Random Möbius dynamics on the unit disc and perturbation theory for Lyapunov exponents

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Abstract

Randomly drawn $2 \times 2$ matrices induce a random dynamics on the Riemann sphere via the Möbius transformation. Considering a situation where this dynamics is restricted to the unit disc and given by a random rotation perturbed by further random terms depending on two competing small parameters, the invariant (Furstenberg) measure of the random dynamical system is determined. The results have applications to the perturbation theory of Lyapunov exponents which are of relevance for one-dimensional discrete random Schrödinger operators.

1 Set-up, intuition and main results

Perturbation theory for Lyapunov exponents associated to products of random matrices is of relevance for spectral analysis, random dynamical systems and quantum dynamics in random media, as well as numerous other physics related questions. Due to the tight connection of Lyapunov exponents and invariant measures on the projective space via the Furstenberg formula, one is naturally led to study these invariant measures in a perturbative regime. If there is a unique invariant measure, it is referred to as the Furstenberg measure and this is known to be the case under a variety of sufficient conditions [1]. When dealing with real $2 \times 2$ matrices, the real projective space is a one-dimensional circle and the analysis of the Furstenberg measures becomes particularly trackable. The first rigorous contribution going back to Pastur and Figotin [12] considers a situation stemming from the one-dimensional Anderson model in which the random $2 \times 2$ matrices are given by a rotation with non-vanishing rotation angle perturbed by a small random term. In this situation, the invariant measure is given by the Lebesgue measure in a weak sense (testing only low frequencies and only up to error terms and provided the frequency is smaller than the rotation angle without perturbation). This is based on the so-called oscillatory phase argument, and the outcome is often also referred to as the random phase property. This technique can be pushed to deal with large deviations [9] and a perturbative analysis of the variance in the central limit theorem [16]. When the rotation angle is trivial so that the real random $2 \times 2$ matrices are given by a random perturbation of the identity matrix, one deals with a so-called anomaly [10] and then the random phase property does not
hold and alternative methods have been developed to deal with this case [2, 18]. Small random perturbations of a Jordan block are of relevance for band edges of random Jacobi matrices and lead to yet another different analytical approach [4, 14]. The perturbative analysis of larger random matrices is quite involved and exhibits a rich variety of dynamical behaviors resulting from the competition between hyperbolic and elliptic parts of the dynamics, see [17, 15, 6] for analytical and [13] for numerical results.

This work considers random $2 \times 2$ matrices with complex entries, again in a perturbative regime. The suitable projective space is then of real dimension 2. Under the stereographic projection it can be identified with the Riemann sphere, on which the dynamics is then given by the Möbius transformation. Of course, it is now more challenging to analyze the invariant measures. Sufficient conditions for their uniqueness can be found in a recent work by Dinh, Kaufmann and Wu [5]. For a perturbative analysis, it is then also natural to consider random matrices depending real analytically on two small parameters $\epsilon$ and $\delta$ (see assumption (i) below), rather than just one. A typical situation of this kind is the single-site Anderson model, in which the parameter $\epsilon$ measures the size of the randomness and the other parameter $\delta$ is the complex part of the energy, see Section 5. When considering real energies in the spectrum, the dynamics at $\epsilon = \delta = 0$ is given by a rotation so that oscillating phases can be used in this situation, while a non-vanishing $\delta$ introduces some hyperbolicity to the dynamics. The competition between these two effects leads to striking differences between the random dynamics and an interesting crossover regime. In this introduction, this will first be described in a qualitative and intuitive manner, and then the main rigorous results are stated.

### 1.1 Möbius action

Let us begin by recalling the framework. The standard action of a matrix $T \in \mathbb{C}^{2 \times 2}$ with $\det(T) = 1$ on the cover $S^{1}_\mathbb{C} = \{x \in \mathbb{C}^2 : \|x\| = 1\}$ of the complex projective space $\mathbb{CP}^1 \cong S^1_\mathbb{C}/U(1)$ is given by

$$T \star x = Tx\|Tx\|^{-1}.$$ 

The stereographic projection $\pi : S^{1}_\mathbb{C} \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1$, which is given by

$$\pi(x) = \begin{cases} \frac{ab^{-1}}{\infty}, & b \neq 0, \\ \infty, & b = 0, \end{cases} \quad x = \begin{pmatrix} a \\ b \end{pmatrix}, \quad (1)$$

satisfies $\pi(T \star x) = T \cdot \pi(x)$. Here, $T \cdot$ denotes the Möbius action given by

$$T \cdot z = \frac{az + b}{cz + d}, \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

when $z \notin \{-dc^{-1}, \infty\}$, for which one sets $T \cdot (-dc^{-1}) = \infty$ and $T \cdot \infty = ac^{-1}$.

Of importance will be two subsets of $\text{SL}(2, \mathbb{C}) = \{T \in \mathbb{C}^{2 \times 2} : \det(T) = 1\}$, namely the Lorentz subgroup

$$\text{SU}(1, 1) = \{T \in \mathbb{C}^{2 \times 2} : T^*JT = J, \det(T) = 1\}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
and the semigroup of sub-Lorentzian matrices

$$\text{SU}_\leq(1, 1) = \{ T \in \mathbb{C}^{2 \times 2} : T^*JT \leq J, \det(T) = 1 \}.$$ 

The Möbius action with $T \in \text{SU}(1, 1)$ leaves the unit circle invariant, i.e., it obeys $T \cdot S^1 = S^1$. Furthermore, the action with $T \in \text{SU}_\leq(1, 1)$ leaves the open unit disc invariant, i.e., it satisfies $T \cdot \mathbb{D} \subset \mathbb{D}$, where $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. The latter fact follows from the inequality

$$\left(1 - |T \cdot z|^2\right)^2 = - \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}^* T \begin{pmatrix} z \\ 1 \end{pmatrix}\right)^2 \geq - \left(\begin{pmatrix} z \\ 1 \end{pmatrix}^* J \begin{pmatrix} z \\ 1 \end{pmatrix}\right) = 1 - |z|^2, \quad (3)$$

holding true for all $z \in \mathbb{D}$ and all $T \in \text{SU}_\leq(1, 1)$. Note that for $T \in \text{SU}(1, 1)$, inequality (3) becomes an equality, hence implying $T \cdot S^1 \subset S^1$.

### 1.2 Random dynamical system

If now a sequence $(T_n)_{n \geq 1}$ of complex $2 \times 2$ matrices with unit determinant is drawn independently and identically from a family $(T_\sigma)_{\sigma \in \Sigma}$ according to a law $\mathbb{P}$ on a probability space $\Sigma$, one obtains a random dynamical system on $\mathbb{C}$ by setting

$$z_n = T_n \cdot z_{n-1}, \quad (4)$$

where $z_0 \in \overline{\mathbb{C}}$ is some initial starting point. As $z_n = \pi(x_n)$, one also has an associated dynamics on $S^1_{\mathbb{C}}$ satisfying $x_n = T_n \ast x_{n-1}$. The average w.r.t. $\mathbb{P}$ is denoted by $\mathbb{E}$. Associated to such a random sequence $(T_n)_{n \geq 1}$, one always has a Lyapunov exponent given by

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \log \left(\|T_n \cdots T_1\|\right) \quad (5)$$

with convergence either almost surely or in expectation [1]. The second objects of interest here are the invariant probability measures $\mu$ on $\overline{\mathbb{C}}$ defined by

$$\int_{\overline{\mathbb{C}}} \mu(dz) f(z) = \mathbb{E} \int_{\overline{\mathbb{C}}} \mu(dz) f(T_\sigma \cdot z), \quad f \in C(\overline{\mathbb{C}}).$$

The corresponding invariant probability measures $\nu$ on $S^1_{\mathbb{C}}$ satisfy $\pi_*(\nu) = \mu$. If $\mu$ is unique, then it is called the Furstenberg measure. In that case, $\nu$ is unique up to some (possibly non-trivially distributed) phase and the Lyapunov exponent can be expressed by the Furstenberg formula

$$\gamma = \mathbb{E} \int_{\overline{\mathbb{C}}} \nu(dx) \log \left(\|T_\sigma x\|\right). \quad (6)$$

Indeed, using natural realifications $S^1_{\mathbb{C}} \to S^3$ and $\text{SL}(2, \mathbb{C}) \to \text{SL}(4, \mathbb{R})$, one can apply Theorem 2.2 in [8] which states that

$$\gamma = \sup \left\{ \mathbb{E} \int_{\overline{\mathbb{C}}} \nu(dx) \log \left(\|T_\sigma x\|\right) : \nu \text{ is a } (T_\sigma \ast)-\text{invariant probability measure on } S^1_{\mathbb{C}} \right\}$$

and the r.h.s. of (6) is equal for all $\nu$ corresponding to the unique $\mu$. 

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1.3 Two-parameter family and list of assumptions

As already stated above, examples such as the ones described in Section 5 lead us to consider the following set-up:

(i) The family \((T_{\sigma}^{\epsilon,\delta})_{\sigma \in \Sigma}\) is real analytic in the parameters \(\epsilon\) and \(\delta\).

(ii) At \(\epsilon = \delta = 0\), one has \(T_{\sigma}^{0,0} = \text{diag}(e^{i\eta\sigma}, e^{-i\eta\sigma})\) for some \(\eta_{\sigma} \in [0, 2\pi)\). Here, \(i = \sqrt{-1}\).

(iii) For \(\delta = 0\), \(T_{\sigma}^{\epsilon,0} \in SU(1,1)\).

(iv) For \(\epsilon = 0\), \(T_{\sigma}^{0,\delta} \in SU(1,1)\).

(v) For \(\delta \geq 0\), \(T_{\sigma}^{\epsilon,\delta} \in SU(1,1)\).

For each pair \(\epsilon, \delta\) of fixed parameters, one now obtains by (4) a random dynamical system, the orbits of which will be denoted by \((z_n)_{n \geq 0}\) and \((x_n)_{n \geq 0}\). Thus their dependence on \(\epsilon\) and \(\delta\) is suppressed. Nevertheless, the dependence of other quantities such as the Lyapunov exponent \(\gamma_{\epsilon,\delta}\) and invariant measure \(\mu_{\epsilon,\delta}\) will be kept as it is precisely the main object of the paper to study their dependence on these parameters. For \(\delta \geq 0\) and an initial condition \(z_0 \in \mathbb{D}\), assumption (v) combined with the comments in Section 1.1 assure that the random dynamics stays in the unit disc \(\mathbb{D}\). Hence one is indeed in the situation described in the title and the abstract. The assumptions (i)-(iv) allow to expand the matrices \(T_{\sigma}^{\epsilon,\delta}\) in a Taylor expansion

\[
T_{\sigma}^{\epsilon,\delta} = R_{\eta_{\sigma}} \exp \left[ \epsilon P_{\sigma} + \epsilon^2 P'_{\sigma} + i\delta Q_{\sigma} + O(\epsilon^3, \epsilon\delta, \delta^2) \right],
\]

where \(R_{\eta_{\sigma}} = \text{diag}(e^{i\eta_{\sigma}}, e^{-i\eta_{\sigma}})\) and \(P_{\sigma}, P'_{\sigma}, Q_{\sigma}\) and the terms of higher order are random variables with values in the Lie algebra

\[
su(1,1) = \{ A \in \mathbb{C}^{2 \times 2} : A^* J = -JA, \ Tr(A) = 0 \}
\]

of \(SU(1,1)\). These coefficient matrices are assumed to satisfy the following properties, which imply, in particular, that the support of \(T_{\sigma}^{\epsilon,\delta}\) is compact:

(vi) The random matrices \(P_{\sigma}, P'_{\sigma}\) and \(Q_{\sigma}\) are uniformly bounded in norm.

(vii) The error term \(O(\epsilon^3, \epsilon\delta, \delta^2)\) in (7) is bounded by \(C(\epsilon^3 + \epsilon\delta + \delta^2)\) for a uniform constant \(C\).

Now, if either \(\epsilon\) or \(\delta\) is non-zero, the following additional assumptions hold in most situations:

(viii) For any finite set \(F \subset \mathbb{C}\), one has \(\text{supp}(T_{\sigma}^{\epsilon,\delta}) \cdot F \nsubseteq F\).

(ix) The semigroup generated by \(\text{supp}(T_{\sigma}^{\epsilon,\delta})\) is not relatively compact.

Under the assumptions (viii) and (ix), the \((T_{\sigma}^{\epsilon,\delta})\)-invariant probability measure \(\mu_{\epsilon,\delta}\) is unique [5].
For $\epsilon = \delta = 0$, the Möbius dynamics is merely given by the multiplication by the random phase $e^{2\eta_\sigma}$. Hence, the points 0 and $\infty$ are fixed points of the action $T^{0,0}_{\sigma}$, and its non-trivial orbits lie on circles $rS^1$, $r > 0$. Neither (viii) nor (ix) hold in this situation. On each such circle $rS^1$, a $(T^{0,0}_{\sigma})$-invariant probability measure is given by the normalized spherical measure, which is unique on $rS^1$ if the support of $\eta_\sigma$ contains an irrational multiple of $\pi$ (if the support of $\eta_\sigma$ is a finite subset of $\pi\mathbb{Q}$, then there are many $(T^{0,0}_{\sigma})$-invariant probability measures on $rS^1$). The Lyapunov exponent vanishes in this case.

For $\delta = 0$ and arbitrary $\epsilon$, the Möbius action of $T^{0,0}_{\sigma} \in SU(1,1)$ leaves the unit circle $\partial\mathbb{D} = S^1$ invariant. Hence, there is a $(T^{0,0}_{\sigma})$-invariant measure supported on $S^1$. Rigorous perturbation theory for the Lyapunov exponent has been carried out in [9, 16] under the two assumptions $\mathbb{E}(e^{2\eta_\sigma}) \neq 1$ and $\mathbb{E}(e^{4\eta_\sigma}) \neq 1$, showing that

$$\gamma^{0,0} = D\epsilon^2 + O(\epsilon^3),$$

where $D \geq 0$ is a constant, the definition of which will be recalled in (9) below. The property $D > 0$ can be characterized (see Proposition 5 below). If the two assumptions $\mathbb{E}(e^{2\eta_\sigma}) \neq 1$ and $\mathbb{E}(e^{4\eta_\sigma}) \neq 1$ do not hold, one has to deal with an anomaly. Nevertheless, a more involved analysis still leads to a quadratic behavior in $\epsilon$ as in (8), but a less explicit formula for the constant $D$ [18, 14].

Now let us assume (viii) and (ix) and come to the novel part of this paper, namely the case $\delta > 0$. We first focus on the simpler case $\epsilon = 0$. Under suitable (weak) conditions on $Q_\sigma$, the Möbius action $T^{0,\delta}_{\sigma}$ then drifts towards the center of $\mathbb{D}$ and has a deterministically attracting region around the origin. This implies that the support of $\mu^{0,\delta}$ is a compact subset of $\mathbb{D}$ (see
Figure 2: Same plot and histogram as in Figure 1, but with $\epsilon = 0.1$ and $\delta = 10^{-3}$. For the plot on the left, $10^5$ iterations were run; for the histogram on the right, $5 \cdot 10^5$ iterations were run. As in Figure 1, the initial condition is $z_0 = 1$. If the system starts with $z_0 = 0$, the plot and the histogram look very similar.

Proposition 16 below). As now $\epsilon$ is increased, this behavior is maintained as long as $\epsilon = o(\delta)$. This is illustrated in Figure 1 by a numerical experiment for a generic model with the properties described above. Figure 1 shows that, starting out with an initial condition at $z_0 = 1$, the orbit rotates with a fixed (thus here non-random) angle, while it spirals towards the attracting region in the center because $\delta > 0$ induces this drift. The concrete form of the random family $(T^\epsilon_{\sigma, \delta})_{\sigma \in \Sigma}$ stems from the Anderson model and is described in detail in Section 5. As $\epsilon$ is increased further, the support of the Furstenberg measure generically grows. However, as long as $\epsilon = o(\delta^{1/2})$, the main weight of $\mu^{\epsilon, \delta}$ is close to the center. This is consistent with the bound (11) in Theorem 1.

