A new matrix formulation of the Maxwell and Dirac equations

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Abstract

Presented in this paper is a new matrix formulation of both the classical electromagnetic Maxwell equations and the relativistic quantum mechanical Dirac equation. These new matrix representations will be referred to as the Maxwell spacetime matrix equation and the Dirac spacetime matrix equation. Both are Lorentz invariant. Key to these new matrix formulations is an 8-by-8 matrix operator referred to here as the spacetime matrix operator. As it turns out, the Dirac spacetime matrix equation is equivalent to four new vector equations, which are similar in form to the four Maxwell vector equations. These new equations will be referred to as the Dirac spacetime vector equations. This allows these new vector equations to be as readily solved as solving a set of Maxwell vector equations. Based on these two new matrix approaches, two computer programs, encoded using Matlab software, have been developed and tested for determining the reflection and transmission characteristics of multilayer optical thin-film structures and multilayer quantum well-and-barrier structures. A listing of these software programs may be found in supplemental material associated with this article. Numerical results obtained based on the use of these computer programs are presented in the results section of this article.

Keywords: Electromagnetism, Quantum mechanics, Special relativity
1. Introduction

In an earlier publication [1] by the authors, a matrix formulation of the Maxwell field equations was presented employing an 8-by-8 matrix operator, referred to then as the spacetime operator. The primary purpose of that article was to employ the methods of matrix calculus to determine electromagnetic fields for arbitrary charge and current distributions. Four-dimensional Fourier transforms, transfer functions, and the convolution theorem were employed. In this article, our emphasis is twofold: Firstly, our motivation is to extend our earlier work to now solve problems numerically dealing with the propagation of electromagnetic waves through multilayer thin-film optical structures. A slightly modified version of the original spacetime operator, now referred to as the spacetime matrix operator, plays a fundamental role in this approach. Secondly, to employ a similar matrix approach in solving relativistic problems numerically involving the propagation of matter waves through multilayer quantum well-and-barrier structures. New matrix formulations of the classical electromagnetic (EM) Maxwell field equations and the quantum mechanical (QM) relativistic Dirac equations will be presented. These new equations will be referred to as the Maxwell and Dirac spacetime matrix equations, respectively.

From the Dirac spacetime matrix equation we were able to form four new relativistic vector equations which highly resemble the four Maxwell vector equations. These equations will be referred to as the Dirac spacetime vector equations.

The new matrix formulations of the Maxwell and Dirac matrix equations were encoded, using the software package Matlab [2]. In particular, two computer programs were written for determining the transmission and reflection characteristics of multilayer optical thin-film structures illuminated by electromagnetic waves as well as multilayer quantum well-and-barrier structures illuminated by matter waves. These two software packages have been tested and results obtained are in excellent agreement with results published in the open scientific literature. Four examples, involving multilayer structures, are presented in the results section of this article. These examples include the interaction of a relativistic neutron with a 3-layer nuclear quantum well-and-barrier structure, the interaction of a non-relativistic electron with an 11-layer periodic quantum well-and-barrier structure, visible light impinging upon a 15-layer optical thin-film longwave pass filter, and visible light impinging upon a 21-layer narrow bandpass optical filter. The Matlab software programs are listed in supplemental material entitled “Supplement EM” and “Supplement QM.” The software program Mathematica [3] is also well suited for implementing these new matrix approaches.
2. Theory

2.1. Spacetime matrix operator

The spacetime matrix operator \( \hat{M} \), defined in Eq. (1), plays a central role in this paper in expressing both the classical electromagnetic Maxwell equations and the relativistic quantum mechanical Dirac equation in a new matrix formulation.

\[
\hat{M} \equiv \begin{bmatrix}
-\partial_4 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\
0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 & -\partial_2 \\
0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 & -\partial_3 \\
0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 & 0 \\
0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 & 0 \\
-\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 & 0 \\
+\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 & 0 \\
-\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4
\end{bmatrix}.
\] (1)

The partial derivative symbols appearing in Eq. (1) are defined by

\[\partial_1 \equiv \partial/\partial x \quad \partial_2 \equiv \partial/\partial y \quad \partial_3 \equiv \partial/\partial z \quad \partial_4 \equiv \partial/\partial icdt.\]

The symbol \( i \) represents the square root of minus one and the symbol \( c \) represents the speed of light in free space. The spacetime matrix operator \( \hat{M} \) may be expressed in the form

\[\hat{M} = M_1 \partial_1 + M_2 \partial_2 + M_3 \partial_3 + M_4 \partial_4.\]

The 8-by-8 spacetime matrices \( M_\mu \), where \( \mu = 1, 2, 3, 4 \), have the following properties:

Each matrix \( M_\mu \) is equal to its own inverse

\[M_\mu = M_\mu^{-1},\]

these matrices satisfy the anti-commutation relation

\[M_\mu M_\nu + M_\nu M_\mu = 2\delta_{\mu\nu} I_8,\]

and each matrix \( M_\mu \) is Hermitian.

