Non-Stationary Streaming PCA

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Abstract

We consider the problem of streaming principal component analysis (PCA) when the observations are noisy and generated in a non-stationary environment. Given $T$, $p$-dimensional noisy observations sampled from a non-stationary variant of the spiked covariance model, our goal is to construct the best linear $k$-dimensional subspace of the terminal observations. We study the effect of non-stationarity by establishing a lower bound on the number of samples and the corresponding recovery error obtained by any algorithm. We establish the convergence behaviour of the noisy power method using a novel proof technique which maybe of independent interest. We conclude that the recovery guarantee of the noisy power method matches the fundamental limit, thereby generalizing existing results on streaming PCA to a non-stationary setting.

1 Introduction

Principal component analysis is one of the most extensively studied methods for constructing linear low-dimensional representation of high-dimensional data. Modern applications such as privacy preserving distributed computations (Hardt and Roth (2013)), covariance estimation of high-frequency data (Chang et al. (2018), Aıt-Sahalia et al. (2010)), detecting power grid attacks (Bienstock and Shukla (2018), Escobar et al. (2018)) etc. require design of sub-linear time algorithms with low storage overhead. Existing work on PCA has focused on design and analysis of single-pass (streaming) algorithms with near-optimal memory and storage complexity assuming stationarity of the underlying data-generating process. However, physical systems generating data for such applications undergo rapid evolution. For example, dynamic market behaviour leads to time-series data with volatile covariance matrices. Our understanding of such physical system crucially relies on accurate estimation of the data generating space. This motivates the design of memory and storage efficient algorithms for analyzing dynamic data. In this work, we consider the streaming PCA problem under the relaxed assumption of non-stationarity of the data generating process.

The streaming PCA problem is concerned with constructing a linear low-dimensional representation of the observed data $\{x_t\}_{t=1}^T$, in a single pass over the dataset with the goal of recovering the best top-$k$, $1 \leq k \leq p$, dimensional space representing the observations. We consider the streaming PCA problem when the observations are sampled from a spiked covariance model and relax the assumption of stationarity of the data generating distribution. Hence, the observations $\{x_t\}_{t=1}^T$ are sampled from a Gaussian
distribution with a time-varying covariance $A_t A_t^\top$ such that $\|A_t A_t^\top - A_{t-1} A_{t-1}^\top\|_2 \leq \gamma$, $0 < \gamma < 1$ and processed in blocks of size $B$ where $x_t$. Define the spectral gap as $\delta = \inf_{t>0} s_k(A_t A_t^\top + \sigma^2 I) - s_{k+1}(A_t A_t^\top + \sigma^2 I)$ where, $s_k(M)$ is the $k$-th largest singular value of matrix $M$. Our goal is to propose a memory and storage efficient algorithm to recover the best $k$-dimensional approximation of the last observed block of data. We define this problem as non-stationary streaming PCA. When $\gamma = 0$, we recover the streaming PCA problem.

### 1.1 Our Contributions

For non-stationary streaming PCA we study the following:

- **Price of non-stationarity**: When the observations are sampled from a non-stationary variant of the spiked covariance model (equation 2) we establish the fundamental performance limits of any algorithm for non-stationary streaming PCA. In section 2 we derive lower bounds on the expected recovery error, $\mathcal{R}(\delta, T, \gamma)$ for any given values of $T$ (the number of collected samples), rate of rotation $\gamma$, the spectral gap $\delta$ and any algorithm $\psi$. Theorem 2.1 establishes $\mathcal{R}(\delta, T, \gamma) = \Omega((\gamma^{1/3})^{1/3}(-\delta^{2}\sigma^2+\epsilon)^{1/3} + (\frac{1}{\sqrt{T}})(\frac{1}{\sqrt{T}})^{1/2})).$ Unlike the stationary case, for the non-stationary case we observe a phase transition in the recovery error as we collect more samples. We show that when the observations are $O(\gamma^{-2/3})$, the error decreases as the inverse of the square root of the number of observations so far. However, the recovery error beyond these observations stagnates to $O(\gamma^{1/3})$ and doesn’t decrease any further upon collecting more samples. In Theorem 3.2, we show that the noisy power method (algorithm 1) can guarantee $O(\gamma^{1/3})$ error when there are sufficient number of samples.

- **Optimal algorithm and analysis**: The noisy power method (algorithm 1) is a memory efficient and storage optimal algorithm for the streaming PCA problem (Mitliagkas et al. (2013)). In section 3 we show that in addition to being memory and storage optimal, it also achieves the fundamental performance limit for the non-stationary streaming PCA problem. However, there are no simple adaptations of existing proof strategies for the non-stationary problem.

To recover the true subspace spanned by the top-$k$ singular vectors of $A$ for the streaming PCA problem, one needs to mitigate the effect of sampling noise and ensure that each iteration improves the estimate of the true subspace. The sampling noise can be controlled using concentration of measure techniques and the estimates improve after every iteration since all the observations provide information about the same underlying space. Existing works such as Allen-Zhu and Li (2017) and Hardt and Price (2014) provide an in-depth analysis of the convergence behaviour of the noisy power method based on this general strategy.

In the non-stationary case in addition to bounding tails from time-varying distributions, there is also the challenge of tracking time-varying spaces which generate the observations. A straightforward generalization of conventional proof strategies to the non-stationary problem is not possible since existing methods require that the distance between the true subspace spanned by top-$k$ singular vectors and the subspace estimated every iteration decreases. Under the non-stationary model, however, the improvement in estimate after every iteration cannot be guaranteed, especially during the initial phase with the random initialization, since the data-generating space rotates and small improvements in estimates maybe negated by the larger rotations. To circumvent this issue, we
introduce a new proof technique based on analyzing the singular value and singular vectors of product of matrices. Precisely, we study the singular values and the left singular vectors of the product

\[(M(L) + \mathcal{E}(L))(M(L - 1) + \mathcal{E}(L - 1)) \cdots (M(1) + \mathcal{E}(1)),\]

where \(M(l)\) is the true covariance matrix at the \(l\)-th iteration and \(\mathcal{E}(l)\) is the noise matrix because of the sampling from time-varying distributions.

We analyze the convergence behaviour of the noisy power method in theorem 3.2. Using lemmas 3.2 and 3.3, we bound singular vectors and singular values of the iterates of the noisy power method and the data-generating subspaces. It turns out that the top-\(k\) left singular vectors of the matrix product are very close to the top-\(k\) left singular vectors of \(M(L)\) and the \(k\)-th largest singular value is much greater than the \(k + 1\)-th largest singular value, when \(\delta\) is large enough compared to the spectral norm of the noise matrices and \(\gamma\). We also recover the results for streaming PCA in corollary 3.1.

### 1.2 Comparison with existing work

This paper introduces non-stationary streaming PCA. To the best of our knowledge, there are no known results for this problem. Past work has exclusively focused on the stationary variant of this problem. Corollary 3.1 guarantees \(\varepsilon\)-accuracy with \(O\left(\frac{p \log(p/\varepsilon)}{\varepsilon^6 \log(\phi)}\right)\) samples (where \(\phi\) is an intricate function of \(\varepsilon, \gamma, \delta\), see A.2) for the streaming PCA problem. It should be noted that although we make assumptions about the spectral norm of the noise matrices, unlike previous work we do not make explicit assumptions on the amount of overlap between the noise matrix and the subspace to be recovered. We instead constraint the ratio of the \(k\)-th and \((k + 1)\)-th largest singular values to facilitate our proof technique.

When the observations are sampled from a generic distribution Hardt and Price (2014) guarantee \(\varepsilon\)-accuracy with \(O\left(\frac{k s_k(A)}{\varepsilon^2 (s_k(A) - s_{k+1}(A))}\right)\) samples. Balcan et al. (2016) improve upon the guarantees of Hardt and Price (2014) by reducing the spectral gap to difference between the \(k\)-th largest singular value and \(q\)-th largest singular value \(k \leq q \leq p\).

The best known convergence results for the streaming PCA problem are by Allen-Zhu and Li (2017). They establish fundamental performance limits and provide the first spectral-gap free global convergence guarantee for streaming PCA using Oja’s algorithm. Their lower bound for the case when observations are sampled from the spiked covariance model is \(\Omega\left(\frac{s_k(A)}{(s_k(A) - s_{k+1}(A))^2} \frac{1}{\delta \gamma T}\right)\) as compared to\(\Omega\left(\frac{(s_{k+1}(A) s_k(A))^{1/2}}{\delta \gamma T}\right)\). For Oja’s algorithm, they show that \(O\left(\frac{s_k(A)}{(s_k(A) - s_{k+1}(A))^2} \frac{1}{\varepsilon^5 \log(\phi)}\right)\) samples are sufficient for \(\varepsilon\)-accurate recovery. They also provide the first spectral-gap free analysis for the streaming PCA problem. A spectral-gap free analysis for the non-stationary streaming PCA problem remains an open question.

### 1.3 Other Related Work

Existing literature on streaming PCA focusses on the stochastic gradient descent based methods or the power method. The power method with a random \(n \times k\) initial matrix is well studied in Halko et al. (2011). Some variants of the power method e.g., Golub and Van Loan (2012) and Musco and Musco
(2015), enhance the convergence speed for the same target error \( \varepsilon \). Oja and Karhunen (1985) was the first work to propose a stochastic approximation based approach for recovering the top eigenvector of a matrix. Recently, Li et al. (2018) analyze Oja’s algorithm for recovering the top eigenvector as a stochastic approximation iteration. For recovering the top eigenvector, Shamir (2016a) provide a gap-dependent analysis which was made gap-independent in Shamir (2016b). The iterative eigenvector computation schemes are instances of stochastic approximation based solution for this problem (Arora et al. (2012)). Most literature on stochastic approximation assumes stationarity of the objective function in the optimization problem. Besbes et al. (2015) were the first to consider a non-stationary variant of stochastic approximation methods. They focus on convex optimization problems and focus on providing statistical guarantees rather than designing efficient algorithms.

### 1.4 Notation and preliminaries

We fix notations and preliminaries used throughout the main body of the paper. Vector spaces are denoted with blackboard bold letters \( \mathbb{R}^n \) representing the \( n \)-dimensional euclidean space and \( \mathbb{S}^{n-1} \) represents the \( n \)-dimensional sphere. Matrices are denoted by bold upper case letters and vectors are denoted by bold lower case letters. For a matrix \( M \in \mathbb{R}^{m \times n} \) we use the following notations. Let \( M^\top \) be the transpose of the matrix \( M \) and \( s_i(M) \) be the \( i \)-th largest singular value. Let \( M_i \) denote the \( i \)-th column of \( M \), \( m_{i,j} \) be the \( (i,j) \)-th element of \( M \) and \( M_{i,j,k,l} \) denote the sub-matrix of \( M \) consisting of elements from row \( i \) to \( j \) and columns \( k \) to \( l \) of the matrix \( M \). The singular value decomposition of \( M \) is defined as \( \text{SVD}(M) = UDV^\top \), where, \( UU^\top = I_{m \times m}, VV^\top = I_{n \times n} \). \( U \in \mathbb{R}^{m \times m}, V \in \mathbb{R}^{n \times n} \). \( D \) is a diagonal matrix with its \( i \)-th diagonal element given by \( s_i(M) \). We assume without loss of generality that the singular values and respective singular vectors are ordered from largest to smallest. \( U, D \) and \( V \) represent the left singular vectors, the diagonal matrix of singular values and the right singular vector of the matrix in context. \( \lambda_i(M) \) represents the \( i \)-th largest eigen value of the matrix \( M \). Let \( \text{span}(M) \) denote the range (column space) of \( M \) and let \( b(M) \) represent an orthonormal basis of \( \text{span}(M) \) when \( m > n \). This can be defined as the column space of \( M(M^\top M)^{-1/2} \) or can be obtained through a QR decomposition of \( M \). The precise use of \( b(\cdot) \) throughout the manuscript will be clear from the context. Let \( M_\perp \) denote the projection matrix of the orthogonal complement of range of \( M \) given by \( M_\perp = I - M(M^\top M)^{-1}M^\top \). The spectral gap of \( M \) is defined as the difference between the \( k \)-th and the \( (k+1) \)-th largest singular value of \( M \) denoted by \( \delta(M) = s_k(M) - s_{k+1}(M) \). In absence of ambiguity we use, \( \delta = \frac{\varepsilon}{2} \delta(M) \). We use \( \prod_{i=1}^T M^{(i)} = M^{(1)}M^{(2)} \cdots M^{(T)} \). We define the distance between the column space of \( M \in \mathbb{R}^{n \times k} \) and \( N \in \mathbb{R}^{n \times k} \) as \( d(M, N) = \| b(M)^\top N_\perp \|_2 \). \( d(M, N) \) is the projection distance on the Grassmannian manifold, \( Gr(k, n) \) (manifold of all \( k \)-dimensional subspaces of \( \mathbb{R}^n \)) (section 2.5 in Golub and Van Loan (2012) and lemma A.1). All constants \( C \) appearing in our results are independent of the problem parameters.

