BOUNDDEDNESS OF THE NUMBER OF NODAL DOMAINS FOR EIGENFUNCTIONS OF GENERIC KALUZA-KLEIN 3-FOLDS

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ABSTRACT. This article concerns the number of nodal domains of eigenfunctions of the Laplacian on special Riemannian 3-manifolds, namely nontrivial principal $S^1$ bundles $P \to X$ over Riemann surfaces equipped with certain $S^1$ invariant metrics, the Kaluza-Klein metrics. We prove for generic Kaluza-Klein metrics that any Laplacian eigenfunction has exactly two nodal domains unless it is invariant under the $S^1$ action.

We also construct an explicit orthonormal eigenbasis on the flat 3-torus $T^3$ for which every non-constant eigenfunction belonging to the basis has two nodal domains.

1. Introduction

This article is concerned with the number of nodal domains of eigenfunctions of the Laplacian on certain 3-dimensional compact smooth Riemannian manifolds $(P,G)$. The manifolds are $S^1 = SO(2)$ bundles $\pi : P \to X$ over a Riemannian surface $(X,g)$, and $G$ is assumed to be a Kaluza-Klein metric adapted to $\pi$, i.e., $G$ is invariant under the free $S^1$ action on $P$ and there exists a splitting $TP = H(P) \oplus V(P)$ of $TP$ so that $d\pi : H_P(P) \to T_{\pi(p)}X$ is isometric and so that the fibers are geodesics. Thus, $\pi : P \to X$ is a special kind of Riemannian submersion with totally geodesic fibers in the sense of [BBB82] (see Definition 1.6). The $S^1$ action commutes with the Laplacian $\Delta_G$ of the Kaluza-Klein metric $G$ and one may separate variables to obtain an orthonormal basis of joint eigenfunctions $\phi_{m,j}$,

$\Delta_G \phi_{m,j} = -\lambda_{m,j} \phi_{m,j}, \quad \frac{\partial}{\partial \theta} \phi_{m,j} = im \phi_{m,j}.$

In Proposition 4.7 we show that for generic choices of $g$ on $X$, these joint eigenfunctions are the only eigenfunctions of $\Delta_G$ up to complex conjugation, i.e. the eigenvalues of $\Delta_G$ are simple.

Our focus is on the nodal sets of the real or imaginary parts

$\phi_{m,j} = u_{m,j} + iv_{m,j}$

and particularly on the number of their nodal domains. Since $\Delta_G$ is a real operator, the real and imaginary parts (1.1) satisfied the modified eigenvalue system,

\[
\begin{cases}
\Delta_G u_{m,j} = -\lambda_{m,j} u_{m,j}, \\
\Delta_G v_{m,j} = -\lambda_{m,j} v_{m,j}, \\
\frac{\partial}{\partial \theta} u_{j} = m v_{j}, \quad \frac{\partial}{\partial \theta} v_{j} = -m u_{j}.
\end{cases}
\]

Our main result is that when 0 is a regular value of $\phi_{m,j}$ for all $(m,j)$, then the number of nodal domains of $u_{m,j}$ and $v_{m,j}$ is bounded above unless $m = 0$, in which case the number is the same as for the corresponding eigenfunctions on $X$. We also prove that it is a generic property of Kaluza-Klein metrics on $S^1$ bundles over Riemann surfaces that 0 is indeed a regular value of $\phi_{m,j}$ for all $(m,j)$. The precise statement requires a discussion of the geometric data underlying a Kaluza-Klein metric and how we allow it to vary when defining ‘genericity’ and is given in Section 4 (see Theorem 4.1 and Lemma 4.10).

Theorem 1.1. Let $(X,g)$ be a Riemannian surface, let $P \to X$ be a non-trivial principal $S^1$ bundle and let $\alpha$ be a connection on $P$. Let $G$ be the Kaluza-Klein metric induced by $(g,\alpha)$. Then, for generic metrics $g$ on $X$, we have:

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• The multiplicity of each eigenvalue $\lambda = \lambda_{m,j}$ of $\Delta_G$ is 1 if $m = 0$, and 2 if $m \neq 0$.
• Every eigenfunction is a joint eigenfunction of $\Delta_G$ and $\frac{\partial}{\partial \theta}$. The eigenspace of $\Delta_G$ corresponding to $\lambda = \lambda_{m,j} = -\lambda_{m,j}$ is spanned by $\phi_{m,j}$ and $\phi_{-m,j} = \overline{\phi_{m,j}}$. In particular, any real eigenfunction with the eigenvalue $\lambda_{m,j}$ is a constant multiple of $T_0(\Phi_{m,j})$, where $T_0$ is the $S^1$ action on $P$ parameterized by $\theta$.
• Invariant eigenfunctions ($m = 0$) of $\Delta_G$ are lifts $\pi^*\psi_j$ of eigenfunctions $\psi_j$ of $\Delta_g$ on the base $X$, and the nodal set of $\pi^*\psi_j$ is the inverse image under $\pi$ of the nodal set of $\psi_j$. The number of nodal domains of $\pi^*\psi_j$ equals the number of nodal domains of $\psi_j$.
• For $m \neq 0$, the nodal sets of $\mathbb{R}\phi_{m,j}$ are connected.
• For $m \neq 0$, the number of nodal domains of $\mathbb{R}\phi_{m,j}$ is 2.

**Remark 1.2.** Instead of varying $g$ we may fix $g$ and vary the connection 1-form on $P$ and obtain the same result.

When $P \to X$ is trivial, and $P \cong S^1 \times X$ is endowed with the product metric, we have $\phi_{m,j} = \psi_je^{im\theta}$ where $\psi_j$ is an eigenfunction of $\Delta_g$ on the base $X$. Hence $\mathbb{R}\phi_{m,j} = \psi_j \cos m\theta$ has many nodal domains, and the last statement in Theorem 1.1 fails.

Note from Weyl law that $\{\lambda_{m,j} < \Lambda\} \sim \Lambda$ and that $\{\lambda_{m,j} < \Lambda\} \sim \Lambda^{3/2}$. Therefore as an immediate consequence of Theorem 1.1 we have the following:

**Corollary 1.3.** Let $P \to X$ be a non-trivial principal $S^1$ bundle with and let $G(g, \alpha)$ be a Kaluza-Klein metric. The for generic choices of $g$, almost all (i.e., along a subsequence of density one) eigenfunctions belonging to any orthonormal basis have exactly two nodal domains.

Theorem 1.1 furnishes the first example of Riemannian manifolds of dimension $> 2$ for which the number of nodal domains and connected components of the nodal set have been counted precisely. The results for $m \neq 0$ may seem rather surprising, since in dimension 2 the only known sequences of eigenfunctions with a bounded number of nodal domains are those constructed in an ingenious way by H. Lewy on the standard $S^2$ and those of Stern on a flat torus [Ste25, CH53] (see also [BH15]). In those cases, the separation-of-variables eigenfunctions have connected nodal sets but the complement of the nodal set has many components, i.e., nodal domains. $\phi_j = \sin 2m\pi x \sin 2m\pi y$ on a flat torus for instance has $4m^2 = \frac{\pi^2}{2} \sim \frac{\pi}{2} j$ nodal domains, where we used Weyl’s law in the last estimate. Compare with the Courant bound that the number of nodal domains of the $j$th eigenfunction (in order of increasing eigenvalue) is $j$.

In the Kaluza-Klein case, all eigenfunctions for generic Kaluza-Klein metrics are separation-of-variables eigenfunctions and have connected nodal sets. But the connectivity is of a different kind than in dimension two and we show that it induces connectivity of nodal domains. Many of the techniques of this paper extend with no major modifications to Kaluza-Klein metrics on principal $S^1$ bundles over manifolds $X$ of any dimension. For simplicity of exposition we restrict to dimension 3.

All of the $S^1$ bundles we consider are unit frame bundles $P_h$ in holomorphic Hermitian line bundles $(L, h) \to X$ (with fixed complex structures on $X$ and $L$). Given a principal $S^1$ bundle $P$ and a character $\chi_m = e^{im\theta}$ of $S^1$ we obtain associated complex line bundles $L^m \to X$, given in a standard notation by,

$$L^m = P_h \times_{\chi_m} \mathbb{C}.$$  

Equivariant functions $\hat{s}$ on $P$ transforming by $\chi_m$ under the $S^1$ action correspond to sections $s$ of $L^m$ (see Section 6.1 for background). In a local frame $e_L$ over an open set $U \subset X$, a second has the form $s = fe_L^n$ where $f$ is a locally defined complex-valued function on $U$. Our strategy in studying nodal sets on $P$ is to relate the nodal sets of sections $s$ on $X$ to their lifts $\hat{s}$ on $P$. For simplicity of exposition we often write the details for surfaces of genus $g \geq 2$ and with $L = K_X$, the canonical line bundle of $X$, whose sections are smooth differentials of type $dz$ and its powers $K_X^n$, the bundle of differentials of type $(dz)^m$. As discussed in Section 6.3 the weight decomposition of $L^2(P)$ under the $S^1$ action corresponds to studying operators $D_m$ on $C^\infty(X, L^m)$. The horizontal part of $\Delta_G$ in the $m$th weight space is equivalent to the Bochner Laplacian $\nabla^*_m \nabla_m$, Eigensections of $\nabla^*_m \nabla_m$ of $L^m$ correspond to the joint eigenfunctions $\phi_{m,j}$.

It is also interesting to study the round metric on $S^3$ and the flat metric on the 3-torus $\mathbb{T}^3$, and we do so in Section 8. As may be expected, the results in special geometries is quite different from the results in generic settings. Nevertheless, we prove the following by constructing an explicit example.

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1 We trust that the notation $g$ for the genus and $g$ for the metric on $X$ will cause no confusion.
**Theorem 1.4.** On the flat 3 torus $\mathbb{T}^3$, one can find an orthonormal eigenbasis for which all nonconstant eigenfunctions have two nodal domains.

To put the result into context, it is proved in varying degrees of generality in [GRS13, JZ16a, JZ16b, Zel16, GRS17, JJ18, JY17, Mag15] that in dimension 2, the number of nodal domains of an orthonormal basis $\{u_j\}$ of Laplace eigenfunctions on certain surfaces with ergodic geodesic flow tends to infinity with the eigenvalue along almost the entire sequence of eigenvalues. By the first item of Theorem 1.1, the same is true for their lifts to the unit tangent bundle $SX$ as invariant eigenfunctions of the Kaluza-Klein metric. But for higher weight eigenfunctions, the situation is virtually the opposite and the number of nodal domains is bounded.

**Remark 1.5.** Note that the geodesic flow on $P$ with a Kaluza–Klein metric never is ergodic since the metric norm $\|\xi\|_G$ generating the geodesic flow commutes with the $S^1$ action.

### 1.1. Adapted Kaluza-Klein metrics

We now describe Kaluza-Klein metrics more precisely so that we can explain what we mean by a generic Kaluza-Klein metric. A more detailed description of such metrics is given in Section 6.

The principal bundles $P$ and Kaluza-Klein metrics are defined from the following geometric data.

1. A Riemannian metric $g$ and a complex structure $J$ on $X$;
2. A nontrivial complex holomorphic line bundle $L \to X$, whose complex structure we denote by $J_L$;
3. A Hermitian metric $h$ on $L$;
4. An $h$-compatible connection $\nabla$ on $L$.

When discussing generic Kaluza-Klein metrics, $J$ and $J_L$ are fixed, while data $(g, h, \nabla)$ may vary. In fact, we mainly consider the case where $h$ is also fixed and only $(g, \nabla)$ vary. The unitary frame bundle for the Hermitian metric $h$ is defined by

$$P_h = \{(z, \lambda) \in L^* : h^*_h(\lambda) = 1\}.$$  

The connection $\nabla$ induces a connection 1-form $\alpha$ on $P_h$ and a splitting $TP_h = H(P_h) \oplus V(P_h)$ into horizontal and vertical spaces; see Section 2 for background.

**Definition 1.6.** The Kaluza-Klein metric on $P_h$ is the $U(1)$-invariant metric $G$ such that the horizontal space $H_p := \ker \alpha$ is isometric to $(T_{\pi(p)}X, g)$, so that $V = \mathbb{R} \frac{\partial}{\partial \lambda}$ is orthogonal to the horizontal sub-bundle $H$, is invariant under the natural $S^1$ action, and so that the fibers are unit speed geodesics.

As mentioned above, the data $(g, h, \nabla)$ induces Bochner Laplacians $\nabla^m_m \nabla_m$ on sections of $L^m$. The data also defines a horizontal Laplacian $\Delta_H$ and a vertical Laplacian $\frac{\partial^2}{\partial \lambda^2}$ on $P_h$, and their sum is the Kaluza-Klein Laplacian $\Delta_G$. Equivariant eigenfunctions of $\Delta_G$ of weight $m$ on $M$ are lifts of eigensections of $\nabla^*_m \nabla_m$. See Lemma 6.2 for details. In proving genericity theorems it is easier to work downstairs on $X$. But the nodal results pertain to the equivariant eigenfunctions on $P_h$.

**Remark 1.7.** The key property we need is that the horizontal and vertical Laplacians commute. Not all $S^1$ invariant metrics on $P$ have this property; the simplest counter-example is a surface of revolution.

The most familiar case is that where the metric $g$ on $X$ is hyperbolic, i.e., of constant curvature $-1$ and $K = T^{1,0}$ is the canonical bundle. Then the total space $P_h = PSL(2, \mathbb{R})/\Gamma$ and the equivariant eigenfunctions of the Kaluza-Klein Laplacian $\Delta$ are the same as joint eigenfunctions of the generator $W$ of $K$ and of the Casimir operator $\Omega$. When the weight $m$ is fixed, one may separate variables and obtain a Maass Laplacian $D_m$ on smooth sections of a complex line bundle $\pi : K^m \to X$, namely the bundle of $m$-differentials of type $(dz)^m$. The eigensections are the usual weight $m$ automorphic Maass eigendifferentials $f_{m,j}(z)(dz)^m$,

$$D_m f_{m,j}(z) = s(1-s)f_{m,j}(z),$$

of the Maass Laplacians

$$D_m = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2i ny \frac{\partial}{\partial x}.$$
1.2. Generic Kaluza-Klein metrics. Theorem 1.1 is valid as long as the eigenvalues of $\Delta_G$ are simple (multiplicity 1, modulo complex conjugation) and the eigensections have 0 as a regular value. In Section 3 we prove that these properties hold for Kaluza-Klein metrics constructed from generic base metrics with fixed $(h, \nabla)$ and also for generic $h$-compatible connections $\nabla \in A_h$ with $g$ fixed. Thus, we get a plentiful supply of metrics for which Theorem 1.1 is valid.

The study of generic properties of eigenvalues and eigensections is based on the work of Uhlenbeck [Uhl76] in the case of scalar Laplacians on Riemannian manifolds. To our knowledge, this work has not been generalized to Bochner Laplacians on twisted complex line bundles, much less for Kaluza-Klein Laplacians. The setting of twisted line bundles gives rise to somewhat different perturbation equations that in the scalar case.

We do not consider the most general types of perturbations of the data $(g, h, \nabla)$. In defining variations of eigenvalues/eigensections we fix $h$ (hence $P_h$) and vary the base metric $g$ and connection $\nabla \in A_h$ compatible with $h$ on $L$. In Section 3 we prove generic simplicity of eigenvalues and that 0 is a regular value of all eigensections for these types of perturbations. For instance, when the genus of $X$ is $\geq 2$, and $L = K_X^m$, the eigensections are eigendifferentials $f_{m,j}(dz)^m$ of the Bochner Laplacian $\nabla^*_m \nabla_m$ on $K_m^m$. For generic base metrics on $X$, and with fixed connection, 0 is a regular value of the eigendifferentials, so that their nodal sets are of dimension 1 for $m \neq 0$ and consist of a finite union of $S^1$ orbits over the zeros of $f_{m,j}(dz)^m$ on $X$.

Unfortunately, we are not able to verify that the hyperbolic metric satisfies the conditions that 0 is a regular value of all eigendifferentials, nor (of course) that the eigenvalues are simple. This would require proving at least that Maass eigendifferentials of higher weight have a finite set of zeros. When weight $m = 1$, this would imply that critical point sets of hyperbolic eigenfunctions form a finite set, and that too is unknown. This cannot be proved by a local argument since, for a hyperbolic surface of revolution with two invariant boundary components and Dirichlet or Neumann boundary conditions, the zero sets on the base are $S^1$ invariant and hence of dimension 1; their lifts have dimension two. See Section 5 for related examples in curvature 1 and 0. For this reason, we study generic metrics rather than hyperbolic metrics.

1.3. Nodal sets. We thus have two versions of the eigenfunctions of the Kaluza-Klein $\Delta_G$, first as scalar complex valued equivariant eigenfunctions on $P_h$ and second as complex eigensections on $X$. In each version we have a nodal set, and we use the base nodal set on $X$ to analyse the nodal set on $P_h$.

We denote the eigensection corresponding to $\phi_{m,j}$ as $f_{m,j}e_L^m$ in a local holomorphic frame. We mainly consider $L = K_X$ and then we write the section as $f_{m,j}(dz)^m$. Let

$$\Re f_{m,j} = a_{m,j}(z), \quad \Im f_{m,j} = b_{m,j}(z).$$

Then,

$$f_{m,j}(z)e^{-im\theta} = (a_{m,j}(z) + ib_{m,j}(z))(\cos m\theta - i \sin m\theta),$$

so that with $\phi_{m,j} = u_{m,j} + iv_{m,j}$,

$$\begin{cases}
u_{m,j} = a_{m,j} \cos m\theta + b_{m,j} \sin m\theta, \\
u_{m,j} = b_{m,j} \cos m\theta - a_{m,j} \sin m\theta.
\end{cases} \tag{1.2}$$

See Section 6.1 for more details.

