The Error Term of the Summatory Euler Phi Function

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1 Introduction

The error term is defined by \( E(x) = \sum_{n \leq x} \varphi(n) - 3\pi^{-2}x \). The earliest estimate for the error term \( E(x) = O(x^{1+\epsilon}) \), with \( \epsilon > 0 \), was computed by Mertens, followed by \( E(x) = O(x \log x) \) computed by Dirichlet. The current error term \( E(x) = O \left( x(\log x)^{2/3}(\log \log x)^{4/3} \right) \) in the mathematical literature is attributed to Walfisz, confer \cite[p. 68]{22}, \cite[p. 102]{29}, \cite[p. 47]{33}, et alii. For a large real number \( x \in \mathbb{R} \), the explicit formula

\[
\sum_{n \leq x} \varphi(n) = \frac{1}{6} + \frac{\delta(x)}{2} \varphi(x) + \frac{3}{\pi^2} x^2 + \sum_{\rho} \frac{\zeta(\rho-1)}{\rho \zeta'(\rho)} x^\rho + \sum_{n \geq 1} \frac{\zeta(-2n-1)}{-2n \zeta'(-2n)} x^{-2n}, \quad (1)
\]

where

\[
\delta(n) = \begin{cases} 
1 & x \in \mathbb{N}, \\
0 & x \notin \mathbb{N}, 
\end{cases} \quad (2)
\]
shows that there is a sharper unconditional error term

\[ E(x) = \delta(x)\varphi(x)/2 + O\left(xe^{-c\sqrt{\log x}}\right). \]  

(3)

The proof of (1) is based on standard analytical techniques and the Perron summation formula, see [22, p. 138], [33, p. 217], and similar references. The result in Theorem 1.1 provides a different and independent proof of this sharper unconditional error term.

**Theorem 1.1.** For a large number \( x \geq 1 \), the average order for the Euler totient function \( \varphi(n) \) has the asymptotic formula

\[ \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2}x^2 + O(x), \]  

unconditionally.

**Theorem 1.2.** Assume the RH. For a large number \( x \geq 1 \), the average order for the Euler totient function \( \varphi(n) \) has the asymptotic formula

\[ \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2}x^2 + \frac{\delta(x)}{2}\varphi(x) + O(x^{1/2}\log^2 x). \]  

(5)

The next few sections provide the elementary background materials, and the last section has the proof of Theorem 1.1. The proof of Theorem 1.2 follows from the explicit formula (1).

**Theorem 1.3.** Let \( x \geq 1 \) be a large number. Then, the average order of the ratio \( 1/\varphi(n) \) is as follows.

\[ \sum_{n \leq x} \frac{1}{\varphi(n)} = c_0 + c_1 \log x + O\left(\frac{\log x}{x}\right), \]  

where \( c_0 \) and \( c_1 \) are constants.

It is shown that this is the best possible, as was determined by Landau over a century ago, in [19, p. 184]. Several different and independent proofs of Theorem 1.3 are possible. A proof based on the convolution method is provided in Sections 6.

## 2 Results For The Mobius Function

The zeta function \( \zeta(s) = \sum_{n \geq 1} n^{-s} \) has a pole at \( s = 1 \), so the inverse series \( 1/\zeta(s) = \sum_{n \geq 1} \mu(n)n^{-s} \) vanishes at \( s = 1 \). For large \( x \geq 1 \), the associated summatory functions are \( \sum_{n \leq x} \mu(n) = o(x) \), and \( \sum_{n \leq x} \mu(n)n^{-1} < 1 \). The true rates of growth and decay respectively of these summatory functions are of considerable interest in number theory.
Lemma 2.1. Let $x \geq 1$ be a large number, and let $\mu(n)$ be the Mobius function. Then

$$\sum_{n \leq x} \mu(n) = O \left( \frac{x}{\log^2 x} \right). \tag{7}$$

Proof. Refer to [22, p. 182], [11] and the literature. \hfill ■

A better unconditional result $\sum_{n \leq x} \mu(n) = O \left( x e^{-c(\log x)^{3/5}(\log \log x)^{1/5}} \right)$, with $c > 0$ an absolute constant, is available in the literature, see [5]. However, the weaker but sufficient result in Lemma 2.1, which has simpler notation, will be used here.

