Is Bimetric Gravity Really Ghost Free?

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ABSTRACT: We perform the Hamiltonian analysis of the bimetric theory of gravity introduced in [arXiv:1109.3515 [hep-th]]. We carefully analyze the requirement of the preservation of all constraints and we find that there is no additional constraint that could eliminate the ghost mode.

KEYWORDS: Bimetric Gravity, Hamiltonian Formalism
1. Introduction and Summary

Recently the new very interesting formulation of the non-linear massive gravity [1, 2] was introduced with significant improvement reached in [3, 4]. This theory was further extended in [5] where the theory was formulated with general reference metric. The most crucial fact that is related to given theory is the proof of the absence of the ghosts that are generally expected in any theory that breaks the diffeomorphism invariance. As is well known the physical content is determined in the Hamiltonian formulation when all constraints are identified together with their nature. This analysis was performed in several papers [6, 7, 8, 9, 10, 11, 12] with the most important results derived in [13, 14] with the outcome that this non-linear massive theory possesses one additional constraint and the resulting constraint structure is sufficient for the elimination of the ghost degree of freedom.

Very interesting extension of given theory was suggested in [15] when the kinetic term for the general reference metric was introduced and hence $\hat{g}_{\mu\nu}$ and $\hat{f}_{\mu\nu}$ come in the symmetric way in the action. Then it was argued in [13] that the resulting theory is the ghost free formulation of the bimetric theory of gravity $^1$.

The goal of this paper is to perform the Hamiltonian analysis of the bimetric theory of gravity in the form introduced in [15]. For simplicity we call this theory as the new bimetric theory of gravity (NBTG). We would like to explicitly determine the structure of the constraints and eventually to prove the absence of the ghosts. Remarkably we find very subtle issue related to NBTG which forces us to doubt whether the ghost could be eliminated in given theory or not. More explicitly, the non-linear massive gravity with general reference metric has the potential that depends on the matrix $H^\nu_\mu \equiv \hat{g}^{\rho\sigma} \hat{f}_{\rho\sigma}$ where $\hat{f}_{\rho\sigma}$ is fixed background metric. There is now no doubt that such theory is ghost free. On the other hand the situation changes in case of the bimetric theory of gravity when we promote $\hat{f}_{\mu\nu}$ as an additional dynamical field with the kinetic term given by Einstein-Hilbert action. Then, since the interaction term between two metrics has the square structure

$^1$For further analysis of given theory, see [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29].
form we follow [13, 14, 15] and perform the redefinition of one shift function that makes the theory linear in $N$ and $M$ which are the lapse functions in $\hat{g}_{\mu\nu}$ and $\hat{f}_{\mu\nu}$ respectively. It is important that the action has the same structure as the non-linear massive gravity action with general reference metric $\hat{f}_{\mu\nu}$ with additional kinetic term for $\hat{f}_{\mu\nu}$. However the fact that $\hat{f}_{\mu\nu}$ is dynamical has crucial impact on the Hamiltonian structure of given theory. Explicitly, the Hamiltonian is given as the linear combination of the constraints as opposite to the case of the non-linear massive gravity where the Hamiltonian does not vanish on the constraint surface. Now the crucial point is that the components of the metric $\hat{f}_{\mu\nu}$ that appear as the fixed parameters in the non-linear massive gravity case should be now considered as Lagrange multipliers whose values are determined by the requirements of the preservation of all constraints during the time evolution of the system. However then we find that the requirement of the preservation of the constraint $C_0$ whose explicit definition will be given below leads to the differential equation for the Lagrangian multiplier $M$ that is related to the $D$. In other words the value of the Lagrange multiplier $M$ is determined by the requirement of the preservation of $C_0$ during the time evolution of the system. In the same way we fix the value of the Lagrange multiplier $N$. Say differently, $C_0$ and $D$ could be interpreted as the second class constraints.

At this place we should compare our result with the known proof of the absence of the ghosts in the bimetric theory. It was shown in the very nice paper [13] that the requirement of the preservation of the constraint $C_0$ implies an additional constraint $C_{(2)}$ (in their notation) given in the e.q. (3.32) in this paper. We see that this constraint contains the covariant derivative of $M$ which, as we argued above, is fixed in case of the non-linear massive gravity theory so that it is really natural to interpret $C_{(2)}$ as an additional constraint. Then this constraint together with $C_0$ are responsible for the elimination of the ghost mode which is crucial for the consistency of non-linear massive gravity. However in case of the bimetric gravity $M$ should be considered as the Lagrange multiplier whose value is fixed by the consistency of given theory.

The fact that $C_0$ and $D$ should be interpreted as the second class constraints has important consequence for the dynamics of the theory. More precisely, since the Hamiltonian is given as the linear combination of the constraints implies that the resulting Hamiltonian vanishes strongly. This is rather puzzling result and we believe that this is a consequence of the redefinition of the shift function [13, 14, 15] which is certainly useful for the non-linear massive gravity where the diffeomorphism invariance is explicitly broken but we are not sure whether it is suitable for the bimetric theory of gravity where all fields are dynamical and the theory is manifestly diffeomorphism invariant under diagonal diffeomorphism. In fact, it is non-trivial task to identify such generator as was shown recently in [19] in case of particular model of bimetric theory of gravity [35]. We are currently analyzing NBTG following [19] and we believe that it is possible to identify four first class constraints that are generators of the diagonal diffeomorphism. On the other hand the analysis performed so far suggests that it is very difficult or even impossible to find an additional constraint that could eliminate the additional mode $^2$.

$^2$This analysis will appear in forthcoming publication.
We should however stress that we have to be very careful with definitive conclusions. We wanted to show that the extension of the non-linear massive gravity to the bimetric theory of gravity as was performed in [15] could be more subtle than we initially thought and that it is not really clear whether given theory is ghost free or not. It is still possible that there exists the way how to find four first class constraints that are generators of diagonal diffeomorphism together with additional constraints that eliminate ghost mode. Clearly more work is needed in order to resolve this issue.

