EXPLICIT REPRESENTATION OF MEMBERSHIP IN POLYNOMIAL IDEALS

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Abstract. We introduce a new division formula on projective space which provides explicit solutions to various polynomial division problems with sharp degree estimates. We consider simple examples such as the classical Macaulay theorem as well as a quite recent result by Hickel, related to the effective Nullstellensatz. We also obtain a related result that generalizes Max Noether’s classical $AF + BG$ theorem.

1. Introduction

Let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^n$ of degrees $d_1 \geq d_2 \geq \ldots \geq d_m$ and assume that $\Phi$ is a polynomial that vanishes on the common zero set of the $F_j$. By Hilbert’s Nullstellensatz one can find polynomials $Q_j$ such that

$$\sum_j F_j Q_j = \Phi^\nu$$

if the power $\nu$ is large enough. A lot of attention has been paid to find effective versions, i.e., control of $\nu$ and the degrees of $Q_j$ in terms of the degrees of $F_j$. In [15] Brownawell obtained bounds on $\nu$ and $\deg Q_j$ not too far from the best possible, using a combination of algebraic and analytic methods. Soon after that Kollár [25] obtained by purely algebraic methods the optimal result:

If $\deg F_j = d_i$ (and $d_i \neq 2$), then (1.1) holds for some $\nu \leq N_{ko}$ and $\deg (F_j Q_j) \leq (1 + \deg \Phi)N_{ko}$, where $N_{ko}$ is $d_1 \cdots d_m$ if $m \leq n$ and $N_{ko} = d_1 \cdots d_{n-1} \cdot d_m$ otherwise.

In particular, if $F_j$ have no common zeros in $\mathbb{C}^n$, then there are polynomials $Q_j$ such that

$$\sum_j F_j Q_j = 1,$$

with

$$\deg F_j Q_j \leq N_{ko}.$$
The restriction \( d_j \neq 2 \) has been removed by Jelonek, \[23\]; even more interesting is that the method he uses, basically elimination theory, actually produces explicitly the desired polynomials \( Q_i \).

A standard way to reformulate problems of this kind is the following. Let \( z = (z_0, \ldots, z_n) \), \( z' = (z_1, \ldots, z_n) \), let \( f_i(z) = z_0^{d_i} F_i(z'/z_0) \) be the homogenizations of \( F_i \), and let \( \phi(z) = z_0^{-\deg\Phi} \Phi(z'/z_0) \). Then there is a solution to

\[
\Phi = \sum_i F_i Q_i
\]

with \( \deg F_i Q_i \leq \rho \) if and only if there are \((\rho - d_i)\)-homogeneous forms \( q_i \) such that

\[
\sum_i f_i q_i = z_0^{\rho - \deg\phi} \psi.
\]

In \[3\], (1.4) is considered as an equation in vector bundles over \( \mathbb{P}^n \) and it is shown that if \( \phi \) annihilates a certain residue current \( R \) on \( \mathbb{P}^n \), then there is indeed a global solution \( q = (q_1, \ldots, q_m) \) provided that \( \rho \) also satisfies an additional estimate from below. In this new paper we introduce an explicit division formula

\[
(1.5) \quad \psi(z) = \sum_i f_i(z) \cdot \int_{\mathbb{P}^n} \mathcal{H}_i U \psi + \int_{\mathbb{P}^n} \mathcal{H} R \psi
\]

that holds for \( \psi = z_0^{\rho - \deg\phi} \phi \) if \( \rho \) is large enough. Here \( U \) is a current that is smooth outside the common zero set \( Z \subset \mathbb{P}^n \) of the \( f_i \), \( R \) is a residue current with support on \( Z \), and \( \mathcal{H}_i \) are smooth in both variables and homogeneous polynomials of degrees \( \rho - d_i \) in \( z \). Thus (1.5) provides an explicit solution (after dehomogenization) to (1.3) if \( R \psi = 0 \). Moreover, in many cases one can choose the same \( \rho \) as in the implicit method, so we indeed get the same degree estimate of the solution.

**Remark 1.** Integral representation of solutions to polynomial division problems was introduced in \[11\] and has been used since then by many authors, see, e.g., \[10\] and the survey article \[29\]. However, in these formulas the integration (or rather the action of a current on a test form) is performed over \( \mathbb{C}^n \) so a size estimate at infinity is needed to get rid of the residue term. The formula (1.5) is intrinsic on \( \mathbb{P}^n \) and so the residue term may vanish for more subtle reasons. A similar formula was introduced already in \[3\] but with less precise degree estimates.

We also notice in this paper that, by a more careful residue calculus, the method in \[3\] admits more general results than stated there. Therefore we start with some examples of results that can be obtained in this way, and for which we now also have explicit representations. If \( \Phi \) belongs to the integral closure of the ideal \((F_1, \ldots, F_m)\) then
it follows from the Briançon-Skoda theorem, [14], that one can take 
\( \nu = \min(m, n) \). The following nice result was proved by Hickel, [22], as an affirmative answer to a conjecture by Berenstein and Yger in [9].

**Theorem 1.1.** [Hickel] If \( \Phi \) is in the integral closure of \((F_1, \ldots, F_m)\), then (1.1) holds with \( \nu \) equal to \( \min(m, n) \) and

\[
\deg(F_i Q_i) \leq \min(m, n)(\deg \Phi + N_{hi}),
\]

where \( N_{hi} \) is \( d_1 \cdots d_m \) if \( m \leq n \) and \( N_{hi} = \min(d_n^m, d_1 \cdots d_m / d_m^{m-n}) \) otherwise.

Notice that, applied to \( \Phi = 1 \), one essentially gets back Kollár’s theorem, except for the factor \( \min(m, n) \) in front of \( N_{hi} \).

The assumption that \( \Phi \) is in the integral closure means (is equivalent to) that \( |\Phi| \leq C|F| \) locally in \( \mathbb{C}^n \). The following slightly stronger statement actually holds:

*If \( \Phi \) is a polynomial such that
\[
|\Phi| \leq C|F|^{\min(m, n)}
\]
locally in \( \mathbb{C}^n \), then (1.3) has a solution such that
\[
\deg(F_i Q_i) \leq \deg \Phi + \min(m, n)N_{hi}.
\]

Notice that the condition (1.7), thanks to the Briançon-Skoda theorem, implies that \( \Phi \) is in the ideal \((F)\) in \( \mathbb{C}^n \).

To prove Theorem 1.1 Hickel introduces a nonnegative rational number \( \nu_\infty \) that is a measure of the “order of contact” of \( Z \) to the hyperplane at infinity, see Section 3 for the precise definition. From the Refined Bezout theorem due to Fulton and MacPherson, see [18], Hickel deduces, by quite rough estimates, that

\[
\nu_\infty \leq N_{hi};
\]
in most cases \( \nu_\infty \) is much smaller. Theorem 1.1 is then an immediate corollary of the following result. As usual, \( \lceil \alpha \rceil \) denotes the least integer that is \( \geq \alpha \).

