Gravitational Radiation Reaction to a Particle Motion II
- Spinning Particle -

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We discuss the leading order correction to the equation of motion of a particle with spin on an arbitrary spacetime. A particle traveling in a curved spacetime is known to trace a geodesic of the background spacetime if the mass of the particle is negligibly small. For a spinning particle, it is known that there appears a term due to the coupling of the spin and the Riemann tensor of the background spacetime. Recently we have found the equation of motion of a non-spinning particle which includes the effect of gravitational radiation reaction. This paper is devoted to discussion of a consistent derivation of the equation of motion which is corrected both by the spin-Riemann coupling and the gravitational radiation reaction.

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I. INTRODUCTION

The gravitational waves from an inspiralling compact binary is one of the most promising sources expected to be detected by the near-future interferometric gravitational wave detectors such as LIGO, VIRGO, GEO, TAMA and LISA. In order to extract the information of a binary from the last inspiralling stage, we need accurate theoretical templates of the gravitational wave forms. As an approach, perturbations of a black hole by an orbiting point particle have been studied.

Most of the previous papers considered the case that the orbiting point particle is structureless and is characterized only by a mass parameter. In those papers, the point particle is defined by using the Dirac delta function, therefore the induced perturbation diverges at the particle. This leads to some conceptual problems in the perturbation study; 1) how far we can incorporate the strong non-linearity around the particle, and 2) how definitely we can define the motion of the particle though the perturbed metric diverges at the particle.

In a previous paper we have considered these problems in the case of a non-spinning particle, extensively using the technique of the matched asymptotic expansion that had been developed by many authors (e.g., D’Eath and Thorne and Hartle). The matched asymptotic expansion is a technique to match physical quantities in different perturbation schemes. In the case of a non-spinning particle, the procedure goes as follows. We first construct the metric in two different regions. We assume the metric around a particle (body’s neighborhood) to be approximated by a Schwarzschild geometry plus its perturbation describing tidal distortions due to the background curvature (internal scheme or body expansion). In this region the metric is expanded in powers of $r/L$, where $r$ is the circumference radius measured in the local inertial frame of the background and $L$ is the characteristic curvature scale of the background spacetime. The metric far from the point particle (external universe) is approximated by a given background spacetime plus the perturbation generated by a point source (external scheme or external-universe expansion). The metric in this region is expanded in powers of $Gm/r$, where $m$ is the mass of the
two Kerr black holes orbiting each other. They assumed that the separation of these two black holes is large enough to neglect the Riemann curvature of the surrounding spacetime. As a specific example of a system, they considered the case of a binary system with large orbital separation when an approximate solution of the Einstein equations of the surrounding spacetime is given.

They found that these integrations contain the "laws of motion and precession" which can be converted into the equations of motion for a non-spinning particle by an axiomatic approach without resorting to the matched asymptotic expansion [8]. However, it is beyond the scope of this paper to discuss the physical meaning of their axioms in the framework of the matched asymptotic expansion.

Having shown that the asymptotic matching can be consistently done with the external metric generated by a non-spinning point particle, we have clarified that the expression of the energy momentum tensor for a point particle used in computing the gravitational radiation reaction to the orbit in previous works [1] can be regarded as a Schwarzschild black hole or any compact object provided that it is kept sufficiently spherically symmetric. Since it is unnecessary to evaluate the external metric at the location of the particle, we do not encounter the divergence even if we use the delta function source. Applying the asymptotic matching, we obtain the equation of motion of the origin of the internal frame with respect to the background spacetime. And hence the equation of motion is understood in a well-defined manner. As a result, we found that the strong equivalence principle holds to the order \( G \| m \| r \) for a non-spinning compact body in the limit \( G \| m \| r \ll 1 \). Here compact body means that it has no typical length scale other than its Schwarzschild radius.

On the other hand, the post-Newtonian calculation shows that the spin of the particle modifies the gravitational wave forms of a realistic binary system [5], which motivates us to study the problem of incorporating a spinning particle into the perturbation theory. For this purpose, we need to know the correction to the equation of motion due to spin and the energy momentum tensor of a spinning particle.

The effect of the spin to the equation of motion was discussed by many authors in various ways [10,11]. Dixon [1] found the covariant description of the energy momentum tensor of an extended body and the equation of motion of it. The resultant equation of motion shows that the coupling of the spin and the Riemann curvature affects the orbit of the body. However, the Dixon’s equation of motion is written in terms of the full metric which can be obtained only after the Einstein equations are completely solved. On the other hand, what we are interested in is to describe the motion of a compact body and the effect of radiation reaction in a given external background. For this purpose, further reduction is necessary. Furthermore, Dixon’s equation of motion does not seem to be applicable to a strongly self-gravitating body like a black hole in a strict sense.