Lemma 2.2. Let $x \geq 1$ be a large number, let $\mu(n)$ be the Mobius function, and let $s \in \mathbb{C}$, with $\text{Re}(s) \geq 1$. Then

$$\sum_{n \leq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O \left( \frac{1}{x^{s-1} \log^2 x} \right). \tag{8}$$

Proof. Use Lemma 2.1 to evaluate the Stieltjes integral

$$\sum_{n \leq x} \frac{\mu(n)}{n^s} = \sum_{n \geq 1} \frac{\mu(n)}{n^s} - \sum_{n \geq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} - \int_x^\infty \frac{1}{t^s} dM(t), \tag{9}$$

where $M(x) = \sum_{n \leq x} \mu(n)$. \hfill ■

3 Finite Fractional Sums

Let the symbol $\{z\} = z - \lfloor z \rfloor$ denotes the fractional part function. The earliest results for the summatory fractional function are due to Dirichlet and del Valle Poussin, see [25] and [23, p. 259] for different proofs.

Theorem 3.1. (Dirichlet) Let $x \geq 1$ be a number, and let $\{x\} = x - \lfloor x \rfloor$ be the fractional part function. Then

$$\sum_{n \leq x} \left\{ \frac{x}{n} \right\} = (1 - \gamma)x + O(x^{1/2}), \tag{10}$$

where $\gamma = 0.5772156649 \ldots$ is Euler constant.

A sharper error term $O \left( x^{1/3} \right)$ can be deduced from the Voronoi estimate for the divisor problem, the proofs appear in [14], [10], [13, Section 13.8], [2, p. 201], [3], and other references.
Lemma 3.1. Let \( x \geq 1 \) be a large number. Then
\[
\sum_{n \leq x} \left\{ \frac{x}{n} \right\} = (1 - \gamma) x + O \left( x^{1/3} \right),
\]  
where \( \gamma = 0.5772156649 \ldots \) is Euler constant.

Proof. Express the sum of fractional parts as difference of the harmonic finite sum, and the average number of divisors:
\[
\sum_{n \leq x} \left\{ \frac{x}{n} \right\} = \sum_{n \leq x} \left( \frac{x}{n} - \left\lfloor \frac{x}{n} \right\rfloor \right)
= x \sum_{n \leq x} \frac{1}{n} - \sum_{n \leq x} d(n). \tag{12}
\]
Using the standard asymptotics for these finite sums, it yields
\[
\sum_{n \leq x} \left\{ \frac{x}{n} \right\} = x \left( \log x + \gamma + O \left( \frac{1}{x} \right) \right) - (x \log x + (2\gamma - 1)x + O(x^{1/3}))
= (1 - \gamma) x + O \left( x^{1/3} \right) \tag{13}
\]
as claimed. \phantom{.} \blacksquare

The conjectured optimum error term is \( O \left( x^{1/4+\epsilon} \right) \), with \( \epsilon > 0 \) any arbitrary small number. This coincides with the best possible error term required in the divisor problem and the circle problem, see \cite{3}.

Lemma 3.2. Let \( x \geq 1 \) be a large number. Then
\[
\sum_{n \leq x} \left\{ \frac{x}{n} \right\} = (1 - \gamma) x + \Omega \pm \left( x^{1/4} \left. \right) \right. \tag{14}
\]

Theorem 3.2. Let \( x \geq 1 \) be a number, and let \( \{x\} = x - \lfloor x \rfloor \) be the fractional part function. Then
\[
\sum_{n \leq x} \frac{1}{n} \{ \frac{x}{n} \} = a_0 \log x + a_1 + O(x^{-1/2}),
\]
where \( a_0 = 1 - \gamma \) and \( a_1 \) are constants.

