Tapping Spin Glasses

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We consider a tapping dynamics, analogous to that in experiments on granular media, on spin glasses and ferromagnets on random thin graphs. Between taps, zero temperature single spin flip dynamics takes the system to a metastable state. Tapping, corresponds to flipping simultaneously any spin with probability $p$. This dynamics leads to a stationary regime with a steady state energy $E(p)$. We analytically solve this dynamics for the one dimensional ferromagnet and ±$J$ spin glass. Numerical simulations for spin glasses and ferromagnets of higher connectivity are carried out, in particular we find a novel first order transition for the ferromagnetic systems.

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Complex systems such as granular media and spin glasses have an exponentially large number $N_{MS}$ of metastable states, also called blocked or jammed configurations, i.e. $N_{MS} = \exp(N_{SEdw})$, where $S_{Edw}$ is the Edwards entropy per particle $\bar{E}$. In granular media the thermal energy available is not sufficient to allow the rearrangement of a single particle and hence the system is effectively at zero temperature in the thermal sense and when not perturbed are stuck in a metastable state. Edwards has proposed $\bar{E}$ that if one slightly perturbs such systems by external forcing such as tapping, the asymptotic measure over the metastable states satisfying the appropriate macroscopic constraints is flat, this idea has recently been further investigated $\bar{E}$. In a, now classic, experiment on granular media $\bar{E}$ dry hard soda glass spheres placed in a glass tube are tapped by using a piston to move the tube vertically through a sine cycle. The tapping parameter $\Gamma$ is defined to be the ratio of the maximal acceleration due to the piston in the cycle to $g$ the acceleration due to gravity. After an initial irreversible curve obtained by increasing the tapping amplitude slowly, the system arrives on a reversible curve where the density is a monotonic function of $\Gamma$, the highest packing densities being obtained at lowest tapping rate. Numerical simulations on granular media $\bar{E}$ reveal similar behavior. There has been a considerable theoretical effort to understand the dynamics of granular media, especially the slow relaxation towards the final asymptotic density under tapping, both theoretically and by simulations on various models $\bar{E}$. Our aim is to gain some understanding of the stationary regime of tapped systems such as granular media.

The possibility of using spin glasses as a paradigm for granular material was first suggested in $\bar{E}$, and recently it has been shown in $\bar{E}$ that spin glasses and ferromagnets on random thin graphs have an extensive Edwards entropy with respect to single spin flip dynamics. In this paper we study a very natural tapping dynamics on these systems: We allow the system to evolve under a random zero temperature single spin flip dynamics where only moves which reduce the energy are allowed (falling). When the system is blocked we tap it with strength $p \in [0,1/2]$, that is to say each spin is flipped with a probability $p$, the updating at this point being parallel. The system is then evolved by the zero temperature dynamics until it becomes once again stuck, the tapping is then repeated. Physically this corresponds to assuming that in granular media the relaxation time to a new metastable state is much shorter than the time between taps. The same tapping dynamics has also been introduced independently in the context of three spin ferromagnetic interactions on thin hypergraphs $\bar{E}$, also in the goal of studying the dynamics of granular media. We find that a stationary regime is reached after a sufficiently large number of taps, characterized by a steady state energy $E(p)$ (analogous to the stationary density – the same analogy as used in $\bar{E}$). We then develop a mean field theory for the dynamics under falling then tapping, which appears to be exact in the case of the above mentioned one dimensional system and one may calculate $E(p)$ within this scheme, the results being in perfect agreement with the numerical simulations. We also examine the tapping of spin glasses and ferromagnets of higher connectivity. In the case of the ferromagnet we find numerically that there exists a critical value $p_c$ of $p$ such that for $p > p_c$, $E(p) > E_{GS}$ where $E_{GS}$ is the energy of the ground state and the inequality is strict, and that for $p < p_c$, $E(p) = E_{GS}$, hence in the ferromagnetic system there is a first order phase transition with the tapping dynamics (in contrast to the usual thermodynamic ferromagnetic transition in these systems which is second order $\bar{E}$). The spin glass is clearly far from a realistic realization of a granular media, however the fact that it has extensive entropy of blocked states and the obviously natural form of the tapping dynamics implemented makes it a natural testing ground for ideas about dynamics and possible thermodynamics of systems such as granular media.
A random thin graph of connectivity $c$ is a collection of $N$ points, each point being linked to $c$ of its neighbors. The spin glass/ferromagnet model we shall consider has the Hamiltonian

$$H = \frac{1}{2} \sum_{j \neq i} J_{ij} n_{ij} S_i S_j$$

(1)

where the $S_i$ are Ising spins, $n_{ij}$ is equal to one if the sites $i$ and $j$ are connected. In the spin glass case the $J_{ij}$ are taken from a binary distribution where $J_{ij} = -1$ with probability half and $J_{ij} = 1$ with probability half. In the ferromagnetic case $J_{ij} = 1$. The definition of the total number of metastable states in this system is the number of states where any single spin flip does not increase the energy of the system.