This paper is organized as follows. In the next section (2) we review the Hamiltonian formulation of the bimetric theory of gravity [15] and identify primary and the secondary constraints. Then in section (3) we calculate the algebra of constraints for the case of the minimal version of the bimetric gravity. Finally in Appendix A we briefly discuss the Hamiltonian formulation of the bimetric $F(R)$ theory of gravity.

### 2. Bimetric Gravity

In this section we review the main properties of the bimetric theory of gravity in the formulation presented in [15]. The starting point is following action

$$
S = M_g^2 \int d^4 x \sqrt{-\hat{g}} R(\hat{g}) + M_f^2 \int d^4 x \sqrt{-\hat{f}} R(\hat{f}) + 2m^2M_{\text{eff}}^2 \int d^4 x \sqrt{-\hat{g}} \sum_{n=0}^{4} \beta_n e_n(\sqrt{-\hat{g} \hat{f}}),
$$

(2.1)

where

$$
M_{\text{eff}}^2 = \left( \frac{1}{M_g^2} + \frac{1}{M_f^2} \right)^{-1},
$$

(2.2)

and where $\hat{g}_{\mu\nu}, \hat{f}_{\mu\nu}$ are four-dimensional metric components with $R(\hat{g}), R(\hat{f})$ corresponding scalar curvatures. Further, $e_k(A)$ are elementary symmetric polynomials of the eigenvalues of $A$. For generic $4 \times 4$ matrix they are given by

$$
\begin{align*}
    e_0(A) &= 1, \\
    e_1(A) &= |A|, \\
    e_2(A) &= \frac{1}{2}(|A|^2 - |A|^2), \\
    e_3(A) &= \frac{1}{6}(|A|^3 - 3|A||A|^2 + 2|A|^3), \\
    e_4(A) &= \frac{1}{24}(|A|^4 - 4|A||A|^2|A|^2 + 3|A|^2|A|^2 + 8|A||A|^3 - 6|A|^4), \\
    e_k(A) &= 0, \text{ for } k > 4,
\end{align*}
$$

(2.3)

where $A^{\mu}_{\nu}$ is $4 \times 4$ matrix and where

$$
|A| = \text{Tr}A = A^\mu_{\mu}.
$$

(2.4)
Of the four \( \beta_n \) two combinations are related to the mass and the cosmological constant while the remaining two combinations are free parameters. If we consider the case when the cosmological constant is zero and the parameter \( m \) is mass, the four \( \beta_n \) are parameterized in terms of the \( \alpha_3 \) and \( \alpha_4 \) of [1, 2]

\[
\beta_n = (-1)^n \left( \frac{1}{2} (4-n)(3-n) - (4-n)\alpha_3 + \alpha_4 \right).
\] (2.5)

The minimal action corresponds to \( \beta_2 = \beta_3 = 0 \) that implies \( \alpha_3 = \alpha_4 = 1 \) and consequently \( \beta_0 = 3 \), \( \beta_1 = -1 \).

Our goal is to find the Hamiltonian formulation of given theory and determine corresponding primary and the secondary constraints. As the first step we introduce 3 + 1 decomposition of both \( \hat{g}_{\mu\nu} \) and \( \hat{f}_{\mu\nu} \) [30, 31]

\begin{align}
\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j, \quad \hat{g}_{0i} = N_i, \quad \hat{g}_{ij} = g_{ij}, \\
\hat{g}^{00} &= -\frac{1}{N^2}, \quad \hat{g}^{0i} = \frac{N^i}{N^2}, \quad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2}.
\end{align}

(2.6)

and

\begin{align}
\hat{f}_{00} &= -M^2 + L_i f^{ij} L_j, \quad \hat{f}_{0i} = L_i, \quad \hat{f}_{ij} = f_{ij}, \\
\hat{f}^{00} &= -\frac{1}{M^2}, \quad \hat{f}^{0i} = \frac{L^i}{M^2}, \quad \hat{f}^{ij} = f^{ij} - \frac{L^i L^j}{M^2}, \quad L^i = L_j f^{ji},
\end{align}

(2.7)

and where we defined \( g^{ij} \) and \( f^{ij} \) as the inverse to \( g_{ij} \) and \( f_{ij} \) respectively

\[
g_{ik} g^{kj} = \delta_i^j, \quad f_{ik} f^{kj} = \delta_i^j.
\] (2.8)

Following [3, 4, 5, 15] we perform following redefinition of the shift function

\[
N^i = M \tilde{n}^i + L^i + N \tilde{D}^i_j \tilde{n}^j
\] (2.9)

so that the resulting action is linear in \( M \) and \( N \). Note that the matrix \( \tilde{D}^i_j \) obeys the equation [3, 4, 5, 15]

\[
\sqrt{x} \tilde{D}^i_j = \sqrt{(g^{ik} - \tilde{D}^i_m \tilde{n}^m \tilde{D}^k_n \tilde{n}^n)} f_{kj}
\] (2.10)

and also following important property

\[
f_{ik} \tilde{D}^k_j = f_{jk} \tilde{D}^i_j.
\] (2.11)

Then after some calculations we derive the bimetric gravity action in the form [3, 4, 5, 15]

\[
S = M_f^2 \int dt d^3x \sqrt{f} [\bar{K}_{ij} \bar{G}^{ijkl} \bar{K}_{kl} + R(f)] + M_g^2 \int dt d^3x N \sqrt{g} [K_{ij} G^{ijkl} K_{kl} + R(g)] + 2m^2 M_{eff}^2 \int dt d^3x \sqrt{g} (MU + NV),
\] (2.12)
where

\[ K_{ij} = \frac{1}{2N} (\partial_t g_{ij} - \nabla_i N_j (\tilde{n}, g) - \nabla_j N_i (\tilde{n}, g)) , \]
\[ \tilde{K}_{ij} = \frac{1}{2M} (\partial_t f_{ij} - \tilde{\nabla}_i L_j - \tilde{\nabla}_j L_i) , \]

