On the Allen-Cahn equation in the Grushin plane:
that is not one-dimensional

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Abstract

We consider solutions of the Allen-Cahn equation in the whole Grushin plane and we show that if they are monotone in the vertical direction, then they are stable and they satisfy a good energy estimate.

However, they are not necessarily one-dimensional, as a counter-example shows.

1 Introduction

We consider here the Grushin plane $G$ (see [II]), that is $\mathbb{R}^2$ endowed with the vector fields $X = \frac{\partial}{\partial x}$ and $Y = x \frac{\partial}{\partial y}$. We also define $T := [X,Y] = \frac{\partial}{\partial y}$.

The Grushin gradient is then $\nabla_G := (X,Y)$ (with the coordinates taken in the $(X,Y)$-frame) and the Grushin Laplacian is $\Delta_G := X^2 + Y^2$.

We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product (when the vectors are taken in the $(X,Y)$-frame), so that, for a smooth function $v$ we have

$$|\nabla_G v(\zeta)| = \sqrt{\langle \nabla_G v(\zeta), \nabla_G v(\zeta) \rangle} = \sqrt{(Xv(\zeta))^2 + (Yv(\zeta))^2},$$

for any $\zeta \in \mathbb{R}^2$.

Moreover, the Grushin norm on $G$ is defined as

$$\|(x, y)\| := \sqrt{|x|^4 + 4|y|^2}$$

for any $(x, y) \in \mathbb{R}^2$, and then the Grushin ball of radius $R > 0$ centered at $\zeta \in \mathbb{R}^2$ is

$$B_R(\zeta) := \{\eta \in \mathbb{R}^2 \text{ s.t. } \|\eta - \zeta\| < R\}.$$

The main purpose of this paper is to study solutions of the Allen-Cahn equation in the Grushin plane, that is

$$\Delta_G u(\zeta) + f(u(\zeta)) = 0$$

(1.1)
for any $\zeta \in \mathbb{R}^2$.

We take, for simplicity, $f \in C^1$, though less regularity is also possible to be dealt with. A particular case of interest is when $f = -W'$ and $W$ is a double-well potential. Namely, through this paper, we denote by $W$ a function with the following properties: $W \in C^2(\mathbb{R})$ is an even function for which $W(\pm 1) = 0 \leq W(r)$ for any $r \in \mathbb{R}$, $W''(0) \neq 0$, $W''(\pm 1) \neq 0$, and such that

$$W'(s) = 0 \text{ if and only if } s \in \{-1, 0, +1\}.$$  

(1.2)

Inspired by a famous conjecture of De Giorgi (see [6]), one may wonder under which conditions the solutions of (1.1) are one-dimensional, i.e., their level sets are straight lines and so, up to rotation, they depend on only one variable (at least when $f = -W'$).

Natural requirements for such symmetry are monotonicity and stability conditions. Namely, if $u$ is a solution of (1.1), we say that $u$ is stable if

$$\int_{\mathbb{R}^2} |\nabla G\phi(x)|^2 - \int_{G_u} f'(u(x))(\phi(x))^2 \, dx \geq 0$$  

(1.3)

for any $\phi \in C^\infty_0(\mathbb{R}^2)$.

Stability is a natural condition in the calculus of variation, since it states that the energy functional associated to (1.1) has non-negative second derivative. The stability condition has thus been widely used in connection with the problems posed by [6] (see, for instance, [1, 7] and references therein).

Also, in the Euclidean setting, the stability condition holds true whenever $u$ is monotone in some direction. The analogy in the Grushin setting is somehow more delicate, since the space is not homogeneous with respect to the choice of a particular direction. Thus, the monotonicity studied in this paper is the following. We are mostly concerned with solutions that are monotone in the $y$-direction, that is for which

$$Tu(\zeta) > 0 \text{ for any } \zeta \in \mathbb{R}^2.$$  

(1.4)

We shall show that (1.4) implies (1.3) (see Proposition 3.1 below).

Symmetry properties for solutions of (1.1) in the Grushin plane have been recently studied in [8]. For instance, [8] pointed out the following result:

**Theorem 1.1.** Let us assume that $u$ is a stable solution of (1.1) in the whole $\mathbb{R}^2$ such that

$$TYuXu(\zeta) - TXuYu(\zeta) \leq 0 \quad \text{for any } \zeta \in \mathbb{R}^2.$$

Suppose that there exists $C_o \geq 1$ in such a way that

$$\int_{B(0,R)} x^2 |\nabla G u|^2 \leq C_o R^4,$$  

(1.5)
for any $R \geq C_o$.  
Assume also that 
\[ \nabla_G u(\zeta) \neq 0 \text{ for any } \zeta \in \mathbb{R}^2. \] (1.6)  
Then, $u$ depends only on the $x$-variable.

We observe that (1.6) is also a sort of monotonicity condition, while (1.5) is an energy growth requirement (and energy bounds are often needed in the Euclidean case too, see [1]). We shall show that (1.4) implies also (1.5), at least when $f = -W'$ (see Theorem 4.2 below).

This said, a natural question arises. Namely, 

**Question 1.2.** Is it true that bounded solutions of (1.1) which satisfy (1.4) are one-dimensional (at least for $f = -W'$)?

Note that one may be quite tempted to answer yes to such a question, since (1.4) implies both the stability condition and the good energy growth in (1.5) (again, see for this Proposition 3.1 and Theorem 4.2 here below).

The main purpose of this paper is in fact to show that the above question has a **negative** answer.

This will be accomplished in Theorem 5.2, by constructing a counter-example which follows the lines of the one in [4].

The paper is organized in the following way. After gathering some elementary observations in Section 2, we point out in Section 3 that the monotonicity condition in (1.4) implies the stability condition in (1.3).

Then, we develop in Section 4 the energy estimates which show that the monotonicity condition in (1.4) also implies the energy growth in (1.5).

Finally, Section 5 contains the construction of the counter-example which shows that Question 1.2 has a negative answer.

## 2 Preliminaries

We collect in this section some elementary, but useful, observations. The expert reader may surely skip this section.

