Abstract. We continue the investigation of Hopf-Galois extensions with central invariants started in [30]. Our objective is not to imitate algebraic geometry using Hopf-Galois extension but to understand their geometric properties.

Let $H$ be a finite-dimensional Hopf algebra over a ground field $k$. Our main object of study is an $H$-Galois extension $U \supseteq O$ such that $O$ is a central subalgebra of $U$. Let us briefly discuss geometric properties of the object. By Kreimer-Takeuchi theorem the module $U_O$ is projective. Thus, it defines a vector bundle of algebras on the spectrum of $O$ by Serre theorem. The fibers carry a structure of Frobenius algebra. A similar structure was of interest to geometers for a while because commutative Frobenius algebras naturally arise in the study of symmetric Poisson brackets of hydro-dynamical type [1]. More recently, a concept of Frobenius manifold was introduced [17]; it is a manifold such that tangent spaces carry a structure of a commutative Frobenius algebra which multiplication has a generating function.

Our set-up is different: we have a vector bundle rather than the tangent bundle and our algebras are not necessarily commutative. However, we have more structure involved: a Hopf-Galois extension may be regarded as a “quantum” principal bundle [51]. (We should point out that the notion of a quantum principal bundle in non-commutative geometry is more involved but, nevertheless, quantum principal bundles with universal differential calculus are the same as Hopf-Galois extensions [10, 16].) If $H$ is commutative (i.e. an algebra of functions on a finite group scheme $G$) then a commutative $H$-Galois extension $U \supseteq O$ is a $G$-principal bundle on the spectrum of $O$.

Finally, we emphasize that centrality of invariants is a crucial property for a geometrical treatment of Hopf-Galois extensions. We will illustrate this claim throughout the paper.
We introduce Hopf-Galois extensions with central invariants and discuss examples in Section 1. The notion of inverse image of a Hopf-Galois extension is discussed in Section 2. A geometric object should become trivial if one looks at it locally. We prove a weak version of this principle in Section 3. A Hopf-Galois extension starts enjoying being a geometrical object on an affine scheme in the next section where we show that it may be pasted from local datum. The next three sections (5,6,7) are devoted to investigation of various features a Hopf-Galois extension carries. In section 8 we give a definition of Hopf-Galois extension (or \( H \)-torsor) over any scheme, and discuss the stack of \( H \)-torsors.

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1. Introduction

Let \( H \) be a finite dimensional Hopf algebra over a ground field \( k \) from now on. One could extend a number of results to the case of a ground ring but it is beyond our interest in the present paper. An associative algebra with unity \( U \) is an \( H \)-comodule algebra if there is a map \( \rho : U \rightarrow U \otimes H \), written as \( \rho(x) = x_0 \otimes x_1 \) such that \( x_0 \varepsilon(x_1) = x \), \( x_0 \otimes \Delta(x_1) = \rho(x_0) \otimes x_1 \), and \( \rho(xy) = \rho(x) \rho(y) \) for all \( x,y \in U \). We use the oxymoronic variant of the Sweedler’s \( \Sigma \)-notation with \( \Sigma \) eliminated. The subalgebra \( O = U^H = \{ x \in U \mid \rho(x) = x \otimes 1 \} \) is called the subalgebra of invariants; one says that \( U \supseteq O \) is an \( H \)-extension. Throughout the paper we make various assumptions on \( O \) as finitely generated, reduced (semiprime), or affine (finitely generated semiprime). An \( H \)-extension is an extension with central invariants if \( O \) is contained in the center of \( U \). An \( H \)-extension is Hopf-Galois (or specifically \( H \)-Galois) if the canonical map \( \text{can} : U \otimes_O U \rightarrow U \otimes H \) defined by \( \text{can}(x \otimes_O y) = (x \otimes 1)(y_0 \otimes y_1) \) is a bijection. [13, 25].

Let us discuss some examples of Hopf-Galois extensions. A Galois extension of fields \( K \supseteq k \) with the Galois group \( G \) is Hopf-Galois for the Hopf algebra \( kG^* \). Geometrically, it means that the point over \( K \) is a \( G \)-principal bundle on the point over \( k \). The vector bundle which structure it carries has the \( k \)-vector space \( K \) as a fiber at the point.

However, a Hopf-Galois extension of fields is not necessarily Galois. A purely inseparable extension may be Hopf-Galois for the dual of a restricted universal enveloping algebra [25]. Moreover, a finite separable not normal field extension may be Hopf-Galois as well [14]. From the geometric prospective it means that the point over \( K \) is a principal bundle on the point over \( k \) for some finite non-reduced group scheme.
A criterion for a finite separable field extension to be Hopf-Galois was proven in [14]. One should consider the normalization $\tilde{K}$ of $K$ over $k$. Let $S(M)$ be the symmetric group of the set $M = \text{Gal}(\tilde{K} \mid k) / \text{Gal}(\tilde{K} \mid K)$. There is a natural embedding $\text{Gal}(\tilde{K} \mid k) \to S(M)$. The extension is Hopf-Galois if and only if there exists a subgroup $N \subseteq S(M)$, normalized by $\text{Gal}(\tilde{K} \mid k)$, such that the left $N$-set $M$ is isomorphic to the left regular $N$-set $N$.

Interesting Hopf-Galois extensions may be constructed as crossed products. More precisely, crossed products constitute all cleft Hopf-Galois extensions. An inquisitive reader should consult [25] for a nice account of the subject. We do not need the general construction. We restrict our attention to twisted products, crossed products with the trivial action, because only they produce central invariants. Let $R$ be a commutative $k$-algebra and $\sigma : H \otimes H \to R$ be a linear map. Let $R_\sigma[H]$ be a vector space $R \otimes H$ with an algebra structure defined by the formula

$$a \otimes g \cdot b \otimes h = ab\sigma(h_1, g_1) \otimes h_2g_2, \quad a, b \in R, g, h \in H$$

(1)

It may not be a structure of an associative algebra in general. The following lemma is straightforward [25, Lemma 7.12].

**Lemma 1.** The formula (1) defines a structure of an associative algebra with an identity element $1 \otimes 1$ if and only if for each $g, h, t \in H$

$$\sigma(h_1 \otimes g_1)\sigma(h_2g_2 \otimes t) = \sigma(g_1 \otimes t_1)\sigma(h \otimes g_2t_2)$$

$$\sigma(h \otimes 1) = \sigma(1 \otimes h) = \varepsilon(h)1$$

If $\sigma$ is invertible with respect to the convolution [25] and satisfies the conditions of Lemma 1 we call it a cocycle, although the corresponding cochain complex has been constructed only for cocommutative $H$ [34]. We use the term twisted product for $R_\sigma[H]$ if $\sigma$ is a cocycle. The natural question is what is happening if the map $\sigma$ is not invertible but nevertheless formula (1) defines a structure of an associative algebra. It may happen if there is a family of cocycles $\sigma_t$ converging in an appropriate sense to a linear map $\sigma_0$; the latter inherits the identities of Lemma 1 but can fail to be convolution invertible. In other words, it is a closed condition to define an associative algebra but it is an open one to be invertible. For such a “non-invertible cocycle” one still obtains an $H$-comodule algebra which is not Hopf-Galois. However, an important piece of the structure theory breaks down in this situation.

**Lemma 2.** Assume that $U \supseteq O$ is an $H$-extension. The following statements are equivalent
1. \( U \) is isomorphic to a twisted product \( O_{\sigma}[H] \) as an \( H \)-comodule algebra.

2. There exists a convolution invertible right \( H \)-comodule map \( \gamma : H \to U \) and the subalgebra \( O \) is central.

3. There is a linear map from \( U \) to \( O \otimes H \) conducting an isomorphism of left \( O \)-modules and right \( H \)-comodules and \( O \) is central.

4. \( U \) is a free \( O \otimes H^* \)-module of rank 1 and \( U^H \) is central.

**Proof.** The equivalence of 1 through 3 follows from Theorems 7.2.2 and 8.2.4 of [25] and Lemma 11 of [30]. 4 follows from 1 because \( 1 \# \Lambda \) for a non-zero left integral \( \Lambda \) is clearly a basis of the left \( O \otimes H^* \)-module \( O_{\sigma}[H] \). Finally, 4 implies 3 since \( O \otimes H \) is a free left \( O \otimes H^* \)-module of rank 1. Thus, there exists an \( O \otimes H^* \)-module isomorphism between \( U \) and \( O \otimes H \) which must be an isomorphism of left \( O \)-modules and right \( H \)-comodules. \( \square \)

Now it is clear why only crossed products with the trivial action are of interest to us: the action must be trivial to produce central invariants.

Let us look at an example. We define the following cocycle on \( \mathbb{C}Z_n \).

