CUTTING RESILIENT NETWORKS

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Abstract

We define the (random) $k$-cut number of a rooted graph to model the difficulty of the destruction of a resilient network. The process is as the cut model of Meir and Moon [29] except now a node must be cut $k$ times before it is destroyed. The $k$-cut number of a path of length $n$, $X_n$, is a generalization of the concept of records in permutations. The first order terms of the expectation and variance of $X_n$, with explicit formula for the constant factors, are proved. We also show that $X_n$, after rescaling, converges in distribution to a limit. The paper concludes with some connections between $X_n$ and $k$-cut numbers with general trees and general graphs.

1 INTRODUCTION

1.1 The $k$-cut number of a graph

Consider $G_n$, a connected graph consisting $n$ nodes with exactly one node labeled as the root, which we call a rooted graph. Let $k$ be an integer. We remove nodes from the graph as follows:

1. Choose a node uniformly at random from the component that contains the root. Cut the selected node once.

2. If this node has been cut $k$ times, remove it from the graph.

3. If the root has been removed, then stop. Otherwise, go to step 1.

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We call the (random) total number of cuts needed to end this procedure the $k$-cut number and denote it by $\mathcal{K}(G_n)$. (Note that in traditional cutting models, nodes are removed as soon as they are cut once, i.e., $k = 1$. But in our model, a node is only removed after being cut $k$ times.)

One can also define an edge version of this process. Instead of cutting nodes, each time we choose an edge uniformly at random from the component that contains the root and cut it once. If the edge has been cut $k$-times then we remove it. The process stops when the root is isolated. We let $\mathcal{K}^e(G_n)$ denote the number of cuts needed for the process to end.

Throughout human history, various secret societies of very different structures existed [14]. Nonetheless, most such societies have a few leaders who are critical for the organizations to function properly. The $k$-cut process can be seen as a simplified model of the destruction process of a resilient secret network. The graph $G_n$ represents the structure of the network and the root node represents the leader. We assume that active members of the network are chosen uniformly at random to be investigated by the authority, and that a member stops operating after having been investigated $k$ times. The network completely breaks down when the root (leader) stops working. Thus the random number $\mathcal{K}(G_n)$ models how much effort it takes to destroy the network.

**Remark 1.** A model of similar flavor was introduced for the destruction of terrorist cells with tree-like structures [15]. In this model, each node of a tree is removed in one step, independently at random with some fixed probability. The quantity being studied is the probability that the root node (leader) is separated from all the leaves (operatives). This model has been studied for deterministic trees [6] and for conditioned Galton-Watson trees [10].

Our model can also be applied to botnets, i.e., malicious computer networks consisting of compromised machines which are often used in spamming or attacks. The nodes in $G_n$ represent the computers in a botnet, and the root represent the bot-master. The effectiveness of a botnet can be measured using the size of the component containing the root, which indicates the resources available to the bot-master [8]. To take down a botnet means to reduce the size of this root component as much as possible. If we assume that we target infected computers uniformly at random and it takes at least $k$ attempts to fix a computer, then the $k$-cut number measures how difficult it is to completely isolate the bot-master.

The case $k = 1$ and $G_n$ being a rooted tree has aroused great interests among mathematicians in the past few decades. The edge version of one-cut was first introduced by Meir and Moon [29] for the uniform random Cayley tree. Janson [23, 24] noticed the equivalence between one-cuts and records in trees and studied them in binary trees and conditional Galton-Watson trees. Later Addario-Berry, Broutin, and Holmgren [1] gave a simpler proof for the limit distribution of one-cuts in conditional Galton-Watson trees. For one-cuts in random recursive trees,
see Meir and Moon [30], Iksanov and Möhle [22], and Drmota, Iksanov, Moehle, and Roesler [12]. For binary search trees and split trees, see Holmgren [19, 20].

1.2 The k-cut number of a tree

One of the most interesting cases is when $G_n = T_n$, where $T_n$ is a rooted tree with $n$ nodes.

There is an equivalent way to define $\mathcal{K}(T_n)$. Imagine that each node is given an alarm clock. At time zero, the alarm clock of node $j$ is set to ring at time $T_{1,j}$, where $(T_{i,j})_{i,j \geq 1}$ are i.i.d. (independent and identically distributed) Exp(1) random variables. After the alarm clock of node $j$ rings the $i$-th time, we set it to ring again at time $T_{i+1,j}$. Due to the memoryless property of exponential random variables (see [13], pp. 134), at any moment, which alarm clock rings next is always uniformly distributed. Thus if we cut a node that is still in the tree when its alarm clock rings, and remove the node with its descendants if it has already been cut $k$-times, then we get exactly the $k$-cut model.

How can we tell if a node is still in the tree? When node $j$’s alarm clock rings for the $r$-th time for some $r \leq k$, and no node above $j$ has already rung $k$ times, we say $j$ has become an $r$-record. And when a node becomes an $r$-record, it must still be in the tree. Thus summing the number of $r$-records over $r \in \{1, \ldots, k\}$, we again get the $k$-cut number $\mathcal{K}(T_n)$. One node can be a 1-record, a 2-record, etc., at the same time, so it can be counted multiple times. Note that if a node is an $r$-record, then it must also be a $j$-record for $j \in \{1, \ldots, r-1\}$. To be more precise, we define $\mathcal{K}(T_n)$ as a function of $(T_{i,j})_{i,j \geq 1}$. Let

$$G_{r,j} \overset{\text{def}}{=} \sum_{i=1}^{r} T_{i,j},$$

i.e., $G_{r,j}$ is the moment when the alarm clock of node $j$ rings for the $r$-th time. Then $G_{r,j}$ has a gamma distribution with parameters $(r, 1)$ (see [13, Thm. 2.1.12]), which we denote by $\text{Gamma}(r)$. In other words, $G_{r,j}$ has the density function,

$$f_{G_{r,j}}(x) = \begin{cases} e^{-x}x^{r-1} \frac{1}{\Gamma(r)} & (x \geq 0), \\ 0 & (x < 0). \end{cases}$$

where $\Gamma(z)$ denotes the gamma function [11, 5.2.1]. Let

$$I_{r,j} \overset{\text{def}}{=} \left[ G_{r,j} < \min\{G_{r,i} : 1 \leq i < j\} \right],$$

where $[.]$ denotes the Iverson bracket, i.e., $[S] = 1$ if the statement $S$ is true and $[S] = 0$ otherwise. In other words, $I_{r,j}$ is the indicator random variable for node $j$ being an $r$-record. Let

$$\mathcal{K}^r(T_n) \overset{\text{def}}{=} \sum_{j=1}^{n} I_{r,j}, \quad \mathcal{K}(T_n) \overset{\text{def}}{=} \sum_{r=1}^{k} \mathcal{K}^r(T_n).$$
Then $K'(T_n)$ is the number of $r$-records and $K(T_n)$ is the total number of records.

1.3 The $k$-cut number of a path

Let $P_n$ be a one-ary tree (a path) consisting of $n$ nodes labeled $1, \ldots, n$ from the root to the leaf. Let $\mathcal{X}_n \overset{\text{def}}{=} K(P_n)$ and $\mathcal{X}_n^r = K'(P_n)$. In this paper, we mainly consider $\mathcal{X}_n$ and we let $k \geq 2$ be a fixed integer.

The first motivation of this choice is that, as shown in section 5, $P_n$ is the fastest to cut among all graphs. (We make this statement precise in Lemma 12.) Thus $\mathcal{X}_n$ provides a universal stochastic lower bound for $K(G_n)$. Moreover, our results on $\mathcal{X}_n$ can be immediately extended to some trees of simple structures: see Section 5. Finally, as shown below, $\mathcal{X}_n$ generalizes the well-known record number in permutations and has very different behavior when $k = 1$, the usual cut-model, and $k \geq 2$, our extended model.

The name record comes from the classic definition of record in random permutations. Let $\sigma_1, \ldots, \sigma_n$ be a uniform random permutation of $\{1, \ldots, n\}$. If $\sigma_i < \min_{1 \leq j < i} \sigma_j$, then $i$ is called a (strictly lower) record. Let $\mathcal{K}_n$ denote the number of records in $\sigma_1, \ldots, \sigma_n$. Let $W_1, \ldots, W_n$ be i.i.d. random variables with a common continuous distribution. Since the relative order of $W_1, \ldots, W_n$ also gives a uniform random permutation, we can equivalently define $\sigma_i$ as the rank of $W_i$. As gamma distributions are continuous, we can in fact let $W_i = G_{k,i}$. Thus being a record in a uniform permutation is equivalent to being a $k$-record and $\mathcal{K}_n \overset{\mathcal{L}}{=} \mathcal{X}_n^k$.

Moreover, when $k = 1$, $\mathcal{K}_n \overset{\mathcal{L}}{=} \mathcal{X}_n$.

Starting from Chandler’s article [7] in 1952, the theory of records has been widely studied due to its applications in statistics, computer science, and physics. For more recent surveys on this topic, see [3], [4], [32] and [5].

A well-known and surprising result of $\mathcal{K}_n$ by Rényi [33] is that $(I_{k,i})_{1 \leq i \leq n}$ are mutually independent. It follows easily that

$$\frac{\mathbb{E}[\mathcal{K}_n]}{\log n} \to 1, \quad \frac{\mathcal{K}_n}{\log n} \overset{a.s.}{\to} 1, \quad \frac{\mathcal{K}_n - \log n}{\sqrt{\log n}} \overset{d}{\to} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution [13, pp. 111].

The following theorem shows that only one-records actually matter.

Theorem 1. We have

$$\mathbb{E}[\mathcal{X}_n^r] \sim \begin{cases} \eta_{k,r} n^{1-\frac{r}{k}} & (1 \leq r < k), \\ \log n & (r = k), \end{cases} \quad (1.4)$$

Here constants $\eta_{k,r}$ are defined by

$$\eta_{k,r} \overset{\text{def}}{=} \frac{(k!)^{\frac{r}{k}} \Gamma\left(\frac{r}{k}\right)}{k - r \Gamma(r)} \quad (1.5)$$
Therefore $\mathbb{E}[X_n] \sim \mathbb{E}[X_n^1]$. Also, for $k = 2$, 

$$
\mathbb{E}[X_n] \sim \mathbb{E}[X_n^1] \sim \sqrt{2\pi n}.
$$

The next theorem gives the variance of $X_n^1$.

**Theorem 2.** We have 

$$
\mathbb{E} \left[ X_n^1 (X_n^1 - 1) \right] \sim \mathbb{E} \left[ (X_n^1)^2 \right] \sim \gamma_k n^{2 - \frac{k}{2}},
$$

where 

$$
\gamma_k = \frac{\Gamma \left( \frac{k}{2} \right) (k!)^{\frac{k}{2}}}{2} + \lambda_k,
$$

and 

$$
\lambda_k = \begin{cases} 
\pi \cot \left( \frac{\pi}{k} \right) \frac{\Gamma \left( \frac{k}{2} \right)}{2} & k > 2, \\
\frac{\pi^2}{4} & k = 2.
\end{cases}
$$

Therefore 

$$
\text{Var} \left( X_n^1 \right) \sim (\gamma_k - \eta_{k,1}^2) n^{2 - \frac{k}{2}}.
$$

In particular, when $k = 2$ 

$$
\text{Var} \left( X_n^1 \right) \sim \left( \frac{\pi^2}{2} + 2 - 2\pi \right) n.
$$

The previous two theorems imply that the correct rescaling parameter should be $n^{1-\frac{k}{2}}$. However, unlike the record number in permutations, the limit distribution of $X_n/n^{1-\frac{k}{2}}$ has a rather complicated representation.

**Theorem 3.** Let $(U_j, E_j)_{j \geq 1}$ be mutually independent random variables with $E_j \stackrel{\text{L}}{=} \text{Exp}(1)$ and $U_j \stackrel{\text{L}}{=} \text{Unif}[0, 1]$. We define the $k$-cut distribution $B_k$ by 

$$
B_k \overset{\text{def}}{=} \sum_{1 \leq p} B_p,
$$

$$
B_p \overset{\text{def}}{=} (1 - U_p) \left( \prod_{1 \leq j < p} U_j \right)^{1 - \frac{1}{k}} S_p,
$$

$$
S_p \overset{\text{def}}{=} \left( k! \sum_{1 \leq s \leq p} \left( \prod_{s \leq j < p} U_j \right) E_s \right)^{\frac{1}{k}}.
$$

(We use the convention that an empty product equals one.) Then 

$$
\frac{X_n}{n^{1-\frac{k}{2}}} \overset{d}{\to} B_k, \quad \mathbb{E} \left[ \frac{X_n}{n^{1-\frac{k}{2}}} \right] \to \mathbb{E} [B_k] = \eta_{k,1} = \frac{(k!)^{\frac{1}{k}} \Gamma \left( \frac{1}{k} \right)}{k - 1}.
$$
Remark 2. An equivalent recursive definition of $S_p$ is

$$S_p = \begin{cases} k!E_1 & (p = 1), \\ (U_{p-1} S_{p-1}^k + k!E_p)^{\frac{1}{k}} & (p \geq 2). \end{cases}$$

Remark 3. It is easy to see that $X_n^e \overset{\text{def}}{=} K_n^e(P_n) \subseteq X_n$ by treating each edge on a length $n + 1$ path as a node on a length $n$ path.

1.4 Outline

In section 2 and 4, we prove Theorem 1, 2, and 3 respectively. In section 5, we discuss some easy extensions of our results to other graphs including binary trees, split trees and Galton-Watson trees. Finally, in section 6, we collect some auxiliary results used in our proofs.

2 THE EXPECTATION

In this section we prove Theorem 1. Since $\mathbb{E}[X_n^k] \sim \log(n)$ is well-known, we only prove (1.4) for $r < k$.

Throughout this paper, we use the notation $O(f(z))$ to denote a function $g(z)$ such that for all $z$ in a given set $S$, there exists a constant $C > 0$, $|g(z)| \leq Cf(z)$. Sometimes we do not explicitly state the set $S$ when it is clear from the context.

Lemma 1. Uniformly for all $i \geq 1$ and $r \in \{1, \ldots, k\}$,

$$\mathbb{E}[I_{r,i+1}] = \left(1 + O\left(i^{1-\frac{1}{r}}\right)\right) \frac{\left(k!\right)^{\frac{1}{k}} \Gamma\left(\frac{r}{k}\right)}{\Gamma(r)} i^{-\frac{r}{k}}.$$

Proof. Conditioning on $G_{r,i+1} = x \geq 0$, for $I_{r,i+1} = 1$, i.e., for node $i + 1$ to be an $r$-record, we need to have $G_{k,1}, \ldots, G_{k,i}$ all greater than $G_{r,i+1}$. Since these $i$ random variables are i.i.d. Gamma($k$), the probability of this event equals $\mathbb{P}\{\text{Gamma}(k) > x \}^i$.

