Compactified Jacobians of curves
with spine decompositions

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1. Introduction

Let $X$ be a curve, that is, a connected, reduced, projective scheme of dimension 1 over an algebraically closed field. If $X$ is smooth, and nonrational, $X$ can be embedded in a canonical Abelian variety, its Jacobian variety $V^0_X$, the moduli scheme for invertible sheaves of degree 0 on $X$. If $X$ is singular, one can still consider $V^0_X$, but the Jacobian variety is no longer projective and, in general, $X$ cannot be embedded in $V^0_X$ nor, of course, in any of its torsors, $V^d_X$, the moduli schemes for degree-$d$ invertible sheaves on $X$.

Compactifications of the $V^d_X$ have been proposed and studied by many authors. The first steps were taken by Igusa [11] and Mayer and Mumford [12], but the first compactification was constructed by D’Souza [7], [8]. Later, Altman and Kleiman [2], [3], [4] gave two different constructions of D’Souza’s compactification, which work for families of integral curves.

If $X$ is reducible, the $V^d_X$ are not even Noetherian. Nevertheless, Oda and Seshadri [14] constructed various compactifications of open subschemes of the $V^d_X$, depending on the choice of polarizations, when $X$ has at most ordinary nodes for singularities. In our more general context, with no conditions imposed on the singularities, the first compactifications were constructed by Seshadri [16], and the case of families has been treated by Simpson [17]. It is worth mentioning as well the compactifications by Caporaso [5] and Pandharipande [15], constructed directly over the moduli space of stable curves.

In [10] there appeared compactifications of open subschemes of the $V^d_X$ that had the important property of being fine, that is, of representing a functor. Those compactifications, dependent on the choice of a point $P$ on the nonsingular locus of $X$, were only shown to be complete. Here we show that they are projective, and give sufficient conditions for when they are isomorphic to Seshadri’s compactifications.

More precisely, let $X_1, \ldots, X_n$ be the irreducible components of $X$. Seshadri’s compactifications depend on the choice of a $n$-tuple $a = (a_1, \ldots, a_n)$ of positive rational numbers, the so-called polarization. Given any integer $\chi$, Seshadri [16] uses Geometric Invariant Theory to construct a projective moduli scheme $U_X(a, \chi)$ for $S$-equivalence classes (see 3.6) of torsion-free, rank-1 sheaves $I$ (see 2.1) of Euler characteristic $\chi$ on $X$ which are $a$-semistable, i.e. such that

$$\chi(I|_Y/(\text{torsion})) \geq a_Y \chi,$$

where $a_Y := \sum_{X_i \subseteq Y} a_i$.

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for every subcurve $Y \subseteq X$.

On the other hand, it is shown in [10] the existence of a complete scheme $J^P_X(a, \chi)$ representing the functor of $P$-quasistable sheaves on $X$ with respect to $a$, that is, the $a$-semistable sheaves $I$ such that the inequality above is strict for every proper subcurve $Y \subset X$ containing $P$. Since $J^P_X(a, \chi)$ admits a universal sheaf and $U_X(a, \chi)$ corepresents the functor of $a$-semistable sheaves, there is a natural map $\Psi : J^P_X(a, \chi) \to U_X(a, \chi)$.

In this paper we show that $J^P_X(a, \chi)$ is projective (see Proposition 2.4) and give sufficient conditions for when $\Psi$ is an isomorphism: Our Theorem 4.4 states that $\Psi$ is an isomorphism if $X$ is locally planar and $a \cdot \chi$ is an integer only if $Y \ni P$ or $Y$ is a spine, i.e. a connected subcurve such that $Y \cap \overline{X-Y}$ consists of separating nodes (ordinary nodes of $X$ whose removal disconnects it). In particular, this is the case when $X$ is of compact type or even treelike.

Most of the statements in the paper can be immediately adapted to families of curves, a task left to the reader. The need for the work contained in this paper arose in [6], where Caporaso, Coelho and I construct and study Abel maps.

Briefly, in Section 2 we introduce the schemes $J^P_X(a, \chi)$, there called $J^P_E$ (see [4] for the connection), and show they are projective. In Section 3, we discuss spines, and show how a curve that decomposes in spines is simpler to study. The various technical results obtained in Section 3 are combined in Section 4 to obtain our main result: sufficient conditions for when $\Psi$ is an isomorphism.

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2. Fine compactified Jacobians

Fix throughout the paper an algebraically closed field $k$, a curve $X$ over $k$, that is, a connected, reduced, projective scheme $X$ of dimension 1 over $k$, and a point $P$ on the nonsingular locus of $X$. Also, denote by $X_1, \ldots, X_n$ the irreducible components of $X$.

All schemes are assumed locally of finite type over $k$. All points are assumed closed, unless stated otherwise.

2.1. (Semistable, torsion-free, rank-1 sheaves) Let $I$ be a coherent sheaf on $X$. We say that $I$ is: torsion-free (or depth-1) if its associated points are generic points of $X$; rank-1 if $I$ is invertible on a dense open subset of $X$; and simple if $\text{End}(I) = k$.

Each invertible sheaf on $X$ is torsion-free, rank-1 and simple.

A subcurve of $X$ is a closed subscheme that is a curve. For each subcurve $Y \subseteq X$, let $I_Y$ denote the restriction of $I$ to $Y$ modulo torsion or, in other words, the image of the natural map

$$I|_Y \longrightarrow \bigoplus_{i=1}^m (I|_Y)_{\xi_i},$$

where $\xi_1, \ldots, \xi_m$ are the generic points of $Y$. If $I$ is a torsion-free (resp. rank-1) sheaf on $X$, so is $I_Y$ on $Y$. 
Let $E$ be a locally free sheaf on $X$ of rank $r > 0$. Set $\mu(E) := \deg(E)/r$, the slope of $E$. If $\mu(E) \in \mathbb{Z}$, we say that $E$ is a polarization. For instance, $\mathcal{O}_X$ is a polarization of $X$ of slope 0, the canonical polarization.

Assume $I$ is torsion-free, rank-1. Call $I$ semistable (resp. stable) with respect to $E$ if $\chi(I \otimes E) = 0$ and $\chi(I_Y \otimes E|_Y) \geq 0$ (resp. $\chi(I_Y \otimes E|_Y) > 0$) for each proper subcurve $Y \subsetneq X$. Since

\begin{equation}
\chi(I_Y \otimes E|_Y) = r\chi(I_Y) + \deg(E|_Y) = r(\chi(I_Y) + \mu(E|_Y)),
\end{equation}

where $\mu(E|_Y) := \deg(E|_Y)/r$, the sheaf $I$ is semistable with respect to $E$ if and only if $\chi(I) = -\mu(E)$, whence $E$ is a polarization, and $\chi(I_Y) \geq -\mu(E|_Y)$ for each proper subcurve $Y \subsetneq X$.

