New Bi-Gravities

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Abstract

We show that the problem of ghosts in critical gravity and its higher dimensional extensions can be resolved by giving dynamics to the symmetric rank two auxiliary field existing in the action of these theories. These New Bi-Gravities, at linear level around the AdS vacuum, are free of Boulware-Deser ghost, kinetic ghost and tachyonic instability within the particular range of parameters. Moreover, we show that the energy and entropy of AdS-Schwarzschild black hole solutions of these new models are positive in the same range of parameters. This may be the sign that these new models are also free of ghost instabilities at the non-linear level.
\section{Introduction}

Despite the perfect agreement of Einstein theory with observational data obtained from the Solar system, this theory is not a consistent theory for large distances. Two examples are disability to explaining the flattening of the galaxy rotation curves \cite{1} and the accelerating expansion of the universe \cite{2}. On the other hand, Einstein gravity is non-renormalizable and there is no well-known method to quantize this theory.

To resolve the first problem, so many proposals are suggested in the literature; among them the dark matter idea seems to be more successful. In these proposals it is common to assume the Einstein gravity is valid at all length scales but there exist invisible amounts of matter in the middle distances which reduce gravitational potential, leading to flattening rotational velocity curves. Some proposals are also presented to resolve the second problem, among them the dark energy hypothesis is mostly accepted. The simplest form of dark energy is achieved by inserting the cosmological constant. Adopting this point of view leads to the celebrated ΛCDM (Λ Cold Dark Matter) scenario which has an incredible agreement with observational data \cite{3}. According to the observational data, and in high contrast with expectations from particle physics perspective, the cosmological constant is very small. In particle physics, a natural value for the vacuum energy is the mass of heaviest field in the theory, which is many order of magnitude higher than the observed cosmological constant.

One may ask why should we assume that the Einstein gravity is valid at all length scales, why the cosmological constant is so small, and/or why we don’t modify the Einstein gravity itself. To answer these questions also many proposals are appeared in the literature. Among them, in this paper we focus on ”massive gravity”, i.e. a Lorentz invariant extension of Einstein gravity in which the gravity is propagated by a massive spin-2 particle. In this type of theories, the gravity becomes exponentially weak at large distances, thereby the problem of flattening of the galaxy rotation curves can be resolved. Moreover, the modified gravitational potential can lead to an accelerating expansion which its rate can be tuned by the mass term.

Fortunately, direct detection of gravitational waves in the recent experiment GW150914, GW151226 \cite{4} by LIGO puts an upper bound on the mass of graviton, i.e. \(m_g < 1.2 \times 10^{-22}\text{eV} \) \cite{5}. The graviton mass may also link to the existence of gravitational wave polarizations.

Due to theoretical and experimental importance of massive gravity, there is a long historical background to explore a consistent theory in this regard. In 1939, Fierz and Pauli (FP) \cite{8} proposed a linear action for describing a free massive graviton. In the next thirty years nothing important happened until 1970 when van Dam, Veltman \cite{9} and

\footnote{Some previous attempts to find different bounds on the mass of graviton can be found in \cite{6,7} and references therein.}
Zakharov [10] separately showed that the FP theory coupled to a source, in the massless limit does not reduce to the Einstein-Hilbert (EH) theory. This phenomenon is known as vDVZ discontinuity.

In 1972, Vainshtein [11] argued that it is not possible to find a radius, $r_V$ around a massive source such that the linear approximation can be trusted inside it; therefore one should consider the full non-linear theory. This opens the possibility to cure the vDVZ discontinuity by the non-linear effects. In the end of 1972, as a quick response to the Vainshtein’s idea, Boulware and Deser [12] argued that the non-linear massive gravities in general possess a scalar field with a wrong sign kinetic term. This unwanted mode is known as Boulware-Deser ghost.

From the point of view of effective field theory, the existence of this ghost mode is not necessarily a problem, unless its mass is smaller than a UV cutoff scale. In 2002, according to this point of view, Arkani-Hamed, Georgi and Schwartz [13] introduced Higgs-like mechanism to give mass to graviton. This idea was followed by Creminelli, Nicolis, Papucci and Trincherini [14], but they showed the scalar ghost appears again in a radius much larger than the Vainshtein radius, $r_V$.

In 2010, de Rham and Gabadadze [15] found a sign mistake in Ref. [14] and together with Tolly, proposed a consistent four dimensional non-linear massive gravity which was free of Boulware-Deser ghost in special limits. The dRGT model was then followed and extended by Hassan and Rosen [16]. Hassan and Rosen [17] then proposed a new model by giving dynamics to the auxiliary spin two field of their massive gravity model. This last model is called HR-Bigravity.

On the other hand, in 2009, a parity invariant higher derivative gravitational model was suggested by Bergshoeff, Hohm and Townsend [18] in three dimensional space-time which at the linear level contained a FP massive spin-2 mode, a massless one and no Boulware-Deser ghost. This theory which is called New Massive Gravity (NMG) is described by

$$I_{NMG} = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left( R - 2\lambda - \frac{1}{m^2} (R^{\mu\nu} R_{\mu\nu} - \frac{3}{8} R^2) \right).$$  \hspace{1cm} (1.1)

Unfortunately, the sign of kinetic terms of massless and massive spin-2 particles are always opposite in this theory. This kind of instability is called kinetic ghost. Moreover, the dual central charges have the same sign as the kinetic term of massless spin-2 particle. This apparent problem is called "bulk-boundary clash". This problem was finally solved in 2013 [19] by introducing an extension of NMG in vierbein formalism. Very recently another extension of NMG in metric formalism [20] is presented, which has the same benefits as NMG and is free of the problem of bulk-boundary clash.

Definitely, the NMG model provides an alternative way through a new consistent mas-
sive gravity in four dimensional space-time along with dRGT and HR-Bigravity models. In this direction, Lü and Pope in 2011 proposed [21] the four dimensional version of NMG, called critical gravity as

\[ I_{\text{CG}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2m^2} (R^\mu{}^\nu R_{\mu\nu} - \frac{1}{3}R^2) \right), \]  

(1.2)

where \( R_{\mu\nu}, R, \Lambda \) and \( m^2 \) are Ricci tensor, Ricci scalar, cosmological constant and a mass parameter, respectively. This theory and its higher dimensional extensions [22], also suffer from the same kinetic ghost problem as in the NMG model. The aim of this paper is to explore a possibility of resolving this problem for these theories in the manner that the Boulware-Deser mode remains non-dynamic. In this way, we introduce a new four dimensional (and higher dimensional) massive gravity model that differs from the well-known dRGT and HR-BiGravity models.

In the next section, we introduce our model and explain its differences with dRGT and HR-BiGravity models. In section 3, we study the AdS-wave solutions of this model. In section 4, we study this model at the linear level and show that it is free of ghosts and tachyons. In sections 5, we provide an evidence for consistency of this model at the full non-linear level. This evidence is positivity of energy and entropy of AdS-Schwarzschild black hole solution in the same parameters range where the model is ghost free and tachyon free. The proposed four dimensional model in this paper can be easily extended to higher dimensions. Studying these higher dimensional extensions is the subject of section 6. In the last section, we discuss about some additional calculations to check more the consistency of this model. We have also reviewed in more details the previous works in the field of massive gravity which enables us to highlight the differences between our model and other well-known massive gravity models. We devoted appendix A to this subject.

2 The Model

In this paper, the main idea to resolve the kinetic ghost problem of critical gravity, (1.2), comes from the recent paper [20] in which a new consistent three dimensional massive gravity model is proposed. To see this idea in four spacetime dimensions, let us start with the four dimensional analogous of NMG which is, in fact, the critical gravity action (1.2). This action described in terms of auxiliary symmetric field \( f_{\mu\nu} \) reads

\[ I_{\text{CG,Aux}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R[g] - 2\Lambda_g + \frac{1}{2} f_{\mu\nu} G^{\mu\nu}[g] + \frac{m^2}{8} (\tilde{f}^{\mu\nu} f_{\mu\nu} - \tilde{f}^2) \right), \]  

(2.1)

\(^2\)Recently some new bi-gravity models [23] are proposed which the Boulware-Deser ghost is absent in them. But these models suffer from kinetic ghosts.
where $G_{\mu\nu}$ is the Einstein tensor (due to the metric $g_{\mu\nu}$), $m^2$ is a mass parameter and the indices in $\tilde{f}$ are raised by the inverse metric $g^{\mu\nu}$. Solving the equations of motion for $f_{\mu\nu}$ and substituting the result into the action (2.1) leads to the original action (1.2).

As we mentioned previously, the theory given by (2.1), or (1.2), is free of Boulware-Deser ghost, although it contains the spin-2 kinetic ghost. For the three dimensional analogous theory, it is shown in [20] that one can overcome the problem by giving dynamics to the auxiliary field via adding a scalar curvature (as well as cosmological constant) term for $f_{\mu\nu}$. Fortunately, the Boulware-Deser ghost remains non-dynamical in this way. Hence, we continue with the same idea of [20] and promote the auxiliary field $f_{\mu\nu}$ to a dynamical field as follows

$$I_{NBG} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R[g] + \frac{1}{2} f_{\mu\nu} G^{\mu\nu}[g] + \frac{m^2}{8} (\tilde{f}^{\mu\nu} f_{\mu\nu} - \tilde{f}^2) - 2\Lambda g \right) +$$

$$+ \frac{1}{16\pi G} \int d^4x \sqrt{-f} \left( R[f] - 2\Lambda f \right), \quad (2.2)$$

where $\tilde{G}$ and $\Lambda_f$ are the Newton constant and cosmological constant for the field $f_{\mu\nu}$.

In the subsequent sections we study different aspects of this model and show that, at the linearized level around AdS vacuum, it is free from ghosts and tachyonic instabilities. Before that, let us emphasize on two differences between this model and the well-known massive gravity models (A.4) and (A.6). First, the models (A.4) and (A.6) are based on the idea of extending the FP mass term through a potential which does not contain derivative terms. However, we will show that in the theory (2.2), the graviton mass has also contributions from the derivative term $f_{\mu\nu} G^{\mu\nu}[g]$.

The second and most important difference is explained by rewriting the HR-BiGravity model, (A.6), in its higher derivative form. As is discussed in [24] for the model (A.6), one can determine $f_{\mu\nu}$ algebraically in term of $g_{\mu\nu}$ and its curvatures $R_{\mu\nu}[g]$. In general, the solution $f_{\mu\nu}(g)$ is a perturbative expansion in powers of $\frac{1}{m^2} R_{\mu\nu}[g]$, where $m^2$ sets the scale of the FP mass. Using this perturbative solution to eliminate $f_{\mu\nu}$ from the action (A.6), one can obtain the higher derivative gravity action, $I_{HD}[g] = I_{HR-BiG}[g, f(g)]$, which at the four-derivative level reads [24]

$$I_{HD}[g] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2m^2} (R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2) \right) + \mathcal{O} \left( \frac{1}{m^4} \right). \quad (2.3)$$

Neglecting higher order terms, $\mathcal{O} \left( \frac{1}{m^4} \right)$, this action is the critical gravity action (1.2). This point is exactly where the difference between the model (A.6) and our model (2.2) is clarified. Hassan, Schmidt-May and von Strauss [24] have shown that the action (2.3), without $\mathcal{O} \left( \frac{1}{m^4} \right)$ terms, gives a massive spin-2 particle which is ghost and its mass differs from the value in associated model (A.6). They argued that the appearance of ghost
with a different mass is the artifact of truncating the original higher derivative theory to a four-derivative action (2.3); therefore to resolve this discrepancy all the higher order terms $\mathcal{O}(\frac{1}{m^n}); n \geq 4$ should be added. In the current work, instead of adding all those higher derivative terms, we rewrite the action (2.3) in its auxiliary form (2.1) and remove the ghost by promoting the auxiliary field in the model to a dynamical field. We show that this way also gives a consistent massive gravity model.

3 AdS wave solutions

In this section, we present one type of solutions of the theory (2.2), known as ”AdS wave” solutions (since they are a special kind of gravitational waves propagating along the AdS spacetime). In general they can be written as

$$g_{\mu\nu} = g^{\text{AdS}}_{\mu\nu} - F k_\mu k_\nu, \tag{3.1}$$

where $k_\mu$ is a null geodesic field, and $F$ is a function which depends on the dynamics of the gravitational theory. Albeit they look like perturbative excitations around the AdS spacetime, one should note that they are solutions of the full non-linear equations of motion. This statement can be understood by the null characteristic of the vector field $k_\mu$.

The AdS wave solutions are studied as a preliminary test of unitarity of the underlying theory. As we will see, the form of the function $F$ is closely related to the particle content of a theory. Non unitarity of a theory may be showed up in the AdS wave solutions. However, if this test is passed, there is no guarantee for the theory to be unitary and one still needs more consistency checks.