The spiked covariance model assumes that observations are lifted from a low-dimensional space and corrupted with high-dimensional Gaussian noise. Let \( x_t \in \mathbb{R}^p \) and \( k, k < p \) be the rank of the low-dimensional space. According to a spiked covariance model, \( x_t \) is generated as:

\[
x_t = Az_t + w_t, \quad i = 1 : T, \quad z_t \sim \mathcal{N}(0_{k \times 1}, I_{k \times k}), \quad w_t \sim \mathcal{N}(0_{p \times 1}, \sigma^2 I_{p \times p}), \quad A \in \mathbb{R}^{p \times k}
\]

The vectors \( z_t \in \mathbb{R}^k \) are independent and identically distributed, sampled from a multivariate Gaussian distribution with zero mean and identity covariance. The homoskedastic noise vectors \( w_t \in \mathbb{R}^n \) are also
independent and identically distributed with zero mean and covariance, \( \sigma^2 I_{p \times p} \). Samples drawn from a spiked covariance model follow a Gaussian distribution \( \mathcal{N}(0, AA^\top + \sigma^2 I) \).

We incorporate non-stationarity by allowing the underlying subspace \( A \) to rotate slowly with time. This subspace rotation characterized by parameter \( \gamma \) is incorporated by allowing the space spanned by its left singular components to rotate slowly. Hence, we assume that \( x_t \) is generated as:

\[
x_t = A_t z_t + w_t, \quad \forall \ t = 1 : T, \quad z_t \sim \mathcal{N}(0_{k \times 1}, I_{k \times k}), \quad w_t \sim \mathcal{N}(0_{p \times 1}, \sigma^2 I_{p \times p}), \quad A_t \in \mathbb{R}^{p \times k}
\]

\[
\| A_t A_t^\top - A_{t-1} A_{t-1}^\top \|_2 \leq \gamma, \quad \forall \ t = 2 : T
\]

We focus on finite-sample recovery guarantees as the performance measure of our algorithms. We are interested in retrieving the column space spanned by the block of observations in the last iteration. Let \( Q \) be the Gaussian distribution where \( U \) is generated as:

\[
x \sim N(0, AA^\top + \sigma^2 I)
\]

\[
\text{Golub and Van Loan (2012))}
\]

In this section, we establish fundamental performance limits of any algorithm \( \psi \) for the non-stationary streaming PCA problem when the observations are sampled from (2) i.e., \( x_t \sim \mathcal{N}(0, A_t A_t^\top + \sigma^2 I) \). To construct the lower bound we consider the family of sequence of matrices \( A(\delta, \gamma, T) \) defined as:

\[
A(\delta, \gamma, T) = \{ (A_1, A_2, \ldots, A_T) : s_k(A_t A_t^\top) \geq \delta, \delta > 0, \forall \ t = 1 : T; \| A_t A_t^\top - A_{t-1} A_{t-1}^\top \|_2 \leq \gamma, 0 < \gamma < 1, \forall \ t = 2 : T \}
\]

Let \( U_{1: \delta}(A_T) \) be the top-\( k \) left singular vectors of \( A_T \). Consider the class of algorithms \( \Psi \) which estimate \( U_{1: \delta}(A_T) \) from observations \( x_1, x_2, \ldots, x_T \) and for \( \psi \in \Psi \) let the estimate be denoted by \( \psi(x_1, x_2, \ldots, x_T) \). The probability measure induced over the space of observations \( x_1, x_2, \ldots, x_T \) by the sequence \( A_1, A_2, \ldots, A_T \) is given by:

\[
P_{A_1, A_2, \ldots, A_T}(x_1 \in \zeta_1, x_2 \in \zeta_2, \ldots, x_T \in \zeta_T) = \Pi_{t=1}^T P_{A_t}(x_i \in \zeta_i)
\]

where, \( \{\zeta_t\}_{t=1}^T \) is a sequence of measurable sets in \( \mathbb{R} \). The minimax risk associated with the problem of inferring \( U_{1: \delta}(A_T) \) from observations \( x_1, x_2, \ldots, x_T \) is:

\[
R(\delta, \gamma, T) = \inf_{\psi \in \Psi} \sup_{(A_1, A_2, \ldots, A_T) \in A(\delta, \gamma, T)} \mathbb{E} \left( d(U_{1: \delta}(A_T), \psi(x_1, x_2, \ldots, x_T)) \right)
\]

where the expectation is taken with respect to \( P_{A_1, A_2, \ldots, A_T}(\cdot) \).

We establish the lower bound on \( R(\delta, \gamma, T) \) using a two hypotheses test. In order to establish the bound it is sufficient to lower bound the minimax probability of error (section 2.2 in Tsybakov (2009)) in recovering the top-\( k \) singular vectors when the observations are generated from family of Gaussian distributions whose covariance are the sum of matrices in two suitably chosen sequences in \( A(\delta, \gamma, T) \) and \( \sigma^2 I \). We
bound the minimax probability of error using lemma A.2 and the KL-divergence between the product distribution generated by these sequences. Consider the following two hypotheses for this purpose:

\[ \mathcal{H}_0 : (A_t^{(0)}, \ldots, A_T^{(0)}) \in \mathbb{R}^{p \times k} \quad t = 1, \ldots, T \]
\[ \mathcal{H}_1 : (A_t^{(1)}, \ldots, A_T^{(1)}) \in \mathbb{R}^{p \times k} \quad t = 1, \ldots, T \]

where,

\[
A_t^{(0)} = \sqrt{\delta} \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \quad \text{and} \quad A_t^{(1)} = \sqrt{\delta} \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \] \quad \forall t = 1, \ldots, T

where \( \theta_t = \max \left\{ 0, \sin^{-1}(2s) - (T - t) \sin^{-1} \left( \frac{2}{\delta} \right) \right\}, \forall t = 1, \ldots, T \) and

\[
s = \left( \frac{\gamma}{\delta} \right) \frac{1}{4} \left( \frac{\sigma^2(\sigma^2 + \delta)}{\delta^2} \right)^{\frac{1}{4}} + \frac{1}{\sqrt{T}} \left( \frac{\sigma^2(\sigma^2 + \delta)}{\delta^2} \right)^{\frac{1}{8}}.\]

We note the following about the proposed hypotheses \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \):

1. The sequence \( \{A_t^{(0)}\}_{t=1}^T \) and \( \{A_t^{(1)}\}_{t=1}^T \) are in \( \mathcal{A}(\delta, \gamma, T) \).
2. For \( \{A_t^{(1)}\}_{t=1}^T \), we have \( \|A_t^{(1)} A_t^{(1)\top} - A_{t-1}^{(1)} A_{t-1}^{(1)\top}\|_2 \leq |\sin(\theta_t - \theta_{t-1})|, \forall t = 2, \ldots, T \)
3. For the sequences \( \{A_t^{(0)}\}_{t=1}^T \) and \( \{A_t^{(1)}\}_{t=1}^T \), we have \( \|A_t^{(1)} A_t^{(1)\top} - A_t^{(0)} A_t^{(0)\top}\|_2 \leq |\sin(\theta_t)|, \forall t = 1, 2, \ldots, T \)

We establish the lower bound for the non-stationary streaming PCA problem in theorem 2.1 using lemma A.2. In order to apply lemma A.2 we bound the KL-divergence between the product distributions of observations generated using \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) by a constant in lemma 2.1.

**Theorem 2.1 (Lower Bound).** *Given \( T \) observations, rate of change \( \gamma \) and the spectral gap \( \delta \) such that \( \frac{\gamma}{\delta} < 1 \). For any algorithm \( \psi \) the minimax estimation error between any two sequences of matrices \( \{M_t\}_{t=1}^T \) and \( \{N_t\}_{t=1}^T \) belonging to \( \mathcal{A}(\delta, \gamma, T) \) is given by:

\[
\mathcal{R}_{\delta, \gamma, T} = \Omega \left( \left( \frac{\gamma}{\delta} \right)^{\frac{1}{4}} \left( \frac{\sigma^2(\sigma^2 + \delta)}{\delta^2} \right)^{\frac{1}{4}} + \frac{1}{\sqrt{T}} \left( \frac{\sigma^2(\sigma^2 + \delta)}{\delta^2} \right)^{\frac{1}{8}} \right).
\]

**Proof.** The theorem follows from lemma A.2. We verify the necessary conditions for the generic reduction scheme outlined in section 2 (also see section 2.2 in Tsybakov (2009)). The application of lemma A.2 relies on applicability of the generic scheme. We then verify the conditions required for lemma A.2 and conclude the result. To this end we need to ensure the following:

1. \( d(A_N^{(0)}, A_N^{(1)}) \geq s \)
2. \( \{A_t^{(0)}\}_{t=1}^T, \{A_t^{(1)}\}_{t=1}^T \in \mathcal{A}(\delta, \gamma, N) \)

3. \( KL(P\|Q) \leq \alpha < \infty \)

The first two properties follow from the construction detailed in section 2. The third property is established in lemma 2.1 with \( \alpha = \frac{C\delta^4}{\sigma^2(\sigma^2 + \delta^2)} \). Putting this into lemma A.2, for minimax probability of error we have,

\[
p_{e,1} \geq \frac{1}{4} \exp \left( -\frac{C\delta^4}{\sigma^2(\sigma^2 + \delta^2)} \right) \]

and therefore:

\[
R_{\delta,\gamma,N} \geq sp_{e,1} \geq \frac{s}{4} \exp \left( -\frac{C\delta^4}{\sigma^2(\sigma^2 + \delta^2)} \right)
\]

\[\square\]

**Lemma 2.1 (Bound on KL-divergence).** Let \( \mathbb{P}, \mathbb{Q} \) denote the measure corresponding to the joint distribution generated by the sequence \( \{A_t^{(0)}\}_{t=1}^T \) and \( \{A_t^{(1)}\}_{t=1}^T \) belonging to \( \mathcal{A}(\delta, \gamma, T) \) through the non-stationary spiked covariance model (equation 2). Then,

\[ KL(\mathbb{P}\|\mathbb{Q}) = O(1). \]

**Proof sketch** Using the properties of the Gaussian distribution and \( \mathcal{A}(\delta, \gamma, T) \), the KL-divergence between \( \mathbb{P} \) and \( \mathbb{Q} \) is \( O\left( \frac{\delta^2}{\sigma^2(\sigma^2 + \delta)} \sum_{t=1}^T \sin^2(\theta_t) \right) \). The sequence \( A_t^{(0)} \) and \( A_t^{(1)} \) is carefully constructed so that the distance between the column space of the sequence of matrices so that \( \sum_{t=1}^T \sin^2(\theta_t) \) is bounded by a parameter-independent constant.

The bound on \( R_{\delta,\gamma,T} \) in theorem 2.1 exhibits a phase transition phenomenon with the first term representing the effect of non-stationarity. Initially, \( R_{\delta,\gamma,T} \) decreases with the rate of \( 1/\sqrt{T} \). However, when \( T \geq \left( \frac{\gamma}{\delta} \right)^{-\frac{2}{3}} \left( \frac{\sigma^2(\sigma^2 + \delta)}{\delta^2} \right)^{\frac{1}{2}} \), the first term of \( R_{\delta,\gamma,T} \) dominates the second term and \( R_{\delta,\gamma,T} \) becomes independent to the number of samples \( T \). This also validates our intuition that in a dynamic environment past information quickly becoming stale. For the streaming PCA problem, we recover the fundamental limit as \( \Omega\left( \frac{\sigma^2(\sigma^2 + \delta)^{1/2}}{\delta \sqrt{T}} \right) \) (\( \gamma = 0 \) in theorem 2.1).

