ON THE FRACTAL DIMENSION OF THE DUFFING ATTRACTOR

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ABSTRACT

The box counting dimension and the correlation dimension change with the number of numerically generated points forming the attractor. At a sufficiently large number of points the fractal dimension tends to a finite value.

Subject headings: Duffing attractor, fractal dimension: box counting dimension — correlation dimension

1. INTRODUCTION

The Duffing pendulum is a kind of a forced pendulum with damping and is governed by a nonlinear equation of the form

\[ \ddot{x} + \delta \dot{x} + \beta x + \alpha x^3 = \gamma \cos(\omega_D t), \]

where \( \delta, \beta, \alpha, \gamma, \omega_D \) are constant parameters. The famous Duffing attractor is visible in a Poincaré surface of section presented in Fig. 1. It is a fine example of a so called strange attractor, due to its fractal structure. Indeed, the Lyapunov spectrum indicates that the phase space volume decreases to a zero Lebesgue measure set (Baker & Gollub 1996; Morbidelli 2002). Also, the attractor is said to be self-similar, i.e. having a fractal structure (see Appendix A). It is a common procedure to verify the fractality of a set by estimating the Hausdorff (fractal) dimension (Ott 2002). The fractal dimension gives information about how much of the space (e.g. the phase space or the space of stroboscopic variables) is covered by the considered set. For instance, a set with a fractal dimension of 1.5 covers the space more densely than an analytical line but not as densely as a regular two-dimensional geometrical figure.

An ideal fractal should be formed of an infinite amount of points, although this is impossible to achieve in numerical computations. Also, the mathematical definition of the Hausdorff dimension does not provide a useful method for calculation. The most commonly used estimates are the box counting dimension and the correlation dimension (Theiler 1990). It is obvious that if the examined set is formed of too few points, the fractal properties could not become apparent. So the set has to be consisted of a large enough number of points. Thus the fractal dimension depends on the amount of points.

This paper is organised in the following manner. Sec. 2 briefly presents numerical methods used for estimating the box counting and correlation dimensions. In Sec. 3 the results of estimating these dimensions for a numerically generated Duffing attractor are presented. In Sec. 4 concluding remarks are given.

2. METHOD

The fractal dimension is estimated in two ways: through the box counting and the correlation dimension. The box counting dimension is defined as

\[ d_C = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}, \]

where \( N(\varepsilon) \) is the number of non-empty boxes (squares) of the size \( \varepsilon \). This dimension is calculated using a computer algebra system Mathematica and the BoxCount function written by Pasquale Nardone (1995).

The correlation dimension is defined as

\[ d_G = \lim_{r \to 0} \frac{\ln C(r)}{\ln r}, \]

with the estimate for the correlation function as

\[ C(r) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} H(r - ||x_i - x_j||), \]

where the Heaviside step function \( H \) adds to \( C(r) \) only points \( x_i \) in a distance smaller than \( r \) from \( x_j \) and vice versa. The total number of points is denoted by \( N \) here. In the following calculations \( N \) is a finite value so the limit in Eq. (3) is omitted. The correlation sum is calculated using a parallel Python program. Both limits in Eq. (2) and (3) are attained by using several small values of \( \varepsilon \) and \( r \) respectively and fitting a straight line to the obtained dependencies. The fractal dimension is estimated as the slope of the linear regression in both cases.

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3. RESULTS

Eq. (1) was integrated using Mathematica’s default method. Although the so called Clean Numerical Simulation (CNS) was recently developed (see Liao 2009, 2013; Liao & Wang 2013 and references therein) in order to avoid truncation and round-off errors in a long time integration interval, herein obtained points forming the Duffing attractor are meant initially to form a set in stroboscopic variables. As is seen in Fig. 1 and A1, A2 in Appendix A the attractor remains stable with the rise of N. Therefore it is claimed that for the purpose of this paper the CNS method is not necessary.

TABLE 1
Fractal Dimensions. The Error in these Quantities, Estimated via the Standard Deviation of the Linear Regressions’ Slope, is Not Greater than 2%.