The one dimensional spin glass/ferromagnet: We remark that by a gauge transformation the one dimensional ferromagnet and $\pm J$ spin glass are equivalent and place ourselves, for transparency, in the context of the ferromagnet. In the initial configuration, we take the probability that a given spin is different to its left neighbor to be $a$. Hence if $a = 0$ we have an initially ferromagnetic configuration, if $a = 1$ it is an antiferromagnetic configuration, the case $a = 1/2$ corresponds to a completely random configuration. The initial energy per spin is $E_0 = -1 + 2a$.

For a given site define $x$ to be the difference between the number of unsatisfied and satisfied bonds. Hence $x$ is the local field on the spin; if $x > 0$ then the spin can flip bringing about the change $x \to -x$. Denote by $P(x, k)$ the probability that the site of interest has local field $x$ after a total of $k$ attempted random single spin flips during the falling dynamics. We define $f_+$ and $f_-$ the probabilities that a neighboring spin can flip conditional on the fact that the bond with the site we are interested in is not satisfied or satisfied respectively. Given that a site has local field $x$, it must have $(c + x)/2$ unsatisfied bonds and $(c-x)/2$ satisfied bonds; using Bayes’ theorem we therefore obtain

$$f_{\pm} = \frac{\sum_x P(x)(c \pm x)}{\sum_x P(x)(c \pm x)}$$

(2)

For the spin considered the possibilities between time $k$ and $k + 1$ are

- $x > 0$. Then the spin can flip and $x$ goes to $-x$;
- A neighboring spin has a positive local field and so can flip. In that case, $x$ goes to $x + 2$ or to $x - 2$;
- Neither the spin considered nor any of its neighbors flips, and so $x$ stays $x$.

Assuming that the distribution at every site is given by $P(x, k)$ and assuming independence between the values of $x$ from site to site (the mean field approximation) we obtain, taking the limit $N \to \infty$ and introducing the continuous time $\tau = k/N$,

$$\frac{dP(x)}{d\tau} = \theta(-x)P(-x) + \theta(-x)P(x) - (c + 1)P(x)$$

(3)

$$+ P(x) \left[ \frac{c + x}{2} (1 - f_+) + \frac{c - x}{2} (1 - f_-) \right]$$

$$+ P(x + 2) \frac{c + x + 2}{2} f_+ + P(x - 2) \frac{c - x + 2}{2} f_-$$

The case where $c = 2$ is accessible to analytic solution. Defining: $u(\tau) \equiv P(-2, \tau)$, $v(\tau) \equiv P(0, \tau)$ and $w(\tau) \equiv P(2, \tau)$, the solution to Eq. (3) is given by

$$w(\tau) = w(0) \exp \left( -\tau + \frac{4}{\lambda(0) + 2} (e^{-\tau} - 1) \right)$$

(4)

$$v(\tau) = -2w(\tau) + w(0)(\lambda(0) + 2) \exp \left( \frac{4}{\lambda(0) + 2} (e^{-\tau} - 1) \right)$$

where $\lambda(0) = v(0)/\omega(0)$. Consequently after the system has fallen into a metastable state we find the final values $v(\infty) = (v(0) + 2w(0))e^{-\frac{4}{\lambda(0) + 2}}$, $\omega(\infty) = 0$ and $u(\infty) = 1 - v(\infty)$.

For initial conditions used here: $u(0) = (1 - a)^2$, $v(0) = 2a(1 - a)$ and $w(0) = a^2$. Hence $E_f$, the average energy of the metastable state into which the system falls, is given by

$$E_f = -1 + v(\infty) = -1 + 2ae^{-2a}$$

(5)

This result can be shown to be exact by a combinatorial calculation and has also been checked in our simulations [13]. For the completely random initial configuration, where $a = 1/2$, $E_f = -0.632121$ is in fact maximal.

This calculation demonstrates two important points:

- The final value of the energy $E_f$ depends strongly on the initial configuration, in addition $E_f$ is not a monotonic function of $E_0$. This means that configurations of higher initial energy can fall into metastable states of lower energy than initial configurations of a lower energy.

