then $Y_L$ is a model for $BA_L$.

3 | $\omega(A_L)$

3.1 | Vanishing of entropy for RAAG’s

**Theorem 3.1.** Let $A_L$ be a RAAG based on a $d$-dimensional flag complex $L$. Suppose that $L$ embeds into a $d$-dimensional contractible complex. Then $\omega(A_L) = 0$.

**Proof.** First, suppose that $L$ is contractible. We use the model $X_L$ of $BA_L$ as in the previous section. In particular, Lemma 2.1 guarantees a $(d + 1)$-dimensional model for $BA_L$ which projects to $L$, and the preimage of a point in $L$ is homotopy equivalent to a torus of some dimension. Therefore, this model satisfies FCA, and hence $\omega(A_L) = 0$ [2].

† A strict fundamental domain for a group action on a CW-complex is a subcomplex which intersects each orbit in a single point.
Now, suppose $L$ is $d$-dimensional and embeds into a $d'$-dimensional contractible complex $L'$. Let $Y_L$ be the toral subcomplex for $L$. Then a model for $B\mathcal{A}_L$ can be obtained by forming the amalgam $X_L = Y_L \cup L' \cup L$ (this follows from Lemma 1.21 and Lemma 1.23 in [6], see the discussion there in Section 1.4). This is $(d+1)$-dimensional, and naturally projects to $L'$. The preimage of a point is homotopy equivalent to a torus of some dimension ($a_0$-torus if the point is in $L' - L$). This shows that this model satisfies FCA, and hence $\omega(A_L) = 0$.

3.2 Nonvanishing of entropy for RAAG’s

We now show that non-vanishing of $\omega$ for RAAG’s follows from work of Sauer on mod $p$-homology growth in residual, finite index normal chains [13], and the calculation of this growth for RAAG’s in [1].

First, assume that $X$ is a simplicial complex with residually finite fundamental group $G$. Let

$$G = \Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \ldots$$

be a chain of finite index normal subgroups with $\cap_k \Gamma_k = 1$, and let $X_k$ be the corresponding covers of $X$ (we say $\Gamma_k$ is a residual chain). We say that $X$ has nonvanishing $\mathbb{F}_p$-homology growth in degree $i$ if

$$\limsup_k \frac{b_i(X_k, \mathbb{F}_p)}{[G : \Gamma_k]} > 0$$

If one instead considers the $\mathbb{Q}$-homology growth, the limsup coincides with the $i$th $L^2$-Betti number of $X$ by a theorem of Lück [10]. If $X = BG$ is aspherical, then we will write $b_i(\Gamma_k, \mathbb{F}_p)$ instead of $b_i(B\Gamma_k, \mathbb{F}_p)$.

We now relate the FCA property to amenable coverings, see Proposition 2.13 of [2]. Suppose that we have a simplicial complex $X$ which satisfies FCA; hence we have a map $f : X \to P$ with $\dim P < \dim X$. If we cover $P$ by open stars of vertices, then we can use $f$ to pull back this cover to $X$. In particular, if we cover $X$ by connected components of preimages of open stars, the multiplicity of this cover is equal to $\dim P + 1 \leq \dim X$. Each open set in the cover deformation retracts to a connected component of $f^{-1}(p)$. Therefore, the image of the fundamental group of each set in the cover is a subexponentially growing subgroup of $\pi_1(X)$. More generally, we say an open set $U$ in a topological space $X$ is amenable if the image subgroup $\pi_1(U)$ in $\pi_1(X)$ is amenable.

The following theorem follows immediately from the proof of Theorem 1.6 in [13].

**Theorem 3.2.** Let $G$ be residually finite, and let $\Gamma_k$ be a residual sequence of finite index normal subgroups. Suppose that there is a finite $BG$ with an amenable cover of multiplicity $n$. Then

$$\lim_{k \to \infty} \frac{b_i(\Gamma_k, \mathbb{F}_p)}{[G : \Gamma_k]} = 0 \quad \text{for } i \geq n.$$
**Corollary 3.3.** Let $G$ be a type $F$, residually finite group with $\operatorname{gdim}(G) = d$. Suppose that $\{\Gamma_k\}$ is a residual chain in $G$ with nontrivial $d$-dimensional $\mathbb{F}_p$-homology growth for some prime $p$. Then any $d$-dimensional model for $BG$ does not have the FCA property.

