Asymptotic Nodal Length and Log-Integrability of Toral Eigenfunctions

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Abstract: We study the nodal set of Laplace eigenfunctions on the flat 2d torus $\mathbb{T}^2$. We prove an asymptotic law for the nodal length of such eigenfunctions, under some growth assumptions on their Fourier coefficients. Moreover, we show that their nodal set is asymptotically equidistributed on $\mathbb{T}^2$. The proofs are based on Bourgain’s derandomisation technique and the main new ingredient, which might be of independent interest, is the integrability of arbitrarily large powers of the doubling index of Laplace eigenfunctions on $\mathbb{T}^2$, based on the work of Nazarov (Algebra Anal 5:3–66, 1993; Summability of large powers of logarithm of classic lacunary series and its simplest consequences https://users.math.msu.edu/users/fedja/prepr.html, 1995).

1. Introduction

1.1. Nodal length of Laplace eigenfunctions and the random wave model. Given a compact $C^\infty$-smooth Riemannian surface $(M, g)$ without boundary, let $\Delta_g$ be the associated Laplace–Beltrami operator. We are interested in the eigenvalue problem

$$\Delta_g f_\lambda + \lambda f_\lambda = 0.$$ 

Since $M$ is compact, the spectrum of $-\Delta_g$ is a discrete subset of $\mathbb{R}$ with only accumulation point at $+\infty$. The eigenfunctions $f_\lambda$ are smooth and their nodal set, that is their zero set, is a smooth 1d sub-manifold outside a finite set of points [11]. In particular, the one dimensional Hausdorff measure of the nodal set is well-defined and called the nodal length

$$\mathcal{L}(f_\lambda) := \mathcal{H}\{x \in M : f_\lambda(x) = 0\}.$$ 

Yau [35], and independently Brüning [8], showed that $\mathcal{L}(f_\lambda) \geq c\lambda^{1/2}$ for some $c = c(M) > 0$. Yau [35] conjectured the matching upper bound

$$c\sqrt{\lambda} \leq \mathcal{L}(f_\lambda) \leq C\sqrt{\lambda},$$ 

where $C$ is a constant depending on $M$. This conjecture was later proven by Sarnak [36] and independently by Katz [18].

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for some $C = C(M) > 0$. Donnelly and Fefferman [12] showed that Yau’s conjecture holds for manifolds of any dimension, provided that the metric is real-analytic. Recently, Logunov [19,20] and Logunov–Malinnikova [21] proved the optimal lower-bound for $C^\infty$ manifolds and gave a polynomial upper-bound.

Some heuristic insight into the behavior of the nodal length can be deduced from a conjecture of Berry [3,4], known as the Random Wave Model (RWM). The RWM asserts that, on a generic chaotic surface, high energy Laplace eigenfunctions restricted to balls of radius $\approx \lambda^{-1/2}$, the so-called Planck scale, should behave like Random Plane Waves (RPW), that is the isotropic Gaussian field $F$ on $\mathbb{R}^2$ with covariance function

$$\mathbb{E}[F(x)F(y)] = J_0(|x - y|),$$

where $J_0(\cdot)$ is the 0-th Bessel function. Berry [3] found the expected nodal length of $F$ on a box $B$ of unit side length to be

$$\mathbb{E}[\mathcal{L}(F, B)] := \mathbb{E}[\mathcal{H}\{x \in B : F(x) = 0\}] = \frac{1}{2\sqrt{2}}.$$

Covering $M$ by balls/boxes of Planck-scale radius, the RWM suggests not only the global behavior

$$\mathcal{L}(f_\lambda) = \frac{\text{Vol}(M)\sqrt{\lambda}}{2\sqrt{2}} (1 + o_{\lambda \to \infty}(1)), \quad (1.1)$$

but also the macroscopic distribution

$$\mathcal{L}(f_\lambda, B) = \frac{\text{Vol}(B)\sqrt{\lambda}}{2\sqrt{2}} (1 + o_{\lambda \to \infty}(1)), \quad (1.2)$$

for any ball $B = B(x, r)$ of fixed, that is independent of $\lambda$, radius $r > 0$ and center $x \in M$. In particular, we expect the nodal set to be asymptotically equidistributed on $M$, see also [36, Chapter 13].

We study a class of deterministic Laplace eigenfunctions on the standard two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with moderate growth of their Fourier coefficients. These are known as “flat” eigenfunctions, see Sect. 1.2 below. The main result is that, within the said class, the asymptotic law (1.2), up to a possibly different leading constant, holds in every ball of fixed radius, along a density one sub-sequence of eigenvalues.\footnote{Let $S \subset \mathbb{R}$ be some (infinite) sequence, a subsequence $S' \subset S$ has density one if $\lim_{X \to \infty} \frac{|\{\lambda \in S' : \lambda \leq X\}|}{|\{\lambda \in S : \lambda \leq X\}|} = 1$. While the behavior of the nodal length of random Laplace eigenfunctions has been intensively studied [2,17,23,29,34], to the best of the author’s knowledge, no other, non-random or non-trivial (e.g. $f_\lambda(x) = \cos(a \cdot x)$ with $|a|^2 = \lambda$), examples of (1.2) or even (1.1) are known. Thus the results of this manuscript seem to be the first to address the asymptotic behavior of the nodal length of deterministic Laplace eigenfunctions.

The proof of the main result is based on the de-randomisation technique pioneered by Bourgain [7] and developed by Buckley–Wigman [9]. Bourgain’s de-randomization asserts that flat eigenfunctions behave according to the RWM in most balls of Planck-scale radius, see Proposition 3.1 below. In order to apply this technique to study the nodal set, it is thus essential to control the zero set in the balls failing the RWM-type behavior. In light of Donnelly–Fefferman work [12], it is well-understood that, in the
real-analytic setting, the nodal set in a ball $B$ can be controlled by the doubling index $N(B)$, a measure of the growth of the function (see Sect. 5.1 below). This leads us to the study, of possible independent interest, of the distribution of the doubling index at Planck-scale, for flat eigenfunctions: Given any $q > 1$, we show that

$$\int_{\mathbb{T}^2} N_{f_\lambda}(B(x, \lambda^{-1/2}))^q dx \leq C,$$

for some $C = C(q) > 0$. This requires a combination of some Fourier-analytic techniques borrowed from the work of Nazarov [25,26], and some arithmetic considerations. We point out that Yau’s conjecture is equivalent to (1.3) with $q = 1$ [28]. In this direction, our work seems to be the first to address the higher-integrability properties of the doubling index.

1.2. Statement of the main results. Before stating our main results we need to introduce some notation pertaining to Laplace eigenfunctions on $\mathbb{T}^2$. The eigenvalues of $-\Delta$ are, up to a factor of $4\pi^2$, integers representable as the sum of two squares $\lambda \in S := \{ \lambda \in \mathbb{Z} : \lambda = \square + \square \}$ and have multiplicity $N = N(\lambda) := |\{ \xi \in \mathbb{Z}^2 : |\xi|^2 = \lambda \}|$ given by the number of lattice points on the circle of radius $\lambda^{1/2}$. Any toral eigenfunction, with eigenvalue $-4\pi^2\lambda$ (we will simply say eigenvalue $\lambda$ from now on), can be expressed as a Fourier sum

$$f_\lambda(x) = f(x) = \sum_{\xi \in \mathbb{Z} \atop |\xi|^2 = \lambda} a_\xi e(\xi \cdot x),$$

where $e(\cdot) = \exp(2\pi i \cdot)$ and the $a_\xi$’s are complex numbers satisfying $a_\xi = a_{-\xi}$ for every $\xi$, making $f_\lambda$ real valued. Moreover, we normalize $f_\lambda$ so that

$$\|f_\lambda\|^2_{L^2(\mathbb{T}^2)} = \sum |a_\xi|^2 = 1.$$  

We first consider the special class of Bourgain’s eigenfunctions, that is functions as in (1.4) whose Fourier coefficients satisfy

$$|a_\xi|^2 = N^{-1},$$

for all $|\xi|^2 = \lambda$. Bourgain’s eigenfunctions are especially important in that they precisely behave as predicted by the RWM, that is they resemble, locally almost everywhere, RPW. In particular, the asymptotic law for their nodal length can be stated directly without the need for extra notation:

**Theorem 1.1.** There exists a density one subsequence $S' \subset S$ such that for $\lambda \in S'$ the following holds: let $B \subset \mathbb{T}^2$ be a fixed ball or $B = \mathbb{T}^2$, then we have

$$\mathcal{L}(f_\lambda, B) = \frac{\text{Vol}(B)}{2\sqrt{2}} (4\pi^2\lambda)^{1/2} (1 + o_{\lambda \to \infty}(1)),$$

uniformly for all $f_\lambda$ as in (1.4) satisfying (1.6).
We point out that the sequence $S' \subset S$ postulated in Theorem 1.1 (and Theorem 1.3 below) can be described explicitly via some conditions of pure arithmetic nature, see Sect. 2.2 below. We also stress that the rate of convergence in Theorem 1.1 (and Theorem 1.3 below) does depend on $B$. However, it is plausible that the techniques developed in this manuscript, combined with some recent work on lattice points [16], could be pushed forward to show that Theorem 1.1 (and Theorem 1.3 below) holds in any ball $B$ of radius larger than the Planck-scale, $r > \lambda^{-1/2+\varepsilon}$. This would imply an essentially optimal equidistribution regime for the nodal length. We leave this question to be addressed elsewhere.

We will now introduce the class of flat toral eigenfunctions and some additional notation required to describe their nodal length.

**Definition 1.2.** Fix some positive function $u : \mathbb{R} \to \mathbb{R}_{>0}$ such that, for every $\varepsilon > 0$, $u(N) = o_{N \to \infty}(N^\varepsilon)$. A function $f_\lambda$ as in (1.4) is said to be flat if

$$\sup_{|\xi|^2 = \lambda} |a_\xi|^2 \leq \frac{u(N)}{N^\varepsilon}.$$ 

Even though the definition of flat eigenfunctions depends on the particular choice of the function $u$, this will only affect the rate of convergence in Theorem 1.3 below. Therefore, in order not to overburden the notation, we fix $u$ throughout the whole manuscript.

As we will see, flat eigenfunction, as Bourgain’s eigenfunction, also behave, locally almost everywhere, as a Gaussian field. However, the covariance structure of the said field, and eventually its nodal length, depend on the measure

$$\mu_f = \sum_\xi |a_\xi|^2 \delta_{\xi/\sqrt{\lambda}}$$

and its Fourier coefficients

$$\hat{\mu}_f(k) = \int_{\mathbb{S}^1} z^k d\mu_f(z),$$

where $\delta_{\xi/\sqrt{\lambda}}$ is the Dirac distribution at the point $\xi/\sqrt{\lambda}$, $k \in \mathbb{Z}$ and $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle.

In order to simplify the exposition of the main result, it will be useful to arrange (sequences) of functions $f_\lambda$ according to the possible weak* limits of $\mu_f$, see [18,30] for a study of the said weak* limits. First, observe that $\mu_f$ is a probability measure with support contained in the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and that the set of probability measures on $\mathbb{S}^1$, equipped with the weak* topology, is compact. Thus, upon passing to a subsequence, we may (and will) assume that

$$\mu_f \rightharpoonup \mu \quad \text{as} \quad N \to \infty,$$

where the convergence is with respect to the weak* topology, for some symmetric$^2$ probability measure $\mu$ on $\mathbb{S}^1$. Moreover, to avoid degeneracies, we assume that the support of $\mu$ is not contained in a line. Sorting (sequences of) functions $f_\lambda$ according to their limiting measure avoids an unnecessary dependence (on $f_\lambda$) of the leading constant in the following result:

$^2$ $\mu(-A) = \mu(A)$ for any measurable set $A \subset \mathbb{S}^1$. 
Theorem 1.3. There exists a density one subsequence $S' \subset S$ such that the following holds. Let $\{f_\lambda\}_{\lambda \in S'}$ be a sequence of flat, in the sense of Definition 1.2, eigenfunctions with limiting measure $\mu$ in the sense of (1.8). Then, for any fixed ball $B \subset \mathbb{T}^2$ or $B = \mathbb{T}^2$, we have

$$\mathcal{L}(f_\lambda, B) = c_1 \text{Vol}(B)(4\pi^2 \lambda)^{1/2}(1 + o_{\lambda \to \infty}(1)),$$

where

$$c_1 = \frac{1 - |\hat{\mu}(2)|^2}{2^{5/2}\pi} \int_0^{2\pi} \frac{1}{(1 - \alpha \cos(2\theta) - \beta \sin(2\theta))^{3/2}} d\theta,$$

and $\hat{\mu}(2) = \alpha + i\beta$.

The dependence of the nodal length of toral eigenfunction on the measure $\mu$, as in (1.8), was already observed, in the random setting, by Kurlberg et al. [17]. They found that the variance of the nodal length depends on the fourth, as opposed to the second, Fourier coefficient of $\mu$, while the expectation is universal. On one hand, Theorem 1.3 shows that the nodal length behavior is much richer than what can be captured by random models. And, on the other hand, it precisely describes how the distribution of lattice points affects the nodal length.

The main new ingredient, instrumental to the proof of (1.2), which will allow us to show (1.3), is the log-integrability of $f$:

**Proposition 1.4.** Let $q \geq 1$ be an integer. Then there exists a density one subsequence $S' = S'(q) \subset S$ and some constant $C = C(q) > 0$ such that for all $\lambda \in S'$ the following holds: for every flat $f_\lambda$, in the sense of Definition 1.2, we have

$$\int_{\mathbb{T}^2} |\log |f_\lambda(x)||^q dx \leq C.$$

The flatness assumption is not essential for the proof of Proposition 1.4 and it can be removed at the cost of a slightly lengthier calculation in Sect. 4.2. For the sake of keeping the exposition as simple as possible and since flatness is essential to Theorems 1.1 and 1.3, we decided to present the proof of Proposition 1.4 under the flatness assumption.

1.3. Notation. To simplify the exposition we adopt the following standard notation: we write $A \lesssim B$ and $A \gtrsim B$ to designate the existence of an absolute constant $C > 0$ such that $A \leq CB$ and $A \geq CB$. If the constant $C$ depends on some auxiliary parameter $\beta$ say, we write $A \lesssim_\beta B$ and $A \gtrsim_\beta B$, respectively. The letters $C, c$ will be used to designate positive constants which may change from line to line. Moreover, for some parameter $\beta > 0$, we write $A = O_\beta(B)$ to mean that there exists some constant $C = C(\beta) > 0$ such that $|A| \leq CB$, if no parameter is specified in the notation, then the constant is absolute. We write $o_{\beta \to \infty}(1)$ for any function that tends to zero as $\beta \to \infty$. Given some function $g : \mathbb{T}^2 \to \mathbb{R}$ and a parameter $t > 0$, we will use the following shorthand notation: $\text{Vol}(x \in \mathbb{T}^2 : g(x) \leq t) =: \text{Vol}(g(x) \leq t)$. 