(2.13)

where

\[ N_i = Mg_{ij} \tilde{n}^j + g_{ij} L^j + N g_{ik} \tilde{D}_j^k \tilde{n}^j , \quad L_i = f_{ij} L^j , \]

(2.14)

and where \( \nabla_i, R^{(g)} \) and \( \tilde{\nabla}_i, R^{(f)} \) are the covariant derivatives and scalar curvatures calculated using \( g_{ij} \) and \( f_{ij} \) respectively. Further, \( G^{ijkl} \) and \( \tilde{G}^{ijkl} \) are de Witt metrics defined as

\[ G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl} , \quad \tilde{G}^{ijkl} = \frac{1}{2} (f^{ik} f^{jl} + f^{il} f^{jk}) - f^{ij} f^{kl} \]

(2.15)

with inverse

\[ G_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{1}{2} g_{ij} g_{kl} , \quad \tilde{G}_{ijkl} = \frac{1}{2} (f_{ik} f_{jl} + f_{il} f_{jk}) - \frac{1}{2} f_{ij} f_{kl} \]

(2.16)

that obey the relation

\[ G_{ijkl} G^{klmn} = \frac{1}{2} (\delta^m_i \delta^n_j + \delta^m_j \delta^n_i) , \quad \tilde{G}_{ijkl} \tilde{G}^{klmn} = \frac{1}{2} (\delta^m_i \delta^n_j + \delta^m_j \delta^n_i) . \]

(2.17)

Finally, \( \mathcal{V} \) and \( \mathcal{U} \) introduced in (2.12) have the form

\[ \mathcal{V} = \beta_0 + \beta_1 \sqrt{x} \tilde{D}^i_i + \beta_2 \frac{1}{2} \sqrt{x} (\tilde{D}^i_i \tilde{D}^j_j - \tilde{D}^i_j \tilde{D}^j_i) + \]
\[ + \frac{1}{6} \beta_3 \sqrt{x} \left[ (\tilde{D}_i^i \tilde{D}_j^j \tilde{D}_k^k - 3 \tilde{D}_i^i \tilde{D}_j^j \tilde{D}_k^k + 2 \tilde{D}_i^j \tilde{D}_j^i \tilde{D}_k^k) \right] , \]
\[ \mathcal{U} = \beta_1 \sqrt{x} + \beta_2 [\sqrt{x} \tilde{D}^i_i + \tilde{n}^i f_{ij} \tilde{D}_k^k \tilde{n}^j] + \]
\[ + \beta_3 [\sqrt{x} (\tilde{D}_i^i \tilde{n}^j f_{ij} \tilde{D}_k^k \tilde{n}^k - \tilde{D}_i^i \tilde{n}^k f_{ij} \tilde{D}_k^j \tilde{n}^i) + \frac{1}{2} \sqrt{x} (\tilde{D}^i_i \tilde{D}^j_j - \tilde{D}^i_j \tilde{D}^j_i)] + \beta_4 \sqrt{\frac{x}{y}} , \]

(2.18)

where

\[ \tilde{x} = 1 - \tilde{n}^i f_{ij} \tilde{n}^j . \]

(2.19)

The action (2.12) is suitable for the Hamiltonian formalism. First we find the momenta conjugate to \( N, \tilde{n}^i \) and \( g_{ij} \)

\[ \pi_N \approx 0 , \quad \pi_i \approx 0 , \quad \pi^{ij} = M_g^2 \sqrt{g} G^{ijkl} K_{kl} \]

(2.20)

together with the momenta conjugate to \( M, L^i \) and \( f_{ij} \)

\[ \rho_M \approx 0 , \quad \rho_i \approx 0 , \quad \rho^{ij} = M_f^2 \sqrt{f} \tilde{G}^{ijkl} \tilde{K}_{kl} . \]

(2.21)
Then after some calculations we find following Hamiltonian

$$H = \int d^3x (\pi^{ij}\partial_t g_{ij} + \rho^{ij}\partial_t f_{ij} - \mathcal{L}) = \int d^3x (N\mathcal{C}_0 + MD + L^i\mathcal{R}_i),$$

where

$$\mathcal{C}_0 = \frac{1}{M^2} \pi^{ij} \tilde{G}_{ijkl}(\pi^{kl} - M^2 \sqrt{g} R^{(g)} + \mathcal{R}^{(g)}_k \tilde{\mathcal{D}}^k \tilde{n}^l - 2m^2 M^{\text{eff}} \sqrt{g} V),$$

$$\mathcal{D} = \frac{1}{M^2} \rho^{ij} \tilde{G}_{ijkl}(\rho^{kl} - M^2 \sqrt{f} R^{(f)} + \tilde{n}^i \mathcal{R}^{(g)}_i - 2m^2 M^{\text{eff}} \sqrt{f} \mathcal{M}),$$

$$\mathcal{R}_i = \mathcal{R}^{(g)}_i + \mathcal{R}^{(f)}_i,$$

(2.22)

where we also denoted

$$\mathcal{R}^{(g)}_i = -2g_{ik} \nabla l \pi^{lk}, \quad \mathcal{R}^{(f)}_i = -2f_{ik} \tilde{\nabla} l \rho^{lk}.$$

(2.23)

From previous analysis we see that we have eight primary constraints

$$\pi_N \approx 0, \quad \pi_i \approx 0, \quad \rho_M \approx 0, \quad \rho_i \approx 0.$$

(2.24)

Then the next step is to analyze the requirement that these constraints are preserved during the time evolution of the system

$$\partial_t \pi_N = \{\pi_N, H\} = -\mathcal{C}_0 \approx 0,$$

$$\partial_t \rho_M = \{\rho_M, H\} = -\mathcal{D} \approx 0,$$

$$\partial_t \pi_i = \{\pi_i, H\} = \mathcal{C}_k \left( M \delta^k_i + N \frac{\delta(\tilde{D}^k_i \tilde{n}^l)}{\delta \tilde{n}^l} \right) \approx 0,$$

$$\partial_t \rho_i = \{\rho_i, H\} = -\mathcal{R}_i \approx 0,$$

(2.25)

