A NEW METHOD FOR THE BOUNDEDNESS OF SEMILINEAR DUFFING EQUATIONS AT RESONANCE

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Abstract. We introduce a new method for the boundedness problem of semilinear Duffing equations at resonance. In particular, it can be used to study a class of semilinear equations at resonance without the polynomial-like growth condition. As an application, we prove the boundedness of all the solutions for the equation \( \ddot{x} + n^2 x + g(x) + \psi(x) = p(t) \) under the Lazer-Leach condition on \( g \) and \( p \), where \( n \in \mathbb{N}^+ \), \( p(t) \) and \( \psi(x) \) are periodic and \( g(x) \) is bounded.

1. Introduction and the main results. The study of semilinear equations at resonance has a long history. The interest in this model is motivated both by its connections to application and by a remarkable richness of the related dynamical systems. It is well known that the linear equation

\[ \ddot{x} + n^2 x = \sin nt, \quad n \in \mathbb{N}^+ \]

has no bounded solutions, where \( \ddot{x} = d^2 x/dt^2 \). Another interesting example was constructed by Ding [5], who proved that each solution of the equation

\[ \ddot{x} + n^2 x + \arctan x = 4 \cos nt, \quad n \in \mathbb{N}^+ \]

is unbounded. Due to these resonance phenomena, the existence of bounded solutions and the boundedness of all the solutions for semilinear equation at resonance are very delicate.

In 1969, Lazer and Leach [9] studied the following semilinear equations:

\[ \ddot{x} + n^2 x + g(x) = p(t), \quad n \in \mathbb{N}^+, \quad (1) \]
where \( p(t + 2\pi) = p(t) \) and \( g \) is continuous and bounded. They proved that if
\[
\left| \int_0^{2\pi} p(t)e^{-int} dt \right| < 2\left( \lim \inf_{x \to +\infty} g - \lim \sup_{x \to -\infty} g \right),
\]
then (1) has at least one \( 2\pi \)-periodic solution. Moreover, they obtained that each solution of (1) is unbounded if
\[
\int_0^{2\pi} p(t)e^{-int} dt \geq 2\left( \sup g - \inf g \right).
\]
Thus if
\[
\lim_{x \to -\infty} g(x) = g(-\infty) \leq g(x) \leq g(+\infty) = \lim_{x \to +\infty} g(x), \quad \forall x \in \mathbb{R},
\]
then condition (2) is sufficient and necessary for the existence of bounded solutions. For this reason, (2) is called Lazer-Leach condition.

In 1996, Alonso and Ortega [1] studied the following equation:
\[
\ddot{x} + n^2x + g(x) + \psi(x) = p(t), \quad n \in \mathbb{N}^+,
\]
where \( g \) and \( p \) are as same as above and the perturbation \( \psi(x) \) will be small at infinity in the following sense:
\[
\lim_{|x| \to \infty} \Psi(x) = 0,
\]
where \( \Psi(x) = \int_0^x \psi(x)dx \). They proved that each solution with large initial condition is unbounded if
\[
\left| \int_0^{2\pi} p(t)e^{-int} dt \right| > 2(H - K),
\]
where
\[
H = \max\{ \lim \sup_{x \to -\infty} g, \lim \sup_{x \to +\infty} g \}, \quad K = \min\{ \lim \inf_{x \to -\infty} g, \lim \inf_{x \to +\infty} g \}.
\]

Other conditions for the existence of bounded and unbounded solutions are described in [1, 2, 6, 8, 15, 16] and their references.

The pioneering work on the boundedness of (1) was due to Ortega [19]. He proved a variant of Moser’s small twist theorem, by which he obtained the boundedness for the equation
\[
\ddot{x} + n^2x + h_L(x) = p(t), \quad p(t) \in C^5(\mathbb{R}/2\pi \mathbb{Z}),
\]
where \( L > 0 \) and \( h_L(x) \) is a piecewise linear function of the form
\[
h_L(x) = \begin{cases} 
  L, & \text{if } x \geq 1, \\
  Lx, & \text{if } -1 \leq x \leq 1, \\
  -L, & \text{if } x \leq -1,
\end{cases}
\]
and \( p(t) \) satisfies
\[
\left| \int_0^{2\pi} p(t)e^{-int} dt \right| < 4L.
\]

Then Liu [11] studied the equation (1) by the assumptions: \( p(t) \in C^7(\mathbb{R}/2\pi \mathbb{Z}), \)
\( g(x) \in C^6(\mathbb{R}) \) satisfying
\[
g(\pm \infty) = \lim_{x \to \pm \infty} g(x) \text{ exist and are finite}, \quad (5)
\]
and
\[
\lim_{|x| \to +\infty} x^k g^{(k)}(x) = 0, \quad 1 \leq k \leq 6.
\]
With Ortega’s small twist theorem, he showed that the Lazer-Leach condition (2) is sufficient for the boundedness of (1). Moreover, if (3) holds true, then Lazer-Leach’s result [9] implies that (2) is also necessary for the boundedness.

One can refer to [11, 12, 13, 14, 19, 23] for the applications of Ortega’s small twist theorem.

All the above mentioned results concern the semilinear equations at resonance with a polynomial-like growth potential, that is, the potential, say \( g(x,t) \), is bounded and satisfies
\[
\lim_{|x| \to +\infty} x^m D_x^m g(x,t) = 0 \tag{6}
\]
for some \( m > 0 \).

In this paper, we study the boundedness of the equation
\[
\ddot{x} + n^2 x + g(x) + \psi(x) = p(t), \quad n \in \mathbb{N}^+,
\tag{7}
\]
where \( g(x) \) is a polynomial-like function, \( p(t + 2\pi) = p(t) \) and \( \psi(x + T) = \psi(x) \) satisfying \( \int_0^T \psi(x)dx = 0. \)

It is easy to see that usually \( \psi(x) \) does not satisfy (6). The most typical example of (7) may be
\[
\ddot{x} + n^2 x + g(x) + \sin x = p(t), \quad n \in \mathbb{N}^+.
\]

So we consider the equation with a mixed potential including not only a polynomial-like growth term but also a periodic term.

We will prove that the Lazer-Leach condition on \( g \) and \( p \) is sufficient for the boundedness of (7). In other words, the periodic term \( \psi \) does not play any role in the boundedness. More precisely, we prove that:

**Theorem 1.1.** Assume \( g(x) \in C^{m_1}(\mathbb{R}), \psi(x) \in C^{m_1}(\mathbb{R}/T\mathbb{Z}) \) and \( p(t) \in C^{m_2}(\mathbb{R}/2\pi \mathbb{Z}) \) with \( m_1 = 18, m_2 = 14 \). Suppose the following conditions hold true:

(A1) \( g(\pm\infty) = \lim_{x \to \pm\infty} g(x) \) exist and are finite,

(A2) \( \lim_{|x| \to +\infty} x^k g^{(k)}(x) = 0, \quad 1 \leq k \leq m_1 \).

Then under the following Lazer-Leach condition:
\[
\left| \int_0^{2\pi} p(t)e^{-int}dt \right| < 2|g(+\infty) - g(-\infty)|, \tag{8}
\]

every solution of (7) is bounded.

On the other hand, under the assumptions in Theorem 1.1, \( g(x) \) is bounded and
\[
\lim_{|x| \to \infty} \frac{\int_0^x \psi(s)ds}{x} = 0,
\]
by \( \int_0^T \psi(x)dx = 0 \). Thus, if
\[
\left| \int_0^{2\pi} p(t)e^{-int}dt \right| > 2|g(+\infty) - g(-\infty)|.
\]

then Alonso-Ortega’s result, cf. Proposition 3.4 in [1] is applicable and it implies the existence of unbounded solutions for (7), also see [11]. Therefore we obtain the following conclusion:

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It is no loss of generality to assume \( \int_0^T \psi(x)dx = 0 \). In fact, if \( \int_0^T \psi(x)dx \neq 0 \), then we can redefine \( \tilde{\psi}(x) = \psi(x) - \frac{1}{T} \int_0^T \psi(x)dx \), \( \tilde{p}(t) = p(t) - \frac{T}{T} \int_0^T \psi(x)dx \).
Corollary 1. Assume \( g(x), \psi(x) \) and \( p(t) \) satisfy the conditions in Theorem 1.1. If
\[
\left| \int_0^{2\pi} p(t)e^{-int}dt \right| \neq 2|g(+\infty) - g(-\infty)|,
\]
then (8) is sufficient and necessary for the boundedness of (7).

Remark 1. When \( n \) in (7) is replaced by a Diophantine irrational, a similar result has been obtained by [7].

The new ingredients in our proof are as follows.

Instead of applying Ortega’s small twist theorem, we use a rotation transformation in Subsection 3.1 to deal with resonance (also see [24]). With such a transformation, the linear term disappears in the new Hamiltonian (and a sublinear one is obtained), and one will not meet the difficulty of resonance any more. For example, when \( \psi(x) = 0 \), Moser’s small twist theorem is directly applicable, see [25].

For the case \( \psi(x) \neq 0 \), however, a new difficulty appears due to the lack of the polynomial-like growth condition. In fact, the estimates on the derivatives of the perturbations are very poor, see Lemma 2.4. Hence the perturbation in the sublinear system can not be reduced to be small enough in \( C^4 \)-topology by the standard method, see Remark 4.

To overcome the difficulty for the case \( \psi(x) \neq 0 \), our observation is that although with the method of solving homological equations, the new perturbation does not become smaller than the old one, the smoothness of the ‘troublesome’ term in the new perturbation of the Hamiltonian on some variable (that is, the old angle variable) become better, see Subsection 4.3. Thus repeatedly solving homological equations will leads to a Hamiltonian whose ‘troublesome’ perturbation term possesses a much better regularity on the angle variable. Then the proof can be finished by following the standard computations of Dieckerhoff-Zehnder and Moser’s theorem, see Subsection 4.5.

It is worth to note that the periodic assumption on \( \psi(x) \) is not necessary. In fact we can show the boundedness holds when \( \psi(x) = \phi(x^{1+\delta}) \) with \( \phi(x) \) periodic and \( \delta > 0 \) small enough. Moreover, \( \psi(x) \) can be replaced by a function \( \psi(x,t) \) which is periodic on both \( x \) and \( t \), see [24]. Thus we show that the classical polynomial-like growth conditions can be considerably weakened. For more references, one can see [10], [22].

The paper is organized as follows. In Section 2, we state some preliminary estimates. In Section 3, we introduce a rotation transformation and then make canonical transformations such that all non-oscillating terms are transformed into normal form possessing desirable properties. The main difficulty in this paper lies in how to deal with oscillating terms caused by \( \psi(x) \). For this purpose, in Section 4 we make canonical transformations to improve estimates on the derivatives of oscillating terms and subsequently change the system into a nearly integrable one. Thus Theorem 1.1 is proved by Moser’s twist theorem in Section 5. The proof of some lemmas can be found in the Appendix.

2. Action-angle coordinates. In the context, we denote \([f](\cdot) = \frac{1}{2\pi} \int_0^{2\pi} f(\cdot, \theta)d\theta \) be the average function of \( f(\cdot, \theta) \) with respect to \( \theta \). Without loss of generality, \( C > 1, \ c < 1 \) are two universal positive constants not concerning their quantities, and \( j, k, l, \nu, \kappa \), etc., are non-negative integers.
Lemma 2.2. It holds that
\[
H(x, y, t) = \frac{1}{2} n(x^2 + y^2) + \frac{1}{n} G(x) - \frac{1}{n} xp(t) + \frac{1}{n} \Psi(x),
\]
where \(G(x) = \int_0^x g(s)ds\), \(\Psi(x) = \int_0^x \psi(s)ds\).

Let \(y = \dot{x}/n\), equation (7) is equivalent to a Hamiltonian system with Hamiltonian
\[
H(x, y, t) = \frac{1}{2} n(x^2 + y^2) + \frac{1}{n} G(x) - \frac{1}{n} xp(t) + \frac{1}{n} \Psi(x),
\]
where \(G(x) = \int_0^x g(s)ds\), \(\Psi(x) = \int_0^x \psi(s)ds\).

Under the action-angle coordinates transformation \((dx \wedge dy = dI \wedge d\theta)\)
\[
\begin{align*}
x &= x(I, \theta) = \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta, \\
y &= y(I, \theta) = \sqrt{\frac{2}{n}} I^\frac{1}{2} \sin n\theta,
\end{align*}
\]
where \(x = x(I, \theta) = \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta\) for simplicity.

Denote \(f_1(I, \theta) = \frac{1}{n} G(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta), f_2(I, \theta, t) = -\frac{1}{n} \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta p(t)\), and \(f_3(I, \theta) = \frac{1}{n} \Psi(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta)\), then (10) is rewritten by
\[
H(I, \theta, t) = I + f_1(I, \theta) + f_2(I, \theta, t) + f_3(I, \theta), \quad (I, \theta, t) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1
\]
with \(\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})\).

Next, we give several lemmas about the estimates on \(f_1(I, \theta)\), \(f_2(I, \theta, t)\) and \(f_3(I, \theta)\) which are basic and similar to those in [11, 24].

Lemma 2.1. For \(I\) large enough, \(\theta \in \mathbb{S}^1\), \(k + j \leq m_1 + 1\), we have
\[
\left|f_1(I, \theta)\right| \leq CI^\frac{1}{2}, \quad \left|\partial_I^k \partial_\theta^j f_1(I, \theta)\right| \leq CI^\frac{1}{2} - k^{(\max(1, j) - 1)};
\]
\[
cI^\frac{1}{2} \leq \left|f_1(I)\right| \leq CI^\frac{1}{2}, \quad cI^\frac{1}{2} - k \leq \left|f_1^{(k)}(I)\right| \leq CI^\frac{1}{2} - k.
\]

Lemma 2.2. It holds that
\[
\lim_{I \to +\infty} I^{-\frac{1}{2}} \cdot |f_1(I)| = \frac{\sqrt{2}}{\pi} n^{-\frac{1}{2}} (g(+\infty) - g(-\infty)),
\]
\[
\lim_{I \to +\infty} I^{\frac{1}{2}} \cdot |f_1'(I)| = \frac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} (g(+\infty) - g(-\infty)),
\]
\[
\lim_{I \to +\infty} I^{\frac{3}{2}} \cdot |f_1''(I)| = -\frac{\sqrt{2}}{4\pi} n^{-\frac{3}{2}} (g(+\infty) - g(-\infty)).
\]

Remark 2. The estimates about Lemmas 2.1 and 2.2 are classic and can be obtained by direct calculations. Thus we omit it. Readers can refer to [11].

