Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy

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Abstract

Generalized Weierstrass representations for generic surfaces conformally immersed into four-dimensional Euclidean and pseudo-Euclidean spaces of different signatures are presented. Integrable deformations of surfaces in these spaces generated by the Davey-Stewartson hierarchy of integrable equations are proposed. Willmore functional of a surface is invariant under such deformations.

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1 Introduction

Surfaces and their deformations (dynamics) were the subject of intensive study for a long time both in mathematics and physics. Theories of immersion and deformations of surfaces have been developed in the classical differential geometry since the last century (see e.g. [1]-[3]). They continue to be an important part of the differential geometry (see [4]-[5]). In physics, surfaces are key ingredients in a broad variety of phenomena from surface waves in hydrodynamics to world-sheets in string theory (see e.g. [6]-[8]).

Analytic methods to study surfaces and their properties are of great interest both in mathematics and in physics. A classical example of such an approach is given by the Weierstrass representation for minimal surfaces (see e.g. [1]-[3]). This representation allows us to construct any minimal surface in the three-dimensional Euclidean space $\mathbb{R}^3$ via two holomorphic functions. It is the most powerful tool for an analysis of minimal surfaces.

Extensions of the old Weierstrass representation to generic nonminimal surfaces in $\mathbb{R}^3$ have been given in [9] and [10]-[11]. Looking differently these two extensions have occurred to be equivalent. The generalized Weierstrass representation proposed in [10]-[11] starts with the linear system

$$
\psi_z = p\varphi, \quad \varphi_{\overline{z}} = -p\overline{\psi}
$$

where $z = x + iy$, bar means complex conjugation, $\psi(z, \overline{z})$, $\varphi(z, \overline{z})$ are complex-valued functions while $p(z, \overline{z})$ is a real-valued one. Then conformal immersion of a surface into $\mathbb{R}^3$ with coordinates $X^1$, $X^2$, $X^3$ is defined by the formulae [10]-[11]

$$
X^2 + iX^1 = i\int_\Gamma \left( \overline{\psi} dz' - \overline{\varphi} d\overline{z}' \right),
$$

$$
X^3 = \int_\Gamma \left( \overline{\psi} \varphi dz' + \overline{\varphi} \psi d\overline{z}' \right) \quad (1.2)
$$

where $\Gamma$ is a contour in $\mathbb{C}$. The induced metric on a surface is given by

$$
ds^2 = \left( |\psi|^2 + |\varphi|^2 \right)^2 dz d\overline{z}, \quad (1.3)
$$

the mean curvature is

$$
H = 2 \frac{p}{|\psi|^2 + |\varphi|^2}. \quad (1.4)
$$
and the Willmore functional (see e.g. [12]) \( W = \int H^2 \, [dS] \) is equal to

\[
W = 4 \int p^2 \, dx \, dy
\]  

(1.5)

An advantage of the generalized Weierstrass formulae (1.1), (1.2) is also that it allows to construct new class of deformations of surfaces in \( R^3 \) \([10]-[11]\). They are generated by evolution of \( p \) via the modified Veselov-Novikov equation and of \( \psi, \varphi \) via certain linear equations. A characteristic feature of these integrable deformations is that the Willmore functional remains invariant.

The generalized Weierstrass representation (1.1), (1.2) has been proved to be an effective tool to study generic surfaces in \( R^3 \) and their deformations. In differential geometry its use has allowed to obtain several interesting results both of local and global character (see e.g. \([13]-[20]\)). In physics, it has been applied to study of various problems in theory of liquid membranes, 2D gravity and string theory \([13],[21]-[23]\). In the string theory the functional \( W = \int H^2 \, [dS] \) is known as the Polyakov extrinsic action and in membrane theory it is the Helfrich free energy \([3]-[8]\).

An extension of the Weierstrass representation to multidimensional spaces would be of a great interest. In physics, a strong motivation lies in the Polyakov string integral over surfaces in multidimensional spaces \([3]-[8]\). Theory of immersion of surfaces into four-dimensional spaces is an important part of the contemporary differential geometry too (see e.g. \([1],[3],[12],[24]-[27]\)).

In this paper we present extensions of the generalized Weierstrass representation (1.1), (1.2) to the cases of generic surfaces immersed into the four-dimensional spaces \( R^4 \), \( R^{3,1} \) and \( R^{2,2} \) with the metrics \( g_{ik} = \text{diag}(1,1,1,1) \), \( g_{ik} = \text{diag}(1,1,1,-1) \) and \( g_{ik} = \text{diag}(1,1,-1,-1) \), respectively. A basic linear system consists of the two two-dimensional Dirac equations while the formulae of immersion are of the type (1.2). The induced metric, mean curvature and Willmore functional are given by formulae similar to (1.3)-(1.5).

