CHARACTERIZATIONS OF INTEGRAL INPUT-TO-STATE STABILITY FOR BILINEAR SYSTEMS IN INFINITE DIMENSIONS

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ABSTRACT. For bilinear infinite-dimensional dynamical systems, we show the equivalence between uniform global asymptotic stability and integral input-to-state stability. We provide two proofs of this fact. One applies to general systems over Banach spaces. The other is restricted to Hilbert spaces, but is more constructive and results in an explicit form of iISS Lyapunov functions.

1. Introduction. Stability and robustness are fundamental for control systems and typically they have been addressed within two different concepts. One is Lyapunov stability characterizing behavior of dynamical systems without inputs near equilibrium points. The other is input-output stability which neglects information about the state of a system and studies response of a system to external inputs. In many cases it is not satisfactory to rely only on one of them. Input-to-state stability (ISS) was introduced in [32] to unify these two concepts and provided powerful tools for systems of ordinary differential equations (ODEs) such as ISS Lyapunov functions [36], design methods for nonlinear control systems [24, 10], and the ISS small-gain theorem [19, 18] that has been extensively utilized in theory and application to establish stability and robustness of interconnections and networks of ISS systems [9, 21]. The usefulness of ISS that was widely recognized first for ODE systems led to generalization, sophistication and abstractions to cover other types of control systems, such as time-delay systems, discrete-time, hybrid systems and trajectory-based systems (e.g. [20, 29, 35, 22] to name a few).

The study of ISS of general infinite-dimensional systems and in particular of partial differential equations (PDEs) started relatively recently. In [17], [7], [8],...
ISS of infinite-dimensional systems

\[ \dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in X, u(t) \in U, \quad (1) \]

has been addressed via methods of semigroup theory [16], [5]. Here the state space \( X \) and the space of input values \( U \) are Banach spaces, \( A : D(A) \to X \) is the generator of a \( C_0 \)-semigroup over \( X \) and \( f : X \times U \to X \) is Lipschitz w.r.t. the first argument. Many classes of evolution PDEs, such as parabolic and hyperbolic PDEs are of this kind [11], [4].

In [7] sufficient conditions, a linearization method and a Lyapunov-based small-gain theorem for ISS of systems (1) have been developed, resulting in an efficient method to construct Lyapunov functions for interconnections of ISS systems and in this way to prove ISS of the interconnections. On the basis of [7], several results within ISS theory for impulsive infinite-dimensional systems have been proposed in [8]. In [17] and [25], systems (1) with linear function \( f \) have been investigated via frequency-domain methods. A substantial effort has been devoted to constructions of ISS Lyapunov functions for nonlinear parabolic systems over \( L_2 \) spaces in [26]. In [30] the construction of ISS-Lyapunov functions for time-variant linear systems of hyperbolic equations (balance laws) has been provided and these results have been applied to design a stabilizing boundary feedback control for the Saint–Venant–Exner equations. In [6] ISS of some classes of monotone parabolic systems has been considered.

In spite of powerful tools developed within ISS theory for ODEs, ISS property is often too restrictive in practical systems, since in many cases boundedness of their trajectories is not guaranteed in the presence of inputs, or their trajectories grow to infinity for large enough inputs. Such a situation is usual in biochemical processes, population dynamics and traffic flows etc. due to saturation and limitations in actuators and processing rates. Such systems are never ISS, but many of them enjoy a weaker robustness property, called integral input-to-state stability (iISS) [33]. In [3] a Lyapunov type necessary and sufficient condition for an ODE system to be iISS has been proved. In [12, 14, 15, 2, 23] small-gain theorems for interconnections whose subsystems are not necessarily ISS have been developed.

It is well-known that for linear ODE systems the notions of ISS and iISS coincide. In [33] it was proved by a direct construction of an iISS Lyapunov function, that bilinear ODE systems with Hurwitz autonomous matrices are always iISS, although many of them are not ISS. Bilinear systems have allowed us to understand a basic class of pure iISS systems and provided clues to dealing with more complicated iISS systems [33, 13].

In this work we are going to generalize the result from [33] to bilinear infinite-dimensional systems (1). On this way several difficulties arise. One of them is that the construction of Lyapunov functions, extending the original technique from [33] directly works only for systems with a Hilbert state space. Therefore in order to prove the the equivalence between iISS and uniform global asymptotic stability for systems over Banach spaces, this paper develops a different method. Another difficulty is a need to use various density arguments, since the direct check of the properties of Lyapunov functions on the whole state space is often not possible. The work leads to an observation that non-uniformity over the spatial variables of a PDE system can make ISS fragile, while the system remains iISS.

The structure of the paper is as follows. Having introduced basic stability notions in Section 2, we define in Section 3 iISS Lyapunov functions and prove that the
existence of an iISS Lyapunov function for a system (1) implies iISS of this system. ISS Lyapunov functions are also defined in a dissipative form so that they become a special case of iISS Lyapunov functions, while ISS Lyapunov functions are defined in an implicative form in [7]. In Section 3, we also show that both formulations of ISS Lyapunov functions coincide, provided that the operator $A$ generates an analytic semigroup and certain additional conditions on a nonlinearity hold. In Section 4 we investigate iISS of bilinear systems. After a short discussion of infinite-dimensional linear systems, we prove that uniformly globally asymptotically stable bilinear systems are necessarily iISS. First we address this question for systems whose state space is an arbitrary Banach space. Next we establish the abovementioned theorem for Hilbert spaces, which results in an explicit construction of an iISS Lyapunov function for bilinear systems. We illustrate our findings on an example of a parabolic system in Section 5 and conclude the paper in Section 6. In Appendix we prove two technical results. The shortened preliminary version of this paper was submitted to the 53rd IEEE Conference on Decision and Control [27].

We use the following notation throughout the paper. For linear normed spaces $X, Y$ let $L(X, Y)$ be the space of bounded linear operators from $X$ to $Y$ and $L(X) := L(X, X)$. A norm in these spaces we denote by $\| \cdot \|$. By $C(X, Y)$ we denote the space of continuous functions from $X$ to $Y$, $C(X) := C(X, X)$ and by $PC(X, Y)$ the space of piecewise right-continuous functions from $X$ to $Y$. Both are equipped with the standard sup-norm.

We define $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{R}_+ := [0, \infty)$. Let $\mathbb{N}$ denote the set of natural numbers. Let $L_p(0, d)$, $p \geq 1$ be a space of $p$-th power integrable functions $f : (0, d) \to \mathbb{R}$ with the norm $\|f\|_{L_p(0, d)} = \left( \int_0^d |f(x)|^p dx \right)^{\frac{1}{p}}$.

2. Problem formulation. Consider a system (1) and assume throughout the paper that $X$ and $U$ are Banach spaces and $f(0, 0) = 0$, i.e., $x \equiv 0$ is an equilibrium point of (1). Let $\phi(t, \phi_0, u)$ denote the state of a system (1) at moment $t \in \mathbb{R}_+$ associated with an initial condition $\phi_0 \in X$ at $t = 0$, and input $u \in U$, where $U$ is a linear normed space of admissible inputs equipped with a norm $\| \cdot \|_{U_c}$.

