Analysis of a Markovian queueing system with single working vacation and impatience of customers

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Abstract. In this paper, we study an infinite capacity single server Markovian queue with a single working vacation and reneging of impatient customers in the queue during working vacation period. Customers arrive to the system following a Poisson distribution. The server goes to vacation when the system is empty and stay in vacation for a random period of time that is exponentially distributed. During the working vacation period, the server still continue providing service with a slow service rate. After the completion of the vacation, the server returns back to the regular service period and continue providing service with the regular busy period service rate if there are one or more customers in the system or it will stay idle until a new customer arrives to the system. During working vacation, customers in the queue get impatient and renege from the system and the reneging time is assumed to follow an exponential distribution. The system is modeled as a quasi-birth-death process and the stationary probabilities of the model are obtained using probability generating function approach. Some numerical analysis is also carried out to show the effect of some of the parameters on some selected performance measures of the system.

Keywords: Single working vacation, Reneging, Stationary probabilities, Probability generating function, Performance measures.

1. Introduction

Mathematical modeling and application of queueing models with server vacations have been studied by various researchers since the 1970’s. Vacation queueing models could conveniently represent congestion problems in various fields of study such as computer networks, telecommunication, production systems etc. Vacation queues have been examined extensively in [1, 2]. $M/M/1/N$ queueing system with balking, reneging, and server vacations was studied by [3] in which they derived the matrix form solution of the steady-state probabilities. Generalization of such queueing models are useful in model building of many real life situations such as digital communication, computer network and production/inventory systems [4, 5]. However, in these studies, the service is assumed to be totally terminated during vacation. Such a vacation is referred as a classical type of vacation. The case where during vacation, the server continues to provide service with a lower service rate is what is referred as a working vacation (WV). Servi and Finn [6] introduced WV’s for the first time and analyzed a single server Markovian queue with multiple working vacation policy. [7] studied an $M/M/1$ queue with multiple working vacation by using the matrix-geometric method and obtained explicit expressions of the performance measures. Analysis of $M/M/1$ queue with single working vacation
was also done by [8] by applying the same method. [9, 10, 11] extended the study by [6] to an $M/G/1/WV$ queue. [12] applied the matrix-geometric solution method to generalize the work of [6] to a $GI/M/1$ queue with general arrival process and multiple working vacations. Later, [11] investigated the $GI/M/1$ queue with single working vacation.

Impatience of customers due to long waiting time is a common phenomena that occurs in many queueing systems. Customers value time and tend to abandon the system when there is too much delay for a service. The behavior of customers that results in leaving the queue due to long waiting time as perceived by the customer is referred as reneging. The situation where customers decide not to join the queueing system due to anticipated waiting delay is known as balking. Patients in an emergency department may decide to renge from the system without receiving any treatment after waiting too long. This is one of the main reasons why emergency cases should either be given high priority or they will be forced to renge and decide go to hospitals where there is relatively small or no queue in the expense of high medical costs. In practice, reneging and balking decisions are highly affected by factors such as conceived information about the queue length, service completions and additional arrivals as stated by [13]. System congestion is also one factor over which customers adapt their patience level as studied through call-center setting by [14]. In this paper we consider customers renge during working vacation. We also assume that a customer who has already begun getting service will not renge from the system. Only customers waiting in the queue tend to get impatient and leave the queue when the system is working with a lower service rate during working vacation. Queueing models with system disasters and impatient customers when system is down have been studied by [15]. [16] considered the impatience of customers only when the servers are on vacation and unavailable for service. [17] presented queues with impatient customers in a fast and slow Markovian random environment. [18] investigated queues with working vacations, balking and reneging. [19] made an extensive analysis of queueing systems with reneging. [20] studied customers impatience in an queue with working vacations. [21] analyzed impatient customers in a single server Markovian queue with single and multiple working vacations.

