From the mass gap in $O(N)$ to the non-Borel-summability in $O(3)$ and $O(4)$ sigma-models

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We give an analytical derivation of the mass gap of the $O(N)$ sigma models and investigate a large-order behavior of the weak coupling asymptotic expansion for the energy. For sufficiently large $N$ the series is sign-oscillating, which is expected from the large $N$ solution of the sigma model. However, for $N = 3$ and $N = 4$ the series are again positive.

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INTRODUCTION

The $O(N)$ sigma-model is often considered as a toy model for the quantum chromodynamics. It is asymptotically free and dynamically generates a mass scale $\Lambda$ (analog of $\Lambda_{QCD}$), although its classical formulation does not contain any dimensionful parameters. It is widely accepted that the asymptotic states of the sigma model are the massive particles in the vector representation of the $O(N)$ group. The mass of the particles should be proportional to the only mass scale of the theory: $m = c \Lambda$.

The mass $m$ is a physical quantity, while $c$ and $\Lambda$ depend on the regularization scheme. The coefficient $c$ cannot be determined from the perturbation theory. Luckily, the $O(N)$ sigma-model can be studied nonperturbatively due to its integrability. The explicit expression for the coefficient $c$ in the $\overline{\text{MS}}$ scheme was found in [1, 2]:

$$c = \left(\frac{8}{\pi^2}\right)^{\frac{1}{N-2}} \frac{1}{\Gamma\left[1 + \frac{1}{N-2}\right]}.$$  (1)

We explain the idea of this calculation in the next section. In order to obtain $c$, one has to solve the integral equation (3) in the weak-coupling regime at the leading and subleading order. At the leading order the solution was found by the application of the generalized Wiener-Hopf method [3]. The subleading order, up to our knowledge, was found only numerically, although with a high precision, which allowed to conjecture the expression (1). The goal of this paper is to solve the integral equation (3) analytically at subleading and higher orders in a recursive manner.

Another motivation for this work comes from the AdS/CFT correspondence where the appearance of the $O(6)$ sigma-model raised the necessity of the solution of the equation (3) and its generalization at next-to the subleading order [4].

Armed with the recursive procedure, we made an estimation for the large-order behavior of the coefficients of expansion and found unexpected properties for $N = 3$ and $N = 4$ (see the discussion session).

INTEGRAL EQUATION

The idea of [1, 2] was to consider the sigma-model in the presence of a magnetic field $h$ coupled to a $U(1)$ charge. When the value of $h$ exceeds the mass gap, a finite density $\rho$ of equally polarized particles is created. At large values of $h$ the free energy of the system can be computed perturbatively due to the asymptotic freedom.

Knowing the free energy density $f[h]$, we can find the energy density $\varepsilon[\rho]$ through the Legendre transform:

$$\varepsilon[\rho] = \min_h (f[h] + \rho h).$$  (2)

It is convenient to introduce a running coupling constant $\alpha[\mu]$ via the relation

$$\frac{1}{\alpha} + (\Delta - 1) \log \alpha = \log \left(\frac{2\pi \mu \Delta}{\Lambda_{\overline{\text{MS}}}}\right), \quad \Delta = \frac{1}{N-2}.$$  (3)

The perturbative QFT predicts the following expansion for the energy density:

$$\frac{\varepsilon[\rho]}{\rho^2} = \pi \Delta \left(\alpha + \frac{1}{2} \alpha^2 + \Delta \sum_{n=3}^{\infty} \chi_n \alpha^n\right) + O\left(\frac{\Lambda_{\overline{\text{MS}}}}{\rho^2}\right).$$  (4)

where $\alpha$ is evaluated at the scale $\mu = \rho$.

The energy of the system in the large volume can be calculated also from the asymptotic Bethe Ansatz, which explicitly contains the mass $m$. The considered energy density $\varepsilon$ is recovered in the thermodynamical limit, in which the number of particles $K$ and the length of the system $L$ both go to infinity with fixed $\rho = K/L$. In this limit the Bethe Ansatz reduces to the integral equation for the density of the rapidity distribution $\chi[\theta]$:

$$\chi[\theta] = \int_{-B}^{B} K[\theta - \theta'] \chi[\theta'] d\theta' = m \cosh[\theta], \quad \theta^2 < B^2;$$

$$K[\theta] = \frac{1}{2\pi i} \frac{d}{d\theta} \log S_0[\theta],$$  (5)