Denoting a generator of \( \mathbb{Z}_n \) by \( t \) we set for each \( 0 \leq k, m < n \)

\[
\sigma(t^k \otimes t^m) = \begin{cases} 
1 & \text{if } k + m < n \\
q & \text{if } k + m \geq n
\end{cases}
\]  

(2)

It is a cocycle if and only if \( q \in O \) is an invertible element. The first choice may be \( q \in O = \mathbb{C}[q, q^{-1}] \). The twisted product \( \mathbb{C}[q, q^{-1}]_{\sigma}[\mathbb{C}Z_n] \) is the small quantum cohomology ring of the projective space \( \mathbb{C}P^{n-1} \). The multiplication coefficients of it in an appropriate basis are 3-point Gromov-Witten invariants. One should consult [24] for a good account of the subject. Another choice one can make is \( q = 0 \in \mathbb{C} \). The corresponding ring \( \mathbb{C}_{\sigma}[\mathbb{C}Z_n] \) is the de Rham cohomology ring of \( \mathbb{C}P^{n-1} \). It is an example of a “non-invertible twisted product”: it may be thought of as a limit of twisted products depending on \( q \in \mathbb{C} \setminus \{0\} \) as \( q \to 0 \). One can make another choice of a ring: let \( q = e^{x_1} \in C^\infty(\mathbb{C}^n) \). After identification \( t^k^{-1} = \frac{\partial}{\partial x_k} \) one obtains a structure of Frobenius manifold on \( \mathbb{C}^n \) [17].

This example demonstrates that one can encounter an interesting geometry studying Hopf-Galois extensions with central invariants. From now on \( U \supseteq O \) is an \( H \)-Galois extension with central invariants unless a contrary is specified. Let \( \Lambda \) be a non-zero left integral of \( H^* \). By the definition \( \Lambda \cdot x = x_0 \Lambda(x_1) \). The following facts were first proven in [19, 1.7, 1.9, 1.10]. One may find an interesting discussion concerning the last two properties in [3].
Lemma 3. The following statements hold under our assumptions.
1. $U_O$ is a projective finitely generated module.
2. $U_O$ is faithfully flat.
3. There exists an $O$ submodule $P \subseteq U$ such that $U = O \oplus P$.
4. There exists an element $x \in U$ such that $\Lambda \cdot x = 1$.

We can now explain why centrality of invariants is essential. One may hope, for instance, that $O$ being commutative suffices to build an interesting geometry. Indeed, there is a one-to-one correspondence between algebraic vector bundles on the spectrum of $O$ and finitely generated projective modules by Serre theorem [32, Corollary 2.4.50]. If $O$ is not central then left and right actions are different. Thus, we get a bi-bundle (i.e. a space carrying two commuting structures of a vector bundle) rather than a vector bundle. Geometry of bi-bundles is less exploded than that of vector bundles. Thus, if we want to remain in the realm of vector bundles we should require invariants being central. In further sections we will see more properties which do not work for Hopf-Galois extensions with non-central invariants.

We finish the introduction with the lemma which is handy if one wants to test whether an $H$-extension is Hopf-Galois. It is a slight generalization of [19, Lemma 1.3].

Lemma 4. If there exists a subalgebra $A \subseteq O$ such that the map $\Gamma : U \otimes_A U \rightarrow U \otimes H$ defined by $\Gamma(x \otimes_A y) = (x \otimes 1)(y_0 \otimes y_1)$ is onto then $U \supseteq O$ is $H$-Galois.

Proof. The map $\Gamma$ factors through the canonical map.

$$
\begin{array}{ccc}
U \otimes_O U & \xrightarrow{\text{can}} & U \otimes H \\
\downarrow \Gamma & & \\
U \otimes_A U & & \\
\end{array}
$$

Thus, the canonical map is onto. It must be also one-to-one by [25, Theorem 8.3.1].

We should notice that $\Gamma$ being one-to-one in the last lemma does not imply $A = O$. A proper embedding $A \hookrightarrow O$ may be an epimorphism in the category of rings making the tensor products $U \otimes_O U$ and $U \otimes_A U$ indistinguishable.
2. Pull-back.

This section is devoted to the crucial geometric property of Hopf-Galois extensions with central invariants, the pull-back. Intuitively, given a map of schemes $X \to Y$ and a geometrical object on $Y$, we should be able to induce a similar object using some kind of the fiber product. So far, we can work only on the level of rings. The following lemma provides the existence of pull-back on the level of rings. This property fails for Hopf-Galois extensions with non-central invariants.

**Lemma 5.** Let $\phi : O \to R$ be a map of commutative algebras. Then $\tilde{U} = U \otimes_O R \supseteq R$ is an $H$-Galois extension with central invariants.

**Proof.** The natural map $R \to \tilde{U}$ is one-to-one by Lemma 3. Indeed, $U = O \oplus P$ as $O$-modules and a split extension is pure. In down-to-earth terms, $\tilde{U} = (O \oplus P) \otimes_O R = O \otimes_O R \oplus P \otimes_O R \supseteq O \otimes_O R \cong R$.

The coaction of $H$ on $\tilde{U}$ is given by $(u \otimes_O r_0 \otimes (u \otimes_O r_1) = (u \otimes_O r_0) \otimes r_1$. Let $S$ be the subalgebra of invariants $\tilde{U}^H$. It is clear that $S \supseteq R$. Let us consider the diagram.

\[
\begin{array}{ccc}
\tilde{U} \otimes_S \tilde{U} & \xrightarrow{\text{can}} & \tilde{U} \otimes H \\
& \downarrow{\Gamma} & \\
\tilde{U} \otimes_O \tilde{U} &
\end{array}
\]

The map $\Gamma$ is onto since the original extension is Hopf-Galois. By Lemma 4 the extension $\tilde{U} \supseteq S$ is $H$-Galois.

It remains to show that $S = R$. Let $x \in U$ be an element such that $\Lambda \cdot x = 1$ which existence is provided by Lemma 3. It is clear that $\Lambda \cdot (x \otimes_O 1) = 1 \otimes_O 1 = 1 \in R$. It implies $\Lambda \cdot \tilde{U} = \tilde{U}^H$. But $\Lambda \cdot \tilde{U} \subseteq \Lambda \cdot ((U \otimes_O 1)R) \subseteq R(\Lambda \cdot U \otimes_O 1) \subseteq R((\Lambda \cdot U) \otimes_O 1) \subseteq R(O \otimes_O 1) \subseteq R$

The module $U_O$ defines a vector bundle of algebras $\tilde{U}$ on Spec $O$. Let us denote by $O_{(x)}$ the local ring of algebraic functions at $x \in \text{Spec } O$, $m_x$ its maximal ideal, and $K_x$ the quotient field $O_{(x)}/m_x$. We remind the main characters of the play.

*The germ algebra* may be defined by

\[ U_{(x)} = U \otimes_O O_{(x)} \supseteq O_{(x)}. \]  

(3)

Here is another way to describe it. $S = O \setminus \chi$ is a multiplicative subset of $U$. The generalized algebra of quotients $US^{-1}$ is isomorphic to $U_{(x)}$. 

The fiber algebra is

\[ U_\chi = U \otimes_O K\chi \supseteq K\chi. \]  

One may introduce the \( n \)-jet algebra generalizing the germs and the fibers. It is important for the study of indecomposable modules in the spirit of \[29\].

\[ U^{(n)}_\chi = U \otimes_O O_{(\chi)}/m^n_{\chi} \subseteq O_{(\chi)}/m^n_{\chi}. \]  

It is obvious that \( U^{(n)}_\chi \cong U_{(\chi)}/U_{(\chi)}m^n_{\chi} \). Conventionally, \( m^\infty_{\chi} = 0 \). Thus, the algebra of \( \infty \)-jets \( U^{(\infty)}_\chi \) is the same as the germ algebra \( U_{(\chi)} \). Another interesting choice is \( n = 1 \): \( U^{(1)}_\chi \) is the fiber algebra \( U_\chi \) at the point \( \chi \). The last algebra we want to introduce is the restriction to the closure of a point \( \chi \):

\[ U_{[\chi]} = U/U\chi \supseteq O/\chi. \]

If the point \( \chi \) is closed then \( U_{[\chi]} \) is isomorphic to \( U_\chi \).

Now we can prove the main theorem of the section improving Theorems 15 and 16 of \[30\].

**Theorem 6.** \( U^H_{[\chi]} \supseteq O/\chi \) and \( U^{(n)}_\chi \supseteq O_{(\chi)}/m^n_{\chi} \) are \( H \)-Galois extensions for each \( \chi \) and \( n \) (including \( n = \infty \)). If the extension \( U \supseteq O \) is cleft then so are the jet extensions and the restrictions.

**Proof.** The first statement is an immediate corollary of Lemma 5. To prove the second statement we assume that \( \gamma : H \longrightarrow U \) is a splitting map. Then the composition \( H \xrightarrow{\gamma} U \longrightarrow U^{(n)}_\chi \) is a splitting map for the jet extension \( U^{(n)}_\chi \supseteq O_{(\chi)}/m^n_{\chi} \). Similarly, the composition \( H \xrightarrow{\gamma} U \longrightarrow U/U\chi \supseteq O/\chi \) is a splitting map for the restriction extension \( U/U\chi \supseteq O/\chi \). □

This theorem is not the best statement one can get. We will prove in the next section that the jet algebras are always cleft for \( n < \infty \) and generically cleft for \( n = \infty \) under mild restrictions on \( O \). Let us give one more definition. Let \( \mathcal{Gal}_H(O) \) be a set of isomorphism classes of \( H \)-Galois extensions \( U \supseteq O \). We understand an isomorphism of \( H \)-comodule algebras by an isomorphism. Following \[30\], we denote the subset of isomorphism classes of cleft \( H \)-Galois extensions by \( \mathfrak{Gal}_H \). The following theorem is an immediate consequence of Lemma 5. It improves \[31\] Theorem 14.

