Conjugate operators to operators of stochastic integration

I.P. Smirnov
Institute of Applied Physics RAS
46 Ul'anova Street, Nizhny Novgorod, Russia

Abstract
The conjugate problem in stochastic optimal control can be formulated in terms of operators conjugated to the operators of stochastic integration [1–3]. In this paper we study some of such operators acting on the spaces of progressively measurable random functions.

Key words: optimal control, stochastic integrals, progressively measurable functions, conjugate operators

1 Introduction

1.1 Functional spaces

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \(\{\mathcal{F}_t, t \in [0,1]\}\) the right continuous stream of subfields of the field \(\mathcal{F}\). We use the following notations: \(\mathcal{B}^n\) is the \(\sigma\)-field of Borel sets on \(R^n\), mes is the Lebesgue measure on \(\mathcal{B}^n\), \(\mathcal{B}^1 \times \mathcal{F}\) and mes \(\times \mathbb{P}\) are the direct products of the fields \(\mathcal{B}^1\), \(\mathcal{F}\) and measures mes, \(\mathbb{P}\), respectively, \(\mathcal{B}\mathcal{F} \subset \mathcal{B}^1 \times \mathcal{F}\) is the \(\sigma\)-field of progressively measurable with respect to \(\{\mathcal{F}_t\}\) subsets of the product \([0,1] \times \Omega\).

Let us introduce the main functional spaces used below. For \(p \in [1, \infty)\) let denote by

1. \(L_p(\Omega)\) the Banach space of \(\mathcal{F}\)-measured random values \(\xi(\omega), \omega \in \Omega\) (i.e. classes of values equivalent to the each other with respect to probability measure \(\mathbb{P}\)) with the norm

\[ \|\xi\|_{L_p(\Omega)} = \{M |\xi|^p\}^{1/p} \]

\((M\) is the expectation under the measure \(\mathbb{P})\);

2. \(E_p\) the set of \(\mathcal{B}\mathcal{F}\)-measurable random functions \(f(t,\omega), t \in [0,1], \omega \in \Omega\) such that \(\mathbb{P}\text{-a.s.}\)

\[ \int_0^1 |f(t,\omega)|^p \, dt < \infty; \]
3. \( L_p \) the Banach space of \( \mathcal{BF} \)-measurable functions \( f(t, \omega) \) (classes of functions equivalent to the each other with respect to measure \( \text{mes} \times \mathcal{P} \)) with the norm
\[
\|f\|_p = \left\{ M \int_0^1 |f(t, \omega)|^p \, dt \right\}^{1/p};
\]

4. \( N_p \) the Banach space of \( \mathcal{BF} \)-measurable functions \( f(t, \omega) \) (classes of functions) with the norm
\[
\|f\|_{N_p} = \left\{ M \left( \int_0^1 |f(t, \omega)|^2 \, dt \right)^{p/2} \right\}^{1/p};
\]

5. \( H_p \) the Banach space of \( \mathcal{BF} \)-measurable functions \( f(t, \omega) \) (classes of functions) with the norm
\[
|f|_p = \left\{ \sup_{0 \leq t \leq 1} M |f(t, \omega)|^p \right\}^{1/p};
\]

6. \( DH_p \) the normed space of \( \mathcal{BF} \)-measurable functions \( f(t, \omega) \) (classes of \( \mathcal{P} \)-indistinguishable functions) having \( \mathcal{P} \)-a.s. trajectories of the class \( D \) with the norm
\[
|f|_{(d)}^p = \left\{ M \sup_{0 \leq t \leq 1} |f(t, \omega)|^p \right\}^{1/p};
\]

7. \( CH_p \) the subset of \( DH_p \) of all continuous by \( t \) a.s. functions \( f(t, \omega) \);

8. \( \Pi (\cdot) \) the \( \sigma \)-finite measure on \( \mathcal{B}^n \).

(a) \( E_p (\Pi) \) the set of all \( \mathcal{BF} \times \mathcal{B}^n \)-measurable functions \( \psi(t, \omega, y), t \in [0, 1], \omega \in \Omega, y \in R^l \) such that
\[
\int_0^1 \int |\psi(t, \omega, y)|^p \Pi(dy) < \infty \text{ a.s.}
\]

(b) \( L_p (\Pi) \) the Banach space \( \mathcal{BF} \times \mathcal{B}^n \)-measurable functions \( \psi(t, \omega, y) \) (classes of functions) with the norm
\[
\|\psi\|_{L_p(\Pi)} = \left\{ M \int_0^1 \left[ \int |\psi|^p \Pi(dy) + \left( \int |\psi|^2 \Pi(dy) \right)^{p/2} \right] \right\}^{1/p};
\]

---

1. A function \( \varphi : [0, 1] \to R^l \) belongs to the class \( D \) if it is right continuous and has finite lower limits everywhere on the segment \([0, 1]\).
We shall consider the embeddings of the type $DH_p \subset H_p \subset L_p$. In this case an element of a slender space is identified with the class of equivalent elements from the wider space. Conversely, if we state that an element of a wide space belongs to a slender space it means that the corresponding class of equivalent function includes an element from the slender space. Note that $DH_p$ and $H_p$ are everywhere dense in $L_p$.

Further, $L_p(\Omega)$, $L_p$, $N_p$, $L_p(\Pi)$ are the Banach ideal spaces of measurable functions with order-type continuous norms (see [4]). Let $\mathcal{H}$ denotes any of such space. As it was proved in [4] there exists a linear isomorphism between the conjugate space $\mathcal{H}^\ast$ and the set $X_{\mathcal{H}}$ of all measurable functions $x$ (classes of $\mu$-equivalent functions) for which

$$\int |x\varphi| \, d\mu < \infty \quad \forall \varphi \in \mathcal{H}.$$ 

This isomorphism has the form

$$X_{\mathcal{H}} \ni x \mapsto f(\cdot) = \int x \cdot d\mu \in \mathcal{H}^\ast.$$ 

Using this fact, we can identify $\mathcal{H}^\ast$ with $X_{\mathcal{H}}$. Particularly,

$$L_p^\ast \equiv X_{L_p} = L_p', \quad N_p^\ast \equiv X_{N_p} = N_p',$$

$$L_2^\ast(\Pi) \equiv X_{L_2(\Pi)} = L_2(\Pi),$$

where $p > 1$, $p' = p/(p - 1)$.