In this paper we aim to study an $M/M/1$ queue with single working vacation and reneging of customers due to working vacation. We assume that a customer who joins the system when the server is idle will directly commence its service with zero waiting time and no possibility of reneging. We make use of the probability generating functions approach to determine the initial probabilities and the sum of steady-state probabilities during working vacation and sum of steady-state probabilities during regular busy period. We also study some performance measures such as the expected queue size and average reneging rate. The effect of model parameters on some system performance measures are also investigated under the numerical analysis section.

2. Model Description
We consider an $M/M/1$ queueing model subject to single working vacation and impatience of customers waiting in queue during working vacation. Arrivals occur according to a Poisson distribution with parameter $\lambda$. The server provides service according to an exponential distribution with parameter $\mu$ and $\eta$ during regular busy period and working vacation periods respectively where ($\eta < \mu$). When the system is empty, the server goes for a single working vacation and the vacation time follows an exponential distribution with parameter $\theta$. When the vacation ends, if there are customers in the system, the server switches back to its regular service rate; Otherwise, it will stay idle until a new customer arrives. A customer waiting for service may get impatient due to delay and decide to renge from the queue. Reneging times are exponentially distributed with parameter $\alpha$. The average reneging rate is $(n - 1)\alpha$, for $n \geq 1$. The queue discipline is first come first served and we assume that the arrival times, service times, the reneging times and the vacation times are all identically and independently distributed.

Let $N$ denote the number of customers in the system at time $t$ and $J$ denotes the state of the
Rewriting equation (6) for $z^n$ in the same manner, multiplying equations (3) and (4) by 1 and $n$ possible values of $\alpha$, we obtain

$$\alpha z(1-z)G'_0(z) + \lambda z^2 - (\lambda + \theta + (n-1)\alpha)z + (\eta - \alpha)\theta P_{0,0} = (1-z)\lambda z P_{0,0} - \mu z P_{1,1}. \quad (6)$$

In the same manner, multiplying equations (3) and (4) by 1 and $z^n$ respectively and adding the equations over all possible values of $n$, we obtain

$$(1-z)(\lambda z - \mu)G_1(z) = \theta z G_0(z) + (1-z)P_{0,1} - \mu z P_{1,1}. \quad (7)$$

Rewriting equation (6) for $z \neq 0$ and $z \neq 1$, we obtain

$$G'_0(z) - \left(\frac{\lambda}{\alpha} + \frac{\theta + \eta - \alpha}{\alpha(1-z)} - \frac{\eta - \alpha}{\alpha z(1-z)}\right)G_0(z) = \frac{(\eta - \alpha)}{\alpha z} P_{0,0} - \frac{\mu}{\alpha(1-z)} P_{1,1}. \quad (8)$$

Multiplying equation (8) from both sides by the integrating factor $e^{\frac{-\lambda x}{\alpha} (1-x)^{\beta}}(\frac{(\eta - \alpha)}{\alpha} x^{\frac{(n-\alpha)}{\alpha}})$, we obtain

$$G_0(z) = \frac{e^{\frac{-\lambda x}{\alpha} (1-z)}}{(1-z)^{\frac{n-\alpha}{\alpha}}} \left[\frac{(\eta - \alpha)}{\alpha} F_1(z) P_{0,0} - \frac{\mu}{\alpha} F_2(z) P_{1,1}\right] \quad (9)$$

where

$$F_1(z) = \int_0^z e^{\frac{-\lambda x}{\alpha} (1-x)^{\beta}} x^{\frac{(n-\alpha)}{\alpha}-1} dx \quad \text{and}$$

$$F_2(z) = \int_0^z e^{\frac{-\lambda x}{\alpha} (1-x)^{\beta}} x^{\frac{(n-\alpha)}{\alpha}} dx.$$

Since $0 \leq G_0(1) = \sum_{n=0}^{\infty} P_{n,0} \leq 1$ and $\lim_{z \to 1} \frac{\theta}{\alpha} = 0$, it must be that

$$\frac{(\eta - \alpha)}{\alpha} F_1(1) P_{0,0} - \frac{\mu}{\alpha} F_2(1) P_{1,1} = 0.$$

