ABSTRACT. Weyl’s unitary matrices, which were introduced in Weyl’s 1927 paper [11] on group theory and quantum mechanics, are \( p \times p \) unitary matrices given by the diagonal matrix whose entries are the \( p \)-th roots of unity and the cyclic shift matrix. Weyl’s unitaries, which we denote by \( u \) and \( v \), satisfy \( u^p = v^p = 1_p \) (the \( p \times p \) identity matrix) and the commutation relation \( uv = \zeta vu \), where \( \zeta \) is a primitive \( p \)-th root of unity. We prove that Weyl’s unitary matrices are universal in the following sense: if \( u \) and \( v \) are any \( d \times d \) unitary matrices such that \( u^p = v^p = 1_d \) and \( uv = \zeta vu \), then there exists a unital completely positive linear map \( \phi : M_p(\mathbb{C}) \to M_d(\mathbb{C}) \) such that \( \phi(u) = u \) and \( \phi(v) = v \). We also show, moreover, that any two pairs of \( p \)-th order unitary matrices that satisfy the Weyl commutation relation are completely order equivalent.

When \( p = 2 \), the Weyl matrices are two of the three Pauli matrices from quantum mechanics. It was recently shown in [7] that \( g \)-tuples of Pauli-Weyl-Brauer unitaries are universal for all \( g \)-tuples of anticommuting selfadjoint unitary matrices; however, we show here that the analogous result fails for positive integers \( p > 2 \).

Finally, we show that the Weyl matrices are extremal in their matrix range, using recent ideas from noncommutative convexity theory.

1. INTRODUCTION

With respect to a positive integer \( p \geq 2 \) and a primitive \( p \)-th root of unity \( \zeta \), a pair of \( d \times d \) unitary matrices \( u \) and \( v \) satisfy the Weyl relations if

\[
(1) \quad u^p = v^p = 1_d \quad \text{(the } d \times d \text{ identity matrix)} \quad \text{and} \quad uv = \zeta vu.
\]

The relation \( uv = \zeta vu \) is referred to as a Weyl commutation relation. The most immediate example of unitary matrices satisfying the Weyl relations comes from H. Weyl’s 1927 paper on quantum mechanics and group theory [11, p. 32], in which \( d = p \) and \( u \) and \( v \) are the \( p \times p \) unitary matrices denoted herein by \( u \) and \( v \), respectively, and are defined by

\[
(2) \quad u = \begin{bmatrix}
1 & \zeta & \zeta^2 & \cdots & \zeta^{p-1} \\
\zeta & 1 & \zeta & \cdots & \zeta^{p-2} \\
\zeta^2 & \zeta & 1 & \cdots & \zeta^{p-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\zeta^{p-1} & \zeta^{p-2} & \zeta^{p-3} & \cdots & 1
\end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}.
\]

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In the case where \( p = 2 \), the Weyl unitaries \( u \) and \( v \) are two (namely, \( \sigma_Z \) and \( \sigma_X \)) of the three Pauli matrices:

\[
\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

For this reason, the Weyl unitaries are often viewed as a generalised form of the Pauli matrices. (In particular, Weyl’s paper \([11]\) introduces his \( p \times p \) unitaries after a fulsome discussion of the Pauli matrices.)

Weyl’s unitary matrices \( u \) and \( v \) in \([2]\) are not just examples of unitary matrices satisfying the relations \([1]\); they are, in fact, universal matrices for all such pairs of unitary matrices. That is, as we shall prove herein, if \( u \) and \( v \) are \( d \times d \) unitary matrices such that \( u^p = v^p = 1_d \), and \( uv = \zeta vu \), then there exists a unital completely positive linear map \( \Phi : M_p(\mathbb{C}) \to M_d(\mathbb{C}) \) such that \( \Phi(u) = u \) and \( \Phi(v) = v \), where \( M_n(\mathbb{C}) \) denotes the algebra of complex \( n \times n \) matrices.

Returning to the case \( p = 2 \), the Weyl matrices \( u \) and \( v \) are the Pauli matrices \( \sigma_Z \) and \( \sigma_X \), respectively. The third Pauli matrix, \( \sigma_Y \), is obtained from the other two via the equation \( \sigma_Y = i\sigma_X\sigma_Z \). Any unitary \( u \in M_d(\mathbb{C}) \) for which \( u^2 = 1_d \) must be selfadjoint, and the condition \( uv = \zeta vu \) is equivalent, in the case \( p = 2 \), to \( uv = -vu \). Therefore, the relations \([1]\) for pairs of unitaries extend easily to \( g \)-tuples of unitaries, thereby describing \( g \)-tuples of anticommuting selfadjoint unitary matrices.

However, for \( p > 2 \), extending the relations in \([1]\) from two unitaries to a higher number of unitaries, say \( g \), has a number of additional considerations. The case of \( g = 2 \) illustrates the situation already: suppose that \( u \) and \( v \) unitaries that satisfy the relations in \([1]\). The commutation relation \( uv = \zeta vu \) implies that \( nu = \zeta^{-1} uv \); it is only in the case where \( p = 2 \) that \( \zeta^{-1} = \zeta \), and so one is required, for \( p > 2 \), to account for the change in the scalar if one swaps the order of the unitary product.

If \( \zeta \) is a primitive \( p \)-th root of unity, then \( \zeta^k \) is also a primitive \( p \)-th root of unity for all \( k \in \mathbb{N} \) that are not divisible by \( p \)—or, stated more conveniently, for any nonzero \( k \in \mathbb{Z}_p \). Thus, one might have commutation relations for unitaries that involve \( \zeta \) and some its powers.

Such considerations lead to the following definition.

**Definition 1.1.** Suppose that \( \zeta \in \mathbb{C} \) is a primitive \( p \)-th root of unity. The Weyl commutation relations for a \( g \)-tuple \( u = (u_1, \ldots, u_g) \) of \( d \times d \) unitary matrices that satisfy \( u_k^p = 1_d \), for each \( k \), are given by the equations

\[
(3)
\]

\[
\quad u_k u_\ell = \zeta^{c_{k\ell}} u_\ell u_k, \quad \text{for all } k, \ell \in \{1, \ldots, g\},
\]

for some skew-symmetric matrix \( C = [c_{k\ell}]_{k,\ell=1}^g \in M_g(\mathbb{Z}_p) \). If \( c_{k\ell} = 1 \) whenever \( k < \ell \), then the commutation relations in \([3]\) are said to be simple.