The effect of the spin was also discussed, for example, in D'Eath [6] and Thorne and Hartle [7], in which the strong self-gravity of the body is taken into account by using the technique of matched asymptotic expansion. Especially Thorne and Hartle [7] established the framework of matched asymptotic expansion of the metric around the particle. Using the metric thus derived, they integrated non-covariant conservation laws written in the inertial frame of the background spacetime (body’s local asymptotic rest frame in [7]) on the world tube enclosing the particle. They found that these integrations contain the ‘laws of motion and precession’ which can be converted into the equations of motion and spin when an approximate solution of the Einstein equations of the surrounding spacetime is given. The laws of motion indicate that the geodesic motion is corrected by the coupling of the spin of the body and the Riemann curvature of the surrounding spacetime. As a specific example of a system, they considered the case of two Kerr black holes orbiting each other. They assumed that the separation of these two black holes is large enough and derived the equations of motion and spin which are accurate up through the 1.5th post-Newtonian order. Since the radiation reaction force is known to give rise from 2.5 post-Newtonian order, the correction due to the radiation reaction derived in [7] did not appear in [6].

The purpose of this paper is to derive the correction to the equation of motion due to the radiation reaction as well as to the spin-Riemann coupling in a unified manner. Again we use the technique of matched asymptotic expansion in order to construct the metric. Recently we have discussed the perturbation due to a spinning point particle [12] and calculated the gravitational waves from a spinning particle orbiting a Kerr black hole up through the 2.5th post-Newtonian order. In this calculation, we have assumed that the spin contribution to the energy momentum tensor is given by a derivative of the delta function. The result agrees with the one obtained by the standard post-Newtonian calculation [6] in a suitable limit. Hence, we assume that metric far from a spinning particle is also approximated by a linear perturbation of a general background spacetime generated by the delta function source, and we assume that the metric around the particle is approximated by that of the Kerr geometry with perturbations, which should/will be justified by the consistency condition of the matching.

The next step is to extract the information of the motion of the particle from the metric thus constructed. At this point we have two methods to obtain the equation of motion; the consistency condition of the matching with an appropriate choice of gauge in the internal scheme [5], or the use of non-covariant conservation laws in the inertial frame of the background spacetime [6]. The former method developed in [5] seems difficult to apply to the present...
case. This is because we do not know a method to distinguish the translational gauge modes of the metric perturbation in a Kerr background. Hence we cannot fix the center of mass condition in the internal scheme when matching the metrics in the internal and external schemes in the overlapping region. Thus we adopt the latter method and integrate the conservation laws as in §II to derive the equation of motion. Relating to the difficulty of the former method, we cannot verify the consistency of matching to the same extent as we could in the case of non-spinning particle. Thus we simply assume it.

This paper is organized as follows. In section 2, we construct the metric in the overlapping region (buffer region), using the covariant expansion method of the tensor Green function [8]. We suppose the reader is familiar with the concept of ‘bi-tensors’ [13] and skip the computation of the tensor Green function [5]. In section 3, we derive the equation of motion of the spinning point particle. Section 4 summarizes the result.

II. MATCHED ASYMPTOTIC EXPANSION

Following the procedure given in [1], we first discuss the external scheme of the metric. We have a general background metric \( g_{\mu\nu}(x) \) which satisfies the vacuum Einstein equations around the particle in the external scheme. We compute its linear perturbation with the Green function method. The perturbation in the external scheme is constructed with an assumption on the perturbation source. (See Eq. (2.3) below.) We will discuss the consistency of this assumption later when we compute the metric in the overlapping region.

We put the metric \( g_{\mu\nu}(x) + h_{\mu\nu}(x) \) into the Einstein equations and solve the linearized equations for \( h_{\mu\nu} \) assuming that the background metric \( g_{\mu\nu} \) satisfies the vacuum Einstein equations around the particle of interest\(^8\). We introduce the trace-reversed metric perturbation \( \psi_{\mu\nu}(x) \), and put the covariant harmonic gauge condition on it.

\[
\psi_{\mu\nu}(x) = h_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) g^{\xi\xi}(x) h_{\xi\xi}(x)
\]

(2.1)

\[
\psi_{\mu\nu,\nu} = 0
\]

(2.2)

We can make use of the tensor Green function of the linearized Einstein equations with the retarded boundary condition derived in Sec. 2 of [8]. Here we assume the metric perturbation is generated by the following source term [12],

\[
T^{\mu\nu}(x) = m \int dT \left\{ v^\mu(x,T) v^\nu(x,T) \frac{\delta^{(4)}(x-z(T))}{\sqrt{-g}} - \nabla_\xi \left( S^{\xi(\mu}(x,T)v^{\nu)}(x,T) \frac{\delta^{(4)}(x-z(T))}{\sqrt{-g}} \right) \right\},
\]