Once $\epsilon^2$ and $\delta$ are of the same order of magnitude, one reaches a crossover regime and the weight of the invariant measure is spread out over the whole closed unit disc. A plot of a typical orbit in that intermediate regime is shown in Figure 2. It is one of the main results of the paper to determine the radial distribution of $\mu^{\epsilon, \delta}$ in a weak perturbative sense and to show how it depends on the ratio of $\frac{\delta}{\epsilon^2}$ (see statement (14) of Theorem 1 below). Actually, as the ratio decreases, the radial distribution is more and more concentrated at the boundary.

As $\epsilon$ grows further so that $\delta = o(\epsilon^2)$, the numerical experiments exhibit a striking feature (see Figure 3): the orbit sticks to the boundary so that the Furstenberg measure apparently has very little weight outside of a small ring touching $\partial \mathbb{D}$. This is consistent with the bound (12) in Theorem 1. Indeed, presumably the measure $\mu^{\epsilon, \delta}$ converges for $\delta \to 0$ weakly to the Furstenberg measure $\mu^{\epsilon, 0}$ supported on $S^1$ described above. However, the results below (statement (14) of Theorem 1) only imply that the radial distribution of $\mu^{\epsilon, \delta}$ converges weakly to the Dirac measure on the radius 1.
Figure 3: Same plot and histogram as in Figure 1, but with $\epsilon = 0.1$ and $\delta = 10^{-5}$ so that $\delta = o(\epsilon^2)$. The number of iterations is $5 \cdot 10^4$. The initial condition was $z_0 = 1$. If one chooses $z_0 = 0$ as initial condition, the orbit takes several hundreds of iterations to attain the boundary, but the histogram after a large number of iterations essentially looks the same.

1.5 Main results on the invariant measures

Let us now state the main result on the Furstenberg measure. Its first two items confirm the numerical results of Figures 1 and 3 in a rather weak form, respectively. More generally, the third item provides an approximation of the radial distribution of $\mu^{\epsilon, \delta}$. To state the results and further ones below, let us introduce the following basis of $\text{su}(1, 1)$:

$$B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The expression for $D$ in terms of $\eta_\sigma$ and the first order term $P_\sigma$ in (7) is

$$D = \frac{1}{2} \mathbb{E}(|\beta_\sigma|^2) + \Re \left( \frac{\mathbb{E}(\beta_\sigma)\mathbb{E}(e^{2i\eta_\sigma} \overline{\beta_\sigma})}{1 - \mathbb{E}(e^{2i\eta_\sigma})} \right),$$

where

$$\beta_\sigma = \frac{1}{2} \text{Tr}((B_1 - iB_2)P_\sigma).$$

Let us also introduce a notation for a further constant that turns out to be relevant in the following:

$$C = \frac{1}{2} \mathbb{E}(\text{Tr}(B_3^* Q_\sigma)).$$

In Remark 7 below, it is shown that assumptions (iii), (iv) and (v) imply $C \geq 0$. If both $C > 0$ and $D > 0$, it is furthermore convenient to use

$$\lambda = 2 \frac{C}{D} \frac{\delta}{\epsilon^2}.$$
Figure 4: Approximate radial density $\varrho_\lambda$ (blue) given by (13) and numerical histogram obtained after $2 \cdot 10^7$ iterations (yellow). The values are $(\epsilon, \delta) = (0.05, 7.5 \cdot 10^{-4})$ on the left, $(\epsilon, \delta) = (0.05, 1.2 \cdot 10^{-4})$ in the middle and $(\epsilon, \delta) = (0.05, 2.5 \cdot 10^{-5})$ on the right.

as a measure of the (crucial) relative size of $\delta$ and $\epsilon^2$.

Furthermore, we denote by $C^1([0,1])$ differentiable functions where the derivatives at the boundary points (and only there) are taken one-sided, and this derivative is a continuous function on $[0,1]$. The classes $C^k([0,1])$ for $k \in \mathbb{N}$ are then defined by iteration in $k$.

**Theorem 1** Assume (i)-(ix) as well as $\mathbb{E}(e^{2\eta_\sigma}) \neq 1$ and $\mathbb{E}(e^{4\eta_\sigma}) \neq 1$. If $C > 0$, one has

$$
\int D \mu_{\epsilon,\delta}^{x,y}(dz) \frac{|z|^2}{z} \approx O(\delta, \epsilon, \epsilon^2) \quad (11)
$$

and, if $D > 0$, one has

$$
\int D \mu_{\epsilon,\delta}^{x,y}(dz) \frac{|z|^2}{z} = 1 + O(\epsilon^{2/3}, \delta^{2/3} \epsilon^{-1}) \quad (12)
$$

Further, if $C > 0$ and $D > 0$, the radial distribution of $\mu_{\epsilon,\delta}$ is approximated in a weak sense by the radial density

$$
\varrho_\lambda(s) = \frac{\lambda}{(1-s)^2} \exp \left( -\frac{\lambda s}{1-s} \right) \quad (13)
$$

with $\lambda$ given by (10), namely more precisely, for all $h \in C^2([0,1])$, one has

$$
\int D \mu_{\epsilon,\delta}^{x,y}(dz) h(|z|^2) = \int_0^1 ds \varrho_\lambda(s) h(s) + O(\epsilon, \epsilon^{-1} \delta, \epsilon^{-2} \delta^2) \quad (14)
$$

**Remark 2** Using the bijection $s \in (0,1) \mapsto x = \frac{s}{1-s} \in (0,\infty)$ the distribution (13) becomes the exponential distribution $\lambda e^{-\lambda x}$ on $[0,\infty)$. Inserting into (14) a smooth approximation $h$ of $\chi_{[0,s]}$ yields an approximation (no claim is made on the error bounds which depend on $h$) to the cumulative radial distribution:

$$
\mu_{\epsilon,\delta}^{x,y}(\{z \in \mathbb{D} : |z|^2 \leq s\}) \approx 1 - \exp \left( -\frac{\lambda s}{1-s} \right)
$$
Hence for $\lambda \to 0$ one has $\mu^{\epsilon,\delta} \to \delta_1$, while for $\lambda \to \infty$ rather $\mu^{\epsilon,\delta} \to \delta_0$, both up to (uncontrolled) error terms. Thus (11) and (12) are consistent with (14). Figure 4 shows a histogram of the values of $|z|^2$ along an orbit for the models stated as well as the density $\rho_{\lambda}$, properly scaled. The agreement is excellent.

Let us note that the histograms in Figures 1, 2 and 3 show the distribution of $r = |z|$ (comparable with the orbit plots). Of course, the approximate distribution of $r$ is given by $2r\rho_{\lambda}(r^2)dr$ and it vanishes linearly as $r \to 0$. ○

**Remark 3** If $f(re^{i\theta}) = \sum_{j=-J}^{J} r^j e^{ij\theta} f_j(r^2)$ is a trigonometric polynomial in the angle with functions $f_0 \in C^2([0,1])$ and $f_j \in C^1([0,1])$ for $j \in \{\pm1, \ldots, \pm J\}$, then the techniques below show that

$$\int_{\mathbb{R}} \mu^{\epsilon,\delta}(dz) f(z) = \int_{\mathbb{R}} \mu^{\epsilon,\delta}(dz) f_0(|z|^2) + O(\epsilon, \delta)$$

$$= \int_0^1 ds \rho_\lambda(s) f_0(s) + O(\epsilon, \epsilon^{-1}, \delta, \epsilon^{-2} \delta^2),$$

provided that $\mathbb{E}(e^{2ij\eta_\sigma}) \neq 1$ for $j = -J, \ldots, J$ (see in particular Lemma 9). ○

### 1.6 Expansion of the Lyapunov exponent

The second main result is an expansion of the Lyapunov exponent up to the order $O(\epsilon^3, \epsilon \delta, \delta^2)$:

**Theorem 4** Assume (i)-(ix) and $\mathbb{E}(e^{2ij\eta_\sigma}) \neq 1$ for $j = 1, 2$. Then, one has

$$\gamma^{\epsilon,\delta} = C \delta + D \epsilon^2 + O(\epsilon^3, \epsilon \delta, \delta^2). \quad (15)$$

The perturbative formula (15) generalizes the result of [16] which considered the case $\delta = 0$. Proposition 5 characterizes the positivity of $D$ that is crucial for many of the results above.

**Proposition 5 ([16])** One has $D \geq 0$ and $D = 0$ if and only one of the following two mutually excluding cases occurs:

(i) Both $e^{2i\eta_\sigma}$ and $\beta_\sigma$ are $\mathbb{P}$-a.s. constant.

(ii) $\mathbb{E}(e^{2n\eta_\sigma}) = 0$ and $\beta_\sigma$ is a constant multiple of $1 - e^{2n\eta_\sigma}$.

**Remark 6** By pushing the techniques of this paper, it is possible to compute also higher order terms in the expansion in $\epsilon$ and $\delta$. This would require to carry out even more cumbersome Taylor expansions in Section 2.1, and we refrained from doing so. A more challenging, but presumably feasible extension is a perturbative formula for the variance in the central limit theorem for the Lyapunov exponent. For the case of real $2 \times 2$ matrices this was achieved in [16] by techniques that would have to be adapted to complex matrices. ○
2 Analysis of the Furstenberg measure

From now on the framework described in Section 1.3 is supposed to hold. Therefore the random dynamical system (4) reads more explicitly

$$ z_n = T_{\sigma_n}^\varepsilon \cdot z_{n-1}, $$

(16)

where $((\sigma_n))_{n \in \mathbb{N}}$ is a random sequence of independent and identically distributed copies of $\sigma$. The sequence $((\sigma_n))_{n \in \mathbb{N}}$ is hence distributed according to $P = \mathbb{P}^{\otimes \mathbb{N}}$. The average w.r.t. $P$ is denoted by $E$. Moreover, any random variable of the form $X_{\sigma_n}$ will simply be denoted by $X_n$. By definition, the Furstenberg measure satisfies

$$ \int_B \mu^{\varepsilon,\delta}(dz) \ f(z) = \int_B \mu^{\varepsilon,\delta}(dz) \ E \ f(T_{\sigma}^\varepsilon \cdot z) $$

(17)

for all continuous functions $f$. Iteration and averaging then shows that all $N \in \mathbb{N}$ obey

$$ \int_B \mu^{\varepsilon,\delta}(dz) \ f(z) = \int_B \mu^{\varepsilon,\delta}(dz) \ E \ \frac{1}{N} \sum_{n=0}^{N-1} f(z_n), $$

(18)

where $z = z_0$ is the initial condition in the definition (16) of the random dynamics. Theorem 1 will be proved by analyzing the Birkhoff sum

$$ E \ \frac{1}{N} \sum_{n=0}^{N-1} f(z_n). $$

2.1 Algebraic preparations

Let us introduce the real-valued random variables $(p_{j,\sigma})_{j=1}^3$, $(p'_{j,\sigma})_{j=1}^3$ and $(q_{j,\sigma})_{j=1}^3$ by

$$ P_\sigma = \sum_{j=1}^3 p_{j,\sigma} B_j, \quad P'_\sigma = \sum_{j=1}^3 p'_{j,\sigma} B_j, \quad Q_\sigma = \sum_{j=1}^3 q_{j,\sigma} B_j. $$

The Baker-Campbell-Hausdorff formula implies the identity

$$ T_{\sigma}^{\varepsilon,\delta} = R_{\eta_a} e^{\varepsilon(p_{3,\sigma} + p'_{3,\sigma}) B_3} e^{\varepsilon(p_{2,\sigma} + p'_{2,\sigma}) B_2} e^{\varepsilon(p_{1,\sigma} + p'_{1,\sigma}) B_1} e^{i \delta q_{1,\sigma} B_3} e^{i \delta q_{2,\sigma} B_2} e^{i \delta q_{1,\sigma} B_1} + O(\varepsilon^3, \delta\varepsilon, \delta^2), $$

(19)

where $(\tilde{p}_{j,\sigma})_{j=1}^3$ are real-valued random variables containing the coefficients of $P'_\sigma$ and the commutators of the terms of the order $O(\varepsilon)$. Due to the assumptions (vi) and (vii), the variables $(\tilde{p}_{j,\sigma})_{j=1}^3$ and the terms of order $O(\varepsilon^3, \delta^2, \varepsilon \delta)$ in (19) have compact support. One is thus led to compute the exponentials of $tB_1$, $tB_2$, $tB_3$, $tB_1$, $tB_2$, $tB_3$ for all $t \in \mathbb{R}$:

$$ e^{tB_1} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}, \quad e^{tB_2} = \begin{pmatrix} \cosh(t) & i \sinh(t) \\ -i \sinh(t) & \cosh(t) \end{pmatrix}, \quad e^{tB_3} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, $$

$$ e^{itB_1} = \begin{pmatrix} \cos(t) & i \sin(t) \\ i \sin(t) & \cos(t) \end{pmatrix}, \quad e^{itB_2} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \quad e^{itB_3} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}. $$

(20)
Remark 7 The assumption (v) guarantees $q_{3, \sigma} \geq 0$ for all $\sigma \in \Sigma$. Thus, $C = \mathbb{E}(q_{3, \sigma}) \geq 0$.

Lemma 8 summarizes explicit expansions of the action $T_{\sigma}^{\epsilon, \delta}$. For this purpose, it will be helpful to introduce further random variables by

$$
\beta_{\sigma} = p_{1, \sigma} - \epsilon p_{2, \sigma}, \quad \beta'_{\sigma} = p'_{1, \sigma} - \epsilon p'_{2, \sigma}, \quad \bar{\beta}_{\sigma} = \bar{p}_{1, \sigma} - \epsilon \bar{p}_{2, \sigma}, \quad \xi_{\sigma} = q_{1, \sigma} - \epsilon q_{2, \sigma}.
$$

Note that $\beta_{\sigma}$ is a rewriting of the definition (9).