$$S_0[\theta] = -\frac{\Gamma \left[\frac{1}{2} + \frac{\theta}{2}\right]}{\Gamma \left[1 + \frac{\theta}{2}\right]} \frac{\Gamma \left[\frac{1}{2} + \Delta + \frac{\theta}{2}\right]}{\Gamma \left[\frac{1}{2} + \Delta + \frac{\theta}{2}\right]} / c.c.
The energy density and the density of particles are given by

\[ \varepsilon = m \int_{-B}^{B} \frac{d\theta}{2\pi} \chi[\theta] \cosh[\theta], \quad \rho = \int_{-B}^{B} \frac{d\theta}{2\pi} \chi[\theta]. \]  

We see that \( \varepsilon \) depends on \( \rho \) through the parameter \( B \).

To compare with the expansion (4) we have to consider the large \( \rho \), or equivalently large \( B \), asymptotics of the integral equation (4).

The integral equation (5) can be rewritten in a nice form in terms of the resolvent for the function \( \chi[\theta] \). For this we first notice that the kernel \( K[\theta] \) can be represented as

\[ K[\theta] = \frac{1}{2\pi i} \left( \frac{D + D^{2\Delta}}{1 + D} - \frac{D^{-1} + D^{-2\Delta}}{1 + D^{-1}} \right) \frac{1}{\theta}, \]

where \( D = e^{i\pi \theta} \) is a shift operator and \( (1 + D^{\pm1})^{-1} = 1 - D^{\pm1} + D^{\pm2} \ldots \). This representation for the kernel can be easily derived if to notice the formal equality \( \Gamma[\theta/2\pi] \simeq (1/\theta)^{i\theta/\pi} \).

The resolvent of \( \chi[\theta] \) defined by

\[ R[\theta] = \int_{-B}^{B} \frac{d\theta}{\theta - \theta'}, \]

is analytic everywhere except on the support \([-B, B]\) of \( \chi[\theta] \). The residue of \( R[\theta] \) at infinity equals to \( 2m/B \). The density distribution \( \chi[\theta] \) can be read from the discontinuity of the resolvent on the interval \([-B, B]\):

\[ \chi[\theta] = -\frac{1}{2\pi i} (R[\theta + i0] - R[\theta - i0]). \]

Using (7), (8), and (9) we can rewrite the integral equation (5) as

\[ \frac{1 - D^{2\Delta}}{1 + D} R[\theta + i0] - \frac{1 - D^{-2\Delta}}{1 + D^{-1}} R[\theta - i0] = -2\pi i m \cosh[\theta], \quad \theta^2 < B^2. \]  

**LEADING ORDER SOLUTION FOR THE ENERGY DENSITY**

In the following we will neglect the terms that give exponentially suppressed contribution to the value of \( \varepsilon \). In this approximation we have

\[ \varepsilon \simeq \int_{0}^{B} e^{\theta} \chi[\theta] \frac{d\theta}{2\pi} \simeq B \int_{-\infty}^{0} e^{\frac{\theta}{2\pi}} \chi[z] \frac{dz}{4\pi}, \quad \theta = B + \frac{z}{2}. \]

In other words, \( \varepsilon \) receives the main contribution from the vicinity of the branch points \( \pm B \). Therefore we will consider the double scaling limit

\[ B, \theta \rightarrow \infty, \quad z = 2(\theta - B) \text{ fixed}. \]

From (9) and (11) we can express the energy density in terms of the inverse Laplace transform of the resolvent:

\[ \varepsilon = \frac{m e^{B}}{4\pi} R[1/2], \quad R[s] = \int_{-i\infty + 0}^{i\infty + 0} \frac{dz}{2\pi i} e^{sz} R[z]. \]

In the double scaling limit the equation (10) is simplified:

\[ \frac{1 - D^{2\Delta}}{1 + D} R[z + i0] - \frac{1 - D^{-2\Delta}}{1 + D^{-1}} R[z - i0] = -\pi i m e^{B} e^{\frac{z}{2}}, \quad z < 0. \]

The inverse Laplace transform of (14) is straightforward:

\[ \sin[2\pi s] \left( \frac{\cos[\pi(s - i0)]}{\cos[\pi(s + i0)]} + \frac{\cos[\pi(s + i0)]}{\cos[\pi(s - i0)]} \right) = \frac{m}{2i} e^{B} \left( \frac{1}{s + \frac{1}{2} + i0} - \frac{1}{s + \frac{1}{2} - i0} \right), \quad s < 0. \]

To find the correct solution to (15) we demand the following analytical properties for \( \hat{R}[s] \) at each order of \( 1/B \) expansion:

- \( \hat{R}[s] \) is analytic everywhere except on the negative real axis,
- \( \hat{R}[s] \) has simple poles at \( s = -n/(2\Delta) \) and simple zeroes at \( s = -1/2 - n \), where \( n \) is a positive integer,
- \( \hat{R}[s] \) is expanded in negative powers of \( s \) at infinity.