(2.3)

where \( v^\mu(x,T) = \tilde{g}^\mu_{\alpha}(x,z(T))\dot{z}^\alpha(T) \) and \( S^{\mu\nu}(x,T) = \tilde{g}^{\mu}_{\alpha}(x,z(T))\tilde{g}^{\nu}_{\beta}(x,z(T))S^{\alpha\beta}(T) \). \( \tilde{g}^\mu_{\alpha}(x,z(T)) \) is a bivector of parallel displacement defined, for example, in Appendix A of [8], and \( S^{\alpha\beta}(T) \) is an anti-symmetric tensor which is called the spin tensor of the particle and is assumed to satisfy the center of mass condition, \( S^{\alpha\beta}(T)\dot{v}^{\beta}(T) = 0 \). A dot over a function means \( T \)-derivative of it, such as \( \dot{z}^\alpha(T) = \frac{d\dot{z}^{\alpha}}{dT} \). The amplitude of the spin tensor \( S^{\alpha\beta}(T) \) is usually assumed to be of order \( m \) in the case of a Kerr black hole, which seems to indicate the 2nd order perturbation theory is necessary when we want to discuss the spin effect of the particle in a consistent expansion with respect to \( m \). However, we will find it not so if we assume that the construction of the metric by the matched asymptotic expansion is consistent. terms of order \( m^2 \), the 2nd order perturbation theory is unnecessary. The exact form of the metric perturbation becomes

\[
\psi^{\mu\nu}(x) = 2Gm\left( \frac{1}{\tilde{\sigma}(x,z(T))} u^{\mu\nu}_{\alpha\beta}(x,z(T))\dot{z}^\alpha(T)\dot{z}^\beta(T) + \frac{\tilde{\sigma}(x,z(T))}{\tilde{\sigma}^3(x,z(T))} u^{\mu\nu}_{\alpha\beta}(x,z(T))\sigma_{\gamma}(x,z(T))S^{\gamma\alpha}(T)\dot{z}^\beta(T) + \frac{1}{\tilde{\sigma}(x,z(T))} u^{\mu\nu}_{\alpha\beta\gamma}(x,z(T))S^{\gamma\alpha}(T)\dot{z}^\beta(T) - \frac{1}{\tilde{\sigma}^2(x,z(T))} \frac{d}{dT} (u^{\mu\nu}_{\alpha\beta}(x,z(T))\sigma_{\gamma}(x,z(T))S^{\gamma\alpha}(T)\dot{z}^\beta(T)) \right)
\]

\(^8\)In this derivation, we follow the notation used in [8]. We assign the indices \( \alpha, \beta, \gamma, \delta \) for the point on the particle trajectory \( z(T) \), and the indices \( \mu, \nu, \xi, \rho \) for the field point \( x \).
where \( \sigma(x,z) \) is a half the squared geodetic interval between \( x \) and \( z \) and is defined, for example, in Appendix A of [5], \( w^{\mu\nu\alpha\beta}(x,z) \) and \( w^{\mu\nu\alpha\beta}(x,z) \) appear in the expression for the tensor Green function and are defined in Sec. 2 of [13]. Eqs. (2.5) and (2.8) show that \( e_{\alpha i}(T) \) are the spatial triad at \( z(T) \) on the 3-hypersurface normal to \( \dot{z}(T) \). The coordinate system for finite \( m \) is determined later.

We denote the components of the metric whose order of magnitude are \( O((Gm)^n/L^m) \) by \((m)_n h_{ab}\).

\[
\text{full}\ g_{ab} = \sum_{m,n=0}^{\infty} \frac{(m)_n h_{ab}}{(n)_m h_{ab}}
\]

We call these terms the \((m)_n\)-components of the metric [3]. Simple dimensional analysis shows

\[
(m)_n h_{ab} \sim |X|(m-n),
\]

where \( |X| = \sqrt{X^iX^j} \). In the next section, we use the Landau-Lifshitz pseudotensor in order to compute the equation of motion, and the metric is expanded by taking the flat Minkowski metric \( \eta_{ab} \) as a background. For its purpose, we raise the indices of the \((m)_n\)-components of the metric by \( \eta_{ab} \), and we define the trace-reversed \((m)_n\)-components of the metric with respect to the flat Minkowski.

\[
(m)_n \tilde{h}_{ab} = \frac{(m)_n h_{ab}}{2} \frac{(m)_n h_{cd}(m)_n h_{cd}}{(n)_m h_{ab}}
\]

The \((m)_n\)-components of the metric are obtained from the background metric \( g_{\mu\nu}(x) \). The background metric in the coordinates \( \{X^a\} \) is partly given in [3]. A further calculation results in

\[
g_{\mu\nu}(x)dx^\mu dx^\nu = \left( \dot{z}^2(T) + 2\dot{z}^a(T)\frac{De_{\alpha i}(T)}{dT}X^i + \frac{De_{\alpha i}(T)}{dT}\frac{De_{\gamma j}(T)}{dT}(T)X^iX^j \right)dt^2 + 2\left( \dot{z}^a(T)e_{\alpha i}(T) + e_{\alpha i}(T)\frac{De_{\gamma j}(T)}{dT}(T)X^j \right)dtX^i + \frac{2}{3} R_{\alpha\beta\gamma\delta}(z(T))e_{\alpha i}(T)X^\beta(T)\dot{z}^\gamma(T)X^\delta(T) + O(|X|^3) + O(|X|^3)\right)dtdX^i
\]

** We assign the indices, \( a,b,c,d \), for the spacetime coordinates i.e. 0, 1, 2, 3, while the indices, \( i,j,k,l \), for the spatial coordinates i.e. 1, 2, 3. Minkowski and Kronecker summation conventions are taken.
We further compute the metric perturbation due to the spin up to $O(1)$. Using (2.5), we can so on get the expression of it up to $O(|X|^3)$

$$
+ \left(e^{\alpha_i(T)}e_{\alpha_j(T)}
+ \frac{1}{3}R_{\alpha\beta\gamma\delta}(z(T))e^{\alpha_i(T)}X^{\delta}(T)e^{\gamma_j(T)}X^{\delta}(T) + O(|X|^3)\right)dX^i dX^j,
$$