Lemma 8 All $z \in \overline{\mathbb{D}}$ satisfy

$$
T_{\sigma}^{\epsilon, \delta} \cdot z = e^{2m_{\sigma} z} \left[ z + \epsilon \left( \beta_{\sigma} + 2 \epsilon p_{3, \sigma} z - \beta_{\sigma} z^2 \right) + \delta \left( \bar{\xi}_{\sigma} - 2 q_{3, \sigma} z - \bar{\xi}_{\sigma} z^2 \right) \right]
\left( 1 + \frac{2}{3} \epsilon \bar{p}_{3, \sigma} + 2 \epsilon^2 \bar{p}_{3, \sigma} \right) z + \beta_{\sigma} z^3
+ O(\epsilon^3, \epsilon \delta, \delta^2)
$$

(21)

and

$$
|T_{\sigma}^{\epsilon, \delta} \cdot z|^2 = |z|^2 + 2 \epsilon \Re(\beta_{\sigma} z) (1 - |z|^2) + \epsilon^2 \left( |\beta_{\sigma}|^2 (1 - |z|^2) + 2 \Re(\bar{\beta}_{\sigma} z - \beta_{\sigma} z^2) \right) (1 - |z|^2)
+ 2 \delta \left[ 3 \Re(\xi_{\sigma} z) [1 + |z|^2] - 2 q_{3, \sigma} |z|^2 \right] + O(\epsilon^3, \epsilon \delta, \delta^2).
$$

Moreover, all $g \in C^2([0, 1])$ and all $z \in \overline{\mathbb{D}}$ satisfy

$$
g \left( |T_{\sigma}^{\epsilon, \delta} \cdot z|^2 \right) = g(|z|^2) + g'(|z|^2) \left[ 2 \epsilon \Re(\beta_{\sigma} z) (1 - |z|^2) + 2 \delta \left[ 3 \Re(\xi_{\sigma} z) [1 + |z|^2] - 2 q_{3, \sigma} |z|^2 \right]
+ \epsilon^2 \left( |\beta_{\sigma}|^2 (1 - |z|^2) + 2 \Re(\bar{\beta}_{\sigma} z - \beta_{\sigma} z^2) \right) (1 - |z|^2) \right]
+ \epsilon^2 g''(|z|^2) \left[ \Re(\beta_{\sigma} z) + |\beta_{\sigma}|^2 |z|^2 \right] (1 - |z|^2)^2 + O(\epsilon^3, \epsilon \delta, \delta^2).
$$

(23)

Proof. For $z \in \overline{\mathbb{D}}$, let us begin by computing the identities

$$
e^{\epsilon(p_{1, \sigma} + \epsilon \bar{p}_{1, \sigma}) B_1} z = z + \epsilon [1 - z^2] p_{1, \sigma} + \epsilon^2 [1 - z^2] [\bar{p}_{1, \sigma} - z p_{2, \sigma}] + O(\epsilon^3),
$$

(24)

$$
e^{\epsilon(p_{2, \sigma} + \epsilon \bar{p}_{2, \sigma}) B_2} z = z + \epsilon [1 + z^2] p_{2, \sigma} + \epsilon^2 [1 + z^2] [\bar{p}_{2, \sigma} - z p_{1, \sigma}] + O(\epsilon^3),
$$

(25)

$$
e^{\epsilon(p_{3, \sigma} + \epsilon \bar{p}_{3, \sigma}) B_3} z = z + 2 \epsilon p_{3, \sigma} z + 2 \epsilon^2 [\bar{p}_{3, \sigma} - p_{3, \sigma}] z + O(\epsilon^3),
$$

(26)

$$
e^{i \beta_{3, \sigma} B_3} e^{i \beta_{2, \sigma} B_2} e^{i \beta_{1, \sigma} B_1} z = z + \delta \left[ i \bar{\xi}_{\sigma} - 2 q_{3, \sigma} z - i \xi_{\sigma} z^2 \right] + O(\delta^2).
$$

(27)

Next, by combining (19), (24), (25) and (27), one obtains for all $z \in \overline{\mathbb{D}}$ the equation

$$
e^{\epsilon(p_{3, \sigma} + \epsilon \bar{p}_{3, \sigma}) B_3} R_{\eta_{\sigma}}^{-1} T_{\sigma}^{\epsilon, \delta} \cdot z = e^{\epsilon(p_{2, \sigma} + \epsilon \bar{p}_{2, \sigma}) B_2} e^{\epsilon(p_{1, \sigma} + \epsilon \bar{p}_{1, \sigma}) B_1} e^{i \beta_{3, \sigma} B_3} e^{i \beta_{2, \sigma} B_2} e^{i \beta_{1, \sigma} B_1} \cdot z
\left( 1 + \frac{2}{3} \epsilon \bar{p}_{3, \sigma} + 2 \epsilon^2 \bar{p}_{3, \sigma} \right) z
+ \delta \left[ i \bar{\xi}_{\sigma} - 2 q_{3, \sigma} z - i \xi_{\sigma} z^2 \right] + O(\epsilon^3, \epsilon \delta, \delta^2).
$$

(28)
As the action $e^{-c(p_{3,\sigma} + \bar{p}_{3,\sigma})B_1} R_{\eta_{\sigma}}^{-1}$ preserves the modulus, (28) implies that all $z \in \overline{D}$ satisfy

$$|T_{\sigma}^{\epsilon, \delta} \cdot z|^2 = |e^{-c(p_{3,\sigma} + \bar{p}_{3,\sigma})B_1} R_{\eta_{\sigma}}^{-1} T_{\sigma}^{\epsilon, \delta} \cdot z|^2$$

$$= |z|^2 + 2\epsilon \Re(\beta_\sigma z)(1 - |z|^2) + \epsilon^2 [\beta_\sigma^2 (1 - |z|^2) + 2 \Re(\bar{\beta}_\sigma z - \beta_\sigma^2 z^2)] (1 - |z|^2)$$

$$+ 2\delta [\Im(\xi_\sigma z)[1 + |z|^2] - 2q_{3,\sigma} |z|^2] + O(\epsilon^3, \epsilon\delta, \delta^2)$$

which proves (22). Moreover, combining (26) with (28) yields

$$R_{\eta_{\sigma}}^{-1} T_{\sigma}^{\epsilon, \delta} \cdot z = z + \epsilon \left[ \frac{\beta_\sigma - \beta_\sigma z^2}{\beta_\sigma - \beta_\sigma z^2} \right] + \epsilon^2 \left[ \frac{\beta_\sigma - \beta_\sigma z^2}{\beta_\sigma - \beta_\sigma z^2} + i \Im(\beta_\sigma^2) z - \beta_\sigma z^2 + \beta_\sigma^2 z^3 \right]$$

$$+ \delta \left[ i \xi_\sigma - 2q_{3,\sigma} z - i \xi_\sigma z^2 \right] + 2i p_{3,\sigma} \left( z + \epsilon \left[ \frac{\beta_\sigma - \beta_\sigma z^2}{\beta_\sigma - \beta_\sigma z^2} \right] + 2\epsilon^2 \left[ \bar{p}_{3,\sigma} - p_{3,\sigma}^2 \right] \right) z$$

$$+ O(\epsilon^3, \epsilon\delta, \delta^2)$$

$$= z + \epsilon \left[ \beta_\sigma + 2i p_{3,\sigma} z - \beta_\sigma z^2 \right] + \delta \left[ i \xi_\sigma - 2q_{3,\sigma} z - i \xi_\sigma z^2 \right]$$

$$+ \epsilon^2 \left[ \frac{\beta_\sigma - \beta_\sigma z^2}{\beta_\sigma - \beta_\sigma z^2} + 2i p_{3,\sigma} (\beta_\sigma - \beta_\sigma z^2) \right]$$

$$- \epsilon^2 \left[ \beta_\sigma^2 + i \Im(\beta_\sigma^2) \right] \left( z + \epsilon \left[ \beta_\sigma + 2i p_{3,\sigma} z - \beta_\sigma z^2 \right] + O(\epsilon^3, \epsilon\delta, \delta^2) \right)$$

for all $z \in \overline{D}$, which implies (21) due to $R_{\eta_{\sigma}} \cdot z = e^{2\eta_{\sigma} z} z$. As for (23), let us use the identity (22) and Taylor’s theorem in the first and second step, respectively, to obtain

$$g(|T_{\sigma}^{\epsilon, \delta} \cdot z|^2) = g(|z|^2 + \epsilon a + \epsilon^2 b + \epsilon^3 c + O(\epsilon^3, \epsilon\delta, \delta^2))$$

$$= g(|z|^2) + \left[ \epsilon a + \epsilon^2 b + \epsilon^3 c \right] g'(|z|^2) + \frac{\epsilon^2}{2} a^2 g''(|z|^2) + O(\epsilon^3, \epsilon\delta, \delta^2), \quad (29)$$

where

$$a = 2 \Re(\beta_\sigma z)(1 - |z|^2), \quad (30)$$

$$b = \left[ \beta_\sigma^2 (1 - |z|^2) + 2 \Re(\bar{\beta}_\sigma z - \beta_\sigma^2 z^2) \right] (1 - |z|^2), \quad (31)$$

$$c = 2 \Im(\xi_\sigma z)[1 + |z|^2] - 2q_{3,\sigma} |z|^2. \quad (32)$$

Inserting equations (30), (31), (32) and

$$a^2 = 4 \Re(\beta_\sigma z)^2 (1 - |z|^2)^2 = 2 \left[ \Re(\beta_\sigma z)^2 + \beta_\sigma^2 |z|^2 \right] (1 - |z|^2)^2$$

into (29) yields the desired identity (23).

\[ \square \]

### 2.2 Oscillatory phase argument to lowest order

At $\epsilon = \delta = 0$, the dynamics is simply a rotation around the origin by the random angle $\eta_{\sigma}$. Due to the assumption $\mathbb{E}(e^{2\eta_{\sigma}}) \neq 1$, there is a proper (non-trivial) average rotation. Hence Birkhoff sums like

$$\mathbb{E}\left[ \frac{1}{N} \sum_{n=0}^{N-1} z_n g(|z_n|^2) \right]$$
for functions $g : [0, 1] \to \mathbb{C}$ tend to zero for large $N$ because the phases $(\theta_n)_{n \geq 1}$ of $(z_n)_{n \geq 1} = (r_n e^{i \theta_n})_{n \geq 1}$ lead to oscillatory summands with constant moduli $(r_n)_{n \geq 1}$. If $\epsilon$ and $\delta$ are non-zero, the same behavior still holds approximately, as stated by Lemma 9 for sufficiently smooth functions $g$. The basic idea of the argument leading to the following statement goes back to Pastur and Figotin [12] and was applied e.g. in [9] and [16].

**Lemma 9** Let $j \in \mathbb{Z} \setminus \{0\}$ and $g \in C^1([0, 1])$. If $E(e^{2ij\eta_\sigma}) \neq 1$, then one has

$$
E \frac{1}{N} \sum_{n=0}^{N-1} z_n^j g(|z_n|^2) = O(\epsilon, \delta, N^{-1})
$$

and

$$
\int_B \mu^\epsilon,\delta(dz) \ z^j g(|z|^2) = O(\epsilon, \delta),
$$

where the constants of the error bounds depend on $j$ and $g$.

**Proof.** As $T^\epsilon,\delta_\sigma \cdot z = e^{2i\eta_\sigma} z + O(\epsilon, \delta)$, Taylor’s theorem implies that for all $n \in \{0, \ldots, N-1\}$

$$
z_{n+1}^j g(|z_{n+1}|^2) = e^{2i\eta_{n+1}} z_n^j g(|z_n|^2) + O(\epsilon, \delta)
$$

and hence

$$
E z_n^j g(|z_{n+1}|^2) = E(e^{2ij\eta_\sigma}) E z_n^j g(|z_n|^2) + O(\epsilon, \delta).
$$

Averaging (35) from $n = 0$ to $N - 1$ yields

$$
E \frac{1}{N} \sum_{n=1}^{N} z_n^j g(|z_n|^2) = E(e^{2ij\eta_\sigma}) E \frac{1}{N} \sum_{n=0}^{N-1} z_n^j g(|z_n|^2) + O(\epsilon, \delta),
$$

which is equivalent to

$$
E \frac{1}{N} \sum_{n=0}^{N-1} z_n^j g(|z_n|^2) = \frac{z_n^j g(|z_0|^2) - E z_n^j g(|z_N|^2)}{1 - E(e^{2ij\eta_\sigma})} + O(\epsilon, \delta) = O(\epsilon, \delta, N^{-1}).
$$

This proves (33), which, in turn, implies (34) by using (18) in the limit $N \to \infty$. $\Box$

In fact, Birkhoff sums like (33) can be computed more precisely by analyzing the terms of the order $O(\epsilon, \delta)$ in (36). Lemma 10 treats the case $j = 1$ with $g \in C^2([0, 1])$.

**Lemma 10** Suppose that $E(e^{2i\eta_\sigma}) \neq 1$ and $E(e^{4i\eta_\sigma}) \neq 1$. Then for all $g \in C^2([0, 1])$ and $N \in \mathbb{N}$

$$
E \frac{1}{N} \sum_{n=0}^{N-1} z_n g(|z_n|^2) = \epsilon \frac{E(e^{2i\eta_\sigma})}{1 - E(e^{2i\eta_\sigma})} E \frac{1}{N} \sum_{n=0}^{N-1} g(|z_n|^2 + |z_n|^2(1 - |z_n|^2)g'(|z_n|^2))

+ O(\epsilon^2, \delta, N^{-1}).
$$

Moreover, all $g \in C^2([0, 1])$ satisfy

$$
\int_B \mu^\epsilon,\delta(dz) \ z g(|z|^2) = \epsilon \frac{E(e^{2i\eta_\sigma})}{1 - E(e^{2i\eta_\sigma})} \int_B \mu^\epsilon,\delta(dz) \left[g(|z|^2) + |z|^2(1 - |z|^2)g'(|z|^2)\right] + O(\epsilon^2, \delta).
$$

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Proof. The identity (21) implies that for all \( n \in \{0, \ldots, N-1\} \)
\[
z_{n+1} = e^{2\text{i}n\eta} + e^{2\text{i}n\eta} \left[ z_n + \epsilon \left[ \beta_{n+1} + 2\text{i}\eta_3 n z_n \right] \right] + \mathcal{O}(\epsilon^2, \delta).
\]
Moreover, the identity (23) imply that for all \( n \in \{0, \ldots, N-1\} \)
\[
g(\varepsilon z_n) = g(\varepsilon z_n) + 2 \epsilon \Re(\beta_{n+1} z_n) \left( 1 - |z_n|^2 \right) g'(|z_n|^2) + \mathcal{O}(\epsilon^2, \delta).
\]
Combining (39) and (40) yields for all \( n \in \{0, \ldots, N-1\} \) the identity
\[
z_{n+1} g(\varepsilon z_n) = e^{2\text{i}n\eta} \left[ z_n g(\varepsilon z_n) + \epsilon \left[ \beta_{n+1} + 2\text{i}\eta_3 n z_n - \beta_{n+1} z_n^2 \right] g(\varepsilon z_n) \right]
\[
+ \beta_{n+1} z_n^2 + \beta_{n+1} z_n^2 \left( 1 - |z_n|^2 \right) g'(\varepsilon z_n)
\]
\[
+ \mathcal{O}(\epsilon^2, \delta)
\]
\[
= e^{2\text{i}n\eta} z_n g(\varepsilon z_n) + \epsilon \left[ e^{2\text{i}n\eta} \beta_{n+1} \left[ g(\varepsilon z_n) + |z_n|^2 \left( 1 - |z_n|^2 \right) g'(\varepsilon z_n) \right] \right]
\]
\[
+ \epsilon \left[ e^{2\text{i}n\eta} \beta_{n+1} z_n^2 \left[ |z_n|^2 \left( 1 - |z_n|^2 \right) g'(\varepsilon z_n) - g(\varepsilon z_n) \right] \right]
\]
\[
+ 2\epsilon \left[ e^{2\text{i}n\eta} \eta_3 n z_n g(\varepsilon z_n) + \mathcal{O}(\epsilon^2, \delta) \right].
\]
Averaging this from \( n = 0 \) to \( N-1 \) yields
\[
\mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} z_{n+1} g(\varepsilon z_n) = \mathbb{E}(e^{2\text{i}n\eta}) \mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} z_n g(\varepsilon z_n)
\]
\[
+ \epsilon \mathbb{E}(e^{2\text{i}n\eta}) \mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} \left[ g(\varepsilon z_n) + |z_n|^2 \left( 1 - |z_n|^2 \right) g'(\varepsilon z_n) \right]
\]
\[
+ \epsilon \mathbb{E}(e^{2\text{i}n\eta}) \mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} \left[ |z_n|^2 \left( 1 - |z_n|^2 \right) g'(\varepsilon z_n) - g(\varepsilon z_n) \right]
\]
\[
+ 2\epsilon \mathbb{E}(e^{2\text{i}n\eta}) \mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} z_n g(\varepsilon z_n) + \mathcal{O}(\epsilon^2, \delta).
\]
According to Lemma 9, the third and the fourth summand are of the order \( \mathcal{O}(\epsilon^2, \delta, N^{-1}\epsilon) \).
Hence
\[
\mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} z_{n+1} g(\varepsilon z_n) = \epsilon \left[ \mathbb{E}(e^{2\text{i}n\eta} \beta_{\sigma}) \right] \mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} \left[ g(\varepsilon z_n) + |z_n|^2 \left( 1 - |z_n|^2 \right) g'(\varepsilon z_n) \right]
\]
\[
+ \frac{z_0 g(\varepsilon z_0) - \mathbb{E} z_N g(\varepsilon z_N)}{[1 - \mathbb{E}(e^{2\text{i}n\eta})] N} + \mathcal{O}(\epsilon^2, \delta, N^{-1}\epsilon)
\]
\[
= \epsilon \left[ \mathbb{E}(e^{2\text{i}n\eta} \beta_{\sigma}) \right] \mathbb{E} \frac{1}{N} \sum_{n=0}^{N-1} \left[ g(\varepsilon z_n) + |z_n|^2 \left( 1 - |z_n|^2 \right) g'(\varepsilon z_n) \right]
\]
\[
+ \mathcal{O}(\epsilon^2, \delta, N^{-1}),
\]
which implies (37), which, in turn, implies (38) by using (18) in the limit \( N \to \infty \).

