“Nonconstant cohomology” of Hietarinta’s two-color solutions to four-simplex equation

I.G. Korepanov

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Abstract
“Nonconstant cohomologies” are introduced for solutions of set-theoretical four-simplex equation (FSE). While usual cohomologies lead to solutions of constant quantum FSE, our “nonconstant cohomologies” lead to solutions of nonconstant quantum FSE. Computer calculations are presented showing that large spaces of such cohomologies exist for all Hietarinta’s two-color linear solutions to set-theoretical FSE. After taking a partial trace of the corresponding quantum operators, combined with one additional trick, this leads to solutions of tetrahedron equation, including those with non-negative matrix elements, and not reducible to a permutation, even with cocycle multipliers.

1 Four-simplex equation

1.1 Set-theoretical FSE
Let there be a set $X$, whose elements are called ‘colors’, and a map

$$R: X \times X \times X \times X \to X \times X \times X \times X$$

from its fourth Cartesian degree into itself. We will need also the tenth Cartesian degree $X^{\times 10}$ of $X$, and, to distinguish between the ten Cartesian factors — copies of $X$ — we will call them also $X_1, \ldots, X_{10}$. Let $1 \leq i < j < k < l \leq 10$; we denote

$$R_{ijkl}: X_i \times X_j \times X_k \times X_l \to X_i \times X_j \times X_k \times X_l$$

the copy of $R$ acting in the product the corresponding copies of $X$, and we also extend $R_{ijkl}$ to the whole product $X^{\times 10}$ by assuming that it acts identically on the six remaining spaces.

The set-theoretical four-simplex equation (set-theoretical FSE) is, by definition, the following equality between two (compositions of) maps $X^{\times 10} \to X^{\times 10}$:

$$R_{1234}R_{1567}R_{2589}R_{3680}R_{4790} = R_{4790}R_{3680}R_{2589}R_{1567}R_{1234}$$ (1)
(here zero stands, of course, for 10).

1.2 Quantum FSE

Another version of FSE is \textit{quantum} equation. This can be written symbolically as the same equation (1), but with a different meaning of the symbols. Namely, let now \( R \) be a linear operator acting in the fourth \textit{tensor} degree \( V^\otimes 4 \) of a linear space \( V \). The quantum FSE is an equality between two operators in \( V^\otimes 10 \). Similarly to Subsection 1.1, we call \( V_{1,\ldots,10} \) the ten copies of \( V \); \( R_{ijkl} \) is the copy of \( R \) acting in \( V_i \otimes V_j \otimes V_k \otimes V_l \) and also in the whole \( V^\otimes 10 \), extended there by multiplying tensorially by identity operators in the remaining spaces.

1.3 Nonconstant quantum FSE

Now we must confess that (1) is far from being the most general form of FSE. Below we call such equations (either set-theoretical or quantum) \textit{constant}, because each of them contains five copies of one and the same non-changing \( R \). While constant \textit{set-theoretical} FSE will be enough for us in this paper, we will be building from it solutions to a more complicated quantum equation, where there are five \textit{different} linear operators — acting, however, in the same tensor products \( V_i \otimes V_j \otimes V_k \otimes V_l \) as before. We will like also to change notations to \textit{calligraphic} letters for the tensor case, in order not to confuse it with the set-theoretical case. So, the \textit{nonconstant} quantum FSE reads, by definition, as follows:

\[
R_{1234}S_{1567}T_{2589}U_{3680}V_{4790} = V_{4790}U_{3680}T_{2589}S_{1567}R_{1234}. \tag{2}
\]

2 From set-theoretical to quantum FSE

2.1 Permutation-type solutions

Let now there be a bijection \( x \mapsto e_x \) between the set \( X \ni x \) and a \textit{basis} of linear space \( V \). If \( R \) is a solution to set-theoretic FSE (1), then we can get, in an obvious way, a solution \( R \) to the constant quantum FSE (where \( V = U = T = S = R \)), setting, by definition,

\[
R(e_x \otimes e_y \otimes e_z \otimes e_t) = e_{x'} \otimes e_{y'} \otimes e_{z'} \otimes e_{t'},
\]

where

\[
(x', y', z', t') = R(x, y, z, t).
\]

