Perturbation and Stability of Continuous Operator Frames in Hilbert $C^*$-Modules

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Frame theory has a great revolution in recent years. This theory has been extended from the Hilbert spaces to Hilbert $C^*$-modules. In this paper, we consider the stability of continuous operator frame and continuous $K$-operator frames in Hilbert $C^*$-modules under perturbation, and we establish some properties.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces is a new theory which was introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series. This theory was reintroduced and developed by Daubechies et al. [2].

In 1993, Ali et al. [3] introduced the concept of continuous frames in Hilbert spaces. Gabardo and Han in [4] called these kinds of frames, frames associated with measurable spaces.

In 2000, Frank and Larson [5] introduced the notion of frames in Hilbert $C^*$-modules as a generalization of frames in Hilbert spaces. The theory of continuous frames has been generalized in Hilbert $C^*$-modules. For more details, see [6–25].

The aim of this paper is to extend the results of Rossafi and Akhlidj [23], given for Hilbert $C^*$-module in a discrete case.

In the following, we briefly recall the definitions and basic properties of $C^*$-algebra and Hilbert $\mathcal{A}$-modules. Our references for $C^*$-algebras are [26, 27]. For $C^*$-algebra $\mathcal{A}$, if $a \in \mathcal{A}$ is positive, we write $a \geq 0$, and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1 (see [26]). Let $\mathcal{A}$ be unital $C^*$-algebra and $\mathcal{H}$ be left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot , \cdot \rangle_\mathcal{A}$, such that it is sesquilinear and positive definite and respects the module action. In the other words,

(i) $\langle x, x \rangle_\mathcal{A} \geq 0$, for all $x \in \mathcal{H}$, and $\langle x, x \rangle_\mathcal{A} = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z \rangle_\mathcal{A} = a \langle x, z \rangle_\mathcal{A} + \langle y, z \rangle_\mathcal{A}$, for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

(iii) $\langle x, y \rangle_\mathcal{A} = \langle y, x \rangle_\mathcal{A}^*$, for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x \rangle_\mathcal{A}\|^{1/2}$. If $\mathcal{H}$ is complete with $\|\cdot\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert $C^*$-module over $\mathcal{A}$.

For every $a$ in $C^*$-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{1/2}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle_\mathcal{A}^{1/2}$, for all $x \in \mathcal{H}$.
Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $A$-modules, a map $T: \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^*: \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle_A = \langle x, T^*y \rangle_A$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We reserve the notation $\text{End}_A^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $\text{End}_A^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $\text{End}_A^*(\mathcal{H})$.

The following lemmas will be used to prove our result.

**Lemma 1** (see [28]). Let $\mathcal{H}$ be a Hilbert $A$-module. If $T \in \text{End}_A^*(\mathcal{H})$, then
\[
\langle Tx,Tx \rangle_A \leq \|T\|^2 \langle x,x \rangle_A, \quad x \in \mathcal{H}.
\]

**Lemma 2** (see [29]). Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $A$-modules and $T \in \text{End}_A^*(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:

(i) $T$ is surjective.

(ii) $T^*$ is bounded below with respect to norm, i.e., there is $\tau > 0$ such that $\|T^*x\| \geq \tau \|x\|$, for all $x \in \mathcal{H}$.

(iii) $T^*$ is bounded below with respect to the inner product, i.e., there is $\zeta > 0$ such that $\langle T^*x, T^*y \rangle_A \geq \zeta \langle x, y \rangle_A$, for all $x \in \mathcal{H}$.

**Lemma 3** (see [30]). Let $(\Omega, \mu)$ be a measure space, $X$ and $Y$ are two Banach spaces, $\lambda: X \to Y$ is a bounded linear operator and $f: \Omega \to X$ measurable function, then
\[
\lambda\left(\int_{\Omega} f \, d\mu\right) = \int_{\Omega} (\lambda f) \, d\mu.
\]

2. Characterisation of Continuous Operator Frame for $\text{End}_A^*(\mathcal{H})$

Let $X$ be a Banach space, $(\Omega, \mu)$ a measure space, and $f: \Omega \to X$ be a measurable function. Integral of Banach-valued function $f$ has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions [30, 31]. Since every $C^*$-algebra and Hilbert $C^*$-module are Banach spaces, we can use this integral and its properties.

Let $(\Omega, \mu)$ be a measure space, $U$ and $V$ be two Hilbert $C^*$-modules over a unital $C^*$-algebra and $\{V_w\}_{w \in \Omega}$ is a family of submodules of $V$. $\text{End}_A^*(U, V_w)$ is the collection of all adjointable $A$-linear maps from $U$ into $V_w$.

We define the following:
\[
\mathcal{P}(\Omega, \{V_w\}_{w \in \Omega}) = \left\{ x = \{x_w\}_{w \in \Omega}: x_w \in V_w, \|x_w\|^2 \, d\mu(w) < \infty \right\}.
\]

For any $x = \{x_w\}_{w \in \Omega}$ and $y = \{y_w\}_{w \in \Omega}$, the $A$-valued inner product is defined by $\langle x, y \rangle_A = \int_\Omega \langle x_w, y_w \rangle_A \, d\mu(w)$ and the norm is defined by $\|x\|^2 = \int_\Omega \|x_w\|^2 \, d\mu(w)$. In this case, $\mathcal{P}(\Omega, \{V_w\}_{w \in \Omega})$ is an Hilbert $C^*$-module [32].

**Definition 2.** We call $\Lambda = \{\Lambda_w \in \text{End}_A^*(\mathcal{H}) : w \in \Omega\}$ a continuous operator frame for $\text{End}_A^*(\mathcal{H})$ if

(a) for any $x \in \mathcal{H}$, the mapping $\tilde{x}: \Omega \to V_w$ defined by $\tilde{x}(w) = \Lambda_w x$ is measurable

(b) there is a pair of constants $0 < \nu, \delta$ such that for any $x \in \mathcal{H}$,
\[
\nu \|x\|^2 \leq \int_{\Omega} \langle \Lambda_w x, \Lambda_w x \rangle_A \, d\mu(w) \leq \delta \|x\|^2,
\]
\[
x \in \mathcal{H}.
\]

The constants $\nu$ and $\delta$ are called continuous operator frame bounds.

If $\nu = \delta$, we call this continuous operator frame a continuous tight operator frame, and if $\nu = \delta = 1$, it is called a continuous Parseval operator frame.