Motivated by the fact that the Pauli matrix \( \sigma_Y \) is a scalar multiple of the product \( \sigma_X\sigma_Z \), one may mimic the construction by considering \( w = \lambda uv \), where \( \lambda \in \mathbb{C} \) and \( u \) and \( v \) are \( d \times d \) unitary matrices such that \( u^p = v^p = 1_d \) and \( uv = \zeta vu \). The condition \( w^p = 1 \) imposes a requirement upon \( \lambda \). Indeed, as \( vu = (1/\zeta)uv \),

\[
(4)
\]

\[
(\lambda uv)^p = u(vu)^{p-1}v = (1/\zeta)^{p-1}u^2(vu)^{p-2}v^2 = (1/\zeta)^{p-1}(1/\zeta)^{p-2}u^3(vu)^{p-3}v^3 = \cdots = (1/\zeta)^{1+2+\cdots+(p-1)}u^pv^p = (1/\zeta)^{\frac{p(p-1)}{2}} 1.
\]
Thus, the condition \( w^p = 1 \) implies that \( \lambda^p = \zeta^{(p-1)i} \), and so \( \lambda \) is uniquely determined: \( \lambda = \zeta^{\frac{i}{p-1}} \). In this case, we have the following commutation relations (in addition to \( uv = \zeta vu \)):

\[
uw = \zeta wu \quad \text{and} \quad vw = \zeta^{-1} vw. 
\]

Thus, if \( (u_1, u_2, u_3) \) were given by \( (u, v, w) \), then the matrix \( C \) for the Weyl commutation relations would be

\[
C = \begin{bmatrix}
0 & 1 & 1 \\
-1 & 0 & -1 \\
-1 & 1 & 0
\end{bmatrix}.
\]

On the other hand, the Weyl commutation relations for \( (u_1, u_2, u_3) = (u, w, v) \) are simple, as in this case the matrix \( C \) is

\[
C = \begin{bmatrix}
0 & 1 & 1 \\
-1 & 0 & 1 \\
-1 & -1 & 0
\end{bmatrix}.
\]

**Definition 1.2.** If \( p \geq 3 \), then the simple Weyl unitary matrices are the three \( p \times p \) unitary matrices \( \omega_u, \omega_b, \) and \( \omega_c \) defined by

\[
\omega_u = u, \quad \omega_b = \zeta^{\frac{i}{p-1}} uv, \quad \omega_c = v.
\]

The triple \( \mathfrak{W} = (\omega_u, \omega_b, \omega_c) \) is called the simple Weyl triple.

In this paper, we establish a result that is stronger than the assertion concerning the universality of the Weyl matrices \( u \) and \( v \). Specifically, we show that any two unitary matrices with the relations (1) are universal – not just the Weyl pair. This result is best phrased in terms of complete order equivalence, which is essentially a special form of isomorphism in the operator system category (see [8] for an overview of the theory of operator systems and completely positive linear maps).

To explain complete order equivalence, suppose that \( x = (x_1, \ldots, x_g) \) is a \( g \)-tuple of complex \( d \times d \) matrices \( x_k \). The operator system of \( x \) is the linear space \( \mathcal{O}_x \) of \( d \times d \) matrices defined by

\[
\mathcal{O}_x = \text{Span}_\mathbb{C} \{ 1_d, x_k, x_k^* | k = 1, \ldots, g \}.
\]

If \( x = (x_1, \ldots, x_g) \) and \( y = (y_1, \ldots, y_g) \) are \( g \)-tuples of matrices \( x_j \in M_d(\mathbb{C}) \) and \( y_k \in M_d(\mathbb{C}) \), then \( x \) and \( y \) are said to be completely order equivalent, denoted by \( x \simeq_{\text{ord}} y \), if there exists a linear isomorphism \( \phi : \mathcal{O}_x \to \mathcal{O}_y \) such that

a) \( \phi \) and \( \phi^{-1} \) are unital completely positive (ucp) linear maps, and

b) \( \phi(x_k) = y_k \), for each \( k = 1, \ldots, g \).

If \( a = (a_1, \ldots, a_g) \) is a \( g \)-tuple of \( d \times d \) matrices, and if \( \gamma : \mathbb{C}^n \to \mathbb{C}^d \) is a linear transformation, then \( \gamma^* a \gamma \) denotes the \( g \)-tuple of \( n \times n \) matrices given by

\[
\gamma^* a \gamma = (\gamma^* a_1 \gamma, \ldots, \gamma^* a_g \gamma).
\]

If \( n = d \) and if \( b = \gamma^* a \gamma \) for a unitary \( \gamma \), then this situation is denoted by

\[
a \simeq_u b
\]

and we say that \( b \) is unitarily equivalent to \( a \).

While it is clear that

\[
x \simeq_u y \implies x \simeq_{\text{ord}} y,
\]

we have established the following theorem.

**Theorem.** If \( x = (x_1, \ldots, x_g) \) is a \( g \)-tuple of complex matrices forming a simple Weyl triple \( \mathfrak{W} \), then

\[
x \simeq_{\text{ord}} y \quad \text{implies} \quad x \simeq_u y.
\]
the converse does not hold in general, for in the complete order equivalence problem for $g$-tuples $x$ and $y$ of matrices, the matrix dimensions $d_1$ and $d_2$ need not be equal. Hence, complete order equivalence is weaker than unitary equivalence.

An answer to the question of when two operator $g$-tuples $x$ and $y$ are completely order equivalent has been given by Davidson, Dor-On, Shalit, and Solel [3]: $x \simeq_{\text{ord}} y$ if and only if $W(x) = W(y)$, where, for a $g$-tuple $z = (z_1, \ldots, z_g)$ of operators $z_k$, $W(z)$ denotes the matrix range of $z$, which is the sequence

$$W(z) = (W^n(z))_{n \in \mathbb{N}}$$

of subsets $W^n(z)$ in $M_n(\mathbb{C}) \times \cdots \times M_n(\mathbb{C})$ ($g$ copies) defined by

$$(4) \quad W^n(z) = \{(\psi(z_1), \ldots, \psi(z_g)) \mid \psi : \mathbb{D}_z \to M_n(\mathbb{C}) \text{ is a ucp map}\}.$$  

This result of Davidson, Dor-On, Shalit, and Solel is especially useful in cases where the matrix ranges can be described; however, such descriptions are not always available.

We shall find it convenient to use the following definition.

**Definition 1.3.** A unitary matrix $u$ is of $p$-th order if $u^p$ is the identity matrix.

Lastly, some notation: if $\mathcal{S}$ is a nonempty subset of $M_d(\mathbb{C})$, then $\text{Alg}(\mathcal{S})$ denotes the complex associative subalgebra of $M_d(\mathbb{C})$ generated by $\mathcal{S}$. In particular, it is a classical result that $M_n(\mathbb{C}) = \text{Alg}(\mathcal{S})$, if $\mathcal{S}$ is the set consisting of the two Weyl unitary matrices $u$ and $v$ [2].

2. **Universality of the Weyl Unitary Matrices $u$ and $v$**

The eigenvalues of the Weyl unitary matrices $u$ and $v$ are precisely the $p$-roots of unity, which is the maximal spectrum of any $p$-th order unitary matrix. It so happens that this spectral property is a consequence of the Weyl commutation relation.

**Lemma 2.1.** If $u$ and $v$ are $p$-th order $d \times d$ unitary matrices in which $uv = \zeta vu$, then

1. $p$ divides $d$,  
2. $\text{Tr}(u) = \text{Tr}(v) = \text{Tr}(uv) = 0$, and  
3. every $p$-root of unity is an eigenvalue of $u$ and an eigenvalue of $v$.

**Proof.** Applying the determinant to the relation $uv = \zeta vu$ leads to

$$\det(u) \det(v) = \det(uv) = \zeta^d \det(vu) = \zeta^d \det(u) \det(v),$$

which implies that $\zeta^d = 1$. Because $\zeta$ is primitive, the positive integer $p$ divides $d$.

