RESOLVING THE QUESTION OF TIME FOR

SEMI-CLASSICAL GRAVITY

D.S. SALOPEK

Department of Physics
University of Alberta
Edmonton, Canada T6G 2J1

ABSTRACT

Hamilton-Jacobi theory provides a natural starting point for a covariant description of the gravitational field. Using a spatial gradient expansion, one may solve for the phase of the wavefunction by using a line-integral in superspace. Each contour of integration corresponds to a particular choice of the time-hypersurface, and each yields the same answer. In this way, one can describe all time choices simultaneously. As a demonstration of the formalism, I will compute large-angle microwave background anisotropies and the galaxy-galaxy correlation function associated with scalar and tensor fluctuations of power-law inflation.

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1. Introduction

Hamilton-Jacobi (HJ) theory is a cornerstone of modern theoretical physics. It may be profitably applied to numerous problems in cosmology, and it is particularly useful in describing the inflationary scenario. Since a full quantum theory of the gravitational field is lacking, we should be content with a semi-classical treatment (at least for now).

HJ theory has been successfully employed in deriving the Zel’dovich approximation (which describes the formation of sheet-like structures in the Universe) and its higher order generalizations from general relativity. Various researchers have employed HJ methods in an attempt to recover the inflaton potential from cosmological observations. Moreover, they can be used to construct inflationary models that yield non-Gaussian primordial fluctuations; such models could possibly resolve the problem of large scale structure.

Here I will focus on one particularly attractive feature of HJ theory: it provides a covariant formulation of the gravitational field. In the semi-classical theory, the answer to the question of time is clear: time is arbitrary. What is required then is a formalism that treats all time choices simultaneously, and I will demonstrate that HJ theory passes the test. Before describing the general technique, I will consider a simple analogy from potential theory which illuminates the essential point.

2. Potential Theory

The fundamental problem in potential theory is: given a force field $g^i(u_k)$ which is a function of $n$ variables $u_k$, what is the potential $\Phi \equiv \Phi(u_k)$ (if it exists) whose gradient returns the force field,

$$\frac{\partial \Phi}{\partial u_i} = g^i(u_k) \quad ?$$

Not all force fields are derivable from a potential. Provided that the force field satisfies the integrability relation,

$$0 = \frac{\partial g^i}{\partial u_j} - \frac{\partial g^j}{\partial u_i} = \left[ \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_i} \right] \Phi,$$

(i.e., it is curl-free), one may find a solution which is conveniently expressed using a
If the two endpoints are fixed, all contours return the same answer. In practice, I will employ the simplest contour that one can imagine: a line connecting the origin to the observation point \( u_k \). Using \( s \), \( 0 \leq s \leq 1 \), to parameterize the contour, the line-integral may be rewritten as

\[
\Phi(u_k) = \sum_{j=1}^{n} \int_{0}^{1} ds \ u_j \ g^j(su_k) .
\]

(4)

Similarly, in solving for the phase of the wavefunctional, I will employ a line-integral in superspace.

### 3. Solving the Hamilton-Jacobi Equation for General Relativity

The Hamilton-Jacobi equation for general relativity is derived using a Hamiltonian formulation of gravity. One first writes the line element using the ADM 3+1 split,

\[
ds^2 = \left( -N^2 + \gamma^{ij} N_i N_j \right) dt^2 + 2N_i dt \ dx^i + \gamma_{ij} \ dx^i \ dx^j ,
\]

(5)

where \( N \) and \( N_i \) are the lapse and shift functions, respectively, and \( \gamma_{ij} \) is the 3-metric. Hilbert’s action for gravity interacting with a scalar field becomes

\[
\mathcal{I} = \int d^4x \left( \pi^\phi \dot{\phi} + \pi^{ij} \dot{\gamma}_{ij} - \mathcal{H} - N^i \mathcal{H}_i \right) .
\]

(6)

The lapse and shift functions are Lagrange multipliers that ensure that the energy constraint \( \mathcal{H}(x) \) and the momentum constraint \( \mathcal{H}_i(x) \) vanish.

The object of chief importance is the generating functional \( S \equiv S[\gamma_{ij}(x), \phi(x)] \). For each universe with field configuration \( [\gamma_{ij}(x), \phi(x)] \) it assigns a number which can be complex. The generating functional is the ‘phase’ of the wavefunctional in the semi-classical approximation: \( \Psi \sim e^{iS} \). For the applications that we are considering, the prefactor before the exponential is not very important, although it has interesting consequences for quantum cosmology. The probability functional, \( P \equiv |\Psi|^2 \), is given by the square of the wavefunctional.

