Simple \( \mathfrak{sl}_{n+1} \)-module structures on \( \mathcal{U}(\mathfrak{h}) \)

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Abstract

We study the category \( \mathcal{M} \) consisting of \( \mathcal{U}(\mathfrak{sl}_{n+1}) \)-modules whose restriction to \( \mathcal{U}(\mathfrak{h}) \) is free of rank 1, in particular we classify isomorphism classes of objects in \( \mathcal{M} \) and determine their submodule structure. This leads to new \( \mathfrak{sl}_{n+1} \)-modules. For \( n = 1 \) we also find the central characters and derive an explicit formula for taking tensor product with a simple finite dimensional module.

1 Introduction and description of the results

Classification of simple modules is an important first step in understanding the representation theory of an algebra. The Lie algebra \( \mathfrak{sl}_2 \) is the only simple finite dimensional complex Lie algebra for which some version of such a classification exists, see [Bl, Maz1]. For other Lie algebras no full classification is known, however, various natural classes of simple modules are classified, in particular, simple finite dimensional modules (see e.g. [Ca, Di]), simple highest weight modules (see e.g. [Di, Hn]), simple weight modules with finite dimensional weight spaces (see [Fc, Ful, Mat]), Whittaker modules (see [Ko]), and Gelfand-Zetlin modules (see [DFO]).

This paper contributes with a new family of simple modules for the Lie algebra \( \mathfrak{sl}_{n+1} = \mathfrak{sl}_n + 1(\mathbb{C}) \), the algebra of \( (n+1) \times (n+1) \) complex matrices with the Lie bracket \([a, b] = ab - ba\). This algebra is important as, by Ado's Theorem (see e.g. [Di]), every finite dimensional complex Lie algebra is isomorphic to a subalgebra of \( \mathfrak{sl}_{n+1} \) for some \( n \). Furthermore, the representation theory of \( \mathfrak{sl}_{n+1} \) has (recently discovered) connection to knot invariants, see e.g. [St, Maz2]. The idea of our construction originates in the attempt to understand whether the general setup for study of Whittaker modules proposed in [BM] can be used to construct some explicit families of simple \( \mathfrak{sl}_{n+1} \)-modules (in analogy as, for example, was done for the Virasoro algebra in [MW, MZ, LLZ]).

Let \( \mathfrak{h} \) denote the standard Cartan subalgebra of \( \mathfrak{sl}_{n+1} \) consisting of all diagonal matrices. One of the most classical families of \( \mathfrak{sl}_{n+1} \)-modules is the family of so-called weight modules which are the modules on which \( \mathfrak{h} \) acts diagonalizably. In the present paper we study the category \( \mathcal{M} \) of \( \mathfrak{sl}_{n+1} \)-modules defined by the "opposite condition", namely, as the full subcategory of \( \mathfrak{sl}_{n+1} \)-mod consisting of modules which are free of rank 1 when restricted to \( \mathcal{U}(\mathfrak{h}) \). Here is a classical \( \mathfrak{sl}_2 \)-example (see [AP]):

**Example 1.** Let \( n = 1 \) and let \( h = \frac{1}{2}(e_{1,1} - e_{2,2}) \). Then \( \mathbb{C}[h] \) becomes an \( \mathfrak{sl}_2 \)-module with the action given by

\[
\begin{align*}
    h \cdot f(h) &= hf(h) \\
    e_{1,2} \cdot f(h) &= hf(h - 1) \\
    e_{2,1} \cdot f(h) &= -hf(h + 1).
\end{align*}
\]

We note that \( \text{Res}^{\mathcal{U}(\mathfrak{sl}_2)}_{\mathcal{U}(\mathfrak{h})} \mathbb{C}[h] \) is isomorphic to \( \mathcal{U}(\mathfrak{h}) \mathcal{U}(\mathfrak{h}) \) (the module \( \mathcal{U}(\mathfrak{h}) \) with the natural left action) so under this action, \( \mathbb{C}[h] \) is free of rank 1 and this module belongs to \( \mathcal{M} \).
Since \( U(\mathfrak{g}) \) acts freely on modules in \( \mathcal{M} \), these modules are infinite dimensional and has no weight vectors. In particular \( \mathcal{M} \) contains no weight modules. We shall later also see that the action of the subalgebra \( \mathfrak{n}_+ \) of upper triangular matrices is generically not locally finite, so the modules in \( \mathcal{M} \) are generically not Whittaker modules in the sense of \([Ko]\), or quotients of such. The modules of \( \mathcal{M} \) are generically not even Whittaker modules in the sense of \([BM]\).

Classifying the objects of \( \mathcal{M} \) is equivalent to finding all possible ways of extending \( U(\mathfrak{g})U(\mathfrak{h}) \) to an \( \mathfrak{sl}_{n+1} \)-module. In Section 3 we focus on the case \( n = 1 \). In Theorem 11 we classify the modules of \( \mathcal{M} \) for \( \mathfrak{sl}_2 \) and determine their Jordan-Hölder composition. It turns out that the situation is analogous to that of Verma modules (see \([Di, Hu]\)) in the following sense: the modules of \( \mathcal{M} \) are generically simple, and each reducible module of \( \mathcal{M} \) has a unique submodule which also belongs to \( \mathcal{M} \), and a corresponding finite dimensional quotient. In Section 3.4 we solve a version of the Clebsch-Gordan problem in \( \mathcal{M} \) for \( n = 1 \): we give an explicit decomposition formula for \( M \otimes E \) where \( M \) is a simple object of \( \mathcal{M} \) and \( E \) is a simple finite dimensional module.

In Section 4 we generalize some of the results to \( \mathfrak{sl}_{n+1} \) for arbitrary \( n \geq 1 \). In particular, we classify isomorphism classes of objects in \( \mathcal{M} \) completely in Theorem 30. Here follows a special case of the result:

**Theorem 2.** For \( 1 \leq k \leq n \), let \( h_k := e_{k,k} - \frac{1}{n+1} \sum_{m=1}^{n+1} e_{m,m} \in \mathfrak{sl}_{n+1} \). Then for each \( b \in \mathbb{C} \), the vector space \( \mathbb{C}[h_1, \ldots, h_n] \) is a simple \( \mathfrak{sl}_{n+1} \)-module under the action

\[
\begin{align*}
  h_i \cdot f(h_1, \ldots, h_n) &= h_i f(h_1, \ldots, h_n) & 1 \leq i \leq n, \\
  e_{i,n+1} \cdot f(h_1, \ldots, h_n) &= (b + \sum_{m=1}^{n+1} h_m)(h_i - b - 1)f(h_1, \ldots, h_i - 1, \ldots, h_n), & 1 \leq i \leq n, \\
  e_{n+1,j} \cdot f(h_1, \ldots, h_n) &= -f(h_1, \ldots, h_j + 1, \ldots, h_n) & 1 \leq j \leq n, \\
  e_{i,j} \cdot f(h_1, \ldots, h_n) &= (h_i - b - 1)f(h_1, \ldots, h_i - 1, \ldots, h_j + 1, \ldots, h_n) & 1 \leq i,j \leq n.
\end{align*}
\]

Moreover, for \( n > 1 \), different choices of \( b \) gives nonisomorphic modules. For details, compare this with Definition 28 where the module structure above corresponds to the module labeled \( M^b_\emptyset \).

In Section 4.3 we determine the submodule structure of the objects in \( \mathcal{M} \). It turns out that the objects generically are simple, while the reducible ones have length 2 with a simple finite dimensional top.

After this paper was finished we heard about some related results \([TZ]\) which are to appear shortly.

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## 2 Preliminaries

In this section we collect some of the basic definitions and results needed for studying our module categories. We denote by \( \mathbb{N} \) and \( \mathbb{N}_0 \) the sets of positive integers and nonnegative integers, respectively.

### 2.1 Some categories of modules

In this subsection let \( \mathfrak{g} \) be any finite dimensional complex Lie algebra admitting a triangular decomposition

\[ \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \]
As usual we let $\mathcal{U}(g)$-Mod be the category of all left $g$ modules while we denote the subcategory of finite dimensional modules by $g$-fmod. We also let $\mathcal{O}$ be the full subcategory of $\mathcal{U}(g)$-Mod consisting of finitely generated weight modules which are locally $\mathcal{U}(n_+)$-finite [BGG, Hu]. Now define $\mathcal{M}$ to be the full subcategory of $g$-Mod consisting of modules whose restriction to $\mathcal{U}(h)$ is finitely generated.

**Proposition 3.**

(i) The category $\mathcal{M}$ is abelian.

(ii) The only weight modules in $\mathcal{M}$ are the finite dimensional modules. In particular $\mathcal{M} \cap \mathcal{O} = g$-fmod.

**Proof.** To prove (i), first note that the category $\mathcal{M}$ is closed under taking quotients and direct sums. Moreover, $\mathcal{U}(h)$ is isomorphic to the polynomial algebra in finitely many variables so it is noetherian. Thus any submodule of a finitely generated module is finitely generated and $\mathcal{M}$ is closed under taking submodules. It follows that $\mathcal{M}$ is abelian.

To prove (ii), first note that any finite-dimensional weight module is generated as an $\mathcal{U}(h)$-module by any basis, so $\mathcal{U}(g)$-fmod $\subset \mathcal{M}$. On the other hand, if $M$ is a weight module and $\{v_i| i \in I\}$ is a basis of $M$ consisting of weight vectors, then as an $\mathcal{U}(h)$-module, $M$ decomposes as a direct sum of one dimensional modules:

$$M = \bigoplus_{i \in I} \mathbb{C}v_i.$$  

In particular, if $I$ is infinite, no finite subset can generate $M$ as a $\mathcal{U}(h)$-module. $\square$

Now define $\mathcal{M}$ to be the full subcategory of $\mathcal{U}(g)$-Mod consisting of objects whose restriction to $\mathcal{U}(h)$ are free of rank 1:

$$\mathcal{M} := \{M \in \mathcal{U}(g)\text{-Mod}| \text{Res}_{\mathcal{U}(h)}^{\mathcal{U}(g)} M \cong_{\mathcal{U}(h)} \mathcal{U}(h)\}.$$  

The goal of this paper is to understand the category $\mathcal{M}$ for $g = \mathfrak{sl}_{n+1}$. We also define

$$\overline{\mathcal{M}} := \{M \in \mathcal{U}(g)\text{-Mod}| \text{Res}_{\mathcal{U}(h)}^{\mathcal{U}(g)} M \text{ is free of finite rank}\}.$$  

Then $\overline{\mathcal{M}}$ is closed under finite direct sums and under taking tensor products with finite dimensional modules. We now have inclusions of full subcategories as follows:

$$\mathcal{M} \subset \overline{\mathcal{M}} \subset \mathcal{M} \subset \mathcal{U}(g)\text{-Mod}.$$  

For $M \in \overline{\mathcal{M}}$ we note that $\mathcal{U}(h)$ acts freely on $M$. Thus the sum of the weight spaces of $M$ is zero and, in particular, $\mathcal{U}(g)$-fmod $\cap \overline{\mathcal{M}} = \emptyset$.

**2.2 A basis of $\mathcal{U}(h)$**

For the rest of this paper, we fix $g = \mathfrak{sl}_{n+1}$. For each $k \in \mathfrak{n} := \{1, 2, \ldots, n\}$ define

$$h_k := e_{k,k} - \frac{1}{n+1} \sum_{i=1}^{n+1} e_{i,i}. \quad (1)$$  

Then $\{h_1, \ldots, h_n\}$ generate $\mathcal{U}(h)$ and we can identify $\mathcal{U}(h) \cong \mathbb{C}[h_1, \ldots, h_n]$. An advantage of this choice of generators is that they satisfy the relations

$$[h_k, e_{i,n+1}] = \delta_{k,i} e_{i,n+1} \quad \text{for all } i, k \in \mathfrak{n}.$$  

We also define

$$\overline{h} := \sum_{i=1}^{n} h_i.$$
and note that \([e_i, e_{i,n+1}] = e_{i,n+1}\) for all \(i \in \mathfrak{n}\). With respect to the basis 
\[
\{e_{i,j} | 1 \leq i, j \leq n+1; i \neq j\} \cup \{h_1, \ldots, h_n\}
\]
of \(\mathfrak{s}_{n+1}\), the bracket operation is given by the following lemma.

**Lemma 4.** For \(i, i', j, j' \in \mathfrak{n} + 1; i \neq j; i' \neq j'\) and \(k, k' \in \mathfrak{n}\) we have
\[
[e_{i,j}, e_{i',j'}] = \delta_{j,i'}e_{i,j'} - \delta_{i,j'}e_{i',j}
\]
\[
[h_k, e_{i,j}] = (\delta_{k,i} - \delta_{k,j})e_{i,j}
\]
\[
[h_k, h_{k'}] = 0.
\]

**Proof.** The first and third identities are obvious and the second identity follows from (1). □

We also introduce some notation for our polynomial rings. Define
\[
\mathcal{P} := \mathbb{C}[h_1, \ldots, h_n],
\]
and for each \(i \in \mathfrak{n}\) define
\[
\mathcal{P}_i := \mathbb{C}[h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n].
\]
Note that \(\mathcal{P} \simeq \mathcal{P}_i[h_i] \simeq (\mathcal{P}_i \cap \mathcal{P}_j)[h_i, h_j]\) and so on.

### 2.3 Gradings

It turns out to be helpful to use some different gradings on \(\mathcal{P}\). For each \(i \in \mathfrak{n}\) we define
\[
\text{deg}_i h_1^{d_1}h_2^{d_2} \cdots h_n^{d_n} := d_i.
\]
In other words, \(\text{deg}_i(f)\) is the degree of \(f\) when considered as a polynomial in a single variable \(h_i\) and with coefficients in \(\mathcal{P}_i\). For convenience we let \(\text{deg}_0 := -1\). We also define
\[
c_i : \mathcal{P} \to \mathcal{P}_i,
\]
to be the map taking the leading coefficient of a given polynomial with respect to the grading \(\text{deg}_i\). For example, for \(f = h_1h_2 + h_2 \in \mathbb{C}[h_1, h_2, h_3]\) we have
\[
c_1(f) = h_2, \quad c_2(f) = h_1 + 1 \quad \text{and} \quad c_3(f) = f.
\]
Note that each map \(c_i\) is nonlinear but multiplicative:
\[
c_i(fg) = c_i(f)c_i(g) \quad \text{for all} \quad f, g \in \mathcal{P}.
\]

### 2.4 Automorphisms of \(\mathcal{U}(\mathfrak{h})\)

For each \(i \in \mathfrak{n}\), define an algebra automorphism
\[
\sigma_i : \mathcal{U}(\mathfrak{h}) \longrightarrow \mathcal{U}(\mathfrak{h})
\]
by
\[
\sigma_i(h_k) := h_k - \delta_{i,k}.
\]
Then \(\sigma_1, \ldots, \sigma_n\) generate an abelian subgroup of \(\text{Aut}(\mathcal{U}(\mathfrak{h}))\). For \(f \in \mathcal{P}\) we explicitly have
\[
\sigma_i(f(h_1, \ldots, h_n)) = f(h_1, \ldots, h_i - 1, \ldots, h_n).
\]
Note also that \(c_i(\sigma_i f) = c_i(f)\).

