Bounds on the *dragging rate* and on the *rotational mass-energy* in slowly and differentially rotating relativistic stars

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Abstract

For relativistic stars rotating slowly and differentially with a positive angular velocity, some properties in relation to the positiveness of the rate of rotational dragging and of the angular momentum density are derived. Also, a new proof for the bounds on the rotational mass-energy is given.

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I. INTRODUCTION

In the prescription for calculating a slowly and differentially rotating relativistic stellar configuration the field equations are expanded in powers of a fluid angular velocity parameter and the perturbations (around a non-rotating configuration) are calculated by retaining only first and second order terms. Hartle has derived these equations of structure in the rigidly rotating case. However, at first order the only effect of the rotation is to drag the inertial frames; at second order it also deforms the star. In fact, the first order equations of structure reduce to the time-angle field equation component (to first order), which is a partial differential equation, linear in the *angular velocity of cumulative dragging* (or *dragging rate*).

This linearity in the dragging rate potential persuades us to write that equation in appropriate coordinates —in order to avoid the coordinate singularity occurring on the axis of rotation in spherical polar coordinates, generally used in the slow rotation approximation— so that in the new coordinates the equation writes in a “regular” form as an elliptic equation with bounded coefficients, and to apply a minimum principle for generalized supersolutions in the whole domain (interior and exterior of the star). Making use of the asymptotic flatness condition, this will lead us directly to the positiveness of the dragging rate, provided that the distribution of angular velocity of the fluid is non-negative everywhere (and non-trivial) and that we start from a reasonable unperturbed (non-rotating) stellar model satisfying the weak energy condition. This and the positive-
ness of other quantities also linear in the angular velocity, like the \textit{angular momentum density}, will be the purpose of this paper.

The \textit{rotational mass-energy}, derived by Hartle\cite{hartle9}, although accurate to second order in the angular velocity, involves only quantities which can be calculated from the first order structure equation (time-angle component of the Einstein equations) as well. A proof of the positivity and an upper bound on this rotational energy is given in the same paper\cite{hartle9}, however using an expansion in eigenfunctions. We shall give here a much simpler proof of these bounds, without using that expansion, and, hence, avoiding the non-trivial mathematical problems on the existence of these eigenfunctions.

The paper is organized as follows. After a description of the relativistic rotating stellar model in Sec. II, and a brief revision of the concepts of angular momentum density and rate of rotational dragging in Sec. III, in Sec. IV we concentrate on the slow rotation approximation, particularly on the first order perturbations of the metric (linear correction of the dragging rate, with description of the unperturbed (zero order) configuration), and explicit expressions for the expansions of the angular momentum density and of the rotational mass-energy are derived. In the same section the null contribution (at first order in the angular velocity) of the integrability condition of the Euler equation is discussed, and the time-angle component of the Einstein equations (to first order) is written in appropriate coordinates, as a background allowing to apply a minimum principle and obtain the first of the properties mentioned above and proved in Sec. V, and consequences of that one. Apart from this, as an independent result, an alternative proof of the bounds on the rotational energy is given. Finally, in Sec. VI, concluding remarks are briefly stated.

II. THE RELATIVISTIC ROTATING STELLAR MODEL

The spacetime of a rotating relativistic star is represented by a Lorentzian 4-manifold $(\mathcal{M}, g)$ which satisfies the following

A. Assumptions

i. the spacetime is stationary in time and axially symmetric, which means that $g$ admits two global Killing vector fields, a time-like future-directed one, $\xi$, and a space-like one, with closed trajectories, $\eta$, except on a time-like 2-surface (defining the axis of rotation) where $\eta$ vanishes;

ii. the spacetime is asymptotically flat; in particular, $g(\xi, \xi) \to -1$, $g(\eta, \eta) \to +\infty$, and $g(\xi, \eta) \to 0$ at spatial infinity (the signature of the metric $g$ being $(-+++)$);

iii. the matter —confined in a compact region in the space (interior), with vacuum on the outside, so that (ii) holds—is perfect fluid, and therefore the energy-momentum tensor (source of the Einstein equations) is written as

$$T = (\varepsilon + p) u^a \otimes u^b + p \, g,$$

where $\varepsilon$ and $p$ denote the energy density and the pressure of the fluid, respectively; and $u^a$ denotes the 1-form equivalent to the 4-velocity of the fluid $u$ (in the exterior $T \equiv 0$; hence, $\varepsilon + p = p = 0$ there);
iv. the fluid velocity is azimuthal (non-convective) \((circularity\ condition)\), i.e.

\[
u^\flat \wedge \xi^\flat \wedge \eta^\flat = 0;
\]

v. \((\mathcal{M}, g)\) satisfies Einstein’s field equations \(G = 8\pi T\) for the energy-momentum tensor \(T\) of a perfect fluid (iii), where \(G \equiv \text{Ric} - \frac{1}{2} R g\) denotes the Einstein tensor — equations which can also be written in the form

\[
\text{Ric} = 8\pi (T - \frac{1}{2} \text{tr}(T) g);
\]

vi. \(\varepsilon\) and \(p\) satisfy a barotropic (one-parameter) equation of state, \(\varepsilon = \varepsilon(p)\);

vii. \(\varepsilon + p \geq 0\) \((weak\ energy\ condition\ for\ perfect\ fluid,\ assuming\ \varepsilon \geq 0)\)

viii. the metric functions are essentially bounded.

B. Form of the metric

Assumptions (i) and (ii) imply that the two Killing fields commute, \([\xi, \eta] = 0\) which is equivalent to the existence of coordinates \(x^0 \equiv t\) and \(x^1 \equiv \phi\) such that \(\xi \equiv \partial_t\) and \(\eta \equiv \partial_\phi\); moreover, by the circularity condition (iv), the spacetime \(g\) admits 2-surfaces orthogonal to the group orbits of the Killing fields (orthogonal transitivity). We may then choose the two remaining coordinates \((x^2, x^3)\) in one of these 2-surfaces and carry them to the whole spacetime along the integral curves of \(\xi\) and \(\eta\); accordingly, the metric can be written in the form

\[
ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta =
\]

\[
= g_{tt} dt^2 + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{22} (dx^2)^2 + 2 g_{23} dx^2 dx^3 + g_{33} (dx^3)^2,
\]

where the metric coefficients are independent of the time \(x^0 \equiv t\) and azimuthal \(x^1 \equiv \phi\) coordinates corresponding to the two Killing fields; that is, \(g_{\alpha\beta} = g_{\alpha\beta}(x^2, x^3)\).

When solving Einstein’s field equations it is convenient to specify the coordinates \(x^2\) and \(x^3\) in such a way as to simplify the task; a particular choice, usually made when studying slowly rotating configurations, is the one which makes \(g_{23} = 0\) and \(g_{33} = g_{\phi\phi} \sin^{-2} x^3\). Hence \(x^2\) and \(x^3\) are chosen so that at large spatial distances the asymptotically flat metric is expressed in terms of spherical polar coordinates in the usual way. In the resulting coordinate system, with the notation \(x^2 \equiv r, x^3 \equiv \theta\), and with new symbols for the metric functions, the line element reads

\[
ds^2 = -H^2 dt^2 + Q^2 dr^2 + r^2 K^2 [d\theta^2 + \sin^2 \theta (d\phi - Adt)^2],
\]

where \(H, Q, K\) and \(A\) are functions of \(r\) and \(\theta\) alone. In these coordinates \((r \geq 0, 0 \leq \theta \leq \pi)\) the spatial infinity is given by \(r \to \infty\), and the axis of rotation \((\partial_\phi \equiv \eta = 0)\) is described by \(\theta \to 0\) or \(\pi\) \((r \geq 0)\).

Notice that the function \(A\) appears in the metric \((2)\) as the non-vanishing of the \((t\phi)\) metric component of a rotating configuration. \(A\) is actually the \(\text{dragging rate\ potential}\) (cf. Sec. III).
C. Differential rotation

According to assumption (iv) of Subsec. II.A —circularity condition—, the fluid 4-velocity \( u \) has the form

\[
u = u^t \partial_t + u^\phi \partial_\phi = u^t(\partial_t + \Omega \partial_\phi), \quad \text{where} \quad \Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi}{dt},
\]

is the angular velocity of the fluid measured in units of \( t \), i.e. as seen by an inertial observer at infinity whose proper time is the same as the coordinate \( t \) (observer \( \partial_t \)), and \( u^t \) is the normalization factor, such that \( g(u,u) = -1 \), i.e. \((u^t)^{-2} = -(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi})\). The star’s matter rotates then in the azimuthal direction \( \phi \).

We consider a star rotating differentially, with a prescribed distribution of angular velocity \( \Omega \equiv \Omega(x^2, x^3) \equiv \Omega(r, \theta) \), an essentially bounded function. However, with the assumptions made (Subsec. II.A), the rotation profile of the fluid cannot be freely chosen, this shows up in the following.

