ON THE NATURE OF THE COSMOLOGICAL CONSTANT PROBLEM

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June 11, 2009

Abstract

General relativity postulates the Minkowski space-time to be the standard flat geometry against which we compare all curved space-times and the gravitational ground state where particles, quantum fields and their vacuum states are primarily conceived. On the other hand, experimental evidences show that there exists a non-zero cosmological constant, which implies in a deSitter space-time, not compatible with the assumed Minkowski structure. Such inconsistency is shown to be a consequence of the lack of a application independent curvature standard in Riemann’s geometry, leading eventually to the cosmological constant problem in general relativity.

We show how the curvature standard in Riemann’s geometry can be fixed by Nash’s theorem on locally embedded Riemannian geometries, which imply in the existence of extra dimensions. The resulting gravitational theory is more general than general relativity, similar to brane-world gravity, but where the propagation of the gravitational field along the extra dimensions is a mathematical necessity, rather than being a postulate. After a brief introduction to Nash’s theorem, we show that the vacuum energy density must remain confined to four-dimensional space-times, but the cosmological constant resulting from the contracted Bianchi identity is a gravitational contribution which propagates in the extra dimensions. Therefore, the comparison between the vacuum energy and the cosmological constant in general

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relativity ceases to be. Instead, the geometrical fix provided by Nash’s theorem suggests that the vacuum energy density contributes to the perturbations of the gravitational field.

1 The Cosmological Constant Problem

Back in the 60’s, Zel’dowich showed how the fluctuations of quantum fields can be described as a perfect fluid with state equation \( p_v = -<\rho_v> = \text{constant} \). Then, using the semi-classical Einstein’s equations, it was shown that such fluid adds a non-trivial contribution to the gravitational field as

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^m_{\mu\nu} + 8\pi G <\rho_v> g_{\mu\nu} \tag{1}
\]

where \( T^m_{\mu\nu} \) denotes the energy-momentum tensor of the classical sources \([1]\). Comparing the constant terms in both sides of this equation we obtain \( \Lambda/8\pi G = <\rho_v> \), or as it is commonly stated, “the cosmological constant is the vacuum energy density”. However, such conclusion still depends on the solution of the cosmological constant problem: Current observations tell that \( \Lambda/8\pi G \approx 10^{-47} \text{ Gev}^4 \) (\( c = 1 \)), while the estimates of the vacuum energy density is \( <\rho_v> \approx 10^{76} \text{ Gev}^4 \). It is somewhat disappointing that this difference cannot be resolved by any known theoretical procedure in quantum field theory \([2]\).

This problem is related to the fact that Riemann’s geometry describes classes of curvature equivalent manifolds, defined by the same curvature tensor \( R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W \) defining the local shape of a manifold, without reference to any previously established standard of shape or curvature. Riemann’s own example is given by a flat manifold defined by \( R(U,V)W = 0 \) which can be a plane, a cylinder, in fact any ruled manifold\([3]\). A solution of such ambiguity was conjectured without proof by L. Schlaeefli in 1873, suggesting that any Riemannian manifold should be embedded into a larger flat manifold, acting as the curvature reference\([4]\). The final proof of that conjecture based on metric perturbations was accomplished by J. Nash in 1956.

The ground state of the gravitational field in general relativity is the Minkowski tangent space-time, taken as the flat-plane standard of curvature, in which particles, quantum fields, their vacuum states and energy are conceived. However, Minkowski’s geometry owes its plane flatness to the Poincaré symmetry of Maxwell’s equations, and not to Riemann’s geometry. On the other hand, modern cosmological observations indicate that the cosmological constant \( \Lambda \) albeit small is not zero. The existence of such
constant requires that the quantum fluctuations of the vacuum should be
defined in a deSitter ground state instead of the Minkowski one. Therefore,
the cosmological constant problem results from the fact that the class of
equivalence of manifolds which contain $\Lambda$ is not the same class of equiva-
rence of manifolds which contain the flat Minkowski’s space-time.

In the following section we apply Nash’s solution of Schlaeffi’s conjecture
to gravitational physics, showing why the emergence of extra dimensions is
necessary to remove Riemann’s curvature ambiguity. In section 3 we apply
this result to the cosmological constant problem.

2 Embedded Riemannian Structures

Consider a $D$-dimensional Riemannian manifold $V_D$ with metric geometry
$G_{AB}$, and another Riemannian manifold $V_n$, $n < D$ with metric $g_{\mu\nu}$. The
local and isometric embedding of $V_n$ in $V_D$ is the map $Z : V_n \to V_D$ with
$D$ components $Z^A(x^\mu, y^a)$ functions of the coordinates $x^m u$ in $V_n$ and the
extra coordinates $y^a$ orthogonal to $V_n$, such that (Index convention: $\mu, \nu =
1 \cdots n, \ a, b = n + 1 \cdots D, \ A, B = 1 \cdots D$)

\[ Z^A_{,\mu} Z^B_{,\nu} G_{AB} = g_{\mu\nu}, \quad Z^A_{,\mu} \eta^B_a G_{AB} = 0, \quad \eta^A_a \eta^B_b G_{AB} = g_{ab} \]

where $\{\eta_a\}$ is a basis of the $N = D - n$-dimensional complementary space,
orthogonal to $V_n$. With this we may construct a Gaussian frame of $V_D$
based on $V_n \{Z^A_{,\mu}, \eta^A_a\}$. The embedded Riemannian geometry differs from
the non-embedded one in that besides the metric there are two additional
variables. The extrinsic curvature defined by $k_{\mu\nu a} = -Z^A_{,\mu} \eta^B_a G_{AB}$ and the
third fundamental form defined by $A_{\mu ab} = \eta^A_a \eta^B_b G_{AB}$.

