Subcomplexes and fixed point sets of isometries of spherical buildings

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Abstract

In this paper we study convex subcomplexes of spherical buildings. We pay special attention to fixed point sets of type-preserving isometries of spherical buildings. This sets are also convex subcomplexes of the natural polyhedral structure of the building. We show, among other things, that if the fixed point set is top-dimensional then it is either a subbuilding or it has circumradius $\leq \frac{\pi}{2}$. If the building is of type $A_n$ or $D_n$, we also show that the same conclusion holds for an arbitrary (top-dimensional in the $D_n$-case) convex subcomplex. This proves a conjecture of Kleiner-Leeb [KL06, Question 1.5] in these cases.

1 Introduction

We are interested in the following geometric question about convex subsets of spherical buildings.

Question 1.1. Let $C$ be a convex subset of a spherical building $B$. Is it true that $C$ is either a subbuilding or it has circumradius $\leq \frac{\pi}{2}$ (i.e. it is contained in a ball of radius $\leq \frac{\pi}{2}$ centered in $C$)?

This question was first asked by Kleiner and Leeb [KL06, Question 1.5] while studying rigidity properties of convex subsets of symmetric spaces of higher rank. A closely related (and weaker) question is Tits’ Center Conjecture, it is concerned with convex subsets, which are also subcomplexes of the natural polyhedral structure of a spherical building, and fixed points of their automorphism groups. We refer to [LRCT11] for more information on the Center Conjecture and its relationship with Question 1.1. This conjecture has been recently proven (see [MT06], [LRCT11], [RC13], [MW13]). This recent success with the Center Conjecture suggests that there might be better prospects for getting an answer to Question 1.1 if we restrict our attention to convex subcomplexes.

The first two of the main results in this paper investigate general convex subcomplexes. If the building is of type $A_n$, then we can answer Question 1.1 for any convex subcomplex.

Theorem A. Let $B$ be a (not necessarily thick) spherical building of type $A_n$. Let $C \subset B$ be a convex subcomplex. Then either $C$ is a subbuilding or it has circumradius $\leq \frac{\pi}{2}$. 
If the building is of type $D_n$, then we can treat the case of top-dimensional convex subcomplexes.

**Theorem B.** Let $B$ be a (not necessarily thick) spherical building of type $D_n$. Let $C \subset B$ be a top-dimensional convex subcomplex. Then either $C$ is a subbuilding or it has circumradius $\leq \frac{\pi}{2}$.

A prominent example (and Tits’ first motivation for the Center Conjecture, cf. [Tit62]) of convex subcomplexes in spherical buildings are fixed point sets of type-preserving isometries. We want to focus now our attention on these kind of convex subsets.

Our first main theorem about fixed point sets of isometries is a positive answer to Question 1.1 in the top-dimensional case.

**Theorem C.** Let $B$ be a spherical building and let $g \in Isom(B)$ be an isometry whose fixed point set $Fix(g) \subset B$ is top-dimensional. Then either $Fix(g)$ is a subbuilding or it has circumradius $\leq \frac{\pi}{2}$.

For fixed point sets of groups of isometries we have the following immediate corollary (Corollary 6.2).

**Corollary D.** Let $H \subset Isom(B)$ be a subgroup of isometries such that the fixed point set $Fix(H)$ is top-dimensional. Suppose that there is an element $g \in H$ such that $Fix(g)$ is not a subbuilding. Then $Fix(H)$ has circumradius $\leq \frac{\pi}{2}$.

If the building has no factors of exceptional type, then we can drop the assumption of top-dimensional fixed point set.

**Theorem E.** Let $B$ be a spherical building without factors of type $F_4, E_6, E_7, E_8$ and let $g \in Isom(B)$ be a type-preserving isometry. Then either $Fix(g)$ is a subbuilding or it has circumradius $\leq \frac{\pi}{2}$.

A main step in the proof of Theorems C and E is to show first the same result for a special kind of isometries with top-dimensional fixed point sets, namely the unipotent isometries (see Theorem 5.1).

As a motivation, let us consider first the example of a spherical building $B = \partial_T X$ which is the Tits boundary of a symmetric space $X = G/K$ of noncompact type. An isometry $g \in Isom_0(X) \cong G$ induces an isometry $g_T$ of the building $B$. If the isometry $g$ is semisimple, then its fixed point set $Fix(g) \subset X$ is a totally geodesic subspace and its boundary at infinity $\partial_T Fix(g) = Fix(g_T)$ is a subbuilding of $B$. If $g$ is parabolic, then $Fix(g_T)$ has circumradius $\leq \frac{\pi}{2}$ (see [Ebe96, Prop. 4.1.1], [BGS85, Lemma 3]). Hence, we obtain a positive answer to Question 1.1 in this case. The fact above about the circumradius of the fixed point set at infinity of a parabolic isometry holds in more generality, e.g. for any CAT(0) space $X$ of finite dimension, this is shown in [CL10] (see also [FNS06]). The proof of this result goes roughly as follows: consider the displacement function $d_g(x) = d(x, gx)$, $x \in X$ of the parabolic isometry $g$. This function is convex and Lipschitz. Now we follow in $X$ a path in the direction of the greatest decrease of the function $d_g$ (e.g. if $X$ is a Riemannian manifold, then we just follow a flow line of minus the gradient of $d_g$). Since $g$ is parabolic, $d_g$ does not attain its infimum in $X$ and this path must have an accumulation point $\xi \in \partial_T X$ at infinity. One then shows that $\xi \in Fix(g_T)$ and for all $\zeta \in Fix(g_T)$ holds $d(\xi, \zeta) \leq \frac{\pi}{2}$. 

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Now suppose that the spherical building \( B = \partial_T X \) is the Tits boundary of a Euclidean building. An isometry \( g \in Isom(X) \) induces again an isometry of \( B \). But in contrast to the symmetric space case, a Euclidean building admits no parabolic isometries (see [Par00], [RC14]). Further, the fixed point sets at infinity of semisimple isometries are not necessarily subbuildings. Hence, we cannot apply the result above to give an answer to Question 1.1 in this case.

We will nevertheless rescue the main idea in the proof mentioned above, that is, to follow the direction of the greatest decrease of a convex function. For this purpose we forget about the CAT(0) space \( X \) and work directly with convex functions defined in a convex subset \( C \) of the building \( B \). For a special family of convex functions, which we call nicely convex in Section 2.3.1, we can assure the existence of a unique point \( x \in C \) where the function attains its minimum and for this point holds \( d(x, y) \leq \frac{\pi}{2} \) for all \( y \in C \) (Lemma 2.3). Thus, the main work will go into finding such functions for the convex subsets in question. Actually, a positive answer to Question 1.1 is equivalent to the existence of such convex functions for convex subsets which are not subbuildings (see Proposition 2.5). This idea is inspired by the approach to the Center Conjecture initiated in [BMR12] using Geometric Invariant Theory.

The functions that we will consider measure essentially the negative of the distance of a point to the boundary of the convex subset (Section 2.4.3). It is easy to see that in the case of convex subsets of spheres such functions satisfy the desired conditions for nice convexity. However, in general, these functions will not be even convex. Our strategy will be as follows, for a given convex subcomplex we find a family of apartments, which is big enough such that any pair of points of the subcomplex is contained in an apartment of the family, and small enough such that the value of the function above for a given apartment of this family at a point of the subcomplex does not depend on the apartment containing the point. This will define a convex function on the convex subcomplex which is nicely convex since its restriction to any apartment of the family (which is just a sphere) is nicely convex. For instance, in the case of fixed point sets of unipotent isometries we will see that we can take the collection of all apartments of the building (Theorem 5.1).

Independently of our interest in Question 1.1 another motivation to study fixed point sets is to investigate the relationship between algebraic properties of an isometry \( g \in Isom(B) \) and the geometry of its fixed point set \( \text{Fix}(g) \subset B \). This will be the main subject in Section 8. For instance, we give a geometric proof of the well known fact (for algebraic groups) that the product of two commuting unipotent elements is again unipotent (see Proposition 8.2). Another question in this direction is to what extent we can read off the fixed point set the Jordan decomposition of an element of an algebraic group. The Jordan decomposition will be discussed in Section 8.3.

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2 Preliminaries

In this paper we consider spherical buildings from the CAT(1) viewpoint as presented in [KL97, Section 3], we refer to it and [Th74, Wei03, AB08] for the basic definitions and facts about spherical Coxeter complexes and spherical buildings. For more information on CAT(1) spaces in general we refer to [BH99].

2.1 Spherical joins

Consider the planes $E_i := \{ x \in \mathbb{R}^{2n} \mid x_k = 0 \text{ if } k \neq 2i-1, 2i \}$ in $\mathbb{R}^{2n}$ for $i = 1, \ldots, n$, and the corresponding unit circles $S^1_i := S^{2n-1} \cap E_i$. For a point $y$ in the unit round sphere $S^{2n-1} \subset \mathbb{R}^{2n}$, let $a_i \geq 0$ be the norm of the projection $\pi_i(y)$ to the plane $E_i$ and let $y^i := \pi_i(y)/a_i \in S^1_i$ if $a_i \neq 0$, then $y = \sum_i a_i y^i$. Hence $S^{2n-1}$ can be thought of as a kind of product of $n$ copies of the unit circle $S^1$ induced by the metric product $\mathbb{R}^n \cong E_1 \times \cdots \times E_n$. The spherical join generalizes this construction to metric spaces.
Let $S^n_+ \subset S^n \subset \mathbb{R}^{n+1}$ denote the points on the round unit $n$-sphere $S^n$ with all their coordinates non-negative. Let $Y_1, \ldots, Y_n$ be metric spaces of diameter $\leq \pi$. The spherical join $Y = Y_1 \circ \cdots \circ Y_n$ is the metric space, that as a set is $S^n_+ \times Y_1 \times \cdots \times Y_n$ modulo the equivalence relation that identifies $(a, y_1, \ldots, y_n) \sim (a', y_1', \ldots, y_n')$ if and only if $a = a' \in S^n_+$, and $y_i = y'_i$ whenever $a_i = a'_i \neq 0$. There is a natural identification (as sets) of $S^{2n-1}$ and the spherical join of $n$ copies of $S^1$ (see above). We define the metric on $S^1 \circ \cdots \circ S^1$ such that this identification is an isometry and use this metric to define the metric of general spherical joins. Let $y = (a, y_1, \ldots, y_n)$, $y' = (a', y'_1, \ldots, y'_n)$ be two points in $Y = Y_1 \circ \cdots \circ Y_n$. Choose points $p_i, p'_i \in S^1$ such that $d(y_i, y'_i) = d(p_i, p'_i)$. Then we define the distance between $y, y'$ as the distance between the points $(a, p_1, \ldots, p_n), (a', p'_1, \ldots, p'_n) \in S^1 \circ \cdots \circ S^1 \simeq S^{2n-1}$.

There are natural isometric embeddings $Y_i \hookrightarrow Y$. Thus, we may think of each $Y_i$ as a subspace of $Y$.

Notice that this definition is made ad-hoc such that the Euclidean cone over $Y$ is canonically isometric to the product of the Euclidean cones over the $Y_i$.

The next lemma follows directly from the definition.

**Lemma 2.1.** Let $Y$ be a metric space of diameter $\leq \pi$. Then the diagonal map $Y \rightarrow Y \circ \cdots \circ Y$ given by $y \mapsto \left[\left((\frac{1}{y_1}, \ldots, \frac{1}{y_n}), (y, \ldots, y)\right)\right]$ is an isometric embedding. □

### 2.2 Circumradius and inradius

Let $y$ be a point in a metric space $Y$. The **circumradius of $Y$ with respect to $y$** is defined as $\text{rad}(Y, y) := \sup_{x \in Y} d(x, y)$ and the **circumradius of $Y$** is $\text{rad}(Y) := \inf_{y \in Y} \text{rad}(Y, y)$, that is, $\text{rad}(Y)$ is the infimum of the radii of balls centered at $Y$ and containing it. A point $y \in Y$, such that $\text{rad}(Y, y) = \text{rad}(Y)$ is called a **circumcenter**.

Let $C \subset Y$ be a subset. The **inradius of $C$ with respect to $y$ in $C$** is defined as $\text{inrad}(C, y) := \sup\{r \geq 0 \mid B_r(y) \subset C\}$ and the **inradius of $C$** is $\text{inrad}(C) := \sup_{y \in C} \text{inrad}(C, y)$, that is, $\text{inrad}(C)$ is the supremum of the radii of balls contained in $C$. A point $y \in Y$, such that $\text{inrad}(Y, y) = \text{inrad}(Y)$ is called an **incenter**.

### 2.3 CAT(1) spaces

A metric space is called a **CAT(1) space** if it is $\pi$-geodesic and geodesic triangles of perimeter less than $2\pi$ are not thicker than those in the round unit sphere.

For points $x, y$ in a CAT(1) space $Y$ at distance $< \pi$, we denote by $xy$ the unique segment connecting both points. Two points at distance $\geq \pi$ are called antipodal. The **link $\Sigma_y Y$ at a point $y \in Y$** is the space of directions at $y$ with the angle metric. It is again a CAT(1) space. If $y \neq x$ and $y$ is not antipodal to $x$, we denote with $\overrightarrow{xy} \in \Sigma_x X$ the direction at $x$ of the segment $xy$.

A subset $C$ of a CAT(1) space is called **convex**, if for any $x, y \in C$ at distance $< \pi$ the segment $xy$ is also contained in $C$. A convex subset of a CAT(1) space is itself CAT(1). The **convex hull $CH(A)$ of a subset $A \subset Y$** is the smallest closed convex subset of $Y$ containing $A$. 
2.3.1 Convex functions on \( \text{CAT}(1) \) spaces

Let \( Y \) be a \( \text{CAT}(1) \) space. A function \( f : Y \to \mathbb{R} \) is said to be (strictly) convex if for any geodesic segment \( c : [0, L] \to Y \) of length \( L < \pi \), the function \( f \circ c : [0, L] \to \mathbb{R} \) is (strictly) convex.

Suppose that \( f : Y \to \mathbb{R} \) is a nonpositive function such that

\[
  f(x) + f(y) \geq 2 \cos \left( \frac{d(x, y)}{2} \right) f(m)
\]

for all \( x, y \in Y \) at distance \( < \pi \), where \( m \in Y \) is the midpoint of the segment \( xy \). Notice that since \( f(m) \leq 0 \), the inequality already implies that \( f \) is convex and also that \( f \) is strictly convex on the convex subset \( \{ f < 0 \} \). Suppose further that \( \{ f < 0 \} \) does not contain pairs of antipodal points. We say that a function with these properties is nicely convex.

Remark 2.2. Our motivation to consider this special kind of convex functions is the following. Suppose that \( Y' \) is the Tits boundary of a \( \text{CAT}(0) \) space \( X \). Let \( h : X \to \mathbb{R} \) be a continuous convex function. Let \( \rho \) be a geodesic ray in \( X \) with \( \rho(\infty) = \xi \). Then \( \text{slope}_h(\xi) := \lim_{t \to \infty} \frac{\rho(h(t))}{t} \) defines a function on \( Y' \), whose restriction to the convex subset \( Y = \{ \text{slope}_h \leq 0 \} \subset Y' \) satisfies the properties above. (cf. [KLM09, Section 3.1], [CL10, Section 4.2].)

Let \( f : Y \to \mathbb{R} \) be nicely convex. If \( y_n \in Y \) is a sequence with \( f(y_n) \to \inf f \leq 0 \), then the inequality implies that \( (y_n) \) is Cauchy. Hence, if \( Y \) is complete, then \( f \) attains its infimum at a unique point \( y_f \in \{ f < 0 \} \).

Lemma 2.3. Let \( Y \) be a complete connected \( \text{CAT}(1) \) space and let \( f : Y \to \mathbb{R} \) be a nonconstant nicely convex function. Then for the unique minimum \( y_f \in Y \) of \( f \) holds

\[
  \text{rad}(Y, y_f) = \sup_{y \in Y} d(y, y_f) \leq \frac{\pi}{2}.
\]

Proof. Let \( y \in Y \) with \( d(y, y_f) < \pi \) and let \( c : [0, L] \to \mathbb{R} \) be the geodesic segment with \( c(0) = y_f \) and \( c(L) = y \). Write \( \tilde{f} := f \circ c \). The inequality for nice convexity implies \( \tilde{f}(t) + \tilde{f}(0) \geq 2 \cos \left( \frac{t}{2} \right) \tilde{f}(\frac{t}{2}) \) for all \( t \in [0, L] \).

