Research Article

On the Classification of Lattices Over $\mathbb{Q}(\sqrt{-3})$ Which Are Even Unimodular $\mathbb{Z}$-Lattices of Rank 32

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Received 13 November 2012; Accepted 28 January 2013

Academic Editor: Frank Werner

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We classify the lattices of rank 16 over the Eisenstein integers which are even unimodular $\mathbb{Z}$-lattices (of dimension 32). There are exactly 80 unitary isometry classes.

1. Introduction

Let $\mathcal{O} = \mathbb{Z}[(1 + \sqrt{-3})/2]$ be the ring of integers in the imaginary quadratic field $K = \mathbb{Q}[\sqrt{-3}]$. An Eisenstein lattice is a positive definite Hermitian $\mathcal{O}$-lattice $(\Lambda, h)$ such that the trace lattice $(\Lambda, q)$ with $q(x, y) := \text{trace}_{K/\mathbb{Q}} h(x, y) = h(x, y) + \overline{h}(x, y)$ is an even unimodular $\mathbb{Z}$-lattice. The rank of the free $\mathcal{O}$-lattice $\Lambda$ is $r = n/2$ where $n = \dim_{\mathbb{R}}(\Lambda)$. Eisenstein lattices (or the more general theta lattices introduced in [1]) are of interest in the theory of modular forms, as their theta series is a modular form of weight $r$ for the full Hermitian modular group with respect to $\mathcal{O}$ (cf. [2]). The paper [2] contains a classification of the Eisenstein lattices for $n = 8, 16,$ and 24. In these cases, one can use the classifications of even unimodular $\mathbb{Z}$-lattices by Kneser and Niemeier and look for automorphisms with minimal polynomial $X^2 - X + 1$.

For $n = 32$, this approach does not work as there are more than $10^9$ isometry classes of even unimodular $\mathbb{Z}$-lattices (cf. [3, Corollary 17]). In this case, we apply a generalisation of Kneser’s neighbour method (compare [4]) over $\mathbb{Z}[(1 + \sqrt{-3})/2]$ to construct enough representatives of Eisenstein lattices and then use the mass formula developed in [2] (and in a more general setting in [1]) to check that the list of lattices is complete.

Given some ring $R$ that contains $\mathcal{O}$, any $R$-module is clearly also an $\mathcal{O}$-module. In particular, the classification of Eisenstein lattices can be used to obtain a classification of even unimodular $\mathbb{Z}$-lattices that are $R$-modules for the maximal order

$$R = \mathcal{M}_{2,\infty} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1 + i + j + ij}{2},$$

$$R = \mathcal{M}_{3,\infty} = \mathbb{Z} + \mathbb{Z}\frac{1 + i\sqrt{3}}{2} + \mathbb{Z}j + \mathbb{Z}\frac{j + ij\sqrt{3}}{2},$$

respectively, where $i^2 = j^2 = -1$, $ij = -ji$, in the rational definite quaternion algebra of discriminant $2^2$ and $3^2$ respectively. For the Hurwitz order $\mathcal{M}_{2,\infty}$, these lattices have been determined in [5], and the classification over $\mathcal{M}_{3,\infty}$ is new (cf. [6]).

2. Statement of Results

Theorem 1. The mass of the genus of Eisenstein lattices of rank 16 is

$$\mu_{16} = \sum_{i=1}^{h} \frac{1}{|U(A_i)|}$$

$$= \frac{16519 \cdot 3617 \cdot 1847 \cdot 809 \cdot 691 \cdot 419 \cdot 47 \cdot 13}{2^{31} \cdot 3^{32} \cdot 5^{4} \cdot 11 \cdot 17} \sim 0.002.$$
There are exactly $h = 80$ isometry classes $[\Lambda_{i}]$ of Eisenstein lattices of rank 16.

Proof. The mass was computed in [2]. The 80 Eisenstein lattices of rank 16 are listed in Table 4 with the order of their unitary automorphism group. These groups have been computed with MAGMA. We also checked that these lattices are pairwise not isometric. Using the mass formula, one verifies that the list is complete. \hfill \square

To obtain the complete list of Eisenstein lattices of rank 16, we first constructed some lattices as orthogonal sums of Eisenstein lattices of rank 12 and 4 and from known 32-dimensional even unimodular lattices. We also applied coding constructions from ternary and quaternary codes in the same spirit as described in [7]. To this list of lattices, we applied Kneser’s neighbor method. For this, we made use of the following facts (cf. [4]): Let $\Gamma$ be an integral $\mathcal{O}$-lattice and $p$ a prime ideal of $\mathcal{O}$ that does not divide the discriminant of $\Gamma$. An integral $\mathcal{O}$-lattice $\Lambda$ is called a $p$-neighbor of $\Gamma$ if

$$\Lambda/\Gamma \cap \Lambda \equiv \mathcal{O}/p \quad \text{and} \quad \Gamma/\Gamma \cap \Lambda \equiv \mathcal{O}/\overline{p}.$$  