\[M_\mu = M_\mu^\dagger.\]

The symbol \( \delta_{\mu\nu} \) is the Kronecker delta and \( I_8 \) represents the 8-by-8 identity matrix.
2.2. The microscopic Maxwell equations

The microscopic Maxwell field equations play a fundamental role in classical electromagnetic theory. The propagation of electromagnetic waves through free space [4], uniform isotropic linear media [5], thin-film optical filters [6], and crystalline materials [7] are just a few examples where the Maxwell field equations play an important role. In Gaussian units [8], the four microscopic Maxwell equations: Gauss’s law of electrostatics (2), Faraday’s law of induction (3), Gauss’s law of magnetism (4), and Ampere’s law (5) are given by

\[ \nabla \cdot \mathbf{E} = +4\pi \rho_e, \]  
(2)  
\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\frac{4\pi}{c} \mathbf{J}_m, \]  
(3)  
\[ \nabla \cdot \mathbf{B} = +4\pi \rho_m, \]  
(4)  
\[ \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = +\frac{4\pi}{c} \mathbf{J}_e. \]  
(5)

The scalar and vector quantities appearing in the microscopic Maxwell equations are: the del operator \( \nabla \), the electric field \( \mathbf{E} \), the magnetic induction \( \mathbf{B} \), the electric current density \( \mathbf{J}_e \), the magnetic current density \( \mathbf{J}_m \), the electric charge density \( \rho_e \), the magnetic charge density \( \rho_m \), and the speed of light \( c \) in free space. Both magnetic charge and magnetic current density [8] have been included in the microscopic Maxwell equations for purposes of completeness. They, of course, may be set equal to zero since magnetic monopoles have not been discovered in nature.

2.3. The Maxwell spacetime matrix equation

Using the spacetime matrix operator \( \hat{M} \), the new modified version (6) of the matrix representation of the microscopic Maxwell equations is given by

\[
\begin{bmatrix}
-\partial_4 & 0 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\
0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 & -\partial_2 \\
0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 & -\partial_3 \\
0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 & 0 \\
0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 & 0 \\
-\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 & 0 \\
+\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 & 0 \\
-\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4
\end{bmatrix}
\begin{bmatrix}
iE_1 \\
iE_2 \\
iE_3 \\
0 \\
B_1 \\
0 \\
B_2 \\
B_3 \\
0
\end{bmatrix}
= \frac{4\pi}{c}
\begin{bmatrix}
J_{e1} \\
J_{e2} \\
J_{e3} \\
c\rho_m \\
iJ_{m1} \\
iJ_{m2} \\
iJ_{m3} \\
-ic\rho_e
\end{bmatrix}.
\]  
(6)

In this paper, Eq. (6) will be referred to as the Maxwell spacetime matrix equation. The compact matrix form (7) of Maxwell spacetime matrix equation is given by

\[ \hat{M} | f \rangle = | j \rangle. \]  
(7)
The ket vector $|f\rangle$ represents the 8-by-1 column vector on the left-hand side of Eq. (6). The ket vector $|j\rangle$ corresponds to the 8-by-1 column vector on the right-hand side of Eq. (6) multiplied by the factor $4\pi/c$. A detailed discussion on the bra $\langle i|$ and ket $|j\rangle$ vector notation may be found in Messiah [9].

### 2.4. The relativistic Dirac equation

The non-relativistic Schrödinger wave equation [10] plays a fundamental role in quantum mechanical phenomena where the spin property of non-relativistic particles may be ignored. This equation is usually first met in introductory modern physics textbooks. However, when particle spin and/or relativistic speeds are involved, the relativistic Dirac equation [11, 12] comes into play. One version of the relativistic Dirac equation (8), in the absence of scalar and vector potentials, is given by

$$
\begin{bmatrix}
+\partial_4 & 0 & -i\partial_3 & -\partial_2 - i\partial_1 \\
0 & +\partial_4 & +\partial_2 - i\partial_1 & +i\partial_3 \\
+i\partial_3 & +\partial_2 + i\partial_1 & -\partial_4 & 0 \\
-\partial_2 + i\partial_1 & -i\partial_3 & 0 & -\partial_4 \\
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\end{bmatrix}
+ \kappa
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\end{bmatrix}
= 0.
$$

(8)

The scalar constant $\kappa$ is defined by

$$
\kappa \equiv m_0 c / \hbar.
$$

Here $m_0$ represents the rest mass of the matter-wave particle under consideration, $c$ again is the speed of light in free space, and $\hbar$ is equal to the Planck constant divided by $2\pi$. Equation (9) represents the compact matrix form of Eq. (8)

$$
\hat{D}|\psi\rangle + \kappa|\psi\rangle = |o\rangle.
$$

(9)