Since $G_{r,i+1} \overset{\text{L}}{=} \text{Gamma}(r)$, using its density function (1.2),

$$\mathbb{E}[I_{r,i+1}] = \int_0^\infty \frac{x^{r-1}e^{-x}}{\Gamma(r)} \mathbb{P}\{\text{Gamma}(k) > x \}^i \, dx$$

$$= \left(1 + O\left(i^{1-\frac{1}{r}}\right)\right) \frac{\left(k!\right)^{\frac{1}{k}} \Gamma\left(\frac{r}{k}\right)}{\Gamma(r)} i^{-\frac{r}{k}}, \quad (2.1)$$

where the estimation of the integral comes from Lemma 16.

Proof of Theorem 1. A simply application of the Euler-Maclaurin formula shows that for $a \in (0,1)$

$$\sum_{1 \leq i \leq n} \frac{1}{i^a} = \frac{1}{1-a} n^{1-a} + O(1). \quad (2.2)$$
Lemma 1. It then follows from Lemma 1 that
\[
E[\mathcal{X}_i] = \sum_{0 \leq i < n} E[I_{r,i+1}]
\]
\[
= \sum_{0 \leq i < n} \left(1 + O\left(i^{-\frac{1}{2n}}\right)\right) \frac{(k!)^\gamma \Gamma\left(\frac{r}{\gamma}\right)}{\Gamma(r)} i^{-\frac{r}{\gamma}}
\]
\[
= \frac{(k!)^\gamma \Gamma\left(\frac{r}{\gamma}\right)}{\Gamma(r)} \left(\frac{1}{1 - \frac{r}{k}} n^{1 - \frac{r}{k}} + O(1)\right) + O\left(n^{1 - \frac{r}{k}}\right)
\]
\[
= \left(1 + O\left(n^{-\frac{1}{2n}}\right)\right) \eta_{k,r} n^{1 - \frac{r}{k}},
\]
where \(\eta_{k,r}\) is defined in (1.5). 

3 THE VARIANCE

In this section we prove Theorem 2.

First we estimate \(E[I_{1,i+1} I_{1,j+1}]\) for \(j > i \geq 0\). For the moment we condition on \(G_{1,i+1} = x\) and \(G_{1,j+1} = y\). For \(I_{1,i+1} I_{1,j+1} = 1\) to happen, both node \(i + 1\) and node \(j + 1\) must be one-records. Recalling the definition of one-records in (1.3), this event can be written as
\[
E_{i,j} = \mathbb{E}\left[\bigcap_{1 \leq s \leq j} G_{s,s} > \max(x, y) \cap \bigcap_{i+2 \leq s \leq j} G_{s,s} > y\right] .
\]
Since \(G_{k,i+1} \equiv \text{Gamma}(k - 1) + x\), \(G_{k,s} \equiv \text{Gamma}(k)\) for \(s \notin \{i + 1, j + 1\}\), and all these random variables are independent, we have
\[
\mathbb{P}\{E_{i,j}\} = \mathbb{P}\{\text{Gamma}(k - 1) + x > y\} \cdot \mathbb{P}\{\text{Gamma}(k) > \max\{x, y\}\} \cdot \mathbb{P}\{\text{Gamma}(k) > y\}^{j-i-1} .
\]
It follows from \(G_{1,i+1}\) and \(G_{1,j+1}\) having Exp(1) distribution that
\[
E[I_{1,i+1} I_{1,j+1}] = \int_{0}^{\infty} \int_{0}^{\infty} e^{-x} e^{-y} \mathbb{P}\{E_{i,j}\} \, dx \, dy
\]
\[
= \int_{0}^{\infty} \int_{y}^{\infty} e^{-x} e^{-y} \mathbb{P}\{E_{i,j}\} \, dx \, dy + \int_{0}^{\infty} \int_{0}^{y} e^{-x} e^{-y} \mathbb{P}\{E_{i,j}\} \, dx \, dy
\]
\[
= A_{1,i,j} + A_{2,i,j} .
\]
Lemma 2. We have
\[
A_{2,i,j} = \left(1 + O\left(j^{-\frac{1}{2n}}\right)\right) \frac{(k!)^\gamma \Gamma\left(\frac{2}{k}\right)}{\Gamma(r)} j^{-\frac{r}{k}} .
\]
Proof. In this case, \(\max(x, y) = y\), thus by (3.1)
\[
A_{2,i,j} = \int_{0}^{\infty} e^{-y} \mathbb{P}\{\text{Gamma}(k) > y\}^{-1} \int_{0}^{y} e^{-y} \mathbb{P}\{\text{Gamma}(k - 1) > y - x\} \, dx \, dy .
\]
Let Poi(c) denote a Poisson distribution with mean c. Using the connection between Poisson processes and gamma distributions [13, Section 3.6.3], the inner integral equals
\[ \int_0^y e^{-x} P \{ \text{Poi}(y - x) < k - 1 \} \, dx = \int_0^y e^{-x} \sum_{0 \leq \ell < k-1} e^{-(y-x)} \frac{(y-x)\ell}{\ell!} \, dx \]
\[ = e^{-y} \sum_{0 \leq \ell < k-1} \frac{y^{\ell+1}}{(\ell+1)!}. \]

It follows from Lemma 16 that
\[ A_{2,i,j} = \sum_{0 \leq \ell < k-1} \int_0^\infty e^{-2y} \frac{y^{\ell+1}}{(\ell+1)!} \mathbb{P} \{ \text{Gamma}(k) > y \}^{j-1} \, dy \]
\[ = \sum_{0 \leq \ell < k-1} \left( 1 + O\left( j^{-\frac{n}{2}} \right) \right) \left( k! \right)^{\frac{1}{k}} \Gamma \left( \frac{\ell + 2}{k} \right) j^{-\frac{n}{2}} \]
\[ = \left( 1 + O\left( j^{-\frac{n}{2}} \right) \right) \frac{k!}{k} \Gamma \left( \frac{2}{k} \right) j^{-\frac{n}{2}}, \]
where the last step uses the fact that only the summand for \( \ell = 0 \) matters.

**Lemma 3.** Let \( a = i \) and \( b = j - i - 1 \). Let \( \beta = \frac{1}{2k(k+1)} \). Let \( x_1 = a^\beta / k! \) and \( y_1 = b^\beta / k! \). Then for all \( a \geq 1 \) and \( b \geq 1 \),
\[ A_{1,i,j} = \zeta_k(a, b) + O\left( \left( a^{-\frac{n}{2k}} + b^{-\frac{n}{2k}} \right) \left( a^{-\frac{n}{2k}} + b^{-\frac{n}{2k}} \right) \right), \]
where
\[ \zeta_k(a, b) \overset{\text{def}}{=} \int_0^\infty \int_y^\infty \exp \left[ -a^{\frac{x^k}{k!}} - b^{\frac{y^k}{k!}} \right] \, dx \, dy. \]

**Proof.** In the case of \( A_{1,i,j} \), \( \max(x, y) = x \) and \( y - x < 0 \). Thus by (3.1),
\[ \mathbb{P} \{ E_{i,j} \} = \mathbb{P} \{ \text{Gamma} (k) > x \}^j \mathbb{P} \{ \text{Gamma} (k) > y \}^{j-i-1}. \]

It follows from Lemma 16 that
\[ A_{1,i,j} = \int_0^\infty \int_y^\infty e^{-x} e^{-y} \mathbb{P} \{ E_{i,j} \} \, dx \, dy \]
\[ = \int_0^\infty \int_y^\infty e^{-x} e^{-y} \left( \frac{\Gamma(k, x)}{\Gamma(k)} \right)^a \left( \frac{\Gamma(k, y)}{\Gamma(k)} \right)^b \, dx \, dy, \]
where \( \Gamma(\ell, z) \) denotes the upper incomplete gamma function [11, 8.2.2].

Let \( S \) be the integral area of (3.3). We will choose \( S_0 \), an appropriate subset of \( S \), in which the integrand of (3.3) can be well approximated by \( \exp \left( -\frac{ax^k + by^k}{k!} \right) \).
Then we will show that the part outside \( S_0 \) can be absorbed by the error term in (3.2). More precisely, let \( x_0 = a^{-\alpha} \) and \( y_0 = b^{-\alpha} \) where \( \alpha = \frac{1}{2} \left( \frac{1}{k} + \frac{1}{k + 1} \right) \). Let

\[
S = \{(x, y) \in \mathbb{R}^2 : 0 < y < x\},
\]
\[
S_0 = S \cap \{(x, y) \in \mathbb{R}^2 : x < x_0, y < y_0\}.
\]

In other words, \( S \) is the integration area of (3.3) and \( S_0 \) is the part of \( S \) in which \( x < x_0 \) and \( y < y_0 \). Note that the shape of \( S_0 \) is different when \( a < b \) and \( a > b \), see Figure 1.

![Figure 1: The integration area of (3.3), \( S = \{(x, y) : 0 < y < x\} \).](image)

We split the integral into two parts

\[
A_{1,i,j} = \iint_{S_0} e^{-x-y} \left( \frac{\Gamma(k, x)}{\Gamma(k)} \right)^a \left( \frac{\Gamma(k, y)}{\Gamma(k)} \right)^b \, dx \, dy
\]
\[
+ \iint_{S \setminus S_0} e^{-x-y} \left( \frac{\Gamma(k, x)}{\Gamma(k)} \right)^a \left( \frac{\Gamma(k, y)}{\Gamma(k)} \right)^b \, dx \, dy \overset{\text{def}}{=} A_{1,1} + A_{1,2}.
\]

Throughout the proof of this lemma, the \( O(f(a, b)) \) notation applies to \( (a, b) \in [1, \infty)^2 \). For \( A_{1,1} \), i.e., the part inside \( S_0 \), we can well approximate the integrand using Lemma 15 and get

\[
A_{1,1} = \left( 1 + O \left( a^{-\frac{k}{k + 1}} + b^{-\frac{k}{k + 1}} \right) \right) \iint_{S_0} \exp \left( -\frac{ax^k + by^k}{k!} \right) \, dx \, dy. \quad (3.4)
\]
Assume for now that \( a > b \), i.e., case (i) in Figure 1. Since \( \exp \left( -\frac{ax^k + by^k}{k!} \right) \) is monotonically decreasing in both \( x \) and \( y \),

\[
\begin{align*}
\int_S \int_{S_0} \exp \left( -\frac{ax^k + by^k}{k!} \right) \, dx \, dy & \leq \exp \left( -\frac{ax_0^k}{2k!} \right) \int_0^\infty \int_0^\infty \exp \left( -\frac{ax^k/2 + by^k}{k!} \right) \, dx \, dy \\
& \overset{\text{def}}{=} e^{-\frac{x_0^k}{2}} \xi_k \left( \frac{a}{2}, b \right) = O \left( e^{-\frac{x_0}{2}} \right), (3.5)
\end{align*}
\]

where the last step uses \( \xi_k \left( \frac{a}{2}, b \right) = O(1) \) by Lemma 18, and that \( ax_0^k / k! = x_1 \).

For the case (ii) in Figure 1, we can further divide \( S \setminus S_0 \) into two parts, as depicted by the dotted line, to show that

\[
\int_S \int_{S \setminus S_0} \exp \left( -\frac{ax^k + by^k}{k!} \right) \, dx \, dy = O \left( e^{-\frac{x_1}{2}} + e^{-\frac{y_1}{2}} \right). \tag{3.6}
\]

Together with (3.5), we can see that (3.6) is valid regardless of the order of \( a \) and \( b \). Putting (3.4) and (3.6) together, we have

\[
A_{1,1} = \left( 1 + O \left( a^{-\frac{1}{2}} + b^{-\frac{1}{2}} \right) \right) \int_S \int_S \exp \left( -\frac{ax^k + by^k}{k!} \right) \, dx \, dy \\
+ O \left( e^{-\frac{x_1}{2}} + e^{-\frac{y_1}{2}} \right) \\
\overset{\text{def}}{=} \left( 1 + O \left( a^{-\frac{1}{2}} + b^{-\frac{1}{2}} \right) \right) \xi_k \left( a, b \right) + O \left( e^{-\frac{x_1}{2}} + e^{-\frac{y_1}{2}} \right). \tag{3.7}
\]

Note that \( \Gamma(k, x) / \Gamma(k) \) is monotonically decreasing in \( x \). Therefore, if \( x > x_0 \), then by Lemma 15, and that \( \Gamma(k, 0) = \Gamma(k) \),

\[
\left( \frac{\Gamma(k, x)}{\Gamma(k)} \right)^a \left( \frac{\Gamma(k, y)}{\Gamma(k)} \right)^b \leq \left( \frac{\Gamma(k, x_0)}{\Gamma(k)} \right)^a = O \left( \exp \left( -\frac{ax_0^k}{k!} \right) \right) = O \left( e^{-x_1} \right).
\]

Thus when \( a > b \), i.e., case (i) in Figure 1,

\[
A_{1,2} = O \left( e^{-x_1} \right) \int_S \int_{S_0} e^{-x-y} \left( \frac{\Gamma(k, x)}{\Gamma(k)} \right)^a \left( \frac{\Gamma(k, y)}{\Gamma(k)} \right)^b \, dx \, dy = O \left( e^{-x_1} \right),
\]

where we again use Lemma 15. Together with a similar analysis for \( a < b \), we can see that, regardless of the order of \( a \) and \( b \),

\[
A_{1,2} = O \left( \left( \frac{\Gamma(k, x_0)}{\Gamma(k)} \right)^{-a} + \left( \frac{\Gamma(k, y_0)}{\Gamma(k)} \right)^{-b} \right) = O \left( e^{-x_1} + e^{-y_1} \right). \tag{3.8}
\]

It follows from (3.7) and (3.8) that

\[
A_{1,1} + A_{1,2} = \left( 1 + O \left( a^{-\frac{1}{2}} + b^{-\frac{1}{2}} \right) \right) \xi_k \left( a, b \right) + O \left( e^{-\frac{x_1}{2}} + e^{-\frac{y_1}{2}} \right) \\
= \xi_k \left( a, b \right) + O \left( \left( a^{-\frac{1}{2}} + b^{-\frac{1}{2}} \right) \left( a^{-\frac{1}{2}} + b^{-\frac{1}{2}} \right) \right),
\]

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where the last step uses \( \check{\xi}_k(a, b) = O\left(a^{-\frac{2}{k}} + b^{-\frac{2}{k}}\right) \) from Lemma 18 and that \( e^{-\frac{y}{2}} \) and \( e^{-\frac{y}{2}} \) are exponentially small.