The presence of zeroes in the resolvent can be directly seen from the equation (15). The solution of the corresponding homogeneous equation should have in addition a zero at \( s = -1/2 \). The origin of the other stated analytical properties are explained in the appendix.

The most general solution of the equation (15) which satisfies stated analyticity properties is the following:

\[ \hat{R}[s] = \left( \frac{A}{s + \frac{1}{2}} + Q[s] \right) \Phi[s], \quad A = \frac{m}{4\Delta \Delta} e^{-\frac{1}{2} + B + \Delta \Gamma[\Delta]}, \]

\[ \Phi[s] = \frac{1}{\sqrt{s}} e^{(1-2\Delta)s \log[s] - 2\Delta s \log[2\Delta]} \frac{\Gamma[2\Delta s + 1]}{\Gamma[s + \frac{1}{2}]}, \]

\[ Q[s] = \frac{1}{B s} \sum_{n,m=0}^{\infty} Q_{n,m} \frac{\log[B]}{B^{m+n} s^{n}}. \]

The dependence of \( Q[s] \) on \( B \) is not a consequence of (14) and can be deduced from the considerations of the next section. We see that \( Q[s] \) does not contribute to the leading order of the \( 1/B \) expansion. This solution should be compared with the expression for \( g_{\pm}[\xi] \) of (3). The leading order of \( \varepsilon \) is given by \( m A \Phi[1/2] e^{B}/(4\pi) \).
THE PARTICLE DENSITY AND SUBLADING CORRECTIONS

First note that if we apply the operator 
\[ D^{-1/2} R[\theta + i0] + D^{-\Delta + 1/2} R[\theta - i0] \] 
to the equation (1) we will get 
\[ (D^{\Delta - 1/2} R[\theta + i0] + D^{-\Delta + 1/2} R[\theta - i0]) = 0. \] (17)

In the previous section we found the most general solution in the double scaling limit. Still we have to fix unknown coefficients \( Q_{n,m} \). For this we consider a different regime. We take again \( B \to \infty \) but now we will be interested in the values of the resolvent \( R[\theta] \) at the distances of the order \( B \) or larger from the branch points \( \theta = \pm B \). In this case the shift operator can be expanded in the Taylor series \( D = 1 + i\pi \theta B - \frac{1}{2} \pi^2 (\theta B)^2 + \ldots \). Solving the equation (17) perturbatively we can expand the resolvent in the inverse powers of \( B \):
\[ R[\theta] = \sum_{n,m=0}^{\infty} \sum_{k=0}^{m+n} \sqrt{B} c_{n,m,k} \frac{[\theta/B]^{k}}{B^{m-n} \theta^2 - B^2} \log \left( \frac{\theta - B}{\theta + B} \right)^k, \] (18)

where \( \epsilon[k] = k \mod 2 \). The perturbative meaning of the expansion (18) is most easily seen in terms of the variable \( u = \theta/B \). The solution (18) gives us the value for the particle density from the residue of the resolvent at infinity:
\[ \rho = \frac{\sqrt{B}}{2\pi} \left( c_{0,0,0} + \sum_{m=1}^{\infty} \frac{c_{0,m,0} - 2c_{0,m,1}}{B^m} \right). \] (19)

If we reexpand (18) in the double scaling limit (12), we should recover the solution obtained in the previous section. This condition uniquely fixes all the coefficients \( c_{n,m,k} \) and \( Q_{n,m} \).

