Paraconistent Reasoning via Quantified Boolean Formulas, I: Axiomatising Signed Systems * **

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Abstract. Signed systems were introduced as a general, syntax-independent framework for paraconsistent reasoning, that is, non-trivialised reasoning from inconsistent information. In this paper, we show how the family of corresponding paraconsistent consequence relations can be axiomatised by means of quantified Boolean formulas. This approach has several benefits. First, it furnishes an axiomatic specification of paraconsistent reasoning within the framework of signed systems. Second, this axiomatisation allows us to identify upper bounds for the complexity of the different signed consequence relations. We strengthen these upper bounds by providing strict complexity results for the considered reasoning tasks. Finally, we obtain an implementation of different forms of paraconsistent reasoning by appeal to the existing system QUIP.

1 Introduction

In view of today’s rapidly growing amount and distribution of information, it is inevitable to encounter inconsistent information. This is why methods for reasoning from inconsistent data are becoming increasingly important. Unfortunately, there is no consensus on which information should be derivable in the presence of a contradiction. Nonetheless, there is a broad class of consistency-based approaches that reconstitute information from inconsistent data by appeal to the notion of consistency. Our overall goal is to provide a uniform basis for these approaches that makes them more transparent and easier to compare. To this end, we take advantage of the framework of quantified Boolean formulas (QBFs). To be more precise, we concentrate here on axiomatising the class of so-called signed systems [2] for paraconsistent reasoning; a second paper will deal with maximal-consistent sets and related approaches (cf. [3,4,5]).

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Our general methodology offers several benefits: First, we obtain uniform axiomatisations of rather different approaches. Second, once such an axiomatisation is available, existing QBF solvers can be used for implementation in a uniform setting. The availability of efficient QBF solvers, like the systems described in \cite{3,11,15}, makes such a rapid prototyping approach practically applicable. Third, these axiomatisations provide a direct access to the complexity of the original approach. Finally, we remark that this approach allows us, in some sense, to express paraconsistent reasoning in (higher order) classical propositional logic and so to harness classical reasoning mechanisms from (a conservative extension of) propositional logic.

Our elaboration of paraconsistent reasoning is part of an encompassing research program, analysing a large spectrum of reasoning mechanisms in Artificial Intelligence, among them nonmonotonic reasoning \cite{8}, (nonmonotonic) modal logics \cite{10}, logic programming \cite{7,17}, abductive reasoning \cite{9}, and belief revision \cite{6}.

In order to keep our paper self-contained, we must carefully introduce the respective techniques. Given the current space limitations, we have thus decided to reduce the motivation and rather concentrate on a thorough formal elaboration. This brings us to the following outline: Section 2 lays down the formal foundations of our work, introducing QBFs and Default Logic. Section 3 is devoted to signed systems as introduced in \cite{2}. Apart from reviewing the basic framework, we provide new unifying characterisations that pave the way for the respective encodings in QBFs, which are the subject of Section 4. This section comprises thus our major contribution: a family of basic QBF axiomatisations that can be assembled in different ways in order to accommodate the variety of paraconsistent inference relations within the framework of signed systems. We further elaborate upon these axiomatisations in Section 5 for analysing the complexity of the respective reasoning tasks. Finally, our axiomatisations are also of great practical value since they allow for a direct implementation in terms of existing QBF-solvers. Such an implementation is described in Section 6, by appeal to the system QUIP \cite{8,7,9}.

2 Foundations

2.1 Preliminary Notation

We deal with propositional languages and use the logical symbols \( \top, \bot, \neg, \lor, \land, \rightarrow, \equiv \) to construct formulas in the standard way. We write \( \mathcal{L}_\Sigma \) to denote a language over an alphabet \( \Sigma \) of propositional variables or atoms. Formulas are denoted by Greek lower-case letters (possibly with subscripts). Finite sets \( T = \{ \phi_1, \ldots, \phi_n \} \) of formulas are usually identified with the conjunction \( \land_{i=1}^n \phi_i \) of its elements. The set of all atoms occurring in a formula \( \phi \) is denoted by \( \text{var}(\phi) \). Similarly, for a set \( S \) of formulas, \( \text{var}(S) = \bigcup_{\phi \in S} \text{var}(\phi) \). The derivability operator, \( \vdash \), is defined in the usual way. The deductive closure of a set \( S \subseteq \mathcal{L}_\Sigma \) of formulas is given by \( \text{Cn}_{\Sigma}(S) = \{ \phi \in \mathcal{L}_\Sigma \mid S \vdash \phi \} \). We say that \( S \) is deductively closed iff \( S = \text{Cn}_{\Sigma}(S) \). Furthermore, \( S \) is consistent iff \( \bot \notin \text{Cn}_{\Sigma}(S) \). If the
language is clear from the context, we usually drop the index \("\Sigma\) from \(Cn_\Sigma(\cdot)\) and simply write \(Cn(\cdot)\) for the deductive closure operator.

For formulas \(\varphi\), \(\phi\), and \(\psi\), we define \textit{positive} and \textit{negative occurrences} as follows:

\begin{itemize}
  \item the occurrence of \(\varphi\) in \(\varphi\) is positive,
  \item if \(\varphi\) occurs positively (negatively) in \(\phi\), then the corresponding occurrence of \(\varphi\) in \(\neg \phi\) and \(\phi \to \psi\) is negative (positive),
  \item if \(\varphi\) occurs positively (negatively) in \(\phi\), then the corresponding occurrence of \(\varphi\) in \(\phi \lor \psi\), \(\phi \land \psi\), and \(\psi \to \phi\) is positive (negative).
\end{itemize}

Given an alphabet \(\Sigma\), we define a disjoint alphabet \(\Sigma^\pm\) as \(\Sigma^\pm = \{ p^+, p^- \mid p \in \Sigma \}\). For \(\alpha \in L_\Sigma\), we define \(\alpha^\pm\) as the formula obtained from \(\alpha\) by replacing each negative occurrence of \(p\) by \(\neg p^-\) and by replacing each positive occurrence of \(p\) by \(p^+\), for each propositional variable \(p\) in \(\Sigma\). For example \((p \land (p \to q))^\pm = p^+ \land (\neg p^- \to q^+)\). This is defined analogously for sets of formulas. Observe that for any set \(T \subseteq L_\Sigma\), \(T^\pm\) is consistent, even if \(T\) is inconsistent.