Direct computations can lead to the following conclusions:

Lemma 2.3. For \(I\) large enough, \(\theta, t \in \mathbb{S}^1\), \(k + j \leq m_1 + 1\) and \(l \leq m_2\), we have
\[
\left|\partial_I^k \partial_\theta^j \partial_t^l f_2(I, \theta, t)\right| \leq CI^{-\frac{1}{2} - k}.
\]

Lemma 2.4. For \(I\) large enough, \(\theta \in \mathbb{S}^1\), \(k + j \leq m_1 + 1\), we have
\[
\left|\partial_I^k \partial_\theta^j f_3(I, \theta)\right| \leq CI^{-\frac{1}{2} + \frac{3}{2}}.
\]
Since $\partial_t H > 1/2$ when $I$ is sufficiently large, we can solve $H(I, \theta, t) = h$ for $I$ as following:

$$I = I(h, t, \theta) = h - R(h, t, \theta), \quad \text{(11)}$$

where $R(h, t, \theta)$ is determined implicitly by the equation

$$R = f_1(h - R, \theta) + f_2(h - R, \theta, t) + f_3(h - R, \theta). \quad \text{(12)}$$

It is clear that $h \to +\infty$ if and only if $I \to +\infty$. From Arnold’s transformation on exchanging the roles of angle and time variables, it is well known that the new Hamiltonian system

$$\begin{aligned}
\begin{cases}
\frac{dt}{d\theta} = -\partial_h I(h, t, \theta), \\
\frac{dh}{d\theta} = \partial_t I(h, t, \theta)
\end{cases}
\end{aligned}$$

is equivalent to the original one, see [3, 10, 11, 24], etc.

We present some estimates on $R(h, t, \theta)$ in (11).

**Lemma 2.5.** For $h$ large enough, $\theta, t \in \mathbb{S}^1$, $k + j \leq m_1 + 1$ and $l \leq m_2$, we have

$$\left| \partial^k_h \partial^j_i \partial^l R \right| \leq C h^{\frac{1}{2} - \frac{1}{2} + \frac{1}{2}(\max\{1, j\} - 1)}. \quad (13)$$

The proof is given in the Appendix.

Moreover, from the identity (12), $R$ has the following form by Taylor’s formula:

$$R = f_1(h, \theta) + f_2(h, t, \theta) + f_3(h - R, \theta) - \int_0^1 \partial_tf_1(h - t\mu R, \theta) Rd\mu - \int_0^1 \partial_tf_2(h - t\mu R, \theta, t) Rd\mu$$

$$= f_1(h, \theta) + f_2(h, \theta, t) + f_3(h - R, \theta) - \partial_tf_1(h, \theta)R - \partial_tf_2(h, \theta, t)R$$

$$+ \int_0^1 \int_0^1 (\partial^2_{f_1}(h - s\mu R, \theta, t) + \partial^2_{f_2}(h - s\mu R, \theta, t))\mu R^2 dsd\mu. \quad (13)$$

(13) yields that

$$R = f_1(h, \theta) + f_2(h, t, \theta) + \frac{1}{n} \Psi(x) - R_{01}(h, t, \theta) - R_{02}(h, t, \theta), \quad (14)$$

where

$$\frac{1}{n} \Psi(x) = f_3(h - R, \theta),$$

$$R_{01}(h, t, \theta) = (\partial_tf_1(h, \theta) + \partial_t f_2(h, \theta, t)) (f_1(h, \theta) + f_2(h, \theta, t)),$$

and

$$R_{02}(h, t, \theta) = (\partial_tf_1(h, \theta) + \partial_t f_2(h, \theta, t)) f_3(h - R, \theta)$$

$$+ (\partial f_1(h, \theta) + \partial f_2(h, \theta, t)) \int_0^1 (\partial_tf_1(h - t\mu R, \theta) + \partial_tf_2(h - t\mu R, \theta, t)) Rd\mu$$

$$- \int_0^1 \int_0^1 (\partial^2_{f_1}(h - s\mu R, \theta, t) + \partial^2_{f_2}(h - s\mu R, \theta, t))\mu R^2 dsd\mu.$$
Lemma 2.6. For $h$ large enough, $\theta$, $t \in \mathbb{S}^1$, $k + j \leq m_1 - 1$, and $l \leq m_2$, it holds that:

$$\left| \partial^k_h \partial^j_{\theta^l} R_{01} \right| \leq Ch^{-k-\frac{1}{2}(\max\{1,j\})-1},$$

and

$$\left| \partial^k_h \partial^l_{\theta^j} R_{02} \right| \leq Ch^{-\frac{1}{2} - \frac{3}{2} + \frac{1}{2}(\max\{1,j\})-1}.$$

The proof is given in the Appendix.

Remark 3. From Lemmas 2.1, 2.3, 2.4 and 2.6, it shows that $f_1, f_2$ and $R_{01}$ satisfy the polynomial-like growth condition (6) with variable $h$, while $-\frac{1}{n}\Psi(x)$ and $R_{02}$ do not satisfy the polynomial-like growth condition due to the oscillating property of the periodic function $\Psi(x)$ which is rather different from the previous works.

3. The normal forms for non-oscillating terms. In this section, we first introduce a rotation transformation to eliminate the linear part of the Hamiltonian which help us to obtain a sublinear function, then obtain the normal form for non-oscillating terms by canonical transformations.

3.1. A rotation transformation. Inspired by the techniques in KAM theory (for example [26]), we construct a rotation transformation $\Phi_1 : (h_1, t_1, \theta) \rightarrow (h, t, \theta)$ by

$$\begin{cases} h &= h_1 \\ t &= t_1 + \theta. \end{cases}$$

Intuitively, we adopt a rotating coordinate system with an angle speed 1.

Under $\Phi_1$, the Hamiltonian $I$ is transformed into $I_1$ as following

$$I_1(h_1, t_1, \theta) = -f_1(h_1, \theta) - f_2(h_1, \theta, t_1 + \theta) \frac{1}{n}\Psi(x) + R_{11}(h_1, t_1, \theta) + R_{12}(h_1, t_1, \theta)$$

with $R_{11}(h_1, t_1, \theta) = R_{01}(h_1, t_1 + \theta, \theta)$, $R_{12}(h_1, t_1, \theta) = R_{02}(h_1, t_1 + \theta, \theta)$.

Lemma 3.1. For $h_1$ large enough, $\theta$, $t_1 \in \mathbb{S}^1$, and $k + j \leq m_1 - 1$, $l \leq m_2$, it holds that:

$$\left| \partial^k_{h_1} \partial^j_{t_1} \partial^l_{\theta} R_{11} \right| \leq Ch_1^{-k-\frac{1}{2}(\max\{1,j\})-1},$$

and

$$\left| \partial^k_{h_1} \partial^l_{\theta^j} R_{12} \right| \leq Ch_1^{-\frac{1}{2} - \frac{3}{2} + \frac{1}{2}(\max\{1,j\})-1}.$$

Proof. It is obtained from Lemma 2.6.

3.2. The normal form with $f_1(h_1, \theta)$. We make a canonical transformation $\Phi_2 : (h_2, t_2, \theta) \rightarrow (h_1, t_1, \theta)$ given by

$$\begin{cases} h_1 &= h_2 \\ t_1 &= t_2 - \partial_{h_2}S_2(h_2, \theta) \end{cases}$$

with the generating function $S_2(h_2, \theta)$ determined by

$$S_2(h_2, \theta) = \int_0^\theta \left( f_1(h_2, \theta) - [f_1](h_2) \right) d\theta. \quad (16)$$

Under $\Phi_2$, the Hamiltonian $I_1$ is transformed into $I_2$ as following

$$I_2(h_2, t_2, \theta) = -f_1(h_2, \theta) - f_2(h_2, \theta, t_2 + \theta - \partial_{h_2}S_2(h_2, \theta)) \frac{1}{n}\Psi(x) + R_{11}(h_2, t_2 - \partial_{h_2}S_2(h_2, \theta), \theta) + R_{12}(h_2, t_2 - \partial_{h_2}S_2(h_2, \theta), \theta) + \partial_{h}S_2(h_2, \theta)$$
\[ [f_1](h_2) - f_1(h_2, \theta, t_2 + \theta) = f_2(h_2, \theta, t_2 + \theta) - \frac{1}{n} \Psi(x) + [f_1](h_2) - f_1(h_2, \theta) + \partial_\theta S_2(h_2, \theta) \]

\[ + \int_0^1 \partial_t f_2(h_2, \theta, t_2 + \theta - \mu \partial_\theta S_2(h_2, \theta)) \partial_{h_2} S_2(h_2, \theta) d\mu \]

\[ + R_{11}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta) + R_{12}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta). \]

It is clear that (16) implies

\[ [f_1](h_2) - f_1(h_2, \theta) + \frac{\partial}{\partial \theta} S_2(h_2, \theta) = 0. \]

Let

\[ R_{21}(h_2, t_2, \theta) = R_{11}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta) \]

\[ - \int_0^1 \partial_t f_2(h_2, \theta, t_2 - \mu \partial_{h_2} S_2(h_2, \theta)) \partial_{h_2} S_2(h_2, \theta) d\mu, \]

\[ R_{22}(h_2, t_2, \theta) = R_{12}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta). \]

Thus, \( I_2 \) is rewritten by

\[ I_2(h_2, t_2, \theta) = -[f_1](h_2) - f_2(h_2, \theta, t_2 + \theta) - \frac{1}{n} \Psi(x) + R_{21}(h_2, t_2, \theta) + R_{22}(h_2, t_2, \theta). \]

**Lemma 3.2.** For \( h_2 \) large enough, \( \theta, t_2 \in S^1 \), it holds that

\[ |\partial_{h_2}^k \partial_{\theta}^j S_2(h_2, \theta)| \leq C h_2^{\frac{1}{2} - k + \frac{1}{2}(\max(2, j) - 2)}, \quad k + j \leq m_1 + 1, \]  

(17)

and

\[ |\partial_{h_2} t_1| \leq C h_2^{-\frac{1}{2}}, \quad \partial_{t_2} t_1 = 1, \quad |\partial_\theta t_1| \leq C h_2^{-\frac{1}{2}}, \]

\[ |\partial_{h_2}^k \partial_{t_2}^l \partial_{\theta}^j t_1| \leq C h_2^{-\frac{1}{2} - k + \frac{1}{2}(\max(2, j) - 2)}, \quad k + l + j \geq 2, \quad k + j \leq m_1. \]

Moreover, for \( k + j \leq m_1 - 1 \), it holds that:

\[ |\partial_{h_2}^k \partial_{t_2}^l | R_{21}| \leq C h_2^{-k + \frac{1}{2}(\max(1, j) - 1)}, \quad l \leq m_2 - 1; \]

\[ |\partial_{h_2}^k \partial_{t_2}^l | R_{22}| \leq C h_2^{-\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max(1, j) - 1)}, \quad l \leq m_2. \]

The proof is given in the Appendix.

### 3.3. The normal form with \( f_2(h_2, \theta, t_2 + \theta) \)

Without causing confusion, we still denote

\[ [f_2](h, t) = \frac{1}{2\pi} \int_0^{2\pi} f_2(h, \theta, t + \theta) d\theta. \]

Then we have

**Lemma 3.3.** For any \( h \in \mathbb{R}^+, t \in S^1 \), it holds that

\[ [f_2](h, t) = -\frac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} h^{\frac{1}{2}} \left\{ \cos(nt) \int_0^{2\pi} p(\tau) \cos(n\tau) d\tau + \sin(nt) \int_0^{2\pi} p(\tau) \sin(n\tau) d\tau \right\}. \]

Moreover,

\[ |[f_2](h, t)| \leq \frac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} h^{\frac{1}{2}} \left| \int_0^{2\pi} p(\tau) e^{int} d\tau \right|. \]  

(18)
Proof.

\[
[f_2](h,t) = \frac{1}{2\pi} \int_0^{2\pi} f_2(h,\theta, t + \theta) d\theta = -\frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} h^\frac{1}{2} \int_0^{2\pi} \cos(n\theta)p(t + \theta) d\theta \\
= -\frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} h^\frac{1}{2} \{\cos(nt) \int_0^{2\pi} p(\tau) \cos(n\tau) d\tau + \sin(nt) \int_0^{2\pi} p(\tau) \sin(n\tau) d\tau\}.
\]

Thus, (18) is obtained by the norm of complex number immediately. □

Now, we make a transformation \(\Phi_3 : (h_3, t_3, \theta) \to (h_2, t_2, \theta)\) implicitly given by

\[
\begin{align*}
  h_2 &= h_3 + \partial_{t_2} S_3(h_3, t_2, \theta) \\
  t_3 &= t_2 + \partial_{h_3} S_3(h_3, t_2, \theta)
\end{align*}
\]

(19)

with the generating function \(S_3(h_3, t_2, \theta)\) determined by

\[
S_3(h_3, t_2, \theta) = \int_0^\theta (f_2(h_3, \theta, t_2 + \theta) - [f_2](h_3, t_2)) d\theta. \quad (20)
\]

Under \(\Phi_3\), the Hamiltonian \(I_2\) is transformed into

\[
I_3(h_3, t_3, \theta) = -[f_1](h_3 + \partial_{t_3} S_3) - f_2(h_3 + \partial_{t_2} S_3, \theta, t_2 + \theta) - \frac{1}{n} \Psi(x) \\
+ R_{21}(h_3 + \partial_{t_2} S_3, t_3 - \partial_{h_3} S_3, \theta) + R_{22}(h_3 + \partial_{t_2} S_3, t_3 - \partial_{h_3} S_3, \theta) + \partial_\theta S_3 \\
= -[f_1](h_3) - [f_2](h_3, t_3) - \frac{1}{n} \Psi(x) \\
+ [f_2](h_3, t_2) - f_2(h_3, \theta, t_2 + \theta) + \partial_\theta S_3 \\
- \int_0^1 [f_1]'(h_3 + \mu \partial_{t_3} S_3) \partial_{t_2} S_3(h_3, t_2, \theta) d\mu \\
- \int_0^1 \partial_\theta f_2(h_3 + \mu \partial_{t_3} S_3, \theta, t_2 + \theta) \partial_{t_2} S_3(h_3, t_2, \theta) d\mu \\
+ \int_0^1 \partial_\theta [f_2](h_3, t_3 - \mu \partial_{h_3} S_3) \partial_{h_3} S_3 d\mu \\
+ R_{21}(h_3 + \partial_{t_2} S_3, t_3 - \partial_{h_3} S_3, \theta) + R_{22}(h_3 + \partial_{t_2} S_3, t_3 - \partial_{h_3} S_3, \theta).
\]

(20) implies

\[
[f_2](h_3, t_2) - f_2(h_3, \theta, t_2 + \theta) + \partial_\theta S_3 = 0.
\]

Let

\[
\alpha(h_3, t_3) = -[f_1](h_3) - [f_2](h_3, t_3);
\]

\[
R_{31}(h_3, t_3, \theta) = R_{21}(h_3 + \partial_{t_2} S_3, t_3 - \partial_{h_3} S_3, \theta) \\
- \int_0^1 [f_1]'(h_3 + \mu \partial_{t_3} S_3) \partial_{t_2} S_3(h_3, t_2, \theta) d\mu \\
- \int_0^1 \partial_\theta f_2(h_3 + \mu \partial_{t_3} S_3, \theta, t_2 + \theta) \partial_{t_2} S_3(h_3, t_2, \theta) d\mu \\
+ \int_0^1 \partial_\theta [f_2](h_3, t_3 - \mu \partial_{h_3} S_3) \partial_{h_3} S_3 d\mu;
\]

\[
R_{32}(h_3, t_3, \theta) = R_{22}(h_3 + \partial_{t_2} S_3, t_3 - \partial_{h_3} S_3, \theta).
\]
Thus we have
\[ I_3(h_3, t_3, \theta) = \alpha(h_3, t_3) - \frac{1}{n} \Psi(x) + R_{31}(h_3, t_3, \theta) + R_{32}(h_3, t_3, \theta). \]

