Gauge Theories on Open Lie Algebra
Non-commutative Spaces

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Abstract

It is shown that non-commutative spaces, which are quotients of associative algebras by ideals generated by highly non-linear relations of a particular type, admit extremely simple formulae for deformed or star products. Explicit construction of these star products is carried out. Quantum gauge theories are formulated on these spaces, and the Seiberg-Witten map is worked out in detail.

Notation: We shall use capital letters \( X^i \) and \( P^i \) to denote non-commuting coordinates. Functions of non-commuting coordinates will also be denoted by capital letters, e.g. \( F(X, P) \). The corresponding commuting coordinates and their functions will be denoted by lower case letters, e.g. \( x^i, p^i \) and \( f(x, p) \).

1 Introduction

In the present paper, we consider a large class of ‘open Lie algebras’ as examples of non-commutative spaces. We use the term ‘open Lie algebras’ to mean associative algebras where the defining relations among the generators are consistent with the Jacobi identity but the commutators of generators do not necessarily close on the set of generators. In the open Lie algebras considered here, the commutators of the generators are allowed to be arbitrary functions of the generators, along with some restrictions which we shall elaborate on later in this section. We find that a judicious choice of ordering on these non-commutative spaces provides one with a surprisingly simple description of deformed products. This allows for a detailed formulation of gauge theories on these non-commutative spaces. We do that following the method outlined by

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Madore et al.\cite{1}, and establish the Seiberg-Witten map \cite{2}. The organization of the paper is as follows. The rest of this section is devoted to the elaboration of our motivation. In the next section we work out the star product. The final section is devoted to the formulation of gauge theories and studying the Seiberg-Witten map.

The study of quantum field theories on non-commutative spaces (for reviews of the subject see \cite{3, 4}) is made possible largely due to success of the paradigm of deformation-quantization \cite{5, 6, 7, 1, 2}. The pioneering work in this field allows us to translate the problem of understanding functions on spaces with operator valued or non-commuting coordinates \cite{8} into statements about functions of commuting variables with deformed (star, $\star$) products. For spaces, where the non-commutativity is in some sense simple, the isomorphism between functions of non-commuting variables and functions of ordinary commuting variables assumes a particularly transparent form because of the availability of simple expressions for star products. The classic example of this is the study of a space which is the algebra formed by generators $'X_i'$ obeying canonical commutation relations,

$$[X_i, X_j] = i\theta^{ij},$$

(1)

where $\theta^{ij}$ is a constant, real, anti-symmetric matrix. For these spaces, the star product of functions $f$ and $g$ of ordinary variables $x^i$, can be expressed by the Groenewold-Moyal \cite{6, 9} formula;

$$f(x) \star g(x) = f(x) \exp \left( \frac{i}{2} \partial_i \theta^{ij} \partial_j \right) g(x).$$

(2)

This deformed product corresponds to symmetric ordering or Weyl quantization.

A measure of the degree of non-commutativity of a space, thought of as an abstract algebra, is encoded in the commutators of the generators. In some sense, the algebra with the relations (3) forms a space that is 'barely non-commutative'. That is so because (4) defines a Lie-algebra, whose non-commutative part (the matrix $\theta^{ij}$) lies in the commutant of the algebra. At the next level of non-commutativity are Lie-algebras with non-vanishing structure constants. Analysis of quantum gauge theories on Lie-algebra non-commutative spaces, using Weyl quantization, was carried out by Madore et al.\cite{1}, and as one might expect, the formulae for star products in such spaces do not admit simple closed forms like (2). The ground breaking work of Kontsevich \cite{5} and Cattaneo et al.\cite{10}, tells us how to extend Weyl quantization to generic non-commutative spaces, where the commutators of the coordinates are not constants, but arbitrary functions of the coordinates.

$$[X^i, X^j] = \theta^{ij}(X).$$

(3)

Compared to (2), the formulae for the deformed products for generic non-commutative spaces (3) are much more involved, and for purposes of doing explicit computations of the sort required in the analysis of quantum field theories, it is certainly worthwhile to look for simplifications of the Kontsevich formula (3).
What is surprising is that for certain non-commutative spaces, whose generators satisfy non-linear relations, there do exist surprisingly explicit formulae for star products.

