Generating Hard Problems of Cellular Automata

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Abstract. We propose two hard problems in cellular automata. In particular the problems are,

– [DDP\textsubscript{\text{M}}\textsubscript{n,p}] Given two randomly chosen configurations \textit{t} and \textit{s} of a cellular automata of length \textit{n}, find the number of transitions \textit{\tau} between \textit{s} and \textit{t}.

– [SDDP\textsubscript{\text{\delta,k,n}}] Given two randomly chosen configurations \textit{s} of a cellular automata of length \textit{n} and \textit{x} of length \textit{k} < \textit{n}, find the configuration \textit{t} such that \textit{k} number of cells of \textit{t} is fixed to \textit{x} and \textit{t} is reachable from \textit{s} within \textit{\delta} transitions.

We show that the discrete logarithm problem over the finite field reduces to DDP\textsubscript{\text{\text{M}}\textsubscript{n,p}} and the short integer solution problem over lattices reduces to SDDP\textsubscript{\text{\delta,k,n}}. The advantage of using such problems as the hardness assumptions in cryptographic protocols is that proving the security of the protocols requires only the reduction from these problems to the designed protocols. We design one such protocol namely a proof-of-work out of SDDP\textsubscript{\text{\delta,k,n}}.

Keywords: Cellular Automata · Discrete Logarithm Problem · Short Integer Solution Problem · Finite Field · Lattices

1 Introduction

Cellular automata CA is a universal model of computation like Turing machines. It has been used in numerous applications ranging from cryptography to coding theory, from VLSI design to memory testing. In most of the applications, it succeeds to achieve some predefined set of desired properties like pseudo-randomness, efficient parallelizability etc. Unfortunately, in the context of cryptography nowadays CAs are often referred to “older crypto”, due to the absence of theorems to prove certain security properties required for any cryptographic schemes. The best that CAs offer for cryptographic design are some conjectures like given a configuration of a CA it is infeasible to find one of its predecessors.

In this paper, we show that (at least) a type of CA, namely linear cyclic hybrid cellular automata LCHCA\textsubscript{\text{M}}\textsubscript{n,p} essentially simulates the computation over the finite field \mathbb{F}_{p^n}. Linear hybrid cellular automata are finite state machines.
that mimics a linear transformation over the vector space defined over a finite field. The mapping between $LCHCA_{n,p}^M$ and finite field $\mathbb{F}_{p^n}$ immediately gives us an edge to convert those longstanding conjectures into theorems.

More importantly as a consequence of this mapping, we pose two new problems, namely Discrete Distance Problem $DDP_{n,p}^M$ and Short Discrete Distance Problem $SDDP_{k,n}^δ$, over $LCHCA_{n,p}^M$. We give two polynomial time reductions,

- the Discrete Logarithm Problem $DLP_{n,p}$ over the finite field $\mathbb{F}_{p^n}$ to $DDP_{n,p}^M$.
- the Short Integer Solution Problem $SIS_{q,β}^{n,m}$ over the lattices to $SDDP_{k,n}^δ$.

This reductions shows that cryptographic protocols based on $DDP_{n,p}^M$ and $SDDP_{k,n}^δ$ are secure as long as $DLP_{n,p}$ and $SIS_{q,β}^{n,m}$ computationally hard. Moreover $SIS_{q,β}^{n,m}$ known to be an average-case hard problem [1]. As $SIS_{q,β}^{n,m}$ reduces to $SDDP_{k,n}^δ$ for randomly chosen instance of $SDDP_{k,n}^δ$, it turns out to be hard in average-case too. As a typical application we design a proof-of-work scheme out of the problem $SDDP_{k,n}^δ$ at the end.

2 Preliminaries

In this section we take a brief review of the tools and techniques that will be required for the rest of the paper.

2.1 Rings and Finite Fields

We take all the definitions related to abstract algebra from this book [3]. A ring is an Abelian group under addition, also having an associative multiplication that is left and right distributive over addition. The rings that allow the multiplication to be commutative is called commutative rings. Suppose $R$ is a commutative ring. A ring of polynomials $R[x] = \{ \sum_{i=0}^{n-1} a_i x^i \}$ where $a_i \in R$ and $n \in \mathbb{Z}^+$, is the set of polynomials whose coefficients are from the ring $R$. A field $\mathbb{F}$ is a commutative ring with unity in which every nonzero element is a unit. As $\mathbb{F}$ is a commutative ring, the set $\mathbb{F}[x]$ is also a ring of polynomials. If $\mathbb{F}_q$ has a finite number $q$ of elements we call $\mathbb{F}_q$ as a finite field of order $q$. It can be shown that every field $\mathbb{F}_q$ to be a finite field if and only if $q = p^n$ where $p$ and $n \in \mathbb{Z}^+$. Every fields of the same order are isomorphic. The nonzero elements of a finite field $\mathbb{F}_q$ form a cyclic multiplicative group $\mathbb{F}_q^*$. Suppose $\alpha$ is one of the generators $\mathbb{F}_q^*$ then $\alpha$ is called a primitive element of the field $\mathbb{F}_q$. We denote $\mathbb{F}_q(\alpha) = \{0, 1, \alpha, \alpha^2 \ldots \alpha^{q-2} \}$ to be the field generated by the primitive element $\alpha$.

We call a subset $\mathbb{F}_d \subset \mathbb{F}_q$ is a subfield of $\mathbb{F}_q$ if $\mathbb{F}_d$ preserves all the operations of $\mathbb{F}_q$. It can be shown that $d$ always divides $n$. We call $\mathbb{F}_p$ as the base field and $\mathbb{F}_d$ and $\mathbb{F}_q$ as the extended fields. We denote these field extensions as $\mathbb{F}_p/\mathbb{F}_d$ and $\mathbb{F}_d/\mathbb{F}_q$.

