Simultaneous All-versus-Nothing Refutation of Local Realism and Noncontextuality by a Single System

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The quantum realms of nonlocality and contextuality are delineated by Bell’s theorem and the Kochen-Specker theorem, respectively, embodying phenomena that surpass the explanatory capacities of classical theories. These realms hold transformative potential for the fields of information and computing technology. In this study, we unveil a “all-versus-nothing” proof that concurrently illustrates the veracity of these two seminal theorems, fostering a more nuanced comprehension of the intricate relationship intertwining quantum nonlocality and contextuality. Leveraging the capabilities of three singlet pairs and a Greenberger-Horne-Zeilinger state analyzer, our proof not only substantiates the conflict between quantum mechanics and hidden-variable theories from another perspective, but can also be readily verifiable utilizing the existing linear optics technology.

Keywords: Bell’s theorem, Kochen-Specker theorem, quantum nonlocality, quantum contextuality

I. INTRODUCTION

Despite the fact that quantum mechanics (QM) is by far the most successful physical theory, its counterintuitive nature remains a conceptual barrier to understanding its foundations. Einstein, Podolsky and Rosen (EPR) discovered one of the physics’ most famous paradoxes [1] among those early efforts on quantum foundations: According to the assumption of local realism, a maximally entangled two-party system appears to allow one to infer definite values of complementary variables (e.g., position and momentum) by using perfect correlations, which contradicts QM. Several hidden-variable theories were inspired by the EPR paradox. To deny hidden variables in QM, two major theorems, Bell’s theorem, [2] and the Kochen-Specker (KS) theorem [3–10], have been developed. Based on EPR’s notion of local realism, the former denies local hidden variables (LHV), whereas the latter poses a more serious challenge to hidden variables. The KS theorem states that it is impossible to assign definite values to each of not all commutative observables in an individual system, a phenomenon known as quantum contextuality. This means that even in the absence of the locality condition, the value of an observable is dependent on the experimental context, thus ruling out noncontextual hidden variables (NCHV). In addition, Klyachko et al. derived inequalities [11, 12] that are valid for NCHV, but violated for individual quantum systems.

The original Bell’s theorem argument is strongly statistical, and perfect two-qubit correlations would not violate Bell’s inequalities. To highlight the contradiction between QM and LHV for definite predictions, the “Bell’s theorem without inequalities”, also known as the Greenberg-Horne-Zeilinger (GHZ) theorem, was developed, which has been demonstrated for the case of three [13–16] or even two [17–20] spacelike separated observers. Because this proof without inequalities is based on perfect correlations, it directly refutes EPR’s notion of local realism, implying a stronger refutation of local realism.

Quantum nonlocality (i.e., violations of Bell’s inequalities) and quantum contextuality both empower certain quantum information tasks [9, 21], according to quantum information science. The search for possible relationships between the two nonclassical phenomena follows naturally. Since the works of Stairs [22], Heywood and Redhead [23], Mermin [24], and R.D. Gill [25, 26], recent years have seen significant advances on the intertwining connections [21] between quantum contextuality and nonlocality in terms of inequalities, such as converting local contextuality into nonlocality [27–29], simultaneously observing contextuality and nonlocality [30, 31], and a trade-off relation between nonlocality and contextuality [32–34].

In this study, we present an demonstration that both Bell’s theorem and the KS theorem, inherently devoid of inequalities, can be illustrated harmoniously within a single system. Drawing inspiration from Mermin’s proof [24] centered on three-qubit Pauli operators for the KS theorem and the “all-versus-nothing” proof of Bell’s theorem [19, 35, 36], our proposal can be viewed as a natural extension of these theorems. It simultaneously rejects both local and non-contextual hidden variables, adopting an all-versus-nothing approach. Within the contemporary landscape of rapid advancements, our proof furnishes compelling evidence supporting the correlation between quantum contextuality and nonlocality, offering a

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fresh lens to perceive their interrelationship.