**Theorem 1.2.** Assume that \( \Phi \) satisfies (1.7).

(i) Then (1.3) has a solution such that

\[
\deg(F_i Q_i) \leq \deg \Phi + \lceil \min(m, n) \nu_\infty \rceil
\]

if \( m \leq n \). In case \( m > n \) the estimate is the maximum of the right hand side and the number \( d_1 + \cdots + d_{n+1} - n \).

(ii) The integral formula provides a solution to (1.3) with

\[
\deg(F_i Q_i) \leq \max \left( \deg \Phi + \lceil \min(m, n) \nu_\infty \rceil, d_1 + \cdots + d_{\min(m, n+1)} - n \right).
\]
Part (i) is a variant of Theorem 2.1' in [22]. By an extra argument due to Hickel one can get rid of the \( \lceil \rceil \) in (1.9) and thus gain one unit when the number inside is not an integer.

We can assume from the beginning that \( \Phi \) is in the ideal \((F_1, \ldots, F_m)\) (but not assuming (1.7)) and ask for an estimate of the degrees of \( Q_j \).

Here is (as far as we know) a new result in this direction:

**Theorem 1.3.** Assume that \( \text{codim} \{ F_1 = \ldots = F_m = 0 \} \geq m \) in \( \mathbb{C}^n \) and that \( \Phi \) is in the ideal \((F_1, \ldots, F_m)\) in \( \mathbb{C}^n \). Then (1.3) has a solution such that

\[
\deg (F_i Q_i) \leq \deg \Phi + \lceil m \nu_\infty \rceil,
\]

whereas the integral formula provides a solution with

\[
\deg (F_i Q_i) \leq \max (\deg \Phi + \lceil m \nu_\infty \rceil, d_1 + \cdots + d_m - n).
\]

**Remark 2.** If \( Z \) has no irreducible component at all contained in the hyperplane at infinity, then \( \nu_\infty \) shall be interpreted as 0 in Theorem 1.3 and then we get back Theorem 1.2 in [3]. In case \( m = n \) this is the classical so-called \( AF + BG \) theorem of Max Noether, [27].

In particular, if \( m \geq n + 1 \) and \( Z \) is empty, then we get back the classical Macaulay theorem, [26], with an explicit formula representing the membership. Such a formula has also been obtained in [12] relying on [13].

We have analogous results for submodules of \( \mathbb{C}[z_1, \ldots, z_n]^r \) rather than just ideals. Let \( F \) be a polynomial mapping \( \mathbb{C}^n \to \text{Hom}(\mathbb{C}^m, \mathbb{C}^r) \) whose columns \( F^j \) have degrees \( \leq d_j, j = 1, \ldots, m \). We also assume that \( F \) is generically surjective in \( \mathbb{C}^n \), which is equivalent to that the ideal \( \text{det} F \), generated by the \( r \times r \) minors of \( F \), is nontrivial. Let \( f \) be the associated matrix whose columns \( f^j \) are \( d_j \)-homogeneous polynomials and let \( Z \) be the set in \( \mathbb{P}^n \) where \( \text{det} f \) is vanishing. Moreover, let \( \nu_\infty \) be associated to the ideal \( \text{det} f \). By the estimate (1.8) above we get (without being too precise)

\[
\nu_\infty \leq (d_1 \cdots d_r)^{\min(n,m!/(m-r)!r!)}.\]

Let \( \Phi \) be an \( r \)-column of polynomials. The proper analogue of (1.7) is, cf., [4], that

\[
(1.11) \quad \| \Phi \| \leq C | \text{det} F |^{\min(n,m-r+1)}
\]

holds locally, where \( \| \Phi \| \) is a somewhat stronger norm than the natural norm \( | \Phi | \); i.e., \( \| \Phi \| \leq | \Phi | \). We have the following generalization of the previous theorems.

**Theorem 1.4.** Assume that \( F \) is an \( r \times m \) matrix of polynomials as above with columns \( F^j \) and \( \Phi \) is an \( r \)-column of polynomials. Assume that either (1.11) holds locally in \( \mathbb{C}^n \), or that

\[
(1.12) \quad \text{codim} \{ \text{det} F = 0 \} \geq m - r + 1
\]
in $\mathbb{C}^n$ and $\Phi$ is in the module generated by the columns $F^j$.

(i) There are polynomials $Q_j$ such that (1.3) holds and

\[(1.13) \quad \deg(F^iQ_i) \leq \deg\Phi + \left\lfloor \min(n, m - r + 1)\nu_\infty \right\rfloor \]

if $m \leq n + r - 1$. In case $m > n$ the estimate is the maximum of the right hand side and the number $d_1 + \cdots + d_{n+r} - n$.

(ii) The division formula provides a solution to (1.3) such that

\[(1.14) \quad \deg(F^iQ_i) \leq \max(\deg\Phi + \left\lfloor \min(n, m - r + 1)\nu_\infty \right\rfloor, d_1 + \cdots + d_{\min(m,n+r)} - n)\]

Notice that for a generic $r \times m$-matrix $F$, the zero set of $\det F$ has codimension $m - r + 1$.

Again $\nu_\infty$ is 0 if (1.12) holds and $Z$ has no irreducible component contained in the hyperplane at infinity.

One can obtain sharper results for special ideals, for instance determinantal ideal, and product ideals, cf., [6]. However we skip precise formulations since our aim is mainly to give examples of various applications of the new representation formulas.

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2. The basic setup

Assume that

\[(2.1) \quad 0 \to E_N \xrightarrow{f_N} \cdots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0\]

is a generically exact complex of Hermitian vector bundles over $\mathbb{P}^n$ and let $Z$ be the algebraic set where (2.1) is not pointwise exact. In [7] were introduced currents $U = U_1 + \cdots + U_N$, $R = R_1 + \cdots + R_N$ associated to (2.1) with the following properties: The current $U$ is smooth outside $Z$, $U_k$ are $(0, k-1)$-currents that take values in $\text{Hom}(E_0, E_k)$, and $R_k$ are $(0, k)$-currents with support on $Z$, taking values in $\text{Hom}(E_0, E_k)$. Moreover, they satisfy the relations

\[(2.2) \quad f_1U_1 = I_{E_0}, \quad f_{k+1}U_{k+1} - \bar{\partial}U_k = -R_k, \quad k \geq 1,\]

which can be compactly written as $\nabla_f U = I_{E_0} - R$ if $\nabla_f = f - \bar{\partial} = f_1 + f_2 + \cdots f_N - \bar{\partial}$. We have the corresponding complex of locally free sheaves

\[0 \to \mathcal{O}(E_N) \xrightarrow{f_N} \cdots \xrightarrow{f_3} \mathcal{O}(E_2) \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0).\]

