The asymptotic expansion of a Mathieu-exponential series

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Abstract

We consider the asymptotic expansion of the functional series

\[ S_{\pm}^\mu(a; \lambda) = \sum_{n=0}^{\infty} \frac{(\pm 1)^n e^{-\lambda n}}{(n^2 + a^2)^\mu} \]

for \( \lambda > 0 \) and \( \mu \geq 0 \) as \( |a| \to \infty \) in the sector \( |\arg a| < \pi/2 \). The approach employed consists of expressing \( S_{\pm}^\mu(a; \lambda) \) as a contour integral combined with suitable deformation of the integration path. Numerical examples are provided to illustrate the accuracy of the various expansions obtained.

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1. Introduction

The functional series

\[ \sum_{n=1}^{\infty} \frac{n}{(n^2 + a^2)^\mu} \]  

(1.1)

is known as a Mathieu series [3], which originally arose (in the case \( \mu = 2 \)) in problems dealing with the elasticity of solid bodies. The asymptotic expansion for large \( a \) of more general functional series of this type has been discussed in [7] and [12]. More recently, Gerhold and Tomovski [1] extended the asymptotic study of (1.1) by introducing the factor \( z^n \), where \( |z| \leq 1 \). From this result they were able to deduce, in particular, the large-\( a \) expansions of the trigonometric Mathieu series

\[ \sum_{n=1}^{\infty} \frac{n \sin nx}{(n^2 + a^2)^\mu}, \quad \sum_{n=1}^{\infty} \frac{n \cos nx}{(n^2 + a^2)^\mu}. \]

Subsequently, the above trigonometric series were generalised in the form [9]

\[ \sum_{n=1}^{\infty} \frac{n^\gamma C_\nu(bn/a)}{(n^2 + a^2)^\mu} \quad (b > 0), \]  

(1.2)
where \( C_\nu \) denotes the oscillatory Bessel functions \( J_\nu(x) \) and \( Y_\nu(x) \) with argument proportional to \( n/a \), and their large-\( a \) asymptotics determined. In addition, this last study also considered the inclusion of the modified Bessel function \( K_\nu(x) \) of similar argument, which contains the decaying exponential as a special case when \( \nu = \frac{1}{2} \), since

\[
K_{1/2}(x) = (\pi/2x)^{1/2}e^{-x}.
\]

In [10], the case where the additional term is a Gaussian exponential, namely the series

\[
\sum_{n=1}^{\infty} \frac{n^\alpha e^{-\lambda n^2/a^2}}{(n^2 + a^2)^\mu} (\lambda > 0),
\]

has been considered. For even integer values of the parameter \( \gamma \) it is found that the large-\( a \) asymptotic expansion consists of an algebraic expansion with a finite number of terms together with a sequence of increasingly subdominant exponentially small contributions. This situation is analogous to the well-known Poisson-Jacobi transformation (corresponding to \( \mu = \gamma = 0 \)) given by [11, p. 124]

\[
\sum_{n=1}^{\infty} e^{-\lambda n^2/a^2} = \frac{a}{2} \sqrt{\pi} - \frac{1}{2} + a \sqrt{\pi} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 a^2 / \lambda}.
\]

The asymptotic expansion we consider here is the alternating Mathieu series coupled with a decaying exponential (depending linearly on the summation index \( n \)) of the form

\[
S_{\mu}^\pm(a; \lambda) := \sum_{n=0}^{\infty} \frac{(\pm 1)^n e^{-\lambda n}}{(a^2 + a^2)^\mu} (\mu \geq 0, \lambda > 0) \tag{1.3}
\]

for \( |a| \to \infty \) in the sector \( |\arg a| < \frac{\pi}{2} \). In [9, §5], the related series with the exponential factor \( e^{-bn/a} \), resulting from [12] with \( C_\nu(ab/a) = K_\nu(bn/a) \) when \( \nu = \gamma = \frac{1}{2} \), was shown to possess the asymptotic expansion

\[
\sum_{n=1}^{\infty} \frac{(\pm 1)^n e^{-bn/a}}{(n^2 + a^2)^\mu} - a^{2\mu-1} \left( \frac{\pi}{2b} \right)^{1/2} \left\{ J_\mu(a; b/a) \begin{array}{c} \frac{1}{e} \\ 0 \end{array} \right\} \sim \sqrt{\frac{2b}{\pi a}} \left( R^+(a; \frac{1}{2}) + R^+(a; -\frac{1}{2}) \right)
\]

as \( |a| \to \infty \) in \( |\arg a| < \frac{\pi}{2} \), where the quantity \( J_\mu(a; b/a) \) is given in (3.1) and

\[
R^\pm(a; w) := \frac{a^{-w-2\mu} b^w}{2^{w+1}} \Gamma(-w) \sum_{k=0}^{\infty} \frac{(-1)^k (\mu)_k}{k! a^{2k}} Z^\pm(-\omega_k) F_k(w, b).
\]

