Interaction of a Point Charge and a Magnet: Comments on
"Hidden Mechanical Momentum Due to Hidden
Nonelectromagnetic Forces"

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Abstract

The interaction of a point charge and a magnetic moment (and by extension a point charge and a solenoid) is explored within well-defined point-charge magnetic-moment models where full calculations are possible. It is shown explicitly how the "hidden mechanical momentum" is introduced by the "hidden" external forces of constraint, requiring a prescribed response (through order $1/c^2$) of the system to electromagnetic forces. These external forces often go unmentioned in the textbook and research literature. The dependence of "hidden mechanical momentum" upon detailed external (nonelectromagnetic) forces may undermine the idea’s usefulness in describing nature. Some statements of dubious validity in the textbook literature are noted.
I. INTRODUCTION

Although the interaction of a point charge and magnet has been discussed repeatedly for decades with some excellent analyses, a convincing account of the interaction remains elusive, particularly when there is relative motion of the objects. Furthermore, some of the ideas based upon the assumption of stationary behavior have been incorporated into the research and textbook literature with statements of dubious validity. We explore this problem yet again. We consider two familiar (point-charge) models for a magnetic moment which allow full calculations for their interaction with a distant point charge. We also consider stacking the magnetic moments to produce solenoids and so revisit the interaction of a point charge and a constant-current solenoid. It is found that the models require external forces of constraint which introduce linear and angular momentum into the systems. Our discussion of simple, explicit models appears to make a sacrifice in generality compared with the earlier treatments. However, we believe that the advantage of the simple models is a gain in clarity, particularly in understanding the changes in the magnetic system and the role of the forces of constraint.

One version for the analysis of the charge-magnet interaction has been particularly influential in the recent literature. It concludes that because of "hidden momentum" a new equation of motion is required for (the center of energy of) a magnetic moment, and the associated ideas have now entered the textbooks of classical electromagnetism in connection with the idea of "hidden mechanical momentum.” However, discussions of "hidden mechanical momentum” are often inaccurate, superficial treatments of complex situations. "Hidden momentum” involves linear momentum terms of order $1/c^2$, and discussions of "hidden momentum” often suggest a form of behavior which is inaccurate even in nonrelativistic physics, because the "hidden momentum” requires detailed (through order $1/c^2$) forces of constraint which often go unmentioned in the textbook and the research literature. In this article, we will not solve the complex problem of the interaction of charges and magnets. Rather, within specific (point-charge) models, we will explore the interaction in detail, noting the conservation laws and the external forces associated with "hidden mechanical momentum.”
A. Outline of the Presentation

We consider two models for a magnetic moment which are widely used in the physics literature. In the first model, a magnetic moment is treated as a charge moving with constant speed in a circular orbit plus a second opposite charge at the center of the orbit. In the second model, a magnetic moment is treated as a charge moving in a circular orbit of fixed radius with an opposite charge at the center of the orbit, but now the speed of the moving particle need not be constant. These two models are used to explore the interaction of a point charge and a magnetic moment, and by extension, the interaction of a point charge and a solenoid.

First we treat the model where the magnetic moment charges move in circles with constant speed. We start our discussion with a point charge outside a constant-current solenoid, and then move to the discussion of a distant point charge outside the orbit of a charged particle in uniform circular motion. We find all the external forces on the systems and evaluate all the conservation laws connected to these external forces. For these constant-speed models, there is no "hidden momentum" in the systems. Next we consider a magnetic moment where the speed of the charge is allowed to change in response to electric forces tangential to its velocity. We note that (as a theorem of mechanics) any point mass $m$ which is constrained to move in an exact circle while subject to a constant perturbing force $\mathbf{F}$ acquires an average linear momentum $\langle \mathbf{p}_{m}^{\text{mech}} \rangle$ in order $1/c^2$ which is related to its angular momentum $\mathbf{L}$ and the perturbing force $\mathbf{F}$ as $\langle \mathbf{p}_{m}^{\text{mech}} \rangle = \frac{1}{1/(2mc^2)} \mathbf{L} \times \mathbf{F}$. This is the "hidden mechanical momentum" which appears in the electromagnetism textbooks (but not in the mechanics texts). We note that this "hidden momentum" is related to the power flow delivered by the perturbing force, but is introduced into the system by the impulse delivered by the external centripetal forces of constraint. We also note that the constrained response of the system which produces "hidden mechanical momentum" in a charged-particle system will necessarily create a speed-dependent electric dipole moment for the system, which does not seem to be mentioned in the literature. Finally we turn to a general discussion of the interaction between charges and magnets. We note the historical context of the discussion, comment on some statements of dubious validity in the textbook literature, and place the problem in the context of previous and continuing experimental work.
II. CONSTANT-SPEED MODEL FOR SOLENOIDS AND MAGNETIC MOMENTS

A. A Point Charge and a Constant-Current Solenoid

When a point charge $q$ is held at rest outside a very long solenoid, it is claimed in some of the literature\cite{10} that there are no forces exerted by one system on the other. Now it is known from elementary electromagnetism that a long, isolated, neutral solenoid has no electric or magnetic fields outside its winding. It is suggested that therefore the solenoid exerts no force on the external point charge $q$. However, the electric fields of the point charge $q$ certainly exert forces on the charge carriers of the solenoid. No one has ever done a convincing calculation of the response of the solenoid charge carriers to the point charge fields, and therefore we do not know the actual behavior of this system. However, we can discuss a related model where the currents of the solenoid are held constant despite the electric force due to the point charge fields experienced by the solenoid charge carriers.\cite{11}

This model requires that there are nonelectromagnetic external forces on the charge carriers of the solenoid which balance the electrical forces due to the point charge fields.

In order to make the situation as symmetrical as possible, the solenoid currents are sometimes regarded as carried by opposite charges of equal magnitude rotating in opposite directions with the cylindrical rings of charged particles differing infinitesimally in radius.\cite{10}

The energy, linear momentum, angular momentum, and energy-times-center-of-energy\cite{12} for our electromagnetic system of a point charge and constant-current solenoid involve the sums of the mechanical and electromagnetic contributions. Since this system is stationary in time, none of the conservation-related quantities is changing in time. For simplicity of calculation here, we assume here that the radius $r$ of the solenoid (with axis along the $z$-axis) is small $r \ll x_q$ compared to the distance $x_q$ from the solenoid axis to the point charge $q$ (located along the $x$-axis at $\mathbf{x}_q = \hat{i}x_q$). Then the integral for the linear momentum $\mathbf{P}_{em}$ in the electromagnetic field (associated with the overlap of the point charge electric field $\mathbf{E}_q$ and the solenoid magnetic field $\mathbf{B}_0$) can be simplified by approximating the value of $\mathbf{E}_q(\mathbf{r})$
as its value \( \mathbf{E}_q(0,0,z) \) on the axis of the solenoid, giving

\[
\mathbf{P}_{em} = \int d^3r \frac{1}{4\pi c} \mathbf{E}_q(\mathbf{r}) \times \mathbf{B}_0 = \int d^3r \frac{1}{4\pi c} \left( \frac{q(\hat{k}z - \hat{i}x_q)}{(z^2 + x_q^2)^{3/2}} \right) \times \hat{k}B_0
\]

\[
= j \frac{qr^2B_0}{2c} = \frac{q}{c} \mathbf{A}(\mathbf{r}_q)
\]

(1)

Here \( \mathbf{A}(\mathbf{r}_q) \) is the vector potential of the infinite solenoid in the Coulomb gauge evaluated at the position of the external charge \( q \). Thus the system of a point charge \( q \) outside a constant-current solenoid contains \textit{linear momentum} in the electromagnetic field. On the other hand, there is no electromagnetic interaction \textit{energy} between the point charge \( q \) and the solenoid since the point charge has only an electric field and the solenoid is electrically neutral.

We notice that all the external forces \( \mathbf{F}_{i}^{ext} \) required to keep the solenoid currents constant are applied to the charges \( e_i \) carrying the currents of the solenoid. Since the positive and negative charges (moving in opposite directions) differ only infinitesimally in location, the nonelectromagnetic external force density and the mechanical momentum density both vanish, and hence the sum of the external forces on the system and the total mechanical momentum are both zero. Thus the conservation law for the total system momentum \( \mathbf{P} = \mathbf{P}_{mech} + \mathbf{P}_{em} \) takes the form

\[
\sum_i \mathbf{F}_{i}^{ext} = 0 = \frac{d\mathbf{P}}{dt} = \frac{d\mathbf{P}_{em}}{dt}
\]

(2)

corresponding to constant linear momentum \( \mathbf{P} = \mathbf{P}_{em} \) for the charge-solenoid system. When switching between the positive and negative charges at a single spatial location, the velocities are reversed along with the forces. Thus the power density due to the external forces does not vanish. However, the power delivered by the external forces \( \mathbf{F}_{i}^{ext} \) on the charges \( e_i \) reverses sign under reflection in the \( xz \)-plane, so that the total power delivered by the external forces is zero

\[
\sum_i \mathbf{F}_{i}^{ext} \cdot \mathbf{v}_i = 0 = \frac{dU}{dt}
\]

(3)

corresponding to constant energy \( U \) for the system. On the other hand, the \( y \)-component of displacement also changes sign under reflection in the \( xz \)-plane so that the power-weighted displacement of the system is not zero. Indeed, we must have\[13\]

\[
\sum_i (\mathbf{F}_{i}^{ext} \cdot \mathbf{v}_i)\mathbf{r}_i = \frac{d(U\mathbf{\hat{X}})}{dt} - c^2 \mathbf{P}
\]

(4)
which becomes $\sum (\mathbf{F}_{i}^{\text{ext}} \cdot \mathbf{v}_{i}) \mathbf{r}_{i} = -c^{2}\mathbf{P}_{\text{em}}$ since neither the energy $U$ nor the center of energy $\mathbf{X}$ is changing with time so that $(d/dt)(U_{\text{em}} \mathbf{X}) = 0$, and $\mathbf{P}_{\text{mech}} = 0$ so that $-c^{2}\mathbf{P} = -c^{2}\mathbf{P}_{\text{em}}$. Later we will calculated explicitly the left-hand side of Eq. (4). However, here we merely emphasize that for this constant-current case, the electromagnetic field momentum must be associated with the power densities of nonrelativistic external forces of constraint. The electromagnetic field momentum is directly related to the introduction of power in the electromagnetic system at one spatial location and its removal at another location.

B. Magnetic Moment Modeled as a Charge with Constant Angular Velocity

In order to simplify the analysis for the interaction of a point charge and a magnet, we consider a solenoid as a stack of magnetic dipoles, and treat the interaction of a point charge and a magnetic dipole. Our model of a magnetic dipole is simplified further by considering only one moving charge and one stationary charge of opposite sign. Specifically, our system consists of a distant point charge $q$ located on the $x$-axis $\mathbf{r}_{q} = \hat{i}x_{q}$, a negative charge $-e$ along the $z$-axis at $z_{e}$, and a particle of positive charge $e$ and mass $m$ in uniform circular motion in the plane $z = z_{e}$ with speed $v_{e} = \omega_{0}r$, the orbit being of radius $r$ and centered on the $z$-axis. The combination of charges $-e$ and $+e$ is electrically neutral and has a magnetic moment given by

$$\mathbf{\mu} = \frac{\hat{k}erv_{e}}{2c}$$

In this case the charge $e$ is moving with constant angular velocity as

$$\mathbf{r}_{e}(t) = r[\hat{i}\cos(\omega_{0}t) + \hat{j}\sin(\omega_{0}t)] + \hat{k}z_{e}$$

$$\mathbf{v}_{e}(t) = r\omega_{0}[-\hat{i}\sin(\omega_{0}t) + \hat{j}\cos(\omega_{0}t)]$$

$$\mathbf{a}_{e} = -r\omega_{0}^{2}[\hat{i}\cos(\omega_{0}t) + \hat{j}\sin(\omega_{0}t)]$$

Now we will calculate the momentum, energy, and angular momentum for this system consisting of point masses and electromagnetic fields. We will then average over the periodic motion to obtain average values for all the quantities. Once again, the constraints imposed upon the system must be provided by nonelectromagnetic external forces $\mathbf{F}_{e}^{\text{ext}}$, $\mathbf{F}_{-e}^{\text{ext}}$, and $\mathbf{F}_{q}^{\text{ext}}$ acting on respectively the charge $e$, the charge $-e$, and the charge $q$. 

