A Bewley-Huggett model with many consumption goods

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Abstract:

We study a pure-exchange incomplete markets model with heterogeneous agents. In each period, the agents choose how much to save and which bundle of goods to consume while their endowments are fluctuating. We focus on a competitive stationary equilibrium (CSE) in which the wealth distribution is invariant, the agents maximize their expected discounted utility, and both the prices of goods and the interest rate are market-clearing. Our main contribution is to extend some general equilibrium results to an incomplete markets setting. Under mild conditions on the agents’ preferences, we show that the aggregate demand for goods depends only on their relative prices and we prove the existence of a CSE. When the agents’ preferences can be represented by a CES (constant elasticity of substitution) utility function with an elasticity of substitution that is higher than or equal to one, we prove that the CSE is unique. Under the same preferences, we show that a higher inequality of endowments does not change the equilibrium prices of goods, and decreases the equilibrium interest rate.

Keywords: Arrow-Debreu model; general equilibrium; heterogeneous agents; Bewley models; dynamic economies.

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1 Introduction

We extend the classic Arrow-Debreu model to a dynamic incomplete markets general equilibrium model with a continuum of agents, in which each agent has an individual state that corresponds to his wealth level. There is an infinite number of periods and in each period agents participate in a pure-exchange Arrow-Debreu model (as in the seminal paper by Arrow and Debreu (1954)). Each agent has a different wealth level and different preferences over consumption bundles, and thus the agents are heterogeneous in the static pure-exchange Arrow-Debreu model. In each period, given the agents’ wealth level and their preferences over consumption bundles, the agents decide how much to spend on a bundle of goods to be consumed in that period, which bundle of goods to consume, and how much to save for future consumption. In the tradition of the Bewley models (see Ljungqvist and Sargent (2012) for a textbook treatment), the markets are incomplete. The agents face uninsurable idiosyncratic risk and can transfer assets from one period to another only by saving in a risk-free bond. In each period, the agents receive a random endowment vector. We assume that all random shocks are idiosyncratic, ruling out aggregate random shocks that are common to all agents. As in Arrow and Debreu (1954), the agents are price takers, that is, the agents take the prices of goods and the risk-free bond’s rate of return as given. In this paper we focus on a pure-exchange economy without production, and hence, the model studied in this paper is closely related to Huggett’s model (Huggett, 1993).

In Huggett’s model, and in most similar Bewley models that are used in applied work, there is only one consumption good. In contrast, in the model presented in this paper, there are many consumption goods and each good has a price. Thus, in addition to the standard inter-temporal savings decision, the agents also make a static decision of how to allocate their spending between the different consumption goods.

This key feature of the model allows us to examine questions relating to wealth distribution, the prices of goods, and the risk-free bond’s rate of return in the framework of a Bewley-Huggett model. Such questions cannot be examined in a Bewley model with one consumption good. Bewley models feature rich heterogeneity and are used to study many economic phenomena.

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1 For similar models with one consumption good see Lucas (1980), Geanakoplos et al. (2014), and Hu and Shmaya (2019).

2 In Bewley (1986) there are multiple consumption goods but the interest rate is fixed and is assumed to be 0 (see also Karatzas et al. (1994)). In this paper the interest rate is determined in equilibrium as in Huggett (1993).

3 There is a vast literature on asset pricing and wealth inequality in different models than the model presented in this paper (for example, Judd et al. (2003), Blume and Easley (2006), Krueger and Lustig (2010), and Kubler and Schmedders (2015), just to name a few).

4 In particular, Bewley models are often used to study wealth inequality (see Heathcote et al. (2009), De Nardi (2013), and Benhabib and Bisin (2010), for surveys).
These include wealth distribution (Benhabib et al., 2015), monetary transmission mechanisms (Kaplan et al., 2018), aggregate demand (Auclert and Rognlie, 2018), and many more. The model presented in this paper, which combines a Bewley model with the classic Arrow-Debreu model, can be used to study the relationship between the prices of goods and other important economic variables in the presence of heterogeneous agents. In this paper we mainly focus on the theoretical analysis of the model, in particular, the existence and the uniqueness of an equilibrium. The paper’s main contribution is to extend some general equilibrium results from the static Arrow-Debreu model to an incomplete markets model in the style of Bewley models.

The solution concept that we study in this paper is the competitive stationary equilibrium. A competitive stationary equilibrium (CSE) consists of a wealth distribution, prices of goods, an interest rate, savings policy functions, and demand functions for goods, such that: (i) given the prices of the goods and the interest rate, the agents choose a savings policy function and a demand function for goods in order to maximize their expected discounted utility; (ii) the wealth distribution induced by the agents’ policy functions is invariant; (iii) the prices of the goods and the interest rate are market-clearing, i.e., for each good, the aggregate supply of that good equals the aggregate demand for that good, and the aggregate supply of savings equals the aggregate demand for savings. This notion of equilibrium is conceptually similar to the notion of mean field equilibrium. In Section 3.4 we compare the solution concept used in this paper - competitive stationary equilibrium - with mean field equilibrium.

Our main theoretical contribution is to provide a proof for the existence of a CSE under fairly general conditions on the agents’ preferences (see Theorem 1). The proof relies on a well-known excess demand approach. We define a natural excess demand function and show that it is continuous, satisfies Walras’ law, and satisfies suitable boundary conditions. In a Bewley model setting, previous existence results assume that there is one consumption good (for example, see Açıkgöz (2018)) or that the interest rate is fixed and is not determined in equilibrium (for example, see Bewley (1986)). These assumptions simplify the analysis of Bewley models considerably as they decouple the aggregate demand for savings and the aggregate demand for goods. We show that some well-known results that apply to the static Arrow-Debreu model also hold in the incomplete markets model studied in this paper. In Proposition 1 we show that the excess demand for savings is homogeneous of degree one in the goods’ prices while the excess demand for each good is homogeneous of degree zero. This result implies that the aggregate demand for goods and the aggregate supply of goods depend only on their relative prices, similar to the static Arrow-Debreu model, and thus the competitive stationary equilibrium depends only on the relative prices of goods. Welfare theorems (e.g., Arrow (1951)) that apply to the static Arrow-Debreu model also hold in the incomplete markets model studied in this paper.
The Arrow-Debreu model do not hold in our setting because of market incompleteness.

We prove the uniqueness of a CSE for the special case that the agents’ preferences over bundles can be represented by a CES (constant elasticity of substitution) utility function with an elasticity of substitution that is equal to or higher than one (see Theorem 2). This assumption on the agents’ preferences implies that the consumption goods are gross substitutes, i.e., the demand for each consumption good increases with the prices of the other consumption goods. We note that the standard argument for proving the uniqueness of an equilibrium in the static Arrow-Debreu model cannot be applied in our setting, since the aggregate demand for consumption goods is not necessarily increasing with the interest rate and the aggregate demand for savings is not necessarily increasing with prices of consumption goods. Thus, the excess demand function does not necessarily satisfy the gross substitutes property.

Our main result regarding the wealth distribution’s influence on the prices of goods and on the interest rate is Theorem 3. We prove that if the agents’ preferences over bundles can be represented by a CES utility function with an elasticity of substitution that is higher than one, then an increase in the risk of the random future endowments (in the sense of the convex stochastic order) changes the CSE in the following way: the interest rate decreases, and the prices of goods do not change. In the classic Arrow-Debreu model the result that the prices of goods do not change when the wealth inequality is higher is intuitive because the demand for each good is linear in wealth. Thus, the demand for each good does not change when the wealth inequality is higher. In our setting, under a CES utility function, the marginal propensity to consume is decreasing, so the demand for each good is concave in wealth. An increase in the risk of the random future endowments increases the aggregate savings because of the precautionary savings effect. Thus, the aggregate demand for each good decreases. At the same time, a decrease in the interest rate decreases the aggregate savings because of the substitution effect. It turns out that in general equilibrium, the precautionary savings effect and the substitution effect offset each other exactly and the prices of goods do not change.

The rest of the paper is organized as follows. Section 2 presents the model. In Section 2.1 we define the CSE. In Section 3 we present the main results of this paper. In Section 3.1 we establish the existence of a CSE. In Section 3.2 we provide conditions that ensure the uniqueness of a CSE. In Section 3.3 we discuss how the wealth distribution influences the prices of goods and the interest rate. In Section 3.4 we compare the current paper to recent work on mean field games. In Section 3.5 we extend the model to ex-ante heterogeneous agents. In Section 4 we provide final remarks, followed by an Appendix containing proofs.

5 See Davila et al. (2012), Shanker (2017), Nuno and Moll (2018), and Park (2018) for a study of welfare maximization in Bewley models.
2 The model

There is a continuum of agents of measure 1. Every agent has an individual state. We assume for now that the agents are ex-ante identical. In Section 3.5 we extend the model to ex-ante heterogeneous agents. There are \( n \) goods. Let \( Y_i \) be a random variable that describes the evolution of good \( i \)'s endowment. We assume that \( Y_i \) has a finite support \( \mathcal{Y}_i \) and a probability mass function \( q_i(y) := \Pr(Y_i = y) \) for all \( 1 \leq i \leq n \) and \( y \in \mathcal{Y}_i \). Let \( \mathcal{Y} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_n \) and let \( q(y) = q(y_1, \ldots, y_n) := \Pr(Y_1 = y_1, \ldots, Y_n = y_n) \) be the joint probability mass function where we denote elements in \( \mathbb{R}^n \) by bold letters. In each period \( t = 1, 2, 3 \ldots \), the agents receive a bundle of goods \( y \in \mathcal{Y} \) with probability \( q(y) \). We assume that \( y \gg 0 \) for all \( y \in \mathcal{Y} \), where \( y \gg 0 \) means that \( y_i > 0 \) for all \( i = 1, \ldots, n \). We refer to \( q \) as the endowments process.

Denote the agents’ wealth at time \( t = 1 \) by \( a(1) \). In each period \( t = 1, 2, 3 \ldots \), the agents receive an endowment vector \( y(t) \). After receiving their endowment vector, the agents choose a bundle of goods to consume in that period \( x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n \) and choose how much to save in a risk-free bond for future consumption. The price of good \( x_i(t) \) is given by \( p_i(t) > 0 \), so the price of a bundle \( x(t) \) is \( p(t) \cdot x(t) \) where \( p(t) \cdot x(t) := \sum p_i(t)x_i(t) \) denotes the scalar product of two elements in \( \mathbb{R}^n \). The agents’ savings rate of return is \( 1 + r(t) \) where \( r(t) \) is the interest rate in period \( t \). The agents are price takers, i.e., they take the sequence of prices \( (p(t), r(t))_{t=1}^{\infty} \) as given. If an agent’s wealth at time \( t = a(t) \), the agent’s wealth at time \( t + 1 \) when \( y(t + 1) \) is the realized endowment vector is

\[
a(t + 1) = (1 + r(t))(a(t) - p(t) \cdot x(t)) + p(t + 1) \cdot y(t + 1).
\]

We assume that the agents can borrow, and the borrowing limit is given by \( b(t) \). Thus, \( a(t) - p(t) \cdot x(t) \geq b(t) \) for each period \( t \). We assume that the borrowing limit in period \( t \) is given by a fraction of the the natural borrowing limit, that is, \( b(t) = (1 - \psi)\min_{y \in \mathcal{Y}} p(t) \cdot y \) where \( 0 < \psi < 1 \) is a tightness parameter. When \( \psi \) is higher then the borrowing constraint is tighter. For a discussion of the natural borrowing constraint see Aiyagari (1994). For now we assume that \( p(t) \gg 0 \) and \( r(t) > 0 \) for all \( t \). We note that under the borrowing constraint \( b(t) \), the equilibrium interest rates satisfy \( r(t) > 0 \) (see more details in Remark 1). In addition the equilibrium prices satisfy \( p(t) \gg 0 \).

We denote by \( C(a, p(t)) = [b(t) \min\{a, \sum_{i=1}^{n} p_i(t) \cdot \bar{b}\}] \) the interval from which an agent may choose his level of savings when his wealth is \( a \) and the prices of goods are \( p(t) \). \( \sum_{i=1}^{n} p_i(t) \cdot \bar{b} \)

\[^6\] All the results in this paper can be extended to the case that \( Y_i \) has a compact support.

\[^7\] As usual, the positive cone of \( \mathbb{R}^n \) is denoted by \( \mathbb{R}_+^n \), i.e., \( \mathbb{R}_+^n = \{x = (x_1, \ldots, x_n) : x_j \geq 0 \text{ holds for all } j = 1, \ldots, n\} \).
is an upper bound on savings that ensure compactness of the state space where \( \bar{b} > 0 \). We assume that the maximal level of savings that an agent can have is bounded to avoid technical difficulties that arise in dynamic programming with unbounded rewards. In a standard income fluctuation problem, one can find sufficient conditions on the utility function that ensure that the upper bound on savings never binds (see Li and Stachurski (2014), Acıkgöz (2018) and references therein).

We assume that the agents’ preferences over bundles are represented by a utility function \( U : \mathbb{R}^n_+ \to \mathbb{R} \). For \( \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n \) we write \( \mathbf{x} \geq \mathbf{x}' \) if \( x_i \geq x'_i \) for all \( i = 1, \ldots, n \). We say that \( U \) is increasing if \( \mathbf{x} \geq \mathbf{x}' \) implies \( U(\mathbf{x}) \geq U(\mathbf{x}') \). We say that \( U \) is strictly increasing if \( \mathbf{x} > \mathbf{x}' \) implies \( U(\mathbf{x}) > U(\mathbf{x}') \).

Throughout the paper, we assume the following conditions on the utility function.

**Assumption 1** The utility function \( U \) is strictly increasing, continuously differentiable, strictly concave, and \( \frac{\partial U(0)}{\partial x_j} = \infty \) for some \( 1 \leq j \leq n \).