We shall later have to solve some equations involving these automorphisms, so we collect some basic facts about them here.
Lemma 5. Let \( f \in \mathcal{P} \) be a nonzero polynomial. Then
\[
\deg_i(\sigma_i(f) - f) = (\deg_i f) - 1.
\]

Proof. Write \( f = \sum_{k=0}^{m} h_i^k f_k \) with \( f_k \in \mathcal{P}_i \) and \( f_m \neq 0 \). Then
\[
\sigma_i(f) - f = \sum_{k=0}^{m} ((h_i - 1)^k - h_i^k) f_k,
\]
and we see that the coefficient at \( h_i^m \) is 0 while the coefficient at \( h_i^{m-1} \) is precisely \( mf_m \) which is nonzero. \(\square\)

Lemma 6. For every \( g \in \mathcal{P} \) the equation
\[
\sigma_i(f) - f = g
\]
has a solution \( f \) and this solution is unique up to addition of elements from \( \mathcal{P}_i \).

Proof. We note that the left side of (2) is linear in \( f \). Thus the general solution is the sum of a particular solution and an arbitrary solution to the corresponding homogeneous equation
\[
\sigma_i(f) - f = 0.
\]
Applying \( \deg_i \) to (3), we obtain \((\deg_i f) - 1 = -1\). This means that any solution to (3) has the form \( f \in \mathcal{P}_i \). Next we claim that \( \sigma_i(f) - f = h_i^k \) has a solution \( f_k \) for each \( k \in \mathbb{N}_0 \). This is true by induction: for \( k = 0 \) a solution is \( f_0 = -h_i \), and supposing it has a solution for all \( p < k \) we note that \( \sigma_i(-\frac{h_i^{k+1}}{k+1}) - (-\frac{h_i^{k+1}}{k+1}) = h_i^k + g \) for some \( g \in \mathbb{C}[h_i] \) with \( \deg g < k \). Thus
\[
f = -\frac{h_i^{k+1}}{k+1} - \tilde{f}
\]
is a solution to \( \sigma_i(f) - f = h_i^k \), where \( \tilde{f} \) is any solution to \( \sigma_i(\tilde{f}) - \tilde{f} = g \). This proves that \( \sigma_i(f) - f = h_i^k \) has a solution \( f_k \) for every \( k \in \mathbb{N}_0 \). Now, finally, we note that for \( f \in \mathcal{P}_i \) we have \( \sigma_i(fp) - fp = f(\sigma_i(p) - p) \), so, writing
\[
g = \sum_{k=0}^{m} g_k h_i^k
\]
for some \( g_k \in \mathcal{P}_i \), a solution to \( \sigma_i(f) - f = g \) is
\[
f = \sum_{k=0}^{m} g_k f_k.
\]
\(\square\)

2.5 Automorphisms of \( \mathcal{U}(\mathfrak{sl}_{n+1}) \)

Denote by \( \tau \) the involutive automorphism of \( \mathcal{U}(\mathfrak{sl}_{n+1}) \) defined by \( \tau : e_{i,j} \mapsto -e_{j,i} \). It is easy to check that, when restricted to \( \mathcal{U}(\mathfrak{h}) \), the automorphism \( \tau \) satisfies
\[
\tau \sigma_i = \sigma_i^{-1} \tau \quad \text{for all } i \in \mathfrak{n}.
\]
Note also that on \( \mathcal{U}(\mathfrak{h}) \) we explicitly have \( \tau(f(h_1, \ldots, h_n)) = f(-h_1, \ldots, -h_n) \).

For each \( a = (a_1, \ldots, a_{n+1}) \in (\mathbb{C}^*)^{n+1} \), we also define an automorphism
\[
\varphi_a : \mathcal{U}(\mathfrak{sl}_{n+1}) \longrightarrow \mathcal{U}(\mathfrak{sl}_{n+1})
\]
by

\[ \varphi_a : e_{i,j} \mapsto \frac{a_i}{a_j} e_{i,j}. \]

This is well defined since

\[
\varphi_a(e_{i,j} e_{k,l} - e_{k,l} e_{i,j}) = \frac{a_i a_k}{a_j a_l}(e_{i,j} e_{k,l} - e_{k,l} e_{i,j}) = \frac{a_i a_k}{a_j a_l} \left( \delta_{k,j} e_{i,l} - \delta_{i,l} e_{k,j} \right)
\]

\[ = \varphi_a(\delta_{k,j} e_{i,l} - \delta_{i,l} e_{k,j}) = \varphi_a([e_{i,j}, e_{k,l}]). \]

Note that

\[ \tau \circ \varphi_a = \varphi_a^{-1} \circ \tau, \]

where the inverse is taken componentwise.

Each automorphism \( \varphi \in U(\mathfrak{sl}_{n+1}) \) induces a functor

\[ F_\varphi : \mathfrak{sl}_{n+1}\text{-Mod} \to \mathfrak{sl}_{n+1}\text{-Mod} \]

which maps each module to itself (as a set) but with a new action \( \cdot \) defined by

\[ x \cdot v := \varphi(x) \cdot v. \]

The functor \( F_\varphi \) maps morphisms to themselves.

We will write \( F_a := F_{\varphi_a} \) and \( \overline{M} := F_\tau(M) \). Note that \( \tau : \overline{U(\mathfrak{h})} \to U(\mathfrak{h}) \) is an isomorphism of left \( U(\mathfrak{h}) \)-modules. This follows from the fact that for \( f, g \in U(\mathfrak{h}) \) we have

\[ \tau(f \cdot g) = \tau(\tau(f) \cdot g) = \tau(\tau(f)) = f \cdot \tau(g). \]

We collect some of the properties of the above functors in a lemma. Multiplication and inversion in \((\mathbb{C}^*)^{n+1}\) are defined pointwise.

**Lemma 7.** The functors \( F_a \) and \( F_\tau \) have the following properties:

\( (i) \) \( F_{(1,1,\ldots,1)} = \text{Id}_{\mathfrak{sl}_{n+1}\text{-Mod}}. \)

\( (ii) \) \( F_a = F_{\lambda a} \) for all \( \lambda \in \mathbb{C}^*. \)

\( (iii) \) \( F_a \circ F_b = F_{a \cdot b}. \)

\( (iv) \) \( F_a^{-1} \simeq F_a^{-1}. \)

\( (v) \) For any \( M \in \mathfrak{sl}_{n+1}\text{-fmod} \) we have \( F_a(M) \simeq M \simeq F_\tau(M). \)

\( (vi) \) \( F_a M = M = F_\tau M. \)

\( (vii) \) For any \( M \in \mathcal{M}, F_a(M) \simeq M \) only if \( a \in \mathbb{C}^*(1,1,\ldots,1). \)

\( (viii) \) \( F_\tau \simeq F_\tau^{-1}. \)

\( (ix) \) \( F_\tau \circ F_a \simeq F_a^{-1} \circ F_\tau. \)

\( (x) \) \( F_a \) and \( F_\tau \) are auto-equivalences.

**Proof.** Claims (i) - (iv) follow directly from the definition of \( F_a \). To prove claim (v) we note that if \( v \) is a weight vector of weight \( \lambda \) in \( M \), then for all \( h \in \mathfrak{h} \), in \( F_a(M) \) we have

\[ h \cdot v = \varphi_a(h) \cdot v = \lambda(h)v, \]
which shows that \( v \) is still a weight vector of weight \( \lambda \) in \( F_a(M) \). Similarly, in \( F_r(M) \) we have
\[
h \cdot v = \tau(h) \cdot v = -h \cdot v = -\lambda(h)v,
\]
so \( v \) is still a weight vector, but now with weight \(-\lambda\). But finite dimensional modules are uniquely determined up to isomorphism by their characters (that is, their occurring weights and the dimension of the corresponding weight spaces), so we have \( F_r(M) \cong M \).

Similarly, we know that the dimension of the weight space of \( \lambda \) and \(-\lambda\) are equal in any finite dimensional module, so we also obtain \( F_r(M) \cong M \). To prove claim (viii), let \( M \in \mathcal{M} \). We first note that \( F_a(M) \) is still equal to \( M \) as a set, and the action of \( h \) is the same since \( \varphi \) fixes \( U(\mathfrak{h}) \) pointwise so \( F_a(M) \) is still free of rank 1 over \( U(\mathfrak{h}) \). Similarly, as we noted before, \( \tau : U(\mathfrak{h}) \to U(\mathfrak{h}) \) is an isomorphism so \( F_r(M) \) is still free of rank 1 over \( U(\mathfrak{h}) \).

To prove claim (viii), suppose \( \Phi : F_a(M) \to M \) is an isomorphism where \( a \notin \mathbb{C}^*(1,1,\ldots,1) \). Since \( \Phi(f) = f \cdot \Phi(1) = f \cdot \Phi(1) \), \( \Phi \) is determined by \( \Phi(1) \) and the same is true for \( \Phi^{-1} \). Since \( 1 = \Phi^{-1}(\Phi(1)) = \Phi^{-1}(1)\Phi(1) \), we obtain \( \Phi(1) = c \in \mathbb{C}^* \) and thus \( \Phi(f) = cf \). Pick indices \( i,j \) such that \( a_i \neq a_j \). Then
\[
c(e_{i,j} \cdot 1) = e_{i,j} \cdot \Phi(1) = \Phi(e_{i,j} \cdot 1) = \Phi(\frac{a_i}{a_j}e_{i,j} \cdot 1) = \frac{a_i}{a_j}(e_{i,j} \cdot 1).
\]
However, \( (e_{i,j} \cdot 1) \) must be nonzero, as otherwise \( [e_{i,j}, e_{j,i}] \cdot 1 = 0 \) which is impossible since \( [e_{i,j}, e_{j,i}] \in \mathfrak{h} \). Thus (H) does not hold, which shows that there exists no such isomorphism \( \Phi \). Claim (vii) and (viii) are obvious from the corresponding relations in \( U(\mathfrak{sl}_{n+1}) \) and claim (H) is a straightforward calculation. Finally, claim (H) is clear since \( F_a \circ F_a \cong \text{Id}_{\mathfrak{sl}_{n+1}} \)-Mod \( \cong F_{a^{-1}} \circ F_a \) and \( F_r \circ F_r \cong \text{Id}_{\mathfrak{sl}_{n+1}} \)-Mod.

### 2.6 Action of Chevalley generators

Let \( M \in \mathcal{M} \). Since \( M \) is free of rank 1 we have an isomorphism \( \varphi : M \to U(\mathfrak{h}) \) in \( U(\mathfrak{h})\)-Mod. By defining \( x \cdot f := \varphi(x \cdot \varphi^{-1}(f)) \) for all \( x \in U(\mathfrak{sl}_{n+1}), f \in U(\mathfrak{h}) \), the space \( U(\mathfrak{h}) \) becomes an \( U(\mathfrak{sl}_{n+1}) \)-module isomorphic to \( M \) via the map \( \varphi \). Thus, to classify the isomorphism classes of objects in \( \mathcal{M} \), we need only consider all possible extensions of the natural left \( U(\mathfrak{h}) \)-action on \( U(\mathfrak{h}) \)-Mod to \( U(\mathfrak{sl}_{n+1}) \).

**Proposition 8.** Let \( M \in \mathcal{M} \). Then, identifying \( M \) as a vector space with \( \mathcal{P} \), the action of \( \mathfrak{sl}_{n+1} \) on \( M \) is completely determined by the action of the Chevalley generators on \( 1 \in M \). Explicitly, for \( f \in \mathcal{P} \) we have
\[
\begin{align*}
h_k \cdot f &= h_k f & k \in \mathbb{N}, \\
e_{i,n+1} \cdot f &= p_i \sigma_i f & i \in \mathbb{N}, \\
e_{n+1,i} \cdot f &= q_j \sigma_j^{-1} f & j \in \mathbb{N}, \\
e_{i,j} \cdot f &= (p_i \sigma_i(q_j) - q_j \sigma_j^{-1}(p_i)) \sigma_i \sigma_j^{-1} f & i,j \in \mathbb{N}, i \neq j,
\end{align*}
\]
where \( p_i := e_{i,n+1} \cdot 1 \) and \( q_j := e_{n+1,j} \cdot 1 \) for all \( i,j \in \mathbb{N} \).

**Proof.** Since \( M \in \mathcal{M} \), we know that, both as a vector space and as an \( \mathfrak{h} \)-module, \( M \) is isomorphic to \( \mathcal{P} \). In other words, the action of \( \mathfrak{h} \) on \( M \) can be written explicitly as \( h_k \cdot f = h_k f \) for all \( f \in M, k \in \mathbb{N} \).

Now for each \( i \in \mathbb{N} \) we define
\[
p_i := e_{i,n+1} \cdot 1 \in \mathcal{P}, \quad q_i := e_{n+1,i} \cdot 1 \in \mathcal{P}.
\]
Since \( \delta_{k,i}(e_{i,n+1} \cdot f) = [h_k, e_{i,n+1}] \cdot f = h_k(e_{i,n+1} \cdot f) - e_{i,n+1} \cdot (h_k f) \), we obtain
\[
e_{i,n+1} \cdot (h_k f) = (h_k - \delta_{k,i})(e_{i,n+1} \cdot f),
\]
which shows that the action of the element \( e_{i,n+1} \) can be determined inductively from its action on \( 1 \). Explicitly we get \( e_{i,n+1} \cdot f = p_i \sigma_i f \), using the operator \( \sigma_i \) from Section 2.4

Analogous calculations show that \( e_{n+1,j} \cdot f = q_j \sigma_j^{-1} f \). But then, for all \( i, j \in n \) with \( i \neq j \) we have

\[
e_{i,j} \cdot f = [e_{i,n+1}, e_{n+1,j}] \cdot f = e_{i,n+1} \cdot e_{n+1,j} \cdot f - e_{n+1,j} \cdot e_{i,n+1} \cdot f.
\]

Explicitly this gives us

\[
e_{i,j} \cdot f = (p_i \sigma_i(q_j) - q_j \sigma_j^{-1}(p_i)) \sigma_i \sigma_j^{-1} f.
\]

This means that the action of \( \mathfrak{sl}_n \) on \( M \) is completely determined by the \((2n)\)-tuple \((p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathcal{P}^{2n} \) as stated in the proposition.