The Dirac matrix operator $\hat{D}$ represents the 4-by-4 matrix operator on the left-hand side of Eq. (8), $|\psi\rangle$ is the 4-by-1 ket vector appearing twice on the left-hand side, and $|o\rangle$ is the 4-by-1 null ket vector appearing on the right-hand side. The Dirac matrix operator may be expressed in the form

$$
\hat{D} = D_1 \partial_1 + D_2 \partial_2 + D_3 \partial_3 + D_4 \partial_4.
$$

The matrices $D_\mu$, where $\mu = 1, 2, 3, 4$, are referred to as the Dirac matrices. There are a number of different matrix representations of the four Dirac matrices. That particular set associated with Eq. (8) is called the Dirac representation. See Appendix 3 in Roman [11] for details. The Dirac matrices have the following properties:

$$
D_\mu = D_\mu^{-1}, \\
D_\mu D_\nu + D_\nu D_\mu = 2\delta_\mu^\nu I_4, \\
D_\mu = D_\mu^\dagger.
$$
Again, the symbol $\delta_{\mu\nu}$ is the Kronecker delta. Here $I_4$ represents the 4-by-4 identity matrix. Notice the properties of the Dirac matrices $D_\mu$ are identical to the properties of the spacetime matrices $M_\mu$.

### 2.5. The Dirac spacetime matrix equation

The spacetime matrix operator $\hat{M}$ has been used by the authors to define a new modified version of the relativistic Dirac equation. In this paper we refer to this new equation as the Dirac spacetime matrix equation (10).

\[
\begin{pmatrix}
-\partial_4 & 0 & 0 & 0 & -\partial_3 & +\partial_2 & -\partial_1 \\
0 & -\partial_4 & 0 & 0 & +\partial_3 & 0 & -\partial_1 \\
0 & 0 & -\partial_4 & 0 & -\partial_2 & +\partial_1 & 0 \\
0 & 0 & 0 & -\partial_4 & +\partial_1 & +\partial_2 & +\partial_3 \\
0 & +\partial_3 & -\partial_2 & +\partial_1 & +\partial_4 & 0 & 0 \\
-\partial_3 & 0 & +\partial_1 & +\partial_2 & 0 & +\partial_4 & 0 \\
+\partial_2 & -\partial_1 & 0 & +\partial_3 & 0 & 0 & +\partial_4 \\
-\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_4
\end{pmatrix}
\begin{pmatrix}
iU_1 \\
iU_2 \\
iU_3 \\
0 \\
0 \\
L_1 \\
L_2 \\
L_3 \\
0
\end{pmatrix}
+ \kappa
\begin{pmatrix}
iU_1 \\
iU_2 \\
iU_3 \\
0 \\
0 \\
L_1 \\
L_2 \\
L_3 \\
0
\end{pmatrix}
= 0.
\]

(10)

Its compact matrix form (11) is given by

\[
\hat{M} |\phi\rangle + \kappa |\phi\rangle = |O\rangle.
\]

(11)

We have simply replaced the Dirac matrix operator $\hat{D}$ in Eq. (8) by the spacetime matrix operator $\hat{M}$. The new wave function $|\phi\rangle$ is an 8-by-1 ket vector. The ket vector $|O\rangle$ represents the 8-by-1 null vector. The wave function $|\phi\rangle$ now contains six scalar components associated with two new vector quantities called $U$ and $L$. In addition, elements (4,1) and (8,1) in $|\phi\rangle$ have purposely been set equal to zero. The wave function $|\phi\rangle$ in the Dirac spacetime matrix equation (10) has the same structure as the ket vector $|f\rangle$ in the Maxwell spacetime matrix equation (6).

### 2.6. Relationship between the spacetime matrices and Dirac matrices

As previously mentioned, the 8-by-8 spacetime matrices $M_\mu$ and the 4-by-4 Dirac matrices $D_\mu$ basically have identical properties. These two sets of matrices can be related through a set of linear transformation equations (12) of the form

\[
Z M_\mu = D_\mu Z \quad \text{where } \mu = 1, 2, 3, 4.
\]

(12)

The matrix $Z$ is a 4-by-8 matrix. Its 32 matrix elements are initially unknown. These elements can be determined by first forming the matrix products according
to Eqs. (12). Through a systematic comparison of corresponding elements on each side of these four matrix equations we find the matrix $Z$ has the general form (13) given by

$$Z = \begin{bmatrix}
0 & 0 & 0 & 0 & +a & -ia & +b & -ib \\
0 & 0 & 0 & 0 & +b & +ib & -a & -ia \\
-a & +ia & -b & +ib & 0 & 0 & 0 & 0 \\
-b & -ib & +a & +ia & 0 & 0 & 0 & 0
\end{bmatrix}$$  \hspace{1cm} (13)$$

where $a$ and $b$ are arbitrary constants.