2. Preliminaries

2.1. Convergence of random fields. The proof of Theorems 1.1 and 1.3 is based on studying the restriction of $f_\lambda$ as in (1.4) to the box $B(x, 1/\sqrt{\lambda})$

$$F_{f_\lambda}(x, y) = f_\lambda \left( x + \frac{y}{\sqrt{\lambda}} \right), \quad (2.1)$$

where $y \in [-1/2, 1/2]^2$, on average as $x$ ranges uniformly over a fixed ball $B \subset \mathbb{T}^2$ (or $B = \mathbb{T}^2$). We observe that $F_{f_\lambda}$ can also be thought of as a random field from the probability space $(B, d \text{ Vol}_B)$, with $d \text{ Vol}_B = d \text{ Vol} / \text{Vol}(B)$, into $C^\infty([-1/2, 1/2]^2)$, the space of infinitely differentiable functions on the unit square $[-1/2, 1/2]^2$. In order to distinguish these two points of view and to keep track of the dependence on $x$, we write $F_{f_\lambda}^B$ for the random field and $F_{f_\lambda}$ for the restriction of $f$ around the point $x \in \mathbb{T}^2$.

Bourgain’s de-randomization asserts that $F_{f_\lambda}^B$, converges in distribution, in the appropriate space of functions, to $F_\mu$, the Gaussian field with spectral measure $\mu$ given by (1.8). In this section, we gather the relevant probabilistic background to rigorously express this claim. We start by briefly collecting some definitions and notation about Gaussian fields (on $\mathbb{R}^2$).

**Gaussian fields.** Let $\Omega$ be an abstract probability space, with probability measure $\mathbb{P}(\cdot)$ and expectation $\mathbb{E}[\cdot]$. A (real-valued) Gaussian field $F$ is a map $F: \Omega \times \mathbb{R}^2 \to \mathbb{R}$ such that $F(\omega, x)$ is continuous in $x \in \mathbb{R}^2$ (for all $\omega \in \Omega$) and all finite dimensional distributions $(F(\cdot, x_1), \ldots, F(\cdot, x_n))$ are multivariate Gaussian vectors. We say that $F$ is **centered** if $\mathbb{E}[F] \equiv 0$ and **stationary** if its law is invariant under translations $x \to x + \tau$ for $\tau \in \mathbb{R}^2$. The **covariance** function of $F$ is

$$\mathbb{E}[F(x)F(y)] = \mathbb{E}[F(x - y)F(0)].$$

Since the covariance is positive definite, by Bochner’s theorem, it is the Fourier transform of a Borel measure $\mu$ on $\mathbb{R}^2$. So we have

$$\mathbb{E}[F(x)F(y)] = \int_{\mathbb{R}^2} e \left( \langle x - y, s \rangle \right) d\mu(s).$$

The measure $\mu$ is called the **spectral measure** of $F$. Since $F$ is real-valued, $\mu$ is symmetric, that is $\mu(-A) = \mu(A)$ for any (measurable) subset $A \subset \mathbb{R}^2$. By Kolmogorov’s theorem, $\mu$ fully determines $F$. From now on, we will simply write $F_\mu$ for the centered, stationary Gaussian field with spectral measure $\mu$. Next, we will describe the metric for the aforementioned convergence of random fields.

**The Lévy–Prokhorov metric.** Let $C^s(V)$ be the space of $s$-times, $s \geq 0$ integer, continuously differentiable functions on $V$, a compact subset of $\mathbb{R}^2$. Since $C^s(V)$ is a separable metric space, Prokhorov’s Theorem, see [5, Chapters 5 and 6], implies that $\mathcal{P}(C^s(V))$, the space of probability measures on $C^s(V)$, is metrizable via the Lévy–Prokhorov metric. This is defined as follows: for a (measurable) subset $A \subset C^s(V)$, denote by $A_{+\epsilon}$ the $\epsilon$-neighborhood of $A$, that is

$$A_{+\epsilon} := \{ p \in C^s(V) \mid \exists q \in A, \| p - q \| < \epsilon \} = \bigcup_{p \in A} B(p, \epsilon),$$
where \( \| \cdot \| \) is the \( C^s \)-norm and \( B(p, \varepsilon) \) is the (open) ball centered at \( p \) of radius \( \varepsilon > 0 \). The Lévy–Prokhorov metric \( d_P : \mathcal{P}(C^s(V)) \times \mathcal{P}(C^s(V)) \to [0, 1] \) is defined for two probability measures \( \mu \) and \( \nu \) as:

\[
d_P(\mu, \nu) := \inf \{ \varepsilon > 0 : \mu(A) \leq \nu(A_{+\varepsilon}) + \varepsilon, \ \nu(A) \leq \mu(A_{+\varepsilon}) + \varepsilon \ \forall \ A \subset C^s(V) \}.
\]

**Convergence of random functions** We are now ready to describe the metric for the convergence of \( F_{f_\lambda}^B \) to \( F_\mu \), with \( \mu \) as in (1.8). Recall that \( B \subset \mathbb{T}^2 \) is a ball or \( B = \mathbb{T}^2 \). Given an integer \( s \geq 1 \), \( F_{f_\lambda}^B \) induces a probability measure on \( C^s([-1/2, 1/2]^2) \) via the push-forward measure

\[
(F_{f_\lambda}^B)_* \text{Vol}_B(A) = \text{Vol}_B([x \in B : F_{f_\lambda}^B(x, \cdot) \in A]),
\]

where \( A \subset C^s([-1/2, 1/2]^2) \) is a measurable subset. Similarly, the push-forward of \( F_\mu \) defines a probability measure on \( C^s([-1/2, 1/2]^2) \) which we denote by \( (F_\mu)_* \mathbb{P} \).

We can now measure the distance between \( F_{f_\lambda}^B \) and \( F_\mu \) as the distance between their push-forward measures in \( \mathcal{P}(C^s([-1/2, 1/2]^2)) \), the space of probability measures on \( C^s([-1/2, 1/2]^2) \), equipped with the Lévy–Prokhorov metric. Therefore, to shorten notation, we will write

\[
d_P(F_{f_\lambda}^B, F_\mu) := d_P((F_{f_\lambda}^B)_* \text{Vol}, (F_\mu)_* \mathbb{P}).
\]

**2.2. Arithmetic background.** In order to study the zero set of \( f_\lambda \) in (1.4), we will need some control over its level sets, \( \{x \in \mathbb{T}^2 : |f(x)| < t \} \) for \( t \in (0, \infty) \). In Sect. 4.2, this will be accomplished by intersecting the said level sets with horizontal and vertical lines. Thus, we will need some information about the restriction of \( f_\lambda \) to horizontal and vertical lines. These are function on \( L^2(\mathbb{T}) := L^2(\mathbb{R}/\mathbb{Z}) \) with spectrum consisting of the projections of the \( \xi \)’s, as in (1.4), onto the first and second coordinate. We collect here some facts about the additive structure of these spectra.

Given \( \lambda \in S \), and some positive integer \( \ell > 0 \), let \( \xi^1, \ldots, \xi^\ell \) be \( \ell \) points on the circle \( |\xi|^2 = \lambda \). We are interested in the number of solutions to the linear equation

\[
\xi^1_i + \cdots + \xi^\ell_i = 0 \quad (2.2)
\]

where \( \xi^j = (\xi^1_j, \xi^2_j) \) and \( i = 1 \) or \( i = 2 \). Solutions to (2.2) are called semi-correlations and were first studied in [10], generalizing an argument of Bombieri and Bourgain [6]. Let \( S_\ell \) be the set of permutations on \( \ell \)-tuples, when \( \ell = 2k \) is even, the set of \( \ell \)-tuples

\[
T_\ell(\lambda, \ell) = \{ \pi(\xi^1_1, -\xi^1_1, \ldots, \xi^k_1, -\xi^k_1) : \pi \in S_\ell \}
\]

is the set of trivial solutions to (2.2), that is the set of \( \ell \)-tuples canceling out in pairs. We call any other solution to (2.2) non-trivial. In particular, when \( \ell \) is odd, we say that there are no trivial solutions, meaning \( T_\ell(\lambda, \ell) = \emptyset \).

For a density one subsequence of \( S \), the number of solutions to (2.2) has been computed precisely in [10, Theorem 1.3]. Although [10, Theorem 1.3] is stated in a form weaker than what we need, the proof gives verbatim the following:
Lemma 2.1. Let $\xi = (\xi_1, \xi_2) \in \mathbb{Z}^2$ and $\ell > 0$ be an integer. Then, for a density one subsequence of $\lambda \in S$, there exists no non-trivial solution to the linear equation

$$\xi_1^1 + \cdots + \xi_1^\ell = 0 \quad |\xi|^2 = \lambda,$$

for either $i = 1$ or $i = 2$. That is, all solutions have the form $\xi_1^1 = -\xi_2^1, \ldots, \xi_1^{\ell-1} = -\xi_2^\ell$, up to permutations. In particular, there are no solutions when $\ell$ is odd.

It will also be important that the number of lattice points on the circle $|\xi|^2 = \lambda$ tends to infinity as $\lambda \to \infty$, this is not always the case as circles with prime (congruent to 1 modulo 4) radius have only 8 lattice points. However, the following consequence of the Erdös–Kac Theorem, see for example [33, Part III Chapter 3] and [32, Lemma 2.3], assures that there are always sufficiently many lattice points.

Lemma 2.2. There exists a density one subsequence $S' \subset S$ such that for all sufficiently large $\lambda \in S'$ we have

$$N \geq (\log \lambda)^{1/8}.$$

In particular, Lemma 2.2 ensures that, up to the rate of convergence, the limits $\lambda \to \infty$ and $N \to \infty$ are equivalent. To shorten the exposition, throughout the manuscript, we assume that every (density one) subsequence $S' \subset S$ satisfies the conclusion of Lemma 2.2.

3. Bourgain’s De-randomisation: Asymptotic Behavior of the Nodal Length

Before embarking on the proof of Theorems 1.1 and 1.3, we will establish some notation and conventions that we will use throughout the rest of the manuscript. First, we observe that, even if $F_{f\lambda}$ as in (2.1) is defined on $B(1)$, the box of side 1 centered at the origin, we can assume that $F_{f\lambda}$ is well defined on $B(R)$, for any fixed parameter $R > 1$. This observation will be useful because we will often need to slightly change the scale at which we study $F_{f\lambda}$ (from the unit box to the box of side (say) 20).

Let $B \subset \mathbb{T}^2$ be a ball, $\mu$ be some symmetric probability measure on unit circle $S^1$ and $F_{f\lambda}^B$ be as in Sect. 2.1. We write

$$\mathcal{L}(F_{f\lambda}^B) := \mathcal{H}(\{y \in [-1/2, 1/2]^2 : F_{f\lambda}^B(\cdot, y) = 0\}),$$

and

$$\mathcal{L}(F_{\mu}) := \mathcal{H}(\{y \in [-1/2, 1/2]^2 : F_{\mu}(y) = 0\}).$$

Note that the function $\mathcal{L}(\cdot)$ always denotes the nodal length in the unit box and $\mathcal{L}(F_{f\lambda}^B)$ is a random variable on $(B, d \text{Vol}_B)$. Finally, in order to shorten some statements, when we say that a function $f\lambda$ as in (1.4) is flat, from now on, we always mean in the sense of Definition 1.2.

The aim of this section is to prove that $\mathcal{L}(F_{f\lambda}^B)$, as a random variable on $(B, d \text{Vol}_B)$, converges in distribution, in the sense of [5, Theorem 2.1], to $\mathcal{L}(F_{\mu})$. In other words, we will prove that $\mathcal{L}(F_{f\lambda}^B)$ is close to $\mathcal{L}(F_{\mu})$ outside a small set of “bad” $x \in B$. Formally, we have the following:
Proposition 3.1. Let, \( \varepsilon > 0 \) and \( F_{f_{\lambda}}^B \) be as in Sect. 2.1. There exists a density one subsequence \( S' = S'(\varepsilon) \subset S \), such that the following holds: let \( \{f_{\lambda}\}_{\lambda \in S'} \) be a sequence of flat eigenfunctions with limiting measure \( \mu \) in the sense of (1.8) then

\[
\mathcal{L}(F_{f_{\lambda}}^B) \xrightarrow{d} \mathcal{L}(F_{\mu}) \quad \lambda \to \infty,
\]

where the convergence is in distribution, uniformly for all balls \( B \subset \mathbb{T}^2 \) of radius \( r > \lambda^{-1/2+\varepsilon} \).

We will use the conclusion of Proposition 3.1 only for fixed balls \( B \subset \mathbb{T}^2 \). However, as the proof of the stronger claim does not require any additional argument, we decided to include it in the manuscript. The first step in the proof of Proposition 3.1 consists of showing that \( F_{f_{\lambda}}^B \) converges, in the sense of Sect. 2.1, to \( F_{\mu} \). This fact has been shown at macroscopic scales in [7,9] and at microscopic scales in [31, Proposition 4.5]:

Lemma 3.2. Let \( R \geq 1, \varepsilon > 0 \) and \( F_{f_{\lambda}}^B \) be as in Sect. 2.1. There exists a density one subsequence \( S' = S'(\varepsilon) \subset S \) such that the following holds: let \( \{f_{\lambda}\}_{\lambda \in S'} \) be a sequence of flat eigenfunctions with limiting measure \( \mu \) in the sense of (1.8) then, recalling the notation in Sect. 2.1, we have

\[
d_P(F_{f_{\lambda}}^B, F_{\mu}) = d_P((F_{f_{\lambda}}^B)_* \text{Vol}_B, (F_{\mu})_* \mathbb{P}) \to 0 \quad \lambda \to \infty,
\]

in the space \( \mathcal{P}(C^2([-R/2, R/2]^2)) \), where the convergence is uniform for all balls \( B \subset \mathbb{T}^2 \) of radius \( r > \lambda^{-1/2+\varepsilon} \), but depends on \( R \).

