Phase transition in Wilson loop correlator from AdS/CFT correspondence

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Abstract

A previous calculation of the phase transition in the Wilson loop correlator in the zero
temperature AdS/CFT correspondence is extended to the case where the loops are concentric
circles of unequal radii. This phase transition occurs due to the instability of the classical
string stretched between the loops. We compute the string action and its expansion in the
distance $h$ between the loops for small $h$. We also find that the connected minimal surface is
subleading or does not even exist when $h = 0$ and the radii are considerably different. This
feature has no analogue in flat space.
The AdS/CFT correspondence [1] allows one to calculate the Wilson loop correlator from the classical string action, with the string propagating in the bulk of $AdS_5 \times S_5$ and the ends attached to the Wilson loops lying on the boundary [2]. It was pointed out by Gross and Ooguri [3] that in the zero temperature case there exists a kind of phase transition, corresponding to a competition between two saddle points. These correspond to a minimal surface which is either an annulus or two disconnected pieces living at the individual loops. In [4] the transition between the two phases was discussed in detail by solving the equations of motion for the case when the two loops are circles of equal radii. The result was that at large distances $h$ between the loops the disconnected surfaces become energetically preferred.

In the present paper we consider the case where the two loops have in general different radii $R_1$ and $R_2$. We find that the equations for the minimal surface can be solved in terms of the elliptic integrals. We construct an expansion in $h$ for small $h$, and discuss the critical values of the parameters. In particular we find that the connected minimal surface is subleading or does not exist if $h = 0$ and $R_1$ and $R_2$ are considerably different. This feature has no analogue in flat space, where the annulus always has smaller area than the two disks if the Wilson loops lie in the same plane.

We should mention that non-analytical behavior in semiclassical amplitudes related to the transition from one saddle point in the path integral to another was encountered in many other problems, such as a false vacuum decay in quantum mechanics, where both first and second order transitions are possible depending on parameters of the problem [7], or sphaleron transitions in quantum field theories with multiple vacua [8]. A minimal surface can be regarded as a world sheet instanton and, in this respect, the phase transition from connected to disconnected surfaces is a generic example of the phenomenon common to many instanton amplitudes.

1 The equations of motion

We will calculate the connected correlation function of two Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory:

$$\langle W(C_1)W(C_2) \rangle_{\text{conn}} = \langle W(C_1)W(C_2) \rangle - \langle W(C_1) \rangle \langle W(C_2) \rangle, \quad (1)$$

where $C_1$ and $C_2$ are concentric circles of radii $R_1$, $R_2$ that lie in the parallel planes separated by distance $h$. At strong coupling, the problem reduces to computation of the minimal area of a surface in $AdS_5$ whose boundaries are $C_1$ and $C_2$:

$$\langle W(C_1)W(C_2) \rangle_{\text{conn}} = \exp \left( -\frac{1}{2\pi} \sqrt{g_{YM}^2 N} S \right), \quad (2)$$

$$S = \int d^2\sigma, \sqrt{\det g_{\mu\nu} \partial_\sigma x^\mu \partial_\sigma x^\nu}, \quad (3)$$

$$ds^2 = \frac{1}{z^2} (dz^2 + dx_\mu^2) \quad (4)$$

*This expansion can be used to get an insight in the structure of electric flux at short distances [3]. The structure of electric flux at large distances was discussed in [4].

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\[
\frac{\delta S}{\delta x^\mu} = 0. \tag{5}
\]

In the given geometry, the minimal surface is axially symmetric and we can choose the coordinate along the symmetry axis and the polar angle as the two parameters on the world sheet. Minimization of the area then reduces to a one-dimensional problem, which can be treated along the lines of ref. [4].

The area of an axially symmetric surface in the metric of \(AdS_5\) is

\[
S = 2\pi \int dx \frac{r}{z^2} \sqrt{1 + (r')^2 + (z')^2}, \tag{6}
\]

The equations following from this action admit an integral of motion due to translational invariance in the \(x\) direction:

\[
r \frac{1}{z^2} \sqrt{1 + (r')^2 + (z')^2} = k. \tag{7}
\]

This integral allows one to get rid of the square root factors in the equations for \(r\) and \(z\) which acquire the following simple form:

\[
r'' - \frac{r}{k^2 z^4} = 0, \tag{8}
\]

\[
z'' + \frac{2r^2}{k^2 z^5} = 0. \tag{9}
\]

We can find the second integral by first rewriting (7) as

\[
(z')^2 + (r')^2 + 1 - \frac{r^2}{k^2 z^4} = 0, \tag{10}
\]

and then adding to it eq. (8) multiplied by \(r\) and eq. (4) multiplied by \(z\):

\[
(r^2 + z^2)'' + 2 = 0, \tag{11}
\]

which yields, upon double integration,

\[
r^2 + z^2 + (x + c)^2 = a^2. \tag{12}
\]

