# Real regulator on $K_1$ of elliptic surfaces

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# 1 Introduction

Let $X$ be a projective nonsingular variety over the complex number field $\mathbb{C}$. Let $H^i_{\text{dr}}(X, \mathbb{Z}(j))$ denotes the motivic cohomology group. It is known that $H^i_{\text{dr}}(X, \mathbb{Q}(j))$ is isomorphic to
Quillen’s $K$-group $K_{2j−i}(X)^{(j)}$. By the theory of higher Chern classes, we have the Beilinson regulator map (higher Chern class map)
\[ \text{reg}_{i,j} : H^j(M(X, Z(j)) \rightarrow H^i(D(X, Z(j))) \]
to the Deligne-Beilinson cohomology group ([S]). The purpose of this paper is to give a certain method for computations of the regulator map for $(i,j) = (3,2)$ (namely $K_1$) and $X$ an elliptic surface.

The cup-product pairing gives rise to a map $C^\times \otimes \text{Pic}(X) \cong C^\times \otimes H^2(H(X, Z(1)) \rightarrow H^3(M(X, Z(2)))$. Its image is called the decomposable part, and the cokernel is called the indecomposable part. The decomposable part does not affect serious difficulty, while the indecomposable part plays the central role in the study of $H^3(M(X, Z(2)))$. According to [GL], we call an element $\xi \in H^3(M(X, Z(2)))$ regulator indecomposable if $\text{reg}_{3,2}(\xi)$ does not lie in the image of $C^\times \otimes \text{NS}(X)$. Obviously regulator indecomposable elements are indecomposable. The converse is also true if the Beilinson-Hodge conjecture for $K_2$ is true. Lewis and Gordon constructed regulator indecomposable elements in case $X$ is a product of ‘general’ elliptic curves ([GL] Theorem 1). There are a lot of other related works, though I don’t catch up all of them. On the other hand, in case that $X$ is defined over a number field, the question is more difficult, and as far as I know there are only a few of such examples (e.g. [R] §12).

The real regulator map $\text{reg}_{3,2}$ is usually written in terms of differential $(1,1)$-forms. Then one of the technical difficulties appears from the fact that it is not easy to describe analytic differential forms explicitly. The key idea in this paper is to use certain “algebraic” 2-forms instead of analytic forms. This makes it easier to describe and compute the real regulator.

This paper is organized as follows. [2] is a quick review of $H^3(M(X, Q(2)))$ and Beilinson regulator. In [3] we provide notations and some elementary results on de Rham cohomology and the Hodge filtration. Especially we introduce “good algebraic 2-forms” which plays a key role in our computations ([3,4]). In [4] we give a method of computations of real regulator on $K_1$ of elliptic surfaces. In [5] we give an example. In particular we construct regulator indecomposable elements for an elliptic surface defined over $Q$ with arbitrary large $p_g$ (Cor.[5,3]). [6] is an appendix providing proofs of some explicit formulas on Gauss-Manin connection.

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2 Real Regulator map on $H^3(M(X, Q(2)))$

For a regular and integral scheme $X$, let $Z_i(X) = Z^{\dim X−i}(X)$ be the free abelian group of irreducible subvarieties of Krull dimension $i$. For an integral scheme $X$, we denote by $\eta_X$ the field of rational functions on $X$. For schemes $X$ and $T$ over a base scheme $S$, we set $X(T) = \text{Mor}_S(T, X)$. and say $x \in X(T)$ a $T$-valued point of $X$. If $T = \text{Spec}R$, then we also write $X(R) = X(\text{Spec}R)$.
2.1 $H_3^M(X, \mathbb{Q}(2))$ and (in)decomposable parts

Let $X$ be a smooth variety over a field $K$. Let $D \subset X$ be an irreducible divisor, and $\tilde{D} \to D$ the normalization. Let $j : \tilde{D} \to D \hookrightarrow X$ be the composition. Then we define $\text{Div}_D(f) := j_* \text{Div}_{\tilde{D}}(f) \in \mathbb{Z}^2(X)$ the push-forward of the Weil divisor on $\tilde{D}$ by $j$. Let

$$
\partial_1 : \bigoplus_{\text{codim } D = 1} \eta^\times_D \longrightarrow \mathbb{Z}^2(X), \quad [f, D] \longmapsto \text{Div}_D(f)
$$

be a homomorphism where we write $[f, D] := \left(\cdots, 1, f, 1, \cdots\right) \in \bigoplus_{\text{codim } D = 1} \eta^\times_D$ (if $f$ is placed in the $D$-component).

Let

$$
\partial_2 : K^M_2(\eta_X) \longrightarrow \bigoplus_{\text{codim } D = 1} \eta^\times_D
$$

be the tame symbol. Then it is well-known that there is the canonical isomorphism

$$
H_3^M(X, \mathbb{Q}(2)) \cong \left(\frac{\text{Ker}(\bigoplus \eta^\times_D \xrightarrow{\partial_1} \mathbb{Z}^2(X))}{\text{Im}(K^M_2(\eta_X) \xrightarrow{\partial_2} \bigoplus \eta^\times_D)}\right) \otimes \mathbb{Q}. \quad (2.1)
$$

In this paper we always identify the motivic cohomology group $H_3^M(X, \mathbb{Q}(2))$ with the group in the right hand side of (2.1).

Let $L/K$ be a finite extension. Write $X_L := X \times_K L$. Then there is the obvious map

$$
L^\times \otimes \mathbb{Z}^1(X_L) \longrightarrow H_3^M(X_L, \mathbb{Q}(2)), \quad \lambda \otimes D \longmapsto [\lambda, D].
$$

Let $N_{L/K} : H_3^M(X_L, \mathbb{Q}(2)) \to H_3^M(X, \mathbb{Q}(2))$ be the norm map on motivic cohomology. Then we put

$$
H_3^M(X, \mathbb{Q}(2))_{\text{dec}} := \sum_{[L : K] < \infty} N_{L/K}(\text{Im}(L^\times \otimes \mathbb{Z}^1(X_L) \to H_3^M(X_L, \mathbb{Q}(2))))
$$

and call it the decomposable part. We put

$$
H_3^M(X, \mathbb{Q}(2))_{\text{ind}} := H_3^M(X, \mathbb{Q}(2))/H_3^M(X, \mathbb{Q}(2))_{\text{dec}}
$$

and call it the indecomposable part. The indecomposable part plays the central role in the study of $H_3^M(X, \mathbb{Q}(2))$.  

3
2.2 Beilinson regulator on indecomposable parts

For a smooth projective variety $X$ over $\mathbb{C}$, we denote by $H^\bullet_B(X, \mathbb{Q}) = H^\bullet_B(X(\mathbb{C}), \mathbb{Q})$ (resp. $H^\bullet_*(X, \mathbb{Q})$) the Betti cohomology (resp. Betti homology). $H^\bullet_{dR}(X) = H^\bullet_{dR}(X/\mathbb{C})$ denotes the de Rham cohomology.

By the theory of universal Chern class, there is the Beilinson regulator map
\[
\text{reg} = \text{reg}_Q : H^3_B(X, \mathbb{Q}(2)) \to H^3_B(X, \mathbb{Q}(2)) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2)))
\]
\[
\cong \frac{H^2_B(X, \mathbb{C})}{F^2 + H^2_B(X, \mathbb{Q}(2))}
\]
(2.2)

to the Deligne-Beilinson cohomology group, which is isomorphic to the Yoneda extension group of mixed Hodge structures where $H^2(X, \mathbb{Q}(2)) = (H^2_B(X, \mathbb{Q}(2)), F^\bullet H^2_{dR}(X))$ denotes the Hodge structure (of weight $-2$). Put
\[
H^2_B(X, \mathbb{Q}(1)) = H^2_B(X, \mathbb{Q}(1))/\text{NS}(X) \otimes \mathbb{Q},
H^2_{dR}(X, \mathbb{Q}(1)) = H^2_{dR}(X)/\text{NS}(X) \otimes \mathbb{C},
\]
\[
H^2(X, \mathbb{Q}(1)) = (H^2_B(X, \mathbb{Q}(1)), F^\bullet H^2_{dR}(X)) = \text{a Hodge structure of weight 0}.
\]

Then (2.2) yields a commutative diagram (2.4)

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
H^3_B(X, \mathbb{Q}(2))_{\text{dec}} & \longrightarrow & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, \text{NS}(X) \otimes \mathbb{Q}(1)) \\
\downarrow & & \downarrow \\
H^3_B(X, \mathbb{Q}(2)) & \overset{\text{reg}}{\longrightarrow} & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))) \\
\downarrow & & \downarrow \\
H^3_B(X, \mathbb{Q}(2))_{\text{ind}} & \overset{\text{reg}}{\longrightarrow} & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2)) \otimes \mathbb{Q}(1)) \\
0 & \longrightarrow & 0
\end{array}
\]

The top arrow is simply written by “log”, namely the composition
\[
\mathbb{C}^\times \otimes \text{Pic}(X) \to H^3_B(X, \mathbb{Q}(2))_{\text{dec}} \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, \text{NS}(X) \otimes \mathbb{Q}(1)) \cong \mathbb{C}/\mathbb{Q}(1) \otimes \text{NS}(X)
\]
is given by $\lambda \otimes Z \mapsto \log(\lambda) \otimes Z$. The bottom arrow $\text{reg}$ plays an important role. Let us describe it in terms of extension of mixed Hodge structures. Let $n = \dim X$. Let $\xi = \sum [f_i, D_i] \in \bigoplus n_{D_i}^\perp$, such that $\partial_1(\xi) = 0$. Let $\text{reg}'$ be the composition
\[
H^3_B(X, \mathbb{Q}(2)) \overset{\text{reg}}{\longrightarrow} \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2)))
\to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))/\langle D_i \rangle)
\cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H_{2n-2}(X, \mathbb{Q}(2-n))/H_{2n-2}(D, \mathbb{Q}(2-n)))
\]
where \( \langle D_i \rangle \) denotes the subgroup generated by the cycle classes of \( D_i \), and the last isomorphism is the Poincare duality. Let \( j : \tilde{D}_i \to D_i \) be the normalization. Let \( \tilde{Z}_i \subset \tilde{D}_i \) be the support of \( \text{Div}_{\tilde{D}_i}(f_i) \). Put
\[
\tilde{Z} := \coprod_i \tilde{Z}_i \subset \tilde{D} := \coprod_i \tilde{D}_i, \quad Z := \bigcup_i j(\tilde{Z}_i) \subset D := \bigcup_i D_i.
\]
Consider a commutative diagram
\[
\begin{array}{ccc}
H^1(\tilde{D} - \tilde{Z}, \mathbb{Z}(1)) & \xrightarrow{a} & H_{2n-3}(\tilde{D}, \mathbb{Z}(2 - n)) \\
0 \xrightarrow{0} H_{2n-3}(\tilde{D}, \mathbb{Z}(2 - n)) & \xrightarrow{j_*} & H_{2n-4}(\tilde{Z}, \mathbb{Z}(2 - n)) \\
& \xrightarrow{\delta_i} & H_{2n-4}(\tilde{Z}, \mathbb{Z}(2 - n)) \\
0 \xrightarrow{0} H_{2n-3}(D, \mathbb{Z}(2 - n)) & \xrightarrow{j_*} & H_{2n-3}(D, \mathbb{Z}(2 - n)) \\
& \xrightarrow{\delta_2} & H_{2n-4}(Z, \mathbb{Z}(2 - n))
\end{array}
\]
with exact rows. Let
\[
\nu := \left( \frac{df_i}{f_i} \right) \in H^1(\tilde{D} - \tilde{Z}, \mathbb{Z}(1)).
\]
Since \( \partial_i(\xi) = 0 \), one has \( j_*\delta_i a(\nu) = 0 \). Therefore \( \nu \) defines \( \nu_\xi \in H_{2n-3}(D, \mathbb{Z}(2 - n)) \) such that \( b(\nu_\xi) = j_* a(\nu) \). Note that \( \nu_\xi \) belongs to the Hodge (0,0)-part because so does \( \nu \). By the exact sequence
\[
\cdots \to H_{2n-2}(X, D; \mathbb{Q}(2 - n)) \xrightarrow{\partial} H_{2n-3}(D, \mathbb{Q}(2 - n)) \xrightarrow{\delta} H_{2n-3}(X, \mathbb{Q}(2 - n)) \xrightarrow{} \cdots
\]
we have an exact sequence
\[
0 \to H_{2n-2}(X, \mathbb{Q}(2 - n))/H_{2n-2}(D, \mathbb{Q}(2 - n)) \to H_{2n-2}(X, D; \mathbb{Q}(2 - n)) \xrightarrow{\partial} \text{Ker}(\delta) \to 0
\]
(2.5)
of mixed Hodge structures. Since the weight of \( H_{2n-3}(X, \mathbb{Q}(2 - n)) \) is \(-1\), the Hodge (0,0)-part of \( H_{2n-3}(D, \mathbb{Q}(2 - n)) \) is contained in the kernel of \( \delta \). In particular we have an exact sequence
\[
0 \to H_{2n-2}(X, \mathbb{Q}(2 - n))/H_{2n-2}(D, \mathbb{Q}(2 - n)) \to H_\xi(X, D) \to \mathbb{Q} \xrightarrow{} 0
\]
(2.6)
by taking the pull-back of \( [2.5] \) via \( \mathbb{Q} \to \text{Ker}(\delta) \), \( 1 \mapsto \nu_\xi \). Then the following is well-known to specialists, proven by using the Riemann-Roch theorem without denominators ([G], see also [AS] Thm. 11.2).

**Theorem 2.1** \( \text{reg}'(\xi) \) corresponds to (2.6) up to sign. In other words, letting
\[
\rho : \mathbb{Q} \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H_{2n-2}(X, \mathbb{Q}(2 - n))/H_{2n-2}(D, \mathbb{Q}(2 - n)))))
\]
be the connecting homomorphism arising from (2.6), one has \( \text{reg}'(\xi) = \pm \rho(1) \).
For the later use, we write down $\rho(1)$ explicitly. Write

$$M := H_{2n-2}(X, \mathbb{Q}(2-n))/H_{2n-2}(D, \mathbb{Q}(2-n)),$$

$$H_{dR}^{2n-2}(X)' := \text{Ker}[H_{dR}^{2n-2}(X) \to H_{dR}^{2n-2}(D)].$$

Then the natural isomorphism $M \otimes_{\mathbb{Q}} \mathbb{C} \cong \text{Hom}(H_{dR}^{2n-2}(X)', \mathbb{C})$ induces

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, M) \cong \text{Coker}[H_{2n-2}(X, \mathbb{Q}(2-n)) \xrightarrow{\Phi} \text{Hom}(F^{n-1}H_{dR}^{2n-2}(X)', \mathbb{C})] \quad (2.7)$$

where

$$\Phi(\Delta) = \left[ \omega \mapsto \int_{\Delta} \omega \right], \quad \omega \in F^{n-1}H_{dR}^{2n-2}(X').$$

Taking the dual of the map $\mathbb{Q} \to H_{2n-3}(D, \mathbb{Q}(2-n)), 1 \mapsto \nu_\ell$, one has $H_{dR}^{2n-3}(D) \to \mathbb{C}$ and this induces

$$0 \to \mathbb{C} \to H_{dR}^{2n-2}(X, D)' \to H_{dR}^{2n-2}(X)' \to 0,$$

which is isomorphic to the dual of (2.6). Let $\omega_{X,D} \in F^{n-1}H_{dR}^{2n-2}(X, D)'$ denotes the element corresponding to $\omega \in F^{n-1}H_{dR}^{2n-2}(X)'$ via the isomorphisms $F^{n-1}H_{dR}^{2n-2}(X, D)' \xrightarrow{\cong} F^{n-1}H_{dR}^{2n-2}(X)'$. Let $\Gamma \in H_{2n-2}(X, D; \mathbb{Q}(2-n))$ be an arbitrary element such that $\partial(\Gamma) = \nu_\ell$. Then we have

$$\rho(1) = \left[ \omega \mapsto \int_{\Gamma} \omega_{X,D} \right] \quad (2.8)$$

under the isomorphism (2.7).

The real regulator map is the composition of $\text{reg}_\mathbb{Q}$ and the canonical map

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))) \to \text{Ext}^1_{\text{R-MHS}}(\mathbb{R}, H^2(X, \mathbb{R}(2)))$$

to the extension group of real mixed Hodge structures, which we denote by $\text{reg}_\mathbb{R}$:

$$\text{reg}_\mathbb{R} : H^3_{dR}(X, \mathbb{Q}(2)) \to \text{Ext}^1_{\text{R-MHS}}(\mathbb{R}, H^2(X, \mathbb{R}(2))) \cong H_B^2(X, \mathbb{R}(1)) \cap H^{1,1}. \quad (2.9)$$

This also induces

$$\text{reg}_\mathbb{R} : H^3_{dR}(X, \mathbb{Q}(2))_{\text{ind}} \to \text{Ext}^1_{\text{R-MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1)) \cong (H_B^2(X)_{\text{ind}} \otimes \mathbb{R}) \cap H^{1,1} \quad (2.10)$$

on the indecomposable part.