Deformed products of functions of commuting variables correspond to isomorphisms between functions of ordinary commuting variables and those of the corresponding non-commuting ones. The isomorphisms are obtained by specifying a choice of ordering for the functions of non-commuting variables. Hence some room for the simplification of the expressions for the deformed products is provided by our freedom in the choice of ordering. Moreover, all deformations of a particular commutative algebra of functions, obtained by different ordering prescriptions are equivalent up to cohomological issues [7, 11]. In the case of finite dimensional non-commutative spaces, which can be coordinatized by global coordinates, the subtle cohomological issues do not matter. In these cases, the cohomological equivalence of orderings translates into a true equivalence, and one is indeed free to choose a prescription of ordering at one’s convenience. For the simplest case [10], the star products are very explicit for almost any reasonable choice of ordering [12, 13]. For a geometric construction of (2) and a review of some star products corresponding to various orderings in (1) we refer to [14, 15].

The non-commutative spaces that we consider in the present paper are associative algebras generated by a finite number of generators (modulo certain non-linear relations). Hence the argument for choosing a preferred ordering does apply here. We coordinatize the space by two sets of generators, labeled $X^i$, and $P^i$, to maintain an analogy with the canonical case. The generators obey the following commutation relations,

$$[P^j, X^i] = \delta_{i,j} \theta_i(X^i),$$

and

$$[X^i, X^j] = [P^i, P^j] = 0.$$  \tag{5}

$\theta_i(X^i)$ (no sum over the repeated index $i$) in (4) corresponds to an arbitrary function of $X^i$. After a suitable choice of ordering is provided, the expression for the deformed product of functions in the non-commutative space of interest to us (4) becomes,

$$f(x, p) \star g(x, p) = f(x, p) \exp \left( \sum_i \frac{\partial}{\partial p^i} \left( \theta^i(x^i) \frac{\partial}{\partial x^i} \right) \right) g(x, p).$$ \tag{6}

This class of non-commutative spaces include several physically interesting examples. The space whose coordinates obey the canonical commutation relations (1), corresponds to the special case $\theta(X) = \text{constant}$. The two dimensional $\kappa$-Minkowski space [16, 17], also falls in this category. In this non-commutative space, the commutation relation among the coordinates $T, X$ is,

$$[T, X] = -\frac{i}{\kappa} X.$$  \tag{7}
This space is obviously of the type considered in (4), with $T$ assuming the role of $P$, and $\theta(X)$ becoming a linear function of $X$. The methods used for calculating the star product can easily be extended to the four dimensional $\kappa$-Minkowski space as well, and the corresponding result is quoted in the next section.

The $h$-deformed plane [18, 19, 20, 21] is another example of a non-commutative space that falls in this category. This space is the algebra generated by two generators $P$, and $X$, satisfying,

$$[P, X] = hX^2. \quad (8)$$

A detailed proof of the expression for the star product (8), and the analysis of gauge theories using this star product is provided in the following sections.

2 The Star Product

For the purpose of simplifying the notation and computations, we shall work with the two dimensional analog of (4) in the rest of the paper; i.e. the case of two non-commuting coordinates $X$ and $P$. The results can be generalized to spaces made of many copies of this two dimensional space in a straightforward manner.

To set up an isomorphism between functions of non-commuting objects $(X, P)$, and functions of ordinary commuting variables $(x, p)$, one requires a rule, $(\Omega)$, for associating a unique function of non-commuting variables to a given function of the ordinary commuting ones. i.e.

$$\Omega : f(x, p) \xrightarrow{\Omega} F(X, P) = \Omega(f(x, p)) \quad (9)$$

This isomorphism corresponds to a quantization. In the present paper we seek to quantize a Poisson manifold with the following Poisson bracket.

$$\{p, x\} = \theta(x) \quad (10)$$

This Poisson bracket is coordinate dependent. But, as long as there exists some coordinate system in which the Poisson bracket can be brought to this form, the analysis goes through.