The equations of motion of the fields $g_{\mu\nu}$ and $f_{\mu\nu}$ in the theory (2.2) are respectively as follows

$$\mathcal{G}[g]_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}^{(1)}[g] + T_{\mu\nu}^{(2)}[g], \quad \mathcal{G}_{\mu\nu}[f] + \Lambda f_{\mu\nu} = T_{\mu\nu}[f], \tag{3.2}$$

where

$$T_{\mu\nu}^{(1)}[g] = -\frac{m^2}{4} \left[ \hat{f}_\mu f_\nu - \hat{f} f_{\mu\nu} - \frac{1}{4} g_{\mu\nu} (\hat{f}^{\rho\sigma} f_{\rho\sigma} - \hat{f}^2) \right],$$

$$T_{\mu\nu}^{(2)}[g] = \frac{1}{2} \left[ -2 \hat{f}_\mu g_{[\nu]_\rho} + \frac{1}{2} f_{\mu\nu} R[g] + \frac{1}{2} R_{\mu\nu}[g] + \frac{1}{2} g_{\mu\nu} f_{\rho\sigma} \mathcal{G}[g]^{\rho\sigma} - \frac{1}{2} \left( \nabla^2 g f_{\mu\nu} - 2 \nabla [g]^{\rho\sigma} \nabla [g]_{(\mu} f_{\nu)\rho} + \nabla [g]_{\mu} \nabla [g]_{(\nu} \hat{f} + (\nabla [g]^{\rho\sigma} \nabla [g]^{\mu\nu}) f_{\rho\sigma} - \nabla^2 [g] \hat{f} g_{\mu\nu} \right) \right],$$
\[ T_{\mu \nu}[f] = \frac{1}{\kappa} \sqrt{g} \left[ \frac{1}{2} f_{\alpha \mu} f_{\beta \nu} G[g]^{\alpha \beta} + \frac{m^2}{4} \left( g^{\sigma \tau} g^{\alpha \beta} - g^{\sigma \tau} g^{\alpha \beta} \right) \left( f_{\sigma \tau} f_{\alpha \mu} f_{\beta \nu} \right) \right], \quad (3.3) \]

in which \( \kappa \equiv \frac{G}{G} \). For two dynamical fields, \( g_{\mu \nu} \) and \( f_{\mu \nu} \), we consider two AdS wave ansatz as follows

\[
\begin{align*}
    ds_g^2 &= \ell^2 g_g \left( dr^2 + dy^2 + 2 dx^+ dx^- - G(r, x^+ dx^2) \right), \\
    ds_f^2 &= \ell^2 f_f \left( dr^2 + dy^2 + 2 dx^+ dx^- - F(r, x^+ dx^2) \right) .
\end{align*}
\]

(3.4)

Plugging the ansatz (3.4) into the equations of motion of \( g_{\mu \nu} \) gives

\[
\begin{align*}
    \Lambda_g \ell_g^2 + 3 &= 0, \\
    -\left( \frac{\ell_g^2}{\ell_f^2} + 2 \right) \left[ \frac{\partial^2 G}{\partial r^2} - \frac{2 \partial G}{r \partial r} \right] + \frac{\ell_f^2}{\ell_g^2} \left[ \frac{\partial^2 F}{\partial r^2} - \frac{2 \partial F}{r \partial r} \right] + 2 \left( \frac{m^2 \ell_f^2}{\ell_g^2} - 3 \right) \left[ \frac{G}{r^2} - \frac{F}{r^2} \right] &= 0 .
\end{align*}
\]

(3.5)

Substituting the same ansatze in the equations of motion of \( f_{\mu \nu} \) gives

\[
\begin{align*}
    \Lambda_f \ell_f^2 + 3 - \frac{3}{2\kappa} \left( 1 - \frac{m^2 \ell_f^2}{2} \right) &= 0, \\
    \left[ \frac{\partial^2 G}{\partial r^2} - \frac{2 \partial G}{r \partial r} \right] - 2\kappa \left[ \frac{\partial^2 F}{\partial r^2} - \frac{2 \partial F}{r \partial r} \right] + 2 \left( 3 - m^2 \ell_f^2 \right) \left[ \frac{G}{r^2} - \frac{F}{r^2} \right] &= 0 .
\end{align*}
\]

(3.6)

A useful class of AdS wave solutions for the equations (3.5) and (3.6) is given by proportionality condition \( \ell_f^2 = \gamma \ell_g^2 \equiv \gamma \ell^2 \). That is because, as we will show in section 4, the model (2.2) has a well-defined mass spectrum around two proportional AdS where the fluctuations \( \delta g_{\mu \nu}, \delta f_{\mu \nu} \) decompose into a massless spin-2 mode and a FP massive spin-2 mode. By using the proportionality condition and assuming power law dependence of the functions \( F \) and \( G \), with respect to the radial coordinate \( r \), the most general solution of the Eqs. (3.5)-(3.6) reads

\[
\begin{align*}
    G(r, x^+) &= f_1(x^+) + f_2(x^+) r^3 + f_3(x^+) r^3 \left( 1 + \sqrt{1 + A} \right) + f_4(x^+) r^3 \left( 1 - \sqrt{1 + A} \right) , \\
    F(r, x^+) &= f_1(x^+) + f_2(x^+) r^3 + \beta f_3(x^+) r^3 \left( 1 + \sqrt{1 + A} \right) + \beta f_4(x^+) r^3 \left( 1 - \sqrt{1 + A} \right) .
\end{align*}
\]

(3.7)
where $f_i(x^+)$’s are arbitrary functions of $x^+$ and

$$A = \frac{8}{9} \left( \gamma m^2 \ell^2 - 3 \right) \frac{(2 + \gamma(2\kappa - 1))}{(4\kappa + \gamma(2\kappa - 1))}, \quad \beta = \frac{2}{\gamma(1 - 2\kappa)}.$$

(3.8)

To explore the particle content of the theory (2.2) by using the AdS wave solutions (3.7), note that the AdS wave solutions are also solutions of linearized equations of motion which are closely related to the particle content of a theory. Comparing to (3.7), the AdS waves of Einstein-Hilbert theory just contain functions $f_1$ and $f_2$. On the other hand, the Einstein-Hilbert theory contains only the massless spin-2 particle. Hence, the AdS wave solutions (3.7) mean that, besides the massless spin-2 particle, the theory (2.2) has other particles in its spectrum, related to the functions $f_3$ and $f_4$. To get information about these new modes a good, although naive, way is comparing the AdS wave solutions (3.7) with solutions of the wave equation for a massive spin-2 particle, i.e.

$$\Box + \frac{2}{\ell^2} - M^2 \right) h_{\mu\nu} = 0,$$

(3.9)

where the D’Alembert operator is defined with an AdS$_4$ background with radius $\ell$. In general, Eq.(3.9) has two independent solutions which can be combined as follows

$$h_{\mu\nu} \sim \ell^2 \left[ a(x^i) r^{\frac{3}{2}} (1 - \sqrt{1 + \frac{4}{9} M^2 \ell^2}) + b(x^i) r^{\frac{3}{2}} (1 + \sqrt{1 + \frac{4}{9} M^2 \ell^2}) \right],$$

(3.10)

where $a, b$ are arbitrary functions of spatial coordinates. It is clear that for the massless particle, $M^2 = 0$, we have $h_{\mu\nu} \sim \ell^2 \left[ a(x^i) + b(x^i) r^3 \right]$. Therefore the AdS wave solutions (3.7), in comparison with (3.10), tells us that the theory (2.2) at least has one massless spin-2 and one massive spin-2 particle with mass

$$M^2 = \frac{9}{4\ell^2} A.$$

(3.11)

We emphasize on the word ”at least”, because the wave equation of massive scalar with mass $M^2$ on the AdS$_4$ spacetime has the same solution as (3.10). However, by this naive analysis we are not sure that the theory (2.2) contains such a scalar particle. To assure about it, one needs to find the action of quadratic fluctuations, which is the subject of section 4.

Now the unitarity, which here means the non-tachyonic nature of excitations, impose the condition $M^2 \geq -\frac{9}{4\ell^2}$ where the lower bound is known as Breitenlohner-Freedman (BF) bound in AdS [26]. Hence, for any value $A \geq -1$, the theory (2.2) is free of tachyonic

As a check point, in the critical gravity limit, $\kappa \rightarrow 0$ and $m^2 \ell^2 \rightarrow 1$, the AdS-wave solutions (3.7) reduce exactly to the AdS-wave solutions of critical gravity [21, 25] which are in the form of (3.13). To see this the first equation in (3.6) should also be used.
spin-2 particles. This last condition can constrain the parameters of the theory according to (3.8). However, note that by this analysis we can not say whether or not the theory contains the kinetic ghosts.

As a further point, from (3.8) we have $A = 0$ for the values

$$\gamma = \frac{3}{m^2 \ell^2}, \quad \text{or} \quad \gamma = -\frac{2}{2\kappa - 1},$$

(3.12)

which converts the massive spin-2 mode to massless one. This phenomena which is common in all critical gravities [25], [21], [27] shows that the theory for the values (3.12) may be non-unitary. The reason is that, for these values the AdS wave solutions of theory (2.2) should be written as

$$G(r, x^+) = \tilde{G}_0[x^+] \log(r) + G_0[x^+] + \tilde{G}_3[x^+] r^3 \log(r) + G_3[x^+] r^3,$$

$$F(r, x^+) = \tilde{F}_0[x^+] \log(r) + F_0[x^+] + \tilde{F}_3[x^+] r^3 \log(r) + F_3[x^+] r^3.$$  

(3.13)

The presence of leading Log term means that the dual theory is a logarithmic conformal field theory (LCFT), i.e. a non-unitary theory. This is also understandable from the fact that the theory (2.2) for the values (3.12) has two massless spin-2 particles in its spectrum. Another kind of logarithmic solutions also occurs for

$$\gamma = \frac{-16m^2 \ell^2 + 15(2\kappa - 1) \pm \tilde{\gamma}}{16m^2 \ell^2 (2\kappa - 1)},$$

(3.14)

with

$$\tilde{\gamma} = \sqrt{(16m^2 \ell^2 - 30\kappa + 15)^2 - 576m^2 \ell^2 (3\kappa - 4)(2\kappa - 1)},$$

which gives $A = -1$. In this case the solutions corresponding to the terms $f_3$ and $f_4$ in (3.7) degenerate and new Log solutions appear as follows

$$G(r, x^+) = G_0[x^+] + \tilde{G}_3[x^+] r^{3/2} \log(r) + G_3[x^+] r^{3/2} + G_3[x^+] r^3,$$

$$F(r, x^+) = F_0[x^+] + \tilde{F}_3[x^+] r^{3/2} \log(r) + F_3[x^+] r^{3/2} + F_3[x^+] r^3.$$  

(3.15)

However, the appearance of these Log solutions is not the sign that the dual theory is a LCFT since the mass $M^2$ is non-zero.
4 Linearization

In this section we want to find the action of quadratic fluctuations around a definite vacuum, in order to show the existence of certain regions in the parameters space where the theory can be unitary. In fact, whether the theory makes sense or not should be studied for each possible vacuum. However, we are interested here in the vacuum solution \( \bar{f}_{\mu\nu} = \gamma \bar{g}_{\mu\nu} \), where \( \bar{g}_{\mu\nu} \) denotes the AdS\(_4\) geometry with curvature radius \( \ell \). From the equations of motion (3.2), one can see for this vacuum solution we should have

\[
\Lambda g_{\ell^2} + 3 = 0, \quad \Lambda f_{\ell^2} + 3 - \frac{3}{2\kappa} \left( 1 - \frac{\gamma}{2} m^2 \ell^2 \right) = 0. \quad (4.1)
\]

The fluctuations around this vacuum can be given as

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad f_{\mu\nu} = \gamma (\bar{g}_{\mu\nu} + \rho_{\mu\nu}). \quad (4.2)
\]

Inserting (4.2) in the action (2.2), and keeping the terms up to the second order terms with respect to the fluctuations, we find the quadratic action as follows

\[
S^{(2)}[h_{\mu\nu}, \rho_{\mu\nu}] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ (1 + \frac{\gamma}{2}) h^{\mu\nu}(\mathbb{G} h)_{\mu\nu} + \kappa \gamma h^{\rho\nu}(\mathbb{G} \rho)_{\mu\nu} - \gamma h^{\mu\nu}(\mathbb{G} \rho)_{\mu\nu} + (h - \rho). (h - \rho) \right\}. \quad (4.3)
\]

In the above action, \( \mathbb{G} \) is the Lichnerowicz operator on the curved AdS\(_4\) background, defined as

\[
p^{\mu\nu}(\mathbb{G} q)_{\mu\nu} \equiv -\frac{1}{4} q_{\rho\sigma} p^{\rho\sigma} p_{\mu\nu} + \frac{1}{2} p_{\mu\nu\rho} q^{\mu\nu, \rho} + \frac{1}{4} p_{\mu\nu} q^{\mu\nu} - \frac{3}{4} p_{\mu\nu} q^{\mu\nu, \rho} - \frac{3}{2} p_{\rho\sigma} p^{\rho\sigma} - \frac{3}{4} p_{\mu\nu} q^{\mu\nu} - \frac{3}{2} p_{\mu\nu}. \quad (4.4)
\]

The dot product in Eq. (4.3) is also defined as

\[
h \cdot h \equiv \chi h_{\mu\nu} h^{\mu\nu} - \xi h^2, \quad (4.5)
\]

where

\[
\chi = \frac{\gamma}{4\ell^2} (3 - \gamma m^2 \ell^2), \quad \xi = \frac{\gamma}{16\ell^2} \left( 6 - \gamma m^2 \ell^2 \right). \quad (4.6)
\]

In comparison with (A.2), the curved Lichnerowicz operator contains the extra term \(-\frac{3}{2\ell^2} (p^{\mu\nu} q_{\mu\nu} - \frac{1}{2} pq)\) which looks like a mass term without the FP tuning. However, by

\footnote{The calculations in this part are done by the xAct [28].}
using some representation theory [29], and after a long discussion about the meaning of massless particle in curved spacetime [30], the FP action in curved AdS$_4$ geometry is given by

$$I_{FP-Curved} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( h^{\mu\nu} (Gh)_{\mu\nu} - \frac{M^2}{4} (h^{\mu\nu} h_{\mu\nu} - h^2) \right),$$  \hspace{1cm} (4.7)$$

where $G$ is defined by (4.4). For most choices of $M^2$, it describes a particle with five propagating degrees of freedom in four dimensions.