(2.12)

where $X^{\alpha}(T) = e^{\alpha_i(T)}X^i$. Because of (2.11), we find

\begin{align}
(0)_{h_{00}} &= \dot{z}^2(T) + O(Gm/L), \\
(0)_{h_{\alpha i}} &= \dot{z}^{\alpha}(T)e_{\alpha i}(T) + O(Gm/L), \\
(0)_{h_{ij}} &= e^{\alpha_i(T)}e_{\alpha_j(T)} + O(Gm/L), \\
(1)_{h_{00}} &= 2\dot{z}^{\alpha}(T)\frac{D e^{\alpha i}}{dT}(T)X^i + O(Gm|X|/L^2), \\
(1)_{h_{\alpha i}} &= e^{\alpha_i}(T)\frac{D e_{\alpha i}}{dT}(T)X^j + O(Gm|X|/L^2), \\
(1)_{h_{ij}} &= O(Gm|X|/L^2), \\
(2)_{h_{00}} &= R_{\alpha\beta\gamma\delta}(z(T))\dot{z}^{\alpha}(T)X^{\beta}(T)\dot{z}^{\gamma}(T)X^{\delta}(T) + O(Gm|X|^2/L^3), \\
(2)_{h_{\alpha i}} &= \frac{2}{3}R_{\alpha\beta\gamma\delta}(z(T))e^{\alpha_i(T)}X^{\beta}(T)\dot{z}^{\gamma}(T)X^{\delta}(T) + O(Gm|X|^2/L^3), \\
(2)_{h_{ij}} &= \frac{1}{3}R_{\alpha\beta\gamma\delta}(z(T))e^{\alpha_i(T)}X^{\beta}(T)e^{\gamma_j(T)}X^{\delta}(T) + O(Gm|X|^2/L^3).
\end{align}

(2.13)-(2.21)

As was argued in [5], the matching condition requires that the $(0)$-components of the metric (2.13)-(2.15) should be equal to those of the flat metric

$$
(0)_{h_{ab}} = \eta_{ab},
$$

(2.22)

which is exactly realized by the coordinate conditions (2.6)-(2.8).

The $(1)$-components of the metric should vanish in order to put the particle in the locally rest frame of the background geometry.

$$
(1)_{p_{ab}} = 0
$$

(2.23)

From (2.16)-(2.18), we thus obtain

$$
\dot{z}^{\alpha}(T)\frac{D e^{\alpha i}}{dT}(T) = O(Gm/L^2),
$$

(2.24)

$$
e^{\alpha_i}(T)\frac{D e_{\alpha i}}{dT}(T) = O(Gm/L^2).
$$

(2.25)

From (2.6), (2.7) and (2.24), we get the geodesic motion of the particle in the background geometry.

$$
\frac{D}{dT}z^{\alpha}(T) = O(Gm/L^2)
$$

(2.26)

(2.24) and (2.27) show the geodesic transform of the spatial triad $e^{\alpha_i}(T)$ in the background geometry.

$$
\frac{D e^{\alpha i}}{dT}(T) = O(Gm/L^2)
$$

(2.27)

The $(0)$-components of the metric and the $(1)$-components of the metric due to the spin of the particle come from both the background metric $g_{\mu\nu}$ and the metric perturbation (2.4). The covariant expansion of the metric perturbation due to the monopole particle is already given in [5]. Using (2.3), we can soon get the expression of it up to $O(mX)$. We further compute the metric perturbation due to the spin up to $O(mS|X|^0)$. 

5
\[
\psi_{\mu\nu}(x)dx^\mu dx^\nu = \left\{ 2Gm \left( \frac{2}{|X|} + \frac{1}{3|X|} R_\alpha\beta\gamma\delta(z(T)) \dot{z}^\alpha(T)X^\beta(T) \dot{z}^\gamma(T)X^\delta(T) \right) \right. \\
-2 \left( V_{\alpha\beta}(T) + V_{\alpha\beta\gamma}(T)X^\gamma(T) \right) \dot{z}^\alpha(T) \dot{z}^\beta(T) + O(m|X|^2, m|S||X|^0) \left\} dT^2 \\
+2 \left\{ -2GmR_{\alpha\beta\gamma\delta}(z(T)) \dot{z}^\alpha(T)X^\beta(T) \dot{z}^\gamma(T)e^\delta_i(T) \\
-2 \left( V_{\alpha\beta}(T) + V_{\alpha\beta\gamma}(T)X^\gamma(T) \right) \dot{z}^\alpha(T)e^\beta_i(T) \\
-2G \frac{m}{|X|^3} S_{\alpha\beta}(T)X^\alpha(T)e^\beta_i(T) + O(m|X|^2, m|S||X|^0) \right\} dTdx^i \\
+ \left\{ -4Gm|X|R_{\alpha\beta\gamma\delta}(z(T)) \dot{z}^\alpha(T)e^\beta_i(T) \dot{z}^\gamma(T)e^\delta_j(T) \\
-2 \left( V_{\alpha\beta}(T) + V_{\alpha\beta\gamma}(T)X^\gamma(T) \right) e^\alpha_i(T)e^\beta_j(T) + O(m|X|^2, m|S||X|^0) \right\} dX^i dX^j, 
\]  
\tag{2.28}