II. THEORY

Consider the arrangement shown in Fig. 1. Initially, Debbie possesses three pairs of antisymmetric singlet Bell states $|\Psi\rangle_{123456} = |\psi^-\rangle_{12} \otimes |\psi^-\rangle_{34} \otimes |\psi^-\rangle_{56}$. Here the singlet state

$$|\psi^-\rangle_{ij} = \frac{1}{\sqrt{2}}(|01\rangle_{ij} - |10\rangle_{ij}),$$

where $|01\rangle_{ij} = |0\rangle_i \otimes |1\rangle_j$. Assume Debbie keeps qubits 2, 4, and 6 in her laboratory, and sends qubits 1, 3, and 5 to Alice, Bob, and Charlie, respectively. Each of these four observers are in a spacelike separated region from the other three observers. The qubits are then measured locally in their respective laboratories. If Debbie performs the GHZ-state measurement, the arrangement considered here is the GHZ-entanglement swapping as experimentally demonstrated in Ref. [37]. In our experimental setup, we assume that each pair of entangled sources is independent. This assumption is crucial, as it restricts our study to exploring quantum nonlocality within the specific context of network scenarios, where the independence of sources plays a key role in the manifestation of nonlocal correlations [38, 39].

We first discuss the negation of local realism by our scheme. To make the following discussion clearly, we define $|\Phi\rangle = |\Phi\rangle_{123456}$, $x_i = \sigma_{xi}$, $y_i = \sigma_{yi}$ ($i = 1, 2, ..., 6$) and $x_2y_4y_6 = \sigma_2\sigma_y\sigma_6$, etc., and apply ($\cdot$) to separate local operators. Thus, one can easily check that the state $|\Phi\rangle$ satisfies the following eigenequations:

$$x_2y_4y_6 \cdot x_1 \cdot y_3 \cdot y_5 |\Psi\rangle = - |\Psi\rangle, \quad (2)$$
$$y_2x_4y_6 \cdot y_1 \cdot x_3 \cdot y_5 |\Psi\rangle = - |\Psi\rangle, \quad (3)$$
$$y_2x_4y_6 \cdot y_1 \cdot y_3 \cdot x_5 |\Psi\rangle = - |\Psi\rangle, \quad (4)$$
$$x_2x_4x_6 \cdot x_1 \cdot x_3 \cdot x_5 |\Psi\rangle = - |\Psi\rangle, \quad (5)$$
$$x_2x_4x_6 \cdot x_2y_4y_6 \cdot y_2x_4y_6 \cdot y_2y_4x_6 |\Psi\rangle = - |\Psi\rangle. \quad (6)$$

The last equation [Eq. (6)] stems from the operator identity $x_2x_4x_6 \cdot x_2y_4y_6 \cdot y_2x_4y_6 \cdot y_2y_4x_6 = -I_{246}$, where $I_{246}$ is the identity operator for three qubits 2, 4, and 6.

Only local operators (Alice’s, Bob’s, Charlie’s, or Debbie’s) are used in the equations (2)-(6), implying that any three of Alice, Bob, Charlie and Debbie can confidently assign values to the local operators of the remaining party by measuring their local observables without disturbing the system of that party in any way. According to EPR’s notion of local realism [1], “If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity”. As a result, according to those who believe in local realistic theories, each local operator in Eqs. (2)-(6) separated by ($\cdot$) can be defined as EPR’s local “elements of reality”. Moreover, all these elements of reality have predetermined values, which are denoted by $v(x_i), v(y_i), v(x_2y_4y_6), v(y_2x_4y_6), v(y_2y_4x_6)$, and $v(x_2x_4x_6)$ with $v = \pm 1$. Therefore, Eqs. (2)-(6) can be rewritten in a local realistic theory as

$$v(x_2y_4y_6)v(x_1)v(y_3)v(x_5) = -1, \quad (7)$$
$$v(y_2x_4y_6)v(y_1)v(x_3)v(y_5) = -1, \quad (8)$$
$$v(y_2y_4x_6)v(y_1)v(y_3)v(x_5) = -1, \quad (9)$$
$$v(x_2x_4x_6)v(x_1)v(x_3)v(x_5) = -1, \quad (10)$$
$$v(x_2x_4x_6)v(x_2y_4y_6)v(y_2x_4y_6)v(y_2y_4x_6) = -1. \quad (11)$$