**Lemma 3.4.** For \( h_3 \) large enough, \( \theta, t_3 \in S^1 \), it holds that
\[
|\partial^k_{h_3} \partial^j_{t_3} S_3(h_3, t_2, \theta)| \leq Ch_3^{\frac{1}{2} - k}, \quad l \leq m_2, \quad \forall \ k, j;
\]
\[
ch_6^\frac{1}{2} \leq |\partial^k_{h_6} \alpha(h_6, t_6)| \leq Ch_6^{\frac{1}{2} - k}, \quad k = 0, 1, 2, \tag{21}
\]
and for \( k + j \leq m_1 - 1, \)
\[
|\partial^k_{h_3} \partial^j_{t_3} \partial^l_{h_6} R_{31}| \leq Ch_3^{-k + \frac{1}{2} (\max\{1, j\} - 1)}, \quad l \leq m_2 - 1;
\]
\[
|\partial^k_{h_3} \partial^j_{t_3} \partial^l_{h_6} R_{32}| \leq Ch_3^{-\frac{1}{2} - \frac{k}{2} + \frac{1}{2} (\max\{1, j\} - 1)}, \quad l \leq m_2 - 1.
\]

Moreover, the map \( \Phi_3 \) satisfies
\[
|\partial_{h_3} t_2| \leq Ch_3^{-k}, \quad \frac{1}{2} \leq |\partial_{h_3} t_2| \leq 2, \quad |\partial_{h_3} t_2| \leq Ch_3^{-\frac{1}{2}},
\]
\[
|\partial^k_{h_3} \partial^j_{t_3} | \leq Ch_3^{-\frac{1}{2} - k}, \quad k + l + j \geq 2, \quad l \leq m_2;
\]
\[
\frac{1}{2} \leq |\partial_{h_3} h_2| \leq 2, \quad |\partial_{h_3} h_2| \leq Ch_3^{\frac{1}{2}}, \quad |\partial_{h_3} h_2| \leq Ch_3^{\frac{1}{2}},
\]
\[
|\partial^k_{h_3} \partial^j_{t_3} \partial^l_{h_3} h_2| \leq Ch_3^{\frac{1}{2} - k}, \quad k + l + j \geq 2, \quad l \leq m_2 - 1.
\]

**Proof.** From (8), Lemmas 2.2 and 3.3, (21) and (22) holds. The rest of the proof is similar to the one for lemma 3.2. \( \square \)

**Remark 4.** For the case \( \Psi(x) = 0 \), the boundedness of the system with Hamiltonian \( I_3 \) can be obtained by a standard method, that is, by solving a series of homological equations on generating functions, see [25]. At the first sight, it seems plausible that this method is still valid if \( \Psi(x) \neq 0 \). However, it is not true, due to the fact that the perturbation of \( I_3 \) does not satisfy the polynomial-like growth condition.

To show this, let us study a simple but similar case. Consider the following Hamiltonian
\[ H(h, t, \theta) = \alpha(h, t) + P(h, t, \theta), \]
where \( \alpha \) is defined as above satisfying (21) and \( P \) satisfies
\[
|\partial^k_{h_3} \partial^j_{t_3} P| \leq Ch_3^{-\frac{1}{2} + \frac{1}{2} (\max\{1, j\} - 1)}.
\]
Suppose \( \Phi_0 \) is a canonical transformation of the form:
\[
\Phi_0 : \begin{cases} 
  h &= \rho + \frac{\partial S_0}{\partial \rho} \\
  \tau &= t + \frac{\partial S_0}{\partial \tau},
\end{cases}
\]
with the generating function \( S_0(\rho, t, \theta) \) satisfying \( \|P(\rho, t) - P(\rho, t, \theta) + \partial_0 S_0 = 0 \). Thus \( S_0 \) satisfies the similar inequalities as \( P \) does. But in the new Hamiltonian expressed in \( (\rho, \tau, t) \), there is a term of the form \( \tilde{P} = \frac{\partial f_{\rho}}{\partial \rho}(h, \tau) \cdot \frac{\partial S_0}{\partial \tau} \). From (18) and estimates on \( S_0 \), we can only have \( \tilde{P} = O(1) \) and no better estimates, say \( \tilde{P} = O(h^{-\frac{1}{2}}) \) as we expect, can be obtained. Thus the new perturbation may be of the same order as the old one!
Lemma 4.1. For \( g(x) \) itself is enough for us to obtain the boundedness with a higher smoothness condition reduce the smooth requirement of the original system. In other words, Proposition 1 and that of Lemma 4.1 is similar except that the latter can help us to using the theory of oscillatory integral \([20]\). It is worthy to note that the function of Section 4.

4. The oscillating terms.

4.1. A canonical transformation for \( \Psi(x) \). In this subsection, we will make a transformation to deal with \( \Psi(x) \). Recall all the transformations we have done before this section:

\[
(x, y, t) \rightarrow (I, \theta, t), \text{ where } x = \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta, \ y = \sqrt{\frac{2}{n}} I^\frac{1}{2} \sin n\theta;
\]

\[
(I, \theta, t) \rightarrow (h, t, \theta), \text{ where } I = h - R(h, t, \theta);
\]

and then

\[
(h, t, \theta) = \Phi_1(h_1, t_1, \theta), \ (h_1, t_1, \theta) = \Phi_2(h_2, t_2, \theta), \ (h_2, t_2, \theta) = \Phi_3(h_3, t_3, \theta).
\]

For convenience, we denote \( \tilde{f}_3(h_3, t_3, \theta) \). The following proposition shows that the average \( \tilde{f}_3(h_3, t_3) \) possesses an estimate better than the one for \( f_3 \) (or \( f_3 \) itself, which is important for us to obtain the boundedness.

**Proposition 1.** For \( h_3 \) large enough, \( \theta, t_3 \in S^1 \) and any \( \epsilon > 0 \), it holds that

\[
|\tilde{f}_3(h_3, t_3)| = \left| \int_0^{2\pi} f_3(h_3 + \partial_{t_3} S_3, \theta) d\theta \right| \leq C h_3^{-\frac{1}{2} + \epsilon};
\]

moreover,

\[
|\partial_{h_3}^k \partial_{t_3}^l \tilde{f}_3(h_3, t_3)| \leq C h_3^{-\frac{1}{2} + \epsilon - \frac{k}{2}}, \ l \leq m_2 - 1, \ k \leq m_1.
\]

**Proof.** The proposition is a special case of Lemma 2.7 in [24]. \( \square \)

**Remark 5.** In Lemma 4.1 we will prove a better result than Proposition 1 by using the theory of oscillatory integral [20]. It is worthy to note that the function of Proposition 1 and that of Lemma 4.1 is similar except that the latter can help us to reduce the smooth requirement of the original system. In other words, Proposition 1 itself is enough for us to obtain the boundedness with a higher smoothness condition on \( g(x) \), \( \psi(x) \), \( p(t) \).

**Lemma 4.1.** For \( h_3 \) large enough, \( \theta, t_3 \in S^1 \), it holds that

\[
|\tilde{f}_3(h_3, t_3)| = \left| \int_0^{2\pi} f_3(h_3 + \partial_{t_3} S_3, \theta) d\theta \right| \leq C h_3^{-\frac{1}{4}};
\]

moreover,

\[
|\partial_{h_3}^k \partial_{t_3}^l \tilde{f}_3(h_3, t_3)| \leq C h_3^{-\frac{1}{4} - \frac{k}{2}}, \ l \leq m_2 - 1, \ k \leq m_1.
\]

The proof is involved, hence we give it in the Appendix.

Now we make a transformation \( \Phi_4 : (h_4, t_4, \theta) \rightarrow (h_3, t_3, \theta) \) implicitly given by

\[
\begin{align*}
\frac{h_3}{h_4} &= h_4 + \partial_{t_3} S_4(h_4, t_3, \theta) \\
\frac{t_4}{t_3} &= t_3 + \partial_{h_4} S_4(h_4, t_3, \theta)
\end{align*}
\]

with the generating function \( S_4(h_4, t_3, \theta) \) determined by

\[
S_4(h_4, t_3, \theta) = \int_0^\theta (\tilde{f}_3(h_4, t_3, \theta) - [\tilde{f}_3](h_4, t_3)) d\theta.
\]
Under $\Phi_4$, the Hamiltonian $I_3$ is transformed into

\[ I_3(h_4, t_4, \theta) = \alpha(h_4 + \partial_{t_3} S_4(t_4 - \partial_{h_4} S_4)) - \int_3(h_4 + \partial_{t_3} S_4(t_3, \theta)) + R_{31}(h_4 + \partial_{t_3} S_4(t_4 - \partial_{h_4} S_4, \theta)) + \partial_{h_4} S_4 \]

\[ = \alpha(h_4, t_4) - \int_3(h_4, t_4) + R_{41}(h_4, t_4, \theta) + R_{42}(h_4, t_4, \theta) + R_{43}(h_4, t_4, \theta), \]

where

\[ R_{41}(h_4, t_4, \theta) = R_{31}(h_4, t_4, \theta); \]

\[ R_{42}(h_4, t_4, \theta) = \int_0^1 \partial_1 \alpha(h_4 + \mu \partial_{t_3} S_4(t_3)) \partial_{t_3} S_4(h_4, t_3, \theta) d\mu \]

\[ - \int_0^1 \int_0^1 \partial_2 \alpha(h_4, t_4 - s \mu \partial_{t_3} S_4) \mu (\partial_{t_3} S_4)^2 d s d\mu \]

\[ - \int_0^1 \partial_{h_4} \int_3(h_4 + \mu \partial_{t_3} S_4(t_3, \theta)) \partial_{t_3} S_4(h_4, t_3, \theta) d\mu \]

\[ + \int_0^1 \partial_{t_3} \left[ \int_3(h_4, t_4 - \mu \partial_{t_3} S_4) \partial_{h_4} S_4 d\mu \right] \]

\[ + \int_0^1 \partial_{h_4} R_{31}(h_4 + \mu \partial_{t_3} S_4(t_3, \theta)) \partial_{t_3} S_4(h_4, t_3, \theta) d\mu \]

\[ - \int_0^1 \partial_{t_3} R_{31}(h_4, t_4 - \mu \partial_{t_3} S_4(h_4, t_3, \theta)) d\mu \]

\[ + R_{43}(h_4 + \partial_{t_3} S_4(t_4 - \partial_{h_4} S_4), \theta); \]

\[ R_{43}(h_4, t_4, \theta) = -\partial_1 \alpha(h_4, t_4) \cdot \partial_{h_4} S_4. \quad (25) \]

Then we have the following estimates.

**Lemma 4.2.** For $h_4$ large enough, $\theta$, $t_4 \in S^1$ and $l \leq m_2 - 1$, it holds that

\[ \left| \partial_{h_4}^k \partial_{t_3}^l S_4(h_4, t_3, \theta) \right| \leq Ch_4^{-\frac{1}{2} + \frac{j}{4}}, \quad k + j \leq m_1; \]

\[ \left| \partial_{h_4}^k \partial_{t_3}^l \partial_{\theta}^j S_4(h_4, t_3, \theta) \right| \leq Ch_4^{-\frac{1}{2} + \frac{j}{4} + \frac{j}{2}}, \quad j \geq 1, \quad k + j \leq m_1; \]

and for $l \leq m_2 - 2$,

\[ \left| \partial_{h_4}^k \partial_{t_3}^l \partial_{\theta}^j R_{41} \right| \leq Ch_4^{-\frac{1}{2} + \frac{j}{4}(\max(1, j) - 1)}, \quad k + j \leq m_1 - 1; \]

\[ \left| \partial_{h_4}^k \partial_{t_3}^l \partial_{\theta}^j R_{42} \right| \leq Ch_4^{-\frac{1}{2} + \frac{j}{4}(\max(1, j) - 1)}, \quad k + j \leq m_1 - 2; \]

\[ \left| \partial_{h_4}^k \partial_{t_3}^l R_{43} \right| \leq Ch_4^{-\frac{1}{2} + \frac{j}{4}}, \quad \left| \partial_{h_4}^k \partial_{t_3}^l \partial_{\theta}^j R_{43} \right| \leq Ch_4^{-\frac{1}{2} + \frac{j}{4} + \frac{j}{2}}, \quad k + j \leq m_1 - 2. \]

**Proof.** The estimates on $R_{41}$, $R_{42}$ are similar to those in Lemma 3.2. Note that

\[ |S_4(h_4, t_3, \theta)| = \left| \int_0^\theta \left( \int_3(h_4, t_3, \theta) - \int_3(h_4, t_3, \theta) \right) d\theta \right|, \]

and

\[ |\partial_{h_4} S_4(h_4, t_3, \theta)| \leq C|\partial_\theta \int_3(h_4, t_3, \theta)|, \]

thus the estimates on $S_4$ are obtained from Lemma 4.1. Finally, the estimates on $R_{43}$ can be obtained directly from (25). \qed
The oscillating terms in $I_4$ include $-\tilde{f}_3$, $R_{42}$ and $R_{43}$ while the “worst” term among them is $R_{43}$. For simplicity, without causing confusion, we still denote the sum $-\tilde{f}_3 + R_{42} + R_{43}$ by $R_{43}$, i.e.

$$I_4(h_4, t_4, \theta) = \alpha(h_4, t_4) + R_{41}(h_4, t_4, \theta) + R_{43}(h_4, t_4, \theta).$$

4.2. A canonical transformation for the perturbation $R_{41}$. Before dealing with the oscillating term $R_{43}$, we first reduce the non-oscillating term $R_{41}$ to be small enough by a canonical transformation.

