On the dynamical generation of the Maxwell term and scale invariance.

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Abstract: Gauge theories with no Maxwell term are investigated in various setups. The dynamical generation of the Maxwell term is correlated to the scale invariance properties of the system. This is discussed mainly in the cases where the gauge coupling carries dimensions. The term is generated when the theory contains a scale explicitly, when it is asymptotically free and in particular also when the scale invariance is spontaneously broken. The terms are not generated when the scale invariance is maintained. Examples studied include the large $N$ limit of the $CP^{N-1}$ model in $(2+\epsilon)$ dimensions, a 3D gauged $\phi^6$ vector model and its supersymmetric extension. In the latter case the generation of the Maxwell term at a fixed point is explored. The phase structure of the $d = 3$ case is investigated in the presence of a Chern-Simons term as well. In the supersymmetric $\phi^6$ model the emergence of the Maxwell term is accompanied by the dynamical generation of the Chern-Simons term and its multiplet and dynamical breaking of the parity symmetry. In some of the phases long range forces emerge which may result in logarithmic confinement. These include a dilaton exchange which plays a role also in the case when the theory has no gauge symmetry. Gauged Lagrangian realizations of the 2D coset models do not lead to emergent Maxwell terms. We discuss a case where the gauge symmetry is anomalous.
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1. Introduction

The idea of a local gauge symmetry and the experimental discovery of several gauge particles have played a crucial role in creating and establishing the standard model. The amazing success of the concept has not prevented over the years casting doubts on how fundamental the concept is. These range from the Kaluza-Klein theories in which the four dimensional gauge invariance is nothing but the shadow of the five dimensional general
covariance, through suggesting that the gauge particles are not themselves elementary objects but only bound states \[^{\text{4, 5, 6}}\] to pointing out that the local symmetry presentation is just a redundancy. In the language of the early 21st century, gauge symmetries could be emergent.

Among the models realizing these ideas is a representation of \( \mathcal{N} = 8 \) supergravity. There are several recent suggestions of indications that the theory although seemingly superficially non renormalizable is actually finite \[^{\text{6}}\]. Even though there are cautionary objections to this claim \[^{\text{7}}\] it would be interesting to speculate on what would be the possible spectrum of such a finite theory. In particular there were suggestions that some, but perhaps not all \[^{\text{5}}\], of its gauge symmetries do get realized by effective propagating gauge bosons the number of which could be appropriate for the standard model. This should happen at some yet to be determined fixed point or fixed surface in some theories of gravity. In this case it would be satisfying if all scales are dynamically generated.

This leads us to reexamine a class of models in which the gauge symmetry itself is present a priori, i.e. is not emergent, but the original Lagrangian does not contain dynamical gauge fields. The gauge fields have no Maxwell term to start with. In general the experience with field theory is that any term which is not forbidden by some symmetry will emerge in the renormalization process even if absent in the classical Lagrangian. Thus also the Maxwell term should emerge in a generic gauge invariant theory.

A symmetry which could enforce the absence of the Maxwell term is scale/conformal invariance in a system in which the gauge coupling is not dimensionless. A particular example of that are the so called coset models of two dimensional Conformal Field Theory(CFT) \[^{\text{9}}\]. Their Lagrangian realizations \[^{\text{10, 11, 12}}\] involve locally gauging some of the global currents of a conformal theory with some algebraic structure, such as WZW models, without adding a Maxwell term for the gauge fields. The theory flows from one CFT to another(with a lower central charge) shedding off its massive confined states, while the conformal symmetry prevents the Maxwell term from emerging.

In this paper we wish to examine also if the Maxwell term emerges in the case that there is a scale symmetry but it is spontaneously broken, such perhaps would be the circumstances if \( \mathcal{N} = 8 \) or some similar theory turn out to be finite and generate dynamically the scales needed for gravity.

The structure of the paper is to first review in section 2 how in the presence of massive dynamical fields the Maxwell term emerges when the gauge coupling carries a dimension.

In section 3 we add to the study of the coset models an examination of the \( CP^{N-1} \) model \[^{\text{13, 14}}\] in \((2 + \epsilon)\) dimensions above the region of power counting renormalizability of the model, in particular at its conformal point.

In sections 4 and 5 we study a gauged version of the \( O(N) \) conformal vector model in three dimensions without \[^{\text{15}}\] and with \[^{\text{16, 17}}\] \( \mathcal{N} = 1 \) supersymmetry (SUSY) respectively. The models exhibit a phase in which the scale symmetry is spontaneously broken. We uncover an extra subtlety in these models: the large \( N \) and the IR do not commute. The dilaton plays a special role, note that this massless particle is a singlet under the group and thus the low energy effective Lagrangian consists only of singlets. Some consequences of these facts are discussed. This is studied also in the presence of a Chern-Simons(CS) term,
see e.g. [21] for a comprehensive review of the subject. While in the nonsupersymmetric case one has the freedom to either introduce the CS term into the action or not, in SUSY it emerges dynamically even if it was absent ab-initio. In particular, the parity symmetry is dynamically broken and the emergence of the Maxwell and CS terms is accompanied by a dynamical generation of their superpartners - the gaugino kinetic and mass terms respectively.

In section 6 we return to the $G/H$ coset models, but this time gauging, what was carefully avoided in the past [22], anomalous subgroups $H$ of $G$. The conclusion section includes a short discussion of the 4D cases in which the gauge coupling is dimensionless.

2. Dynamical generation of the Yang-Mills term in $d$ dimensions.

In this section we consider a gauge invariant action of massive complex scalar fields $\Phi$ in a $d$-dimensional spacetime which does not contain a Yang-Mills term for the gauge fields. The action is

$$S(\Phi, A) = \int \left[ D_\mu \Phi D^\mu \Phi + m^2 \Phi \Phi \right] d^d x, \quad (2.1)$$

where $D_\mu = \partial_\mu + i A_\mu$ is the covariant derivative and $m$ is the mass of the field $\Phi$. In particular, for $SU(N_c) A_\mu = A_\mu^a T^a$, $1 \leq a \leq N_c^2 - 1$, with the generators $\{T^a\}$ normalized by

$$[T^a, T^b] = i f^{abc} T^c, \quad \text{tr}(T^a T^b) = \frac{1}{2} \delta_{ab}, \quad (2.2)$$

where $f^{abc}$ are the structure constants and are antisymmetric in all indices. In the presence of only a covariant derivative the coupling constant could have been and was absorbed in the definition of the gauge field. Quantum loop corrections generate the Yang-Mills term for the gauge field $A_\mu$ when the theory is superrenormilizable. The gauge coupling turns out to be proportional to the mass of the scalar field raised to the power determined by dimensional analysis.

Integrating out the $\Phi$ degrees of freedom leads to a non-local effective action for the gauge field $A_\mu$

$$e^{-S_{eff}(A)} = \int D\Phi \overline{D\Phi} e^{-S(\Phi, A)}, \quad (2.3)$$

where

$$S_{eff}(A) = \text{Tr} \ln (-D_\mu D^\mu + m^2). \quad (2.4)$$

Expanding $S_{eff}(A)$ about $A_\mu = 0$ and keeping only quadratic terms in the field gives\(^1\) (see Appendix A for details of the derivation)

$$S_{eff}(A) = \frac{\Gamma(2 - d/2)}{6(4\pi)^{d/2}} \text{tr} \int \frac{d^d p}{(2\pi)^d} A_\mu(p) A_\nu(-p) \left[ \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right] p^{d-2} \times \left( \frac{1}{4} + \frac{m^2}{p^2} \right)^{d/2-2} 2 \text{F1} \left( \frac{3}{2}, \frac{4 - d}{2}, \frac{5}{2}; \frac{1}{1 + 4 m^2/p^2} \right) + \mathcal{O}(A^3). \quad (2.5)$$

\(^1\)Throughout this section lower case “tr” represents trace over color indices.
In particular, choosing the Lorentz gauge fixing term

\[ S_{GF} = m^{d-4} \frac{\Gamma(2-d/2)}{6(4\pi)^{d/2}} \int d^d x (\partial_\mu A^\mu)^2 = m^{d-4} \frac{\Gamma(2-d/2)}{6(4\pi)^{d/2}} \int p_p p_\nu A^\mu(p) A_\nu(-p), \quad (2.6) \]
leads to the following propagator for the gauge field

\[ \langle A^\mu_\nu(p)A^\mu_\nu(-p) \rangle = \frac{6(4\pi)^{d/2}}{\Gamma(2-d/2)} \frac{m^{4-d}}{p^2 \gamma(p)} \delta^{ab} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} [1 - \gamma(p)] \right), \quad (2.7) \]
where

\[ \gamma(p) \equiv \left( 1 + \frac{p^2}{4m^2} \right)^{d/2-2} ~ _2F_1 \left( \frac{3}{2}, \frac{4-d}{2}, \frac{5}{2}; \frac{1}{1+4m^2/p^2} \right). \quad (2.8) \]

In the limit \( p \ll m \), we obtain

\[ S_{eff}(A) = m^{d-4} \frac{\Gamma(2-d/2)}{6(4\pi)^{d/2}} \text{tr} \int \frac{d^d p}{(2\pi)^d} A_\mu(p) A_\nu(-p) \left[ \delta^{\mu\nu} p^2 - p^\mu p^\nu \right] + \ldots = m^{d-4} \frac{\Gamma(2-d/2)}{12(4\pi)^{d/2}} \int d^d x \frac{1}{2} (F^a_{\mu\nu})^2 + \ldots \quad (2.9) \]

with \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu \). In the last equality we used gauge invariance of the action to supplement the quadratic part with appropriate self-interaction terms of \( A^a_\mu \) that make gauge invariance explicit. Although \( A^a_\mu \) were originally introduced as dummy fields a Yang-Mills term is dynamically generated for all of them. The fact that the Yang-Mills term is generated for all \( A^a_\mu \) fields rather than for a smaller subset results from the gauge invariance of the action, and the gauge coupling scales appropriately with \( m \).

A comment should be made regarding the situation when the action depends on a collection of distinct complex scalar fields charged under the gauge group. In this case, according to the above discussion each massive field contributes to the Yang-Mills term. In particular, if an \( SU(N_f) \) vector multiplet \( \Phi \) is introduced into the action, then the coupling charge decreases as \( 1/\sqrt{N_f} \) and becomes arbitrarily small in the large \( N_f \) limit.

In the ultrarelativistic limit \( p >> m \) or equivalently in the massless case \( m = 0 \), we obtain

\[ S_{eff}(A) = - \frac{\Gamma(1-d/2)\Gamma[d/2]}{2(16\pi)^{d+1} \Gamma[d/2 + 1/2]} \text{tr} \int \frac{d^d p}{(2\pi)^d} A_\mu(p) A_\nu(-p) \left[ \delta^{\mu\nu} p^2 - p^\mu p^\nu \right] p^{d-4} + \ldots = \frac{\sqrt{\pi}}{16(2\pi)^d} \frac{\Gamma(d/2 - 1)\Gamma(d - 2)}{\Gamma[d/2 + 1/2]} \int d^d x d^d y \frac{(F^a)^{\mu\nu}(x) (F^a)^{\mu\nu}(y)}{2|x - y|^{2d-4}} + \ldots \quad (2.10) \]

Hence, the theory explicitly manifests a non-local character. However, this conclusion is a reflection of the ultrarelativistic limit. In this case we probe the short wavelength physics. The gauge particles are introduced into the action as dummy variables and eventually are built out of the scalar \( \Phi \) particle degrees of freedom. As a result, one indeed might expect that the short distance behavior of the gauge field will reveal a non-local structure.
For those cases in which the gauge coupling is a dimensionful parameter the Yang-Mills term was generated. This suggests that once there is no scale in the problem or all the scales at hand are irrelevant, the Yang-Mills term will not be generated.

More generally, if the theory possesses a symmetry, for example a scale/conformal symmetry, which forbids any scale generation then it also excludes dynamical generation of the long-range force mediated by the gauge particles originally introduced into the action as dummy fields. At least as long as a symmetry is not spontaneously broken.

In what follows, we demonstrate how this general argument manifests itself in the case of large $N$ vector models with a $U(1)$ gauge symmetry.

3. The CP$^{N-1}$ model in $(2 + \epsilon)$-dimensions.

In this section we discuss the non emergence of the Maxwell term at a fixed point. This behavior is exhibited already in two dimensions in the so called coset models \cite{11, 10, 12} mentioned in the introduction. Here we study the $CP^{N-1}$ model in $(2 + \epsilon)$ dimensions at its fixed point \cite{17}. While in two dimensions this asymptotically free model served as an example how the dynamical mass generation leads to a long range force, that is a Maxwell term for a gauge field, in $(2 + \epsilon)$ dimensions the model has a fixed point at which it is scale invariant \cite{23}. This fixed point can be reliably analyzed for large $N$ using the $\epsilon$ expansion.

We review in section 3.1 and 3.2 the essential properties of the model starting by recalling the gap equations and showing that their solutions reveal three possible phases of the system - the weakly coupled phase, where the global $SU(N)$ symmetry is spontaneously broken and the $U(1)$ gauge potential is short-ranged; the phase in which the $SU(N)$ symmetry is unbroken and the $U(1)$ gauge field confines; and the third phase which is a scale invariant theory at the fixed point separating the two other phases. We describe the renormalization of a four-point function and reaffirm the existence of a fixed point \cite{17}.

In subsection 3.3 we further analyze the massive phase of the system. It is shown that in the low energy limit a Maxwell term for the auxiliary field $A_\mu$ emerges. However, it disappears from the action when the coupling constant is tuned to its critical value.

The Lagrangian as written in terms of constrained fields is

$$\mathcal{L} = \partial_\mu z^\dagger \partial^\mu z + \frac{g_0}{4N} (z^\dagger \partial_\mu z)^2,$$

with $z$ being an $N$- component column vector satisfying the following constraint

$$z^\dagger \cdot z = \frac{N}{g_0}.$$

(3.1)

This system exhibits invariance under local gauge transformations $z \rightarrow e^{i\alpha(x)} z$. Following \cite{18} we use it to fix $z_N$ to be real for all spacetime points $x$.