### 2.1 An integration by parts

We now point out a variation of Green formula, complicated here by the non-homogeneous Grushin scaling.
Lemma 2.1. Let \( u \in \Lambda^2(\mathbb{R}^2), v \in \Lambda^1(\mathbb{R}^2) \). Suppose that \( |\nabla_G u| \in L^\infty(\mathbb{R}^2) \) and that \( v \geq 0 \).

Then
\[
\int_{B_R(0)} \left( <\nabla_G u, \nabla_G v > + \Delta_G uv \right) \geq -R^2 \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} v(RX, R^2Y) \, d\mathcal{H}^1(X, Y). \tag{2.1}
\]

Proof. We set \( U(X, Y) := u(RX, R^2Y), V(X, Y) := v(RX, R^2Y) \). Then, by changing variable, we have
\[
\int_{B_R(0)} \left( <\nabla_G u, \nabla_G v > + \Delta_G uv \right) = \int_{B_R(0)} \left( \partial_x u(x, y) \partial_x v(x, y) + x^2 \partial_y u(x, y) \partial_y v(x, y) \right. \\
+ \partial_{xx} u(x, y) v(x, y) + x^2 \partial_{yy} u(x, y) v(x, y) \left. \right) \, dx \, dy \\
= \frac{1}{R^2} \int_{B_R(0)} \left( \partial_x U \left( \frac{x}{R}, \frac{y}{R^2} \right) \partial_x V \left( \frac{x}{R}, \frac{y}{R^2} \right) \\
+ \frac{x^2}{R^2} \partial_y U \left( \frac{x}{R}, \frac{y}{R^2} \right) \partial_y V \left( \frac{x}{R}, \frac{y}{R^2} \right) + \partial_{xx} U \left( \frac{x}{R}, \frac{y}{R^2} \right) V \left( \frac{x}{R}, \frac{y}{R^2} \right) \\
+ \frac{x^2}{R^2} \partial_{yy} U \left( \frac{x}{R}, \frac{y}{R^2} \right) V \left( \frac{x}{R}, \frac{y}{R^2} \right) \right) \, dx \, dy \\
= R \int_{B_1(0)} \partial_x U(X, Y) \partial_x V(X, Y) + X^2 \partial_y U(X, Y) \partial_y V(X, Y) \\
+ \partial_{xx} U(X, Y) V(X, Y) + X^2 \partial_{yy} U(X, Y) V(X, Y) \, dX \, dY.
\]

Thence, by the standard Euclidean Divergence Theorem,
\[
\int_{B_R(0)} \left( <\nabla_G u, \nabla_G v > + \Delta_G uv \right) = R \int_{B_1(0)} \text{div} \left[ V(X, Y) \left( \partial_x U(X, Y), X^2 \partial_y U(X, Y) \right) \right] \, dX \, dY \tag{2.2}
\]
\[
= R \int_{\partial B_1(0)} V(X, Y) \left( \partial_x U(X, Y), X^2 \partial_y U(X, Y) \right) \cdot \nu^E(X, Y) \, d\mathcal{H}^1(X, Y),
\]
where “\( \cdot \)” denotes the standard Euclidean scalar product and “\( \nu^E \)” is the standard Euclidean outward normal of \( \partial B_1(0) \).

We write (2.2) as
\[
\int_{B_R(0)} \left( <\nabla_G u, \nabla_G v > + \Delta_G uv \right) = R^2 \int_{\partial B_1(0)} v(RX, R^2Y) \left( \partial_x u(RX, R^2Y), RX^2 \partial_y u(RX, R^2Y) \right) \\
\cdot \nu^E(X, Y) \, d\mathcal{H}^1(X, Y)
\]
and so, since the Euclidean norm of \( v^E \) is 1,
\[
\int_{B_R(0)} \left( <\nabla_G u, \nabla_G v> + \Delta_G uv \right) \geq - R^2 \int_{\partial B_1(0)} v(RX, R^2Y) \left| \left( \partial_x u(RX, R^2Y), RX^2 \partial_y u(RX, R^2Y) \right) \right|_E dH^1(X, Y),
\tag{2.3}
\]
where \( |\cdot|_E \) is the Euclidean norm.

We now observe that, for any \((X, Y) \in \partial B_1(0)\), we have \(|X| \leq 1\) and
\[
\begin{align*}
\left| \left( \partial_x u(RX, R^2Y), RX^2 \partial_y u(RX, R^2Y) \right) \right|_E^2 &= \left( \partial_x u(RX, R^2Y) \right)^2 + R^2 X^4 \left( \partial_y u(RX, R^2Y) \right)^2 \\
&\leq \left( \partial_x u(RX, R^2Y) \right)^2 + R^2 X^2 \left( \partial_y u(RX, R^2Y) \right)^2 \\
&= |\nabla_G u(RX, R^2Y)| \\
&\leq \|\nabla_G u\|_{L^\infty(\mathbb{R}^2)}.
\end{align*}
\]
From this and (2.3) we get (2.1).

2.2 An interpolation inequality

We point out the following elementary estimate:

**Lemma 2.2.** Let \( h \in C^2(\mathbb{R}) \). Then,
\[
\|h'\|_{L^\infty(\mathbb{R})} \leq 2 \left( \|h\|_{L^\infty(\mathbb{R})} + \|h''\|_{L^\infty(\mathbb{R})} \right).
\tag{2.4}
\]

*Proof.* We may assume that both \( \|h\|_{L^\infty(\mathbb{R})} \) and \( \|h''\|_{L^\infty(\mathbb{R})} \) are finite, otherwise (2.4) is void.