| Length of time series | Number of points forming the attractor | Box counting dimension | Correlation dimension |
|-----------------------|---------------------------------------|------------------------|-----------------------|
| 1. $1 \cdot 10^1$    | 160                                   | 1.25164                | 1.45068               |
| 2. $1 \cdot 10^4$    | 1592                                  | 1.33008                | 1.35716               |
| 3. $5 \cdot 10^4$    | 7958                                  | 1.38010                | 1.39567               |
| 4. $1 \cdot 10^5$    | 15 916                                | 1.39157                | 1.40250               |
| 5. $2 \cdot 10^5$    | 31 831                                | 1.39461                | 1.38567               |
| 6. $4 \cdot 10^5$    | 63 662                                | 1.41441                | 1.37372               |
| 7. $6 \cdot 10^5$    | 95 493                                | 1.42120                | 1.37633               |
| 8. $8 \cdot 10^5$    | 127 324                               | 1.41623                | 1.37722               |
| 9. $1 \cdot 10^6$    | 150 155                               | 1.42120                | 1.37764               |
| 10. $1.5 \cdot 10^6$ | 238 733                               | 1.42848                | 1.37895               |
| 11. $1.8 \cdot 10^6$ | 286 478                               | 1.42515                | —                     |

The following parameters were used:

$$\{\alpha, \beta, \delta, \gamma, \omega_D\} = \{1., -1., 0.2, 0.3, 1.\}. \quad (5)$$

The initial conditions were preselected to be $$(x_0, \dot{x}_0) = (1., 1.)$$. The attractor was formed by taking the values $$(x, \dot{x})$$ in stroboscopic variables with a step equal to $$2\pi$$ due to the forcing frequency $$\omega_D = 1$$. The maximum length of the time series was equal to $$1.8 \cdot 10^6$$ and lead to the attractor shown in Fig. 1. Table 1 presents all lengths of obtained time series and the corresponding box counting dimension $d_C$ and the correlation dimension $d_G$. Note this is a semi-log plot.

4. CONCLUSIONS

The fractal dimension was estimated for the Duffing attractor via the box counting and the correlation dimension. The values obtained are dependent on the number of points forming the examined set. In the box counting procedure one covers the space with boxes of a given size and counts how many of them are filled with points. This method relies only on global distribution of the points. On the other hand, the correlation dimension takes into account the local point density. Therefore, these two estimates can be expected to behave differently with the number of points varied.

The box counting dimension, in general, continually rose with the number $N$, although the bigger the $N$, the slower the rise was. It can be predicted that the $d_C$ value would finally reach its limit for $N$ large enough. The maximum $N = 286 478$ appears to be a value large enough to give a reliable estimate of $d_C \approx 1.43$.

The rise of $d_C$ in Fig. 2 reaches its local maximum at $N = 95 493$, followed by an oscillatory behavior, after which one can observe a decline at the maximum used value of $N = 286 478$. This is due to the arbitrariness of the choice which part of the log $N(\varepsilon)$ vs. log$(1/\varepsilon)$ plot was linear (see Appendix B). Including or excluding one point from each fitting the monotonical behavior may be retained, although the linear regressions herein were performed so that the standard deviation of the slope was minimal. On the other hand, the relative difference in this case is less than 1%.

The correlation dimension even for an extremely small $N = 160$ gave a value not greater than a few percent than the final one obtained for $N = 238 214$, which was $d_G \approx 1.38$. What is a significant observation for numerical computations is that the $d_G$ for relatively small $N$ (starting herein from $N = 1592$) does not differ much from its final value. After $N = 63662$ the relative changes in the $d_G$ value are not significant. Although the correlation dimension does not act monotonically on $N$ (as does not the box counting dimension), it appears to manifest some oscillating behavior for small $N$ and tends to a limit value for higher numbers of points. Therefore the final value of $d_G \approx 1.38$ is a good estimate for the fractal dimension.

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APPENDIX

A. STRUCTURE OF THE DUFFING ATTRACTION

Fig. A1 and A2 reveal the self-similar property of the Duffing attractor. This is a fundamental feature of all fractals and justifies the estimation of the fractal dimension in this paper.

![Fig. A1. Magnification of a part of Fig. 1 with a self-similar structure.](image1)

![Fig. A2. A further magnification of the Duffing attractor.](image2)

while the blue one’s slope is $1.28012 \pm 0.03560$, which was obtained by taking into account one point more than for the red line. This point is indicated by an arrow. The relative increase of the standard deviation is 21%, which is a significant value, although the relative decrease of the fractal dimension is only 4%. This means that within the error both values are equally reliable, however the criterion that was used to estimate the fractal dimension for each $N$ was so that the standard deviation was minimal.

![Fig. B1. Linear regressions for the attractor formed of $N = 1592$ points that lead to the box counting dimension value $d_C = 1.33008$.](image3)

All linear regressions, starting from $N = 1592$, conducted in order to estimate the correlation dimension, had a form presented in Fig. B2.

![Fig. B2. The linear regression that was used to obtain the $d_G$ value for $N = 1592$.](image4)

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