The relation $uv = \zeta vu$ is equivalent to $v^* uv = \zeta u$. Thus, by applying the trace, we have that $\text{Tr}(u) = \zeta \text{Tr}(u)$, which is true only if $\text{Tr}(u) = 0$. Analogous reasoning leads to $\text{Tr}(v) = 0$. Finally,

$$\text{Tr}(vu) = \text{Tr}(uv) = \zeta \text{Tr}(vu),$$

which leads to $\zeta = 1$ or $\text{Tr}(uv) = 0$. As the former cannot hold, it must be that the latter does.

We next show that every $p$-th root of unity is an eigenvalue of $u$. Because the spectrum of a matrix is invariant under unitary similarity, the relation $v^* uv = \zeta u$ implies that $\zeta \lambda$ is an eigenvalue of $u$ whenever $\lambda$ is an eigenvalue of $u$. Applying this observation to the eigenvalue $\zeta \lambda$, we see that $\zeta^2 \lambda$ is an eigenvalue of $u$. Hence,
by iteration of the argument, \( \zeta^k \lambda \) is an eigenvalue of \( u \) for every \( k = 1, \ldots, p \). Because the map \( \alpha \mapsto \alpha \lambda \) is a bijection of \( \mathbb{C} \) onto itself, the spectrum of \( u \) must contain at least \( p \) elements. But on the other hand, the spectrum of \( u \) cannot contain more than \( p \) elements; hence, the spectrum of \( u \) must coincide with the set of \( p \)-th roots of unity. A similar argument applies to \( v \).

The next lemma determines the set of all \( p \)-th order unitaries \( v \), given a unitary \( u \) with the spectral structure described in Lemma 2.1, for which \( uv = \zeta vu \).

**Lemma 2.2.** If \( d = pn \) and if \( u, v \in \mathcal{M}_d(\mathbb{C}) \) are \( p \)-th order unitary matrices such that \( uv = \zeta vu \), then there exist a unitary matrix \( y \in \mathcal{M}_d(\mathbb{C}) \) and unitary matrices \( v_2, \ldots, v_p \in \mathcal{M}_n(\mathbb{C}) \) such that \( y^* y = \bar{u} \) and \( y^* y = \bar{v} \), where

\[
\bar{u} = \begin{bmatrix} 1_n & \zeta 1_n & \zeta^2 1_n & \cdots & \zeta^{p-1} 1_n \end{bmatrix}
\]

and

\[
\bar{v} = \begin{bmatrix} 0 & 0 & \cdots & v_1 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \vdots \\
v_p & \cdots & 0 & v_p \end{bmatrix},
\]

where \( 1_n \) denotes the \( n \times n \) identity matrix, and where \( v_1 \) is given by \( v_1 = v_2^* v_3^* \cdots v_p^* \)

**Proof.** Lemma 2.1 indicates that every \( p \)-th root of unity is a spectral element of \( u \) and \( \text{Tr}(u) = 0 \). Therefore, using \( d = pn \), each eigenvalue \( \zeta^k \) of \( u \) must appear with multiplicity \( n \). Hence, \( u \) admits a diagonalisation as that given in (5).

Now let \( \bar{v} = y^* y u \); in accordance with the structure of \( \bar{u} \), the operator \( \bar{v} \) can be expressed as a \( p \times p \) matrix of \( n \times n \) matrices \( v_{ij} \). The relation \( \bar{u} \bar{v} = \zeta^p \bar{v} \bar{u} \) holds only if the operators \( v_{ij} \) are \( 0 \) whenever \( (i, j) \notin \{(1, p), (k, k-1) \mid k = 2, \ldots, p \} \). Thus, denote \( v_{ip} \) by \( v_i \) and \( v_{k,k-1} \) by \( v_k \). Because \( v^* v = v v^* = 1_d \), each \( v_k \) satisfies \( v_k^* v_k = v_k v_k^* = 1_n \). Furthermore, \( v^p = 1_d \) implies that \( v_p v_{p-1} \cdots v_2 v_1 = 1_n \), and so \( v_1 = v_2^* v_3^* \cdots v_p^* \).

As a consequence of the result above, it is possible to construct a path-connected set of \( p \)-th order unitaries \( v \) such that \( uv = \zeta vu \) from a certain \( p \)-th order unitary matrix \( u \).

**Corollary 2.3.** If \( u \in \mathcal{M}_d(\mathbb{C}) \) is a \( p \)-th order unitary matrix such that every \( p \)-th root of unity is an eigenvalue of \( u \) of multiplicity \( d/p \), then the set of all \( p \)-th order unitary matrices \( v \in \mathcal{M}_d(\mathbb{C}) \) for which \( uv = \zeta vu \) is homeomorphic to the Cartesian product of \( p - 1 \) copies of the unitary group \( U_p \).

The next result is crucial for establishing an explicit complete order equivalence in our main result, Theorem 2.10 below.

**Lemma 2.4.** If \( d = pn \) and if \( u, v \in \mathcal{M}_d(\mathbb{C}) \) are unitaries of the form (5), for some unitaries \( v_2, \ldots, v_p \in \mathcal{M}_n(\mathbb{C}) \), then a \( p \times p \) matrix \( Z = [z_{ij}]_{i,j=1}^p \) of \( n \times n \) matrices \( z_{ij} \) is an element of \( \text{Alg}([u, v]) \) if and only if there exist \( p^2 \) scalars \( \lambda_{ij} \) such that:

\[
\begin{align*}
z_{ii} &= \lambda_{ii} 1_n, \quad \text{for all } i; \\
z_{ip} &= \lambda_{ip} v_i \text{ and } z_{k,k-1} = \lambda_{k,k-1} v_k, \quad \text{for all } k = 2, \ldots, p; \\
z_{pi} &= \lambda_{pi} v_i^* \text{ and } z_{k,k+1} = \lambda_{k,k+1} v_{k+1}^*, \quad \text{for all } k = 1, \ldots, p - 1; \\
z_{ij} &= \lambda_{ij} (v_i^* \cdots v_{i+1}^*), \quad \text{for all } i > j \text{ with } |i - j| \geq 2; \\
z_{ij} &= \lambda_{ij} (v_{i+1}^* \cdots v_j^*), \quad \text{for all } i < j \text{ with } |i - j| \geq 2.
\end{align*}
\]
Proof. First note that if a matrix $b$ commutes with a unitary $w$, then $b$ also commutes with $w^{-1} = w^*$, implying that $b^*$ commutes with $w$; hence, the set \{\{w\}\} of matrices commuting with $w$ is closed under the adjoint $\star$. Second, as the inverse $w^{-1}$ of a unitary matrix $w$ is a polynomial in $w$, then algebra generated by one or more unitary matrices is $\star$-closed. Hence, by von Neumann’s Double Commutant Theorem,

$$\text{Alg}([u,v]) = \{u,v\}''',$$

where $S''$ denotes the double commutant $(S')'$ of a set $S$. We begin, therefore, by showing that

$$\langle u,v \rangle' = \left\{ \left( \bigoplus_{k=1}^{p-1} (v_p \cdots v_{k+1})^* x(v_p \cdots v_{k+1}) \right) \bigotimes x \mid x \in \mathbb{M}_n(\mathbb{C}) \right\}.$$  