Let \( A \) be the set of possible wealth levels that an agent can have, and let \( A^t := A \times \ldots \times A \) \( t \) times. A strategy \( \pi \) for the agents is a function that assigns to every finite history \( \mathbf{a}^t = (a(1), \ldots, a(t)) \in A^t \) a feasible bundle \( \mathbf{x}(t) \). A strategy \( \pi \) induces a probability measure over the space of all infinite histories.\(^8\) We denote the expectation with respect to that probability measure by \( E_\pi \).

When the agents follow a strategy \( \pi \) and the sequence of prices is given by \( (p(t), r(t)) \), their expected present discounted value is

\[
V_\pi(a) = E_\pi \left( \sum_{t=1}^{\infty} \beta^{t-1} U(\pi(a(1), \ldots, a(t))) \right),
\]

where \( a(1) = a \) is the initial wealth and \( 0 < \beta < 1 \) is the agents’ discount factor. Denote

\[
V(a) = \sup_{\pi} V_\pi(a).
\]

That is, \( V(a) \) is the maximal expected utility that an agent can have when his initial wealth is \( a \). We call \( V \) the value function.

\(^8\) The probability measure on the space of all infinite histories \( A^\mathbb{N} \) is uniquely defined (see for example Bertsekas and Shreve (1978)).
2.1 Competitive stationary equilibrium

In this section we define a competitive stationary equilibrium (CSE). We first introduce some notations that are necessary in order to define a CSE. In a CSE the prices of goods and the interest rate are constant over time. For the rest of the section we assume that \((p(t), r(t)) = (p, r)\) for all \(t \in \mathbb{N}\).

We denote by \(b\) the agents’ savings in the next period. When the agents’ wealth is \(a\), their next period’s savings are \(b \in C(a, p)\), and the prices of goods are \(p\), then the set of consumption bundles available to the agents is given by \(X(a - b, p) = \{x \in \mathbb{R}_+^n : p \cdot x = a - b\}\). We sometimes change variables and define \(c = a - b\) to the total consumption of the agents.

The minimal level of wealth that an agent can have when \(p \gg 0\) and \(r > 0\) is \(\underline{a}(p, r) = (1 + r)b + \min_{y \in Y} p \cdot y\) and the maximal level of wealth that an agent can have is \(\overline{a}(p, r) := (1 + r)\sum_{i=1}^n p_i b + \max_{y \in Y} p \cdot y\), so the set of possible wealth levels that an agent can have \(A(p, r) = [\underline{a}(p, r), \overline{a}(p, r)]\) is compact for all \(p \gg 0\) and \(r > 0\). For the rest of the section we assume that \(p \gg 0\) and \(r > 0\).

Since for any \(p \gg 0\) and \(r > 0\) the value function is bounded on the compact set \(A(p, r)\), we can use standard dynamic programming arguments to solve the agents’ problem. Let \(B(A)\) be the space of all bounded real-valued functions defined on a set \(A\). For any \(p \gg 0\) and \(r > 0\), define the operator \(T : B(A(p, r)) \to B(A(p, r))\) by

\[
Tf(a, p, r) = \max_{b \in C(a, p)} \max_{x \in X(a - b, p)} U(x) + \beta \sum_{y \in Y} q(y) f((1 + r)b + p \cdot y, p, r).
\]

The value function \(V\) is the unique fixed point of \(T\), i.e., there is a unique function \(V \in B(A(p, r))\) such that \(TV = V\).

We denote by \(x^*(a - b, p)\) the demand function of an agent, i.e.,

\[
x^*(a - b, p) = \arg\max_{x \in X(a - b, p)} U(x).
\]

Note that given the choice of the next’s period savings \(b\), the decision of how to distribute the spending \(a - b\) between the different consumption goods is a static decision. Also note that \(x^*\) is single-valued since \(U\) is strictly concave and continuous. We denote by \(g(a, p, r)\) the savings

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9 The Banach-fixed point theorem (see Theorem 3.48 in Aliprantis and Border (2006)) shows that \(T\) has a unique fixed point. Standard dynamic programming arguments (e.g., Blackwell (1965)) show that the value function \(V\) is the unique fixed point of \(T\).
policy function, i.e.,

$$g(a, p, r) = \arg\max_{b \in C(a, p)} U(x^*(a - b, p)) + \beta \sum_{y \in Y} q(y)V((1 + r)b + p \cdot y, p, r).$$  \hspace{1cm} (1)$$

For a set $K \subseteq \mathbb{R}^n$ we denote by $\mathcal{P}(K)$ the set of all probability measures on $K$ and by $\mathcal{B}(K)$ the Borel sigma-algebra on $K$. Define

$$M\mu(D; p, r) = \int_{A(p, r)} \sum_{y \in Y} q(y)1_D((1 + r)g(a, p, r) + p \cdot y)\mu(da; p, r),$$  \hspace{1cm} (2)$$

for any $D \in \mathcal{B}(A(p, r))$ where $1_D$ is the indicator function of the set $D \in \mathcal{B}(A(p, r))$. $M\mu \in \mathcal{P}(A(p, r))$ describes the next period’s wealth distribution, given that the current wealth distribution is $\mu \in \mathcal{P}(A(p, r))$ and the prices are $(p, r)$. A wealth distribution $\mu \in \mathcal{P}(A(p, r))$ is called an invariant wealth distribution if $\mu = M\mu$.

We now define a CSE.

**Definition 1** A competitive stationary equilibrium consists of prices $(p, r)$, a savings policy function $g$, a demand function $x^*$, and a wealth distribution $\mu \in \mathcal{P}(A(p, r))$ such that

(i) Given the prices $(p, r)$, the savings policy function $g$ and the demand function $x^*$ are optimal for the agents. That is, $g$ satisfies equation (1) and

$$x^*(a - g(a, p, r), p) = \arg\max_{x \in X(a - g(a, p, r), p)} U(x).$$

(ii) Given the prices $(p, r)$, $\mu$ is an invariant wealth distribution. That is, $\mu \in \mathcal{P}(A(p, r))$ satisfies $\mu = M\mu$.

(iii) For each good $1 \leq i \leq n$, the aggregate supply of good $i$ equals the aggregate demand for good $i$:

$$\int_{A(p, r)} x_i^*(a - g(a, p, r), p)\mu(da; p, r) = \sum_{y_i} q_i(y_i)y_i$$

(iv) The aggregate supply of savings equals the aggregate demand for savings:

$$\int_{A(p, r)} g(a, p, r)\mu(da; p, r) = 0.$$
The third equilibrium condition says that the aggregate demand for good $i$ equals the aggregate supply of good $i$. The fourth equilibrium condition says that the aggregate savings in the economy are 0, i.e., the supply of savings equals the demand for savings. The third and fourth equilibrium conditions require that the prices $(p, r)$ are market-clearing prices. The natural interpretation of the stationary equilibrium prices are that the prices represent average prices (see Huggett (1993)). An alternative to CSE is a competitive recursive equilibrium (see Miao (2006)). A competitive recursive equilibrium is a sequence of prices $(p(t), r(t))$, and a sequence of measures $(\lambda(t))$ such that the savings and consumption decisions are optimal for the agents; the prices $(p(t), r(t))$ are market-clearing prices for every period $t$; and the wealth distribution follows the law of motion defined by equation (2). Clearly, if the initial agents’ wealth distribution is invariant, then the CSE is also a competitive recursive equilibrium. In this paper we focus on a CSE. The existence result presented in the next section can be applied to the competitive recursive equilibrium case as well. The analysis and computation of a competitive recursive equilibrium is generally much harder than the analysis and computation of a CSE.

We note that the model presented in this paper is closely related to Huggett’s model (Huggett, 1993) and Bewley’s model (Bewley, 1986). In Huggett’s model there is only one consumption good and only the interest rate is determined in equilibrium. In Bewley’s model there are many consumption goods and their prices are determined in equilibrium but the interest rate is fixed and is not determined in equilibrium. In the model presented in this paper, however, there are many consumption goods, and both their prices and the interest rate are determined in equilibrium. Our model also generalizes the static pure-exchange Arrow-Debreu model to an incomplete markets setting where the agents can transfer assets from one period to another only by investing in a risk-free bond.

## 3 Main results

In this section we present our main results. In Section 3.1 we state our existence result and provide the main idea behind the proof. In Section 3.2 we provide conditions that ensure the uniqueness of a competitive stationary equilibrium (CSE). In Section 3.3 we discuss how an increase in the risk of the endowments process influences the equilibrium prices of goods and the equilibrium interest rate when the agents’ preferences can be represented by a CES utility function. In Section 3.4 we compare our model to mean field equilibrium models. In Section 3.5

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10 For general existence results of a competitive recursive equilibrium with aggregate shocks see Brumm et al. (2017).
we extend the model to include ex-ante heterogeneous agents.

3.1 Existence of a CSE

The main theorem of this section is the following:

Theorem 1 Suppose that Assumption 1 holds. Then, there exists a competitive stationary equilibrium.

To prove the theorem, we use a well-known excess demand approach (e.g., Debreu (1982)). Under Assumption 1, the savings policy function $g(a, p, r)$ is single-valued and continuous. Furthermore, there exists a unique invariant wealth distribution $\mu(da; p, r)$ for all $(p, r) \in P$ where $P$ is the non-empty and convex set defined in equation (3) below. We define an excess demand function $\zeta(p, r)$ from $P \subseteq \mathbb{R}^{n+1}$ into $\mathbb{R}^{n+1}$, where

$$\zeta_i(p, r) = \int_{A(p, r)} x_i^a(a - g(a, p, r), p) \mu(da; p, r) - \sum_{y_i \in Y_i} q_i(y_i) y_i$$

is the excess demand for good $i$, $i = 1, ..., n$, and

$$\zeta_{n+1}(p, r) = -\int_{A(p, r)} g(a, p, r) \mu(da; p, r)$$

is the excess demand for savings. The excess demand function $\zeta : P \rightarrow \mathbb{R}^{n+1}$ is defined by

$$\zeta(p, r) = (\zeta_1(p, r), \ldots, \zeta_n(p, r), \zeta_{n+1}(p, r)).$$

Note that if $\zeta(p, r) = 0$ then $(p, r)$ are equilibrium prices, $\mu(\cdot : p, r)$ is the equilibrium invariant wealth distribution, $x^a(a - g(a, p, r), p)$ is the equilibrium demand function, and $g(a, p, r)$ is the equilibrium savings policy function.

We extend a well-known result from the static Arrow-Debreu model to the incomplete markets model studied in this paper. We show that the aggregate demand for goods and the aggregate supply of goods depend only on their relative prices. In particular, the next Proposition shows that if $(p, r)$ are equilibrium prices then $(\theta p, r)$ are also equilibrium prices for all $\theta > 0$. This result is trivial in the standard static Arrow-Debreu model, however, in the incomplete markets Arrow-Debreu model this result is not immediate.

Proposition 1 Fix $p \gg 0$, $r > 0$ with $(1 + r)\beta < 1$, and $\theta > 0$. Then
(i) $\theta g(a, p, r) = g(\theta a, \theta p, r)$ and $x^*(a - g(a, p, r), p) = x^*(\theta a - g(\theta a, \theta p, r), \theta p)$ for all $a$.

(ii) $\zeta_i(\theta p, r) = \zeta_i(p, r)$ for $1 \leq i \leq n$ and $\zeta_{n+1}(\theta p, r) = \theta \zeta_{n+1}(p, r)$.

Thus, if $(p, r)$ are equilibrium prices then $(\theta p, r)$ are also equilibrium prices.

We note that the excess demand for savings is homogeneous of degree one in the prices of goods, while the excess demand for good $k$ is homogeneous of degree zero for all $1 \leq k \leq n$. These results rely on the fact that $\theta C(a, p) = C(\theta a, \theta p)$ for all $\theta > 0$ where

$$C(a, p) = [(1 - \psi)\min_{y \in Y} p \cdot y, \min\{a, \sum_{i=1}^{n} p_i \beta\}]$$

is the interval from which an agent may choose his level of savings and $\theta A = \{\theta x : x \in A\}$ for any set $A$. That is, if the agent can save an amount $b$ given the wealth level $a$ and the prices $p$, then the agent can save an amount $\theta b$ given the wealth level $\theta a$ and the prices $\theta p$. This is reasonable in our setting since all the prices in our model are real prices.

Also note that if the bonds are not in zero net supply than it is not true that if $(p, r)$ are equilibrium prices then $(\theta p, r)$ are also equilibrium prices for all $\theta > 0$. This follows since the aggregate demand for savings is homogeneous of degree one in the prices of goods.

From Proposition 1, if $(p, r)$ are equilibrium prices then $(\theta p, r)$ are also equilibrium prices for all $\theta > 0$. Thus, we can normalize the prices of the goods. More precisely, the search for equilibrium prices can be confined to sets that contain at least one element from the half-ray $\{\theta p : \theta > 0\}$.

We define the sets $\Lambda = \{(p, r) \in \mathbb{R}_+^n \times \mathbb{R}_+ : \sum_{i=1}^{n} p_i + r = \frac{1}{\beta} - 1\}$ and

$$P = \{(p, r) \in \Lambda : p \gg 0, r > 0\}.$$  \hspace{1cm} (3)

Note that if $(p, r) \in P$ then $11 (1 + r)\beta < 1$

In order to prove the existence of a CSE, that is, to prove that there are prices $(p, r) \in P$ such that $\zeta(p, r) = 0$, we show that the excess demand function is continuous, satisfies Walras’ law and satisfies suitable boundary conditions, and we apply a well-known Proposition that guarantees the existence of at least one vector $(p, r) \in P$ that satisfies $\zeta(p, r) = 0$ (see Proposition 2 in the Appendix).