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C. Conservation-Related Quantities for the Magnetic Moment

The values (mechanical plus electromagnetic) for the system energy, linear momentum, angular momentum, and energy times center of energy can all be obtained through order $1/c^2$ by integrating over the field densities. This leads to the values which are obtained from the Darwin Lagrangian. We will time-average over these values, retaining only through first-order terms in $r/x_q$. Then the displacement from the charge $q$ to the moving charge $e$ is given by

$$
|\mathbf{r}_e - \mathbf{r}_q| = \sqrt{x_q^2 - 2r_x \cos(\omega_0 t) + r^2 + z_e^2 + 1 - [nr_x \cos(\omega_0 t)]/(x_q^2 + z_e^2)}.
$$

The time-averages (indicated by brackets $\langle \rangle$) give

$$
\langle \sin^2(\omega_0 t) \rangle = \langle \cos^2(\omega_0 t) \rangle = 1/2,
$$

and

$$
\langle \sin(\omega_0 t) \cos(\omega_0 t) \rangle = 0.
$$

The total energy of the system includes mechanical plus electromagnetic energy and gives an average value

$$
\langle U \rangle = \left< m\gamma_e c^2 - \frac{e^2}{r} + \frac{eq}{[x_q^2 - 2r \cos(\omega_0 t) + r^2 + z_e^2]^{1/2}} - \frac{eq}{[x_q^2 + z_e^2]^{1/2}} + M_e c^2 + M_q c^2 \right>
= m\gamma_e c^2 - \frac{e^2}{r} + M_e c^2 + M_q c^2
$$

(9)

where we have written $\gamma_e = (1 - v_e^2/c^2)^{-1/2}$ with $v_e = \omega_0 r$. Here we are ignoring electric quadrupole terms of order $r^2/x_q^2$. We notice that there is no time-average interaction energy between the stationary charge $q$ and the magnetic dipole.

Using the expression for the velocity of charge $e$ given in Eq. (7), the linear momentum
\[ \langle \mathbf{P}_\mu \rangle = \left\langle m\gamma_e \mathbf{v}_e + \frac{qe}{2c^2} \left( \frac{\mathbf{v}_e}{r_{qe}} + \frac{(\mathbf{v}_e \cdot \mathbf{r}_{qe})\mathbf{r}_{qe}}{r_{qe}^3} \right) \right\rangle \]

\[ = \frac{qe}{m\gamma_e} \mathbf{v}_e + \frac{qe}{2c^2} \left\{ \frac{x_q^2}{x_q^2 - 2r x_q \cos(\omega t) + r^2 + z_e^2} \right\}^{1/2} \]

\[ + \frac{v_e}{2c^2} \left\{ \left[ x_q^2 - 2r x_q \cos(\omega t) + r^2 + z_e^2 \right]^{3/2} \right\} \left\{ i[r \cos(\omega t) - x_q] + j r \sin(\omega t) + k z_e \right\} \right\}^{\langle \text{av} \rangle} / \]

\[ = \frac{av}{qe} \left\{ \left[ x_q^2 - 2r x_q \cos(\omega t) + r^2 + z_e^2 \right]^{3/2} \right\} \left\{ i[r \cos(\omega t) - x_q] + j r \sin(\omega t) + k z_e \right\} \right\}^{\langle \text{av} \rangle} / \]

\[ = \frac{j qev c x_q}{2c^2 [x_q^2 + z_e^2]^{3/2}} = \frac{j qev c x_q}{c [x_q^2 + z_e^2]^{3/2}} = \frac{1}{c} \mathbf{E}_q(0, 0, z_e) \times \vec{\mu} \]

(10)

where \( \vec{\mu} = \vec{kerv}_e/(2c) \) is the magnetic moment produced by the moving charge \( e \), and \( \mathbf{E}_q(0, 0, z_e) \) is the electric field of the charge \( q \) evaluated at the position of the magnetic moment. Since the charge \( e \) moves with uniform circular motion and so has no average linear momentum, the average system momentum is contained entirely in the electromagnetic field.

The angular momentum about the origin can also be evaluated using the calculation above since we recognize the same average as was needed in Eq. (10). The system angular momentum contains both the mechanical angular momentum of the orbiting charge \( e \) and the electromagnetic field angular momentum:

\[ \langle \mathbf{L} \rangle = \left\langle \mathbf{r}_e \times m\gamma_e \mathbf{v}_e + \frac{qe}{2c^2} \mathbf{i} x_q \times \left( \frac{\mathbf{v}_e}{r_{qe}} + \frac{(\mathbf{v}_e \cdot \mathbf{r}_{qe})\mathbf{r}_{qe}}{r_{qe}^3} \right) \right\rangle \]

\[ = \vec{k}m\gamma ev + \mathbf{i} x_q \times \frac{j qev c x_q}{2c^2 [x_q^2 + z_e^2]^{3/2}} \]

\[ = \vec{k}m\gamma ev + \frac{j qev c x_q^2}{2c^2 [x_q^2 + z_e^2]^{3/2}} = \vec{k}m\gamma ev + \frac{\mu q x_q^2}{c [x_q^2 + z_e^2]^{3/2}} \]

\[ = \mathbf{L}_{\text{mech}} + \mathbf{r}_q \times \left( \frac{1}{c} \mathbf{E}_q(0, 0, z_e) \times \vec{\mu} \right) \]

(11)
D. Conservation-Related Quantities for a Solenoid

From the results for the conservation-related quantities for a point charge and a constant-speed magnetic moment, we can use integration to obtain the corresponding quantity for the system of a point charge and a constant-current solenoid. In the present model calculation, only one set of charges carries the solenoid currents, in contrast to the discussion above of a solenoid with moving positive and negative charge carriers; however, the electromagnetic conservation-related quantities depend only upon the fields and are unchanged. The average energy in Eq. (9) shows no average interaction energy between a point charge and a magnetic moment, and we expect none between a point charge and a constant-current solenoid. The linear momentum associated with the interaction of a point charge and a solenoid can be obtained by integrating Eq. (10) over a stack of magnetic moments using the replacement

\[ \frac{e rv_e}{2c} = \mu \rightarrow dz \left[ \frac{\pi r^2 B_0}{4\pi} \right] \] since \( \frac{\pi r^2 B_0}{4\pi} \) is the dipole moment per unit length for a solenoid. Thus the momentum in the point charge-solenoid system is

\[
P_{\mu\rightarrow \text{sol}} = \hat{j} \int_{-\infty}^{\infty} dz \frac{qx_q}{c[x_q^2 + z^2]^{3/2}} \left( \frac{\pi r^2 B_0}{4\pi} \right)
\]

which is identical with the result obtained earlier in Eq. (1).

The angular momentum of the interacting point charge and constant-current solenoid of finite length \( L \) can also be obtained by integrating over Eq. (11) for the angular momentum of the interacting point charge and magnetic moment. The mechanical angular momentum \( \vec{k} m \gamma e rv_e \) associated with the particle in uniform circular motion does not reflect the charge-solenoid interaction and will diverge for an infinite solenoid. The contribution from the angular momentum in the electromagnetic field for a solenoid of length \( L \) is given by the integral

\[
\mathbf{L}_{\mu\rightarrow \text{sol}} = \vec{k} \int_{-L/2}^{L/2} dz \frac{qx_q^2}{c[x_q^2 + z^2]^{3/2}} \left( \frac{\pi r^2 B_0}{4\pi} \right)
\]

which is

\[
\mathbf{L}_{\mu\rightarrow \text{sol}} = \vec{k} \frac{q r^2 B_0}{2c} \left( \frac{L/2}{[x_q^2 + (L/2)^2]^{1/2}} \right)
\] (13)

In the limit of an infinite solenoid \( L \rightarrow \infty \), the field angular momentum \( \mathbf{L}_{\mu\rightarrow \text{sol}} \) in Eq. (13)
becomes
\[
L_{\mu\rightarrow\text{sol}} = \vec{k} qr^2 B_0/(2c) = r_q \times A(r_q) \tag{14}
\]
and is independent of the separation \(x_q\) of the point charge and the solenoid\[16\]. Any stack of magnetic moments of \textit{finite} length, as in Eq. (13), will have an electromagnetic angular momentum which depends upon the distance \(x_q\) from the solenoid to the charge \(q\), and which vanishes when the point charge \(q\) is removed infinitely far from the stack of magnetic moments.

E. External Forces of Constraint for the Constant-Speed Magnetic Moment

The quantities obtained in Eqs. (9)-(13) are those traditionally obtained in the textbooks and the research literature. However, the literature sometimes makes no reference to the role of the nonelectromagnetic external forces acting on the charged particles of our electromagnetic system. Nevertheless, the external forces of constraint are absolutely crucial in producing these conservation-related quantities. Accordingly, we now wish to investigate the forces \(F^\text{ext}_e\), \(F^\text{ext}_{-e}\), and \(F^\text{ext}_q\).

The equation of motion for the charge \(e\) in the magnetic moment of Eq. (5) is given by
\[
(d/dt)(m\gamma_e v_e) = F^\text{ext}_e + eE_{-e}(r_e) + eE_q(r_e) \tag{15}
\]
where we have written \(\gamma_e = (1 - v_e^2/c^2)^{-1/2}\) with \(v_e = \omega_0r\), and have approximated the electric field due to \(q\) as its value at the center of the orbit since \(r << x_q\). We can solve for \(F^\text{ext}_e\) in Eq. (15) and then average over a period so as to obtain \(\langle F^\text{ext}_e \rangle\), \(\langle F^\text{ext}_e \cdot v_e \rangle\), \(\langle r_e \times F^\text{ext}_e \rangle\) and \(\langle (F^\text{ext}_e \cdot v_e) r_e \rangle\). We find
\[
\langle F^\text{ext}_e \rangle = \frac{eq(i x_q - \hat{k} z_e)}{(z_e^2 + x_q^2)^{3/2}} \tag{16}
\]
\[
\langle F_e \cdot v_e \rangle = 0 \tag{17}
\]
\[
\langle r_e \times F^\text{ext}_e \rangle = 0 \tag{18}
\]
and

\[ \langle (\mathbf{F}_{e}^{\text{ext}} \cdot \mathbf{v}_{e}) \mathbf{r}_{e} \rangle = \frac{\langle \mathbf{F}_{e}^{\text{ext}} \rangle}{v} / \left( \frac{\hat{r} \cdot \mathbf{F}_{e}^{\text{ext}}}{(z_e^2 + x_e^2)^{3/2}} \right) \cdot \left[ -\hat{r} \sin(\omega_0 t) + \hat{j} \cos(\omega_0 t) \right] \varepsilon + \hat{k} z_e \right\}_{av} \]

\[ = -\hat{j} \frac{eqx q v}{2(z_e^2 + x_e^2)^{3/2}} \] \( (19) \)