In order to re-write the Lagrangian in terms of unconstrained fields it is necessary to go to a non-linear representation of the coset space \cite{13}

$$CP^{N-1} \simeq SU(N)/S(U(1) \times U(N - 1)).$$

(3.3)
Rather than presenting the details of such a transformation we address the reader to \cite{16,18} and references therein. The action of the $CP^{N-1}$ model in terms of the unconstrained variables is given by

$$S\left(\vec{\phi}, \vec{\phi}^\dagger\right) = \int d^{2+\epsilon}x \left[ \frac{\partial_\mu \phi_i^\dagger \partial^\mu \phi_i}{\left(1 + \frac{g_0 4N}{4N} \phi_i \phi_i^\dagger\right)^2} + \frac{g_0}{4N} \left(\phi_i^\dagger \partial_\mu \phi_i\right)^2 \right] ,$$

(3.4)

where $i = 1, \ldots, N - 1$ and

$$z_i = \frac{\phi_i}{1 + \frac{g_0}{4N} \vec{\phi} \cdot \vec{\phi}} .$$

(3.5)

The real $z_N$ field is determined through constraint (3.2). Note that the original gauge invariance of the model is only partially fixed. For configurations in which the $N$'s component of $\vec{z}$ is non vanishing the gauge invariance is indeed fixed. However, when it vanishes, i.e. on a subset of the constrained $\vec{\phi}$ fields which obey

$$\vec{\phi}^\dagger \cdot \vec{\phi} = \frac{4N}{g_0} ,$$

(3.6)

there remains a residual gauge symmetry \cite{18}, by analogy referred as a Gribov ambiguity. The choice of having the $z_N$ component real constitutes a gauge fixing only as long as this component is non vanishing. When it does vanish the gauge choice is performed by choosing the first non vanishing component of $\vec{z}$ down the line to be real. For all these choices there are Gribov ambiguities which lead to the same physics result. A similar statement can be made when the gauge fixing is centered around attempting to choose any other component, $z_i$, as real first.

As will be shown below, in the large $N$ limit one out of three accessible phases of the system dynamically emphasizes this subset and it is accompanied by an appearance of the Maxwell term for the U(1) gauge field. In the other phases the Gribov ambiguity plays no important role.

### 3.1 Generating functional and the gap equations.

Rescaling the fields according to

$$\vec{\phi} = Z_{\phi}^{1/2} \vec{\phi}_r ,$$

(3.7)

where $Z_{\phi}$ is an arbitrary real constant, we obtain

$$S\left(\vec{\phi}_r, \vec{\phi}_r^\dagger\right) = \int d^{2+\epsilon}x \left[ \frac{Z_\phi \partial_\mu \phi_r^\dagger \partial^\mu \phi_r}{\left(1 + \frac{Z_\phi}{4N} g_0 \phi_r \phi_r^\dagger\right)^2} + \frac{g_0}{4N} \phi_r^\dagger \partial_\mu \phi_r \phi_r^\dagger \right] .$$

(3.8)

Introducing the following identities

$$1 \sim \int DA_\mu D\eta_\mu \left(N_{\eta_\mu} + i \phi_r^\dagger \partial_\mu \phi_r\right) \sim \int DA_\mu D\eta_\mu \frac{i f A_\mu}{N_{\eta_\mu} + i \phi_r^\dagger \partial_\mu \phi_r} d^{2+\epsilon}x ,$$

(3.9)
and
\[ 1 \sim \int D\rho \delta(\vec{\phi}_r \cdot \vec{\phi}^\dagger_r - N\rho) \sim \int D\rho \, D\lambda \, e^{-i \int \lambda(\vec{\phi}_r \cdot \vec{\phi}^\dagger_r - N\rho) d^3x}, \] (3.10)
into the generating functional for \( \vec{\phi}_r \) yields\(^2\)
\[ Z[\vec{J}, \vec{J}^\dagger] = \int D\rho D\lambda DA_\mu D\eta^\mu e^{-S_{\text{eff}}(\rho, \lambda, A, \eta)} \]
\[ \times e^{\int \vec{J}^\dagger(x) \left[-Z_\phi \partial_\mu \left(1 + \frac{Z_\phi}{4} g_0 \rho\right)^{-2} \partial^\mu + i\lambda + A^\mu \leftrightarrow \partial_\mu \right]^{-1} \vec{J}(y) \, d^2x \, d^2y,} \] (3.11)
where
\[ S_{\text{eff}}(\rho, \lambda, A, \eta) = -N \int d^2x \left[i\lambda \rho + i \eta_\mu A^\mu + \frac{g_0}{4} Z_\phi^2 \frac{\eta_\mu \eta^\mu}{\left(1 + \frac{Z_\phi}{4} g_0 \rho\right)^2} \right] \]
\[ + N \text{Tr} \ln \left[-Z_\phi \partial_\mu \left(1 + \frac{Z_\phi}{4} g_0 \rho\right)^{-2} \partial^\mu + i\lambda + A^\mu \leftrightarrow \partial_\mu \right]. \] (3.12)

The above action is both quadratic in the auxiliary field \( \eta_\mu \) and does not contain its derivatives. Hence, \( \eta_\mu \) can be eliminated by using its equations of motion, and one arrives at
\[ S_{\text{eff}}(\rho, \lambda, A) = -N \int d^2x \left[i\lambda \rho + \frac{\left(1 + \frac{Z_\phi}{4} g_0 \rho\right)^4}{g_0 Z_\phi^2} A_\mu A^\mu \right] \]
\[ + N \text{Tr} \ln \left[-Z_\phi \partial_\mu \left(1 + \frac{Z_\phi}{4} g_0 \rho\right)^{-2} \partial^\mu + i\lambda + A^\mu \leftrightarrow \partial_\mu \right]. \] (3.13)

In the next section we show that the vector field \( A_\mu \) plays a role of the \( U(1) \) gauge field as a consequence of the Gribov ambiguity mentioned earlier.

The Lorentz invariant gap equations are given for the large \( N \) by the stationary phase approximation\(^3\)
\[ \langle A_\mu \rangle = 0, \]
\[ i\bar{\lambda} = g_0 \frac{Z_\phi^2}{2} \left(1 + \frac{Z_\phi}{4} g_0 \rho\right)^{-3} \text{tr} \left[ \frac{\partial^2}{-Z_\phi \left(1 + \frac{Z_\phi}{4} g_0 \rho\right)^{-2} \partial^2 + i\bar{\lambda}} \right], \]
\[ \bar{\rho} = \text{tr} \left[ \frac{1}{-Z_\phi \left(1 + \frac{Z_\phi}{4} g_0 \rho\right)^{-2} \partial^2 + i\bar{\lambda}} \right]. \] (3.14)

Choosing
\[ Z_\phi^{1/2} = \frac{2}{1 + (1 - g_0 \bar{\rho})^{1/2}}, \] (3.15)

---

\(^2\)Index \( r \) is suppressed for brevity and \( N \sim N - 1 \) is used.

\(^3\)Note that the original gauge symmetry of the model was fixed up to a possible Gribov ambiguity.
in order to fix the residue of the scalar field propagator to 1, simplifies the gap equations as follows

\[ m^2 = -g_0 \frac{Z_{\phi}^{1/2}}{2} \text{tr} \left[ \frac{-\partial^2}{-\partial^2 + m^2} \right] = m^2 g_0 \frac{Z_{\phi}^{1/2}}{2} \rho, \]

\[ \bar{\rho} = \text{tr} \left[ \frac{1}{-\partial^2 + m^2} \right] = \frac{\Gamma(-\epsilon/2)}{(4\pi)^{1+\epsilon/2}} (m^2)^{\epsilon/2}, \] (3.16)

where \( m^2 = i\tilde{\lambda} \) stands for the mass of the scalar field, and the dimensional regularization has been used to evaluate the divergent loop integral.

From the first equation we recover two out of three possible phases of the system: the weak coupling and the strong coupling ones. These phases were studied in \([16]\) for the nonlinear \(\sigma\) model and in \([18, 17]\) for \(CP^{N-1}\)

**Phase 1:** \( m^2 = 0, \bar{\rho} = 0, Z_\phi = 1 \),

**Phase 2:** \( m^2 = \frac{(4\pi)^{1+\epsilon/2}}{g_0 \Gamma(-\epsilon/2)} \frac{2^\epsilon}{2^\epsilon}, \bar{\rho} = \frac{2}{g_0 Z_\phi^{1/2}} = \frac{1}{g_0}, Z_\phi = 4 \). (3.17)

The third phase is derived and explored in the next section. It corresponds to a scale invariant fixed point that separates between the phases.

### 3.2 \( \phi\phi \rightarrow \phi\phi \) scattering amplitude and RG flow.

It will be useful to explicitly obtain the various propagators as well as a four-point proper vertex. For this purpose we use the generating functional (3.11) and differentiate it with respect to the sources \( \vec{J} \) and \( \vec{J}^\dagger \)

\[ \langle \phi_i(x_1)\phi_j^\dagger(x_2)\phi_k(x_3)\phi_l^\dagger(x_4) \rangle \simeq \frac{\delta^4}{\delta \vec{J}_i(x_1)\delta \vec{J}_j(x_2)\delta \vec{J}_k(x_3)\delta \vec{J}_l(x_4)} Z \left[ \vec{J}, \vec{J}^\dagger \right] \bigg|_{\vec{J}=\vec{J}^\dagger=0}. \] (3.18)

Henceforth we shall deal with \( \delta_{ij}\delta_{kl} \) part of the fully connected one-particle irreducible amputated four-point function shown on figure \[\text{4}\]. This part is associated with the \( s\)–channel and applying appropriate changes \( (s \rightarrow t, \delta_{ij}\delta_{kl} \rightarrow \delta_{il}\delta_{jk} \text{ and } p_1 \leftrightarrow p_3) \) one gets similar results for the \( t\)–channel.

In what follows we omit the various delta-functions, redefine \( i\lambda \rightarrow \lambda \) as well as adopt the notation of Appendix \[A\]

\[ \int_x \equiv \int d^{2+\epsilon}x, \quad \int_p \equiv \int \frac{d^{2+\epsilon}p}{(2\pi)^d}, \]

\[ \hat{G} \equiv (-\partial^2 + m^2)^{-1} \iff G(x-y) = \langle x|\hat{G}|y \rangle = \int \frac{\delta^4(x-y)}{p^2 + m^2}. \] (3.19)

where \( x \) represents the spacetime coordinate and \( p \) is the momentum \( (2 + \epsilon)\)-vector.
Carrying out the differentiation with respect to the sources one gets a pair of propagators inside the path integral. Each such propagator has the following form

\[
\langle x_m | \left[ -Z_\phi \partial_\mu \left( 1 + \frac{Z_\phi}{4} g_0 \rho \right)^{-2} \partial^\mu + \lambda + A^{\mu \nu} \partial_\mu \partial_\nu \right]^{-1} | x_n \rangle = G(x_m - x_n)
\]

\[
+ \frac{g_0 \sqrt{Z_\phi}}{2} \delta^{\mu \nu} \int_z \frac{\partial}{\partial x_m^\mu} G(x_m - z) \rho(z) \frac{\partial}{\partial x_n^\nu} G(x_n - z) - \int_z G(x_m - z) \lambda(z) G(x_n - z) + \int_G(x_m - z) A^{\mu}(z) \frac{\partial}{\partial x_n^\nu} G(x_n - z) \int_z \frac{\partial}{\partial x_n^\nu} G(x_m - z) A^{\mu}(z) G(x_n - z) + \ldots , \quad (3.20)
\]

where \((m, n)\) equals either \((1, 2)\) or \((3, 4)\) and we have expanded around the solutions of the gap equations \((3.16)\): \(\rho \rightarrow \bar{\rho} + \rho, \lambda \rightarrow \bar{\lambda} + \lambda\), keeping only linear terms in the small perturbations \(\rho, \lambda\).

It turns out, that these terms are sufficient to compute any correlation function to leading order in the \(1/N\) expansion. Indeed, terms beyond linear order will require to introduce extra propagators of the auxiliary fields in the computation of a given correlation function. However, as we shall see below, each such propagator is inversely proportional to \(N\) and therefore the contribution of higher order terms in the above expansion will be suppressed by a power of \(1/N\) relative to the result based on the linear terms only.

Accordingly, in order to compute the four-point function it is enough to evaluate the propagators of the auxiliary fields \(\rho, \lambda\) and \(A_\mu\). For this purpose we expand the effective action \((3.13)\) around the solutions \(\bar{\rho}, \bar{\lambda}\) of the gap equations \((3.16)\) and keep only quadratic part in the perturbations. The \(\rho - A_\mu\) and \(\lambda - A_\mu\) mixing terms turn out to vanish. Hence, \(A_\mu\) is decoupled from the \(\rho - \lambda\) sector. Using the gap equations, one finds that linear terms vanish as well.