To avoid the cross terms in Eq.(4.3), it is useful to utilize a new basis for the fluctuations, i.e.

$$h = a_1 h^{(0)} + a_2 h^{(m)}, \quad \rho = h^{(0)} + h^{(m)}. \hspace{1cm} (4.8)$$

Using this new basis (4.8) in (4.3) gives

$$S^{(2)}[h^{(0)}, h^{(m)}] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ A h^{(0)}{}_{\mu\nu} (Gh^{(0)})_{\mu\nu} + B h^{(m)}{}_{\mu\nu} (Gh^{(m)})_{\mu\nu} + M_1^{(0)} (h^{(0)}{}_{\mu\nu} h^{(0)}{}_{\mu\nu} + h^{(0)}{}_{\mu\nu} h^{(m)}{}_{\mu\nu} + h^{(m)}{}_{\mu\nu} h^{(0)}{}_{\mu\nu} + h^{(m)}{}_{\mu\nu} h^{(m)}{}_{\mu\nu}) + M_2^{(0)} (h^{(m)}{}_{\mu\nu} h^{(m)}{}_{\mu\nu}) \right\},$$  \hspace{1cm} (4.9)$$

where the coefficients $A, B, C$ are as follows

$$A = \kappa \gamma + a_1 \left[ a_1 + \left( \frac{a_1}{2} - 1 \right) \gamma \right], \quad B = A[a_1 \rightarrow a_2], \quad C = 2\kappa \gamma + 2a_1a_2 + \gamma (a_1a_2 - a_1 - a_2),$$

and the coefficients $M_1^{(0)}, M_1^{(m)}, M_1^{(0,m)}$ are given by

$$M_1^{(0)} = \frac{\gamma}{4\ell^2} (a_1 - 1)^2 (3 - \gamma m^2 \ell^2), \quad M_1^{(0)} = \frac{3}{4\gamma m^2 \ell^2 - 3}, \quad M_1^{(m)} = M_1^{(0)}[a_1 \rightarrow a_2], \quad M_1^{(m)} = M_1^{(0)},$$

$$M_1^{(0,m)} = \frac{\gamma}{2\ell^2} (a_1 - 1)(a_2 - 1)(3 - \gamma m^2 \ell^2), \quad M_2^{(0,m)} = \frac{\gamma}{8\ell^2} (a_1 - 1)(a_2 - 1)(\gamma m^2 \ell^2 - 6).$$

Now FP tuning of the quadratic action (4.9) for the absence of scalar ghost implies that $M_2^{(0)} = M_2^{(m)} = -1$ which gives

$$\gamma = \frac{2}{m^2 \ell^2}. \hspace{1cm} (4.10)$$
By using this value of $\gamma$, the coefficients of mixing terms become

$$C = -\frac{2}{m^2 \ell^2} \left( a_1 + a_2 - a_1 a_2 (m^2 \ell^2 + 1) - 2\kappa \right),$$

$$M_1^{(0,m)} = -M_2^{(0,m)} = \frac{1}{m^2 \ell^4} (a_1 - 1)(a_2 - 1),$$

which should vanish to decouple the modes. Vanishing of $M_1^{(0,m)}, M_2^{(0,m)}$ implies

$$a_1 = 1 \quad \text{or} \quad a_2 = 1 \quad \text{or} \quad a_1 = a_2 = 1.$$  \hspace{1cm} (4.12)

For the case $a_1 = a_2 = 1$, $C$ vanishes if $m^2 \ell^2 - 1 + 2\kappa = 0$. This is exactly the condition for appearance of Log-solution (3.13), hence this choice should be discarded. The theory is symmetric with respect to $a_1 \rightarrow a_2$, so in the following we restrict ourselves to the case $a_1 = 1$. Vanishing of the coefficient $C$ for this case implies

$$a_2 = -\frac{(2\kappa - 1)}{m^2 \ell^2}.$$ \hspace{1cm} (4.13)

To this end, the proper basis in which the scalar ghost is absent and two modes are decoupled is

$$h = h^{(0)} - \frac{(2\kappa - 1)}{m^2 \ell^2} h^{(m)}, \quad \rho = h^{(0)} + h^{(m)}.$$ \hspace{1cm} (4.14)

In this basis, the action (4.9) reads

$$S^{(2)}[h^{(0)}, h^{(m)}] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ A_0 h^{(0)}{}^{\mu\nu} \left( G h^{(0)} \right)_{\mu\nu} + \right.$$  

$$+ A_m \left\{ h^{(m)}{}^{\mu\nu} \left( G h^{(m)} \right)_{\mu\nu} - \frac{M^2}{4} \left( h^{(m)}{}_{\mu\nu} h^{(m)}{}^{\mu\nu} - (h^{(m)})^2 \right) \right\},$$ \hspace{1cm} (4.15)

where

$$A_0 = 1 + \frac{2\kappa - 1}{m^2 \ell^2}, \quad A_m = A_0 \frac{2\kappa (m^2 \ell^2 + 1) - 1}{m^4 \ell^4},$$

$$M^2 = -A_0 \frac{2m^2}{2\kappa (m^2 \ell^2 + 1) - 1}.$$ \hspace{1cm} (4.16)

It is illustrative to take a closer look at the above results in the limits, $\kappa \to 0$ and $m^2 \ell^2 \to 1$, where the theory (2.2) reduces to the critical gravity (1.2). In these limits, the quadratic action (4.15) vanishes which seems to be in contradiction with the critical gravity in the linearized level. However, the correct way to taking the limits, $\kappa \to 0$
and $m^2 \ell^2 \rightarrow 1$, is imposing them in Eq.(4.11) to find $a_1 = a_2 = 1$ upon demanding the absence of the mixing terms. As stated before, this is discarded due to appearance of Log solutions (3.13).

Now, the kinetic ghost free condition together with the absence of tachyonic mode implies that

$$A_0 > 0, \quad A_m > 0, \quad M^2 \geq -\frac{9}{4\ell^2}. \quad (4.17)$$

The allowed values for $(m^2, \kappa)$, which satisfy the above conditions, are presented in Fig.1. Also for these allowed values, the massive spin-2 particle acquires masses in the range $-\frac{9}{4\ell^2} \leq M^2 < 0$ as depicted in Fig.2.

![Figure 1: Unitary region for $m^2$ and $\kappa$. Note that here we set $\ell = 1$.](image)

In the allowed region, Fig.1, it exists a subregion in which the mass of spin-2 particle becomes a special value, $M^2 = -\frac{2}{\ell^2}$. The theory (2.2) at this subregion has an interesting feature that is appearance of a new gauge symmetry which eliminates the helicity 0 part of the massive spin-2 field, leaving behind only four propagating modes. These four remaining degrees of freedom collectively present a Partially Massless spin-2 particle (PM mode) [30] and [31]. That new gauge symmetry is

$$\delta_{\xi} h^{(m)}_{\mu \nu} = \left( \nabla_{\mu} \nabla_{\nu} - \frac{1}{\ell^2} \bar{g}_{\mu \nu} \right) \xi(x), \quad (4.18)$$

where the $\bar{g}_{\mu \nu}$ is AdS$_4$ background and covariant derivative is defined also by this metric.
Also the quadratic action (4.15) in that subregion reduces to
\[
S^{(2)}_{PM}[h^{(0)}, h^{(m)}] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ h^{(0)\mu\nu}(G h^{(0)})_{\mu\nu} + \frac{1}{m^2 \ell^2} \left\{ h^{(m)\mu\nu}(G h^{(m)})_{\mu\nu} + \frac{1}{2\ell^2} \left( h^{(m)\mu\nu} h^{(m)}_{\mu\nu} - (h^{(m)})^2 \right) \right\} \right]. \tag{4.19}
\]

The point should be noticed is that this action is the same as four dimensional Conformal Gravity [32] at the linearized level. This means that the Weyl symmetry, in that special subregion, comes back to the theory (2.2), at least at the linearized level. If this structure can be extended to all orders, then the theory (2.2) can provide a non-linear theory for PM particles.

Before closing this section, let us try to answer a natural question which might be asked: How much the analysis in this section depends to the background metric (vacuum solution). The crucial point that should be considered to answer this question is explicitly absence of Riemann tensor in the action (2.2) and equations of motion (3.2). This means that around the backgrounds (vacuum solutions) where they have a same Ricci tensor and Ricci-Scalar tensor, the spectrum of particles and unitary regions would be exactly the same. An example of such backgrounds are the AdS\(_4\) spacetime and Schwarzschild-AdS black hole where the Ricci tensor and Ricci-Scalar tensor for both of them are \(-\frac{3}{\ell^2} g_{\mu\nu}\) and \(-\frac{12}{\ell^2}\), respectively. In next section we check the existence of these black hole solutions.

\footnote{We have checked the linearized analysis for these two different backgrounds explicitly and found the same spectrum and unitary regions.}
for the model (2.2).

5 Black Hole Solution

In this section, we present one black hole solution for the theory (2.2) and calculate its energy and entropy. Consider the following Schwarzschild-AdS ansatz

$$ds_g^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad ds_f^2 = \gamma ds_g^2,$$ (5.1)

with $F(r) = 1 - \frac{\mu}{r} + \frac{r^2}{\ell^2}$, where $\mu$ is a mass parameter that can be expressed in terms of the horizon radius $r_h$ and the curvature radius $\ell$ as follows

$$\mu = r_h(1 + \frac{r_h^2}{\ell^2}).$$ (5.2)

The metric (5.1) reduces to AdS$_4$ for $\mu = 0$ and the standard Schwarzschild solution for $\ell \to \infty$. Substituting the ansatz (5.1) in the equations of motion (3.2) gives

$$\Lambda_g = -\frac{3}{\ell^2}, \quad \Lambda_f = -\frac{3\gamma m^2 \ell^2 + 6(2\kappa - 1)}{4\kappa \gamma \ell^2},$$ (5.3)

which are exactly the same conditions as (4.1) for two proportional AdS$_4$. The reason for this coincidence is equivalence of the Ricci tensor and Ricci-Scalar of the Schwarzschild-AdS black hole (5.1) with pure AdS$_4$ metric.

5.1 Energy and Entropy of black hole

In this subsection we find the energy and entropy of black hole solutions (5.1) using the "renormalized on-shell action". To have a well-defined variational principle, depending on the boundary conditions, one needs the appropriate Gibbons-Hawking terms. The black hole solutions (5.1) are obtained according to the Dirichlet boundary condition. However, the theory (2.2) may have other solutions which are obtained by different boundary conditions, such as Log solutions (3.13) where the variational principle, as well as the Gibbons-Hawking terms, should be modified appropriately. Here, we are interested in some ranges of parameters of the theory (2.2) where these solutions do not exist. On the other hand, the on-shell action in general may contain divergences which should be removed by adding suitable counterterms. This is what we mean by the word "renormalized".

In the theory (2.2), the boundary terms which may invalidate the variational principle
for Dirichlet boundary condition emerge from the following terms

$$\delta I_{\delta g, \delta f} = \frac{1}{16 \pi G} \int d^4 x \left( \sqrt{-g} \delta R[g] + \frac{1}{2} \sqrt{-g} f_{\mu \nu} \delta G^{\mu \nu}[g] + \kappa \sqrt{-f} \delta R[f] \right).$$  \hspace{1cm} (5.4)

These terms can be simplified in the following form

$$\delta I_{\delta g, \delta f} = \frac{1}{16 \pi G} \int d^4 x \left( \sqrt{-g} A_{\mu \nu} \delta R^{\mu \nu}[g] + \kappa \sqrt{-f} f_{\mu \nu} \delta R^{\mu \nu}[f] \right),$$  \hspace{1cm} (5.5)

where

$$A_{\mu \nu} = g_{\mu \nu} + \frac{1}{2} (f_{\mu \nu} - \frac{1}{2} f_{\alpha \beta} g^{\alpha \beta} g_{\mu \nu}).$$  \hspace{1cm} (5.6)

The Eq.(5.5) can be simplified more according to proportionality of the fields $g_{\mu \nu}$ and $f_{\mu \nu}$ in the black hole solution (5.1). The result of this simplification is

$$\delta I_{\delta g, \delta f} = \frac{1}{16 \pi G} \int d^4 x \left( \sqrt{-g} (1 - \frac{\gamma}{2}) g_{\mu \nu} \delta R^{\mu \nu}[g] + \kappa \sqrt{-f} f_{\mu \nu} \delta R^{\mu \nu}[f] \right).$$  \hspace{1cm} (5.7)

Hence, the proper Gibbons-Hawking terms can be suggested as follows

$$I_{\text{GH}} = -\frac{2(1 - \frac{\gamma}{3})}{16 \pi G} \int d^3 x \sqrt{-\eta} \eta^{ij} K[g]_{ij} - \frac{2}{16 \pi G} \int d^3 x \sqrt{-\eta} \eta^{ij} K[f]_{ij},$$  \hspace{1cm} (5.8)

where $\eta_{gij}$ and $\eta_{fij}$ are the induced metrics, on the boundary, associated with the metrics $g$ and $f$, respectively. By having the Gibbons-Hawking action (5.8) at hand, one can find the on-shell action by plugging the metric (5.1), in its Euclidean signature ($\tau = -it$), in the sum of bulk action (2.2) and boundary action (5.8) and then performing the integration over $r(r_h \to \mathcal{R})$, $\tau(0 \to \beta)$, $\theta(0 \to \pi)$ and $\phi(0 \to 2\pi)$. The resulted on-shell action is

$$I_0 + I_{\text{GH}} = \frac{a}{16 \pi G} \left( \frac{\mathcal{R}^3}{\ell^2} + \mathcal{R} + b \right),$$  \hspace{1cm} (5.9)

where $\mathcal{R} \gg r_h$ is a cutoff and

$$a = 8 \pi \beta (2 + (2\kappa - 1)\gamma), \quad b = -\frac{3}{4} r_h (1 + \frac{r_h^2}{3\ell^2}).$$  \hspace{1cm} (5.10)

It is clear that the on-shell action (5.9) is divergent due to the infinite volume limit. Therefore, one needs proper boundary counterterms to render the action finite. One can easily check that these needed boundary terms are

$$I_{\text{ct}} = \frac{1}{16 \pi G} \int d^3 x \sqrt{\eta} \left( c_1 + c_2 R[\eta] \right),$$  \hspace{1cm} (5.11)
where
\[ c_1 = \frac{2}{\ell} (2 + (2\kappa - 1)\gamma), \quad c_2 = \frac{\ell^2}{4} c_1. \] (5.12)

Just as a consistency check notice that, for the case \( \gamma = 0 \), the above counterterm action reduces correctly to the result for Einstein-Hilbert theory \([33, 34]\). Putting everything together, the renormalized on-shell action reads
\[ I_{\text{ren}} = \frac{\tilde{a}_{\text{fin}}}{32\pi G} (2 + (2\kappa - 1)\gamma), \quad \tilde{a}_{\text{fin}} = \frac{4\pi}{\ell^2} \beta r_h (r_h^2 - \ell^2). \] (5.13)

The energy and entropy of the gravitational system can be found from the renormalized on-shell action as
\[ E = -\frac{\partial}{\partial \beta} I_{\text{ren}}, \quad S = (1 - \beta \frac{\partial}{\partial \beta}) I_{\text{ren}}. \] (5.14)

To use the above formula, we need the relation between \( r_h \) and the temperature. The Hawking temperature is given by
\[ T_H \equiv \frac{1}{\beta} = \frac{1}{4\pi} \partial_r F|_{r=r_h} = \frac{r_h}{4\pi} \left( \frac{3}{\ell^2} + \frac{1}{r_h^2} \right), \] (5.15)

which can be inverted to find \( r_h \) with respect to \( \beta \) as
\[ r_h = \frac{2}{3} \frac{\pi \ell^2}{\beta} \left( 1 \pm \sqrt{1 - \frac{3\beta^2}{4\pi^2\ell^2}} \right). \] (5.16)

As the free energy would be smaller for the smaller value of \( r_h \), in what follows we continue with the minus sign in (5.16). Plugging the smaller \( r_h \) in (5.13) and using (5.14) gives
\[ E = \left( 2 + (2\kappa - 1)\gamma \right) \frac{r_h (r_h^2 + \ell^2)}{4\ell^2 G}, \quad S = \left( 2 + (2\kappa - 1)\gamma \right) \frac{\pi r_h^2}{2G}. \] (5.17)

In section 4, we showed that the absence of scalar ghost implies that \( \gamma = \frac{2m^2}{\ell^2} \). For this value of \( \gamma \), the expressions for energy and entropy in (5.17) are
\[ E = \left( 1 + \frac{2\kappa - 1}{m^2\ell^2} \right) \frac{r_h (r_h^2 + \ell^2)}{2G\ell^2}, \quad S = \left( 1 + \frac{2\kappa - 1}{m^2\ell^2} \right) \frac{\pi r_h^2}{G}. \] (5.18)

As a consistency check, in the limits \( \kappa \to 0 \) and \( m^2\ell^2 \to 1 \), where the theory (2.2) reduces to the critical gravity (1.2), the energy and entropy of Schwarzschild-AdS black hole solutions (5.18) vanish. This is in complete agreement with vanishing of energy and entropy of Schwarzschild-AdS black hole solutions in critical gravity \([21]\) and \([35]\). Interestingly, the obtained energy and entropy in (5.18) could be written with respect to
the coefficient of the kinetic term of massless graviton \((4.16)\) as below

\[
E = \frac{r_h(E^2 + r_h^2)}{2G\ell^2} \kappa_0, \quad S = \frac{\pi r_h^2}{G} \kappa_0. \quad (5.19)
\]

This means that the absence of scalar ghost and positivity of kinetic term of massless graviton, imply that the energy and entropy of Schwarzschild-AdS black hole solution \((5.1)\) in the theory \((2.2)\) are positive. These positive values for black hole solutions might be the sign that the theory \((2.2)\) at full non-linear level is also healthy.