Integrable deformations of surfaces are generated by the Davey-Stewartson hierarchy of 2 + 1-dimensional soliton equations. These deformations of surfaces inherit all remarkable properties of the soliton equations. Geometrically, such deformations are characterized by the invariance of an infinite set of functionals over surfaces. The simplest of them is given by the Willmore functional.
Note that the Weierstrass type representations for particular classes of surfaces in $R^4$ have been discussed in [25],[28] and recently in [29]. Formulae of the Weierstrass type for immersion of surfaces into $R^4$ can be derived also within the quaternionic approach [30].

2 Generalized Weierstrass representations

Main steps in our construction of the generalized Weierstrass formulae for surfaces in four-dimensional spaces are basically the same as those in $R^3$ (see also [29]).

Proposition 1 The generalized Weierstrass formulae

$$X^1 + iX^2 = \int_{\Gamma} (-\varphi_1 \varphi_2 dz' + \psi_1 \psi_2 d\bar{z}'),$$
$$X^1 - iX^2 = \int_{\Gamma} (\bar{\psi}_1 \bar{\psi}_2 dz' - \bar{\varphi}_1 \bar{\varphi}_2 d\bar{z}'),$$
$$X^3 + iX^4 = \int_{\Gamma} (\bar{\psi}_2 \varphi_1 dz' + \psi_1 \varphi_2 d\bar{z}'),$$
$$X^3 - iX^4 = \int_{\Gamma} (\bar{\psi}_1 \varphi_2 dz' + \psi_2 \varphi_1 d\bar{z}').$$ (2.1)

where

$$\psi_{1z} = p\varphi_1, \quad \psi_{2z} = \bar{p}\varphi_2, \quad \varphi_{1\bar{z}} = -\bar{p}\psi_1, \quad \varphi_{2\bar{z}} = -p\psi_2,$$ (2.2)

$\psi_\alpha, \varphi_\alpha (\alpha = 1, 2), p$ are complex-valued functions of $z, \bar{z}, \Gamma$ is a contour in $C$, define the conformal immersion of a surface into $R^4$: $X^i(z, \bar{z}) : C \to R^4$. The induced metric of a surface is of the form

$$ds^2 = u_1 u_2 dz d\bar{z}$$ (2.3)

where $u_\alpha = |\psi_\alpha|^2 + |\varphi_\alpha|^2, \alpha = 1, 2$, the Gaussian and mean curvatures are respectively

$$K = -\frac{2}{u_1 u_2} [\log (u_1 u_2)]_{z\bar{z}} , \quad \bar{H}^2 = \frac{4|p|^2}{u_1 u_2},$$ (2.4)

the total squared mean curvature $W = \int \bar{H}^2 [dS]$ (Willmore functional) is

$$W = 4 \int |p|^2 dx dy.$$ (2.5)
Indeed, equations (2.2) imply that

\[(\psi_1 \psi_2)_z = -(\varphi_1 \varphi_2)_z, \quad (\psi_1 \psi_2)_\bar{z} = (\varphi_1 \varphi_2)_\bar{z}.\]  

(2.6)

In virtue of (2.6) the r.h.s. of (2.1) do not depend on the contour \(\Gamma\) and, hence, the coordinates \(X^i\) \((i = 1, 2, 3, 4)\) given by (2.1) are defined uniquely up to the displacement constants. Induced metric and Gaussian curvature are calculated straightforwardly. The mean curvature vector \(\vec{H} = \frac{\vec{X}_z}{g_{zz}}\) is given by

\[\vec{H} = 2 \frac{u_1 u_2}{|p\psi_2 \varphi_1 + \overline{p\psi_1 \varphi_2}|} | \psi_1|^2 + |\varphi_1|^2 | \psi_2|^2 + |\varphi_2|^2 \]

(2.7)

where \(Re, Im\) denote the real and imaginary parts, respectively. The formula (2.7) yields the expression (2.4) for \(\vec{H}^2\) and, finally, (2.5).

Since equations (2.2) contain two arbitrary real-valued functions of two variables \((Re(p)\) and \(Im(p)\)) the Weierstrass representation (2.1)-(2.2) provides us a generic surface in \(R^4\).