According to the results described in Appendix A we get the following expression for the term quadratic in \(A_\mu\) in the effective action

\[
\mathcal{O}(A^2) = N \left( \bar{\rho} - \frac{1}{g_0} \right) \int_p A_\mu(p) A^{\mu}(-p)
+ \frac{\epsilon N \bar{\rho}}{12} \int_p A_\mu(p) A_\nu(-p) \left[ \delta^{\mu \nu} - \frac{p^{\mu} p^{\nu}}{p^2} \right] \left( \frac{p^2}{m^2} \right) \left( \frac{1}{4 + \frac{m^2}{p^2}} \right) \frac{2}{1 + \frac{4 m^2}{p^2}} . \quad (3.21)
\]

Adding altogether the quadratic terms in the action leads to

\[
S_{eff}(\rho, \lambda, A) = \frac{N}{2} \int_p \left[ \rho(p) \rho(-p) T_{\rho \rho}(p) + 2 \lambda(p) \rho(-p) T_{\rho \lambda}(p) + \lambda(p) \lambda(-p) T_{\lambda \lambda}(p) \right]
+ N \left( \bar{\rho} - \frac{1}{g_0} \right) \int_p A_\mu(p) A^{\mu}(-p)
+ \frac{N \epsilon}{12} \int_p A_\mu(p) A_\nu(-p) \left[ \delta^{\mu \nu} - \frac{p^{\mu} p^{\nu}}{p^2} \right] \left( \frac{p^2}{m^2} \right) \left( \frac{1}{4 + \frac{m^2}{p^2}} \right) \frac{2}{1 + \frac{4 m^2}{p^2}} + \ldots \quad (3.22)
\]
\[ \Gamma_{ijkl} = \Gamma_s \delta_{ij} \delta_{kl} + \Gamma_t \delta_{il} \delta_{jk} = \Gamma(1 - \frac{\epsilon}{2}) \frac{\phi}{(4\pi)^{1+\epsilon/2}} \left( \frac{p^2}{4} + m^2 \right)^{\epsilon/2} 2F_1 \left( \frac{1}{2}, \frac{2 - \epsilon}{2}, \frac{3}{2}; \frac{1}{1 + 4m^2/p^2} \right) \]

\[ \Gamma(1 - \frac{\epsilon}{2}) \frac{\phi}{(4\pi)^{1+\epsilon/2}} \left( \frac{p^2}{4} + m^2 \right)^{\epsilon/2} 2F_1 \left( \frac{1}{2}, \frac{2 - \epsilon}{2}, \frac{3}{2}; \frac{1}{1 + 4m^2/p^2} \right) \]

Figure 1: Diagramatic representation of the full connected one-particle irreducible amputated four-point function.

where

\[ T_{\rho\rho}(p) = -\frac{g_0^2 Z_\phi}{4} \left( p^\mu p^\nu I_{\mu\nu}(p) - 2m^2 p^\mu I_\mu(p) - p^2 \bar{\rho} + m^4 I_0(p) - \frac{1}{2} m^2 \bar{\rho} \right) \]

\[ T_{\rho\lambda}(p) = g_0 \sqrt{Z_\phi} \left( p^\mu I_\mu(p) - m^2 I_0(p) + \bar{\rho} \right) - 1 \]

\[ T_{\lambda\lambda}(p) = -I_0(p) \]

(3.23)

with

\[ I_0(p^2) \equiv \int_q \frac{1}{[q^2 + m^2][(q + p)^2 + m^2]} \]

\[ = \frac{\Gamma(1 - \epsilon/2)}{(4\pi)^{1+\epsilon/2}} \left( \frac{p^2}{4} + m^2 \right)^{\epsilon/2-1} 2F_1 \left( \frac{1}{2}, \frac{2 - \epsilon}{2}, \frac{3}{2}; \frac{1}{1 + 4m^2/p^2} \right) \]

(3.24)

\[ I_{\mu}(p) \equiv \int_q \frac{q_\mu}{[q^2 + m^2][(q + p)^2 + m^2]} = -\frac{1}{2} I_0(p) p_\mu \]

(3.25)

and

\[ I_{\mu\nu}(p) \equiv \int_q \frac{q_\mu q_\nu}{[q^2 + m^2][(q + p)^2 + m^2]} \]

\[ = \delta_{\mu\nu} \frac{\Gamma(-\epsilon/2)}{2(4\pi)^{1+\epsilon/2}} \left( \frac{p^2}{4} + m^2 \right)^{\epsilon/2} 2F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{1}{1 + 4m^2/p^2} \right) \]

\[ + p_\mu p_\nu \frac{\Gamma(1 - \epsilon/2)}{4(4\pi)^{1+\epsilon/2}} \left( \frac{p^2}{4} + m^2 \right)^{\epsilon/2-1} \]

\[ \times \left[ 2F_1 \left( \frac{1}{2}, \frac{2 - \epsilon}{2}, \frac{3}{2}; \frac{1}{1 + 4m^2/p^2} \right) + \frac{1}{3} 2F_1 \left( \frac{3}{2}, \frac{2 - \epsilon}{2}, \frac{5}{2}; \frac{1}{1 + 4m^2/p^2} \right) \right] \]

(3.26)

Note also that based on the above definitions the following useful identity holds

\[ p^\mu p^\nu I_{\mu\nu}(p) = \frac{p^4}{4} I_0(p^2) + \frac{p^2}{2} \bar{\rho} \]

(3.27)
Since $\rho-\lambda$ sector is decoupled from $A_\mu$ we compute their contributions to the four-point proper vertex separately. We start from $\rho-\lambda$. Inverting the matrix $T$, we obtain

$$D^{(1)}_{\rho\rho}(p) = \frac{I_0}{N(1 + \frac{g_0^2}{2}p^2I_0)}; \quad D^{(1)}_{\lambda\lambda}(p) = \frac{g_0^2}{16} \frac{p^4I_0}{N(1 + \frac{g_0^2}{2}p^2I_0)}; \quad D^{(1)}_{\lambda\rho}(p) = -\frac{1 + \frac{g_0^2}{2}p^2I_0}{N(1 + \frac{g_0^2}{2}p^2I_0)},$$

(3.28)

and

$$D^{(2)}_{\rho\rho}(p) = \frac{2}{N} \frac{1}{g_0^2 p^2 + m^2}; \quad D^{(2)}_{\lambda\lambda}(p) = \frac{g_0^2}{2N} \frac{p^4}{p^2 + m^2} + \frac{2g_0m^2}{N} \frac{1}{NI_0};$$

$$D^{(2)}_{\lambda\rho}(p) = -\frac{1}{N} \frac{m^2}{N(p^2 + m^2)},$$

(3.29)

for the Higgs and confinement phases respectively. The upper index indicates the phase number, whereas lower indices indicate what kind of propagator is concerned.

The feature of (3.29) is that to leading order in $1/N$ the $\rho$-propagator consists only of a simple pole when rotated back to the Minkowski space. That is $\tilde{\phi}\tilde{\phi}^\dagger$ creates a single-particle state when acting on the vacuum. This particle is degenerate in mass with the $\phi$'s and gets bound by the confining potential into the singlet and the adjoint representations of $SU(N)$ [18]. This corresponds to a full restoration of the global $SU(N)$ symmetry in the confined phase.

Let us next use (3.20) and the propagators of the auxiliary fields to calculate the contribution of the $\rho-\lambda$ sector to the $s$-channel of the connected one-particle irreducible amputated four-point function $\Gamma_s^{(r)}$

$$\Gamma_s^{(r)} \bigg|_{\rho\lambda} = \frac{g_0^2 Z_\phi}{4} (p_1 \cdot p_2)(p_3 \cdot p_4) D^{(r)}_{\rho\rho}(s) + D^{(r)}_{\lambda\lambda}(s) + \frac{g_0 \sqrt{Z_\phi}}{2} (p_1 \cdot p_2 + p_3 \cdot p_4) D^{(r)}_{\lambda\rho}(s),$$

(3.30)

where $r = 1, 2$ serves to indicate the different phases and the vertical bar with a subscript "$\rho\lambda$" indicates restriction of the full one-particle irreducible amputated four-point function $\Gamma_s^{(r)}$ to the $\rho-\lambda$ sector. Substituting (3.28), (3.29) yields

$$\Gamma_s^{(1)} \bigg|_{p_1^2 = 0}\bigg|_{\rho\lambda} = \frac{-g_0s}{N(2 + g_0sI_0(s))},$$

(3.31)

for the on-shell vertex in the Higgs phase and

$$\Gamma_s^{(2)} \bigg|_{p_1^2 = 0}\bigg|_{\rho\lambda} = \frac{g_0}{2N} (s - m^2 - p_1^2 - p_2^2) \frac{1}{s + m^2} (s - m^2 - p_3^2 - p_4^2)$$

$$+ \frac{g_0}{2N} \left(-s + 3m^2 + \sum_{i=1}^{4} p_i^2 \right) - \frac{1}{NI_0(s)},$$

(3.32)

for the off-shell vertex in the confinement phase.

Let us complete the above expression by adding the contribution from the exchange auxiliary field $A_\mu$. For this purpose we need to compute its propagator in the two accessible phases separately. In the Higgs phase according to (3.17) and (3.21), we obtain

$$D^{(1)}_{\mu\mu}(p) = -\frac{g_0}{2N} \left(1 + g_0 f^{(1)} p^\epsilon\right)^{-1} \left(\delta_{\mu\nu} + g_0 f^{(1)} p^\epsilon p_\mu p_\nu p^2\right),$$

(3.33)
with
\[ f^{(1)} = -\epsilon \frac{\Gamma(-\epsilon/2)\Gamma(\epsilon/2)}{4^{2+\epsilon}\pi^{\epsilon/2+1/2}\Gamma(3/2 + \epsilon/2)} = \frac{1}{(1 + \epsilon)g_c}, \] (3.34)

where
\[ g_c = 2(4\pi)^{1+\epsilon/2} \frac{\Gamma(\epsilon)}{\Gamma(1 - \frac{1}{2}\epsilon)\Gamma(\epsilon/2)} , \] (3.35)

and we show below that \( g_c \) is an ultraviolet-stable fixed point of the renormalization group flow. The propagator of the auxiliary field (3.33) has no pole at \( p = 0 \). Thus we denote this phase as a Higgs phase.

However, in the confinement phase \( A_\mu \) behaves like a gauge vector field and it will be necessary to further fix the gauge in order to proceed. The need to further fix the gauge is tightly related to the Gribov ambiguity mentioned earlier. In fact, it occurs because the vacuum has chosen to localize on the ambiguous field configurations. Indeed, in the confinement phase according to (3.17), we have
\[ \langle \vec{\phi}_r^\dagger \cdot \vec{\phi}_r \rangle = N\bar{\rho} = \frac{N}{g_0} \Leftrightarrow \langle \vec{\phi}_r^\dagger \cdot \vec{\phi} \rangle = \frac{4N}{g_0} . \] (3.36)

Therefore this phase corresponds to those field configurations (3.6) where the residual gauge invariance manifests itself. In contrast, the Higgs phase of scalars corresponds to the field configuration with vanishing vacuum expectation value of \( \vec{\phi}_r^\dagger \vec{\phi}_r \) and thus the residual gauge invariance does not play a role. Choosing the Lorentz gauge
\[ S_{GF} = \frac{N\epsilon}{12m^2g_0} \int_x (\partial_\mu A_\mu)^2 = \frac{N\epsilon}{12m^2g_0} \int p_\mu p_\nu A_\mu(p)A_\nu(-p) . \] (3.37)

and combining it with (3.22), leads to the following propagator
\[ D^{(2)}_{\mu\nu}(p) = \frac{6g_0}{N\epsilon} \left[ \frac{m^2 p_\mu p_\nu}{p^2} + \frac{1}{f^{(2)}} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \right] , \] (3.38)

where
\[ f^{(2)} \equiv \left( \frac{p}{m} \right)^\epsilon \left( \frac{1}{4} + \frac{m^2}{p^2} \right)^{\epsilon/2-1} {}_2F_1 \left( \frac{3}{2}, \frac{2 - \epsilon}{2}, \frac{5}{2}; \frac{1}{1 + 4m^2/p^2} \right) . \] (3.39)

In the low-energy limit \( p \ll m \) one obtains
\[ D^{(2)}_{\mu\nu}(p) \xrightarrow{p \ll m} \frac{6m^2g_0}{N\epsilon} \frac{\delta_{\mu\nu}}{p^2} . \] (3.40)

Using (3.20) the full s–channel of the connected one-particle irreducible amputated four-point function is then obtained by adding
\[ \Gamma^{(r)}_{s} \bigg|_{A_2} = -(p_1 - p_2)^\mu D^{(r)}_{\mu\nu}(s)(p_3 - p_4)^\nu , \] (3.41)

to (3.31),(3.32). For the Higgs phase we get
\[ \Gamma^{(1)}_{s} (p_i^2 = 0) = -\frac{g_0}{N(2 + g_0 s I_0(s))} (p_1 \cdot p_2 + p_3 \cdot p_4) + \frac{g_0}{2N} \frac{1}{1 + g_0 f^{(1)}(s)^{\epsilon/2}} (p_1 - p_2) \cdot (p_3 - p_4) . \] (3.42)
We next adopt the approach of [16], and show that $g_c$ is an ultraviolet fixed point and the system is within the Higgs phase if $g < g_c$ and within the confinement phase if $g > g_c$, where $g$ is the renormalized running coupling constant defined by

$$g_0 = g Z_g \mu^{-\epsilon},$$  \hspace{1cm} (3.43)\]

which renders $g$ dimensionless by introducing an arbitrary scale $\mu$.

We impose the following renormalization condition

$$\Gamma^{(1)}_{s}(\lambda; s = \mu^2, p_i^2 = 0) = -\frac{g \mu^{-\epsilon}}{2N}(p_1 \cdot p_2 + p_3 \cdot p_4 - (p_1 - p_2)(p_3 - p_4)) .$$  \hspace{1cm} (3.44)\]

Combining (3.42) with the above condition leads to

$$\frac{Z_g}{1 + (g/g_c)Z_g} = 1 \Rightarrow Z_g = \left(1 - \frac{g}{g_c}\right)^{-1} .$$  \hspace{1cm} (3.45)\]

Since the bare coupling $g_0$ is scale independent, we get

$$\mu \frac{dg_0}{d\mu} = 0 \Rightarrow \beta_g = \mu \frac{dg}{d\mu} = \epsilon g \left(1 - \frac{g}{g_c}\right)^{-1} .$$  \hspace{1cm} (3.46)\]

Moreover if $g < g_c$ it follows that $Z_g > 0$ and thus one must choose the Higgs phase as the solution to the gap equation (3.16), since the massive solution in (3.17) would be self-contradictory. On the other hand, for $g > g_c$, the theory exists only in the confinement phase, since if we let $g > g_c$ in (3.42), there would appear a tachyon pole at

$$s = \mu^2 \left(1 - \frac{g_c}{g}\right)^{2/\epsilon} .$$  \hspace{1cm} (3.47)\]

Before turning to analyze the properties of the system at the fixed point, we wish to discuss a subtlety which is there in the $CP^{N-1}$ model both in $d = 2$ and $d = 2 + \epsilon$ dimensions. This will accompany the analysis also in the other cases discussed later in this work. In the phase where the global symmetry is unbroken for $d = 2 + \epsilon$, which is the only phase at $d = 2$, one notes that the IR limit and the large $N$ limit do not commute. This could leave one with the impression that one is free to make a choice of the way in which to order the limits. This is wrong. One clue comes from the $d = 2$ case, taking first the large $N$ limit suggests that the theory describes 2N massive particles in the vector representation of $O(2N)$ if this was the case the global $SU(N)$ symmetry apparent in equation (3.3) would not have been restored. This contradicts, in particular, the consequences of Colemans theorem according to which one would expect that the lowest excitations of the system to be massive particles in the adjoint representation of $SU(N)$ or at least they should be neutral under $Z(N)$ [18]. Having the lowest mass particles in the fundamental representation would require them to be non perturbative states of such solitons which does seem likely. A Maxwell term could have allowed to bind the states back into the appropriate representations of $SU(N)$ but it is down by $1/N$. The correct description of the physics should thus include Maxwell’s term even though its presence is
non leading in the $1/N$ expansion. The necessity to follow this order of limits can also be inferred from a more general point of view.

