Liouville distorted Brownian motion

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Abstract. The Liouville Brownian motion was introduced in [3] as a time changed process $B_{A^{-1}}$ of a planar Brownian motion $(B_t)_{t \geq 0}$, where $(A_t)_{t \geq 0}$ is the positive continuous additive functional of $(B_t)_{t \geq 0}$ in the strict sense w.r.t. the Liouville measure. We first consider a distorted Brownian motion $(X_t)_{t \geq 0}$ starting from all points in $\mathbb{R}^2$ associated to a Dirichlet form $(\mathbb{E}, D(\mathbb{E}))$ (see [7]). We show that the positive continuous additive functional $(F_t)_{t \geq 0}$ of $(X_t)_{t \geq 0}$ in the strict sense w.r.t. the Liouville distorted measure can be constructed.

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1 Introduction

The Liouville Brownian motion, introduced by C. Garban, R. Rhodes, V. Vargaz in [3], is a Markov process defined as a time changed process on $\mathbb{R}^2$. By classical theory of Gaussian multiplicative chaos (cf. [5]), the Liouville measure $M_\gamma$, $\gamma \in (0, 2)$ is well defined (see Section 2). In [3] the positive continuous additive functional $(A_t)_{t \geq 0}$ of a planar Brownian motion $(B_t)_{t \geq 0}$ in the strict sense w.r.t. the Liouville measure $M_\gamma$ is constructed and then the Liouville Brownian motion is defined as $B_{A^{-1}}$.

In this paper we are concerned with the extension of the Liouville Brownian motion $B_{A^{-1}}$ to more general Markov processes. Note that the planar Brownian motion $(B_t)_{t \geq 0}$ is associated with the Dirichlet form $(\mathbb{E}', D(\mathbb{E}'))$ defined on $L^2(\mathbb{R}^2, d\rho)$, where $\rho(x) := |x|^\alpha$, $\alpha \in (-2, \infty)$ (see [7]). It is known from [6] and [7, Section 3] that there exists the distorted Brownian motion $(X_t)_{t \geq 0}$ starting from all points in $\mathbb{R}^2$ associated with the Dirichlet form $(\mathbb{E}, D(\mathbb{E}))$. Then using the estimates of the resolvent kernel and part Dirichlet form method, we can construct the positive continuous additive functional $(F_t)_{t \geq 0}$ of $(X_t)_{t \geq 0}$ in the strict sense w.r.t. the Liouville distorted measure $M_\rho^\alpha$ (see Section 2). Similarly to the Liouville Brownian motion, the Liouville distorted Brownian motion is defined as $X_{F^{-1}}$.

Notations:
For any open set $U$ in $\mathbb{R}^2$, we denote the set of all Borel measurable functions and the set of
2 Massive Gaussian free field and Gaussian multiplicative chaos

We first state the definition of the massive Gaussian free field as stated in [3]. The massive Gaussian free field on \( \mathbb{R}^2 \) is a centered Gaussian random distribution (in the sense of Schwartz) on a probability space \((\Omega, \mathcal{A}, P)\) with covariance function given by the Green function \(G^{(m)}\) of the operator \(m^2 - \Delta\), \(m > 0\), i.e.

\[
(m^2 - \Delta)G^{(m)}(x, \cdot) = 2\pi \delta_x, \quad x \in \mathbb{R}^2,
\]

where \(\delta_x\) stands for the Dirac mass at \(x\). The massive Green function with the operator \((m^2 - \Delta)\) can be written as

\[
G^{(m)}(x, y) = \int_0^\infty e^{-\frac{x^2 + y^2}{2s}} \frac{ds}{2s} = \int_0^\infty \frac{k_m(s(x-y))}{s} ds, \quad x, y \in \mathbb{R}^2,
\]

where

\[
k_m(z) = \frac{1}{2} \int_0^\infty e^{-\frac{|z|^2 + s^2}{2}} ds.
\]

Let \((c_n)_{n \geq 1}\) be an unbounded strictly increasing sequence such that \(c_1 = 1\) and \((Y_n)_{n \geq 1}\) be a family of independent centered continuous Gaussian fields on \(\mathbb{R}^2\) on the probability space \((\Omega, \mathcal{A}, P)\) with covariance kernel given by

\[
E[Y_n(x) Y_n(y)] = \int_{c_{n-1}}^{c_n} \frac{k_m(s(x-y))}{s} ds.
\]