1.2 Operators

Let $w(t, \omega), \ t \in [0, 1]$ be a standard Wiener process concurrent with the stream $\{\tilde{\mathcal{G}}_t\}$, having for $s > t$ independent of $\tilde{\mathcal{G}}_t$ increments $w(s) - w(t)$; let $\nu(t, A)$ be a random Poisson measure on $R^{d+1}$ with the parameter $\Pi(A) \ t$ ($\Pi(\cdot)$ is a $\sigma$-finite measure on $R^d$) concurrent with $\{\tilde{\mathcal{G}}_t\}$, having for $s > t$ independent of $\tilde{\mathcal{G}}_t$ increments $\nu(s, A) - \nu(t, A)$; let $\tilde{\nu}(t, A) \equiv \nu(t, A) - \Pi(A) \ t$ be the centered Poisson random measure.

Let us consider the following linear operations:

$$\mathcal{L}(f)(t, \omega) \equiv \int_0^t f(s, \omega) \, ds, \ f \in E_1,$$

$$\mathcal{J}(\varphi)(t, \omega) \equiv \int_0^t \varphi(s, \omega) \, dw(s, \omega), \ \varphi \in E_2,$$

$$\mathcal{P}(\alpha)(t, \omega) \equiv \int_0^t \int \alpha(s, \omega, y) \, \tilde{\nu}(ds, dy), \ \alpha \in E_2(\Pi).$$

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Integrals in (1) are interpreted respectively as the Lebesgue integral over trajectories of random functions, the stochastic Ito integral and the stochastic integral with respect to Poisson measure. Below we study the operators generated by operations (1) on the above declared spaces of random functions.

1.3 Some preliminary facts

Let us review some facts from the theory of martingales and stochastic integrals [5]. Using the completeness of \((\Omega, \mathcal{F}, P)\), we shall consider only separable modifications of stochastic processes. Let \(\mu(t), t \in [0, 1]\) be a martingale with respect to the stream \(\{\mathcal{F}_t\}\). As the stream is right-continuous, then the process \(\mu(t)\) has a modification with its trajectories from \(D\). If \(M|\mu(t)|^p < \infty, p \in (1, \infty)\), then

\[
M \sup_{0 \leq t \leq 1} |\mu(t)|^p \leq (p')^p \sup_{0 \leq t \leq 1} M|\mu(t)|^p = (p')^p M|\mu(1)|^p. \tag{2}
\]

Let \(\mathcal{M}_2\) be the class of all martingales on \([0, 1]\) with \(M\mu(1) < \infty\); \(\mathcal{M}_2\) is a Hilbert space with the inner product

\[
(\mu_1, \mu_2) = M\mu_1(1)\mu_2(1).
\]

A closed subspaces in \(\mathcal{M}_2\) form the sets of martingales of the following type

\[
\mathcal{M}_2^w = \left\{ \int_0^t \varphi \, dw, \varphi \in L_2 \right\},
\]

\[
\mathcal{M}_2^\psi = \left\{ \int_0^t \int \psi \tilde{\nu} \, (ds, dy), \psi \in L_2(\Pi) \right\}.
\]

Let \(\varphi \in N_p, p \in (1, \infty), \psi \in L_2(\Pi)\). The following inequalities for the stochastic integrals are well known:

\[
A_p^p \|\varphi\|_{N_p}^p \leq M \left| \int_0^1 \varphi(s) \, dw(s) \right|^p \leq B_p^p \|\varphi\|_{N_p}^p, \tag{3}
\]

\[
M \left| \int_0^1 \int \psi(s, y) \tilde{\nu} \, (ds, dy) \right|^2 = M \int_0^1 \int \psi^2(s, y) \, ds\Pi(dy). \tag{4}
\]

Here \(A_p, B_p\) are the positive values depending only on \(p\), \(A_2 = B_2 = 1\). According to [6] \(A_p\) has the following property: \(\forall r > 2\) there exists \(r_1 < r\) such that \(\hat{A}_r \equiv \inf_{p \in (r_1, r)} A_p > 0\). The right one of the inequalities (3) holds for \(p \in (0, 1]\) also. Moreover, if \(\varphi \in N_1\), then the random function \(\mathcal{J}(\varphi)(t, \omega)\) is a martingale and \(M\mathcal{J}(\varphi)(t, \omega) = 0\).

\(^2\)As all other martingales in this paper.
Let $\mathfrak{F}^w_t = \sigma[w(s), s \leq t]$. Denote by $N_p(w)$ a subspace of $N_p$ of all random functions progressively measurable with respect to the stream $\{\mathfrak{F}^w_t\}$. If a random value $\xi(\omega)$ is $\mathfrak{F}^w_t$-measurable and belongs to $L^2(\Omega)$ then there exists a function $\lambda \in N_2(w)$ such that

$$\xi(\omega) = M\xi(\omega) + \int_0^1 \lambda(s, \omega) \, dw(s, \omega) \quad a.s.$$  \hspace{1cm} (5)

(Ito-Clark theorem). Using inequalities \[3\] we generalize formula \[5\] for a more wide class of random values.

**Theorem 1** Let a random value $\xi(\omega)$ is $F^w_t$-measurable and $M|\xi|^p < \infty$, where $p \in (1, \infty)$. Then there exit a function $\lambda \in N_p(\omega)$ such that the equality \[2\] is fulfilled. Moreover,

$$A_p \|\lambda\|_{N_p} \leq \|\xi - M\xi\|_p \leq B_p \|\lambda\|_{N_p}.$$  \hspace{1cm} (6)

In particular case, if a martingale $\mu(t)$ is concurrent with the stream $\{\mathfrak{F}^w_t\}$ and $M|\mu(1)|^p < \infty$, then $\mu(t)$ can be identically represented in the form

$$\mu(t) = M\mu(t) + \int_0^t \lambda(s, \omega) \, dw(s, \omega), \lambda \in N_p(w).$$  \hspace{1cm} (7)

**Proof.** Denote by $P(p)$ the statement claiming that the theorem holds for a given $p$. First prove the following two statements.

**Lemma 2** $P(p) \Rightarrow P(r) \forall r \in (1, p).$

**Proof.** Let us use the fact that $L_p(w)$ is everywhere dense in $L_r(w)$ for $p > r$. Let $\xi \in L_r(w)$, $\{\xi_n\} \subset L_p(w)$ and $\|\xi - \xi_n\|_r \to 0$ as $n \to \infty$. Also let $\{\lambda_n\} \subset N_p(w) \subset N_r(w)$ be a sequence of functions from \[5\] corresponding to the sequence $\{\xi_n\}$. According to \[4\]

$$A_r \|\lambda_n - \lambda_m\|_{N_r} \leq \|\xi_n - M\xi_n - \xi_m + M\xi_m\|_r \leq 2 \|\xi_n - \xi_m\|_r \to 0$$

as $n, m \to \infty$. We see that the sequence $\{\lambda_n\}$ is fundamental in $N_r$; therefore, it converges to some element $\lambda \in N_r(w)$. Let $\xi \equiv M\xi + J(\lambda)$ \[1\]. Using the right inequality in \[4\], we get

$$\|\xi - \xi\|_r \leq \|\xi - \xi_n\|_r + \|\xi_n - \xi\|_r,$$

$$\|\xi_n - \xi\|_r \leq \|M(\xi_n - \xi)\|_r + \left\| \int_0^1 (\lambda_n - \lambda) \, dw \right\|_r \leq$$

$$\leq \|\xi_n - \xi\|_r + B_r \|\lambda_n - \lambda\|_{N_r}.$$  

Taking limits in these inequalities as $n \to \infty$, we receive $\xi = \tilde{\xi}$ a.s. That proves Lemma 2. The uniqueness of the representation \[5\] can be easily received from the left of inequalities in \[4\].