Let $\Phi_5 : (h_5, t_5, \theta) \to (h_4, t_4, \theta)$ be implicitly given by

$$\begin{aligned}
&\begin{cases}
  h_4 = h_5 + \partial_t S_5(h_5, t_4, \theta) \\
  t_5 = t_4 + \partial_h S_5(h_5, t_4, \theta)
\end{cases}
\end{aligned}$$

with the generating function $S_5(h_5, t_4, \theta)$ determined by

$$S_5(h_5, t_4, \theta) = -\int_0^\theta (R_{41}(h_5, t_4, \theta) - [R_{41}](h_5, t_4)) d\theta.$$ 

Under $\Phi_5$, the Hamiltonian $I_4$ is transformed into

$$I_5(h_5, t_5, \theta) = \alpha(h_5 + \partial_t S_5, t_5 - \partial_h S_5) + R_{43}(h_5 + \partial_t S_5, t_5 - \partial_h S_5, \theta) + R_{41}(h_5 + \partial_t S_5, t_4, \theta) + \partial_h S_5$$

$$= \alpha(h_5, t_5) + [R_{41}](h_5, t_5) + R_5(h_5, t_5, \theta), \quad (26)$$

where

$$R_5(h_5, t_5, \theta) \overset{\text{def}}{=} \int_0^1 \partial_t \alpha(h_5 + \mu \partial_t S_5, t_4) \cdot \partial_t S_5 d\mu - \int_0^1 \partial_t \alpha(h_5, t_5 - \mu \partial_h S_5) \cdot \partial_h S_5 d\mu + \int_0^1 \partial_t [R_{41}](h_5, t_5 - \mu \partial_h S_5) \cdot \partial_h S_5 d\mu + R_{43}(h_5 + \partial_t S_5, t_5 - \partial_h S_5, \theta).$$

**Lemma 4.3.** Assume $h_5$ large enough, $\theta$, $t_5 \in S^1$. Then it holds that for $k + j \leq m_1 - 1$, $l \leq m_2 - 2$,

$$\left| \partial^k_{h_5} \partial^j_t \partial^l_{\theta} S_5(h_5, t_4, \theta) \right| \leq Ch_5^{-k + \frac{j}{2}(\max(2, j) - 2)},$$

and for $k + j \leq m_1 - 1$, $l \leq m - 2$,

$$\left| \partial^k_{h_5} \partial^l_{\theta} [R_{41}](h_5, t_5) \right| \leq Ch_5^{-k},$$

$$\left| \partial^k_{h_5} \partial^j_t R_5 \right| \leq Ch_5^{-\frac{j}{2} - \frac{k}{2}},$$

$$\left| \partial^k_{h_5} \partial^l_{\theta} \partial^j_t R_5 \right| \leq Ch_5^{-\frac{j}{2} + \frac{k}{2}}, \quad j \geq 1.$$ 

**Proof.** Following Lemma 4.2 and similar to the proof of Lemma 3.2, the estimates are obtained by a direct computation. \qed

Without causing confusion, $\alpha(h_5, t_5) + [R_{41}](h_5, t_5)$ is still denoted by $\alpha(h_5, t_5)$, therefore (26) is rewritten as

$$I_5(h_5, t_5, \theta) = \alpha(h_5, t_5) + R_5(h_5, t_5, \theta) \quad (27)$$

with

$$ch_5^{\frac{j}{2} - k} \leq \left| \partial^k_{h_5} \alpha(h_5, t_5) \right| \leq Ch_5^{\frac{j}{2} - k}, \quad k = 0, 1, 2, \quad (28)$$

$$\left| \partial^k_{h_5} \alpha(h_5, t_5) \right| \leq Ch_5^{\frac{j}{2} - k}, \quad k = 0, 1, 2, \quad (28)$$
Then there exists a transformation $\Phi_{6,\nu} : (h_6, t_6, \theta) \to (h_5, t_5, \theta)$, such that
\begin{equation}
I_6(h_6, t_6, \theta) = I_5 \circ \Phi_{6,\nu}(h_6, t_6, \theta) = \alpha(h_6, t_6) + R_6(h_6, t_6, \theta),
\end{equation}
and for $h_6$ large enough, $\theta$, $t_6 \in S^1$, $l \leq m_2 - \nu - 3$, $k + j \leq m_1 - \nu - 2$, it holds that
\begin{equation}
\left| \partial_{h_6}^k \partial_{t_6}^l \alpha(h_5, t_5) \right| \leq Ch_5^{-\frac{1}{2} - \frac{k}{2}}, \quad k \leq m_1 - 1, \quad l \leq m_2 - 2.
\end{equation}

### 4.3. Improvement of estimates on derivatives of the oscillating terms.

We improve the estimates on the oscillating term $R_5$ in this subsection. It will help us to obtain a nearly integrable superlinear system in Subsection 4.5.

**Lemma 4.4.** Given $\nu \in \mathbb{Z}^+$, there exists a transformation $\Phi_{6,\nu} : (h_6, t_6, \theta) \to (h_5, t_5, \theta)$, such that
\begin{equation}
I_6(h_6, t_6, \theta) = I_5 \circ \Phi_{6,\nu}(h_6, t_6, \theta) = \alpha(h_6, t_6) + R_6(h_6, t_6, \theta),
\end{equation}
and for $h_6$ large enough, $\theta$, $t_6 \in S^1$, $l \leq m_2 - \nu - 3$, $k + j \leq m_1 - \nu - 2$, it holds that
\begin{equation}
\left| \partial_{h_6}^k \partial_{t_6}^l \partial_{\theta}^j R_6 \right| \leq Ch_6^{-\frac{1}{4} - \frac{k}{2}}, \quad j = 0, 1, \ldots, \nu.
\end{equation}

**Remark 6.** Lemma 4.4 shows that with the cost of reducing the smoothness on $t$, the smoothness of the perturbation on $\theta$ and the corresponding estimates can be improved.

To prove Lemma 4.4, we give the following iteration lemma firstly.

**Lemma 4.5.** Assume
\begin{equation}
I = \alpha(h, t) + R(h, t, \theta)
\end{equation}
with $\alpha$ defined in (27) and $R(h, t, \theta)$ satisfying that for $h$ large enough, $\theta$, $t \in S^1$, $l \leq m_2 - i - 3$, $k + j \leq m_1 - i - 2$,
\begin{equation}
\left| \partial_{h}^k \partial_{t}^l \partial_{\theta}^j R \right| \leq Ch^{-\frac{1}{4} - \frac{k}{2}}, \quad j = 0, 1, \ldots, i.
\end{equation}
Then there exists a transformation $\Phi_+ : (h_+, t_+, \theta) \to (h, t, \theta)$, such that
\begin{equation}
I_+ (h_+, t_+, \theta) = I \circ \Phi_+(h_+, t_+, \theta) = \alpha(h_+, t_+) + R_+(h_+, t_+, \theta).
\end{equation}
Moreover for $h_+ \gg 1$, $\theta$, $t_+ \in S^1$, $l \leq m_2 - (i + 1) - 3$, $k + j \leq m_1 - (i + 1) - 2$, it holds that
\begin{equation}
\left| \partial_{h_+}^k \partial_{t_+}^l \partial_{\theta}^j R_+ \right| \leq Ch_+^{-\frac{1}{4} - \frac{k}{2}}, \quad j = 0, 1, \ldots, i + 1.
\end{equation}

**Proof.** Set $\Phi_+ : (h_+, t_+, \theta) \to (h, t, \theta)$ implicitly given by
\begin{equation}
\begin{cases}
h = h_+ + \partial_t S_+(h_+, t, \theta) \\
t_+ = t + \partial_{h_+} S_+(h_+, t, \theta)
\end{cases}
\end{equation}
with the generating function $S_+(h_+, t, \theta)$ determined by
\begin{equation}
S_+(h_+, t, \theta) = - \int_0^\theta \left( R(h_+, t, \theta) - [R](h_+, t) \right) d\theta.
\end{equation}
It is easy to show that, for $k + j \leq m_1 - (i + 1) - 2$, $l \leq m_2 - (i + 1) - 3$,
\begin{equation}
\left| \partial_{h_+}^k \partial_{t_+}^l \partial_{\theta}^j S_+(h_+, t, \theta) \right| \leq Ch_+^{-\frac{1}{4} - \frac{k}{2}}, \quad j = 0, 1, \ldots, i + 1,
\end{equation}
which means that the smoothness of $S_+(h_+, t, \theta)$ depending on $\theta$ is better than $R$.

Under $\Phi_+$, the Hamiltonian $I$ is transformed into
\begin{align}
I_+(h_+, t_+, \theta) &= \alpha(h_+ + \partial_t S_+, t_+ - \partial_{h_+} S_+) + R(h_+ + \partial_t S_+, t, \theta) + \partial_{\theta} S_+ \\
&= \alpha(h_+, t_+) + R_+(h_+, t_+, \theta),
\end{align}
where $R_+$ determines the estimate of $R_+$. Therefore, $I_+$ is also superlinear.
where
\[
R_+(h_+, t_+, \theta)
\]
\[
= \left[ R \right](h_+, t_+)
\]
\[
+ \int_0^1 \partial_t \alpha(h_+ + \mu \partial_t S_+, t) \partial_t S_+ \, d\mu - \int_0^1 \partial_t \alpha(h_+, t_+ - \mu \partial_{h_+} S_+) \partial_{h_+} S_+ \, d\mu
\]
\[
+ \int_0^1 \partial_t [R](h_+, t_+ - \mu \partial_{h_+} S_+) \partial_{h_+} S_+ \, d\mu + \int_0^1 \partial_h R(h_+ + \mu \partial_t S_+, t, \theta) \partial_t S_+ \, d\mu.
\]
Note that the worst term in \( R_+ \) is \( \int_0^1 \partial_t \beta(h_+, t_+ - \mu \partial_{h_+} S_+) \partial_{h_+} S_+ \, d\mu \), therefore the estimates are calculated directly.

**Proof of Lemma 4.4.** The proof of Lemma 4.4 is completed by repeatedly using Lemma 4.5 \( \nu \) times and \( \Phi_{h, \nu} : (h_6, t_6) \rightarrow (h, t) \) is the composition of \( \nu \) corresponding transformations.

\section*{4.4. Exchange the roles of \((h_6, t_6)\) and \((I_6, \theta)\).}

With Lemma 4.4, we have better estimates about derivatives of the new perturbation with respect to \( \theta \) than those for the old one. Thus the method of exchanging the roles of angle and time will work now.

Consider the Hamiltonian (30). Assume \( h_6 = N(\rho, t_6) \) be the inverse function of \( \rho = \alpha(h_6, t_6) \) with respect to the variable \( \rho \). Noting that \( \partial_{h_6} I_6 > ch_6^{-\frac{3}{2}} > 0 \) as \( h_6 \to \infty \), for large \( h_6 \) we can solve (30) for it as the following form:
\[
h_6(I_6, \theta, t_6) = N(I_6, t_6) + P(I_6, \theta, t_6).
\]
(33)

With (30) and (33), we have
\[
I_6 = \alpha(N + P, t_6) + R_6(N + P, t_6, \theta)
\]
\[
= \alpha(N, t_6) + \partial_{h_6} \alpha(N, t_6) P + \int_0^1 \int_0^1 \partial_{h_6}^2 \alpha(N + s \mu P, t_6) \mu P^2 dsd\mu
\]
\[
+ R_6(N + P, t_6, \theta).
\]
(34)

Note that \( I_6 = \alpha(N, t_6) \), then
\[
0 = \partial_{h_6} \alpha(N, t_6) P + \int_0^1 \int_0^1 \partial_{h_6}^2 \alpha(N + s \mu P, t_6) \mu P^2 dsd\mu + R_6(N + P, t_6, \theta).
\]
Implicitly,
\[
P = -\frac{1}{\partial_{h_6} \alpha(N, t_6)} \{ R_6(N + P, t_6, \theta) + \int_0^1 \int_0^1 \partial_{h_6}^2 \alpha(N + s \mu P, t_6) \mu P^2 dsd\mu \}.
\]
(35)

In the following, we give the estimates on \( N(I_6, t_6) \) and \( P(I_6, \theta, t_6) \).

**Lemma 4.6.** For \( I_6 \) large enough, \( \theta, \ t_6 \in S^1 \) and \( \nu \) defined as in Lemma 4.4, it holds that
\[
\left| \partial_{t_6}^k N(I_6, t_6) \right| \leq CI_6^{2-k}, \quad k = 0, 1, 2;
\]
(36)
\[
\left| \partial_{h_6}^k \partial_{t_6} I_6 N(I_6, t_6) \right| \leq CI_6^{2-k}, \quad k \leq m_1 - 1, \quad l \leq m_2 - 2;
\]
(37)
and
\[
\left| \partial_{t_6}^k \partial_{\theta}^j \partial_{t_6}^l P(I_6, \theta, t_6) \right| \leq CI_6^{\frac{1}{2}}, \quad k + j \leq m_1 - \nu - 2, \quad j \leq \nu, \quad l \leq m_2 - \nu - 3.
\]
(38)
Proof. Although the proof is similar to the one of Lemma 2.5, the details are different. In Lemma 2.5, the polynomial-like growth condition is available, while in this lemma it is not the case. We show a complete proof as follows. (i) Firstly, we estimate \( N(I_6, t_6) \). Note that \( \alpha(N(I_6, t_6), t_6) \equiv I_6 \), then
\[
cl^2_6 \leq |N| \leq CI^2_6,
\]
and
\[
\partial_{h_6} \alpha \cdot \partial_{t_6} N = 1, \quad \partial_{h_6} \alpha \cdot \partial_{t_6} N + \partial_{t_6} \alpha = 0.
\]
Thus from (28) and (29), it follows that
\[
cI_6 \leq \left| \partial_{t_6} N \right| \leq CI^2_6, \quad \left| \partial_{t_6} N \right| \leq CI^2_6.
\]
Generally, for \( 2 \leq k + j \leq m_1 - 1 \) and \( l \leq m_2 - 2 \),
\[
\partial_{t_6}^k \partial_{t_6}^l \alpha(N(I_6, t_6), t_6) = 0.
\]
Using Leibniz’s rule, \( \partial_{t_6}^k \partial_{t_6}^l \alpha(N(I_6, t_6), t_6) \) is the sum of terms
\[
(\partial_{h_6}^u \partial_{t_6}^v \alpha(\alpha(N(I_6, t_6), t_6), t_6)) \Pi_{i=1}^u \partial_{t_6}^{k_i} \partial_{t_6}^{l_i} N
\]
with \( 1 \leq u + v \leq k + l \), \( \sum_{i=1}^u k_i = k, v + \sum_{i=1}^u l_i = l \), and \( k_i + l_i \geq 1, \ i = 1, \ldots, u \).
Following (28) and (29), (36) and (37) are obtained inductively.

(ii) Secondly, from (35) we obtain \( |P| \leq CI^2_6 \) and
\[
-\partial_{h_6} \alpha(N(t_6), t_6) \cdot P = R_6(N + P, t_6, \theta) + \int_0^1 \int_0^1 \partial_{h_6}^2 \alpha(N + s\mu P, t_6) \mu P^2 ds d\mu. \tag{39}
\]
Suppose
\[
\left| \partial_{t_6}^k \partial_{t_6}^l \alpha(N(t_6), t_6) \right| \leq CI^2_6 \tag{40}
\]
hold for \( k + j + l < m \), \( k + j \leq m_1 - \nu - 2 \), \( l \leq m_2 - \nu - 3 \).
When \( k + j + l = m \), \( k + j \leq m_1 - \nu - 2 \), \( l \leq m_2 - \nu - 3 \), consider the left hand side of (39). We claim that
\[
\left| \partial_{t_6}^k \partial_{t_6}^l \alpha(N(t_6), t_6) \right| \leq CI_6^{-1-k}. \tag{41}
\]
In fact, by Leibniz’s rule, \( \partial_{t_6}^k \partial_{t_6}^l \alpha(N(t_6), t_6) \) is the sum of terms
\[
(\partial_{h_6}^{u+1} \partial_{t_6}^v \alpha(\alpha(N(t_6), t_6), t_6)) \Pi_{i=1}^u \partial_{t_6}^{k_i} \partial_{t_6}^{l_i} N,
\]
with \( 1 \leq u + v \leq k + l \), \( \sum_{i=1}^u k_i = k, v + \sum_{i=1}^u l_i = l \), and \( k_i + l_i \geq 1, \ i = 1, \ldots, u \).
Then,
\[
\left| \partial_{t_6}^k \partial_{t_6}^l (\partial_{h_6} \alpha(N(t_6), t_6)) \right| \leq CI_6^{-1-2u+2u-k} \leq CI_6^{-1-k}.
\]
Thus, differentiating the left hand side of (39), we have
\[
\partial_{t_6}^k \partial_{t_6}^l \partial_{h_6} \alpha(N(t_6), t_6) \cdot P = \partial_{h_6} \alpha(N(t_6), t_6) \cdot \partial_{t_6}^k \partial_{t_6}^l \partial_{t_6} \partial_{t_6} P + \bar{P}. \tag{42}
\]
where \( \bar{P} \) is the sum of terms
\[
\partial_{t_6}^k \partial_{t_6}^l (\partial_{h_6} \alpha(N(t_6), t_6)) \cdot \partial_{t_6}^{k_2} \partial_{t_6}^{l_2} P
\]
with \( k_1 + k_2 = k, l_1 + l_2 = l, k_2 + l_2 + j < m \). From (40), (41), it holds that \( |\bar{P}| \leq I^{-\frac{3}{2}} \).
Now consider the right hand side of (39),
\[
\partial_{t_6}^k \partial_{t_6}^l \partial_{t_6} R_6(N + P, t_6, \theta)
\]
is the sum of terms
\[ \partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r R_6(h_0, t_0, \theta) \Pi_{i=1}^{p} \partial_{t_i}^{k_i} \partial_{\theta}^{j_i} \partial_{t_i}^{l_i} (N + P), \]
with \( 1 \leq p + q + r \leq k + j + l \), \( \sum_{i=1}^{p} k_i = k, q + \sum_{i=1}^{p} j_i = j, r + \sum_{i=1}^{p} l_i = l \), and \( k_i + j_i + l_i \geq 1, i = 1, \ldots, p \).

