Semiclassical almost isometry

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Abstract

Let \((M, \omega, J)\) be a compact and connected polarized Hodge manifold, \(\mathcal{S}\) an isodrastic leaf of half-weighted Bohr-Sommerfeld Lagrangian submanifolds. We study the relation between the Weinstein symplectic structure of \(\mathcal{S}\) and the asymptotics of the pull-back of the Fubini-Study form under the projectivization of the so-called BPU maps on \(\mathcal{S}\).

Keywords: Hodge manifold, Bohr-Sommerfeld Lagrangian submanifolds, isodrastic leaf, BPU map, Fourier integral operator, asymptotic expansions

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1 Introduction

Let \((M,J)\) be an irreducible \(n\)-dimensional complex projective manifold, \(A \to M\) an ample line bundle, \(h\) an Hermitian metric on \(A\) such that the curvature of the unique compatible covariant derivative is \(-2\pi i\omega\), where \(\omega\) is a Kähler form on \(M\). By the Tian-Zelditch almost isometry theorem [11], [13], the projective embeddings \(\varphi_k =: \varphi_{A \otimes k} : M \to \mathbb{P}(H^0(M, A^\otimes k)^*)\) are asymptotically symplectic as \(k \to +\infty\), in an appropriate rescaled sense. Thus the symplectic structure of the classical phase space \((M, \omega)\) is encapsulated in the asymptotics of its quantizations \(H^0(M, A^\otimes k)\). However, in light of the uncertainty principle and of the WKB method, the geometric objects most naturally associated to quantum physical states (the true points of phase space) are Lagrangian submanifolds of \((M, \omega)\) [1], [6], [7], [12]. Motivating the study of quantization and reduction, this point of view led to Weinstein’s discovery of a natural symplectic structure on isodrastic leaves of weighted Lagrangian submanifolds [12]. One may then ask whether almost isometry still holds when \((M, \omega)\) is replaced by an isodrastic leaf of compact and connected half-weighted Bohr-Sommerfeld Lagrangian submanifolds, endowed with a closed 2-form \(\Omega\) of Weinstein type, and the \(\varphi_k\)’s by their semiclassical analogues taking value in \(\mathbb{P}H^0(M, A^\otimes k) \cong \mathbb{P}N_k\), and denoted \(\Phi_k\) below; these are the (projectivisation of the) maps introduced in [2] (and called BPU maps in [5]).
Let $\mathcal{S}$ be the manifold of all half-weighted Bohr-Sommerfeld Lagrangian submanifolds $(L, \lambda)$ of $(M, \omega)$ such that $L$ is isodraistically equivalent to a given $L_0$ \( \Theta \mathcal{S} \). Besides $\Omega$, the isodraastic leaf $\mathcal{S}$ also carries a natural semidefinite Riemannian metric $G$. In fact, $G$ and $\Omega$ are non-degenerate and compatible, hence define an almost Kähler structure, on the open subset $\mathcal{S}' \subseteq \mathcal{S}$ of all pairs $(L, \lambda) \in \mathcal{S}$ with $\lambda$ nowhere vanishing on $L$. Given the Kähler structure $(M, J, \omega)$, the smooth tangent space $T_{(L, \lambda)} \mathcal{S}$ to $\mathcal{S}$ at $(L, \lambda)$ is naturally isomorphic to the space of pairs $(f, \ell)$, where $f \in C^\infty(L)$ and $\ell$ is a $C^\infty$ half-density on $L$, satisfying $\int_L f \lambda \cdot \lambda = \int_L \ell \cdot \lambda = 0$ (statement i) of Proposition 2.2. Let $W(f, \ell)$ be the tangent vector associated to a pair $(f, \ell)$. For every $k \gg 0$ suitably divisible, let $\Phi_k : U_k \rightarrow \mathbb{P}^\infty$ be the $k$-th projectivized BPU map (\[ \mathcal{S} \]); $U_k \subseteq \mathcal{S}$ is an appropriate open subset, and $(L, \lambda) \in U_k$ for any $(L, \lambda) \in \mathcal{S}$ and for all suitably divisible $k \gg 0$. Thus, $\mathcal{S} = \bigcup_k U_k$.

In Weinstein’s setting, almost isometry does not hold literally (if anything because an isodraastic leaf is infinite dimensional). Nonetheless, $\Omega$ and $G$ can be extracted from the asymptotics of BPU maps:

**Theorem 1.** Let $\mathcal{S}$ be an isodraastic leaf of half-weighted compact and connected Bohr-Sommerfeld Lagrangian submanifolds of $(M, \omega)$. For every $(L, \lambda) \in \mathcal{S}$ and $W(f, \ell), W(f', \ell') \in T_{(L, \lambda)} \mathcal{S}$, the following asymptotic expansions hold as $k = lr$ and $l \rightarrow +\infty$:

$$
\Phi_k^*(\omega_{FS}(L, \lambda))(W(f, k\ell), W(f', k\ell')) \sim k^2 \cdot \Omega_{(L, \lambda)} \left( W(f, l), W(f', \ell) \right) + \sum_{h \geq 1} b_h(L, \lambda, f, \ell, f', \ell') k^{2-h/2},
$$

$$
\Phi_k^*(g_{FS}(L, \lambda))(W(f, k\ell), W(f', k\ell')) \sim k^2 \cdot G_{(L, \lambda)} \left( W(f, l), W(f', \ell) \right) + \sum_{h \geq 1} c_h(L, \lambda, f, \ell, f', \ell') k^{2-h/2}.
$$

Here, $r \in \mathbb{N}$ is the an invariant of $\mathcal{S}$ given by order of the image in $S^1$ of the holonomy representation of $L$ for the unit circle bundle $X \subseteq A^*$. If $s \geq 1$, $S\mathcal{S} \subseteq T\mathcal{S}$ is the unit sphere bundle (for any given smooth metric), and $K \subseteq S\mathcal{S}$ is the image of a smooth map from a compact subset of $\mathbb{R}^s$, these asymptotic expansions are uniform on the set of pairs of tangent vectors multiples of elements of $K$.

We hint at two possible extensions: to the category of compact almost Kähler manifolds, on the hand, in view of the microlocal description of almost complex Szegö kernels in \[ \mathcal{S} \], and on the other allowing $L$ to be open, but requiring the half-weights to be compactly supported. We shall leave these generalizations to the interested reader.

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2 Preliminaries

We shall denote by $\mathcal{D}^\infty(Z)$ (resp., $\mathcal{D}_{(1/2)}^\infty(Z)$) the space of all $\mathcal{C}^\infty$ real-valued densities (resp., half-densities) on a manifold $Z$. There is a natural commutative product $\bullet : \mathcal{D}_{(1/2)}^\infty(Z) \otimes \mathcal{D}_{(1/2)}^\infty(Z) \to \mathcal{D}^\infty(Z)$, $\lambda \otimes \eta \mapsto \lambda \bullet \eta$, given by pointwise multiplication of functions on frame bundles. All densities and half-densities will be understood to be real-valued. Given a Riemannian structure on $Z$, $\text{dens}_Z$ (resp., $\text{dens}^{(1/2)}_Z$) will denote the corresponding volume density (resp., half-density).

2.1 Weighted Lagrangian and Planckian submanifolds

The space $\mathbf{L} = \mathbf{L}(M,\omega)$ of all compact and connected Lagrangian submanifolds of $(M,\omega)$ is an infinite-dimensional manifold. The (smooth) tangent space of $\mathbf{L}$ at any $L \in \mathbf{L}$ is $T_L \mathbf{L} = \mathcal{Z}^1(L)$, the vector space of all closed 1-forms on $L$. Furthermore, $\mathbf{L}$ carries a natural integrable distribution $\mathfrak{B} \subseteq TL$, whose value at any $L \in \mathbf{L}$ is the subspace $\mathcal{B}^1(L) \subseteq \mathcal{Z}^1(L)$ of all exact 1-forms on $L$. $\mathfrak{B}$ is called the isodrastic distribution, and its leaves the isodrastic leaves of $\mathbf{L}$. Lagrangian submanifolds $L, L' \in \mathbf{L}$ belong to the same isodrastic leaf (in which case they are called isodrastically equivalent) if and only if $L'$ can be deformed into $L$ by flowing it along globally defined Hamiltonian vector fields [12]. A compact and connected weighted Lagrangian submanifold of $(M,\omega)$ is a pair $(L,\varrho)$, where $L \in \mathbf{L}$ and $\varrho \in \mathcal{D}^\infty(L)$ is a weight on $L$, that is, $\int_L \varrho = 1$. We shall denote by $\mathbf{WL} = \mathbf{WL}(M,\omega)$ the manifold of all such pairs. Given the natural forgetful projection, $p : \mathbf{WL} \to \mathbf{L}$, for any isodrastic leaf $\mathfrak{J} \subseteq \mathbf{L}$ set $W\mathfrak{J} = p^{-1}(\mathfrak{J})$ (really an immersed submanifold). It is the infinite dimensional manifold $W\mathfrak{J}$ that carries a built-in symplectic structure $\Omega_{\text{Wein}}$ (§3 of [12]).