Proof. Let \( F(t) = \sum_{n \leq t} \{t/n\} \). Using Theorem 3.1 and partial summation yield
\[
\sum_{n \leq x} \frac{1}{n} \left\{ \frac{x}{n} \right\} = \int_1^x \frac{1}{t} dF(t)
= \frac{F(x)}{x} - F(1) + \int_1^x \frac{F(t)}{t^2} dt
= \frac{(1 - \gamma)x + O(x^{1/2})}{x} - F(1) \tag{15}
+ \int_1^x \left( \frac{(1 - \gamma)t + O(t^{1/2})}{t^2} \right) dt
= a_0 \log x + a_1 + O(x^{-1/2}),
\]
where \( a_0 = 1 - \gamma \) and \( a_1 \) are constants. \phantom{.} \blacksquare
### 3.1 Twisted Finite Fractional Sums

The next Lemmas give estimates for the twisted summatory fractional Mobius function.

**Lemma 3.3.** Let \( x \geq 1 \) be a large number, let \( \mu(n) \) be the Mobius function, and let \( \{x\} = x - \lfloor x \rfloor \) be the fractional part function. Then,

\[
\sum_{n \leq x} \mu(n) \left\{ \frac{x}{n} \right\} = -1 + O \left( \frac{x}{\log^2 x} \right). \tag{16}
\]

**Proof.** Let \( F(x) = [x] \) be the largest integer function, and let \( G(x) = 1 \) in Lemma 2.2. Next, replace the integer part-fractional part identity:

\[
1 = \sum_{n \leq x} \mu(n) \left\{ \frac{x}{n} \right\} = \sum_{n \leq x} \mu(n) \left( \frac{x}{n} - \left\{ \frac{x}{n} \right\} \right) = x \sum_{n \leq x} \frac{\mu(n)}{n} - \sum_{n \leq x} \mu(n) \left\{ \frac{x}{n} \right\}
\]

\[
= O \left( \frac{x}{\log^2 x} \right) - \sum_{n \leq x} \mu(n) \left\{ \frac{x}{n} \right\},
\]

where the last line follows from Lemma 2.2. Now, solve for the fractional Mobius sum. \[\blacksquare\]

Almost the same calculation appears in [18, p. 590].

**Lemma 3.4.** Let \( x \geq 1 \) be a large number, let \( \mu(n) \) be the Mobius function, and let \( \{z\} \) be the fractional part function. Then,

\[
\sum_{n \leq x} \frac{\mu(n)}{n} \left\{ \frac{x}{n} \right\} = O(1). \tag{18}
\]

**Proof.** Let \( V(x) = \sum_{n \leq x} \mu(n) \{x/n\} = -1 + O \left( x \log^{-2} x \right) \), see Lemma 3.3. The integral representation yields

\[
\sum_{n \leq x} \frac{\mu(n)}{n} \left\{ \frac{x}{n} \right\} = \int_1^x \frac{1}{t} dV(t)
\]

\[
= \frac{V(x)}{x} - V(1) + \int_1^x \frac{V(t)}{t^2} dt
\]

\[
= O(1).
\]

Note that the integral

\[
\int_1^x \frac{V(t)}{t^2} dt = \int_1^x \frac{1}{t^2} dt + O \left( \frac{t \log^{-2} t}{t^2} \right) dt = O \left( \frac{1}{\log x} \right).
\]

This verifies the claim. \[\blacksquare\]
Similar calculations as in Lemmas 3.3 and 3.4 are given in [22, p. 248].

### 3.2 Quasibalanced And Balanced Average Orders

The average order of the *quasibalanced* fractional part function \( \psi(x) = \{x\} - 1/2 \) appears in several problems, see [35, 31, 7], et alii. But the applications for *balanced* fractional part function \( \psi_0(x) = \{x\} - (1 - \gamma) \) are very rare.

**Lemma 3.5.** (Quasibalanced Average) Let \( x \geq 1 \) be a number, and let \( \{x\} = x - \lfloor x \rfloor \) be the fractional part function. Then

\[
\sum_{n \leq x} \frac{\{x/n\} - 1/2}{n} = \frac{1}{2} - \gamma \log x + c_0 + O \left( \frac{1}{x^{1/2}} \right),
\]

where \( c_0 \) is a constant.