Here \( Z^+(x) := \zeta(x) \), \( Z^-(x) := (1 - 2^{1-x}) \zeta(x) \), \( \omega_k := \frac{1}{2} + w + 2k \), with \( \zeta(s) \) being the Riemann zeta function, and \( F_k(w, b) \) are polynomials expressed as a terminating hypergeometric series defined by

\[
F_k(w, b) := \frac{1}{2^{k+1}} \binom{-w+k}{k} \zeta(-k) - \frac{1}{2^{k+1}} \binom{-w+k}{k} \zeta(-k) - b^2 / 4.
\]

However, if we set \( b = \lambda a \), where \( \lambda > 0 \) is finite, to obtain a series equivalent to that in (1.3), it is seen that the polynomials \( F_k(w, b) = F_k(w, \lambda a) \) with the consequence that the formal series \( R(a; \pm 1/2) \) lose their asymptotic character.
Rather than adopting a Mellin transform approach used in [9], we express $S^\pm_\mu(a; \lambda)$ as a contour integral combined with suitable integration path deformation. Such an approach has been employed by Olver in his well-known book [5, p. 303], who showed that in the particular case $\lambda = 0$

$$
\sum_{n=0}^{\infty} \frac{(-1)^n}{(n^2 + a^2)^\mu} = \frac{1}{2a^{2\mu}} + \frac{2^{\frac{1}{2} - \mu}}{a^{2\mu-1}} \sum_{k=0}^{\infty} K_{\frac{1}{2} - \mu}((2k + 1)\pi a)\frac{1}{((2k + 1)\pi a)^{\frac{1}{2} - \mu}}
$$

(1.4)

for $\mu > 0$. For large values of $a$ in $|\arg a| < \frac{1}{2}\pi$, the sum of Bessel functions decays exponentially fast. Series of the form (1.4) have arisen in aerodynamic interference calculations; see [4].

We shall find that when $\lambda > 0$ there is an analogous sum of $K$-Bessel functions (with complex argument) together with an additional term possessing an algebraic asymptotic expansion when $a$ is large.

We note at this point the special evaluations for non-negative integer $\mu$ given by

$$
S^+_0(a; \lambda) = \frac{e^\lambda}{e^\lambda + 1}
$$

(1.5)

and

$$
S^+_1(a; \lambda) = \frac{F_1}{2a^2}, \quad S^+_2(a; \lambda) = \frac{1}{4a^2}(F_1 + F_2),
$$

$$
S^+_3(a; \lambda) = \frac{1}{16a^6}(3F_1 + 3F_2 + 2F_3),
$$

$$
S^+_4(a; \lambda) = \frac{1}{32a^8}(5F_1 + 5F_2 + 4F_3 + 2F_4),
$$

$$
S^+_5(a; \lambda) = \frac{1}{256a^{10}}(35F_1 + 35F_2 + 30F_3 + 20F_4 + 8F_5),\ldots,
$$

(1.6)

where

$$
F_n = F_n(a) + F_n(-a), \quad F_n(a) = n+1F_n\left(\begin{array}{c}1, ia, \ldots, ia \\ 1 + ia, \ldots, 1 + ia\end{array} ; \pm e^{-\lambda}\right)
$$

and $n+1F_n$ is the generalised hypergeometric function.