The massive Gaussian free field is the Gaussian distribution defined by

\[
X(x) = \sum_{k \geq 1} Y_k(x).
\]

We define \(n\)-regularized field by

\[
X_n(x) = \sum_{k \geq 1} Y_k(x), \quad n \geq 1
\]

and the associated \(n\)-regularized measure by

\[
M_{\gamma, n}(dz) = \exp \left( \gamma X_n(z) - \frac{\gamma^2}{2} E[X_n(z)^2] \right) \rho(z) dz, \quad \gamma \in (0, 2),
\]
where \(\rho dz\) is a positive Radon measure on \(\mathbb{R}^2\). By the classical theory of Gaussian multiplicative chaos (see [5]), \(P\)-a.s. the family \((M_{n,y})_{n\geq 1}\) weakly converges to the Liouville distorted measure

\[
M^\rho_t(dz) = \exp \left( g X(z) - \frac{\gamma^2}{2} E[X(z)^2] \right) \rho(z) dz.
\]

It is known from [5] that \(M^\rho_t\) is a Radon measure on \(\mathbb{R}^2\). If \(\rho = 1\), we denote the n-regularized Liouville measure \(M^\rho_{n,y}\) and the Liouville measure \(M^\rho_t\) by \(M_{n,y}\) and \(M_t\), respectively.

## 3 Liouville distorted Brownian motion

We consider \(\rho(x) := |x|^\alpha, \quad \alpha \in (-2, \infty)\) and the symmetric bilinear form

\[
E(f, g) := \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla f(x), \nabla g(x) \rangle m(dx), \quad f, g \in C_0^\infty(\mathbb{R}^2),
\]

where \(m := \rho dx\). It is known that \((E, C_0^\infty(\mathbb{R}^2))\) is closable and its closure is a strongly local, regular Dirichlet form \((E, D(E))\) on \(L^2(\mathbb{R}^2, m)\) (cf. e.g. [4]). Let \((T_t)_{t\geq 0}\) and \((G_\beta)_{\beta\geq 0}\) be the \(L^2(\mathbb{R}^2, m)\)-semigroup and resolvent associated to \((E, D(E))\) (see [4]). By [6] and [7, Section 3] there exists the distorted Brownian motion associated with the Dirichlet form \((E, D(E))\)

\[
\mathcal{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \zeta, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^2})
\]

with transition function \((P_t)_{t\geq 0}\) where \(\zeta\) is the lifetime. Moreover, it is known from [7, Section 3] that there exists a jointly continuous transition kernel density \(p_t(x, y)\) such that

\[
P_t f(x) := \int_{\mathbb{R}^2} p_t(x, y) f(y) m(dy), \quad t > 0, \ x, y \in \mathbb{R}^2, \ f \in \mathcal{B}_b(\mathbb{R}^2)
\]

is an \(m\)-version of \(T_t f\) if \(f \in L^2(\mathbb{R}^2, m)_b\). We set \(P_0 := \text{id}\). Taking the Laplace transform of \(p(x, y)\), we obtain a \(\mathcal{B}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R}^2)\) measurable non-negative resolvent kernel density \(r_\beta(x, y)\) such that

\[
R_\beta f(x) := \int_{\mathbb{R}^2} r_\beta(x, y) f(y) m(dy), \quad \beta > 0, \ x \in \mathbb{R}^2, \ f \in \mathcal{B}_b(\mathbb{R}^2),
\]

is an \(m\)-version of \(G_\beta f\) if \(f \in L^2(\mathbb{R}^2, m)_b\).

