Deviations from Scale Invariance near a General Conformal Background

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Abstract

Deviations from scale invariance resulting from small perturbations of a general two dimensional conformal field theory are studied. They are expressed in terms of beta functions for renormalization of general couplings under local change of scale. The beta functions for homogeneous background are given perturbatively in terms of the data of the original conformal theory without any specific assumptions on its nature. The renormalization of couplings to primary operators and to first descendents is considered as well as that of couplings of a dilatonic type which involve explicit dependence on world sheet curvature.

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1
1 Introduction

The condition for a consistent background for motion of a critical string is that the corresponding world sheet model should be a conformal field theory with a specified total anomaly. This means that the theory is a fixed point of the renormalization group with vanishing $\beta$-functions. For a string moving in $d$-dimensional target space this requirement was shown long ago to imply Einstein equation on the geometry of target space to leading order in the deviation from a flat space [1]. It was also found [2], that the target space field corresponding to the dilaton particle is expressed in this context as an additional coupling in the world sheet theory which involves explicitly the world sheet curvature. This type of coupling is induced by renormalization and through the requirement of vanishing of $\beta$-function also the dilaton field acquires its appropriate target space dynamics.

A flat $d$-dimensional target space is just a very special case of a conformal field theory. Although there is no general classification of all these theories, there is a general framework which assigns to each conformal theory some characterizing data like its central charge, chiral algebra, spectrum of primary fields and operator product coefficients [3], [4]. Each such a theory can consist a part of a consistent background for motion of a string. It is natural then to generalize the $\sigma$-model approach described above and ask for the deviation from scale invariance resulting from small general perturbation around any conformal field theory. This deviation should involve $\beta$-functions expressed universally in terms of the conformal data of the perturbed theory without reference to its specific nature.

Concentrating on the flow of the coupling to the Virasoro primary fields, which are most relevant, these $\beta$-functions were written to second order in the perturbations by Zamolodchikov [5] and Cardy and Ludwig [6]. They found the simple expression

$$\beta^i = \epsilon_i g^i + \sum_{j,k} C_{jk}^{i} g^j g^k \quad (1)$$

where $g^i$ is the coupling corresponding to a primary operator of dimension $2 - \epsilon_i$. For almost marginal perturbation eq. (1) can be used to find approximate new fixed points
near the original one \cite{5,6}.

The $\sigma$-model renormalization equations of ref. \cite{1} should actually be a special application of eq. (1) which deals with a general conformal field theory. However, since eq. (1) stops at the level of primary fields the full structure of the sigma model is not revealed, in particular dilaton couplings are not included. It is the purpose of this work to extend eq. (1) to the first descendent level and to take into account operators which are explicitly sensitive to the world sheet metric being proportional to its curvature. We will still ignore higher level descendents and higher powers and derivatives of the world sheet curvature, which are also induced by the renormalization flow in a form dictated by the particular regularization scheme being used, but are less relevant. Since we do not commit ourselves to any special property of the conformal theory perturbed around, we cannot use regularizations which depend on such properties, e.g. schemes that rely on Feynmann integrals. Rather, following \cite{5} and \cite{6}, we will use the straightforward and intuitive cut off scheme which does not allow interaction vertices to get too close to each other on the world sheet.

Unlike \cite{5} and \cite{6} we need a renormalization scheme which is sensitive to the world sheet geometry. We will, therefore, adopt the point of view that the renormalization flow is induced by a local conformal change of the metric rather than by a global change in cut off scale \cite{7,8}. In other words we are actually calculating the trace of the energy momentum tensor of the perturbed model, including quantum corrections. The dependence of the action on the background metric is dictated by its required general covariance. Of course, once the change of scale is made local, the responding renormalization of coupling constant is local as well resulting in a theory with couplings which vary from point to point on the world sheet. Also operators of nonzero spin are induced which couple to derivatives of the curvature. We will choose our perturbed background to include only constant coupling on a flat world sheet geometry, and study the infinitesimal renormalization response of this background to local change of scale. For the purpose of identifying new approximate conformal points, for which this response vanishes, that should be enough, since at a conformal point there is no dependence on the background world sheet metric and it can always be brought into a flat form. We will follow the renormalization of primary fields and of first level descendents including dilatonic interactions proportional to the scalar world sheet curvature. Higher powers and derivatives of the curvature will not be followed being
less relevant. Since no derivatives of the curvature are involved, only scalar operators, of total spin 0, are induced.

The kind of analysis done here has also bearing on a more ambitious problem. Suppose we try to couple the non conformal perturbed model to quantum two dimensional gravity. Then one should integrate over all local scales where for each metric the matter action is the running one, the solution of the local renormalization group. So just to formulate the action of the matter plus gravity, the local renormalization group including all the metric dependent terms should be solved. Of course our crude approximation including just the most relevant terms near flat geometry is still very far from that goal.

In the next section a general conformal model perturbed by primary fields will be studied. We will show that renormalization induces changes in the coupling constants as well as new perturbing non primary operators and dilatonic coupling. Then in section 3 we will include such perturbations in the original action and calculate the full renormalization flow to that level.

2 Primary Perturbations

Let \( S_0 \) be the action of some conformal field theory. Let \( O_i(z, \bar{z}) \) be primary fields within the theory with a respect to both left and right hand side Virasoro algebra of dimensions \( h_i \) and \( \bar{h}_i \) respectively. Besides the spectrum of dimensions the theory is characterized by operator product coefficients \( C_{ij}^k \), namely for any quantity \( A \) and operators \( O_i \) and \( O_j \), the expectation value in the \( S_0 \) theory of \( < O_i(z, \bar{z})O_j(\omega, \bar{\omega})A >_0 \) satisfies the expansion

\[
< O_i(z, \bar{z})O_j(\omega, \bar{\omega})A >_0 = \sum_k C_{ij}^k (z - \omega)^{h_k - h_i - h_j}(\bar{z} - \bar{\omega})^{\bar{h}_k - \bar{h}_i - \bar{h}_j} < O_k(\omega, \bar{\omega})A >_0
\]

\[
+ \sum_{\{m\},\{\bar{m}\}} \gamma_{\{m\}}^{(ijk)}(z - \omega)^{|m|}(\bar{z} - \bar{\omega})^{|\bar{m}|} < L_{-\{m\}} \bar{L}_{-\{\bar{m}\}} O_k(\omega, \bar{\omega})A >_0
\]

Here \( \{m\} \) stands for a series of positive integers \( m_1, m_2, \ldots, m_n, |m| = \sum_i m_i \) and \( L_{-\{m\}} = L_{-m_1}L_{-m_2} \ldots L_{-m_n} \). The coefficients \( \gamma_{\{m\}}^{(ijk)} \) are determined algebraically in terms of \( h_i, h_j \) and
Let us perturb the conformal theory by adding to its action some small combination of its primary operators

\[
S = S_0 + \int d^2z \sum_i a^{-\epsilon_i} g^i O_i(z, \bar{z})
\]  

(4)

Here \(\epsilon_i = 2 - h_i - \bar{h}_i\), \(a\) is some short distance cut off scale which is needed to define the theory since after the perturbation it is not conformal any more, and \(g^i\) is small dimensionless coupling constants. Eq. (4) defines the theory on the flat background metric. Since after perturbation the theory is not conformal its formulation depends also on the world sheet metric. Choose the conformal gauge

\[
ds^2 = e^{2\sigma} dz d\bar{z}
\]  

(5)

Note that in this gauge the only nonvanishing components of the connection are

\[
\Gamma^x_{zz} = 2\partial \sigma \quad \Gamma^z_{z\bar{z}} = 2\bar{\partial} \sigma
\]  

(6)

and the scalar curvature is

\[
R = -e^{-2\sigma} \partial \bar{\partial} \sigma
\]  

(7)

where we put \(\partial = \frac{\partial}{\partial z}\), \(\bar{\partial} = \frac{\partial}{\partial \bar{z}}\).