The average force on the central negative charge is simply that due to the charge \( q \)

\[ \langle \mathbf{F}_{e}^{\text{ext}} \rangle = -eq(\hat{x}_q - \hat{k} z_e) \]

\[ \frac{1}{z_e^2 + x_e^2)^{3/2}} \] \( (20) \)

With these results, we can evaluate the various conservation laws. The net external force vanishes for the magnetic moment system of charges \( e \) and \( -e \), \( \langle \mathbf{F}_{e}^{\text{ext}} \rangle + \langle \mathbf{F}_{-e}^{\text{ext}} \rangle = 0 \). The average force on the charge \( q \) vanishes \( \langle \mathbf{F}_{q}^{\text{ext}} \rangle = 0 \), when we neglect the average quadrupole moment of the system of charges \( e \) and \( -e \). Thus the average of the sum of the external forces vanishes \( \langle \mathbf{F}_{e}^{\text{ext}} \rangle + \langle \mathbf{F}_{-e}^{\text{ext}} \rangle + \langle \mathbf{F}_{q}^{\text{ext}} \rangle = 0 \), and the average system momentum given in Eq. (10) is a constant in time. The average power delivered by the external forces also vanishes \( \langle \mathbf{F}_{e}^{\text{ext}} \cdot \mathbf{v}_{e} \rangle + \langle \mathbf{F}_{-e}^{\text{ext}} \cdot \mathbf{v}_{-e} \rangle + \langle \mathbf{F}_{q}^{\text{ext}} \cdot \mathbf{v}_{q} \rangle = \langle \mathbf{F}_{e}^{\text{ext}} \cdot \mathbf{v}_{e} \rangle + \langle \mathbf{F}_{-e}^{\text{ext}} \cdot 0 \rangle + \langle \mathbf{F}_{q}^{\text{ext}} \cdot 0 \rangle = 0 \), and the system energy is constant in time. The average total torque about the origin supplied by the sum of the forces vanishes \( \langle \mathbf{r}_{e} \times \mathbf{F}_{e}^{\text{ext}} \rangle + \langle \mathbf{r}_{-e} \times \mathbf{F}_{-e}^{\text{ext}} \rangle + \langle \mathbf{r}_{q} \times \mathbf{F}_{q}^{\text{ext}} \rangle = \langle \mathbf{r}_{e} \times \mathbf{F}_{e}^{\text{ext}} \rangle + \langle \mathbf{r}_{-e} \times 0 \rangle + \langle \mathbf{r}_{q} \times 0 \rangle = 0 \), and the average total angular momentum is constant in time. However, from Eq. (19), the power-weighted displacement \( \langle (\mathbf{F}_{e}^{\text{ext}} \cdot \mathbf{v}_{e}) \mathbf{r}_{e} \rangle + \langle (\mathbf{F}_{-e}^{\text{ext}} \cdot \mathbf{v}_{-e}) \mathbf{r}_{-e} \rangle + \langle (\mathbf{F}_{q}^{\text{ext}} \cdot \mathbf{v}_{q}) \mathbf{r}_{q} \rangle = \langle (\mathbf{F}_{e}^{\text{ext}} \cdot \mathbf{v}_{e}) \rangle + \langle (\mathbf{F}_{-e}^{\text{ext}} \cdot 0) \rangle + \langle (\mathbf{F}_{q}^{\text{ext}} \cdot 0) \rangle = -\hat{j} eqx q v/[2(z_e^2 + x_e^2)^{3/2}] \) is not zero and is indeed equal to \( c^2 \) times the negative of the system linear momentum in Eq. (10), as required by the time average of the relation in Eq. (4).

F. External Forces of Constraint for a Constant-Current Solenoid

From the expressions obtained in Eqs. (16)-(20), we can obtain the corresponding expressions for a point charge \( q \) interacting with a constant-current solenoid. Thus we can evaluate \( \sum (\mathbf{F}_{i}^{\text{ext}} \cdot \mathbf{v}_{i}) \mathbf{r}_{i} \) by integrating the time average in Eq. (19) over the length of the solenoid. The magnetic moment is replaced by \( dz \) times the magnetic moment per unit
length \( \mu = erv/(2c) \rightarrow dz \pi r^2 B_0/(4\pi) \)

\[
\sum_i (F^\text{ext}_i \cdot v_i) r_i = -j \int_{-\infty}^{\infty} dz \frac{c\pi r^2 B_0}{4\pi} \frac{qx_q}{(z^2 + x_q^2)^{3/2}}
\]

\[
= -j \frac{cr^2 B_0 qx_q}{4} \left( \frac{z}{x_q^2 (z^2 + x_q^2)^{1/2}} \right) \bigg|_{-\infty}^{\infty}
\]

\[
= -j \frac{cr^2 B_0 q}{2x_q} \quad (21)
\]

From Eqs. (1) and (21), we find that indeed equation (4) holds true for the constant-current solenoid. It seems striking that work done by nonrelativistic external forces in Eq. (21) is related directly to the relativistic expression (1) for the linear momentum in the associated electromagnetic fields.

**G. Quasistatic Changes for the Point Charge Near the Magnetic Moment**

The connection between the nonelectromagnetic external forces of constraint and the electromagnetic field momentum and angular momentum can be made more emphatic by bringing the charge \( q \) quasistatically along the \( x \)-axis from spatial infinity up to its final position \( r_q = \hat{i}x_q \). During the quasistatic change, the external nonelectromagnetic forces must provide the rates of change for the system energy, linear momentum, and angular momentum. The magnetic forces associated with the moving charges \( e \) and \( q \) provide nonvanishing impulses when averaged over the uniform circular motion of the charge \( e \) of the magnetic moment. The average external forces of constraint can be evaluated alternatively as the time average over the external forces required to maintain the uniform circular motion of the charge \( e \) and the quasistatic motion of the charge \( q \) in the presence of the magnetic forces, or, more simply, as the external forces needed to balance the magnetic forces on the magnetic moment \( \rightarrow -\vec{\mu} \) and on the charge \( q \). We will present the simpler calculations here.

When the charge \( q \) is moved quasistatically, it has a velocity \( v_q = \hat{i}v_q = \hat{i}(dx_q/dt) \) which may be taken as small as desired. The magnetic moment \( \vec{\mu} = \hat{k}\mu \) creates a magnetic field \( \mathbf{B}_\mu \) which places a magnetic force \( F^\text{mag}_q \) on the charge \( q \). This force on \( q \) must be balanced
by a nonelectromagnetic external force $\Delta F_{q}^{ext-mag} = -F_{q}^{mag}$ giving

$$
-\Delta F_{q}^{ext-mag} = F_{q}^{mag} = q(v_{q}/c) \times B_{\mu}(r_{q})
= q \frac{dx_{q}}{dt} \left( \frac{3[\mu k \cdot (ix_{q} - kze_{c})]}{(x_{q}^{2} + z_{e}^{2})^{5/2}} - \frac{\mu k}{(x_{q}^{2} + z_{e}^{2})^{3/2}} \right)
= -\frac{j}{c(x_{q}^{2} + z_{e}^{2})^{5/2}} \frac{dx_{q}}{dt} + \frac{j}{c(x_{q}^{2} + z_{e}^{2})^{3/2}} \frac{dx_{q}}{dt}
$$

(22)

The impulse $I_{q}^{ext}$ delivered by the external force when the charge $q$ is moved from spatial infinity to the position $r_{q} = \hat{x}_{q}$ follows from Eq. (22) as

$$
I_{q}^{ext} = \int dt \Delta F_{q}^{ext-mag} = -\jmath \frac{q \mu}{c} \int dt \frac{dx_{q}}{dt} \left( -\frac{3z_{e}^{2}}{(x_{q}^{2} + z_{e}^{2})^{5/2}} + \frac{1}{(x_{q}^{2} + z_{e}^{2})^{3/2}} \right)
= -\jmath \frac{q \mu}{c} \int_{\infty}^{x_{q}} dx \left( -\frac{3z_{e}^{2}}{(x_{q}^{2} + z_{e}^{2})^{5/2}} + \frac{1}{(x_{q}^{2} + z_{e}^{2})^{3/2}} \right)
= \jmath \frac{q \mu}{c} \left[ \frac{x_{q}}{(x_{q}^{2} + z_{e}^{2})^{3/2}} + \frac{1}{z_{e}} \left( \frac{x_{q}}{(x_{q}^{2} + z_{e}^{2})^{1/2}} - 1 \right) \right]
$$

(23)

Furthermore, when the charge $q$ is moved quasistatically, it generates a magnetic field $B_{q}$ which acts on the magnetic moment $\mu$ with a force $F_{\mu}^{mag}$. This magnetic force on $\mu$ (located at $r_{\mu} = \hat{k}z_{e}$) must be balance by a nonelectromagnetic external force $\Delta F_{\mu}^{ext-mag} = -F_{\mu}^{mag}$,

$$
-\Delta F_{\mu}^{ext-mag} = F_{\mu}^{mag} = \nabla[\mu \cdot B_{q}(r)]_{r_{\mu}}
= \nabla \left[ \mu k \cdot \left( \frac{v_{q}}{c} \times \left( \frac{\hat{i}(x - x_{q}) + \hat{j}y + \hat{k}z}{[(x - x_{q})^{2} + y^{2} + z^{2}]^{3/2}} \right) \right) \right]_{r_{\mu}}
= \nabla \left[ \frac{q \mu y}{c[(x - x_{q})^{2} + y^{2} + z^{2}]^{3/2}} \frac{dx_{q}}{dt} \right]_{r_{\mu}} = \jmath \frac{q \mu}{c(z_{e}^{2} + x_{q}^{2})^{3/2}} \frac{dx_{q}}{dt}
$$

(24)

The impulse $I_{\mu}^{ext}$ delivered by the external force when the charge $q$ is moved from spatial infinity to the position $r_{q} = \hat{x}_{q}$ follows from Eq. (24) as

$$
I_{\mu}^{ext} = \int dt \Delta F_{\mu}^{ext-mag} = -\jmath \frac{q \mu}{c} \int dt \frac{dx_{q}}{dt} \frac{1}{(z_{e}^{2} + x_{q}^{2})^{3/2}}
= -\jmath \frac{q \mu}{c} \int_{\infty}^{x_{q}} dx \frac{1}{(z_{e}^{2} + x_{q}^{2})^{3/2}} = -\jmath \frac{1}{z_{e}} \left( \frac{x_{q}}{(x_{q}^{2} + z_{e}^{2})^{1/2}} - 1 \right)
$$

(25)

If we add together the results of Eqs. (23) and (25) to obtain the total impulse delivered to the electromagnetic system of point charge $q$ and magnetic moment $\vec{\mu}$, we find

$$
I_{\text{total}}^{ext} = I_{q}^{ext} + I_{\mu}^{ext} = \jmath \frac{q \mu}{c} \frac{x_{q}}{(x_{q}^{2} + z_{e}^{2})^{3/2}} = \frac{1}{c} E_{q}(0, 0, z_{e}) \times \vec{\mu}
$$

(26)
which is exactly the average electromagnetic field momentum calculated in Eq. (10). Thus for quasistatic charge motions, the system momentum located in the electromagnetic field is introduced by the nonelectromagnetic external forces.