### 2. Derivation of the expansion for $S^-_\mu(a; \lambda)$

We first consider the sum $S^-_\mu(a; \lambda)$. Following [5] p. 303], we have

$$
S^-_\mu(a; \lambda) = \frac{1}{2\pi} \int_{\infty}^{(0+)} e^{-\lambda t} \frac{dt}{\sin \pi t (a^2 + t^2)^\mu} \quad (|\arg a| < \frac{1}{2}\pi),
$$

(2.1)

where the integration path encloses only the poles of the integrand situated at $t = 0, 1, 2, \ldots$. Let us first consider the case $a > 0$. The integrand then has branch points at $t = \pm ia$ with cuts along the imaginary axis emanating from these points and passing to infinity. Since $\lambda > 0$, the loop contour may be deformed to coincide with the imaginary axis between $[-i(a-\rho), i(a-\rho)]$ (with an indentation of radius $\rho$ around the origin on the left-hand side) together with the portions of the imaginary axis situated above $ia$ and below $-ia$ on the right-hand side of the cuts; see Fig. 1. Provided $\mu < 1$, the contribution from the indentations of radius $\rho$ round the branch points vanishes as $\rho \to 0$. 

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2.1 The contribution between the branch points

We first deal with the integral between the branch points which we denote by $I_1$. We have

$$I_1 = -\frac{a^{1-2\mu}}{2\pi} \int_{-1+\rho}^{1-\rho} \frac{e^{-\lambda at}}{\sinh \pi at} \frac{dt}{(1-t^2)^\mu},$$

where the integration path is indented to the left at the origin, so that

$$I_1 = \frac{1}{2a^{2\mu}} + H^{-}_\mu(a; \lambda), \quad H^{-}_\mu(a; \lambda) := a^{1-2\mu} \int_0^1 \frac{\sin \lambda at}{\sinh \pi at} \frac{dt}{(1-t^2)^\mu}$$

when $\rho \to 0$. Now

$$\frac{\sin \lambda x}{\sinh \pi x} = \frac{\lambda}{\pi} \sum_{k=0}^\infty (-1)^k A_k x^{2k} \quad (|x| < \infty),$$

where

$$A_0 = 1, \quad A_1 = \frac{1}{6}(\lambda^2 + \pi^2), \quad A_2 = \frac{1}{360}(3\lambda^4 + 10\lambda^2\pi^2 + 7\pi^4),$$

$$A_3 = \frac{1}{15120}(3\lambda^6 + 21\lambda^4\pi^2 + 49\lambda^2\pi^4 - 31\pi^6),$$

$$A_4 = \frac{1}{1814400}(5\lambda^8 + 60\lambda^6\pi^2 + 294\lambda^4\pi^4 + 620\lambda^2\pi^6 + 381\pi^8), \ldots.$$

Then we find

$$H^{-}_\mu(a; \lambda) = \frac{\lambda a^{1-2\mu}}{\pi} \sum_{k=0}^\infty (-1)^k A_k a^{2k} \int_0^1 \frac{t^{2k}}{(1-t^2)^\mu} dt.$$
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\[ a^{1-2\mu} \Gamma(1-\mu) \sum_{k=0}^{\infty} (-1)^k A_k \Gamma(k+\frac{1}{2}) \Gamma(k+\frac{3}{2}-\mu) a^{2k} \quad (\mu < 1). \]  

(2.3)

This form of expansion is suitable for computation when \( a \) is not large.

To deal with the situation when \( a \to \infty \), we expand the factor \((1-t^2)^{-\mu}\) appearing in (2.2) by the binomial theorem to obtain

\[
H_{-\mu}(a; \lambda) = a^{1-2\mu} \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \int_0^1 t^{2k} \frac{\sin \lambda t}{\sinh \pi a} dt.
\]

Since \( \sinh \pi a t \) decays exponentially for \( t \) bounded away from the origin, we can replace the upper limit 1 by \( \infty \), thereby introducing an error of \( O(e^{-\pi a}) \). From the result [2, (3.524.1)], we have

\[
B_k := \int_0^\infty \tau^{2k} \frac{\sin \lambda \tau}{\sinh \pi \tau} d\tau = \frac{i(2k)!}{(2\pi)^{2k+1}} \left\{ \zeta(2k+1, \frac{1}{2} + \frac{i\lambda}{2\pi}) - \zeta(2k+1, \frac{1}{2} - \frac{i\lambda}{2\pi}) \right\}
\]

where \( \zeta(a, s) = \sum_{n=0}^{\infty} (n+a)^{-s} \) is the Hurwitz zeta function. Using the fact that \( \zeta(2k+1, z) = -\psi^{(2k)}(z)/(2k)! \), where \( \psi(z) \) is the logarithmic derivative of the gamma function, and [6, (5.15.6)]

\[
\psi^{(2k)}(1-z) - \psi^{(2k)}(z) = \pi \left( \frac{d}{dz} \right)^{2k} \cot \pi z,
\]

we find that

\[
B_k = \frac{i}{(2\pi)^{2k+1}} \left\{ \psi^{(2k)}(1/2) - \psi^{(2k)}(1/2 + i\lambda/2\pi) - \psi^{(2k)}(1/2 + i\lambda/2\pi) - \psi^{(2k)}(1/2) \right\} = (-1)^k \frac{1}{2^{2k+1}} \left( \frac{d}{dx} \right)^{2k} \tanh x \bigg|_{x=\lambda/2}.
\]