Let \( g_0(t) := 1 \) and \( g_{n+1}(t) := 2 \cos \left( \frac{t}{2} \right) g_n(t) - 1 \). We show inductively that \( \tilde{f}(t) \geq \tilde{f}(0) g_n(t) \) for all \( n \geq 0 \) and \( t \in [0, L] \): For \( n = 0 \) this is clear since \( \tilde{f}(0) \) is the minimum of \( f \). By induction we have \( \tilde{f}(t) + \tilde{f}(0) \geq 2 \cos \left( \frac{t}{2} \right) \tilde{f}(\frac{t}{2}) \geq 2 \cos \left( \frac{t}{2} \right) \tilde{f}(0) g_{n-1}(\frac{t}{2}) \) and this in turn implies that \( \tilde{f}(t) \geq \tilde{f}(0) (2 \cos \left( \frac{t}{2} \right) g_{n-1}(\frac{t}{2}) - 1) = \tilde{f}(0) g_n(t) \).

Now we claim that \( g_n(t) \xrightarrow{\text{n} \to \infty} \cos(t) \). Indeed, let us compute

\[
  |\cos t - g_n(t)| = |(2 \cos^2 \left( \frac{t}{2} \right) - 1) - (2 \cos \left( \frac{t}{2} \right) g_{n-1}(\frac{t}{2}) - 1)| = 2 \cos \left( \frac{t}{2} \right) |\cos \left( \frac{t}{2} \right) - g_{n-1}(\frac{t}{2})| = \ldots = 2^n \cos \left( \frac{t}{2^n} \right) \ldots \cos \left( \frac{1}{2^{n-1}} \right) (1 - \cos \frac{t}{2^n}) \leq 2^n (1 - \cos \frac{t}{2^n}).
\]

But \( 2^n (1 - \cos \frac{t}{2^n}) \to 0 \) as can be seen e.g. with L'Hôpital.

Therefore we obtain \( 0 \leq \tilde{f}(L) \leq \tilde{f}(0) \cos L \). Hence, \( \cos L \geq 0 \) and we conclude that \( L \leq \frac{\pi}{2} \). Thus, for any \( y \in Y \) holds \( d(y, y_f) \leq \frac{\pi}{2} \) or \( d(y, y_f) \geq \pi \). Since \( Y \) is connected, we conclude that \( \sup_{y \in Y} d(y, y_f) \leq \frac{\pi}{2} \). \( \square \)

Remark 2.4. Let \( Y \) be a complete \( \text{CAT}(1) \) space and \( C \subset Y \) a connected, closed, convex subset. Lemma 2.3 implies that if \( C \) admits a nonconstant nicely convex function, then \( \text{rad}(C) = \inf_{x \in C} \text{rad}(C, x) \leq \frac{\pi}{2} \). Conversely, if \( \text{rad}(C) \leq \frac{\pi}{2} \) and \( C \) has finite dimension, then
by [BL05] Lemma 3.3] C has a circumcenter $x_0$, that is, $\text{rad}(C, x_0) = \text{rad}(C) \leq \frac{\pi}{2}$. The function $-\cos(d(\cdot, x_0))$ is nicely convex in C by Lemma 2.6 and triangle comparison.

This observation applied to convex subsets of spherical buildings generalizes [BMR12 Thm. 4.5]:

**Proposition 2.5.** A closed convex subset of positive dimension of a spherical building has circumradius at most $\frac{\pi}{2}$ if and only if it admits a nonconstant nicely convex function. \hfill \qed

Now we see a special example of a nicely convex function on a sphere, that we will use later.

**Lemma 2.6.** Let $H \subset S^n \subset \mathbb{R}^{n+1}$ be a hemisphere of the round unit sphere $S^n$. The function $f : H \to \mathbb{R}$ given by $f(x) = -\sin(d(x, \partial H))$ is nicely convex.

**Proof.** Let $x_0$ be the center of the hemisphere $H$ and let $\langle , \rangle$ denote the standard scalar product in $\mathbb{R}^{n+1}$. Then $f(x) = -\sin(d(x, \partial H)) = -\cos d(x, x_0) = -\langle x, x_0 \rangle$. Let $x, y \in H$ with $d(x, y) < \pi$. Then $f(x) + f(y) = -\langle x + y, x_0 \rangle = -\|x + y\|\langle \frac{x + y}{\|x + y\|}, x_0 \rangle = \|x + y\|f(m) = 2\cos d(x, y) f(m)$, where $m \in H$ is the midpoint of the segment $xy \subset S^n$. And since the interior of $H$ does not contain antipodal points, it follows that $f$ is nicely convex. Alternatively, the lemma follows from Remark 2.2 and the fact that $f$ is the slope of the convex function $-\langle \cdot, x_0 \rangle$ in $\mathbb{R}^{n+1}$. \hfill \qed

**Corollary 2.7.** [cf. [Whi67], Lemma 1] Let $C \subset S^n \subset \mathbb{R}^{n+1}$ be a closed convex subset with non-empty interior. Then $C$ has a unique incenter. That is, there is a unique $x_0 \in C$ such that $d(x_0, \partial C) = \sup_{x \in C} d(x, \partial C)$. Moreover, $\text{rad}(C, x_0) \leq \frac{\pi}{2}$.

**Proof.** The function $-\sin(d(\cdot, \partial C)) = \sup_{x \in C} \{-\sin(d(\cdot, \partial H_x))\}$, where $H_x$ is a hemisphere with $x \in \partial H_x$ and $C \subset H_x$, is nicely convex by the previous lemma. The unique minimum of this function is the incenter of $C$. \hfill \qed

**Remark 2.8.** The conclusion of Corollary 2.7 is not true anymore for convex subsets of spherical buildings as we will see later. In general, the function $-\sin(d(\cdot, \partial C))$ is not even convex.

### 2.4 Spherical Coxeter complexes

A spherical Coxeter complex $(S, W)$ is a pair consisting in a unit round sphere $S = S^n \subset \mathbb{R}^{n+1}$ together with a finite group of isometries $W$, called the Weyl group, generated by linear reflections at hyperplanes.

The spheres of codimension one in $S$, that are the fixed point sets of the reflections in $W$ are called the walls. The Weyl chambers or just chambers are the closures of the connected components of $S$ minus the union of all the walls. A Weyl chamber is a convex spherical polyhedron, they are fundamental domains for the action of the Weyl group on $S$ and therefore isometric to the model Weyl chamber $\triangle_{\text{mod}} := S/W$. A root is a top-dimensional hemisphere bounded by a wall. A singular sphere is an intersection of walls. A face is the intersection of a Weyl chamber and a singular sphere. The codimension one faces of a Weyl chamber are called panels. The center of a root is called a point of root-type.
The geometry of a spherical Coxeter complex can be encoded in a graph, the so-called Dynkin diagram. We say that \((S,W)\) is of simply-laced type if its Dynkin diagram has no loops, that is, if its irreducible factors are of type \(A_n, D_n, E_n\). A labelling by an index set \(I\) of the vertices of the Dynkin diagram induces a labelling of the vertices of the model Weyl chamber \(\Delta_{mod}\). We say that a vertex in \(S\) is of type \(i\) or that it is an \(i\)-vertex for \(i \in I\), if its projection under \(S \to S/W = \Delta_{mod}\) has label \(i\).

Suppose \((S,W)\) is irreducible (i.e. its Dynkin diagram is connected). If \((S,W)\) is simply-laced, then there is only one \(W\)-orbit of points of root-type. Their possible mutual distances are \(0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\). If \((S,W)\) is not simply-laced, then there are two \(W\)-orbits of root-type points. Root-type points in different orbits have mutual distances \(\frac{\pi}{7}, \frac{\pi}{2}, \frac{2\pi}{7}, \pi\). If \((S,W)\) is of type \(B_n\), then the possible mutual distances between root-type points in one of the \(W\)-orbits are \(0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\), and in the other orbit are \(0, \frac{\pi}{2}, \pi\). If \((S,W)\) is of type \(F_4\), then root-type points in the same orbits have mutual distances \(0, \frac{\pi}{7}, \frac{\pi}{2}, \frac{2\pi}{7}, \pi\). For more information on possible distances between vertices in a spherical Coxeter complex we refer to [LRC11 Section 2.2] and [RCT13 Section 3].

2.4.1 Root systems

Let \(\Phi\) be an irreducible root system in \(\mathbb{R}^k\) (we refer to [Bou02 Chapter VI] for the definition). A root \(\alpha \in \Phi\) is called reduced if \(2\alpha \notin \Phi\). A root system, whose roots are all reduced is also called reduced. The reduced root systems of rank \(k \geq 3\) are of type \(A_k, B_k, C_k, D_k, E_6, E_7, E_8, F_4\). The non-reduced root systems are of type \(BC_k\). A root \(\alpha \in \Phi\) is called divisible if \(\alpha/2 \in \Phi\). Notice that for a root, non-reduced implies indivisible. We say that a root \(\alpha \in \Phi\) is a short root if its length is minimal among roots in \(\Phi\) and a long root if its length is maximal. If \(\Phi\) is non-reduced (i.e. of type \(BC_k\)), then for a short root \(\alpha\) it holds that \(\alpha\) is non-reduced and \(2\alpha\) is a long root.

We suppose that \(\Phi\) is always so normalized that a short root \(\alpha \in \Phi\) has norm \(\|\alpha\| = 1\). In particular, the possible norms of roots are \(\{1, \sqrt{2}, 2\}\).

The reflections on hyperplanes orthogonal to roots in \(\Phi\) generate a finite group of isometries \(W\) of \(\mathbb{R}^k\). If we restrict the action of \(W\) to the unit sphere \(S = S^{k-1} \subset \mathbb{R}^k\) we obtain a spherical Coxeter complex \((S,W)\).

Let \(\alpha \in \Phi\) be an indivisible root. We denote also with \(\alpha \subset S\) the root (i.e. the hemisphere) \(\{x \in S \mid \langle \alpha, x \rangle \geq 0\} \subset S\). Conversely, if \(\alpha \subset S\) is a root, then we denote again with \(\alpha \in \Phi\) the corresponding indivisible root. There should be no confusion with this abuse of notation.

2.4.2 Convex subcomplexes of spherical Coxeter complexes

Let \(K \subset S\) be a convex subcomplex, that is, \(K\) is an intersection of roots in \(S\). Let \(s \subset S\) be the singular sphere of the same dimension as \(K\) containing \(K\). We define \(\Lambda_K\) to be the set of the singular hemispheres \(h \subset s\) containing \(K\). Then \(\Lambda_K\) is the largest set of singular hemispheres such that \(K = \bigcap_{h \in \Lambda_K} h\). Let \(\Lambda_K^{min} \subset \Lambda_K\) be the set of singular hemispheres \(h \in \Lambda_K\) such that \((-h) \cap K\) has codimension one in \(s\), where \(-h\) is the other hemisphere in \(s\) with \(\partial (-h) = \partial h\). That is, each \(h \in \Lambda_K^{min}\) determines a boundary component in \(\partial K\) of codimension one, hence, \(\Lambda_K^{min}\) is the minimal set of singular hemispheres in \(s\) such that \(K = \bigcap_{h \in \Lambda_K^{min}} h\). Notice that if \(K\) is top-dimensional, then \(\Lambda_K\) is a set of roots in \(S\).
2.4.3 Weighted incenter of a top-dimensional subcomplex

Let \( \Phi \) be a (not necessarily irreducible) root system in \( \mathbb{R}^{n+1} \) and let \((S = S^n, W)\) be its associated spherical Coxeter complex. We choose weights \( \mu_\alpha > 0 \) for each root \( \alpha \subset S \) as follows. If the corresponding indivisible root \( \alpha \in \Phi \) is reduced, then we set \( \mu_\alpha = ||\alpha|| \). If \( \alpha \in \Phi \) is non-reduced, then we can choose \( \mu_\alpha \in \{1, 2\} \). Notice that we only have a flexibility on the choice of the weights \( \mu_\alpha \) if the root system \( \Phi \) is non-reduced.

For a root \( \alpha \subset S \), let \( x_\alpha \in S \) denote the center of \( \alpha \). If the corresponding indivisible root \( \alpha \in \Phi \) is reduced, then \( \mu_\alpha x_\alpha = \alpha \in \Phi \); if it is non-reduced, then \( \mu_\alpha x_\alpha \in \{\alpha, 2\alpha\} \subset \Phi \).

Let \( K \subset S \) be a proper top-dimensional convex subcomplex. We define the function \( f_K \) in \( K \) as

\[
 f_K(x) := \max_{\alpha \in \Lambda_K} \{-\mu_\alpha \sin(d(x, \partial \alpha))\}.
\]

The function \( f_K \) is nicely convex by Lemma 2.6 and therefore has a unique minimum, which we call the weighted incenter of \( K \). Notice that if \((S, W)\) is of simply-laced type, then \( \mu_\alpha = 1 \) for all roots and the weighted incenter is the same as the incenter of \( K \).

Notice that \( \mu_\alpha \sin(d(x, \partial \alpha)) = \mu_\alpha \cos(d(x, \alpha)) = \langle x, \mu_\alpha \alpha \rangle \). Hence, the function \( f_K(x) \) is given by \( \max_{\alpha \in \Lambda_K} \{-\langle x, \mu_\alpha \alpha \rangle\} \).

The next lemma shows that we can define the function \( f_K \) using the smaller set of roots \( \Lambda_K^{\min} \) (or any set of roots between \( \Lambda_K \) and \( \Lambda_K^{\min} \)).

Lemma 2.9. With the notation above, \( f_K(x) = \max_{\alpha \in \Lambda_K^{\min}} \{-\mu_\alpha \sin(d(x, \partial \alpha))\} \).

Proof. Clearly \( f_K(x) \geq f_K^{\min}(x) := \max_{\alpha \in \Lambda_K^{\min}} \{-\mu_\alpha \sin(d(x, \partial \alpha))\} \). For any root \( \beta \in \Lambda_K - \Lambda_K^{\min} \) holds that \( d(x, \partial \beta) = \frac{\pi}{2} \); or, if \( d(x, \partial \beta) < \frac{\pi}{2} \), then the segment between \( x \) and its projection to \( \partial \beta \) must cross the boundary of \( K \), in particular, it must intersect a wall \( \partial \alpha_0 \) for some \( \alpha_0 \in \Lambda_K^{\min} \). The next Lemma implies \(-\mu_\beta \sin(d(x, \partial \beta)) \leq -\mu_{\alpha_0} \sin(d(x, \partial \alpha_0)) \) and therefore \( f_K(x) \leq f_K^{\min}(x) \).

Lemma 2.10. Let \( x \in S \) and let \( \alpha, \beta \subset S \) be two roots containing \( x \) such that \( d(x, \partial \beta) = \frac{\pi}{2} \); or, \( d(x, \partial \beta) < \frac{\pi}{2} \) and the segment between \( x \) and its projection to \( \partial \beta \) intersects \( \partial \alpha \). Then \( \mu_\beta \sin(d(x, \partial \beta)) \geq \mu_\alpha \sin(d(x, \partial \alpha)) \).

Proof. It follows from the conditions that \( d(x, \partial \beta) \geq d(x, \partial \alpha) \). If \( \mu_\beta \geq \mu_\alpha \), then the assertion follows. So suppose that \( \mu_\beta < \mu_\alpha \). In particular, \((S, W)\) is not of simply-laced type. If its root system \( \Phi \) is reduced this implies that \((\mu_\alpha, \mu_\beta) = (\sqrt{2}, 1) \) and if \( \Phi \) is non-reduced, then \((\mu_\alpha, \mu_\beta) = (2, \sqrt{2}), (2, 1), (\sqrt{2}, 1) \).

