On Dynamical (B/Gd) Neutron Cancer Therapy by Accelerator Based Two Opposing Neutron Beams

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Keywords: Accelerator Based Modulated Neutron Sources; One-Speed Neutron Diffusion; Two Opposing Neutron Beams; Dynamical NCT; Neutron-Density Waves

Introduction

In a previous paper [1] we analyzed dynamical (B/Gd) neutron cancer therapy (B/Gd) NCT, using a single modulated neutron beam, in a BNCT [2-4] and/or GdNCT [5] setup. This paper generalizes the results, obtained in [1], to the case of two opposing beams of slow neutrons, produced by accelerator sources and directed onto a cancerous region R, through two tumor-free regions Λ and Π. The beam transport is performed by means of collimators, hollow neutron guides, and possibly by solid neutron fibers, that penetrate a region Λ, to the left of R, and a region Π to its right.

An investigation of the dynamical neutronics of such a system is the main purpose of this paper. For the sake of simplicity of the analysis, we shall focus our attention on neutrons of only one speed (as in [1]) and on individual cosinusoidal time-modulation of the two opposing neutron beams, and on possible relative time advance between them. The rest of this paper is organized as follows. The next section contains the details of a one-dimensional setup formulation of the posed dynamical boundary value problem (BVP) for the modulated and time-advanced right neutron beam source, which is appropriately mapped in space and time. The section that follows addresses the solution to the dynamical boundary value problem (BVP) for the modulated and time-advanced right neutron beam source, that stands in the diffusion equation context of composite region coupling of 2.1 in [1], happen to enable the generalization of the analytical developments of the neutron flux \( \phi(x,t) \), due the left source \( S(x,t) \), to \( \psi(z,\tau) \) corresponding to this \( S(z,\tau) \) right source, that stands in the diffusion equation

\[
\frac{1}{v} \frac{\partial}{\partial \tau} \psi(z,\tau) - D \frac{\partial^2}{\partial z^2} \psi(z,\tau) + \sum_{m=1}^{\infty} \psi_m(z,\tau) = S(z,\tau), 0 \leq z \leq l, \tag{5}
\]

One-Dimensional Problem Formulation for the Right Neutron Beam

The work in [1] studied the distribution of the neutron flux \( \phi(x,t) \) that is established inside R due to the left source \( S(x,t) \). The same description applies here with regard to this source in Figure 1. In addition to this flux, we shall consider a neutron flux \( \psi(z,\tau) \) to be associated with the opposing right source \( S(z,\tau) \) where \( z \) and \( \tau \) are appropriately designed space and time variables. These are namely

\[
z = l - x, \tag{1}
\]

so as when \( z = 0, x = l \) and when \( z = l, x = 0 \); and

\[
\tau = t + \epsilon, \tag{2}
\]

so as when \( \tau = 0, t = -\epsilon \) and when \( \tau = \epsilon, t = 0 \). Here \( \epsilon \) stands for possible time advance between the cosinusoidally modulated sources of the two opposing neutron beams. Hence

\[
S(z,\tau) = \begin{cases} \frac{b_0}{2} + \sum_{m=1}^{\infty} b_m \cos \sigma \tau; & z = 0 \\ 0; & z \geq 0, \end{cases}
\]

With \( S(\tau) \) assumed to be a periodic function, of period \( b \), with even symmetry (like \( S(t) \) of the opposing first neutron beam) and having a modulation frequency

\[
\sigma = \frac{2\pi}{b},
\]

which can be varied freely of \( \omega \). These facts are apparent from

\[
S(x,t) = \begin{cases} \frac{b_0}{2} + \sum_{m=1}^{\infty} b_m \cos \omega (t+\epsilon); & x=l \\ 0; & x \leq l. \end{cases}
\]