- The system does not fall into a state of energy corresponding to the maximum of $N_{MS}(E)$, the total number of metastable states of energy $E$ per spin. In [10,11] it was shown that $N_{MS}(E) \sim \exp(Ns_{Edw}(E))$ where $s_{Edw}(E)$ is a concave function peaked at $E^* = -1/\sqrt{5} \approx 0.44721$. Hence even if the total number of metastable states is dominated (in the thermodynamics sense) by those of energy $E^*$, generic initial conditions always seem to lead to an energy lower than this $E^*$.

Tapping the system with probability $p$, starting from the values $(u(\infty), v(\infty), w(\infty))$, we obtain the new tapped values $(u'(0), v'(0), w'(0))$. Defining $q \equiv (1 - p)$,
the relations between the old and tapped probabilities are:

\[
\begin{align*}
&u'(0) = (1 - 3pq) u(\infty) + pq v(\infty) \\
n\nu'(0) = 2pq u(\infty) + (1 - 2pq) v(\infty) \\
&\nu w'(0) = pq
\end{align*}
\]

After another zero temperature evolution of the system, it reaches a new local energy probability distribution with \(w'(\infty) = 0, \nu'(\infty)\) and \(\nu'(\infty) = 1 - \nu'(\infty)\). The fixed point of this dynamics is given by

\[
v_s(p) = (4pq + v_s(p)(1 - 4pq)) \exp \left(-\frac{4pq}{4pq + v_s(p)(1 - 4pq)}\right)
\]

the subscript \(s\) indicating steady state. We remark that this solution contains one of the main features of our numerical simulations, that is reversibility. Here the asymptotic distribution is independent of the initial conditions and depends only on \(p\). Eq. (6) can be solved numerically and the result is shown in Fig. (1) in comparison with the numerical simulations which we see is excellent. The small \(p\) behavior of \(E(p)\) from (6) is:

\[
E(p) = -1 + \sqrt{2p + O(p)}
\]

**Systems with \(c > 2\): cases.** In an annealed approximation to the Edwards entropy per spin of metastable states at fixed energy \(E\), \(s_{Edw}(E) = \ln(\langle N_{MS}(E) \rangle / N)\) was carried out for spin glasses and ferromagnets on random thin graphs. There is an energy threshold \(E^*\) above which the results are the same for the \(\pm J\) spin glass and the ferromagnet; below \(E^*\) the ferromagnet has more metastable states (which aquire a non-zero magnetization \([\tilde{1}]\)). Hence, as far as metastable states are concerned, both ferromagnet and spin glass are the same above \(E^*\), that is the effect of loop frustration is negligible. In this regime, one suspects that the zero temperature dynamics are the same. In particular, numerical simulations with 100 samples of \(N = 10000\) sites for connectivities of 3, 4 and 5 have found the same \(E_f\) for the spin glass and ferromagnet to very good accuracy (the relative error is about \(10^{-6}\)). The results of tapping experiments on systems with \(c = 3\) are displayed in Fig.(2). There is a critical tapping rate \(p_c\) above which the curves of \(E(p)\) versus \(p\) are the same for the spin glass and ferromagnet. At \(p_c \approx 0.249\) the ferromagnet undergoes a phase transition such that for \(p < p_c\), the steady state reached is the ground state. Finite size effects have been studied and reveal that the transition appears to be first order (in as far that \(E(p_c^+) \neq E(p_c^-)\)). As shown in Fig.(3), there is a coexistence of two phases at \(p_c\): one sees two separated peaks in the distribution of the internal energy and not a single peak as one would expect for a second order transition. In [\(\tilde{1}\)] it was shown that for the ferromagnet the annealed approximation to the Edwards entropy as a function of \(E\) is concave for \(E > E^*\) and convex for \(E < E^*\). The value of \(E(p_c^\pm)\) estimated from the tapping experiments are very close to those obtained for \(E^*\) in [\(\tilde{1}\)] the energy at which \(s_{Edw}(E)\) becomes convex (for \(c = 3, E^* = -1.0714\) analytically and \(E(p_c^+) \approx -1.075 \pm 0.005\) from the simulation). Encouraged by this observation we will try to make a tentative link with a possible thermodynamics for such systems. We consider a partition function inspired by the flat Edwards measure over metastable states [\(\tilde{3}\)]

\[
Z = \int dE N_{MS}(E) \exp(-\beta NE)\]