If only the right-hand inequality of (4) is satisfied, we call $\Lambda = \{\Lambda_w \}_{w \in \Omega}$ the continuous Bessel operator frame for $\text{End}_A^*(\mathcal{H})$ with Bessel bound $\delta$.

The continuous frame operator $S$ of $\Lambda$ on $\mathcal{H}$ is defined by
\[
Sx = \int_{\Omega} \Lambda_w^* \Lambda_w x \, d\mu(w), \quad x \in \mathcal{H}.
\]

The continuous frame operator $S$ is a bounded, positive, self-adjoint, and invertible.

**Theorem 1.** Let $\Lambda = \{\Lambda_w \in \text{End}_A^*(\mathcal{H}) : w \in \Omega\}$. $\Lambda$ is a continuous operator frame for $\text{End}_A^*(\mathcal{H})$ if and only if there exist constants $0 < \nu, \delta$ such that for any $x \in \mathcal{H}$,
\[
\nu \|x\|^2 \leq \int_{\Omega} \langle \Lambda_w x, \Lambda_w x \rangle_A \, d\mu(w) \leq \delta \|x\|^2.
\]

3. Perturbation and Stability of Continuous Operator Frame for $\text{End}_A^*(\mathcal{H})$

**Theorem 2.** Let $\{T_w\}_{w \in \Omega}$ be a continuous operator frame for $\text{End}_A^*(\mathcal{H})$ with bounds $\nu$ and $\delta$. If $\{R_w\}_{w \in \Omega} \subset \text{End}_A^*(\mathcal{H})$ is a continuous operator Bessel family with bound $\xi < \nu$, then $\{T_w + R_w\}_{w \in \Omega}$ is a continuous operator frame for $\text{End}_A^*(\mathcal{H})$.

**Proof.** We just prove the case that $\{T_w + R_w\}_{w \in \Omega}$ is a continuous operator frame for $\text{End}_A^*(\mathcal{H})$.

On the one hand, for each $x \in \mathcal{H}$, we have
\[ \| (T_w + R_w)x \|_{\omega \in \Omega} = \left( \int_{\Omega} \langle (T_w + R_w)x, (T_w + R_w)x \rangle_{\omega} d\mu(w) \right)^{1/2} \]
\[
\leq \left\| (T_w x)_{\omega \in \Omega} \right\| + \left\| (R_w x)_{\omega \in \Omega} \right\|
\leq \int_{\Omega} \langle T_w x, T_w x \rangle_{\omega} d\mu(w) \left( \int_{\Omega} \langle R_w x, R_w x \rangle_{\omega} d\mu(w) \right)^{1/2} + \int_{\Omega} \langle R_w x, R_w x \rangle_{\omega} d\mu(w) \left( \int_{\Omega} \langle T_w x, T_w x \rangle_{\omega} d\mu(w) \right)^{1/2}
\leq \sqrt{\delta} \|x\| + \sqrt{\xi} \|x\|. \quad (7) \]

Hence,
\[
\left\| \int_{\Omega} \langle (T_w + R_w)x, (T_w + R_w)x \rangle_{\omega} d\mu(w) \right\|^{1/2} \leq (\sqrt{\delta} + \sqrt{\xi}) \|x\|. \quad (8) \]

\[
\left\| (T_w + R_w)x \right\|_{\omega \in \Omega} = \left( \int_{\Omega} \langle (T_w + R_w)x, (T_w + R_w)x \rangle_{\omega} d\mu(w) \right)^{1/2}
\geq \left\| (T_w x)_{\omega \in \Omega} \right\| - \left\| (R_w x)_{\omega \in \Omega} \right\|
\geq \int_{\Omega} \langle T_w x, T_w x \rangle_{\omega} d\mu(w) \left( \int_{\Omega} \langle R_w x, R_w x \rangle_{\omega} d\mu(w) \right)^{1/2} - \int_{\Omega} \langle R_w x, R_w x \rangle_{\omega} d\mu(w) \left( \int_{\Omega} \langle T_w x, T_w x \rangle_{\omega} d\mu(w) \right)^{1/2}
\geq \sqrt{\nu} \|x\| - \sqrt{\xi} \|x\|. \quad (9) \]

Then,
\[
\left\| \int_{\Omega} \langle (T_w + R_w)x, (T_w + R_w)x \rangle_{\omega} d\mu(w) \right\|^{1/2} \geq (\sqrt{\nu} - \sqrt{\xi}) \|x\|. \quad (10) \]

\[
(\sqrt{\nu} - \sqrt{\xi})^2 \|x\|^2 \leq \int_{\Omega} \langle (T_w + R_w)x, (T_w + R_w)x \rangle_{\omega} d\mu(w) \leq (\sqrt{\delta} + \sqrt{\xi})^2 \|x\|^2. \quad (11) \]

Therefore, \( (T_w + R_w)_{\omega \in \Omega} \) is a continuous operator frame for \( \text{End}_{\omega}^d(\mathcal{H}) \). \( \square \)

**Theorem 3.** Let \( (T_w)_{\omega \in \Omega} \) be a continuous operator frame for \( \text{End}_{\omega}^d(\mathcal{H}) \) with bounds \( \nu \) and \( \delta \) and let \( (R_w)_{\omega \in \Omega} \subset \text{End}_{\omega}^d(\mathcal{H}) \). The following statements are equivalent:

(i) \( (R_w)_{\omega \in \Omega} \) is a continuous operator frame for \( \text{End}_{\omega}^d(\mathcal{H}) \).