In this model for the neutron source, it is assumed that even before the modulation starts at \( t = 0 \), a steady state (stationary mode) exists with a level equaling to \( \frac{b_0}{2} \). The variables of (1) and (2) in the context of composite region coupling of 2.1 in [1], happen to enable the generalization of the analytical developments of the neutron flux \( \phi(x,t) \), due the left source \( S(x,t) \), to \( \psi(z,\tau) \) corresponding to this \( S(z,\tau) \) right source, that stands in the diffusion equation

\[
\frac{1}{v} \frac{\partial}{\partial \tau} \psi(z,\tau) - D \frac{\partial^2}{\partial z^2} \psi(z,\tau) + \sum_{m=1}^{\infty} \psi_m(z,\tau) = S(z,\tau), 0 \leq z \leq l, \tag{5}
\]
where \((z,τ)=vN(z,τ),\) and \(N(z,τ)\) is the pertaining neutron density. The other constants are the same as in [1] or in the classical literature on reactor engineering [6]. \(Σ_a\) is the macroscopic absorption cross section of these neutrons in \(R\) which is loaded e.g. with B or Gd. If \(Σ_aW\) and \(Σ_aB\) are the cross sections corresponding to light water and natural boron, then the boron loading factor, [7], is

\[
ϕ = \sum Σ_a W + \sum Σ_a B.
\]

The clinically admissible value of \(ϕ\) which allows for the validity of the neutron diffusion model, [7], is not expected to exceed 5. The diffusion coefficient of \(R\) is however

\[
D = \frac{1}{3} \sum Σ_v α_v.
\]

with \(Σ_v\) as the neutron transport cross section. The point \(z=0\) shall be the point of our \(S(z,τ)\) source application, \(z=a\) is the physical boundary of the slab \(R\), while \(z=l\) shall be its extrapolated boundary; where

\[
l-a = 0.667.
\]

Furthermore, the diffusion length \(L\) of neutrons in \(R\), satisfies:

\[
L = \frac{1}{3} \sum Σ_v α_v ϕ,
\]

and for rather small cancers, it may be assumed

that the cancerous slab width satisfies \(a \leq (2-3) L\).

The source term in (3) is discontinuous in \(z\) and is generated by a moderator attached to an accelerator target that can be \((ε, δ)\) modulated in time. The accelerator generates pulses of fast neutron emissions \(S(τ)\) at \(z_ε\) of pulse width \(δ\) in the range 10 \(μs < δ < 1000 \(μs\). At the end of the moderator, i.e. at \(z_ε\), the thermal neutron source becomes \(S(τ)\). This same source of thermal neutrons is transported by a system of hollow neutron guides or solid neutron fibers, [1], through \(Π\) to emerge at \(z = 0\) with a reduced amplitude represented by \(S(τ)\), as sketched in Figure 1. In this setup, the ratio \(\frac{\delta}{D} \rightarrow 0\), when \(b \rightarrow \infty\), i.e. when \(m \rightarrow \infty\). Moreover, to distinguish a dynamical therapy from a steady state of therapy, \(ϕ\) should exceed the life time \(T_a\), of thermal neutrons in \(R\), i.e. \(T_a < cϕ\) and

\[
\frac{2π}{T_a} > m >> 0
\]
a condition that needs to be satisfied by \(m\) as well.

Obviously, for any \(ε (-∞,∞)\)

\[
b_m = \int \sum Σ_1 (τ) cosm d τ = \frac{π}{T_a} \sum Σ_1 (τ) cosm d τ
\]

(7)

Composite region coupling by a source at the common boundary

As in [1], it would be assumed that the neutron flux intensity \(I(0,τ)\) at \(z=0\) should satisfy:

\[
I(0\ast,τ) = \chi_{R,Π} S(z,τ)
\]

(8)

Where

\[
\chi_{R,Π} = \frac{\varrho_{\Pi}}{\varrho_{R} + \varrho_{Π}}
\]

(9)

is the coupling factor between the \(R\) region and the adjacent \(Π\) region, whose physical boundary is at \(z=b\) (i.e. \(x=l-b\)) and extrapolated boundary is at \(z=l_l\) (i.e. \(x=l_l\)).

Here \(ϕ_l\) is the albedo [8] for \(Π\), \(ϕ_l = \left[1-2 \frac{D_n \cosh D}{L_Π} \frac{L_Π}{L_Π} \right]\) and \(\chi_{R,Π}\) accounts for boosting up \(I(0\ast,τ)\) via reflection of neutrons from region \(Π\) to region \(R\).

To simplify notation in the forthcoming analysis, as in [1], we shall throughout consider:

\[
\hat{a}_m = \chi_{R,Π} a_m, m=0,1,2,3,\ldots
\]

\[
\hat{b}_m = \chi_{R,Π} b_m, m=0,1,2,3,\ldots
\]

(10)

The PDE (5) is subjected to satisfaction of (i) a zero flux at the extrapolated boundary \(z=l\), and (ii) modulated neutron flux intensity at \(z=0\) (instead of at \(z=0\) in a physically acceptable ad-hoc sense).