If $\psi$ is a holomorphic section of $\mathcal{O}(E_0)$ that annihilates $R$, i.e., the current $R\phi$ vanishes, then $\psi$ is in the sheaf $\mathcal{F} = \text{Im} f_1$, see [7], Proposition 2.3.
In this paper we will only consider bundles that are direct sums of line bundles. Let $\mathcal{O}(\ell)$ be the holomorphic line bundle over $\mathbb{P}^n$ whose sections are (naturally identified with) $\ell$-homogeneous functions in $\mathbb{C}^{n+1}$. Moreover, let $E^j_k$ be disjoint trivial line bundles over $\mathbb{P}^n$ with basis elements $e_{k,j}$, and let
\begin{equation}
E_k = (E^1_k \otimes \mathcal{O}(-d^1_k)) \oplus \cdots \oplus (E^r_k \otimes \mathcal{O}(-d^r_k)).
\end{equation}
Then
\[
\mathbf{f}_k = \sum_{ij} f^{ij}_k e_{k-1,i} \otimes e^*_k,j
\]
are matrices of homogeneous forms, here $e^*_k,j$ are the dual elements, and
\[
\deg f^{ij}_k = d^i_{k-1} - d^j_k.
\]
We equip $E_k$ with the natural Hermitian metric, i.e., such that
\[
|\xi(z)|^2_{E_k} = \sum_{j=1}^{r_k} |\xi_j(z)|^2 |z|^{2d^j_k},
\]
if $\xi = (\xi_1, \ldots, \xi_{r_k})$.

The Dolbeault cohomology $H^{0,q}(\mathbb{P}^n, \mathcal{O}(r))$ vanishes for all $r$ if $1 \leq q \leq n - 1$ and also for $q = n$ if $r \geq -n$. From (2.2) and a simple homological argument we get the following proposition, see [7] for details.

**Proposition 2.1.** Assume that $\psi$ is a holomorphic section of $\mathcal{O}(\rho) \otimes E_0$ such that $R\psi = 0$. If
\begin{equation}
N \leq n \quad \text{or} \quad \rho - \max_i d^i_{n+1} \geq -n,
\end{equation}
then there is a global holomorphic section $q$ of $\mathcal{O}(\rho) \otimes E_1$ such that $f_1q = \psi$.

Given an ideal sheaf $\mathcal{J} = (f^1, \ldots, f^m)$ as above one can find a complex (2.1) such that $\psi$ annihilates the associated residue current $R$ if and only if $\psi$ is in $\mathcal{J}$, see [7] Section 7. However, in general the numbers $d^i_{n+1}$ may be very big and (2.4) then reflects the obstruction for global solvability. For the purpose of this paper we need more specialized complexes. The Koszul complex will be of particular importance.

**Example 1 (The Koszul complex).** Let $f^1, \ldots, f^m$ be our given homogeneous forms of degrees $d_i$, assume that $E^0$ is the trivial line bundle, and let
\[
E = (E^1 \otimes \mathcal{O}(-d_1)) \oplus \cdots \oplus (E^m \otimes \mathcal{O}(-d_m)),
\]
where $E^i$ are trivial line bundles. Let $e_i$ be basis elements for $E^i$ and let $e^*_i$ be the dual basis elements. We take
\[
E_k = \Lambda^k E = \sum_{|I|=k} \mathcal{O}(-(d_{I_1} + \cdots + d_{I_k})) E^{I_1} \otimes \cdots \otimes E^{I_k}
\]
and \( f_k \) as interior multiplication with \( f = \sum f^j e^*_j \). Now
\[
\sigma = \sum_j \frac{\bar{f}^j(z) e_j}{|f(z)|^2} \left/ |f|^2_{E^*} \right. ,
\]
is the section of \( E \) with minimal norm such that \( f \cdot \sigma = 1 \) outside \( Z \), where
\[
|f(z)|^2_{E^*} = \sum_1^m |f^j(z)|^2 |z|^{-2d_j}.
\]
Moreover,
\[
U = |f|^{2\lambda}_{E^*} \sum_1^m \left( \sigma \wedge (\bar{\partial} \sigma)^{k-1} \right)_{\lambda=0}
\]
and
\[
(2.5) \quad R = \bar{\partial} |f|^{2\lambda}_{E^*} \wedge \sum_1^m \left( \sigma \wedge (\bar{\partial} \sigma)^{k-1} \right)_{\lambda=0},
\]
cf., \cite{2} and \cite{3}; here \( |_{\lambda} \) means evaluation at \( \lambda = 0 \) after analytic continuation (the existence of which is nontrivial and part of the statement).

The condition (2.4) in this case is
\[
(2.6) \quad m \leq n \quad \text{or} \quad \rho - (d_1 + \ldots + d_n) \geq -n.
\]
If \( \text{codim} Z = m \), then the resulting residue current \( R \) only consists of the term \( R_m \) (cf., Proposition 2.2 below); it coincides with the classical Coleff-Herrera product, cf., \cite{3} p. 112, and hence the duality theorem, \cite{28} and \cite{16}, holds, i.e., locally \( R \psi = 0 \) if and only if \( \psi \) (locally) belongs to the ideal sheaf \( \mathcal{J} \) generated by \( f \). \( \square \)

**Example 2.** These formulas become much simpler if we assume that all \( d_j = d \). Then
\[
\sigma = \sum_1^m \frac{\bar{f}^j(z) e_j}{|f(z)|^2}.
\]
If \( \bar{f} \cdot e = \sum \bar{f}^j e_j \) and \( d \bar{f} \cdot e = \sum d \bar{f}^j \wedge e_j \), then
\[
\sigma \wedge (\bar{\partial} \sigma)^{k-1} = \frac{\bar{f} \cdot e \wedge (d \bar{f} \cdot e)^{k-1}}{|f|^{2k}}
\]
and it is easy to check that we can replace \( |f|^{2\lambda}_{E^*} \) by \( |f|^{2\lambda} \) in the definition of \( U \) and \( R \). Thus
\[
(2.7) \quad U = |f|^{2\lambda} \sum_1^m \left( \frac{\bar{f} \cdot e \wedge (d \bar{f} \cdot e)^{k-1}}{|f|^{2k}} \right)_{\lambda=0}
\]
and
\[
R = \bar{\partial} |f|^{2\lambda} \wedge \sum_1^m \left( \frac{\bar{f} \cdot e \wedge (d \bar{f} \cdot e)^{k-1}}{|f|^{2k}} \right)_{\lambda=0}.
\]
\( \square \)
In [8] was introduced the sheaf of pseudomeromorphic currents $\mathcal{P}M$ on a complex manifold $X$. It is a module over the sheaf of smooth forms and closed under $\bar{\partial}$. For any $T \in \mathcal{P}M$ and variety $V$ there exists a “restriction” $1_V T$ that is in $\mathcal{P}M$ and has support on $V$. The current $1_{X\setminus V} T = T - 1_V T$ is determined by the natural restriction of $T$ to the open set $X \setminus V$ or equivalently, $T = 1_V T$ if and only if $T$ has support on $V$. If $h$ is any tuple of holomorphic functions with common zero set $V$, then

$$1_{X \setminus V} T = |h|^{2\lambda} T|_{\lambda=0},$$

where, as before, the right hand side is the evaluation at the origin of a current-valued holomorphic function. We also have the following important fact (Corollary 2.4 in [8]):

**Proposition 2.2.** If $T \in \mathcal{P}M$ has bidegree $(*, p)$ and support on a variety $T$ of codimension $k > p$, then $T = 0$.