If $I$ is semistable and $\chi(I_Y \otimes E|_Y) > 0$ for each proper subcurve $Y \subset X$ containing $P$ we say that $I$ is $P$-quasistable. Of course, if $I$ is stable then $I$ is $P$-quasistable.

2.2. (The fine compactified Jacobians) There is a scheme $J_X$ parametrizing torsion-free, rank-1, simple sheaves on the curve $X$; see [10] Thm. B, p. 3048. More precisely, $J_X$ represents the functor that associates to each scheme $T$ the set of $T$-flat coherent sheaves $\mathcal{I}$ on $X \times T$ such that $\mathcal{I}|_{X \times t}$ is torsion-free, rank-1 and simple for each $t \in T$, modulo equivalence $\sim$. We say that such sheaves $\mathcal{I}$ are torsion-free, rank-1 and simple on $X \times T/T$. We say that two such sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ are equivalent, and denote $\mathcal{I}_1 \sim \mathcal{I}_2$, if there is an invertible sheaf $\mathcal{N}$ on $T$ such that $\mathcal{I}_1 \cong \mathcal{I}_2 \otimes p_2^* \mathcal{N}$, where $p_2: X \times T \to T$ is the projection map.

If $T$ is a connected scheme, and $\mathcal{I}$ is a torsion-free, rank-1 sheaf on $X \times T/T$, then $\chi = \chi(\mathcal{I}|_{X \times t})$ does not depend on the choice of $t \in T$; we say that $\mathcal{I}$ is a sheaf of Euler characteristic $\chi$ on $X \times T/T$. So, there is a natural decomposition

\[ J_X = \bigsqcup_{\chi \in \mathbb{Z}} J_X^\chi, \]

where $J_X^\chi$ is the subscheme of $J_X$ parametrizing sheaves of Euler characteristic $\chi$.

Fix an integer $\chi$. The scheme $J_X^\chi$ is universally closed over $k$; see [10] Thm. 32, (2), p. 3068. However, in general, $J_X^\chi$ is neither of finite type nor separated.

Let $E$ be a locally free sheaf on $X$ of slope $-\chi$. By [10] Prop. 34, p. 3071, the subschemes $J_E^{ss}$ (resp. $J_E^s$, resp. $J_E^p$) of $J_X^\chi$ parametrizing simple and semistable (resp. stable, resp. $P$-quasistable) sheaves on $X$ with respect to $E$ are open. By [10] Thm. A, p. 3047, $J_E^{ss}$ is of finite type and universally closed, $J_E^s$ is separated and $J_E^p$ is complete over $k$. We call $J_E^p$ a fine compactified Jacobian.

2.3. (Theta divisors) For each scheme $S$ and $S$-flat coherent sheaf $\mathcal{F}$ on $X \times S$, there is an associated invertible sheaf $\mathcal{D}(\mathcal{F})$ on $S$, called the determinant of cohomology of $\mathcal{F}$. If $\chi(\mathcal{F}|_{X \times s}) = 0$ for every $s \in S$, then there is an associated global section $\sigma_\mathcal{F}$ of $\mathcal{D}(\mathcal{F})$, whose zero scheme parametrizes those $s \in S$ for which $\mathcal{F}|_{X \times s}$ admits a nonzero global section. For the construction and basic properties of $\mathcal{D}(\mathcal{F})$ and $\sigma_\mathcal{F}$; see [10] §6.1, p. 3076 and Prop. 44, p. 3078.

Fix an integer $\chi$, and recall that $J_X^\chi$ is the scheme parametrizing simple, torsion-free, rank-1 sheaves of Euler characteristic $\chi$ on $X$; see 2.2. Let $\mathcal{I}$ be a universal sheaf
on $X \times J_X^p/J_X^s$. To each locally free sheaf $E$ on $X$ with $\mu(E) = -\chi$, we associate the invertible sheaf $\mathcal{L}_E := \mathcal{D}(\mathcal{I} \otimes p_1^*E)$ on $J_X^s$ and its global section $\theta_E := \sigma_{\mathcal{I} \otimes p_1^*E}$, where $p_1: X \times J_X^s \to X$ is the projection. Recall that $\mathcal{I}$ is unique up to tensoring with an invertible sheaf from $J_X^s$. So, by [10] Prop. 44 (3), p. 3078 (the projection property of the determinant of cohomology), $\mathcal{L}_E$ and $\theta_E$ are well-defined, modulo isomorphism, and so is the zero scheme $\Theta_E$ of $\theta_E$. We call $\Theta_E$ a theta subscheme.

Let $\mathcal{L}^n := \mathcal{L}^{\otimes n}|_{J_E^p}$. In [10], we proved that, if $n$ is large enough, then $\mathcal{L}^n$ is generated by its global sections; and the morphism these sections define,

$$\Psi^n: J_E^p \to \mathbb{P}(H^0(J_E^p; \mathcal{L}^n)),$$

restricts to an embedding on $J_E^s$. So $J_E^s$ is quasiprojective. In general, $\Psi^n$ is not an embedding. Nevertheless, $J_E^s$ is projective; see below.

**Proposition 2.4.** Let $E$ be a polarization on $X$. Then $J_E^p$ is projective.

**Proof.** Without loss of generality, assume that $P \in X_1$. Let $r$ be the rank of $E$. Let $F$ be any locally free sheaf on $X$ of rank $rn$ such that

$$\deg(F|_{X_1}) = n\deg(E|_{X_1}) - (n - 1),$$

$$\deg(F|_{X_1}) = n\deg(E|_{X_1}) + 1 \quad \text{for each } i = 2, \ldots, n.$$

For instance, $F$ can be constructed from $E^{\otimes n}$ by a sequence of elementary transformations centered at one nonsingular point of $X$ on each of $X_2, \ldots, X_n$ and at $n - 1$ nonsingular points of $X$ on $X_1$. (These can be chosen to be all equal to $P$.)

We claim that $J_E^p \subseteq J_E^s$. Indeed, let $I$ be any $P$-quasistable sheaf with respect to $E$. Then $\chi(I \otimes E) = 0$ and

$$r\chi(I_Y) - \deg(E|_Y) \geq 0$$

for every proper subcurve $Y \subset X$, with equality only if $P \not\in Y$. First, since $E$ and $F$ have the same slope, $\chi(I \otimes F) = 0$. Second, let $Y$ be any proper subcurve of $X$, and $m$ its number of irreducible components. If $P \not\in Y$ then

$$\chi(I_Y \otimes F|_Y) = r\chi(I_Y) + \deg(F|_Y) \geq -n \deg(E|_Y) + n \deg(E|_Y) + m = m > 0.$$ 

And if $P \in Y$ then

$$\chi(I_Y \otimes F|_Y) = r\chi(I_Y) + n \deg(E|_Y) + m - n \geq n + m - n = m > 0.$$ 

Either way, $\chi(I_Y \otimes F|_Y) > 0$. So, $I$ is stable with respect to $F$, proving our claim.