Accordingly, for \(z=0\), the posing mixed-type BVP becomes

\[
\frac{1}{D} \frac{∂}{∂τ} ψ(z,τ) - D \frac{∂^2}{∂z^2} ψ(z,τ) + \sum Σ_a ψ(z,τ) = 0,
\]

\[
(i) ψ(l,τ) = 0,
\]

\[
(ii) ψ(z,τ) \mid_{z=0} = -\frac{b_m}{2D} \sum \hat{b}_m cosmτ,
\]

\[
(iii) ψ(z,0) = Ω(z),
\]

(11)

where \(Ω(z)\) satisfies the auxiliary ordinary BVP:

\[
\frac{d^2}{dz^2} φ(z) - \sum a_m φ(z) = 0,
\]

(12)

(i) \(ϕ(l)=0\),

(ii) \(ϕ(z) = 0\).
(ii) \[ \frac{d}{dz} \phi(z)|_{z=0} = -\frac{b_0}{2D}, \quad (12) \]
with the condition (ii) as the natural temporal boundary condition corresponding to the \( m=0 \) mode. This implies that even before the source modulation starts at \( \tau=0 \), a steady state source exists with a level equalling to \( \frac{b_0}{2} \).

**Solution of the auxiliary BVP**

If \( \mu = \sqrt{\sum z^2} \), the solution to (12) is

\[ \phi(z) = \frac{b_0}{2D} \sinh \mu (1-z) \cosh \mu l, \quad (13) \]
with \( \phi(0) = \frac{b_0}{2D} \tan \mu l, \) while \( \phi(l) = 0 \).

**Solution of the dynamical BVP**

Each \( m \geq 1 \) temporal mode of the dynamical BVP solution shall be denoted as \( \psi_m(z,\tau) \) and should satisfy a corresponding modal partial BVP viz

\[ \frac{\partial}{\partial \tau} \psi_m(z,\tau) - vD \frac{d^2}{dz^2} \psi_m(z,\tau) + v \sum \psi_m(z,\tau) = 0, \quad z \geq 0, \]

(i) \( \psi_m(l,\tau) = 0 \),

(ii) \( \psi_m(z,0) = \mathcal{O}_m(z) = 0 \),

(iii) \( \psi_m(z,0) = \mathcal{O}_m(z) = 0 \).

With respect to BC(iii), \( \mathcal{O}(z) = \mathcal{O}(z); \) hence \( \mathcal{O}_m(z) = 0, \) \( \forall \ m \geq 1 \).

This BVP happens to quite simplify by application of the Laplace transformation in the \( \tau \) domain. So after invoking the Laplace transform pair \( \psi_m(z,\tau) \leftrightarrow \bar{\psi}_m(z,s) \), it is possible to write

\[ \frac{\partial}{\partial \tau} \bar{\psi}_m(z,s) \leftrightarrow s \bar{\psi}_m(z,s), \quad (15) \]
which leads to the ODE of the following modal ordinary BVP

\[ s \bar{\psi}_m(z,s) - vD s^2 \bar{\psi}_m(z,s) + v \sum \bar{\psi}_m(z,s) = 0, \quad z \geq 0, \]

(i) \( \bar{\psi}_m(l,s) = 0 \),

(ii) \( \bar{\psi}_m(z,0) = -\frac{b_0}{D} \frac{s}{s^2 + m^2 \sigma^2}. \)

Utilization of (15) in (16), after assuming:

\[ \bar{\alpha}^2 = \left(s + \sum_{m=0}^\infty \frac{\beta_m}{l} \right) / vD \]
leads to

\[ \bar{\psi}_m(z,s) = \frac{b_0}{D} \frac{\sinh \bar{\alpha} (1-z)}{\cosh \bar{\alpha} z} \frac{s}{s^2 + m^2 \sigma^2}. \quad (17) \]

After adoption of the notation

\[ \mathcal{F}(z,s) = \frac{\sinh \bar{\alpha} (1-z)}{\cosh \bar{\alpha} z} \leftrightarrow F(z,\tau), \]
we may utilize the real convolution property of the Laplace transformation to write:

\[ \psi_m(z,s) = \frac{b_0}{D} \mathcal{F}(z,s) \frac{s}{s^2 + m^2 \sigma^2}. \]