Which in turn gives

$$P_{1,1} = \frac{(\eta - \alpha)}{\mu} \frac{F_1(1)}{F_2(1)} P_{0,0}. \quad (10)$$
Evaluating equation (6) at \( z = 1 \) and using equation (10), we obtain

\[
\theta G_0(1) = \mu P_{1,1} = \frac{(\eta - \alpha)F_1(1)}{F_2(1)}P_{0,0}.
\]  

(11)

Using equation (10), equation (9) can be rewritten as

\[
G_0(z) = \frac{(\eta - \alpha)e^{\frac{\lambda z}{\alpha(1-z)^{\frac{\eta-\alpha}{\alpha}}}}}{\alpha z - \alpha} \left[ F_1(z) - F_1(1) F_2(z) \right] P_{0,0}
\]  

(12)

From equation (6), we have for \( z \neq 0 \) and \( z \neq 1 \),

\[
G_0'(z) = \frac{(1 - z)(\eta - \alpha)P_{0,0} - [\lambda z^2 - (\lambda + \theta + \eta - \alpha)z + (\eta - \alpha)]G_0(z) - \mu z P_{1,1}}{\alpha z(1-z)}.
\]  

(13)

Applying Li hospital’s rule on equation (13), we obtain \( G_0'(1) \) as

\[
G_0'(1) = \frac{(\lambda - (\eta - \alpha))G_0(1) + (\eta - \alpha)P_{0,0}}{\alpha + \theta}.
\]  

(14)

From equation (7), we have for \( z \neq 1 \),

\[
G_1(z) = \frac{\theta z G_0(z) - \mu(1-z)P_{0,1} - \mu z P_{1,1}}{(1-z)(\lambda z - \mu)}.
\]  

(15)

Applying Li hospital’s rule on equation (15), we obtain \( G_1(1) \) as

\[
G_1(1) = \frac{\theta G_0'(1) + \mu P_{0,1}}{\mu - \lambda}.
\]  

(16)

It is easy to see from equation (3) that

\[
P_{0,1} = \frac{\theta P_{0,0}}{\lambda}.
\]  

(17)

From the normalization condition, we have that \( G_0(1) + G_1(1) = \sum_{n=0}^{\infty} P_{n,0} + \sum_{n=0}^{\infty} P_{n,1} = 1 \).

Thus, we have, using equations (11), (14), (16) and (17),

\[
P_{0,0} = \left\{ \left( \frac{(\eta - \alpha)F_1(1)}{\theta F_2(1)} \right) + \frac{(\lambda - (\eta - \alpha))(\eta - \alpha)F_1(1)}{(\alpha + \theta)(\mu - \lambda)F_2(1)} + \frac{\theta(\eta - \alpha)}{(\alpha + \theta)(\mu - \lambda)} + \frac{\mu \theta}{\lambda} \right\}^{-1}.
\]  

(18)

3. Performance Measures

- Expected number of customers in the system during working vacation, \( E[N_{WV}] \) is given by

\[
E[N_{WV}] = G_0'(1) = \frac{(\lambda - (\eta - \alpha))G_0(1) + (\eta - \alpha)P_{0,0}}{\alpha + \theta},
\]  

(19)

where \( G_0(1) \) and \( P_{0,0} \) are as given in equations (11) and (18) respectively.