We denote by $Z_{f_{m,j}}$ the zero set of the eigensection $f_{m,j}e_L^m$ on $X$:

$$Z_{f_{m,j}} = \{z \in X : f_{m,j}(z) = 0\}.$$

In general, it is not obvious whether or not the zero set of $f_{m,j}$ is discrete in $X$. Since the frame $e_L$ is non-vanishing, the zero set $Z_{\phi_{m,j}}$ of $\phi_{m,j}$ is the inverse image of $Z_{f_{m,j}}$ under the natural projection $\pi$: Let

$$\Sigma := \pi^{-1}Z_{f_{m,j}} = Z_{\phi_{m,j}} \tag{1.3}$$

be the inverse image of the base nodal points. It is a union of fibers and is a finite union of fibers if and only if $f_{m,j}$ has a finite number of zeros. We refer to $\Sigma$ as the ‘singular fibers’ or singular set.

Our main focus is on the nodal sets of the real and imaginary parts of the lift, not to be confused with the lifts of the real and imaginary parts of the local expression $f_{m,j}$ of the section (since the frame $e_L^m$ must also be taken into account).

We denote the nodal sets of the real, resp. imaginary, parts of the lift by

$$N_{\Re \phi_{m,j}} = \{p \in P_h : \Re \phi_{m,j}(p) = 0\}, \text{ resp. } N_{\Im \phi_{m,j}} = \{p \in P_h : \Im \phi_{m,j}(p) = 0\}.$$
The analysis is the same for real and imaginary parts and we generally work with the imaginary part, following the tradition for quadratic differentials.

A special case is \( m = 0 \), in which case we have the obvious (but interesting)

**Proposition 1.8.** For \( m = 0 \), the real invariant eigenfunctions of the Kaluza-Klein Laplacian are just pullbacks of the eigenfunctions of the base Laplacian, and their nodal sets are inverse images of nodal sets on the base. Hence the number of nodal domains of ‘invariant’ Kaluza-Klein eigenfunctions is the number for the corresponding eigenfunction on the base.

Henceforth we always assume \( m \neq 0 \). The nodal set of the real and imaginary parts of the lift \( \phi_{m,j} \) of \( f_{m,j} e^{\psi} \) is very different over nodal versus non-nodal points of \( f_{m,j} \).

We denote by \( X \setminus Z_{f_{m,j}} \) the punctured Riemann surface in which the zero set of \( f_{m,j} \) is deleted. A key statement in the nodal analysis is the following:

**Proposition 1.9.** For \( m \neq 0 \), the maps

\[
\pi : N_{\phi_{m,j}} \to X \setminus Z_{f_{m,j}}, \quad N_{\phi_{m,j}} \to X \setminus Z_{f_{m,j}}
\]

is an \( m \)-fold covering space.

It follows that the topology of the nodal set is entirely determined by the combinatorics of gluing the sheets along the singular fibers. In fact, the gluing is rather simple and easily yields the following

**Theorem 1.10.** For all \( m \neq 0 \), the nodal set \( N_{\phi_{m,j}} \) is connected.

To count nodal domains, we need to make the assumption that there are just a finite number of zeros of \( f_{m,j} \) and that at least one of them is regular. To this end, we prove:

**Theorem 1.11.** For generic metrics \( g \) on \( X \), all of the eigenfunctions \( f_{m,j} \) (for all \( (m,j) \)) have isolated zeros of multiplicity 1, i.e., zero is a regular value. Hence, \( Z_{f_{m,j}} \) is a finite set of points.

When the zero set is transverse to the zero section, then the sum of the indices of the zeros is the first Chern class of \( K^m \), and in particular is non-empty when the genus of \( X \) is \( 0 \), i.e., when \( X \) is not a torus.

For metrics satisfying Theorem 1.11 we prove Theorem 1.1 by using Proposition 1.9 together with some geometric/combinatorial arguments in Section 7.

### 2. Geometric background

In this section we discuss the geometric data that goes into the construction of Kaluza-Klein metrics, which are defined in Definition 1.6. They are also the data needed to define Bochner Laplacians \( \nabla^* \nabla \) and Kaluza-Klein Laplacians \( \Delta_G \).

#### 2.1. Riemannian metrics on \( X \) and Hermitian metrics on \( L \)

Let \((X,J,g)\) denote a Riemann surface with complex structure \( J \) and Riemannian metric \( g \). We write \( g_{1\bar{1}} = g(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) \) and \( g^{1\bar{1}} = g^*(dz, d\bar{z}) \), where \( g^* \) is the dual metric. The complex structure gives a decomposition of \( T^*X \otimes \mathbb{C} = T^{*(1,0)} \oplus T^{*(0,1)} \) into \((1,0)\) resp. \((0,1)\) parts. We denote the area form of \( g \) by

\[
dA_g = \omega = ig_{1\bar{1}} dz \wedge d\bar{z},
\]

where the Kähler form \( \omega \) is the \((1,1)\) form defined by \( g_J(X,Y) = \omega(JX,Y) \).

Choice of a complex structure \( J \) on \( X \) is equivalent to choice of a conformal class \( \text{Conf}(g_0) \) of metrics. In each conformal class, we may pick a background metric \( g_0 \) and represent other metrics in the form

\[
\text{Conf}(g_0) = \{ e^{2\sigma} g_0 : \sigma \in C^\infty(X) \}.
\]

The Riemannian metrics in \( \text{Conf}(g_0) \) are Kähler metrics, and may also be parameterized by their local Kähler potentials \( \psi \), with \( d\bar{\partial}\psi = \omega \). Here \( \bar{\partial} = \frac{1}{2}(\partial - i\bar{\partial}) \). Then, \( \omega_h = i\partial\bar{\partial}\log h = dd^c\psi = \Delta_g \psi L(dz) \), where \( L(dz) = idz \wedge d\bar{z} \). Relative to the background form \( \omega_0 = dA_0 \), any Kähler metric has a global relative potential \( \phi \), so that the area form \( \omega_\phi \) of the Kähler metric is related to that of the reference metric by

\[
\omega_\phi = \omega_0 + i\partial\bar{\partial}\phi.
\]

The only difference in the two parameterizations of conformal metrics is that the area of metrics in \( K_\omega \) is fixed while it may vary in \( \text{Conf}(g_0) \). Thus \( \text{Conf}(g_0) \cong K_\omega \times \mathbb{R} \).
2.2. Complex line bundles $L \to X$, connections and curvature. We have fixed a complex structure $J$ and now fix a holomorphic line bundle $L \to X$, i.e. we equip $L$ with a complex structure $J_L$. A Hermitian metric $h$ on $L$ is determined by the length of a local holomorphic frame $e_L$ (i.e., a local holomorphic nonvanishing section) of $L$ over an open set $U \subset M$ by $e^{-\psi} = \|e_L\|^2_h$, where $\|e_L\|_h = h(e_L,e_L)^{1/2}$ denotes the $h$-norm of $e_L$.

In the real setting, a connection on a vector bundle $E$ defines a covariant derivative
\[
\nabla : C^\infty(X,E) \to C^\infty(X,E \otimes T^*X).
\]
When $L$ is a holomorphic Hermitian line bundle, a connection $\nabla$, is determined in a local frame $e_L$ by a 1-form, $\nabla e_L = \alpha \otimes e_L$. We denote the $(1,0)$ resp. $(0,1)$ parts of $\nabla$ by $\nabla^{1,0}$, resp. $\nabla^{0,1}$. We consider several types of compatibility conditions between this data:

- An $h$-connection $\nabla_h$ is one compatible with $h$. In a unitary frame, the connection 1-form is $i\mathbb{R}$ valued and is denoted by $i\alpha$. We denote the space of $h$-compatible connections by $\mathcal{A}_h$. They will be fundamental in this article.

- A $J_L$-compatible connection is one for which the connection 1-form $\alpha$ in a local holomorphic frame is of type $(1,0)$. We denote the space of $J_L$-compatible connections by $\mathcal{A}_C$. Suppose that $\nabla \in \mathcal{A}_C$. Then if $s = fe$ with $e$ a local holomorphic frame,
\[
\begin{align*}
\nabla^{(1,0)}(fe) &= (\partial f + \alpha f) \otimes e, \\
\nabla^{(0,1)}(fe) &= \overline{\partial} f \otimes e.
\end{align*}
\]

- The Chern connection $\nabla$ associated to the Hermitian metric $h$ is the unique metric connection
\[
\nabla : C^\infty(X,L) \to C^\infty(M, X \otimes T^*X)
\]
whose connection 1-form in a holomorphic frame $e_L$ has type $(1,0)$. The connection 1-form is given by $\nabla e_L = \alpha \otimes e_L$ with $\alpha = \partial \log |h|$.

The same data may be described ‘upstairs’ on the principal bundle as follows:

- The Hermitian metric $h$ induces the principal bundle of $h$-unitary frames $P_h = \{(z,\lambda) \in L^* : |\lambda|_h = 1\}$.

- An $h$-compatible connection $\nabla \in \mathcal{A}_h$ induces a real 1-form $\alpha$ on $P_h$.

- Connections $\mathcal{A}_C$ determine complex-valued 1-forms on $L^*$.

The holomorphic line bundle $L$ also has a natural Cauchy-Riemann operator,
\[
\overline{\partial}_L : C^\infty(M,L) \to C^\infty_1(M,L).
\]
In a local holomorphic frame $e$, we write a smooth section $s = fe$ and then
\[
\overline{\partial}_L s = \overline{\partial} f \otimes e.
\]
It is well-defined since if $e'$ is another holomorphic frame and $e = ge'$, then $s = fge'$ and $\overline{\partial}_L s = \overline{\partial} f \otimes ge' = \overline{\partial} f \otimes e$.

The metric $g^*$ is a Hermitian metric on $K$. Any Hermitian metric $h$ on a line bundle $L$ induces metrics $h^m = e^{-m\overline{\partial}}$ on the tensor powers $L^m$ in the local frame $e^m_L$. The Hermitian metric and complex structure determine a Chern connection $\partial \log h$ whose curvature 2-form $\Theta_h$ is given locally by
\[
\Theta_h = -\partial \overline{\partial} \log \|e_L\|_h^2,
\]
and we say that $(L,h)$ is positive if the (real) 2-form $\sqrt{-1} \Theta_h$ is positive.

Given a connection $\nabla$ on $L$ and a vector field $V$ on $X$, the covariant derivative of a section $s$ is defined by $\nabla_V s = \langle \nabla s, V \rangle$. The curvature is the 2-form $\Omega$ defined by $\Omega(V,W) = [\nabla_V, \nabla_W] - \nabla_{[V,W]}$. If $e_L$ is a local frame and $\nabla e_L = \alpha \otimes e_L$ then $\Omega = d\alpha$. 
2.3. Examples.

- Let $F^*X$ be the unit co-frame bundle of $(X, g)$, consisting of orthonormal frames of $T^*X$. Then $S^m F^*X$ is the bundle of real $m$-differentials, i.e., homogeneous polynomials of degree $m$ in $dx, dy$ or $dz, d\bar{z}$.

- When $X$ is given a complex structure, we may decompose $T^*X \otimes \mathbb{C} = T^{*,(1,0)} \oplus T^{*,(0,1)}$ into covectors $fdz$ of type $(1,0)$ and $g dz$ of type $(0,1)$. The holomorphic tangent bundle is usually denoted by $K = K_X = T^{*(1,0)}$ and is called the canonical bundle. Its tensor powers $K^m$ are bundles of differentials of type $f(dz)^m$ with $f(z, \bar{z})$ a smooth function.

- When the genus is 0, i.e., $X = S^2$, $K_X$ is a negative line bundle and has no holomorphic sections. However $K_X^{-1}$ is ample. The associated circle bundle of frames is $SO(3) \cong \mathbb{R}P^3 = S^3/\pm 1$.

- When the genus is 1, then $K_X$ and $T^*X$ are trivial and $F_h \cong \mathbb{T}^2 \times S^1$. There is an ample line bundle $L \to \mathbb{T}^2$ whose holomorphic sections are theta functions. The associated principal $S^1$ bundle is the reduced Heisenberg group, the quotient of the simply connected Heisenberg group by the integer lattice.

- When the genus is $\geq 1$ we may twist $L$ by a flat line bundle. This is not particularly relevant for this article, and we usually ignore this additional degree of freedom.

- When the genus is $\geq 2$ then $X = \mathbb{H}^2/\Gamma$ where $\Gamma \subset PSL(2, \mathbb{R})$. The associated $S^1$ bundle is $SL_2(\mathbb{R})/\Gamma$. $K_X$ is ample and for $m$ large there are many holomorphic sections of $K^m_X$.

2.4. Canonical bundle and $m$-differentials. We mainly work with the canonical bundle $K_X = T^{*,(1,0)}X$ of $(1,0)$ forms $fdz$ with genus $g \geq 2$. Given a metric $g$ on $X$, there exists a Hermitian metric $h$ on $K$ with curvature form $i\partial \bar{\partial} \log h = \omega_g$. Here, $g(X, Y) = \omega_g(X, JY)$. The co-metric $\ast$ defines metric coefficients on $T^*X \otimes \mathbb{C}$ by extending $g^\ast$ by complex linearity and induces the Hermitian metric,

$$\|dz\|_{g^\ast} = g^{11},$$

on $K_X$. The curvature $(1,1)$ form is therefore $\partial \bar{\partial} \log g^{11}$. This should be distinguished from the curvature of $g$, which is given by

$$dd^c \log g^{11} = K \omega_h,$$

where $K$ is the scalar curvature.

In terms of the Hermitian metric $h = e^{-\phi_0}$ on $K$, $|dz|_g = e^{-\phi_0} = g^{11}$. Also,

$$\partial_{\bar{z}} \log \omega_0 = \partial_{\bar{z}} \log (1 - \Delta_0 \phi), \quad \omega_\phi = (\omega_0 + dd^c \phi) = ((1 - \Delta_0 \phi)\omega_0).$$

When $\dim X = 2$ we write the area form as $dA_\phi = \sqrt{g}dx$ or as $\omega_g$. The metric $g$ induces metrics $g$ on $TX, T^*X, T^{*,(1,0)}X = K_X$ and on powers such as $K^m$. On $K^m$, the Hermitian metric induced by $g$ is $\|dz\|_g^2 = e^{-\phi} = g^{11}$ so $\phi = -\log g^{11}$. The Chern connection on $K$ is the same as the Riemannian connection. More precisely, consider the complex line vector bundles $T^{1,0}_X$ and $(TX, J)$. They are isomorphic under the map

$$\xi : T_X \to T^{1,0}_X, \quad v \to \frac{1}{2}(v - iJv).$$

Then, under the isomorphism $\xi \in T_X \to T^{1,0}_X$, the Chern connection $D$ on the holomorphic tangent bundle $T^{1,0}$ is the Levi-Civita connection $\nabla$.

2.5. Hilbert spaces of sections. Let $(L, h) \to X$ be a Hermitian holomorphic line bundle. We thus have a pair of metrics, $h$ resp. $g$ (with Kähler form $\omega_\phi$) on $L$ resp. $TX$.

To each pair $(h, g)$ of metrics we associate Hilbert space inner products $\text{Hilb}_m(h, g)$ on sections $s \in L^2_{m,h}(X, L^m)$ of the form

$$\|s\|^2_{h,m} := \int_X |s(z)|^2_{h,m} \omega_g,$$

where $|s(z)|^2_{h,m}$ is the pointwise Hermitian norm-squared of the section $s$ in the metric $h^m$. In a local holomorphic frame $e_L$, we write

$$\|e_L\|^2_h = e^{-\psi}.$$
In local coordinates $z$ and the local frame $e_L^m$ of $L^m$, we may write $s = f e_L^m$ and then
\[ |s(z)|^2_{h_m} = |f(z)|^2 e^{-m \varphi(z)} \|e_L\|_{h_0}^{2m}. \]

Henceforth we write
\[ \|f e_L^m\|_{h_m}^2 := \int_X |f(z)|^2 e^{-m \varphi} \|e_L\|_{h_0}^{2m} \omega_\phi. \]

Locally we may also write $\|e_L\|_{h_0}^2 = e^{-\psi_0}.

In the special case where $L = K_X$, we may use the frame $dz$ in a local holomorphic coordinate $z$. In the local frames $(dz)^m$ of $K^m$ we may write sections as $s = f(dz)^m$ and then $|s(z)|^2_{h_m} = |f(z)|^2 e^{-m \varphi(z)} |dz|_{h_0}^{2m}$ and then,
\[ \|f(dz)^m\|_{h_m}^2 := \int_X |f(z)|^2 e^{-m(\psi_0 + \varphi)} dA_g, \]

where $dA_g = \omega_\phi$ is the area form of $g$.

3. **Bochner Laplacians on line bundles**

In this section, we give explicit local formulae for Bochner Laplacians $\nabla^*_h \nabla$ on $L^2(X, L)$ equipped with the data $(g, h, J, J_L, \nabla)$, where $(L, h) \to X$ is a Hermitian holomorphic line bundle, $g$ is a metric on $X$, $\nabla$ is a connection on $L$. In a local frame $e_L$ of $L$, with $s = f e_L$, the inner product $\text{Hilb}(g, h)$ on $L^2(X, L)$ takes the form,
\[ \|s\|^2_{\text{Hilb}(g, h)} = \int_X |f|^2 e^{-\varphi} dV_g, \quad (\text{where } \|e_L(z)\|^2_h = e^{-\psi(z)}). \]

The inner product on $L^2(X, L \otimes T^*X)$ has the form,
\[ \|s \otimes \eta\|^2_{\text{Hilb}(g, h)} = \int_X |f|^2 \|\eta\|^2 e^{-\varphi} dV_g, \quad (\text{where } \|e_L(z)\|^2_h = e^{-\psi(z)}). \]

With no loss of generality, we fix $J$ on $X$ and assume that $(g, J, \omega)$ is a Kähler metric with $g(X, Y) = \omega(X, JY)$. Then $g(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = 0 = g(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z})$. There is only one metric coefficient, $g^{11} = G(dz, d\bar{z})$. It is a Hermitian metric on $T^{1,0}X$ and is compatible with $J$. We also denote the Riemannian volume form by $dV_g = \omega = dd^c \log g^{11}$.

**Remark 3.1.** Notational remark: We use $G$ rather than $g^{-1}$ or $g^*$ for the dual co-metric on 1-forms, because it is a convenient notation for later variations.