To this end, suppose that $X = [x_{ij}]_{i,j=1}^p \in \mathbb{M}_d(\mathbb{C})$, where $x_{ij} \in \mathbb{M}_n(\mathbb{C})$ for all $i$ and $j$, commutes with $u$ and $v$. Then $Xu = uX$ implies that $\zeta^{i-1}x_{ij} = \zeta^{j-1}x_{ij}$ for all $i,j$ and so $x_{ij} = 0$ for all $i$ and $j$ with $j \neq i$. The equation $Xv = vX$ yields $x_{11}v_1 = v_1 x_{pp}$ and $v_k x_{k-1,k-1} = x_{kk} v_k$ for $k = 2,\ldots,p$. Thus,

$$X_{pp} = v_1^* x_{11} v_1, \quad x_{11} = v_2^* x_{22} v_2, \quad x_{22} = v_3^* x_{33} v_3, \ldots, \quad x_{p-1,p-1} = v_p^* x_{pp} v_p,$$

which yields

$$x_{kk} = (v_k \cdots v_1) x_{pp} (v_1^* \cdots v_k^*),$$

for all $k = 1,\ldots,p$. Therefore, once $x_{pp}$ is specified, all other diagonal entries of $X$ are determined.

Suppose next that $Z = [z_{ij}]_{i,j=1}^p \in \mathbb{M}_d(\mathbb{C})$, where $z_{ij} \in \mathbb{M}_n(\mathbb{C})$ for all $i$ and $j$, commutes with every $X \in \{u,v\}'$. Write the diagonal entries of $X \in \{u,v\}'$ as $x_1,\ldots,x_p$, and recall from equation (9) that

$$x_k = (v_k \cdots v_1) x_p (v_1^* \cdots v_k^*),$$

for all $k = 1,\ldots,p$. The equation $XZ = ZX$ implies that $x_i z_{ij} = z_{ij} x_j$ for all $i,j$. For $i = j$, these relations imply that $z_{ii}$ commutes with every $n \times n$ matrix; hence, $z_{ii} = \lambda_i 1_n$, for some $\lambda_i \in \mathbb{C}$.

If $k \in \{2,\ldots,p\}$, then $x_k z_{k,k-1} = z_{k,k-1} x_{k-1}$ is written, using the equations (8), as

$$x_k z_{k,k-1} = z_{k,k-1} (v_k^* x_k v_k),$$

and so $x_k (z_{k,k-1} v_k^*) = (z_{k,k-1} v_k^*) x_k$. As this holds for all $x_k \in \mathbb{M}_n(\mathbb{C})$, $z_{k,k-1} v_k^* = \lambda_k 1_n$, for some $\lambda_k \in \mathbb{C}$; that is, $z_{k,k-1} = v_k$, for $k = 2,\ldots,p$. The same type of calculation leads to $z_1 p = \lambda_1 p v_1$.

Similarly, if $k \in \{1,\ldots,p-1\}$, then $x_k z_{k,k+1} = z_{k,k+1} x_{k+1}$ is, using the equations (8), equivalent to the equation

$$x_{k+1} (v_k + 12 z_{k,k+1}) = (v_{k+1} z_{k,k+1}) x_{k+1},$$

which implies that $z_{k,k+1} = \lambda_{k+1} v_{k+1}^*$, as the equation above must hold for all $x_{k+1}$. Likewise, $z_{p+1} = v_p^*$.

Consider now the entries of $Z$ for which $|i-j| \geq 2$. Assume first the cases where $i > j$. By equations (9),

$$x_i = (v_i \cdots v_{j+1})(v_j \cdots v_1) x_p (v_1^* \cdots v_j^*) (v_{j+1}^* \cdots v_p^*) = (v_i \cdots v_{j+1}) x_j (v_{j+1}^* \cdots v_p^*).$$
Thus, the equation $x_i z_{ij} = z_{ij} x_i$ is
\[ x_i z_{ij} = z_{ij} (v_i \cdots v_{i+1})^* x_i (v_i \cdots v_{i+1}), \]
which is equivalent to
\[ x_i (z_{ij} (v_i \cdots v_{i+1})^* ) = (z_{ij} (v_i \cdots v_{i+1})^* ) x_i. \]
As $x_i \in M_n(\mathbb{C})$ can be arbitrary, $z_{ij} = \lambda_{ij} 1_n$ for some $\lambda_{ij} \in \mathbb{C}$, which implies that
\[ z_{ij} = \lambda_{ij} (v_i \cdots v_{i+1}), \text{ for all } i > j \text{ with } |i - j| \geq 2. \]
In the cases where $|i - j| \geq 2$ and $i < j$, the same type of arguments lead to
\[ z_{ij} = \lambda_{ij} (v_{i+1}^* \cdots v_j^*), \text{ for all } i < j \text{ with } |i - j| \geq 2, \]
which completes the proof. \qed

As an example of the matrix structure indicated Lemma 2.4, consider the case
\[ \begin{bmatrix} 1_n & \zeta 1_n & \zeta^2 1_n & \zeta^3 1_n & \zeta^4 1_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & v_2 & 0 & v_4 & 0 \\ v_2 & 0 & v_3 & 0 & v_5 \\ 0 & v_3 & 0 & v_4 & 0 \\ v_4 & 0 & v_5 & 0 & v_1 \\ v_1 & v_5 & v_4 & v_3 & v_2 \end{bmatrix}, \]
By Lemma 2.4, $z \in \text{Alg}(\langle u, v \rangle)$ if and only if
\[ z = \begin{bmatrix} \lambda_{11} 1_n & \lambda_{12} v_2^* & \lambda_{13} (v_2^* v_3^*) & \lambda_{14} (v_2^* v_3^* v_4^*) & \lambda_{15} v_1 \\ \lambda_{21} v_2 & \lambda_{22} 1_n & \lambda_{23} v_3^* & \lambda_{24} (v_3^* v_4^*) & \lambda_{25} (v_3^* v_4^* v_5^*) \\ \lambda_{31} (v_3 v_2) & \lambda_{32} v_3 & \lambda_{33} 1_n & \lambda_{34} v_4^* & \lambda_{35} (v_4^* v_5^*) \\ \lambda_{41} (v_4 v_3 v_2) & \lambda_{42} (v_4 v_3) & \lambda_{43} v_4 & \lambda_{44} 1_n & \lambda_{45} v_5^* \\ \lambda_{51} (v_5 v_4 v_3 v_2) & \lambda_{52} (v_5 v_4 v_3) & \lambda_{53} (v_5 v_4) & \lambda_{54} v_5 & \lambda_{55} 1_n \end{bmatrix}, \]
for some $\lambda_{ij} \in \mathbb{C}$.

Let us also make note of the following useful consequence of Lemma 2.4.