where \(\beta\) is a Lagrange multiplier corresponding to the inverse Edwards temperature which depends solely on \(p\), and not on \(E\), and is a monotonically decreasing function of \(p\) for \(p \in [0^+, 1/2]\). The monotonic hypothesis is supported by the simulation results that \(E(p)\) decreases with decreasing \(p\). Clearly the energy which dominates in the sum is that obeying \(\frac{\partial s_{Edw}(E)}{\partial E} - \beta = 0\). However if this saddle point gives a true maximum of the action the Edwards entropy must be concave for the energy considered to be thermodynamically stable. Hence one is lead to conclude that for \(E < E^*\) the only stable energy is the ground state. Finally we remark for \(c \geq 3\) that the mean field equations of the previous section may be numerically solved and give reasonable results for \(E(p)\) for \(p > p_c\) [\(\tilde{13}\)], which again reinforces the assertion that for higher energies the spin glass and ferromagnetic dynamics are the same (the mean field theory does not distinguish between the two)..

No observable singularities are seen in our simulations of spin glasses, but again we found that for all the systems studied that \(E(p)\) decreased as a function of \(p\). As \(p \to 0\) we found that generically \(E(p) \sim E(0^+) + A p^p\). For the one dimensional case we know analytically that \(\theta = 1/2\), however for \(c = 3, 4,\) and 5 we found that \(\theta = 1\). We also carried out simulations on the totally connected Sherrington Kirkpatrick \(\pm J\) spin glass system of size 400, 600 and 800 spins where an average was taken over 1000 samples. Here we used a deterministic sequential (rather than random) update for the zero temperature dynamical evolution – as used by Parisi in [\(\tilde{14}\)] where the numerical evaluation of \(E_f\) was undertaken. Interestingly we found that the finite size scaling introduced by Parisi for \(E_f\) worked extremely well for \(E(p)\) that is \(E(p, \infty) \sim E(N, p) - 1/4N^\theta\), for all \(p\). In addition we found that for \(p\) small \(E(p) - E(0^+) \sim A p^\theta\) with \(\theta \approx 0.4 \pm 0.1\) [\(\tilde{13}\)]. We emphasize that all of the numerical simulations found here (for spin glasses and ferromagnets) exhibit perfect reversibility for \(E(p)\). In addition the values of \(E_f\) found in all systems were less than the value of the energy dominating thermodynamically in \(s_{Edw}(E)\) (calculated in [\(\tilde{0}, \tilde{1}\)], showing that the falling dynamics in strongly non ergodic \([\tilde{3}\]).

Granular media are examples of systems having an extensive entropy of metastable states. In such systems the role of thermal fluctuations are negligible and in order to evolve one must apply some external tapping mechanism.
One would ultimately like to be able to formulate some sort of thermodynamics for such systems. The proposition of Edwards [1] for such a thermodynamics is an important step in this direction and has had some success [2] but in the same paper it was shown not generally to be true. A more general understanding of the asymptotic states of tapped systems has also far reaching implications for computer science as the tapping mechanism studied here is similar to certain algorithms used in optimization problems. We have presented what appears to be the exact solution to the problem of tapping a one dimensional spin glass or ferromagnet. The fixed point equation for Eq.(7) may have a thermodynamic interpretation which presents an open challenge to find. In a wide context of models we confirm the observations of [4–6], that if one reduces the strength of tapping, then the compaction process, corresponding here to the reduction of the energy of the system, becomes more efficient. The existence of a first order type phase transition for tapped ferromagnets on random thin graphs is of great interest, the possible explanation using the calculations of [11] on the Edwards entropy for this system indicates the possibility of constructing a general theory for the thermodynamics and phase transitions in tapped systems.

Figure Captions

Fig. 1. Comparison between numerical simulations of tapping experiments (b) and the analytical result (a) obtained with (7).

Fig. 2. Numerical simulations of tapping experiments for the spin glass (c) and the ferromagnet ((a), (b) and (d)) for \( c = 3 \) for \( N = 1000 \) ((c) and (d)), \( N = 2000 \) (b) and \( N = 10000 \) (a). The inset shows the scaling \( E(N) = f(N(p - p_c)) \) for \( p \approx p_c \) for \( N = 400 \), \( N = 1000 \) and \( N = 2000 \).

Fig. 3. Histogram of the distribution \( p(E) \) of energy during a tapping simulation at \( p \approx p_c \) for a single run of 50000 taps and \( N = 5000 \) spins.

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