\textsuperscript{11}It is well known that if $(1 + r)\beta \geq 1$ then the aggregate savings tend to the upper bound on savings (Aivagari, 1994).
Remark 1 We prove the existence of an equilibrium with a strictly positive interest rate. An equilibrium with a strictly positive interest rate exists since we assume that the borrowing constraint tends to minus infinity as the interest rate tends to zero (a similar observation is made on page 673 in Aiyagari (1994)).

3.2 Uniqueness of a CSE

In this section we prove the uniqueness of a CSE for the special case where the utility function is given by: $U(x) = \sum_{i=1}^{n} \alpha_i x_i^\gamma$ for some $0 < \gamma < 1$, $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$, i.e., the agents’ preferences over bundles can be represented by a CES utility function with an elasticity of substitution that is higher than one.

There is a vast literature that provides sufficient conditions to ensure the uniqueness of an equilibrium in the standard static pure-exchange Arrow-Debreu model. The property that the demand for each good increases with the prices of the other goods (”gross substitutes property”) usually plays a crucial role in proving the uniqueness of an equilibrium in the static Arrow-Debreu model. Given the gross substitutes property, an easy argument shows that the equilibrium must be unique. This fact led most of the previous literature on the uniqueness of an equilibrium to find conditions on agents’ preferences that ensure that the gross substitutes property holds. While the gross substitutes property remains a crucial property in proving the uniqueness of an equilibrium in the dynamic incomplete markets Arrow-Debreu model considered in this paper also, the standard argument that proves the uniqueness of an equilibrium does not apply. The reason is that the aggregate demand for goods does not necessarily increase with the interest rate, and the aggregate demand for savings does not necessarily increase with the prices of goods. Thus, the excess demand function does not necessarily have the gross substitutes property. We show that when the agents’ preferences are represented by a CES utility function with an elasticity of substitution that is higher than one the CSE is unique even when the excess demand does not have the gross substitutes property.

It is well known and easy to check that when the agents’ preferences are represented by a CES utility function, the indirect utility function

$$v(a, b, p) = \max_{x \in X(a-b, p)} U(x)$$

is given by a constant relative risk aversion (CRRA) utility function. The CRRA utility function

12 For a survey of the work done on the uniqueness of equilibrium, see Arrow and Hahn (1971), Mas-Colell (1991), and Kehoe (1998). For recent results, see Toda and Walsh (2017) and Geanakoplos and Walsh (2018), and references therein.
tion is popular in the applied literature and is often used in numerical analysis of incomplete markets heterogeneous agent models. The uniqueness of an equilibrium in these models with one consumption good and a CRRA utility function has recently been studied in Light (2017) and Proehl (2018). The next theorem generalizes the results in Light (2017) to a model with many consumption goods.

**Theorem 2** Assume that \( U(x) = \sum_{i=1}^{n} \alpha_i x_i^\gamma \) for some \( 0 < \gamma < 1, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1 \). Then there exists a unique competitive stationary equilibrium.

If the agents’ preferences can be represented by a Cobb-Douglas utility function, i.e., \( U(x) = \sum_{i=1}^{n} \alpha_i \ln(x_i) \) for \( \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1 \), then the same proof as the proof of Theorem 2 shows that there exists a unique CSE in this case, as well. Note that in this case, the indirect utility function corresponds to a log utility function, which is often used in the quantitative literature (for example, see Aiyagari (1994) and Krusell et al. (2010)).

The conditions on the agents’ preferences that ensure uniqueness are restrictive. However, uniqueness results in Bewley models are rare, and a multiplicity of equilibria can arise even under the standard specifications of the model (for examples of the multiplicity of equilibria see Toda (2017) and Acıkgöz (2018)). Even in a static Arrow-Debreu model, a multiplicity of equilibria can easily arise. Kubler and Schmedders (2010a) and Kubler and Schmedders (2010b) provide examples of multiplicity in the case that the agents’ preferences can be represented by a CES utility function, and also provide a general method of finding all the equilibria in semi-algebraic Arrow-Debreu models.

### 3.3 The price of goods and wealth inequality

In this section we show that if the agents’ preferences are represented by a CES utility function with an elasticity of substitution that is higher than one, then an increase in the risk of the endowments process (in the sense of the convex stochastic order) does not change the equilibrium prices of goods, and decreases the equilibrium interest rate.

In response to an increase in the risk of the endowments process, we show that the partial equilibrium wealth inequality is higher in the sense of the convex stochastic order. That is, for a fixed interest rate and fixed prices of goods, the wealth inequality is higher when the endowments...
process is riskier. This follows from the facts that agents save more when the future endowment is riskier and that the savings policy function is convex in wealth, i.e., the marginal propensity to consume is decreasing. In addition, for a fixed interest rate \( r \) and prices of goods \( p \), the precautionary savings effect increases the aggregate savings, and thus the aggregate expenditure on goods decreases. Since the goods are normal, the decrease in the aggregate expenditure on goods implies that the aggregate demand for each good decreases.

We note that this is different from the static Arrow-Debreu model where riskier endowments do not change the demand for each good. In the static Arrow-Debreu model the demand for each good is linear in wealth while in our setting the demand for each good is concave in wealth since the marginal propensity to consume is decreasing. Thus, in the dynamic incomplete markets Arrow-Debreu model the equilibrium prices might change in response to an increase in the risk of the endowments process. The prices of goods, however, do not change at all in the new CSE. While the interest rate decreases, the negative effect of this decrease on the aggregate savings is exactly offset by the positive effect on the aggregate savings of an increase in the risk of the endowments process. In other words, the negative substitution effect on the aggregate savings and the positive precautionary effect on the aggregate savings are equal.

We now introduce notations that are needed to state the main theorem of this section.

For two probability measures \( \lambda_1, \lambda_2 \) we define the partial order \( \succeq_{CX} \) by

\[
\int f(a)\lambda_2(da) \geq \int f(a)\lambda_1(da)
\]

for every convex and increasing function \( f \). Similarly, we write \( \lambda_2 \succeq_{CX} \lambda_1 \) if and only if \( \int f(a)\lambda_2(da) \geq \int f(a)\lambda_1(da) \) for every convex function \( f \).

We say that the endowments process \( q \) is riskier than the endowments process \( q' \) if \( q \succeq_{CX} q' \). With slight abuse of notation, we add the argument \( q \) to the functions defined above, when \( q(y) \) is the probability of receiving the endowment vector \( y \in Y \). For example, we write \( \mu(\cdot; p, r, q) \) for the invariant wealth distribution, \( g(a, p, r, q) \) for the savings policy function, and \( x^*(a - g(a, p, r, q), p) \) for the demand function.

**Theorem 3** Assume that \( U(x) = \sum_{i=1}^{n} \alpha_i x_i^\gamma \) for some \( 0 < \gamma < 1, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1 \). Assume that the endowments process \( q \) is riskier than the endowments process \( q' \). Then

(i) The partial equilibrium wealth inequality is higher under \( q \) than under \( q' \), i.e., \( \mu(\cdot; p, r, q) \succeq_{CX} \mu(\cdot; p, r, q') \) for all \( (p, r) \in P \). In addition, if \( (p(q), r(q)) \) are equilibrium prices under the endowments process \( q \) then \( \mu(\cdot; p(q), r(q), q) \succeq_{CX} \mu(\cdot; p(q), r(q), q') \).

(ii) The equilibrium prices of goods do not change, i.e., \( p(q) = p(q') \). The equilibrium interest rate is lower under \( q \) than under \( q' \), i.e., \( r(q') \geq r(q) \).

\(^{14}\) The convexity of the savings policy function follows from the CES assumption and is not easy to establish for general utility functions.
We note that when the endowments process $q$ is riskier than the endowments process $q'$ then the total supply of each consumption good does not change and the relative total supply of each consumption good does not change either (see more details in the proof of Theorem 3). This fact plays a major role in the proof of Theorem 3, in particular, in proving that the prices of consumption goods do not change.

When the agents’ preferences are not represented by a CES utility function, Theorem 3 does not necessarily hold. Since the excess demand function does not satisfy a gross substitute property we are not able to prove comparative statics results for a utility function that is not a CES utility function. It would be interesting to explore the connection between the prices of goods and the risk of the endowment process for different utility functions.

3.4 Comparison to mean field equilibrium models

Mean field equilibrium models have been popularized in the recent literature in operations research, economics and optimal control. In a mean field model, the agents’ utility functions and the evolution of the agents’ states depend on the distribution of the other agents’ states. In a mean field equilibrium, each agent maximizes his expected discounted payoff, assuming that the distribution of the other agents’ states is fixed. Given the agents’ strategy, the distribution of the agents’ states is an invariant distribution of the Markov process that governs the dynamics of the agents’ states. While the notion of a mean field equilibrium is conceptually similar to the notion of a CSE, we cannot write the dynamic incomplete markets Arrow-Debreu model studied in this paper as a discrete-time mean field model. This is because the market-clearing conditions (see conditions (iii) and (iv) in Definition 1) are not consistent with the definition of a mean field equilibrium. Thus, we cannot apply the recent existence, uniqueness and comparative statics results developed for discrete-time mean field equilibrium models (e.g., Adlakha and Johari (2013), Acemoglu and Jensen (2015), and Light and Weintraub (2018)).

3.5 Ex-ante heterogeneous agents

In this section we extend the model described in Section 2 to the case of ex-ante heterogeneous agents. We assume that the agents are heterogeneous in their preferences over consumption bundles as well as in their endowments. Assume that before the process starts, each agent has a type $\theta \in \Theta$. For simplicity we assume that $\Theta$ is a finite set. Each agent’s type is fixed throughout the horizon. An agent with type $\theta \in \Theta$ has preferences that are represented by preferences that are represented by preferences that are represented by

\begin{footnote}
\footnote{For example, see Lasry and Lions (2007), Weintraub et al. (2008), and Iyer et al. (2014).}
\end{footnote}
a utility function \( U(x, \theta) \) and receives an endowment \( y(\theta) \) with a probability \( q(y(\theta)) \) in each period. Let \( \phi \) be the probability mass function over the type space; \( \phi(\theta) \) is the mass of agents whose type is \( \theta \in \Theta \). Adding the argument \( \theta \in \Theta \) to the functions defined in Section 2, we can modify the definitions of Section 2 to include the ex-ante heterogeneity of agents. For example, \( g(a, p, r, \theta) \) is the savings policy function of type \( \theta \) agents and \( x^*(a - g(a, p, r, \theta), p, \theta) \) is the demand function of type \( \theta \) agents.

Let \( A_h = \mathbb{R} \times \Theta \) be an extended state space for the model with ex-ante heterogeneous agents. If an agent’s extended state is \( a_h = (a, \theta) \in A_h \) then the agent’s wealth is \( a \) and his type is \( \theta \). Let \( \lambda_h \) be a probability measure over the extended state space, i.e., \( \lambda_h \in \mathcal{P}(A_h) \).

Define the Markov kernel

\[
Q_h((a, \theta), D \times E) = \sum_{y \in Y} q(y(\theta)) 1_D((1 + r)g(a, p, r, \theta) + p \cdot y(\theta)) 1_E(\theta)
\]

for any \( D \times E \in \mathcal{B}(\mathbb{R}) \times 2^\Theta \). The Markov kernel \( Q_h \) describes the evolution of the extended state. That is, when the agent’s wealth is \( a \) and his type is \( \theta \), the probability that the next period’s pair of wealth-type will lie in \( D \times E \in \mathcal{B}(\mathbb{R}) \times 2^\Theta \) is given by \( Q_h((a, \theta), D \times E) \).

Define

\[
M\lambda_h(D \times E; p, r) = \int \sum_{y \in Y} q(y(\theta)) 1_D((1 + r)g(a, p, r, \theta) + p \cdot y(\theta)) 1_E(\theta) \lambda_h(d(a, \theta); p, r),
\]

for any \( D \times E \in \mathcal{B}(\mathbb{R}) \times 2^\Theta \). \( M\lambda_h \in \mathcal{P}(A_h) \) describes the next period’s wealth-types distribution, given that the current wealth-types distribution is \( \lambda_h \in \mathcal{P}(A_h) \) and the prices are \( (p, r) \). A wealth-types distribution \( \mu_h \in \mathcal{P}(A_h) \) is called an invariant wealth-types distribution if \( \mu_h = M\mu_h \).

These definitions map the model with ex-ante heterogeneous agents to the model with ex-ante homogeneous agents that we considered in Section 2. We can define a competitive stationary equilibrium as in Definition 1. A competitive stationary equilibrium consists of prices \( (p, r) \), savings policy functions \( g \), demand functions \( x^* \), and a wealth-types distribution \( \mu_h \in \mathcal{P}(A_h) \) such that the savings policy function \( g \) and the demand function \( x^* \) are optimal for each type \( \theta \), the wealth-types distribution \( \mu_h \) is invariant, and the prices of goods and the interest rate are market-clearing (see conditions (iii) and (iv) in Definition 1).

We note that if \( U(x, \theta) \) satisfies Assumption 1 for each \( \theta \), then Theorem 1 holds and there exists a CSE. The proof is similar to the proof of Theorem 1 so we omit the details.
4 Final remarks

In this paper we study a dynamic incomplete markets Arrow-Debreu model which combines a Huggett-Bewley model with the classic static pure-exchange Arrow-Debreu model. We study a competitive stationary equilibrium where the prices of consumption goods and the interest rate are market clearing. Under mild conditions on the agents’ preferences, we prove that the aggregate demand for consumption goods is homogeneous of degree 0, while the aggregate demand for savings is homogeneous of degree 1 (see Proposition 1). We prove the existence of a competitive stationary equilibrium (CSE) (see Theorem 1). We provide conditions that ensure the uniqueness of a CSE. Under a CES utility function, we discuss how a riskier endowments process affects wealth inequality, the prices of goods and the interest rate. We prove that if the agents’ preferences can be represented by a CES utility function with an elasticity of substitution that is equal to or higher than one, then there exists a unique CSE (see Theorem 2), and that a riskier endowments process increases the partial equilibrium wealth inequality, decreases the equilibrium interest rate, and does not change the equilibrium prices of goods (see Theorem 3). It remains an open question whether Theorem 2 and Theorem 3 can be extended to different utility functions. Many other open questions remain concerning the CSE. For example, studying the stability of a CSE awaits future research.