An ideal is a subset $I$ of elements in a ring $R$ that forms an additive group such that, for $x \in R$ and $y \in I$, then $xy \in I$ and $yx \in I$. We call $R/I$ as a
Suppose \( f(x) \in \mathbb{F}_p[x] \) is a polynomial over the field \( \mathbb{F}_p \). We call \( f(x) \) to be an irreducible polynomial if and only if \( f(x) \) cannot be factored into non-constant polynomials over the field \( \mathbb{F}_p \). As \( \mathbb{F}_p[x] \) is a ring \( \langle f(x) \rangle \) is an ideal in \( \mathbb{F}_p[x] \). The ideal \( \langle f(x) \rangle \) is maximal if and only if \( f(x) \) is irreducible over \( \mathbb{F}_p \). Therefore, the quotient ring \( \mathbb{F}_p[x]/\langle f(x) \rangle \) is a field if and only if \( f(x) \) is irreducible.

2.2 Discrete Logarithm Problem

**Definition 1.** (Discrete Logarithm Problem DLP). Given a finite cyclic multiplicatively written group \( G = \langle g \rangle \), a generator \( g \) of \( G \), and an element \( h \in G \), the DLP\(_{n,p} \) is to find an integer \( x \), unique modulo the order of \( G \), such that \( h = g^x \).

For certain groups \( G \), the DLP\(_{n,p} \) turns out to be a computationally intensive problem. With the relevance to our current context we mention such a group.

**Definition 2.** (DLP over Finite Fields DLP\(_{n,p} \)). Let \( g \) be a generator of \( \mathbb{F}_p^* \), and \( a \in \mathbb{F}_p^* \) for some prime \( p \). There is an integer \( x \), unique modulo \( p^n - 1 \), such that \( a = g^x \in \mathbb{F}_p^* \). The DLP\(_{n,p} \) is the problem of determining \( x \) from the pair \((g,a)\).

2.3 Lattices

An \( n \)-dimensional lattice is the set of all integer combinations \( \{ \sum_{i=1}^{n} x_i \beta_i \mid x_i \in \mathbb{Z} \} \) of \( n \) linearly independent vectors \( \{ \beta_1, \ldots, \beta_n \} \) in \( \mathbb{R}^n \). The set of vectors \( \{ \beta_1, \ldots, \beta_n \} \) is called a basis for the lattice, and can be compactly represented by the matrix \( B = [\beta_1, \ldots, \beta_n] \in \mathbb{R}^{n \times n} \) having the basis vectors as columns. The lattice generated by \( B \) is denoted by \( \mathcal{L}(B) \).

The minimum distance of a lattice \( \mathcal{L}(B) \) is the minimum distance between any two (distinct) lattice points and equals the length of the shortest nonzero lattice vector. The minimum distance can be defined with respect to any norm. For any \( p \geq 1 \), the \( p \)-norm of a vector \( \bar{x} \) is defined by \( \| \bar{x} \|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} \) corresponding minimum distance is denoted,

\[
\lambda_1^p(\mathcal{L}(B)) = \min\{\| \bar{x} - \bar{y} \|_p \mid \bar{x} \neq \bar{y} \in \mathcal{L}(B)\} = \min\{\| \bar{x} \|_p \mid \bar{x} \in \mathcal{L}(B) \setminus \{0\}\}.
\]

Without loss of generality the \( \ell_2 \)-norm is used in rest of the paper.

**Definition 3.** (\( \ell_2 \)-norm). The \( \ell_2 \)-norm \( \|(z_1, z_2, \ldots, z_n)\| = \sqrt{z_1^2 + z_2^2 + \ldots + z_n^2} \).

**Definition 4.** (\( \gamma \)-approximate Shortest Vector Problem SVP\(_\gamma \)). Given a lattice \( \mathcal{L}(B) \), find a nonzero vector \( \bar{x} \in \mathcal{L}(B) \) such that \( \| \bar{x} \| \leq \gamma \lambda_1(\mathcal{L}(B)) \).
SVP\(^\gamma\) is known to be hard for \(\gamma = \text{poly}(n)\).

**Definition 5. ((Inhomogeneous) Short Integer Solution Problem SIS\(_{n,m}^q,\beta\)).** Given \(m\) uniformly random vectors \(\vec{a}_i \in \mathbb{Z}_q^n\), forming the columns of a matrix \(A \in \mathbb{Z}_q^{n \times m}\), and a uniformly random vector \(\vec{v} \in \mathbb{Z}_q^n\), find a nonzero integer vector \(\vec{z} \in \mathbb{Z}_m^q\) such that,

1. the \(\ell_2\)-norm \(\|\vec{z}\| \leq \beta\),
2. \(A\vec{z} = \sum \vec{a}_i \cdot z_i = \vec{v} \in \mathbb{Z}_q^n\).

The above-mentioned definition of SIS\(_{n,m}^q,\beta\) known to be the inhomogeneous version of SIS\(_{n,m}^q,\beta\). Historically, SIS\(_{n,m}^q,\beta\) made its debut with its homogeneous version fixing \(\vec{v} = \vec{0} \in \mathbb{Z}_n^q\) [1]. It is shown that SVP\(^{\text{poly}(n)}\) in the worst case reduces to SIS\(_{n,m}^q,\beta\) even if \(A\) is chosen uniformly at random i.e., in the average case. Both these versions are equally hard for typical parameters for \(m \geq \lceil n \log q \rceil\) and \(q \gg \beta\), however \(\beta > \sqrt{n \log q}\) is required to guarantee that there exists at least one solution.

### 2.4 Cellular Automata

Cellular automata is a universal model of computation like Turing machine.