\( \square \)

Now we are ready to finish the proof of Theorem 2. Since

\[
X_n^1 \left( X_n^1 - 1 \right) = \sum_{1 \leq i \leq n} I_{1,i} \left( \sum_{1 \leq j \leq n} I_{1,j} - 1 \right)
= \sum_{1 \leq i \neq j \leq n} I_{1,i} I_{1,j} + \sum_{1 \leq i \leq n} I_{1,i}^2 - \sum_{1 \leq i \leq n} I_{1,i} = 2 \sum_{1 \leq i < j \leq n} I_{1,i} I_{1,j},
\]

we have

\[
E \left[ X_n^1 \left( X_n^1 - 1 \right) \right] = 2 \sum_{1 \leq i < n} \sum_{i+1 < j \leq n} E \left[ I_{1,i} I_{1,j} \right]
= 2 \sum_{1 \leq i < n} \sum_{i+1 < j \leq n} \left( A_{1,i,j} + A_{2,i,j} \right).
\] (3.9)

By (2.2), we have

\[
\sum_{1 \leq i < n} \sum_{i+1 < j \leq n} j^{-\frac{2}{k}} = \sum_{1 \leq j+1 \leq n} \sum_{1 \leq i < j+1} j^{-\frac{2}{k}} = \sum_{1 \leq j \leq n} j^{1-\frac{2}{k}} = \frac{kn^{2-\frac{2}{k}}}{2(k-1)} + O(1).
\]

Thus by Lemma 2,

\[
\sum_{1 \leq i < n} \sum_{i+1 < j \leq n} A_{2,i,j} = \sum_{1 \leq i < n} \sum_{i+1 < j \leq n} \left[ \frac{(k!)^\frac{2}{k} \Gamma \left( \frac{2}{k} \right)}{k} j^{-\frac{2}{k}} + O\left(j^{-\frac{2}{k}}\right) \right]
= \frac{(k!)^\frac{2}{k} \Gamma \left( \frac{2}{k} \right)}{2(k-1)} n^{2-\frac{2}{k}} + O\left(n^{2-\frac{2}{k}}\right).
\] (3.10)

For \( A_{1,i,j} \), it follows from Lemma 3 that

\[
\sum_{1 \leq i < n} \sum_{i+1 < j \leq n} A_{1,i,j} = \sum_{1 \leq a < n} \sum_{1 \leq b \leq n-a} \check{\xi}_k(a, b) + O\left(n^{2-\frac{2}{k}}\right)
= \int_0^n \int_0^{n-a} \check{\xi}_k(a, b) \, db \, da + O\left(n^{2-\frac{2}{k}}\right)
= n^{2-\frac{2}{k}} \int_0^1 \int_0^{1-s} \check{\xi}_k(s, t) \, dt \, ds + O\left(n^{2-\frac{2}{k}}\right)
= \lambda_k n^{2-\frac{2}{k}} + O\left(n^{2-\frac{2}{k}}\right),
\] (3.11)

where last step follows from Lemma 19. Also, in the above computation, we can approximate the double sum by an integral because \( \check{\xi}_k(a, b) \) is monotonically decreasing in both \( a \) and \( b \) (see Lemma 18). Plug (3.10), (3.11) into (3.9),

\[
E \left[ X_n^1 \left( X_n^1 - 1 \right) \right] = \left( \frac{(k!)^\frac{2}{k} \Gamma \left( \frac{2}{k} \right)}{k-1} + 2\lambda_k \right) n^{2-\frac{2}{k}} + O\left(n^{2-\frac{2}{k}}\right)
\defeq \left( 1 + O\left(n^{-\frac{1}{k}}\right) \right) \gamma_k n^{2-\frac{2}{k}},
\]

where \( \gamma_k \) and \( \lambda_k \) are defined in Theorem 2.
By Theorem 1 and Markov’s inequality [13, Thm. 1.6.4], \( X_n^r / n^{1 - 1/r} \rightarrow P \) for \( r \in \{2, \ldots, k\} \). So instead of proving Theorem 3 for \( X_n \), it suffices to prove it for \( X_n^1 \).

The idea of the proof is to condition on the positions and values of the \( k \)-records, and study the distribution of the number of one-records between two consecutive \( k \)-records.

We use \( (R_{n,j})_{j \geq 1} \) to denote the \( k \)-record values and \( (P_{n,j})_{j \geq 1} \) the positions of these \( k \)-records. To define them more precisely, recall that \( G_{r,j} \) is the moment when the alarm clock of \( j \) rings for the \( r \)-th time, see (1.1). Let \( R_{n,0} \overset{\text{def}}{=} 0 \), and \( P_{n,0} = n + 1 \). For \( p \geq 1 \), if \( P_{n,p-1} > 1 \), then let

\[
R_{n,p} \overset{\text{def}}{=} \min\{G_{k,j} : 1 \leq j < P_{n,p-1}\}, \quad P_{n,p} \overset{\text{def}}{=} \arg\min\{G_{k,j} : 1 \leq j < P_{n,p-1}\} ;
\]

otherwise let \( P_{n,p} = 1 \) and \( R_{n,p} = \infty \). Note that \( R_{n,1} \) is simply the minimum of \( n \) i.i.d. Gamma(\( k \)) random variables.

Recall that \([S] = 1\) if \( S \) is true and \([S] = 0\) otherwise. According to \( (P_{n,j})_{j \geq 1} \), we can split \( X_n^1 \) into the following sum

\[
X_n^1 = \sum_{1 \leq j \leq n} I_{1,j} = X_n^1 + \sum_{1 \leq p \leq j} \sum_{1 \leq j} [P_{n,p-1} > j > P_{n,p}] I_{1,j} \overset{\text{def}}{=} X_n^k + \sum_{1 \leq p} B_{n,p}, \tag{4.2}
\]

where \( I_{1,j} \) is the indicator for \( j \) being a one-record and we also use the fact that a \( k \)-record must be a one record. Figure 2 gives an example of \( (B_{n,p})_{p \geq 1} \) for \( n = 12 \). It depicts the positions of the \( k \)-records and the one-records. It also shows the values and the summation ranges for \( (B_{n,p})_{p \geq 1} \).

![Figure 2: An example of \( (B_{n,p})_{p \geq 1} \) for \( n = 12 \).](image)

Recall that \( T_{i,j} \) is the lapse of time between the alarm clock of \( j \) rings for the \((i - 1)\)-th time and the \( i \)-th time, see (1.1). Conditioning on \( (P_{n,j})_{j \geq 1} \) and \( (R_{n,j})_{j \geq 1} \), for \( j \in (P_{n,p}, P_{n,p-1}) \), we must have \( \sum_{1 \leq i \leq k} T_{i,j} < R_{n,p-1} \). (Otherwise \( j \) would have
become a $k$-record.) And for $j$ to be a one-record, we need $T_{1,j} < R_{n,p}$. Since $(T_{i,j})_{i \geq 1, j \geq 1}$ are i.i.d. Exp(1) before the conditioning, we have

$$\mathbb{E} [I_{1,j}] = \mathbb{P} \left\{ T_{1,j} < R_{n,p} \left| \sum_{i=1}^{k} T_{i,j} > R_{n,p-1} \right. \right\}$$

$$= \frac{\mathbb{P} \left\{ \text{Exp}(1) < R_{n,p} \right\}}{\mathbb{P} \left\{ \text{Gamma}(k) > R_{n,p-1} \right\}} = \left( 1 - e^{-R_{n,p}} \right) \frac{\Gamma(k)}{\Gamma(k, R_{n,p-1})},$$

where the last step uses Lemma 15. Then the distribution of $B_{n,p}$ is just

$$\text{Bin} \left( P_{n,p-1} - P_{n,p} - 1, \left( 1 - e^{-R_{n,p}} \right) \frac{\Gamma(k)}{\Gamma(k, R_{n,p-1})} \right),$$

where Bin$(m, p)$ denotes a binomial $(m, p)$ distribution. When $R_{n,p-1}$ is small and $P_{n,p-1} - P_{n,p}$ is large, this is roughly

$$\text{Bin} \left( P_{n,p-1} - P_{n,p}, \mathbb{P} \left\{ \text{Exp}(1) < R_{n,p} \right\} \right) \overset{\mathcal{L}}{=} \text{Bin} \left( P_{n,p-1} - P_{n,p}, 1 - e^{-R_{n,p}} \right). \quad (4.3)$$

Therefore, to simplify the computations, we first study a slightly modified model. Let $(T_{i,j}^*)_{i \geq 1, j \geq 1}$ be i.i.d. Exp(1) which are also independent from $(T_{i,j})_{i \geq 1, j \geq 1}$. Let

$$I_j^* \overset{\text{def}}{=} \lfloor T_{i,j} < \min \{ G_{k,i} : 1 \leq i \leq j \} \rfloor, \quad X_n^* \overset{\text{def}}{=} \sum_{1 \leq j \leq n} I_j^* \quad (4.4)$$

We say a node $j$ is an alt-one-record if $I_j^* = 1$. As in (4.2), we can write

$$X_n^* = \sum_{1 \leq j \leq n} I_j^* = \sum_{1 \leq p \leq n} \sum_{1 \leq j} [P_{n,p-1} > j \geq P_{n,p}] I_j^* \overset{\text{def}}{=} \sum_{1 \leq p} B_{n,p}^*. \quad (4.5)$$

Then conditioning on $(R_{n,p}, P_{n,j})_{n \geq 1, j \geq 1}$, $B_{n,p}^*$ has exactly the distribution as (4.3). Figure 3 gives an example of $(B_{n,p}^*)_{p \geq 1}$ for $n = 12$. It shows the positions of alt-one-records, as well as the values and the summation ranges of$(B_{n,p}^*)_{p \geq 1}$. Note that the positions and the number of one-records and alt-one-records are not necessarily the same.

The most part of this section is devoted to showing that

$$\frac{X_n^*}{n^{1-\frac{1}{k}}} \overset{\mathcal{D}}{=} \frac{\sum_{1 \leq j} B_{n,j}^*}{n^{1-\frac{1}{k}}} \overset{\text{d}}{=} \sum_{1 \leq j} B_j^* \overset{\text{d}}{=} B_k. \quad (4.6)$$

We will argue at the end of this section that $X_n^*/n^{1-\frac{1}{k}}$ and $X_n^*/n^{1-\frac{1}{k}}$ converge to the same limit.

To prove (4.6), we first show the following proposition:

**Proposition 1.** For all fixed $p \in \mathbb{N}$,

$$\left( \frac{B_{n,1}^*}{n^{1-\frac{1}{k}}}, \ldots, \frac{B_{n,p}^*}{n^{1-\frac{1}{k}}} \right) \overset{\mathcal{D}}{=} (B_1, \ldots, B_p),$$

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which implies that

$$\sum_{1 \leq j \leq p} \frac{B^*_{n,j}}{n^{1-\frac{1}{k}}} \xrightarrow{d} \sum_{1 \leq j \leq p} B_j, \quad (4.7)$$

by the Cramér–Wold device [17, pp. 245].

Then we show that if we choose \( p \) large enough, then the leftovers, i.e., \( \sum_{p < j} B_j \) and \( \sum_{p < j} B^*_{n,j}/n^{1-\frac{1}{k}} \), are negligible.

### 4.1 Proof of Proposition 1

The first step to prove (4.7) is to construct a coupling by defining all the random variables that we are studying in one probability space. Let

$$P_{n,p} = \max \left\{ \left\lceil U_p \left( P_{n,p-1} - 1 \right) \right\rceil, 1 \right\}, \quad (4.8)$$

for \( p \geq 1 \), where \( (U_i)_{i \geq 1} \) are i.i.d. Unif[0, 1] random variables. This does not change the distribution of \( P_{n,p} \) formerly defined in (4.1), which is Unif\{1, \ldots, \max\{P_{n,p-1} - 1, 1\}\}. Note that this implies that for all \( p \in \mathbb{N} \)

$$\frac{P_{n,p}}{n} \xrightarrow{a.s.} \prod_{1 \leq s \leq p} U_s, \quad (4.9)$$

since \( P_{n,0} = n + 1 \). And we do not change the definition of any other random variables.

Recall that \( R_{m,1} \) is the minimum of \( m \) independent Gamma\((k)\) random variables (see (4.1)). Let \( M(m, t) \overset{\mathcal{L}}{=} (R_{m,1} | R_{m,1} > t) \). In other words, \( M(m, t) \) has the distribution of \( R_{m,1} \) conditioned on \( R_{m,1} > t \).