### 2.2 Quantified Boolean Formulas

Quantified Boolean formulas (QBFs) generalise ordinary propositional formulas by the admission of quantifications over propositional variables (QBFs are denoted by Greek upper-case letters). Informally, a QBF of form \(\forall p \exists q \Phi\) means that for all truth assignments of \(q\) there is a truth assignment of \(p\) such that \(\Phi\) is true. For instance, it is easily seen that the QBF \(\exists p \exists q ((p \to q) \land \forall r (r \to q))\) evaluates to true.

The precise semantical meaning of QBFs is defined as follows. First, some ancillary notation. An occurrence of a propositional variable \(p\) in a QBF \(\Phi\) is \textit{free} iff it does not appear in the scope of a quantifier \(Qp\) (\(Q \in \{\forall, \exists\}\)), otherwise the occurrence of \(p\) is \textit{bound}. If \(\Phi\) contains no free variable occurrences, then \(\Phi\) is \textit{closed}, otherwise \(\Phi\) is \textit{open}. Furthermore, we write \(\Phi[p_1/\phi_1, \ldots, p_n/\phi_n]\) to denote the result of uniformly substituting each free occurrence of a variable \(p_i\) in \(\Phi\) by a formula \(\phi_i\), for \(1 \leq i \leq n\).

By an \textit{interpretation}, \(M\), we understand a set of atoms. Informally, an atom \(p\) is true under \(M\) iff \(p \in M\). In general, the truth value, \(\nu_M(\Phi)\), of a QBF \(\Phi\) under an interpretation \(M\) is recursively defined as follows:

\begin{enumerate}
  \item if \(\Phi = \top\), then \(\nu_M(\Phi) = 1\);
  \item if \(\Phi = p\) is an atom, then \(\nu_M(\Phi) = 1\) if \(p \in M\), and \(\nu_M(\Phi) = 0\) otherwise;
  \item if \(\Phi = \neg \Psi\), then \(\nu_M(\Phi) = 1 - \nu_M(\Psi)\);
  \item if \(\Phi = (\Phi_1 \land \Phi_2)\), then \(\nu_M(\Phi) = \min\{\nu_M(\Phi_1), \nu_M(\Phi_2)\}\);
  \item if \(\Phi = \forall p \Psi\), then \(\nu_M(\Phi) = \nu_M(\Psi[p/\top] \land \Psi[p/\bot])\);
  \item if \(\Phi = \exists p \Psi\), then \(\nu_M(\Phi) = \nu_M(\Psi[p/\top] \lor \Psi[p/\bot])\).
\end{enumerate}

The truth conditions for \(\bot, \lor, \neg, \equiv\) follow from the above in the usual way. We say that \(\Phi\) is \textit{true under} \(M\) iff \(\nu_M(\Phi) = 1\), otherwise \(\Phi\) is \textit{false under} \(M\). If \(\nu_M(\Phi) = 1\), then \(M\) is a \textit{model} of \(\Phi\). If \(\Phi\) has some model, then \(\Phi\) is said to be
satisfiable. If $\Phi$ is true under any interpretation, then $\Phi$ is valid. As usual, we write $\models \Phi$ to express that $\Phi$ is valid. Observe that a closed QBF is either valid or unsatisfiable, because closed QBFs are either true under each interpretation or false under each interpretation. Hence, for closed QBFs, there is no need to refer to particular interpretations. Two sets of QBFs (or ordinary formulas) are logically equivalent iff they possess the same models.

In the sequel, we use the following abbreviations in the context of QBFs: For a set $P = \{p_1, \ldots, p_n\}$ of propositional variables and a quantifier $Q \in \{\forall, \exists\}$, we let $Q^P \Phi$ stand for the formula $Q p_1 Q p_2 \cdots Q p_n \Phi$. Furthermore, for indexed sets $S = \{\phi_1, \ldots, \phi_n\}$ and $T = \{\psi_1, \ldots, \psi_n\}$ of formulas, $S \leq T$ abbreviates $\wedge_{i=1}^n (\phi_i \rightarrow \psi_i)$.

The operator $\leq$ is a fundamental tool for expressing certain tests on sets of formulas in terms of QBFs. In particular, we use $\leq$ for expressing the following task:

Given finite sets $S$ and $T$ of formulas, compute all subsets $R \subseteq S$ such that $T \cup R$ is consistent.

This problem can be encoded by a QBF in the following way:

**Proposition 1.** Let $S = \{\phi_1, \ldots, \phi_n\}$ and $T$ be finite sets of formulas, let $P = \text{var}(S \cup T)$, and let $G = \{g_1, \ldots, g_n\}$ be a set of new variables. Furthermore, for any $S' \subseteq S$, define the interpretation $M_{S'} \subseteq G$ such that $\phi_i \in S'$ iff $g_i \in M_{S'}$, for $1 \leq i \leq n$.

Then, $T \cup S'$ is consistent iff the QBF

$$C[T,S] = \exists P (T \land (G \leq S))$$

is true under $M_{S'}$.

**Theorem 1.** Given the prerequisites of Proposition 1, we have that $S'$ is a maximal subset of $S$ consistent with $T$ iff $M_{S'}$ is a model of the QBF

$$C[T,S] \land \bigwedge_{i=1}^n (-g_i \rightarrow -C[T \cup \{\phi_i\}, S \setminus \{\phi_i\}]).$$

### 2.3 Default Logic

The primary technical means for dealing with “signed theories” is default logic [18], whose central concepts are default rules along with their induced extensions of an initial set of premises. A default rule (or default for short)

$$\frac{\alpha}{\gamma} : \beta$$

has two types of antecedents: a prerequisite $\alpha$ which is established if $\alpha$ is derivable and a justification $\beta$ which is established if $\beta$ is consistent. If both conditions
hold, the consequent \( \gamma \) is concluded by default. For convenience, we denote the prerequisite of a default \( \delta \) by \( p(\delta) \), its justification by \( j(\delta) \), and its consequent by \( c(\delta) \). Accordingly, for a set of defaults \( D \), we define \( p(D) = \{ p(\delta) \mid \delta \in D \} \), \( j(D) = \{ j(\delta) \mid \delta \in D \} \), and \( c(D) = \{ c(\delta) \mid \delta \in D \} \).