**Proof.** Applications of Theorem 3.2 and the asymptotic formula for the harmonic finite sum yield

\[
\sum_{n \leq x} \frac{\{x/n\} - 1/2}{n} = \sum_{n \leq x} \frac{1}{n} \{x/n\} - \frac{1}{2} \sum_{n \leq x} \frac{1}{n} \tag{22}
\]

\[
= \left( (1 - \gamma) \log x + c_1 + O \left( \frac{1}{x^{1/2}} \right) \right) - \frac{1}{2} \left( \log x + \gamma + O \left( \frac{1}{x} \right) \right) \n\]

\[
= \left( \frac{1}{2} - \gamma \right) \log x + c_2 + O \left( \frac{1}{x^{1/2}} \right),
\]

where \( c_1 \) and \( c_2 \) are constants.

**Lemma 3.6.** (Balanced Average) Let \( x \geq 1 \) be a number, and let \( \{x\} = x - \lfloor x \rfloor \) be the fractional part function. Then

\[
\sum_{n \leq x} \frac{\{x/n\} - (1 - \gamma)}{n} = c + O \left( \frac{1}{x^{1/2}} \right),
\]

where \( c \) is a constant.

**Proof.** Applications of Theorem 3.2 and the asymptotic formula for the harmonic finite sum yield

\[
\sum_{n \leq x} \frac{\{x/n\} - (1 - \gamma)}{n} = \sum_{n \leq x} \frac{1}{n} \{x/n\} - (1 - \gamma) \sum_{n \leq x} \frac{1}{n} \tag{24}
\]

\[
= \left( (1 - \gamma) \log x + c_1 + O \left( \frac{1}{x^{1/2}} \right) \right) - \left( (1 - \gamma) \log x + \gamma + O \left( \frac{1}{x} \right) \right) \n\]

\[
= c_0 + O \left( \frac{1}{x^{1/2}} \right),
\]

where \( c_0 \) is a constant.
3.3 Variance

The average and the quasibalanced fractional function $\psi(x) = \{x/n\} - 1/2$ is slightly different from the uniform random variable on the interval $[-1/2, 1/2]$. However, the variance almost the same as a uniform random variable on the interval $[-1/2, 1/2]$.

**Lemma 3.7.** (QuasiBalanced Variance) Let $x \geq 1$ be a number, and let $\{x\} = x - [x]$ be the fractional part function. Then

$$\sum_{n \leq x} (\{\alpha n\} - 1/2)^2 = \frac{1}{12} x + O(x^\varepsilon),$$

(25)

where $\varepsilon > 0$ is arbitrarily small constant depending on the irrational $\alpha > 0$.

**Proof.** This requires the Fourier series of the Bernoulli polynomial $\psi(x)^2 = (\{x\} - 1/2)^2 - 1/12$. Summing the Fourier series over the range of integers yields

$$\sum_{n \leq x} (\{\alpha n\} - 1/2)^2 = \sum_{n \leq x} \left( \frac{1}{12} + \frac{1}{2\pi^2} \sum_{m \geq 1} \frac{e^{i2\pi\alpha mn}}{m^2} \right)$$

$$= \frac{1}{12} x + \frac{1}{2\pi^2} \sum_{m \geq 1} \frac{1}{m^2} \sum_{n \leq x} e^{i2\pi\alpha mn}$$

$$= \frac{1}{12} x + O(x^\varepsilon).$$

(26)

(27)

The error term $O(x^\varepsilon)$ depends on the irrational $\alpha \in \mathbb{R}$; numbers with unbounded partial quotients have the largest error term.

This proof was known quite sometimes ago, see [9].

3.4 Comparison Of Dirichlet And Walfisz Results

The estimate of the quasibalanced fractional sum

$$\sum_{n \leq x} \frac{x/n - 1/2}{n} = (1/2 - \gamma) \log x + c + O\left( \frac{1}{x^{1/2}} \right),$$

(28)

see Lemma 3.5 for a proof, is based on Dirichlet or delaValle Poussin result in Theorem 3.1. This result seems to contradict a well known result described below.

