Unipotent Schottky bundles on Riemann surfaces and complex tori

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Abstract

We study a natural map from representations of a free (resp. free abelian) group of rank \( g \) in \( GL_r(\mathbb{C}) \), to holomorphic vector bundles of degree zero over a compact Riemann surface \( X \) of genus \( g \) (resp. complex torus \( X \) of dimension \( g \)). This map defines what is called a Schottky functor. Our main result is that this functor induces an equivalence between the category of unipotent representations of Schottky groups and the category of unipotent vector bundles on \( X \). We also show that, over a complex torus, any vector or principal bundle with a flat holomorphic connection is Schottky.

1 Introduction

Let \( X \) be a compact Kähler manifold. In the context of the so-called non-abelian Hodge theory, Simpson has established in [Sim92, Lemma 3.5] an equivalence of categories between the category of flat bundles over \( X \) and the category of Higgs bundles on \( X \) which are extensions of stable bundles of degree zero with vanishing first and second Chern classes. In this article, we are particularly interested in unipotent objects, and establish equivalences of categories between certain categories of flat unipotent modules and categories of unipotent holomorphic bundles.

To describe these results, let \( X \) be a complex manifold, \( \mathcal{O}_X \) its structure sheaf and denote by \( \pi_1(X) \) the fundamental group of \( X \) (for some base point). In this situation there is a well known functor

\[
\mathbb{C}\pi_1(X)\text{-mod} \longrightarrow \mathcal{O}_X\text{-mod},
\]

that assigns to a module over the group ring \( \mathbb{C}\pi_1(X) \) an \( \mathcal{O}_X \)-module on \( X \). If \( M \) is a \( \mathbb{C}\pi_1(X) \)-module and if \( M \) is of finite rank \( r \) as a \( \mathbb{C} \)-module, then \( M \) may be identified with a representation \( \rho : \pi_1(X) \longrightarrow Aut_{\mathbb{C}}(M) \), and after a choice of basis, \( Aut_{\mathbb{C}}(M) \) may be identified with \( GL_r(\mathbb{C}) \). In this case the assigned \( \mathcal{O}_X \)-module \( E_{\rho} \) is a holomorphic vector bundle on \( X \).

An interesting subclass of these \( \mathbb{C}\pi_1(X) \)-modules are the so-called Schottky modules. Let \( \Sigma \) be a free (or free abelian) group and let \( \alpha : \pi_1(X) \longrightarrow \Sigma \) be a surjective homomorphism. A \( \mathbb{C}\pi_1(X) \)-module \( M \) is called \( \Sigma \)-Schottky if it is induced by a \( \mathbb{C}\Sigma \)-module via the canonical morphism \( \tilde{\alpha} : \mathbb{C}\pi_1(X) \longrightarrow \mathbb{C}\Sigma \).
If one restricts the functor defined above to $\pi_1(X)$-modules that are Schottky, then one obtains a new functor $S$ (called the Schottky functor)

$$S : \Sigma\text{-mod} \rightarrow O_X\text{-mod}.$$ 

We will also call a $O_X$-module $E$ on $X$ Schottky, if there is a $\Sigma$-module $M$ such that $S(M)$ is isomorphic to $E$. See Section 2 for more details.

Let us now define $\text{Un}_{\Sigma}$ as the category of unipotent $\Sigma$-modules and $\text{Un}_{O_X}$ as the category of unipotent $O_X$-modules on $X$. The Schottky functor is exact, so induces a well defined functor $S : \text{Un}_{\Sigma} \rightarrow \text{Un}_{O_X}$ that we will still denote by $S$. Note that a unipotent $O_X$-module of level $r$ is always a vector bundle of rank $r$ (cf. Section 4).

Two special cases of Schottky representations are of particular interest in our paper:

If $X$ is a complex torus of dimension $g$, defined as a quotient $V/\Lambda$ of a $g$-dimensional $\mathbb{C}$-vector space $V$ by a lattice $\Lambda$, then by [Gun67, S 8 p. 143] there exists a $\mathbb{C}$-linear basis $e_1,\ldots,e_g$ of $V$, and a basis $\lambda_1,\ldots,\lambda_{2g}$ of $\Lambda$, such that $\Pi$, the period matrix of $X$, defined by $\lambda_i = \sum_{j=1}^g \pi_{ij} \cdot e_j$, is of the form $\Pi = (Z,I)$, where $Z \in M(g \times g, \mathbb{C})$ is a symmetric invertible matrix, and $I$ is the identity matrix in dimension $g$. In this case we can let $\Sigma$ be the free abelian group with $g$ generators $B_1,\ldots,B_g$ and define $\alpha : \pi_1(X) :\rightarrow \Sigma$ as the surjective homomorphism sending $\lambda_i$ to $B_i$, and $\lambda_{g+i}$ to the identity in $\Sigma$, for all $i = 1,\ldots,g$.

Our main result for complex tori is the following (see Section 5):

**Theorem 1.1.** Let $X$ be a complex torus of dimension $g$, and let $\Sigma$ be a free abelian group of rank $g$. The Schottky functor $S$ induces an equivalence of categories

$$S : \text{Un}_{\Sigma} \simeq \text{Un}_{O_X},$$

between the category of unipotent $\Sigma$-modules, and the category of unipotent $O_X$-modules on $X$.

One can deduce from Simpson’s correspondence that unipotent vector bundles admit a flat connection. Then we parametrize explicitly the set of representations that give rise to isomorphic unipotent vector bundles. This method allows us to show that the Schottky functor is essentially surjective.

If $X$ is a compact Riemann surface of genus $g$, then let us fix a canonical basis [FaKr92 III 1.] of $\pi_1(X)$: elements $a_1,\ldots,a_g,b_1,\ldots,b_g$ that generate $\pi_1(X)$, subject to the single relation $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1$. Let $\Sigma := F_g$ be the free group of $g$ generators $B_1,\ldots,B_g$, and let $\alpha : \pi_1(X) \rightarrow \Sigma$ be the homomorphism defined by $\alpha(a_i) = 1$, $\alpha(b_i) = B_i$ for $i = 1,\ldots,g$. This is the classical case that justifies the use of the term “Schottky functor”.

Our main result for Riemann surfaces, proved in Section 6 is as follows:

**Theorem 1.2.** Let $X$ be a compact Riemann surface of genus $g$, and let $\Sigma$ be a free group of rank $g$. The Schottky functor $S$ induces an equivalence of categories

$$S : \text{Un}_{\Sigma} \simeq \text{Un}_{O_X},$$

between the category of unipotent $\Sigma$-modules, and the category of unipotent $O_X$-modules on $X$.  

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Of course, both theorems above include the case of elliptic curves \((g = 1)\). In the level (or rank) two case, the result basically follows from the fact that the Schottky functor \(S\) induces an isomorphism of Yoneda Ext groups \(S_* : YExt^1_{\mathcal{C}\Sigma}(\mathbb{C}, \mathbb{C}) \rightarrow YExt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)\). The general case then results from the induction on the level, as well as the fact that \(S\) is compatible with Yoneda Ext of \(\mathcal{C}\Sigma\)-modules and \(\mathcal{O}_X\)-modules. The relevant definitions and techniques from homological algebra that are needed will be described in Section 3.

We note that, in the broader context of Simpson’s correspondence, the category of Higgs bundles which are extensions of stable bundles of degree zero with vanishing first and second Chern classes, contains all unipotent Higgs bundles (successive extensions of trivial Higgs line bundles). The latter category contains the unipotent vector bundles as a full subcategory by setting the Higgs fields equal to zero. In a recent paper S. Lekaus [Lek05] has classified those unipotent \(\pi_1(X)\)-representations that give rise to unipotent Higgs bundles with zero Higgs fields under the correspondence between flat bundles and their monodromy representations. The method used for this classification differs from ours, and it would be interesting to know whether one can find a relationship between her result and our results of Theorem 1.1 and Theorem 1.2.

The structure of flat vector bundles on complex tori is well understood, mainly due to the work of Matsushima [Mat59] and Morimoto [Mor59]. In particular, such a vector bundle admits a flat connection if and only if every indecomposable component is a tensor product of a line bundle of degree zero with a flat unipotent vector bundle. For elliptic curves this result was already known due to the work of Atiyah [Ati57]. An analogous result in the case of principal bundles over complex tori was shown by Biswas-Gomez in [BiGo08], and similar descriptions were obtained recently by M. Brion [Br11] for algebraic homogeneous vector and principal bundles over abelian varieties.

Based on the classification of Matsushima and Morimoto, and Theorem 1.1 we could then show the following theorem, which generalizes [Flo01, Theorem 6] (see Section 7).

**Theorem 1.3.** Let \(E\) be a vector bundle over a complex torus. Then, \(E\) admits a flat holomorphic connection if and only if \(E\) is a Schottky vector bundle.

We also prove the corresponding result in the case of principal bundles. Let \(G\) be any connected linear algebraic group defined over \(\mathbb{C}\). As an application of Theorem 1.3 we have (see Section 8.3):

**Theorem 1.4.** Let \(P\) be a principal \(G\)-bundle over a complex torus. Then, \(P\) admits a flat holomorphic connection if and only if \(P\) is Schottky.