A default theory is a pair \((D, W)\) where \( D \) is a set of default rules and \( W \) a set of formulas. A set of conclusions (sanctioned by a given set of default rules and by means of classical logic) is called an extension of an initial set of facts.

More formally, extensions are defined as follows:

**Definition 1** (\[18\]). Let \((D, W)\) be a default theory and let \( E \) be a set of formulas. Define \( E_1 = W \) and, for \( n \geq 1 \),
\[
E_{n+1} = Cn(E_n) \cup \left\{ \gamma \mid \frac{\alpha \cdot \beta}{\gamma} \in D, \alpha \in E_n, \neg \beta \notin E \right\}.
\]

Then, \( E \) is an extension of \((D, W)\) iff \( E = \bigcup_{n \in \omega} E_n \).

### 3 Signed Systems

The basic idea of signed systems is to transform an inconsistent theory into a consistent one by renaming propositional variables and then to extend the resulting signed theory by equivalences using default logic.

#### 3.1 Basic Approach

Starting with a possibly inconsistent finite theory \( W \subseteq L_{\Sigma} \), we consider the default theory obtained from \( W^\pm \) and a set of default rules \( D_{\Sigma} = \{ \delta_p \mid p \in \Sigma \} \) defined in the following way. For each propositional letter \( p \) in \( \Sigma \), we define
\[
\delta_p = \frac{p^+ \equiv \neg p^-}{(p \equiv p^+) \land (\neg p \equiv p^-)}.
\]

Using this definition, we define the first family of paraconsistent consequence relations:

**Definition 2.** Let \( W \) be a finite set of formulas in \( L_{\Sigma} \) and let \( \varphi \) be a formula in \( L_{\Sigma} \). Let \( E \) be the set of all extensions of \((D_{\Sigma}, W^\pm)\). For each set of formulas \( S \subseteq L_{\Sigma^\pm} \), let
\[
\Pi_S = \{ c(\delta_p) \mid p \in \Sigma, \neg j(\delta_p) \notin S \}.
\]

Then, we define
\[
W \vdash_c \varphi \quad \text{iff} \quad \varphi \in \bigcup_{E \in E} Cn(W^\pm \cup \Pi_E) \quad \text{(credulous unsigned)}
\]
\[
W \vdash_s \varphi \quad \text{iff} \quad \varphi \in \bigcap_{E \in E} Cn(W^\pm \cup \Pi_E) \quad \text{(skeptical unsigned)}
\]
\[
W \vdash_p \varphi \quad \text{iff} \quad \varphi \in Cn(W^\pm \cup \bigcap_{E \in E} \Pi_E) \quad \text{(prudent unsigned)}
\]

\(^1\) The term “unsigned” indicates that only unsigned formulas are taken into account.
For illustration, consider the inconsistent theory

\[ W = \{ p, q, \neg p \lor \neg q \}. \]  

(2)

For obtaining the above paraconsistent consequence relations, \( W \) is turned into the default theory

\[ (D, W^\pm) = \{(\delta_p, \delta_q), \{p^+, q^+, p^- \lor q^- \}\}. \]

We obtain two extensions, viz. \( Cn(W^\pm \cup \{c(\delta_p)\}) \) and \( Cn(W^\pm \cup \{c(\delta_q)\}) \). The following relations show how the different consequence relations behave:

\[ W \vdash_c p, \quad W \nvdash_s p, \quad W \nvdash_p p, \]

but, for instance,

\[ W \vdash_c p \lor q, \quad W \vdash_s p \lor q, \quad W \nvdash_p p \lor q. \]

For a complement, the following “signed” counterparts are defined.

**Definition 3.** Given the prerequisites of Definition 3, we define

\[
\begin{align*}
W \vdash_c^\pm \varphi & \iff \varphi \in \bigcup_{E \in E} Cn(W^\pm \cup \Pi_E) & \text{(credulous signed consequence)} \\
W \vdash_s^\pm \varphi & \iff \varphi \in \bigcap_{E \in E} Cn(W^\pm \cup \Pi_E) & \text{(skeptical signed consequence)} \\
W \vdash_p^\pm \varphi & \iff \varphi \in Cn(W^\pm \cup \bigcap_{E \in E} \Pi_E) & \text{(prudent signed consequence)}
\end{align*}
\]

### 3.2 Formal Properties

As shown in 3, these relations compare to each other in the following way.

**Theorem 2.** Let \( C_i \) be the operator corresponding to \( C_i(W) = \{ \varphi \mid W \vdash_i \varphi \} \) where \( i \) ranges over \( \{p, s, c\} \), and similarly for \( C_i^\pm \). Then, we have

1. \( C_i(W) \subseteq C_i^\pm(W) \);
2. \( C_p(W) \subseteq C_s(W) \subseteq C_c(W) \) and \( C_p^\pm(W) \subseteq C_s^\pm(W) \subseteq C_c^\pm(W) \).

That is, signed derivability gives more conclusions than unsigned derivability and within each series of consequence relations the strength of the relation is increasing.

Moreover, they enjoy the following logical properties:

**Theorem 3.** Let \( C_i \) be the operator corresponding to \( C_i(W) = \{ \varphi \mid W \vdash_i \varphi \} \) where \( i \) ranges over \( \{p, s, c\} \), and similarly for \( C_i^\pm \). Then, we have

3. \( W \subseteq C_i^\pm(W) \);
4. \( C_p(W) = Cn(C_p(W)) \) and \( C_s(W) = Cn(C_s(W)) \);
5. \( C_i^\pm(W) = C_i^\pm(C_i^\pm(W)) \);
6. \( Cn(W) \neq \mathcal{L}_\Sigma \) only if \( Cn(W) = C_i(W) = C_i^\pm(W) \);
7. \( C_i(W) \neq \mathcal{L}_\Sigma \) and \( C_i^\pm(W) \neq \mathcal{L}_\Sigma \);
8. \( W \subseteq W' \) does not imply \( C_i(W) \subseteq C_i(W') \), and \( W \subseteq W' \) does not imply \( C_i^\pm(W) \subseteq C_i^\pm(W') \).