For instance, the currents $R$ and $U$ above are pseudomeromorphic.

3. **Proofs of Theorems 1.2 and 1.3**

Let $\mathcal{J}$ denote the sheaf over $\mathbb{P}^n$ generated by our homogeneous forms $f^1, \ldots, f^m$, let

$$\nu: X_+ \to \mathbb{P}^n$$

be the normalization of the blowup along $\mathcal{J}$, and let

$$Y^+ = \sum_i m_i^+ Y_i^+$$

be the associated divisor in $X_+$, where $Y_i^+$ are its irreducible components. The varieties $\nu(Y_i^+)$ in $\mathbb{P}^n$ are the so-called distinguished varieties associated to $\mathcal{J}$. For each $Y_i^+$ such that $\nu(Y_i^+)$ is entirely contained in the hyperplane at infinity, $H = \{z_0 = 0\}$, we let $e_i$ be the order of vanishing of $\nu^* h$ at $Y_i^+$, where $h(z) = z_0$, and define

$$\nu_\infty = \max_i \frac{m_i^+}{e_i},$$

where the maximum is taken over all such indices $i$.

Let us now consider the residue current $R$ in (2.5) obtained from the Koszul complex. Assume that $\pi: \tilde{X} \to \mathbb{P}^n$ is a log resolution of $\mathcal{J}$, i.e., $\tilde{X}$ is a smooth modification such that $\pi^* \mathcal{J}$ is principal and its zero set $Y$ has normal crossings. Then

$$\pi^* f = f^0 f',$$

where $f^0$ is a holomorphic section of a line bundle $L \to \tilde{X}$ that defines the divisor

$$Y = \sum_i m_i Y_i$$
and $f'$ is a non-vanishing section of $\pi^*E^* \otimes L^{-1}$. Moreover,

$$L = \otimes_i L_i^m,$$

and

$$f^0 = \otimes_i s_i^m,$$

where $s_i$ are sections of $L_i$ that vanish to the first order; the normal crossing assumption means that in local trivializations the $s_i$ are part of a local holomorphic coordinate system. It follows that

$$\pi^* \sigma = (1/f^0) \sigma',$$

where $1/f^0$ is a meromorphic section of $L^{-1}$ and $\sigma'$ is a smooth section of $\pi^*E \otimes L$. Now (in this section $| \cdot |$ always refers to vector bundle norm)

$$\partial|\pi^*f|^{2\lambda} \wedge \pi^* u_k = \partial|\pi^*f|^{2\lambda} \wedge \frac{1}{(f^0)_k} u'_k,$$

where

$$u'_k = \sigma' \wedge (\bar\partial \sigma')^{k-1}$$

is smooth. It follows that (3.2) has an analytic continuation to $\lambda = 0$ and the value there is

$$\tilde{R} = \partial \frac{1}{(f^0)_k} \wedge u'_k;$$

moreover, cf., (2.5),

$$R_k = \pi_* \tilde{R}_k.$$

Since we have normal crossings we also have the decomposition

$$\tilde{R}_k = \sum_j \tilde{R}_{kj} = \sum_j \partial \frac{1}{s_j^km_j} \wedge \otimes_{i \neq j} \frac{1}{s_i^km_i} u'_k$$

and hence

$$R_k = \sum_j R_{kj} = \sum_j \pi_* \tilde{R}_{kj}.$$

Claim: $R_{kj}$ vanishes unless $\pi(Y_j)$ is a distinguished variety of $J$.

To see this first notice that in $X_+$,

$$\nu^* f = f^0 f'_+,$$

where $f^0$ is a holomorphic section of the line bundle $L_+ \to X_+$ that defines the divisor $Y^+$ and $f'_+$ is a non-vanishing section of $\nu^*E^* \otimes L^{-1}$. The log resolution $\pi$ factorizes over $\nu$, i.e., we have

$$\tilde{X} \xrightarrow{\tilde{\nu}} X_+ \xrightarrow{\nu} \mathbb{P}^n.$$

If $\pi(Y_j)$ is not a distinguished variety of $J$, then $\tilde{\nu}(Y_j)$ has at least codimension 2 in $X_+$. Therefore

$$\tilde{\nu}_* \left[ \partial \frac{1}{s_j^km_j} \wedge \otimes_{i \neq j} \frac{1}{s_i^km_i} \right]$$
must vanish according to Proposition 2.2 since it is a pseudo meromorphic \((0, 1)\)-current in \(X_+\) with support on a variety with codimension at least 2. (In general \(X_+\) is not smooth but the proof of Proposition 2.2 goes through verbatim even in the non-smooth case.) Notice that
\[\nu^*u_k = \frac{1}{(f^0)^k} u'_{+,k},\]
where \(u'_{+,k}\) is smooth, and that \(\tilde{u}'_k = \nu^*u'_{+,k}\). It follows that \(\tilde{\nu}^*R_{kj} = 0\) and hence \(\pi^*R_{kj} = \nu^*\tilde{\nu}^*R_{kj} = 0\) as claimed.

The resulting decomposition of \(R_k\) with respect to the distinguished varieties is inspired by [30] and [24], where it plays a fundamental role.

We are now ready for the proofs. We have already observed that Theorem 1.1 follows from Theorem 1.2.

**Proof of Theorem 1.2.** Let \(\phi\) be the homogenization of \(\Phi\) and let \(\mu = \min(m, n)\). By the definition of \(\nu_\infty\), we have that \(\nu^*h^{\nu_\infty}\) must vanish to order \(\mu m_i^+\) on each divisor \(Y_i^+\) such that \(\nu(Y_i^+) \subset H\). On the other hand, it follows from (1.7) that \(\nu^*\phi\) vanishes to order \(\mu m_i^+\) on any other divisor \(Y_i^+\). Thus \(\nu^*(\phi h^{\nu_\infty})\) vanishes to order \(\mu m_i^+\) on each \(Y_i^+\) and therefore

\[
|\phi h^{\nu_\infty}| \leq C |f|^\mu
\]
on \(\mathbb{P}^n\). Thus \(\pi^*(\phi h^{\nu_\infty})\) must contain the factor \(s_i^m j\), which implies that it annihilates \(\tilde{R}_{kj}\), cf., \((3.3)\) and \((3.4)\), for each \(k \leq \mu\). It follows that \(\phi h^{\nu_\infty}\) annihilates the current \(R\).