**Lemma 4.** Assume that \( \frac{m}{t^k} \to 1 \) and \( t \in [0, \infty) \). Then as \( m \to \infty \),

$$H_m \overset{\text{def}}{=} \frac{1}{m} \cdot M \left( m, \frac{t}{r_m} \right) \xrightarrow{d} H \overset{\text{def}}{=} \left( t^k + k!E \right)^{\frac{1}{k}},$$
where \( E \triangleq \text{Exp}(1) \). In particular,

\[
m_{m}^{1/2} M(m,0) \xrightarrow{d} (k!)^{1/2}.
\]

Moreover, the density function of \( H_{m} \) converges point-wise to the density function of \( H \). The lemma also holds if we replace \( H_{m} \) by

\[
H_{m}' \overset{\text{def}}{=} r_{m}'^{1/2} \cdot \left(1 - \exp \left(-M\left(m, \frac{t}{r_{m}'}\right)\right)\right).
\]

Proof. We only prove the lemma for \( H_{m} \). Similar argument works for \( H_{m}' \). We show that for all fixed \( x \geq t \), \( \mathbb{P}\{H_{m} > x\} \) converges to

\[
\mathbb{P}\{H > x\} = \mathbb{P}\left\{ \left(\frac{t^{k} + k!E}{x}\right)^{1/2} > x\right\} = \mathbb{P}\left\{ E > \frac{x^k - t^k}{k!}\right\} = \exp\left(-\frac{x^k - t^k}{k!}\right).
\]

Let \( y_{m} = x/r_{m}'^{1/2} \) and let \( s_{m} = t/r_{m}'^{1/2} \). By the definition of \( H_{m} \) and Lemma 15, we have

\[
\mathbb{P}\{H_{m} > x\} = \mathbb{P}\{M(m, s_{m}) \geq y_{m}\} = \frac{\mathbb{P}\{R_{m,1} \geq y_{m}\}}{\mathbb{P}\{R_{m,1} \geq s_{m}\}} = \left(\frac{\Gamma(k, y_{m})}{\Gamma(k, s_{m})}\right)^{m} \sim \exp\left(m\left(-\frac{y_{m}^k - s_{m}^k}{k!}\right)\right)
\]

\[
\rightarrow \exp\left(-\frac{x^k - t^k}{k!}\right), \quad (4.10)
\]

where \( \Gamma(\ell, z) \) denotes the upper incomplete gamma function [11, 8.2.2]. Using (4.10) and the derivative formula for the incomplete gamma function [11, 8.8.13], it is straightforward to verify that, as \( m \rightarrow \infty \),

\[
\frac{d}{dx} \mathbb{P}\{H_{m} > x\} = \frac{d}{dx}\left(\frac{\Gamma(k, y_{m})}{\Gamma(k, s_{m})}\right)^{m} \frac{d}{dx} \exp\left(-\frac{x^k - t^k}{k!}\right) \rightarrow \frac{d}{dx} \mathbb{P}\{H > x\}.
\]

Thus we have the point-wise convergence of density functions. \( \square \)

We define the auxiliary random variables

\[
L_{n,p}^{*} \overset{\text{def}}{=} \left(\prod_{1 \leq j < p} U_{j}\right)^{1/2}, \quad L_{n,p} \overset{\text{def}}{=} (P_{n,p} - 1)^{1/2},
\]

\[
S_{n,p}^{*} \overset{\text{def}}{=} L_{n,p}^{*} R_{n,p}, \quad S_{n,p} \overset{\text{def}}{=} L_{n,p} (1 - e^{-R_{n,p}}).
\]

The second step is to show that

**Lemma 5.** For all fixed \( p \in \mathbb{N} \),

\[
(S_{n,1}, S_{n,2}, \ldots, S_{n,p}) \overset{d}{\rightarrow} (S_{1}, S_{2}, \ldots, S_{p}), \quad (4.12)
\]

where \( (S_{p})_{p \geq 1} \) are defined by (1.8) in Theorem 3. Moreover, the joint density function of \( (S_{n,1}, \ldots, S_{n,p}) \) converges point-wise to the joint density function of \( (S_{1}, \ldots, S_{p}) \). The lemma also holds if we replace \( S_{n,j} \) by \( S_{n,j}^{*} \).
Proof. We only prove the lemma for $S_{n,1}$. The same argument works for $S_{n,j}^*$.

Let $\mathcal{F} = \sigma((U_j)_{j\geq 1})$ denote the sigma algebra generated by $(U_j)_{j\geq 1}$. To prove Lemma 5, we will condition on $\mathcal{F}$ and treat $(U_p, P_{n,p}, L_{n,p}^*, L_{n,p})_{p \geq 0, n \geq 1}$ as deterministic numbers. If we can show the convergence of distribution in (4.7) conditioning on $\mathcal{F}$, i.e., if for all $(x_1, \ldots, x_p) \in \mathbb{R}^p$,

$$\left| \mathbb{P} \{ S_{n,1} > x_1, \ldots, S_{n,p} > x_p \mid \mathcal{F} \} - \mathbb{P} \{ S_1 > x_1, \ldots, S_p > x_p \mid \mathcal{F} \} \right| \rightarrow 0,$$

then we have, for all fixed $(x_1, \ldots, x_p)$,

$$\left| \mathbb{P} \{ S_{n,1} > x_1, \ldots, S_{n,p} > x_p \} - \mathbb{P} \{ S_1 > x_1, \ldots, S_p > x_p \} \right| \rightarrow 0.$$

Recall that $R_{n,1}$ is the minimum of $n$ i.i.d. Gamma($k$) random variables and $P_{n,0} = n + 1$, see (4.1). Then

$$L_{n,1}^* \overset{\text{def}}{=} (P_{n,0} - 1)^\frac{1}{k} = n^\frac{1}{k}, \quad R_{n,1} \overset{\text{def}}{=} \min \{ G_{k,j} : 1 \leq j < P_{n,0} \} \overset{\text{def}}{=} M(n, 0).$$

Let $f_{n,1}(\cdot)$ and $f_1(\cdot)$ denote the density functions of $S_{n,1}$ and $S_1$ respectively. It follows from Lemma 4 that

$$S_{n,1} \overset{\text{def}}{=} L_{n,1}^* R_{n,1} \overset{\text{def}}{=} n^\frac{1}{k} M(n, 0) \overset{\text{d}}{\rightarrow} (k! E_1)^\frac{1}{k} \overset{\text{d}}{=} S_1,$$

where $(E_j)_{j \geq 1}$ are i.i.d. Exp(1) random variables, and for all $y_1 \in \mathbb{R}$

$$f_{n,1}(y_1) \rightarrow f_p(y_1). \quad \quad (4.13)$$

For $p > 1$, we condition on $S_{p-1} = y_{p-1} \in [0, \infty)$. We will apply Lemma 4 to $P_{n,p}$ by taking

$$m = t^k_{n,p}, \quad r_m = \left( L_{n,p}^* \right)^k, \quad t = S_{p-1} U_{p-1}^\frac{1}{k} = y_{p-1} U_{p-1}^\frac{1}{k}.$$

Recall that $R_{n,p}$ is the minimum of $(P_{p-1} - 1)$ i.i.d. Gamma($k$) random variables restricted to $(R_{n,p-1}, \infty)$, see (4.1). Thus

$$R_{n,p} \overset{\text{d}}{=} M \left( P_{n,p-1} - 1, R_{n,p-1} \right) = M \left( k \frac{R_{n,p-1} L_{n,p}}{L_{n,p}^*} \right) = M \left( m, \frac{t}{r_m} \right),$$

where we use the definition of $L_{n,p}^*$ and $S_{n,p}$ in (4.11) to get

$$R_{n,p-1} L_{n,p}^* = R_{n,p-1} L_{n,p-1}^* U_{p-1}^\frac{1}{k} = S_{n,p-1} U_{p-1}^\frac{1}{k} = y_{p-1} U_{p-1}^\frac{1}{k} = t.$$

Also note that by (4.9),

$$\frac{r_m}{m} = \left( \frac{L_{n,p}^*}{L_{n,p}} \right)^k = \frac{n \prod_{1 \leq j < p} U_j}{P_{n,p-1} - 1} \rightarrow 1.$$
Let \( f_{n,p}(\cdot|y_{p-1}) \) and \( f_p(\cdot|y_{p-1}) \) denote the density function of \( S_{n,p}|S_{n,p-1} = y_{p-1} \), and \( S_p|S_{p-1} = y_{p-1} \) respectively. It follows from Lemma 4 that

\[
S_{n,p} \overset{\text{def}}{=} L_{n,p}^* R_{n,p} = r_m M \left( m, \frac{t}{r_m^2} \right)
\]

\[
\overset{d}{\to} \left( y_{p-1}^k U_{p-1} + k! E_p \right)^t = \left( S_{p-1}^k U_{p-1} + k! E_p \right)^t \overset{\text{def}}{=} S_p,
\]

and for all \( y_p \in [0, \infty) \)

\[
f_{n,p}(y_p|y_{p-1}) \to f_p(y_p|y_{p-1}). \quad (4.14)
\]

Then by (4.13) and (4.14), for all \( y_1, \ldots, y_p \in [0, \infty)^p \),

\[
g_{n,p}(y_1, \ldots, y_p) \equiv f_{n,p}(y_p|y_{p-1}) f_{n,p-1}(y_{p-1}|y_{p-2}) \cdots f_{n,1}(y_1) \to f_p(y_p|y_{p-1}) f_{p-1}(y_{p-1}|y_{p-2}) \cdots f_1(y_1) \overset{\text{def}}{=} g_p(y_1, \ldots, y_p).
\]

In other words, the joint density function of \( (S_{n,1}, \ldots, S_{n,p}) \) converges point-wise to the joint density function of \( (S_1, \ldots, S_p) \). Thus by Scheffé’s lemma [17, pp. 227], we have the convergence in distribution in (4.12).

Now it is quite easy to finish the proof of Proposition 1 using the following lemma

**Lemma 6.** Let \( W_m \overset{\text{def}}{=} \text{Bin}(m, p_m) \). If \( \ell_m p_m \to c \in (0, \infty) \) and \( m/\ell_m \to \infty \), then \( \ell_m W_m / m \overset{p}{\to} c \).

**Proof.** For all fixed \( \varepsilon > 0 \),

\[
P \left\{ \left| \frac{\ell_m W_m}{m} - c \right| > \varepsilon \right\} \leq P \left\{ |W_m - mp_m| > \frac{\varepsilon m}{2\ell_m} \right\} + P \left\{ |p_m \ell_m - c| > \frac{\varepsilon}{2} \right\}
\]

\[
\leq 2 \exp \left( - \frac{\varepsilon^2 m^2}{4\ell_m^2} \right) \frac{1}{3mp_m} + P \left\{ |p_m \ell_m - c| > \frac{\varepsilon}{2} \right\} \to 0,
\]

where the second step uses Chernoff’s bound [31, pp. 43].

By the same argument in the proof of Lemma 5, we can condition on \( \mathcal{F} = \sigma(\{U_j\}_{j \geq 1}) \) and treat \( (U_j, P_{n,j}, L_{n,j})_{j \geq 0, n \geq 1} \) as deterministic numbers. Define the event \( A \) by

\[
A(y_1, \ldots, y_p) = \left[ (S_{n,1}^*, \ldots, S_{n,p}^*) = (y_1, \ldots, y_p) \right]. \quad (4.15)
\]

Let \( j \in \{1, \ldots, p - 1\} \). Recall that \( S_{n,j}^* \overset{\text{def}}{=} L_{n,j}^* (1 - e^{-R_{n,j}}) \). Thus by (4.3), conditioning on the event \( A(y_1, \ldots, y_p) \), \( B_{n,1}^*, \ldots, B_{n,p}^* \) are independent and for \( j \in \{1, \ldots, p\} \),

\[
B_{n,j}^* \overset{\text{def}}{=} \text{Bin} \left( P_{n,j-1} - P_{n,j}, 1 - e^{-R_{n,j}} \right) = \text{Bin} \left( P_{n,j-1} - P_{n,j}, \frac{S_{n,j}^*}{L_{n,j}^*} \right)
\]

\[
= \text{Bin} \left( P_{n,j-1} - P_{n,j}, \frac{y_j}{L_{n,j}^*} \right).
\]
We will apply Lemma 6 to $B_{n,j}^*$ by taking

$$m = P_{n,j-1} - P_{n,j}, \quad \ell_m = L_{n,j}^*, \quad p_m = \frac{y_j}{L_{n,j}^*}, \quad c = y_j.$$  

Note that by (4.9) and $L_{n,p}^* \equiv \left( n \prod_{1 \leq j < p} U_j \right)^{\frac{1}{k}}$

$$\frac{m}{n^{1 - \frac{k}{2}} \ell_m} = \frac{P_{n,j-1} - P_{n,j}}{n^{1 - \frac{k}{2}} L_{n,j}^*} \xrightarrow{p} \frac{1}{(\prod_{1 \leq s < j} U_s)^{\frac{1}{k}}} \to (1 - U_j) \left( \prod_{1 \leq s < j} U_s \right)^{1 - \frac{1}{k}}.$$  

It follows from Lemma 6 that conditioning on $A \left( y_1, \ldots, y_p \right)$ defined in (4.15)

$$\frac{L_{n,j}^* B_{n,j}^*}{P_{n,j-1} - P_{n,j}} = \frac{\ell_m B_{n,j}^*}{m} \to y_j.$$  

Combining the two above expressions,

$$\frac{B_{n,j}^*}{n^{1 - \frac{k}{2}}} = \frac{L_{n,j}^* B_{n,j}^*}{P_{n,j-1} - P_{n,j}} \cdot \frac{P_{n,j-1} - P_{n,j}}{n^{1 - \frac{k}{2}} L_{n,j}^*} \to (1 - U_j) \left( \prod_{1 \leq s < j} U_s \right)^{1 - \frac{1}{k}} y_j. \quad (4.16)$$

Let $g_{n,p}^*(y_1, y_2, \ldots, y_p)$ and $g_p^*(y_1, y_2, \ldots, y_p)$ be the joint density functions of $(S_{n,1}^*, \ldots, S_{n,p}^*)$ and $(S_1, \ldots, S_p)$ respectively. Then for all $(x_1, \ldots, x_p) \in [0, \infty)^p$,

$$\mathbb{P} \left\{ \bigcap_{j=1}^p \left[ \frac{B_{n,j}^*}{n^{1 - \frac{k}{2}}} > x_j \right] \right\} = \int_0^\infty \cdots \int_0^\infty g_{n,p}^* \left( y_1, \ldots, y_p \right) \times \prod_{1 \leq j \leq p} \mathbb{P} \left\{ \frac{B_{n,j}^*}{n^{1 - \frac{k}{2}}} > x_j \bigg| A \left( y_1, \ldots, y_p \right) \right\} dy_1 \cdots dy_p. \quad (4.17)$$

By (4.16) and Lemma 5, the integrand converges point-wise to

$$g_p^* \left( y_1, \ldots, y_p \right) \prod_{1 \leq j \leq p} \left[ (1 - U_j) \left( \prod_{1 \leq s < j} U_s \right)^{1 - \frac{1}{k}} y_j > x_j \right],$$

where $[S] = 1$ if $S$ is true and $[S] = 0$ otherwise. Then by Scheffé's Lemma [17, pp. 227], (4.17) converges to

$$\int_0^\infty \cdots \int_0^\infty g_p^* \left( y_1, \ldots, y_p \right) \prod_{j=1}^p \left[ (1 - U_j) \left( \prod_{1 \leq s < j} U_s \right)^{1 - \frac{1}{k}} y_j > x_j \right] dy_1 \cdots dy_p$$

$$= \mathbb{P} \left\{ \bigcap_{j=1}^p \left[ (1 - U_j) \left( \prod_{1 \leq s < j} U_s \right)^{1 - \frac{1}{k}} S_j > x_j \right] \right\} = \mathbb{P} \left\{ \bigcap_{j=1}^p [B_j > x_j] \right\}.$$
In other words, jointly, conditioning on $\mathcal{F} = \sigma((U_i)_{i \geq 1})$, 
\[
\left( \frac{B_{n,1}^*}{n^{1-\frac{1}{r}}}, \ldots, \frac{B_{n,p}^*}{n^{1-\frac{1}{r}}} \right) \xrightarrow{d} (B_1, \ldots, B_p),
\]
and the convergence also holds without conditioning on $\mathcal{F}$ by the same argument for Lemma 5. Thus we are done proving Proposition 1.