In general, once obtaining the leading term of the effective action in the $1/N$ expansion, one should inspect that it contains all the relevant operators allowed by the symmetries of the problem. Note, that anomalous dimensions encoded in the theory have to be taken into account when determining whether a given term is relevant or not. If allowed operators are absent, one should continue the expansion to the higher orders in $1/N$ until all the missing terms show up. To ensure the stability of the theory, it is essential to add all such terms to the effective action even if the first time they appear is at a higher order in $1/N$ than the leading contribution\[4\].

### 3.3 The fate of the Maxwell term.

In the current subsection we investigate the generation and subsequent fate of the Maxwell term when the system starts in the confinement phase and is gradually driven into a fixed point.

The long distance behavior of (3.21) in the confinement phase, $p << m$, is given by

$$
O(A^2) \simeq \frac{\epsilon N}{24g_0 m^2} \int_x F_{\mu\nu} F^{\mu\nu} .
$$

where (3.17) has been used. As a result, the Maxwell term for the auxiliary field $A_\mu$ is generated. In order to figure out what happens to this term at a fixed point, note first that according to (3.17) and (3.45) we get for the mass in the strong-coupling phase

$$
m^2 = \mu^2 \left[ \left( 1 - \frac{g_c}{g} \right) \frac{\Gamma^2(1 + \epsilon/2)}{\Gamma(1 + \epsilon)} \right]^{2/\epsilon} .
$$

Hence, if we set $g = g_c$ (with $\mu$ being fixed), we get $m^2 = 0$ and thus the condition $p << m$ is not valid anymore. In particular, (3.48) must be modified. Relying on (3.21), we get in this case

$$
O(A^2) \rightarrow \frac{N}{2(16\pi)^{1+\epsilon}} \frac{\Gamma[1 + \epsilon/2][-\epsilon/2]}{\Gamma[3/2 + \epsilon/2]} \int_p A_\mu(p) A_\nu(-p) \left[ \delta_{\mu\nu} p^2 - p^\mu p^\nu \right] p^{\epsilon-2}
$$

$$
= \frac{\sqrt{\pi}}{16(2\pi)^{2+\epsilon}} \frac{\Gamma(\epsilon/2)\Gamma(\epsilon)}{\Gamma[\epsilon/2 + 3/2]} \int_x \int_y \frac{F^\alpha_{\mu\nu}(x) F^\alpha_{\nu\mu}(y)}{|x - y|^{2\epsilon}} .
$$

Therefore neither the Maxwell term nor the corresponding long-range force appear in this case.

Alternatively, one could start at some definite value of the coupling constant $g(\mu_0) > g_c$ and gradually increase the energy scale $\mu$. Solving (3.46) leads to

$$
\left( \frac{\mu_0}{\mu} \right)^\epsilon \left( 1 - \frac{g_c}{g(\mu_0)} \right) = 1 - \frac{g_c}{g(\mu)} .
$$

\[4\] We thank David Kutasov for sharpening this understanding.
Substituting this result back into (3.49) yields a finite mass generated by the dimensional transmutation
\[
m^2 = \mu_0^2 \left[ \left( 1 - \frac{g_c}{g(\mu_0)} \right) \frac{\Gamma^2(1+\epsilon/2)}{\Gamma(1+\epsilon)} \right]^{2/\epsilon}.
\]
(3.52)

However, even though the mass is finite, when \( \mu/\mu_0 \to \infty \) the coupling constant is driven to the critical value \( g_c \) while the typical momentum satisfies \( p \gg m \). This in turn leads to the same result (3.50) and the Maxwell term does not emerge.

4. The 3D gauged and ungauged \( \phi^6 \) \( O(N) \) vector model.

In this section we investigate whether a Maxwell term is generated in the case when the gauged system does possess a scale/conformal symmetry that is spontaneously broken. It turns out that the cases of spontaneous breaking of scale invariance and asymptotically free theories are very similar as far as their generating Maxwell terms for the gauge particles is concerned. We use the \( \phi^6 \) vector model in which the non-perturbative dynamics can be studied for a purely bosonic CFT \( [19] \). Before proceeding to the various issues associated with the gauged system only, we turn to refine the properties which are shared by both the gauged and the ungauged cases. In particular, we reexamine the phase with spontaneous breaking of scale invariance where the massless dilaton emerges and a confining logarithmic potential is generated. The latter feature of confinement by the dilaton exchange was not brought up in previous investigations \( [18] \). We explore it in this work.

Recall that the large \( N \) limit of the \( \phi^6 \) \( O(N) \) is a very useful setup to study the exact behavior of conformal theories in higher than two dimensions. It does have the limitation that the large \( N \) limit does not commute with the limit of removing the ultraviolet cutoff of the theory. To obtain a scale invariant theory of only scalars in this dimension one needs to take the large \( N \) limit first for a fixed UV cutoff and only then remove the cutoff. As shown in \( [18] \), in this particular order of limits, the theory is scale invariant and exhibits a phase with spontaneously broken scale invariance. A Goldstone massless particle, the dilaton, emerges. However, such a massless particle in 3D generates a long distance confining logarithmic potential which is accompanied by a \( 1/N \) overall coefficient, and thus also the IR limit does not commute with the \( 1/N \) expansion. Taking first \( N \) to infinity may seem to lead to a free theory containing one massless particle and \( N \) massive particles in the vector representation of \( O(N) \). In the other order of limits the lowest massive excitations are the symmetric and singlet representations of \( O(N) \times O(N) \).

This intricacy resembles the situation in the \( CP^{N-1} \) model in \( d = 2 \), where the IR and the large \( N \) limits do not commute either\(^5\). However, there is a difference between the two: in the \( CP^{N-1} \) model the large \( N \) and the UV limits do commute, and thus the leading contribution to the force due to the gauge field can be reliably obtained although it comes with a \( 1/N \) overall coefficient.

Nevertheless, in light of the discussion at the end of subsection 3.2 one may attempt to adopt the following procedure. First remove the IR cutoff and only next resort to the

\(^5\)Another manifestation of this was found on the lattice the strong coupling and the large \( N \) limit do not commute \( [25] \).
large $N$ limit and lastly remove the UV cutoff. Therefore in the phase where the scalars $\tilde{\phi}$ are massive and the massless dilaton is formed, one can consider a non relativistic limit in which the potential picture makes sense, as it did in 2d $CP^{N-1}$ model, and the properties of the theory can be investigated as if the binding force operates in a CFT arena.

The dilaton on its own would form massive two-body bound states of the fields $\tilde{\phi}$, some of them would be the lowest massive excitations of the system. As the bound states have no “neutrality” properties one could expect also higher mass more complicated bound states. We are aware that this order of limits may result in less than logarithmically confining potential.

The uncovering of the ABJM models [26] has led to a discovery of a large class of three dimensional scale invariant field theories. Some of these contain massless dilatons [27]. In those cases which are weakly coupled the dilaton couples to itself like a Goldstone boson and transmits a logarithmically confining potential among massive excitations\(^6\). This without obviously suffering from the necessity to define a certain order of limits.

Next let us anticipate the results in the case when one gauges a $U(1)$ subgroup of the global symmetry without introducing a Maxwell term. This can be done for an $O(2N)$ global symmetry. The details are presented in the next sections. Although in $d < 4$ the gauge coupling is dimensionful and the Maxwell term is relevant, in the absence of a Maxwell term the conformal symmetry is maintained.

The emergence of a Maxwell term presents the similar challenge to the one described above when studying the long distance forces. The value of the effective electric charge strictly vanishes for infinite $N$. Hence, like in the ungauged $\phi^6$ for infinite $N$ the degrees of freedom do not interact. For finite $N$ the theory is not conformal and the analysis is invalid.

However, once the Maxwell term emerges, the large $N$ and the IR limit need to be taken in the same order as in the case of the ungauged $\phi^6$. In the absence of the CS term the gauge field will confine at least logarithmically the $U(1)$ charge and only the bound states which are neutral under $U(1)$ will be formed. They will be in the adjoint representation of $SU(N)$.

When the CS term is introduced either ab-initio or it emerges, it will be only the dilaton which provides a confining potential. The CS term on its own has no bulk degrees of freedom, but when coupled to a Maxwell term it effectively generates a mass for the gauge field. Hence, the photon becomes massive and the long-range force it carried disappears.

As a result, the CS parameter splits the phase with spontaneously broken scale invariance into two distinct phases: the phase where only the dilaton binds the particles into an irreducible representation of $O(2N) \times O(2N)$ and the phase with neutral bound states belonging to an irreducible representation of $SU(N)$. Figure\(^2\) demonstrates the enrichment of the phase structure when the CS term is considered. Phases $I$ and $III$ in the figure are unstable [19], whereas phase $II$ corresponds to a massless conformal phase without a Maxwell term being generated.

There has been a suggestion [28] to relate a $d = 3$ $O(N)$ CFT vector model on the

\(^6\)We thank Ofer Aharony for a discussion on this point.
\(\theta - g_6\) plane. \(g_c = (4\pi)^2\) represents the critical value of the coupling constant. Phases I and III are unstable as argued in [19]. Phase II corresponds to a massless conformal phase, where the Maxwell term does not emerge. The critical line \(g_6 = g_c\) represents two distinct phases of the spontaneously broken scale invariance: the dashed line (phase IV) is associated with the phase where \(\theta \neq 0\) and the particles are bound by dilaton only, whereas the bold point (phase V) corresponds to \(\theta = 0\) where only the states which are neutral under \(U(1)\) will be formed.

Boundary to higher spin bulk theories. It was pointed out [29] that in order to maintain, as needed, only the \(O(N)\) singlet sector, one needs to study the IR limit of the \(\phi^6\) vector model. At its critical point when the scale invariance is spontaneously broken the IR limit consists of only one massless field - the \(O(N)\) singlet dilaton. This remains also in the cases studied here. If one wants to have a theory containing only \(O(N)\) singlets both massive and massless, one needs to gauge the full global symmetry so that the states in the theory are all singlets.

4.1 The gap equations.

The gauged 3-dimensional model considered in this section is given by

\[
S(\vec{\phi}, \phi^\dagger, A) = \int \left[ D_\mu \phi_i D^\mu \phi_i + \frac{g_6}{3N^2} (\phi_i \phi_i^\dagger)^3 + \frac{N\theta}{2} i \epsilon^{\alpha\mu\nu} A_\alpha F_{\mu\nu} \right] d^3x , \tag{4.1}
\]

The ungauged case and its supersymmetric extension were extensively studied in a number of works [24]. In particular, the phase with spontaneously broken scale invariance was explored in [14, 29]. Recently, time-dependent rolling of the system in the conformal potential unbounded from below was solved exactly in the large \(N\) limit [30]. Moreover, the effects of the CS coupling on the high-energy behavior of the model was considered in [31]. On the other hand, the long wavelength physics of the model was not analyzed and this is the main purpose of the current section. Without the CS term the action is invariant under reflections. Therefore integrating out various degrees of freedom will not
introduce the CS term into the effective action unless it appears in the model from the very beginning.

We demonstrate that once the system is in a phase with spontaneously broken scale invariance, the Maxwell term for the dummy gauge field $A_\mu$ is generated. While it is shown that the CS term does not alter the saddle-point equations, it does affect the long-distance physics. In particular, it screens that part of the confining potential for which the gauge field is responsible. It does so by introducing a mass for the gauge field. As an outcome, the bound states of the system with and without the CS term will not be the same. In fact, there will be less bound states in the presence of a CS term if at all since the long-range force associated with the Maxwell term does not confine in this case. However, without the CS term, only neutral states will be present in the spectrum.

The generating functional of this model is given by

$$Z[J, J^\dagger] = \int D\phi D\phi^\dagger DA_\mu \exp \left[ -S(\phi, \phi^\dagger, A) - NS_{GF}(A) + \int d^3x \left( J^\dagger \cdot \phi + J \cdot \phi^\dagger \right) \right].$$

(4.2)

where a gauge fixing term, $S_{GF}(A)$, was introduced into the action in order to make the generating functional and consequently the Green’s functions well-defined. We choose the Lorentz gauge condition

$$S_{GF}(A) = \frac{1}{2\alpha} \int d^3x (\partial_\mu A^\mu)^2.$$  

(4.3)

Note also, that in our conventions the dimension of the arbitrary parameter $\alpha$ is 1. Hence, in what follows we choose the Landau gauge, $\alpha \to 0$, in order to eliminate the unphysical scale associated with $\alpha$.

Inserting (3.10) and integrating over $\phi_i$, $\phi^\dagger_i$ yields

$$Z[J, J^\dagger] = \int D\rho D\lambda DA_\mu e^{-NS_{eff}(\rho, \lambda, A)} e^{\int J^\dagger(x)(-D_\mu D^\mu + i\lambda)\frac{1}{2\alpha}J(y)d^3xd^3y},$$

(4.4)

where

$$S_{eff}(\rho, \lambda, A) = \int \left[ \frac{g_6}{3} \rho^3 - i\lambda\rho + \frac{\theta}{2} \varepsilon^{\mu\nu}A_\mu F_{\nu\rho} + \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \right] d^3x + \text{Tr ln} \left( -D_\mu D^\mu + i\lambda \right).$$

(4.5)

When $N$ is large, the saddle point method is used. In the Landau gauge for the Lorentz invariant vacuum $\langle A_\mu \rangle = 0$ there is no gauge field contribution to the gap equations. Hence, varying the effective action with respect to the auxiliary fields $\rho$ and $\lambda$, yields

$$\bar{\lambda} = g_6 \rho^2, \quad \overline{\rho} = \text{tr} \frac{1}{-\partial^2 + i\lambda} = -\frac{\sqrt{i\lambda}}{4\pi} = -\frac{m}{4\pi},$$

(4.6)

where barred quantities denote the solution of the gap equation and $m$ will assume a role of a mass. Moreover, we have used the dimensional regularization procedure in order to define the divergent loop. Reinserting the gap equation for $\rho$ into the gap equation for $\lambda$ leads to

$$m^2 = g_6 \frac{m^2}{(4\pi)^2}. $$

(4.7)

\footnote{We use here the definition $\text{Tr} = \int d^3x \text{tr}$}
The theory is in a conformal invariant phase when \( 0 \leq g_6 < (4\pi)^2 \) or when \( g_6 = (4\pi)^2 \) and one chooses \( m = 0 \). The theory is in a spontaneously broken scale invariant phase for \( g_6 = (4\pi)^2 \) and dynamically generated nonzero mass \( m \). In the cases when the coupling is either greater than \( (4\pi)^2 \) or less than 0 the theory is unstable as argued in \([13]\).