With the use of Eqs. (9), (11), and (12), we will now establish the key relationship between the wave functions $|\phi\rangle$ and $|\psi\rangle$. We start with Eq. (11) and multiply both sides of this equation by the matrix $Z$. This gives

$$Z \hat{M} |\phi\rangle + \kappa Z |\phi\rangle = Z |O\rangle.$$  \hspace{1cm} (14)$$

With the use of Eqs. (12) we can write

$$Z \hat{M} |\phi\rangle = \hat{D} Z |\phi\rangle.$$  \hspace{1cm} (15)$$

In addition

$$|o\rangle = Z |O\rangle.$$  \hspace{1cm} (16)$$

This implies

$$\hat{D} Z |\phi\rangle + \kappa Z |\phi\rangle = |o\rangle.$$  \hspace{1cm} (17)$$

Next we defined, using Eq. (14), the wave function $|\psi\rangle$ in terms of the wave function $|\phi\rangle$ through the linear transformation matrix $Z$, that is

$$|\psi\rangle \equiv Z |\phi\rangle$$  \hspace{1cm} (18)$$

which now implies

$$\hat{D} |\psi\rangle + \kappa |\psi\rangle = |o\rangle.$$  \hspace{1cm} (19)$$

This is just the relativistic Dirac equation in compact matrix form (9). We have thus established a very important relationship between the Dirac spacetime matrix equation (10) and the relativistic Dirac equation (8). And that is, solutions $|\phi\rangle$ of the Dirac spacetime matrix equation can be easily transformed into solutions $|\psi\rangle$ satisfying the relativistic Dirac equation simply by using Eq. (14).

For numerical computations, the authors chose the arbitrary constants $a$ and $b$, appearing in Eq. (13), equal to $\sqrt{2}/2$. Reason: If the eight 8-by-1 vector solutions of the Dirac spacetime matrix equation, for a given application, form an orthonormal set, then the four 4-by-1 vector solutions satisfying the relativistic Dirac equation,
formed from Eq. (14), will also form an orthonormal set. For this choice of the constants $a$ and $b$, the matrix $Z$ of interest (15) is now given by

$$Z = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & +1 & -i & +1 & -i \\ 0 & 0 & 0 & 0 & +1 & +i & -1 & -i \\ -1 & +i & -1 & +i & 0 & 0 & 0 & 0 \\ -1 & -i & +1 & +i & 0 & 0 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (15)

2.7. The Dirac vector and Maxwell vector equations for free space

The Dirac spacetime matrix equation (10) when expanded is equivalent to the four partial differential vector equations. In the absence of scalar and vector potentials, we will refer to these equations as the Dirac spacetime vector equations for free space. These equations, as we will see, resemble the four Maxwell vector equations for free space. The simplest solutions of these two sets of vector equations are time-harmonic plane-wave solutions. Shortly we will compare the properties of the electromagnetic plane-wave solutions with those of the quantum mechanical plane-wave solutions.

The Dirac spacetime matrix equation (10) when expanded is equivalent to the four vector equations (16), (17), (18), and (19). Again, we refer to these equations as the Dirac spacetime vector equations for free space. In the absence of scalar and vector potentials they are given by

$$\nabla \cdot \mathbf{U} = 0,$$  \hspace{1cm} (16)

$$\nabla \times \mathbf{U} + \frac{1}{c} \frac{\partial \mathbf{L}}{\partial t} + i \kappa \mathbf{L} = 0,$$  \hspace{1cm} (17)

$$\nabla \cdot \mathbf{L} = 0,$$  \hspace{1cm} (18)

$$\nabla \times \mathbf{L} - \frac{1}{c} \frac{\partial \mathbf{U}}{\partial t} + i \kappa \mathbf{U} = 0.$$  \hspace{1cm} (19)

The microscopic Maxwell equations, for the special case when charge and current densities are set equal to zero, yield the following Maxwell vector equations (20), (21), (22), and (23) for free space

$$\nabla \cdot \mathbf{E} = 0,$$  \hspace{1cm} (20)

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0,$$  \hspace{1cm} (21)

$$\nabla \cdot \mathbf{B} = 0,$$  \hspace{1cm} (22)

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = 0.$$  \hspace{1cm} (23)

The only real difference between these two sets of equations, except for vector symbols, is the two factors in the Dirac vector curl equations involving the constant $\kappa$. Recall the constant $\kappa$ was defined in terms of the rest mass $m_0$ of the matter-wave particle of interest. The quantum of electromagnetic radiation, namely the photon,
has zero rest mass. Therefore, the Maxwell equations have no terms involving rest mass.