\[
V_{\alpha\beta}(T) = Gm \int_{-\infty}^T dT' \left( \epsilon^{\alpha\beta}_{\alpha'\beta'}(z(T), z(T')) \dot{z}^\alpha(T') \dot{z}^\beta(T') \\
+ \epsilon^{\alpha\beta}_{\alpha'\beta'}(z(T), z(T')) S_{\alpha'\beta'}(T') \dot{z}^\beta(T') \right), 
\tag{2.29}
\]

\[
V_{\alpha\beta\gamma}(T) = Gm \int_{-\infty}^T dT' \left( \epsilon^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'}(z(T), z(T')) \dot{z}^\alpha(T') \dot{z}^\beta(T') \dot{z}^\gamma(T') \\
+ \epsilon^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'}(z(T), z(T')) S_{\alpha'\beta'\gamma'}(T') \dot{z}^\beta(T') \dot{z}^\gamma(T') \right), 
\tag{2.30}
\]

We first discuss the \((1)_i\)-components and the \((0)_i\)-components of the metric. From \(2.28\), we obtain

\[
\begin{align*}
\begin{bmatrix} (0)_i & (1)_i \end{bmatrix} h_{00} &= \frac{4Gm}{|X|^3}, 
\tag{2.31} \\
\begin{bmatrix} (0)_i & (1)_i \end{bmatrix} h_{0i} &= 0, 
\tag{2.32} \\
\begin{bmatrix} (0)_i & (1)_i \end{bmatrix} h_{ij} &= 0, 
\tag{2.33} \\
\begin{bmatrix} (0)_i & (1)_i \end{bmatrix}_{\text{spin}} &= 0, 
\tag{2.34} \\
\begin{bmatrix} (0)_i & (2)_i \end{bmatrix}_{\text{spin}} &= -2Gm \frac{m}{|X|^3} S_{\alpha\beta}(T)X^\alpha(T)e^\beta_i(T) + O((Gm)^3/L|X|^2), 
\tag{2.35} \\
\begin{bmatrix} (0)_i & (2)_i \end{bmatrix}_{\text{spin}} &= 0. 
\tag{2.36}
\end{align*}
\]

where \([ \text{ spin} ]\) means that only the spin contribution is taken into account. In view of the internal scheme, the \((1)_i\)-components of the metric describe the background metric, thus the consistency condition of the matched asymptotic expansion requires that these components realize the asymptotic gravitational field of a Kerr black hole. \cite{5} argues the \((1)_i\)-components of the metric fit the asymptotic metric of a Schwarzschild black hole with mass parameter \(m\) and we have shown in \(2.28\) that the spin contribution to the \((1)_i\)-components of the metric correctly realizes the asymptotic field of a Kerr black hole. Therefore the metric thus constructed in the external scheme matches the metric in the internal scheme and the use of \(2.23\) is justified in computing the leading order contribution of the mass \(m\) and the spin \(S\). The monopole contribution to the \((1)_i\)-components of the metric cannot be computed in the external scheme unless we develop the 2nd order perturbation formalism. However, the consistency condition of the matched asymptotic expansion requires that these components are those of the asymptotic expansion of a Schwarzschild black hole. Thus we finally obtain

\[
\begin{align*}
\begin{bmatrix} (0)_i & (2)_i \end{bmatrix} h_{00} &= \frac{(Gm)^2}{|X|^2}, 
\tag{2.37} \\
\begin{bmatrix} (0)_i & (2)_i \end{bmatrix} h_{0i} &= -2Gm \frac{m}{|X|^3} S_{\alpha\beta}(T)X^\alpha(T)e^\beta_i(T) + O((Gm)^3/L|X|^2), 
\tag{2.38}
\end{align*}
\]

6
We next discuss the \( (1) \)-components and the \( (2) \)-components of the metric. As is argued in \cite{7}, we can take the coordinate system such that all the \( (1) \)-components vanish as long as the \( (0) \)-components of the metric vanish. In view of the internal scheme, the \( (1) \)-components of the metric describe a linear perturbation of the background metric. This perturbation is induced by the curvature of the external universe, i.e. by the \( (2) \)-components. However, we have seen that the \( (1) \)-components should vanish because of the coordinate condition. Thus the \( (1) \)-components of the metric have no physical mode, and we can set them zero with a suitable choice of coordinates. As discussed by Thorne and Hartle \cite{3}, it is essential to put them zero for reducing the uncertainties in the definitions of the mass, momentum and spin of the particle when deriving the equation of motion. Hence we require \( (1) \)-components of the metric.