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Lemma 3 Let \( r \geq 4 \) and for every \( \xi \in L_r(w) \) there exists a function \( \lambda \in N_{r-2}(w) \) such that (7) holds. Then \( P(r) \).

Proof. Let take the martingale \( \eta(t) = \mathcal{J}(\lambda)(t) \), \( \eta(1) = \xi \). Applying formula Ito, we obtain for all \( t \in [0,1] \)

\[
\eta^2(t) = \int_0^t \lambda^2 \, ds + 2 \int_0^t \eta \lambda \, dw, \quad (8)
\]

\[
\eta^3(t) = 6 \int_0^t \eta^2 \lambda \, ds + 4 \int_0^t \eta^3 \lambda \, dw \quad a.s. \quad (9)
\]

Prove that \( \lambda \in N_{p_n}, p_n = r - 2^{-n}, n = -1, 0, 1, 2, \ldots \) For \( n = -1 \) this statement follows from the assumption of Lemma 3. Let us prove that from realizability of Lemma 3 for \( n = k - 1, k \geq 0 \) the realizability of it for \( n = k \) follows. Then, using the induction on \( n \) we shall prove Lemma 3.

Using Hölder inequality and (2), we receive

\[
M \left| \int_0^t \eta^6 \lambda^2 \, ds \right|^{p_k/8} \leq M \sup_{0 \leq t \leq 1} |\eta(t)|^{3p_k/4} \left( \int_0^t \lambda^2 \, ds \right) \leq \left( \frac{d_k}{d_k - 1} \right)^{3p_k/4} \left( M |\xi|^{d_k} \right)^{3p_k/4d_k} \left( M \int_0^t \lambda^2 \, ds \right)^{p_k/4p_{k-1}} < \infty,
\]

\[
d_k = \frac{3p_k p_{k-1}}{4p_{k-1} - p_k} \leq r, \ k = 0, 1, 2 \ldots
\]

Thus, due to (8) and (9) we get

\[
M \left| \int_0^1 \eta^3 \lambda \, dw \right|^{p_k/4} \leq B_{p_k/4}^{p_k/4} M \left( \int_0^1 \eta^6 \lambda^2 \, ds \right)^{p_k/8}.
\]

\[
M \left( \int_0^1 \eta^2 \lambda^2 \, ds \right)^{p_k/4} \leq M \left[ \xi^4 - 4 \int_0^1 \eta^3 \lambda \, dw \right] < \infty.
\]

Next using (8) and (3), we get

\[
M \left| \int_0^1 \lambda^2 \, ds \right|^{p_k/2} \leq 2^{p_k/2 - 1} \left[ M |\xi|^{p_k} + 2^{p_k/2} B_{p_k/2}^{p_k/4} M \left( \int_0^1 \eta^2 \lambda^2 \, ds \right)^{p_k/4} \right] < \infty,
\]

that proves the statement. Therefore, \( \lambda \in N_{p_n} \) and according to (3)

\[
A_{p_n} \|\lambda\|_{N_{p_n}} \leq 2 \|\xi\|_{p_n} \leq 2 \|\xi\|_r.
\]
Taking in the left-hand side of the equality the limit as \( n \to \infty \) and using of lower continuity of function \( p \to \|\lambda\|_{N_p} \), we obtain
\[
\tilde{A}_r \|\lambda\|_{N_r} \leq 2 \|\xi\|_r,
\]
so, \( \lambda \in N_r \). This proves Lemma 3.

Using of the lemmas, we can now prove Theorem 1. Really, from Lemma 3 we have:
\[
P(p) \Rightarrow P(p+2) \quad \forall p \geq 4.
\]
But \( P(4) \) follows from Ito-Clark theorem (5). So, for all \( n = 2, 3, \ldots \), the statement \( P(2^n) \) holds. Using of Lemma 2, we receive \( P(r) \) for all \( r \in (1, 2^n) \) that is equivalent to the first statement of Theorem 1. For receiving of the inequalities (6) it is sufficiently to apply the inequalities (3) to (5). To get (7) it is sufficiently to represent \( \mu(1) \) in the form (5). Then
\[
\mu(t) = M \{ \mu(1) | \tilde{\beta}_t \} = M \mu(1) + M \left\{ \int_0^t \varphi(s) dw(s) \right\} =
\]
\[
= M \mu(1) + \int_0^t \varphi(s) dw(s).
\]
This concludes the proof of Theorem 1.

**Corollary 4** Let \( M|\xi|^\gamma < \infty \) and \( M|\xi|^{\gamma+\epsilon} = \infty \forall \epsilon > 0 \). Then in (3) the function \( \lambda \in N_r(w) \) but \( \lambda \notin N_{r+\epsilon}(w), \epsilon > 0 \).

**Corollary 5** Let \( \mu(t) = J(\lambda)(t) \), where \( \lambda \in E_2 \), the process \( \mu(t) \) is concurrent with the stream \( \{\tilde{\beta}_t\} \) and \( M \sup_{0 \leq t \leq 1} |\eta(t)|^p < \infty \) for some \( p \in (1, \infty) \). Then \( \mu(t) \) is a martingale and \( \|\lambda\|_{N_p} < \infty \).

**Lemma 6** Let \( \varphi, \kappa \in N_p, p > 1 \). Then
\[
M \int_0^t \varphi(s) dw(s) \int_0^t \kappa(s) dw(s) = M \int_0^t \kappa(s) \varphi(s) ds, \quad t \in [0, 1].
\]

**Proof.** Denote by
\[
\eta_1(t) = \int_0^t \varphi(s) dw(s),
\]
\[
\eta_2(t) = \int_0^t \kappa(s) dw(s).
\]
Using Ito’s formula we have

$$\eta_1(t) \eta_2(t) = \int_0^t \eta_1 \kappa \, dw + \int_0^t \eta_2 \varphi \, dw + \int_0^t \varphi \, ds.$$ 

Now we must only prove that

$$M \int_0^t \eta_1 \kappa \, dw = M \int_0^t \eta_2 \varphi \, dw = 0. \quad (10)$$

Applying to $\eta_1(t)$ the inequalities \([2], [3]\) we receive

$$M \sup_{0 \leq t \leq 1} |\eta_1(t)|^p \leq (p')^p M |\eta(1)|^p \leq (p')^p B_p^p \|\varphi\|^p_{N_p} < \infty.$$ 

Hence,

$$M \left( \int_0^1 |\eta_1 \kappa|^2 \, ds \right)^{1/2} \leq M \sup_{0 \leq t \leq 1} |\eta_1(t)| \left( \int_0^1 |\kappa|^2 \, ds \right)^{1/2} \leq$$

$$\leq \left( M \sup_{0 \leq t \leq 1} |\eta_1(t)|^p \right)^{1/p} \|\kappa\|_{N_{p'}} < \infty,$$

so, $\eta_1 \kappa \in N_1$. Analogously, $\eta_2 \varphi \in N_1$. This proves (10) and the lemma.