Note that not every tuple \((p_1, \ldots, p_n, q_1, \ldots, q_n) \) determines an \( \mathfrak{sl}_{n+1} \)-module. We now turn to the converse problem: determine which choices of \((p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathcal{P}^{2n} \) give rise to a module structure on \( P \) by the definition of the action as in Proposition 8.

**Proposition 9.** Suppose that a \((2n)\)-tuple \((p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathcal{P}^{2n} \) gives a \( \mathfrak{sl}_{n+1} \)-module via the action in Proposition 8. Then so does the \((2n)\)-tuple

\[
(-\tau(q_1), \ldots, -\tau(q_n), -\tau(p_1), \ldots, -\tau(p_n)).
\]

**Proof.** Let \( M \) be the module defined by \((p_1, \ldots, p_n, q_1, \ldots, q_n) \). Applying the functor \( F_\tau \) from Section 2.4 to \( M \) we obtain a module \( \overline{M} \) which is isomorphic to \( \mathcal{U}(\mathfrak{h}) \) and hence to \( \mathcal{U}(\mathfrak{h}) \) as a left \( \mathcal{U}(\mathfrak{h}) \) module via the restriction of \( \tau \) to \( \mathcal{U}(\mathfrak{h}) \) (where we identify \( \overline{M} \) with \( \mathcal{U}(\mathfrak{h}) \) as sets). Transferring the action of \( \mathcal{U}(\mathfrak{sl}_{n+1}) \) on \( \overline{M} \) to \( \mathcal{U}(\mathfrak{h}) \) via this map \( \tau \) we explicitly obtain a \( \mathcal{U}(\mathfrak{sl}_{n+1}) \)-module structure on \( \mathcal{U}(\mathfrak{h}) \) given explicitly by

\[
e_{i,j} \ast f = \tau(e_{i,j} \cdot \tau(f)) = \tau(\tau(e_{i,j} \cdot \tau(f))).
\]

In particular, we obtain \( h_k \ast f = h_k f \) and

\[
e_{i,n+1} \ast f = \tau(e_{i,n+1} \cdot \tau(f)) = \tau(-e_{n+1,i} \cdot \tau(f)) = \tau(-q_i \sigma_i^{-1}(\tau(f))) = -\tau(q_i) \sigma_i(f),
\]

as well as

\[
e_{n+1,j} \ast f = \tau(e_{n+1,j} \cdot \tau(f)) = \tau(-e_{j,n+1} \cdot \tau(f)) = \tau(-p_i \sigma_i(\tau(f))) = -\tau(p_i) \sigma_i^{-1}(f),
\]

for all \( i, j, k \in n \). Thus \( \overline{M} \) is isomorphic to the module in \( \mathcal{M} \) determined by

\[
(-\tau(q_1), \ldots, -\tau(q_n), -\tau(p_1), \ldots, -\tau(p_n))
\]

via the action in Proposition 8.

Before considering the problem of classification for arbitrary \( n \), we first solve it for \( n = 1 \). The solution turns out to be a prototype for the general solution.

### 3 The \( \mathfrak{sl}_2 \) case

In this section we consider the case \( n = 1 \). We classify all the objects of \( \mathcal{M} \), determine their submodule structure, find their central character, and derive an explicit formula for taking tensor product with any simple finite-dimensional module.
3.1 Classification of objects in $\mathcal{M}$

We consider the case $n = 1$. As before, we identify modules in $\mathcal{M}$ (as sets) with $\mathbb{C}[h]$ where $h := h_1 = \frac{1}{2}(e_{1,1} - e_{2,2})$. We also write $\sigma$ for $\sigma_1$.

**Definition 10.** For each $b \in \mathbb{C}$ define $M_b$ to be the vector space $\mathbb{C}[h]$ equipped with the following $\mathfrak{sl}_2$-action:

\[
\begin{align*}
    h \cdot f &= hf, \\
    e_{1,2} \cdot f &= (h + b)\sigma f, \\
    e_{2,1} \cdot f &= -(h - b)\sigma^{-1} f.
\end{align*}
\]

Similarly, define $M'_b$ to be the vector space $\mathbb{C}[h]$ equipped with the following $\mathfrak{sl}_2$-action:

\[
\begin{align*}
    h \cdot f &= hf, \\
    e_{1,2} \cdot f &= \sigma f, \\
    e_{2,1} \cdot f &= -(h + b + 1)(h - b)\sigma^{-1} f.
\end{align*}
\]

**Theorem 11.** (i) For all $b \in \mathbb{C}$ the above defines on $M_b$ and $M'_b$ the structure of $\mathfrak{sl}_2$-modules.

(ii) The set of modules

\[
\{ F_{(\alpha,1)}(M_b) | a \in \mathbb{C}^*, b \in \mathbb{C} \} 
\]

\[
\cup \{ F_{(\alpha,1)}(M'_b) | a \in \mathbb{C}^*, b \in \mathbb{C}_{\geq \frac{1}{2}} \} 
\]

\[
\cup \{ F_{(\alpha,1)} \circ F_{\tau}(M'_b) | a \in \mathbb{C}^*, b \in \mathbb{C}_{\geq \frac{1}{2}} \}
\]

where $\mathbb{C}_{\geq \frac{1}{2}} = \{ z \in \mathbb{C} | \text{Re}(z) \geq \frac{-1}{2} \}$ is a skeleton of $\mathcal{M}$.

**Proof.** To prove claim (i) We first check that $M_b$ is an $\mathfrak{sl}_2$-module. We check the following relations:

\[
\begin{align*}
    h \cdot e_{1,2} \cdot f - e_{1,2} \cdot h \cdot f = h((h + b)\sigma f) - (h + b)\sigma (hf) &= (h + b)\sigma f - [h, e_{1,2}] \cdot f \\
    h \cdot e_{2,1} \cdot f - e_{2,1} \cdot h \cdot f = h(- (h - b)\sigma^{-1} f) + (h - b)\sigma^{-1} (hf) &= (h - b)\sigma^{-1} f - [h, e_{2,1}] \cdot f \\
    e_{1,2} \cdot e_{2,1} \cdot f - e_{2,1} \cdot e_{1,2} \cdot f &= -(h + b)\sigma(h - b - 1 + (h - b)(h + b + 1))\sigma^{-1} f \\
    &= (2h) f = [e_{1,2}, e_{2,1}] \cdot f.
\end{align*}
\]

The remaining relations are obvious. Thus $M_b$ is an $\mathfrak{sl}_2$-module. It is left to the reader to verify that also $M'_b$ is an $\mathfrak{sl}_2$-module.

To prove claim (ii), let $M \in \mathcal{M}$ and define

\[
p := p_1 = e_{1,2} \cdot 1 \in \mathbb{C}[h] \quad \text{and} \quad q := q_1 = e_{2,1} \cdot 1 \in \mathbb{C}[h].
\]

Then, in accordance with Proposition [8] the action of $\mathfrak{sl}_2$ on $M = \mathbb{C}[h]$ is given by

\[
\begin{align*}
    h \cdot f &= hf, \\
    e_{1,2} \cdot f &= p\sigma f, \\
    e_{2,1} \cdot f &= q\sigma^{-1} f.
\end{align*}
\]

Since $M$ is a module, we have, in particular, $(2h) f = [e_{1,2}, e_{2,1}] \cdot f = e_{1,2} \cdot e_{2,1} \cdot f - e_{2,1} \cdot e_{1,2} \cdot f$ for all $f \in \mathbb{C}[h]$. This is equivalent to $p \sigma(q) - q \sigma^{-1} p = 2h$. The latter transforms to $\sigma(g) - g = 2h$ by letting $g = q \sigma^{-1} (p)$. The equation $\sigma(g) - g = 2h$ has the form discussed
in Lemma [3] and the set of solutions is \( \left\{ -h(h+1)+c \mid c \in \mathbb{C} \right\} \). Now, every \( g \) in this set has a factorization into linear factors of the form \( g = -(h+b+1)(h-b) \) for some \( b \in \mathbb{C} \). Thus we try to find all polynomials \( p, q \) for which \( g \sigma^{-1}(p) = -(h+b+1)(h-b) \). Since \( \mathbb{C}[h] \) is a UFD, we only have three cases, corresponding to \( (\deg p, \deg q) \in \{(2,0), (0,2), (1,1)\} \).

If \( (\deg p, \deg q) = (0, 2) \), then

\[
(p, q) = (a, -a^{-1}(h + b + 1)(h - b)) \quad \text{for some} \quad a \in \mathbb{C}^*
\]

and hence \( M \cong F_{(a,1)}(M'_b) \). If \( (\deg p, \deg q) = (2,0) \), then

\[
(p, q) = (a(h + b)(h - b - 1), -a^{-1}) \quad \text{for some} \quad a \in \mathbb{C}^*
\]

and hence \( M \cong F_{(a^{-1},1)} \circ F_{\tau}(M'_b) \). Finally, if \( (\deg p, \deg q) = (1,1) \), we obtain either

\[
(p, q) = (a(h + b), -a^{-1}(h - b)) \quad \text{for some} \quad a \in \mathbb{C}^*,
\]

or

\[
(p, q) = (a(h - b - 1), -a^{-1}(h + b + 1)) \quad \text{for some} \quad a \in \mathbb{C}^*.
\]

In this case either \( M \cong F_{(a,1)}(M_b) \) or \( M \cong F_{(a,1)}(M_{b-1}) \).

The above argument shows that any module in \( \mathcal{M} \) is isomorphic to either \( F_a(M_b) \), \( F_a(M'_b) \), or \( F_a \circ F_{\tau}(M_b') \) for some \( a \in \mathbb{C}^* \) and \( b \in \mathbb{C} \). We now show that these cases are essentially mutually exclusive. First note that any morphism \( \Phi \) in \( \mathcal{M} \) is determined by its value at 1 since \( \Phi(f) = f \Phi(1) \). This also shows that any invertible morphism \( \Phi \) must have \( \Phi(1) \) invertible, and thus \( \Phi \) is multiplication by a nonzero constant: \( \Phi(f) = cf \). Now, let \( \Phi : M \rightarrow M' \) be an isomorphism in \( \mathcal{M} \). Let \( p := e_{1,2} \cdot 1 \in M \) and \( p' := e_{1,2} \cdot 1 \in M' \). Then

\[
cp' = p' \sigma \Phi(1) = e_{1,2} \Phi(1) = \Phi(e_{1,2} \cdot 1) = \Phi(p) = p \Phi(1) = cp
\]

which gives \( p = p' \). Thus two modules can only be isomorphic if \( e_{1,2} \cdot 1 \) is the same in both modules. However, given two modules from the set \([3]\), we see that \( (e_{1,2} \cdot 1) \) is determined by its degree, its leading coefficient and its set of zeros. All these three properties coincide nontrivially only in the pairs

\[
(F_{(a,1)}(M'_b), F_{(a,1)}(M'_{b-1})) \quad \text{and} \quad (F_{(a,1)} \circ F_{\tau}(M'_b), F_{(a,1)} \circ F_{\tau}(M'_{b-1})).
\]

But unless they coincide, precisely one of \( b \) and \( -b - 1 \) lies in the set \( \mathbb{C}_{\geq -\frac{1}{2}} \). This shows that no nontrivial isomorphisms exists between the objects listed in \([3]\). \( \square \)

### 3.2 Submodules and quotients

Now we turn to some properties of the objects in \( \mathcal{M} \).

**Lemma 12.** The modules \( F_{(a,1)}(M'_b) \) and \( F_{\tau} \circ F_{(a,1)}(M'_b) \) from the classification in the previous section are simple for all \( b \in \mathbb{C} \) and \( a \in \mathbb{C}^* \).

**Proof.** Since \( F_{\tau} \) and \( F_{(a,1)} \) are auto-equivalences it suffices to prove that \( M'_b \) is simple for all \( b \in \mathbb{C} \). Let \( S \) be a nonzero submodule of \( M'_b \) and let \( f \) be a nonzero polynomial in \( S \). We let \( N(f) \) be the set of zeros of \( f \); this is a finite subset of \( \mathbb{C} \). From the definition of the module structure we see that

\[
N(e_{1,2} \cdot f) = N(f) + 1,
\]

and inductively we obtain

\[
N(e^k_{1,2} \cdot f) = N(f) + k.
\]
Now take $k$ large enough so that $(N(f)+k) \cap N(f) = \emptyset$. Then $f$ and $e_{1,2}^k \cdot f$ are relatively prime elements of $S$, and we can find $g_1, g_2 \in \mathbb{C}[h]$ such that

$$g_1 f + g_2 e_{1,2}^k \cdot f = 1 \in S.$$ 

Then we have $S = \mathbb{C}[h] = M'_b$ which shows that $M'_b$ is simple for each $b \in \mathbb{C}$. \hfill \square

The last type of modules is more interesting.

**Lemma 13.**

(i) For $2b \not\in \mathbb{N}_0$ the module $F_{(a,1)}(M_b)$ is simple.