**Theorem 3.3.** (Walfisz) Let $x \geq 1$ be a number, and let $\{x\} = x - [x]$ be the fractional part function. Then

$$\sum_{n \leq x} \frac{x/n - 1/2}{n} = O\left( \frac{\log x}{\log \log x} \right).$$

(29)
The detailed and lengthy proof is provided in [35, pp. 72–78]. It is based on the Fourier series
\[ \psi(x) = -\frac{1}{\pi} \sum_{m \geq 1} \frac{\sin(2\pi mx)}{m} = \begin{cases} \{x\} - 1/2 & x \notin \mathbb{Z}, \\ 0 & x \in \mathbb{Z}, \end{cases} \quad (30) \]
of the function \( \psi(x) = \{x\} - 1/2 \), and an exponential sum estimate similar to
\[ \sum_{n \leq x} e^{i\pi mx/n} = o(x). \quad (31) \]

About forty years later, the estimate in (29) was improved to
\[ \sum_{n \leq x} \frac{\{x/n\} - 1/2}{n} = -\frac{1}{\pi} \sum_{n \leq x} \frac{1}{n} \sum_{m \geq 1} \frac{\sin(2\pi mx/n)}{m} = O\left(\log^{2/3} x\right). \quad (32) \]

This version and the associated results are often quoted in the literature, see [30], [32], [33, p. 46], [7, Section 2], et alii.

4 Inversion Identities

**Lemma 4.1.** (Mobius summatory inversion) Let \( F, G : \mathbb{N} \rightarrow \mathbb{C} \) be complex-valued arithmetic functions. Then
\[ F(x) = \sum_{n \leq x} G(x/n) \quad \text{and} \quad G(x) = \sum_{n \leq x} \mu(n)F(x/n) \quad (33) \]
are a Mobius inversion pair.

**Proof.** Refer to [8, p. 237], [22, p. 36], [27, p. 25], [29, p. 62], [33, p. 35], et alii. ■

5 Results For The Ratio \( \varphi(n)/n \)

The corresponding normalized summatory totient function \( \varphi(n)/n \) has the well known asymptotic formula \( \sum_{n \leq x} \varphi(n)/n = 6\pi^{-2}x + O(\log x) \), confer [22, p. 36], and [23, p. 229]. Some earlier works on this problem appear in [21], [24], [6], [15], and similar references. An improved error term for the normalized summatory totient function is considered first.

**Theorem 5.1.** For large number \( x \geq 1 \), the average order for the normalized Euler totient function \( \varphi(n)/n \) has the asymptotic formula
\[ \sum_{n \leq x} \frac{\varphi(n)}{n} = \frac{6}{\pi^2}x + O(1). \quad (34) \]
Proof. The analysis proceeds as usual, but improves on the last steps:

\[
\sum_{n \leq x} \frac{\varphi(n)}{n} = \sum_{n \leq x} \sum_{d|n} \frac{\mu(n)}{d} \tag{35}
\]

\[
= \sum_{d \leq x} \frac{\mu(n)}{d} \sum_{n \leq x, d|n} 1
\]

\[
= \sum_{d \leq x} \frac{\mu(n)}{d} \left\lfloor \frac{x}{d} \right\rfloor ,
\]

where \(\mu(n) \in \{-1, 0, 1\}\) is the Mobius function. Substituting the integer part/fractional part functions identity leads to

\[
\sum_{n \leq x} \frac{\varphi(n)}{n} = \sum_{d \leq x} \frac{\mu(d)}{d} \left( \frac{x}{d} - \left\{ \frac{x}{d} \right\} \right) \tag{36}
\]

\[
= x \sum_{d \leq x} \frac{\mu(d)}{d^2} - \sum_{d \leq x} \frac{\mu(d)}{d} \left\{ \frac{x}{d} \right\}.
\]

Now, using Lemmas [2.2 and 3.4] yields

\[
\sum_{n \leq x} \frac{\varphi(n)}{n} = x \left( \frac{6}{\pi^2} + O \left( \frac{1}{x \log^2 x} \right) \right) + O(1)
\]

\[
= \frac{6}{\pi^2} x + O(1). \tag{37}
\]

This proves the claim. \qed

The standard proof for the average order \(\sum_{n \leq x} \varphi(n) = 3\pi^{-2}x^2 + O(x \log x)\) of \(\varphi(n)\) are due to Mertens, [20], [18, p. 591]. Currently, it is claimed that \(\sum_{n \leq x} = 3\pi^{-2}x^2 + O(x (\log x)^{2/3} (\log \log x)^{4/3})\), see [34, p. 99?], [22, p. 36], [33, p. 46], and other authors.