4.2 The leftovers

In this section, we show that for $p$ large enough, $\sum_{s > p} B_s$, $\sum_{s > p} B_{n,s}^*/n^{1-\frac{1}{r}}$, and $\sum_{s > p} B_{n,s}/n^{1-\frac{1}{r}}$ are all negligible.

Lemma 7. For all $\varepsilon > 0$ and $\delta > 0$, there exists an $p \in \mathbb{N}$ such that 
\[
P\left\{ \sum_{p < s} B_s \geq \varepsilon \right\} < \delta.
\]

Proof. Let $(U'_j, E'_j)_{j \geq 1}$ be mutually independent such that for $j \geq 1$, $U'_j \sim \text{Unif}[0,1]$ and $E'_j \sim \text{Exp}(1)$. By definition of $B_s$ (see (1.7) and (1.8)), we have 
\[
(B_s)^k \leq \left( \prod_{1 \leq j \leq s} U'_j \right) \left( k! \sum_{1 \leq j \leq s} E'_j \right),
\]
i.e., the left hand side is stochastically dominated by the right hand side [18, pp. 68]. Let $W_s$ and $W'_s$ be independent Gamma$(s)$. Then
\[
- \log \left( \prod_{1 \leq j \leq s} U'_j \right) \overset{\mathcal{L}}{=} W_s, \quad \sum_{1 \leq j \leq s} E'_j \overset{\mathcal{L}}{=} W'_s.
\]
(See [17, pp. 115].) It is well known that $E\left[ (W_s - s)^4 \right] = 3s^2 + 6s$ [26, pp. 339]. It follows from Markov’s inequality that for $s \geq 1$,
\[
P \left\{ |W_s - s| \geq \frac{s}{2} \right\} \leq \frac{E\left[ (W_s - s)^4 \right]}{(s^2/16)^4} = \frac{3s^2 + 6s}{s^4/16} = \frac{9s^2}{s^4/16} \leq \frac{144}{s^2}.
\]
Therefore
\[
P \left\{ B_s \geq \left( k! \frac{3}{2} e^{-s/2} \right)^{\frac{1}{2}} \right\} \leq P \left\{ \left( \prod_{1 \leq j \leq s} U'_j \right) \left( k! \sum_{1 \leq j \leq s} E'_j \right) \geq k! \frac{3}{2} e^{-s/2} \right\}
\]
\[
\leq P \left\{ \prod_{1 \leq j \leq s} U'_j \geq e^{-s/2} \right\} + P \left\{ \sum_{1 \leq j \leq s} E'_j \geq \frac{3}{2}s \right\}
\]
\[
= P \left\{ W_s \leq \frac{s}{2} \right\} + P \left\{ W'_s \geq \frac{3s}{2} \right\} = O\left( \frac{1}{s^2} \right).
\]
We are done since
\[ \sum_{s>p} \frac{1}{s^2} = O(p^{-1}) \quad \text{and} \quad \sum_{s>p} \left( k! \frac{3}{2} s e^{-s/2} \right)^{1/k} = O\left( -e^{p/2} \right). \]

To deal with \( \sum_{s>p} B_{n,s}^* \) and \( \sum_{s>p} B_{n,s} \), the next lemma allows us to choose an appropriate \( p \).

**Lemma 8.** Uniformly for all \( p \in \mathbb{N} \) and \( n \in \mathbb{N} \),
\[ \mathbb{P} \left\{ \frac{P_{n,p}}{n} \in \left[ e^{-3p/2}, e^{-p/2} \right] \cap \left[ P_{n,p}R_{n,p}^k < k!^{3p/2} \right] \right\} = 1 - O\left( \frac{1}{p^2} \right). \] (4.19)

**Proof.** By (4.8), \( P_{n,p}/n \overset{a.s.}{\to} \prod_{1 \leq s \leq p} U_s \). Thus
\[ \mathbb{P} \left\{ \frac{P_{n,p}}{n} \notin \left[ e^{-3p/2}, e^{-p/2} \right] \right\} \to \mathbb{P} \left\{ \left| p - \sum_{1 \leq s \leq p} \log U_s \right| > \frac{p}{2} \right\} = O\left( \frac{1}{p^2} \right), \] (4.20)
where the last steps uses (4.18) in Lemma 7.

On the other hand, by Lemma 5,
\[ P_{n,p}R_{n,p}^k = \frac{P_{n,p}}{(L_{n,p}^*)^k} \left( L_{n,p}^* R_{n,p} \right)^k = \frac{P_{n,p}}{n \prod_{1 \leq s \leq p} U_s} \left( S_{n,p} \right)^k \overset{d}{\to} S_p^k. \]

Recall that \( \preceq \) denotes stochastically smaller than. Then by the definition of \( S_p \) (see (1.8)), \( S_p^k \preceq k!W_p \), where \( W_p \leq \text{Gamma}(p) \). It follows from (4.18) that
\[ \mathbb{P} \left\{ P_{n,p}R_{n,p}^k \geq k!^{3p/2} \right\} \to \mathbb{P} \left\{ S_p^k \geq k!^{3p/2} \right\} \leq \mathbb{P} \left\{ W_p \geq \frac{3p}{2} \right\} = O\left( \frac{1}{p^2} \right). \] (4.21)

Combining (4.20) and (4.21) gives the lemma. \( \Box \)

**Lemma 9.** For all \( \varepsilon > 0 \) and \( \delta > 0 \), there exists \( p \in \mathbb{N} \) and \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \),
\[ \mathbb{P} \left\{ \frac{\sum_{s>p} B_{n,s}^*}{n^{1-\frac{1}{k}}} \geq \varepsilon \right\} < \delta. \]

**Proof.** Let \( A_p \) denote the event in (4.19) for a \( p \) chosen later. We condition on
\[ A_p(m, y) \overset{\text{def}}{=} A_p \cap [P_{n,p} = m, R_{n,p} = y], \]
(4.22)
for \((m, y)\) satisfying
\[ (m, y) \in \mathcal{S}^* \overset{\text{def}}{=} \left\{ (m, y) \in \mathbb{R}^2 : ne^{-3p/2} \leq m \leq ne^{-p/2}, my^k \leq k!^{3p/2} \right\}. \] (4.23)

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(If \((m,y) \notin S^*\), the event \(A_p(m,y)\) is empty.) Note that this changes the distribution of \(G_{r,j}\) for \(j < m\), from \(\text{Gamma}(k)\) to \(\text{Gamma}(k)\) restricted to \((y, \infty)\). Thus by the definition of \(I_j^*\) in (4.4)

\[
\mathbb{E} \left[ I_j^* \mid A_p(m,y) \right] = \mathbb{P} \left\{ T_{1,j}^* < \min \{ G_{k,i} : 1 \leq i \leq j \} \mid \cap_{1 \leq i \leq j} [G_{k,i} > y] \right\} = \int_y^\infty e^{-x} \, dx + \int_y^\infty e^{-x} \left( \frac{\mathbb{P} \{ \text{Gamma}(k) > x \}}{\mathbb{P} \{ \text{Gamma}(k) > y \}} \right)^j \, dx
\]

\[
\leq y + \left( \frac{\Gamma(k)}{\Gamma(k,y)} \right)^j \int_0^\infty e^{-x} \left( \frac{\Gamma(k,x)}{\Gamma(k)} \right)^j \, dx, \tag{4.24}
\]

where the last step follows from Lemma 15. Thus for \(n\) large enough, for all \((m,y) \in S^*\),

\[
\left( \frac{\Gamma(k)}{\Gamma(k,y)} \right)^j \leq \left( \frac{\Gamma(k)}{\Gamma(k,y)} \right)^m \leq \frac{1}{(1-y)^m} \leq 2my^k \leq 3k!p,
\]

where the second inequality uses [11, 8.10.11]. Together with (2.1) in Lemma 1,

\[
\mathbb{E} \left[ I_j^* \mid A_p(m,y) \right] \leq y + 3k!p\mathbb{E} \left[ I_{p,j+1} \right].
\]

Thus for \(n\) large enough, by Theorem 1, for all \((m,y) \in S^*\),

\[
\mathbb{E} \left[ \sum_{p<s} B_{n,s}^* \left| A_p(m,y) \right. \right] = \sum_{1 \leq i \leq m} \mathbb{E} \left[ I_j^* \mid A_p(m,y) \right] \leq my + 3k!p\mathbb{E} \left[ A_{m+1}^* \right]
\]

\[
\leq \left( k! \left( \frac{3p}{2} \right)^{\frac{1}{k}} m^{1-\frac{1}{k}} + 6k!p\eta_{k,1} \cdot m^{1-\frac{1}{k}} \right)^{1-1/k}.
\]

Then by Markov’s inequality,

\[
\sup_{(m,y) \in S^*} \mathbb{P} \left\{ \frac{\sum_{p<s} B_{n,s}^*}{n^{1-\frac{1}{k}}} > \epsilon \mid A_p(m,y) \right\} \leq \frac{\mathbb{E} \left[ \sum_{p<s} B_{n,s}^* \mid A_p(m,y) \right]}{\epsilon n^{1-\frac{1}{k}}}
\]

\[
= \left[ \left( k! \left( \frac{3p}{2} \right)^{\frac{1}{k}} + 6k!\eta_{k,1} \right)^{1-1/k} \right] (e^{-p/2})^{1-1/k} \leq O \left( e^{-\frac{p}{2}} \right),
\]

if we take \(p\) large enough. This implies that there exists an \(p \in \mathbb{N}\) and \(n_0\) such that for all \(n > n_0\),

\[
\mathbb{P} \left\{ \frac{\sum_{p<s} B_{n,s}^*}{n^{1-\frac{1}{k}}} > \epsilon \mid A_p \right\} \leq O \left( e^{-\frac{p}{2}} \right).
\]

Now we are done since by Lemma 8, \(\mathbb{P} \left\{ A_p \right\} = O(p^{-2}). \)
Lemma 10. For all $\varepsilon > 0$ and $\delta > 0$, there exists $p \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\mathbb{P}\left\{ \frac{\sum_{j>p} B_{n,j}}{n^{1-\frac{1}{2}}} \geq \varepsilon \right\} < \delta.$$ 

Proof. We again condition on $A_p(m, y)$, as defined by (4.22) in Lemma 9, for $(m, y)$ satisfying (4.23).

Let $(E'_i)_{i \geq 1}$ be i.i.d. Exp(1) random variables. By our conditioning, for $j < m$, the distribution of $T_{1,j+1}$ has changed from $E'_1$ to $E'_1$ conditioning on $E'_1 + \cdots + E'_k > y$. Let $f(x)$ be the density function of $T_{1,j+1}$ conditioning on $A_p(m, y)$. Then by Lemma 15, for $x \geq y$,

$$f(x) = \frac{e^{-x} \mathbb{P}\{\Gamma(k-1) > y - x\}}{\mathbb{P}\{\Gamma(k) > y\}} = \frac{e^{-x}}{\mathbb{P}\{\Gamma(k) > y\}} \frac{\Gamma(k)}{\Gamma(k,y)},$$

and for $x < y$

$$f(x) = \frac{e^{-x} \mathbb{P}\{\Gamma(k-1) > y - x\}}{\mathbb{P}\{\Gamma(k) > y\}} \leq \frac{e^{-x}}{\mathbb{P}\{\Gamma(k) > y\}} = \frac{\Gamma(k)}{\Gamma(k,y)}.$$

By (4.23), $my^k < k!3p/2$ and $m \geq e^{-3p/2n}$. So for $n$ large enough $y < 1/2$. Thus by [11, 8.10.11],

$$\frac{\Gamma(k)}{\Gamma(k,y)} \leq \frac{1}{(1-y^k)} \leq 2.$$

In other words $f(x) \leq 2e^{-x}$. Thus

$$\mathbb{E}\left[ I_{1,j+1} \mid A_p(m, y) \right] = \int_0^y f(x) \, dx + \int_y^\infty f(x) \left( \frac{\mathbb{P}\{\Gamma(k) > x\}}{\mathbb{P}\{\Gamma(k) > y\}} \right)^j \, dx \leq 2\mathbb{E}\left[ I_j^* \right],$$

by (4.24) in Lemma 9. From now on the proof simply follows the same argument as Lemma 9. \square

4.3 Finishing the proof Theorem of 3

By Proposition 1 and Lemma 9, for all $x > 0$ and $\delta > 0$, there exists $\varepsilon > 0$, $p \in \mathbb{N}$ and $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$,

$$\mathbb{P}\left\{ \frac{\sum_{1 \leq j \leq p} B_{n,j}^*}{n^{1-\frac{1}{2}}} > x \right\} \leq \mathbb{P}\left\{ \sum_{1 \leq j \leq p} \frac{B_{n,j}^*}{n^{1-\frac{1}{2}}} > x - \varepsilon \right\} + \mathbb{P}\left\{ \sum_{p < j} \frac{B_{n,j}^*}{n^{1-\frac{1}{2}}} > \varepsilon \right\} \leq \mathbb{P}\left\{ \sum_{1 \leq j \leq p} B_j > x - \varepsilon \right\} + \frac{\delta}{3} + \frac{\delta}{3} \leq \mathbb{P}\left\{ \sum_{1 \leq j} B_j > x \right\} + \delta.$$
On the other hand, we can choose $\epsilon$ small enough such that

$$\mathbb{P}\left\{ \sum_{1 \leq j} B_j > x + \epsilon \right\} > \mathbb{P}\left\{ \sum_{1 \leq j} B_j > x \right\} - \frac{\delta}{3}. $$

And by Lemma 7, we can choose $p$ such that $\mathbb{P}\left\{ \sum_{p < j} B_j \geq \epsilon \right\} < \frac{\delta}{3}$. Thus by Proposition 1,

$$\mathbb{P}\left\{ \sum_{1 \leq j} B_{n,j} > x \right\} = \mathbb{P}\left\{ \sum_{1 \leq j \leq p} B_{n,j} > x \right\} - \mathbb{P}\left\{ \sum_{p < j} B_j > \epsilon \right\} - \frac{\delta}{3}.$$

In other words, we have

$$X^*_n \overset{d}{=} \sum_{1 \leq j} \frac{B_{n,j}}{n^{1-k}} \rightarrow \sum_{1 \leq j} B_j = B_k. \quad (4.25)$$

Now we fill the gap between $X^*_n$ and $X^1_n$ as we promised.