\( F(z,\tau) \) can be derived from existing tables. [9] of Laplace transform pairs to be

\[ F(z,\tau) = 2vD \sum_{l=1}^\infty (-1)^{l-1} e^{-\beta_v \tau} \sin (2n-1) \frac{(1-z)}{l}, \]
with

\[ \beta_v = vD(2n-1)^2 \frac{1}{4l^2} + v \sum_s. \]

Apparenty

\[ \sin (2n-1) \frac{(1-z)}{l} = (-1)^{n-1} \cos (2n-1) \frac{(1-z)}{l} = (-1)^{n-1} Q_n(z), \]
and that allows for rewriting

\[ F(z,\tau) = 2vD \sum_{l=1}^\infty e^{-\beta_v \tau} Q_n(z), \]
Then

\[ \psi_m(z,\tau) = 2vD \sum_{l=1}^\infty \{ \beta_l \left[ \cos \omega_t + m \sin \omega_t \right] \} Q_n(z), \]

\[ -2vD \sum_{l=1}^\infty \{ \beta_l \left[ \cos \omega_t + m \sin \omega_t \right] \} e^{-\beta_v \tau} Q_n(z), \quad (24) \]
By summing up over all the \( m \) temporal harmonics we arrive at the proposing BVP solution \( \psi(z,\tau) = \sum_{m=0}^\infty \psi_m(z,\tau) \) which upon division by \( v \) yields the associated, with \( S(z,\tau) \), neutron density distribution

\[ S(z,\tau) = \sum_{m=0}^\infty \psi_m(z,\tau) = 2vD \sum_{l=1}^\infty \left[ \left\{ \beta_l \cos \omega_t + m \sin \omega_t \right\} \right] Q_n(z), \]

\[ -2vD \sum_{l=1}^\infty \left[ \left\{ \beta_l \cos \omega_t + m \sin \omega_t \right\} \right] e^{-\beta_v \tau} Q_n(z), \]
(25) which decomposes, like \( N(x,t) \), into the three superimposed distinct effects viz

\[ N(z,\tau) = N_a(z,\tau) + N_b(z,\tau) + N_c(z), \]
where \( N_a(z,\tau) \) is periodic, \( N_b(z,\tau) \) is dissipative and \( N_c(z) \) is stationary.

**The Neutron Sensory Distribution From The Right Beam In The xt –Domain**

Back substitution of the maps of the (1) and (2) for \( z \) and \( \tau \) in (29) invokes

\[ Q_a(z) = \cos (2n-1) \frac{\pi x}{2l} = \cos (2n-1) \frac{(1-x)}{2l} \]

\[ = (-1)^{(n-1)} \sin (2n-1) \frac{\pi x}{2l} = (-1)^{(n-1)} G_n(x), \]

\[ e^{-\beta_v \tau} = e^{-\beta_v \tau} e^{-\beta_v \tau} \]

\[ \sinh(1-z) = \sinh x, \]
\[ \left[ \beta_l \cos \omega_t + m \sin \omega_t \right] = \beta_l \cos(\omega_t + m \sin \omega_t), \]
\[ = \left[ \beta_l \cos(\omega_t + m \sin \omega_t) \right] \cos \omega_t + \beta_l \sin(\omega_t + m \sin \omega_t) \]
\[ = \left[ \Xi_m(\omega,\epsilon) \right] \cos \omega_t + \Xi_m(\omega,\epsilon) \sin \omega_t \]
to yield
\[ N(x,t) = \sum_{n=0}^{\infty} N_n(x,t) = 2 \sum_{n=1}^{\infty} p_n \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{l} \left[ \Xi_{nm}^+(\sigma, \epsilon) \cos \sigma t + \Xi_{nm}^- (\sigma, \epsilon) \sin \sigma t \right] G_n(x) \]

\[ = \frac{1}{l} \sum_{n=1}^{\infty} p_n \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{l} \frac{\beta_n}{(\beta_n^2 + m^2 \sigma^2)} e^{-\beta_n t} G_n(x) \]

\[ = -2 \sum_{n=1}^{\infty} \bar{p}_n \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\beta_n}{(\beta_n^2 + m^2 \sigma^2)} e^{-\beta_n t} G_n(x) \]