We shall think of the functions $f(x, p)$ of the commuting variables as formal power series of the form,

$$f(x, p) = \sum_{m,n} a_{m,n} x^m p^n. \quad (11)$$

The functions of the non-commuting variables $F(X, P)$ will also be thought of as elements of the space of formal power series generated by the two letters $X, P$, modulo the relations that characterize the particular non-commutative space of interest. In the non-commutative space of interest to us, the relations between the two coordinates (denoted by $P$ and $X$) are,

$$[P, X] = \theta(X) \quad (12)$$
These commutation relations can be solved by representing $X$ by the multiplication operator $x$ and $P$ by $\theta(x) \frac{\partial}{\partial x}$. Assuming that $\theta(x)$ admits a power series expansion, we have,

$$P = \theta(x) \frac{\partial}{\partial x} = \sum_n \theta_n x^n \frac{\partial}{\partial x}$$  

(13)

In the general case too we can solve the commutation relations as,

$$P^i = \theta^i(x^i) \frac{\partial}{\partial x^i},$$  

(14)

where no sum is implied over the index $i$.

Specifying $\Omega$ amounts to choosing an ordering, and for the present problem, we shall chose the ‘standard ordering’; for which,

$$\Omega(x^m p^n) = X^m P^n = \Omega(p^n x^m)$$  

(15)

To complete the isomorphism between the functions of commuting variables and those of non-commuting ones, we need a $\Omega$ dependent deformation of the ordinary point-wise multiplication of functions of the commuting variables. The deformed product is defined as,

$$f(x, p) \star g(x, p) = \Omega^{-1}(\Omega(f(x, p))\Omega(g(x, p)))$$  

(16)

To work out the star product corresponding to standard ordering, let us first consider standard ordered operator valued functions

$$F = X^a P^b = \Omega(f(x, p) = x^a p^b),$$  

(17)

and,

$$G = X^r P^s = \Omega(g(x, p) = x^r p^s).$$  

(18)

These functions are elements of a ‘basis’ for expanding an arbitrary standard ordered function as a formal power series. So the extension of the star product to arbitrary functions can be had once it is worked out for the basis elements.

The product of two basis elements,

$$FG = X^a P^b X^r P^s = X^{a+r} P^{b+s} + X^a [P^b, X^r] P^s,$$  

(19)

can be written as a standard ordered function, if the commutation relation $[P^a, X^r]$ can be expressed in the standard form. This can indeed be done using the representation of $P$ as a differential operator described above.

Using this representation, it can be shown that,

**Proposition 1**

$$P^b X^r = \sum_{n_1, \ldots, n_l = 0}^{\infty} C_l(\theta, r, b, n_1, \ldots n_{l-1}) X^{(r+(n_1-1)+(n_2-1)+\ldots(n_{l}-1))} P^{b-l},$$  

(20)
where,

$$C_l(\theta, r, b, n_1, \ldots, n_{l-1}) = \binom{b}{l} \theta_{n_1} \ldots \theta_{n_l} r(r+(n_1-1)) \ldots (r+(n_{l-1}-1)) \ldots (r+(n_l-1)) \ldots (r+(n_{l-1}-1)).$$  \hspace{1cm} (21)

In the expression for the coefficients $C_l(\theta, r, b, n_1, \ldots, n_{l-1})$ in the proposition above, no sum is implied over the repeated indices $n_1, \ldots, n_{l-1}$. Hence,

$$FG = X^a P^b X^r P^s = \sum_{l=0}^{b} C_l(\theta, r, b, n_1, \ldots, n_{l-1}) X^{r+(n_1-1)} \ldots X^{r+(n_{l-1}-1)} P^{b+s-l} \ldots$$  \hspace{1cm} (22)

From the definition of the star product \([11]\), it now follows that

$$f \ast g = \Omega^{-1}(\Omega(f)\Omega(g)) = \sum_{l=0}^{b} \frac{1}{l!} \left( \frac{\partial^l}{\partial p^l} x^a p^b \right) \left( (\theta(x) \frac{\partial}{\partial x})^l x^r p^s \right).$$  \hspace{1cm} (23)

Extending this analysis to arbitrary differentiable functions $f, g$, we have.

$$f \ast g = f e^{\frac{\theta}{\kappa} (\theta(x) \frac{\partial}{\partial x})} g $$  \hspace{1cm} (24)

A straightforward generalization to algebras containing several pairs of the generators $P^i$ and $X^i$ and the relations (4) gives us (6).