Notice that since the segment between \( x \) and its projection to \( \partial \beta \) intersects \( \partial \alpha \) we must have \( d(x, \alpha \cap \partial \beta) \geq d(x, (-\alpha) \cap \partial \beta) \). Let \( C := \{y \in \beta \mid d(y, \alpha \cap \partial \beta) \geq d(y, (-\alpha) \cap \partial \beta)\} \). Then \( x \in \alpha \cap C \neq \emptyset \). This implies that \( 0 < d(x, \alpha \cap \partial \beta) \leq \frac{\pi}{2} \) (where \( x_\alpha, x_\beta \) are the centers of the respective roots). If \( d(x, \alpha \cap \partial \beta) = \frac{\pi}{2} \), then \( x \in \alpha \cap C \subset \partial \alpha \) and the assertion follows because \( \mu_\alpha \sin(d(x, \partial \alpha)) = 0 \). Thus, we may assume \( 0 < d(x, \alpha \cap \partial \beta) < \frac{\pi}{2} \). This in particular excludes the case \((\mu_\alpha, \mu_\beta) = (2, 1) \) because in this case \( \Phi \) has a factor of type \( BC_n \) and \( \alpha, \beta \in \Phi \) are short roots, which all have mutual distances in \( \{0, \frac{\pi}{2}, \pi\} \). In the remaining cases we have that \( \mu_\alpha = \sqrt{2} \mu_\beta \) and \( \alpha \) and \( \beta \) are of different type, hence, \( d(x, \alpha \cap \partial \beta) \in \{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\} \). It follows that \( d(x, \alpha \cap \partial \beta) = \frac{\pi}{4} \). Then \( \mu_\alpha x_\alpha \in \Phi \) and \( \gamma := \mu_\beta x_\beta - \mu_\alpha x_\alpha = \mu_\beta(x_\beta - \sqrt{2}x_\alpha) \in \Phi \) are long roots as can be seen in the root system. Also notice that \( C = \gamma \cap \beta \).
Since \( x \in C \subset \gamma \subset A \), we obtain \( 0 \leq \langle x, \gamma \rangle = \langle x, \mu_\beta x_\beta - \mu_\alpha x_\alpha \rangle \). This in turn implies \( \mu_\beta \sin d(x, \partial \beta) = \langle x, \mu_\beta x_\beta \rangle \geq \langle x, \mu_\alpha x_\alpha \rangle = \mu_\alpha \sin d(x, \partial \alpha) \).

The following observation will be used in the proof of Theorem 6.1.

**Lemma 2.11.** Let \( K_1 \supseteq K_2 \) be two top-dimensional subcomplexes, then \( f_{K_1}(x) \leq f_{K_2}(x) \) for all \( x \in K_2 \) whenever the weights defining the functions \( f_{K_i} \) coincide. This occurs in particular if the root system is reduced.

On the other hand, if \( f_{K_1}(x) > f_{K_2}(x) \) for some \( x \in K_2 \) and \( \alpha \in \Lambda_{K_1}^{\text{min}} \) is a root such that \( f_{K_1}(x) = -\mu_\alpha \sin d(x, \partial \alpha) \), then \( \alpha \in \Lambda_{K_2}^{\text{min}} \) and the weight \( \mu_\alpha \) for the root \( \alpha \) corresponding to the function \( f_{K_1} \) must be \( \mu_\alpha = \mu_\alpha \).

**Proof.** The first assertion follows directly from \( \Lambda_{K_1} \subset \Lambda_{K_2} \).

For the second assertion let \( \mu_{i,\beta} \) be the weights for the roots \( \beta \in S \) defining the function \( f_{K_i} \). Let \( \bar{f}_{K_2} \) be the function in \( K_2 \) defined by the weights \( \mu_\beta := \mu_{2,\beta} \) if \( \beta \neq \alpha \) and \( \mu_\alpha := \mu_{1,\alpha} \). Notice that \( \alpha \in \Lambda_{K_1} \subset \Lambda_{K_2} \). Then \( \bar{f}_{K_1}(x) < \bar{f}_{K_2}(x) = \max_{\beta \in \Lambda_{K_2}^{\text{min}}} \{-\mu_\beta \sin d(x, \partial \beta)\} \).

Since \( \mu_\beta = \mu_{2,\beta} \) for all roots but \( \alpha \), it follows that \( \alpha \in \Lambda_{K_2}^{\text{min}} \) and \( \mu_\alpha = \mu_{1,\alpha} < \mu_{2,\alpha} \). In particular, \( \mu_{i,\alpha} = \).

Let \( \alpha \in S \) be a root and let \( \tau \subset \partial \alpha \) be a face. Let \( y \in \alpha \) and let \( z \) be the projection of \( y \) to \( \partial \alpha \). Let \( x \) be the projection of \( y \) to the singular sphere \( s \subset S \) spanned by \( \tau \). Then the sine rule of spherical triangles applied to the triangle \( (x, y, z) \) implies \( \sin d(y, \partial \alpha) = \sin d(y, z) = \sin d(y, x) \sin \angle_x(y, z) = \sin d(y, s) \sin d(s, \partial (\Sigma_\tau \alpha)) \). We apply this observation in the following situation. This will be used for an induction argument in Sections 4 and 7.

Let \( \tau \) be a face in the boundary of the subcomplex \( K \). The weights \( \mu_\alpha \) induce weights in the spherical Coxeter complex \( (\Sigma_\tau S, \text{Stab}_W(\tau)) \). Thus, we have an induced convex function \( f_{\Sigma, K} \) in \( \Sigma_\tau K \).

**Lemma 2.12.** Let \( \tau \) be a face in the boundary of the subcomplex \( K \) and let \( s \subset S \) be the singular sphere spanned by \( \tau \). Let \( y \in K \). Then \( f_{CH(K, s)}(y) = \sin d(y, s) f_{\Sigma, K}(s) \).

**Proof.** It follows directly from the observation above after noticing that \( CH(K, s) \) is the intersection of the roots in \( \Lambda_{K}^{\text{min}} \) containing \( s \) in their boundaries.

The next lemmata describe the possible values of the function \( f_K \) on vertices for the Coxeter complexes of non-exceptional type. These results will be used in Sections 4.1, 4.2 and 7.

**Lemma 2.13.** Let \( (S, W) \) be the spherical Coxeter complex of type \( A_n \). Let \( \alpha \subset S \) be a root and let \( v \in \alpha \) be a vertex in its interior. Then \( \lambda_v := \sin d(v, \partial \alpha) = \cos d(v, x_\alpha) = \langle v, x_\alpha \rangle > 0 \) depends only on the type of the vertex \( v \). In particular, if \( K \subset S \) is a top-dimensional subcomplex containing \( v \) in its interior, then \( f_K(v) = -\lambda_v \) is independent of \( K \).
Proof. We use the vector space realization of the Coxeter complex of type $A_n$ given in [LRC11, Section 2.2.4]. Then modulo the action of the Weyl group, a vertex of type $n-k$ is $v = \frac{1}{\sqrt{k(n+1-k)(n+1)}}(n-1-k, \ldots, n+1-k, -k, \ldots, -k)$ and the center of a root is $x_\alpha = \frac{1}{\sqrt{n}}(e_i - e_j)$ with $i \neq j$ and where $(e_i)$ is the standard basis of $\mathbb{R}^{n+1}$. If $0 \leq d(v, x_\alpha) < \frac{\pi}{2}$, it readily follows that $\lambda_v = \langle v, x_\alpha \rangle = \sqrt{\frac{n+1}{2k(n+1-k)}}$. \hfill \Box

For $n \geq 4$ we consider the Dynkin diagram of type $D_n$ with the following labelling $\cdot \cdot \cdot n \cdot \cdot \cdot$. Let $\lambda_i := \frac{1}{\sqrt{2(n+1-i)}}$ for $i \neq 2$ and $\lambda_2 := \lambda_1$.

Lemma 2.14. Let $(S, W)$ be the spherical Coxeter complex of type $D_n$. Let $\alpha \subset S$ be a root and let $v \in \alpha$ be a vertex of type $i$ in its interior. Then $\sin d(v, \partial \alpha) = \cos d(v, x_\alpha) = \langle v, x_\alpha \rangle \in \{\lambda_i, 2\lambda_i\}$. Moreover, if $i = n$, then $\sin d(v, \partial \alpha) = \lambda_n$ and if $i \in \{1, 2\}$, then $\sin d(v, \partial \alpha) = 2\lambda_1 = 2\lambda_2$.

Proof. We use the vector space realization of the Coxeter complex of type $D_n$ given in [LRC11, Section 2.2.4]. Let $(e_i)$ be the standard basis of $\mathbb{R}^n$. Then modulo the action of the Weyl group, a vertex of type $i$ is $v = \frac{1}{\sqrt{n+1-i}}(e_i + e_{i+1} + \cdots + e_n)$ if $i \neq 2$ and $v = \frac{1}{\sqrt{n}}(-e_1 + e_2 + e_3 + \cdots + e_n)$ if $i = 2$. The center of a root is a vertex of type $n-1$ and therefore has the form $x_\alpha = \frac{1}{\sqrt{2}}(e_j \pm e_k)$ for $j \neq k$. Since $\langle v, x_\alpha \rangle > 0$, the assertion follows. \hfill \Box

Lemma 2.15. Let $(S, W)$ be the spherical Coxeter complex of type $D_n$. Let $\alpha \subset S$ be a root and let $v_i, v_j \in \alpha$ be two adjacent vertices of type $i < j$. If $j \geq 3$ and $\sin d(v_j, \partial \alpha) = 2\lambda_j$, then $\sin d(v_i, \partial \alpha) = 2\lambda_i$.

Proof. With the notation in the proof of Lemma 2.14 the hypotheses imply $v_j = \frac{1}{\sqrt{n+1-j}}(e_j + \cdots + e_n)$ and $x_\alpha = \frac{1}{\sqrt{2}}(e_k + e_i)$ with $k, l \geq j > i$. It follows: $\sin d(v_i, \partial \alpha) = 2\lambda_i$. \hfill \Box

For $n \geq 2$ we consider the Dynkin diagram of type $B_n$ with the following labelling $\cdot \cdot \cdot n \cdot \cdot \cdot$. We have analogous results to Lemmata 2.14 and 2.15 for the Coxeter complex of type $B_n$. Their proofs are similar and we omit them here.

Lemma 2.16. Let $\Phi$ be the root system of type $B_n, C_n$ or $BC_n$ and let $(S, W)$ be its associated spherical Coxeter complex. Let $\alpha \subset S$ be a root and let $v \in \alpha$ be a vertex of type $i$ in its interior. Then $\mu_\alpha \sin d(v, \partial \alpha) = \mu_\alpha \cos d(v, x_\alpha) = \langle v, \mu_\alpha x_\alpha \rangle \in \{\lambda_i, 2\lambda_i\}$. Where $\lambda_i$ depends only on the type of the root system. \hfill \Box

Lemma 2.17. Let $\Phi$ be the root system of type $B_n, C_n$ or $BC_n$ and let $(S, W)$ be its associated spherical Coxeter complex. Let $\alpha \subset S$ be a root and let $v_i, v_j \in \alpha$ be two adjacent vertices of type $i < j$. If $\mu_\alpha \sin d(v_j, \partial \alpha) = 2\lambda_j$, then $\mu_\alpha \sin d(v_i, \partial \alpha) = 2\lambda_i$. \hfill \Box

The next Lemma provides a generalization of the Lemma 2.11 in the case of vertices in subcomplexes of positive codimension of spherical buildings of non-exceptional type. This Lemma will be used in the proof of Theorem 7.3.

Lemma 2.18. Let $\Phi$ be the root system of type $A_n, B_n, C_n, D_n$ or $BC_n$ and let $(S, W)$ be its associated Coxeter complex. Let $s \subset S$ be a singular sphere. Let $K_1, K_2 \neq S$ be two top-dimensional convex subcomplexes, such that the interior of $K_i \cap s$ lies in the interior of
Let $K_i$ and $K_2 \cap s \subset K_1 \cap s$. Let $f_{K_j}$ for $j = 1, 2$ be the functions defined above (with possibly distinct set of weights $\{\mu_{j,\alpha}\}$ for the roots of $S$ if $\Phi$ is non-reduced).

Let $x$ be a vertex in $K_2 \cap s$. If $f_{K_2}(x) < f_{K_1}(x)$ and $\alpha \in \Lambda_{K_1}^{\min}$ is a root such that $f_{K_1}(x) = -\mu_{1,\alpha} \sin d(x, \partial \alpha)$, then $\alpha \cap s \in \Lambda_{K_1}^{\min} \cap \Lambda_{K_2}^{\min}$.

**Proof.** First notice that for $\Phi$ of type $A_n$, the inequality $f_{K_2}(x) < f_{K_1}(x)$ cannot hold by Lemma 2.13. Hence we may assume that $\Phi$ is not of type $A_n$.

Let $i$ be the type of the vertex $x$. The inequality $f_{K_2}(x) < f_{K_1}(x)$ implies $f_{K_1}(x) = -j\lambda_i$, where $\lambda_i$ takes the respective value depending on the type of $\Phi$. We use again the usual identification of $S$ with the unit round sphere in $\mathbb{R}^n$. Then modulo the action of the Weyl group and normalization of the vector (for simplicity we will always omit the normalization factors of the corresponding vectors), we have $x = x_i + \cdots + x_n$. Let $x_\alpha$ be the center of the root $\alpha$. Then $f_{K_1}(x) = -\mu_{1,\alpha} \cos d(x, x_\alpha) = -\lambda_i$ implies that $x_\alpha = \varepsilon x_{\alpha_1} + x_{\alpha_2}$ with $\alpha_1 < i \leq \alpha_2$ and $\varepsilon = 0, \pm 1$ ($\varepsilon = 0$ occurs only in the cases $B_n, BC_n$). After multiplying with an element of the Weyl group, we may assume that $x_\alpha = \varepsilon x_{i-1} + x_i$ and $\varepsilon = 0, 1$.

Consider the sets of roots $\Pi = \{\gamma \subset S \mid \mu_{2,\gamma} \cos d(x, x_\gamma) = 2\lambda_i\}$ and $\Upsilon = \{\gamma \subset S \mid s \subset \gamma\}$. Then $s = \bigcap_{\gamma \in \Pi} \gamma$. Let $M := \bigcap_{\gamma \in \Upsilon} \gamma$. Since $f_{K_2}(x) = -2\lambda_i$, we obtain $M \subset K_2$, which in turn implies the inclusions $s \cap M \subset s \cap K_2 \subset s \cap K_1 \subset \alpha$. Observe that if $\gamma \in \Pi$, then $x_\gamma$ is of the form $e_j$ or $e_j + e_k$ for $i \leq j < k$ and if $\gamma \in \Upsilon$, then $\langle x, x_\gamma \rangle = 0$ and therefore $x_\gamma$ is of the form $e_j$ or $\pm (e_j \pm e_k)$ for $j < k < i$ or $\pm (e_j - e_k)$ for $i \leq j < k$.

Suppose first that $\varepsilon = 1$, that is, $x_\alpha = x_{i-1} + x_i$. Consider a point in $S$ with $(i-1)$-coordinate negative and let $y \neq 0$ be its projection to the subspace $\{z \in \mathbb{R}^n \mid \langle z, e_j \rangle = 0, j \geq i\}$. Then $y \notin \alpha$ and $y \notin M$, therefore $y \notin s$. It follows that for any point in $s$, its $(i-1)$-coordinate must be zero. Hence $\alpha \cap s = \{\langle e_i, \cdot \rangle \geq 0\} \cap s$.

Let now $y = -e_i + e_{i+1} + \cdots + e_n$, then $y \notin \alpha$. It follows that $y \notin s$ or $y \notin M$. In the former case, there is a root in $\Upsilon$ with center $e_i - e_j$ for some $j > i$, then for the root $\beta \in \Pi$ centered at $e_i + e_j$ holds $\beta \cap s = \{\langle e_i + e_j, \cdot \rangle \geq 0\} \cap s = \{\langle e_i, \cdot \rangle \geq 0\} \cap s = \alpha \cap s$. In the latter case, there must be a root in $\Pi$ not containing $y$, the only possibility is the root $\beta$ centered at $e_i$. In this case we also get $\beta \cap s = \{\langle e_i, \cdot \rangle \geq 0\} \cap s = \alpha \cap s$.