2 Operator $\mathcal{L}$

2.1 Definition of operator $\mathcal{L}$

Let $f \in L_1$. Random functions $\mathcal{L} \left( \tilde{f} \right) (t, \omega)$ corresponding to different members $\tilde{f}$ of the class $f$ are continuous a.s. and indistinguishable from the each other. So, we can define the operator $\mathcal{L} : L_p \to CH_p$.

Lemma 7 The linear operator $\mathcal{L} : L_p \to CH_p$, $p \in [1, \infty)$ is bounded.

Proof. Using of Hölder inequality, we get the inequality

$$\sup_{0 \leq t \leq 1} \left| \int_0^t f(s) \, ds \right|^p \leq \int_0^1 |f(s)|^p \, ds.$$ 

By averaging of it by the measure $\mathbb{P}$ we then receive

$$|\mathcal{L}(f)|^p_p \leq \|f\|_p$$

that completes the proof.
2.2 Conjugate operator $\mathcal{L}^*$

Denote by $\mathcal{L}^* : (CH_p)^* \to L_p^*$ conjugate to the operator $\mathcal{L}$. We'll obtain the explicit representation for the restriction of $\mathcal{L}^*$ on a subspace $L_{p'} \equiv (L_p)^*$ of the space $(CH_p)^*$.

**Theorem 8** Let $f \in L_{p'}$. Then there exist selections of conditional expectations such that

$$\mathcal{L}^* (f) (t, \omega) = M \left\{ \int_0^1 f (s, \omega) \, ds \right\} \bar{F}_t .$$

More precisely: all $BF$-measurable functions from the right-hand side of (11) belong to the class $L_{p'} (f)$.

**Proof.** In accordance to definition of the conjugate operator $\mathcal{L}^* (f)$ is an element $h$ of $L_{p'}$ such that for all $\phi \in L_p$

$$\langle f, \mathcal{L} (\phi) \rangle = \langle h, \phi \rangle = M \int_0^1 h (s) \phi (s) \, ds .$$

As the random function $\int_0^1 f (s, \omega) \, ds$ is $\mathcal{B}^1 \times \mathcal{F}$-measurable (Fubini theorem), then (see [7]) there exist selections of conditional expectations such that the random function $\tilde{h} (t)$ from right-hand side of (11) is $\mathcal{B} \mathcal{F}$-measurable. It is easy to prove that $\tilde{h} \in L_{p'}$. Using Fubini theorem we get for any $\phi \in L_p$

$$M \int_0^1 \tilde{h} (s) \phi (s) \, ds = M \int_0^1 \int_0^1 f (t, \omega) \, dt \varphi (s) \, ds =
$$

$$= M \int_0^1 \int_0^1 f (t, \omega) \, dt \varphi (s) \, ds = \langle f, \mathcal{L} (\varphi) \rangle .$$

This completes the proof of the theorem.

**Corollary 9** The restriction of the operator $\mathcal{L}^*$ on $L_q, q \in (1, \infty)$ is bounded as an operator from $L_q$ to $DH_q$.

**Proof.** Using (11), we get for any $f \in L_q$

$$\mathcal{L}^* (f) (t, \omega) = M \left\{ \int_0^1 f (s, \omega) \, ds \right\} \bar{F}_t - \int_0^t f (s, \omega) \, ds \equiv \mu (t, \omega) - \mathcal{L} (f) (t, \omega) .$$
As a martingale the process \( \mu(t) \) has a modification with trajectories from \( D \).

Using (2), we get

\[
M \sup_{0 \leq t \leq 1} |\mu(t)|^q \leq (q')^q M |\mu(1)|^q \leq (q')^q \|f\|^q.
\]

Taking into account the inequalities and the properties of operator \( L \) also we obtain the corollary from (12) ■

Corollary 10 Suppose that the stream \( \{F_t\} \) is continuous in \([0, 1]\). Then the restriction of \( L^\ast \) on \( L_q, q \in (1, \infty) \) is bounded as an operator from \( L_q \) to \( CH_q \).

Proof. As in this case the martingale \( \mu(t) \) has a continuous modification, then the corollary leads immediately from (12) ■

3 Operator \( J \)

3.1 Definition of \( J \)

Let \( \varphi \in N_1 \). We consider only continuous modifications of Ito integral. In this case random functions \( J(\hat{\varphi})(t, \omega) \) corresponding to different members \( \hat{\varphi} \) of the class \( \varphi \) are indistinguishable from the each other. So, we can define the operator \( J : N_p \rightarrow CH_p \).

Lemma 11 The linear operator \( J : N_p \rightarrow CH_p, p \in [2, \infty) \) is bounded.

Proof. The random function \( J(\varphi)(t, \omega) \) is a martingale for any \( \varphi \in N_p \). Using (2) and (3), we get

\[
M \sup_{0 \leq t \leq 1} |J(\varphi)(t, \omega)|^p \leq (p')^p M |J(\varphi)(1, \omega)|^p \leq (p')^p B_p \|\varphi\|_{N_p}^p.
\]

This inequality can be rewrite in the form

\[
|J(\varphi)(t, \omega)|_{(d)} \leq p'B_p \|\varphi\|_{N_p}
\]

that proves the lemma ■

Corollary 12 The operator \( J : L_p \rightarrow CH_p, p \in [2, \infty) \) is bounded.

3.2 Operator \( \tilde{J} \)

Let \( J^\ast : (CH_p)^\ast \rightarrow N_p^\ast \) be conjugate to the operator \( J \). Let obtain the explicit representation for the restriction of \( J^\ast \) to a subspace \( L_{p'} \equiv (L_p)^\ast \subset (CH_p)^\ast \). For this purpose let firstly perform some auxiliary constructions.