2.8. Time-harmonic plane-wave solutions

For quantum mechanical time-harmonic plane-wave solutions (24), the vectors \( U(\mathbf{r}, t) \) and \( L(\mathbf{r}, t) \) may be expressed as

\[
U(\mathbf{r}, t) = U_0 \exp\{i(\mathbf{p} \cdot \mathbf{r} - Et)/\hbar\} \quad L(\mathbf{r}, t) = L_0 \exp\{i(\mathbf{p} \cdot \mathbf{r} - Et)/\hbar\}. \tag{24}
\]

The quantities \( \mathbf{p} \) and \( E \) correspond to the linear momentum and the total energy of the associated matter-wave particle; \( \mathbf{r} \) and \( t \) represent the position vector and the instantaneous time. For matter-wave particles, with non-zero rest mass \( m_0 \), the following special theory of relativity equations [13] may also be useful

\[
E = \gamma m_0 c^2 \quad p = \gamma m_0 v \quad \gamma = 1/\sqrt{1 - \beta^2} \quad \beta = v/c.
\]

The quantities \( \gamma \) and \( \beta \) are the Lorentz factor and the speed parameter, respectively. The symbol \( v \) represents the relativistic speed of the matter-wave particle.

For electromagnetic time-harmonic plane-wave solutions (25), the vectors \( \mathbf{E}(\mathbf{r}, t) \) and \( \mathbf{B}(\mathbf{r}, t) \) may be represented by

\[
E(\mathbf{r}, t) = E_0 \exp\{i(\mathbf{p} \cdot \mathbf{r} - Et)/\hbar\} \quad B(\mathbf{r}, t) = B_0 \exp\{i(\mathbf{p} \cdot \mathbf{r} - Et)/\hbar\}. \tag{25}
\]

Here \( \mathbf{p} \) represents the linear momentum of the associated electromagnetic quantum and \( E \) its energy. If one prefers to use the wave vector \( \mathbf{k} \) and the angular frequency \( \omega \), traditionally used for electromagnetic waves, then the following substitutions may be used in Eqs. (25)

\[
\mathbf{p} = \hbar \mathbf{k} \quad E = \hbar \omega.
\]

2.9. Time-harmonic plane-wave properties

Substitution of Eqs. (24) back into Eqs. (16), (17), (18) and (19) yield the following set of vector equations for matter waves

\[
\begin{align*}
\mathbf{p} \cdot \mathbf{U}_0 &= 0, \\
\mathbf{p} \times \mathbf{U}_0 &= +E \frac{\gamma - 1}{\gamma} \mathbf{L}_0, \\
\mathbf{p} \cdot \mathbf{L}_0 &= 0, \\
\mathbf{p} \times \mathbf{L}_0 &= -E \frac{\gamma + 1}{\gamma} \mathbf{U}_0.
\end{align*}
\]

Similarly, substitution of Eqs. (25) back into Eqs. (20), (21), (22) and (23) yield the following set of vector equations for electromagnetic waves
It can be easily shown these two preceding sets of equations yield the following

\[ U_0 \cdot L_o = 0 \quad E_o \cdot B_o = 0. \]

From the above results we see the electromagnetic field vectors \( E_o \) and \( B_o \) are transverse and orthogonal. These results are well known established facts [4] in the field of electromagnetic plane wave propagation through free space. In addition, the vectors \( U_o \) and \( L_o \) and their corresponding linear momentum vector \( p \) are also mutually perpendicular. So we have

\[ p \perp U_o \quad U_o \perp L_o \quad p \perp L_o \quad p \perp E_o \quad E_o \perp B_o \quad p \perp B_o. \]

In addition, we also obtain the important results

\[ (\gamma + 1) U_o^2 = (\gamma - 1) L_o^2 \quad E_o^2 = B_o^2. \]

So for free space, the magnitudes of the electromagnetic field vectors \( E_o \) and \( B_o \) are equal, a well known result in electromagnetic wave propagation. Recall we are using Gaussian units. The magnitudes of the vectors \( U_o \) and \( L_o \) are related through the Lorentz factor \( \gamma \), which depends on the speed parameter \( \beta \), which depends on the speed \( v \) of the non-zero rest-mass particle. Note, for \( \gamma \) much greater than unity, characteristic of a relativistic particle, the magnitudes of the vectors \( U_o \) and \( L_o \) are nearly equal. On the other hand, for \( \gamma \) close to unity, characteristic of a non-relativistic particle, the magnitude of the vector \( L_o \) is much greater than the magnitude of the vector \( U_o \).

Two other important results for the quantum mechanical and the electromagnetic cases, respectively, are

\[ E^2 = p^2 c^2 + m_o^2 c^4 \quad \Rightarrow \quad E = \pm \sqrt{p^2 c^2 + m_o^2 c^4}, \]

\[ E^2 = \beta^2 c^2 \quad \Rightarrow \quad E = pc. \]

The total energy \( E \) of the electromagnetic quantum, namely the photon, is a positive quantity. The \( \pm \) sign associated with the quantum mechanical energy \( E \) of a matter-wave particle, like an electron, was first interpreted by Paul A.M. Dirac. He recognized the negative energy levels predicted by his relativistic equation could not be ignored. This led to his concept of a hole theory of positrons. Refer to Schiff [12] for a more detailed discussion on negative energy states.
2.10. Real fields or ghost fields

The vector fields $\mathbf{E}$ and $\mathbf{B}$ previously described represent real fields. That is, these fields exert real forces on electrically charged particles like electrons, positrons, protons, and anti-protons. In addition, electromagnetic waves consist of real $\mathbf{E}$ and $\mathbf{B}$ vector fields oscillating with time.