The last item simply says that all of our consequence relations are nonmonotonic. For instance, we have \( C_i(\{A, A \rightarrow B\}) = C_i^\pm(\{A, A \rightarrow B\}) = Cn(\{A, A\}) \), while neither \( C_i(\{A, \neg A, A \rightarrow B\}) \) nor \( C_i^\pm(\{A, \neg A, A \rightarrow B\}) \) contains \( B. \)

\[ \text{For simplicity, we omitted all } \delta_x \text{ for } x \in \Sigma \setminus \{p, q\}. \]
3.3 Refinements

The previous relations embody a somewhat global approach in restoring semantic links between positive and negative literals. In fact, the application of a rule $\delta_p$ re-establishes the semantic link between all occurrences of proposition $p$ and its negation $\neg p$ at once. A more fine-grained approach is to establish the connections between complementary occurrences of an atom individually.

Formally, for a given $W$ and an index set $I$ assigning different indices to all occurrences of all atoms in $W$, define

$$\delta_{i,j}^p = \begin{cases} 
(p \equiv p_i^+ ) \land (\neg p \equiv p_j^-) 
& \text{if } i, j \in I \text{ and } i, j \text{ are complementary occurrences of } p \text{ in } W, \\
\delta_p & \text{otherwise}.
\end{cases}$$

for all $p \in \Sigma$ and all $i, j \in I$, provided that $i$ and $j$ refer to complementary occurrences of $p$ in $W$, otherwise set $\delta_{i,j}^p = \delta_p$. Denote by $D_1^1$ this set of defaults and by $W_{i}^\pm$ the result of replacing each $p^+ \in W^\pm$ (resp., $p^- \in W^\pm$) by $p_i^+$ (resp., $p_i^-$) where $i$ is the index assigned to the corresponding occurrence, provided that there are complementary occurrences of $p$ in $W$.

Finally, abandoning the restoration of semantical links and foremost restoring original (unsigned) literals leads to the most adventurous approach to signed inferences. Consider the following set of defaults, defined for all $p \in \Sigma$ and $i, j \in I$,

$$\delta_{i}^+ = \begin{cases} 
(p \equiv p_i^+ ) & \\
\delta_p & \text{otherwise}
\end{cases} \quad \delta_{j}^- = \begin{cases} 
(\neg p \equiv p_j^-) & \\
\delta_p & \text{otherwise}
\end{cases}$$

for all positive and negative occurrences of $p$, respectively. As above, we use these defaults provided that there are complementary occurrences of $p$ in $W$, otherwise use $\delta_p$. A set of defaults of form (4) with respect to $W$ is denoted by $D_2^2$.

Thus, further consequence relations are defined when $(D_\Sigma, W^\pm)$ in Definition 3 is replaced by $(D_1^1, W_i^\pm)$ or by $(D_2^2, W_i^\pm)$. Similar results to Theorem 2 and 3 can be shown for these families of consequence relations.

In the following, we identify all introduced default theories as follows. Given a finite set $W \subseteq L_\Sigma$, the class $D(W)$ contains $(D_\Sigma, W)$, as well as $(D_1^1, W_i^\pm)$ and $(D_2^2, W_i^\pm)$ for any index set $I$. Furthermore, $D = \bigcup_{W \subseteq L_\Sigma} D(W)$ denotes the class of all possible default theories under consideration.

3.4 Hierarchic Extensions

Whenever a problem instance may give rise to several solutions, it is useful to provide a preference criterion for selecting a subset of preferred solutions. This is accomplished in [3] by means of a ranking function $\varrho : \Sigma \rightarrow \mathbb{N}$ on the alphabet $\Sigma$ for inducing a hierarchy on the default rules in $D_\Sigma$:

**Definition 4.** Let $\varrho : \Sigma \rightarrow \mathbb{N}$ be some ranking function on alphabet $\Sigma$, and $(D, V) \in D$. We define the hierarchy of $D$ with respect to $\varrho$ as the partition
\begin{equation}
\langle D_n \rangle_{n \in \omega} \text{ of } D \text{ such that for each } \delta \in D \text{ with } \delta \text{ of form } \delta_p, \delta^i_j, \delta^{i+}_p, \delta^{i-}_p, \text{ for } p \in \Sigma \text{ and } i, j \in I, \delta \in D_n \text{ iff } \varphi(p) = n \text{ holds.}
\end{equation}

Strictly speaking, \( \langle D_n \rangle_{n \in \omega} \) is not always a genuine partition, since \( D_n \) may be the empty set for some values of \( n \).

Such rankings are used for inducing so-called **hierarchic extensions**. This concept has been introduced to deal with a given partition on the defaults \( D \) of a default theory \((D, V)\).

**Definition 5.** Let \( W \) be a finite set of formulas in \( L_\Sigma \), \((D, V) \in \mathcal{D}(W)\), and \( E \) a set of formulas. Let \( \langle D_n \rangle_{n \in \omega} \) be the hierarchy of \( D \) with respect to some ranking function \( \varphi \).

Then, \( E = \bigcup_{n \in \omega} E_n \) is a hierarchic extension of \((D, V)\) relative to \( \varphi \) if \( E_1 = V \) and \( E_{n+1} \) is an extension of \((D_n, E_n)\) for all \( n \geq 1 \).

Let \( \langle D_n \rangle_{n \in \omega} \) be the hierarchy of \( D \) with respect to some ranking function \( \varphi \), and let \( E \) be the set of all hierarchic extensions of a default theory \((D, V) \in \mathcal{D}\) in Definition 2. Then, we immediately get corresponding consequence relations \( \vdash_{ch}, \vdash_{sh}, \text{ and } \vdash_{ph} \). Furthermore, applying hierarchic extensions on default theories \((D^\Sigma, W^\pm)\) in accordance to Definition 2 yields new relations \( \vdash_{ch}^\pm, \vdash_{sh}^\pm, \text{ and } \vdash_{ph}^\pm \).