**Theorem 7.** \( \mathcal{Gal}_H \) is a covariant functor from the category of commutative \( k \)-algebras to the category of sets and \( \mathfrak{Gal}_H \) is its subfunctor.
3. Local structure.

Usually if one looks at a geometrical object locally it seems trivial. A trivial $H$-Galois extension is the tensor product $O \otimes H \supseteq O$. It is false that an $H$-Galois extension is locally trivial: if $U \supseteq O$ is a proper Galois extensions of fields with Galois group $G$ then $O$ is already localized, nevertheless $U$ is not isomorphic to $O \otimes kG^*$. However, one could expect that a Hopf-Galois extension is locally cleft. We don’t know whether it holds. The following question is interesting from this prospective. In this section we prove a weaker version that a Hopf-Galois extension is generically locally cleft.

**Question.** Let $U \supseteq O$ be an $H$-Galois with central $O$ and finite-dimensional $H$. Assume that $O$ is a local algebra. Is the extension cleft?

First, we want to understand what is happening if the vector bundle defined by a Hopf-Galois extension is trivial. If we managed to show that this implied that the Hopf-Galois extension were cleft it would be very nice. Indeed, any vector bundle is locally trivial. The following fact is the best we can do. It first appeared in [2, Lemma 2.9]. We should mention that trivial bundles correspond to free modules on the algebraic side.

**Lemma 8.** If $U_O$ is a free module of rank $n$ then $U^n$ is a free left $O \otimes H^*$-module of rank $n$.

**Proof.** Using the canonical map we get a chain of isomorphisms:

$$U^n \cong U \otimes O U \cong U \otimes H \cong U \otimes_O (O \otimes H) \cong (O \otimes H)^n \cong (O \otimes H^*)^n \quad \square$$

Thus, $U$ is a projective module of rank 1 if the rank is well-defined for projective $O \otimes H^*$-modules. The next idea is to put some constraints on $O$ and $H$ to ensure that any projective $O \otimes H^*$-module of rank 1 is free. A reader may supplement the next corollary with other similar statements providing cleftness.

**Corollary 9.** $U$ is cleft if one of the following conditions holds:

1. $O$ is artinian and $U_O$ is free;
2. $O$ is a local artinian ring;
3. $O$ is a field;
4. $O \otimes H^*$ is self-injective and $U_O$ is free;
5. Spec$O$ is a toric variety and $O \otimes H^*$ is self-injective.

**Proof.** Each of the conditions 1-5 implies the module $U_O$ is free. It is explicitly stated in 1 and 4. Any finitely generated projective module over local ring or field is free. Algebra of functions on a toric variety
is a semigroup algebra of a normal monoid. A finitely generated projective module is free over such an algebra according to the Anderson conjecture proved in [15, Theorem 2.1].

Using Lemma 8, it suffices to understand some kind of uniqueness of decomposition theorem in each of the cases. In 1, 2, and 3 one can use the Krull-Remark-Schmidt theorem [11, 5.18.11]. Indeed, $O \otimes H^*$ is artiniian if so is $O$: since $H$ is finite dimensional $O \otimes H^*$ is even an artiniian $O$-module. Thus the $O \otimes H^*$-module $U^n$ must have a finite length which implies the uniqueness of the decomposition.

We can use a stronger version of the Krull-Remark-Schmidt theorem in 4 and 5 [11, 5.19.18]. $U^n$ is injective $O \otimes H^*$-module in this case; so are its direct summands. The endomorphism ring of an injective indecomposable module is local and we can use the Krull-Remark-Schmidt theorem. 

Now we can state and prove the local trivialization theorem.

**Theorem 10.** The germ extension is cleft for generic $\chi \in \text{Spec } O$. If $O$ is finitely generated then the $n$-jet extension is cleft for every $\chi \in \text{Spec } O$ and $n < \infty$.

**Proof.** The second statement follows from Corollary 9.2 since the $n$-jets of functions $O_{(\chi)}/m^n_{\chi}$ is a local artiniian ring (even finite dimensional) in the case of finitely generated $O$.

The first statement can be reduced to a component of the spectrum: let $I$ be a minimal prime ideal of $O$. We replace the extension $U \supseteq O$ with $U \otimes_O O/I \supseteq O/I$. Thus, we can assume $O$ is a domain without loss of generality. This argument becomes sloppy unless $O$ is finitely generated. Otherwise, no minimal prime ideal can exist. However, the term generic is unclear in this case. Thus, we can make up the definition of a generic point and think that the theorem still holds. For example, generic could mean generic in closure of each point: it entitles us to replace $O$ with $O/I$ where $I$ is an arbitrary prime ideal.

We should use the field fractions $Q(O)$ to comprehend the first statement. Geometrically, it is a field of rational functions on the spectrum. By Corollary 9.3 the extension $U \otimes_O Q(O) \supseteq Q(O)$ is cleft. Let $\gamma : H \rightarrow U \otimes_O Q(O)$ be a splitting map. The algebra $U \otimes_O Q(O)$ can be realized as a generalized ring of quotients of $U$ by the multiplicative set $O \setminus \{0\}$. Let $h_i$ be a base of $H$. Let $\mathcal{V}$ be the complement of zeroes of the denominators of $\gamma(h_i)$. It is obvious that $U_{(\chi)} \supseteq O_{(\chi)}$ is cleft for each $\chi \in \mathcal{V}$ 

$\square$
4. Pasting.

We do algebraic geometry of $H$-Galois extensions on affine schemes on the level of rings till Section 8 where we sheafify the enterprise. After Section 2, given an $H$-Galois extension of an affine scheme $X$ and an open cover $\bigcup V_i$ of $X$, we can restrict the extension to each set $V_i$. Now we are interested in the inverse problem: given $H$-Galois extensions on each $V_i$, compatible on intersections, we want to paste an $H$-Galois extension on $X$.

We are working in the flat topology since it is sufficiently general for a number of applications. A covering of an algebra $O$ is a finite set of $O$-algebras $O_i$ such that the product $\prod_i O_i$ (denoted $\overline{O}$ from now on) is a finitely presented $O$-algebra which is a faithfully flat $O$-module. The last condition provides that the natural map of spectra $\prod_i \text{Spec } O_i \rightarrow \text{Spec } O$ is onto.

The standard example of a covering can be constructed using the partition of unity. Given a finite set of elements $\{f_i\}$ of $O$ generating the trivial ideal $O$ (i.e. $\sum_i x_i f_i = 1$ for some $x_i$), we set $O_i = Of_i^{-1}$. It is standard to check that we get a covering this way. We also point out that partitions of unity are the same as coverings in Zariski topology.

We introduce another piece of notation: given a covering $O_i$ we define $O_{ij} = O_i \otimes O_j$ and $O_{ijk} = O_i \otimes O_j \otimes O_k$. These algebras should be thought of as double and triple intersections in the covering.

We consider a covering $O_i$ of $O$ and a collection of $H$-Galois extensions $U_i \supseteq O_i$. We assume there are isomorphisms of $H$-comodule algebras $\phi_{ij} : U_i \otimes O_i O_{ij} \rightarrow U_j \otimes O_j O_{ij}$. The datum $(O_i, U_i, \phi_{ij})$ is called an $H$-structure on the algebra $O$ if $\phi_{ij}$, restricted to $O_{ij}$, is the identity map and the cocycle condition is satisfied: $(\phi_{jk} \otimes O_{jk} \text{Id}_{O_{ijk}}) \circ (\phi_{ij} \otimes O_{ij} \text{Id}_{O_{ijk}}) = \phi_{ik} \otimes O_{ik} \text{Id}_{O_{ijk}}$ for each $i, j, k$.

**Theorem 11.** For each $H$-structure on an algebra $O$ there exists a unique up to an isomorphism $H$-Galois extension with central invariants $U \supseteq O$ such that $U \otimes O_i \cong U_i$.

This theorem allows to glue Hopf-Galois extensions from local data. However, this locality property is insufficient to conclude $\mathcal{GAL}_H$ is a sheaf in the flat topology in the category of $k$-algebras because there can be non-trivial automorphisms. For instance, if $G$ is a group with non-trivial center $Z$ and $U \supseteq O$ is a $G$-Galois field extension then $Z$ acts by $kG^*$-comodule algebra automorphisms. Nevertheless, $\mathcal{GAL}_H$ can made into a stack which will be discussed in Section 8.

A natural question is to understand what kind of local property the subfunctor $\mathcal{Gal}_H$ inherits. Given an $H$-structure with split extensions
$U_i$, the global extension $U \supseteq O$, whose existence is ensured by Theorem 11, is not necessarily split. Indeed, let $\gamma_i : H \to U_i$ be a bunch of splittings (i.e. convolution-invertible right $H$-comodule maps); one has to glue a splitting $\gamma : H \to U$. It can be done if $\gamma_i$ agree on “intersections” $U_{ij}$ but it is unclear how to do it without any restrictions on behavior on intersections.