5 Appendix

5.1 Homogeneity of the excess demand function

In this section we prove Proposition 1.

Recall that for a set $K$ we denote by $\mathcal{P}(K)$ the set of all probability measures defined on $K$. We endow $\mathcal{P}(\mathbb{R})$ with the topology of weak convergence. We say that $\lambda_n \in \mathcal{P}(\mathbb{R})$ converges weakly to $\lambda \in \mathcal{P}(\mathbb{R})$ if for all bounded and continuous functions $f : \mathbb{R} \to \mathbb{R}$ we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(a) \lambda_n(da) = \int_{\mathbb{R}} f(a) \lambda(da).$$

**Proposition 1.** Fix $p \gg 0$, $r > 0$ with $(1 + r)\beta < 1$, and $\theta > 0$. Then

(i) $\theta g(a, p, r) = g(\theta a, \theta p, r)$ and $x^*(a - g(a, p, r), p) = x^*(\theta a - g(\theta a, \theta p, r), \theta p)$ for all $a$.

(ii) $\zeta_i(\theta p, r) = \zeta_i(p, r)$ for $1 \leq i \leq n$ and $\zeta_{n+1}(\theta p, r) = \theta \zeta_{n+1}(p, r)$.

Thus, if $(p, r)$ are equilibrium prices then $(\theta p, r)$ are also equilibrium prices.

**Proof.** (i) Recall that a function $f(a, p, r)$ is homogeneous of degree $l \geq 0$ in $(a, p)$ if $f(\theta a, \theta p, r) = \theta^l f(a, p, r)$. Then

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(a) \lambda_n(da) = \int_{\mathbb{R}} f(a) \lambda(da).$$
$\theta^f(a,p,r)$ for all $\theta > 0$. We now show that $g$ is homogeneous of degree 1 in $(a,p)$.

Fix $p \gg 0$, $\theta > 0$, $a \in \mathbb{R}$, and $r \in (0, \frac{1}{\beta} - 1)$.

Assume that $f(a,p,r)$ is homogeneous of degree 0 in $(a,p)$. We have

$$T_f(a,p,r) = \max_{b \in C(a,p)} \max_{x \in X(a-b,p)} U(x) + \beta \sum_{y \in Y} q(y) f((1+r)b + p \cdot y, p, r)$$

$$= \max_{\theta b \in C(\theta a, \theta p)} \max_{x \in X(\theta a-\theta b, \theta p)} U(x) + \beta \sum_{y \in Y} q(y) f((1+r)\theta b + \theta p \cdot y, \theta p, r)$$

$$= \max_{z \in C(\theta a, \theta p)} \max_{x \in X(\theta a-z, \theta p)} U(x) + \beta \sum_{y \in Y} q(y) f((1+r)z + \theta p \cdot y, \theta p, r)$$

$$= T_f(\theta a, \theta p, r).$$

Thus, $T_f(a,p,r)$ is homogeneous of degree 0 in $(a,p)$. The first and fourth equalities follow from the definition of $T_f$. The second equality follows from the facts that $X(a-b,p) = X(\theta a-\theta b, \theta p)$, $b \in C(a,p)$ if and only if $\theta b \in C(\theta a, \theta p)$ and $f((1+r)\theta b + \theta p \cdot y, \theta p, r) = f((1+r)b + p \cdot y, p, r)$ for $\theta > 0$.

We conclude that for all $n = 1, 2, 3, \ldots$, $T^n f$ is homogeneous of degree 0. From standard dynamic programming arguments, $T^n f$ converges to $V$ uniformly. Since the set of functions that are homogeneous of degree 0 is closed under uniform convergence, $V$ is homogeneous of degree zero.

Let

$$h(a, b, p, r, f) := v(a, b, p) + \beta \sum_{y \in Y} q(y) f((1+r)b + p \cdot y, p, r),$$

where $v(a, b, p) = \max_{x \in X(a-b,p)} U(x)$.

Note that $v$ is homogeneous of degree 0 in $(a, b, p)$ since $X(a-b,p) = X(\theta a-\theta b, \theta p)$. Thus, $h(a, b, p, r, f)$ is homogeneous of degree 0 in $(a, b, p)$ whenever $f$ is homogeneous of degree 0 in $(a, p)$. Since $V$ is homogeneous of degree zero in $(a, p)$ we conclude that $h(a, b, p, r, V)$ is

---

$^{16}$ Recall that $C(a, p) = [(1 - \psi)\min_{y \in Y} p \cdot y, \min\{a, \sum_{i=1}^n p_i b_i\}]$. 

---
homogeneous of degree 0 in \((a, b, p)\). We have
\[
\begin{aligned}
h(\theta a, \theta g(a, p, r), \theta p, r, V) &= h(a, g(a, p, r), p, r, V) \\
&= \max_{b \in C(a, p)} h(a, b, p, r, V) \\
&= V(a, p, r) \\
&= V(\theta a, \theta p, r) \\
&= h(\theta a, g(\theta a, \theta p, r), \theta p, r, V).
\end{aligned}
\]

The single-valuedness of the savings policy function \(g\) (see Lemma 1) implies that \(g(\theta a, \theta p, r) = \theta g(a, p, r)\). We conclude that \(g\) is homogeneous of degree 1 in \((a, p)\).

The following chain of equalities show that the demand function \(x^*\) is homogeneous of degree 0 in \((a, p)\):
\[
\begin{aligned}
x^*(a - g(a, p, r), p) &= \arg\max_{x \in X(a - g(a, p, r), p)} U(x) \\
&= \arg\max_{x \in X(\theta a - g(\theta a, \theta p, r), \theta p)} U(x) \\
&= \arg\max_{x \in X(\theta a - g(\theta a, \theta p, r), \theta p)} U(x) \\
&= x^*(\theta a - g(\theta a, \theta p, r), \theta p).
\end{aligned}
\]

(ii) We say that a probability measure \(\lambda(\cdot; p, r)\) is homogeneous of degree \(l \geq 0\) in \(p\) if for every continuous and bounded function \(f(a, p)\) that is homogeneous of degree \(l\) in \((a, p)\) and all \(\theta > 0\) we have
\[
\int f(a, \theta p) \lambda(da; \theta p, r) = \theta^l \int f(a, p) \lambda(da; p, r).
\]
(4)

We now show that the invariant wealth distribution \(\mu\) is homogeneous of degree \(l \geq 0\) in \(p\).

Assume that the probability measure \(\lambda(\cdot; p, r)\) is homogeneous of degree \(l\) in \(p\). Let \(f(a, p)\) be a continuous and bounded function that is homogeneous of degree \(l\) in \((a, p)\) and let \(\theta > 0\),
\( p \gg 0, \) and \( r \in (0, \frac{1}{\beta} - 1) \). We have

\[
\theta^l \int f(a, p) M \lambda(da; p, r) = \theta^l \int \sum_{y \in \mathcal{Y}} q(y) f((1 + r)g(a, p, r) + p \cdot y, p) \lambda(da; p, r)
\]
\[
= \int \sum_{y \in \mathcal{Y}} q(y) f((1 + r)g(a, \theta p, r) + \theta p \cdot y, \theta p) \lambda(da; \theta p, r)
\]
\[
= \int f(a, \theta p) M \lambda(da; \theta p, r).
\]

The first and last equalities follow from Equation (5) (see Lemma 4). The second equality follows from the facts that \( \tilde{f}(a, p) := \sum_{y \in \mathcal{Y}} q(y) f((1 + r)g(a, p, r) + p \cdot y, p) \) is homogeneous of degree \( l \) in \( (a, p) \) and \( \lambda \) is homogeneous of degree \( l \) in \( p \). We conclude that for all \( k \), \( M^k \lambda \) is homogeneous of degree \( l \) in \( p \).

From Lemma 2, \( M^k \lambda \) converges weakly to \( \mu \) for all \((p, r) \) such that \( p \gg 0, \) and \( r \in (0, \frac{1}{\beta} - 1) \). For every continuous and bounded function \( f(a, p) \) that is homogeneous of degree \( l \) in \( (a, p) \), we have

\[
\int f(a, \theta p) \mu(da; \theta p, r) = \lim_{k \to \infty} \int f(a, \theta p) M^k \lambda(da; \theta p, r)
\]
\[
= \lim_{k \to \infty} \theta^l \int f(a, p) M^k \lambda(da; p, r)
\]
\[
= \theta^l \int f(a, p) \mu(da; p, r).
\]

Thus, \( \mu \) is homogeneous of degree \( l \) in \( p \).

From the fact that \( g(a, p, r) \) is a continuous function on \( \mathbb{R} \times \mathbb{R}^n_+ \times (0, \frac{1}{\beta} - 1) \) (see Lemma 1) that is homogeneous of degree 1 in \( (a, p) \) and the fact that \( A(p, r) \) is compact for all \( p \gg 0, \) \( r > 0 \) we have

\[
\zeta_{n+1}(\theta p, r) = \int g(a, \theta p, r) \mu(da; \theta p, r) = \theta \int g(a, p, r) \mu(da; p, r) = \theta \zeta_{n+1}(p, r).
\]

Similarly, since for all \( 1 \leq i \leq n \) the function \( x_i^*(a - g(a, p, r), p) \) is a continuous function that
is homogeneous of degree 0 in \((a, p, r)\), we have

\[
\zeta_i(\theta p, r) = \int x_i^*(a - g(a, \theta p, r), \theta p) \mu(da; \theta p, r) - \sum_{y_i \in Y_i} q_i(y_i) y_i
\]

\[
= \int x_i^*(a - g(a, p, r), p) \mu(da; p, r) - \sum_{y_i \in Y_i} q_i(y_i) y_i = \zeta_i(p, r).
\]

Thus, if \((p, r)\) are equilibrium prices, i.e., \(\zeta(p, r) = 0\), then \(\zeta(\theta p, r) = 0\); and so \((\theta p, r)\) are also equilibrium prices.

### 5.2 The existence of a competitive stationary equilibrium

In this section we prove the existence of a competitive stationary equilibrium.

**Theorem 1.** Suppose that Assumption 1 holds. Then, there exists a competitive stationary equilibrium.

Recall that the sets \(\Lambda\) and \(P\) are given by \(\Lambda = \{ (p, r) \in \mathbb{R}_+^n \times \mathbb{R}_+ : \sum_{i=1}^n p_i + r = \frac{1}{\beta} - 1 \}\) and

\[
P = \{ (p, r) \in \Lambda : p \gg 0, r > 0 \}.
\]

The excess demand function \(\zeta : P \to \mathbb{R}^{n+1}\) is given by

\[
\zeta(p, r) = (\zeta_1(p, r), \ldots, \zeta_n(p, r), \zeta_{n+1}(p, r))
\]

where for \(i = 1, \ldots, n,\)

\[
\zeta_i(p, r) = \int_{A(p,r)} x_i^*(a - g(a, p, r), p) \mu(da; p, r) - \sum_{y_i \in Y_i} q_i(y_i) y_i
\]

is the excess demand for good \(i\), and

\[
\zeta_{n+1}(p, r) = - \int_{A(p,r)} g(a, p, r) \mu(da; p, r)
\]

is the excess demand for savings. Note that if \(\zeta(p, r) = 0\) then \((p, r)\) are equilibrium prices, \(\mu(\cdot; p, r)\) is the equilibrium wealth distribution, \(x^*(a - g(a, p, r), p)\) is the equilibrium demand function, and \(g(a, p, r)\) is the equilibrium savings policy function.

For a proof of the following well-known proposition, see, for example, Theorem 1.4.8 in
For a function \( \zeta(\cdot) = (\zeta_1(\cdot), ..., \zeta_{n+1}(\cdot)) \) from \( P \) into \( \mathbb{R}^{n+1} \) assume that:

(i) \( \zeta \) is continuous.

(ii) \( \zeta(p, r) \) satisfies Walras’ law, i.e., \( (p, r) \cdot \zeta(p, r) = 0 \) for all \( (p, r) \in P \).

(iii) \( \{p_q, r_q\} \rightarrow (p, r) \in P \) imply \( \lim_{q \to \infty} \|\zeta(p_q, r_q)\|_1 = \infty \).

(iv) \( \{p_q, r_q\} \subseteq P, (p_q, r_q) \rightarrow (p, r) = (p_1, ..., p_n, r) \) and \( p_k > 0 \) imply that the sequence \( \{\zeta_k(p_q, r_q)\} \) of the \( k^{th} \) components of \( \{\zeta(p_q, r_q)\} \) is bounded. Similarly, \( r > 0 \) implies that the sequence \( \{\zeta_{n+1}(p_q, r_q)\} \) is bounded.

(v) The excess demand function \( \zeta(p, r) \) is bounded from below, i.e., there exists \( \xi > 0 \) such that \( \zeta_i(p, r) \geq -\xi \) for all \( 1 \leq i \leq n+1 \) and all \( (p, r) \in P \).

Then there exists at least one vector \( (p, r) \in P \) that satisfies \( \zeta(p, r) = 0 \).