Let \( N_r(\mathcal{B} \times \mathcal{F}), r \in (1, \infty) \) be the Banach space of \( \mathcal{B} \times \mathcal{F} \)-measurable functions \( \lambda(t, s, \omega), 0 \leq t \leq 1, 0 \leq s \leq t, \omega \in \Omega \) (classes of equivalent functions) with the norm

\[
|||\lambda|||_r = \left\{ M \int_0^1 \left( \int_0^1 \lambda^2(t, s, \omega) \right)^{r/2} \right\}^{1/r}.
\]
Consider a linear manifold $T_r$ in $N_r (B \times BF)$ consisting of functions of the type
\[
\lambda (t, s, \omega) = \sum_{k=0}^{n-1} \chi_{\Delta_k} (t) \alpha_k(s, \omega), \ t \in [0, 1], \ s \in [0, t], \ \omega \in \Omega, \tag{13}
\]
where $\Delta_k \equiv [t_k, t_{k-1})$, $0 = t_0 < t_1 < \ldots < t_n = 1$, functions $\alpha_k(s, \omega)$ are $BF$-measurable, $\alpha_0 \equiv 0$, $\alpha_k(s, \omega) = 0$ for $s \in (t_k, 1]$, $k \geq 1$,
\[
M \left( \int_0^{t_k} \alpha_k^2 (s, \omega) \, ds \right)^{r/2} < \infty,
\]
$\chi_A(t)$ is the indicator of a set $A$.

Set for $\lambda \in T_r$
\[
\tilde{J}(\lambda)(t, \omega) \equiv \sum_{k=0}^{n-1} \chi_{\Delta_k} (t) \int_0^{t_k} \alpha_k(s, \omega) \, dw(s, \omega).
\]

It is easy to prove that the operator $\tilde{J}$ is linear on the manifold $T_r$. Applying inequalities (14), we then obtain the inequalities
\[
A_r |||\lambda|||_r \leq |||\tilde{J}(\lambda)|||_r \leq B_r |||\lambda|||_r. \tag{14}
\]

Let $\tilde{T}_r$ be the closure of $T_r$ in $N_r (B \times BF)$. Using of the right inequality in (14), we can perform the expansion of operator $\tilde{J}$ on $\tilde{T}_r$ by continuity; this expansion also keep the inequalities (14).

**Lemma 13** The set $L_r (w) \equiv \{ \tilde{J}(\lambda), \ \lambda \in \tilde{T}_r \}$ is closed in $L_r$.

**Proof.** Let $\{ \xi_n = \tilde{J}(\lambda_n), \ \lambda_n \in \tilde{T}_r \} \subset L_r (w)$ and $||\xi_n - \xi_m||_r \to 0$ as $n, m \to \infty$. Using the left inequality in (14) we obtain
\[
A_r |||\lambda_n - \lambda_m|||_r \leq |||\tilde{J}(\lambda)|||_r \to 0, \ n, m \to \infty.
\]
The sequence $\{\lambda_n\}$ is fundamental in $N_r (B \times BF)$, so it converges to some element $\lambda \in N_r (B \times BF)$. As $\tilde{T}_r$ is closed, then $\lambda \in \tilde{T}_r$. Let $\xi = \tilde{J}(\lambda)$. From the right inequality in (14) we get
\[
||\xi_n - \xi||_r \leq B_r |||\lambda_n - \lambda|||_r \to 0, \ n \to \infty,
\]
that proves the lemma $\blacksquare$.

Let study the operator $\tilde{J}$.

**Lemma 14** Let $\lambda \in \tilde{T}_r$, $\eta = \tilde{J}(\lambda)$. Then
a) $M\eta(t) = 0$ for almost all $t \in [0, 1]$ ;
b) \( \forall t \in [0,1] \) almost all \( s \in [t,1] \) a.s.

\[
M \{ \eta(s) \mid F_t \} = \int_0^t \lambda(t,s) \, dw(s,\omega);
\]

c) if \( \varphi \in N_\varphi \), then for almost all \( t \in [0,1] \)

\[
M \eta(t) \mathcal{J}(\varphi)(t) = M \int_0^t \lambda(t,s) \varphi(s) \, ds;
\]

d) if \( r = 2 \) and \( \tau \in (0,1] \), then

\[
\int_0^\tau \eta(s) \, ds = \int_0^\tau \left( \int_0^t \lambda(t,s) \, ds \right) \, dw(t,\omega) \quad a.s.
\]

**Proof.** Since \( \lambda \in N_r (B \times BF) \), it follows that \( \lambda(t,\cdot) \in N_r ([0,t] \times \Omega) \) for almost all \( t \in [0,1] \). So, we can define the value

\[
\xi_t(\omega) = \int_0^t \lambda(t,s) \, dw(s,\omega),
\]

where the integral in the right-hand side is of Ito type. Let us show that for almost all \( t \)

\[
\xi_t(\omega) = \eta(t,\omega) \quad a.s.
\] (15)

Indeed, we have \( \eta = \lim_{n \to \infty} (L_r) \tilde{\mathcal{J}}(\lambda_n) \), where \( \{\lambda_n\} \subset T_r \). Thus, for almost all \( t \) and for some sequence \( n_k \to \infty \)

\[
M |\eta_{n_k}(t) - \eta(t)|^r \to 0, \eta_k = \tilde{\mathcal{J}}(\lambda_k).
\]

In other hand, for some subsequence of the sequence \( n_k \)

\[
M \left| \eta_{nk_m}(t) - \xi(t) \right|^r \leq B_r^r M \left( \int_0^t |\lambda(t,s) - \lambda_{nk_m}(t,s)|^2 \, ds \right)^{r/2} \to 0, \ m \to \infty
\]

for almost all \( t \). Hence, for almost all \( t \in [0,1] \) we have \( M |\eta(t) - \xi(t)|^r = 0 \) that implies (15). Using this result, the Lemma (14) and Ito’s integral properties we obtain items (a)-(c) of the lemma.

Let prove the last item. Denote by \( \alpha \) and \( \beta \), respectively, random values in right-hand and left-hand sides in item (d). Let also

\[
\alpha_n \equiv \int_0^\tau \eta_n(s) \, ds, \ \beta_n \equiv \int_0^\tau \left( \int_0^t \lambda_n(s,t) \, ds \right) \, dw(t).
\]
Using the construction of $\lambda_n \in T$, it is easy to verify that
\[
\alpha_n = \sum_{k=0}^{n-1} \text{mes} (\Delta_k \cap [0, \tau]) \int_0^{t_k} \alpha_k(s) \, dw(s) = \beta_n \text{ a.s.}
\]
Further,
\[
M |\alpha_n - \alpha|^2 \leq \|\eta_n - \eta\|^2_2 \to 0,
\]
\[
M |\beta_n - \beta|^2 = M \int_0^1 \left| \int_0^t (\lambda_n(s,t) - \lambda(s,t)) \, ds \right| \, dt \leq
\]
\[
\leq M \int_0^1 \int_0^t |\lambda_n(s,t) - \lambda(s,t)|^2 \, ds \, dt \equiv \|\lambda_n - \lambda\|^2_2 \to 0, \quad n \to \infty.
\]
So, we have $M |\alpha - \beta|^2 = 0$ that completes the proof. $\blacksquare$