First, we observe that, for any \( j \in \mathbb{Z} \), there exists \( t_j \in [j, j + 1] \) in such a way that
\[
|h'(t_j)| \leq 2\|h\|_{L^\infty(\mathbb{R})}.
\tag{2.5}
\]

Indeed, if (2.5) were false, there would exist \( j_o \in \mathbb{Z} \) such that \( |h'(t)| > 2\|h\|_{L^\infty(\mathbb{R})} \) for any \( t \in [j_o, j_o + 1] \). Since \( h' \) is continuous, this means that either \( h'(t) > 2\|h\|_{L^\infty(\mathbb{R})} \) or \( h'(t) < -2\|h\|_{L^\infty(\mathbb{R})} \) for any \( t \in [j_o, j_o + 1] \). We assume that the second possibility holds (the first case is analogous). Then,
\[
-2\|h\|_{L^\infty(\mathbb{R})} \leq h(j_o + 1) - h(j_o) = \int_{j_o}^{j_o + 1} h'(t) \, dt < \int_{j_o}^{j_o + 1} (-2\|h\|_{L^\infty(\mathbb{R})}) \, dt = -2\|h\|_{L^\infty(\mathbb{R})}.
\]
This contradiction proves (2.5).
Then, making use of (2.5), given any \( j \in \mathbb{Z} \) and any \( t \in [j, j+1] \),

\[
|h'(t)| \leq |h'(t_j)| + \left| \int_{t_j}^{t} h''(s) \, ds \right| \\
\leq 2\|h\|_{L^\infty(\mathbb{R})} + \|h'\|_{L^\infty(\mathbb{R})}|t - t_j| \leq 2\|h\|_{L^\infty(\mathbb{R})} + \|h'\|_{L^\infty(\mathbb{R})}.
\]

\( \square \)

2.3 ODE analysis

The scope of this section is an elementary analysis of the solutions \( h \in C^2(\mathbb{R}) \) of

\[ h''(t) = W'(h(t)) \quad \text{for any } t \in \mathbb{R}. \] (2.6)

Recall that for any any \( C^2 \) solution of (2.6) and any \( s, t \in \mathbb{R} \),

\[
\frac{|h'(s)|^2}{2} - W(h(s)) = \frac{|h'(t)|^2}{2} - W(h(t)).
\] (2.7)

Furthermore

**Lemma 2.3.** Let \( h \) be bounded. Then,

\[
\|h\|_{C^2(\mathbb{R})} \leq C,
\] (2.8)

for a suitable \( C > 0 \), possibly depending on \( \|h\|_{L^\infty(\mathbb{R})} \).

Also, for any \( t \in \mathbb{R} \),

\[-W\left(\inf_{\mathbb{R}} h\right) = -W\left(\sup_{\mathbb{R}} h\right) = \frac{|h'(t)|^2}{2} - W(h(t)).\] (2.9)

**Proof.** By construction,

\[
|h''(t)| \leq \max_{[-\|h\|_{L^\infty(\mathbb{R})},\|h\|_{L^\infty(\mathbb{R})}]} |W'|
\]

and so, by (2.4), we get (2.8).

We take

\[
\sigma \in \left\{ \inf_{\mathbb{R}} h, \sup_{\mathbb{R}} h \right\}.
\]

Let also \( t_n \) be a sequence for which

\[
\lim_{n \to +\infty} h(t_n) = \sigma.
\]

Let \( w_n(t) := h(t + t_n) \). From (2.8), we have that \( w_n \) converges, up to subsequence, in \( C^1_{\text{loc}}(\mathbb{R}) \) to some function \( w \in C^1(\mathbb{R}) \).
We now suppose that $\sigma = \inf_{\mathbb{R}} h$ (for this argument, the case $\sigma = \sup_{\mathbb{R}} h$ is completely analogous). Then,

$$w(0) = \lim_{n \to +\infty} w_n(0) = \lim_{n \to +\infty} h(t_n) = \sigma \leq \lim_{n \to +\infty} h(t + t_n) = w(t)$$

for any $t \in \mathbb{R}$, so $w'(0) = 0$ and therefore, by (2.7), for any $t \in \mathbb{R},$

$$\frac{|h'(t)|^2}{2} - W(h(t)) = \lim_{n \to +\infty} \frac{|h'(t_n)|^2}{2} - W(h(t_n)) = \lim_{n \to +\infty} \frac{|w_n'(0)|^2}{2} - W(w(0)) = -W(\sigma).$$

**Lemma 2.4.** If $h'(t_0) = 0$, then $h$ is symmetric with respect to $t = t_0$, that is $h(t_0 - t) = h(t_0 + t)$ for any $t \in \mathbb{R}$.

**Proof.** We set $h_\pm(t) := h(t_0 \pm t)$. Since $h_\pm''(t) = W'(h_\pm(t))$ for any $t \in \mathbb{R}$, $h_\pm(0) = h(t_0)$ and $h_\pm'(0) = 0$, we deduce from Cauchy Uniqueness Theorem that $h_+(t) = h_-(t)$.

**Lemma 2.5.** If $h$ has two or more critical points, then it is periodic.

**Proof.** Suppose that $h'(a) = h'(b) = 0$ with $b > a$ and let $T := b - a$. Then, utilizing Lemma 2.4,

$$h(t + T) = h(b + (t - a)) = h(b - (t - a)) = h(a - (t - b)) = h(a + (t - b)) = h(t - T)$$

for any $t \in \mathbb{R}$, and so $h$ has period $2T$.

**Lemma 2.6.** If $|h| \leq 1$, then

$$\sup_{\mathbb{R}} h = -\inf_{\mathbb{R}} h.$$

**Proof.** Let

$$m := \inf_{\mathbb{R}} h \quad \text{and} \quad M := \sup_{\mathbb{R}} h.$$

By (2.9), we have

$$W(m) = W(M). \tag{2.10}$$

Thus, by Rolle’s Theorem, there exists $\xi \in (m, M)$ such that $h'(\xi) = 0$. From (1.2), we deduce that $\xi \in \{-1, 0, +1\}$. But since, by assumption, both $m$ and $M$ lie in $[-1, 1]$, we have that $\xi \in (m, M) \subseteq (-1, 1)$ and so $\xi = 0$.

This says that $m < 0 < M$. Thus the claim follows from (1.2) and (2.10).

**Lemma 2.7.** Suppose that $h$ is either non-periodic or it is constant but not zero. Then,

$$W(\inf_{\mathbb{R}} h) = W(\sup_{\mathbb{R}} h) = 0. \tag{2.11}$$
Proof. If \( h \) is constant but not zero, then either \( h \) is constantly equal to \(-1\) or it is constantly equal to \(+1\), because of (1.2).