The idea of the proof of Theorem 1 is to show that the conditions of Proposition 2 hold. Lemmas 1 and 2 show that the excess demand function \( \zeta(p, r) \) is a well-defined function. In Lemmas 4 and 5 we prove that the excess demand function is continuous. In Lemma 6 we prove that the excess demand function satisfies Walras’ law. In Lemma 7 and Lemma 8 we prove that the excess demand function satisfies the boundness and boundary conditions (conditions (iii), (iv) and (v) of Proposition 2). Thus, all the conditions of Proposition 2 hold and there exists a CSE.

Lemma 1 The savings policy function \( g(a, p, r) \) is single-valued and continuous in \( (a, p, r) \). The value function \( V(a, p, r) \) is continuous in \( (a, p, r) \), increasing in \( a \), and strictly concave in \( a \).

Proof. Note that \( v(a, b, p) = \max_{x \in X(a-b, p)} U(x) \) is strictly concave in \( (a, b) \). To see this, let \( (a_1, b_1) \in \mathbb{R}^2, (a_2, b_2) \in \mathbb{R}^2, \gamma \in [0, 1], a_\gamma = \gamma a_1 + (1 - \gamma)a_2, \) and \( b_\gamma = \gamma b_1 + (1 - \gamma)b_2 \).

We have

\[
v(a_\gamma, b_\gamma, p) = \max_{x \in X(a_\gamma-b_\gamma, p)} U(x)
\]

\[
\geq U(\gamma x^*(a_1 - b_1, p) + (1 - \gamma)x^*(a_2 - b_2, p))
\]

\[
> \gamma U(x^*(a_1 - b_1, p)) + (1 - \gamma)U(x^*(a_2 - b_2, p))
\]

\[
= \gamma v(a_1, b_1, p) + (1 - \gamma)v(a_2, b_2, p).
\]

17 For a more general version of this proposition, see Debreu (1982) and Hildenbrand and Kirman (2014).

18 For \( x \in \mathbb{R}^n \) we write \( \|x\|_1 = \sum_{j=1}^n |x_i| \).
The first inequality follows from the fact that \( \gamma x^*(a_1 - b_1, p) + (1 - \gamma)x^*(a_2 - b_2, p) \in X(a, b, p) \). The second inequality follows from the fact that \( U \) is strictly concave. We conclude that \( v \) is strictly concave in \((a, b)\). Furthermore, since \( U \) is continuous and \( X(a - b, p) \) is a continuous correspondence, i.e., \( X \) is upper hemicontinuous and lower hemicontinuous, the maximum theorem (see Theorem 17.31 in Aliprantis and Border (2006)) implies that \( v(a, b, p) \) is continuous. Since \( U \) is increasing, \( v \) is increasing in \( a \). Now standard dynamic programming arguments show that \( g(a, p, r) \) is single-valued and continuous in \((a, p, r)\) and that \( V(a, p, r) \) is continuous, as well as strictly concave and increasing in \( a \) (see Chapter 9 in Stokey and Lucas (1989)). □

**Lemma 2** For every \((p, r) \in P\) there exists a unique invariant wealth distribution \( \mu(\cdot; p, r) \in \mathcal{P}(A(p, r)) \). Furthermore, for all \( \lambda(\cdot; p, r) \in \mathcal{P}(A(p, r)) \), the sequence of measures \( \{M^n\lambda\} \) converges weakly to \( \mu(\cdot; p, r) \in \mathcal{P}(A(p, r)) \).

**Proof.** Fix \((p, r) \in P\). Define the Markov chain

\[
Q(a, D) = \sum_{y \in Y} q(y)1_D((1 + r)g(a, p, r) + p \cdot y).
\]

for any \( D \in \mathcal{B}(A(p, r)) \) where \( 1_D \) is the indicator function of the set \( D \in \mathcal{B}(A(p, r)) \).

We prove a more general result than the result stated in Lemma 2. We show that the Markov chain \( Q \) is uniformly ergodic. \(^{19}\) The proof follows a similar line to the proofs in Rabault (2002) and in Benhabib et al. (2015), so we only provide a sketch of the proof. \(^{20}\)

The Markov chain \( Q \) is said to satisfy the Doeblin condition if there exists a positive integer \( n_0, \epsilon > 0 \) and a probability measure \( v \) on \( A(p, r) \) such that \( Q^{n_0}(a, D) \geq \epsilon v(D) \) for all \( a \in A(p, r) \) and all \( D \in \mathcal{B}(A(p, r)) \). Under Assumption 1, the arguments in Proposition 3.1 in Rabault (2002) yield that the borrowing constraint binds with positive probability after a finite number of periods for any initial wealth level \( a \in A(p, r) \). In other words, for any initial wealth level \( a \in A(p, r) \), we have \( g(a, p, r) = b \) with a positive probability after a finite number of periods. Thus, if we define the probability measure \( v(D) = \sum_{y \in Y} q(y)1_D((1 + r)b + p \cdot y) \), we can find a positive integer \( n_0 \) and \( \epsilon > 0 \) such that \( Q^{n_0}(a, D) \geq \epsilon v(D) \) for all \( a \in A(p, r) \) and all \( D \in \mathcal{B}(A(p, r)) \). We conclude that \( Q \) satisfies the Doeblin condition. From the facts that \( M : \mathcal{P}(A(p, r)) \to \mathcal{P}(A(p, r)) \) is continuous (see a more general result in Lemma 2) and

\[^{19}\] Recall that the Markov chain \( Q \) is called uniformly ergodic if it has an invariant distribution \( \mu \) and \( \sup_{D \in \mathcal{B}(A(p, r))} |Q^n(a, D) - \mu(D)| \leq M\rho^n \) for some \( \rho < 1, M < \infty \) and for all \( n \in \mathbb{N}, a \in A(p, r) \). Clearly, if \( Q \) is uniformly ergodic then Lemma 2 holds.

\[^{20}\] See also Schechtman and Escudero (1977), Ma et al. (2018), and Foss et al. (2018).
\( \mathcal{P}(A(p, r)) \) is compact in the weak topology (since \( A(p, r) \) is compact), Schauder’s fixed-point theorem (see Corollary 17.56 in Aliprantis and Border (2006)) implies that \( M \) has at least one fixed point. That is, \( Q \) has at least one invariant distribution. A Markov chain that has an invariant distribution and satisfies the Doeblin condition is uniformly ergodic (see Theorem 8 in Roberts et al. (2004)). This completes the proof the Lemma.

We say that \( w_n : \mathbb{R} \to \mathbb{R} \) converges continuously to \( w \) if \( w_n(a_n) \to w(a) \) whenever \( a_n \to a \). Lemma 3 provides a bounded convergence theorem with varying measures. For a proof, see Theorem 3.3 in Serfozo (1982). 21 We will use this Lemma to prove the continuity of the excess demand function.

**Lemma 3** Assume that \( w_n : \mathbb{R} \to \mathbb{R} \) is a uniformly bounded sequence of functions. If \( w_n : \mathbb{R} \to \mathbb{R} \) converges continuously to \( w \) and \( \lambda_n \in \mathcal{P}(\mathbb{R}) \) converges weakly to \( \lambda \in \mathcal{P}(\mathbb{R}) \) then

\[
\lim_{n \to \infty} \int w_n(a)\lambda_n(da) = \int w(a)\lambda(da).
\]

**Lemma 4** The unique invariant wealth distribution \( \mu \) is continuous in \((p, r)\) on \( P \), i.e., if \( \{p_n, r_n\} \) converges to \((p, r)\), then \( \mu(p_n, r_n) \) converges weakly to \( \mu(p, r) \).

**Proof.** First note that for every bounded and measurable function \( f : \mathbb{R} \to \mathbb{R} \) and for all \((p, r)\) such that \( p \gg 0 \) and \( r > 0 \) we have

\[
\int f(a)M\lambda(da; p, r) = \int \sum_{y \in \mathcal{Y}} q(y)f((1 + r)g(a, p, r) + p \cdot y)\lambda(da; p, r).
\]

To see this, note that if \( f = 1_D \) then Equality (5) holds from the definition of \( M \). A standard argument shows that Equality (5) holds for any bounded and measurable \( f \).

Assume that \( \{p_n, r_n\} \subseteq P \) converges to \((p, r)\) \( \in P \). Let \( \{\mu(p_k, r_k)\} \) be a subsequence of \( \{\mu(p_n, r_n)\} \) that converges weakly to \( \overline{\mu}(p, r) \). Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded and continuous function. From the continuity of the savings policy function \( g \), we have

\[
\lim_{k \to \infty} f((1 + r_k)g(a_k, p_k, r_k) + p_k \cdot y) = f((1 + r)g(a, p, r) + p \cdot y)
\]

whenever \( \lim_{k \to \infty}(a_k, p_k, r_k) = (a, p, r) \).

Let us define \( w_k(a) = \sum_{y \in \mathcal{Y}} q(y)f((1 + r_k)g(a, p_k, r_k) + p_k \cdot y) \) and \( w(a) = \sum_{y \in \mathcal{Y}} q(y)f((1 + r)g(a, p, r) + p \cdot y) \). Then, \( w_k(a) \) is a uniformly bounded sequence of functions that converges continuously to \( w(a) \).

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21See Feinberg et al. (2019) for a more general result of this type.
Applying Lemma 3 and using Equality (5) twice yield
\[
\lim_{k \to \infty} \int f(a) \mu(da; p_k, r_k) = \lim_{k \to \infty} \int \sum_{y \in Y} q(y) f((1 + r_k)g(a, p_k, r_k) + p_k \cdot y) \mu(da; p_k, r_k) \\
= \lim_{k \to \infty} \int w_k(a) \mu(da; p_k, r_k) \\
= \int w(a) \overline{\mu}(da; p, r) \\
= \int f(a) M \overline{\mu}(da; p, r).
\]

Since \( \mu(p_k, r_k) \) converges weakly to \( \overline{\mu}(p, r) \), we also have
\[
\lim_{k \to \infty} \int f(a) \mu(da; p_k, r_k) = \int f(a) \overline{\mu}(da; p, r).
\]
Thus,
\[
\int f(a) M \overline{\mu}(da; p, r) = \int f(a) \overline{\mu}(da; p, r)
\]
which implies that \( \overline{\mu} = M \overline{\mu} \), since \( \overline{\mu} \) and \( M \overline{\mu} \) are Borel probability measures that agree on all open sets. From Lemma 2 \( \mu \) is the unique fixed point of \( M \), and thus, \( \overline{\mu} = \mu \).

We conclude that each subsequence of \( \{ \mu(p_n, r_n) \} \) that converges weakly at all converges weakly to \( \mu(p, r) \). Furthermore, since \( A(p, r) \) is compact, for all \( (p, r) \in P \) we can assume that the supports of \( \mu(p_n, r_n) \) and \( \mu(p, r) \) are contained in a compact set so the sequence \( \{ \mu(p_n, r_n) \} \) is a tight sequence of probability measures. Thus, \( \mu(p_n, r_n) \) converges weakly to \( \mu(p, r) \) (see the Corollary after Theorem 25.10 in Billingsley (2008)).

\textbf{Lemma 5} \( \zeta(p, r) \) is continuous on \( P \).

\textbf{Proof.} Assume that the sequence \( \{ p_n, r_n \} \subseteq P \) converges to \( (p, r) \in P \). Fix \( i \) such that \( 1 \leq i \leq n \). Define \( w_n(a) = x_i^*(a - g(a, p_n, r_n), p_n) \) and \( w(a) = x_i^*(a - g(a, p, r), p) \). The continuity of \( x_i^* \) and of \( g \) imply that \( w_n \) converges continuously to \( w \), i.e., \( w_n(a_n) \to w(a) \) whenever \( a_n \to a \). The sequence of functions \( \{ w_n(a) \} \) is bounded (see Lemma 8). Using Lemma
and the fact that \( \mu(p_n, r_n) \) converges weakly to \( \mu(p, r) \) (see Lemma 4) yield

\[
\lim_{n \to \infty} \zeta_i(p_n, r_n) = \lim_{n \to \infty} \int w_n(a) \mu(da; p_n, r_n) - \sum_{y_i \in Y_i} q_i(y_i) y_i
= \int w(a) \mu(da; p, r) - \sum_{y_i \in Y_i} q_i(y_i) y_i
= \zeta_i(p, r).
\]

Thus, \( \zeta_i(p, r) \) is continuous for \( 1 \leq i \leq n \). A similar argument shows that \( \zeta_{n+1}(p, r) \) is continuous. We conclude that \( \zeta(p, r) \) is continuous. \( \blacksquare \)

**Lemma 6** \( \zeta(p, r) \) satisfies Walras’ law, i.e., \( (p, r) \cdot \zeta(p, r) = 0 \) for all \( (p, r) \in P \).

**Proof.** Fix \( (p, r) \in P \). Equation (5) implies that

\[
\int a \mu(da; p, r) = \int \sum_{y \in Y} q(y)((1 + r)g(a, p, r) + p \cdot y) \mu(da; p, r)
= (1 + r) \int g(a, p, r) \mu(da; p, r) + \sum_{y \in Y} q(y)p \cdot y.
\]

Note that \( \sum_{y \in Y} q(y)p \cdot y = \sum_{i=1}^{n} p_i \sum_{y_i \in Y_i} q_i(y_i) y_i \). To see this, let \( Y = \{ y^1, \ldots, y^l \} \) and reason as follows:

\[
\sum_{y \in Y} q(y)p \cdot y = q(y^1)p \cdot y^1 + \ldots + q(y^l)p \cdot y^l = \sum_{i=1}^{n} p_i \sum_{j=1}^{l} q(y^j)y_i^j = \sum_{i=1}^{n} p_i \sum_{y_i \in Y_i} q_i(y_i) y_i.
\]

From the agents’ budget constraints, we have \( p \cdot x^*(a - g(a, p, r), p) = a - g(a, p, r) \).