**Lemma 11.** There exists a coupling such that

$$\mathbb{P}_n - \mathbb{P}_n^1 \underset{n \to \infty}{\to} 0.$$

Note that in the following proof, we construct $(P_{n,j}, R_{n,j})_{j \geq 0}$ as in (4.1). In other words, we do not use the coupling constructed in subsection 4.1.

**Proof.** Recall that $(T_{i,j})_{i \geq 1, j \geq 1}$ are i.i.d. Exp(1) random variables that we used, together with $(P_{n,j}, R_{n,j})_{j \geq 0}$ to define $X^*_n$. Now we modify $(T_{i,j})_{i \geq 1, j \geq 1}$ by letting $T_{i,j} = T^*_i$ for all $i \in \mathbb{N}$ and $j \not\in \{P_{n,j}\}_{j \geq 0}$, unless there is a discrepancy, i.e., if for some $p \geq 1$,

$$P_n - p < j < P_n, \quad \text{and} \quad \sum_{i=1}^k T^*_i < R_{n,p}.$$

This may change the value of $(B_{n,j})_{j \geq 1}$ but not its distribution.

Let $J_{n,p}$ denote the number of discrepancies between $P_{n,p-1}$ and $P_{n,p}$ i.e.,

$$J_{n,p} = \sum_{j \geq 1} [P_{n,p-1} < j < P_{n,p}] [R_{n,p} > \sum_{1 \leq i \leq k} T^*_i].$$
Recall that (see (4.2) and (4.5))

\[ \lambda_n^1 \overset{\text{def}}{=} \lambda_n^k + \sum_{p \geq 1} B_{n,p} = \lambda_n^k + \sum_{j \geq 1} \sum_{p \geq 1} [P_{n,p-1} < j < P_{n,p}] [T_{1,j} < R_{n,p}], \]

\[ \lambda_n^* = \sum_{p \geq 1} B_{n,p}^* = \sum_{j \geq 1} \sum_{p \geq 1} [P_{n,p-1} < j \leq P_{n,p}] [T_{1,j}^* < R_{n,p}]. \]

Then with the above coupling, for all fixed \( p \in \mathbb{N} \),

\[ |\lambda_n^1 - \lambda_n^*| \leq \sum_{1 \leq j \leq p} J_{n,j} + 2\lambda_n^k + \sum_{j > p} B_{n,p} + \sum_{j > p} B_{n,p}^*. \]  

(4.26)

By Theorem 1, we have \( \lambda_n^k / n^{1-\frac{1}{k}} \overset{p}{\rightarrow} 0 \). It follows from Lemma 9 and Lemma 10 that by choosing \( p \) large enough, the last two terms of (4.26) divided by \( n^{1-\frac{1}{k}} \) are all negligible. Thus, it suffices to only consider \( \sum_{1 \leq j \leq p} J_{n,j} \).

Conditioning on \( (R_{n,j}, P_{n,j})_{n \geq 1, j \geq 0} \)

\[ J_{n,p} \overset{\text{def}}{=} \text{Bin} \left( P_{n,p-1} - P_{n,p} - 1, \mathbb{P} \{ \text{Gamma} \left( k, 1 \right) < R_{n,p} \} \right). \]

Therefore, it follows from Lemma 15 and the series expansion of the incomplete gamma function (see (6.1)) that

\[ \mathbb{E} \left[ J_{n,p} \mid (R_{n,j}, P_{n,j})_{n \geq 1, j \geq 0} \right] \leq (P_{n,p-1} - P_{n,p}) \cdot \left( 1 - \frac{\Gamma(k, R_{n,p})}{\Gamma(k)} \right) \leq P_{n,p-1} R_{n,p}^k \leq 2 \frac{P_{n,p-1}}{(L_{n,p}^\ast R_{n,p})^k} \]

\[ a.s. \rightarrow 2S_p^k \leq 2k! \text{Gamma}(k), \]

for \( n \) large enough, where \( \preceq \) denotes stochastically smaller than. In other words, for all fixed \( p \in \mathbb{N} \), \( \sup_{n \geq 1} \mathbb{E} \left[ J_{n,p} \right] < \infty \). Thus

\[ \sum_{1 \leq j \leq p} \frac{J_{n,j}}{n^{1-\frac{1}{k}}} \rightarrow 0. \]

Thus by (4.26) we are done. \( \Box \)

By Lemma 11, \( |\lambda_n^* - \lambda_n^1| / n^{1-\frac{1}{k}} \overset{d}{\rightarrow} 0 \). Together with \( \lambda_n^* / n^{1-\frac{1}{k}} \overset{p}{\rightarrow} \mathcal{B}_k \) (see (4.25)),

\[ \frac{\lambda_n^1}{n^{1-\frac{1}{k}}} = \frac{\lambda_n^*}{n^{1-\frac{1}{k}}} + \frac{\lambda_n^1 - \lambda_n^*}{n^{1-\frac{1}{k}}} \overset{d}{\rightarrow} \mathcal{B}_k. \]

By Theorem 1,

\[ \mathbb{E} \left[ \left( \frac{\lambda_n^1}{n^{1-\frac{1}{k}}} \right)^2 \right] \rightarrow \gamma_k. \]
Thus \((X_n^1/n^{1-k})_{n \geq 1}\) is uniformly integrable [13, pp. 221]. Therefore by Theorem 5.5.2 of [13],

\[
\mathbb{E} \left[ \frac{X_n^1}{n^{1-k}} \right] \to \mathbb{E} [B_k] = \eta_{k,1} = \frac{(kt)^{k} \Gamma \left( \frac{1}{k} \right)}{k-1},
\]

and we are done with Theorem 3.

**Remark 4.** It is not obvious how to directly compute \(\mathbb{E} [B_k]\) from its representation \(B_k = \sum_{1 \leq p} B_p\) (see (1.6)). In fact, it is not difficult to show that

\[
\sup_{n \geq 1} \mathbb{E} \left[ \left( \frac{X_n^1}{n^{1-k}} \right)^3 \right] < \infty.
\]

Thus \(((X_n^1/n^{1-k})^2)_{n \geq 1}\) is uniformly integrable and by Theorem 2

\[
\mathbb{E} \left[ \left( \frac{X_n^1}{n^{1-k}} \right)^2 \right] \to \mathbb{E} [B_k^2] = \gamma_k.
\]

We leave the details to the reader.

5 **SOME EXTENSIONS**

In this section we briefly discuss some easy implications of our main results.

5.1 **A lower bound and an upper bound for general graphs**

Let \(G_n\) be the set of rooted graphs with \(n\) nodes. Recall that for \(G_n \in G_n, \mathcal{K}(G_n)\) denotes the \(k\)-cut number of \(G_n\). The following lemma shows that \(P_n\), a path of length \(n\) with one endpoint as the root, is the easiest to break down among all graphs in \(G_n\).

**Lemma 12.** For all \(G_n \in G_n,\)

\[
\mathcal{X}_n \overset{\text{def}}{=} \mathcal{K}(P_n) \preceq \mathcal{K}(G_n),
\]

i.e., the left hand side is stochastically dominated by the right hand side [18, pp. 68]. Therefore,

\[
\min_{G_n \in G_n} \mathbb{E} \mathcal{K}(G_n) \geq \mathbb{E} \mathcal{X}_n \sim \eta_{k,1} n^{1-k},
\]

where \(\eta_{k,1}\) is as in Theorem 1.

**Proof.** Let \(T_n\) be an arbitrary spanning tree of \(G_n\) with the root of \(G_n\) marked as the root of \(T_n\). It is not difficult to see that \(\mathcal{K}(T_n) \preceq \mathcal{K}(G_n)\) — adding edges to the graph certainly would not decrease the \(k\)-cut number.
Consider the simple case when \( T_n \) is a tree that consists of only two paths connected to the root. If we disconnect one of these two paths from the root and connect it to the leaf of the other path, we can only decrease the number of records in the tree. In other words, we have a coupling which implies \( K(P_n) \preceq K(T_n) \).

For a more complicated \( T_n \), we can repeat the above transformation for sub-trees that consist of a root and two paths connected to the root, until the whole tree becomes a path. In other words, for all trees \( P_n \in G_n \), \( K(P_n) \preceq K(T_n) \). This proves (5.1). The second result of the lemma follows trivially from Theorem 1 (see e.g., [18, Theorem 2.15, pp. 71]).

The most resilient graph is obviously \( K_n \), the complete graph with \( n \) vertices. Thus we have the following upper bound.

**Lemma 13.** Let \( Y \overset{d}{=} \text{Gamma}(k, 1) \). Then

\[
\frac{K(K_n)}{n} \overset{d}{\rightarrow} \mathbb{E}[\max(\text{Poi}(Y), k) \mid Y].
\]

Therefore,

\[
\max_{G_n \in G_n} \mathbb{E}G_n \leq \mathbb{E}K_n \sim k \left(1 - \frac{1}{2^k \binom{2k}{k}}\right)n.
\]

**Proof.** Let \( S_n \) be the tree of \( n \) nodes with one root and \( n - 1 \) leaves. Obviously \( K(K_n) \overset{d}{=} K(S_n) \). So we can prove the lemma for \( K(S_n) \) instead.

Let \( Y \) be the time when the alarm clock of the root rings for the \( k \)-th time. Let \( W_{1,n}, \ldots, W_{n-1,n} \) be the number of cuts each leaf receives. Conditioning on the event \( Y = y \), \( W_{1,n}, \ldots, W_{n-1,n} \) are i.i.d. with the same distribution \( \max(\text{Poi}(y), k) \) (each node can receive at most \( k \) cuts). In other words, conditioning on \( Y = y \), by the law of large numbers,

\[
\frac{K(S_n)}{n} = \frac{k + \sum_{i=1}^{n-1} W_{i,n}}{n} \Rightarrow \mathbb{E}[\max(\text{Poi}(y), k)].
\]

Therefore, (5.2) follows. Since \( K(S_n)/n \leq k \), i.e., it is bounded, we also have

\[
\mathbb{E}\left[\frac{K(S_n)}{n}\right] \rightarrow \mathbb{E}[\mathbb{E}[\max(\text{Poi}(Y), k) \mid Y]] = k \left(1 - \frac{1}{2^k \binom{2k}{k}}\right),
\]

where we omit the computation for the last step. \( \square \)

#### 5.2 Path-like graphs

If a graph \( G_n \) consists of only long paths, then the limit distribution \( K(G_n) \) should be related to \( B_k \), the limit distribution of \( K(P_n)/n^{1-\frac{k}{2}} \) (see Theorem 3). We give a few simple examples in this section whose details we leave to the reader.
Example 1 (Long path). Let \((G_n)_{n \geq 1}\) be a sequence of rooted graphs such that \(G_n\) contains a path of length \(m(n)\) starting from the root with \(n - m(n) = o(n^{1-\frac{1}{k}})\). Since it takes at most \(k(n - m(n))\) cuts to remove all the nodes outside the long path,

\[
\mathcal{K}(P_{m(n)}) \leq \mathcal{K}(G_n) \leq \mathcal{K}(P_{m(n)}) + ko\left(n^{1-1/k}\right).
\]

Together with Lemma 12, this implies that \(\mathcal{K}(G_n)/n^{1-\frac{1}{k}}\) converges in distribution to \(B_k\).

Example 2 (Caterpillar). Let \(\ell \geq 2\) be a fixed integer. Let \(T^{[\ell]}_n\) be a tree with \(n\) nodes. If \(T^{[\ell]}_n\) consists of only a path connected to the root such that \(\ell - 1\) leaves are attached to each node on this path, except the last one which may have between 1 and \(\ell\) leaves attached, then we call \(T^{[\ell]}_n\) an \(\ell\)-caterpillar. We call this path of length \(n/\ell + O(1)\) the spine. It is easy to see that the number of one-records in \(T^{[\ell]}_n\) is about \(\ell\) times the number of one-records in \(P_{\lceil n/\ell \rceil}\). Therefore, it is not difficult to show that

\[
\frac{\mathcal{K}(T^{[\ell]}_n)}{(n/\ell)^{1-\frac{1}{\ell}}} \overset{d}{\to} \ell B_k.
\]

Example 3 (Curtain). Let \(\ell \geq 2\) be a fixed integer. Let \(T^{(\ell)}_n\) be a graph that contains of only \(\ell\) paths connected to the root, with the first \(\ell - 1\) of them having length \(\lceil n-1/\ell \rceil\). We call \(T^{(\ell)}_n\) an \(\ell\)-curtain. It is easy to see that cutting \(T^{(\ell)}_n\) is very similar to cutting \(\ell\) separated paths of length \(\lceil n/\ell \rceil\). Therefore, we can show that

\[
\frac{\mathcal{K}(T^{(\ell)}_n)}{(n/\ell)^{1-\frac{1}{\ell}}} \overset{d}{\to} \sum_{j=1}^{\ell} B_k^{[j]},
\]

where \(B_k^{[1]}, \ldots, B_k^{[\ell]}\) are i.i.d. copies of \(B_k\).

5.3 Deterministic and random trees

The approximation given in Lemma 1 can be used to compute the expectation of \(k\)-cut numbers in many deterministic or random trees. We give three examples: complete binary trees, split trees, and Galton-Watson trees.

5.3.1 Complete binary trees

Let \(T^{\text{bi}}_n\) be a complete binary tree of with \(n = 2^{m+1} - 1\) nodes, i.e., its height is \(m\). Observe that for a node at depth \(i\), i.e., at distance \(i\) to the root, to be an \(r\)-record we require that it has been cut \(r\) times before any of the \(i\) nodes above it have been cut \(k\) times. This has exactly the same probability for the \(i + 1\)-th node in a path to
be an \( r \)-record. Hence the random variable \( I_{r,i+1} \) in Lemma 1 is also the indicator that a node at depth \( i \) is an \( r \)-record. Then by Lemma 1, for \( r \leq k \),

\[
\mathbb{E}K(T^{bi}_n) = \sum_{i=0}^{m} 2^i \mathbb{E}I_{r,i+1}
\]

\[= 1 + \sum_{i=1}^{m} 2^i \left( 1 + O\left( \frac{1}{i} \right) \right) \frac{(k!)^{3/2}}{k} \Gamma \left( \frac{3}{2} \right) j^{-3/2} \sim \frac{(k!)^{3/2}}{k} \Gamma \left( \frac{3}{2} \right) \frac{2^{m+1}}{m^3}.
\]

Thus only the one-records matter as in the case of \( P_n \) and

\[
\mathbb{E}K(T^{bi}_n) \sim \mathbb{E}K(T^{bi}_n) \sim \frac{(k!)^{3/2}}{k} \Gamma \left( \frac{3}{2} \right) \frac{2^{m+1}}{m^3} \sim \frac{(k!)^{3/2}}{k} \Gamma \left( \frac{3}{2} \right) \frac{n}{(\log_2 n)^{3/2}}.
\]

### 5.3.2 Split trees

Split trees were first defined by Devroye [9] to encompass many families of trees that are frequently used in algorithm analysis, e.g., binary search trees. The random split tree \( T_{n}^{sp} \) has parameters \( b,s,s_0,s_1,V \) and \( n \) which are required to satisfy the inequalities

\[
2 \leq b, \quad 0 < s, \quad 0 \leq s_0 \leq s, \quad 0 \leq bs_1 \leq s + 1 - s_0, \quad (5.3)
\]

and \( V = (V_1, \ldots, V_b) \) is a random non-negative vector with \( \sum_{i=1}^{b} V_i = 1 \).