We have found a root $\beta \in \Pi$ such that $\beta \cap s = \alpha \cap s$. Let now $y$ be a point with $j$-coordinate equal 0 if $j \leq i$, or if $e_i - e_j \notin \Upsilon$ and $j > i$, and with all other coordinates equal 1. Suppose $y \neq 0$, that is, it defines a point in $S$. It follows that $y \in s \cap M$ and $y \in \{\langle e_i, \cdot \rangle = 0\} \cap s = \partial(\beta \cap s)$. If there is another root $\gamma \in \Pi$ such that $y \in \partial(\gamma \cap s)$, then it must hold $\partial(\gamma \cap s) = \{\langle e_i, \cdot \rangle = 0\} \cap s = \partial(\beta \cap s)$. Therefore $\beta \cap s \in \Lambda_{M/\Gamma s}^{\min}$. If $y = 0$, then $\partial(\gamma \cap s) = \{\langle e_i, \cdot \rangle = 0\} \cap s = \partial(\beta \cap s)$ holds for any root $\gamma \in \Pi$. We can again conclude that $\beta \cap s \in \Lambda_{M/\Gamma s}^{\min}$.

Since $\alpha \cap s = \beta \cap s \in \Lambda_{M/\Gamma s}^{\min}$ and we have the inclusions $M \cap s \subset K_2 \cap s \subset K_1 \cap s \subset \alpha \cap s$ it follows that $\alpha \cap s = \beta \cap s \in \Lambda_{K_1/\Gamma s}^{\min} \cap \Lambda_{K_2/\Gamma s}^{\min}$.

**2.5 Spherical buildings**

A spherical building $B$ modelled on a spherical Coxeter complex $(S, W)$ is a CAT(1) space together with an atlas of isometric embeddings $S \rightarrow B$ (the images of these embeddings are called apartments) with the following properties: any two points in $B$ are contained in a common apartment, the atlas is closed under precomposition with isometries in $W$ and
the coordinate changes are restrictions of isometries in $W$. We consider the empty set as a spherical building.

The objects (walls, roots,...) defined for spherical Coxeter complexes can be defined for the building $B$ as the corresponding images in $B$.

A spherical building has a unique decomposition as a join of spherical buildings and a sphere, whose buildings factors cannot be decomposed further. We say that the building is irreducible if it is not a sphere and this decomposition is trivial.

A building is called thick if every wall is the boundary of at least three different roots. A spherical building has a canonical thick structure (depending only on its isometry type) which results from restricting to a subgroup of its Weyl group ([KL97, Sec. 3.7]).

We say that an isometry of a spherical building is type preserving if it induces the identity on the model Weyl chamber with respect to its thick structure. We denote with $\text{Isom}_0(B)$ the group of type preserving isometries. It is a normal subgroup of the isometry group $\text{Isom}(B)$ and the quotient group $\text{Isom}(B)/\text{Isom}_0(B)$ naturally embeds as a subgroup of the isometry group of the model Weyl chamber (in particular, it is finite if $B$ does not split off a spherical factor).

A subbuilding is a convex subset $B'$ of a building, such that any two points in $B'$ are contained in a convex sphere $s \subset B'$ of the same dimension as $B'$. A subbuilding carries a natural structure as a spherical building induced by its ambient building (cf. [LRC11, Proposition 2.13]).

For any point $x \in B$, the link $\Sigma_x B$ is again a spherical building. It decomposes as the join of a sphere of dimension $\dim(\tau) - 1$, where $\tau$ is the smallest face of $B$ containing $x$, and a spherical building $\Sigma_\tau B$ (which we call the link of the face $\tau$).

Let $K \subset B$ be a top-dimensional convex subcomplex (e.g., an apartment) and let $\tau \subset K$ be a face. We denote with $St_\tau(K) \subset K$ the union of all chambers in $K$ containing $\tau$. We call $St_\tau(K)$ the star of $\tau$ in $K$.

A point $x \in C \subset B$ in a convex subset of a spherical building is said to be an interior point if $\Sigma_x C \subset \Sigma_x B$ is a subbuilding and a boundary point otherwise. The set of boundary points is denoted with $\partial C$.

### 2.5.1 Root groups

Let $B$ be a spherical building and let $\alpha \subset B$ be a root. The root group $U_\alpha$ associated to $\alpha$ is the group of isometries of $B$ fixing $\alpha$ pointwise and every chamber $\sigma$ such that $\sigma \cap \alpha$ is a panel not contained in the boundary wall of $\alpha$. Notice that $U_\alpha$ consists on type preserving isometries.

The building $B$ is called Moufang if for all roots $\alpha \subset B$, the root group $U_\alpha$ acts transitively on apartments containing $\alpha$.

It is a fundamental result of Tits [Tit74] that irreducible spherical buildings of dimension at least 2 are Moufang. In this case, the root group $U_\alpha$ acts simply transitively on apartments containing $\alpha$.

Let $\sigma \subset \partial \alpha$ be a face in the boundary wall of a root $\alpha \subset B$. The set $\Sigma_\sigma \alpha$ is a root of the building $\Sigma_\sigma B$. Then there is a natural restriction homomorphism $U_\alpha \rightarrow U_{\Sigma_\sigma \alpha}$. This homomorphism implies that the links of Moufang buildings are again Moufang. If $B$ is
irreducible, then by the simply transitivity of the action of \( U_\alpha \) on apartments containing \( \alpha \), this homomorphism must be injective. If \( \Sigma_\alpha B \) is irreducible, then by the simply transitivity of the action of \( U_{\Sigma_\alpha} \), the homomorphism must be surjective. In particular, if both \( B \) and \( \Sigma_\alpha B \) are irreducible, then the root groups \( U_\alpha \) and \( U_{\Sigma_\alpha} \) are canonically isomorphic.

### 2.5.3 Parabolic and unipotent subgroups

The fact that root groups of \( B \) can be canonically identified with the root groups of the links of \( B \) allows us to translate computations on the root groups in computations on root groups of buildings of lower dimension. In particular, we can use the commutator relations given in [Tit94] for Moufang polygons to deduce the commutator relations of root groups of irreducible spherical buildings of dimension \( \geq 2 \). These relations also follow from the classification of spherical buildings, but this is a much stronger result.

Let \( B \) be a spherical building with associated spherical Coxeter complex \((S = S^n, W)\). Let \( \Phi \) in \( \mathbb{R}^{n+1} \) be a root system with the same associated Coxeter complex \((S, W)\). Let \( \alpha \in \Phi \) be an indivisible root. Given a chart \((S, W) \hookrightarrow B\) for an apartment \( A = \iota(S) \subset B \), we also denote with \( \alpha \) the root \( \iota(\alpha) \subset B \) (cf. Section 2.4.1). Conversely, if \( \alpha \subset B \) is a root, then we denote again with \( \alpha \) the corresponding indivisible root. There should be no confusion with this abuse of notation.

We can now explain the commutator relations for the root groups of \( B \) (cf. [Tit94] and [Tim00] Section 3).

**Theorem 2.19.** For an irreducible spherical building \( B \) of dimension \( n \geq 2 \), there exists a (possibly non-reduced) root system \( \Phi \) with the same associated Coxeter complex \((S, W)\) as \( B \), such that the following holds. If \((S, W) \hookrightarrow B\) is a chart for an apartment \( A \subset B \), and \( \alpha, \beta \in \Phi \) are two roots, then

\[
[U_\alpha, U_\beta] \subset (U_\gamma \mid \gamma = a\alpha + b\beta \in \Phi, a, b \in \mathbb{N}).
\]

If the root system \( \Phi \) is non-reduced (i.e. \( \Phi \) is of type \( BC_n \) and \( (S, W) \) is of type \( B_n \)) and \( \alpha \in \Phi \) is a non-reduced root, then part of the assertion of the Theorem is the existence of a subgroup \( U_{2\alpha} \) of the root group \( U_\alpha \) such that \( 1 \neq [U_\alpha, U_\alpha] \subset U_{2\alpha} \subset Z(U_\alpha) \), where \( Z(U_\alpha) \) denotes the center of \( U_\alpha \).

**Remark 2.20.** As an application of our results we will see that we can define \( U_{2\alpha} \subset U_\alpha \) geometrically as the pointwise stabilizer of the ball of radius \( \frac{1}{2} \) containing \( \alpha \subset B \). (See Proposition 8.1.)

### 2.5.2 Commutator relations

For the rest of this section, let \( B \) be an irreducible spherical building of dimension \( \geq 2 \). We denote with \( G = G_B \) the group of isometries generated by the root groups of \( B \). Then \( G \) acts transitively on pairs \((\sigma, A)\), where \( \sigma \) is a chamber contained in the apartment \( A \subset B \). Moreover, \( G \) is normal in \( Isom_0(B) \) and \( Isom_0(B) = G \cdot \hat{H} \), where \( \hat{H} = Fix_{Isom(B)}(A) \) is the pointwise stabilizer in \( Isom_0(B) \) of an apartment \( A \subset B \) (see e.g. [Tim00, 3.15], [Wei03 Thm. 11.36]).

Let \( \sigma \subset B \) be a Weyl chamber and \( A \subset B \) an apartment containing \( \sigma \). The set of positive roots \( \Lambda_+ \) with respect to \((\sigma, A)\) is the set of roots contained in \( A \) containing \( \sigma \). For a root
\(\alpha \subset A\), we denote with \(-\alpha\) the other root in \(A\) with the same boundary wall as \(\alpha\). Let now \(\tau \subset \sigma\) be a face. We call the stabilizer \(P_\tau := Stab_G(\tau)\) of \(\tau\) in \(G\) the parabolic subgroup associated to \(\tau\). We denote the group \(U_\tau := \langle U_\alpha \mid \tau \subset \alpha, \tau \not\subset (-\alpha) \rangle \subset P_\tau\) the unipotent subgroup associated to \(\tau\). The unipotent subgroup is independent of the chosen apartment \(A\). Notice that \(U_\sigma = \langle U_\alpha \mid \alpha \in \Lambda_+ \rangle\). The unipotent subgroup \(U_\sigma\) acts simply transitively on the apartments containing \(\sigma\). Further, let \(L_\tau = \langle U_\alpha, U_{-\alpha} \mid \tau \subset \alpha \cap (-\alpha) \rangle \subset P_\tau\). Observe that \(L_\tau\) is trivial. Finally, let \(\hat{P}_\tau := Stab_{Isom_0(B)}(\tau)\) be the stabilizer of \(\tau\) in \(Isom_0(B)\). We have the following result relating these subgroups (see [Tim00, 3.12]).

**Proposition 2.21.** For a face \(\tau\) of a Weyl chamber \(\sigma\) in an apartment \(A \subset B\) holds

(i) \(U_\tau\) is normal in \(\hat{P}_\tau\); 

(ii) \(P_\tau = U_\tau L_\tau H\), in particular, \(P_\sigma = U_\sigma H\); 

(iii) \(\hat{P}_\tau = U_\tau L_\tau \hat{H}\), in particular, \(\hat{P}_\sigma = U_\sigma \hat{H}\), 

where \(H = Fix_G(A)\) and \(\hat{H} = Fix_{Isom(B)}(A)\).

**Remark 2.22.** The product decomposition \(g = ulh \in \hat{P}_\tau = U_\tau L_\tau \hat{H}\) depends on the apartment \(A\). We can read off the factor \(u \in U_\tau\) from the action of \(g\) on \(A\) as follows. Let \(\check{\tau} \subset A\) be the face in \(A\) antipodal to \(\tau\). By the definition of \(U_\tau\), we see that \(St_\tau(A) \subset Fix(u)\). Hence, the convex hull of \(g\check{\tau} = ulh\check{\tau} = u\check{\tau}\) and \(St_\tau(A)\) is the apartment \(A' = uA\). Then \(u\) is the unique element in \(U_\tau \subset U_\sigma\) mapping the apartment \(A\) to \(CH(St_\tau(A), g\check{\tau})\).

Consider now a unipotent isometry \(g\), that is, \(g \in U_\sigma\) for some chamber \(\sigma \subset B\). Let again \(A \subset B\) be an apartment containing \(\sigma\) and let \(\check{\sigma} \subset A\) be the chamber in \(A\) antipodal to \(\sigma\). Let \(\Gamma = (\sigma_0 = \sigma, \sigma_1, \ldots, \sigma_d = \check{\sigma})\) be a minimal gallery between \(\sigma\) and \(\check{\sigma}\), that is, a sequence of chambers of minimal length such that \(\sigma_i \cap \sigma_{i+1}\) is a panel. The chambers \(\sigma_i\) must be all contained in \(A\). Then there is a unique representation \(g = g_1 \ldots g_d\) as the product of \(g_i \in U_{i,j} := U_{\alpha_i}, \) where \(\alpha_i \subset A\) is the positive root such that the panel \(\sigma_i \cap \sigma_{i-1}\) is contained in \(\partial \alpha_i\) (see [Wei03, Prop. 11.11]). We can say more about this product representation of \(g\) if we consider its fixed point set, cf. Proposition 2.23.

Let \(K \subset A\) be a proper top-dimensional convex subcomplex. Recall the definitions of the sets of roots \(\Lambda_K^{min} \subset \Lambda_K\) in Section 2.12.

**Proposition 2.23.** Let \(g \in U_\sigma\) be a unipotent element and let \(A \subset B\) be an apartment containing \(\sigma\). Then \(g \in U_{\Lambda_{Fix}(g)\cap A} := \langle U_\alpha \mid \alpha \in \Lambda_{Fix}(g)\cap A \rangle\). More precisely, if \(g = g_1 \ldots g_d\) is the product representation with respect to the minimal gallery \(\Gamma\) (cf. above), then \(g_j = 1\) if \(\alpha_j \notin \Lambda_{Fix}(g)\cap A\). Moreover, if \(\alpha_i \in \Lambda_{Fix}(g)\cap A\) then we can read off the i-coordinate \(g_i\) from the action of \(g\) on \(\Sigma_i B\), where \(\mu\) is any panel contained in \(\partial \alpha_i \cap Fix(g)\). In particular, the i-coordinate \(g_i\) is independent of the chosen minimal gallery \(\Gamma\).

**Proof.** Let \(k \geq 1\) be the largest number such that \(g_k \notin U_{\Lambda_{Fix}(g)\cap A}\). In particular, \(\alpha_k \notin \Lambda_{Fix}(g)\cap A\) and \(g_k \neq 1\). It follows that there exists a chamber \(\nu \subset Fix(g) \cap A\) such that \(\nu \cap \alpha_k\) is a panel. This implies that \(g_k \ldots g_m \nu = g_k \nu = : \nu' \not\subset A\). Let \(\Psi = (\nu_0 = \nu, \nu_1, \ldots, \nu_r = \sigma)\) be a minimal gallery with \(\nu \cap \nu_1 \subset \partial \alpha_k\). Since \(g_1 \ldots g_k \sigma = \sigma\) and \(g_1 \ldots g_k \nu = g \nu = \nu\) we deduce that \(g_1 \ldots g_k \Psi = \Psi\).