Since we use a different formulation from [7,9,31], we will briefly justify Lemma 3.2:

Proof of Lemma 3.2. Let \( \Omega \) be the abstract probability space where \( F_{\mu} \) is defined and let \( \delta > 0 \) be given. Under the assumptions of Lemma 3.2, [31, Lemma 4.4] and [31, Proposition 4.5] state that, for all sufficiently large \( \lambda \in S' \), there exists a map \( \tau : \Omega \to \mathcal{B} \) and a subset \( \Omega' \subset \Omega \), both independent of \( f_{\lambda} \), such that:

1. For any measurable \( A \subset \Omega \), \( \text{Vol}(\tau(A)) = \text{Vol}(\mathcal{B})\mathbb{P}(A) \),
2. \( \mathbb{P}(\Omega') \leq \delta \),
3. For all \( \omega \not\in \Omega' \),

\[
||F_{\mu}(\omega, y) - F_{f_{\lambda}}^B(\tau(\omega), y)||_{C^2([-R/2, R/2]^2)} \leq R^2 ||F_{\mu}(\omega, Ry) - F_{f_{\lambda}}^B(\tau(\omega), Ry)||_{C^2([-1/2, 1/2]^2)} \leq R^2 \delta.
\]

Therefore, given a measurable set \( A \subset C^2([-1/2, 1/2]^2) \), we have

\[
(F_{\mu})_* \mathbb{P}(A) = \mathbb{P}(F_{\mu}(\omega, \cdot) \in A) = \mathbb{P}(F_{\mu}(\omega, \cdot) \in A, \omega \in \Omega') + \mathbb{P}(F_{\mu}(\omega, \cdot) \in A, \omega \not\in \Omega') \leq \mathbb{P}(F_{\mu}(\omega, \cdot) \in A, \omega \not\in \Omega') + \delta \leq (F_{\lambda})_* \text{Vol}_B(A_\delta) + R^2 \delta + \delta
\]

Similarly, we have \( (F_{f_{\lambda}})_* \text{Vol}_B(A) \leq (F_{\mu})_* \mathbb{P}(A_\delta) + 2R^2 \delta \). Hence, since \( \delta \) is arbitrary and \( R \) is fixed, we obtain \( d_P(F_{f_{\lambda}}^B, F_{\mu}) \to 0 \), as required. \( \square \)
The second step in the proof of Proposition 3.1 consists of showing that we can pass from the convergence of $F_{f_\lambda}$ to $F_{\mu}$ to the convergence of their nodal sets. The following lemma shows that the nodal length is a continuous functional on the appropriate (open) subspace of $C^2$, see [24] and [32, Lemma 6.1]. The precise form of this fact, as stated below, can be found in [27, Lemma 6.1]:

**Lemma 3.3.** Let $B \subset \mathbb{R}^2$ be a ball/box, let $2B$ be the concentric ball/box of twice the radius/sample and let $C^2_*(2B) := \{ g \in C^2(2B) : |g| + |\nabla g| > 0 \}$. Then $L(g, B) = \mathcal{H}(\{ x \in B : g(x) = 0 \})$ is a continuous functional on $C^2_*(2B)$.

In light of Lemma 3.3, Proposition 3.1 would follow from Lemma 3.2 via the Continuous Mapping Theorem, provided that $F_{\mu} \in C^2_*$. This is a well-known result of Bulinskaya, see [24, Lemma 6].

**Lemma 3.4.** (Bulinskaya’s lemma) Let $F = F_{\mu}$, with $\mu$ a symmetric measure supported on $S^1$ and $B(2) \subset \mathbb{R}^2$ be the box of side 2 centered at zero. If $\mu$ is not supported on a line, that is $(F, \nabla F)$ is non-degenerate, then $F \in C^2_*(B(2))$ almost surely, with $C^2_*(B(2))$ as in Lemma 3.3.

We are finally ready to prove Proposition 3.1

**Proof of Proposition 3.1.** Let $S' \subset S$ be given by Lemma 3.2 with (say) $\varepsilon = 1/4$. First, applying Lemma 3.2 with (say) $R = 4$, we obtain

$$d_P(F_{f_\lambda}, F_{\mu}) \to 0,$$

with respect to the $C^2(B(2))$ metric. Moreover, since the support of $\mu$ is not contained in a line, Lemma 3.4 implies that $F_{\mu} \in C^2_*(B(2))$ almost surely. Hence, Lemma 3.3 together with the Continuous Mapping Theorem [5, Theorem 2.7] imply Proposition 3.1, as required.

**4. Log-Integrability and Level-Sets Estimates**

We formulate (a slightly stronger version of) Proposition 1.4 in terms of level sets estimates as follows:

**Proposition 4.1.** Let $q \geq 1$ be an integer. There exist a density one subsequence of $S' = S'(q) \subset S$, $\lambda_0 = \lambda_0(q) > 0$ and $\alpha = \alpha(q) > 0$ such that the following holds: uniformly for all flat $f_\lambda$ in (1.4), with $\lambda > \lambda_0$ in $S'$, and all $t \in (0, \infty)$, we have

$$\text{Vol} \left( x \in \mathbb{T}^2 : \log |f_\lambda(x)| < -t^{1/q} \right) \lesssim q t^{-1-\alpha}.$$

We are now going to prove Proposition 1.4 assuming Proposition 4.1.

**Proof of Proposition 1.4 assuming Proposition 4.1.** As mentioned in Sect. 1.3, given $g : \mathbb{T}^2 \to \mathbb{R}$ and a parameter $t > 0$, we will use the shorthand notation

$$\text{Vol}(x \in \mathbb{T}^2 : g(x) \leq t) =: \text{Vol}(g(x) \leq t).$$
We are now ready to begin the proof of Proposition 1.4. Let \( q \geq 1 \) be given, write \( f = f_\lambda \) and let \( S' \) be given by Proposition 4.1. First, we observe that, by a straightforward integration by parts,\(^3\) we have
\[
\int_{\mathbb{T}^2} |\log |f(x)||^q \, dx = \int_0^\infty t \, d\text{Vol}(|\log |f(x)||^q \leq t)
\]
\[
= -\int_0^\infty t \, d\text{Vol}(|\log |f(x)||^q \geq t)
\]
\[
= \int_0^\infty \left( \text{Vol}\left( |\log |f(x)|| \geq t^{1/q}\right) \right) \, dt + O(1). \tag{4.1}
\]

Since \( ||f||_{L^2} = 1 \), Chebyshev’s inequality gives
\[\text{Vol}\left( |f(x)| \geq \exp(t^{1/q}) \right) \leq \exp(-2t^{1/q}),\]
thus the first term on the r.h.s. of (4.1) is bounded by some constant depending on \( q \) only. Proposition 4.1 implies that the second term on the r.h.s. of (4.1) is also bounded by some constant depending on \( q \) only, for all sufficiently large \( \lambda \in S' \). By discarding at most finitely many elements of \( S' \), we may assume that the claimed bound holds for all \( \lambda \in S' \). This concludes the proof of Proposition 1.4. \( \square \)

The rest of the section is dedicated to the proof of Proposition 4.1.

4.1. Nazarov’s result: \( \Lambda(p) \)-systems and level-sets estimates. The aim of this section is to present (some of) the results of [25, Chapter 3] and [26] in a form that will be useful to prove Proposition 4.1, we claim no originality and refer the reader to see directly [25,26].

We need to first introduce some definitions: given some \( g \in L^2(\mathbb{T}) \), the spectrum of \( g \) is
\[\text{Spec}(g) := \left\{ n \in \mathbb{Z} : \hat{g}(n) := \int_{\mathbb{T}} e(-n \cdot x) g(x) \, dx \neq 0 \right\}.\]

We say that a (possibly finite) set \( V = \{n_i\}_i \subset \mathbb{Z} \) is a \( \Lambda(p) \)-system for some \( p \geq 2 \) if, for every \( g \in L^2(\mathbb{T}) \) with \( \text{Spec}(g) \subset V \), there exists some constant \( C_0 = C_0(V, p) > 0 \), independent of \( g \), such that
\[
||g||_{L^p(\mathbb{T})} \leq C_0 ||g||_{L^2(\mathbb{T})}. \tag{4.2}
\]

We say that a set \( V \subset \mathbb{Z} \) is symmetric if \( n \in V \) implies \( -n \in V \). We now prove the following sufficient condition for a symmetric set to be a \( \Lambda(p) \)-system:

**Claim 4.2.** Let \( V = \{n_i\}_i \subset \mathbb{Z} \) be a symmetric set. Suppose that, for some even \( p \geq 2 \), the only solutions to
\[n_{i_1} + n_{i_2} + \cdots + n_{i_p} = 0\]
are trivial, that is \( n_{i_1} = -n_{i_2}, \ldots \) up to permutations. Then, \( V \) is a \( \Lambda(p) \)-system with constant \( C_0(p) = c(p) \) independent of \( V \).

\(^3\) Note that, by Cauchy–Schwarz and (1.5), we have \( \sup_x |f(x)| \leq \sqrt{N} \) so \( f \) cannot assume arbitrarily large values.
Proof. Let \( g \in L^2(\mathbb{T}) \) with \( \text{Spec}(g) \subset V \), we may write \( g \) as

\[
g(x) = \sum_i a_i e(n_i \cdot x),
\]

for some \( a_i \in \mathbb{C} \). Normalizing \( g \), we may assume that \( ||g||_{L^2} = \sum_i |a_i|^2 = 1 \). Now, expanding the \( p \)-th power of \( g \), we have

\[
||g||_{L^p}^p = \sum_{i_1, \ldots, i_p} a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_p} \int_{\mathbb{T}} e(\langle n_{i_1} - n_{i_2} + \cdots - n_{i_p}, x \rangle) dx.
\]

Using the orthogonality of the exponentials and the assumptions of Claim 4.2 (note that, since \( V \) is symmetric, the choice of signs in the sum is irrelevant) we deduce

\[
||g||_{L^p}^p = \sum_{i_1, \ldots, i_p} a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_p} = c(p) \left( \sum_i |a_i|^2 \right)^{p/2} = c(p)
\]

where \( c(p) \) is the number of permutations of \( n_{i_1} = -n_{i_2}, \ldots \) and therefore independent of \( g \) and of \( V \), as required. \( \square \)

The last piece of notation that we need is the following: given some \( V = \{n_i\}_i \subset \mathbb{Z} \), we denote

\[
R(V) := \sup_{\substack{r \in \mathbb{Z} \setminus 0 \ni r \neq 0}} |\{(n_i, n_j) \in V^2 : n_i - n_j = r\}|, \quad D(V) := \{n_i - n_j \in \mathbb{Z} : i \neq j\}.
\]

(4.3)

With the above notation, we have the following theorem from [26] whose proof will be given, for completeness, in “Appendix A”.

**Theorem 4.3** (Nazarov). Let \( \varepsilon > 0 \), \( V \subset \mathbb{Z} \) and \( R(V) \), \( D(V) \) be as in (4.3). Suppose that \( R(V) < \infty \) and \( D(V) \) is a \( \Lambda(p) \)-system for some integer \( p > 2 \) with \( C_0 = C_0(V, p) \) as in (4.2). Then there exists some constant \( C = C(C_0, \varepsilon, R(V)) > 0 \) such that, uniformly for all \( g \in L^2(\mathbb{T}) \) with spectrum contained in \( V \) and any set \( U \subset \mathbb{T} \) of positive measure, we have

\[
||g||_{L^2(\mathbb{T})}^2 \leq \exp \left( \frac{C}{\rho(U)^{\frac{4}{p} + \varepsilon}} \right) \int_U |g(x)|^2 dx,
\]

where \( \rho(\cdot) \) is the (normalized) Lebesgue measure on \( \mathbb{T} \).

We will need the following corollary:

**Corollary 4.4.** Under the assumptions of Theorem 4.3 and maintaining the same notation, there exists some constant \( C = C(C_0, \varepsilon, R(V)) > 0 \) such that, uniformly for all \( g \in L^2(\mathbb{T}) \) with spectrum contained in \( V \) and satisfying \( ||g||_{L^2}^2 \geq 1/2 \), we have

\[
\rho \left( x \in \mathbb{T} : \log |g(x)| \leq -t^{\frac{4}{p} + \varepsilon} \right) \leq Ct^{-1 - \varepsilon/3}.
\]

The constant \( 1/2 \) in the postulated lower bound for \( ||g||_{L^2}^2 \) in the statement of Corollary 4.4 is arbitrary and it could be substituted by any other (absolute) constant.
Proof. Let \( 0 < \delta < 1 \) and define the set \( U_\delta = \{ x \in \mathbb{T} : |g(x)| \leq \delta \} \). Theorem 4.3, applied to \( U = U_\delta \) with some \( \varepsilon_1 > 0 \) to be chosen later, implies that there exists some \( C = C(C_0, \varepsilon_1, R(V)) > 0 \) such that

\[
1 \leq 2 \exp \left( \frac{C}{\rho(U_\delta)\frac{4}{p} + \varepsilon_1} \right) \rho(U_\delta) \delta^2 \leq \exp \left( \frac{100C}{\rho(U_\delta)\frac{4}{p} + \varepsilon_1} \right) \delta^2.
\]

Therefore, for some \( C_1 = C_1(C, \varepsilon_1) > 0 \) and some \( c = c(p) > 0 \), we have

\[
\rho(U_\delta) \leq C_1(- \log \delta)^{-\frac{p}{4} + \varepsilon_1}.
\]

Taking \( \delta = \exp(-t \frac{4}{p} + \varepsilon^2) \) and choosing \( \varepsilon_1 \) appropriately in terms of \( \varepsilon \) and \( p \), we deduce

\[
\rho \left( x \in \mathbb{T} : \log |g(x)| \leq -t \frac{4}{p} \right) \leq C_1 \rho^{1-\varepsilon/3},
\]

as required. \( \square \)

4.2. Preliminaries for the proof of Proposition 4.1. Before embarking on the proof of Proposition 4.1, we will need to set up some relevant notation and make a couple of observations, which, for convenience, we collect in this section. Given \( t \in (0, \infty) \) and an integer \( q \geq 1 \), we will bound the volume of the set

\[
A = A_{t,q} := \{ x \in \mathbb{T}^2 : \log |f_\lambda(x)| < -t^{1/q} \}
\]

by using Corollary 4.4 to estimate the Lebesgue measure of its intersection with horizontal lines. To this end, we need to introduce some notation. First, for \( \xi \in \mathbb{Z}^2 \) and \( x \in \mathbb{T}^2 \), we write \( \xi = (\xi_1, \xi_2) \) and \( x = (x_1, x_2) \). Second, for \( f_\lambda = f \) as in (1.4) and some fixed \( x_2 \in \mathbb{T} \), we write

\[
H_{x_2}(f)(\cdot) = f(\cdot, x_2) := \sum_{\xi_1} b_{\xi_1} e(\xi_1 \cdot),
\]

where

\[
b_{\xi_1} = b(\xi, x_2) := a_{(\xi_1, \xi_2)} e(\xi_2 \cdot x_2) + a_{(\xi_1, -\xi_2)} e(-\xi_2 \cdot x_2).
\]

That is, \( H_{x_2}f \) is \( f \) considered as a function of the first coordinate only and therefore \( H_{x_2}f \in L^2(\mathbb{T}) \). In particular, the \( L^2 \)-norm of \( H_{x_2}f \) is a function of \( x_2 \) only. We denote (the square of) this function by \( P \):

\[
P(x_2) = P_f(x_2) := ||H_{x_2}f||_{L_2}^2 = \int_{\mathbb{T}} |f(x_1, x_2)|^2 dx_1 = \sum_{\xi_1} |b_{\xi_1}|^2
\]

\[
= \sum_{\xi_1} \left( |a_{(\xi_1, \xi_2)}|^2 + |a_{(\xi_1, -\xi_2)}|^2 \right) + Q(x_2) = 1 + Q(x_2),
\]

where

\[
Q(x_2) = Q_f(x_2) := \sum_{\xi_2} d_{\xi_2} e(2\xi_2 \cdot x_2) \quad d_{\xi_2} = d(\xi) := a_{(\xi_1, \xi_2)} a_{(-\xi_1, \xi_2)}.
\]
and in (4.6) we have used the normalization \( \sum_\xi |a_\xi|^2 = 1 \) and the fact that \( a_\xi = a_{-\xi} \).