All $p$-neighbors of a given $\mathcal{O}$-lattice $\Gamma$ can be constructed as

$$\Gamma(p, x) := \mathcal{O}^{-1}x + \Gamma_x, \quad \Gamma_x := \{y \in \Gamma \mid h(x, y) \in p\},$$  

where $x \in \Gamma \setminus \mathcal{O}p^\ast$ with $h(x, x) \in \mathcal{O}p^\ast$ (such a vector is called admissible). We computed (almost random) neighbors (after rescaling the already computed lattices to make them integral) for the prime elements 2, $2 - \sqrt{-3}$, and $4 - \sqrt{-3}$ by randomly choosing admissible vectors $x$ from a set of representatives and constructing $\Gamma(p, x)$ or all integral overlattices of $\Gamma_x$ of suitable index. For details of the construction, we refer to [4].

**Corollary 2.** There are exactly 83 isometry classes of $\mathcal{M}_{3,\infty}$-lattices of rank 8 that yield even unimodular $\mathbb{Z}$-lattices of rank 32.
Table 4: Continued.

| no. | \( R \) | \#Aut | \( \mathfrak{M}_{3,\infty} \) | \( \mathfrak{M}_{2,\infty} \) |
|-----|-------|------|-----------------|-----------------|
| 48  | 4A_2  | 4199040 | 2               |                 |
| 49  | 4A_2  | 1399680 | 1               |                 |
| 50  | 4A_2  | 314928  |                 |                 |
| 51  | 4A_2  | 139968  | 1               |                 |
| 52  | 4A_2  | 69984   | 3               |                 |
| 53  | D_4   | 660320461920 | L_6(\mathfrak{B}) |                 |
| 54  | D_4   | 1813985280 |                 |                 |
| 55  | D_4   | 87091200  |                 |                 |
| 56  | D_4   | 1990656  |                 |                 |
| 57  | 3A_2  | 58320   |                 |                 |
| 58  | 3A_2  | 15552   |                 |                 |
| 59  | 2A_2  | 606528  |                 |                 |
| 60  | 2A_2  | 186624  | 1               |                 |
| 61  | 2A_2  | 41472   | 1               |                 |
| 62  | 2A_2  | 25920   |                 |                 |
| 63  | 2A_2  | 181444  | 2               |                 |
| 64  | 2A_2  | 181444  | 2               |                 |
| 65  | 2A_2  | 16200   | 4               |                 |
| 66  | A_2   | 2204496 |                 |                 |
| 67  | A_2   | 108864  |                 |                 |
| 68  | A_2   | 3888    |                 |                 |
| 69  | A_2   | 2916    |                 |                 |
| 70  | 0     | 30321676920 | 2               | \( \mathcal{B}W_{32}, \Lambda_{32}^{n} \) |
| 71  | 0     | 15552000 | 5               | \( \Lambda_{32}^{n} \) |
| 72  | 0     | 9289728  | 3               | \( \Lambda_{32} \) |
| 73  | 0     | 1658880  | 1               |                 |
| 74  | 0     | 387072  | 3               |                 |
| 75  | 0     | 29376   | 2               |                 |
| 76  | 0     | 10368   | 1               |                 |
| 77  | 0     | 8064    | 2               |                 |
| 78  | 0     | 5760    | 4               |                 |
| 79  | 0     | 4608    | 2               |                 |
| 80  | 0     | 2592    | 3               |                 |

Then we need to find representatives of all conjugacy classes of elements \( \tau \in U'(\Gamma) \) such that

\[
\tau^2 = -1, \quad \tau \sigma = -\sigma^2 \tau. \tag{7}
\]

This can be shown as in [8] in the case of the Gaussian integers.

Alternatively, one can classify these lattices directly using the neighbor method and a mass formula, which can be derived from the mass formula in [9] as in [5]. The results are contained in [6]. For details on the neighbor method in a quaternionic setting, we refer to [10].

The Eisenstein lattices of rank up to 16 are listed in Tables 1–4 ordered by the number of roots. For the sake of completeness, we have included the results from [2] in rank 4, 8 and 12. \( R \) denotes the root system of the corresponding even unimodular \( \mathbb{Z} \)-lattice (cf. [II, Chapter 4]). In the column \#Aut, the order of the unitary automorphism group is given.

The next column contains the number of lattices of the order \( \mathfrak{M}_{3,\infty} \). For lattices with a structure over the Hurwitz quaternions \( \mathfrak{M}_{2,\infty} \) (note that \((i+j+j)^2 = -3\), so all lattices with a structure over \( \mathfrak{M}_{2,\infty} \) have a structure over \( \mathfrak{M} \)), the name of the corresponding Hurwitz lattice used in [5] is given in the last column.

A list of the Gram matrices of the lattices is given in [12].

Remark 3. We have the following.

(a) The 80 corresponding \( \mathbb{Z} \)-lattices belong to mutually different \( \mathbb{Z} \)-isometry classes.

(b) Each of the lattices listed previously is isometric to its conjugate. Hence the associated Hermitian theta series are symmetric Hermitian modular forms (cf. [1]).

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