The Bochner Laplacian is the Laplacian on $L^2(X, L)$ determined by the quadratic form,
\[ q_{g, h, \nabla}(s) = \int_X |\nabla s|^2_{\otimes g} dV_g = \langle \nabla^*_h \nabla s, s \rangle_{\text{Hilb}(g, h)}. \tag{3.1} \]

Throughout we assume that $g$ is $J$-compatible. In a local frame $e_L$ of $L$, with $s = f e_L$, $\nabla(f e_L) = (df + f \alpha) \otimes e_L$ and with $\|e_L(z)\|^2_h = e^{-\psi(z)}$, and the quadratic form is given by
\[ q_{g, h, \nabla}(f e_L) = \int_X |df + f \alpha|^2 g e^{-\psi} dV_g. \]

The adjoints are taken with respect to the volume form $e^{-\psi} dV_g$.

We give local formulae for $\nabla^*_h \nabla$ under several assumptions on $\nabla$ and in correspondingly adapted frames (equivalently, choosing a gauge for $\nabla$):

(i) $\nabla$ is $h$-compatible (see Section 3.1); in this case, we compute in a local unitary frame. Fixing $h$ is equivalent to fixing the principal $S^1$ bundle $P_h \to X$, and varying the connection 1-forms $\alpha \in A_h$ on $P_h$ (with fixed $g$).

(ii) $\nabla \in A_C$ is compatible with a fixed complex structure $J_L$ on $L$ (see Section 3.2), in this case we compute in a local holomorphic frame. In the next section we fix $J_L$ and vary $h$ (with fixed $g$).

---

\(^3\)Although $\dim M = 2$, we use the notation $dV_g$ and the term ‘volume form’ to avoid clashing with the notation $A$ for connections and area form.
Thus, we get

\[ h \text{ is the Hermitian norm-squared, so} \]

Let Proposition 3.2. In a unitary frame, the connection 1-form is

\[ \nabla \] and as before \( \{ \nabla \} \) is the set \( \nabla \) of connections on \( L \) which are compatible with the Hermitian metric is the affine space \( \{ \alpha = A_0 + \alpha : \alpha \in \Omega^1(X) \} \) where \( A_0 \) is a fixed background connection and \( \Omega^1(X) \) are the real 1-forms on \( X \). The Hermitian metric determines the principal \( U(1) \) bundle \( P_h \) of unitary frames of \( L \) and as before \( A_1 \) determines a connection 1-form \( \alpha_1 \) on \( P_h \). On the base \( X \), the connection 1-form is \( \imath \alpha \)-valued in a unitary frame and we write it as \( \imath \alpha \) with a real-valued \( \alpha \).

### 3.1. Calculation in a unitary frame.

In this section we assume that \( \nabla \) is compatible with \( h \). We recall from Section 3.1 that on a Hermitian line bundle \( (L,h) \), the set \( A_h \) of connections on \( L \) which are compatible with the Hermitian metric is the affine space \( \{ \alpha = A_0 + \alpha : \alpha \in \Omega^1(X) \} \) where \( A_0 \) is a fixed background connection and \( \Omega^1(X) \) are the real 1-forms on \( X \). The Hermitian metric determines the principal \( U(1) \) bundle \( P_h \) of unitary frames of \( L \) and as before \( A_1 \) determines a connection 1-form \( \alpha_1 \) on \( P_h \). On the base \( X \), the connection 1-form is \( \imath \alpha \)-valued in a unitary frame and we write it as \( \imath \alpha \) with a real-valued \( \alpha \).

#### Proposition 3.2.

Let \( (X,g) \) be a Riemannian manifold and let \( (L,h) \) be a Hermitian line bundle with \( h \)-compatible connection \( \nabla \). Let \( \nabla(fe_L) = (df + if\alpha) \otimes e_L \) with \( \alpha \in \mathbb{R} \) in a unitary frame \( e_L \). Then,

\[ \nabla^* \nabla(fe_L) = (-\Delta_g f - 2iG(df,\alpha) + ifd^*_\alpha + G(\alpha,\alpha)f) e_L. \]

where \( \Delta_g \) is the scalar Laplace operator.

**Proof.** In a unitary frame, \( |e_L|^2_h = e^{-\psi} = 1 \) and this factor drops out. We leave it in until the last step for purposes of later comparison to other frames. Since \( \nabla(fe_L) = df \otimes e_L + if\alpha e_L \), and by (3.1),

\[ q_{g,h,\nabla}(s) = \int_X |df + if\alpha|^2_h e^{-\psi} dV_g. \]

Note that \( |df + if\alpha|^2_h = G(df + if\alpha,df + if\alpha) = |df|^2 + 2\Re \bar{f}G(df,-if\alpha) + G(\alpha,\alpha)|f|^2 \) is the Hermitian norm-squared, so

\[ q_{g,h,\nabla}(s) = \int_X (|df|^2 + 2\Re \bar{f}G(df,-if\alpha) + G(\alpha,\alpha)|f|^2) e^{-\psi} dV_g \]

\[ = \int_X (-2\Re \bar{f}G(df,i\alpha) + G(\alpha,\alpha)|f|^2) e^{-\psi} dV_g - \int_X \bar{f}d^*_\alpha(e^{-\psi} df) dV_g \]

\[ = \int_X (-2\Re \bar{f}G(df,i\alpha) + G(\alpha,\alpha)|f|^2) dV_g - \int_X \bar{f}(d^*_\alpha df) dV_g, \]

where in the last line we use that \( \psi = 0 \) in a unitary frame. Since \( \alpha \) is real-valued,

\[-2\Re \bar{f}G(df,i\alpha) = -i(\bar{f}G(df,\alpha) - fG(df,\alpha)).\]

Recall that \( d^*_\alpha(f\alpha) = -*d(*f\alpha) = -G(df,\alpha) + f\bar{d}^*_\alpha \). Replacing \( iG(\bar{f},\alpha) \) by \( -i(d^*_\alpha(f\alpha) - \bar{d}^*_\alpha) \) and integrating the \( d^*_\alpha \) by parts gives

\[ \int_X (-i\bar{f}G(df,\alpha) - \bar{f}G(df,\alpha) + if|f|^2d^*_\alpha + G(\alpha,\alpha)|f|^2) dV_g - \int_X \bar{f}(d^*_\alpha df) dV_g, \]

Thus, we get

\[ \nabla^* \nabla f = -\Delta_g f - 2iG(df,\alpha) + ifd^*_\alpha + G(\alpha,\alpha)f. \]
3.2. Holomorphic line bundles: \( J_L \)-compatible connections. In this section we give a local formula for the Bochner Laplacian when \( L \to X \) is a holomorphic line bundle and \( \nabla \) is compatible with the complex structure. Thus, complex structures \( J \) on \( X \) and \( J_L \) on \( L \) are fixed. In a holomorphic frame, \( \|e_L\| = e^{-\psi} \neq 1 \) and \( \nabla e_L = \alpha \otimes e_L \), where \( \alpha \) is of type \((1,0)\). We write \( \nabla = \partial \nabla + \overline{\nabla} \) for the decomposition of a connection into its \((1,0)\) resp. \((0,1)\) parts, with \( \partial \nabla = \nabla^{(1,0)}, \overline{\nabla} = \nabla^{(0,1)} \).

The Bochner-Kodaira identity relates \( \nabla^* \nabla \) to \( \partial J_L \), where
\[
\begin{align*}
\partial L(f e_L) & := \nabla^{(1,0)}(f e_L) = (\partial f + \alpha f) \otimes e_L, \\
\overline{\partial} L(f e_L) & := \nabla^{(0,1)}(f e_L) = \overline{\partial} f \otimes e_L.
\end{align*}
\]

The analogue of Proposition 3.2 is

**Proposition 3.3.** If \( \nabla \) is compatible with \( J_L \) with connection 1-form \( \nabla e_L = \alpha \otimes e_L \) with \( \alpha \) of type \((1,0)\) in the holomorphic frame \( e_L \), then
\[
\nabla^* \nabla (f e_L) = (-\Delta_g f + G(d\psi + \alpha, df) - f G(d\psi, \alpha) + f d^* \alpha + G(\alpha, \overline{\alpha}) f) e_L.
\]

**Proof.** The proof is similar to that of Proposition 3.2, with two differences: (i) We use a holomorphic frame rather than a unitary frame and \( |e_L|^2 = e^{-\psi} \) is not equal to 1; (ii) \( \alpha \) is of type \((1,0)\) rather than being \( i\mathbb{R} \)-valued. Note that
\[
|df + f \alpha|^2 = G(df + f \alpha, df + f \alpha) + 2Rf G(df, \alpha) + G(\alpha, \overline{\alpha}) |f|^2
\]

By \((3.1)\), and integrating by parts the \( |df|^2 \) term, and with \( |\cdot|^2 \) denoting the Hermitian metric, we get
\[
\begin{align*}
g_{g,\nabla}(s) & = \int_X |df + f \alpha|^2 e^{-\psi} dV_g \\
& = \int_X (|df|^2 + 2Rf G(df, \alpha) + G(\alpha, \overline{\alpha}) |f|^2) e^{-\psi} dV_g \\
& = \int_X (2Rf G(df, \alpha) + G(\alpha, \overline{\alpha}) |f|^2) e^{-\psi} dV_g - \int_X f (\overline{d^* df}) dV_g \\
& = \int_X (2Rf G(df, \alpha) + G(\alpha, \overline{\alpha}) |f|^2) e^{-\psi} dV_g - \int_X f (\overline{d^* df} - G(df, df)) e^{-\psi} dV_g
\end{align*}
\]

Further,
\[
2Rf G(df, \alpha) = f G(df, \alpha) + f G(df, \alpha).
\]

We simplify the \( f G(df, \alpha) \) term using that \( d^* (f \alpha) = -* d^* (f \alpha) = -G(df, \alpha) + \overline{f} d^* \alpha \) so that \( G(df, \alpha) = -d^* (f \alpha) + \overline{f} d^* \alpha \). Integrating the \( d^* \) by parts gives
\[
\begin{align*}
& = \int_X (2Rf G(df, \alpha) + G(\alpha, \overline{\alpha}) |f|^2) e^{-\psi} dV_g \\
& = \int_X (\overline{f} G(df, \alpha) + f G(df, \alpha) + G(\alpha, \overline{\alpha}) |f|^2) e^{-\psi} dV_g \\
& = \int_X (\overline{f} G(df, \alpha) - f d^* (\overline{f} \alpha) + |f|^2 d^* \alpha + G(\alpha, \overline{\alpha}) |f|^2) e^{-\psi} dV_g
\end{align*}
\]

Combining with the term \(- \int_X f (\overline{d^* df} - G(df, df)) e^{-\psi} dV_g\), we get
\[
\nabla^* \nabla f = -\Delta_g f + G(df + \alpha - \alpha, df) - f G(d\psi, \alpha) + f d^* \alpha + G(\alpha, \overline{\alpha}) f
\]

\( \square \)

3.3. Chern connection. In this section we assume \( \nabla \) is both \( h \)-compatible and \( J_L \)-compatible, i.e., that it is the Chern connection with connection 1-form \( \partial \psi \). One can then compute \( \nabla^* \nabla \) using the relation
\[
\nabla^* \nabla f = -\Delta_g f + G(d\psi + \alpha - \alpha, df) - f G(d\psi, \alpha) + f d^* \alpha + G(\alpha, \overline{\alpha}) f
\]

between the Kodaira and Bochner Laplacians.

Note that \( d\psi \) is real and \( \partial \psi = \alpha \), so \( d\psi = \alpha + \overline{\alpha} \) and \( G(d\psi + \alpha - \alpha, df) = G(\alpha, \overline{\alpha}) f \) above. Also, \( G(d\psi, \alpha) = G(\alpha, \alpha) \), so the terms \(- f G(d\psi, \alpha) + G(\alpha, \overline{\alpha}) f \) cancel and from the preceding Proposition we get
\[
\nabla^* \nabla (f e_L) = -\Delta_g f + G(\alpha, \overline{\alpha} f) + f d^* \alpha.
\]
We now prove this directly.

**Proposition 3.4.** Let $\nabla$ be the Chern connection for $(L, h)$. Then,
\[
\nabla^* \nabla (f e_L) = (-\Delta_g f + G(\partial \psi, \bar{\partial} f) + f(i * \Omega^\nabla)) e_L.
\]

**Proof.** Using (3.2) and $\bar{\partial}^\nabla (f e_L) = (\bar{\partial} f \otimes e_L)$, we have
\[
(\bar{\partial}_L^* \bar{\partial}_L s)_h := (\bar{\partial}_L s, s)_h.
\]

We rewrite $G(\bar{\partial} f, \partial f)$ term using that $d^* (\bar{\partial} f) = -d^* (\partial f) = -G(d \bar{\partial} \psi, \partial f) + \bar{\partial} f \partial f$ so that $G(\bar{\partial} f, \partial f) = -d^* (\bar{\partial} f) + \bar{\partial} f \partial f$. Integrating the $d^*$ by parts gives
\[
\int_X G(\bar{\partial} f, \partial f) e^{-\psi} \omega_g = \int_X (-d^* (\bar{\partial} f) + \bar{\partial} f \partial f) e^{-\psi} \omega_g
= \int_X G(\partial f, \bar{\partial} \psi) - \Delta f) \bar{\partial} f e^{-\psi} \omega_g.
\]

Adding the curvature term adds $f(i * \Omega^\nabla)$. \qed

**Remark 3.5.** Proposition 3.4 and 3.2 are consistent by the following calculation: If $\alpha = \partial \psi$ is a Chern connection 1-form, then
\[
\langle d^*_g \alpha, \omega_g \rangle = \Omega^\nabla = i \partial \bar{\partial} \psi = (\Delta_g \psi) \omega_g, \quad d^*_g \alpha = \Delta_g \psi.
\]

Indeed, in terms of the Hermitian inner product,
\[
\langle d^*_g \alpha, f \rangle_{L^2} = \langle \alpha, df \rangle_{L^2} = \langle \alpha, \partial f \rangle_{L^2} = \int_X \partial \alpha \cdot \partial \bar{\partial} f \omega_g
= i \int_X \partial \psi \partial \bar{\partial} f dz \bar{\partial} f dz \bar{\partial} f dz \bar{\partial} f dz
= \int_X \partial^2 \psi \partial \bar{\partial} f dz \bar{\partial} f dz \bar{\partial} f dz \bar{\partial} f dz
= -\langle \Delta \psi, f \rangle_{L^2}.
\]

3.4. **Canonical bundle: $L = K$ and $\nabla$ is the Riemannian connection.** Let $z$ be a local holomorphic coordinate and let $dz$ be the associated section of $K$. Differentials of type $(dz)^m$ are sections of $K^m$, the $m$-th power of the canonical bundle. The Riemannian metric on $X$ induces a Hermitian metric $h^m$ on $K^m$, namely $|dz|^h = |dz|^g$ where $g$ is the co-metric. $dz = dx + idy$ and at $x + iy$, $|dz| = y$.

The metric $g$ on $TX$ endows a Hermitian metric $g^*$ on $K$ and the associated Riemannian connection $\nabla_g$ is the Chern connection with connection 1-form $\alpha = -\partial \log g^{11}$ in the frame $dz$. For simplicity of notation we write $\phi = -\log g^{11}$. It induces connections and Hermitian metrics on $K^m$ with connection 1-forms $m \alpha$ for $m = 0, 1, 2, \ldots$.

The associated Bochner Laplacian $\nabla_{m, g}^\nabla \nabla_{m, g}$ on $K^m$ corresponds to the quadratic form
\[
q_{m, g}(s) = \int_X |\nabla_{m, g} s|^2_{m, g} \omega_g = \int_X |d f + m \partial \phi|^2 (dz)^m \|g_m \omega_g \|
= \int_X |d f + m \partial \phi|^2 e^{-m \phi} \omega_g.
\]

Note that $\omega_g = \frac{i}{2} g_{11} dz \wedge d \bar{\partial} f$ and the Laplacian on scalar functions is given by $\Delta_0 f = g^{11} \partial^2 f / \partial z \partial \bar{\partial} f$.

**Proposition 3.6.** Let $\nabla$ be the Chern connection for $(K^m, g^m)$. Then,
\[
\nabla^* m \nabla m (f (dz)^m) = (-\Delta_g f + m G(\partial \phi, \bar{\partial} f) + m K f) (dz)^m.
\]

**Proof.** This follows from Proposition 3.4. We give a direct proof. By the Bochner-Kodaira formula (3.2), it suffices to prove
\[
\bar{\partial}_m (f (dz)^m) = \left( g^{11} \partial^2 f / \partial z \partial \bar{\partial} f - m \left( \partial \phi / \partial z \right) g^{11} \right) (dz)^m,
\]

where $\phi(z) = -\log |dz|^g = -\log g^{11}$.