For general covariance the perturbing Lagrangian density should transform as a scalar density. The operator \(O_i\) of dimension \((h_i, \bar{h}_i)\) is actually a tensor with \(h_i\) indices of type \(z\) and \(\bar{h}_i\) of type \(\bar{z}\). To construct a scalar we must have \(h_i = \bar{h}_i\) i.e. \(O_i\) should have spin \(S_i = h_i - \bar{h}_i = 0\), and then contract the \(h_i\) indices by \((g^{zz})^{h_i} = e^{-2h_i} \sigma\). Further multiplying by \(\sqrt{g} = e^{2\sigma}\) to give a density, the perturbation on a general background metric becomes
\[ S = S_0 + \int d^2z \sum_i a^{-\epsilon_i} e^{\epsilon_i \sigma(z, \bar{z})} g^i O_i(z, \bar{z}) \] (8)

Let \( A \) be an arbitrary operator and \(< A >\) its expectation value in the perturbed theory. Up to second order in the couplings we have

\[
< A >= < A >_0 + \sum_i a^{-\epsilon_i} g_i \int d^2z \ e^{\epsilon_i \sigma(z, \bar{z})} < O_i(z, \bar{z}) A >_0 + \\
\frac{1}{2} \sum_{ij} a^{-(\epsilon_i + \epsilon_j)} g^i g^j \int d^2z d^2\omega \ e^{\epsilon_i \sigma(z, \bar{z})} + \epsilon_j \sigma(\omega, \bar{\omega}) \times \\
\times < O_i(z, \bar{z}) O_j(\omega, \bar{\omega}) A >_0 \theta(d(z, \omega) - a)
\] (9)

The \( \theta \)-function in (9) contains our coordinate independent regularization prescription which avoids the singularity in \(< O_i(z) O_j(\omega) A >\) at the coincidence of \( z \) and \( \omega \) by demanding that \( d(z, \omega) \), the geodesic distance between the points \( z \) and \( \omega \) in the metric (5) is larger than the cut off \( a \). Under an arbitrary small variation of the conformal factor of the metric(5), \( \sigma \rightarrow \sigma + \delta \sigma \), the variation of \(< A >\) is

\[
\delta < A > = \frac{c}{6\pi} \int d^2z \ e^{2\sigma} R(z, \bar{z}) < A >_0 \delta \sigma(z, \bar{z}) + \\
+ \sum_i \int d^2z \ G(z, \epsilon_i) \epsilon_i g^i < O_i(z, \bar{z}) A >_0 \delta \sigma(z, \bar{z}) + \\
+ \frac{1}{2} \sum_{ij} g^i g^j \int d^2z d^2\omega \ G(z, \epsilon_i) G(\omega, \epsilon_j) (\epsilon_i \delta \sigma(z, \bar{z}) + \epsilon_j \delta \sigma(\omega, \bar{\omega})) \times \\
\times < O_i(z, \bar{z}) O_j(\omega, \bar{\omega}) A >_0 \theta(d(z, \omega) - a) + \\
+ \frac{1}{2} \sum_{ij} g^i g^j \int d^2z d^2\omega \ G(z, \epsilon_i) G(\omega, \epsilon_j) < O_i(z, \bar{z}) O_j(\omega, \bar{\omega}) A >_0 \times \\
\times \delta(d(z, \omega) - a) \delta d(z, \omega)
\] (10)

Here and further the following notation is used

\[
G(z, \epsilon) = e^{\epsilon \sigma(z, \bar{z})} a^{-\epsilon}
\]
The first term in (10) which is of zero order in the couplings expresses the fact that even before the perturbation the original conformal theory has some dependence on \( \sigma \) through the conformal anomaly. In (10) \( c \) is the central charge of the unperturbed conformal theory. The second term in (10), first order in the coupling, is the classical scale dependance of the model resulting from its dimensionful couplings \( g^i a^{-\epsilon_i} \). The third term in (10) is just the second order iteration of this classical dependence. The last term is an extra quantum mechanical scale dependence arising from the presence of the cut off in the theory. Here \( \delta d(z, \omega) \) is the variation in the geodesic distance of \( z \) from \( \omega \) resulting from the variation of \( \sigma \). To avoid any necessity to specify the \( \sigma \) dependence of the operator \( A \), we can always chose the arbitrary variation \( \delta \sigma \) to vanish on the support of \( A \).

Up to first order in \( z \) and also \( \bar{z} \) derivatives express \( d(z, \omega) \) in terms of \( \sigma \) near \( \omega \) as (see appendix)

\[
d(z, \omega) = e^{\sigma(\omega)}|z - \omega|(1 + \frac{1}{2}(\partial \sigma(\omega)(z - \omega) + \bar{\partial} \sigma(\omega)(\bar{z} - \bar{\omega})) + \frac{1}{3}\bar{\partial} \partial \sigma(\omega)|z - \omega|^2 + \frac{1}{4}\bar{\partial} \sigma(\omega)\bar{\partial} \sigma(\omega)|z - \omega|^2)
\]

which gives to this order for \( \delta d(z, \omega) \) of (10)

\[
\delta d(z, \omega) = e^{\sigma(\omega)}\left\{y\left[1 + \frac{1}{2}y(e^{i\theta} \partial \sigma(\omega) + e^{-i\theta} \bar{\partial} \sigma(\omega)) + \frac{1}{3}y^2\bar{\partial} \partial \sigma(\omega) + \frac{1}{4}y^2 \bar{\partial} \sigma(\omega)\bar{\partial} \sigma(\omega)\right]\right. + \frac{1}{2}y^2(e^{i\theta} \partial \delta \sigma(\omega) + e^{-i\theta} \bar{\partial} \delta \sigma(\omega)) + \frac{1}{3}y^3\partial \partial \delta \sigma(\omega)
\]

where the real variables \( y \) and \( \theta \) stand for

\[
z - \omega = ye^{i\theta}
\]

The same change of variables can be made in the last term of (10). For any function of \( y \) and \( \theta \), \( f(y, \theta) \),
\[
\int d^2z d^2\omega (d(z, \omega) - a) f(y, \theta) = \int dydyd\theta \delta(y - a) \frac{\partial y(d, \theta)}{\partial d} |_{d=a} f(y, \theta) \\
= \int d\theta y(a, \theta) \frac{\partial y(a, \theta)}{\partial a} f(a, \theta)
\]

Here \(y(a, \theta)\) is the value of \(|z - y|\) for which \(d(z, \omega) = a\) where the phase of \(z - \omega\) is \(\theta\).