- To evaluate the expected number of customers in the system during regular busy period, denoted by \( E[N_B] \), we differentiate equation (7) with respect to \( z \) and applying L’Hospital’s rule, we obtain the limit of \( G_1'(z) \) at \( z = 1 \) as

\[
E[N_B] = G_1'(1) = \frac{\theta G_0''(1)}{2(\mu - \lambda)} + \frac{\mu \theta G_0'(1)}{(\mu - \lambda)^2} + \frac{\mu \theta P_{0,0}}{(\mu - \lambda)^2}.
\]  

(20)
where $G_0'(1)$ is computed by taking the derivative of equation (6) with respect to $z$ and applying L’Hospital’s rule at $z = 1$ as

$$G_0'(1) = \frac{2(\lambda - \theta - \eta)G_0'(1) + 2\lambda G_0(1)}{2\alpha + \theta}$$

(21)

such that $G_0'(1)$ and $G_0(1)$ are given in equations (14) and (11) respectively.

- Expected number of customers in the system, $E[N]$, is thus given as

$$E[N] = E[N_{WV}] + E[N_B]$$

(22)

- Expected reneging rate of the system, denoted by $E[RR]$ is given as

$$E[RR] = \sum_{n=1}^{\infty} \alpha(n-1)P_{n,0} = \alpha(G_0'(1) - G_0(1) + P_{0,0})$$

(23)

4. Numerical Analysis

Numerical computations have been carried out to study the parameter impact on the system performance and a few of those are presented in the form of tables and graphs. We consider the parameters as $\lambda = 2$, $\eta = 3$, $\mu = 5$, $\theta = 3$ and $\alpha = 0.7$ for all the figures and tables, unless they are considered as variables or their values are mentioned in the respective figures and tables.

From Table 1, we can see that with the increase in the rate of arrival($\lambda$) results in the increase of performance measures $E[N_{WV}]$, $E[N_B]$, $E[N]$ and $E[RR]$. Also the percent variation in the last row indicates the increasing trend. That is, as the arrival rate increases, the number of customers joining the system increases, and it is obvious that probability that the system is idle during working vacation($P_{0,0}$) and regular busy period($P_{0,1}$) decrease.

Table 2 shows the effect of the service rate during working vacation on the performance measures. With the increase in the service rate $\eta$, $E[N_B]$, $E[N_{WV}]$, $E[N]$, $E[RR]$ and percent variation in the last row decrease. This is logical since higher service rate implies faster service, hence results in small queue size. On the other hand, $P_{0,0}$ and $P_{0,1}$ increase.

Table 3 shows the effect of vacation rate ($\theta$) on the performance measures. As $\theta$ increases, the mean vacation time decreases and the server spends lesser duration of time in working vacations. Hence $E[N_{WV}]$, $E[N]$, $E[RR]$ and $P_{0,0}$ decrease. Also the percent variation in the last row indicates decreasing trend. On the other hand, $E[N_B]$ and $P_{0,1}$ increase.

In Table 4, the effect of $\eta$ on various performance measures for different values of $\alpha$ is presented. It is observed that for a fixed $\eta$, increasing $\alpha$ results in the decrease of $E[N_{WV}]$, $E[N_B]$, $E[N]$. While the performance measures $E[RR]$, $P_{0,0}$ and $P_{0,1}$ increase with the increase of $\alpha$. This follows from the fact that reneging results in the decrease of the queue size regardless of what. On the other hand, the decreasing queue size implies higher tendency or probability that the system gets empty. Hence the increase in the probabilities $P_{0,0}$ and $P_{0,1}$. Obviously $E[RR]$ increases with the increase in $\alpha$.

The impact of arrival rate ($\lambda$) on $E[N_B]$ is shown in Figure 1 for different busy period service rates ($\mu$). It can be observed that $E[N_B]$ increases with increase in $\lambda$. For any given $\lambda$, $E[N_B]$ decreases as $\mu$ increases because customers served per unit time increases.
Figure 1. Effect of $\lambda$ on $E[N_B]$ for different $\mu$