From (31), it is easy to show that
\[ |\partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r R_6(h_0, t_0, \theta)| \leq C h_6^{-\frac{1}{2} - \frac{r}{2}} \sim CI_6^{-\frac{1}{2} - p}, \]
which, combining with (37), implies that
\[ |\partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r R_6(h_0, t_0, \theta) \Pi_{i=1}^{p} \partial_{t_i}^{k_i} \partial_{\theta}^{j_i} \partial_{t_i}^{l_i} N| \leq CI_6^{-\frac{1}{2} - p + 2p - k} \leq CI_6^{-\frac{1}{2}}. \]

Finally, consider
\[ \begin{align*}
\sum_{k_i + j_i + l_i \leq k+j+l} \partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r R_6 \Pi_{i=1}^{p} \partial_{t_i}^{k_i} \partial_{\theta}^{j_i} \partial_{t_i}^{l_i} P &= \partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r R_6 \Pi_{i=1}^{p} \partial_{t_i}^{k_i} \partial_{\theta}^{j_i} \partial_{t_i}^{l_i} P + \sum_{k_i + j_i + l_i \leq k+j+l} \partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r R_6 \Pi_{i=1}^{p} \partial_{t_i}^{k_i} \partial_{\theta}^{j_i} \partial_{t_i}^{l_i} P.
\end{align*} \]
From the assumption, it follows that
\[ \sum_{k_i + j_i + l_i \leq k+j+l} \partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r R_6 \Pi_{i=1}^{p} \partial_{t_i}^{k_i} \partial_{\theta}^{j_i} \partial_{t_i}^{l_i} P \leq CI_6^{-\frac{1}{2} - p + \frac{r}{2}} \leq CI_6^{-\frac{1}{2}}. \]

Hence
\[ \partial_{t_0}^p \partial_{\theta}^q \partial_{t_0}^r R_6(N + P, t_0, \theta) = \partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r P + \tilde{P} \]
with \( |\tilde{P}| \leq CI_6^{-\frac{1}{2}}. \)

Finally, consider
\[ \partial_{h_0}^p \partial_{\theta}^q \partial_{t_0}^r (\partial_{h_0}^2 \alpha(N + s \mu P, t_0) \mu P^2). \]
By the same method, we have
\[ \begin{align*}
\partial_{h_0}^2 \alpha(N + s \mu P, t_0) \mu P^2 &= (\partial_{h_0}^2 \alpha(N + s \mu P, t_0) \mu P^2 + 2 \partial_{h_0}^2 \alpha(N + s \mu P, t_0) \mu P) \cdot \partial_{t_0}^k \partial_{\theta}^q \partial_{t_0}^r P + \tilde{P}
\end{align*} \]
with \( |\tilde{P}| \leq CI_6^{-2}. \)
With (42), (43) and (44), by induction, we get (38).

4.5. A nearly integrable system. For convenience, we redefine the variables as \((I_0, \theta, t_0, h_0) \rightarrow (\rho, \theta, \tau, h)\), and (33) is rewritten by
\[ h(\rho, \theta, \tau) = N(\rho, \tau) + P(\rho, \theta, \tau). \]
Inductively, consider the Hamiltonian
\[ h_s(\rho_s, \theta_s, \tau) = N(\rho_s, \tau) + M_s(\rho_s, \tau) + P_s(\rho_s, \theta_s, \tau), \quad s = 0, 1, 2, \ldots \]
with
\[ (\rho_0, \theta_0) = (\rho, \theta), \quad M_0 = 0, \quad P_0 = P; \]
and \( M_s(\rho_s, \tau), P_s(\rho_s, \theta_s, \tau) \) satisfying
\[ |\partial_{\rho_s}^l \partial_{\theta_s}^m M_s(\rho_s, \tau)| \leq C \rho_s^{-\frac{1}{2}}, \quad k + j \leq m_1 - \nu - s - 1, \quad l \leq m_2 - \nu - s - 2; \]
We omit it.

\[ \left| \partial^j \rho \partial^k \theta \partial^l P_s(\rho, \theta, \tau) \right| \leq C \rho_\delta^{2^{-\frac{j}{2}}} s, \ j \leq \nu, \ k + j \leq m_1 - \nu - s - 2, \]  

(47)

\( l \leq m_2 - \nu - s - 3, \) for \( \rho_s \) large enough. Thus we have

**Lemma 4.7.** Suppose the Hamiltonian \( h_s \) with \( s = \kappa - 1 \) satisfies (46), (47). Then there exists a canonical transformation \( \Psi_s : (\rho_s, \theta_s, \tau) \to (\rho_{s-1}, \theta_{s-1}, \tau) \) such that the new Hamiltonian \( h_{s-1} \) with \( s = \kappa \) satisfies (46), (47).

**Proof.** Suppose \( h_s \) with (46), (47) holds for \( s = \kappa - 1 \) (case \( \kappa = 1 \) is already satisfied by (38)). Let \( \Psi_s : (\rho_s, \theta_s, \tau) \to (\rho_{s-1}, \theta_{s-1}, \tau) \) be defined implicitly by

\[ \Psi_s : \begin{cases} 
\rho_{s-1} = \rho_s + \theta_{s-1} Q_s(\rho_s, \theta_{s-1}, \tau) \\
\theta_{s-1} = \theta_s + \partial_{\rho_s} Q_s(\rho_s, \theta_{s-1}, \tau)
\end{cases} \]  

(48)

with the generating function \( Q_s(\rho_s, \theta_{s-1}, \tau) \) given by

\[ Q_s(\rho_s, \theta_{s-1}, \tau) = -\int_0^{\theta_{s-1}} \frac{1}{\partial_{\rho_s} N(\rho_s, \tau)} (P_{s-1}(\rho_s, \theta_{s-1}, \tau) - [P_{s-1}]_{\rho_s}(\rho_s, \tau)) d\theta_{s-1}. \]  

(49)

Under \( \Psi_s \), the Hamiltonian \( h_{s-1} \) is transformed into

\[ h_s(\rho_s, \theta_s, \tau) = N(\rho_s + \theta_{s-1} Q_s(\rho_s, \theta_{s-1}, \tau) + M_{s-1}(\rho_s, \theta_{s-1}, \tau) + \partial_{\rho_s} Q_s(\rho_s, \theta_{s-1}, \tau) + P_{s-1}(\rho_s, \theta_{s-1}, \tau) + P_s(\rho_s, \theta_s, \tau), \]  

where \( M_s(\rho_s, \tau) = M_{s-1}(\rho_s, \tau) + [P_{s-1}]_{\rho_s}(\rho_s, \tau) \) and

\[ P_s(\rho_s, \theta_s, \tau) = \int_0^{1} \partial_{\rho_{s-1}}^l M_{s-1}(\rho_s + \theta_{s-1} Q_s(\rho_s, \theta_{s-1}, \tau) + \mu \partial_{\theta_{s-1}} \theta_{s-1} Q_s(\rho_s, \theta_{s-1}, \tau)) d\mu \]

\[ + \int_0^{1} \int_0^{1} \partial_{\rho_{s-1}}^l N(\rho_s + \mu \partial_{\theta_{s-1}} \theta_{s-1} Q_s(\rho_s, \theta_{s-1}, \tau)) \mu \partial_{\theta_{s-1}}^2 Q_s(\rho_s, \theta_{s-1}, \tau) d\mu \]  

From (36), (47), (49), it follows that

\[ \left| \partial^j \rho \partial^k \theta \partial^l Q_s(\rho_s, \theta_{s-1}, \tau) \right| \leq C \rho_s^{2^{-\frac{j}{2}}}, \ j \leq \nu, \ k + j \leq m_1 - \nu - \kappa - 1, \ l \leq m_2 - \nu - \kappa - 2 \]  

for \( \rho_s \) large enough.

Finally, the estimates on \( M_s, P_s \) are similar to those in the proof of lemma 3.2. We omit it.

Now, under a series of transformations as \( \Psi_1, \Psi_2, \ldots, \Psi_s, \) \( h_s \) is transformed into \( h_{s-1} \) with

\[ h_{s-1}(\rho_s, \theta_s, \tau) = N(\rho_s, \tau) + M_{s-1}(\rho_s, \tau) + P_{s-1}(\rho_s, \theta_{s-1}, \tau) + P_s(\rho_s, \theta_s, \tau) \]  

satisfying (46) and (47) with \( s = \kappa. \)

From (47), we find that

\[ m_2 - \nu - \kappa \geq 3. \]  

(50)
5. Existence of invariant curves. Consider the system with Hamiltonian \( h_\kappa(\rho_\kappa, \theta_\kappa, \tau) \), that is

\[
\begin{align*}
\frac{d\theta_\kappa}{d\tau} & = \partial_{\rho_\kappa}(N + M_\kappa) + \partial_{\rho_\kappa} P_\kappa, \\
\frac{d\rho_\kappa}{d\tau} & = -\partial_{\theta_\kappa} P_\kappa.
\end{align*}
\]  

(51)

The Poincaré map \( P \) of (51) is of the form

\[
\begin{align*}
\theta_{\kappa+} &= \theta_\kappa + \gamma(\rho_\kappa) + F_1(\rho_\kappa, \theta_\kappa), \\
\rho_{\kappa+} &= \rho_\kappa + F_2(\rho_\kappa, \theta_\kappa).
\end{align*}
\]  

(52)

where \((\rho_\kappa, \theta_\kappa) = (\rho_\kappa(0), \theta_\kappa(0))\), and

\[
\gamma(\rho_\kappa) = \int_0^{2\pi} \partial_{\rho_\kappa}(N + M_\kappa)(\rho_\kappa, \tau) d\tau;
\]

\[
F_1(\rho_\kappa, \theta_\kappa) = \int_0^{2\pi} \partial_{\rho_\kappa} P_\kappa(\rho_\kappa(\tau)), \theta_\kappa(\tau), \tau) d\tau \\
+ \int_0^{2\pi} \partial_{\rho_\kappa}(N + M_\kappa)(\rho_\kappa(\tau), \tau) d\tau - \int_0^{2\pi} \partial_{\rho_\kappa}(N + M_\kappa)(\rho_\kappa, \tau) d\tau;
\]

\[
F_2(\rho_\kappa, \theta_\kappa) = -\int_0^{2\pi} \partial_{\theta_\kappa} P_\kappa(\rho_\kappa(\tau)), \theta_\kappa(\tau), \tau) d\tau.
\]

Denote \( \gamma(\rho_\kappa, \tau) = \int_0^\tau \partial_{\rho_\kappa}(N + M_\kappa)(\rho_\kappa, s) ds, \) and \( \gamma(\rho_\kappa) = \gamma(\rho_\kappa, 1) \). By direct computations, we have

**Lemma 5.1.** Given \( \rho_\kappa \) large enough and \( \theta_\kappa \in \mathbb{S}^1 \), it holds that

\[
c \rho_\kappa \leq \gamma(\rho_\kappa) \leq C \rho_\kappa, \quad \left| \gamma^{(k)}(\rho_\kappa) \right| \leq C \rho_\kappa^{\frac{k}{2}}, \quad k \leq m_1 - \nu - \kappa - 2;
\]

and for \( k + j \leq m_1 - \nu - \kappa - 3, \) \( j \leq \nu - 1, \)

\[
\left| F_1(\rho_\kappa, \theta_\kappa) \right| \leq C \rho_\kappa^{\frac{1}{2} - \frac{j}{2}}, \quad \left| F_2(\rho_\kappa, \theta_\kappa) \right| \leq C \rho_\kappa^{\frac{1}{2} - \frac{j}{2}};
\]

\[
\left| \partial^{k}_{\rho_\kappa} \partial^j_{\theta_\kappa} F_1(\rho_\kappa, \theta_\kappa) \right| \leq C \rho_\kappa^{\frac{1}{2} - \frac{3}{2} - \frac{j}{2}}, \quad \left| \partial^{k}_{\rho_\kappa} \partial^j_{\theta_\kappa} F_2(\rho_\kappa, \theta_\kappa) \right| \leq C \rho_\kappa^{\frac{1}{2} - \frac{3}{2} - \frac{j}{2}}.
\]

Moreover, the following twist condition holds true.

**Lemma 5.2.** Given \( A_1 \) large enough, there exists an interval \([A_1, A_1 + A_1^{-\frac{2}{3}}]\) such that for \( \rho_\kappa \in [A_1, A_1 + A_1^{-\frac{2}{3}}] \) and \( \theta_\kappa \in \mathbb{S}^1 \) it holds that

\[
c \leq \left| \gamma'(\rho_\kappa) \right| \leq C.
\]

**Proof.** Denote \( \gamma(\rho_\kappa) = \gamma_1(\rho_\kappa) + \gamma_2(\rho_\kappa) \) with

\[
\gamma_1(\rho_\kappa) = \int_0^{2\pi} \partial_{\rho_\kappa} N(\rho_\kappa, s) ds, \quad \gamma_2(\rho_\kappa) = \int_0^{2\pi} \partial_{\rho_\kappa} M_\kappa(\rho_\kappa, s) ds.
\]

It is easy to verify that

\[
c \leq \left| \gamma'_1(\rho_\kappa) \right| \leq C,
\]
and
\[ \left| \gamma_2^{(k)}(\rho_\kappa) \right| \leq C \rho_\kappa^{\frac{1}{2}}, \quad k \leq m_1 - \nu - \kappa - 2. \]

Note that, for \( A \) large enough,
\[ \gamma(\rho_\kappa)|_{A}^{2A} = \int_{A}^{2A} \gamma_1'(\rho_\kappa) d\rho_\kappa + \gamma_2(\rho_\kappa)|_{A}^{2A} \]
\[ \geq cA - CA^{\frac{1}{2}} \geq c_1 A. \]

By Mean Value Theorem of the integral, there exists a point \( \xi \in (A, 2A) \), such that \( \gamma'(\xi) \geq c > 0 \). Note that \( |\gamma''(\rho_\kappa)| \leq C \rho_\kappa^{\frac{3}{2}} \). Therefore, we can choose \( A_1 \) such that \( \xi \in [A_1, A_1 + A_1^{-\frac{3}{2}}] \subset [A, 2A] \). This ends the proof. \( \square \)

Finally, for \( A_1 \gg 1 \), let \( \gamma(\rho_\kappa) = (A_1) + A_1^{-1} \lambda, \lambda \in [1, 2] \). Denote \( \rho_\kappa = \rho_\kappa(\lambda), \lambda \in [1, 2] \). Then \( \rho_\kappa(\lambda) \subset [A_1, A_1 + A_1^{-\frac{3}{2}}], \lambda \in [1, 2] \), and
\[ |\rho_\kappa^{(k)}(\lambda)| \leq CA_1^{-\frac{k+1}{4}}, \quad k \leq m_1 - \nu - \kappa - 2. \]

In fact, for \( k = 1 \), \( \gamma'(\rho_\kappa)\rho_\kappa''(\lambda) = A_1^{\frac{1}{2}} - 1 \). Thus from Lemma 5.2, we have \( |\rho_\kappa'(\lambda)| \leq CA_1^{-\frac{1}{2}} \). For \( k = 2 \), \( \gamma''(\rho_\kappa)\rho_\kappa''(\lambda)^2 + \gamma'(\rho_\kappa)\rho_\kappa''(\lambda) = 0 \), then from Lemma 5.2 we have \( |\rho_\kappa''(\lambda)| \leq CA_1^{-\frac{3}{2}} \). Similarly, (53) is obtained inductively.