For \(l = 1, \ldots, k\) let \(s_l\) be the length of the chain \(g_l \ldots g_k \Psi \cap \Psi\) and write \(r_l := r - s_l\). Observe that \(g_l \ldots g_k \Psi \cap \Psi = (\nu_r, \ldots, \nu_r)\). We have just seen that \(r_1 = 0\) and since \(g_k \Psi = \Psi\) for \(l = 1, \ldots, k\), we conclude that \(g_k \Psi \cap \Psi = \Psi\). Therefore, \(g_k \Psi = 1\).
Proof. Let us prove the claim. As induction basis we take \( l = k \). In this case we have \( r_k = 1 \) and \( \nu_0 \cap \nu_t \subset \partial \alpha_k \). For the induction step let us consider \( l - 1 \). If \( \alpha_{l-1} \) contains the chamber \( \nu_{r_{l-1}} \) then it also contains \( \nu_t \) and the isometry \( \gamma_1 \in U_{l-1} \) must fix \( \gamma_1 \ldots \gamma_k \nu_{r_{l-1}} \). In this case, it follows that \( r_{l-1} = r_l \) and \( \nu_{r_{l-1}-1} \cap \nu_{r_{l-1}} = \nu_{r_{l-1}} \cap \nu_t \subset \partial \alpha_l \) with \( t \geq l > l - 1 \) by induction. If \( \alpha_{l-1} \) does not contain the chamber \( \nu_{r_{l-1}} \) then there is a \( m \) with \( r_l - 1 \leq m < r \) such that \( \nu_m \not\subset \alpha_{l-1} \) and \( \nu_{m+1} \subset \alpha_{l-1} \), in particular, \( \nu_m \cap \nu_{m+1} \subset \partial \alpha_{l-1} \). By induction \( \nu_{r_{l-1}} \cap \nu_t \subset \partial \alpha_l \) with \( t \geq l \), thus, \( r_l \leq m \). It follows that \( \gamma_1 \ldots \gamma_m \nu_m = \gamma_1 \nu_m \neq \nu_m \) and therefore \( \gamma_1 \gamma_2 \ldots \gamma_k \Psi \cap \Psi = (\nu_{m+1}, \ldots, \nu_r) \). In this case, it follows that \( r_{l-1} = m + 1 > r_l \) and \( \nu_{r_{l-1}-1} \cap \nu_{r_{l-1}} = \nu_{m+1} \cap \nu_m \subset \partial \alpha_{l-1} \). This proves the claim and the first assertion of the proposition, that is, \( \gamma_k \in U_{\Lambda_{\text{Fix}(g)} \cap A} \) for all \( k = 1, \ldots, d \) and \( g \in U_{\Lambda_{\text{Fix}(g)} \cap A} \).

Corollary 2.24. Let \( g \in U_{\sigma} \) be a unipotent isometry. Then \( g \in U_{\sigma'} \) for all chambers \( \sigma' \in \text{Fix}(g) \).

Proof. Take an apartment \( A \) containing \( \sigma \) and \( \sigma' \). Then \( g \in U_{\Lambda_{\text{Fix}(g)} \cap A} \subset U_{\sigma} \cap U_{\sigma'} \) by Proposition 2.23.

Corollary 2.25. Let \( g \) be a unipotent isometry. Then whenever \( \tau \subset \text{Fix}(g) \) is a panel not contained in the boundary of \( \text{Fix}(g) \), it holds \( St \tau B \subset \text{Fix}(g) \).

Proof. The desired property follows from Proposition 2.23 because \( g \) is a product of root elements for roots \( \alpha \) such that \( \tau \not\subset \partial \alpha \) and each root element fixes \( St \tau B \) by definition.

3 Reducing to the irreducible case

Let \( B = B_1 \circ \cdots \circ B_n \) be the decomposition of the spherical building \( B \) as a join of its irreducible components. Notice that some of the factors of \( B \) may be isometric and can be permuted by an isometry of \( B \).

Let \( g \in \text{Isom}(B) \) be an isometry and let \( k \geq 1 \) be the smallest integer with \( g^k(B_1) = B_1 \). Then \( g \) induces an isometry of \( B' = B_1 \circ g(B_1) \circ \cdots \circ g^{k-1}(B_1) \subset B \). Let \( x \in B' \) be a fixed point of \( g \) and let \( \alpha = (a, x_0, \ldots, x_{k-1}) \) be its representation as element of the spherical join. Then \( x = gx = \cdots = g^{k-1}x \) implies that \( a_j = 1/1 \) for \( j = 1, \ldots, k \), \( g^i x_0 = x_i \) for \( i = 0, \ldots, k - 1 \) and \( g^k x_0 = x_0 \). In particular, \( x_0 \in B_1 \) is a fixed point of \( g^k \). It follows from Lemma 2.1 that the fixed point set of \( g \) in \( B' \) is isometric to the fixed point set of \( g^k \) in \( B_1 \). This shows the following proposition, which allows us to restrict our attention to irreducible spherical buildings.

Lemma 3.1. Let \( g \) be an isometry of a spherical building \( B \). Then the fixed point set \( \text{Fix}(g) \subset B \) decomposes as a spherical join, whose factors are isometric to fixed point sets of isometries of irreducible spherical buildings.
In particular, if $\text{Fix}(g)$ is not a subbuilding, then at least one of the factors given by the Proposition is not a subbuilding either. On the other hand, if one of these factors has circumradius $\leq \frac{5}{4}$, then the same is true for $\text{Fix}(g)$.

4 Convex subcomplexes of spherical buildings

Let $C \subset B$ be a convex subcomplex of a spherical building. We say that an apartment $A \subset B$ supports $C$ if $\dim(C \cap A) = \dim C$ and $\partial(C \cap A) = \partial C \cap A$.

**Lemma 4.1.** Let $C \subset B$ be a convex subcomplex and let $C' \subset C$ be a spherical convex subset. Then there is an apartment $A$ supporting $C$ such that $C' \subset A$. In particular, any two points in $C$ are contained in an apartment supporting $C$.

**Proof.** Let $D \subset C$ be a maximal (under inclusion) spherical convex subcomplex containing $C'$. We claim that any apartment $A$ containing $D$ supports $C$: First observe that $C \cap A = D$ by maximality. Since $D$ is spherical, there is a singular sphere $s$ of dimension $k := \dim D$ containing $D$. If $k < \dim C = m$, there exists a $m$-dimensional face $\sigma \subset C$ such that $D \cap \sigma = s \cap \sigma$ has dimension $k$. The subset $s \cup \sigma \subset C$ is contained in an apartment $A'$ and $D \not\subseteq A' \cap C$, contradicting the maximality of $D$. Thus $k = m$. If $\partial D \not\subset \partial C$, then there is a singular hemisphere $h \subset s$ containing $D$ and a $m$-dimensional face $\sigma \subset C$ such that $h \cap \sigma = D \cap \sigma$ has dimension $m - 1$. There is an apartment $A'$ containing the subset $h \cup \sigma$. and $D \not\subseteq A' \cap C$, contradicting again maximality.

4.1 Buildings of type $A_n$

In the case of buildings of type $A_n$ we are able to prove that any subcomplex (not just a fixed point set) which is not a subbuilding has circumradius $\leq \frac{5}{4}$.

**Theorem 4.2.** Let $B$ be a (not necessarily thick) spherical building of type $A_n$. Let $C \subset B$ be a convex subcomplex. Then either $C$ is a subbuilding or it has circumradius $\leq \frac{5}{4}$.

**Proof.** Let $x \in C$ and let $\tau$ be the face containing $x$ in its interior. Hence, $\tau \subset C$ because $C$ is a subcomplex. Let $V = \{v_1, \ldots, v_k\}$ be the set of vertices of $\tau$, then after identifying $\tau$ with a subset of the unit round sphere in $\mathbb{R}^k$, we can write $x = \sum_{i=1}^{k} a_i v_i$ with $a_i \geq 0$.

Let $A$ be an apartment supporting $C$ containing $x$. Let $L := C \cap A$ and let $K \subset A$ be the smallest top-dimensional subcomplex such that the interior of $L$ is contained in the interior of $K$. Then a point $y \in L$ is in the boundary of $L$ if and only if it is in the boundary of $K$ and since $A$ supports $C$, this is also equivalent to $y$ being in the boundary of $C$.

Consider the function $f_K(x)$ defined in Section 2.4.3. If $\alpha \in \Lambda_K^{\text{min}}$, then $\sin d(x, \partial \alpha) = \cos d(x, x_\alpha) = \sum_{i=1}^{k} a_i (v_i, x_\alpha) = \sum_{\alpha \notin \partial \alpha} a_i \lambda_i$, where $\lambda_i := \lambda_{v_i}$ is the constant given by Lemma 2.13. It follows that $f_K(x) = \max_{\alpha \in \Lambda_K^{\text{min}}} \{ -\sin d(x, \partial \alpha) \} = \max_{\alpha \in \Lambda_K^{\text{min}}} \{ -\sum_{\alpha \notin \partial \alpha} a_i \lambda_i \} = \max_{\alpha \in \Lambda_K^{\text{min}}} \{ -\sum_{v_i \in V - F} a_i \lambda_i \}$, where the last maximum is taken over all maximal subsets $F \subset V$ such that the face spanned by the vertices in $F$ is contained in the boundary of $K$, or equivalently, contained in the boundary of $C$. This implies that the function $f(x) := f_K(x)$ is independent of the apartment $A$ supporting $C$. 

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Let $f$ be a (not necessarily thick) spherical building of type $D_n$. Let $C \subset B$ be a top-dimensional convex subcomplex which is not a subbuilding. Let $x \in C$ be a vertex and $C$ be another apartment containing $x$ and supporting $C$. By Lemma 4.4, we can find a sequence $A = A_0, \ldots, A_m = A'$ of apartments supporting $C$ and containing $x$ such that $A_i \cap A_{i+1}$ is a root. Thus, we may assume that $\alpha := A \cap A'$ is a root.

Suppose that $f_{C \cap A}(x) > f_{C \cap A'}(x)$. Let $i$ be the type of the vertex $x$. Then by Lemma 2.14, $f_{C \cap A'}(x) = -\lambda_i > -2\lambda_i = f_{C \cap A}(x)$ and $3 \leq i \leq n-1$. We identify the apartments $A, A'$ simultaneously with the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that the centers of roots correspond to points $\frac{1}{\sqrt{2}}(\pm e_j \pm e_k)$, the identifications coincide in $\alpha = A \cap A'$ and $x$ corresponds to the point $\frac{1}{\sqrt{n+1-i}}(e_i + e_{i+1} + \cdots + e_n)$. To simplify the notation, we omit the normalizing factor, so we write $x = e_i + e_{i+1} + \cdots + e_n$. Observe that $x \in \alpha$ implies that the center $x_\alpha$ of $\alpha$ is of the form $e_j \pm e_k$ for $i \leq j, k$ and $j \neq k$; $\pm e_j + e_k$ for $j < i < k$; or $\pm e_j \pm e_k$ for $j < k < i$. Since $f_{C \cap A'}(x) = -\lambda_i$, there must be a root $\beta \in \Lambda_{C \cap A'}$ centered at $x_\beta$ such that $\cos d(x, x_\beta) = \lambda_i$. It follows that $x_\beta$ has the form $\pm e_r + e_s$ with $1 \leq r < i$ and $i \leq s \leq n$. Suppose $x_\beta = e_r + e_s$, the other case is similar. On the other hand, $f_{C \cap A}(x) = -2\lambda_i$ implies that $\bigcap_{\gamma \in \Pi_i} \gamma \subset C \cap A$ for $\Pi_i := \{ (e_j + e_k, \gamma) \geq 0 \mid i \leq j < k \}$. It follows that $\alpha \cap (\bigcap_{\gamma \in \Pi_i} \gamma) \subset C \cap A \cap A' \subset C \cap A'$.

Notice that since $A$ and $A'$ both support the subcomplex $C$, the roots $\alpha$ and $\beta$ must be different. This implies (see the different possibilities for $x_\alpha$ above) that there exists a point $y \in \alpha \cap \{ (v_1, \ldots, v_n) \in S^{n-1} \mid v_r = -2, v_i = v_{i+1} = \cdots = v_n = 1 \} \neq \emptyset$. Then $y \notin \beta$ and $y \in \alpha \cap (\bigcap_{\gamma \in \Pi_i} \gamma)$. We get a contradiction. Hence, $f_{C \cap A'}(x) \leq f_{C \cap A}(x)$. Interchanging the roles of $A$ and $A'$ we obtain the equality $f_{C \cap A'}(x) = f_{C \cap A}(x)$. \hfill \qed

Theorem 4.5. Let $B$ be a (not necessarily thick) spherical building of type $D_n$. Let $C \subset B$ be a top-dimensional convex subcomplex which is not a subbuilding. Then $-\sin d(\cdot, \partial C)$ is a nicely convex function on $K$ and $K$ has a unique incenter $x_0$. In particular, $C$ has circumradius $\leq \frac{\pi}{2}$.
Proof. Let \( x \in C \) and let \( \tau \subset C \) be the face containing \( x \) in its interior. Let \( v_{i_1}, \ldots, v_{i_k} \) be the vertices of \( \tau \) with \( v_{i_j} \) of type \( i_j \) and \( i_1 < \cdots < i_k \). After identifying \( \tau \) with a subset of the unit round sphere in \( \mathbb{R}^k \), we can write \( x = \sum_{j=1}^k a_j v_{i_j} \) with \( a_j \geq 0 \).

Let \( A \subset B \) be an apartment supporting \( C \) and \( x \in A \). We want to prove that the function \( f(x) := f_{C \cap A}(x) \) as defined in Section 2.4.3 does not depend on the choice of the apartment \( A \). The conclusion of the theorem then follows from Lemmata 4.1 and 2.6. So, let \( A' \) be another such apartment and suppose \( f_{C \cap A}(x) < f_{C \cap A'}(x) \).

Let \( \alpha \in \Lambda_{C \cap A'} \) be a root such that \( f_{C \cap A'}(x) = -\sin d(x, \partial \alpha) \). Suppose first that there is a vertex \( v = v_{i_j} \) of \( \tau \) with \( v \in \partial \alpha \). Then by Lemma 2.12 we have \( f_{C \cap A'} = \sin d(x, v) f_{\Sigma_{\alpha}(C \cap A')}(v) \). Since the links of a building of type \( D_n \) are spherical joins of buildings of type \( D \) and \( A \), using induction on \( n \) and Theorem 1.2, we obtain \( f_{\Sigma_{\alpha}(C \cap A')}(v) = f_{\Sigma_{\alpha}(C \cap A)}(v) \). It follows again by Lemma 2.12 that \( f_{C \cap A}(x) \geq \sin d(x, v) f_{\Sigma_{\alpha}(C \cap A)}(v) = \sin d(x, v) f_{\Sigma_{\alpha}(C \cap A)}(v) = f_{C \cap A'}(x) \), contradicting our first assumption \( f_{C \cap A}(x) < f_{C \cap A'}(x) \). Hence, \( \tau \) is contained in the interior of \( \alpha \).

Recall that by Lemma 2.14, \( \sin d(v_{i_j}, \partial \alpha) \) can only take at most the two values \( \lambda_{i_j}, 2\lambda_{i_j} \). Let \( r \) be the smallest number such that \( \sin d(v_{i_r}, \partial \alpha) = \lambda_{i_r} \). In particular, \( i_r \geq 3 \) by Lemma 2.14. Then by Lemma 2.15 \( \sin d(v_{i_j}, \partial \alpha) = \lambda_{i_j} \) for all \( j \geq r \). Therefore

\[
\begin{align*}
 f_{C \cap A'}(x) &= -\sin d(x, \partial \alpha) = -\sum_{j=1}^k a_j \sin d(v_{i_j}, \partial \alpha) = -2 \sum_{j=1}^{r-1} a_j \lambda_{i_j} - \sum_{i=r}^k a_j \lambda_{i_j} \\
 &> f_{C \cap A}(x) = \max_{\beta \in \Lambda_{C \cap A}} \{-\sin d(x, \partial \beta)\} = \max_{\beta \in \Lambda_{C \cap A}} \{-\sum_{j=1}^k a_j \sin d(v_{i_j}, \partial \beta)\}.
\end{align*}
\]

It follows that for each \( \beta \in \Lambda_{C \cap A} \), there must be a \( j \geq r \) such that \( \sin d(v_{i_j}, \partial \beta) = 2\lambda_{i_j} \). Then again by Lemma 2.15 \( \sin d(v_{i_r}, \partial \beta) = 2\lambda_{i_r} \) for all \( \beta \in \Lambda_{C \cap A} \). Thus, \( f_{C \cap A}(v_{i_r}) = -2\lambda_{i_r} < -\lambda_{i_r} = -\sin d(v_{i_r}, \partial \alpha) \leq f_{C \cap A'}(v_{i_r}) \), contradicting Lemma 4.3.