The periodic component of the space-time transient \( N(x,t) \) is
\[ N_p(x,t) = 2 \sum_{n=1}^{\infty} \bar{p}_n \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{l} \frac{\beta_n}{(\beta_n^2 + m^2 \sigma^2)} e^{-\beta_n t} G_n(x) \]

\[ = \frac{\bar{b}_n}{l} \sinh \mu x \frac{\sin \mu x}{2 \nu_y \cos hyl} \]

\[ = \frac{\bar{b}_n}{l} \sinh \mu x \frac{\sin \mu x}{2 \nu_y \cos hyl} \]

The dissipative neutron density wave

The dynamical component
\[ N_d(x,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \bar{M}_{nm}(x,t) \]

with
\[ \bar{M}_{nm}(x,t) = 2 \frac{\bar{b}_n}{l} (-1)^{n-1} \frac{\beta_n}{(\beta_n^2 + m^2 \sigma^2)} e^{-\beta_n t} G_n(x) \]

vanishes asymptotically with time and increasingly faster with increasing \( \Sigma \), i.e. by increasing the B/Gd content of \( R \).

Apart from the implicit dependence of its \( b_m \)'s on \( \sigma \), \( N_d(x,t) \) itself is rather sensitive to variations in \( \sigma \). However, when \( \omega \rightarrow \infty \) all the \( b_m \)'s are zeros and the entire \( N_d(x,t) \) disappears.

The periodic neutron-density wave

The dispersive term of
\[ N(x,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N_{nm}(x,t) \]

with an \( mn \)-double indexed modal term
\[ N_{nm}(x,t) = 2 \frac{\bar{b}_n}{l} (-1)^{n-1} \frac{\beta_n}{(\beta_n^2 + m^2 \sigma^2)} e^{-\beta_n t} \]

\[ = N_{nm} \sin[\sigma t + \theta_{nm}(\sigma)] \]

with \( N_{nm} = \sqrt{\beta_n^2 + m^2 \sigma^2} \) and
\[ \theta_{nm}(\sigma) = \tan^{-1} \frac{\beta_n}{m \sigma} \]

which approximates with
\[ \theta_{nm}(\sigma) = \frac{\beta_n}{m \sigma} \text{ as } \sigma \rightarrow \infty \]

Clearly then
\[ \Xi_{nm}^+(\sigma, \epsilon) \cos \sigma t + \Xi_{nm}^- (\sigma, \epsilon) \sin \sigma t = N_{nm} \sin[\sigma t + \theta_{nm}(\sigma) + \nu_{nm}] \]

which in view of
\[ \Omega_{nm}(\sigma, \epsilon) = \theta_{nm}(\sigma) + m \sigma \epsilon = \tan^{-1} \frac{\beta_n}{m \sigma} + m \sigma \epsilon \]

becomes
\[ \Xi_{nm}^+(\sigma, \epsilon) \cos \sigma t + \Xi_{nm}^- (\sigma, \epsilon) \sin \sigma t = N_{nm} \sin[\sigma t + \Omega_{nm}(\sigma, \epsilon)] \]

\[ \Xi_{nm}(\sigma, \epsilon) = \theta_{nm}(\sigma) + m \sigma \epsilon = \tan^{-1} \frac{\beta_n}{m \sigma} + m \sigma \epsilon \]

of a fixed (with varying \( m \)) shape function \( G_n(x) = \sin(2n-1) \frac{x}{2} \pi \), but with a varying (with varying \( \sigma \) and \( \epsilon \)) amplitude \( E_{nm}(\sigma) \Pi_{nm}(\sigma, t, \epsilon) \), whose \( \Pi_{nm}(m, t, \epsilon) = \sin[\sigma t + \Omega_{nm}(m, \epsilon)] \) factor is also time-dependent, while
\[ E_{nm}(\sigma) = -2 \frac{\bar{b}_n}{l} (-1)^{n-1} \frac{1}{\sqrt{\beta_n^2 + m^2 \sigma^2}} \]

\[ G_n(x) = \text{a standing (in time) wave with a number of nodes } \nu(n) \text{ that depends on } n. \text{ Obviously, there is always, and for all } n, \text{ a node at } x = 0; \text{ and a summary of } \nu(n) \text{ for the first 10 } n \text{ is given in Table 1.} \]