Definition 2.1. Let $X \subseteq A^*$ be the unit circle bundle, with projection $\pi : X \to M$, so that the connection 1-form $\alpha$ on $X$ is a contact structure, and $d\alpha = \pi^\ast \omega$. A submanifold $P \subseteq X$ is Planckian if it is Legendrian and furthermore (by restriction of $\pi$) an unramified cover of a Lagrangian submanifold $L = \pi(P) \subseteq M$. $\mathbf{P} = \mathbf{P}(X,\alpha)$ will denote the collection of all compact and connected Planckian submanifolds of $(X,\alpha)$.

Definition 2.2. A submanifold $L \subseteq M$ is a Bohr-Sommerfeld Lagrangian submanifold (BSL for short) if $L = \pi(P)$ for some Planckian submanifold $P \subseteq X$. Let $\mathbf{L}_{\text{BS}} \subseteq \mathbf{L}$ be the subspace of all compact and connected BSL submanifolds.

Remark 2.1. Suppose $L \in \mathbf{L}$; then $L \in \mathbf{L}_{\text{BS}}$ if and only if the image of the holonomy representation $\pi_1(L) \to S^1$ for the principal $S^1$-bundle $X$ is a finite subgroup $\text{Hol}(L) \subseteq S^1$. If $P \in \mathbf{P}$ is such that $L = \pi(P)$, the projection $P \to L$ is an unramified cover of degree $r = |\text{Hol}(L)|$.

The property of being BSL is invariant under isodrastic deformations [12], hence $\mathbf{L}_{\text{BS}}$ is a union of isodrastic leaves. Define $\bar{\pi} : \mathbf{P} \to \mathbf{L}_{\text{BS}}$ by $\bar{\pi}(P) = : \pi(P)$. 

For any isodrastic leaf \( \mathcal{I} \subseteq \mathcal{L}_{BS} \), \( \mathbf{P}_3 =: \pi^{-1}(\mathcal{I}) \) is an infinite-dimensional manifold, and \( T_P(\mathbf{P}_3) \cong C^\infty(L) \) for any \( P \in \mathbf{P}_3 \), where \( L = \pi(P) \) \(^{[12]}\), Lemma 4.1). Furthermore, the image of the holonomy representation \( \pi_1(L) \to S^1 \) associated to the principal \( S^1 \)-bundle \( X \) is the same \( \forall L \in \mathcal{I} \); its cardinality equals the degree of the unramified cover \( P \to L =: \pi(P), \forall P \in \mathcal{I} \). Denote this image by \( \text{Hol}(\mathcal{I}) \subseteq S^1 \), and set \( G_\mathcal{I} =: S^1/\text{Hol}(\mathcal{I}) \cong S^1 \). Thus \( \mathbf{W}P_3 =: \mathbf{P}_3 \times G_\mathcal{I} \) consists of all pairs \((P, \varrho)\), where \( P \in \mathbf{P}_3 \) and \( \varrho \) is a weight on \( \pi(P) \in \mathcal{I} \). The projection \( \tilde{\pi} : \mathbf{W}P_3 \to \mathbf{W}\mathcal{I} \), given by \((P, \varrho) \mapsto (\pi(P), \varrho)\), is a principal \( G_\mathcal{I} \)-bundle, and has an intrinsic universal connection in the terminology of \(^{[12]}\); the normalized curvature of this connection is the symplectic structure \( \Omega_{\text{Wein}} \) on \( \mathbf{W}\mathcal{I} \) \(^{Proposition 4.3 of [12]}\). Heuristically, the circle bundle \( \tilde{\pi} : \mathbf{W}P_3 \to \mathbf{W}\mathcal{I} \) is a semiclassical analogue of the circle bundle \( \pi : X \to M \). In the present Kähler context the theory of \(^{[12]}\) implies that the tangent space \( T_{(L, \varrho)}\mathbf{W}\mathcal{I} \) has a simple intrinsic description. Pairing the proof of Theorem 3.32 of \(^{[9]}\) with that of Lemma 3.14 of \(^{[11]}\), yields the following:

**Lemma 2.1.** Let \( L \subseteq M \) be a Lagrangian submanifold, \( T^*L \) its cotangent bundle, with projection \( q : T^*L \to L \), and canonical symplectic structure \( \omega_{\text{can}} \). Let \( \mathcal{O}(L) =: \{(l, 0) : l \in L\} \subseteq T^*L \). Then there exist open neighborhoods \( L \subseteq U \subseteq M \) and \( \mathcal{O}(L) \subseteq V \subseteq T^*L \), and a natural choice of a symplectomorphism \( \gamma : U \to V \), such that \( \gamma(l) = l \), and the inverse image \( \gamma^{-1}\{(q^{-1}(l)) \subseteq U \) of \( q^{-1}(l) \subseteq T^*L \) is perpendicular to \( L \) at \( l \) (for the Kähler metric), \( \forall l \in L \).

We shall call \( \gamma \) the normal cotangent structure of \( M \) near \( L \), and denote the projection by \( \beta : U \to L \). The discussion surrounding equation (3) of \(^{[12]}\) implies:

**Proposition 2.1.** Given the (almost) Kähler structure \((M, \omega, J)\),

i): \( \forall (L, \varrho) \in \mathbf{W}\mathcal{I}, T_{(L, \varrho)}(\mathbf{W}\mathcal{I}) \) is naturally isomorphic to the vector space of all pairs \((f, \phi) \in C^\infty(L) \times \mathcal{D}^\infty(L) \) satisfying \( \int_L f \varrho = \int_L \phi = 0 \);

ii): \( \forall (P, \varrho) \in \mathbf{W}P_3, T_{(P, \varrho)}(\mathbf{W}P_3) \) is naturally isomorphic to the vector space of all pairs \((f, \phi) \in C^\infty(L) \times \mathcal{D}^\infty(L) \) satisfying \( \int_L f \phi = 0 \).

In i), \( f \) is implicitly extended by pull-back to \( U \) under \( \beta \), and thus defines a Hamiltonian vector field \( \nu_f \) on \( U \). The flow of \( \nu_f \) determines a path \( L_t, |t| < \epsilon \), of Lagrangian submanifolds with \( L_0 = L \). The condition on \( f \) is a renormalization that fixes it uniquely. Restricting \( \beta \) determines diffeomorphisms \( \beta_t : L \to L \). If \( \rho_t \) is a family of weights on \( L \) with \( \rho_0 = \varrho \), we obtain pull-back weights \( \varrho_t = \beta_t^*(-\rho_t) \) on \( L_t \). The first order datum at \( t = 0 \) corresponding to the family of weights \( \varrho_t \) is the density \( \phi = \rho_t(0) \) on \( L \), so that \( \rho_t = \varrho + t\cdot \phi + O(t^2) \); the condition on \( \phi \) results from differentiating the constraint \( \int_{L_t} \varrho_t = \int_{L_t} \rho_t = 1 \) at \( t = 0 \). If \((L, \varrho) \in \mathbf{W}\mathcal{I} \) and \( V_i = V(f_i, \phi_i) \in T_{(L, \varrho)}(\mathbf{W}\mathcal{I}), i = 1, 2 \), by (3) on page 139 of \(^{[12]}\) their Weinstein symplectic pairing is

\[
(\Omega_{\text{Wein}})_{(L, \varrho)}(V_1, V_2) = \int_L (f_1 \phi_2 - f_2 \phi_1). \tag{1}
\]
Definition 2.3. Suppose \( L \in \mathcal{L}, \ f \in \mathcal{C}^\infty(L) \). For sufficiently small \( t \), let \( \phi_t \) be the flow of \( \psi_f \), defined locally near \( L \), and \( L_t := \phi_t(L) \). Let \( \text{dens}^{(1/2)}_{L_t} \) be the Riemannian half-density on \( L_t \), \( \psi_t : L \to L_t \) the diffeomorphism \( m \mapsto \phi_t(m) \), and \( \gamma_t := \psi_t^*(\text{dens}^{(1/2)}_{L_t}) \). Thus \( \gamma_t = G_t \cdot \text{dens}^{(1/2)}_L \) for a unique \( G_t \in \mathcal{C}^\infty(L) \). Then \( \Gamma(L, f) := \frac{\partial G_{t}}{\partial t} |_{t=0} \in \mathcal{C}^\infty(L) \) depends linearly on \( f \).