Denote \( \phi = \theta_\kappa \), then the map (52) is changed into
\[
\begin{cases}
\phi_+ = \phi + \gamma(A_1) + A_1^{-1} \lambda + \widetilde{F}_1(\lambda, \phi) \\
\lambda_+ = \lambda + \widetilde{F}_2(\lambda, \phi)
\end{cases}
\]
(54)

with
\[ \widetilde{F}_1(\lambda, \phi) = F_1(\rho_\kappa(\lambda), \phi), \]
\[ \widetilde{F}_2(\lambda, \phi) = A_1 \left( \gamma(\rho_\kappa(\lambda) + A_2(\rho_\kappa(\lambda), \phi)) - \gamma(\rho_\kappa(\lambda)) \right). \]

For large \( A_1 \), by a direct computation, from (53) and Lemma 5.1 we have
\[ \left| \widetilde{F}_1(\lambda, \phi) \right| \leq CA_1^{\frac{3}{2} - \frac{1}{2}}, \quad \left| \widetilde{F}_2(\lambda, \phi) \right| \leq CA_1^{\frac{3}{2} - \frac{1}{2}}, \]
and for \( k + j \leq m_1 - \nu - \kappa - 3, \ j \leq \nu - 1, \)
\[ \left| \partial_\phi^k \partial_\phi^j \widetilde{F}_1(\lambda, \phi) \right| \leq CA_1^{\frac{3}{2} - \frac{1}{2} - \frac{j+1}{2}}, \quad \left| \partial_\phi^k \partial_\phi^j \widetilde{F}_2(\lambda, \phi) \right| \leq CA_1^{\frac{3}{2} - \frac{1}{2} - \frac{j+1}{2}}. \]

**Proof of Theorem 1.1.** From (50), we have that for \( \lambda \in [1, 2] \), the map (54) is close to a small twist map in \( C^4 \) topology provided that \( m_1 \geq 18 \) and \( m_2 \geq 14 \). Moreover, it has the intersection property, thus the assumptions of Moser’s Small Twist Theorem [17, 21] are met. More precisely, given any number \( \lambda \in [1, 2] \) satisfying
\[ \left| \gamma(A_1) + A_1^{-1} \lambda - \frac{p}{q} \right| > c|q|^{-5/2} \]
for all integers \( p \) and \( q \neq 0 \), there exists a \( \mu(\varphi) \in C^{3}(\mathbb{R} \setminus 2\pi\mathbb{Z}) \) such that the curve \( \Gamma = \{ (\varphi, \mu(\varphi)) \} \) is invariant under the mapping (54). The image of a point on \( \Gamma \) is obtained by replacing \( \varphi \) by \( \varphi + \gamma(A_1) + A_1^{-1} \lambda \). Hence the system with Hamiltonian \( h_6 \) has invariant curve with frequency \( \gamma(A_1) + A_1^{-1} \lambda \). Then the system with Hamiltonian \( h_6 \) has an invariant curve with the frequency \( (\gamma(A_1) + A_1^{-1} \lambda)^{-1} \), which implies the systems \( I \) has an invariant curve with the frequency \( 1 + (\gamma(A_1) + A_1^{-1} \lambda)^{-1} \).
When $\phi$ frequency Lemma A.1. [(20) pp. 226-230, Theorem 10] on oscillatory integrals in finitely smooth topology later. Finally, we obtain that the original Hamiltonian $H$ (9) has an invariant curve with the frequency $\omega = \frac{\gamma(A_1) + A_1^{-1} \lambda}{1 + \gamma(A_1) + A_1^{-1} \lambda}$.

Note that $I \to \infty$ as $A_1 \to \infty$. It means that we have found arbitrarily large amplitude invariant cylinders in $(x, y, t)$-space, which implies the boundedness of all the solutions. Thus the proof of Theorem 1.1 is finished. \hfill $\square$

Appendix A. The theory of oscillatory integral. We need the following theorem on oscillatory integrals in finitely smooth topology later.

Lemma A.1. [(20) pp. 226-230, Theorem 10] Suppose $0 < \lambda < 1$, $0 < \mu < 1$, $\varphi(x) \in C^\nu[a, b]$, $f(x) \in C^\nu[a, b]$ satisfying

$$f'(x) = (x - a)^{\rho - 1}(b - x)^{\sigma - 1}f_1(x)$$

where $\rho \geq 1$, $\sigma \geq 1$, $f_1(x) > 0$, $x \in [a, b]$, then

$$I = \int_a^b \varphi(x)e^{\inf f(x)}(x - a)^{\lambda - 1}(b - x)^{\mu - 1}dx = B(n) - A(n),$$

where

$$A(n) = A_\nu(n) + R_\nu(n), \quad B(n) = B_\nu(n) + Q_\nu(n),$$

and

$$A_\nu(n) = \sum_{k=0}^{\nu-1} \frac{h^{(k)}(0)}{k!\rho} \Gamma\left(\frac{k + \lambda}{\rho}\right)e^{\frac{i(k+\lambda)}{\rho} n - \frac{k + \lambda}{\rho} e^{\inf f(a)}}, \quad \left|R_\nu(n)\right| \leq Cn^{-\frac{\nu}{\pi}},$$

$$B_\nu(n) = \sum_{k=0}^{\nu-1} \frac{l^{(k)}(0)}{k!\sigma} \Gamma\left(\frac{k + \mu}{\sigma}\right)e^{-\frac{i(k+\mu)}{\sigma} n - \frac{k + \mu}{\sigma} e^{\inf f(b)}}, \quad \left|Q_\nu(n)\right| \leq Cn^{-\frac{\nu}{\pi}}.$$

Remark 7. When $\lambda = \mu = 1$, then it holds that

$$c|B_{\nu-1}(n) - A_{\nu-1}(n)| < \int_a^b \varphi(x)e^{\inf f(x)}dx < C|B_{\nu-1}(n) - A_{\nu-1}(n)|.$$ 

Appendix B. Proof of Lemma 2.5. Suppose $k + j \leq m_1 + 1$ and $l \leq m_2$.

i) When $k + j + l = 0$, the conclusion follows from Lemmas 2.1, 2.3 and 2.4.

ii) When $k + j + l = 1$, define

$$g_1(h, t, \theta) = \partial_t f_1(h - R, \theta) + \partial_t f_2(h - R, t, \theta) + \partial_1 f_3(h - R, \theta);$$

$$g_2(h, t, \theta) = \partial_t f_2(h - R, t, \theta);$$

$$g_3(h, t, \theta) = \partial_\theta f_1(h - R, \theta) + \partial_\theta f_2(h - R, t, \theta) + \partial_\theta f_3(h - R, \theta);$$

$$\Delta(h, t, \theta) = 1 + \partial_t f_1(h - R, \theta) + \partial_1 f_2(h - R, t, \theta) + \partial_\theta f_3(h - R, \theta).$$

Obviously, $\Delta(h, t, \theta) \geq 1/2$ for $h \gg 1$ and

$$\Delta \cdot \partial_t R(h, t, \theta) = g_1(h, t, \theta), \quad \Delta \cdot \partial_t R(h, t, \theta) = g_2(h, t, \theta),$$

$$\Delta \cdot \partial_\theta R(h, t, \theta) = g_3(h, t, \theta).$$

(55)
From Lemmas 2.1-2.4, we obtain
\[
\frac{1}{2} |\partial_h R(h, t, \theta)| \leq |\Delta \cdot \partial_h R(h, t, \theta)|
\]
\[
= |\partial_1 f_1(h - R, \theta) + \partial_1 f_2(h - R, t, \theta) + \partial_1 f_3(h - R, \theta)|
\]
\[
\leq C(h - R)^{-\frac{1}{2}} \leq Ch^{-\frac{1}{2}},
\]
\[
\frac{1}{2} |\partial_t R(h, t, \theta)| \leq |\Delta \cdot \partial_t R(h, t, \theta)| = |\partial_t f_2(h - R, t, \theta)|
\]
\[
\leq C(h - R)^{\frac{1}{2}} \leq Ch^{\frac{1}{2}},
\]
and
\[
\frac{1}{2} |\partial_\theta R(h, t, \theta)| \leq |\Delta \cdot \partial_\theta R(h, t, \theta)|
\]
\[
= |\partial_\theta f_1(h - R, \theta) + \partial_\theta f_2(h - R, t, \theta) + \partial_\theta f_3(h - R, \theta)|
\]
\[
\leq C(h - R)^{\frac{1}{2}} \leq Ch^{\frac{1}{2}}.
\]

iii) When \(k + j + l = 2\), From i) and ii), we have
\[
|\partial_h g_1(h, t, \theta)| \leq Ch^{-1}, \quad |\partial_h g_1(h, t, \theta)| \leq Ch^{-\frac{1}{2}}, \quad |\partial_h g_1(h, t, \theta)| \leq Ch^0;
\]
\[
|\partial_h g_2(h, t, \theta)| \leq Ch^{-\frac{1}{2}}, \quad |\partial_h g_2(h, t, \theta)| \leq Ch^{\frac{1}{2}}, \quad |\partial_h g_2(h, t, \theta)| \leq Ch^2;
\]
\[
|\partial_h g_3(h, t, \theta)| \leq Ch^0, \quad |\partial_h g_3(h, t, \theta)| \leq Ch^{\frac{1}{2}}, \quad |\partial_h g_3(h, t, \theta)| \leq Ch^1;
\]
and
\[
|\partial_\theta \Delta(h, t, \theta)| \leq Ch^{-1}, \quad |\partial_t \Delta(h, t, \theta)| \leq Ch^{-\frac{1}{2}}, \quad |\partial_\theta \Delta(h, t, \theta)| \leq Ch^0.
\]

From (55), differentiating on both sides of the equations, we obtain:
\[
\Delta \cdot \partial_h^2 R(h, t, \theta) = \partial_h g_1(h, t, \theta) - \partial_h \Delta \cdot \partial_h R(h, t, \theta),
\]
\[
\Delta \cdot \partial_t^2 R(h, t, \theta) = \partial_t g_2(h, t, \theta) - \partial_t \Delta \cdot \partial_t R(h, t, \theta),
\]
\[
\Delta \cdot \partial_\theta^2 R(h, t, \theta) = \partial_\theta g_3(h, t, \theta) - \partial_\theta \Delta \cdot \partial_\theta R(h, t, \theta),
\]
\[
\Delta \cdot \partial_h \partial_t R(h, t, \theta) = \partial_{ht} g_1(h, t, \theta) - \partial_{ht} \Delta \cdot \partial_h R(h, t, \theta),
\]
\[
\Delta \cdot \partial_h \partial_\theta R(h, t, \theta) = \partial_{ht} g_1(h, t, \theta) - \partial_{ht} \Delta \cdot \partial_\theta R(h, t, \theta),
\]
\[
\Delta \cdot \partial_\theta \partial_t R(h, t, \theta) = \partial_{ht} g_1(h, t, \theta) - \partial_{ht} \Delta \cdot \partial_\theta R(h, t, \theta).
\]

It follows that
\[
\frac{1}{2} |\partial_h^2 R(h, t, \theta)| \leq \left| \partial_h g_1(h, t, \theta) \right| + \left| \partial_h \Delta \cdot \partial_h R(h, t, \theta) \right| \leq Ch^{-1},
\]
\[
\frac{1}{2} |\partial_t^2 R(h, t, \theta)| \leq \left| \partial_t g_2(h, t, \theta) \right| + \left| \partial_t \Delta \cdot \partial_t R(h, t, \theta) \right| \leq Ch^{\frac{1}{2}},
\]
\[
\frac{1}{2} |\partial_\theta^2 R(h, t, \theta)| \leq \left| \partial_\theta g_3(h, t, \theta) \right| + \left| \partial_\theta \Delta \cdot \partial_\theta R(h, t, \theta) \right| \leq Ch^1,
\]
\[
\frac{1}{2} |\partial_h \partial_t R(h, t, \theta)| \leq \left| \partial_{ht} g_1(h, t, \theta) \right| + \left| \partial_{ht} \Delta \cdot \partial_h R(h, t, \theta) \right| \leq Ch^{-\frac{1}{2}},
\]
\[
\frac{1}{2} |\partial_\theta \partial_\theta R(h, t, \theta)| \leq \left| \partial_{ht} g_1(h, t, \theta) \right| + \left| \partial_{ht} \Delta \cdot \partial_\theta R(h, t, \theta) \right| \leq Ch^0,
\]
\[ \frac{1}{2} |\partial_t \partial_\theta R(h, t, \theta)| \leq |\partial_\theta g_2(h, t, \theta)| + |\partial_\theta \Delta \cdot \partial_t R(h, t, t, \theta)| \leq Ch^{\frac{1}{2}}. \]

Generally, if
\[ |\partial^k_h \partial^l_\theta \partial^\ell_t R(h, t, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max\{1,j\} - 1)}, \text{ for } 1 \leq j + k + l \leq m, \]
then
\[ |\partial^k_h \partial^l_\theta \partial^\ell_t g_1(h, t, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}}, \quad |\partial^k_h \partial^l_\theta \partial^\ell_t g_2(h, t, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2}}, \]
\[ |\partial^k_h \partial^l_\theta \partial^\ell_t g_3(h, t, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}}, \quad |\partial^k_h \partial^l_\theta \partial^\ell_t \Delta(h, t, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}}. \]

The proof of these estimates is based on Leibniz’s rule and a direct computation. Consequently, by induction and Leibniz’s rule again to (55), we obtain
\[ |\partial^k_h \partial^l_\theta \partial^\ell_t R(h, t, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max\{1,j\} - 1)}, \text{ for } 1 \leq j + k + l \leq m + 1. \]

\[\Box\]

**Appendix C. Proof of Lemma 2.6.** The estimates on \(R_{01}\) are based on a direct computation and Lemmas 2.1 and 2.3.

The estimates on \(R_{02}\) are based on Leibniz’s rule, Lemmas 2.1–2.5, and the following claim. Readers can also refer to [11] (Lemma 3.4).

