Stability Analysis of Dual-rate Haptics Controller Using Two Control Architectures

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Abstract. Digital implementation of uniform-rate haptics controller limits the range of virtual environment parameters that can be stably implemented, particularly at higher sampling rates. Dual-rate sampling scheme alleviates this limitation and enlarges the range of these parameters that can be stably implemented. In this paper, theoretical stability analysis of the dual-rate haptics controller is presented using two control architectures. Analytical conditions that guarantee the closed-loop stability of the system are developed using linear control theory tools. The stability criteria constrains the virtual environment parameters as a function of the ratio of sampling periods of the controller, and thereby establish limits on the performance of these controllers. The developed criteria can be seen as the generalization of existing criterion available in the literature.

Keywords – Impedance haptic interfaces, Closed-loop stability, Dual-rate sampling.

1. Introduction
Haptic interfaces enable a human operator to interact with virtual or tele-operated environments through the sense of touch. Most of the commercially available and custom-made haptic interfaces belong to a class of impedance based devices that use motion commands of the operator as input and respond with a force feedback. Virtual walls being the fundamental building block of any complex virtual environments, are typically modelled as having certain impedance, i.e., stiffness and damping. When a human operator interacts with the virtual wall, a feedback force is exerted, which is typically a function of the device position and velocity (see Fig. 1).

Prior research has demonstrated that impedance based haptics interfaces are capable of rendering contact with compliant walls stably. However, they often exhibit unstable behavior when attempts are made to simulate stiff virtual walls [1], [2]. One of the factors that has a bearing on the limited performance of these devices is the sampling rate at which they are controlled [1], [2], [3]. Although in principle, higher sampling rates should enhance their impedance range (known as $Z$-width in the conventional haptics literature [1]), however, in practice, they do just the opposite. This is mainly due to the digital implementation of the virtual environments. Other factors that contribute to the limited performance of haptic interfaces include physical damping, sensor quantization, time delay, vibration modes of the device, velocity estimation, bandwidth of device amplifiers, etc. [1], [2], [4], [5].

Rendering stiff virtual walls without jeopardizing the interaction stability has thus been a persistent problem in impedance based haptics interfaces. Many techniques have been proposed to address the same [6], [7], [8]. Multi-rate control strategy forms one of such techniques that
focuses on the controller part of the haptic interface. Various strategies focusing on the multi-
rate control architecture have thus been proposed, e.g., [9], [10], [11], [12], with each focussed
on improving the performance of haptic interfaces, particularly at higher sampling rates of the
controller.

Lee and Lee [9] for instance, proposed a multi-rate control architecture in which a high-
frequency controller was added to a relatively slow conventional controller to reduce the effect of
time discretization. Similarly, in [11] a multi-rate control strategy was proposed for stable haptic
dental training system. Koul et. al [10] proposed a dual-rate sampling scheme to alleviates the
stability issues at higher sampling rates. The scheme de-couples the sampling of position and
the velocity loops in the conventional uniform-rate haptic controllers. The strategy allowed
simulating stiffer virtual walls while simultaneously maintaining the stability of the haptic
interface.

Unlike robotic manipulators, haptic interfaces share a common workspace and operate in
direct physical contact with a human operator, placing stringent requirements on the stability
of these devices. Moreover, instabilities arising during haptic interaction can lead to unrealistic
touch sensations and also damage the hardware. Ensuring the stability of these devices is
therefore essential. Stability analysis of haptic interfaces is complicated as the overall system is
non-linear and hybrid (sampled-data) in nature. The presence of human operator in the loop
further adds to the complexity, as human impedance changes dynamically and radically.

Several authors have used linear control tools to analyze the stability of haptics controllers.
Minsky et al. [3] established an expression to guarantee the stability of a haptic system after
theoretical formulations and experimental analysis. Gil et al. [14] performed stability analysis of
a 1 degree-of-freedom (1-DOF) haptic device by direct application of Routh-Hurwitz criterion.
Their work was extended in [4], [15] to study the effect of physical damping, time delay and
human operator on the stability of haptic devices. Lee and Lee [9] carried out the stability
analysis of their multi-rate controller in the frequency domain and obtained an upper bound
on the virtual stiffness that can be stably rendered. Recently Mashayekhi et al [16] carried
out stability analysis of haptic interfaces and obtained analytical expressions using frequency
response function analysis. The expressions were obtained without making any assumptions
about the virtual damping and time delay.