The last equation implies

\[
\sum_{i=1}^{n} p_i \int x^*_i(a - g(a, p, r), p) \mu(da; p, r) = \int (a - g(a, p, r)) \mu(da; p, r).
\]
Thus,
\[
(p, r) : \zeta(p, r) = \sum_{i=1}^{n} p_i \int x_i^r(a - g(a, p, r), p) \mu(da; p, r) - \sum_{i=1}^{n} p_i \sum_{y_i \in \mathcal{Y}_i} q_i(y_i)y_i - r \int g(a, p, r) \mu(da; p, r)
\]
\[
= \int (a - g(a, p, r)) \mu(da; p, r) - \sum_{i=1}^{n} p_i \sum_{y_i \in \mathcal{Y}_i} q_i(y_i)y_i - r \int g(a, p, r) \mu(da; p, r)
\]
\[
= \int a \mu(da; p, r) - (1 + r) \int g(a, p, r) \mu(da; p, r) - \sum_{y \in \mathcal{Y}} q(y) p \cdot y = 0,
\]
which proves that \(\zeta(p, r)\) satisfies Walras’ law. ■

**Lemma 7** The excess demand function \(\zeta(p, r)\) is bounded from below, i.e., there exists \(\xi > 0\) such that \(\zeta_i(p, r) \geq -\xi\) for all \(1 \leq i \leq n + 1\) and all \((p, r) \in P\).

**Proof.** We have \(\zeta_i(p, r) \geq -\sum_{y_i \in \mathcal{Y}_i} q_i(y_i)y_i\) for all \(1 \leq i \leq n\) and all \((p, r) \in P\). Thus, \(\zeta_i\) is bounded from below for all \(1 \leq i \leq n\).

Since \(g(a, p, r)\) is bounded from above by \(\sum_{i=1}^{n} p_i b\) we have \(\int g(a, p, r) \mu(da; p, r) \leq \sum_{i=1}^{n} p_i b\), so
\[
\zeta_{n+1}(p, r) = -\int g(a, p, r) \mu(da; p, r) \geq -\sum_{i=1}^{n} p_i b \geq -b(1/\beta - 1)
\]
for all \((p, r) \in P\). We conclude that the excess demand function is bounded from below. ■

**Lemma 8** (i) \((p_q, r_q) \to (p, r) \in \Lambda \setminus P\) with \(\{p_q, r_q\} \subseteq P\) imply \(\lim_{q \to \infty} \|\zeta(p_q, r_q)\|_1 = \infty\).

(ii) \(\{p_q, r_q\} \subseteq P\), \((p_q, r_q) \to (p, r) = (p_1, \ldots, p_n, r)\) and \(p_k > 0\) imply that the sequence \(\{\zeta_k(p_q, r_q)\}\) of the \(k\)th components of \(\{\zeta(p_q, r_q)\}\) is bounded. Similarly, \(r > 0\) implies that the sequence \(\{\zeta_{n+1}(p_q, r_q)\}\) is bounded.

**Proof.** (i) Suppose that \((p_q, r_q) \to (p, r) = (p_1, \ldots, p_n, r)\) where \((p, r) \in \Lambda \setminus P\). We consider two different cases.

Case I: We have \(r_q \to r = 0\). In this case the borrowing constraint tends to minus infinity and it follows from the same arguments as in page 673 in [Aiyagari (1994)] that
\[
\lim_{q \to \infty} \int g(a, p_q, r_q) \mu(da; p_q, r_q) = -\infty.
\]
Thus, we have \(\lim_{q \to \infty} \zeta_{n+1}(p_q, r_q) = \infty\) which implies that \(\lim_{q \to \infty} \|\zeta(p_q, r_q)\|_1 = \infty\).
Case II: We have \( r > 0 \). In this case \( (p, r) \in \Lambda \setminus \mathcal{P} \) implies that \( p_k = 0 \) for some \( 1 \leq k \leq n \). Since the utility function \( U \) is strictly increasing, a standard argument shows that the demand for some good tends to infinity, and thus, \( \lim_{q \to \infty} \sum_{i=1}^{n} x_i (a - g(a, p_q, r_q), p_q) = \infty \) (see for example Theorem 1.4.6 in Aliprantis et al. [1990]). We conclude that \( \lim_{q \to \infty} \| \zeta(p_q, r_q) \|_1 = \infty \).

(ii) Assume that \( \{p_q, r_q\} \) is a sequence of strictly positive prices satisfying the conditions of the Lemma where \( p_q = (p^1_q, \ldots, p^n_q) \). Since \( p_k > 0 \) for some \( 1 \leq k \leq n \) and \( (p_q, r_q) \to (p, r) \), we infer that there exists some \( \epsilon > 0 \) such that \( p_k^q > \epsilon \) for all \( q \). We can assume that \( \sum_{y \in Y} q(y) p_q \cdot y \leq M \) for all \( q \).

We have
\[
p_k^q \int_{A(p_q, r_q)} x_k^*(a - g(a, p_q, r_q), p_q) \mu(da; p_q, r_q) \leq \int_{A(p_q, r_q)} (a - g(a, p_q, r_q)) \mu(da; p_q, r_q).
\]

The last inequality implies that
\[
\int_{A(p_q, r_q)} x_k^*(a - g(a, p_q, r_q), p_q) \mu(da; p_q, r_q) \leq \frac{\int_{A(p_q, r_q)} (a - g(a, p_q, r_q)) \mu(da; p_q, r_q)}{p_k^q} \leq \frac{\int_{A(p_q, r_q)} r_q g(a, p_q, r_q) \mu(da; p_q, r_q) + \sum_{y \in Y} q(y) p_q \cdot y}{p_k^q} \leq \frac{1}{\beta - 1} \frac{\beta^2 b + M}{\epsilon}.
\]

The equality follows from Equation (5). The second inequality follows since \( g(a, p_q, r_q) \leq \sum p_i^q b \) for all \( a \in A(p_q, r_q) \). Therefore, the sequence \( \{\zeta_k(p_q, r_q)\} \) is bounded for \( 1 \leq k \leq n \).

Now assume that \( r > 0 \). In this case, we can assume that there exists \( \delta > 0 \) such that \( r_k > \delta \) for all \( q \). We can also assume that \( (1 - \psi) \min_{y \in Y} p_q \cdot y \leq M \) for all \( q \).

Using the borrowing constraint, we have
\[
- \int_{A(p_q, r_q)} g(a, p_q, r_q) \mu(da; p_q, r_q) \leq (1 - \psi) \min_{y \in Y} p_q y \leq \frac{M}{\delta}.
\]

Therefore the sequence \( \{\zeta_{n+1}(p_q, r_q)\} \) is bounded from above. From Lemma \( \{\zeta_{n+1}(p_q, r_q)\} \) is bounded from below. The proof of the Lemma is completed.

We proved that the excess demand function \( \zeta \) satisfies the properties of Proposition. Thus, a CSE exists.
5.3 The uniqueness of a competitive stationary equilibrium

In this section we prove Theorem 2.

**Theorem 2.** Assume that \( U(x) = \sum_{i=1}^{n} \alpha_i x_i^\gamma \) for some \( 0 < \gamma < 1, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1 \). Then there exists a unique competitive stationary equilibrium.

**Proof.** Since the savings policy function \( g \), the demand function \( x^* \), and the invariant wealth distribution \( \mu \) are unique given fixed prices \((p, r)\), it is enough to show that the prices \((p, r)\) that clear the market are unique in order to prove the uniqueness of a CSE. The proof involves a number of steps.

**Step 1.** If \((p, r)\) and \((p', r')\) are equilibrium prices, then \( p = p' \). Suppose, in contradiction, that there are equilibrium prices \((p, r)\) and \((p', r')\) such that \( p' \neq p \), and \( p \) and \( p' \) are not linearly independent. From Proposition 11 we can normalize the prices such that \( p \geq p' \) and \( p'_k = p_k = 1 \) for some \( 1 \leq k \leq n \). We have

\[
\int a\mu(da; p, r) = \sum_{y \in \mathcal{Y}} q(y)((1 + r) \int g(a, p, r)\mu(da; p, r) + p \cdot y) = \sum_{y \in \mathcal{Y}} q(y)p \cdot y.
\]

The first equality follows from Equation (5). The second equality follows from the fact that \((p, r)\) are equilibrium prices.

Similarly, \( \int a\mu(da; p', r') = \sum_{y \in \mathcal{Y}} q(y)p' \cdot y \). Since \( y \gg 0 \) we conclude that \( \int a\mu(da; p', r') < \int a\mu(da; p, r) \).

Using the fact that \( \int g(a, p', r')\mu(da; p', r') = \int g(a, p, r)\mu(da; p, r) = 0 \) we have

\[
\int (a - g(a, p', r'))\mu(da; p', r') < \int (a - g(a, p, r))\mu(da; p, r).
\]

Since the utility function is in the constant elasticity of substitution class, it is well known and easy to check that \( x^*(a - g(a, p, r), p) = (z_1(p)(a - g(a, p, r)), \ldots, z_n(p)(a - g(a, p, r)) \) where \( z_i(p) \) is a positive function for each \( i = 1, \ldots, n \). Thus, \( x^*_i \) is linear in the total expenditure \( a - g(a, p, r) \) for all \( i \). From the assumption that the elasticity of substitution is higher than one, the demand for each good increases with the prices of the other goods. Since \( p \geq p' \) and \( p'_k = p_k = 1 \) we have \( z_k(p) \geq z_k(p') \).
We have
\[
\int x_k^*(a - g(a, p, r), p) \mu(da; p, r) = \int z_k(p)(a - g(a, p, r)) \mu(da; p, r) > \int z_k(p')(a - g(a, p', r')) \mu(da; p', r')
\]
which leads to the contradiction \(0 = \zeta_k(p, r) > \zeta_k(p', r') = 0\).

**Step 2.** \(g(a, p, r)\) is increasing and convex in \(a\) for all \((p, r) \in P\). It is easy to check that the indirect utility function \(v(a, b, p) = \max_{x \in X(a-b,p)} U(x)\) is given by \(v(a, b, p) = (a - b)^\gamma z(p)\) where \(z(p)\) is a positive function. The indirect utility function is a constant relative risk aversion utility function and thus the savings policy function is convex in \(a\) (for example, we can apply Theorem 4 in Jensen (2017) or the results in Huggett (2004)).

To show that \(g\) is increasing in \(a\), note that \(v(a, b, p)\) has increasing differences in \((a, b)\) (recall that a function \(v\) is said to have increasing differences in \((a, b)\) if for all \(a_2 \geq a_1\) and \(b_2 \geq b_1\) we have \(v(a_2, b_2, p) - v(a_2, b_1, p) \geq v(a_1, b_2, p) - v(a_1, b_1, p)\)). Thus, the function

\[
v(a, b, p) + \beta \sum_{y \in Y} q(y)V((1 + r)b + p \cdot y, p, r)
\]

has increasing differences in \((a, b)\) as the sum of functions with increasing differences. Now Theorem 6.1 in Topkis (1978) implies that \(g(a, p, r)\) is increasing in \(a\).

**Step 3.** \(g(a, p, r)\) is increasing in \(r\) on \(I = (0, 1/\beta - 1)\) for all \((a, p)\). The proof of this result follows from similar arguments to the arguments in the proof of Theorem 1 in Light (2017). Since the current setting is different from the setting in Light (2017) we provide the proof here.

Assume that \(f(a, p, r)\) is a bounded function that is increasing, concave and continuously differentiable in \(a\) with the following properties: (i) \(f\) has increasing differences in \((a, r)\); (ii) \(af_a(a, p, r)\) is increasing in \(a\) on \(\mathbb{R}_+\) (for a function \(f\) we denote by \(f_a\) the derivative of \(f\) with respect to \(a\)). Let \(r > r'\). We have

\[
(1 + r)f_a((1 + r)b + p \cdot y, p, r) \geq (1 + r')f_a((1 + r')b + p \cdot y, p, r) \geq (1 + r')f_a((1 + r')b + p \cdot y, p, r').
\]

The first inequality follows from property (ii) if\(^{22}\) \(b > 0\), and from the concavity of \(f\) if \(b \leq 0\).

\(^{22}\) To see this, let \(a = (1 + z)b + p \cdot y\). Then \(af_a(a, p, r) = b(1 + z)f_a((1 + z)b + p \cdot y, p, r) + p \cdot yf_a((1 + z)b + p \cdot y, p, r)\).
Thus, \( T_f \) holds in our case, because of Assumption 1. \( T_f \) is differentiable and \( C_r \) is increasing in \( (b,r) \). Topkis (1978) implies that \( C \) choose his level of savings. Note that \( y \) has increasing differences in \( (b,r) \). The second inequality follows from property (i). Thus, the derivative of the function \( v((1 + r)b + p \cdot y, p, r) \) with respect to \( b \) is increasing in \( r \). We conclude that \( f((1 + r)b + p \cdot y, p, r) \) has increasing differences in \( (b, r) \). Thus, the function

\[
v(a, b, p) + \beta \sum_{y \in Y} q(y) f((1 + r)b + p \cdot y, p, r)
\]

has increasing differences in \( (b, r) \) as the sum of functions with increasing differences. Recall that \( C(a, p, r) = [(1 - \psi) - \min_{y \in Y} p_{y} \cdot y, \min\{a, \sum_{i=1}^{n} p_{y_{i}}\}] \) is the interval from which an agent may choose his level of savings. Note that \( C \) is ascending in \( r \) (i.e., \( r_2 \geq r_1, b \in C(a, p, r_1), \) and \( b' \in C(a, p, r_2) \) imply \( \max\{b, b'\} \in C(a, p, r_2) \) and \( \min\{b, b'\} \in C(a, p, r_1) \)). Theorem 6.1 in [Topkis (1978)] implies that

\[
g^f(a, p, r) := \arg\max_{b \in C(a, p, r)} v(a, b, p) + \beta \sum_{y \in Y} q(y) f((1 + r)b + p \cdot y, p, r)
\]

is increasing in \( r \). The envelope theorem (see Benveniste and Scheinkman (1979)) implies that \( T_f \) is differentiable and \( (T_f)_a(a, p, r) = v_a(a, g^f(a, p, r, p)) \) when \( a - g(a, p, r) > 0 \) (which always holds in our case, because of Assumption [1]).