**Lemma 15** For every $\varphi \in L_r(\omega)$, $r \in (1, 2]$ there exist an element $\lambda \in \bar{T}_r$ such that $\text{mes} \times P$-a.s.
\[
\varphi(t,\omega) = M\varphi(t,\omega) + J(\lambda)(t,\omega).
\]
*Linear operator $\mathcal{K} : \lambda = \mathcal{K}\varphi$ is bounded from $L_r(\omega)$ to $N_r(\mathcal{B} \times \mathcal{B} \mathcal{F})$.*

**Proof.** Let $S_r$ is the set of step-functions in $L_r(\omega)$, i.e., the set of all functions of the type
\[
\psi(t,\omega) = \sum_{k=0}^{n-1} \chi_{\Delta_k}(t) \psi_k(\omega), \psi_k \in L_r(\Omega),
\]
where the random values $\psi_k(\omega), k \geq 0$ are measurable with respect to $\sigma$-algebra $\mathcal{F}_{t_k}^\omega$. It is well known that $\bar{S}_r = L_r(\omega)$ \cite{13}. Let prove that any $\psi \in S_r$ can be represented in the form \cite{16} with some $\lambda \in \bar{T}_r$. In fact, according to Theorem \cite{14} the random function $\psi_k(\omega)$ can be represented in the form
\[
\psi_k(\omega) = M\psi_k(\omega) + \int_0^{t_k} \alpha_k(s,\omega) \, dw(s,\omega),
\]
where $M\left(\int_0^{t_k} \alpha_k^2(s) \, ds\right)^{r/2}$. Extending functions $\alpha_k(s,\omega)$ by zero to $[0, 1] \times \Omega$ we obtain
\[
\psi(t,\omega) = M\psi(t,\omega) + \int_0^t \alpha_k(s,\omega) \, dw(s,\omega),
\]
where the function $\lambda$ has the form \cite{13}. Further, let $\varphi$ be an arbitrary function from $L_r(\omega)$ and $\|\varphi - \psi_n\|_r \to 0, n \to \infty$, where $\{\psi_n\} \subset S_r$. Let $\lambda_n = \mathcal{K}\psi_n$. Applying the left-hand bound in \cite{14} and taking into account that the sequence $\{\psi_n\}$ is fundamental we receive
\[
A_r \|\lambda_n - \lambda_m\|_r \leq \|J(\lambda_n - \lambda_m)\|_r \leq 2\|\psi_n - \psi_m\|_r \to 0
\]
as $n, m \to \infty$. Hence, $\{\lambda_n\}$ is a fundamental sequence in $N_r (B \times BF)$, so, it converges to some $\lambda \in \overline{T}_r$. Let $\varphi_1 = M\varphi + \tilde{J}(\lambda)$. We get

$$
\|\varphi_1 - \varphi\|_r \leq \|\varphi - \psi_n\|_r + \|\varphi_1 - \psi_n\|_r \leq \\
\leq \|\varphi - \psi_n\|_r + \|M(\varphi - \psi_n)\|_r + \\
+ \|\tilde{J}(\lambda_n - \lambda)\|_r \leq 2\|\varphi - \psi_n\|_r + B_r \|\lambda_n - \lambda\|_r \to 0, \ n \to \infty
$$

that proves (16). Applying to (16) the left-hand bound in (14) we obtain

$$
A_r \|\lambda\|_r \leq \|\tilde{J}(\lambda)\|_r = \|\varphi - M\varphi\|_r \leq 2\|\varphi\|_r.
$$

This shows the uniqueness of the representation (16) and the boundedness of operator $K$. This completes the proof. 

### 3.3 Conjugate operator $J^*$

Let study the structure of operator $J^*$. Let $L_2^\perp (w)$ be the orthogonal complement of $L_2 (w)$ to Hilbert space $L_2$: $L_2 = L_2 (w) \oplus L_2^\perp (w)$.

**Theorem 16** The restrictions of the operator $J^* : (CH_p)^* \to N_p^*, \ p \in [2, \infty)$ on the subspace $L_2 \subset (CH_p)^*$ and also on the subspace $L_{p'} (w) \subset (CH_p)^*$ have the form

$$
J^* (\chi) (t, \omega) = \int_0^1 \lambda (s, t, \omega) \ ds \tag{17}
$$

where $\chi = \chi_1 + \tilde{J}(\lambda), \chi_1 \in L_2^\perp (w), \lambda \in \overline{T}_2, \lambda = K\chi$.

**Proof.** Let first of all prove the theorem for $\chi \in L_{p'} (w), p > 2$. Let $\varphi \in N_p$. Expanding the function $\chi$ into the sum (16) and taking into account item (b) of Lemma 16 we receive on Fubini theorem

$$
\langle \chi, J^* (\varphi) \rangle = M \int_0^1 \left( M\chi (t) + \tilde{J}(\lambda) (t) \right) J^* (\varphi) (t) \ dt = \\
= \int_0^1 M \int_0^t \lambda (t, s) \varphi (s) \ ds \ dt = M \int_0^1 \left( \int_0^1 \lambda (t, s) \ ds \right) \varphi (t) \ dt = \langle J^* \chi, \varphi \rangle.
$$

As $\varphi$ is an arbitrary function, then this completes the proof of theorem for $p > 2$. Case $\chi \in L_2$ can be studied analogously. Note, that if $\chi \in L_2 (w) \subset L_2 \cap L_{p'} (w)$, then in accordance to (16) we have $\chi_1 = M\chi \in L_2^\perp (w)$.

**Corollary 17** The restrictions of the operators listed in the theorem are bounded from $L_2$ to $L_2$ and from $L_q (w)$ to $L_q (w), q \in [p', 2]$ respectively.
Proof. Really, we get from formula (17)
\[
\| J^* \chi \|_q^q = M \int_0^1 \left( \int_0^1 \lambda (s, t) \, ds \right)^q \, dt \leq M \int_0^1 \int_0^1 | \lambda (s, t) |^q \, ds \, dt = M \int_0^1 \int_0^1 | \lambda (s, t) |^q \, ds \, dt \leq \| | |^q \|_q \leq (2A_q)^{-q} \| \chi \|_q^q.
\]

The first statement of the corollary can be proved analogously \( \blacksquare \)

4 Operator \( \mathcal{P} \)

4.1 Definition of \( \mathcal{P} \)

Let \( \psi \in L_2 (\Pi) \). Since the random functions \( \mathcal{P} \tilde{\psi} (t, \omega) \) corresponding to different members \( \tilde{\psi} \) of the class \( \psi \) a.s. have trajectories from \( D \) and are indistinguishable from the each other, then it is possible to define operators \( \mathcal{P} : L_p (\Pi) \rightarrow DH_p, \ p \geq 2 \).