Most of the works on stability analysis reported in the haptics literature are restricted to
uniform-rate controllers. The present work is concerned with the stability analysis of the dual-
rate haptics controller proposed by Koul et. al [10]. The proposed controller is complicated

![Figure 1. Schematic of human interaction with a virtual environment.](image-url)
and the stability analysis of the same is therefore of interest. We use linear control tools to obtain analytical expressions that guarantee the stability of the proposed dual-rate haptics controller. The controller is particularly analyzed using two control architectures, detailed later. The derived stability criteria are shown to reduce to the well-known criterion for uniform-rate haptics controller available in the literature.

2. System Architecture

For analytical purposes, we consider two architectures for the dual-rate haptics controller and the same are depicted in Figures 2 and 3. These two architectures will henceforth be referred to as variable position sampling architecture and variable velocity sampling architecture respectively. In the variable position sampling architecture, position loop is sampled every \( T/N \) instants (\( N \) being an integer) and velocity is sampled at \( T \). This allows us to investigate the effect of varying the position loop sample period while maintaining that of the velocity loop as constant. In the variable velocity sampling architecture, the position is sampled every \( T \) instants and velocity loop is sampled at \( NT \). This allows us to investigate the effect of varying the velocity loop sample period while maintaining that of the position loop as constant.

The 1 degree-of-freedom (1-DOF) haptic device is modeled as an inertia (\( I \)) with viscous damping (\( b \)). The virtual wall is implemented as the typical viscoelastic parallel spring-damper (Kelvin-Voight) model, which in essence is a discrete proportional-derivative controller. The stiffness and damping of the virtual wall are denoted by \( K \) and \( B \) respectively. It is assumed that the device is equipped with only a single position sensor and a single actuator that provides the necessary feedback force to the operator. The sampled position data (\( \theta_d \)) is differentiated digitally using a backward difference to obtain the velocity estimate. The haptic device is acted upon by two torques (or equivalently forces): the torque due to the human operator, \( \tau_H \), and the feedback torque from the virtual wall, \( \tau_{VE} \). The feedback torque is relayed to the operator after being held in the zero-order hold (ZOH) blocks. The unilateral constraint represents the hard nonlinearity of the haptic interaction and ensures that no forces/torques are fed to the user during free space motion i.e. when the operator is not in contact with the wall. As the main focus of the current work is to analyze the effect of dual-rate sampling on stability, hence nonlinear effects like position sensor quantization, device Coulomb friction and actuator saturation are omitted from the analysis.
3. Stability Analysis of Dual-rate Haptics Controller

In this section, we develop stability criteria for dual-rate haptic rendering using linear control techniques. The criteria are in the form of inequalities that give stability bounds of the virtual wall parameters as a function of device parameters and the ratio of sample periods of the controller. The stability analysis for both the control architectures follows a similar procedure, however, we provide a detailed derivation for the variable position sampling architecture only, whereas the results are provided for both. The theoretical analyses are conducted by assuming no human interaction with the device. This approach leads to the worst case stability conditions, as the human operator has a stabilizing effect on the haptic interaction [14], [15].

The presence of two sampling rates in the system requires the dynamics of each loop be represented by a unique discrete-time variable ‘z’ [18]. In case of the variable position sampling architecture (Fig. 2), \( z_1 \) and \( z_2 \) are used to represent the dynamics of the position and the velocity loops respectively. The discrete-time variables \( z_1 \) and \( z_2 \) are related to their continuous-time counterparts through Tustin transformation [17], [18] as:

\[
\begin{align*}
    z_1 &= \frac{1 + \frac{sT}{2N}}{1 - \frac{sT}{2N}} \\
    z_2 &= \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}
\end{align*}
\]  

(1)

where ‘s’ is the continuous-time (Laplace) variable.

Elimination of \( s \) from Eq. 1 yields the following relation between the discrete time variables

\[
    z_1 = \frac{z_2(N + 1) + (N - 1)}{z_2(N - 1) + (N + 1)}
\]

(2)

Equation 2 can be used to transform a discrete-time system with sample period \( T/N \) to another discrete-time system with sample period \( T \). In other words, this equation can be used to resample a discrete-time system from sample period \( T/N \) to \( T \).

The stability of the dual-rate haptics controller is characterized by the roots of its characteristic equation, be it in continuous domain or discrete domain. Due to the sampled-data nature of the haptic system, the determination of characteristic equation requires continuous to discrete and discrete to continuous transformations during the analysis. To arrive at the characteristic equation following procedure is adopted here: In the first step, the device transfer function, \( D(s) \) is discretized using the ZOH method [17] with sample period \( T/N \). Once the discretized model of the plant i.e, \( D(z_1) \) is obtained, the position feedback loop is closed to get \( G(z_1) \). For conformity with the velocity loop, \( G(z_1) \) is then resampled using Eq. 2 and \( G(z_2) \) is obtained. Subsequently, the velocity feedback loop is closed and the overall system transfer function is obtained. The denominator of the overall transfer function yields the discrete-time characteristic equation.