The first few coefficients \( B_k \) are given by:

\[
B_0 = \frac{1}{2} \tanh x, \quad B_1 = \frac{\sinh x}{4 \cosh^4 x}, \quad B_2 = \frac{\sinh x}{4 \cosh^6 x} (2 - \sinh^2 x),
\]

\[
B_3 = \frac{\sinh x}{8 \cosh^8 x} (17 - 26 \sinh^2 x + 2 \sinh^4 x),
\]

\[
B_4 = \frac{\sinh x}{4 \cosh^{10} x} (62 - 192 \sinh^2 x + 60 \sinh^4 x - \sinh^6 x), \ldots,
\]

where \( x = \lambda/2 \). Hence we obtain the expansion

\[
H_{-\mu}(a; \lambda) \sim a^{-2\mu} \sum_{k=0}^{\infty} \frac{(\mu)_k B_k}{k! a^{2k}}
\]

as \( a \to +\infty \).

2.2 The contribution beyond the branch points

Consider the contribution to \( S_{-\mu}(a; \lambda) \) resulting from the part of the integration path situated in the interval \([ia, i\infty)\). We have

\[
I_2 = -\frac{1}{2i} \int_{ia}^{i\infty} e^{-\lambda t} \frac{dt}{\sin \pi t (a^2 + t^2)^\mu} = -\frac{a^{1-2\mu} e^{-\pi i \mu}}{2i} \int_{1}^{\infty} e^{-\lambda i a t} \frac{dt}{\sinh \pi a t (t^2 - 1)^\mu}.
\]
Now on $t \in [1, \infty)$
\[
\cosec \pi at = 2e^{-\pi at} - e^{-2\pi at})^{-1} = 2 \sum_{k=0}^{\infty} e^{-(2k+1)\pi at},
\]
so that substitution in $I_2$ followed by term-by-term integration yields
\[
I_2 = ie^{-\pi \mu} a^{1-2\mu} \sum_{k=0}^{\infty} \int_{1}^{\infty} \frac{e^{-X_k t}}{(t^2 - 1)^{\mu}} dt, \quad X_k := (2k + 1)\pi a + \lambda ia
\]
where the integral has been evaluated in terms of a modified Bessel function by [5, p. 254] when $\mu < 1$.

The contribution from the path $[-ia, -i\infty)$ is the conjugate of the above expression. Hence we find
\[
I_2 + \mathcal{I}_2 = T_\mu(a; \lambda) := \frac{2^{\frac{3}{2} - \mu} \sqrt{\pi}}{a^{2\mu - 1} \Gamma(\mu)} \sum_{k=0}^{\infty} \frac{\sin(\pi \mu - \theta_k)}{\sin \pi \mu} \left| K_{\frac{1}{2} - \mu}(X_k) \right|,
\]
where
\[
\theta_k := \arg \left( X_k^{\mu - \frac{1}{2}} K_{\frac{1}{2} - \mu}(X_k) \right).
\]

Collecting together the results in (2.2) and (2.7), we finally obtain the expansion
\[
S^- \mu (a; \lambda) = \frac{1}{2a^{2\mu}} + H^- \mu (a; \lambda) + \frac{2^{\frac{3}{2} - \mu} \sqrt{\pi}}{a^{2\mu - 1} \Gamma(\mu)} \sum_{k=0}^{\infty} \frac{\sin(\pi \mu - \theta_k)}{\sin \pi \mu} \left| K_{\frac{1}{2} - \mu}(X_k) \right|,
\]
where $X_k$ is defined in (2.6). When $\lambda = 0$, we have $H^- \mu (a; \lambda) \equiv 0$, $\theta_k \equiv 0$ and (2.8) reduces to Olver’s expansion stated in (1.3). However, the result (2.8) has been established provided $\mu < 1$. It does not appear possible to extend the validity of (2.8) to $\mu \geq 1$ by analytic continuation, as was the case in Olver’s treatment when $\lambda = 0$. When $a$ is large, $S^- \mu (a; \lambda)$ is seen to consist of an algebraic expansion given in (2.4) together with an exponentially small contribution from the infinite sum of modified Bessel functions.