Lemma 18 Linear operator \( \mathcal{P} : L_p (\Pi) \rightarrow DH_p, \ p \geq 2 \) (\( p \) is an even number) is bounded.

Proof. For \( p = 2 \) the lemma follows from [I]. Let \( p > 2 \). Prove the boundedness of operator \( \mathcal{P} : L_p (\Pi) \rightarrow DH_p \). Let use the estimations from [8, p. 150]. For \( \alpha \in L_p (\Pi) \)

\[
M | \mathcal{P} (\alpha) (t) |^p \leq K_1 (p) M \int_0^t \sum_{k=2}^p v_k^{p/k} (s) \, ds,
\]

\[
v_k (s) \equiv \int | \alpha (s, y) |^k \Pi (dy).
\]

Using Hölder inequality and binomial formula of Newton we get

\[
M | \mathcal{P} (\alpha) (t) |^p \leq K_2 (p) M \sum_{k=2}^p \left( \int_0^t v_2^{p/2} (s) \, ds \right)^{2(p-k)} \left( \int_0^t v_p (s) \, ds \right)^{2k/p} \leq \]

\[
\leq K_2 (p) M \sum_{k=2}^p \left[ \left( \int_0^t v_2^{p/2} (s) \, ds \right)^{2(p-k)} + \left( \int_0^t v_p (s) \, ds \right)^{2k/p} \right] \leq \]

\[
\leq K_2 (p) \sum_{k=2}^p 2^{(p-2)k-1} M \int_0^t \left[ v_2^{p/2} (s) + v_p (s) \right] \, ds \leq K_p^p \| \alpha \|_{L_p (\Pi)}^p.
\]
Hence,

$$|\mathcal{P}(\alpha)(t)|_p \leq K_p \|\alpha\|_{L^p(\Pi)}$$

where $K_1(p), K_2(p), K_p$ are functions of $p$. Because $\mathcal{P}(\alpha)(t)$ is martingale, then according to (2)

$$|\mathcal{P}(\alpha)|^{(d)}_p \leq p' |\mathcal{P}(\alpha)|_p \leq p' K_p \|\alpha\|_{L^p(\Pi)},$$

that completes the proof. ■

### 4.2 Operator $\tilde{P}$

To study the structure of operator $\mathcal{P}^*$ let define in analogous to $N_2(B \times BF \times B')$ the Hilbert space $N_2(B \times BF \times B^l)$ of $B \times BF \times B^l$-measurable functions $\mu(t, s, \omega, y)$, $t \in [0, 1]$, $s \in [0, t]$, $\omega \in \Omega$, $y \in R^l$ (classes of equivalent functions with respect to measure $\text{mes} \times \text{mes} \times \mathcal{P} \times \Pi$) with the norm

$$|||\mu|||_\Pi = \left\{ \frac{1}{2} \int_0^1 \int_0^t \int_0^{\mu_2(t, s, \omega, y)} \Pi(dy) ds dt \right\}^{1/2}.$$

In analogous to $T_r$ and $\tilde{J}$ define the lineal $T_2(\nu) \subset N_2(B \times BF \times B^l)$ and the operator $\tilde{P}$ on it; then extend this operator to the closer $T_2(\nu)$ by the continuity. Instead (14) we use now the equality

$$|||\tilde{P}(\mu)|||_2 = |||\mu|||_\Pi, \mu \in T_2(\nu).$$

Proofs of the next sentences are similar to the proofs of lemmas [13] and [14]

**Lemma 19** Let $\mu \in \overline{T_2(\nu)}$, $\eta(t) = \tilde{P}(\psi)(t)$. Then

a) $M\eta(t) = 0$ for almost all $t \in [0, 1]$;

b) $\forall t \in [0, 1]$ and almost all $s \in [t, 1]$ a.s.

$$M\{\eta(s)|\mathcal{F}_t\} = \int_0^t \mu(s, \tau, \omega, y) \tilde{\nu}(d\tau, dy);$$

c) if $\alpha \in L^2(\Pi)$, then for almost all $t$

$$M\eta(t) \mathcal{P}(\alpha)(t) = M \int_0^t \mu(t, \tau, \omega, y) \alpha(\tau, \omega, y) \Pi(dy) d\tau;$$

d) $\forall \tau \in (0, 1]$ a.s.

$$\int_0^\tau \eta(s) ds = \int_0^\tau \left( \int_0^\tau \mu(s, t, \omega, y) ds \right) \tilde{\nu}(dt, dy).$$

**Lemma 20** The set $L_2(\nu) \equiv \{\tilde{P}(\mu), \mu \in \overline{T_2(\nu)}\}$ is a closed subset of $L_2$. 

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4.3 Conjugate operator \( P^* \)

Let \( L_2^\perp (\nu) \) is the orthogonal complement of \( L_2 (\nu) \) to Hilbert space \( L_2 = L_2 (\nu) \oplus L_2^\perp (\nu) \). Using lemmas 19, 20, we can prove the following theorem.

**Theorem 21**  The restriction of operator \( P^* : (DH_p)^* \to (L_p (\Pi))^* \), \( p \geq 2 \) (\( p \) is an even number) to the space \( L_2 \subset (DH_p)^* \) is bounded from \( L_2 \) to \( L_2 (\Pi) \); it can be represented in the form

\[
P^* (\chi) (t, \omega, y) = \int_0^t \mu (s, t, \omega, y) \, ds,
\]

where \( \chi = \chi_1 + \tilde{\Psi} (\mu), \chi_1 \in L_2^\perp (\nu), \mu \in T_2 (\nu) \).