For the computation of the Lyapunov exponent, the special case \( g(s) = \log(1 + s) \) in (38) will be relevant, and it is also one of the elements in the proof of Theorem 4:

**Corollary 11** The Furstenberg measure \( \mu^{\epsilon, \delta} \) satisfies

\[
\int_{\mathbb{D}} \mu^{\epsilon, \delta}(dz) \frac{z}{1 + |z|^2} = \epsilon \mathbb{E}(e^{2\eta_0} \beta_0) \int_{\mathbb{D}} \mu^{\epsilon, \delta}(dz) \frac{1 + |z|^4}{(1 + |z|^2)^2} + \mathcal{O}(\epsilon^2, \delta). \tag{41}
\]

**2.3 Oscillatory phase argument to second order**

The section applies the oscillation argument to functions that only depend on the modulus of their argument. This allows to complete the proof of Theorem 1.

**Lemma 12** Suppose that \( \mathbb{E}(e^{2\eta_0}) \neq 1 \) and \( \mathbb{E}(e^{4\eta_0}) \neq 1 \) hold. Then all \( g \in C^3([0,1]) \) satisfy

\[
2C \delta \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} |z_n|^2 g'(|z_n|^2)
= \mathcal{D} \epsilon^2 \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} (1 - |z_n|^2)^2 \left(g'(|z_n|^2) + |z_n|^2 g''(|z_n|^2)\right) + \mathcal{O}(\epsilon^3, \epsilon \delta, \delta^2, N^{-1}) \tag{42}
\]

for all \( N \in \mathbb{N} \), and

\[
\int_{\mathbb{D}} \mu^{\epsilon, \delta}(dz) \left[2C \delta |z|^2 g'(|z|^2) - \mathcal{D} \epsilon^2 (1 - |z|^2)^2 \left(g'(|z|^2) + |z|^2 g''(|z|^2)\right)\right] = \mathcal{O}(\epsilon^3, \epsilon \delta, \delta^2). \tag{43}
\]

**Proof.** Let us start out with (23) for each point of the orbit \( (z_n)_{n \in \mathbb{N}} \). Taking the average along the orbit leads to

\[
\mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} g(|z_{n+1}|^2) = \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} g(|z_n|^2) + 2 \epsilon \Re \left( \mathbb{E}(\beta_0) \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} z_n (1 - |z_n|^2) g'(|z_n|^2) \right)
- 4C \delta \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} |z_n|^2 g'(|z_n|^2) + \epsilon^2 \mathbb{E}(|\beta_0|^2) \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} (1 - |z_n|^2)^2 g'(|z_n|^2)
+ \epsilon^2 \mathbb{E}(|\beta_0|^2) \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} |z_n|^2 (1 - |z_n|^2)^2 g''(|z_n|^2)
+ 2 \delta \Im \left( \mathbb{E}(\xi_0) \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} z_n [1 + |z_n|^2] g'(|z_n|^2) \right)
+ 2 \epsilon^2 \Re \left( \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} \left[ \mathbb{E}(\beta_0) z_n - \mathbb{E}(\beta_0^2) z_n^2 \right] (1 - |z_n|^2) g'(|z_n|^2) \right)
+ \epsilon^2 \Re \left( \mathbb{E}(\beta_0^2) \mathbb{E}\frac{1}{N} \sum_{n=0}^{N-1} z_n^2 (1 - |z_n|^2)^2 g''(|z_n|^2) \right) + \mathcal{O}(\epsilon^3, \epsilon \delta, \delta^2).
\]

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According to Lemma 9, the last three lines are of the order $O(\epsilon^3, \epsilon\delta, \epsilon^2N^{-1}, \delta N^{-1})$. Thus

$$2C\delta E\frac{1}{N} \sum_{n=0}^{N-1} |z_n|^2 g'(|z_n|^2)$$

$$= \frac{g(|z_0|^2) - E g(|zN|^2)}{N} + \epsilon \Re \left( E(\beta_\sigma) E \frac{1}{N} \sum_{n=0}^{N-1} z_n (1 - |z_n|^2) g'(|z_n|^2) \right)$$

$$+ \epsilon^2 \frac{1}{2} E(|\beta_\sigma|^2) E \frac{1}{N} \sum_{n=0}^{N-1} (1 - |z_n|^2)^2 \left[ g'(|z_n|^2) + |z_n|^2 g''(|z_n|^2) \right] + O(\epsilon^3, \epsilon\delta, \epsilon^2N^{-1}, \delta N^{-1})$$

$$= \epsilon \Re \left( E(\beta_\sigma) E \frac{1}{N} \sum_{n=0}^{N-1} z_n (1 - |z_n|^2) g'(|z_n|^2) \right)$$

$$+ \epsilon^2 \frac{1}{2} E(|\beta_\sigma|^2) E \frac{1}{N} \sum_{n=0}^{N-1} (1 - |z_n|^2)^2 \left[ g'(|z_n|^2) + |z_n|^2 g''(|z_n|^2) \right] + O(\epsilon^3, \epsilon\delta, \epsilon^2, N^{-1}).$$

In view of Lemma 10 and due to the identity

$$(1 - s)g'(s) + s(1 - s)\partial_s((1 - s)g'(s)) = (1 - x)^2(g'(s) + sg''(s))$$

for $s = |z|^2$, this implies (42). In the limit $N \to \infty$, one infers (43) by using (18). \hfill \Box

Inserting $g(s) = \log(1 + s)$ in (43) yields an equation used in the proof of Theorem 4:

**Corollary 13** The Furstenberg measure $\mu^{\epsilon, \delta}$ satisfies the identity

$$2C\delta \int_B d\mu^{\epsilon, \delta}(z) \frac{|z|^2}{1 + |z|^2} - D \epsilon^2 \int_B d\mu^{\epsilon, \delta}(z) \left[ \frac{1 - |z|^2}{1 + |z|^2} \right]^2 = O(\epsilon^3, \epsilon\delta, \epsilon^2). \quad (44)$$

Based on the statement (43) of Lemma 12, it is now possible to proceed with the proof of Theorem 1. Before going into technical details, let us give some intuition though, basically based on the general strategy outlined in [14]. For that purpose, let us set $s = |z|^2$ and suppose that $\mu^{\epsilon, \delta}$ is absolutely continuous on radial functions, namely that there exists a probability density $\rho^{\epsilon, \delta} : [0, 1] \to [0, \infty]$ such that

$$\int_B \mu^{\epsilon, \delta}(dz) g(|z|^2) = \int_0^1 ds \rho^{\epsilon, \delta}(s) g(s).$$

Supposing, moreover, that $D > 0$, one can rewrite (43) divided by $D\epsilon^2$ as

$$\int_0^1 ds \rho^{\epsilon, \delta}(s) \left( \lambda sg'(s) - (1 - s)^2(g'(s) + sg''(s)) \right) = O(\epsilon, \epsilon^{-1}\delta, \epsilon^{-2}\delta^2), \quad (45)$$

with $\lambda$ defined as in (10). Given this link between $\epsilon$ and $\delta$, let us set $\rho_\lambda = \rho^{\epsilon, \delta}$. Therefore it is of interest to define a second-order differential operator $L_\lambda : C^2([0, 1]) \to C([0, 1])$ by

$$L_\lambda = (\lambda - (1 - s)^2\partial_s)s\partial_s = -s(1 - s)^2\partial_s - (\lambda s - (1 - s)^2)\partial_s.$$
Then (45) states that functions in the image of \( \mathcal{L}_\lambda \) have a small expectation w.r.t. \( \varrho_\lambda \). Furthermore, let us introduce a formal adjoint \( \mathcal{L}_\lambda^* : C^2([0, 1]) \to C([0, 1]) \) of \( \mathcal{L} \) by

\[
\mathcal{L}_\lambda^* = -\partial_s s (\lambda + \partial_s (1 - s)^2) .
\]

Supposing that \( \varrho_\lambda \) is also in \( C^2([0, 1]) \), partial integration leads to

\[
\int_0^1 ds \varrho_\lambda(s) (\mathcal{L}_\lambda g)(s) = \int_0^1 ds (\mathcal{L}_\lambda^* \varrho_\lambda)(s) g(s) + \lambda \varrho_\lambda(1) g(1) .
\]

Hence by the above, this is of order \( O(\epsilon, \epsilon^{-1} \delta, \epsilon^{-2} \delta^2) \) for all \( g \in C^3([0, 1]) \). This suggests that \( \varrho_\lambda(1) = 0 \). One is thus led to determine the non-negative elements of the kernel of \( \mathcal{L}_\lambda^* \) which vanish at 1. The corresponding subspace contains the normalized function \( \varrho_\lambda \) given by (13). It actually already lies in the kernel of the first order operator \( (\lambda s + \partial_s (1 - s)^2) \) which is part of \( \mathcal{L}_\lambda^* \). Also note that \( \varrho_\lambda(1) = 0 \) and that L'Hôpital’s rule allows to compute the limits \( s \to 0 \) and \( s \to 1 \) of \( \varrho_\lambda \) and its derivatives, implying that \( \varrho_\lambda \in C^2([0, 1]) \). Of course, at this point these formal arguments have to be completed. For example, it is necessary to show that the kernel of \( \mathcal{L}_\lambda^* \) is one-dimensional. This and other analytical issues have to deal with the fact that both \( \mathcal{L}_\lambda \) and \( \mathcal{L}_\lambda^* \) are singular elliptic in the sense that the highest order term \( -s(1 - s)^2 \partial_s^2 \) has a coefficient function that vanishes at the boundary points \( s = 0 \) and \( s = 1 \) (this can be dealt with by the techniques of the appendix in [14]). Here the proof of Theorem 1 rather follows a more direct approach.

**Proof of Theorem 1.** Inserting \( g = 1 \) into (43) yields

\[
\mathcal{D} \epsilon^2 \int_\mathcal{D} \mu^\epsilon \delta (dz) (1 - |z|^2)^2 - 2 \mathcal{C} \delta \int_\mathcal{D} \mu^\epsilon \delta (dz) |z|^2 = O(\epsilon^3, \epsilon \delta, \delta^2) ,
\]

which implies the first statement (11) if \( \mathcal{C} > 0 \). Moreover, if \( \mathcal{D} > 0 \), the identity (46) also implies

\[
\int_\mathcal{D} \mu^\epsilon \delta (dz) (1 - |z|^2)^2 = O(\epsilon, \delta \epsilon^{-2}) ,
\]

from which one infers the second statement (12) by using Jensen’s inequality:

\[
\int_\mathcal{D} \mu^\epsilon \delta (dz) |z|^2 = 1 - \int_\mathcal{D} \mu^\epsilon \delta (dz) (1 - |z|^2) \geq 1 - \left( \int_\mathcal{D} \mu^\epsilon \delta (dz) (1 - |z|^2)^2 \right)^{\frac{1}{2}} .
\]

Let us now come to the third statement (14). In view of the identity (43) of Lemma 12, the task is to find a function \( g \in C^3([0, 1]) \) that satisfies \( \mathcal{L}_\lambda g = \tilde{h} \), where \( \tilde{h} \in C^2([0, 1]) \) is defined in terms of \( h \) and (13) by

\[
\tilde{h}(s) = h(s) - \int_0^1 dx \varrho_\lambda(x) h(x) .
\]

For this purpose, let us first solve the first order differential equation

\[
(\lambda - (1 - s)^2 \partial_s) F(s) = \tilde{h}(s) ,
\]

(48)
in the open interval $(0, 1)$ by the method of variation of constants:

\[ F(s) = \exp \left[ \frac{\lambda}{1-s} \int_0^s dx \frac{\tilde{h}(x)}{(1-x)^2} \exp \left[ -\frac{\lambda}{1-x} \right] \right]. \]

The function $F : (0, 1) \to \mathbb{R}$ lies in $C^3((0, 1))$ and so does $G$ given by $G(s) = F(s) s^{-1}$. Due to (48), one has

\[ (\lambda - (1-s)^2 \partial_s) s G(s) = \tilde{h}(s). \quad (49) \]

Actually, it turns out that $G$ has a continuous extension in $C^2([0, 1])$ (see Lemma 14 below). Hence, its integral $g : [0, 1] \to \mathbb{R}$ given by

\[ g(s) = \int_0^s dx G(x) \quad (50) \]

lies in $C^3([0, 1])$. Now, (49) and (50) imply

\[ (\mathcal{L}_s g)(s) = (\lambda - (1-s)^2 \partial_s) s G(s) = (\lambda - (1-s)^2 \partial_s) s G(s) = \tilde{h}(s). \]

Therefore the identity (43) of Lemma 12 yields

\[ \mathcal{D} \epsilon^2 \int_{\mathbb{D}} \mu^\epsilon (dz) \tilde{h}(|z|^2) = \mathcal{D} \epsilon^2 \int_{\mathbb{D}} \mu^\epsilon(dz) (\mathcal{L}_s g)(|z|^2) = O(\epsilon^3, \epsilon \delta, \delta^2). \]

Together with (47) one infers

\[ \int_{\mathbb{D}} \mu^\epsilon (dz) h(|z|^2) = \int_0^1 ds g_\lambda(s) h(s) + \int_{\mathbb{D}} \mu^\epsilon (dz) \tilde{h}(|z|^2) \]

\[ = \int_0^1 ds g_\lambda(s) h(s) + O(\epsilon, \epsilon^{-1} \delta, \epsilon^{-2} \delta). \]

This proves (14). \hfill \Box

**Lemma 14** The map $G : (0, 1) \to \mathbb{R}$, defined by

\[ G(s) = \frac{1}{s} \exp \left[ \frac{\lambda}{1-s} \int_0^s dx \frac{\tilde{h}(x)}{(1-x)^2} \exp \left[ -\frac{\lambda}{1-x} \right] \right]. \]

where $\tilde{h}$ is given by (47), has a continuous extension that lies in $C^2([0, 1])$.