Figure 2. Effect of $\lambda$ on $E[N]$ for different $\theta$

Figure 3. Effect of $\mu$ on $E[N]$ for different $\eta$
Table 1. Effect of $\lambda$ on performance measures

| $\lambda$ | $E[N_{WV}]$ | $E[N_B]$ | $E[N]$ | $E[RR]$ | $P_{0.0}$ | $P_{0.1}$ |
|-----------|--------------|----------|--------|--------|----------|----------|
| 2.0       | 0.116025     | 0.610454 | 0.726479 | 0.0202933 | 0.2277   | 0.34155  |
| 2.2       | 0.129999     | 0.725251 | 0.855249 | 0.024583 | 0.222947 | 0.304018 |
| 2.4       | 0.142631     | 0.859897 | 1.00253  | 0.0289139 | 0.215668 | 0.269585 |
| 2.6       | 0.153516     | 1.01926  | 1.17278  | 0.0331322 | 0.206214 | 0.237939 |
| Percentage variation | 3.74 | 40.88 | 44.63 | 1.28 | -2.14 | -10.36 |

Table 2. Effect of $\eta$ on performance measures

| $\eta$ | $E[N_{WV}]$ | $E[N_B]$ | $E[N]$ | $E[RR]$ | $P_{0.0}$ | $P_{0.1}$ |
|--------|--------------|----------|--------|--------|----------|----------|
| 3.2    | 0.112473     | 0.60512  | 0.717593 | 0.019273 | 0.22931  | 0.343966 |
| 3.4    | 0.109094     | 0.600107 | 0.709201 | 0.0183142 | 0.23085  | 0.346275 |
| 3.6    | 0.105878     | 0.595393 | 0.701271 | 0.0174177 | 0.232322 | 0.348483 |
| 3.8    | 0.102815     | 0.590955 | 0.693771 | 0.0165791 | 0.23373  | 0.350595 |
| Percentage variation | -0.96 | -1.41 | -2.38 | -0.26 | 0.44 | 0.66 |

Table 3. Effect of $\theta$ on performance measures

| $\theta$ | $E[N_{WV}]$ | $E[N_B]$ | $E[N]$ | $E[RR]$ | $P_{0.0}$ | $P_{0.1}$ |
|----------|--------------|----------|--------|--------|----------|----------|
| 4        | 0.0781052    | 0.630551 | 0.708657 | 0.0120423 | 0.192683 | 0.385365 |
| 5        | 0.0561273    | 0.641629 | 0.697756 | 0.00774796 | 0.166722 | 0.416804 |
| 6        | 0.0422646    | 0.648343 | 0.690608 | 0.00528698 | 0.146793 | 0.44038 |
| 7        | 0.0329646    | 0.652704 | 0.685669 | 0.00377258 | 0.13105 | 0.458675 |
| Percentage variation | -4.51 | 2.21 | -2.29 | -0.82 | -6.16 | 7.33 |

Table 4: Effect of $\eta$ on performance measures for different $\alpha$

| $\eta$ = 3 | $\alpha$ = 0.6 | $\alpha$ = 0.7 | $\alpha$ = 0.8 |
|------------|----------------|----------------|----------------|
| $E[N_{WV}]$ | 0.116702       | 0.116025       | 0.115375       |
| $E[N_B]$    | 0.612303       | 0.610454       | 0.608705       |
| $E[N]$      | 0.729005       | 0.726479       | 0.72408        |
| $E[RR]$     | 0.0177606      | 0.0202933      | 0.022739       |
| $P_{0.0}$   | 0.227485       | 0.2277         | 0.227907       |
| $P_{0.1}$   | 0.341227       | 0.34155        | 0.34186        |

| $\eta$ = 4 | $\alpha$ = 0.6 | $\alpha$ = 0.7 | $\alpha$ = 0.8 |
|------------|----------------|----------------|----------------|
| $E[N_{WV}]$ | 0.1004        | 0.099897       | 0.0994129      |
| $E[N_B]$    | 0.588052      | 0.586775       | 0.58556        |
| $E[N]$      | 0.688452      | 0.686672       | 0.684973       |
| $E[RR]$     | 0.013791      | 0.0157938      | 0.0177261      |
| $P_{0.0}$   | 0.23491       | 0.235077       | 0.235238       |
| $P_{0.1}$   | 0.352635      | 0.352615       | 0.352857       |