The article finishes by putting together Theorems 1.3 and 1.4 and the main result of [BiGo08] to show that not only the three notions of flat, Schottky and homogeneous are equivalent for a given principal bundle \(P\) over a complex torus, but also, when the first and second Chern class of \(P\) both vanish, these are equivalent to the corresponding notions for the adjoint vector bundle \(\text{Ad}P\).

Finally, we expect that the methods used in our paper are not only limited to the complex case, but may also be applied in the \(p\)-adic case, i.e., to the functors studied by Deninger-Werner [DeWe05], [DeWe10] and Faltings [Fal05], or also for the characteristic \(p\) case.
2 Schottky functors

Let $X$ be a connected and locally simply connected topological space. Let $R$ be a commutative ring with unit, denote by $R_X$ the constant sheaf of rings on $X$ defined by $R$, and let $\mathcal{O}_X$ be a sheaf of rings on $X$. We further assume that there is a morphism of sheaves of rings $i : R_X \to \mathcal{O}_X$, i.e., $\mathcal{O}_X$ is an $R_X$-algebra. Let $\pi_1(X)$ denote the fundamental group of $X$ (for some base point), and let $\Sigma$ be an arbitrary group, together with a group homomorphism $\alpha : \pi_1(X) \to \Sigma$.

We are going to describe a functor $S : R\Sigma\text{-}\underline{\text{mod}} \to \mathcal{O}_X\text{-}\underline{\text{mod}}$, from the category of modules over the group ring $R\Sigma$ to the category of $\mathcal{O}_X$-modules on $X$.

Construction 2.1. The group homomorphism $\alpha$ induces a morphism of group rings $\tilde{\alpha} : R\pi_1(X) \to R\Sigma$. If $\rho$ is an $R\Sigma$ module, then the change of rings functor $\tilde{\alpha}^* \circ i^* \circ RH \circ \tilde{\alpha}^*(\rho)$ defines a functor $i^*$ from the category of locally constant sheaves of $R$-modules to the category of $\mathcal{O}_X$-modules.

Proposition 2.3. If $R = \mathbb{C}$ and $X$ is a complex manifold, then the Schottky functor $S$ is faithful, exact, additive, and compatible with direct sums and tensor products.

Proof. It suffices to show these properties (if they apply) for each of the functors $\tilde{\alpha}^*, RH$ and $i^*$. The change of rings functor $\tilde{\alpha}^*$ is additive and exact by [Rot09, CH 8. Proposition 8.33], faithful by proof of [Rot09, CH 8. Proposition 8.33], and it is easy to see that it is compatible with direct sums and tensor products. The equivalence of categories functor $RH$ is additive, exact, and compatible with direct sums and tensor products because the inverse functor $RH^{-1}$ is induced by the fiber functor mapping a sheaf $\mathcal{F}$ to its stalk $\mathcal{F}_x$ at $x$ [Sza09, Theorem 2.5.14], and this functor satisfies all these properties. $RH$ is obviously faithful as well since $RH$ is an equivalence. Finally, it can be checked directly that the functor $i^*$ is additive and compatible with direct sums and tensor products. Exactness and faithfulness can be seen by looking at the stalks. □

As mentioned in the introduction, two cases are especially important here. Let $\Sigma$ be a free group on $g$ generators. The classical case is when $X$ is a Riemann surface of genus $g$ and $\alpha$ is defined as before Theorem 1.2. The other case is when $X$ is a complex torus of dimension $g$ and $\alpha$ is as given just before the statement of Theorem 1.1.
3 Schottky functors and cohomology

Let us assume now that $\Sigma$ is an arbitrary group and that $X$ is a complex manifold, so that $S$ is additive and exact. In both abelian categories, $\Sigma$-mod and $\mathcal{O}_X$-mod, one can compute $\text{Ext}$ groups using injective resolutions. We will investigate the Schottky functor in this context.

Let $A$ and $B$ be two $\Sigma$-modules, let $I$ denote an injective resolution of $B$ and let $J$ denote an injective resolution of $S(B)$. Then, since $J$ is injective and $S(I)$ is an exact sequence, the identity $f := \text{id}_{S(B)}$ can be extended to a chain map $F : S(I) \to J$ [Rot09 Theorem 6.16]. Then, after applying $\text{Hom}$, we obtain the following map of chain complexes

$$
\text{Hom}_{\Sigma}(A, I) \xrightarrow{S} \text{Hom}_{\mathcal{O}_X}(S(A), S(I)) \xrightarrow{F} \text{Hom}_{\mathcal{O}_X}(S(A), J).
$$

**Proposition 3.1.** The previous construction defines a morphism $S_* := S \circ F_*$ of graded $\mathbb{C}$-vector spaces

$$
S_* : \text{Ext}^*_\Sigma(A, B) \to \text{Ext}^*_\mathcal{O}_X(S(A), S(B))
$$

that is natural in the following sense: Let

$$
0 \to B' \xrightarrow{i} B \xrightarrow{p} B'' \to 0
$$

be an exact sequence of $\Sigma$-modules. Then there is a commutative diagram of $\mathbb{C}$-vector spaces with exact rows,

$$
\begin{array}{cccccc}
\text{Ext}^n_{\Sigma}(A, B') & \xrightarrow{i_*} & \text{Ext}^n_{\Sigma}(A, B) & \xrightarrow{p_*} & \text{Ext}^n_{\Sigma}(A, B'') & \xrightarrow{\delta} & \text{Ext}^{n+1}_{\Sigma}(A, B') \\
S_* \downarrow & & S_* \downarrow & & S_* \downarrow & & S_* \downarrow \\
\text{Ext}^n_{\mathcal{O}_X}(S(A), S(B')) & \xrightarrow{S(i)_*} & \text{Ext}^n_{\mathcal{O}_X}(S(A), S(B)) & \xrightarrow{S(p)_*} & \text{Ext}^n_{\mathcal{O}_X}(S(A), S(B'')) & \xrightarrow{\delta} & \text{Ext}^{n+1}_{\mathcal{O}_X}(S(A), S(B')).
\end{array}
$$

**Proof.** Using the injective version of the Horseshoe lemma [Rot09 Proposition 6.24] we may find injective resolutions $I'$, $I$ and $I''$ of $B'$, $B$ and $B''$, respectively, and lifts $i$ and $p$ of $i$ and $p$, respectively, such that they fit into a normal exact sequence (cf. [CaEi56 V, Section 2])

$$
0 \to I' \xrightarrow{i} I \xrightarrow{p} I'' \to 0.
$$

We note that applying $S$ to this normal exact sequence of complexes again gives normal exact sequences. Using again the Horseshoe lemma we may find injective resolutions $J'$, $J$ and $J''$ of $S(B')$, $S(B)$ and $S(B'')$, respectively, and lifts $S(i)$ and $S(p)$ of $S(i)$ and $S(p)$, respectively, such that they fit into an exact sequence

$$
0 \to J' \xrightarrow{S(i)} J \xrightarrow{S(p)} J'' \to 0.
$$

As above, we may also find chain maps $F' : S(I') \to J'$ and $F'' : S(I'') \to J''$, lifting the identities $f' = \text{id}_{S(B')}$ and $f'' = \text{id}_{S(B'')}$, respectively. Furthermore, since $J'$ is injective and $S(I')$ is acyclic, we can apply the injective version of [CaEi56 V §2 Proposition 2.3] and find a chain map $F : S(I) \to J$ above the
identity map \( f = \text{id}_{S(B)} \), in such a way that there is a commutative diagram of complexes

\[
0 \rightarrow S(I') \xrightarrow{S(\tilde{i})} S(I) \xrightarrow{S(\tilde{p})} S(I'') \rightarrow 0
\]

After applying \( \text{Hom} \) and adding the initial sequence, we obtain the following commutative diagram

\[
0 \rightarrow \text{Hom}_{\mathbb{C}^\Sigma}(A, I') \xrightarrow{\tilde{i}_*} \text{Hom}_{\mathbb{C}^\Sigma}(A, I) \xrightarrow{\tilde{p}_*} \text{Hom}_{\mathbb{C}^\Sigma}(A, I'') \rightarrow 0
\]

The top and the bottom sequences are exact, since \( I' \) and \( J' \) are sequences of injective objects. Now we can omit the horizontal sequence in the middle and consider the vertical composite maps. Then we can apply [Rot09, Theorem 6.13] which shows the claim.