Suppose that \( f \) is flat as in Definition 1.2, then the \( L^2 \)-norm of \( P \) is

\[
\|P\|^2_{L^2} = 1 + \int_T Q(x_2)dx_2 + \int_T \overline{Q}(x_2)dx_2 + \|Q\|^2_{L^2} = 1 + \|Q\|^2_{L^2} + O(u(N) \cdot N^{-1/2}),
\]

where the error term comes from (possible) terms with \( \xi_2 = 0 \) and the flatness assumption in Definition 1.2. Now, again using the flatness assumption in Definition 1.2, we compute

\[
\|Q\|^2_{L^2} = \sum_\xi |d_{\xi_2}|^2 \lesssim u(N)^2 N^{-1} \lesssim N^{-1/2}.
\]

Therefore, in light of Lemma 2.2, we have shown the following claim:

**Claim 4.5.** There exists a density one subsequence \( S' \subset S \) such that for all sufficiently large \( \lambda \in S' \) the following holds: suppose that \( f_{\lambda} \) in (1.4) is flat then we have

\[
\|P\|^2_{L^2(T)} > 1/2,
\]

where \( P \) is as in (4.6).

We will also need to be able to apply Theorem 4.3 to the sets

\[
V_{i,\lambda} = V_i := \{ 2\xi_i : \xi = (\xi_1, \xi_2), \ |\xi|^2 = \lambda \} \cup \{0\} \quad i = 1, 2.
\]

Lemma 2.1 and Claim 4.2 imply that (for a density one sub-sequence) \( V_i \) is a \( \Lambda(p) \)-system with constant \( C_0 \) independent of \( V \), for any fixed \( p \geq 2 \). Unfortunately, \( D(V_i) \), the difference set, may contain trivial linear relations. So Claim 4.2 is not directly applicable to conclude that \( D(V_i) \) is also a \( \Lambda(p) \)-system with constant \( C_0 \) independent of \( V \).

A “well-known” fact, mentioned in [26], is that if \( V_i \) has no non-trivial relations of length \( 2p \), then \( D(V_i) \) is a \( \Lambda(p) \)-system with constant \( C_0 = c(p) \) independent of \( V \). This, together with Lemma 2.1, directly gives the following claim.

**Claim 4.6.** Let \( p \geq 2 \) be a positive even integer, for \( i = 1, 2 \) define the sets

\[
V_{i,\lambda} = V_i := \{ 2\xi_i : \xi = (\xi_1, \xi_2), \ |\xi|^2 = \lambda \} \cup \{0\}
\]

and let \( D(V_i) \) be as in (4.3). Then, there exists a density one subsequence \( S' = S'(p) \subset S \) such that, for all sufficiently large \( \lambda \in S' \), \( D(V_i) \) is a \( \Lambda(p) \)-system with constant \( C_0 = C_0(p) \) independent of \( V_i \).

Since we could not find a precise reference for the aforementioned fact about \( \Lambda(p) \)-systems, we include a proof in “Appendix B”, that is Lemma B.2.

### 4.3. Concluding the proof of Proposition 4.1

We are now ready to prove Proposition 4.1:

**Proof of Proposition 4.1.** We choose the subsequence \( S' \) to be the intersection of three sub-sequences \( S_1, S_2 \) and \( S_3 \) as follows. We pick \( S_1 = S_1(q) \) so that the conclusion of Lemma 2.1 holds for all \( \ell \leq 10p \) for some \( p = p(q) \) to be chosen later. We choose \( S_2 \) so that the conclusion of Claim 4.5 holds and \( S_3 \) so that the conclusion of Claim 4.6 holds. Having prescribed \( S' \), we begin the proof of Proposition 4.1.
Given \( t \geq 0 \) and an integer \( q > 0 \), let \( A \) be as in (4.4). Observe that
\[
\Vol(A) = \Vol(A \cap E) + \Vol(A \cap F),
\]
where
\[
E = E_{t,q} := \{ x = (x_1, x_2) \in \mathbb{T}^2 : \log |P(x_2)| > -t^{1/q} \}
\]
and \( F \) is the complement of \( E \) and \( P(\cdot) \) is as in (4.6). First, we are going to bound the first term on the r.h.s. of (4.7). Writing \( \rho(\cdot) \) for the (normalized) Lebesgue measure on \( \mathbb{T} \), by Fubini we have
\[
\Vol(A \cap E) \leq \int_{\tilde{E}} \rho \left( x_1 : \log |f(x_1, x_2)| \leq -t^{1/q} \right) dx_2,
\]
where \( \tilde{E} \) is the projection of \( E \) onto the second coordinate. Using the trivial bound \( \rho(\tilde{E}) \leq 1 \) and in light of the notation introduced in (4.5), we deduce
\[
\Vol(A \cap E) \leq \sup_{x_2 \in \tilde{E}} \rho \left( x_1 : \log |f(x_1, x_2)| \leq -t^{1/q} \right)
\]
\[
= \sup_{x_2 \in \tilde{E}} \rho \left( x_1 : \log |(H_{x_2} f)(x_1)| \leq -t^{1/q} \right)
\]
\[
= \sup_{x_2 \in \tilde{E}} \rho \left( x_1 : \log |(H_{x_2} f)(x_1)| - \log(|P(x_2)|^{1/2}) \leq -t^{1/q} - \log(|P(x_2)|^{1/2}) \right)
\]
\[
\leq \sup_{x_2 \in \tilde{E}} \rho \left( x_1 : \log \left| \frac{(H_{x_2} f)(x_1)}{P^{1/2}(x_2)} \right| \leq -\frac{t^{1/q}}{2} \right). \tag{4.8}
\]
To estimate the r.h.s. of (4.8), we wish to apply Corollary 4.4 to the function
\[
g(x_1) = g_{x_2}(x_1) := \frac{H_{x_2} f(x_1)}{P^{1/2}(x_2)}.
\]
Observe that, for \( x_2 \in \tilde{E} \), \( g \) is well-defined and, by (4.6), we also have
\[
||g||_{L^2} = 1.
\]
Therefore, in order to apply Corollary 4.4, it is enough to verify the assumptions of Theorem 4.3 for the set \( V_1 = V_{1,\lambda} = \{ \xi_1 : \xi = (\xi_1, \xi_2), \ |\xi|^2 = \lambda \} \), which is a symmetric set. Recall
\[
R(V_1) = \sup_{r \in \mathbb{Z}, r \neq 0} |\{(n_i, n_j) \in V_1^2 : n_i - n_j = r\}|.
\]
Then Lemma 2.1 says, for any \( r \in \mathbb{Z} \setminus \{0\} \), that
\[
n_i - n_j = r = n_i' - n_j',
\]
has at most 3 solutions. Thus, \( R(V_1) \leq 3 \). Moreover, let \( D(V_1) \) be as in (4.3), then Claim 4.6 says that \( D(V_1) \) is a \( \Lambda(p) \)-system with constant \( C_0 = C_0(p) \) independent of \( V_1 \).
Thus, we are in the position of applying Corollary 4.4 with (say) \( p = 8q \) and \( \varepsilon = 1/q \), to find some constant \( C = C(q) \), independent of \( x_2 \), such that

\[
\sup_{x_2 \in \tilde{E}} \rho \left( x_1 : \log \left| \frac{(H_{x_2} f)(x_1)}{P^{1/2}} \right| \leq -\frac{t^{1/q}}{2} \right) \leq Ct^{-1-1/(3q)}.
\]

(4.9)

We are now going to bound the second term on the r.h.s. of (4.7). Observe that

\[
\text{Vol}(A \cap F) \leq \text{Vol}(F) \leq \rho \left( x_2 : \log |P(x_2)| \leq -t^{1/q} \right).
\]

(4.10)

Similar to the above argument, we are going to use Corollary 4.4 to bound the r.h.s. of (4.10). Thanks to Claims 4.5 and 4.6, we may apply Corollary 4.4 to the function \( P \) with \( p = 8q \) and \( \varepsilon = 1/q \) to see that there exist constants \( \lambda_0 = \tilde{\lambda}_0(q) \) and \( \tilde{C} = \tilde{\tilde{C}}(q) \) such that for all \( \lambda > \lambda_0 \) in \( S' \), we have

\[
\rho \left( x_2 : \log |P(x_2)| \leq -t^{1/q} \right) \leq \tilde{C}t^{-1-1/(3q)}.
\]

(4.11)

Hence, Proposition 4.1, with \( \alpha = 1/(3q) \), follows by combining (4.7), (4.8), (4.9), (4.10) and (4.11).

\( \square \)

5. Moments of \( \mathcal{L}(F_{f_\lambda}) \)

The aim of this section is to apply Proposition 4.1 to show that arbitrarily high moments of \( \mathcal{L}(F_{f_\lambda}) \), with \( F_{f_\lambda} \) as in (2.1), are finite. In particular, this will show that the “bad” set of \( x \in \mathcal{B}, \) coming from Proposition 3.1, does not significantly contribute to the moments of \( \mathcal{L}(F_{f_\lambda}) \), leading to the fundamental Proposition 5.4 below. The main result of this section is the following:

**Proposition 5.1.** Let \( q \geq 1 \) be an integer. There exists a density one subsequence \( S' = S'(q) \subset S \) and some constant \( C = C(q) > 0 \) such that for all \( \lambda \in S' \) the following holds: suppose that \( f_\lambda \) in (1.4) is flat, then we have

\[
\int_{\mathbb{T}^2} \mathcal{L}(F_{f_\lambda}(x, \cdot)) q \, dx \leq C,
\]

where \( F_{f_\lambda} \) is as in (2.1).

The following corollary is a direct consequence of Proposition 5.1:

**Corollary 5.2.** Let \( q \geq 1 \) be an integer. There exists a density one subsequence \( S' = S'(q) \subset S \) such that for all \( \lambda \in S' \) and all fixed balls \( B \subset \mathbb{T}^2 \) or \( B = \mathbb{T}^2 \) the following holds: there exists some constant \( C = C(q, B) > 0 \) such that if \( f_\lambda \) in (1.4) is flat then

\[
\frac{1}{\text{Vol } B} \int_B \mathcal{L}(F_{f_\lambda}(x, \cdot)) q \, dx \leq C,
\]

where \( F_{f_\lambda} \) is as in (2.1).

Thanks to Corollary 5.2 we can “upgrade” the convergence in distribution in Proposition 3.1 to convergence of expectations. Formally, we will need the following well-known fact about uniform integrability [5, Theorem 3.5]:
Lemma 5.3. Let \( X_n \) be a sequence of random variables such that \( X_n \overset{d}{\to} X \), that is, convergence in distribution. Suppose that there exists some \( \alpha > 0 \) such that \( \mathbb{E}[|X_n|^{1+\alpha}] \leq C < \infty \) for some \( C > 0 \), uniformly for all \( n \geq 1 \). Then,

\[ \mathbb{E}[X_n] \to \mathbb{E}[X] \quad n \to \infty. \]

We are finally ready to state (and prove) the main consequence of Proposition 5.1:

**Proposition 5.4.** There exists a density one subsequence \( S' \subset S \) such that the following holds: let \( \{f_\lambda\}_{\lambda \in S} \) be a sequence of flat eigenfunctions with limiting measure \( \mu \) in the sense of (1.8) and \( B \subset \mathbb{T}^2 \) be a fixed ball or \( B = \mathbb{T}^2 \), then

\[ \frac{1}{\text{Vol} B} \int_B \mathcal{L}(F_{f_\lambda}(x, \cdot))dx \to \mathbb{E}[\mathcal{L}(F_\mu)] \quad \lambda \to \infty, \]

where \( F_{f_\lambda} \) is as in (2.1).

**Proof.** Let \( S' \) be the intersection of the sub-sequence given by Proposition 3.1 and the sub-sequence in Corollary 5.2 applied with (say) \( q = 2 \). Then, under the assumptions of Proposition 5.4, we have

\[ \mathcal{L}(F_{f_\lambda}^B) \overset{d}{\to} \mathcal{L}(F_\mu). \]

Now, Proposition 5.4 follows from Corollary 5.2 via Lemma 5.3. \( \square \)

The rest of this section is dedicated to the proof of Proposition 5.1.

5.1. **Proof of Proposition 5.1.** In this section, given a box \( B \subset \mathbb{R}^n \) and \( r > 0 \), we write \( rB \) for the concentric box of \( r \)-times the side. It is a well-known fact, see [12, Proposition 6.7], that the nodal length of Laplace eigenfunction, on real analytic manifolds, can be bounded in terms of the doubling index. Given a (say) \( C^3 \) function \( g : 3B \to \mathbb{R} \), the doubling index of \( g \) on the box \( B \) is defined by

\[ N_g(B) := \log \frac{\sup_{2B} |g|}{\sup_B |g|} + 1. \quad (5.1) \]

We added 1 in (5.1) to ensure that the doubling index is strictly greater than zero.

In order to prove Proposition 5.1, we will use the following lemma, see [22, Lemma 2.6.1] and [12, Proposition 6.7], to control the (local) nodal length:

**Lemma 5.5.** Let \( \tilde{B} \subset \mathbb{R}^3 \) be the unit box, suppose that \( h : 5\tilde{B} \to \mathbb{R} \) is an harmonic function \( (\Delta h = 0) \), then

\[ \mathcal{V}(h, \tilde{B}) \lesssim N_h(2\tilde{B}), \]

where \( \mathcal{V}(h, \tilde{B}) = \mathcal{H}^2(\{x \in \tilde{B}; h(x) = 0\}) \).