### 2.3 Q-structure on determinant of $H^3_{dR}(X/\mathbb{R}, \mathbb{R}(2))$

Suppose that $X$ is a projective smooth variety over $\mathbb{Q}$. Write $X_\mathbb{C} := X \times_{\mathbb{Q}} \mathbb{C}$. The infinite Frobenius map $F_\infty$ is defined to be the anti-holomorphic map on $X(\mathbb{C}) = \text{Mor}_\mathbb{Q}(\text{Spec} \mathbb{C}, X)$ induced from the complex conjugation on $\text{Spec} \mathbb{C}$. For a subring $A \subset \mathbb{R}$, the infinite Frobenius map acts on the Deligne-Beilinson complex $A_X(j)_\mathbb{D}$ in a canonical way, so that we have the involution on $H^\bullet_{dR}(X_\mathbb{C}, A(j))$, which we denote by the same notation $F_\infty$. We define

$$H^\bullet_{dR}(X/\mathbb{R}, A(j)) := H^\bullet_{dR}(X_\mathbb{C}, A(j))^{F_\infty=1}$$
the fixed part by \( F_\infty \). We call it the real Deligne-Beilinson cohomology. Since the action of \( F_\infty \) is compatible via the Beilinson regulator map, we have

\[
\operatorname{reg}_R : H^3_{\dR}(X, \mathbb{Q}(2)) \longrightarrow H_{\dR} := \operatorname{Ext}^1_{\mathbb{R}\text{-MHS}}(\mathbb{R}, H^2(X, \mathbb{R}(2)))^{F_\infty = 1} \cong \frac{H^2_B(X, \mathbb{R}(1))^{F_\infty = 1}}{F^2 H^2_{\dR}(X/\mathbb{R})},
\]

(2.11)

and

\[
\overline{\operatorname{reg}}_R : H^3_{\dR}(X, \mathbb{Q}(2)) \longrightarrow H_{\dR, \text{ind}} := \operatorname{Ext}^1_{\mathbb{R}\text{-MHS}}(\mathbb{R}, H^2(X, \mathbb{R})_{\text{ind}} \otimes \mathbb{R}(1))^{F_\infty = 1} \cong \frac{[H^2_B(X, \mathbb{R})_{\text{ind}} \otimes \mathbb{R}]^{F_\infty = 1}}{F^2 H^2_{\dR}(X/\mathbb{R})}.
\]

(2.12)

There are the canonical \( \mathbb{Q} \)-structures \( e_\mathbb{Q} \) and \( e_{\text{ind, } \mathbb{Q}} \) on the determinant vector spaces \( \det H_{\dR} \) and \( \det H_{\dR, \text{ind}} \):

\[
\mathbb{R} \cdot e_\mathbb{Q} = \det H_{\dR}, \quad \mathbb{R} \cdot e_{\text{ind, } \mathbb{Q}} = \det H_{\dR, \text{ind}}.
\]

Here we recall the definition. The isomorphisms (2.12) and (2.14) induce

\[
\det H_{\dR} \cong \det[H^2_B(X, \mathbb{R}(1))^{F_\infty = 1}] \otimes [\det F^2 H^2_{\dR}(X/\mathbb{R})]^{-1},
\]

(2.13)

and

\[
\det H_{\dR, \text{ind}} \cong \det[(H^2_B(X, \mathbb{R})_{\text{ind}} \otimes \mathbb{R})^{F_\infty = 1}] \otimes [\det F^2 H^2_{\dR}(X/\mathbb{R})]^{-1}.
\]

(2.14)

The right hand sides of (2.15) and (2.16) have the \( \mathbb{Q} \)-structures induced from the \( \mathbb{Q} \)-structures

\[
H^2_B(X, \mathbb{R}(1))^{F_\infty = 1}, \quad H^2_B(X, \mathbb{R})_{\text{ind}}^{F_\infty = 1}, \quad F^2 H^2_{\dR}(X/\mathbb{Q}).
\]

The \( \mathbb{Q} \)-structures \( e_\mathbb{Q} \) and \( e_{\text{ind, } \mathbb{Q}} \) are defined to be the corresponding one:

\[
\mathbb{Q} \cdot e_\mathbb{Q} \cong \det[H^2_B(X, \mathbb{Q}(1))^{F_\infty = 1}] \otimes [\det F^2 H^2_{\dR}(X/\mathbb{Q})]^{-1},
\]

(2.15)

and

\[
\mathbb{Q} \cdot e_{\text{ind, } \mathbb{Q}} \cong \det[H^2_B(X, \mathbb{R})_{\text{ind}}^{F_\infty = 1}] \otimes [\det F^2 H^2_{\dR}(X/\mathbb{Q})]^{-1}.
\]

(2.16)

\section{2.4 \( e_{\mathbb{Q}}^{\text{false}} \) and \( e_{\text{ind, } \mathbb{Q}}^{\text{false}} \)}

We introduce other \( \mathbb{Q} \)-structures \( e_{\mathbb{Q}}^{\text{false}} \) and \( e_{\text{ind, } \mathbb{Q}}^{\text{false}} \) on \( \det H_{\dR} \) and \( \det H_{\dR, \text{ind}} \). For simplicity, we assume \( \dim X = 2 \). Put

\[
H_2(X, \mathbb{Q})_{\text{ind}} := H_2(X, \mathbb{Q})/(\text{NS}(X) \otimes \mathbb{Q}(1)) \cong H^2_B(X, \mathbb{R})_{\text{ind}} \otimes \mathbb{Q}(1),
\]

\[
H^2_{\dR}(X/\mathbb{Q})_{\text{ind}} := \operatorname{Coim}(H^2_{\dR}(X/\mathbb{Q}) \longrightarrow H^2_{\dR}(X/\mathbb{C})/(\text{NS}(X) \otimes \mathbb{C})).
\]

Note that \( H^2_{\dR}(X/\mathbb{Q})_{\text{ind}} \otimes \mathbb{C} \cong H^2_{\dR}(X/\mathbb{C})/(\text{NS}(X) \otimes \mathbb{C}) \). There are exact sequences

\[
0 \longrightarrow H_2(X, \mathbb{R})^{F_\infty = 1} \longrightarrow \operatorname{Hom}(F^1 H^2_{\dR}(X/\mathbb{Q}), \mathbb{R}) \longrightarrow H_{\dR} \longrightarrow 0
\]

(2.17)

\[
0 \longrightarrow H_2(X, \mathbb{R})_{\text{ind}}^{F_\infty = 1} \longrightarrow \operatorname{Hom}(F^1 H^2_{\dR}(X/\mathbb{Q})_{\text{ind}}, \mathbb{R}) \longrightarrow H_{\dR, \text{ind}} \longrightarrow 0
\]

(2.18)
under the canonical isomorphisms
\[ H^2_B(X_C, \mathbb{C}) \cong H^2_{\text{dR}}(X/\mathbb{C}), \quad H^2_B(X_C, \mathbb{Q}(2)) \cong H_2(X_C, \mathbb{Q}). \] (2.21)

Then the \( \mathbb{Q} \)-structures
\[ H_2(X_C, \mathbb{Q})^\varpi = H^2_{\text{dR}}(X/\mathbb{Q}), \quad H^2_{\text{dR}}(X/\mathbb{Q})_{\text{ind}}, \]
induce \( e^{\text{false}}_Q \) and \( e^{\text{false}}_{\text{ind}, Q} \):
\[ Q \cdot e^{\text{false}}_Q \cong [\det H_2(X_C, \mathbb{Q})^\varpi]^{-1} \otimes [\det F^1 H^2_{\text{dR}}(X/\mathbb{Q})]^{-1}, \] (2.22)
\[ Q \cdot e^{\text{false}}_{\text{ind}, Q} \cong [\det H_2(X_C, \mathbb{Q})_{\text{ind}}^\varpi]^{-1} \otimes [\det F^1 H^2_{\text{dR}}(X/\mathbb{Q})_{\text{ind}}]^{-1}. \] (2.23)

**Proposition 2.2** Put
\[ r := \dim H_2(X_C, \mathbb{Q})^\varpi = \dim H^2_B(X_C, \mathbb{Q}(1))^{\varpi}, \]
\[ s := \dim H_2(X_C, \mathbb{Q})_{\text{ind}}^\varpi = \dim H^2_B(X_C)_{\text{ind}}^{\varpi} = r - \dim \text{NS}(X_C)^{\varpi}. \]
Write
\[ H_B := H^2_B(X_C, \mathbb{Q}(1)), \quad H_{B, \text{ind}} := H^2_B(X_C)_{\text{ind}}, \]
\[ F^* H_{\text{dR}} := F^* H^2_{\text{dR}}(X/\mathbb{Q}), \quad F^* H_{\text{dR}, \text{ind}} := F^* H^2_{\text{dR}}(X/\mathbb{Q})_{\text{ind}} \]
simply. Then
\[ Q \cdot e^{\text{false}}_Q = Q \cdot e \otimes Q(-r) \otimes \det H_{\text{dR}} \otimes [\det H_B]^{-1}, \]
\[ Q \cdot e^{\text{false}}_{\text{ind}, Q} = Q \cdot e_{\text{ind}, Q} \otimes Q(-s) \otimes \det H_{\text{dR}, \text{ind}} \otimes [\det H_{B, \text{ind}}]^{-1}, \]
where we mean
\[ \det H_{\text{dR}} \otimes [\det H_B]^{-1} \subset \det H^2_{\text{dR}}(X/\mathbb{C}) \otimes [\det H^2_B(X_C, \mathbb{C})]^{-1} \cong \mathbb{C}, \text{ etc.} \]

**Proof.** By the Poincare duality,
\[ \det F^1 H_{\text{dR}} = [\det H_{\text{dR}}]^{-1} \otimes \det F^2 H_{\text{dR}}, \quad \det F^1 H_{\text{dR}, \text{ind}} = [\det H_{\text{dR}, \text{ind}}]^{-1} \otimes \det F^2 H_{\text{dR}}. \]
Moreover
\[ \det [H_2(X_C, \mathbb{Q})]^{\varpi} = \det [H_2^2(X_C, \mathbb{Q}(2))]^{\varpi} = \det [H_B^{F_{\text{dR}}} \otimes (1)] = \mathbb{Q}(r) \otimes \det H_B^{F_{\text{dR}}} \]
and
\[ \det [H_2(X_C, \mathbb{Q})_{\text{ind}}]^{\varpi} = \det [H_B^{F_{\text{dR}, \text{ind}}} \otimes (1)] = \mathbb{Q}(s) \otimes \det H_B^{F_{\text{dR}, \text{ind}}} \]
Therefore we have
\[ Q \cdot e^{\text{false}}_Q \otimes e^{-1} = Q(-r) \otimes [\det H_B^{F_{\text{dR}}}]^{-1} \otimes [\det H_{\text{dR}}]^{-1} \otimes [\det H_B]^{-1} = Q(-r) \otimes [\det H_B]^{-1} \otimes [\det H_{\text{dR}}] \]
by (2.17) and (2.22), and
\[ Q \cdot e^{\text{false}}_{\text{ind}, Q} \otimes e^{-1}_{\text{ind}, Q} = Q(-s) \otimes [\det H_B^{F_{\text{dR}, \text{ind}}}]^{-1} \otimes [\det H_{\text{dR}, \text{ind}}]^{-1} \otimes [\det H_{B, \text{ind}}]^{-1} = Q(-s) \otimes [\det H_{B, \text{ind}}]^{-1} \otimes [\det H_{\text{dR}, \text{ind}}] \]
by (2.18) and (2.23). This completes the proof. \( \square \)
Remark 2.3 The Poincare duality implies
\[ (\det H_B)^{\otimes 2} \cong H^4_B(X_\mathbb{C}, \mathbb{Q}(2))^{\otimes m} \cong H^4_{dR}(X/\mathbb{Q})^{\otimes m} \cong (\det H_{dR})^{\otimes 2}, \]
and
\[ (\det H_{B, \ind})^{\otimes 2} \cong H^4_B(X_\mathbb{C}, \mathbb{Q}(2))^{\otimes m'} \cong H^4_{dR}(X/\mathbb{Q})^{\otimes m'} \cong (\det H_{dR, \ind})^{\otimes 2}. \]
Therefore \((\det H_{dR} \otimes [\det H_B]^{-1})\) and \((\det H_{dR, \ind} \otimes [\det H_{B, \ind}]^{-1})\) are contained in \(\sqrt{\mathbb{Q}}\) (possibly rational numbers).

3 Elliptic surface and good algebraic 2-forms

3.1 Notations

Let \(K\) be a field of characteristic 0. Let \(f : X \to C\) be an elliptic surface with a section \(e : C \to X\). This means that \(X\) (resp. \(C\)) is a projective smooth surface (resp. curve) over \(K\), and the generic fiber of \(f\) is an elliptic curve. Hereafter we assume that the \(j\)-invariant of \(f\) is not constant, namely, \(f\) is not isotrivial.

Throughout \(\S\) and \(\S\) we use the following notations.

- \(D \subset X\) is the sum of the multiplicative fibers. Put \(T_m = f(D) \subset C\). Note \(T_m \neq \emptyset\) by the assumption.
- \(E \subset X\) is the sum of the additive fibers. Put \(T_a = f(E) \subset C\).
- \(S = C - (T_m + T_a)\) and \(U = f^{-1}(S) = X - (E + D)\).
- \(\overline{S} = C - T_a\) and \(\overline{U} = f^{-1}(\overline{S}) = X - E\).
- Let \(F \subset S\) be the support of the cokernel of the \(\mathcal{O}_S\)-linear map
  \[ \nabla : f_*\Omega^1_{U/S} \longrightarrow \Omega^1_{\overline{S}} \otimes f_*\mathcal{O}_U \] (3.1)
induced from the Gauss-Manin connection. (By Cor. 6.2, this is a set of finite closed points.) Hence \(\nabla\) is an isomorphism outside \(F\).
- Put \(S^o := S - F = C - (T_m + T_a + F)\) and \(U^o := f^{-1}(S^o)\).
- \(\overline{S}^o := S^o + T_m = C - (T_a + F)\) and \(\overline{U}^o := f^{-1}(\overline{S}^o)\).
- Write \(X_{\overline{K}} = X \times_K \overline{K}\). Let \(\text{NF}(X_{\overline{K}}) \subset \text{NS}(X_{\overline{K}})\) denotes the subgroup of the Neron-Severi group generated by \(e(C)\) and irreducible components of \(D_{\overline{K}} + E_{\overline{K}}\).
- \(\text{NF}_{dR}(X) = H^2_{dR}(X/K) \cap (\text{NF}(X_{\overline{K}}) \otimes_{\overline{Z}} K) \subset H^2_{dR}(X_{\overline{K}}/K)\).
Remark 3.1  The intersection pairing $NF(X_K) \otimes NF(X_K) \to \mathbb{Q}$ is non-degenerate. This is proven on a case-by-case analysis by using the classification of degenerations (see [S] IV, Thm. 8.2 for the classification).

Remark 3.2  $NF_{dR}(X) \otimes_K K = NF(X_K) \otimes_{\mathbb{Z}} K$ in $H^2_{dR}(X_K/K)$. This is proven by using [AEC] II Lemma 5.8.1.

Remark 3.3  By Cor. 6.2, $F$ is described in the following way. Around a neighborhood of $s \in S$, $f$ is written by a Weierstrass form $y^2 = 4x^3 - g_2x - g_3$ ($g_2, g_3 \in \mathcal{O}_{S,s}$, $\Delta = g_2^3 - 27g_3^2 \in \mathcal{O}^x_{S,s}$). Let $j = 1728g_2^3/(g_2^3 - 27g_3^2)$ be the $j$-invariant. Then $s \in F$ if and only if

$$
\frac{\partial}{\partial j} \in \mathcal{O}_{S,s} \otimes \Omega^1_S \cong \mathcal{O}_{S,s} (\Omega^1_S := \Omega^1_{S/K})
$$

is a free $\mathcal{O}_{S,s}$-basis.

Proposition 3.4  Let $Q \subset C$ be a non-empty open set, and $V := f^{-1}(Q)$. We put

$$
H^2_{dR}(V) := \text{Ker}[H^2_{dR}(V) \to \prod_{s \in Q} H^2_{dR}(f^{-1}(s)) \times H^2_{dR}(V \cap e(C))].
$$

When $V = X$, we also write $NF_{dR}(X) \perp = H^2_{dR}(X) = 0$ (the orthogonal complements of $NF_{dR}(X)$ in $H^2_{dR}(X)$ with respect to the cup-product pairing). Then the following hold.

1. If $V \neq X$, then $H^2_{dR}(V)_0 = \text{Im} [\Gamma(V, \Omega^2_V) \to H^2_{dR}(V)]$.

2. Let $Q_1 \supset Q_2$ and $V_i = f^{-1}(Q_i)$. Then there is an exact sequence

$$
0 \to H^2_{dR}(V_1)_0 \to H^2_{dR}(V_2)_0 \to \bigoplus_{s \in Q_1 - Q_2} H^1_{dR}(f^{-1}(s)).
$$

3. $NF_{dR}(X) \perp \to H^2_{dR}(\mathcal{U})_0$.

Proof. Note that $NF_{dR}(X) \perp \otimes_K K = (NF(X_K) \otimes_{\mathbb{Z}} K) \perp$ by Rem. 3.2. Therefore we may assume $K = K$ throughout the proof pf Prop. 3.4.

We consider a spectral sequence

$$
E^{pq}_1 = H^q(V, \Omega^p_V) \Rightarrow H^{p+q}_{dR}(V).
$$

Since $Q$ is affine by the assumption, $E^{pq}_0 = H^q(V, \Omega^p_V) = \Gamma(Q, R^q f_* \mathcal{O}^p_V) = 0$ unless $p \leq 2$ and $q \leq 1$, so that we have

$$
E^{20}_0 = E^{20}_\infty = \text{Im} \Gamma(V, \Omega^2_V), \quad E^{11}_0 = E^{11}_\infty, \quad E^{02}_0 = 0,
$$

$$
0 \to \text{Im} \Gamma(V, \Omega^2_V) \to H^2_{dR}(V) \to E^{11}_\infty \to 0.
$$
**Lemma 3.5** $E_2^{11} = E_\infty^{11}$ is generated by the image of the cycle classes of $e(C)$ and irreducible components of each fiber $f^{-1}(s)$ as $K$-module (note we assumed $K = \overline{K}$ throughout the proof).

**Proof.** Let $Q^o := Q \cap S^o$ and $j : V^o := f^{-1}(Q^o) \hookrightarrow V$ be the open immersion. Consider a commutative diagram

$$
\begin{array}{ccc}
\Gamma(V, j_*\Omega_{V^o}/\Omega_V) & \xrightarrow{\delta} & H^1(V, \Omega_V^1) \\
\downarrow d & & \downarrow d \\
H^1(V, \Omega_V^1) & \xrightarrow{j^*} & H^1(V^o, \Omega_{V^o}^1)
\end{array}
$$

(3.1) is an isomorphism on $V^o$. Since the characteristic of $K$ is zero, the kernel of it is one-dimensional over $K$. This means Ker $\delta$ is generated by the cycle class $[e(C)]$. Thus $x' := x - c[e(C)]$ for some $c \in K$ is contained in Ker $j^* = \text{Im} \delta$. However, as is well-known, the image of $\delta$ is generated by the cycle classes of the irreducible components of $V - V^o$. This shows that $x$ is a linear combination of the cycle classes of $e(C)$ and $D$. Since Ker$(d)$ is generated by the cycle classes of $e(C)$ and $D$ as $K$-module, so is $E_2^{11}$. □

Let $(e(C), f^{-1}(s))_{s \in Q} \subset H^2_{\text{dR}}(V)$ denotes the $K$-module generated by the cycle classes of $e(C)$ and irreducible components of $f^{-1}(s)$. Consider the composition of maps

$$
\langle e(C), f^{-1}(s) \rangle_{s \in Q} \longrightarrow H^2_{\text{dR}}(V/K) \longrightarrow \prod_{s \in Q} H^2_{\text{dR}}(f^{-1}(s)).
$$

(3.2)

This is given by intersection pairing. Then it is not hard to show that (3.2) is injective. Moreover since the composition

$$
\Gamma(V, \Omega_V^1) \longrightarrow H^2_{\text{dR}}(V/K) \longrightarrow \prod_{s \in Q} H^2_{\text{dR}}(f^{-1}(s))
$$

is obviously zero, the second arrow in (3.2) factors through $E_2^{11} = E_\infty^{11}$. Summing up this and Lem. 3.5 we have a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Im}\Gamma(V, \Omega_V^1) \longrightarrow H^2_{\text{dR}}(V/K) \longrightarrow E_\infty^{11} \longrightarrow 0 \\
& & \downarrow \cong \\
& & \langle e(C), f^{-1}(s) \rangle_K \longrightarrow \prod_{s \in Q} H^2_{\text{dR}}(f^{-1}(s))
\end{array}
$$

(3.3)

with an exact row. This shows (1).