Corollary 2.5. If $u, v \in M_p(\mathbb{C})$ are unitaries of the form (5), for some complex numbers $v_2, \ldots, v_p$, then $\langle u, v \rangle' = \{ \lambda 1_p \mid \lambda \in \mathbb{C} \}$ and $\text{Alg}(\langle u, v \rangle) = M_p(\mathbb{C})$.

The structure of $\text{Alg}(\langle u, v \rangle)$ provided by Lemma 2.4 gives an explicit *-isomorphism of matrix algebras:

Proposition 2.6. If $d = p n$ and if $u, v \in M_d(\mathbb{C})$ are unitaries of the form (5), for some unitaries $v_2, \ldots, v_p \in M_n(\mathbb{C})$, then the function $\rho : M_p(\mathbb{C}) \to M_d(\mathbb{C})$ defined by
\[ \rho \left( [\lambda_{ij}]_{i,j=1}^p \right) = [z_{ij}]_{i,j=1}^p, \]
where the matrices $z_{ij} \in M_n(\mathbb{C})$ are given as in equations (6), is a unital *-isomorphism of $M_p(\mathbb{C})$ and $\text{Alg}(\langle u, v \rangle)$.

Proof. By Lemma 2.4, the function $\rho$ maps $M_p(\mathbb{C})$ onto $C^*(u, v)$. Furthermore, $\rho$ is plainly linear and $\ker \rho = \{0\}$, so $\rho$ is a linear isomorphism. It remains to show that $\rho$ is a *-homomorphism. Equations (6) indicate that $\rho(\Lambda^*) = \rho(\Lambda)^*$, for all $\Lambda \in M_p(\mathbb{C})$, and so the multiplicativity of $\rho$ is the only property left to confirm.
To this end, let \( A = [\alpha_{ij}]_{i,j} \) and \( B = [\beta_{ij}]_{i,j} \) be elements of \( \mathcal{M}_p(\mathbb{C}) \). We shall compare the entries of \( \rho(A)\rho(B) \) with those of \( \rho(AB) \). If, for example, \( i > j \) and \( |i - j| \geq 2 \), then the entries of row \( i \) of \( \rho(A) \) are
\[
\alpha_{ii}(v_1 \cdots v_{i-1} v_i v_{i+1} \cdots v_p), \quad \alpha_{i1}(v_i v_{i+1}), \quad \ldots, \quad \alpha_{i1}^{(i+1)}(v_{i+1} v_{i+2}), \quad \ldots, \quad \alpha_{ip}^{(i+p)}(v_{i+1} \cdots v_p),
\]
while the entries of column \( j \) of \( \rho(B) \) are
\[
\beta_{jj}(v_1^2 \cdots v_{j-1} v_j v_{j+1} \cdots v_p), \quad \beta_{j1}(v_j v_{j+1}), \quad \ldots, \quad \beta_{j1}^{(j+1)}(v_{j+1} v_{j+2}), \quad \ldots, \quad \beta_{jp}^{(j+p)}(v_{j+1} \cdots v_p).
\]
Therefore, the \((i,j)\)-entry of \( \rho(A)\rho(B) \) is \( \sum_{k=1}^{p} \alpha_{ik} \beta_{kj}(v_i \cdots v_{j-1} v_j v_{j+1} \cdots v_p) \), which is the \((i,j)\)-entry of \( \rho(AB) \). The arguments for all other choices of \( i \) and \( j \) are similar. \( \square \)

**Theorem 2.7.** If two \( p \)-th order unitary matrices \( u \) and \( v \) satisfy the Weyl commutation relation \( uv = \zeta v u \), then \( (u, v) \simeq_{\text{ord}} (u, v) \).

**Proof.** If \( u, v \in \mathcal{M}_d(\mathbb{C}) \), then Lemma 2.4 asserts that \( p \) divides \( d \) and that there is a unitary matrix \( y \in \mathcal{M}_d(\mathbb{C}) \) such that \( y^*uy \) and \( y^*vy \) are the matrices in (5). As the map \( x \mapsto y^*xy \) is a \(*\)-automorphism of \( \mathcal{M}_d(\mathbb{C}) \), it is enough to assume that \( u \) and \( v \) are in this form already. In that regard, the isomorphism \( \rho : \mathcal{M}_p(\mathbb{C}) \rightarrow \mathcal{M}_d(\mathbb{C}) \) in Theorem 2.6 satisfies \( \rho(u) = u \) and \( \rho(v) = v \). Furthermore, as \( \rho \) is a unital \(*\)-isomorphism, its restriction \( \phi \) to the operator system generated by the Weyl matrices \( u \) and \( v \) has the property that both \( \phi \) and \( \phi^{-1} \) are completely positive. \( \square \)

**Corollary 2.8.** The Weyl unitary matrices \( u \) and \( v \) are universal for the commutation relation \( uv = \zeta vu \).

**Proof.** Suppose that \( u \) and \( v \) are \( p \)-th order \( d \times d \) unitary matrices that satisfy the Weyl commutation relation \( uv = \zeta vu \). By Theorem 2.10, the linear isomorphism \( \Phi : \mathcal{O}_{(u,v)} \rightarrow \mathcal{O}_{(u,v)} \) is completely positive. Viewing \( \Phi \) as a ucp from the operator system \( \mathcal{O}_{(u,v)} \) into the matrix algebra \( \mathcal{M}_d(\mathbb{C}) \), the map \( \Phi \) admits a ucp extension \( \Phi \) to \( \mathcal{M}_p(\mathbb{C}) \), by the Arveson Extension Theorem 1.8. Clearly the map \( \Phi \) sends \( u \) to \( u \) and \( v \) to \( v \). \( \square \)

**Corollary 2.9.** If two \( p \)-th order unitary matrices \( u \) and \( v \) satisfy the Weyl commutation relation \( uv = \zeta vu \), then the pair \( (u, v) \) is universal for all \( p \)-th order unitary matrices that satisfy the Weyl commutation relation.

Our final observation is that if the Weyl commutation relations are satisfied by \( p \)-th order \( p \times p \) unitaries, then these unitaries must be unitarily equivalent to the Weyl unitaries.

**Corollary 2.10.** If two \( p \)-th order \( p \times p \) unitary matrices \( u \) and \( v \) satisfy the Weyl commutation relation \( uv = \zeta vu \), then \( (u, v) \simeq_u (u, v) \).

**Proof.** By hypothesis, there is a unital completely positive bijection \( \phi : \mathcal{O}_{(u,v)} \rightarrow \mathcal{O}_{(u,v)} \) in which \( \phi(u) = u \), \( \phi(v) = v \), and \( \phi^{-1} \) is completely positive. Let \( \Phi = \Phi_1 \circ \Phi \) be extensions of \( \phi \) and \( \phi^{-1} \), respectively, to ucp maps \( \mathcal{M}_p(\mathbb{C}) \rightarrow \mathcal{M}_p(\mathbb{C}) \). The ucp map \( \Phi_1 \circ \Phi \) fixes every element of the irreducible operator system \( \mathcal{O}_{(u,v)} \), and so by Arveson’s Boundary Theorem 2.6, \( \Phi_1 \circ \Phi \) is the identity map of \( \mathcal{M}_p(\mathbb{C}) \). Hence, \( \Phi \) is a ucp map of \( \mathcal{M}_p(\mathbb{C}) \) with a completely positive inverse, which by Wigner’s Theorem implies that \( \Phi \) is a unitary similarity transformation \( x \mapsto w^*xw \) for some unitary \( w \in \mathcal{M}_p(\mathbb{C}) \). \( \square \)
3. WEYL-BRAUER UNITARIES

Consider the Weyl triple \( \mathcal{W} = (\omega_a, \omega_a, \omega_c) \) of \( p \times p \) unitary matrices, where
\[
\omega_a = u, \quad \omega_b = \zeta^{-1}uw, \quad \omega_c = v,
\]
and where \( u \) and \( v \) are the Weyl unitary matrices. Recall that the triple \( \mathcal{W} = (\omega_a, \omega_a, \omega_c) \) satisfies the simple Weyl commutation relations.