**Question:** Do the vectors $\mathbf{U}$ and $\mathbf{L}$ represent real or ghost fields? **Answer:** Unknown.

In the de Broglie–Bohm picture of quantum mechanics, Hardy [14] and Bell [15] suggest empty waves represented by wave functions propagating in spacetime, but not carrying energy or momentum [16], can exist. This same concept was called ghost waves or ghost fields by Albert Einstein, see Selleri [16]. The controversy as to whether matter waves correspond to real waves or ghost waves has been and is still a subject of debate and controversy.

Much of the material presented in this paper is a result of the authors interest in matter waves impinging upon multilayer quantum well-and-barrier structures. Regardless, whether the vectors $\mathbf{U}$ and $\mathbf{L}$ correspond to real fields or ghost fields, the important thing to emphasize here is the theory presented in this paper still serves as an important tool in predicting how a relativistic particle and its associated matter wave interact with multilayer quantum well-and-barrier structures.

3. Results

3.1. Multilayer thin-film structures

As previously indicated, the material presented in this paper is a result of the authors interest in electromagnetic waves and matter waves reflected from and transmitted through multilayer structures. A computer software program for determining the reflectance and the transmittance characteristics of multilayer thin-film coplanar structures, illuminated by time-harmonic electromagnetic plane waves using the Maxwell spacetime matrix equation, has been written by one of the authors (RPB). Each thin-film is characterized by a thickness and an index of refraction. Appropriate boundary conditions have been taken into account. With slight modifications to this program, a second software program for computing the reflectance and the transmittance properties of multilayer quantum well-and-barrier structures, illuminated by time-harmonic matter plane waves, using the Dirac spacetime matrix equation, with scalar potentials and appropriate boundary conditions included, was also written. These two software programs were written using the Matlab [2] code (see Supplement EM and Supplement QM). The results of computations using these
programs have been compared to those results published by others in the open scientific literature with excellent agreement.

Figure 1 depicts a multilayer structure consisting of a number of thin-films with coplanar surfaces. The structure is illuminated by an incident beam which gives rise to both a reflected and a transmitted beam. Following are four examples illustrating the capabilities of the aforementioned computer programs in predicting the reflectance and the transmittance characteristics of a variety of different multilayer structures.

3.2. Example 1: a neutron impinging upon a 3-layer structure

The Dirac spacetime matrix equation is well suited for analyzing problems involving either non-relativistic or relativistic particles moving through a region of space characterized by a scalar potential \( V \) and vector potential \( \mathbf{A} \). According to Schiff [12], to introduce scalar and vector potentials into the relativistic Dirac equation for the case of a particle with electric charge \( e \), we simply make the following substitutions

\[
E \to E - eV \quad \mathbf{cp} \to \mathbf{cp} - e\mathbf{A}.
\]

For our interests, each of the thin-films in a quantum well-and-barrier structure is characterized by having either a constant scalar electrical potential energy \( P = eV \), or a scalar nuclear potential energy \( P \), as well as a thickness \( \Delta \).

The first example we wish to consider involves a neutron (zero electric charge) moving at a relativistic speed impinging upon a quantum mechanical structure containing a single nuclear potential well surrounded by two identical nuclear potential barriers. This problem was of particular interest for the following reasons: a) to determine the validity of the new Dirac spacetime matrix equation, b) to show an example of a particle tunneling through a quantum mechanical structure at a relativistic speed, c) to show how to determine the discrete energy levels of a finite potential well of nuclear dimensions, and d) to illustrate the usefulness of matrix based software packages like Matlab in solving scientific problems of this nature.

We became interested in this problem after coming across a lecture by Jenkins [17] at the University of Maryland describing a neutron trapped in a finite nuclear potential
Figure 2. A neutron inside a one-dimensional finite nuclear potential well having four discrete energy levels. The neutron makes a transition from the energy level \( n = 3 \) to the ground state level \( n = 1 \). An energetic gamma ray is emitted from the well conserving energy.

Figure 3. A neutron with kinetic energy \( K \) impinging upon a 3-layer nuclear quantum well-and-barrier structure. A neutron having kinetic energy \( K \) impinges upon the 3-layer nuclear quantum well-and-barrier structure at normal incidence. Our goal is to predict the reflectance and the transmittance of the nuclear quantum structure as a function of neutron kinetic energy \( K \) for various values of the two identical barrier widths \( w \). For this example, the Matlab software program written and based on the Dirac spacetime matrix equation was employed.