Lemma 5.5 has the following direct consequence:

**Lemma 5.6.** Let \( F_{f_\lambda} \) be as in (2.1). We have the following bound:

\[ \mathcal{L}(F_{f_\lambda}(x, \cdot)) \lesssim N_{F_{f_\lambda}}(B), \]

where \( B = B(2) \subset \mathbb{R}^2 \) is the box of side 2 centered at zero.
Proof. First, observe that, by trivially extending the domain of $F_{f_{\lambda}}$, we may assume that $F_{f_{\lambda}}$ is well-defined on the box of side (say) 20 centered at 0. Let $B = [-1, 1]^2$ and $\tilde{B} = [-1, 1]^2 \times [-1, 1]$. Then the “harmonic lift” of $F_{f_{\lambda}}$

$$h(y, s) := F_{f_{\lambda}}(\cdot, y)e^{2\pi s} : 5\tilde{B} \to \mathbb{R},$$

is harmonic, that is $\Delta h = 0$. Therefore, Lemma 5.5 gives

$$\mathcal{V}\left(h, \frac{1}{2}\tilde{B}\right) = \mathcal{H}^2(\{x \in 2^{-1}\tilde{B}; h(x) = 0\}) \lesssim \log \frac{\sup_{2\tilde{B}} |h|}{\sup_{\tilde{B}} |h|} + 1.$$  

Observe that if $F_{f_{\lambda}}$ vanishes at some point $y$ then $h$ vanishes on the line $\{y\} \times [-1, 1]$, thus

$$\mathcal{L}(F_{f_{\lambda}}) \lesssim \mathcal{V}\left(h, \frac{1}{2}\tilde{B}\right).$$

Moreover, since $\sup_{2\tilde{B}} |h| = e^{4\pi} \sup_B |F_{f_{\lambda}}|$ and $\sup_{\tilde{B}} |h| = e^{2\pi} \sup_B |F_{f_{\lambda}}|$, we also have

$$\mathcal{V}\left(h, \frac{1}{2}\tilde{B}\right) \lesssim 1 + \log \frac{\sup_{y \in 2B} |F_{f_{\lambda}}(\cdot, y)|}{\sup_{y \in B} |F_{f_{\lambda}}(\cdot, y)|},$$

as required. \qed

Lemma 5.6, up to scaling factors, reduces Proposition 5.1 to (1.3) which, we are now going to show, follows from Proposition 4.1. In the proof of (1.3), we will need a standard consequence of the elliptic estimates for harmonic functions [14, Page 332]. The elliptic estimates state that any $L^p$-norm, for $2 \leq p \leq \infty$, of a harmonic function in a ball/box $B$ is bounded by its $L^2$-norm on (say) $\frac{1}{2}B$. More precisely, we have the following fact:

**Lemma 5.7.** Let $F_{f_{\lambda}}$ be as in (2.1), $h(y, s) = F_{f_{\lambda}}(\cdot, y)e^{2\pi s}$ as in (the proof of) Lemma 5.6, $B = [-1, 1]^2$ and $\tilde{B} = [-1, 1]^2 \times [-1, 1]$. Then (uniformly for all $x \in \mathbb{T}^2$), we have

$$\left(\sup_{y \in 2B} |F_{f_{\lambda}}(\cdot, y)|\right)^2 \leq \left(\sup_{2\tilde{B}} |h|\right)^2 \lesssim ||h||_{L^2(3\tilde{B})}^2.$$  

We are finally ready to prove Proposition 5.1.

**Proof of Proposition 5.1.** Given $q \geq 1$, let $S' \subset S$ be given by Proposition 4.1. Moreover, by trivially extending the domain of $F_{f_{\lambda}}$, we may assume that $F_{f_{\lambda}}$ is well-defined on the box of side (say) 20 centered at 0. Thanks to Lemma 5.6, it is enough to show that there exists some constant $C = C(q)$ such that

$$\int_{\mathbb{T}^2} \left(\log \frac{\sup_{y \in 2B} |F_{f_{\lambda}}(x, y)|}{\sup_{y \in B} |F_{f_{\lambda}}(x, y)|}\right)^q dx \leq C,$$
where \( B = B(2) \subset \mathbb{R}^2 \) is the box of side 2 centered at zero. Using the inequality \((X + Y)^q \lesssim_q X^q + Y^q\) it is enough to show that

\[
\int_{\mathbb{T}^2} \log \sup_{y \in 2B} |F_{f_k}(x, y)|^q \, dx \leq C, \\
\int_{\mathbb{T}^2} \log \sup_{y \in B} |F_{f_k}(x, y)|^q \, dx \leq C.
\]

We are just going to show the first claimed bound, the proof of the second one being identical. First, as in the proof of Proposition 1.4, recalling the notation in Sect. 1.3, we may write

\[
\int_{\mathbb{T}^2} \left| \log \sup_{y \in 2B} |F_{f_k}(x, y)| \right|^q \, dx = \int_{10}^{\infty} \text{Vol} \left( \log \sup_{y \in 2B} |F_{f_k}(x, y)| > t^{1/q} \right) dt \\
+ \int_{10}^{\infty} \text{Vol} \left( \log \sup_{y \in 2B} |F_{f_k}(x, y)| \leq -t^{1/q} \right) dt \\
+ O(1).
\]

Since \( \sup_{y \in 2B} |F_{f_k}(x, y)| \geq |f(x)| \), the second term on the r.h.s. of (5.2) is bounded by some constant depending on \( q \) only by Proposition 4.1. Therefore, in order to prove Proposition 5.1, it is enough to show that the first term on the r.h.s. of (5.2) is bounded by some constant depending on \( q \) only.

Writing \( h \) to be the harmonic lift of \( F_{f_k} \) introduced in the proof of Lemma 5.6, \( B = [-1, 1]^2 \) and \( \widetilde{B} = [\sqrt{3}, 1]^2 \times [-1, 1] \), Lemma 5.7 gives

\[
\left( \sup_{y \in 2B} |F_{f_k}(\cdot, y)| \right)^2 \leq \left( \sup_{2B} |h| \right)^2 \lesssim ||h||^2_{L^2(3\widetilde{B})} \lesssim \int_{3\widetilde{B}} |F_{f_k}(\cdot, y)|^2 dy.
\]

Since

\[
\int_{\mathbb{T}^2} ||F_{f_k}(x, y)||^2_{L^2(3\widetilde{B})} dx = \sum_{\xi, \eta} a_\xi \overline{a_\eta} \int_{\mathbb{T}^2} e((\xi - \eta)x) dx \int_{3\widetilde{B}} e((\xi - \eta)\lambda^{-1/2} y) dy \\
= O(1),
\]

Chebyshev’s inequality gives

\[
\text{Vol} \left( x : \sup_{y \in 2B} |F_{f_k}(x, y)| > \exp(t^{1/p}) \right) \lesssim \exp(-2t^{1/p}).
\]

Therefore the first term on the r.h.s. of (5.2) is bounded. This concludes the proof of Proposition 5.1.

6. Concluding the Proofs of the Main Results

6.1. Nodal length of Gaussian random fields. In this section we collect a few preliminary results towards the proofs of Theorems 1.1 and 1.3.
**Lemma 6.1.** Let \( \mu \) be a symmetric probability measure supported on \( \mathbb{S}^1 \), and not supported on a line. Then we have
\[
\mathbb{E}[\mathcal{L}(F_{\mu})] = c_1 \cdot (2\pi),
\]
where
\[
c_1 = \frac{1 - |\hat{\mu}(2)|^2}{2^{5/2}\pi} \int_0^{2\pi} \frac{1}{(1 - \alpha \cos(2\theta) - \beta \sin(2\theta))^{3/2}} d\theta,
\]
and
\[
\hat{\mu}(2) = \int_{\mathbb{S}^1} z^2 d\mu(z) = \alpha + i\beta.
\]
The proof of Lemma 6.1 follows by a standard use of the Kac–Rice formula [1, Theorem 6.2].

**Proof.** We write \( F = F_{\mu} \). Since \( \mu \) is not supported on a line, \( (F, \nabla F) \) is non-degenerate, thus we apply the Kac–Rice formula [1, Theorem 6.2] to see that
\[
\mathbb{E}[\mathcal{L}(F_{\mu}, B)] = \int_B \mathbb{E}[|\nabla F(y)||F(y) = 0] \phi_{F(y)}(0) dy,
\]
where \( \phi_{F(y)}(\cdot) \) is the density of the random variable \( F(y) \). Since \( F(y) \) is Gaussian with mean zero and variance 1, \( \phi_{F(y)}(0) = 1/\sqrt{2\pi} \). As \( F \) and \( \nabla F \) are independent (this can be seen directly differentiating \( \mathbb{E}[F(y)^2] = 1 \)) and using stationarity, we also have
\[
\mathbb{E}[|\nabla F(y)||F(y) = 0] = \mathbb{E}[|\nabla F(y)|] = \mathbb{E}[|\nabla F(0)|].
\]
Thus, (6.1) simplifies to
\[
\mathbb{E}[\mathcal{L}(F_{\mu})] = \frac{1}{\sqrt{2\pi}} \cdot \mathbb{E}[|\nabla F(0)|] = \frac{2\pi}{\sqrt{2\pi}} \mathbb{E}[(2\pi)^{-1}|\nabla F(0)|]. \quad (6.1)
\]
Now, we compute the covariance matrix of \( \nabla F \). First we write
\[
\hat{\mu}(2) = \alpha + i\beta := \int_0^1 \cos(2\theta) d\mu(e(\theta)) + i \int_0^1 \sin(2\theta) d\mu(e(\theta))
\]
and
\[
\mathbb{E}[F(x)F(y)] = \int_{\mathbb{S}^2} e(\langle x - y, s \rangle) d\mu(s).
\]
By using the relations \( \cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \) and \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \) and writing \( s = (s_1, s_2) \) and \( x = (x_1, x_2) \), we have
\[
(2\pi)^{-2} \mathbb{E}[\partial_{x_1}^2 F(x)F(y)]_{x=y} = \int_{\mathbb{R}^2} s_1^2 d\mu(s) = \int_0^1 \cos^2(\theta) d\mu(e(\theta)) = \frac{1}{2} + \frac{\alpha}{2},
\]
\[
(2\pi)^{-2} \mathbb{E}[\partial_{x_2}^2 F(x)F(y)]_{x=y} = \int_{\mathbb{R}^2} s_2^2 d\mu(s) = \int_0^1 \sin^2(\theta) d\mu(e(\theta)) = \frac{1}{2} - \frac{\alpha}{2},
\]
\[
(2\pi)^{-2} \mathbb{E}[\partial_{x_1} \partial_{y_2} F(x)F(y)]_{x=y} = \int_{\mathbb{R}^2} s_1 s_2 d\mu(s) = \int_0^1 \cos(\theta) \sin(\theta) d\mu(e(\theta)) = \frac{\beta}{2},
\]
and
\[
(2\pi)^{-2} \mathbb{E}[\partial_{x_1} \partial_{y_1} F(x)F(y)]_{x=y} = \int_{\mathbb{R}^2} s_1^2 d\mu(s) = \int_0^1 \cos^2(\theta) d\mu(e(\theta)) = \frac{1}{2} + \frac{\alpha}{2},
\]
\[
(2\pi)^{-2} \mathbb{E}[\partial_{x_2} \partial_{y_2} F(x)F(y)]_{x=y} = \int_{\mathbb{R}^2} s_2^2 d\mu(s) = \int_0^1 \sin^2(\theta) d\mu(e(\theta)) = \frac{1}{2} - \frac{\alpha}{2},
\]
\[
(2\pi)^{-2} \mathbb{E}[\partial_{x_1} \partial_{y_3} F(x)F(y)]_{x=y} = \int_{\mathbb{R}^2} s_1 s_3 d\mu(s) = \int_0^1 \cos(\theta) \sin(\theta) d\mu(e(\theta)) = \frac{\beta}{2},
\]
and
\[
(2\pi)^{-2} \mathbb{E}[\partial_{x_2} \partial_{y_3} F(x)F(y)]_{x=y} = \int_{\mathbb{R}^2} s_2 s_3 d\mu(s) = \int_0^1 \sin(\theta) \sin(\theta) d\mu(e(\theta)) = \frac{\beta}{2}.
\]
Therefore, the covariance matrix of \((2\pi)^{-1}\nabla F\) is
\[
L = \begin{bmatrix}
\frac{1}{2} + \frac{\alpha}{2} & \frac{\beta}{2} \\
\frac{\beta}{2} & \frac{1}{2} - \frac{\alpha}{2}
\end{bmatrix}
\]  
\[
\det(L) = \frac{1}{4} \left(1 - \alpha^2 - \beta^2\right).
\]

Since \((2\pi)^{-1}\nabla F(0)\) is a bi-variate Gaussian with mean 0 and covariance \(L\), given in (6.1), we have
\[
\mathbb{E}[|(2\pi)^{-1}\nabla F(0)|] = \frac{1}{\pi(1 - \alpha^2 - \beta^2)^{1/2}}
\int_{\mathbb{R}^2} \sqrt{x^2 + y^2} \exp\left(-\frac{x^2(1 - \alpha) + y^2(1 + \alpha) - 2\beta xy}{(1 - \alpha^2 - \beta^2)}\right) \, dx \, dy.
\] (6.2)

Finally, by passing to polar coordinates in (6.2) we have:
\[
\mathbb{E}[|(2\pi)^{-1}\nabla F|] = \frac{1}{\pi(1 - \alpha^2 - \beta^2)^{1/2}}
\int_0^{2\pi} d\theta \int_0^{\infty} r^2 \exp\left(-\frac{r^2}{(1 - \alpha^2 - \beta^2)} (1 - \alpha \cos(2\theta) - \beta \sin(2\theta))\right) \, dr.
\]

Substituting \(r = (\eta y)^{1/2}\), where \(\eta = \eta(\theta) = (1 - \alpha \cos(2\theta) - \beta \sin(2\theta))^{-1}(1 - \alpha^2 - \beta^2)\), we deduce
\[
\mathbb{E}[|(2\pi)^{-1}\nabla F|] = \frac{1}{2\pi(1 - \alpha^2 - \beta^2)^{1/2}} \int_0^{2\pi} \eta^{3/2} d\theta \int_0^{\infty} y^{1/2} e^{-y} \, dy
\]
\[
= \frac{1}{2\pi} \Gamma \left(1 + \frac{1}{2}\right) (1 - \alpha^2 - \beta^2) \int_0^{2\pi} \frac{1}{(1 - \alpha \cos(2\theta) - \beta \sin(2\theta))^{3/2}} \, d\theta.
\] (6.3)

As \(\Gamma(3/2) = \sqrt{\pi}/2\), Lemma 6.1 follows from (6.1) and (6.3).

We will also need the following lemma:

\textbf{Lemma 6.2.} There exists a density one subsequence \(S' \subset S\) such that the following holds: let \(B \subset \mathbb{T}^2\) be a ball of radius \(r > 0\), suppose that \(f_\lambda\) in (1.4), with \(\lambda \in S'\), is flat then we have
\[
\mathcal{L}(f_\lambda, B) = \lambda^{1/2} \int_B \mathcal{L}(F_{f_\lambda}(x, \cdot)) \, dx + O \left(r^{1/2} \lambda^{1/4}\right),
\]
where \(F_{f_\lambda}\) is as in (2.1).