Here, \(a\) and \(c\) are the integration constants that are determined by the boundary conditions:

\[
z(0) = 0 = z(h) \tag{13}
\]

\[
r(0) = R_2, \quad r(h) = R_1. \tag{14}
\]

The latter are satisfied by

\[
c = \frac{R_2^2 - R_1^2}{2h} - \frac{h}{2}, \tag{15}
\]

\[
a^2 = c^2 + R_2^2. \tag{16}
\]
The trigonometric parameterization:

\[ r = \sqrt{a^2 - x^2} \cos \theta, \]
\[ z = \sqrt{a^2 - x^2} \sin \theta, \]

(17)

allows to separate variables in equation (10) and, after some calculations, it takes the form:

\[ \theta' = \pm \frac{a}{a^2 - (x + c)^2} \sqrt{\frac{\cos^2 \theta}{k^2 a^2 \sin^4 \theta}} - 1. \]

(18)

The explicit solution for the minimal surface is obtained by integrating this equation with the boundary conditions:

\[ \theta(0) = 0 = \theta(h). \]

(19)

Besides, the solution must satisfy an inequality \( 0 \leq \theta \leq \pi / 2 \), which follows from positivity of the AdS radial coordinate \( z \). It is easy to see that \( \theta(x) \) grows at small \( x \). Then it reaches a maximum at some \( x = x_0 \), so the positive root in (18) should be chosen for \( 0 < x < x_0 \). For \( x > x_0 \), the solution monotonously decreases. Hence, the negative root must be chosen at \( x_0 < x < h \). By continuity, \( \theta'(x_0) = 0 \). Equation (18) then fixes \( \theta(x_0) \):

\[ \theta(x_0) \equiv \theta_0 = \arccos \left( \frac{\sqrt{k^2 a^2 + 1} - 1}{2ka} \right). \]

(20)

Integration of eq. (18) from 0 to \( x_0 \) and from \( x_0 \) to \( h \) gives two equations that determine \( x_0 \) and \( k \):

\[ \frac{1}{2} \ln \frac{a + x_0 + c}{a - x_0 - c} - \frac{1}{2} \ln \frac{a + c}{a - c} = ka \int_0^{\theta_0} \frac{d\phi \sin^2 \phi}{\sqrt{\cos^2 \phi - k^2 a^2 \sin^4 \phi}}, \]
\[ \frac{1}{2} \ln \frac{a + h + c}{a - h - c} - \frac{1}{2} \ln \frac{a + x_0 + c}{a - x_0 - c} = ka \int_0^{\theta_0} \frac{d\phi \sin^2 \phi}{\sqrt{\cos^2 \phi - k^2 a^2 \sin^4 \phi}}. \]

(21)

Adding these equations and defining

\[ F(ka) = ka \int_0^{\theta_0} \frac{d\phi \sin^2 \phi}{\sqrt{\cos^2 \phi - k^2 a^2 \sin^4 \phi}} = \frac{ka}{2} \int_0^{\theta_0} \frac{dy y^{1/2}}{\sqrt{(1 - y)(1 - y - (ka)^2 y^2))}} \]
\[ = \frac{ka}{2} \int_0^1 \frac{du}{\sqrt{u \left(2(ka)^2 + 1 - \sqrt{1 + 4(ka)^2(1 - u)} - 1\right) \sqrt{1 + 4(ka)^2(1 - u)}}}, \]

we get:

\[ F(ka) = \frac{1}{4} \ln \frac{a + h + c}{a - h - c} - \frac{1}{4} \ln \frac{a + c}{a - c} = \frac{1}{4} \ln \left( \frac{a + \frac{h}{2}}{a - \frac{h}{2}} \right)^2 - \left( \frac{R_1^2 - R_2^2}{2h} \right)^2 \]
\[ = \frac{1}{2} \ln \frac{R_1^2 + R_2^2 + h^2 + \sqrt{(R_2^2 - R_1^2)^2 + h^4 + 2h^2(R_1^2 + R_2^2)}}{2R_1 R_2}. \]

(23)
In the last two steps in (22) we have made the substitutions \( y = \sin^2 \phi \) and \( u = 1 - y - (ka)^2 y^2 \), respectively. The last form of \( F(ka) \) has the advantage that its derivatives with respect to \( ka \) are explicitly finite.

There is an ambiguity in solving the last equation for \( ka \), because generically there are two roots. It could imply that there are two minimal surfaces with a given boundary, but in fact only one of these surfaces is a true minimum of the string action, while the other is a saddle point. It appears that the true solution has larger \( ka \).

