Optimal non-linear transformations for large scale structure statistics

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ABSTRACT

Recently, several studies proposed non-linear transformations, such as a logarithmic or Gaussianization transformation, as efficient tools to recapture information about the (Gaussian) initial conditions. During non-linear evolution, part of the cosmologically relevant information leaks out from the second moment of the distribution. This information is accessible only through complex higher order moments or, in the worst case, becomes inaccessible to the hierarchy. The focus of this work is to investigate these transformations in the framework of Fisher information using cosmological perturbation theory of the matter field with Gaussian initial conditions. We show that at each order in perturbation theory, there is a polynomial of corresponding order exhausting the information on a given parameter. This polynomial can be interpreted as the Taylor expansion of the maximally efficient “sufficient” observable in the non-linear regime. We determine explicitly this maximally efficient observable for local transformations. Remarkably, this optimal transform is essentially the simple power transform with an exponent related to the slope of the power spectrum; when this is $-1$, it is indistinguishable from the logarithmic transform. This transform Gaussianizes the distribution, and recovers the linear density contrast. Thus a direct connection is revealed between undoing of the non-linear dynamics and the efficient capture of Fisher information. Our analytical results were compared with measurements from the Millennium Simulation density field. We found that our transforms remain very close to optimal even in the deeply non-linear regime with $\sigma^2 \sim 10$.

Key words: large-scale structure of Universe, cosmology: theory, methods: statistical

1 INTRODUCTION

The non-linear regime of structure formation in the Universe is rich in cosmological information, although the extraction of this information is a serious challenge. Traditional observables in galaxy or weak-lensing surveys, such as power spectra or two-point correlation functions, are optimal in the linear, Gaussian regime, but their statistical power decreases due to the emergence of correlations between Fourier modes (Meiksin & White 1999; Rimes & Hamilton 2002; Nevrinck et al. 2006) from non-linear dynamics. The long non-linear tails in the distribution of density fluctuations and the corresponding cosmic variance reduce the ability of observables based on moments of the field to capture the information efficiently. It has been suggested and tested with numerical simulations that non-linear transformations of the field, such as a logarithmic or a Gaussianizing map, are able to capture more efficiently this information (Nevrinck et al. 2006; Nevrinck 2011; Seo et al. 2011; Yu et al. 2011; Joachimi et al. 2011; Seo et al. 2012; Carron 2012), at least in the high signal to noise regime. This effect is magnified to dramatic extent in the lognormal model of the density field (Coles & Jones 1991). In this case it can be shown that a large fraction of the information escapes entirely the hierarchy of $N$-point moments in the large variance regime (Carron 2011; Carron & Nevrinck 2012).

Our principal aim is to build an analytic theory to quantify the ability and optimality of these transforms to capture information within the matter fluctuation field. To move beyond phenomenological models or simulations, we will use cosmological perturbation theory (Bernardeau et al. 2002). Information is of course a broad concept, and optimality must refer to some simple criteria. When it comes to inference on model parameters, the ideal measure is the Fisher information. The field possesses some definite Fisher information and in the linear, Gaussian regime, this
The information matrix of a set of observables is given by the ubiquitous Fisher matrix for Gaussian variables (Vogeley & Szalay 1996; Tegmark et al. 1997). This information is itself entirely contained within the two-point statistics of the field. Our goal is to investigate how this simple situation changes in the weakly non-linear regime.

As discussed below in some detail, Fisher information efficient observables generically strongly depend on the model parameters of interest. Presently, we restrict our investigations to local transformations. In that case, it is enough to perform the analysis of the information content of the one-point probability density function $p(\delta)$ and to determine how to capture this information efficiently with generic observables $\langle f(\delta) \rangle$. This restriction simplifies drastically the analysis and singles out the variance of $\xi$ as the sole parameter of relevance. Nevertheless, several qualitative conclusions of this work on the information in the quasi-linear regime depends only on the structure of the correlations induced by gravity with Gaussian initial conditions,

$$\xi_n \propto \xi_n^{\text{2-halo}} + \text{loop corrections},$$

and will remain unchanged in the case of a spatially correlated random field. Quantitative results in the correlated case are more involved and left for future work. In the case of the one-point probability distribution, the cumulants $\langle \delta \rangle_{\alpha}$ are given by

$$\langle \delta^n \rangle_{\alpha} = S_n \sigma_{\alpha}^{(n-1)} + \text{loop corrections},$$

where $\sigma_{\alpha}^2$ is the linear variance and the loop corrections involve only even powers of $\sigma_L$. In the hierarchical model, $\sigma_L$ is identified with the variance $\sigma$ and the parameters $S_n$ are constants. Throughout the text, the explicit expansion parameter is the non-linear, true variance of $\delta$, which makes the notation much simpler, without changing our conclusions.

In section 2 after introducing notations and definitions we discuss in general terms the information efficient observables, and their explicit form in terms of $p(\delta)$. In section 3 we show that the form of the moments induce a simple structure within the Fisher information content of the field. We discuss how to make use of this structure to obtain the observables exhausting this information, and present the leading component of this observable. In section 4 we test our findings on the Millennium Simulation density field $p(\delta)$. We end with a discussion in section 5. A set of appendices collect technical details that would break the continuity of the main text.

2 ON INFORMATION EFFICIENT OBSERVABLES

The information matrix of a set of observables $(f_1(\delta), f_2(\delta), \cdots)$ is the following

$$\sum_{ij} \frac{\partial (f_i)}{\partial \alpha} (\Sigma^{-1})_{ij} \frac{\partial (f_j)}{\partial \beta} \leq F_{\alpha\beta} = \left\langle \frac{\partial \ln p}{\partial \alpha} \frac{\partial \ln p}{\partial \beta} \right\rangle,$$

where $\Sigma_{ij} = \langle f_i f_j \rangle - \langle f_i \rangle \langle f_j \rangle$ is the covariance matrix, $F_{\alpha\beta}$ is the Fisher information content of $\delta$ for the parameters of interests, and the inequality is the Cramé-Rao inequality (an inequality between positive definite quadratic forms). In the perturbative regime, it is reasonable to expect that the information of $p$ is the same as that of the moment series. We have in that case

$$F_{\alpha\beta} = \lim_{N \to \infty} \sum_{i,j=1}^{N} \frac{\partial m_i}{\partial \alpha} [\Sigma_{N}^{-1}]_{ij} \frac{\partial m_j}{\partial \beta}, \quad m_i = \langle \delta \rangle^{(i)}$$

an equality that we assume throughout the analytical part of this work. This is equivalent to assume that the functions $\partial \alpha \ln p(\delta)$ can be expanded in powers of $\delta$ over the full range of $p$ (Carron 2011; Carron & Nevrines 2012). While this assumption appears justified, its validity lies in the (unobservable) decay rate of $p(\delta)$ at infinity: it must not be much shallower than exponential. Careful study of the analyticity properties of the moment generating function of $p(\delta)$ in Valageas (2002) and their impact on the tail suggests that strict equality does not hold in [1], as for the lognormal distribution, if the slope $n$ of the power spectrum $P(k) \propto k^n$ is negative enough. As long as $\sigma$ is small enough, the mismatch in [1] should be negligible for all practical purposes.