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Next we show (2). We first prove it in case $V_1 \neq X$ and $Q_2 \subset S$. Consider a commutative diagram

$$
\begin{array}{c}
0 \\
H^2_{\text{dr}}(V_1)_0 \overset{a_1}{\longrightarrow} H^2_{\text{dr}}(V_2)_0 \\
\downarrow \quad \downarrow \\
(f^{-1}(s))_{s \in Q_1 - Q_2} \overset{b}{\longrightarrow} H^2_{\text{dr}}(V_1) \overset{a_2}{\longrightarrow} H^2_{\text{dr}}(V_2) \oplus_{s \in Q_1 - Q_2} H_{1,\text{dr}}(f^{-1}(s)) \\
\downarrow \quad \downarrow \\
\prod_{s \in Q_2} H^2_{\text{dr}}(f^{-1}(s)) \overset{a_3}{\longrightarrow} H^2_{\text{dr}}(E_s)
\end{array}
$$

where $E_s \subset V_2$ is a fixed smooth fiber and $a_3$ is a projection. As we have seen in the proof of (1), the composition $cb$ is injective. Moreover $\text{Im}(c) \cong E_2^{11}$ is generated by the image of the cycle classes of $e(C)$ and irreducible components of $f^{-1}(s)$ with $s \in Q_1$ (Lem. 3.5). Therefore codimension of $\text{Im}(cb)$ in $\text{Im}(c)$ is at most one. The kernel of $\text{Im}(c) \to H^2_{\text{dr}}(E_s)$ is of codimension 1 because the cycle class $[e(C)]$ goes to non-zero via $a_3$. Hence we have $\text{Im}(c) \cap \text{Ker}(a_3) = \text{Im}(cb)$. Now (2) follows from the snake lemma. In case $V_1 \neq X$ and $V_1 \supset V_2$ is arbitrary, we take $V_3 \subset V_2 \cap U$. Then a diagram chase of a commutative diagram

$$
\begin{array}{c}
0 \\
H^2_{\text{dr}}(V_1)_0 \longrightarrow H^2_{\text{dr}}(V_2)_0 \longrightarrow \bigoplus_{s \in Q_1 - Q_3} H_{1,\text{dr}}(f^{-1}(s)) \\
\downarrow \quad \downarrow \\
0 \\
H^2_{\text{dr}}(V_2)_0 \longrightarrow H^2_{\text{dr}}(V_3)_0 \longrightarrow \bigoplus_{s \in Q_2 - Q_3} H_{1,\text{dr}}(f^{-1}(s))
\end{array}
$$

yields the assertion. There remains the case $V_1 = X$. However it is easy to see that there is an exact sequence

$$0 \longrightarrow H^2_{\text{dr}}(X)_0 \longrightarrow H^2_{\text{dr}}(X - E_s)_0 \longrightarrow H_{1,\text{dr}}(E_s)$$

where $E_s = f^{-1}(s)$ is a smooth fiber. Then the rest of the argument is similar to the above.

Finally we show (3). Consider a commutative diagram

$$
\begin{array}{c}
0 \\
H^2_{\text{dr}}(U)_0 \longrightarrow H^2_{\text{dr}}(U) \quad H^2_{\text{dr}}(D) \\
\downarrow \quad \downarrow \quad \downarrow \\
NF^1_{\text{dr}}(X) \quad NF^2_{\text{dr}}(X) \quad NF^3_{\text{dr}}(X) \longrightarrow 0
\end{array}
$$

(3.4)

with exact rows. Since $X - U = E$ are additive fibers, $a$ is surjective. Therefore it is enough to show that $\text{Ker}(a) \to \text{Ker}(b)$ is bijective. $\text{Ker}(a)$ is the sub $K$-module generated by the irreducible components of $E$. This implies $NF^1_{\text{dr}}(X) \cap \text{Ker}(a) = 0$ and hence $\text{Ker}(a) \to \text{Ker}(b)$ is injective. On the other hand, since $\text{Ker}(b)$ is generated by the irreducible components of $E$, $\text{Ker}(a) \to \text{Ker}(b)$ is surjective. This completes the proof of (3). □
3.2 Hodge filtration

By taking the embedded resolution of singularities if necessary, we can assume that \( E_{\text{red}} \) is a NCD. We then consider the de Rham cohomology groups

\[
H^q_{\text{dR}}(U) = H^q_{\text{zar}}(U, \Omega^\bullet_U) \cong H^q_{\text{zar}}(X, \Omega^\bullet_X(\log D + E))
\]

with the Hodge filtration

\[
F^p H^q_{\text{dR}}(U) := \text{Im}[H^q(X, \Omega^{\leq p}_X(\log D + E)) \hookrightarrow H^q(X, \Omega^\bullet_X(\log D + E))].
\]

Let \( T := T_m + T_a \). Define a sheaf \( \Omega^1_{X/C}(\log D + E) \) by the exact sequence

\[
0 \longrightarrow f^* \Omega^1_C(\log T) \longrightarrow \Omega^1_X(\log D + E) \longrightarrow \Omega^1_{X/C}(\log D + E) \longrightarrow 0.
\]

This is a locally free sheaf of rank one. Put

\[
\mathcal{H}_e := R^1 f_* \Omega^\bullet_{X/C}(\log D + E), \quad \mathcal{H}_e^{1,0} := f_* \Omega^1_{X/C}(\log D + E), \quad \mathcal{H}_e^{0,1} := R^1 f_* \mathcal{O}_X.
\]

Then the Gauss-Manin connection

\[
\nabla : \mathcal{H}_e \longrightarrow \Omega^1_C(\log T) \otimes \mathcal{H}_e
\]

is defined to be the connecting homomorphism arising from an exact sequence

\[
0 \longrightarrow f^* \Omega^1_C(\log T) \otimes \Omega^\bullet_{X/C}(\log D + E) \longrightarrow \Omega^\bullet_X(\log D + E) \longrightarrow \Omega^\bullet_{X/C}(\log D + E) \longrightarrow 0 \quad (3.5)
\]

(see Appendix for a remark on sign.) Write

\[
H^2_{\text{dR}}(C, \mathcal{H}_e) := H^q_{\text{zar}}(C, \Omega^\bullet_C(\log T) \otimes \mathcal{H}_e).
\]

**Theorem 3.6 (cf. [SZ] §5)** Let us put \( H^2_{\text{dR}}(U)_0 := \text{Ker}[H^2_{\text{dR}}(U) \hookrightarrow H^2_{\text{dR}}(E_s)] \) where \( E_s = f^{-1}(s) \) is a smooth fiber contained in \( U \). Then there is the natural isomorphism

\[
H^1_{\text{dR}}(C, \mathcal{H}_e) \cong H^2_{\text{dR}}(U)_0. \quad (3.6)
\]

Moreover under the above isomorphism, the Hodge filtration corresponds in the following way.

\[
F^1 H^2_{\text{dR}}(U)_0 \cong H^1_{\text{zar}}(C, \mathcal{H}_e^{1,0} \rightarrow \Omega^1_C(\log T) \otimes \mathcal{H}_e^{1,0})) \quad (3.7)
\]
\[
F^2 H^2_{\text{dR}}(U)_0 \cong H^0_{\text{zar}}(C, \Omega^\bullet_C(\log T) \otimes \mathcal{H}_e^{1,0}) \quad (3.8)
\]
\[
\text{Gr}_F^0 H^2_{\text{dR}}(U)_0 \cong H^1_{\text{zar}}(C, \mathcal{H}_e^{0,1}) \quad (3.9)
\]
Proof. The exact sequence \((3.5)\) gives rise to a spectral sequence
\[
E_2^{pq} = H^p_{dR}(C, R^q f_* \Omega^\bullet_{X/C}(\log D)) \implies H^{p+q}_{dR}(U).
\]
This yields
\[
0 \longrightarrow H^1_{dR}(C, \mathcal{H}_e) \longrightarrow H^2_{dR}(U) \longrightarrow H^0_{dR}(C, R^2 f_* \Omega^\bullet_{X/C}(\log D + E)) \longrightarrow 0.
\]
Since the last term is one-dimensional, isomorphic to \(H^1_{dR}(E_s)\), we have \((3.6)\).

\((3.5)\) induces an exact sequence
\[
0 \longrightarrow f^* \Omega^1_C(\log T) \otimes \Omega^{p-2} \longrightarrow \Omega^1_C(\log T) \otimes R^1 f_* \Omega^\bullet_{X/C}(\log D + E) \longrightarrow \Omega^\bullet_{X/C}(\log D + E) \longrightarrow 0.
\]
and this yields
\[
H^1_{zar}(C, R^1 f_* \omega^{\geq p}_{X/C}) \rightarrow \Omega^1_C(\log T) \otimes \Omega^\bullet_{X/C}(\log D + E)
\]
where \(\omega^{\geq p}_{X/C} := \Omega^{\geq p}_{X/C}(\log D + E)\). Now \((3.7)\), \((3.8)\) and \((3.9)\) easily follow from this. \(\square\)

A basis of the locally free sheaf \(\mathcal{H}_e\) is given in the following way. Let \(s \in C(K)\). We choose a minimal Weierstrass equation
\[
y^2 = 4x^3 - g_2x - g_3, \quad \Delta := g_3^2 - 27g_2^3
\]
of \(X\) around a (sufficiently small) neighborhood of a fiber \(f^{-1}(s)\). Let \(\omega \) and \(\omega^*\) be the following elements of \(\mathcal{O}_{C,s} \otimes \mathcal{H}_e\) (see \((6.1)\) and \((6.2)\) in Appendix for the notation):
\[
\omega := (0) \times \left( \frac{dx}{y}, \frac{dx}{y} \right), \quad \omega^* := \left( \frac{x^2}{y} \right) \times \left( \frac{y^2}{2y}, \frac{(2g_2x^2 + 3g_3x)dx}{2y^3} \right).
\]

Then
\[
\bullet \text{ If } f^{-1}(s) \text{ is smooth or multiplicative, then } \{\omega, \omega^*\} \text{ is a free basis of } \mathcal{O}_{C,s} \otimes \mathcal{H}_e.
\]
\[
\bullet \text{ If } f^{-1}(s) \text{ is additive, then } \{t\omega, \omega^*\} \text{ is a basis where } t \in \mathcal{O}_{C,s} \text{ is a uniformizer.}
\]

The following theorem is useful.

**Theorem 3.7 (Canonical bundle formula)** Let

\[
\begin{array}{c|cccccccc}
\epsilon_s & 0 & b & 2 & 3 & 4 & b+6 & 10 & 9 & 8 \\
\hline
f^{-1}(s) & \text{smooth} & I_b & II & III & IV & I_b & II* & III* & IV*
\end{array}
\]

and put
\[
\epsilon := \frac{1}{12} \sum_{s \in C(K)} \epsilon_s \in \mathbb{Z}.
\]

Then there is an invertible sheaf \(\mathcal{L}\) on \(C\) of degree \(\epsilon\) such that
\[
K_X \cong f^*(K_C \otimes \mathcal{L}), \quad R^1 f_* \mathcal{O}_X \cong \mathcal{L}^{-1}.
\]
Moreover let \(a\) be the number of additive fibers in the fibration \(f : X_K \to C_K\). Then one has
\[
\deg(\mathcal{H}_e^{1,0}) = \epsilon - a, \quad \deg(\mathcal{H}_e^{0,1}) = -\epsilon.
\]
3.3 Relative cohomology and Extra terms

For a smooth manifold $M$, we denote by $A^q(M)$ the space of smooth differential $q$-forms on $M$ with coefficients in $\mathbb{C}$.

Suppose $K = \mathbb{C}$. Let $D_0$ be a union of some multiplicative fibers. Let $\rho : \widetilde{D}_0 \to D_0$ be the normalization and $\Sigma \subset D_0$ the set of singular points. Let $s : \Sigma := \rho^{-1}(\Sigma) \hookrightarrow \tilde{D}_0$. There is the exact sequence

$$0 \to \mathcal{O}_{D_0} \xrightarrow{\rho^*} \mathcal{O}_{\tilde{D}_0} \xrightarrow{s^*} \mathcal{C}_{\Sigma}/\mathcal{C}_\Sigma \to 0$$

where $\mathcal{C}_\Sigma = \text{Maps}(\Sigma, \mathbb{C}) = \text{Hom}(\mathbb{Z}\Sigma, \mathbb{C})$ etc. and $\rho^*$ and $s^*$ are the pull-back. We define $\mathcal{A}^\bullet(D_0)$ to be the mapping fiber of $s^* : \mathcal{A}^\bullet(\tilde{D}_0) \to \mathcal{C}_{\Sigma}/\mathcal{C}_\Sigma$:

$$\mathcal{A}^0(\tilde{D}_0) \xrightarrow{s^* \oplus d} \mathcal{C}_{\Sigma}/\mathcal{C}_\Sigma \oplus \mathcal{A}^1(\tilde{D}_0) \xrightarrow{0 \oplus d} \mathcal{A}^2(\tilde{D}_0)$$

where the first term is placed in degree 0. Then

$$H^q_{\text{dR}}(D_0) = H^q(\mathcal{A}^\bullet(D_0))$$

is the de Rham cohomology of $D_0$, which fits into the exact sequence

$$\cdots \to H^0_{\text{dR}}(\tilde{D}_0) \to \mathcal{C}_{\Sigma}/\mathcal{C}_\Sigma \to H^1_{\text{dR}}(D_0) \to H^1_{\text{dR}}(\tilde{D}_0) \to \cdots.$$ 

There is the natural pairing

$$H_1(D_0, \mathbb{Z}) \otimes H^1_{\text{dR}}(D_0) \to \mathbb{C}, \quad \gamma \otimes z \mapsto \int_\gamma z := \int_\gamma \eta - c(\partial(\rho^{-1}\gamma))$$

(3.13)

where $z = (c, \eta) \in \mathcal{C}_{\Sigma}/\mathcal{C}_\Sigma \oplus \mathcal{A}^1(\tilde{D}_0)$ with $d\eta = 0$ and $\partial$ denotes the boundary of homology cycles.

Let $V \subset X$ be a Zariski open set containing $D_0$. We define $\mathcal{A}^\bullet(V, D_0)$ to be the mapping fiber of $j^* : \mathcal{A}^\bullet(V) \to \mathcal{A}^\bullet(D_0)$ the pull-back of $j : D_0 \hookrightarrow V$:

$$\mathcal{A}^0(V) \xrightarrow{\mathcal{D}_0} \mathcal{A}^0(\tilde{D}_0) \oplus \mathcal{A}^1(V) \xrightarrow{\mathcal{D}_1} \mathcal{C}_{\Sigma}/\mathcal{C}_\Sigma \oplus \mathcal{A}^1(\tilde{D}_0) \oplus \mathcal{A}^2(V) \xrightarrow{\mathcal{D}_2} \cdots$$

where

$$\mathcal{D}_0 = j^* \oplus d, \quad \mathcal{D}_1 = \left(\begin{array}{c} -s^* \oplus d \\ j^* \\
\end{array}\right), \quad \mathcal{D}_2 = \left(\begin{array}{c} -(0 \oplus d) \\ j^* \end{array}\right), \ldots$$

Then

$$H^q_{\text{dR}}(V, D_0) = H^q(\mathcal{A}^\bullet(V, D_0))$$

is the de Rham cohomology which fits into the exact sequence

$$\cdots \to H^{q-1}_{\text{dR}}(D_0) \to H^q_{\text{dR}}(V, D_0) \to H^q_{\text{dR}}(V) \to H^q_{\text{dR}}(D_0) \to \cdots.$$ 

(3.14)
In particular, an element of $H^2_{dR}(V, D_0)$ is described by $z = (c, \eta, \omega) \in \mathbb{C}_\Sigma / \mathbb{C}_\Sigma \oplus \mathcal{A}^1(\tilde{D}_0) \oplus \mathcal{A}^2(V)$ with $j^* \omega = d\eta$ and $d\omega = 0$ which are subject to relations $(s^* f, df, 0) = 0$ and $(0, j^* \theta, d\theta) = 0$ for $f \in \mathcal{A}^0(\tilde{D}_0)$ and $\theta \in \mathcal{A}^1(V)$. The natural pairing

$$H_2(V, D_0; \mathbb{C}) \otimes H^2_{dR}(V, D_0) \rightarrow \mathbb{C}, \quad \Gamma \otimes z \mapsto \int_\Gamma z$$

is given by

$$\int_\Gamma z := \int_\Gamma \omega - \int_{\partial \Gamma} (c, \eta) = \int_\Gamma \omega - \int_{\partial \Gamma} \eta + c(\rho^{-1}(\partial \Gamma)). \quad (3.16)$$

There are canonical maps

$$\Gamma(V, \Omega^2_V) \rightarrow H^2_{dR}(V) = H^2(\mathcal{A}^*(V)), \quad \omega \mapsto \overline{\omega}. \quad (3.17)$$

$$\Gamma(V, \Omega^2_V) \rightarrow H^2_{dR}(V, D_0), \quad \omega \mapsto (0, 0, \omega). \quad (3.18)$$

Define $G\Gamma(V, \Omega^2_V) \subset \Gamma(V, \Omega^2_V)$ to be the inverse image of $F^1H^2_{dR}(V)$ via the natural map $\Gamma(V, \Omega^2_V) \rightarrow H^2_{dR}(V)$ where $F^*$ denotes the Hodge filtration. We define a map $\text{ex}_{D_0}$ by a commutative diagram

$$
\begin{array}{c}
0 & \rightarrow & G\Gamma(V, \Omega^2_V) & \rightarrow & \Gamma(V, \Omega^2) & \rightarrow & \text{Gr}_F^0H^2_{dR}(V) \\
& | & \downarrow & & | & | & | \\
0 & \rightarrow & H^1_{dR}(D_0) & \rightarrow & \text{Gr}_F^0H^2_{dR}(V, D_0) & \rightarrow & 0 \\
\end{array}
$$

with exact rows. Here $i : H^1_{dR}(D_0) \rightarrow H^2_{dR}(V, D_0)$ is the map appearing in (3.14) and $i^0$ denotes the induced map on the graded piece. We call $\text{ex}_{D_0}(\omega)$ the extra term of $\omega$ at $D_0$.