### 3 Non-Stationary Streaming PCA

The noisy power method (algorithm 1) is an iterative algorithm for computing the top-\( k \) eigen vectors of a given matrix. It is initialized with a random \( k \)-dimensional basis of \( \mathbb{R}^p \), \( Q^{(0)} \in \mathbb{R}^{p \times k} \) (line 3 in algorithm 1). Each iteration computes a representative orthogonal basis upon observing a new block of data, \( Q^{(l+1)} = b\left( \frac{1}{B} \sum_{t=(l-1)B}^l X_t X_t^T \right)Q^{(l)} \) (line 7 in algorithm 1), where \( b(M) \) denote an orthonormal matrix whose columns span the column space of \( M \) (e.g. it can be computed through the QR decomposition of \( M \)).

We begin by considering the problem of recovering the top-\( k \) singular vectors of a symmetric matrix \( A \in \mathbb{R}^{p \times p} \) using the power method. Let \( SVD(A) = UDV^T \). Starting with a random initial matrix
Algorithm 1 Noisy Power Method

1: **Input**: $x_1, x_2, \ldots, x_T$, block size: $B$, iterations: $L = \frac{T}{B}$, accuracy: $\epsilon$
2: **Output**: $Q^{(L)}$
3: $Q^{(0)} \leftarrow p \times k$ random matrix
4: for $l = 0 : L - 1$ do
5:      $N^{(l+1)} \leftarrow 0$
6:      for $t = lB+1: (l+1)B$ do
7:              $N^{(l+1)} \leftarrow N^{(l+1)} + \frac{1}{B} x_t x_t^T Q^{(l)}$
8:      end for
9: $Q^{(l+1)}$ is an orthonormal basis for span($N^{(l+1)}$) (e.g., obtained through QR decomposition)
10: end for

$Q^{(0)}$, in the $l^{th}$ iteration the power method computes, $Q^{(l+1)} = b(AQ^{(l)})$. The column space of $Q^{(L)}$ is equivalent to the column space of $A^L Q^{(0)}$ since,

$$Q^{(L)} = b(A(b(A \ldots (b(AQ^{(0)})))) = A^L Q^{(0)}R^{(1)} R^{(2)} \ldots R^{(L)}$$

where, $R^{(l)}$ is the $\mathbb{R}^{k \times k}$ matrix computed each time from the QR decomposition of $AQ^{(l)}$. Hence, $d(U_{1:k}, Q^{(l)}) = d(U_{1:k}, A^L Q^{(0)})$. We analyze the distance between the output of the power method $Q^{(L)}$ and the space spanned by $A$, $d(U_{1:k}, A^L Q^{(0)})$ from the singular values of $A^L$. In Theorem 3.1, we quantify the decrease in the error in estimation of the top-$k$ singular vectors of $A$. After each iteration of the power method, the desired singular vector $U_{1:k}$ is amplified by at least $s_k(A)$ where as the remaining are amplified by at most $s_{k+1}(A)$. Since the power iterations start with a random initialization $Q^{(0)} \in \mathbb{R}^{p \times k}$, we have (Hardt and Price (2014)):

$$\frac{s_1(Q^{(0)}U_{k+1:p})}{s_k(Q^{(0)})U_{1:k}} \leq p^2.$$ 

Theorem 3.1 bounds the distance between the top-$k$ singular vectors of $A^{(L)}$ as:

$$d(U_{1:k}, A^{(L)}Q^{(0)}) \leq \frac{s_{k+1}(A)^L s_1(U_{k+1:p}Q^{(0)})}{s_k(A)^L s_k(U_{1:k}Q^{(0)})}$$

Therefore,

$$\|U_{1:k}Q^{(L)}\|_2 \leq p^2 \left( \frac{s_k(A)}{s_{k+1}(A)} \right)^L$$

**Theorem 3.1** (Perturbation by multiplication). Let $M \in \mathbb{R}^{m \times n}$ and $SVD(M) = UDV^T$. Let $N \in \mathbb{R}^{n \times k}$ ($m \geq n \geq k$) and $Y = MN$. Assume, $s_k(M) > 0$ and $s_k(V_{1:k}^T N) > 0$. Then,

$$d(U_{1:k}, Y) \leq \frac{s_{k+1}(M) s_1(V_{k+1:n}^T N)}{s_k(M) s_k(V_{1:k}^T N)}$$

**Proof sketch** Bounding the distance between the column spaces of $U_{1:k}$ and the output $Y$ is equivalent to bounding the largest singular value of the projection of the space orthogonal to $U_{1:k}$ onto $Y$. This projects $MN$ onto the space spanned by the last $k+1$ right singular vectors $U_{k+1:n}^T$. The projection of $MN$ onto the $(k+1)$ right singular vectors is bounded using lemma A.5.
We now establish the convergence behaviour of the noisy power method (algorithm 1) in presence of noise and non-stationarity. We observe \( T = LB \) vectors in blocks of size \( B \), \( \{ \{ x_t \}^{lB}_{t=(l-1)B} \}^L_{l=1} \). For all \( l = 0, 2, \ldots, L - 1 \) let \( E(l) \) be the deviation of the empirical covariance from the expected empirical covariance of the vector at the end of every block of computation \( M(l) = \mathbb{E}(x_{lB}x_{lB}^\top) \):

\[
\frac{1}{B} \sum_{t=(l-1)B}^{lB} x_t x_t^\top = \mathbb{E}(x_{lB}x_{lB}^\top) + E(l) \\
= A_{lB}A_{lB}^\top + \sigma^2 I + E(l) \\
= M(l) + E(l)
\]  

(3)

In presence of non-stationarity: (i) we need to bound for sampling noise from time-varying distributions \( \|E(l)\| \) and (ii) analyse the distance between the time-varying underlying subspaces and the output of the noisy power method. In lemma 3.1, we obtain a bound on the spectral norm of the noise matrix using concentration inequalities to bound the spectral norm of the noise matrix due to sampling from time-varying distributions.

**Lemma 3.1 (Spectral norm of noise).** Given \( 0 < \gamma < 1 \), spectral gap \( \delta \), observations \( \{x_t\}_{t=1}^T \) generated according to 2 with \( \{A_t\}_{t=1}^T \in A(\delta, \gamma, T) \), with probability \( 1 - \frac{1}{T} \),

\[
\max_{1 \leq i \leq L} \|E(l)\|_2 \leq \sqrt{\frac{C_p \log(T)}{B}} + \frac{B \gamma}{2}.
\]

We now bound the distance between the output of the noisy power method and the time varying underlying subspaces. Let \( \mathcal{M}^{(L)} = \prod_{l=1}^L (M(l) + E(l)) \). The output of the noisy power method (algorithm 1) is equivalent to computing an orthonormal basis for the product \( \mathcal{M}^{(L)}Q_{(0)} \). In order to analyse the convergence behaviour we bound the distance between the output of the noisy power method, \( \hat{U}_{1:k}(L) \) and the subspace to be recovered, \( U_{1:k}(L) \) However, due to non-stationarity there doesn’t exist a fixed reference subspace with respect to which we can bound these distances. In lieu of a fixed reference subspace, we identify a sequence of \( (n-k) \) and \( k \) dimensional subspaces of \( \mathbb{R}^n \), arising from the observed vectors \( \{\{x_t\}_{t=(l-1)B}^{lB}\}_{l=1}^L \), such that the sequence of \( k \)-dimensional subspaces remains close to the underlying subspace every iteration and the first subspaces of the sequence are mutually orthogonal. We identify this sequence of \( k \) and \( n-k \) dimensional subspaces, through the column space of a sequence of matrices \( \{N(l)\}_{l=1}^L \in \mathbb{R}^{n \times (n-k)} \) and \( \{W(l)\}_{l=1}^L \in \mathbb{R}^{n \times k} \) such that span\(N(1)\) is orthogonal to span\(W(1)\).

We need to ensure that the the identified \( k \)-dimensional subspaces are close to the true subspace and their distance from the subspace estimates of the noisy power method is not too large. We also need to ensure that the identified \( (n-k) \)-dimensional subspaces are far from the true subspace and the estimates of the noisy power method. Then, using the sequence of identified subspaces as a reference we can relate the distance between the \( \hat{U}(L) \) and \( U_{1:k}(L) \). We formalize this idea in theorem 3.2 to establish the convergence behaviour of the noisy power method for non-stationary streaming PCA.

Before stating the construction of the sequence of subspaces and the convergence analysis we state our assumptions. Let \( U(l)D(l)U(l)^\top \) be the singular value decomposition of \( M(l) \), \( \delta(l) = s_k(M(l)) - s_{k+1}(M(l)) \) and \( \delta = \inf_{l \geq 0} \delta(l) \). By equation 2 and triangle inequality we have that:

\[
\|M(l) - M(l + 1)\|_2 \leq B \gamma \quad \forall \ l \geq 0
\]
Let $\eta = \frac{B \gamma}{\delta - B \gamma}$. Using Davis-Kahan sin $\theta$ theorem (lemma A.3), we have:

$$d(U_{1:k}(l), U_{1:k}(l+1)) \leq \eta \quad \text{for all} \quad \forall l \geq 0.$$  

For all $l \geq 0$ we have $s_k(M(l)) \geq \delta + \sigma^2$ and $s_{k+1}(M(l)) = \sigma^2$. Note that when $\delta \geq \frac{1}{8} \sigma^2$, for

$$16(Cp \log(T))^{\frac{1}{4} \frac{1}{\delta}} \leq \varepsilon \leq \frac{1}{4},$$

with

$$B = \frac{64Cp \log(T)}{\varepsilon^2 \delta^2},$$

with probability $1 - \frac{1}{T}$, for all $l = 1, 2, \ldots, L$ the following holds:

A.1 $\|E(l)\|_2 \leq \Delta$ where $\Delta = \frac{\varepsilon}{4} \delta$

A.2 $\phi > 1$ where $\phi := \inf_{t \geq 0} \frac{(\delta + \sigma^2) - \Delta}{\sigma^2} \sqrt{1 - (\varepsilon + \eta)^2}(1 - (\varepsilon + \eta)^2)$

A.3 $\sigma^2 + 0.754 \geq \frac{(\varepsilon + \eta) \sqrt{1 - \varepsilon^2}}{\varepsilon \sqrt{1 - (\varepsilon + \eta)^2}}$

A.4 $\varepsilon > \frac{32}{\delta^2} \eta$

Theorem 3.2, lemma 3.3 and lemma 3.2 are based on assumptions A.1, A.2, A.3 and A.4.

**Theorem 3.2** (Iteration). Assume that $\delta \geq \frac{1}{8} \sigma^2$. Let $\hat{U}$ denote the output of the noisy power method (algorithm 1) when the observations $\{x_i\}_{i=1}^T$ are sampled from the non-stationary spiked covariance model (equation 2). For $16(Cp \log(T))^{\frac{1}{4} \frac{1}{\delta}} \leq \varepsilon \leq \frac{1}{4}$, there exists a block size $B = \frac{64Cp \log(T)}{\varepsilon^2 \delta^2}$ such that

$$\|\hat{U}_{1:k}(L) U_{k+1:n}(L)^T\|_2 \leq \varepsilon + O\left(\frac{\sqrt{\phi}^{-L}}{\sqrt{p - \sqrt{k - 1}}}ight)$$

with probability $1 - e^{\Omega(p - k + 1)} - \exp(-\Omega(p))$.