\[
\begin{align*}
\sigma_\alpha(x, z(T)) + e_\alpha(T)X^i &= 0 \quad (2.43) \\
\dot{z}^2(T) &= -1 + V_{\alpha\beta}(T)\left( g^{\alpha\beta}(z(T)) + 2\dot{z}^\alpha(T)\dot{z}^\beta(T) \right) + O((Gm)^2/L^2) \quad (2.44) \\
\dot{z}^\alpha(T)e_\alpha(T) &= -2V_{\alpha\beta}(T)\dot{z}^\alpha(T)e_\beta(T) + O((Gm)^2/L^2) \quad (2.45) \\
e^{\alpha i}(T)e_{\alpha j}(T) &= \delta_{ij} + V_{\alpha\beta}(T)\left( -\delta_{ij}\dot{z}^{\alpha\beta}(z(T)) + 2e^{\alpha i}(T)e^{\beta j}(T) \right) + O((Gm)^2/L^2) \quad (2.46)
\end{align*}
\]

Setting the \( (1) \)-components of the metric zero will give us the coordinate conditions to the order \( O((Gm)^2/L^2) \), but we do not need them for the present purpose.

The metric derived by the matched asymptotic expansion is summarized as follows.

\[
\begin{align*}
^{(0)}\bar{h}_{ab} &= \eta_{ab}, \quad (2.47) \\
^{(1)}\bar{h}_{ab} &= 0, \quad (2.48) \\
^{(2)}\bar{h}_{00} &= \frac{2}{3}R_{\alpha\beta\gamma\delta}(z(T))\dot{z}^\alpha(T)X^\beta(T)\dot{z}^\gamma(T)X^\delta(T) + O(Gm|X|^2/L^3), \quad (2.49) \\
^{(2)}\bar{h}_{0i} &= \frac{2}{3}R_{\alpha\beta\gamma\delta}(z(T))e^{\alpha i}(T)X^\beta(T)\dot{z}^\gamma(T)X^\delta(T) + O(Gm|X|^2/L^3), \quad (2.50) \\
^{(2)}\bar{h}_{ij} &= \frac{1}{3}R_{\alpha\beta\gamma\delta}(z(T))X^\beta(T)X^\delta(T)(e^{\alpha i}(T)e^{\gamma j}(T) + \delta_{ij}\dot{z}^\alpha(T)\dot{z}^\gamma(T)) + O(Gm|X|^2/L^3), \quad (2.51) \\
^{(0)}\bar{h}_{00} &= \frac{4Gm}{|X|}, \quad (2.52) \\
^{(0)}\bar{h}_{0i} &= 0, \quad (2.53) \\
^{(0)}\bar{h}_{ij} &= 0, \quad (2.54) \\
^{(1)}\bar{h}_{ab} &= 0, \quad (2.55) \\
^{(2)}\bar{h}_{00} &= \dot{z}^\alpha(T)\frac{D}{dT}e_{\alpha i}(T)X^i(T)
\end{align*}
\]
\[
\begin{align*}
+ \frac{10Gm}{3|X|} R_{\alpha\beta\gamma\delta}(z(T)) \dot{z}^{\alpha}(T) X^{\beta}(T) \dot{z}^{\gamma}(T) X^{\delta}(T) \\
-2V_{\alpha\beta}(T) \dot{z}^{\alpha}(T) \dot{z}^{\beta}(T) X^{\gamma}(T) + O((Gm)^2 |X|/L^3),
\end{align*}
\]
(2.56)

\[
\begin{align*}
(2)_1 \tilde{h}_{0i} &= \epsilon^a_i(T) \frac{D}{dt} \epsilon_{a_j}(T) X^j \\
+2GmR_{\alpha\beta\gamma\delta}(z(T)) X^{\beta}(T) \dot{z}^{\gamma}(T) \left( -\dot{z}^{\alpha}(T) \epsilon^a_i(T) - \frac{2}{3|X|} \epsilon^a_i(T) X^{\delta}(T) \right) \\
-2V_{\alpha\beta}(T) \dot{z}^{\alpha}(T) \epsilon^a_i(T) X^{\beta}(T) + O((Gm)^2 |X|/L^3),
\end{align*}
\]
(2.57)

\[
\begin{align*}
(2)_1 \tilde{h}_{ij} &= \delta_{ij} \dot{z}^{\alpha}(T) \frac{D}{dt} \epsilon_{a_k}(T) X^k \\
+2GmR_{\alpha\beta\gamma\delta}(z(T)) \left( \frac{1}{3|X|} X^{\beta}(T) X^{\delta}(T) \right) (\epsilon^a_i(T) \epsilon^{\gamma}_j(T) - \delta_{ij} \dot{z}^{\alpha}(T) \dot{z}^{\gamma}(T)) \\
-2|X| \dot{z}^{\alpha}(T) \epsilon^a_i(T) \dot{z}^{\beta}(T) e^{\delta}(T),
\end{align*}
\]
(2.58)

\[
\begin{align*}
(0)_2 \tilde{h}_{00} &= \frac{(Gm)^2}{|X|^2},
\end{align*}
\]
(2.59)

\[
\begin{align*}
(0)_2 \tilde{h}_{0i} &= -\frac{2Gm}{|X|^2} S_{\alpha\beta}(T) X^{\alpha}(T) \epsilon^a_i(T) + O((Gm)^3/L|X|^2),
\end{align*}
\]
(2.60)

\[
\begin{align*}
(0)_2 \tilde{h}_{ij} &= \frac{(Gm)^2}{|X|^2} \left( -2\delta_{ij} + \frac{X^i X^j}{|X|^2} \right),
\end{align*}
\]
(2.61)

\[
\begin{align*}
(1)_2 \tilde{h}_{ab} &= 0.
\end{align*}
\]
(2.62)