**Proposition 3.8** Let $\omega \in G\Gamma(V, \Omega^2_V)$. Then $\omega_{V, D_0} := (0, 0, \omega) - i\text{ex}_{D_0}(\omega) \in F^1H^2_{dR}(V, D_0)$ is the unique element corresponding to $\overline{\omega} \in H^2_{dR}(V)$ via the natural map $F^1H^2_{dR}(V, D_0) \rightarrow F^1H^2_{dR}(V)$. Moreover

$$\int_\Gamma \omega_{V, D_0} = \int_\Gamma \omega - \int_{\partial \Gamma} \text{ex}_{D_0}(\omega). \quad (3.20)$$

**Proof.** It follows from the construction that $\omega - i\text{ex}_{D_0}(\omega)$ belongs to $F^1H^2_{dR}(V, D_0)$. The uniqueness follows from the injectivity of the map $F^1H^2_{dR}(V, D_0) \rightarrow F^1H^2_{dR}(V)$. (3.20) follows from (3.16). \qed

The map “$\text{ex}_{D_0}$” can be defined in an algebraic way. Let us denote by $(\check{C}^*(\mathcal{F}), \delta)$ the Cech complex of a sheaf $\mathcal{F}$. Then $H^1_{dR}(D_0)$ is isomorphic to the cohomology of the complex

$$(\check{C}^0(\mathcal{D}_0) \xrightarrow{\delta} \check{C}^1(\mathcal{D}_0) \times \check{C}^0(\mathbb{C}_\Sigma / \mathbb{C}_\Sigma \oplus \Omega^1_{D_0}) \xrightarrow{\delta} \check{C}^2(\mathcal{D}_0) \times \check{C}^1(\mathbb{C}_\Sigma / \mathbb{C}_\Sigma \oplus \Omega^1_{D_0}))$$

at the middle term where

$$\mathcal{D}_0 = \delta \times (s^* \oplus d), \quad \mathcal{D}_1 := \begin{pmatrix} \delta & -(s^* \oplus d) \\ \delta & \end{pmatrix}. \quad 16$$
Moreover $H^2_{\text{dR}}(V)$ and $H^2_{\text{dR}}(V, D_0)$ are isomorphic to the cohomology of the following complexes

$$
\check{C}^1(\mathcal{O}_V) \times \check{C}^0(\Omega^1_V) \xrightarrow{\partial_2} \check{C}^2(\mathcal{O}_V) \times \check{C}^1(\Omega^1_V) \times \check{C}^0(\Omega^2_V) \xrightarrow{\partial_3} \check{C}^3(\mathcal{O}_V) \times \check{C}^2(\Omega^1_V) \times \check{C}^1(\Omega^2_V)
$$

$$
\check{C}^1(\mathcal{O}_V) \times \check{C}^0(\mathcal{O}_{\tilde{D}_0} \oplus \Omega^1_V) \xrightarrow{\partial_2} \check{C}^2(\mathcal{O}_V) \times \check{C}^1(\mathcal{O}_{\tilde{D}_0} \oplus \Omega^1_V) \times \check{C}^0(\mathbb{C}/\mathbb{C} \oplus \Omega^1_{\tilde{D}_0} \oplus \Omega^2_{\tilde{D}_0})
$$

at the middle terms respectively,

\[
\partial_2 = \begin{pmatrix} \delta & -d & \delta \\ \delta & d & -\delta \\ \delta & \delta & -d \end{pmatrix}, \quad \partial_3 = \begin{pmatrix} \delta & d & -\delta \\ \delta & -d & \delta \\ \delta & \delta & -d \end{pmatrix}, \quad \partial_4 = \begin{pmatrix} \delta & -(j^* + d) & -T \\ \delta & -d & \delta \\ \delta & \delta & -d \end{pmatrix}, \quad \partial_5 = \begin{pmatrix} \delta & j^* + d & \delta \\ \delta & -d & \delta \\ \delta & \delta & -d \end{pmatrix}.
\]

For a $\omega \in \Gamma(V, \Omega^2_V)$, we simply write $\omega = (0) \times (0) \times (\omega) \in \check{C}^2(\mathcal{O}_V) \times \check{C}^1(\Omega^1_V) \times \check{C}^0(\Omega^2_V)$. There is a Cech cocycle

$$
\xi = (0) \times (\eta_{ij}) \times (\omega_i) \in \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega^1_X) \times \check{C}^0(\Omega^2_X)
$$

such that $\xi \equiv \omega$ in $H^2_{\text{dR}}(V)$ and $\xi$ belongs to $F^1 H^2_{\text{dR}}(X)$ and the kernel of $H^2_{\text{dR}}(X) \rightarrow H^2_{\text{dR}}(D_0)$ (Prop [3.4](1)). Then there is a unique Cech cocycle

$$
\xi_{X, D_0} = (0) \times (0, \eta_{ij}) \times (0, \tilde{\eta}_i, \omega_i) \in \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\mathcal{O}_{\tilde{D}_0} \oplus \Omega^1_X) \times \check{C}^0(\mathbb{C}/\mathbb{C} \oplus \Omega^1_{\tilde{D}_0} \oplus \Omega^2_X)
$$

such that $\eta_{ij}|_{\tilde{D}_0} = \tilde{\eta}_j - \tilde{\eta}_i$. This belongs to $F^1 H^2_{\text{dR}}(X, D_0)$ by definition. Since $\xi \equiv \omega$ in $H^2_{\text{dR}}(V)$, there is a Cech cycle $z = (f_{ij}) \times (\nu_i) \in \check{C}^1(\mathcal{O}_V) \times \check{C}^0(\Omega^1_V)$ such that

$$
\omega - \xi = (0) \times (0) \times (0, \omega) \times (\omega - \omega_i) = (f_{jk} - f_{ik} + f_{ij}) \times (-df_{ij} + \nu_j - \nu_i) \times (d\nu_i)
$$

in $\check{C}^2(\mathcal{O}_V) \times \check{C}^1(\Omega^1_V) \times \check{C}^0(\Omega^2_V)$. Hence

$$
(0) \times (0, 0) \times (0, 0, \omega) - \xi_{X, D_0} = (0) \times (0, -df_{ij} + \nu_j - \nu_i) \times (0, -\tilde{\eta}_i, d\nu_i)
$$

\[
\equiv (0) \times (f_{ij}|_{\tilde{D}_0}, 0) \times (0, -\tilde{\eta}_i - \nu_i|_{\tilde{D}_0}, 0)
\]

\[
\in \check{C}^2(\mathcal{O}_V) \times \check{C}^1(\mathcal{O}_{\tilde{D}_0} \oplus \Omega^1_V) \times \check{C}^0(\mathbb{C}/\mathbb{C} \oplus \Omega^1_{\tilde{D}_0} \oplus \Omega^2_{\tilde{D}_0})
\]

modulo the image of $\check{C}^1(\mathcal{O}_V) \times \check{C}^0(\mathcal{O}_{\tilde{D}_0} \oplus \Omega^1_V)$. This shows $\text{ex}_{D_0}(\omega) = (f_{ij}|_{\tilde{D}_0}) \times (0, -\tilde{\eta}_i - \nu_i|_{\tilde{D}_0})$ in $H^1_{\text{dR}}(D_0)$. There is $(h_i) \in \check{C}^0(\mathcal{O}_{\tilde{D}_0})$ such that $f_{ij}|_{\tilde{D}_0} = h_j - h_i$. Then

$$
\text{ex}_{D_0}(\omega) = (f_{ij}|_{\tilde{D}_0}) \times (0, -\tilde{\eta}_i - \nu_i|_{\tilde{D}_0}) \equiv (0) \times (s^* h_i, 0) \in H^1_{\text{dR}}(D_0).
$$
3.4 Good algebraic 2-forms

We introduce two subspaces

$$\Lambda^2(U) \subset \Lambda^1(U) \subset \Gamma(S^o, \Omega^1_{\text{So}} \otimes \mathcal{H}_e) = \Gamma(U^o, \Omega^2_{U^o}),$$

which we call the spaces of good algebraic 2-forms. Define

$$\Lambda^2(U) := \text{Im}[\Gamma(C, \Omega^1_C(\log T) \otimes \mathcal{H}_e^1.0) = \Gamma(X, \Omega^2_X(\log D + E)) \hookrightarrow \Gamma(U^o, \Omega^2_{U^o})].$$

We define $$\Lambda^1(U)$$ in the following way. Let us consider a diagram

$$\begin{array}{c}
\Omega^1_{\text{So}} \otimes \mathcal{H}_e^{1.0}|_{\text{So}} \\
\downarrow \\
\mathcal{H}_e^{1.0}|_{\text{So}} \downarrow \mathcal{H}_e^{0.1}|_{\text{So}} \\
\downarrow \\
0
\end{array}$$

It follows from the definition of $$S^o$$ and Cor. 6.2 that the bottom arrow is isomorphism. This yields an isomorphism

$$\Omega^1_{\text{So}} \otimes \mathcal{H}_e^{1.0}|_{\text{So}} \xrightarrow{\cong} \Omega^1_{\text{So}} \otimes \mathcal{H}_e|_{\text{So}} / \text{Im}(\mathcal{H}_e^{1.0}|_{\text{So}})$$

and

$$H^1_{\text{zar}}(C, \mathcal{H}_e^{1.0}) \rightarrow \Gamma(S^o, \Omega^1_{\text{So}} \otimes \mathcal{H}_e) \xrightarrow{\cong} \Gamma(S^o, \Omega^1_{\text{So}} \otimes \mathcal{H}_e/\text{Im}(\mathcal{H}_e^{1.0}|_{\text{So}})) \xrightarrow{\cong} \Gamma(S^o, \Omega^1_{\text{So}} \otimes \mathcal{H}_e^{1.0}) = \Gamma(U^o, \Omega^2_{U^o}).$$

Define $$\Lambda^1(U)$$ to be the image of the composition of the above maps:

$$\Lambda^1(U) := \text{Im}[H^1_{\text{zar}}(C, \mathcal{H}_e^{1.0}) \rightarrow \Omega^1_{\text{C}}(\log T) \otimes \mathcal{H}_e) \rightarrow \Gamma(U^o, \Omega^2_{U^o})].$$

Proposition 3.9

$$H^1_{\text{zar}}(C, \mathcal{H}_e^{1.0}) \rightarrow \Omega^1_{\text{C}}(\log T) \otimes \mathcal{H}_e \xrightarrow{\cong} \Lambda^1(U), \quad \Gamma(X, \Omega^2_X(\log D + E)) \xrightarrow{\cong} \Lambda^2(U).$$

Hence

$$\Lambda^1(U) \xrightarrow{\cong} F^1H^2_{\text{dR}}(U)_0, \quad \Lambda^2(U) \xrightarrow{\cong} F^2H^2_{\text{dR}}(U)_0$$

by Thm 3.6.
Proof. There is nothing to show other than the injectivity of $H^1_{\text{zar}}(C, \mathcal{H}_e^{1,0} \to \Omega^1_{\text{C}}(\log T) \otimes \mathcal{H}_e) \to \Lambda^1(U)$. However this follows from the fact that $F^1H^2_{\text{dR}}(U) \to F^1H^2(U^o)$ is injective. □

**Lemma 3.10** Along $D$, good algebraic 2-forms have at most log poles. Namely

$$\Lambda^1(U) \subset \Gamma(U^o, \Omega^2_{\text{log}}(\log D)). \quad (3.25)$$

**Proof.** We may replace $K$ with $\overline{K}$. Let us consider a diagram

$$\begin{array}{ccc}
0 & \rightarrow & \Omega^1_{\overline{S}}(\log T_m) \otimes \mathcal{H}_e^{1,0}|_{\overline{S}} \\
\downarrow & & \downarrow \\
\mathcal{H}_e^{1,0}|_{\overline{S}} \rightarrow & \Omega^1_{\overline{S}}(\log T_m) \otimes \mathcal{H}_e|_{\overline{S}} & \rightarrow \\
\downarrow & & \downarrow \\
\mathcal{H}_e^{1,0}|_{\overline{S}} \rightarrow & \Omega^1_{\overline{S}}(\log T_m) \otimes \mathcal{H}_e^{0,1}|_{\overline{S}} & \rightarrow \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array} \quad (3.26)$$

Around a multiplicative fiber $D_0 = f^{-1}(s_0)$, $X$ can be written by a Weierstrass form $y^2 = 4x^3 - g_2x - g_3$ with $\text{ord}_{s_0}(g_2^3 - 27g_3^2) > 0$ and $\text{ord}_{s_0}(g_2) = \text{ord}_{s_0}(g_3) = 0$ where $\text{ord}_{s_0}$ denotes the valuation order on $\mathcal{O}_{C,s_0}$ (cf. Tate’s algorithm). Thus Cor. 6.2 implies that the bottom arrow $\nabla$ in (3.26) is an isomorphism. Then we have

$$H^1_{\text{zar}}(C, \mathcal{H}_e^{1,0} \to \Omega^1_{\text{C}}(\log T) \otimes \mathcal{H}_e) \rightarrow H^1_{\text{zar}}(S^o, \mathcal{H}_e^{1,0} \to \Omega^1_{\overline{S}}(\log T_m) \otimes \mathcal{H}_e) \xrightarrow{\cong} \Gamma(S^o, \Omega^1_{\overline{S}}(\log T_m) \otimes \mathcal{H}_e/\text{Im}(\mathcal{H}_e^{1,0}|_{\overline{S}})) \xleftarrow{\cong} \Gamma(U^o, \Omega^2_{\overline{U}}(\log D))$$

and this shows (3.25). □

By Lem 3.10, one can have the residue map

$$\text{Res}_D : \Lambda^1(U) \rightarrow H^1_{\text{dR}}(D) \quad (3.27)$$

along $D$. We define

$$\Lambda^1(U) := \Lambda^1(U) \cap \text{Ker}(\text{Res}_D) \supset \Lambda^2(U) := \Lambda^2(U) \cap \text{Ker}(\text{Res}_D). \quad (3.28)$$

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Lemma 3.11 \( \Lambda^1(U) \subset \Gamma(U^\sigma, \Omega^2_{\overline{U^\sigma}}) \).

Proof. There is the weight filtration \( W_i \Omega^2_{\overline{U^\sigma}}(\log D) \) such that \( \text{Gr}_1^W \Omega^2_{\overline{U^\sigma}}(\log D) = \Omega^2_{\overline{U^\sigma}} \), \( \text{Gr}_1^W \Omega^2_{\overline{U^\sigma}}(\log D) = \Omega^1_D \) and \( \text{Gr}_2^W \Omega^2_{\overline{U^\sigma}}(\log D) = \Omega^2_{\overline{E^\sigma}} \). By definition one has \( \Lambda^1(U) \subset \Gamma(U^\sigma, \Omega^2_{\overline{U^\sigma}}(\log D)) \). However since \( D \) is a union of \( \mathbb{P}^1 \), one has \( \Gamma(U^\sigma, \text{Gr}_1^W \Omega^2_{\overline{U^\sigma}}(\log D)) = \Gamma(U^\sigma, \Omega^1_D) = 0 \) and hence \( \Gamma(U^\sigma, W_i \Omega^2_{\overline{U^\sigma}}(\log D)) = \Gamma(U^\sigma, W_0 \Omega^2_{\overline{U^\sigma}}(\log D)) = \Gamma(U^\sigma, \Omega^2_{\overline{U^\sigma}}) \).

By Prop. 3.4 (1), the image of \( \Lambda^1(U) \) via the natural map \( \Gamma(U^\sigma, \Omega^2_{\overline{U^\sigma}}) \rightarrow H^2_{\text{dR}}(U^\sigma) \) is contained in \( H^2_{\text{dR}}(U^\sigma)_0 \).

Proposition 3.12

\[ \Lambda^1(U) \cong F^1 H^2_{\text{dR}}(U)_0, \quad \Lambda^2(U) \cong F^2 H^2_{\text{dR}}(U)_0. \]

Proof. Prop. 3.4 (2) and the definition of \( \Lambda^1(U) \) give rise to a commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \Lambda^1(U) & \rightarrow \Lambda^1(U) & \rightarrow H_{1,\text{dR}}(D) \\
| & | & | & | & |
0 & \rightarrow & F^1 H^2_{\text{dR}}(U)_0 & \rightarrow F^1 H^2_{\text{dR}}(U)_0 & \rightarrow H_{1,\text{dR}}(D)
\end{array}
\]

with exact rows. Now the assertion follows from Prop. 3.9.

The following proposition is the motivation by which we introduced the good algebraic 2-forms.

Proposition 3.13 Let \( \omega \in \Lambda^1(U) \) be a good 2-form. Suppose that \( D_0 = f^{-1}(s_0) \) is an irreducible multiplicative fiber (i.e. type \( I_1 \)). Then the extra term \( \text{ex}_{D_0}(\omega) \) is zero.

Remark 3.14 Prop. 3.13 seems true for a fiber of type \( I_n \) for arbitrary \( n \geq 1 \).

Proof. We may replace \( K \) with \( \overline{K} \). Put \( E^* = E + f^{-1}(F) \). We use the description of \( H^*_{\text{dR}}(X) \) etc. by the Čech complexes. Let

\[
(\alpha_{ij}) \times (\beta_i) \in \check{C}^1(\mathcal{H}^{1,0}_c) \times \check{C}^0(\Omega^1_C(\log T) \otimes \mathcal{H}_e)
\]

be a corresponding Čech cocycle to \( \omega \), and this defines

\[
z := (0) \times (\eta_{ij}) \times (\pi_i) \in \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega^1_X(\log D + E^*)) \times \check{C}^0(\Omega^2_X(\log D + E^*))
\]

in a natural way. The proof of Lem. 3.10 shows that there is \( y_0 \in \check{C}^0(\mathcal{H}_{c|\overline{U^\sigma}}) \) such that

\[
(0) \times (\omega) = (\alpha_{ij}) \times (\beta_i) - \mathcal{D}_0(y_0)
\]

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where $\mathcal{D}_0 : \mathcal{C}^0(\mathcal{H}_e) \to \mathcal{C}^1(\mathcal{H}_e^{1,0}) \times \mathcal{C}^0(\Omega_{\mathcal{E}}(\log T) \otimes \mathcal{H}_e)$. This means that there is $y = (0) \times (\nu_i) \in \mathcal{C}^1(\mathcal{O}_{\mathcal{E}}) \times \mathcal{C}^0(\Omega_{\mathcal{E}}(\log D))$ such that
\[
|_{\mathcal{D}} - \mathcal{D}(y) = (0) \times (\eta_{ij} - (\nu_j - \nu_i)) \times (\pi_i - d\nu_i) = (0) \times (0) \times (\omega) \tag{3.29}
\]
where
\[
\mathcal{D} : \mathcal{C}^1(\mathcal{O}_X) \times \mathcal{C}^1(\Omega_X^1(\log D + E^*)) \to \mathcal{C}^2(\mathcal{O}_X) \times \mathcal{C}^1(\Omega_X^1(\log D + E^*) \times \mathcal{C}^0(\Omega_X^2(\log D + E^*)).
\]
On the other hand, there is a Čech cocycle $w = (0) \times (*) \times (*) \in \mathcal{C}^2(\mathcal{O}_X) \times \mathcal{C}^1(\Omega_X^1(\log E^*)) \times \mathcal{C}^0(\Omega_X^2(\log E^*))$ such that $[w] \in F^1 H^2_{\text{dr}}(U_0) \otimes H^2_{\text{dr}}(D)$ and $[w]|_V = [z]$ in $H^2_{\text{dr}}(U)$. Since $[w]|_V = [z] \in F^1 H^2_{\text{dr}}(U) \otimes H^2(X, \Omega_X^2(\log D + E^*))$, this means that there is $\tilde{y} = (0) \times (\tilde{\nu}_i) \in \mathcal{C}^1(\mathcal{O}_X) \times \mathcal{C}^0(\Omega_X^2(\log D + E^*))$ such that
\[
w = z - \mathcal{D}(\tilde{y}) = (0) \times (\eta_{ij} - (\tilde{\nu}_j - \tilde{\nu}_i)) \times (\pi_i - d\tilde{\nu}_i) \tag{3.30}
\]
and this belongs to $\mathcal{C}^2(\mathcal{O}_X) \times \mathcal{C}^1(\Omega_X^1(\log E^*)) \times \mathcal{C}^0(\Omega_X^2(\log E^*))$.