We can also calculate the angular impulse due to the nonelectromagnetic external forces when the charge \( q \) is moved quasistatically in from spatial infinity. The external torque \( \Delta \vec{\Gamma}^{\text{ext-mag}}_q \) about the origin due to the force \( \Delta \mathbf{F}^{\text{ext-mag}}_q \) on the charge \( q \) is given by

\[
\Delta \vec{\Gamma}^{\text{ext-mag}}_q = \mathbf{r}_q \times \Delta \mathbf{F}^{\text{ext-mag}}_q = \hat{i} x_q \times \left( \hat{j} \frac{3q\mu x^2}{c(x^2 + z^2)^{5/2}} \frac{dx_q}{dt} - \hat{j} \frac{q\mu}{c(x^2 + z^2)^{3/2}} \frac{dx_q}{dt} \right)
\]

\[
= \hat{k} q\mu x \frac{dx_q}{dt} \left( \frac{3z^2}{(x^2 + z^2)^{5/2}} - \frac{1}{(x^2 + z^2)^{3/2}} \right)
\]

The external torque on the magnetic moment is due to the nonelectromagnetic external forces on the charges \( e \) and \(-\dot{e}\). In the calculation through the magnetic moment vector \( \mu \), the external torque \( \Delta \vec{\Gamma}^{\text{ext-mag}}_\mu \) about the origin is due both to the external force \( \Delta \mathbf{F}^{\text{ext-mag}}_\mu \) acting on \( \mu \) and also to the external torque on \( \mu \) needed to balance the torque \( \mu \times \mathbf{B}_q(\mathbf{r}_q) \) due to the magnetic field \( \mathbf{B}_q(\mathbf{r}_q) \) of the moving charge \( q \). Thus the average torque about the origin due to the forces on the charges \( e \) and \(-e\) is

\[
\Delta \vec{\Gamma}^{\text{ext-mag}}_\mu = \mathbf{r}_\mu \times \Delta \mathbf{F}^{\text{ext-mag}}_\mu - \mu \times \mathbf{B}_q(\mathbf{r}_q)
\]

\[
= \hat{k} z \times \left( -\hat{j} \frac{q\mu}{c(z^2 + x^2)^{3/2}} \frac{dx_q}{dt} \right) - \mu \times \left( \frac{v_q}{c} \times \hat{i} x_q + \hat{k} z \right)
\]

\[
= 0
\]

Then the total angular impulse \( \vec{I}^{\text{ext}} \) delivered to the system by the nonelectromagnetic forces is that due to the external force \( \Delta \mathbf{F}^{\text{ext-mag}}_q \) on the charge \( q \) alone and is given by

\[
\vec{I}^{\text{ext}} = \int dt \Delta \vec{\Gamma}^{\text{ext-mag}}_q = \int dt \hat{k} \frac{q\mu x}{c} \left( \frac{3z^2}{(x^2 + z^2)^{5/2}} - \frac{1}{(x^2 + z^2)^{3/2}} \right)
\]

\[
= \int_{\infty}^{x_q} dx \hat{k} \frac{q\mu x^2}{c} \left( \frac{3z^2}{(x^2 + z^2)^{5/2}} - \frac{1}{(x^2 + z^2)^{3/2}} \right)
\]

\[
= \hat{k} \frac{q\mu}{c} \left( \frac{x^2}{x^2 + z^2} \right) = \mathbf{r}_q \times \left( \frac{1}{c} \mathbf{E}_q(0, 0, z_e) \times \mu \right)
\]

which agrees exactly with the electromagnetic angular momentum calculated in Eq. (11).
H. Quasistatic Changes for a Point Charge Near a Constant-Current Solenoid

It is interesting to note how these results (22)-(29) for a constant-speed magnetic moment go over to those for an constant-current infinite solenoid. The situation of an infinite solenoid can again be obtained by adding the contributions of a stack of magnetic moments. Thus the force $\Delta F^\text{ext-sol}_q$ can be obtained by replacing the magnetic moment $\mu$ by $dz \frac{\pi r^2 B_0}{(4\pi)}$ in Eq. (22) and integrating

$$\Delta F^\text{ext-sol}_q = \int_{-\infty}^{\infty} dz \left( \frac{\hat{y}}{c(x_q^2 + z^2)^{5/2}} \frac{dx_q}{dt} - \frac{\hat{x}}{c(x_q^2 + z^2)^{3/2}} \frac{dx_q}{dt} \right) \frac{\pi r^2 B_0}{4\pi} = 0 \quad (30)$$

Thus in the limit of an infinitely long constant-current solenoid, there is no external force needed on the charge $q$ since there is no magnetic force on the charge $q$. The external force $\Delta F^\text{ext-sol}_\mu$ on the solenoid needed to balance the magnetic force on the solenoid due to the moving charge $q$ can be obtained analogously by integrating over the magnetic moment result in Eq. (24) as

$$\Delta F^\text{ext-sol}_\mu = \int_{-\infty}^{\infty} dz \left( -\frac{\hat{y}}{2cx_q^2} \right) \frac{q}{c} \frac{dx_q}{dt} \frac{\pi r^2 B_0}{4\pi} = -\frac{\hat{y} qr^2 B_0}{2c x_q} \quad (31)$$

The impulse $I^\mu_\mu-sol$ delivered to the system by this nonelectromagnetic external force $\Delta F^\text{ext-sol}_\mu$ is

$$I^\mu_\mu-sol = \int dt \Delta F^\text{ext}_\mu-sol = \int_{-\infty}^{\infty} dx \left( -\frac{\hat{y} qr^2 B_0}{2c x_q} \right) = -\frac{\hat{y} qr^2 B_0}{2c x_q} \quad (32)$$

which agrees with the linear momentum in the electromagnetic field calculated in Eq. (1). Thus it is the nonelectromagnetic external force which introduces the field linear momentum for quasistatic changes in the system of a point charge and a constant-current solenoid.

The angular impulse $\vec{T}^\mu_\mu-sol$ delivered by the nonelectromagnetic external forces can also be calculated. Because of the delicate limit for angular momentum, we give first the result for a solenoid of finite length $L$ following from Eq. (29)

$$\vec{T}^{(L)}_\mu_\mu-sol = \int_{-L/2}^{L/2} dz \hat{\kappa} \frac{q}{c} \frac{x_q^2}{(x_q^2 + z^2)^{3/2}} \frac{\pi r^2 B_0}{4\pi} = \hat{\kappa} \frac{qr^2 B_0}{2c} \frac{L}{\left(x_q^2 + (L/2)^2\right)^{3/2}} \quad (33)$$
which is in agreement with Eq. (13). In the limit of an infinite solenoid, we have from Eq. (33) that
\[ \vec{I}_{\mu-sol} = \hat{k}qr^2B_0/(2c) \] (34)
which agrees with the field angular momentum found in Eq. (14).

It is interesting to see the varying roles played by the external forces as the length of the solenoid increases. Thus as the solenoid becomes longer, the magnetic force on the distant charge \( q \) becomes smaller so that all the electromagnetic field momentum is accounted for by the external force on the solenoid. However, none of the angular momentum is introduced into the system by the external force on the solenoid. Rather it is the external force on the charge \( q \) which accounts of the electromagnetic angular momentum, even in the limit where the solenoid becomes infinite; because of the extra factor of \( x_q \), the angular impulse of the torque due to the external force on \( q \) does not vanish even though the linear impulse of the external force on \( q \) does indeed vanish.

### III. VARYING-SPEED MODEL OF A MAGNETIC MOMENT

#### A. "Hidden Mechanical Momentum" in a Magnetic Moment Model

In the discussion above, the magnetic moment model involved a charge \( e \) moving in its circular orbit with constant speed. A variation on this situation requires that the nonelectromagnetic forces of constraint on the charge \( e \) provide only a centripetal acceleration for the circular orbit. Thus the charge \( e \) is allowed to change its speed due to the electric field of the point charge \( q \). In this case, the nonelectromagnetic external forces on the system \( F_{e-cent}^e, F_{e}^{ext}, \) and \( F_{q}^{ext} \) do no work; this follows since the charge \( q \) and the charge \( -e \) are at rest, while the force \( F_{e-cent}^e \) on the charge \( e \) is always perpendicular to the velocity of the charge. The displacement of the charge \( e \) is still that of a circular orbit
\[ \mathbf{r}_e(t) = r(\hat{i}\cos\phi + \hat{j}\sin\phi) + \hat{k}z_e \] (35)
but the velocity must be written as
\[ \mathbf{v}_e(t) = r\frac{d\phi}{dt}(-\hat{i}\sin\phi + \hat{j}\cos\phi) \] (36)
and the acceleration
\[ \mathbf{a}_e = -r \left( \frac{d\phi}{dt} \right)^2 (\hat{i}\cos\phi + \hat{j}\sin\phi) + r \frac{d^2\phi}{dt^2} (-\hat{i}\sin\phi + \hat{j}\cos\phi) \] (37)
Here we are allowing a *response* to the perturbation caused by the distant charge $q$, and these changes will in general involve changes in particle position, velocity, and acceleration. However, these *changes* from the unperturbed values will all be first order in the perturbation when we are carrying out calculations only through first order in the perturbing field due to the distant charge $q$. Thus for this new magnetic-moment model, the electromagnetic energy, linear momentum, and angular momentum (which were already first order in the perturbation) will remain unchanged from the values obtained in Part II above. What we are now allowing to change are the mechanical energy, linear momentum and angular momentum associated with the mass $m$.

It is a result from mechanics (to be derived in the next section) that a small constant perturbing force $F$ acting on a mass $m$ constrained to move in a circle leads to a non-zero average (relativistic) mechanical momentum $\langle p_{\text{mech}} \rangle$ for the particle given by

$$\langle p_{\text{mech}} \rangle = \frac{1}{2mc^2}L \times F$$

(38)

where $L$ is the average angular momentum of the particle in its circular motion and the calculation is through first order in the perturbing force $F$ and through order $v^2/c^2$ in the particle speed $v$. The angular momentum $L$ of a particle in orbit is related to the magnetic moment $\vec{\mu}$ by

$$\vec{\mu} = \frac{e}{2mc}L$$

(39)

so that the mechanical momentum given in Eq. (38) can be rewritten in the case of a charged particle $e$ in the form

$$\langle p_{\text{mech}} \rangle = \frac{1}{2mc^2}L \times F = \frac{1}{c} \frac{\vec{\mu}}{e} \times F$$

(40)

or, in the case that the force $F$ is provided by a constant electric field $F = eE$, the relation becomes

$$\langle p_{\text{mech}} \rangle = \frac{1}{c} \frac{\vec{\mu}}{e} \times E = \frac{1}{c} \vec{\mu} \times E$$

(41)

In the case of the magnetic moment treated earlier, the electric field $E_q$ acting on the charge $e$ provides the constant force in the approximation $r << x_q$ that the perturbing electric field $E_q$ due to the charge $q$ is uniform across the orbital radius $r$ of the charged particle $e$. Thus in the modified magnetic moment model, where the forces of constraint do no work, there is an average mechanical linear momentum which is equal in magnitude and opposite in sign to the electromagnetic field momentum found in Eq. (10). For the case of a varying-speed
magnetic moment, this mechanical momentum in Eq. (41) is termed "hidden mechanical momentum" in the literature. In this article, we emphasize that this "hidden mechanical momentum" is due to the external forces which provide the conditions of constraint and does not necessarily have anything to do with electromagnetism.