We use the following classes of comparison functions

$\mathcal{P} := \{ \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0 \}$

$\mathcal{K} := \{ \gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing} \}$

$\mathcal{K}_\infty := \{ \gamma \in \mathcal{K} \mid \gamma \text{ is unbounded} \}$

$\mathcal{L} := \{ \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t \to \infty} \gamma(t) = 0 \}$

$\mathcal{KL} := \{ \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K}, \forall t \geq 0, \beta(r, \cdot) \in \mathcal{L}, \forall r > 0 \}$

Under solutions of (1) we understand solutions of an integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(x(s), u(s))ds.$$  \hspace{1cm} (2)

belonging to the class $C([0, \tau], X)$ for all $\tau > 0$ (so-called weak solutions).

Definition 1. We call $f : X \times U \to X$ Lipschitz continuous on bounded subsets of $X$, uniformly w.r.t. the second argument if $\forall w > 0 \exists L(w) > 0$, such that $\forall x, y : \|x\|, \|y\| \leq w$, $\forall v \in U$

$$\|f(y, v) - f(x, v)\|_X \leq L(w)\|y - x\|_X.$$ \hspace{1cm} (3)

We will use the following assumption concerning nonlinearity $f$ throughout the paper.
Assumption 1. We assume that $f : X \times U \to X$ is Lipschitz continuous on bounded subsets of $X$, uniformly w.r.t. the second argument and that $f(x, \cdot)$ is continuous for all $x \in X$.

Assumption 1 ensures that the weak solution of (1) exists and is unique, according to a variation of a classical existence and uniqueness theorem [4, Proposition 4.3.3].

Next we introduce stability properties for the system (1).

Definition 2. System (1) is globally asymptotically stable at zero uniformly with respect to state (0-UGASs), if $\exists \beta \in KL$, such that $\forall \phi_0 \in X$, $\forall t \geq 0$ it holds

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t).$$

(4)

To study stability properties of (1) with respect to external inputs, we use the notion of input-to-state stability [7]:

Definition 3. System (1) is called input-to-state stable (ISS) w.r.t. space of inputs $U_c$, if there exist $\beta \in KL$ and $\gamma \in K$ such that the inequality

$$\|\phi(t, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma(\|u\|_{U_c}).$$

holds $\forall \phi_0 \in X$, $\forall u \in U_c$ and $\forall t \geq 0$.

We emphasize that the above definition does not yet exactly correspond to ISS of finite dimensional systems [34] since Definition 3 allows the flexibility of the choice $U_c$. A system (1) is said to be ISS, without expressing the normed space of inputs explicitly, if it is ISS w.r.t. $U_c = C(\mathbb{R}_+, U)$ endowed with a usual supremum norm. This terminology follows that of ISS for finite dimensional systems.

The following notion is central in this paper

Definition 4. System (1) is called integral input-to-state stable (iISS) if there exist $\theta \in K_\infty$, $\mu \in K$ and $\beta \in KL$ such that the inequality

$$\|\phi(t, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \theta \left(\int_0^t \mu(\|u(s)\|_{U_c}) ds\right)$$

holds $\forall \phi_0 \in X$, $\forall u \in U_c = C(\mathbb{R}_+, U)$ and $\forall t \geq 0$.

In the next section we will provide a Lyapunov sufficient condition for iISS of systems (1).

3. Lyapunov characterization of iISS.

Definition 5. A continuous function $V : X \to \mathbb{R}_+$ is called an iISS Lyapunov function, if there exist $\psi_1, \psi_2 \in K_\infty$, $\alpha \in \mathcal{P}$ and $\sigma \in K$ such that

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \; \forall x \in X$$

(7)

and system (1) satisfies

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_{U_c})$$

(8)

for all $x \in X$ and $u \in U_c$, where the Lie derivative of $V$ corresponding to the input $u$ is defined by

$$\dot{V}_u(x) = \lim_{t \to +0} \frac{1}{t} (V(\phi(t, x, u)) - V(x)).$$

(9)

Furthermore, if

$$\lim_{\tau \to \infty} \alpha(\tau) = \infty \text{ or } \liminf_{\tau \to \infty} \alpha(\tau) \geq \lim_{\tau \to \infty} \sigma(\tau)$$

(10)

holds, system $V$ is called an ISS Lyapunov function.
The following Lyapunov-type theorem provides us with a way to prove iISS of (1).

**Proposition 1.** If there exist an iISS (resp. ISS) Lyapunov function for (1), then (1) is iISS (resp. ISS).

**Proof.** Let $\tau > 0$ be such that $[0, \tau)$ is the maximal time interval over which a system (1) admits a solution. For a given initial condition $\phi_0 \in X$ and a given input $u \in U$, let $y(t) = V(x(t))$, where $x(t) = \phi(t, \phi_0, u)$. With the help of (7), definition (9) and property (8) imply

$$\dot{y}(t) \leq -\alpha(\psi_2^{-1}(y(t))) + \sigma(\|u(t)\|_U)$$

for each $t \in [0, \tau)$. From [3, Corollary IV.3] it follows the existence of $\hat{\beta} \in KL$ satisfying

$$y(t) \leq \hat{\beta}(y(0), t) + \int_0^t 2\sigma(\|u(s)\|_U)ds.$$ 

Again, using (7) we have

$$\psi_1(\|x(t)\|_X) \leq \tilde{\beta}(\|x(0)\|_X, t) + \int_0^t 2\sigma(\|u(s)\|_U)ds,$$  \hspace{1cm} (11)

where $\tilde{\beta} = \hat{\beta}(\psi_2(\cdot), \cdot) \in KL$. Since $\psi_1$ is of class $K_\infty$ and satisfies $\psi_1^{-1}(a + b) \leq \psi_1^{-1}(2a) + \psi_1^{-1}(2b)$ for any $a, b \in \mathbb{R}_+$, we arrive at (6) for all $t \in [0, \tau]$ with $\theta(s) = \psi_1^{-1}(2s)$ and $\mu(s) = 2\sigma(s)$, and $\beta(s, \tau) = \psi_1^{-1}(2\hat{\beta}(s, \tau))$.

Although $\tau$ representing the endpoint of the maximal interval was assumed to be finite, property (11) implies that $\tau$ cannot be finite, because otherwise the solution $x(\cdot)$ will be unbounded near $t = \tau$ (under Assumption 1), due to a variation of [4, Theorem 4.3.4]. Therefore, property (6) must hold for all $t \geq 0$, and thus (1) is iISS.

Finally, we can prove ISS when (10) holds as in the finite-dimensional case [36], with the help of [7, Theorem 1 and Proposition 5]. \qed

In Definition 5 the ISS Lyapunov function is introduced in a so-called dissipative form in parallel to the iISS Lyapunov function. In the preceding study [7] on ISS, however, a so-called implicative form has been used as follows:

**Definition 6.** A continuous function $V : X \to \mathbb{R}_+$ is called an ISS Lyapunov function in an implicative form, if there exist $\psi_1, \psi_2 \in K_\infty$, $\eta \in K_\infty$ and $\gamma \in K$ such that (7) holds and system (1) satisfies

$$\|x\|_X \geq \gamma(\|u(0)\|_U) \Rightarrow \dot{V}_a(x) \leq -\eta(\|x\|_X)$$  \hspace{1cm} (12)

for all $x \in X$ and $u \in U$.

The dissipative and implicative definitions coincide for finite-dimensional systems (see [36, Remark 2.4, p. 353] and [31]). For infinite-dimensional systems, it can be verified that they are equivalent under some additional assumptions on nonlinearity $f$.