**Proof.** It is useful to factorize $G$ as

\[ G(s) = \frac{G_1(s)}{G_2(s)}, \quad (51) \]
where
\[ G_1(s) = \int_0^s dx \frac{\tilde{h}(x)}{(1-x)^2} \exp \left[\frac{-\lambda}{1-x}\right] \quad \text{and} \quad G_2(s) = s \exp \left[\frac{-\lambda}{1-s}\right]. \]

Clearly \(G_1\) and \(G_2\) satisfy
\[ \lim_{s \downarrow 0} G_1(s) = \lim_{s \downarrow 0} G_2(s) = 0 \quad \text{and} \quad \lim_{s \uparrow 1} G_2(s) = 0, \tag{52} \]
and (47) implies that
\[ \lim_{s \downarrow 1} G_1(s) = \int_0^1 dx \frac{\tilde{h}(x)}{(1-x)^2} \exp \left[\frac{-\lambda}{1-x}\right] = [\lambda e^\lambda]^{-1} \int_0^1 ds \varrho_\lambda(s) \tilde{h}(s) = 0, \tag{53} \]

since \(\varrho_\lambda\) is normalized. Moreover, the derivatives of \(G_1\) and \(G_2\) are given by
\[ G_1'(s) = \frac{\tilde{h}(s)}{(1-s)^2} \exp \left[\frac{-\lambda}{1-s}\right] \quad \text{and} \quad G_2'(s) = \left[1 - \frac{\lambda s}{(1-s)^2}\right] \exp \left[\frac{-\lambda}{1-s}\right]. \tag{54} \]

Using (52), (53) and (54) allows to apply L’Hôpital’s rule to show
\[ \lim_{s \downarrow 0} G(s) = \lim_{s \downarrow 0} G_1'(s) G_2'(s)^{-1} = \tilde{h}(0), \]
\[ \lim_{s \uparrow 1} G(s) = \lim_{s \uparrow 1} G_1'(s) G_2'(s)^{-1} = -\lambda^{-1} \tilde{h}(1). \]

In particular, the limits \(\lim_{s \downarrow 0} G(s)\) and \(\lim_{s \uparrow 1} G(s)\) exist so that \(G\) has a continuous extension to \([0, 1]\).

Next let us compute the derivative of \(G\) for \(s \in (0, 1)\) and factorize it as follows:
\[ G'(s) = \frac{\tilde{G}_1(s)}{\tilde{G}_2(s)}, \]

where
\[ \tilde{G}_1(s) = \tilde{h}(s) + [\lambda s - (1-s)^2] G(s) \quad \text{and} \quad \tilde{G}_2(s) = s(1-s)^2. \tag{55} \]

Moreover,
\[
\frac{d}{ds} \left( \tilde{G}_1(s) s \exp \left[\frac{-\lambda}{1-s}\right] \right) \\
= \left( \left[ \tilde{h}'(s) + (\lambda + 2[1-s]) G(s) \right] s + \left[ 1 - \frac{\lambda s}{(1-s)^2} \right] \left[ \tilde{G}_1(s) - \tilde{G}_2(s) G'(s) \right] \right) \exp \left[\frac{-\lambda}{1-s}\right] \\
= \left[ \tilde{h}'(s) + (\lambda + 2[1-s]) G(s) \right] s \exp \left[\frac{-\lambda}{1-s}\right], \tag{56} \]

where (55) was used in the second step, and
\[
\frac{d}{ds} \left( \tilde{G}_2(s) s \exp \left[\frac{-\lambda}{1-s}\right] \right) = \left[ 2 (1 - 3s + 2s^2) - \lambda s \right] s \exp \left[\frac{-\lambda}{1-s}\right]. \tag{57} \]
Now the conditions
\[
\lim_{s \downarrow 0} \tilde{G}_1(s) s \exp \left[ -\frac{\lambda}{1-s} \right] = 0 \quad \text{and} \quad \lim_{s \uparrow 1} \tilde{G}_1(s) s \exp \left[ -\frac{\lambda}{1-s} \right] = 0,
\]
which hold for \( l = 1 \) and \( l = 2 \), allow to apply L'Hôpital's rule again to infer
\[
\lim_{s \downarrow 0} G'(s) = \lim_{s \downarrow 0} \left[ \frac{d}{ds} \left( \tilde{G}_1(s) s \exp \left[ -\frac{\lambda}{1-s} \right] \right) \right] \left[ \frac{d}{ds} \left( \tilde{G}_2(s) s \exp \left[ -\frac{\lambda}{1-s} \right] \right) \right]^{-1} = \frac{1}{2} \left[ \tilde{h}'(0) + (2 + \lambda) \lim_{s \downarrow 0} G(s) \right]
\]
and
\[
\lim_{s \uparrow 1} G'(s) = \lim_{s \uparrow 1} \left[ \frac{d}{ds} \left( \tilde{G}_1(s) s \exp \left[ -\frac{\lambda}{1-s} \right] \right) \right] \left[ \frac{d}{ds} \left( \tilde{G}_2(s) s \exp \left[ -\frac{\lambda}{1-s} \right] \right) \right]^{-1} = -\frac{1}{2} \left[ \lambda^{-1} \tilde{h}'(1) + \lim_{s \uparrow 1} G(s) \right]
\]
by using (55), (56) and (57). In particular, \( \lim_{s \downarrow 0} G'(s) \) and \( \lim_{s \uparrow 1} G'(s) \) exist so that \( G' \) has a continuous extension to \([0, 1]\). This implies that the continuous extension of \( G \) lies in \( C^1([0, 1]) \).

Finally let us consider the second derivative of \( G \). Due to (55) it is given by
\[
G''(s) = \frac{\tilde{G}_1(s)}{\tilde{G}_2(s)} , \quad s \in (0, 1) , \tag{58}
\]
where
\[
\tilde{G}_1(s) = \tilde{G}_1'(s) \tilde{G}_2(s) - \tilde{G}_1(s) \tilde{G}_2'(s) \quad \text{and} \quad \tilde{G}_2(s) = \tilde{G}_2(s)^2. \tag{59}
\]
Moreover,
\[
\frac{d}{ds} \left( \tilde{G}_1'(s) \tilde{G}_2(s) s \exp \left[ -\frac{\lambda}{1-s} \right] \right) = \left( \left[ 1 - \frac{\lambda s}{(1-s)^2} \right] \tilde{G}_1'(s) \tilde{G}_2(s) - G''(s) \tilde{G}_2(s)^2 + \tilde{G}_1'(s) \tilde{G}_2'(s) \right) \exp \left[ -\frac{\lambda}{1-s} \right] + \tilde{h}''(s) - 2 \tilde{G}(s) + 2 \left[ \lambda + 2(1-s) \right] G'(s) \tilde{G}_2(s) s \exp \left[ -\frac{\lambda}{1-s} \right] \tag{60}
\]
and
\[
\frac{d}{ds} \left( \tilde{G}_1(s) \tilde{G}_2'(s) s \exp \left[ -\frac{\lambda}{1-s} \right] \right) = \left( \left[ 1 - \frac{\lambda s}{(1-s)^2} \right] \tilde{G}_1(s) \tilde{G}_2'(s) + \tilde{G}_1'(s) \tilde{G}_2'(s) \right) \exp \left[ -\frac{\lambda}{1-s} \right] + G'(s) \tilde{G}_2'(s) \tilde{G}_2(s) s \exp \left[ -\frac{\lambda}{1-s} \right] . \tag{61}
\]
Due to (58) and (59), the first summands on the right sides of (60) and (61) are equal. Thus,
\[
\frac{d}{ds} \left( \hat{G}_1(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] \right) = 0
\]
and
\[
\frac{d}{ds} \left( \hat{G}_2(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] \right) = 0.
\]
Moreover, one has
\[
\frac{d}{ds} \left( \hat{G}_1(s) \hat{G}_2(s) - \hat{G}_1(s) \hat{G}_2'(s) \right) s \exp \left[ -\frac{-\lambda}{1 - s} \right] = \left[ \hat{h}''(s) - 2 \hat{G}(s) + \left[ 2\lambda + 4 (1 - s) - \hat{G}''(s) \right] \hat{G}'(s) \right] \hat{G}_2(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right].
\]

Moreover, one has
\[
\frac{d}{ds} \left( \hat{G}_2(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] \right) = \left[ 3 (1 - s)^2 + \lambda s - 4 s (1 - s) \right] \hat{G}_2(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right].
\]

Now the conditions
\[
\lim_{s \downarrow 1} \hat{G}_1(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] = 0 \quad \text{and} \quad \lim_{s \uparrow 1} \hat{G}_1(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] = 0,
\]
which hold for \( l = 1 \) and \( l = 2 \), allow to apply L'Hôpital's rule a third time to infer
\[
\lim_{s \downarrow 0} G''(s) = \lim_{s \downarrow 0} \left[ \frac{d}{ds} \left( \hat{G}_1(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] \right) \right] \left[ \frac{d}{ds} \left( \hat{G}_2(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] \right) \right]^{-1}
= \frac{1}{3} \left[ \hat{h}''(0) - 2 \lim_{s \downarrow 0} G(s) + 2(\lambda + 4) \lim_{s \downarrow 0} G'(s) \right]
\]
and
\[
\lim_{s \uparrow 1} G''(s) = \lim_{s \uparrow 1} \left[ \frac{d}{ds} \left( \hat{G}_1(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] \right) \right] \left[ \frac{d}{ds} \left( \hat{G}_2(s) s \exp \left[ -\frac{-\lambda}{1 - s} \right] \right) \right]^{-1}
= \frac{1}{\lambda} \left[ \hat{h}''(1) - 2 \lim_{s \uparrow 1} G(s) + 2(\lambda - 1) \lim_{s \uparrow 1} G'(s) \right].
\]

In particular, \( \lim_{s \downarrow 0} G''(s) \) and \( \lim_{s \uparrow 1} G''(s) \) exist so that \( G'' \) has a continuous extension to \([0, 1]\). This implies that the continuous extension of \( G' \) lies in \( C^1([0, 1]) \) and, all in all, that the continuous extension of \( G \) to \([0, 1]\) lies indeed in \( C^2([0, 1]) \).

\[\square\]

3 The Lyapunov exponent

The Lyapunov exponent \( \gamma^{c, \delta} \) can be expressed by the Furstenberg formula
\[
\gamma^{c, \delta} = \int_{S_{\mathbb{C}}^1} \nu^{c, \delta}(dx) \mathbb{E} \log \| T^{c, \delta}_\sigma x \|,
\]
(62)

where \( \nu^{c, \delta} \) is some invariant probability measure on \( S_{\mathbb{C}}^1 \) corresponding to \( \mu^{c, \delta} \) (and satisfying \( \pi \ast (\nu^{c, \delta}) = \mu^{c, \delta} \)). For the proof of Theorem 4, one has to express the term \( \log \| T^{c, \delta}_\sigma x \| \) appearing in (62) in terms of the stereographic projection of \( x \). This is carried out in Lemma 15.
Lemma 15 Let \( x \in \mathbb{S}_C^1 \) and \( z = \pi(x) \). Then,
\[
\log \| T^{x,\delta}_\sigma x \| = 2 \epsilon \frac{\Re(\beta_\sigma z)}{1 + |z|^2} + e^2 |\beta_\sigma|^2 \left( \frac{1 + |z|^4}{1 + |z|^2} + \delta q_{3,\sigma} \frac{1 - |z|^2}{1 + |z|^2} \right) + 2 e^2 \left[ \frac{\Re(\beta_\sigma^2 z^2)}{1 + |z|^2} - \frac{\Re(\beta_\sigma^2 z^2)}{(1 + |z|^2)^2} \right] + O(\epsilon^3, \epsilon \delta, \delta^2).
\]

Proof. Let us begin by computing
\[
\| T^{x,\delta}_\sigma x \|^2 = \| \exp \left[ \epsilon P_\sigma + e^2 P_\sigma' + i \delta Q_\sigma + O(\epsilon^3, \epsilon \delta, \delta^2) \right] x \|^2
\]
\[
= \left\langle x, \left[ 1 + \epsilon (P_\sigma)_s + e^2 (P_\sigma')_s + \delta (i Q_\sigma)_s + \frac{1}{2} e^2 (P_\sigma^2)_s + e^2 |P_\sigma|^2 \right] x \right\rangle + O(\epsilon^3, \epsilon \delta, \delta^2),
\]
where the notation \((A)_s = A + A^* A\) and \( |A|^2 = A^* A \) was used. Next, one verifies the formulae
\[
(P_\sigma)_s = 2 \left[ p_{1,\sigma} B_1 + p_{2,\sigma} B_2 \right], \quad (P_\sigma')_s = 2 \left[ p'_{1,\sigma} B_1 + p'_{2,\sigma} B_2 \right], \quad (i Q_\sigma)_s = 2 q_{3,\sigma} B_3,
\]
\[
(P_\sigma^2)_s = 2 (|\beta_\sigma|^2 - p_{3,\sigma}^2) 1, \quad |P_\sigma|^2 = (|\beta_\sigma|^2 + p_{3,\sigma}^2) 1 - 2 p_{3,\sigma} [p_{1,\sigma} B_2 - p_{2,\sigma} B_1]
\]
and
\[
\langle v, B_1 v \rangle = \frac{2 \Re(z)}{1 + |z|^2}, \quad \langle v, B_2 v \rangle = \frac{2 \Im(z)}{1 + |z|^2}, \quad \langle v, B_3 v \rangle = \frac{|z|^2 - 1}{1 + |z|^2}.
\]
Combining them yields
\[
\left\langle x, (P_\sigma)_s x \right\rangle = \frac{4 \Re(\beta_\sigma z)}{1 + |z|^2}, \quad \left\langle x, (P_\sigma')_s x \right\rangle = \frac{4 \Re(\beta_\sigma' z)}{1 + |z|^2}, \quad \left\langle x, (i Q_\sigma)_s x \right\rangle = \frac{2q_{3,\sigma}}{1 + |z|^2},
\]
\[
\left\langle x, (P_\sigma^2)_s x \right\rangle = 2 (|\beta_\sigma|^2 - p_{3,\sigma}^2), \quad \left\langle x, |P_\sigma|^2 x \right\rangle = |\beta_\sigma|^2 + p_{3,\sigma}^2 - \frac{4 p_{3,\sigma} \Im(\beta_\sigma z)}{1 + |z|^2}.
\]
Using these identities one can now rewrite (64) as
\[
\| T^{x,\delta}_\sigma x \|^2 = 1 + 4 \epsilon \frac{\Re(\beta_\sigma z)}{1 + |z|^2} + 2 \delta q_{3,\sigma} \frac{1 - |z|^2}{1 + |z|^2} + 2 e^2 \left[ |\beta_\sigma|^2 + 2 \frac{\Re(\beta_\sigma' + i p_{3,\sigma} \beta_\sigma z)}{1 + |z|^2} \right] + O(\epsilon^3, \epsilon \delta, \delta^2).
\]
In view of \( \log(1 + a) = a - \frac{a^2}{2} + O(a^3) \), this implies
\[
\log \left[ \| T^{x,\delta}_\sigma x \|^2 \right] = 4 \epsilon \frac{\Re(\beta_\sigma z)}{1 + |z|^2} - 8 \epsilon^2 \frac{\Re(\beta_\sigma z)^2}{(1 + |z|^2)^2} + 2 \delta q_{3,\sigma} \frac{1 - |z|^2}{1 + |z|^2}
\]
\[
+ 2 e^2 \left[ |\beta_\sigma|^2 + 2 \frac{\Re(\beta_\sigma' + i p_{3,\sigma} \beta_\sigma z)}{1 + |z|^2} \right] + O(\epsilon^3, \epsilon \delta, \delta^2).
\]
This implies (63) because \( \Re(\beta_\sigma z)^2 = \frac{1}{2} \Re(\beta_\sigma^2 z^2) + \frac{1}{2} |\beta_\sigma|^2 |z|^2 \) and \( 1 - \frac{2 |z|^2}{(1 + |z|^2)^2} = \frac{1 + |z|^4}{(1 + |z|^2)^2} \).
Proof of Theorem 4. Thanks to Lemma 15, equation (62) can be rewritten as
\[
\gamma^{\epsilon, \delta} = 2\epsilon \Re \left( \mathbb{E}(\beta_{\sigma}) \int_{D} \mu^{\epsilon, \delta}(dz) \frac{z}{1+|z|^2} \right) + \epsilon^2 \mathbb{E}(|\beta_{\sigma}|^2) \int_{D} \mu^{\epsilon, \delta}(dz) \frac{1+|z|^4}{(1+|z|^2)^2}
\]
\[
+ \delta C \int_{D} \mu^{\epsilon, \delta}(dz) \frac{1-|z|^2}{1+|z|^2} + 2\epsilon^2 \Re \left( \mathbb{E}(\beta'_{\sigma} + \nu p_{3,\sigma} \beta_{\sigma}) \int_{D} \mu^{\epsilon, \delta}(dz) \frac{z}{1+|z|^2} \right) \quad (65)
\]
\[
- 2\epsilon^2 \Re \left( \mathbb{E}(\beta_{\sigma}^2) \int_{D} \mu^{\epsilon, \delta}(dz) \frac{z^2}{1+|z|^2} \right) + O(\epsilon^3, \epsilon \delta, \delta^2).
\]
According to Lemma 9, the fourth and the fifth summand of the r.h.s. of (65) are of the order \(O(\epsilon^3, \epsilon^2 \delta)\). Together with equation (41), which is applicable to the first summand of the r.h.s. of (65), this implies
\[
\gamma^{\epsilon, \delta} = D \epsilon^2 \int_{D} \mu^{\epsilon, \delta}(dz) \frac{2(1+|z|^4)}{(1+|z|^2)^2} + C \delta \int_{D} \mu^{\epsilon, \delta}(dz) \frac{1-|z|^2}{1+|z|^2} + O(\epsilon^3, \epsilon \delta, \delta^2)
\]
\[
= D \epsilon^2 \int_{D} \mu^{\epsilon, \delta}(dz) \left[ 1 + \left| \frac{1-|z|^2}{1+|z|^2} \right|^2 \right] + C \delta \int_{D} \mu^{\epsilon, \delta}(dz) \left[ 1 - \frac{2|z|^2}{1+|z|^2} \right] + O(\epsilon^3, \epsilon \delta, \delta^2).
\]
Due to (44), this implies (15). \(\square\)

4 The support of the Furstenberg measure

The statement (14) of Theorem 1 approximates the radial distribution of \(\mu^{\epsilon, \delta}\) as long as \(\delta = o(\epsilon)\). Since the approximate radial density \(q_{\lambda}\) given by (13) is supported on \([0, 1]\), it is natural to presume that \(\mu^{\epsilon, \delta}\) is supported by the whole closed unit disc \(\overline{D}\) in that case. Of course, statement (14) does not imply that presumption. In the complementary case \(\epsilon = o(\delta)\), however, the support of \(\mu^{\epsilon, \delta}\) can be proven to be a strict subset of the unit disc under some supplementary assumption.