**Claim.** For \(h\) large enough, \(\theta, t \in \mathbb{S}^1\), \(k + j \leq m_1 - 1\) and \(l \leq m_2\), it holds that:

\[ |\partial^k_h \partial^l_\theta \partial^\ell_t f_1(h - sR, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max\{1,j\} - 1)}; \quad (56) \]
\[ |\partial^k_h \partial^l_\theta \partial^\ell_t f_2(h - sR, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max\{1,j\} - 1)}; \quad (57) \]
\[ |\partial^k_h \partial^l_\theta \partial^\ell_t f_2(h - sR, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max\{1,j\} - 1)}; \quad (58) \]
\[ |\partial^k_h \partial^l_\theta \partial^\ell_t f_3(h - sR, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max\{1,j\} - 1)}; \quad (59) \]

and

\[ |\partial^k_h \partial^l_\theta \partial^\ell_t f_3(h - R, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max\{1,j\} - 1)}; \quad (60) \]

**Proof of (56).** When \(k + l + j = 0\), then
\[ |\partial_t f_1(h - sR, \theta)| \leq Ch^{-\frac{1}{2}}. \]

For \(k + l + j > 0\), using Leibniz’s rule, \(\partial^k_h \partial^l_\theta \partial^\ell_t f_1(h - sR, \theta)\) is the sum of terms
\[ (\partial^p_h \partial^q_\theta \partial^r_t f_1) \cdot \Pi_{i=1}^u \partial^{k_i}_h \partial^{l_i}_\theta \partial^{\ell_i}_t (h - sR), \]
with \(1 \leq u + v \leq j + k + l, \sum_{i=1}^u k_i = k, \sum_{i=1}^u l_i = l, v + \sum_{i=1}^u j_i = j\), and \(k_i + j_i + l_i \geq 1, i = 1, \ldots, u\). Thus from Lemmas 2.1 and 2.5, it holds that
\[ |\partial^k_h \partial^l_\theta \partial^\ell_t f_1(h - sR, \theta)| \leq Ch^{\frac{1}{2} - \frac{k}{2} + \frac{1}{2}(\max\{1,j\} - 1)}. \]

The proofs of (57–60) are similar to the one of (56). We omit it here. \[\Box\]
Appendix D. Proof of Lemma 3.2. (17) follows from (16) and Lemma 2.1.

From (15), it is easy to see
\[ |\partial_h t_1| \leq Ch_2^{-\frac{3}{2}}, \quad |\partial_y t_1| \leq Ch_2^{-\frac{3}{2}}. \]

By a direct computation, for \(k + l + j \geq 2\) and \(k + j \leq m_1\),
\[ |\partial_h^k \partial_t^j \partial_y^l t_1| = |\partial_h^k \partial_t^j \partial_y^l (t_2 - \partial_h S_2(h_2, \theta))| = |\partial_h^{k+1} \partial_t^j \partial_y^l S_2(h_2, \theta)| \leq Ch_2^{-\frac{1}{2} - k + \frac{j}{2} + \frac{1}{2}(\max(2j) - 2)}. \]

Next, we consider the estimates on \(R_{21}\).

Obviously, it holds that \(|R_{21}| \leq C\). Suppose \(k + j \leq m_1 - 1\).

i) Consider \(\partial_h^k \partial_t^j \partial_y^l R_{11}(h_2, t_2 - \partial_h S_2(h_2, \theta), \theta)\). From (17) and By Leibniz’s rule, it is the summation of terms
\[ (\partial_h^k \partial_t^j \partial_y^l R_{11})(\Pi_{i=1}^q \partial_h^{k_i} \partial_t^{j_i} \partial_y^{l_i} t_1) \]
with \(1 \leq p + q + r \leq j + k + l\), \(p + \sum_{i=1}^q k_i = k\), \(\sum_{i=1}^q l_i = l\), \(r + \sum_{i=1}^q j_i = j\), and \(k_i + j_i + l_i \geq 1\), \(i = 1, \ldots, q\), which implies that
\[ |\partial_h^k \partial_t^j \partial_y^l R_{11}(h_2, t_2 - \partial_h S_2(h_2, \theta), \theta)| \leq Ch_2^{-k + \frac{j}{2} + \frac{1}{2}(\max(1, j) - 1)}, \text{ for } l \leq m_2. \]

ii) Similar to the part i), with Lemma 2.3 we have
\[ |\partial_h^k \partial_t^j \partial_y^l (\partial_h f_2(h_2, \theta, t_2 + \theta - \mu \partial_h S_2))| \leq Ch_2^{-k + \frac{j}{2} + \frac{1}{2}(\max(2j) - 2)}, \text{ for } l \leq m_2 - 1. \]

By Leibniz’s rule and the estimates on \(\partial_h^k \partial_t^j \partial_y^l S_2(h_2, \theta)\), it holds that
\[ |\partial_h^k \partial_t^j \partial_y^l (\partial_h f_2(h_2, \theta, t_2 + \theta - \mu \partial_h S_2) \partial_h S_2)| \leq Ch_2^{-\frac{1}{2} - k + \frac{j}{2} + \frac{1}{2}(\max(2j) - 2)}, \text{ for } l \leq m_2 - 1, \]
which, together with parts i) and ii), implies that
\[ |\partial_h^k \partial_t^j \partial_y^l R_{21}| \leq Ch_2^{-k + \frac{j}{2} + \frac{1}{2}(\max(1, j) - 1)}, \text{ for } l \leq m_2 - 1. \]

The proof of \(R_{22}\) is similar to the one of \(R_{21}\), we omit it.

Appendix E. Proof of Lemma 4.1. Note that
\[ \Psi(x) = \Psi\left(\sqrt{\frac{2}{n}} t \cos n\theta\right) = \Psi\left(\sqrt{\frac{2}{n}} (h - R) \frac{1}{2} \cos n\theta\right) = \cdots \]
\[ = \Psi\left(\sqrt{\frac{2}{n}} (h_3)^{\frac{1}{2}} (1 + Q(h_3, t_3, \theta)) \frac{1}{2} \cos n\theta\right), \]
where \(Q(h_3, t_3, \theta) = h_3^{-1}(h_2(h_3, t_3, \theta) - h_3 - R(h_2(h_3, t_3, \theta), t(h_3, t_3, \theta), \theta)). \)

To prove Lemma 4.1, we first give the estimates on \(Q\).

Lemma E.1. For \(h_3\) large enough, \(\theta, t_3 \in S^1\), \(k + j \leq m_1\), and \(l \leq m_2 - 1\), it holds that:
\[ |\partial_h^k \partial_t^j \partial_y^l Q(h_3, t_3, \theta)| \leq Ch_3^{-\frac{1}{2} - k + \frac{j}{2} + \frac{1}{2}(\max(1, j) - 1)}. \] (61)

Moreover, the following equation holds true:
\[ \partial_y^l Q(h_3, t_3, \theta) = q_1(h_3, t_3, \theta) + q_2(h_3, t_3, \theta) \sin^2 n\theta \] (62)
with \(|q_1| \leq Ch_3^{-\frac{1}{2}}\), \(|q_2| \leq C\).
Proof. The proof is based on a direct computation.

First of all, \( Q = h_3^{-1}(h_2 - h_3) - h_3^{-1}R \) implies that
\[
|Q| \leq Ch_3^{-\frac{1}{2}}.
\]

Now we consider the estimates on derivatives.

Suppose that \( l \leq m_2 - 1, \, k + j \leq m_1 \). Using Leibniz’s rules, \( \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (h_3^{-1} h_2) \) is the summation of terms \( \partial_{h_3}^{k_1} h_3^{-1} \cdot \partial_{h_3}^{k_2} \partial_{t_3}^{l_1} \partial_{\theta}^{j_1} h_2 \), where \( h_2 = h_2(h_3, t_3, \theta) \). Following Lemma 3.4, we have
\[
\left| \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (h_3^{-1} h_2) \right| \leq Ch_3^{-\frac{1}{2} - k}, \, k + l + j \geq 1. \tag{63}
\]

Next, consider \( \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (h_3^{-1} R(h_3, t_3, \theta)) \). Note that \( h = h_1 = h_2 = h_2(h_3, t_3, \theta) \) (see Lemma 3.4 for the estimates), and \( t = t(h_3, t_3, \theta) \). We first estimate \( t = t(h_3, t_3, \theta) \), which can be regarded as the composition of \( t = t(h_2, t_2, \theta) \) and \( h_2 = h_2(h_3, t_3, \theta) \), \( t_2 = t_2(h_3, t_3, \theta) \).

**Step 1.** Consider \( t = t(h_2, t_2, \theta) \) be the composition of \( t = t_1 + \theta, \, t_1 = t_1(h_2, t_2, \theta) \).

Following Leibniz’s rule, \( \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (t_1) \) is the summation of terms
\[
(\partial_{h_3}^p \partial_{t_3}^q \partial_{\theta}^r)(\Pi_{i=1}^q \partial_{h_3}^{k_i} \partial_{t_3}^{l_i} \partial_{\theta}^{j_i} t_1),
\]
with \( 1 \leq q + r \leq j + k + l, \, \sum_{i=1}^q k_i = k, \, \sum_{i=1}^q l_i = l, \, r + \sum_{i=1}^q j_i = j \), and \( k_i + j_i + l_i \geq 1, \, i = 1, \ldots, q \).

Following Lemma 3.2, we have
\[
\left| \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (t_1) \right| \leq Ch_2^{-\frac{1}{2} - k, \, k + j} \leq 2, \tag{64}
\]

**Step 2.** Consider \( t = t(h_3, t_3, \theta) \) be the composition of \( t = t(h_2, t_2, \theta), \, h_2 = h_2(h_3, t_3, \theta), \, t_2 = t_2(h_3, t_3, \theta) \). Following Leibniz’s rule, \( \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j R(h_3, t_3, \theta) \) is the summation of terms
\[
(\partial_{h_3}^p \partial_{t_3}^q \partial_{\theta}^r)(\Pi_{i=1}^q \partial_{h_3}^{k_i} \partial_{t_3}^{l_i} \partial_{\theta}^{j_i} h_2)(\Pi_{i=p+1}^{p+q} \partial_{h_3}^{k_i} \partial_{t_3}^{l_i} \partial_{\theta}^{j_i} t_2),
\]
with \( 1 \leq p + q + r \leq j + k + l, \, \sum_{i=1}^{p+q} k_i = k, \, \sum_{i=1}^{p+q} l_i = l, \, r + \sum_{i=1}^{p+q} j_i = j \), and \( k_i + j_i + l_i \geq 1, \, i = 1, \ldots, p + q \).

With Lemma 3.4 and the estimates (64), we obtain directly the estimates on the function \( t(h_3, t_3, \theta) \) as follows:
\[
\left| \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (h_3) \right| \leq Ch_3^{-\frac{1}{2}}, \, c \leq \left| \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (t_3) \right| \leq C, \, c \leq \left| \partial_{\theta}^j (t_3) \right| \leq C,
\]
\[
\left| \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (R) \right| \leq Ch_3^{-\frac{1}{2} - k, \, k + j} \leq 2. \tag{65}
\]

Then, let us consider \( \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j R(h_3, t_3, \theta) \). By Leibniz’s rule, it is the summation of terms
\[
(\partial_{h_3}^p \partial_{t_3}^q \partial_{\theta}^r)(\Pi_{i=1}^q \partial_{h_3}^{k_i} \partial_{t_3}^{l_i} \partial_{\theta}^{j_i} h_2)(\Pi_{i=p+1}^{p+q} \partial_{h_3}^{k_i} \partial_{t_3}^{l_i} \partial_{\theta}^{j_i} t),
\]
with \( 1 \leq p + q + r \leq j + k + l, \, \sum_{i=1}^{p+q} k_i = k, \, \sum_{i=1}^{p+q} l_i = l, \, r + \sum_{i=1}^{p+q} j_i = j \), and \( k_i + j_i + l_i \geq 1, \, i = 1, \ldots, p + q \).

With Lemmas 2.5, 3.4 and the estimates (65), we obtain directly the estimates as following:
\[
\left| \partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j (R) \right| \leq Ch_3^{\frac{1}{2} + \frac{1}{2}(\max(1,j)-1)}, \, l \leq m_2 - 1, \, k + j \leq m_1.
\]
Therefore,
\[
\left| \partial^k_{h_3} \partial^l_{t_3} \partial^m_{\theta} (h_3^{-1}R) \right| \leq Ch_3^{-\frac{1}{2} + \frac{l}{2} + \frac{1}{2} (\max(1, j) - 1)}, \quad l \leq m_2 - 1, \quad k + j \leq m_1,
\]
which, together with (63), implies that (61) holds.

Finally, we consider the expression of \( \partial^3_\theta Q \). Note that \( Q = h_3^{-1}(h_2 - h_3) - h_3^{-1}R \). We have
\[
\partial_\theta(h_2 - h_3) = \partial_\theta(\partial_3_{t_3}S_3(h_3, t_2, \theta)) = \partial^2_{t_3}S_3(h_3, t_2, \theta) t_2, \theta + \partial_\theta t_2, \theta,
\]
with \( t_2, \theta = \partial_\theta t_2 \), and thus,
\[
\partial^3_\theta(h_2 - h_3) = \partial_\theta(\partial^2_{t_3}S_3(h_3, t_2, \theta) t_2, \theta + \partial_\theta t_2, \theta)
\]
\[
= \partial^3_{t_2}S_3(h_3, t_2, \theta)(t_2, \theta)^2 + 2\partial_\theta \partial^2_{t_3}S_3(h_3, t_2, \theta) t_2, \theta + \partial^2_{t_2}S_3(h_3, t_2, \theta) t_2, \theta.
\]
Then, from Lemma 3.4, it holds that
\[
\left| \partial^3_\theta(h_2 - h_3) \right| \leq Ch_3^\frac{1}{2}.
\]
Recall
\[
R(h_3, t_3, \theta) = f_1(h, \theta) + f_2(h, t, \theta) + \frac{1}{n} \sum_{x} \left[ (h_2 - h_3) - \frac{1}{n} \sum_{x} \phi(x) \right],
\]
where \( f_1(h, \theta) = \frac{1}{n} \sum_{x} \left( \partial_\theta \left( G\left( \sqrt{h^2 + \frac{\cos n\theta}{n}} \right) \right) \right) \), \( f_2(h, t, \theta) = -\frac{1}{n} \sum_{x} \left( \partial_\theta \left( G\left( \sqrt{h^2 - \frac{\cos n\theta}{n}} \right) \right) \right) \), \( h = h_2 = h_3(t_3, \theta) \), and \( t = t(h, \theta) \). The estimates of \( R \) are divided into the following five parts:

(i) Consider \( f_1(h, \theta) = \frac{1}{n} \sum_{x} \left( \partial_\theta \left( G\left( \sqrt{h^2 + \frac{\cos n\theta}{n}} \right) \right) \right) \). We have
\[
n \partial_\theta \left( f_1 \big|_{(h_3, t_3, \theta)} \right) = G'(x)(x_h h_2, \theta + x_\theta),
\]
and
\[
n \partial^2_\theta \left( f_1 \big|_{(h_3, t_3, \theta)} \right) = G''(x)(x_h h_2, \theta + x_\theta)^2 + G'(x)(x_h h_2, \theta)^2 + 2x_h h_2, \theta + x_\theta h_2, \theta + x_\theta \theta)
\]
\[
= g_1(h_3, t_3, \theta) + g_2(h_3, t_3, \theta) \sin^2 n\theta
\]
with \(|g_1| \leq Ch_3^{\frac{1}{2}}, |g_2| \leq Ch_3\) by Lemma 3.4.