We consider the equation of hydrostatic equilibrium, \( T^{\alpha\beta}_{\beta} = 0 \) (integrability conditions of the field equations), particularly, its part orthogonal to the fluid 4-velocity \( u \), i.e. the Euler equation,

\[
\frac{dp}{dt} = -\left(\varepsilon + p\right) a,
\]

where \( a \) is the 4-acceleration of the fluid, \( a^2 = \nabla_u u \), specifically,

\[
a = -d\ln u^t + u^tu_\phi d\Omega.
\]

And the integrability condition of Eq. (3), taking into account (vi) of Subsec. II.A, \( \varepsilon = \varepsilon(p) \), is \( da = 0 \); following, from (4), \( d(u^t u_\phi) \wedge d\Omega = 0 \); in other words, the fluid angular velocity, \( \Omega \), is functionally related to the specific angular momentum times \( u^t \),

\[
u^t u_\phi = F(\Omega).
\]

Nevertheless, as will be seen in Subsec. IV.B, in the slow rotation approximation, at first order in the angular velocity, Eq. (5) is no restriction on \( \Omega(r, \theta) \).

III. ANGULAR MOMENTUM DENSITY AND DRAGGING RATE

The total angular momentum of a rotating relativistic star can be defined from the variational principle for general relativity —for an isolated system which is not radiating gravitational waves—, but this is shown to coincide with the geometrical definition—from the asymptotic form of the metric at large space-like distances from the rotating fluid— (analog to the ADM mass), which for stationary and axisymmetric (asymptotically flat) spacetimes is given by the Komar integral for the angular momentum,

\[
J = \int_I T^t_{\alpha\beta} \eta^\alpha n_\beta dv = \int_I T^t_{\phi} n_{\phi} dv = \int_I T^t_{\phi}(-g)^{1/2} d^3x,
\]

where

\[
\eta^\alpha n_\beta = \delta^\alpha_{\beta} - \eta^\alpha_{\beta} = -\delta^\alpha_{\beta}.
\]
where $T$ is the energy-momentum tensor of perfect fluid, $\eta$ is the Killing field corresponding to the axial symmetry, $n$ is the unit time-like and future pointing normal to the hypersurface of constant $t$, i.e. $n = n_t dt$, with $n_t > 0$, and $dv$ is the proper volume element of the surface $t = \text{const.}$, i.e. $\int_I dv = \text{Vol}$, the volume of the body of the star, $I \equiv \text{interior of the star (} t = \text{const.)}$. $g \equiv \det(g)$. The invariantly defined integrand of this volume integral $\eta^\beta n_\beta$, is what one would naturally define as angular momentum density —coinciding with the standard form in special relativity—, and can be calculated

\[
T^\alpha_\beta \eta^\alpha n_\beta = n_t T^t_\phi = n_t (\varepsilon + p)u^t u_\phi \quad [u_\phi = u^t (g_{t\phi} + \Omega g_{\phi\phi})]
\]

\[
= n_t (\varepsilon + p)(u^t)^2 (g_{t\phi} + \Omega g_{\phi\phi})
\]

\[
= n_t (\varepsilon + p)(u^t)^2 g_{\phi\phi}(\Omega - A), \quad \text{with } n_t = H = \left(\frac{-g_{t\phi} g_{\phi\phi} + g_{t\phi}^2}{g_{\phi\phi}}\right)^{1/2},
\]

where $A$ is the metric function (cf. (2)) such that

\[
g_{t\phi} = -A g_{\phi\phi}.
\]

It is remarkable that, since $n_t > 0$, $g_{\phi\phi} \geq 0$, and we are assuming the energy condition $\varepsilon + p \geq 0$ ((vii) in Subsec. II.A), the sign of the angular momentum density (8) is determined by the sign of the difference $\Omega - A$.

The metric function $A$ is indeed the angular velocity of a particle which is dragged along in the gravitational field of the star, as seen from a non-rotating observer at spatial infinity ($\partial_t$), so that it has zero angular momentum relative to the axis, $p_\phi = 0$,

\[
d\phi \over dt = p^\phi \over p^t = g^{t\phi} p_t \over g^{t\phi} = g^{t\phi} \over g_{\phi\phi} = -g_{t\phi} \over g_{\phi\phi}; \quad g_{t\phi} + \left(\frac{d\phi}{dt}\right) g_{\phi\phi} = 0; \quad \frac{d\phi}{dt} = A.
\]

$A$ is called angular velocity of cumulative dragging (shortly called dragging rate). One of the purposes of this work is precisely to find appropriate bounds on the uniformly non-negative distribution of angular velocity, $\Omega \equiv \Omega(x^2, x^3) \geq 0$, of a slowly differentially rotating star, so that $\Omega - A \geq 0$ holds; from where the positivity of the angular momentum density (8) follows (Property (c) in Sec. V).

Observe, in the special relativistic limit ($g_{t\phi} \to 0$, using coordinates $(x^2, x^3)$ which go at spatial infinity to the usual flat coordinates, cf. Subsec. II.B), if the fluid rotates uniformly with angular velocity $\Omega$ positive (negative), then the angular momentum density, Eq. (8), is uniformly positive (negative).

**IV. SLOWLY DIFFERENTIALLY ROTATING STARS. FIRST ORDER PER-TURBATIONS**

By slow rotation we mean that the absolute value of the angular velocity is much smaller than the critical value $\Omega_{\text{crit}} \equiv (M/R^3)^{1/2}$ (taking units $c = G = 1$), where $M$ is the total mass of the unperturbed (non-rotating) configuration, and $R$, its radius; $|\Omega(x^2, x^3)|/\Omega_{\text{crit}} \ll 1$. Thus, stars which rotate slowly can be studied by expanding the Einstein field equations for a fully relativistic differentially rotating star in powers of the dimensionless ratio
\[ \frac{\Omega_{\text{max}}}{\Omega_{\text{crit}}} =: \mu, \quad (10) \]

where \( \Omega_{\text{max}} \) is the maximum value of \( |\Omega(x^2, x^3)| \) (at the interior of the star).

### A. The metric and the energy density and pressure of the fluid

We assume that the star is slowly rotating, with angular velocity

\[ \Omega(r, \theta) \equiv \Omega(x^2, x^3) = O(\mu), \]

parameter \( \mu \) given e.g. by (10). Because the star (stationary in time and axially symmetric) rotates in the \( \phi \) direction ((iv) of Subsec. II.A), a time reversal \((t \to -t)\) would change the sense of rotation, as well as an inversion in the \( \phi \) direction \((\phi \to -\phi)\) would do. As a result, the metric coefficients \( H, Q \) and \( K \) (in (2)) and the energy density will not change sign under one of these inversions, whereas \( A \) will do. Therefore, an expansion of \( H, Q \) and \( K \), as well as of the energy density, \( \varepsilon \), and, hence, of the pressure, \( p \), in powers of the angular velocity parameter \( \mu \) will contain only even powers, while an expansion of \( A \) will have only odd ones.

Accordingly, at first order in the angular velocity, \( O(\mu) \), the metric coefficients, and fluid energy density and pressure, are

\[
\begin{align*}
H &= H_0 + O(\mu^2) \\
Q &= Q_0 + O(\mu^2) \\
K &= K_0 + O(\mu^2) \\
\varepsilon &= \varepsilon_0 + O(\mu^2) \\
p &= p_0 + O(\mu^2)
\end{align*}
\]

but \( A = \omega + O(\mu^3) \),

\( (11) \)

where \( H_0, Q_0 \) and \( K_0 \) are the coefficients of the unperturbed (non-rotating) configuration, and \( \omega \) denotes the linear (first order) correction in \( \mu \) of the dragging rate \( A \), so that, from Eq. (2),

\[ g_{t\phi} = -\omega (g_{\phi\phi})_0 + O(\mu^3). \quad (12) \]

We shall keep here only the effects linear in the angular velocity. The only first order \( O(\mu) \) perturbation brought about by the rotation is the dragging of the inertial frames; the star is still spherical, because the “potential functions” which deform the shape of the star are \( O(\mu^2) \).