Our principal results are based on the assertion that for any model parameter $\alpha$ it is always possible, at least in principle, to design a single observable that will exhaust the Fisher information of the data tied to $\alpha$, represented by the row $F_{\alpha\beta}$. Likewise, it is always formally possible to design a single observable that contains as much information as the entire moment hierarchy.

Note that the above statement is significantly stronger than the more familiar assertion that in the family of Gaussian fields the power spectrum or two-point function exhausts the information on any parameters. The spectrum is a large collection of observables, one for each Fourier mode. On the contrary, we state that inference on a given number of parameters can be performed optimally with only the same number of observables. These observables are fine-tuned to the particular set of parameters, unlike the power spectrum.

2.1 Formal considerations

We can give the explicit form of the advertised designer or sufficient observables as follows. Consider the observable $o_\alpha(\delta) = \partial \alpha \ln p(\delta)$, with the understanding that $o_\alpha$ is a function of $\delta$ only, evaluated at the fiducial value of the parameter space. By definition, we have that

$$\frac{\partial (o_\alpha(\delta))}{\partial \beta} = F_{\alpha\beta},$$

This equality holds because $o_\alpha$ is treated as a constant in parameter space and $\partial \alpha p = p \partial \alpha \ln p$. Similarly, $o_\alpha(\delta)$ has zero mean, and its variance is then

$$\langle o_\alpha^2(\delta) \rangle - \langle o_\alpha(\delta) \rangle^2 = F_{\alpha\alpha}$$

by definition of the Fisher information. All in all, the information content of $o_\alpha$ is thus

$$\frac{1}{\langle o_\alpha^2(\delta) \rangle - \langle o_\alpha(\delta) \rangle^2} \frac{\partial (o_\alpha(\delta))}{\partial \alpha} \frac{\partial (o_\alpha(\delta))}{\partial \beta} = F_{\alpha\beta}.$$

We recovered the corresponding row of the Fisher matrix, showing that $o_\alpha(\delta)$ is a maximally efficient observable for
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3 INFORMATION IN THE QUASI-LINEAR REGIME

A brute force approach presents itself to explicitly determine the observable. We can expand directly expressions (4), (11) or (12) in powers of the variance. This is possible, though rather tedious and not especially enlightening. Since the optimal observables can in principle be read out from $\partial_\alpha \ln P$, the most convenient approach, exposed in the following, turns out to be the Edgeworth series of the logarithm $\ln P$ of the probability density function. This might be surprising at first. It is well known that the Edgeworth series does not necessarily converge, and if truncated at a finite order it might not even be a sensible probability density function. However, the perturbation series obtained in that way for the quantities of interest are exact, identical to those obtained from the moments, equations (4), (11) and (12). Indeed, regardless of the question of its convergence or of the behavior of the probability density it represents, the Edgeworth series produces the correct series of moments by construction. Since we are assuming the information to be entirely within the moments, this is the only relevant property of the series. It should be viewed as a formal generating function, Fisher information and undoing the non-linear dynamics. In fact, this simple argument will turn out to be remarkably successful.

2.2 The linear density contrast as a sufficient statistic

Let us make an educated guess on the plausible form of our optimal observable. It is well known that the saddle-point approximation to $p(\delta)$ is of the form (Bernardeau 1994; Bernardeau et al 2002)

$$\ln p(\delta) = \frac{\tau^2(\delta)}{2\sigma^2} - \frac{1}{2} \ln \sigma^2 + c(\delta),$$

where $-\tau$ is the linear density contrast, and $c(\delta)$ collects terms that do not depend on $\sigma^2$ and are thus irrelevant for the information. It follows that

$$\frac{\partial \ln p}{\partial \alpha} = \frac{1}{2} \frac{\partial \ln \sigma^2}{\partial \alpha} \left( \frac{\tau^2(\delta)}{\sigma^2} - 1 \right)$$

up to irrelevant constants. Thus, under this approximation, an optimal observable is simply given by

$$o_\alpha(\delta) = \tau^2(\delta)$$

where we used our freedom to subtract constants and rescale by constants. The optimal transform is then simply the mapping recovering the linear density contrast. We will be interested in power-law spectra $P(k) \propto k^n$. In this case, using the approximate form $G(\delta) = (1 + \sigma^2/\tau^2)^{-3/2} - 1$ for the vertex generating function and the relation $G_3(\tau) = g_3 (\tau [1 + g_3^2]^{-3/2} - 1)$ (Bernardeau 1994) implementing smoothing effects, one has

$$\tau(\delta) = \frac{3}{2} (1 + \delta)^{(n+3)/6} \left[ (1 + \delta)^{-2/3} - 1 \right].$$

Just as our simple examples above, the optimal observable can be chosen as the square of a Gaussian variable, and independently of the variance of the field. This illustrates in a very explicit manner the tight connection between Gaussianization, Fisher information and undoing the non-linear dynamics. In fact, this simple argument will turn out to be remarkably successful.

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In $\sigma^2$ only. The Fisher information on derived parameters is given by

$$ F_{ij} = \frac{\partial \ln \sigma^2}{\partial \alpha} \frac{\partial \ln \sigma^2}{\partial \beta} F $$

with $F = F_{\alpha=\ln \sigma^2, \beta=\ln \sigma^2}$. Due to the Gaussian initial conditions, we have

$$ F = \frac{1}{2} + \text{corrections}. \quad (20) $$

### 3.1 Structure of the information

It is remarkable, and of key significance for our purpose, that the Edgeworth series of $\ln p(\delta)$ is in fact much simpler than that of $p(\delta)$. Recall that the general form of the expansion reads (Blinnikov & Moessner 1998, see also appendix B)

$$ p(\delta) = \frac{1}{\sqrt{2\pi}e^{\delta^2/2}} \left( 1 + \Delta p(\sigma, \nu) \right) \quad (21) $$

with $\nu = \delta / \sigma$, where formally

$$ \Delta p(\sigma, \nu) = \sum_{n=1}^{\infty} \sum_{k} \left( \frac{S_k}{H_{n+k}} \right)^2 R \left( \nu \right) $$

$$ = \sigma S_6/6 H_3(\nu) + \sigma^2 \left( S_4/24 H_4(\nu) + S_3^2/72 H_6(\nu) \right) + \cdots \quad (22) $$