(ii) For $2b \in \mathbb{N}_0$ the module $F_{(a,1)}(M_b)$ has a unique proper submodule which is isomorphic to $F_{(a,1)}(M_{-b-1})$, and the corresponding simple quotient is isomorphic to the simple finite dimensional module $L(2b)$ with highest weight $2b$ and dimension $2b+1$. In other words, we have a nonsplit short exact sequence:

$$0 \rightarrow F_{(a,1)}(M_{-b-1}) \rightarrow F_{(a,1)}(M_b) \rightarrow L(2b) \rightarrow 0.$$

**Proof.** We first prove the two statements for $M_b$ using an argument similar to that of Lemma 12. We see that

$$N(e_{1,2} \cdot f) = \{-b\} \cup (N(f) + 1),$$

so inductively we obtain

$$N(e_{1,2}^k \cdot f) = \{-b, -b+1, -b+2, \ldots, -b+k-1\} \cup (N(f) + k).$$

Similarly,

$$N(e_{2,1} \cdot f) = \{b\} \cup (N(f) - 1),$$

which implies

$$N(e_{2,1}^k \cdot f) = \{b, b-1, b-2, \ldots, b-k+1\} \cup (N(f) - k).$$

Now for large integers $k$,

$$(N(f) - k) \cap \{\{-b, -b+1, -b+2, \ldots, -b+k-1\} \cup (N(f) + k)\} = \emptyset,$$

and

$$(N(f) + k) \cap \{\{b, b-1, b-2, \ldots, b-k+1\} \cup (N(f) - k)\} = \emptyset.$$

Note that for $2b \not\in \mathbb{N}_0$, we also have

$$\{-b, -b+1, -b+2, \ldots, -b+k-1\} \cap \{b, b-1, b-2, \ldots, b-l+1\} = \emptyset$$

for all natural numbers $k$ and $l$, so as in the argument in Lemma 12 we see that $e_{1,2}^k \cdot f$ and $e_{2,1}^k \cdot f$ are relatively prime for large enough $k$ so any submodule containing a nonzero polynomial $f$ also contains 1 and the submodule is all of $M_b$.

Finally, suppose $2b \in \mathbb{N}_0$. We claim that

$$S := \mathbb{C}[h] \prod_{j=0}^{2b} (h+b-j)$$
is a proper submodule of $M_b$. Clearly $\mathfrak{h}S \subset S$. We now calculate explicitly
\[
e_{1,2} \cdot \prod_{j=0}^{2b} (h+b-j)f = (h+b)\sigma \prod_{j=0}^{2b} (h+b-j)f
\]
\[= (h+b) \prod_{j=0}^{2b} (h+b-j-1)\sigma f
\]
\[= (h+b) \prod_{k=1}^{2b+1} (h+b-k)\sigma f
\]
\[= ((h-b-1)\sigma f)(h+b) \prod_{k=1}^{2b} (h+b-k)
\]
\[= ((h-b-1)\sigma f) \prod_{k=0}^{2b} (h+b-k),
\]
which shows that $e_{1,2}S \subset S$. Analogous calculations show that $e_{2,1}S \subset S$ and thus $S$ is a proper submodule. Now write $Q := \prod_{j=0}^{2b} (h+b-j)$. Then $S = \{ pQ \mid p \in \mathbb{C}[h] \}$. An explicit calculation gives
\[
h \cdot Qf := Qh f
\]
\[e_{1,2} \cdot Qf := Q((h-b-1)\sigma f)
\]
\[e_{2,1} \cdot Qf := Q(-(h+b+1)\sigma^{-1} f).
\]
Thus we immediately see that $f \mapsto fQ$ is an isomorphism $M_{-b-1} \to S$. Note that $S$ is simple since $-b-1 \notin \mathbb{N}_0$. Next we look at the quotient $M_b/S$. Define
\[v := \prod_{j=0}^{2b-1} (h+b-j) + S.
\]
Then $e_{1,2} \cdot v = 0$ and $(e_{1,1} - e_{2,2}) \cdot v = 2b \cdot v = 2bv$ so $v$ is a highest weight vector of weight $2b$. Hence, since $\dim M_b/S = 2b+1$, $M_b/S$ is isomorphic to $L(2b)$. We thus have a nonsplit short exact sequence:
\[0 \to M_{-b-1} \to M_b \to L(2b) \to 0.
\]
This proves the statements for $M_b$. Since the functor $F_{(a,1)}$ is an auto-equivalence it maps simples to simples, it follows that $F_{(a,1)}(M_b)$ is simple for $2b \notin \mathbb{N}_0$. Application of the exact functor $F_{(a,1)}$ to our short exact sequence, we get the corresponding sequence
\[0 \to F_{(a,1)}(M_{-b-1}) \to F_{(a,1)}(M_b) \to F_{(a,1)}(L(2b)) \to 0,
\]
and since $F_{(a,1)}$ is the identity functor on finite dimensional modules by Lemma [a][b][c], we have an exact sequence
\[0 \to F_{(a,1)}(M_{-b-1}) \to F_{(a,1)}(M_b) \to L(2b) \to 0,
\]
as claimed.

\[\square\]
\[\textbf{Remark 14.} \text{ The above lemma shows that every finite dimensional simple \mathfrak{sl}_2\text{-module can be expressed as a quotient of two (infinite dimensional) modules from \mathcal{M}. This is similar to what holds for Verma modules: if } 2b \in \mathbb{N}_0 \text{ there exists a short nonsplit exact sequence}
\]
\[0 \to M(-2b-2) \to M(2b) \to L(2b) \to 0,
\]
\[\text{where } M(\lambda) \text{ is the Verma module of highest weight } \lambda.
\]
3.3 Central character

The simple modules $F_{(a,1)}(M_b), F_{(a,1)}(M'_b)$ and $F_r \circ F_{(a,1)}(M'_b)$ have central characters by Schur’s lemma, that is there exist algebra homomorphisms $\chi_N : Z(\mathfrak{sl}_2) \to \mathbb{C}$ such that

$$z \cdot v = \chi_N(z)v \quad \text{for all } z \in Z(\mathfrak{sl}_2) \text{ and } v \in N,$$

for each $N \in \{F_{(a,1)}(M_b), F_{(a,1)}(M'_b), F_r \circ F_{(a,1)}(M'_b)\}$.

**Proposition 15.** Let $N \in \{F_{(a,1)}(M_b), F_{(a,1)}(M'_b), F_r \circ F_{(a,1)}(M'_b)\}$. The central character of $N$ is determined by

$$\chi_{M_b}(e_2) = \chi_{M'_b}(e_2) = \chi_{M''_b}(e_2) = 2b(b + 1),$$

where $e_2 = 2h^2 + e_{1,2}e_{2,1} + e_{2,1}e_{1,2}$.

**Proof.** Since the center of $U(\mathfrak{sl}_2)$ is $\mathbb{C}[e_2]$, the central character of a module is determined by the single scalar

$$\chi(e_2) = e_2 \cdot 1 = (2h^2 + e_{1,2}e_{2,1} + e_{2,1}e_{1,2}) \cdot 1. \quad (5)$$

Since we know that the right hand side of (5) is a scalar, we need only consider the constant term on the left hand side of (5). An explicit calculation gives the central characters as stated in the proposition. □

3.4 Tensoring with finite dimensional modules

Let $L$ be the natural $\mathfrak{sl}_2$-module. It has basis $\{e_1, e_2\}$ and the action is given by

$$e_{i,j} \cdot e_k = \delta_{j,k}e_i.$$ 

In the basis $\{e_1, e_2\}$ we explicitly have:

$$
\begin{align*}
  h \cdot e_1 &= \frac{1}{2}e_1 \\
  e_{1,2} \cdot e_1 &= 0 \\
  e_{2,1} \cdot e_1 &= e_2 \\
  e_1 \cdot e_1 &= 0 \\
  e_{1,2} \cdot e_2 &= e_1 \\
  e_{2,1} \cdot e_2 &= 0.
\end{align*}
$$

Being simple and 2-dimensional, $L$ is isomorphic to $L(1)$, the simple highest weight module of highest weight 1, and from here on we shall identify the two. Let $N \in \{M, M'\}$ and consider the module $N_b \otimes L(1)$. A basis for $N_b \otimes L(1)$ is $\{h^k \otimes e_1 | k \geq 0\} \cup \{h^k \otimes e_2 | k \geq 0\}$, so, in particular, every element of $N_b \otimes L(1)$ has a unique expression of the form

$$(f, g) := f \otimes e_1 + g \otimes e_2,$$

for some $f, g \in \mathbb{C}[h]$. In this notation, the action of $\mathfrak{sl}_2$ on $N_b \otimes L(1)$ is given by the following lemma which is easily proved by a straightforward computation.

**Lemma 16.** The action of $\mathfrak{sl}_2$ on $M_b \otimes L(1)$ is given by:

$$
\begin{align*}
  h \cdot (f, g) &= ((h + \frac{1}{2})f, (h - \frac{1}{2})g), \\
  e_{1,2} \cdot (f, g) &= ((h + b)\sigma(f) + g, (h + b)\sigma(g)), \\
  e_{2,1} \cdot (f, g) &= (- (h + b)\sigma^{-1}(f), -(h + b)\sigma^{-1}(g) + f).
\end{align*}
$$

The action of $\mathfrak{sl}_2$ on $M'_b \otimes L(1)$ is given by:

$$
\begin{align*}
  h \cdot (f, g) &= ((h + \frac{3}{2})f, (h - \frac{3}{2})g), \\
  e_{1,2} \cdot (f, g) &= (\sigma(f) + g, \sigma(g)), \\
  e_{2,1} \cdot (f, g) &= (- (h + b + 1)(h - b)\sigma^{-1}(f), -(h + b + 1)(h - b)\sigma^{-1}(g) + f).
\end{align*}
$$
We can now derive an explicit decomposition formula.

**Proposition 17.** Let $N \in \{M, M'\}$.

(i) For $2b \neq -1$, we have
\[ N_b \otimes L(1) \simeq N_{b - \frac{1}{2}} \oplus N_{b + \frac{1}{2}}. \]

(ii) For $2b = -1$, we have a nonsplit short exact sequence
\[ 0 \to N_0 \to N_{-\frac{1}{2}} \otimes L(1) \to N_{-1} \to 0. \]

**Proof.** We first consider the case $N = M'$. We determine the submodules of $M'_b \otimes L(1)$.

Let $S$ be a nonzero submodule containing a nonzero element $(g_1, g_2)$. Assume first that $g_2 \neq 0$. Since
\[ (e_{1,2} - 1) \cdot (g_1, g_2) = (\sigma(g_1) + g_2 - g_1, \sigma(g_2) - g_2), \]
by Lemma 5 we see that $(e_{1,2} - 1)$ acts by decreasing the degree by 1 in the second component. Thus $c(e_{1,2} - 1)^{deg \cdot g_2} \cdot (g_1, g_2) = (f, 1) \in S$, for some $f \in \mathbb{C}[h]$ and some $c \in \mathbb{C}^*$. Acting again by $(e_{1,2} - 1)$ we obtain $(\sigma(f) - f + 1, 0) \in S$. Assume $\sigma(f) - f + 1 \neq 0$. Then, noting that $(e_{1,2} - 1) \cdot (g, 0) = (\sigma(g) - g, 0)$, we obtain
\[ c(e_{1,2} - 1)^{deg \cdot \sigma(f) - f + 1} \cdot (\sigma(f) - f + 1, 0) = (1, 0) \in S, \]
for some $c \in \mathbb{C}^*$. But then, since 1 and $(h + \frac{1}{2})$ generate the algebra $\mathbb{C}[h]$, by acting on $(1, 0)$ by $\mathbb{C}[h]$ we obtain $(\mathbb{C}[h], 0) \subset S$. Since $(g_1, g_2) \in S$, we also obtain $(0, g_2) \in S$ which as above gives us $(0, 1) \in S$ and $(0, \mathbb{C}[h]) \subset S$. Thus $M'_b \otimes N = (\mathbb{C}[h], \mathbb{C}[h]) \subset S$ and $S = M'_b \otimes N$. On the other hand, if $g_2 = 0$ to begin with, by the same argument we immediately get $(1, 0) \in S$ and $(0, \mathbb{C}[h]) \subset S$. Now $e_{2,1} \cdot (1, 0) = (-(h + b + 1)(h - b), 1) \in S$ so we again obtain $(0, 1) \in S$ and $(0, \mathbb{C}[h]) \subset S$, so again $S = M'_b \otimes N$.

The only case remaining is when $(f, 1) \in S$ where $\sigma(f) - f + 1 = 0$. The equation $\sigma(f) - f = -1$ has the form discussed in Lemma 4 and we know how to solve it. The solutions are precisely $f = h + c$, where $c$ is some constant. Explicit calculations show that the submodule generated by $(h + c, 1)$ is proper if and only if $c \in \{-b, b + 1\}$. Thus we define $S_1$ to be the submodule of $M'_b$ generated by $(h - b, 1)$, and we define $S_2$ to be the submodule of $M'_b$ generated by $(h + b + 1, 1)$.

We define two linear maps $\Phi_1 : \mathcal{P} \to S_1$ and $\Phi_2 : \mathcal{P} \to S_2$ by
\[ \Phi_1(f(h)) := ((h - b)f(h + \frac{1}{2}), f(h - \frac{1}{2})) \quad \text{and} \quad \Phi_2(f(h)) := ((h + b + 1)f(h + \frac{1}{2}), f(h - \frac{1}{2})). \]

Explicit calculations show that
\[
\begin{align*}
    h \cdot \Phi_1(f) &= \Phi_1(hf) \\
    e_{1,2} \cdot \Phi_1(f) &= \Phi_1(\sigma f) \\
    e_{1,2} \cdot \Phi_1(f) &= \Phi_1(-(h + (b - \frac{1}{2}) + 1)(h - (b - \frac{1}{2}))\sigma^{-1} f),
\end{align*}
\]
and also
\[
\begin{align*}
    h \cdot \Phi_2(f) &= \Phi_2(hf) \\
    e_{1,2} \cdot \Phi_2(f) &= \Phi_2(\sigma f) \\
    e_{1,2} \cdot \Phi_2(f) &= \Phi_2(-(h + (b + \frac{1}{2}) + 1)(h - (b + \frac{1}{2}))\sigma^{-1} f).
\end{align*}
\]
This shows that $\Phi_1$ is an isomorphism $M'_{b - \frac{1}{2}} \to S_1$ and that $\Phi_2$ is an isomorphism $M'_{b + \frac{1}{2}} \to S_2$. Finally, we show that for $b \neq -\frac{1}{2}$, the submodules $S_1$ and $S_2$ are complementary. If $(f, g) \in S_1 \cap S_2$, then $(h - b)\sigma^{-1}(g) = f = (h + b + 1)\sigma^{-1}(g)$, so if $2b + 1 \neq 0$ we necessarily have $f = g = 0$ and $S_1 \cap S_2 = 0$. 