### 4.2 Dynamical generation of the Maxwell term.

The aim of this subsection is to examine if in the phase with spontaneously broken scale invariance there is a dynamical generation of the Maxwell term for the \( U(1) \) gauge field introduced above.

Consider the energy, \( E \), is much less than the dynamically generated mass \( m \)
\[
E \ll m. \tag{4.8}
\]

Then according to equation (2.9), once the 3–dimensional effective action (4.5) is expanded in the vicinity of the saddle point \( \rho, \lambda, A_\mu = 0 \), we get
\[
S_{\text{eff}}(\rho, \lambda, A) = \int \left[ \frac{g_6}{3} \rho^3 + g_6 \rho \rho^2 - i \lambda \rho + \frac{i \theta}{2} \varepsilon^{\alpha\mu\nu} A_\alpha F_{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right] d^3x \\
+ \frac{1}{2} \int d^3x \int d^3y G^2(x, y) \lambda(x) \lambda(y) + \frac{1}{96\pi m} \int d^3x F_{\mu\nu} F_{\mu\nu} + \ldots, \tag{4.9}
\]

where
\[
G(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip(x-y)}}{p^2 + m^2}. \tag{4.10}
\]
and the constant terms associated with \( \rho, \lambda \) are omitted from (4.12), whereas ellipsis there denote various interactions of the gauge field \( A_\mu \) with the scalar field \( \lambda \) and their self-interactions. To diagonalize the quadratic part of the effective action, let us apply the following shift
\[
\rho \rightarrow \rho + \frac{i \lambda}{2g_6 \rho}, \tag{4.11}
\]
then
\[
S_{\text{eff}}(\rho, \lambda, A) = \int \left[ \frac{g_6}{3} \left( \rho + \frac{i \lambda}{2g_6 \rho} \right)^3 + g_6 \rho \rho^2 + \frac{\lambda^2}{4g_6 \rho} + \frac{i \theta}{2} \varepsilon^{\alpha\mu\nu} A_\alpha F_{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right] d^3x \\
+ \frac{1}{2} \int d^3x \int d^3y G^2(x, y) \lambda(x) \lambda(y) + \frac{1}{96\pi m} \int d^3x F_{\mu\nu} F_{\mu\nu} + \ldots. \tag{4.12}
\]

There are several effects associated with the spontaneously broken scale invariant phase. A mass is generated for the scalar particles\(^8\) and the Maxwell term is generated for the gauge field \((4.12)\). The effective charge of the particles is fixed by the dynamically generated scale, and according to \((4.12)\), is given by \( \sqrt{24\pi m/N} \).

If, on the other hand, \( 0 \leq g_6 < g_c \), then according to \((4.7)\) \( m = 0 \) and the Maxwell term does not emerge\(^9\). This time expanding around the saddle point and taking the limit \( m \rightarrow 0 \) will lead to a nonlocal gauge invariant term \((2.10)\) with \( d = 3 \). The long-range potential is weaker in this case and does not lead to a confinement.

\(^8\)The \( \phi \)-propagator is obtained by differentiating the partition function \((4.4)\) with respect to the source \( \vec{J} \) and therefore (to leading order in \( 1/N \)) \( i \lambda = m^2 \) plays the role of the physical mass.

\(^9\)Outside \( 0 \leq g_6 \leq g_c \) range the system is unstable \([13]\).
4.3 Confinement.

In general, a massless particle in three spacetime dimensions generates a logarithmic confining potential. For a compact \( U(1) \) gauge symmetry, the nonperturbative effects turn the logarithmic confining potential into a linear confining potential \([32]\). The \( U(1) \) symmetry in this problem is a subgroup of the compact group \( SU(2) \). Nevertheless, for our purposes it is enough that the potential is confining.

In the previous section, we showed that in the phase with spontaneously broken scale invariance a massless gauge particle emerges, and thus it binds the scalar degrees of freedom into neutral states. Furthermore, since the scale symmetry is spontaneously broken there is an associated Goldstone particle - the massless dilaton. Hence, the dilaton on its own would confine the particles as well. In the case of ungauged \( O(N) \) vector model \([19]\) the latter observation was not addressed and therefore we find it instructive to shed light on the confining phenomenon in the current manuscript.

In the effective action (4.12), the dilaton is represented by the scalar field \( i\lambda \), and one can readily verify that \( i\lambda \) is massless by examining its propagator

\[
D_\lambda(x - y) = \langle i\lambda(x) i\lambda(y) \rangle = \frac{8\pi m}{N} \int \frac{d^3 p}{(2\pi)^3} \frac{e^{i p(x-y)}}{1 - 8\pi m B(p)} = \frac{96\pi m^3}{N} \int \frac{d^3 p}{(2\pi)^3} \frac{e^{i p(x-y)}}{p^2} \left( 1 + \frac{3}{20} \frac{p^2}{m^2} + O\left( \frac{p^4}{m^4} \right) \right), \tag{4.13}
\]

where

\[
B(q) = \int \frac{d^3 p}{(2\pi)^3} G(p)G(p + q) = \frac{1}{4\pi q} \arctan \left( \frac{q}{2m} \right). \tag{4.14}
\]

In the absence of CS term (\( \theta = 0 \)) the gauge particle is massless. Hence, in two space dimensions both, the dilaton and the gauge particle, contribute to the logarithmic potential which we now turn to compute.

Differentiating (4.12) with respect to the source \( J \) and \( J^\dagger \) twice and then setting \( J \) to zero leads to a path integral with two insertions of \((-D_\mu D^\mu + i\lambda)^{-1}\). Expanding each such factor around the solution of the gap equation and keeping only linear terms in small perturbations\(^{10} \) leaves us with the following expression for the four-point function (in what follows we adopt the notation of Appendix A and there is no summation on the repeated indices \( a \) and \( b \))

\[
\langle \phi_a(x_1) \phi^\dagger_b(x_2) \phi_a(x_3) \phi^\dagger_b(x_4) \rangle = \int_u \int_w G(x_1 - w)G(w - x_2)(i\lambda(w)i\lambda(u))G(x_3 - u)G(u - x_4) - \left( \frac{\partial}{\partial x_1^\mu} - \frac{\partial}{\partial x_2^\mu} \right) \left( \frac{\partial}{\partial x_3^\nu} - \frac{\partial}{\partial x_4^\nu} \right) \int_u \int_w G(x_1 - w)G(w - x_2)(A^\mu(w)A^\nu(u))G(x_3 - u)G(u - x_4). \tag{4.15}
\]

\(^{10}\)Higher order terms will contribute to the \( 1/N \) correction, since propagators of \( \lambda \) and \( A_\mu \) carry \( 1/N \) factor.
The $A^\mu$ propagator can be read off the quadratic part of the effective action (4.12) and we get in the low energy limit (recall that we consider now $\theta = 0$ case)

$$
\langle A^\mu(x) A^\nu(y) \rangle \simeq \frac{24\pi m}{N} \int_p e^{ip(x-y)} \frac{\delta^{\mu\nu}}{p^2}, \quad (4.16)
$$

where $\alpha = 24\pi m$ has been used in the gauge fixing term.

The diagrams in figure 3 contribute to the nonrelativistic $\phi\phi \rightarrow \phi\phi$ scattering amplitude and thus according to (4.15) we get in this limit

$$
A_{\phi^+\phi^+ \rightarrow \phi^+\phi^+} = A_{\phi^-\phi^- \rightarrow \phi^-\phi^-} \simeq \frac{96\pi m^3}{N} \frac{1}{(p_1 - p_1')^2} + \frac{24\pi m}{N} \frac{(p_1 + p_1')_\mu(p_2 + p_2')^\mu}{(p_1 - p_1')^2} = O(m),
$$

$$
A_{\phi^+\phi^- \rightarrow \phi^+\phi^-} \simeq \frac{192\pi m^3}{N} \frac{1}{(p_1 - p_1')^2}.
$$

(4.17)

Hence, to leading order the electromagnetic repulsion of the similarly charged scalar particles is neutralized by the attractive force due to the dilaton exchange. In contrast, if the particles are oppositely charged, then they are confined by the logarithmic potential

$$
V(r) = \frac{24}{N} m \log(r).
$$

(4.18)

Hence, the bound states of the system must be neutral in this case.

However, it turns out that appearance of the $\theta$-term in the action (4.12) changes the above conclusion. In the $\theta \neq 0$ case dealt within the next section, the gauge field is massive and does not confine in the IR. Thus, in this case the confinement is due only to the dilaton exchange. It is insensitive to the electric charge which is anyhow screened. The latter is reflected in a phase diagram.

Before leaving this subsection it is instructive to examine the coupling of the dilaton to the original scalar degrees of freedom in (4.1). For this purpose, let us examine the leading
$N$ effective action of the full theory in the low energy limit, see the details in Appendix B

\[ \Gamma[\vec{\phi},\psi,A_\mu] = \int_x \left[ \bar{\phi}(-D_\mu D^\mu + m^2)\phi + \frac{1}{2} \psi \left( 1 + \frac{3}{20} \frac{\partial^2}{m^2} + \ldots \right) (-\partial^2 \psi) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right. \\
+ \sqrt{\frac{96\pi m^3}{N}} \bar{\phi} \cdot \phi + \frac{i\theta}{2} \varepsilon^{\alpha\mu\nu} A_\alpha F_{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 \right] + \ldots \] (4.19)

where we have rescaled the Maxwell field and the coupling constants according to $A_\mu \to \sqrt{24\pi m/N} A_\mu$, $\alpha \to 24\pi m \alpha$, $\theta \to \theta/(24\pi m)$; the scalar field $\psi = N^{1/2}(96\pi m^3)^{-1/2} i\lambda$ carries the canonical mass dimension $1/2$, and its low energy propagator possesses a canonical form; $D_\mu = \partial_\mu + i\sqrt{24\pi m/N} A_\mu$ is the covariant derivative, whereas the ellipsis denote the $1/N$ corrections.

As seen from the above expression, the interaction term between the scalars of the model and the emerging dilaton is a relevant operator of dimension $3/2$ and therefore, as argued in section 3.2, is essential for the stability of the system although it is subleading in $1/N$. The same is true for the gauge coupling.

Apparently, the above expression is singular in the limit $m \to 0$. However, in this case one has to examine the full propagator (4.13) instead, since neither the limit $p \ll m$ nor the rescaling of $i\lambda$ are legitimate. In fact, from (4.13) we learn that in this limit the dilaton (as expected) disappears.

### 4.4 The mass of the gauge field in the presence of a CS term.

Let us now illustrate that the CS term introduces a mass also for a gauge field whose Maxwell term was dynamically generated as it does when the Maxwell term is there ab initio. This results in the screening of the logarithmic Coulomb potential computed in the previous section.

Building on (2.5) and (4.5) we get the following expression for the full quadratic in $A_\mu$ piece of the effective action

\[ O(A^2) = N \int p A_\mu(-p)A_\nu(p) \left[ \frac{p^2 \delta^{\mu\nu} - p^\mu p^\nu}{16\pi p} \left( -\frac{m^2}{p} + \frac{b^2 + 4m^2}{p^2} \arctan \frac{p}{m} \right) \right. \\
+ \left. \frac{1}{2\alpha} \frac{p^\mu p^\nu - \theta \varepsilon^{\mu\nu\alpha} p_\alpha}{p^2} \right] \] (4.20)

The propagator of the photon is given by the inverse of the quadratic part

\[ D_{\mu\nu}(p) = \frac{1}{N} \frac{\Gamma}{\Gamma^2 + 8\theta^2 p^2} \left[ \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{2}{\Gamma} \frac{p^2 + 4m^2}{8\pi p} \varepsilon^{\mu\nu\alpha} \frac{p_\alpha}{p^2} \right] \right] + \ldots \] (4.21)

where

\[ \Gamma(p^2) = \frac{m^2}{4\pi p} + \frac{\theta^2 + 4m^2}{8\pi p} \arctan \frac{p}{m} . \] (4.22)

Expanding this propagator in the low energy limit $p \ll m$ and applying the Landau gauge $\alpha \to 0$, yields

\[ D_{\mu\nu}(p) \simeq \frac{24\pi m}{N} \frac{1}{p^2 + M^2} \left[ \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{M^2}{\sqrt{2}} \frac{\varepsilon_{\mu\nu\alpha}}{p^2} \right] . \] (4.23)
where $M_\gamma = 48\sqrt{2}\pi m \theta$.

Due to the presence of the CS term the gauge field becomes massive with $M_\gamma$ being its mass. As a result, the CS term screens the Coulomb part of the spatial confining potential and we get

$$V(r) = \frac{12}{N} m \log(r) \pm \frac{12m}{N} K_0(r M_\gamma) \xrightarrow{r M_\gamma >> 1} \frac{12}{N} m \log(r) \pm m \frac{e^{-r M_\gamma}}{M_\gamma} ,$$

where upper (lower) sign corresponds to the similarly (oppositely) charged particles. Note that when $\theta \to 0$ we recover the previous result, since in the vicinity of zero we have $K_0(x) \simeq -\log x$.

5. A supersymmetric extension of the 3D gauged $\phi^6$ model.

The main goal of the current section is to demonstrate that most of the properties studied in the previous sections are maintained in the presence of SUSY and in addition some new features arise. In particular, using the example of the massive $N = 1$ gauged 3D SUSY model we show that the generation of the Maxwell term is not prevented by supersymmetry. Moreover, the CS term emerges dynamically as well, and there is no need to introduce it by hand into the action. The latter is a consequence of the fact that a fermion mass term is not invariant under reflection and violates parity conservation in odd dimensions. As a result, the long-range $U(1)$ force is screened and the confinement is entirely due to the dilaton and its superpartner dilatino. The Maxwell and the CS terms are accompanied by the superpartners that are generated dynamically: the gaugino kinetic and mass terms respectively.

Furthermore, in the case of the supersymmetric extension of the $\phi^6$ model studied in the previous section, we illustrate that all these terms are dynamically generated if the system is in a special phase where the scale invariance is spontaneously broken. In particular, the parity violation emerges dynamically in this case as it is driven by dynamically generated fermion mass.