Consider now an extension of \( A \) by \( B \) in the category of \( \mathbb{C}^\Sigma \)-modules, i.e., an exact sequence

\[
0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0.
\]

Such extensions are classified by the Yoneda Ext groups (cf. [Rot09, Ch. 7.2] or [Mit65, Ch VII]) that we will denote by \( \text{YExt}^1(A, B) \). Let us recall the comparison isomorphism (cf. [Rot09, Ch 7.2 Lemma 7.27 and Theorem 7.35] or the injective version in [Mit65, Ch VII, Section 7]) between \( \text{Ext}^1(A, B) \) and \( \text{YExt}^1(A, B) \). Let \( I \) be an injective resolution of \( B \). Then the identity \( \phi = \text{id}_B \) can be lifted to a chain map \( \Phi \)

\[
0 \rightarrow B \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \ldots
\]

Because of the quadrant on the right the morphism \( \Phi_1 \) represents a cocycle class \([\Phi_1]\) in \( \text{Ext}^1_{\mathbb{C}^\Sigma}(A, B) \). An analogous reasoning applies also to extensions in the category of \( \mathcal{O}_X \)-modules. Moreover, the functor \( S \) defines a canonical morphism \( \text{YExt}^1_{\mathbb{C}^\Sigma}(A, B) \rightarrow \text{YExt}^1_{\mathcal{O}_X}(S(A), S(B)) \) that we will also denote by \( S_* \). We have the following comparison result:

**Proposition 3.2.** The following diagram is commutative:

\[
\begin{array}{c}
\text{YExt}^1_{\mathbb{C}^\Sigma}(A, B) \xrightarrow{\text{C}_{\mathbb{C}^\Sigma}} \text{Ext}^1_{\mathbb{C}^\Sigma}(A, B) \\
\downarrow S_* \\
\text{YExt}^1_{\mathcal{O}_X}(S(A), S(B)) \xrightarrow{\text{C}_{\mathcal{O}_X}} \text{Ext}^1_{\mathcal{O}_X}(S(A), S(B))
\end{array}
\]
where \( C_{\Sigma} \) and \( C_{\Sigma_X} \) are the comparison isomorphisms.

**Proof.** Let \( 0 \to B \to E \to A \to 0 \) represent an extension class \([E]\) in \( YExt_{\Sigma}(A, B)\). Then \( C_{\Sigma} \) maps \([E]\) to the class \([\Phi_1]\), and \( S_n \) maps \([\Phi_1]\) to the class \([F_1 \circ S(\Phi_1)]\), where \( F_1 \circ S(\Phi_1) \) is the horizontal map in the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & S(B) \\
id_{S(B)} & & F_0 \\
\downarrow & & \downarrow F_1 \\
0 & \longrightarrow & S(I^0) \\
S(id_B) & & S(\Phi_0) \\
\downarrow & & \downarrow S(\Phi_1) \\
0 & \longrightarrow & S(E) \\
& & S(A) \\
\end{array}
\]

On the other side \( \psi = id_{S(B)} \) can be extended to a chain map \( \Psi \)

\[
\begin{array}{ccc}
0 & \longrightarrow & S(B) \\
id_{S(B)} & & \Psi_0 \\
\downarrow & & \downarrow \Psi_1 \\
0 & \longrightarrow & S(E) \\
& & S(A) \\
\end{array}
\]

and so the class \([E]\) under the map \( C_{\Sigma_X} \circ S_n \) is mapped to the class \([\Psi_1]\).

Since \( \Psi \) and \( F \circ S(\Phi) \) are two chain maps over \( id_{S(B)} \), and because the comparison isomorphism \( C_{\Sigma_X} \) is well defined, the two morphisms \( \Psi_1 \) and \( F_1 \circ S(\Phi_1) \) must represent the same class in \( Ext^1_{\Sigma_X}(S(A), S(B)) \). Therefore, the diagram commutes. \( \square \)

**Remark 3.3.** Although it is not necessary here, the results of this chapter can be stated in more generality. We would like to refer the interested reader to an earlier version (arxiv.org/abs/1102.3006v2, Sections 2 and 3) of this preprint for further details.

Finally, we calculate the dimensions of some Ext groups:

**Proposition 3.4.**

a) If \( X \) is a complex torus of dimension \( g \), then \( Ext^n_{\Sigma_X}(\mathcal{O}_X, \mathcal{O}_X) \)

is a \( \mathbb{C} \)-vector space of dimension 1, if \( n = 0 \), and of dimension \( g \), if \( n = 1 \).

b) If \( X \) is a compact Riemann surface of genus \( g \), then \( Ext^n_{\Sigma_X}(\mathcal{O}_X, \mathcal{O}_X) \)

is a \( \mathbb{C} \)-vector space of dimension 1, if \( n = 0 \), and of dimension \( g \), if \( n = 1 \). For any sheaf \( F \) of \( \mathcal{O}_X \)-modules, the \( \mathbb{C} \)-vector spaces \( Ext^n_{\Sigma_X}(\mathcal{O}_X, F) \) vanish for all \( n \geq 2 \).

c) If \( \Sigma \) is a free abelian group of rank \( g \), then \( Ext^n_{\Sigma}(\mathbb{C}, \mathbb{C}) \)

is a \( \mathbb{C} \)-vector space of dimension 1, if \( n = 0 \), and of dimension \( g \), if \( n = 1 \).

d) If \( \Sigma \) is a free group of \( g \) free generators, then \( Ext^n_{\Sigma}(\mathbb{C}, \mathbb{C}) \)

is a free \( \mathbb{C} \)-module of rank 1, and \( Ext^1_{\Sigma}(\mathbb{C}, \mathbb{C}) \) is a free \( \mathbb{C} \)-module of rank \( g \). For any \( \mathbb{C} \Sigma \)-module \( M \) the \( \mathbb{C} \)-vector spaces \( Ext^n_{\Sigma}(\mathbb{C}, M) \) vanish for all \( n \geq 2 \).

**Proof.** The assertions in a) and b) are well known (See e.g. [Gun67] or [BiLa04]). For c) and d) the claims for \( n = 0, 1 \) follow from [CaEi56] X §4, (5) and (6)], and the second claim of d) follows from [CaEi56] X §5. \( \square \)
4 Unipotent bundles and unipotent representations

Let \( R \) be a commutative ring with unit, let \( \Sigma \) be a group, and let \((X, \mathcal{O}_X)\) be a ringed space (commutative with unit).

**Definition 4.1.**

a) A \( R\Sigma \)-module \( M \) is called unipotent of level \( r \) \((r \in \mathbb{N})\), if there exists a filtration of \( R\Sigma \)-modules \( 0 = M_0 \subset \ldots \subset M_{r-1} \subset M_r = M \), such that \( M_{i+1}/M_i \cong R \) for \( i = 0, \ldots, r - 1 \). We denote the collection of unipotent \( R\Sigma \)-modules of level \( r \) by \( \text{Un}^r_{\mathcal{O}_X} \) and by \( \text{Un}^r_{R\Sigma} \) its union over all \( r \geq 0 \). These are full subcategories of the category of \( R\Sigma \)-modules.

b) A \( \mathcal{O}_X \)-module \( F \) on \( X \) is called unipotent of level \( r \) \((r \in \mathbb{N})\), if there exists a filtration of sub \( \mathcal{O}_X \)-modules \( 0 = F_0 \subset \ldots \subset F_{r-1} \subset F_r = F \), such that \( F_{i+1}/F_i \cong \mathcal{O}_X \) for \( i = 0, \ldots, r - 1 \). We denote collection of unipotent \( \mathcal{O}_X \)-modules of level \( r \) by \( \text{Un}^r_{\mathcal{O}_X} \) and \( \text{Un}^r_{\mathcal{O}_X} \) its union over all \( r \geq 0 \). These are full subcategories of the category of \( \mathcal{O}_X \)-modules.

**Remark 4.2.** If \( M \) is a unipotent \( R\Sigma \)-module of level \( r \), then \( M \) is also a free \( R \)-module of rank \( r \). If \( F \) is a unipotent \( \mathcal{O}_X \)-module of level \( r \) on \( X \), then \( F \) is a locally free \( \mathcal{O}_X \)-module of rank \( r \) (also called vector bundle). Both properties follow by induction using [EGA, 4 I Ch 0 (5.4.9)].

**Remark 4.3.** If \( M \in \text{Un}^r_{R\Sigma} \), then after choosing a basis of \( M \), we may interpret \( M \) as a representation \( \rho : \Sigma \to \text{GL}_r(R) \). If \( R = \mathbb{C} \) is the field of complex numbers, then by Kolchins’s Theorem [Ser92, Part I, Chapter V] one may choose the basis of \( M \) in such a way that for all \( \sigma \in \Sigma \), the matrix \( \rho(\sigma) \) is upper triangular with ones on the diagonal. In this case one can show [Lek01, proof of Proposition 11.4] that there is a unipotent \( \mathbb{C}\Sigma \)-module \( M_{r-1} \) of level \( r - 1 \) and an exact sequence

\[
0 \to \mathbb{C} \to M \to M_{r-1} \to 0. \tag{1}
\]

This alternative description of unipotent representations will be used later.

In [Sim92], an important correspondence between flat bundles on compact Kähler manifolds and certain classes of Higgs bundles is obtained. When considering unipotent vector bundles, this result has the following consequence.

**Proposition 4.4.** (Simpson, [Sim92]) Let \( Y \) be a complex compact Kähler manifold. Then every unipotent vector bundle \( U \) on \( Y \) admits a flat holomorphic connection.