\textbf{Proof.} Let the postulated subsequence be given by Proposition 5.1 with \(q = 2\). Let us write the ball \(B\) as \(B = B(z, r) = B(r)\) for some \(z \in \mathbb{T}^2\) and \(r > 0\). Moreover, we will write \(r' = 1/\lambda^{1/2}\) and \(B(x, r')\) for the box of side \(r'\) centered at \(x\). First, we observe that
\[
\lambda^{1/2} \int_{B(r - 2r')} \mathcal{L}(F_{f_\lambda}(x, \cdot)) \, dx \leq \mathcal{L}(f_\lambda, B) \leq \lambda^{1/2} \int_{B(r + 2r')} \mathcal{L}(F_{f_\lambda}(x, \cdot)) \, dx,
\] (6.4)
Indeed, by definition of $\mathcal{L}(\cdot)$ and Fubini,\textsuperscript{4} we have
\[
\int_{\mathcal{B}(r)} \mathcal{L}(f, B(x, r'))dx = \int_{\mathcal{B}(r)} \int_{\mathcal{B}(r+2r')} 1_{B(x,r')}(y) 1_{f^{-1}(0)}(y)d\mathcal{H}(y)dx.
\]
\[
= \int_{\mathcal{B}(r+2r')} 1_{f^{-1}(0)}(y) \operatorname{Vol} (B(y, r') \cap B(r)) d\mathcal{H}(y),
\]
where $1$ is the indicator function and $\mathcal{H}$ the Hausdorff measure. Thus (6.4) follows from the scaling property $\mathcal{L}(f, B(x, r')) = \lambda^{-1/2} \mathcal{L}(F_{f_\lambda}(x, \cdot))$, upon noticing
\[
1_{B(r-2r')}(\cdot) \leq \frac{\operatorname{Vol} (B(\cdot, r') \cap B(r))}{\operatorname{Vol} B(r')} \leq 1_{B(r+2r')}(\cdot).
\]
Finally, by Proposition 5.1 with $q = 2$ and the Cauchy–Schwarz inequality, for a density one subsequence of $\lambda \in S$, we have
\[
\left( \int_{\mathcal{B}(r)} - \int_{\mathcal{B}(r\pm 2r')} \right) \mathcal{L}(F_{f_\lambda}(x, \cdot))dx \lesssim (r \cdot r')^{1/2}. \tag{6.5}
\]
Hence Lemma 6.2 follows from (6.4) and (6.5).

6.2. Proof of Theorems 1.1 and 1.3. In order to prove Theorem 1.1, we will need the following fact, see [13,15].

**Lemma 6.3.** There exists a density one subsequence of $S' \subset S$ such that, for any sequence of $f_\lambda$ in (1.4) with $\lambda \in S'$ and satisfying the assumption (1.6), we have
\[
\mu_f \to \rho,
\]
where $\mu_f$ is as in (1.7) and $\rho$ is the uniform measure on the unit circle $\mathbb{S}^1$.

We are finally ready to carry out the proof of Theorems 1.1 and 1.3:

**Proof of Theorem 1.1.** Let $S' \subset S$ be such that the conclusions of Lemma 6.2, Lemma 6.3 and Proposition 5.4 hold. Let $B$ be a fixed ball, that is the radius of $B$ does not depend on $\lambda$, or $B = \mathbb{T}^2$, then by Lemma 6.2 and Proposition 5.4, we have
\[
\mathcal{L}(f, B) = \lambda^{1/2} \operatorname{Vol}(B) \cdot \frac{1}{\operatorname{Vol}(B)} \int_{\mathcal{B}} \mathcal{L}(F_{f_\lambda}(x, \cdot))dx + O(\lambda^{1/4})
\]
\[
= \lambda^{1/2} \operatorname{Vol}(B) \mathbb{E}[\mathcal{L}(F_\rho)](1 + o_{\lambda \to \infty}(1)). \tag{6.6}
\]
where, thanks to Lemma 6.3, $\rho$ is the uniform measure on the unit circle $\mathbb{S}^1$. Using the explicit formula in Proposition 6.1 with $\mu = \rho$ (that is $\alpha = \beta = 0$) we obtain
\[
\mathbb{E}[\mathcal{L}(F_\rho)] = \frac{2\pi}{2\sqrt{2}}.
\]
Hence, Theorem 1.1 follows from (6.6).

\textsuperscript{4} Note that, in general, the Hausdorff measure may not be $\sigma$-finite. Therefore Fubini does not hold. However, since $f^{-1}(0)$ is the zero set of a real-analytic function, the one dimensional Hausdorff measure is nothing else but the arc-length on the curve $f^{-1}(0)$. Thus, we may still apply Fubini.
Proof of Theorem 1.3. The proof is very similar to the proof of Theorem 1.1. Indeed, for a density one subsequence of \( \lambda \in S \), the asymptotic law (6.6) holds. Now, the r.h.s. of (6.6) can be computed via Proposition 6.1 concluding the proof of Theorem 1.3. □

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Appendix A: Log-integrability

For the sake of completeness, in this section we provide the proof of Theorem 4.3. The proof is based on [25,26], we claim no originality.

A.1. Proof of Theorem 4.3. The main ingredient in the proof of Theorem 4.3 is the following Lemma, which we will prove in Sect. 6.2 below, see [25, Corollary 3.5] and [26].

Lemma A.1 (Spreading Lemma). Let \( V = \{n_i\}_i \subset \mathbb{Z} \) be a set such that \( R(V) < \infty \) with \( R(V) \) as in (4.3). Moreover, let \( U \subset \mathbb{T} \) be a positive measure set with \( \rho(U) \leq 4R(V)/(4R(V) + 1) \), where \( \rho(\cdot) \) is the Lebesgue measure on \( \mathbb{T} \). Suppose that there exists some integer \( m \geq 1 \) such that

\[
\frac{4}{\rho(U')^2} \sum_{n_i \neq n_j} |\mathbbm{1}_{U'}(n_i - n_j)|^2 \leq m + 1,
\]

for all subsets \( U' \subset U \) of measure \( \rho(U') \geq \rho(U)/2 \), where \( \mathbbm{1}_{U'} \) is the indicator function of the set \( U' \). Then, there exists a set \( U_1 \supset U \) such that

1. The measure of \( U_1 \) satisfies

\[
\rho(U_1 \setminus U) \geq \frac{\rho(U)}{4m}.
\]

2. Uniformly for all \( g \in L^2(\mathbb{T}) \) with \( \text{Spec}(g) \subset V \), we have

\[
\int_{U_1} |g(x)|^2 dx \leq \left(C \frac{m^5}{\rho(U)^2}\right)^{3m} \int_{U} |g(x)|^2 dx,
\]

for some absolute constant \( C > 0 \).

In order to apply Lemma A.1, we will need the following two claims:
Claim A.2. Under the assumptions of Theorem 4.3 and maintaining the same notation, the integer $m > 0$ in Lemma A.1 can be taken to be

$$m = \left\lfloor \frac{C_0^2 R(V)}{\rho(U')^{\frac{2}{p}}} \right\rfloor = \left\lfloor \frac{G}{\rho(U')^{\frac{2}{p}}} \right\rfloor$$

where $\lfloor \cdot \rfloor$ is the integer part.

Proof. By definition of the $L^2(\mathbb{T})$ norm, we have

$$\left( \sum_{n_i \neq n_j} |\hat{1}_{U'}(n_i - n_j)|^2 \right)^{1/2} \leq R(V)^{1/2} \left( \sum_{r \in D(V)} |\hat{1}_{U'}(r)|^2 \right)^{1/2}$$

$$= R(V)^{1/2} \sup \left\{ \left| \int_{U'} \bar{h}(x) dx \right| : ||h||_{L^2(\mathbb{T})} \leq 1, \ Spec(h) \subset D(V) \right\}, \quad (A.1)$$

with $D(V)$ as in (4.3). Now, since $D(V)$ is a $\Lambda(p)$-system, we can bound the right hand side of (A.1) using Hölder’s inequality as follows:

$$R(V)^{-1/2} \text{r.h.s}(A.1) \leq \rho(U')^{-1 - \frac{1}{p}} \sup \left\{ ||h||_{L^p(\mathbb{T})} : ||h||_{L^2(\mathbb{T})} \leq 1, \ Spec(h) \subset D(V) \right\}$$

$$\leq C_0 \rho(U')^{1 - \frac{1}{p}},$$

for some constant $C_0 = C_0(V, p) > 0$ as in (4.2). Therefore, in light of (A.1), we obtain

$$\frac{4}{\rho(U')^2} \sum_{n_i \neq n_j} |\hat{1}_{U'}(n_i - n_j)|^2 \leq C_0^2 R(V) \rho(U')^{-\frac{2}{p}},$$

as required. \hfill \Box

Claim A.3. Under the assumptions of Theorem 4.3 and maintaining the same notation, let $g \in L^2(\mathbb{T})$ with $\text{Spec}(g) \subset V = \{n_i\}_i$, and $U \subset \mathbb{T}$ be a measurable subset. If $\rho(U) \geq 4R(V)/(4R(V) + 1)$ then

$$||g||^2_{L^2(\mathbb{T})} \leq \frac{2}{\rho(U)} \int_U |g(x)|^2 dx.$$

Proof. First, we may write

$$g(x) = \sum_i \widehat{g}(n_i) e(n_i \cdot x).$$

Thus, separating the diagonal terms from the others, we have

$$\int_U |g(x)|^2 dx = \rho(U) \sum_i |\widehat{g}(n_i)|^2 + \sum_{i \neq j} \widehat{1}_{U'}(n_i - n_j)\widehat{g}(n_i)\overline{\widehat{g}(n_j)}$$

$$= \rho(U) ||g||^2_{L^2(\mathbb{T})} + \langle QUg, g \rangle, \quad (A.2)$$
where \( Q_U = (q_{ij}) \) is an operator on \( L^2(\mathbb{T}) \) with matrix representation, in the base \( \{ e(nx) \}_{n \in \mathbb{Z}} \), given by
\[
q_{ij} = \begin{cases} 
\hat{1}_U(n_i - n_j) & n_i \neq n_j \\
0 & \text{otherwise} \end{cases}.
\]

(A.3)

Since \( \hat{1}_U(\cdot) \) is real-valued, \( \hat{1}_U(-n) = \overline{\hat{1}_U(n)} \), thus \( Q_U \) is a self-adjoint operator whose Hilbert–Schmidt norm is bounded by
\[
||Q_U|| \leq R(V)^{1/2} \left( \sum_{n \neq 0} |\hat{1}_U(n)|^2 \right)^{1/2} = (R(V)\rho(U)(1 - \rho(U)))^{1/2}.
\]

(A.4)

In particular, if \( \rho(U) \geq 4R(V)/(4R(V) + 1) \), we have \( (R(V)\rho(U)(1 - \rho(U)))^{1/2} \leq \rho(U)/2 \), thus (A.4) together with (A.2) give Claim A.3.

We are finally ready to prove Theorem 4.3:

**Proof of Theorem 4.3.** Let \( 0 < \nu \leq 4R(V)/(4R(V) + 1) \) be some parameter and denote by \( A(\nu) \) the smallest constant such that
\[
||g||^2_{L^2(\mathbb{T})} \leq A(\nu) \int_U |g(x)|^2 dx,
\]
uniformly for all \( g \in L^2(\mathbb{T}) \) with \( \text{Spec}(g) \subset V \) and any set \( U \subset \mathbb{T} \) with \( \rho(U) \geq \nu \). Moreover, let \( \varphi(\nu) = \log A(\nu), \Delta(\nu) = \nu^{1+2/p}(4G)^{-1} \) with \( G \) given by Claim A.2.

Applying Lemma A.1, bearing in mind that \( m \leq GV^{-2/p} \), we obtain a set \( U_1 \subset \mathbb{T} \) of measure \( \rho(U_1) \geq \nu + \Delta(\nu) \) such that
\[
\int_{U_1} |g(x)|^2 dx \leq \left( \frac{CG^S}{\nu^{2+\frac{10}{p}}} \right)^{3GV^{-2/p}} \int_U |g(x)|^2 dx.
\]

for some constant \( C > 0 \). Since, by definition of \( A(\cdot) \),
\[
||g||^2_{L^2(\mathbb{T}^2)} \leq A(\nu + \Delta(\nu)) \int_{U_1} |g(x)|^2 dx,
\]
we have
\[
A(\nu) \leq A(\nu + \Delta(\nu)) \left( \frac{CG^S}{\nu^{2+\frac{10}{p}}} \right)^{3GV^{-2/p}},
\]
and taking the logarithm of both sides, we finally deduce
\[
\frac{\varphi(\nu) - \varphi(\nu + \Delta(\nu))}{\Delta(\nu)} \leq \frac{12G^2}{\nu^{1+\frac{4}{p}}} \log \frac{CG^S}{\nu^{2+\frac{10}{p}}} \leq \frac{C_1(\epsilon)G^3}{\nu^{1+\frac{4}{p}+\epsilon}}, \quad (A.5)
\]
for some constant $C_1(\varepsilon) > 0$. Comparing (A.5) with the differential inequality
\[
d\varphi(v)/dv \leq C(\varepsilon)Gv^{-1-\frac{4}{p}-\varepsilon},
\]
in light of the fact that $A(v)$ is decreasing, we deduce that
\[
\varphi(v) \leq C(\varepsilon)C_0R(V)^3v^{-1-\frac{4}{p}-\varepsilon},
\]
where we have used the definition of $G$ given by Claim A.2. If $v(U) \geq 4R(V)/(4R(V) + 1)$, then Claim A.3 shows that the conclusion of Theorem 4.3 is still satisfied. \qed

A.2. Proof of Lemma A.1. In this section we prove Lemma A.1. The proof follows closely the arguments in [25, Section 3.4], again we claim no originality. We will need the following definition:

**Definition A.4.** Let $m$ be a positive integer and let $\tau, \kappa > 0$ be some parameters. Given $g \in L^2(T)$, we say that $g \in EP_m^{\text{loc}}(\tau, \kappa)$ if for every $t \in (0, \tau)$ there exist constants $a_0(t), \ldots, a_m(t) \in \mathbb{C}$ such that $\sum |a_k|^2 = 1$ and
\[
\left\| \sum_{k=0}^{m} a_k(t)g_{kt} \right\|_{L^2(T)} \leq \kappa,
\]
where $g_{kt}(\cdot) := g(\cdot + kt)$.

We refer the reader to [25, Sections 3.1–3.4] for an accurate description of the class $EP_m^{\text{loc}}(\tau, \kappa)$. Intuitively, functions in $EP_m^{\text{loc}}(\tau, \kappa)$ “behave like” trigonometric polynomials of degree $m$ in intervals of length $\tau$ up to an error $\kappa$. The key estimate that we will need is the following [25, Corollary 3.5']:

**Lemma A.5.** Let $g \in EP_m^{\text{loc}}(\tau, \kappa)$ for some integer $m > 0$ and some $\tau, \kappa > 0$. Moreover, let $U \subset T \subset \mathbb{R}^2$ be a set of positive measure and $v := \rho(e(m\tau)U \setminus U)$. There exists a set $U_1 \supset U$ of measure $\rho(U_1 \setminus U) \geq \frac{\kappa}{2}$ such that
\[
\int_{U_1} |g(x)|^2dx \leq \left( \frac{Cm^3}{\nu^2} \right)^{2m} \left( \int_{U} |g(x)|^2dx + \kappa^2 \right),
\]
for some constant $C > 0$.

We will also need the following two claims:

**Claim A.6.** Let $U \subset T$ be a measurable subset and let $m$ be as in Lemma A.1. Then, there exists a subspace $V_m$ of $L^2(T)$ of dimension at most $m$ such that for all $g \in L^2(T)$ orthogonal to $V_m$, we have
\[
||g||_{L^2(T)}^2 \leq \frac{2}{\rho(U')} \int_{U'} |g(x)|^2dx
\]
for all subsets $U' \subset U$ with $\rho(U') \geq \rho(U)/2$. 