Hietarinta \cite{1} calls this \textit{permutation-type} solutions to (constant) quantum FSE.
2.2 Cohomologies

Permutation-type solutions to constant quantum FSE can be generalized as follows. Set

\[ \mathcal{R}(e_x \otimes e_y \otimes e_z \otimes e_t) = \phi(x, y, z, t)e_{x'} \otimes e_{y'} \otimes e_{z'} \otimes e_{t'}, \]

where \( \phi \) is some scalar function. Operator (3) will satisfy the constant quantum equation provided function \( \phi \) satisfies a system of as many as \((\#X)^{10}\) equations (that is, 1024 if \( X \) contains two elements), see [3]. We do not write them out here; what is important is that they are all of the multiplicative form

\[ \phi(\ldots)\phi(\ldots)\phi(\ldots) = \phi(\ldots)\phi(\ldots)\phi(\ldots), \]

where dots stay for some quadruples of arguments depending on \( \mathcal{R} \).

First, such are constant functions (taking just one fixed value). So, it makes sense to consider cocycles taken up to a constant (nonzero) factor, call them reduced cocycles.

Second, such are coboundaries, that is, functions of the form

\[ \phi(x, y, z, t) = \frac{\psi(x') \psi(y') \psi(z') \psi(t')}{\psi(x) \psi(y) \psi(z) \psi(t)}. \]

So, of interest can be reduced cocycles modulo coboundaries, they may be called reduced homologies. It seems, however, that nontrivial reduced homologies seldom exist in this setting.

2.3 “Nonconstant cohomologies”

The situation changes if we allow our quantum \( \mathcal{R} \)-operators to be different, as in equation (2). Let there be \( \mathcal{R} \), \( \mathcal{S} \), \( \mathcal{T} \), \( \mathcal{U} \), \( \mathcal{V} \) functions \( \phi_{\mathcal{R}}, \ldots, \phi_{\mathcal{V}} \), and set, instead of (3),

\[ \mathcal{R}(e_x \otimes e_y \otimes e_z \otimes e_t) = \phi_{\mathcal{R}}(x, y, z, t)e_{x'} \otimes e_{y'} \otimes e_{z'} \otimes e_{t'}, \]
\[ \mathcal{S}(e_x \otimes e_y \otimes e_z \otimes e_t) = \phi_{\mathcal{S}}(x, y, z, t)e_{x'} \otimes e_{y'} \otimes e_{z'} \otimes e_{t'}, \]
\[ \mathcal{T}(e_x \otimes e_y \otimes e_z \otimes e_t) = \phi_{\mathcal{T}}(x, y, z, t)e_{x'} \otimes e_{y'} \otimes e_{z'} \otimes e_{t'}, \]
\[ \mathcal{U}(e_x \otimes e_y \otimes e_z \otimes e_t) = \phi_{\mathcal{U}}(x, y, z, t)e_{x'} \otimes e_{y'} \otimes e_{z'} \otimes e_{t'}, \]
\[ \mathcal{V}(e_x \otimes e_y \otimes e_z \otimes e_t) = \phi_{\mathcal{V}}(x, y, z, t)e_{x'} \otimes e_{y'} \otimes e_{z'} \otimes e_{t'}. \]

We call a quintuple \( \phi_{\mathcal{R}}, \ldots, \phi_{\mathcal{V}} \) cocycle if (2) holds; this happens provided it satisfies a system of \((\#X)^{10}\) equations of the form

\[ \phi_{\mathcal{R}}(\ldots)\phi_{\mathcal{S}}(\ldots)\phi_{\mathcal{T}}(\ldots)\phi_{\mathcal{U}}(\ldots)\phi_{\mathcal{V}}(\ldots) = \phi_{\mathcal{V}}(\ldots)\phi_{\mathcal{U}}(\ldots)\phi_{\mathcal{T}}(\ldots)\phi_{\mathcal{S}}(\ldots)\phi_{\mathcal{R}}(\ldots). \]