From (11) we read
\[
y(a, \theta) = e^{-\sigma(\omega)} a \left( 1 - \frac{1}{2}ae^{-\sigma(\omega)}(\partial \sigma e^{i\theta} + \bar{\partial} \sigma e^{-i\theta}) - \frac{a^2}{3} e^{-2\sigma(\omega)} \partial \bar{\partial} \sigma \right)
\]

+ higher order in \(\partial \sigma, \bar{\partial} \sigma\)

and
\[
\frac{\partial y(a, \theta)}{\partial a} = e^{-\sigma(\omega)} \left[ 1 - ae^{-\sigma(\omega)}(\partial \sigma e^{i\theta} + \bar{\partial} \sigma e^{-i\theta}) - a^2 e^{-2\sigma(\omega)} \partial \bar{\partial} \sigma(\omega) \right]
\]

The last term of (10) becomes, after applying (13) with the substitution of (12), (14) and (15)
\[
\frac{1}{2} \sum_{ij} g^i g^j \int d^2\omega G(z, \epsilon_i) G(\omega, \epsilon_j) < O_i(z, \bar{z}) O_j(\omega, \bar{\omega}) A >_0 \times
\]
\[
\times e^{-\sigma(\omega)} a \left[ 1 - \frac{1}{2}ae^{-\sigma(\omega)}(\partial \sigma e^{i\theta} + \bar{\partial} \sigma e^{-i\theta}) - \frac{a^2}{3} e^{-2\sigma(\omega)} \partial \bar{\partial} \sigma \right] e^{-\sigma(\omega)} \times
\]
\[
\times \left[ 1 - ae^{-\sigma(\omega)}(\partial \sigma e^{i\theta} + \bar{\partial} \sigma e^{-i\theta}) - a^2 e^{-2\sigma(\omega)} \partial \bar{\partial} \sigma \right] \times
\]
\[
\times \left[ a \partial \delta \bar{\sigma} + \frac{1}{2} a^2 e^{-\sigma(\omega)}(e^{i\theta} \partial \delta \sigma + e^{-i\theta} \bar{\partial} \delta \sigma) - \right.
\]
\[
- \frac{1}{4} a^3 e^{-2\sigma(\omega)}(\partial \delta \sigma \bar{\partial} \delta \sigma + \partial \sigma \bar{\partial} \delta \sigma) + \frac{1}{3} a^3 e^{-2\sigma(\omega)} \partial \bar{\partial} \delta \sigma(\omega) \right]
\]

Expanding in (14) \(G(z, \epsilon_i)\) and the operator product around \(\omega\) using (2), the variation (10) becomes
\[
\delta < A > = \frac{c}{6\pi} \int d^2\omega e^{2\sigma} R(\omega, \bar{\omega}) \delta \sigma(\omega, \bar{\omega}) +
\]
\[
+ \sum_i \int d^2\omega G(\omega, \epsilon_i) \epsilon_i g^i < O_i(\omega, \bar{\omega}) A >_0 \delta \sigma(\omega, \bar{\omega}) +
\]
Here γ\textsubscript{ijk} stands for γ\textsuperscript{(ijk)}\textsubscript{1} of eq. (3) namely

\begin{equation}
\gamma_{ijk} = \frac{h_i - h_j + h_k}{2h_k}
\end{equation}

and similarly for \overline{\gamma}_{ijk}. Also α\textsubscript{ijk} is a short notation for

\begin{equation}
α_{ijk} = h_k + \overline{h}_k - h_i - \overline{h}_i - h_j - \overline{h}_j
\end{equation}

and \textit{S} \textsubscript{k} = h_k - \overline{h}_k is the spin of the operator \textit{O} \textsubscript{k}. \textit{e} \textsubscript{k} = 2 - h_k - \overline{h}_k expresses the deviation of \textit{O} \textsubscript{k} from marginality.

Eq. (17) is valid up to first order in \partial σ, \overline{∂} σ and \partial \overline{∂} σ, and up to the first descendents of \textit{O} \textsubscript{k} under left and right hand side Virasoro algebra. In the same spirit we shall take into account operators \textit{O} \textsubscript{k} such that \textit{S} \textsubscript{k} = 0, +1, −1. To this order integrating by parts the \partial \overline{∂} σ terms and performing the \theta integration we get
\[ \delta < A > = \frac{c}{6\pi} \int d^2 \omega \ e^{2\sigma} R(\omega, \bar{\omega}) \delta \sigma(\omega, \bar{\omega}) + \]
\[ + \sum_i \int d^2 \omega \ G(\omega, \epsilon_i) \epsilon_i g^i < O_i(\omega, \bar{\omega}) A >_0 \delta \sigma(\omega, \bar{\omega}) + \]
\[ + \frac{1}{2} \sum_{ij} g^i g^j \int d^2 z d^2 \omega G(z, \epsilon_i) G(\omega, \epsilon_j) \times \]
\[ \times (\epsilon_i \delta \sigma(z, \bar{z}) + \epsilon_j \delta \sigma(\omega, \bar{\omega})) < O_i(z, \bar{z}) O_j(\omega, \bar{\omega}) A >_0 \theta(d(z, \omega) - a) + \]
\[ + \pi \sum_{ijk} C^i_{kj} \int d^2 \omega g^i g^j \left\{ G(\omega, \epsilon_k) < O_k(\omega) A >_0 + \right. \]
\[ \left. + (\gamma^2_{ijk} - \gamma_{ijk} + \frac{1}{3}) G(\omega, \epsilon_k - 2) \partial_\omega \partial_\sigma < O_k(\omega) A >_0 - \right. \]
\[ - 2h^i_k (\gamma^2_{ijk} - \gamma_{ijk} + \frac{1}{3}) G(\omega, \epsilon_k - 2) (\partial \sigma(\omega) \bar{\partial} < O_k(\omega) A >_0 + \bar{\partial} \sigma(\omega) \partial < O_k(\omega) A >_0) + \]
\[ + \frac{1}{6} (\epsilon_i + \epsilon_j + \epsilon_k - 2) G(\omega, \epsilon_k - 2) \partial \partial \sigma(\omega) < O_k(\omega) A >_0 \right] \delta S_{k,0} + \]
\[ + G(\omega, \epsilon_k - 1) \left[ \frac{\bar{h}_i - \bar{h}_j}{2h_k} \bar{\partial} < O_k(\omega) A >_0 - (\bar{h}_i - \bar{h}_j) \bar{\partial} \sigma < O_k(\omega) A >_0 \right] \delta S_{k,1} + \]
\[ + G(\omega, \epsilon_k - 1) \left[ \frac{h_i - h_j}{2h_k} \partial < O_k(\omega) A >_0 - (h_i - h_j) \partial \sigma < O_k(\omega) A >_0 \right] \delta S_{k,-1} \} \delta \sigma(\omega) \]

In the gauge (3), due to (6), the covariant derivative of \( O_k \) is

\[ DO_k = \partial O_k - 2h_k \partial \sigma O_k \] (21)

and, to first order in \( \sigma \) derivatives,

\[ D \bar{D} O_k = \bar{\partial} \bar{\partial} O_k - 2h_k \bar{\partial} \sigma \bar{\partial} O_k - 2\bar{h}_k \partial \sigma \partial O_k - 2\bar{h}_k \bar{\partial} \sigma \bar{\partial} O_k \] (22)