Note that in [Theorem 1.6, [13]], Sauer shows that if an aspherical $n$-manifold $M$ has an open cover by amenable sets of multiplicity $n$, then the $\mathbb{F}_p$-homology growth of $\pi_1(M^n)$ vanishes in all degrees (for all finite index normal chains). To do this, Sauer constructs complexes $S(k)$ which homotopy retract\(^1\) onto $\bar{M}^n / \Gamma_k$ and have sublinear (in $[G : \Gamma_k]$) number of $n$-cells. The construction of these complexes does not require the manifold structure. Therefore, the same argument shows that if an aspherical $n$-complex $X$ has an open cover by amenable sets of multiplicity $n$, then the $\mathbb{F}_p$-homology growth of $\pi_1(X)$ vanishes in degree $n$ (for the usual $L^2$-Betti numbers, this is Theorem C of [12]).

In [1], Avramidi, Okun, and the second author computed the $\mathbb{F}_p$-homology growth of RAAG’s; the analogous theorem for $b_i^{(2)}(A_L)$ was proved earlier by Davis and Leary [5].

**Theorem 3.4.** Let $A_L$ be a RAAG based on a flag complex $L$. Let $\{\Gamma_k\}_{k \in \mathbb{N}}$ be a residual chain of finite index normal subgroups. Then

$$\lim_{k \to \infty} \frac{b_i(\Gamma_k, \mathbb{F}_p)}{[A_L : \Gamma_k]} = \bar{b}_{i-1}(L, \mathbb{F}_p),$$

where $\bar{b}_{i-1}(L, \mathbb{F}_p)$ is the reduced Betti number of $L$ with $\mathbb{F}_p$-coefficients.

Combining these results gives our main theorem characterizing the nonvanishing of $\omega(A_L)$.

**Theorem 3.5.** Suppose that $A_L$ is a RAAG based on a $d$-dimensional flag complex $L$. If $H_d(L, \mathbb{F}_p) \neq 0$ for some $p$, then $\omega(A_L) > 0$. Hence if $H_d(L, \mathbb{Z}) \neq 0$, then $\omega(A_L) > 0$.

**Proof.** By Theorem 3.4, if $H_d(L, \mathbb{F}_p) \neq 0$, then $A_L$ has nonvanishing $\mathbb{F}_p$-homology growth in dimension $d + 1$. By Corollary 3.3, any minimal dimensional classifying space for $A_L$ will not have FCA. Therefore, any minimal dimensional classifying space has FNCA, and by Babenko and Sabourau’s results (plus the fact that RAAG’s have uniform uniform exponential growth) we obtain $\omega(A_L) > 0$.

**Remark.** Matt Clay pointed out to us that Proposition 3.9 in [4] implies that $\omega(A_L) > \epsilon$ for $\epsilon$ only depending on $\dim L$.

**Remark.** The computation of $\mathbb{F}_p$-homology growth in [1] was extended in [11]. The correct context for these computations seems to be a group $G$ acting on a contractible complex with strict fundamental domain $Q$. For example, a similar computation works for any residually finite Artin group which satisfies the $K(\pi, 1)$-conjecture. By the same argument as above, if the nerve $L$ of these Artin groups has $H^{\dim L}(L, \mathbb{Z}) \neq 0$, then the minimal volume entropy will be strictly positive. The corresponding vanishing result for Artin groups seems harder to prove.

\(^1\)That is, there are maps $f : \bar{M}^n / \Gamma_k \to S(k)$ and $g : S(k) \to \bar{M}^n / \Gamma_k$ so that $g \circ f \sim \operatorname{id}_{\mathbb{R}^n / \Gamma_k}$. 
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