There are, again, two kinds of trivial cocycles. First kind appears when each of \( \phi_{\mathcal{R}}, \ldots, \phi_{\mathcal{V}} \) in constant (but, in contrast to Subsection 2.2, there can be now five
different constants). Second kind appears from ten scalar functions on $X$, call them $\psi_1, \ldots, \psi_{10}$, and reads

\[ \varphi_R(x, y, z, t) = \frac{\psi_1(x') \psi_2(y') \psi_3(z') \psi_4(t')}{\psi_1(x) \psi_2(y) \psi_3(z) \psi_4(t)}, \]  

\[ \varphi_S(x, y, z, t) = \frac{\psi_1(x') \psi_5(y') \psi_6(z') \psi_7(t')}{\psi_1(x) \psi_5(y) \psi_6(z) \psi_7(t)}, \]  

\[ \varphi_T(x, y, z, t) = \frac{\psi_2(x') \psi_5(y') \psi_8(z') \psi_9(t')}{\psi_2(x) \psi_5(y) \psi_8(z) \psi_9(t)}, \]  

\[ \varphi_U(x, y, z, t) = \frac{\psi_3(x') \psi_6(y') \psi_8(z') \psi_{10}(t')}{\psi_3(x) \psi_6(y) \psi_8(z) \psi_{10}(t)}, \]  

\[ \varphi_V(x, y, z, t) = \frac{\psi_4(x') \psi_7(y') \psi_9(z') \psi_{10}(t')}{\psi_4(x) \psi_7(y) \psi_9(z) \psi_{10}(t)}. \]  

### 3 Calculations for Hietarinta’s solutions

#### 3.1 Two-color $\mathbb{F}_2$-linear solutions

Let now $X$ be the field $X = \mathbb{F}_2$ of two elements, that is, as a set, $X = \{0, 1\}$. Assume also that $R$ is $\mathbb{F}_2$-linear. In this situation, Hietarinta [1, Subsection 6.21] calculated in 1997 all maps $R$ enjoying (11). These linear maps are given by the following matrices:

\[ A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]  

\[ A_3 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]  

and their transposes $A_1^T, A_2^T, A_3^T$ and $A_4^T$.

#### 3.2 Linear systems for logarithms

In terms of logarithms $\log \varphi_\ldots$, all equations (6) become linear. So do also (7)–(11), in terms of $\log \varphi_\ldots$ and $\log \psi_\ldots$. To specify each of five functions $\varphi_\ldots$, we must specify its 16 values for 16 possible quadruples of arguments. So, equations (6) make a system of 1024 linear (in terms of logarithms) equations on 80 variables. Nontrivial solutions appear after factorizing modulo five additive constants for all $\log \varphi_\ldots$, and modulo those $\varphi_\ldots$ that can be obtained in the form (7)–(11).
3.3 Computer calculations

The following two dimensions of linear spaces have been calculated using computer algebra system Maxima[1]:

- the number \( n \) of independent equations among the 1024 equations (6),
- the dimension \( d \) of the linear space of quintuples \( \log \phi_R, \ldots, \log \phi_V \) that can be obtained in the form (7)–(11).

Then, the space of our “nonconstant homologies”, implying nontrivial solutions to system of 1024 equations (6), has dimension

\[
h = 80 - n - d - 5,
\]

where 5 stands for five additive constants mentioned above in Subsection 3.2.

The results are as follows.