Note that the coefficients in (20) of \( \partial < O_k A > \) and \( \partial \sigma < O_k A > \) and also of \( \partial \bar{\partial} < O_k A > \) and \( \partial \sigma \bar{\partial} < O_k A > \) are related precisely such that their combination are expressible in terms of covariant derivatives. Also the term involving \( \bar{\partial} \partial \sigma \), is expressible in terms of the scalar world sheet curvature due to (5). This results of course from our coordinate independent regularization procedure.
The variation $\delta \sigma$ induces a change in the local cut off scale $ae^{-\sigma}$. In the spirit of the renormalization group, one looks for a corresponding local change in the coupling constants $g^i \rightarrow g^i + \beta^i \delta \sigma$ which will compensate for the change of scale such that $< A >$ calculated with the new couplings for the new cut off scale will equal to $< A >$ with the old couplings for the original scale. The appearance in $\delta < A >$ of (20) of terms involving $D < O_k A >$, $D \bar{D} < O_k A >$ and $R < O_k A >$ implies that the renormalization should not only change the values of existing couplings to primary operators but also induce new types of couplings to descendent operators of the type $DO_k$ and $D \bar{D}O_k$ and to dilatonic type operators of the form $RO_k$. Let us denote the coupling to the operator $DO_k$ by $g^k_z$, that of $D \bar{D}O_k$ by $g^k_{\bar{z}}$, and the dilaton coupling to $RO_k$ by $\bar{g}^k$. The requirement of the local renormalization flow to preserve physical quantity under change of scale is expressed by the corresponding Callan Symanzik equation

$$
\delta < A > + \int d^2 \omega \sum_k \left( \beta^k \frac{\delta}{\delta g^k} + \beta^k \frac{\delta}{\delta g^k_{\bar{z}}} + \beta^k \frac{\delta}{\delta g^k_z} + \beta^k \frac{\delta}{\delta g^k_{\bar{z}}}ight) < A > \delta \sigma(\omega) = 0
$$

Since $\frac{\delta < A >}{\delta g^k} = -a^{-\epsilon_k} e^{\epsilon_k \sigma(\omega)} < O_k(\omega)A >$, eq. (23) reads

$$
\delta < A > = \int d^2 \omega \sum_k \left\{ \beta^k G(\omega, \epsilon_k) < O_k(\omega)A > + G(\omega, \epsilon_k - 1) (\beta^k D_\omega < O_k(\omega)A > + \beta^k D_\bar{\omega} < O_k(\omega)A >) + \beta^k \times G(\omega, \epsilon_k - 2) D_\omega D_\bar{\omega} < O_k(\omega)A > - \bar{g}^k G(\omega, \epsilon_k - 2) \partial \bar{\partial} \sigma(\omega) < O_k(\omega)A > \right\} \delta \sigma(\omega)
$$

We have already an expression for $\delta < A >$ up to second order in the perturbation in eq.(20). Expand also each $\beta$-function and each expectation value in the r.h.s. of (24) to this order and equate term by term to (20) to get,

$$
\beta^k = \epsilon_k g^k + \delta_{S_k,0} \pi \sum_{ij} C_{ij} g^i g^j
$$

$$
\beta^k_z = \delta_{S_k,-1} \pi \sum_{ij} C_{ij} g^i g^j h_i - h_j \frac{2 h_k}{2h_k}
$$
\[ \beta^k_z = \delta S_{k,1} \pi \sum_{ij} C_{ij}^k g^i \bar{h}_i - \bar{h}_j \]

\[ \beta^{k}_{zz} = \delta S_{k,0} \pi \sum_{ij} C_{ij}^k g^i \bar{g}^j (\gamma_{ijk}^2 - \gamma_{ijk} + \frac{1}{3}) \]

\[ \bar{\beta}^k = \frac{c}{6\pi} \delta_{k,0} - \delta S_{k,0} \pi \sum_{ij} C_{ij}^k g^i \bar{g}^j \left[ \frac{1}{6} (\epsilon_i + \epsilon_j - \epsilon_k - 2) + 2h_k (\gamma_{ijk}^2 - \gamma_{ijk} + \frac{1}{3}) \right] \]

where \( O_0 \) stands for the unit operator.

The first equation in (25) is the renormalization of primary couplings found in [5, 6], the other equations are its generalization to leading descendent and dilatonic operators. We see from our analysis, that even if originally the perturbation is primary, descendent operators, in principle of any order, and couplings to any power of the world sheet curvature are induced by the renormalization group. Induced derivative type operators like \( DO_k \) or \( D\bar{D}O_k \) are certainly significant in the local renormalization framework where they have non constant coupling. In our way of regularization they have non trivial influence beyond first order even for a background with constant couplings in spite of their appearance as total derivatives. This is because of the boundary terms at the edge of the small disc of radius \( a \) present in this scheme.

### 3 Inclusion of non primary couplings

Since non primary and dilatonic couplings are induced in the process of renormalization flow, in order to follow this flow one has to know the \( \beta^- \) function for initial perturbations of this type. Here we will repeat the calculation of the previous section for a perturbation more general than that of eq.(8). We take the perturbed action to be

\[ S = S_0 + \int d^2 z \sum_i \left( e^{\epsilon_i \sigma(z)} a^{-\epsilon_i} g^i O_i(z, \bar{z}) + e^{(\epsilon_i-1)\sigma(z)} a^{1-\epsilon_i} (g^i_+ D_+ O_i(z, \bar{z}) + g^i_- D_- O_i(z, \bar{z})) \right) \]

\[ + g^i_+ D_+ O_i(z, \bar{z}) + g^i_- D_- O_i(z, \bar{z}) + g^i e^{\epsilon_i \sigma(z)} a^{2-\epsilon_i} R(z, \bar{z}) O_i(z, \bar{z}) \]

We assume constant couplings, for general covariance of the action they should couple to
scalars. Hence $g^i_z$ is non zero only for an operator $O_i$ of spin -1, $g^i_z$ exists only for $O_i$ of spin 1 and all the remaining couplings in (26) involve operators of spin 0.

Up to second order in the couplings and first order in $z$ and $\bar{z}$ derivatives, the expectation of an arbitrary operator $A$ is

$$< A > = < A >_0 + \int d^2\omega \sum_i \left[ G(\omega, \epsilon_i) g^i < O_i(\omega)A >_0 + \right.$$  
$$+ G(\omega, \epsilon_i - 1)(g^i_z D_\omega < O_i(\omega)A >_0 + g^i_{\bar{z}} D_{\bar{\omega}} < O_i(\omega)A >_0) + $$  
$$+ G(\omega, \epsilon_i - 2)(g^i_{zz} D_\omega D_{\bar{\omega}} < O_i(\omega)A >_0 - g^i D_\omega < O_i(\omega)A >_0) \right] + $$  
$$+ \sum_{ij} \int d^2z d^2\omega \theta(d(z, \omega) - a) \left[ \frac{1}{2} G(z, \epsilon_i) G(\omega, \epsilon_j) g^i g^j < O_i(z)O_j(\omega)A >_0 + \right.$$  
$$+ G(z, \epsilon_i) G(\omega, \epsilon_j - 1) g^i (g^j_z D_\omega + g^j_{\bar{z}} D_{\bar{\omega}}) < O_i(z)O_j(\omega)A >_0 + $$  
$$+ G(z, \epsilon_i) G(\omega, \epsilon_j - 2) [g^i g^j_{zz} D_\omega D_{\bar{\omega}} < O_i(z)O_j(\omega)A >_0 - $$  
$$- g^i \bar{g}^j \partial \bar{\partial} \sigma(\omega) < O_i(z)O_j(\omega)A >_0] + $$  
$$+ g^j_z g^j_{\bar{z}} G(z, \epsilon_i - 1) G(\omega, \epsilon_j - 1) D_z D_{\bar{\omega}} < O_i(z)O_j(\omega)A >_0 \right\}.$$