Proposition 16 Suppose that \(q_{3,\sigma} > 0\) holds for all \(\sigma \in \Sigma\). Then, one has
\[
supp(\mu^{\epsilon, \delta}) \subset \left\{ z \in \overline{D} : |z|^2 \leq \esssup_{\sigma \in \Sigma} |\xi_{\sigma}| q_{3,\sigma}^{-1} + O(\epsilon \delta^{-1}, \delta) \right\}. \quad (66)
\]
Proof. One may assume that
\[
\zeta = \esssup_{\sigma \in \Sigma} \left[ q_{\sigma} - |\xi_{\sigma}| \right] > 0,
\]
as (66) is trivial otherwise. For \(n \in \mathbb{N}_0\), let us compute by using (19) and (20)
\[
|z_{n+1}|^2 - |z_n|^2 = 2\delta \Re m(\xi_{n+1} z_n) [1 + |z_n|^2] - 4\delta q_{3, n+1} |z_n|^2 + O(\delta^2, \epsilon),
\]
which implies
\[
|z_{n+1}|^2 = |z_n|^2 (1 - 2\delta [q_{3, n+1} - \Re m(\xi_{n+1} z_n)]) + 2\delta \left[ \Re m(\xi_{n+1} z_n) - q_{3, n+1} |z_n|^2 \right] + O(\delta^2, \epsilon)
\]
\[
\leq |z_n|^2 (1 - 2\delta \zeta) + 2\delta \left[ |\xi_{n+1}| - q_{3, n+1} |z_n|^2 \right] + O(\delta^2, \epsilon).
\]
This shows
\[ |z_{n+1}|^2 \leq |z_n|^2(1 - 2\delta \zeta) \] (67)
whenever
\[ 2\delta \left[ |\xi_{n+1}| - q_{3,n+1}|z_n|^2 \right] + \mathcal{O}(\delta^2, \epsilon) \leq 0, \]
which is equivalent to
\[ |z_n|^2 \geq |\xi_{n+1}|q_{3,n+1}^{-1} + \mathcal{O}(\delta, \epsilon \delta^{-1}). \] (68)

In conclusion, if \( z_n \) is not contained in the r.h.s. of (66), then it obeys (68) and thus (67), i.e., the modulus is properly decreased by a uniform factor. Hence, the dynamics \( (z_n)_{n \in \mathbb{N}} \) runs deterministically into the set on the r.h.s. of (66) for all starting points \( z_0 \in \mathbb{D} \). This implies (66). \( \square \)

In general, however, the relation of \( \epsilon \) and \( \delta \) does not shrink the support of \( \mu^{\epsilon, \delta} \) in any manner. To illustrate this, an elementary example is given in Proposition 17, in which the Furstenberg measure is supported by the whole (closed) unit disc, regardless of the relation of \( \epsilon > 0 \) and \( \delta > 0 \). The assumptions enforce \( C > 0 \), but both \( \mathcal{D} = 0 \) and \( \mathcal{D} > 0 \) is possible.

**Proposition 17** Let \( \epsilon, \delta > 0 \) and \( p, q > 0 \) and \( k \in \mathbb{R} \setminus \pi \mathbb{Q} \) and suppose that
\[ T^{\epsilon, \delta}_\sigma = R_{\eta_\sigma} \exp[\epsilon P_\sigma + i\delta Q_\sigma], \quad \text{where} \quad P_\sigma = p_{1,\sigma}B_1, \quad Q_\sigma = q_{3,\sigma}B_3, \]
where \( \{(k,0,0),(0,p,0),(0,0,q)\} \subset \text{supp}(\{(\eta_\sigma,p_{1,\sigma},q_{3,\sigma})\}) \). Then, the \( (T^{\epsilon, \delta}_\sigma) \)-invariant probability measure is uniquely given by the Furstenberg measure \( \mu^{\epsilon, \delta} \) and \( \mu^{\epsilon, \delta} \) satisfies \( \text{supp}(\mu^{\epsilon, \delta}) = \overline{\mathbb{D}} \).

**Proof.** Due to the assumption \( \{(k,0,0),(0,p,0),(0,0,q)\} \subset \text{supp}(\{(\eta_\sigma,p_{1,\sigma},q_{3,\sigma})\}) \), the matrices
\[ R_k = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix}, \quad \exp[\epsilon p B_1] = \begin{pmatrix} \cosh(\epsilon p) & \sinh(\epsilon p) \\ \sinh(\epsilon p) & \cosh(\epsilon p) \end{pmatrix}, \quad \exp[i\delta q B_3] = \begin{pmatrix} e^{-\delta q} & 0 \\ 0 & e^{\delta q} \end{pmatrix}, \]
lie in the support of \( T^{\epsilon, \delta}_\sigma \).

**Step 1.** The \( (T^{\epsilon, \delta}_\sigma) \)-invariant probability measure is unique.
Since \( k \in \mathbb{R} \setminus \pi \mathbb{Q} \), one has \( e^{ikm_1} \neq e^{ikm_2} \) whenever \( m_1, m_2 \in \mathbb{N} \) and \( m_1 \neq m_2 \). Therefore, the set \( \{(R_k)^m \cdot z \}_{m \in \mathbb{N}} = z \left(e^{2im} \right)_{m \in \mathbb{N}} \) is infinite if \( z \in \mathbb{C} \setminus \{0\} \). But \( \exp[\epsilon p B_1] \cdot 0 = \tanh(\epsilon p) \) and \( \exp[\epsilon p B_1] \cdot \infty = \coth(\epsilon p) \) lie in \( \mathbb{C} \setminus \{0\} \). Hence, no finite subset of \( \overline{\mathbb{C}} \) is left invariant under the Möbius action of \( \text{supp}(T^{\epsilon, \delta}_\sigma) \), i.e., \( T^{\epsilon, \delta}_\sigma \) fulfills condition (viii). Moreover, one has \( \|\exp[i\delta q B_3]^n\| = e^{\delta q n} \to \infty \) as \( n \to \infty \). Therefore, the semigroup generated by \( \text{supp}(T^{\epsilon, \delta}_\sigma) \) is not relatively compact, i.e., \( T^{\epsilon, \delta}_\sigma \) fulfills condition (ix). All in all, (viii) and (ix) imply that the \( (T^{\epsilon, \delta}_\sigma) \)-invariant probability measure is uniquely given by the Furstenberg measure \( \mu^{\epsilon, \delta} \) (see [5]). \( \square \)

**Step 2.** The support of \( \mu^{\epsilon, \delta} \) is a subset of \( \overline{\mathbb{D}} \).
Given some Borel probability measure \( \varrho_0 \) on \( \overline{\mathbb{C}} \) as initial distribution, each weak limit point of
\[ (\xi_N)_{N \in \mathbb{N}}, \quad \text{where} \quad \xi_N = \frac{1}{N} \sum_{n=1}^{N} \varrho_n, \quad \text{with} \quad \varrho_n = \left((T^{\epsilon, \delta}_\sigma)^{(n)}\right)(\varrho_0), \] (69)
is again a Borel probability measure on $\mathbb{C}$ and is, moreover, invariant under the Möbius action of $T^e_\sigma$ (see [1], Part A, Chapter I, Lemma 3.5). Here, $((T^e_\sigma)^n)$ is the $n$-th iterate of the pushforward of the Möbius action of $T^e_\sigma$. Due to the compactness of $\mathbb{C}$, (69) has a weakly convergent subsequence and, therefore, each initial distribution $\rho_0$ produces at least one $(T^e_\sigma)^n$-invariant Borel probability measure in that manner. In fact, it was shown in Step 1 that $\mu^e_\sigma$ is the unique $(T^e_\sigma)^n$-invariant probability measure and, therefore, $\mu^e_\sigma$ is the only weak limit point of $(\xi_N)_{N \in \mathbb{N}}$, regardless of the choice of the initial distribution $\rho_0$.

Now, if one chooses the initial distribution $\rho_0$ to be supported by a subset of the unit disc $\mathbb{D}$, then, all $\xi_N$ are supported by a subset of $\mathbb{D}$, since the Möbius action of $T^e_\sigma$ leaves $\mathbb{D}$ invariant. In particular, the weak limit point $\mu^e_\sigma$ of $(\xi_N)_{N \in \mathbb{N}}$ is supported by a subset of the closed unit disc $\overline{\mathbb{D}}$.

Step 3. For all $\zeta > 0$ and all $z, z' \in \mathbb{D}$, there exists some $\vartheta > 0$ and $N \in \mathbb{N}$ such that one has

$$\mathbb{P}\left(\{(T_N \cdots T_1) \cdot \tilde{z} \in B_\zeta(z')\}\right) > 0 \quad \forall \zeta \in B_\vartheta(z),$$

where $(T_n)_{n=1}^N$ are independent copies of $T^e_\sigma$ and $B_\vartheta(z)$ and $B_\zeta(z')$ are balls of radius $\vartheta$ and $\zeta$ around $z$ and $z'$, respectively.

First, every angle shift can be approximated arbitrarily well while the radius is preserved:

Step 3a. For all $\kappa > 0$, $r \in [0, 1]$ and $\Delta \varphi \in [0, 2\pi)$, there exists a number $N_{\Delta \varphi} \in \mathbb{N}$ such that

$$\left| R_k^{N_{\Delta \varphi}} \cdot (r e^{i \varphi}) - r e^{i \varphi + \Delta \varphi} \right| < \kappa.$$  

(71)

Indeed, since $k \not\equiv \pi \varnothing$, the sequence $((2nk) \mod (2\pi))_{n \in \mathbb{N}}$ lies dense in $[0, 2\pi)$, which implies (71).

Second, within $\mathbb{D}$, arbitrary radius growth is possible at the expense of some angle change:

Step 3b. For all $r \in [0, 1]$ and $C \in 1 + \sinh(\epsilon p)^2[0, 1]$, there exists an angle $\varphi \in [0, 2\pi)$ such that

$$1 - \left| e^{\epsilon p B_1} \cdot (r e^{i \varphi}) \right|^2 = [1 - r^2]C^{-1}.$$  

(72)

To prove (72), let us observe that all $r \in [0, 1]$ and $\varphi \in [0, 2\pi)$ satisfy

$$1 - \frac{r^2}{1 - \left| e^{\epsilon p B_1} \cdot (r e^{i \varphi}) \right|^2} = 1 + r \cos(\varphi) \sinh(2\epsilon p) + [1 + r^2] \sinh(\epsilon p)^2 =: h(\cos(\varphi))$$

(73)

and one has

$$h(\pm 1) = 1 \pm r \sinh(2\epsilon p) + [1 + r^2] \sinh(\epsilon p)^2,$$

which implies

$$h(1) \geq 1 + \sinh(\epsilon p)^2 \geq 1 \geq h(-1).$$

(74)

In view of (74) and the continuity of $h$, there is some $\varphi \in [0, 2\pi)$ obeying $h(\cos \varphi) = C$. With this choice, (73) implies that (72) is indeed satisfied.

Third, from any point in $\overline{\mathbb{D}}$, the origin can be approached arbitrarily closely:

Step 3c. For all $\kappa > 0$ and $z \in \mathbb{D}$, there exists a number $N \in \mathbb{N}$ such that

$$\left| \exp [i \delta q B_3]^N \cdot z \right| < \kappa.$$  

(75)

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Inequality (75) follows from \( \exp[i\delta qB_3]^N \cdot z = e^{-\delta qN}|z| \) for arbitrarily large \( N \).

Now, let \( \zeta > 0 \) and \( z, z' \in \mathbb{D} \). In view of Step 3a, Step 3b, Step 3c and due to the continuity of the Möbius action, there exists a finite sequence \( (T_n)_{n=1}^{N} \subset \{R_k, \exp[\epsilon \rho B_1], \exp[i\delta qB_3]\} \) and a positive number \( \vartheta > 0 \) for which all \( \tilde{z} \in B_{\vartheta}(z) \) satisfy

\[
T_N \cdots T_1 \cdot \tilde{z} \in B_{\vartheta}^{\vartheta}(z').
\]  

(76)

Since the matrices \( R_k, \exp[\epsilon \rho B_1] \) and \( \exp[i\delta qB_3] \) lie in the support of \( T_{\vartheta}^{\vartheta} \), the inclusion (76) allows to infer (70), again by taking the continuity of the Möbius action into account.

\( \diamond \)

Step 4. The support of \( \mu_{\vartheta}^{\vartheta} \) is a superset of \( \overline{\mathbb{D}} \).

Let \( z' \in \mathbb{D} \) and \( \zeta > 0 \). By Step 2, the (non-empty) support of \( \mu_{\vartheta}^{\vartheta} \) is a subset of \( \mathbb{D} \). Therefore, one can pick some \( z \in \text{supp}(\mu_{\vartheta}^{\vartheta}) \), for which the statement of Step 3 implies the existence of some \( \vartheta > 0 \) and some \( N \in \mathbb{N} \) that satisfy (70). Now, since \( z \in \text{supp}(\mu_{\vartheta}^{\vartheta}) \), one has \( \mu_{\vartheta}^{\vartheta}(B_{\vartheta}(z)) > 0 \). Combined with (70) and the invariance property of \( \mu_{\vartheta}^{\vartheta} \), this implies

\[
\mu_{\vartheta}^{\vartheta}(B_{\vartheta}(z')) = \int_{\mathbb{D}} d\mu_{\vartheta}^{\vartheta}(\tilde{z}) P(T_N \cdots T_1 \cdot \tilde{z} \in B_{\vartheta}(z'))
\geq \int_{B_{\vartheta}(z)} d\mu_{\vartheta}^{\vartheta}(\tilde{z}) P(T_N \cdots T_1 \cdot \tilde{z} \in B_{\vartheta}(z')) > 0.
\]  

(77)

Since \( \zeta > 0 \) was arbitrary, (77) implies that \( z' \) lies in the support of \( \mu_{\vartheta}^{\vartheta} \). Now, since \( z' \in \mathbb{D} \) was also arbitrary, the (closed) support of \( \mu_{\vartheta}^{\vartheta} \) is a superset of the closure \( \overline{\mathbb{D}} \) of \( \mathbb{D} \).