**The case of the $\kappa$-Minkowski space:** As an aside, it is worth mentioning that a similar ordering prescription leads to a simple formula for the star product in the $\kappa$-Minkowski space. The four dimensional $\kappa$-Minkowski space is generated by the generators, $T$ and $X^i$, satisfying the following relations.

$$[T, X^i] = -\frac{i}{\kappa} X^i. \hspace{1cm} (25)$$

The $X^i$'s commute among themselves. We pick 'standard ordering' between $X^i$ and $T$ as the preferred ordering prescription in this non-commutative space. Ordering among the $X$ variables does not matter as they commute with each other. More specifically,

$$\Omega \left( \prod_{i=1}^{3} (x^i)^{r_i} t^j \right) = \Omega \left( t^j \prod_{i=1}^{3} (x^i)^{r_i} \right) = \prod_{i=1}^{3} (X^i)^{r_i} T^j. \hspace{1cm} (26)$$

Some recent advances in the non-commutative geometry of $\kappa$-Minkowski spaces from a similar point of view can be found in \([22, 23]\).

A standard ordered function on the $\kappa$-Minkowski space, $F(T, X^1, X^2, X^3)$ can be thought of as a formal power series,

$$F(T, X^1, X^2, X^3) = \sum_{r_1, r_2, r_3, i=0}^{\infty} f_{r_1, r_2, r_3, i} (X^1)^{r_1} (X^2)^{r_2} (X^3)^{r_3} T^i = \Omega(f(t, x^1, x^2, x^3)) \hspace{1cm} (27)$$
Once this choice of ordering is made, we can, as was done in (20), bring products of standard ordered functions to standard forms by commuting $T$ through the $X^i$s using (25) repeatedly. It is tedious but straightforward to see that the resulting expression for star product in the $\kappa$-Minkowski space is,

$$(f \ast g)(t, x^1, x^2, x^3) = f(t, x^1, x^2, x^3) \prod_{j=1}^{3} e^{\kappa \frac{\partial}{\partial x^j}} g(t, x^1, x^2, x^3).$$  \hfill (28)

In the argument of the exponential in the above equation, no sum is implied over the repeated index $j$.

**Proof of Proposition 1:**

We present a proof based on induction on the index $b$ in (20). For $b = 1$, it is easy to see that,

$$PX^r = \sum_{n_1} \theta_{n_1} x^{n_1} \partial_x x^r = \sum_{n_1} \theta_{n_1} r X^{r+(n_1-1)} + X^r P.$$  \hfill (29)

Hence the formula holds for $b = 1$.

Using the representation of $P$ \hfill (13), it is again straightforward to verify that, for $b = 2$,

$$P^2 X^r = \sum_{n_1, n_2} \theta_{n_1} \theta_{n_2} r (r + (n_1 - 1)) X^{r+(n_1-1)+(n_2-1)} + 2 \sum_{n_1} \theta_{n_1} r X^{r+(n_1-1)} P + X^r P^2.$$  \hfill (30)

Hence the formula holds for $b = 2$. Assuming, that (20) hold for a particular value of $b$, one can use \hfill (13) once again to get,

$$P^{b+1} X^r = \sum_{l} C_l (\theta, r, b, n_1, \ldots, n_l) \theta_{n_{l+1}} (r + (n_1 - 1) + \ldots + (n_l - 1)) \times \hfill (31)$$

$$X^{r+(n_1-1)+\ldots+(n_{l+1}-1)} P^{b-l} + \sum_{l} C_l (\theta, r, b, n_1, \ldots, n_l) X^{r+(n_1-1)+\ldots+(n_l-1)} P^{b-l+1}.$$  \hfill (32)

We now note that the coefficients satisfy the following recursion relation,

$$C_l (\theta, r, n_1, \ldots, n_{l-1}) \theta_{n_{l+1}} (r + (n_1 - 1) + \ldots + (n_l - 1)) + \hfill (33)$$

$$C_{l+1} (\theta, r, b, n_1, \ldots n_l) = C_{l+1} (\theta, r, b+1, n_1, \ldots n_l).$$  \hfill (34)

Hence, by combining the coefficients for each value of $l$ ($0 < l < b$) from the second series in \hfill (32) with that of the $l-1$th term from the first one, we obtain,

$$P^{b+1} X^r = \sum_{l=0}^{b+1} C_l (\theta, r, b+1, n_1, \ldots, n_{l-1}) X^{r+(n_1-1)+\ldots+(n_l-1)} P^{b+1-l}$$  \hfill (35)
3 Construction of Gauge theories, and the Seiberg-Witten map