**Corollary 1** Surfaces of a constant mean curvature \(H^2 = H^2 = \text{constant}\) in \(R^4\) are generated by the formulae (2.1) where \(\psi_\alpha, \varphi_\alpha\) \((\alpha = 1, 2)\) obey the system of equations

\[\psi_\alpha = \frac{H}{2} e^{i\theta} \sqrt{(|\psi_1|^2 + |\varphi_1|^2) (|\psi_2|^2 + |\varphi_2|^2)} \varphi_\alpha, \]

\[\varphi_\alpha = -\frac{H}{2} e^{-i\theta} \sqrt{(|\psi_1|^2 + |\varphi_1|^2) (|\psi_2|^2 + |\varphi_2|^2)} \psi_\alpha, \quad \alpha = 1, 2\]  

(2.8)

where \(\theta(z, \bar{z})\) is an arbitrary function.

At the particular case \(\overline{p} = p\) the formulae (2.1)-(2.3) are reduced to those derived in [29] with the substitution \(X^1 \leftrightarrow X^2, X^3 \leftrightarrow -X^3\). This case corresponds to a special class of surfaces in \(R^4\). The reduction \(\psi_1 = \pm \psi_2, \varphi_1 = \pm \varphi_2\) (consequently \(\overline{p} = p\)) converts (2.1)-(2.3) into the generalized Weierstrass representation (1.1)-(1.2) for generic surfaces in \(R^3\).

Note that a linear system of the form (2.2) arises also as the restriction of the Dirac equation to a surface in \(R^4\) [31].

The case of the pseudo-Euclidean space \(R^{2,2}\) with the metric \(g_{ik} = \text{diag}(1, 1, -1, -1)\) is rather similar to that of \(R^4\).
Proposition 2  The generalized Weierstrass formulae

\[ \begin{align*}
X^1 + iX^2 &= \int_\Gamma (\varphi_1 \varphi_2 d\bar{z}' + \psi_1 \psi_2 d\bar{z}), \\
X^1 - iX^2 &= \int_\Gamma (\overline{\psi_1} \overline{\varphi_2} d\bar{z}' + \overline{\varphi_1} \overline{\psi_2} d\bar{z}), \\
X^3 + iX^4 &= i \int_\Gamma (\overline{\psi_1} \varphi_2 d\bar{z}' + \psi_2 \overline{\varphi_1} d\bar{z}'), \\
X^3 - iX^4 &= -i \int_\Gamma (\overline{\varphi_1} \psi_2 d\bar{z}' + \varphi_2 \overline{\psi_1} d\bar{z}')
\end{align*} \]

(2.9)

where

\[ \begin{align*}
\psi_1 &= p \varphi_1, & \psi_2 &= \overline{p} \varphi_2, \\
\varphi_1 &= \overline{p} \psi_1, & \varphi_2 &= p \psi_2
\end{align*} \]

(2.10)

\( \psi_\alpha, \varphi_\alpha, p \) are complex-valued functions, \( \Gamma \) is a contour in \( \mathbb{C} \), define the conformal immersion \( \vec{X} : \mathbb{C} \to \mathbb{R}^{2,2} \) of a surface into the space \( \mathbb{R}^{2,2} \). The induced metric is

\[ ds^2 = v_1 v_2 d\bar{z}d\bar{z} \]

(2.11)

where \( v_\alpha = |\psi_\alpha|^2 - |\varphi_\alpha|^2, \alpha = 1, 2, \) the Gaussian and mean curvature are of the form

\[ \begin{align*}
K &= -\frac{2}{v_1 v_2} \left[ \log (v_1 v_2) \right]_{\bar{z} \bar{z}}, & \bar{H}^2 &= -\frac{4|p|^2}{v_1 v_2}
\end{align*} \]

(2.12)

and the Willmore functional \( W = \int \bar{H}^2 |dS| \) is given by

\[ W = -4 \int |p|^2 dx dy \]  

(2.13)

The proof is similar to the case of \( \mathbb{R}^4 \), only now the equations (2.10) give \( (\psi_1 \psi_2)_z = (\varphi_1 \varphi_2)_\bar{z} \) and \( (\psi_1 \varphi_2)_z = (\varphi_1 \psi_2)_\bar{z} \). In the particular case \( \overline{p} = p \) the formulae (2.9)-(2.13) are reduced to those obtained in [29] with the substitution \( X^1 \leftrightarrow X^2, X^3 \leftrightarrow X^4 \). In contrast to [29] the formulae (2.9)-(2.10) allow to represent an arbitrary surface in \( \mathbb{R}^{2,2} \).