(ii) There exists a constant \( \xi > 0 \), such that for all \( x \) in \( \mathcal{H} \), we have

\[
\| (T_w - R_w)x, (T_w - R_w)x \|_{\omega \in \Omega} \leq \xi \cdot \min \left( \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\omega} d\mu(w) \right\|, \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\omega} d\mu(w) \right\| \right). \quad (12) \]

**Proof.** Suppose that \( (R_w)_{\omega \in \Omega} \) is a continuous operator frame for \( \text{End}_{\omega}^d(\mathcal{H}) \) with bound \( \eta \) and \( \rho \). Then for all \( x \in \mathcal{H} \), we have
\[
\left\| (T_w - R_w)x \right\|_{w \in \Omega} = \left\| \int_{\Omega} \langle (T_w - R_w)x, (T_w - R_w)x \rangle_{sY} d\mu(w) \right\|^{1/2}
\]
\[
\leq \left\| (T_w x)_{w \in \Omega} \right\| + \left\| (R_w x)_{w \in \Omega} \right\|
\]
\[
= \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{sY} d\mu(w) \right\|^{1/2} + \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{sY} d\mu(w) \right\|^{1/2}
\]
\[
\leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{sY} d\mu(w) \right\|^{1/2} + \sqrt{\rho} \| x \|
\]
\[
\leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{sY} d\mu(w) \right\|^{1/2} + \sqrt{\rho} \| \int_{\Omega} \langle T_w x, T_w x \rangle_{sY} d\mu(w) \right\|^{1/2}
\]
\[
= \left( 1 + \sqrt{\rho} \right) \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{sY} d\mu(w) \right\|^{1/2}.
\]

In the same way, we have

\[
\left\| \int_{\Omega} \langle (T_w - R_w)x, (T_w - R_w)x \rangle_{sY} d\mu(w) \right\|^{1/2} \leq \left( 1 + \sqrt{\delta/\eta} \right) \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{sY} d\mu(w) \right\|^{1/2}.
\]

(13)

For (12), we take \( \xi = \min (1 + \sqrt{\delta/\eta}, 1 + \sqrt{\rho/\gamma}) \).

Now we assume that (12) holds. For each \( x \in \mathcal{H} \), we have

\[
\sqrt{\gamma} \| x \| \leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{sY} d\mu(w) \right\|^{1/2}
\]
\[
= \left\| (T_w x)_{w \in \Omega} \right\|
\]
\[
\leq \left\| (T_w - R_w)x \right\|_{w \in \Omega} + \left\| (R_w x)_{w \in \Omega} \right\|
\]
\[
= \left\| \int_{\Omega} \langle (T_w - R_w)x, (T_w - R_w)x \rangle_{sY} d\mu(w) \right\|^{1/2} + \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{sY} d\mu(w) \right\|^{1/2}
\]

(15)

From (12), we have

\[
\left\| \int_{\Omega} \langle (T_w - R_w)x, (T_w - R_w)x \rangle_{sY} d\mu(w) \right\| \leq \xi \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{sY} d\mu(w) \right\|
\]

(16)

Then,

\[
\left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{sY} d\mu(w) \right\|^{1/2} \leq \sqrt{\xi} \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{sY} d\mu(w) \right\|^{1/2} + \sqrt{\xi} \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{sY} d\mu(w) \right\|^{1/2}
\]

(17)
Hence,

\[
\sqrt{\nu} \|x\| \leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} d\mu(w) \right\|^{(1/2)} \leq (1 + \sqrt{\xi}) \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} d\mu(w) \right\|^{(1/2)}.
\]  

Also, we have

\[
\left\| \{R_w x\}_{w \in \Omega} \right\| = \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} d\mu(w) \right\|^{(1/2)} = \left\| \{(R_w x - T_w x) + T_w x\}_{w \in \Omega} \right\|^{(1/2)} = \left\| \int_{\Omega} \langle (T_w - R_w) x, (T_w - R_w) x \rangle_{\mathcal{H}} d\mu(w) \right\|^{(1/2)} + \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} d\mu(w) \right\|^{(1/2)}.
\]

From (12), we have

\[
\left\| \int_{\Omega} \langle (T_w - R_w) x, (T_w - R_w) x \rangle_{\mathcal{H}} d\mu(w) \right\| \leq \xi \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} d\mu(w) \right\|.
\]

Then,

\[
\left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} d\mu(w) \right\|^{(1/2)} \leq (1 + \sqrt{\xi}) \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} d\mu(w) \right\|^{(1/2)}.
\]

So,

\[
\left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} d\mu(w) \right\|^{(1/2)} \leq (1 + \sqrt{\xi}) \sqrt{\delta} \|x\|.
\]  

From (18) and (22), we give that

\[
\frac{\nu}{(1 + \sqrt{\xi})^2} \|x\|^2 \leq \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} d\mu(w) \right\| \leq \delta (1 + \sqrt{\xi})^2 \|x\|^2.
\]  

Therefore, \( \{R_w\}_{w \in \Omega} \) is a continuous operator frame for \( \text{End}_{\mathcal{H}}^n(\mathcal{H}) \).

**Theorem 4.** Let \( \{T_{k,w}\}_{w \in \Omega} \subset \text{End}_{\mathcal{H}}^n(\mathcal{H}), k = 1, 2, \ldots, n \) be a continuous operator frames for \( \text{End}_{\mathcal{H}}^n(\mathcal{H}) \) with bounds \( \nu_k \) and \( \delta_k \) and let \( \{\alpha_k\}_{k=1}^n \) be any scalars. If there exists a constant \( \lambda > 0 \) and some \( p \in \{1, 2, \ldots, n\} \) such that

\[
\lambda \left\| \{T_{p,w}\}_{w \in \Omega} \right\| \leq \sum_{k=1}^n |\alpha_k T_{k,w} x|, \quad x \in \mathcal{H},
\]

then \( \left\{ \sum_{k=1}^n \alpha_k T_{k,w} \right\}_{w \in \Omega} \) is a continuous operator frame for \( \text{End}_{\mathcal{H}}^n(\mathcal{H}) \) and conversely.

**Proof.** For every \( x \in \mathcal{H} \), we have

\[
\sqrt{\nu p} \lambda \|x, x\| \leq \left\| \{T_{p,w} x\}_{w \in \Omega} \right\| \leq \left\| \left\{ \sum_{k=1}^n |\alpha_k T_{k,w} x| \right\}_{w \in \Omega} \right\| \leq \left( \max_{1 \leq k \leq n} |\alpha_k| \right) \sum_{k=1}^n \| T_{k,w} x \|_{w \in \Omega} \leq \left( \max_{1 \leq k \leq n} |\alpha_k| \right) \left( \sum_{k=1}^n \sqrt{\delta_k} \right) \|x, x\|^{(1/2)}.
\]
Hence
\[
y_p \lambda^2 \| \langle x, x \rangle_{sf} \| \leq \left\| \sum_{k=1}^{n} a_k T_{k,w} x \right\|_{w \in \Omega}^2 \leq \left( \max_{k \in \Omega} |a_k| \right)^2 \left( \sum_{k=1}^{n} \frac{1}{\sqrt{\delta_k}} \right)^2 \| \langle x, x \rangle_{sf} \|. \tag{26}
\]

Therefore, \( \left\{ \sum_{k=1}^{n} a_k T_{k,w} x \right\}_{w \in \Omega} \) is a continuous operator frame for \( \text{End}_f^* (\mathcal{H}) \).