The trigonometric identity \( 2 \sin A \sin B = \cos(A-B) - \cos(A+B) \) allows for a travelling wave representation for \( N_{nm}(x,t) \), which is namely
\[ N_{nm}(x,t) = E_{nm}(\sigma) \Pi_{nm}(\sigma, t, \epsilon) G_n(x) = A_{nm}(x, \sigma, t, \Omega_{nm}) + B_{nm}(x, -t, \sigma - \Omega_{nm}) \]

where

| n | Node Location | \( \nu(n) \) |
|---|---|---|
| 1 | 0 | 1 |
| 2 | 2/3 | 1 |
| 3 | 2/5 | 1 |
| 4 | 2/7 | 1 |
| 5 | 2/3, 2/9 | 2 |
| 6 | 2/11 | 1 |
| 7 | 2/13 | 1 |
| 8 | 2/3, 2/5, 2/15 | 3 |
| 9 | 2/17 | 1 |
| 10 | 2/19 | 1 |

Table 1: Nodes for some \( G_n(x) \).
\[ \Omega_{\text{st}} (x, t, \sigma, \Omega_{\text{st}}) = \frac{\pi}{2} \cos [2n - 1] + m \sigma t + \Omega_{\text{st}} (\sigma, \epsilon)] \]

Relation (42) is a d'Alembert-like modal solution, with \( \Omega_{\text{st}} \) and \( B_{\text{st}} \) as two different modal waves, one progressing to the left, while the other is progressing to the right of the \( x \)-axis. Summing up over all \( m \) indices defines two periodic neutron density travelling waves, \[ \Omega_{\text{st}} (x, t, \sigma, \Omega_{\text{st}}) \]

\[ B_{\text{st}} (x, -t, \sigma, -\Omega_{\text{st}}) = \frac{1}{2} F_{\text{st}} (\sigma) \cos [2n - 1] - \frac{\pi x}{2} t - m \sigma t - \Omega_{\text{st}} (\sigma, \epsilon)] \]

The Neutron-Density Distribution From The Two Opposing Beams

Consider now the total neutron flux \( f(x, t) \) inside \( R \) resulting from the combined application of both left and right beam sources \( S(x, t) \) and \( S(x, t) \), respectively, i.e.

\[ f(x, t) = \phi(x, t) + \psi(x, t) \]

The pertaining neutron density distribution is

\[ g(x, t) = N(x, t) + N(x, t) \]

and substitution of relations (30), of [1], and (31) for \( N(x, t) \) leads to

\[ g(x, t) = \sum_{n=0}^{\infty} [N_{n}(x, t) + \bar{N}_{n}(x, t)] \]

The first term (with figurative brackets), in the relation above, is the periodic part of the neutron-density wave generated inside \( R \) by the two opposing beams. It can also be represented as

\[ g_{n}(x, t) = 2 \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} \left[ \frac{\hat{\beta}_{m} \sin \theta + \Omega_{\text{st}} (\sigma, \epsilon)}{\sqrt{\beta_{m}^{2} + m^{2} \sigma^{2}}} \right] \cos [2n - 1] \frac{\pi x}{2} t \]

which is explicitly a weighted sum of two quasi-standing waves having distinct frequencies \( \omega \) and \( \epsilon \). With respect to time periodicity, \( g_{n}(x, t) \) can be periodic only when the ratio \( \omega/\epsilon \) is commensurate.

The dissipative part of this wave is:

\[ g_{x}(x, t) = -2 \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} \left[ \frac{\hat{\beta}_{m} \sin \theta + \Omega_{\text{st}} (\sigma, \epsilon)}{\sqrt{\beta_{m}^{2} + m^{2} \sigma^{2}}} \right] \cos [2n - 1] \frac{\pi x}{2} t \]

\[ + \left[ \frac{1}{2D_{\text{st}} \cosh \mu l} \left[ \hat{\beta}_{m} \sin \mu (l - x) + \bar{\beta}_{m} \sin \mu x \right] \right] \]

Utility and Ballistic Indices For Neutrons From The Two Opposing Beams

The neutron-density wave utility index

For the opposing beams arrangement, the ratio
Inside (58), let us contemplate the following modal component ratio, which undoubtedly affects $\zeta(\omega, \sigma, \epsilon, x, t)$ in some indirect sense.

\[
\frac{M_{mn}(x, t) + \dot{M}_{mn}(x, t)}{N_{mn}(x, t) + \dot{N}_{mn}(x, t)} = \frac{\beta_{\nu}}{(\beta_{\nu} + m^2 \sigma^2)} e^{-\beta_{\nu} \frac{2}{3}} \sin \left( 2n - 1 \right) \frac{\pi x}{2} = e^{-\beta_{\nu} \frac{2}{3}} \sin \left( 2n - 1 \right) \frac{\pi x}{2} \tag{58}
\]

which is additionally dependent on $\sigma$ and $\epsilon$, in addition to $\omega$.