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Moreover, we have $S_1 + S_2 \ni (h + b + 1, 1) - (h - b, 1) = (2b + 1, 0)$, so if $2b + 1 \neq 0$, $S_1 + S_2 \supset (P, 0)$ which then gives $(0, 1) \in S_1 + S_2$, $(0, P) \subset S_1 + S_2$, and $S_1 + S_2 = (P, P) = M_b' \otimes N$.

Thus, if $2b + 1 \neq 0$ we have

$$M_b' \otimes N = S_1 \oplus S_2 \simeq M_{b - \frac{1}{2}} \oplus M_{b + \frac{1}{2}},$$

as claimed in (i).

On the other hand, if $2b + 1 = 0$, then $S_1 = S_2$ is the unique submodule of $M_b' \otimes N$, and explicit calculations show that the quotient module also is isomorphic to $S_1$ and $S_2$. Thus in this case we have a nonsplit exact sequence in $M$:

$$0 \to M_0' \to M_{-\frac{1}{2}}' \otimes N \to M_{-1}' \to 0.$$

Thus (i) and (ii) are proved for $N_b = M_b'$.

For $N_b = M_b$ take instead $S_1 := \langle (1, 1) \rangle$ and $S_2 := \langle (h - b, h + b) \rangle$ in $M_b \otimes L(1)$. Then

$$f(h) \mapsto (f(h + \frac{1}{2}), f(h - \frac{1}{2}))$$

is an isomorphism $M_{b - \frac{1}{2}} \to S_1$, and

$$f(h) \mapsto ((h - b)f(h + \frac{1}{2}), (h + b)f(h - \frac{1}{2}))$$

is an isomorphism $M_{b + \frac{1}{2}} \to S_2$, and for $b \neq -\frac{1}{2}$ we have $M_b = S_1 \oplus S_2$. On the other hand, for $b = -\frac{1}{2}$ the sum is not direct, but we still have $(M_{-\frac{1}{2}} \otimes L(1))/S_1 \simeq M_{-1}$, so by Lemma 13 the multiset of Jordan-Hölder components of $M_{-\frac{1}{2}} \otimes L(1)$ is precisely \{L(0), M_{-1}, M_{-1}\}. \hfill \Box

We can now describe explicitly the decomposition of modules obtained by taking the tensor product of a module from $M$ with a finite dimensional module. For this it suffices to consider only the simple finite dimensional module $L(k)$.

**Remark 18.** The Clebsch-Gordan formula for simple finite dimensional $\mathfrak{sl}_2$-modules is well known, see for example Theorem 1.39 in [Maz1]. It states states that for $m, n \in \mathbb{N}_0$ with $m \geq n$ we have

$$L(m) \otimes L(n) = L(m + n) \oplus L(m + n - 2) \oplus \cdots \oplus L(m - n).$$

**Corollary 19.** For all $2b \in \mathbb{C} \setminus \mathbb{N}_0$ and for $N_b \in \{M_b, M_b'\}$ we have

$$N_b \otimes L(k) \simeq \bigoplus_{i=0}^{k} N_{b + \frac{i}{2}}.$$

**Proof.** We proceed by induction on $k$, the case $k = 0, 1$ holding by Proposition 17. Assume the claim of the Corollary holds for $k$ and for $k - 1$ and apply $- \otimes L(1)$ to both sides of $N_b \otimes L(k) \simeq \bigoplus_{i=0}^{k} N_{b + \frac{i}{2}}$. Using associativity of the tensor product and applying the Clebsch-Gordan formula from Remark 13 on the left, we obtain

$$N_b \otimes (L(k + 1) \oplus L(k - 1)) \simeq \bigoplus_{i=0}^{k} N_{b + \frac{i}{2}} \otimes L(1).$$
Lemma 20. Using the distributive property of the tensor product, this simplifies to

\[(N_b \otimes L(k + 1)) \oplus (N_b \otimes L(k - 1)) \simeq \bigoplus_{i=0}^{k} (N_{b+\frac{i-2}{2}} \otimes L(1)).\]

Now, since \(b + \frac{k-2}{2} \not\in \mathbb{Z}\) for all \(i, k \in \mathbb{Z}\), we can apply our formula to both sides giving

\[(N_b \otimes L(k + 1)) \oplus \bigoplus_{i=0}^{k-1} N_{b+\frac{i-2}{2}} \simeq \bigoplus_{i=0}^{k-1} N_{b+\frac{i-2}{2}} \oplus \bigoplus_{i=0}^{k} N_{b+\frac{k-2}{2}}.\]

Writing

\[\bigoplus_{i=0}^{k} N_{b+\frac{i-2}{2}} = N_{b+\frac{k-2}{2}} \oplus \bigoplus_{i=0}^{k-1} N_{b+\frac{i-2}{2}},\]

and inserting this in the above formula, we obtain

\[(N_b \otimes L(k + 1)) \oplus \bigoplus_{i=0}^{k-1} N_{b+\frac{i-2}{2}} \simeq N_{b+\frac{k-2}{2}} \oplus \bigoplus_{i=0}^{k-1} N_{b+\frac{i-2}{2}} \oplus \bigoplus_{i=0}^{k} N_{b+\frac{k-2}{2}}.\]

Now we can cancel corresponding equal direct summands on each side resulting in

\[N_b \otimes L(k + 1) \simeq N_{b+\frac{k-2}{2}} \oplus \bigoplus_{i=0}^{k} N_{b+\frac{k-2}{2}} \simeq N_{b+\frac{(k+1)-2}{2}}.\]

Thus the formula holds for \(k + 1\), and the claim of the corollary follows by induction. \(\square\)

4 The \(\mathfrak{sl}_{n+1}\) case

We now try to generalize the above results to the general case. We are trying to find all \((p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathcal{P}^{2n}\) such that \(\mathcal{P}\) becomes an \(\mathfrak{sl}_{n+1}\) module under the action:

\[
\begin{align*}
    h_k \cdot f &= h_k f, \\
    e_{i,n+1} \cdot f &= p_i \sigma_i f, \\
    e_{n+1,j} \cdot f &= q_j \sigma^{-1}_j f, \\
    e_{i,j} \cdot f &= (p_i \sigma_i (q_j) - q_j \sigma^{-1}_j (p_i)) \sigma_i \sigma^{-1}_j f.
\end{align*}
\]

From here on, assume that \((p_1, \ldots, p_n, q_1, \ldots, q_n)\) yields a module. We shall find a number of necessary relations amongst these polynomials.

4.1 Technical Lemmas

Lemma 20. For all \(i, j \in \mathfrak{n}\) we have

\[
\begin{align*}
    (i) \quad p_i \sigma_i (p_j) &= p_j \sigma_j (p_i), \\
    (ii) \quad q_i \sigma^{-1}_i (q_j) &= q_j \sigma^{-1}_j (q_i), \\
    (iii) \quad \sigma^{-1}_i (p_i) q_i &= -h_i (h_i + 1) + \tilde{g}_i, \text{ where } \tilde{g}_i \in \mathcal{P}_i \text{ for each } i.
\end{align*}
\]

Proof. Statement (i) and (ii) are equivalent to the two identities

\[e_{i,n+1} \cdot e_{j,n+1} \cdot f - e_{j,n+1} \cdot e_{i,n+1} \cdot f = [e_{i,n+1}, e_{j,n+1}] \cdot f = 0\]
and 
\[ e_{n+1,i} \cdot e_{n+1,j} \cdot f - e_{n+1,j} \cdot e_{n+1,i} \cdot f = [e_{n+1,i}, e_{n+1,j}] \cdot f = 0, \]
for \( i, j \in \mathbb{n} \).

For (iii), consider the identity
\[ [e_{i,n+1}, e_{n+1,i}] \cdot f = e_{i,n+1} \cdot e_{n+1,i} \cdot f - e_{n+1,i} \cdot e_{i,n+1} \cdot f. \]

Using our explicit choice of basis in \( \mathcal{P} \), we have \( e_{i,i} - e_{n+1,n+1} = h_i + \overline{h} \) so (6) becomes
\[ (h_i + \overline{h}) \cdot f = p_i \sigma_i(q_i) \cdot f - q_i \sigma_i^{-1}(p_i) \cdot f, \]
or, equivalently,
\[ h_i + \overline{h} = p_i \sigma_i(q_i) - q_i \sigma_i^{-1}(p_i). \]

Substituting \( f := q_i \sigma_i^{-1}(p_i) \), it reads \( \sigma_i(f) - f = h_i + \overline{h} \). This equation is of the form discussed in Lemma 6 so we know how to solve it. The set of solutions is precisely \( \{-h_i(\overline{h} + 1) + \tilde{g}_i \mid \tilde{g}_i \in \mathcal{P}_i\} \), as claimed in the lemma. \( \square \)

**Remark 21.** Note that claims (i) and (iii) of Lemma 20 are equivalent to
\[ p_i(\sigma_i(p_j) - p_j) = p_j(\sigma_j(p_i) - p_i) \]
and
\[ q_i(\sigma_i^{-1}(q_j) - q_j) = q_j(\sigma_j^{-1}(q_i) - q_i). \]

**Lemma 22.** The polynomials \( p_1, \ldots, p_n, q_1, \ldots, q_n \) satisfy the following:

(i) \( \deg_i p_i, \deg_i q_i \in \{0, 1, 2\} \).

(ii) \( \deg_i p_i + \deg_i q_i = 2 \).

(iii) If \( \deg_k p_k = 2 \), then \( p_k \) has a nontrivial factorization. Similarly for \( q \).

(iv) If \( \deg_k p_k = 1 \), then \( p_k \) is irreducible. Similarly for \( q \).

(v) Suppose \( \deg_k p_k = 1 \) and \( \deg_i p_i = 2 \). Then \( p_k \) divides \( p_i \). Similarly for \( q \).

(vi) Let \( S = \{\deg_i p_i \mid i \in \mathbb{n}\} \). Then either \( S \subset \{0, 1\} \) or \( S \subset \{1, 2\} \). Similarly for \( q \).

(vii) Suppose \( \deg_k p_k = 2 = \deg_i p_i \). Then \( p_i \) and \( p_k \) share a common factor. Similarly for \( q \).

**Proof.** First note that since \( \tau \) is an algebra automorphism of \( \mathcal{U}(\mathfrak{h}) \), by Proposition 9 it suffices to prove statements (iii) - (vii) for the polynomials \( p \).

By part (iii) of Lemma 20 we have \( p_i, q_i \neq 0 \). Applying \( \deg_i \) to the same equality we get
\[ \deg_i(p_i) + \deg_i(q_i) = \deg_i(p_i q_i) = \deg_i(\sigma_i^{-1}(p_i) q_i) = \deg_i(-h_i(\overline{h} + 1)) = 2, \]
which proves claims (i) and (ii).

We now look at claim (iii). The case \( n = 1 \) is obvious so let \( n \geq 2 \). Assume that \( \deg_k p_k = 2 \) with \( p_k \) irreducible. Consider the equality \( \sigma_k^{-1}(p_k) q_k = -h_k(\overline{h} + 1) + \tilde{g}_k \) from Lemma 20. By comparing the coefficients of \( h_k^2 \) (or by application of \( c_k \)) on both sides, we see that \( q_k = c \in \mathbb{C}^* \) so we have
\[ (p_k, q_k) = (-c^{-1}((h_k - 1)\overline{h} - \tilde{g}_k), c). \]

Now let \( i \) be an index different from \( k \). We have \( p_i(\sigma_i(p_k) - p_k) = p_k(\sigma_k(p_i) - p_i) \) from formula (7). Since \( p_k \) is irreducible, it divides one of the factors on the left. However, \( p_k \)
does not divide $\sigma_i(p_k) - p_k$ since the latter is a nonzero polynomial with lower $i$-degree than $p_k$. Thus $p_k$ divides $p_i$, which implies that $\sigma_i^{-1}(p_k)$ divides $\sigma_i^{-1}(p_i)$ which, in turn, divides $-h_i(h_i + 1) + \tilde{g}_i$. Thus we have

$$(h_k(h_i + 1) - \tilde{g}_k)f = -h_i(h_i + 1) + \tilde{g}_i$$

(9)

for some $f \in \mathcal{P}$. On the right hand side of (9) we have the term $-h_i^2$ which comes from the product $\tilde{g}_k f$. However, if $\deg_i f > 0$, then we get terms of the form $h_k h_i^{1+\deg_i f}$ on the left of (9) which does not appear on the right. Thus $\deg_i f = 0$ and $\deg_i \tilde{g}_k = 2$ and applying $c_i$ to (9) gives $c_i(\tilde{g}_k)c_i(f) = -1$. In particular, this shows that $c_i(f) = f$ is invertible and we have $p_i = \alpha p_k$ for some $\alpha \in \mathbb{C}^*$. But then $p_i \sigma_i(p_k) = p_k \sigma_k(p_i)$ simplifies to $\sigma_i(p_k) = \sigma_k(p_i)$ which contradicts the form of $p_k$ above. Thus $p_k$ is reducible. The argument for $q_k$ is analogous.

For claim (X), let $\deg_k p_k = 1$. Then by (II) we also have $\deg_k q_k = 1$ and thus

$$(p_k, q_k) = (f_1 h_k + g_1, f_2 h_k + g_2)$$

for some $f_1, f_2, g_1, g_2 \in \mathcal{P}_k$. But then, since $\sigma_k^{-1}(p_k)q_k = -h_k(h_i + 1) + \tilde{g}_k$ and the coefficient of $h_k^2$ on the right is $-1$, we have $f_1, f_2 \in \mathbb{C}^*$ and then, clearly, $p_k$ and $q_k$ are both irreducible.

To prove claim (VII) we consider again the equality $p_i(\sigma_i(p_k) - p_k) = p_k(\sigma_k(p_i) - p_i)$ given by formula (11). Since $\deg_k p_k = 1$, the polynomial $p_k$ is irreducible by claim (VII) so $p_k$ divides either $p_i$ or $(\sigma_i(p_k) - p_k)$. However, considering the $i$-degree, we have that $p_k(\sigma_i(p_k) - p_k)$ only if $(\sigma_i(p_k) - p_k) = 0$. But the right hand side of (11) is nonzero since we know that $p_i$ has the form $-\frac{1}{i}(h_i - 1)\tilde{g}_i$. Thus the only remaining possibility is that $p_k|p_i$.