We present some definitions and properties concerning \((E, D(E))\). We will refer to [3] till the end, hence some of its standard notations may be adopted below without definition. For any set \(A \subset \mathbb{R}^2\) the capacity of \(A\) is defined as

\[
\text{Cap}(A) := \inf_{\alpha \in \mathcal{B} \subset \mathbb{R}^2} \inf_{(f, g) \in \mathcal{E}} \frac{\mathcal{E}(f, f) + (f, f)}{L^2(\mathbb{R}^2, m)}.
\]

**Definition 3.1.** Let \(B\) be an open set in \(\mathbb{R}^2\). For \(x \in B, t \geq 0, \beta > 0\) and \(p \in [1, \infty)\) let

- \(\sigma_p := \inf\{t > 0 | X_t \in B\}, \quad D_p := \inf\{t \geq 0 | X_t \in B\}\),
- \(\mathbb{P}^B f(x) := \mathbb{E}_x [f(X_t); t < \sigma_p]\), \(f \in \mathcal{B}_b(B)\),
- \(\rho^B_t f(x) := \mathbb{E}_x \left[ \int_0^t e^{-\beta s} f(X_s) ds \right], \quad f \in \mathcal{B}_b(B)\),
- \(D(E^B) := \{ u \in D(E) | u = 0 \text{ E-q.e on } B^c\}\),
- \(E^B := \mathcal{E} |_{D(E^B) \times D(E^B)}\).
• \( \mathcal{E}_f^p(f,g) := \mathcal{E}_f^p(f,g) + \int_B fg \, dm, \quad f, g \in D(\mathcal{E}_f^p) \).

• \( \|f\|_{D(\mathcal{E}_f^p)} := \mathcal{E}_f^p(f,f)^{1/2}, \quad f \in D(\mathcal{E}_f^p) \).

It is known that \((\mathcal{E}_f^p, D(\mathcal{E}_f^p))\) is a regular Dirichlet form on \(L^2(B,m)\), which is called the part Dirichlet form of \((\mathcal{E}, D(\mathcal{E}))\) on \(B\) (cf. [2, Section 4.4]). Let \((T_t^B)_{t \geq 0}\) and \((G_t^B)_{t \geq 0}\) be the \(L^2(B,m)\)-semigroup and resolvent associated to \((\mathcal{E}_f^p, D(\mathcal{E}_f^p))\). Then \(T_t^B f, G_t^B f\) is an \(m\)-version of \(T_t f, G_t f\), respectively for any \(f \in L^2(B,m)\). Since \(P_t^B 1_A(x) \leq P_t 1_A(x)\) for any \(A \in \mathcal{B}(B), x \in B\) and \(m\) has full support on \(E, A \mapsto P_t^B 1_A(x), A \in \mathcal{B}(B)\) is absolutely continuous with respect to \(1_B \cdot m\).

Hence there exists a (measurable) transition kernel density \(p_t^B(x,y), x,y \in B\), such that

\[
P_t^B f(x) = \int_B p_t^B(x,y) f(y) m(dy), \quad t > 0, \quad x \in B
\]

(3.1)

for \(f \in \mathcal{B}(B)\). Correspondingly, there exists a (measurable) resolvent kernel density \(r_t^B(x,y)\), such that

\[
R_\beta^B f(x) = \int_B r_\beta^B(x,y) f(y) m(dy), \quad \beta > 0, \quad x \in B
\]

for \(f \in \mathcal{B}(B)\). For a signed Radon measure \(\mu\) on \(B\), let us define

\[
R_\beta^B \mu(x) = \int_B r_\beta^B(x,y) \mu(dy), \quad \beta > 0, \quad x \in B
\]

whenever this makes sense. The process defined by

\[
X_t^\mu(\omega) = \begin{cases} X_t(\omega), & 0 \leq t < D^\mu(\omega) \\ \Delta, & t \geq D^\mu(\omega) \end{cases}
\]

is called the part process associated to \(\mathcal{E}_f^p\) and is denoted by \(\mathcal{M}_\mu\). The part process \(\mathcal{M}_\mu\) is a Hunt process on \(B\) (see [2, p.174 and Theorem A.2.10]). In particular, by [3, Theorem 4.1] \(\mathcal{M}_\mu\) satisfies the absolute continuity condition on \(B\).