B. "Hidden Mechanical Momentum" for a Constrained Circular Orbit

We now turn to a derivation of this mechanical momentum of a particle moving along a prescribed path under an external perturbing force. In order to emphasize that this mechanical momentum need not be related to electromagnetism, we will phrase our discussion of the particle motion in terms of a general perturbing force $F$ rather than in terms of a force $eE$ due to an electric field $E$. We consider a particle of mass $m$ constrained by external centripetal forces to move in a circle of radius $r$ parallel to the $xy$-plane while subjected to a small constant perturbing force $F = -\hat{i}f + \hat{j}F_z$ which has a component $-f$ parallel to the $x$-axis. The expressions for the displacement, velocity, and acceleration of the mass $m$ are taken exactly as given above in Eqs. (35)-(37), except that we drop the subscript $e$.

The speed of the mass $m$ follows from energy conservation as

$$m\gamma_0 c^2 = m\gamma c^2 + r f \cos(\phi)$$  \hspace{1cm} (42)

where $\gamma = [1 - (v^2/c^2)]^{-1/2}$. Then writing $v = v_0 + \Delta v$ and retaining terms through first order in $v_0^2/c^2$ and first order in $rf/(mv_0^2)$, we may replace $\phi$ in Eq. (42) by the unperturbed value $\omega_0 t$ and obtain

$$v = r \frac{d\phi}{dt} = v_0 - \frac{rf}{mv_0} \left(1 - \frac{3v_0^2}{2c^2}\right) \cos(\omega_0 t)$$  \hspace{1cm} (43)

We can integrate equation (43) once with respect to time to obtain $\phi = \omega_0 t + \Delta \phi$, given through first order in the perturbation $f$ as

$$\phi(t) = \omega_0 t - \frac{f}{mv_0 \omega_0} \left(1 - \frac{3v_0^2}{2c^2}\right) \sin(\omega_0 t)$$  \hspace{1cm} (44)

The direction $(-\hat{i}\sin \phi + \hat{j}\cos \phi)$ of the tangential velocity $\mathbf{v}$ can be evaluated from Eq. (44) for $\phi$ and the use of the first order expansions

$$\sin(\phi + \Delta \phi) = \sin(\phi) + \Delta \phi \cos(\phi)$$  \hspace{1cm} (45)
\[ \cos(\phi + \Delta \phi) = \cos(\phi) - \Delta \phi \sin(\phi) \] (46)

giving

\[ [\hat{i} \sin \phi + \hat{j} \cos \phi] = -\hat{i} \sin(\omega_0 t) + \hat{j} \cos(\omega_0 t) \]

\[ + \frac{f}{mv_0 \omega_0} \left(1 - \frac{3 v_0^2}{c^2}\right) \sin(\omega_0 t) \hat{i} \cos(\omega_0 t) + \hat{j} \sin(\omega_0 t) \] (47)

We can now calculate the average mechanical linear momentum of the mass \( m \) which is moving in a circular orbit. Through first order in the perturbing force \( f \) and first order in \( v_0^2/c^2 \), the mechanical momentum \( \mathbf{p}_{\text{mech}} \) is given from Eqs. (36), (42), and (47) by

\[
\mathbf{p}_{\text{mech}} = m \gamma_e v_e \\
= \left( m \gamma_0 - \frac{rf}{c^2} \cos(\omega_0 t) \right) \left( v_0 - \frac{rf}{mv_0} \left(1 - \frac{3 v_0^2}{c^2}\right) \cos(\omega_0 t) \right) [\hat{i} \sin \phi + \hat{j} \cos \phi] \\
= \left( m \gamma_0 v_0 - \frac{v_0 rf}{c^2} \cos(\omega_0 t) + \frac{\gamma_0 rv_0 f}{v_0} \left(1 - \frac{3 v_0^2}{c^2}\right) \cos(\omega_0 t) \right) [-\hat{i} \sin(\omega_0 t) + \hat{j} \cos(\omega_0 t)] \\
+ m \gamma_0 v_0 \frac{f}{mv_0 \omega_0} \left(1 - \frac{3 v_0^2}{c^2}\right) \sin(\omega_0 t) \hat{i} \cos(\omega_0 t) + \hat{j} \sin(\omega_0 t) \] (48)

Then averaging in time gives us an average mechanical momentum for the mass \( m \)

\[ \langle \mathbf{p}_{\text{mech}} \rangle = -\hat{j} \frac{v_0 rf}{2c^2} = -\hat{j} \frac{L f}{2mc^2} = \frac{1}{2mc^2} \mathbf{L} \times \mathbf{F} \] (49)

where \( \mathbf{L} = \hat{k} r m v_0 \) and \( \mathbf{F} = -\hat{i} f + \hat{j} F_z \) in agreement with Eq. (38).

We notice that there is no change in the average angular momentum due to the perturbing force \( \mathbf{F} \). Thus the average angular momentum is

\[ \langle \mathbf{L} \rangle = \langle \hat{k} r m \gamma v \rangle = \hat{k} r m \left\langle v + \frac{v_0^3}{2c^2} \right\rangle \\
= \hat{k} r \left\langle v_0 + \frac{v_0^3}{2c^2} + \Delta v \left(1 + \frac{3 v_0^2}{2c^2}\right) \right\rangle = \mathbf{L}_0 \] (50)

since from Eq. (43) \( \langle \Delta v \rangle = 0 \).

C. External Forces for the "Hidden Mechanical Momentum"

The mechanical system discussed in the previous section consists of one point mass \( m \) in a circular orbit, and the energy, linear momentum, and angular momentum are all associated...
with this point mass alone. Thus as far as our mechanical system is concerned, all forces on the mass $m$ are external forces, both the constant perturbing force $\mathbf{F} = -\hat{i} f + \hat{j} F_z$ and the centripetal constraining force $\mathbf{F}_{\text{cent}}$. The time-average values $\langle \mathbf{F} \cdot \mathbf{v} \rangle$, $\langle \mathbf{r} \times \mathbf{F} \rangle$, and $\langle (\mathbf{F} \cdot \mathbf{v}) \mathbf{r} \rangle$ can all be calculated from the results of Eqs. (43) - (47). By symmetry alone, we can easily see that $\langle \mathbf{F} \cdot \mathbf{v} \rangle = 0$, and $\langle \mathbf{r} \times \mathbf{F} \rangle = 0$. However, from Eqs. (43) - (47), we see that through first order in the perturbation $f$

$$\mathbf{F} \cdot \mathbf{v} = (-\hat{i} f + \hat{j} F_z) \cdot \{[-\hat{i} \sin(\omega_0 t) + \hat{j} \cos(\omega_0 t)] \left( v_0 - \frac{rf}{mv_0} \left( 1 - \frac{3v_0^2}{2c^2} \right) \cos(\omega_0 t) \right)$$

$$+ \frac{f}{mv_0 \omega_0} \left( 1 - \frac{3v_0^2}{2c^2} \right) \sin(\omega_0 t) [\hat{i} \cos(\omega_0 t) + \hat{j} \sin(\omega_0 t)] v_0 \}$$

$$= -fv_0 \sin(\omega_0 t)$$

and so

$$\langle (\mathbf{F} \cdot \mathbf{v}) \mathbf{r} \rangle =_{av} \langle f v_0 \sin(\omega_0 t) \rangle r \{ [\hat{i} \cos(\omega_0 t) + \hat{j} \sin(\omega_0 t)] + \hat{k} z_e \} \rangle_{av}$$

$$= j \frac{rfv_0}{2}$$

The centripetal constraining force $\mathbf{F}_{\text{cent}}$ can be found from the equation of motion for the mass $m$, $(d/dt)(m\gamma \mathbf{v}) = \mathbf{F}_{\text{cent}} + \mathbf{F}$, giving

$$\mathbf{F}_{\text{cent}} = \frac{d}{dt}(m\gamma \mathbf{v}) - \mathbf{F}$$

(53)

Since the particle motion is periodic in time and the perturbing force $\mathbf{F}$ is constant, it follows from Eq. (53) that on time average $\mathbf{F}_{\text{cent}}$ balances the perturbing force, $\langle \mathbf{F}_{\text{cent}} \rangle = \mathbf{F}$. Also, since $\mathbf{F}_{\text{cent}}$ is a centripetal force, we find vanishing values for $\mathbf{F}_{\text{cent}} \cdot \mathbf{v} = 0$, $\mathbf{r} \times \mathbf{F}_{\text{cent}} = 0$, and $\langle (\mathbf{F}_{\text{cent}} \cdot \mathbf{v}) \mathbf{r} \rangle = 0$. These results allow us to confirm (on time average) the conservation laws for energy, linear momentum, and angular momentum for the system consisting of the mass $m$. The one interesting result follows from relativistic symmetry corresponding to Eq. (4). In the present case, there is no average change in energy or center of energy so that the time average of Eq. (4) involves the average power-weighted displacement given by Eq. (52) balancing the average linear momentum in Eq. (49)

$$\langle (\mathbf{F} \cdot \mathbf{v}) \mathbf{r} \rangle = j \frac{rfv_0}{2} = -c^2 \langle p_{\text{mech}}^m \rangle$$

(54)
D. Quasistatic Increase of the Perturbing Force

It seems somewhat surprising to find that a particle which is required by centripetal forces of constraint to remain in a circular orbit despite the influence of a small constant perturbing force should acquire an average linear momentum in the plane of the orbit in a direction perpendicular to the perturbing force, corresponding to Eq. (49). In order to confirm our result, we will consider the average impulse introduced by the external forces of constraint when the perturbing force $F = -\hat{i}f + \hat{j}F_z$ is increased quasistatically from zero. Thus we will write the changing $x$-component of the perturbing force as

$$\alpha t$$

where $\alpha$ is taken as very small and the total time $t_f$ of increase is large so that the final value is $f = \alpha t_f$. The equation of motion of the mass $m$ is

$$(d/dt)(m\gamma v) = F^{cent} + F$$

where $F^{cent}$ is the centripetal force of constraint which keeps the mass $m$ in a circular orbit but is always perpendicular to the orbit, and $F = -\hat{i}at + \hat{j}F_z$ is the slowly-changing, spatially uniform perturbing force on the orbit. The point mass $m$ remains in a plane parallel to the $xy$-plane, and therefore the $z$-component of $F^{cent}$ simply balances the $z$-component of $F$, $F_{z^{cent}} = -F_z$.

In the plane of the particle motion, the equation of motion becomes

$$\frac{d}{dt}(m\gamma v) = - (i\cos \phi + j\sin \phi)F_{xy^{cent}} - \hat{i}at$$

(55)

If we take the inner product of this equation with the particle velocity $v$, then we obtain the energy equation

$$v \cdot \frac{d}{dt}(m\gamma v) = \frac{d}{dt}(m\gamma c^2) = v \cdot (-i\sin \phi + j\cos \phi)F_{xy^{cent}} - v \cdot \hat{i}at = v\alpha t \sin \phi$$

(56)

since $F^{cent}$ is always perpendicular to the particle velocity. Since the orbital radius $r$ is not changing, this equation can be rewritten through order $v^2/c^2$ as

$$mv \left(1 + \frac{3v^2}{2c^2}\right) r \frac{d^2 \phi}{dt^2} = v\alpha t \sin \phi$$

(57)

or

$$\frac{d^2 \phi}{dt^2} = \left(1 - \frac{3v^2}{2c^2}\right) \frac{\alpha t}{mr} \sin \phi$$

(58)

Since the right-hand side of Eq. (58) is already first order in the perturbation $\alpha$, we may replace $\phi$ on the right-hand side by $\phi = \omega_0 t$ and $v$ by $v = v_0$. We can then integrate to obtain the angular velocity

$$\frac{d\phi}{dt} = \frac{v}{r} = \omega_0 + \left(1 - \frac{3v_0^2}{2c^2}\right) \frac{\alpha}{m\omega_0 r} \left(\frac{\sin(\omega_0 t)}{\omega_0^2} - \frac{t \cos(\omega_0 t)}{\omega_0}\right)$$

(59)
and the angle $\phi$

$$\phi = \omega_0 t + \left(1 - \frac{3v_0^2}{2c^2}\right) \frac{\alpha}{mr} \left(-\frac{2\cos(\omega_0 t)}{\omega_0^3} - \frac{t\sin(\omega_0 t)}{\omega_0^2}\right)$$  \hspace{1cm} (60)

We notice that when we take the time $t = t_f$ with the limit of $\alpha$ vanishingly small but $f = \alpha t_f$, we obtain the results in Eqs. (43) and (44).