In concluding this section, let us briefly recapitulate all paraconsistent consequence relations introduced so far. As a basic classification, we have credulous, skeptical and prudent consequence. For each of these relations, we defined unsigned operators, which are invokable on three different classes of default theories (viz. on \((D^\Sigma, W^\pm), (D^1_\Sigma, W^\pm_I), \text{ and } (D^2_\Sigma, W^\pm_I)\)), either on ordinary extensions \( \vdash \) or on hierarchic extensions \( \vdash_{ih} \), and, on the other hand, signed operators also relying on ordinary extensions \( \vdash^\pm \) or hierarchic extensions \( \vdash_{ih}^\pm \) of the default theory \((D^\Sigma, W^\pm)\). This gives in total 18 unsigned and 6 signed paraconsistent consequence relations, which shall all be considered in the following two sections.

### 4 Reductions

In this section, we show how the above introduced consequence relations can be mapped into quantified Boolean formulas in polynomial time.

Recall the set \( \mathcal{D}(W) \) for finite \( W \subseteq L_\Sigma \). In what follows, we use finite default theories

\[
\mathcal{D}^*(W) = \{ (D_W, V) \mid (D, V) \in \mathcal{D}(W) \}
\]

where \( D_W = \{ \delta \in D \mid \text{var}(\delta) \cap \text{var}(W) \neq \emptyset \} \). Hence, \( D_W \) contains each default from \( D \) having an unsigned atom which also occurs in \( W \).

The next subsection shows the adequacy of these default theories. Afterwards, Section 4.2 gives QBF-reductions based on the finite default theories \( \mathcal{D}^*(W) \).
4.1 Finitary Characterisations

**Lemma 1.** Let $W \subseteq L_\Sigma$ be a finite set of formulas and $(D,V) \in D(W)$ a default theory. Moreover, let $C \subseteq D$ and $C_W = \{ \delta \in C \mid \text{var}(\delta) \cap \text{var}(W) \neq \emptyset \}$. Then,

1. $Cn(V \cup c(C_W)) \cap L_\Sigma = Cn(V \cup c(C)) \cap L_\Sigma$; and
2. for each $\varphi^\pm \in L_{\Sigma^+}$, $\varphi^\pm \in Cn(V \cup c(C))$ iff $\varphi^\pm \in Cn(V \cup c(C_W) \cup c(D_\varphi))$

where $D_\varphi = \{ \delta \mid p \in \text{var}(\varphi) \setminus \text{var}(W) \}$. Both results show that having computed a (possibly hierarchic) extension, one has a finite set of generating defaults sufficient for deciding whether a paraconsistent consequence relation holds. The following result shows that these sets are also sufficient to compute the underlying extensions themselves.

**Theorem 4.** Let $W$, $(D,V)$, $C$, and $C_W$ be as in Lemma 4, and let $D_W = \{ \delta \in D \mid \text{var}(\delta) \cap \text{var}(W) \neq \emptyset \}$.

Then, there is a one-to-one correspondence between the extensions of $(D,V)$ and the extensions of $(D_W,V)$. In particular, $Cn(V \cup c(C))$ is an extension of $(D,V)$ iff $Cn(V \cup c(C_W))$ is an extension of $(D_W,V)$. Similar relations hold for hierarchic extensions as well.

The next result gives a uniform characterisation for all default theories under consideration. It follows from the fact that, for each $\delta_p$, the consequent $(p \equiv p^+)^\land (\neg p \equiv p^-)$ is actually equivalent to $(p^+ \equiv \neg p^-) \land (p \equiv p^+)$, and, furthermore, that defaults of form [8] and [9] share the property that their justifications and consequents are identical. Hence, given $W$ and $I$ as usual, it holds that $c(\delta) = j(\delta)$, for each $\delta \in D$, with $(D,V) \in D^*(W)$.

**Theorem 5.** Let $W \subseteq L_\Sigma$ be a finite set of formulas, let $(D,V) \in D^*(W)$ be a default theory, and let $C_W \subseteq D$.

Then, $Cn(V \cup c(C))$ is an extension of $(D,V)$ iff $j(C)$ is a maximal subset of $j(D)$ consistent with $V$.

Note that the subsequent QBF reductions, obtained on the basis of the above result, represent a more compact axiomatics than the encodings given in [8] for arbitrary default theories.

We derive an analogous characterisation for hierarchic extensions. In fact, each hierarchic extension is also an extension (but not vice versa) [8]. Thus, we can characterise hierarchic extensions of a default theory $(D,V)$ as ordinary extensions, viz. by $Cn(W \cup c(C))$ with $C \subseteq D$ suitably chosen. The following result generalises Theorem [8] with respect to a given partition on the defaults. In particular, if $(D_n)_{n \in \omega} = \langle D \rangle$, Theorem [8] corresponds to Theorem [8].

**Theorem 6.** Let $W$, $(D,V)$, and $C$ be given as in Theorem [8].

Then, $Cn(V \cup c(C))$ is a hierarchic extension of $(D,V)$ with respect to partition $(D_n)_{n \in \omega}$ on $D$ iff for each $i \in \omega$, $j(D_i \cap C)$ is a maximal subset of $j(D_i)$ consistent with $V \cup \bigcup_{j<i} c(D_j \cap C)$. 

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Finally, in order to relate extensions of default theories to paraconsistent consequence operators, we note the following straightforward observations.

Let $\Pi_S$ be as in Definition 2. Then, for each extension $E$ of $(D, V) \in \mathcal{D}(W)$, there exists a $C \subseteq D$ such that $c(C) = \Pi_E$. However, since we have to check whether a given formula is contained in some $Cn(V \cup \Pi_E)$, by Lemma 1 it is obviously sufficient to consider just the generating defaults of an extension of the corresponding restricted default theory from $\mathcal{D}^*(W)$. In view of Theorems 5 and 6, this immediately implies that all paraconsistent consequence relations introduced so far can be characterised by maximal subsets of the consequences $c(D)$ of the corresponding default theory $(D, V) \in \mathcal{D}^*(W)$. More specifically, credulous and skeptical paraconsistent consequence reduces to checking whether a given formula is contained in at least one or respectively all such maximal subsets. Additionally, prudent consequence enjoys the following property.

**Lemma 2.** Let $W \subseteq \mathcal{L}_\Sigma$ be a finite set of formulas, and $(D, V) \in \mathcal{D}^*(W)$.

Then, for each $\varphi \in \mathcal{L}_\Sigma$, we have that $W \not\vdash_p \varphi$ (resp., $W \not\vdash_{ph} \varphi$) iff there exists a set $C \subseteq D$ such that $\varphi \notin Cn(V \cup c(C))$ and, for each $\delta \in D \setminus C$, there is some extension (resp., hierarchic extension) $E$ of $(D, V)$ such that $c(\delta) \notin E$. An analogous result holds for relations $\vdash_p^\pm$ and $\vdash_{ph}^\pm$.