Now \(\phi h^{\nu_\infty}\) is a section of \(O(\rho)\) with \(\rho = \deg \Phi + \nu_\infty\). If necessary we raise the power of \(h\) further so that (2.6) holds. Then the first part of Theorem 1.2 follows from Proposition 2.1 after dehomogenization. The second part concerning explicit representation follows from Section 4. □

**Remark 3.** Let \(\bar{J}\) denote the integral closure sheaf generated by \(J\). It is well known that \(\xi \in O_{\mathbb{P}^n}\) is in \(\bar{J}\) if and only if \(\nu^*\xi\) vanishes to order (at least) \(m_i^+\) on \(Y_i^+\), i.e.,
\[\bar{J} = \nu_*(O(-Y^+)).\]
In the same way, \((3.6)\) means that \(\phi h^{\nu_\infty}\) belongs to the integral closure of \(J^\mu\). □

**Remark 4.** One can get rid of \(\nu^\gamma\) in (1.9) by the following trick. Assume that \(\mu \nu_\infty = a/b\), for integers \(a, b\). Let \(\tilde{f}(z) = f(z_0^b, \ldots, z_n^b)\) and similarly with \(\phi\). Now
\[|h^a \hat{\phi}| \leq C |\hat{f}|^\mu\]
and as before we then have a solution \(\Psi_j\) to
\[
\sum_j \hat{f}^j \Psi_j = \hat{\phi}
\]
with
\[ \deg \hat{F}^j \Psi_j \leq \deg \hat{\Phi} + a = b \deg \Phi + a. \]
However, one can then choose \( \Psi \) of the form \( \Psi = \hat{Q} \), and it follows that
\[ \deg F^j Q_j \leq \deg \Phi + a/b \]
as desired. See [22] for details. \( \square \)

In the previous proof we killed the residue by size estimates in \( \mathbb{C}^n \) as well as on the hyperplane \( H \). In the proof of Theorem 1.3 the residue calculus is somewhat more involved because then we will kill different parts of the residue in different ways.

Proof of Theorem 1.3. We begin with the same set-up as in the previous proof. Since \( R \) is pseudomeromorphic, cf., Section 2 above, we have the decomposion
\[ R = 1_{\mathbb{C}^n}R + 1_H R, \]
where \( 1_{\mathbb{C}^n}R \) is the natural extension to \( \mathbb{P}^n \) of the restriction of \( R \) to \( \mathbb{C}^n \). Since \( f \) is a complete intersection here, \( R \) coincides with the Coleff-Herrera product, and by the duality theorem, it follows that \( h^{\rho-\deg \Phi} 1_{\mathbb{C}^n} R = 0 \).

Assume now that \( Z \) has no irreducible component contained in \( H \). Then \( Z \) has codimension \( m \) in \( \mathbb{P}^n \) and by Proposition [2.2], hence \( R = R_m \). Thus \( 1_H R = 1_H R_m \) has bidegree \((0,m)\) and support on \( H \cap Z \) that has codimension strictly larger than \( m \). By Proposition [2.2] it must therefore vanish. It follows that \( h^{\rho-\deg \Phi} R = 0 \) and since (2.6) is satisfied, the membership follows.

We now consider the general case. We can choose the log resolution so that also \( \pi h \) is a monomial in \( s_i \). Notice, cf., (3.3) and (3.4), that
\[ |h|^{2\lambda} R_{jk} = \pi_* (|\pi^* h|^{2\lambda} \tilde{R}_{jk}) \]
vanishes when \( \Re \lambda \) is large and hence for \( \lambda = 0 \) if \( s_j \) is a factor in \( \pi^* h \), whereas the value at \( \lambda = 0 \) is \( R_{jk} \) if \( s_j \) is not a factor in \( \pi^* h \). In view of (2.8) it follows that \( 1_H R \) is the sum of \( R_{kj} \) such that \( \pi(Y_j) \) is contained in \( H \). Moreover, we know that only \( j \) corresponding to distinguished varieties give a contribution. Take such a \( j \) and assume that \( \tilde{\nu}(Y_j) = Y_i^+ \). We know that \( \nu^* h^{\mu \nu \infty \gamma} \) vanishes at least to the same order as \( \nu^* f^\mu \) does on \( Y_i^+ \), and hence \( \pi^* h^{\mu \nu \infty \gamma} \) must vanish at least to the same order as \( \pi^* f^\mu \) on \( Y_j \), i.e., \( \pi^* h^{\mu \nu \infty \gamma} \) contains the factor \( s_j^{\mu m} \).

It follows that \( \tilde{R}_{kj} \pi^* h^{\mu \nu \infty \gamma} = 0 \).

Summing up, we have that \( R h^{\mu \nu \infty \gamma} \phi = 0 \), and hence the first part of the theorem is proved. The second part, again, is treated in Section 4. \( \square \)

The proof of Theorem 1.4 is analogous, but one has to use the so-called Buchsbaum-Rim complex which is a generalization of the Koszul
complex. See [4] where the associated currents are discussed, and special cases of Theorem [1,4] are proved.

4. Explicit representation

We first discuss, following [1], [5], [19], and [20], how one can generate representation formulas for holomorphic sections of a vector bundle $F \to \mathbb{P}^n$. Let $F_z$ denote the pull-back of $F$ to $\mathbb{P}_\zeta \times \mathbb{P}_z$ under the natural projection $\mathbb{P}_\zeta \times \mathbb{P}_z \to \mathbb{P}_z$ and define $F_\zeta$ analogously. Notice that

$$\eta = 2\pi i \sum_0^n z_i \frac{\partial}{\partial \zeta_i}$$

is a section of the bundle $\mathcal{O}_z(1) \otimes \mathcal{O}_\zeta(-1) \otimes T_{1,0}(\mathbb{P}_\zeta)$ over $\mathbb{P}_\zeta \times \mathbb{P}_z$. If we express a projective form in homogeneous coordinates and contract with $\eta$ we get a new projective form, i.e., we have a mapping

$$\delta_\eta : \mathcal{D}'_{\ell+1,q}(\mathcal{O}_\zeta(k) \otimes \mathcal{O}_z(j)) \to \mathcal{D}'_{\ell,q}(\mathcal{O}_\zeta(k-1) \otimes \mathcal{O}_z(j+1)),$$

where $\mathcal{D}'_{\ell,q}(\mathcal{O}_\zeta(k) \otimes \mathcal{O}_z(j))$ denotes the sheaf of currents on $\mathbb{P}_\zeta \times \mathbb{P}_z$ of bidegree $(\ell, q)$ in $\zeta$ and $(0, 0)$ in $z$ that take values in $\mathcal{O}_\zeta(k) \otimes \mathcal{O}_z(j)$. Given a vector bundle $L \to \mathbb{P}_\zeta \times \mathbb{P}_z$, let

$$L^\nu(L) = \bigoplus_j \mathcal{D}'_{j,j+\nu}(\mathcal{O}_\zeta(j) \otimes \mathcal{O}_z(-j) \otimes L).$$

If $\nabla_\eta = \delta_\eta - \bar{\partial}$, where $\bar{\partial} = \bar{\partial}_\zeta$, then $\nabla_\eta : L^\nu(L) \to L^{\nu+1}(L)$. Furthermore, $\nabla_\eta$ obeys Leibniz’ rule, and $\nabla_\eta^2 = 0$.