For the converse, let \( \left\{ \sum_{k=1}^{n} a_k T_{k,w} x \right\}_{w \in \Omega} \) be a continuous operator frame for \( \text{End}_f^* (\mathcal{H}) \) with bounds \( \nu, \delta \) and let any \( k \in \{1, 2, \ldots, n\} \).

Since \( \left\{ T_{p,w} \right\}_{w \in \Omega} \) is a continuous operator frame for \( \text{End}_f^* (\mathcal{H}) \) with bounds \( \nu_p \) and \( \delta_p \), then for any \( x \in \mathcal{H} \), we have
\[
y_p \| \langle x, x \rangle_{sf} \| \leq \left\| \left\{ T_{p,w} \right\}_{w \in \Omega} \right\| \| \langle x, x \rangle_{sf} \|. \tag{27}
\]
Hence,
\[
\delta_p^{-1} \left\| \left\{ T_{p,w} \right\}_{w \in \Omega} \right\| \leq \| \langle x, x \rangle_{sf} \|. \tag{28}
\]
Also, we have
\[
\nu \| \langle x, x \rangle_{sf} \| \leq \left\| \left\{ \sum_{k=1}^{n} a_k T_{k,w} x \right\}_{w \in \Omega} \right\|_{w \in \Omega}^2, \quad x \in \mathcal{H}. \tag{29}
\]
Then,
\[
\| \langle x, x \rangle_{sf} \| \leq \nu^{-1} \left\| \left\{ \sum_{k=1}^{n} a_k T_{k,w} x \right\}_{w \in \Omega} \right\|_{w \in \Omega}^2, \quad x \in \mathcal{H}. \tag{30}
\]
So,
\[
\left\| \int_{\Omega} \langle T_{k,w} - R_{k,w}, (T_{k,w} - R_{k,w}) x \rangle_{sf} d\mu(w) \right\| \leq \lambda \int_{\Omega} \langle T_{w,x}, T_{w,x} \rangle_{sf} d\mu(w). \tag{34}
\]
Then \( \left\{ \sum_{k=1}^{n} R_{k,w} \right\}_{w \in \Omega} \) is a continuous operator frame for \( \text{End}_f^* (\mathcal{H}) \).

Proof. For all \( x \in \mathcal{H} \), we have
\[
\left\| \sum_{k=1}^{n} R_{k,w} x \right\|_{w \in \Omega} \leq \left\| \sum_{k=1}^{n} \left\| T_{k,w} x \right\|_{w \in \Omega} \right\| \leq \left\| \sum_{k=1}^{n} \left\{ T_{k,w} - R_{k,w} \right\}_{w \in \Omega} \right\| + \left\| \sum_{k=1}^{n} R_{k,w} x \right\|_{w \in \Omega} \leq (1 + \sqrt{\lambda}) \left( \sum_{k=1}^{n} \frac{1}{\sqrt{\delta_k}} \right) \| \langle x, x \rangle_{sf} \|^{(1/2)}. \tag{35}
\]
Since, for any \( x \in \mathcal{H} \), we have
\[
\left\| L \left( \sum_{k=1}^{n} R_{k,w} \right) \right\| = \left\| \{ T_{p,w} \}_{w \in \Omega} \right\|. \tag{36}
\]

Then
\[
\sqrt{\frac{\| p \|}{\| L \|}} \| \langle x, x \rangle \|_{\mathcal{H}}^{1/2} \leq \left\| \sum_{k=1}^{n} R_{k,w} \right\|_{w \in \Omega}, \quad x \in \mathcal{H}. \tag{37}
\]

Hence
\[
\frac{\sqrt{\| p \|}}{\| L \|} \| \langle x, x \rangle \|^{1/2} \leq \left\| \sum_{k=1}^{n} R_{k,w} \right\|_{w \in \Omega}. \tag{38}
\]

Therefore
\[
\frac{\sqrt{\| p \|}}{\| L \|} \| \langle x, x \rangle \|^{1/2} \leq \left( 1 + \sqrt{\lambda} \right) \left\| \sum_{k=1}^{n} \sqrt{\| \delta_k \|} \right\| \| \langle x, x \rangle \|^{1/2}. \tag{39}
\]

This gives \( \sum_{k=1}^{n} R_{k,w} \) \( w \in \Omega \) is a continuous operator frame for \( \text{End}_{\mathcal{Y}}^{*} (\mathcal{H}) \).

\[\Box\]

4. Characterisation of Continuous K-Operator Frames for \( \text{End}_{\mathcal{Y}}^{*} (\mathcal{H}) \)

**Definition 3.** Let \( K \in \text{End}_{\mathcal{Y}}^{*} (\mathcal{H}) \). A family of adjointable operators \( \{ T_{w} \}_{w \in \Omega} \) on a Hilbert \( \mathcal{Y} \)-module \( \mathcal{H} \) is said to be a continuous K-operator frame for \( \text{End}_{\mathcal{Y}}^{*} (\mathcal{H}) \), if there exists two positive constants \( \nu, \delta > 0 \) such that
\[
\delta \| K^{*} x, K^{*} x \|_{\mathcal{Y}} \leq \int_{\Omega} \langle T_{w} x, T_{w} x \rangle_{\mathcal{Y}} \mathrm{d} \mu (w) \leq \delta \| x, x \|_{\mathcal{Y}}, \quad x \in \mathcal{H}. \tag{40}
\]

The numbers \( \nu \) and \( \delta \) are called, respectively, lower and upper bound of the continuous K-operator frame.

The continuous K-operator frame is called a \( \nu \)-tight if:
\[
\nu \| K \|^{2} \| K^{*} x, K^{*} x \|_{\mathcal{Y}} \leq \nu \| x, x \|_{\mathcal{Y}} \leq \int_{\Omega} \langle T_{w} x, T_{w} x \rangle_{\mathcal{Y}} \mathrm{d} \mu (w) \leq \delta \| x, x \|_{\mathcal{Y}}, \quad x \in \mathcal{H}. \tag{43}
\]

Hence \( \{ T_{w} \}_{w \in \Omega} \) is a continuous K-operator frame with bounds \( \nu \| K \|^{2} \) and \( \delta \).