**Theorem 1.** Let

1. $f$ be Lipschitz on bounded subsets of $X$ uniformly w.r.t. the second argument.
2. $A$ generates an analytic semigroup.
3. $V$ be Fréchet differentiable in $X$ and its derivative $\frac{\partial V}{\partial x}$ be bounded on bounded balls.
4. \( f \) and \( V \) admit the existence of \( p \in \mathcal{K} \) and \( q \in \mathcal{K}_\infty \) satisfying
\[
\left\| \frac{\partial V}{\partial x} f(0, v) \right\|_X \leq p(\|x\|_X) + q(\|v\|_U), \quad \forall x \in X, \, v \in U.
\] (13)

Then \( V \) is an ISS Lyapunov function in an implicative form iff it is an ISS Lyapunov function in a dissipative form.

The proof is given in Appendix 7.2.

Remark 1. Importantly, for infinite-dimensional systems, posing (13) is fundamentally less restrictive than \( \|f(0, v)\|_X \leq q(\|v\|_U) \) since norms are in general not equivalent, as opposed to finite-dimensional systems.

In many cases it is hard to compute the derivative of a Lyapunov function for all \( x \in X \) directly, but it is much more convenient to differentiate \( V \) on some dense subspaces of \( X \) and \( U_c \), and to verify dissipation inequality (8) on the dense subspaces. This directly leads to iISS/ISS of a system only on the dense subspaces. We will show next, that under a natural assumption, this already implies iISS of the system on the original state and input spaces. For this purpose, let \( \Sigma := (X, U_c, \phi) \) denote system (1) defined with the transition map \( \phi \) corresponding to the spaces \( X, U_c \). Let \( \tilde{\Sigma} := (\tilde{X}, \tilde{U}_c, \tilde{\phi}) \) be system (1) defined with the same transition map \( \phi \) but restricted to the state space \( \tilde{X} \) and the input space \( \tilde{U}_c \) which are dense linear normed subspaces of \( X \) and \( U_c \), respectively, and endowed with the norms in original spaces \( \|\cdot\|_X \) and \( \|\cdot\|_{U_c} \), respectively.

When considering the approximations of dynamical systems, the following property is useful

Definition 7. We say that \( \Sigma \) depends continuously on inputs and on initial states, if \( \forall x \in X, \forall u \in U_c, \forall \tau > 0 \) and \( \forall \varepsilon > 0 \) there exist \( \delta = \delta(x, u, \tau, \varepsilon) > 0 \), such that
\[
\forall x' \in X : \|x - x'\|_X < \delta \quad \text{and} \quad \forall u' \in U_c : \|u - u'\|_{U_c} < \delta \quad \text{it holds}
\]
\[
\|\phi(t, x, u) - \phi(t, x', u')\|_X < \varepsilon, \quad \forall t \in [0, \tau].
\]

If under the same assumptions
\[
\|\phi(t, x, u) - \phi(t, x', u)\|_X < \varepsilon, \quad \forall t \in [0, \tau].
\]
holds, \( \Sigma \) is called continuously dependent on initial states.

The following result will be used frequently in the sequel.

Proposition 2. Let \( \Sigma \) depend continuously on inputs and on initial states and let \( \tilde{\Sigma} \) be iISS. Then \( \Sigma \) is also iISS with the same \( \beta, \theta \) and \( \mu \) in the estimate (6).

The proof is given in Appendix 7.1.

Often only approximations of the state space are needed, therefore we state another proposition whose proof is analogous to the proof of Proposition 2.

Proposition 3. Let \( \Sigma \) depend continuously on initial states, \( \tilde{\Sigma} \) be iISS and \( \tilde{U} = U \). Then \( \Sigma \) is also iISS with the same \( \beta, \theta \) and \( \mu \) in the estimate (6).

This result will be used several times in this paper.

4. iISS and ISS of bilinear systems. Throughout this section, let \( \mathcal{T} = \{T(t), t \geq 0\} \) be a \( C_0 \)-semigroup on a Banach space \( X \) with an infinitesimal generator \( A : D(A) \to X, Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x) \), whose domain of definition \( D(A) \) consists of those \( x \in X \), for which this limit exists.
4.1. **Linear systems.** We will begin with a class of linear systems. Consider system (1) with $f(x(t), u(t)) = Bu(t)$:

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
x(0) &= \phi_0,
\end{align*}
$$

where $x : \mathbb{R}_+ \to X$, $u : \mathbb{R}_+ \to U$, and $B : U \to X$ is a linear operator.

It is well-known that for linear finite-dimensional systems 0-GAS (local stability + global attractivity), 0-UGASs, ISS and iISS are identical properties [32, 33]. For infinite-dimensional systems, 0-GAS is not equivalent to 0-UGASs in general. In [7, Proposition 3] it was shown that 0-UGASs is equivalent to ISS provided that $B$ is a bounded operator. Moreover, in [7, p. 8] and [26, p. 247] examples of 0-GAS infinite-dimensional systems are provided whose solutions go to infinity even for inputs of arbitrarily small magnitude. Next we investigate relations between ISS and iISS for systems (14).

**Proposition 4.** Let $B \in L(U, X)$. Then, (14) is 0-UGASs $\iff$ (14) is ISS $\iff$ (14) is ISS w.r.t. $L_p(\mathbb{R}_+, U)$ for some $p \geq 1$.

**Proof.** The fact that (14) is 0-UGASs $\iff$ (14) is ISS has been proved in [7, Proposition 3]. Next, by definition, it is obvious that if (14) is ISS w.r.t. $L_p(\mathbb{R}_+, U)$ for a real number $p \geq 1$, then it is 0-UGASs. To prove the converse, let (14) be 0-UGASs. Then $T$ is an exponentially stable semigroup, see e.g. [7, Lemma 1]. Thus $\exists M, \lambda > 0 : \|T(t)\| \leq Me^{-\lambda t}$.

We estimate the solution $x(t) = \phi(t, \phi_0, u)$ of (14):

$$
\|x(t)\|_X = \|T(t)\phi_0 + \int_0^t T(t - r)Bu(r)dr\|_X \\
\leq \|T(t)\|\|\phi_0\|_X + \int_0^t \|T(t - r)\||B||u(r)||_U dr, \\
\leq Me^{-\lambda t}\|\phi_0\|_X + M\|B\| \int_0^t e^{-\lambda (t - r)}||u(r)||_U dr.
$$

Now, estimating $e^{-\lambda (t - r)} \leq 1$, $r \leq t$ we obtain that (14) is ISS w.r.t. $L_1(\mathbb{R}_+, U)$.

To prove the claim for $p > 1$, pick any $q \geq 1$ so that $\frac{1}{p} + \frac{1}{q} = 1$. We continue the above estimates, using the Hölder’s inequality

$$
\|x(t)\|_X \leq Me^{-\lambda t}\|\phi_0\|_X + M\|B\| \left( \int_0^t e^{-\frac{\lambda}{2} (t - r)} e^{-\frac{\lambda}{2} (t - r)} ||u(r)||_U dr \right)^{\frac{1}{2}} \\
\leq Me^{-\lambda t}\|\phi_0\|_X + M\|B\| \left( \int_0^t e^{-\frac{\lambda}{q} (t - r)} ||u(r)||_U^q dr \right)^{\frac{1}{q}} \\
\leq Me^{-\lambda t}\|\phi_0\|_X + M\|B\| \left( \frac{2}{q\lambda} \right)^{\frac{1}{q}} \left( \int_0^t ||u(r)||_U^q dr \right)^{\frac{1}{q}} \\
= Me^{-\lambda t}\|\phi_0\|_X + w\|u\|_{L_p(\mathbb{R}_+, U)},
$$

where $w := M\|B\| \left( \frac{2}{q\lambda} \right)^{\frac{1}{q}}$. This means that (14) is ISS w.r.t. $L_p(\mathbb{R}_+, U)$, $p \geq 2$.