The rest of the section is devoted to the proof of Theorem 11. The major tool is the following lemma [1.1.3.14]. If $f : A \to B$ is a morphism of rings then a linear map $h : M \to N$ from an $A$-module $M$ to a $B$-module $N$ is called adapted to $f$ if $f(am) = h(a)f(m)$ for each $a \in A, m \in M$.

Lemma 12. Let $R$ be a commutative ring and $B$ be a faithfully flat commutative $R$-algebra. Let us consider modules $J, K, L$ over $B$, $B \otimes_R B$, and $B \otimes_R B \otimes_R B$ with homomorphisms

\[
\begin{array}{c}
J & \xrightarrow{u_0} & K & \xrightarrow{u'_0} & L \\
\xrightarrow{u_1} & & \xrightarrow{u'_1} & & \\
& & \xrightarrow{u'_2} & & \\
\end{array}
\]

adapted to corresponding ring homomorphisms

\[
\begin{array}{c}
B & \xrightarrow{d_0} & B \otimes_R B & \xrightarrow{d'_0} & B \otimes_R B \otimes_R B \\
\xrightarrow{d_1} & & \xrightarrow{d'_1} & & \\
& & \xrightarrow{d'_2} & & \\
\end{array}
\]

such that $d_0(b) = b \otimes 1, d_1(b) = 1 \otimes b, d'_0(b \otimes c) = b \otimes c \otimes 1, d'_1(b \otimes c) = b \otimes 1 \otimes c$, and $d'_2(b \otimes c) = 1 \otimes b \otimes c$ for each $b, c \in B$. We also assume that $u'_0u_0 = u'_1u_0, u'_0u_1 = u'_2u_0$, and $u'_1u_1 = u'_2u_1$. If $I = \text{Ker}(u_0, u_1)$ then the embedding $I \hookrightarrow J$ induces an isomorphism $I \otimes_R B \to J$.

Proof. Let us first look at the special case $O = \prod O_i$. It is easy to see that $U = \prod_i U_i$ satisfies the conditions of the theorem. If $\bar{U}$ is another extension satisfying the conditions of the theorem then $\bar{U} \cong \bar{U} \otimes_O O \cong \prod_i \bar{U} \otimes_O O_i \cong \prod_i U_i \cong U$. It is easy to see that all isomorphisms are those of $H$-comodule algebras. We have just established an important property of $\mathcal{GA}_H$ that for finite products

\[ \mathcal{GA}_H(\prod O_i) = \prod_i \mathcal{GA}_H(O_i). \]

Now we want to use Lemma 12 with $R = O, B = \overline{O} = \prod_i O_i, J = \prod_i U_i, K = J \otimes^2_{\overline{O}}(\overline{O} \otimes_O \overline{O})$ where 2 means that the tensor product is taken using the embedding $d_1$ to the second factor $b \mapsto 1 \otimes b$, and, finally, $L = J \otimes^3_{\overline{O}}(\overline{O} \otimes_O \overline{O} \otimes_O \overline{O})$ given by the embedding to the third
factor \( b \mapsto 1 \otimes 1 \otimes b \). We also need some maps, three of which are easy to describe:
\[ u_1(x) = x \otimes 1 \otimes 1, \quad u_1'(x \otimes a \otimes b) = x \otimes a \otimes 1 \otimes b, \quad u_2'(x \otimes a \otimes b) = x \otimes 1 \otimes a \otimes b. \]

It is straightforward to see that \( u_1, u_1', \) and \( u_2' \) are adapted to \( d_1, d_1', \) and \( d_2' \) correspondently. Let us define
\[ \phi = \prod_{i,j} \phi_{ij} : J \otimes_1 (\mathcal{O} \otimes_0 \mathcal{O}) \to J \otimes_2 (\mathcal{O} \otimes_0 \mathcal{O}). \]

\( J \supseteq \mathcal{O} \) is an \( H \)-Galois extension because the theorem has already been proved for direct products. \( \phi \) is an isomorphism of \( H \)-Galois extensions. We can define
\[ u_0(x) = \phi(x \otimes 1 \otimes 1). \]

Finally we need an isomorphism
\[ \prod_{i,j,k} \phi_{ijk} \otimes O_{ijk} \text{Id}_{O_{ijk}} : J \otimes_1 (\mathcal{O} \otimes_0 \mathcal{O} \otimes_0 \mathcal{O}) \to J \otimes_3 (\mathcal{O} \otimes_0 \mathcal{O} \otimes_0 \mathcal{O}) \]
to define
\[ u_0'(x \otimes a \otimes b) = \prod_{i,j,k} \phi_{ijk} \otimes O_{ijk} \text{Id}_{O_{ijk}} (\phi^{-1}(x \otimes a \otimes b) \otimes 1). \]

It is straightforward to check that \( u_0 \) and \( u_0' \) are adapted to \( d_0 \) and \( d_0' \) correspondently. We have to compute the condition for compositions to apply Lemma 12. It is easy to see that
\[ u_1'(u_1(x)) = x \otimes (1 \otimes 1 \otimes 1) = u_2'(u_1(x)). \]

Assuming \( \phi(x \otimes 1 \otimes 1) = \sum_t y_t \otimes a_t \otimes b_t \), we compute that
\[ u_0'(u_0(x)) = \sum_t y_t \otimes (b_t \otimes 1 \otimes c_t) = u_1'(u_0(x)). \]

Finally,
\[ u_2'(u_0(x)) = \sum_t y_t \otimes (1 \otimes a_t \otimes b_t). \]

Assuming \( \phi^{-1}(x \otimes 1 \otimes 1) = \sum_t z_t \otimes c_t \otimes d_t \), we can write
\[ u_0'(u_1(x)) = \prod_{i,j,k} \phi_{ijk} \otimes O_{ijk} \text{Id}_{O_{ijk}} (\sum_t z_t \otimes (c_t \otimes d_t \otimes 1)). \]

The results of the last two calculations because of the cocycle condition.

Now we define \( U = \text{Ker}(u_0, u_1) = \{ x \in J \mid \phi(x \otimes (1 \otimes 1)) = x \otimes (1 \otimes 1) \} \). \( U \) is a subalgebra of \( J \), closed under \( H^* \)-action, since \( \phi \) is a map of \( H \)-comodule algebras. \( U^H = U \cap J^H = U \cap \mathcal{O} = \{ x \in \mathcal{O} \mid \phi(x \otimes (1 \otimes 1)) = x \otimes (1 \otimes 1) \} = O \). Finally, \( \text{can}_J = \text{can}_U \otimes_0 \text{Id}_\mathcal{O} \) modulo natural identifications \( (U \otimes H) \otimes_0 \mathcal{O} \cong (U \otimes_0 \mathcal{O}) \otimes H \cong J \otimes H \) and \( (U \otimes_0 U) \otimes_0 \mathcal{O} \cong (U \otimes_0 \mathcal{O}) \otimes_0 (U \otimes_0 \mathcal{O}) \cong J \otimes \mathcal{O} \). The second identification works through the map \( (a \otimes x) \otimes (b \otimes y) \mapsto (a \otimes b) \otimes xy \).
Since $\mathcal{O}_O$ is faithfully flat and $\text{can}_I$ is an isomorphism the map $\text{can}_O$ is an isomorphism too and $U \supseteq O$ is $H$-Galois.

Let $\check{U}$ be another extension with the given properties. Then $\check{U} \otimes O \cong \prod \check{U} \otimes O_i \cong \prod U_i \cong J$. Thus, there is the natural map $\check{U} \to J$ which factors through $U$ because of the universal property of the kernel of a pair. The map $\check{U} \to U$ is of projective $O$-modules which is a local isomorphism (on the covering $O_i$). Thus, its kernel and cokernel have trivial support and the map in an isomorphism. Noticing that it is a map of $H$-comodule algebras finishes the proof. $\blacksquare$

The argument about the canonical maps fails for Hopf-Galois extensions which invariants are not central because of the lack of the identifications.

5. Frobenius form.

A Frobenius manifold has a Riemannian metric as a part of its structure. Similarly, Hopf-Galois extensions carry a canonical (up to a scalar) non-degenerate associative form which can fail to be symmetric. Chosen $\Lambda$, a non-zero left integral of $H^*$, we construct an $O$-bilinear form $\langle \cdot, \cdot \rangle : U \times U \to O$ by $\langle x, y \rangle = \Lambda(x_0y_0\Lambda(x_1y_1))$ for each $x, y \in U$.

Lemma 13. [19, 1.7.5] The form $\langle \cdot, \cdot \rangle$ is non-degenerate.

Non-degeneracy means that the map $\zeta : U \to \text{Hom}(U, O)$ given by $\zeta(u)(v) = \langle u, v \rangle$ is an isomorphism. The following corollary is immediate from Lemma 13, Theorem 6, and the definition of a Frobenius algebra.

Corollary 14. The algebras $U_\chi$ are Frobenius $K_\chi$-algebras.