A weight with respect to $F \to \mathbb{P}^n$ and a point $z \in \mathbb{P}^n$ is a section $g$ of $L^0(\text{Hom}(F_\zeta, F_z))$ such that $\nabla_\eta g = 0$, $g$ is smooth for $\zeta$ close to $z$, and $g_{0,0} = I_F$ when $\zeta = z$, where $g_{0,0}$ denotes the component of $g$ with bidegree $(0, 0)$, and $I_F$ is the identity endomorphism on $F$.

**Proposition 4.1.** Let $g$ be a weight with respect to $F \to \mathbb{P}^n$ and $z$, and assume that $\psi$ is a holomorphic section of $F \otimes \mathcal{O}(-n)$. We then have the representation formula

$$\psi(z) = \int_{\mathbb{P}^n} g_{n,n} \psi.$$

This formula appeared in [19] (Proposition 5.5), and it can be deduced from [20]; however, for the reader’s convenience we sketch a direct argument.

**Proof.** Notice that

$$b = \frac{1}{2\pi i} \frac{|\zeta|^2 \bar{z} \cdot d\zeta - (\bar{z} \cdot \zeta) \bar{\zeta} \cdot d\zeta}{|\zeta|^2|z|^2 - |\zeta \cdot z|^2}$$
is a $(1,0)$-form with values in $O_\zeta(-1) \otimes O_\zeta(1)$ that is smooth outside
the diagonal $\Delta$ in $\mathbb{P}^n_{\zeta} \times \mathbb{P}^n_{\bar{\zeta}}$. Fix the point $z$ and let $B = b + b \wedge \bar{\partial} b + \cdots + b \wedge (\bar{\partial} b)^{n-1}$. Then $\nabla_\eta B = 1$ outside $z$, and we claim that

$$\nabla_\eta B = 1 - [z]$$

in the current sense, where $[z]$ is the current such that

$$\int_{\mathbb{P}^n_{\zeta}} [z] \xi(\zeta) = \xi(z)$$

for each smooth section $\xi$ of $O(-n)$. Because of rotational invariance it is enough to choose affine coordinates $\zeta = (1, \zeta') = (1, \zeta'_1, \ldots, \zeta'_n)$ and assume that $z = (1, 0, \ldots, 0)$. Then $b$ becomes

$$b' = \frac{1}{2\pi i} \frac{\zeta' \cdot d\zeta'}{|\zeta'|^2},$$

and it is elementary to check, cf., [1] p. 5, that $(\delta_{\zeta'} - \bar{\partial})(b' + b' \wedge \bar{\partial} b' + \cdots + b' \wedge (\bar{\partial} b')^{n-1}) = 1 - [0]$. Now (4.1) follows. Thus

$$\nabla_\eta (B \wedge g) = g - [z] \wedge g = g - [z] g_{0,0} = g - [z] I_F,$$

and identifying the top degree terms we get

$$\bar{\partial} (B \wedge g)_{n,n-1} = [z] I_F - g_{n,n}.$$

Multiplying with $\psi$, the proposition follows from Stokes’ formula. □

Consider now a complex (2.1). In order to represent membership in $J = \text{Im } f_1$, we will find a weight $g$ that contains $f_1(z)$ as a factor and apply Proposition 4.1. To this end we introduce a generalization of so-called Hefer forms, inspired by [5] and [7], to the case of non-trivial vector bundles.

**Definition 1.** We say that $H = (H_{\ell}^k)$ is a Hefer morphism for the complex $E_\bullet$ in (2.1) if $H_{\ell}^k$ are smooth sections of

$$L^{-k+\ell} (\text{Hom} (E_{\zeta, k}^{\ell}, E_{z, \ell}^k))$$

that are holomorphic in $z$, $H_k^\ell = 0$ for $k < \ell$, the term $(H_0^0)_{0,0}$ of bidegree $(0,0)$ is the identity $I_{E_\ell}$ on the diagonal $\Delta$, and

$$(4.2) \quad \nabla_\eta H_k^\ell = H_{k-1}^\ell f_k - f_{k+1}(z) H_k^{\ell+1},$$

where $f_k$ stands for $f_k(\zeta)$.

Notice that we do not require $H$ to be holomorphic in $\zeta$.

**Remark 5.** We can always find a Hefer morphism locally. To begin with we can easily find $H_0^\ell$ with the stated properties locally. Since $\eta$ is a complete intersection, the sheaf complex induced by $\nabla_\eta$ is exact except at $L^0$, whereas a $\nabla_\eta$-closed section $\xi$ of $L^0$ is locally exact if and only if $\xi_{0,0}$ vanishes on the diagonal (see, e.g., [1], Proposition 4.1). This latter condition is fulfilled by the right hand side of $\nabla_\eta H_{\ell+1}^\ell = \cdots$
$-f_{\ell+1}(z)H_{\ell+1}^{\ell+1} + H_{\ell}^\ell f_{\ell+1}$ so there is locally a solution $H_{\ell+1}^\ell$. One can now proceed by induction. □

Assume that $H$ is a Hefer morphism for $E_\bullet$ and let $U$ and $R$ be the associated currents. We can then form the currents $H_1^1 U = \sum_j H_j^1 U_j$ and $H_0^0 R = \sum_j H_j^0 R_j$. To be precise with the signs one has to introduce a superbundle structure on $E = \oplus E_k$; then for instance $f$ is mapping of even order since it maps $E_k \to E_{k-1}$ (and therefore $f$ anti-commutes with odd order forms) whereas, e.g., $H$ is even since $H_0^\ell$ is a form of degree $k - \ell (\mod 2)$ that takes values in Hom $(E_\ell, E_k)$, giving another factor $k - \ell (\mod 2)$. See Section 5 in [5] for details.

Proposition 4.2. Assume that $H$ is a Hefer morphism for the complex $E_\bullet$. Then the current

\[ f_1(z)H^1 U + H^0 R, \]

is a weight with respect to $E_0$ and $z$ outside $Z$. If $\psi$ is a holomorphic section of $E_0 \otimes O(-n)$, then we have the representation

\[ \psi(z) = f_1(z) \int_{\mathbb{P}^n} (H^1 U)_{n,n} \psi + \int_{\mathbb{P}^n} (H^0 R)_{n,n} \psi, \quad z \in \mathbb{P}^n. \]

If $R\psi = 0$ we thus have the explicit holomorphic solution

\[ q(z) = \int_{\mathbb{P}^n} (H^1 U)_{n,n} \psi \]

to $f_1 q = \psi$.