**Proof.** Since \( U \) is a successive extension of the trivial bundle \( \mathcal{O}_Y \), it follows by induction that all the rational characteristic classes of \( U \) of positive degree vanish. Moreover, \( U \) is also a successive extension of the stable vector bundle \( \mathcal{O}_Y \). Therefore, it follows from [Sim92 Section 3 - Examples, remarks following Lemma 3.5, p. 36/37] that \( U \) admits a flat holomorphic connection. \( \square \)

**Remark 4.5.** Note also that any unipotent vector bundle over a compact Kähler manifold is always semi-stable.
5 The case of complex tori

As in the introduction let \( X = V / \Lambda \) be a complex torus of dimension \( g \), where \( \Lambda \) is canonically identified with \( \pi_1(X) \).

**Lemma 5.1.** Let \( U \) be a unipotent \( O_X \)-module of level \( r \) on \( X \). Then there exists a unipotent \( C\Lambda \)-module \( \rho \) of level \( r \), such that \( E_\rho \) is isomorphic to \( U \).

**Proof.** Since a complex torus is a compact Kähler manifold, by Proposition 4.4 the vector bundle \( U \) over \( X \) admits a flat holomorphic connection. So, there exists a \( C\Lambda \)-module \( \rho \), such that \( E_\rho \cong U \), and we need to show that one can choose such a module that is unipotent. We claim that each indecomposable component of \( U \) is also unipotent: By [Mat59, Théorème 3], each indecomposable component of \( U \) is isomorphic to a vector bundle of the form \( L_c \otimes U_c \), where \( L_c \) is a line bundle of degree zero, and \( U_c \) is an indecomposable unipotent vector bundle. If we have such a (nonzero) component, then we obtain an injection \( L_c \hookrightarrow U \), and since \( U \) is unipotent, we can form the following commutative diagram

\[
\begin{array}{c}
0 \rightarrow U_{r-1} \rightarrow U \rightarrow \mathcal{O}_X \rightarrow 0,
\end{array}
\]

where \( U_{r-1} \) is unipotent of level \( r - 1 \), and the sequence is exact. If \( L_c \) is nontrivial, then the diagonal map must be zero since \( \text{Hom}_{\mathcal{O}_X}(L_c, \mathcal{O}_X) \) would vanish in this case. Therefore the inclusion \( L_c \hookrightarrow U \) must factor over the kernel \( U_{r-1} \), so we obtain a new inclusion \( L_c \hookrightarrow U_{r-1} \). By inverse induction we could find then an injection \( L_c \hookrightarrow 0 \) what is clearly a contradiction. This shows that each indecomposable component of \( U \) is unipotent, and henceforth we may assume that \( U \) is indecomposable. In this case the \( C\Lambda \) module \( \rho \) is indecomposable as well (use that \( S \) is compatible with direct sums). Since \( \Lambda \) is abelian, such an indecomposable \( C\Lambda \)-module must be isomorphic to the tensor product \( \alpha \otimes \beta \), where \( \alpha \) is a \( C\Lambda \)-module of \( C \)-dimension one, and \( \beta \) is a unipotent \( C\Lambda \)-module. (This can be seen analogously as in [Lud09a, Corollary 2.7]) By the same reasoning as above we see that the line bundle \( E_\alpha \) must be trivial since \( U \cong E_\alpha \otimes E_\beta \) is indecomposable and unipotent. Therefore we may replace \( \rho \) by \( \alpha^{-1} \otimes \beta \), and so we have found a \( \rho \) that is unipotent and satisfies \( E_\rho \cong U \), this shows our claim.

Consider now the morphism \( \alpha : \pi_1(X) = \Lambda \rightarrow \Sigma \) defined just before Theorem 1.1. Using [Mor59, Lemme 5.1] we can show the following proposition:

**Proposition 5.2.** The Schottky functor \( S \) is essentially surjective for unipotent objects, i.e., for each unipotent \( O_X \)-module \( U \) on \( X \) of level \( r \), there exists a unipotent \( C\Sigma \)-module \( \rho \) of level \( r \), such that \( E_\rho = S(\rho) \) is isomorphic to \( U \).

**Proof.** It follows from Lemma 5.1 that there exists a unipotent representation \( \rho : \Lambda \rightarrow GL(W) \), such that \( E_\rho \) is isomorphic to \( U \), and we need to find such a representation that factors over \( \Sigma \). Denote by \( Un(W) \) the unipotent subgroup of \( GL(W) \). Let \( H \) be the smallest commutative Lie group containing \( \rho(\Lambda) \) inside \( Un(W) \), so that:

\[
\rho(\Lambda) \subset H \subset Un(W).
\]
Note that any holomorphic function \( f : V \to H \), due to the commutativity of \( H \), verifies

\[
f(\lambda + z) \cdot \rho(\lambda) \cdot f(z)^{-1} = f(\lambda + z) \cdot f(z)^{-1} \cdot \rho(\lambda).
\]

Also because \( H \) is commutative, there is a well defined notion of logarithm: \( \log : H \to \mathfrak{h} \) where \( \mathfrak{h} \) is the Lie algebra of \( H \). Let \( A_j := -\log(\rho(\lambda_j + j)) \in \mathfrak{h} \), \( j = 1, \ldots, g \), for the generators \( \lambda_j + j \in I \cdot \mathbb{Z}^g \subset \Lambda \), and consider the \( \mathfrak{h} \)-valued 1-form \( \omega = A_1 dz_1 + \cdots + A_g dz_g \) where \( z_1, \ldots, z_g \) are coordinates on \( V \) dual to the basis \( e_1, \ldots, e_g \). Let \( f(z) := \exp(\int_0^z \omega) \). One easily checks that this is well defined and belongs to \( GL(W) \). Then \( f(\lambda + z) \cdot f(z)^{-1} = \exp(\int_0^z \lambda + \omega) \) and this expression is independent of \( z \) as \( \omega \) has constant coefficients. So, for \( \lambda \in I \cdot \mathbb{Z}^g \) we can uniquely write \( \lambda = c_1 e_1 + \cdots + c_g e_g \) with \( c_j \in \mathbb{Z} \) and we have

\[
f(\lambda + z) \cdot f(z)^{-1} = \exp(\int_0^\lambda \omega)
\]

\[
= \exp(\int_0^\lambda A_1 dz_1 + \cdots + A_g dz_g)
\]

\[
= \exp(c_1 A_1 + \cdots + c_g A_g)
\]

\[
= \exp(A_1)^{c_1} \cdots \exp(A_g)^{c_g}
\]

\[
= \rho(e_1)^{-c_1} \cdots \rho(e_g)^{-c_g}
\]

\[
= \rho(c_1 e_1 + \cdots + c_g e_g)^{-1}
\]

\[
= \rho(\lambda)^{-1}
\]

where we used \( \int_{e_j} dz_k = \delta_{jk} \). This means that, defining

\[
\sigma(\lambda) = f(\lambda + z) \cdot f(z)^{-1} \cdot \rho(\lambda),
\]

we have shown that \( \sigma \) is a Schottky representation w.r.t. \( \alpha \), because \( \sigma(\lambda) = 1 \) for all \( \lambda \in I \cdot \mathbb{Z}^g \), i.e., \( \sigma \) factors over \( \Sigma \). Moreover, \( \rho \) and \( \sigma \) define isomorphic vector bundles, by [Mor59, Lemme 5.1].

**Lemma 5.3.** The map

\[
S_* : Y\text{Ext}^1_{\mathbb{C}^\Sigma}(\mathbb{C}, \mathbb{C}) \longrightarrow Y\text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)
\]

is an isomorphism of \( g \)-dimensional \( \mathbb{C} \)-vector spaces.

**Proof.** Both spaces are \( \mathbb{C} \)-vector spaces of dimension \( g \) by Proposition 3.3 a) and c), and because of the comparison isomorphisms \( C_{\Sigma} \) and \( C_{\mathcal{O}_X} \) from Proposition 3.2 So, it suffices to show that \( S_* \) is injective. Let \( Y := (\mathbb{C} \hookrightarrow \rho \to \mathbb{C}) \) represent an extension class \([Y]\) in \( Y\text{Ext}^1_{\mathbb{C}^\Sigma}(\mathbb{C}, \mathbb{C}) \), and assume that \( S_*(\mathcal{Y}) \) is trivial. Then \( S_*(\mathcal{Y}) \) splits and so \( S(\rho) \) is isomorphic to the trivial rank two \( \mathcal{O}_X \)-module. After choosing a basis, we may assume that \( \rho \) is a representation into \( GL_2(\mathbb{C}) \) and given in matrix form by

\[
\rho(\lambda) = \begin{pmatrix} 1 & \rho'(\lambda) \\ 0 & 1 \end{pmatrix},
\]

for a group homomorphism \( \rho' : \Lambda \to \mathbb{C} \). It follows then from [Mat59, Lemme 6.3] that \( \rho' \in \text{Hom}_\mathbb{C}(V, \mathbb{C}) = \text{Hom}_\mathbb{Z}(I \cdot \mathbb{Z}^g, \mathbb{C}) \) because \( S(\rho) \) is decomposable.
But by definition of $\alpha$, we must have $\rho' \in \text{Hom}_C(Z \cdot Z^g, C)$ so $\rho'$ and $\rho$ must be trivial. Finally, since $\rho$ is trivial, $Y$ may be considered as an exact sequence of vector spaces, and such a sequences splits, so $Y$ must represent the trivial extension class.