**Definition 6. (Cellular Automata).** A cellular automata \(\text{CA}\), having \(n\) cells and \(\mathcal{N}\)-neighborhoodness, is a tuple \(\text{CA} = \langle Q, n, s, F, \mathcal{N}, \delta \rangle\) with the following meaning,

1. \(Q\) is the finite and nonempty set of states and also tape alphabet.
2. \(n\) is the length of the CA i.e., the number of cells.
3. \(s \in Q^n\) is the initial configuration.
4. \(F\) is the set of halting states.
5. Given the index \(i\) of a cell, the neighborhood \(\mathcal{N}\) is the set of relative offsets from the \(i\)-th cell, from which the local transition function assume inputs i.e., \(f_i : Q^\mathcal{N} \rightarrow Q\).
6. \(\delta : Q^n \rightarrow Q^n\) is the global transition function which is an ensemble of \(n\) number of local transition functions \(\{f_i : Q^\mathcal{N} \rightarrow Q\}\).

Given the index \(i\) of a cell we denote its neighborhood set \(\mathcal{N}_i \subseteq \{-i, -i + 1, \ldots, -1, 0, +1, \ldots, n - i - 2, n - i - 1\}\) where 0 corresponds to the \(i\)-th cell itself, starting from the leftmost cell. When the neighborhoods \(\mathcal{N}_i\) are regular for all the cells, we drop the subscript \(i\) and the cardinality of the set \(\mathcal{N}\) is called the neighborhoodness of the CA. For example, the three-neighborhoodness \(\mathcal{N} = \{-1, 0, +1\}\) denotes the neighborhood of \(\{i - 1, i, i + 1\}\) for any \(1 < i < n\).

Any configuration \(t = \{f_i(\mathcal{N}_i)\} \in Q^n\) is an ensemble of the images of \(f_i\) on the states of \(\mathcal{N}_i\). A unit of computations is considered to be a single transition \(t' \leftarrow\)

\(^1\) Configuration is often referred as state of CA. We prefer to use state as the state of a control unit not of the entire CA.
\(\delta(t)\). Thus a CA, initialized with a global configuration \(s\), makes its transitions through the sequence of configurations \(\{s, \delta(s), \delta^2(s), \ldots, \delta^{\lfloor Q^n - 1 \rfloor}(s)\}\).

Given a CA there may exist set of global configurations \(\Delta \subset Q^n\) such that \(\delta(s) = s\) if \(s \in \Delta\). These configurations \(\Delta\) are called as dead configurations.

**Theorem 1.** If the initial configuration \(s\) of a cellular automata \(\text{CA} = (Q, n, s, F, \mathcal{N}, \delta)\) is chosen uniformly at random i.e., \(s \in_R Q^n\), then for any \(0 < \tau < |Q|^n\), all the cells of the subsequent configuration \(\delta^\tau(s)\) remain independent of one another.

**Proof.** We proceed by induction on \(\tau\). For \(\tau = 0\), the result follows from the initialization hypothesis as \(s \in_R Q^n\). For the inductive step, suppose that all the cells of the state \(t = \delta^\tau(s)\) are independent at some time \(\tau \geq 0\) and \(t' = \delta(t) = \delta^{\tau + 1}(s)\). Assuming \(v[i]\) denotes the \(i\)-th cell of the configuration \(v\), it suffices to show that

\[
Pr[t'[i] = a \mid t'[j] = b] = Pr[t'[i] = a]
\]

for all \(i, j\) with \(i \neq j\), and for all \(a, b \in Q\).

We denote \(\mathcal{N}_i\) as the neighborhood of the \(i\)-th cell irrespective of the configuration. Trivially Eq. (1) holds when \(\mathcal{N}_i \cap \mathcal{N}_j = \{\emptyset\}\). So assume that \(\mathcal{N}_i \cap \mathcal{N}_j \neq \{\emptyset\}\), we have

\[
\begin{align*}
\begin{aligned}
t'[i] &= f_i(\{\mathcal{N}_i \cap \mathcal{N}_j\} \cup \{\mathcal{N}_i \setminus \mathcal{N}_j\}) \\
t'[j] &= f_j(\{\mathcal{N}_i \cap \mathcal{N}_j\} \cup \{\mathcal{N}_j \setminus \mathcal{N}_i\})
\end{aligned}
\end{align*}
\]

The cell \(t'[i]\) depends on \(\{\mathcal{N}_i \setminus \mathcal{N}_j\}\) but not on \(\{\mathcal{N}_j \setminus \mathcal{N}_i\}\). Similarly, the bit \(t'[j]\) depends on \(\{\mathcal{N}_j \setminus \mathcal{N}_i\}\) but not on \(\{\mathcal{N}_i \setminus \mathcal{N}_j\}\). Evidently, \(\{\mathcal{N}_i \setminus \mathcal{N}_j\} \neq \{\mathcal{N}_j \setminus \mathcal{N}_i\} \neq \emptyset\).

By induction hypothesis, all the cells of the configuration \(t\) are independent of each other, so \(\mathcal{N}_i \setminus \mathcal{N}_j\) and \(\mathcal{N}_j \setminus \mathcal{N}_i\) are independent of each other. Therefore Eq. (1) holds.

**Definition 7.** (Cyclic Cellular Automata). A CA \(\text{CA} = (Q, n, s, F, \mathcal{N}, \delta)\) with \(\Delta\) dead states is a cyclic CA if and only if its initial configuration \(s = \delta^k(s)\) for some \(k \leq |Q|^n - 1 - |\Delta|\) and \(s \neq \delta^i(s)\) for all \(i < k\).

Note that \(k\) is an invariant of the initial configuration \(s\) but depends only on \(\delta\). Essentially the transition graph of these CA looks like a cycle, however, there may be multiple such disjoint cycles. Based upon the length of this transition cycle we may classify CAs into these two categories.