The second sum runs over all set of positive integers $k = (k_1, k_2, \cdots)$ such that $\sum_i (i - 2) k_i = k_3 + 2k_4 + \cdots = n$, and $|k| = \sum k_i$. The polynomials $H_n(\nu)$ are the Hermite polynomials. At each power of $n$, the polynomial of lowest degree is $H_{n+2}$ and that of highest degree is $H_{2n}$. One therefore naively expect the term of order $\sigma^n$ of $\ln p$ to be a polynomial of degree $3n$ as well. This would mean that high order moments contribute very quickly to the information. It is known (Takemura & Takeuchi 1988), however, that remarkable cancellations occur in the expansion of $\ln p(\delta)$ producing a polynomial of degree $n+2$, the lowest degree in the corresponding term in the expansion of $p(\delta)$. To second order we obtain from (22), discarding some irrelevant constants,

$$ \ln p = -\frac{\nu^2}{2} - \frac{1}{2} \ln \sigma^2 + \frac{S_6}{6} H_3(\nu) $$

$$ + \sigma^2 \left[ \frac{S_4}{24} H_4(\nu) + \frac{S_3^2}{72} (H_6(\nu) - H_3^2(\nu)) \right] + \cdots. \quad (23) $$

The term $H_6 - H_3^2$ is a polynomial of degree 4, so that the entire $\sigma^2$ term is a polynomial of degree 4. The expansion of $\ln p$ was studied in detail in the univariate and multivariate case by Takemura & Takeuchi (1988), to which we refer for details and a proof of these cancellations. To summarize, we have formally

$$ \ln p = -\frac{\nu^2}{2} - \frac{1}{2} \ln \sigma^2 + \sum_{n=1}^{\infty} \sigma^n g_{n+2}(\nu) \quad (24) $$

where $g_n(\nu)$ is a polynomial of degree $k$, which, as $H_k(\nu)$, contains only powers of $\nu$ of the same parity as $k$. We are interested in the derivatives of $\ln p$. We have

$$ \frac{\partial \ln p}{\partial \ln \sigma^2} = \frac{1}{2} H_2(\nu) + \sum_{n=1}^{\infty} \left( \frac{\sigma^n}{2} h_{n+2}(\nu) - \frac{\nu}{2} h_{n+2}(\nu) \right) $$

$$ = \sum_{n=0}^{\infty} \sigma^n r_{n+2}(\nu) \quad (25) $$

where $r_n(\nu)$ is again some polynomial of degree $k$ that includes only powers with the same parity as $k$. If loop corrections are taken into account, it is easy to see that the same result holds with $\sigma_L$ in place of $\sigma$ (with of course different polynomials $r_{n+2}$).

Let us discuss some immediate consequences of (25).

(i) The expansion of the Fisher information matrix contains only even powers of $\sigma$. It is thus truly an expansion in powers of the variance $\sigma^2$, unlike the Edgeworth series which is an expansion in $\sigma$. This follows directly from the parity properties of the polynomials entering $p$ and $\partial \ln p$.

(ii) Comparing equation (25) to its exact form

$$ \frac{\partial \ln p}{\partial \ln \sigma^2} = \frac{1}{2} H_2(\nu) + \sum_{n=1}^{\infty} \left( \frac{\sigma^n}{2} h_{n+2}(\nu) - \frac{\nu}{2} h_{n+2}(\nu) \right) $$

we infer that the leading term of $s_n$ is of order $\sigma^{n-2}$. If it were lower, then $\partial \ln p$ would contain at that lower order a term proportional to $\nu^n$, but we have seen that it does not. Besides, the expansion of $s_n$ can contain only powers of the same parity as $n$. Otherwise, the expansion of the information would also contain odd powers in the variance. We see that the independent contribution $F_{\alpha} = s^2$ of the $n$th moment to the information is $\sigma^{2n-4}$ and higher. Thus, for any $n$,

$$ F = \sum_{i,j=1}^{n+2} \frac{\partial m_{ij}}{\partial \ln \sigma^2} \left[ \sum_{n=1}^{\infty} \frac{1}{i!} \frac{1}{j!} \right] \frac{\partial m_{ij}}{\partial \ln \sigma^2} + O \left( \sigma^{2n+2} \right). \quad (26) $$

In other words, the $n$ first terms (the Gaussian information being the $0$th term) of the total information are entirely within the first $n + 2$ moments, a rather remarkable structure not obvious at first sight. It also follows immediately that we can always devise a polynomial of order $n + 2$ that captures that entire information, neglecting terms in $\sigma^{2n+4}$ and higher.

(iii) The form (25) is not limited to the case of a single variable. This property of the polynomial expansion of the joint probability $\ln p(\delta(x_1)\delta(x_2)\cdots)$ holds whenever the leading term of the joint cumulants $\xi_N$ is proportional to some expansion parameter of the adequate power,

$$ \xi_N(\delta(x_1)\cdots\delta(x_N)) \propto \epsilon^{N-1} + \cdots. \quad (27) $$

where the expansion parameter coincide with the variance in the one-dimensional case. Even though there is no such simple and explicit expansion parameter for the $N$-point functions, this is nonetheless the natural expansion in our case since $\xi_N \propto \xi_{N-1}^2 + \text{loop corrections in perturbation theory}$. We can conclude that for any $n$, the information in a spatially correlated field is as above contained within the first $n$ point functions when neglecting terms of order $2n + 2$ and higher. Again, one can define a (multivariate) polynomial of order $n + 2$ in the field that captures that entire information.

Let us now proceed with the discussion of the information efficient observables. We work out the general
structure of these observables, assuming all series of interests do actually converge. We define \( R_{nk} \) to be the coefficient of \( \nu^k \) in \( r_n(\nu) \), and similarly \( G_{nk} \) is the coefficient of \( \nu^k \) in \( g_n(\nu) \). We reorganize the series \(^2\) using \( \nu = \delta/\sigma \), multiply by \( 2\sigma^2 \) such that we will recover \( \delta^2 \) for the Gaussian distribution, and ignore the irrelevant constant term. Following these steps we obtain the following expression

\[
o(\delta) = \sum_{i=0}^{\infty} \sigma^i \left( 2 \sum_{n=1}^{\infty} R_{n+i,n} \delta^n \right).
\]

(28)

The series in parenthesis defines some function \( f_i \) of \( \delta \), so that we can write

\[
o(\delta) = f_0(\delta) + \sigma^2 f_2(\delta) + \sigma^4 f_4(\delta) + \cdots
\]

(29)

Only even powers occur due to the parity properties of the \( r \) polynomials. The function \( f_0 \) is given by the leading terms of the \( r \) polynomials, \( f_2 \) the next to leading, and so forth. Recall that \( r_n(\nu) \) is obtained from \( g_n(\nu) \) according to equation \(^2\).