2.2 Half-weighted Lagrangian submanifolds

If \( L \in \mathcal{L}, \ \lambda \in \mathcal{D}^{(1/2)}(L) \) is a half-weight if \( \lambda \ast \lambda \) is a weight on \( L \). Let \( \mathcal{W}_h \mathcal{L} = \mathcal{W}_h \mathcal{L}(M, \omega) \) be the space of all pairs \((M, \omega)\), and by \( \mathcal{W}_h \mathcal{P} = \mathcal{W}_h \mathcal{P}(X, \alpha) \) the space of all pairs \((P, \lambda)\), where \( P \in \mathcal{P} \) and \( \lambda \) is a half-weight on \( \pi(P) \). Given an isodstratic leaf \( \mathcal{J} \subseteq \mathcal{L}_{\text{BS}} \), we have leaves \( \mathcal{W}_h \mathcal{J} \) and \( \mathcal{W}_h \mathcal{P}_3 \). The analogue of Proposition 2.1 is:

Proposition 2.2. Given the (almost) Kähler structure \((M, \omega, J)\),

\begin{itemize}
  \item[i)] \( \forall (L, \lambda) \in \mathcal{W}_h \mathcal{J}, \ T_{(L, \lambda)}(\mathcal{W}_h \mathcal{J}) \) is naturally isomorphic to the vector space of all pairs \((f, \ell) \in \mathcal{C}^\infty(L) \times \mathcal{D}^\infty(L) \) satisfying \( \int_L f \lambda \ast \lambda = \int_L \ell \ast \lambda = 0 \);
  \item[ii)] \( \forall (P, \lambda) \in \mathcal{W}_h \mathcal{P}_3, \ T_{(P, \lambda)}(\mathcal{W}_h \mathcal{P}_3) \) is naturally isomorphic to the vector space of all pairs \((f, \ell) \in \mathcal{C}^\infty(L) \times \mathcal{D}^\infty(L) \) satisfying \( \int_L \ell \ast \lambda = 0 \).
\end{itemize}

Definition 2.4. If \((f, \ell) \in \mathcal{C}^\infty(L) \times \mathcal{D}^\infty(L) \) satisfy \( \int_L f \lambda \ast \lambda = \int_L \ell \ast \lambda = 0 \), let \( W(f, \ell) \in T_{(L, \lambda)}(\mathcal{W}_h \mathcal{J}) \) be the tangent vector associated to \((f, \ell)\). Similarly, if \((f, \ell) \in \mathcal{C}^\infty(L) \times \mathcal{D}^\infty(L) \) satisfy \( \int_L \ell \ast \lambda = 0 \), let \( \tilde{W}(f, \ell) \in T_{(P, \lambda)}(\mathcal{W}_h \mathcal{P}_3) \) be the tangent vector associated to \((f, \ell)\). If \( P \in \mathcal{P}_3 \) is Planckian and \( L = \pi(P) \) any \( W(f, \ell) \in T_{(L, \lambda)}(\mathcal{W}_h \mathcal{J}) \) lifts to \( \tilde{W}(f, \ell) \in T_{(P, \lambda)}(\mathcal{W}_h \mathcal{P}_3) \).

Define \( \Psi : \mathcal{W}_h \mathcal{J} \to \mathcal{W}_3 \) by \((L, \lambda) \mapsto (L, \lambda \ast \lambda)\). The differential \( d_{(L, \lambda)} \Psi : T_{(L, \lambda)}(\mathcal{W}_h \mathcal{J}) \to T_{(L, \lambda \ast \lambda)}(\mathcal{W}_3) \) is given by \( d_{(L, \lambda)} \Psi((f, \ell)) = (f, 2\ell \ast \lambda) \). Clearly, \( \mathcal{W}_h \mathcal{P}_3 \) is the principal \( S^1 \)-bundle on \( \mathcal{W}_h \mathcal{J} \) obtained by pulling back \( \bar{\pi} : \mathcal{W}_3 \to \mathcal{W}_3 \) by \( \Psi \). We shall also write \( \bar{\pi} \) the projection \( \mathcal{W}_h \mathcal{P}_3 \to \mathcal{W}_h \mathcal{J} \). Consider the closed 2-form \( \Omega = : \Psi^* (\Omega_{\text{Wein}}) \) on \( \mathcal{W}_h \mathcal{J} \). If \((L, \lambda) \in \mathcal{W}_h \mathcal{J}, W = W(f, \ell), W' = W(f', \ell') \in T_{(L, \lambda)}(\mathcal{W}_h \mathcal{J})\), we have:

\begin{equation}
\Omega_{(L, \lambda)}(W, W') = 2 \int_L (f \cdot \ell' - f' \cdot \ell) \ast \lambda.
\end{equation}

Now on \( \mathcal{W}_h \mathcal{J} \) we also have a positive semi-definite Riemannian structure, given by

\begin{equation}
G_{(L, \lambda)}(W, W') = 2 \int_L (f f' \ast \lambda \ast \ell + \ell \ast \ell').
\end{equation}

Endowed with \( \Omega \) and \( G \), the open subset \( \mathcal{W}_h' \mathcal{J} \subseteq \mathcal{W}_h \mathcal{J} \) of pairs \((L, \lambda) \in \mathcal{W}_h \mathcal{J} \) such that \( \lambda \) is nowhere vanishing is an infinite-dimensional almost Kähler manifold. In fact, \( \Omega \) and \( G \) are non-degenerate on \( \mathcal{W}_h' \mathcal{J} \), and related by the almost complex structure \( J(W(f, g \lambda)) = W(-g, f \lambda) \).
2.3 Good coordinates along a Legendrian submanifold

For \( r \in \mathbb{N} \) and \( \epsilon > 0 \), let \( B^r_\epsilon \subseteq \mathbb{R}^r \) be the open ball of radius \( \epsilon \) centered at the origin. Let \((p,q,\theta)\) be the standard linear coordinates on \( \mathbb{R}^{2n+1} \cong \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \); if \( \psi : B^{2n+1}_\epsilon \to V =: \psi(B^{2n+1}_\epsilon) \subseteq X \) is a local chart, \((p,q,\theta) : V \to \mathbb{R}^{2n+1} \) will also denote the induced local coordinates, and \( \frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial \theta} \) the corresponding vector fields on \( V \).

The compatible connection defines a direct sum decomposition \( TX = \text{Hor}(X) \oplus \text{Ver}(X) \), where \( \text{Hor}(X) = \ker(\alpha), \text{Ver}(X) = \ker(d\pi) \). To define a Riemannian structure \( g_X \) on \( X \), we declare this to be an orthogonal direct sum, take the pull-back of the Riemannian structure on \( M \) as a metric on \( \text{Hor}(X) \), and require the generator of the \( S^1 \)-action to have norm \( \frac{1}{2\pi} \). The \( S^1 \)-orbits have unit length for \( g_X \), and for the corresponding Riemannian density \( \text{dens}_X \), the natural isomorphism \( H^{0}(M, \Lambda^{\otimes k}) \cong H(X)_k \subseteq L^2(X) \) is unitary; here \( H(X)_k \) is the level \( k \) Hardy space of \( X \). Given \( x \in X \), let \( || \cdot ||_x \) be associated the norm on \( T_x X \).

**Proposition 2.3.** Let \( P \subseteq X \) be a Legendrian submanifold. For any \( x \in P \), there exists a local chart \( \psi : B^{2n+1}_\epsilon \to X \) for \( X \), centered at \( x \) and such that:

1. \( P \cap V \) is defined by the conditions \( p = 0 \) and \( \theta = 0 \), where \( V =: \psi(B^{2n+1}_\epsilon) \);
2. \( e^{i\theta} \cdot \psi(p,q,\theta) = \psi(p,q,\theta + \vartheta) \), whenever all terms are defined;
3. at any \( y \in V \cap P \), \( \text{Hor}(X/M)_y = \text{span} \left\{ \frac{\partial}{\partial q_i}|_y, \frac{\partial}{\partial p_i}|_y : 1 \leq i \leq n \right\}; \)
4. for every \( y \in V \cap P \), one has
\[
\text{span} \left\{ \frac{\partial}{\partial p_i}|_y : 1 \leq i \leq n \right\} = J_y \left( \text{span} \left\{ \frac{\partial}{\partial q_i}|_y : 1 \leq i \leq n \right\} \right) = J_y(T_y P),
\]
where \( J_y \in \text{End}(\text{Hor}(X/M)_y) \) is induced by the complex structure of \( M \);
5. if \( y \in V \cap P \) and \( \eta_1, \ldots, \eta_n \in \mathbb{R} \), then
\[
\left\| \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial p_j}|_y \right\|_y^2 = \sum_{j=1}^{n} \eta_j^2.
\]

**Proof.** In the following, \( \epsilon > 0 \) is allowed to vary from line to line. By §2 of [4], for any \( y \in P \) there exists a system of Heisenberg local coordinates \((p^{(y)}, q^{(y)}, \theta^{(y)})\) adapted to \( P \) at \( y \). This means that \((p^{(y)}, q^{(y)}, \theta^{(y)})\) are local Heisenberg coordinates for \( X \) centered at \( y \), in the sense of [10], and that \( P \) is tangent to the locus \( p^{(y)} = 0 \) at \( y \). This construction may be deformed smoothly with \( y \in P \): \( \forall x \in P \) there exist an open neighborhood \( x \in P' \subseteq P \), \( \epsilon > 0 \), and a smooth map \( \Psi : P' \times B^{2n}_\epsilon \to S^1 \) such that \( \forall y \in P' \) the partial map \( \Psi(y, \cdot) : B^{2n}_\epsilon \to X \) is an Heisenberg local chart adapted to \( P \) at \( y \).

We may assume without loss that \( P' \) is the image of a local chart \( \phi : B^{n}_\epsilon \to P' \) for \( P \) centered at \( x \). Let \( q = (q_i) \) denote the linear coordinates on \( \mathbb{R}^n \). Define
ψ : $B^{2n+1}_r \to X$ by $ψ(p, q, θ) = (φ(q), (p, 0, θ))$. By definition of Heisenberg local coordinates, ψ is a local chart for X satisfying all the conditions in the statement of the Proposition.

**Definition 2.5.** A system of local coordinates defined as in Proposition 2.3 will be called a system of good local coordinates for X along P.