We are now able to prove Theorem 1.1.

Proof. By Proposition 5.2 we know that all unipotent vector bundles lie in the essential image of the functor $S$. Moreover, since $C$ is a field, we know by Proposition 2.3 c) that $S$ is also faithful. So, it remains to show that $S$ is full as well. If $M_1, M_2$ are two unipotent $C\Sigma$-modules, then $S_*$ defines an injective morphism of $C$-vector spaces $\text{Hom}_{C\Sigma}(M_1, M_2) \rightarrow \text{Hom}_{O_X}(S(M_1), S(M_2))$, since $S$ is faithful. So, it suffices to show that both have the same dimension, or equivalently that $\text{Hom}_{C\Sigma}(C, M_1 \otimes M_2)$ and $\text{Hom}_{O_X}(S(M_1)^* \otimes S(M_2))$ have the same dimension ($^*$ denotes dual). It is also easy to see that $M_1 \otimes M_2$ is unipotent. So, our claim follows if we can show that if $U_r$ is a unipotent $C\Sigma$-module, then $S_*$: $\text{Hom}_{C\Sigma}(C, U_r) \rightarrow \text{Hom}_{O_X}(S(C), S(U_r))$ is an isomorphism. We will show this by induction on the level. For $r = 0, 1$ this is trivial, and so let us assume that this is known for all unipotent bundles of level less than $r$. By Section 4, Equation 1, there exists a unipotent $C\Sigma$-module of level $r-1$ and an exact sequence

$$0 \rightarrow C \rightarrow U_r \rightarrow U_{r-1} \rightarrow 0.$$  

Moreover by Proposition 3.1 we have the following commutative diagram in which the horizontal sequences are exact:

$$
\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}_{C\Sigma}(C, C) & \rightarrow & \text{Hom}_{C\Sigma}(C, U_r) & \rightarrow & \text{Hom}_{C\Sigma}(C, U_{r-1}) & \rightarrow & \text{Ext}^1_{C\Sigma}(C, C) \\
& & \downarrow S_* & & \downarrow S_* & & \downarrow S_* & & \\
0 & \rightarrow & \text{Hom}_{O_X}(S(C), S(C)) & \rightarrow & \text{Hom}_{O_X}(S(C), S(U_r)) & \rightarrow & \text{Hom}_{O_X}(S(C), S(U_{r-1})) & \rightarrow & \text{Ext}^1_{O_X}(S(C), S(C))
\end{array}
$$

The second and the fourth vertical arrows are isomorphisms by the induction hypothesis, and it follows from Lemma 5.3 and Proposition 3.2 that the fifth vertical arrow is an isomorphism. It follows then by the five lemma that the middle arrow is an isomorphism as well, and our claim follows by induction. This shows that $S$ is full as well, and so by abstract category it follows that $S$ induces an equivalence.

Remark 5.4. a) If $X$ is an elliptic curve, then it was shown that in [Flo01, Lemma 5] that all unipotent bundles lie in the essential image of the Schottky functor $S$. In the rank two case, an analogous result was already shown in [Mat59, Section 6].

b) In the $p$-adic case, an analogous result was shown in [Lud09b, Lemma 4.3] by a different method.

6 The case of Riemann Surfaces

As in the introduction let $X$ be a compact Riemann Surface of genus $g$, and consider the morphism $\alpha: \pi_1(X) \rightarrow \Sigma$.

We will need the following lemma:
Lemma 6.1. Let $\rho$ be a unipotent $C_{\pi_1(X)}$-module of rank 2, and assume that $\rho$ is induced by a $C_{\Sigma}$-module. If $S(\rho)$ is isomorphic to the trivial rank two vector bundle on $X$, then $\rho$ is trivial.

Proof. (The proof is analogous to [Mat59, Lemme 6.3]) We may assume that $\rho$ is a representation into $GL_2(C)$ and given in matrix form by

$$\rho(\gamma) = \begin{pmatrix} 1 & \rho'(\gamma) \\ 0 & 1 \end{pmatrix},$$

for a group homomorphism $\rho' : \Sigma \to C$. Then since $S(\rho)$ is trivial, we can apply [Flo01, 4 Lemma 2], and so there exists a holomorphic map $f$ from the universal covering $\tilde{X}$ of $X$ into $GL_2(C)$, such that $f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)$. We may fix a point $\tilde{x}_0$ above a base point $x_0$ of $X$, and replace $f(x)$ by $f(x) \cdot f(\tilde{x}_0)^{-1}$ so that we can assume $f(\tilde{x}_0) = I$. Let us set

$$f(x) = \begin{pmatrix} u(x) & v(x) \\ w(x) & z(x) \end{pmatrix}. \tag{5}$$

From the equation $f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)$, one can deduce the following equations: $w(\gamma \cdot x) = w(x)$ and $z(\gamma \cdot x) = z(x)$. Therefore $w$ and $z$ are invariant under the $\pi_1(X)$-action, and so define global holomorphic sections of $X$, and so they must be constant. From our normalization condition we see that $w = 0$ and $z = 1$. This implies also the equation $u(\gamma \cdot x) = u(x)$, so by the same reasoning as before, we can see that $u = 1$. From these computations we obtain the equation $v(\gamma \cdot x) = v(x) + \rho'(\gamma)$. Therefore, the holomorphic differential $dv$ is invariant under the $\pi_1(X)$-action, and so it descends to a holomorphic differential $\omega$ on $X$. We have the following equation:

$$\int_{\gamma} \omega = \int_{\tilde{x}_0}^{\gamma \tilde{x}_0} dv = v(\gamma \tilde{x}_0) - v(\tilde{x}_0) = \rho'(\gamma).$$

By our initial assumptions on $\rho$, the homomorphism $\rho'$ must vanish for the $a$-periods $a_1, \ldots, a_g$. On the other side, there exists a basis $\zeta_1, \ldots, \zeta_g$ of holomorphic differentials on $X$ such that $\int_{a_i} \zeta_k = \delta_{ik}$ [FaKr92, III 2.8]. This implies that $\omega$ must be the trivial differential, and so $\rho'$ and $\rho$ must be trivial.

Lemma 6.2. The map

$$S_* : Y\text{Ext}^1_{C_{\Sigma}}(C, C) \to Y\text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

is an isomorphism of $g$-dimensional $C$-vector spaces.

Proof. Both $C$-vector spaces are of dimension $g$ by Proposition 5.4 b) and d) and the comparison isomorphisms $C_{\Sigma}$ and $C_{\mathcal{O}_X}$ from Proposition 5.2. So it suffices to show that $S_*$ is injective. The proof is now analogous to Lemma 6.1 with the difference that we can apply Lemma 6.1 instead of [Mat59, Lemme 6.3].

We are now able to show Theorem 1.2
Proof. Using the same arguments as in the proof of Theorem\ref{thm1}, we can deduce that $S$ is fully faithful (for unipotent representations). Let us now show using induction that $S$ is essentially surjective. We first are going to show that the map $S_\ast : \Ext^1_{\mathrm{C}\Sigma}(\mathbb{C},U) \to \Ext^1_{\mathrm{C}\Sigma}(\mathbb{C},S(U))$ is an isomorphism for all unipotent $\mathrm{C}\Sigma$-modules $U$. For level 1, i.e., $U = \mathbb{C}$ this follows from Lemma \ref{lem6.2} and Proposition \ref{prop3.2}. So, let us assume that this map is an isomorphism for all unipotent $\mathrm{C}\Sigma$-modules of level less than $r$, and let us fix a unipotent $\mathrm{C}\Sigma$-module $U_r$ of rank $r$. Then since $U_r$ is unipotent, there exists a unipotent $\mathrm{C}\Sigma$-module $U_{r-1}$ of level $r-1$ and an exact sequence

$$0 \to U_{r-1} \to U_r \to \mathbb{C} \to 0. \quad (6)$$

Moreover, by Propositions \ref{prop3.1} and \ref{prop3.2} we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
\Hom(\mathbb{C},\mathbb{C}) & \xrightarrow{\delta} & \Ext^1(\mathbb{C},U_{r-1}) & \xrightarrow{S_r} & \Ext^1(\mathbb{C},U_r) & \xrightarrow{S_r} & \Ext^1(\mathbb{C},\mathbb{C}) & \xrightarrow{S_r} & 0 \\
\Hom(\mathbb{C},S(\mathbb{C})) & \xrightarrow{\delta} & \Ext^1(S(\mathbb{C}),S(U_{r-1})) & \xrightarrow{S_r} & \Ext^1(S(\mathbb{C}),S(U_r)) & \xrightarrow{S_r} & \Ext^1(S(\mathbb{C}),S(\mathbb{C})) & \xrightarrow{S_r} & 0 \\
\end{array}
\]