#### 1. The (unperturbed) non-rotating configuration

The starting non-rotating equilibrium configuration is described by the spherically symmetric metric in the Schwarzschild form
\[ ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]
\[ \equiv -H_0^2 dt^2 + Q_0^2 dr^2 + r^2 K_0^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (A_0 = 0), \]
with \( \lambda(r) \), or equivalently, the mass \( m(r) \) interior to a given radial coordinate \( r \), given by
\[ 1 - \frac{2m(r)}{r} = e^{-\lambda(r)}, \]
and \( \nu(r) \), together with the pressure \( p_0(r) \), and the energy density \( \varepsilon_0(r) \), solutions of the system of equations of general relativistic hydrostatics, which for a non-rotating configuration are: the equation of hydrostatic equilibrium (Tolman-Oppenheimer-Volkoff equation),
\[ \frac{dp_0}{dr}(r) = - \frac{[\varepsilon_0(r) + p_0(r)][m(r) + 4\pi r^3 p_0(r)]}{r^2 [1 - 2m(r)/r]}, \]
the mass equation,
\[ \frac{dm}{dr}(r) = 4\pi r^2 \varepsilon_0(r), \]
and the source equation for \( \nu \),
\[ \frac{d\nu}{dr}(r) = - \frac{2}{\varepsilon_0(r) + p_0(r)} \frac{dp_0}{dr}(r), \]
with the initial boundary conditions
\[ 0 < p_0(0) = p_{0c} < \infty \quad (\text{central pressure}), \]
\[ m(0) = 0, \quad \text{and} \]
\[ \nu(0) = \nu_c \quad (\text{constant fixed by the asymptotic condition at infinity}), \]
this being the prescription for the interior of the star, that is, inside the fluid, \( r \leq R \); \( R \equiv \) radius of the surface of the star, determined by \( p_0(R) = 0 \). Furthermore, we assume \( p_0 \) and \( \varepsilon_0 \) related to each other by a barotropic equation of state,
\[ \varepsilon_0 = \varepsilon_0(p_0), \]
\[ p_0 \mapsto \varepsilon_0(p_0) \quad \text{a bounded function on any closed interval, and satisfying the weak energy condition} \]
\[ \varepsilon_0 + p_0 \geq 0. \]
Observe, from Eqs. (15) and (19), \( p_0 \) is a decreasing function from the center, \( r = 0 \), to the star’s surface, \( r = R \); in particular, \( p_0 \geq 0 \) and attains its maximum value \( p_{0c} \) at the center.

In the exterior (vacuum) the geometry is described by the same line element (13), but with the metric function \( \nu \) specified and related to \( \lambda \) by
\[ e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{2M}{r}, \quad \forall r > R, \]
where \( M \equiv m(R) \) is the star’s total mass.
Consider the first integral of the Euler equation (3), namely, \( r, \theta \). It will be important to note that at first order the integrability condition of the Euler equation is \( \theta \). and (19) for the starting non-rotating configuration are (vi) and (vii) of Subsec. II.A

from where the effect of the rotation can be seen as given by the term \( d\phi - Adt \) in the place of \( d\phi \).

Note, since, as has been seen above in Subsec. IV.A, at first order in the angular velocity there is still no effect on the pressure, and on the energy density, conditions (18) and (19) for the starting non-rotating configuration are (vi) and (vii) of Subsec. II.A (at first order).

### B. Euler equation

It will be important to note that at first order the integrability condition of the Euler equation and, hence, Eq. (5) are no restriction on \( \Omega(r, \theta) \), that shows up in the following. Consider the first integral of the Euler equation (3), namely,

\[
\int_0^\infty \frac{d\bar{p}}{\bar{p} + \bar{p}} + \frac{1}{2} \ln[(u')^{-2}] \bigg|_{(r, \theta)} + \int_{\Omega_0}^{\Omega(r, \theta)} F(\Omega) d\Omega = \text{const.},
\]

where Eqs. (1) and (5) have been used, and \( \Omega_0 \) is a given constant (changing the value of \( \Omega_0 \) simply modifies the value of the constant on the right hand side). The first term in Eq. (21) is a function of the pressure, which is, to this approximation, a function of \( r \), i.e. \( O(1) \) with respect to the angular velocity; on the other hand,

\[
(u')^{-2} = -(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}) = H^2 - K^2 r^2 \sin^2 \theta (\Omega - A)^2 = O(1 - (\Omega - A)^2),
\]

so the second term is \( O((\Omega - A)^2) \) and, hence, \( O(\mu^2) \); also, since

\[
u t u_\phi = (u')^2 g_{\phi\phi} (\Omega - A) = \frac{K^2 r^2 \sin^2 \theta (\Omega - A)}{H^2 - K^2 r^2 \sin^2 \theta (\Omega - A)^2} = O(\Omega - A),
\]

\( F(\Omega) = u t u_\phi = O(\Omega - A) \), thus, the third term is \( O((\Omega - A)^2) \) as well, and, hence, \( O(\mu^2) \). Consequently, to \( O(\mu) \), the Euler equation reduces to its static (non-rotating) case, and indeed we have presumably already used it to get the starting unperturbed solution. Therefore, at this order in the angular velocity, Eq. (5) is no restriction on \( \Omega(r, \theta) \).

### C. The angular momentum density

Using the definition of \( \omega \), linear correction of the dragging rate, via the expansion of the metric coefficient \( g_{\phi\phi} \), Eq. (12), and the metric coefficients of the non-rotating configuration (13), we obtain the expansion for the angular momentum density (8)

\[
T^\alpha_\beta \eta^\alpha n_\beta = n_t T^{t}_\phi = n_t (\varepsilon + p)(u')^2 (g_{t\phi} + \Omega g_{\phi\phi}) \]

\[
= (n_t) (\varepsilon_0 + p_0) \left[ (-g_{tt})^{-1} (-\omega (g_{\phi\phi})_0 + \Omega (g_{\phi\phi})_0) + O(\mu^3) \right] \]

\[
= e^{\nu/2} (\varepsilon_0 + p_0) e^{-\nu} r^2 \sin^2 \theta (\Omega - \omega) + O(\mu^3) \]

\[
= (\varepsilon_0 + p_0) e^{-\nu/2} r^2 \sin^2 \theta (\Omega - \omega) + O(\mu^3). \quad (22)
\]
Thus showing also for the first order rotational perturbation that, since we are assuming the energy condition \( \varepsilon_0 + p_0 \geq 0 \), the sign of the angular momentum density \( \varepsilon_0 + p_0 \geq 0 \) to \( O(\mu) \) is determined by the sign of \( \Omega - \omega \).

**D. The rotational mass-energy**

In Ref. 2 Hartle has derived the difference in total mass-energy, \( M_{\text{rot}} \), between a slowly and differentially rotating relativistic star and a non-rotating star with the same number of baryons and the same distribution of entropy, namely,

\[
M_{\text{rot}} = \frac{1}{2} \int I \cdot \theta \, dJ + O(\mu^4),
\]

where \( dJ \) is the angular momentum of a fluid element in the star (to first order in the angular velocity), i.e. from \( I \),

\[
dJ = T^t_\phi (\theta - g)^{1/2} \, d^3x |_{O(\mu)};
\]

taking into account \( \ref{22} \) and \( \ref{13} \), we obtain an explicit expression for the expansion of \( M_{\text{rot}} \) in powers of the angular velocity parameter \( \mu \),

\[
M_{\text{rot}} = \frac{1}{2} \int_0^R \int_0^\pi \left( \epsilon(r) + p(r) \right) r^4 e^{\frac{(\lambda - \nu)}{2}} \sin^3 \theta \, \Omega (\omega - \Omega) + O(\mu^4). \tag{23}
\]

**E. The time-angle component of the Einstein equation**

The \( (t\phi) \) field equation component retaining only first order terms in the angular velocity, i.e. from Eq. \( \ref{1} \),

\[
R^t_\phi = 8\pi T^t_\phi + O(\mu^3),
\]

takes the form

\[
\partial_r \left[ r^4 j(r) \partial_r \omega \right] + \frac{r^2 k(r)}{\sin^2 \theta} \partial_\theta [\sin^3 \theta \partial_\theta \omega] - 16\pi r^4 k(r) \left[ \epsilon(r) + p_0(r) \right] \left[ \omega - \Omega \right] = 0, \tag{24}
\]

where we have introduced the abbreviations

\[
 j(r) \equiv e^{-[\lambda(r) + \nu(r)]/2} \quad \text{and} \quad k(r) \equiv e^{[\lambda(r) - \nu(r)]/2}. \tag{25}
\]

As outlined in Ref. 1, using the 0-order field equations, \( \ref{14} \), \( \ref{15} \), \( \ref{16} \) and \( \ref{17} \), it follows

\[
4\pi r \left[ \epsilon(r) + p_0(r) \right] k(r) = -j'(r), \tag{26}
\]

(where \( ' \equiv d/dr \) which, substituted into Eq. \( \ref{24} \), yields

\[
\partial_r \left[ r^4 j(r) \partial_r \omega \right] + \frac{r^2 k(r)}{\sin^2 \theta} \partial_\theta [\sin^3 \theta \partial_\theta \omega] + 4 r^3 j'(r) \omega = 4 r^3 j'(r) \Omega(r, \theta). \tag{27}
\]

We write this differential equation for the dragging rate in the abbreviated form

\[
\bar{L} \omega = -\Psi^2 \Omega, \tag{28}
\]

9
with the linear second order partial differential operator \( \bar{L} \equiv \bar{L}_0 - \Psi^2 \), where

\[
\bar{L}_0 \omega := \frac{1}{r^4 j(r)} \partial_r \left[ r^4 j(r) \partial_r \omega \right] + \frac{k(r)}{r^2 j(r) \sin^3 \theta} \partial_\theta \left[ \sin^3 \theta \partial_\theta \omega \right] \quad \text{and} \quad (29)
\]

\[
\Psi^2(r) := -\frac{4}{r} \frac{j'(r)}{j(r)} = 16\pi \left( \varepsilon_0(r) + p_0(r) \right) \frac{k(r)}{j(r)} \geq 0 \quad \forall r \geq 0. \quad (30)
\]

Equation (26) has been used in (30), and the sign follows from the assumed energy condition (19), the functions \( j \) and \( k \) (25) are always positive. \( \Psi^2 \equiv 0 \) in the exterior (\( \forall r \in [R, \infty[ \)), where vacuum (\( \varepsilon_0 = p_0 = 0 \), cf. (iii) in Subsec. II.A) is considered.