Again, vary $\sigma$ into $\sigma + \delta \sigma$ with arbitrary $\delta \sigma$ which vanishes on the support of the operator $A$, find $\delta < A >$ to second order and substitute into the Callan Symanzik equation (23) to determine the $\beta$ -functions perturbatively. To first order one has the dimension counting result

$$\beta^k = \epsilon_k g^k$$

$$\tilde{\beta}^k = \frac{c}{6\pi} \delta_{k,0} + (\epsilon_k - 2) \tilde{g}^k$$

$$\beta^k_z = \beta^k_{\bar{z}} = 0$$

$$\beta^k_{zz} = -\tilde{g}^k$$

Note that the explicit $\sigma$ dependence in the coefficient $e^{(\epsilon_i - 1)\sigma}$ of $g^i_z DO_i$ exactly cancels against the extra $\sigma$ dependence in the covariant derivative, eq. (21), to give no $g^i_z$ dependence in $\beta$ to first order, as should be expected since to this order the coupling $DO_k$ is
indeed a spurious total derivative. The explicit $\sigma$ dependence in the $R$ term contributes to the first order $\beta$-function of $g^k_{zz}$.

The presence of the dilatonic and descendent couplings in (26) introduces additional sources of $\sigma$ dependence compared to that of the previous section. For example let us consider the second order contribution to $\delta < A >$ proportional to $g^i\tilde{g}^j$. Varying this term in (27) we get

$$\delta < A >= -\sum_{ij} g^i\tilde{g}^j \int d^2z d^2\omega G(z,\epsilon_i)G(\omega,\epsilon_j - 2) \left[ (\epsilon_i \delta\sigma(z) + (\epsilon_j - 2)\delta\sigma(\omega)) \times \right. \left. (29) \right.$$

$$\times \partial\bar{\partial}\sigma(\omega) - O_i(z)O_j(\omega)A >_0 \theta(d(z,\omega) - a) + \partial\bar{\partial}\sigma(\omega) - O_i(z)O_j(\omega)A >_0 \times$$

$$\times \delta(d(z,\omega) - a)\delta d(z,\omega) + \partial\bar{\partial}\delta\sigma(\omega) - O_i(z)O_j(\omega)A >_0 \theta(d(z,\omega) - a) \right] +$$

$$+ \text{other terms not proportional to } g^i\tilde{g}^j$$

The first two terms are familiar from previous section. The first term of $\delta < A >$ in (29) is taken care of by the first order piece of the $\beta$-functions $\beta^i$ and $\tilde{\beta}^j$ of eq. (28) when substituted into the Callan Symanzik equation (24). The second term which involves $\delta d(z,\omega)$ is treated exactly as in previous section and we can copy from eq. (20) after integrating over the $\delta$- function and utilizing operator product expansion the form for this term,

$$-2\pi \sum_{ijk} C^k_{ij} \int d^2\omega \ g^i\tilde{g}^j G(\omega,\epsilon_k - 2)\partial\bar{\partial}\sigma(\omega) < O_k(\omega)A >_0 \ (30)$$

thus contributing to $\tilde{\beta}^k$.

The third term in (29) comes from the explicit dependence of the dilatonic coupling on $\sigma$ through the curvature factor. Integrating $\partial_\omega \partial_\omega \delta\sigma$ by parts this term gives rise to

$$-\sum_{ij} g^i\tilde{g}^j \int d^2z d^2\omega \ G(z,\epsilon_i)G(\omega,\epsilon_j - 2) \times \ (31)$$

$$\times \left[ D_\omega \bar{D}_\omega < O_i(z)O_j(\omega)A >_0 \theta(d(z,\omega) - a) + D_\omega < O_i(z)O_j(\omega)A >_0 \bar{D}_\omega \theta(d(z,\omega) - a) + \right.$$

$$\left. + \bar{D}_\omega < O_i(z)O_j(\omega)A >_0 \partial_\omega \theta(d(z,\omega) - a) + < O_i(z)O_j(\omega)A >_0 \partial_\omega \bar{D}_\omega \theta(d(z,\omega) - a) \right]$$

14
The first term in (31), which involves double integration over \( z \) and \( \omega \) with no \( \delta \)-function, does not induce a new second order \( \beta \)-function. Rather, it is taken care of by the first order contribution to \( \beta_{zz}^{2} \) in (28), multiplied by the first order correction to \( DD < O_{j}A > \) in the term \( \beta_{zz}^{2} D\bar{D} < O_{j}A > \) of the Callan Symanzik equation (24). For the remaining 3 terms in (31) use the identities

\[
\partial_{\omega} \theta(d(z, \omega) - a) = \delta(d(z, \omega) - a) \partial_{\omega} d(z, \omega) \\
\partial_{\omega} \bar{\partial}_{\bar{c}} \theta(d(z, \omega) - a) = \delta'(d(z, \omega) - a) \partial_{\omega} d(z, \omega) + \delta(d(z, \omega) - a) \partial_{\omega} \bar{\partial}_{\bar{c}} d(z, \omega)
\]

The \( \delta \)-functions on the r.h.s. of (32) make sure that in the last 3 terms in (31) the double integral on \( z \) and \( \omega \) is effectively only integrals over \( \omega \) which can be canceled by local second order counter terms in \( \beta \)-functions. To compute the required terms pass again to the variables \( ye^{i\theta} = z - \omega \) and integrate the \( \delta \)-function using (13) and the formula

\[
\int dy f(y, \theta) \delta'(d(y, \theta) - a) = -\frac{\partial^2 y(a, \theta)}{\partial a^2} f(y(a, \theta)) - \left( \frac{\partial y(a, \theta)}{\partial a} \right)^2 f'(y(a, \theta))
\]

valid for any function of \( y \) and \( \theta \). For the quantities \( \partial_{\omega} d(z, \omega) \) and \( \partial_{\omega} \bar{\partial}_{\bar{c}} d(z, \omega) \) appearing in (32), derive from (11) their expansion in \( \sigma \) derivatives

\[
\partial_{\omega} d(z, \omega) = -\frac{1}{2} e^{\sigma(\omega)} e^{-i\theta} \left[ 1 - \frac{y}{2} (\partial \sigma(\omega)e^{i\theta} - \bar{\partial} \sigma(\omega)e^{-i\theta}) \right] \\
\partial_{\omega} \bar{\partial}_{\bar{c}} d(z, \omega) = \frac{e^{\sigma(\omega)}}{4y} \left[ 1 - \frac{y}{2} (\partial \sigma(\omega)e^{i\theta} + \bar{\partial} \sigma(\omega)e^{-i\theta}) + y^2 \partial \bar{\partial} \sigma(\omega) \right]
\]

As an example consider the second term in (31) involving \( D_{\omega} < O_{i}(z)O_{j}(\omega)A >_0 \). We have

\[
- \sum_{ij} \int d^2z d^2\omega G(z, \epsilon_i)G(\omega, \epsilon_j - 2) D_{\omega} < O_{i}(z)O_{j}(\omega)A >_0 \bar{\partial}_{\omega} \theta(d(z, \omega) - a) \delta \sigma(\omega)
\]
In terms of $y$ and $\theta$

$$\partial_\omega = -\frac{e^{-i\theta}}{2} \frac{\partial}{\partial y} - \frac{1}{2y} \frac{\partial}{\partial e^{i\theta}}$$