**Proof sketch** We use lemma 3.2 to identify and iteratively construct the sequence $\{N^{(l)}\}_{l=1}^L$ to ensure that $N^{(l)}$ has small overlap generating the observations in the $l^{th}$ iteration and the space spanned by top-$k$ singular vectors generating those observations. Consequently, the column space of $M^{(L)} N^{(1)}$ lies in the column space of $U_{k+1:n}(L)$ and the largest singular value of the product of $M^{(L)}$ and $N^{(1)}$ is decreases exponentially i.e.

$$\text{span}(M^{(L)} N^{(1)}) \subseteq \text{span}(U_{k+1:n}(L)) \quad \text{and} \quad \|M^{(L)} N^{(1)}\|_2 \leq \prod_{l=1}^{L} \left(\frac{\delta + \sigma^2}{\sqrt{1 - (\varepsilon + \eta)^2}}\right) \quad (4)$$

Similarly, using lemma 3.3 we identify and iteratively construct the sequence $\{W^{(l)}\}_{l=1}^L$. We ensure that

$$\text{for all} \ 0 \leq l \leq L, \text{the} \ \text{column space of} \ W^{(l)} \ \text{is} \ \varepsilon \ \text{away from} \ U_{1:k}(l) \ \text{and} \ \text{the least singular value of} \ M^{(L)} W^{(1)} \ \text{is at least} \ \sqrt{1 - (\varepsilon + \eta)^2(\delta + \sigma^2)} - \Delta. \ \text{Therefore, the distance between the true underlying space} \ U_{1:k}(L) \ \text{and} \ M^{(L)} W^{(1)} \ \text{is less than} \ \varepsilon \ \text{and} \ \text{the least singular value of the product} \ M^{(L)} W^{(1)} \ \text{increases exponentially i.e.}$$

$$d(M^{(L)} W^{(1)}, U_{1:k}(L)) \leq \varepsilon \ \text{and} \ \ s_k \left(M^{(L)} W^{(1)}\right) \geq \prod_{l=1}^{L} \left(\sqrt{1 - (\varepsilon + \eta)^2(\delta + \sigma^2)} - \Delta\right) \quad (5)$$
We bound the distance between \( U_{1:k}(L) \) and \( \tilde{U}_{1:k}(L) \) through \( M(L)W^{(1)} \). The construction ensures that the distance between \( U_{1:k}(L) \) and \( M(L)W^{(1)} \) is at most \( \varepsilon \). Due to the power iterations, the distance between \( \tilde{U}_{1:k}(L) \) and \( M(L)W^{(1)} \) is identical to the distance between \( M(L)	ilde{U}_{1:k}(0) \) and \( M(L)W^{(1)} \). We can bound the distance between \( M(L)W^{(1)} \) and \( M(L)	ilde{U}_{1:k}(0) \) by projecting \( M(L)	ilde{U}_{1:k}(0) \) onto space spanned by \( N^{(1)} \) and its orthogonal complement and using properties 4 and 5.

The sequences of subspaces \( \{N^{(l)}\}_{l=1}^{L} \) and \( \{W^{(l)}\}_{l=1}^{L} \) in theorem 3.2 are identified using lemma 3.2 and lemma 3.3 respectively. Beginning with \( N^{(L+1)} = U_{k+1:n}(L) \), in order to establish property 4, it is necessary to ensure that in every iteration \( 0 \leq l \leq L \), span\((M(l) + \mathcal{E}(l))N^{(l)}\) \( \subseteq N^{(l)} \) and 
\[
s_1((U_{1:k}(l))^\top N^{(l)}) \leq \varepsilon \quad \text{and} \quad s_1((M(l) + \mathcal{E}(l))N^{(l)}) \leq \frac{\sigma^2 + \Delta}{\sqrt{1-(\varepsilon + \eta)^2}}.
\]
Lemma 3.2 we show that, if \((M(l) + \mathcal{E}(l))x\) is almost orthogonal to \( U_{1:k}(l) \), then \( x \) has to be almost orthogonal to \( U_{1:k}(l) \) thereby facilitating the iterative construction of \( \{N^{(l)}\}_{l=1}^{L} \).

**Lemma 3.2 (Orthogonal projection after perturbation).** Let \( M \in \mathbb{R}^{n \times n} \) be a positive definite matrix and \( SVD(M) = UDU^\top \). Let \( 0 < k < n \) and let \( \mathcal{Y} \) be the set of \( Y \in \mathbb{R}^{n \times (n - k)} \) matrices with orthonormal columns such that \( s_1(U_{1:k}^\top Y) \leq \varepsilon + \eta \) with \( \varepsilon \leq \frac{1}{4} \). Under the conditions of theorem 3.2, for every given \( Y \in \mathcal{Y} \) there exists a \( n \times (n - k) \) orthonormal matrix \( \overline{N} \) such that
\[
\begin{align*}
\text{span} ((M + \mathcal{E})\overline{N}) &\subseteq \text{span} (Y), \quad s_1(U_{1:k}^\top \overline{N}) \leq \varepsilon, \quad (6) \\
s_1((M + \mathcal{E})\overline{N}) &\leq \frac{(\sigma^2 + \Delta)}{\sqrt{1-(\varepsilon + \eta)^2}}. \quad (7)
\end{align*}
\]

**Proof sketch** The proof of existence of the orthonormal \( n \times (n - k) \) matrix \( \overline{N} \) is involved and is established case-by-case depending on the rank of \((M + \mathcal{E})\). The second claim follows since assumption A.3 constrains the ratio of singular values of projection of any vector in the column space of \((M + \mathcal{E})\overline{N}\) onto \( U_{1:k}^\top \) and \( U_{1:k} \). To obtain the third claim from the second claim project the column space of \((M + \mathcal{E})\overline{N}\) onto the column space of \( U_{1:k}^\top \) and its orthogonal complement and note that column space of \( Y \) are almost orthogonal to those of \( U_{1:k} \) and \( \| U_{k+1:n}^\top (M + \mathcal{E})\|_2 \leq s_{k+1}(M) + \Delta \).

To identify the sequence \( \{W^{(l)}\}_{l=1}^{L} \) as the space orthogonal to \( N^{(l)} \), we must show for all \( 0 \leq l \leq L \), \( s_k((M(l) + \mathcal{E}(l))W^{(l)}) \geq \sqrt{1-(\varepsilon + \eta)^2}(\delta + \sigma^2) - \Delta \) and \( d(U_{1:k}(l), W^{(l+1)}) \leq \varepsilon \). This is established with the help of lemma 3.3 which shows that the span of a matrix \( W \) upon multiplication with \((M(l) + \mathcal{E}(l))\) remains close to the span of \( U_{1:k}(l) \) if the span of matrix \( W \) is close to the linear span of the top-\( k \) singular vectors \( U_{1:k}(l) \).

**Lemma 3.3 (Projection after perturbation).** Let \( M \in \mathbb{R}^{n \times n} \) be a positive semi-definite matrix, \( SVD(M) = UDU^\top \) and \( W \) be a \( n \times k \) matrix which consists of orthonormal vectors. When \( \varepsilon \leq \frac{1}{4} \) and \( d(U_{1:k}, W) \leq \varepsilon + \eta \), under the conditions of theorem 3.2,
\[
\begin{align*}
s_k((M + \mathcal{E})W) &\geq \sqrt{1-(\varepsilon + \eta)^2}(\delta + \sigma^2) - \Delta \quad \text{and} \\
d(U_{1:k}, (M + \mathcal{E})W) &\leq \varepsilon.
\end{align*}
\]

**Proof sketch** The first claim can be established by observing that the maximum singular value of the product \( \mathcal{E}W \) is \( \Delta \) and \( s_k(U_{1:k}W) \leq \sqrt{1-(\varepsilon + \eta)^2} \). To establish the second claim, we show that
distance of the space orthogonal to the span of \((M + \mathcal{E})W\) from \(U_{1:k}\) is at most \(1 - \epsilon^2\). This follows by carefully controlling the ratio of length of projection of vector \(x\) in the column space of \((M + \mathcal{E})W\) onto the spaces spanned by the top-\(k\) singular vectors of \(M\) and its orthogonal complement through assumption A.3.

Theorem 3.2 shows that the noisy power method can achieve a recovery error of \(O(\frac{\gamma^{1/3}}{\delta})\) which is order-wise identical with respect to \(\gamma\) and \(\delta\) to the fundamental limit of \(\Omega(\frac{\gamma^{1/3}}{\delta})\) established in theorem 2.1. The non-stationary streaming PCA problem reduces to the streaming PCA problem for \(\gamma = 0\). For the streaming PCA problem we guarantee \(\epsilon\)-accurate recovery with \(O\left(\frac{p\log(\sqrt{p} - \sqrt{k} - 1)}{\epsilon^2 \delta^2 \log(\phi)}\right)\) samples in corollary 3.1.

**Corollary 3.1 (Streaming PCA).** Let \(\delta \geq \frac{1}{8}\sigma^2\), \(\epsilon \leq \frac{1}{4}\), \(\|\mathcal{E}(l)\|_2 \leq \frac{\epsilon}{4}(s_k(A) - s_{k+1}(A))\) and \(s_{k+1}(A) > s_k(A)\left(\frac{2 - \epsilon^2 - \sqrt{1 - \epsilon^2}}{\sqrt{1 - \epsilon^2}}\right)\). Let \(\hat{U}\) denote the output of the noisy power method (algorithm 1) when the observations \(\{x_t\}_{t=1}^T\) are sampled from a spiked covariance model (equation 1), with \(B = \frac{64Cp\log(T)}{\epsilon^2 \delta^2}\), we have

\[
\|\hat{U}_{1:k}(L)U_{k+1:n}(L)\|_2 \leq \epsilon + O\left(\frac{\sqrt{p}\phi - L}{\sqrt{p} - \sqrt{k} - 1}\right)
\]

with probability \(1 - c^{\Omega(p-k+1)} - \exp(-\Omega(p))\).

## 4 Conclusion and Future Work

In this paper we propose non-stationary streaming PCA. We establish fundamental limits on the performance of any algorithm for this problem and illustrate a phase transition phenomenon characterizing the effect of non-stationarity. We also establish that the noisy power method achieves the fundamental limit for this problem. An interesting future direction would be to make provide a spectral spectral-gap free analysis of this problem.
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A Appendix

A.1 Preliminary Results

We begin by collecting some preliminary results which are used extensively throughout the proofs and aid in keeping the manuscript self-contained.

**Lemma A.1** (Theorem 2.6.1, Golub and Van Loan (2012)). Let $S_1$ and $S_2$ be two subspaces of $\mathbb{R}^n$, such that $\text{dim}(S_1) = \text{dim}(S_2)$. We define the distance between these two subspaces by $\text{dist}(S_1, S_2) = \|P_1 - P_2\|_2$, where $P_i, i = 1, 2$ is the orthogonal projection onto $S_i, i = 1, 2$. Suppose: $W = [W_1 \ W_2], Z = [Z_1 \ Z_2]$ are $n \times n$ orthonormal matrices. If $S_1 = \text{ran}(W_1)$ and $S_2 = \text{ran}(Z_1)$ then,

$$\text{dist}(S_1, S_2) = \|W_1^T Z_2\|_2 = \|Z_1^T W_2\|_2$$

**Lemma A.2** (Theorem 2.2 (iii), Tsybakov (2009)). Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on a measurable space $(\Omega, \mathcal{F})$. If $KL(P || Q) \leq \beta < \infty$, then,

$$p_{\epsilon, 1} \geq \max \left( \frac{1}{4} \exp(-\beta), \frac{1 - \sqrt{\beta}}{2} \right)$$

Given a matrix $M$ and a subset $S \in \mathbb{R}$, let $P_M(S)$ denote the orthogonal projection onto the subspace spanned by eigenvectors of $M$ corresponding to those eigenvalues that lie in $S$.

**Lemma A.3** (Davis-Kahan $\sin(\theta)$ theorem; Theorem VII.3.1, Bhatia (2013)). Let $A, B \in \mathbb{R}^{n \times n}$, $S_1 = [a, b]$ and $S_2 = \mathbb{R} \setminus [a - \delta, b + \delta]$. Let $P_1 := P_A(S_1), P_2 := P_B(S_2)$, then for every unitary invariant norm $\|P_1 P_2\|_2 \leq \frac{1}{\delta} \|P_1 (A - B) P_2\|_2 \leq \frac{1}{\delta} \|A - B\|_2$.