Nash’s theorem starts with the analysis of smoothing operators on manifolds[6],
specifying that the embedding map must be differentiable and regular. The
second and most important part of the theorem tells how to generate a Rie-
mannian manifold by an infinitesimal deformations (or perturbations) of $V_n$
along the extra dimensions given by

\[ \delta \bar{g}_{\mu\nu} = -2 \bar{k}_{\mu\nu a} \delta y^a \quad (2) \]

Therefore, for a given metric $\bar{g}_{\mu\nu}$, we obtain a perturbed metric $g_{\mu\nu} =
\bar{g}_{\mu\nu} + \delta \bar{g}_{\mu\nu}$ and by a successive repetition of this procedure, any different-
tiable embedded Riemannian manifold may be generated. Assuming that
the embedding is regular, then we may use the inverse function theorem
to un-embed the perturbed geometry, obtaining a purely intrinsic Riemannian geometry. (More details on Nash’s theorem and its applications to cosmology and quantum gravity can be found in [5].)

The 4-dimensionality of the space-time is a direct consequence of the Hodge duality of the gauge fields: Denoting $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$, $F_{\mu\nu} = [D_\mu, D_\nu]$, $D_\mu = \partial_\mu + A_\mu$, then the Yang-Mills equations can be written as $D \wedge F = 0$ and $D \wedge F^* = 4\pi j^*$ where $D = D_\mu dx^\mu$ and $F^*$ is the dual of $F$: $F^* = F_{\mu\nu} dx^\mu \wedge dx^\nu$, $F^*_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$. Such duality holds only when a 3-form is isomorphic to a 1-form, which is typical of four dimensional space-times. On the other hand, [2] implies that the gravitational field propagates along the extra dimensions of the embedding space.

In principle, the embedding space can be any Riemannian manifold. However, the differentiable condition required by Nash’s theorem suggests that the geometry of that space must obey the Einstein-Hilbert principle, stating that the metric geometry must be the smoothest possible. Consequently, the reference geometry (the bulk geometry) is defined by the local Einstein’s equations

$$R_{AB} - \frac{1}{2} R g_{AB} = \alpha_5 T^*_{AB}, \quad A, B = 1 \cdots D$$ (3)

where $\alpha_5$ denotes the $D$-dimensional energy scale and $T^*_{AB}$ is the energy-momentum tensor of the known sources, which we assume to be composed of 4-dimensional confined fields and ordinary matter. In this case, the confinement can be generically expressed as $\alpha_5 T^*_{\mu\nu} = 8\pi GT_{\mu\nu}$, $T^*_{\mu a} = T^*_{ab} = 0$.

In the following we will consider the simpler case of a flat 5-dimensional embedding space such that $R_{ABCD} = 0$. This is sufficient to embed the standard cosmological model [5] and it is sufficient to our analysis of the cosmological constant problem. The gravitational equations for the embedded geometry are obtained the equations (3) written in the mentioned Gaussian frame

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + Q_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad k^\rho_{\mu;\rho} - h_{\mu;\rho} = 0$$ (4)

where we have denoted $k_{\mu\nu5} = k_{\mu\nu}$, $h = g^{\mu\nu} k_{\nu\nu}$, $K^2 = k^{\mu\nu} k_{\mu\nu}$ and the conserved extrinsic geometric term

$$Q_{\mu\nu} = -k^\rho_{\mu} k_{\rho\nu} + h k^\rho_{\mu} + \frac{1}{2} (K^2 - h^2) g_{\mu\nu}, \quad Q^\mu_{\nu;\nu} = 0$$ (5)

which is a consequence of the embedding equations [5]. The cosmological constant $\Lambda$ appears after the contracted Bianchi identity.
Now, we may return to the cosmological constant problem. Assuming again the validity of the semi-classical regime for the contribution of the quantum fluctuations of the confined vacuum, then equation (4) becomes

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + Q_{\mu\nu} = 8\pi G T^m_{\mu\nu} + 8\pi G <\rho_v> g_{\mu\nu} \quad (6)$$

However, in contrast with the case of general relativity, $\Lambda$ and $<\rho_v>$ have different meanings and dynamical behavior: While $\Lambda$ represents a direct contribution to the gravitational field, the vacuum energy $<\rho_v>$ remains confined. Furthermore, except in the cases where $k_{\mu\nu} = 0$, it is not possible to cancel $\Lambda/8\pi G$ with $<\rho_v>$, because $Q_{\mu\nu}$ is a function of $k_{\mu\nu}$, so that it also contributes to the propagation of the gravitational field in the extra dimensions according to (2).

The overall conclusion is that the cosmological constant problem is proper of general relativity, which inherits the ambiguity associated with the equivalence classes of Riemann’s curvature. Either we take the Minkowski’s ground state as the standard flat plane, where $<\rho_v>$ is defined, or else, in face of the observations of a non-zero $\Lambda$, we take the deSitter ground state. The solution of such conflict comes from the fix to Riemannian geometry suggested by Schlafli and demonstrated by Nash, placing the curvature standard in the bulk geometry defined by the Einstein-Hilbert principle. The presence of this higher-dimensional embedding space provides a geometric standard of curvature, making it possible to contemplate the confined vacuum structure and the deSitter space-time without the cosmological constant conflict.

References

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