Specifically, we are only interested in solutions \( \omega \equiv \omega(r, \theta) \) of Eq. (28) in \([0, \infty[ \times [0, \pi] \), which satisfy the boundary conditions

\[
\omega \quad \text{asymptotically flat} \quad \left( \lim_{r \to \infty} \omega = 0 \right), \quad (31)
\]

\[
\omega \quad \text{C}^1\text{-regular} \quad \text{on the axis of rotation}, \quad (32)
\]

and a matching condition, namely, to be at least a class \( C^1 \) function on the surface of the star—which is spherical at first order rotational perturbations—

\[
\omega(., \theta) \quad \text{class} \quad C^1 \quad \text{across} \quad r = R. \quad (33)
\]

Notice, (31) follows from our star model (condition (ii) in Subsec. II.A), and it can be easily seen that (33) follows from the equation itself, provided that \( \omega(., \theta) \) and \( \Omega(., \theta) \) are at least essentially bounded (\( \in L^\infty \))—as has been assumed—i.e. even if they have a jump discontinuity.

At the star’s surface, \( r = R \), higher regularity of \( \omega(., \theta) \) is not guaranteed by the equation, due to a jump discontinuity of the function \( \Psi^2 \) at this point. For this reason, we shall be considering (in the following section) \emph{generalized} (\( \in W^{1,2} \)) solutions \( \omega \) of Eq. (28) in the whole domain (interior and exterior).

\textbf{“Coordinate change”}.

In order to avoid the coordinate singularity occurring on the axis in polar coordinates \((r, \theta)\), and wishing instead to have in the differential operator (29) a Laplacian in some higher dimension, we consider the following “change of coordinates”.

Firstly, we introduce \emph{isotropic} cylindrical coordinates in the meridian plane,

\[
(r, \theta) \mapsto \left( \rho := h(r) \sin \theta, \ z := h(r) \cos \theta \right) \in \mathbb{R}_0^+ \times \mathbb{R}, \quad (34)
\]

with the function \( h \) satisfying the following ordinary differential equation of first order with separated coefficients

\[
\frac{h'(r)}{h(r)} = \frac{e^{\lambda(r)/2}}{r} \quad (35)
\]

(which makes the coefficient of the crossed derivatives in the operator (29) after the change (34) to vanish), and the \textit{boundary condition}

\[
\lim_{r \to \infty} \frac{h(r)}{r} = 1, \quad (36)
\]
i.e. so that the isotropic radius \( h(r) \equiv \tilde{r} \) approaches \( r \) at spatial infinity, because far away from the source we assume to have euclidean geometry. This leads us to the definition of the function

\[
w(\rho, z) := \omega \left( h^{-1} \left( \sqrt{\rho^2 + z^2} \right), \arctan \left( \frac{\rho}{z} \right) \right),
\]

or inversely, \( w \) such that

\[
\omega(r, \theta) = w(h(r) \sin \theta, h(r) \cos \theta).
\]

Secondly, (in the spirit of Ref. 10) we define (with \( w(\rho, z) \)) the 5-lift of \( w : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R} \) in flat \( \mathbb{R}^5 \), axisymmetric around the \( x_5 \)-axis, by

\[
w \mapsto \tilde{\omega} \quad \text{such that} \quad \tilde{\omega}(x) \equiv \tilde{\omega}(x_1, x_2, x_3, x_4, x_5) := w \left( \rho = \left( \sum_{i=1}^{4} x_i^2 \right)^{1/2}, \ z = x_5 \right),
\]

and, for every function \( \tilde{\omega} : \mathbb{R}^5 \to \mathbb{R} \), the meridional cut (in direction \( x_1 \)) of \( \tilde{\omega} \)

\[
\tilde{\omega} \mapsto w \quad \text{such that} \quad w(\rho, z) := \tilde{\omega}(\rho, 0, 0, 0, z).
\]

For axisymmetric functions these are isometric operations inverse to each other.\(^{10} \) After considering the change of variable \(^{32} \) with \(^{37} \) in the differential operator \( \tilde{L}_0 \)

\[
\tilde{L}_0 w = \frac{e^{\lambda(r)} h(r)^2}{r^2} \left\{ \partial_{\rho\rho} w + \partial_{zz} w + \frac{3}{\rho} \partial_{\rho} w + H(r) \frac{\partial_{\rho} w + z \partial_z w}{h(r)} \right\},
\]

where

\[
H(r) := -\frac{e^{-\frac{\lambda(r)}{2}} \left( -6 + 6 e^{\frac{\lambda(r)}{2}} + r \nu'(r) \right)}{2 h(r)}.
\]

But, through the 5-lift \(^{38} \), the flat Laplacian in 5 dimensions of the “lifted” function \( \tilde{\omega} \) gives exactly

\[
\Delta \tilde{\omega} \equiv \sum_{i=1}^{5} \partial_i \tilde{\omega} = \partial_{\rho\rho} w + \partial_{zz} w + \frac{3}{\rho} \partial_{\rho} w,
\]

first three terms in the bracket of \(^{39} \). Furthermore, as outlined in Ref. 10, \( n \)-lift and meridional cut (of axisymmetric functions) leave the regularity properties and the norm invariant; and axisymmetric operations, like multiplication, \( \partial_{\rho} \left( \tilde{r} \equiv h(r) = (\rho^2 + z^2)^{1/2} = \left( \sum_{i=1}^{4} x_i^2 \right)^{1/2} \right) \), and scalar product, commute with \( n \)-lift and meridional cut. In particular, in the fourth term in the bracket of \(^{39} \) the factor

\[
\rho \partial_{\rho} w + z \partial_z w = \partial_{\tilde{r}} \tilde{\omega} = \sum_{i=1}^{5} x_i \partial_i \tilde{\omega}.
\]

Therefore, substituting \(^{41} \) and \(^{42} \) into \(^{39} \), Eq. \(^{28} \) in the form \( \tilde{L}_0 \tilde{\omega} = -\Psi^2 (\tilde{\Omega} - \tilde{\omega}) \) (with \( \tilde{\Omega} \) defined from \( \Omega \) as it was \( \tilde{\omega} \) from \( \omega \), and \( \Psi^2 = e^{\lambda} 16\pi |\varepsilon_0 + p_0| \) now writes

\[
\tilde{L}_0 \tilde{\omega} \equiv \frac{e^{\lambda(r)} h(r)^2}{r^2} \left\{ \Delta \tilde{\omega} + H(r) \sum_{i=1}^{5} x_i \partial_i \tilde{\omega} \right\} = e^{\lambda(r)} \left\{ -16\pi |\varepsilon_0(r) + p_0(r)|[\tilde{\Omega} - \tilde{\omega}] \right\},
\]
\[
\Delta \tilde{\omega} + H(r) \frac{\sum_{i=1}^{5} x_i \partial_i \tilde{\omega}}{h(r)} = -16\pi \frac{r^2}{h(r)^2} [\varepsilon_0(r) + p_0(r)][\tilde{\Omega} - \tilde{\omega}].
\] (43)

V. PROPERTIES

With the assumptions made in Subsec. II.A for this slowly rotating configuration (starting from a non-rotating one as described in Subsec. IV.A.1; particularly, satisfying the energy condition \( \varepsilon_0 + p_0 \geq 0 \)), and considering only solutions of Eq. (28) satisfying the boundary and matching conditions (31), (32), and (33), the following results hold

**Property (a) (Positiveness of the dragging rate)**

If the distribution of angular velocity of the fluid is non-negative (and non-trivial), then the dragging rate (to first order in the fluid angular velocity) is positive everywhere,

\[ \Omega \geq 0, \quad \Omega \neq 0 \implies \omega > 0. \]

*Proof.* We have seen in the former section that Eq. (28) for \( \omega \) is equivalent to Eq. (43) for \( \tilde{\omega} \) using the coordinate change (34) and the 5-lift (38). The isotropic radius \( \bar{r} \equiv h(r) \) is Gaussian coordinate with respect to the star’s surface, \( \bar{r} = h(R) \), and, thus, \( \tilde{\omega} \) is at least class \( C^1 \) across this surface; therefore, from conditions on the stellar model, \( \tilde{\omega} \in W^{1,2}(\mathbb{R}^5) \cap C^1(\mathbb{R}^5) \), and, hence, Eq. (43), i.e.