As above, we calculate the adjoint to be
\[
\bar{\partial}_m^* (f (dz)^m \otimes d \bar{\partial} f) = \left( e^{m \phi} g_{11} \partial \phi / \partial z \right) (f (dz)^m)
= g^{11} \partial f / \partial z - m f(z) g^{11} \partial \phi / \partial z.
\]
It follows that
\[
\bar{\partial}_m \partial_m (f(dz)^m) = \bar{\partial}_m^* \frac{\partial f}{\partial \bar{z}} (dz)^m \otimes (d\bar{z}) \\
= \left( e^{m\phi \omega_1} \frac{\partial f}{\partial \bar{z}} g_{11}^{1\bar{1}} e^{-m\phi \omega_0} \right) \\
= g_{11}^{1\bar{1}} \frac{\partial^2 f}{\partial z \partial \bar{z}} - m \left( \frac{\partial f}{\partial \bar{z}} g_{11}^{1\bar{1}} \right) \frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial z} g_{11}^{1\bar{1}} \frac{\partial \log g_{11}^{1\bar{1}}}{\partial z} + \left( \frac{\partial f}{\partial \bar{z}} g_{11}^{1\bar{1}} \right) \frac{\partial \log \omega_0}{\partial z},
\]
where we used \( \frac{\partial \log g_{11}^{1\bar{1}}}{\partial z} + \frac{\partial \log \omega_0}{\partial z} = 0. \)

\[\square\]

4. Perturbation theory and genericity

In this section we prove generic properties of the eigenvalues and eigensections of Bochner Laplacians \( \nabla_{g,h}^* \nabla \) on complex holomorphic Hermitian line bundles \( (L,h) \to X \). Our ultimate goal is to deduce generic properties of Kaluza-Klein Laplacians on the principal \( U(1) \) frame bundles \( P_h \to X \) associated to \( h \). First we discuss generic properties of Bochner Laplacians on the line bundles and then we draw conclusions for the Kaluza-Klein Laplacians. We prove that for generic data \( (g, h, \nabla) \) (with fixed \( (J_L, J) \)), eigenvalues of Bochner Laplacians \( \nabla_{g,h}^* \nabla \) are simple (multiplicity one) and all eigensections intersect the zero section transversally (i.e., have 0 as a regular value). This immediately implies that for the associated Kaluza-Klein Laplacians \( \Delta_G \) on \( P_h \), all joint eigenfunctions of the \( U(1) \) action and \( \Delta_G \) have simple joint spectrum and have 0 as a regular value. In Section 4.6 we discuss the multiplicity of the spectrum of \( \Delta_G \), hence proving a part of Theorem 1.1.

The main result of this section is:

**Theorem 4.1.** For generic `admissible data' described below, and for every \( m \), the spectrum of each Bochner Laplacians \( \nabla_{g,h}^* \nabla \) on \( C^k(X, L^m) \) is simple and all of its eigensections have zero as a regular value. Moreover, if we lift sections to equivariant eigenfunctions \( \phi \), then \( \Re \phi \) and \( \Im \phi \) have zero as a regular value. The generic admissible data is of the following kinds:

(i) We fix \( h, g \) and vary the connection \( \nabla \) in \( A_h \). Fixing \( h \) is equivalent to fixing the principal \( U(1) \) bundle \( P_h \to X \), and varying the connection 1-forms.

(ii) We fix \( (J, J_L, g) \) and vary both \( h \) and \( \nabla \), assuming that \( \nabla \in A_C \) is compatible with \( J_L \) on \( L \) but not necessarily with \( h \).

(iii) We fix \( (g, J_L, J) \) and vary \( (h, \nabla) \) assuming that \( \nabla \) is compatible with both \( (h, J_L) \), hence is the Chern connection of \( (L, J_L, h) \).

(iv) We fix \( L = K^m \) and also fix \( J \) and vary \( g \) in the conformal class associated to \( J \). We assume that \( h \) is the Hermitian metric induced by \( g \) and that \( \nabla \) is the Levi-Civita connection.

The proofs in each of the cases are given in separate sections.

Note that the functions relevant to this article are smooth sections of a complex line bundle \( L \), and may locally be represented as complex valued functions \( u \). We will prove that \( u : M \to \mathbb{C} \) has zero as a regular value, i.e., that \( du_p = d\Re u + id\Im u \) is surjective. It follows that \( \Re u, \Im u \) are independent and nowhere vanishing on their zero sets, and that each has zero as a regular value.

**Remark 4.2.** The Uhlenbeck approach to genericity through infinite dimensional transversality theory is in most ways more powerful than the traditional approach of first order Rayleigh-Schroedinger perturbation of eigenvalues. However, the traditional approach gives somewhat different formulae for splitting eigenvalues and therefore is not superceded by the Uhlenbeck approach. We use both to prove Theorem 1.1; see Section 4.6 for an application of the traditional approach.
4.1. **The Uhlenbeck framework.** To study generic properties of the spectrum, we follow [Uhl76] and work with $C^r$ spaces of metrics and connections. We use the following notation:

- We denote by $\mathcal{G}^r(X)$ the Banach space of $C^r$ metrics on $X$. Since $X$ is a surface and we usually fix the complex structure $J$, we only work with $C^r$ metrics in the associated conformal class $\text{Conf}(J)$ and represent them in the usual Weyl gauge $g = e^u g_0$ relative to a fixed background metric $g_0 \in \text{Conf}(J)$. Thus, we may identify $\mathcal{G}^r(X) \cong C^r(X)$. We may also fix the area of the metrics with no loss of generality and then $\text{Conf}(J)$ may be identified with the space $\mathcal{K}_\omega$ of $\text{Kähler}$ metrics on $X$ in a fixed cohomology class. This is simply a different choice of gauge in which we write the $\text{Kähler}$ forms as $\omega_\phi = \omega_0 + i \partial \bar{\partial} \phi$ and use the potentials $\phi$ rather than the Weyl gauge $u$ to parameterize metrics.

- We denote by $\mathcal{H}^r(L)$ the Banach space of $C^r$ Hermitian metrics on $L$. Once we fix a local frame $e_L$ we may identify $h \in \mathcal{H}^r(L)$ with the function $\psi$ such that $||e_L(z)||_h^2 = e^{-\psi(z)}$, and $\mathcal{H}^r(L)$ is then equivalent to $C^r(X)$ except of course that the identification is frame dependent and the frame is only local (defined on the complement of a smooth closed curve in $X$, e.g.).

- We denote by $\mathcal{A}^r(L)$ the space of connections with $C^r$ connection forms. As before, we also denote by $\mathcal{A}_h^r$, resp. $\mathcal{A}_C^r$, the $h$-compatible (resp. $J_L$-compatible) $C^r$ connections.

- We denote by $C^r(X,L)$ the $C^r$ sections of $L$. We also denote by $H^s(X,L)$ the Sobolev space of sections with $s$ derivatives in $L^2$.

We define

$$\Phi_L : \mathcal{G}^r(X) \times \mathcal{H}^r(L) \times \mathcal{A}^r(L) \times H^2(X,L) \times \mathbb{C} \rightarrow L^2(X,L),$$

by

$$\Phi_L(g,h,\nabla,s,\lambda) = (\nabla^*_g h, \nabla - \lambda)s.$$

Here, the eigenvalue parameter $\lambda$ in the domain is allowed to be complex even though at zeros of $\Phi_L$ it is always real. This does not change the arguments in [Uhl76] but is needed so that $\lambda s$ spans the eigenspace when $s$ is an eigensection. In [Uhl76] the eigenfunctions were real-valued, so this issue did not arise.

Recall that a linear map between Banach spaces is Fredholm if it has closed image and finite dimensional kernel and cokernel. The index of a Fredholm operator is the difference of the dimensions of its kernel and cokernel. A nonlinear map $\Phi : N \rightarrow Y$ of Banach manifolds is Fredholm if its derivative $D\Phi_n$ is Fredholm for every $n \in N$.

Our first goal, roughly speaking, is to prove that $\Phi$ is a Fredholm map of index 0, i.e., to prove surjectivity of the differentials $D_2\Phi$ from tangent spaces of

$$Q = \{(g,h,\nabla,s,\lambda) : \Phi_L(g,h,\nabla,\lambda) = 0\}$$

to $L^2(X,L)$. It is sufficient to pick the relevant types of frames and calculate the Bochner Laplacians in the frame as in Section 3.

Regarding the surjectivity, we need to prove density of the image and that the image is closed. Some care needs to be taken because sections of complex line bundles are ‘vector-valued’, i.e., have two real components. As explained in [EPS12], there are pitfalls to avoid when generalizing the arguments of [Uhl76] to the vector-valued case. But sections of line bundles are locally complex-valued functions and are essentially scalar functions, albeit with scalars in $\mathbb{C}$.

4.2. **Uhlenbeck’s argument.** We briefly review Uhlenbeck’s proof that for generic metrics on compact $C^r$ Riemannian manifolds, all eigenvalues are simple and all eigenfunctions have 0 as a regular value.

Her framework is quite general and therefore uses the notation $B$ for the relevant space of metrics or other geometric data, and $L_b$ for the Laplacian associated to $b$. The relevant functions are denoted by $u$ and the space of such functions on a manifold $M$ is denoted by $C^k(M)$, even though they could be sections of a bundle over $M$. Then define

$$\Phi(u,\lambda,b) = (L_b + \lambda)u,$$

and put

$$Q := \{(u,\lambda,g) \in C^k(X) \times \mathbb{R}_+ \times B : \Phi(u,\lambda,b) = 0\}.$$
We often write

Then,

Further, let \( D \)

Proposition 4.4. \([\text{Uhl76}, \text{Proposition 2.10}]\).

Theorem 4.3. Assume that \( \alpha \)

Also define \( J \) to be the image of \( D_2 \Phi \),

\[
J = \text{Im}D_2\Phi_{(u, \lambda, b)} = \{ \hat{\Delta}u : \hat{\Delta} \text{ is a variation of } \Delta \text{ along a curve of metrics} \}.
\]

We use the following ‘abstract genericity’ result of [Uhl76, Theorem 1]- [Uhl76, Lemmas 2.7-2.8].

**Theorem 4.3.** Assume that \( \Phi \) is \( C^k \) and has zero as a regular value. Then the eigenspaces of \( L_\Phi \) are one-dimensional. If additionally, \( \alpha : Q \times M \to \mathbb{C} \) has zero as a regular value, then additionally

\[
\{ b \in B : \text{the eigenfunctions of } L_\Phi \text{ have } 0 \text{ as a regular value} \} \text{ is residual in } B.
\]

The key proposition is the following procedure for verifying the first hypothesis of Theorem 4.3 (see [Uhl76, Proposition 2.10]).

**Proposition 4.4.** Let \( J = \text{Im}D_2\Phi \) and assume that for \( W \in L^1(M) \) and \( W \in C^2(M - \{ y \}) \), the property \( \int_M W(x)j(x)d\mu_x = 0 \) for all \( j \in J \) implies \( W = 0 \). Then \( \phi \) is \( C^k \) and has zero as a regular value.

For the sake of completeness, we briefly review the main steps in proving Theorem 4.3. The main input are two transversality theorems. The first is: Let \( \phi : H \times B \to E \) be a \( C^k \) map where \( H, B, E \) are Banach manifolds. If 0 is a regular value of \( \phi \) and \( \phi_b(\cdot) := \phi(\cdot, b) \) is a Fredholm map of index \( < k \), then the set \( \{ b \in B : 0 \text{ is a regular value of } \phi_b \} \) is residual in \( B \).

The second statement follows from [Uhl76, Lemma 2.7]: Let \( \pi : Q \to B \) be a \( C^k \) Fredholm map of index 0. Then if \( f : Q \times X \to Y \) is a \( C^k \) map for \( k \) sufficiently large and if \( f \) is transverse to \( Y' \) then \( \{ b \in B : f_b := f|_{\pi^{-1}(b)} \text{ is transverse to } Y' \} \) is residual in \( B \). Let

\[
\alpha : f^{-1}(Y') \to B = \alpha : f^{-1}(Y') \subset Q \to B.
\]

**Lemma 4.5.** The eigenfunctions of \( L_\Phi \) have zero as a regular value if \( b \) is a regular value of \( \pi \) and if 0 is a regular value of \( \alpha|_{f^{-1}(b)} \times M := \alpha_b \).

Eigenfunctions and eigenvalues move continuously under perturbations of the operator. So it is easy to show that the set of metrics with for which the \( j \)th eigenvalue is simple is open. The difficulty is to prove that this set is dense.

To prove the first statement in Theorem 4.3 we need to verify the hypotheses of Theorem 4.3 and therefore need to prove Proposition 4.4 i.e., to determine the range of \( D_2 \phi \).

**Proposition 4.6.** For each of the admissible types of perturbation, \( D_2\Phi_m \) is surjective from \( T_{(u, \lambda, \phi)}Q_m \to C^{k-2} \).

4.3. **Base metric variations.** In this section we fix \((h, \nabla, J, J_L)\) and vary only \( g = e^\theta \). Equivalently, we consider Kaluza-Klein metrics on a fixed \( U(1) \) bundle \( P_h \to M \) with a fixed connection \( \alpha \) and vary the base metric \( g \).

**Proposition 4.7.** Suppose that \((L, h, J) \to X \) is a Hermitian holomorphic line bundle with \( h \)-compatible connection \( \nabla \). Let \( \nabla(f \psi_L) = (df + if \alpha) \otimes e_L \) with \( \alpha \in \mathbb{R} \) in a unitary frame \( e_L \). Then for generic Riemannian metrics \( g = e^\theta g_0 \) in the conformal class of \( J \), all of the eigenvalues of \( \nabla_{g,h} \) are simple and all of the eigensections have 0 as a regular value.
Proof. By Proposition 3.2
\[ \nabla^* \nabla (fe_L) = (-\Delta_g f - 2iG(df, \alpha) + ifd^*_g \alpha + G(\alpha, \alpha)f) e_L, \]
where \( \Delta_g f \) is the scalar Laplace operator, where \( g = e^\rho \).

Taking the variation \( \delta \) with respect to \( \rho \) (and designating the variation with a dot),
\[ \delta \nabla^* \nabla (fe_L) = (-\dot{\Delta_g} f - 2i\dot{G}(df, \alpha) + i\dot{f}d^*_g \alpha + \dot{G}(\alpha, \alpha)f) e_L. \]
But each term is conformal to that of \( g \) with conformal factor \( e^{-\rho} \). Hence
\[ \delta \nabla^* \nabla (fe_L) = -\frac{1}{2} \rho \Delta f(x) - 2i\rho G(df, \alpha) + (ipd^*_g \alpha + \rho G(\theta, \theta)) \] \[ f e_L = \rho \nabla^* \nabla (fe_L). \]

If \( \nabla^* \nabla (fe_L) \) is flat, then \( f e_L \equiv 0 \). If \( \nabla^* \nabla (fe_L) \) is non-flat, or if it is flat, that \( \alpha \neq 0 \). Then for generic gauge equivalence classes \( \alpha \in \mathcal{A}_h \cong \Omega^1(X), \) all of the eigenvalues of \( \nabla^*_{g,h} \nabla_h \) are simple and all of the eigenspaces have \( 0 \) as a regular value.

Proposition 4.8. Suppose that \((L, h, J) \to X\) is a Hermitian holomorphic line bundle and let \( \nabla \in \mathcal{A}_h \) be given by \( \nabla (fe_L) = (df + if \alpha) \otimes e_L \) with \( \alpha \in \mathcal{R} \) in a unitary frame \( e_L \). Suppose that \( L \) is non-flat, or if it is flat, that \( \alpha \neq 0 \). Then for generic gauge equivalence classes \( \alpha \in \mathcal{A}_h \cong \Omega^1(X), \) all of the eigenvalues of \( \nabla^*_{g,h} \nabla_h \) are simple and all of the eigenspaces have \( 0 \) as a regular value.

Proof. Again by Proposition 3.2
\[ \nabla^* \nabla (fe_L) = (-\Delta_g f - 2iG(df, \alpha) + ifd^*_g \alpha + G(\alpha, \alpha)f) e_L, \]
where \( \Delta_g f \) is the scalar Laplace operator. Taking the variation with respect to \( \alpha \) gives,
\[ \delta \nabla^* \nabla (fe_L) = (-2iG(df, \hat{\alpha}) + ifd^*_g \hat{\alpha} + 2G(\hat{\alpha}, \alpha)f) e_L. \]

If the image is not dense, there exists \( W = Fe_L \) so that
\[ \int_X (-2iG(df, \hat{\alpha}) + ifd^*_g \hat{\alpha} + 2G(\hat{\alpha}, \alpha)f) \bar{F} e^{-\psi} dV_g = 0, \]
for all \( \hat{\alpha} \in \Omega^1(X) \). We integrate \( d^*_g \) by parts to get,
\[ \int_X ((-2iG(df, \hat{\alpha}) + 2G(\hat{\alpha}, \alpha)f) \bar{F} + iG(\hat{\alpha}, d(f \bar{F}))) e^{-\psi} dV_g = 0. \]

We may assume that the frame \( e_L \) is unitary so that \( \psi = 0 \). If \( \beta \in \Omega^1(M, \mathbb{C}) \) and \( \int_X G(\beta, \nu) dV_g = 0 \) for all \( \nu \in \Omega^1(M, \mathbb{R}) \), then \( \beta = 0 \). Indeed, we may consider \( \nu \) of the types \( \nu = \nu_1 dx, \nu_2 dy \) separately to get orthogonality of the components \( \beta_j \) with \( \nu_j \). This reduces matters to the fact that if \( u, \nu \) are complex-valued and \( \int u \nu dV_g = 0 \) for all \( u \), then \( u \equiv 0 \). We conclude that
\[ (-2idf + 2\alpha f) \bar{F} + i\nu d(\bar{F}) = 0 \iff (-idf + 2\alpha f) \bar{F} + i\nu dF = 0. \]
On any open set $U$ where $f, F \neq 0$ we may divide by $i f \tilde{F}$ and write the solution as,
\[
\frac{d\tilde{F}}{F} = -\left(-\frac{df}{f} + 2i\alpha\right).
\]
This implies that
\[
d\log \frac{\tilde{F}}{f} = -2i\alpha \implies d\alpha = 0
\]
on a dense open set and since \( \alpha \in C^\infty \), it is everywhere closed and hence the curvature of \((L, h)\) is zero. This is impossible unless \( L \) is a topologically trivial line bundle, and the contradiction implies that \( F \equiv 0 \) except when \( d\alpha = 0 \).

4.5. **Proof of Theorem 4.1.** Eigenfunctions move continuously under perturbations of the operator. So it is easy to show that the set of metrics with for which the \( j \)th eigenvalue is simple is open. The difficulty is to prove that this set is dense.

To prove the first statement in Theorem 4.1 we need to verify the hypotheses of Theorem 4.3 and therefore need to prove Proposition 4.4, i.e., to determine the range \( J \) of \( D_2\phi \).