(2.26)

where

$$\mathcal{C}_i = \mathcal{R}^{(g)}_i + 2m^2 M^{\text{eff}} \sqrt{g} \tilde{\nabla} l \rho^{pm} \left[ \beta_1 \delta^m_i + \beta_2 [\delta^m_i \tilde{D}_i - \tilde{D}^m] + \beta_3 \sqrt{x} \left( \frac{1}{2} \delta^m_i (\tilde{D}_m \tilde{D}_p - \tilde{D}^m \tilde{D}^p) + \tilde{D}^m \tilde{D}_i - \tilde{D}^m \tilde{D}_i n + \tilde{D}^m \tilde{D}_i \right),

(2.27)

and where we used the canonical Poisson brackets

$$\{N(x), \pi_N(y)\} = \delta(x - y), \quad \{\tilde{n}^i(x), \pi_j(y)\} = \delta^i_j \delta(x - y),$$

$$\{g_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2} (\delta^i_k \delta^j_l + \delta^i_l \delta^j_k) \delta(x - y),$$

$$\{M(x), \rho_M(y)\} = \delta(x - y), \quad \{L^i(x), \rho_j(y)\} = \delta^i_j \delta(x - y),$$

$$\{f_{ij}(x), \rho^{kl}(y)\} = \frac{1}{2} (\delta^i_k \delta^j_l + \delta^i_l \delta^j_k) \delta(x - y)$$

(2.28)
and also following important relations \[13\]

\[
\frac{\delta \sqrt{x} \bar{D}^i}{\delta \bar{n}^i} = -\frac{1}{\sqrt{x}} \bar{n}^n f_{mn} \frac{\delta (\bar{D}^m p \bar{n}^p)}{\delta \bar{n}^i},
\]

\[
\frac{\partial}{\partial \bar{n}^i} \text{Tr}(\sqrt{x} \bar{D})^2 = -2 \bar{n}^p f_{pm} \bar{D}^m_k \frac{\delta (\bar{D}^k p \bar{n}^n)}{\delta \bar{n}^i},
\]

\[
\frac{\delta}{\delta \bar{n}^i} \text{Tr}(\sqrt{x} \bar{D})^3 = -3 \sqrt{x} \bar{n}^k f_{km} \bar{D}^m_k \bar{D}^n_p \frac{\delta (\bar{D}^p n \bar{n}^n)}{\delta \bar{n}^i}.
\]

(2.29)

In summary we have following 16 constraints

primary : \( \pi_N \approx 0 \), \( \pi_i \approx 0 \), \( \rho_M \approx 0 \), \( \rho_i \approx 0 \),

secondary : \( C_0 \approx 0 \), \( D \approx 0 \), \( C_i \approx 0 \), \( R_i \approx 0 \).

(2.30)

Now we have to check the stability of all constraints when the total Hamiltonian takes the form

\[
H_T = \int d^3x (NC_0 + MD + L^i R_i + u^N \pi_N + u^i \pi_i + v^M \rho_M + v^i \rho_i + \Sigma^i C_i),
\]

(2.31)

where \( N, M, L^i, u^N, u^i, v^M, v^i, \Sigma^i \) are Lagrange multipliers related to the constraints \(2.30\). For simplicity we restrict ourselves to the case of the minimal bi-metric theory.

3. Preservation of Constraints in Case of Minimal Bimetric Gravity

The minimal bimetric theory is defined by the following choice of parameters

\[
\beta_0 = 3 \ , \beta_1 = -1 \ , \beta_2 = 0 \ , \beta_3 = 0 \ , \beta_4 = 1 \ .
\]

(3.1)

Now we proceed to the analysis of the preservation of all constraints given in \(2.30\). It is easy to see that the constraint \( \pi_N \approx 0 \) is trivial preserved. On the other hand the requirement of the preservation of the constraint \( \pi_i \approx 0 \) takes the form

\[
\partial_t \pi_i = \{ \pi_i, H_T \} = -\left( M \delta^k_i + \frac{\partial (\bar{D}^k j \bar{n}^j)}{\partial \bar{n}^i} \right) C_k + \int d^3x \Sigma^j \{ \pi_i, C_j(x) \} = 0 ,
\]

(3.2)

where

\[
\{ \pi_i(x), C_j(y) \} = \left[ \frac{1}{\sqrt{x}} f_{ij} + \frac{\bar{n}^k f_{ki} \bar{n}^l f_{lj}}{\sqrt{x}} \right] \delta(x - y) \equiv \Delta_{\pi_i, C_j}(x - y) .
\]

(3.3)
Since
\[ \det \Delta_{\pi,C_j} = \det \left( \frac{f_{ik}}{\sqrt{x}} \right) \det \left( \delta^k_j + \frac{1}{x} \tilde{n}^k \tilde{n}^m f_{mj} \right) = \frac{1}{x^{5/2}} \det f_{ij} \neq 0 \]
we find that \( \Delta_{\pi,C_j} \) is non-singular matrix on the whole phase space. However this fact also implies that the equation \( \Box \) has trivial solution
\[ \Sigma^i = 0. \]

In the same we proceed with the analysis of the time evolution of the constraint \( C_i \)

\[ \partial_t C_i(x) = \{ C_i(x), H_T \} = \int d^3 y \ (N(y) \{ C_i(x), C_0(y) \} + M(y) \{ C_i(x), D(y) \} + v^i(y) \{ C_i(x), \pi_j(y) \}) = 0. \]

According to \( \Box \) we see that \( \Box \) can be solved for \( v^i \) as functions of the canonical variables and \( N, M \). Say differently, \( \pi_i \) and \( C_i \) are the second class constraints.