When $a$ is allowed to take on complex values in the sector $|\arg a| < \frac{1}{2} \pi$, the branch points $\pm ia$ move off the imaginary axis to the points $ae^{(\phi \pm \frac{1}{2} \pi)i}$, where $\phi = \arg a$, as indicated in [5, p. 303]. The analysis in this case follows that given above and we conclude that the expansion (2.8) holds for $|a| \to \infty$ in the sector $|\arg a| < \frac{1}{2} \pi$.

3. Derivation of the expansion for $S^+ \mu (a; \lambda)$

Since the residue of $\cot \pi t$ at any integer $n$ is $1/\pi$, we have
\[
S^+ \mu (a; \lambda) = \sum_{n=0}^{\infty} \frac{e^{-\lambda n}}{(n^2 + a^2)^{\mu}} = \frac{1}{2i} \int_{\infty}^{(0+)} \cot \pi t e^{-\lambda t} (t^2 + a^2)^{\mu} dt.
\]
where the path encloses only the poles at $t = 0, 1, 2, \ldots$. The integration path can be deformed as illustrated in Fig. 1; the upper and lower parts of this path are denoted by $C_1$ and $C_2$, respectively. We have (see [5, p. 305])

$$
\cot \frac{\pi t}{2i} = \begin{cases} 
-\frac{1}{2} + \frac{e^{\pi it}}{2i \sin \pi t} & \text{(on } C_1) \\
\frac{1}{2} + \frac{e^{-\pi it}}{2i \sin \pi t} & \text{(on } C_2),
\end{cases}
$$

so that

$$
S^+_\mu(a; \lambda) = J_\mu(a; \lambda) - \frac{1}{2i} \int_{C_1} \frac{e^{-\lambda t + \pi it}}{\sin \pi t (t^2 + a^2)^\mu} dt + \frac{1}{2i} \int_{C_2} \frac{e^{-\lambda t - \pi it}}{\sin \pi t (t^2 + a^2)^\mu} dt.
$$

The integral $J_\mu(a; \lambda)$ is given by

$$
J_\mu(a; \lambda) := \int_0^\infty \frac{e^{-\lambda t}}{(t^2 + a^2)^\mu} dt = \sqrt{\pi} a^{1-2\mu} \Gamma(1-\mu) \frac{2}{(\frac{1}{2} \lambda a)^{\frac{1}{2}-\mu}} K_{\frac{1}{2}-\mu}(\lambda a),
$$

where $K_\nu(x) := H_\nu(x) - Y_\nu(x)$ is the Struve function defined in [6, (11.5.2)].

The contribution between the branch points $\pm ia$ is

$$
\frac{1}{2a^{2\mu}} + H^+_\mu(a; \lambda),
$$

where

$$
H^+_\mu(a; \lambda) = a^{1-2\mu} \int_0^1 e^{-\pi au} \frac{\sin \lambda au}{\sinh \pi au} \frac{du}{(1-u^2)^\mu}.
$$

This integral may be evaluated as in Section 2 by expanding $\sin \lambda au / \sinh \pi au$. However the resulting integrals are expressible in terms of two $2F_2$ hypergeometric functions and so will not be presented here. Our main interest is the estimation of $H^+_\mu(a; \lambda)$ for large $a$. Proceeding as in Section 2 we find that

$$
H^+_\mu(a; \lambda) \sim a^{1-2\mu} \sum_{k=0}^\infty \frac{(\mu)_k}{k!} \int_0^\infty e^{-\pi au} \frac{\sin \lambda au}{\sinh \pi au} u^{2k} du
$$

$$
= a^{1-2\mu} \sum_{k=0}^\infty \frac{(\mu)_k}{k!(\pi a)^{2k+1}} \int_0^\infty e^{-aw} \frac{\sin \lambda w/\pi}{\sinh w} w^{2k} dw.
$$