Using the facts that \( v \) has increasing differences in \( (a, b) \) and that \( g^f(a, p, r) \geq g^f(a, p, r') \) yield

\[
(T_f)_a(a, p, r) = v_a(a, g^f(a, p, r, p)) \geq v_a(a, g^f(a, p, r', p)) = (T_f)_a(a, p, r').
\]

Thus, \( T_f \) has increasing differences in \( (a, r) \). Let \( a \geq 0 \). We have

\[
a(T_f)_a(a, p, r) = av_a(a, g^f(a, p, r, p)) = a\gamma(a - g^f(a, p, r))^{-1} \gamma z(p)
\]

\[
= \frac{a}{a - g^f(a, p, r)} \gamma(a - g^f(a, p, r))^{-1} \gamma z(p).
\]

Since \( T_f \) is concave in \( a \) (see Lemma [1]) for \( a \geq a' \) we have

\[
\gamma(a - g^f(a, p, r))^{-1} \gamma z(p) = (T_f)_a(a, p, r) \leq (T_f)_a(a', p, r) = \gamma(a' - g^f(a', p, r))^{-1} \gamma z(p)
\]

which implies that the function \( a - g^f(a, p, r) \) is increasing in \( a \). We conclude that the function \( y, p, r \). The facts that \( a f_a(a, p, r) \) is increasing in \( a \) on \( \mathbb{R}_+ \) and \( f_a \) is decreasing in \( a \) imply that \( (1 + z)f_a((1 + z)b + p \cdot y, p, r) \) is increasing in \( z \) on \( f \). Note that if \( f_a \) is strictly decreasing, then \( (1 + r)f_a((1 + r)b + p \cdot y, p, r) \) is strictly increasing in \( r \).
The function \( (1 + g^r(a, p, r)) \) is increasing in \( a \). Furthermore, the function \( \frac{a}{a - g^r(a, p, r)} \) is increasing in \( a \) on \( \mathbb{R}_+ \).

Thus, \( a(Tf)_a(a, p, r) \) is increasing on \( \mathbb{R}_+ \) as the product of two positive increasing functions.

Define \( f^n = T^n f := T(T^{n-1} f) \) for \( n = 1, 2, \ldots \) where \( T^0 f := f \). We conclude that \( f^n(a, p, r) \) is bounded, concave, increasing, and differentiable in \( a \) with increasing differences in \( (a, r) \), and that \( af^n(a, p, r) \) is increasing in \( a \) on \( \mathbb{R}_+ \) for all \( n \). The argument above shows that \( g^{f^n}(a, p, r) \) is increasing in \( r \) for all \( n \). Theorem 3.8 and Theorem 9.9 in Stokey and Lucas (1989) show that \( g^{f^n} \) converges pointwise to \( g \). Thus, the savings policy function \( g \) is increasing in \( r \). Furthermore,

\[
\lim_{n \to \infty} f^n_a(a, p, r) = \lim_{n \to \infty} \gamma(a - g^{f^n}(a, p, r))^{\gamma-1} z(p) = \gamma(a - g(a, p, r))^{\gamma-1} z(p) = V_a(a, p, r).
\]

Thus, \( aV_a(a, p, r) \) is increasing in \( a \) on \( \mathbb{R}_+ \) and has increasing differences in \( (a, r) \). The same argument as the argument above shows that the savings policy function \( g \) is increasing in \( r \).

**Step 4.** If \( (p, r) \) and \( (p', r') \) are equilibrium prices with \( r > r' \) then \( \int g(a, p, r) \mu(da; p, r) > \int g(a, p, r') \mu(da; p, r') \).

Let \( r > r' \). We first show that \( g(a, p, r) > g(a, p, r') \) for all \( a \in \tilde{A} \), and all \( p \gg 0 \) where \( \tilde{A} = \{ a : g(a, p, r') \in \text{int} C(a, p, r') \} \) is the set of wealth levels such that the optimal savings decision is interior. Suppose, in contradiction, that \( g(a, p, r') = g(a, p, r) \) for some \( a \in \tilde{A} \). Since \( V \) is differentiable and strictly concave in \( a \) (see Lemma 1), the arguments in Step 3 imply that the function \( (1 + r)V_a((1 + r)b + p \cdot y, p, r) \) is strictly increasing in \( r \). The first order condition implies that

\[
0 = -z(p)\gamma(a - g(a, p, r'))^{\gamma-1} + \beta(1 + r') \sum_{y \in Y} q(y)V_a((1 + r')g(a, p, r') + p \cdot y, p, r')
\]

\[
- z(p)\gamma(a - g(a, p, r))^{\gamma-1} + \beta(1 + r) \sum_{y \in Y} q(y)V_a((1 + r)g(a, p, r) + p \cdot y, p, r) \leq 0,
\]

which is a contradiction.

For \( \tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathcal{P}(\mathbb{R}) \) we define the partial order \( \succeq I \) by \( \tilde{\lambda}_2 \succeq I \tilde{\lambda}_1 \) if and only if \( \int f(a)\lambda_2(da) \geq \int f(a)\lambda_1(da) \).

---

23 To see this, note that \( a - g^f(a, p, r) := k(a) \) is concave since \( g^f \) is convex in \( a \). Thus, for \( a' > a \geq 0 \) we have

\[
\frac{k(a') - k(a)}{a' - a} \leq \frac{k(a') - k(0)}{a' - 0}.
\]

Rearranging and using the fact that \( k(0) > 0 \) yield

\[
\frac{a'}{k(a')} \geq \frac{a}{k(a)}.
\]
\[ \int f(a) \lambda_1(da) \text{ for every increasing function } f. \]

Assume that \( \lambda(\cdot, \mathbf{p}, r) \succeq_I \lambda(\cdot, \mathbf{p}, r'). \) Then, for every increasing function \( f \) we have

\[
\int f(a) M \lambda(da; \mathbf{p}, r) = \int \sum_{y \in \mathcal{Y}} q(y) f((1 + r) g(a, \mathbf{p}, r) + \mathbf{p} \cdot y) \lambda(da; \mathbf{p}, r) \\
\geq \int \sum_{y \in \mathcal{Y}} q(y) f((1 + r') g(a, \mathbf{p}, r') + \mathbf{p} \cdot y) \lambda(da; \mathbf{p}, r) \\
\geq \int \sum_{y \in \mathcal{Y}} q(y) f((1 + r') g(a, \mathbf{p}, r') + \mathbf{p} \cdot y) \lambda(da; \mathbf{p}, r') \\
= \int f(a) M \lambda(da; \mathbf{p}, r').
\]

The equalities follow from Equation (5) (see Lemma 4). The first inequality follows from the fact that \( g \) is increasing in \( r \). The second inequality follows from the facts that \( g \) is increasing in \( a \) and \( \lambda(\cdot; \mathbf{p}, r) \succeq_I \lambda(\cdot; \mathbf{p}, r') \). We conclude that \( M^k \lambda(\cdot; \mathbf{p}, r) \succeq_I M^k \lambda(\cdot; \mathbf{p}, r') \) for all \( k = 1, 2, \ldots \).

From Lemma 2, the sequence \( \{M^k \lambda\} \) converges weakly to \( \mu \) for all \( (\mathbf{p}, r) \). Since \( \succeq_I \) is closed under weak convergence, we conclude that \( \mu(\cdot; \mathbf{p}, r) \succeq_I \mu(\cdot; \mathbf{p}, r') \).

Suppose that \( (\mathbf{p}, r) \) and \( (\mathbf{p}, r') \) are equilibrium prices with \( r > r' \). We have

\[
\int g(a, \mathbf{p}, r) \mu(da; \mathbf{p}, r) > \int g(a, \mathbf{p}, r') \mu(da; \mathbf{p}, r) \geq \int g(a, \mathbf{p}, r') \mu(da; \mathbf{p}, r').
\]

The first inequality follows from the fact that \( g \) is strictly increasing in \( r \) on \( \tilde{\mathcal{A}} \) (and we have \( \mu(A; \mathbf{p}, r) > 0 \) since \( (\mathbf{p}, r) \) are equilibrium prices). The second inequality follows from the facts that \( g \) is increasing in \( a \) and \( \mu(\cdot; \mathbf{p}, r) \succeq_I \mu(\cdot; \mathbf{p}, r') \).

**Step 5.** Suppose that \( (\mathbf{p}, r) \) and \( (\mathbf{p}', r') \) are equilibrium prices. From Step 1, we know that \( \mathbf{p}' = \mathbf{p} \). From Step 4, if \( r > r' \) then \( 0 = \int g(a, \mathbf{p}, r) \mu(da; \mathbf{p}, r) > \int g(a, \mathbf{p}, r') \mu(da; \mathbf{p}, r') = 0 \) which is a contradiction. We conclude that \( (\mathbf{p}, r) = (\mathbf{p}', r') \). Thus, there is at most one CSE. It easy to see that Assumptions 1 is satisfied so Theorem 1 implies that there exists at least one CSE. We conclude that there is a unique CSE.

### 5.4 Proof of Theorem 3

In this section we prove Theorem 3.

**Theorem 3** Assume that \( U(x) = \sum_{i=1}^{n} \alpha_i x_i^\gamma \) for some \( 0 < \gamma < 1, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1. \) Assume that the endowments process \( q \) is riskier than the endowments process \( q' \). Then

(i) The partial equilibrium wealth inequality is higher under \( q \) than under \( q' \), i.e., \( \mu(\cdot; \mathbf{p}, r, q) \succeq_I \mu(\cdot; \mathbf{p}, r, q') \).
\( \mu(\cdot; p, r, q') \) for all \((p, r) \in P\). In addition, if \((p(q), r(q))\) are equilibrium prices under the endowments process \(q\), then \(\mu(\cdot; p(q), r(q), q) \succeq_{\text{Cox}} \mu(\cdot; p(q), r(q), q')\).

(ii) The equilibrium prices of goods do not change, i.e., \(p(q) = p(q')\). The equilibrium interest rate is lower under \(q\) than under \(q'\), i.e., \(r(q') \geq r(q)\).

**Proof.** (i) Fix \((p, r) \in P\). Assume that the endowments process \(q\) is riskier than the endowments process \(q'\). From Theorem 2 in [Light (2018)], we can show that \(g(a, p, r, q) \geq g(a, p, r, q')\) for all \((a, p, r) \in A \times P\).

Suppose that \(\lambda(\cdot; p, r, q) \succeq_{\text{Cox}} \lambda(\cdot; p, r, q')\). Then for every convex and increasing function \(f\) we have

\[
\int f(a)M \lambda(da; p, r, q) = \int \sum_{y \in Y} q(y)f((1 + r)g(a, p, r, q) + p \cdot y)\lambda(da; p, r, q) \\
\geq \int \sum_{y \in Y} q'(y)f((1 + r)g(a, p, r, q) + p \cdot y)\lambda(da; p, r, q) \\
\geq \int \sum_{y \in Y} q'(y)f((1 + r)g(a, p, r, q') + p \cdot y)\lambda(da; p, r, q') \\
\geq \int \sum_{y \in Y} q'(y)f((1 + r)g(a, p, r, q') + p \cdot y)\lambda(da; p, r, q') \\
= \int f(a)M \lambda(da; p, r, q').
\]

The equalities follow from Equation (5) (see Lemma 4). The first inequality follows from the fact that \(f((1 + r)g(a, p, r, q) + p \cdot y)\) is convex in \(y\) as the composition of a convex and increasing function with a convex function. The second inequality follows from the facts that \(g(a, p, r, q) \geq g(a, p, r, q')\) and \(f\) is increasing. The third inequality follows from the fact that \(g\) is convex and increasing in \(a\) (see Step 2 in the proof of Theorem 2), which implies that \(f((1 + r)g(a, p, r) + p \cdot y)\) is convex and increasing in \(a\), and from the fact that \(\lambda(\cdot; p, r, q) \succeq_{\text{Cox}} \lambda(\cdot; p, r, q')\).

We conclude that \(M^k \lambda(\cdot; p, r, q) \succeq_{\text{Cox}} M^k \lambda(\cdot; p, r, q')\) for all \(k = 1, 2, \ldots\). From Lemma 2 the sequence \(\{M^k \lambda\}\) converges weakly to \(\mu\) for all \((p, r)\). Since under our assumptions (see Theorem 1.5.9 in [Müller and Stoyan (2002)]) \(\succeq_{\text{Cox}} \mu(\cdot; p, r, q')\).

Now assume that \((p(q), r(q))\) are equilibrium prices under the endowment process \(q\), so

\[
\int g(a, p(q), r(q))\mu(da; p(q), r(q), q) = 0.
\]
\( q \succeq_{C_X} q' \) and the linearity of the function \( p \cdot y \) imply that \( \sum q(y)p \cdot y = \sum q'(y)p \cdot y \). We have

\[
\int a\mu(da; p(q), r(q), q) = \sum_{y \in \mathcal{Y}} q(y)((1 + r(q)) \int g(a, p(q), r(q))\mu(da; p(q), r(q), q) + p(q) \cdot y) \\
= \sum_{y \in \mathcal{Y}} q(y)p(q) \cdot y \\
= \sum_{y \in \mathcal{Y}} q'(y)p(q) \cdot y \\
= \int a\mu(da; p(q), r(q), q').
\]

We proved that \( \mu(\cdot; p(q), r(q), q) \succeq_{I-C_X} \mu(\cdot; p(q), r(q), q') \) and

\[
\int a\mu(da; p(q), r(q), q) = \int a\mu(da; p(q), r(q), q').
\]

This implies that \( \mu(\cdot; p(q), r(q), q) \succeq_{C_X} \mu(\cdot; p(q), r(q), q') \) (see Theorem 1.5.3 in Müller and Stoyan (2002)).