To construct a gauge theory on the non-commutative space discussed above, we shall use the approach of Madore et al [1]. We shall now summarize some of their results in the context of a general non-commutative space coordinatized by the non-commuting variables $X^i$, obeying the relation,

$$[X^i, X^j] = \theta^{ij}(X).$$  \hspace{1cm} (36)

In this approach, the fields $\phi$, are taken to be elements of the space of formal power series generated by $X^i$, modulo the defining relations given above. The effect of a gauge transformation on the fields is taken to be of the form,

$$\delta_\alpha \phi(X) = i\alpha(X)\phi(X).$$  \hspace{1cm} (37)

The coordinates $(X^i)$ themselves are taken to be invariant under the gauge transformations. Since left multiplication by a coordinate is not a gauge covariant operation, generalized coordinates,

$$Q^i = X^i + A^i$$  \hspace{1cm} (38)

are introduced, and it is required that left multiplication by the generalized coordinates be a gauge covariant operation. i.e

$$\delta_\alpha(Q^i)\phi(X) = i\alpha(X)Q^i\phi(X).$$  \hspace{1cm} (39)

This restriction implies that,

$$\delta_\alpha(A^i)(X) = i[\alpha(X), A^i(X) + X^i].$$  \hspace{1cm} (40)

One can also construct a tensor $T^{ij}$:

$$T^{ij} = [Q^i, Q^j] - i\theta^{ij}(X) = [A^i, X^j] + [X^i, A^j] + [A^i, A^j]$$  \hspace{1cm} (41)

This tensor is gauge-covariant, and satisfies,

$$\delta_\alpha(T^{ij}) = i[\alpha(X), T^{ij}].$$  \hspace{1cm} (42)

Clearly, $A^i(X)$ and $T^{ij}(X)$ are the generalizations of the Yang-Mills connection and curvature to non-commutative geometry.

One can now use the $\Omega$ correspondence [10] to translate the statements made above into statements about functions of commuting variables $x^i$ as follows.

$$\delta_\alpha(x)\phi(x) = [\alpha(x) \ast \phi(x)]$$

$$\delta_\alpha(x)A^i(x) = [\alpha(x) \ast (A^i(x) + x^i)]$$

$$T^{ij} = [A^i(x), A^j(x)] + [x^i \ast A^j(x)] + [A^i(x), x^j]$$

$$\delta_\alpha(x)T^{ij}(x) = i[\alpha(x) \ast T^{ij}(x)]$$  \hspace{1cm} (43)
In fact more is possible. Just as the algebra of operator valued functions can be thought of as the algebra of functions of commuting variables with a deformed product, there exists a map between gauge theories on commutative spaces and gauge theories on non-commutative ones. This is the Seiberg-Witten map [2, 4, 24, 25, 26].

Denoting the connection on commutative spaces by \( a_i(x) \), and the infinitesimal gauge parameter by \( \epsilon(x) \), we recall that the gauge transformation is given by
\[
\delta_{\epsilon(x)} a_i(x) = \partial_i \epsilon(x) + [\epsilon(x), a_i(x)].
\] (44)

The connection \( A_i(x) \) and the gauge parameter \( \alpha(x) \) on the non-commutative space are now to be viewed as non-linear functions of \( a_i(x) \), \( \epsilon(x) \) and \( \theta_{ij}(x) \) such that the law for gauge transformation in the non-commutative world (40) follows from the ordinary gauge transformation for \( a_i(x) \).

We now present an explicit construction of the Seiberg-Witten map (up to \( O(\theta^2) \) for the non-commutative space defined by (12). We shall concentrate on the abelian gauge theories in this paper. For this specific case, we have,
\[
[f \ast g] = \theta(x) \left( \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial g}{\partial p} \frac{\partial f}{\partial x} \right) + \frac{1}{2} \theta(x)^2 \left( \frac{\partial^2 f}{\partial p^2} \frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial p^2} \frac{\partial^2 f}{\partial x^2} \right) + \frac{1}{2} \theta(x) \frac{\partial \theta(x)}{\partial x} \left( \frac{\partial^2 f}{\partial p^2} \frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial p^2} \frac{\partial f}{\partial x} \right) + O(\theta(x)^3). \] (45)