Proof. (Theorem 1.1) By Theorem 5.1, \(W(x) = \sum_{n \leq x} \varphi(n)/n = 6\pi^{-2}x + O(1)\), and summation by part yields

\[
\sum_{n \leq x} \varphi(n) = \sum_{n \leq x} n \cdot \frac{\varphi(n)}{n}
\]

\[
= \int_1^x t \, dW(t)
\]

\[
= xW(x) + O(1) - \int_1^x W(t) \, dt \tag{38}
\]

\[
= x \left( \frac{6}{\pi^2} x + O(1) \right) - \int_1^x \left( \frac{6}{\pi^2} t + O(1) \right) \, dt
\]

\[
= \frac{3}{\pi^2} x^2 + O(x).
\]

Quod erat demonstrandum. \qed

9
6 Results For The Ratio $n/\varphi(n)$

This section continues with the analysis of the error term of the average order for the reciprocal $1/\varphi(n)$ of the Euler totient function $\varphi(n)$. It proves that the best error term is the same as that determined by Landau over a century ago, in [19, p. 184]. The simpler analysis for the ratio $n/\varphi(n)$ is considered first.

Theorem 6.1. Let $x \geq 1$ be a large number. Then, the average order of the ratio $n/\varphi(n)$ has the asymptotic formula

$$\sum_{n \leq x} \frac{n}{\varphi(n)} = a_0 x + O(\log x),$$  \hfill (39)

where $a_0 = \frac{\zeta(2)\zeta(3)}{\zeta(6)}$ is a constant.

Proof. The result is derived using the identity $\sum_{d|n} \mu^2(d)/\varphi(d)$. Substituting this formula, and reversing the order of summation yield

$$\sum_{n \leq x} \frac{n}{\varphi(n)} = \sum_{n \leq x} \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$$

$$= \sum_{d \leq x} \frac{\mu^2(d)}{\varphi(d)} \sum_{n \leq x, \ d|n} 1$$

$$= \sum_{d \leq x} \frac{\mu^2(d)}{\varphi(d)} \left( \frac{x}{d} - \left\{ \frac{x}{d} \right\} \right)$$

$$= x \sum_{d \leq x} \frac{\mu^2(d)}{d\varphi(d)} - \sum_{d \leq x} \frac{\mu^2(d)}{\varphi(d)} \left\{ \frac{x}{d} \right\}.$$ \hfill (40)

The first finite sum

$$x \sum_{n \leq x} \frac{\mu^2(n)}{n\varphi(n)} = x \left( \sum_{n \geq 1} \frac{\mu^2(n)}{n\varphi(n)} - \sum_{n > x} \frac{\mu^2(n)}{n\varphi(n)} \right)$$

$$= c_0 x + O \left( \frac{1}{x} \right).$$ \hfill (41)

The constant $a_0 > 0$ is expressable in terms of zeta functions as

$$\sum_{n \geq 1} \frac{\mu^2(n)}{n\varphi(n)} = \prod_{p \geq 2} \left( 1 + \frac{1}{p(p-1)} \right) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}. $$ \hfill (42)

The second finite sum

$$\sum_{n \leq x} \frac{\mu^2(n)}{\varphi(n)} \left\{ \frac{x}{n} \right\} \gg \log x$$ \hfill (43)

is always positive and exhibits no cancellations. \hfill ■
The form of the error term in (43) concretely proves that it cannot be improved, see also Subsection 3.4 for a related discussion.

**Theorem 6.2.** (Same as Theorem 1.3) Let $x \geq 1$ be a large number. Then, the average order of the ratio $1/\varphi(n)$ is as follows.

\[
\sum_{n \leq x} \frac{1}{\varphi(n)} = c_0 + c_1 \log x + O\left(\frac{\log x}{x}\right), \quad (44)
\]

where $c_0$ and $c_1$ are constants.