**Definition 8.** (Maximum-Length Vs. Group Cellular Automata). A cyclic cellular automata \(\text{CA} = (Q, n, s, F, \mathcal{N}, \delta)\) with \(\Delta\) dead states is a Maximum-Length CA if and only if its initial configuration \(s = \delta^k(s)\) for \(k = |Q|^n - 1 - |\Delta|\) and \(s \neq \delta^i(s)\) for all \(i < k\). On the other hand, it is a group CA if its initial configuration \(s = \delta^k(s)\) for some \(k < |Q|^n - 1 - |\Delta|\) and \(s \neq \delta^i(s)\) for all \(i < k\).
Here, the term “Maximum-Length” corresponds to the length of the cycle in the transition graph of the CA. In Sect 3.5 we will see that for group CAs, \( k \mid (|Q|^n - 1 - |\Delta|) \) i.e., all the disjoint cycles have equal length.

Like Turing machines, CAs are also required to be encoded. The easiest way to do it is to use a bijection \( Q \rightarrow \mathbb{Z}_{|Q|} \) so that the transition functions \( \delta : \mathbb{Z}_{|Q|}^n \rightarrow \mathbb{Z}_{|Q|}^n \) can be defined mathematically over the vector space \( \mathbb{Z}_{|Q|}^n \). In some cases, \( \delta \) results into functions having algebraic closed forms, however, may not be possible always. Depending upon the algebraic closed form (if available) of \( \delta \), CAs can be characterized as,

**Definition 9. (Linear Cellular Automata).** A cellular automata \( \text{CA} = \langle \mathbb{Z}_{|Q|}, n, s, F, N, \delta \rangle \) is linear if and only if its transition function \( \delta \) can be represented by a linear operator \( g : \mathbb{Z}_{|Q|}^n \rightarrow \mathbb{Z}_{|Q|}^n \).

It means for any \( \vec{v}_i \in \mathbb{Z}_{|Q|}^n \) and any \( k_i \in \mathbb{Z}_{|Q|} \) we have \( g(\sum_i k_i \vec{v}_i) = \sum_i k_i \cdot g(\vec{v}_i) \).

**Transition Matrix \( M \) of a linear CA** As \( g \) is a linear operator over the vector space \( \mathbb{Z}_{|Q|}^n \), there is a matrix \( M \in \mathbb{Z}_{|Q|}^{n \times n} \) such that for any \( \vec{v} \in \mathbb{Z}_{|Q|}^n \), we have \( g(\vec{v}) = M \times \vec{v} \). Therefore, given an initial configuration \( \vec{s} \in \mathbb{Z}_{|Q|}^n \) of a linear CA, we have the sequence of configurations \( \{\vec{s}, M \vec{s}, M^2 \vec{s}, \ldots, M^{|Q|^n-1} \vec{s}\} \).

Traditionally \( M \) is called as the transition matrix of the linear CA. The \( \delta \) of a linear CA can be represented with its transition matrix \( M \in \mathbb{Z}_{|Q|}^{n \times n} \). For any linear CA, \( \vec{s} = \delta(\vec{s}) \) if and only if \( \vec{s} = \vec{0} \), so \( \Delta = \{0\} \).

**Theorem 2.** If the initial configuration of a linear cellular automata \( \text{CA} = \langle Q, n, \vec{s}, F, N, M \rangle \) is chosen uniformly at random i.e., \( \vec{s} \in \mathbb{R} \mathbb{Z}_{|Q|}^n \), then for any \( 0 < \tau < |Q|^n \), all the coordinates of the subsequent configuration \( \vec{t} = M^\tau \vec{s} \) remain uniformly unbiased.

**Proof.** We proceed by induction on \( \tau \). For \( \tau = 0 \), the result follows from the initialization hypothesis as \( \vec{s} \in \mathbb{R} \mathbb{Z}_{|Q|}^n \). For the inductive step, suppose that all the coordinates of the configuration \( \vec{t} = M^\tau \vec{s} \) are unbiased at some time \( \tau \geq 0 \) and \( \vec{t} = M^\tau \vec{s} = M^{\tau+1} \vec{s} \). Assuming \( \vec{v}[i] \) denotes the \( i \)-th coordinate of the configuration \( \vec{v} \), it suffices to show that,

\[
Pr[t'[i] = a \mid t[i] = b] = Pr[t'[j] = a] = \frac{1}{|Q|},
\]

for all \( i, j \) including \( i = j \), and for all \( a, b \in \mathbb{Z}_{|Q|} \).

Suppose \( M = \{m_{ij}\} \) then \( t'[i] = \sum_{j=1}^{n} m_{ij}t[j] \). Observe that addition and multiplication over the set \( \mathbb{Z}_{|Q|} \) are unbiased because of the modulo reduction. For each of the operations, there are exactly \( |Q|^2 \) number of pairs which are mapped to \( |Q| \) number of elements through modulo reduction. Therefore each of these operations is unbiased with probability \( \frac{|Q|}{|Q|^2} = \frac{1}{|Q|} \). Further by the
induction hypothesis, \( t[j]\)s are unbiased. As a result \( t'[i] = \sum_{j=1}^{n} m_{ij}t[j] \) becomes unbiased. It does not matter if the non-zero \( m_{ij}\)s are biased as they are fixed already.

By Theorem 1, all the coordinates are independent of each other. Putting these two together, for all \( i, j \) including \( i = j \), and for all \( a, b \in \mathbb{Z}_{|Q|} \) for Eq. \( 2 \) holds true.

Theorem 2 implies that there is no leakage of information from one configuration to the next one. One must evaluate all of the coordinates to obtain the next configuration from the present configuration. Theorem 1 and Theorem 2 together indicate that this sequence of configurations \( \{ \vec{s}, M\vec{s}, M^2\vec{s}, \ldots, M^{|Q|^n-1}\vec{s} \} \) for a linear \( \text{CA} \) looks like a pseudo-random sequence of vectors chosen uniformly at random from the vector space \( \mathbb{Z}_{|Q|}^n \) with a periodicity of \( (|Q|^n-1) \) as \( \Delta = \{0\} \).