Observe that \( v = \zeta^{-1}u^{-1}w \), and so \( v \) is in the associative algebra generated by \( \omega_a \) and \( \omega_b \). Because \( \mathcal{M}_p(\mathbb{C}) \) is generated as an algebra by \( u \) and \( v \), this means that the algebra generated \( \omega_a \) and \( \omega_b \) is also \( \mathcal{M}_p(\mathbb{C}) \). Hence,
\[
\text{Alg} (\Omega_{1-}) = \text{Alg} (\Omega_{1-}) = \mathcal{M}_p(\mathbb{C}),
\]
where \( \Omega_1 = \{\omega_a, \omega_b, \omega_c\} \) and \( \Omega_{1-} = \{\omega_a, \omega_b\} \).

As in [10, Definition 6.63] and by adapting the method of the proof of Theorem 4.3 in [7], we shall invoke an iteration whereby we produce, from \( m \) invertible matrices \( x_1, \ldots, x_m \), a set of \( m + 2 \) invertible matrices:
\[
x_1 \otimes 1_p, \ldots, x_{m-1} \otimes 1_p, x_m \otimes \omega_a, x_m \otimes \omega_b, x_m \otimes \omega_c.
\]
Specifically, in taking \( x_1, x_2, \) and \( x_3 \) to be \( \omega_a, \omega_b, \) and \( \omega_c \), respectively, the iteration yields a set \( \Omega_2 \subseteq \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_p(\mathbb{C}) \) of 5 elements:
\[
\Omega_2 = \{\omega_a \otimes 1, \omega_b \otimes 1, \omega_c \otimes \omega_a, \omega_c \otimes \omega_b, \omega_c \otimes \omega_c\} = \Omega_{2-} \cup \{\omega_c \otimes \omega_c\},
\]
where \( \Omega_{2-} = \Omega_2 \setminus \{\omega_c \otimes \omega_c\} \). The matrices in \( \Omega_2 \) satisfy the Weyl commutation relations \( \tilde{\omega} \tilde{v} = \zeta \tilde{v} \tilde{u} \) when \( \tilde{u} \) is selected before \( \tilde{v} \) in \( \Omega_2 \) and the set \( \Omega_2 \) is considered as an ordered list.

Another iteration of the construction generates a set \( \Omega_3 \) consisting of 7 elements:
\[
\Omega_3 = \Omega_{3-} \cup \{\omega_c \otimes \omega_c \otimes \omega_c\},
\]
where \( \Omega_{3-} = \{\omega_a \otimes 1 \otimes 1, \omega_b \otimes 1 \otimes 1, \omega_c \otimes \omega_a \otimes 1, \omega_c \otimes \omega_b \otimes 1, \omega_c \otimes \omega_c \otimes \omega_a, \omega_c \otimes \omega_c \otimes \omega_b\} \).

Once again, the matrices in \( \Omega_3 \) satisfy the Weyl commutation relations \( \tilde{\omega} \tilde{v} = \zeta \tilde{v} \tilde{u} \) when \( \tilde{u} \) is selected before \( \tilde{v} \) in \( \Omega_3 \) and the set \( \Omega_3 \) is considered as an ordered list.

Repeated iteration produces, for each positive integer \( k \), a set \( \Omega_{k-} \) of cardinality \( 2k \) and a set \( \Omega_k \) with one additional element, namely
\[
\Omega_k = \Omega_{k-} \cup \left\{ \bigotimes_{1}^{k} \omega_c \right\},
\]
such that the matrices in \( \Omega_k \) satisfy the Weyl commutation relations \( \tilde{\omega} \tilde{v} = \zeta \tilde{v} \tilde{u} \) when \( \tilde{u} \) is selected before \( \tilde{v} \) in \( \Omega_k \) and the set \( \Omega_k \) is considered as an ordered list.

The \( 2k \) elements of \( \Omega_{k-} \) consist of \( k \) pairs such that, in the order given by the iterative construction, the product of each pair is a product tensor in which all factors are the identity matrix and one tensor factor is \( \omega_a \omega_b \). More specifically, if
\[
\Omega_{k-} = \{z_1, z_2, z_3, z_4, \ldots, z_{2k-1}, z_{2k}\} \subset \bigotimes_{1}^{k} \mathcal{M}_p(\mathbb{C}),
\]
then
\[
\begin{align*}
zhzhz &= (\omega_a \omega_b) \otimes 1 \otimes 1 \cdots \otimes 1 = \zeta^{1/p} (\omega_c \otimes 1 \otimes 1 \cdots 1) \\
zhz &= 1 \otimes (\omega_a \omega_b) \otimes 1 \cdots \otimes 1 = \zeta^{1/p} (1 \otimes \omega_c \otimes 1 \cdots 1) \\
& \vdots \quad = \vdots \\
zhzh &= 1 \otimes 1 \otimes 1 \cdots \otimes (\omega_a \omega_b) = \zeta^{1/p} (1 \otimes 1 \otimes 1 \cdots \otimes \omega_c).
\end{align*}
\]

Hence,
\[
\bigotimes_1^k \omega_c = (\zeta^{1/p})^{-k} \prod_{j=1}^k w_{2j-1}w_{2j} \in \text{Alg}(\Omega_{k,-}),
\]
which shows that
\[
\text{Alg}(\Omega_{k,-}) = \text{Alg}(\Omega_k),
\]
for every \(k \in \mathbb{N}\).