For matrix \( A_1 \):

\[
n = 50, \quad d = 9, \quad h = 16.
\]

For matrix \( A_2 \):

\[
n = 37, \quad d = 7, \quad h = 31.
\]

For matrix \( A_3 \):

\[
n = 54, \quad d = 9, \quad h = 12.
\]

For matrix \( A_4 \):

\[
n = 50, \quad d = 9, \quad h = 16.
\]

For matrix \( A_1^T \):

\[
n = 50, \quad d = 9, \quad h = 16.
\]

For matrix \( A_2^T \):

\[
n = 50, \quad d = 9, \quad h = 16.
\]

For matrix \( A_3^T \):

\[
n = 40, \quad d = 7, \quad h = 28.
\]

[1]http://maxima.sourceforge.net/
For matrix $A_1^T$:

\[ n = 50, \quad d = 9, \quad h = 16. \]

4 From quantum FSE to quantum tetrahedron equation

Our results in Subsection 3.3 show that there exist large families of interesting solutions to quantum four-simplex equation (2). Now we can make from them solutions to the lower-dimensional quantum $n$-simplex equation, namely Zamolodchikov tetrahedron (3-simplex) equation, by taking partial traces of our $R$-operators in a well-known way. Even more interesting things come out if we apply more ingenuity and modify the traces by multiplying $R$-operators by some additional operators, as we are going to explain. We will content ourselves with doing our construction, in this paper, only for the matrix $A_1$, see (12).

As all our vector spaces always have fixed bases, we do not make difference between operators and their matrices.

4.1 Special cocycles leaving the last operator pure permutation

We construct such cocycles $(\varphi_R, \varphi_S, \varphi_T, \varphi_U, \varphi_V)$ for the permutation-type $R$-operators generated by matrix $A_1$ that $\varphi_V \equiv 1$, that is, in terms of logarithms, find such subspace of the corresponding linear space where $\log \varphi_V \equiv 0$. Our calculation using Maxima shows that its dimension is 14, while the dimension of the coboundary logarithms in the sense (7)–(10) is 6. There are also 4 constant cocycles, so, there remain $14 - 6 - 4 = 4$ essential parameters.

Operator $V = V_{4790}$ remains thus pure permutation, which implies the following important fact: $V$ commutes with the operator

\[ P = P_{4790} = \left( \begin{array}{cccc} 1 & 1 \\ 1 & 1 \end{array} \right)_4 \otimes \left( \begin{array}{cccc} 1 & 1 \\ 1 & 1 \end{array} \right)_7 \otimes \left( \begin{array}{cccc} 1 & 1 \\ 1 & 1 \end{array} \right)_9 \otimes \left( \begin{array}{cccc} 1 & 1 \\ 1 & 1 \end{array} \right)_0. \] (14)

This is because $P$ can be also represented as a tensor product of a column matrix consisting of unities and a row matrix consisting of unities, and any permutation does not change a vector of unities. The subscripts in (14) mean of course the numbers of spaces, and tensor multiplication is implied by the identity matrices/operators $\left( \begin{array}{ccc} 1 & 0 \\ 0 & 1 \end{array} \right)$ in the remaining six spaces.

Also, our calculations show that, for any individual operator $R$, $S$, $T$ or $U$, cocycles give a 6-dimensional linear space of parameters, and coboundaries give a 3-dimensional space. As there is also one trivial multiplicative constant, there remain two essential parameters for an individual $R$-operator.
4.2 Tetrahedron solutions from special cocycles

It follows from (2) and the commutativity between $V$ and $P$ that

$$P_{4790} R_{1234} S_{1567} T_{2589} U_{3680} = V_{4790} P_{4790} U_{3680} T_{2589} S_{1567} R_{1234} V^{-1}_{4790}.$$  \hspace{0.5cm} (15)

Taking the partial trace of (15) in spaces 4, 7, 9 and 0, we arrive at the tetrahedron equation (with a somewhat nonstandard numbering of spaces)

$$K_{123} L_{156} M_{258} N_{368} = N_{368} M_{258} L_{156} K_{123}$$ \hspace{0.5cm} (16)

for operators

$$K_{123} = \text{Trace}_4 \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)^4 R_{1234}, \quad L_{156} = \text{Trace}_7 \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)^7 S_{1567},$$

$$M_{258} = \text{Trace}_9 \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)^9 T_{2589}, \quad N_{368} = \text{Trace}_0 \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)^0 U_{3680}. $$

Here $\text{Trace}_k$ means of course the partial trace in the $k$-th space.