Since in such cases (2.11) is obvious, we focus on the case in which \( h \) is not periodic. Then, by Lemma 2.5,

\[
h \text{ has at most one critical point.} \tag{2.12}
\]

In particular, \( h \) attains either its sup or its inf at either \(+\infty\) or \(-\infty\). So, let us assume, for definiteness that

\[
\sup_{\mathbb{R}} h = \lim_{t \to +\infty} h(t), \tag{2.13}
\]

the other cases being analogous.

By (2.8), we obtain that the limit in (2.13) holds in \(C^1\), therefore

\[
0 = \lim_{t \to +\infty} \int_{\mathbb{R}} h'(s + t)\phi'(s) + W'(h(s + t))\phi(s) \, ds = \int_{\mathbb{R}} W'(\sup_{\mathbb{R}} h)\phi(s) \, ds,
\]

for any \( \phi \in C^\infty_0(\mathbb{R}) \).

This says that

\[
W'(\sup_{\mathbb{R}} h) = 0
\]

and so, by (1.2),

\[
\sup_{\mathbb{R}} h \in \{-1, 0, +1\}.
\]

If \( \sup_{\mathbb{R}} h \in \{-1, +1\} \), then (2.11) holds true, recalling (2.9).

Thus, we consider the case

\[
\sup_{\mathbb{R}} h = 0.
\]

Then, recalling (2.12), we have two possibilities: either

\[
\inf_{\mathbb{R}} h = \lim_{t \to -\infty} h(t), \tag{2.14}
\]

or there exists \( t_o \in \mathbb{R} \) such that

\[
h(t_o) = \inf_{\mathbb{R}} h. \tag{2.15}
\]

Now, if (2.14) holds, we repeat the argument after (2.13) to obtain that

\[
W'(\inf_{\mathbb{R}} h) = 0
\]

and so, by (1.2),

\[
\inf_{\mathbb{R}} h \in \{-1, 0, +1\}. \tag{2.16}
\]

Since \( h \) is not constantly equal to zero, we have that

\[
0 = \sup_{\mathbb{R}} h > \inf_{\mathbb{R}} h
\]
and so (2.16) means that $\inf_{\mathbb{R}} h = -1$. This implies that (2.11) holds true, recalling (2.9).

Thus, we have only to deal with the case in which (2.15) holds, which we now show that is impossible. Indeed, if (2.15) were true, we would have $h'(t_o) = 0$ and so, by (2.9),

$$W(0) = W\left(\sup_{\mathbb{R}} h\right) = W(h(t_o)).$$

Thus, by (1.2), we would have that $\sup_{\mathbb{R}} h = 0 = \inf_{\mathbb{R}} h$,

in contradiction with our assumptions. \qed

**Lemma 2.8.** Let $a < b$. If $h$ is monotone in $(a, b)$ then

$$\int_a^b |h'(t)| \, dt \leq 2 \sup_{\mathbb{R}} |h|.$$

**Proof.** We have

$$\int_a^b |h'(t)| \, dt = \left| \int_a^b h'(t) \, dt \right| = |h(b) - h(a)| \leq 2 \sup_{\mathbb{R}} |h|. \qed$$

**Lemma 2.9.** Suppose $|h| \leq 1$. Then, if $h$ is not periodic,

$$\int_{-\infty}^{+\infty} |h'(t)| \, dt \leq 4.$$

**Proof.** By Lemma 2.8, we see that only two cases hold: either $h'$ never vanish or $h'$ has only one zero.

In any case, there exists $c \in \mathbb{R}$ in such a way that $h'(t) \neq 0$ for any $t \in (-\infty, c) \cup (c, +\infty)$.

Consequently, by Lemma 2.8, for any $a < c < b$,

$$\int_a^b |h'(t)| \, dt = \int_a^c |h'(t)| \, dt + \int_c^b |h'(t)| \, dt \leq 2 \sup_{\mathbb{R}} |h| + 2 \sup_{\mathbb{R}} |h|.$$

Then, the desired result follows by sending $a \to -\infty$ and $b \to +\infty$. \qed

**Lemma 2.10.** Let

$$\sigma \in \left\{ \inf_{\mathbb{R}} h, \sup_{\mathbb{R}} h \right\}.$$

Suppose $|h| \leq 1$. Then,

$$\int_{-\infty}^{+\infty} \frac{|h'(t)|^2}{2} + W(h(t)) - W(\sigma) \, dt \leq C,$$  

(2.17)

for a suitable structural constant $C > 0$, unless $h$ is periodic and non-constant.
Proof. Since (2.17) is obvious for $h$ constant, we focus on the case in which $h$ is not periodic.

We exploit Lemma 2.7, (2.8) and (2.9) to conclude that
\[
\int_{-\infty}^{+\infty} \left| h'(t) \right|^2 \frac{1}{2} + W(h(t)) - W(\sigma) \, dt \\
= \int_{-\infty}^{+\infty} \left| h'(t) \right|^2 \, dt \\
\leq C \int_{-\infty}^{+\infty} \left| h'(t) \right| \, dt \\
\leq 4C.
\]

\[\square\]

2.4 A compactness result

We now point out a useful compactness criterion:

Lemma 2.11. Let $u_k$ be a sequence of solutions of (1.1) in the whole $\mathbb{R}^2$. Then, up to subsequence, $u_k$ converges locally uniformly to some $u$ which is also a solution of (1.1) in the whole $\mathbb{R}^2$.

Proof. By Grushin-elliptic regularity (see, e.g., [9, 10, 13, 14]), we have that
\[
\|u_k\|_{C^\alpha(\mathbb{R}^2)} \leq \bar{C}
\]
for some $\bar{C} > 0$, therefore up to subsequence, $u_k$ converges locally uniformly to some $u$, and so
\[
\text{u is continuous.} \tag{2.18}
\]
Moreover, for any $a \in (0,1)$, the Grushin operator $\Delta_G$ is uniformly elliptic in $D_a := \{ |x| \in (a,1/a) \}$, therefore standard elliptic estimates give that
\[
\|u_k\|_{C^{2,\beta_a}(D_a)} \leq \bar{C}_a
\]
and so
\[
\Delta_G u(x) = W'(u(x)) \tag{2.19}
\]
for any $x \in D_a$.
Since $a$ can be taken arbitrarily small, we have that (2.19) holds for any $x \in \mathbb{R}^2 \setminus \{0\}$.
But then, since the map $x \mapsto W'(u(x))$ is continuous, by means of (2.18), it follows that $\Delta_G u$ is continuous too and so (2.19) holds for any $x \in \mathbb{R}^2$. \[\square\]

2.5 Basic spectral theory

Lemma 2.12. Fix $a \in (0,1)$, $R \geq 1$. Let $x_o = a + R$ and $\Omega := B_R(x_o,0)$. 
Then, there exists $\lambda \in \mathbb{R}$ for which there exists a non-trivial solution $\phi$ of
\[
\begin{cases}
\Delta_G \phi + \lambda \phi = 0 & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega.
\end{cases}
\] (2.20)

Moreover, we can take
\[
\lambda \in \left(0, \frac{C}{R^2}\right],
\] (2.21)
for a suitable $C > 0$.