The ZOH equivalent \( D(z_1) \) of the device transfer function, \( D(s) = \frac{1}{Is^2 + bs} \) is given by [17]

\[
D(z_1) = \mathcal{Z}[\text{ZOH}_{T/N}.D(s)] = \mathcal{Z}\left[\mathcal{L}^{-1}\left(\frac{1 - e^{-sT/N}}{s} \frac{1}{Is^2 + bs}\right)\right]
\]

(3)

where \( \mathcal{Z} \) and \( \mathcal{L} \) are the Z-transform and Laplace transform operators respectively.

Eq. 3 upon simplification yields

\[
D(z_1) = \frac{I(\dot{z}_1 + \dot{e})}{b^2(z_1 - 1)(z_1 - e^n)}
\]

(4)
where, \( e^n = e^\frac{4\pi}{T} \), \( \dot{e} = e^n - 1 + \frac{\beta T}{T} \) and \( \ddot{e} = 1 - (1 + \frac{\beta T}{T})e^n \)

The position feedback loop is then closed, resulting in the transfer function

\[
G(z_1) = \frac{D(z_1)}{1 + KD(z_1)}
\]

\[
= \frac{I(\ddot{e}z_1 + \dot{e})}{b^2(z_1 - 1)(z_1 - e^n) + KI(\ddot{e}z_1 + \dot{e})}
\]

Next, \( G(z_1) \) is resampled to sample period \( T \) using Eq. 2. The transfer function so obtained is denoted by \( G(z_2) \) and is given by

\[
G(z_2) = \frac{I(z_2^2h_2h_3 + z_2(h_1h_3 + h_2h_4) + h_1h_4)}{g_2z_2^2 + g_1z_2 + g_0}
\]

where, \( h_1 = N + 1 \); \( h_2 = N - 1 \); \( h_3 = \ddot{e}h_1 + \dot{e}h_2 \); \( h_4 = \ddot{e}h_2 + \dot{e}h_1 \);

\( g_2 = 2b^2(h_1 - e^n h_2) + IKh_2h_3; \)

\( g_1 = 2b^2(h_2 - h_1 - e^n h_1 + e^n h_2) + IK(h_1 h_3 + h_2 h_4); \)

\( g_0 = -2b^2(h_2 - e^n h_1) + IKh_1 h_4. \)

Now, the overall closed-loop transfer function can be written as

\[
\frac{G(z_2)}{1 + B^2(z_2 - 1)G(z_2)}
\]

From Eq. 8 the characteristic equation of the system can be identified as

\[
\Delta(z_2) = 1 + B\frac{(z_2 - 1)}{Tz_2}G(z_2) = 0
\]

\[
= \alpha_d z_2^3 + \beta_d z_2^2 + \gamma_d z_2 + \delta_d = 0
\]

where, \( \alpha_d = Tg_2 + BIh_2h_3; \)

\( \beta_d = Tg_1 + BI(h_2h_4 + h_1h_3) - BIh_2h_3 = Tg_1 + BIh_3(h_1 - h_2) + BIh_2h_4; \)

\( \gamma_d = Tg_0 + BIh_1h_4 - BI(h_2h_4 + h_1h_3) = Tg_0 + BIh_4(h_1 - h_2) - BIh_1h_3; \)

and \( \delta_d = -BIh_1h_4. \)

The discrete-time characteristic equation given by Eq. 10 can be analyzed for stability using Jury’s criterion [13]. An alternative approach however, is to transform the discrete-time equation into the continuous domain and then use the Routh-Hurwitz criterion [14]. The latter approach is employed here. To transform Eq. 10 into continuous domain \( \text{Tustin mapping} \) (Eq. 2) is used and the equation so obtained is given by

\[
T^3\alpha_c s^3 + 2T^2\beta_c s^2 + 4T\gamma cs + 8\delta_c = 0
\]

where \( \alpha_c = \alpha_d - \beta_d + \gamma_d - \delta_d; \beta_c = 3\alpha_d - \beta_d - \gamma_d + 3\delta_d; \gamma_c = 3\alpha_d + \beta_d - \gamma_d - 3\delta_d; \) and \( \delta_c = \alpha_d + \beta_d + \gamma_d + \delta_d. \)

Based on Eq. 11 the most stringent condition for stability according to the Routh-Hurwitz criterion is

\[
\beta_c\gamma_c - \alpha_c\delta_c > 0
\]

which can be simplified to

\[
\alpha^2 - \delta^2 - \alpha\gamma + \beta\delta > 0
\]
Declare system/loop parameters and set up a counter

Discretize plant using ZOH method

Set $B = B_{\text{Initial}}$

Increment counter by 1

Set $K = K_{\text{Initial}}$

Close position feedback loop

Resample system using Tustin method

Close velocity feedback loop and obtain system characteristic equation

Is stability criterion satisfied?