(The two terms $\Ext^2(\mathbb{C},U_{r-1})$ and $\Ext^2(S(\mathbb{C}),S(U_{r-1}))$ on the right hand side vanish, because of Proposition \ref{prop3.4} b) and d)). It is clear that the first vertical arrow is an isomorphism, and the second and fourth vertical arrows are isomorphisms by induction hypothesis. Now, it follows from the five lemma that the third vertical arrow is an isomorphism, so our claim follows by induction. We can now show that $S$ is essentially surjective. This is known in the level one case, so let us assume that all unipotent vector bundles of level less than $r$ are in the essential image of $S$. Let us fix a unipotent vector bundle $G_r$ of level $r$. Then since $G_r$ is unipotent, there exists a unipotent vector bundle $G_{r-1}$ of level $r-1$ and an exact sequence

$$0 \to G_{r-1} \to G_r \to \mathcal{O}_X \to 0, \quad (7)$$

defining a class in $Y\Ext^1(\mathcal{O}_X,G_{r-1})$. By induction hypothesis there exists an unipotent representation $U_{r-1}$ with $S(U_{r-1}) = G_{r-1}$. Then by Proposition \ref{prop3.2} the morphism $S_\ast : Y\Ext^1(\mathbb{C},U_{r-1}) \to Y\Ext^1(\mathcal{O}_X,G_{r-1})$ is surjective and so there exists an exact sequence of unipotent $\mathrm{C}\Sigma$-modules

$$0 \to U_{r-1} \to U_r \to \mathbb{C} \to 0, \quad (8)$$

and a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{S(U_{r-1})} & S(U_r) & \xrightarrow{S(\mathbb{C})} & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{G_{r-1}} & G_r & \xrightarrow{\mathcal{O}_X} & 0. \\
\end{array}
\]

By the five lemma the middle arrow must be an isomorphism so $G_r$ lies in the essential image of $S$, and by induction it follows that $S$ is essentially surjective. Finally, since $S$ is fully faithful and essentially surjective, it follows by general category theory that $S$ induces an equivalence of categories for unipotent objects. \hfill $\square$
7 Flat and Schottky bundles over complex tori

Let $X$ be a complex torus, and let $E$ be a vector bundle over $X$. The bundle $E$ is called homogeneous if $t_a^*E \cong E$ for all $a \in X$, where $t_a : X \to X$, $t_a(x) = x + a$ denotes the translation-by-$a$ morphism. The theorem of Matsushima and Morimoto can be stated as follows (see [Mat59] and [Mor59]).

**Theorem 7.1** (Matsushima, Morimoto). Let $E$ be a vector bundle over a complex torus $X$. Then the following properties are equivalent:

a) $E$ admits a flat connection.

b) $E$ admits a flat holomorphic connection.

c) $E$ is homogeneous.

d) Every indecomposable component of $E$ has the form $L \otimes E_\rho$ where $L$ is a line bundle of degree zero over $X$ and $\rho$ is a unipotent representation.

As in the introduction, let us write $X = V/\Lambda$, and consider the morphism $\alpha : \Lambda = \pi_1(X) \to \Sigma$ from which Schottky bundles over $X$ are defined.

**Lemma 7.2.** Every line bundle of degree zero over a complex torus is Schottky.

**Proof.** The proof is entirely analogous to that of Proposition 5.2. Any degree zero line bundle on $X = V/\Lambda$ is necessarily flat, so it is of the form $L_\rho$, for some $\rho : \pi_1(X) = \Lambda \to GL_1(\mathbb{C}) = \mathbb{C}^*$. As $\exp$ is surjective, we can find $A_j \in \mathbb{C}$ for $j = 1, \ldots, g$, such that $\exp(A_j) = \rho(e_j)^{-1}$, and we can define $\omega := A_1dz_1 + \ldots + A_gdz_g$ and $f(z) = \exp(\int_0^z \omega)$. Then, the same computation as in Proposition 5.2 gives $f(\lambda + z) \cdot f(z)^{-1} = \rho(\lambda)^{-1}$ for all $\lambda \in I \cdot \mathbb{Z}^g$ which implies that the representation defined by $\sigma(\lambda) := f(\lambda + z) \cdot f(z)^{-1} \cdot \rho(\lambda)^{-1}$ factors over $\Sigma$, and produces a line bundle isomorphic to $L_\rho$. 

As a consequence we can prove **Theorem 1.3**:

**Proof.** Clearly, if $E$ is Schottky then it admits a flat connection. Conversely, if $E$ admits a flat connection, then using Theorem 7.1, each of its indecomposable components is of the form $L \otimes E_\rho$ where $\rho$ is unipotent and $L$ is a line bundle of degree zero. Thus, by Theorem 1.1 and Lemma 7.2, $E_\rho$ and $L$ are both Schottky. The theorem then follows since tensor products and direct sums of Schottky bundles are Schottky. 

8 Unipotent and Schottky principal $G$-bundles

8.1 Principal and flat $G$-bundles

Let $X$ be a complex manifold. Let $G$ be an affine algebraic group over $\mathbb{C}$ (see for example [Bor91, Chapter I]) and let $P$ be a holomorphic principal $G$-bundle over $X$. For definitions and elementary properties of holomorphic principal bundles, see for example [Bor91].

Recall that principal $G$-bundles over $X$ form a category and in particular, the notion of isomorphism between principal $G$-bundles is naturally defined.
Recall also a few important constructions. Given another complex affine algebraic group $H$, a principal $H$-bundle over $X$ and a group homomorphism $\phi : H \to G$, we can construct a principal $G$-bundle $\phi_* P$ over the same manifold $X$, by defining it as $P \times G / \sim$ where $(p, g) \sim (p \cdot h, \phi(h)^{-1}g)$ for all $p \in P$, $h \in H$ and $g \in G$. The bundle $\phi_* P$ is said to be the $G$-bundle induced by the morphism $\phi$.

One other related construction is that of an associated vector bundle. Given a representation of $H$, as a map $\psi : H \to GL(M)$, where $M$ is some finite dimensional vector space, we can construct the vector bundle $\psi_* P$ defined by $P \times M / \sim$ where now $(p, m) \sim (p \cdot h, \psi(h)^{-1}m)$. The bundle $\psi_* P$ is called the vector bundle associated to the representation $\psi$. Since the two above constructions are intimately related, it is clear that $\psi_* P$ can be also regarded as a principal $GL(M)$-bundle.

**Definition 8.1.** Given a $G$-bundle $P$ and a subgroup $i : H \hookrightarrow G$, we say that $P$ admits a reduction of structure group to $H$ (or a $H$-structure) if there exists a $H$-bundle $P_H$ such that $i_* P_H \cong P$ (as principal $G$-bundles). The bundle $P_H$ is then called a reduction of $P$ to a $H$-bundle.

For example, a principal $G$-bundle is trivializable, in the sense that it is isomorphic to the trivial principal $G$-bundle given by the projection onto the second factor $G \times X \to X$, if and only if it admits a trivial structure, that is a reduction of structure group to the trivial subgroup.

**Lemma 8.2.** Let $H \subset G$ be a subgroup and $G'$ another group. Let $P$ be a $G$-bundle and $\phi : G \to G'$ a homomorphism. If $P$ has a $H$-structure, then $\phi_* P$ has a $\phi(H)$-structure.

**Proof.** Consider the commutative diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\phi} & \phi(H) \\
i & \downarrow & \downarrow j \\
G & \xrightarrow{\phi} & G'.
\end{array}
$$

If $P$ has a $H$-structure then there exists $P_H$, a $H$-bundle such that $i_* P_H \cong P$ as $G$-bundles. Then, $\phi_* P_H$ is a $\phi(H)$-bundle and $j_*$ of it is isomorphic to $\phi_* P$ as $G'$-bundles because

$$j_* (\phi_* P_H) = (j \circ \phi)_* P_H = (\phi \circ i)_* P_H = \phi_* i_* P_H \cong \phi_* P.$$

So, $\phi_* P$ has a $\phi(H)$-structure as wanted. \hfill $\square$

Let $\pi_1(X)$ denote the fundamental group of $X$ (for some base point of $X$).

In the same way as with vector bundles, one can define natural bijections between: (a) principal $G$-bundles with a flat connection (flat $G$-bundles, for short), (b) $G$-local systems, and (c) representations of $\pi_1(X)$ into $G$. By connection we will always mean holomorphic connection.

In particular, one has the following natural construction. For a given representation $\rho \in \text{Hom}(\pi_1(X), G)$, we can associate a principal $G$-bundle $P_\rho$ with a flat connection. It can be defined using the principal $\pi_1(X)$-bundle $\tilde{X} \to X$, the universal cover of $X$, as the $G$-bundle associated to the morphism $\rho : \pi_1(X) \to G$.
(the same construction as before, which works even though $\pi_1(X)$ is, of course, not an affine algebraic group). Conversely, given a principal $G$-bundle $P$ with a flat connection, its holonomy representation provides a natural representation (modulo conjugation by an element in $G$). The construction of a vector bundle $E_P$ from a representation $\rho : \pi_1(X) \to GL(M)$ is an instance of this construction.