This allows us to write in an explicit form the variation of the connection under the ‘star’ gauge transformations(40). These are given by,
\[
\delta_{\alpha(x)} A_x = i\theta(x) \left( \frac{\partial \alpha}{\partial p} \left( 1 + \frac{\partial A_x}{\partial x} \right) - \frac{\partial A_x}{\partial p} \frac{\partial \alpha}{\partial x} \right) + \frac{i}{2} \theta(x)^2 \left( \frac{\partial^2 \alpha}{\partial p^2} \frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial p^2} \frac{\partial^2 \alpha}{\partial x^2} \right) + \frac{i}{2} \theta(x) \frac{\partial \theta(x)}{\partial x} \left( \frac{\partial^2 \alpha}{\partial p^2} \frac{\partial A_x}{\partial x} - \frac{\partial^2 A_x}{\partial p^2} \frac{\partial \alpha}{\partial x} \right) + O(\theta(x)^3),
\] (46)

and,
\[
\delta_{\alpha(x)} A_p = i\theta(x) \left( \frac{\partial \alpha}{\partial p} \frac{\partial A_p}{\partial x} - \left( 1 + \frac{\partial A_p}{\partial x} \right) \frac{\partial \alpha}{\partial x} \right) + \frac{i}{2} \theta(x)^2 \left( \frac{\partial^2 \alpha}{\partial p^2} \frac{\partial^2 A_p}{\partial x^2} - \frac{\partial^2 A_p}{\partial p^2} \frac{\partial^2 \alpha}{\partial x^2} \right) + \frac{i}{2} \theta(x) \frac{\partial \theta(x)}{\partial x} \left( \frac{\partial^2 \alpha}{\partial p^2} \frac{\partial A_p}{\partial x} - \frac{\partial^2 A_p}{\partial p^2} \frac{\partial \alpha}{\partial x} \right) + O(\theta(x)^3). \] (47)

We now make the following ansatz,
\[
\alpha = \epsilon + \gamma(\theta(x), \epsilon, a) + O(\theta(x)^2)
\]
\[
A_x = \theta(x)a_p + g_x(\theta(x), a) + O(\theta(x)^3)
\]
\[
A_p = -\theta(x)a_x + g_p(\theta(x), a) + O(\theta(x)^3)
\]  
(48)

This ansatz guarantees that to the first order on \(\theta(x)\), the variation of \(A\) is obtained by the variation of \(a\). It is implied that \(\gamma\) is of \(O(\theta(x))\) and that \(g_x, g_p\) are both of \(O(\theta(x)^2)\). To go beyond the leading order, we need to put the ansatz in equations (46) and (47). This gives us expressions for the variations of \(g_x\) and \(g_p\); which are,

\[
\delta_\alpha g_x = i\theta(x)^2 \left( \frac{\partial \epsilon}{\partial a_p} \frac{\partial a_p}{\partial x} - \frac{\partial a_p}{\partial p} \frac{\partial \epsilon}{\partial x} \right) + \theta(x) \frac{\partial \theta(x)}{\partial x} \left( \frac{\partial \epsilon}{\partial p} a_p + \frac{1}{2} \frac{\partial^2 \epsilon}{\partial p^2} \right) + i\theta(x) \frac{\partial \gamma}{\partial p} + O(\theta(x)^3),
\]  
(49)

and

\[
\delta_\alpha g_p = i\theta(x)^2 \left( \frac{\partial \epsilon}{\partial a_x} \frac{\partial a_x}{\partial p} - \frac{\partial \epsilon}{\partial p} \frac{\partial a_x}{\partial x} \right) - i\theta(x) \frac{\partial \theta(x)}{\partial x} \left( \frac{\partial \epsilon}{\partial a_p} a_p \right) - i\theta(x) \frac{\partial \gamma}{\partial x}.
\]  
(50)

If we now allow \(\gamma\) to have the following form,

\[
\gamma = \theta(x) \frac{\partial \epsilon}{\partial a_p} = \theta(x)(\delta \epsilon a_x) a_p,
\]  
(51)

it then follows that,

\[
\delta_\alpha g_x = \delta \epsilon \left( \frac{1}{2} i\theta(x) \frac{\partial \theta(x)}{\partial x} \left( a_p^2 + \frac{\partial a_p}{\partial p} \right) + i\theta(x)^2 \left( a_p \frac{\partial a_p}{\partial x} \right) \right),
\]

\[
\delta_\alpha g_p = \delta \epsilon \left( -i\theta(x)^2 \left( a_p \frac{\partial a_x}{\partial x} \right) - i\theta(x) \frac{\partial \theta(x)}{\partial x} (a_x a_p) \right).
\]  
(52)