Component-wise, this relation gives

\[
R_{n+2,k} = \frac{1}{2} (n-k) G_{n+2,k}, \quad n \geq 1.
\]

(30)

Using this relation, in terms of the matrices \( G \) the different observables \( f_i \) takes the simple form

\[
f_i(\delta) = (i-2) \sum_{n=1}^{\infty} G_{n+i,n} \delta^n.
\]

(31)

In particular, \( f_2(\delta) \) is zero, such that

\[
o(\delta) = f_0(\delta) + \sigma^4 f_4(\delta) + \cdots
\]

(32)

Note that the powers of \( \sigma \) in \(^2\) do not reflect at which order they enter the information. We will see below that \( f_4(\delta) \) only contributes to \( \sigma^6 \). The leading component \( f_0 \) is

\[
f_0(\delta) = \delta^2 - 2 \sum_{n=3}^{\infty} G_{n,0} \delta^n.
\]

(33)

It does contain only the leading hierarchical cumulants, and no term linear in \( \delta \).

For further reference, we have directly from \(^2\)

\[
g_3(\nu) = \frac{S_3}{6} (\nu^3 - 3\nu)
\]

\[
g_4(\nu) = \frac{S_4}{24} (\nu^4 - 6\nu^2 + 3) - \frac{S_2^2}{24} (3\nu^4 - 12\nu^2 + 5),
\]

(34)

loop corrections entering only \( g_2 \) and higher. In appendix \[^3\] we give \( g_5 \) and \( g_6 \) within the hierarchical model.

### 3.2 Constructing the optimal observable

Given the structure displayed above, it is simple to obtain the information content including all terms below a given power of the variance, as well as the observables exhausting entirely this information. These quantities can be read out from the polynomials \( g_n(\nu) \) entering \( \ln p \). To obtain the information content including up to \( \sigma^6 \), one has to keep track of the polynomials \( g_3(\nu) \) up to \( g_{2+n}(\nu) \). Interestingly, to obtain the optimal observable capturing that same information, it is enough to truncate at \( g_{2+n}(\nu) \). For instance, with the explicit form of \( g_3 \) and \( g_4 \) in equation \(^3\), we can obtain the first, \( \sigma^2 \) term of the total information \( F \), but also the observable capturing the first two terms of \( F \), proportional to \( \sigma^2 \) and \( \sigma^4 \). In this respect, it is twice as simple to obtain the optimal observable than the total information content.

We prove this non-trivial but convenient fact in appendix \[^4\].

These considerations lead immediately to one of the main results of this paper. Reading out \( g_3(\nu) \) and \( g_4(\nu) \) in \(^2\), we have that

\[
o(\delta) = \delta^2 - \frac{S_3}{3} \delta^3 + \frac{1}{12} (3S_3^2 - S_4) \delta^4
\]

(35)

captures the entire information, when neglecting terms of order \( \sigma^6 \) and higher in the expansion of \( F \). Remarkably, to that order only \( f_0(\delta) \) contributes to the information, such that the optimal observable is still independent of the variance of the field and of loop corrections. This situation changes only when interested in capturing the \( \sigma^6 \) term or higher in the information, where \( f_4(\delta) \) or higher are necessary. We discuss \( f_4(\delta) \) in appendix \[^5\] and give there its first two coefficients, proportional to \( \delta \) and \( \delta^2 \), in the hierarchical model.

Since \( f_0(\delta) \) completely dominates the information, we focus on this observable in the following. We derive in appendix \[^6\]

\[
f_0(\delta) = \sum_a a_n \delta^n_{\text{with}}
\]

\[
a_n = \frac{2}{n!} \sum_{|k|=0}^{n} (-1)^{|k|} (n - 2 + |k|) \prod_{i=0}^{12} \frac{S_i^{k_i}}{(i - 1)^{k_i} k_i!}
\]

(36)

where the second sum runs over all vectors of positive integers \( k = (k_3, k_4, \cdots) \) of any dimension such that \( \sum_{i=0}^{12} i k_i = n - 2 \), and where \( |k| \) stands for \( \sum_{i=0}^{12} k_i \). We already derived \( a_n \) for \( n = 0 \) to \( n = 4 \). Further, for \( n = 5 \), contributing are \( k = (0,0,1),(1,1),(3) \), and for \( n = 6 \), \( k = (0,0,0,1),(1,0,1),(0,2),(2,1),(4) \). These coefficients can also be read out from \( g_5 \) and \( g_6 \) in equation \(^3\). The full list of the first six Taylor coefficients are given by

\[
a_0 = a_1 = 0, \quad a_2 = 1
\]

\[
a_3 = \frac{S_3}{3}, \quad a_4 = \frac{1}{12} (3S_3^2 - S_4)
\]

\[
a_5 = \frac{S_5}{60} + \frac{S_6}{6} \quad a_6 = \frac{S_6}{360} + \frac{S_6}{24} + \frac{S_4^2}{36} - \frac{7}{24} S_3^2 S_4 + \frac{7}{24} S_4^2.
\]

(37)

In Fig. 1 \[^7\] we show the value of these coefficients for a power-law spectrum \( P(k) \propto k^n \), using the values of \( S_3 \) to \( S_6 \) from Bernardeau (1994). It is clear from the figure that the leading optimal observable is a strong function of \( n \), and the smaller \( n \) is, the stronger function of \( \delta \). Remember that \( f_0(\delta) \) does not contain a linear term, so that we can tentatively write it as the square of the non-linear transformation. The cumulants are very small for \( n \sim 2 \), and it is known that the lognormal distribution is a good match to \( p(\delta) \) for \( n \sim -1 \). The following observable therefore suggests itself

\[
\omega_n^2(\delta) = \left( \frac{(1 + \delta(n+1)/3 - 1)}{(n+1)/3} \right)^2.
\]

(38)
that interpolates between no transformation, \( \omega_2(\delta) = \delta \) for \( n = 2 \), and the exact logarithmic mapping, \( \omega_{-1}(\delta) = \ln(1 + \delta) \) for \( n = -1 \). Note that \( \omega_n(\delta) \) is simply the power (Box-Cox) transformation of \( \delta \) with exponent \((n + 1)/3\).

The agreement between \( \omega_n(\delta) \), shown as the dotted line on the figure and the exact coefficients is remarkable for any value of interest. As discussed earlier, one should expect according to the saddle-point approximation to \( p(\delta) \) the leading observable to be \( \tau^2(\delta) \), where \( \tau \) is the linear density. The dashed line on figure 1 shows the coefficients of \( \tau^2 \), according to the approximation (18). The agreement is again excellent, confirming our expectations.