For $k \geq 1$, we have

$$
\int_0^\infty \frac{e^{-w(1 \pm i\lambda/\pi)}}{\sinh w} w^{2k} dw = \frac{-i(2k)!}{2^{2k+1}} \zeta(2k + 1, 1 \pm \frac{i\lambda}{2\pi})
$$

upon use of the result [2, (3.552.1)]

$$
\int_0^\infty \frac{x^{\alpha-1} e^{-\beta x}}{\sinh x} dx = 2^{1-\alpha} \Gamma(\alpha) \zeta(2k + 1, \frac{1}{2} + \frac{1}{2} \beta)
$$
valid for $\Re(\alpha) > 1$, $\Re(\beta) > -1$. Then, we define the coefficients $\hat{B}_k$ by

$$\hat{B}_k := \frac{i(-1)^{k-1}(2k)!}{(2\pi)^{2k+1}} \left\{ \frac{\zeta(2k+1, 1 - i\lambda/2\pi)}{2\pi} - \frac{\zeta(2k+1, 1 + i\lambda/2\pi)}{2\pi} \right\}$$

$$= \frac{i(-1)^{k-1}}{(2\pi)^{2k+1}} \left\{ \psi^{(2k)}(1 + i\lambda/2\pi) - \psi^{(2k)}(1 - i\lambda/2\pi) - \frac{i(-1)^k(2k)!}{(\lambda/2\pi)^{2k+1}} \right\},$$

where we have employed the result $\psi^{(2k)}(1 + z) = \psi^{(2k)}(z) + (2k)!/z^{2k+1}$ [6 (15.5.5)]. Application of (2.2) then yields

$$\hat{B}_k = 2^{-2k-1} \left\{ \left( \frac{d}{dx} \right)^{2k} \coth x - \frac{(2k)!}{x^{2k+1}} \right\}_{x = \lambda/2}.$$ 

The first few coefficients are given by:

$$\hat{B}_0 = \frac{1}{2} \left( \coth x - \frac{1}{x} \right), \quad \hat{B}_1 = \frac{1}{4} \left( \frac{\cosh x}{\sinh^3 x} - \frac{1}{x^3} \right), \quad \hat{B}_2 = \frac{1}{4} \left( \frac{\cosh x}{\sinh^5 x} \left( 2 + \cosh^2 x \right) - \frac{3}{x^5} \right),$$

$$\hat{B}_3 = \frac{1}{8} \left( \frac{\cosh x}{\sinh^7 x} \left( 17 + 26 \cosh^2 x + 2 \cosh^4 x \right) - \frac{45}{x^7} \right),$$

$$\hat{B}_4 = \frac{1}{4} \left( \frac{\cosh x}{\sinh^9 x} \left( 62 + 192 \cosh^2 x + 60 \cosh^4 x + \cosh^6 x \right) - \frac{315}{x^9} \right), \ldots ,$$

where $x = \lambda/2$. Then we have the expansion

$$H^+_\mu(a; \lambda) \sim a^{-2\mu} \sum_{k=0}^{\infty} \frac{(-1)^k(\mu)\hat{B}_k}{k! a^{2k}}$$

as $a \to \infty$.

Finally, the contribution from $[ia, \infty i]$ is

$$I_2 = -e^{-\pi i\mu} a^{1-2\mu} \int_1^\infty \frac{e^{-(\pi + i\lambda)at}}{\sinh \pi at} \frac{dt}{(t^2 - 1)^\mu} = i e^{-\pi i\mu} a^{1-2\mu} \sum_{k=0}^{\infty} \int_1^\infty \frac{e^{-\hat{X}_k t}}{(t^2 - 1)^\mu} dt$$

$$= \frac{i\pi e^{-\pi i\mu}}{(2a^2)^{1-\mu} \Gamma(\mu) \sin \pi \mu} \sum_{k=0}^{\infty} \frac{K_{\frac{1}{2}\mu}(\hat{X}_k)}{\hat{X}_k^{\frac{\mu}{2} - \mu}},$$

where

$$\hat{X}_k := (2k + 2)\pi a + \lambda ia.$$  (3.3)