However, it is simple to see that Eqs. (7)-(11) are mutually incompatible. Because each $v$-value appears twice on the left-hand sides of Eqs. (7)-(11), the product of all the left-hand sides must equal $+1$, which contradicts the result $-1$ obtained by multiplying all the right-hand sides. This contradiction demonstrates that any local realistic theory based on EPR’s notion of local realism is incapable of reproducing the results of perfect correlations predicted by QM as in Eqs. (2)-(6). This brings the proof of all-versus-nothing nonlocality to a close. This type of logical impossibility proof is known as Bell’s theorem without inequalities. Our scheme’s experimental issues (particularly the measurement strategy) will be discussed further below.

Bell’s theorem without inequalities has been demonstrated for the case of three [13–16] or even two [17–20] spacelike separated observers. It is not surprising, then, that our argument using four spacelike separated observers can demonstrate Bell’s theorem without inequalities. However, our argument differs from previous ones in
that local realism and noncontextuality can be denied simultaneously by one and the same experimental system. This type of argument sheds new light on the relationship between Bell’s theorem and the KS theorem.

To deny noncontextuality in our scheme, we only need to focus on Debbie’s system of three qubits 2, 4, and 6. Consider the ten operators \( x_2, y_2, x_4, y_4, x_6, y_6, x_2 y_4 y_6, y_2 x_4 y_6, y_2 y_4 x_6 \), and \( x_2 x_4 x_6 \) of this system. Now we have five operator identities [24]

\[
\begin{align*}
x_2 y_4 y_6 \cdot x_2 \cdot y_4 \cdot y_6 &= I_{246}, \\
y_2 x_4 y_6 \cdot y_2 \cdot x_4 \cdot y_6 &= I_{246}, \\
y_2 y_4 x_6 \cdot y_2 \cdot y_4 \cdot x_6 &= I_{246}, \\
x_2 x_4 x_6 \cdot x_2 \cdot x_4 \cdot x_6 &= I_{246}, \\
x_2 x_4 x_6 \cdot x_2 y_4 y_6 \cdot y_2 x_4 y_6 \cdot y_2 y_4 x_6 &= -I_{246}.
\end{align*}
\]

These identities are quantum mechanical predictions that are independent of the state of the system. Note that, in each of Eqs. (12)-(16), these operators separated by \( \cdot \) have predetermined values, but here without appealing to EPR’s local realism. Let us denote these predetermined values by \( v(x_i), v(y_i) \), \( v(x_2 y_4 y_6), v(y_2 x_4 y_6), v(y_2 y_4 x_6) \), and \( v(x_2 x_4 x_6) \) with \( v = \pm 1 \). Therefore, Eqs. (12)-(16) can be rewritten in an NCHV theory as

\[
\begin{align*}
v(x_2 y_4 y_6)v(x_2)v(y_4)v(y_6) &= 1, \\
v(y_2 x_4 y_6)v(y_2)v(x_4)v(y_6) &= 1, \\
v(y_2 y_4 x_6)v(y_2)v(y_4)v(x_6) &= 1, \\
v(x_2 x_4 x_6)v(x_2)v(x_4)v(x_6) &= 1, \\
v(x_2 x_4 x_6)v(x_2 y_4 y_6)v(y_2 x_4 y_6)v(y_2 y_4 x_6) &= -1.
\end{align*}
\]