The following statement is a consequence of Proposition 4 (by taking $\theta := \text{id}$ and $\mu := c \cdot \text{id}$ for large enough $c > 0$ in Definition 4):

**Corollary 1.** System (14) is ISS iff it is iISS.
For finite-dimensional systems, in the presence of nonlinearities which are locally Lipschitz w.r.t. state, 0-GAS implies local ISS [37, Lemma I.1], i.e., the ISS property for initial states and inputs with a sufficiently small norm. In contrast to this finite-dimensional fact, we next show an infinite-dimensional linear system illustrating that for unbounded operator $B$, 0-UGASs implies neither ISS nor iISS, even if the initial state and the input is restricted to sufficiently small neighborhoods of the origin.

**Example 2.** Consider the following ODE ensemble defined on the interval $(0, \pi/2)$ of the spatial variable $l$:

$$
\dot{x}(l, t) = -x(l, t) + (\tan l)\dot{x}(l, t), \quad l \in (0, \pi/2).
$$

Let $X = C(0, \pi/2)$ be a space of bounded continuous functions on $(0, \pi/2)$. The functions $x(l, t)$ and $u(l, t)$ are scalar-valued. The input operator $B : D(B) \to X$ for (15) is defined by $(Bv)(l) = (\tan l)\dot{x}(l)$ which is unbounded with a domain of definition

$$
D(B) = \{v \in C(0, \pi/2) : \sup_{l \in (0, \pi/2)} |(\tan l)\dot{x}(l)| < \infty\}.
$$

Since $x(\cdot, t) = e^{-t}x(\cdot, 0)$ holds for $u(\cdot, t) = 0$, $\forall t \geq 0$, system (15) is 0-UGASs. But it is neither ISS nor iISS for $U = D(B)$. To verify this fact, consider an input $u(l, t) = \hat{u}(l)$ given by

$$
\hat{u}(l) = \begin{cases} 
  b 
  & l < \arctan(e^b) \\
  bc(\tan l) - \frac{\pi}{2} 
  & l \geq \arctan(e^b)
\end{cases}
$$

for real $b, c > 0$. It is easy to see that $\hat{u} \in D(B)$ and $\|\hat{u}\|_U = b$ from $\|B\hat{u}\|_X = \sup_{t \in (0, \pi/2)} |\hat{u}(l)(\tan l)\dot{x}| = bc$ and the definition of $\hat{u}$. The solution of (15) for $\phi_0 = 0$ is computed as $\phi(t, 0, u) = \int_0^t e^{-(t-r)}\hat{u}(l)(\tan l)\dot{x} dr = (1 - e^{-t})\hat{u}(l)(\tan l)\dot{x}$.

Thus, by definition, the solution satisfies

$$
\sup_{t \in (0, \pi/2)} \phi(t, 0, u) = bc(1 - e^{-t}).
$$

Now, assume that system (15) is iISS. From Definition 4 it follows that there exist $\alpha, \mu \in \mathcal{K}_{\infty}$ satisfying $\|\phi(t, 0, u)\|_X \leq \alpha^{-1}(t\mu(b))$ for $t \geq 0$. Clearly, for any given $\alpha, \mu \in \mathcal{K}_{\infty}$ and any $t > 0$, one can find $c > 0$ so that $bc(1 - e^{-t}) > \alpha^{-1}(t\mu(b))$. Since $\|\hat{u}\|_U = b$ is satisfied for any $c > 0$, system (15) is not iISS.

Next, suppose that system (15) is ISS. Definition 3 implies the existence of $\gamma \in \mathcal{K}_{\infty}$ satisfying $\|\phi(t, 0, u)\|_X \leq \gamma(b)$ for $t \geq 0$ when $u = \hat{u}$ is applied to (15). For any given $\gamma \in \mathcal{K}_{\infty}$ and any $t > 0$, there exists $c > 0$ such that $bc(1 - e^{-t}) > \gamma(b)$. Hence, system (15) is not ISS either. Since we can take $\phi_0 = 0$ and $b > 0$ is arbitrary, the system (15) is neither iISS nor ISS even if the initial state and the input are restricted to arbitrarily small neighborhoods of the origin.

**Remark 2.** The construction of a control $\hat{u}$ in Example 2 indicates that the notions of global and local ISS coincide for systems (14). More precisely, system (14) is locally ISS, i.e. there exist $\delta > 0$ s.t. the inequality (5) holds only for $\phi_0 \in X : \|\phi_0\|_X \leq \delta$ and $u \in U_c : \|u\|_{U_c} \leq \delta$ iff (14) is ISS.

In the following example the same equation (15) is studied as in Example 2. It highlights the dependency of ISS and iISS on the choice of spaces.
Example 3. Consider the system (15) again. The system is ISS if we choose $X = L_2(0, \pi/2)$, $U = L_4(0, \pi/2)$.

Choose

$$V(x) = \int_0^{\pi/2} x^2(l) dl = \|x\|^2_{L_2(0, \pi/2)}$$

For the solutions $x(\cdot,t) = \phi(t, \phi_0, u)$ of (15) we obtain

$$\frac{d}{dt} V(x) = 2 \int_0^{\pi/2} x(l,t) \left( -x(l,t) + u(l,t)(\tan l) \frac{d}{dl} \right) dl$$

$$\leq -2V(x) + wV(x) + \frac{1}{w} \int_0^{\pi/2} u(l,t)^2(\tan l)^2 dl$$

$$\leq (-2 + w)V(x) + \frac{K}{w} \|u(\cdot,t)\|^2_{L_4(0,\pi)}$$

for any $w > 0$ (between lines 1 and 2 Young’s inequality has been used). Here, it is important to notice that $K := \int_0^{\pi/2} (\tan l)^2 dl < \infty$. Hence, taking $w < 2$, Proposition 1 proves that system (15) is ISS for $X = L_2(0, \pi/2)$ and $U = L_4(0, \pi/2)$.

4.2. iISS of generalized bilinear systems. While for linear infinite-dimensional systems with bounded input operators the properties of ISS and iISS coincide, the difference between these two properties arises for bilinear systems which is one of the simplest classes of nonlinear systems. For finite-dimensional bilinear systems, Sontag [33] demonstrated that 0-GAS systems are seldom ISS (for example, one-dimensional bilinear systems are never ISS), and that a system is 0-GAS if and only if it is iISS. To generalize this result to infinite-dimensional systems, consider

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)),$$

$$x(0) = \phi_0,$$  \hspace{1cm} (16)

where $B \in L(U, X)$, and $C : X \times U \to X$ is such that there exist $K > 0$ and $\xi \in K$ so that

$$\|C(x, u)\|_X \leq K \|x\|_X \xi(\|u\|_U).$$  \hspace{1cm} (17)

for all $x \in X$ and all $u \in U$.

Remark 3. This class of includes systems with $C$ linear in both variables and bounded in the sense that $\|C\| := \sup_{\|x\|_X = 1, \|u\|_U = 1} \|C(x, u)\|_X < \infty$ (then $K = \|C\|$ and $\xi(r) = r$ for all $r \in \mathbb{R}_+$).

In this class, there are systems which are not ISS. The following is a simple example of such a system.