**III. EQUATION OF MOTION**

We apply Eqs. (2.2a), (2.2b), (2.3a) and (2.3b) of \([6]\) to derive the equation of motion, which are

\[
P^a(T, r) = \frac{1}{16\pi G} \int_{|X|=r} d^2 S_j H^{ab0j} b
\]
(3.1)

\[
J^{ij}(T, r) = \frac{1}{16\pi G} \int_{|X|=r} d^2 S_k \left( X^i H^{j0k} a - X^j H^{ia0k} a + H^{ik0j} - H^{j0i} \right)
\]
(3.2)

\[
\frac{d}{dT} P^a(T, r) = -\frac{1}{G} \int_S d^2 S_k t^{0j}(X)
\]
(3.3)

\[
\frac{d}{dT} J^{ij}(T, r) = -\frac{1}{G} \int_S d^2 S_k \left( X^i t^{jk}(X) - X^j t^{ik}(X) \right)
\]
(3.4)

As in \([6]\), we adopt the choice of Landau-Lifshitz for \(H^{abcd}\) and \(t^{ab}\).

\[
H^{abcd} = H^{abcd}_{L-L},
\]
(3.5)

\[
t^{ab} = (-g)^{ab}_{L-L}.
\]
(3.6)

The explicit expressions are given in standard texts, such as, Sec.20 of Misner-Thorne-Wheeler \([14]\) or Sec.96 of Landau-Lifshitz \([13]\). The integrals are taken over a 2-surface of constant \(|X|\) in the overlapping region for definiteness, but we focus on the \(r\)-independent terms in the integrals since it is enough only to consider them in order to derive the leading order correction to the orbit \(z(T)\).

Before considering \((3.1)\) and \((3.3)\), we first show that the spin tensor \(S_{\alpha\beta}(T)\) is geodetic transported in the background geometry \(g_{\alpha\beta}(z(T))\) in the test particle limit \(m \to 0\). Eq. \((3.2)\) has a dimension of \((\text{mass}) \times \text{(length)}\) and we consider the terms of order \((Gm)^2\). Power counting of \(X\) shows that there will be contributions linear in the \((\alpha)\)-components of the metric and those from bilinear combinations of the \((\gamma)\)- and \((\delta)\)-components of the metric. We obtain

\[
J^{ij}(T, r) = mS_{\alpha\beta}(T) e^{\alpha i} e^{\beta j} + O(G^2 m^3/L) + (r\text{-dependent terms}).
\]
(3.7)
Eq. (3.4) has a dimension of (mass)\(^1\) and we consider the terms of order \(Gm^2/L\) in the same way. Power counting of \(X\) shows that there will be contributions from bilinear combinations of the \((\frac{1}{2})\)- and \((\frac{3}{2})\)-components of the metric, and we soon find that (3.4) vanishes.

\[
\frac{d}{d\tau} J^{ij}(T, r) = O(G^2 m^3/L^2) + (r\text{-}dependent \text{ terms})
\]  

(3.8)

Since the spatial triad are geodetic transported in the background geometry to the leading order, (3.7) and (3.8) result in

\[
\frac{DS^{\alpha\beta}}{dT}(T) = 0.
\]

(3.9)

We next consider (3.1) and (3.3). Eq. (3.1) has a dimension of (mass)\(^1\) and we consider the terms of order \(m\) and \(Gm^2/L\). We find that there will be linear contributions from \((\frac{1}{2})\)-, \((\frac{3}{2})\)-components of the metric, and bilinear contributions from pairs of \((\frac{1}{2})\)- and \((\frac{3}{2})\)-components of the metric. We obtain

\[
P^0(T, r) = m + O(G^2 m^3/L^2) + (r\text{-}dependent \text{ terms}),
\]

(3.10)

\[
P^i(T, r) = O(G^2 m^3/L^2) + (r\text{-}dependent \text{ terms}).
\]

(3.11)

Eq. (3.3) has a dimension of (mass)/(length) and we consider the terms of order \(m/L\) and \(Gm^2/L^2\). There will be bilinear contributions from pairs of the \((\frac{1}{2})\)- and \((\frac{3}{2})\)-components of the metric. The former pairs give the leading correction to the motion due to the spin of the particle, and the latter pairs give the radiation reaction of motion. A straightforward computation results in

\[
\frac{d}{dT} P^0(T, r) = O(G^2 m^3/L^3) + (r\text{-}dependent \text{ terms}),
\]