The expansion (18) in the double scaling limit organizes at each order of \( 1/B \) as a \( 1/z \) expansion. Therefore we should compare it with the Laplace transform of the small \( s \) expansion of \( \hat{R}[s] \). As an illustration, we give here the terms of these expansions which are relevant for calculation of the leading and the subleading orders of \( \rho \) and \( \epsilon \):
\[ R = \frac{c_{0,0,0}}{\sqrt{z}} + \frac{c_{1,0,0} + c_{1,0,1}}{z^{3/2}} \log \left( \frac{z}{17} \right) - \frac{\sqrt{z}}{8B} c_{0,0,0} + \frac{8c_{0,1,0} - 3c_{1,0,0} - 2c_{1,0,1} + (8c_{0,1,1} + c_{1,0,1})}{z^{3/2}} \log \left( \frac{z}{17} \right), \]
\[ \int_0^\infty ds \ e^{-sz} \hat{R}[s] = \frac{2A}{\sqrt{z}} - A \frac{2\Delta \log \left( \frac{z^2}{2} \right) + 1 + (1 - 2\Delta) \log z}{z^{5/2}} + \frac{Q_{0,0,\sqrt{z}}}{B} \left( -2 + \frac{2\Delta \log \left( \frac{z^2}{2} \right) + 1 - (1 - 2\Delta) \log z}{z} \right). \] (20)

RESULTS AND DISCUSSIONS

From the results of the previous sections we find the expressions for \( \rho \) and \( \epsilon \) at the leading and subleading orders:
\[ \frac{\epsilon[B]}{m^2} = \frac{\epsilon^2 B^{2\Delta - 1}}{16\pi \Delta^{2\Delta - 1}} \Gamma[\Delta]^2 \left( 1 + \frac{1}{4B} \right), \] (21)
\[ \frac{\rho[B]}{m} = \frac{\epsilon^2 B^{\Delta - 1}}{4\pi \Delta \Gamma[\Delta]} \left( \sqrt{B} - \frac{1}{2} B + (1 - 2\Delta) \log 4B \right). \] (22)

Resolving the parametric dependence we obtain exactly \( \alpha \) and \( \alpha^2 \) terms in the expansion (4) if \( \alpha \) is defined as
\[ \frac{1}{\alpha} + (\Delta - 1) \log \alpha = \log \left( \frac{\Delta \pi^2}{e} \right) \frac{\Gamma[\Delta]}{\Gamma[\Delta - 1]}, \] (22)

Comparing (22) with (3) we confirm the result (1). Comparing the solutions (10) and (18) one can find the higher order corrections to the energy density. At first four loops they are given by:
\[ \chi_3 = \frac{1}{2}, \quad \chi_4 = -\frac{1}{32} (24\zeta(3) \Delta^2 + 8\Delta^2 - 42\zeta(3) \Delta - 28\Delta + 21\zeta(3) - 8) \] (23)
\[ \chi_5 = -\frac{1}{96} (456\zeta(3) \Delta^3 + 24\zeta(3)^3 - 918\zeta(3) \Delta^2 - 60\Delta^2 + 609\zeta(3) \Delta - 140\Delta - 105\zeta(3) - 24). \] (24)

The two-loop result \( \chi_3 \) coincides with the field theory calculations in [4]. The fact that the energy can be expanded in power series over the running coupling constant \( \epsilon \) is a nontrivial property of the integral equation (4). This is a strong check of the validity of the bootstrap approach. This hidden renorm-group dynamics of the integral equation was explained in [5]. We have found analytical expressions for \( \chi_n \) up to \( n = 26 \) [6]. This allowed us to estimate the large \( n \) behavior of \( \chi_n \):
\[ \chi_n \approx \frac{\Gamma[n]}{2^{n-1} a_n \Delta}. \] (24)

For \( \Delta = 0 \) we have \( a_n[0] = (-1)^{n-1} \). This result is consistent with the fact that in the large \( N \) limit the leading IR renormalon pole in the Borel plane is absent [3]. The large-order behavior of the coefficients \( \chi_n \) is therefore governed by the leading UV renormalon pole leading to the Borel-summable oscillating behavior (22).

Surprisingly, \( a_n \approx 1.09 \) for \( \Delta = 1 \) and \( a_{n+1} \approx n^{-1}(2.09 - 0.43(n \mod 2)), i.e. \) the series is non-Borel-summable. The Borel ambiguity is of the order \( \Delta^2/\rho^2 \) as it should be from the field theory point of view (see [4]). For arbitrary \( \Delta > 1 \) the behavior of the coefficients \( a_n \) interpolates between those for \( \Delta = 0, 1/2 \) and 1. We estimate the sign oscillation of the coefficients for sufficiently large \( n \) and \( \Delta < \Delta_c \approx 0.4 \). For \( 1 \leq \Delta \leq \)}
\[ \Delta, \] all the \( a_n \) are positive. For \( \Delta > 1 \) the asymptotic behavior of \( a_n \) is given by \( a_n \approx -\alpha_1 n^\alpha_2 \Delta^{n-2} \), where \( \alpha_1 \) is positive.