To prove claim (VI) we suppose that there exist indices $i, k$ such that $\deg_i p_i = 2$ and $\deg_k p_k = 0$. Then, as in the proof of claim (III), we know that $q_i$ is a constant and $\deg_k q_k = 2$. But then the equation $q_i \sigma_i^{-1}(q_k) = q_k \sigma_k^{-1}(q_i)$ from Lemma 20 simplifies to $\sigma_i^{-1}(q_k) = q_k$ which does not hold since $q_k$ depends on $i$.

We now turn to claim (VII). Let $p_i = \alpha_1\alpha_2$ and $p_k = \beta_1\beta_2$ be the corresponding decompositions into prime polynomials. Then the equation $p_i \sigma_i(p_k) = p_k \sigma_k(p_i)$ from Lemma 20 is equivalent to

$$\alpha_1 \alpha_2 \sigma_i(\beta_1 \beta_2) = \beta_1 \beta_2 \sigma_k(\alpha_1 \alpha_2).$$

Suppose $p_i$ and $p_k$ do not share a common factor. Then we have $\alpha_1 \alpha_2 = c \sigma_k(\alpha_1 \alpha_2)$ for some $c \in \mathbb{C}^*$. By applying $c_i$ to both sides, we obtain $c = 1$ and $p_i = \sigma_k p_i$ which is not possible since $p_i$ has the form $c(-h_i \tilde{g}_i + \tilde{g}_i)$ where $\tilde{g}_i \in \mathcal{P}_i$ and thus depends on $k$. Therefore $p_i$ and $p_k$ share a common factor. \hfill \Box

**Lemma 23.** With respect to the grading $\deg_i$, the leading coefficients of both $p_i$ and $q_i$ are invertible, that is $c_i(p_i), c_i(q_i) \in \mathbb{C}^*$.

**Proof.** By Lemma 20 we have $\sigma_i^{-1}(p_i)q_i = -h_i(h_i + 1) + \tilde{g}_i$ for some $\tilde{g}_i \in \mathcal{P}_i$. Applying $c_i$ to this, we get $c_i(\sigma_i^{-1}(p_i))c_i(q_i) = -1$, and $c_i(\sigma_i^{-1}(p_i)) = c_i(p_i)$ which shows that $c_i(p_i), c_i(q_i) \in \mathbb{C}^*$ as stated. \hfill \Box

**Remark 24.** Note that, if a module is determined by $p_1, \ldots, p_n, q_1, \ldots, q_n$, we can apply the functor $F_a$ with

$$a = (c_1(p_1)^{-1}, c_2(p_2)^{-1}, \ldots, c_n(p_n)^{-1}, 1)$$

to obtain a module where the leading coefficient of $p_i$ is 1 for all $i \in n$. Thus from here on we will assume that the leading coefficient of each $p_i$ is 1. All other module structures can then be obtained by applying the functors $F_a$.  

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Lemma 25. For each $i \in n$ we have:

(i) The irreducible components of $p_i$ have the form $h_i + \beta_i$ where $\beta_i \in P_i$.

(ii) $\beta_i = \sum_{j \neq i} c_j^{(i)} h_j + c_0^{(i)}$ for some constants $c_j^{(i)}$.

(iii) $c_j^{(i)} \in \{0, 1\}$ for all $i \in n$, $j \in n \setminus \{i\}$.

Proof. If $\text{deg}_i p_i$ is 0 or 1, then claim (i) follows from Lemma 22. If $\text{deg}_i p_i = 2$, then by Lemma 22(iii) we know that $\sigma_i^{-1}(p_i)$ and thus also $p_i$ has a nontrivial factorization, say $p_i = fg$. But then $2 = \text{deg}_i(f) + \text{deg}_i(g)$, and $1 = c_i(f) c_i(g)$, which shows that $\text{deg}_i(f) = 1 = \text{deg}_i(g)$ since the factorization was nontrivial, and thus both $f$ and $g$ have the prescribed form.

To prove (ii), we need only show that for each $k \in n \setminus \{i\}$ we either have $\text{deg}_k p_i = 0$ or we have $\text{deg}_k p_i = 1$ and $c_k(p_i) \in \mathbb{C}$. Suppose first that $\text{deg}_i p_i = 1$, so that $p_i = h_i + \beta_i$. We consider equation (7):

$$p_i(\sigma_i(p_k) - p_k) = p_k(\sigma_k(p_i) - p_i).$$

If $\text{deg}_k p_k = 0$, then $p_k = 1$ and (7) reads $0 = \sigma_k(p_i) - p_i$. This shows that $\text{deg}_k p_i = 0$ and we are done. If $\text{deg}_k p_k = 1$, then claim (i) and equation (7) give

$$(h_i + \beta_i)(\sigma_i(\beta_k) - \beta_k) = (h_k + \beta_k)(\sigma_k(\beta_i) - \beta_i).$$

Since $h_i + \beta_i$ is prime by Lemma 22, this term either divides the factor $(\sigma_i(\beta_k) - \beta_k)$, which implies $(\sigma_i(\beta_k) - \beta_k) = 0$ and thus $\text{deg}_k p_i = \text{deg}_k \beta_i = 0$ and we are done; or $h_i + \beta_i$ divides $h_k + \beta_k$ which is also prime and thus $h_i + \beta_i = c(h_k + \beta_k)$ for some $c \in \mathbb{C}^\ast$. Define

$$\gamma := h_i - c \beta_k = ch_k - \beta_i.$$

Then $\gamma \in P_i \cap P_k$, and we have $\beta_i = ch_k - \gamma$ as required.

If $\text{deg}_k p_k = 2$, then by Lemma 22(vi) we know that $p_k$ shares a common factor with $p_i$, and again we have $h_i + \beta_i = c(h_k + \beta_k)$ for some nonzero $c$ and the same argument works. Suppose now instead that $\text{deg}_k p_i = 2$ and let $(h_i + \beta_i)$ be a factor of $p_i$. Then we have $p_i = (h_i + \beta_i)(h_i + \beta_i)$, where by Lemma 20 we have $q_i = -1$, and

$$\beta_i = \sum_{j \neq i} h_j - \beta_i - 1.$$

Let $\text{deg}_k p_k$ have degree 1 or 2. Then by Lemma 22 we know that $p_k$ shares a common factor with $p_i$ and we again get an equality of the form $h_i + \beta_i = c(h_k + \beta_k)$ for some $c \in \mathbb{C}^\ast$. By the same argument as above we again see that $\beta_i$ has the stated form. But $\beta_i$ has the prescribed form if and only if $\beta_i$ does, so we are done.

To prove claim (iii) we fix an index $i \in n$. We shall prove that $c_k^{(i)} \in \{0, 1\}$ for each $k \in n \setminus \{i\}$. Suppose first that $\text{deg}_i p_i = 1$. Then we have

$$p_i = (h_i + \sum_{j \neq i} c_j^{(i)} h_j + c_0^{(i)}).$$

and, by Lemma 20, we have

$$q_i = -(h_i + \sum_{j \neq i} (1 - c_j^{(i)}) h_j - c_0^{(i)}).$$

(10)

In the proof of claim (iii) for $(\text{deg}_i p_i, \text{deg}_k p_k) = (1, 0)$ we had $\text{deg}_k p_i = 0$ and thus $c_k^{(i)} = 0$.  


For \((\deg_i p_i, \deg_k p_k) = (1,1)\), we have
\[
p_k = (h_k + \sum_{j \neq k} c_j^{(k)} h_j + c_0^{(k)})
\]
and equation (11) in this case reads
\[
c_i^{(k)} (h_i + \sum_{j \neq i} c_j^{(i)} h_j + c_0^{(i)}) = c_k^{(i)} (h_k + \sum_{j \neq k} c_j^{(k)} h_j + c_0^{(k)}).
\]

Applying \(c_i\) and \(c_k\) separately to this equation we get
\[
\begin{cases}
c_i^{(k)} = c_k^{(i)} c_i^{(k)}, \\
c_i^{(i)} = c_i^{(k)} c_k^{(i)},
\end{cases}
\]
which has solutions \((c_i^{(i)}, c_k^{(k)}) \in \{(0,0), (1,1)\}\). Thus, in particular, \(c_k^{(i)} \in \{0,1\}\).

Next, suppose \((\deg_i p_i, \deg_k p_k) = (1,2)\). Then \(q_k = -1\) and, by Lemma 20, \(q_i\) does not depend on \(h_k\). But then from our explicit form (10) of \(q_i\) above we see that \(1 - c_k^{(i)} = 0\). We have now proved that any \(p_i\) with \(\deg_i p_i = 1\) has the prescribed form.

Next, let \(\deg_i p_i = 2\). Then
\[
p_i = (h_i + \sum_{j \neq i} c_j^{(i)} h_j + c_0^{(i)})(h_i + \sum_{j \neq i} (1 - c_j^{(i)}) h_j - c_0^{(i)} - 1)
\]
and \(q_i = -1\). By assumption, \(\deg_k p_k \neq 0\). For \(\deg_k p_k = 1\), the polynomial \(p_k\) satisfies (iii) and divides \(p_i\) by Lemma 22. Since the coefficient at \(h_k\) in \(p_k\) is 1, we obtain either \(c_k^{(i)} = 1\) or \(1 - c_k^{(i)} = 1\), as desired. Finally, we consider the case \((\deg_i p_i, \deg_k p_k) = (2,2)\).

We shall again show that \(c_k^{(i)} \in \{0,1\}\). Let
\[
p_i = \alpha_1 \alpha_2, \quad p_k = \beta_1 \beta_2 \quad \text{and} \quad q_i = q_k = -1,
\]
where
\[
\begin{cases}
\alpha_1 = h_i + \sum_{j \neq i} c_j^{(i)} h_j + c_0^{(i)}, \\
\alpha_2 = h_i + \sum_{j \neq i} (1 - c_j^{(i)}) h_j - c_0^{(i)} - 1, \\
\beta_1 = h_k + \sum_{j \neq k} c_j^{(k)} h_j + c_0^{(k)}, \\
\beta_2 = h_k + \sum_{j \neq k} (1 - c_j^{(k)}) h_j - c_0^{(k)} - 1.
\end{cases}
\]

Then the equality \(p_i \sigma_i p_k = p_k \sigma_k p_i\) from Lemma 20 becomes
\[
\alpha_1 \alpha_2 (\beta_1 - c_i^{(k)})(\beta_2 - (1 - c_i^{(k)})) = \beta_1 \beta_2 (\alpha_1 - c_k^{(i)})(\alpha_2 - (1 - c_k^{(i)})).
\]

If \(\alpha_1 \alpha_2 | \beta_1 \beta_2\), formula (12) becomes \(\sigma_i(p_i) = \sigma_k(p_i)\) which implies
\[
p_i \in \mathbb{C}[h_i + h_k, h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{k-1}, h_{k+1}, \ldots, h_n] \simeq (\mathcal{P}_i \cap \mathcal{P}_k)[h_i + h_k].
\]

Here we view \((\mathcal{P}_i \cap \mathcal{P}_k)[h_i + h_k]\) as the subring of \(\mathcal{P}\) consisting of polynomials in the single variable \((h_i + h_k)\) with coefficients in \(\mathcal{P}_i \cap \mathcal{P}_k\). This contradicts the fact that \(\sigma_i^{-1}(p_i) q_i = -h_i (\mathcal{P} + 1) + \tilde{g}_i\) from Lemma 20. Thus, without loss of generality, \(\alpha_1\) does not divide \(\beta_1 \beta_2\). Moreover, \(\alpha_1\) only divides \((\alpha_1 - c_k^{(i)})\) if \(c_k^{(i)} = 0\), so we may assume that \(\alpha_1\) divides \((\alpha_2 - (1 - c_k^{(i)})\)) and \(\alpha_2\) divides \((\alpha_1 - c_k^{(i)}\)). Considering the coefficient at \(h_i\), this happens only if \(\alpha_1 = \alpha_2 - (1 - c_k^{(i)})\), so formula (12) becomes
\[
\alpha_2 (\beta_1 - c_i^{(k)})(\beta_2 - (1 - c_i^{(k)})) = \beta_1 \beta_2 (\alpha_2 - 1).
\]
This shows that $\alpha_2$ divides $\beta_1$ or $\beta_2$. Without loss of generality we may assume that $\alpha_2 = c\beta_2$ for some $c \in C^s$. Comparing the coefficient of $h_k$ in our explicit expressions of $\alpha_2$ and $\beta_1$ in formula (11), we get $c = (1 - c_k^{(i)}) \neq 0$ and thus

$$\left(\beta_1 - c_k^{(i)}\right)\left(\beta_2 - (1 - c_k^{(i)})\right) = \beta_2 \left(\beta_1 - (1 - c_k^{(i)})^{-1}\right).$$

(13)

The four prime factors occurring in (13) all have $h_k$-coefficient 1, so one divides another if and only if they are equal. Now $\beta_2 = (\beta_2 - (1 - c_k^{(i)}))$ gives $c_k^{(i)} = 1$ and thus we have $\left(\beta_1 - 1\right) = \left(\beta_1 - (1 - c_k^{(i)})^{-1}\right)$, which gives $c_k^{(i)} = 0$. On the other hand, $\beta_2 = (\beta_1 - c_k^{(i)})$ gives $\left(\beta_1 - 1\right) = \left(\beta_1 - (1 - c_k^{(i)})^{-1}\right)$, which also implies $c_k^{(i)} = 0$. We have now considered all cases, and can conclude that $c_k^{(i)} \in \{0, 1\}$ always.

From here on, we shall assume that $\deg_j p_i \in \{1, 2\}$ for all $i \in n$. By Proposition 9 all other module structures can then be obtained by application of the functor $F_r$. Define a binary relation $\sim$ on $n$ by

$$i \sim j \iff p_i \text{ and } p_j \text{ share a common prime divisor.}$$

(14)

Note that $\sim$ is symmetric and reflexive (since we assume that all $\deg_j p_i \geq 1$), but it is in general not transitive: for example we would have $h_1 \sim h_1h_2 \sim h_2$ while $h_1 \not\sim h_2$.

**Proposition 26.** If $\deg_k p_k = 1$, then for some $b_k \in C$ we have

$$p_k = \left(\sum_{j \sim k} h_j + b_k\right) \quad \text{and} \quad q_k = -(h_k + \sum_{j \not\sim k} h_j - b_k).$$

**Proof.** Since $\deg_k p_k = 1$ by Lemma 22 the polynomial $p_k$ divides all polynomials $p_j$ of degree 2 and hence the coefficient at $h_j$ in $p_k$ is 1 for all $j$ with $\deg_j p_j = 2$. Assume now instead that $p_j$ has degree 1 as well. Then equation (7) becomes $c_k^{(j)} p_k = c_j^{(k)} p_j$. Now if $j \sim k$, then $p_j = p_k$ and $c_j^{(k)} = 1$. On the other hand, if $j \not\sim k$, then $p_j \not= p_k$ which implies $c_j^{(k)} = 0$, so the formula for $p_k$ in the proposition is correct and $q_k$ is uniquely determined by $p_k$ (see formula (10) in the proof of Lemma 23).