In the notation of the Proposition, $π(V) \subseteq M$ is an open subset, $π(P \cap V) \subseteq π(V)$ is a Lagrangian submanifold, and $(p, q)$ is naturally a local coordinate chart on $π(V)$, in which $π(P \cap V)$ is defined by the condition $p = 0$. The Heisenberg local charts $Ψ(y, ·, ·)$ appearing in the proof are defined on $B^{2n}_e \times (−π, π)$, but the good coordinate chart is defined on $B^{2n+1}_r \subseteq B^{2n}_e \times (−π, π)$. This ensures that $P$ intersect each $S^1$-orbit at most once in the given chart, and the image of $P \cap V$ in X is a submanifold. Now suppose that actually $P \in P$. Let $r \in \mathbb{N}$ be the degree of the unramified cover $P \to L =: π(P) \subseteq M$. Then $e^{iθ} \cdot P = P$ if $e^{iθ} \in \mathbb{Z}_r = \langle e^{2ni/r} \rangle \subseteq S^1$, and $e^{iθ} \cdot P \cap P = 0$ if $e^{iθ} \notin \mathbb{Z}_r$. In fact, $\mathbb{Z}_r$ acts as a group of Riemannian covering maps for $P \to L =: π(P)$, and Proposition 2.4 may be strengthened:

**Proposition 2.4.** Suppose $P \in P$. For any $x \in P$, there exists a local chart $ψ : B^{2n}_r \times (−π, π) \to X$ for X, such that $x = ψ(0, 0, θ_1)$ for some $θ_1 \in (−π, π)$, satisfying conditions 2., 3., and 4. of Proposition 2.3 with $P$ in place of $Λ$, and such that in addition condition 1 is replaced by:

1a let $V := ψ\left(B^{2n}_r \times (−π, π)\right)$; then $P \cap V$ is defined by the conditions $p = 0$ and $θ ∈ \{θ_1, · · ·, θ_r\}$, where $θ_j \in (−π, π)$ are all distinct;

1b $P \cap V = π^{-1}(P \cap V) \cap Λ$.

### 2.4 Projectivized BPU maps

Given the volume form $\text{vol}_X = α \wedge π^*(ω)^n$, we shall identify functions, densities and half-densities on X. Any $(P, λ) \in W_1P_2$ induces a generalized half-density $δ(P, λ) ∈ D'(X)$, essentially the delta function determined by $(P, λ)$; here, λ is implicitly viewed by pull-back as a half-density on P. To express this, given $γ ∈ D^∞_{(1/2)}(X)$ write $λ = S_λ \cdot \text{dens}^{(1/2)}_{P}$ and $γ = T_γ \cdot \text{dens}^{(1/2)}_{X}$ for unique $S_λ ∈ C^∞(P)$ and $T_γ ∈ C^∞(X)$; then $δ(P, λ), γ = \int_P S_λ T_γ \cdot \text{dens}_P$. Now $δ(P, λ)$ is a Lagrangian distribution: Let $φ = (p, q, θ) : U \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be local coordinates for X, centered at some $x_0 ∈ P$, and defined on an open neighborhood $U \ni x_0$. Suppose that $P \cap U = \{p = P(q), θ = Θ(q)\} \subseteq U$, where $(P, Θ) : V =: φ(U) → \mathbb{R}^n \times \mathbb{R}$ is $C^∞$. Then (q) restricts to a system of local coordinates for P, defined on $P \cap U$ and centered at $x_0$; accordingly, we shall write $\text{dens}^{(1/2)}_{P} = D_P \cdot \sqrt{|dq|}$, for a unique $C^∞$ positive function $D_P$ on $P \cap U$. Then if γ is supported in U, we have
\[ \langle \delta(P, \lambda), \gamma \rangle = \int_{\mathbb{R}^n} S_\lambda(q) \, T_r(\mathcal{P}(q), q, \Theta(q)) \cdot D_P(q)^2 \, dq . \]

On \( \mathcal{C}_0^\infty(V) \), therefore, \( \delta(P, \lambda) \) is the Fourier integral distribution

\[
\frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i(\tau F + \eta H)} \, S_\lambda(q) \, D_P(q)^2 \, d\tau \, d\eta ,
\]

where \( F(p, q, \theta) = \theta - \Theta(q) \), \( H(p, q, \theta) = p - \mathcal{P}(q) \).

By its microlocal structure, the Szegö projector of \( X \) extends to \( \Pi : \mathcal{D}'(X) \rightarrow \mathcal{H}(X) \), where \( \mathcal{H}(X) \subseteq \mathcal{D}'(X) \) is the subspace of those distributions all of whose Fourier components belong to the Hardy space. Define \( \Delta : \mathbf{W}_h \mathcal{P}_3 \rightarrow \mathcal{D}'(X) \) by \( (P, \lambda) \mapsto \delta(P, \lambda) \). Set \( u(P, \lambda) =: \Pi \circ \Delta(P, \lambda) = \Pi(\delta(P, \lambda)) \in \mathcal{H}(X) \), and for \( k \in \mathbb{N} \) consider the Fourier components, \( u(P, \lambda), k =: \Pi_k \circ \Delta(P, \lambda) \in H(X)_k \); here \( H(X)_k \cong H^0(M, \Lambda^\otimes k) \) is the level-\( k \) Hardy space of \( X \), and \( \Pi_k : \mathcal{D}'(X) \rightarrow H(X)_k \) the (extension of the) \( L^2 \)-orthogonal projector. This is the level-\( k \) BPU map, \( \tilde{\Phi}_k =: \Pi_k \circ \Delta : \mathbf{W}_h \mathcal{P}_3 \rightarrow H(X)_k \). Since by construction \( \delta(P, \lambda) \) is \( \mathbb{Z}_r \)-invariant, so is \( u(P, \lambda) \); therefore \( u(P, \lambda), k = 0 \), hence \( \tilde{\Phi}_k = 0 \), unless \( r \mid k \). Next suppose \( k = l \cdot r \), \( l \in \mathbb{N} \).

By i) of Corollary 1.1 of [4], \( u(P, \lambda), k(x) = O(k^{-\infty}) \) whenever \( x \not\in S^1 \cdot P \). If on the other hand \( x \in S^1 \cdot P \), by ii) of the same Corollary in local Heisenberg coordinates for \( X \) adapted to \( P \) at \( x \) and \( \forall w \in T_m M \cong \mathbb{C}^n \), \( m =: \pi(x) \in L \subseteq M \), there is an asymptotic expansion:

\[
u(P, \lambda), k(x + w/\sqrt{k}) \sim k^{n/2} \cdot r \cdot \left( \frac{2}{\pi} \right)^{n/2} e^{-ik\theta(x)} \, S_\lambda(m) \, e^{-\|w\|^2/2 - i\omega_m(w^+, w^\perp)} + \sum_{f \geq 1} k^{(n-f)/2} c_f(x, w).
\]

Here, \( w^\perp \in T_m L \), \( w^\perp \in (T_m L)^\perp \) denote the orthogonal components of \( w \), and \( \omega_m(w^+, w^\perp) \) their symplectic pairing. Furthermore, \( e^{i\theta(x)} \in S^1 \) is such that \( e^{i\theta(x)} \cdot x \in P \), and \( \lambda = S_\lambda \cdot \text{dens}_L^{(1/2)} \), where \( \text{dens}_L^{(1/2)} \).

**Remark 2.2.** By the arguments surrounding equations (54)-(58), Lemma 3.7 and Claim 3.2 of [4], \( c_f(x, w) = \tilde{z}_f(x, w) = e^{-\|w\|^2/2} \), where \( \tilde{z}_f(x, w) \) is a rapidly decaying function of \( w^\perp \). More precisely, up to some constant factor and oscillating term, \( \tilde{z}_f(x, w) \) is the evaluation in \( w^+ \) of the Fourier transform of the rapidly decreasing function \( Z_j \) in statement ii) of Lemma 3.7 of [4] (there \( w^\perp = p_w, w^\parallel = q_w \)).

In particular, \( \tilde{\Phi}_k(P, \lambda) \neq 0 \) if \( (P, \lambda) \in \mathbf{W}_h \mathcal{P}_3 \) and \( r \mid k \), \( k \gg 0 \). For \( k = lr \), \( l \in \mathbb{N} \), let \( \mathfrak{U}_k \subseteq \mathbf{W}_h \mathcal{P}_3 \) be the \( S^1 \)-invariant open subset where \( \tilde{\Phi}_k \neq 0 \). Thus, \( \mathbf{W}_h \mathcal{P}_3 = \bigcup_k \mathfrak{U}_k \). Similarly, let \( \mathfrak{U}_k =: \tilde{\pi}(\hat{\mathfrak{U}}_k) = \mathfrak{U}_k / S^1 \subseteq \mathbf{W}_h \tilde{\mathcal{J}} \); thus \( \mathfrak{U}_k \) is open in \( \mathbf{W}_h \tilde{\mathcal{J}} \), and \( \mathbf{W}_h \tilde{\mathcal{J}} = \bigcup_{r \mid k} \mathfrak{U}_k \).