For linear \( \text{CA}s \), it is not necessary to have a common linear form for all the local transition functions \( f_i : Q^N \rightarrow Q \). For example, an \( f_i \) may ignore one of its neighbors \( k \in N \) by making \( m_{i,i+k} = 0 \) while another \( f_j \) includes the same offset \( k \in N \) keeping \( m_{j,j+k} \neq 0 \).

**Definition 10. (Hybrid Vs. Uniform Cellular Automata).** A linear cellular automata \( \text{CA} = \langle Q, n, s, F, N, M \rangle \) is a uniform \( \text{CA} \) if and only if given a row \( i \) of the transition matrix \( M \) any other row \( j > i \) can be obtained by \( j - i \) number of right shifts of the i-th. A \( \text{CA} \) is hybrid if it is not uniform.

**The Characteristic Polynomial of \( M \)** The characteristic polynomial \( f_M(x) \) of the transition matrix helps us to identify the uniformity of a linear \( \text{CA} \). If \( f_M(x) \) is reducible over the set \( \mathbb{Z}_{|Q|} \), then the \( \text{CA} \) is a uniform one \([2]\), else if \( f_M(x) \) is irreducible over the set \( \mathbb{Z}_{|Q|} \) then \( M \) generates a hybrid \( \text{CA} \). Additionally, if \( f_M(x) \) is a primitive (also irreducible) polynomial over the set \( \mathbb{Z}_{|Q|} \) then \( M \) generates a maximum-length hybrid \( \text{CA} \).

### 3 Hard Problems on Cellular Automata

In this section, we will define two computational problems based on the transition of configurations of a particular type of linear cyclic cellular automata. As \( \mathbb{Z}_{|Q|} \) becomes a field when \( |Q| = p \) is a prime, we define our final characterization of \( \text{CA}s \) as follows,

**Definition 11. (Linear Cyclic Hybrid Cellular Automata over \( \mathbb{F}_p \), \( \text{LCHCA}^M_{n,p} \)).** A cellular automata \( \text{LCHCA}^M_{n,p} = \langle \mathbb{F}_p, n, \vec{s}, F, N, M \rangle \) is linear cyclic hybrid cellular automata over the field \( \mathbb{F}_p \), if and only if it is linear, cyclic, hybrid and \( Q = \mathbb{F}_p \) for some prime \( p \).

**3.1 Discrete Distance Problem**

We observe that the exponent \( \tau \) of \( M \) in the expression \( \vec{t} = M^\tau \vec{s} \) acts as an discrete distance between two vectors \( \vec{t} \) and \( \vec{s} \). So we name this problem as **Discrete Distance Problem**.
**Definition 12.** (Discrete Distance Problem $\text{DDP}^M_{n,p}$). Given an initial configuration $\vec{s} \in \mathbb{F}_p^n$ and a fixed configuration $\vec{t} \in \mathbb{F}_p^n$ of a linear cyclic hybrid cellular automata over the field $\mathbb{F}_p$ $\text{LCHCA}^M_{n,p}$, find the discrete distance $\tau$ such that $\vec{t} = M^n \vec{s}$.

For a maximum-length $\text{LCHCA}^M_{n,p}$, such a $\tau$ is unique modulo $p^n - 1$. However, for a group $\text{LCHCA}^M_{n,p}$ if $\vec{t} \notin \{M\vec{s}, M^2\vec{s}, \ldots, M^{d-1}\vec{s}\}$ where $M^d = I$ the identity matrix then $\vec{t}$ is not reachable from $\vec{s}$ at all. In this case, $\tau = \infty$.

### 3.2 DDP and DLP are Equivalent

We start with a different but relevant issue of minimal polynomial $\mu_M(x)$ of the transition matrix $M$ in order to show that $M$ is always diagonalizable. Finding $\mu_M(x)$ needs no effort since the transition matrix $M$ of any linear finite state machine is non-derogatory [2]. For non-derogatory matrices the characteristic polynomial $f_M(x) = \mu_M(x)$. As a linear CA is a special case of linear finite state machines the minimal polynomial of $\mu_M(x) = f_M(x)$.

However, we can show this explicitly as the following lemma.

**Lemma 1.** For every $\text{LCHCA}^M_{n,p}$, the minimal polynomial $\mu_M(x)$ and the characteristic polynomial $f_M(x)$ of its transition matrix $M$ are equal.

**Proof.** For every $\text{LCHCA}^M_{n,p}$, the characteristic polynomial is an irreducible polynomial of degree $n$ over the field $\mathbb{F}_p$. Suppose, $\{\alpha, \alpha^p, \ldots, \alpha^{p^{n-1}}\}$ is the normal basis for the extended field $\mathbb{F}_{p^n}$. Therefore, the minimal polynomial can be factorized as $\mu_M(x) = \prod_{i=0}^{p^n-1} (x - \alpha^i) = (x - \alpha)(x - \alpha^p)\ldots(x - \alpha^{p^{n-1}})$ where $\alpha$ is a generator of the field $\mathbb{F}_{p^n}$. So, the degree of $\mu_M(x) = n$. We know that minimal polynomial divides the characteristic polynomial i.e., $\mu_M(x) | f_M(x)$ [3]. So, these two following conditions need to be satisfied simultaneously,

1. $\deg(\mu_M(x)) = \deg(f_M(x))$ where $\deg(\cdot)$ denotes the degree of a polynomial.
2. $\mu_M(x) | f_M(x)$ where both $\mu_M(x)$ and $f_M(x)$ are monic irreducible monomials.

Both of these conditions together implies that $\mu_M(x) = f_M(x)$.

Now we show that $M$ is always diagonalizable.