\[
L \tilde{\omega} := \Delta \tilde{\omega} + \sum_{i=1}^{5} H(r) \frac{x_i}{h(r)} \partial_i \tilde{\omega} - 16\pi [\varepsilon_0 + p_0] \frac{r^2}{h(r)^2} \tilde{\omega} = -16\pi [\varepsilon_0 + p_0] \frac{r^2}{h(r)^2} \tilde{\Omega},
\] (44)

is satisfied in \( \mathbb{R}^5 \) in a generalized sense (cf. Appendix B). Equation (44) may be obviously written in divergence form

\[
L \tilde{\omega} \equiv \partial_i [a^{ij}(x) \partial_j \tilde{\omega} + a^i(x) \tilde{\omega}] + b^i(x) \partial_i \tilde{\omega} + c(x) \tilde{\omega} = g(x)
\]

(45)

(where repeated indices denote summation over the index), with the coefficients

\[
a^{ij}(x) \equiv \delta_{ij} \quad (= 1 \text{ if } i = j, \text{ and } = 0 \text{ otherwise}),
\]

\[
a^i(x) = 0,
\]

\[
b^i(x) = H(r) \frac{x_i}{h(r)} \quad (\forall \ i, \ j \in \{1, \ldots, 5\}), \quad \text{and}
\]

\[
c(x) = -16\pi [\varepsilon_0(r) + p_0(r)] \frac{r^2}{h(r)^2} \quad (\leq 0),
\]

and

\[
g(x) = c(x) \tilde{\Omega}(x). \quad (46)
\]

We shall consider the domain \( G \) defined by a ball in \( \mathbb{R}^5 \) centered at the origin \( x = 0 \) (\( \bar{r} = 0 \)) and of arbitrary large radius \( \sigma \),

\[
G := B_{\sigma}(0) \subset \mathbb{R}^5.
\] (47)

Notice, whenever \( \tilde{\Omega} \geq 0 \), we have, by (46), \( g \leq 0 \) (because \( c \leq 0 \)), and, hence, \( L \tilde{\omega} \leq 0 \), specifically \( \tilde{\omega} \in W^{1,2}(G) \cap C^1(G) \) is a generalized supersolution relative to the operator
L, in (44), and the domain G, (47). We look at the requirements for a minimum principle to be applied (Appendix B). The Laplacian operator is obviously strictly elliptic, and the coefficients (45) are measurable and bounded functions on G, this shows up in the following: the mapping \( r \mapsto h(r) \) is bounded from above and below everywhere in \([0, \infty[\) (Appendix A); \( \varepsilon_0 + p_0 \) is also bounded, since \( p_0 \) is bounded and \( p_0 \mapsto \varepsilon_0(p_0) \) is bounded in any closed interval (Subsec. IV.A.1); consequently, the coefficient \( c \) is bounded (from above and below); the coefficients of the first derivatives, \( b^i \), are also bounded (from above and below), because the function \( H \) is bounded everywhere (Appendix A) and since \((\forall i = 1, \ldots, 5)\) \( x_i^2 \leq \sum_{j=1}^{5} x_j^2 = h(r)^2 \), we have \((\forall i = 1, \ldots, 5)\) \( x_i^2 / h(r)^2 \leq 1 \).

Thus, all conditions of a minimum principle for generalized supersolutions relative to the differential operator \( L \) and the domain \( G \) hold, and, as a result of the weak minimum principle (Theorem 1 in Appendix B), we have

\[
\inf_{G} \bar{\omega} \geq \inf_{\partial G} \bar{\omega}^-, \quad (\bar{\omega}^- \equiv \min(\bar{\omega}, 0)).
\]  

But, since the radius of the ball \( G \) is arbitrary, we can make it sufficiently large \((\sigma \to \infty)\) so that, by asymptotic flatness \((\lim_{r \to \infty} \bar{\omega} = 0, \text{from condition (31)} \) and \( \lim_{r \to \infty} h(r) = 1 \)), \( \bar{\omega} \) is arbitrary small at \( \partial G \), following, from (48), \( \bar{\omega} \geq 0 \). Actually, the positivity is strict, because if \( \bar{\omega}(x_0) = 0 \) for some \( x_0 \in G \) (interior point), then \( \bar{\omega}(x_0) = \min \bar{\omega} \) (since \( \bar{\omega} \geq 0 \)), and, by the strong minimum principle (Theorem 2 in Appendix B), \( \bar{\omega} \) would be constant in \( G \); in this case, \( \bar{\omega} \equiv \text{const.} = 0 \) in \( G \) (i.e. everywhere); but \( \bar{\omega} \equiv 0 \) yields, by Eq. (44), \( \bar{\Omega} \equiv 0 \), or, equivalently, \( \Omega \equiv 0 \), and we are assuming that \( \Omega \) is non-trivial. We conclude then \( \omega > 0 \) everywhere. □

**Property (b)**

Suppose we perturb the non-rotating configuration (in particular, with a given equation of state) with two (small) different distributions of angular velocity \( \Omega_1 \) and \( \Omega_2 \), and integrate Eq. (28) to obtain their respective solutions for the dragging rate, \( \omega_1 \) and \( \omega_2 \), then

\[
\Omega_1 \geq \Omega_2, \quad \Omega_1 \neq \Omega_2 \quad \Rightarrow \quad \omega_1 > \omega_2.
\]

**Proof.** This follows form the linearity of Eq. (28) and Property (a). □

We are already in position to get a result about the positiveness of the difference \( \Omega - \omega \), and, hence, of the angular momentum density (22). However, in order to first do this more specific and concrete, we shall make use of a property for the particular case of rigid rotation (RR), which can be found in Ref. 1, Sec. IV.

**Property RR**

For the slowly rotating configuration,

\[
\omega(r, \theta) = \omega(r)
\]

\[
\Omega(r, \theta) = \text{const.} \equiv \hat{\Omega} > 0 \quad \Rightarrow \quad 0 < \omega(r) < \hat{\Omega} \quad \text{in} \ [0, R]
\]

\[
(in \ [0, R] \times [0, \pi]) \quad \omega > 0 \quad \text{in} \ [0, \infty[, \quad \omega' < 0 \quad \text{in} \ [0, \infty[, \quad \omega'(0) = 0.
\]
Property (c) (Positiveness of the angular momentum density)

For the slowly rotating configuration, with a given equation of state, its dragging rate $\omega$ will satisfy

$$\omega(r, \theta) < \Omega(r, \theta)$$

if $\Omega \equiv \Omega(r, \theta) \geq 0$ is bounded in the form

$$\Omega(\theta, \theta) \equiv \Omega \leq \Omega(r, \theta) \leq \Omega,$$

(in $[0, R] \times [0, \pi]$) where $\Omega$ is an arbitrary positive constant $0 < \Omega (\ll \Omega_{\text{crit}})$ (Sec. IV), and $\Omega = \Omega(0)$, $\Omega$ solution of Eq. (28) with $\Omega(0, \theta) = \text{const.} = \Omega$, and with the same 0-order coefficients (same starting non-rotating configuration) as the ones considered for our slowly rotating configuration, in particular with the same equation of state.

Or, more generally, if (with that notation)

$$\omega(r) \leq \Omega(r, \theta) \leq \Omega.$$

Notice, $\omega < \Omega$ means that the angular momentum density to first order in the fluid angular velocity, (22), of this configuration (with the energy condition (19)) is $\geq 0$, vanishing on the axis.

(Remarkably, the upper bound required on $\Omega$ is not restrictive, because for $\Omega$ continuous, $\Omega$ is essentially bounded ($\in L^\infty$) there, and $\Omega/\|\Omega\|\_\infty \leq 1$.)

Proof. We give a practical method of construction in two steps:

1st step: Consider $\Omega(r, \theta) := \Omega = \text{const.} > 0$, and solve the corresponding Eq. (28) for $\omega$. Then, by Property RR, the solution satisfies

$$\omega(r, \theta) = \omega(r), \quad 0 < \omega(r) < \Omega \text{ in } [0, R], \quad \omega > 0 \text{ in } [0, \infty[, \quad \omega < 0 \text{ in } ]0, \infty[, \quad \omega(0) = 0.$$

2nd step: Consider a slowly rotating configuration starting from the same non-rotating configuration (as in the first step) with a fluid angular velocity distribution $\Omega(r, \theta)$ such that

$$\omega(0) =: \Omega \leq \Omega(r, \theta) \leq \Omega.$$

Observe, we are always allowed to do this because of (50). Or, more generally, such that $\omega(r) \leq \Omega(r, \theta) \leq \Omega$; notice, from (51), $\omega$ is a (positive) decreasing function; in particular, $\omega(0) \geq \omega(r) > 0$, $\forall r \in [0, \infty[.$

Now, form the second inequality in (52), i.e. from $\Omega(r, \theta) \leq \Omega$, and, since we have the same starting unperturbed configuration (same 0-order coefficients) for these both slowly rotating configurations, it follows, by Property (b), that their corresponding solutions (of Eq. (28)) satisfy

$$\omega(r, \theta) < \omega(r),$$
where we have used (59). On the other hand, the first inequality in (52), and (53) yield
\[ \overline{\omega}(r) \leq \underline{\omega}(0) := \Omega \leq \Omega(r, \theta), \quad (54) \]
and consequently, from (53) and (54),
\[ \omega(r, \theta) < \Omega(r, \theta). \]
\[ \square \]

**Remark.** Notice, the same argument also assures that, given a slowly rotating configuration with \( \Omega(r, \theta) \) such that the corresponding dragging rate \( \underline{\omega}(r, \theta) < \Omega(r, \theta) \), we shall have the same positivity result (Property (c)) for any slowly rotating configuration, starting from the same unperturbed configuration (in particular, with the same equation of state), with an angular velocity distribution \( \Omega(r, \theta) \) such that
\[ \Omega(r, \theta) := \underline{\omega}(r, \theta) \leq \Omega(r, \theta) \leq \overline{\Omega}(r, \theta), \]
because we obtain, form the last inequality and Property (b), \( \omega(r, \theta) < \underline{\omega}(r, \theta) \), and, hence, \( \omega(r, \theta) < \Omega(r, \theta) \).