Finally, since $J_E^s$ is quasiprojective by [10] Thm. C (4), p. 3048, so is $J_E^p$. But, since $J_E^p$ is complete, $J_E^p$ is projective. \[\square\]

**Remark 2.5.** Keep the notation used in the proof of Proposition 2.4. We claim that $J_E^p \supseteq J_E^s$, and thus $J_E^p = J_E^s = J_E$. Indeed, let $I$ be a semistable sheaf with respect to $F$. Since $\mu(E) = \mu(F)$, we have $\chi(I \otimes E) = 0$. Now, let $Y$ be a proper subcurve of $X$, and $m$ its number of irreducible components. If $P \in Y$ then

$$\chi(I_Y \otimes E|_Y) = r\chi(I_Y) + \deg(E|_Y) = r\chi(I_Y) + \frac{\deg(F|_Y) - m + n}{n} \geq \frac{n - m}{n} > 0.$$
On the other hand, suppose $P \notin Y$. Since $I$ is semistable with respect to $F$, 
\[ \chi(I_Y) \geq -\frac{\deg(F|_Y)}{rn}. \]
Let $s$ be the smallest nonnegative integer such that $rn$ divides $\deg(F|_Y) - s$. Then 
\[ \chi(I_Y) \geq -\frac{\deg(F|_Y) - s}{rn}. \]
Now, since $\deg(F|_Y) - m = n \deg(E|_Y)$, also $s - m$ is divisible by $n$. But $m \leq n - 1$. Then 
$s \geq m$, and hence 
\[ \chi(I_Y \otimes E|_Y) = r\chi(I_Y) + \deg(E|_Y) \geq -\frac{\deg(F|_Y) - m}{n} + \deg(E|_Y) = 0. \]
Either way, it follows that $I$ is $P$-quasistable with respect to $E$.

3. Curves with spine decompositions

3.1. (Spines) A point $N \in X$ is called a separating node if $N$ is an ordinary node of $X$ and $X - N$ is not connected. Since $X$ is itself connected, $X - N$ would have two connected components. Their closures are called the tails attached to $N$.

A (connected) subcurve $Y$ of $X$ is called a spine if every point in $Y \cap X - Y$ is a separating node. Then each connected component $Z$ of $X - Y$ is a tail intersecting $Y$ transversally at a single point on the nonsingular loci of $Y$ and $Z$.

If $Y$ is a union of spines of $X$ then any connected component of $Y$ or $X - Y$ is a spine. Two spines $Y_1$ and $Y_2$ of $X$ with no common component intersect transversally at a single point on the smooth loci of $Y_1$ and $Y_2$.

A tuple $\mathfrak{Z} := (Z_1, \ldots, Z_q)$ of spines $Z_i$ covering $X$ with finite pairwise intersection is called a spine decomposition of $X$.

Proposition 3.2. Let $\mathfrak{Z} := (Z_1, \ldots, Z_q)$ be a spine decomposition of $X$. Then there is an isomorphism 
\[ u : J_X \longrightarrow J_{Z_1} \times \cdots \times J_{Z_q} \]
sending $[I]$ to $([I|_{Z_1}], \ldots, [I|_{Z_q}])$. Furthermore, for each integer $\chi$,
\[ u(J^\chi_X) = \bigcup_{\chi_1 + \cdots + \chi_q = \chi + q - 1} J_{Z_1}^{\chi_1} \times \cdots \times J_{Z_q}^{\chi_q}. \]

Proof. The statements are clearly true if $q = 1$. Assume $q \geq 2$. Since $Z_i$ and $Z_j$ intersect at at most one point, for $i \neq j$, there is at least one $Z_i$ which is a tail of $X$, say for $i = 1$. Set $Y_1 := Z_1$ and $Y_2 := X - Z_2$. By induction, since $(Z_2, \ldots, Z_q)$ is a spine decomposition of $Y_2$, it will be enough to show that there is an isomorphism 
\[ u' : J_X \rightarrow J_{Y_1} \times J_{Y_2} \]
sending $[I]$ to the pair $([I|_{Y_1}], [I|_{Y_2}])$ such that 
\[ u'(J^\chi_X) = \bigcup_{\chi_1 + \chi_2 = \chi + 1} J_{Y_1}^{\chi_1} \times J_{Y_2}^{\chi_2} \]
for each integer $\chi$. 

We need to show that \( u' \) is well-defined. First of all, a simple torsion-free, rank-1 sheaf \( I \) on \( X \) must be invertible at \( N \), because otherwise \( I \cong I_{Y_1} \oplus I_{Y_2} \). So the restrictions \( I|_{Y_1} \) and \( I|_{Y_2} \) are torsion-free, rank-1 sheaves. The sum of their Euler characteristics is \( \chi + 1 \), where \( \chi \) is the Euler characteristic of \( I \), as it follows from applying \( \chi(\cdot) \) to the natural exact sequence,

\[
0 \rightarrow I|_{Y_2} \otimes O_{Y_2}(-N) \rightarrow I \rightarrow I|_{Y_1} \rightarrow 0.
\]

We claim that a torsion-free, rank-1 sheaf \( I \) on \( X \), invertible at \( N \), is simple if and only if the restrictions \( I|_{Y_1} \) and \( I|_{Y_2} \) are simple. Indeed, since \( N \) is a node, and \( I \) is invertible at \( N \), there is a natural isomorphism

\[
\text{End}(I) \rightarrow \text{End}(I|_{Y_1}) \times_{\text{End}(I|_{N})} \text{End}(I|_{Y_2}),
\]

given by restriction, and hence

\[
\dim \text{End}(I) = \dim \text{End}(I|_{Y_1}) \cdot \dim \text{End}(I|_{Y_2}).
\]

So \( \dim \text{End}(I) = 1 \) if and only if \( \dim \text{End}(I|_{Y_i}) = 1 \) for \( i = 1 \) and \( i = 2 \).

Let \( \mathcal{I} \) be a universal simple, torsion-free rank-1 sheaf on \( X \times J_X \). Then, as seen above, \( \mathcal{I}|_{X \times s} \) is invertible at \( N \times s \) for each \( s \in J_X \), and hence \( \mathcal{I} \) is invertible along \( N \times J_X \). So, for \( i = 1, 2 \), the restriction \( \mathcal{I}|_{Y_i \times J_X} \) is also invertible along \( N \times J_X \). Since \( \mathcal{I}|_{Y_i \times J_X} \) agrees with \( \mathcal{I} \) on \( (Y_i - N) \times J_X \), it follows that \( \mathcal{I}|_{Y_i \times J_X} \) is torsion-free, rank-1 on \( Y_i \times J_X \). Moreover, since \( \mathcal{I}|_{X \times s} \) is simple, so is \( \mathcal{I}|_{Y_i \times s} \) for \( i = 1, 2 \), by our claim above, for each \( s \in J_X \). The pair \( (\mathcal{I}|_{Y_1 \times J_X}, \mathcal{I}|_{Y_2 \times J_X}) \) defines \( u' \).