### 3.3 Leading non-Gaussian information

It is worth discussing the total information content of \( p(\delta) \) in order to make a connection to previous results in the literature and to illustrate ours. One way to obtain the total information is from the decomposition (25) together with \( F = \partial \alpha_n / \partial \ln p \), with \( \partial \alpha_n / \partial \ln p \) fixed in parameter space. Expanding the polynomials in terms of their matrix elements gives us

\[
F = \sum_{n,k=0}^{\infty} \sigma^n R_{n,k+2} \frac{1}{\sigma^k} \frac{\partial m_k}{\partial \ln \sigma^2}.
\]

With \( g_3 \) and \( g_4 \) in (34), and \( r_2 = H_2/2 \), we get

\[
F = R_{22} + \sigma^2 (6R_{44} + 2S_3 R_{33}) + O(\sigma^4)
\]

\[
= \frac{1}{2} - \frac{1}{4} \sigma^2 (S_4 - S_3^2) + \sigma^2 S_3^2/6 + O(\sigma^4)
\]

For clarity, we separated in the last line the contribution to the information from the second and third moments. The first term, 1/2, is the Gaussian information, the second represents the change in the covariance of the second moment due to non-Gaussianity. The presence of the kurtosis \( S_4 \) is expected, since it enters directly the variance of \( \delta^2 \). One way to understand the modulation with \( -S_3^2 \) is that a linear piece could be added to \( \delta^2 \), with zero mean but reducing its variance. The third term is the independent information content of the third moment, derived in the multivariate setting in Taylor & Watts (2001); our second term is a generalization to their results.

The reason for the difference is that Taylor & Watts (2001) truncate the expansion of \( p(\delta) \) after the first order term. We expand \( F \) in powers of the variance, therefore we include all terms up to order \( \sigma^2 \). As a consequence, in contrast to the conclusions of Taylor & Watts (2001), the leading change in \( F \) is not necessarily positive, but can have any sign depending on the value of the cumulants \( S_3 \) and \( S_4 \).

According to our reasoning in the last section, the observable \( \delta^2 - S_3^2 \delta^4 \) captures the entire expression in (40). This can be illustrated simply, providing us with a simple sanity check of our methods and results. Consider more generically the observable

\[
f(\delta) = \delta^2 + a_3 \delta^4
\]

as a function of \( a_3 \). We have \( \langle f \rangle = \sigma^2 + a_3 S_3 \sigma^4 \). It follows

\[
\left( \frac{\partial \langle f \rangle}{\partial \ln \sigma^2} \right)^2 = \sigma^4 \left( 1 + 4a_3 S_3 \sigma^2 + O(\sigma^4) \right)
\]

\[
\langle f^2 \rangle - \langle f \rangle^2 = m_4 + 2a_3 m_3 + a_3^2 m_2 - (\sigma^2 + a_3 S_3 \sigma^4)^2
\]

\[
= \sigma^4 \left[ 2 + \sigma^2 (S_4 + 18S_3 a_3 + 15a_3^2) + O(\sigma^4) \right].
\]

Building the ratio, the information content of \( f \) becomes

\[
\frac{1}{2} + \sigma^2 \left( \frac{S_4}{4} - \frac{5}{2} S_3 - \frac{15}{4} a_3^2 \right) + O(\sigma^4)
\]

\[
= \frac{1}{2} - \frac{1}{4} \sigma^2 \left( S_4 - S_3^2 \right) + \sigma^2 S_3^2/6
\]

\[
- \frac{15}{4} \sigma^2 \left( a_3 + S_3/3 \right)^2 + O(\sigma^4).
\]

Clearly, this expression reaches a maximum precisely when \( a_3 = S_3/3 \), when we recover the total information (40). Equation (40) is still independent of loop corrections. In appendix C we give the next term in the expansion of \( F \) within the hierarchical model.

### 4 TESTS TO SIMULATIONS

We used the publicly available matter density field from the Millennium Simulation (Springel et al. 2005) to estimate the information content of our optimal observables. We calculated the probability distribution function \( p(\delta) \) in the \( z = 0 \) \( \Lambda \)CDM dark matter field of 500h\(^{-1}\)Mpc box size on a 256\(^3\) grid running the cumulative grid algorithm (see Szapudi 2004, for details) on several scales \( i \times 1.95h^{-1}\)Mpc, with \( i = 1 \ldots 29 \). In addition, we measured moments, negative and log moments, and cumulants directly from the grid. Our accuracy for \( p(\delta) d\delta \) was \( 6 \times 10^{-6} \) for each scale with \( d\delta = 0.001 \), and we checked that from \( p(\delta) \) we can recover the grid-direct moments, negative and log moments to sub-percent accuracy. Note that Poisson noise was negligible even on our smallest base scale, 1.95h\(^{-1}\)Mpc, with average...
count of about 600 particles per cell. We then obtained the derivatives $\partial_{\ln \sigma^2} \ln p(\delta)$ and the slope $n = -3 - \partial_{\ln R} \ln \sigma^2$ at each scale, using finite differences. With these derivatives we evaluate the total information content $F = \langle (\partial_{\ln \sigma^2} \ln p)^2 \rangle$ and that of our observables, implementing straightforwardly the formulae in Eq. 3.

The crosses on figure 3 show the total information of $p(\delta)$. It might be surprising that it does not asymptote to the Gaussian value 1/2, but is in fact slightly higher, $F \approx 0.6$. We found $p(\delta)$ had significant cosmic variance on these scales due to the relatively small volume of the simulations. For this reason, the shape of $p(\delta)$ contains some artificial features, deviating slightly from an exact Gaussian. These features propagate to the derivatives of $p(\delta)$. Any feature in the derivatives contributes to the total information, which is why $F$ is slightly higher than the Gaussian value. These features however do not contribute to the information within the smooth observables $\omega^2_n(\delta)$, $\tau^2(\delta)$, and $\ln^2(1 + \delta)$, shown as the upper solid, dashed and dotted lines. These curves tend to 1/2 as expected. It is striking, and somewhat unexpected, that these three observables remain in fact essentially optimal throughout the entire dynamic range. They still capture as much as 90% of the total information when $\sigma^2 \approx 10$. On the other hand, the information of $\delta^2$ and the combined information of $\delta^2$ and $\delta^3$ (lower solid lines). They show the steep decay characteristic of distributions with heavy tails. On the largest scales, the total information is noticeably higher than the Gaussian value 1/2. This is due to artificial information originating from cosmic variance, entering high order statistics but not the smooth observables shown on the figure (See Section 4 in the text).