**Proof.** The result is derived Theorem 6.1 by partial summation. More precisely, let $R(t) = \sum_{n \leq t} n/\varphi(n) = a_0x + O(\log x)$. Then

\[
\sum_{n \leq x} \frac{1}{\varphi(n)} = \sum_{n \leq x} \frac{1}{n/\varphi(n)} = \int_1^x \frac{1}{t} dR(t) = \frac{R(t)}{t} \bigg|_1^x + \int_1^x \frac{R(t)}{t^2} dt, \quad (45)
\]

\[
= \frac{a_0 x + O(\log x)}{x} + a_1 + \int_1^x \frac{a_0 t + O(\log t)}{t^2} dt,
\]

\[
= c_0 + c_1 \log x + O\left(\frac{\log x}{x}\right),
\]

where $a_0 = \zeta(2)\zeta(3)/\zeta(6)$, $a_1 = -R(1)$, $c_0$, and $c_1$ are constants. \[\square\]

The work in \[30\] is devoted to improving the error term from $O((\log x)/x)$ to $O((\log x)^{2/3}/x)$. This analysis was based on the estimate

\[
\sum_{n \leq x} \frac{\{x/n\} - 1/2}{n} = O\left(\left(\log x\right)^{2/3}\right), \quad (46)
\]

Refer to subsection 3.4 for a discussion on this estimate. However, by Theorem 6.2, this seems to be impossible since the error term in (44) satisfies $\gg (\log x)/x$, confer (43).

### 7 Abridged History Of The Error Term

Recall that by definition, the error term is given by

\[
E(x) = \sum_{n \leq x} \varphi(n) - \frac{3}{\pi^2}x^2, \quad (47)
\]
The earliest estimate of the error term $E(x) = O(x^{1+\epsilon})$ was computed by Dirichlet, followed by Mertens as $E(x) = O(x \log x)$, see [20], and later $E(x) = O(x(\log x)^{2/3}(\log \log x)^{1/3})$ was computed by Walfisz, see [34, p. 99]. The assertions that $E(x) \neq o(x \log \log \log x)$, and $E(x) = \Omega(x \log \log \log \log x)$, appear in [24], and [6] respectively. Moreover, there is a conjecture that $R(x) = O(x \log \log x)$, and the omega estimate

$$\sum_{n \leq x} \varphi(n) - \frac{3}{\pi^2} x^2 = \Omega_{\pm} (x \sqrt{\log \log x})$$

(48)

coner [21]. The analysis given in those papers are for the error terms over a short interval, namely, $R(x) = \sum_{x \leq n \leq x+y} \varphi(n)$, where $y = O(\log \log x)$, not for $E(x) = \sum_{n \leq x} \varphi(n) - 3\pi^{-1} x^2$.

Furthermore, are a few other recent result such as the smoothed omega estimate

$$\sum_{n \leq x} \varphi(n) \log \left( \frac{x}{n} \right) - \frac{3}{\pi^2} x^2 = \Omega_{\pm} (x^{1/2} \log \log \log x)$$

(49)

was proved in [15].
8 Problems

1. Find the exact value of the main term of the finite sum

\[
\sum_{n \leq x} \frac{\mu^2(n)}{\varphi(n)} \left\{ \frac{x}{n} \right\}
\]  

(50)

2. Let \( \psi(x) = \{x\} - 1/2 \). Compute the Fourier series

\[
\psi(x) = -\frac{1}{\pi} \sum_{n \geq 1} \frac{\sin(2\pi xn)}{n}.
\]  

(51)

3. Let \( \psi(x) = \{x\} - 1/2 \). Compute the Fourier series

\[
\psi(x)^2 = \frac{1}{12} + \frac{1}{2\pi^2} \sum_{n \geq 1} \frac{e^{i2\pi xn}}{n^2}.
\]  

(52)

4. Let \( \alpha \in \mathbb{R} - \mathbb{Z} \) be irrational, with unbounded partial quotients, estimate the error term \( \sum_{n \leq x} (\{\alpha n\} - 1/2)^2 - x/12 = O(x^\epsilon) \) in Lemma 3.7.
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