As for the inverse to \( u' \), we construct a map

\[
u': J_{Y_1} \times J_{Y_2} \rightarrow J_X
\]
as follows: For each \( i = 1, 2 \), let \( \mathcal{I}_i \) be a universal simple, torsion-free rank-1 sheaf on \( Y_i \times J_{Y_i} \). We may assume that \( \mathcal{I}_i|_{Y_i \times J_{Y_i}} \) is trivial (a rigidification). Let \( \mathcal{M}_i \) be the pullback of \( \mathcal{I}_i \) to \( Y_i \times J_{Y_1} \times J_{Y_2} \) under the projection map. Since

\[
\mathcal{M}_1|_{N \times J_{Y_1} \times J_{Y_2}} \cong O_{N \times J_{Y_1} \times J_{Y_2}} \cong \mathcal{M}_2|_{N \times J_{Y_1} \times J_{Y_2}},
\]

and since \( N \) is a node of \( X \), we may glue \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) along \( N \times J_{Y_1} \times J_{Y_2} \) to obtain a sheaf \( \mathcal{M} \) on \( X \times J_{Y_1} \times J_{Y_2} \), invertible along \( N \times J_{Y_1} \times J_{Y_2} \), such that \( \mathcal{M}|_{Y_1 \times J_{Y_1} \times J_{Y_2}} \cong \mathcal{M}_i \) for \( i = 1, 2 \). Since \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are flat over \( J_{Y_1} \times J_{Y_2} \), so is \( \mathcal{M} \). Also, since \( \mathcal{M}_i|_{Y_i \times s} \) is simple, torsion-free and rank-1 on \( Y_i \) for each \( i = 1, 2 \), so is \( \mathcal{M}|_{Y_i \times s} \), for each \( s \in J_{Y_1} \times J_{Y_2} \). Let \( \mathcal{M} \) define \( u' \).

It is not hard to check that \( u' \) and \( u' \) are indeed inverse to each other. \( \square \)

3.3. (Spine decomposition) Let \( \mathcal{Z} := (Z_1, \ldots, Z_q) \) be a spine decomposition of \( X \) and \( c := (\chi_1, \ldots, \chi_q) \) a \( q \)-tuple of integers. For each \( \sigma \in S_q \), where \( S_q \) is the permutation group of \( \{1, \ldots, q\} \), there is a natural open and closed embedding

\[
v^{c\sigma}_\mathcal{Z}: J_{Z_1}^{\chi_1} \times \cdots \times J_{Z_q}^{\chi_q} \rightarrow J_X^{\chi_1+\cdots+\chi_q}
\]
sending ([I_1], \ldots, [I_q]) to [I], where I is the unique (modulo isomorphism) simple, torsion-free, rank-1 sheaf on X such that

\[ I|_{Z_{a(i)}} \cong I_{\sigma(i)} \otimes \mathcal{O}_{Z_{a(i)}} \left( \sum_{j=i+1}^{q} \sum_{N \in Z_{a(i)} \cap Z_{a(j)}} N \right) \text{ for } i = 1, \ldots, q. \]

In fact, \( v_3^{c,\sigma} \) is, after translations, a restriction of \( u^{-1} \) to an open subscheme, where \( u \) is the isomorphism of Proposition 3.2. To see that \( I \) has indeed Euler characteristic \( \chi_1 + \cdots + \chi_q \), just observe that \( I \) has a filtration

\[ 0 = K_0 \supsetneq K_1 \supsetneq \cdots \supsetneq K_{q-1} \supsetneq K_q = I \]

where \( K_j/K_{j-1} \cong I_{\sigma(j)} \) for \( j = 1, \ldots, q \). We set \( v_3^c := v_3^{c,\text{id}} \).

Let \( E \) be a polarization of \( X \) such that \( \mu(E|_{Z_i}) = -\chi_i \) for \( i = 1, \ldots, q \). Then

\[ J^s_E = \bigcup_{\sigma \in S_q} v_3^{c,\sigma}(J^s_E|_{Z_1} \times \cdots \times J^s_E|_{Z_q}). \]

Indeed, a simple torsion-free, rank-1 sheaf \( I \) on \( X \) satisfying (3.3.1) is semistable if the successive quotients of (3.3.2) are semistable. Conversely, let \( I \) be a semistable sheaf on \( X \). By induction, it is enough to prove that \( I|_{Z_i} \) is semistable for a certain \( i \). Now, since \( \#(Z_i \cap Z_j) \leq 1 \) for \( i \neq j \), there is a spine \( Z_j \) that is a tail. If \( I|_{Z_j} \) is semistable we are done. If not, then \( \chi(I|_{Z_j}) \geq 1 \), and hence \( \chi(I|_{Z'_j}) = \chi_j \), where \( Z'_j := \bigcup_{\ell \neq j} Z_\ell \) and \( \chi_j := \sum_{\ell \neq j} \chi_\ell \), implying that \( I|_{Z'_j} \) is semistable. Rinse and repeat.

Notice that, as a consequence, if \( q \geq 2 \) then \( J^s_E = \emptyset \).

**Definition 3.4.** A polarization \( E \) of \( X \) is called integer at a (connected) proper subcurve \( Y \) of \( X \) if \( \mu(E|_Y) = \mu(E|_Z) \) for every connected component \( Z \) of \( X - Y \), are integers.

**Proposition 3.5.** Let \( E \) be a polarization of \( X \). If \( E \) is not integer at any subcurve of \( X \) then \( J^s_E = J_E \). If \( E \) is integer only at subcurves of \( X \) containing \( P \) then \( J^P_E = J^s_E \).

**Proof.** Let \( I \) be a semistable sheaf on \( X \) which is not stable. Then there is a (connected) proper subcurve \( Y \) of \( X \) such that \( \chi(I_Y) = -\mu(E|_Y) \). If \( I \) is \( P \)-quasi-stable then \( P \notin Y \). We need only show that \( \mu(E|_Z) \) is an integer for each connected component \( Z \) of \( X - Y \).

Now, let \( K \) denote the kernel of the natural surjection \( I \to I_Y \). By the additivity of the Euler characteristic, \( \chi(K \otimes E) = 0 \). Let \( Z_1, \ldots, Z_q \) be the connected components of \( X - Y \). For each \( i = 1, \ldots, q \), set \( K_i := K|_{Z_i} \) and \( N_i := \text{Ker}(K \to K_i) \). Then \( K = K_1 \oplus \cdots \oplus K_q \). Moreover, since \( N_i \) is a subsheaf of \( I \), and \( I \) is semistable, \( \chi(N_i \otimes E) \leq 0 \), and hence \( \chi(K_i \otimes E) \geq 0 \). But

\[ 0 = \chi(K \otimes E) = \chi(K_1 \otimes E) + \cdots + \chi(K_q \otimes E). \]

So \( \chi(K_i \otimes E) = 0 \), and hence \( \mu(E|_{Z_i}) \) is an integer for each \( i = 1, \ldots, q \). \( \square \)
3.6. (The S-equivalence) Let $E$ be a polarization of $X$ and $I$ a semistable sheaf on $X$. Then there are (connected) subcurves $Z_1, \ldots, Z_q$ covering $X$ and a filtration

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{q-1} \subsetneq I_q = I$$

such that the quotient $I_j/I_{j-1}$ is a stable sheaf on $Z_j$ with respect to $E|_{Z_j}$ for each $j = 1, \ldots, q$. The above filtration is called a Jordan–Hölder filtration. The sheaf $I$ may have many Jordan–Hölder filtrations but the collection of subcurves $S(I) := \{Z_1, \ldots, Z_q\}$ and the isomorphism class of the sheaf

$$\text{Gr}(I) := I_1/I_0 \oplus I_2/I_1 \oplus \cdots \oplus I_q/I_{q-1}$$

depend only on $I$, by the Jordan–Hölder theorem.