5 Some relations between operators \( \mathcal{L} \), \( \mathcal{J} \) and \( \mathcal{P} \)

Hilbert spaces \( \mathcal{M} \) and \( L_2 \) can be expand to the direct sums of mutually orthogonal closed subspaces:

\[
\mathcal{M} = \mathcal{M}_2 \oplus \mathcal{M}_2' \oplus \mathcal{M}_2^\perp,
\]

\[
L_2 = L_2 (w) \oplus L_2^\perp (w) = L_2 (\nu) \oplus L_2^\perp (\nu) = L_2 (w) \oplus L_2 (\nu) \oplus L_2^\perp.
\]

**Lemma 22**  Let \( \chi \in L_2, \theta \equiv \mathcal{J}^* (\chi), \zeta \equiv \mathcal{J}^* (\chi), \alpha \equiv \mathcal{P}^* (\chi) \). Then there exist a martingale \( h \in \mathcal{M}_2^\perp \) such that

\[
\theta (t) = - \int_0^t \chi (s) \, ds + \int_0^t \zeta (s) \, dw (s) + \int_0^t \alpha (s, y) \, d\hat{\nu} (ds, dy) + h (t), \quad t \in [0, 1].
\]

**Proof.** Using formula (12) we expand martingale \( \mu \in \mathcal{M}_2 \) to the sum

\[
\mu (t) = \mathcal{J} (\zeta_1) (t) + \mathcal{P} (\alpha_1) (t) + h (t),
\]

where \( \zeta_1 \in L_2, \alpha_1 \in L_2 (\Pi), h \in \mathcal{M}_2^\perp \). Let show that \( \zeta_1 = \zeta, \alpha_1 = \alpha \). Using Fubini theorem we get for any \( \varphi \in L_2 \)

\[
\langle \zeta_1, \varphi \rangle = M \int_0^1 \zeta_1 \varphi \, ds = M \int_0^1 \zeta_1 \, dw \int_0^1 \varphi \, dw = M \mu (1) \int_0^1 \varphi \, dw =
\]

\[
= M \int_0^1 \chi \, ds \int_0^1 \varphi \, dw = \int_0^1 MM \left\{ \chi (s) \int_0^1 \varphi \, dw \right| F_s \right\} ds =
\]

\[
= M \int_0^1 \chi (s) \int_0^1 \varphi \, dw \, ds = \langle \chi, \mathcal{J} (\varphi) \rangle.
\]
This implies $\varkappa_1 = J^*(\chi) = \varkappa$. The equality $\alpha_1 = \alpha$ can be established analogously. 

**Lemma 23** Let $\mathcal{F}_t \equiv \mathcal{F}_t^w$, $\chi \in L_r$, $r \in (1, 2]$, $\theta = \mathcal{L}^*(\chi)$, $\varkappa = J^*(\chi)$. Then for every $t \in [0, 1]$

$$\theta(t) = -\int_0^t \chi(s) \, ds + \int_0^t \varkappa(s) \, dw(s).$$

**Proof.** According to Theorem 1 the martingale $\mu(t)$ can be represented in the form

$$\mu(t) = M\mu(t) + \int_0^t \varkappa_1(s) \, dw(s),$$

where $\varkappa_1 \in N_r$. The equality $\varkappa_1 = \varkappa$ can be established similar to preceding lemma (using also Lemma 6). 

**Lemma 24** Let $\chi \in L_2$, $\theta = \mathcal{L}^*(\chi)$. Expand $\theta \in L_2$ to the sum

$$\theta = J^*(\lambda) + \tilde{P}(\mu) + \varkappa_1,$$

where $\lambda \in \mathcal{T}_2$, $\mu \in \mathcal{T}_2(\nu)$, $\varkappa_1 \in L_2^\perp$. Then functions $\lambda(t, \tau, \omega)$, $\mu(t, \tau, \omega, y)$ are continuous for $t \geq \tau$ and a.s.

$$J^*(\chi)(t, \omega) = \lambda(t, t, \omega),$$

$$\mathcal{P}^*(\chi)(t, \omega, y) = \mu(t, t, \omega, y).$$

**Proof.** Note that $\mathcal{T}_2 = N_2(\mathcal{B} \times \mathcal{B}F)$. According to theorem 4 we have $\chi = \chi_1 + \tilde{J}(\lambda_1)$, where $\lambda_1 \in \mathcal{T}_2$, $\chi_1 \in L_2^\perp(w)$. Thus, $\theta = \theta_2 + \mathcal{L}^*(\tilde{J}(\lambda_1), \theta_2 = \mathcal{L}^*(\chi_1)$. According to (12)

$$\mathcal{L}^* \tilde{J}(\lambda_1)(t) = M \left\{ \int_0^1 \tilde{J}(\lambda_1) ds \right| \mathcal{F}_t \right\} - \int_0^t \tilde{J}(\lambda_1)(s) \, ds. $$

Transform the right-hand side of this equality

$$M \left\{ \int_0^1 \left( \int \lambda_1(s, \tau) \, ds \right) \right| \mathcal{F}_t \right\} - \int_0^t \left( \int \lambda_1(s, \tau) \, ds \right) \, dw(\tau) =$$

$$= \int_0^1 \left( \int \lambda_1(s, \tau) \, ds \right) \, dw(\tau) - \int_0^t \left( \int \lambda_1(s, \tau) \, ds \right) \, dw(\tau) =$$

$$= \int_0^t \left( \int \lambda_1(s, \tau) \, ds \right) \, dw(\tau), \, t \in [0, 1]$$

(see item (d) of Lemma 14). It is easy to prove that function

$$\lambda_2(t, \tau, \omega) \equiv \int_0^1 \lambda_1(s, \tau, \omega) \, ds \in N_2(\mathcal{B} \times \mathcal{B}F) = \mathcal{T}_2.$$
Thus, as it was mentioned in the proof of lemma 6,

\[ \mathcal{L}^* \mathcal{J} (\lambda_1) (t) = \int_0^t \lambda_2 (t, \tau, \omega) \, dw (\tau) = \mathcal{J} (\lambda_2) (t) \]

for almost all \( t \) a.s. Let us next show that \( \theta_2 \in L^2_\perp (w) \). Indeed, if \( \phi = \mathcal{J} (\psi) \in L_2 (w) \) then for almost all \( t \) a.s.

\[ \mathcal{L} (\phi) (t) = \int_0^t \left( \int_\tau^t \psi (s, \tau, \omega) \, ds \right) \, dw (\tau) = \mathcal{J} (\mu_1) (t) \in L_2 (w), \]

\[ \mu_1 (t, \tau, \omega) = \int_\tau^t \psi (s, \tau, \omega) \, ds \in N_2 (\mathcal{B} \times \mathcal{B} \mathcal{F}). \]

Since \( \chi_1 \in L^2_\perp (w) \), it follows

\[ \langle \phi, \theta_2 \rangle = \langle \phi, \mathcal{L}^* (\chi_1) \rangle = \langle \mathcal{L} (\phi), \chi_1 \rangle = 0 \]

that prove our statement. Thus, we prove that

\[ \theta = \mathcal{J} (\lambda_2) + \theta_2, \quad \theta_2 \in L^2_\perp (w). \]

Comparing this formula with the initial we receive

\[ \lambda (t, \tau, \omega) = \lambda_2 (t, \tau, \omega) = \int_1^t \lambda_1 (s, \tau, \omega) \, ds. \]

But in the other hand according to the theorem 2

\[ \mathcal{J}^* (\chi) (t, \omega) = \int_1^t \lambda_1 (s, t, \omega) \, ds. \]

Combining these two equalities, we obtain the first statement of the lemma. The second statement can be obtained analogously \( \blacksquare \)

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