Let \( \{ T_{w} \}_{w \in \Omega} \) be a continuous K-operator for \( \text{End}_{\mathcal{Y}}^{*} (\mathcal{H}) \). We define the operator
\[
\mathcal{R} : \mathcal{H} \rightarrow \ell^{2} (\mathcal{H}), \quad x \mapsto \mathcal{R} x = \{ T_{w} x \}_{w \in \Omega}. \tag{44}
\]

The operator \( \mathcal{R} \) is called the analysis operator of the continuous K-operator frame \( \{ T_{w} \}_{w \in \Omega} \), and its adjoint is defined as follows:

\[
\mathcal{R}^{*} : \ell^{2} (\mathcal{H}) \rightarrow \mathcal{H}, \quad \{ x_{w} \}_{w \in \Omega} \mapsto \mathcal{R}^{*} \{ x_{w} \}_{w \in \Omega} = \sum_{w \in \Omega} T_{w} x_{w} \mathrm{d} \mu (w). \tag{45}
\]

The operators \( \mathcal{R} \) is called the synthesis operator of the continuous K-operator frame \( \{ T_{w} \}_{w \in \Omega} \).

By composing \( \mathcal{R} \) and \( \mathcal{R}^{*} \), we obtain the operator
\[
\mathcal{S}^{*} : \mathcal{H} \rightarrow \mathcal{H}, \quad x \mapsto \mathcal{S}^{*} x = \mathcal{R}^{*} \mathcal{R} x = \sum_{w \in \Omega} T_{w}^{*} T_{w} x \mathrm{d} \mu (w). \tag{46}
\]
It is easy to show that the operator $S_{\mathcal{H}}$ is positive and self-adjoint.

**Theorem 6.** Let $\{T_w\}_{w \in \Omega}$ be a family of adjointable operators on a Hilbert $\mathcal{A}$-module $\mathcal{H}$. Assume that 

$$\int_\Omega \langle T_w x, T_w x \rangle_d \mu(w) \text{ converges in norm for all } x \in \mathcal{H}. \quad (47)$$

Then $\{T_w\}_{w \in \Omega}$ is a continuous K-operator frame for $\text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$ if and only if there exists two positive constants $\nu, \delta > 0$ such that

$$\nu \|K^*x\|^2 \leq \int_\Omega \langle T_w x, T_w x \rangle_d \mu(w) \leq \delta \|x\|^2. \quad (47)$$

**Proof.** Suppose that $\{T_w\}_{w \in \Omega}$ is a continuous K-operator frame.

From the definition of continuous K-operator frame, (47) holds.

Conversely, assume that (47) holds. The frame operator $S_{\mathcal{H}}$ is positive and self-adjoint; then

$$\langle S^{(1/2)}_{\mathcal{H}} x, S^{(1/2)}_{\mathcal{H}} x \rangle \leq \langle S_{\mathcal{H}} x, S_{\mathcal{H}} x \rangle \leq \int_\Omega \langle T_w x, T_w x \rangle_d \mu(w). \quad (48)$$

Then $\{T_w\}_{w \in \Omega}$ is a continuous K-operator frame for $\text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$.

We have for any $x \in \mathcal{H}$,

$$\|K^*x\| \leq \|S^{(1/2)}_{\mathcal{H}} x\| \leq \sqrt{\delta} \|x\|. \quad (49)$$

Using Lemma 2, there exist two constants $\tau, \xi > 0$ such that

$$\tau \langle K^* x, K^* x \rangle \leq \langle S^{(1/2)}_{\mathcal{H}} x, S^{(1/2)}_{\mathcal{H}} x \rangle \leq \tau \langle x, x \rangle. \quad (50)$$

This proves that $\{T_w\}_{w \in \Omega}$ is a continuous K-operator frame for $\text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$. \hfill $\Box$

### 5. Perturbation and Stability of Continuous K-Operator Frames for $\text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$

**Theorem 7.** Let $\{T_w\}_{w \in \Omega}$ be a continuous K-operator frame for $\text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$ with bounds $A$ and $B$, let $\{R_w\}_{w \in \Omega} \subset \text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$ and $\{\beta_w\}_{w \in \Omega}, \{\mu_w\}_{w \in \Omega} \in \mathbb{R}$ be two positively families. If there exist two constants $0 \leq \lambda, \mu < 1$ such that for any $x \in \mathcal{H}$, we have

\[
\|\beta_w R_w x\|_{w \in \Omega} \leq (1 + \lambda) \|\alpha_w T_w x\|_{w \in \Omega} \quad \text{and} \quad \|\alpha_w T_w x\|_{w \in \Omega} \leq (1 + \mu) \|\beta_w R_w x\|_{w \in \Omega}. \quad (51)
\]

Then, $\{R_w\}_{w \in \Omega}$ is a continuous K-operator frame for $\text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$.

**Proof.** For every $x \in \mathcal{H}$, we have

\[
\|R_w x\|_{w \in \Omega} \leq (1 + \lambda) \sup_{w \in \Omega} (\alpha_w) \|T_w x\|_{w \in \Omega}. \quad (55)
\]

Hence

\[
\|R_w x\|_{w \in \Omega} \leq \frac{(1 + \lambda) \sup_{w \in \Omega} (\alpha_w)}{(1 - \mu) \inf_{w \in \Omega} (\beta_w)} \|T_w x\|_{w \in \Omega}. \quad (56)
\]

Also, for all $x \in \mathcal{H}$, we have

\[
(1 - \mu) \inf_{w \in \Omega} (\beta_w) \|R_w x\|_{w \in \Omega} \leq (1 + \lambda) \sup_{w \in \Omega} (\alpha_w) \|T_w x\|_{w \in \Omega}. \quad (54)
\]

\[
\|\alpha_w T_w x\|_{w \in \Omega} \leq \|\alpha_w T_w x\|_{w \in \Omega} + \|\beta_w R_w x\|_{w \in \Omega} \leq \|\alpha_w T_w x\|_{w \in \Omega} + \|\beta_w R_w x\|_{w \in \Omega}. \quad (56)
\]
then
\[(1-\lambda)\|\alpha_w T_w x\|_{w \in \Omega} \leq (1+\mu)\|\beta_w R_w x\|_{w \in \Omega} \].