This can equivalently be stated as:

For given symmetric matrices $A, B$ with singular value decomposition $A = UDU$ and $A + B = \hat{U}D\hat{U}$, then,

$$\|U_{1:k} U_{1:k}^T - \hat{U}_{1:k} \hat{U}_{1:k}^T\|_2 \leq \frac{\|B\|_2}{s_k(A) - s_{k+1}(A) + \|B\|_2}$$

**Lemma A.4** (Weyl’s Theorem). For any $A, B \in \mathbb{R}^{n \times n}$, $s_i(A + B) \leq s_i(A) + s_1(B)$

**Lemma A.5** (Matrix-norm Inequality). Let $M \in \mathbb{R}^{m \times n}$ and $N \in \mathbb{R}^{n \times p}$ then

$$\|MN\|_2 \leq \|M\|_2 \|N\|_2$$

where, $\|M\|_2 = \sup_{x \in \mathbb{S}^{n-1}} \|Mx\|_2$. Further, if the rank of matrix is at least $k$ then,

$$\|MN\|_k \geq \|M\|_k \|N\|_k$$

**Lemma A.6** (Theorem 2.1 in Vershynin (2012)). Consider independent random vectors $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^p$, $n \geq p$ from a sub-gaussian distribution with parameter $1$. Then $\forall \alpha > 0$ with probability $1 - \alpha$ we have $\|\frac{1}{n} \sum_{i=1}^n x_i x_i^T - \mathbb{E}(x_i x_i^T)\|_2 \leq C \sqrt{\frac{\log(2/\alpha)}{n}}$
A.2 Results from the main text

Lemma 2.1 (Bound on KL-divergence). Let $\mathbb{P}, \mathbb{Q}$ denote the measure corresponding to the joint distribution generated by the sequence $\{A_t^{(0)}\}_{t=1}^T$ and $\{A_t^{(1)}\}_{t=1}^T$ belonging to $\mathcal{A}(\delta, \gamma, T)$ through the non-stationary spiked covariance model (equation 2). Then,

$$KL(\mathbb{P}||\mathbb{Q}) = O(1).$$

Theorem 3.1 (Perturbation by multiplication). Let $M \in \mathbb{R}^{m \times n}$ and $SVD(M) = UDV^T$. Let $N \in \mathbb{R}^{n \times k}$ ($m \geq n \geq k$) and $Y = MN$. Assume, $s_k(M) > 0$ and $s_k(V_{1:k}^T N) > 0$. Then,

$$d(U_{1:k}, Y) \leq \frac{s_{k+1}(M) s_1(V_{k+1:n}^T N)}{s_k(M) s_k(V_{1:k}^T N)}$$

Lemma 3.1 (Spectral norm of noise). Given $0 < \gamma < 1$, spectral gap $\delta$, observations $\{x_t\}_{t=1}^T$ generated according to 2 with $\{A_t\}_{t=1}^T \in \mathcal{A}(\delta, \gamma, T)$, with probability $1 - \frac{1}{T}$,

$$\max_{1 \leq l \leq L} \|E(l)\|_2 \leq \sqrt{\frac{Cp \log(T)}{B}} + \frac{B \gamma}{2}.$$ 

Theorem 3.2 (Iteration). Assume that $\delta \geq \frac{1}{8} \sigma^2$. Let $\tilde{U}$ denote the output of the noisy power method (algorithm 1) when the observations $\{x_t\}_{t=1}^T$ are sampled from the non-stationary spiked covariance model (equation 2). For $16(Cp \log(T))^\frac{1}{2} \gamma \delta \leq \varepsilon \leq \frac{1}{4}$, there exists a block size $B = \frac{64Cp \log(T)}{\varepsilon^2 \delta^2}$ such that

$$\|\tilde{U}_{1:k}(L) U_{k+1:n}^T(L)\|_2 \leq \varepsilon + O\left(\frac{\sqrt{p} \phi^{-L}}{\sqrt{B} - \sqrt{k} - 1}\right)$$

with probability $1 - c^{\Omega(p-k+1)} - \exp(-\Omega(p))$.

Lemma 3.2 (Orthogonal projection after perturbation). Let $M \in \mathbb{R}^{n \times n}$ be a positive definite matrix and $SVD(M) = UDU^T$. Let $0 < k < n$ and let $Y$ be the set of $Y \in \mathbb{R}^{n \times (n-k)}$ matrices with orthonormal columns such that $s_1(U_{1:k}^T Y) \leq \varepsilon + \eta$ with $\varepsilon \leq \frac{1}{4}$. Under the conditions of theorem 3.2, for every given $Y \in \mathcal{Y}$ there exists a $n \times (n-k)$ orthonormal matrix $\mathbf{N}$ such that

$$\text{span} \left((M + E)\mathbf{N}\right) \subseteq \text{span}(Y), \quad s_1(U_{1:k}^T \mathbf{N}) \leq \varepsilon,$$

$$s_1((M + E)\mathbf{N}) \leq \frac{(\sigma^2 + \Delta)}{1 - (\varepsilon + \eta)^2}. \quad (6) \quad (7)$$

Lemma 3.3 (Projection after perturbation). Let $M \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix, $SVD(M) = UDU^T$ and $W$ be a $n \times k$ matrix which consists of orthonormal vectors. When $\varepsilon \leq \frac{1}{4}$ and $d(U_{1:k}, W) \leq \varepsilon + \eta$, under the conditions of theorem 3.2,

$$s_k((M + E)W) \geq \sqrt{1 - (\varepsilon + \eta)^2}(\delta + \sigma^2) - \Delta \quad \text{and} \quad d(U_{1:k}, (M + E)W) \leq \varepsilon.$$
Corollary 3.1 (Streaming PCA). Let $\delta \geq \frac{1}{8}\sigma^2$, $\epsilon \leq \frac{1}{4}$, $\|\mathcal{E}(l)\|_2 \leq \frac{\epsilon}{4}(s_k(A) - s_{k+1}(A))$ and $s_{k+1}(A) > s_k(A)\left(\frac{2-\epsilon^2-\sqrt{1-\epsilon^2}}{\sqrt{1-\epsilon^2}}\right)$. Let $\hat{U}$ denote the output of the noisy power method (algorithm 1) when the observations $\{x_t\}_{t=1}^T$ are sampled from a spiked covariance model (equation 1), with $B = \frac{64C_p\log(T)}{\epsilon^2\delta^2}$, we have

$$\|\hat{U}_{1:k}(L)U_{k+1:n}(L)\|_2 \leq \epsilon + O\left(\frac{\sqrt{p}\phi_L}{\sqrt{p} - \sqrt{k-1}}\right)$$

with probability $1 - c^{\Omega(p-k+1)} - \exp(-\Omega(p))$. 
A.3 Proof of lemma 2.1

Let the distributions induced by \( \{A_t^{(0)}\}_{t=1}^T \) and \( \{A_t^{(1)}\}_{t=1}^T \) be given by \( \mathbb{P} \) and \( \mathbb{Q} \) respectively. Then,

\[
KL(P\|Q) = \sum_{t=1}^T \left( \log \frac{\Sigma_t^{(1)}}{\Sigma_t^{(0)}} - p + Tr(\Sigma_t^{(1)-1}\Sigma_t^{(0)}) \right)
\]

\[
= \sum_{t=1}^T \log \left( \frac{|A_t^{(1)}A_t^{(1)\top} + \sigma^2 I|}{|A_t^{(0)}A_t^{(0)\top} + \sigma^2 I|} \right) - p + Tr \left( (A_t^{(1)}A_t^{(1)\top} + \sigma^2 I)^{-1}(A_t^{(0)}A_t^{(0)\top} + \sigma^2 I) \right)
\]

\[
= \sum_{t=1}^T -p + Tr \left( \frac{1}{\sigma^2}(I - \frac{1}{(\sigma^2 + \delta)}A_t^{(1)}A_t^{(1)\top})(\sigma^2 I + A_t^{(0)}A_t^{(0)\top}) \right)
\]

\[
= \sum_{t=1}^T -p + \sigma^2 Tr \left( A_t^{(0)}A_t^{(0)\top} - \frac{\sigma^2}{\sigma^2 + \delta}A_t^{(1)}A_t^{(1)\top} - \frac{1}{(\sigma^2 + \delta)}A_t^{(1)}A_t^{(1)\top}A_t^{(0)A_t^{(0)\top}} \right)
\]

\[
= \sum_{t=1}^T \frac{1}{\sigma^2} Tr \left( A_t^{(1)}A_t^{(1)\top} - \frac{\sigma^2}{\sigma^2 + \delta}A_t^{(1)}A_t^{(1)\top} - \frac{1}{(\sigma^2 + \delta)}A_t^{(1)}A_t^{(1)\top}A_t^{(0)A_t^{(0)\top}} \right)
\]

\[
= \sum_{t=1}^T \frac{1}{\sigma^2} Tr \left( \frac{\delta}{\sigma^2 + \delta}A_t^{(1)}A_t^{(1)\top} - \frac{1}{\sigma^2 + \delta}A_t^{(1)}A_t^{(1)\top}A_t^{(0)A_t^{(0)\top}} \right)
\]

\[
= \sum_{t=1}^T \frac{1}{\sigma^2(\sigma^2 + \delta)} \sum_{t=1}^T Tr \left( A_t^{(1)}A_t^{(1)\top}(\delta I - A_t^{(0)}A_t^{(0)\top}) \right)
\]

\[
= \sum_{t=1}^T \frac{\delta^2}{\sigma^2(\sigma^2 + \delta)} \sum_{t=1}^T \|U_t^{(1)}U_t^{(1)\top}(I - U_t^{(0)}U_t^{(0)\top})\|_2^2
\]

\[
= \sum_{t=1}^T \frac{\delta^2}{\sigma^2(\sigma^2 + \delta)} \sum_{t=1}^T \|U_t^{(1)}U_t^{(1)\top} - U_t^{(0)}U_t^{(0)\top}\|_2^2
\]

\[
= \sum_{t=1}^T \frac{\delta^2}{\sigma^2(\sigma^2 + \delta)} \sum_{t=1}^T \sin^2(\theta_t)
\]

\[
= O(1)
\]

where, \((i), (ii)\) follow since the distributions are Gaussian and property of conditional entropy. \((iii)\) follows by observing that the eigenvectors of \((A_t^{(1)}A_t^{(1)\top} + \sigma^2 I)^{-1}\) are same as those of \(A_t^{(1)}A_t^{(1)\top} + \sigma^2 I\) and the eigenvalues are \((\delta + \sigma^2)^{-1}\). \((iv)\) follows since \(Tr(I) = p\), \((vi)\) follows since \(Tr(A_t^{(1)}A_t^{(1)\top}) = Tr(A_t^{(0)}A_t^{(0)\top}) = k\delta\). \((ix)\) follows since \(A_t^{(1)}A_t^{(1)\top}\) and \(I - A_t^{(0)}A_t^{(0)\top}\) are projection matrices and \((x)\) follows by lemma A.1. \((xii)\) follows by the property of the sequences \(\{A_t^{(1)}\}_{t=1}^T\) and \(\{A_t^{(0)}\}_{t=1}^T\) and \((xii)\) follows from lemma A.7.
Lemma A.7. Given $T, \gamma, \delta$ such that $\frac{\gamma}{\delta} < 1$. Let $\phi_T = \sin^{-1}(2s), \phi_\gamma = \sin\left(\frac{\gamma}{\delta}\right)$ and $\theta_t = \max\{0, \phi_T - (T - i)\phi_\gamma\}$. For sequences $\{A^{(0)}\}_{t=1}^{T}$ and $\{A^{(1)}\}_{t=1}^{T}$ generated by hypothesis $H_0$ and $H_1$ there exists a finite positive constant $C$ independent of $T, \gamma, \delta$ such that $\sum_{t=1}^{T} \sin^2(\theta_t) \leq \frac{C\sigma^2(\sigma^2 + \delta)}{\delta^2}$.