Proof. We have
\[
\int_\Omega |\nabla_G v|^2 \leq \int_\Omega |\partial_x v|^2 + (x_o + R)^2 |\partial_y v|^2 \\
\leq (1 + a + 2R)^2 \int_\Omega |\nabla v|^2.
\] (2.22)
Therefore, by standard Poincaré inequality,
\[
\int_\Omega |\nabla_G v|^2 \leq C(a, R) \int_\Omega |v|^2,
\] (2.23)
for any $v \in C^\infty_0(\Omega)$.
Moreover,
\[
\int_\Omega |\nabla_G v|^2 \geq \int_\Omega |\partial_x v|^2 + a |\partial_y v|^2 \\
\geq \min\{1, a\} \int_\Omega |\nabla v|^2.
\] (2.24)

From (2.23), we thus follow the standard minimization argument, taking
\[
\lambda = \inf_{v \in C^\infty_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla_G v|^2}{\int_\Omega |v|^2}
\] (2.25)
and we recover compactness from (2.21) and the classical embeddings, proving (2.20).

Now, if $\lambda$ and $\phi$ satisfy (2.20), we may suppose that
\[
\int_\Omega \phi^2 = 1
\]
and so
\[
\int_\Omega |\nabla_G \phi|^2 = \lambda
\]
which gives that $\lambda > 0$. 

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Also, from (2.25) and a change of variable,

\[ \lambda = \inf_{v \in C^\infty_0(B_R(x_0,0) \setminus \{0\})} \frac{\int_{\{(x-x_o)^4+4y^2 \leq R^4\}} |\partial_x v(x, y)|^2 + |x|^2|\partial_y v(x, y)|^2 \, d(x, y)}{\int_{\{(x-x_o)^4+4y^2 \leq R^4\}} |v(x, y)|^2 \, d(x, y)} \]

\[ = \inf_{\psi \in C^\infty_0(B_1) \setminus \{0\}} \frac{\int_{B_1} R^{-2}\left(\partial_w \psi(w, z)\right)^2 + R^{-4}|x_o + Rw|^2\left(\partial_z \psi(w, z)\right)^2}{\int_{B_1} \psi^2} \]

\[ \leq \inf_{\psi \in C^\infty_0(B_1) \setminus \{0\}} \frac{\int_{B_1} R^{-2}\left(\partial_w \psi(w, z)\right)^2 + R^{-4}(3R)^2\left(\partial_z \psi(w, z)\right)^2}{\int_{B_1} \psi^2} \]

\[ \leq 10R^{-2} \inf_{\psi \in C^\infty_0(B_1) \setminus \{0\}} \frac{\int_{B_1} |\nabla \psi|^2}{\int_{B_1} \psi^2}. \]

This and the classical Poincaré inequality imply (2.21). \( \square \)

### 2.6 Extension of bounded harmonic functions

**Lemma 2.13.** Let \( u \) be \( \Delta_G \)-harmonic in \( B_r \setminus \{0\} \). Suppose that \( u \) is bounded in \( B_r \setminus \{0\} \). Then, it may be extended to a \( \Delta_G \)-harmonic in \( B_r \).

**Proof.** The fundamental solution of \( \Delta_G \) is \( \psi(x, y) = (x^4 + 4y^2)^{-\frac{1}{4}} \) (see Theorem 3.1 of \[3\] for a formula for generalized Grushin operators). Thus, the argument on pages 16–17 of \[12\] may be repeated verbatim. \( \square \)

### 3 Monotonicity and stability

We show that (1.4) is sufficient for stability. This is in analogy with the fact that monotonicity in any direction implies stability in the Euclidean setting (see \[1\]) – but in the Grushin plane the directions do not play the same role, thus (1.4) somehow selects the good direction for stability.

**Proposition 3.1.** Let \( u \in C^2(\mathbb{R}^2) \) be a solution of (1.1) satisfying (1.4). Then, \( u \) is stable.

**Proof.** The argument we present here is a modification of a classical one (see \[1\] and also Section 7 in \[7\] for a general result). We recall that we need to prove that for any smooth \( \phi \), compactly supported

\[ 0 \leq \int_{\mathbb{R}^n} |\nabla_G \phi|^2 - f'(u)\phi^2 \, dx. \]
For any $\varphi$ smooth and compactly supported, we have

$$\int_{\mathbb{R}^2} f'(u) Tu \varphi = \int_{\mathbb{R}^2} \partial_y (f(u)) \varphi = - \int_{\mathbb{R}^2} f(u) \partial_y \varphi$$

$$\int_{\mathbb{R}^2} \Delta_G u \partial_y \varphi = - \int_{\mathbb{R}^2} < \nabla_G u, \nabla \partial_y \varphi >$$

$$= \int_{\mathbb{R}^2} < T \nabla u, \nabla_G \varphi > = \int_{\mathbb{R}^2} < \nabla (Tu), \nabla_G \varphi > .$$

Therefore, by taking $\varphi := \phi^2 / (Tu)$, and making use of the Cauchy-Schwarz inequality,

$$0 \int_{\mathbb{R}^n} \frac{2\phi \nabla_G (Tu), \nabla_G \phi}{Tu} - \frac{\phi^2 |\nabla_G (Tu)|^2}{(Tu)^2} - f'(u) \phi^2 \, dx$$

$$\leq \int_{\mathbb{R}^n} |\nabla_G \phi|^2 - f'(u) \phi^2 \, dx. \quad \square$$

### 4 Energy estimates

We follow here some ideas of [1] to estimate the energy

$$\mathcal{F}_R(u) := \int_{B_R(0)} \frac{|\nabla_G u(\xi)|^2}{2} + W(u(\xi)) \, d\xi.$$ 

For this, for any $t \in \mathbb{R}$, we define the translation

$$u^t(x, y) := u(x, y + t)$$

and the translated energy

$$\mathcal{E}_R(t) := \mathcal{F}_R(u^t).$$

Of course, $\mathcal{E}_R(0) = \mathcal{F}_R(u)$.