6 Higher Dimensional New Bi-Gravities

In this section we extend our previous discussions to higher dimensions.

6.1 Action

The D-dimensional extension of NMG in its axially form is given by \([22]\]

\[
I = \frac{1}{16\pi G} \int d^Dx \sqrt{-g} \left( R[g] - 2\Lambda_g + \frac{1}{D-2} f_{\mu\nu} G^{\mu\nu}[g] + \frac{m^2}{4(D-2)} (\tilde{f}^{\mu\nu} f_{\mu\nu} - \tilde{f}^2) \right). \quad (6.1)
\]

Here we have used a notation in which \(\tilde{f}^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} f_{\alpha\beta}, \tilde{f} \equiv g^{\mu\nu} f_{\mu\nu}.\) Solving the equations of motion for the field \(f_{\mu\nu}\) gives

\[
f_{\mu\nu} = -\frac{2}{m^2} \left( R_{\mu\nu}[g] - \frac{1}{2(D-1)} R[g] g_{\mu\nu} \right). \quad (6.2)
\]

Substituting back this expression in \((6.1)\) gives the action of critical gravity \([22, 36]\), in its higher derivative form,

\[
I = \frac{1}{16\pi G} \int d^Dx \sqrt{-g} \left( R - 2\Lambda_g - \frac{1}{m^2(D-2)} (\tilde{R}^{\mu\nu} R_{\mu\nu} - \frac{D}{4(D-1)} R^2) \right). \quad (6.3)
\]

One can find the dynamical degrees of freedom, at the linearized level, by performing the linearization of the theory \((6.1)\) around a maximally symmetric vacuum with \(AdS_D\) geometry.

For generic values of the parameters, it is shown that the theory \((6.1)\), around this background, describes one massless spin-2 and one massive spin-2 particle with mass \([22]\]

\[
M^2 = (D-2) \left( m^2 + \frac{\Lambda}{(D-1)} \right). \quad (6.4)
\]

Moreover, the kinetic terms of massless and massive modes have opposite signs and
therefore there is no way to get rid of ghosts. To avoid these ghosts, one can choose a special value for \( \Lambda \) [22] as

\[
m^2 + \frac{\Lambda}{(D - 1)} = 0. \tag{6.5}
\]

Now another problem arises. The linearized equation of motion for this value of \( \Lambda \) reads

\[
\left( \Box - \frac{4\Lambda}{(D - 1)(D - 2)} \right)^2 h_{\mu\nu} = 0, \tag{6.6}
\]

which its quadratic nature implies logarithmic modes. Hence, the dual field theory is a LCFT which is non-unitary [22, 25, 35]. To conclude, the theory (6.1) always contains ghosts.

Based on the idea of Ref. [20], to find an extension for the theory (6.1) which has a consistent unitary and tachyon free spectrum, we promote the axially field \( f_{\mu\nu} \) to a dynamical field by adding kinetic and cosmological terms for it. We present the new action as

\[
I = \frac{1}{16\pi G} \int d^Dx \sqrt{-g} \left( R[g] - 2\Lambda_g + \frac{1}{D - 2} f_{\mu\nu} G^{\mu\nu}[g] + \frac{m^2}{4(D - 2)} (\tilde{f}^{\mu\nu} f_{\mu\nu} - \tilde{f}^2) \right) +
+ \frac{1}{16\pi G} \int d^Dx \sqrt{-f} \left( R[f] - 2\Lambda_f \right). \tag{6.7}
\]

The equations of motion for the two fields \( g_{\mu\nu} \) and \( f_{\mu\nu} \) can be obtained by varying the action (6.7) as

\[
G_{\mu\nu}[g] + \Lambda_g g_{\mu\nu} = \frac{1}{(D - 2)} \left( T^{(1)}_{\mu\nu}[g] + T^{(2)}_{\mu\nu}[g] \right),
\]

\[
G_{\mu\nu}[f] + \Lambda_f f_{\mu\nu} = \frac{1}{(D - 2)} T^T_{\mu\nu}[f], \tag{6.8}
\]

where \( \kappa = \frac{G}{G} \) and

\[
T^{(1)}_{\mu\nu}[g] = -\frac{m^2}{2} \left[ \tilde{f}^{\rho\nu}_{\mu\rho} - \tilde{f}_{\mu\nu} - \frac{1}{4} g_{\mu\nu}(\tilde{f}^{\rho\sigma} f_{\rho\sigma} - \tilde{f}^2) \right],
\]

\[
T^{(2)}_{\mu\nu}[g] = \left[ -2 \tilde{f} (\mu G[g]_{\nu})_\rho - \frac{1}{2} f_{\mu\nu} R[g] + \frac{1}{2} \tilde{f} R_{\mu\nu}[g] + \frac{1}{2} g_{\mu\nu} f_{\rho\sigma} G[g]^{\rho\sigma} - \frac{1}{2} \left( \nabla^2[g] f_{\mu\nu} - 2 \nabla[g]^{\rho} \nabla[g]_{(\mu} f_{\nu)\rho} + \nabla[g]_{\mu} \nabla[g]_{\nu} \tilde{f} + (\nabla[g]^{\rho} \nabla[g]^{\sigma} f_{\rho\sigma} - \nabla^2[g] \tilde{f}) g_{\mu\nu} \right) \right].
\]
\[ T_{\mu\nu}[f] = \frac{1}{\kappa} \sqrt{\frac{g}{f}} \left[ f_{\alpha\mu} f_{\beta\nu} G[g]^{\alpha\beta} + \frac{m^2}{2} (g^{\sigma\alpha} g^{\tau\beta} - g^{\sigma\tau} g^{\alpha\beta}) (f_{\sigma\tau} f_{\alpha\mu} f_{\beta\nu}) \right]. \] (6.9)

Note that we did not call the \( g_{\mu\nu} \) or \( f_{\mu\nu} \) as metric. Because for the moment it is not clear which of them is the source for the energy-momentum tensor or equivalently which of them corresponds to the massless graviton. In fact, as we see in subsection 6.3, the real metric may be a proper combination of both at linear level.

### 6.2 AdS wave solutions

In this subsection, we present the general AdS wave solutions for equations (6.8). These are important since they solve the linearized equations of motion, as well. Due to complexity of equations (6.8), one should be careful about the form of the ansatz given. We consider the following ansatz for the fields \( g_{\mu\nu} \) and \( f_{\mu\nu} \)

\[
\begin{align*}
\text{for } g_{\mu\nu} & : & d\bar{s}_g^2 &= \frac{\ell_g^2}{r^2} \left( dr^2 + dx_i^2 - 2dx^+dx^- - G(r,x^+)dx^+ dx^- \right), \\
\text{for } f_{\mu\nu} & : & d\bar{s}_f^2 &= \frac{\ell_f^2}{r^2} \left( dr^2 + dx_i^2 - 2dx^+dx^- - F(r,x^+)dx^+ dx^- \right).
\end{align*}
\] (6.10)

Plugging these ansatzs in the equation of motion of \( g_{\mu\nu} \) gives the following relations

\[ \Lambda_g \ell_g^2 + \frac{1}{2} (D-1)(D-2) - \frac{\ell_f^2}{4} \left( D-1 \right) \left( D-4 \right) \left[ 1 - \frac{m^2}{2(D-2)} \ell_f^2 \right] = 0, \] (6.11)

and

\[
\begin{align*}
\left( (D-6) \frac{\ell_f^2}{2 \ell_g^2} - (D-2) \right) \left[ \frac{\partial^2 G}{\partial r^2} - (D-2) \frac{1}{r} \frac{\partial G}{\partial r} \right] + \frac{\ell_f^2}{\ell_g^2} \left[ \frac{\partial^2 F}{\partial r^2} - (D-2) \frac{1}{r} \frac{\partial F}{\partial r} \right] - \\
- \frac{\ell_f^2}{\ell_g^2} (D-2) (D-1 - m^2 \ell_f^2) \left[ \frac{G}{r^2} - \frac{F}{r^2} \right] &= 0.
\end{align*}
\] (6.12)

Similarly the equation of motion of \( f_{\mu\nu} \) gives

\[ \Lambda_f \ell_f^2 + \frac{1}{2} (D-1)(D-2) + \frac{1}{2\kappa} \left( \frac{D-1}{D-2} \right) \left( 2 - D + m^2 \ell_f^2 \right) \left( \frac{\ell_g}{\ell_f} \right)^{D-4} = 0, \] (6.13)

and

\[
\begin{align*}
\left[ \frac{\partial^2 G}{\partial r^2} - (D-2) \frac{1}{r} \frac{\partial G}{\partial r} \right] - \kappa (D-2) \left( \frac{\ell_f}{\ell_g} \right)^{D-4} \left[ \frac{\partial^2 F}{\partial r^2} - (D-2) \frac{1}{r} \frac{\partial F}{\partial r} \right] + \\
\left[ \frac{\partial^2 F}{\partial r^2} - (D-2) \frac{1}{r} \frac{\partial F}{\partial r} \right] - \kappa (D-2) \left( \frac{\ell_f}{\ell_g} \right)^{D-4} \left[ \frac{\partial^2 G}{\partial r^2} - (D-2) \frac{1}{r} \frac{\partial G}{\partial r} \right] &= 0.
\end{align*}
\]
\[ \frac{G}{r^2} - \frac{F}{r^2} = 0. \]  

(6.14)

Considering the power law solutions of the form \( r^\alpha \) for functions \( G \) and \( F \) in Eqs.(6.12) and (6.14), we find \( \alpha = 0, D - 1, \frac{D-1}{2} (1 \pm \sqrt{1+A}) \) in which

\[
A = -4 \frac{(D-2)^2(D-1-m^2\ell_g^2)(\gamma - 2 - 2\kappa\gamma^{D/2-1})}{(D-1)^2(2\gamma + \kappa\gamma^{D-2}(D-2)(-6\gamma + (\gamma - 2)D + 4))},
\]

(6.15)

where \( \ell_g^2 = \gamma \ell_g^2 \). Hence, the most general solutions for functions \( G \) and \( F \) are

\[
G(r, \mathbf{x}^+) = g_1(\mathbf{x}^+) + g_2(\mathbf{x}^+) r^{D-1} + g_3(\mathbf{x}^+) r^{\frac{D-1}{2}(1+\sqrt{1+A})} + g_4(\mathbf{x}^+) r^{\frac{D-1}{2}(1-\sqrt{1+A})},
\]

\[
F(r, \mathbf{x}^+) = g_1(\mathbf{x}^+) + g_2(\mathbf{x}^+) r^{D-1} + \beta g_3(\mathbf{x}^+) r^{\frac{D-1}{2}(1+\sqrt{1+A})} + \beta g_4(\mathbf{x}^+) r^{\frac{D-1}{2}(1-\sqrt{1+A})},
\]

(6.16)

where \( g_a(\mathbf{x}^+) \)'s are arbitrary functions of \( \mathbf{x}^+ \) and \( \beta \) is given by

\[
\beta = -\frac{1}{2} \left( \frac{-4\gamma + (\gamma - 2)D + 4}{\gamma - (D-2)\kappa\gamma^{D/2-1}} \right).
\]

(6.17)

As for the case of four dimensions, it may happen that for some special values of parameters, we have logarithmic AdS-wave solutions; which some of them must be avoided because of unitarity. The log-solutions arise at \( A = 0 \) and \( A = -1 \). The case \( A = 0 \) implies

\[
\gamma = \frac{1}{m^2\ell_g^2}(D - 1), \quad \text{or} \quad 1 - \frac{\gamma}{2} + \kappa\gamma^{D/2-1} = 0,
\]

(6.18)

and gives

\[
\begin{align*}
\left. ds_g^2 \right| = & \frac{\ell_g^2}{r^2} \left( dr^2 + dx_i^2 - 2dx^+dx^- - g(r, \mathbf{x}^+) dx^{+2} \right), \\
\left. ds_f^2 \right| = & \frac{\gamma\ell_g^2}{r^2} \left( dr^2 + dx_i^2 - 2dx^+dx^- - f(r, \mathbf{x}^+) dx^{+2} \right),
\end{align*}
\]

(6.19)

with

\[
\begin{align*}
g(r, \mathbf{x}^+) = & \tilde{G}_0[\mathbf{x}^+] \log(r) + G_0[\mathbf{x}^+] + \tilde{G}_{D-1}[\mathbf{x}^+] r^{D-1} \log(r) + G_{D-1}[\mathbf{x}^+] r^{D-1},
\end{align*}
\]
\[ f(r, x^+) = \tilde{F}_0[x^+] \log(r) + F_0[x^+] + \tilde{F}_{D-1}[x^+] r^{D-1} \log(r) + F_{D-1}[x^+] r^{D-1}. \]

For the case \( A = -1 \), the solutions are of the form (6.19) with
\[
g(r, x^+) = G_0[x^+] + \tilde{G}_{D-1}[x^+] r^{\frac{D-1}{2}} \log(r) + G_{D-1}[x^+] r^{\frac{D-1}{2}} + G_{D-1}[x^+] r^{D-1},
\]
\[
f(r, x^+) = F_0[x^+] + \tilde{F}_{D-1}[x^+] r^{\frac{D-1}{2}} \log(r) + F_{D-1}[x^+] r^{\frac{D-1}{2}} + F_{D-1}[x^+] r^{D-1}.
\]

Therefore the dangerous values of the parameters which causes non-unitary logarithmic solutions, and should be avoided, are given by Eq.(6.18).