Example 4. Consider the following parabolic PDE:

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial l^2}(l, t) + x(l, t)u(l, t), \ (l, t) \in (0, 1) \times (0, \infty)$$

$$x(0, t) = x(1, t) = 0.$$  \hspace{1cm} (18)

with a state space $X = C_0[0, 1]$ (functions, continuous on $[0, 1]$ and zero on the boundary) and input space $U = C(0, 1)$. A closure $A_c$ of an operator $A_c = \frac{\partial^2}{\partial l^2} + cl$ with a domain of definition $D(A) = C^2(0, 1) \cap C_0[0, 1]$ generates an analytic semigroup and the system (18) is bilinear in the above sense.
Operator $A_c$ has a spectrum $\{-(\pi n)^2 + c\mid n \geq 1\}$. Taking $u \equiv c > \pi^2$, we obtain a system whose spectral bound is positive, and thus there are solutions of (18), which grow exponentially to infinity. Thus (18) is not ISS.

Next we prove the equivalence between 0-UGASs and iISS for the general system (16) in Banach spaces. For infinite-dimensions, we employ the notion of 0-UGASs instead of 0-GAS. To establish iISS from 0-GAS for finite-dimensional systems, Sontag [33] constructed Lyapunov functions for systems with Hurwitz $A$ by means of the Lyapunov equation. To the best of authors’ knowledge, there is no generalization of the method for construction of Lyapunov functions by means of the Lyapunov equation for linear systems on Banach spaces, although for systems on Hilbert spaces such a construction exists. Thus, to allow for Banach spaces, we employ another method to prove that iISS is equivalent to 0-UGASs.

**Theorem 5.** System (16) is iISS $\iff$ (16) is 0-UGASs.

**Proof.** Clearly, iISS implies 0-UGASs. To prove the converse, assume that (16) be 0-UGASs, that is let $T$ be an exponentially stable semigroup. Integrating (16), we obtain

$$x(t) = T(t)x(0) + \int_0^t T(t-r)(Bu(r) + C(x(r), u(r)))\,dr.$$  

From $B \in L(U, X)$, inequality (17) and since $T$ is an exponentially stable semigroup it follows for some $K, M, \lambda > 0$, that

$$\|x(t)\|_X \leq \|T(t)\|\|x(0)\|_X + \int_0^t \|T(t-r)\| (\|B\|\|u(r)\|_U + \|C(x(r), u(r))\|_X)\,dr$$

$$\leq Me^{-\lambda t}\|x(0)\|_X + \int_0^t Me^{-\lambda(t-r)} (\|B\|\|u(r)\|_U + K\|x(r)\|_X\|u(r)\|_U)\,dr.$$  

We multiply both sides of the inequality by $e^{\lambda t}$ and define $z(t) = x(t)e^{\lambda t}$. From $\lambda > 0$ we obtain

$$\|z(t)\|_X \leq \|z(0)\|_X + \|B\| \int_0^t e^{\lambda r}\|u(r)\|_U\,dr + \int_0^t MK\|z(r)\|_X\|u(r)\|_U\,dr.$$  

Since $q : t \mapsto M(\|z(0)\|_X + \|B\| \int_0^t e^{\lambda r}\|u(r)\|_U\,dr)$ is a non-decreasing function, Gronwall’s inequality (see e.g. [38, Lemma 2.7, p.42]) yields

$$\|z(t)\|_X \leq \|z(0)\|_X + \|B\| \int_0^t e^{\lambda r}\|u(r)\|_U\,dr e^{0}MK\|z(0)\|_X\|u(r)\|_U\,dr.$$  

Coming back to original variables and using $\lambda > 0$, we have

$$\|x(t)\|_X \leq \|e^{-\lambda t}\|\|x(0)\|_X + \|B\| \int_0^t e^{-\lambda(t-r)}\|u(r)\|_U\,dr e^{0}MK\|z(0)\|_X\|u(r)\|_U\,dr$$

$$\leq \|e^{-\lambda t}\|\|x(0)\|_X + \|B\| \int_0^t \|u(r)\|_U\,dr e^{0}MK\|z(0)\|_X\|u(r)\|_U\,dr.$$  

Applying to the both sides the function $\alpha \in K_\infty$ defined for all $r \geq 0$ as $\alpha(r) = \ln(1 + r)$ results in

$$\alpha(\|x(t)\|_X) \leq \ln \left(1 + M(\|x(0)\|_X + \|B\| \int_0^t \|u(r)\|_U\,dr) e^{0}MK\|z(0)\|_X\|u(r)\|_U\,dr\right).$$  

Now since for all $a, b \in \mathbb{R}_+$ it holds that

$$\ln(1 + ae^b) \leq \ln((1 + a)e^b) = \ln(1 + a) + b,$$  

we have
we obtain
\[
\alpha(\|x(t)\|_X) \leq \ln \left( 1 + M \left( e^{-\lambda t} \|x(0)\|_X + \|B\| \int_0^t \|u(r)\|_U dr \right) \right) \\
+ \int_0^t MK\xi(\|u(r)\|_V) dr.
\]
Moreover, for all \(a,b \in \mathbb{R}^+_+\) it holds that
\[
\ln(1 + a + b) \leq \ln((1 + a)(1 + b)) = \ln(1 + a) + \ln(1 + b),
\]
which implies
\[
\alpha(\|x(t)\|_X) \leq \ln \left( 1 + Me^{-\lambda t} \|x(0)\|_X \right) \\
+ \ln \left( 1 + M \|B\| \int_0^t \|u(r)\|_U dr \right) + \int_0^t MK\xi(\|u(r)\|_V) dr.
\]
Since \(\beta : (s,t) \mapsto \ln(1 + Me^{-\lambda s})\) is a \(KL\)-function, with the help of \(\alpha^{-1}(a + b) \leq \alpha^{-1}(2a) + \alpha^{-1}(2b)\) holding for any \(a,b \in \mathbb{R}^+_+\), the above estimate shows us that (16) is iISS as defined in Definition 3.

4.3. Lyapunov functions for generalized bilinear systems. This section develops a method to construct an iISS Lyapunov function for the infinite-dimensional system (16) analogous to the finite-dimensional case [33]. For this purpose, in this section, let \(X\) be a Hilbert space with a scalar product \(\langle \cdot, \cdot \rangle\). Note that if (16) is 0-UGASs, the operator \(A\) generates exponentially stable semigroup [7, Lemma 1]. Since \(X\) is a Hilbert space, the exponential stability of this semigroup is equivalent to existence of a positive self-adjoint operator \(P \in L(X)\) satisfying the Lyapunov equation
\[
\langle Ax, P x \rangle + \langle Px, Ax \rangle = -\|x\|_X^2, \quad \forall x \in D(A) \tag{19}
\]
see [5, Theorem 5.1.3, p. 217]. Recall that a self-adjoint operator \(P \in L(X)\) is said to be positive if \(\langle P x, x \rangle > 0\) holds for all \(x \in X \setminus \{0\}\). A positive operator \(P \in L(X)\) is called coercive if there exists \(k > 0\) such that
\[
\langle P x, x \rangle \geq k\|x\|_X^2 \quad \forall x \in D(P).
\]

**Theorem 6.** Consider a system (16) over a Hilbert space \(X\). Let Assumption 1 hold and let there exists a coercive positive self-adjoint operator \(P \in L(X)\) satisfying (19). If \(A\) generates an analytic semigroup, then a system (16) is iISS and its iISS Lyapunov function can be constructed as
\[
W(x) = \ln \left( 1 + \langle P x, x \rangle \right). \tag{20}
\]
If \(A\) doesn’t necessarily generate an analytic semigroup, but all the trajectories, emanating from \(D(A)\) under arbitrary input \(u \in \mathcal{U}_c\) remain in \(D(A)\), then (16) is still iISS and \(W\) is its iISS Lyapunov function on \(D(A)\).