Next we prove that either all the first degree polynomials $p_i$ coincide, or they are pairwise different.

**Lemma 27.** Let

$$n_1 := \{i \in n | \deg_i p_i = 1\} \quad \text{and let} \quad n_2 := \{j \in n | \deg_j p_j = 2\} = n \setminus n_1.$$

Then either $p_i = p_j$ for all $i, j \in n_1$ or $p_i \not= p_j$ for all distinct $i, j \in n_1$.

**Proof.** For each $k \in n$ define

$$A_k := \{t \in n | t \sim k\}.$$

Suppose there exists indices $i, k \in n_1$ with $p_i \not= p_k$. The statement of Proposition 26 can now be written as follows:

$$q_k = -(h_k + \sum_{t \in n \setminus A_k} h_t - b_k).$$

(15)

But now, since $i \not\sim k$, we have $i \in n \setminus A_k$ and $k \in n \setminus A_i$, so equality (15) becomes just $q_i = q_k$. Using our explicit expressions for $q_i$ and $q_k$ from (15) we see that this is equivalent to $b_i = b_k$ and $\{i\} \cup (n \setminus A_i) = \{k\} \cup (n \setminus A_k)$, which simplifies to

$$A_i \setminus \{i\} = A_k \setminus \{k\}.$$

(16)

Now for $j \in n_2$ we always have $i \sim j$ and for $j \in n_1$ we have $i \sim j$ only if $p_i = p_j$. Thus $A_i \cap n_1$ and $A_k \cap n_1$ are disjoint, so (16) implies $A_i = n_2 \cup \{i\}$ and $A_k = n_2 \cup \{k\}$. In particular, $p_i \not= p_j$ for any $j \in n_1$ with $j \not= k$. This shows that all the polynomials $p_j$ with $j \in n_1$ are pairwise distinct.
4.2 Classifications of objects in \( \mathcal{M} \)

We are now ready to classify objects in \( \mathcal{M} \) for \( \mathfrak{sl}_{n+1} \).

**Definition 28.** Let \( S \subset n \) and \( b \in \mathbb{C} \). Define \( M^S_b \) to be the set \( \mathcal{P} \) equipped with the following \( \mathfrak{sl}_{n+1} \)-action:

\[
\begin{align*}
    h_k \cdot f &= h_k f, \quad k \in n; \\
    e_{i,n+1} \cdot f &= \begin{cases} 
      (h + b)\sigma_i f, & i \in S, \\
      (h + b)(h_i - b - 1)\sigma_i f, & i \not\in S;
    \end{cases} \\
    e_{n+1,j} \cdot f &= \begin{cases} 
      -(h_j - b)\sigma_j^{-1} f, & j \in S, \\
      -\sigma_j^{-1} f, & j \not\in S;
    \end{cases} \\
    e_{i,j} \cdot f &= \begin{cases} 
      (h_i - b)\sigma_i\sigma_j^{-1} f, & i \in S, j \not\in S, \\
      \sigma_i\sigma_j^{-1} f, & i \not\in S, j \in S, \\
      (h_i - b - 1)(h_j - b)\sigma_i\sigma_j^{-1} f, & i \not\in S, j \not\in S, \\
      (h_i - b - 1)\sigma_i\sigma_j^{-1} f, & i \not\in S, j \not\in S.
    \end{cases}
\end{align*}
\]

To write this more compactly we introduce the indicator functions \( \delta_P \) where \( P \) is some statement, and \( \delta_P = 1 \) if \( P \) is true and \( \delta_P = 0 \) if \( P \) is false. Then the above can be written as follows.

\[
\begin{align*}
    h_k \cdot f &= h_k f, \\
    e_{i,n+1} \cdot f &= (h + b)(\delta_{i \in S} + \delta_{j \not\in S}(h_i - b - 1))\sigma_i f, \\
    e_{n+1,j} \cdot f &= -(\delta_{j \not\in S}(h_j - b) + \delta_{j \not\in S})\sigma_j^{-1} f, \\
    e_{i,j} \cdot f &= (\delta_{i \in S} + \delta_{j \not\in S}(h_i - b - 1))(\delta_{j \not\in S}(h_j - b) + \delta_{j \not\in S})\sigma_i\sigma_j^{-1} f.
\end{align*}
\]

**Theorem 29.** Equipped with the action of Definition 28, \( M^S_b \) is a \( \mathfrak{sl}_{n+1} \)-module for all \( b \in \mathbb{C} \) and all \( S \subset n \).

**Proof.** First note that for all \( k \in n \) and all \( i, j \in n + 1 \) with \( i \neq j \) we have \( e_{i,j} \cdot h_k = h_k - \delta_{i,k} + \delta_{j,k} \). But then for all \( f \in M^S_b \) we have

\[
\begin{align*}
    e_{i,j} \cdot h_k \cdot f - h_k \cdot e_{i,j} \cdot f &= e_{i,j} \cdot (h_k f) - h_k (e_{i,j} \cdot f) \\
    &= (h_k - \delta_{i,k} + \delta_{j,k})(e_{i,j} \cdot f) - h_k (e_{i,j} \cdot f) \\
    &= (\delta_{i,k} - \delta_{j,k})e_{i,j} \cdot f \\
    &= [e_{i,j}, h_k] \cdot f
\end{align*}
\]

where we used Lemma 4 in the last step. Thus the relation

\[
[e_{i,j}, h_k] \cdot f = e_{i,j} \cdot h_k \cdot f - h_k \cdot e_{i,j} \cdot f
\]

holds for all \( k \in n \) and all \( i, j \in n + 1 \) with \( i \neq j \).

The remaining relations are more time consuming to check. We first check that for all \( i, j, k \in n \) (with \( i \neq j \)) and all \( f \in M^S_b \) we have

\[
e_{n+1,k} \cdot e_{i,j} \cdot f - e_{i,j} \cdot e_{n+1,k} \cdot f = [e_{n+1,k}, e_{i,j}] \cdot f.
\]

(17)
Expressing the left side of (17) explicitly we get

\[ e_{n+1,k} \cdot e_{i,j} \cdot f - e_{i,j} \cdot e_{n+1,k} \cdot f \]

\[ = - \left[ \delta_{k \in S}(h_k - b) + \delta_{k \notin S}(h_k - b - 1) \right] \left( \delta_{j \in S}(h_j - b) + \delta_{j \notin S}(h_j - b - 1) \right) \sigma_i \sigma_j \sigma_{k}^{-1} f \]

\[ + \delta_{j \in S}(h_j - b) + \delta_{j \notin S}(h_j - b - 1) \right] \left( \delta_{j \in S}(h_j - b) + \delta_{j \notin S}(h_j - b - 1) \right) \sigma_i \sigma_j \sigma_{k}^{-1} f \]

Now, when expanding this expression, all terms not containing \( \delta_{i,k} \) or \( \delta_{j,k} \) will cancel by symmetry. Thus, by factoring out \( \delta_{i,k} \), \( \delta_{j,k} \) and \( \delta_{i,k} \delta_{j,k} \) separately, we can rewrite this as

\[ e_{n+1,k} \cdot e_{i,j} \cdot f - e_{i,j} \cdot e_{n+1,k} \cdot f \]

\[ = \delta_{i,k} \left[ \left( \delta_{i \in S}(h_i - b) + \delta_{i \notin S}(h_i - b - 1) \right) \left( \delta_{j \in S}(h_j - b) + \delta_{j \notin S}(h_j - b - 1) \right) \sigma_i \sigma_j \sigma_{k}^{-1} f \right] \]

\[ + \delta_{j,k} \left[ \left( \delta_{i \in S}(h_i - b) + \delta_{i \notin S}(h_i - b - 1) \right) \left( \delta_{j \in S}(h_j - b) + \delta_{j \notin S}(h_j - b - 1) \right) \sigma_i \sigma_j \sigma_{k}^{-1} f \right] \]

\[ + \delta_{i,k} \delta_{j,k} \left[ \left( \delta_{i \in S}(h_i - b) + \delta_{i \notin S}(h_i - b - 1) \right) \left( \delta_{j \in S}(h_j - b) + \delta_{j \notin S}(h_j - b - 1) \right) \sigma_i \sigma_j \sigma_{k}^{-1} f \right] \]

Now, since \( i \neq j \), we have \( \delta_{i,k} \delta_{j,k} = 0 \) so the last term is zero. Using the fact that \( \delta_{P \wedge Q} = \delta_{P} \delta_{Q} \) and that \( \delta_{\neg P} = 1 - \delta_{P} \), the above expression can be further simplified to

\[ e_{n+1,k} \cdot e_{i,j} \cdot f - e_{i,j} \cdot e_{n+1,k} \cdot f \]

\[ = \delta_{i,k} \left[ \left( \delta_{i \in S}(h_i - b) + \delta_{i \notin S}(h_i - b - 1) \right) \left( \delta_{j \in S}(h_j - b) + \delta_{j \notin S}(h_j - b - 1) \right) \sigma_i \sigma_j \sigma_{k}^{-1} f \right] \]

Thus (17) holds for all \( i, j, k \in \mathfrak{n} \) (with \( i \neq j \)) and all \( f \in M_b^S \) as required.
The remaining relations lead to similar equations which are left to the reader to verify.

**Theorem 30.** For \( n > 1 \), let \( S \) be the full subcategory of \( \mathcal{M} \) consisting of all modules of form
\[
F_a(M_b^S) \quad \text{and} \quad F_a \circ F_c(M_b^S)
\]
for all \( a = (a_1, a_2, \ldots, a_n, 1) \in (\mathbb{C}^*)^n \times \{1\} \), \( S \subset \mathfrak{n} \) and \( b \in \mathbb{C} \). Then \( S \) is a skeleton in \( \mathcal{M} \).

**Proof.** Suppose we are given a module structure on \( \mathcal{P} \) determined by \( (p_1, \ldots, p_n, q_1, \ldots, q_n) \) in accordance with Proposition 8. First assume that \( \deg_k p_k \in \{1, 2\} \) for all \( k \in \mathfrak{n} \). By Lemma 23 and Remark 24 it suffices to prove that \( c_k(p_k) = 1 \) for all \( k \in \mathfrak{n} \) implies that the module is isomorphic to either \( F_a(M_b^S) \) or \( F_a \circ F_c(M_b^S) \) for suitable choices of \( a \in (\mathbb{C}^*)^{n+1}, b \in \mathbb{C} \) and \( S \subset \mathfrak{n} \). Thus we assume that the leading coefficient of each \( p_k \) is 1 and \( \deg_k p_k \in \{1, 2\} \). Note that for \( n = 1 \) we obtain the same set of modules as in Section 3. Define
\[
N := \# \{ i \in \mathfrak{n} | \deg_i p_i = 2 \}.
\]
We consider first the case \( N = 0 \) when all \( p_k \) have \( k \)-degree 1. By Proposition 26 each \( p_k \) and thus the module structure is completely determined by the relation \( \sim \) from (14), which in this case becomes an equivalence relation:
\[
i \sim k \iff p_i = p_k.
\]
By Lemma 27 either all \( p_i \) are pairwise distinct or they all coincide. Using the explicit expression of \( p_i \) from Proposition 26 in the first case (i.e. all \( p_i \) are distinct) we have
\[
p_i = h_i + b_i \quad \text{for all} \quad i \in \mathfrak{n},
\]
which implies that \( q_i = \overline{h} - b_i \) for all \( i \) and by Remark 21 all the scalars \( b_i \) coincide (write \( b := b_1 \)) and the module is isomorphic to \( F_{(1,1,\ldots,1,-1)} \circ F_{(M_b^S)} \). In the other case (i.e. all \( p_i \) coincide) we get
\[
p_i = \overline{h} + b \quad \text{for all} \quad i \in \mathfrak{n},
\]
which makes the module isomorphic to \( M_b^n \).

Next, suppose \( N = 1 \) and let \( \deg_k p_k = 2 \). For \( n = 2 \), the unique \( p_i \) of degree 1 divides both \( p_i \) and \( p_k \), so the module is isomorphic to \( M_b^{(i)} \) for some \( b \in \mathbb{C} \). For \( n = 3 \), write \( \{1, 2, 3\} = \{i, j, k\} \). If \( p_i = p_j \), then the module is isomorphic to \( M_b^{(i,k)} \) for some \( b \in \mathbb{C} \). If \( p_i \neq p_j \), since both divide \( p_k \), we get \( p_k = p_ip_j \) and \( q_k = -1 \). This implies that \( p_i = h_i + h_k + b_i \) and \( p_j = h_j + h_k + b_j \) and, by Lemma 20, we get first \( b_i + b_j + 1 = 0 \) and then \( b_i = b_j = -\frac{1}{2} \). But this would determine the module structure completely and one can check that we for example would have \( e_2,4 \cdot e_1,3 \cdot 1 - e_1,3 \cdot e_2,4 \cdot 1 \neq [e_2,4, e_1,3] \cdot 1 \), so this module structure is not possible.

Finally, for \( n > 3 \) we have at least three degree 1 polynomials dividing \( p_k \) and the latter polynomial has two prime factors. Thus at least two of the divisors coincide and hence they all coincide by Lemma 27. This gives a module isomorphic to \( M_b^{(i,k)} \) for some \( b \in \mathbb{C} \). Thus the statement of the theorem holds for \( N = 1 \).

We now turn to the case \( 2 \leq N < n \). Let \( i, k \in \mathfrak{n} \) be distinct indices such that \( \deg_i p_i = 2 = \deg_k p_k \). Then \( p_i \) and \( p_k \) share a common prime factor \( \alpha \) by Lemma 22. So we have \( p_i = \alpha \beta_i \) and \( p_k = \alpha \beta_k \) where \( \alpha, \beta_i, \beta_k \) are pairwise distinct. But then all \( p_j \) for \( j \in \mathfrak{n}_1 \) share a factor with both \( p_i \) and \( p_k \), so \( p_j = \alpha \) for all \( j \in \mathfrak{n}_1 \). Thus \( \alpha | p_j \) for all \( j \in \mathfrak{n} \) and our module is isomorphic to \( M_b^{(i)} \) for some \( b \in \mathbb{C} \).