NCHV theories can explain the equations (17)-(20). According to NCHV theories, the predetermined values of these operators can be measured without disturbance, regardless of the context in which they are contained. In other words, no matter how an operator \( A \) is measured for a specific system, the result of measuring \( A \) will always be \( v(A) \). Therefore, \( v(x_2 y_4 y_6) = v(x_2)v(y_4)v(x_6) \) in Eq. (17) because \( x_2 y_4 y_6 \) can be measured by separately measuring \( x_2, y_4 \) and \( y_6 \) and multiplying their results. Alternatively, because they are mutually commuting operators, \( x_2 y_4 y_6, x_2, y_4 \) and \( y_6 \) can be co-measured by a single apparatus, yielding the product of the four operators to be \( +1 \). Likewise, Eqs. (18)-(20) are also consequences of noncontextuality. Finally, the left-hand side of Eq. (21) is the product of mutually commuting operators \( x_2 y_4 y_6, y_2 x_4 y_6, y_2 y_4 x_6 \) and \( x_2 x_4 x_6 \), which can be simultaneously measured by a single measurement apparatus, to be considered below. Now, the contradiction between NCHV theories and QM can be seen by combining Eqs. (17)-(21).

Because each \( v \)-value appears twice on the left-hand sides of Eqs. (17)-(21) and can only be assigned the value \(+1\) or \(-1\), the product of the left-hand sides must be \(+1\), whereas the product of the right-hand sides must be \(-1\). As a result, the attempt to assign values to these ten operators, as shown in Eqs. (17)-(21), must fail. This failure implies that any NCHV theory contradicts the quantum mechanical predictions in Eqs. (12)-(16). The contradiction occurs on a nonstatistical level in predictions for these operators. As a result, our scheme completes the demonstration of the all-versus-nothing noncontextuality without specifying any state of the system. Namely, our proof of contextuality is state-independent.

Note that Mermin [24], generalizing a state-dependent argument due to Peres [40], made use of the same set of operator identities in Eqs. (12)-(16) for his argument on the KS theorem. To validate his GHZ argument, the three qubits involved must be spacelike separated. In contrast, for both the KS and GHZ arguments, the three qubits are always kept on Debbie’s side in our proof. Therefore, the current work represents a genuinely simultaneous all-versus-nothing refutation of local realism and noncontextuality by a single system.

In an ideal experimental situation, each of Eqs. (2)-(6) would be proven true by experimental data when demonstrating the negation of local realism. This demonstrates that QM can withstand the experiment’s scrutiny. Experiments would reject any local realistic theory that is fundamentally contradictory to QM when combined with the preceding discussion. Unfortunately, perfect correlations and ideal measurement devices are difficult to prepare. Therefore, to address these less-than-ideal experimental situations, one can introduce the operator \[ O = x_2 y_4 y_6 \cdot x_1 \cdot y_3 \cdot y_5 + y_2 x_4 y_6 \cdot y_1 \cdot x_3 \cdot y_5 + y_2 y_4 x_6 \cdot y_1 \cdot y_3 \cdot x_5 + x_2 x_4 x_6 \cdot x_1 \cdot x_3 \cdot x_5 \], in which \(-x_2 x_4 x_6 \cdot x_2 y_4 y_6 \cdot y_2 x_4 y_6 \cdot y_2 y_4 x_6 \) being an identity operator is not included. Combined with Eqs. (2)-(6), one can easily check that \( |\Psi\rangle \) is an eigenstate of the operator \( O \), thus having

\[ O|\Psi\rangle = -4|\Psi\rangle. \]

Because of the operator identity in Eq. (16), one of the observed values of these four operators \( x_2 y_4 y_6, y_2 x_4 y_6, y_2 y_4 x_6 \) and \( x_2 x_4 x_6 \) is always different from the other three. As a result, any local realistic theory predicts the observed values of the operator \( O \)

\[ |\langle O\rangle_{\text{LRT}}| \leq 2, \]

which is a Bell-Mermin inequality [41] for \( |\Psi\rangle \). The inequality (23) also shows a contradiction with the prediction (22) of QM, and can thus be verified in a less-than-ideal experimental situation. Suppose that we have a noisy singlet state \( F |\psi^-angle (|\psi^-\rangle + 1/\sqrt{2} I \rangle, \) where \( I \) is the unit operator in the two-qubit state space. Then the violation of the inequality (23) can be observed as long as the fidelity \( F \) of the noisy singlet state is greater than \( \sqrt{1/2} \approx 0.7071 \). While our experimental design is indeed based on logical contradictions to demonstrate the discrepancies between local realism and quantum mechanical predictions, it inherently does not negate the
necessity for statistical analysis. In this context, the Bell-Mermin inequality links theory with the empirical evidence needed, highlighting the essential role of statistics in experimental physics to approximate theoretical ideals.