5 CONCLUSIONS

Let us discuss our main results and future prospects. We presented a rigorous approach to understanding the information content of the density field evolving from Gaussian initial conditions. We described how one can obtain the maximally efficient, "sufficient", observables. We showed that the structure of the moments under the action of gravity in the quasi-linear regime makes a clear prediction of the shape of these observables, and of the associated non-linear transformations. To a very good approximation, optimal observables can be chosen independently of the variance of the field. They are for this reason fundamental observables associated to a set of hierarchical cumulants, and can be applied to data with minimal fine-tuning, a most desirable property. The optimal mapping depends on the slope of the power spectrum, coinciding with the logarithmic mapping only if $n = -1$. We found with the help of the Millennium Simulation that in practice the slope is close enough to $-1$ for the scales of interest so that the logarithmic mapping remains essentially optimal. We established in this way a direct connection between the optimal transformation, the linear density contrast $-\tau$, the well-known logarithmic mapping $\ln(1 + \delta)$, and its generalization to other slopes, the Box-Cox transformation $\omega_n(\delta)$. The success of these transforms lies within perturbation theory. Three different facets of non-linear transforms, the undoing of the non-linear dynamics, the capture of information on parameters, and the Gaussianization of the field become unified in this picture. The methods we expose in this work are fairly general and
we conjecture that they can be useful in a variety of situations. We have concentrated on the one-point \( p(\delta) \), with the justification that we were interested in local transformations. A next logical step would be to investigate in more detail the case of a spatially correlated field, making similar use of the high level of structure displayed in \( \ln p \). The required calculations are more tedious but entirely analogous to what has been presented. In fact, a first approximation for the total Fisher information can be obtained by simply multiplying our results with \( n_p \), the number of pixels (or the number of effective pixels). Our techniques can be relatively straightforwardly adapted to other random fields, such as weak lensing convergence, or CMB maps. The theory is general enough that optimal observables can be constructed for Poisson or sub-Poisson scatter encountered in galaxy catalogs and simulations, or to deal with practical issues such as redshift distortions, bias (especially in the context of the halo model), and projection effects for 2-dimensional surveys. Also, non-Gaussian initial conditions can be implemented in this approach. These and other possible generalizations are left for subsequent research.

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APPENDIX A: PERTURBATIVE EXPANSION OF THE OPTIMAL OBSERVABLE

In order to calculate the perturbation series of the Fisher information including \( \sigma^n \), it is necessary to keep track of the terms of that same order in \( \partial_{\alpha} \ln p \). Nevertheless, we used the fact in the text that in order to read out observables capturing that same information, it is enough to truncate the series of \( \partial_{\alpha} \ln p \) at \( \sigma^n \). We prove this assertion next.

Writing schematically the expansion of \( p(\delta) \) and \( \partial_{\alpha} \ln p(\delta) \) as

\[
p = p_G \left( 1 + \sum_{i,j>0} p^{(i)} \right) \frac{\partial \ln p}{\partial \alpha} = \sum_{i,j>0} \frac{\partial \ln p^{(i)}}{\partial \alpha},
\]

consider the observable obtained by truncating the series at order \( n \),

\[
o_{\alpha}(\delta) = \sum_{k=0}^{n} \frac{\partial \ln p^{(k)}}{\partial \alpha}.
\]

We have then formally

\[
\frac{\partial \langle \phi \rangle}{\partial \alpha} = \sum_{i,j=0}^{n} \sum_{k=0}^{n} p_{ijk},
\]

where we discarded summation indices higher than \( 2n \), and

\[
p_{ijk} = \langle p^{(i)} \left( \frac{\partial \ln p^{(j)}}{\partial \alpha} \right) \left( \frac{\partial \ln p^{(k)}}{\partial \alpha} \right) \rangle.
\]

Similarly, neglecting such higher order terms, its variance reduces to

\[
\langle \sigma_{\alpha}^{2}(\delta) \rangle - \langle o_{\alpha}(\delta) \rangle^{2} = \sum_{i=0}^{2n} \sum_{j,k=0}^{n} p_{ijk}.
\]

Writing now

\[
\sum_{i,j=0}^{2n} \sum_{k=0}^{n} p_{ijk} = \sum_{i=0}^{2n} \sum_{j,k=0}^{n} p_{ijk} + \sum_{i=0}^{2n} \sum_{j=0}^{n} \sum_{k=0}^{n} p_{ijk}
\]

Its information content becomes

\[
\frac{\langle \sigma_{\alpha}^{2}(\delta) \rangle - \langle o_{\alpha}(\delta) \rangle^{2}}{\langle o_{\alpha}^{2}(\delta) \rangle} = \sum_{i=0}^{2n} \sum_{j,k=0}^{n} p_{ijk} + 2 \sum_{i=0}^{2n} \sum_{j>0}^{n} \sum_{k=0}^{n} p_{ijk}.
\]
where again we suppressed terms explicitly higher than \(2n\). Since \(p_{i,j,k}\) is symmetric in the last two indices, we can conclude that its information \((\text{A}6)\) is simply

\[
\sum_{i,j,k=0}^{2n} p_{i,j,k} = \sum_{i+j+k=2n} p_{i,j,k} + O(\sigma^{2n+1}) \tag{A7}
\]

which is exactly the Fisher information content \(\langle p(\partial, \ln p)^2 \rangle\) of \(p\) to order \(2n\).

**APPENDIX B: MATRIX APPROACH TO THE INFORMATION CONTENT AND OPTIMAL OBSERVABLES**

We used in the main text the Edgeworth series of \(\ln p(\delta)\) around the Gaussian distribution to derive our results. We describe in this appendix a method that is more involved though completely general, based uniquely on the explicit expressions of the moments. We discuss then why these two methods are in our case equivalent, irrespectively of the behavior of the function represented by the Edgeworth series.

This method is based on the expressions \((\text{B1})\) and \((\text{B4})\), which is the expansion of the information into the orthonormal series of polynomials, valid whenever the moment series determine uniquely \(p(\delta)\). Expanding the polynomials in powers of \(\delta\), we have

\[
P_n(\delta) = \sum_{k=0}^{n} C_{n,k} \delta^k \quad (\text{B1})
\]

and

\[
s^{\alpha}_n = \sum_{k=1}^{n} C_{n,k} \delta k \quad (\text{B2}).
\]

For each \(N\), the triangular matrix \(C_{n,k} = n, k = 0, \ldots, N\) is the Cholesky factor of the inverse moment matrix of the same size,

\[
M^T C = M^{-1} \quad (\text{B3})
\]

Thus, if we obtain a perturbation series for the matrix \(C\), we can find both \(P_n(\delta)\) and \(s^{\alpha}_n\). The optimal observable and Fisher information are then obtained by the relations \((\text{B7})\) and \((\text{B3})\). To obtain a perturbation series of the matrix \(C\), we can proceed as follows. In a first step, consider that the total moment matrix is as in our case the sum \(M = \bar{M} + \delta M\) of a reference moment matrix \(\bar{M}\), whose inverse has Cholesky factorisation \(\bar{M}^{-1} = C^T \bar{C}\), plus a perturbation \(\delta M\). We can write

\[
M^{-1} = (\bar{M} + \delta M)^{-1} = \bar{C}^T \left(1 + \bar{C} \delta M \bar{C}^T\right)^{-1} \bar{C}. \tag{B3}
\]

If the perturbation is small enough follows

\[
M^{-1} = \bar{C}^T \sum_{k=0}^{\infty} (-1)^k A^k \bar{C}, \tag{B4}
\]

where we defined \(A = \bar{C} \delta M \bar{C}^T\). Note that \(A\) also has the following more insightful interpretation. It holds

\[
A_{nm} = \langle P_n P_m \rangle - \delta_{nm}, \tag{B5}
\]

where the average is with respect to the true probability density function. This is easily shown using the fact that the reference polynomials are given by \(\bar{P}_n(\delta) = \sum_k \bar{C}_{nk} \delta^k\). Thus, the matrix \(A\) directly measures the deviations from orthogonality of the reference polynomials.