In the case of equal radii there is an upper limit on \( h \), and the same is true in the present case. To see this, let us take \( R_1/h = \alpha R_2/h \equiv \alpha r \). Then we obtain from (23)

\[
r^2 = \frac{1}{2\alpha(1 + 2 \sinh^2 F(ka)) - \alpha^2 - 1}.
\] (24)

From the condition \( r^2 > 0 \) we obtain

\[
1 + 2 \sinh^2 F - 2 \sinh F \cosh F < \alpha < 1 + 2 \sinh^2 F + 2 \sinh F \cosh F.
\] (25)

The lowest value of \( F \) is zero, obtained for \( ka \to \infty \). In this case only \( \alpha = 1 \) is possible. From (24) we then obtain \( r \to \infty \), and since \( r = R_2/h \) this corresponds to \( h \to 0 \). In general, for some value for \( F \) we get from (24)

\[
h = R_2\sqrt{2\alpha(1 + 2 \sinh^2 F) - \alpha^2 - 1}.
\] (26)

The maximum of \( h \) therefore occurs for

\[
\alpha = R_1/R_2 = 1 + 2 \sinh^2 F,
\] (27)

and hence

\[
h_{\text{max}} = R_2 \sinh(2F(ka)).
\] (28)

Since \( F \) is bounded by \( \sinh F \approx 0.52 \) for \( ka \approx 0.58 \), we obtain the result that \( h \) cannot exceed 1.172 \( R_2 \). If \( R_1 = R_2 \) (\( \alpha = 1 \)) we get from (24) \( h = 2R_2 \sinh F(ka) \), in accordance with a result derived in [4].

### 2 Computation of the area

The area of the minimal surface can be computed with the help of the equations of motion:

\[
S = \int_0^h dx \frac{r}{z^2} \sqrt{1 + (r')^2 + (z')^2} = \frac{2\pi}{k} \int_0^h dx \frac{r^2}{z^4} = \frac{2\pi}{k} \int_0^h dx \frac{dz}{a^2 - (x + c)^2 \cos^2 \theta \sin^4 \theta}
\] (29)

where the factor of two comes from the two branches of \( \theta(x) \).

The above unregularized area is ill defined because of the divergency at the boundary. It needs to be regularized by the shift of the surface into the interior of AdS:

\[
z(0) = \epsilon = z(h).
\] (30)
Then, since
\[ \theta = \arctan \frac{z}{r}, \]  
the boundary conditions for \( \theta \) are
\[ \theta(0) = \arctan \frac{\epsilon}{R_2} \approx \frac{\epsilon}{R_2}, \]
\[ \theta(h) = \arctan \frac{\epsilon}{R_1} \approx \frac{\epsilon}{R_1}. \]  
(32)

After a change of variables
\[ \tan \theta = \left( \frac{\sqrt{4k^2a^2 + 1} - 1}{2} \right)^{-1/2} \sin \psi, \]
the regularized area takes the form:
\[ S = \frac{2\pi(R_1 + R_2)}{\epsilon} - 4\pi \frac{\alpha}{\sqrt{\alpha - 1}} \int_0^{\pi/2} \frac{d\psi}{1 + \alpha \sin^2 \psi + \sqrt{1 + \alpha \sin^2 \psi}}, \]  
(33)

where
\[ \alpha = \frac{1 + 2k^2a^2 + \sqrt{1 + 4k^2a^2}}{2k^2a^2}. \]  
(34)

The area appears to be a universal function of the parameter \( ka \), which is determined by geometric data according to eq. (23).

As an example of an application of this result, let us consider the case when \( \alpha \to \infty \), corresponding to \( F \) being small. Then using the above equations we obtain
\[ S \approx -\frac{16\pi^4}{\Gamma(1/4)^4} \sqrt{\frac{R_1R_2}{(R_1 - R_2)^2 + h^2}}, \]  
(35)

Here \( R_1 \approx R_2 \) and \( h \) is small.

It is possible to gain some insight in the behavior of the action as a function of \( h \) for small values of \( h \). Starting from (23) we get
\[ F(ka) \approx -\frac{1}{2} \ln \frac{R_2}{R_1} + \frac{h^2}{2(R_1^2 - R_2^2)} + O(h^4), \]  
(36)

where we assumed \( R_1 > R_2 \) and
\[ h^2 \ll 2R_1^2 + 2R_2^2 \quad \text{and} \quad h^2 \ll \frac{R_1^2 - R_2^2}{2(1 + R_2^2/R_1^2)}. \]  
(37)

If \( h = 0 \) we determine a value \( k_0 \) from
\[ F(k_0a) = -\frac{1}{2} \ln \frac{R_2}{R_1}. \]  
(38)
Expanding $F$ around this point, and assuming that $k_0 a$ does not correspond to the maximum of $F$, we obtain from (36)

$$k - k_0)a \approx \frac{\hbar^2}{2(R_1^2 - R_2^2)F'(k_0 a)} + O(h^4).$$

(39)