**Theorem 3.** Transition matrix $M$ of any $\text{LCHCA}^M_{n,p}$ is diagonalizable over the field $\mathbb{F}_{p^n}/\langle \mu_M(x) \rangle$ where $\mu_M(x)$ is the minimal polynomial of $M$ over the polynomial ring $\mathbb{F}_p[x]$.

**Proof.** For any $\text{LCHCA}^M_{n,p}$, the transition matrix $M$ is in $\mathbb{F}_{p}^{n \times n}$, so its minimal polynomial $\mu_M(x)$ is defined over the polynomial ring $\mathbb{F}_p[x]$. Using Lemma 1 the characteristic polynomial $f_M(x) = \mu_M(x) = (x - \alpha)(x - \alpha^p)\ldots(x - \alpha^{p^{n-1}})$. By Kronecker’s Theorem, $\mathbb{F}_{p^n}[x]/\langle f_M(x) \rangle$ is also a field having $f_M(x) = 0$. Therefore, $f_M(x)$ has $n$ distinct roots $\{\alpha, \alpha^p, \ldots, \alpha^{p^{n-1}}\}$ over the field $\mathbb{F}_{p^n}/\langle f_M(x) \rangle$. 

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The eigenvalues of a matrix are the roots of its minimal polynomial. As \( f_M(x) = \mu_M(x) \), \( M \) has \( n \) distinct eigenvalues \( \{\alpha, \alpha^p, \ldots, \alpha^{p^n-1}\} \) over the field \( \mathbb{F}_{p^n}/(\mu_M(x)) \).

Since a matrix is diagonalizable over the field \( \mathbb{F}_{p^n} \) if and only if the factors of its minimal polynomial has \( n \) distinct roots over \( \mathbb{F}_{p^n} \), \( M \) is diagonalizable over the field \( \mathbb{F}_{p^n}/(\mu_M(x)) \).

### 3.3 The Reduction DDP\(_{\leq p}\)DLP

**Theorem 4.** If there exists an algorithm \( \mathcal{D} \) that efficiently solves the problem \( \text{DDP}_{n,p} \) over the field \( \mathbb{F}_{p^n} \), then there exists an algorithm \( \mathcal{D}' \) that efficiently solves the problem of \( \text{DDP}_{n,p}^M \).

**Proof.** \( \mathcal{D} \) takes a generator \( g \in \mathbb{F}_{p^n}^* \) and an element \( a \in \mathbb{F}_{p^n}^* \) as its input and outputs an \( x \) such that \( a = g^x \). The inputs of a \( \text{DDP}_{n,p}^M \) instance are two configurations \( \vec{s}, \vec{t} \in \mathbb{F}_{p^n}^* \). \( \mathcal{D} \) needs to output a \( \tau \leq p^n - 1 \) such that \( \vec{t} = M^\tau \vec{s} \).

Using lemma \( 3 \) \( \mathcal{D}' \) first diagonalizes \( M = QAQ^{-1} \) uniquely, such that,

\[
M = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} \quad \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \alpha^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha^{p^n-1} \end{pmatrix} \quad \begin{pmatrix} q_{11}' & q_{12}' & \cdots & q_{1n}' \\ q_{21}' & q_{22}' & \cdots & q_{2n}' \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}' & q_{n2}' & \cdots & q_{nn}' \end{pmatrix} \]

\[
M^\tau = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} \alpha^\tau & 0 & \cdots & 0 \\ 0 & (\alpha^\tau)^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\alpha^\tau)^{p^n-1} \end{pmatrix} \begin{pmatrix} q_{11}' & q_{12}' & \cdots & q_{1n}' \\ q_{21}' & q_{22}' & \cdots & q_{2n}' \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}' & q_{n2}' & \cdots & q_{nn}' \end{pmatrix}
\]

Therefore, assuming \( x[i] \) denotes the \( i \)-th coordinate of any vector \( \vec{x} \),

\[
\vec{t} = M \vec{s}
\]

\[
\begin{pmatrix} t[1] \\ t[2] \\ \vdots \\ t[n] \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} \alpha^\tau & 0 & \cdots & 0 \\ 0 & (\alpha^\tau)^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\alpha^\tau)^{p^n-1} \end{pmatrix} \begin{pmatrix} q_{11}' & q_{12}' & \cdots & q_{1n}' \\ q_{21}' & q_{22}' & \cdots & q_{2n}' \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}' & q_{n2}' & \cdots & q_{nn}' \end{pmatrix} \begin{pmatrix} s[1] \\ s[2] \\ \vdots \\ s[n] \end{pmatrix}
\]

\[
\begin{pmatrix} t[1] \\ t[2] \\ \vdots \\ t[n] \end{pmatrix} = \begin{pmatrix} q_{11}s[1]q_{11}' + \cdots + q_{1n}s[n]q_{1n}' \\ q_{21}s[1]q_{21}' + \cdots + q_{2n}s[n]q_{2n}' \\ \vdots \\ q_{n1}s[n]q_{n1}' + \cdots + q_{nn}s[n]q_{nn}' \end{pmatrix} \begin{pmatrix} \alpha^\tau \\ (\alpha^p)^\tau \\ \vdots \\ (\alpha^{p^n-1})^\tau \end{pmatrix}
\]
Given \( \vec{t} \) and \( \hat{s} \), \( \mathcal{D}' \) finds all the \( \alpha \tau \) by Gaussian elimination. Then \( \mathcal{D}' \) passes the pair \((\alpha, \alpha \tau)\) to \( \mathcal{D} \). When \( \mathcal{D} \) outputs \( \tau \), \( \mathcal{D}' \) outputs \( \tau \).

### 3.4 The Reduction DLP \( \leq_p \text{DDP} \)

This reduction is much more straightforward than the previous one. The main difference between this reduction with the previous one is that in this case given the generator \( g \in \mathbb{F}_{p^n} \) for DLP\(_{n,p} \) we need to find an 

\[
M = Q \Lambda Q^{-1}
\]

such that the diagonal entries of the diagonal matrix \( \Lambda \) become \( g^p \). We do it by the following lemma.