Since the center of $R$ (i.e. $x=l/2$) is the most critical point, with respect to effectiveness of this kind of neutron therapy, then there is a need to focus attention on the amplitude $M_{mn}(x, 0) + \dot{M}_{mn}(x, 0)$ at $x=l/2$, which in view of (1) $\frac{2\pi}{2} = 1$ writes as

\[
\frac{M_{mn}(l/2, 0) + \dot{M}_{mn}(l/2, 0)}{N_{mn}(l/2, 0) + \dot{N}_{mn}(l/2, 0)} = \frac{\beta_{\nu}}{(\beta_{\nu} + m^2 \sigma^2)} e^{-\beta_{\nu} \frac{2}{3}} \sin \left( 2n - 1 \right) \frac{\pi x}{2} \tag{59}
\]

which is obviously the same as

\[
\frac{M_{mn}(l/2, 0) + \dot{M}_{mn}(l/2, 0)}{N_{mn}(l/2, 0) + \dot{N}_{mn}(l/2, 0)} = \frac{\beta_{\nu}}{(\beta_{\nu} + m^2 \sigma^2)} e^{-\beta_{\nu} \frac{2}{3}} \sin \left( 2n - 1 \right) \frac{\pi x}{2} \tag{60}
\]

Due to symmetry of the two opposing beams therapeutic setup, $\sigma$ and $\omega$ are free to satisfy

\[
\sigma \geq \omega \tag{62}
\]

Therefore we may assume in the remaining part of this paper, without loss of generality, that

\[
\sigma > \omega \tag{63}
\]

and write
\( \eta(\omega, \sigma, \epsilon) \) undergo a cyclic behaviour, of period 4, with varying \( n \).

**Remark 1.** It should be noted, moreover, that a hoped for applicability of conclusions based on (65)-(66) to (70) can obviously only be restricted to very small \( M \) and \( N \). For instance, an interesting situation emerges from (68) when the ratio \( \frac{a}{\omega} = \sqrt[\omega]{C_{mn}(\omega, \sigma, \epsilon)} \) \( \forall M, N \) and \( r \), is commensurate, i.e. when \( \frac{a}{\omega} = k \in \mathbb{N} \). This corresponds to a situation when \( \sigma - \omega = (k-1)\omega \), especially if \( \sigma \) is close to \( \omega \). Is this going to be some kind of tenuous beat effect [10], for the neutron-density waves?

The existence of such a maximum \( \omega_{\text{opt}} \) vector in the \( \omega \)-space. The search of \( \omega_{\text{opt}} \) satisfying (68) is close to \( \omega \). Is this going to situate when \( \omega_{\text{opt}} / \omega - \omega = (k-1)\omega \), especially if \( \omega_{\text{opt}} / \omega \) and \( \omega \) are of the same order of magnitude, we can write

\[
\eta_{\omega_{\text{opt}}}(\omega) = (-1)^{n-1} \frac{1}{(\beta_n^2 + \mu^2 - \omega^2)} \left[ 1 - e^{-(\beta_n^2 + \mu^2 - \omega^2)} \right].
\]  

(73)

Substitution of these relations in (71) transforms it to:

\[
\tilde{a}(\omega, \sigma, \epsilon) = \frac{3}{D} \left( \tilde{a}_0 + \tilde{b}_0 \right) \left( 1 - \text{sech} \, \mu \ell \right)
\]

- \( \frac{4}{\pi} \sum_{m=1}^{M} \sum_{n=1}^{N} \left( \frac{1}{2n-1} \right) \left( \tilde{a}_m \eta_{\omega_{\text{opt}}}(\omega) + \tilde{b}_m e^{-\beta_n \epsilon} \eta_{\omega_{\text{opt}}}(\sigma) \right) \)

\[
+ \tilde{b}_0 \eta_{\omega_{\text{opt}}}(\omega) \right)
\]

which is a unique rational nonlinear function of three independent temporal variables \( (\omega, \sigma, \epsilon) \) for every set \( \{\mathcal{X}, D, \ell\} \) of composition and geometric parameters of the region \( R \). Obviously, the existence of a critical point \( \eta_{\omega_{\text{opt}}}(\omega) = (\omega', \sigma', \epsilon') \) of \( \tilde{a}(\omega, \sigma, \epsilon) \), at \( \omega = \omega(\sigma') \). The existence of such a maximum requires additionally that all the eigenvalues of the Hessian matrix \( H(\tilde{a}(\omega, \sigma, \epsilon)) \) to be negative [11]. Moreover if \( H(\tilde{a}(\omega, \sigma, \epsilon)) \) turns out to have mixed sign eigenvalues, then \( \omega_{\text{opt}} \) is a saddle point.