Surfaces in \( \mathbb{R}^{2,2} \) with the constant \( \bar{H}^2 \) are generated by (2.9) where \( \psi_\alpha, \varphi_\alpha \) obey the system (2.8) with obvious change of signs. Note that the space \( \mathbb{R}^{2,2} \) is of importance also in string theory [32].

Conformal immersions into the Minkowski space \( \mathbb{R}^{3,1} \) are given by slightly different formulae.
Proposition 3  The Weierstrass type formulae

\[
X^1 + iX^2 = \int_{\Gamma} \left( \overline{\psi}_2 \varphi_1 dz' + \psi_1 \overline{\varphi}_2 d\overline{z}' \right),
\]
\[
X^1 - iX^2 = \int_{\Gamma} \left( \overline{\psi}_1 \varphi_2 dz' + \psi_2 \overline{\varphi}_1 d\overline{z}' \right),
\]
\[
X^3 + X^4 = \int_{\Gamma} \left( \overline{\psi}_1 \varphi_1 dz' + \psi_1 \overline{\varphi}_1 d\overline{z}' \right),
\]
\[
X^3 - X^4 = -\int_{\Gamma} \left( \overline{\psi}_2 \varphi_2 dz' + \psi_2 \overline{\varphi}_2 d\overline{z}' \right)
\]

(2.14)

where

\[
\psi_{\alpha z} = p \varphi_{\alpha}, \quad \varphi_{\alpha \overline{z}} = q \psi_{\alpha}, \quad \alpha = 1, 2
\]

(2.15)

p and q are real-valued functions, \( \Gamma \) is a contour in \( \mathbb{C} \), define the conformal immersion of a surface into the Minkowski space \( \mathbb{X} : \mathbb{C} \rightarrow R^{3,1} \). The induced metric on a surface is

\[
ds^2 = |\psi_1 \varphi_2 - \psi_2 \varphi_1|^2 dz d\overline{z},
\]

(2.16)

the mean curvature \( \overline{H}^2 \) and the Willmore functional are given respectively by

\[
\overline{H}^2 = -\frac{4pq}{|\psi_1 \varphi_2 - \psi_2 \varphi_1|^2}, \quad W = -4 \int pq dx dy.
\]

(2.17)

In this case the linear system (2.15) implies that

\[
\left( \psi_{\alpha \overline{z}} \right)_z = \left( \varphi_{\alpha \overline{z}} \right)_\overline{z}, \quad \alpha, \beta = 1, 2
\]

that guarantee an independence of the r.h.s. of (2.14) on the choice of the contour \( \Gamma \) of integration. The rest is straightforward.

Since again one has two arbitrary real-valued functions \( p \) and \( q \), the Weierstrass type formulae (2.14), (2.15) allow us to construct any surface immersed into \( R^{3,1} \).

Surfaces of constant \( \overline{H}^2 \) are generated by the formulae (2.14), where \( \psi_\alpha \) and \( \varphi_\alpha \) obey the system of equations (2.15) with the constraint

\[
pq = -\frac{1}{4} \overline{H}^2 |\psi_1 \varphi_2 - \psi_2 \varphi_1|^2.
\]

(2.18)
Special classes of surfaces which correspond to the cases $q = \text{constant}$ or $p = \pm q$ could be of particular interest.

Differential versions of all three generalized Weierstrass representations given above can be written in the following common form

$$d \left( \sum_{i=1}^{4} \tau_i X^i \right) = \Phi_2^\dagger \begin{pmatrix} 0 & dz \\ d\bar{z} & 0 \end{pmatrix} \Phi_1$$

(2.19)

where $\dagger$ denotes Hermitian conjugation. In the case of immersion into $R^4$ one has

$$\tau_1 = \sigma_1 , \quad \tau_2 = \sigma_2 , \quad \tau_3 = \sigma_3 , \quad \tau_4 = i\sigma_4$$

and

$$\Phi_\alpha = \begin{pmatrix} \psi_\alpha & -\bar{\varphi}_\alpha \\ \varphi_\alpha & \psi_\alpha \end{pmatrix} , \quad \alpha = 1, 2$$

(2.20)

where $\sigma_1, \sigma_2, \sigma_3$ are the standard Pauli matrices and $\sigma_4$ is an identical $2 \times 2$ matrix. At the $R^{2,2}$ case

$$\tau_1 = \sigma_1 , \quad \tau_2 = \sigma_2 , \quad \tau_3 = i\sigma_3 , \quad \tau_4 = \sigma_4$$

and

$$\Phi_\alpha = \begin{pmatrix} \psi_\alpha & -\bar{\varphi}_\alpha \\ \varphi_\alpha & \psi_\alpha \end{pmatrix} , \quad \alpha = 1, 2$$