Hence
\[(1-\lambda) \inf_{\omega \in \Omega} (\alpha_w) \|T_w x\|_{w \in \Omega} \leq (1+\mu) \sup_{\omega \in \Omega} (\beta_w) \|R_w x\|_{w \in \Omega} \].

Thus
\[
\frac{(1-\lambda) \inf_{\omega \in \Omega} (\alpha_w)}{(1+\mu) \sup_{\omega \in \Omega} (\beta_w)} \|T_w x\|_{w \in \Omega} \leq \|R_w x\|_{w \in \Omega} \].

Therefore

\[\nu \left( \frac{(1-\lambda) \inf_{\omega \in \Omega} (\alpha_w)}{(1+\mu) \sup_{\omega \in \Omega} (\beta_w)} \right)^2 \|\langle x, x \rangle_{A^1}\| \leq \left( \frac{(1-\lambda) \inf_{\omega \in \Omega} (\alpha_w)}{(1+\mu) \sup_{\omega \in \Omega} (\beta_w)} \right)^2 \|T_w x\|_{w \in \Omega}^2 \leq \|R_w x\|_{w \in \Omega}^2.\]

So,

\[\|R_w x\|_{w \in \Omega} \leq \left( \frac{(1+\lambda) \sup_{\omega \in \Omega} (\alpha_w)}{(1-\mu) \inf_{\omega \in \Omega} (\beta_w)} \right) \|T_w x\|_{w \in \Omega} \leq \delta \left( \frac{(1+\lambda) \sup_{\omega \in \Omega} (\alpha_w)}{(1-\mu) \inf_{\omega \in \Omega} (\beta_w)} \right) \|\langle x, x \rangle_{A^1}\|.\]

Hence

\[
\nu \left( \frac{(1-\lambda) \inf_{\omega \in \Omega} (\alpha_w)}{(1+\mu) \sup_{\omega \in \Omega} (\beta_w)} \right)^2 \|\langle x, x \rangle_{A^1}\| \leq \int_{\Omega} \langle R_w x, R_w x \rangle d\mu(\omega) \leq \delta \left( \frac{(1+\lambda) \sup_{\omega \in \Omega} (\alpha_w)}{(1-\mu) \inf_{\omega \in \Omega} (\beta_w)} \right) \|\langle x, x \rangle_{A^1}\|.\]

This gives that \( R_w \) is a continuous \( K \)-operator frame for \( \text{End}_{A^1}(\mathcal{H}) \).

**Theorem 8.** Let \( T_w \) be a continuous \( K \)-operator frame for \( \text{End}_{A^1}(\mathcal{H}) \) with bounds \( \nu \) and \( \delta \). Let \( R_w \in \text{End}_{A^1}(\mathcal{H}) \) and \( 0 \leq \alpha < (\beta/\gamma) < 1 \) such that for all \( x \in \mathcal{H} \), we have

\[\int_{\Omega} \langle T_w x, T_w x \rangle d\mu(\omega) \leq \alpha \|\langle x, x \rangle_{A^1}\| + \beta \|\langle x, x \rangle_{A^1}\|.\]

Then \( R_w \) is a continuous \( K \)-operator frame with bounds \( \nu(1-\sqrt{\alpha+(\beta/\gamma)})^2 \) and \( \delta(1+\sqrt{\alpha+(\beta/\gamma)})^2 \).

**Proof.** Let \( T_w \) be a continuous \( K \)-operator frame with bounds \( \nu \) and \( \delta \). Then for any \( x \in \mathcal{H} \), we have

\[
\|T_w x\|_{w \in \Omega} \leq \|T_w x\|_{w \in \Omega} + \|R_w x\|_{w \in \Omega} \leq \left( \alpha \int_{\Omega} \langle T_w x, T_w x \rangle d\mu(\omega) + \beta \|\langle x, x \rangle_{A^1}\| \right)^{\frac{1}{2}}
\]

\[
+ \|\int_{\Omega} \langle R_w x, R_w x \rangle d\mu(\omega) \|^{\frac{1}{2}} \leq \left( \alpha \int_{\Omega} \langle T_w x, T_w x \rangle d\mu(\omega) + \beta \|\langle x, x \rangle_{A^1}\| \right)^{\frac{1}{2}}
\]

\[
+ \|\int_{\Omega} \langle R_w x, R_w x \rangle d\mu(\omega) \|^\frac{1}{2} = \sqrt{\alpha + \frac{\beta}{\gamma}} \|T_w x\|_{w \in \Omega} + \|\int_{\Omega} \langle R_w x, R_w x \rangle d\mu(\omega) \|^\frac{1}{2}.
\]
Therefore

\[
\left( 1 - \sqrt{\alpha + \frac{\beta}{\nu}} \right) \left\| \{T_w x \}_{w \in \Omega} \right\| \leq \left( \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{S}} d\mu(\omega) \right)^{1/2}.
\]

(65)

Thus

\[
\nu \left( 1 - \sqrt{\alpha + \frac{\beta}{\nu}} \right)^2 \left\| \langle K^* x, K^* x \rangle_{\mathcal{S}} \right\| \leq \left( 1 - \sqrt{\alpha + \frac{\beta}{\nu}} \right)^2 \left( \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{S}} d\mu(\omega) \right)^{1/2}.
\]

(66)

Also, we have

\[
\left\| \{R_w x \}_{w \in \Omega} \right\| \leq \left\| \{T_w x - R_w x \}_{w \in \Omega} \right\| + \left\| \{T_w x \}_{w \in \Omega} \right\|
\]

\[
\leq \sqrt{\alpha + \frac{\beta}{\nu}} \left\| \{T_w x \}_{w \in \Omega} \right\| + \left\| \{T_w x \}_{w \in \Omega} \right\| = \left( 1 + \sqrt{\alpha + \frac{\beta}{\nu}} \right) \left\| \{T_w x \}_{w \in \Omega} \right\|
\]

(67)

\[
\leq \sqrt{\delta} \left( 1 + \sqrt{\alpha + \frac{\beta}{\nu}} \right) \langle x, x \rangle_{\mathcal{S}}^{1/2}.
\]

Hence

\[
\left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{S}} d\mu(\omega) \right\| \leq \delta \left( 1 + \sqrt{\alpha + \frac{\beta}{\nu}} \right)^2 \langle x, x \rangle_{\mathcal{S}}.
\]