(ii) Consider \( f_2(h, t, \theta) = -\frac{1}{n} \sum_{x} \phi(t) \). We have
\[
-n \partial_\theta \left( f_2 \big|_{(h_3, t_3, \theta)} \right) = (x_h h_2, \theta + x_\theta) p(t) + x_\theta p'(t) t_\theta,
\]
and
\[
n \partial^2_\theta \left( f_2 \big|_{(h_3, t_3, \theta)} \right) = p(t)(x_h h_2, \theta)^2 + 2x_h h_2, \theta + x_\theta h_2, \theta + x_\theta \theta)
\]
\[
+ (x_h h_2, \theta + x_\theta) p'(t) t_\theta + x_\theta p'(t) t_\theta + x_\theta p(t) t_\theta.
\]
By Lemma 3.4 and (65), it holds that
\[
\left| \partial^2_\theta \left( f_2 \big|_{(h_3, t_3, \theta)} \right) \right| \leq Ch_3^{\frac{1}{2}}.
\]
(iii) Consider \( \Psi(x) \) with \( x = \sqrt{\frac{2}{n}} h^2 \cos n\theta \). Similar to part (i), it holds that
\[
\partial_\theta \left( \Psi(x) \bigg|_{(h_3,t_3,\theta)} \right) = \psi(x)(x_hh_2,\theta + x_\theta),
\]
and
\[
\partial_\theta^2 \left( \Psi(x) \bigg|_{(h_3,t_3,\theta)} \right) = -\Psi(x)(x_hh_2,\theta + x_\theta)^2 + \psi(x)(x_hh_2,\theta + x_\theta)^2 + 2x_hh_2,\theta + x_hh_2,\theta + x_\theta \tag{69}
\]
\[
g_3(h_3, t_3, \theta) + g_4(h_3, t_3, \theta) \sin^2 n\theta \tag{70}
\]
with \( |g_3| \leq Ch_3^2, \ |g_4| \leq Ch_3 \) by Lemma 3.4.

(iv) Consider \( R_{01}(h, t, \theta) \),
\[
\partial_\theta \left( R_{01} \bigg|_{(h_3,t_3,\theta)} \right) = R_{01,hh_2,\theta} + R_{01,t_\theta} + R_{01,\theta},
\]
and
\[
\partial_\theta^2 \left( R_{01} \bigg|_{(h_3,t_3,\theta)} \right) = (R_{01,hh_2,\theta} + R_{01,t_\theta} + R_{01,h_0})h_2,\theta + (R_{01,h_0}h_2,\theta + R_{01,\theta}t_\theta + R_{01,\theta})t_\theta + R_{01,h_0}h_2,\theta + R_{01,\theta}t_\theta + R_{01,h_0}h_2,\theta + R_{01,\theta}t_\theta
\]
By Lemmas 2.6, 3.4 and (65), it holds that
\[
\left| \partial_\theta^2 \left( R_{01} \bigg|_{(h_3,t_3,\theta)} \right) \right| \leq Ch_3^2 \tag{71}
\]
(v) Similar to case (iv), we have
\[
\left| \partial_\theta^2 \left( R_{02} \bigg|_{(h_3,t_3,\theta)} \right) \right| \leq C. \tag{72}
\]
From (67)-(72), we have
\[
\partial_\theta^2 \left( R \bigg|_{(h_3,t_3,\theta)} \right) = g(h_3, t_3, \theta) + g_6(h_3, t_3, \theta) \sin^2 n\theta \tag{73}
\]
with \( |g| \leq Ch_3^2, \ |g_6| \leq Ch_3 \).

Finally, with (66) and (73), (62) is proved.

Proof of Lemma 4.1. Note that \( \Psi(x) \in C^m, \ +1(\mathbb{R}/(T\mathbb{Z})) \) and \( \int_0^T \Psi(x)dx = 0 \) by the assumption, it follows that
\[
\Psi(x) = \sum_{m=1}^{+\infty} (\Psi_m^1 \sin \frac{2m\pi}{T} x + \Psi_m^2 \cos \frac{2m\pi}{T} x), \tag{74}
\]
where the Fourier coefficients satisfy, integrated by parts,
\[
\left| \Psi_m^1 \right| = \left| \frac{2}{T} \int_0^T \Psi(x) \sin \frac{2m\pi}{T} x dx \right| \leq m^{-m_1-1}, \tag{75}
\]
\[
\left| \Psi_m^2 \right| = \left| \frac{2}{T} \int_0^T \Psi(x) \cos \frac{2m\pi}{T} x dx \right| \leq m^{-m_1-1}. \tag{76}
\]
For given \( m \), consider the estimates on
\[
\int_0^{2\pi} \sin\left(\frac{2m\pi}{T} x\right) d\theta = \int_0^{2\pi} \sin\left(\frac{2m\pi}{T} \sqrt{\frac{2}{n} h^\frac{3}{2} u(h_3, t_3, \theta) \cos n\theta}\right) d\theta
\]
with \( u = (1 + Q(h_3, t_3, \theta))^{\frac{1}{2}} \).

**Step 1.** Note that \( \frac{1}{2} \leq |u(h_3, t_3, \theta)| < C \). And for \( k + j \leq m_1, l \leq m_2 - 1, k + l + j \geq 1 \), we have
\[
|\partial_h^k \partial_l^j \partial_\theta^m u(h_3, t_3, \theta)| \leq Ch_3^{-\frac{k}{2} - \frac{j}{2} + \frac{1}{2}(\max\{1,j\}-1)}
\]
by Leibniz’s rule and Lemma E.1.
Moreover, for \( k \leq m_1, l \leq m_2 - 1 \) and \( k + l \geq 1 \), it holds that
\[
|\partial_h^k \partial_l^j \sin \frac{2m\pi}{T} x| \leq Ch_3^{-\frac{k}{2}}.
\]

**Step 2.** Let \( v(h_3, t_3, \theta) = \frac{2\pi}{T} \sqrt{\frac{2}{n}} u(h_3, t_3, \theta) \cos n\theta \), then
\[
\partial_\theta v(h_3, t_3, \theta) = \frac{2\pi}{T} \sqrt{\frac{2}{n}} \{ \partial_\theta u(h_3, t_3, \theta) \cos n\theta - nu(h_3, t_3, \theta) \sin n\theta \}.
\]
We claim that \( \partial_\theta v(h_3, t_3, \theta) \) has exactly \( 2n \) zero points in \([0, 2\pi)\) for sufficiently large \( h_3 \).
In fact, note that \( \partial_\theta v(h_3, t_3, \theta) \) is \( 2\pi \)-periodic in \( \theta \), we may consider \( \partial_\theta v(h_3, t_3, \theta) \) in the interval \([-\frac{\pi}{2n}, 2\pi - \frac{\pi}{2n})\) equivalently. Let \( \theta_k = \frac{k\pi}{n}, k = 0, 1, \cdots, 2n - 1 \) be \( 2n \) zero points of \( \sin n\theta \) in \([-\frac{\pi}{2n}, 2\pi - \frac{\pi}{2n})\), then
\[
[-\frac{\pi}{2n}, 2\pi - \frac{\pi}{2n}) = \bigcup_{k=0}^{n-1} I_k
\]
with \( I_k = [\theta_k - \frac{\pi}{2n}, \theta_k + \frac{\pi}{2n}) \).
Next, we will prove that \( \partial_\theta v(h_3, t_3, \theta) \) has exactly one zero point in each \( I_k \).

For any sufficiently small \( \epsilon > 0 \), there exists \( \delta > 0 \), such that \( |\sin n\theta| < \epsilon \) for \( \theta \in \Delta_k = [\theta_k - \delta, \theta_k + \delta] \). Then, following (77) and (79), there exists sufficiently large \( h \) such that if \( h_3 > h \) then \( \partial_\theta v(h_3, t_3, \theta) \neq 0 \) for \( \theta \in I_k \setminus \Delta_k \), and \( \partial_\theta v(h_3, t_3, \theta_k - \delta) \cdot \partial_\theta v(h_3, t_3, \theta_k + \delta) < 0 \) which means that \( \partial_\theta v(h_3, t_3, \theta) \) has at least one zero point \( \theta_k^* \) in \( \Delta_k \).
Moreover, for \( \theta \in \Delta_k \),
\[
\partial_\theta^2 v(h_3, t_3, \theta) = \frac{2\pi}{T} \sqrt{\frac{2}{n}} \{ \partial_\theta^2 u(h_3, t_3, \theta) \cos n\theta - 2n \partial_\theta u(h_3, t_3, \theta) \sin n\theta - n^2 u(h_3, t_3, \theta) \cos n\theta) \}.
\]
Note that
\[
\partial_\theta u(h_3, t_3, \theta) = \frac{1}{2}(1 + Q)^{-\frac{1}{2}} \partial_\theta Q,
\]
and
\[
\partial_\theta^2 u(h_3, t_3, \theta) = -\frac{1}{4}(1 + Q)^{-\frac{3}{2}} (\partial_\theta Q)^2 + \frac{1}{2}(1 + Q)^{-\frac{1}{2}} \partial_\theta^2 Q.
\]
It follows from (61) and (62) that
\[
\partial_\theta^2 u(h_3, t_3, \theta) = q_3(h_3, t_3, \theta) + q_4(h_3, t_3, \theta) \sin^2 n\theta
\]
with $|q_3| \leq C h_3^{-\frac{1}{2}}$, $|q_4| \leq C$.

Therefore it shows that for $\theta \in \Delta_k$,

$$|\partial_\theta^3 v(h_3, t_3, \theta)|$$

$$\geq \frac{2\pi}{T} \sqrt{\frac{2}{n}} \{ n^2 |u(h_3, t_3, \theta) \cos n\theta| - |\partial_\theta^3 u(h_3, t_3, \theta) \cos n\theta| - 2n|\partial_\theta u(h_3, t_3, \theta) \sin n\theta| \}$$

$$\geq \frac{2\pi}{T} \sqrt{\frac{2}{n}} \{ \frac{1}{2} n^2 (1 - \epsilon) - Cc^2 - Ch_3^{-\frac{1}{2}} - 2nCh_3^{-\frac{1}{2}} \epsilon \}$$

$$\geq \frac{\sqrt{2\pi}}{2T} n^2 > 0.$$  

Therefore, the zero point of $\partial_\theta v$ in $\Delta_k$ is unique.

Finally, we have shown that for any given $(h_3, t_3)$ with $h_3(\geq \bar{h})$ large enough, $v(h_3, t_3, \theta)$ has exactly $2n$ isolated critical points in interval $[0, 2\pi]$.

**Step 3.** Without loss of generality, for given $(h_3, t_3)$, suppose $[a, b] \subset [0, 2\pi]$ is an interval where $a$, $b$ are the only two critical points of $v(h_3, t_3, \theta)$ in $[a, b]$. Following Lemma A.1 and Remark 7 in the Appendix, with $\lambda = \mu = 1$, $\rho = \sigma = 2$, $\nu = 2$, we have, for $m h_3^2 \gg 1$,

$$|B_1(mh_3^3) - A_1(mh_3^3)| \leq |\int_a^b e^{imh_3^2 v(h_3, t_3, \theta)} d\theta| < C |B_1(mh_3^3) - A_1(mh_3^3)|,$$

where

$$A_1(mh_3^3) = -C_1 e^{i(mh_3^2 \cos h_3^3 + \frac{\pi}{4})} m^{-\frac{1}{4}} h_3^{-\frac{1}{2}};$$

$$B_1(mh_3^3) = -C_2 e^{i(mh_3^2 \cos h_3^3 + \frac{\pi}{4})} m^{-\frac{1}{4}} h_3^{-\frac{1}{2}}$$

with $C_1, C_2$ independent of $m$.

Note that Lemma A.1 does not involve parameters, while the function $v(h_3, t_3, \theta)$ we are dealing with here depends on two parameters $h_3, t_3$. But in this situation, we can easily prove a similar result as Lemma A.1 for the following reason. On one hand, $t_3$ is a variable on a circle. On the other hand, for the dependence of $v(h_3, t_3, \theta)$ with respect to $h_3$, it is of the form $(1 + O(h_3^{-1/2})) \cos n\theta$ as $h_3 \to \infty$. Thus from the point of view of classical Oscillatory Integrals, there is no difference between our case with parameters $h_3, t_3$ and the classical one. Since the proof is similar to the one for Lemma A.1, we omit it here.

Then we have

$$\left| \int_a^b \sin(mh_3^3 v(h_3, t_3, \theta)) d\theta \right| \leq C m^{-\frac{1}{4}} h_3^{-\frac{1}{2}}. \tag{81}$$

In the same way, we can prove

$$\left| \int_a^b \cos(mh_3^3 v(h_3, t_3, \theta)) d\theta \right| \leq C m^{-\frac{1}{4}} h_3^{-\frac{1}{2}}. \tag{82}$$

Together with (74)-(76), (81) and (82), we obtain that

$$\left| \langle \tilde{f}_3 \rangle(h_3, t_3) \right| = \frac{1}{n} \left| \int_0^{2\pi} \Psi(x) d\theta \right|$$
Then repeating Step 1, Step 2 & Step 3 above, with the help of Lemma A.1, we obtain (24).

Hence (23) is proved.

To prove (24), note that

\[ I_{kl} = \partial^k h_3 \partial^l_3 \phi_3(h_3, t_3) = \int_0^{2\pi} \partial^k h_3 \partial^l_3 f_3(h_3 + \partial_t S_3, \theta) d\theta \]

with \( \partial^k h_3 \partial^l_3 f_3(h_3 + \partial_t S_3, \theta) = \frac{1}{n} \partial^k h_3 \partial^l_3 \psi(\sqrt{2} \pi h_3^\frac{1}{2} u(h_3, t_3, \theta) \cos n\theta) \). From (74), it follows that

\[ \partial^k h_3 \partial^l_3 \psi(x) = \sum_{m=1}^{+\infty} (\Psi_1^m \partial^k h_3 \partial^l_3 \sin \frac{2m\pi}{T} x + \Psi_2^m \partial^k h_3 \partial^l_3 \cos \frac{2m\pi}{T} x) \]

with \( x = \sqrt{2} \pi h_3^\frac{1}{2} u(h_3, t_3, \theta) \cos n\theta \).

By Leibniz’s rule, each term \( \partial^k h_3 \partial^l_3 \sin \frac{2m\pi}{T} x \) is of the form

\[ \varphi^i_{mkl}(h_3, t_3, \theta) \sim \frac{2m\pi}{T} x + \varphi^i_{mkl}(h_3, t_3, \theta) \cos \frac{2m\pi}{T} x. \]

Form (78), for \( k \leq m_1, l \leq m_2 - 1 \), it holds that

\[ |\varphi^i_{mkl}(h_3, t_3, \theta)| \leq C h_3^{-\frac{1}{2}}, \quad i = 1, 2. \]

Let \( \varphi^i(\theta) = \frac{1}{n} h_3^{\frac{1}{2}} \varphi^i_{mkl}(h_3, t_3, \theta) \) for \( i = 1, 2 \), then

\[ I_{mkl} = \int_0^{2\pi} \partial^k h_3 \partial^l_3 \sin \frac{2m\pi}{T} x d\theta \]

\[ = h_3^{-\frac{1}{2}} \int_0^{2\pi} (\varphi^1(\theta) \sin \frac{2m\pi}{T} x + \varphi^2(\theta) \cos \frac{2m\pi}{T} x) d\theta. \]

Then repeating Step 1, Step 2 & Step 3 above, with the help of Lemma A.1, we obtain (24).

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