Remark 6. In [7] occur more general currents $U_\ell^k$ and $R_\ell^k$ taking values in Hom $(E_k, E_\ell)$. With the same proof as below we get the more general formula

\[ \psi(z) = f_{\ell+1}(z) \int_{\mathbb{P}^n} (H^{\ell+1} U)_{n,n} \psi + \int_{\mathbb{P}^n} (H^\ell R)_{n,n} \psi + \int_{\mathbb{P}^n} (H^\ell U)_{n,n} f_{\ell+1} \psi \]

for holomorphic sections of $E_\ell \otimes O(-n + \ell)$. If $f_{\ell+1} \psi = 0$ and $R\psi = 0$ we thus get an explicit holomorphic solution to $f_{\ell+1} q = \psi$. □

Proof of Proposition 4.2. The first part of the proposition follows in the same way as the corresponding statement (5.4) in [5]. However, we will provide an argument for a more general statement, which also implies the more general formula in Remark 6.

To this end let $R$ and $U$ denote the “full” currents mentioned in the remark. Then

\[ \nabla_f \circ U + U \circ \nabla_f = I_E - R, \]

see [7] Section 2. Let

\[ \tilde{g} = H - \nabla_\eta (HU) \]
and let \( g \) be the components that take values in \( \oplus \text{Hom}(E_{\zeta,\ell}, E_{z,\ell}). \)

Since \( (H_{\ell})_{0,0} \) is the identity on \( \Delta \) it follows that \( g \) is indeed a weight with respect to \( E \). Recalling that \( H \) has even order and \( f \) is odd, and using (4.5) we have

\[
\tilde{g} = H - (-f(z)H + Hf)U - H\nabla_i U = \\
H + f(z)HU - HfU + H\partial U = \\
H + f(z)HU - H(fU + Uf - \partial U - Uf) = \\
H + f(z)HU - H(I_{E_{\zeta}} - R) + HUf.
\]

Now (4.3) is the component of the last term that takes values in \( \text{Hom}(E_{\zeta,0}, E_{z,0}) \), and hence (4.3) is a weight.

The division formula (4.4) now follows from Proposition 4.1 for \( z \) outside \( Z \), and hence in general since both sides of (4.4) are holomorphic. One gets the formula in Remark 6 from the component of \( \tilde{g} \) that takes values in \( \text{Hom}(E_{\zeta,\ell}, E_{z,\ell}). \)

\[\square\]

Assume now that \( E_\bullet \) is a complex with \( E_k \) of the form (2.3) and choose \( \kappa \) such that \( \kappa \geq d_i^k \) for all \( i,k \). We can then construct a Hefer morphism for the complex \( E_\bullet \otimes \mathcal{O}(\kappa). \) Notice that the currents \( U \) and \( R \) that are associated to \( E_\bullet \) are also the associated currents to \( E_\bullet \otimes \mathcal{O}(\kappa). \) We thus obtain a division formula for sections \( \psi \) of \( E_0 \otimes \mathcal{O}(\kappa - n). \)

Let \( E'_\bullet \) denote the complex of trivial bundles over \( \mathbb{C}^{n+1} \) that we get from \( E_\bullet \), and let \( F_k \) denote the corresponding mappings (which are formally the original matrices \( f_k \)). Let \( \delta_{w-z} \) denote interior multiplication with

\[
2\pi i \sum_0^n (w_j - z_j) \frac{\partial}{\partial w_j}
\]

in \( \mathbb{C}^{n+1}_w \times \mathbb{C}^{n+1}_z. \)

**Proposition 4.3.** There exist \((k - \ell, 0)\)-form-valued mappings

\[
h_k^\ell = \sum_{ij} (h_k^\ell)_{ij} e_i^* \otimes e_j^* : \mathbb{C}^{n+1}_w \times \mathbb{C}^{n+1}_z \rightarrow \text{Hom}(E_k, E'_\ell),
\]

such that \( h_k^\ell = 0 \) for \( k < \ell, \) \( h_\ell^\ell = I_{E'_\ell}, \) and

\[
(4.7) \quad \delta_{w-z} h_k^\ell = h_{k-1}^\ell F_k(w) - F_{\ell+1}(z) h_k^{\ell+1},
\]

and the coefficients in the form \((h_k^\ell)_{ij}\) are homogeneous polynomials of degree \( d_k^i - d_{\ell}^i - (k - \ell). \)

For a proof, see [21]. In Section 4 in [5] there is an explicit formula that provides \( h_k^\ell. \) One has to verify, though, that they get the desired degrees and homogeneities.

Notice that

\[
\alpha = \alpha_{0,0} + \alpha_{1,1} = \frac{z \cdot \bar{\zeta}}{|\zeta|^2} - \bar{\partial} \frac{\bar{\zeta} \cdot d\zeta}{2\pi i |\zeta|^2}
\]
is a well-defined smooth form in $L^0(\text{Hom}(\mathcal{O}_\zeta(1), \mathcal{O}_z(1)))$, such that
\[
\nabla_\eta \alpha = 0,
\]
and $\alpha_{0,0}$ is equal to $I_{\mathcal{O}(1)}$ on the diagonal. Thus $\alpha$ is weight with respect to $\mathcal{O}(1)$. Furthermore,

\[
\gamma_j = d\zeta_j - \frac{\bar{\zeta} \cdot d\zeta_j}{|\zeta|^2} \zeta_j
\]
is a projective form and
\[
\nabla_\eta \gamma_j = 2\pi i (z_j - \alpha \zeta_j).
\]
If $h(w, z)$ is a homogeneous form in $\mathbb{C}_w^{n+1} \times \mathbb{C}_z^{n+1}$ with differentials $dw$ and polynomial coefficients, we let $\tau^* h$ be the projective form obtained by replacing $w$ by $\alpha \zeta$ and $dw_j$ by $\gamma_j$. We then have
\[
\nabla_\eta \tau^* h = \tau^*(\delta_{w-z} h),
\]
in light of (4.9) and (4.8).

**Proposition 4.4.** Assuming that $\kappa \geq d^j_k$ for all $k$ and $j$ we define
\[
H^\ell_k = \sum_{ij} (\tau^* h^\ell_k)_{ij} \wedge \alpha^{\kappa-d^j_k} \epsilon^{*}_{\ell,i} \otimes \epsilon^{*}_{k,j},
\]
Then $(H^\ell_k)$ is a Hefer morphism for the complex $E_\cdot \otimes \mathcal{O}(\kappa)$.