**Lemma 3.15** Fix an arbitrary multiplicative fiber $D_0 = f^{-1}(s_0)$, and choose a (sufficiently small) neighborhood $V$ of $D_0$. Then there is a constant $c$ such that
\[
\theta_i := \nu_i|_V - \tilde{\nu}_i|_V - c \frac{dt}{t - s_0}
\]
has no log pole along $D_0$.

**Proof.** There is the exact sequence
\[
0 \to \Omega^1_V \to \Omega^1_V(\log D_0) \to \mathcal{O}_{\mathcal{D}_0} \to 0.
\]
Since neither $z|_{\mathcal{D}} - \mathcal{D}(y)$ or $z - \mathcal{D}(\tilde{y})$ has log pole along $D_0$,
\[
\text{Res}(\eta_{ij}) = \text{Res}(\nu_j) - \text{Res}(\nu_i) = \text{Res}(\tilde{\nu}_j) - \text{Res}(\tilde{\nu}_i) \in \mathcal{C}^1(\mathcal{O}_{\mathcal{D}_1}).
\]
Since $D_0$ is irreducible, $\text{Ker}[\mathcal{C}^0(\mathcal{O}_{\mathcal{D}_0}) \to \mathcal{C}^1(\mathcal{O}_{\mathcal{D}_0})]$ is one-dimensional, and hence $\text{Res}(\nu_i) - \text{Res}(\tilde{\nu}_i)$ is a constant $c$. This implies that
\[
\theta_i := \nu_i|_V - \tilde{\nu}_i|_V - c \frac{dt}{t - s_0}
\]
has no log pole. \qed

We turn to the proof of Prop. 3.13. By Lem. 3.15 and (3.29) and (3.30), one has
\[
z - \mathcal{D}(\tilde{y})|_V = (0) \times ((\eta_{ij} - (\nu_j - \nu_i))|_V - (\theta_j - \theta_i)) \times ((\pi_i - d\nu_i)|_V - d\theta_i)
= (0) \times ((\theta_j - \theta_i)) \times (\omega - d\theta_i).
\]
Let $z_{X,D_0} \in F^1 H^2_{\text{dr}}(U_0, D_0)$ be the corresponding Čech cocycle to $z - \mathcal{D}(\tilde{y})$ via the injection $F^1 H^2_{\text{dr}}(U_0, D_0) \to F^1 H^2_{\text{dr}}(U_0)$. Then
\[
z_{X,D_0}|_V = (0) \times (0, (0, 0), (\theta_j - \theta_i)) \times (0, (0, 0), (\omega - d\theta_i))
\equiv (0) \times (0, 0) \times (0, 0, \omega|_V - d\theta_i)
\in H^2_{\text{dr}}(V, D_0).
\]
This belongs to $F^1 H^2_{\text{dr}}(V, D_0)$ since so does $z_{X,D_0}$. Hence $(0) \times (0, 0) \times (0, 0, \omega|_V) \in F^1 H^2_{\text{dr}}(V, D_0)$ and this means $\text{ex}_{D_0}(\omega) = 0$. \qed
4 Explicit computations of regulator on $K_1$ of elliptic surfaces

We keep the notations in §3.1. The base field $K$ is $\mathbb{C}$ and we assume $D \neq \emptyset$ throughout this section.

4.1 1-Extension of MHS’s arising from a multiplicative fiber

For each $\gamma \in H_1(D, \mathbb{Q})$, there is a corresponding element $\xi_\gamma \in H_3^M(X, \mathbb{Q}(2))$ which is unique up to the decomposable part. It is given in the following way. Let $D_k = \sum_{i=1}^n D_{k}^{(i)}$ be a multiplicative fiber over a point $P_k \in C$ and $Q_i$ the intersection points. There are rational functions $f_i$ on $D_k^{(i)}$ such that $\text{Div}_{D_k^{(i)}}(f_i) = Q_{i+1} - Q_i$. Then we put

$$\xi_{D_k} := \sum_{i=1}^n [f_i, D_{k}^{(i)}] \in H_3^M(X, \mathbb{Q}(2)).$$

If $\gamma = (m_1, \ldots, m_s) \in H_1(D, \mathbb{Z}) = \bigoplus_k H_1(D_k, \mathbb{Z}) \cong \mathbb{Z}^{\oplus k}$ then we put $\xi_{\gamma} := m_1 \xi_{D_1} + \cdots + m_s \xi_{D_s}$. This depends on the choice of $f_i$’s, though the ambiguity is killed by the decomposable part.

Let us recall the regulator $\text{reg}(\xi_\gamma)$ from §2.2. Let

$$\text{NF}^B(X) := \text{Im}(H_2(D, \mathbb{Q}) \oplus H_2(E, \mathbb{Q}) \oplus H_2(e(C), \mathbb{Q}) \longrightarrow H_2(X, \mathbb{Q})).$$

Then there are the natural isomorphisms

$$H_2(X, \mathbb{Q})/\text{NF}^B(X) \cong (\text{NF}(X)^\perp)^* \cong \text{NF}(X)^\perp \otimes \mathbb{Q}(2). \quad (4.1)$$

The exact sequence

$$0 \longrightarrow H_2(X, \mathbb{Q})/H_2(D, \mathbb{Q}) \longrightarrow H_2(X, D, \mathbb{Q}) \overset{\partial}{\longrightarrow} H_1(D, \mathbb{Q}) \longrightarrow 0 \quad (4.2)$$

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of mixed Hodge structures gives rise to a map
\[ \rho : H_1(D, \mathbb{Q}) \to \text{Ext}^1(\mathbb{Q}, H_2(X, \mathbb{Q})/H_2(D, \mathbb{Q})) \]
\[ \to \text{Ext}^1(\mathbb{Q}, H_2(X, \mathbb{Q})/\text{NF}^B(X)) \]
\[ \xleftarrow{\cong} \text{Ext}^1(\mathbb{Q}, \text{NF}(X)^+ \otimes \mathbb{Q}(2)) \]
where \( \text{NF}(X)^+ \subset H^2(X, \mathbb{Q}) \) is a Hodge structure of weight 2. Then we have from Thm 2.1
\[ \text{reg}(\xi_\gamma) = \pm \rho(\gamma). \]  
(4.3)

In this section, we shall use a slight modification of (4.2).

**Lemma 4.1** \( \partial : H_2(U, D; \mathbb{Q}) \to H_1(D, \mathbb{Q}) \) is surjective.

**Proof.** The assertion is equivalent to saying that \( H^1(U, D; \mathbb{Q}) \to H^1(U, \mathbb{Q}) \) is surjective. Since the functional \( j \)-invariant of \( U/S \) is not constant (by the assumption), one has
\[ \Gamma(S, R^1f_*\mathbb{Q}) = H^1(f^{-1}(t), \mathbb{Q})^{\pi_1(S,t)} = 0 \]
and hence \( H^1(U, \mathbb{Q}) = H^1(S, \mathbb{Q}) \). This and a commutative diagram
\[
\begin{array}{ccccccc}
0 = H^1(U, \mathbb{Q}) & \to & H^1(U, \mathbb{Q}) & \to & H^1(U, \mathbb{Q}) & \to & H^2(D, \mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = H^1_m(S, \mathbb{Q}) & \to & H^1(S, \mathbb{Q}) & \to & H^1(S, \mathbb{Q}) & \to & H^2_m(S, \mathbb{Q}) \\
\end{array}
\]
yield
\[ H^1(S, \mathbb{Q}) \xleftarrow{\cong} H^1(U, \mathbb{Q}). \]  
(4.4)

Therefore, to show the surjectivity of \( H^1(U, D; \mathbb{Q}) \to H^1(U, \mathbb{Q}) \) it is enough to show that \( H^1(S, T_m; \mathbb{Q}) \to H^1(S, \mathbb{Q}) \) is surjective. However it is clear because \( H^1(T_m, \mathbb{Q}) = 0. \) \( \square \)

\[ \text{Ext}^1(\mathbb{Q}, \text{NF}(X)^+ \otimes \mathbb{Q}(2)) = \text{Coker}[\text{NF}(X)^+ \otimes \mathbb{Q}(2) \to \text{NF}(X)^+ \otimes \mathbb{C}/F^2] \]  
(4.5)
\[ \cong \text{Coker}[H_2(X, \mathbb{Q})/\text{NF}^B(X) \to \text{Hom}(F^1H^2_{\text{dR}}(U)_0, \mathbb{C})] \]  
(4.6)
\[ \cong \text{Coker}[H_2(U^\sigma, \mathbb{Q}) \to \text{Hom}(F^1H^2_{\text{dR}}(U)_0, \mathbb{C})] \]  
(4.7)
\[ \cong \text{Coker}[H_2(U^\sigma, \mathbb{Q}) \to \text{Hom}(\Lambda^1(U^\sigma), \mathbb{C})] \]  
(4.8)

where (4.6) follows from Prop 3.4 (3) and (4.1), and (4.7) follows from the surjectivity of \( H_2(U^\sigma, \mathbb{Q}) \to H_2(X, \mathbb{Q})/\text{NF}^B(X) \) and (4.8) follows from Prop 3.12. The map \( \Phi \) in (4.8) is given by
\[ \Delta \mapsto \left[ \omega \mapsto \int_\Delta \omega \right], \quad \omega \in \Lambda^1(U^\sigma). \]

\footnote{Note that, since \( \Lambda^1(U^\sigma) \subset \Gamma(U^\sigma, \Omega^2_{U^\sigma}) \), one can a priori define "\( \int_\Delta \omega \)" only for \( \Delta \in H_2(U^\sigma, \mathbb{Q}). \)}
Next consider a commutative diagram

\[
\begin{array}{c}
0 \rightarrow H_2(X, \mathbb{Q})/H_2(D, \mathbb{Q}) \rightarrow H_2(X, D; \mathbb{Q}) \rightarrow H_1(D, \mathbb{Q}) \rightarrow 0 \\
\text{surj.} \quad \square \quad \text{surj.} \\
0 \rightarrow H_2(X, \mathbb{Q})/\text{NF}^B(X) \rightarrow H_2(X, D; \mathbb{Q}) \rightarrow H_1(D, \mathbb{Q}) \rightarrow 0 \\
\text{surj.} \quad \square \quad \text{surj.} \\
0 \rightarrow H_2(U^o, \mathbb{Q})/H_2(D) \rightarrow H_2(U^o, D; \mathbb{Q}) \rightarrow H_1(D, \mathbb{Q}) \rightarrow 0 
\end{array}
\]

where the surjectivity of the right arrows follows from Lem. 4.1. The middle row gives the regulator class (4.3). Let us describe it explicitly under the identification (4.8). There is an isomorphism

\[
F^1 H^2_{\text{dR}}(U, D) \cong F^1 H^2_{\text{dR}}(U) \cong \Lambda^1(U).
\]

We denote by \(\omega_{U, D} \in F^1 H^2_{\text{dR}}(U, D)\) the corresponding element of \(\omega \in \Lambda^1(U)\) via (4.10).

Fix a \(\Gamma \in H_2(U^o, D, \mathbb{Q})\) such that \(\gamma = \partial(\Gamma)\). Then, by Thm. 2.1 (2.8) and Prop. 3.8 we have

\[
\text{reg}(\xi_0) = \left[ \omega \mapsto \int_{\Gamma} \omega_{U, D} = \int_{\Gamma} \omega - \int_{\partial \Gamma} \text{ex}_D(\omega) \right] \in \text{Hom}(\Lambda^1(U), \mathbb{C})/\text{Im}\Phi
\]

under the identification (4.8).

4.2 \(E(U^o, D; \mathbb{Z})\) and \(E(U^o, \mathbb{Z})\)

Take a path \(\gamma : [0, 1] \rightarrow S^0(\mathbb{C}), t \mapsto \gamma_t\) such that \(\gamma_t \in S^0(\mathbb{C})\) for \(t \neq 0, 1\). Take a cycle \(\varepsilon \in H_1(f^{-1}(\gamma_{t_0}), \mathbb{Z})\) for some (fixed) \(t_0 \in [0, 1]\). Then it extends to a flat section \(\varepsilon_t \in H_1(f^{-1}(\gamma_t), \mathbb{Z})\) over \(t \in [0, 1]\) in a unique way. We denote by \(\Gamma(\varepsilon, \gamma)\) the fibration over the path \(\gamma\) whose fiber is \(\varepsilon_t\).

Then

\[
\Gamma(\varepsilon, \gamma) \in H_2(U^o, f^{-1}(\gamma_{t_0}) \cup f^{-1}(\gamma_1); \mathbb{Z}),
\]

with \(\partial(\Gamma(\varepsilon, \gamma)) = \varepsilon_1 - \varepsilon_0 \in H_1(f^{-1}(\gamma_{t_0}) \cup f^{-1}(\gamma_1), \mathbb{Z})\).
Define $E(U^o, D; \mathbb{Z}) \subset H_2(U^o, D; \mathbb{Z})$ the subgroup generated by $\Gamma(\varepsilon, \gamma)$’s where $\gamma$ and $\varepsilon$ run over as above such that $\gamma_0, \gamma_1 \in T_m = S^o - S^o$. Define $E(U^o, \mathbb{Z})$ by an exact sequence

$$0 \rightarrow E(U^o, \mathbb{Z}) \rightarrow E(U^o, D; \mathbb{Z}) \stackrel{\partial}{\rightarrow} H_1(D, \mathbb{Z}).$$

Write $E(U^o, D; \mathbb{Q}) := E(U^o, D; \mathbb{Z}) \otimes \mathbb{Q}$ etc.

**Proposition 4.2** We have

$$E(U^o, D; \mathbb{Q}) = H_2(U^o, D; \mathbb{Q}), \quad (4.12)$$

$$E(U^o, \mathbb{Q}) \cong H_2(U^o, \mathbb{Q}) / H_2(D, \mathbb{Q}) \cong (H^2(U^o)_0)^*. \quad (4.13)$$

Hence we have

$$0 \rightarrow E(U^o, \mathbb{Q}) \rightarrow E(U^o, D; \mathbb{Q}) \stackrel{\partial}{\rightarrow} H_1(D, \mathbb{Q}) \rightarrow 0$$

**Lemma 4.3** The sequence

$$H_2(U^o, \mathbb{Q}) \rightarrow H_2(U^o, D; \mathbb{Q}) \stackrel{\partial}{\rightarrow} H_1(D, \mathbb{Q}) \rightarrow 0 \quad (4.14)$$

is exact.

**Proof.** The surjectivity of $\partial$ is proven in the same way as Lem. 4.1. We only show

$$\text{Im}(H_2(U^o, \mathbb{Q}) \rightarrow H_2(U^o, D; \mathbb{Q})) = \text{Im}(H_2(U^o, \mathbb{Q}) \rightarrow H_2(U^o, D; \mathbb{Q})). \quad (4.15)$$

Consider a diagram

$$H_2(D, \mathbb{Q}) \rightarrow H_2(U^o, \mathbb{Q}) \rightarrow H_2(U^o, D; \mathbb{Q}) \stackrel{\partial}{\rightarrow} H_1(D, \mathbb{Q}) \rightarrow H_1(U^o, \mathbb{Q}) \rightarrow H_1(U^o, \mathbb{Q})$$

with exact row and column. Hence it is enough to show $\text{Im}(ba) = \text{Im}(b)$ or equivalently $\dim \text{Coker}(ba) = \dim \text{Coker}(b)(= \dim \text{Ker}(c))$. Since $ba$ is given by the intersection pairing, a direct calculation shows that $\dim \text{Coker}(ba) = \dim H_0(T_m)$. On the other hand,

$$\text{Ker}(H_1(U^o, \mathbb{Q}) \rightarrow H_1(U^o, \mathbb{Q})) \cong \text{Ker}(H_1(S^o, \mathbb{Q}) \rightarrow H_1(S^o, \mathbb{Q}) \cong H_0(T_m)$$

(cf. (4.4)), so we are done. \qed
Proof of Prop. 4.2. Let $\mathcal{L}$ be the local system on $S^0(\mathbb{C})$ whose fiber is $H_1(f^{-1}(t), \mathbb{Q})$. Then the image of $H_2(U, \mathbb{Q})$ in $H_2(U^0, D; \mathbb{Q})$ coincides with that of $H_1(S^0, \mathcal{L})$. The homology group $H_1(S^0, \mathcal{L})$ is generated by $\Gamma(\varepsilon, \gamma)$’s such that $\gamma_0 = \gamma_1 \in S^0$ and $\varepsilon_0 = \varepsilon_1$. Take a path $\delta$ such that $\delta_0 \in T_m = \overline{S^0} - S^0$ and $\delta_1 = \gamma_0 = \gamma_1$. Put $\tilde{\gamma} = \delta \cdot \gamma \cdot \delta^{-1}$. Then the image of $\Gamma(\varepsilon, \gamma)$ in $H_2(U^0, D; \mathbb{Q})$ coincides with $\Gamma(\varepsilon, \tilde{\gamma})$, and this is an element of $\mathbb{E}(U^0, D; \mathbb{Q})$. There remains to show the surjectivity of $\mathbb{E}(U^0, D; \mathbb{Q}) \rightarrow H_1(D, \mathbb{Q})$ (this gives an alternative proof of Lem. 4.1). To do this, it is enough to show that for each $p \in T_m$, there is a path $\nu$ such that $\nu_0 = p$ and $\nu_1 \in T_m$, and there is a cycle $\alpha_1 \in H_1(f^{-1}(\nu_1), \mathbb{Q})$ such that $\alpha_0 \neq 0$ and $\alpha_1 = 0$. Since

$$\partial(\Gamma(\alpha, \nu)) = (\cdots, 0, \alpha_0, 0, \cdots) \in H_1(D, \mathbb{Q}) = \bigoplus_{x \in T_m} H_1(f^{-1}(x), \mathbb{Q})$$

this implies the surjectivity of $\mathbb{E}(U^0, D; \mathbb{Q}) \rightarrow H_1(D, \mathbb{Q})$. Fix paths $\nu'$ and $\nu''$ such that $\nu'_0 = p$, $\nu'_1 = \nu''_q = q \in S^0$ and $\nu'_0 \in T_m$. Fix $\alpha' \in H_1(f^{-1}(q), \mathbb{Q})$ such that $\alpha'$ goes to a nonzero cycle as $q \rightarrow p$, and $\alpha'' \in H_1(f^{-1}(q), \mathbb{Q})$ such that $\alpha''$ goes to zero as $q \rightarrow \nu''_0$. Since $H_1(f^{-1}(q), \mathbb{Q})$ is an irreducible $\mathbb{Q}[\pi_1(S^0, q)]$-module, there is $g \in \mathbb{Q}[\pi_1(S^0, q)]$ such that $g(\alpha') = \alpha''$. Joining $\Gamma(\alpha', \nu')$, $\Gamma(\alpha', g)$ and $\Gamma(\alpha'', \nu'')$, we obtain $\Gamma(\alpha, \nu)$ as desired. This completes the proof of Prop. 4.2.