(2.21)

Finally, the immersion into the Minkowski space $R^{3,1}$ corresponds to

$$\tau_i = \sigma_i \quad (i = 1, 2, 3, 4)$$

and

$$\Phi_1 = \Phi_2 = \begin{pmatrix} \psi_1 & \psi_2 \\ \varphi_1 & \varphi_2 \end{pmatrix} .$$

(2.22)

In fact, one can start with the formulae (2.19) to derive the Weierstrass representations in the forms (2.1)-(2.2), (2.9)-(2.10) and (2.14)-(2.15). Indeed, one can show that the $1$–form in the r.h.s. of (2.19) is closed if the $2 \times 2$ matrices $\Phi_1, \Phi_2$ obey the Dirac equations

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \Phi_1 = \begin{pmatrix} u & p \\ q & v \end{pmatrix} \Phi_1 , \quad \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \Phi_2 = \begin{pmatrix} -\bar{\varphi} & \varphi \\ \bar{\psi} & -\psi \end{pmatrix} \Phi_2$$

(2.23)
where \( p, q, u, v \) are arbitrary complex-valued functions. Functions \( u \) and \( v \) always can be converted to zeros by gauge transformation (redefinition of \( \Phi \)). Then the reality conditions for \( X^i \) are satisfied if matrices \( \Phi_\alpha \) have the form (2.20), (2.21) or (2.22) while the functions \( p, q \) should obey the constraints 
\[
p + \overline{q} = 0, \quad p - \overline{q} = 0, \quad p = \overline{p}, \quad q = \overline{q},
\]
respectively. Consequently, the corresponding formula (2.19) gives rise to the Weierstrass representations considered above.

A formula of the type (2.19) appears naturally [30] in the quaternionic approach to surfaces (see also [33]–[35]) which could provide an invariant formulation of the construction presented above.

Following to [29], one can extend the generalized Weierstrass representations to generic surfaces immersed into four-dimensional Riemann spaces. In particular, in the cases of spaces of constant curvature and conformally-flat spaces the Willmore functional still has the form (2.3), (2.13) and (2.17).

Explicit construction of surfaces via the Weierstrass formulae requires the resolving the linear systems (2.2), (2.10) and (2.15). The methods of solving these type of problems are well-developed now within the theory of \((2 + 1)\)–dimensional integrable (soliton) equations (see e.g. [36]–[38]). Using these methods one can construct broad classes of surfaces explicitly. The results will be presented in a separate paper.

### 3 Integrable deformations

Now, following the general approach of [10]–[11], we will construct integrable deformations of surfaces generated by the Weierstrass type formulae. The basic idea is to use deformations of the functions \( p, q, \psi, \varphi \) compatible with the linear equations (2.2), (2.10) and (2.15).

All of them are the particular cases of the linear system

\[
\begin{align*}
\psi_z &= p \varphi, \\
\varphi_z &= q \psi
\end{align*}
\]

(3.1)

where \( p \) and \( q \) are in general complex-valued functions. In soliton theory this system is known as the Davey-Stewartson II (DSII) linear problem (see e.g. [39]–[41]).

An infinite hierarchy of nonlinear differential equations associated with (3.1) is referred as the DSII hierarchy. It arises as the compatibility conditions
of (3.1) with the systems \[36\]-\[38\]

\[
\begin{align*}
\psi_{t_n} &= A_n \psi + B_n \varphi , \\
\varphi_{t_n} &= C_n \psi + D_n \varphi .
\end{align*}
\] (3.2)

where \(t_n\) are new (deformation) variables and \(A_n, B_n, C_n, D_n\) are differential operators of \(n\)-th order. At \(n = 1\) one gets the linear system

\[
\begin{align*}
pt_1 &= \alpha p + \gamma p_z , \\
qt_1 &= \gamma q_z + \alpha q .
\end{align*}
\] (3.3)

where \(\alpha, \gamma\) are arbitrary constants. The corresponding operators in (3.2) are

\[
\begin{align*}
A_1 &= \alpha \partial_z , \\
B_1 &= \gamma p , \\
C_1 &= \alpha q , \\
D_1 &= \gamma \partial_z .
\end{align*}
\] (3.4)