**Proof.** First notice that $\alpha$ is in $L^0(\mathcal{O}_\zeta(1) \otimes \mathcal{O}_\zeta(-1))$ and hence $\alpha \zeta \nu$ and $z_\nu$ are in $L^0(\mathcal{O}_z(1) \otimes \mathcal{O}_\zeta(0))$. It follows that $\tau^* (h^\ell_k)_{ij}$ is a section of $L^{-k+\ell} (\mathcal{O}_z(d^j_k - d^j_k) \otimes \mathcal{O}_\zeta(0))$ and hence $\alpha^{\kappa-d^j_k} (\tau^* h^\ell_k)_{ij}$ is a section of $L^{-k+\ell} (\mathcal{O}_z(\kappa - d^j_k) \otimes \mathcal{O}_\zeta(-\kappa + d^j_k))$. This means that $H^\ell_k$ is indeed a section of $L^{-k+\ell} (\text{Hom}(E_{\zeta,k} \otimes \mathcal{O}_\zeta(\kappa), E_{z,\ell} \otimes \mathcal{O}_z(\kappa)))$.

It is readily checked that $(H^\ell_k)_{0,0}$ equals $I_{E_\zeta} = I_{E_z \otimes \mathcal{O}(\kappa)}$ on the diagonal. We now show that (4.12) holds. Using (4.9), (4.7) and (4.8) we have that
\[
(\nabla_\eta H^\ell_k)_{ij} = \nabla_\eta [\alpha^{\kappa-d^j_k} \wedge \tau^* (h^\ell_k)_{ij}] = \alpha^{\kappa-d^j_k} \wedge \tau^* \delta_{w-z} (h^\ell_k)_{jk} = \alpha^{\kappa-d^j_k} \wedge \tau^* \sum_{\nu} (F_{\ell+1})_{i\nu} (z)(h^\ell_{k+1})_{\nu j} + \alpha^{\kappa-d^j_k} \wedge \tau^* \sum_{\nu} (h^\ell_{k-1})_{i\nu} (F_k(w))_{\nu j}.
\]
The next to last term is
\[
-[(f(z)_{\ell+1} H^\ell_k)_{ij}]
\]
and since
\[
\tau^* (F_k(w)) = \alpha^{d^j_k - d^j_{k-1}} f_k(\zeta),
\]
the last term is
\[
[H^\ell_{k-1} f_k]_{ij}.
\]
Thus the proposition follows. \qed
5. Examples

Let us compute a solution formula corresponding to the Koszul complex, cf., Example 1. Then we first have to find a Hefer morphism. Let \( \tilde{h}_j(w,z) \) be \((1,0)\)-forms in \( \mathbb{C}^{n+1} \) of polynomial degrees \( d_j - 1 \) such that

\[
\delta w - z \tilde{h}_j = f^j(w) - f^j(z)
\]

and let \( h_j = \tau^* \tilde{h}_j \). We only have to care about \( k \leq \min(m,n+1) \) so we assume that

\[
\kappa \geq d_1 + \ldots + d_{\min(m,n+1)}.
\]

Then

\[
H^0_k = \sum'_{|J|=k} \sum' \pm h_{J_1} \land \ldots \land h_{J_{k-\ell}} \land e_I \otimes e^*_J \land \alpha^\kappa \land (d_{J_1} + \ldots + d_{J_{k-\ell}} + d_{I_1} + \ldots + d_{J_k})
\]

is a Hefer morphism. The components of most interest for us are \( H^0_k \) and \( H^1_k \). Since

\[
H^0_k = \sum'_{|J|=k} \pm h_{J_1} \land \ldots \land h_{J_{k-\ell}} \land e_I \otimes e^*_J \land \alpha^\kappa \land N(\delta h)_k - 1
\]

it can be more compactly written as

\[
H^0_k = \alpha^\kappa \land (\delta h)_k
\]

where \( \delta h \) denotes formal interior multiplication with

\[
\hat{h} = \sum \alpha^{-d_j} h_j \land e^*_j
\]

and \( (\delta h)_k = \delta_h^k / k! \). In the same way

\[
H^1_k = \alpha^\kappa \land N(\delta h)_{k-1},
\]

where

\[
N = \sum \alpha^{-d_j} e_j \otimes e^*_j.
\]

Assume now that \( \psi \) is a section of \( \mathcal{O}(\rho) \) where \( \rho = \kappa - n \). We then have the decomposition (4.4). If in addition \( R\phi = 0 \) we thus have that

\[
\psi(z) = \sum f^j(z) \cdot q_j(z) = \delta f(z) q(z)
\]

where, cf., Example 1

\[
(5.1) \quad q(z) = \int_{\mathbb{P}^{m,n+1}} \alpha^\kappa \land N(\delta h)_{k-1} (\sigma \land (\partial \sigma)^{k-1}) \psi.
\]

With the assumptions in Theorem 1.2 or 1.3, the proofs in Section 3 show that \( \psi = z^{\rho - \deg \Phi} \phi \) annihilates the residue \( R \), for appropriate choice of \( \rho \), and hence (5.1) is an explicit solution to the division problem (after dehomogenization).
Example 3. As in Example 2 let us assume that $d_j = d$. If $h = \sum h_j \wedge e^*_j$, then

$$H_k^1 = \alpha^{\kappa - dk} (\delta_h)_{k-1}.$$ 

Then our solution $q$ in (5.1) takes values in $O(\rho - d) = O(\kappa - n - d)$, and it can be written as

$$q = \sum_{k=1}^{\min(m,n+1)} \int_{\mathbb{P}^n} \alpha^{\rho + n - dk} \wedge (\delta_h)_{k-1} \frac{\bar{\mathcal{f}} \cdot e \wedge (d \bar{f} \cdot e)_{k-1}}{|f|^{2k}} \psi.$$

If $f$ has no zeros at all, then (5.1) provides a completely explicit representation of Macaulay’s theorem and the integrand is smooth. If $m \leq n$ and $|\psi| \leq |f|^{\min(m,n)}$, then $U\psi$ is integrable so (5.1) (and in particular (5.2)) is a convergent integral. In general, however, $U\psi$ may be a distribution of higher order than zero, and then (5.1) must be regarded as a principal value. For instance one can multiply by $|f|^{2\lambda}$ and put $\lambda = 0$, cf., Example 1. One can just as well define it as a classical principal value, see, e.g., [2],

$$\lim_{\epsilon \to 0} \sum_{k=1}^{\min(m,n+1)} \int_{\mathbb{P}^n} \chi(|f|/\epsilon) \alpha^{\kappa} \wedge (\delta_h)_{k-1} (\sigma \wedge (\bar{\sigma})_{k-1}) \psi,$$

where $\chi(t)$ is (a smooth approximand of) the characteristic function for the interval $[1, \infty)$.

Remark 7. If $f^j$ are rational we can choose rational Hefer polynomials $\tilde{h}_j$, and then $h_j$ are rational expressions in $\alpha \zeta$, $z$, and $\gamma_i$. It is possible that the resulting solution then actually is rational if $\psi$ is rational but we have no argument.

Also in the case of the Buchsbaum-Rim complex one can find quite simple expressions for a Hefer morphism for the corresponding homogeneous complex, see Section 6 in [5]. One then obtain the projective Hefer morphism following Section 4 above. Therefore we get an explicit representation for the solutions in Theorem 1.4 as well; however we omit the details.

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