Proof. Indeed, let \( |\sigma_1| \leq |\sigma_2| \leq \cdots \) be the eigenvalues of the operator \( Q_{U'} \) defined in (A.3) with \( U' \) instead of \( U \). Then we take \( V_m \) to be the subspace generated by the eigenvectors with eigenvalues \( \sigma_1, \ldots, \sigma_m \). We are now going to show that \( V_m \) has the claimed property. By definition of \( m \), we have

\[
\sum_i |\sigma_i|^2 = ||Q_{U'}||^2 \leq \sum_{i \neq j} |q_{ij}|^2 = \sum_{n_i \neq n_j} |\hat{U'}(n_i - n_j)|^2 \leq \frac{\rho(U')^2(m + 1)}{4}.
\]

Thus,

\[
|\sigma_{m+1}|^2 \leq \frac{1}{m+1} \cdot \frac{\rho(U')^2(m + 1)}{4} \leq \frac{\rho(U')^2}{4}.
\]

Therefore Claim A.6 follows from that fact that the norm of \( Q_{U'} \) restricted to \( L^2(\mathbb{T}) \setminus V_m \) is at most \( |\sigma_{m+1}| \leq \rho(U')/2 \) and an analogous argument to Claim A.3. \( \square \)

Now, if \( V \) is finite we let \( N = |V| \), if \( V \) is infinite we can ignore the dependence on \( N \) in the rest of the argument. With this notation, we claim the following:

**Claim A.7.** Let \( U \subset \mathbb{T} \) be a measurable set, \( m \) be as in Lemma A.1 and, if \( V \) is finite, suppose that \( m < N \), moreover let \( g \in L^2(\mathbb{T}) \) with \( \text{Spec}(g) \subset V \). Then there exists some \( \sigma \in (0, 1) \) such that \( g \in EP_{\text{loc}}^n(\tau, x) \) where \( x^- = \frac{4}{\rho(U')}(m + 1) \int_U |g(x)|^2 dx \), \( \tau = \sigma/2m \) and, moreover \( v := \rho(e(mt\tau)U \setminus U) \geq \rho(U)/2m \).

**Proof.** Let \( t \in [0, 1) \) be given, since exponentials with different frequencies are linearly independent\(^5\) in \( L^2(\mathbb{T}) \), we can choose coefficients \( a_k(t) \), so that \( \sum_k |a_k|^2 = 1 \) and the function

\[
h(\cdot) = \sum_{k=0}^{m} a_k(t) g_{kt}(\cdot),
\]

where \( g_{kt}(x) = \sum a_{ni} e(n_i kt) e(n_i x) \), is orthogonal to \( V_m \), given in Claim A.6, provided that \( m < N \). Therefore, Claim A.6 gives

\[
||h||^2_{L^2(\mathbb{T})} \leq \frac{2}{\rho(U')} \int_{U'} |h(x)|^2 dx,
\]

for all \( U' \subset U \) with \( \rho(U') \geq \rho(U)/2 \).

We are now going to choose an appropriate set \( U' \) in order to estimate the r.h.s. of (A.6). Let \( t \geq 0 \) and take \( U'_t = U_t := \cap_{k=0}^{m} e^{-kt} U \), since the function \( t \to \rho(U_t \setminus U) \) is continuous and takes value 0 at \( t = 0 \), we can find some sufficiently small \( \tau > 0 \) so that, for all \( t \in (0, \tau) \), the set \( U_t := \cap_{k=0}^{m} e^{-kt} U \) has measure at least \( \rho(U)/2 \). To estimate the r.h.s. of (A.6), we observe that, for every \( k = 0, \ldots, m \), we have

\[
\int_{U_t} |g_{kt}(x)|^2 dx \leq \int_{e^{-kt}U} |g_{kt}(x)|^2 dx = \int_{U} |g(x)|^2 dx.
\]

\(^5\) Suppose that \( n_i \neq n_j \) for \( i \neq j \) and \( \sum_i a_{ni} e(n_i x) = 0 \). Multiplying both sides by \( e(-n_1 x) \) and integrating for \( x \in \mathbb{T} \), we see that \( a_1 = 0 \). Repeating the argument, we get \( a_i = 0 \) for all \( i \).
Thus, the Cauchy–Schwarz inequality gives
\[
\int_{U_t} |h(x)|^2 dx \leq \left( \sum_{k=0}^{m} \int_{U_t} |g_{kt}(x)|^2 dx \right) \leq (m + 1) \int_{U} |g(x)|^2 dx.
\] (A.7)

Hence, (A.6) together with (A.7), bearing in mind that \( \rho(U_t) \geq \rho(U)/2 \), give that for all \( t \in (0, \tau) \) there exists coefficients \( a_1(t), \ldots, a_m(t) \) such that \( \sum_k |a_k|^2 = 1 \) and
\[
\left\| \sum_{k=0}^{m} a_k(t) g_{kt} \right\|_{L^2(U)}^2 \leq \frac{4(m + 1)}{\rho(U)} \int_{U} |g(x)|^2 dx.
\]

We are now left with proving the claimed estimates on \( \tau \) and \( \nu \). Let \( \psi(s) = \rho(e(s)U \setminus U) \), bearing in mind that \( \rho(U) \leq R(V)/(4R(V) + 1) \) so that, by (A.1), \( \rho(T \setminus U) \geq (4R(V) + 1)^{-1} \geq (2m)^{-1} \), we have
\[
\int_0^1 \psi(s) ds = \rho(U) \rho(T \setminus U) \geq \frac{\rho(U)}{2m}.
\]

Thus, since \( \psi(s) \) is non-negative and continuous, there exists some \( \sigma \in (0, 1) \) such that for all \( s \leq \sigma \) we have \( \rho(e(s)U \setminus U) \leq \rho(U)/2m \). We now verify that such \( \tau = \sigma/m \) satisfies \( \rho(U_t) \geq \rho(U)/2 \) for all \( t \in (0, \tau) \). Indeed, bearing in mind that \( kt \in (0, m\tau) \), we have
\[
\rho(U_t) = \rho(\cap_{k=0}^{m} e(-kt)U) \\
\geq \rho(U) - \sum_{k=1}^{m} \rho(e(kt)U \setminus U) \geq \rho(U) - m \frac{\rho(U)}{2m} \geq \rho(U)/2,
\]
concluding the proof of Claim A.7. \( \square \)

We are finally ready to present the proof of Lemma A.1:

**Proof of Lemma A.1.** Suppose that \( m < N \), then, applying Lemma A.5 with the choice of parameters given by Claim A.7, we obtain part (1) of Lemma A.1. For part (2), Lemma A.5 gives
\[
\int_{U_1} |g(x)|^2 dx \leq \left( \frac{C m^5}{\rho(U)^2} \right)^{2m} \left( \frac{4(m + 1)}{\rho(U)} + 1 \right) \int_{U} |g(x)|^2 dx \\
\leq \left( \frac{C m^5}{\rho(U)^2} \right)^3 \int_{U} |g(x)|^2 dx,
\] (A.8)
as required.

Let us now suppose that \( m \geq N \), then the Nazarov–Turán Lemma [25, Theorem 1], for any set \( U_1 \subset T \) of measure \( \rho(U_1) = \rho(U) + \rho(U)/4m \), gives
\[
\int_{U_1} |g(x)|^2 dx \leq \left( \frac{C \rho(U_1)}{\rho(U)} \right)^{N-1} \int_{U} |g(x)|^2 dx \\
\leq \left( C + \frac{C}{4m} \right)^{N-1} \int_{U} |g(x)|^2 dx \\
\leq 100 m \int_{U} |g(x)|^2 dx
\]
and \((A.8)\) follows, up noticing that \(\rho(U) \leq 1.\)

\[\square\]

**Appendix B: On \(\Lambda(p)\)-systems**

In order to state the main results in this appendix let us first recall some notation. Let \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\) be the one dimension torus and \(V \subset \mathbb{Z}\). We say that subset \(V \subset \mathbb{Z}\) is symmetric if \(n \in V\) implies \(-n \in V\). We recall the notation \(D(V) := \{n_i - n_j \in \mathbb{Z} : i \neq j, \ n_i, n_j \in V\}\). It will be convenient to make the following definition:

**Definition B.1.** Let \(p \geq 2\) be some even integer. A symmetric subset \(V \subset \mathbb{Z}\) is called a \(\Lambda(p)^*\)-system if all the solutions to the linear equation

\[
n_1 + \cdots + n_p = 0 \quad n_i \in V, \quad \text{for} \ i \in \{1, \ldots, p\}
\]

are trivial, that is \(n_1 = -n_2, \ldots, n_{p-1} = -n_p\) up to permutations.

The aim of this section is to prove the following result, mentioned in [26].

**Lemma B.2.** Let \(V = \{n_i\}_i \subset \mathbb{Z}\) be a symmetric set and let \(p \geq 2\) be an integer. Suppose that \(V\) is a \(\Lambda(2p)^*\)-system. Then, \(D(V)\) is a \(\Lambda(p)\)-system with constant \(C_0(p) = c(p)\) independent of \(V\).

We observe that, by Riesz’s Theorem, see for example [?, Page 2], it is enough to check the definition of \(\Lambda(p)\) systems (cf. Sect. 4.1) for polynomials \(g \in L^2(\mathbb{T})\) with \(\text{Spec}(g) \subset V\) (called \(V\)-polynomial in [?]).

We begin with following simple case of Lemma B.2 below. We single out this case because the combinatorial argument is easier to explain and it contains all the arguments of the more general one.

**Claim B.3.** Let \(V = \{n_i\}_i \subset \mathbb{Z}\) be a symmetric set. Suppose that \(V\) is a \(\Lambda(8)^*\)-system. Then, \(D(V)\) is a \(\Lambda(4)\)-system with constant \(C_0 > 0\) independent of \(V\).

**Proof.** Let as write \(D(V) = D\) and, for \(m_k \in D\), let us write \(m_k = n_{ik} - n_{jk}\). We now split \(D\) into at most 5 sets \(D(a, b)\) for \((a, b) \in \{\pm 1\}^3\) and \(D_1(0)\) defined as follows. Given \((a, b), m_k \in D(a, b)\) if and only if \(a \cdot n_{ik} > 0\) and \(b \cdot n_{jk} > 0\) for all \(k\). So, for example, given \(D(1, 1)\) we have \(n_{ik} > 0\) and \(n_{jk} > 0\). We denote by \(D_1(0) := \{m_k \in D : m_k = -n_{jk}\}\), that is \(n_{ik} = 0\) for all \(k\) and observe that, as \(V\) is symmetric, \(D_1(0) = \{m_k \in D : m_k = n_{ik}\}\), that is \(n_{jk} = 0\) for all \(k\).

Now, given \(f \in L^2(\mathbb{T})\) with \(\text{Spec}(f) \subset V\), which in, light of the observation under Lemma B.2, we may assume is a \(V\)-polynomial, let us write

\[
f(x) := \sum_{m \in D} a_m e(m \cdot x) = \left( \sum_{m \in D_1(0)} + \sum_{(a, b, c) \in D(a, b, c)} \right) a_m e(m \cdot x), \quad (B.1)
\]

for some \(a_m \in \mathbb{C}\) which are zero outside a finite set of \(m\)’s. We may also normalize \(f\) so that

\[
\|f\|_{L^2} = 1.
\]

In order to prove Claim B.3, by the triangle inequality, it is enough to estimate the \(L^4\) norm of each sum on the r.h.s. of (B.1) individually. Moreover, by considering the real and imaginary parts of \(f\) separately, we may also assume that

\[
\overline{a_m} = a_{-m}. \quad (B.2)
\]
Finally, before embarking in the proof of Claim (B.3), we make one last observation. Let us write 
\[ a(i, j) = a_m \] for the coefficient associate to \( m = n_i - n_j \) with \( m \notin D_1(0) \), and observe the following. Since \( V \) is a \( \Lambda(4)^* \)-system and bearing in mind (B.2), we have the following bound
\[
\sum_{i, j} |a(i, j)|^2 \leq \sum_m |a_m|^2 = 1.
\]
Since \( V \) is a \( \Lambda(4)^* \) system, we readily see that
\[
\left\| \sum_{m \in D_1(0)} a_m e(m \cdot x) \right\|_{L^4} \leq C \|f\|_{L^2},
\]
for some constant \( C > 0 \) independent of \( D_1(0) \).

Let us now estimate the sums over \( D(a, b) \) on the r.h.s. of (B.1). Since \( V \) is symmetric and by definition of \( D(V) \), it is enough to consider the cases \((a, b) = (1, -1)\) and \((a, b) = (1, 1)\). Let us begin with the case \((a, b) = (1, -1)\), by expanding the \( L^4 \)-norm and using the orthogonality of the exponentials, we have
\[
\left\| \sum_{m \in D_1((1,1,-1))} a_m e(m \cdot x) \right\|_{L^4}^4 = \sum_{m_1 - m_2 + m_3 - m_4 = 0} a_{m_1} a_{m_2} a_{m_3} a_{m_4}
\]
Let us no write \( m_k = n_{ik} - n_{jk} \) with \( n_{ik} \neq n_{jk} \), then the equation
\[
m_1 - m_2 + m_3 - m_4 = 0,
\]
can be re-written as
\[
n_{i_1} - n_{j_1} + n_{i_3} - n_{j_3} - (n_{i_2} - n_{j_2} + n_{i_4} - n_{j_4}) = 0 \quad \text{(B.3)}
\]
Since the \( n_j \)'s are negative, there are no cancellations in the sum \( n_{i_1} - n_{j_1} + n_{i_3} - n_{j_3} \) in (B.3). Thus, since \( V \) is a \( \Lambda(8)^* \) system, we must have that the set \( \{i_2, j_2, i_4, j_4\} \) is a permutation of the set \( \{i_1, j_1, i_3, j_3\} \) (note that the set \( \{i_1, j_1, i_3, j_3\} \) may contain repetitions). We are now going to bound the following sum
\[
\left| \sum_{m_1 - m_2 + m_3 - m_4 = 0} a_{m_1} a_{m_2} a_{m_3} a_{m_4} \right| \leq \sum_{i_1, j_1, i_3, j_3} |a(i_1, j_1)a(i_3, j_3)a(\sigma(i_1), \sigma(j_1))a(\sigma(i_3), \sigma(j_3))| \quad \text{(B.4)}
\]
where \( a_{(i_k,j_k)} \) is the coefficients associated to \( m_{(i_k,j_k)} := n_{i_k} - n_{j_k} \) and \( \sigma \) is some permutation on the set of 4 elements. Using the Cauchy–Schwartz inequality, we obtain

\[
\left| \sum_{i_1,j_1,i_3,j_3} |a_{(i_1,j_1)}a_{(i_3,j_3)}a_{(\sigma(i_1),\sigma(j_1))}a_{(\sigma(i_3),\sigma(j_3))}| \right|^2 \\
\leq \sum_{i_1,j_1,i_3,j_3} |a_{(i_1,j_1)}a_{(i_3,j_3)}|^2 \sum_{i_1,j_1,i_3,j_3} |a_{(\sigma(i_1),\sigma(j_1))}a_{(\sigma(i_3),\sigma(j_3))}|^2 \\
\leq \left( \sum_{i,j} |a_{(i,j)}|^2 \right)^2 \sum_{i_1,j_1,i_3,j_3} |a_{(\sigma(i_1),\sigma(j_1))}a_{(\sigma(i_3),\sigma(j_3))}|^2 \\
\leq \sum_{i_1,j_1,i_3,j_3} |a_{(\sigma(i_1),\sigma(j_1))}a_{(\sigma(i_3),\sigma(j_3))}|^2 
\]

where we used the fact that \( \sum_{i,j} |a_{(i,j)}|^2 \leq \sum_m |a_m|^2 = 1 \). Since, \( i_k \neq j_k \), for all \( k \), we have \( \sigma(i_k) \neq \sigma(j_k) \), thus we have also have

\[
\sum_{i_1,j_1,i_3,j_3} |a_{(\sigma(i_1),\sigma(j_1))}a_{(\sigma(i_3),\sigma(j_3))}|^2 \leq \left( \sum_{i,j} |a_{(i,j)}|^2 \right)^2 \leq 1. 
\]