The question when this form is symmetric is quite interesting. The following theorem is a sufficient condition for being symmetrical. It is a slight generalization of [30, Theorem 17].

Theorem 15. Let $H$ be unimodular with the antipode of order 2. The form $\langle \cdot, \cdot \rangle$ is symmetric if one of the following holds:

1. $U \supseteq O$ is cleft.
2. $O$ is semiprime.

Proof. Let us treat the cleft case first. We assume that $U$ is isomorphic to a twisted product $O_\sigma[H]$. We notice that $h_1\Lambda(h_2) = \Lambda(h)1$ for each $h \in H$. Indeed, the equality $\alpha(h_1\Lambda(h_2)) = \alpha(\Lambda(h)1)$ holds for each $\alpha \in H^*$ since $\alpha(h_1\Lambda(h_2)) = \alpha(h_1)\Lambda(h_2) = \alpha(1)\Lambda(h)$. It was proved in [27] that $H$ is unimodular with the antipode of order 2 if and only if $\Lambda(xy) = \Lambda(yx)$ for each $x, y \in H$. Therefore,
\( \langle a \otimes h, b \otimes g \rangle = abh_1g_1\Lambda(h_2g_2) = ab\Lambda(hg) = ba\Lambda(gh) = \langle b \otimes g, a \otimes h \rangle \) for each \( b \otimes g, a \otimes h \in U \).

The element \( \langle u, v \rangle - \langle v, u \rangle \) belongs to every maximal ideal for each \( u, v \in U \) by Theorem 10 and the consideration above. If \( O \) is semiprime the intersection of maximal ideals (i.e. the radical) is zero. Thus, the form is symmetric. \( \square \)

The following question seems to be of interest. Assume that the ground field \( k \) is formally real, i.e. zero cannot be presented as a sum of squares. Under which conditions is the form positive definite. Over \( \mathbb{R} \) it is a question of having a Riemannian structure.

**Proposition 16.** Let \( k \) be formally real. If the form is positive definite then \( H \) is cosemisimple.

**Proof.** \( 0 < \langle 1, 1 \rangle = \Lambda(1)1 \) Thus, \( \Lambda(1) \neq 0 \) and \( H^* \) is cosemisimple by the Sweedler criterion. \( \square \)

We may also notice that a formally real field has zero characteristic. Thus, \( H \) must be as well semisimple by the Larson-Radford theorem.

The form allows us to introduce two more features. The map \( C : U \otimes_O U \otimes_O U \to k \) given by \( C(u \otimes v \otimes w) = \langle uv, w \rangle \) encodes the multiplication. There is also the Nakayama automorphism \([26]\) defined by \( \langle u, v \rangle = \langle v, \text{Nak}(u) \rangle \). It measures the failure of the form to be symmetric. In particular, \( \text{Nak} \) is inner if and only if \( U \) admits a symmetric associative bilinear form.

6. Connections.

The study of connections on both principal and vector bundles is an important part of modern geometry. Connections on quantum principal bundles were actively studied \([16]\). A Hopf-Galois extension carries also a structure of a vector bundle which connections may have some significance. Moreover, the notion of connection on a vector bundle is less technical than that on a principal bundle yet to mention quantum principal bundles. Throughout this section we prove some interesting propositions, involving connections and Hopf theoretical features of the extensions, towards a conjectural existence and uniqueness theorem for a some kind of connections similar to the existence and uniqueness of the torsion-free Riemannian connection on the tangent bundle of a Riemannian manifold.

We study bilinear pairings \( \nabla : \mathcal{L} \times U \to U \) where \( \mathcal{L} \) is the Lie algebra of derivations of \( O \) (vector fields on the spectrum). We denote the result of the pairing by \( \nabla_Xu \) with \( X \) being a vector field. \( \nabla \) is called a connection if

\[
\nabla_X(au) = X(a)u + a\nabla_Xu \quad \text{and} \quad \nabla_{aX}u = a\nabla_Xu.
\]
for each \( a \in O, \) \( X \in \mathcal{L}(O) \), and \( u \in U \). A connection \( \nabla \) is called a Frobenius connection if
\[
X\langle u, v \rangle = \langle \nabla_X u, v \rangle + \langle u, \nabla_X v \rangle.
\]
A connection \( \nabla \) is called multiplicative if
\[
\nabla_X (uv) = (\nabla_X u)v + u \nabla_X v.
\]
A connection \( \nabla \) is called equivariant if
\[
\nabla_X u_0 \otimes u_1 = (\nabla_X u)_0 \otimes (\nabla_X u)_1
\]
A connection \( \nabla \) is called Nakayama if
\[
\text{Nak}^{-1}(\nabla_X \text{Nak}(u)) = \nabla_X u.
\]

Roughly speaking Frobenius connection is the one along which the form is covariant constant. Indeed, being Frobenius is equivalent to \( \nabla_X \langle , \rangle = 0 \). A multiplicative connection may be thought of as a way to extent derivation of \( O \) to derivations of \( U \). The properties we have just defined are not independent.

**Proposition 17.** A multiplicative equivariant connection is Frobenius. A Frobenius connection is Nakayama.

**Proof.** For each \( u, v \in U \) we have \( X(\langle u, v \rangle) = \nabla_X (u_0v_0\Lambda(u_1v_1)) = \nabla_X (u_0v_0\Lambda(u_1v_1) + u_0\nabla_X (v_0)\Lambda(u_1v_1) + u_0v_0X(\Lambda(u_1v_1)) = (\nabla_X u_0)v_0\Lambda((\nabla_X u)_1v_1) + u_0(\nabla_X v_0)\Lambda(u_1(\nabla_X v)_1) = \langle \nabla_X u, v \rangle + \langle u, \nabla_X v \rangle \)
which proves the first claim. To show the second one it suffices to notice the following sequence of equalities for all \( u, v \in U \) and \( X \in \mathcal{L} \)
\[
\langle \nabla_X u, v \rangle = X(\langle u, v \rangle) - \langle u, \nabla_X v \rangle = X(\langle v, \text{Nak}(u) \rangle) - \langle \nabla_X v, \text{Nak}(u) \rangle = \langle v, \nabla_X \text{Nak}(u) \rangle - \langle \text{Nak}^{-1}(\nabla_X \text{Nak}(u)), v \rangle
\]

The next three propositions deal with the issue of existence of a connection with certain properties.

**Proposition 18.** A connection always exists.

**Proof.** \( U \) is direct summand of a free \( O \)-module \( O^n \). We identify \( U \) with the submodule of \( O^n \). Let \( \pi : O^n \to U \) be a projection. Let \( e_i \) be a base of \( O^n \). The free module admits a trivial connection making \( e_i \) covariant constants. We want to restrict this connection to \( U \). Given \( u = u^i e_i \in U \), we define \( \nabla_X u = \pi(X(u^i)e_i) \). The map \( \nabla \) is obviously a connection: for instance, \( \nabla_X au = \pi(X(a^i u^i)e_i) = \pi(X(a)u^i e_i) + \pi(a X(u^i) e_i) = X(a)\pi(u^i e_i) + a \pi(X(u^i) e_i) = X(a)u + a \nabla_X u \) for \( X \in \mathcal{L}, a \in O, u \in U \). \( \square \)
Proposition 19. If the characteristic of $k$ is not 2 and there exists a Nakayama connection then there exists a Frobenius connection.

Proof. Let $\nabla$ be a Nakayama connection. Let us define $T$ by $\langle T_X u, v \rangle = X((u, v)) - \langle \nabla_X u, v \rangle - \langle u, \nabla_X v \rangle$. Then $\langle u, T_X v \rangle = \langle T_X v, \text{Nak}(u) \rangle = X((v, \text{Nak}(u))) - \langle \nabla_X v, \text{Nak}(u) \rangle - \langle v, \nabla \text{Nak}(u) \rangle = X((u, v)) - \langle u, \nabla_X v \rangle - \langle \text{Nak}^{-1}(\nabla_X \text{Nak}(u)), v \rangle$.

It is easy to show that $\tilde{\nabla} = \nabla + \frac{1}{2}T$ is a connection: $\langle \tilde{\nabla}_X bu, v \rangle = \langle \nabla a_X bu, v \rangle + \frac{1}{2}[aX((bu, v)) - \langle \nabla a_X bu, v \rangle - \langle bu, \nabla a_X v \rangle] = ab\langle \nabla_X u, v \rangle + aX(b)\langle u, v \rangle + \frac{1}{2}[aX(b)(u, v) + abX((u, v)) - aX(b)(u, v) - ab\langle \nabla_X u, v \rangle - ab(u, \nabla_X v)] = ab\langle \tilde{\nabla}_X u, v \rangle + aX(b)(u, v) + \frac{1}{2}ab(T_X u, v)$.

$\tilde{\nabla}$ is Frobenius: $\langle \tilde{\nabla}_X u, v \rangle + \langle u, \tilde{\nabla}_X v \rangle = \langle \nabla_X u, v \rangle + \frac{1}{2}[X((u, v)) - \langle \nabla_X u, v \rangle - \langle u, \nabla_X v \rangle] + \langle u, \nabla_X v \rangle + \frac{1}{2}[X((u, v)) - \langle \nabla_X u, v \rangle - \langle u, \nabla_X v \rangle]$. \qed

The next lemma describes another trick which may be performed with connections. It may be restated that a certain affine hyperplane of $H^*$ is acting on the space of connections.