Replacing the conjugate momenta by functional derivatives of \( S \) with respect to the fields,

\[
\pi^{ij}(x) = \frac{\delta S}{\delta \gamma_{ij}(x)} , \quad \pi^{\phi}(x) = \frac{\delta S}{\delta \phi(x)} ,
\]

(7)
and substituting into the energy constraint, one obtains the Hamilton-Jacobi equation,

\[
\mathcal{H}(x) = \gamma^{-1/2} \frac{\delta S}{\delta \gamma_{ij}(x)} \frac{\delta S}{\delta \gamma_{kl}(x)} \left[ 2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x) \right] + \frac{1}{2} \gamma^{-1/2} \left( \frac{\delta S}{\delta \phi(x)} \right)^2 + \gamma^{1/2} V(\phi(x)) - \frac{1}{2} \gamma^{1/2} R + \frac{1}{2} \gamma^{1/2} \gamma_{ij} \phi, i \phi, j = 0 ,
\]

which describes how \( S \) evolves in superspace. \( R \) is the Ricci scalar associated with the 3-metric, and \( V(\phi) \) is the scalar field potential. In addition, one must also satisfy the momentum constraint

\[
\mathcal{H}_i(x) = -2 \left( \gamma_{ik} \frac{\delta S}{\delta \gamma_{kj}(x)} \right)_j + \frac{\delta S}{\delta \gamma_{lk}(x)} \gamma_{ki} + \frac{\delta S}{\delta \phi(x)} \phi, i = 0 ,
\]

which legislates that \( S \) be invariant under reparametrizations of the spatial coordinates. (Units are chosen so that \( c = 8\pi G = \bar{\hbar} = 1 \). Since neither the lapse function nor the shift function appears in eqs.\( (8,9) \) the temporal and spatial coordinates are arbitrary: HJ theory is covariant.

In order to solve eqs.\( (8,9) \), I will expand the generating functional

\[
S = S^{(0)} + S^{(2)} + S^{(4)} + \ldots ,
\]

in a series of terms according to the number of spatial gradients that they contain. As a result, the Hamilton-Jacobi equation can likewise be grouped into terms with an even number of spatial derivatives:

\[
\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \mathcal{H}^{(4)} + \ldots .
\]

The invariance of the generating functional under spatial coordinate transformations suggests a solution of the form,

\[
S^{(0)}[\gamma_{ij}(x), \phi(x)] = -2 \int d^3 x \gamma^{1/2} H[\phi(x)] ,
\]

for the zeroth order term \( S^{(0)} \). The function \( H \equiv H(\phi) \) satisfies the separated HJ equation of order zero,

\[
H^2 = \frac{2}{3} \left( \frac{\partial H}{\partial \phi} \right)^2 + \frac{1}{3} V(\phi) ,
\]

which is an ordinary differential equation. Note that \( S^{(0)} \) contains no spatial gradients.
In order to compute the higher order terms, one introduces a change of variables, 
$(\gamma_{ij}, \phi) \rightarrow (f_{ij}, u)$:

$$u = \int \frac{d\phi}{-2\partial H}, \quad f_{ij} = \Omega^{-2}(u) \gamma_{ij}, \quad (14)$$

where the conformal factor $\Omega \equiv \Omega(u)$ is defined through

$$\frac{d \ln \Omega}{du} \equiv -2 \frac{\partial H}{\partial \phi} \frac{\partial \ln \Omega}{\partial \phi} = H. \quad (15)$$

in which case the equation for $S^{(2m)}$ becomes

$$\frac{\delta S^{(2m)}}{\delta u(x)} \bigg|_{f_{ij}} + R^{(2m)}[u(x), f_{ij}(x)] = 0. \quad (16)$$

The remainder term $R^{(2m)}$ depends on some quadratic combination of the previous order terms (it may be written explicitly). For example, for $m = 1$, it is

$$R^{(2)} = \frac{1}{2} \gamma^{1/2} \gamma_{ij} \phi_i \phi_j - \frac{1}{2} \gamma^{1/2} R. \quad (17)$$

Eq.(14) has the form of an infinite dimensional gradient. It may integrated using a line integral analogous to eq.(4):

$$S^{(2m)} = -\int d^3x \int_0^1 ds \ u(x) \ R^{(2m)}[su(x), f_{ij}(x)]; \quad (18)$$

the conformal 3-metric $f_{ij}(x)$ is held constant during the integration which may be performed explicitly in many cases of interest.

The integrability condition for the HJ equation follows from the Poisson bracket of the energy constraints evaluated at spatial points $x$ and $x'$,

$$\{\mathcal{H}(x^k), \mathcal{H}(x'^k)\} = [\gamma^{ij}(x^k) \mathcal{H}_j(x^k) + \gamma^{ij}(x'^k) \mathcal{H}_j(x'^k)] \delta^3_{ij}(x^k - x'^k). \quad (19)$$

In fact, alternative contours describing the line-integral will correspond to different time-hypersurface choices. Provided that the generating functional is invariant under reparametrizations of the spatial coordinates, (e.g., $\mathcal{H}_i$ vanishes in the right-hand-side of eq.(15)), different time-hypersurface choices will lead to the same generating functional. Hypersurface invariance is closely related to gauge-invariance. Hence the line-integral solution goes a long way in understanding the role of time in semi-classical gravity.
4. Computing Large-Angle Microwave Background Fluctuations and Galaxy Correlations

In order to describe the fluctuations arising during the inflationary epoch, it is necessary to sum an infinite subset of the terms $S^{(2m)}$. In this case, one considers all terms which are quadratic in the Ricci tensor $\tilde{R}_{ij}$ of the conformal 3-metric $f_{ij}(x)$. Once again, no explicit choice of time hypersurface is made.