**Proof.** Let assumptions of the theorem hold. Consider a function \(V : x \mapsto \langle P x, x \rangle\). Since \(P\) is bounded and coercive, for some \(k > 0\) it holds
\[
k\|x\|_X^2 \leq V(x) \leq \|P\|\|x\|_X^2, \quad \forall x \in X,
\]
and property (7) is verified. Let us compute the Lie derivative of \( V \) with respect to system (1). For \( x \in D(A) \) we have

\[
\dot{V}(x) = \langle P \dot{x}, x \rangle + \langle Px, \dot{x} \rangle \\
= \langle PAx + Bu + C(x, u), x \rangle \\
+ \langle Px, Ax + Bu + C(x, u) \rangle \\
= \langle PAx, x \rangle + \langle Px, Ax \rangle \\
+ \langle (Bu + C(x, u)), x \rangle + \langle Px, Bu + C(x, u) \rangle .
\]

From \( \langle PAx, x \rangle = \langle Ax, Px \rangle \), (17) and (19) with the help of Cauchy-Schwarz inequality, we obtain

\[
\dot{V}(x) \leq - \|x\|^2_\mathcal{X} + \|P(Bu + C(x, u))\|_X \|x\|_X \\
+ \|Px\|_X \|Bu + C(x, u)\|_X \\
\leq - \|x\|^2_\mathcal{X} + \|P\| \|Bu + C(x, u)\|_X \|x\|_X \\
+ \|P\| \|x\|_X \|Bu + C(x, u)\|_X \\
\leq - \|x\|^2_\mathcal{X} + 2\|P\| \|x\|^2_\mathcal{X} \xi(\|u\|_U) + 2\|P\| \|B\| \|x\|_X \|u\|_U.
\]

Let \( \varepsilon > 0 \). Using Young’s inequality

\[
2\|x\|_X \|u\|_U \leq \varepsilon \|x\|^2_\mathcal{X} + \frac{1}{\varepsilon} \|u\|^2_U,
\]

we can continue the above estimates as

\[
\dot{V}(x) \leq - \left(1 - \varepsilon \|P\| \|B\| \right) \|x\|^2_\mathcal{X} + 2\varepsilon \|P\| \|x\|^2_\mathcal{X} \xi(\|u\|_U) \\
+ \frac{\|P\| \|B\|}{\varepsilon} \|u\|^2_U.
\]

Defining \( W \) as in (20) yields

\[
\dot{W}(x) \leq \frac{1}{1 + V(x)} \left[ - \left(1 - \varepsilon \|P\| \|B\| \right) \|x\|^2_\mathcal{X} \\
+ 2\varepsilon \|P\| \|x\|^2_\mathcal{X} \xi(\|u\|_U) + \frac{\|P\| \|B\|}{\varepsilon} \|u\|^2_U \right] \\
\leq - \left(1 - \varepsilon \|P\| \|B\| \right) \frac{\|x\|^2_\mathcal{X}}{1 + \dot{V}(x)} \\
+ \frac{2\varepsilon \|P\| \|x\|^2_\mathcal{X} \xi(\|u\|_U)}{1 + k \|x\|^2_\mathcal{X}} + \frac{\|P\| \|B\|}{\varepsilon} \|u\|^2_U,
\]

which finally leads to

\[
\dot{W}(x) \leq - \left(1 - \varepsilon \|P\| \|B\| \right) \frac{\|x\|^2_\mathcal{X}}{1 + \|P\| \|x\|^2_\mathcal{X}} \\
+ \frac{2\varepsilon \|P\| \xi(\|u\|_U)}{k} + \frac{\|P\| \|B\|}{\varepsilon} \|u\|^2_U. \tag{21}
\]

These derivations hold for \( x \in D(A) \subset X \). If \( x \notin D(A) \), then for all admissible \( u \) the solution \( x(t) \in D(A) \) and \( t \to W(x(t)) \) is a continuously differentiable function for
all $t > 0$ (these properties follow from the properties of solutions $x(t)$, see Theorem 3.3.3 in [11]). Therefore, by the mean-value theorem, $\forall t > 0 \exists t_* \in (0, t)$

$$\frac{1}{t} (W(x(t)) - W(x)) = \dot{W}(x(t_*)),$$

where $x = x(0)$. Taking the limit when $t \to +0$ we obtain that (21) holds for all $x \in X$. Pick $\varepsilon > 0$ such that $\varepsilon < 1/(\|P\|\|B\|)$. According to Proposition 1, system (16) is ISS and $W$ is an ISS Lyapunov function.

Let now all the trajectories, emanating from $D(A)$ under arbitrary input $u \in U_c$ remain in $D(A)$. This means, that (16) generates a control system on $D(A)$ with inputs $U_c$. According to above argument $W$ is an ISS Lyapunov function for this restricted system, which shows its ISS. Similarly to [4, Proposition 4.3.7] one can prove, that Assumption 1 implies that (16) depends continuously on initial data. Thus, due to Proposition 3 ISS of (16) for state from $D(A)$ and inputs belonging to $U$ implies ISS of (16) on all spaces $X, U$.

As we see the proof of the previous theorem consists basically of two parts: the verification that $W$ is an ISS Lyapunov function for $x \in D(A)$ and then use of a density argument for analytic semigroups. We will see that this strategy will be useful also in the proof of Theorem 1.

Remark 4. Note, that in the case when $A$ generates a nonanalytic semigroup we do not claim that $W$ is an ISS Lyapunov function on the whole space $X$, since we do not know the Lie derivative $\dot{W}_u$ for $x \notin D(A)$.

5. An example. In this concluding section we give an example of an ISS system. Let $c > 0$ and $L > 0$. Consider the following reaction-diffusion system

$$\begin{cases}
\frac{\partial x}{\partial t} (l, t) = c \frac{\partial^2 x}{\partial l^2} (l, t) + \frac{x(l, t)}{1 + |l - 1| x(l, t)^2} u(l, t), \\
x(0, t) = x(L, t) = 0;
\end{cases} \tag{22}$$

on the region $(l, t) \in (0, L) \times (0, \infty)$ of the $\mathbb{R}$-valued functions $x(l, t)$ and $u(l, t)$.

Let $X = L^2(0, L)$ and $U = C(0, L)$. It is easy to see that the above system is bilinear since its nonlinearity satisfies inequality (17). Clearly, this system is UGASs, therefore it is ISS for any $L > 0$. Below we give an explicit construction of an ISS Lyapunov function for this system. Afterwards we will prove that this system is ISS for $L < 1$.