(ii) Assume that \((p(q), r(q))\) and \((p(q'), r(q'))\) are equilibrium prices. Suppose, in contradiction, that \( p(q') \neq p(q) \). From Proposition 1 we can normalize the prices and set \( p(q) \geq p(q') \) and \( p_k' = p_k = 1 \) for some \( 1 \leq k \leq n \).

We have \( x^*(a - g(a, p, r, q), p) = (z_1(p))(a - g(a, p, r, q)), ..., z_n(p))(a - g(a, p, r, q)) \) where \( z_i(p) \) is a positive function and \( z_1(p) \geq z_1(p') \) (see Step 1 of the Proof of Theorem 2).

Since \( \int a\mu(da; p(q), r(q), q) = \int a\mu(da; p(q), r(q), q') \) we have

\[
\int x^*_k(a - g(a, p(q), r(q), q), p)\mu(da; p(q), r(q), q) = z_k(p(q)) \int (a - g(a, p(q), r(q), q))\mu(da; p(q), r(q), q) \\
= z_k(p(q)) \int a\mu(da; p(q), r(q), q) \\
= z_k(p(q)) \int a\mu(da; p(q), r(q), q') \\
> z_k(p(q')) \int a\mu(da; p(q'), r(q'), q') \\
= \int x^*_k(a - g(a, p(q'), r(q'), q'), p')\mu(da; p(q'), r(q'), q').
\]

The inequality follows from the same argument as in Step 1 of the proof of Theorem 2. Since \( q \succeq_{C_X} q' \), we have \( q_i \succeq_{C_X} q_i' \) for all \( 1 \leq i \leq n \) (see Theorem 3.4.4. In Müller and Stoyan (2002)). Recall that \( q_i \succeq_{C_X} q_i' \) implies that \( \sum q_i'(y_i)y_i = \sum q_i(y_i)y_i \). Thus, \( 0 = \zeta_k(p(q), r(q), q) > \zeta_k(p(q'), r(q'), q') = 0 \) which is a contradiction. We conclude that \( p(q) = p(q') \).
Now assume, in contradiction, that \( r(q) > r(q') \). We have

\[
0 = \int g(a, p(p), r(q), q)\mu(da; p(p), r(q), q) > \int g(a, p(q), r(q'), q)\mu(da; p(q), r(q'), q) \\
\geq \int g(a, p(q), r(q'), q)\mu(da; p(q), r(q'), q') \\
\geq \int g(a, p(q'), r(q'), q')\mu(da; p(q'), r(q'), q') = 0
\]

which is a contradiction. The first (strict) inequality follows from Step 4 of the proof of Theorem 2. The second inequality follows from the facts that \( g \) is convex in \( a \) and \( \mu(\cdot; p, r, q) \geq_{CX} \mu(\cdot; p, r, q') \). The third inequality follows from the facts that \( g(a, p, r, q) \geq g(a, p, r, q') \) and \( \mu(\cdot; p, r, q') = \mu(\cdot; p, r, q) \). We conclude that \( r(q') \geq r(q) \). ■

References

ACEMOGLU, D. AND M. K. JENSEN (2015): “Robust Comparative Statics in Large Dynamic Economies,” *Journal of Political Economy*, 587–640.

AÇIKGÖZ, Ö. T. (2018): “On the existence and uniqueness of stationary equilibrium in Bewley economies with production,” *Journal of Economic Theory*, 18–55.

ADLAKHA, S. AND R. JOHARI (2013): “Mean field equilibrium in dynamic games with strategic complementarities,” *Operations Research*, 971–989.

AIYAGARI, S. R. (1994): “Uninsured idiosyncratic risk and aggregate saving,” *The Quarterly Journal of Economics*, 659–684.

ALIPRANTIS, C. D. AND K. BORDER (2006): *Infinite dimensional analysis: A hitchhiker’s guide*, Springer.

ALIPRANTIS, C. D., D. J. BROWN, AND O. BURKINSHAW (1990): *Existence and optimality of competitive equilibria*, Springer Science & Business Media.

ARROW, K. AND F. HAHN (1971): “Competitive equilibrium analysis,” *San Francisco, Holden-Day*.

ARROW, K. J. (1951): “An Extension of the Basic Theorems of Classical Welfare Economics,” in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, The Regents of the University of California.

ARROW, K. J., H. D. BLOCK, AND L. HURWICZ (1959): “On the stability of the competitive equilibrium, II,” *Econometrica*, 82–109.
Arrow, K. J. and G. Debreu (1954): “Existence of an equilibrium for a competitive economy,” *Econometrica*, 265–290.

Arrow, K. J. and L. Hurwicz (1958): “On the stability of the competitive equilibrium, I,” *Econometrica*, 522–552.

Auclert, A. and M. Rognlie (2018): “Inequality and aggregate demand,” *Working paper*.

Benhabib, J. and A. Bisin (2016): “Skewed wealth distributions: Theory and empirics,” *Working paper*.

Benhabib, J., A. Bisin, and S. Zhu (2015): “The wealth distribution in Bewley economies with capital income risk,” *Journal of Economic Theory*, 489–515.

Benveniste, L. M. and J. A. Scheinkman (1979): “On the differentiability of the value function in dynamic models of economics,” *Econometrica*, 727–732.

Bertsekas, D. P. and S. E. Shreve (1978): *Stochastic optimal control: The discrete time case*, Academic Press New York.

Bewley, T. (1986): “Stationary monetary equilibrium with a continuum of independently fluctuating consumers,” *Contributions to mathematical economics in honor of Gérard Debreu*.

Billingsley, P. (2008): *Probability and measure*, John Wiley & Sons, Inc., Hoboken, NJ, USA, 3rd edition, 1995.

Blackwell, D. (1965): “Discounted dynamic programming,” *The Annals of Mathematical Statistics*, 226–235.

Blume, L. and D. Easley (2006): “If you’re so smart, why aren’t you rich? Belief selection in complete and incomplete markets,” *Econometrica*, 929–966.

Brumm, J., D. Kryczka, and F. Kubler (2017): “Recursive equilibria in dynamic economies with stochastic production,” *Econometrica*, 1467–1499.

Davila, J., J. H. Hong, P. Krusell, and J.-V. Ríos-Rull (2012): “Constrained efficiency in the neoclassical growth model with uninsurable idiosyncratic shocks,” *Econometrica*, 80, 2431–2467.

De Nardi, M. (2015): “Quantitative models of wealth inequality: A survey,” *Working paper*.

Debreu, G. (1970): “Economies with a finite set of equilibria,” *Econometrica*, 387–392.

——— (1982): “Existence of competitive equilibrium: K.J. Arrow and M. D. Intriligator (Eds.),” *Handbook of mathematical economics*, 697–743.

Feinberg, E. A., P. O. Kasyanov, and Y. Liang (2019): “Fatou’s Lemma in its classic form and Lebesgue’s convergence theorems for varying measures with applications to MDPs,” *Working paper*.

Foss, S., V. Shneer, J. P. Thomas, and T. Worrall (2018): “Stochastic stability of
monotone economies in regenerative environments,” *Journal of Economic Theory*, 334–360.

Geanakoplos, J., I. Karatzas, M. Shubik, and W. D. Sudderth (2014): “Inflationary equilibrium in a stochastic economy with independent agents,” *Journal of Mathematical Economics*, 52, 1–11.

Geanakoplos, J. and K. J. Walsh (2018): “Uniqueness and stability of equilibrium in economies with two goods,” *Journal of Economic Theory*, 261–272.

Heathcote, J., K. Storesletten, and G. L. Violante (2009): “Quantitative macroeconomics with heterogeneous households,” *Annual Reviews in Economics*, 319–352.

Hildenbrand, W. and A. P. Kirman (2014): *Introduction to equilibrium analysis: variations on themes by Edgeworth and Walras*, Elsevier.

Hu, T.-W. and E. Shmaya (2019): “Unique monetary equilibrium with inflation in a stationary Bewley–Aiyagari model,” *Journal of Economic Theory*.

Huggett, M. (1993): “The risk-free rate in heterogeneous agent incomplete insurance economies,” *Journal of Economic Dynamics and Control*, 953–969.

——— (2004): “Precautionary wealth accumulation,” *The Review of Economic Studies*, 769–781.

Iyer, K., R. Johari, and M. Sundararajan (2014): “Mean field equilibria of dynamic auctions with learning,” *Management Science*, 2949–2970.

Jensen, M. K. (2017): “Distributional comparative statics,” *The Review of Economic Studies*, 581–610.

Judd, K. L., F. Kubler, and K. Schmedders (2003): “Asset trading volume with dynamically complete markets and heterogeneous agents,” *The Journal of Finance*, 2203–2217.

Kaplan, G., B. Moll, and G. L. Violante (2018): “Monetary policy according to HANK,” *American Economic Review*, 697–743.

Karatzas, I., M. Shubik, and W. D. Sudderth (1994): “Construction of stationary Markov equilibria in a strategic market game,” *Mathematics of Operations Research*, 19, 975–1006.

Kehoe, T. J. (1998): “Uniqueness and stability,” in *Elements of General Equilibrium Analysis*, Basil Blackwell.

Krueger, D. and H. Lustig (2010): “When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)?” *Journal of Economic Theory*, 145, 1–41.

Krusell, P., T. Mukoyama, and A. Şahin (2010): “Labour-market matching with precautionary savings and aggregate fluctuations,” *The Review of Economic Studies*, 77, 1477–1507.

Kubler, F. and K. Schmedders (2010a): “Competitive equilibria in semi-algebraic
economies,” *Journal of Economic Theory*, 301–330.

——— (2010b): “Tackling multiplicity of equilibria with Gröbner bases,” *Operations research*, 1037–1050.

——— (2015): “Life-cycle portfolio choice, the wealth distribution and asset prices,” Working paper.

LASRY, J.-M. AND P.-L. LIONS (2007): “Mean field games,” *Japanese journal of mathematics*, 229–260.

LI, H. AND J. STACHURSKI (2014): “Solving the income fluctuation problem with unbounded rewards,” *Journal of Economic Dynamics and Control*, 353–365.

LIGHT, B. (2017): “Uniqueness of equilibrium in a Bewley-Aiyagari model,” Working paper.

——— (2018): “Precautionary saving in a Markovian earnings environment,” *Review of Economic Dynamics*, 138–147.

LIGHT, B. AND G. Y. WEINTRAUB (2018): “Mean Field Equilibrium: Uniqueness, Existence, and Comparative Statics,” Working paper.

LJUNGQVIST, L. AND T. J. SARGENT (2012): *Recursive macroeconomic theory*, MIT press.

LUCAS, R. E. (1980): “Equilibrium in a pure currency economy,” *Economic inquiry*, 203–220.

MA, Q., J. STACHURSKI, AND A. A. TODA (2018): “The income fluctuation problem with capital income risk: optimality and stability,” Working paper.

MAS-COLELL, A. (1991): “On the uniqueness of equilibrium once again,” *Equilibrium Theory and Applications*, 275–296.

MIAO, J. (2006): “Competitive equilibria of economies with a continuum of consumers and aggregate shocks,” *Journal of Economic Theory*, 274–298.

MÜLLER, A. AND D. STOYAN (2002): *Comparison methods for stochastic models and risks*, Wiley.

NUÑO, G. AND B. MOLL (2018): “Social optima in economies with heterogeneous agents,” *Review of Economic Dynamics*, 28, 150–180.

PARK, Y. (2018): “Constrained efficiency in a human capital model,” *American Economic Journal: Macroeconomics*, 10, 179–214.

PROEHL, E. (2018): “Existence and Uniqueness of Recursive Equilibria With Aggregate and Idiosyncratic Risk,” Working paper.

RABAULT, G. (2002): “When do borrowing constraints bind? Some new results on the income fluctuation problem,” *Journal of Economic Dynamics and Control*, 217–245.

ROBERTS, G. O., J. S. ROSENTHAL, ET AL. (2004): “General state space Markov chains and MCMC algorithms,” *Probability Surveys*, 20–71.
Schechtman, J. and V. L. Escudero (1977): “Some results on an income fluctuation problem,” *Journal of Economic Theory*, 151–166.

Serfozo, R. (1982): “Convergence of Lebesgue integrals with varying measures,” *Sankhyā: The Indian Journal of Statistics, Series A*, 380–402.

Shanker, A. (2017): “Existence of recursive constrained optima in the heterogeneous agent neoclassical growth model,” *Working paper*.

Stokey, N. and R. Lucas (1989): *Recursive methods in economic dynamics*, Harvard University Press.

Toda, A. A. (2017): “Huggett economies with multiple stationary equilibria,” *Journal of Economic Dynamics and Control*, 77–90.

Toda, A. A. and K. J. Walsh (2017): “Edgeworth box economies with multiple equilibria,” *Economic Theory Bulletin*, 65–80.

Topkis, D. M. (1978): “Minimizing a submodular function on a lattice,” *Operations research*, 305–321.

Weintraub, G. Y., C. L. Benkard, and B. Van Roy (2008): “Markov perfect industry dynamics with many firms,” *Econometrica*, 1375–1411.