5.1 $N = 1$ 3D gauged massive SUSY.

First we show that for a massive $N = 1$ gauged SUSY theory the Maxwell term emerges. Consider a complex scalar superfield

$$\Phi(x, \theta) = A(x) + \theta \psi(x) - \theta^2 F(x) ,$$

with $A$, $\theta^\alpha$ and $\psi^\alpha$ being respectively complex scalar, real Majorana two-component spinor and a two-component complex spinor, while $F$ is a complex auxiliary scalar field. We

12The notations and conventions adopted throughout this section are those of [33]. The representation for the $\gamma$-matrices is taken as $\gamma^\mu = (\sigma_y, i\sigma_z, i\sigma_x)$ and $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$, with the metric being given by $\eta^{\mu\nu} = \text{diag}(-++)$, $\theta^2 = 1/2(\theta^\alpha \theta^\alpha)$ and for an arbitrary two-component complex spinor $\tilde{\psi} = (\psi^\dagger)^*$. First half of the Greek letters $\alpha, \beta, ...$ denotes spinor indices, whereas second half $\mu, \nu, ...$ stands for the Euclidean space indices.
now examine the Lagrangian describing the minimal supersymmetric coupling of a gauge vector particle to a complex massive scalar multiplet

\[ S = -\frac{1}{2} \int d^3x \, d^2\theta \, \left[ (D^\alpha + i\Gamma^\alpha)\Phi(D_\alpha - i\Gamma_\alpha)\Phi - 2m|\Phi|^2 \right], \tag{5.2} \]

where \( D_\alpha = \partial_\alpha + i\partial^\beta \partial_{\alpha\beta} \) is the covariant derivative on a superspace, and the real spinor gauge superfield is given by

\[ \Gamma^\alpha(x,\theta) = \chi^\alpha(x) - \theta^\gamma \left[ \delta^\alpha_\gamma B(x) - iV^\alpha_\gamma(x) \right] - \theta^2 \left[ 2\lambda^\alpha(x) + i\partial^\gamma \chi^\alpha(x) \right], \tag{5.3} \]

with \( \chi^\alpha \) and \( \lambda^\alpha \) being Majorana spinors, \( B \) is a real scalar, whereas \( V^\alpha_\beta = V_\mu(\gamma^\mu)_\beta^\alpha \) is a traceless second-rank spinor corresponding to the vector gauge field \( V_\mu \).

This action can be rewritten in terms of covariant components of \( \Phi(x,\theta) \) defined by covariant projection \[ A'(x) = \Phi(x,\theta)| = A(x), \]
\[ \psi^\alpha(x) = (D_\alpha - i\Gamma_\alpha)\Phi(x,\theta)| = \psi^\alpha(x) - i\chi^\alpha(x)A(x), \]
\[ F'(x) = (D_\alpha - i\Gamma_\alpha)\Phi(x,\theta)| = F(x) - iB(x)A(x) - i\chi(x)\psi(x) - \chi^2(x)A(x), \tag{5.4} \]

where vertical bar means evaluation at \( \theta = 0 \). Omitting primes for simplicity of notation, eliminating auxiliary field \( F \) by using its algebraic equation of motion and performing Euclidean continuation, yields

\[ S = \int d^3x \left[ -\bar{\psi}(\partial - iV + m)\psi - (i\bar{\psi}\gamma A + h.c.) - \bar{A}(\partial_\mu - iV_\mu)^2A + m^2\bar{A}A \right], \tag{5.5} \]

where the Euclidean \( \gamma \)-matrices are taken as \( \gamma_E^\mu = -\sigma_y, \sigma_z, \sigma_x \) and \( \{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu} \).

We now assume that typical momentum is much less than the mass of the complex scalar superfield and integrate out \( \psi \) and \( A \) fields to get an effective (Euclidean) theory of the gauge vector particle and gaugino

\[ S_{\text{eff}}(V_\mu, \lambda) = \text{Tr} \log \left[ -(\partial_\mu - iV_\mu)^2 + m^2 + \lambda(\partial - iV + m)^{-1}\lambda \right] - \text{Tr} \log(\partial - iV + m). \tag{5.6} \]

Expanding this expression in the weak field approximation leads to the following expression for the quadratic part of the effective action

\[ S_{\text{eff}}(\lambda, V_\mu) = \int \frac{d^3p}{(2\pi)^3} V^\mu(-p)V^\nu(p) \frac{p^2\delta_{\mu\nu} - p_\mu p_\nu}{16\pi|p|} \left( -2\frac{|m|}{|p|} + \left( 1 + 4\frac{m^2}{|p|^2} \right) \arctan \left( \frac{|p|}{2|m|} \right) \right) \]

\[ + \int_x \int_y G(x - y)\bar{\lambda}(y)|\lambda + m|^{-1}x\lambda(x) \]

\[ + \int_p V^\mu(-p)V^\nu(p) \frac{p^2\delta_{\mu\nu} - p_\mu p_\nu}{16\pi|p|^2} \left[ 2\frac{|m|}{|p|} + \left( 1 - 4\frac{m^2}{|p|^2} \right) \arctan \left( \frac{|p|}{2|m|} \right) \right] \]

\[ - \frac{m}{4\pi} \varepsilon^{\mu\nu} \int_p V_\mu(-p)V_\nu(p) \frac{p_\lambda}{|p|} \arctan \left( \frac{|p|}{2|m|} \right) + \ldots \tag{5.7} \]
where the first term represents a contribution of the 3D bosonic functional determinant (2.5), the second term is associated with figure 4, whereas the third and fourth terms emerge entirely from the fermionic functional determinant (see Appendix C for details).

Using Feynman parametrization and dimensional regularization (A.5), (A.13), yields

\[ \int G(x - y) \lambda(y) \langle y | (\partial + m)^{-1} | x \rangle \lambda(x) = \int \frac{\lambda(p)}{8 \pi} \left( i \frac{\not{p} + 2 m}{p} \right) \lambda(-p) \arctan \frac{|p|}{2|m|}. \] (5.8)

Combining altogether, we finally obtain

\[ S_{\text{eff}}(\lambda, V^\mu) = \int V^\mu(-p) V^\nu(p) \frac{p^2 \delta_{\mu\nu} - p^\mu p^\nu}{8 \pi |p|} \arctan \frac{|p|}{2|m|} - \frac{1}{8 \pi} \int \lambda(p) \left( i \frac{\not{p} + 2 m}{p} \right) \lambda(-p) \arctan \frac{|p|}{2|m|} - \frac{m}{4 \pi} \varepsilon^{\alpha\mu\nu} \int V^\mu(-p) V^\nu(p) \frac{p_\alpha}{|p|} \arctan \frac{|p|}{2|m|} + \ldots \] (5.9)

In the long wavelength limit \( p \ll m \), one can expand the integrands in the above expression and get

\[ S_{\text{eff}}(\lambda, V^\mu) = \frac{1}{16 \pi |m|} \int \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - i m \varepsilon^{\alpha\mu\nu} V_\alpha F_{\mu\nu} - \lambda(x) (\partial + 2m) \lambda(x) \right] + \ldots \] (5.10)

To make SUSY manifest, we rewrite this action in terms of superfield \( \Gamma^\alpha \) as follows

\[ S_{\text{eff}}(\Gamma^\alpha) = \frac{1}{32 \pi |m|} \int d^3 x d^2 \theta \left( \frac{1}{4} D^\gamma D^\alpha \Gamma_\gamma + m \Gamma^\alpha \right) D^3 D_\alpha \Gamma_\beta + \ldots \] (5.11)

**5.2 Gauged \( \mathcal{N} = 1 \) supersymmetric model in the large \( N \) limit.**

Let us now explore the supersymmetric realization of the \( SU(N) \) invariant \( \phi^6 \) model studied in the previous section and examine the emergence of the Maxwell term. In terms of superfields the action of such a system is given by

\[ S = \frac{1}{2} \int d^3 x d^2 \theta \left[ -(D^\alpha + i \Gamma^\alpha) \bar{\Phi}(D_\alpha - i \Gamma_\alpha) \Phi + \frac{g}{N} (\Phi \bar{\Phi})^2 \right], \] (5.12)
here $\Phi$ is an $N$-component vector superfield and $\Gamma_{\alpha}$ is $SU(N)$ singlet. In component form Euclidean counterpart of this action is given by

$$
S = \int d^3x \left[ -\bar{\psi}(\partial - iV)\psi - (i\bar{\psi}\lambda A + h.c.) - \bar{A}(\partial_\mu - iV_\mu)^2 A \\
+ \frac{g^2}{N^2}(\bar{A}A)^3 - \frac{g}{N}(\bar{A}A)(\bar{\psi}\psi - \bar{\psi}\psi) - \frac{g}{2N}(A\bar{\psi} + \psi\bar{A})(A\bar{\psi} + \psi\bar{A}) \right], \quad (5.13)
$$

Using the following two identities

$$
1 \sim \int D\rho D\sigma e^{-\frac{i}{2}f(\bar{A}A - N\rho)} \sim \int D\rho D\sigma e^{-\frac{i}{2}f(\bar{A}A - N\rho)d^3x} e^{-\frac{i}{2}\xi M\xi} = (\det M)^\frac{1}{2} \int d\eta e^{-\frac{i}{2}n^{-1}\eta + \eta\xi}, \quad (5.14)
$$

where $\xi, \eta$ are two auxiliary Majorana fields, yields

$$
S = \int d^3x \left[ -\bar{\psi}(\partial - iV + g\rho)\psi + [\bar{\psi}(\eta - i\lambda) A + h.c.] - \bar{A} \left[ (\partial_\mu - iV_\mu)^2 - i\sigma \right] A \\
+ N g^2\rho^3 - iN\rho\sigma - \frac{N}{g}\eta^2 \right], \quad (5.16)
$$

Integrating out $\psi$ and $A$, we obtain

$$
S_{eff} = N\text{Tr} \log \left[ -(\partial_\mu - iV_\mu)^2 + i\sigma + (\eta + i\lambda)(\partial - iV + m)^{-1}(\eta - i\lambda) \right] \\
- N\text{Tr} \log(\bar{\psi} - iV + g\rho) + N \int d^3x \left[ g^2\rho^3 - \frac{1}{g}\eta^2 - i\rho\sigma \right]. \quad (5.17)
$$

The last form of the action suggests a saddle point evaluation. The Lorentz invariant gap equations are

$$
\lambda = V_\mu = \xi = \eta = 0, \\
i\sigma = 3g^2\rho^2 - g\text{tr} \frac{1}{\partial + g\tilde{\rho}}, \\
\tilde{\rho} = \text{tr} \frac{1}{-\partial^2 + i\tilde{\sigma}}. \quad (5.18)
$$

These set of equations is unaltered by gauging, namely it is identical to the ungauged case \cite{20}. In particular, the masses of the complex scalar and complex fermion are given respectively by $m_A^2 = i\sigma$ and $m_\psi = g\tilde{\rho}$, therefore one can show that SUSY is maintained, i.e. $m_A^2 = m_\psi^2$. Furthermore, the theory is conformal and possesses two $SU(N)$ invariant phases, one with $g \neq \pm(4\pi)$ or $g = \pm(4\pi)$ and vanishing mass, and the other with spontaneously broken scale invariance and a dynamically generated arbitrary mass for $g_6 = \pm(4\pi)$. In the latter phase the parity violation is dynamically generated.

\footnote{As mentioned, the theory violates parity. However, the space reflection is equivalent to the change $g \rightarrow -g$, and this in turn reveals the origin of the +/- sign above, see e.g. \cite{20}.}
Expanding (5.17) around the solution of the gap equations, \( \rho \to \bar{\rho} + \rho, \sigma \to \bar{\sigma} + \sigma \), keeping only quadratic terms in small perturbations and using (4.14), (5.9) one obtains

\[
S_{\text{eff}} = N \int \frac{d^4p}{(2\pi)^4} \left[ \frac{p^2}{8\pi|p|} \delta_{\mu\nu} - p_\mu p_\nu \right] \text{arctan} \frac{|p|}{2|m|} - \frac{N}{8\pi} \int \rho(p)\rho(-p)|p| \left( \frac{|m|}{|p|} + \left( 1 + \frac{m^2}{p^2} \right) \text{arctan} \frac{|p|}{2|m|} \right) \\
+ \frac{g^2 N}{8\pi} \int \rho(p)\rho(-p)|p| \left( \frac{|m|}{|p|} + \left( 1 + \frac{m^2}{p^2} \right) \text{arctan} \frac{|p|}{2|m|} \right) \\
- \frac{m N}{4\pi} \varepsilon^{\alpha\mu\nu} \int \rho(p)\rho(-p)|p| \left( \frac{|m|}{|p|} + \left( 1 + \frac{m^2}{p^2} \right) \text{arctan} \frac{|p|}{2|m|} \right) \\
+ \frac{N}{2} \int \frac{\sigma(p)\sigma(-p)}{4\pi|p|} \text{arctan} \left( \frac{|p|}{2|m|} \right) \\
+ N \int d^3x \left[ \frac{3g^2}{8\pi} \bar{\rho} \rho - \frac{1}{4} \eta^2 - i\rho \sigma \right] + \ldots \ (5.19)
\]

here \( m = m_{\psi} = m_A \).

Considering the phase with spontaneously broken scale invariance, expanding the above result in the long wavelength limit \( |p| \ll |m| \) and rescaling the fields

\[
\rho \to \tilde{\rho} \sqrt{\frac{3|m|}{2\pi}},
\]

\[
i\sigma \to 8|m| \tilde{\sigma} \sqrt{3\pi|m|}, \quad \text{(5.20)}
\]
yields

\[
S_{\text{eff}}(\lambda, V_\mu) = \frac{N}{16\pi|m|} \int \frac{1}{2} F_{\mu\nu} F_{\mu\nu} - i m \varepsilon^{\alpha\mu\nu} V_\alpha F_{\mu\nu} - \lambda(\bar{\eta} + 2m)\lambda - \eta \bar{\eta} \] \\
+ N \int d^3x \left[ -\tilde{\rho} \left( \partial^2 + 6m^2 \right) \tilde{\rho} - \tilde{\sigma} \left( \partial^2 + 12m^2 \right) \tilde{\sigma} - 12\sqrt{2} m^2 \tilde{\rho} \tilde{\sigma} \right] + \ldots \ (5.21)
\]

The mass matrix of the second line possesses zero eigenvalue associated with the Goldstone boson - the massless dilaton. It necessarily appears in the spectrum since the model exhibits spontaneous breakdown of the scale invariance. The last term in the first line corresponds to the massless dilatino - the superpartner of the dilaton. Finally, the first three terms represent dynamically generated gauge vector particle, gaugino and their appropriate masses.