Lemma 20. Let $h$ be an element of $H^*$ such that $h(1) = 1$ and $\nabla$ be a connection. The pairing $h \cdot \nabla$ defined by $(h \cdot \nabla)_X u = h_1 \cdot (\nabla_X h_2 \cdot u)$ is a connection.

Proof. It is clear that $(h \cdot \nabla)_f X u = h_1 \cdot (\nabla_f X h_2 \cdot u) = h_1 \cdot (f \nabla_X h_2 \cdot u) = (h_1 \cdot f)h_2 \cdot (\nabla_X h_3 \cdot u) = f(h \cdot \nabla)_X u$ for each $X \in \mathcal{L}, u \in U$.

It suffices to check $(h \cdot \nabla)_X f u = h_1 \cdot [\nabla_X (h_2 \cdot f) u] = h_1 \cdot [\nabla_X ((h_3 \cdot f)(h_2 \cdot u))] = h_1 \cdot [\nabla_X f (h_2 \cdot u)] = h_1 \cdot [X(f)(h_2 \cdot u)] + f \nabla_X (h_2 \cdot u)] = h_1 \cdot [X(f)(h_2 \cdot u)] + (h_2 \cdot f) \nabla_X (h_2 \cdot u) = (h_1 X(f)) u + f(h \cdot \nabla)_X u = X(f) u + f(h \cdot \nabla)_X u$. \qed

We remind that $H$ being cosemisimple is equivalent to $\Lambda(1) \neq 0$ for some integral $\Lambda$. Without loss of generality we may assume that $\Lambda(1) = 1$. This allows us to integrate connections: given $\nabla$, we obtain $\Lambda \cdot \nabla$. We also remind that in zero characteristic a cosemisimple Hopf algebra is involutive and unimodular (and even semisimple) [20, 21].

Proposition 21. If $H$ is cosemisimple then $\Lambda \cdot \nabla$ is an equivariant connection for any connection $\nabla$. If $\nabla$ is multiplicative then so is $\Lambda \cdot \nabla$ provided $H$ is unimodular and involutive.

Proof. We utilize [20, Formula 3] with $a = S^{-1}(h) \in H^*$ to prove the first statement:

$$h\Lambda_1 \otimes \Lambda_2 = \Lambda_1 \otimes S^{-1}(h)\Lambda_2.$$ 

Let us apply $\text{Id} \otimes S$ to this:

$$h\Lambda_1 \otimes S(\Lambda_2) = \Lambda_1 \otimes S(\Lambda_2)h.$$
Now, \( h \cdot [(\Lambda \cdot \nabla)_X u] = h\Lambda_1 \cdot [\nabla_X (S(\Lambda_2) \cdot u)] = \Lambda_1 \cdot [\nabla_X (S(\Lambda_2) h \cdot u)] = (\Lambda \cdot \nabla)_X h \cdot u \) which is reformulation of the equivariance condition in the language of \( H^* \)-action.

\( H \) being unimodular and involutive is equivalent to \( \Lambda \) being cocommutative. It easily implies that \( \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3 \otimes \Lambda_4 = \Lambda_4 \otimes \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3 \). If \( f \in O \) then \( (\Lambda \cdot \nabla)_X (f) = \Lambda_1 \cdot \nabla_X (S\Lambda_2 \cdot f) = \Lambda(1)_X (f) = X(f) \). \( H^* \) is involutive since so is \( H \). We are ready to check that \( \Lambda \cdot \nabla \) is multiplicative provided so is \( \nabla \):

\[
(\Lambda \cdot \nabla)_X (uv) = \Lambda_1 \cdot \nabla_X (S\Lambda_2 \cdot uv) = \Lambda_1 \cdot \nabla_X [(S\Lambda_3 \cdot u)(S\Lambda_2 \cdot v)] = \\
= [\Lambda_1 \cdot \nabla_X (S\Lambda_4 \cdot u)](\Lambda_2 S\Lambda_3 \cdot v) + \Lambda_1 S\Lambda_4 \cdot u[\Lambda_2 \cdot \nabla_X (S\Lambda_3 \cdot v)] = \\
= \Lambda_1 \cdot [\nabla_X (S\Lambda_2 \cdot u)] v + \Lambda_2 S^{-1} \Lambda_1 \cdot u[\Lambda_3 \cdot \nabla_X (S\Lambda_4 \cdot v)] = \\
= [(\Lambda \cdot \nabla)_X u] v + u(\Lambda \cdot \nabla)_X v. \quad \square
\]

Now we would like to address an issue of uniqueness of connection. Let us assume that we are given two connections \( \nabla^1 \) and \( \nabla^2 \). Let \( D = \nabla^1 - \nabla^2 \). It is straightforward to see that \( D_X \) is an endomorphism of the \( O \)-module \( U \). Thus, two connections differ by an element of \( \text{Hom}_O(\mathcal{L}, \text{End}_O U) \). According to [23, 8.3.3], \( \text{End}_O U \cong U \# H^* \), the smash product, as algebras. In particular, \( \text{Hom}_O(\mathcal{L}, \text{End}_O U) \cong \text{Hom}_O(\mathcal{L}, U \# H^*) \cong \text{Hom}_O(\mathcal{L}, U)^n \) where \( n \) is the dimension of \( H \). We have just proved the following proposition.

**Proposition 22.** The space of connections is a \( \text{Hom}_O(\mathcal{L}, U)^n \)-torsor (i.e. a space with free transitive action of \( \text{Hom}_O(\mathcal{L}, U)^n \)).

If the connections \( \nabla^1 \) and \( \nabla^2 \) are equivariant then the difference \( D = \nabla^1 - \nabla^2 \) is an element of \( \text{Hom}_O(\mathcal{L}, \text{End}_O \otimes H^* U) \) and the space of equivariant connections is a torsor over this group. It allows a nice interpretation if \( U \) is cleft: let us assume \( U \cong O_\sigma [H] \) is given. Now, \( \text{End}_O \otimes H^* U \cong \text{Comod}(H, U) \). Right \( H \)-comodule maps from \( H \) to \( U \) (or integrals in the terminology of [3, 9]) were actively studied. We think that the fact we have just noticed is worth writing as a proposition.

**Proposition 23.** Given an isomorphism \( U \cong O_\sigma [H] \), the space of equivariant connections becomes a \( \text{Hom}_O(\mathcal{L}, \text{Comod}(H, U))^n \)-torsor.

Adding other special properties in this spirit will further refine the space of connections. For example, if \( \nabla^1 \) and \( \nabla^2 \) are multiplicative connections then \( \nabla^1_X - \nabla^2_X \) is a differentiation of \( U \) for each \( X \). Such process may eventually lead to an interesting uniqueness theorem.

The last proposition is just a reformulation of Frobenius property involving the map \( C \) defined in Section 5.

**Proposition 24.** We assume that \( \nabla \) is Frobenius. Given \( X \), \( \nabla_X \) is a derivation of \( U \) if and only if \( \nabla_X \cdot C = 0 \).
Proof. Being a derivative means the equality
\[ \langle \nabla_X(uv), w \rangle = \langle \nabla_X(u)v, w \rangle + \langle u \nabla_X(v), w \rangle \]
for each \( u, v, w \in U \). Since the connection is Frobenius it may be rewritten
\[ X(C(u \otimes v \otimes w)) - \langle uv, \nabla_X w \rangle = \langle \nabla_X(u)v, w \rangle + \langle u \nabla_X(v), w \rangle \]
By the definition,
\[ \nabla_X \cdot C(u \otimes v \otimes w) = X(C(u \otimes v \otimes w)) - C(\nabla_X u \otimes v \otimes w) - C(u \otimes \nabla_X v \otimes w) - C(u \otimes v \otimes \nabla_X w) \]
This proves the proposition. \( \Box \)

Corollary 25. A Frobenius connection is multiplicative if and only if \( \nabla \cdot C = 0 \).

It may be also interesting to investigate a significance of flatness. A connection is flat if the curvature form \( R_{X,Y}u = [\nabla_X, \nabla_Y]u - \nabla_{[X,Y]}u \) is zero which is equivalent to \( \nabla \) defining a representation of Lie algebra \( \mathcal{L} \) on \( U \).

7. Miyashita-Ulbrich action.

Miyashita-Ulbrich action was introduced in [9]. If \( U \subseteq O \) is an \( H \)-Galois extension with not necessarily central \( O \) we can define the Miyashita-Ulbrich action for any algebra morphism \( \alpha : U \to B \). We denote \( B^0 \) and \( B^U \) the centralizer of \( \alpha(O) \) and \( \alpha(U) \) in \( B \). Miyashita-Ulbrich action is a right \( H \)-action on \( B^O \) defined by the formula \( c \overset{\alpha}{\longleftarrow} x = \sum \alpha(a_i)c\alpha(b_i) \) for any \( x \in H \) so that \( \sum a_i \otimes b_i = \text{can}^{-1}(1 \otimes x) \). It has a property that for \( c \in B^O \) and \( b \in U \) the formula \( c\alpha(b) = \alpha(b_0)(c \overset{\alpha}{\longleftarrow} b_1) \) holds. Furthermore, the invariants of the Miyashita-Ulbrich action is precisely \( B^U \) [9].