If $P$ is a principal bundle with a flat connection, then for any group morphism $\phi : G \to G'$ the associated bundle $\phi_*P$ receives an induced flat connection.

8.2 Principal unipotent bundles

Let us recall the definition of (algebraic) unipotent groups. Let $H$ be an affine algebraic group over $C$. It has a faithful linear representation into some general linear group, $\varphi : H \to GL(M)$, for some finite dimensional $C$-vector space $M$ (see [Bor91]). An element $h \in H$ is called unipotent if there is a natural number $n$ such that $(\varphi(h) - 1)^n = 0$, where $1 \in GL(M)$ is the identity. Then $H$ is called unipotent if all its elements are unipotent. It is known that any unipotent algebraic group is isomorphic to a closed subgroup of the group of upper triangular matrices with diagonal entries 1, and conversely any such subgroup is unipotent (cf. Remark 4.3).

Definition 8.3. A principal $G$-bundle $P$ is called unipotent if there is a reduction of structure group from $G$ to a unipotent subgroup $U \subset G$.

Denote by $Ad : G \to GL(g)$ the adjoint representation of $G$ into its Lie algebra $g$. Given a principal $G$-bundle $P$, the associated vector bundle $AdP = Ad_*P$ will now play an important role. Let $ZG$ denote the center of $G$, and note that $G/ZG$ is again an affine algebraic group. It is clear that $Ad$ factors through $G/ZG$, and denote by $Ad_0 : G/ZG \to GL(g)$ the induced representation.

Recall that, when $G$ is connected, the homomorphism $Ad_0 : G/ZG \to GL(g)$ is faithful, since for a connected algebraic group $\ker Ad = ZG$.

Proposition 8.4. Let $P$ be a principal $G$-bundle and assume that $ZG$ is trivial. Then, $P$ is unipotent if and only if $AdP$ is a unipotent vector bundle.

Proof. Since $G$ has no center, $Ad$ is faithful. Let $g_{ij} : U_{ij} \to G$ be (holomorphic) transition functions for $P$, so that $Ad(g_{ij})$ are transition functions for $AdP$. Suppose that $AdP$ is a unipotent vector bundle, i.e., a successive extension of the trivial line bundle $O_X$ over $X$. Then, wlog we can suppose that there is a basis of $g$, the fiber of $AdP$, with respect to which $Ad(g_{ij})$ is upper triangular with 1’s in the diagonal. Then, $(Ad(g) - 1)^n = 0$ which, by definition implies that all the transition functions $g_{ij}$ have values in a unipotent subgroup $U \subset G$. In turn, this means that $P$ has a $U$-structure and is therefore unipotent. Conversely, if $P$ has a reduction of structure group to a unipotent subgroup $U \subset G$ then $AdP$ has a reduction of structure group to $AdU$ because of Lemma 8.2. And $AdU$ is clearly a unipotent subgroup of $GL(g)$. In this direction, we do not need to impose the condition $ZG = 1$.

We now introduce the following definition.

Definition 8.5. Let $H$ be an affine algebraic group. An element $h \in H$ is called $Ad$-unipotent if $(Ad(h) - 1)^n = 0$ for some natural number $n$. $H$ is called an
Ad-unipotent group if all its elements are Ad-unipotent. If $P$ is a principal $G$-bundle, then $P$ is called Ad-unipotent if there is a reduction of structure group from $G$ to an Ad-unipotent subgroup $H \subset G$.

Now, we can show an analogue of the previous proposition, without requiring that $G$ has a trivial center.

**Proposition 8.6.** Let $P$ be a principal $G$-bundle. Then, $P$ is Ad-unipotent if and only if $AdP$ is a unipotent vector bundle.

**Proof.** If $P$ is Ad-unipotent, with reduction of structure group to an Ad-unipotent subgroup $H \subset G$, then Ad$P$ has a reduction of structure group to Ad$H \subset GL(g)$, which is easily verified to be a unipotent subgroup. So Ad$P$ is a unipotent vector bundle. Conversely, let $g_{ij} : U_{ij} \to G$ be (holomorphic) transition functions for $P$, so that Ad$(g_{ij})$ are transition functions for Ad$P$. As before, if Ad$P$ is a unipotent vector bundle, then, wlog we can suppose that there is a basis of $g$, the fiber of Ad$P$, with respect to which Ad$(g_{ij})$ is upper triangular with 1’s in the diagonal. Then, $(Ad(g) - I)^n = 0$ which, by definition implies that all the transition functions $g_{ij}$ have values in a Ad-unipotent subgroup $H \subset G$, which means that $P$ has a $H$-structure and is therefore Ad-unipotent.

Recall that, given an element $g \in G$, the Jordan decomposition theorem enables us to write $g = su$ where $s$ is semisimple and $u$ is unipotent.

**Proposition 8.7.** Let $X$ be a complex torus and let $E$ be a homogeneous vector bundle over $X$ isomorphic to Ad$P$ for a principal $G$-bundle $P$. Then, $E$ is unipotent.

**Proof.** Let $E$ be homogeneous. By Theorem 7.4 $E$ is flat, and since it is isomorphic to Ad$P$ for some principal $G$-bundle $P$, we can write $E = E_ρ$ for some representation $ρ = Adσ : π_1(X) → GL(g)$ with $σ : π_1(X) → G$.

Assume now that $E$ is indecomposable. Then by Theorem 8.1 $E$ is isomorphic to $L_θ ⊗ U$ where $L$ is a degree zero line bundle and $U$ is unipotent. This means that $E = E_ρ = L_θ ⊗ U$ for some representations $θ : π_1(X) → Z(GL(g))$, $τ : π_1(X) → U(GL(g))$ where $Z(GL(g))$, $U(GL(g))$ denote, respectively the center of $GL(g)$ and the subgroup of upper triangular matrices with 1’s in the diagonal (upon choosing a basis for $g$). It is clear that, for any $γ \in π_1(X)$, $ρ(γ) = θ(γ)τ(γ)$ is the Jordan decomposition of $ρ(γ)$ inside $GL(g)$. Let $σ(γ) = su$, with $s$ semisimple and $u$ unipotent in $G$. Since the Jordan decomposition is preserved under morphisms and $ρ = Adσ$, we obtain $Ad$s = $θ(γ)$ and $Adu = τ(γ)$. An easy computation shows that if $Adσ \in Z(GL(g))$ then $s \in ZG$. But then $1 = Ads = θ(γ)$. Since this is true for all $γ \in π_1(X)$ we conclude that $L_θ$ is the trivial line bundle, so $E$ is unipotent.

Finally, if $E$ is not necessarily indecomposable, the previous argument shows $E$ is a direct sum of unipotent vector bundles, so it is unipotent.

### 8.3 Principal Schottky bundles

Let $P$ be a principal $G$-bundle over a complex manifold $X$ with fundamental group $π_1(X)$. As before let $P_ρ$ be the flat $G$-bundle associated with the representation $ρ ∈ Hom(π_1(X), G)$. Consider a surjective morphism $α : π_1(X) → Σ$ onto some free or free abelian group $Σ$. Recall that $ZG$ denotes the center of a group $G$. 


Definition 8.8. We say that $P$ is a $G$-principal Schottky bundle with respect to $\alpha$ (or $G$-Schottky bundle for short) if $P$ is isomorphic to $P_\rho$ for some representation $\rho$ that satisfies:

$$\rho(\gamma) \in ZG, \quad \text{for all } \gamma \in \ker \alpha.$$ 

Such a representation is called a Schottky representation (w.r.t. $\alpha$).

For example, when $X$ is a Riemann surface of genus $g$, and the morphism $\alpha : \pi_1(X) \to \Sigma$ is the one defined in the introduction, then a representation $\rho : \pi_1(X) \to G$ is Schottky if and only if $\rho(a_j) \in ZG$ for all $j = 1, \ldots, g$. This is the motivating example for the previous definition.

Remark 8.9. By the correspondence between representations of $\pi_1(X)$ into $G$ and principal $G$-bundles with a flat connection, it is clear that a necessary condition for a principal $G$-bundle to be Schottky is that it admits a flat connection.

Proposition 8.10. Let $P$ be a principal $G$-bundle. If $P$ is $G$-Schottky (w.r.t. $\alpha$) then $\text{Ad}P$ is a Schottky vector bundle (w.r.t $\alpha$). Moreover, if we assume that $P$ admits a flat connection, then the converse is also true.