**Example 4.6.** Consider a building of type \( D_4 \) and the convex subcomplex \( C \) consisting of a segment \( c_1 \) with vertices of type 31313 and a segment \( c_2 \) with vertices 131, which intersect in their midpoints. Let \( x \) be their common midpoint. Let \( A_i \) be apartments containing \( c_i \). Then \( A_i \) supports \( C \). Let \( K_1 \subset A_i \) be the smallest top-dimensional subcomplex such that the interior of \( c_i \) is contained in the interior of \( K_1 \). Then \( K_1 \) is a root and \( x \) is its center. It follows that \( f_{K_1}(x) = -1 < f_{K_2}(x) \).

## 5 Fixed point sets of unipotent isometries

Let \( B \) be an irreducible spherical building of dimension at least 2 and let \( g \neq 1 \) be a unipotent isometry. Let \( A \subset B \) be an apartment such that \( \text{Fix}(g) \cap A \) is a top-dimensional subset. Then by Corollary 2.27 \( \text{Fix}(g) \cap A \) has a unique incenter. If \( B \) is of simply-laced type we will prove that \( \text{Fix}(g) \) also has a unique incenter. However, this is no longer true for other types of buildings. Nevertheless, we will show that \( \text{Fix}(g) \) has always a unique weighted incenter in the sense of Section 2.4.3.

Let \( g \) be an unipotent isometry of \( B \). Let \( \Phi \) the root system associated to \( B \) by Theorem 2.19. We consider now the top-dimensional convex subcomplex \( K := \text{Fix}(g) \cap A \subset A \) for some apartment \( A \subset B \). We want to define the weighted incenter of \( K \) as in Section 2.4.3.
For this, we have to define the corresponding weights for non-reduced roots. Let \( \alpha \subset A \) be a root such that the corresponding indivisible root \( \alpha \in \Phi \) is non-reduced. We define the weight \( \mu_\alpha \) in dependency on \( g \) as follows. First, if \( \alpha \notin \Lambda_K \), we set \( \mu_\alpha := 2 = \|2\alpha\| \); and if \( \alpha \in \Lambda_K - \Lambda_K^{\text{min}} \), we set \( \mu_\alpha := 1 = |\alpha| \). Finally, if \( \alpha \in \Lambda_K^{\text{min}} \), we set the weight \( \mu_\alpha := 2 \), if the \( \alpha \)-coordinate \( g_\alpha \) of \( g \) (see Proposition 2.23) is in \( U_{2\alpha} \) (see Theorem 2.19); and we set \( \mu_\alpha := 1 \), if \( g_\alpha \in U_{\alpha} - U_{2\alpha} \). Notice that by Proposition 2.23 the weights \( \mu_\alpha \) do not depend on the apartment \( A \) containing \( \alpha \).

**Theorem 5.1.** Let \( B \) be an irreducible spherical building of dimension at least 2 and let \( g \neq 1 \) be a unipotent isometry. Let \( x \in \text{Fix}(g) \) and let \( A \subset B \) be an apartment containing \( x \). Then the function \( f(x) := f_{\text{Fix}(g) \cap A}(x) \) as defined in Section 2.4.3 with the weights given above does not depend on the choice of the apartment \( A \). In particular, \( f \) defines a nicely convex function in \( \text{Fix}(g) \) and it has a unique minimum \( x_0 \in \text{Fix}(g) \), the weighted incenter of \( \text{Fix}(g) \). Moreover, \( \text{rad}(\text{Fix}(g), x_0) \leq \frac{\pi}{2} \).

**Proof.** Let \( A' \subset B \) be another apartment containing \( x \). We may assume that there is a chamber \( \sigma \subset \text{Fix}(g) \) with \( x \in \sigma \subset A' \cap A \). Then there is a unipotent element \( u \in U_\sigma \) such that \( uA = A' \). Let \( \Lambda_K \) be the set of the positive roots in \( A \) containing \( K := \text{Fix}(g) \cap A \) and let \( \Lambda_K' \) be the set of the positive roots in \( A' \) containing \( K' := \text{Fix}(g) \cap A' \).

Let \( \Pi' \) be a set of positive roots in \( A' \) such that \( g \in U_{\Pi'} = \langle U_{\alpha} \mid \alpha \in \Pi' \rangle \). Then \( M = \bigcap_{\alpha \in \Pi'} \alpha \subset \text{Fix}(g) \cap A' = K' \) and by Lemma 2.9

\[
f_M(x) = \max_{\alpha \in \Lambda_M} \left\{ -\mu_\alpha \sin d(x, \partial \alpha) \right\} = \max_{\alpha \in \Pi'} \left\{ -\mu_\alpha \sin d(x, \partial \alpha) \right\} = \max_{\alpha \in \Lambda_M} \left\{ -\mu_\alpha \sin d(x, \partial \alpha) \right\} \geq f_{K'}(x).
\]

Our goal is to find a \( \Pi' \) as above such that \( f_K(x) \geq -\mu_\alpha \sin d(x, \partial \alpha) \) for all \( \alpha \in \Lambda_M^{\text{min}} \subset \Pi' \). From this, it follows that \( f_K(x) \geq f_M(x) \geq f_{K'}(x) \). Switching the roles of \( A, A' \) we also deduce \( f_{K'}(x) \geq f_K(x) \) and therefore we obtain the equality \( f_K(x) = f_{K'}(x) \).

Without loss of generality we may assume that \( u =: g_0 \in U_{\alpha_0} \) for some positive root \( \alpha_0 \subset A \). Choose some minimal gallery from \( \sigma \) to its antipodal chamber in \( A \) and let \( g = g_1 \ldots g_d \) with \( g_i \in U_{\alpha_i} \) be the product representation of \( g \) with respect to this gallery. Then Proposition 2.23 implies that for \( i = 1, \ldots, d \) if \( \alpha_i \notin \Lambda_K \) then \( g_i = 1 \). For \( i = 0, 1, \ldots, d \) write \( \beta_i = 2\alpha_i \) if \( \alpha_i \) is non-reduced and \( g_i \in U_{2\alpha_i} \), (see Theorem 2.19) and \( \beta_i = \alpha_i \) otherwise. By Theorem 2.19 we have \( u^{-1}g_\gamma u = g_\gamma h_i \) with \( h_i \in \langle U_\gamma \mid \gamma = a_\beta 0 + b_\beta, \Phi, a, b \in \mathbb{N} \rangle \). Let \( \Pi \) be the set of roots \( \delta \subset A \) such that for the corresponding indivisible root \( \delta \in \Phi \) holds that \( \delta \) or \( 2\delta \) is in \( \{ \gamma \in \Phi \mid \gamma = a_\beta 0 + b_\beta, \Phi, \alpha_i \in \Lambda_K; a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{N} \} \). It follows that \( u^{-1}g_\gamma u \in \langle U_\gamma \mid \gamma \in \Pi \rangle \). Notice that \( \Lambda_K \subset \Pi \).

Let \( \Pi' = \{ u\gamma \mid \gamma \in \Pi \} \). Then \( \Pi' \) is a set of roots in \( A' \) and since \( U_{u\gamma} = uU_\gamma u^{-1} \), we obtain \( g \in U_{\Pi'} = \langle U_{\alpha} \mid \alpha \in \Pi' \rangle \). We now verify that \( \Pi' \) has the desired properties, that is, \( f_K(x) \geq -\mu_\alpha \sin d(x, \partial \alpha) \) for all \( \alpha \in \Lambda_M^{\text{min}} \subset \Pi' \).

We give first the argument for \( \Phi \) reduced because it is much simpler, although we could just omit it, since the argument in the non-reduced case works in general. So suppose \( \Phi \) is reduced. Take \( u\gamma \in \Pi' \). Let \( x_\gamma, x_{\alpha_i} \) be the centers of the respective roots. In the reduced case we have \( \beta_i = \alpha_i \) and \( \gamma = a_\alpha 0 + b_\alpha \) for some \( \alpha_j \in \Lambda_K \). Further, \( \mu_{u\gamma} = \mu_\gamma \). Identify as usual the apartment \( A \) with the unit sphere. Then \( \mu_{u\gamma} x_{\alpha_j} = \gamma, \mu_\gamma x_{\alpha_j} = \alpha_j \in \Phi \). It follows that \( \langle x, \mu_{u\gamma} x_\gamma \rangle - \langle x, \mu_\gamma x_{\alpha_j} \rangle = \langle x, \gamma - \alpha_j \rangle = \langle x, a_\alpha 0 + (b - 1)\alpha_j \rangle \geq 0 \) because
\[ \alpha_0, \alpha_j \text{ are positive roots and } a \geq 0, b \geq 1. \text{ This implies that } f_K(x) \geq -\mu_{\alpha_j} \sin d(x, \partial \alpha_j) = -\langle x, \mu_{\alpha_j} x_{\alpha_j} \rangle \geq -\langle x, \mu_{\alpha_j} x \rangle = -\mu_{\alpha_j} \sin d(x, \partial \gamma) = -\mu_{\alpha_j} \sin d(x, \partial (u\gamma)). \]

We consider now the general case. Take \( u \gamma \in \Lambda_{min}^\gamma \subset \Pi^j \) and let \( x, x_{\alpha_j} \) be the centers of the respective roots. In this case we have several possibilities: \( \mu_{u\gamma} x_{\alpha_j}, \beta_j \in \{ \alpha_j, 2\alpha_j \} \subset \Phi \) for some \( \alpha_j \in \Lambda_K \) and \( \mu_{u\gamma} x_{\alpha_j}, \beta_j \in \{ \alpha_j, 2\alpha_j \} \subset \Phi \). It follows that \( \mu_{u\gamma} x_{\alpha_j} = c(a\beta_0 + b\beta_j) \) and \( \mu_{u\gamma} x_{\alpha_j} = c'\beta_j \) for some \( c, c' \in \{1/2, 1, 2\} \). Hence, \( f_K(x) + \mu_{u\gamma} \sin d(x, \partial (u\gamma)) \geq -\mu_{\alpha_j} \sin d(x, \partial \alpha_j) + \mu_{u\gamma} \sin d(x, \partial (u\gamma)) = \langle x, \mu_{u\gamma} x_{\alpha_j} \rangle - \langle x, \mu_{\alpha_j} x_{\alpha_j} \rangle = \langle x, ca\beta_0 + (cb - c')\beta_j \rangle. \)

Since \( \alpha_0, \alpha_j \) are positive roots and \( a \geq 0, b \geq 1 \), it suffices to show that \( cb - c' \geq 0 \).

Suppose first that \( c' = 2 \). Then \( \alpha_j \in \Phi \) is non-reduced and \( \mu_{\alpha_j} x_{\alpha_j} = 2\beta_j = 2\alpha_j \). In particular, \( \mu_{\alpha_j} = 2 \). By the definition of the weights, it follows that \( \alpha_j \notin \Lambda_K \); or, \( \alpha_j \in \Lambda_{min}^\gamma \) and \( g_{\alpha_j} = g_j \in U_{2\alpha_j} \). The former cannot happen by the definition of \( \Pi \) and the latter implies \( \beta_j = 2\alpha_j \) by the definition of \( \beta_j \). We get a contradiction, thus, \( c \leq 1 \).

Suppose now that \( c = \frac{1}{2} \). Then \( \gamma \in \Phi \) is non-reduced and \( \mu_{u\gamma} x_{\alpha_j} = \gamma = \frac{1}{2}(a\beta_0 + b\beta_j) \). In particular, \( \mu_{\alpha_j} = 1 \). This implies that \( u\gamma \in \Lambda_K' - \Lambda_{min}^\gamma \); or, \( u\gamma \in \Lambda_{min}^\gamma \) and \( g_{\alpha_j} \in U_{\alpha_j} \). The former cannot happen because \( u\gamma \in \Lambda_{min}^\gamma \) and \( M \subset K' \). Thus, \( u\gamma \in \Lambda_{min}^\gamma \cap \Lambda_{min}^{\gamma'} \). Let \( \tau \subset A' \) be a panel, then \( \langle u\gamma \rangle \cap (M \cap K') \). Then by Proposition 2.23 we can read off the element \( g_{\alpha_j} \) from the action of \( g \) on \( \Sigma_{\tau} B \). Since \( \tau \subset \langle u\gamma \rangle \cap M \) lies on the boundary of \( M \) and \( M = \bigcap_{\alpha \in \Pi^j} U_\alpha \), the only root group \( U_\delta \) for \( \delta \in \Pi' \) acts non-trivially on \( \Sigma_{\tau} B \) is \( U_{\alpha_j} \).

Recall that \( g \in \langle u\delta^j \rangle \delta = p\beta_0 + q\beta_q \in \Phi; \alpha_j \in \Lambda_K; p \in \mathbb{Z}_{\geq 0}; q \in \mathbb{N} \). Suppose that for all \( k \) such that \( p\beta_0 + q\beta_k \in \{ \gamma, 2\gamma \} \) for some \( p \geq 0, q \geq 1 \) follows that \( p\beta_0 + q\beta_k = 2\gamma \). This would imply that \( g \) is a product of elements in \( U_\delta \) for \( \delta \in \Pi' - \{ u\gamma \} \) and elements in \( U_{\alpha_j} \). In turn would imply that the action of \( g \) on \( \Sigma_{\tau} B \) is given by the action of an element in \( U_{\alpha_j} \). This contradicts the fact \( g_{\alpha_j} \in U_{\alpha_j} \). Hence, there is a \( k \) such that \( p\beta_0 + q\beta_k = \gamma \). From this we see that after replacing \( j \) with \( k \) we may assume that \( c \geq 1 \).

Finally we can see that \( cb - c' \geq 1b - 1 \geq 1 - 1 = 0 \) and conclude that \( f_K(x) + \mu_{\alpha_j} \sin d(x, \partial (u\gamma)) \geq 0 \). This is what remained to be proved.

The remaining assertions of the theorem are just a consequence of Lemma 2.3.

**Remark 5.2.** Although we are mainly interested in buildings of dimension \( \geq 2 \), the results of this section remain valid for Moufang generalized triangles and quadragles with the proofs unchanged. The reason is that the commutator relations are still valid in these cases ([Tit94], [Tim00]).

## 6 Top-dimensional fixed point sets

In this section let again \( B \) be an irreducible spherical building of dimension at least 2. Let \( g \in Isom(B) \) be an isometry such that \( Fix(g) \subset B \) is a top-dimensional subcomplex which is not a subbuilding. Let \( A \subset B \) be an apartment such that \( Fix(g) \cap A \) is top-dimensional. Then we can define the function \( f_{Fix(g) \cap A} \) in \( Fix(g) \cap A \) as in Section 4 but it is no longer true that \( f_{Fix(g) \cap A} \) does not depend on the apartment \( A \) containing \( x \in Fix(g) \). However, we can rescue the argument if we consider only some specials apartments.

If \( A \) is an apartment with \( Fix(g) \cap A \) top-dimensional, we say that \( A \) supports \( g \) if \( A \) supports \( Fix(g) \) (cf. Section 4) and additionally the following holds: if \( \alpha \subset A \) is a non-reduced root in \( \Lambda_{Fix(g) \cap A}^\min \) and \( \tau \subset \partial \alpha \cap Fix(g) \) is a panel, then if there is an apartment \( A' \) containing \( \alpha \) such that the unique element in \( U_\alpha \) sending \( \Sigma_{\tau} A' \) to \( \Sigma_{\tau} gA' \) lies in \( U_{2\alpha} \), then the
unique element in $U_\alpha$ sending $\Sigma_\tau A$ to $\Sigma_\tau gA$ also lies in $U_{2\alpha}$. Lemma 2.1 readily generalizes to apartments supporting $g$, since in the notation above clearly $A'$ also supports $Fix(g)$.

Observe that if $u$ is a unipotent isometry, then any apartment, which intersects $Fix(u)$ in a top-dimensional set, supports $u$.