As the next step we consider the constraint \( R_i \). It turns out that is convenient to extend it by the expression \( \partial_i \tilde{n}^j \pi_j + \partial_j (\tilde{n}^j \pi_i) \) and consider its smeared form
\[ T_S(N^i) = \int d^3 x N^i(R_i^{(g)} + R_i^{(f)}) + p_g \partial_i \phi + \partial_i \tilde{n}^j \pi_j + \partial_j (\tilde{n}^j \pi_i)) \equiv \int d^3 x N^i \tilde{R}_i. \]

Then using the canonical Poisson brackets we find
\[ \{ T_S(N^i), g_{ij}(x) \} = -\partial_k g_{ij}(x) N^k(x) - \partial_i N^k(x) g_{kj}(x) - g_{ik}(x) \partial_j N^k(x), \]
\[ \{ T_S(N^i), \pi^j(x) \} = -\partial_k (N^k(x) \pi^j(x)) + \partial_k N^i(x) \pi^k(x) + \pi^k(x) \partial_k N^j(x), \]
\[ \{ T_S(N^i), f_{ij}(x) \} = -\partial_k f_{ij}(x) N^k(x) - \partial_i N^k(x) f_{kj}(x) - f_{ik}(x) \partial_j N^k(x), \]
\[ \{ T_S(N^i), \rho^j(x) \} = -\partial_k (N^k(x) \rho^j(x)) + \partial_k N^i(x) \rho^k(x) + \rho^k(x) \partial_k N^j(x), \]
\[ \{ T_S(N^i), \tilde{n}^i(x) \} = -N^k(x) \partial_k \tilde{n}^i(x) + \partial_j N^i(x) \tilde{n}^j(x), \]
\[ \{ T_S(N^i), \pi_i(x) \} = -\partial_k (N^k(x) \pi_i(x)) - \partial_i N^k(x) \pi_k(x). \]

Then we easily find the familiar result
\[ \{ T_S(N^i), T_S(M^j) \} = T_S((N^j \partial_j M^i - M^j \partial_j N^i)). \]

To proceed further we need to know the Poisson bracket between \( T_S(N^i) \) and \( \tilde{D}_j^i \) which can be determined when we know the explicit form of \( \tilde{D}_j^i \).

\[ \tilde{D}_j^i = \sqrt{g^{ik} f_{km} Q^m_n (Q^{-1})^n_j}, \]
where
\[ Q^i_j = \tilde{x} \delta^i_j + \tilde{\eta}^i \tilde{\eta}^k f_{kj}, (Q^{-1})^j_k = \frac{1}{\tilde{x}} (\delta^j_k - \tilde{\eta}^i \tilde{\eta}^m f_{mk}) . \] (3.11)

Using the explicit form of \( \tilde{D}^i_j \) given in (3.10) we see that the Poisson bracket between \( T_S(N^i) \) and \( \tilde{D}^i_j \) is determined by the Poisson brackets between \( T_S(N^i) \) and \( g_{ij}, f_{ij} \) and \( Q^i_j \). The Poisson brackets between \( T_S(N^i) \) and \( g_{ij} \) and \( f_{ij} \) were given in (3.8) and the Poisson bracket between \( T_S(N^i) \) and \( Q^i_j \) can be easily determined using (3.11) and (3.11)

\[ \{ T_S(N^i), Q^i_j \} = -\partial_k Q^i_j N^k + \partial_k N^i \tilde{\eta}^k f_{mj} - \tilde{\eta}^i \tilde{\eta}^m f_{mk} \partial_j N^k = -\partial_k Q^i_j N^k + \partial_k N^i Q^j_k - Q^i_k \partial_j N^k . \] (3.12)

Then with the help of this result we find

\[ \{ T_S(N^i), \tilde{D}^i_j \} = -\partial_k \tilde{D}^i_j N^k + \partial_k N^i \tilde{D}^k_j - \tilde{D}^i_k \partial_j N^k \] (3.13)

and finally collecting all these results we obtain

\[ \{ T_S(N^i), C_0 \} = -\partial_i C_0 N^i - \partial_i N^i C_0 , \]
\[ \{ T_S(N^i), D \} = -\partial_i DN^i - \partial_i N^i D , \]
\[ \{ T_S(N^i), C_i \} = -\partial_j N^i C_i - N^i \partial_j C_i - \partial_i N^j C_j . \] (3.14)

Then it is easy to see that \( T_S(N^i) \) is preserved during the time evolution of the system and that it corresponds to the generator of the spatial diffeomorphism. In other words \( \tilde{R}_i \) are first class constraints.

Now we come to the calculation of the Poisson brackets between the constraints \( C_0 \) and \( D \). It turns out that it is useful to introduce the smeared form of these constraints

\[ C(N) = \int d^3 x N(x) C_0(x) , \quad D(M) = \int d^3 x M(x) D(x) . \] (3.15)

We begin with the Poisson bracket between \( C_0(x) \) and \( C_0(y) \). Since \( C_0 \) does not depend on \( \rho^i \) we immediately find that the Poisson bracket between \( C_0(x) \) and \( C_0(y) \) has the same form as in (3.13) which means that it vanishes on the constraint surface

\[ \{ C_0(x), C_0(y) \} \approx 0 . \] (3.16)