A positive Radon measure \(\mu\) on \(B\) is said to be of finite energy integral if

\[
\int_B |f(x)| \mu(dx) \leq C \mathcal{E}_f^p(f,f), \quad f \in D(\mathcal{E}_f^p) \cap C_0(B),
\]

where \(C\) is some constant independent of \(f\) and \(C_0(B)\) is the set of all compactly supported continuous functions on \(B\). A positive Radon measure \(\mu\) on \(B\) is of finite energy integral (on \(B\)) if and only if there exists a unique function \(U_\mu \mu \in D(\mathcal{E}_f^p)\) such that

\[
\mathcal{E}_f^p(U_\mu \mu, f) = \int_B f(x) \mu(dx),
\]

for all \(f \in D(\mathcal{E}_f^p) \cap C_0(B)\). \(U_\mu \mu\) is called 1-potential of \(\mu\). In particular, \(R_\mu \mu\) is a version of \(U_\mu \mu\) (see e.g. [2, Exercise 4.2.2]). The measures of finite energy integral are denoted by \(S_\mu^0\). We further define \(S_\infty^\mu := \{\mu \in S_0^\mu | \mu(B) < \infty, \|U_\mu \mu\|_{L^{\infty}(\mathcal{B}, m)} < \infty\}\).

If \(\mu \in S_\infty^\mu\), then there exists a unique \(A \in A_{\mu,1}^B\) with \(\mu = \mu_A\), i.e. \(\mu\) is the Revuz measure of \(A\) (see [2, Theorem 5.1.6]). Here, \(A_{\mu,1}^B\) denotes the positive continuous additive functionals on \(B\) in the strict sense.
We define 
\[ E_k := \{ x \in \mathbb{R}^2 \mid 1/k < |x| < k \}, \] and 
\[ E := \bigcup_{k \geq 1} E_k = \mathbb{R}^2 \setminus \{0\}. \]

The following proposition recalls some properties of the Dirichlet form \((E, D(E))\) as stated in [7, Theorem 2.10, Lemma 3.13]:

**Proposition 3.2.**

(i) For any \( \alpha \in (-2, \infty) \), the Dirichlet form \((E, D(E))\) is conservative.

(ii) For any \( \alpha \in [0, \infty) \), \( \text{Cap}\left(\{0\}\right) = 0 \).

(iii) For any \( \alpha \in [0, \infty) \) and any \( x \in E \)
\[ \mathbb{P}\left( \lim_{k \to \infty} D_{E_k} = \infty \right) = \mathbb{P}\left( \lim_{k \to \infty} \sigma_{E_k} = \infty \right) = 1. \]

From now on till the end of this paper, we consider \( \rho(x) = |x|^{\alpha}, \ \alpha \in [2, \infty) \).

**Lemma 3.3.** Almost surely in \( X \), for any relatively compact open set \( G \subset E \),
\[ M_{\rho}^{\gamma}(G) \leq \sup_{x \in G} \rho(x) M_{\gamma}(G). \]

**Proof.** The statement follows from
\[ M_{\rho}^{\gamma}(G) \leq \sup_{x \in G} \rho(x) \int_G \exp\left( \gamma X_n(z) - \frac{\gamma^2}{2} E[X_n(z)^2] \right) dz. \]

**Lemma 3.4.** Let \( G \) be any relatively compact open set in \( E \) with \( G \subset \overline{G} \subset E \). For any \( x, y \in G \) and any \( \delta > 0 \)
\[ r_{1, \delta}^G(x, y) \leq c_1 \frac{1}{|x - y|^{\delta}}, \]
where \( c_1 > 0 \) is some constant.

**Proof.** By [7, Lemma 3.10] for \( m\text{-a.e.} \), \( x, y \in G \) and any \( \delta > 0 \)
\[ r_{1, \delta}^G(x, y) \leq c_1 \frac{1}{|x - y|^{\delta}}, \]
where \( c_1 > 0 \) is some constant. Since \( p_{\rho}(x, y) \) is jointly continuous, the statement holds for all \( x, y \in G \). \( \square \)

**Theorem 3.5.** Almost surely in \( X \), for any relatively compact open set \( G \subset \overline{G} \subset E \), \( 1_G \cdot M_{\rho}^{\gamma} \in S_{\infty}^G \).