### 4.2 Main Construction

We start with some basic QBF-modules. To this end, recall the schema $\mathcal{C}[-,-]$ from Proposition 3.

**Definition 6.** Let $W \subseteq \mathcal{L}_\Sigma$ be a finite set of formulas and $\varphi \in \mathcal{L}_\Sigma$. For each finite default theory $T = (D, V) \in \mathcal{D}^*(W)$, let $D = \{\delta_1, \ldots, \delta_n\}$, and define

\[
\text{Ext}[T] = \mathcal{C}[V, j(D)] \wedge \bigwedge_{i=1}^{n} \left( \neg g_i \rightarrow \neg \mathcal{C}[V \cup \{j(\delta_i)\}, j(D \setminus \{\delta_i\})] \right);
\]

\[
\text{Conseq}[T, \varphi] = \forall P \left( V \wedge (G \leq c(D)) \rightarrow \varphi \right),
\]

where $P$ denotes the set of atoms occurring in $T$ or $\varphi$, and $G = \{g_i \mid \delta_i \in D\}$ is an indexed set of globally new variables corresponding to $D$.

**Lemma 3.** Let $W$, $T = (D, V)$, and $G$ be as in Definition 6. Furthermore, for any set $C \subseteq D$, define the interpretation $M_C \subseteq G$ such that $g_i \in M_C$ iff $\delta_i \in C$, for $1 \leq i \leq n$.

Then, the following relations hold:

1. $Cn(V \cup c(C))$ is an extension of $T$ iff $\text{Ext}[T]$ is true under $M_C$; and
2. $\varphi \in Cn(V \cup c(C))$ iff $\text{Conseq}[T, \varphi]$ is true under $M_C$, for any formula $\varphi$ in $\mathcal{L}_\Sigma$. 
Observe that the correctness of Condition 1 follows directly from Theorem 1 since we have that $\text{Ext}(T)$ is true under $MC$ iff $j(C)$ is a maximal subset of $j(D)$ consistent with $V$, and, in view of Theorem 3, the latter holds iff $Cn(V \cup c(C))$ is an extension of $T$. Moreover, Condition 2 is actually reducible to Proposition 1.

Combining these two QBF-modules, we obtain encodings for the basic inference tasks as follows:

**Theorem 7.** Let $W \subseteq L_\Sigma$ be a finite set of formulas, $T = (D, V)$ a default theory from $D^+(W)$ with $D = \{\delta_1, \ldots, \delta_n\}$, $\varphi$ a formula in $L_\Sigma$, and $G = \{g_1, \ldots, g_n\}$ the indexed set of variables occurring in $\text{Ext}(T)$ and $\text{Conseq}(T, \varphi)$.

Then, paraconsistent credulous and skeptical consequence relations can be axiomatised by means of QBFs as follows:

1. $W \vdash_c \varphi \iff \exists G(\text{Ext}(T) \land \text{Conseq}(T, \varphi))$; and
2. $W \vdash_s \varphi \iff \neg \exists G(\text{Ext}(T) \land \neg \text{Conseq}(T, \varphi))$.

Moreover, for prudent consequence, let $G' = \{g'_i \mid g_i \in G\}$ be an additional set of globally new variables and

$$\Psi = \bigwedge_{i=1}^n \left( \neg g'_i \rightarrow \exists G(\text{Ext}(T) \land \neg \text{Conseq}(T, c(\delta_i))) \right).$$

Then,

3. $W \vdash_p \varphi \iff \neg \exists G'(-\text{Conseq}_{G-G'}(T, \varphi) \land \Psi)$,

where $\text{Conseq}_{G-G'}(T, \varphi)$ denotes the QBF obtained from $\text{Conseq}(T, \varphi)$ by replacing each occurrence of an atom $g \in G$ in $\text{Conseq}(T, \varphi)$ by $g'$.

In what follows, we discuss the remaining consequence relations under consideration. We start with signed consequence. Here, we just have to adopt the calls to $\text{Conseq}((D_\Sigma, W^\pm), \varphi)$ with respect to Lemma 2 by adding those defaults $\delta_p$ to $W^\pm$ such that $p \in \text{var}(\varphi) \setminus \text{var}(W)$. Observe that in the following theorem this addition is not necessary for the module $\Psi$. Furthermore, recall that signed consequence is applied only to default theories $(D_\Sigma, W^\pm)$.

**Theorem 8.** Let $W \subseteq L_\Sigma$ be a finite set of formulas and $\varphi$ a formula in $L_\Sigma$. Moreover, let $D_\varphi = \{\delta_p \mid p \in \text{var}(\varphi)\}$ and $D_\varphi = \{\delta_p \mid p \in \text{var}(\varphi) \setminus \text{var}(W)\}$, with the corresponding default theories $T = (D_\Sigma, W^\pm)$ and $T' = (D_\varphi, W^\pm \cup c(D_\varphi))$, and let $G$, $G'$, and $\Psi$ be as in Theorem 7.

Then, paraconsistent signed consequence relations can be axiomatised by means of QBFs as follows:

1. $W \vdash_p ^\pm \varphi \iff \exists G(\text{Ext}(T) \land \text{Conseq}(T', \varphi^\pm))$;
2. $W \vdash_p ^\pm \varphi \iff \neg \exists G(\text{Ext}(T) \land \neg \text{Conseq}(T', \varphi^\pm))$; and
3. $W \vdash_p ^\pm \varphi \iff \neg \exists G'(\Psi \land \neg \text{Conseq}_{G-G'}(T', \varphi^\pm))$,

where, as above, $\text{Conseq}_{G-G'}(\cdot, \cdot)$ replaces each $g$ by $g'$.
It remains to consider the consequence relations based on hierarchical extensions. To this end, we exploit the characterisation of Theorem 6.

**Definition 7.** Let \( W \subseteq \mathcal{L}_\Sigma \) be a finite set of formulas, \( T = (D, V) \) a default theory from \( \mathcal{D}^*(W) \) with \( D = \{ \delta_1, \ldots, \delta_n \} \), and \( P = \langle D_n \rangle_{n \in \omega} \) a partition on \( D \).