Similarly, in an experimental demonstration of contextuality, one can define the operator \( O' = x_2 y_4 y_6 \cdot x_2 \cdot y_4 \cdot y_6 + y_2 x_4 y_6 \cdot y_2 \cdot x_4 \cdot y_6 + y_2 y_4 x_6 \cdot y_2 \cdot y_4 \cdot x_6 + x_2 x_4 x_6 \cdot x_2 \cdot x_4 \cdot x_6 \), for which any NCHV theory must predict in a less-than-ideal experimental situation

\[
|\langle O' \rangle_{\text{NCHV}}| \leq 2. \tag{24}
\]

In contrast, the maximal value of quantum mechanical prediction of \( |\langle O' \rangle_{\text{QM}}| \) is 4. This difference allows a statistical test of NCHV theories in a less-than-ideal experimental situation. Suppose that we have a noisy singlet state \( F |\psi^-\rangle (\psi^-) + \frac{1-F}{2} I \), where \( I \) is the unit operator in the two-qubit state space. Then the violation of the inequality (24) can be observed as long as the fidelity \( F \) of the noisy singlet state is greater than \( \sqrt{1/2} \approx 0.7071 \).

We have discussed how our scheme demonstrates simultaneous refutation of local realism and noncontextuality by one and the same experimental system in a less-than-ideal experimental situation. Here we further emphasize that these four operators \( x_2 y_4 y_6, y_2 x_4 y_6, y_2 y_4 x_6 \) and \( x_2 x_4 x_6 \) are measured with one and the same apparatus, which is crucial for demonstrating the negation of local realism [19]. In our argument of the negation of local realism, each of the four operators \( x_2 y_4 y_6, y_2 x_4 y_6, y_2 y_4 x_6 \) and \( x_2 x_4 x_6 \) appears twice in Eqs. (2)-(6). For example, the operator \( x_2 y_4 y_6 \) appears in Eqs. (2) and (6), whereas Eqs. (2) and (6) involve two different experimental contexts for measuring \( x_2 y_4 y_6 \). LHV, with emphasis, does not assume noncontextuality. In other words, LHV only emphasizes that all operators have predetermined values, but it does not rule out the possibility of measuring an operator causing a disturbance in the system, thereby modifying the values of other operators. As a result, those who believe in local realistic theory can argue that the contradiction in Eqs. (7)-(11) is caused by contextuality, rather than EPR’s local elements of reality. To avoid this problem, one either adds the additional assumption of noncontextuality, or measures the four operators \( x_2 y_4 y_6, y_2 x_4 y_6, y_2 y_4 x_6 \) and \( x_2 x_4 x_6 \) with one and the same apparatus such that they are measured within a single context. Note that our argument on the negation of noncontextuality does not require such apparatuses, since noncontextuality is already assumed in this argument.

### III. CONCEPTUALIZATION OF AN EXPERIMENTAL SCHEME

As a nonlocality argument, one of course has to avoid the introduction of additional assumption (i.e., noncontextuality). Then one needs to measure the four operators \( x_2 y_4 y_6, y_2 x_4 y_6, y_2 y_4 x_6 \) and \( x_2 x_4 x_6 \) with the same apparatus. A possible measurement strategy of our nonlocality argument is to perform the GHZ-state measurement [42] on photonic qubits 2, 4, and 6 of Debbie [Fig. 2(a)]. With emphasis, photonic GHZ-state discrimination realized with linear optics can only identify two out of the eight maximally entangled GHZ states, namely, \( \Phi^+_0 = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle|0\rangle \pm |1\rangle|1\rangle|1\rangle) \), where \( |0\rangle = |H\rangle \) for horizontal polarization and \( |1\rangle = |V\rangle \) for vertical polarization. See Appendix A for a detailed discussion. Fortunately, this does not affect our argument, because one only needs to postselect the corresponding experimental data when \( \Phi^+_0 = \frac{1}{\sqrt{2}} (|0\rangle|0\rangle|0\rangle \pm |1\rangle|1\rangle|1\rangle) \) is measured. To verify that this post-selection is desirable and it does not affect our conclusions, one can expand \( |\Psi\rangle \) into the form of the GHZ-entanglement swapping [37]
can measure therein the four operators argument as stated above. To verify Eq. (16), Debbie there is no way to measure all 10 operators in Eqs. (12)-