Higher equations are nonlinear ones. At \(n = 2\) one has the system \[36\]-\[38\]

\[
\begin{align*}
p_{t_2} &= \alpha_2 (p_{zz} + p_{zzz} + up) , \\
q_{t_2} &= -\alpha_2 (q_{zz} + q_{zzz} + uq) , \\
u_{z\pi} &= -2(pq)_{zz} - 2(pq)_{z\pi}.
\end{align*}
\] (3.5)

where \(\alpha_2\) is an arbitrary constant. For the system \[35\]

\[
\begin{align*}
A_2 &= \alpha_2 (\partial_z^2 + w_1) , \\
B_2 &= \alpha_2 (p_z - p \partial_z) , \\
C_2 &= -\alpha_2 (q \partial_z - q \partial_z^2) , \\
D_2 &= -\alpha_2 (\partial_z^2 + w_2) .
\end{align*}
\] (3.6)

where

\[
\begin{align*}
w_{1z} &= -2(pq)_z , \\
w_{z\pi} &= -2(pq)_{\pi} , \\
u &= w_1 - w_2 .
\end{align*}
\]

For the \(t_3\)-deformations one has (see e.g. \[37\])

\[
\begin{align*}
p_{t_3} &= p_{zzz} + p_{z\pi\pi} + 3p_z \partial_z^{-1} (pq)_z + 3p \partial_z^{-1} (pq)_\pi + 3p \partial_z^{-1} (qp)_z + 3p \partial_z^{-1} (qp)_{\pi}\pi , \\
q_{t_3} &= q_{zzz} + q_{z\pi\pi} + 3q_z \partial_z^{-1} (pq)_z + 3q \partial_z^{-1} (pq)_\pi + 3q \partial_z^{-1} (pq)_{\pi\pi} + 3q \partial_z^{-1} (pq)_{z\pi} .
\end{align*}
\] (3.7)

In this case

\[
\begin{align*}
A_3 &= \partial_z^2 + 3 [\partial_z^{-1} (pq)_{z\pi}] \partial_z + 3 [\partial_z^{-1} (pq)_{z\pi}] , \\
B_3 &= -p \partial_z^2 + p_z \partial_z - p_{zzz} - 3p [\partial_z^{-1} (pq)_z] , \\
C_3 &= -q \partial_z^2 + q_z \partial_z - q_{zzz} - 3q [\partial_z^{-1} (pq)_{z\pi}] , \\
D_3 &= \partial_z^3 + 3 [\partial_z^{-1} (pq)_z] \partial_z + 3 [\partial_z^{-1} (pq)_{z\pi}] .
\end{align*}
\] (3.8)
All equations of this DSII hierarchy, known also as the 2–component KP hierarchy, are the \( (2+1) \)–dimensional soliton equations. They are integrable by the inverse spectral transform method (see e.g. \([10]-[18]\)) and have all remarkable properties typical for soliton equations: namely, they possess infinite classes of explicit solutions, including multi-soliton solutions, infinite symmetry algebra, Darboux and Backlund transformations etc. They have an infinite set of integrals of motion. The simplest of them is

\[
C_1 = \int pq \, dxdy
\]  

(3.9)

while other integrals of motion are non-local. Note that (3.9) is the integral of motion for the whole DSII hierarchy \((C_{1n} = 0)\).

So, now we assume that all quantities (except \(z, \bar{z}\)) in the systems (2.2), (2.10) and (2.15) and, consequently, coordinates \(X^i\) depend on the deformation parameters \(t_n (n = 1, 2, \ldots)\) (times). Then, we assume that this dependence on \(t_n\) is such that there are operators \(A_n, B_n, C_n, D_n\) such that (3.2) holds. The compatibility conditions of (2.2), (2.10) or (2.15) with (3.2) fix the dependence of \(\psi, \varphi\) and \(p, q\) on \(t_n\) and, consequently, define the deformations of surfaces. Concrete cases are governed by different specialization (reductions) of the DSII hierarchy.

Let us consider first immersions into the Minkowski space \(R^{3,1}\). In this case \(p\) and \(q\) are real-valued functions. The corresponding deformations are generated by the ”real” DSII hierarchy (with real-valued \(p\) and \(q\)). In particular, in equations (3.3), (3.5) the constants \(\alpha, \gamma, \alpha_2\) are real. Since (3.9) is obviously the integral of motion also for real-valued \(p\) and \(q\), then, in virtue of (2.17), the Willmore functional \(W\) remains invariant under these DSII deformations.