In case of \( D \) we find

\[ \{ D(M), D(N) \} = \int d^3 x (\partial_i M N - \partial_i N M)[f^{ij}(R^{(f)}_j + R^{(g)}_j) + (\tilde{\eta}^i \tilde{\eta}^j - f^{ij}) R^{(g)}_j + \tilde{\eta}^i 2m^2 M_{eff}^2 \sqrt{\tilde{g}} \sqrt{\tilde{x}} ] = \int d^3 x (\partial_i M N - \partial_i N M)[f^{ij} R_j + (\tilde{\eta}^i \tilde{\eta}^j - f^{ij}) C_j] . \] (3.17)
We see that the right side vanishes on the constraint surface. Finally we come to the calculation of the Poisson bracket between $C(M)$ and $D(N)$

$$
\{C(N), D(M)\} = - \int d^3 x M \tilde{n}^i \partial_i N C_0 + \int d^3 x N M \left( \frac{4m^2 M_{\epsilon f f}^2}{M_\gamma} \tilde{n}^{ij} G_{ijkl} U^{kl} + \right.
$$

$$
\left. \tilde{D}_m \tilde{n}^m \partial_j \tilde{n}^i \mathcal{R}(g) - \tilde{n}^i \partial_j (\tilde{D}_m \tilde{n}^m \mathcal{R}(g)) + 2 \mathcal{R}(g) \frac{\delta (\tilde{D}_m \tilde{n}^m)}{\delta f_{ij}} \frac{1}{M_f^2} G_{ijkl} \rho^{kl} \right)
$$

$$
+ 2m^2 M_{\epsilon f f}^2 \tilde{n}^i \partial_i V + 4m^2 M_{\epsilon f f}^2 \sqrt{g} \tilde{V}_{mn} \tilde{g}_{mnkl} \rho^{kl} \right)
$$

$$
+ \int d^3 x [N \tilde{D}_m^i \tilde{n}^m \tilde{n}^i \mathcal{R}(g) \partial_j M - M \tilde{n}^j \partial_j N \tilde{D}_m^i \tilde{n}^m \mathcal{R}(g)] - 4m^2 M_{\epsilon f f}^2 \int d^3 x [NV^{kl} \nabla_l (M \tilde{n}^i) g_{ik} - \nabla_p (N \tilde{D}_m \tilde{n}^i) g_{km} U^{mp} M],
$$

(3.18)

where

$$
U^{kl} = \frac{\delta (\sqrt{g} kl)}{\delta g^{kl}}, \tilde{V}_{mn} = \frac{\delta V}{\delta f^{mn}}, V^{kl} = \sqrt{g} \frac{\delta V}{\delta g^{kl}},
$$

(3.19)

Let us analyze the Poisson bracket calculated above in more details. First of all we see that the first expression vanishes on the constraint surface $C_0 \approx 0$ which is desired result. On the other hand in order to analyze the time evolution of the local constraint $C_0$ it is useful to express the local form of the Poisson bracket from (3.18) that can be schematically written as

$$
\{C(N), D(M)\} = \int d^3 z (N(z) M(z) F(z) + \partial_z N(z) V^i(z) M(z) + N(z) \partial_z M(z) W^i(z)),
$$

(3.20)

where the explicit form of $F, V^i, W^i$ follow from (3.18). Let us now write

$$
N(z) = \int d^3 x N(x) \delta (x - z), M(z) = \int d^3 y M(y) \delta (y - z)
$$

(3.21)

and insert it to the right side of the Poisson bracket (3.20). Then after some calculation we find that it is equal to

$$
\int d^3 x d^3 y N(x) M(y) [\delta (x - y) F(x) + \frac{\partial}{\partial y^i} \delta (x - y) V^i(y) + \frac{\partial}{\partial x^i} \delta (x - y) W^i(x)].
$$

(3.22)

On the other hand we have

$$
\{C(N), D(M)\} = \int d^3 x d^3 y N(x) M(y) \{C(x), D(y)\}.
$$

(3.23)
Since (3.22) and (3.23) have to match for any \(N(x), M(y)\) we obtain
\[
\{\mathcal{C}(x), \mathcal{D}(y)\} = \delta(x - y)F(x) + \frac{\partial}{\partial y^i}\delta(x - y)\mathbf{V}^i(y) + \frac{\partial}{\partial x^i}\delta(x - y)W^i(x).
\tag{3.24}
\]

Now using this expression we can easily determine the requirement of the preservation of the constraint \(\mathcal{C}_0\) during the time evolution of the system
\[
\partial_t \mathcal{C}_0 = \{\mathcal{C}_0(x), H_T\} \approx \int d^3y M(y) \{\mathcal{C}(x), \mathcal{D}(y)\} = \\
= M(x)[F(x) - \partial_i \mathbf{V}^i(x)] + \frac{\partial M(x)}{\partial x^i}[W^i(x) - \mathbf{V}^i(x)] = 0
\tag{3.25}
\]

This is the most crucial point of our calculation that deserves careful explanation. Let us imagine that we have \(\mathbf{V}^i = \mathbf{W}^i\). Then (3.25) has solution either \(M(x) = 0\) or \(F(x) - \partial_i \mathbf{V}^i(x) = 0\). In fact, the first case occurs when the expression \(F(x) - \partial_i \mathbf{V}^i(x)\) is non-zero on the whole phase space, as for example in case when this expression is constant. On the other hand when \(F(x) - \partial_i \mathbf{V}^i(x)\) depends on the phase space variables it is more natural to impose the condition \(\mathcal{C}_0^{(II)} \equiv F(x) - \partial_i \mathbf{V}^i(x) = 0\) as an additional constraint. This would be the desired result since now we would have two second class constraints \(\mathcal{C}_0, \mathcal{C}_0^{(II)}\) that would be sufficient for elimination of the ghost mode. Unfortunately as we can see from (3.18) \(\mathbf{V}^i \neq \mathbf{W}^i\) and the situation is completely different since the equation (3.25) cannot leave \(M\) undetermined. Rather we should interpret (3.25) as equation that can be solved for \(M\) as function of the phase space variables. In fact, in the same way we can analyze the requirement of the preservation of the constraint \(\mathcal{D}\) that again leads to the differential equation that can be solved for \(N\). In other words we mean that it is now natural to interpret \(\mathcal{C}_0\) together with \(\mathcal{D}\) as the second class constraints. Certainly this is very strange result. In particular, now we find that the total Hamiltonian strongly vanishes up the diffeomorphism constraint. Of course, we know that this cannot be right since the theory possesses the overall diffeomorphism invariance and hence there should be four the first call constraints that are generators of this diffeomorphism. The way how to find such generators for bimetric theory of gravity was suggested in [19] at least for particular bimetric gravity model. The extension of this work to the case of the non-linear bimetric gravity is currently under consideration. Then the result derived in this section suggests that the redefinition of the shift function which is very useful in the case of the non-linear massive gravity may not be the right way in the case of the bimetric theory of gravity.