The problem we investigated involves Figure 3 depicting two identical nuclear potential barriers, each having a height of 50 MeV and a width \( w \). These barriers surround a finite nuclear potential well of width 8.0 fm and depth 50 MeV, the same as the finite nuclear potential well in Figure 2. The total energy \( E \) of a neutron is given by the special theory of relativity equations

\[
E = \gamma m_o c^2 = m_o c^2 + (\gamma - 1) m_o c^2
\]

or equivalently

\[
E = E_o + K.
\]

Here \( m_o \), \( E_o \), and \( K \) represent the rest mass, rest-mass energy, and kinetic energy of the neutron, respectively. Again, the symbol \( \gamma \) represents the Lorentz factor and \( c \) the speed of light in free space.
The rest-mass energy $E_0$ of a neutron is equal to 939.56 MeV. The kinetic energy $K$ of the incident neutron was varied between 0 MeV and 100 MeV. The barrier widths $w$ were chosen to have values between 0.125 fm and 4.000 fm. It is noted that a neutron having a 100 MeV kinetic energy is moving with a relativistic speed $v$ equal to 0.43c. Figures 4a through 4f indicate how the reflectance and the transmittance of the 3-layer nuclear quantum well-and-barrier structure vary with the kinetic energy $K$ of the neutron and the barrier widths $w$. Notice as the barrier widths $w$ increase in value, it becomes more and more difficult for the neutron to tunnel through the structure, particularly for $K$ less than 50 MeV. However, the neutron is still able to tunnel through the quantum well-and-barrier structure quite easily for $K$, less than 50 MeV, but near four energy levels approaching the four discrete energy levels of the finite nuclear potential well described in Figure 2. There the potential well can be thought of as being bounded by two potential barriers having infinite width. The four symbols $\phi$ in each of the following six figures denote the location of these four discrete energy levels of the quantum well of Figure 2 having a depth of 50 MeV for the case when the bottom of the well is chosen as the 0 MeV level.

For Figures 4e and 4f, the neutron kinetic energy $K$ was again varied from 0 MeV to 100 MeV in increments of 0.0001 MeV in the Matlab program. This corresponds to one million data points for the reflectance plot and the transmittance plot for these two figures. Decreasing the increment size improves the resolution of the plots. For improved resolution, you will find the transmittance values near the tunneling energies are 100% and the reflectance values near these same four energy levels are 0%.

Based on the results displayed in Figures 4a through 4f, if you know the discrete energy levels of a nuclear quantum well of finite depth, then it is possible to use this knowledge to design a 3-layer quantum well structure that will pass a particle (in this case a neutron) at only certain kinetic energies for finite barrier widths, thus acting like an energy filter or a velocity selector. Likewise, you can determine the allowed energy levels of a quantum well of finite depth surrounded by two barriers of infinite width by turning the problem into one involving matter wave propagation through a 3-layer quantum mechanical structure as we have done in this example.

### 3.3. Example 2: an electron impinging upon an 11-layer structure

The Matlab software program used in Example 1 has been successfully used in analyzing multilayer quantum well-and-barrier structures whose thin-films are on the order of nm (nanometer) thicknesses when the incident matter-wave particle beams have kinetic energies on the order of several electron-volts (eV). Minor scaling of thin-film thicknesses, kinetic energy and rest-mass energy of the incident
Figure 4. Reflectance and transmittance plots for a single neutron, whose kinetic energy varies between 0 MeV and 100 MeV, impinging upon a 3-layer nuclear quantum well-and-barrier structure at normal incidence. The quantum well width is equal to 8.000 fm, each barrier has a height equal to 50.0 MeV and each barrier has a width \( w \). Each of the above six figures represent different values of \( w \). In particular, (a) corresponds to \( w = 0.125 \) fm, (b) corresponds to \( w = 0.250 \) fm, (c) corresponds to \( w = 0.500 \) fm, (d) corresponds to \( w = 1.000 \) fm, (e) corresponds to \( w = 2.000 \) fm, and (f) corresponds to \( w = 4.000 \) fm.
Figure 5. This figure represents a single non-relativistic electron, whose kinetic energy $K$ varies between 0.0 eV and 4.0 eV, impinging upon an 11-layer periodic quantum well-and-barrier structure at normal incidence. The diagram (a) on the left-hand side of this figure indicates the potential barrier widths and the potential well widths are equal to 0.3 nm. Each barrier height is equal to 2.0 eV and the bottom of each quantum well is located at the 0.0 eV level. The upper diagram (b) on the right-hand side corresponds to the plot of the reflectance (%) of the multilayer structure versus the kinetic energy (eV) of the electron. The lower diagram (c) on the right-hand side corresponds to the plot of the transmittance (%) of the multilayer structure versus the kinetic energy (eV) of the electron.

particle must be taken into account in the program. In this example we consider a non-relativistic electron whose kinetic energy varies between 0.0 eV and 4.0 eV impinging upon an 11-layer periodic structure. Each thin-film in the structure has a thickness of 0.30 nm. The electron has a rest-mass energy equal to 0.511 MeV. The results of our numerical computations are presented in Figures 5a–5c. These results are in excellent agreement with results published in the open literature [18] describing the same periodic structure.