**Lemma 2.** The left multiplication by any element \( a \in \mathbb{F}_{p^n} \) translates to a linear transformation \( T_a \) over the field \( \mathbb{F}_{p^n} \).

**Proof.** Suppose \( \mathbb{B} = \{\beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1}\} \) form an \( \mathbb{F}_{p^n} \)-basis of \( \mathbb{F}_{p^n} \). For example, we can represent \( \mathbb{F}_{p^n} = \mathbb{F}_p[x]/(f(x)) \), and consider the polynomial basis \( 1, x, x^2, \ldots, x^{n-1} \). Every \( \beta_i \beta_j \) can be written as an \( \mathbb{F}_{p^n} \)-linear combination of the basis elements.

Fix an element \( a = \alpha_0 \beta_0 + \alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_{n-1} \beta_{n-1} \in \mathbb{F}_{p^n}^* \), where, \( \alpha_i \in \mathbb{F}_{p^n} \). Now, take an arbitrary element

\[
b = \gamma_0 \beta_0 + \gamma_1 \beta_1 + \gamma_2 \beta_2 + \cdots + \gamma_{n-1} \beta_{n-1} = (\beta_0 \beta_1 \beta_2 \cdots \beta_{n-1})^{(4)}
\]

assuming \( \gamma_i \in \mathbb{F}_{p^n} \). Expand the product \( ab \), and replace each \( \beta_i \beta_j \) as the (known) linear combination of the basis elements. This lets us write

\[
ab = (\beta_0 \beta_1 \beta_2 \cdots \beta_{n-1})T_a \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{n-1} \end{pmatrix}
\]

(5)

where \( T_a \in \mathbb{F}_{p^n}^{n \times n} \). That is, multiplication by \( a \) can be viewed as the linear transformation \( T_a \).

Conversely, given a matrix \( T_a \) can be obtained efficiently by solving the equation \( \text{det}(xI - T_a) \). In particular, \( a \) is one of the eigenvalues of \( T_a \).

**Theorem 5.** If there is an algorithm \( \mathcal{D}' \) that efficiently solves the problem DDP\(_{n,p} \), then there is an efficient algorithm \( \mathcal{D} \) that solves the problem of DLP\(_{n,p} \).

**Proof.** Given two arbitrary configurations \( \vec{s}, \vec{t} \in \mathbb{F}_p^n \) of an LCHCA\(_{n,p} \), \( \mathcal{D}' \) outputs a \( \tau \) such that \( \vec{t} = M\vec{s} \). Given an instance \((g, a)\) of DLP\(_{n,p} \), \( \mathcal{D} \) finds the linear
transformation $T_\alpha$ and $T_n$ using Lemma 4. $D$ takes an arbitrary nonzero $\vec{s} \in \mathbb{F}_p^n$ and obtains $\vec{t} = T_\alpha \vec{s}$. $D$ feeds $\vec{s}$, $\vec{t}$ and $T_\alpha$ to $\mathcal{D'}$ to have the solution $\tau$ as $\vec{t} = T_\alpha \vec{s} = (T_\alpha)^\tau \vec{s}$. 

We did not construct $M = QAQ^{-1}$ where $A_{ij} = g^{p^i}$ because $Q_{ij} = f(g)^{q^i}$. So the cost of reduction would have been more. Similarly, linear transformations were not used for the previous reduction as given a $\vec{t}$ and $\vec{s}$, to find the $M^\tau$ we required to solve the system of matrix equations $M^\tau \vec{t} = M^\tau M^\tau \vec{s}$, but it consumes more time.

3.5 Explaining The Equivalence

Here we show that $\mathbb{F}_p(\alpha)$ and $\mathbb{F}_p(M)$ are isomorphic and this is the key reason that $DLP_{n,p}$ and $DDPM_{n,p}$ equivalent.

As already mentioned that the characteristic polynomial $f_M(x)$ of any $\text{LCHA}_{n,p}^M$ is irreducible. Since $f_M(x)$ is irreducible polynomial of degree $n$, the polynomial ring $\mathbb{F}_p[x]/\langle f_M(x) \rangle$ must be isomorphic to the field $\mathbb{F}_p^n$. Suppose $\alpha$ is one of the roots of $f_M(x)$ over the field $\mathbb{F}_p^n$ such that $M = T_\alpha$, then the field $\mathbb{F}_p(\alpha) = \{0, 1, \alpha, \ldots, \alpha^{p^n-2}\}$ must be a subfield of $\mathbb{F}_p[x]/\langle f_M(x) \rangle$ where $d | n$. When $d = n$ and $\alpha' \neq 1$ for all $i < p^n - 1$ then $\alpha$ is called a primitive element of the field $\mathbb{F}_p^n$ and $f_M(x)$ is called a primitive polynomial. As $f_M(x)$ is the characteristic polynomial of the matrix $M$ by Caley-Hamilton theorem $f(M) = 0$ over the field $\mathbb{F}_p$. And using Lemma 2 we can replace $\alpha$ with the matrix $M = T_\alpha$. Therefore, $\mathbb{F}_p(\alpha) \simeq \mathbb{F}_p(M) = \{0, I, M, \ldots, M^{p^n-2}\}$. In fact, using the matrices of $\mathbb{F}_p(M)$ and a state $\vec{s}_i \in \mathbb{F}_p^n$ we can have a set of disjoint sequences $S_i = \langle 0, I, \vec{s}_i, M, \vec{s}_i, \ldots, M^{p^n-2}, \vec{s}_i \rangle$. We have $\bigcap S_i = \emptyset$ and $\bigcup S_i = \mathbb{F}_p^n$. Essentially, the sequences $S_i$ are the cycles of a linear cyclic hybrid group cellular automata. When $f_M(x)$ is a primitive polynomial, $M$ becomes a primitive element of the field $\mathbb{F}_p(M)$ yielding a maximum length linear cyclic hybrid cellular automata.