**Lemma 4.1.** Suppose that $u \in C^2(\mathbb{R}^2, [-1, 1])$, with $Tu > 0$ and $|\nabla_G u| \in L^\infty(\mathbb{R}^2)$, is a solution of

$$\Delta_G u(\xi) = W'(u(\xi)) \quad \text{for any } \xi \in \mathbb{R}^2.$$ 

Then, there exists a structural constant $C$ in such a way that

$$\mathcal{E}_R(0) \leq \mathcal{E}_R(t) + CR^2, \quad (4.1)$$

for any $t \in \mathbb{R}$ and any $R > 0$.

**Proof.** We prove (4.1) for $t > 0$ (this is enough, since $u(x, -y)$ is also a solution).
We have, recalling Lemma 2.1
\[
\frac{d}{dt} \mathcal{E}_R(t) = \int_{B_R(0)} < \nabla_G u^t, \nabla_G T u^t > + W'(u^t) T u^t \, d\xi
\]
\[
\geq \int_{B_R(0)} \left( - \Delta_G (T u^t) + W'(u^t) \right) T u^t \, d\xi
\]
\[
- R^2 \| \nabla_G u^t \|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} T u^t (RX, R^2 Y) \, d\mathcal{H}^1(X, Y)
\]
\[
= 0 - R^2 \| \nabla_G u \|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} \partial_y u(RX, R^2 Y + t) \, d\mathcal{H}^1(X, Y).
\]

Hence, fixed any \( \tau > 0 \),
\[
\mathcal{E}_R(\tau) - \mathcal{E}_R(0) = \int_0^\tau \frac{d}{dt} \mathcal{E}_R(t) \, dt
\]
\[
\geq - R^2 \| \nabla_G u \|_{L^\infty(\mathbb{R}^2)} \int_0^\tau \int_{\partial B_1(0)} \partial_y u(RX, R^2 Y + t) \, d\mathcal{H}^1(X, Y) \, dt
\]
\[
= - R^2 \| \nabla_G u \|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} \left( \int_0^\tau \partial_y u(RX, R^2 Y + t) \, dt \right) \, d\mathcal{H}^1(X, Y)
\]
\[
= - R^2 \| \nabla_G u \|_{L^\infty(\mathbb{R}^2)} \int_{\partial B_1(0)} u(RX, R^2 Y + \tau) - u(RX, R^2 Y) \, d\mathcal{H}^1(X, Y)
\]
\[
\geq - 2 R^2 \| \nabla_G u \|_{L^\infty(\mathbb{R}^2)} \| u \|_{L^\infty(\mathbb{R}^2)} \mathcal{H}^1(\partial B_1(0)),
\]
which gives (4.1). \( \square \)

**Theorem 4.2.** Suppose that \( u \in C^2(\mathbb{R}^2, [-1, 1]) \), with \( |\nabla_G u| \in L^\infty(\mathbb{R}^2) \) is a solution of
\[
\Delta_G u(\xi) = W'(u(\xi)) \quad \text{for any } \xi \in \mathbb{R}^2.
\]
Assume that (1.4) holds true.

Then, there exists a structural constant \( C \) in such a way that
\[
\mathcal{F}_R(u) \leq CR^2
\]
for any \( R > 0 \).

As a consequence, (1.3) holds true.

**Proof.** We have that \( u \) is bounded and monotone in \( y \), thanks to (1.4). Thus, we may define
\[
u^\pm(x) := \lim_{y \to \pm\infty} u(x, y).
\]

Then, from Lemma 2.11 we have that
\[
\Delta_G u^\pm(x) = W'(u^\pm(x))
\]
In fact, since \( u \) does not depend on \( y \), we may write (4.3) as
\[
(u^\pm)''(x) = W'(u^\pm(x)) \tag{4.4}
\]
and so we may apply to \( u^\pm \) the ODE analysis developed in Section 2.3.

For this, we observe that
\[
\text{at least one between } u^+ \text{ and } u^- \text{ is either constant or non-periodic.} \tag{4.5}
\]

To prove (4.5), we argue by contradiction, supposing that \( u^+ \) and \( u^- \) are both periodic and non-constant. In particular, by Cauchy Uniqueness Theorem, \(|u^\pm| < 1\) and then, by Lemma 2.6, we would have that
\[
\max_{\mathbb{R}} u^\pm = - \min_{\mathbb{R}} u^\pm. \tag{4.6}
\]

But from (1.4), we know that
\[
u^+(x) > u^-(x) \text{ for any } x \in \mathbb{R} \tag{4.7}
\]
and so, if we set \( x_{\min}^\pm, x_{\max}^\pm \) be such that

\[
u^\pm(x_{\min}^\pm) = \min_{\mathbb{R}} u^\pm \text{ and } \nu^\pm(x_{\max}^\pm) = \max_{\mathbb{R}} u^\pm,
\]
we deduce from (4.6) and (4.7) that
\[
u^-(x_{\min}^+) \geq u^-(x_{\min}^-) = -u^-(x_{\max}^-) > -u^+(x_{\max}^-) \\
\geq -u^+(x_{\min}^+) = u^+(x_{\min}^+) > u^-(x_{\min}^-).
\]

This contradiction proves (4.5).