Note also, that since the gauge field is massive it does not confine in the IR, and thus the bound states emerge due to the logarithmic confining potential generated by either the dilaton or the dilatino. The former binds the particles having the same statistics and thus creates the bosonic bound states, whereas the latter binds the particles possessing different statistics and therefore is responsible for the generation of the fermionic bound states. This behavior is a reflection of SUSY.
6. An anomalous 2D coset model.

Requiring theories to be free of anomalies of local symmetries imposes constraints on the matter content of both gauge theories, general coordinate invariant ones and theories with local scale invariance. One may be tempted to investigate the properties of such theories when they are left to be anomalous. There are general ideas on what should go wrong, but this was not done yet explicitly for theories with Weyl or general coordinate invariance anomalies, it was done for some cases of gauge theories.

A theory with an anomalous gauge symmetry will behave differently in different gauges, this does not imply that the theory is inconsistent in all gauge choices. Consistent refers to unitarity. In fact, in the gauge \( A_0 = 0 \) the theory is unitary, however it is expected that for anomalous theories Lorentz invariance will not be restored as it is in the anomaly free theories. The idea that such a theory will be consistent is implied already in Dirac’s work (his Yeshiva lectures [34]). He discusses the possibility to implement in quantum mechanics a condition that both the coordinate operator \( x \) and its conjugate momentum \( p_x \) annihilate a state although their commutator is a \( c \) number. His answer is that indeed it may happen in the case when the Hamiltonian has neither \( x \) nor \( p_x \) dependence. For an anomalous gauge theory what happens is that the commutator of Gauss’s law at different points contains a \( c \) number.

Two dimensional anomalous gauge theories have been studied in some detail in [22], and we refer the reader to the original paper for many of the details. The anomalous Schwinger model is parameterized by an effective fermion carrying a right handed electric charge \( e_R \) and a left-handed electric charge \( e_L \). The model can be diagonalized in the gauge \( A_0 = 0 \) and its spectrum was obtained. It is indeed unitary and contains particles whose spectrum is not Lorentz invariant. These are described below.

Next let us point out that the Lagrangian realizations of \( G/H \) coset models as mentioned above are conformal theories. Essentially the dynamics is that all states which are not massless have infinite mass in the absence of a Maxwell term for the gauge field, or alternatively in the presence of a Maxwell term the theory becomes very strongly coupled and flows in the infrared to the conformal \( G/H \) theory. Care was always taken that the gauged group \( H \) be anomaly free. This was also important for the geometrical interpretation of such systems. Here we ignore this warning and consider what would go wrong if one gauges an anomalous group \( H \) in the coset construction. That would be tantamount to taking the strong coupling limit in the dispersion relations.

In the \( A_0 = 0 \) gauge when \( G \) and \( H \) are both \( U(1) \), what happens is that the states with a non relativistic dispersion relation obtain an infinite energy and all what survives at finite energy are states with relativistic dispersion relations. The speed of light is renormalized. This we show below. Gauging an anomalous gauge passed with impunity. This may have interesting implications on the geometrical interpretations of the \( G/H \) models leading also to the singular ones.

As shown in [22], the dispersion relation is determined by the positive solutions of the following equation

\[
E^2(E^2 - k^2 - e_L^2 - e_R^2)^2 - k^2(e_L^2 - e_R^2)^2 = 0 ,
\]  

(6.1)
where $e_L$ and $e_R$ denote the total left handed and the total right handed electric charges in the underlying Schwinger model. For $e_L = e_R$ the system is anomaly free. The general solution of this equation is given by

$$E^2 = \frac{a^2}{18u} - \frac{a}{3} + \frac{u}{2}, \quad (6.2)$$

where

$$a = -2(k^2 + e_L^2 + e_R^2),$$

$$u = \left[4k^2(e_L^2 - e_R^2)^2 + \frac{a^3}{27} + k(e_L^2 - e_R^2)\sqrt{16k^2(e_L^2 - e_R^2)^2 + \frac{8}{27}a^3}\right]^{1/3}. \quad (6.3)$$

The above expression for $u$ generates up to three distinct solutions\textsuperscript{14} of $(6.1)$. In the particular case $e_R = 0$, three solutions are given by

$$E_\pm(k) = \sqrt{\frac{k^2}{4} + e_L^2} \pm \frac{k}{2}, \quad (6.4)$$

$$E_R(k) = k. \quad (6.5)$$

In the limit $e_L \gg k$ with $e_R = \alpha e_L + \gamma$, where $\alpha$ and $\gamma$ are fixed, the solutions $(6.2)$ of $(6.1)$ can be expanded as follows

$$E_\pm = e_L\sqrt{1 + \alpha^2} \pm \frac{k}{2}\sqrt{\frac{\alpha^2 - 1}{\alpha^2 + 1}} + \frac{\alpha}{\sqrt{\alpha^2 + 1}} \gamma + \mathcal{O}(k^2/e_L),$$

$$E_R = k\left|\frac{\alpha^2 - 1}{\alpha^2 + 1}\right| + \mathcal{O}(k^2/e_L). \quad (6.6)$$

Let us show how the original symmetry $e_L \leftrightarrow e_R$ of $(6.1)$ manifests itself in the above expression. Inverting the initial ansatz leads to

$$e_L = \frac{1}{\alpha}e_R - \frac{\gamma}{\alpha}, \quad (6.7)$$

Substituting it into $(6.6)$ yields

$$E_\pm = e_R\sqrt{1 + \alpha^{-2}} \pm \frac{k}{2}\sqrt{\frac{\alpha^{-2} - 1}{\alpha^{-2} + 1}} - \frac{\gamma}{\alpha^2\sqrt{\alpha^{-2} + 1}} + \mathcal{O}(k^2/e_L),$$

$$E_R = k\left|\frac{\alpha^{-2} - 1}{\alpha^{-2} + 1}\right| + \mathcal{O}(k^2/e_L). \quad (6.8)$$

As expected from the symmetry $e_L \leftrightarrow e_R$, these solutions are obtained from $(6.6)$ by applying the following replacements (see $(6.7)$)

$$e_L \rightarrow e_R, \quad \alpha \rightarrow \frac{1}{\alpha}, \quad \gamma \rightarrow -\frac{\gamma}{\alpha}. \quad (6.9)$$

\textsuperscript{14} There are three cubic roots related by a factor which is one of the two non-real cubic roots of one, and two square roots of any sign; but these 6 expressions can generate only 3 distinct solutions.
The same results can be obtained by expanding the cubic equation (6.1) rather than its full solution (6.2). Indeed, assuming $e_L \gg k$ with $e_R/e_L = \alpha$ fixed ($\gamma = 0$), yields

$$E^2\left(\frac{k^2}{e^2_L} + \alpha^2 + 1 - E^2/e^2_L\right)^2 - k^2(\alpha^2 - 1)^2 = 0 ,$$

(6.10)

therefore to leading order in $k/e_L$

$$E^2 = k^2 \left(\frac{\alpha^2 - 1}{\alpha^2 + 1}\right)^2 \quad \text{or} \quad E^2 = e^2_L(\alpha^2 + 1) .$$

(6.11)

Hence, in this particular limit the spectrum consists of a single massless particle in a relativistic theory with a modified speed of light. In the example above, the Maxwell term did not emerge just as it did not emerge when the guaging involved in a non anomalous system.

In contrast, in the limit $e_L \gg k$ with $e_R - e_L = \gamma$ fixed ($\alpha = 1$), one gets

$$E^2 \left(\frac{E^2 - k^2}{2e_L + \gamma} - \frac{\gamma^2}{2e_L + \gamma} - \frac{e_L + \gamma}{1 + \gamma/2e_L}\right)^2 - k^2\gamma^2 = 0 ,$$

(6.12)

therefore to leading order in $k/e_L$, the solution is either zero or $E^2 \simeq 2e^2_L$, and thus the model exhibits no finite non trivial spectrum in this particular limit.

7. Concluding Remarks.

We have studied a variety of gauge invariant theories without a Maxwell term. In those theories in which the gauge coupling carried dimensions we found that the term was generated essentially when expected. That is whenever the theory had a scale be it generated dynamically, be it formed in an asymptotic free theory or be it present ab-initio.

Of particular interest to us was the case when the scale symmetry was broken spontaneously and the low energy spectrum consisted only of a massless singlet field of the appropriate group - the dilaton. $\mathcal{N} = 1$ supersymmetry did not obstruct this feature. Whenever a scale was absent the Maxwell term failed to emerge. We have studied in this context aspects of the structure of gauge theories with and without the initial presence of a Chern-Simons term. Some interesting patterns emerged.

In four dimensions the coupling is classically dimensionless and the absent Maxwell term is a classically marginal operator. One way to view its absence which is useful in lower dimensions is to consider this case as the infinite gauge coupling limit. In lower dimensions this is valid as even in the ab-initio presence of a Maxwell term the coupling becomes infinite in the deep IR. In the IR non trivial conformal theories, we saw that as long as the scale symmetry is not broken the Maxwell term does not emerge, while in an asymptotically free theories the coupling is not a free parameter and the term emerges.

Another way to consider this is in the strong coupling limit of a lattice gauge theory. From both points of view the theory will confine and the Maxwell term should emerge.

These general arguments are also true in the case of a dimensionless gauge coupling when the theory without a Maxwell term resides at a fixed point/surface at which the scale
invariance is not spontaneously broken\textsuperscript{15}. An example for such a fixed point is \cite{35}. If the conformal window starting to open before the fixed end point is accompanied also by a formation of a moduli space such as in the $\phi^6$ vector model studied above, then the scale invariance can be spontaneously broken along the moduli space resulting in the emergence of a Maxwell term. This we have shown to occur in the gauged $\phi^6$ vector theory.

For the 4D, $\mathcal{N} = 4$ super conformal SUSY Yang-Mills theory, an $\mathcal{N} = 4$ super symmetrical removal of the Maxwell term does not leave any dynamical terms. For the special gauge theories in $d = 5$ we also expect such a term to emerge at a fixed point which is a part of a moduli space where the scale symmetry can be broken.

In the process of analyzing the various systems we had visited some sideways. It is there where surprises were encountered. We found an additional twist in the 3D $\phi^6$ theory, a logarithmic confinement by dilatons in addition to the confining phase induced by the gauge fields when they are massless. We have found a Lorentz invariant coset model where the gauged subgroup of the global symmetry was anomalous.

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Appendices

A. Expansion of the bosonic functional determinant in the presence of a gauge field.

In this appendix we expand (2.4) about $A_\mu = A^A_\mu T^A = 0$ and verify (2.5). For simplicity, let us denote

$$
\hat{G} \equiv (-\nabla^2 + m^2)^{-1} \iff G(x - y) = \langle x | \hat{G} | y \rangle = \int_p \frac{e^{ip(x-y)}}{p^2 + m^2} . \quad (A.1)
$$

where $x$ represents the spacetime coordinate and $p$ is the momentum $d$-vector.

\textsuperscript{15}We thank Zohar Komargodski for a discussion on this point.
Then one can write

\[ S_{\text{eff}}(A) \equiv \text{Tr} \ln (-D \mu D^\mu + m^2) = \text{Tr} \ln(G^{-1}) + \text{Tr} \hat{G} (A_\mu A^\mu - 2iA_\mu \partial^\mu - i\partial^\mu A_\mu) \]

\[ - \frac{1}{2} \text{Tr} \hat{G} (A_\mu A^\mu - 2iA_\mu \partial^\mu - i\partial^\mu A_\mu) \hat{G} (A_\nu A^\nu - 2iA_\nu \partial^\nu - i\partial^\nu A_\nu) + \ldots \]  

(A.2)

The first term in the above expansion is just a constant and thus can be discarded from the action. Linear terms in \( A_\mu \) are vanishing since \( \text{Tr}(T^A) = 0 \), or alternatively they reveal a total derivative and thus can be suppressed as well\(^{16}\). As a result, the expansion starts from the quadratic terms, which are given by

\[ S_{\text{eff}}(A) = \text{Tr} \hat{G} (A_\mu A^\mu) + \frac{1}{2} \text{Tr} \hat{G} (2A_\mu \partial^\mu + \partial^\mu A_\mu) \hat{G} (2A_\nu \partial^\nu + \partial^\nu A_\nu) + \ldots \]  

(A.3)

In what follows we compute this expression term by term, we start with\(^{17}\)

\[ I_0 \equiv \text{Tr} \hat{G} (A_\mu A^\mu) = \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} m^{d/2} \text{tr} \int_x A_\mu(x) A^\mu(x) , \] 

(A.4)

where the dimensional regularization formula

\[ \int_p \frac{1}{(p^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \Delta^{d/2-n} , \]  

(A.5)

has been used in order to evaluate \( G(x,x) \).

Next term we consider is given by

\[ I_1 \equiv \text{tr} \int_x \int_y \langle x | \hat{G} \partial^\mu A_\mu | y \rangle \langle y | \hat{G} \partial^\nu A_\nu | x \rangle = \text{tr} \int_x \int_y G^2(x-y) \partial^\mu A_\mu(x) \partial^\nu A_\nu(y) . \]  

(A.6)

Building on the definition of \( \hat{G} \) yields

\[ G^2(x-y) = \int_p \int_q \frac{e^{ip(x-y)}}{(q^2 + m^2)((q+p)^2 + m^2)} = \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} \int_p e^{ip(x-y)} p^{d-4} \]

\[ \times \left( \frac{1}{4} + \frac{m^2}{p^2} \right)^{d/2-2} \quad _2F_1 \left( \frac{1}{2}, \frac{4 - d}{2}, \frac{3}{2}; \frac{1}{1 + 4m^2/p^2} \right) . \]  

(A.7)

To evaluate the above loop integral over momentum \( q \) we have used Feynman parametrization and dimensional regularization. As a result, we deduce

\[ I_1 = \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} \text{tr} \int_x \int_y A_\mu(x) A^\nu(y) \int_p e^{ip(x-y)} p^{d-4} p_\mu p_\nu \]

\[ \times \left( \frac{1}{4} + \frac{m^2}{p^2} \right)^{d/2-2} \quad _2F_1 \left( \frac{1}{2}, \frac{4 - d}{2}, \frac{3}{2}; \frac{1}{1 + 4m^2/p^2} \right) . \]  

(A.8)

\(^{16}\)This argument is worthwhile in the case of U(1) where the tracelessness of generators is not applicable.