Define

$$W(x) = \int_0^L x^2(l) dl = \|x\|^2_{L^2(0, L)}.$$

Since $1 + |l - 1| x(l, t)^2 \geq 1$, we obtain

$$W(x) = 2 \int_0^L x(l) \left( c \frac{\partial^2 x}{\partial l^2} (l, t) + \frac{x(l, t)}{1 + |l - 1| x(l, t)^2} u(l, t) \right) dl \leq -2c \int_0^L \left( \frac{\partial x}{\partial l} (l, t) \right)^2 dl + 2 \int_0^L x^2(l, t) |u(l, t)| dl.$$

Using the Friedrich’s inequality (see e.g. [28, p. 67]) in the first term, we continue estimates:

$$W(x) \leq -2c \left( \frac{\pi}{L} \right)^2 W(x) + 2W(x)\|u\|_{C(0, L)}$$
Choosing
\[ V(x) = \ln(1 + W(x)) \] (23)
yields
\[
\dot{V}(x) \leq -2c\left(\frac{\pi}{L}\right)^2 \frac{W(x)}{1 + W(x)} + 2 \frac{W(x)}{1 + W(x)} \|u\|_{C(0,L)}
\]
\[
\leq -2c\left(\frac{\pi}{L}\right)^2 \frac{\|x\|_{L^2(0,L)}^2}{1 + \|x\|_{L^2(0,L)}^2} + 2\|u\|_{C(0,L)},
\]
\[
= -\alpha(\|x\|_{L^2(0,L)}) + \sigma(\|u\|_{C(0,L)}),
\]
where
\[
\alpha(s) = 2c\left(\frac{\pi}{L}\right)^2 \frac{s^2}{1 + s^2}, \quad \sigma(s) = 2s.
\] (25)
Thus, Proposition 1 establishes iISS of (22) irrespective of a value of \( L \).

**Remark 5.** Since \( x(\cdot, t) \in L^2(0, L) \), the spatial derivative of \( x \) above may not exist. However, the above derivations hold for smooth enough functions \( x \), and the general result for all \( x(\cdot, t) \in L^2(0, L) \) will follow due to Proposition 2.

Interestingly, when \( L < 1 \), the system (22) is ISS for the input space \( U = C(0, L) \) as well as \( U = L^2(0, L) \). To verify this, we first note that
\[
\sup_{s \in \mathbb{R}} \left| \frac{s}{1 + |l - 1|s^2} \right| = \frac{1}{2\sqrt{1 - l}}
\] (26)
holds for \( l < 1 \). Assume \( L < 1 \). Using the same Lyapunov function \( W \) we obtain
\[
\dot{W}(x) \leq 2\int_0^L x(l, t) \frac{\partial^2 x}{\partial l^2}(l, t) dl + 2\int_0^L \frac{1}{2\sqrt{1 - l}}|x(l, t)u(l, t)| dl
\]
\[
\leq -2c\left(\frac{\pi}{L}\right)^2 \frac{\|x\|_{L^2(0,L)}^2}{\sqrt{1 - L}} + \frac{1}{\sqrt{1 - L}} \|x\|_{L^2(0,L)} \|u\|_{L^2(0,L)}
\]
\[
\leq -\left(2c\left(\frac{\pi}{L}\right)^2 - w\right) \|x\|_{L^2(0,L)}^2 + \frac{1}{4(1 - L)} \|u\|_{L^2(0,L)}^2
\] (27)
for \( 0 < w < 2c(\pi/L)^2 \). Recall that \( \|u\|_{L^2(0,L)}^2 \leq L \|u\|_{C(0,L)}^2 \). Thus, by virtue of Proposition 1, system (22) is ISS whenever \( L < 1 \). It is stressed that the coefficient of \( \|u\|_{L^2(0,L)}^2 \) in (27) goes to \( \infty \) as \( L \) tends to \( 1 \) from below. Hence, the ISS estimate (27) is valid only if \( L < 1 \).

For the choice of input space \( U = L^p(0,1) \) with \( p \geq 1 \), the case of \( L \geq 1 \) does not allow us to have an ISS estimate like (27). In fact, if \( L \geq 1 \) and \( U = L^p(0,1) \) for any \( p \geq 1 \), the right hand side of (22) system is undefined. To see this take \( u : l \mapsto |l - 1|^{-\frac{1}{2p}} \in L^p(0,L) \) and \( x : l \mapsto |l - 1|^{-\frac{1}{2p} + \frac{1}{p}} \in L^2(0,L) \). Then \( f(x,u) : l \mapsto \frac{x(l,t)}{|l - 1| |x(l,t)|^2} u(l,t) \notin L^2(0,L) \). Thus, according to our formulation of (1), the system (22) is not well-defined for \( U = L^p(0,L) \) for any real \( p \geq 1 \).

For the choice of input space \( U = C(0,L) \), we expect that the system (22) is not ISS for \( L \geq 1 \), but we have not proved it at this time. The blow-up of the \( \sigma \)-term in (27) corresponding to the dissipation inequality (12) for \( V = W \) suggests the absence of ISS for the system (22) provided \( L \geq 1 \). The notion of iISS describes the absence of ISS with a bounded decay rate \( \alpha \) satisfying \( \lim_{s \to \infty} \alpha(s) < \lim_{s \to \infty} \sigma(s) \), and it allows the dissipation inequality (12) to be well-defined for all \( u \in U \) and
all \( x \in X \) as in the iISS estimate (24), i.e., (25) that is valid uniformly over all \( L > 0 \). Being able to uniformly characterize iISS irrespectively of whether systems are ISS or not should be advantageous in many applications. For instance, ISS of subsystems is not always necessary for stability of their interconnections, and there are examples of UGAS interconnections involving iISS systems which are not ISS [12, 14, 2, 23].

6. Conclusion. We have proved that infinite-dimensional bilinear systems described by differential equations in Banach spaces are integral input-to-state stable provided they are uniformly globally asymptotically stable. For the systems whose state space is Hilbert we have obtained under some additional restrictions, another proof of this result, which leads to a construction of an iISS Lyapunov function for the system.

The possible directions for future research are investigation of iISS of more general nonlinear control systems and development of novel methods for construction of iISS Lyapunov functions for such systems. Another challenging problem is a study of interconnected infinite-dimensional systems, whose subsystems are iISS or ISS.

7. Appendix.

7.1. Proof of Proposition 2. Since \( \hat{\Sigma} \) is iISS, there exist \( \beta \in \mathcal{K} \mathcal{L} \) and \( \mu \in \mathcal{K} \), \( \theta \in \mathcal{K}_\infty \), such that \( \forall \hat{x} \in \hat{X}, \forall \hat{u} \in \hat{U}_c \) and \( \forall t \geq 0 \) it holds that

\[
\| \phi(t, \hat{x}, \hat{u}) \|_X \leq \beta(\| \hat{x} \|_X, t) + \theta \left( \int_0^t \mu(\| \hat{u}(s) \|_U) ds \right). \tag{28}
\]

Let \( \Sigma \) be not iISS with the same \( \beta, \mu, \theta \). Then there exist \( t^* > 0, x \in X, u \in U_c \):

\[
\| \phi(t^*, x, u) \|_X = \beta(\| x \|_X, t^*) + \theta \left( \int_0^{t^*} \mu(\| u(s) \|_U) ds \right) + r,
\]

where \( r = r(t^*, x, u) > 0 \). From (28) and (29) we obtain

\[
\begin{align*}
\| \phi(t^*, x, u) \|_X - \| \phi(t^*, \hat{x}, \hat{u}) \|_X & \geq \beta(\| x \|_X, t^*) - \beta(\| \hat{x} \|_X, t^*) \\
& \quad + \theta \left( \int_0^{t^*} \mu(\| u(s) \|_U) ds \right) - \theta \left( \int_0^{t^*} \mu(\| \hat{u}(s) \|_U) ds \right) + r. \tag{30}
\end{align*}
\]