Despite of the fact that the total Hamiltonian vanishes it is instructive to count the number of the physical degrees of freedom. Recall that phase space variables are \(N, \pi_N, \tilde{\pi}^i, \pi_i, M, \rho_M, L^i, \rho_i, f_{ij}, \rho^{ij}, g_{ij}, \pi^{ij}\) so that the total number of the phase space degrees of freedom is \(N_{p.s.d.f.} = 40\). On the other hand we have \(N_{f.c.} = 8\) first class constraints \(\pi_N \approx 0, \rho_M \approx 0, \rho_i \approx 0, \tilde{\mathcal{R}}_i \approx 0\). Finally we have \(N_{s.c.} = 8\) second class constraints \(\mathcal{C}_0 \approx 0, \mathcal{D} \approx 0, \pi_i \approx 0, \mathcal{C}_i \approx 0\). Then the number of the physical degrees of freedom is equal to
\[
N_{f.d.f.} = N_{p.s.d.f.} - 2N_{f.c.} - N_{s.c.} = 16.
\tag{3.26}
\]
At linearized level we can identify four degrees of freedom corresponding to the massless graviton, ten degrees of freedom corresponding to the massive graviton and two additional degrees of freedom corresponding to the ghost mode. It is important to stress that the same result can be found when we identify four first class constraints corresponding to the diagonal diffeomorphism and also additional eight second class constraints as in case of the bimetric gravity model analyzed in [19]. Of course, the square root structure of the potential has remarkable property in case of the non-linear massive gravity and maybe it could be useful in case of the bimetric gravity as well. We only say that the step from the non-linear massive gravity to the bimetric gravity is not straightforward as it seems to be.

A. Hamiltonian Analysis of $F(R)$ Bimetric Gravity

In this appendix we briefly perform the Hamiltonian formulation of $F(R)$ bimetric theory of gravity which was introduced by S.Odintsov and Nojiri in [22].

The starting point is the action for the non-linear bimetric gravity theory

$$S = M_g^2 \int d^4x \sqrt{-g} R(g) + M_f^2 \int d^4x \sqrt{-f} R(f) + 2m^2M_{eff}^2 \int d^4x \sqrt{-\hat{g}} \sum_{n=0}^4 \beta_n e_n(\sqrt{-\hat{g}}\hat{f}) \,.$$  \hspace{1cm} (A.1)

Then in order to construct the $F(R)$ analogue of the bimetric massive gravity we add following expression to the action (A.1)

$$S_1 = -M_g^2 \int d^4x \sqrt{-g} \left( \frac{3}{2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right) \,.$$  \hspace{1cm} (A.2)

Then with the help of the Weyl transformation

$$\hat{g}^\prime_{\mu\nu} = e^\phi \hat{g}_{\mu\nu} \,,$$

$$R[\hat{g}] = e^{\phi}(R[\hat{g}]) - \frac{3}{2} \hat{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + 3\hat{g}^{\mu\nu} \nabla_\mu \nabla_\nu \phi)$$  \hspace{1cm} (A.3)

we find that $S_{tot} = S_{bi} + S_1$ takes the form

$$S_{FR} = M_f^2 \int d^4x \sqrt{-f} R(f) + M_g^2 \int d^4x \sqrt{-g} [e^{-\phi} R[\hat{g}'] - e^{-2\phi}V(\phi)] + 2m^2M_{eff}^2 \int d^4x \sqrt{-\hat{g}} \sum_{n=0}^4 \beta_n e_{n}(\frac{3}{2}-2)\phi e_n(\sqrt{-\hat{g}'\hat{f}})$$  \hspace{1cm} (A.4)

using

$$e_n(\sqrt{-\hat{g}'\hat{f}}) = e^{\phi/2} e_n(\sqrt{-\hat{g}'\hat{f}}) \,.$$  \hspace{1cm} (A.5)
In what follows we will consider (A.4) as the definition of the F(R) bimetric theory of gravity. Of course we should be able to solve the equation of motion for \( \phi \) at least in principle so that we could express \( \phi \) as the function of \( g', \hat{f}' \). Then inserting back to the action (A.4) we obtain the action that is non-linear function of \( R[g'] \) and hence has the form of the \( F(R) \) theory of gravity \(^3\). For that reason we can name (A.4) as bimetric \( F(R) \) theory of gravity even if its definition using the scalar field is more natural. Finally, in the following we omit \( \hat{'} \) over \( \hat{g}_{\mu\nu}, \hat{f}_{\mu\nu} \).

To proceed further we perform the redefinition of the shift \( N^i \) as in section (2) so that we find the \( F(R) \) bigravity action in the form

\[
S_{FR} = M_f^2 \int dt d^3x [M \sqrt{f} \hat{K}_{ij} \hat{G}^{ijkl} \hat{K}_{kl} + \sqrt{f} M R_f] +
+ M_g^2 \int dt d^3x \sqrt{g} N [e^{-\phi} K_{ij} G^{ijkl} K_{kl} + e^{-\phi} R(g) - e^{-2\phi} V(\phi)] +
+ 2m^2 M_{eff}^2 \int dt d^3x \sqrt{g} N (M \dot{U} + N V).
\]

(A.6)