Keeping in mind that \( U \supseteq O \) is an extension with central invariants we consider the identity map from \( U \) to \( U \) as \( \alpha \). This defines a right \( H \)-action on \( U \) such that the center of \( U_\chi \) is the subalgebra of invariants. We use a shorthand notation \( \overset{\alpha}{\longleftarrow} \) rather than \( \overset{\text{id}}{\longleftarrow} \) for this action. Using antipode one can get a left Miyashita-Ulbrich as well. By doing so one obtains a left action of the quantum double \( D(H) \) in the case of cocommutative \( H \) [3].

Miyashita-Ulbrich action seems to be an important piece of structure in our set-up. We think of \( U_\chi \) as a deformation of right \( H \)-module. One may also think of \( U_\chi \) as a deformation of an algebra structure: it was done in [30].

Let \( E_\chi = \text{End}_k U_\chi \). Then \( E_\chi \) carries a structure of \( H-H \)-bimodule by \( h\phi h'(u) = \phi(u \overset{h}{\longleftarrow} h') \). We are interested in the first Hochschild
cohomology $\text{HH}^1(H, E_\chi)$ because normalized first Hochschild cocycles with coefficients in $E_\chi$ constitute infinitesimals of deformations of module structure. Indeed, let $D = \mathbb{k}[\epsilon]/(\epsilon^2)$ be the ring of dual numbers. An infinitesimal deformation is a right $H$-module structure $U_\chi[\epsilon] \otimes_D H \to U_\chi[\epsilon]$ of the form:

$$a \otimes h \mapsto a \leftarrow h + \mu(a, h)\epsilon$$

for $a \in U_\chi, h \in H$. The following is the associativity condition:

$$a \leftarrow hh' + \mu(a, hh')\epsilon = a \leftarrow h \leftarrow h' + \mu(a, h) \leftarrow h'\epsilon + \mu(a \leftarrow h, h')\epsilon.$$  

It may be rewritten as $\mu(a, hh') = \mu(a, h) \leftarrow h' + \mu(a \leftarrow h, h')$ which is the Hochschild 1-cocycle condition for the map $\tilde{\mu} : H \to E_\chi$ given by $\tilde{\mu}(h)(a) = \mu(a, h)$. The unitary condition $\mu(a, 1) = 0$ corresponds to the normalization condition on the cocycle.

We should point out that this consideration never uses specific features of the situation: it works for any algebra $H$ and a family of right $H$-modules $U_\chi$. The following theorem is an adaptation of [13, theorem 3.2] for module deformations. One can probably prove a stronger version of this theorem utilizing other ideas of [13].

**Theorem 26.** We assume $\text{HH}^1(H, E_\chi) = 0$ for some $\chi$. If $C$ is a curve in $\text{Spec} \mathcal{O}$ smooth at the point $\chi$ then there exists an open neighborhood $W$ of $\chi$ in $C$ such that $U_\chi \otimes L \cong U_\chi \otimes L$ as $H$-modules under Miyashita-Ulbrich action for each $\eta \in W$ and some field extension $L \supseteq \mathbb{k}$.

**Proof.** Let $t$ be a local parameter on $C$ at the point $\chi$. It gives us a generic point $\beta(t)$ of $C$ such that $\beta(0) = \chi$. The generic point may be thought of as an element of $\text{hom}(\mathcal{O}, \mathbb{K}[[t]])$ where $\mathbb{K}$ is the field $\mathcal{O}/\chi$.

By Lemma 5, $U \otimes_\mathcal{O} \mathbb{K}[[t]]$ is an $H$-Galois extension with central invariants. Therefore, it experiences the Miyashita-Ulbrich action which may be written as a formal deformation of that of $U_\chi$:

$$\phi : U_\chi[[t]] \otimes_{\mathbb{K}[[t]]} H \longrightarrow U_\chi[[t]].$$

$\phi$ may be given through a bunch of $\phi_n : U_\chi \times H \to U_\chi$ such that $\phi(a \otimes h) = \sum_n \phi_n(a, h)t^n$. Clearly, $\phi_0(a \otimes h) = a \leftarrow h$. The non-zero $\phi_n$ with the smallest positive $n$ is a first normalized Hochschild 1-cocycle with coefficients in $E_\chi$ by the argument similar to the one about infinitesimals.

But every cocycle is a coboundary! Thus, there exists $\Theta \in E_\chi$ such that $\phi_n(a, h) = \Theta(a \leftarrow h) - \Theta(a) \leftarrow h$. We consider an $H$-module isomorphism $F_\chi : U_\chi[[t]] \to U_\chi[[t]]$ given by $F_\chi(a) = a + \Theta(a)t^n$. It is clear that $F_\chi^{-1}(a) = a - \Theta(a)t^n + o(t^n)$. Let us compute

$$F_\chi^{-1}(F_\chi(a)h) = F_\chi^{-1}(a \leftarrow h + \Theta(a) \leftarrow ht^n + \phi_n(a, h)t^n + o(t^n)) =$$
\[ a \leftarrow h + [−\Theta(a \leftarrow h) + \Theta(a) \leftarrow h + \phi_n(a, h)]t^n + o(t^n) = a \leftarrow h + o(t^n). \]

Thus, \( F_n \) “kills” the \( t^n \)-term of the deformation. We may go on defining \( F_{n+1}, F_{n+2}, \ldots \) in the similar fashion. The product \( F = \cdots F_{n+1} \circ F_n \), though infinite, is well-defined because the coefficient at \( t^k \) is computed in the finite product \( F_k \circ \cdots \circ F_n \). It is clear that

\[ F^{-1}(F(a)h) = a \leftarrow h \]

and, therefore, \( F \) performs an \( H \)-module isomorphism between \( U_\chi[[t]] \) and the trivial deformation \( U_\chi \otimes_K K[[t]] \). But \( U_\chi[[t]] \) may be specialized to \( U_\eta \) for \( \eta \) from some open subset \( W \) of \( C \). Thus, \( U_\eta \otimes_{O/\eta} K((t)) \) must be isomorphic to \( U_\chi \otimes_K K((t)) \) as right \( H \)-modules.

Theorem 26 gives a fine tool for calculations. Let \( \mathfrak{g} \) be a semisimple classical Lie algebra of dimension \( n \) and rank \( r \). We assume that \( k \) is algebraically closed of good characteristic (i.e. the quotient of the root lattice by any sublattice generated by a root subsystem has no \( p \)-torsion). The list of bad primes is 2 for \( B_n, C_n, D_n, 2 \) and \( 3 \) for \( E_6, E_7, F_4, G_2 \), and \( 2, 3 \), and \( 5 \) for \( E_8 \). One should consult [30, 3.2] or [12] for the definition of reduced enveloping algebras. Roughly speaking, they are algebras \( U_\chi \) as \( U = U(\mathfrak{g}) \) and \( O = O(\mathfrak{g}) \cong O(\mathfrak{g}^{*1}) \). We consider a line \( \alpha\chi \) with \( \alpha \in k \) and regular semisimple \( \chi \). \( U_0 \) is just restricted enveloping algebra. By [12, Corollary 3.6] \( U_{\alpha\chi} \) for \( \alpha \neq 0 \) is a direct sum of \( p^r \) copies of the algebra of \( p^{n-r} \times p^{n-r} \)-matrices. Since the dimension of the center does not change with a field extension (by an elementary argument as in [30, Theorem 26]) we obtain the following corollary of Theorem 26.

**Corollary 27.** Let \( \mathfrak{g} \) be a classical semisimple Lie algebra. We assume \( p \) is good. If \( \text{HH}^1(u(\mathfrak{g}), \text{End}_ku(\mathfrak{g})) = 0 \) where \( u(\mathfrak{g}) \) is the module over itself under the right adjoint action then the center of \( u(\mathfrak{g}) \) has dimension \( p^r \).

It follows from the results of [18] that the dimension of the center is at least \( p^r \). To be precise one needs slightly stronger restriction on \( p \): \( p \) has to be good and should not divide \( n + 1 \) if \( \mathfrak{g} \) has \( A_n \) as a direct summand. To the best of our knowledge, it is unknown whether the dimension is precisely \( p^r \) even for \( sl_2 \). We don’t know how to compute \( \text{HH}^1(u(\mathfrak{g}), \text{End}_ku(\mathfrak{g})) \) but it looks like a possible way to put hands on the center. Not only could it settle the question about the center of \( u(\mathfrak{g}) \) but it could also help to understand the centers of \( U_\chi \) for nilpotent \( \chi \). Jacobson-Morozov theorem provides that \( \chi \) and \( \alpha\chi \) are conjugate under coadjoint action for \( \alpha \neq 0 \) which implies \( U_{\alpha\chi} \cong U_\chi \). We can use Theorem 26 to the curve \( k\chi \) at 0.
Corollary 28. Let $\mathfrak{g}$ be a classical semisimple Lie algebra with the Coxeter number $h$. If $p > 3h - 1$ and $\text{HH}^1(u(\mathfrak{g}), \text{End}_k u(\mathfrak{g})) = 0$ where $u(\mathfrak{g})$ is the module over itself under the right adjoint action then the center of $U_{\chi}$ has dimension $p^r$ for every nilpotent $\chi$.