Yes

Save $K$

Break $K$ loop

No

Decrement $K$ by $K_{\text{Inc}}$

Increment $B$ by $B_{\text{Inc}}$

Yes

No

Is $B = B_{\text{Final}}$

End

Figure 4. Flowchart depicting the steps in the MATLAB program.

which after further simplification leads to

$$T^2g_2 \left[ 4N(1 - e^n)(b^2 - IK) + 4bTKe^n \right] - 16N^2B^2b^2T(1 - e^n)\hat{e} + TBI \left[ h_2h_3(2g_2 - g_0) - 2g_2h_4 + g_2h_1h_3 - g_1h_1h_4 \right] > 0 \quad (14)$$

This is the stability criterion for the dual-rate haptics controller with variable position sampling architecture.

For $N = 1$, i.e., for uniform-rate sampling Eq. 14 reduces to

$$Tb^2(b^2 - IK)(1 - e^n) + b^2KT^2(b + B)e^n + B^2b^2T(1 - e^n)e^n + BFI \left( IK - Bb - b^2 \right)(1 - e^n)^2 + BbT \left( b^2 - IK \right)(1 - e^n)(1 + e^n) > 0 \quad (15)$$

Equation 15 is the same stability condition as is reported by Gil et al in [14] for uniform-rate sampling.

The theoretical analysis presented in this section can be automated using MATLAB and
the stability bounds can be obtained with much ease. The automated procedure is particularly
advantageous for systems which involve more than two sampling operations. We obtained
similar results with the automated procedure as were obtained using the theoretical analysis
(shown in the next section). The flowchart for the MATLAB program is shown in Fig. 4.
For the variable velocity sampling architecture, the procedure for carrying out the stability
analysis remains the same as outlined previously for the variable position sampling architecture,
however, the below mentioned coefficients are redefined as
\[ e^n = e^{-\frac{bT}{I}}; \hat{e} = e^n - 1 + \frac{bT}{I} e^n \]
and \[ \hat{e} = 1 - (1 + \frac{bT}{I}) e^n \].

The stability condition for this case is given as
\[
N^2T^2g_2 \left[ 4N(1 - e^n)(b^2 - IK) + 4bNTKe^n \right] - 16N^3B^2hT(1 - e^n)\hat{e} + NTBI \left[ h_2h_3(2g_2 - g_0) - 2g_2h_4 + g_2h_1h_3 - g_1h_1h_4 \right] > 0 \quad (16)
\]
which for \( N = 1 \) again leads to the stability condition as reported by Gil et al in [14].

4. Results and Discussions
The stability conditions derived in the previous section constrain the maximum value of \( K \) that
can be rendered as a function of \( B \) and \( N \). To obtain the stability boundaries, the values of \( B \) and \( K \)
were incremented with fixed steps and for each combination Eq. 14 (for variable position sampling) was analyzed. At a particular value of \( B \), the value of \( K \) was incremented until the stability condition was violated. The highest value of \( K \) was noted and the procedure
was repeated for different values of \( B \) and \( N \). To automate this iterative procedure, a MATLAB
program was written. The values of \( K \) and \( B \) which satisfied Eq. 14 were plotted with respect
to the sampling period ratio as shown in Fig. 5. Figure 6 depicts the Z-width of the dual-rate
haptics controller. The device parameters used were based on a custom made 1-DOF haptic
device developed in [10] and are listed in Table 1. A similar procedure was adopted for the

| Table 1. Device Parameters. |
|-----------------------------|
| Inertia \( (kgm^2) \) | Viscous damping coefficient \( (Nms/rad) \) | Coulomb damping coefficient \( (Nms/rad) \) |
| 0.000929 | 0.0264 | 0.0015 |

variable velocity sampling architecture. The results obtained are shown in Figs. 7 and 8.

The results obtained using the two architectures signify the fact that the dual-rate controller
has an improved Z-width compared to the uniform-rate controller. The range of stable virtual
stiffness increases as the sampling rate increases in both the control architectures. However, an
interesting difference can be observed: for the variable position sampling architecture, the range
Figure 5. Stable $K$ and $B$ values as a function of $N$ - Variable position sampling.

Figure 6. Z-width Plot - Variable position sampling.