In a second step, we seek to find the matrix \(C\) associated to \(P\). It is convenient to introduce the following notation, borrowed from Stewart (1997) who discusses the first order perturbation to the Cholesky decomposition. Given a matrix \(A\), the matrix \(L_{1/2}A\) is the lower triangular matrix obtained by zeroing the part above the diagonal, and multiplying by 1/2 the diagonal itself. Likewise, \(U_{1/2}A\) is the upper triangular matrix obtained by zeroing the entries below the diagonal and multiplying by 1/2 the diagonal. If \(A\) is symmetric we have that \([L_{1/2}A]^T = U_{1/2}A\). Putting formally

\[
C = \left(1 + \sum_{k=1}^{\infty} L_{1/2}Q^{(k)}(1)\right) \bar{C} \tag{B6}
\]

for some yet unknown symmetric matrices \(Q^{(k)}\), the requirement \(C^T C = M^{-1}\) gives us from equation \((\text{B3})\) the following relation at order \(k\),

\[
(-1)^k A^k = U_{1/2}Q^{(k)} + L_{1/2}Q^{(k)} + \sum_{i+j=k} U_{1/2}Q^{(i)} L_{1/2}Q^{(j)} \tag{B7}
\]

By definition of \(L_{1/2}\) and \(U_{1/2}\), we have \(L_{1/2}Q^{(k)} + U_{1/2}Q^{(k)} = Q^{(k)}\). We obtain thus the recursion relations

\[
Q^{(1)} = -A \tag{B8}
\]

\[
Q^{(k)} = (-1)^k A^k - \sum_{i+j=k} U_{1/2}Q^{(i)} L_{1/2}Q^{(j)}. \tag{B8}
\]

These relations together with \((\text{B6})\) allow the perturbative calculation of the polynomials. We have the formal series

\[
P_n(\delta) = \bar{P}_n(\delta) + \sum_{i=1}^{\infty} \sum_{m=0}^{n} L_{1/2}Q^{(i)}(1) \bar{P}_m(\delta). \tag{B9}
\]

Note that in general not all entries of the perturbation \(\delta M\) are of the same order, so that the perturbation series must be further reorganized in order to obtain a consistent expansion.

A connection to the Edgeworth series is the following. In our case, the reference distribution is the Gaussian with zero mean and variance \(\sigma^2\). The orthonormal polynomials are the (suitably rescaled) Hermite polynomials

\[
\bar{P}_n(\delta) = \frac{1}{\sqrt{n!}} H_n(\nu). \tag{B10}
\]

It is then an interesting if somewhat lengthy exercise of algebra to derive the matrix \(1 + A = \bar{C} M \bar{C}^T\), using only the explicit expression of the moments

\[
m_n = \sum_{2k_1 + 2k_2 + \cdots + n} \sigma^{2n-2|k|} \prod_{i \geq 2} \frac{\phi_i^{k_i}}{(k_i!^2)}, \tag{B11}
\]

(with the understanding that \(S_2 = 1\), and \(|k| = \sum_i k_i\)) and that of the Hermite polynomials

\[
H_n(\nu) = n! \sum_{2k \leq n} \frac{(-1)^k \nu^{n-2k}}{k!(n-2k)!}. \tag{B12}
\]
The result can be written as follows

$$\langle CMCG^T \rangle_{ij} = \frac{1}{\sqrt{1!j!}} \sum_{n=0} \sigma^n \sum_{k_3,k_4,\ldots} (i; j; n+2|k|) \prod_{p=3} \left( \frac{S_p}{p!^{\nu^2 p} k_p!} \right)$$

(B13)

where the sum runs over all set positive integers $(k_3,k_4,\ldots)$ with the condition $\sum (i - 2)k_i = n$, and the symbol $(i; j; k)$ is the integral of three Hermite polynomials with respect to the Gaussian distribution of unit variance,

$$(i; j; k) = (H_i(\nu) H_j(\nu) H_k(\nu))_{G}.$$  

(B14)

In this representation, it can be readily verified that all sums contain only a finite number of terms and thus do not suffer any ambiguities. This is because this integral of three Hermite polynomials satisfy the triangle conditions, non zero only for $|i - j| \leq k \leq |i + j|$ at fixed $i$ and $j$. The values of $n$ covers only a finite range at each $i$ and $j$. However, using the integral representation \[134\] of the symbols, one can try put the full sums under the integral $\langle \cdots \rangle_G$, getting precisely

$$\langle CMCG^T \rangle_{ij} = \frac{1}{\sqrt{1!j!}} \langle H_i(\nu) H_j(\nu) (1 + \Delta p(\sigma, \nu)) \rangle_G$$  

(B15)

where $\Delta p$ is the Edgeworth series as in equation \[21\]. According to \[135\], $\langle CMCG^T \rangle_{ij}$ must be equal to $\langle H_i H_j \rangle / \sqrt{1!j!}$, where the average is with respect to the true $p(\delta)$, or a $p(\delta)$ with the correct series of moments when this is not unique. Comparing with \[135\], we have thus recovered here the Edgeworth representation of $p(\delta)$ in the case of convergence of that series. If the series for $\Delta p$ does not converge, then the representation \[135\] does not make sense, but the exact \[135\] still does, coinciding with the formal expansion of the Edgeworth series in \[134\]. This justifies the use of the Edgeworth series in this work, since it reproduces the correct perturbation series in all cases.

**APPENDIX C: EDGEWORTH SERIES AND INFORMATION**

We provided in the text the first two terms of the Edgeworth series of $\ln p$, allowing the calculation of the Fisher information including $\sigma^2$, and of the observable capturing the information including $\sigma^4$. We list here for completeness the next two terms of $\ln p$ as well, allowing us to read out the next two terms in the optimal observables and the next term in the total information. We then obtain the Taylor expansion of the information dominant observable $f_0(\delta)$ at all orders. We work for simplicity within the hierarchical model. Loop corrections would enter $g_3(\nu)$ and $g_6(\nu)$. The corresponding adaptations of $f_3(\delta)$ and of the $\sigma^4$ terms of $F$ would be required if their inclusion is desired.