3.4. Example 3: light illuminating a 15-layer optical longwave filter

In this example we consider a beam of visible light illuminating a multilayer optical thin-film structure. For this case, the Maxwell spacetime matrix equation was employed and encoded using Matlab software. Each thin-film in the multilayer structure is characterized by having a constant index of refraction $n$ and thickness $\Delta$. The results shown in Figures 6a–6c correspond to visible light, whose wavelength varies between 400 and 700 nm, incident upon a 15-layer optical longwave pass filter at an angle of incidence equal to 30 degrees. The red and blue curves of the plots on the right-hand side of this figure are associated with the transverse electric (TE) and transverse magnetic (TM) polarization states. This example shows that angles of incidence other than 0 degrees (normal incidence) may be used in the analysis of optical thin-film structures.
This figure represents visible light, whose wavelength varies between 400 nm and 700 nm, impinging upon a 15-layer optical longwave pass filter at an angle of incidence equal to 30 degrees. The diagram on the left-hand side (a) depicts the index of refraction and thickness (nm) of each thin-film in the multilayer thin-film filter. The upper diagram (b) on the right-hand side corresponds to a plot of the reflectance (%) of the filter versus the wavelength (nm) of the visible light. The lower diagram (c) on the right-hand side corresponds to a plot of the transmittance (%) of the filter versus the wavelength (nm) of the visible light.

3.5. Example 4: light illuminating a 21-layer optical bandpass filter

The same software program employed in the analysis of the 15-layer longwave filter described in Example 3 was used to determine the reflectance and the transmittance properties of a 21-layer optical bandpass filter. Figures 7a–7c show the results of visible light, whose wavelength was also varied between 400 nm and 700 nm, incident upon a 21-layer optical bandpass filter at an angle of incidence equal to 0 degrees. The pass band was designed to be centered at 550 nm. The results presented here are in excellent agreement with those published [6] in the open literature for the same optical bandpass filter.

It is noted in passing, this electromagnetic software program allows for angles of incidence between 0 and 90 degrees. The program can also calculate the reflectance and the transmittance characteristics of multilayer optical thin-film structures for both the transverse electric (TE) and the transverse magnetic (TM) states of polarization of the incident light.

4. Discussion and conclusions

1. The four classical electromagnetic microscopic Maxwell field equations have been rewritten as a single matrix equation, referred to in this paper as the Maxwell
Figure 7. This figure represents visible light, whose wavelength varies between 400 nm and 700 nm, impinging upon a 21-layer optical narrow bandpass filter at an angle of incidence equal to 0 degrees. The diagram on the left-hand side (a) depicts the index of refraction and thickness (nm) of each thin-film in the multilayer thin-film filter. The upper diagram (b) on the right-hand side corresponds to a plot of the reflectance (%) of the bandpass filter versus the wavelength (nm) of the visible light. The lower diagram (c) on the right-hand side corresponds to a plot of the transmittance (%) of the bandpass filter versus the wavelength (nm) of the visible light.

1. The spacetime matrix equation, using the spacetime matrix operator $\hat{M}$. The Maxwell spacetime matrix equation is relativistic invariant under a Lorenz transformation.

2. The relativistic Dirac equation has been expressed as a new matrix equation, referred to as the Dirac spacetime matrix equation, using the same spacetime matrix operator $\hat{M}$. The Dirac spacetime matrix equation is also relativistic invariant under a Lorenz transformation.

3. Solutions of the new Dirac spacetime matrix equation can be easily transformed into solutions satisfying the traditional relativistic Dirac equation using the linear transformation matrix $Z$.

4. The Dirac spacetime matrix equation is equivalent to four new relativistic quantum mechanical vector equations. In the absence of scalar and vector potentials, these vector equations resemble the four classical electromagnetic microscopic Maxwell field vector equations in the absence of charge and current densities.

5. Two computer software programs, employing Matlab matrix code, have been written using both the new Maxwell spacetime matrix equation and the Dirac spacetime matrix equation. These programs have been used by the authors to analyze both electromagnetic wave propagation through multilayer optical thin-film structures and matter wave propagation through multilayer quantum well-and-barrier structures. These software programs are listed in supplemental material entitled “Supplement EM” and “Supplement QM.”
6. Numerical results predicted by these matrix based software programs are in excellent agreement with results published by others in the open scientific literature.

Declarations

Author contribution statement

Richard P. Bocker, B. Roy Frieden: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

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