To define the random split tree consider an infinite \( b \)-ary tree \( \mathcal{U} \). The split tree \( T_{n}^{sp} \) is constructed by distributing \( n \) balls among nodes of \( \mathcal{U} \). For a node \( u \), let \( n_u \) be the number of balls stored in the subtree rooted at \( u \). Once \( n_u \) are all decided, we take \( T_{n}^{sp} \) to be the largest subtree of \( \mathcal{U} \) such that \( n_u > 0 \) for all \( u \). Let \( V_u = (V_{u,1}, \ldots, V_{u,b}) \) be the independent copy of \( V \) assigned to \( u \). Let \( u_1, \ldots, u_b \) be the child nodes of \( u \). Conditioning on \( n_u \) and \( V_u \), if \( n_u \leq s \), then \( n_{u_i} = 0 \) for all \( i \); if \( n_u > s \), then

\[
(n_{u_1}, \ldots, n_{u_b}) \sim \text{Mult}(n - s_0 - bs_1, V_{u,1}, \ldots, V_{u,b}) + (s_1, s_1, \ldots, s_1),
\]

where \( \text{Mult} \) denotes multinomial distribution, and \( b,s,s_0,s_1 \) are integers satisfying (5.3).

In the setup of split trees (and other random trees), we obtain \( \mathcal{K}(T_{n}^{sp}) \) by picking a random tree \( T_{n}^{sp} \) and a random \( k \)-cut of it. We let \( \mathcal{K}'(T_{n}^{sp}) \) be the total number of \( r \)-records, just as we did for fixed trees.

In the study of split trees, the following condition is often assumed:

**Condition A.** The split vector \( V \) is permutation invariant. Moreover, \( P \{ V_1 = 1 \} = 0 \), \( P \{ V_1 = 0 \} = 0 \), and that \( -\log(V_1) \) is non-lattice.

Let \( N \) be the number of nodes in \( T_{n}^{sp} \). Assuming condition A, Holmgren [21, Thm. 1.1] showed that there exists a constant \( \alpha \) such that

\[
\mathbb{E}N \sim an.
\]
Holmgren [20, Thm. 1.1] also showed that for \( k = 1 \), condition A implies that \( \mathcal{K}(T_{n}^{\text{sp}}) \) converges to a weakly 1-stable distribution after normalization, and that

\[
\mathbb{E}\mathcal{K}(T_{n}^{\text{sp}}) \sim \frac{\mu an}{\log n}, \tag{5.4}
\]

where \( \mu \overset{\text{def}}{=} b \mathbb{E}[-V_{1} \log V_{1}] \). We extend (5.4) for \( k \geq 2 \).

**Lemma 14.** Let \( T_{n}^{\text{sp}} \) be a split tree defined as above. Assuming condition A, we have

\[
\mathbb{E}[\mathcal{K}^{r}(T_{n}^{\text{sp}})] \sim \frac{(k!)^{\frac{r}{k}} \Gamma \left( \frac{r}{k} \right)}{\Gamma(r)} \frac{an}{(\log n)^{\frac{r}{k}}} \quad (1 \leq r \leq k), \tag{5.5}
\]

\[
\mathbb{E}[\mathcal{K}(T_{n}^{\text{sp}})] \sim \frac{(k!)^{\frac{1}{k}}}{} \frac{\Gamma \left( \frac{1}{k} \right)}{\Gamma \left( \frac{1}{k} \right)} \frac{an}{(\log n)^{\frac{1}{k}}}.
\]

**Proof.** We say a node \( v \) is **good** if it has depth \( d(v) \) where

\[
\left| d(v) - \frac{1}{\mu} \log n \right| \leq \log^{0.6} n,
\]

otherwise we say it is **bad**. Let \( B_{n}^{\text{sp}} \) be the number of bad nodes in \( T_{n}^{\text{sp}} \). It is known that there are not so many bad nodes. More specifically, by [21][Thm. 1.2],

\[
\mathbb{E}B_{n}^{\text{sp}} = O \left( \frac{n}{(\log n)^{3}} \right). \tag{5.6}
\]

Let \( X_{n}^{\text{sp}} \) be the number of \( r \)-records that are also good nodes. By (5.6),

\[
\mathbb{E}[\mathcal{K}^{r}(T_{n}^{\text{sp}}) - X_{n}^{\text{sp}}] \leq \mathbb{E}[kB_{n}^{\text{sp}}] = O \left( \frac{n}{(\log n)^{3}} \right),
\]

which is negligible. Thus it suffices to prove the lemma for \( X_{n}^{\text{sp}} \).

By Lemma 1 and the definition of good nodes, we have

\[
\mathbb{E}[X_{n}^{\text{sp}} | T_{n}^{\text{sp}}] = (N - B_{n}^{\text{sp}}) \frac{(k!)^{\frac{1}{k}} \Gamma \left( \frac{1}{k} \right)}{\Gamma \left( \frac{1}{k} \right)} \left[ \frac{1}{\mu} \log n + O(\log^{0.6} n) \right]^{-\frac{1}{k}} (1 + O(\log^{-\frac{1}{k}} n))
\]

\[
= (N - B_{n}^{\text{sp}}) \frac{(k!)^{\frac{1}{k}} \Gamma \left( \frac{1}{k} \right)}{\Gamma \left( \frac{1}{k} \right)} \frac{1}{\Gamma \left( \frac{1}{k} \right)} (1 + O(\log^{-\frac{1}{k}} n)).
\]

Taking expectations, we get (5.5). \qed

### 5.3.3 Galton-Watson trees

A Galton-Watson tree \( T_{n}^{\text{gw}} \) is a random tree that starts with the root node and recursively attaches a random number of children to each node in the tree, where the numbers of children are drawn independently from the same distribution \( \xi \). A
conditional Galton-Watson tree $T_n^{gw}$ is $T_n^{gw}$ restricted to size $n$. Conditional Galton-Watson trees have been well-studied, see, e.g., [25]. We assume throughout that $\mathbb{E} \xi = 1$ and $\sigma^2 \triangleq \text{Var} (\xi) \in (0, \infty)$.

Let $Z_i(T_n^{gw})$ be the number of nodes of depth $i$ (at distance $i$ to the root). Let $H(T_n^{gw})$ be the height of $T_n^{gw}$. Then, by Lemma 1, conditioning on $T_n^{gw}$,

$$
\mathbb{E} [\mathcal{K}'(T_n^{gw})|T_n^{gw}] = \sum_{i=1}^{H(T_n^{gw})} Z_i(T_n^{gw}) O \left( \frac{1}{i^{-\frac{1}{2}}} \right).
$$

It has been shown that [24, Theorem 1.13]

$$
\mathbb{E} Z_i(T_n^{gw}) = O(i),
$$

uniformly for all $i \geq 1$ and $n \geq 1$. It is also well-known that $H(T_n^{gw})$ is of the order $\sqrt{n}$. More precisely, there exist constants $C'$ and $c'$ such that

$$
\mathbb{P} \{ H(T_n^{gw}) \geq h \} \leq C' e^{-c'h^2/n},
$$

for all $n \geq 1$ and $h \geq 1$ [2, Theorem 1.2]. Using (5.7) and (5.8), we have

$$
\mathbb{E} [\mathcal{K}'(T_n^{gw})] = O \left( (\log n)^{2-\frac{3}{4}} n^{1-\frac{1}{4}} \right), \quad \mathbb{E} [\mathcal{K}(T_n^{gw})] = O \left( (\log n)^{2-\frac{3}{4}} n^{1-\frac{1}{4}} \right),
$$

uniformly for all $n \geq 1$.

In fact, we conjecture that $n^{1-\frac{1}{4}}$ is actually the right order of $\mathbb{E} \mathcal{K}(T_n^{gw})$. Let $v_0, \ldots, v_{n-1}$ be the nodes of $T_n$ in depth first order. (In other words, $v_0$ is the root of the tree. Assuming that $v_0$ has $d$ subtrees $T_1, \ldots, T_d$ attached to it, then $v_1, \ldots, v_{n-1}$ are the nodes of $T_1$ in depth first order, followed by nodes in $T_2$ in depth first order, and so on, until $T_d$. This continues recursively.) Let $D_n(i)$ be the depth of $v_i$. As is well-known, when $\mathbb{E} \xi = 1$ and $\sigma^2 \triangleq \text{Var} (\xi) \in (0, \infty)$,

$$
\left( \frac{D_n(nt)}{\sqrt{n}} \right)_{t \in (0,1)} \overset{d}{\to} \left( \frac{2e(t)}{\sigma} \right)_{t \in (0,1)},
$$

where

$$
D_n(nt) = D_n(i) + (nt - i) * (D_n(i + 1) - D_n(i)), \quad i \leq nt < i + 1,
$$

and $e(t)$ is a Brownian excursion. See [28] for details. Therefore, the expected number of $k$-records in $T_n^{gw}$ conditioned on $T_n^{gw}$ satisfies

$$
\frac{\mathbb{E} [\mathcal{K}^k(T_n^{gw})|T_n^{gw}]}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \frac{1}{D_n(i) + 1}
$$

$$
= \sum_{i=0}^{n-1} \frac{1}{n} \left( \frac{D_n \left( \frac{in}{n} \right) + 1}{\sqrt{n}} \right)^{-1} \overset{d}{\to} \int_0^1 \left( \frac{2e(t)}{\sigma} \right)^{-1} \, dt.
$$
Thus it is natural to expect that
\[
\mathbb{E} K^k(T_n^{gw}) \sqrt{n} \to \mathbb{E} \left[ \int_0^1 \left( \frac{2e(t)}{\sigma} \right)^{-1} \, dt \right] = \sigma \sqrt{\frac{\pi}{2}}.
\]
This has indeed been proved with other methods [24, Theorem A.1]. As a result, we conjecture that for \( r \in \{1, \ldots, k\}, \)
\[
\mathbb{E} K^r(T_n^{gw}) \to \frac{(k!)^{\frac{r}{k}}}{\Gamma(r)} \mathbb{E} \left[ \int_0^1 \left( \frac{2e(t)}{\sigma} \right)^{-\frac{r}{k}} \, dt \right] = \frac{(k!)^{\frac{r}{k}}}{\Gamma(r)} \frac{\Gamma \left( 1 - \frac{r}{2k} \right)}{\Gamma \left( \frac{r}{2k} \right)} \sigma \sqrt{\frac{\pi}{2}},
\]
and as a result, we further conjecture that
\[
\mathbb{E} K(T_n^{gw}) \sim \mathbb{E} K^1(T_n^{gw}) \sim \frac{(k!)^{\frac{1}{k}}}{\Gamma \left( \frac{1}{k} \right)} \Gamma \left( 1 - \frac{r}{2k} \right) \left( \frac{\sigma}{\sqrt{2}} \right)^{\frac{1}{k}} n \frac{1}{2n}.
\]

6 Some Auxiliary Results

In this section, we collect some lemmas that are used in previous sections.

Lemma 15. Let \( a \overset{\text{def}}{=} \frac{1}{2} \left( \frac{1}{k} + \frac{1}{k+1} \right) \) and \( x_0 \overset{\text{def}}{=} m^{-a}. \) Then uniformly for all \( x \in [0, x_0], \)
\[
\mathbb{P} \{ \Gamma(k) > x \}^m = \left( \frac{\Gamma(k, x)}{\Gamma(k)} \right)^m = \left( 1 + O \left( m^{-\frac{1}{2k}} \right) \right) \exp \left( -\frac{mx^k}{k!} \right),
\]
where \( \Gamma(\ell, z) \) denotes the upper incomplete gamma function [11, 8.2.2].