(68)

\[
\nu \left( 1 - \sqrt{\alpha + \frac{\beta}{\nu}} \right)^2 \left\| \langle K^* x, K^* x \rangle_{\mathcal{S}} \right\| \leq \left( \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{S}} d\mu(\omega) \right)^{1/2} \leq \delta \left( 1 + \sqrt{\alpha + \frac{\beta}{\nu}} \right)^2 \langle x, x \rangle_{\mathcal{S}}.
\]

(69)

Corollary 1. Let \{T_w \}_{w \in \Omega} be a continuous K-operator frame for \text{End}_{\mathcal{S}}(\mathcal{H}) with bounds \nu and \delta. Let \{R_w \}_{w \in \Omega} \subset \text{End}_{\mathcal{S}}(\mathcal{H}) and 0 \leq a. If 0 \leq a < \nu such that

\[
\nu(1 + \sqrt{\alpha - (\beta/\nu)})^2 \quad \text{and} \quad \delta(1 + \sqrt{\alpha + (\beta/\nu)})^2.
\]

Hence \{R_w \}_{w \in \Omega} is a continuous K-operator frame with bounds \nu(1 + \sqrt{\alpha - (\beta/\nu)})^2 and \delta(1 + \sqrt{\alpha + (\beta/\nu)})^2.
\[ \left\| \int_{\Omega} \langle (T_w - R_w)x, (T_w - R_w)x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \leq a \left\| \langle K^* x, K^* x \rangle_{\mathcal{H}} \right\|, \quad x \in \mathcal{H}, \]  \tag{70}

Then \( \{R_w\}_{w \in \Omega} \) is a continuous K-operator frame with bounds \( \nu (1 - \sqrt{\alpha/\gamma})^2 \) and \( \delta (1 + \sqrt{\alpha/\gamma})^2 \).

**Proof.** The proof comes from the previous theorem. \( \square \)

**Theorem 9.** Let \( \{T_w\}_{w \in \Omega} \) be a continuous K-operator frame for \( \text{End}^*_g(\mathcal{H}) \) with bounds \( \nu \) and \( \delta \). Let \( \{R_w\}_{w \in \Omega} \subset \text{End}^*_g(\mathcal{H}) \). If there exists \( \xi > 0 \) such that for any \( x \in \mathcal{H} \), we have

\[ \left\| \int_{\Omega} \langle (T_w - R_w)x, (T_w - R_w)x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \leq \xi \min \left( \left\| \int_{\Omega} \langle R_wx, R_wx \rangle_{\mathcal{H}} \, d\mu(\omega) \right\|, \left\| \int_{\Omega} \langle R_wx, R_wx \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \right). \]  \tag{71}

Then \( \{R_w\}_{w \in \Omega} \) is a continuous K-operator frame for \( \text{End}^*_g(\mathcal{H}) \). The converse is true for any surjective operator \( K \) such that in particular if \( K \) is co-isometry.

**Proof.** Assume that (71) holds. On the one hand, we have for any \( x \in \mathcal{H} \)

\[ \sqrt{\nu} \| K^* x \| \leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| = \left\| \int_{\Omega} \langle T_w x, T_w x - R_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \]
\[ \leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| + \left\| \int_{\Omega} \langle T_w x - R_w x, T_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \]
\[ \leq \sqrt{\nu} \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \]
\[ = (1 + \sqrt{\nu}) \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\|. \]  \tag{72}

On the other hand, we have

\[ \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \]
\[ \leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| + \left\| \int_{\Omega} \langle T_w x - R_w x, T_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \]
\[ \leq \left\| \int_{\Omega} \langle T_w x, T_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| + \sqrt{\delta} \left\| \langle x, x \rangle_{\mathcal{H}} \right\|. \]  \tag{74}

Then

\[ \left\| \int_{\Omega} \langle R_w x, R_w x \rangle_{\mathcal{H}} \, d\mu(\omega) \right\| \leq \sqrt{\nu} \left( 1 + \sqrt{\nu} \right) \left\| \langle x, x \rangle_{\mathcal{H}} \right\|. \]  \tag{73}
From (73) and (75), we obtain

\[
\frac{\nu}{(1 + \sqrt{\xi})^2} \| K^* x \|^2 \leq \left\| \int_\Omega \langle R_w x, R_w x \rangle_\omega d\mu(\omega) \right\| \leq \delta (1 + \sqrt{\xi})^2 \| \langle x, x \rangle_\omega \|^2.
\]

\[
(76)
\]

Hence \( \{ R_w \}_{w \in \Omega} \) is a continuous K-operator frame for \( \text{End}^*_\nu(\mathcal{H}) \).

For the converse, if \( \{ R_w \}_{w \in \Omega} \) is a continuous K-operator frame for \( \text{End}^*_\nu(\mathcal{H}) \) with bound \( \eta \) and \( \rho \), and \( K \) verify that, i.e, \( \| x \| \leq \| K^* x \| \), then for every \( x \in \mathcal{H} \), we have

\[
\| (T_w - R_w) x \|_{w \in \Omega} = \left\| \int_\Omega \langle T_w x, T_w x \rangle_\omega d\mu(\omega) \right\|^{1/2} \leq \left\| \int_\Omega \langle T_w x, T_w x \rangle_\omega d\mu(\omega) \right\|^{1/2} + \sqrt{\nu} \| x \|,
\]

\[
(77)
\]

Similarly we can obtain

\[
\int_\Omega \langle T_w x, T_w x \rangle_\omega d\mu(\omega) \leq \left( 1 + \sqrt{\frac{\delta}{\eta}} \right) \int_\Omega \langle R_w x, R_w x \rangle_\omega d\mu(\omega).
\]

\[
(78)
\]

We take \( \xi = \min \{ 1 + \sqrt{\rho/\nu}, 1 + \sqrt{\delta/\eta} \} \), then (5.1) is verified.