**Definition 2.6.** For \( k = lr \), \( l \in \mathbb{N} \), define \( \Phi_k : \mathfrak{U}_k \rightarrow \mathcal{PH}(X)_k \) by \( (L, \lambda) \mapsto \left[ \tilde{\Phi}_k(P, \lambda) \right] \), for any \( P \in \mathcal{P} \) covering \( L \). \( \Phi_k \) is the level-\( k \) projectivized BPU map.
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Now we shall give an asymptotic expansion for certain scaling limits of \( d_{(P,\lambda)} \Phi_k : T_{(P,\lambda)}W_0P_3 \to H(X)_k \), at a given \((P,\lambda) \in W_0P_3\). We may assume without loss that \( k = r \cdot l \), \( l \in \mathbb{N} \). Assuming that \( \Delta \) is differentiable, since \( \Pi_k \) is linear we have

\[
d_{(P,\lambda)} \Phi_k(W) = \Pi_k \left( d_{(P,\lambda)} \Delta(W) / W \in T_{(P,\lambda)}W_0P_3 \right). \tag{6}
\]

We shall first determine \( d_{(P,\lambda)} \Delta(W) \). To this end, set \( L = \pi(P) \), and suppose \( \tilde{W} = \tilde{W}(f,\ell) \), with \( \int_L \ell \cdot \lambda = 0 \) (Proposition 2.2). Identify \( \lambda \) and \( \ell \) with their pull-backs to \( P \), and write \( \lambda = S_\lambda \cdot \text{dens}^{(1/2)}_P \) and \( \ell = S_t \cdot \text{dens}^{(1/2)}_P \); the smooth functions \( S_\lambda, S_t \) on \( P \) and \( S \) descend on \( L \). Locally near some \( x_0 \in P \), fix good local coordinates \((p,q,\theta)\) for \( X \) along \( P \) centered at \( x_0 \), defined on \( X' \subseteq X \) (Definition 2.5). Then \( (p,q) \) are naturally local coordinates for \( M \) centered at \( m_0 =: \pi(x_0) \), defined on \( M' =: \pi(X') \); the projection \( X' \to M' \) is represented by \((p,q,\theta) \mapsto (p,q)\). Set \( P' =: X' \cap P = \{p = 0, \theta = 0\} \), \( L' = L \cap M' =: \{p = 0\} \). We may view \( (q) \) as local coordinates on \( P' \). Perhaps after restricting \( X', \pi_{|P'} : P' \to L' \) is an isometric diffeomorphism.

**Proposition 3.1.** In the notation of the preceding discussion, the following holds:

1. Let us extend \( f \) to some tubular neighborhood of \( L \) by the normal cotangent structure, and let \( v_f \) be its Hamiltonian vector field. Then, \( \forall m \in L' \), we have \( v_f(m) = \sum_{j=1}^n a_j(m) \cdot \frac{\partial}{\partial p_j} |_m \), for unique \( a_j \in \mathcal{C}^\infty(L') \).

2. Let \( \Gamma(L, f) \) be as in Definition 2.5 \( a = (a_j) \). Locally near \( x_0 \), \( d_{(P,\lambda)} \Delta(W) \in \mathcal{D}'(X) \) is the Fourier integral

\[
\frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^n} e^{i(\tau \theta + \eta \phi)} \left[ \left( S_t + S_\lambda \cdot \Gamma(L, f) \right) - i(\tau f + \eta \cdot a) S_\lambda \right] D_P d\tau d\eta,
\]

where \( \text{dens}^{(1/2)}_P = D_P \cdot \sqrt{|dq|} \) is the Riemannian half-density on \( P \) (or \( L \)).

**Proof of Proposition 3.1.** By the discussion surrounding Proposition 2.3 \( v_f(m) \in (T_mL)^\perp = J_m(T_mL), \forall m \in L \). This proves 1., in view of Proposition 2.3.

For some \( \epsilon > 0 \), suppose \( \gamma : (-\epsilon, \epsilon) \to W_0P_3, \gamma(t) = (P_t, \lambda_t) \), is \( \mathcal{C}^\infty \) with \( \gamma(0) = (P, \lambda) \), \( \gamma'(0) = \tilde{W}. \) Then \( L_t =: \pi(P_t) \in \mathcal{J} \forall t \in (-\epsilon, \epsilon). \) If \( \phi_t \) is the local flow of \( v_f \), then \( L_t = \phi_t(L) \) to first order in \( t \). Let \( v_f^2 \) be horizontal lift of \( v_f \) to \( X \), and \( \frac{\partial}{\partial \theta} \) is the generator of the \( S^1 \)-action. Then \( \tilde{v}_f =: v_f^2 - f \cdot \frac{\partial}{\partial \theta} \) is a contact vector field on \( \pi^{-1}(M') \supseteq P \), whose local flow \( \tilde{\phi}_t \) covers \( \phi_t \), and \( P_t = \tilde{\phi}_t(P) \) to first order in \( t \). Next, let \( \beta_t : L_t \to L \) be induced by the normal cotangent structure near \( L \). There is a smooth path \( \eta_t \) of half-weights on \( L \), such that \( \lambda_t = \beta_t^* (\eta_t) \) for every \( t \). Then \( \ell = \eta_t(0) \), so that \( \eta_t = \lambda + t \cdot \ell + O(t^2) \). By 1., we have
\[ \tilde{v}_f(x) = \sum_{j=1}^{n} a_j(q) \cdot \frac{\partial}{\partial p_j} \bigg|_{x} - f(q) \cdot \frac{\partial}{\partial \eta} \bigg|_{x}, \] if \( x \in P' \) has local coordinates \((0, q, 0)\).

Thus, for \( t \sim 0, P_t \subseteq X \) is locally defined by \( p_j = t \cdot a_j(q) + O(t^2) \) (\( j = 1, \ldots, n \)), and \( \theta = - t \cdot f(q) + O(t^2) \). Write \( \lambda_t = S_{\lambda_t} \cdot \text{dens}^{(1/2)} \), for unique \( S_{\lambda_t} \in C^\infty(L_t) \). By \( \ref{2.2} \), \( \delta_{(P_t, \lambda_t)} = \Delta(P_t, \lambda_t) \) is locally near \( x_0 \) the Fourier integral

\[
\frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\tau F_t + \eta H_t)} S_{\lambda_t}(q) \, D_{P_t}(q)^2 \, d\tau \, d\eta, \tag{7}
\]

where \( F_t(p, q, \theta) = \theta - tf(q) + O(t^2), \) \( H_t(p, q, \theta) = p - ta(q) + O(t^2) \). Here \( p = (p_j), a = (a_j) \) (cfr Lemma 2.2 of \( \ref{4} \)). Furthermore, \( q = (q_j) \) restrict to local coordinates on \( P_t \), and \( S_{\lambda_t}(q), D_{P_t}(q) \) are meant in this local coordinate system (thus, \( \text{dens}^{(1/2)} = D_{P_t} \cdot \sqrt{|dq|} \)). By §7.8 of \( \ref{5} \), the \( t \)-derivative of \( \Delta(P_t, \lambda_t) \) may be computed by differentiating with respect to \( t \) under the integral sign in (7).

**Lemma 3.1.** **A:** When \( q \) is adopted as a system of local coordinates for both \( L = L_0 \) and \( L_t \), we have \( \beta_t(q) = q + O(t^2) \). **B:** Let \( \phi_t|_L : L \to L_t = \phi_t(L) \) be the diffeomorphism induced by the flow of \( \nu_f \). Let \( (\phi_t|_L)^{-1} : L_t \to L \) be the inverse diffeomorphism. Then \( (\phi_t|_L)^{-1} = \beta_t + O(t^2) \).

**Proof.** Let \( U \subseteq M \) be the open tubular neighborhood of \( L \) produced in the construction of the normal cotangent structure, and let \( \hat{\pi} : U \to L \) be the normal cotangent projection. Let \( \hat{\pi}' : U \to L \) be the locally defined projection \( M' \to L' \) which is given in good local coordinates by \( (p, q) \mapsto q \). By the properties of good local coordinates, the fibers of both \( \hat{\pi} \) and \( \hat{\pi}' \) meet \( L \) perpendicularly at each \( m \in M' \). It follows that, in local coordinates, \( \hat{\pi}(m) - \hat{\pi}'(m) = O(\text{dist}(m, L)^2) \) \( (m \in M') \). Restricting to \( P_t \), this implies \( \textbf{A} \), and the proof of \( \textbf{B} \) is similar. \( \square \)

As in the previous discussion, let \( \eta_t \) be the half-weight on \( L \) such that \( \beta_t^*(\eta_t) = \lambda_t \). Let us write \( \eta_t = S_{\eta_t} \cdot \text{dens}^{(1/2)}_L \), \( \ell = S_{\ell} \cdot \text{dens}^{(1/2)}_L \) for uniquely determined \( C^\infty \) functions \( S_{\eta_t} \) and \( S_{\ell} \) on \( L \). Therefore, \( S_{\eta_t} = S_{\lambda_t} + t \, S_{\ell} + O(t^2) \). Notice that \( S_{\lambda_t} \) is a smooth function on \( P_t \), while \( S_{\eta_t} \) is a smooth function on \( P = P_0 \). Since \( q \) restricts to a system of local coordinates on both \( P \) and \( P_t \), \( t \sim 0 \), we can consider the local expressions \( S_{\lambda_t}(q) \) and \( S_{\eta_t}(q) \). By Definition 2.3 and Lemma 3.1, \( D_{P_t}(q)/D_P(q) = 1 + t \Gamma(L, f) + O(t^2) \). In view of Corollary 3.1, \( \beta_t^*(\sqrt{|dq|}) = \sqrt{|dq|} + O(t^2) \), and \( (\beta_t^*g)(q) = g(q) + O(t^2) \) for every locally defined function \( g \). Therefore,

\[
\lambda_t = \beta_t^*(S_{\eta_t} \cdot \text{dens}_P) = \beta_t^*(S_{\eta_t} \cdot D_P \cdot \sqrt{|dq|}) = S_{\eta_t} \cdot D_P \cdot \sqrt{|dq|} + O(t^2)
\]

\[
= S_{\eta_t} \frac{D_P}{D_{P_t}} \cdot D_{P_t} \cdot \sqrt{|dq|} + O(t^2) = S_{\eta_t} \frac{D_P}{D_{P_t}} \cdot \text{dens}_{P_t} + O(t^2).
\]

We deduce that

\[
S_{\lambda_t} = S_{\eta_t} \frac{D_P}{D_{P_t}} + O(t^2) = \left( S_{\lambda_t} + t S_{\ell} \right) \cdot \left( 1 - t \, \Gamma(L, f) \right) + O(t^2)
\]

\[
= S_{\lambda_t} + t \left( S_{\ell} - S_{\lambda_t} \, \Gamma(L, f) \right) + O(t^2),
\]

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whence $S_\lambda \cdot D_h^2 = D_p^2 \cdot \left[ S_\lambda + t \left( S_\ell + S_\lambda \cdot \Gamma(L, f) \right) \right] + O(t^2)$. The proof of the second statement of Proposition 3.1 is completed by inserting this equality in (7) and differentiating with respect to $t$ under the integral sign at $t = 0$.  