The force $F^{\text{cent}}$ provides the centripetal acceleration following from Eq. (55) as

$$F^{\text{cent}}_{xy} = -\left\{i\cos(\omega_0 t) + \hat{j}\sin(\omega_0 t)\right\}\frac{m v_0^2}{r} \left[\frac{v_0^2}{1 + \frac{v_0^2}{2c^2}} + 2v_0 \Delta v \left(1 + \frac{v_0^2}{2c^2}\right)\right]$$

$$+ \left\{i\sin(\omega_0 t) + \hat{j}\cos(\omega_0 t)\right\} \Delta \phi \frac{mv_0^2}{r} \left[1 + \frac{v_0^2}{2c^2}\right] + \left[i\cos(\omega_0 t) + \hat{j}\sin(\omega_0 t)\right] \alpha t \cos \phi$$

$$= -\left\{i\cos(\omega_0 t) + \hat{j}\sin(\omega_0 t)\right\}\frac{m v_0^2}{r} \left[\frac{v_0^2}{1 + \frac{v_0^2}{2c^2}}\right] + \left[i\cos(\omega_0 t) + \hat{j}\sin(\omega_0 t)\right] \alpha t \cos \phi$$

$$= -\left\{i\cos(\omega_0 t) + \hat{j}\sin(\omega_0 t)\right\}\frac{m v_0^2}{r} \left[\frac{v_0^2}{1 + \frac{v_0^2}{2c^2}}\right] + \left[i\cos(\omega_0 t) + \hat{j}\sin(\omega_0 t)\right] \alpha t \cos \phi$$

$$+ \left[i\sin(\omega_0 t) + \hat{j}\cos(\omega_0 t)\right] \frac{mv_0^2}{r} \left[1 + \frac{v_0^2}{2c^2}\right] \left(1 - \frac{3v_0^2}{2c^2}\right) \frac{\alpha}{m \omega_0} \left(\frac{\sin(\omega_0 t)}{\omega_0^3} - \frac{t\cos(\omega_0 t)}{\omega_0}\right)$$

$$+ \left[i\sin(\omega_0 t) + \hat{j}\cos(\omega_0 t)\right] \frac{mv_0^2}{r} \left[1 + \frac{v_0^2}{2c^2}\right] \left(1 - \frac{3v_0^2}{2c^2}\right) \frac{\alpha}{m r} \left(-\frac{2\cos(\omega_0 t)}{\omega_0^3} - \frac{t\sin(\omega_0 t)}{\omega_0^2}\right)$$

$$+ \left[i\sin(\omega_0 t) + \hat{j}\cos(\omega_0 t)\right] \frac{mv_0^2}{r} \left[1 + \frac{v_0^2}{2c^2}\right] \left(1 - \frac{3v_0^2}{2c^2}\right) \frac{\alpha}{m r} \left(-\frac{2\cos(\omega_0 t)}{\omega_0^3} - \frac{t\sin(\omega_0 t)}{\omega_0^2}\right)$$

$$+ \left[i\sin(\omega_0 t) + \hat{j}\cos(\omega_0 t)\right] \frac{mv_0^2}{r} \left[1 + \frac{v_0^2}{2c^2}\right] \left(1 - \frac{3v_0^2}{2c^2}\right) \frac{\alpha}{m r} \left(-\frac{2\cos(\omega_0 t)}{\omega_0^3} - \frac{t\sin(\omega_0 t)}{\omega_0^2}\right)$$

$$+ \left[i\sin(\omega_0 t) + \hat{j}\cos(\omega_0 t)\right] \frac{mv_0^2}{r} \left[1 + \frac{v_0^2}{2c^2}\right] \left(1 - \frac{3v_0^2}{2c^2}\right) \frac{\alpha}{m r} \left(-\frac{2\cos(\omega_0 t)}{\omega_0^3} - \frac{t\sin(\omega_0 t)}{\omega_0^2}\right)$$

(62)

If we now average over time, we find that Eq. (62) becomes

$$\langle F^{\text{cent}}_{xy} \rangle = i\alpha t - \hat{j}\frac{r v_0}{2c^2} \alpha = i f - \hat{j}\frac{r v_0}{2c^2} \frac{df}{dt}$$  \hspace{1cm} (63)

We see from Eq. (63) that the average external force on the mass $m$ balances the perturbing force $F$ and also has a second contribution proportional to the rate of change of the perturbing force $F$

$$\langle F^{\text{cent}} \rangle = -\bar{F} + \frac{1}{2mc^2} \mathbf{L} \times \frac{d\mathbf{F}}{dt}$$  \hspace{1cm} (64)

If the perturbing force $F$ is increased quasistatically, then (on time average) the constraint-maintaining force $F^{\text{cent}}$ balances the perturbing force $F$ and also introduces a net unbalanced impulse $I^{\text{cent-F}}$

$$I^{\text{cent-F}} = \int dt \langle F^{\text{cent}} \rangle = \int dt \frac{1}{2mc^2} \mathbf{L} \times \frac{d\mathbf{F}}{dt}$$

$$= \frac{1}{2mc^2} \mathbf{L} \times \mathbf{F}$$  \hspace{1cm} (65)
which exactly accounts for the average linear momentum stored in the constrained motion as given in Eq. (49). It is emphasized that this mechanical momentum is introduced by the external force of constraint \( F^{\text{cent}} \) which keeps the charge \( e \) moving precisely in a circular orbit, and not by the perturbing force \( F \) which is in a direction perpendicular to the average momentum. On the other hand, the power delivered by the perturbing force \( F \) is associated with the average linear momentum as in Eq. (52).

**E. Electric Dipole Moment Produced by External Forces of Constraint in the "Hidden Momentum" Case**

In addition to producing a "hidden momentum," the nonelectromagnetic external forces of constraint in our varying-speed model for a magnetic moment also produce an average electric dipole moment which seems to go unmentioned in the literature. Thus if a current is constant but the speed of the charges is varying in space, then the charge density must also vary in space. For our example involving circular motion, the time-average linear charge density \( \lambda(\phi) \) on the circular path followed by the charge \( e \) is inversely related to the speed of the charge at the angle \( \phi \). Then from Eq. (44), with the reinsertion of \( f = eE_q \), we have to first order in the perturbing field \( E_q \)

\[
\lambda(\phi) = \frac{e}{2\pi r} \langle v \rangle = \frac{e}{2\pi r} \left[ 1 + \frac{reE_q}{mv_0^2} \left( 1 - \frac{3v_0^2}{2c^2} \right) \cos \phi \right]
\]

Then the time-average electric dipole moment for our varying-speed magnetic moment is

\[
\langle \bar{p} \rangle = \int (d\phi \, r) r \lambda(\phi)
= \int (d\phi \, r) (\hat{i} \cos \phi + \hat{j} \sin \phi) \frac{e}{2\pi r} \left[ 1 + \frac{reE_q}{mv_0^2} \left( 1 - \frac{3v_0^2}{2c^2} \right) \cos \phi \right]
= \frac{e^2 r E_q}{2mv_0^2} \left( 1 - \frac{3v_0^2}{2c^2} \right)
\]

(67)

Alternatively, we can calculate the electric dipole moment in Eq. (67) by taking the time average of \( er_\phi = e(\hat{i} \cos \phi + \hat{j} \sin \phi) = e(\hat{i} \cos \omega_0 t + \hat{j} \sin \omega_0 t) + e\Delta \phi(\hat{i} \cos \omega_0 t + \hat{j} \sin \omega_0 t) \), where \( \Delta \phi = \phi - \omega_0 t \) is read off from Eq. (44). We note that this average electric dipole moment involves nonrelativistic terms as well as terms in order \( 1/c^2 \). The nonrelativistic terms give an electric dipole moment in the opposite direction from that expected when discussing the polarization of a conductor due to an external charge \( q \). The expression
diverges as the speed of the charge $e$ decrease toward zero; however, this zero-speed limit is inconsistent with our assumption that we are expanding in a small perturbation where $erE_q/(mv_0^2) \ll 1$. This electric dipole will put a force on the distant charge $q$ which is causing the polarization of the magnetic moment.

It seems worth noting that any proposal for "hidden mechanical momentum" which depends upon the electrical current density $\mathbf{J} = ne\mathbf{v}$ remaining constant while the speed $v$ varies along a prescribed path requires that the density $n$ of charges $e$ must vary along the path and so must produce an electric dipole moment connected to the perturbing electric field and dependent upon the unperturbed speed $v_0$ analogous to the result in Eq. (67). It is also worth emphasizing that the direction of this magnetic moment depends explicitly upon the presence of external centripetal forces of constraint. In the case of a purely electromagnetic magnetic moment based upon a classical hydrogen atom, the magnetic moment also develops an electric dipole moment due to a perturbing electric field, but this dipole moment is in a direction perpendicular (not parallel) to the perturbing field.

F. Momentum Associated with Work Done by External Forces in Magnetic Moment Models

In the models which we have considered, the existence of linear momentum is associated with work done by external forces according to the law given in Eq. (4). In the example of Part II where the particle is moving with constant speed in a circular orbit, the work is done by the external forces $\mathbf{F}_e^{\text{ext}}$ which maintain the constant speed of the particle. In this case, the system consists of both electromagnetic fields and mechanical masses so that the electric field is internal to the system. In this case, the work of the electric field is done against the external force of constraint which removes energy from the system when the particle $e$ has positive values of coordinate $y$ and introduces energy into the system when the particle $e$ has negative values of coordinate $y$. Thus the value of $\sum (\mathbf{F}_i^{\text{ext}} \cdot \mathbf{v}_i) \mathbf{r}_i$ is in the negative $y$-direction as in Eq. (19). On the other hand, in the example for the "hidden mechanical momentum" in Parts IIB and IIC where the particle $m$ moved in a circle with varying speed, the system consists of the mass $m$ alone so that both the perturbing force $\mathbf{F}$ and the centripetal forces of constraint are external to the system. In this case, it is the perturbing force $\mathbf{F}$ which is an external force doing work on the system, introducing kinetic energy.
when the particle has positive values of coordinate $y$ and removing kinetic energy when the particle has negative values of coordinate $y$. Thus the value of $\sum (\mathbf{F}^\text{ext}_i \cdot \mathbf{v}_i)\mathbf{r}_i = \langle (\mathbf{F} \cdot \mathbf{v})\mathbf{r} \rangle$ is in the positive $y$-direction in this second case as in Eq. (52).