4.3 A formula of Beilinson regulator on $H^3_{ad}(X, \mathbb{Q}(2))$

We summarize all of the results in §4.1 and §4.2 together with Thm. 2.1 in the following theorem.

**Theorem 4.4** Let the notations be as in §3.1, §3.4 and §4.2. Suppose $K = \mathbb{C}$ and $D \neq \emptyset$. Then we have

$$\text{Ext}^1(\mathbb{Q}, NF(X) \cap \mathbb{Q}(2)) \cong \text{Coker} \left[ \Phi : \mathbb{E}(U^0, \mathbb{Q}) \rightarrow \text{Hom}(\Lambda^1(U), \mathbb{C}) \right]$$

where

$$\Phi(\Delta) = \left[ \omega \mapsto \int_\Delta \omega \right], \quad \omega \in \Lambda^1(U).$$

For $\gamma \in H_1(D, \mathbb{Q})$, fix $\Gamma \in \mathbb{E}(U^0, D; \mathbb{Q})$ with $\partial(\Gamma) = \gamma$. Then

$$\text{reg}(\xi_\gamma) = \pm \left[ \omega \mapsto \int_\Gamma \omega_{\Gamma \cap D} = \int_\Gamma \omega - \int_{\Gamma} \text{ex}_D(\omega) \right] \in \text{Hom}(\Lambda^1(U), \mathbb{C})/\text{Im}\Phi$$

(see Prop. 3.8 and 3.19 for “$\text{ex}_D$”). Moreover if $\gamma \in H_1(D_0, \mathbb{Q})$ with $D_0$ a union of irreducible multiplicative fibers, then $\text{ex}_D(\omega) = 0$ by Prop. 5.7.3 so that we have simply

$$\text{reg}(\xi_\gamma) = \pm \left[ \omega \mapsto \int_\Gamma \omega \right].$$

The point is that “$\omega \in \Lambda^1(U)$” is an algebraic 2-form. This makes it easier to compute the regulator. To carry out the computation practically, we need the following data.
(a) A basis of $\Lambda^1(U)$,

(b) A basis of $H_2(X, D; \mathbb{Q})$ (see (4.9)),

(c) Computation of the extra term $e_{x_D}$ (however see Rem. 3.14).

One can obtain (a) by a direct computation of $H^1_{dR}(C, \mathcal{H}_e)$ and by explicit formula of Gauss-Manin connection (Appendix). See §5.2 for an example of the computation. A basis (b) can be constructed from $H_2(U_0, D; \mathbb{Q}) \cong \mathbb{E}(U_0, D; \mathbb{Q})$. It is not hard to obtain a basis of $\mathbb{E}(U_0, D; \mathbb{Q})/\mathbb{E}(U_0, \mathbb{Q})$. To obtain a basis of $H_2(X, \mathbb{Q})/\text{NF}_B(X)$ we assume that the precision of the values of integrations can be raised as many as one likes. Then, by using the basis of $\Lambda^1(U)$ together with the fact that there is an embedding $\text{Im}[\mathbb{E}(U_0, \mathbb{Q})] \hookrightarrow H_2(X, \mathbb{Q})/\text{NF}_B(X) \hookrightarrow \text{Hom}(\Lambda^1(U), \mathbb{C})$, one can prove the linear independence of given cycles in $\mathbb{E}(U_0, \mathbb{Q})$ if they were linear independent. Hence one can eventually obtain a basis of $\text{Im}\mathbb{E}(U_0, \mathbb{Q})$.

We shall apply the above method to an example in the next section.

5 Example : $3y^2 + x^3 + (3x + 4t^1)^2$

Let $l \geq 1$ be an integer. We consider a minimal elliptic surface

$$f : X \longrightarrow \mathbb{P}^1, \quad f^{-1}(t) : 3y^2 + x^3 + (3x + 4t^1)^2 = 0$$

defined over $\mathbb{Q}$. There is the section $e : \mathbb{P}^1 \to X$ of “$\infty$”. Write $X_C := X \times_{\mathbb{Q}} \mathbb{C}$.

The purpose of this section is to compute the real regulator

$$\text{reg}_{\mathbb{R}} : H^3_\#(X, \mathbb{Q}(2)) \longrightarrow \text{Ext}^1_{\mathbb{R}-\text{MHS}}(\mathbb{R}, H^2(X_{\text{ind}}) \otimes \mathbb{R}(1))_{F_\infty = 1},$$

where $H^2(X)_{\text{ind}} := H^2(X_C, \mathbb{Q}(1))/\text{NS}(X_C)$, especially for an element

$$\xi_{D_1} = \begin{bmatrix} y - (x + 4) \\ y + (x + 4) \end{bmatrix}, D_1 \in H^3_\#(X, \mathbb{Q}(2)) \quad (D_1 := f^{-1}(1))$$

arising from a split multiplicative fiber $D_1$ of type $I_1$. We note that if $(l, 6) = 1$ then $\xi_{D_1}$ is an “integral” element, in the sense that it comes from the motivic cohomology of a proper flat regular model of $X$ over $\mathbb{Z}$ (see [Sch] 1.1.6 for “unconditional” definition of integral elements).

5.1 Basic data of $X$

The following is easy to show (the proof is left to the reader).

- The Hodge numbers are as follows:

$$h^{10} = h^{01} = 0, \quad h^{20} = h^{02} = \left\lfloor \frac{l - 1}{3} \right\rfloor, \quad h^{11} = 10(1 + h^{20}).$$

In particular, $H^2_B(X)_{\text{ind}} := H^2_B(X(\mathbb{C}), \mathbb{Q})/\text{NS}(X_C) \otimes \mathbb{Q} \neq 0$ if and only if $l \geq 4$. (If $1 \leq l \leq 3$, then $X$ is a rational surface.)
• There are \((l + 1)\)-multiplicative fibers:

\[
D_0 := f^{-1}(0) (= \text{type } I_{3l}), \quad D_i := f^{-1}(\zeta_i^{-1}) (= \text{type } I_1),
\]

where \(\zeta_i = \exp(2\pi i/l)\) and \(1 \leq i \leq l\). Moreover \(D_1 = f^{-1}(1)\) is the unique split multiplicative fiber.

• If \(3|l\), then there is no additive fiber. If \((3, l) = 1\) then \(E = f^{-1}(\infty)\) is the unique additive fiber (type \(IV^*\) if \(l \equiv 1 \mod 3\), and type \(IV\) if \(l \equiv 2 \mod 3\)). In particular, \(E \neq \emptyset\) if and only if \((l, 3) = 1\).

• \(\text{NF}(X_C) \otimes \mathbb{Q} = \text{NS}(X_C) \otimes \mathbb{Q}\) if and only if \(l\) is odd ([St] Example 4).

• There is an automorphism \(\sigma : X \to X\) given by \(\sigma(x, y, t) = (x, y, \zeta_l t)\).

Hereafter we assume \((l, 6) = 1\). Then

\[
\text{NF}_B(X_C) \perp = \text{NS}(X_C) \perp \xrightarrow{\cong} H^2_B(X)_{\text{ind}} := H^2_B(X(C), \mathbb{Q}(1))/\text{NS}(X_C) \otimes \mathbb{Q}.
\]

Since \(U = U^0\) in this case one has

\[
\dim \mathbb{E}(\mathcal{U}, \mathbb{Q}) = \dim \text{NF}(X) \perp = l - 1 \quad \text{by (4.13) and Prop. 3.4 (3)),} 
\]

\[
dim \mathbb{E}(\mathcal{U}, D; \mathbb{Q}) = (l - 1) + \dim H_1(D; \mathbb{Q}) = 2l - 1. \tag{5.2}
\]

### 5.2 Good algebraic 2-forms \(\Lambda^1(\mathcal{U})\) and \(\Lambda^2(\mathcal{U})\)

Suppose \((l, 6) = 0\). We use the same notations in [3.1]: \(D = D_0 + \cdots + D_i, U = X - (D + E), \mathcal{U} = X - E, T_m = \{0, \zeta^3_i\}, T_a = \{\infty\}\) and \(T = T_m + T_a\). Let \(H^2_{\text{dr}}(\mathcal{U})_0 := \text{Ker}[H^2_{\text{dr}}(\mathcal{U}) \to H^2_{\text{dr}}(D)]\). This is isomorphic to \(\text{NF}_{\text{dr}}(X) \perp\) by Prop. 3.4 (3).

As is easily seen, one has

\[
\Lambda^2(\mathcal{U}) = \langle t^{i - 1} \frac{dt dx}{y} \mid 1 \leq i \leq \left\lfloor \frac{l - 1}{3} \right\rfloor \rangle_{\mathbb{Q}} \xrightarrow{\cong} F^2 H^2_{\text{dr}}(\mathcal{U})_0. \tag{5.3}
\]

Let us compute \(\Lambda^1(\mathcal{U})\). Since \(H^1_{\text{zar}}(\mathbb{P}^1, \mathcal{H}^{1,0}_e) = 0\), \(\Lambda^1(\mathcal{U})\) is the image of the composition

\[
\Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathcal{H}_e) \longrightarrow \Gamma(A_1, \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathcal{H}_e)/\nabla(\mathcal{H}^{1,0}_e) \\
\xleftarrow{\cong} \Gamma(A_1, \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathcal{H}_e^{1,0}) \\
= \Gamma(A_1, \Omega^2_{A_1}(\log T_m))
\]

where \(A_1 = \mathbb{P}^1 - \{\infty\}\). Using the basis \(\omega\) and \(\omega^*\) in Appendix (6.1), (6.2), one easily sees that \(\Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathcal{H}_e)\) is generated by the following elements.

\[
\frac{t^i dt}{t(t^l - 1)} \otimes \omega \ (0 \leq i \leq l + \left\lfloor \frac{l - 1}{3} \right\rfloor), \quad \frac{t^j dt}{t(t^l - 1)} \otimes \omega^* \ (0 \leq j \leq l - 1 - \left\lfloor \frac{l - 1}{3} \right\rfloor).
\]
By using Thm. 6.1 (6.3) and (6.4), we can compute their image into \( \Gamma(A^1, \Omega^2_{\mathcal{A}_1}(\log T_m)) \) directly. We thus have

\[
\Lambda^1(U) = \left\langle \frac{t^i dt}{t(t^i - 1)} \frac{dx}{y} \mid 0 \leq i \leq 2l - 1 - \left\lfloor \frac{l - 1}{3} \right\rfloor \right\rangle.
\] (5.4)

Therefore

\[
\Lambda^1(U) = \left\langle t^{i-1} \frac{dt dx}{y} \mid 1 \leq i \leq l - 1 - \left\lfloor \frac{l - 1}{3} \right\rfloor \right\rangle.
\] (5.5)

By Prop. 3.13, the extra term for \( D_k \ (1 \leq k \leq l) \) vanishes:

\[
\text{ex}_{D_k} \left( t^{i-1} \frac{dt dx}{y} \right) = 0 \text{ for } 1 \leq i \leq l - 1 - \left\lfloor \frac{l - 1}{3} \right\rfloor, \ 1 \leq k \leq l.
\] (5.6)

**Example 5.1** One can show that

\[
(l + 3i)(2l + 3i)t^{i-1} \frac{dt dx}{y} \equiv 9i^2 t^{i-1} \frac{dt dx}{y}
\]

in \( H^2_{dR}(U) \), however

\[
\int_\epsilon \text{ex}_{D_1} \left( (l + 3i)(2l + 3i)t^{i-1} \frac{dt dx}{y} - 9i^2 t^{i-1} \frac{dt dx}{y} \right) = \pm 9l \neq 0
\]

by using Prop. 6.6 (6.13), where \( \epsilon \in H_1(D_1, \mathbb{Z}) \cong \mathbb{Z} \) is the generator. In particular \( \text{ex}_{D_1} \left( t^{i-1} \frac{dt dx}{y} \right) \neq 0 \).

**5.3 Cycles \( \Delta \) and \( \Gamma \)**

Let \( \delta_0 \) (resp. \( \delta_1 \)) be the homology cycle in \( H_1(f^{-1}(t), \mathbb{Z}) \) which vanishes as \( t \to 0 \) (resp. \( t \to 1 \)). Define \( \Delta \) and \( \Gamma \) to be fibrations over the segment \([0, 1] \subset \mathbb{P}^1(\mathbb{C})\) whose fibers are the vanishing cycles \( \delta_1 \) and \( \delta_0 \) respectively.

\[
\Delta \in H_2(U, D_0; \mathbb{Z}), \quad \Gamma \in H_2(U, D_1; \mathbb{Z}).
\]

The boundary \( \partial \Delta \) (resp. \( \partial \Gamma \)) is a generator of the homology group \( H_1(D_0, \mathbb{Z}) \cong \mathbb{Z} \) (resp. \( H_1(D_1, \mathbb{Z}) \cong \mathbb{Z} \)).
Let $r_1(t) < r_2(t) < r_3(t)$ be the real roots of $x^3 + (3x + 4t^4)^2$ for $0 < t < 1$. Then one has

\[
\int_{\Delta} t^{j-1} dt \frac{dx}{y} = 2\sqrt{-3} \left( \int_{0}^{1} t^{j-1} dt \int_{r_1(t)}^{r_2(t)} \frac{dx}{\sqrt{x^3 + (3x + 4t^4)^2}} \right) \in \sqrt{-1} \mathbb{R}_{>0} \quad (5.7)
\]

purely imaginary number $\neq 0$

\[
\int_{\Gamma} t^{j-1} dt \frac{dx}{y} = 2\sqrt{-3} \left( \int_{0}^{1} t^{j-1} dt \int_{r_2(t)}^{r_3(t)} \frac{dx}{\sqrt{x^3 + (3x + 4t^4)^2}} \right) \in \mathbb{R}_{>0}. \quad (5.8)
\]

Let $\sigma : X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ be an automorphism given by $\sigma(x, y, t) = (x, y, \zeta_t t)$. Since

\[
\sigma^* t^{j-1} dt \frac{dx}{y} = \zeta_t^j t^{j-1} dt \frac{dx}{y}.
\]
one has
\[ \int_{\sigma^k \Delta} t^{j-1} dt \frac{dx}{y} = \zeta_{kj}^{l-1} \int_{\Delta} t^{j-1} dt \frac{dx}{y}. \]

This and an elementary calculation show that
\[ \Delta - \sigma, \Delta - \sigma^2 \Delta, \ldots, \Delta - \sigma^{l-1} \Delta \]
are linearly independent in \( \mathbb{E}(\mathcal{U}, \mathbb{Q}) \). Since it is \((l - 1)\)-dimensional by (5.1), the above is a basis of \( \mathbb{E}(\mathcal{U}, \mathbb{Q}) \). \( \mathbb{E}(\mathcal{U}, D; \mathbb{Q}) / \mathbb{E}(\mathcal{U}, \mathbb{Q}) \) is \( l \)-dimensional with a basis \( \Gamma, \sigma \Gamma, \ldots, \sigma^{l-1} \Gamma \).

Let \( F_\infty \) denotes the infinite Frobenius morphism. Then
\[ F_\infty(\Delta) = -\Delta, \quad F_\infty(\Gamma) = \Gamma. \]

By Thm.4.4 and the above computations we have the following.

**Theorem 5.2** Suppose \((l, 6) = 1\). Put \( h = \dim F^1 \mathcal{V}_{dR} = l - \lfloor \frac{l-1}{3} \rfloor - 1 \) and \( \zeta = \exp(2\pi i/l) \).

Let
\[ A = \left( (\zeta^{pq} - \zeta^{-pq}) \int_{\Delta} t^{p-1} dt \frac{dx}{y} \right)_{1 \leq p \leq h, \ 1 \leq q \leq (l-1)/2} \]
be \( h \times (l-1)/2 \)-matrix (the entries are real numbers by (5.1)). Then
\[ \text{Ext}^1_{\text{MHS}}(\mathbb{R}, H^2(\mathcal{X})_{\text{ind}} \otimes \mathbb{R}(1))^{F_\infty = 1} \cong \text{Coker}[A : \mathbb{R}^{(l-1)/2} \rightarrow \mathbb{R}^h]. \]

and we have
\[ \text{reg}_{\mathbb{R}}(\xi_{D_1}) = \pm \left( \int_{\Gamma} \frac{dx}{y}, \ldots, \int_{\Gamma} \frac{t^{h-1} dt \frac{dx}{y}}{y} \right) \in \mathbb{R}^h / \text{Im}A \]
under the above isomorphism.

**Corollary 5.3** Suppose \((l, 6) = 1\). Then
\[ \text{reg}_{\mathbb{R}}(\xi_{D_1}) \neq 0 \in \text{Ext}^1_{\text{MHS}}(\mathbb{R}, H^2(\mathcal{X})_{\text{ind}} \otimes \mathbb{R}(1))^{F_\infty = 1}. \]

In particular \( \xi_{D_1} \) is regulator indecomposable.