However, when one compares with observations, there are indeed preferred gauges. The phase transition where photons decouple from matter occurs essentially on a uniform temperature slice, $T \sim 4000K$, when protons combine with electrons to form neutral hydrogen. For adiabatic perturbations at large wavelengths, this is the same as a comoving, synchronous time slice which was the choice made by Sachs and Wolfe in the computation of large-angle temperature anisotropies.

Here I will be content to graphically display the final results for the power-law inflationary model where the scalar field potential is given by

$$V(\phi) = V_0 \exp \left( -\frac{2}{p} \phi \right)$$

where $p$ is a constant that determines the steepness of the potential. This model is of high interest because it may produce copious amounts of primordial gravitational radiation, which is in essence a quantum gravitational effect.

For various values of $p$, Fig.(1) shows the power spectrum, $P_\zeta(k)$, for $\zeta$ which parametrizes the primordial scalar perturbations associated with the comoving wavenumber $k$. (The present value of the Hubble parameter is assumed to be $H_0 = 50$ km s$^{-1}$ Mpc$^{-1}$.) The power spectra have been normalized using the 2-year data set of the DMR (Differential Microwave Radiometer) experiment on board of the COBE (Cosmic Background Explorer) satellite:

$$\sigma_{\text{sky}}(10^0) = 30.5 \pm 2.7 \mu K \ (68\% \text{ confidence level}).$$

The spectral index $n_s$ for scalar perturbations is defined by

$$n_s \equiv 1 + \frac{d \log_{10} P_\zeta(k)}{d \log_{10}(k)} = 1 - \frac{2}{(p - 1)}.$$  \hspace{1cm} (22)

As $p \to \infty$, one recovers the flat Zel’dovich spectrum $n_s = 1$, and there is no graviton production. As $p$ decreases, the spectrum tilts giving more power at larger scales, and the graviton production increases, contributing 50% to COBE’s signal at $n_s = 0.8$. (For comparison, if there were no graviton production, essentially all the curves would join at $k = 10^{-4}$Mpc which is the length scale effectively probed by COBE).
In Fig.(2), I plot the linear density contrast at the present epoch for the same set of models, assuming the cold-dark-matter model. The bold line shows the observed clustering of galaxies: \( \xi_{gg}(r) = \left( \frac{r}{r_0} \right)^{-\gamma} \) where \( r_0 = 10 \) Mpc, \( \gamma = 1.8 \). In order that there be enough fluctuations to seed galaxies, one requires that the biasing parameter \( b_\rho \) be less than 2 which implies that

\[
ns > 0.8, \quad \text{(power - law inflation).} \tag{23}
\]

This constraint was more stringent than the lower limit given by COBE’s first year data set: \( ns = 1.1^{+0.45}_{-0.32} \) (68% confidence level).

Using two years of data, the COBE DMR team now reports \( ns = 1.10 \pm 0.32 \) (68% confidence level). Here they have included the quadrupole amplitude in their computations. However in determining the quadrupole, one must subtract out the contribution of the Milky Way which is quite tricky to do in practice. For this reason, one may wish to exclude the quadrupole component in which case they obtain \( ns = 0.87 \pm 0.36 \). Either way, their results are still consistent with the simplest models of inflation which yield \( ns < 1 \). In addition, they are consistent with the limit for power-law inflation eq.(23).
5. Conclusions

The question of the choice of time is an extremely difficult one, particularly for the quantum theory of the gravitational field. Here I have shed some light on the semiclassical problem. The choice of time is arbitrary, and using Hamilton-Jacobi theory one may construct a covariant formalism which treats all time choices on an equal footing. A line-integral in superspace allows one to solve for the phase of the wavefunctional. Different contours of integration lead to the same answer provided gauge-invariance is maintained.

![Power Spectra for Density Perturbation](image)

**Fig. 2.** For the present epoch, the power spectra for the linear density perturbation $\delta \rho / \rho$ in comoving synchronous gauge are shown. The dark line depicts the observed two-point correlation function describing galaxy clustering. If there is no biasing, $n_s = 0.9$ gives a good fit to the data near $k = 10^{-1} Mpc^{-1}$. In order that there be enough fluctuations to seed galaxies, one requires that the biasing parameter $b_\rho$ be less than 2 which implies that $n_s > 0.8$. As a result for power-law inflation, at most $\sim 50\%$ of COBE’s signal may arise from gravitational waves.

As an application, I briefly summarized computations of large-angle microwave background anisotropies and the galaxy-galaxy correlation function for power-law inflation used in conjunction with the cold-dark-matter model. This inflation model is particularly interesting because it may produce large amounts of primordial gravitational waves. As a result, the spectral index is restricted to be $n_s > 0.8$ otherwise there are not enough fluctuations to seed galaxies.
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