Notice that also $\text{Gr}(I)$ is semistable, and

$$\text{Gr}(I) \cong \bigoplus_{Z \in S(I)} \text{Gr}(I)_Z.$$

We say that two semistable sheaves $I$ and $J$ on $X$ are $S$-equivalent if $S(I) = S(J)$ and $\text{Gr}(I) \cong \text{Gr}(J)$.

**Lemma 3.7.** Let $E$ be a polarization of $X$, and $I$ and $K$ two $S$-equivalent $P$-quasistable sheaves on $X$. If $S(I)$ is a collection of spines of $X$, then $I \cong K$.

**Proof.** Since the map $u$ in Proposition [3,2] is an isomorphism, to show that $I \cong K$, we need only show that $I|_Z \cong K|_Z$ for each $Z \in S(I)$.

For each $Z \in S(I)$, let $L_Z := \text{Gr}(I)_Z$, and let $\Delta^P_Z$ be the set of points in $Z \cap X - \overline{Z}$ on the connected components of $X - \overline{Z}$ not containing $P$. We need only show that

$$(3.7.1) \quad I|_Z \cong L_Z \otimes O_Z \left( \sum_{N \in \Delta^P_Z} N \right).$$

To prove (3.7.1), let

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{q-1} \subsetneq I_q = I$$

be a Jordan–Hölder filtration of $I$. For each $j = 1, \ldots, q$, let $Z_j$ be the support of $I_j/I_{j-1}$, and put $Y_j := Z_1 \cup \cdots \cup Z_j$. It follows from [10] Prop. 6, p. 3053, that $P \in Z_1$ and that all the $Y_j$ are connected. So, for each $j = 1, \ldots, q$,

$$\Delta^P_{Z_j} = Z_j \cap X - Y_j.$$

On the other hand, since

$$I_j \cong I_{j+1}|_{Y_j} \otimes O_{Y_j} \left( - \sum_{N \in Y_j \cap Z_{j+1}} N \right)$$

for each $j = 1, \ldots, q - 1$, it follows by induction that

$$I_j \cong I|_{Y_j} \otimes O_{Y_j} \left( - \sum_{N \in Y_j \cap X - Y_j} N \right).$$
for each \( j = 1, \ldots, q \). So

\[
I|_{Z_j} = (I|_{Y_j})|_{Z_j} \cong (I_j)|_{Z_j} \otimes O_{Z_j} \left( \sum_{N \in Z_j \cap X-Y_j} N \right) \cong L_{Z_j} \otimes O_{Z_j} \left( \sum_{N \in \Delta_{Z_j}^q} N \right)
\]

for each \( j = 1, \ldots, q \).

**Proposition 3.8.** Let \( E \) be a polarization of \( X \). Assume \( E \) is integer only at sub-curves of \( X \) that are spines or contain \( P \). Then any two \( S \)-equivalent \( P \)-quasistable sheaves are isomorphic.

*Proof.* For every semistable sheaf \( I \) on \( X \), the polarization \( E \) is integer at every \( Z \in \mathcal{S}(I) \). By hypothesis, every \( Z \in \mathcal{S}(I) \) but the one containing \( P \) is a spine. But the subcurve \( Z \in \mathcal{S}(I) \) containing \( P \) is also a spine because \( \overline{X-Z} \) is a union of spines. Then apply Lemma 3.10. \( \square \)

**Proposition 3.9.** Let \( \mathfrak{Z} = (Z_1, \ldots, Z_q) \) be a spine decomposition of \( X \), and let \( \mathfrak{c} = (\chi_1, \ldots, \chi_q) \) be a \( q \)-tuple of integers. Let

\[
\nu_{\mathfrak{Z}}^* : J^X_{Z_1} \times \cdots \times J^X_{Z_q} \longrightarrow J^X
\]

be the associated embedding. Then, for each locally free sheaf \( E \) on \( X \) satisfying \( \mu(E|_{Z_i}) = -\chi_i \) for \( i = 1, \ldots, q \), we have

\[
(3.9.1) \quad (\nu_{\mathfrak{Z}}^*)^{-1}\Theta_E = \sum_{i=1}^{q} J^X_{Z_i} \times \cdots \times J^X_{Z_{i-1}} \times \Theta_{E|_{Z_i}} \times J^X_{Z_{i+1}} \times \cdots \times J^X_{Z_q}.
\]

(The sum is the subscheme defined by the product of the sheaves of ideals of the summands.)

*Proof.* Set \( J := J^X_{Z_1} \times \cdots \times J^X_{Z_q} \), and denote by \( q: X \times J \rightarrow X \) the projection. Let \( \mathcal{I} \) be a universal sheaf on \( X \times J^X/J^X_{\mathfrak{Z}} \). Set \( v := \nu_{\mathfrak{Z}}^* \). For each \( j = 1, \ldots, q \), let

\[
\mathcal{I}_j := (1_X, v)^* \mathcal{I}|_{X \times J} \otimes O_{Z_j \times J} \left( - \sum_{\ell=j+1}^{q} \sum_{N \in \Delta_{Z_j}^q} N \times J \right).
\]

Then \( \mathcal{I}_j \) is equivalent to \( p_j^* \mathcal{L}_j \), where \( \mathcal{L}_j \) is a universal sheaf on \( Z_j \times J_{Z_j}/J_{Z_j}^X \) and

\[
p_j : Z_j \times J_{Z_j}^X \times \cdots \times J_{Z_q}^X \longrightarrow Z_j \times J_{Z_j}^X
\]

is the projection onto the indicated factors. Also, there is a filtration

\[
0 = \mathcal{K}_0 \subsetneq \mathcal{K}_1 \subsetneq \cdots \subsetneq \mathcal{K}_{q-1} \subsetneq \mathcal{K}_q = (1_X, v)^* \mathcal{I}
\]

such that \( \mathcal{K}_j/\mathcal{K}_{j-1} \cong \mathcal{I}_j \) for \( j = 1, \ldots, q \).