Hence, since there are (loosely) at most \( 2^4 \cdot 4! \) sums of the form (B.4) and each one is bounded by at most 1, we have shown that

\[
\left\| \sum_{m \in D_i((1,1,-1))} a_m e(m \cdot x) \right\|_{L^4}^4 \leq 2^4 \cdot 4!.
\]

Let us now consider the case \((a, b) = (1, 1)\). Expanding the \( L^4 \) norm and using orthogonality, we need to study solutions to the equation \( m_1 - m_2 + m_3 - m_4 = 0 \), which, this time, we re-write as

\[
n_{i_1} + n_{j_2} + n_{i_3} + n_{j_4} - (n_{j_1} + n_{i_2} + n_{j_3} + n_{i_4}) = 0.
\]

Since the \( n_i \)'s and the \( n_j \)'s are positive and \( V \) is a \( \Lambda(8)^* \) system, we must have that the set \( \{i_1, j_2, i_3, j_4\} \) is a permutation of the set \( \{j_1, i_2, j_3, i_4\} \). Now, we wish to bound the sum

\[
\left| \sum_{m_1 - m_2 + m_3 - m_4 = 0} a_{m_1} \overline{a_{m_2}} \overline{a_{m_3}} \overline{a_{m_4}} \right| \leq \sum_{i_1,j_2,i_3,j_4} |a_{(i_1,\sigma(i_1))}a_{(j_2,\sigma(j_2))}a_{(i_3,\sigma(i_3))}a_{(j_4,\sigma(j_4))}| 
\]

(B.5)

To this end, let us for convenience, write \( \{i_1, j_2, i_3, j_4\} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \), suppose that the \( \alpha_i \) are distinct and consider the cycle structure of \( \sigma \). Since \( \sigma(\alpha_i) \neq \alpha_i \) (as the coefficients \( a_{(\sigma,\alpha)} \) do not appear in the sum in (B.5)) sigma does not have any fixed point.
so, up to relabeling, we have $\sigma = (12)(34)$ or $\sigma = (1234)$. Thus, bearing in mind that $|a(\alpha_1, \alpha_2)| = |a(\alpha_2, \alpha_1)|$, we may bound the r.h.s. of (B.5), in the first case, as

$$
\sum_{i_1, j_2, i_3, j_4} |a(i_1, \sigma(i_1))a(\sigma(j_2), j_2)a(i_3, \sigma(i_3))a(\sigma(j_4), j_4)| \\
\leq \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_1, \alpha_2)a(\alpha_2, \alpha_1)a(\alpha_3, \alpha_4)a(\alpha_4, \alpha_3)| \\
\leq \left( \sum_{\alpha_1, \alpha_2} |a(\alpha_1, \alpha_2)a(\alpha_2, \alpha_1)|^2 \right)^{1/2} \leq 1.
$$

Note that, as $\sigma$ is the product of two cycles, the sum splits as a product over these cycles.

In the second case, using the Cauchy–Schwartz inequality, we have

$$
\sum_{i_1, j_2, i_3, j_4} |a(i_1, \sigma(i_1))a(\sigma(j_2), j_2)a(i_3, \sigma(i_3))a(\sigma(j_4), j_4)| \\
\leq \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_1, \alpha_2)a(\alpha_2, \alpha_3)a(\alpha_3, \alpha_4)a(\alpha_4, \alpha_1)| \\
\leq \left( \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_1, \alpha_2)a(\alpha_3, \alpha_4)|^2 \cdot \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_2, \alpha_3)a(\alpha_4, \alpha_1)|^2 \right)^{1/2} \leq 1.
$$

If there were repetitions among the $\alpha_i$’s, the set $\{\alpha_i\}$ can be identified as the quotient of the symmetric group $S_4$ by one, or more, of its stabilizers. Thus, the resulting sums are sub-sums of one of the above. Another (equivalent) way to see this is the following.

We add dummy variables to the set $\{\alpha_i\}$ until we get a set with four elements and carry the relations among the $\alpha_i$ under the sum. For example, say that $\alpha_1 = \alpha_3$ (in the second case above), then

$$
\sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_1, \alpha_2)a(\alpha_2, \alpha_1)a(\alpha_1, \alpha_4)a(\alpha_4, \alpha_1)| = \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_1, \alpha_2)a(\alpha_3, \alpha_4)a(\alpha_2, \alpha_1)a(\alpha_4, \alpha_3)| \\
\leq \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_1, \alpha_2)a(\alpha_3, \alpha_4)|^2 \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_2, \alpha_1)a(\alpha_4, \alpha_3)|^2 \\
\leq \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_1, \alpha_2)a(\alpha_3, \alpha_4)|^2 \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |a(\alpha_2, \alpha_1)a(\alpha_4, \alpha_3)|^2 \leq 1.
$$

Hence, all in all, we have the bound

$$
\left\| \sum_{m \in D(1, 1)} a_m e(m \cdot x) \right\|_{L^4}^4 \leq 2^4 \cdot 4!,
$$

which concludes the proof. $\square$

We now move onto the general lemma:
Proof of Lemma B.2. Let \( D(a, b) \) for \((a, b) \in (\pm 1)^2 \) and \( D_1(0) \) be as in the proof of Claim B.3. By the same reduction argument in Claim B.3, it is enough to show that

\[
\left\| \sum_{m \in D(1,-1)} a_m e(m \cdot x) \right\|_{L^p}^p \leq 2^p \cdot p!, \tag{B.6}
\]

and

\[
\left\| \sum_{m \in D(1,1)} a_m e(m \cdot x) \right\|_{L^p}^p \leq 2^p \cdot p!, \tag{B.7}
\]

assuming that

\[
\sum_m |a_m|^2 = \|f\|_{L^2} = 1 \quad \text{and} \quad \overline{a_m} = a_{-m}.
\]

We begin with showing (B.6). As in the proof of Claim B.3, we may write the relation

\[
m_1 - m_2 + \cdots - m_p = 0
\]

as

\[
\sum_{k \text{ even}} (n_{i_k} - n_{j_k}) - \left( \sum_{k \text{ odd}} (n_{i_k} - n_{j_k}) \right) = 0.
\]

Therefore, since \( V \) is a \( \Lambda(2p)^* \)-system and the \( n_{j_k} \)'s are negative, we deduce that the set \( \{i_k, j_k\}_{k \text{ even}} \) is a permutation of the set \( \{i_k, j_k\}_{k \text{ odd}} \). Using the Cauchy–Schwartz inequality, we see that

\[
\left( \sum_{i, j} |a_{i,j}|^2 \right)^p \cdot \left( \sum_{i, j} |a_{i,j}|^2 \right)^p \leq \left( \sum_{i, j} |a_{i,j}|^2 \right)^p \cdot \left( \sum_{i, j} |a_{i,j}|^2 \right)^p \leq 1.
\]

Since there are \( p! \) choices for \( \sigma \) and taking into account possible repetitions of the indices (which can be bounded by the same argument as repeated indices do not affect the Cauchy–Schwartz inequality), (B.6) follows.

In order to see (B.7), we re-write the relation

\[
m_1 - m_2 + \cdots - m_p = 0
\]

as

\[
\sum_k (n_{i_{2k-1}} + n_{j_{2k}}) - \left( \sum_k (n_{j_{2k-1}} + n_{i_{2k}}) \right) = 0.
\]
As above, we deduce that the set \( \{ j_{2k-1}, i_{2k} \}_{k=1}^{P} \) determines, up to permutations the set \( \{ j_{2k-1}, i_{2k} \}_{k=1}^{P} = \{ \alpha_1, \ldots, \alpha_P \} \) and claim the bound

\[
\sum_{\alpha_1, \ldots, \alpha_{\ell}} \prod_{k=1}^{\ell} |a(\alpha_k, \sigma(\alpha_k))| \leq 1, 
\]  
(B.8)

where \( \sigma \) is any permutation on the set of \( \ell \) elements with no fixed points, and the sum is subject to \( \sum_{i,j} |a_{i,j}|^2 \leq 1 \) and \( |a_{i,j}| = |a_{j,i}| \).

Let’s begin with some preliminary observations/simplification. Since any permutation can be written as a product of cyclic permutations and the sum in (B.8) splits as a product over the said cycles, it is enough to show (B.8) for cyclic permutations. Moreover, as above, we may assume that the \( \alpha_i \) are distinct (if not we are considering a sum over quotients of \( S_\ell \) by stabilizers, which is a sub-sum of one of the sums with distinct indices). Furthermore, up to relabeling, we may assume that \( \sigma = (12 \cdots \ell) \). Let us first suppose that \( \ell \) is even, writing for notational convenience \( \alpha_{\ell+1} = \alpha_{\ell} \) and using Cauchy–Schwartz, we have the bound

\[
\left| \sum_{\alpha_1, \ldots, \alpha_{\ell}} \prod_{k=1}^{\ell} |a(\alpha_k, \alpha_{k+1})| \right|^2 = \sum_{\alpha_1, \ldots, \alpha_{\ell}} \prod_{k=1}^{\ell} |a(\alpha_{2k-1}, \alpha_{2k})| \prod_{k=1}^{\ell} |a(\alpha_{2k}, \alpha_{2k+1})| ^2 
\leq \sum_{\alpha_1, \ldots, \alpha_{\ell}} \prod_{k=1}^{\ell} |a(\alpha_{2k-1}, \alpha_{2k})|^2 \cdot \sum_{\alpha_1, \ldots, \alpha_{\ell}} \prod_{k=1}^{\ell} |a(\alpha_{2k}, \alpha_{2k+1})|^2 
\leq \left( \sum_{i,j} |a_{i,j}|^2 \right)^{\ell/2} \cdot \left( \sum_{i,j} |a_{i,j}|^2 \right)^{\ell/2} \leq 1.
\]

As a side note, repeated indices do not affect the above inequalities.

Let us now suppose that \( \ell \) is odd, to treat this case we need to introduce some notation. Let us define the matrix \(^6\)

\[
A = (a_{i,j})_{i,j}
\]

with \( a_{i,i} = 0 \), that is the diagonal is zero. Moreover, let us write \( B = (b_{i,j})_{i,j} = A^2 \), and observe that

\[
b_{i,j} = \sum_k |a_{i,k}a_{k,j}|
\]

\[
\sum_{i,j} |b_{i,j}|^2 = ||B||_F^2 \leq ||A||_F^4 = \left( \sum_{i,j} |a_{i,j}|^2 \right)^4 \leq ||f||_L^4 \leq 1,
\]

where we write \( \langle \cdot, \cdot \rangle_F \) for the Frobenius inner product and \( || \cdot ||_F \) for the associated norm which is sub-multiplicative. Having introduce the relevant notation, we may proceed to

---

\(^6\) Which is indeed a finite matrix as \( f \) is a \( V \) polynomial.
the proof of (B.8). First, we write
\[
\sum_{\alpha_1, \ldots, \alpha_\ell} \prod_{k=1}^\ell |a(\alpha_k, \alpha_{k+1})| = \sum_{\alpha_k, \alpha_{k+1} \text{ odd}} \prod_{k=1}^\ell |a(\alpha_{k+1}, \alpha_{k+2})| \cdot |a(\alpha_\ell, \alpha_1)|
\]
\[= \sum_{\alpha_1, \alpha_3, \ldots, \alpha_\ell} |b(\alpha_1, \alpha_3) b(\alpha_3, \alpha_5) \cdots b(\alpha_{\ell-2}, \alpha_\ell) a(\alpha_\ell, \alpha_1)| \quad (B.9)
\]
Note that if there were repeated indices, then (B.9) would be an inequality as \(|a(\alpha_i, \alpha_j)| \leq b(\alpha_i, \alpha_j)|. Writing \(\ell = 2u + 1\), we have two cases. If \(u\) is odd, the inner product in (B.9) contains an even number of elements, thus we may bound them, using the Cauchy–Schwartz inequality, as
\[
\left| \sum_{\alpha_1, \alpha_3, \ldots, \alpha_\ell} \left( \prod_{k=1}^u |b(\alpha_{4k-3}, \alpha_{4k-1})| \right) \cdot \left( \prod_{k=1}^u |b(\alpha_{4k-1}, \alpha_{4k+1}) a(\alpha_\ell, \alpha_1)| \right) \right|^2 \leq \sum_{\alpha_1, \alpha_3, \ldots, \alpha_\ell} \left( \prod_{k=1}^u |b(\alpha_{4k-3}, \alpha_{4k-1})|^2 \right) \sum_{\alpha_1, \alpha_3, \ldots, \alpha_\ell} \left( \prod_{k=1}^u |b(\alpha_{4k-1}, \alpha_{4k+1}) a(\alpha_\ell, \alpha_1)|^2 \right)
\]
\[\leq \left( \sum_{i,j} |b(i,j)|^2 \right)^u \cdot \left( \sum_{i,j} |b(i,j)|^2 \right)^{u-1} \cdot \left( \sum_{i,j} |a(i,j)|^2 \right) \leq 1,
\]
where, in the last inequality, we have used the fact that
\[
\sum_{i,j} |b(i,j)|^2 = \|B\|_F^2 \leq \|A\|_F^4 \leq 1.
\]
If \(u\) is even, we write
\[
C := B^2 = (c_{i,j})_{i,j} = \left( \sum_k b(i,k) b(k,j) \right)_{i,j} \quad \sum_{i,j} |c_{i,j}|^2 = \|C\|_F^2 \leq \|A\|_F^8 \leq 1
\]
and regroup the sums in (B.9) accordingly (as we did with \(B\)). We repeat this procedure until we end up with an even number of summands, at that point we may apply Cauchy–Schwartz. This concludes the proof of (B.8) which implies (B.7) and thus Lemma B.2. \(\square\)

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