### 6.3 Linearization

In subsection 6.2, we showed that the theory (6.7) has vacuum solution with two proportional AdS metrics. In the following, we determine the spectrum of propagating modes of the theory (6.7) around this vacuum solution, where \( \bar{f}^{\mu\nu} = \gamma \bar{g}^{\mu\nu} \). Inserting these AdS solutions in the equations of motion (6.8) gives (similar to Eqs.(6.11) and (6.13))
\[
\Lambda g^{\ell^2} + \frac{1}{2} \frac{(D - 1)(D - 2) - \gamma^2 (D - 1)(D - 4)}{2(D - 2)} \left[ 1 - \frac{m^2}{2(D - 2)} \gamma \ell^2 \right] = 0,
\]
\[
\Lambda f \gamma \ell^2 + \frac{1}{2} \frac{(D - 1)(D - 2) + \frac{(D - 1)(D - 1)}{2\kappa (D - 2)} (2 - D + \gamma m^2 \ell^2) \gamma^{(2 - D)} = 0, \tag{6.20}
\]
where \( \ell \) is the radius of \( \bar{g}_{\mu\nu} \). Consider the general form of fluctuations as follows
\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad f_{\mu\nu} = \gamma (\bar{g}_{\mu\nu} + \rho_{\mu\nu}). \tag{6.21}
\]

We expand various tensor terms of the action (6.7) up to second order in the perturbations \( h_{\mu\nu} \) and \( \rho_{\mu\nu} \). The quadratic action in perturbations emerges as
\[
S^{(2)}[h_{\mu\nu}, \rho_{\mu\nu}] = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left\{ \frac{2(D - 2) - (D - 6) \gamma}{2(D - 2)} h^{\mu\nu} (\mathcal{G} h)_{\mu\nu} + \kappa \frac{\gamma}{2} h^{\mu\nu} (\mathcal{G} \rho)_{\mu\nu} \right. \]
\[
- \frac{2\gamma}{(D - 2)} h^{\mu\nu} (\mathcal{G} \rho)_{\mu\nu} + (h - \rho) \cdot (h - \rho) \right\}, \tag{6.22}
\]
where \( \mathcal{G} \) is the Pauli-Fierz operator on the curved AdS\(_D\) background which is defined as
\[
p^{\mu\nu} (\mathcal{G} q)_{\mu\nu} \equiv -\frac{1}{4} p_{\rho\mu} q^{\rho\mu} + \frac{1}{2} p_{\mu\nu} q^{\nu\rho} - \frac{1}{4} q^{\rho\mu} p^{\nu\rho}. \]

\(^6\)These expansions are calculated by using the xAct package of Mathematica.
\[-\frac{(D-1)}{2\ell^2} (p^{\mu\nu} q_{\mu\nu} - \frac{1}{2} pq), \quad (6.23)\]

and the dot product is defined as

\[A \cdot A \equiv \chi A_{\mu\nu} A^{\mu\nu} - \xi A^2, \quad (6.24)\]

with

\[\chi = \frac{\gamma}{4\ell^2} (D - 1 - \gamma m^2 \ell^2), \quad \xi = \frac{\gamma}{8\ell^2} \left( \frac{D^2 + 2 - (D - 3)\gamma m^2 \ell^2 - 3D}{D - 2} \right). \quad (6.25)\]

To investigate the presence of ghost instabilities, we need to omit the cross terms in the quadratic action (6.22). For this reason, it is useful to utilize a new basis for the fluctuations as follows

\[h = a_1 h^{(0)} + a_2 h^{(m)}, \quad \rho = h^{(0)} + h^{(m)}. \quad (6.26)\]

Hence, the quadratic action reads\(^7\)

\[S^{(2)}[h^{(0)}_{\mu\nu}, h^{(m)}_{\mu\nu}] = \frac{1}{16\pi G} \int d^Dx \sqrt{-g} \left\{ Ah^{(0)\mu\nu}(G h^{(0)})_{\mu\nu} + Bh^{(m)\mu\nu}(G h^{(m)})_{\mu\nu} + M_1^{(0)} (h^{(0)\mu\nu} h^{(0)\nu\mu} + h^{(0)\mu\nu} h^{(0)\nu\mu}) + M_1^{(0,m)} (h^{(m)\mu\nu} h^{(m)\nu\mu} + h^{(m)\mu\nu} h^{(m)\nu\mu}) + Ch^{(m)\mu\nu}(G h^{(0)})_{\mu\nu} + M_1^{(0,m)} h^{(0)\mu\nu} h^{(m)\nu\mu} + M_2^{(0,m)} h^{(0)\mu\nu} h^{(m)\nu\mu} \right\}, \quad (6.27)\]

where the precise expressions for \(A, B, C, M_1^{(0)}, M_1^{(m)}, M_2^{(0,m)}\) are as below

\[A = \kappa \gamma \frac{D}{2D - 1} + \frac{a_1}{2(D - 2)} \left( 2a_1(D - 2) - [4 + a_1(D - 6)] \gamma \right), \quad B = A[a_1 \rightarrow a_2],\]

\[C = 2\kappa \gamma \frac{D}{2D - 1} + \frac{1}{D - 2} \left( 2a_1 a_2(D - 2) - [2(a_1 + a_2) + a_1 a_2(D - 6)] \gamma \right), \quad (6.28)\]

and

\[M_1^{(0)} = \frac{\gamma}{4\ell^2} (a_1 - 1)^2 (D - 1 - \gamma m^2 \ell^2), \quad M_2^{(0)} = \frac{(D - 2)(D - 1) - (D - 3)\gamma m^2 \ell^2}{2(D - 2)(\gamma m^2 \ell^2 - D + 1)},\]

\[M_1^{(m)} = M_1^{(0)}[a_1 \rightarrow a_2], \quad M_2^{(m)} = M_2^{(0)}, \quad M_1^{(0,m)} = \frac{\gamma}{2\ell^2} (a_1 - 1)(a_2 - 1) \left( D - 1 - \gamma m^2 \ell^2 \right),\]

\(^7\)The calculations in this part also are done by the xAct [28].
\[ M_{2}^{(0,m)} = -\frac{\gamma}{4\ell^2(D-2)}(a_1 - 1)(a_2 - 1)\left((D-2)(D-1) - (D-3)\gamma m^2\ell^2\right). \] (6.29)

The absence of scalar ghost (Boulware-Deser ghost [12]) in propagating modes \( h^{(0)} \) and \( h^{(m)} \) implies that \( M_{2}^{(0)} = M_{2}^{(m)} = -1 \). The last condition gives

\[ \gamma = \frac{1}{m^2\ell^2}(D-2). \] (6.30)

Using this value for \( \gamma \), gives

\[
C = 2\kappa \left( \frac{D-2}{m^2\ell^2} \right)^{\frac{D-4}{2}} - \frac{1}{m^2\ell^2} \left( 2(a_1 + a_2) + a_1 a_2 (D - 6 - 2m^2\ell^2) \right),
\]

\[
M_{1}^{(0,m)} = -M_{2}^{(0,m)} = -\frac{1}{2m^2\ell^4}(a_1 - 1)(a_2 - 1)(D-2).
\] (6.31)

To have decoupled modes, \( C, M_{1}^{(0,m)}, M_{2}^{(0,m)} \) should vanish. Vanishing of \( M_{1}^{(0,m)}, M_{2}^{(0,m)} \) implies that

\[ a_1 = 1 \quad \text{or} \quad a_2 = 1 \quad \text{or} \quad a_1 = a_2 = 1. \] (6.32)

We discard the case \( a_1 = a_2 = 1 \) since for this case vanishing of \( C \) implies that

\[ 2m^2\ell^2 - (D-2) + 2\kappa m^2\ell^2 \left( \frac{D-2}{m^2\ell^2} \right)^{\frac{D-4}{2}} = 0, \]

which is exactly the condition for appearance of logarithmic solution (6.19). Moreover, the symmetry of the model under interchange of \( a_1 \) and \( a_2 \) implies that the two cases \( a_1 = 1 \) and \( a_2 = 1 \) are physically equivalent. Therefore, in the following we restrict ourselves to the case \( a_1 = 1 \). Vanishing of the coefficient \( C \) for this case implies

\[ a_2 = -\frac{2}{(D-2)} \left( \frac{D - 2 - \kappa m^2\ell^4 \left( \frac{D-2}{m^2\ell^2} \right)^{\frac{D-4}{2}}}{(D - 4 - 2m^2\ell^2)} \right). \] (6.33)

To this end, the proper basis in which the theory is scalar ghost free and we have two decoupled propagating modes is the following

\[ h = h^{(0)} - \frac{2}{(D-2)} \left( \frac{D - 2 - \kappa m^2\ell^4 \left( \frac{D-2}{m^2\ell^2} \right)^{\frac{D-4}{2}}}{(D - 4 - 2m^2\ell^2)} \right) h^{(m)}, \quad \rho = h^{(0)} + h^{(m)}. \] (6.34)
In this basis, the action (6.27) becomes

\[ S^{(2)}[h^{(0)},h^{(m)}] = \frac{1}{16\pi G} \int d^Dx \sqrt{-g} \left[ \mathcal{A}_0 h^{(0)\mu\nu} (G h^{(0)})_{\mu\nu} + \mathcal{A}_m \left( h^{(m)\mu\nu} (G h^{(m)})_{\mu\nu} - \frac{M^2}{4} \left( h^{(m)\mu\nu} h^{(m)\mu\nu} - (h^{(m)})^2 \right) \right) \right] , \quad (6.35) \]

where

\[ \mathcal{A}_0 = 1 - \frac{1}{2m^2\ell^2} (D - 2) + \kappa \left( \frac{D - 2}{m^2\ell^2} \right)^{\frac{D}{2} - 1}, \]
\[ \mathcal{A}_m = -2 \left( 2 + \kappa m^2\ell^2 (D - 6 - 2m^2\ell^2) \left( \frac{D - 2}{m^2\ell^2} \right)^{\frac{D}{2} - 1} \right) \frac{A_0}{(D - 4 - 2m^2\ell^2)^2}, \]
\[ M^2 = -\frac{4m^2(D - 2) A_0^2}{(D - 4 - 2m^2\ell^2)^2 A_m} . \quad (6.36) \]

Ghost free condition together with absence of tachyonic mode implies that

\[ \mathcal{A}_0 > 0, \quad \mathcal{A}_m > 0, \quad M^2 \geq -\frac{(D - 1)^2}{4\ell^2}. \quad (6.37) \]

According to the above conditions, for \( D = 5, 6 \) and \( 7 \), we presented the allowed values of \((m^2, \kappa)\) in Fig.3. For these values, the massive mode acquires different masses in the range \(-\frac{(D - 1)^2}{4\ell^2} \leq M^2 < 0\) as depicted in Fig.4.

### 6.4 Schwarzschild-AdS Black Hole Solutions

In this subsection, we present one of the black hole solutions for the theory (6.7). We consider two following proportional Schwarzschild-AdS solutions

\[ ds^2_g = -F(r)dt^2 + \frac{1}{F(r)} dr^2 + r^2 d\Omega^2_{D-2}, \quad ds^2_f = \gamma ds^2_g, \quad (6.38) \]

where \( F(r) = 1 + \frac{r^2}{\ell^2} - (\frac{\ell}{r})^{D-3} \). The parameter \( \mu \) is related to the mass of the black hole and can be expressed in terms of the horizon radius \( r_h \) as follows

\[ \mu = r_h \left( 1 + \frac{r_h^2}{\ell^2} \right)^{\frac{1}{D-3}}. \quad (6.39) \]

The metric (6.38) reduces to D-dimensional anti-de Sitter spacetimes with radius of curvature \( \ell \) for \( \mu = 0 \), and converts to the standard Schwarzschild solution for \( \ell \to \infty \). Substituting this ansatz (6.38) into the equations of motion (6.8) and (6.9) gives the
same equations as (6.20) for two proportional AdS. The reason is that the Ricci tensor and Ricci-Scalar tensor, which are the only tensors present in the equations of motion (6.8) and (6.9), are the same for Schwarzschild-AdS black hole and pure AdS spacetime.