4.3 Explicit form of tetrahedral $R$-operators

Direct calculations using Maxima lead to the following remarkable results. First, the number of essential parameters in our solution of equation (16) remains equal to 4 if we impose on matrices $K, L, M, N$ the additional requirement of being symmetric: $K = K^T, \ldots, N = N^T$. With this condition, it turns out that

- $K$ and $L$ are proportional, or simply can be taken equal, and have the following form:

$$K = L = \begin{pmatrix} a & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & c \\ b & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & b & 0 & 0 & c & 0 & 0 \\ 0 & b & 0 & 0 & c & 0 & 0 & 0 \end{pmatrix},$$  \hspace{0.5cm} (17)

- the same applies to matrices $M$ and $N$:

$$M = N = \begin{pmatrix} a' & 0 & 0 & 0 & 0 & b' & 0 & 0 \\ 0 & 0 & b' & 0 & 0 & 0 & 0 & c' \\ 0 & b' & 0 & 0 & a' & 0 & 0 & 0 \\ 0 & 0 & 0 & c' & 0 & 0 & b' & 0 \\ 0 & a' & 0 & 0 & 0 & 0 & b' & 0 \\ b' & 0 & 0 & 0 & c' & 0 & 0 & 0 \\ 0 & 0 & b' & 0 & 0 & a' & 0 & 0 \\ 0 & c' & 0 & 0 & b' & 0 & 0 & 0 \end{pmatrix},$$  \hspace{0.5cm} (18)
equality \( (16) \) imposes no dependencies on parameters \( a, b, c, a', b', c' \). So, six of them minus two scalings in (17) and (18) give four essential parameters as promised.

5 Algebraic nontriviality, but simple thermodynamical behavior

5.1 Genuine three-dimensioneness

Let us speak, for concreteness, of operator \( \mathcal{K} \). It acts in the tensor product \( V_1 \otimes V_2 \otimes V_3 \) of three two-dimensional linear spaces with fixed bases. Here we show that there is no such vector

\[
\begin{pmatrix} 1 \\ u \end{pmatrix} \in V_1 \tag{19}
\]

that

\[
\begin{pmatrix} 1 \\ u \end{pmatrix} \otimes X \mapsto \begin{pmatrix} 1 \\ u \end{pmatrix} \otimes Y \tag{20}
\]

for any vector \( X \in V_2 \otimes V_3 \) in the tensor product of two remaining spaces. Of course, similar facts can be also proved for \( \begin{pmatrix} 1 \\ v \end{pmatrix} \in V_2 \) and \( \begin{pmatrix} 1 \\ w \end{pmatrix} \in V_3 \).

If (20) held true, then \( \mathcal{K} \) could be thought of as not completely three-dimensional (in the sense “well suited for three-dimensional lattice integrable models in mathematical physics”), because at least if vector (19) is on its first input, then \( \mathcal{K} \) would reduce to the two-dimensional operator \( \mathcal{K}_{23} : X \mapsto Y \). More formally, \( \mathcal{K} \) would acquire block structure

\[
\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}
\]

if we change the basis in \( V_1 \) properly; \( A, B \) and \( C \) are here \( 4 \times 4 \) blocks. This block structure would, in its turn, imply that some nontrivial linear combination of \( 4 \times 4 \) blocks in (17) is zero matrix — but this is obviously impossible.