**Series expansion.** \( M_{\text{rot}}. \)

Before we prove next property, we first stress that, since \( \Omega \) and \( \omega \) transform like vectors under rotation, Eq. (27) may be separated by expanding them as
\[ \Omega(r, \theta) \equiv \Omega(r, x) \sim \sum_{l=1}^{\infty} \Omega_l(r) y_l(x) \quad \text{and} \quad (55) \]
\[ \omega(r, \theta) \equiv \omega(r, x) \sim \sum_{l=1}^{\infty} \omega_l(r) y_l(x), \quad (56) \]
with the change of variable \( \theta \mapsto x := \cos \theta \), and
\[ y_l(x) := \frac{dP_l}{dx} \quad \forall x \in [-1, 1] \quad (\theta \in [0, \pi]), \quad P_l \equiv \text{Legendre polynomial of degree } l. \quad (57) \]

Then the equation for \( \omega_l \) takes the form
\[ \frac{d}{dr}[r^4 j(r) \omega_l'] + [4r^3 j'(r) - r^2 k(r) \lambda_l] \omega_l = 4r^3 j'(r) \Omega_l(r), \quad (58) \]
with \( \lambda_l := l(l + 1) - 2, \quad l \in \mathbb{N}, \quad l \neq 0 \), and \( j \) and \( k \) defined in (25).

From conditions (31) and (32) on \( \omega \), we have the respective boundary conditions on \( \omega_l \)
\[ \lim_{r \to \infty} \omega_l(r) = 0, \quad (59) \]
\[ \omega_l \ C^1\text{-regular at the origin}, \quad (60) \]
and, from (33), the matching condition
\[ \omega_l \text{ class } C^1 \text{ across } r = R. \quad (61) \]
In Subsec. IV.D an explicit expression for the expansion of the rotational mass-energy $M_{\text{rot}}$ in powers or the angular velocity parameter was obtained (23), or, using Eq. (26),

$$M_{\text{rot}} = -\frac{1}{4} \int_0^R dr \, r^3 j' \int_0^\pi d\theta \sin^3\theta \, \Omega(\Omega - \omega) + O(\mu^4). \quad (62)$$

Now, using the series expansions of $\Omega$ and $\omega$ (55) and (56), and the fact that the system $\{ y_l \}_{l=1}^\infty$ is orthogonal in the Hilbert space $L^2((1, 1))$, with respect to the weight function $\rho(x) := 1 - x^2,$ $x \in [-1, 1]$, and have norm $\|y_l\|^2_\rho = 2l(l + 1)/(2l + 1)$, the integral over $\theta$ in (62) may be expressed as the sum

$$\int_0^\pi d\theta \sin^3\theta \, \Omega(r, \theta) \left[ \Omega(r, \theta) - \omega_l(r) \right] = \sum_{l=1}^\infty \frac{2l(l + 1)}{2l + 1} \Omega_l(r) \left[ \Omega_l(r) - \omega_l(r) \right], \quad (63)$$

and, consequently, the rotational mass-energy (62) can be expressed as a sum of integrals (over $r$)

$$M_{\text{rot}} = \sum_{l=1}^\infty \frac{l(l + 1)}{2(2l + 1)} M_l + O(\mu^4), \quad (64)$$

with $M_l := \int_0^R f^2(r) \Omega_l(r) \left[ \Omega_l(r) - \omega_l(r) \right] \, dr,$ $f^2(r) := -r^3 j'(r) \, (\geq 0). \quad (65)$

**Property (d) (Positivity and upper bound on the rotational energy $M_{\text{rot}}$)**

We consider Eq. (58), which can be written

$$\frac{d}{dr} \left( r^4 j \, \omega'_l \right) - r^2 \, k \lambda_l \, \omega_l = -4 \, f^2 \, (\Omega_l - \omega_l). \quad (66)$$

The main observation is that, multiplying both sides of Eq. (66) by $\omega_l$ and integrating from $r = 0$ to $r = \infty$, and taking into account that $f^2 = -r^3 j' = 4\pi r^4 (\varepsilon_0 + \rho_0) \, k \equiv 0 \, \forall r > R,$

$$\int_0^\infty \left[ \frac{d}{dr} (r^4 j \, \omega'_l) \omega_l - r^2 \, k \lambda_l \, \omega_l^2 \right] \, dr = -4 \int_0^R f^2 \omega_l \, (\Omega_l - \omega_l) \, dr$$

(note, the integral on the left hand side converges, because an asymptotically flat (59) solution of Eq. (58) must behave as $r \to \infty \, \omega_l = O(r^{-l-2})$, and $\omega'_l = O(r^{-l-3})$, $l \geq 1$); and, after integrating once by parts the first term on the left hand side,

$$r^4 j \, \omega'_l \omega_l \bigg|_0^\infty - \int_0^\infty [r^4 j \, (\omega'_l)^2 + r^2 \, k \lambda_l \, \omega_l^2] \, dr = -4 \int_0^R f^2 \omega_l \, (\Omega_l - \omega_l) \, dr.$$

Now, the first term vanishes because $\omega_l$ falls off rapidly enough at $r \to \infty$, and the second term (minus the integral on the left hand side) is non-positive (since $j$ and $k$ are always positive), therefore

$$\int_0^R f^2 \omega_l \, (\Omega_l - \omega_l) \, dr \geq 0. \quad (67)$$
We introduce now the sesquilinear (indeed bilinear) form

\[
\langle u, v \rangle_f := \int_0^R f^2(r) \ u(r) \ v(r) \ dr \quad u, v \in C^0([0, R]),
\]

and the induced semi-norm \( \| \cdot \|_f := (\langle \cdot, \cdot \rangle_f)^{1/2} \). With this definition, \( M_l \) (65) may be written

\[
M_l = \langle \Omega_l, \Omega_l - \omega_l \rangle_f . \tag{68}
\]

We have (67), which now reads \( \langle \omega_l, \Omega_l - \omega_l \rangle_f \geq 0 \), i.e.

\[
\langle \omega_l, \Omega_l \rangle_f \geq \| \omega_l \|_f^2 , \tag{69}
\]

in particular

\[
\langle \Omega_l, \omega_l \rangle_f = \langle \omega_l, \Omega_l \rangle_f \geq 0 . \tag{70}
\]

Using the Cauchy-Schwarz inequality,

\[
\langle \omega_l, \Omega_l \rangle_f \leq \| \omega_l \|_f \| \Omega_l \|_f ,
\]

which, together with (69), yields \( \| \omega_l \|_f^2 \leq \| \omega_l \|_f \| \Omega_l \|_f \) and, hence, \( \| \Omega_l \|_f \geq \| \omega_l \|_f \) for \( \| \omega_l \|_f \neq 0 \), but this inequality is still true for \( \| \omega_l \|_f = 0 \), because then, by Eq. (58), \( \| \Omega_l \|_f = 0 \); therefore

\[
\| \Omega_l \|_f \geq \| \omega_l \|_f \tag{71}
\]

holds in general. Multiplying (71) by \( \| \Omega_l \|_f \), we find, again using the Cauchy-Schwarz inequality,

\[
\| \Omega_l \|_f^2 \geq \langle \Omega_l, \omega_l \rangle_f ,
\]

but

\[
M_l = \langle \Omega_l, \Omega_l - \omega_l \rangle_f = \| \Omega_l \|_f^2 - \langle \Omega_l, \omega_l \rangle_f , \tag{72}
\]

thus showing

\[
M_l \geq 0 . \tag{73}
\]

Furthermore, since (by (71)) \( \langle \Omega_l, \omega_l \rangle_f \geq 0 \), from (72) we also have the upper bound

\[
M_l \leq \| \Omega_l \|_f^2 , \quad \text{i.e.}
\]

\[
M_l \leq \int_0^R f^2(r) \ \Omega_l^{-2}(r) \ dr . \tag{74}
\]