We define

\[
Ext_h[T, P] = \bigwedge_{i \in \omega} \left( Ext[(V \land \bigwedge_{\delta \in D_i \cup \ldots \cup D_{i-1}} (g_j \rightarrow c(\delta_j)), D_i)] \right),
\]

where \( G = \{ g_i \mid \delta_i \in D \} \) is the same indexed set of globally new variables corresponding to \( D \) as above appearing in each \( Ext[\cdot] \).

**Lemma 4.** Let \( W, (D, V), G, \) and \( P \) be as in Definition 7. Furthermore, for any set \( C \subseteq D \), define the interpretation \( M_C \subseteq G \) such that \( g_i \in M_C \) iff \( \delta_i \in C \), for \( 1 \leq i \leq n \).

Then, \( Cn(V \cup c(C)) \) is a hierarchic extension of \( T \) with respect to \( P \) iff \( Ext_h[T, P] \) is true under \( M_C \).

**Theorem 9.** Paraconsistent consequence relations \( \vdash_{ch}, \vdash_{\pm ch}, \vdash_{sh}, \vdash_{\pm sh}, \vdash_{ph} \), and \( \vdash_{\pm ph} \) are expressible in the same manner as in Theorems 7 and 8 by replacing \( Ext[T] \) with \( Ext_h[T, P] \).

This concludes the reductions to QBFs. Observe that all these reductions are solely built from simple QBF-modules like \( Ext[\cdot] \) and \( Conseq[\cdot, \cdot] \) and are constructible in polynomial time.

## 5 Complexity Issues

In what follows, we assume the reader familiar with the basic concepts of complexity theory (cf. e.g., [16] for a comprehensive textbook on this subject). Relevant for our purposes are the complexity classes \( \Sigma^p_2 \) and \( \Pi^p_2 \). \( \Sigma^p_2 \) is the class of all problems solvable on a nondeterministic Turing machine in polynomial time having access to an oracle for problems in NP (the class NP consists of all decision problems which can be solved with a nondeterministic Turing machine working in polynomial time), and \( \Pi^p_2 \) consists of the problems which are complementary to the problems in \( \Sigma^p_2 \), i.e., \( \Pi^p_2 = \text{co-}\Sigma^p_2 \). Recall that both classes are part of the polynomial hierarchy.

In the sequel, we derive complexity results for deciding paraconsistent consequence in all variants discussed previously. We show that all considered tasks are located at the second level of the polynomial hierarchy. This is in some sense not surprising, because the current approach relies on deciding whether a given formula is contained in an extension of a suitably constructed default theory. This problem was shown to be \( \Sigma^p_2 \)-complete by Gottlob [8], even if normal default theories are considered. However, this completeness result is not directly
applicable here because of the specialised default theories in the present setting. Furthermore, for dealing with hierarchic extensions, it turns out that the complexity remains at the second level of the polynomial hierarchy as well. This result is interesting, since the definition of hierarchic extensions is somewhat more elaborate than standard extensions. In any case, this observation mirrors in some sense complexity results derived for cumulative default logic (cf. [14]).

In the same way as the satisfiability problem of classical propositional logic is the “prototypical” problem of NP, i.e., being an NP-complete problem, the satisfiability problem of QBFs in prenex form possessing \( k \) quantifier alternations is the “prototypical” problem of the \( k \)-th level of the polynomial hierarchy.

**Proposition 2** ([19]). Given a propositional formula \( \phi \) whose atoms are partitioned into \( i \geq 1 \) sets \( P_1, \ldots, P_i \), deciding whether \( \exists P_1 \forall P_2 \ldots Q_i P_i \phi \) is true is \( \Sigma^P_i \)-complete, where \( Q_i = \exists \) if \( i \) is odd and \( Q_i = \forall \) if \( i \) is even. Dually, deciding whether \( \forall P_1 \exists P_2 \ldots Q'_i P_i \phi \) is true is \( \Pi^P_i \)-complete, where \( Q'_i = \forall \) if \( i \) is odd and \( Q'_i = \exists \) if \( i \) is even.

Given the above characterisations, we can estimate upper complexity bounds for the reasoning problems discussed in Section 3 simply by inspecting the quantifier order of the respective QBF encodings. This can be argued as follows. First of all, by applying quantifier transformation rules similar to ones in first-order logic, each of the above QBF encodings can be transformed in polynomial time into a QBF in prenex form having exactly one quantifier alternation. Then, by invoking Proposition 2 and observing that completeness of a decision problem \( D \) for a complexity class \( C \) implies membership of \( D \) in \( C \), the quantifier order of the resultant QBFs determines in which class of the polynomial hierarchy the corresponding reasoning task belongs to.

Applying this method to our considered tasks, we obtain that credulous paraconsistent reasoning lies in \( \Sigma^P_2 \), whilst skeptical and prudent paraconsistent reasoning are in \( \Pi^P_2 \). Furthermore, note that the QBFs expressing paraconsistent reasoning using the concept of hierarchical extensions share exactly the same quantifier structures as those using ordinary extensions.

Concerning lower complexity bounds, it turns out that most of the above given estimations are strict, i.e., the considered decision problems are hard for the respective complexity classes. The results are summarised in Table 1. There, all entries denote completeness results, except where a membership relation is explicitly stated. The following theorem summarises these relations:

**Theorem 10.** The complexity results in Table 1 hold both for ordinary as well as for hierarchical extensions of \( T_i \) (\( i = 0, 1, 2 \)) as underlying inference principle.

Some of these complexity results have already been shown elsewhere. As pointed out in [2], prudent consequence, \( W \vdash_p \varphi \), on the basis of the default theory \( (D, W^0) \) captures the notion of free-consequences as introduced in [1]. This formalism was shown to be \( \Pi^P_2 \)-complete in [3].