Although both the field momentum of Eq. (10) and the ”hidden mechanical momentum” of Eq. (41) are associated with the work done by external forces, it should be emphasized that the ”hidden mechanical momentum” actually arises from a different level of approximation than the electromagnetic field momentum which appeared above. The electromagnetic field momentum due to the charge $q$ and the magnetic moment arises in a calculation based upon the unperturbed motion of the magnetic moment and the point charge, and therefore is present in the magnetic moment examples of both Parts II and III discussed above. This is not the case with the ”hidden mechanical momentum.” For the ”hidden mechanical momentum,” the motion of the mass $m$ must respond to the perturbing force before the new mechanical linear momentum appears. Thus in the earlier discussions where the magnetic moment was produced by a charge undergoing uniform circular motion, there is no such ”hidden momentum” at all. The distinction which is being made here is by no means trivial. In the case of a magnetic moment modeled as a hydrogen atom, the electromagnetic field momentum indeed arises because it appears from the unperturbed motion; however, the ”hidden mechanical momentum” is not found because the nonrelativistic perturbed motion is quite different from that of a rigid circular orbit.

IV. DISCUSSION

A. Overview of Charge-Magnet Interactions

At this point we have seem to have reached the end of our careful analysis for the two simple models for magnetic systems, and it is appropriate to place the analysis in perspective. The sharp increase interest regarding the interaction of charges and magnets goes back at least to the 1960s, and particularly to the paradox emphasized by Shockley and James in their article entitled, ”'Try simplest cases' discovery of 'hidden momentum' forces on 'magnetic currents.'” In an oft-cited response, Coleman and Van Vleck point out that the electromagnetic momentum is of order $1/c^2$, and therefore the mechanical behavior of the system must be treated relativistically to the same order. These authors show that the
conservation law for linear momentum need not be violated, without providing a detailed account for the behavior of the charge-magnet system. They mention in a foot note exactly the varying-speed model for a magnetic moment which is discussed in the present article in Part III (but they make no reference to the forces of constraint). Furry’s extensive analysis followed shortly thereafter.

Discussions of the interaction of a charge and a magnet usually assume that the system is stationary and closed (having no external forces of constraint). These two assumptions are unlikely to be compatible physically. The charge-magnet system may well be unstable, just as a point charge outside a conductor is unstable. However, these treatments (which assume closed, stationary behavior) note that the total system momentum must vanish; therefore there must be momentum present to balance any arising from the electric field of the distant charge $q$ and the magnetic field of the magnet. The theorem does not specify the nature of the momentum. It is interesting that Johnson, Cragin, and Hodges avoid reference to "hidden mechanical momentum.” Rather, they repeatedly follow the flow of electromagnetic energy within the systems which they discuss. Furry does refer to “material momentum” and ”hidden momentum” but never insists that the momentum associated with the energy flow is "mechanical” momentum.

In any case, the "hidden momentum’ forces” mentioned in the title of Shockley and James’s article have now become the ”hidden mechanical momentum” described in the most recent editions of electromagnetism textbooks. It seems curious that this concept of "hidden mechanical momentum” does not seem to appear in any mechanics textbook but rather only in electromagnetism texts. The reason for this may be conjectured. Although the "hidden mechanical momentum” appearing in Eqs. (38), (41) and (49) may be an appropriate subject for mechanics, the ideas and dependence upon external forces of constraint (constraints accurate to order $1/c^2$) seem quite contrived for a realistic discussion of mechanical forces and special relativistic effects. However, electromagnetism incorporates relativity as soon as one passes beyond electrostatics. Furthermore, electromagnetism often calculates electromagnetic fields from prescribed charge and current distributions without inquiring about the forces of constraint necessary to produce the prescribed distributions. Finally, electromagnetism is far more poorly understood than mechanics, and therefore contrived apparent solutions may be more tolerable when facing unsolved problems. The idea of "hidden mechanical momentum” has passed into the electromagnetism literature in its
present form because of our inability to describe in detail the interaction of a point charge and a solenoid. Furthermore, there are now physicists who have strong vested interests in certain points of view regarding these interactions.

B. The Accepted View as Related to Controversy Over the Aharonov-Casher Phase Shift

The motivating force for the presently accepted description of charge-magnet interactions comes from controversy over the Aharonov-Casher effect suggested in 1984 and observed experimentally in 1989. This effect is claimed to be the dual of the Aharonov-Bohm effect suggested in 1959 and first observed experimentally in 1960. Both experiments involve the interactions of charges and magnets. Although both observed experimental effects involve the shift of a particle interference pattern, the theoretical basis for the shifts is in dispute. On one side of the dispute are those who claim that these phase shifts represent quantum topological effects with no classical analogue. On the other side are those who claim that the phase shifts may well arise from the classical electromagnetic interactions of charges and magnets. This controversy had existed earlier in connection with the Aharonov-Bohm effect, but became acute in 1987 when it was pointed out that the calculation of Aharonov and Casher assumed a magnetic moment made of magnetic charges; for a current-loop magnetic moment, naive ideas of classical electromagnetism and Newtonian mechanics would account for the phase shift of the Aharonov-Casher effect. In order to counter this classical electromagnetic argument, proponents of the quantum topological view suggested that "hidden mechanical momentum" rendered the naive, current-loop classical analysis invalid. In the late 1980s both sides of the controversy submitted articles to both the Physical Review and to the American Journal of Physics. It seems curious that at that time The Physical Review accepted the quantum article and rejected the classical article whereas the American Journal of Physics did just the opposite. However, in 1990, the American Journal of Physics accepted the "hidden mechanical momentum" analysis of the charge-magnet interaction which was favored by the quantum side of the dispute, and this point of view now dominates all the influential literature and textbooks to the exclusion of the classical point of view.

The presently accepted view is that there is no force between a charge and a long mag-
net, and that "hidden mechanical momentum" plays a significant role in realizing this situation. Within the accepted view, the crucial role played by "hidden mechanical momentum" can be understood as follows. When a point charge and a long but finite-length magnet are far away, there is no linear momentum of interaction between them. However, if the charge is in motion toward the magnet, then, when the charge approaches the magnet, (due to the overlap of the electric field of the charge and the magnetic field of the magnet) there will be electromagnetic linear momentum in the system. The change in this field linear momentum suggests that there must have been an interaction between the charge and the magnet which might be compensated by motions of the centers of energy of the charge and/or of the magnet. Indeed, the magnetic field of the moving charge exerts an obvious Lorentz force on the magnet. Now such an interaction violates the view that there is no force (causing a change in motion) between a charge and a long magnet. Thus here is where "hidden mechanical momentum" comes in. The presently accepted view claims that the (current-loop) magnet acquires a "hidden mechanical momentum" which exactly "compensates" the electromagnetic field momentum associated with the Lorentz force on the magnet. In other words, the momentum transferred to the magnet by the magnetic Lorentz force is in the form of "hidden mechanical momentum" which involves no motion of the magnet’s center of energy. Therefore the total momentum of the system remains zero, consistent with the original Aharonov-Casher view that there is no force (causing a change in motion) between the charge and the magnet. This view is contained in the claim that the "force" on a magnetic moment \( \vec{\mu} \) due to a magnetic field is not just the Lorentz force \( \nabla(\vec{\mu} \cdot \vec{B}) \) as reported in the previous editions of the electromagnetism textbooks but rather is \( \nabla(\vec{\mu} \cdot \vec{B}) + (1/c^2)(d/dt)(\vec{E} \times \vec{\mu}) \) so that for a constant magnetic moment the "force" on a magnetic moment becomes \( \vec{\mu} \cdot \nabla \vec{B} \), that of a magnetic moment formed from magnetic charges.

The present article casts a skeptical eye on the accepted point of view that there are no forces exchanged between charges and long magnets, and that the force on a magnetic moment is that of the magnetic-charge model. It encourages skepticism for four basic reasons. 1) In the analysis above where we looked at two standard models for a (current-loop) magnetic moment, we have found that these models require the presence of external forces of constraint, and that it is precisely these external forces which introduce the "hidden mechanical momentum" into the charge-magnet system. How can we be sure that a
magnet appearing in nature will actually have nonelectromagnetic forces which respond in precisely the fashion demanded by the presently accepted view? It should be emphasized that although electromagnetic linear momentum arises from the unperturbed behavior of a charge and magnet, the "hidden mechanical momentum" arises from a perturbation of the magnetic moment behavior, and this perturbation must take a rigidly prescribed form. 2) The accepted view involving "hidden mechanical momentum" is certainly in error because it does not properly account for the angular momentum in the electromagnetic field for a point charge which approaches a long but finite-length magnet from spatial infinity. Thus in Section IIIG we found that upon quasistatic motion of the charge $q$, the external forces of constraint (which balanced the magnetic forces between the charge and magnet) introduced electromagnetic field angular momentum into the charge-magnet system. This electromagnetic field angular momentum is based upon the unperturbed behavior of the charge and of the magnetic moment, and therefore is, in lowest order of approximation, also present even when the external forces are absent. However, the external forces of constraint in Section IIIB which introduce the "hidden mechanical momentum" do not introduce any angular momentum. Thus the contrived argument about "hidden mechanical momentum" balancing the electromagnetic field linear momentum may allow the total linear momentum to remain zero as the charge approaches the magnet, but it can not serve this function for angular momentum which vanished when the charge was very far away from a (finite-length) magnet. Thus there is (at the very least) a missing link in the presently-accepted statements that there is no interaction between a charge and a long magnet, and that the force on a magnetic moment is properly given as $\nabla(\vec{\mu} \cdot \vec{B}) + (1/c^2)(d/dt)(\vec{E} \times \vec{\mu})$. 3) The accepted view prescribes a perturbed motion for the magnetic moment which unavoidably produces an electric dipole moment. This electric dipole moment will apply a (zero-order in $1/c^2$) current-dependent electrostatic force back on the distant charge $q$, in contradiction to the claim that there is no force between a charge and a long magnet. 4) The presently accepted view of charge-magnet interactions demands a nonrelativistic behavior which is wildly different from that found in the simplest model of a magnetic moment. The simplest model of a magnetic moment has no external forces and corresponds to a classical hydrogen atom. This version of a magnetic moment can be obtained from the models considered above by choosing the central charge $-e$ to have large mass compared to the mass $m$ of the orbiting charge $e$ and by choosing the speed of the charge $e$ so that $m\gamma v^2/r = e^2/r^2$.
placing the light mass (charge $e$) in Coulomb orbit about the heavy mass (charge $-e$). The behavior of this purely electromagnetic magnetic moment has been analyzed by Solem \cite{18} and is found to have a nonrelativistic behavior totally different from that demanded by the promoters of "hidden momentum." The interaction between this magnetic moment and a point charge has been calculated in some detail, \cite{19} and there is indeed an electric force on the charge $q$ due to the magnetic moment which is proportional to the magnitude of the magnetic moment. This calculation contradicts the presently accepted view that there is no exchange of forces between a charge and a long magnet formed by stacking magnetic moments.

C. Dubious Statements in Recent Electromagnetism Textbooks

Although there are many statements in books and articles regarding "hidden momentum" which are of dubious validity, we wish to comment on remarks in only two outstanding electromagnetism textbooks. Thus in a fine undergraduate text on electrodynamics, \cite{32} we find the following statement leading up to the idea of "hidden momentum": "In fact, if the center of mass of a localized system is at rest, its total momentum must be zero." This statement needs a qualification. In our system of a point charge outside a solenoid (which can be taken of finite length) with constant currents, we found that the center of energy \cite{33} was not changing and yet there was nonzero total momentum in the system given by Eq. (1) (or by Eq. (10)). This contradicts the statement of the textbook. If we refer back to equation (4) above, we see that one requires a condition that no power is introduced by external forces and transferred through space, in addition to the condition on the center of energy [center of mass]. Indeed, in our system of a point charge outside a magnetic moment where the particle in circular motion is allowed to change speed, there is no local power delivered by the nonelectromagnetic external (centripetal) forces, and so indeed the total system momentum vanishes, the electromagnetic field momentum being equal in magnitude and opposite in sign to the "hidden mechanical momentum." In this case, there is work done on the charge $e$ by the electric force of the charge $q$; however, both the charge $q$ and the charge $e$ are included within the system and so the work done by the electric force simply transfers energy within the system.