Note that (A.6) has similar form as the action (2.12) up to presence of the additional scalar potential \( V(\phi) \) and powers of the factor \( e^\phi \). Explicitly, we have

\[
V = \beta_0 e^{-2\phi} + \beta_1 e^{-2\phi} \sqrt{x} \hat{D}_i \hat{D}_i + \beta_2 e^{-\phi} \frac{1}{2} \sqrt{x} \left( (\hat{D}_i \hat{D}_j - \hat{D}_j \hat{D}_i) + \frac{1}{6} \beta_3 e^{-\phi} \sqrt{x} \left[ \hat{D}_i \hat{D}_j \hat{D}_k \hat{D}_l - 3 \hat{D}_i \hat{D}_j \hat{D}_k \hat{D}_j + 2 \hat{D}_j \hat{D}_j \hat{D}_k \hat{D}_i \right] \right),
\]

\[
U = \beta_1 e^{-\phi} \sqrt{x} + \beta_3 e^{-\phi} \left[ \sqrt{x} (\hat{D}_i \hat{n}_i \hat{f}_{ij} \hat{D}_j \hat{n}_j - \hat{D}_i \hat{n}_j \hat{n}_i \hat{f}_{ij} \hat{D}_j \hat{n}_i) + \frac{1}{2} \sqrt{x} \left( (\hat{D}_i \hat{D}_j - \hat{D}_j \hat{D}_i) \right) \right] + \beta_4 \frac{\sqrt{\hat{f}}}{\sqrt{g}}.
\]

(A.7)

Now using the action (A.4) we can find the corresponding Hamiltonian. Firstly we find the momenta conjugate to \( N, \hat{n}_i \) and \( g_{ij} \)

\[
\pi_N \approx 0 , \quad \pi_i \approx 0 , \quad \pi^{ij} = M_g^2 \sqrt{g} e^{-\phi} G^{ijkl} K_{kl}
\]

(A.8)

together with the momenta conjugate to \( N, L_i \) and \( f_{ij} \)

\[
\rho_M \approx 0 , \quad \rho^i \approx 0 , \quad \rho^{ij} = M_f^2 \sqrt{f} \hat{G}^{ijkl} \hat{K}_{kl}.
\]

(A.9)

Since the action (A.4) does not contain the time derivative of \( \phi \) we find that the momentum conjugate to \( \phi \) is zero

\[
p_\phi \approx 0.
\]

(A.10)

As a result we find following Hamiltonian

\[
H = \int d^3x (\pi^{ij} \partial_t g_{ij} + \rho^{ij} \partial_t f_{ij} - \mathcal{L}) =
\]

\(^3\)For review, see [3, 5].
\[ \int d^3 x (NC_0 + MD + L^i R_i) , \]  
\text{(A.11)}

where
\[ C_0 = e^\phi \frac{1}{M_g^2} \pi^i G_{ijkl} \pi^{kl} - e^{-\phi} \sqrt{g} M_p^2 R^{(g)} + e^{-2\phi} \sqrt{g} M_p^2 V + \pi^{(g)} \tilde{D}_i \pi^i - 2m^2 M^2_{\text{eff}} \sqrt{g} \mathcal{V} , \]
\[ D = \frac{1}{\sqrt{f}} M_f^2 \pi^i G_{ijkl} \pi^{kl} - M_f^2 \sqrt{f} R^{(f)} + \pi^i R^{(g)} - 2m^2 M^2_{\text{eff}} \sqrt{g} \mathcal{U} , \]
\[ R = R^{(f)} + R^{(g)} . \]
\text{(A.12)}

Comparing with the situation in the second section we see that there is an additional primary constraint \( p_\phi \approx 0 \). Then again the requirement of the preservation of the primary constraints implies the secondary constraints that have the same form as in case of pure bimetric theory of gravity. There is however an additional constraint \( \mathcal{G} \) that follows from the requirement of the preservation of the constraint \( p_\phi \approx 0 \)
\[ \partial_t p_\phi = \{ p_\phi , H \} = N \left( \frac{e^\phi}{M_g^2} \pi^i G_{ijkl} \pi^{kl} + e^{-\phi} \sqrt{g} M_p^2 R^{(g)} - 2e^{-2\phi} \sqrt{g} M_p^2 \frac{dV}{d\phi} - 2m^2 M^2_{\text{eff}} \sqrt{g} \frac{\delta \mathcal{V}}{\delta \phi} \right) \equiv -N \mathcal{G} \approx 0 . \]
\text{(A.13)}

In summary we have following set of 18 constraints

primary: \( \pi_N \approx 0 , \pi_i \approx 0 , \rho_M \approx 0 , \rho_i \approx 0 , p_\phi \approx 0 \)
secondary: \( C_0 \approx 0 , D \approx 0 , C_i \approx 0 , R_i \approx 0 , \mathcal{G} \approx 0 \).
\text{(A.14)}

Note that now the constraint \( C_i \) has explicit form
\[ C_i = R_i^{(g)} + 2m^2 M^2_{\text{eff}} \sqrt{g} \times \]
\[ \times \left[ \beta_1 e^{-3/2} \delta^m_i + \beta_2 e^{-\phi} [\delta^m_i \tilde{D}_l - \tilde{D}^m_l] + \beta_3 e^{-\phi/2} \sqrt{x} (\frac{1}{2} \delta^m_l (\tilde{D}_n \tilde{D}_m - \tilde{D}^n_m) + \tilde{D}^m_i \tilde{D}_l - \tilde{D}^m_i \tilde{D}_l) \right] . \]
\text{(A.15)}

Now we should check the stability of all constraints when the total Hamiltonian takes the form
\[ H_T = \int d^3 x (NC_0 + MD + L^i R_i + u_\phi p_\phi + u^N \pi_N + u^i \pi_i + v^M \rho_M + v^i \rho_i + u_\phi^i \mathcal{G} + \Sigma^i C_i) . \]
\text{(A.16)}
It is easy to see that the Hamiltonian structure of $F(R)$ bimetric theory of gravity is almost the same as the structure of NBTG analyzed in previous two sections with small exception that there are two additional constraints $p_\phi \approx 0, \mathcal{G} \approx 0$. They are the second class constraints that vanish strongly and can be explicitly solved with respect to $p_\phi$ and $\phi$ at least in principle. On the other hand they do not affect the analysis of all remaining constraints so that the constraint structure of given theory is the same as in case of non-linear bimetric gravity. For that reason we will not repeat the calculations performed in section (3).

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