5.2 No disguised permutation

Our tetrahedral \( \mathcal{R} \)-operators do not turn into a permutation with possible cocycle multipliers (tetrahedral analogue of (3); below “permutation with cocycle” for short) under any “gauge” transformation. That is, if we speak again of operator \( \mathcal{K} \), let \( F, G, H \) be any invertible \( 2 \times 2 \) matrices, then

\[
(F \otimes G \otimes H) \mathcal{K} (F \otimes G \otimes H)^{-1}
\]

is not a permutation with cocycle.
If $\mathcal{K}$ were such a permutation with cocycle, this could be reformulated as follows: there are vectors
\[ f_i = \begin{pmatrix} 1 \\ u_i \end{pmatrix} \in V_1, \quad g_j = \begin{pmatrix} 1 \\ v_j \end{pmatrix} \in V_2, \quad h_k = \begin{pmatrix} 1 \\ w_k \end{pmatrix} \in V_3, \quad i, j, k = 1, 2, \]
forming bases in their corresponding spaces and such that
\[ \mathcal{K}(u_i \otimes v_j \otimes w_k) = \varphi_{ijk}(u_i' \otimes v_j' \otimes w_k') \tag{22} \]
for any $i, j, k$ — that is, vectors \([21]\) would be “vacuum vectors” of $\mathcal{K}$ in the Krichever’s \([4]\) sense.

Some degree of any permutation makes identical mapping. Considering the 1st and 6th components of vectors in \([22]\) in one case and the 4th and 7th components in the other, and taking into account the explicit form \([17]\) of $\mathcal{K}$, we see that \(\begin{pmatrix} 1 \\ u_i w_j \end{pmatrix}\) and \(\begin{pmatrix} w_k \\ u_i \end{pmatrix}\) are eigenvectors for some degree of matrix \(\begin{pmatrix} a & b \\ b & c \end{pmatrix}\) and hence (as $a, b, c$ are free parameters) for this matrix itself, for all $i, j, k, l$.

Simple analysis, using the fact that the mentioned matrix has not more than two eigenvectors, shows that this cannot be.

### 5.3 Thermodynamical behavior

Two obvious eigenvectors of operator \([17]\) are
\[ \Omega_{1,2} = \begin{pmatrix} 1 \\ u_{1,2} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]
where \(\begin{pmatrix} 1 \\ u_{1,2} \end{pmatrix}\) are two eigenvectors of matrix \(\begin{pmatrix} a & b \\ b & c \end{pmatrix}\). Let now $a, b$ and $c$ in \([17]\) be all positive, then exactly one of $u_{1,2}$ is also positive, let it be $u_1$, and call the corresponding eigenvalue $\lambda$.

Let there be a statistical physical model on a cubic lattice, with operator $\mathcal{K}$ in each vertex. Cutting the lattice into slices by planes orthogonal to a space diagonal, we obtain a “hedgehog” transfer matrix in each slice, see \([2]\). The positive eigenvector of such transfer matrix — the only one essential for the thermodynamical limit — is a tensor product of vectors $\Omega_1$, and this leads to the simple fact that the free energy per site (vertex) in the thermodynamical limit is $\log \lambda$.

### 6 Discussion

Our general idea for searching of solutions to simplex equations can now be stated as follows. Take a higher simplex equation, find a Hietarinta-style “linear
permutation” solution for it, then calculate its “nonconstant homologies”, and then descend to the required simplex dimension by taking partial traces and possibly doing additional tricks, like in our Section 4. We did it for 3-simplex equation and using 4-simplex equation, and this already looks like a useful step in the right direction: the obtained solutions of tetrahedron equation, as explained in Section 5, look significantly richer than simple permutations. Our hope is that even more impressive results could be obtained if we employ simplex equations of great dimensions, for instance, $n = 100500$ or even $n = \infty$, as Hietarinta already proposed in [1, Section 7]. Perhaps, interesting statistical physical models, with nonnegative Boltzmann weights and nontrivial thermodynamical behavior, are waiting for us on this way.

References

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