A.4 Proof of lemma A.7

Recall that $\phi_\gamma = \sin^{-1}\left(\frac{\gamma}{\delta}\right) = \frac{\gamma}{\delta} + O\left(\left(\frac{\gamma}{\delta}\right)^3\right)$ (8)

Simplifying the given expression, (9)

$$\sum_{t=0}^{T} \sin^2(\theta_t) \leq \int_{\theta_0}^{\theta_T} \sin^2(\phi_T - x\phi_\gamma) dx \leq \frac{1}{\phi_\gamma} \int_{\theta_0}^{\theta_T} \sin^2 y dy \leq \frac{1}{\phi_\gamma} \left(\frac{(\theta_T - \theta_0)}{2} - \left(\sin(2\theta_T) - \sin(2\theta_0)\right)\right)$$

$$\leq \frac{1}{2\phi_\gamma} \left(\frac{(\theta_T - \theta_0)}{2} - \sin(\theta_T - \theta_0) \cos(\theta_T + \theta_0)\right)$$

(10)

Note that $\theta_N = \phi_N$. We consider the following cases:

1. Case 1: $N > \phi_\gamma^{-2/3} \left(\frac{\sigma^2(\sigma^2 + \delta)}{\delta^2}\right)^{\frac{1}{3}}$.

   With $\phi_N = O\left(\phi_\gamma^{1/3}\right)$ we have $N\phi_\gamma > \phi_\gamma^{1/3} > \phi_N$. Hence, $\theta_0 = 0$.

   $$\frac{1}{\phi_\gamma} \left(\frac{(\theta_T - \theta_0)}{2} - \sin(\theta_T - \theta_0) \cos(\theta_T + \theta_0)\right) = \frac{2\phi_N - \sin(2\phi_N)}{4\phi_\gamma} = \frac{2\phi_N - \left(2\phi_N + O\left(\phi_N^3\right)\right)}{4\phi_\gamma} = O\left(\frac{\phi_N^3}{\phi_\gamma}\right) = O\left(\frac{\sigma^2(\sigma^2 + \delta)}{\delta^2}\right)$$

2. Case 2: $N < \phi_\gamma^{-2/3} \left(\frac{\sigma^2(\sigma^2 + \delta)}{\delta^2}\right)^{\frac{1}{3}}$

   By the assumptions of this case and the definition of $s$,

   $$\sum_{t=0}^{N} \sin^2(\theta_t) < N \sin^2(\phi_N) < N(2s)^2 = O\left(\frac{\sigma^2(\sigma^2 + \delta)}{\delta^2}\right).$$
A.5 Proof of Theorem 3.1

\[
D(U_{1:k}, Y) \overset{(i)}{=} D(Y, U_{1:k}) \\
\leq \overset{(ii)}{\|((MN)^\top MN)^{-1/2}(MN)^\top U_{k+1:n} U_{k+1:n}^\top\|_2} \\
\leq \overset{(iii)}{\|((MN)^\top MN)^{-1/2}\|_2 \|N^\top M^\top U_{k+1:n} U_{k+1:n}^\top\|_2} \\
\leq \overset{(iv)}{\|N^\top M^\top U_{k+1:n}\|_2} \\
\leq \overset{(v)}{\|N^\top V_{k+1:n}^\top D_{k+1:n}\|_2} \\
\leq \overset{(vi)}{\frac{s_{k+1}(M)\|N^\top V_{k+1:n}\|_2}{s_k(M)}} \\
\leq \overset{(vii)}{\frac{s_{k+1}(M)s_k(V_{k+1:n}^\top N)}{s_k(M) s_k(V_{1:k}^\top N)}},
\]

where, (i), (ii) (iii) follow from symmetry and definition of the distance operator and Cauchy-Schwarz inequality. (iv) follows from the singular value decomposition of MN and noting that the columns of \(U_{k+1:n}\) are orthonormal. Precisely, let \(G = N^\top M^\top U_{k+1:n}, \quad x \in \mathbb{R}^k\) then,

\[
\|x^\top G U_{k+1:n}^\top\|_2 = \|y^\top U_{k+1:n}\|_2 = \|y\|_2 = \|x^\top G\|_2
\]

This also follows from lemma A.5. (v) and (vi) follow from the singular value decomposition of M and lemma A.5. (vii) can be obtained as follows:

\[
s_k(MN) \overset{(vii),(a)}{=} s_k(M V_{1:k}^\top V_{1:k}^\top N + M V_{k+1:n}^\top V_{k+1:n}^\top N) \\
\geq \overset{(vii),(b)}{s_k(M V_{1:k}^\top V_{1:k}^\top N)} \\
\geq \overset{(vii),(c)}{s_k(MV_{1:k}) s_k(V_{1:k}^\top N)} \\
\geq \overset{(vii),(d)}{s_k(M) s_k(V_{1:k}^\top N)}
\]

where, (vii),(a) follows since \(I = V_{1:k}^\top V_{1:k} + V_{k+1:n}^\top V_{k+1:n}\). Observe that \(MV_{1:k} V_{1:k}^\top N + MV_{k+1:n} V_{k+1:n}^\top N\) have orthogonal columns, implying that for any \(c \in \mathbb{R}^k, \quad y = H c\) and \(z = H^\top c, \quad y, z\) are orthogonal, where \(H = AV_{1:k}V_{1:k}^\top X\) and \(H' = AV_{k+1:n}V_{k+1:n}^\top X, \quad H \in \mathbb{R}^{m \times k}, \quad H' \in \mathbb{R}^{m \times k}\). Since \(N\) has \(k\) columns, by the definition of the least singular value: \(s_k(H + H') = \min_{x \in \mathbb{R}^{k-1}} \|H + H'x\|_2\). By Pythagoras theorem, \(\|Hx + H'x\|_2^2 = \|Hx\|_2^2 + \|H'x\|_2^2 \geq \|Hx\|_2^2 \geq s_k(H) \forall x \in \mathbb{R}^{k-1}\) leading to (vii),(b). To obtain (vii),(c) let \(D = AV_{1:k}\) and \(D' = V_{1:k}^\top X, \quad D \in \mathbb{R}^{m \times k}, \quad D' \in \mathbb{R}^{k \times k}\). Since \(N\) has \(k\) columns, by the definition of the least singular value we have:

\[
s_k(DD') = \min_{x \in \mathbb{R}^{k-1}} \|DD'x\|_2 = \min_{y \in \mathbb{R}^k} \frac{\|Dy\|_2}{\|y\|_2} \|y\|_2, \quad y = \|D'x\|_2
\]

Further, \(\|y\|_2 \geq s_k(D')\) and therefore, \(s_k(DD') \geq s_k(D)s_k(D')\). Finally, (vii),(d) follows by the singular value decomposition of \(M\).
A.6 Proof of Lemma 3.1

For all \(1 \leq l \leq L\), we have,

\[
\sum_{i=(l-1)B+1}^{lB} x_i x_i^\top = \mathbb{E} (x_i^B x_i^B) + \frac{1}{B} \sum_{i=(l-1)B+1}^{lB} \left( x_i x_i^\top - \mathbb{E}(x_i^B x_i^B) \right)
\]

\[
= \mathbb{E} (x_i^B x_i^B) + \frac{1}{B} \sum_{i=(l-1)B+1}^{lB} \left( x_i x_i^\top - \mathbb{E}(x_i^B x_i^B) + \mathbb{E}(x_i x_i^\top) - \mathbb{E}(x_i^B x_i^B) \right)
\]

By lemma A.6, there exists a constant \(C\) such that with probability at least \(1 - \frac{1}{T^2}\) we have:

\[
\left\| \frac{1}{B} \sum_{i=(l-1)B+1}^{lB} \left( (x_i x_i^\top - \mathbb{E}(x_i x_i)) + (\mathbb{E}(x_i x_i^\top) - \mathbb{E}(x_i^B x_i^B)) \right) \right\|_2 \leq \left\| \frac{1}{B} \sum_{i=(l-1)B+1}^{lB} (x_i x_i^\top - \mathbb{E}(x_i x_i)) \right\|_2 + \left\| \frac{1}{B} \sum_{t=(l-1)B+1}^{lB} (\mathbb{E}(x_i x_i^\top) - \mathbb{E}(x_i^B x_i^B)) \right\|_2 \leq \sqrt{\frac{Cp \log(T)}{B}} + \frac{B\gamma}{2}.
\]

From the union bound and the above inequality, with probability \(1 - \frac{1}{T}\),

\[
\max_{1 \leq l \leq L} \| \mathcal{E}(l) \|_2 \leq \sqrt{\frac{Cp \log(T)}{B}} + \frac{B\gamma}{2}.
\]

A.7 Proof of Theorem 3.2

We split the proof of theorem 3.2 into two steps. In the first step, using lemma 3.3 and 3.2 we identify an appropriate \(N^{(l)}\) and \(W^{(l)}\). Then, in the second step we bound the distance between the output of the noisy power method and the span of \(N^{(l)}\) and \(W^{(l)}\).

**Step 1: Identifying \(N^{(l)}\) and \(W^{(l)}\)**

In this step we identify \(N^{(l)}\) and \(W^{(l)}\) using a sequence of matrices \(\{N^{(l)}, W^{(l)}\}_{l=1}^{L+1}, N^l \in \mathbb{R}^{n \times (n-k)}, W^l \in \mathbb{R}^{n \times k}\) from the observed vectors \(\{(x_i)_{i=lB}^{(l+1)B}\}_{l=1}^{L}\). We construct the sequence \(\{N^{(l)}\}_{1 \leq l \leq L+1}, N^{(l)} \in \mathbb{R}^{n \times (n-k)}\) so that the following is satisfied:

**N.1** \(N^{(L+1)} = U_{k+1:n} (L)\)

**N.2** \(\text{span}((M(l) + \mathcal{E}(l)) N^{(l)}) \subseteq \text{span}(N^{(l+1)})\) for all \(1 \leq l \leq L\)

**N.3** For all \(1 \leq l \leq L\),

\[
S_1 \left( (U_{1:k}(l))^\top N^{(l)} \right) \leq \varepsilon \quad \text{and} \quad S_1 \left( (M(l) + \mathcal{E}(l)) N^{(l)} \right) \leq \frac{S_{k+1}(M(l)) + \Delta}{\sqrt{1 - (\varepsilon + \eta)^2}}.\quad (11)
\]
\[ s_1((U_{1:k}(l-1))^\top N(l)) \leq \varepsilon + \eta \text{ for all } 1 \leq l \leq L. \]

To show the existence of \( \{N\}_{i=1}^{L+1} \) satisfying \textbf{N.1}, \textbf{N.2}, \textbf{N.3} and \textbf{N.4} we use lemma 3.2 and backward mathematical induction.

\textbf{Base case:} At \( t = L + 1 \), \( N^{(L+1)} = U_{k+1:n}(L) \), therefore \textbf{N.4} holds from the model assumption \( \| (U_{k+1:n}(I))^\top U_{1:k}(l-1) \|_2 \leq \eta \). Other conditions are required for \( t \leq L \) and hence \( N^{(L+1)} \) exists.

\textbf{Inductive Hypothesis}: Assume that there exists \( N^{(l+1)} \) satisfying \textbf{N.1} - \textbf{N.4}. We show that there exists an \( N^{(l)} \). Define \( N^{(l)} \) to be the matrix identified as \( \bar{W} \) in lemma 3.2 with \( A = M(l) \), \( E = E(l) \) and \( Y = N^{(l+1)} \). Then, lemma 3.2 shows that \( N^{(l)} \) satisfies \textbf{N.2} and \textbf{N.3}. Further, since

\[ s_1(U_{1:k}(l-1)N^{(l)}) \overset{(i)}{=} \| U_{1:k}(l-1)U_{1:k}^\top(l-1) - (I - N^{(l)})(N^{(l)})^\top \|_2 \]

\[ \overset{(ii)}{\leq} \| U_{1:k}(l-1)U_{1:k}^\top(l-1) - U_{1:k}(l)U_{1:k}^\top(l) \|_2 + \| U_{1:k}(l)U_{1:k}^\top(l) - (I - N^{(l)})(N^{(l)})^\top \|_2 \]

\[ \overset{(iii)}{\leq} \eta + \varepsilon \]

where, (i) follows from lemma A.1, (ii) follows from triangle inequality and (iii) follows since \( \| (U_{k+1:n}(I))^\top U_{1:k}(l-1) \|_2 \leq \eta \).

Therefore, there exists \( \{N^{(l)}\}_{1 \leq l \leq L+1} \) such that \( N^{(L+1)} \) satisfies \textbf{N.1} and for \( 1 \leq l \leq L \), \( N^{(l)} \) we have \textbf{N.2}, \textbf{N.3}, and \textbf{N.4}. From the properties \textbf{N.1}, \textbf{N.2}, and \textbf{N.3} of \( \{N^{(l)}\}_{1 \leq l \leq L+1} \) we have

\[ \text{span}(M^{(L)}N^{(1)}) \subseteq \text{span}(U_{k+1:n}(L)) \quad \text{and} \quad \| M^{(L)}N^{(1)} \|_2 \leq \prod_{i=1}^{L} \left( \frac{s_{k+1}(M(l)) + \Delta}{\sqrt{1 - (\varepsilon + \eta)^2}} \right). \tag{12} \]

The first inequality above follows from \textbf{N.2}. The second inequality in 12 is obtained from \textbf{N.3} by applying the following argument \( L \) times: for \( y \in \mathbb{R}^n \), \( y^{(l)} = (M(l) + E(l))y \), we have \( \| y^{(l)} \|_2 \leq \frac{s_{k+1}(M(l) + \Delta)}{\sqrt{1 - (\varepsilon + \eta)^2}} \).