| $\eta$ = 5 | $\alpha$ = 0.6 | $\alpha$ = 0.7 | $\alpha$ = 0.8 |
|------------|----------------|----------------|----------------|
| $E[N_{WV}]$ | 0.0875782     | 0.0872018      | 0.0868383      |
| $E[N_B]$    | 0.570041      | 0.56938        | 0.56674        |
| $E[N]$      | 0.657619      | 0.65634        | 0.655113       |
| $E[RR]$     | 0.0109301     | 0.0125405      | 0.0140996      |
| $P_{0.0}$   | 0.240874      | 0.241003       | 0.241128       |
| $P_{0.1}$   | 0.361312      | 0.361505       | 0.361692       |

The impact of arrival rate ($\lambda$) on $E[N]$ is shown in Figure 2 for different vacation rates ($\theta$). It can be observed that $E[N]$ increases with increase in $\lambda$. For any given $\lambda$, $E[N]$ decreases with increase of vacation rate. This is because the vacation rate increases means the average vacation time decreases or the average busy time increases.

Figure 3 depicts the impact of service rate during a regular busy period ($\mu$) on mean system size $E[N]$ for different service rates during working vacation ($\eta$). We observed that when $\mu$
increases, $E[N]$ decreases. In addition, for a fixed $\mu$, $E[N]$ decreases when $\eta$ increases.

5. Conclusion
In this paper, we have studied an M/M/1 queueing system with single working vacation and customers’ impatience. We have derived probability generating functions of the number of customers in the system and the corresponding mean system sizes. We have obtained the mean system size and other performance measures. The effects of some parameters on the performance measures of the system have been investigated numerically and presented graphically.

References
[1] Ke J C, Wu C H and Zhang Z G 2010 Inter. Jour. of Oper. Res. 7 3-8.
[2] Tian N and Zhang G 2006 Springer-Verlag, New York
[3] Yue D, Zhang Y and Yue W 2006 Inter. Jour. of Pure and App. Math. 28 101-115
[4] Doshi B T 1986 Queueing sys. 1 29–66
[5] Takagi H 1991 Vaca. and pri. syst. Vol. I. Part I. North-Holland, Amsterdam
[6] Servi L D and Finn S G 2002 Perf. Eval. 50 41–52
[7] Liu W Y, Xu X L and Tian N S 2007 Oper. Res. Let. 35 595–600
[8] Tian N, Zhao X and Wang K 2008 Inter. Jour. of Infor. and Mon. Sci. 19 621–634
[9] Kim J, Choi D and Chae K 2003 Inter. Conf. on Stat. and Rel. Fields, Hawaii
[10] Wu D. and Takagi H. 2006 Perf. Eval. 63 654–681
[11] Li J, Tian N, Zhang ZG and Luh HP 2011 Queu. Sys. 61 139-166
[12] Baba Y 2005 Oper. Res. Lett. 33 201–209
[13] Batt R J and Terwiesh C 2015 Manag. Sci. 61 39–59
[14] Zohar E, Mandelbaum A and Shimkin N 2002 Manag. Sci. 48 566–583
[15] Yechiali U 2007 Queu. Syst. 56 195-202
[16] Altman E and Yechiali U 2006 Queueing Systems 52 261-279
[17] Perel N and Yechiali U 2010 Europ. Jour. of Ope. Res. 201 247-258
[18] Laxmi P V, Goswami V and Jyothsna K 2013 Inter. Jour. of Strategic Dec. Sci. 4 1-24
[19] Bocquet S 2005 Defence Sci. and Tech. Org.
[20] Yue D, Yue W and Xu G, 2012 Journal of Industrial and Management Optimization 8 895-908
[21] Selvaraju N and Goswami C 2013 Comp. and Indus. Eng. 65 207-215