**Proof.** Clearly, \( M_{\rho}^{\gamma}(G) < \infty \). By Lemma 3.3 and Lemma 3.4 for any \( x \in G \) and \( \delta > 0 \),
\[ R_{1, \delta}^G(1_G \cdot M_{\rho}^{\gamma})(x) \leq c_1 \int_{G} \frac{1}{|x - y|^{\delta}} M_{\rho}^{\gamma}(dy) \leq c_1 \sup_{y \in G} \rho(y) \int_{G} \frac{1}{|x - y|^{\delta}} M_{\gamma}(dy). \]
Since $G$ is a relatively compact open set, we can find a constant $R > 0$ such that $G \subset B_R(0)$. By [3] Theorem 2.2, there exist a constant $c_2 > 0$ and $\alpha > 0$ (depending on $R > 0$ and $\gamma$) such that for all $r \in (0, R)$ and $x \in B_R(0)$

\[ M_r(B_r(x)) \leq c_2 r^\alpha. \]

By taking $0 < \delta < \alpha$, we obtain

\[
\int_G \frac{1}{|x-y|^\delta} M_r(dy) \leq \int_{B_R(0)} \frac{1}{|x-y|^\delta} M_r(dy) \leq \sum_{n \geq 0} \int_{B_R(0) \cap \{2^{-n}R < |x-y| \leq 2^{-n+1}R\}} \frac{1}{|x-y|^\delta} M_r(dy)
\]

\[
\leq \sum_{n \geq 0} 2^{n \delta} R^{-\delta} M_r(B_{2^{-n+1}R}(x)) \leq c_2 \sum_{n \geq 0} 2^{n \delta - \alpha} < \infty.
\]

Hence,

\[ \|R^E_1(1_G \cdot M'_r)\|_{L^\infty(E,\mu)} < \infty, \]

and

\[ \int_G \int_G \frac{1}{|x-y|^\delta} M'_r(dy)M'_r(dx) < \infty. \quad (3.2) \]

By [2] Exercise 4.2.2, (3.2) implies that $1_G \cdot M'_r \in S^G_0$. Therefore, $1_G \cdot M'_r \in S^G_{00}$. \(\square\)

**Corollary 3.6.** Almost surely in $X$, $M'_r$ does not charge capacity zero sets.

**Proof.** Let $N \subset \mathbb{R}^2$ be an open set such that $\text{Cap}(N) = 0$. Note that by [2] Lemma 2.2.3, $M'_r(1_{E_k} \cap N) = 0$ (see Theorem 3.5). Then the statement follows from

\[ M'_r(N) \leq \sum_{k \geq 1} M'_r(1_{E_k} \cap N) + M'_r([0]) = 0. \]

\(\square\)

The proof of the following theorem is a slight modification of [6] Lemma 5.11 in our setting.

**Theorem 3.7.** Let $F^t_1$ be the positive continuous additive functional of $(X_{E_k}^t)_{t \geq 0}$ in the strict sense associated to $1_{E_k} \cdot M'_r \in S^G_{00}$. Then, $F^t_1 = F^t_{E_1} + 1 \cdot \sigma_{E_k}^c \text{E}_k$-a.s. for all $x \in E_k$. In particular, $F_t := \lim_{t \to \infty} F^t_1$, $t \geq 0$, is well defined in $A_{e,x}^{\cdot j}$, and related to $M'_r$ via the Revuz correspondence.

**Proof.** We denote the set of all bounded, non-negative Borel measurable functions on $E_k$ by $B^*_E(E_k)$. Fix $f \in B^*_E(E_k)$ and for $x \in E_{k+1}$ define

\[ f_k(x) := E_{x'} \left[ \int_0^{\sigma_{E_k}^c} e^{-s} f(X_s) dF^t_{E_k} \right]. \]

Since $f_k \in D(E^t_{E_k+1})$ and $f_k = 0$ $E^t_{E_k+1}$-q.e. on $E_{k+1}^c$, we have $f_k \in D(E^t_{E_k})$. For $x \in E_k$

\[ R^t_{E_k} (f 1_{E_k} \cdot M'_r)(x) = E_{x'} \left[ \int_0^{\sigma_{E_k}^c} e^{-s} f(X_s) dF^t_{E_k} \right]. \]

Then, for $g \in B^*_E(E_k) \cap L^2(E_k,m)$

\[ E^t_{E_k} \left( f_k, R^t_{E_k} g \right) = E^t_{E_k+1} \left( f_k, R^t_{E_k} g \right) \]

\[ = \int_{E_k} R^t_{E_k} g f 1_{E_{k+1}} \cdot M'_r = E_{x'}^t (R^t_{E_k} (f 1_{E_k} \cdot M'_r), R^t_{E_k} g). \]