The contribution from the path $[-ia, -i\infty]$ is the conjugate of the above expression. Hence we find

$$I_2 + I_2^* = \hat{T}_{\mu}(a; \lambda) := \frac{\sqrt{\pi}}{a^{2-\mu} \Gamma(\mu)} \sum_{k=0}^{\infty} \frac{\sin(\pi \mu - \hat{\theta}_k)}{\sin \pi \mu} \left| \frac{K_{\frac{1}{2}\mu}(\hat{X}_k)}{(\hat{X}_k)^{\frac{\mu}{2} - \mu}} \right|,$$

where

$$\hat{\theta}_k := \arg \left( \hat{X}_k^{\frac{\mu}{2} - \mu} K_{\frac{1}{2}\mu}(\hat{X}_k) \right).$$  (3.4)
Collecting together the results in (3.1) – (3.4), we finally obtain the expansion

$$S_\mu^+(a; \lambda) = \frac{1}{2a^{2\mu}} + J_\mu(a; \lambda) + H_\mu^+(a; \lambda) + \frac{2^{\frac{3}{2} - \mu} \sqrt{\pi}}{a^{2\mu-1} \Gamma(\mu)} \sum_{k=0}^{\infty} \frac{\sin(\pi \mu - \hat{\theta}_k)}{\sin \pi \mu} \left| \frac{K_{\frac{1}{2} - \mu}(\hat{X}_k)}{(X_k)^{\frac{3}{2} - \mu}} \right|$$

valid when $\mu < 1$, where $\hat{X}_k$ is defined in (3.3). The exponentially small sum of Bessel functions is seen to be $O(a^{1-2\mu} e^{-2\pi a})$ for large $a$, which is smaller than the equivalent sum arising in $S_\mu^-(a; \lambda)$ of $O(a^{1-2\mu} e^{-\pi a})$.

When $a$ is complex satisfying $| \arg a | < \frac{1}{2} \pi$, the analysis follows the same procedure and we conclude that the expansion (3.5) holds for $|a| \to \infty$ in the sector $| \arg a | < \frac{1}{2} \pi$.

4. Numerical verification

To demonstrate the validity of the sum $T_\mu(a; \lambda)$ involving the modified Bessel functions appearing in (2.8), we define

$$S := S_\mu^-(a; \lambda) - H_\mu^-(a; \lambda) - \frac{1}{2a^{2\mu}},$$

where the contribution $H_\mu^-(a; \lambda)$ is either computed for small $a$ by (2.3), or for larger $a$ from (2.2) using high-precision numerical evaluation of the integral. The value of $S$ so obtained is then compared to $T_\mu(a; \lambda)$ defined in (2.7). For example, when $a = 3$, $\lambda = 1$ and $\mu = \frac{1}{3}$, we find $S \approx -6.35783 \times 10^5$ with $T_\mu(a; \lambda) \approx -6.35783 \times 10^5$ in exact agreement at this level of precision.

In Table 1 we show the absolute relative error in the expansion of $S_\mu^-(a; \lambda)$ based on the algebraic component

$$S_\mu^-(a; \lambda) \sim \frac{1}{2a^{2\mu}} + H_\mu^-(a; \lambda)$$

for different values of $a$ and truncation index $k$ in the expansion of $H_\mu^-(a; \lambda)$ in (2.5). The final row shows the value of $S_\mu^-(a; \lambda)$ obtained by high-precision evaluation of (1.3). Table 2 shows the absolute relative error for complex $a$ when $a = 6e^{i\phi}$ for different $\phi$, $\mu$ and $\lambda$.

The algebraic component of $S_\mu^+(a; \lambda)$ is given by

$$S_\mu^+(a; \lambda) \sim \frac{1}{2a^{2\mu}} + J_\mu(a; \lambda) + H_\mu^+(a; \lambda),$$

where the expansion of $H_\mu^+(a; \lambda)$ is given by (3.2). The expansion of $J_\mu(a; \lambda)$ can be obtained from that of $K_\nu(z)$ given in [6] (11.6.1) to yield the asymptotic expansion

$$J_\mu(a; \lambda) \sim \frac{a^{1-2\mu}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})_k (\mu)_k}{(\frac{a}{\lambda})^{2k+1}}$$

for $|a| \to \infty$ in $| \arg a | < \frac{1}{2} \pi$ (with $\lambda$ bounded away from zero). The absolute relative error in the expansion of $S_\mu^+(a; \lambda)$ for different values of $a$ and common truncation index $k$ in the expansions of $H_\mu^+(a; \lambda)$ and $J_\mu(a; \lambda)$ are presented in Table 3, where the final row shows the value of $S_\mu^+(a; \lambda)$. 
It is not obvious how this result can be analytically continued into \( \mu \). One approach might be to exploit the fact that

\[
\lambda = 0, \quad \text{our result in the case of } S^+_{\mu}(a; \lambda) \text{ reduces to the expression given by Olver stated in (1.4)}.
\]