Apply this to the operator product formula,

$$< O_i(z)O_j(\omega)A >_0 = \sum_k C_{ij}^k y^{\alpha_{ijk}} e^{i\theta S_{ijk}} [< O_k(\omega)A >_0 +$$

$$+ y(\gamma_{ijk} e^{i\theta} \partial_\omega < O_k(\omega)A >_0 + \tilde{\gamma}_{ijk} e^{-i\theta} \partial_\omega < O_k(\omega)A >_0 + y^2 \gamma_{ijk} \tilde{\gamma}_{ijk} \partial_\omega \tilde{\partial}_\omega < O_k(\omega)A >_0]$$

with $S_{ijk} = S_k - S_i - S_j$ to get

$$D_\omega < O_i(z)O_j(\omega)A >_0 = (\partial_\omega - 2h_j \partial \sigma(\omega)) < O_i(z)O_j(\omega)A >_0 =$$

$$= -\sum_k C_{ij}^k \left\{ \left[ \frac{1}{2} (\alpha_{ijk} + S_{ijk} + 2h_j y e^{i\theta} \partial \sigma(\omega)) \right] < O_k(\omega)A >_0 +$$

$$+ \left[ \left( \frac{1}{2} (\alpha_{ijk} + S_{ijk} + 2) + 2h_j y e^{i\theta} \partial \sigma(\omega) \right) \gamma_{ijk} - 1 \right] y e^{i\theta} \partial_\omega < O_k(\omega)A >_0 +$$

$$+ \left( \frac{1}{2} (\alpha_{ijk} + S_{ijk} + 2) + 2h_j y e^{i\theta} \partial \sigma(\omega) \right) \tilde{\gamma}_{ijk} y e^{-i\theta} \tilde{\partial}_\omega < O_k(\omega)A >_0 +$$

$$+ y^2 \left[ \left( \frac{1}{2} (\alpha_{ijk} + S_{ijk} + 2) + 2h_j y e^{i\theta} \partial \sigma(\omega) \right) \gamma_{ijk} \tilde{\gamma}_{ijk} - (\gamma_{ijk} e^{2i\theta} + \tilde{\gamma}_{ijk}) \right] \times$$

$$\times \partial_\omega \tilde{\partial} < O_k(\omega)A >_0 \right\} y^{\alpha_{ijk}-1} e^{i\theta(S_{ijk}-1)}$$

We have then for (35)

$$- \sum_{ij} g_i g_j \int d^2zd^2\omega G(\omega, \epsilon_i + \epsilon_j - 2) \int d\theta \left[ 1 + \epsilon_i y(a, \theta) (\partial \sigma(\omega) e^{i\theta} + \partial \sigma(\omega) e^{-i\theta}) +$$

$$+ \epsilon_i y^2(a, \theta) \partial \tilde{\sigma} \sigma(\omega) \right] e^{i\theta} \left( -\frac{e^{\sigma(\omega)}}{2} \right) \left[ 1 + \frac{y(a, \theta)}{2} (\partial \sigma(\omega) e^{i\theta} - \partial \sigma(\omega) e^{-i\theta}) \right] \times$$

$$\times e^{-\sigma(\omega)} \left[ 1 - e^{-\sigma(\omega)} (\partial e^{i\theta} + \tilde{\partial} e^{-i\theta}) - e^{-2\sigma(\omega)} \partial \tilde{\sigma} \right] (-C_{ij}^k) y^{\alpha_{ijk}}(a, \theta) e^{i\theta(S_{ijk}-1)} \times$$
The first factor in (37) is the expansion of $e^{i\sigma(z)}$ about $\omega$, the second factor is $\bar{\partial}d(z, \omega)$ eq. (34). The third factor is $\frac{\partial y}{\partial \sigma}$. Again, as they should be, the coefficients of $y$ of (37) we have to substitute (21) and (22) to form covariant world sheet derivatives and (38) becomes:

$$\times \left[ \left( \frac{1}{2} (\alpha_{ijk} + S_{ijk}) - 2h_jy(a, \theta)e^{i\theta} \partial \sigma \right) < O_k(\omega)A > + \right. \right.$$ 

$$+ \left[ \left( \frac{1}{2} (\alpha_{ijk} + S_{ijk} + 2) + 2h_jy(a, \theta)e^{i\theta} \partial \sigma \right) \gamma_{ijk} - 1 \right] y(a, \theta)e^{i\theta} \partial \omega < O_k(\omega)A > + \right.$$ 

$$+ \left[ \left( \frac{1}{2} (\alpha_{ijk} + S_{ijk} + 2) + 2h_jy(a, \theta)e^{i\theta} \partial \sigma \right) \gamma_{ijk}\bar{\gamma}_{ijk} - \gamma_{ijk}e^{2i\theta} - \gamma_{ijk} \right] \times$$

$$\times y^2(a, \theta)\partial \omega < O_k(\omega)A > \right]$$

The first factor in (37) is the expansion of $e^{i\sigma(z)}$ about $\omega$, the second factor is $\bar{\partial}d(z, \omega)$ from eq. (34), the Jacobian resulting from integrating $\delta(d(y, \theta) - a)$ over $y$. The last factor is $D_\omega < O_i(z)O_j(\omega)A >$ from (36) multiplied by an extra $y$ from the measure $yd\theta$ of polar integration. Due to the $\delta(d(y, \theta) - a)$ for every $y$ of (37) we have to substitute $y(a, \theta)$ of eq. (14), the value of the coordinate $y$ for which $d(y, \theta) = a$. Substituting (14) and performing the $\theta$ integration, keeping operators of spin 0, +1, -1, and first order $\sigma$ derivatives, we get for (37),

$$-(2\pi) \sum_{ijk} C_{ij}^k g^i g^j \int d^2\omega \left\{ \left[ \frac{\alpha_{ijk}}{4} G(\omega, \epsilon_k) < O_k(\omega)A > + \right. \right.$$ 

$$+ \frac{1}{4} \left( (\alpha_{ijk} + 2) \gamma_{ijk} - 2 \right) \bar{\gamma}_{ijk} G(\omega, \epsilon_k - 2) \partial \bar{\partial} < O_k(\omega)A > + \right.$$ 

$$+ \frac{1}{4} \left( (\alpha_{ijk} + 2) \gamma_{ijk} - 2 \right) \left( \epsilon_i - \frac{\alpha_{ijk} + 4}{2} \right) G(\omega, \epsilon_k - 2) \partial \sigma \partial \bar{\partial} < O_k(\omega)A > + \partial \bar{\sigma} \partial \omega < O_k(\omega)A > + \right.$$ 

$$+ \frac{1}{4} \left( (\alpha_{ijk} + 1) \gamma_{ijk} - 2 \right) G(\omega, \epsilon_k - 1) \partial \omega < O_k(\omega)A > + \right.$$ 

$$+ \frac{1}{4} \left( (\alpha_{ijk} - 1) \left( \epsilon_i - \frac{\alpha_{ijk} + 1}{2} \right) - 4h_j \right) G(\omega, \epsilon_k - 1) \partial \sigma < O_k(\omega)A > + \right.$$ 

$$+ \frac{1}{4} \left( (\alpha_{ijk} + 1) \gamma_{ijk} G(\omega, \epsilon_k - 1) \partial \omega < O_k(\omega)A > + \right.$$ 

$$+ \frac{1}{4} (\alpha_{ijk} + 1) \left( \epsilon_i - \frac{\alpha_{ijk} + 3}{2} \right) G(\omega, \epsilon_k - 1) \partial \sigma < O_k(\omega)A > + \right\} \delta \sigma(\omega)$$