We now show that \(\text{Alg}(\Omega_{k,-}) = \text{Alg}(\Omega_k) = \bigotimes_1^k M_p(\mathbb{C})\). Of course, it is sufficient to show this for \(\text{Alg}(\Omega_{k,-})\). The claim holds for \(k = 1\) because \(\Omega_{1,-} = \{\omega_a, \omega_b\}\) generates \(M_p(\mathbb{C})\). Looking at the case \(k = 2\),
\[
\Omega_{2,-} = \{\omega_a \otimes 1, \omega_b \otimes 1, \omega_c \otimes \omega_a, \omega_c \otimes \omega_b\}.
\]
The algebra generated by \(\{\omega_a \otimes 1, \omega_b \otimes 1\}\) consists of all matrices of the form \(s \otimes 1\), for \(s \in M_p(\mathbb{C})\), whereas the algebra generated by \(\{\omega_c \otimes \omega_a, \omega_c \otimes \omega_b\}\) consists of all matrices of the form \(\omega_c \otimes t\), for \(t \in M_p(\mathbb{C})\). Because the set of all products of matrices of these two types is the set of all elementary tensors in \(M_p(\mathbb{C}) \otimes M_p(\mathbb{C})\), we see that the claim holds for \(k = 2\). In general, using induction, if we consider matrices of the form given by the construction
\[
x_1 \otimes 1_p, \ldots, x_{m-1} \otimes 1_p, x_m \otimes \omega_a, x_m \otimes \omega_b, x_m \otimes \omega_c,
\]
where the algebra generated by invertible matrices \(x_1, \ldots, x_{m-1}\) is a full matrix algebra \(M_d(\mathbb{C})\), then our argument here shows that \(x_1 \otimes 1_p, \ldots, x_{m-1} \otimes 1_p\) generate matrices of the form \(s \otimes 1_p\) while \(x_m \otimes \omega_a\) and \(x_m \otimes \omega_b\) generate all matrices of the form \(x_m \otimes t\). Thus, collectively, these matrices generate \(M_d(\mathbb{C}) \otimes M_p(\mathbb{C})\).

The construction above proves the following theorem.

**Theorem 3.1.** For every positive integer \(k\) there exist \(p\)-th order unitaries \(u_1, \ldots, u_{2k+1}\) that satisfy the simple Weyl commutation relations and are such that
\[
\text{Alg}(\{u_1, \ldots, u_{2k}\}) = \text{Alg}(\{u_1, \ldots, u_{2k}, u_{2k+1}\}) = \bigotimes_1^k M_p(\mathbb{C}).
\]

As mentioned in [7], it is often the case that spin systems arise in quantum theory; for this reason, the following modification of the construction above is worth a brief mention.

**Theorem 3.2.** If \(\mathcal{H}\) is an infinite-dimensional complex Hilbert space, then there exists a countable sequence \(\{u_n\}_{n \in \mathbb{N}}\) of \(p\)-th order unitary operators \(u_n\) such that \(u_ku_l = \zeta_{k,l}u_k\) whenever \(k < l\) and such that the norm-closed algebra generated by the sequence \(\{u_n\}_{n \in \mathbb{N}}\) is isomorphic to the \(C^*\)-algebra \(\bigotimes_1^\infty M_p(\mathbb{C})\).
Proof. Let $\mathcal{H} = \bigotimes_{k=1}^{\infty} \mathbb{C}^p$, which is the direct limit of the finite-dimensional Hilbert spaces $\mathcal{H}_k = \bigotimes_{k=1}^{k} \mathbb{C}^p$. On each $\mathcal{H}_k$ construction the Weyl-Brauer unitaries, and then form the tensor product of these unitaries with infinitely many copies of the $p \times p$ identity matrix so as to produce unitary operators $u_1, \ldots, u_{2k}$ on $\mathcal{H}$ of order $p$ that satisfy the simple Weyl commutation relations. This construction also shows that the algebra $A_k$ generated by $u_1, \ldots, u_{2k}$ is isomorphic to $\bigotimes_{k=1}^{k} M_p(\mathbb{C})$ and that $A_k$ is a unital subalgebra of $A_{k+1}$. Hence, the norm-closed algebra generated by $\{u_n\}_{n \in \mathbb{N}}$ coincides with the norm-closure of $\bigcup_{k \in \mathbb{N}} A_k$, which is precisely $\bigotimes_{k=1}^{\infty} M_p(\mathbb{C})$. □

4. Weyl-Brauer Unitaries Are Not Universal

Although the Pauli-Weyl-Brauer matrices are universal for selfadjoint anticommuting unitaries [7], the analogous result fails for $p > 2$.

Proposition 4.1. Assume that $p \geq 3$ and let $\mathfrak{M} = (\omega_a, \omega_b, \omega_c)$ denote the triple of simple Weyl-Brauer matrices. Let $x, y, z \in M_p(\mathbb{C})$ be the $p$-th order unitary matrices given by $x = \omega_a$, $z = \omega_c$, and

$$
(10) \quad y = \begin{bmatrix}
0 & \zeta^2 & 0 & \cdots & 0 \\
\zeta^3 & 0 & \cdots & \zeta^{p-2} & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& & \zeta^{p-1} & \zeta^{p-2} & \zeta^{p-1} & 0 \\
& & & \zeta^{(1-p)/2} & 0 & \cdots & 1
\end{bmatrix}.
$$

The triple $(x, y, z)$ satisfies the simple Weyl commutation relations, but there does not exist any unital completely positive linear map $\phi : M_p(\mathbb{C}) \to M_p(\mathbb{C})$ in which $\phi(\omega_a) = x$, $\phi(\omega_b) = y$, and $\phi(\omega_c) = z$.

Proof. If $\lambda_p$ denotes the $(p, p-1)$-entry of $y$, then the condition $y^p = 1_p$ implies that

$$
\lambda_p = (1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1})^{-1} = \zeta^{-\sum_{k=1}^{p-1} k} = \zeta^{(1-p)/2},
$$

furthermore, matrix multiplication confirms that the triple $(x, y, z)$ satisfies the simple Weyl commutation relations. Hence, $(x, y, z)$ is a simple Weyl triple.

Assume that a unital completely positive linear map $\phi : M_p(\mathbb{C}) \to M_p(\mathbb{C})$ in which $\phi(\omega_a) = x$, $\phi(\omega_b) = y$, and $\phi(\omega_c) = z$ does exist. The fixed point set $\mathfrak{F}_\phi = \{s \in M_d(\mathbb{C}) \mid \phi(s) = s\}$ of $\phi$ is an operator system that contains the Weyl unitaries $\omega_a = u$ and $\omega_c = v$; because $(u, v) = C 1_d$, the operator system $\mathfrak{F}_\phi$ is irreducible.

Let $\psi : \mathfrak{F}_\phi \to M_p(\mathbb{C})$ be the ucp map $\psi(s) = s$, for all $s \in \mathfrak{F}_\phi$. Thus, $\phi$ is a ucp extension of $\psi$ from $\mathfrak{F}_\phi$ to $M_p(\mathbb{C})$. However, as $\mathfrak{F}_\phi$ is irreducible, $\psi$ has unique completely positive extension to $M_p(\mathbb{C})$, by Arveson’s Boundary Theorem [2] [6]. Therefore, $\phi$ can only be the identity map, as the identity map on $M_d(\mathbb{C})$ is one ucp
extension of $\psi$. However, as $\phi(\omega_a) = x \neq \omega_a$, $\phi$ is not the identity map. Hence, this contradiction leads us to conclude that $\phi$ is not completely positive. \hfill \Box

Corollary 4.2. The Weyl-Brauer unitaries are not universal for $g$-tuples (where $g \geq 3$) of $p$-th order unitaries that satisfy the simple Weyl commutation relations.

Proof. If universality were to hold for some $g > 3$, then it would need to hold for $g = 3$. However, Proposition 4.1 indicates that universality fails for $g = 3$. \hfill \Box

5. The Matrix Range of the Weyl Unitaries

The work of Arveson [2] and Davidson, Dor-On, Shalit, and Solel [3] demonstrate the role of the matrix range in questions such as those we have considered herein. For this reason, it is of interest to consider the matrix range of the Weyl unitaries, especially in connection with the geometry of the matrix range from the perspective of free convexity [4].