Next we expand the action around the point $k_0 a$ and to lowest order we then obtain (ignoring the infinite part of $S$)

$$S \approx G(k_0 a) + \frac{G'(k_0 a)}{2(R_1^2 - R_2^2)F'(k_0 a)} h^2 + O(h^4),$$

(40)

where

$$G(k_0 a) = -4\pi \frac{\alpha}{\sqrt{\alpha - 1}} \int_0^{\pi/2} \frac{d\psi}{1 + \alpha \sin^2 \psi \sqrt{1 + \alpha \sin^2 \psi}}.$$ (41)

Numerical evaluation shows that the two derivatives in (40) are both negative, and hence the first order $h^2$ correction is positive.

As a numerical example we can take $k_0 a = 2.6$, corresponding to $F(k_0 a) \approx 0.34$. From (38) we then get $R_1 \approx 2R_2$. Furthermore, $F'(2.6) \approx -0.054$ and $G'(2.6) \approx -1.77$, leading to

$$S(h) \approx -14.8 + 21.6 \frac{h^2}{R_1^2} + \ldots, \text{ with } R_1 \approx 2R_2.$$ (42)

Numerical studies of eq.(40) show that the coefficient of $h^2/R_1^2$ is quite sensitive to the ratio $R_1/R_2$.

### 3 Critical behavior

As was mentioned in the discussion following eq. (23), the classical string world sheet with the topology of annulus exists only in a certain range of parameters $R_1$, $R_2$ and $h$. Outside this range, the minimal surface is disconnected. Actually, the area of the connected solution starts to exceed the area of two surfaces that span individual Wilson loops before the connected solution ceases to exist. In fig. [1], we plot the areas of the stable and the unstable branches of the connected solution as a function of $h$ for $R_1 = R_2 = 1$ vs. the area of the disconnected surface, which is $-4\pi$ after subtraction of an infinite term, as shown in [9, 10].

The shift from one saddle point in the string action to the other leads to the phase transition in the semiclassical amplitude. In the exact amplitude, the transition is smoothened by corrections that are non-perturbative in $\alpha' = 1/\sqrt{g_{YM}^2 N}$. [1], but if $\alpha'$ is sufficiently small, the transition is still rather sharp. The phase diagram in the $r_1$, $r_2$ plane is shown in fig. [2], where $r_1 = R_1/h$ and $r_2 = R_2/h$. It is interesting that the phase transition survives the $h \to 0$ limit, which corresponds to $r_1, r_2 \to \infty$. Therefore, the connected minimal surface is subleading or even does not exist if $h = 0$, but $R_1$ and $R_2$ differ considerably. This behavior has no analogue in the flat space where the annulus always has smaller area than the two disks if the Wilson loops lie in the same plane.
Figure 1: The areas of the stable and the unstable branches (bold lines) of the connected surface and the area of the disconnected surface (thin horizontal line $= -4\pi$) as a functions of $h$.

Figure 2: The connected minimal surface as function of $r_1 = R_1/h$ and $r_2 = R_2/h$ exists to the right of the thin line and is the globally stable to the right of the bold line.

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References

[1] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998). [hep-th/9711200]; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B428, 105 (1998) [hep-th/9802109]; E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].
[2] J. Maldacena, Phys. Rev. Lett. 80, 4859 (1998) hep-th/9803002. S. Rey and J. Yee, hep-th/9803001.

[3] D. J. Gross and H. Ooguri, Phys. Rev. D58, 106002 (1998) hep-th/9805129.

[4] K. Zarembo, Phys. Lett. B459, 527 (1999) hep-th/9904149.

[5] J. Greensite and P. Olesen, hep-th/0008080.

[6] J. Erickson, G.W. Semenoff and K. Zarembo, Phys. Lett. B466, 239 (1999) hep-th/9906211.

[7] E.M. Chudnovsky, Phys. Rev. A46, 8011 (1992).

[8] S. Habib, E. Mottola and P. Tinyakov, Phys. Rev. D54 7774 (1996) hep-ph/9608327; K.L. Frost and L.G. Yaffe, Phys. Rev. D60 105021 (1999) hep-ph/9905224; Phys. Rev. D59 065013 (1999) hep-ph/9807324; G.F. Bonini, S. Habib, E. Mottola, C. Rebbi, R. Singleton and P.G. Tinyakov, Phys. Lett. B474 113 (2000) hep-ph/9905243.

[9] D. Berenstein, R. Corrado, W. Fischler and J. Maldacena, hep-th/9809188.

[10] N. Drukker, D.J. Gross and H. Ooguri, Phys. Rev. D60, 125006 (1999) hep-th/9904191.