Now, tensor the above filtration with \( q^*E \), and take determinants of cohomology. From the base-change, functorial, projection and additive properties of determinants of cohomology and associated sections, [10] Prop. 44, p. 3078, we get (3.9.1). \( \square \)

**Lemma 3.10.** Let \( Z_1 \) and \( Z_2 \) be proper complementary subcurves of \( X \). Let \( L \) be an invertible sheaf on \( X \). Let \( F_1 \) and \( F_2 \) be locally free sheaves on \( Z_1 \) and \( Z_2 \), respectively, of the same rank such that \( \det F_i \cong L|_{Z_i} \) for \( i = 1, 2 \). Then there is a locally free sheaf \( F \) on \( X \) with \( \det F \cong L \) such that \( F|_{Z_i} = F_i \) for \( i = 1, 2 \).
Proof. Let $r$ be the common rank of $F_1$ and $F_2$. Fix a very ample invertible sheaf $\mathcal{O}_X(1)$ on $X$. Let $m$ be an integer such that the twist $F_i(m)$ is generated by global sections for each $i = 1, 2$. Then, for each $i = 1, 2$ there is an exact sequence of the form

\[
0 \to \mathcal{O}_Z(-m)^{\oplus r-1} \to F_i \to M_i \to 0,
\]

where $M_i := (\det F_i)(m(r - 1))$. The extension (3.10.1) is represented by an element $v_i \in H^1(Z_i, M_i(m))^{\oplus r-1}$. Let $M := L(m(r - 1))$. Since $\det F_i \cong L|Z_i$ for $i = 1, 2$, we have $M|Z_i \cong M_i$ for $i = 1, 2$. The induced map $M \to M_1 \oplus M_2$ is injective with finite length cokernel, and hence induces a surjection

\[
H^1(X, M(m))^{\oplus r-1} \to H^1(Z_1, M_1(m))^{\oplus r-1} \oplus H^1(Z_2, M_2(m))^{\oplus r-1}.
\]

Let $v \in H^1(X, M(m))^{\oplus r-1}$ mapping to $(v_1, v_2)$. Then $v$ corresponds to an exact sequence

\[
0 \to \mathcal{O}_X(-m)^{\oplus r-1} \to F \to M \to 0
\]

restricting to (3.10.1) on $Z_i$ for each $i = 1, 2$. So $F|Z_i = F_i$ for $i = 1, 2$. Also, $F$ is locally free of rank $r$ with $\det F \cong M(-m(r - 1)) \cong L$. \hfill \Box

Lemma 3.11. Let $E$ be a polarization of $X$, and $\chi := -\mu(E)$. Let $I$ be a simple, semistable sheaf on $X$ such that $\mathcal{S}(I)$ is a collection of spines. Then, for each $v \in T_{J^2, [1]}$ nonzero there are an integer $m \geq 2$ and a locally free sheaf $F$ on $X$ with $\text{rk}(F) = \text{mrk}(E)$ and $\det(F) \cong \det(E)^{\otimes m}$ such that $[I] \in \Theta_F$ but $v \not\in T_{\Theta_F, [1]}$.

Proof. Let $3 := (Z_1, \ldots, Z_q)$ be a spine decomposition with $\mathcal{S}(I) = \{Z_1, \ldots, Z_q\}$. For each $i = 1, \ldots, q$, set $E_i := E|_{Z_i}$ and $I_i := \text{Gr}(I)_{Z_i}$, and put $\chi_i := \chi(I_i)$ for $i = 1, \ldots, q$. Notice that $\mu(E_i) = -\chi_i$ and that $I_i$ is stable with respect to $E_i$. Put $c := (\chi_1, \ldots, \chi_q)$. As shown in [3,3], up to reordering the elements of $\mathcal{S}(I)$, we may assume that $[I] = v_3^{[c]}([I_1], \ldots, [I_q])$. Moreover, since $v_3^{[c]}$ is an open embedding, there are $v_i \in T_{J^2_{3}, [I]}$ for $i = 1, \ldots, q$ such that $d((v_3^{[c]}([I_1], \ldots, [I_q]))(v_1 + \cdots + v_q) = v$. Since $v \neq 0$, there is $i$ such that $v_i \neq 0$. Fix such an $i$.

Since $I_i$ is stable with respect to $E_i$, it follows from [9] Lemma 12, p. 583, that there are an integer $m \geq 2$ and a locally free sheaf $F_i$ on $Z_i$ with $\text{rk}(F_i) = \text{mrk}(E)$ and $\det(F_i) \cong \det(E_i)^{\otimes m}$ such that $[I_i] \in \Theta_{F_i}$ but $v_i \not\in T_{\Theta_{F_i}, [I_i]}$. On the other hand, for each $j \neq i$, it follows from [10] Thm. 11, p. 3057, that there is a locally free sheaf $F_j$ on $Z_j$ with rank $\text{mrk}(E)$ and determinant $\det(F_j) \cong \det(E_j)^{\otimes m}$ such that $[I_j] \not\in \Theta_{F_j}$. By a repeated application of Lemma 3.10 there is a locally free sheaf $F$ on $X$ with $\det(F) \cong \det(E)^{\otimes m}$ such that $F|Z_i = F_i$ for $i = 1, \ldots, q$. Now, applying Proposition 3.9, we get that $F$ satisfies the statement of the lemma. \hfill \Box

Proposition 3.12. Let $E$ be a polarization of $X$, and $\chi := -\mu(E)$. Assume that $E$ is integer only at subcurves of $X$ that are spines (resp. are spines or contain $P$). Then, for every simple, semistable (resp. $P$-quasistable) sheaf $I$ on $X$ and each nonzero $v \in T_{J^2, [1]}$ there are an integer $m \geq 2$ and a locally free sheaf $F$ on $X$ with rank $\text{mrk}E$ and determinant $\det(E)^{\otimes m}$ such that $[I] \in \Theta_F$ but $v \not\in T_{\Theta_F, [1]}$.

Proof. Just observe that the hypothesis implies that $\mathcal{S}(I)$ is a collection of spines for every semistable sheaf $I$. Then apply Lemma 3.11. \hfill \Box
4. Coarse compactified Jacobians

Throughout this section, let $\chi$ be an integer and $a = (a_1, \ldots, a_n)$ be an $n$-tuple of rational numbers summing up to 1. For each subcurve $Y$ of $X$, set $a_Y := \sum_{X_i \subseteq Y} a_i$.

4.1. (The coarse compactified Jacobians) According to Seshadri [16] Déf. 9 and Remarques on p. 153, a torsion-free, rank-1 sheaf $I$ on $X$ is $a$-semistable if

$$\chi(I_Y) \geq a_Y \chi(I)$$

for each proper subcurve $Y$ of $X$. In addition, $I$ is called $a$-stable if the inequalities are strict. (Seshadri worked in higher rank as well, what we will not do here.)