Before we continue, let us explain our motivation to consider this special kind of apartments. Let $X = G/K$ be a symmetric space of noncompact type and let $g \in G = Isom_0(X)$ be an isometry. Then $g$ has a Jordan decomposition $g = su$ such that $s$ is semisimple, $u$ is unipotent and they commute with each other. The minimal set $Min(s)$ (i.e. the set where the displacement function of $s$ attains its minimum) is a totally geodesic subspace. The boundary at infinity $\partial T Min(s)$ is a subbuilding of $\partial T X$ and $u$ acts on $\partial T Min(s)$ as a unipotent isometry. Further, $Fix_\infty(g) = Fix_\infty(s) \cap Fix_\infty(u) = \partial T Min(s) \cap Fix_\infty(u)$, where $Fix_\infty$ denotes the fixed point set in $\partial T X$ (see [Ebe96, Proposition 4.1.5] and the discussion in Section 2.4.3). We can then apply Theorem 3.1 to the action of $u$ on the building $\partial T Min(s)$ to conclude that $Fix_\infty(g)$ has circumradius $\leq \frac{\tau}{2}$. In general, we do not have a Jordan decomposition, so we use the union of the apartments supporting $g$ as a substitute of the subbuilding $\partial T Min(s)$.

We now return to our original discussion. Let now $A$ be an apartment supporting $g$. By Proposition 2.21 we can write $g = uh$ with $u \in U_\sigma$ for all $\sigma \subset Fix(g) \cap A$ and $h \in H = Fix_{Isom(B)}(A)$. In particular, $Fix(g) \cap A = Fix(u) \cap A$ and $gA = uA$. Now if $\alpha \subset A$ is a non-reduced root, then we define the weight $\mu_\alpha$ as the corresponding one for the unipotent isometry $u$. With these weights we obtain the functions $f_{Fix(g) \cap A}(x)$ as defined in Section 2.4.3.

Theorem 6.1. Let $B$ be an irreducible spherical building of dimension at least 2 and let $g \in Isom(B)$ be an isometry such that $Fix(g) \subset B$ is a top-dimensional subcomplex which is not a subbuilding. Let $x \in Fix(g)$ and let $A \subset B$ be an apartment containing $x$ and supporting $g$. Then the function $f(x) := f_{Fix(g) \cap A}(x)$ does not depend on the choice of the apartment $A$ supporting $g$. In particular, $f$ defines a nicely convex function in $Fix(g)$ and it has a unique minimum $x_0 \in Fix(g)$, the weighted incenter of $Fix(g)$. Moreover, $rad(Fix(g), x_0) \leq \frac{\tau}{2}$.

Proof. Let $A'$ be another apartment supporting $g$ and containing $x \in Fix(g)$. We may assume that $A \cap A'$ contains a chamber $x \in \sigma \subset Fix(g)$. Write $g = uh$ with $u \in U_\sigma$ and $h \in H = Fix_{Isom(B)}(A)$ and $g = u'h'$ with $u' \in U_{\sigma}$ and $h' \in H' = Fix_{Isom(B)}(A')$.

Consider $Fix(u) \cap A'$. We claim that it is contained in $Fix(h) \cap A'$: Suppose not, then there is a positive root $\alpha \subset A'$ such that $\alpha \in \Lambda_{Fix(h) \cap A'}^\text{min}$ and $\alpha \notin \Lambda_{Fix(u) \cap A'}^\text{min}$. Let $\tau$ be a panel in $\partial \alpha \cap Fix(u) \cap Fix(h)$. Let $\nu_0, \nu_1$ be the chambers in $-\alpha \subset A'$ and $\alpha$ respectively containing $\tau$. Then $\nu_1 \subset Fix(h) \cap Fix(u)$, thus $\nu_1 \subset Fix(g)$. Further, $\alpha \notin \Lambda_{Fix(u) \cap A'}^\text{min}$ implies that $\nu_0 \subset Fix(u)$, this in turn implies that $u$ fixes every chamber containing $\tau$ by Corollary 2.25. Also, $\alpha \in \Lambda_{Fix(h) \cap A'}^\text{min}$ implies that $\nu_0 \notin Fix(h)$. It follows that $gu_0 = uhv_0 = hv_0 \neq \nu_0$ and $\nu_0 \notin Fix(g)$. Therefore $\tau \subset \partial(Fix(g) \cap A) \subset \partial Fix(g)$ because $A'$ supports $Fix(g)$. On the other hand, $Fix(h) \subset B$ is a subbuilding and therefore there exists a chamber $\nu' \subset Fix(h)$ such that $\nu' \cap \nu_1 = \tau$. But $u$ must also fix $\nu'$, hence, $g = uh$ fixes $\nu'$ as well. This contradicts the fact that $\tau \subset \partial Fix(g)$. So we conclude that $Fix(u) \cap A' \subset Fix(h) \cap A'$ and in particular $Fix(u) \cap A' \subset Fix(g) \cap A'$.

In the non-reduced case we have to pay special attention in how the weights are defined. For $\beta \subset A'$, let $\bar{\mu}_\beta$ be the weight as defined for the unipotent isometry $u$ and let...
\[ \tilde{f}_{\text{Fix}(g)\cap A'}(x) := \max_{\beta \in \Lambda_{\text{Fix}(g)\cap A'}} \{-\bar{\mu}_\beta \sin d(x, \partial \beta)\}. \]

Then Theorem [5.1] applied to \( u \) implies

\[ f_{\text{Fix}(g)\cap A}(x) = f_{\text{Fix}(u)\cap A}(x) = \tilde{f}_{\text{Fix}(u)\cap A'}(x). \]

By Lemma [2.9] and the fact \( \text{Fix}(u) \cap A' \subset \text{Fix}(g) \cap A' = \text{Fix}(u') \cap A' \) (cf. Lemma [2.11]), if we want to show that \( \tilde{f}_{\text{Fix}(u)\cap A'}(x) \geq f_{\text{Fix}(g)\cap A'} = f_{\text{Fix}(u')\cap A'} \), we just have to show that for every root \( \beta \in \Lambda^\text{min}_{\text{Fix}(u')\cap A'} \cap \Lambda^\text{min}_{\text{Fix}(u)\cap A'} \) holds \( \mu_\beta \geq \bar{\mu}_\beta \). This is clear unless \( \beta \) is non-reduced. Let us consider this case. Since \( \text{Fix}(u) \cap A' \subset \text{Fix}(h) \cap A' \), we can take a chamber \( \omega \subset \text{Fix}(h) \) such that \( \omega \cap \beta \) is a panel \( \pi \). Let also \( \omega' \subset A' \subset \text{Fix}(h') \) be the chamber such that \( \omega' \cap \beta = \pi \). Recall the definition of the weights: We have \( \mu_\beta = 2 \) if the element \( (\omega' \mapsto g\omega' = u'\omega') \in U_\beta \) lies in \( U_{2\beta} \) and \( \mu_\beta = 1 \) otherwise; similarly, \( \bar{\mu}_\beta = 2 \) if the element \( (\omega \mapsto g\omega = u\omega) \in U_\beta \) lies in \( U_{2\beta} \) and \( \bar{\mu}_\beta = 1 \) otherwise. Therefore \( \bar{\mu}_\beta = 2 \) implies \( \mu_\beta = 2 \) by the condition about non-reduced roots in the definition of an apartment supporting \( g \) and because \( A' \) supports \( g \). Hence, \( \mu_\beta \geq \bar{\mu}_\beta \). So we can conclude

\[ f_{\text{Fix}(g)\cap A} = f_{\text{Fix}(u)\cap A} = \tilde{f}_{\text{Fix}(u)\cap A'} \geq f_{\text{Fix}(g)\cap A'}. \]

Exchanging the roles of \( A, A' \) we obtain \( f_{\text{Fix}(g)\cap A} = f_{\text{Fix}(g)\cap A'} \). The second part of the assertion follows from Lemmata [4.1] and [2.3].

Theorem [C] from the introduction follows directly from Theorem [6.1] and Lemma [2.1].

**Corollary 6.2.** Let \( H \subset \text{Isom}(B) \) be a subgroup of isometries such that the fixed point set \( \text{Fix}(H) \) is top-dimensional. Suppose that there is an element \( g \in H \) such that \( \text{Fix}(g) \) is not a subbuilding. Then \( \text{rad}(\text{Fix}(H)) \leq \frac{\pi}{2} \).

**Proof.** By Theorem [6.1] \( \text{Fix}(g) \) has a circumcenter \( x_0 \) such that \( \text{rad}(\text{Fix}(g), x_0) \leq \frac{\pi}{2} \). Since \( \text{Fix}(H) \subset \text{Fix}(g) \), it follows that \( \text{Fix}(H) \subset B_{\frac{\pi}{2}}(x_0) \). The distance from \( x_0 \) to points in \( \text{Fix}(H) \) cannot be constant \( \frac{\pi}{2} \) because \( \text{Fix}(H) \) is top-dimensional. This implies that the projection \( \bar{x}_0 \) of \( x_0 \) into \( \text{Fix}(H) \) is well defined and \( \text{rad}(\text{Fix}(H), \bar{x}_0) \leq \frac{\pi}{2} \). \( \square \)

## 7 Fixed point sets in non-exceptional buildings

In this section we consider fixed point sets of any codimension and show that fixed point sets of isometries of spherical buildings without factors of exceptional type are either subbuildings or have circumradius \( \leq \frac{\pi}{2} \). The proof relies on Lemma [2.18]. This Lemma does not hold for the Coxeter complex of type \( F_4 \). We have not find counterexamples for the types \( E_k \), \( k = 6, 7, 8 \), but we also have no reason to believe that they do not exist. Hence, the proof of Theorem [7.4] cannot be extended for the buildings of exceptional type.

Let again \( B \) be an irreducible spherical building of dimension \( \geq 2 \). Let \( g \in \text{Isom}_0(B) \) be a type-preserving isometry with \( m \)-dimensional fixed point set \( \text{Fix}(g) \) which is not a subbuilding.

**Lemma 7.1.** Let \( g = uk \) with \( u \) unipotent be the decomposition given in Proposition [2.27] with respect to an apartment \( A \), that is, \( \text{Fix}(k) \cap A = s \) is a singular sphere of the same dimension as \( \text{Fix}(g) \) and \( \text{Fix}(g) \cap A = \text{Fix}(u) \cap s \not\subset \partial \text{Fix}(u) \). Then \( g = uk \) is also the decomposition of \( g \) with respect to any apartment containing \( \text{Fix}(k) \cap A \).
Proof. Let \( A' \) be an apartment containing \( s = \text{Fix}(k) \cap A \) and let \( g = u'k' \) be the decomposition with respect to \( A' \). Then \( \text{Fix}(k') \cap A' = s \) and \( k^{-1}k' \) must fix \( s \). On the other hand, \( u \) and \( u' \) fix a neighborhood of \( \text{Fix}(g) \cap s \) by Proposition 2.21 and \( u^{-1}u' \) is unipotent by Corollary 2.24. It follows that the unipotent isometry \( u^{-1}u' = k^{-1}k' \) fixes an apartment and therefore must be the identity.

Let \( \tau \subset B \) be a \((m-1)\)-dimensional face in \( \partial \text{Fix}(g) \). Let \( \{\xi\} := \Sigma_\tau \text{Fix}(g) \). The point \( \xi \) is a vertex in \( \Sigma_\tau B \). Let \( \xi' \in \Sigma_\tau B \) be a vertex antipodal to \( \xi \). Choose an apartment \( A \subset B \) containing \( \tau \) and such that \( \xi, \xi' \in \Sigma_\tau A \). Let \( g = uk \) be the decomposition of \( g \) with respect to the apartment \( A \) given in Proposition 2.21, that is, \( u \) is unipotent and \( \text{Fix}(k) \cap A \) is a singular sphere of the same dimension as \( \text{Fix}(g) \). The weights for roots in \( A \) defined by the unipotent isometry \( u \) as in Section 5 induce weights for the roots in \( \Sigma_\tau A \) and we obtain a convex function \( f_{\Sigma_\tau(\text{Fix}(u) \cap A)} \). We define \( \lambda_{g,\tau,\xi} := -f_{\Sigma_\tau(\text{Fix}(u) \cap A)}(\xi) \). Notice that \( \lambda_{g,\tau,\xi} \) depends only on \( \hat{\xi} \) and not in the chosen apartment \( A \), this follows from Lemma 7.1 and Theorem 5.1 applied to the isometry \( \Sigma_\tau B \) induced by \( g \). We also define the number \( \lambda_{g,\tau} := \max\{\lambda_{g,\tau,\xi} \mid \xi \text{ antipode of } \xi \text{ in } \Sigma_\tau B\} \).

We say that an apartment \( A \) supports \( g \) if the following holds:

(i) \( A \) supports the convex subcomplex \( \text{Fix}(g) \).

(ii) Let \( s \subset A \) be the \( m \)-dimensional singular sphere containing \( \text{Fix}(g) \cap A \). If \( \tau \) is a \((m-1)\)-dimensional face in \( \partial(\text{Fix}(g) \cap A) \subset \partial \text{Fix}(g) \), then with the notation above holds \( \lambda_{g,\tau,\xi} = \lambda_{g,\tau} \), where \( \{\xi\} := \Sigma_\tau \text{Fix}(g) \) and \( \{\xi, \xi'\} := \Sigma_\tau s \).

Remark 7.2. Condition (ii) coincides with the condition about non-reduced roots in the definition of apartments supporting \( g \) for top-dimensional fixed point sets in Section 6. Hence, this definition generalizes the top-dimensional case.

Lemma 7.3. Any two points in \( \text{Fix}(g) \) are contained in an apartment supporting \( g \).

Proof. By Lemma 4.1, for any two points in \( \text{Fix}(g) \) there is an apartment \( A \) supporting \( \text{Fix}(g) \) containing them. Let \( \tau \) be a \((m - 1)\)-dimensional face in \( \partial(\text{Fix}(g) \cap A) \subset \partial \text{Fix}(g) \) and let \( \{\xi\} = \Sigma_\tau \text{Fix}(g) \). Let \( \xi \in \Sigma_\tau B \) be an antipode of \( \xi \) such that \( \lambda_{g,\tau,\xi} = \lambda_{g,\tau} \). Let \( h \subset A \) be the singular hemisphere of dimension \( m \) containing \( \text{Fix}(g) \cap A \) and such that \( \tau \subset \partial h \). Notice that \( \Sigma_\tau h = \Sigma_\tau \text{Fix}(g) = \{\xi\} \). There exists an apartment \( A' \subset B \) containing \( h \) and such that \( \hat{\xi} \in \Sigma_\tau A' \). Then \( A' \) is an apartment supporting \( \text{Fix}(g) \) and satisfying condition (ii) for the face \( \tau \).

Let \( s' \) be the singular sphere of dimension \( m \) in \( A' \) containing \( h \). Suppose that \( \tau' \) is a face in a different codimension one boundary component of \( \partial(\text{Fix}(g) \cap A) \) from \( \tau \), and observe that \( \Sigma_\tau h = \Sigma_\tau' s' \). Therefore if \( A \) already satisfied condition (ii) for \( \tau' \), then \( A' \) still satisfied condition (ii) for \( \tau' \). We can repeat the construction until we get an apartment supporting \( \text{Fix}(g) \) satisfying condition (ii) for all codimension one faces. \( \square \)

Theorem 7.4. Let \( B \) be an irreducible spherical building of dimension at least 2 and not of type \( F_4, E_6, E_7, E_8 \). Let \( g \in \text{Isom}_0(B) \) be an isometry such that \( \text{Fix}(g) \subset B \) is not a subbuilding. Let \( x \in \text{Fix}(g) \) and let \( A \subset B \) be an apartment containing \( x \) and supporting \( g \). Let \( u \) be the unipotent part of \( g \) with respect to \( A \) as in Proposition 2.24. Then the function \( f(x) := f_{\text{Fix}(u) \cap A}(x) \) as defined in Section 3 does not depend on the choice of the apartment \( A \) supporting \( g \). In particular, \( f \) defines a nicely convex function in \( \text{Fix}(g) \) and it has a unique minimum \( x_0 \in \text{Fix}(g) \), the weighted incenter of \( \text{Fix}(g) \). Moreover, \( \text{rad}(\text{Fix}(g), x_0) \leq \frac{\pi}{2} \).
Proof. Let $A'$ be another apartment supporting $g$ and containing $x \in Fix(g)$. Write $g = uk$ with respect to $A$ and $g = u'k'$ with respect to $A'$. Let $s = Fix(k) \cap A$ and $s' = Fix(k') \cap A'$.