\( \diamond \)

The statements of Step 1, Step 2 and Step 4 imply the claim.

\( \square \)

5 Complex energies for random Jacobi matrices

A random Jacobi matrix is a family \((H_\omega)_{\omega \in \Omega}\) of Jacobi operators on \( \ell^2(\mathbb{Z}) \) indexed by a compact dynamical system \((\Omega, \tau, Z, \mathbb{P})\) specified by a compact set \( \Omega \) equipped with a \( Z \) action \( \tau \) and a \( \tau \)-invariant and ergodic probability measure \( \mathbb{P} \) on \( \Omega \), which satisfies the covariance relation

\[
U_n^* H_\omega U_n = H_{\tau_n \omega}.
\]

Here \( U_n \) is the left shift on \( \ell^2(\mathbb{Z}) \) by \( n \). A Jacobi operator is a selfadjoint tridiagonal operator, namely it is of the form

\[
(H_\omega \psi)(n) = -t_\omega(n+1)\psi(n+1) + v_\omega(n)\psi(n) - t_\omega(n)\psi(n-1), \quad \psi \in \ell^2(\mathbb{Z}),
\]  

(78)

with \((t_\omega(n))_{n \in \mathbb{Z}}\) and \((v_\omega(n))_{n \in \mathbb{Z}}\) sequences of compactly supported, positive and real random numbers, respectively, called the hopping and potential values. Here, we will focus on particular kinds of random Jacobi matrices, namely so-called random polymer models [9, 7]. In these models, \( H_\omega \) is built from independently drawn blocks of random length \( K \). Each such block is called a polymer and is given by the data \( \sigma = (K, \hat{t}_\sigma(1), \ldots, \hat{t}_\sigma(K), \hat{v}_\sigma(1), \ldots, \hat{v}_\sigma(K)) \) containing the length, as well as the hopping and potential values of the polymer. Hence, \( \sigma \in \Sigma \) with
\[ \Sigma \subset \bigcup_{K=1}^{L} \{K\} \times \mathbb{R}^{K}_+ \times \mathbb{R}^{K}, \] which is supposed to be compact and equipped with a probability measure \( \mathbb{P}_\Sigma \). How to construct the dynamical system \( \Omega \) as a Palm measure is explained in detail in [9, 7], but this is not relevant for the following. The best known example is the Anderson model in which \( K = 1 \) and \( t_\omega(n) = 1 \) and only the potential values are random and given by an i.i.d. sequence \( (v_\omega(n))_{n \in \mathbb{Z}} \) of compactly supported, real-valued random variables.

Solutions of the Schrödinger equation \( H_\omega \psi = E \psi \) for \( E \in \mathbb{C} \) are usually [1, 3] studied using transfer matrices

\[ S_{v-E,t}^\sigma = \frac{1}{t} \begin{pmatrix} v - E & -t^2 \\ 1 & 0 \end{pmatrix}. \]

In case of real energies \( E \in \mathbb{R} \), the matrices \( S_{v-E,t}^\sigma \) lie in \( \text{SL}(2, \mathbb{R}) = \{ A \in \mathbb{R}^{2 \times 2} : \det(A) = 1 \} \). A basis of the Lie algebra \( \text{sl}(2, \mathbb{R}) = \{ A \in \mathbb{R}^{2 \times 2} : \text{Tr}(A) = 0 \} \) of \( \text{SL}(2, \mathbb{R}) \) is given by \( \{ B_1, tB_2, tB_3 \} \).

For polymer models it is then natural to consider the polymer transfer matrices \( S^E_\sigma \) defined by

\[ S^E_\sigma = \prod_{k=1}^{K} S_{\hat{v}_\sigma(k)-E,\hat{t}_\sigma(k)}. \] (79)

**Definition 18 ([9])** An energy \( E_c \in \mathbb{R} \) is called a critical energy for the random family \( (H_\omega)_\omega \in \Omega \) of polymer Hamiltonians if the polymer transfer matrices \( S^E_\sigma \) commute for all \( \sigma, \sigma' \in \Sigma \):

\[ [S^E_\sigma, S^E_{\sigma'}] = 0. \] (80)

The critical energy is called elliptic if for all \( \sigma \) one has either \( |\text{Tr}(S^E_\sigma)| < 2 \) or \( S^E_\sigma = \pm 1 \).

The definition of an elliptic critical energy implies that there is a basis change \( M' \in \text{SL}(2, \mathbb{R}) \) that transforms all polymer transfer matrices simultaneously into rotations:

\[ M'S^E_\sigma (M')^{-1} = \begin{pmatrix} \cos(\eta_\sigma) & -\sin(\eta_\sigma) \\ \sin(\eta_\sigma) & \cos(\eta_\sigma) \end{pmatrix}. \] (81)

For further use let us next introduce the notations

\[ J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \sqrt{-\frac{i}{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}. \] (82)

The matrix \( C \) is also referred to as the Cayley transform. It satisfies \( iI = C^*JC \) so that \( C \text{SL}(2, \mathbb{R}) C^* = \text{SU}(1, 1) \). Here it yields as basis change \( M = CM' \) that transforms all polymer transfer matrices simultaneously into diagonal matrices:

\[ MS^{E_c}_\sigma M^{-1} = R_{\eta_\sigma}. \] (83)
Now, for energies $E = E_c + \epsilon - i\delta$ in the vicinity of $E_c$, let us estimate for all $k \in \{1, \ldots, K\}$

$$[MS_{\tilde{\nu}_a}^{-1}(E_c + \epsilon - i\delta), \tilde{\imath}_a(k)]^\dagger J [MS_{\tilde{\nu}_a}^{-1}(E_c + \epsilon - i\delta), \tilde{\imath}_a(k)] M^{-1}$$

$$= (M^{-1})^* [S_{\tilde{\nu}_a}^{-1}(E_c + \epsilon - i\delta), \tilde{\imath}_a(k)] N^* C^* J C N S_{\tilde{\nu}_a}^{-1}(E_c + \epsilon - i\delta), \tilde{\imath}_a(k)] M^{-1}$$

$$= (M^{-1})^* [S_{\tilde{\nu}_a}^{-1}(E_c + \epsilon - i\delta), \tilde{\imath}_a(k)] N^* INS_{\tilde{\nu}_a}^{-1}(E_c + \epsilon - i\delta), \tilde{\imath}_a(k)] M^{-1}$$

$$= (M^{-1})^* [\tau I - 2\delta \hat{\imath}_\sigma(k)^{-2} \text{diag}(1,0)] M^{-1}$$

$$\leq (C^{-1})^*(N^{-1})^* \text{IN}^{-1} C^{-1} - 2\delta \hat{\imath}_\sigma(k)^{-2} (M^{-1})^* \text{diag}(1,0) M^{-1}$$

$$= J - 2\delta \hat{\imath}_\sigma(k)^{-2} (M^{-1})^* \text{diag}(1,0) M^{-1}$$

where $N^* \text{IN} = I$ was used. The estimate (84) implies that the matrices $MS_{\tilde{\nu}_a}^{-1}(E_c + \epsilon - i\delta), \tilde{\imath}_a(k)] M^{-1}$ are $\text{SU}_{\lesssim}(1,1)$-valued if $\delta \geq 0$ and are even $\text{SU}(1,1)$-valued if $\delta = 0$. Since $\text{SU}_{\lesssim}(1,1)$ and $\text{SU}(1,1)$ are semi-groups, one has

$$T_\sigma^{\epsilon, \delta} = MS_{\sigma}^{E_c + \epsilon - i\delta} M^{-1} \in \text{SU}_{\lesssim}(1,1) \quad \text{if} \quad \delta \geq 0$$

and

$$MS_{\sigma}^{E_c + \epsilon + i\delta} M^{-1} \in \text{SU}(1,1)$$

and

$$MS_{\sigma}^{E_c - (i\delta)} M^{-1} \in \text{SU}(1,1).$$

All in all, if $E_c$ is an elliptic critical energy of a random polymer model, then the random matrices

$$T_\sigma^{\epsilon, \delta} = MS_{\sigma}^{E_c + \epsilon - i\delta} M^{-1}$$

satisfy the assumptions (i)-(vii) stated in the introduction, where the (negative) imaginary part of the energy plays the role of the parameter $\delta$.

For a system stemming from a random Jacobi matrix, the Lyapunov exponent can be obtained from the density of states $N$ via the so-called Thouless formula (see [3], p. 376):

$$\gamma^{\epsilon, \delta} = \int N(d\lambda) \log(|E - \lambda|) + \text{const.}, \quad E = E_c + \epsilon - i\delta. \quad (86)$$

Due to (86), the Lyapunov exponent increases if the energy diverges from the real line. Indeed,

$$\gamma^{\epsilon, \delta} - \gamma^{\epsilon, 0} = \frac{1}{2} \int N(d\lambda) \log \left(1 + \frac{\delta^2}{(E_c + \epsilon - \lambda)^2}\right) > 0, \quad E \in \mathbb{R}, \quad \delta > 0. \quad (87)$$

Theorem 4 makes a more precise statement on the l.h.s. of (87).

Explicit examples of random polymer models with $K \geq 2$ that have such an elliptic critical energy are given in [9]. For the Anderson model where $K = 1$ and $t_a(n) = 1$ for all $n$, all energies in the interval $(-2,2)$ are critical in the sense of Definition 18. One can thus also work
with the family (85). There is, however, a more interesting choice when the potentials are small
and of the form \( \hat{v}_\sigma(1) = \epsilon w_\sigma \), where \( w_\sigma \) is a compactly supported, real-valued random variable,
which is independent from \( \epsilon \) and \( \delta \). Then for any fixed \( E \in (-2, 2) \) also

\[
T^{\epsilon, \delta}_\sigma = M \begin{pmatrix} \epsilon w_\sigma - E + i\delta & -1 \\ 1 & 0 \end{pmatrix} M^{-1}
\]

(88)
satisfies the assumptions (i)-(v). Let us also give the explicit form of \( M \) in this case. First set

\[
M' = \frac{1}{\sqrt{\sin(k)}} \begin{pmatrix} \sin(k) & 0 \\ -\cos(k) & 1 \end{pmatrix}, \quad k = \arccos \left( -\frac{E}{2} \right) \in (0, \pi)
\]

and deduce

\[
M' \begin{pmatrix} \epsilon w_\sigma - E + i\delta & -1 \\ 1 & 0 \end{pmatrix} (M')^{-1} = \begin{pmatrix} \cos(k) & -\sin(k) \\ \sin(k) & \cos(k) \end{pmatrix} \exp \left[ -i\delta + \epsilon w_\sigma \frac{0}{\sin(k)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right].
\]

(89)

A further conjugation by the Cayley transform given by (82) yields

\[
T^{\epsilon, \delta}_\sigma = M \begin{pmatrix} \epsilon w_\sigma - E + 2\delta & -1 \\ 1 & 0 \end{pmatrix} M^{-1} = \begin{pmatrix} e^{-ik} & 0 \\ 0 & e^{ik} \end{pmatrix} \exp \left[ i\delta + \epsilon w_\sigma \frac{2}{2\sin(k)} \left( B_2 + B_3 \right) \right],
\]

(90)

where \( M = CM' \). Then, one has

\[
\eta_\sigma = -k, \quad P_\sigma = \frac{w_\sigma}{2\sin(k)} (B_2 + B_3), \quad P'_\sigma = 0, \quad Q_\sigma = \frac{1}{2\sin(k)} (B_2 + B_3)
\]

without any terms of higher order in the exponent of (90). Therefore, (90) also satisfies the
assumptions (vi) and (vii) stated in the introduction.

The numerics in Figures 1-3 were done with the single-site Anderson model (88) with the
choice \( E = -2 \cos(2) \) and the random variable \( w_\sigma \) being distributed uniformly on \([-1, 1]\). In
that case, one readily computes \( C = [2 \sin(2)]^{-1} \) and \( D = [24 \sin(2)^2]^{-1} \).

It is next shown in Proposition 19 that the positivity of the coefficients \( q_{3,\sigma} \) as given in
Remark 7 can be strengthened for matrices \( T^{\epsilon, \delta}_\sigma \) that arise from random polymer models. We
encourage the reader to have a look at Proposition 16 again and compare with Proposition 19.

**Proposition 19** If \( T^{\epsilon, \delta}_\sigma \) arises from a random polymer model, then all \( \sigma \in \Sigma \) satisfy \( q_{3,\sigma} \geq |\xi_\sigma| \).

**Proof.** The statement follows by mimicking the proof of Proposition 3 in [7]. For this, let us
observe that one has $M^*JM = iI$ due to $M \in C[\text{SL}(2, \mathbb{R})]$. This allows to compute

$$-(T^{0,0}_\sigma)^* J \partial_\delta T^{0,\delta}_\sigma \bigg|_{\delta=0} = -i (S_{\sigma}^{E_c} M^{-1})^* I (\partial_\delta S_{\sigma}^{E_c+i\delta}) M^{-1} \bigg|_{\delta=0} = (S_{\sigma}^{E_c} M^{-1})^* I \sum_{k=1}^{K} \prod_{m=k+1}^{K} S_{\nu_\sigma(m)-E_c-i\delta, \bar{\iota}_\sigma(m)} \begin{pmatrix} \tilde{\iota}_\sigma(k) & 0 \\ 0 & 0 \end{pmatrix} \prod_{m=1}^{k-1} S_{\nu_\sigma(m)-E_c-i\delta, \bar{\iota}_\sigma(m)} M^{-1} \bigg|_{\delta=0} = (M^{-1})^* \sum_{k=1}^{K} \prod_{m=1}^{k-1} S_{\nu_\sigma(m)-E_c, \bar{\iota}_\sigma(m)} \begin{pmatrix} \tilde{\iota}_\sigma(k) & 0 \\ 0 & 0 \end{pmatrix} \prod_{m=1}^{k-1} S_{\nu_\sigma(m)-E_c, \bar{\iota}_\sigma(m)} M^{-1} = (M^{-1})^* \sum_{k=1}^{K} \prod_{m=1}^{k-1} S_{\nu_\sigma(m)-E_c, \bar{\iota}_\sigma(m)} \begin{pmatrix} \tilde{\iota}_\sigma(k) & 0 \\ 0 & 0 \end{pmatrix} \prod_{m=1}^{k-1} S_{\nu_\sigma(m)-E_c, \bar{\iota}_\sigma(m)} M^{-1} = (M^{-1})^* \sum_{k=1}^{K} \prod_{m=1}^{k-1} S_{\nu_\sigma(m)-E_c, \bar{\iota}_\sigma(m)} \begin{pmatrix} \tilde{\iota}_\sigma(k) & 0 \\ 0 & 0 \end{pmatrix} \prod_{m=1}^{k-1} S_{\nu_\sigma(m)-E_c, \bar{\iota}_\sigma(m)} M^{-1} .$$

Clearly, the r.h.s. of this equation is non-negative, and thus also the l.h.s. Thus, its determinant

$$\det \left( -(T^{0,0}_\sigma)^* J \partial_\delta T^{0,\delta}_\sigma \bigg|_{\delta=0} \right) = \det \left( -i R_{\eta_\sigma}^{*} J R_{\eta_\sigma} Q_\sigma \right) = \det \left( -i J Q_\sigma \right) = \det \begin{pmatrix} q_{3,\sigma} & -\xi_\sigma \\ \xi_\sigma & q_{3,\sigma} \end{pmatrix} = q_{3,\sigma}^2 - |\xi_\sigma|^2$$

and its trace $\text{Tr} \left( -(T^{0,0}_\sigma)^* J \partial_\delta T^{0,\delta}_\sigma \bigg|_{\delta=0} \right) = 2q_{3,\sigma}$ are non-negative, which implies $q_{3,\sigma} \geq |\xi_\sigma|$.