Figure 7. Stable $K$ and $B$ values as a function of $N$ - Variable velocity sampling.

Figure 8. Z-width Plot - Variable velocity sampling.

of stable virtual damping was different for different values of $N$ and it increased as the value of $N$ increased; however, for the variable velocity sampling architecture the range of stable virtual damping remained constant and was independent of $N$.

These results were somewhat counter-intuitive, as one might expect the stable virtual damping range to remain constant for the variable position sampling architecture rather than for the variable velocity sampling architecture. This would be owing to the fact that the sampling period of the velocity loop remained constant in the former case, and not in the latter. Upon investigation, it was observed that this discrepancy in the results was due to the sequence of transformations that were applied during the theoretical analysis. A different choice of transformation methods in the theoretical analysis would yield a different shape and size of the stability region. This outcome brings out a scope into further analyses of such systems using various other techniques present in the literature. The same was beyond the scope of this work and shall be attempted in our future work in this area.

It is noteworthy to mention here that the stability regions shown in Fig. 6 and Fig. 8 would be enhanced if the influence of human operator was considered in the analysis. Further, non-linearities like actuator saturation, Coulomb friction, and sensor quantization etc. present in
the system, however, would reduce the actual stability bounds in practice.

5. Conclusions
In this paper, we presented theoretical stability analysis of a dual-rate haptics controller, proposed elsewhere, using linear control tools. In particular, the analyses were carried out using two control architectures, namely, *variable position sampling* and *variable velocity sampling* architectures, respectively. Stability criteria were derived for both the architectures that constrain the virtual environment parameters as a function of sampling period ratio. In particular, the analyses revealed that the stability region obtained significantly depend on the sequence of transformation methods used in the analyses.

It is important to mention that the methodology employed could be extended to study the combined effect of dual-rate sampling and other dynamic effects like time-delay, velocity estimation methods, velocity filtering, different virtual wall implementations etc. on the stability, provided that the system remains linear.

References

[1] Colgate J E and Brown J M 1994 *Proc. Int. Conf. on Robotics and Automation* IEEE pp 3205-10
[2] Abbott J J and Okamura A M 2005 *IEEE Transactions on Robotics* 21(5) pp 952-64
[3] Minsky M, Ming O Y, Steele O, Brooks Jr F P and Behensky M 1990 *ACM SIGGRAPH Computer Graphics* 24(2) pp 235-41
[4] Gil J J, Sanchez E, Hulin T, Preusche C and Hirzinger G 2009 *Journal of Computing and Information Science in Engineering* 9(1) p 011005
[5] Díaz I and Gil J J 2010 *IEEE Transactions on Robotics* 26(1) pp 160-5
[6] Mehling J S, Colgate J E and Peshkin M A 2005 *1st Joint Eurohaptics Conf. and Symp. on Haptic Interfaces for Virtual Environment and Teleoperator Systems* pp 257-62
[7] Weir D W, Colgate J E and Peshkin M A 2008 *Symp. Haptic interfaces for virtual environment and teleoperator systems* pp 169-75
[8] Srikanth M B, Vasudevan H and Muniyandi M 2008 *Int. Conf. on Human Haptic Sensing and Touch Enabled Computer Applications* Springer pp 53-62
[9] Lee K and Lee D Y 2004 *Int. Conf. on systems, man and cybernetics* IEEE 3 pp 2542-47
[10] Koul M H, Manivannan M and Saha S K 2013 *Proc. of Conf. on Advances In Robotics* ACM pp 1-6
[11] Dai X, Zhang Y and Cao Y 2008 *Int. Conf. on Intelligent Robotics and Applications - Springer* pp 27-35
[12] Mahvash M and Hayward V 2004 *Computer Graphics and Applications* IEEE 24(2) pp 48-55
[13] Love L J and Book W J 1995 (Georgia: Georgia Institute of Technology)
[14] Gil J J, Avello A, Rubio A and Florez J 2004 *IEEE Transactions on control systems technology* 12(4) pp 583-8
[15] Hulin T, Preusche C and Hirzinger G 2008 *Int. Conf. on Intelligent Robots and Systems* IEEE/RSJ pp 3483-88
[16] Mashayekhi A, Behbahani S, Ficuciello F and Siciliano B 2018 *IEEE/ASME Transactions on Mechatronics* 23(2) pp 596-603
[17] Ogata K 1995 *Discrete-time Control Systems* (Englewood Cliffs, NJ: Prentice Hall)
[18] Kuo B 1992 *Digital Control Systems - Oxford Series in Electrical and Computer Engineering* (United Kingdom: Oxford University Press)