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Therefore, \( f_k = R^{E_k}_{\alpha}(f 1_{E_k} \cdot M^\alpha) \) \( m \)-a.e. Since \( R^{E_k}_{\alpha}(f 1_{E_k} \cdot M^\alpha) \) is \( 1 \)-excessive for \( (R^{E_k}_{\alpha})_{\alpha>0} \), we obtain for any \( x \in E_k \)

\[
R^{E_k}_{\alpha}(f 1_{E_k} \cdot M^\alpha)(x) = \lim_{\alpha \to 0} \alpha R^{E_k}_{\alpha+1}(R^{E_k}_{\alpha}(f 1_{E_k} \cdot M^\alpha))(x)
= \lim_{\alpha \to 0} \int_{E_k} r^{E_k}_{\alpha+1}(x,y) R^{E_k}_{\alpha}(f 1_{E_k} \cdot M^\alpha)(y) \, m(dy)
= \lim_{\alpha \to 0} \int_{E_k} r^{E_k}_{\alpha+1}(x,y) f_k(y) \, m(dy) = \lim_{\alpha \to 0} \alpha R^{E_k}_{\alpha+1}f_k(x).
\]

Using in particular the strong Markov property, we obtain by direct calculation that the right hand limit equals \( f_k(x) \) for any \( x \in E_k \). Thus, we showed for all \( x \in E_k \)

\[
E_x \left[ \int_0^{\sigma_{E_k}} e^{-t} f(X_t) \, dF_t \right] = E_x \left[ \int_0^{\sigma_{E_k}} e^{-t} f(X_t) \, dF^{\alpha}_{t+1} \right].
\]

This implies that \( F^\alpha_t = F^{\alpha+1}_t, \forall t < \sigma_{E_k} \) \( P_z \)-a.s. for all \( x \in E_k \) (see e.g. [1] IV. (2.12) Proposition), and \( (F_t)_{t \geq 0} \) is well defined in \( A^{E_k}_{\alpha,t} \). Moreover, \( (F_t)_{t \geq 0} \) is associated with \( M^\alpha_k \) via the Revuz correspondence.

Finally, almost surely in \( X \), the time changed process \((Z_t)_{t \geq 0} \) on \( E \) can be defined as

\[
Z_t = X_{F_t^{-1}}, \quad t \geq 0,
\]

where \( F_t^{-1} := \inf \{ s > 0 \mid F_s > t \} \), which is called the Liouville distorted Brownian motion.

**Remark 3.8.** In [2] we considered the following assumptions: \( \rho(x) = |x|^\alpha, \alpha \in [2, \infty) \) and a symmetric (possibly) degenerate (uniformly weighted) elliptic \( 2 \times 2 \) matrix \( A = (a_{ij})_{i,j \in \mathbb{Z}^2} \), that is \( a_{ij} \in L^1_\text{loc}(\mathbb{R}^2,dx) \) and there exists a constant \( \lambda \geq 1 \) such that for \( dx \)-a.e. \( x \in \mathbb{R}^2 \)

\[
\lambda^{-1} \rho(x) \| \xi \|^2 \leq \langle A(x)\xi,\xi \rangle \leq \lambda \rho(x) \| \xi \|^2, \quad \forall \xi \in \mathbb{R}^2,
\]

and the symmetric bilinear form

\[
\mathcal{E}^A(f,g) = \frac{1}{2} \int_{\mathbb{R}^2} \langle A \nabla f, \nabla g \rangle \, dx, \quad f, g \in C^0_0(\mathbb{R}^2).
\]

The closure \((\mathcal{E}^A, D(\mathcal{E}^A)) \) of \((\mathcal{E}^A, C^0_0(\mathbb{R}^2)) \) is a strongly local, regular, symmetric Dirichlet form. It is known from [2] that there exists a Hunt process \((Y_t)_{t \geq 0} \) starting from all points in \( \mathbb{R}^2 \) associated with the Dirichlet form \((\mathcal{E}^A, D(\mathcal{E}^A)) \). Following the methods and techniques as in this section, we can construct the positive continuous additive functional \((H_t)_{t \geq 0} \) of \((Y_t)_{t \geq 0} \) in the strict sense w.r.t. \( M^\alpha_t \) and then the time changed process \( Y_t^{E_k^{-1}}, t \geq 0 \) on \( E \) can be defined in the same way.

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