**Theorem 10.** Let \( K \in \text{End}^*_\nu(\mathcal{H}) \). For \( k = 1, 2, \ldots, n \), let \( \{ T_{k,w} \}_{w \in \Omega} \subset \text{End}^*_\nu(\mathcal{H}) \) be a continuous K-operator frame for \( \text{End}^*_\nu(\mathcal{H}) \) with bounds \( \nu_k \) and \( \delta_k \), \( \{ \alpha_k \} \) be any scalars. If there exists a constant \( \lambda > 0 \) and \( p \in \{ 1, 2, \ldots, n \} \) such that

\[
\sqrt{\nu_p} \lambda \left\| \langle K^* x, K^* x \rangle_\omega \right\|^{1/2} \leq \lambda \left\| \{ T_{p,w} x \}_{w \in \Omega} \right\| \leq \sum_{k=1}^n |\alpha_k| \left\| \{ T_{k,w} x \}_{w \in \Omega} \right\| \leq \sum_{k=1}^n |\alpha_k| \left\| \{ T_{k,w} x \}_{w \in \Omega} \right\| \leq \max_{1 \leq k \leq n} |\alpha_k| \left\| \sum_{k=1}^n \alpha_k \right\| \left\| \{ T_{k,w} x \}_{w \in \Omega} \right\| \leq \max_{1 \leq k \leq n} |\alpha_k| \left( \sum_{k=1}^n \delta_k \right) \left\| \langle x, x \rangle_\omega \right\|^{1/2}.
\]

\[
(80)
\]
Hence for any $x \in \mathcal{H}$, we have

$$\sqrt{\nu_p}\|\langle K^*x, K^*x \rangle_{\mathcal{H}}\|^{(1/2)} \leq \sqrt{\sum_{k=1}^{n} \left\langle \alpha_k T_{k,w}x \right\rangle_{\mathcal{H}}} \leq \max_{1 \leq k \leq n} |\alpha_k| \left( \sum_{k=1}^{n} \delta_k \right) \|\langle x, x \rangle_{\mathcal{H}}\|^{(1/2)}. \quad (81)$$

Then

$$\nu_p \delta_p^2 \|\langle K^*x, K^*x \rangle_{\mathcal{H}}\| \leq \left\langle \sum_{k=1}^{n} \alpha_k T_{k,w}x \right\rangle_{\mathcal{H}} \leq \left( \max_{1 \leq k \leq n} |\alpha_k| \right)^2 \left( \sum_{k=1}^{n} \delta_k \right)^2 \|\langle x, x \rangle_{\mathcal{H}}\|. \quad (82)$$

This gives that $\{\sum_{k=1}^{n} \alpha_k T_{k,w}x\}_{w \in \Omega}$ is a continuous K-operator frame for $\text{End}_{\mathcal{H}}^d(\mathcal{H})$. For the converse, let $K$ be a co-isometric operator on $\mathcal{H}$, let $\{\sum_{k=1}^{n} \alpha_k T_{k,w}x\}_{w}$ be a continuous K-operator frame for $\text{End}_{\mathcal{H}}^d(\mathcal{H})$ with bounds $\nu$ and $\delta$ and let for all $p \in \{1, 2, \ldots, n\}$, $\{T_{k,w}x\}_{w}$ be a continuous K-operator frame for $\text{End}_{\mathcal{H}}^d(\mathcal{H})$ with bounds $\nu_p$ and $\delta_p$. Then, for every $x \in \mathcal{H}$, we have

$$\nu_p \|\langle K^*x, K^*x \rangle_{\mathcal{H}}\| \leq \left\langle \sum_{k=1}^{n} \alpha_k T_{k,w}x \right\rangle_{\mathcal{H}} \leq \|\langle x, x \rangle_{\mathcal{H}}\|. \quad (83)$$

Then

$$\frac{1}{\delta_p^2} \left\langle T_{k,w}x \right\rangle_{w} \leq \|\langle x, x \rangle_{\mathcal{H}}\|. \quad (84)$$

Also, we have

$$\nu \|\langle K^*x, K^*x \rangle_{\mathcal{H}}\| \leq \left\langle \sum_{k=1}^{n} \alpha_k T_{k,w}x \right\rangle_{\mathcal{H}} \leq \|\langle x, x \rangle_{\mathcal{H}}\|. \quad (85)$$

Since $K$ is a co-isometric operator, then

$$\|\langle x, x \rangle_{\mathcal{H}}\| = \|\langle K^*x, K^*x \rangle_{\mathcal{H}}\| \leq \frac{1}{\nu} \left\langle \sum_{k=1}^{n} \alpha_k T_{k,w}x \right\rangle_{\mathcal{H}}, \quad x \in \mathcal{H}. \quad (86)$$

So

$$\frac{\nu}{\delta_p^2} \left\langle T_{k,w}x \right\rangle_{w} \leq \left\langle \sum_{k=1}^{n} \alpha_k T_{k,w}x \right\rangle_{w}, \quad x \in \mathcal{H}. \quad (87)$$

Therefore, for $\lambda = (\nu/\delta_p)$, we have

$$\lambda \left\langle T_{k,w}x \right\rangle_{w} \leq \left\langle \sum_{k=1}^{n} \alpha_k T_{k,w}x \right\rangle_{w}, \quad x \in \mathcal{H}. \quad (88)$$

\[ \text{Theorem 11. Let } K \in \text{End}_{\mathcal{H}}^d(\mathcal{H}). \text{ For } k = 1, 2, \ldots, n, \text{ let } \{T_{k,w}\}_{w \in \Omega} \subset \text{End}_{\mathcal{H}}^d(\mathcal{H}) \text{ be a continuous K-operator frame for } \]
\[
\sqrt{\Psi} \| \langle K^* x, K^* x \rangle_{\mathcal{A}} \|_{1/2} \\
\leq \| \langle T_{p,\omega} x \rangle_{\mathcal{A}} \| = \| S \left\{ \sum_{k=1}^{n} R_{k,\omega} x \right\}_{\mathcal{A}} \| \\
\leq \| S \| \| \sum_{k=1}^{n} R_{k,\omega} x \|_{\mathcal{A}}, \quad x \in \mathcal{H}.
\] 

\text{(92)}

Thus

\[
\sqrt{\Psi} \| \langle K^* x, K^* x \rangle_{\mathcal{A}} \|_{1/2} \leq \left\| \sum_{k=1}^{n} R_{k,\omega} x \right\|_{\mathcal{A}}^2 \\
\leq (1 + \sqrt{\lambda}) \left( \sum_{k=1}^{n} \delta_k \right) \| \langle x, x \rangle_{\mathcal{A}} \|_{1/2}, \quad x \in \mathcal{H}.
\] 

\text{(94)}

This gives that \( \{ \sum_{k=1}^{n} R_{k,\omega} \}_{\mathcal{A}} \) is a continuous K-operator frame for \( \text{End}^*_{\mathcal{A}} (\mathcal{H}) \).

\[ \square \]

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

On behalf of all authors, the corresponding author states that there are no conflicts of interest.

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