Seshadri’s notion of stability is encompassed by ours. In other words, there is a locally free sheaf $E$ on $X$ such that $a$-semistability (resp. $a$-stability) for torsion-free, rank-1 sheaves of Euler characteristic $\chi$ is equivalent to semistability (resp. stability) with respect to $E$; see [9] Obs. 13, p. 584. In fact, any locally free sheaf $E$ on $X$ such that

$$(4.1.1) \mu(E|_{X_i}) = -a_i \chi \quad \text{for each } i = 1, \ldots, n$$

has this property. We let $J_X(a, \chi) := J_{E_X}^{ss}$ and $J_{E_X}(a, \chi) := J_{E_X}^{P}$ for any such $E$.

In [16] Thm. 15, p. 155, Seshadri constructs a projective scheme $U_X(a, \chi)$ corepresenting the functor $U$ that associates to each scheme $T$ the set of torsion-free, rank-1 sheaves $I$ on $X \times T/T$ such that $I|_{X \times t}$ is $a$-semistable and of Euler characteristic $\chi$ for each $t \in T$. The points on $U_X(a, \chi)$ are in one-to-one correspondence with the $S$-equivalence classes of semistable sheaves. For a sketch of this construction, see the proof of Proposition 4.3 below.

Since $J_X(a, \chi)$ represents a functor, there exists a universal $a$-semistable sheaf of Euler characteristic $\chi$ on $X \times J_X(a, \chi)/J_X(a, \chi)$, and hence a naturally induced map

$$(4.1.2) \Phi: J_X(a, \chi) \rightarrow U_X(a, \chi).$$

This map is surjective and its fibers parametrize $S$-equivalence classes of simple $a$-semistable sheaves.

**Proposition 4.2.** If $X$ is locally planar then $U_X(a, \chi)$ is reduced.

**Proof.** Fix an ample invertible sheaf $O_X(1)$ on $X$; let $d$ denote its degree. Since the family of all $a$-semistable sheaves is bounded, there is an integer $t$ such that the twist $I^*(t) := I^* \otimes O_X(t)$ is generated by global sections and $h^1(X, I^*(t)) = 0$ for every $a$-semistable sheaf $I$ of Euler characteristic $\chi$ on $X$. Then there is an injective homomorphism $O_X \rightarrow I^*(t)$. Taking duals, we obtain an injection $I(-t) \rightarrow O_X$, which defines a closed subscheme $Y$ of $X$ of length $\ell := \chi(O_X) - td - \chi$. Let $\text{Hilb}_X^{\ell}$ be the Hilbert scheme of $X$, parametrizing subschemes of length $\ell$. Consider the rational map

$$\alpha: \text{Hilb}_X^\ell \rightarrow U_X(a, \chi)$$

sending $[Y]$ to $[\mathcal{I}_{Y/X}(t)]$, where $\mathcal{I}_{Y/X}$ denotes the sheaf of ideals of $Y$ in $X$. The map $\alpha$ is defined on the open locus $V$ parametrizing subschemes $Y \subset X$ such that $\mathcal{I}_{Y/X}(t)$ is $a$-semistable. As shown above $\alpha|_V$ is a surjection. By [11] Cor. 7, p. 7, since $X$ is locally planar, $\text{Hilb}_X^\ell$ is reduced, and hence $\alpha(V) \subseteq U_X(a, d)_{\text{red}}$. 


Now, let $S$ be a scheme and $\mathcal{I}$ a torsion-free, rank-1 sheaf on $X \times S/S$ such that $\mathcal{I}|_{X \times S}$ is $a$-semistable for every $s \in S$. Let $\nu: S \to U_X(a,d)$ be the induced morphism. Since $U_X(a,d)$ corepresents the functor $\mathcal{U}$, it is enough to show that $\nu(S) \subseteq U_X(a,d)_\text{red}$. This is a local condition: we need only show that for each $s \in S$ there is an open neighborhood $W \subseteq S$ of $s$ such that $\nu(W) \subseteq U_X(a,d)_\text{red}$. So, let $s \in S$ and set $I := \mathcal{I}|_{X \times S}$. Since $I^*(t)$ is generated by global sections, there is an injection $\mathcal{O}_X \to I^*(t)$. Since $h^1(X, I^*(t)) = 0$, up to passing to an open neighborhood of $s$, the injection lifts to a homomorphism $\mathcal{O}_{X \times S} \to I^*(t)$, which is injective on the fibers of $X \times S/S$. Taking duals, we obtain a homomorphism $I^*(-t) \to \mathcal{O}_{X \times S}$, which is again injective on the fibers of $X \times S/S$. So, we have a well-defined morphism $\mu: S \to \text{Hilb}^d_X$ such that $\mu(S) \subseteq V$ and $\nu = \alpha \circ \mu$. Since $\alpha|_V$ factors through $U_X(a,d)_\text{red}$, so does $\nu$.

**Proposition 4.3.** Let $E$ be a polarization of $X$ such that $\mu(E|_{X_i}) = -a_i\chi$ for $i = 1, \ldots, n$. Let $m$ be an integer greater than 1 and $F$ a locally free sheaf on $X$ with $\text{rk}(F) = mrk(E)$ and $\text{det}(F) \cong \text{det}(E)^\otimes m$. Then there is a subscheme $\Theta_F \subseteq U_X(a,\chi)$ whose inverse image under the natural map $\Phi: J_X(a,\chi) \to U_X(a,\chi)$ is the theta subscheme $\Theta_F \subseteq J_X(a,\chi)$.

**Proof.** We will need to recall the construction of $U_X(a,\chi)$. Seshadri fixes an ample invertible sheaf $\mathcal{O}_X(1)$ on $X$; let $d$ denote its degree. Then he chooses and integer $t$ large enough that the twist $I(t) := I \otimes \mathcal{O}(t)$ is generated by global sections and $h^1(X, I(t)) = 0$ for every $a$-semistable torsion-free, rank-1 sheaf $I$ on $X$ of Euler characteristic $\chi$. Set $c := dt + \chi$, and let $S$ be the scheme parametrizing $a$-semistable, torsion-free, rank-1 quotients $I$ with Euler characteristic $\chi$ of $\mathcal{O}_X^{\oplus c}(-t)$ such that the induced map $H^0(X, \mathcal{O}_X^{\oplus c}) \to H^0(X, I(t))$ is an isomorphism. The scheme $S$ is an open subscheme of Grothendieck’s scheme of quotients of $\mathcal{O}_X^{\oplus c}(-t)$; so there is a universal quotient $b: p_1^*(\mathcal{O}_X^{\oplus c}(-t)) \to \mathcal{I}$ on $X \times S$, where $p_1: X \times S \to X$ is the projection.

Let $G := \text{SL}(c)$. Then there is an action $\mu: G \times S \to S$ given by the natural action of $G$ on $\mathcal{O}_X^{\oplus c}$, which induces one on $\mathcal{O}_X^{\oplus c}$. Notice that, from the description of $\mu$, and the universal property of the quotient $b$, there is an isomorphism

\[(4.3.1) \quad (1_X, \mu)^*\mathcal{I} \cong (1_X, q_2)^*\mathcal{I}\]

satisfying the cocycle condition, where $q_2: G \times S \to S$ is the second projection. (For the statement of the cocycle condition for invertible sheaves, see [13] Def. 1.6, p. 30. The same statement can be made for all sheaves.)