Thus, whenever a GHZ state, e.g., $\Phi^-_0$, satisfies Eqs. (2)-(6). Within the reach of the current quantum optical technology, one to demonstrate our scheme using the GHZ post-measurement values of the four operators, whose eigenstates, and then feed qubits 2, 4, and 6 into the GHZ-state analyzer to measure $x_2y_4y_6$. Figure 3(a) shows as an example the case where, before entering the GHZ-state analyzer, qubits 2, 4, and 6 collapse to the state $|+RR\rangle$, with $|+\rangle (|R\rangle)$ being the eigenstate in the $X (Y)$ basis and subscripts omitted for simplicity. This means that the measurement results of $x_2$, $y_4$, and $y_6$ are all +1. Meanwhile, $|+RR\rangle$ can be expanded in terms of the GHZ states (see Appendix B)

$$|+RR\rangle = \frac{1}{2}(|\Phi^-_0\rangle + |\Phi^-_1\rangle + i|\Phi^+_2\rangle + i|\Phi^+_3\rangle).$$ (26)

When combined with Table I the joint measurement of $x_2y_4y_6$ for $|+RR\rangle$ must yield +1. As a result, as long as Debbie observes coincidence in the GHZ-state analyzer, Eq. (12) is verified. With emphasis, this verification method is not 100% successful because the probability of qubits 2, 4, and 6 all entering the GHZ-state analyzer is only $1/8$, and the GHZ-state analyzer can only identify two of the eight maximally entangled GHZ states. Again, this does not affect our argument. Similarly, Fig. 3(b-d) are used to verify Eqs. (13)-(15), respectively. Before entering the GHZ-state analyzer in Fig. 3(b-d), qubits 2, 4, and 6 collapse into $|R + R\rangle$, $|RR+\rangle$, and $|+++\rangle$, respectively. Here

$$|R + R\rangle = \frac{1}{2}(|\Phi^-_0\rangle + i|\Phi^+_1\rangle + |\Phi^+_2\rangle + i|\Phi^+_3\rangle)$$

$$|RR+\rangle = \frac{1}{2}(|\Phi^-_0\rangle + i|\Phi^+_1\rangle + i|\Phi^+_2\rangle + |\Phi^+_3\rangle)$$

$$|+++\rangle = \frac{1}{2}(|\Phi^+_1\rangle + |\Phi^+_2\rangle + |\Phi^+_3\rangle).$$ (27)

which, together with Table I, confirm Eqs. (13)-(15). Thus, following the above measurement strategy Eqs. (12)-(16) can be tested quantum mechanically, but are not consistent with NCHV.

**IV. CONCLUSION**

Previous research has chiefly advanced our understanding of the intriguing connections between quantum noncontextuality and nonlocality, primarily focusing on the role

| $x_2x_4y_6$ | $x_2y_4y_6$ | $y_2x_4y_6$ | $y_2y_4x_6$ |
|----------|----------|----------|----------|
| $\Phi^-_0$ | +1 | -1 | -1 | -1 |
| $\Phi^-_1$ | -1 | +1 | +1 | +1 |
| $\Phi^+_2$ | +1 | -1 | +1 | +1 |
| $\Phi^+_3$ | -1 | -1 | +1 | +1 |

TABLE I. The results of jointly measuring $x_2x_4y_6$, $x_2y_4y_6$, $y_2x_4y_6$ and $y_2y_4x_6$ upon different GHZ states.
of inequalities. Our study showcases the feasibility of demonstrating both quantum contextuality and nonlocality, devoid of inequalities, within a singular system. It is worth noting that Ref. [30] illustrated the concurrent observation of quantum contextuality and nonlocality utilizing inequalities. Our approach, however, surpasses the strategy adopted in Ref. [30] by eschewing the dependency on inequalities, echoing the superiority of the GHZ theorem over Bell’s theorem.