Next, we define the sequence \( \{W\}_{i=1}^{L+1} \), \( W_i \in \mathbb{R}^{n \times k} \) as follows:

\textbf{W.1} \( W^{(1)} \in \mathbb{R}^{n \times k} \) be a matrix consisting of orthonormal vectors such that \( \| (W^{(1)})^\top N^{(1)} \|_2 = 0. \)

\textbf{W.2} \( W^{(l+1)} = (M(l) + E(l))W^{(l)} \) for all \( 1 \leq l \leq L. \)

From \textbf{N.4} and the triangle inequality, we have \( d(U_{1:k}(1), W^{(1)}) \leq \varepsilon + \eta \) and for all \( 1 \leq l \leq L \), lemma 3.3 then implies that:

\textbf{CW.1} \( s_k((M(l) + E(l))W^{(l)}) \geq \sqrt{1 - (\varepsilon + \eta)^2}s_k(M(l)) - \Delta. \)

\textbf{CW.2} \( d(U_{1:k}(l), W^{(l+1)}) \leq \varepsilon \) and since \( \| U_{1:k}(l)^\top U_{1:k}(l-1) \|_2 \leq \eta \), we have \( d(U_{1:k}(l), W^{(l)}) \leq \varepsilon + \eta. \)

From \textbf{CW.1} and \textbf{CW.2}, we have

\[ d(M^{(L)}W^{(1)}, U_{1:k}(L)) \leq \varepsilon \quad \text{and} \quad s_k(M^{(L)}W^{(1)}) \geq \prod_{i=1}^{L} \left( \sqrt{1 - (\varepsilon + \eta)^2}s_k(M(l)) - \Delta \right). \tag{13} \]
This establishes the existence and properties of the sequence \( \{N^{(l)}, W^{(l)}\}_{l=1}^{L} \). We now use this characterization to bound the distance between the \( k \)-dimensional subspace of \( \mathbb{R}^n \) and \( M^{(L)}Q_{(0)} \). Since, \( N.3 \) bounds the distance between the \( (n - k) \) dimensional subspace of \( \mathbb{R}^n \) and \( M^{(L)}Q_{(0)} \), we consider bound the distance between \( M^{(L)}Q_{(0)} \) and \( W^{(1)} \).

**Step 2: Distance between actual and recovered spaces**

Now, we upper bound the distance between the output of the power method \( \hat{U}_{1:k}(L) \) and the last \( n - k - 1 \) singular vectors of the true underlying subspace \( U_{k+1:n}(L) \), \( d(U_{1:k}(L), \hat{U}_{1:k}(L)) \). From the triangle inequality we have,

\[
d(U_{1:k}(L), \hat{U}_{1:k}(L)) \leq d(U_{1:k}(L), M^{(L)}W^{(1)}) + d(M^{(L)}W^{(1)}, \hat{U}_{1:k}(L)).
\]

From 13, we have

\[
d(U_{1:k}(L), M^{(L)}W^{(1)}) \leq \varepsilon.
\]

To bound the second term in the RHS of 14 consider the following,

\[
d(M^{(L)}W^{(1)}, \hat{U}_{1:k}(L)) \overset{(i)}{=} \left\| (M^{(L)}W^{(1)})_{\perp} \hat{U}_{1:k}(L) \right\|_2
\]

\[
\overset{(ii)}{=} \left\| (M^{(L)}W^{(1)})_{\perp} M^{(L)}(\hat{U}_{1:k}(0))((M^{(L)}\hat{U}_{1:k}(0))^T M^{(L)}\hat{U}_{1:k}(0))^{-1/2} \right\|_2
\]

\[
\overset{(iii)}{\leq} \left\| (M^{(L)}W^{(1)})_{\perp} M^{(L)}\hat{U}_{1:k}(0) \right\|_2 \left\| ((M^{(L)}\hat{U}_{1:k}(0))^T M^{(L)}\hat{U}_{1:k}(0))^{-1/2} \right\|_2
\]

\[
\overset{(iv)}{=} \frac{\left\| (M^{(L)}W^{(1)})_{\perp} M^{(L)}(N^{(1)}(N^{(1)})^T \hat{U}_{1:k}(0)) \right\|_2}{s_k(M^{(L)}\hat{U}_{1:k}(0))}
\]

\[
\overset{(v)}{=} \left\| (M^{(L)}W^{(1)})_{\perp} M^{(L)}(N^{(1)})^T \hat{U}_{1:k}(0) \right\|_2 \left\| (N^{(1)})^T \hat{U}_{1:k}(0) \right\|_2
\]

where, \((i)\) follows by substituting the definition of distance function and \((ii)\) follows by observing that due to the power iterations \( \hat{U}_{1:k}(L) \) is an orthonormal basis of \( M^{(L)}\hat{U}_{1:k}(0) \) and therefore can be written as \( b(M^{(L)}\hat{U}_{1:k}(0)) \). \((iii)\) follows by using Cauchy-Schwarz inequality for matrix norms inequality. \((iv)\) follows by noting that \( \left\| ((M^{(L)}\hat{U}_{1:k}(0))^T M^{(L)}\hat{U}_{1:k}(0))^{-1/2} \right\|_2 \leq \frac{1}{s_k(M^{(L)}U_{1:k}(0))} \). To obtain \((v)\), decompose the numerator of \((iv)\) as \( (M^{(L)}W^{(1)})_{\perp} M^{(L)}U_{1:k}(0) = (M^{(L)}W^{(1)})_{\perp} M^{(L)}(W_1W_1^T + N_1N_1^T)M^{(L)}U_{1:k}(0) \) and note that by orthogonality of \( (M^{(L)}W^{(1)})_{\perp} \) and \( (M^{(L)}W_1) \), \( (M^{(L)}W^{(1)})_{\perp} (M^{(L)}W_1)W_1^T U_{1:k}(0) = 0 \). Finally, \((vi)\) follows from the repeated application of Cauchy-Schwarz inequality for matrix norms. Further, When \( s_k(M^{(L)}W^{(1)})s_k((W^{(1)})^T \hat{U}_{1:k}(0)) - \)
\[
\| \mathcal{M}^{(L)} \mathbf{N}^{(1)} \|_2 \left\| (\mathbf{N}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0) \right\|_2 > 0,
\]

\[
d(\mathcal{M}^{(L)} \mathbf{W}^{(1)}, \hat{\mathbf{U}}_{1:k}(L)) \overset{\text{(vii)}}{=} \frac{\| \mathcal{M}^{(L)} \mathbf{N}^{(1)} \|_2 \left\| (\mathbf{N}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0) \right\|_2}{s_k(\mathcal{M}^{(L)} \mathbf{W}^{(1)}) s_k(( (\mathbf{W}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0)) - \| \mathcal{M}^{(L)} \mathbf{N}^{(1)} \|_2 \left\| (\mathbf{N}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0) \right\|_2}
\]

\[
\overset{\text{(viii)}}{\leq} \frac{\| \mathcal{M}^{(L)} \mathbf{N}^{(1)} \|_2 \left\| (\mathbf{N}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0) \right\|_2}{s_k(\mathcal{M}^{(L)} \mathbf{W}^{(1)}) s_k(( (\mathbf{W}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0))}
\]

\[
1 - \phi^{-L} \frac{\left\| (\mathbf{N}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0) \right\|_2}{s_k(( (\mathbf{W}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0))}
\]

\[
\overset{\text{(ix)}}{\leq} \frac{\phi^{-L} \left\| (\mathbf{N}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0) \right\|_2}{s_k(( (\mathbf{W}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0))}
\]

(\text{viii}) \text{ follows since } \| (\mathcal{M}^{(L)} \mathbf{W}^{(1)})_\perp \|_2 = 1 \text{ since } (\mathcal{M}^{(L)} \mathbf{W}^{(1)})_\perp \text{ is a projection matrix and (ix) stems from (12) and (13).}

Putting (15) and (16) onto (14), we have

\[
d(\mathbf{U}_{1:k}, \hat{\mathbf{U}}_{1:k}) \leq \varepsilon + \min \left\{ 1, 2\phi^{-L} \frac{\left\| (\mathbf{N}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0) \right\|_2}{s_k(( (\mathbf{W}^{(1)})^\top \hat{\mathbf{U}}_{1:k}(0))} \right\}.
\]

Further, by lemma 2.5 in Hardt and Price (2014), we have

\[
\frac{\| (\mathbf{N}^{(1)})\hat{\mathbf{U}}_{1:k}(0) \|_2}{s_k(( (\mathbf{W}^{(1)})\hat{\mathbf{U}}_{1:k}(0)) \leq \frac{c\sqrt{p}}{\sqrt{p} - \sqrt{k} - 1}
\]

with probability \(1 - c^{\Omega(p-k+1)} - c^{-\Omega(p)}\). Therefore when,

\[
L > \frac{\log \left( \frac{c\sqrt{p}}{\sqrt{p} - \sqrt{k} - 1} \right)}{\log(\phi)}
\]

we have,

\[
d(\mathbf{U}_{1:k}, \hat{\mathbf{U}}_{1:k}) \leq \varepsilon + O\left( \frac{\phi^{-L} \sqrt{p}}{\sqrt{p} - \sqrt{k} - 1} \right)
\]
### A.8 Proof of Lemma 3.2

We first show that for any positive definite matrix $M \in \mathbb{R}^{n \times n}$, $E \in \mathbb{R}^{n \times n}$ and $Y \in \mathcal{Y}$, there exists orthonormal matrix $\overline{N} \in \mathbb{R}^{n \times (n-k)}$ such that $\text{span } ((M + E)\overline{N}) \subseteq \text{span } (Y)$ as follows:

1. When $M + E$ is a full-rank matrix, $\text{span } ((M + E)\overline{N}) = \text{span } (Y)$ with $\overline{N}$ = basis $((M + E)^{-1}Y)$.

2. When the rank of $M + E$ is $r \leq k$, every $\overline{N}$ such that $(M + E)\overline{N} = 0$ satisfies that $\text{span } ((M + E)\overline{N}) = \emptyset \subseteq \text{span } (Y)$.

3. Assume that the rank of $M + E$ is $r$, $k < r < n$. We identify $\overline{N}$ in parts by identifying the first $(r-k)$ columns and then the remaining columns. Let $(M + E) = UD\tilde{V}^T$ and $Y^T\tilde{U}_{1:r} = UD\tilde{V}^T$ be the singular value decomposition of $(M + E)$ and $Y^T\tilde{U}_{1:r}$ respectively.

Observe that $Y$ has $(n-k)$ columns and $\tilde{U}_{1:r}$ has $r$ columns and these vectors form a basis for $(n-k)$ dimensional subspace and $r$ dimensional subspace of $\mathbb{R}^n$ respectively. Since $(n-k)+r > n$, the column spaces of $Y$ and $\tilde{U}_{1:r}$ overlap on a subspace of dimension at least $r - k$. Therefore, we can find $(r-k)$ orthonormal vectors in this shared subspace, say, $v_1, v_2, \ldots, v_{r-k} \in \mathbb{R}^n$. For $1 \leq j \leq r$, let $x_j \in \mathbb{R}^i$ be such that

$$\tilde{U}_{1:r}x_j = v_j$$

i.e. $x_j = \tilde{U}_{1:r}^Tv_j$. Thus the $x_j$ are orthonormal, and since $\{v_j\}_{j=1}^{r-k}$ are orthonormal and contained in the column space of $Y$, for $1 \leq j \leq r$ we have $1 = \|Y^Tv_j\|_2 = \|Y^T\tilde{U}_{1:r}x_j\|_2 = \|\tilde{U}_{1:r}x_j\|_2$. Thus, $\{x_j\}_{j=1}^{r-k}$ are right-singular vectors of $Y^T\tilde{U}_{1:r}$ with singular value 1 (which is the maximum singular value of $Y^T\tilde{U}_{1:r}$) and therefore without loss of generality, they are the first $r - k$ columns of $V_Y$. To identify these $v_j$ we use the above paragraph, that is to say,

$$(v_j)_{j=1}^{r-k} = \tilde{U}_{1:r-k}(\hat{V})_{1:r} = (M + E)(\tilde{V}^{-1})_{1;r}(\hat{V})_{1;r-k}.$$

Hence, since the $v_j$ are spanned by the columns of $Y$,

$$(M + E)(\tilde{V}^{-1})_{1;i}(\hat{V})_{1;i-k} \subseteq \text{span } (Y).$$

Define $\{z\}_{i=1}^{r-k}$ to be an orthonormal basis of the column space of $(\tilde{V}^{-1})_{1;r}(V_Y)_{1;r-k}$ i.e. $\{z\}_{i=1}^{r-k} = b((\tilde{V}^{-1})_{1;r}(V_Y)_{1;r-k})$. The first $r - k$ columns of $\overline{N}$ are defined to be $\{z\}_{i=1}^{r-k}$.