In distinction from the therapeutic optimization problem of [1], we have here a control variable vector \( \omega \) for the (B/Gd) NCT nonlinear optimization problem of the two opposing neutron beams with an optimal control vector \( \omega_{\text{opt}} = (\omega', \sigma', \epsilon') \) of \( \mathbb{N} \) satisfying

Maximize \( \tilde{\eta}(\omega) \), of (70),

\[
\text{Subject to: } \| \nabla \tilde{a}(\omega) \|_{\omega} \leq \epsilon \omega, \quad \epsilon \omega < 2\pi,
\]

\[
\frac{2\pi}{\epsilon \omega} \geq \mathcal{X}_{\text{opt}}(\omega, \sigma, \epsilon) \quad \omega > 0,
\]

\[\text{Where } \epsilon \omega \text{ is the tolerance associated with truncating the infinite harmonic sums in (74) after the } M \text{ and } N \text{ numbers, and } \| \nabla \tilde{a}(\omega) \|_{\omega} \text{ is some norm of the gradient } \nabla \tilde{a}(\omega) \text{ vector in the } \omega \text{-space. The search for such a point } \Omega, \text{ for the optimal } \omega^*, \text{ is expected to be carried out numerically via conjugate gradients or variable metric methods [12], which both require computations of the gradient } \nabla\tilde{a}(\Omega).\text{ The nonlinear inequality constraint } \epsilon \omega < 2\pi \text{ means that the time advance should not exceed the period of the modulated reference neutron beam, otherwise it becomes redundant. Although a direct solution of (71) is possible analytically (essentially due to simplicity of (74)), but the emerging expressions turn out to become formidable long to be put down in writing. This situation happens to simplify considerably when } \omega \text{ for the left beam is fixed as } \omega^* \text{ (in a special optimization process) to reduce the size of the control vector } (\omega^*, \sigma, \epsilon) \text{ to } (\omega, \sigma, \epsilon) \text{.}

**Remark 2.** In some cancer patients, transport of thermal neutrons...
by neutron guides or neutron optical fibres through the regions Λ and Π (with respective thicknesses $l_\Lambda$ and $l_\Pi$ and neutron macroscopic removal cross sections $\Sigma_\Lambda$ and $\Sigma_\Pi$), may turn out to be medically unfeasible. As an approximate substitute to solving the composite RU(AUIG) regional neutronics problem in such cases, one can simply assume a planar attenuation towards R, of all (or some of) the sources $S(x, t)$ and $S(x, t)$ respectively to $S^\Lambda(x, t) = V_\Lambda S(x, t) = e^{-\Sigma_\Lambda l_\Lambda} S(x, t)$ and $S^\Pi(x, t) = V_\Pi S(x, t) = e^{-\Sigma_\Pi l_\Pi} S(x, t)$.

As a result, the entire analysis, reported in this paper, should hold true if we replace each affected $\hat{a}_m = \chi_{R,\Lambda} a_m$ and $\hat{b}_m = \chi_{R,\Pi} b_m$ respectively to $\hat{a}_m = V_\Lambda a_m$ and $\hat{b}_m = V_\Pi b_m$.

**Concluding Remarks**

The basic result of this paper has been its demonstration, for the first time, that $(\omega, \sigma, \epsilon)$ is employable as a control vector in the formulation of a nonlinear optimization process that can maximize the therapeutic $\varpi$, $\epsilon^*$ objective can possibly also be achieved via a clinically feasible profiling of the (B/Gd) uptake inside the tumor. Indeed, a spatially-dependent $\Sigma(x)$, instead of the present constant $\Sigma$, that is maximal at $x = l/2$ and minimal at $x = 0$ and $x = l$, could turn out to be ideal for boosting up the ballistic index of this NCT. Clearly then, research into the associated pharmacology and biochemistry of B/Gd distributed uptakes, and into histology of pertaining tissue-engineered tumors, could be not only relevant, but even quite essential.

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