A problem with the representation (2.8) is that it has been derived only when \( 0 \leq \mu < 1 \). It is not obvious how this result can be analytically continued into \( \mu \geq 1 \), as is the case in (1.3). One approach might be to exploit the fact that

\[
S_{\mu+1}(a; \lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-\lambda n}}{(n^2 + a^2)^{\mu+1}} = -\frac{1}{2\mu a} \frac{\partial}{\partial a} S^-_{\mu}(a; \lambda) \quad (0 < \mu < 1).
\]

Differentiation of the right-hand side of (2.8) using the series expansion (2.3) for \( H^-_{\mu}(a; \lambda) \) and the properties of the \( K \)-Bessel function shows that its value is given by (2.8) with \( \mu \) replaced by \( \mu + 1 \). This enables us to extend the representation to \( 1 < \mu < 2 \), and by continuation to higher ranges of \( \mu \). The representations when \( \mu = 1, 2, \ldots, 5 \) are displayed in (1.3). Thus,
Table 3: The absolute relative error in the computation of $S^+_\mu(a;\lambda)$ from (4.2) for different $a$ and truncation index $k$ in the asymptotic expansions of $H^+_\mu(a;\lambda)$ and $J_\mu(a;\lambda)$ when $\lambda = 1$ and $\mu = \frac{1}{4}$.

| $k$ | $a = 10$       | $a = 15$       | $a = 20$       |
|-----|----------------|----------------|----------------|
| 0   | $2.959 \times 10^{-3}$ | $1.358 \times 10^{-3}$ | $7.736 \times 10^{-4}$ |
| 1   | $1.991 \times 10^{-4}$ | $4.293 \times 10^{-5}$ | $1.408 \times 10^{-5}$ |
| 2   | $3.864 \times 10^{-5}$ | $3.962 \times 10^{-6}$ | $7.525 \times 10^{-7}$ |
| 3   | $1.485 \times 10^{-5}$ | $7.268 \times 10^{-7}$ | $8.054 \times 10^{-8}$ |
| 4   | $9.491 \times 10^{-6}$ | $2.214 \times 10^{-7}$ | $1.433 \times 10^{-8}$ |
| 5   | $9.129 \times 10^{-6}$ | $1.010 \times 10^{-7}$ | $3.817 \times 10^{-9}$ |

$S^+_\mu(a;\lambda)$

| $k$ | $a = 10$       | $a = 15$       | $a = 20$       |
|-----|----------------|----------------|----------------|
| 0   | $4.98789 \times 10^{-1}$ | $4.07911 \times 10^{-1}$ | $3.53467 \times 10^{-1}$ |

The algebraic part of the expansion is given by

$$S^-_\mu(a;\lambda) \sim \frac{1}{2a^{2\mu}} + H^-_\mu(a;\lambda) \sim \frac{1}{2a^{2\mu}} + \frac{1}{a^{2\mu}} \sum_{k=0}^{\infty} \frac{(\mu)_k B_k}{k! a^{2k}} \quad (\mu \geq 0)$$

as $|a| \to \infty$ in $|\arg a| < \frac{1}{2}\pi$. When $\mu = 0$, we note that the right-hand side of the above expression reduces to $e^\lambda/(1 + e^\lambda)$ as stated in [12]. Similar considerations can be brought to bear on the sum $S^+_\mu(a;\lambda)$.

Finally, we observe that when $\lambda = 0$, the quantity $J_\mu(a;\lambda)$ defined in (3.1) reduces to

$$J_\mu(a;0) = \frac{\sqrt{\pi} \Gamma\left(\mu - \frac{1}{2}\right)}{2a^{2\mu-1}\Gamma(\mu)}.$$

Since $H^+_\mu(a;0) \equiv 0$, we consequently find that

$$\sum_{n=0}^{\infty} e^{-\lambda n} \frac{e^{-\lambda n}}{(n^2 + a^2)\mu} = \frac{1}{2a^{2\mu}} + \frac{\sqrt{\pi} \Gamma\left(\mu - \frac{1}{2}\right)}{2a^{2\mu-1}\Gamma(\mu)} + \frac{2^{\frac{3}{2} - \mu}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{K_{\frac{1}{2} - \mu}}{(2k + 2)\pi a} \frac{((2k + 2)\pi a)^{\frac{1}{2} - \mu}}{a^{2\mu-1}\Gamma(\mu)}$$

for $\mu > 0$ and $|\arg a| < \frac{1}{2}\pi$, which complements Olver’s result in [14].

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