Proof. Using the density of gamma distributions (see (1.2)),
\[
\mathbb{P} \{ \Gamma(k) > x \} = \frac{\int_x^\infty e^{-x} x^{k-1} \, dx}{\Gamma(k)} = \frac{\Gamma(k, x)}{\Gamma(k)}.
\]
By [11, 8.7.3], for \( c \neq 0, -1, -2, \ldots, \)
\[
\Gamma(c, z) = \Gamma(c) \left( 1 - z^c e^{-z} \sum_{\ell=0}^\infty \frac{z^\ell}{\Gamma(c + \ell + 1)} \right).
\]
Thus uniformly for all \( x \leq x_0, \)
\[
\left( \frac{\Gamma(k, x)}{\Gamma(k)} \right)^m = \left( 1 - \frac{x^k}{k!} + O \left( x_0^{k+1} \right) \right)^m
\]
\[
= \left( 1 + O \left( m x_0^{k+1} \right) \right) \exp \left( -\frac{mx^k}{k!} \right)
\]
\[
= \left( 1 + O \left( m^{-\frac{1}{2k}} \right) \right) \exp \left( -\frac{mx^k}{k!} \right),
\]
where we use that \( -a(k+1) + 1 = -\frac{1}{2k}. \) \( \square \)
Lemma 16. Let $a \geq 0$ and $b \geq 1$ be fixed. Then uniformly for $m \geq 1$,

$$
\int_0^\infty x^{b-1}e^{-ax}\mathbb{P}\{\text{Gamma}(k) > x\}^m \, dx
= \left(1 + O\left(m^{-\frac{1}{2}}\right)\right) \frac{(k!)^\frac{b}{k} \Gamma \left(\frac{b}{k}\right)}{k} m^{-m^\frac{b}{k}}. 
$$

(6.3)

Proof. Let $x_0 = m^{-a}$ and $\alpha = \frac{1}{2} \left(\frac{b}{k} + \frac{1}{x+1}\right)$, as in Lemma 15. Then by Lemma 15, the left-hand-side of (6.3) equals

$$
\int_0^\infty x^{b-1}e^{-ax} \left(\frac{\Gamma(k,x)}{\Gamma(k)}\right)^m \, dx
= \int_0^{x_0} x^{b-1}e^{-ax} \left(\frac{\Gamma(k,x)}{\Gamma(k)}\right)^m \, dx + \int_{x_0}^\infty x^{b-1}e^{-ax} \left(\frac{\Gamma(k,x)}{\Gamma(k)}\right)^m \, dx \overset{\text{def}}{=} A_1 + A_2.
$$

Then

$$
A_1 = \left(1 + O\left(m^{-\frac{1}{2}}\right)\right) \int_0^{x_0} x^{b-1}e^{-ax} \exp \left(-\frac{mx^k}{k!}\right) \, dx
= \left(1 + O\left(m^{-\frac{1}{2}}\right)\right) \frac{(k!)^\frac{b}{k} m^{-\frac{b}{k}}}{k} \int_0^{w_0} w^{\frac{b}{k}}e^{-w} \, dw
= \left(1 + O\left(m^{-\frac{1}{2}}\right)\right) \frac{(k!)^\frac{b}{k}}{k} \left(\Gamma \left(\frac{b}{k}\right) - \Gamma \left(\frac{b}{k}, w_0\right)\right) m^{-m^\frac{b}{k}},
$$

where we change variable by $w = \frac{mx^k}{k!}$ and let $w_0 = \frac{mx_0^k}{k!}$. By [11, 8.11.i],

$$
\Gamma \left(\frac{b}{k}, w_0\right) \leq w_0^{\frac{b}{k}}e^{-w_0} \left(1 + \left|\frac{1 - \frac{b}{k}}{w_0}\right|\right) = O\left(e^{-\frac{w_0}{k}}\right).
$$

Since $1 - ak = \frac{1}{2(k+1)} > 0$, $w_0 = O\left(m^{2(k+1)}\right)$ and $\Gamma \left(\frac{b}{k}, w_0\right)$ is exponentially small. Thus by (6.2)

$$
A_1 = \left(1 + O\left(m^{-\frac{1}{2}}\right)\right) \frac{(k!)^\frac{b}{k} \Gamma \left(\frac{b}{k}\right)}{k} m^{-\frac{b}{k}}.
$$

To bound $A_2$, note first that for $x > x_0$, by (6.2),

$$
\left(\frac{\Gamma(k,x)}{\Gamma(k)}\right)^m \leq \left(\frac{\Gamma(k,x_0)}{\Gamma(k)}\right)^m = \left(1 + O\left(m^{-\frac{1}{2}}\right)\right) e^{-w_0} = O\left(e^{-\frac{w_0}{k}}\right).
$$

Therefore

$$
A_2 = \int_{x_0}^\infty x^{b-1}e^{-x} \left(\frac{\Gamma(k,x)}{\Gamma(k)}\right)^m \, dx \leq O\left(e^{-\frac{w_0}{k}}\right) \int_{x_0}^\infty x^{b-1}e^{-x} \, dx = O\left(e^{-\frac{w_0}{k}}\right),
$$

which is exponentially small. Thus we have

$$
A_1 + A_2 = \left(1 + O\left(m^{-\frac{1}{2}}\right)\right) \frac{(k!)^\frac{b}{k} \Gamma \left(\frac{b}{k}\right)}{k} m^{-m^\frac{b}{k}}.
$$

$\square$
Lemma 17. For $a > 0$, $b > 0$ and $k \geq 2$,
\[
\xi_k(a, b) \overset{\text{def}}{=} \int_0^\infty \int_0^\infty \exp \left[ -\frac{a x^k}{k!} - \frac{b y^k}{k!} \right] \, dx \, dy
\]
\[
= \frac{\Gamma \left( \frac{2}{k} \right)}{k} \left( \frac{k!}{a} \right) \frac{2}{k} \, F \left( \frac{2}{k}, \frac{1}{k}; 1 + \frac{1}{k}, -\frac{b}{a} \right),
\]
where $F$ denotes the hypergeometric function \cite{11, 15.2.1}. In particular,
\[
\xi_2(a, b) = \frac{\arctan \left( \sqrt{\frac{b}{a}} \right)}{\sqrt{ab}}.
\]

Proof. Changing to polar system by $x = r \cos(\theta)$ and $y = r \sin(\theta)$,
\[
\xi_k(a, b) = \int_0^{\pi/4} \int_0^\infty \exp \left[ -r^k \left( \frac{a \cos(\theta)^k}{k!} + \frac{b \sin(\theta)^k}{k!} \right) \right] \, r \, dr \, d\theta
\]
\[
= \int_0^{\pi/4} \left( \frac{a \cos(\theta)^k}{k!} + \frac{b \sin(\theta)^k}{k!} \right)^{-\frac{2}{k}} \frac{\Gamma \left( \frac{2}{k} \right)}{k} \, d\theta
\]
\[
= \frac{\Gamma \left( \frac{2}{k} \right)}{k} \left( \frac{k!}{a} \right)^{\frac{2}{k}} \int_0^{\pi/4} \left( 1 + \frac{b}{a} \tan(\theta) \right)^{-\frac{2}{k}} \, d\theta.
\]
Changing variable again by letting $u = \tan(\theta)^k$, the above equals
\[
\frac{\Gamma \left( \frac{2}{k} \right)}{k^2} \left( \frac{k!}{a} \right)^{\frac{2}{k}} \int_0^1 u^{\frac{1}{k}-1} \left( 1 + \frac{b}{a} u \right)^{-\frac{2}{k}} \, du.
\]
It follows from an integral representation of the hypergeometric function (see \cite{11, 15.6.1}) that
\[
\int_0^1 u^{\frac{1}{k}-1} \left( 1 + \frac{b}{a} u \right)^{-\frac{2}{k}} \, du = \frac{\Gamma \left( \frac{2}{k} \right)}{k^2} \frac{\Gamma \left( \frac{1}{k} \right)}{\Gamma \left( \frac{1}{k} \right)} F \left( \frac{2}{k}, \frac{1}{k}; 1 + \frac{1}{k}, -\frac{b}{a} \right).
\]
Plug the above identity into (6.6) and we get (6.4). For (6.5), see \cite{11, 15.4.3}. \qed

Lemma 18. For $a > 0$, $b > 0$ and $k \geq 2$,
\[
(a + b)^{-\frac{2}{k}} \leq \frac{k}{\Gamma \left( \frac{2}{k} \right) (k!)^\frac{2}{k}} \xi_k(a, b) \leq a^{-\frac{2}{k}} + b^{-\frac{2}{k}}.
\]
Moreover, $\xi_k(a, b)$ is monotonically decreasing in both $a$ and $b$.

Proof. Let
\[
\tilde{\xi}_k(a, b) \overset{\text{def}}{=} \frac{2}{k} \frac{1}{k} \xi_k(a, b) \overset{\text{def}}{=} a^{-\frac{2}{k}} F \left( \frac{2}{k}, \frac{1}{k}; 1 + \frac{1}{k}, -\frac{b}{a} \right).
\]
By a well-known identity for the hypergeometric function \([11, 15.8.1]\),

\[
\xi_k^*(a, b) = (a + b)^{-\frac{k}{2}} F \left( \frac{2}{k}, 1; 1 + \frac{1}{k}; \frac{b}{a + b} \right). \tag{6.8}
\]

Let \(\alpha_1 = \frac{k}{k + 1}\). By Karp \([27, \text{cor. 2}]\), for \(x \in (0, 1)\),

\[
(1 - \alpha_1 x)^{-\frac{2}{k}} \leq F \left( \frac{2}{k}, 1; 1 + \frac{1}{k}; x \right) \leq 1 - \alpha_1 + \alpha_1 (1 - x)^{-\frac{2}{k}}.
\]

Together with (6.8), we have

\[
(a + b)^{-\frac{2}{k}} \leq \xi_k^*(a, b) \leq a^{-\frac{2}{k}} + b^{-\frac{2}{k}},
\]

which gives us (6.7).

For monotonicity, using the formula of derivatives for hypergeometric functions \([11, 15.5.1]\), it is easy to verify that for \(a > 0\) and \(b > 0\)

\[
\frac{\partial}{\partial a} \xi_k^*(a, b) < 0, \quad \text{and} \quad \frac{\partial}{\partial b} \xi_k^*(a, b) < 0. \quad \square
\]

**Lemma 19.** For \(k \geq 2\), let

\[
\lambda_k \overset{\text{def}}{=} \int_0^1 \int_0^{1-s} \xi_k(s, t) \, dt \, ds.
\]

Then

\[
\lambda_k = \begin{cases} 
\frac{\pi \cot \left( \frac{\pi}{k} \right) \Gamma \left( \frac{1}{k} \right) \left( k! \right)^{1/2}}{2 (k - 2) (k - 1)} & k > 2, \\
\frac{\pi^2}{4} & k = 2.
\end{cases}
\]

**Proof.** When \(k = 2\), applying (6.5) and changing to polar system by letting \(s = (r \cot(\theta))^2\) and \(t = (r \sin(\theta))^2\),

\[
\xi_2 = \int_0^1 \int_0^{1-s} \frac{\arctan \left( \sqrt{\frac{t}{s}} \right)}{\sqrt{st}} \, dt \, ds = \int_0^{\frac{\pi}{2}} \int_0^1 4r^2 \, dr \, d\theta = \frac{\pi^2}{4}.
\]

For \(k \geq 3\), by Lemma 17, it suffices to show that

\[
\int_0^1 s^{-\frac{k}{2}} \int_0^{1-s} F \left( \frac{2}{k}, 1; \frac{1}{k}; 1 + \frac{1}{k}; -\frac{t}{s} \right) \, dt \, ds = \frac{k \pi \cot \left( \frac{\pi}{k} \right)}{2 (k - 2) (k - 1)}. \tag{6.9}
\]

It is easy to verify this using Mathematica, but we give a human proof here for suspicious readers.
Let \((x), \overset{\text{def}}{=} x(x + 1) \ldots (x + r - 1)\). Recall that the hypergeometric function is defined by (see [11, 15.6.1])

\[
F(a, b; c; z) \overset{\text{def}}{=} \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r r!} z^r,
\]

for \(|z| < 1\), and by analytic continuation elsewhere. Therefore, for \(s \in (1/2, 1)\),

\[
\int_{0}^{1-s} F\left(a, b; c; -\frac{t}{s}\right) dt = \sum_{r=0}^{\infty} \int_{0}^{1-s} \frac{(a)_r(b)_r}{(c)_r r!} \left(-\frac{t}{s}\right)^r dt
\]

\[
= \sum_{r=0}^{\infty} \frac{-s(a)_r(b)_r}{(c)_r(r+1)!} \left(1 - \frac{1}{s}\right)^{r+1}
\]

\[
= \frac{-s(c-1)}{(a-1)(b-1)} \left(F\left(a-1, b-1; c-1; 1 - \frac{1}{s}\right) - 1\right).
\]

Therefore for \(s \in (\frac{1}{2}, 1)\),

\[
\int_{0}^{1-s} F\left(\frac{2}{k}, \frac{1}{k}; 1 + \frac{1}{k}; -\frac{t}{s}\right) dt = \frac{-ks}{(k-1)(k-2)} \left(F\left(\frac{2}{k} - 1, \frac{1}{k}; 1 - \frac{1}{s}\right) - 1\right).
\]

By continuity, the equality also holds for \(s \in (0, \frac{1}{2}]\).

Thus it suffices to show that

\[
\int_{0}^{1} s^{1-\frac{2}{k}} F\left(\frac{2}{k} - 1, \frac{1}{k}; 1 - \frac{1}{s}\right) ds = \frac{1}{2} \left(\frac{k}{k - 1} - \pi \cot \left(\frac{\pi}{k}\right)\right). \tag{6.10}
\]

Changing variable by \(z = 1 - s\), using [11, 15.8.1] and applying [16, 7.512.6], the integral equals

\[
\int_{0}^{1} (1-z)^{1-\frac{2}{k}} F\left(\frac{2}{k} - 1, \frac{1}{k}; 1 - \frac{z}{1-z}\right) dz
\]

\[
= \int_{0}^{1} F\left(\frac{2}{k} - 1, \frac{1}{k}; z\right) dz
\]

\[
= 3F_2\left(1, -1 + \frac{2}{k}; 1, \frac{1}{k}; 2, 1\right)
\]

\[
= 1 + \frac{2 - k}{6(k+3)} 3F_{2}\left(2, \frac{2}{k} + 1, 2; \frac{1}{k} + 2, 4; 1\right)
\]

\[
- \frac{k-2}{2} 3F_2\left(1, \frac{2}{k}; 1; 1 + \frac{1}{k}; 2, 1\right), \tag{6.11}
\]

where \(3F_2(z)\) denotes the generalized hypergeometric function [11, 16.2.1], and the verification of last step is by elementary arithmetic.

Applying [11, 16.4.6] and [11, 5.5.3], we have

\[
3F_2\left(1, \frac{2}{k}; 1; 1 + \frac{1}{k}; 2; 1\right) = \frac{\Gamma(\frac{1}{2}) \Gamma\left(\frac{k}{2}\right)}{\Gamma(1)} \frac{\Gamma\left(1 + \frac{1}{k}\right) \Gamma\left(1 - \frac{1}{k}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{k}\right) \Gamma\left(\frac{1}{2} - \frac{1}{k}\right)} = \frac{\pi \cot\left(\frac{\pi}{k}\right)}{k - 2}.
\]

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Again by \[11, 16.4.6\],
\[
\mathbf{3}_2 \left( 2, \frac{2}{k} + 1, 2; \frac{1}{k} + 2, 4; 1 \right) = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1}{k} + 2 \right) \Gamma \left( 1 - \frac{1}{k} \right)}{\Gamma \left( \frac{3}{2} \right) \Gamma \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{3}{2} \right) \Gamma \left( 2 - \frac{1}{k} \right)} = \frac{3(k + 1)}{k - 1}.
\]

Putting the above two equations into (6.11) gives (6.10) and we are done. \(\square\)

**Remark 5.** In an attempt to prove Lemma 19, we discovered the following identity
\[
\int_{0}^{\infty} (w + 1)^{\frac{2}{k} - 2} F \left( \frac{2}{k}, \frac{1}{k} + 1; 1 + \frac{1}{k}; -w \right) \, dw = \frac{\pi \cot \left( \frac{\pi}{k} \right)}{k - 2}, \quad (k \geq 3),
\]
which we have not found in the literature. The proof follows from changing to polar system in the left-hand-side of (6.9) by letting \(s = (r \cos(\theta))^k\) and \(t = (r \sin(\theta))^k\).

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