Thus, locally near $x_0$, we have $d_{(P,\lambda)} \Delta(\bar{W}) = \sum_{j=1}^4 d_{(P,\lambda)} \Delta(\bar{W})_j$, where

$$d_{(P,\lambda)} \Delta(\bar{W})_j = \frac{1}{(2\pi)^n+1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\gamma_\delta + \eta \cdot p)} b_j D_p^2 d\gamma d\eta,$$

with $b_1 =: S_\ell$, $b_2 =: S_\lambda \cdot \Gamma(L, f)$, $b_3 =: -i\tau f S_\lambda$, $b_4 =: -i(\eta \cdot a) S_\lambda$. Applying the level-$k$ Szegő kernel, by (9) we obtain $d_{(P,\lambda)} \Phi_k(\bar{W}) = \sum_{j=1}^4 d_{(P,\lambda)} \Phi_k(\bar{W})_j$, where $d_{(P,\lambda)} \Phi_k(\bar{W})_j =: \Pi_k (d_{(P,\lambda)} \Delta(\bar{W})_j)$, $1 \leq j \leq 4$. Let us now consider the transverse scaling asymptotics for $d_{(P,\lambda)} \Phi_k(\bar{W})_j$ near $x_0$. Given $x \in P'$ with good local coordinates $(0, q, 0)$, and $w \in \mathbb{R}^n$, $x + w$ will mean the point in $P'$ having good local coordinates $(w, q, 0)$. The real n-space $\mathbb{R}^n \subseteq \mathbb{C}^n$ is unitarily identified with the orthocomplement $(T_m L)^\perp$, $m =: \pi(x)$, hence with a subspace of $\text{Hor}(X)_x \subseteq T_x X$.

**Lemma 3.2.** Suppose $x_0 \in P' \subseteq P$ is a sufficiently small open neighborhood. Uniformly in $x \in S^1 \cdot P'$ and $w \in \mathbb{R}^n$ of bounded norm, for $j = 1, 2$ the following asymptotic expansion holds as $t \to +\infty$ and $k = t \cdot r$:

$$d_{(P,\lambda)} \Phi_k(\bar{W})_j \left( x + \frac{w}{\sqrt{k}} \right) \sim k^{n/2} \cdot r \left( \frac{2}{\pi} \right)^{n/2} e^{-ik\vartheta(x)} b_j(m) e^{-\|w\|^2} + \sum_{h \geq 1} C_{hj}(x, w) k^{(n-h)/2},$$

where $m =: \pi(x) \in L$, and $\vartheta(x) \in (-\pi, \pi]$ is such that $e^{i\vartheta} \cdot x \in P'$. Similarly,

$$d_{(P,\lambda)} \Phi_k(\bar{W})_3 \left( x + \frac{w}{\sqrt{k}} \right) \sim -ik^{1+n/2} \cdot r \left( \frac{2}{\pi} \right)^{n/2} e^{-ik\vartheta(x)} f(m) S_\lambda(m) e^{-\|w\|^2} + \sum_{h \geq 1} C_{h3}(x, w) k^{1+(n-h)/2},$$

and

$$d_{(P,\lambda)} \Phi_k(\bar{W})_4 \left( x + \frac{w}{\sqrt{k}} \right) \sim \sum_{h \geq 0} C_{h4}(x, w) k^{(1+n-h)/2}.$$  

Furthermore, for $j = 1, 2, 3, 4$ and $h \geq 1$, $C_{hj}(x, w) = \tilde{C}_{hj}(x, w) e^{-\|w\|^2/2}$, where $\tilde{C}_{hj}(x, w)$ is a rapidly decaying function of $w$.

**Proof.** We may assume $x \in P$. For $j = 1, 2$, we have $d_{(P,\lambda)} \Phi_k(\bar{W})_j = \Pi_k (\delta_{(P,\sigma_j)})$, where $\sigma_j$ is the half-density (not necessarily a half-weight) on $L$ defined by $\sigma_j = b_j \cdot \text{dens}_L^{(1/2)}$. By Corollary 1.1 of [H], the scaling asymptotics of
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\[ \Pi_k(\delta_{(P, \sigma)}) \] at \( x \in P \) are given in Heisenberg local coordinates adapted to \( P \) at \( x \) by asymptotic expansions akin to (5), with \( b_j \) in place of \( S_\lambda \). In (5), \( w \) is allowed to vary in \( \mathbb{C}^n \), and \( x + w \) denotes the point of \( X \) having adapted Heisenberg local coordinates \((w, 0)\). On the other hand, good local coordinates along \( P \) are constructed by gluing moving systems of transverse Heisenberg local coordinates along a system of arbitrary local coordinates along \( P \) (this is made precise in Proposition 2.3). Since in (9) \( w \) is required to be a real vector, the expression \( x + \sqrt{k} w \) (in the given system of good local coordinates) represents a transverse displacement from \( P \) which is also represented by the expression \( x + \frac{w}{\sqrt{k}} \) in a system of Heisenberg local coordinates adapted to \( P \) at \( x \). Thus, an expansion of type (5) still holds in good local coordinates, so far as the rescaling occurs in the transverse direction only.

The proof of (10) is similar, but we need to explain the extra factor of \( k \). To this end, we remark that \( d_{(P, \lambda)} \tilde{\Phi}_k(\tilde{W})_3 = \Pi_k \left( d_{(P, \lambda)} \Delta(\tilde{W})_3 \right) \). Now \( d_{(P, \lambda)} \Delta(W)_3 \) is the Fourier integral (8), with \( b_3 = -i \tau f S_\lambda \) (introduce a cut-off to make this compactly supported near \( x \)). Due to the factor \( \tau \) appearing in the amplitude, this is not of the form \( \delta_{(P, \sigma)} \) for a \( \mathcal{C}^\infty \) half-density on \( P \). However, the techniques in the proof of (5) can still be applied. Namely, one applies to (8) the Boutet de Monvel - Sjöstrand parametrix for the Szegö kernel, and then takes the \( k \)-th Fourier component. After suitably rescaling the integration variables involved, this yields an oscillatory integral to which the stationary phase Lemma may be applied. By Claim 3.2 of [4], this leads to a unique stationary point where \( \tau = 1 \) and \( \eta = 0 \). The rescaling involved in \( \tau \) is \( \tau \mapsto k \tau \), and this accounts for the extra factor of \( k \) in (10).

Let us consider (11). Now \( d_{(P, \lambda)} \tilde{\Phi}_k(\tilde{W})_4 = \Pi_k \left( d_{(P, \lambda)} \Delta(\tilde{W})_4 \right) \), and \( d_{(P, \lambda)} \Delta(\tilde{W})_4 \) is the Fourier integral (8), with \( b_4 = -i(\eta \cdot a) S_\lambda \). The same arguments used in the previous paragraph apply, so that \( \Pi_k \left( d_{(P, \lambda)} \Delta(\tilde{W})_4 \right) \left( x + \frac{w}{\sqrt{k}} \right) \) is an oscillatory integral to which the stationary phase Lemma may be applied. By the arguments in the proof of Theorem 1.1 of [4], the rescaling in \( \eta \) is \( \eta \mapsto k^{3/2} \eta \), hence the leading order term of the resulting asymptotic expansion has degree at most \( k^{(n+3)/2} \). However, as mentioned the stationary point of the phase has \( \eta = 0 \). Since by Theorem 7.7.5 of [8] the first term involving derivatives of the amplitude has degree \( (3 + n)/2 - 1 = (1 + n)/2 \), the terms in \( k^{(n+3)/2} \) and \( k^{1+n/2} \) both vanish.