For the Weierstrass representations in the spaces \(R^4\) and \(R^{2,2}\) we have the linear system (3.1) with the following reductions:

\[
\begin{pmatrix}
0 & p_1 \\
q_1 & 0
\end{pmatrix} =
\begin{pmatrix}
0 & \varepsilon p \\
\varepsilon \bar{p} & 0
\end{pmatrix} \quad \text{for } \Phi_1
\]  

(3.10)

and

\[
\begin{pmatrix}
0 & p_2 \\
q_2 & 0
\end{pmatrix} =
\begin{pmatrix}
0 & \varepsilon \bar{p} \\
\varepsilon p & 0
\end{pmatrix} \quad \text{for } \Phi_2
\]  

(3.11)

where \(\varepsilon = -1\) for \(R^4\) and \(\varepsilon = 1\) for \(R^{2,2}\). Both the reductions (3.10) and (3.11) are admissible by all equations of the DSII hierarchy if one chooses
(see e.g. \[30\]-\[33\])

\[
A_{2n-1} = \partial_z^{2n-1} + \ldots, \quad D_{2n-1} = \partial_z^{2n-1} + \ldots, \\
A_{2n} = \pm i \partial_z^{2n} + \ldots, \quad D_{2n} = \mp i \partial_z^{2n} + \ldots \quad (n = 1, 2, 3, \ldots)
\]

In our case we have different linear problems for \(\psi_1, \varphi_1\) and \(\psi_2, \varphi_2\). To have the same equations for \(p\) it is sufficient to take

\[
A_{2n-1} = \partial_z^{2n-1} + \ldots, \quad D_{2n-1} = \partial_z^{2n-1} + \ldots, \\
A_{2n} = \pm i \partial_z^{2n} + \ldots, \quad D_{2n} = \mp i \partial_z^{2n} + \ldots \quad (3.12)
\]

in equations (3.2) for \(\psi_1, \varphi_1\) and

\[
A_{2n-1} = \partial_z^{2n-1} + \ldots, \quad D_{2n-1} = \partial_z^{2n-1} + \ldots, \\
A_{2n} = \mp i \partial_z^{2n} + \ldots, \quad D_{2n} = \pm i \partial_z^{2n} + \ldots \quad (3.13)
\]

in the case of \(\psi_2, \varphi_2\). In particular, one gets

\[
p_{t_2} = i \left( p_{zz} + p_{\bar{z}z} + u p \right), \quad u_{z\bar{z}} = -2\varepsilon |p|_{z\bar{z}}^2 - 2\varepsilon |p|_{z\bar{z}}^2
\]

and

\[
\psi_{1t_2} = i \left( \partial_z^2 + w_1 \right) \psi_1 + i \left( p_z - p \partial_z \right) \varphi_1, \\
\varphi_{1t_2} = -i \varepsilon \left( \bar{p}_z - \bar{p} \partial_z \right) \psi_1 - i \left( \partial_z^2 + w_2 \right) \varphi_1 \quad (3.15)
\]

while

\[
\psi_{2t_2} = -i \left( \partial_z^2 + w_1 \right) \psi_2 - i \left( \bar{p}_z - \bar{p} \partial_z \right) \varphi_2, \\
\varphi_{2t_2} = i \varepsilon \left( p_{\bar{z}z} - p \partial_z \right) \psi_2 + i \left( \partial_z^2 + w_2 \right) \varphi_2 \quad (3.16)
\]

where \(w_{1z} = -2\varepsilon |p|_{z\bar{z}}^2, \quad w_{1\bar{z}} = 2\varepsilon |p|_{z\bar{z}}^2\).

The \(t_3\) deformation is given now by the equation

\[
p_t = p_{zzz} + p_{\bar{z}zz} + 3\varepsilon p_z \partial_z^{-1}(|p|_z^2) + 3\varepsilon p \partial^{-1}(\bar{p} |p|_z^2) + 3\varepsilon p \partial^{-1}(\bar{p} |p|_{z\bar{z}}) \quad (3.17)
\]

and the deformations of \(\psi_1, \varphi_1\) and \(\psi_2, \varphi_2\) are given by (3.3), (3.8) with the reduction (3.10) and (3.11), respectively.

Thus, in the cases of \(R^4\) and \(R^{2,2}\) deformations of surfaces are generated by the proper DSII equation (3.14) and the corresponding hierarchy. Properties of solutions of the DSII equation (3.14) are essentially different for different signs of \(\varepsilon\). Consequently, the properties of deformations of surfaces in \(R^4\) and \(R^{2,2}\) will differ too.
In both cases $C_1 = \int |p|^2 dx dy$ is the integral of motion for the whole hierarchy. Hence, the Willmore functional $W$ for surfaces immersed into $R^4$ and $R^{2.2}$ is invariant under deformations generated by the DSII hierarchy.