Again, as they should be, the coefficients of $\partial < O_kA >$ and of $\partial \sigma < O_kA >$ are related by (21) and (22) to form covariant world sheet derivatives and (38) becomes:

17
\[-(2\pi) \sum_{ijk} C^k_{ij} g^i g^j \int d^2 \omega \left\{ \left[ \frac{\alpha_{ijk}}{4} G(\omega, \epsilon_k) < O_k(\omega) A > + \right. \right. \]
\[+ \frac{1}{4} \left( (\alpha_{ijk} + 2) \gamma_{ijk} - 2 \right) \tilde{g}_{ijk} G(\omega, \epsilon_k - 2) \bar{D} \tilde{D} < O_k(\omega) A > + \]
\[+ \frac{1}{4} \left( \alpha_{ijk} \left( \epsilon_i - \frac{\alpha_{ijk} + 3}{3} \right) - (\alpha_{ijk} + 2) \gamma_{ijk} - 2 \right) \times \]
\[\times \left( \epsilon_i - \frac{\alpha_{ijk} + 4}{2} \right) G(\omega, \epsilon_k - 2) (\partial \bar{\partial} \sigma < O_k(\omega) A > \right] \delta S_{ijk,0} + \]
\[+ \frac{1}{4} \left( (\alpha_{ijk} + 1) \gamma_{ijk} - 2 \right) G(\omega, \epsilon_k - 1) D < O_k(\omega) A > \delta S_{ijk,-1} + \]
\[+ \frac{1}{4} \left( \alpha_{ijk} + 1 \right) \tilde{g}_{ijk} G(\omega, \epsilon_k - 1) \bar{D} < O_k(\omega) A > \delta S_{ijk,1} \right\} \delta \sigma(\omega) \]

From (39) the contribution of this particular term in (31) to the various second order \( \beta \)-functions are explicitly read off. The remaining terms in (31) are treated in the same way.

In a similar fashion the terms in \(< A >\) of eq. (27) involving \( g^i g^j, g^i z \bar{g}^j \) or \( g^i g^j z \bar{z} \) have also extra contributions to \( \delta < A >\) besides that coming from \( \delta d(z, \omega)\) discussed in previous section. All these terms contain \( \sigma \) derivatives inside the covariant derivatives, which contribute derivatives of \( \delta \sigma \) after the variation. Exactly as in eq. (31) when these \( \delta \sigma \) derivatives are integrated by parts, one gets boundary terms from the edges of integration region at the cut off circle of radius \( a \). These boundary terms are treated in the same way as we did to eq. (31) to extract their contributions to \( \beta \)-functions.

Collecting all contributions we have our final results for the \( \beta \)-functions induced by a perturbation of a general conformal theory by primary, first order descendent, and first order dilatonic operators with uniform coupling constants,

\[ \beta^k = \epsilon_k g^k + 2\pi \sum_{ij} C^k_{ij} \left[ \frac{1}{2} g^i g^j + \frac{\alpha}{4} g^i \bar{g}^j + \frac{1}{4} \alpha (\alpha - \epsilon_j + 2) g^i g^j \right] + \]
\[+ \frac{1}{4} \left( \epsilon_j - \alpha - 2 \right) g^i (g^j + g^j z \bar{z}) + \frac{1}{4} \left( (\epsilon_i + \epsilon_j - 2) - \alpha \right) \alpha g^i g^j \right] , \]

\[ \beta^k_{zz} = \tilde{g}^k + 2\pi \sum_{ij} C^k_{ij} \left[ \frac{1}{2} \left( \gamma^2 - \gamma + \frac{1}{3} \right) g^i g^j + \left[ \frac{\gamma^2 (\alpha + 2)}{4} - \gamma \right] g^i \bar{g}^j + \right] \]

18
\[
\beta^k = \frac{c}{6\pi} \delta_{k,0} + (\epsilon_k - 2) \tilde{g}^k + 2\pi \sum \left\{ \left[ (1 - \gamma) \frac{\alpha + 2}{2} - (\epsilon_j - 2) \gamma \frac{\alpha + 2}{4} + \frac{\alpha^2}{12} \right] g^i g^j + \left[ \left( 1 - \gamma \right) \frac{\alpha + 2}{2} + \frac{\alpha^2}{12} \right] g^i g^j + \left[ \left( 1 - \gamma \right) \frac{\alpha + 2}{2} - \frac{\alpha^2}{12} \right] g^i g^j \right\}
\]

\[
\beta^k = \frac{c}{6\pi} \delta_{k,0} + (\epsilon_k - 2) \tilde{g}^k + 2\pi \sum \left\{ \left[ (1 - \gamma) \frac{\alpha + 2}{2} - (\epsilon_j - 2) \gamma \frac{\alpha + 2}{4} + \frac{\alpha^2}{12} \right] g^i g^j + \left[ \left( 1 - \gamma \right) \frac{\alpha + 2}{2} + \frac{\alpha^2}{12} \right] g^i g^j + \left[ \left( 1 - \gamma \right) \frac{\alpha + 2}{2} - \frac{\alpha^2}{12} \right] g^i g^j \right\}
\]

\[
\beta^k = \frac{c}{6\pi} \delta_{k,0} + (\epsilon_k - 2) \tilde{g}^k + 2\pi \sum \left\{ \left[ (1 - \gamma) \frac{\alpha + 2}{2} - (\epsilon_j - 2) \gamma \frac{\alpha + 2}{4} + \frac{\alpha^2}{12} \right] g^i g^j + \left[ \left( 1 - \gamma \right) \frac{\alpha + 2}{2} + \frac{\alpha^2}{12} \right] g^i g^j + \left[ \left( 1 - \gamma \right) \frac{\alpha + 2}{2} - \frac{\alpha^2}{12} \right] g^i g^j \right\}
\]

\[
\tilde{g}^k = \frac{c}{6\pi} \delta_{k,0} + (\epsilon_k - 2) \tilde{g}^k + 2\pi \sum \left\{ \left[ (1 - \gamma) \frac{\alpha + 2}{2} - (\epsilon_j - 2) \gamma \frac{\alpha + 2}{4} + \frac{\alpha^2}{12} \right] g^i g^j + \left[ \left( 1 - \gamma \right) \frac{\alpha + 2}{2} + \frac{\alpha^2}{12} \right] g^i g^j + \left[ \left( 1 - \gamma \right) \frac{\alpha + 2}{2} - \frac{\alpha^2}{12} \right] g^i g^j \right\}
\]
\[
\beta^k_z = 2\pi \sum_{ij} C^k_{ij} \left\{ \left( \bar{\gamma} - \frac{1}{2} \right) g^i g^j + \left[ \bar{\gamma} + \frac{\bar{\gamma} (\alpha + 1)}{2} - 1 \right] g^i z g^j + \frac{\alpha + 1}{2} \left[ \gamma \left( \frac{\alpha + 1}{2} - \frac{\epsilon_j - 1}{2} \right) \right] g^i z^2 + \frac{\alpha}{4} \bar{\gamma} \left( \frac{\alpha + 1}{2} + \frac{\epsilon_j - 1}{2} \right) g^i g^j + \frac{\alpha + 1}{2} \left( 1 + \frac{\alpha - 1}{4} - \gamma \frac{\alpha + 1}{2} \right) + (\epsilon_j - 1)^2 \right\} g^i g^j \},
\]

where we use the notation

\[
\epsilon_i = 2 - h_i - \bar{h}_i,
\]
\[
\gamma = \gamma_{ijk} = \frac{h_k + h_i - h_j}{2h_k},
\]
\[
\bar{\gamma} = \bar{\gamma}_{ijk} = \frac{\bar{h}_k + \bar{h}_i - \bar{h}_j}{2\bar{h}_k},
\]
\[
\alpha = \alpha_{ijk} = h_k + \bar{h}_k - h_i - \bar{h}_i - h_j - \bar{h}_j
\]