Finally, we consider the case \( N = n \) where all polynomials \( p_i \) have degree 2. For \( n = 2 \), the polynomials \( p_1 \) and \( p_2 \) share a common factor which then divides all \( p_i \) for \( i \in \{1, 2\} \).
and the module is isomorphic to $M_b^S$ for some $b \in \mathbb{C}$. For $n \geq 3$, suppose that not all polynomials $p_i$ share the same factor. Then there exist distinct indices $i, j, k$ such that

$$p_i = \alpha \beta, \quad p_j = \alpha \gamma \quad \text{and} \quad p_k = \beta \gamma$$

for some pairwise distinct $\alpha, \beta, \gamma$. But then a fourth $p_j$ cannot share a common factor with both $p_i, p_j, p_k$ so for $n > 3$ all $p_i$ share a fixed factor $\alpha$ for all $i \in \mathfrak{n}$, and the module is isomorphic to $M_b^S$ for some $b \in \mathbb{C}$. Thus the only remaining case is $n = 3$ and

$$p_1 = \alpha \beta, \quad p_2 = \alpha \gamma \quad \text{and} \quad p_3 = \beta \gamma.$$

But then we explicitly have

$$\alpha = h_1 + h_2 + b_1, \quad \beta = h_1 + h_3 + b_2 \quad \text{and} \quad \gamma = h_2 + h_3 + b_3.$$

Moreover, $q_1 = q_2 = q_3 = -1$, so Lemma \[23] implies that $b_1 = b_2 = b_3 = -\frac{1}{2}$. But this does not give a module structure, since, for example, $e_{2,4} \cdot e_{1,3} \cdot 1 - e_{1,3} \cdot e_{2,4} \cdot 1 \neq [e_{2,4}, e_{1,3}] \cdot 1$.

We have now proved the theorem in case $\deg_q p_k \in \{1, 2\}$ for all $k \in \mathfrak{n}$. Suppose now this is not the case for some module $M$. By Lemma \[24] we then have $\deg_q p_k \in \{0, 1\}$ for all $k \in \mathfrak{n}$ and $\deg_q q_k \in \{1, 2\}$ for all $k \in \mathfrak{n}$. Thus we apply the theorem above to $F_\tau(M)$ instead, giving either $F_\tau(M) \cong F_a(M_b^S)$ or $F_\tau(M) \cong F_\tau \circ F_a(M_b^S)$ for some $a, b, S$.

Applying $F_\tau$ again we get either $M \cong F_{a^{-1}} \circ F_\tau(M_b^S)$ or $M \cong F_{a^{-1}}(M_b^S)$.

Thus we have proved that every module in $\mathcal{M}$ is isomorphic to one of the representatives in the theorem. It remains to show that different module structures are not isomorphic.

Any morphism $\varphi$ in $\mathcal{M}$ is determined by its value at 1 since $\varphi(f) = f \varphi(1)$. Suppose now that $\varphi : M \to M'$ is an isomorphism, where we identify $M$ and $M'$ with $\mathcal{P}$ as $\mathcal{U}(\mathfrak{h})$-modules. Then

$$1 = \varphi(\varphi^{-1}(1)) = \varphi^{-1}(1)\varphi(1),$$

which shows that $\varphi(1) \in \mathbb{C}^*$ and $\varphi = c \mathbf{1}$ for some $c \in \mathbb{C}^*$. For all $i \in \mathfrak{n}$ we define $p_i := e_{i,n+1} \cdot 1 \in M$ and $p'_i := e_{i,n+1} \cdot 1 \in M'$. Then we have

$$cp'_i = p'_i \sigma_i \varphi(1) = e_{i,n+1} \cdot \varphi(1) = \varphi(e_{i,n+1} \cdot 1) = \varphi(p_i) = p_i \varphi(1) = cp_i,$$

which gives $p_i = p'_i$. Thus two modules can be isomorphic only if for every $i \in \mathfrak{n}$ the action of the element $e_{i,n+1}$ on these two modules is given by the same polynomial. Similarly, the polynomials $q_i$ must coincide in isomorphic modules. Now in a module $F_a(M_b^S)$, the set of $p_i$’s are uniquely determined by $a, b, S$ so there are no nontrivial isomorphisms between objects of this form, and hence the same also holds for objects of form $F_a \circ F_\tau(M_b^S)$. Finally, for the module $F_a(M_b^S)$, since $n > 1$ each $p_i$ has an irreducible component $(\mathfrak{h} + b)$ which $q_i$ does not have. This shows that there are no isomorphisms $F_a \circ F_\tau(M_b^S) \to F_a'(M_b^S)$.

\[ \square \]

**Remark 31.** Note that the theorem applies also to $n = 1$, except that in this case we have isomorphisms $M_b^1 \cong F_{(1,-1)} \circ F_\tau(M_b^1)$ for each $b \in \mathbb{C}$. Via the relations $M_b \cong M_b^1$ and $M_b' \cong F_\tau(M_b^S)$, we recover again the results of Theorem \[21].

### 4.3 Simples and subquotients

We now show that the modules of $\mathcal{M}$ generically are simple. We start with some sufficient conditions for simplicity.

**Theorem 32.**  (i) For $b \in \mathbb{C}$ with $(n+1)b \not\in \mathbb{N}_0$, the module $M_b^n$ is simple.

(ii) For $S \neq \mathfrak{n}$, the module $M_b^S$ is simple.
Proof. Fix a subset $S \subset \mathfrak{n}$. We start by constructing a new basis of $\mathcal{U}(\mathfrak{h})$. For each $i \in \mathfrak{n}$ and for all integers $k \geq -1$, we define

$$H_{k}^{(i)} := \prod_{j=0}^{k} (h_{i} - b + j).$$

Then the set

$$B := \{H_{k_{1}, k_{2}, \ldots, k_{n}} := H_{k_{1}}^{(1)} \cdots H_{k_{n}}^{(n)} | k_{1}, \ldots, k_{n} \geq -1\}$$

is a basis for $\mathcal{U}(\mathfrak{h})$. For each $i \in \mathfrak{n}$ we also define

$$A_{i} := \begin{cases} e_{n+1,i} + (h_{i} - b) & i \in S \\ (h_{i} - b)(e_{n+1,i} + 1) & i \notin S \end{cases} \cdot$$

Then for each $i \in \mathfrak{n}$ we have

$$A_{i} \cdot H_{k}^{(i)} = -(k + 1)H_{k}^{(i)},$$

and since $A_{i}$ commutes with $H_{k_{j}}^{(j)}$ for all $j \neq i$ we deduce that

$$A_{i} \cdot H_{k_{1}, k_{2}, \ldots, k_{n}} = -(k_{i} + 1)H_{k_{1}, k_{2}, \ldots, k_{n}},$$

so $H_{k_{1}, k_{2}, \ldots, k_{n}}$ is an eigenvector of $A_{i}$ with eigenvalue $-(k_{i} + 1)$. This shows that the elements $A_{1}, \ldots, A_{n}$ act diagonally in the basis $B$, where each generalized eigenspace is of dimension 1. We conclude that any submodule of $M_{b}^{S}$ is the span of some subset of elements from $B$.

Let $V$ be a nonzero submodule of $M_{b}^{S}$, and let $f$ be a nonzero element of $V$.

Now let $H_{k_{1}, k_{2}, \ldots, k_{n}}$ be a basis element occurring with nonzero coefficient in $f$ expressed in the basis $B$. Then $H_{k_{1}, k_{2}, \ldots, k_{n}} \in V$. We shall show that for each $i \in \mathfrak{n}$, if $k_{i} \neq -1$, we have also $H_{k_{1}, \ldots, k_{i}-1, \ldots, k_{n}} \in V$; it will then follow by induction that $1 = H_{-1,\ldots,-1} \in V$ which implies $V = M_{b}^{S}$.

Fix $i \in \mathfrak{n}$. If $i \notin S$ we note that

$$(e_{n+1,i} + 1) \cdot H_{k_{1}, k_{2}, \ldots, k_{n}} = -(k_{i} + 1)\sigma_{i}^{-1}(H_{k_{1}, \ldots, k_{i}-1, \ldots, k_{n}}),$$

so for $k_{i} \geq 0$, by considering the $i$-degree, we see that when we express this in the basis $B$ the coefficient of $H_{k_{1}, \ldots, k_{i}-1, \ldots, k_{n}}$ is nonzero which implies that $H_{k_{1}, \ldots, k_{i}-1, \ldots, k_{n}} \in V$ as required.

On the other hand, if $i \in S$, we act by $e_{i,n+1}$ on $H_{k_{1}, k_{2}, \ldots, k_{n}}$ and express the result in
our basis $B$:

\[
e_{i,n+1} \cdot H_{k_1,k_2,\ldots,k_n} = (\mathcal{H} + b)(H_{k_1,\ldots,k_n} - (k_i + 1)H_{k_1,\ldots,k_{i-1},\ldots,k_n})
\]

\[
= \mathcal{H}H_{k_1,\ldots,k_n} - (k_i + 1)\mathcal{H}H_{k_1,\ldots,k_{i-1},\ldots,k_n}
+ b(H_{k_1,\ldots,k_n} - (k_i + 1)H_{k_1,\ldots,k_{i-1},\ldots,k_n})
\]

\[
= \sum_{j=1}^{n} (h_j - b + k_j + 1))H_{k_1,\ldots,k_n} + (nb - \sum_{j=1}^{n} (k_j + 1))H_{k_1,\ldots,k_n}
- (k_i + 1)((h_i - b + k_i) + \sum_{j\neq i} (h_j - b + k_j + 1))H_{k_1,\ldots,k_{i-1},\ldots,k_n}
- (k_i + 1)(nb - k_i - \sum_{j\neq i} (k_j + 1))H_{k_1,\ldots,k_{i-1},\ldots,k_n}
+ b(H_{k_1,\ldots,k_n} - (k_i + 1)H_{k_1,\ldots,k_{i-1},\ldots,k_n})
\]

Thus we see that the coefficient of $H_{k_1,\ldots,k_{i-1},\ldots,k_n}$ in $e_{i,n+1} \cdot H_{k_1,k_2,\ldots,k_n}$ is precisely

\[-(k_i + 1)((n + 1)b - k_i - \sum_{j\neq i} (k_j + 1)).\]

Now if $(n+1)b$ is not a natural number, this quantity is nonzero since $k_i + \sum_{j\neq i} (k_j + 1) \in \mathbb{N}_0$ while $(n+1)b \notin \mathbb{N}_0$. This proves the induction step and implies the simplicity of $M_b^S$ for $(n+1)b \notin \mathbb{N}_0$, which in particular proves (ii).

To prove (iii), assume that $S \neq \mathbf{n}$. Suppose again that $H_{k_1,\ldots,k_n} \in V$ with $k_i \neq -1$ for some $i$. We observe from the calculation above that the coefficient of $H_{k_1,\ldots,k_{i-1},\ldots,k+1,\ldots,k_n}$ in $e_{i,n+1} \cdot H_{k_1,\ldots,k_n}$ is nonzero for each $j \neq i$, thus $H_{k_1,\ldots,k_{i-1},\ldots,k_{j+1},\ldots,k_n}$ belongs to $V$ also.

Acting repeatedly by $e_{i,n+1}$ for all $i \in S$ we obtain finally $H_{k_1',\ldots,k_n'} \in V$ where $k'_j = -1$ for all $j \in S$. Acting repeatedly by $(e_{n+1,k} + 1)$ for all $k \notin S$ we finally obtain $1 \in V$ so $V$ is simple.

Since the functors $F_a$ and $F_\tau$ from Section 2.3 are equivalences we also have the following corollary.

**Corollary 33.** For $(n+1)b \notin \mathbb{N}_0$ or $S \neq \mathbf{n}$, the modules $F_a(M_b^S)$ and $F_a \circ F_\tau(M_b^S)$ are simple.

It turns out that any simple module in $\mathcal{M}$ are of the form in the above corollary. The only case remaining is when both $S = \mathbf{n}$ and $(n+1)b$ is a natural number, this is covered in the following theorem.

**Theorem 34.** For $(n+1)b \in \mathbb{N}_0$, the module $M_b^n$ has a unique proper submodule $W$ which is simple and belongs to $\mathcal{M}$ but not to $\mathcal{M}$. The corresponding simple quotient has dimension $\binom{(n+1)b+n}{n}$ and has lowest weight $\lambda$, where $\lambda(h_i) = b - \delta_i,1(n+1)b$.  

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Proof. Define
\[ W := \text{span}\{H_{k_1, \ldots, k_n} \mid \sum_{i=1}^n k_i \geq (n + 1)b - (n - 1)\} . \]

This is a submodule of \( M^b_n \). From the calculation in the proof of Theorem 32 it is clear that any vector in \( M^b_n / W \) can be reduced to 1 so the module \( M^b_n / W \) is simple. Similarly one shows that \( W \) is simple. Define
\[ v := \prod_{k=0}^{(n+1)b-1} (h_1 - b + k) = H_{(n+1)b-1,-1,-1,-1,-1} \in M^b_n / W. \]

Now \((h_1 - b + (n + 1)b) \cdot v = H_{(n+1)b-1,-1,-1,-1,-1} = 0\) in the quotient. Similarly, for \( i > 1 \) we have \((h_i - b) \cdot v = 0\), so \( v \) is a weight vector. From Definition 28 we see that \( e_{n+1,j} \cdot v = -(h_j - b)\sigma_j^{-1}v = 0 \) for all \( j \in n \). For \( 1 \leq j < i \leq n \) we also obtain \( e_{i,j} \cdot v = (h_j - b)\sigma_i \sigma_j^{-1}v = 0 \), since \( i \neq 1 \). This means that \( v \) is killed by \( n \).

Thus we have showed that \( v \) is a lowest weight vector in \( M^b_n / W \) of weight \( \lambda \) where \( \lambda(h_i) = b - \delta_{i,1}(n+1)b \) as stated in the theorem. The dimension of \( M^b_n / W \) equals the number of ways to choose \( n \) integers \( k_i \geq -1 \) such that their sum is less than \((n + 1)b - (n - 1)\). Thus we obtain
\[ \dim M^b_n / W = \binom{(n + 1)b + n}{n}. \]

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