To complete the proof of Theorem 4.1 it suffices to prove:

**Proposition 4.9.** For each \( m \), \( D_1\alpha_m \) is surjective to \( \mathbb{C} \).

**Proof.** Let \( G_{m,\lambda}(z, w) \) be the kernel of the Green’s function \( G_{m,\lambda} : [\ker(D_m + \lambda)]^1 \to [\ker(D_m + \lambda)]^1 \) for \( D_m(g) + \lambda \) for a given background metric \( g \). As above, one may use the Hermitian metric \( h \) on \( K \) or the associated Kähler metric \( g = \omega_J \) as the parameter space of metrics.

We need to show that for each \( x \in M \),
\[
\alpha_m : Q \times \{ x \} \to \mathbb{C} : \alpha(u, \lambda, g, x) = u(x)
\]
has \( 0 \in \mathbb{C} \) as a regular value, i.e., that
\[
D_1\alpha(\cdot, x) : T_{u,\lambda,b}(Q) \to \mathbb{C}, \quad D_1\alpha(\cdot, x)(u(x), 0, c, 0) = \delta u(x)
\]
is surjective to \( \mathbb{C} \), where \( D_1 \) is the differential along \( Q \) with \( x \in M \) held fixed. Since \( x \) is fixed we may use a local coordinate \( z \) and frame \((dz)^m \) as above and identify local sections of \( K^m \) with complex-valued functions \( u : U \to \mathbb{C} \), where \( U \) is an open set containing \( x \).

The constraint equation for \((v, 0, c, 0) \in T^*_{u,\lambda,b}Q\) is
\[
(D_m(g) + \lambda)v + (\hat{D}_m(g) + \hat{\lambda})u = 0,
\]
and we can solve for \( v \perp \ker(D_m(g) + \lambda) \) as
\[
v(x) = -\int_M G_{m,\lambda}(x, y)\Pi^{\perp}_{\lambda}[(\hat{D}_m + \hat{\lambda})u](y)dV(y).
\]
By Proposition 4.6 the range of \( D_2\Phi \), i.e., the set of functions \([\hat{D}_m + \hat{\lambda})u] \), spans \( L^2 \). Therefore, the image \( \Pi^{\perp}_{\lambda}[(\hat{D}_m + \hat{\lambda})u] \) spans \( [\ker(D_m + \lambda)]^1 \). It follows that the possible values of \( v \) are all functions of the form,
\[
v(x) = \int_M G_{m,\lambda}(x, y)f(y)dV(y),
\]
where \( f \perp \ker(D_m(g) + \lambda) \). Thus, \( D_1\alpha \) is surjective to \( \mathbb{C} \) unless for all \( j \perp \ker(D_m(g) + \lambda) \), either the real or imaginary parts of
\[
G_{m,\lambda}(j)(x) = \int_M G_{m,\lambda}(x, y)j(y)dV(y)
\]
vanish (or both) for every such \( j \).

Since \( j = [D_m(g) + \lambda]f \) where \( \int f = 0 \) we would get the absurd conclusion that
\[
f(x) = 0, \quad \forall f \perp \ker(D_m(g) + \lambda).
\]
Equivalently,
\[
G_{m,\lambda}(x, y) + u_\lambda(x) = 0.
\]
This is not possible and the contradiction ends the proof. \( \square \)
4.6. Multiplicity of the spectrum of $\Delta_g$. In this section we use the Rayleigh-Shrödinger perturbation theory. We begin by observing that $\lambda_{m,j} = \lambda_{-m,j}$ and that $\phi_{-m,j} = \bar{\phi}_{m,j}$. Then any real eigenfunction which is a linear combination of $\phi_{m,j}$ and $\phi_{-m,j}$ is

$$ \Re (e^{i\theta_0}\phi_{m,j}) $$

for some constant $\theta_0$. In local coordinates, $\phi_{m,j}$ is $\tilde{\phi}(z)e^{im\theta}$, and therefore we see that

$$ \Re (e^{i\theta_0}\phi_{m,j}) = \Re (T_{\theta_0}\phi_{m,j}) = T_{\theta_0}\Re (\phi_{m,j}). $$

For $m_1$ and $m_2$ such that $|m_1| \neq |m_2|$, we argue that $\lambda_{m_1,j_1} \neq \lambda_{m_2,j_2}$ is satisfied for an open dense subset of metric $G$. This immediately implies the first, third, and the fourth statement of Theorem 1.1. Note that the eigenvalue moves continuously with respect to $G$. So it is sufficient to prove that

**Lemma 4.10.** Let $P \to X$ be a non-trivial principal $S^1$ bundle. Fix integers $m_1$ and $m_2$ such that $|m_1| \neq |m_2|$. Among all $S^1$-invariant metric $G$ on $P$, $G$ satisfying $\lambda_{m_1,j_1} \neq \lambda_{m_2,j_2}$ is dense.

**Proof.** The deformation of the base of the Kaluza-Klein metric does not touch the vertical operator $\frac{\partial}{\partial \theta}$ and therefore the first order perturbation equations for infinitesimal deformations of the base metric $g$ gives,

$$ (\Delta_H + \lambda_{m,j})\phi_{m,j} = (\Delta_H + \lambda_{m,j} + m^2)\phi_{m,j} $$

Taking the inner product with $\phi_{m,j}$ gives

$$ -\lambda_{m,j} = (\Delta_H\phi_{m,j}, \phi_{m,j}). $$

If there exist weights $m_1 \neq m_2$ for which we cannot split the eigenvalue $\lambda_{m_1,j_1} = \lambda_{m_2,j_2}$ then for all infinitesimal base perturbations $\rho$ we get

$$ \lambda_{m_1,j_1} = \lambda_{m_2,j_2} $$

$$ \iff (\Delta_H\phi_{m_1,j_1}, \phi_{m_1,j_1}) = (\Delta_H\phi_{m_2,j_2}, \phi_{m_2,j_2}). $$

Write $\phi_{m,j} = f_{m,j}(dz)^m$. Differentiation of the eigenvalue equation therefore gives the well-known formula

$$ \langle D_{m_1} f_{m_1,j_1}, f_{m_1,j_1} \rangle = \langle D_{m_2} g_{m_2,j_2}, g_{m_2,j_2} \rangle $$

for every variation of $g$, where the inner product is that of $g_0$.

Recall from previous section that $\Delta_H = \rho\Delta_H$. Because $-\Delta_H\phi_{m,j} = (\lambda_{m,j} - m^2)\phi_{m,j}$ we have for any $\rho \in C^\infty(X)$,

$$ (\lambda_{m_1,j_1} - m_1^2) \int_X \rho |f_{m_1,j_1}|^2 e^{-m_1\phi} dA_0 = (\lambda_{m_2,j_2} - m_2^2) \int_X \rho |f_{m_2,j_2}|^2 e^{-m_2\phi} dA_0. $$

Thus,

$$ (\lambda_{m_1,j_1} - m_1^2)|f_{m_1,j_1}|^2 e^{-m_1\phi} = (\lambda_{m_2,j_2} - m_2^2)|f_{m_2,j_2}|^2 e^{-m_2\phi}. $$

Integrating both sides against $dV_g$ and using that both eigenfunctions are $L^2$ normalized gives

$$ (\lambda_{m_1,j_1} - m_1^2) = (\lambda_{m_2,j_2} - m_2^2), $$

i.e., $|m_1| = |m_2|$.

$\square$

5. Local structure of eigensections at zeros

To study the nodal sets of real and imaginary parts of Kaluza-Klein Laplacians, we first study the zeros of the associated sections of the line bundles. For simplicity of exposition, we assume that $L = K$ and describe the zero sets of eigen-$m$-differentials. Essentially the same discussion is valid for other line bundles.

We follow the notation and terminology in the theory of holomorphic quadratic differentials, even though our eigendifferentials are $C^\infty$, usually not holomorphic and of general weight $m$. Following a standard terminology for quadratic differentials, we call a point $z$ such that $f_{m,r_j}(z) \neq 0$ a “regular point” and a point where $f_{m,r_j}(z) = 0$ a “critical point” or a “singular point”.

After the first version of this article was written, we located some recent articles generalizing the geometric properties of quadratic differentials on Riemann surfaces to $C^\infty$ higher order differentials $\text{FNNP12, AM17}$ and to other line bundles. We now use the terminology and results of these articles but have retained some from our first version since it is important for us to lift to $P_h$. 
5.1. **Trajectories of eigen-differentials.** The real and imaginary parts of the eigendifferentials \( \omega_{m,j} = f_{m,j}(z)(dz)^m \) are called binary differentials of degree \( m \) and the equation for the zero set of \( \Im \omega_{m,j} \) is called a bi-\( m \)-periodic differential equation of degree \( m \) \cite{FNnB12}. It is traditional to consider the nodal set \( \Im f_{m,j}(z)(dz)^m = 0 \). If there exist exactly \( m \) solutions at a regular point where \( \omega_{m,j}(z) \neq 0 \) then \( \omega_{m,j} \) is called totally real in \cite{FNnB12}. Our \( m \)-differentials are of a special type since they are real and imaginary parts of \( f_{m,j}(z)(dz)^m \) and therefore only have terms of the form \((dz)^m \) or \((d\bar{z})^m \). The following is the key input into Proposition 4.1.

**Lemma 5.1.** \( \Im f_{m,j}(dz)^m \) is a totally real \( m \)-differential. At a regular point \( z \), there exist \( m \) distinct solutions \( v \) of \( \Im f_{m,j}(dz)^m(v) = 0 \) in \( T_z X \).

**Proof.** If \( v = (\cos \phi, \sin \phi) \), then in the notation of \((1.2)\), the equation is

\[
(a_{m,j} \ell_m - b_{m,j} \ell_m)(\cos \theta, \sin \theta) = 0.
\]

Here \( \ell_m = \Re(\cos \theta + i \sin \theta)^m = \cos m\theta \), and the equation is

\[
a_{m,j}(z) \cos m\theta - b_{m,j}(z) \sin m\theta = 0 \iff \tan m\theta = \frac{a_{m,j}}{b_{m,j}},
\]

where \( a_{m,j}, b_{m,j} \in \mathbb{R} \) and where we assume with no loss of generality that \( b_{m,j} \neq 0 \). Since the principal branch of \( \tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2) \) is one-to-one, there exists precisely one solution \( \ell_0 \) of \( \tan m\theta = \frac{a_{m,j}}{b_{m,j}} \) with \( m\theta \in (-\pi/2, \pi/2) \), namely the principal branch of \( \tan^{-1}(\frac{a_{m,j}}{b_{m,j}}) \). Since \( \tan \theta \) is \( \pi \)-periodic, \( \tan m\theta \) is \( \frac{\pi}{m} \)-periodic, and the full set of solutions is \( \ell_0 + k\frac{\pi}{m} \) with \( k = 0, \ldots, m-1 \). \( \square \)

The \( m \) line fields defines a web of \( m \) transverse singular foliations, whose leaves are the trajectories.

**Definition 5.2.** The kernel of \( \Im f_{m,j}(dz)^m \) defines a smooth \( m \)-valued distribution on \( X \) with singularities where \( \omega_{m,j} = f_{m,j}(dz)^m = 0 \). The integral curves are the trajectories, i.e., are curves \( \gamma(t) \) in \( X \) along which \( \Im \phi_{m,j}(\gamma(t), \gamma'(t)) = 0 \). Trajectories naturally lift to curves in \( P_h \).

**Remark 5.3.** A trajectory in this sense of this article is called a ‘horizontal trajectory’ in \cite{Str84} Definition 5.5.3. They are illustrated in \cite{Str84} Section 7 for holomorphic quadratic differentials. Illustrations of webs for higher order real differentials can be found in \cite{FNnB12}.

Trajectories downstairs on \( X \) lift to \( P_h \) by their tangent vectors. A trajectory \( \gamma_{z_0,0}(t) \) downstairs is a smooth curve along which

\[
\Im(\phi_{m,j}(\gamma_{z_0,0}(t), \dot{\gamma}_{z_0,0}(t))) = 0.
\]

It lifts to a smooth curve \( (\gamma_{z_0,0}(t), \dot{\gamma}_{z_0,0}(t)) \) in the nodal set upstream. Since \( d\pi \) is an isomorphism, the trajectories are special curves on the nodal set \( \Im \phi_{m,j} = 0 \).

5.2. **Non-degenerate singular points.** The structure of the trajectories through a singular (zero) may be complicated in general if no conditions are placed on the degeneracy of the zeros. The purpose of Theorem 4.1 is to allow us to assume that the zeros are of first order, so that they are isolated and non-degenerate.

The structure of the trajectories of a totally real \( m \)-differential near an isolated singular point is discussed in \cite{FNnB12}. As with vector fields, the key topological invariant of the singular point is its **index**

**Definition 5.4.** The **index** of a singular point \( z_0 \) where \( f_{m,j}(z_0)(dz)^m = 0 \) is related to the degree of the circle map defined by \( \delta(t) = z_0 + re^{it} \rightarrow \frac{f_{m,j}(\delta(t))}{f_{m,j}(\dot{\delta}(t))} \) on a small circle around \( z_0 \) to \( S^1 \) by

\[
\text{ind}(z_0) = \frac{\pm 1}{m} \deg \frac{f_{m,j}(\delta(t))}{f_{m,j}(\dot{\delta}(t))}.
\]

Equivalently, in a small circle \( C \) around \( z_0 \), choose a unit vector \( X(0) \in \ker \Im f_{m,j}(dz)^m|_{C(0)} \) where \( C(t) : [0, 2\pi] \rightarrow X \) is a constant speed parametrization of \( C \) and let \( \ell = L(C) \) be its length. Let \( X(t) \) be a smooth extension of \( X(0) \) along \( C(t) \). After a complete turn, \( X(2\pi) \) must be one of the \( 2m \) solutions of \( \omega(X) = 0 \). After \( 2m \) turns \( X(2m\ell) = 0 \). Let \( \theta(t) \) be a smooth determination of the angle between the tangent line to \( C \) and \( X(t) \). Then \( \theta(2m\ell) \) and \( \theta(0) \) differ by an integer multiple of \( 2\pi \). The index of \( z_0 \) is defined by

\[
\text{ind}(\omega, z_0) = \frac{\theta(2m\ell) - \theta(0)}{2\pi m}.
\]
Thus, the index has the form $\frac{s}{2m}$ with $s \in \mathbb{Z}$. The following Lemma shows that singular points must exist when the genus of $X$ is non-zero.

**Lemma 5.5.** If $f_{m,j}(dz)^m$ has isolated non-degenerate zeros, then the sum of the indices of the zeros is the Chern class of $K_X^\wedge$.

**Lemma 5.6.** If $z_0$ is a non-degenerate singular point (zero of order 1) of $f_{m,j}(dz)^m$, then $\text{ind}(\omega_{m,j}, z_0) = \frac{\pm 1}{m}$.

**Proof.** This follows from the fact that $f_{m,j}$ is linear in this case and hence the degree of the associated circle map is $\pm 1$.

**Proposition 5.7.** For a generic Riemannian metric $g$ on $X$, all singular points of all eigendifferentials of $\nabla^*\nabla$ on $K^m$ have index $\frac{\pm 1}{m}$ for all $m \neq 0$.

**Proof.** It is part of Theorem 4.1 all singular points are non-degenerate. To prove this it suffices to show that the coefficients $f_{m,j}$ are linear near each singular point. This follows from the Bers local formula for eigensections around a zero.

We use Proposition 5.3 to Taylor expand the operator

$$D_m = \nabla_m^* \nabla_m = 2g^{11} \frac{\partial^2}{\partial z \partial \bar{z}} f - 2m[\frac{\partial f}{\partial x} g^{11}] \frac{\partial \phi}{\partial x} + Kf,$$

around a nodal point.

Let $p$ be a nodal point of $f_{m,j}$. We Taylor expand the coefficients in Kähler normal coordinates for $(J,g)$ in a disc $z \in D(p,r)$ to get

- $g^{11} = 1 + K(p)|z|^2 + \cdots$;

- $\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial z}(p) + \omega_p z + \cdots = z + \cdots$.

Thus, the osculating constant coefficient operator is

$$D_m f = 2\frac{\partial^2}{\partial z \partial \bar{z}} f + K(p)f.$$

Let $P_k$ denote homogeneous polynomials of degree $k$ in $z, \bar{z}$. It is better to arrange the terms of the Taylor expansion of $D_m$ at $p$ into terms

$$D_m = L_{-2} + L_{-1} + L_0 + L_1 + \cdots,$$

where $L_j : P_k \to P_{k+j}$. Thus, $L_{-2} = \frac{\partial^2}{\partial z \partial \bar{z}}, L_{-1} = 0, L_0 = K[2 \frac{\partial^2}{\partial z \partial \bar{z}} - 2m \frac{\partial f}{\partial x}] \bar{z} + K(p) \text{ etc}$. Note that $L_{-1} = 0$ because $dg^{11}(p) = 0$ and $\frac{\partial \phi}{\partial z}(p) = 0$, so neither the second or first derivative terms contribute at this order.

Also expand

$$f(z) = f_1(p)z + f_1(p)\bar{z} + f_{11}(p)|z|^2 + f_{11}(\bar{z})^2 + \cdots + f_{[k]} + \cdots,$$

where $f_{[k]} \in P_k$ is homogeneous of order $k$.

The following is the generalization of the Bers local expansion theorem to complex line bundles.

**Lemma 5.8.** Let $z_0$ be a zero of $f_{m,j}(dz)^m$. The first non-zero homogeneous term $f_{[n]}$ of the Taylor expansion of an eigenfunction is a harmonic homogeneous polynomial. If the order of vanishing is $n$, $f_{[n]}(z) = aRz^n + ib\bar{z}z^n$. In particular, at a non-degenerate zero, the first homogeneous term is $f_{m,j} = aRz + ib\bar{z}$.

**Proof.** It is evident that $L_{-2} = \frac{\partial^2}{\partial z \partial \bar{z}} : P_k \to P_{k-2}$. If $f_{[k]}$ is the term of lowest degree in the expansion of $f$ then $\frac{\partial^2}{\partial z \partial \bar{z}} f_{[k]} = 0$, i.e., $f_{[k]}$ is a homogeneous harmonic polynomial. In real dimension 2 the only possibilities are linear combinations of the real and imaginary parts of $z^k$. By a well-known argument, the nodal set of the real and imaginary parts of $f$ are topologically equivalent to those of the leading order homogeneous term.