In deriving these expressions we have used the relation,

\[
\delta \epsilon \left( \frac{\partial a_x}{\partial p} - \frac{\partial a_p}{\partial x} \right) = \frac{\partial^2 \epsilon}{\partial x \partial p} - \frac{\partial^2 \epsilon}{\partial x \partial p} = 0.
\]  
(53)

Equations (51) and (52) provide us with the following explicit expressions for the Seiberg-Witten map at the first non-trivial order,

\[
\alpha = \epsilon + \theta(x) \frac{\partial \epsilon}{\partial x} a_p + O(\theta(x)^2).
\]  
(54)
\[
A_x = \theta(x) a_p + \left( \frac{1}{2} i \theta(x) \frac{\partial \theta(x)}{\partial x} \left( a_p^2 + \frac{\partial a_p}{\partial p} \right) + i \theta(x)^2 \left( a_p \frac{\partial a_p}{\partial x} \right) \right) + O(\theta(x)^3).
\]

\[
A_p = -\theta(x) a_x + \left( -i \theta(x)^2 \left( a_p \frac{\partial a_x}{\partial x} \right) - i \theta(x) \frac{\partial \theta(x)}{\partial x} \left( a_x a_p \right) \right) + O(\theta(x)^3).
\]

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**References**

[1] J.Madore, S.Schraml, P.Schupp, and J.Wess, Eur. Phys. J. **C16**, 161 (2000).

[2] N.Seiberg and E.Witten, JHEP **9909**, 032 (1999).

[3] A.Konechny and A.Schwarz, Phys. Rept **360**, 353 (2002).

[4] R.J.Szabo, hep-th/0109162.

[5] M.Kontsevich, q-alg/9709040.

[6] J.Moyal, Proc. Camb. Phil. Soc **45**, 99 (1949).

[7] F.Bayen, M.Flato, M.Fronsdal, C.Lichnerowicz, and D.Sternheimer, Ann. Phys **111**, 61 (1978).

[8] A.Connes, *Noncommutative Geometry* (San Diego: Academic Press, 1994).

[9] H.J.Groenwold, Physica **12**, 405 (1946).

[10] A.S.Cattaneo and G.Felder, Commun. Math. Phys **212**, 591 (2000).

[11] I.Bakas and A.C.Kakas, Class. Quantum Grav **4**, L67 (1987).

[12] G.Agarwal and E.Wolf, Phys. Rev. D **2**, 2187 (1970).

[13] G.V.Dunne, J. Phys. A **21**, 2321 (1988).

[14] C.Zachos, J.Math.Phys **41**, 5129 (2000).

[15] C.Zachos, hep-th/0008010.

[16] S.Majid and H.Ruegg, Phys. Lett. B **334**, 348 (1994).

[17] J.Likierski, H.Ruegg, and W.J.Zakrzewski, Ann. Phys **143**, 90 (1995).

[18] Y.Manin, *Topics in Noncommutative Geometry* (Princeton University Press, Princeton, 1991).

[19] S.Cho, J.Madore, and K.Park, J. Phys. A: Math. Gen **31**, 2639 (1998).
[20] E. Demidov, Y. Manin, E. Mukhin, and D. Zhdanovich, Prog. Theor. Phys (Suppl) 102, 203 (1990).

[21] J. Madore and H. Steinacker, J. Phys. A: Math. Gen 33, 327 (2000).

[22] G. Amelino-Camelia and M. Arzano, Phys. Rev. D 65, 084044 (2002).

[23] A. Agostini, F. Lizzi, and A. Zampini, Mod. Phys. Lett. A 17, 2105 (2002).

[24] B. Jurco, P. Schupp, and J. Wess, Nucl. Phys. B 584, 784 (2000).

[25] B. Jurco and P. Schupp, Eur. Phys. J. C 14, 367 (2000).

[26] B. Jurco, P. Schupp, and J. Wess, Nucl. Phys. B 604, 148 (2001).