\(^{17}\)“\( \text{tr} \)” represents trace over color indices.
Finally, the last two terms which are necessary for the computation are

\[ I_2 \equiv \text{tr} \int_x \int_y \langle x | \hat{G} A_\mu \partial^\mu | y \rangle \langle y | \hat{G} A_\nu \partial^\nu | x \rangle \]

\[ = \text{tr} \int_x \int_y \left( \frac{\partial}{\partial y^\mu} G(x - y) A^\mu(y) + G(x - y) \partial_\mu A^\mu(y) \right) \times \left( \frac{\partial}{\partial x^\nu} G(y - x) A^\nu(x) + G(y - x) \partial_\nu A^\nu(x) \right), \tag{A.9} \]

and

\[ I_3 \equiv \text{tr} \int_x \int_y \langle x | \hat{G} A_\mu \partial^\mu | y \rangle \langle y | \hat{G} (\partial^\nu A_\nu) | x \rangle \]

\[ = -\text{tr} \int_x \int_y \left( \frac{\partial}{\partial y^\mu} G(x - y) A^\mu(y) + G(x - y) \partial_\mu A^\mu(y) \right) G(y - x) \partial^\nu A_\nu(x). \tag{A.10} \]

With this definitions at hand, the quadratic part of the effective action for \( A_\mu \) can be written as follows

\[ S_{\text{eff}}(A) = I_0 + \frac{I_1}{2} + 2(I_2 + I_3) + \ldots \tag{A.11} \]

Combining (A.9) and (A.10) yields

\[ I_2 + I_3 = \text{tr} \int_x \int_y \left( \frac{\partial}{\partial y^\mu} G(x - y) A^\mu(y) + G(x - y) \partial_\mu A^\mu(y) \right) \frac{\partial}{\partial x^\nu} G(x - y) A^\nu(x) \]

\[ = -\text{tr} \int_x \int_y A^\mu(y) A^\nu(x) \int_p e^{ip(x - y)} \int_q \frac{g_\mu g_\nu}{[q^2 + m^2][(q + p)^2 + m^2]}, \tag{A.12} \]

Using Feynman parametrization, (A.5) and

\[ \int_p \left( \frac{p^2}{p^2 + \Delta} \right)^m = \frac{d/2}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)} \Delta^{d/2 - n + 1}, \tag{A.13} \]

yields

\[ I_2 + I_3 = -\frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} \int_x \int_y A^\mu(y) A^\nu(x) \int_p e^{ip(x - y)} p^{d - 2} \left[ \delta_{\mu\nu} f_1(m/p) + \frac{p_\mu p_\nu}{p^2} f_2(m/p) \right], \tag{A.14} \]

where

\[ f_1(m/p) = \frac{1}{2} \left( \frac{1}{4} + \frac{m^2}{p^2} \right)^{d/2 - 1} \quad 2F_1 \left( \frac{1}{2} ; \frac{2 - d}{2} , \frac{3}{2} ; \frac{1}{1 + 4m^2/p^2} \right), \tag{A.15} \]

and

\[ f_2(m/p) = -\frac{d - 2}{24} \left( \frac{1}{4} + \frac{m^2}{p^2} \right)^{d/2 - 2} \]

\[ \times (3 \cdot 2F_1 \left[ 1/2 , 2 - d/2 , 3/2 ; (1 + 4m^2/p^2)^{-1} \right] + \gray{2F_1 \left[ 3/2 , 2 - d/2 , 5/2 ; (1 + 4m^2/p^2)^{-1} \right]}). \tag{A.16} \]

Substituting now (A.4), (A.8) and (A.14) into (A.11) we recover (P.5).
B. Effective action of the gauged \((\phi^2)^3\) model to leading order in 1/\(N\).

In this appendix we derive the effective action of the gauged \((\phi^2)^3\) model \([4.4]\) to leading order in 1/\(N\). The derivation is carried out when the system is in the phase with spontaneously broken scale invariance. In particular, the coupling of the dilaton and the \(U(1)\) gauge field to the scalar field \(\phi\) is elucidated.

By definition, the effective action is given by

\[
\Gamma[\tilde{\phi}_{cl}] = W(\vec{J}, \vec{J}^\dagger) - \int_x \left[ \vec{J}^\dagger \cdot \tilde{\phi}_{cl} + \vec{J} \cdot \tilde{\phi}^\dagger_{cl} \right], \quad (B.1)
\]

where as usual

\[
e^{-W(\vec{J}, \vec{J}^\dagger)} = Z[\vec{J}, \vec{J}^\dagger], \quad \tilde{\phi}_{cl}(x) = \frac{\delta W(\vec{J}, \vec{J}^\dagger)}{\delta \vec{J}^\dagger(x)}, \quad \tilde{\phi}^\dagger_{cl}(x) = \frac{\delta W(\vec{J}, \vec{J}^\dagger)}{\delta \vec{J}(x)}. \quad (B.2)
\]

From \([4.3]\), we learn that to leading order in 1/\(N\)

\[
W(\vec{J}, \vec{J}^\dagger) = - \int_x \int_y \int_{x_1} \int_{x_2} J_x G_{xy} J_y + \frac{1}{2} \prod_{i=1}^{i=4} \int_x \left[ \vec{J}^\dagger_{x_1} \cdot \frac{\partial J_{x_2}^{\nu}}{\partial x_1^\mu} - \vec{J}^\dagger_{x_2} \cdot \frac{\partial J_{x_1}^{\nu}}{\partial x_2^\mu} \right] \left( \vec{J} \cdot \frac{\partial J_{x_3}^{\mu}}{\partial x_2^\nu} - \vec{J} \cdot \frac{\partial J_{x_4}^{\mu}}{\partial x_3^\nu} \right)
\]

\[
\times \int_w \int_u G_{x_1,w} G_{w,x_2} D^{\mu\nu}(w - u) G_{x_3,u} G_{u,x_4}
\]

\[
- \frac{1}{2} \prod_{i=1}^{i=4} \int_w \int_u \int_{x_1} \int_{x_2} \vec{J}^\dagger_{x_1} \cdot \vec{J}_{x_2} G_{x_1,w} G_{w,x_2} D_\lambda(w - u) G_{x_3,u} G_{u,x_4} \vec{J} \cdot \vec{J} + \ldots \quad (B.3)
\]

where the notation of \([4.13]\) and \([4.21]\) is used to denote propagators of the dilaton and the gauge particle respectively. The subscripts in the above formula indicate the coordinate(s) on which a given quantity depends, while the ellipsis here and thereafter denote higher order terms in the external source and in 1/\(N\).

Hence,

\[
\tilde{\phi}_{cl}(x) = - \int_y G_{xy} J_y - \prod_{i=1}^{i=3} \int_x \left[ J_{x_2} \cdot \frac{\partial J_{x_1}^{\nu}}{\partial x_2^\mu} - J_{x_1} \cdot \frac{\partial J_{x_2}^{\nu}}{\partial x_1^\mu} \right] \left( J_{x_3} \cdot \frac{\partial J_{x_1}^{\mu}}{\partial x_2^\nu} + \frac{\partial J_{x_1}^{\mu}}{\partial x_3^\nu} \right)
\]

\[
\times \int_u G_{x_1,u} G_{u,x_2} D^{\mu\nu}(w - u) G_{x_3,u} G_{u,x_4}
\]

\[
- \prod_{i=1}^{i=3} \int_x \int_w \int_u \vec{J}^\dagger_{x_1} \cdot \vec{J}_{x_2} G_{x_1,u} G_{w,x_2} D_\lambda(w - u) G_{u,x_3} G_{x_3,u} + \ldots \quad (B.4)
\]

or equivalently

\[
\vec{J}(x) = (\partial^2 - m^2) \tilde{\phi}_{cl}(x) + \int_w \tilde{\phi}_{cl}(x) D_\lambda(x - w) \tilde{\phi}(w) \cdot \tilde{\phi}^\dagger_{cl}(w) \quad (B.5)
\]

\[
+ \int_u \int_w \left( \tilde{\phi} \cdot \frac{\partial \tilde{\phi}^\dagger}{\partial w^\mu} - \tilde{\phi}^\dagger \cdot \frac{\partial \tilde{\phi}}{\partial w^\mu} \right) \left( \tilde{\phi} \cdot \frac{\partial \tilde{\phi}}{\partial w^\nu} + \frac{\partial \tilde{\phi}}{\partial w^\nu} \right) D^{\mu\nu}(w - u) \delta(x - u) + \ldots
\]
and similarly for $\tilde{\phi}_d^\dagger$ and $\tilde{J}^\dagger$. Substituting back into (B.1), yields

$$
\Gamma[\tilde{\phi}_d] = \int_x \tilde{\phi}_d^\dagger(-\partial^2 + m^2)\tilde{\phi}_d - \frac{1}{2} \int_u \int_w \tilde{\phi}_d(u) \cdot \tilde{\phi}_d^\dagger(u) \cdot D_\lambda(u-w) \tilde{\phi}_d(w) \cdot \tilde{\phi}_d^\dagger(w)
+ \frac{1}{2} \int_u \int_w \left( \frac{\partial \tilde{\phi}_d}{\partial w^\mu} - \frac{\partial \tilde{\phi}_d^\dagger}{\partial w^\mu} \right) \left( \frac{\partial \tilde{\phi}_d}{\partial x^\nu} - \frac{\partial \tilde{\phi}_d^\dagger}{\partial x^\nu} \right) D^{\mu\nu}(u-w) + \ldots (B.6)
$$

Using (4.21) and reintroducing $A_\mu$ and $i\lambda$, this can be written as follows

$$
\Gamma[\tilde{\phi}_d, i\lambda, A_\mu] = \int_x \left[ \tilde{\phi}_d^\dagger(-D_\mu D^\mu + m^2)\tilde{\phi}_d + i\lambda \tilde{\phi}_d \cdot \tilde{\phi}_d^\dagger \right]
+ N \int_p A_\mu(-p)A_\nu(p) \left[ \frac{p^2 \delta^{\mu\nu} - p^\mu p^\nu}{16\pi p} \left( -\frac{2m}{p} + \frac{p^2 + 4m^2}{p^2} \arctan \frac{p}{2m} \right) \right]
+ \frac{1}{2\alpha} p^\mu p^\nu - \theta \epsilon^{\mu\nu\rho\sigma} p_\rho \right] + \frac{1}{2} \int_u \int_w i\lambda(u) D_\lambda^{-1}(u-w) i\lambda(w) + \ldots (B.7)
$$

Substituting (4.13), rescaling the fields $i\lambda \rightarrow \sqrt{96\pi m^3/N} \psi$, $A_\mu \rightarrow \sqrt{24\pi m/N} A_\mu$ to establish the canonical form of the propagators and redefining the coupling constants $\alpha \rightarrow 24\pi m/\alpha$, $\theta \rightarrow \theta/(24\pi m)$, we obtain the low energy effective action

$$
\Gamma[\tilde{\phi}_d, \psi, A_\mu] = \int_x \left[ \tilde{\phi}_d^\dagger(-D_\mu D^\mu + m^2)\tilde{\phi}_d + \frac{1}{2} \psi \left( 1 + \frac{3}{20m^2} \partial^2 + \ldots \right) (-\partial^2 \psi) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu}
+ \sqrt{\frac{96\pi m^3}{N}} \phi_d \cdot \tilde{\phi}_d \psi + \frac{i\theta}{2} \epsilon^{\alpha\mu\nu} A_\alpha F_{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right] + \ldots (B.8)
$$

with $D_\mu = \partial_\mu + i\sqrt{\frac{24\pi m}{N}} A_\mu$.

C. Expansion of the 3D fermionic functional determinant in the presence of a gauge field.

The aim of this appendix is to compute the contribution of the fermionic functional determinant $\text{Tr} \log(\underline{\partial} - i\mathbb{Y} + m)$ to the quadratic part of the effective action (5.7).

Expanding around $\mathbb{Y} = 0$ yields

$$
\text{Tr} \log(\underline{\partial} - i\mathbb{Y} + m) = \text{Tr} \log(\underline{\partial} + m) + \frac{1}{2} \text{Tr} \left[ (\underline{\partial} + m)^{-1} \mathbb{V}(\underline{\partial} + m)^{-1} \mathbb{V} \right] + \ldots
= \text{Tr} \log(\underline{\partial} + m) - \int_p \mathbb{V}^\mu (-p) \mathbb{V}^\nu(p) \int_q 2q_\mu (p + q)_\nu - 2m q_\alpha \epsilon^{\alpha\mu\nu} - [m^2 + q(p + q)] \delta^{\mu\nu} + \ldots
$$

(C.1)

where we performed a shift in the integration variable $p \rightarrow p + q$ and used the following identities

$$
\langle x | (\underline{\partial} + m)^{-1} | y \rangle = \int_p \frac{m - ip_\nu}{p^2 + m^2} e^{ip(x-y)},
\text{tr} (\gamma^\mu \gamma^\nu \gamma^\alpha) = -2i \epsilon^{\mu\nu\alpha} \left( \epsilon^{012} = 1 \right),
\text{Tr} \sigma_1 \sigma_2 \sigma_3 \sigma_4 = 2[a_1 \cdot a_2 a_3 \cdot a_4 - a_1 \cdot a_3 a_2 \cdot a_4 + a_1 \cdot a_4 a_2 \cdot a_3] \ .
$$

(C.2)
Using Feynman parametrization and the dimensional regularization scheme (A.5), (A.13), yields

\[
\text{Tr} \log(\partial - iV + m) = \text{Tr} \log(\partial + m) - \int_p V^\mu(-p)V^\nu(p) \frac{p^2 \delta_{\mu\nu} - p^\mu p^\nu}{16\pi |p|} \left[ 2 \frac{|m|}{|p|} + \left( 1 - 4 \frac{m^2}{p^2} \right) \arctan \frac{|p|}{2|m|} \right] + \frac{m}{4\pi} \varepsilon^{\alpha\mu\nu} \int_p V_\mu(-p)V_\nu(p) \frac{p_\alpha}{|p|} \arctan \frac{|p|}{2|m|} + \ldots \quad (C.3)
\]

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