Moreover, if $\epsilon$ is non-zero and the potential $w_\sigma$ is non-trivial, then (90) fulfills also conditions (viii) and (ix) so that the $(T^{\epsilon,\delta}_\sigma)$-invariant probability measure is unique. In fact, to insure condition (ix), it is also sufficient that the imaginary part of the energy is non-zero. All this is carried out in the following Propositions 20 and 21.

**Proposition 20** Suppose that $\epsilon \neq 0$ and $\text{card} (\text{supp}(w_\sigma)) > 1$. Then, (90) fulfills condition (viii).

**Proof.** By assumption, there exist $\sigma_1, \sigma_2 \in \Sigma$ such that $w_{\sigma_1}, w_{\sigma_2} \in \text{supp}(w_\sigma)$ satisfy $w_{\sigma_1} \neq w_{\sigma_2}$. **Case 1:** Both $T^{\epsilon,\delta}_{\sigma_1}$ and $T^{\epsilon,\delta}_{\sigma_2}$ are diagonalizable.

In that case, there exist some $U_{\sigma_1} \in \text{SL}(2, \mathbb{C})$ and $U_{\sigma_2} \in \text{SL}(2, \mathbb{C})$ for which one has

$$\hat{T}^{\epsilon,\delta}_{\sigma_1} := U_{\sigma_1}^{-1} T^{\epsilon,\delta}_{\sigma_1} U_{\sigma_1} = \begin{pmatrix} r_{\sigma_1} e^{i \varphi_{\sigma_1}} & 0 \\ 0 & r_{\sigma_1}^{-1} e^{-i \varphi_{\sigma_1}} \end{pmatrix}, \quad r_{\sigma_1} > 0, \quad \varphi_{\sigma_1} \in [0, 2\pi), \quad l \in \{1, 2\} .$$

Now, if $(\varphi_l, r_l) \not\in \pi \mathbb{Q} \times \{1\}$, the singeltons $\{0\}$ and $\{\infty\}$ specify all finite orbits of $\hat{T}^{\epsilon,\delta}_{\sigma_1}$. Otherwise, one has for each couple $(\varrho, \vartheta) \in (0, \infty) \times [0, 2\pi)$ a further finite orbit $\varrho \exp[i(\vartheta + \varphi_l N)]$. By symmetry, a finite subset of $\mathbb{C}$ is an orbit of $\hat{T}^{\epsilon,\delta}_{\sigma_1}$, and if only if it is an orbit of $(\hat{T}^{\epsilon,\delta}_{\sigma_1})^{-1}$. Accordingly, a finite subset of $\mathbb{C}$ is an orbit of $T^{\epsilon,\delta}_{\sigma_1}$ if and only if it is an orbit of $(T^{\epsilon,\delta}_{\sigma_1})^{-1}$. Moreover, any finite $(T^{\epsilon,\delta}_{\sigma_1})^{\pm 1}$-invariant set is a union of finitely many finite $(T^{\epsilon,\delta}_{\sigma_1})^{\pm 1}$-invariant orbits.
Hence, if $F$ were a finite subset of $\mathbb{C}$ invariant both under $T_{\sigma_1}^{\epsilon, \delta}$ and $T_{\sigma_2}^{\epsilon, \delta}$, then, it would also be invariant under $(T_{\sigma_1}^{\epsilon, \delta})^{-1}$ and $(T_{\sigma_2}^{\epsilon, \delta})^{-1}$. and, in particular, under the Möbius action of

$$(T_{\sigma_2}^{\epsilon, \delta})^{-1} T_{\sigma_1}^{\epsilon, \delta} = \begin{pmatrix} 1 - i\xi & -i\xi \\ i\xi & 1 + i\xi \end{pmatrix},$$

where $\xi = \epsilon[w_{\sigma_2} - w_{\sigma_1}]\sin(k)^{-1} \neq 0$.

Therefore, $C^{-1} \cdot F$ (would also be finite and) would be invariant under the Möbius action of

$$C^{-1}(T_{\sigma_2}^{\epsilon, \delta})^{-1} T_{\sigma_1}^{\epsilon, \delta} C = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix},$$

which clearly satisfies

$$\lim_{N \to \infty} \left[ C^{-1}(T_{\sigma_2}^{\epsilon, \delta})^{-1} T_{\sigma_1}^{\epsilon, \delta} C \right]^N \cdot z = 0 \quad \forall \ z \in \mathbb{C} \quad \text{and} \quad \left( C^{-1}(T_{\sigma_2}^{\epsilon, \delta})^{-1} T_{\sigma_1}^{\epsilon, \delta} C \cdot z = 0 \iff z = 0 \right).$$

These properties, the finiteness of $C^{-1} \cdot F$ and its invariance under the Möbius action of (91) would imply $C^{-1} \cdot F = \emptyset$, i.e., $F = \{C \cdot 0\} = \{-1\}$. But, $T_{\sigma}^{\epsilon, \delta} \cdot (-1) = -e^{2ik} \neq -1$, as $k \in (0, \pi)$.

**Case 2:** For some $l \in \{1, 2\}$, the matrix $T_{\sigma}^{\epsilon, \delta}$ is not diagonalizable.

First of all, let us remark that this case is of minor relevance, since it can only occur if $\delta = 0$ and $|\epsilon|$ is sufficiently large. One can assume without loss of generality that $T_{\sigma_1}^{\epsilon, 0}$ is not diagonalizable. Then, $T_{\sigma_1}^{\epsilon, 0}$ has either $+1$ or $-1$ as its only eigenvalue, namely with geometric multiplicity 1. Moreover, the conjugation of $T_{\sigma_1}^{\epsilon, 0}$ with some $U \in \text{SL}(2, \mathbb{C})$ yields the Jordan form of $T_{\sigma_1}^{\epsilon, 0}$,

$$U^{-1} T_{\sigma_1}^{\epsilon, 0} U = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$$

which clearly satisfies

$$\lim_{N \to \infty} \left( U T_{\sigma_1}^{\epsilon, 0} U^{-1} \right)^N \cdot z = \infty \quad \forall \ z \in \mathbb{C} \quad \text{and} \quad \left(U T_{\sigma_1}^{\epsilon, 0} U^{-1} \cdot z = \infty \iff z = \infty \right).$$

Thus, the only finite $(T_{\sigma_1}^{\epsilon, 0})$-invariant subset of $\mathbb{C}$ is given by $\{U^{-1} \cdot \infty\}$. Now, if $\{U^{-1} \cdot \infty\}$ were also $(T_{\sigma_2}^{\epsilon, 0})$-invariant, then one would have $T_{\sigma_2}^{\epsilon, 0} \cdot (U^{-1} \cdot \infty) = U^{-1} \cdot \infty$, which is equivalent to

$$U(T_{\sigma_2}^{\epsilon, 0})^{-1} U^{-1} \cdot \infty = \infty$$

(92)

Combining (92) with $U T_{\sigma_1}^{\epsilon, 0} U^{-1} \cdot \infty = \infty$ would yield $U(T_{\sigma_2}^{\epsilon, 0})^{-1} T_{\sigma_1}^{\epsilon, 0} U^{-1} \cdot \infty = \infty$ and, due to (91),

$$UC \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} (UC)^{-1} \cdot \infty = \infty$$

or, equivalently,

$$\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \cdot ((UC)^{-1} \cdot \infty) = (UC)^{-1} \cdot \infty.$$  (93)

Since 0 is the only fixed point of the Möbius action of (91), the equation (93) would imply $(UC)^{-1} \cdot \infty = 0$, which is equivalent to $U^{-1} \cdot \infty = C \cdot 0$. It would follow that $C \cdot 0 = -1$ is a fixed point of $T_{\sigma_1}^{\epsilon, 0}$ and $T_{\sigma_2}^{\epsilon, 0}$. But, $T_{\sigma}^{\epsilon, \delta} \cdot (-1) = -e^{2ik} \neq -1$, as $k \in (0, \pi)$.   \qed
Proposition 21 If $\delta \neq 0$ or $\epsilon \neq 0$ and $\text{card}(\text{supp}(w_\sigma)) > 1$, then (90) fulfills condition (ix).

Proof.

Case 1: One has either $\delta \neq 0$ or $\epsilon \neq 0$ and $|\epsilon w - E| > 2$ for some $w \in \text{supp}(w_\sigma)$.

In that case, there is some $T \in \text{supp}(T_{\sigma}^{\epsilon, \delta})$ such that $T$ has an eigenvalue whose modulus is larger than 1. Therefore, one has $\|T^N\| \to \infty$ as $N \to \infty$. In particular, the semigroup generated by $\text{supp}(T_{\sigma}^{\epsilon, \delta})$ is not relatively compact. $\diamond$

Case 2: One has $\delta = 0$ and $\epsilon \neq 0$ and $|\epsilon w - E| = 2$ for some $w \in \text{supp}(w_\sigma)$.

In that case, there is some $T \in \text{supp}(T_{\sigma}^{\epsilon, 0})$ such that for either $+$ or $-$, it holds that

$$M^{-1}TM = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad M^{-1}TM = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix},$$

which implies

$$M^{-1}T^N M = \begin{pmatrix} 1 + N & -N \\ N & 1 - N \end{pmatrix} \quad \text{or} \quad M^{-1}T^N M = (-1)^N \begin{pmatrix} 1 + N & N \\ -N & 1 - N \end{pmatrix}. $$

Therefore, $\|T^N\| = [1 + (1 + 4N^2)^{\frac{1}{2}}]^2 \to \infty$ as $N \to \infty$. In particular, the semigroup generated by $\text{supp}(T_{\sigma}^{\epsilon, 0})$ is not relatively compact. $\diamond$

Case 3: One has $\delta = 0$ and $\epsilon \neq 0$ and $\text{card}(\text{supp}(w_\sigma)) > 1$ and $|\epsilon w - E| < 2$ for all $w \in \text{supp}(w_\sigma)$.

By assumption, there exist $\sigma_1, \sigma_2 \in \Sigma$ such that $w_{\sigma_1}, w_{\sigma_2} \in \text{supp}(w_\sigma)$ satisfy $w_{\sigma_1} \neq w_{\sigma_2}$. Since $|\epsilon w - E| < 2$ for all $w \in \text{supp}(w_\sigma)$, one can assume without loss of generality that $w_{\sigma_1} = 0$ and $w_{\sigma_2} \neq 0$, as a shift of the energy by $-\epsilon w_{\sigma_1}$ is possible without the violation of $E \in (-2, 2)$.

Then, one has

$$T_{\sigma_1}^{\epsilon, 0} = \begin{pmatrix} e^{-i\kappa} & 0 \\ 0 & e^{i\kappa} \end{pmatrix}, \quad T_{\sigma_2}^{\epsilon, 0} = \begin{pmatrix} e^{-i\kappa} & 0 \\ 0 & e^{i\kappa} \end{pmatrix}\exp\left[\frac{\epsilon w_{\sigma_2}(B_2 + B_3)}{2\sin(k)}\right] = T_{\sigma_1}^{\epsilon, 0} \begin{pmatrix} 1 + \frac{\epsilon \kappa^2}{2} & \frac{\epsilon \kappa}{2} \\ -\frac{\epsilon \kappa}{2} & 1 - \frac{\epsilon \kappa}{2} \end{pmatrix},$$

where $\kappa = \epsilon w_{\sigma_2}\sin(k)^{-1} \neq 0$. Next, let us observe that all $m \in \mathbb{N}$ satisfy

$$\|[(T_{\sigma_1}^{\epsilon, 0})^{-1}T_{\sigma_2}^{\epsilon, 0}]^m\| = \left\| \begin{pmatrix} 1 + \frac{\kappa^2m^2}{2} & \frac{\kappa m}{2} \\ -\frac{\kappa m}{2} & 1 - \frac{\kappa^2m^2}{2} \end{pmatrix} \right\| = \left(1 + \frac{\kappa^2m^2}{2} + \frac{\kappa m}{2}\right)^\frac{1}{2}$$

and, therefore, $\|[(T_{\sigma_1}^{\epsilon, 0})^{-1}T_{\sigma_2}^{\epsilon, 0}]^m\|$ is increasing in $m$ and diverges as $m \to \infty$.

Now, for all $\zeta > 0$, there exists some $\Xi \in \mathbb{N}$ such that $|e^{ik\Xi} - 1| \leq \zeta$. Thus, one can pick a sequence $\{\Xi_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ for which the inequality $|e^{ik\Xi} - 1| \leq 2^{-n-1}\|[(T_{\sigma_1}^{\epsilon, 0})^{-1}T_{\sigma_2}^{\epsilon, 0}]^n\|$ and hence

$$\|[(T_{\sigma_1}^{\epsilon, 0})^{\Xi_n} - 1\| \leq 2^{-n-1}\|[(T_{\sigma_1}^{\epsilon, 0})^{-1}T_{\sigma_2}^{\epsilon, 0}]^n\| \quad (95)$$

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holds. Further, as $T_{\sigma_1}^{\epsilon,0}$ is unitary, all $n \in \mathbb{N}$ obey $\|(T_{\sigma_1}^{\epsilon,0})^{\Xi_n}\| = 1$. Together with (95), this implies

\[
\left\| \prod_{n=1}^{N} (T_{\sigma_1}^{\epsilon,0})^{\Xi_n - 1} T_{\sigma_2}^{\epsilon,0} - \left[ (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \right]^{N} \right\| \\
= \sum_{m=1}^{N} \left\| \prod_{n=m+1}^{N} (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \left[ \prod_{n=1}^{m} (T_{\sigma_1}^{\epsilon,0})^{\Xi_n - 1} T_{\sigma_2}^{\epsilon,0} \right] - \prod_{n=m}^{N} (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \right\| \\
\leq \sum_{m=1}^{N} \left\| \prod_{n=m+1}^{N} (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \left[ (T_{\sigma_1}^{\epsilon,0})^{\Xi_m} - 1 \right] \prod_{n=1}^{m} (T_{\sigma_1}^{\epsilon,0})^{\Xi_n - 1} T_{\sigma_2}^{\epsilon,0} \right\| \\
\leq \sum_{m=1}^{N} 2^{-m-1} \left\| \left[ (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \right]^{N-m} \left[ (T_{\sigma_1}^{\epsilon,0})^{\Xi_m} - 1 \right] \right\| \left\| (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \right\|^m \\
\leq \frac{1}{2} \left\| \left[ (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \right]^{N} \right\| \\
\text{for all } N \in \mathbb{N} \text{ and, therefore,} \\
\left\| \prod_{n=1}^{N} (T_{\sigma_1}^{\epsilon,0})^{\Xi_n - 1} T_{\sigma_2}^{\epsilon,0} \right\| \geq \left\| \left[ (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \right]^{N} \right\| - \frac{1}{2} \left\| \left[ (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \right]^{N} \right\| = \frac{1}{2} \left\| \left[ (T_{\sigma_1}^{\epsilon,0})^{-1} T_{\sigma_2}^{\epsilon,0} \right]^{N} \right\| \rightarrow \infty \\
\text{as } N \rightarrow \infty \text{ due to (94). Thus the semigroup generated by } \text{supp}(T_{\sigma}^{\epsilon,0}) \text{ is not relatively compact.} \diamond

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