**Proof.** Put \( h := l - \lfloor \frac{l-1}{3} \rfloor - 1 \) and \( \zeta := \exp(2\pi i/l) \). Put
\[ I_p := \int_{\Delta} t^{p-1} dt \frac{dx}{y}, \quad J_p := \int_{\Gamma} t^{p-1} dt \frac{dx}{y}. \]

Then
\[ \text{reg}_{\mathbb{R}}(\xi_{D_1}) \neq 0 \in \text{Ext}^1_{\text{MHS}}(\mathbb{R}, H^2(\mathcal{X})_{\text{ind}} \otimes \mathbb{R}(1))^{F_\infty = 1}. \]
if and only if the rank of a matrix

\[
\begin{pmatrix}
(\zeta - \zeta^{-1})I_1 & (\zeta^2 - \zeta^{-2})I_1 & \cdots & (\zeta^{l_1-1} - \zeta^{-(l_1-1)})I_1 & J_1 \\
(\zeta^2 - \zeta^{-2})I_2 & (\zeta^4 - \zeta^{-4})I_2 & \cdots & (\zeta^{l_2-1} - \zeta^{-(l_2-1)})I_2 & J_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(\zeta^h - \zeta^{-h})I_h & (\zeta^{2h} - \zeta^{-2h})I_h & \cdots & (\zeta^{h(l_1-2)} - \zeta^{-h(l_1-2)})I_h & J_h
\end{pmatrix}
\]

(5.9)
is maximal. Thus it is enough to show that

\[
\det
\begin{pmatrix}
(\zeta - \zeta^{-1}) & (\zeta^2 - \zeta^{-2}) & \cdots & (\zeta^{l_1-1} - \zeta^{-(l_1-1)}) & J_1/I_1 \\
(\zeta^2 - \zeta^{-2}) & (\zeta^4 - \zeta^{-4}) & \cdots & (\zeta^{l_2-1} - \zeta^{-(l_2-1)}) & J_2/I_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(\zeta^k - \zeta^{-k}) & (\zeta^{2k} - \zeta^{-2k}) & \cdots & (\zeta^{k(l_1-2)} - \zeta^{-k(l_1-2)}) & J_k/I_k
\end{pmatrix}
\]

(5.10)
is nonzero where \( k = (l + 1)/2 \). Since the sum of the \((k - 1)\)-th row and \(k\)-th row is \((0, \cdots , 0, J_{k-1}/I_{k-1} + J_k/I_k)\), one has

\[
(5.10) = (J_{k-1}/I_{k-1} + J_k/I_k) \times \det(\zeta^{pq} - \zeta^{-pq})_{1 \leq p, q \leq (l-1)/2}
\]

\[= (J_{k-1}/I_{k-1} + J_k/I_k) \times \sqrt{(-1)^{(l-1)/2}}.
\]

Since \(J_p/I_p \in i\mathbb{R}_{>0}\) by (5.7) and (5.8), this is non-zero. \(\square\)

### 5.4 Another description of \(\int_\Delta\) and \(\int_\Gamma\)

When \(l = 1\), \(f : X \to \mathbb{P}^1\) is the universal elliptic curve over \(X_1(3)\). Using this, one can obtain another description of the real regulator.

Let \(q = \exp(2\pi i z)\) and

\[
E_{3a}(z) := 1 - 9 \sum_{n=1}^\infty \left( \sum_{k|n} \left( \frac{k}{3} \right) k^2 \right) q^n,
\]

\[
E_{3b}(z) := \sum_{n=1}^\infty \left( \sum_{k|n} \left( \frac{n/k}{3} \right) k^2 \right) q^n
\]

be the Eisenstein series of weight 3 for \(\Gamma_1(3)\), where \(\left( \frac{\cdot}{3} \right)\) denotes the Legendre symbol. Then

\[
t^l = \frac{E_{3a}}{E_{3a} + 27E_{3b}}
\]

and

\[
l \frac{dt}{t} \frac{dx}{y} = -27E_{3b} \frac{du}{u} \frac{dq}{q}, \quad \frac{lt^{l-1}}{t-1} \frac{dt}{y} = E_{3a} \frac{du}{u} \frac{dq}{q}
\]

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where “$du/u$” denotes the canonical invariant 1-form of the Tate curve around the cusp $z = i\infty$ ($t = 1$). Therefore we have
\begin{align*}
\int_{\Delta} \tau^{j-1} dt \frac{dx}{y} &= \frac{-27}{l} \times (2\pi i)^2 \int_{i\infty}^{\infty} t^j E_{3b}(z) dz \\
\int_{\Gamma} \tau^{j-1} dt \frac{dx}{y} &= \frac{-27}{l} \times (2\pi i)^2 \int_{i\infty}^{i\infty} t^j E_{3b}(z)z dz. \quad (5.11)
\end{align*}

On the other hand there are formulas
\begin{align*}
E_{3a} E_{3b} - 27 E_{3b} &= 27 E_{3b} E_{3a} + 27 E_{3b} \\
E_{3a} E_{3b} - 3 \sqrt{3} i z^3 &= 27 E_{3b} E_{3a} + 27 E_{3b} \\
27 E_{3b} &= 3 \sqrt{3} i z^3 E_{3a} \quad (5.13)
\end{align*}
on the Eisenstein series. Applying (5.13) to (5.11) and (5.12), we have the following theorem.

**Theorem 5.4** Put $c := \exp(-2\pi/\sqrt{3}) = 0.026579933\cdots$. Define rational numbers $a_n(j)$ and $b_n(j)$ by
\begin{align*}
E_{3b} \left( \frac{E_{3a}}{E_{3a} + 27 E_{3b}} \right)^{j/l} &= \sum_{n=1}^{\infty} a_n(j) q^n \\
&= q + \left( 3 - 27 \frac{j}{l} \right) q^2 + \left( 9 - \frac{81 j}{2} + \frac{729}{2} \left( \frac{j}{l} \right)^2 \right) q^3 + \cdots,
\end{align*}
\begin{align*}
E_{3a} \left( \frac{E_{3b}}{q(E_{3a} + 27 E_{3b})} \right)^{j/l} &= \sum_{n=0}^{\infty} b_n(j) q^n \\
&= 1 + \left( -9 - 15 \frac{j}{l} \right) q + \left( 27 + \frac{387 j}{2} + \frac{225}{2} \left( \frac{j}{l} \right)^2 \right) q^2 + \cdots.
\end{align*}

Put
\begin{align*}
I(j) &= \sum_{n=1}^{\infty} \frac{a_n(j)}{n} q^n + 3^{3j/l-3} \sum_{n=0}^{\infty} b_n(j) \left( \frac{1}{n+j/l} + \frac{\sqrt{3}}{2\pi(n+j/l)^2} \right) q^{n+j/l} \\
J(j) &= \sum_{n=1}^{\infty} a_n(j) \left( \frac{2\pi}{\sqrt{3}n} + \frac{1}{n^2} \right) q^n + 2\pi \cdot 3^{3j/l-7/2} \sum_{n=0}^{\infty} b_n(j) q^{n+j/l},
\end{align*}

Then we have
\begin{align*}
\int_{\Delta} \tau^{j-1} dt \frac{dx}{y} &= \frac{54\pi i}{l} I(j), \\
\int_{\Gamma} \tau^{j-1} dt \frac{dx}{y} &= \frac{-27}{l} J(j).
\end{align*}
for $1 \leq j \leq l - 1$. 

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This is useful since the series $I(j)$ and $J(j)$ converge rapidly!

**Example 5.5** Suppose $l = 5$. Then $X$ is a K3 surface. By Thm.5.4 one has

| $j$ | $I(j)$          | $J(j)$          |
|-----|----------------|----------------|
| $1$ | $0.42745977255318$ | $0.717696894965804$ |
| $2$ | $0.151180394233147$ | $0.37715912066032$ |
| $3$ | $0.0871841692346256$ | $0.261572572611421$ |
| $4$ | $0.0603840144077692$ | $0.202670503662525$ |

$\text{Ext}^1_{\mathbb{R}\text{-MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1))^{F_{\text{ind}} = 1} \cong \text{Coker}(\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^3)$. Since this is 1-dimensional, this has the canonical base $e_{\text{ind}, \mathbb{Q}}$ (up to $\mathbb{Q}^\times$) and a different base $e_{\text{false}_{\text{ind}, \mathbb{Q}}}$ \[^{[22]}\]. With respect to $e_{\text{false}_{\text{ind}, \mathbb{Q}}}$ one has

$$\text{reg}_{\mathbb{R}}(\xi_{D_1}) = \pi^2 \begin{vmatrix} i(\zeta - \zeta^{-1})I(1) & i(\zeta^2 - \zeta^{-2})I(1) & J(1) \\ i(\zeta^2 - \zeta^{-2})I(2) & i(\zeta^4 - \zeta^{-4})I(2) & J(2) \\ i(\zeta^3 - \zeta^{-3})I(3) & i(\zeta^6 - \zeta^{-6})I(3) & J(3) \end{vmatrix} \mod \mathbb{Q}^\times$$

$$= \left(\frac{J(2)}{I(2)} + \frac{J(3)}{I(3)}\right) \mod \mathbb{Q}^\times$$

with respect to $e_{\text{ind}, \mathbb{Q}}$ by Prop.2.2.

**Example 5.6** Suppose $l = 7$. Then $h^{20}(X) = h^{02}(X) = 2$, $h^{11}(X) = 30$. 

| $j$ | $I(j)$          | $J(j)$          |
|-----|----------------|----------------|
| $1$ | $0.740059830730164$ | $0.987994510350351$ |
| $2$ | $0.24646699651114$ | $0.51401702238944$ |
| $3$ | $0.137265313181901$ | $0.354195498081428$ |
| $4$ | $0.0929578147374374$ | $0.273237679671921$ |
| $5$ | $0.0696363855176379$ | $0.224004116344261$ |
| $6$ | $0.0554349861351089$ | $0.19073921727221$ |

$\text{Ext}^1_{\mathbb{R}\text{-MHS}}(\mathbb{R}, H^2(X)_{\text{ind}} \otimes \mathbb{R}(1))^{F_{\text{ind}} = 1} \cong \text{Coker}(\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^4)$.  

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Since \( s = (l - 1)/2 = 3 \) and \( \det H^3_{\text{dR}}(X/\mathbb{Q})_{\text{ind}} \otimes [\det H^3_{\text{dR}}(X_{\mathbb{C}})_{\text{ind}}]^{-1} = \sqrt{-7} \), one has

\[
\overline{\text{reg}}_{\text{gr}}(\xi_{D_1}) = \frac{\sqrt{7}}{\pi^3} \cdot \pi^3 \begin{vmatrix}
i(\zeta - \zeta^{-1})I(1) & i(\zeta^2 - \zeta^{-2})I(1) & i(\zeta^3 - \zeta^{-3})I(1) & J(1) \\
i(\zeta^2 - \zeta^{-2})I(2) & i(\zeta^4 - \zeta^{-4})I(2) & i(\zeta^6 - \zeta^{-6})I(2) & J(2) \\
i(\zeta^3 - \zeta^{-3})I(3) & i(\zeta^6 - \zeta^{-6})I(3) & i(\zeta^9 - \zeta^{-9})I(3) & J(3) \\
i(\zeta^4 - \zeta^{-4})I(4) & i(\zeta^8 - \zeta^{-8})I(4) & i(\zeta^{12} - \zeta^{-12})I(4) & J(4) \\
\end{vmatrix}
\]

\[
= 49I(1)I(2)I(3)I(4) \left( \frac{J(3)}{I(3)} + \frac{J(4)}{I(4)} \right)
\]

\[
= 0.629487860860585 \mod \mathbb{Q}^\times (\zeta := \exp(2\pi i/7))
\]

with respect to the canonical \( \mathbb{Q} \)-structure \( e_{\text{ind}, \mathbb{Q}} \).

**Remark 5.7** According to the Beilinson conjecture, \( \overline{\text{reg}}_{\text{gr}}(\xi_{D_1}) \) in Example 5.5 or 5.6 is expected to be the value of the \( L \)-function \( \Lambda(h^2(X)_{\text{ind}}, s) \) at \( s = 1 \) (\( \mathbb{S} \)).

## 6 Appendix : Gauss-Manin connection

Let \( R \) be an integral domain of characteristic 0 in which 6 is invertible. For a smooth scheme \( Y \) over \( T \), we denote by \( \Omega^q_{Y/T} = \wedge^q \Omega^1_{Y/T} \) the sheaf of relative differential \( q \)-forms on \( Y \) over \( T \). If \( T = \text{Spec} R \), we simply write \( \Omega^q_Y = \Omega^q_{Y/R} \).

### 6.1 Explicit formulas

Let \( S \) be an irreducible affine smooth scheme over \( R \) of relative dimension one. Let \( g_2, g_3 \in \mathcal{O}_S(S) = \Gamma(S, \mathcal{O}_S) \) satisfy \( \Delta := g_2^3 - 27g_3^2 \in \mathcal{O}_S(S)^\times \). Let \( f : U \to S \) be a projective smooth family of elliptic curves whose affine form is given by a Weierstrass equation \( y^2 = 4x^3 - g_2x - g_3 \). More precisely letting

\[
U_0 = \text{Spec} \mathcal{O}_S(S)[x, y]/(y^2 - 4x^3 + g_2x + g_3),
\]

\[
U_\infty = \text{Spec} \mathcal{O}_S(S)[u, z]/(z - 4u^3 + g_2uz^2 + g_3z^3),
\]

\( U \) is obtained by gluing \( U_0 \) and \( U_\infty \) via identification \( u = x/y, \ z = 1/y \). Let \( e : S \to U \) be a section given by \( (u, z) = (0, 0) \). To describe the de Rham cohomology \( H^q_{\text{dR}}(U/S) := H^q(U, \Omega^q_{U/S}) \) we use the Cech complex. Write

\[
\check{\mathcal{C}}^0(\mathcal{F}) := \Gamma(U_0, \mathcal{F}) \oplus \Gamma(U_\infty, \mathcal{F}), \quad \check{\mathcal{C}}^1(\mathcal{F}) := \Gamma(U_0 \cap U_\infty, \mathcal{F})
\]

for a (Zariski) sheaf \( \mathcal{F} \). Then the double complex

\[
\begin{array}{ccc}
\check{\mathcal{C}}^0(\mathcal{O}_U) & d \rightarrow & \check{\mathcal{C}}^0(\Omega^1_{U/S}) \\
\downarrow \delta & & \downarrow \delta \\
\check{\mathcal{C}}^1(\mathcal{O}_U) & d \rightarrow & \check{\mathcal{C}}^1(\Omega^1_{U/S})
\end{array}
\]

\[
(x_0, x_\infty) \quad \text{(dx0, dx_\infty)}
\]

\[
x_0 - x_\infty
\]
gives rise to the total complex
\[ \hat{C}^\bullet(U/S) : \hat{C}^0(\mathcal{O}_U) \xrightarrow{\delta \times d} \hat{C}^1(\mathcal{O}_U) \rightarrow \hat{C}^1(\Omega^1_{U/S}) \rightarrow \cdots \]
of \( R \)-modules starting from degree 0, and the cohomology of it is the de Rham cohomology \( H^\bullet_{\text{dr}}(U/S) \):
\[ H^q_{\text{dr}}(U/S) = H^q(\hat{C}^\bullet(U/S)), \quad q \geq 0. \]
Elements of \( H^1_{\text{dr}}(U/S) \) are represented by cocycles \((f) \times (x_0, x_\infty)\) with \( df = x_0 - x_\infty \).

The purpose of Appendix is to write down the Gauss-Manin connection
\[ \nabla : H^1_{\text{dr}}(U/S) \rightarrow \Omega^1_S \otimes H^1_{\text{dr}}(U/S) \]
(we use the same symbol “\( \Omega^1_S \)” for \( \Gamma(S, \Omega^1_S) \) since it will be clear from the context which is meant). This is defined in the following way (cf. [H] Ch.III, §4). By applying \( Rf_* \) on an exact sequence
\[ 0 \rightarrow f^*\Omega^1_S \otimes \Omega^{\bullet-1}_{U/S} \rightarrow \Omega^\bullet_U \rightarrow \Omega^\bullet_{U/S} \rightarrow 0, \]
one has the connecting homomorphism \( R^1f_*\Omega^\bullet_{U/S} \rightarrow R^2f_*(f^*\Omega^1_S \otimes \Omega^{\bullet-1}_{U/S}) \cong \Omega^1_S \otimes R^2f_*(\Omega^{\bullet-1}_{U/S}) \).
By identifying \( R^2f_*(\Omega^{\bullet-1}_{U/S}) \) with \( R^1f_*\Omega^\bullet_{U/S} \), one gets the Gauss-Manin connection \( \nabla \). Here we should be careful about “sign” because the differential of the complex \( \Omega^{\bullet-1}_{U/S} \) is “\(-d\)”:
\[ \Omega^{\bullet-1}_{U/S} : \mathcal{O}_U \xrightarrow{-d} \Omega^1_{U/S} \]
where the first term is placed in degree 1. So we need to choose an isomorphism between \( R^qf_*\Omega^\bullet_{U/S} \) and \( R^{q+1}f_*\Omega^{\bullet-1}_{U/S} \) because the natural one is unique up to sign. Here we choose it by
\[ \begin{array}{ccc}
\mathcal{O}_U & \xrightarrow{d} & \Omega^1_{U/S} \\
-\text{id} & & \text{id} \\
\mathcal{O}_U & \xrightarrow{-d} & \Omega^1_{U/S}
\end{array} \]
Then \( \nabla \) satisfies the usual Leibniz rule
\[ \nabla(fe) = df \otimes e + f\nabla(e), \quad e \in H^1_{\text{dr}}(U/S), \ f \in \mathcal{O}(S). \]

**Theorem 6.1** Suppose that \( \Omega^1_S \) is a free \( \mathcal{O}_S \)-module with a base \( dt \in \Gamma(S, \Omega^1_S) \). For \( f \in \mathcal{O}_S(S) \), we define \( f' \in \mathcal{O}_S(S) \) by \( df = f'dt \). Let
\[ \omega := (0) \times \left( \frac{dx}{y}, \frac{dy}{y} \right) \]
(6.1)
\[ \omega^* := \left( \frac{x^2}{y} \right) \times \left( \frac{2x^2 + 3x}{2y^3} \right) \] (6.2)

be elements in \( H^1_{\text{dr}}(U/S) \). Then we have

\[ \nabla(\omega) = \left( \frac{6g_2g_3' - 9g_2g_3}{\Delta} dt \otimes \omega^* - \frac{\Delta'}{12\Delta} dt \otimes \omega \right) \in \Omega^1_S \otimes H^1_{\text{dr}}(U/S), \] (6.3)

\[ \nabla(\omega^*) = \left( \frac{\Delta'}{12\Delta} dt \otimes \omega^* - \frac{g_2(2g_2g_3' - 3g_2g_3)}{16\Delta} dt \otimes \omega \right) \in \Omega^1_S \otimes H^1_{\text{dr}}(U/S). \] (6.4)

Note that \( \omega \) and \( \omega^* \) are basis of the free \( \mathcal{O}(S) \)-module \( H^1_{\text{dr}}(U/S) \):

\[ H^1_{\text{dr}}(U/S) = \mathcal{O}(S)\omega \oplus \mathcal{O}(S)\omega^*. \]

The following is straightforward from Thm. 6.1.