The last statement is proved arguing as in Remark 2.2. □

The \( \mathcal{C}^\infty \mathbb{R}^n \)-valued function \( a \) on \( P \) appearing in \( b_4 \) depends linearly on \( f \) and is independent of \( \ell \), while \( S_\ell \) depends linearly on \( \ell \), and is independent of \( f \). Let us write \( W = W(f, \ell) = W(f, 0) + W(0, \ell) \in T_{(P, \lambda)} \mathbb{W}_h \mathbb{P}_3 \). With \( x \in S^1 \cdot P \),
Proposition 3.2. If \( \tilde{W} = \tilde{W}(f, \ell) \in T_{(P, \lambda)}W_kP_3 \) and \( k = 1, 2, \ldots \), set \( \tilde{W}_k = \tilde{W}(f, k \ell) \). Thus, \( d_{(P, \lambda)}\tilde{\Phi}_k(\tilde{W}_k) = d_{(P, \lambda)}\tilde{\Phi}_k(\tilde{W}(f, 0)) + k d_{(P, \lambda)}\tilde{\Phi}_k(\tilde{W}(0, \ell)) \). Given Lemma 3.2 summing over \( j \) we obtain:

**Corollary 3.1.** Suppose \( (P, \lambda) \in W_kP_3 \), \( \tilde{W} \in T_{(P, \lambda)}W_kP_3 \). Then:

- If \( x \not\in S^1 \cdot P \), then \( d_{(P, \lambda)}\tilde{\Phi}_k(\tilde{W}_k)(x) = O(k^{-\infty}) \), uniformly in \( x \) on compact subsets of \( X \setminus S^1 \cdot P \).

- Uniformly in \( x \in S^1 \cdot P \) and in \( w \in T_{\pi(x)}L^1 \subseteq T_{\pi(x)}M \) of bounded norm, the following asymptotic expansion holds as \( l \to +\infty \) and \( k = l \cdot r \):

\[
d_{(P, \lambda)}\tilde{\Phi}_k(\tilde{W}_k) \left( x + \frac{w}{\sqrt{k}} \right) \sim k^{1+n/2} \cdot r \left( \frac{2}{\pi} \right)^{n/2} e^{-ik\theta(x)} \gamma_{\ell f}(m) e^{-\|w\|^2}
+ \sum_{h \geq 1} H_h(x, w) k^{1+(n-h)/2},
\]

where \( \gamma_{\ell f} = S_{\ell} - i f S_{\lambda} \), and \( \forall h \geq 1 \) we have \( H_h(x, w) = \hat{H}_h(x, w) e^{-\|w\|^2/2} \), \( \hat{H}_h \) being a rapidly decaying function of \( w \).

We can now prove:

**Proposition 3.2.** If \( (P, \lambda) \in W_kP_3 \), as \( l \to +\infty \) and \( k = l \cdot r \) we have:

\[
\left( \tilde{\Phi}_k(P, \lambda), \tilde{\Phi}_k(P, \lambda) \right)_{L^2(X)} \sim k^{n/2} \left( \frac{2}{\pi} \right)^{n/2} r^2 + \sum_{h \geq 1} q_h \cdot k^{(n-h)/2}.
\]

Given tangent vectors \( \tilde{W} = \tilde{W}(f, \ell), \tilde{W}' = \tilde{W}(f', \ell') \in T_{(P, \lambda)}W_kP_3 \), set

\[
F(\tilde{W}, \tilde{W}') = (S_{\ell} S_{\ell'} + f f' S_{\lambda}^2) + i(S_{\ell} f' - f S_{\ell'}) S_{\lambda}.
\]
Then the following asymptotic expansions hold as \( l \to +\infty \) and \( k = l \cdot r \):

\[
\left( d_{(P, \lambda)} \Phi_k(\tilde{W}_k), d_{(P, \lambda)} \Phi_k(\tilde{W}'_k) \right)_{L^2(X)} \sim k^{2+n/2} \cdot r^2 \left( \frac{2}{\pi} \right)^{n/2} \int_\Lambda F(\tilde{W}, \tilde{W}') \cdot \text{dens}_L \\
+ \sum_{h \geq 1} r_h k^{2(n-h)/2},
\]

(15)

\[
\left( \Phi_k(P, \lambda), d_{(P, \lambda)} \Phi_k(\tilde{W}_k) \right)_{L^2(X)} \sim k^{1+n/2} \cdot r^2 \left( \frac{2}{\pi} \right)^{n/2} \int_\Lambda S^2_\lambda f \cdot \text{dens}_L \\
+ \sum_{h \geq 1} s_h k^{1(n-h)/2}.
\]

(16)

An estimate similar to (14) was first proved in [2], where BPU maps where originally phrased using Fourier-Hermite distributions and symplectic spinors.

**Remark 3.1.** Suppose \((P, \lambda) \in \mathcal{P}_3\), and set \( L = \pi(P) \). If \( W = W(f, \ell), W' = W'(f', \ell') \in T_{(L, \lambda)}W_3\), and \( \tilde{W} = \tilde{W}(f, \ell), \tilde{W}' = \tilde{W}'(f', \ell') \in T_{(P, \lambda)}W_3\) are their lifts, then \( \int_{\Lambda} F(\tilde{W}, \tilde{W}') \cdot \text{dens}_L = G_{(L, \lambda)}(\tilde{W} \cdot \tilde{W}') + i \Omega_{(L, \lambda)}(\tilde{W} \cdot \tilde{W}'). \)

**Proof of Proposition 3.2.** Since \( \Phi_k(P, \lambda) = \Pi_k(\delta_{P, \lambda}) \), arguing as in (75) of [4] we have \((\Phi_k(P, \lambda), \Phi_k(P, \lambda)\big)_{L^2(X)} = \int_P S_{\lambda} \Phi_k(P, \lambda) \cdot \text{dens}_P\). Now [5] with \( w = 0 \) yields an asymptotic expansion for \( \Phi_k(P, \lambda)(x), x \in P \); when \( x \in P \), we may actually assume \( \vartheta(x) = 0 \) in [5]. Inserting the latter asymptotic expansion in the former integral proves (14), since \( \int_P S^2_\lambda \cdot \text{dens}_P = r \int_\Lambda S^2_\lambda \cdot \text{dens}_L = r \int_\Lambda \lambda \cdot \lambda = r, \) because \( P \to L \) is a Riemannian covering of degree \( r \), and \( \lambda = S_\lambda \cdot \text{dens}_{L(1/2)} \) is a half-weight on \( L \). The proof of (15) is similar, except that we now need to use the asymptotic expansion in Corollary 3.1 and recall that \( \int_\Lambda S_\lambda \cdot \text{dens}_L = \int_\Lambda \lambda \cdot \ell = 0 \).

Let us now consider (15). Let \( U \supseteq \Lambda = \pi(P) \) be a suitably small tubular neighborhood of \( L \) in \( M \), so that \( T =: \pi^{-1}(U) \subseteq X \) is an \( S^1 \)-invariant open neighborhood of \( P \). In view of the first statement of Corollary 3.1 we have:

\[
\left( d_{(P, \lambda)} \Phi_k(\tilde{W}_k), d_{(P, \lambda)} \Phi_k(\tilde{W}'_k) \right)_{L^2(X)} \sim \int_T d_{(P, \lambda)} \Phi_k(\tilde{W}_k) \cdot d_{(P, \lambda)} \Phi_k(\tilde{W}'_k) \cdot \text{dens}_X.
\]

(17)

Suppose that \( U = \bigcup U_j \) is an open cover of \( U \), such that on each \( T_j =: \pi^{-1}(U_j) \) there is a system of good local coordinates for \( X \) near \( P \), in the stronger sense of Proposition 2.3 we may as well assume that the \( T_j \)'s are finitely many. Let \( \{ \varphi_j \} \) be a partition of unity on \( U \) subordinate to the open cover \( \{ U_j \} \), so that \( \{ \varphi_j \} \) is a partition of unity on \( T \) for the open cover \( \{ T_j \} \), where \( \varphi_j =: \varphi_j \circ \pi \). Given (17), we obtain

\[
\left( d_{(P, \lambda)} \Phi_k(\tilde{W}_k), d_{(P, \lambda)} \Phi_k(\tilde{W}'_k) \right)_{L^2(X)} \sim \sum_j A_{jk}, \text{ where we have set } A_{jk} =: \int_T \varphi_j d_{(P, \lambda)} \Phi_k(\tilde{W}_k) \cdot d_{(P, \lambda)} \Phi_k(\tilde{W}'_k) \cdot \text{dens}_X. \text{ Let us now evaluate each } A_{jkapis} asymptotically as } k \to +\infty.
Let us set \( R_k := d_{(P, \lambda)} \Phi_k(W_k) d_{(P, \lambda)} \Phi_k(W'_k) \). In good local coordinates, \( A_{jk} = \int_{T_j} \varphi_j \cdot R_k(p, q, \theta) \cdot D_P \cdot dp \cdot dq \cdot d\theta = k^{-n/2} \int \int \varphi_j \cdot R_k \cdot D_X^2 (\frac{u}{\sqrt{k}}, q, \theta) \cdot dw \cdot dq \cdot d\theta \), where we have performed the rescaling \( p = \frac{u}{\sqrt{k}} \), and written \( \text{dens}_X = \frac{1}{2\pi} D_X^2 \cdot |dp \cdot dq \cdot d\theta| \). Clearly, \( (\frac{u}{\sqrt{k}}, q, \theta) \) corresponds to \( x + \frac{u}{\sqrt{k}} \), where \( x \in (S^1 \cdot P) \cap T_j \) has good local coordinates \((0, q, \theta)\). On the other hand, by the second statement of Corollary 3.1 working in good local coordinates we have