4 Conclusions

We have expressed the deviations from scale invariance resulting from a small perturbation of a general conformal theory by the $\beta$-functions of eq. (40). These are given in terms of conformal data of an unperturbed model, the spectrum of dimensions and the operator product coefficients. We have seen that besides the primary fields the renormalization induced descendent field and also dilatonic terms with explicit world sheet metric dependence. These are of particular importance for non unitary theories where these terms may still be relevant. This explicit coupling to the background metric has to be taken into account when this metric becomes dynamical, that is when trying to couple the perturbed theory to quantum gravity.
The specific form of the $\beta$-functions in eq. (40) depends of course on the specific regularization chosen, since the precise meaning of adding an operator to the action with a certain coupling strength depends on the regularization scheme. For an example of such dependence, suppose that the sharp cut off scheme of eq. (9) is replaced by a smooth regularization by substituting

$$\theta(d(z, \omega) - a) \rightarrow \int_0^\infty d\lambda \theta(d(z, \omega) - \lambda a) f(\lambda)$$

where $f(\lambda)$ is some fixed smooth function which vanishes fast enough when $\lambda \to 0$ and $\lambda \to \infty$. In order to preserve the meaning of the coupling strengths $g^k$, normalize $f$ to unity

$$\int_0^\infty d\lambda f(\lambda) = 1$$

(42)

Repeating our analysis with this smooth regularization we see that the contribution in (40) to the $\beta$-function of operator of dimension $h_k$ coming from the product of two perturbing operators of dimensions $h_i$ and $h_j$ gets multiplied by the quantity

$$\int_0^\infty d\lambda f(\lambda) \lambda^{\epsilon_i + \epsilon_j - \epsilon_k}$$

(43)

i.e. by the $h_k - h_i - h_j + 2$ moment of the function $f$. In the particular case $\epsilon_i + \epsilon_j = \epsilon_k$ which corresponds to a logarithmic singularity in the integration of the location of $O_j$ near that of $O_i$, the $\beta$-function coefficient is universal due to (42).

**Appendix**

Here we shall calculate the geodesical distance between points $z$ and $\omega$ on two dimensional surface with conformal metric $g_{\alpha\beta} = e^{\sigma(z)} diag(1, 1)$. The geodesical distance is

$$d(z, \omega) = \frac{1}{\sqrt{2}} \int_0^1 d\tau [g_{\alpha\beta} \dot{v}^\alpha \dot{v}^\beta]^{\frac{1}{2}} = \int_0^1 d\tau e^{\sigma(\nu(\tau))} |\dot{v}(\tau)|$$

(44)
where \( \tau \) is parameter which varies from 0 to 1 along the geodesical curve \( v(\tau) \) and dot means the derivative with respect to this parameter. The equation for geodesic curve

\[
\ddot{v}(\tau) + 2\dot{\sigma}(v(\tau))\dot{v}^2(\tau) = 0 \tag{45}
\]

and analogous equation for \( \bar{v}(\tau) \) may be solved in perturbations over \( \sigma \). In the zeroth order we have

\[
v_0(\tau) = \tau z + (1 - \tau)\omega \tag{46}
\]

Let us call by \( v_1, v_2 \) contributions of the first and the second order in \( \sigma \) to \( v(\tau) \). Expansion in the Taylor series gives

\[
|\dot{v}_0 + \dot{v}_1 + \dot{v}_2| = |\dot{v}_0| \left( 1 + \frac{\dot{v}_1}{2\dot{v}_0} + \frac{\dot{v}_2}{2\dot{v}_0} - \frac{\dot{v}_1^2}{8\dot{v}_0^2} + \right. \\
\left. + \frac{|\dot{v}_1|^2}{|\dot{v}_0|^2} + \text{complex conjugated} \right) \tag{47}
\]

Substituting this and (46) into expression (44) for \( d \) we get

\[
d(z, \omega) = |z - \omega| \left[ 1 + \int_0^1 d\tau \sigma(v_0(\tau)) + \int_0^1 d\tau \left( \frac{\dot{v}_1}{2(z - \omega)} + \text{c.c.} \right) + \frac{1}{2} \int_0^1 d\tau \sigma^2(v_0(\tau)) + \\
+ \int_0^1 d\tau (\partial_\sigma(v_0(\tau))v_1(\tau) + \text{c.c.}) + \frac{1}{2} \int_0^1 d\tau \sigma(v_0(\tau)) \left( \frac{\dot{v}_1}{z - \omega} + \text{c.c.} \right) + \\
+ \frac{1}{4} \int_0^1 d\tau \frac{|\dot{v}_1|^2}{|v_0|^2} + \frac{1}{2} \int_0^1 d\tau \left( \frac{\dot{v}_2}{2\dot{v}_0} + \text{c.c.} \right) - \frac{1}{8} \int_0^1 d\tau \left( \frac{\dot{v}_2^2}{\dot{v}_0^2} + \text{c.c.} \right) \right] \tag{48}
\]

The last and before last integrals may be dropped out since last is higher order in derivative of \( \sigma \), and before last is total derivative over \( \tau \). So in considered order of perturbation we need only to know \( v_1 \), which may be found from (45):
Substituting this into (48) after integration over $\tau$ we get the answer

$$d(z, \omega) = |z - \omega| \left( 1 + \sigma(\omega) + \frac{1}{2} (\partial \sigma(\omega)(z - \omega) + \bar{\partial} \sigma(\omega)(\bar{z} - \bar{\omega})) + \frac{1}{3} \partial \bar{\partial} \sigma(\omega)|z - \omega|^2 + \frac{1}{2} \sigma^2(\omega) + \frac{1}{2} \sigma(\omega) \partial \sigma(\omega)(z - \omega) + \frac{1}{2} \sigma(\omega) \bar{\partial} \sigma(\omega)(\bar{z} - \bar{\omega}) + \frac{1}{3} \sigma(\omega) \partial \bar{\partial} \sigma(\omega) \right) =$$

$$ = e^{\sigma(\omega)} |z - \omega| \left( 1 + \frac{1}{2} (\partial \sigma(\omega)(z - \omega) + \bar{\partial} \sigma(\omega)(\bar{z} - \bar{\omega})) + \frac{1}{3} \partial \bar{\partial} \sigma(\omega)|z - \omega|^2 + \frac{1}{4} \partial \sigma(\omega) \bar{\partial} \sigma(\omega) \right)$$

where $y = |z - \omega|, \theta = arg(z - \omega)$ and we collected all powers of $\sigma$ into exponential in the considered second order of perturbation in $\sigma$.

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