6.5 Energy and Entropy of Black Hole Solutions

In this subsection, we find the energy and entropy of the Schwarzschild-AdS black hole solutions (6.38) in the theory (6.7). The standard way to do this is using the free energy function. Choosing the Dirichlet boundary condition for the Schwarzschild-AdS black hole solutions (6.38), the corresponding boundary terms which may violate the variational principle emerge from the following terms

\[
\delta I_{\delta g, \delta \theta, \delta f} = \frac{1}{16 \pi G} \int d^Dx \left( \sqrt{-g} \delta R[g] + \frac{1}{D-2} \sqrt{-g} f_{\mu\nu} \delta G^{\mu\nu}[g] + \kappa \sqrt{-f} \delta R[f] \right),
\]
which can be written as

\[
\delta I_{\delta \alpha, g, \delta \alpha, f} = \frac{1}{16\pi G} \int d^D x \left( \sqrt{-g} A_{\mu \nu} \delta R^{\mu \nu}[g] + \kappa \sqrt{-f} f_{\mu \nu} \delta R^{\mu \nu}[f] \right),
\]

where

\[
A_{\mu \nu} = g_{\mu \nu} + \frac{1}{D-2} \left( f_{\mu \nu} - \frac{1}{2} f_{\alpha \beta} g^{\alpha \beta} g_{\mu \nu} \right).
\]

In general, the appropriate Gibbons-Hawking term for the variations (6.41) is difficult to find. We can simplify the problem by considering the particular subspace of the space of solutions in which the two fields are proportional, i.e. \( f_{\mu \nu} = \gamma g_{\mu \nu} \). For this case, the
variation terms (6.41) can be simplified as
\[
\delta I_{\delta \theta_5, \delta \theta_6, f} = \frac{1}{16\pi G} \int d^D x \left( \sqrt{-g} (1 - \frac{\gamma}{2}) g_{\mu \nu} \delta R^{\mu \nu} [g] + \kappa \sqrt{-f} f_{\mu \nu} \delta R^{\mu \nu} [f] \right),
\]
(6.43)
from which the proper Gibbons-Hawking terms can be suggested as follows
\[
I_{GH} = - \frac{2(1 - \frac{\gamma}{2})}{16\pi G} \int d^{D-1} x \sqrt{-g} K[g] - \frac{2}{16\pi G} \int d^{D-1} x \sqrt{-f} K[f],
\]
(6.44)
where \( \eta_{gij} \) and \( \eta_{fij} \) are the induced metrics, on the boundary, associated with the metrics \( g \) and \( f \) and \( K[g] = \eta^{ij} K[g]_{ij}, K[f] = \eta^{ij} K[f]_{ij} \). To find the general form of Gibbons-Hawking term one can use the methodology of Ref. [37].

To this end, a well-defined variational principle implies that the Gibbons-Hawking terms of Eq.(6.44) should be added to the original bulk action (6.7) (named \( I_0 \)). Let us substitute the Schwarzschild-AdS black hole solutions (6.38) in this total action. By changing to Euclidean signature, \( t \to i \tau \), and integrating over \( r(r_h \to R), \tau(0 \to \beta) \) and angular parameters we arrive at
\[
I_0 + I_{GH} = \frac{a}{4G} \frac{(D - 2)(\sqrt{\pi})^{D-3}}{\Gamma(\frac{D-1}{2})} \left( \frac{\mathcal{R}^{D-1}}{\ell^2} + \mathcal{R}^{D-3} + b \right),
\]
(6.45)
where \( \mathcal{R} \gg r_h \) is a cutoff and
\[
a = \beta \left( 1 - \frac{\gamma}{2} + \kappa \frac{D}{2} - 1 \right), \quad b = -\frac{r_h^{D-3}}{2(D - 2)} \left( D - 1 + (D - 3) \frac{r_h^2}{\ell^2} \right).
\]
(6.46)
It is clear that the above on-shell action is divergent due to the infinite volume limit, so appropriate counterterms are needed to remove the divergent terms. One can easily check that the suitable counterterms are
\[
I_{ct} = \frac{1}{16\pi G} \int d^{D-1} x \sqrt{\eta} \left( c_1 + c_2 R[\eta_g] + c_3 R^{ij}[\eta_g] R_{ij}[\eta_g] + c_4 R[\eta_g]^2 + \ldots \right),
\]
(6.47)
with
\[
c_1 = -\frac{2(D - 2)}{\ell} \left( 1 - \frac{\gamma}{2} + \kappa \frac{D}{2} - 1 \right), \quad c_2 = \frac{\ell^2}{2(D - 2)(D - 3)} c_1,
\]
\[
c_3 = -\left( (D - 2) + \frac{(D - 5)}{2(D - 1)} \ell c_1 \right), \quad c_4 = \frac{(D - 1)\ell^3}{4(D - 2)(D - 3)^2(D - 5)}.
\]
(6.48)
It should be noticed that the counterterms in (6.47) render the renormalized action finite up to \( D = 7 \). For \( D = 5 \) one needs just \( c_1 \) and \( c_2 \) terms and for \( D = 6 \) and 7 one needs
all of $c_i$ ($i < 5$) terms. As a check point the counterterm action (6.48) for the case $\gamma = 0$ becomes

\[
I_{\text{ct}} = -\frac{1}{16\pi G} \int d^{D-1}x \sqrt{\eta_g} \left[ \frac{2}{\ell} (D - 2) + \frac{\ell}{(D - 3)} R[\eta_g] + \frac{\ell^3}{(D - 3)^2(D - 5)} \left( R^{ij[\eta_g]} R_{ij[\eta_g]} - \frac{(D - 1)}{4(D - 2)} R^2[\eta_g] \right) + ... \right],
\]

(6.49)

which is the standard counterterms for Einstein-Hilbert action of Ref. [33] by identifying $D = d + 1^8$. Putting everything together, the renormalized on-shell action reads

\[
I_{\text{ren}} = I_0 + I_{\text{GH}} + I_{\text{ct}} = \tilde{a} \frac{16\pi G}{16\pi G} \left( 1 - \frac{\gamma}{2} + \kappa \frac{D}{2} \right),
\]

\[
\tilde{a}_{\text{even}} = 2\beta \frac{\sqrt{\pi}}{\ell^2} \left( r_h^2 - \ell^2 r_h^{D-3} \right),
\]

\[
\tilde{a}_{\text{odd}} = \tilde{a}_{\text{even}} + 2\pi \beta \frac{D - 2}{(D - 3)^2} \left( -\pi \ell^2 \right)^{D-3},
\]

(6.50)

where the index even and odd refers to the bulk dimension. The Hawking temperature is then given by

\[
T_H = \frac{1}{\beta} = \frac{1}{4\pi} \partial_r F \Big|_{r = r_h} = \frac{1}{4\pi} \left( (D - 3) \frac{1}{r_h} + \frac{(D - 1)}{\ell^2} \ell r_h \right),
\]

(6.51)

which can be used to explain $r_h$ in terms of $\beta$ as

\[
r_h = \frac{2\pi \ell^2}{(D - 1)\beta} \left( 1 \pm \sqrt{1 - \frac{(D - 1)(D - 3)\beta^2}{4\pi^2 \ell^2}} \right).
\]

(6.52)

We continue with the minus sign since it leads to a smaller free energy. Considering the relations among partition function, renormalized on-shell action and energy, i.e.

\[
\log Z = I_{\text{ren}}, \quad E = -\frac{\partial}{\partial \beta} \log Z,
\]

(6.53)

the energy of the black hole solution (6.38) is derived as

\[
E_{\text{BH}} = \frac{\mathcal{E}}{16\ell^2 G} \left( 1 - \frac{\gamma}{2} + \kappa \frac{D}{2} \right),
\]

\[
\mathcal{E}_{\text{even}} = (D - 1)(D - 2) \frac{\sqrt{\pi}}{\Gamma \left( \frac{D + 1}{2} \right)} \left( r_h^2 + \ell^2 \right) r_h^{D-3},
\]

Note that our curvature convention differs by a minus sign with the convention of [33].
$$\mathcal{E}_{\text{odd}} = \mathcal{E}_{\text{even}} - 2\ell^2 \frac{(D-2)}{(D-3)^2} (-\pi \ell^2)^{\frac{D-3}{2}}, \quad (6.54)$$

where again the index even and odd refers dimension of the bulk spacetime. Furthermore, according to definition of entropy, $S = \beta E + \log Z$, we have

$$S_{\text{BH}} = \frac{S}{G} \left(1 - \frac{\gamma}{2} + \kappa \gamma \frac{D-1}{D} \right) \quad \text{with} \quad S = \frac{(D-1)}{4} \left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{D+1}{2}\right)}\right)^{D-2}. \quad (6.55)$$

In subsection 6.3, we showed that absence of Boulware-Deser scalar ghost implies $\gamma = \frac{1}{m^2 \ell^2} (D-2)$. For this value of $\gamma$, the expressions for energy and entropy of black hole solutions (6.38) become

$$E = \frac{\mathcal{E}}{16\ell^2 G} \left(1 - \frac{1}{2m^2 \ell^2} (D-2) + \kappa \frac{(D-2)}{m^2 \ell^2} \right)^{\frac{D-1}{2}},$$

$$S = \frac{S}{G} \left(1 - \frac{1}{2m^2 \ell^2} (D-2) + \kappa \frac{(D-2)}{m^2 \ell^2} \right)^{\frac{D-1}{2}}, \quad (6.56)$$

where $\mathcal{E}$ and $S$ are given in Eqs.:(6.54) and (6.55). Interestingly, these values could be written in terms of the coefficient of the kinetic term of massless graviton (6.36) as

$$E = \frac{\mathcal{E}}{16G \ell^2} A_0, \quad S = \frac{S}{G} A_0. \quad (6.57)$$

According to Eqs.(6.54) and (6.55), $E$ and $S$ are positive in even as well as odd bulk dimensions. Fortunately, the absence of Boulware-Deser scalar ghost and positivity of kinetic term of massless graviton, imply that the energy and entropy of Schwarzschild-AdS black hole solution (6.38) are also positive.

### 7 Conclusion and Discussion

The key word in studying field theories is "consistency". The most important criteria which determine the consistency of a gravitational theory are: 1) Absence of Boulware-Deser ghost, 2) Absence of kinetic ghost, 3) Absence of superluminal modes (i.e. tachyons) and 4) Predictability (i.e. absence of local closed time-like curves).

In this paper we showed that the problem of ghost in critical gravity and its higher dimensional extensions can be resolved by giving dynamics to the symmetric rank two auxiliary field appearing in the action of these theories. The new models, at linear level around the AdS vacuum, are free of Boulware-Deser ghost, kinetic ghost and tachyonic instability within the particular ranges of parameters. Note that for Lorentz invariant theories the conditions 3 and 4 are equivalent. Moreover, we showed that the energy
and entropy of the AdS-Schwarzschild black hole solutions in our model are positive in the same range of parameters. This might be the sign that the model is free of ghost instabilities at the non-linear level as well.

A natural and very important question which can be asked is as follows. Is it possible that the Boulware-Deser ghost appears again or the non-tachyonic mode changes to tachyonic one at the full non-linear level? In general, the answer may be Yes, however, it needs to be checked explicitly to assure about the answer No.

Let us remind that the dRGT model (see Eq.(A.4)) and HR- Bigravity model (Eq.(A.6)) despite passing the consistency conditions 1 and 2 suffer from violating the conditions 3 and 4 at the full non-linear level. These inconsistencies are shown by the method of characteristics [38]. In this approach, the absence of superluminal propagating modes means that the characteristic matrix determinant does not vanish anywhere. Also absence of zero and negative norm states implies that this determinant should be non-degenerate [39]. Moreover, the predictability (absence of local CTC) implies that this determinant shows the lack of space-like characteristic surfaces. All these mean that the characteristic matrix determinant also should be calculated for the theory (2.2); it is the subject of our future works.

In a different point of view, to explain the cosmological constant problem, the mass of spin-2 particle in theories (A.4), (A.6) and (2.2) should be small. The interesting feature of all these models is that this small mass, if able to generate a late-time cosmic acceleration, would be protected against large quantum corrections because of restoring the diffeomorphism symmetry in small mass limit, $M^2 \to 0$. It is shown that flat and closed Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological solutions do not exist in the dRGT model (A.4) with a flat reference metric [40]. Also its open FLRW solutions (and cosmological solutions with general reference metrics) suffer from either Higuchi [41] ghost at the level of linear perturbations or from a new non-linear ghost [42]. Unlike the dRGT model, its bimetric generalization (A.6) is able to provide the accelerating solutions [43] but unfortunately they also suffer from ghost and/or gradient instabilities [44]. Hence, exploring the cosmological solutions for our model (2.2) is important and can be another subject for our future works.

The crucial point is that the main reason for vanishing of the characteristic matrix determinant, existence of spacelike characteristic surfaces and absence of cosmological solutions in models (A.4) and (A.6) is the constraint which removes the Boulware-Deser ghost at the full non-linear level. It is argued that the only possible way to remove the problems caused by this condition is the existence of PM modes [38] and [45]. Actually the PM action has an enhanced symmetry which can protect the mass of spin-2 particle against receiving the large non-linear corrections. Unfortunately, this hope is also ex-
cluded for (A.4) and (A.6) theories precisely at the non-linear level [46]$^9$ (even though their linearization has PM mode [24]). Interestingly, the theory (2.2) has PM modes in its spectrum, at least at the linear level. Therefore another important study which should be done on the theory (2.2) is checking the existence of PM modes at the full non-linear level that is also the subject of our future works.

Another study which can be done on the theory (2.2) is understanding the mechanism by which the mass appears in this theory.

8 Acknowledgment

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Appendix

A Review: Massive Gravities

The Lorentz invariant massive gravity models belong to two general categories. In the first category, the Einstein-Hilbert (EH) action is modified by adding some special interaction terms which do not contain any derivative of metric. The models in this category are called ”non linear massive gravities”. In the second category, the EH action is modified by higher powers of scalar curvatures which explicitly contain the derivatives of metric. The models in this category are called ”higher derivative massive gravities”. As reviewing the models in these two categories would help us to transfer better our main idea and motivations for proposing a new model, in the following we present a summary of them$^{10}$.

A.1 Non-Linear Massive Gravities

The simplest way to have a Lorentz invariant massive graviton is adding an explicit Lorentz invariant mass term to the EH action. This mass term, which is actually a potential term, involves just the metric without any derivative. Historically, the linear version of this massive gravity theory is proposed by Fierz and Pauli in 1939 [8]. They

$^9$The criticisms on these works can be found in [47].

$^{10}$Three long review articles on the subject have already been written by Hinterbichler [48], de Rham [49], Schmidt-May and von Strauss [50] which we encourage an eager reader to see them.
argued that the one single Lorentz invariant massive spin-2 field that propagates in the flat spacetimes is described by a unique classically consistent action which is

\[
S_{\text{FP}} = \frac{1}{16\pi G} \int d^4x \sqrt{-\eta} \left( h^{\mu\nu}(G h)_{\mu\nu} - \frac{M^2}{4} (h^{\mu\nu} h_{\mu\nu} - h^2) \right). \tag{A.1}
\]

In this action, \( G \) is the Lichnerowicz operator on the flat background \(^{11}\)

\[
h^{\mu\nu}(G h)_{\mu\nu} \equiv -\frac{1}{4} h_{\nu\rho\mu} h^{\nu\rho\mu} + \frac{1}{2} h_{\nu\rho\;\sigma} h^{\nu\sigma\rho\;\mu} - \frac{1}{2} h_{\nu\;\rho} h^{\nu\rho\;\mu} + \frac{1}{4} h_{\mu\nu} h^{\mu\nu}. \tag{A.2}
\]

The first point about FP action is the special combination of \( h^{\mu\nu} h_{\mu\nu} \) with \( h^2 \) in FP mass term. This combination is fixed uniquely by demanding the Lorentz invariance and absence of ghost instability. For example, if we choose another combination as

\[
-\frac{M^2}{4} \left( h^{\mu\nu} h_{\mu\nu} - [1 - a] h^2 \right),
\]

apart from one massive spin-2 field we have a massive scalar field with mass \( m^2 = (\frac{3}{4} - 1)M^2 \), which its kinetic term has a negative sign. By choosing the FP tuning, the scalar ghost becomes very heavy and decouples from the dynamical part.