Proof. Consider $P = P_\rho$ for some $\rho$ satisfying Definition 8.8 for some $\alpha : \pi_1(X) \to \Sigma$. By construction, $\text{Ad}P_\rho$ is the vector bundle $E_{\text{Ad}\rho}$ associated to the adjoint representation $\text{Ad}\rho : \pi_1(X) \to \text{GL}(g)$, where $g$ is the Lie algebra of $G$. By the properties of $\rho$, $\text{Ad}\rho(\gamma)$ acts as the identity on $g$, for all $\gamma \in \ker \alpha$. So $\text{Ad}P_\rho$ is a Schottky vector bundle with respect to $\alpha$. To prove the second statement, assume that $P$ has a flat connection so that $P = P_\rho$ where $\rho$ is some representation $\rho : \pi_1(X) \to G$. Then $\text{Ad}P \cong E_{\text{Ad}\rho}$. Suppose that $\text{Ad}P$ is Schottky, so that $\text{Ad}\rho(\gamma) = 1$ for all $\gamma \in \Sigma$. This implies that $\rho(\gamma)$ is in the kernel of $\text{Ad}$, which is $ZG$. So, $P$ is a $G$-Schottky bundle. □

Remark 8.11. Let $P$ be a $C^\ast$-bundle corresponding to a line bundle of non-zero degree. Then, even though $\text{Ad}P$ is trivial and has the trivial flat connection, $P$ cannot admit a flat connection, so the hypothesis that $P$ is flat is important in the second statement of Proposition 8.10.

The question whether all $G$-bundles which admit a flat connection are Schottky is an important open problem, whose answer is not known even in the case of Riemann surfaces. The situation is better when $\pi_1(X)$ is abelian: when $X$ is an elliptic curve, all flat vector bundles over $X$ are Schottky.

Let $X$ be a complex torus. We can now prove Theorem 1.4.

Proof. By definition Schottky bundles are flat. Conversely, let $P$ have a flat connection. Then $\text{Ad}P$ has the induced flat connection, as remarked earlier. Therefore, by Theorem 1.3 $\text{Ad}P$ is a Schottky vector bundle. So $P$ is a Schottky $G$-bundle, by Proposition 8.10 this shows the claim. □

In particular, all flat principal bundles over elliptic curves are Schottky.
8.4 Other equivalent conditions for flat bundles

From the work of Biswas-Gómez [BiGo08], we can give a criterion for flatness using the adjoint vector bundle as follows. Given a degree $k$ polynomial $q \in \text{Sym}^k(\mathfrak{g}^*)$ invariant under the adjoint action of $G$, we obtain a characteristic class of degree $k$ of the $G$-bundle $P$, $c_q(P) \in H^{2k}(X, \mathbb{C})$, using the Chern-Weil construction.

**Proposition 8.12.** Let $P$ be a principal $G$-bundle over a complex torus $X$ with vanishing characteristic classes of degree one and two. Then, $P$ is flat (resp. homogeneous, resp. Schottky) if and only if $\text{Ad}P$ is flat (resp. homogeneous, resp. Schottky).

*Proof.* If $P$ admits a flat connection (resp. is homogeneous or Schottky), $\text{Ad}P$ has the induced flat connection (resp. is homogeneous or Schottky) (in this direction, we do not need the vanishing of characteristic classes). Conversely, let $\text{Ad}P$ be flat (equivalently homogeneous or Schottky). Then, by Theorem 7.1 $\text{Ad}P$ is a direct sum of vector bundles of the form $L \otimes U$, where $L$ is a line bundle of degree zero and $U$ is unipotent. Therefore $\text{Ad}P$ is pseudostable (see [BiGo08]). Therefore, by definition, $P$ is pseudostable. Then, by [BiGo08, Theorem 4.1], and using the vanishing of the characteristic classes, $P$ is flat (and is homogeneous, and Schottky).

Note that the one dimensional case (i.e, bundles over an elliptic curve) was obtained by Azad-Biswas [AzBi01]. Note also, that Biswas-Subramanian [BiSu04] showed that if $P$ be a principal $G$-bundle with vanishing characteristic classes of degree one and two (over any complex projective manifold) and $P$ is polystable, then $P$ is flat (with a unitary connection!). In the Proposition above, we are not assuming polystability.

**Corollary 8.13.** Let $P$ be a principal $G$-bundle over a complex torus. If $P$ is flat then it is $\text{Ad}$-unipotent. Moreover, if the first and second Chern classes of $P$ vanish, the converse also holds.

*Proof.* If $P$ is flat then $\text{Ad}P$ is flat, since it has the induced connection. By Theorem 7.1 $\text{Ad}P$ is a homogeneous. By Proposition 8.6 this implies that $\text{Ad}P$ is unipotent. By Proposition 8.7 we obtain that $P$ is $\text{Ad}$-unipotent. Conversely, assume that $P$ is $\text{Ad}$-unipotent. Then $\text{Ad}P$ is unipotent, by the same Proposition, which means $\text{Ad}P$ is pseudostable. So, by [BiGo08, Theorem 4.1], we conclude that $P$ is flat.

*Remark 8.14.* Note that if $P$ is unipotent, then it is flat, by [BiGo08, paragraph after Corollary 3.8].

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**References**

[Ati57] M. G. Atiyah, *Vector Bundles over an Elliptic Curve*, Proc. London Math. Soc. s3-7 (1), p. 414-452 (1957).
[AzBi01] H. Azad, I. Biswas, *On holomorphic principal bundles over a compact Riemann surface admitting a flat connection*, Math. Ann., Volume 322, Number 2, 333-346, (2001).

[Ba09] V. Balaji, *Lectures on principal bundles. Moduli spaces and vector bundles*, 2-28, London Math. Soc. Lecture Note Ser., 359, Cambridge Univ. Press, Cambridge, (2009).

[BiGo08] I. Biswas, T. Gómez, *Connections and Higgs fields on a principal bundle*, Ann Glob Anal Geom 33, p 19-46 (2008).

[BiLa04] C. Birkenhake, H. Lange, *Complex Abelian Varieties*, Springer GMW 302, 629 p (2004).

[BiSu04] I. Biswas, S. Subramanian, *Flat holomorphic connections on principal bundles over a projective manifold*, Trans. Amer. Math. Soc. 356, no. 10, p. 3995-4018 (2004).

[Bor91] A. Borel, *Linear Algebraic Groups*, Springer, Berlin GTM 126 2nd, 308 p (1991).

[Br11] M. Brion, *Homogeneous bundles over abelian varieties*, Preprint 2011, arXiv:1101.2771

[CaEi56] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press (1956).

[DeWe05] C. Deninger, A. Werner, *Line bundles and p-adic characters*, G. van der Geer, B. Moonen, R. Schoof (eds.), Number Fields and Function Fields - Two Parallel Worlds, Progress in Mathematics 239, 101-131, Birkhäuser (2005).

[DeWe10] C. Deninger, A. Werner, *Vector bundles on p-adic curves and parallel transport II*, In I. Nakamura, L. Weng (eds): Algebraic and Arithmetic Structures of Moduli Spaces. Sapporo 2007. Advanced Studies in Pure Mathematics 58 1-26 (2010).

[FaKr92] H. Farkas, I. Kra, *Riemann surfaces*, GTM Springer (1992).

[Fal05] G. Faltings, *A p-adic Simpson correspondence*, Advances in Mathematics 198 (2005).

[Flo01] C. Florentino, *Schottky uniformization and vector bundles over Riemann surfaces*, Manuscripta Math 105 (2001).

[Fre03] P. Freyd, *Abelian categories*, Reprints in Theory and Applications of Categories, No. 3, (2003).

[Gun67] R. Gunning, *Lectures on Riemann Surfaces*, Princeton Academic Press, 254 p (1967).

[EGA] A. Grothendieck, *Éléments de Géométrie Algébrique*, I-IV, Inst. Hautes Études Sci. Publ. Math.:4,8,11,17,20,24,28,32 (1960-1967).
[Lek01] S. Lekaus, *Vector bundles of degree zero over an elliptic curve, flat bundles and Higgs bundles over a compact Kähler manifold*, Dissertation Univ. Duisburg-Essen (2001).

[Lek05] S. Lekaus, *Unipotent flat bundles and Higgs bundles over compact Kähler manifolds*, Trans. Amer. Math. Soc. 357 (2005).

[Lud09a] T. Ludsteck, *Homogeneous p-adic vector bundles on abelian varieties that are analytic tori*, Archiv der Mathematik 92-6 (2009).

[Lud09b] T. Ludsteck, *Homogeneous p-adic vector bundles on abelian varieties that are analytic tori II*, Archiv der Mathematik 93-4 (2009).

[Mat59] Y. Matsushima, *Fibrés holomorphes sur un tore complexe*, Nagoya Math. J. Volume 14, 1-24 (1959).

[Mit65] B. Mitchell, *Theory of Categories*, Boston, MA: Academic Press (1965).

[Mor59] A. Morimoto, *Sur la classifications des espaces fibrés vectoriels holomorphes sur un tore complexe admettant des connexions holomorphes*, Nagoya Math. J.15, 83-154 (1959).

[Rot09] J. Rotman, *An Introduction to Homological Algebra*, Springer, Second Edition (2009).

[Ser92] J.-P. Serre, *Lie algebras and Lie groups*, Lect. Notes in Math., Springer, 1500 (1992).

[Sim92] C. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math 75, p. 5-95 (1992).

[Sza09] T. Szamuely, *Galois Groups and Fundamental Groups*, Cambridge Studies in Advanced Mathematics (No. 117) (2009).