\[
R_k \left( x + \frac{w}{\sqrt{k}} \right) \sim k^{2+n} \cdot r^2 \left( \frac{2}{\pi} \right)^n F(W, W') e^{-2\|w\|^2} + \sum_{h \geq 1} t_h(x, w) k^{2+n-h/2},
\]

where for every \( h \geq 1 \) we have \( t_h(x, w) = \hat{t}_h(x, w) e^{-\|w\|^2} \), with \( \hat{t}_h \) a rapidly decaying function of \( w \). Now we can perform the integration over \( T_j \) by first integrating in \( dw \) over \( \mathbb{R}^n \) and in \( \frac{1}{\sqrt{k}} d\theta \) over \((-\pi, \pi)\), and then in \( dq \) (viewed now as a system of local coordinates on \( L \)). To perform the former, we remark that by the construction of good local coordinates, we have \( D_X \left( x + \frac{w}{\sqrt{k}} \right)^2 = D_X \left( \frac{w}{\sqrt{k}}, q, \theta \right)^2 = D_L(q)^2 + O(k^{-1/2}) \), where \( \text{dens}_L = D_L^2 \cdot |dq| \). Since \( \int_{\mathbb{R}^n} e^{-2\|p\|^2} dp = (\pi/2)^n/2 \), given \( 18 \) we obtain \( A_{jk} \sim k^{2+n/2} \cdot r^2 \left( \frac{2}{\pi} \right)^n / 2 \int_{\mathbb{R}^n} \varphi_j F(W, W') \cdot \text{dens}_L + \sum_{h \geq 1} u_j h k^{2+n-h/2} \). Summing over \( j \), we get \( 15 \). □

4 Proof of Theorem

Suppose that \( \mathcal{J} \subseteq \mathcal{L}_{BS} \) is the isodrastic leaf such that \( \mathcal{G} = \mathcal{W}_h \mathcal{J} \). Given \((L, \lambda) \in \mathcal{W}_h \mathcal{J} \), we shall now consider the asymptotics of the derivative of the projectivized BPU map, \( d_{(L, \lambda)} \Phi_k : T_{(L, \lambda)} \mathcal{W}_h \mathcal{J} \rightarrow T_{(L, \lambda)} \mathcal{P} H(X)_k \), for \( k = lr \) and \( l \rightarrow +\infty \).

Let \((V, \langle \cdot, \cdot \rangle)\) be a finite-dimensional unitary vector space. Let \( \varsigma : V \setminus \{0\} \rightarrow \mathbb{P} V \), \( v \mapsto [v] \) be the projection. If \( v \in V \setminus \{0\} \), the differential \( d_v \varsigma : V \rightarrow T_{[v]} \mathbb{P} V \) induces an algebraic isomorphism \( v^+ \rightarrow T_{[v]} \mathbb{P} V \), where \( v^+ \subseteq V \) is the unitary orthocomplement of \( v \). The Fubini-Study metric is determined by

\[
\langle d_v \varsigma(v'), d_v \varsigma(v'') \rangle_{[v]} = \frac{\langle v', v'' \rangle_{[v]} \cdot \langle v', v'' \rangle_{[v]}}{\langle v, v \rangle_{[v]}}, \quad (v', v'') \in v^+. \tag{19}
\]

Suppose \( W = W(f, \ell) \in T_{(L, \lambda)} \mathcal{W}_h \mathcal{J} \), where \( \int_L f \lambda \bullet \lambda = \int_L \ell \bullet \lambda = 0 \), and set \( W_k := W(f, k\ell) \). Since by assumption \( L \) is a BSL submanifold, by definition \( \exists \mathcal{P} \subseteq X \) compact and connected Planckian submanifold with \( L = \pi(P) \). Thus \((P, \lambda) \in \mathcal{W}_h \mathcal{P}_\mathcal{J} \) lies over \((L, \lambda) \). Now \( W_k \) naturally lifts to \( \tilde{W}_k \in T_{(P, \lambda)} \mathcal{W}_h \mathcal{P}_\mathcal{J} \), hence \( W_k = d_{(P, \lambda)} \tilde{\pi}(\tilde{W}_k) \), where \( \tilde{\pi} : \mathcal{W}_h \mathcal{P}_\mathcal{J} \rightarrow \mathcal{W}_h \mathcal{J} \) is the projection. If \( \varsigma : H(X)_k \setminus \{0\} \rightarrow \mathbb{P} H(X)_k \) is the projection, then \( \Phi_k \circ \tilde{\pi} = \varsigma \circ \Phi_k \). Therefore,

\[
d_{(L, \lambda)} \Phi_k(W_k) = d_{(L, \lambda)} \Phi_k \circ d_{(P, \lambda)} \tilde{\pi}(\tilde{W}_k) = d_{\Phi_k(P, \lambda)} \varsigma \left(d_{(P, \lambda)} \Phi_k(\tilde{W}_k)\right). \tag{20}
\]
Now \( \left( \Phi_k(P, \lambda), d_{(P, \lambda)} \Phi_k(\tilde{W}_k) \right)_{L^2(X)} \sim \sum_{h \geq 0} s_h k^{(1+n-h)/2} \), in view of (16) and the conditions on \( f \) and \( \ell \). Define \( Z_k \in H(X)_k, k \gg 0 \), by

\[
Z_k =: d_{(P, \lambda)} \Phi_k(\tilde{W}_k) - \left[ \frac{\Phi_k(P, \lambda), d_{(P, \lambda)} \Phi_k(\tilde{W}_k)}{\Phi_k(P, \lambda), \Phi_k(P, \lambda)} \right] \cdot \Phi_k(P, \lambda).
\] (21)

Thus, \( (Z_k, \Phi_k(P, \lambda))_{L^2(X)} = 0 \), and furthermore \( d_{(L, \lambda)} \Phi_k(W_k) = d_{\Phi_k(P, \lambda)}(Z_k) \) by (20). Suppose now that \( W' = W(f', \ell') \in T_{(L, \lambda)} W_0, f_0 \) is a second tangent vector, and let \( W'_k \) and \( Z'_k \) be defined as \( W_k \) and \( Z_k \), starting from \( W' \). Then using Proposition 5.2 and the above we obtain an asymptotic expansion

\[
(Z_k, Z'_k)_{L^2(X)} \sim k^{2+n/2} \cdot r^2 \left( \frac{2}{\pi} \right)^{n/2} \int_L F(\tilde{W}, \tilde{W}') \cdot \text{dens}_L + \sum_{h \geq 1} L_h k^{2+(n-h)/2}.
\]

Again in view of Proposition 5.2 we deduce from (19):

\[
\left( d_{(L, \lambda)} \Phi_k(W_k), d_{(L, \lambda)} \Phi_k(W'_k) \right)_{\Phi_k(L, \lambda)} = \left( d_{\Phi_k(P, \lambda)}(Z_k), d_{\Phi_k(P, \lambda)}(Z'_k) \right)_{[\Phi_k(P, \lambda)]} (22)
\]

\[
= \frac{(Z_k, Z'_k)_{L^2(X)}}{\Phi_k(P, \lambda), \Phi_k(P, \lambda)} \sim k^2 \cdot \int_L F(\tilde{W}, \tilde{W}') \cdot \text{dens}_L + \sum_{h \geq 1} L_h k^{2-h/2}.
\]

Given Remark 3.1 to complete the proof of Theorem 1 we need only take real and imaginary parts in (22). \( \square \)

**References**

[1] S. Bates, A. Weinstein, *Lectures on the geometry of quantization*, Berkeley Mathematics Lecture Notes 8, AMS 1997

[2] D. Borthwick, T. Paul, A. Uribe, *Legendrian distributions with applications to relative Poincaré series*, Invent. Math. 122 (1995), no. 2, 359–402

[3] L. Boutet de Monvel, J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegö*, Astérisque 34-35 (1976), 123–164

[4] M. Debernardi, R. Paoletti, *Equivariant asymptotics for Bohr-Sommerfeld Lagrangian submanifolds*, to appear in Comm. Math. Phys.

[5] A. L. Gorodentsev, A. N. Tyurin, *Abelian Lagrangian algebraic geometry*, Izv. Ross. Akad. Nauk Ser. Mat. 65:3 (2001), 15–50; English transl., Izv. Math. 65 (2001), 437–467

[6] V. Guillemin, S. Sternberg, *Geometric quantization and multiplicities of group representations*, Inven. Math. 67 (1982), 515-538
| Reference | Author(s) | Title | Details |
|-----------|-----------|-------|---------|
| 7 | V. Guillemin, S. Sternberg | *The Gelfand-Cetlin system and quantization of the complex flag manifold*, J. Func. Anal. **52** (1983), 106–128 |
| 8 | L. Hörmander | *The analysis of partial differential operators I*, Springer-Verlag 1990 |
| 9 | D. McDuff, D. Salamon | *Introduction to symplectic topology*, Clarendon Press Oxford 1995 |
| 10 | B. Shiffman, S. Zelditch | *Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds*, J. Reine Angew. Math. **544** (2002), 181–222 |
| 11 | G. Tian | *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), no. 1, 99–130 |
| 12 | A. Weinstein | *Connections of Berry and Hannay type for moving Lagrangian submanifolds*, Adv. Math. **82** (1990), 133–159 |
| 13 | S. Zelditch | *Szegő kernels and a theorem of Tian*, Int. Math. Res. Not. **6** (1998), 317–331 |