Since \( \hat{X} \) and \( \hat{U}_c \) are dense in \( X \) and \( U_c \) respectively, and since operator \( u \mapsto \theta \left( \int_0^{t^*} \mu(\| u(s) \|_U) ds \right) \) is continuous, we can find sequences \( \{ \hat{x}_i \} \subset \hat{X} \) : \( \| x - \hat{x}_i \|_X \to 0 \) and \( \{ \hat{u}_i \} \subset \hat{U}_c \) : \( \| u - \hat{u}_i \|_U \to 0 \). From (30) it follows that for each arbitrary \( \varepsilon > 0 \), there exist \( \hat{x}_i \) and \( \hat{u}_i \) such that

\[
\| \phi(t^*, x, u) - \phi(t^*, \hat{x}_i, \hat{u}_i) \|_X \geq | | \phi(t^*, x, u) \|_X - \| \phi(t^*, \hat{x}_i, \hat{u}_i) \|_X | \geq r - 2\varepsilon.
\]

This contradicts to the assumption of continuous dependence of \( \Sigma \) on initial states and inputs. Thus, \( \Sigma \) is iISS with the same \( \theta, \beta \) and \( \mu \) in (6).

7.2. Proof of Theorem 1. First, assume that \( V \) is an ISS Lyapunov function in an implicitive formulation for (1). Pick any \( x \in X \) and \( u \in U_c \) s.t. \( \| x \|_X \leq \gamma(\| u(0) \|_U) \). Since \( V \) is Fréchet differentiable in \( X \) and since for \( x \in D(A) \) the trajectory \( \phi(\cdot, x, u) \).
is differentiable, \( V(x(t)) \) is also differentiable (see [1, par. 2.2]) and can be computed as

\[
\dot{V}_u(x) = \frac{\partial V}{\partial x}(x)(Ax + f(x, u(0))) \\
= \frac{\partial V}{\partial x}(x)(Ax + f(x, 0)) + \frac{\partial V}{\partial x}(x)(f(x, u(0)) - f(x, 0)) \\
\leq -\eta(\|x\|_X) + \left\| \frac{\partial V}{\partial x}(x)(f(x, u(0)) - f(x, 0)) \right\|_X,
\]

(31)

Here \( \frac{\partial V}{\partial x}(x) \) denotes a Fréchet derivative of \( V \) at point \( x \in X \) (which is a bounded linear operator from \( X \) to \( \mathbb{R} \) with an operator norm \( \| \cdot \| \)). Since \( \frac{\partial V}{\partial x}(x) \) is bounded on bounded balls, there exists \( q_2 \in \mathcal{K} \) such that

\[
\left\| \frac{\partial V}{\partial x}(x) \right\| \leq q_2(\|x\|_X).
\]

Moreover, since \( f \) is Lipschitz w.r.t. \( x \) on bounded subsets of \( X \), we have

\[
\|f(x, u(0)) - f(0, u(0))\|_X \leq w(\|x\|_X)\|x\|_X
\]

for some continuous non-decreasing function \( w : \mathbb{R}_+ \to \mathbb{R}_+ \). Thus, we have

\[
\left\| \frac{\partial V}{\partial x}(x)(f(x, u(0)) - f(0, u(0))) \right\|_X \leq \hat{w}(\|x\|_X)\|x\|_X
\]

for some \( \hat{w} \in \mathcal{K} \). This implies

\[
\left\| \frac{\partial V}{\partial x}(x)(f(x, u(0)) - f(0, u(0))) \right\|_X \leq \hat{w}(\|x\|_X)\|x\|_X + \left\| \frac{\partial V}{\partial x}(x)(f(x, u(0)) - f(0, u(0))) \right\|_X
\]

\[
\leq 2\hat{w}(\|x\|_X)\|x\|_X + \left\| \frac{\partial V}{\partial x}(x)(f(x, u(0)) - f(0, u(0))) \right\|_X.
\]

Due to (13) we proceed from (31) to

\[
\dot{V}_u(x) \leq -\eta(\|x\|_X) + 2\hat{w}(\|x\|_X)\|x\|_X + p(\|x\|_X) + q(\|u(0)\|_U).
\]

And for \( \|x\|_X \leq \gamma(\|u(0)\|_U) \) we obtain finally

\[
\dot{V}_u(x) \leq -\eta(\|x\|_X) + \sigma(\|u(0)\|_U)
\]

(32)

with \( \sigma(r) := 2\hat{w}(\gamma(r))\gamma(r) + p(\gamma(r)) + q(r) \) which is of class \( \mathcal{K}_\infty \). Pick \( \alpha = \eta \), which is of class \( K_\infty \) due to Definition 6. Combining (32) with the implication (12) we obtain that for all \( x \in D(A) \) and all \( u \in U_c \)

\[
\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U),
\]

i.e. \( V \) is an ISS Lyapunov function in a dissipative form (8) for states from \( D(A) \).

Since \( A \) generated an analytic semigroup, we can apply a density argument for analytic semigroups as in the proof of Theorem 6 to prove that \( V \) is an ISS Lyapunov function in a dissipative form for (1) on the whole \( X \).

Now let us prove the converse. Basically we can follow the argument for finite-dimensional systems. Suppose that \( V \) is an ISS Lyapunov function in a dissipative
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formulation, i.e., (8). Due to $\sigma \in \mathcal{K}$, property (10) ensures the existence of $\hat{\alpha} \in \mathcal{K}$ such that

$$\hat{\alpha}(s) \leq \alpha(s), \quad \forall s \in \mathbb{R}_+$$

$$\lim_{\tau \to \infty} \hat{\alpha}(\tau) \geq \lim_{\tau \to \infty} \sigma(\tau).$$

If either $\lim_{\tau \to \infty} \hat{\alpha}(\tau) = \infty$ or $\lim_{\tau \to \infty} \hat{\alpha}(\tau) > \lim_{\tau \to \infty} \sigma(\tau)$ holds, we can pick a constant $K > 1$ such that $\lim_{\tau \to \infty} \hat{\alpha}(\tau) \geq K \lim_{\tau \to \infty} \sigma(\tau)$. Then $V$ achieves (12) with $\gamma = \hat{\alpha}^{-1} - K\sigma \in \mathcal{K}$ and $\eta = (1 - 1/K)\hat{\alpha} \in \mathcal{K}$. In the case of $\infty > \lim_{\tau \to \infty} \hat{\alpha}(\tau) = \lim_{\tau \to \infty} \sigma(\tau)$, there exists a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $id + \omega \in \mathcal{K}_\infty$ and

$$\omega(\sigma(s)) > 0, \quad s \in (0, \infty)$$

$$\lim_{\tau \to \infty} \omega(\sigma(\tau)) = 0.$$

Then property (8) with $\gamma = \hat{\alpha}^{-1} - (id + \omega) \circ \sigma \in \mathcal{K}_\infty$ yields (12) with $\eta = (id - (id + \omega)^{-1}) \circ \hat{\alpha} \in \mathcal{P}$. The function $\eta$ obtained in the above two cases is only guaranteed to satisfy either $\eta \in \mathcal{K}$ or $\eta \in \mathcal{P}$. The function $V$ can be transformed into another continuous function $\hat{V} : X \to \mathbb{R}_+$ by applying the technique in [31] to obtain $\eta \in \mathcal{K}_\infty$ in (12). The transformed $\hat{V}$ is an ISS Lyapunov function of (1) in an implicit form.

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