This completes the proof of Proposition 5.7.
6. Adapted Kaluza-Klein metrics

So far, we have discussed spectral theory of Bochner-Kodaira Laplacians on line bundles. It is now time to relate that to the spectral theory of Kaluza-Klein Laplacians. As discussed in the Introduction, all of the Kaluza-Klein metrics are Riemannian metrics on principal \( S^1 \) bundles \( P_h \rightarrow X \) associated to \( C^\infty \) complex Hermitian line bundles \( L \rightarrow X \). The Hermitian metric \( h \) determines \( P_h \) as follows: Let \( D^*_1 \subset L^* \) be the unit co-disc bundle with respect to \( h \); then \( P_h = \partial D^*_1 \) is its boundary, an \( S^1 \) bundle \( \pi : P_h \rightarrow X \). Let \( T = \frac{\partial}{\partial \alpha} \) generate the \( S^1 \cong U(1) \) action. We endow \( P_h \) with a connection \( \alpha \), that is, an \( S^1 \) invariant 1-form on \( P_h \) such that \( \alpha(T) = 1 \). The connection defines a splitting

\[
T_p P_h = H_p \oplus V_p
\]

into horizontal and vertical spaces. The vertical space is given by orbits of the \( S^1 \) action. The horizontal space is defined by \( H_p = \ker \alpha \) and is isomorphic under \( d\pi_p \) to \( T_z X \) where \( \pi(p) = z \). The Kaluza-Klein metrics are defined in Definition 1.6.

6.1. Lifts to \( P_h \). The natural inner product on \( L^2(P_h, dV_G) \) is given by

\[
\langle f, f \rangle = \int_{P_h} |f|^2 dV_G.
\]

Sections \( s \) of \( L^m \) naturally lift to \( L^* \) and \( F_h \) by

\[
\hat{s}(z, \lambda) := \lambda(s(z)).
\]

It is straightforward to check that the lift of \( s \in C(X, L^m) \) satisfies \( \hat{s}(r \theta x) = e^{im\theta} \hat{s}(x) \) and that

\[
\int_{P_h} |\hat{s}(x)|^2 dV_G = \int_X \|s(z)\|^2_{h^m} dA_g.
\]

Indeed, if \( x = r_\theta \frac{e^{i\lambda}(z)}{|e^{i\lambda}(z)|} \) then \( \hat{s}(x) = e^{im\theta} |s(z)|^2_{h^m} \).

In the case of \( L = K_X \), the lift has the form,

\[
f(dz)^m(Y) = f(dz(Y))^m.
\]

We define an orthonormal frame of \( T^*X \) by \( \omega_1 = e^{-\phi} dz := \frac{dz}{|dz|_h} \) as above, and let \( \| \frac{\partial}{\partial z} \|^{-1} \frac{\partial}{\partial z} = e^{\phi} \frac{\partial}{\partial z} \) be the dual frame. In local coordinates \( z, \bar{z} \) on \( X \) and in this local frame we define local coordinates \( (z, \bar{z}, \theta) \) on \( SX \) corresponding to the point \( e^{i\theta} e^{\phi} \frac{\partial}{\partial z} \).

Then \( (dz)^m \) lifts to the function

\[
e_m(z, \bar{z}, \theta) = (dz)^m(e^{i\theta} e^{\phi} \frac{\partial}{\partial z}) = e^{im\theta} e^{m\phi(z)}.
\]

Consequently, the eigendifferential \( f_{m,j}(dz)^m \) lifts to

\[
\phi_{m,j}(z, \bar{z}, \theta) = f_{m,j}(z) e^{im\theta} e^{m\phi(z)}.
\]

In (1.1) we decomposed the lift into real and imaginary parts. We now relate them to the real and imaginary parts of \( f_{m,j} \).

If we take the inner product of \( u_{m,j} \) and \( v_{m,j} \) just along the fiber and use orthogonality of \( \cos m\theta, \sin m\theta \) and that \( \int_0^{2\pi} (\cos^2 m\theta - \sin^2 m\theta) d\theta = 0 \), and then integrate in \( dA(z) \) we get

**Lemma 6.1.** \( \langle u_{m,j}, v_{m,j} \rangle = 0 \).

6.2. Eigenspace decompositions. The Kaluza-Klein Laplacian has the form

\[
\Delta_G = \Delta_H + \frac{\partial^2}{\partial \theta^2}, \quad \text{where } \Delta_H = \xi_1^2 + \xi_2^2
\]

is the horizontal Laplacian. The fact that the fiber Laplacian is \( \frac{\partial^2}{\partial \theta^2} \) reflects the fact that \( S^1 \) orbits are geodesics isometric to \( \mathbb{R}/2\pi\mathbb{Z} \).

The weight spaces are \( \Delta_H \)-invariant, i.e., as an unbounded self-adjoint operator,

\[
\Delta_H : \mathcal{H}_m \rightarrow \mathcal{H}_m.
\]

Under the canonical identification

\[
\mathcal{H}_m \cong L^2(X, L^m)
\]
using the lifting map and \( \Delta_H|_{\mathcal{H}_m} \cong D_m - m^2I \) under the lifting map.

We then consider joint eigenfunctions \( \phi_{m,j} \) of the Kaluza-Klein Laplacian \( \Delta_G \) and of \( \frac{\partial}{\partial \theta} \). The commutation relations show that \( [\Delta_G, \frac{\partial^2}{\partial \theta^2}] = 0 \).

**Lemma 6.2.** The Bochner Laplacian agrees with the horizontal Laplacian \( \Delta_H \). In the above local coordinates and frame,

\[
\nabla_m^* \nabla_m (f(dz)^m) = \Delta_H (f(dz)^m).
\]

Note that except for the last identity, these statements are true for any isometric \( S^1 \) action, not just for adapted Kaluza-Klein metrics.

### 6.3. Equivariant decomposition.

Since \( S^1 \) acts isometrically on \( (M,G) \) we may decompose into its weight spaces,

\[
L^2(M,dV_G) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_m,
\]

where

\[
\mathcal{H}_m = \{ f : M \to \mathbb{C} : f(e^{i\theta}x) = e^{im\theta}f(x) \}.
\]

The weight spaces are \( \Delta_H \)-invariant, i.e., as an unbounded self-adjoint operator,

\[
\Delta_H : \mathcal{H}_m \to \mathcal{H}_m.
\]

The lifting map gives a canonical identification

\[
\mathcal{H}_m \cong L^2(X,L^m).
\]

### 7. Nodal sets and nodal domains of real and imaginary parts

#### 7.1. Connectivity of nodal sets of Kaluza-Klein eigenfunctions: Proofs of Theorem [1.9] and Theorem [1.10]

#### 7.2. Proof of Theorem [1.9]

The following is an immediate consequence of Lemma [5.1]

**Lemma 7.1.** If \( 0 \) is a regular value, then \( \mathcal{N}_{h,m,j} \subset SX \) is a singular \( 2m \)-fold cover of \( X \) with blow-down singularities over points where \( f_{m,j}(z)(dz)^m = 0 \).

Indeed, the \( 2m \) zeros of \( \Im \omega_{m,j}(v) = 0 \) in \( S_xX \) give \( 2m \) points on the fiber \( \pi^{-1}(z) \) in \( P_h \). Since locally there exist \( 2m \) smooth determinations of the zeros, the nodal set is a covering map away from the singular points.

#### 7.3. The nodal set of Kaluza-Klein eigenfunctions of \( S^*X \) is connected.

**Proof.** If we puncture out the set \( \Sigma \) zeros of \( \phi_{m,j} \) as a 1-form \( \omega_{m,j} \) we get a punctured Riemann surface. Let \( \mathcal{N}_{m,j} \) be the nodal set of \( \Im \phi_{m,j} \). By Proposition [1.9] \( \mathcal{N}_{m,j} \backslash (\mathcal{N}_{m,j} \cap \pi^{-1}(\Sigma)) \to X \backslash \mathcal{Z}_{m,j} \) is an \( m \)-sheeted cover. Thus, over the deleted Riemann surface, the sheets are disjoint.

However, every sheet contains the full fiber at a point of \( \Sigma \). Therefore all of the sheets intersect at each singular fiber. It follows that the nodal set is connected.

#### 7.4. Proof of Theorem [1.1]

By Proposition [1.9] and [1.3], \( \Im \phi_{m,j} \backslash (\Im \phi_{m,j} \cap \Sigma) \to X \backslash \mathcal{Z}_{m,j} \) is a \( 2m \)-sheeted cover. Moreover, \( P_h \backslash \Sigma \to X \) is an \( S^1 \) bundle and

\[
(P_h \backslash \Sigma) \backslash \mathcal{N}_{m,j} \to X
\]

is a fiber bundle whose fibers consist of the punctured fibers \( \pi^{-1}(z) \backslash \Im \phi_{m,j} \). The connected components of each punctured fiber consist of 'arcs' along which \( \Im \phi_{m,j} \) has a constant sign. We therefore express it as

\[
(P_h \backslash \Sigma) \backslash \mathcal{N}_{m,j} = P_+ \bigcup P_-
\]

where \( \text{sign} \Im \phi_{m,j} = \pm \) in \( P_\pm \). Each \( \pi : P_\pm \to X \) is a fiber bundle whose fiber consists of \( m \) arcs of the fibers of \( \pi : P_h \to X \). Since the number of zeros in each regular fiber is \( 2m \), the number of connected components of \( P_\pm \) is \( \leq m \). When we take the closure of these sets (i.e., add in the singular fibers, on which \( \Im \phi_{m,j} = 0 \), the connected components of the fiber are the nodal domains. It follows that there are \( \leq 2m \) nodal domains. We now argue that the closure of \( P_\pm \) is connected, so that there exist exactly \( 2 \) nodal domains.
We now use the local analysis in Section 5 of eigendifferentials of generic Bochner Laplacians around their zeros to determine how the sheets are connected at the singular fibers \( C_j = \pi^{-1}(z_j) \), corresponding to singular points (i.e., zeros) of \( f_{m,j}(dz)^m \) i.e., we consider the maximal components \( \mathcal{P}_{\pm,j} \) of

\[
\mathcal{P}_{\pm,j} \bigcup_{j=1}^{m} C_j = \bigcup_{j=1}^{m} \mathcal{P}_{\pm,j},
\]

in which \( \exists \phi_{m,j} \) has a single sign. When we union the left side with \( \bigcup_{j=1}^{m} C_j \) we glue together some of these domains along intervals of the singular fibers.

The gluing rule for the nodal domains is determined by the gluing rule for the nodal set, since the boundary of each nodal domain is the nodal set. From the downstairs point of view, the gluing rule is the monodromy of the cover \( N_{u,m,j} \rightarrow X \setminus Z(\omega_{m,j}) \) If we fix a singular point \( z_0 \), then we get a monodromy representation

\[
\rho : \pi_1(X \setminus Z(\omega_{m,j})) \rightarrow \text{Aut}(\pi^{-1}(z_0)),
\]

determining how the sheets of the nodal set are changed as the point circles around \( z_0 \).

By Proposition 5.7, the index of the singular points \( z_0 \) is \( \frac{1}{m} \). In terms of the monodromy, this means precisely that each turn around a circle \( C \) enclosing \( z_0 \) lifts to an arc from one vector in the fiber to its nearest neighbor with the same sign of \( \Re \phi_{m,j} \) (i.e., skipping the neighboring vector of the opposite sign).

It follows that both the + region and − region is connected in \( P_h \). Hence there are just two nodal domains.

### 7.5. Counting the number of nodal domains

We now give a more detailed presentation.

Let \( D \) be an open disc. We first study connectivity of a certain graph that arise from a pair of partitions of \( D \).

Let \( P \) and \( Q \) be partitions of \( D \), i.e., \( P \) (resp. \( Q \)) is a collection of disjoint open-sets \( \Omega_P(1), \ldots, \Omega_P(n_P) \subseteq D \) (resp. \( \Omega_Q(1), \ldots, \Omega_Q(n_Q) \subseteq D \)) such that

\[
\bigcup_{k=1}^{n_P} \Omega_P(k) = D \quad \text{(resp. } \bigcup_{k=1}^{n_Q} \Omega_Q(k) = D).\]

Let \( c_P : P \rightarrow \{0, 1\} \) and \( c_Q : P \rightarrow \{0, 1\} \) be colorings of \( P \) and \( Q \), and define the inversions of \( c_P \) and \( c_Q \) by \( c'_P = 1 - c_P \) and \( c'_Q = 1 - c_Q \).

We now define a graph \( G_m(P, Q, c_P, c_Q) \) as follows:

The vertex set is

\[
\begin{align*}
&v_{1,1}, \quad v_{1,2}, \quad \cdots \quad v_{1,n_P}, \\
v_{2,1}, \quad v_{2,2}, \quad \cdots \quad v_{2,n_Q}, \\
v_{3,1}, \quad v_{3,2}, \quad \cdots \quad v_{3,n_P}, \\
v_{4,1}, \quad v_{4,2}, \quad \cdots \quad v_{4,n_Q}, \\
&\vdots \\
v_{4m,1}, \quad v_{4m,2}, \quad \cdots \quad v_{4m,n_Q}
\end{align*}
\]

and edges are

\[
\begin{align*}
\{v_{4j+a}, v_{4j+1,b}\} & \text{ such that } \Omega_Q(a) \cap \Omega_P(b) \neq \emptyset, \text{ and } c'_Q(\Omega_Q(a)) = c_P(\Omega_P(b)), \\
\{v_{4j+1,a}, v_{4j+2,b}\} & \text{ such that } \Omega_P(a) \cap \Omega_Q(b) \neq \emptyset, \text{ and } c_P(\Omega_P(a)) = c_Q(\Omega_Q(b)), \\
\{v_{4j+2,a}, v_{4j+3,b}\} & \text{ such that } \Omega_Q(a) \cap \Omega_P(b) \neq \emptyset, \text{ and } c_Q(\Omega_Q(a)) = c'_P(\Omega_P(b)), \\
&\vdots \\
\{v_{4j+3,a}, v_{4j+4,b}\} & \text{ such that } \Omega_P(a) \cap \Omega_Q(b) \neq \emptyset, \text{ and } c'_P(\Omega_P(a)) = c'_Q(\Omega_Q(b))
\end{align*}
\]

for \( j = 0, 1, \ldots, m - 1 \) with the identification \( v_{0,a} = v_{4m,a} \).

**Definition 7.2.** We say a pair of partitions \((P, Q)\) generic, if

\[
D - \left( \bigcup_{k=1}^{n_P} \Omega_P(k) \cup \bigcup_{k=1}^{n_Q} \Omega_Q(k) \right)
\]

does not contain a closed curve.

**Lemma 7.3.** For a generic pair of partitions \((P, Q)\) with any given colorings \( c_P \) and \( c_Q \), any connected component of \( G_m(P, Q, c_P, c_Q) \) contains at least one of the following 2m vertices:

\[
v_{1,1}, \quad v_{3,1}, \ldots, \quad v_{4m-3}, \quad v_{4m-1}.
\]

In particular, \( G_m(P, Q, c_P, c_Q) \) has at most 2m connected components.
Proof. We first consider the case $m = 1$. To claim $\Gamma_1(P,Q,c_P,c_Q)$ has only 2 connected components, it is sufficient to prove that if $c_P(\Omega_P(a_1)) = c_P(\Omega_P(a_2))$, then $v_{1,a_1}$ and $v_{1,a_2}$ are path-connected.

Because $(P,Q)$ is a generic pair, one can find a chain of open-sets

$$\Omega_P(a_1) = \Omega_P(c_1), \Omega_Q(b_1), \Omega_P(c_2), \Omega_Q(b_2), \ldots, \Omega_P(c_k) = \Omega_P(a_2)$$

such that two adjacent open-sets have non-trivial intersection.

Observe that if $\Omega_P(c) \cap \Omega_Q(b) \neq \emptyset$, then either

$$\{v_{1,c}, v_{2,b}\}, \text{ or } \{v_{1,c}, v_{4,b}\}$$

is an edge, and likewise either

$$\{v_{3,c}, v_{2,b}\}, \text{ or } \{v_{3,c}, v_{4,b}\}$$

is an edge.

Therefore the above chain of open-sets corresponds to a path connecting $v_{1,a_1}$ with either $v_{1,a_2}$ or $v_{3,a_2}$. However, from the assumption $c_P(\Omega_P(a_1)) = c_P(\Omega_P(a_2))$, and from the construction of $\Gamma_1(P,Q,c_P,c_Q)$, $v_{1,a_1}$ cannot be connected to $v_{3,a_2}$, hence is connected to $v_{1,a_2}$.

Now for the rest, note that $\Gamma_m$ is an $m$-covering of $\Gamma_1$, and because $v_{1,1}$ and $v_{1,3}$ belongs to the different connected components of $\Gamma_1$, any connected components of $\Gamma_m$ must contain at least one vertex of the fiber of $v_{1,1}$ or $v_{1,3}$.

For a large class of colorings, we can deduce a much stronger result.

**Lemma 7.4.** Let $(P,Q)$ be a generic pair of partitions. Assume that we are given with a pair of colorings $c_P$ and $c_Q$:

There exist four open sets $\Omega_P(a_1), \Omega_P(a_2), \Omega_Q(b_1), \Omega_Q(b_2)$ such that

$$\Omega_P(a_i) \cap \Omega_Q(b_j) \neq \emptyset$$

for $i = 1, 2$ and $j = 1, 2$, and that $c_P(\Omega_P(a_1)) + c_P(\Omega_P(a_2)) = c_Q(\Omega_Q(b_1)) + c_Q(\Omega_Q(b_2)) = 1$.

Then the graph $\Gamma_m(P,Q,c_P,c_Q)$ has 2 connected components.

Proof. Note that any connected component of $\Gamma_m$ must contain either one of $v_{4j,a_1}$ or one of $v_{4j,a_2}$ with $j = 1, \ldots, m$, because $\Gamma_1$ has only two connected components.