We now claim that
\[
\text{either } u^+ \text{ or } u^- \text{ is non-periodic or constant but not zero.} \tag{4.8}
\]

To prove this, we argue by contradiction. Suppose (4.8) is false. Then, both \( u^- \) and \( u^+ \) are periodic. Then, at least one, say \( u^+ \) is constant, because of (4.5). If \( u^+ \) were not equal to zero, then (4.8) would be true, thus we have to say that \( u^+ \) is constantly equal to zero and that \( u^- \) is periodic. But then \( u^- \) cannot be constant, otherwise (4.4), (1.2) and (1.4) would say that \( u^- \) is constantly equal to \(-1\) and (4.8) would be true. Thence, we are forced to the case in which \( u^+ \) is identically zero and \( u^- \) is periodic and non-constant. Thus, by (1.4),
\[
\sup_{\mathbb{R}} u^- \leq 0
\]
and so, by Lemma 2.6,
\[
\inf_{\mathbb{R}} u^- = - \sup_{\mathbb{R}} u^- \geq 0 \geq \sup_{\mathbb{R}} u^-.
\]
This would say that \( u^- \) is constant, while we know it is not the case. This contradiction proves (4.8).

By means of (4.8), up to a sign change, we may suppose that \( u^+ \) is either constant but not zero or it is non-periodic. Consequently, by Lemma 2.10,

\[
\int_{-\infty}^{+\infty} \frac{|(u^+)'(t)|^2}{2} + W(u^+(t)) - W(\sigma^+) \, dt \leq C^+,
\]

(4.9)

for a suitable \( C^+ > 0 \), with

\[
\sigma^+ \in \left\{ \inf_{\mathbb{R}} u^+, \sup_{\mathbb{R}} u^+ \right\}.
\]

In fact, (2.11) and (4.9) give that

\[
\int_{-\infty}^{+\infty} \frac{|(u^+)'(t)|^2}{2} + W(u^+(t)) \, dt \leq C^+.
\]

(4.10)

Moreover, by (4.1),

\[
\mathcal{E}_R(0) - CR^2 \leq \lim_{t \to +\infty} \mathcal{E}_R(t)
= \lim_{t \to +\infty} \int_{B_R(0)} \frac{|
abla_G u(x, y + t)|^2}{2} + W(u(x, y + t)) \, d(x, y)
= \int_{B_R(0)} \frac{|
abla_G u^+(x)|^2}{2} + W(u^+(x)) \, d(x, y)
\leq \int_{-R/2}^{R/2} \int_{-\infty}^{+\infty} \frac{|
abla_G u^+(x)|^2}{2} + W(u^+(x)) \, dx \, dy
= R^2 \int_{-\infty}^{+\infty} \frac{|
abla_G u^+(x)|^2}{2} + W(u^+(x)) \, dx.
\]

Thus, by (4.10),

\[
\mathcal{E}_R(0) - CR^2 \leq C^+ R^2.
\]

\[\square\]

5  The counter-example

5.1  Monotonicity and Maximum Principle

For any \( s \in \mathbb{R} \) and \( \xi \in \mathbb{R}^2 \), let

\[
T_s \xi := \xi + (0, s).
\]

A domain \( \Omega \subset \mathbb{R}^2 \) is said to be T-convex if for any \( \xi_1 \in \Omega \) and any \( \alpha > 0 \) such that \( T_\alpha \xi_1 \in \Omega \) one has that \( T_s \xi_1 \in \Omega \) for every \( s \in (0, \alpha) \).

That is, \( \Omega \) is T-convex when vertical segments joining two points of \( \Omega \) lie in \( \Omega \).
Theorem 5.1. Let $\Omega$ be an arbitrary bounded domain of $\mathbb{R}^2$ which is $T$-convex. Let $u \in \Lambda^2(\Omega) \cap C(\overline{\Omega})$ be a solution of

\[
\begin{align*}
\Delta_G u + f(u) &= 0 \quad \text{in } \Omega \\
u &= \psi \quad \text{on } \partial \Omega
\end{align*}
\]

(5.1)

where $f$ is a Lipschitz continuous function. Assume that for any $\xi_1, \xi_2 \in \partial \Omega$, such that $\xi_2 = T_\alpha \xi_1$ for some $\alpha > 0$, we have, for each $s \in (0, \alpha)$ either

\[
\psi(\xi_1) < u(T_s \xi_1) < \psi(\xi_2) \quad \text{if } T_s \xi_1 \in \Omega \quad (5.2)
\]

or

\[
\psi(\xi_1) < \psi(T_s \xi_1) < \psi(\xi_2) \quad \text{if } T_s \xi_1 \in \partial \Omega. \quad (5.3)
\]

Then $u$ satisfies

\[
u(T_{s_1} \xi) < u(T_{s_2} \xi) \quad (5.4)
\]

for any $0 < s_1 < s_2 < \alpha$ and for every $\xi \in \Omega$.

Moreover, $u$ is the unique solution of (5.1) in $\Lambda^2(\Omega) \cap C(\overline{\Omega})$ satisfying (5.2).

The proof of this result is done through the sliding method introduced in [2] for uniformly elliptic equations. This method uses two fundamental ingredients: the Maximum Principle in small domains and the invariance of the operator with respect to “sliding”. In [5] the equivalent of Theorem 5.1 was proved for sub-elliptic equations in nilpotent Lie groups. There, the key “new” ingredient being a Hölder estimate for Hörmander type operators proved in [13] that allowed to prove the Maximum Principle in small domains.

The operator is invariant by $T_s$ translations and our equation satisfies the hypotheses of [13], hence the proof of Theorem 5.1 proceeds exactly like the one given in [5], and we omit it.

5.2 Existence of monotone solutions that are not one-dimensional

The following result shows that Question 1.2 has a negative answer:

Theorem 5.2. There exists a solution of

\[
\Delta_G u - W'(u) = 0
\]

in $\mathbb{R}^2$ such that $Tu = \partial_y u > 0$.

Also, such $u$ is not one-dimensional.
Proof. We follow the two steps of \cite{[4]}.  

Step 1. Construction of a monotone solution in a bounded set. 
Let $M > 0$ be greater than the Lipschitz constant of $f$, let $g(u) := f(u) + Mu$, $Q^+_R := (-R, R) \times [0, R^2]$ and $Q^-_R := (-R, R) \times (-R^2, 0)$. 

We consider the operator $T$ on $C^{\alpha}$ such that $T v = u$ is the classical solution of 

\[
\begin{cases}
\Delta_G u - Mu = -g(v) & \text{in } Q^+_R \\
u(x, 0) = 0, \ u(x, R^2) = 1, \ u(-R, y) = \psi(y), \ u(R, y) = \psi(y) & \end{cases}
\]

where $0 \leq \psi \leq 1$ with $\psi(0) = 0$ and $\psi(R^2) = 1$. 