(3.12)

\[
\frac{d}{dT} P^i(T, r) = \frac{m}{2} R_{\alpha\beta\gamma\delta}(z(T)) e^i(T) \dot{z}^\beta(T) S^{\gamma\delta}(T)
\]

\[
- \frac{m}{2} V_{\alpha\beta\gamma}(T) e^\gamma(T) \left(2 \dot{z}^\alpha(T) \dot{z}^\beta(T) + g^{\alpha\beta}(z(T))\right)
\]

\[
+ m \dot{z}^\alpha(T) D_{\alpha\beta}(r) e_{\alpha\beta}(T) + O(G^2 m^3/L^3) + (r\text{-}dependent \text{ terms}).
\]

(3.13)

Taking the \(T\)-derivative of (2.44) and (2.45), we finally get the equation of motion,

\[
\frac{D}{d\tau} \dot{z}^\alpha(T) = \ddot{V}_{\alpha\beta\gamma}(T) \left(2 \dot{z}^\alpha(T) g^{\alpha\gamma}(z(T)) \dot{z}^\beta(T) - \dot{z}^\beta(T) \dot{z}^\gamma(T) g^{\alpha\delta}(z(T))\right)
\]

\[
+ \frac{1}{2} R_{\alpha\beta\gamma\delta}(z(T)) \dot{z}^\beta(T) S^{\gamma\delta}(T) + O(G^2 m^2/L^3),
\]

(3.14)

where we have defined \(\ddot{V}_{\alpha\beta\gamma} = V_{\alpha\beta\gamma} - 1/2 g_{\alpha\beta} V^\delta_{\delta\gamma}\). Introducing the proper time \(\tau\) of the orbit,

\[
\frac{d\tau}{d\tau} = 1 - \ddot{V}_{\alpha\beta\gamma}(T) \dot{z}^\alpha(T) \dot{z}^\beta(T),
\]

(3.15)

we finally get

\[
\frac{D}{d\tau} \frac{dz^\alpha}{d\tau}(\tau) = \ddot{V}_{\beta\gamma\delta} (\tau) \left(\frac{dz^\alpha}{d\tau} \frac{dz^\beta}{d\tau} \frac{dz^\gamma}{d\tau} \frac{dz^\delta}{d\tau} + 2 \frac{dz^\alpha}{d\tau} g^{\alpha\gamma}(z(\tau)) \frac{dz^\beta}{d\tau} - \frac{dz^\beta}{d\tau} \frac{dz^\gamma}{d\tau} g^{\alpha\delta}(z(\tau))\right)
\]

\[
+ \frac{1}{2} R_{\alpha\beta\gamma\delta}(z(\tau)) \frac{dz^\beta}{d\tau} S^{\gamma\delta}(\tau) + O(G^2 m^2/L^3).
\]

(3.16)

IV. CONCLUSION

In this paper, we have derived the equation of motion of a spinning particle on a given background spacetime. We have constructed the metric using the technique of matched asymptotic expansion and integrated the conservation laws introduced in [7] to extract the information of the equations of motion. One possible drawback of the present derivation of the equation of motion could be that the consistency of the matched asymptotic expansion has not been
explicitly demonstrated. To do so, we need to extract the translational gauge modes from the metric perturbation in the internal scheme and examine if the condition of their disappearance leads to the equation of motion, as done in the previous derivation of the equation of motion for a non-spinning particle in \[5\]. Unfortunately, as mentioned in Introduction, since we have no theory of the metric perturbation in the Kerr background, it is at present impossible for us to perform this procedure. However, since we find no logical flaw in the method of using the conservation laws, we believe the present derivation of the equation of motion is perfectly legitimate.

The conclusion of our previous paper \[5\] is that the non-spinning particle moves along a geodesic of the regularized perturbed space,

\[
\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + h_{(v)\mu\nu}(x),
\]

where \(h_{(v)\mu\nu}(x)\) is the tail part of the metric perturbation \[2.4\],

\[
h_{(v)\mu\nu}(x) = \psi_{(v)\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)g^{\xi\rho}(x)\psi_{(v)\xi\rho}(x),
\]

\[
\psi_{(v)\mu\nu}(x) = -2Gm \int_{-\infty}^{T_{\text{Ric}}(x)} dT \left( \epsilon^{\mu\alpha\beta}(x, z(T))\dot{z}^\alpha(T)\dot{z}^\beta(T) + \delta^{\mu\alpha\beta}(x, z(T))S^{\gamma\alpha}(T)\dot{z}^\beta(T) \right).
\]

We have found that the spin of the particle gives rise to an effective force in addition to this, which is the same as derived in \[10,11,7\]. Hence in terms of the regularized metric \(\tilde{g}_{\mu\nu}\) and renormalized proper time \(\tilde{\tau}\), the equation of motion coincides with the one derived in \[10,11,7\]:

\[
\frac{d}{d\tilde{\tau}}\tilde{z}(\tilde{\tau}) = \frac{1}{2}R^{\alpha\beta\gamma\delta}(z(\tilde{\tau}))\dot{z}^{\beta}(\tilde{\tau})S^{\gamma\delta}(\tilde{\tau}).
\]

As was commented in \[5\], it is still under investigation to find a method to explicitly compute \(h_{(v)\mu\nu}(x)\) even for well-known background geometries, such as a Kerr geometry.

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