In the same undergraduate electromagnetism textbook, there is a calculation \cite{34} of "hid-
den momentum” analogous to that given here in Section IIIB. The statement appears: "Thus a magnetic dipole in an electric field carries linear momentum, *even though it is not moving!* This so-called hidden momentum is strictly relativistic, and purely mechanical; it precisely cancels the electromagnetic momentum stored in the fields ...” [35] The idea of the ”hidden mechanical momentum” canceling the electromagnetic field momentum is made twice in the text.[36] It is not at all clear that the momentum of the example is actually ”mechanical” if there are a large number of charges moving around the rigid circuit, as shown in the figure of the text; closely-spaced charges would not accelerate as isolated charges and might well carry some of the momentum as electromagnetic momentum associated with electrostatic fields between the charges. Furthermore, the ”hidden mechanical momentum” does not in any sense ”cancel” the electromagnetic field angular momentum whose density is closely related to that of the field linear momentum. In addition, the example in the text seems incomplete since there is no mention of the forces of constraint required to produce the ”hidden mechanical momentum.” Finally, the textbook’s example seems misleading since there is no mention of the electric dipole moment (analogous to that discussed above in Section IIIE and obviously present in the figure of the text) which must develop in the textbook’s model for a magnetic moment with ”hidden mechanical momentum.” The electric forces associated with this electric dipole are of zero order in $1/c^2$ and might dominate the order-$1/c^2$ considerations related to ”hidden mechanical momentum.”

The leading graduate textbook of classical electromagnetism also makes remarks regarding ”hidden momentum” which must be regarded with suspicion. The text states,[37] ”This force $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$] represents the rate of change of the total mechanical momentum, including the ’hidden momentum’ associated with the presence of electromagnetic momentum.” This sentence might suggest that ”hidden momentum” is always associated with electromagnetic momentum. This is certainly not the case. ”Hidden mechanical momentum” requires cohesive forces within the magnetic moment which respond to a perturbing force in a very specific way. If the cohesive forces respond so as to keep the speeds constant as in our Part II above, then there is no ”hidden mechanical momentum,” nor is there any in the more-believable case of a hydrogen-atom magnetic moment, although both of these models do included electromagnetic field linear momentum. Only if the nonelectromagnetic external forces respond so as to keep the orbit fixed in shape through order $1/c^2$ and arrange changes in particle speeds according to one-particle energy conservation (as in
our one-particle examples of Part III) do we find "hidden mechanical momentum." Calculations of "hidden mechanical momentum" of this sort appear in the problems of this graduate text, but there is no mention of the required external forces of constraint, or of multiparticle interactions, or of the electric dipole moment (of the sort appearing in our Eq. (67)) which must be associated with this kind of motion. Furthermore, the text continues, "The effective force in Newton’s equation of motion of mass times acceleration is $[\mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B})]$, augmented by $(1/c^2)\frac{d}{dt}(\mathbf{E} \times \mathbf{m})$, where $\mathbf{E}$ is the external electric field at the position of the dipole." This statement seems of dubious validity because it assumes the existence of cohesive forces responding in such a fashion as to give "hidden mechanical momentum" according to the currently accept view. The statement takes no account of the possibility of nonmechanical momentum, nor of the angular momentum balance, nor of the associated electric dipole moment. The statement of the text arises without adequate explanation; it merely echoes part of the presently accepted view which claims that there is no exchange of forces between charges and long magnets.

D. Understanding the Physics of the Experiments

Understanding of the interaction of charges and magnets is fundamental to the interpretation of the experimentally observed Aharonov-Bohm and Aharonov-Casher phase shifts. Do these phase shifts represent a new phenomenon with no classical analogue or are they the result of classical electromagnetic interactions? Most of the treatments of "hidden momentum" have been in the context of discussions claiming that the observed phase shifts can not possibly arise based upon classical electromagnetic interactions, and this point of view has been maintained recently with such dogmatic certainty that any suggestion to the contrary has been rejected by the referees and editors at the leading physics journals. However, now a new set of experiments is being undertaken which should explore new aspects of the phase shifts, and so the theoretical classical electromagnetic aspects are being explored anew. In order to interpret the new (and old) experimental results accurately, it is important that the errors and uncertainties in the theoretical literature be recognized as such. I believe that the idea of "hidden mechanical momentum," which now appears in the electromagnetism textbooks, may represent a misleading distraction regarding an interaction which is still not properly understood. The subject should be described not
as "hidden mechanical momentum” but rather as "hidden mechanical momentum due to hidden nonelectromagnetic forces.” Although the idea of "hidden mechanical momentum" certainly allows curious calculations in textbooks, it may be irrelevant to our efforts to describe nature.

[1] W. Shockley and R. P. James, "'Try simplest cases' discovery of 'hidden momentum' forces on 'magnetic currents',” Phys. Rev. Lett. 18, 876-879 (1967).
[2] S. Coleman and J. H. Van Vleck, "Origin of 'hidden momentum forces' on magnets,” Phys. Rev. 171, 1370-1375 (1968).
[3] W. H. Furry, "Examples of momentum distributions in the electromagnetic field and in matter,” Am. J. Phys. 37, 621-636 (1969).
[4] R. S. Johnson, B. L. Cragin, and R. R. Hodges, "Am. J. Phys. 62, 33-41 (1994).
[5] Y. Aharonov, P. Pearle, and L. Vaidman, "Comment on 'proposed Aharonov-Casher effect: another example of an Aharonov-Bohm effect arising from a classical lag,”” Phys. Rev. 115, 485-491 (1988).
[6] L. Vaidman, "Torque and force on a magnetic dipole,” Am. J. Phys. 58,978-983 (1990).
[7] H. Hnizdo, "Conservation of linear and angular momentum and the interaction of a moving charge with a magnetic dipole,” Am. J. Phys. 60, 242-246 (1992).
[8] D. J. Griffiths, Introduction to Electrodynamics 3rd edn (Prentice-Hall, Upper Saddle River,NJ 1999).
[9] J. D. Jackson, Classical Electrodynamics 3rd edn (Wiley, New York 1999).
[10] Y. Aharonov and D. Rohrlich, Quantum Paradoxes: Quantum Theory for the Perplexed (Wiley-VCH, Weinheim 2005), Chapter 13.
[11] See, for example, T. H. Boyer, "Classical electromagnetic interaction of a charged particle with a constant-current solenoid,” Phys. Rev. D 8, 1667-1679 (1973).
[12] T. H. Boyer, "Illustrations of the relativistic conservation law for the center of energy,” Am. J. Phys. 73, 953-961 (2005), Eq. (14). The generator of Lorentz transformations is the system energy times the center of energy.
[13] See Eq. (14) of the work by Boyer in ref. 10.
[14] J. D. Jackson, Classical Electrodynamics 2nd edn (Wiley, New York 1975), p. 183.
[15] L. Page and N. I. Adams, "Action and reaction between moving charges," Am. J. Phys. 13, 141-147 (1945).

[16] In a private communication, Dr. Vladimir Hnizdo has called my attention to the paradoxical nature of the angular momentum associated with the charge-magnet system, and to an error in my earlier work listed in ref. 11. In endnote 17 of ref. 11, I incorrectly stated that the relation $L_{em} = (q/c)r_q \times A(r_q)$ (where $r_q$ is the displacement of the external charge $q$ from the center of the solenoid) was in error. This correct relation appeared in the work of G. T. Trammel [Phys. Rev. 134B, 1183 (1964)]. If one takes the limit of an infinite solenoid before carrying out the integral over the field angular momentum density, then the integral omits the contributions of the return magnetic fields at spatial infinity, and the integral gives a vanishing result for the field angular momentum, $L_{em} = 0$, as appears in my work of ref. 11. These considerations reappear in ref. 4, in the undergraduate textbook problem 8.14 of ref. 8, and in the work of Kirk McDonald, "McKenna’s Paradox: Charged Particle Exiting the Side of a Solenoid Magnet." I wish to thank Dr. Hnizdo for bringing this information to my attention.

[17] Although the result is much more general, we are interested here in the situation of motion in a circle. V. Hnizdo noted that the result from the magnetic moment situation could be extended to a stationary mass current in a gravitational field in his article, "Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems," Am J. Phys. 65, 5151-518 (1997).

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[19] T. H. Boyer, "Darwin-Lagrangian analysis for the interaction of a point charge and a magnet: considerations related to the controversy regarding the Aharonov-Bohm and Aharonov-Casher phase shifts," J. Phys. A: Math. Gen. 39, 3455-3477 (2006).

[20] R. H. Romer, "Angular momentum of static electromagnetic field," Am. J. Phys. 34, 772-778 (1966), "Electromagnetic angular momentum,' ibid. 35, 445-446 (1967).

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[22] A. Cimmino, G. I. Opat, A. G. Klein, H. Kaiser, S. A. Werner, M. Arif, and R. Clothier, "Observation of the topological Aharonov-Casher phase shift by neutron interferometry," Phys. Rev. Lett. 63, 380-383 (1989).

[23] Y. Aharonov and D. Bohm, Significance of electromagnetic potentials in quantum theory," Phys. Rev. 115, 485-491 (1959).

[24] R. G. Chambers, "Shift of an electron interference pattern by enclosed magnetic flux," Phys. Rev. Lett. 5, 3-5 (1960).

[25] See for example, reference 8.

[26] T. H. Boyer, "Classical electromagnetic deflections and lag effects associated with quantum interference pattern shifts: considerations related to the Aharonov-Bohm effect," Phys. Rev. D 8, 1679-1693 (1973).

[27] T. H. Boyer, "Proposed Aharonov-Casher effect: another example of an Aharonov-Bohm effect arising from a classical lag," Phys. Rev. 36, 5083-5086 (1987).

[28] Indeed in ref. 10, problem 13.15 at the end of Chapter 13 acknowledges that naive classical electromagnetic forces would account for the Aharonov-Casher phase shift but then asks the reader to show that this would lead to violations of energy conservation.

[29] The article accepted by the Physical Review corresponds to ref 5. The article accepted by the American Journal of Physics was by T. H. Boyer, "The force on a magnetic dipole," Am. J. Phys. 56, 688-692 (1988).

[30] The classical point of view is presented by T. H. Boyer, "Proposed experimental test for the paradoxical forces associated with the Aharonov-Bohm phase shift," Found. Phys. Lett. 19, 491-498 (2006).

[31] This view is presented emphatically in ref. 10.

[32] See ref. 8, p. 357.

[33] The designation "center of energy" seems to me more descriptive than "center of mass." The two terms mean the same thing in the relativistic context.

[34] See ref. 8, pp. 520-521.

[35] See ref. 8, p. 521.

[36] See ref. 8, pp. 357 and 521.

[37] See ref. 9, p. 189.

[38] See ref. 9, pp. 286 and 618.
[39] A. Caprez, B. Barwick, and H. Batelaan, "A macroscopic test of the Aharonov-Bohm effect," (preprint).

[40] G. Gronniger, Z. Simmons, S. Gilbert, A. Caprez, and H. Batelaan, "The Aharonov-Bohm effect, phase or force," (preprint).