Definition 5.1. A sequence $K = (K_n)_{n \in \mathbb{N}}$ of subsets $K_n$ in the Cartesian product $M_n(\mathbb{C})^g$ of $g$ copies of $M_n(\mathbb{C})$ is matrix convex if

\begin{equation}
\sum_{j=1}^{m} \gamma_j^* a_j \gamma_j \in K_n
\end{equation}

for all $m \in \mathbb{N}$, all $a_j \in K_{n_j}$, and all linear transformations $\gamma_j : \mathbb{C}^{n_j} \to \mathbb{C}^{n_j}$ for which

\begin{equation}
\sum_{j=1}^{m} \gamma_j^* \gamma_j = 1_n.
\end{equation}

Linear transformations $\gamma_j$ that satisfy (12) are called matrix convex coefficients and elements of the form (11) are called matrix convex combinations of the elements $a_j$.

We are interested in the following notions [4] of extremal element in the context of matrix convexity.

Definition 5.2. If $K = (K_n)_{n \in \mathbb{N}}$ is matrix convex, where $K_n \subseteq M_n(\mathbb{C})^g$ for each $n$, then an element $b \in K_n$ is:

1. an absolute extreme point of $K$ if whenever $b$ is a matrix convex combination (11) of elements $a_j \in K_{n_j}$ such that each matrix convex coefficient $\gamma_j$ is nonzero, then, for each $j$, either (i) $n_j = n$ and $a_j \succeq_u b$ or (ii) $n_j > n$ and there exists a $c_j$ such that $a_j \succeq_u b \oplus c_j$;

2. a matrix extreme point of $K$ if whenever $b$ is a matrix convex combination (11) of elements $a_j \in K_{n_j}$ such that each matrix convex coefficient $\gamma_j$ is surjective, then, for each $j$, $n_j = n$ and $a_j \succeq_u b$.

Theorem 5.3. The Weyl pair $(u, v)$ is a matrix extreme point of its matrix range.

Proof. Let $b = (u, v)$ and suppose that $b = \sum_{j=1}^{m} \gamma_j^* a_j \gamma_j$ for some surjective matrix convex coefficients $\gamma_j$ and matrix pairs $a_j = (a_{j1}, a_{j2}) \in W^{n_{j1}}(b)$. For each $j$ there is a ucp map $\psi_j : O_{(u, v)} \to M_{n_{j1}}(\mathbb{C})$ such that $a_{j1} = \psi_j(u)$ and $a_{j2} = \psi_j(v)$. Let $\Psi_j$
be a ucp extension of \( \psi_j \) to a ucp map \( \Psi_j : M_p(\mathbb{C}) \to M_{n_j}(\mathbb{C}) \), for each \( j \), and let
\[
\Phi : M_p(\mathbb{C}) \to M_p(\mathbb{C})
\]
be given by
\[
\Phi = \sum_{j=1}^{m} \gamma_j^* \psi_j \gamma_j.
\]
Note that \( \Phi | O_{(u,v)} \) is the identity map on \( O_{(u,v)} \); hence, in considering the identity map in \( O_{(u,v)} \) as a ucp map from \( O_{(u,v)} \) into \( M_p(\mathbb{C}) \), \( \Phi \) is a ucp extension of that map. Because the operator system \( O_{(u,v)} \) is irreducible, Arveson’s Boundary Theorem \cite{2, 6} implies that \( \Phi \) is the identity map on \( M_p(\mathbb{C}) \). That is,
\[
id_{M_p(\mathbb{C})} = \sum_{j=1}^{m} \gamma_j^* \psi_j \gamma_j.
\]
Now because the identity map of \( M_p(\mathbb{C}) \) is a pure matrix state of \( M_p(\mathbb{C}) \), it is also a matrix extreme point of its matrix state space \cite{5}. Hence, \( n_j = p \) for every \( j \) and there are unitaries \( w_j \in M_p(\mathbb{C}) \) such that \( \psi_j(x) = w_j^* x w_j \) for every \( x \in M_p(\mathbb{C}) \). In particular, \( u = w_j^* a_j w_j \) and \( v = w_j^* a_j v \) for all \( j \), which yields \( a_j \simeq_u (u,v) = b \) for all \( j \).

**Theorem 5.4.** The Weyl pair \( (u,v) \) is an absolute extreme point of its matrix range.

**Proof.** Let \( b = (u,v) \) and suppose that, for some \( \ell \in \mathbb{N} \), the pair
\[
(a_1, a_2) = \left( \begin{bmatrix} u & r_1 \\ s_1 & t_1 \end{bmatrix}, \begin{bmatrix} v & r_2 \\ s_2 & t_2 \end{bmatrix} \right) \in W^{p+\ell}(b),
\]
for some matrices \( r_1, s_1, t_1 \) of appropriate sizes. We claim that the matrix pair above \( (a_1, a_2) \) can be an element of \( W^{p+\ell}(b) \) only if the off-diagonal matrices \( r_i \) and \( s_i \) are zero, for \( i = 1, 2 \). The reason for this is straightforward. Because each \( a_1 \) is a ucp image of a Weyl unitary, \( \|a_i\| \leq 1 \) for \( i = 1, 2 \). Therefore, no row or column in \( a_i \) can have norm (in \( \mathbb{C}^{p+\ell} \)) exceeding 1. However, because each row of \( u \) and \( v \) has exactly one nonzero entry and this entry is of modulus 1, a nonzero entry in \( r_1 \) or \( r_2 \) would cause a row in \( a_1 \) or \( a_2 \) to have norm exceeding 1, which we noted cannot happen. Thus, \( r_1 \) and \( r_2 \) are zero matrices. Using a similar argument for the columns, we deduce that \( s_1 \) and \( s_2 \) are also zero matrices.

In the language of matrix convexity, the previous paragraph proves that the pair \( (u,v) \) is an Arveson extreme point of its matrix range \( W(b) \). By \cite[Theorem 1.1(3)]{4}, if an Arveson extreme point of a matrix convex set is irreducible, then it is an absolute extreme point. Since the commutant \( \{u,v\}' \) is 1-dimensional, the Weyl pair is irreducible and, hence, an absolute extreme point of its matrix range. \( \square \)

The proofs of Theorems \ref{5} and \ref{4} extend beyond Weyl pairs to all Weyl-Brauer unitaries, once it has been shown that the Weyl-Brauer matrices generate irreducible operator systems. The details are left to the reader (using, if one wishes, the method of proof in \cite{7} that showed the irreducibility of the operator system generated by the Pauli-Weyl-Brauer unitaries). However, at the very least, we state below the version of this theorem for the three basic Weyl-Brauer unitaries \( \omega_a, \omega_b, \) and \( \omega_c \).

**Theorem 5.5.** If \( \mathbb{M} = (\omega_a, \omega_b, \omega_c) \) is the Weyl-Brauer triple, then \( \mathbb{M} \) is a matrix extreme point and an absolute extreme point of its matrix range.
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