While Bell’s theorem utilizes statistical methods to challenge LHV, raising potential doubts regarding confidence levels, the GHZ theorem seeks to obliterate this statistical refutation, providing a conclusive rebuttal of LHV through singular, precise measurements under optimal conditions. Likewise, the methodology in Ref. [30] observes a coincidence of classical and quantum theories at “trivial points” of perfect correlations, thereby instigating a hunt for discrepancies between QB and LHV (or NCHV) amidst a myriad of imperfectly correlated points, ultimately culminating in complex schemes. Moreover, this approach finds it challenging to pinpoint any contradictions between QB and LHV (or NCHV) for definite predictions.

In contrast, our technique significantly amplifies entanglement correlations by employing three pairs of antisymmetric singlet Bell states [Eqs. (25)], thereby simplifying the framework to a great extent. The result is that we can demonstrate the contradiction between QB and LHV (or NCHV) even for certain predictions, namely, just a logical contradiction between a few equations is enough to reveal the contradiction between QB and LHV (or NCHV).

Furthermore, we have devised a complementary experimental scheme aligned with our theory, viable with current linear optics technology. Quantum nonlocality and contextuality are pivotal resources empowering various quantum information tasks to transcend classical boundaries. Nevertheless, leveraging them synergistically remains a complex issue. Our efforts offer new insights into this dilemma, with the hope of providing additional ideas for the development of novel quantum protocols. Future endeavours should pivot towards empirical validations to affirm the concurrent negation of local realism and non-contextuality within unified systems, along with an exploration into the fascinating interplay between these two quantum resources.

In conclusion, we introduce an approach that demonstrates quantum contextuality and nonlocality without relying on inequalities, contrasting with traditional methods that use statistical analysis and inequalities. Our method employs three pairs of antisymmetric singlet Bell states and GHZ-state measurements, simplifying the experimental framework significantly. This approach allows for the clear demonstration of contradictions between quantum mechanics and local hidden variables, even with specific predictions. Our experimental scheme, compatible with current linear optics technology, offers new perspectives in quantum information science, highlighting the interplay between nonlocality and contextuality and their potential applications in quantum information tasks.

**CREDIT AUTHORSHIP CONTRIBUTION STATEMENT**

Min-Gang Zhou: Writing–original draft, Visualization, Investigation. Hua-Lei Yin: Writing–review editing, Validation, Supervision, Funding acquisition. Zeng-Bing Chen: Writing–review editing, Validation, Supervision, Funding acquisition, Conceptualization.

**DECLARATION OF COMPETING INTEREST**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**DATA AVAILABILITY**

No data was used for the research described in the article.

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**Appendix A: IMPLEMENTATION AND ANALYSIS OF GHZ STATE**

Projecting photons into GHZ states is a necessary means for many quantum information tasks. However, due to the difficulty in implementing controlled-NOT (CNOT) gate required for GHZ state preparation through linear optics and single photons, current photonic GHZ-state analyzers probabilistically distinguish between two out of the $2^N$ maximally entangled GHZ states formed by $N$ particles.
In this work, photonic qubits 2, 4, and 6 of Debbie are spectrally indistinguishable identical photons. Therefore, eight polarization GHZ states could be written as

\[
|\Phi_0^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle|0\rangle \pm |1\rangle|1\rangle|1\rangle), \\
|\Phi_0^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle|0\rangle|0\rangle \pm |0\rangle|1\rangle|1\rangle), \\
|\Phi_2^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle|0\rangle \pm |1\rangle|0\rangle|1\rangle), \\
|\Phi_2^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle|1\rangle \pm |1\rangle|1\rangle|0\rangle),
\]