Thus, though the deformations for surfaces in $R^4$, $R^{3.1}$ and $R^{2.2}$ are governed by different nonlinear integrable equations, they have the following common property.

**Proposition 4** The DSII hierarchy generates integrable deformations of surfaces immersed into $R^4$, $R^{3.1}$ and $R^{2.2}$ via the generalized Weierstrass representations. The Willmore functionals $W$ for surfaces in $R^4$, $R^{3.1}$ and $R^{2.2}$ are invariant under the corresponding deformations ($W_{t_n} = 0$).

DSII hierarchy of integrable equations is well studied (see e.g. [36]-[38]). This provides us a broad class of deformations of surfaces in $R^4$, $R^{3.1}$ and $R^{2.2}$ given explicitly. Moreover, since the inverse spectral transform method allows us to linearize the initial-value problem

$$p(z, \bar{z}, t_n = 0), q(z, \bar{z}, t_n = 0) \rightarrow p(z, \bar{z}, t_n), q(z, \bar{z}, t_n)$$

for soliton equations of the DSII hierarchy (see e.g. [36]-[38]), then the generalized Weierstrass formulae allows us to linearize the initial-value problem for the deformation of surfaces $\tilde{X}(z, \bar{z}, 0) \rightarrow \tilde{X}(z, \bar{z}, t_n)$. In virtue of all that, the deformations generated by the DSII hierarchy can be referred as integrable one.

Higher integrals of motion for the DSII hierarchy are also certain functionals on surfaces invariant under deformations generated by the DSII hierarchy. Since the Willmore functional $W$ is invariant under the conformal transformations in four-dimensional spaces, then it is quite natural to suggest

**Conjecture 1** Higher integrals of motion for the DSII hierarchy are functionals on surfaces in $R^4$, $R^{3.1}$ and $R^{2.2}$ which are invariant under conformal transformations in these spaces.

For surfaces in $R^3$ an analogous conjecture has been proved in [17].

In the particular case $p = \bar{p}$ the Weierstrass representations for $R^4$ and $R^{2.2}$ are reduced to those derived in [24]. The reduction $p = \bar{p}$ is admissible only by deformations associated with odd times $t_{2n-1}$. In this case equation (3.17) is nothing but the modified Veselov-Novikov equation and the DSII
sub-hierarchy associated with only odd times is converted into the mVN hierarchy. This specialized class of surfaces in $R^4$ and $R^{2,2}$ and their deformations via the mVN hierarchy have been considered in [29].

For the case of Minkowski space $R^{3,1}$ there is a class of surfaces for which $q = 1$. Integrable deformations of such class of surfaces are generated by the DSII hierarchy under the reduction $q = 1$. This reduction is compatible only with odd times deformations. For instance, the $t_3$–deformation is of the form (reduction $q = 1$ of the system (3.7))

$$p_t = p_{zzz} + p_{zzzz} + 3 \left[ p \partial_z^{-1}(p_z) \right]_z + 3 \left[ p \partial_z^{-1}(p_z) \right]_{zz}$$

(3.18)

that is the Veselov-Novikov equation [30]–[38]. So, integrable deformations of surfaces with $q = 1$ are generated by the Veselov-Novikov hierarchy.

Generalized Weierstrass formulae with specialized $p = p(x)$ and $q = q(x)$ ($x = \text{Re}(z)$) give rise obviously to surfaces of ”revolution” in four dimensional spaces. In this case the DSII hierarchy is reduced to the AKNS hierarchy of the $(1+1)$–dimensional soliton equations (see e.g. [36]–[38]). Equation (3.14) is converted into the well-known nonlinear Schroedinger (NLS) equation

$$ip_{t_2} + \frac{1}{2}p_{xx} - 2\varepsilon|p|^2p = 0$$

(3.19)

So, integrable deformations of surfaces of revolution in $R^4$ and $R^{2,2}$ are generated by the NLS hierarchy. The NLS equation arises also in the description of integrable motion of space curves in $R^3$ [39]. So, there is an intimate connection between integrable deformations of surfaces of revolution in $R^4$ and integrable motions of space curves in $R^3$.

Finally, we note that in the particular case $\psi_1 = \pm \psi_2$, $\varphi_1 = \pm \varphi_2$ the Weierstrass representation (2.1)–(2.2) is reduced to that in $R^3$ [10]–[11]. Since then $\overline{p} = p$, the DSII hierarchy is reduced to the mVN hierarchy which generates integrable deformations of surfaces in $R^3$ [10]–[11].

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