On the other hand, as $f_M(M) = 0$, we define a subfield $\mathbb{F}(M) = \{0, I, M, \ldots, M^{d-1}\}$ of $\mathbb{F}_p[x]/\langle f_M(x) \rangle$. In particular, when $f_M(x)$ is a primitive polynomial the primitive roots $\alpha$ and $M$ generate the entire field $\mathbb{F}_p[x]/\langle f_M(x) \rangle$ as $d = n$. Otherwise, when $d | n$, $\mathbb{F}(\alpha)$ and $\mathbb{F}(M)$ are subfields of $\mathbb{F}_p[x]/\langle f_M(x) \rangle$. We know that two fields of the same characteristics and same order are isomorphic. Hence, $\mathbb{F}(\alpha)$ and $\mathbb{F}(M)$ are isomorphic. Any isomorphism between two finite fields is efficient to compute 4. Therefore, if one among the problems $DDPM_{n,p}$ and $DLP_{n,p}$ was easier than the other one, then we could have always turned down the harder problem into the easier one using this isomorphism. So, the equivalence between these two could not be proven.
4 Two Important Variants of DDP

Definition 13. (Fixed (coordinates) DDP $\text{FDP}_{k,n}$). For a fixed $k < n$, given a configuration $\vec{s} \in \mathbb{F}_p^n$ of an $n$-cell LCHCA$_{n,p}^M$, and a vector $\vec{x} \in \mathbb{F}_p^k$, the problem of $\text{FDP}_{k,n}$ is to find a $\tau$ such that $M^r \vec{s} \in \{(\vec{x}|\vec{a}) \mid \forall \vec{a} \in \mathbb{F}_p^{n-k}\}$.

By the vector $(\vec{x}|\vec{a})$ we mean the augmentation of two vectors $\vec{x}$ and $\vec{a}$. So $\{(\vec{x}|\vec{a}) \mid \forall \vec{a} \in \mathbb{F}_p^{n-k}\}$ is the set of vectors whose $k$ coordinates are fixed to $\vec{x}$. As the coordinates are independent of each other so the problem $\text{FDP}_{k,n}$ also remains independent of the choice of the $k$ coordinates and the choice for $\vec{x}$ too.

The Reduction $\text{FDP}_{k,n} \leq_p \text{DDP}_{n,p}^M$. This reduction is straightforward as we are free to choose any $\vec{t} \in \{(\vec{x}|\vec{a}) \mid \forall \vec{a} \in \mathbb{F}_p^{n-k}\}$ and call the $\text{DDP}_{n,p}^M$ oracle $G'$ on the inputs $\vec{t}$ and $\vec{s}$. The reply $\tau$ will be a valid solution of the $\text{FDP}_{k,n}$ instance as $M^r \vec{s} = \vec{t}$.

Does $\text{DDP}_{n,p}^M \leq_p \text{FDP}_{k,n}$? Presently, we have no answer for this question except the belief that the answer is a no. We reason this "no" with the following difference between $\text{FDP}_{k,n}$ and $\text{DDP}_{n,p}^M$. The solution of $\text{DDP}_{n,p}^M$ is unique modulo $p^r - 1$ while there exist $p^r - k - 1$ solutions for $\text{FDP}_{k,n}$. This lack of uniqueness of solutions makes $\text{FDP}_{k,n}$ easier than $\text{DDP}_{n,p}^M$. Thus this reduction should not be possible.

Definition 14. (Short (Fixed) DDP $\text{SDDP}^\delta_{k,n}$). For a fixed $k < n$ and a $\delta \in \mathbb{Z}^+$, given a configuration $\vec{s} \in \mathbb{F}_p^n$ of an $n$-cell LCHCA$_{n,p}^M$, and a vector $\vec{x} \in \mathbb{F}_p^k$, the problem of $\text{SDDP}^\delta_{k,n}$ is to find a $\tau < \delta$ such that $M^r \vec{s} \in \{(\vec{x}|\vec{a}) \mid \forall \vec{a} \in \mathbb{F}_p^{n-k}\}$.

By the above argument $\text{SDDP}^\delta_{k,n}$ also remains independent of the choice for $k$ coordinates and the vector $\vec{x}$.

Due to the restriction $\tau < \delta$, unlike $\text{DDP}_{n,p}^M$, we have no hint if $\text{SDDP}^\delta_{k,n}$ reduces to $\text{DDP}_{n,p}^M$ if at all. A famous problem that becomes hard due to the presence of such restriction on the norm-bound of its solution is short integer solution problem $\text{SIS}^\delta_{n,m}$ [1].

Our goal is to determine the hardness of the problem $\text{SDDP}^\delta_{k,n}$. It appears that the reduction $\text{SIS}^\delta_{n,m} \leq_p \text{SDDP}^\delta_{k,n}$ is not immediate. Solutions of $\text{SIS}^\delta_{n,m}$ are integer vectors while that of $\text{SDDP}^\delta_{k,n}$ are vectors over $\mathbb{F}_p$. Therefore, we define another problem, namely Short Lacunary Solution Problem ($\text{SLS}^\delta_{n,m}$), as the missing link between $\text{SDDP}^\delta_{k,n}$ and $\text{SIS}^\delta_{n,m}$. In particular, we will show that $\text{SIS}^\delta_{n,m} \leq_p \text{SLS}^\delta_{n,m}$ followed by $\text{SLS}^\delta_{n,m} \leq_p \text{SDDP}^\delta_{k,n}$. As Turing reductions are transitive, we will conclude that $\text{SIS}^\delta_{n,m} \leq_p \text{SDDP}^\delta_{k,n}$.

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