8. $H$-torsors on schema.

An $H$-extension of a $k$-ringed space $(X, \mathcal{R})$ (i.e. a topological ring with a sheaf of algebras) is a sheaf of $H$-comodule algebras $\mathcal{U}$ on $X$ (i.e. $H$ coacts on $\mathcal{U}(V)$ for each open set $V$ and restriction maps are $H$-equivariant) such that $\mathcal{R}$ is a subsheaf of $\mathcal{U}$ and $\mathcal{R}(V) = \mathcal{U}(V)^H$ for each open set $V$.

This definition is not very helpful. The problem is that we cannot glue such objects from local data. To check that we have an $H$-extension we must look at each open set. For instance, if $X$ is an affine scheme and $\mathcal{R}$ is a sheaf of algebraic functions $\mathcal{O}$ then a Hopf-Galois extension of the algebra $\mathcal{O}(X)$ does not necessarily give a Hopf-Galois extension of $(X, \mathcal{O})$. However, if we restrict our attention to $H$-Galois extensions with central invariants then it suffices to work with open sets from some affine covering.

We are interested only in a scheme $S$ over $k$ and the sheaf of regular functions $\mathcal{O}$. By an $H$-torsor on the scheme $S$ we understand an $H$-extension $\mathcal{U}$ of $(S, \mathcal{O})$ with central invariants which is locally $H$-Galois. The latter means that there exists an affine covering $\coprod V_i$ of $S$ in the flat topology such that $\mathcal{U}(V_i)$ is a Hopf-Galois extension of $\mathcal{O}(V_i)$ for each $i$. We should point out that the extension $\mathcal{U}(V_i) \supseteq \mathcal{O}(V_i)$ has central invariants automatically.

One can use the usual gluing procedure to paste $H$-torsors. $H$ torsors on an affine scheme is the same as $H$-Galois extensions of the algebra of global algebraic functions. An $H$-structure on a scheme $S$ is the following data: an affine covering $\coprod V_i$, a collection of $H$-Galois extensions $\mathcal{U}_i \supseteq \mathcal{O}(V_i)$, and a collection of isomorphisms $\phi_{ij} : \mathcal{U}(V_i) \otimes_{\mathcal{O}(V_i)} \mathcal{O}(V_i \times_S V_j) \to \mathcal{U}(V_j) \otimes_{\mathcal{O}(V_j)} \mathcal{O}(V_i \times_S V_j)$ which satisfy the cocycle condition on triple intersections and give the identity map upon restriction to $\mathcal{O}(V_i \times_S V_j)$.

Similarly to Theorem 11 an $H$-structure determines the $H$-torsor. However, an $H$-torsor is not necessarily a sheaf of $H$-Galois extensions. The problem is that the extension of sections on a bad open subset can fail to be Galois. For each scheme $S$ there is a natural map $\psi_S : S \to \text{Spec } \mathcal{O}(S)$. We call the scheme good if the natural map $\psi_S$ is faithfully flat. Both affine and projective schemes are good but there are schemes which are not. If, for instance, $S$ is the complement of a point in $\mathbb{A}^2$.
then \( \text{Spec} \mathcal{O}(S) \cong \mathbb{A}^2 \) and \( \psi_S \) is the natural embedding which is not surjective and, therefore, not faithfully flat.

**Theorem 29.** If \( V \) is a good open subscheme of \( S \) and \( \mathcal{U} \) is an \( H \)-torsor on \( S \) then the extension \( \mathcal{U}(V) \supseteq \mathcal{O}(V) \) is \( H \)-Galois.

**Proof.** Let \( \prod_i V_i \) is an affine open covering of \( S \) such that \( \mathcal{U}(V_i) \supseteq \mathcal{O}(V_i) \) is \( H \)-Galois for each \( i \). Let \( \prod_j W_{ij} \) be an affine open covering of \( V \times_S V_i \). The extensions of sections on each \( W_{ij} \) is \( H \)-Galois. Now the composition \( \prod_{ij} W_{ij} \to V \to \text{Spec} \mathcal{O}(V) \) is faithfully flat since so are both maps. Theorem 11, applied to the composition, proves Theorem 26. \( \square \)

We need the notion of pull-back of \( H \)-torsors to continue the discussion. Given a map of schemes \( f : S' \to S \) and an \( H \)-torsor \( \mathcal{U} \) on \( S \), there exists a pull-back \( H \)-torsor \( f^* \mathcal{U} \). It is constructed by choosing compatible coverings as in Theorem 29 and applying tensor products (Lemma 5). We obtain an \( H \)-structure which determines the \( H \)-torsor. The definition is independent of the choice of covering: given two coverings, one can use their common refinement to construct the pull-back. This construction also gives a canonical isomorphism between the two \( H \)-torsors constructed by the two coverings.

A natural question is to classify \( H \)-torsors on a scheme \( S \). As one could expect this problem leads to theory of stacks \([1, 35]\). Let us build a category \( \mathfrak{GAL}_{H,S} \). The objects of \( \mathfrak{GAL}_{H,S} \) are pairs \((X, \mathcal{U})\) where \( X \) is a scheme over \( k \) and \( \mathcal{U} \) is an \( H \)-torsor on \( X \times S \). Morphisms \( F : (X', \mathcal{U}') \to (X, \mathcal{U}) \) are maps \( f : X' \to X \) such that \((\text{Id}_X \times f)^* \mathcal{U} \cong \mathcal{U}'\).

**Theorem 30.** The category \( \mathfrak{GAL}_{H,S} \) together with the forgetful functor to \( \pi \) the category of scheme (i.e. \( \pi(X, \mathcal{U}) = X \)) is a stack in the flat topology over the category of schemes over \( k \).

**Proof.** We go over the four defining properties of stacks as defined in \([1, 35]\). The first two properties determine a groupoid over the category of schemes, the others provide that a groupoid is a stack.

1. If \( f : X \to Y \) is a map of schemes and \( (Y, \mathcal{U}) \) is an object of \( \mathfrak{GAL}_{H,S} \) then \( f \) is lifted to a map in \( \mathfrak{GAL}_{H,S} \) from \( (X, (\text{Id}_X \times f)^* \mathcal{U}) \) to \( (Y, \mathcal{U}) \).

2. If \( c : X \to Y \) and \( b : Y \to Z \) are maps of schemes and \( a = b \circ c \) and \( (X, \mathcal{U}), (Y, \mathcal{U}_1), \) and \( (Z, \mathcal{U}_2) \) are objects in \( \mathfrak{GAL}_{H,S} \) such that \( a \) and \( b \) define morphisms in \( \mathfrak{GAL}_{H,S} \) between the corresponding objects then \( c \) also defines the unique (uniqueness follows from the definition of the morphism in \( \mathfrak{GAL}_{H,S} \)) morphism from \( (X, \mathcal{U}) \) to \( (Y, \mathcal{U}_1) \) because \((\text{Id}_X \times c)^* \circ (\text{Id}_X \times b)^* \mathcal{U}_2 \cong (\text{Id}_X \times b \circ c)^* \mathcal{U}_3 \cong (\text{Id}_X \times a)^* \mathcal{U}_3 \). This holds because the pull-back is defined by tensor products and tensor product is associative.
3. For each scheme $X$ and each two objects $(X, \mathcal{U}_1)$ and $(X, \mathcal{U}_2)$ of $\mathfrak{AL}_{H,S}$ the functor $I$ from the category of schemes over $X$ to the category of sets such that $I(Y \xrightarrow{f} X)$ is a set of isomorphisms in $\mathfrak{AL}_{H,S}$ between $f^*(\mathcal{U}_1)$ and $f^*(\mathcal{U}_2)$ must be a sheaf in the flat topology. Indeed, let $\coprod_i g_i : \coprod_i Y_i \rightarrow Y$ be a covering. We consider a bunch of isomorphisms

$$\Omega_i : (fg_i)^*(\mathcal{U}_1) \rightarrow (fg_i)^*(\mathcal{U}_2)$$

which agree on double intersections. They are given by isomorphisms $\omega_i : Y_i \rightarrow Y_i$ such that $fg_i = fg_i\omega_i$. Since $\omega_i$ agree on double intersections they can be glued into a map $\omega : Y \rightarrow Y$ such that $f = f\omega$. We have to check that $\omega$ determines a morphism in $\mathfrak{AL}_{H,S}$ from $f^*(\mathcal{U}_1)$ to $f^*(\mathcal{U}_2)$ which is equivalent to the fact that $\omega^*(f^*(\mathcal{U}_2)) \cong f^*(\mathcal{U}_1)$. This condition is local but we know that locally $\omega_i^*((fg_i)^*(\mathcal{U}_2)) \cong (fg_i)^*(\mathcal{U}_1)$.

4. The last property is that an $H$-structure determines an $H$-torsor. 

\[ \square \]

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