The bounds (73) and (74) on \( M_l \) yield respective bounds on \( M_{\text{rot}} \) (64),

\[
0 \leq M_{\text{rot}} \leq \sum_{l=1}^{\infty} \frac{l(l+1)}{2(2l+1)} \int_0^R f^2(r) \ \Omega_l^{-2}(r) \ dr + O(\mu^4) ,
\]

or, writing the sum as integral over \( \theta \) (as in (63)),

\[
0 \leq M_{\text{rot}} \leq \frac{1}{4} \int_0^R \ f^2(r) \int_0^\pi d\theta \sin^3 \theta \ [\Omega(r, \theta)]^2 + O(\mu^4) , \tag{75}
\]

where \( f^2 := -r^3 j' = 4\pi r^4 (\varepsilon_0 + p_0) e^{(\lambda-\nu)/2} . \)
VI. CONCLUDING REMARKS

Summing up, it has been seen that relativistic stars rotating slowly and differentially, with a non-negative (and non-trivial) angular velocity distribution, \( \Omega(x_2, x_3) \geq 0 \) (\( \neq 0 \)), and satisfying the energy condition \( \varepsilon + p \geq 0 \), have positive rate of rotational dragging \( \omega > 0 \) (Property (a) in Sec. V); and certain upper and lower bounds on \( \Omega \) assure also the positivity of the difference \( \Omega - \omega \) and, hence, of the angular momentum density, this later vanishing on the axis, (Property (c)). We also observe that, the rotational mass-energy, (from Property (d)) non-negative and (as expected) “increased” by a (slow) angular velocity of the fluid, \( \Omega \), is “decreased” by the dragging effect (over what it would be if this effect were neglected), i.e. is decreasing with respect to dragging rate, \( \omega \), despite of (as shown in Property (b)) \( \omega \) being an “increasing function” of \( \Omega \).

We have given here a much simpler proof of Property (d) than in Ref. 2; this alternative proof can be even generalized outside the slow rotation limit. Property (b) and, hence, also Property (c), however are based on the linearity of the time-angle field equation component to first order in the fluid angular velocity. Moreover, in the general differentially rotating case the rotation profile \( \Omega \) cannot be freely chosen, but is restricted by the integrability condition of the Euler equation, i.e. by Eq. (5). This makes unlikely a generalization of Property (c) outside the slow rotation limit, other than in the form given in Ref. 12, Subsec. IV.B.

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APPENDIX A: Boundedness of some functions in \([0, \infty[ \ni r\]

The ratio radius - isotropic radius \( \chi(r) := \frac{r}{h(r)} \)

We have

\[
\frac{h'(r)}{h(r)} = \frac{e^{\lambda(r)/2}}{r},
\]

(A 1)

with

\[
e^{-\lambda(r)} = 1 - \frac{2m(r)}{r},
\]

(A 2)

where

\[
m(r) := \begin{cases} 
4\pi \int_0^r \varepsilon_0(s)s^2ds & : r \in [0, R] \\
M \equiv 4\pi \int_0^R \varepsilon_0(s)s^2ds & : r \in ]R, \infty[ 
\end{cases}
\]

(A 3)

if we denote the stellar radius of the static model by \( R > 0 \). As we start with a (physically) regular (i.e. non-collapsed) static solution, we assume that \( 2m(r) < r \) (for all \( r \in [0, R] \)), and \( 2M < R \).
Integrating Eq. (A 1) and using Eq. (A 2) we get

\[ h(r) = h(R) \exp \left( \int_R^r \frac{ds}{\sqrt{s(s-2m(s))}} \right). \]  

(Note, the constant \( h(R) \) is determined by the asymptotic condition

\[ \lim_{r \to \infty} \frac{h(r)}{r} = 1; \]

see below.) Let

\[ g(r) := \int_R^r \frac{ds}{\sqrt{s(s-2m(s))}} \quad \forall r > 0. \]  

(A 5)

With this definition the solution, (A 4), now writes

\[ h(r) = h(R) \exp (g(r)). \]  

(A 6)

Due to the assumptions made for \( m \), \( g \) is obviously a continuous function in the open interval \( ]0, \infty[ \); consequently, by (A 6), \( h \) is also a continuous function there, and, in particular, \( h(r) \) cannot be zero in \( ]0, \infty[ \) (unless \( h(R) = 0 \), however this would contradict asymptotic flatness); therefore \( \chi : r \mapsto \chi(r) := \frac{r}{h(r)} \) is continuous in \( ]0, \infty[ \) as well. Choose an \( \epsilon \in ]0, R[ \) and an \( \epsilon' \in ]R, \infty[ \), then \( \chi \) is bounded below and above in the interval \( ]\epsilon, \epsilon'[ \) (where the upper and lower bound depend on the selected \( \epsilon \) and \( \epsilon' \), of course). Let us now consider the intervals \( [0, \epsilon] \) and \( ]\epsilon', \infty[ \) separately:

**On \( ]\epsilon', \infty[ \):** We have

\[ 0 \leq m(r) \equiv M. \]

Then

\[ \frac{1}{s} \leq \frac{1}{\sqrt{s(s-2m(s))}} \equiv \frac{1}{\sqrt{s(s-2M)}} \quad \forall r \in ]\epsilon', \infty[. \]

As in this interval \( r \geq R \), we find, with Eq. (A 5),

\[ \ln \left( \frac{r}{R} \right) = \int_R^r \frac{ds}{s} \leq g(r) \equiv \int_R^r \frac{ds}{\sqrt{s(s-2m(s))}} = 2\ln \left( \frac{\sqrt{r} + \sqrt{r-2M}}{\sqrt{R} + \sqrt{R-2M}} \right). \]

Inserting it into Eq. (A 6), yields (since exp is a monotonically increasing function)

\[ \frac{h(R)}{R} \leq h(r) \equiv h(R) \left( \frac{\sqrt{r} + \sqrt{r-2M}}{\sqrt{R} + \sqrt{R-2M}} \right)^2 \leq \frac{4h(R)}{(\sqrt{R} + \sqrt{R-2M})^2} r. \]  

(A 7)

Note especially that \( \lim_{r \to \infty} \frac{h(r)}{r} = \frac{4h(R)}{(\sqrt{R} + \sqrt{R-2M})^2} \), but, by asymptotic flatness,

\[ \lim_{r \to \infty} \frac{h(r)}{r} = 1; \]

therefore \( h(R) = \frac{1}{4} \left( \sqrt{R} + \sqrt{R-2M} \right)^2 > 0. \) Thus, \( h(R) > 0 \), and, from Eq. (A 7),

\[ 0 < \frac{\left( \sqrt{R} + \sqrt{R-2M} \right)^2}{4h(R)} \leq \chi(r) \leq \frac{R}{h(R)} < \infty \quad \forall r \in ]\epsilon', \infty[. \]  

(A 8)
We have

\[ 0 \leq m(r) = 4\pi \int_0^r \varepsilon_0(s)s^2 \, ds \leq \frac{4\pi}{3} \hat{\varepsilon}_0 r^3 =: \frac{c_0}{2} r^3, \]

where \( \hat{\varepsilon}_0 := \sup_{r \in [0, R]} \varepsilon_0(r) > 0 \). Now, choose \( \epsilon > 0 \), such that \( 1 - c_0 r^2 > 0 \) on \( [0, \epsilon] \).

Now, choose \( \epsilon > 0 \), such that \( 1 - c_0 r^2 > 0 \) on \( [0, \epsilon] \) (e.g., \( \epsilon := (2\sqrt{c_0})^{-1} \)). Then

\[ \frac{1}{s} \leq \frac{1}{s\sqrt{s - 2m(s)}} \leq \frac{1}{s\sqrt{1 - c_0 s^2}} \forall r \in [0, \epsilon], \]

and, since in this interval \( r \leq R \), we find, with Eq. (A 5),

\[ \ln \left( \frac{R}{r} \right) = \int_r^R \frac{ds}{s} \leq g(r) \leq \int_r^R \frac{ds}{s\sqrt{1 - c_0 s^2}} = \ln \left( \frac{R}{r} \right) + \ln \left( \frac{1 + \sqrt{1 - c_0 r^2}}{1 + 1 - c_0 R^2} \right). \]

Again, inserting it into Eq. (A 6), yields

\[ \frac{h(R)}{R} r \geq h(r) \geq \frac{h(R)}{R} \frac{1 + \sqrt{1 - c_0 R^2}}{1 + 1 - c_0 r^2} r \geq \frac{h(R)}{R} \frac{1 + \sqrt{1 - c_0 R^2}}{2R} r, \]

and, hence,

\[ 0 < \frac{R}{h(R)} \leq \chi(r) \leq \frac{2R}{h(R) (1 + \sqrt{1 - c_0 R^2})} < \infty \forall r \in [0, \epsilon]. \quad (A 9) \]

We can therefore conclude that, since \( \mathbb{R}^+_0 = [0, \epsilon] \cup [\epsilon, \epsilon'] \cup [\epsilon, \infty] \) and \( \chi \) is bounded (from above and below) in each of these subintervals, \( \chi \) is bounded (from above and below) in \( \mathbb{R}^+_0 \). \[ \square \]