The second point about FP action is the number of degrees of freedom for this massive spin-2 particle. In four dimensional spacetimes, the irreducible representations of little group of spacetime symmetry group can be used to classify the particles \(^{52}\). In this review, we assume that the symmetry group of spacetime is Lorentz group, more precisely Poincare group. Therefore particles are irreducible representations of the \( SU(2) \) group which is the little group of the Lorentz group. The spin-2 representation of \( SU(2) \) for massless and massive particles respectively has two and five degrees of freedom (helicity). The helicities \( \pm2 \) for massless one and \( \pm2, \pm1, 0 \) for massive one. Just to check that the model (A.1) describes one massive spin-2 field, note that the equations of motion of this model imply three equations

\[
\Box h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0, \quad h = 0. \tag{A.3}
\]

The first one is a massive wave equation, while the latter two are constraints on components of \( h_{\mu\nu} \). The perturbation \( h_{\mu\nu} \) is symmetric and has 10 independent components. The divergence free and traceless conditions remove 5 degrees, therefore just 5 degrees of freedom are remained which are the correct ones for a massive spin-2 particle.

However, in 1970, around thirty years after the FP paper, van Dam and Veltman \(^9\) and Zakharov \(^10\) separately discovered that the FP theory cannot be a consistent theory

\(^{11}\)The generic operator in D dimensions and for (A)ds backgrounds can be found in [51].
for describing the massive spin-2 particle. They argued that the FP theory coupled to a matter, in the massless limit, \( M^2 \to 0 \), does not reduce to the (linearized) EH theory. This discontinuity is named as vDVZ discontinuity\(^{12}\).

Fortunately in 1972, two years after discovering the vDVZ discontinuity, Vainshtein [11] argued that this discontinuity is an artifact of linear analysis and it could be resolved by considering the non-linear effects. He argued that around any massive source of mass \( m \), such as Sun, there is a new length scale \( r_V \sim (m \frac{G}{M^4})^{\frac{1}{5}} \). At distances \( r \leq r_V \), the non-linearities begin to dominate and the predictions of the FP action cannot be trusted. Interestingly in the massless limit, \( M^2 \to 0 \), we have \( r_V \to \infty \) that means that it does not exist any radius within it one can use the FP action and inevitably a full non-linear theory should be considered.

In the end of 1972, as a quick response to Vainshtein’s idea, Boulware and Deser [12] argued that at the non-linear level the scalar ghost which was absent in the FP theory comes back to the dynamical spectrum and therefore no consistent non-linear theory of massive spin-2 field can exist. Their result was based on two main assumptions. First, they assumed that the general non-linear extension of FP action contains just the arbitrary function of FP mass term, \( f(h^{\mu\nu}h_{\mu\nu} - h^2) \). Second, in their Hamiltonian analysis they insisted on the same number of primary constraints as EH theory. Based on their conclusion, the program of non-linear massive gravity was halted for almost thirty years.

Whether the assumptions of Boulware and Deser were correct or not, from an effective field theory point of view, one can claim that existence of the ghost is not necessarily a problem until its mass is above the UV cutoff of effective theory. In 2002, according to this point of view, Arkani-Hamed, Georgi and Schwartz [13] proposed a new perspective to give mass to a spin-2 particle. This perspective was applying the Goldstone boson equivalent theorem for gravitational theories. The important point is that since a graviton mass is originated from spontaneous breaking of diffeomorphism symmetry, the Goldstone modes themselves might be ghosts.

Unfortunately in 2005, after returning the hope by the above approach, Creminelli, Nicolis, Papucci and Trincherini [14] argued that the Boulware-Deser ghost comes back again in the radius \( r_{\text{CNPT}} \) which is much larger than the Vainshtein radius \( r_V \). Just to have some numbers in mind, for the Sun as a massive source, \( r_V \sim 10^{16}\text{km} \) and \( r_{\text{CNPT}} \sim 10^{22}\text{km} \) where the first one is of order of Milky way and the second one is of order of cosmological horizon. This implies that the effective field theory approach is not advantageous and we need a UV completion. This was the second time where the program of non-linear massive gravity was halted.

In 2009, Gabadadze [54] proposed a new model which not only contained the massive

\(^{12}\)Note that around a curved spacetime, \( \Lambda(dS) \), the vDVZ discontinuity is absent, though there is a ghost instead in the de Sitter theory [53].
spin-2 particle but also potentially it was free of the Boulware-Deser ghost. In this model, which is known as "auxiliary extra dimension (AuXD)" model, the extrinsic curvature of four dimensions embedded in five dimensions is responsible for creating the mass term. Very soon after introducing this model, de Rham and Gabadadze [55, 56] verified that this model in the decoupling limit\(^ \text{13} \) is free of Boulware-Deser ghost to 3rd order\(^ \text{14} \). Unfortunately, in 2011, Hassan and Rosen [57] showed that in the decoupling limit, at the 4th order the Boulware-Deser ghost comes back again.

In 2010, in that atmosphere of doubt and despair, de Rham and Gabadadze [15] found a sign mistake in [14], that actually was originated from [13] (for which the sign was not important). They showed that in the decoupling limit, up to the fifth order in the non-linearities, the approach of [13] can give a theory free from Boulware-Deser ghost and the conclusion of [14] is not correct. Very soon after that in the late 2010, de Rham, Gabadadze and Tolley [58] extended the calculations of [15] and found a consistent four dimensional non-linear massive gravity which was free of Boulware-Deser ghost in the decoupling limit to all orders. They also showed that this extended model, away from the decoupling limit is also ghost free at least up to quartic order in the non-linearities. This model, which is known as "dRGT" model, is described by this action

\[
I_{\text{dRGT}} = \frac{1}{16\pi G} \int d^4x \sqrt{g} \left( R[g] + \frac{m^2}{2} \sum_{n=2}^{4} \alpha_n e_n(K) n!(4-n)! \right), \tag{A.4}
\]

where \( m^2 \) is the mass parameter, \( \alpha_2 = 1 \), the two remaining \( \alpha_n \) are arbitrary parameters and the \( e_n(K) \) are the elementary symmetric polynomials constructed out of the matrix \( K^\mu_\nu = \delta^\mu_\nu - [\sqrt{g^{-1}} \eta^\mu_\nu] \).

In 2011, Hassan and Rosen [16] by using the basic properties of those elementary symmetric polynomials, reformulated the dRGT model (A.4) as below

\[
I_{HR} = \frac{1}{16\pi G} \int d^4x \sqrt{g} \left( R[g] + 2m^2 \sum_{n=0}^{4} \beta_n e_n(\sqrt{g^{-1}} f) \right), \tag{A.5}
\]

and furthermore they proved [59] that it is free of the Boulware-Deser ghost by doing the Hamiltonian analysis based on the ADM formalism [60]. In this reformulated form, \( \beta_n \) are five arbitrary parameters and also in compression with (A.4), the reference metric \( \eta^\mu_\nu \) is substituted by the generic auxiliary field tensor \( f^\mu_\nu \). As is clear from (A.5), to

\(^{13}\)The decoupling limit here means: Taking the graviton mass \( M^2 \rightarrow 0 \) and Newton constant \( G \rightarrow 0 \) while keeping \( \frac{M^2}{G} \) fixed. To be more precise, in this limit, we just retain the terms which are linear in \( h^\mu_\nu \) but to all orders in the non-linear St"uckelberg field \( \pi \). In other worlds, in this limit effectively one considers the non-linearities which are most relevant to Boulware-Deser ghost problem.

\(^{14}\)The 3rd order here means: The AuXD model has two coupled equations, one for the field contents in the four dimensions and the other for the extra dimension. One can solve these coupled equations perturbatively, for example to 3rd order.
construct a non-linear massive gravity (extension of FP action), introducing of auxiliary tensor $f_{\mu\nu}$ is inevitable. Hassan and Rosen then extended their previous result (A.5) by giving dynamics to the metric $f_{\mu\nu}$ [17]. This model, which is known as "HR bigravity" model, is described by the following action

$$I_{HR-BiG} = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} R[g] + \frac{1}{16\pi \tilde{G}} \int d^4x \sqrt{|f|} R[f] +$$

$$+ \frac{2m^2}{G_{eff}} \int d^4x \sqrt{g} \sum_{n=0}^{4} \beta_n \epsilon_n (\sqrt{g^{-1}} f), \quad (A.6)$$

where a new Newton constant $\tilde{G}$ for the field $f_{\mu\nu}$ and an effective Newton constant, which is just a combination of "$G, \tilde{G}$", for the interaction term are introduced. They showed that this model has a one massless spin-2 field and one massive spin-2 field\textsuperscript{15}. Further consistency check of this model is done in [62]. All these mean that, the program of finding a consistent four dimensional Lorentz invariant non-linear massive gravity, from extending the FP action, is almost completed by finding the action (A.6)\textsuperscript{16}.

Before closing this section and reviewing the completely different way to find a massive spin-2 field in the next section, let us explain, at least by using the above results, why the conclusion of Boulware and Deser [12] in general is not correct. First, it is clear from the action (A.6) that the full non-linear interaction term which is responsible for producing the mass is completely different from the potential term that Boulware and Deser assumed in their paper. Remember that for them it was just the arbitrary function of FP mass term, $f(h^{\mu\nu}h_{\mu\nu} - h^2)$. Second, the Hamiltonian analysis of [62] showed that it is not a lapse equation which is responsible for removing the scalar ghost; rather it is the combination of laps and shift equations that gives the correct constraint.

### A.2 Higher Derivative Massive Gravities

As we reviewed in previous section, in the non-linear massive gravity program, the studies were concentrated on finding a consistent non-linear extension of the FP action. In the FP action, which is a linear action, the diffeomorphism symmetry is broken explicitly by the mass term. According to the dRGT idea, one needs to add extra non-derivative terms which they are special combinations of dynamical field $g_{\mu\nu}$ and (a priori arbitrary) reference metric $f_{\mu\nu}$. But using this fixed absolute frame in some sense is not compatible with our intuition from general relativity which says that the physics should be independent of our coordinate choice. To overcome this problem, the reference metric $f_{\mu\nu}$

\textsuperscript{15}It is shown [61] that the consistent theory of just interacting massless spin-2 fields does not exist. The consistent theory, apart from a massless spin-2 field should also have a massive spin-2 field.

\textsuperscript{16}The problem of vDVZ discontinuity in massive gravity/bigravity (dRGT and its extensions), is studied well in [63] and [64].
in dRGT model upgraded to a dynamical field by introducing an EH term for it in the HR-Bigravity model. A natural question which can be asked is that: Is it possible that we find a FP action by a reverse approach, indeed we start with a fully diffeomorphism invariant action and then see that at the linear level it reduces to the FP action? The answer is Yes, and it is achievable in higher derivative gravities. Apart from this point, there is another reason for studying higher derivative gravities. Albeit the pure GR is not renormalizable, the higher derivative gravities can be renormalizable [65]. Unlike these two good reasons, up to 2009, the higher derivative gravities were not considered too much because of ghost degree of freedom.

In 2009, Bergshoeff, Hohm and Townsend [18] suggested a parity invariant higher derivative action in three dimensions, which at the linear level (around Minkowski or A(dS) vacuum) contains a massless spin-2 particle together with a FP massive spin-2 one. This action which is called "New Massive Gravity" (NMG) is

\[
I_{\text{NMG}} = \frac{1}{16 \pi G_3} \int d^3 x \sqrt{-g} \left( R - 2\lambda - \frac{1}{m^2} (R_{\mu \nu} R_{\mu \nu} - \frac{3}{8} R^2) \right).
\]

One important point is that in this fully diffeomorphism invariant theory the graviton acquires mass after linearization. What is the origin of mass here? Does there exist a Higgs-type mechanism? To answer this questions, Tekin and Dengiz [74] firstly constructed a Weyl invariant extension of NMG. They have shown that if the scalar field in this extended action obtains a vacuum expectation value, one can recover the NMG action. Around AdS background this non-zero value for the expectation value comes from choosing an on-shell (classical) value for scalar field and around flat spacetime it comes from two-loop effective action. Therefore in NMG, the mass of graviton comes from the breaking of Weyl symmetry in analogy with the Higgs mechanism in the standard quantum field theory.

Another important point is the absence of ghost in this higher derivative action. In general, the kinetic terms of massless and massive spin-2 particles in this theory can not be positive simultaneously and therefore this theory contains ghost. But in three dimensions the massless spin-2 particle does not propagate, therefore by choosing the wrong sign for its kinetic term, and therefore correct sign for massive one, one can obtain the unitary model.

17 In addition, this model does not contain the Boulware-Deser ghost at non-linear level [66] and its renormalizability is studied in [67]. Supergravity extension is also proposed in [68].
18 Another higher derivative model in three dimensions which gives a massive graviton around an AdS vacuum is known as Topologically Massive Gravity (TMG) [69]. In this model the parity invariance is broken and massive graviton has just one helicity. This model has the same benefits and problems as NMG.
19 By demanding the existence of a holographic c-function [70], one can find the higher derivative extensions of NMG. For this see [71–73].
Definitely, the linear analysis is one part in the program of checking the consistency of a theory. One another part is checking the non-perturbative structure of a theory. The non-perturbative structure of gravitational theories can be explored by different methods which the simplest one is finding the energy and entropy of black hole solutions and checking their positivity\textsuperscript{20}. It is argued that [75], by choosing the wrong sign for EH term the energy and entropy of NMG’s black hole solutions (those respect to the boundary conditions for finding the vacuum) cannot be positive. According to AdS\textsubscript{3}/CFT\textsubscript{2} [76] and Cardy formula [77], the negativity of entropy results in negativity of dual 2D central charges. This conflict between the absence of ghost in the bulk and positivity of the dual central charges, which NMG has it, is called ”Bulk-Boundary Clash”.