(A1)

where \(|0\rangle = |H\rangle\) for horizontal polarization and \(|1\rangle = |V\rangle\) for vertical polarization. The three photons are now fed into the GHZ analyzer by modes A, B, and C (Fig. 2(a)). Since the PBSs transmit \(H\) and reflect \(V\) polarizations, only when all photons are transmitted or reflected can a coincidence detection at the three outputs be generated. The result is that two GHZ states, i.e. \(|\Phi_0^+\rangle\), can be distinguished from these eight GHZ states. Further, an HWP can be placed after each PBS to distinguish \(|\Phi_0^-\rangle\).

In this case, the \(|\Phi_0^+\rangle\) state will result in coincidences \(+ + +\), \(+ + -\), \(- + +\), and \(- + -\), and \(+ - +\), \(+ - -\), \(- + +\), \(- - +\). In this way, photonic qubits 2, 4, and 6 of Debbie are projected into one of the GHZ states and the corresponding physical values of \(x_2y_4y_6\), \(y_2x_4y_6\), \(y_2y_4x_6\) and \(x_2x_4x_6\) are obtained.

**Appendix B: Expansion of GHZ-states on Other Polarization Basis**

The eight maximally entangled GHZ states formed by three photons can be expanded on other polarization basis. For example,

\[
|\Phi_1^+\rangle = \frac{1}{2}(|+ + +\rangle + |+ - -\rangle + |- + +\rangle + |- - -\rangle), \\
= \frac{1}{2}(|RL+\rangle + |LR+\rangle - |RR-\rangle - |LL-\rangle),
\]

(B3)

\[
|\Phi_1^-\rangle = \frac{1}{2}(|+ + +\rangle + |+ - -\rangle - |- + +\rangle - |- - -\rangle), \\
= \frac{1}{2}(|RL+\rangle - |LR+\rangle + |RR-\rangle + |LL-\rangle),
\]

(B4)

\[
|\Phi_2^+\rangle = \frac{1}{2}(|+ + +\rangle - |+ - +\rangle - |- + -\rangle + |- - +\rangle), \\
= \frac{1}{2}(|RL-\rangle + |LR-\rangle + |RR+\rangle + |LL+\rangle),
\]

(B5)

\[
|\Phi_2^-\rangle = \frac{1}{2}(|+ + +\rangle - |+ - +\rangle + |- + -\rangle + |- - +\rangle), \\
= \frac{1}{2}(|RL-\rangle - |LR-\rangle - |RR+\rangle - |LL+\rangle),
\]

(B6)

\[
|\Phi_3^+\rangle = \frac{1}{2}(|+ + +\rangle - |+ - +\rangle - |- + -\rangle + |- - +\rangle), \\
= \frac{1}{2}(|RR+\rangle + |LL+\rangle - |LR-\rangle - |RL-\rangle),
\]

(B7)

\[
|\Phi_3^-\rangle = \frac{1}{2}(|+ + +\rangle + |+ - +\rangle - |- + -\rangle - |- - +\rangle), \\
= \frac{1}{2}(|RL+\rangle - |LR+\rangle + |RR-\rangle + |LL-\rangle).
\]
\[
\Phi_5 = \frac{1}{2} (| + + + - \rangle + | + - + + \rangle - | - + + - \rangle - | - - + + \rangle)
\]
\[
= \frac{1}{2} (| + \ L R \rangle - | + \ R L \rangle + | - \ R R \rangle - | - \ L L \rangle)
\]
\[
= \frac{1}{2} (|L + R\rangle - |R + L\rangle + |R - R\rangle - |L - L\rangle)
\]
\[
= \frac{1}{2} (|RR+\rangle + |LL+\rangle - |RL-\rangle - |LR-\rangle),
\]
(B8)

where \(|+\rangle\) and \(|-\rangle\) (|R\rangle and |L\rangle) are the eigenstates in the X (Y) basis. The Eqs. 26, Eqs. 27 and Table I can be calculated according to the above formulas, thus the feasibility of the scheme in Fig. 3 can be verified.

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