Without loss of generality, assume that

$$c_P(\Omega_P(a_1)) = c_Q(\Omega_Q(b_1)).$$

Then from the construction of the graph and from the assumption of the lemma

$$\{v_{4j,a_1}, v_{4j+1,b_1}\},$$

$$\{v_{4j+1,b_1}, v_{4j+2,a_2}\},$$

$$\{v_{4j+2,a_2}, v_{4j+3,b_2}\},$$

and $$\{v_{4j+3,b_2}, v_{4j+4,a_1}\}$$

are edges, hence $v_{4j,a_1}$ and $v_{4j+4,a_1}$ are connected. Likewise, $v_{4j,a_2}$ and $v_{4j+4,a_2}$ are connected. Therefore any connected component of $\Gamma_m$ must contain either $v_{4j,a_1}$ or $v_{4j,a_2}$. \quad \square

7.6. **The number of nodal domains of generic eigenfunctions.** Let $P$ be a principal $S^1$ bundle over a connected smooth compact Riemannian surface $X$ with the covering map $\pi : P \to X$. Let $m$ be a fixed integer, and assume that $\phi \in C^1(M)$ satisfies the following conditions:

**Condition 7.5.** For any small open $U \subset X$ such that $\pi^{-1}U \cong U \times S^1$, there exists a local coordinate $(x, \theta)$ of $\pi^{-1}U$ such that

(i) $\phi(x, \theta) = f(x)e^{im\theta}$,

(ii) the zero set of $Re f$ (resp. $Im f$) gives rise to a partition $P = PU$ (resp. $Q = QU$) of $U$, and

(iii) $(PU, QU)$ is a generic pair of partitions of $U$.

In this section, we prove the following theorem.

**Theorem 7.6.** Fix any point $x \in X$ such that $\phi(x, \theta) \neq 0$. Then any nodal domain of $Re \phi$ has a nonempty intersection with $\pi^{-1}x$. In particular, the number of nodal domains of $Re \phi$ is $\leq 2m$. Assume further that $\phi$ has a regular zero. Then the number of nodal domains of $Re \phi$ is 2.
We begin with few observations in terms of fixed $U$ and a local coordinate $(x, \theta)$ of $\pi^{-1}U$.

**Proposition 7.7.** If $\Re \phi$ is positive on two open sets $U_1 \subset \pi^{-1}U \cap \{\theta = \frac{k\pi}{2m}\}$ and $U_2 \subset \pi^{-1}U \cap \{\theta = \frac{(k+1)\pi}{2m}\}$ for some integer $k$, and if $\pi U_1 \cap \pi U_2 \neq \emptyset$, then $U_1$ and $U_2$ are contained in the same nodal domain of $\Re \phi$.

**Proof.** Let $x_0$ be a point in the intersection $\pi U_1 \cap \pi U_2$. Then from the equation

$$\Re \phi(x_0, \theta) = \Re f(x_0) \cos(m\theta) + \Im f(x_0) \sin(m\theta),$$

we see that $\Re \phi$ is positive along the curve

$$\{(x_0, \theta) : \frac{k\pi}{2m} \leq \theta \leq \frac{(k+1)\pi}{2m}\},$$

which connects $U_1$ and $U_2$. Therefore $U_1$ and $U_2$ are contained in the same nodal domain. \hfill \Box

**Proposition 7.8.** Any nodal domain of $\Re \phi|_{\pi^{-1}U}$ must intersect $\pi^{-1}U \cap \{\theta = \frac{k\pi}{2m}\}$ nontrivially for some integer $k \in \mathbb{Z}$.

**Proof.** Assume for contradiction that $\Omega$ is a nodal domain of $\Re \phi|_{\pi^{-1}U}$ that is contained in

$$\pi^{-1}U \cap \left\{ \frac{k\pi}{2m} \leq \theta < \frac{(k+1)\pi}{2m} \right\}.$$

From the equation

$$\Re \phi(x, \theta) = \Re f(x) \cos(m\theta) + \Im f(x) \sin(m\theta),$$

we see that for each fixed $x$, $\Re \phi(x, \theta)$ either vanishes identically or has at most one sign change along the curve

$$\{(x, \theta) : \frac{k\pi}{2m} \leq \theta < \frac{(k+1)\pi}{2m}\}.$$

This implies that if $x \in \pi \Omega$, then

$$\Re \phi(x, \frac{k\pi}{2m}) = \Re \phi(x, \frac{(k+1)\pi}{2m}) = 0,$$

which contradicts the assumption that the zero set of $\Re f$ gives rise to a partition of $U$. \hfill \Box

From these two propositions, we see that the nodal domains of $\Re \phi|_{\pi^{-1}U}$ can be understood from the nodal domains of the restrictions of $\Re \phi|_{\pi^{-1}U}$ to the 4m-hypersurfaces

$$\pi^{-1}U \cap \{\theta = \frac{k\pi}{2m}\}, \quad k = 0, 1, 2, \ldots, 4m - 1.$$

In particular, if we define $c_{P_U}$ and $c_{Q_U}$ in terms of the sign of $\Re f$ and $\Im f$, then the number of connected components of $G_m(P_U, Q_U, c_{P_U}, c_{Q_U})$ is equal to the number of nodal domains of $\Re \phi|_{\pi^{-1}U}$.

**Proof of Theorem 7.6.** Let $x \in X$ be a point where $\phi(x, \theta) \neq 0$, and let $U$ be a sufficiently small neighborhood of $x$. We may assume without loss of generality that the vertices $v_{1,1}$, $v_{3,1}$, \ldots, $v_{4m-3}$, $v_{4m-1}$ of $G_m(P_U, Q_U, c_{P_U}, c_{Q_U})$ correspond to the nodal domains of the restrictions of $\Re \phi|_{\pi^{-1}U}$ to the hypersurfaces

$$\pi^{-1}U \cap \{\theta = \frac{k\pi}{2m}\}, \quad k = 1, 3, \ldots, 4m - 3, 4m - 1,$$

that intersect the fiber $\pi^{-1}x$. Then Lemma 7.3 implies that any nodal domain of $\Re \phi|_{\pi^{-1}U}$ must intersect $\pi^{-1}x$.

Now assume that $x'$ is another point in $U$. Then we may restate this as “any nodal domain of $\Re \phi|_{\pi^{-1}U}$ that intersect $\pi^{-1}x'$ must intersect $\pi^{-1}x$”, and equivalently, “any nodal domain of $\Re \phi$ that intersect $\pi^{-1}x'$ must intersect $\pi^{-1}x$”. Because we assumed that $X$ is connected, by the freedom of choice of the pair of points $x$ and $x'$, any nodal domain of $\Re \phi$ must intersect $\pi^{-1}x$. This proves the first part of the theorem.

For the latter part of the theorem, let $p$ be a regular zero of $\phi$, i.e.,

$$d\phi : T_p P \to \mathbb{C}$$

is a surjection. Choose a sufficiently small neighborhood $U \subset X$ of $\pi p$, and let $f$ be the function that satisfies

$$\phi(x, \theta) = f(x)e^{im\theta} = \Re f \cos(m\theta) + \Im f \sin(m\theta) + i(\Im f \cos(m\theta) - \Re f \sin(m\theta)) = \Re \phi + i\Im \phi.$$
If $d\Re f$ and $d\Im f$ are linearly dependent, then a straightforward computation implies that $d\phi$ has rank $\leq 1$, so $d\Re f$ and $d\Im f$ are linearly independent.

This implies that $\pi p$ is a regular zero of both $\Re f$ and $\Im f$. Also, linear independency implies that locally around $\pi p$, $\Re f = 0$ and $\Im f = 0$ define two curves intersecting transversally at $\pi p$. From this, we may find four open sets near $p$ that are required for Lemma 7.4 and we infer that the number of nodal domains of $\Re \phi_{|_{\pi^{-1}U}}$ is two.

Now because any nodal domain of $\Re \phi$ must intersect with $\pi^{-1}x$ for some $x \in U$, any nodal domain of $\Re \phi$ must contain one of the nodal domains of $\Re \phi_{|_{x^{-1}U}}$, from which we conclude that $\Re \phi$ has only two nodal domains.

We are ready to prove our main theorem, Theorem 1.1.

**Proof.** It is sufficient to verify the assumptions in Theorem 1.1 is satisfied. The first condition is trivial to verify. For the other conditions, note from the assumption that $P \to X$ is non-trivial, $Z_{f_{m,j}}$ is non-empty, and Theorem 4.1 implies that it is discrete and consists only of regular zeros.

**Remark 7.9.** If $Z_{f_{m,j}}$ contains a closed curve that divides $X$ into two connected components, then the number of nodal domain can be large. For instance, if $f_{m,j}$ vanishes on the boundary of small open disc $U \subset X$, and if it does not vanish on $U$, then $\Re \phi_{m,j}$ vanish identically on $\partial (\pi^{-1}U)$, and therein, $\Re \phi_{m,j}$ has $2m$-distinct nodal domains. In particular, Theorem 7.7 fails even if $f_{m,j}$ has a regular zero elsewhere.

### 8. Surfaces of constant curvature

In this section, we illustrate the geometry of Kaluza-Klein metrics and the Kaluza-Klein eigenvalue problem on unit tangent bundles of surfaces of constant curvature.

#### 8.1. Flat tori

Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We use coordinates $z = x_1 + i x_2$. Its unit tangent bundle is $ST^2 = T^2 \times S^1$. The connection is flat and $\Delta_H = \Delta$ is simply the Laplacian of $T^2$. The Kaluza-Klein Laplacian is that $\Delta_G = \Delta + \frac{d^2}{d\theta^2}$ on $T^2 \times S^1$. The Kaluza-Klein eigenfunctions are linear combinations of the product eigenfunctions,

$$\phi_{m,k}(x_1, x_2, \theta) = e^{2\pi i (k \cdot \bar{x})} e^{2\pi i m \theta}, \quad \Delta_G \phi_{m,k} = -4\pi^2 (|k|^2 + m^2) \phi_{m,k}.$$

The multiplicity of the eigenvalue is the same as on $T^2$, i.e., the number of ways of representing $|k|^2$ as a sum of two squares. They correspond to eigendifferentials

$$f_{m,k}(z)(dz)^m = e^{2\pi i (k \cdot \bar{x})} (dz)^m.$$

In the notation 1.1,

$$\begin{align*}
\Re \phi_{m,k}(x_1, x_2, \theta) &= u_{m,k}(x_1, x_2, \theta) = \cos 2\pi (\langle k, \bar{x} \rangle + m \theta), \\
\Im \phi_{m,k}(x_1, x_2, \theta) &= v_{m,k}(x_1, x_2, \theta) = \sin 2\pi (\langle k, \bar{x} \rangle + m \theta).
\end{align*}$$

The nodal sets of the imaginary part are given by,

$$Z_{v_{m,k}} = \{(x_1, x_2, \theta) : \langle k, \bar{x} \rangle + m \theta = \frac{\ell}{2m}, \ell = 1, \ldots, m\}.$$

$Z_{v_{m,k}}$ contains the set

$$\{(x_1, x_2, \theta) : \langle k, \bar{x} \rangle \in \frac{1}{2} \mathbb{Z}, \theta = \frac{\ell}{2m}, \ell = 1, \ldots, m\}.$$

Note that $\phi_{m,k}(x_1, x_2, \theta)$ has no zeros on $T^2 \times S^1$ and $f_{m,k}(z)(dz)^m$ has no zeros as an $m$-differential on $T^2$.

If we change the lattice to a general lattice $L \subset \mathbb{R}^2$, the eigenfunctions of $T^2$ change to $e_{\vec{\lambda}}(\bar{x}) = e^{2\pi i (\bar{\lambda} \cdot \bar{x})}$ where $\vec{\lambda} \in \Lambda = L^*$, the dual lattice. For generic $L$, the eigenvalues have multiplicity 2 and the eigenspaces are spanned by the real and imaginary parts of $e_{\vec{\lambda}}$ or equivalently by $e_{\vec{\lambda}}$ and its complex conjugate $e_{-\vec{\lambda}}$. The same is true of the Kaluza-Klein eigenfunctions $\phi_{m,\bar{\lambda}} = e^{2\pi i (\bar{\lambda} \cdot \bar{x})} e^{im \theta}$. Again, $\phi_{m,\bar{\lambda}}$ has no zeros. Using the bifurcation of nodal sets of eigenfunctions under generic paths of metrics of [Uhl76], one can show that

**Proposition 8.1.** For generic Kaluza-Klein metrics on $ST^2$, the joint eigenfunctions $\phi_{m,j}$ have no zeros.
We now give an explicit orthonormal eigenbasis of $\mathbb{T}^3$ such that all of them have exactly two nodal domains, hence proving Theorem 1.4.

To begin with, let $f_1(x) = \cos(2\pi x)$ and $f_0(x) = \sin(2\pi x)$. Then
\[
\{ f_{j_1}(m_1x_1)f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) : j_k = 0 \text{ or } 1, \ m_k \in \mathbb{Z}_{\geq 0} \}
\]
is an orthogonal eigenbasis of $\mathbb{T}^3$. We consider four cases.

Case 1: $m_1m_2m_3 > 0$. We first have
\[
\langle \{ f_{j_1}(m_1x_1)f_{j_2}(m_2x_2)f_{j_3}(m_3x_3), f_{1-j_1}(m_1x_1)f_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3) \} \rangle
= \langle \{ f_{j_1}(m_1x_1)f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) \pm f_{1-j_1}(m_1x_1)f_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3) \} \rangle
\]
Assume without loss of generality that $j_1 = 0$. Then
\[
f_{j_1}(m_1x_1)f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) \pm f_{1-j_1}(m_1x_1)f_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3)
= \Re \left( (f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) \pm i f_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3)) e^{2\pi i m_1 x_1} \right),
\]
has two nodal domains by Theorem 7.6, because
\[
f_{j_2}(m_2x_2)f_{j_3}(m_3x_3) \pm i f_{1-j_2}(m_2x_2)f_{1-j_3}(m_3x_3)
\]
has a regular zero.

Case 2: exactly one $m_k$ is zero, and the other two are different. From the same reasoning, each eigenfunction in the new basis in the following has two nodal domains:
\[
\langle \{ f_{j_1}(m_1x_1)f_{j_2}(m_2x_2), f_{j_1}(m_1x_1)f_{j_3}(m_2x_3) : j_k = 0 \text{ or } 1 \} \rangle
= \langle \{ f_{j_1}(m_1x_1)f_{j_2}(m_2x_2) \pm f_{1-j_1}(m_1x_1)f_{j_3}(m_2x_3) : j_k = 0 \text{ or } 1 \} \rangle,
\]
\[
\langle \{ f_{j_2}(m_1x_1)f_{j_1}(m_2x_1), f_{j_2}(m_1x_1)f_{j_3}(m_2x_3) : j_k = 0 \text{ or } 1 \} \rangle
= \langle \{ f_{j_2}(m_1x_1)f_{j_1}(m_2x_1) \pm f_{1-j_2}(m_1x_2)f_{j_3}(m_2x_3) : j_k = 0 \text{ or } 1 \} \rangle,
\]
and
\[
\langle \{ f_{j_3}(m_1x_1)f_{j_1}(m_2x_1), f_{j_3}(m_1x_1)f_{j_2}(m_2x_2) : j_k = 0 \text{ or } 1 \} \rangle
= \langle \{ f_{j_3}(m_1x_1)f_{j_1}(m_2x_1) \pm f_{1-j_3}(m_1x_3)f_{j_2}(m_2x_2) : j_k = 0 \text{ or } 1 \} \rangle.
\]

Case 3: exactly one $m_k$ is zero, and the other two are equal. Again by the same reasoning, each of the following
\[
f_0(mx_1)f_0(mx_2) \pm f_1(mx_1)f_0(mx_3),
f_0(mx_2)f_0(mx_3) \pm f_1(mx_2)f_0(mx_1),
f_0(mx_3)f_0(mx_1) \pm f_1(mx_3)f_0(mx_2),
f_1(mx_1)f_1(mx_2) \pm f_1(mx_3)f_0(mx_1),
f_1(mx_2)f_1(mx_3) \pm f_1(mx_1)f_0(mx_2),
f_1(mx_3)f_1(mx_1) \pm f_1(mx_2)f_0(mx_3)
\]
has two nodal domains, and these are the basis of
\[
\langle \{ f_{j_1}(mx_1)f_{j_2}(mx_2), f_{j_1}(mx_1)f_{j_3}(mx_3), f_{j_2}(mx_2)f_{j_3}(mx_3) : j_k = 0 \text{ or } 1 \} \rangle.
\]
Case 4: exactly one \( m_k \) is nonzero. In this case, we consider orthogonal eigenfunctions
\[
\begin{align*}
  f_0(mx_1) + f_0(mx_2) - \frac{1}{2}f_0(mx_3), \\
  f_0(mx_1) + f_0(mx_3) - \frac{1}{2}f_0(mx_2), \\
  f_0(mx_2) + f_0(mx_3) - \frac{1}{2}f_0(mx_1), \\
  f_1(mx_1) + f_1(mx_2) - \frac{1}{2}f_1(mx_3), \\
  f_1(mx_1) + f_1(mx_3) - \frac{1}{2}f_1(mx_2), \\
  f_1(mx_2) + f_1(mx_3) - \frac{1}{2}f_1(mx_1),
\end{align*}
\]
which span \( \langle \{ f_j(mx_1), f_j(mx_2), f_j(mx_3) : j = 0 \text{ or } 1 \} \rangle \).
Each of these has only two nodal domains from the following lemma.

**Lemma 8.2.** Let \( m \) be a positive integer. Then
\[
\cos(mx_1) + \cos(mx_2) - \frac{1}{2} \cos(mx_3)
\]
has only two nodal domains.