At this point we have identified only $r - k$ columns for $\overline{N}$. The remaining $(n - i)$ columns are picked from the null space of $(M + E)$. A vector $x$ in the null space $(M + E)x = 0$ is also a right singular vector of $(M + E)$ whose singular value is 0. Since $M + E$ has rank $r$, there are $n - r$ right singular vectors of $M + E$ with zero singular value and we use them to define the remaining $r - k$ columns of $\overline{N}$.

Thus, when

$$\overline{N} = \left[b \left((\tilde{V}^{-1})_{1;i}(\hat{V})_{1;i-k}\right), \tilde{V}_{i+1:n}\right],$$

we have $\text{span } ((M + E)\overline{N}) \subseteq \text{span } (Y)$.
We establish the second part of (6) by contradiction. To show that

\[ s_1 \left( U_{1:k}^\top \mathbf{N} \right) \leq \varepsilon \quad \text{if} \quad \text{span} \left( (M + \mathcal{E})\mathbf{N} \right) \subseteq \text{span} \left( \mathbf{Y} \right), \]

we will show that if

\[ x \in \text{span}(\mathbf{N}), \quad \|x\|_2 = 1, \quad \text{and} \quad \|U_{1:k}^\top x\| > \varepsilon \quad \text{then} \quad (M + \mathcal{E})x \notin \text{span}(Y) \]

When \( \|U_{1:k}^\top x\| > \varepsilon, \)

\[
\|U_{1:k}^\top (M + \mathcal{E})x\|_2 \geq \|U_{1:k}^\top Mx\|_2 - \|U_{1:k}^\top \mathcal{E}x\|_2 \\
> s_k(M) \varepsilon - \Delta \quad \text{and} \\
\|U_{k+1:n}^\top (M + \mathcal{E})x\|_2 \leq \|U_{k+1:n}^\top Mx\|_2 + \|U_{k+1:n}^\top \mathcal{E}x\|_2 \\
\leq s_{k+1}(M) \sqrt{1 - \varepsilon^2} + \Delta.
\]

(i) and (ii) follows from triangle inequality for matrix norms. \( \|U_{1:k}^\top Y\|_2 \leq \varepsilon + \eta \) is equivalent to

\[
\frac{\varepsilon}{\sqrt{1 - \varepsilon^2}} s_k(M) - \Delta \sqrt{1 - (\varepsilon + \eta)^2} \geq \frac{\varepsilon}{\sqrt{1 - \varepsilon^2}} s_{k+1}(M) + \frac{\Delta}{\sqrt{1 - (\varepsilon + \eta)^2}}.
\]

We can derive (7) from (6) as follows:

\[
s_1 \left( (M + \mathcal{E})\mathbf{N} \right) \\
\overset{(iv)}{=} \sup_{y \in \mathbb{R}^{n-k} : \|y\|_2 = 1} \| (M + \mathcal{E})\mathbf{N} y \|_2 \\
\overset{(v)}{\leq} \sup_{y \in \mathbb{R}^{n-k} : \|y\|_2 = 1} \left\| \frac{U_{k+1:n}^\top (M + \mathcal{E})\mathbf{N} y}{\sqrt{1 - (\varepsilon + \eta)^2}} \right\|_2 \\
\overset{(vi)}{\leq} \frac{\left\| U_{k+1:n}^\top (M + \mathcal{E}) \right\|_2}{\sqrt{1 - (\varepsilon + \eta)^2}} \\
\overset{(vii)}{=} \frac{s_{k+1}(M) + \Delta}{\sqrt{1 - (\varepsilon + \eta)^2}},
\]

(iv) to (v) follows since \( (M + \mathcal{E})\mathbf{N} y \in \text{span}(Y), \) \( \|U_{1:k}^\top (M + \mathcal{E} \mathbf{N} y)\|_2 \leq (\varepsilon + \eta) \| (M + \mathcal{E})\mathbf{N} y \|_2, \)

\[
\left\| U_{k+1:n}^\top (M + \mathcal{E})\mathbf{N} y \right\|_2^2 = \| (M + \mathcal{E})\mathbf{N} y \|_2^2 - \| U_{1:k}^\top (M + \mathcal{E})\mathbf{N} y \|_2^2 \quad \text{therefore,} \quad \| (M + \mathcal{E})\mathbf{N} y \|_2 \leq \frac{\| U_{k+1:n}^\top (M + \mathcal{E})\mathbf{N} y \|_2}{\sqrt{1 - (\varepsilon + \eta)^2}}, \quad (v) \quad \text{to} \quad (vii) \quad \text{follows since} \quad \mathbf{N} \quad \text{has orthonormal columns and} \quad (vii) \quad \text{follows from} \quad (6) \quad \text{where we have} \quad \left\| U_{k+1:n}^\top (M + \mathcal{E}) \right\|_2 \leq s_{k+1}(M) + \Delta.
\]

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A.9 Proof of Lemma 3.3

First, we show that \( s_k((M + X)W) \geq \sqrt{1 - (\varepsilon + \eta)^2} s_k(M) - \Delta. \)

\[
\begin{align*}
    s_k((M + \mathcal{E})W) & \overset{(i)}{=} s_k(U_{1:k}^T(M + \mathcal{E})W) \\
    & \overset{(ii)}{=} s_k(U_{1:k}^T MW) - \|\mathcal{E}W\|_2 \\
    & \overset{(iii)}{=} s_k(M) s_k(U_{1:k}^T W) - \|\mathcal{E}W\|_2 \\
    & \overset{(iv)}{=} s_k(M) \sqrt{1 - (\varepsilon + \eta)^2} - \|\mathcal{E}W\|_2.
\end{align*}
\]

(i) follows since, \( I = U_{1:k}^T U_{1:k} + U_{k+1:n}^T U_{k+1:n} \). (ii) follows since

\[
\|A + B\|_k \leq \|A\|_k + \|B\|_2
\]

with \( A = U_{1:k}^T(M + \mathcal{E})W \) and \( B = -U_{1:k}^T\mathcal{E}W \). (iii) follows from singular value decomposition of \( M \) and noting that \( M \) is positive semidefinite and lemma A.5. To obtain (iv), let columns of \( \mathcal{W} \) represent the space orthogonal to column space of \( W \) and note that both \( W \) and \( \tilde{W} \) have orthonormal columns. Then,

\[
\|x^T U_{1:k}^T(W + \tilde{W})\|_2^2 = \|x^T U_{1:k}^T W\|_2^2 + \|x^T U_{1:k}^T \tilde{W}\|_2^2 = 1
\]

and therefore,

\[
\min_{x \in \mathbb{S}^{k-1}} \|x^T U_{1:k}^T W\|_2^2 = 1 - \max_{x \in \mathbb{S}^{k-1}} \|x^T U_{1:k}^T \tilde{W}\|_2^2 = 1 - (d(U_{1:k} \tilde{W}))^2
\]

(iv) now follows from the definition of largest singular value and the assumptions on the lemma.

We now prove that \( d(U_{1:k}, (M + \mathcal{E})W) \leq \varepsilon \) or equivalently \( s_k(U_{1:k}, b((M + \mathcal{E})W))^2 \geq 1 - \varepsilon^2 \), since

\[
d(U_{1:k}, (M + \mathcal{E})W) = s_k(U_{1:k}, b((M + \mathcal{E})W)_\perp) = \sqrt{1 - s_k(U_{1:k}, b((M + \mathcal{E})W))^2}.
\]

Further,

\[
\begin{align*}
    s_k\left(U_{1:k}^T b((M + \mathcal{E})W)^2\right) & \overset{(vi)}{=} \min_{x \in \mathbb{S}^{k-1}} \frac{\|U_{1:k}^T(M + \mathcal{E})Wx\|_2^2}{\|M + \mathcal{E}W\|_2^2} \\
    & \overset{(vii)}{=} \min_{x \in \mathbb{S}^{k-1}} \frac{\|U_{1:k}^T(M + \mathcal{E})Wx\|_2^2}{\|M + \mathcal{E}W\|_2^2} \\
    & \overset{(viii)}{=} \min_{x \in \mathbb{S}^{k-1}} \frac{\|U_{1:k}^T(M + \mathcal{E})Wx\|_2^2}{\|U_{1:k}^T(M + \mathcal{E})W\|_2^2 + \|U_{k+1:n}^T(M + \mathcal{E})W\|_2^2}
\end{align*}
\]

To obtain (vi) note that by definition of \( b((M + \mathcal{E})W) \) and \( (M + \mathcal{E})W \) share the same column space and therefore, \( \forall x \in \mathbb{R}^k, \exists y \in \mathbb{R}^k \) such that \( b((M + \mathcal{E})W)x = (M + \mathcal{E})Wy \|M + \mathcal{E}Wy\|_2^2 \). (vi) then follows from the definition of the largest singular value. (vii) and (viii) follow by projecting \( M + \mathcal{E}W \) onto the column
spaces of $U_{1:k}$ and $U_{k+1:n}$ and noting that $\|Uy\|_2 = \|y\|_2$ when $U$ has orthonormal columns. Using this decomposition it now suffices to show:

$$\min_{x \in S_{k-1}} \frac{\|U_{1:k}^T (M+E)Wx\|_2^2}{\|U_{1:k}^T (M+E)Wx\|_2^2 + \|U_{k+1:n}^T (M+E)Wx\|_2^2} \geq 1 - \varepsilon^2$$  \hspace{1cm} (18)

To obtain (18) observe that its left hand side is of the form $\min_{x \in S_{k-1}} \frac{p}{1+p}$ with $p = \frac{\|U_{1:k}^T (M+E)Wx\|_2}{\|U_{k+1:n}^T (M+E)Wx\|_2}$ and it monotonically increases in $p$ and therefore the minimum is attained at the smallest possible value of $p$. Therefore, we bound $p$ from below as

$$\min_{x \in S_{k-1}} \frac{\|U_{1:k}^T (M+E)Wx\|_2}{\|U_{k+1:n}^T (M+E)Wx\|_2} \geq \frac{(viii) \frac{s_k(M) - \Delta/\sqrt{1-(\varepsilon+\eta)^2} \sqrt{1-(\varepsilon+\eta)^2}}{s_{k+1}(M) + \Delta/(\varepsilon+\eta)} \varepsilon + \eta}{(ix) \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}},$$

where $(viii)$ stems from the fact that for all given unit vector $x \in \mathbb{R}^k$,

$$\|U_{1:k}^T (M+E)Wx\|_2 \geq \|U_{1:k}^T MWx\|_2 - \|U_{1:k}^T E Wx\|_2 \geq s_k(M) \sqrt{1-(\varepsilon+\eta)^2} - \Delta$$

and

$$\|U_{k+1:n}^T (M+E)Wx\|_2 \leq \|U_{k+1:n}^T MWx\|_2 + \|U_{k+1:n}^T E Wx\|_2 \leq s_{k+1}(M)(\varepsilon + \eta) + \Delta$$

and $(ix)$ can be obtained from the definition of $\Delta$ and the assumption $\varepsilon \leq \frac{1}{4}$. Substituting, $p > \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}$ gives the desired lower bound in lemma 18.

A.10 Proof of Corollary 3.1

For $\gamma = 0$, assumptions A.1-A.4 become $\varepsilon \leq \frac{1}{4}$, $\|E(l)\|_2 \leq \frac{\varepsilon}{4}(s_k - s_{k+1})$ and $s_{k+1}(A) > s_k(A) \left( \frac{2-\varepsilon^2 \sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2}} \right)$. The block size $B$ can be obtained as $\sqrt{\frac{Cp \log(2/\alpha)}{B}} = \frac{e\delta}{4}$, $B = \left( \frac{4Cp \log(2/\alpha)}{e\delta} \right)^2$. Putting into theorem 3.2 we get the result.