The non-Borel-summability in the \( O(N) \) sigma models for \( N = 3 \) and \( N = 4 \) is a quite unexpected property and should be understood better. For \( N = 3 \) one might try to explain it by the presence of the instantons. For \( N = 4 \) our observation is supported by the non-Borel-summability in the SU(\( N \)) principal chiral field model (PCF) at large \( N \) [4](since the \( O(4) \) sigma-model can be viewed as an \( SU(2) \) PCF model).

The method used in this paper was developed in the papers [7, 8] and with no doubts can be applied to similar systems such as \( \mathbb{CP}^n \), PCF or Gross-Neveu sigma-models.

**APPENDIX: ANALYTICAL PROPERTIES OF THE RESOLVENTS**

There is yet another representation of the integral equation (\ref{eq:resolvent}). To derive it we have to extend this equation to the whole real axis. For this we introduce a new function \( \chi_h[\theta] \) (density of holes) which has the support complementary to \([-B, B]\). The function \( \chi_h[\theta] \) is defined to be such that the equation

\[
\chi_h[\theta'] + \chi[\theta'] - \int_B^{\infty} d\theta'' K[\theta' - \theta''] \chi[\theta''] = m \cosh[\theta'] \Theta[B^2 - (\theta')^2]
\]

is valid for any real \( \theta' \).

Integrating the equation (26) with the Cauchy kernel \( \frac{1}{\pi i (s - \theta)} \) and is analytic for \( \text{Re}[s] > 0 \).

where \( R_h \) is the resolvent for \( \chi_h \) and \( T \) is the resolvent for the r.h.s of (26).

The function \( \hat{R}[s] \) is defined via the inverse Laplace transform (see (24)) and is analytic for \( \text{Re}[s] > 0 \). We define the analytical continuation of \( \hat{R}[s] \) to \( \text{Re}[s] < 0 \) via the path that does not cross the ray \( s < 0 \). To study the analytical properties of \( \hat{R}[s] \) for all values of \( \text{Re}[s], \text{Im}[s] < 0 \) we consider the equation (26) for \( \text{Im}[\theta] \) in the double scaling limit and apply the integral \( \int_{\infty}^{-\infty} \frac{dz}{2\pi i} e^{sz} \). The result can be written as

\[
\hat{R}[s] = \frac{1 + e^{-i\pi s}}{1 - e^{-i2\pi s}} \left( \hat{T}[s] + \frac{m}{2} e^{sB} \frac{1}{s + \frac{1}{2}} \right), \quad \text{Im}[s] < 0,
\]

\[
\hat{T}[s] = \int_{\infty}^{-\infty} \frac{dz}{2\pi i} e^{sz} T[z].
\]

Since \( T[z] \) is analytic everywhere except on the ray \( z > 0 \), the function \( \hat{T}[s] \) is analytic for \( \text{Re}[s] < 0 \). We conclude that \( \hat{R}[s] \) is analytic everywhere in the considered region and has poles on the ray \( s < 0 \).

Similarly we consider the region \( \text{Re}[s] < 0, \text{Im}[s] > 0 \). This explains the pole structure of the resolvent.

Since \( K[\theta - \theta'] \) analytic at \( \theta = B \), \( \chi[\theta] \) analytic as well. This means that expansion of \( \hat{R}[z] \) at \( z = 0 \) is given by \( \hat{R}[z] = \log[z](\beta_0 + z\beta_1 + z^2\beta_2 + \ldots) \). The inverse Laplace transform of this expansion gives us the stated behavior of \( \hat{R}[s] \) at infinity.

Note that the stated analytical properties of \( \hat{R}[s] \) are valid only in the double scaling limit. The inverse Laplace transform of \( \hat{R}[\theta] \) at finite \( B \) is an entire function in the \( s \)-plane. This phenomena is illustrated by the function \( f[s] = \frac{1}{\pi} \frac{1}{s(1 - e^{-2\pi s})} \). This is an entire function. But since we are interested in \( f[1/2] \) in the large \( B \) limit the exponentially suppressed term can be neglected and effectively we get a pole.

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