The following properties hold: 
(P1) $T$ is well defined, see \cite{[14],[13]}. 
(P2) It is monotone, i.e. $0 \leq v_1 \leq v_2 \leq 1$ implies $Tv_1 \leq Tv_2$. This is just the Maximum Principle, because with our choice of $M$ we get that 

\[v_1 \leq v_2 \implies g(v_1) \geq g(v_2)\]. 

(P3) If $0 \leq v \leq 1$ then $0 \leq Tv \leq 1$ (again by Maximum Principle). 
(P4) For $R$ sufficiently large there exists $v_o > 0$ in some fixed subset of $Q^+_R$ such that if $u_k := T^k(v_o)$ then $u_k \geq v_o$ for any $k \in \mathbb{N}$. 

Let us prove (P4). 

Let 

\[l := \lim_{s \to 0} \frac{|W'(s)|}{s}\]

and let $R_o$ be sufficiently large that $Q^+_{R_o}$ contains a ball $B$ such that $B \cap \{x = 0\} = \emptyset$ and $\lambda_o$ the principal eigenvalue of $-\Delta_G$ in $B$ satisfies 

\[\lambda_o \leq \frac{l}{2}\]. 

We remark that we can take such a $\lambda_o$ in the light of Lemma \ref{Lemma2.12}. 

Let $\varphi_o$ be the corresponding eigenfunction normalized by $\sup \varphi_o = 1$. Our choice of $\lambda_o$ implies that there exists $\varepsilon > 0$ such that 

\[\lambda_o \varepsilon \varphi_o \leq |W'(\varepsilon \varphi_o)|.\] \hspace{1cm} (5.5)

Now we define 

\[v_o = \begin{cases}
\varepsilon \varphi_o & \text{in } B \\
0 & \text{in } Q^+_R \setminus B.
\end{cases}\] \hspace{1cm} (5.6)

Observe that, in $B$, the Grushin operator $\Delta_G$ is uniformly elliptic, and so by standard estimates we know that $v \in C^{\alpha}(Q^+_R)$. Using (5.5), (P2) and (P3), we get that $u_o = T(v_o) \geq v_o$. 

So, iteratively, $T^k(v_o) \geq v_o$ for any $k \in \mathbb{N}$. This proves (P4).
By Lemma 2.11 we may and do suppose that $u_k := T^k(u_o)$ converges to a solution $\tilde{u} = \tilde{u}_R$ of

\[
\begin{cases}
\Delta_G \tilde{u} - W'(\tilde{u}) = 0 & \text{in } Q^+_R \\
\tilde{u}(x,0) = 0, \quad \tilde{u}(x,R^2) = 1 & \tilde{u}(-R,y) = \psi(y), \quad \tilde{u}(R,y) = \psi(y)
\end{cases}
\]

Note that $\tilde{u}$ satisfies $0 \leq \tilde{u} \leq 1$. Hence, using Theorem 5.1, we know that

$$\partial_y \tilde{u} > 0. \quad (5.7)$$

Finally, we extend the solution to $Q_R = \overline{Q^+_R} \cup \overline{Q^-_R}$ by taking

$$v_R(x,y) := \begin{cases} 
\tilde{u}(x,y) & \text{for } (x,y) \in \overline{Q^+_R} \\
-\tilde{u}(x,-y) & \text{for } (x,y) \in \overline{Q^-_R}
\end{cases}$$

Clearly, $v_R$ is a solution in $Q^+_R \cup Q^-_R$. Also the solution $u$ is $C^2$ up to the boundary for $x \neq 0$. Hence we get that $v_R$ is a solution in $Q_R \setminus \{(0,0)\}$. To check that $v_R$ is a solution in all of $Q_R$. Observe that the map $\zeta \mapsto W'(v_R(\zeta))$ is in $C^{\alpha}(Q_R)$, hence there exists $w \in C^{\alpha}_{\text{loc}}(Q_R) \cap C(Q_R)$ solution of

$$\Delta_G w - W'(v_R) = 0 \quad \text{in } Q_R.$$ 

Then $w - v_R$ is $\Delta_G$-harmonic in $Q_R \setminus \{(0,0)\}$ and it is bounded. Thus, by Lemma 2.13 it is $\Delta_G$-harmonic in $Q_R$, and so $v_R$ is a solution in all of $Q_R$.

Furthermore $v_R$ is monotone in $T$, in the sense that $Tv_R = \partial_y v_R > 0$, because of (5.7).

**Step 2.** Let $R \to \infty$. Then, by Lemma 2.11, $v_R$ locally uniformly converges to some $u$, which is a solution of

$$\Delta_G u - W'(u) = 0 \quad \text{in } \mathbb{R}^2.$$ 

Furthermore, in $Q^+_R$,

$$v_R = \tilde{u} = \lim_{k \to +\infty} u_k \geq v_0,$$

due to (P4) and so, by (5.6), $u \neq 0$ in $Q^+_R$.

Then, $u$ is monotone i.e. $\partial_y u > 0$, and it is therefore the counter-example we are looking for.

Indeed, $u$ is not one-dimensional; suppose, by contradiction, that there exists a function $g$ such that

$$u(x,y) = g(ax + by),$$

for any $(x,y) \in \mathbb{R}^2$.

Then, the strict monotonicity in $T$ of $u$ implies that

$$b \neq 0. \quad (5.8)$$

Clearly $g$ would be a solution of

$$(a^2 + b^2 x^2)g''(ax + by) - W'(g(ax + by)) = 0, \quad (5.9)$$
for any \((x, y) \in \mathbb{R}^2\).

This implies that for any \(t\) along the lines \(ax + by = t\),

\[
(a^2 + b^2 x^2)g''(t) - W'(g(t)) = 0.
\]

Since \(b \neq 0\), this implies that \(g'' \equiv 0\). Hence \(W'(g(t)) = 0\) for any \(t\) and so \(g\) would be constant, in contradiction with the fact that \(Tu > 0\). \hfill \square

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