Viewed as a power series in $\sigma$, the relation between $1 + \Delta p$ in equation \[21\] and $\ln p$ is the same as the relation between a moment generating function and a cumulant generating function. The first four terms of $\Delta p$ reads

$$\Delta p = \sigma \frac{S_3}{6} H_3 + \sigma^2 \left( \frac{S_4}{1296} H_3 + \frac{S_4}{144} H_7 + \frac{S_4}{120} H_5 \right)$$

$$+ \sigma^3 \left( \frac{S_4}{31104} H_{12} + \frac{S_4}{1728} H_{10} + \frac{S_4}{1152} H_8 + \frac{S_4}{720} H_8 + \frac{S_4}{720} H_6 \right)$$

$$+ \sigma^4 \left( \frac{S_4}{5} H_{16} + \frac{S_4}{4} S_4 H_{12} + \frac{S_4}{2} S_4 H_8 + \frac{S_4}{2} S_4 H_8 + \frac{S_4}{2} S_4 H_6 \right).$$

From this, we get

$$\ln p = -\frac{\nu^2}{2} - \frac{1}{2} \ln (2\pi \sigma^2) + \sum_{n=1}^4 \sigma^n g_{2+n}(\nu) + O(\sigma^5),$$

with

$$g_3(\nu) = \frac{S_3}{6} (\nu^3 - 1)$$

$$g_4(\nu) = \frac{S_4}{24} (\nu^4 - 6\nu^2 + 3) - \frac{S_4^2}{24} (3\nu^4 - 12\nu^2 + 5)$$

$$g_5(\nu) = \frac{S_5}{120} (\nu^5 - 10\nu^3 + 15\nu) - \frac{S_5 S_4}{12} (\nu^5 - 7\nu^3 + 8\nu)$$

$$+ \frac{S_5^2}{24} (3\nu^5 - 16\nu^3 + 15\nu)$$

$$g_6(\nu) = \frac{S_6}{720} (\nu^6 - 15\nu^4 + 45\nu^2 - 15)$$

$$- \frac{S_6 S_4}{48} (\nu^6 - 11\nu^4 + 25\nu^2 - 7)$$

$$- \frac{S_6^2}{144} (2\nu^6 - 21\nu^4 + 48\nu^2 - 12)$$

$$+ \frac{S_5 S_4}{48} (7\nu^6 - 59\nu^4 + 109\nu^2 - 25)$$

$$- \frac{S_6^2}{48} (7\nu^6 - 48\nu^4 + 75\nu^2 - 15),$$

(C1)

in perfect agreement with \[136\]. From these polynomials we read out the observable optimal

$$a(\delta) = f_0(\delta) + \sigma^4 f_4(\delta)$$

(C2)

where $f_0 = \delta^2 + \sum_{n=3} a_n \delta^n$ is given explicitly in the main text, and

$$f_4(\delta) = \delta \left( \frac{S_4}{4} - \frac{4}{3} S_4 S_3 + \frac{5}{4} S_3^2 \right)$$

$$+ \delta^2 \left( \frac{S_6}{8} - \frac{25}{24} S_4 S_3 - \frac{2}{3} S_3^4 + \frac{109}{24} S_4 S_3^3 - \frac{25}{8} S_3^4 \right)$$

(C3)

capturing the entire information including $\sigma^8$ and lower order terms. On the other hand, we can obtain the total information from these polynomials including $\sigma^4$. Separating the independent contribution from each moment

$$F = F_2 + F_3 + F_4 + O(\sigma^6), \quad F_n = s_n^2,$$

(C4)

we get

$$F_2 = \frac{1}{2} - \frac{\sigma^2}{4} (S_4 - S_3^2) + \frac{\sigma^4}{8} (S_4 - S_3^2)^2 + O(\sigma^6)$$

$$F_3 = \sigma^2 \frac{S_3^2}{6} + \sigma^4 \left( \frac{S_3^4}{6} - \frac{11}{12} S_3 S_3^3 + \frac{3}{4} S_3^4 \right) + O(\sigma^6)$$

(C5)

$$F_4 = \frac{\sigma^4}{24} (S_4 - 3S_3^2)^2 + O(\sigma^6)$$

We can give explicitly the leading term of $F_n$, simply from reading the leading coefficients of the $g$ polynomials,

$$F_n = n! G_{2n}^g \sigma^{2n-4} + O(\sigma^{2n-2}).$$

(C6)

This follows from comparing $F_n$ to its exact form \[23\], as in point (ii) in section \[24\]. The leading term of $P_n(\nu)$ is $H_n(\nu)/\sqrt{n!}$ (see \[137\]). Since $H_n(\nu) = \nu^n + \cdots$ we infer

$$s_n = \sqrt{n! R_{n\nu}} \sigma^{n-2} + O(\sigma^n).$$

(C7)
From $R_{nn} = -G_{nn}$ and $F_n = s_n^2$ follows our claim (C6).

We now turn to the derivation of the variance independent, loop corrections independent

$$f_0(\delta) = \sum_{n \geq 2} a_n \delta^n.$$ From our results in the main text, we have

$$a_n = -2G_{nn}, \quad (C8)$$

where $G_{nn}$ is the leading coefficient of the polynomial $g_n(\nu)$ entering the Edgeworth series of $\ln p$. Again, we can make use of the results of Takemura & Takeuchi (1988), who obtained in their equations (2.25) and (2.34) the necessary ingredients. In their notation and conventions, their $\beta_i$ corresponds for us to

$$\beta_i = \sigma^{i-2} \frac{S_i}{\sigma^i}. \quad (C9)$$

They derive that the leading coefficient of the polynomials accompanying $\beta_3^{k_3} \beta_4^{k_4} \cdots$ is given by (writing $k = (k_3, k_4, \cdots)$)

$$c(k) = (-1)^{|k|-1} \left( \prod_{i \geq 3} i^{k_i} \right) \prod_{j \geq 0} \left( \sum_{i \geq 3} (i-1)k_i - j \right), \quad (C10)$$

where the product is unity if $|k| = 2$. If $|k| = 1$ the leading coefficient is 1. The power of $\sigma$ accompanying this term is according to (C9) equal to $\sum_{i \geq 3} (i-2)k_i$. In the series for $\ln p$, the polynomial $g_n$ multiply $\sigma^{n-2}$. Fixing thus the order $n - 2$, we get that the leading coefficient of $g_n$ is given by

$$G_{nn} = \sum_k \prod_{i \geq 3} \left( \frac{S_{k_i}}{i^{k_i} k_i!} \right) c(k), \quad (C11)$$

where the sum includes all vectors of positive integers $k$ such that

$$\sum_{i \geq 3} (i-2)k_i = n - 2. \quad (C12)$$

Elementary manipulations leads to

$$G_{nn} = \frac{1}{n!} \sum_k (-)^{|k|-1} (n - 2 + |k|)! \prod_{i \geq 3} \left( \frac{S_{k_i}^{k_i}}{(i-1)! k_i!} \right), \quad (C13)$$

Equation (C13) follows.