If we don’t insist on having a massive spin-2 particle in NMG, it exists a special value for the mass parameter $m^2$ in (A.7), by which both of the spin-2 particles become massless and the Bulk-Boundary clash disappears. But another problem would be arisen for NMG at this special value of $m^2$. The new type solutions, ”Logarithmic Solutions”, appear which existence of them is the sign for non-unitarity of a theory. It is shown that the dual theory of NMG at this special value of $m^2$ is a Logarithmic Conformal Field Theory(LCFT) [78]\textsuperscript{21}. All these mean that, NMG by itself does not seem to be a good candidate for unitary quantum gravity in three dimensions and it needs some extensions. These extensions are provided in [19], [20] which respectively they are based on the first order formalism and metric formalism\textsuperscript{22}.

Discovering the NMG, and its extensions [19], [20], in three dimensions opens a new window by which one can become hopeful to find a consistent massive gravity model in four dimensions that it differs with (A.4) and (A.6) models. In this line, Lü and Pope in 2011 [21] studied the higher derivative action in four spacetime dimensions

$$I = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left( R - 2\Lambda_g + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2 \right), \quad (A.8)$$

which is renormalizable [65], and contains one massless spin-2 particle, one massive spin-2 particle and one massive scalar in its spectrum. Note that the product of two Riemann tensors in four dimensions is not needed to render the EH action finite because this product term can be absorbed in the Gauss-Bonnet term which the latter one is just a topological term. For $\alpha = -3\beta \textsuperscript{23}$, the scalar mode decouples but around flat (AdS) background the theory contains ghost because the kinetic terms of massless and massive spin-2 particles cannot be positive simultaneously. This is exactly the one happens for

\textsuperscript{20}Another methods which can be used are for example studying the wave solutions and doing the Hamiltonian analysis.

\textsuperscript{21}To see the same discussions around TMG model, see [79]

\textsuperscript{22}The previous attempts to find a consistent three dimensional bi-gravity can be found in [80] and references therein.

\textsuperscript{23}If we omit the square of the Ricci tensor, then we gain unitarity but lose the renormalizability [81].
three dimensional model NMG. The strategy to get rid of the ghost for NMG at this level was choosing the wrong sign for EH action which itself gives the negative sign to the kinetic term of massless spin-2 particle. As the massless spin-2 particle does not propagate in three dimensions therefore we are remained with a healthy massive spin-2 particle. But this strategy is not applicable in four dimensions because the massless spin-2 particle is a propagating mode in this dimension.

The only possibility which is remained for the model (A.8) with $\alpha = -3\beta$, is choosing the parameters of the model appropriately such that the massive spin-2 particle becomes massless. This is achievable for $\beta = \frac{\ell^2}{6}$ where $\ell$ is the radius of AdS$_4$ solution. The obtained action is called ”Critical Gravity” [21]

$$I_{CG} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2m^2} (R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2) \right), \quad (A.9)$$

which instead of $\ell^2$ a mass parameter $m^2 = \frac{1}{\ell^2}$ is introduced$^{24}$. Unfortunately another problem, similar to the three dimensions, happens for the model (A.9) that is the appearance of non-unitary logarithmic modes [21] and [25]. All these mean that the model (A.9), is not a consistent unitary theory and it needs an extension. Providing this extension is the goal of this paper and is presented in (2.2). The new model (2.2) differs with well-known models (A.4) and (A.6).

References

[1] V. C. Rubin, N. Thonnard, and W. K. Jr Ford, SA through SC-1978. Astrophys.J.225 L107-L111; V. C. Rubin, N. Thonnard, and W. K. Jr Ford, Astrophys.J.238 L471-L487.

[2] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998).

[3] R. Adam et al. [Planck Collaboration], arXiv:1502.01582 [astro-ph.CO].

[4] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 6, 061102 (2016); B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 24, 241103 (2016).

[5] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 22, 221101 (2016).

$^{24}$Dropping the first two terms in (A.9), one obtains the action for conformal gravity. This action modulo the Gauss-Bonnet term is the square of the Weyl tensor and is invariant under Weyl scalings of the metric. The conformal gravity action is studied widely in the literature and in general it contains ghost. For recent discussions about Conformal Gravity see [82].

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[6] C. de Rham, J. T. Deskins, A. J. Tolley and S. Y. Zhou, arXiv:1606.08462 [astro-ph.CO].

[7] E. Berti, J. Gair and A. Sesana, Phys. Rev. D 84, 101501 (2011); A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. 82, 939 (2010).

[8] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173, 211 (1939).

[9] H. van Dam and M. J. G. Veltman, Nucl. Phys. B 22, 397 (1970).

[10] V. I. Zakharov, JETP Lett. 12, 312 (1970) [Pisma Zh. Eksp. Teor. Fiz. 12, 447 (1970)].

[11] A. I. Vainshtein, Phys. Lett. B 39, 393 (1972).

[12] D. G. Boulware and S. Deser, Phys. Rev. D 6, 3368 (1972).

[13] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Annals Phys. 305, 96 (2003).

[14] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, JHEP 0509, 003 (2005).

[15] C. de Rham and G. Gabadadze, Phys. Rev. D 82, 044020 (2010).

[16] S. F. Hassan and R. A. Rosen, JHEP 1107, 009 (2011); S. F. Hassan, R. A. Rosen and A. Schmidt-May, JHEP 1202, 026 (2012).

[17] S. F. Hassan and R. A. Rosen, JHEP 1202, 126 (2012).

[18] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. 102, 201301 (2009).

[19] E. A. Bergshoeff, S. de Haan, O. Hohm, W. Merbis and P. K. Townsend, Phys. Rev. Lett. 111, no. 11, 111102 (2013) Erratum: [Phys. Rev. Lett. 111, no. 25, 259902 (2013)]; H. R. Afshar, E. A. Bergshoeff and W. Merbis, JHEP 1408, 115 (2014).

[20] A. Akhavan, M. Alishahiha, A. Naseh, A. Nemati and A. Shirzad, JHEP 1605, 006 (2016).

[21] H. Lu and C. N. Pope, Phys. Rev. Lett. 106, 181302 (2011).

[22] E. A. Bergshoeff, O. Hohm, J. Rosseel and P. K. Townsend, Phys. Rev. D 83, 104038 (2011).

[23] W. Li, arXiv:1508.03246 [gr-qc].

[24] S. F. Hassan, A. Schmidt-May and M. von Strauss, Universe 1, no. 2, 92 (2015).
[25] M. Alishahiha and R. Fareghbal, Phys. Rev. D 83, 084052 (2011); I. Gullu, M. Gurses, T. C. Sisman and B. Tekin, Phys. Rev. D 83, 084015 (2011).

[26] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B 115, 197 (1982).

[27] D. Grumiller, W. Riedler, J. Rosseel and T. Zojer, J. Phys. A 46, 494002 (2013).

[28] J.M. Martin-Garcia, xAct: http://metric.iem.csic.es/Martin-Garcia/xAct.

[29] N. T. Evans, Journal of Mathematical Physics 8 (Feb., 1967) 170184.

[30] S. Deser and R. I. Nepomechie, Annals Phys. 154, 396 (1984); S. Deser and A. Waldron, Phys. Lett. B 508, 347 (2001).

[31] S. Deser and A. Waldron, Phys. Rev. Lett. 87, 031601 (2001); S. Deser and A. Waldron, Nucl. Phys. B 607, 577 (2001); S. Deser and A. Waldron, Phys. Lett. B 513, 137 (2001).

[32] H. Lu, Y. Pang and C. N. Pope, Phys. Rev. D 84, 064001 (2011).

[33] S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. 217, 595 (2001).

[34] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208, 413 (1999).

[35] N. Johansson, A. Naseh and T. Zojer, JHEP 1209, 114 (2012).

[36] S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman and B. Tekin, Phys. Rev. D 83, 061502 (2011).

[37] O. Hohm and E. Tonni, JHEP 1004, 093 (2010).

[38] S. Deser and A. Waldron, Phys. Rev. Lett. 110, no. 11, 111101 (2013); K. Izumi and Y. C. Ong, Class. Quant. Grav. 30, 184008 (2013); S. Deser, K. Izumi, Y. C. Ong and A. Waldron, arXiv:1312.1115 [hep-th].

[39] K. Johnson and E. C. G. Sudarshan, Annals Phys. 13, 126 (1961).

[40] G. D’Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava and A. J. Tolley, Phys. Rev. D 84, 124046 (2011).

[41] A. Higuchi, Nucl. Phys. B 282, 397 (1987).

[42] A. De Felice, A. E. Gmrkolu, C. Lin and S. Mukohyama, Class. Quant. Grav. 30, 184004 (2013); M. Fasiello and A. J. Tolley, JCAP 1211, 035 (2012).
[43] Y. Akrami, T. S. Koivisto and M. Sandstad, JHEP 1303, 099 (2013); M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell and S. F. Hassan, JCAP 1203, 042 (2012);

[44] F. Koennig, Y. Akrami, L. Amendola, M. Motta and A. R. Solomon, Phys. Rev. D 90, 124014 (2014); D. Comelli, M. Crisostomi and L. Pilo, JHEP 1206, 085 (2012).

[45] S. Deser, K. Izumi, Y. C. Ong and A. Waldron, Phys. Lett. B 726, 544 (2013).

[46] S. Deser, M. Sandora and A. Waldron, Phys. Rev. D 88, 081501 (2013); E. Joung, W. Li and M. Taronna, Phys. Rev. Lett. 113, 091101 (2014).

[47] S. F. Hassan, A. Schmidt-May and M. von Strauss, Int. J. Mod. Phys. D 23, no. 13, 1443002 (2014); S. F. Hassan, A. Schmidt-May and M. von Strauss, Class. Quant. Grav. 33, no. 1, 015011 (2016).

[48] K. Hinterbichler, Rev. Mod. Phys. 84, 671 (2012).

[49] C. de Rham, Living Rev. Rel. 17, 7 (2014).

[50] A. Schmidt-May and M. von Strauss, J. Phys. A 49, no. 18, 183001 (2016).

[51] B. Tekin, Phys. Rev. D 93, no. 10, 101502 (2016).

[52] Steven. Weinberg, Vol.1: Foundations (Cambridge University Press) (2005).

[53] M. Porrati, Phys. Lett. B 498, 92 (2001); A. Karch, E. Katz and L. Randall, JHEP 0112, 016 (2001); I. I. Kogan, S. Mouslopoulos and A. Papazoglou, Phys. Lett. B 503, 173 (2001).

[54] G. Gabadadze, Phys. Lett. B 681, 89 (2009).

[55] C. de Rham, Phys. Lett. B 688, 137 (2010).

[56] C. de Rham and G. Gabadadze, Phys. Lett. B 693, 334 (2010).

[57] S. F. Hassan and R. A. Rosen, Phys. Lett. B 702, 90 (2011).

[58] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. 106, 231101 (2011).

[59] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. 108, 041101 (2012).

[60] R. L. Arnowitt, S. Deser and C. W. Misner, Gen. Rel. Grav. 40, 1997 (2008).

[61] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, Nucl. Phys. B 597, 127 (2001).

[62] S. F. Hassan and R. A. Rosen, JHEP 1204, 123 (2012).
[63] C. de Rham, A. J. Tolley and S. Y. Zhou, JHEP 1604, 188 (2016)
[64] C. de Rham, A. J. Tolley and S. Y. Zhou, Phys. Lett. B 760, 579 (2016)
[65] K. S. Stelle, Phys. Rev. D 16, 953 (1977).
[66] C. de Rham, G. Gabadadze, D. Pirtskhalava, A. J. Tolley and I. Yavin, JHEP 1106, 028 (2011).
[67] I. Oda, JHEP 0905, 064 (2009); E. A. Bergshoeff, O. Hohm and P. K. Townsend, Springer Proc. Phys. 137, 291 (2011).
[68] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin and P. K. Townsend, Class. Quant. Grav. 27, 025010 (2010); E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin and P. K. Townsend, Class. Quant. Grav. 28, 015002 (2011).
[69] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48, 975 (1982); S. Deser, R. Jackiw and S. Templeton, Annals Phys. 140, 372 (1982) [Annals Phys. 281, 409 (2000)] Erratum: [Annals Phys. 185, 406 (1988)].
[70] R. C. Myers and A. Sinha, Phys. Rev. D 82, 046006 (2010); R. C. Myers and A. Sinha, JHEP 1101, 125 (2011).
[71] A. Sinha, JHEP 1006, 061 (2010). M. F. Paulos, Phys. Rev. D 82, 084042 (2010).
[72] I. Gullu, T. C. Sisman and B. Tekin, Class. Quant. Grav. 27, 162001 (2010).
[73] I. Gullu, T. C. Sisman and B. Tekin, Phys. Rev. D 82, 024032 (2010).
[74] S. Dengiz and B. Tekin, Phys. Rev. D 84, 024033 (2011).
[75] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. D 79, 124042 (2009).
[76] J. M. Maldacena, Int. J. Theor. Phys. 38, 1113 (1999) [Adv. Theor. Math. Phys. 2, 231 (1998)]; E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998); S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998).
[77] J. L. Cardy, Nucl. Phys. B 270, 186 (1986); S. Carlip, Class. Quant. Grav. 22, R85 (2005).
[78] D. Grumiller and O. Hohm, Phys. Lett. B 686, 264 (2010); M. Alishahiha and A. Naseh, Phys. Rev. D 82, 104043 (2010).
[79] D. Grumiller and N. Johansson, JHEP 0807, 134 (2008); A. Maloney, W. Song and A. Strominger, Phys. Rev. D 81, 064007 (2010); K. Skenderis, M. Taylor and B. C. van Rees, JHEP 0909, 045 (2009).
[80] H. R. Afshar, M. Alishahiha and A. Naseh, Phys. Rev. D 81, 044029 (2010).

[81] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lst and A. Riotto, Fortsch. Phys. 64, no. 2-3, 176 (2016).

[82] J. Maldacena, arXiv:1105.5632 [hep-th]; D. Grumiller, M. Irakleidou, I. Lovrekovic and R. McNees, Phys. Rev. Lett. 112, 111102 (2014); A. Ghodsi, B. Khavari and A. Naseh, JHEP 1501, 137 (2015).