**Corollary 6.2** Put \( \mathcal{H} := R^1 f_* \Omega^\bullet_{U/S} \) and

\[ \mathcal{H}^{1,0} := f_* \Omega^1_{U/S} = \mathcal{O}S \omega, \quad \mathcal{H}^{0,1} := \mathcal{H} / \mathcal{H}^{1,0} \cong \mathcal{O}\omega^*. \]

Then the \( \mathcal{O}(S) \)-linear map

\[ \nabla : \mathcal{H}^{1,0} \rightarrow \Omega^1_S \otimes \mathcal{H}^{0,1} \] (6.5)

induced from the Gauss-Manin connection \( \nabla \) is described as follows.

\[ \nabla(\omega) = \frac{6g_2g_3' - 9g_2g_3}{\Delta} dt \otimes \omega^*. \]

In particular, noting

\[ \frac{j'}{j} = 27 \cdot \frac{g_3}{g_2} \cdot \frac{2g_2g_3' - 3g_2g_3}{\Delta}, \quad j := \frac{1728g_2^3}{g_2^3 - 27g_3^3}, \]

(6.5) is bijective if and only if

\[ \frac{g_2 d j}{g_3 j} \in \Omega^1_S \]

is a base of \( \mathcal{O}(S) \)-module.

Let us consider a diagram

\[ \begin{array}{ccc}
0 & \rightarrow & \Omega^1_S \otimes \mathcal{H}^{1,0} \\
\downarrow & & \downarrow \\
\mathcal{H} & \xrightarrow{\nabla} & \Omega^1_S \otimes \mathcal{H}^{0,1} \\
\downarrow & & \downarrow \\
\mathcal{H} & \xrightarrow{\tilde{\nabla}} & 0
\end{array} \] (6.6)
Let $S^o \subset S$ be a Zariski open set such that $\nabla$ is bijective on $S^o$. Then it gives rise to an exact sequence
\[
\text{Ker} \tilde{\nabla}|_{S^o} \to \Omega^1_{S^o} \otimes \mathcal{H}^{0,1}|_{S^o} \to \text{Coker} \tilde{\nabla}|_{S^o} \to 0.
\] (6.7)

Since the natural map $\text{Ker} \tilde{\nabla}|_{S^o} \to H^0(S^o, \Omega^1_{S^o})$ is bijective, we have an exact sequence
\[
0 \to \Gamma(S^o, \mathcal{H}^{0,1}) \xrightarrow{\text{PF}} \Gamma(S^o, \Omega^1_{S^o} \otimes H^1, \mathcal{H}^{0,1}) \xrightarrow{\text{Coker} \tilde{\nabla}|_{S^o}} 0 \to H^1_{dR}(S^o, \mathcal{H}).
\] (6.8)

The map $\text{PF}$ in (6.8) is called the Picard-Fuchs operator.

**Corollary 6.3** Suppose that $\Omega^1_{S^o}$ is a free $\mathcal{O}_{S^o}$-module with a base $dt \in \Gamma(S^o, \Omega^1_{S^o})$. Write $f'dt = df$ for $f \in \mathcal{O}(S^o)$. Put
\[
A := -\Delta \frac{g_2 g_3'}{6g_2 g_3'} - \frac{9}{g_2 g_3'}
\]
\[
B := \frac{1}{48} \left( \frac{g_2 (g_2')^2 - 12(g_3')^2}{2g_2 g_3' - 3g_2 g_3'} - \frac{4\Delta'}{3(2g_2 g_3' - 3g_2 g_3')} \right) .
\]

Then the Picard-Fuchs operator is described as follows.
\[
\text{PF}(f(t)\omega^*) = (f''A + f'(t)A' + fB)dt \otimes \omega, \quad f \in \mathcal{O}(S^o)
\]

**Proof.** Let
\[
z := f(t)\omega^* - \frac{\Delta}{6g_2 g_3' - 9g_2 g_3'}(f'(t) + \frac{\Delta'}{12\Delta} f(t))\omega \in \Gamma(S^o, \mathcal{H}).
\]
This belongs to the kernel of $\tilde{\nabla}$ by (6.3) and (6.4). Then
\[
\text{PF}(f(t)\omega^*) = \nabla(z)
\]
and apply (6.3) and (6.4) again to the RHS. □

### 6.2 Proof of Theorem 6.1

**Lemma 6.4**
\[
\left(\frac{x^i}{y^j}\right) \times (0, 0) \equiv (0) \times (0, d(x^i/y^j)) \quad (0 \leq i \leq j)
\]
\[
(x^i y^j) \times (0, 0) \equiv (0) \times (-d(x^i y^j), 0) \quad (i, j \geq 0)
\]
in $\check{\mathcal{C}}^1(\mathcal{O}_U) \times \check{\mathcal{C}}^0(\Omega^1_{U/S})$ where “$\equiv$” denote modulo $\text{Im}\check{\mathcal{C}}^0(\mathcal{O}_U)$.

**Proof.** Straightforward from the definition. □
Lemma 6.5 Let \( \eta_U \) be the generic point of \( U \). We think of \( dx \) and \( dt \) as elements in \( \Gamma(\eta_U, \Omega^1_U) \). Then
\[
\frac{dx}{y} \in \Gamma(U_\infty, \Omega^1_U),
\]
\[
\hat{\frac{dx}{y}} := \frac{dx}{y} - \frac{(6g_2x^2 - 9g_3x - g_2^3)(g_2'x + g_3')}{\Delta} dt \in \Gamma(U_0, \Omega^1_U).
\]

**Proof.**

\[
\frac{dx}{y} = du - \frac{u}{z}((12a^2 - g_2z^2)du - (2g_2uz + 3g_3z^2)dz - (g_2'uz^2 + g_3'z^3)dt)
\]
\[
= -3(1 + g_2uz + g_3z^2)du
\]
\[
\equiv 0 \mod \Gamma(U_\infty, \Omega^1_U).
\]

Hence (6.9) follows. Next we show (6.10). Since \( f(x) = 4x^3 - g_2x - g_3 \) is prime to \( f'(x) = 12x^2 - g_2 \), there are \( a(x) \) and \( b(x) \) such that
\[
a(x)f(x) + b(x)f'(x) = 1.
\]

Explicitly, they are given as follows.

\[
a(x) = \frac{9(3g_3 - 2g_2x)}{\Delta}, \quad b(x) = \frac{6g_2x^2 - 9g_3x - g_2^3}{\Delta}.
\]

Now

\[
\frac{dx}{y} = a(x)f(x)dx + b(x)f'(x)dx
\]
\[
= \frac{a(x)y^2dx + b(x)(2ydy + (g_2'x + g_3')dt)}{y}
\]
\[
= a(x)ydx + 2b(x)dy + \frac{b(x)(g_2'x + g_3')}{y}dt \in \Gamma(U_0 \cap U_\infty, \Omega^1_U)
\]

and hence we have
\[
\frac{\hat{dx}}{y} = \frac{dx}{y} - \frac{b(x)(g_2'x + g_3')}{y}dt = a(x)ydx + 2b(x)dy \in \Gamma(U_0, \Omega^1_U).
\]

Let us prove (6.3). Let
\[
\tilde{\omega} := (0) \times \left( \frac{\hat{dx}}{y}, \frac{dx}{y} \right) \in \tilde{C}^1(\mathcal{O}_U) \times \tilde{C}^0(\Omega^1_U)
\]

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be a lifting of \( \omega \) where \( \widehat{\frac{dx}{y}} \) is as in (6.10). Let

\[
\begin{array}{c}
\tilde{C}^0(\mathcal{O}_U) \xrightarrow{d} \tilde{C}^0(\Omega_U^1) \xrightarrow{\delta} \tilde{C}^0(\Omega_U^2) \\
\tilde{C}^1(\mathcal{O}_U) \xrightarrow{d} \tilde{C}^1(\Omega_U^1) \xrightarrow{\delta} \tilde{C}^1(\Omega_U^2)
\end{array}
\]

be the double complex and

\[
\begin{array}{c}
\tilde{C}^0(\mathcal{O}_U) \xrightarrow{\delta \times d} \tilde{C}^1(\mathcal{O}_U) \xrightarrow{d} \tilde{C}^0(\Omega_U^1) \\
\tilde{C}^1(\mathcal{O}_U) \xrightarrow{d} \tilde{C}^1(\Omega_U^1) \xrightarrow{\delta \times d} \tilde{C}^1(\Omega_U^2)
\end{array}
\]

\( \mathcal{D} : (\alpha, \beta) \mapsto (-d\alpha + \delta(\beta), d\beta) \)

the associated total complex. It gives the de Rham cohomology \( H^\bullet_{dR}(U) \) together with a natural map

\[
\Omega^1_S \otimes H^1_{dR}(U/S) \longrightarrow H^2_{dR}(U), \quad dt \otimes [(f) \times (z_0, z_\infty)] \mapsto (-f dt) \times (dt \wedge z_0, dt \wedge z_\infty).
\]

Now

\[
\mathcal{D} : \widehat{\omega} \mapsto \left( \frac{dx}{y} - \frac{dy}{y} \right) \times \left( d \left( \frac{dx}{y} \right), \frac{dx dy}{y^2} \right)
\]

\[
= \left( -\frac{(6g_2x^2 - 9g_3x - g_3^2)(g_2'x + g_3') dt}{y^2} \right) \times \left( d \left( \frac{dx}{y} \right), \frac{dx dy}{y^2} \right)
\]

\[
= (F dt) \times (G_1 \frac{dt dx}{y}, G_2 \frac{dt dx}{y}) \in \tilde{C}^1(\Omega_U^1) \times \tilde{C}^0(\Omega_U^2)
\]

where

\[
F := \frac{-(6g_2g_3' - 9g_2g_3)(6x^2 - g_2) - \Delta' x - 9g_2g_3' y^2}{6y \Delta},
\]

\[
G_1 := \frac{18g_2g_3'^2 y^2 + (6g_2g_3' - 9g_2g_3)x - 2g_2g_2'g_3' + 9g_3g_3'}{2 \Delta}, \quad G_2 := \frac{g_2' x + g_3'}{2 y^2}.
\]

This means

\[
\nabla(\omega) = dt \otimes \left( -F \right) \times \left( G_1 \frac{dx}{y}, G_2 \frac{dx}{y} \right) \in \Omega^3_S \otimes H^3_{dR}(U/S).
\]

By Lemma 6.4 we get

\[
dt \otimes \left( -F \right) \times \left( G_1 \frac{dx}{y}, G_2 \frac{dx}{y} \right) \equiv dt \otimes \left( -\bar{F} \right) \times \left( \bar{G}_1 \frac{dx}{y}, \bar{G}_2 \frac{dx}{y} \right) \mod \text{Im}\tilde{C}^0(\mathcal{O}_U)
\]

where

\[
\bar{F} = \frac{6g_2g_3' - 9g_2g_3 x^2}{\Delta}
\]
\[ \bar{G}_1 = -\frac{\Delta'}{12\Delta} + \frac{6g_2'g_3 - 9g_2'g_3 x}{2}, \quad \bar{G}_2 = -\frac{\Delta'}{12\Delta} + \frac{6g_2'g_3 - 9g_2'g_3 2g_2 x^2 + 3g_3 x}{2y} \]

and the RHS is equal to
\[ \frac{6g_2'g_3 - 9g_2'g_3 dt \otimes \omega^* - \frac{\Delta'}{12\Delta} dt \otimes \omega}. \]

This completes the proof of (6.3).

Next we show (6.4). The proof goes in the same way as above. Let \( \bar{\omega}^* := \left( \frac{x^2}{y} \right) \times \left( \frac{x dx}{2y} \frac{(2g_2 x^2 + 3g_3 x) dx}{2y^3} \right) \in \tilde{C}^1(\mathcal{O}_U) \times \tilde{C}^0(\Omega^1_U) \) be a lifting of \( \omega^* \). Then
\[
\mathcal{D}(\bar{\omega}^*) = \left( -d\left( \frac{x^2}{y} \right) + \frac{x dx}{2y} - \frac{2g_2 x^2 + 3g_3 x) dx}{2y^3} \right) \times \left( d \left( \frac{x dx}{2y} \right), d \left( \frac{2g_2 x^2 + 3g_3 x) dx}{2y^3} \right) \right) \\
= (F dt) \times (G_1 \frac{dt dx}{y}, G_2 \frac{dt dx}{y}) \in \tilde{C}^1(\Omega^1_U) \times \tilde{C}^0(\Omega^2_U)
\]

where
\[
h_1 = -(g_2 x + 3g_3)(2g_2 g_3' - 3g_2' g_3), \quad h_2 = -4x^2(g_2' x + g_3') \Delta \\
F = \frac{1}{8\Delta} \left( -6g_2'g_3 - 9g_2'g_3 + 6g_2'g_3 x + \frac{h_1}{y} + \frac{h_2}{y^3} \right) - \frac{\Delta'}{12\Delta y} x^2 \\
G_1 = \frac{30g_2'g_3 - (4g_2'g_3 + 9g_3 g_3') x + 9(2g_2 g_3' - 3g_2' g_3) x^2 - 2g_2 g_3' x}{y^3} \\
G_2 = \frac{6g_2'g_3 - 4g_2'g_3 x^2 + (9g_3 g_3' + 6g_2' g_3 x^2 + 6g_2' x y^2 + 9g_3 g_3' x}{4y^4}
\]

This means
\[ \nabla (\omega^*) = dt \otimes \left( -F \times (G_1 \frac{dx}{y}, G_2 \frac{dx}{y}) \right) \in \Omega^1_5 \otimes H^0_{dr}(U/S). \]

By Lemma 6.4 again, we get
\[ dt \otimes \left( -F \times (G_1 \frac{dx}{y}, G_2 \frac{dx}{y}) \right) \equiv dt \otimes \left( -\tilde{F} \times (\tilde{G}_1 \frac{dx}{y}, \tilde{G}_2 \frac{dx}{y}) \right) \quad \text{mod Im} \tilde{C}^0(\mathcal{O}_U) \]
\[ = \left( \frac{\Delta'}{12\Delta} dt \otimes \omega^* - \frac{g_2(2g_2 g_3' - 3g_2' g_3)}{16\Delta} dt \otimes \omega \right) \]

where
\[
\tilde{F} = -\frac{\Delta'}{12\Delta} x^2, \quad \tilde{G}_1 = \frac{\Delta'}{12\Delta} x - \frac{g_2(2g_2 g_3' - 3g_2' g_3)}{16\Delta} \\
\tilde{G}_2 = \frac{\Delta'}{12\Delta} \frac{2g_2 x^2 + 3g_3 x}{2y^2} - \frac{g_2(2g_2 g_3' - 3g_2' g_3)}{16\Delta}
\]

This completes the proof of (6.4). QED.

The above computation shows the following.
Proposition 6.6 Let
\[
\left( \frac{dx}{y} \right)_0 := \frac{\Delta x}{y} + \frac{3}{2} g_2 y \frac{dt}{\Delta} = \frac{dx}{y} - \frac{(6x^2 - g_2)(6g_2 g'_3 - 9g'_2 g_3) + \Delta' x dt}{6y}
\]
\[
\left( \frac{dx}{y} \right)_{\infty} := \frac{dx}{y} - \frac{g_2(6g_2 g'_3 - 9g'_2 g_3) x + \Delta' x dt}{6y}
\]
and
\[
\tilde{\omega} := (0) \times \left( \left( \frac{dx}{y} \right)_0, \left( \frac{dx}{y} \right)_{\infty} \right) \in \tilde{C}^1(\mathcal{O}_U) \times \tilde{C}^0(\Omega^1_U).
\]
Let
\[
\tilde{\omega}^* = \left( \frac{x^2}{y} \right) \times \left( \frac{x}{2} \frac{dx}{y} + F_1 dt, \frac{2g_2 x^2 + 3g_3 x}{2y^3} + F_2 dt \right) \in \tilde{C}^1(\mathcal{O}_U) \times \tilde{C}^0(\Omega^1_U).
\]
where
\[
F_1 = -\frac{\Delta' x^2}{12 \Delta y} - \frac{(2g_2 g'_3 - 3g'_2 g_3)(g_2 x + 3g_3)}{8 \Delta y},
\]
\[
F_2 = -\frac{x^2(g'_2 x + g'_3)}{2y^3} - \frac{(2g_2 g'_3 - 3g'_2 g_3)(g_2 x + 3g_3)}{8 \Delta y}.
\]
Let \(\mathcal{D} : \tilde{C}^1(\mathcal{O}_U) \times \tilde{C}^0(\Omega^1_U) \to \tilde{C}^1(\Omega^1_U) \times \tilde{C}^0(\Omega^2_U)\) be as before. Then
\[
\mathcal{D} \tilde{\omega} = \frac{6g_2 g'_3 - 9g'_2 g_3}{\Delta} (dt \otimes \omega^*)' - \frac{\Delta'}{12 \Delta} (dt \otimes \omega)' \tag{6.11}
\]
\[
\mathcal{D} \tilde{\omega}^* = \frac{\Delta'}{12 \Delta} (dt \otimes \omega)^* - \frac{g_2(2g_2 g'_3 - 3g'_2 g_3)}{16 \Delta} (dt \otimes \omega)' \tag{6.12}
\]
\[
\mathcal{D} \left( f(t) \left[ \tilde{\omega}^* - \frac{\Delta'}{36(2g_2 g'_3 - 3g'_2 g_3)} \tilde{\omega} \right] - \frac{f'(t) \Delta}{6g_2 g'_3 - 9g'_2 g_3} \tilde{\omega} \right) = (f'' A + f'A' + f B)(dt \otimes \omega)'. \tag{6.13}
\]
Here \(A, B\) are as in Cor. 6.3 and we denote
\[
(dt \otimes \omega)^* = \left( \frac{x^2}{y} dt \right) \times \left( \frac{x dt dx}{2y}, \frac{2g_2 x^2 + 3g_3 x}{2y^3} \right)
\]
\[
(dt \otimes \omega)' = (0) \times \left( \frac{dt dx}{y}, \frac{dt dx}{y} \right)
\]

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