**The function \( H \)**

We have

\[ H(r) := \frac{-e^{-\lambda(r)} \left[-6 + 6e^{\frac{\lambda(r)}{2}} + r\nu'(r)\right]}{2h(r)}, \]

and

\[ e^{-\lambda(r)/2} = \sqrt{1 - \frac{2m(r)}{r}}, \]

\[ r\nu'(r) = \frac{2m(r) + 8\pi r^3 p_0(r)}{r - 2m(r)} = \frac{2m(r)}{r} + \frac{8\pi r^2 p_0(r)}{1 - \frac{2m(r)}{r}}. \]

Thus,

\[ H(r) = \frac{1}{2h(r)} \left\{ 6 \left[ \sqrt{1 - \frac{2m(r)}{r}} - 1 \right] \left[ \frac{2m(r)}{r} + 8\pi r^2 p_0(r) \right] \left( \sqrt{1 - \frac{2m(r)}{r}} \right)^{-1} \right\}. \quad (A 10) \]
Now, using the Cauchy-Schwarz inequality in (A 10), and the following estimates in \( r \in [0, \epsilon] \), for some \( \epsilon \in [0, R] \) small, (see former Subsec. in Appendix A)

\[
0 \leq m(r) \leq \frac{c_0}{2} r^3 \tag{A 11}
\]

\[
0 \leq \sqrt{1-x} \leq 1 - \frac{x}{2} \quad \forall x \in [0,1] \tag{A 12}
\]

\[
0 \leq c_1 r \leq h(r) \leq c_2 r \tag{A 13}
\]

\[
0 \leq p_0(r) \leq p_0 := \sup_{r \in [0,R]} p_0(r) \quad \forall r \geq 0 , \tag{A 14}
\]

where the constants \( c_i \ (i=0,...,2) \) are all strictly positive (and finite), we get

\[
|H(r)| \leq \frac{1}{2h(r)} \left\{ 6 \left| \sqrt{1 - \frac{2m(r)}{r}} - 1 \right| + \left| \frac{2m(r)}{r} + 8\pi r^2 p_0(r) \right| \left( \sqrt{1 - \frac{2m(r)}{r}} \right)^{-1} \right\}
\]

\[
\leq \frac{1}{2c_1 r} \left\{ 6 \left[ \frac{c_0}{2} r^2 \right] + [c_0 r^2 + 8\pi \hat{p}_0 r^2] \left( \sqrt{1 - c_0 \epsilon^2} \right)^{-1} \right\}
\]

\[
=: \frac{c_3 r^2}{c_1 r} =: c_4 r , \tag{A 15}
\]

with \( 0 < c_3, c_4 < \infty \). Therefore \( H \) is bounded in \([0, \epsilon]\). (Especially, due to Eq. (A 15), \( H(0) = 0 \).) And, since, by Eq. (A 10), \( H \) is also continuous in the open interval \([0, \infty]\) and \( \lim_{r \to \infty} H(r) = 0 \) (because \( \lim_{r \to \infty} \frac{h(r)}{r} = 1 \)), \( H \) is bounded everywhere in \([0, \infty]\). \( \square \)

**APPENDIX B: The minimum principle for generalized supersolutions**

Consider in a domain (open and connected set) \( G \subset \mathbb{R}^n \ (n \geq 2) \) the differential operator with principal part of divergence form, defined by

\[
Lu = \partial_i [a_{ij}(x) \partial_j u + a_i(x) u] + b_i(x) \partial_i u + c(x) u ,
\]

with \( a_{ij} = a_{ji} \). Notice, an operator \( L \) of the general form \( Lu = \tilde{a}_{ij}(x) \partial_{ij} u + \tilde{b}_i(x) \partial_i u + \tilde{c}(x) u \) may be written in divergence form provided its principal coefficients \( \tilde{a}_{ij} \) are differentiable. If furthermore the \( \tilde{a}_{ij} \) are constants, then even with coinciding coefficients (\( a_{ij} = \tilde{a}_{ij}, b_i = \tilde{b}_i, c = \tilde{c} \)) and \( a_i \equiv 0 \). Let us assume that

1. \( L \) is strictly elliptic in \( G \), i.e. \( \exists \) a constant \( \lambda > 0 \) such that \( \lambda \leq \) the minimum eigenvalue of the principal coefficient matrix \([a_{ij}(x)]\),

\[
\lambda |y|^2 \leq a_{ij}(x) y_i y_j \quad \forall y \in \mathbb{R}^n , \quad \forall x \in G ; \tag{B 1}
\]

2. \( a_{ij}, a_i, b_i, \) and \( c \) are measurable and bounded functions in \( G \),

\[
|a_{ij}| < \infty , \quad |a_i| < \infty , \quad |b_i| < \infty , \quad |c| < \infty \quad \text{in} \quad G \quad (i, j \in \{1, \ldots, n\}) . \tag{B 2}
\]

By definition, for a function \( u \) which is only assumed to be weakly differentiable and such that the functions \( a_{ij} \partial_{ij} u + a_i u \) and \( b_i \partial_i u + cu, \ i = 1, \ldots, n \) are locally integrable (in particular, for \( u \) belonging to the Sobolev space \( W^{1,2}(G) \)), \( u \) is said to satisfy \( Lu = g \),
in \( G \) in a generalized (or weak) sense (\( g \) also a locally integrable function in \( G \)) if it satisfies

\[
\mathcal{L}(u, \varphi; G) := \int_G \left\{ (a_{ij} \partial_j u + a_i u) \partial_i \varphi - (b_i \partial_i u + cu) \varphi \right\} dx
\]

\[
= -\int_G g \varphi dx, \quad \forall \varphi \geq 0 \quad \varphi \in C^1_c(G)
\]

(where \( C^1_c(G) \) is the set of functions in \( C^1(G) \) with compact support in \( G \)).

Notice, \( u \) is generalized supersolution relative to a differential operator \( L \) and the domain \( G \) (i.e. satisfies \( Lu \leq 0 \) in \( G \) in a generalized sense) if it satisfies \( \mathcal{L}(u, \varphi; G) \geq 0 \), \( \forall \varphi \geq 0 \ \varphi \in C^1_c(G) \).

**Theorem 1:** (weak minimum principle)

Let \( u \in W^{1,2}(G), \ G \) a bounded domain, satisfy \( Lu \leq 0 \) in \( G \) in a generalized sense with

\[
\int_G (c \varphi - a^i \partial_i \varphi) dx \leq 0, \quad \forall \varphi \geq 0 \quad \varphi \in C^1_c(G).
\]

and conditions (B 1) and (B 2) above, then

\[
\min_G u \geq \min_{\partial G} u^- , \quad (u^- \equiv \min(u, 0)).
\]

(A proof of this theorem can be found in Ref. 13, Theorem 8.1.)

**Theorem 2:** (strong minimum principle)

Let \( u \in W^{1,2}(G) \cap C^0(G) \) satisfy \( Lu \leq 0 \) in \( G \) in a generalized sense, with the operator \( L \) satisfying conditions (B 1), (B 2), and (B 3), then \( u \) cannot achieve a non-positive minimum in the interior of \( G \), unless \( u \equiv \text{const.} \)

(A proof of this theorem can be found in Ref. 13, Theorem 8.19.) Note that the weak minimum principle, Theorem 1, for \( C^0(G) \) supersolutions is a direct consequence.

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Indeed, the first order terms (in the fluid angular velocity) of the general metric, as given in Ref. 12, yield (omitting here the ‘˜’ symbol for all 5-lifted functions)

\[ H(r) = (3 \partial_i B - 4 \partial_i U)|_{h(r)}, \]

where \( \bar{r} = h(r) \), given by Eq. (35). (This can be seen by a straightforward calculation, using \( e^{2U(h(r))} = e^{\nu(r)}, h(r)^2 e^{2[B(h(r)) - U(h(r))] = r^2} \), and their differentiations with respect to \( r \)). And since (from \( h(r) = (\sum_{i=1}^{5} x_i^2)^{1/2} \) \( \partial_i h(r) = x_i h(r)^{-1} \), we have \( H(r) x_i h(r)^{-1} = (3 \partial_i B - 4 \partial_i U)|_{O(\Omega)} \), and, hence, \( \langle 3 \nabla B - 4 \nabla U, \nabla \omega \rangle|_{O(\Omega)} \) as second term in Eq. (43). Also, since to first order (spherical; \( K = B \)) \( e^{2K} N^{-1} = e^{2(B - U)} = r^2 h^{-2} \), the coefficient of the right hand side of Eq. (43) is actually \( -\Psi(r)^2 e^{-\lambda(r)} r^2 h(r)^{-2} = -\psi^2|_{O(\Omega)} \), in the notation of Ref. 12.

11 M.J. Pareja, preprint gr-qc/0309073

12 D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer, Berlin and Heidelberg, 1977).