Seshadri uses Geometric Invariant Theory to show that there is a categorical quotient of $S$ under $\tau$, and that the quotient is projective. Then it is not difficult to show that this quotient corepresents the functor $\mathcal{U}$ described in [4.1].

Fix an isomorphism $\mathcal{D}(E) \to \mathcal{O}_{\text{Spec}(k)}$. Let $\mathcal{L}_E(\mathcal{I})$ be the determinant of cohomology of $p_1^*E \otimes \mathcal{I}$ with respect to the second projection $p_2: X \times S \to S$. The invertible sheaf $\mathcal{L}_E(\mathcal{I})$ is equipped with a $G$-linearization. Indeed, by the functorial properties of the determinant of cohomology the isomorphism $\mathcal{L}_E(\mathcal{I}) \otimes \ell$ induces an isomorphism $\mu^*\mathcal{L}_E(\mathcal{I}) \to q_2^*\mathcal{L}_E(\mathcal{I})$ satisfying the cocycle condition. Of course, $\mathcal{L}_E(\mathcal{I})^\otimes \ell$ comes with an induced $G$-linearization for each integer $\ell$. 

For each positive integer \( \ell \), each locally free sheaf \( H \) on \( X \) with rank \( \ell \text{rk}(E) \), and each isomorphisms \( \det(H) \to \det(E) \otimes \ell \) and \( D(H) \to O_{\text{Spec}(k)} \), we produce a global section \( \theta_H(E, I) \) of \( \mathcal{L}_E(I)^{\otimes \ell} \) as follows: First of all, using the identification \( \det(E^{\otimes \ell}) = \det(E)^{\otimes \ell} \), the isomorphism \( \det(H) \to \det(E)^{\otimes \ell} \) induces an isomorphism,

\[
\tau: D(I \otimes p_1^* H) \otimes D(E)^{\otimes \ell} \to D(I \otimes p_1^*(E^{\otimes \ell})) \otimes D(H);
\]

see [10], Lemma 46, p. 3082. Combining \( \tau \) with the isomorphisms \( D(E) \to O_{\text{Spec}(k)} \) and \( D(H) \to O_{\text{Spec}(k)} \), and the identification \( D(I \otimes p_1^*(E^{\otimes \ell})) = D(I \otimes p_1^* E)^{\otimes \ell} \), we get an isomorphism \( D(I \otimes p_1^* H) \to \mathcal{L}_E(I)^{\otimes \ell} \). Then we let \( \theta_H(E, I) \) be the section corresponding to \( \sigma_{I \otimes p_1^* H} \) under this isomorphism. By the functorial properties of \( \mu \), the section \( \theta_H(E, I) \) of \( \mathcal{L}_E(I)^{\otimes \ell} \) is \( G \)-invariant, that is, \( \mu^* \theta_H(E, I) \) is carried to \( p_2^* \theta_H(E, I) \) under the \( G \)-linearization of \( \mathcal{L}_E(I)^{\otimes \ell} \) mentioned above.

Set \( F_0 := F \). It follows from [10] Thm. 11, p. 3057, that there are locally free sheaves \( F_1, \ldots, F_r \) on \( X \) of rank \( m \text{rk}(E) \) and determinant \( (\det E)^{\otimes m} \) such that, for each \( \alpha \)-semistable torsion-free, rank-1 sheaf \( I \) on \( X \) of Euler characteristic \( \chi \), there is \( j \) such that \( h^0(X, I \otimes F_j) = 0 \). Fix isomorphisms \( \det(F_j) \to (\det E)^{\otimes m} \) and \( D(F_j) \to O_{\text{Spec}(k)} \) for each \( j = 0, \ldots, r \), and consider the well-defined \( G \)-invariant map

\[
(\theta_{F_0}(E, I), \ldots, \theta_{F_n}(E, I)) : S \to P^r.
\]

Since \( U_X(a, d) \) is the categorical quotient of \( S \) by \( G \), the above map factors through the quotient map \( \pi: S \to U_X(a, d) \). It follows that there is a subscheme \( \overline{\Theta}_F \) of \( U_X(a, d) \) such that \( \pi^{-1}(\overline{\Theta}_F) = \Theta_F \) is the zero scheme \( Z_F = \pi^{-1}(\overline{\Theta}_F) \) of \( \pi_F(E, I) \), or of \( \sigma_{I \otimes p_1^* F} \).

We claim that \( \Phi^{-1}(\overline{\Theta}_F) = \Theta_F \). Indeed, choose a universal sheaf on \( X \times J^s_E \) and determinant \( (\det E)^{\otimes s} \); call it \( \mathcal{N} \). Let \( z \in J^s_E \) and \( I := \mathcal{N}|_{X \times z} \). Since \( I \) is \( \alpha \)-semistable of Euler characteristic \( \chi \), the twist \( I(t) \) is globally spanned, \( h^1(X, I(t)) = 0 \) and \( h^0(X, I(t)) = c \), and there is a surjection \( O_X^c(-t) \to I \) inducing an isomorphism \( H^0(X, O_X^c) \to H^0(X, I(t)) \). The same applies to all points \( z' \) on a neighborhood \( W \subseteq J^s_E \) of \( z \). So there is a map \( \lambda: W \to S \) such that \( (1_X, \lambda)^* I \cong \mathcal{N}|_{X \times W} \). Using the base-change property of the determinant of cohomology and its associated global section, it follows that \( \lambda^{-1}(Z_F) = \Theta_F \cap W \). Since \( \Phi = \pi \circ \lambda \), and \( z \) was any point of \( J^s_E \), it follows that \( \Phi^{-1}(\overline{\Theta}_F) = \Theta_F \), as claimed.

**Theorem 4.4.** Assume that every subcurve \( Y \) of \( X \) with \( a_Y \chi \in \mathbb{Z} \) is a spine or contains \( P \). Then the natural map \( \Phi: J_X(a, \chi) \to U_X(a, \chi) \) restricts to a bijective closed embedding \( \Psi: J^s_X(a, \chi) \to U_X(a, \chi) \). Furthermore, if \( X \) is locally planar then \( \Psi \) is an isomorphism.

**Proof.** By Proposition 3.18 the restriction \( \Psi := \Phi|_{J^s_X(a, \chi)} \) is injective. It is also surjective, by [10] Thm. 7, p. 3054. In addition, by Propositions 3.12 and 4.3, it is an immersion. Since \( J^s_X(a, \chi) \) is complete, \( \Psi \) is proper, and thus \( \Psi \) is a bijective embedding. If \( X \) is locally planar then \( U_X(a, \chi) \) is reduced by Proposition 4.2 and hence \( \Psi \) is an isomorphism. \( \square \)
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