**Proof.** Let \( x_1 - x_2 = a, x_1 - x_3 = b, \) and \( x_2 + x_3 = c. \) Then
\[
e^{2\pi imx_1} + e^{2\pi imx_2} - \frac{1}{2}e^{2\pi imx_3} = \left( e^{\pi im} e^{\pi imb} + e^{-\pi im} e^{-\pi imb} - \frac{1}{2} e^{\pi im} e^{-\pi imb} \right) e^{2\pi imc},
\]
and from Theorem 7.6 it is sufficient to prove that
\[
e^{\pi im} e^{\pi imb} + e^{-\pi im} e^{-\pi imb} - \frac{1}{2} e^{\pi im} e^{-\pi imb}
\]
has a regular zero. Let \( \pi m(a + b) = x \) and \( \pi m(a - b) = y, \) then this is equivalent to
\[
\cos x + \frac{1}{2} \cos y + i \left( \sin x - \frac{3}{2} \sin y \right)
\]
having a regular zero. Since \( \cos x + \frac{1}{2} \cos y \) and \( \sin x - \frac{3}{2} \sin y \) do not have singular points, it is sufficient to check if these two functions have a common zero, in other words, if
\[
\cos x + \frac{1}{2} \cos y + i \left( \sin x - \frac{3}{2} \sin y \right) = 0
\]
has a solution. Note that this is equivalent to
\[
e^{ix} = -\frac{1}{2} \cos y + i \frac{3}{2} \sin y. \tag{8.1}
\]
Because
\[
\left| -\frac{1}{2} \cos y + i \frac{3}{2} \sin y \right| = \frac{1}{4} + 2 \sin^2 y,
\]
for \( y \) such that \( \frac{1}{4} + 2 \sin^2 y = 1, \) there is \( x \) satisfying (8.1), and this completes the proof. \( \square \)

### 8.2. \( S^2 \) of constant curvature.
Let \( (S^2, g_0) \) be the 2-sphere with its standard metric of curvature 1. Then its unit tangent \( SS^2 = SO(3) = \mathbb{RP}^3 = S^3/\pm 1 \) and the Kaluza-Klein metric is the standard metric of constant sectional curvature 1 on \( S^3 \) (divided by the antipodal group \( \mathbb{Z}_2 \)). The Kaluza-Klein Laplacian is therefore the standard Laplacian \( \Delta_{S^3} \) on \( \mathbb{Z}_2 \)-invariant functions.

We use the following Euler angles coordinates on \( S^3 \):
\[
\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r \begin{pmatrix} \sin \theta_1 \sin \theta_2 \cos \theta_3 \\ \sin \theta_1 \sin \theta_2 \sin \theta_3 \\ \sin \theta_3 \cos \theta_2 \\ \cos \theta_3 \end{pmatrix}.
\]
Here $0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_1, \theta_2 \leq \pi$.

Let $K \subset SO(3)$ be rotations in the fibers of $S(S^2) \to S^2$. It is generated by $\frac{\partial}{\partial \theta_3}$.

One may also use Hopf coordinates $\alpha, \theta, \phi$ on $S^3$ defined by

$$
\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r \begin{pmatrix} \sin \alpha \cos \phi \\ \sin \alpha \sin \phi \\ \cos \alpha \cos \theta \\ \cos \alpha \sin \theta \end{pmatrix}.
$$

Here $0 \leq \alpha \leq \pi/2, 0 \leq \theta, \phi \leq 2\pi$. This corresponds to writing

$$
z_1 = e^{i\phi} \sin \alpha, \quad z_2 = e^{i\theta} \cos \alpha.
$$

The Hopf fibration arises by writing $\mathbb{R}^4 = \mathbb{C}^2, \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ under $(x_1 + i x_2, x_3 + i x_4)$, resp. $(x_1 + i x_2, x_3)$. The Hopf fibration is the map

$$
\pi(z_1, z_2) = (2z_1z_2 - |z_1|^2 - |z_2|^2).
$$

The Hopf map $\pi : S^3 \to S^2$ in rectangular coordinates is given by

$$
\pi(x_1, x_2, x_3, x_4) = (2(x_1x_4 - x_2x_3, 2(x_2x_4 - x_1x_3), x_1^2 + x_2^2 - x_3^2 - x_4^2).
$$

If one writes

$$
z_1 = e^{i\xi_1 + \xi_2} \sin \eta, \quad z_2 = e^{i\xi_2 - \xi_1} \cos \eta
$$

then the Hopf map is

$$
\pi(\xi_1, \xi_2, \eta) = (\sin(2\eta) \cos \xi_1, \sin(2\eta) \sin \xi_1, \cos(2\eta)).
$$

Thus, $\xi_2$ runs over the unit tangent circle, $\eta$ is the azimuthal angle on $S^2$ and $\xi_1$ is the rotational angle around the third axis.

There exist two commuting isometric $S^1$ actions generated by the Killing vector fields

$$
X = \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \xi_2}, \quad Y = \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \xi_2}.
$$

**Moral** The $S^1$ action in the fibers is generated by $Y = \frac{\partial}{\partial \xi_2}$ with $\xi_2 = \phi - \theta$.

The identification $S^3 = SU(2)$ is given by

$$
(\alpha, \theta, \phi) \to \begin{pmatrix} \cos \alpha e^{i\theta} & i \sin \alpha e^{i\phi} \\ i \sin \alpha e^{-i\phi} & \cos \alpha e^{-i\theta} \end{pmatrix}.
$$

Here, $SU(2) \to SO(3)$ is a double cover.

The identification with the Euler angles $(\alpha, \beta, \gamma)$ is given by

$$
\alpha = \frac{\beta}{2}, \quad \theta = \frac{\alpha + \gamma}{2}, \quad \phi = \frac{\alpha - \gamma}{2}.
$$

### 8.2.1. Kaluza-Klein eigenfunctions

Since $S^3$ is a group, $L^2(S^3) = \bigoplus_{N=0}^{\infty} V_N \otimes V_N$ where $V_N$ is an irreducible representation of $S^3$ of dimension $N + 1$ and of the type of homogeneous holomorphic polynomials of degree $N$ on $\mathbb{C}^2$. This may be concretely realized by considering $S^3 = \partial B^2$ where $B^2 \subset \mathbb{C}^2$ is the unit ball. Alternatively, the eigenfunctions of $S^3$ are harmonic homogeneous polynomials on $\mathbb{R}^3$. Moreover, $\Delta|V_N \otimes V_N = N(N + 2)(N + 1)^2 - 1$. The eigenfunctions of $\mathbb{R}^3$ are those where $N$ is even.

An explicit basis of spherical harmonics of degree $N$ on $S^3$ are the $Y_N^{m,k}$ defined in spherical coordinates $(\theta_1, \theta_2, \theta_3)$ by

$$
Y_N^{m,k}(\theta_1, \theta_2, \theta_3) = C_{m,k}^N Y_N^{m}(\theta_1, \theta_2) C_{N+1}^{m,k}(\cos \theta_3) \sin^N(\theta_3),
$$

where $Y_N^m$ are the standard spherical harmonics on $S^2$, where $C_{m,k}^N$ are Gegenbauer polynomials, and where

$$
C_{m,k}^N = 2^N N! \sqrt{\frac{2(k+1)(k-N)}{\pi(k+N+1)}}.
$$

The metric is $(\partial \alpha)^2 + (\cos \alpha \partial \theta)^2 + (\sin \alpha \partial \phi)^2$. In these coordinates one has an orthogonal basis of eigenfunctions given by

$$
\Phi_N^{m_+,m_-}(\alpha, \phi, \theta) = \frac{m_+ - m_-}{2} e^{i(m_+ + m_-)\phi} e^{i(m_+ - m_-)\theta},
$$

$$
\cdot (1 - \cos 2\alpha)^{m_+ - m_-}(1 + \cos 2\alpha)^{m_- - m_+} P_{\frac{m_+ + m_-}{2} - m_+}^{m_+ + m_- - m_-}(\cos 2\alpha),
$$

where $P_L^M(x)$ are the Legendre polynomials.
where \( P_{N}^{(a,b)} \) is a Jacobi polynomial and where

\[
|m| \leq \frac{N}{2}, \frac{N}{2} - m \in \mathbb{N}.
\]

**Moral** \( m_{-} = m \) in our notation. Weight \( m \) means \( e^{im_{-}(\phi - \theta)} \).

Then \( \Phi_{N}^{m_{+},m_{-}} \) corresponds to the “Wigner D-functions” on \( SU(2) \). Another expression is

\[
T_{N}^{m_{1},m_{2}} = C_{N}^{m_{1},m_{2}} \left( \cos \alpha e^{i\theta} \right)^{m_{1}+m_{2}} \sin \alpha e^{i\phi} \left( \cos(2\alpha) \right).
\]

These are manifestly joint eigenfunctions of \( \Delta_{S} \) and of \( \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta} \).

**Moral** Here \( m_{1} = m \). There is something about the \((1 \pm \cos 2\alpha) \) versus \( \cos \alpha, \sin \alpha \). But clearly, the only factors with zeros are the \( \alpha \)-functions. These have roughly \( m \) discrete zeros in \( \alpha \). Hence, the complex nodal set is a union

\[
\{(\theta, \phi, \alpha) : (\cos \alpha)^{m_{1}+m_{2}} (\sin \alpha)^{m_{1}-m_{2}} P_{N/2-m_{2}}^{m_{1},m_{2}+m_{1}} (\cos(2\alpha)) = 0\},
\]

and thus has real dimension 2.

### 8.3. Hyperbolic surfaces \( \mathbb{H}^{2} \)

Let us start with a finite area hyperbolic real Riemann surface of constant negative curvature \(-1\). Then \( X = S^{*}_{\eta} M = \Gamma \backslash G \) where \( G = PSL(2, \mathbb{R}) \). The total space \( X \) carries a Lorentz Cartan-Killing metric with indefinite Laplacian the Casimir operator \( \Omega \). It is well known that \( \Omega = H^{2} + V^{2} - W^{2} \). We now change the sign of the third term to get the Kaluza-Klein Laplacian \( \Delta_{X} = H^{2} + V^{2} + W^{2} \). The associated metric defines a Riemannian submersion \( \pi : X \rightarrow M \) with fibers given by \( K \)-orbits. They are necessarily totally geodesic. It follows that the horizontal Laplacian \( H^{2} + V^{2} \) commutes with the vertical Laplacian \( W^{2} \). This is obvious because \( 0 = [\Omega, W^{2}] = [H^{2} + V^{2}, W^{2}] = 0 \).

The joint eigenfunctions of \( \Omega, W \) are denoted by \( \phi_{m,j} \). When \( m = 0 \) they are pullbacks of eigenfunctions of \( M = \Gamma \backslash G/K \).

In particular the number of nodal domains of \( \phi_{j,0} \) on \( X \) is the same as the number of nodal domains of \( \phi_{j} \) on \( M \). The former nodal sets are \( K \)-invariant and in the case of regular nodal components are 2-tori over circles.

The lift of weight \( m \) of an \( m \)-differential \( f(dx)^{m} \) is given by

\[
\Phi(x, y, \theta) = y^{m/2} f(x + iy)e^{-im\theta}.
\]

Here, the Kähler potential is \( \phi = \log y, \; d\phi = \frac{dy}{y}, \; \Delta \phi = y^{2} (\log y)'' = -1 \). Also, \( ||d\phi||^{2} = y^{2} ||\frac{dy}{y}||^{2} = 1 \) and \( *d\phi = *(\phi_{x} dx + \phi_{y} dy) = (-\phi_{x} dy + \phi_{y} dx) y \). The Maass operator is

\[
D_{m} = y^{2} \left( \frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}} \right) - 2iym \frac{\partial}{\partial x}
\]

and

\[
D_{m}f_{m,j} = s(1 - s)f_{m,j}.
\]

Breaking up into real and imaginary parts gives the system,

\[
\begin{cases}
\Delta \Re f_{m,j} + 2my \frac{\partial}{\partial x} \Im f_{m,j} = s(1 - s)\Re f_{m,j}, \\
\Delta \Im f_{m,j} - 2my \frac{\partial}{\partial x} \Re f_{m,j} = s(1 - s)\Im f_{m,j}.
\end{cases}
\]

The raising/lowering operators are the Maass operators defined by

\[
\begin{cases}
K_{k} = (z - \bar{z}) \frac{\partial}{\partial z} + k = 2iy^{1-k} \frac{\partial}{\partial z} y^{k}, \\
L_{k} = (z - \bar{z}) \frac{\partial}{\partial z} - k = -2iy^{1+k} \frac{\partial}{\partial z} y^{-k} = \bar{K}_{-k}.
\end{cases}
\]

Then,

\[
K_{k} : \mathcal{S}_{k} \rightarrow \mathcal{S}_{k+1}, \quad L_{k} : \mathcal{S}_{k} \rightarrow \mathcal{S}_{k-1},
\]

and

\[
D_{k+1}K_{k} = K_{k}D_{k}, D_{k}L_{k+1} = L_{k+1}D_{k},
\]

and

\[
D_{k} = L_{k+1}K_{k} + k(k + 1) = K_{k-1}L_{k} + k(k - 1).
\]
8.3.1. **Automorphic forms on the full modular group.** Now we consider the case when $\Gamma = SL_2(\mathbb{Z})$.

**Theorem 8.3.** Let $X = SL_2(\mathbb{Z}) \backslash \mathbb{H}$, and let $\phi_{m,ir}$ be a weight $m$ Maass–Hecke cusp form on $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$. Assume that the zeros of $\phi_{m,ir}$ are isolated. Then $\Re \phi_{m,ir}$ has only two nodal domains.

*Proof.* The first statement of Condition [7.3](#) follows from the definition of Maass–Hecke cusp form, and the second statement follow from the fact that $\phi_{m,ir} : X \rightarrow \mathbb{C}$ can not be scaled to a real-valued function, and that $\phi_{m,ir}$ is analytic. The third statement follows from the assumption.

Now, because the first Hecke eigenvalue is 1, the first Fourier coefficient of $\phi_{m,ir}$ at the cusp does not vanish, meaning that $i\infty$ is a regular zero of $\phi_{m,ir}$. We conclude the proof by applying Theorem [7.6](#). \qed

**Remark 8.4.** It is not hard to see that in the constant curvature case, the nodal set of $\phi_{2,ir}$ consists of the fibers over the critical point set $C_{\phi_{ir}}$ of $\phi_{ir}$. At this time, it does not seem to be known whether $C_{\phi_{ir}}$ is necessarily a discrete set of points in the case of hyperbolic surfaces. This cannot be proved by a purely local calculation, since the critical point set of rotationally invariant Dirichlet/Neumann eigenfunctions on a compact rotationally invariant submanifold $C_R$ of a hyperbolic cylinder $\mathbb{H}^2/\langle \gamma_0 \rangle$ consists of a union of $S^1$ orbits. Here, $\gamma_0$ is a hyperbolic element and $\langle \gamma_0 \rangle$ is the cyclic group it generates. Thus, negative curvature does not rule out codimension 1 critical point sets. One can put any negatively curved $S^1$ invariant metric on $C_R$ and obtain the same result, so it is not an effect of constant curvature. We conjecture that for compact hyperbolic surfaces without boundary, $C_{\phi_{ir}}$ is a finite set for every eigenfunction.

When we have holomorphicity of $\phi$, we may remove the assumption that the zeros of $\phi$ being isolated. For instance, we have:

**Theorem 8.5.** Let $X = SL_2(\mathbb{Z}) \backslash \mathbb{H}$, and let $\phi_{m,0}$ be a Laplacian eigenfunction on $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$ corresponding to a holomorphic Hecke eigenform $F$ of weight $m$. Then $\Re \phi_{m,0}$ has only two nodal domains.

*Proof.* We first note that $\phi_{m,0}(z, \theta) = y^{m/2} F(z) e^{-im\theta}$, and $F$ is holomorphic. Therefore Condition [7.5](#) is satisfied.

Because we assumed that $F$ is a Hecke eigenform, the first Hecke eigenvalue is 1. Therefore $i\infty$ is a regular zero of $\phi_{m,0}$, and now the theorem follows from Theorem [7.6](#). \qed

**Corollary 8.6.** There exist eigenfunctions on $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$ that have only two nodal domains but with arbitrarily large eigenvalues.

We remark here that Theorem [8.3](#) is false, without the assumption that $F$ is a Hecke eigenform. To construct a counter example, let $\Delta(z)$ be the discriminant modular form given by

$$
\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \ldots,
$$

where $q = e^{2\pi i z}$. This is a weight 12 modular form on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$. Thus $\Delta(z)^2$ is a modular form of weight 24 on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$, and

$$
\Phi = \Re (y^{12} \Delta(z)^2 e^{-24i\theta})
$$

is a Laplacian eigenfunction on $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$ of weight 24. To count the number of nodal domains of this eigenfunction, we let

$$
\mathcal{F} = \{ x + iy : |x| \leq \frac{1}{2}, x^2 + y^2 \geq 1 \} \subset \mathbb{H}
$$

be the fundamental domain of $SL_2(\mathbb{Z}) \backslash \mathbb{H}$, and let $M_0 = \mathcal{F} \times \{ \theta : 0 \leq \theta < 2\pi \}$.

We then consider the restrictions of $\Phi$ to the top $\theta = 2\pi$, side $x = -1/2$, and front $x^2 + y^2 = 1$ of the solid $M_0$.

It can be shown that the nodal set of $\Phi$ on the side is that of $\cos(24\theta) = 0$, and on the front is that of $\cos(12(\varphi + 2\theta)) = 0$, where we define $\varphi = \arccos(x)$. We compute the nodal set of the restriction to the top numerically using Mathematica.
The nodal set of $\Phi$ on the front, the side, and the top of the solid $M_0$.

Note that we may obtain $\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})$ from $M_0$ by gluing the sides via $(x, y, \theta) = (x + 1, y, \theta)$ (corresponding to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$), the top and the bottom via $(x, y, \theta) = (x, y, \theta + 2\pi)$ (corresponding to $k(\theta) = k(\theta + 2\pi)$), and then the front with itself via $(\varphi, \theta) = (\pi - \varphi, \theta + \varphi)$ and $(\varphi, \theta) = (\varphi, \theta + 2\pi)$ (corresponding to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $k(\theta) = k(\theta + 2\pi)$).

From these, one can verify that $\Phi$ has exactly four nodal domains, where in the pictures above, two positive nodal domains are colored differently with red and orange.

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