Finally, [3] considers the complexity of a number of different paraconsistent reasoning principles, among them the completeness results for \( \vdash_s \) and \( \vdash^+ \). More-
Table 1. Complexity results for all paraconsistent consequence relations.

|   | $T_0 = (D_\Sigma, W^\pm)$ | $T_1 = (D_\Sigma^1, W^\pm_1)$ | $T_2 = (D_\Sigma^2, W^\pm_2)$ |
|---|-----------------------------|-----------------------------|-----------------------------|
| $\Gamma_c$ | $\Sigma_2^P$ | $\Sigma_2^P$ | $\Sigma_2^P$ |
| $\Gamma_s$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_p$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_\pm$ | $\Sigma_2^P$ in $\Pi_2^P$ | $\Sigma_2^P$ in $\Pi_2^P$ | |
| $\Gamma_{\pm 2}$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_{\pm 3}$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_{\pm 4}$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_{\pm 6}$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_{\pm 8}$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_{\pm 12}$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_{\pm 16}$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\Gamma_{\pm 24}$ | $\Pi_2^P$ | $\Pi_2^P$ | $\Pi_2^P$ |

over, that paper extends the intractability results to some restricted subclasses as well.

6 Discussion

We have shown how paraconsistent inference problems within the framework of signed systems can be axiomatised by means of quantified Boolean formulas. This approach has several benefits: First, the given axiomatisation provides us with further insight about how paraconsistent reasoning works within the framework of signed systems. Second, this axiomatisation allows us to furnish upper bounds for precise complexity results, going beyond those presented in [5]. Last but not least, we obtain a straightforward implementation technique of paraconsistent reasoning in signed systems by appeal to existing QBF solvers.

For implementing our approach, we rely on the existing system QUIP [8, 7]. The general architecture of QUIP consists of three parts, namely the filter program, a QBF-evaluator, and the interpreter int. The input filter translates the given problem description (in our case, a signed system and a specified reasoning task) into the corresponding quantified Boolean formula, which is then sent to the QBF-evaluator. The current version of QUIP provides interfaces to most of the currently available QBF-solvers. The result of the QBF-evaluator is interpreted by int. Depending on the capabilities of the employed QBF-evaluator, int provides an explanation in terms of the underlying problem instance. This task relies on a protocol mapping of internal variables of the generated QBF into concepts of the problem description.

References
1. S. Benferhat, D. Dubois, and H. Prade. Argumentative Inference in Uncertain and Inconsistent Knowledge Bases. In *Proceedings of the Ninth Conference on Uncertainty in Artificial Intelligence*, pages 411–419, 1993.

2. P. Besnard and T. Schaub. Signed Systems for Paraconsistent Reasoning. *Journal of Automated Reasoning*, 20:191–213, 1998.

3. M. Cadoli, A. Giovanardi, and M. Schaerf. An Algorithm to Evaluate Quantified Boolean Formulae. In *Proceedings of the Fifteenth National Conference on Artificial Intelligence (AAAI-98)*, pages 262–267. AAAI Press, 1998.

4. C. Cayrol, M. Lagasquie-Schiex, and T. Schiex. Nonmonotonic Reasoning: From Complexity to Algorithms. *Annals of Mathematics and Artificial Intelligence*, 22(3–4):207–236, 1998.

5. S. Coste-Marquis and P. Marquis. Complexity Results for Paraconsistent Inference Relations. In *Proceedings of the Eighth International Conference on Principles of Knowledge Representation and Reasoning (KR-02)*, pages 61–72, 2002.

6. J. Delgrande, T. Schaub, H. Tompits, and S. Woltran. On Computing Solutions to Belief Change Scenarios. In *Proceedings of the Sixth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU-01)*, pages 510–521. Springer Verlag, 2001.

7. U. Egly, T. Eiter, V. Klotz, H. Tompits, and S. Woltran. Computing Stable Models with Quantified Boolean Formulas: Some Experimental Results. In *Proceedings of the AAAI 2001 Spring Symposium on Answer Set Programming*, pages 53–59, 2001.

8. U. Egly, T. Eiter, H. Tompits, and S. Woltran. Solving Advanced Reasoning Tasks Using Quantified Boolean Formulas. In *Proceedings of the Seventeenth National Conference of Artificial Intelligence (AAAI-00)*, pages 417–422. AAAI Press, 2000.

9. U. Egly, V. Klotz, H. Tompits, and S. Woltran. A Toolbox for Abduction: Preliminary Report. In *Proceedings of the IJCAR 2001 Workshop on Theory and Applications of Quantified Boolean Formulas*, pages 29–39, 2001.

10. T. Eiter, V. Klotz, H. Tompits, and S. Woltran. Modal Nonmonotonic Logics Revisited: Efficient Encodings for the Basic Reasoning Tasks. In *Proceedings of the Eleventh Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX-02)*, 2002. To appear.

11. R. Feldmann, B. Monien, and S. Schamberger. A Distributed Algorithm to Evaluate Quantified Boolean Formula. In *Proceedings of the Seventeenth National Conference of Artificial Intelligence (AAAI-00)*, pages 285–290. AAAI Press, 2000.

12. E. Giunchiglia, M. Narizzano, and A. Tacchella. QuBE: A System for Deciding Quantified Boolean Formulas Satisfiability. In *Proceedings of the International Joint Conference on Automated Reasoning (IJCAR-01)*, pages 364–369. Springer Verlag, 2001.

13. G. Gottlob. Complexity Results for Nonmonotonic Logics. *Journal of Logic and Computation*, 2(3):397–425, 1992.

14. G. Gottlob and Z. Mingyi. Cumulative Default Logic: Finite Characterization, Algorithms, and Complexity. *Artificial Intelligence*, 69(1–2):329–345, 1994.

15. R. Letz. Advances in Decision Procedures for Quantified Boolean Formulas. In *Proceedings of the IJCAR 2001 Workshop on Theory and Applications of Quantified Boolean Formulas*, pages 55–64, 2001.

16. C. H. Papadimitriou. *Computational Complexity*. Addison-Wesley, New York, 1994.

17. D. Pearce, H. Tompits, and S. Woltran. Encodings for Equilibrium Logic and Logic Programs with Nested Expressions. In *Proceedings of the Tenth Portuguese Conference on Artificial Intelligence (EPIA-01)*, pages 306–320. Springer Verlag, 2001.
18. R. Reiter. A Logic for Default Reasoning. *Artificial Intelligence*, 13(1–2):81–132, 1980.

19. C. Wrathall. Complete Sets and the Polynomial-Time Hierarchy. *Theoretical Computer Science*, 3(1):23–33, 1976.