As in the top-dimensional case, we want to show that $Fix(u) \cap s' \subset Fix(k) \cap A' = Fix(k) \cap s'$. The proof is the same, using the fact that $A'$ supports $Fix(g)$. Suppose not, then there is a face $\tau$ of full dimension in $\partial Fix(k) \cap s'$ and two faces $v_0, v_1 \subset Fix(u) \cap s'$ of full dimension in $s'$ such that $v_0 \cap v_1 = \tau$, $v_0 \not\subset Fix(k) \cap s'$ and $v_1 \subset Fix(k) \cap s'$. This implies that $u$ fixes $St_\tau B$ pointwise. It follows, $g\nu_0 = u(k\nu_0) = k\nu_0 \neq \nu_0$, hence, $\nu_0 \not\subset Fix(g)$. Therefore $\tau \subset \partial Fix(g) \cap A' \subset \partial Fix(g)$ because $A'$ supports $Fix(g)$. On the other hand, $Fix(k) \subset B$ is a subbuilding and therefore there exists a chamber $\nu' \subset Fix(k)$ such that $\nu' \cap \nu_1 = \tau$. But $u$ must also fix $\nu'$, hence, $g\nu' = u\nu' = \nu' \neq \nu' \subset Fix(g)$. This contradicts the fact that $\tau \subset \partial Fix(g)$. So we conclude that $Fix(u) \cap s' \subset Fix(k) \cap s'$ and in particular $Fix(u) \cap s' \subset Fix(g) \cap A' = Fix(u') \cap s'$.

Let us consider first the case when $x \in Fix(g)$ is a vertex. Suppose that $f_{Fix(u') \cap A'}(x) > f_{Fix(u) \cap A}(x)$. Let $f_{Fix(u) \cap A}(x)$ be the function in $Fix(u) \cap A'$ with the weights defined by $u$. Then $f_{Fix(u) \cap A}(x) = f_{CH(K_1, \partial h)}(x)$ by Theorem 5.1. Now we apply Lemma 2.18 to $K_1 = Fix(u') \cap A'$ and $K_2 = Fix(u) \cap A'$, let $\alpha \in \Lambda_{min}$ be a root such that $f_{K_1}(x) = -\mu_\alpha sin d(x, \partial \alpha)$, then the Lemma implies that $h := \alpha \cap s' \subset \Lambda_{min} \cap \Lambda_{K_2}$. Notice that $h \subset CH(K_1, \partial h)$ and therefore $f_{K_1}(x) = f_{CH(K_1, \partial h)}(x)$. Let $\xi := \Sigma_{\partial h}(K_1 \cap s') \subset \Sigma_{\partial h}(Fix(g) \cap A') = \Sigma_{\partial h}(Fix(g))$. By Lemma 2.12 we have $f_{K_1}(x) = f_{CH(K_1, \partial h)}(x) = sin d(x, \partial h)f_{\Sigma_{\partial h}K_1}(\xi)$ and $f_{K_2}(x) = f_{CH(K_2, \partial h)}(x) = sin d(x, \partial h)f_{\Sigma_{\partial h}K_2}(\xi)$. Our assumption $f_{K_1}(x) > f_{K_2}(x)$ implies $f_{\Sigma_{\partial h}K_1}(\xi) > f_{\Sigma_{\partial h}K_2}(\xi)$.

Let $\tau$ be a face in $\partial h \cap (K_1 \cap s') \cap (K_2 \cap s') \subset \partial Fix(g)$. Let $\xi_1 \in \Sigma_{\tau} B$ be the vertex such that $\Sigma_{\tau} s' = \{\xi_1, \xi_1\}$. Then $-\lambda_{g,\tau, \xi_1} = f_{\Sigma_{\tau} Fix(u') \cap A'}(\xi_1) = f_{\Sigma_{\partial h}K_1}(\xi_1)$. Since $K_2 \cap s' = Fix(u) \cap s' \subset Fix(k) \cap A'$ and $Fix(k)$ is a subbuilding, we find an apartment $A'' \subset Fix(k)$ containing $\tau$ and $\xi \in \Sigma_{\tau} A''$. Let $\xi_2 \in \Sigma_{\tau} A''$ be the antipode of $\xi$. Then $-\lambda_{g,\tau, \xi_2} = f_{\Sigma_{\tau} Fix(u) \cap A''}(\xi_2) = f_{\Sigma_{\tau} Fix(u) \cap A''}(\xi) = f_{\Sigma_{\tau} Fix(u) \cap A''}(\xi) = f_{\Sigma_{\partial h}K_2}(\xi)$ (the second equality follows from Theorem 5.1). It follows that $\lambda_{g,\tau, \xi_2} > -\lambda_{g,\tau, \xi_1}$. Since $A'$ supports $g$, condition (ii) implies that $\lambda_{g,\tau, \xi_1} = \lambda_{g,\tau}$, hence $\lambda_{g,\tau, \xi_2} > \lambda_{g,\tau}$, contradicting the definition of $\lambda_{g,\tau}$. Thus, $f_{Fix(u') \cap A''}(x) \leq f_{Fix(u) \cap A}(x)$ and interchanging the roles of $A$ and $A'$, we obtain $f_{Fix(u') \cap A'}(x) = f_{Fix(u') \cap A}(x)$. We have shown the theorem in the case when $x \in Fix(g)$ is a vertex.

Now we proceed as in the proof of Theorem 4.5 to show the general case. Let $x \in Fix(g)$ and let $\tau \subset Fix(g)$ be the face containing $x$ in its interior. Let $v_1, \ldots, v_k$ be the vertices of $\tau$ with $v_{i_j}$ of type $i_j$ and $i_1 < \cdots < i_k$. After identifying $\tau$ with a subset of the unit round sphere in $\mathbb{R}^k$, we can write $x = \sum_{j=1}^k a_j v_{i_j}$ with $a_j \geq 0$. Suppose again that $f_{Fix(u') \cap A'}(x) > f_{Fix(u) \cap A}(x)$.

Let $\alpha \in \Lambda_{Fix(u') \cap A'}$ be a root such that $f_{Fix(u') \cap A'}(x) = -\mu_\alpha sin d(x, \partial \alpha)$. Suppose first that there is a vertex $v = v_{i_j}$ of $\tau$ with $v \in \partial \alpha$. Then by Lemma 2.12 we have $f_{Fix(u) \cap A'} = sin d(x, \partial \alpha)f_{\Sigma_{\partial h}Fix(u) \cap A'}$. The link $\Sigma_{\partial h} B$ has again no factors of exceptional type, then using induction on the rank of the building applied to the isometry of $\Sigma_{\partial h} B$ induced by $g$, we obtain $f_{\Sigma_{\partial h}Fix(u) \cap A'}(\overline{e}) = f_{\Sigma_{\partial h}Fix(u) \cap A'}(\overline{e})$. It follows again by Lemma 2.12 that $f_{Fix(u') \cap A}(x) \geq sin d(x, \partial \alpha)f_{\Sigma_{\partial h}Fix(u) \cap A'}(\overline{e}) = sin d(x, \partial \alpha)f_{\Sigma_{\partial h}Fix(u) \cap A'}(\overline{e}) = f_{Fix(u') \cap A'}(x)$, contradicting our assumption $f_{Fix(u) \cap A}(x) < f_{Fix(u') \cap A'}(x)$. Hence, $\tau$ is contained in the interior of $\alpha$.

Recall that by Lemmata 2.16 and 2.14 $\mu_\alpha sin d(v_{i_j}, \partial \alpha)$ can only take at most the two values $\lambda_{i_j}, 2\lambda_{i_j}$. Let $r$ be the smallest number such that $\mu_\alpha sin d(v_{i_j}, \partial \alpha) = \lambda_{i_j}$. Then by
Lemmata 2.17 and 2.15, \( \mu_\alpha \sin d(v_{ij}, \partial \alpha) = \lambda_{ij} \) for all \( j \geq r \). Therefore

\[
\begin{align*}
  f_{\text{Fix}(u') \cap A'}(x) &= -\mu_\alpha \sin d(x, \partial \alpha) = -\sum_{j=1}^{k} a_j \mu_\alpha \sin d(v_{ij}, \partial \alpha) = -2 \sum_{j=1}^{r-1} a_j \lambda_{ij} - \sum_{j=r}^{k} a_j \lambda_{ij} \\
  > f_{\text{Fix}(u) \cap A}(x) &= \max_{\beta \in \Lambda_{\text{Fix}(u) \cap A}} \{-\mu_\beta \sin d(x, \partial \beta)\} \\
  &= \max_{\beta \in \Lambda_{\text{Fix}(u) \cap A}} \{-\sum_{j=1}^{k} a_j \mu_\beta \sin d(v_{ij}, \partial \beta)\}.
\end{align*}
\]

It follows that for each \( \beta \in \Lambda_{\text{Fix}(u) \cap A} \), there must be a \( j \geq r \) such that \( \mu_\beta \sin d(v_{ij}, \partial \beta) = 2\lambda_{ij} \). Then again by Lemma 2.15, \( \mu_\beta \sin d(v_{ir}, \partial \beta) = 2\lambda_{ir} \) for all \( \beta \in \Lambda_{\text{Fix}(u) \cap A} \). Thus, \( f_{\text{Fix}(u) \cap A}(v_{ir}) = -2\lambda_{ir} < -\lambda_{ir} = -\mu_\alpha \sin d(v_{ij}, \partial \alpha) \leq f_{\text{Fix}(u') \cap A'}(v_{ir}) \), contradicting the case for vertices in \( \text{Fix}(g) \). We conclude in the general case that \( f_{\text{Fix}(u') \cap A'}(x) = f_{\text{Fix}(u) \cap A}(x) \). \( \Box \)

8 Some special cases and applications

8.1 Long root subgroups

Recall from Theorem 2.19, that if \( \alpha \) is a non-reduced root, then we have some flexibility in defining the root group \( U_{2\alpha} \subset U_\alpha \). The next proposition implies that there is a unique maximal such subgroup and it coincides with the pointwise stabilizer of the ball of radius \( \frac{\pi}{2} \) containing \( \alpha \subset B \). This gives a geometric definition of \( U_{2\alpha} \).

**Proposition 8.1.** Let \( B \) be an irreducible spherical building of dimension at least two, a Moufang generalized triangle or a Moufang generalized quadrangle with associated root system \( \Phi \). Let \( \alpha \subset B \) be a root and let \( x_\alpha \) be its center.

If \( \Phi \) is reduced and \( \alpha \in \Phi \) is a long root, then \( U_\alpha = \text{Fix}_{\text{Isom}(B)}(B_{\frac{\pi}{2}}(x_\alpha)) \).

If \( \Phi \) is non-reduced and \( \alpha \in \Phi \) is a non-reduced root, then \( U_{2\alpha} \subset \text{Fix}_{\text{Isom}(B)}(B_{\frac{\pi}{2}}(x_\alpha)) \subset U_\alpha \). Moreover, we can replace \( U_{2\alpha} \) with \( \text{Fix}_{\text{Isom}(B)}(B_{\frac{\pi}{2}}(x_\alpha)) \) and Theorem 2.19 remains valid.

**Proof.** By the definition of root subgroup, it is clear that \( \text{Fix}_{\text{Isom}(B)}(B_{\frac{\pi}{2}}(x_\alpha)) \subset U_\alpha \) for any root \( \alpha \). Let now \( 1 \neq g \in U_\alpha \) (or \( U_{2\alpha} \) in the non-reduced case) as in the statement of the proposition. Let \( f \) be the function on \( \text{Fix}(g) \) given by Theorem 5.1. Let \( A \) be an apartment containing the root \( \alpha \). Then \( f(x_\alpha) = f_{\text{Fix}(g) \cap A}(x_\alpha) = f_\alpha(x_\alpha) = -\mu_\alpha \). By the hypothesis of the proposition \( \mu_\alpha \) is the norm of the longest root in \( \Phi \). Now for any other apartment \( A' \) containing \( x_\alpha \) we have \( -\mu_\alpha = f(x_\alpha) = f_{\text{Fix}(g) \cap A'}(x_\alpha) = \max_{\beta \in \Lambda_{\text{Fix}(g) \cap A'}} \{-\mu_\beta \sin d(x_\alpha, \partial \beta)\} \geq \max_{\beta \in \Lambda_{\text{Fix}(g) \cap A'}} \{-\mu_\beta\} \geq -\mu_\alpha \). The equality implies that \( x_\alpha \) must be the center of every root in \( \Lambda_{\text{Fix}(g) \cap A'} \). That is, \( \text{Fix}(g) \cap A' = B_{\frac{\pi}{2}}(x_\alpha) \cap A' \). It follows that \( \text{Fix}(g) = B_{\frac{\pi}{2}}(x_\alpha) \).

Now we prove the second assertion. For a non-reduced root \( \alpha \), let \( \tilde{U}_{2\alpha} := \text{Fix}_{\text{Isom}(B)}(B_{\frac{\pi}{2}}(x_\alpha)) \).
We have to verify the commutator relations for these subgroups. Let \( \beta \) be another root and let \( x_\beta \) be its center. Then \( d(x_\alpha, x_\beta) \in \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\} \).

Case \( d(x_\alpha, x_\beta) = \frac{\pi}{4} \): Then \([\tilde{U}_{2\alpha}, U_\beta] \subset [U_\alpha, U_\beta] = 1\).

Case \( d(x_\alpha, x_\beta) = \frac{\pi}{2} \): Let \( g \in \tilde{U}_{2\alpha} \) and \( h \in U_\beta \). Then \( h \) fixes \( x_\alpha \) and therefore stabilizes \( B_{\frac{\pi}{2}}(x_\alpha) \). Since \( g \) fixes \( B_{\frac{\pi}{2}}(x_\alpha) \) pointwise, it follows that \([g, h] \) also fixes \( B_{\frac{\pi}{2}}(x_\alpha) \) pointwise. Similarly, \( g \) fixes \( x_\beta \) and therefore stabilizes \( St_{x_\beta}B \). On the other hand \( h \)
fixes $St_{x_i}B$ pointwise and it follows that $[g, h]$ also fixes $St_{x_i}B$ pointwise. We conclude that $Fix([g, h]) \supset B_{\frac{\pi}{2}}(x_i) \cup St_{x_i}B$. But the convex hull of $B_{\frac{\pi}{2}}(x_i) \cup St_{x_i}B$ is $B$, hence $Fix([g, h]) = B$ and $[g, h] = 1$. We obtain $[\tilde{U}_{2\alpha}, U_\beta] = 1$.

Case $d(x_\alpha, x_\beta) = \frac{3\pi}{4}$: Let $g \in \tilde{U}_{2\alpha}$ and $h \in U_\beta$. Then $[h, g] = g_1g_2$ with $g_i \in U_{\beta + \alpha}$ (see e.g. [Wei03, Prop. 11.17]). By [Tit94, Lemma 2.1], $g_i$ is conjugated to $g$ and therefore it fixes a ball of radius $\frac{\pi}{2}$. This implies that $g_1 \in \tilde{U}_{2(\beta + \alpha)}$. Hence $[\tilde{U}_{2\alpha}, U_\beta] \subset \langle \tilde{U}_{2(\beta + \alpha)}, U_{\beta + 2\alpha} \rangle$.

Case $d(x_\alpha, x_\beta) = 0$: Let $g \in \tilde{U}_{2\alpha}$ and $h \in U_\alpha$. Choose a root $\gamma$ with $d(x_\alpha, x_\gamma) = \frac{\pi}{2}$ and let $u \in U_\gamma$. Then by [Tit94, Lemma 2.1], there is a $g' \in U_{\alpha - \gamma}$ conjugate to $g$ (and therefore $g' \in \tilde{U}_{2(\alpha - \gamma)}$) and a $k \in U_{2\alpha - \gamma}$ such that $[u, g'] = gk$. Since $[U_\gamma, U_\alpha] = 1$ and $[U_{2\alpha - \gamma}, U_\alpha] = 1$, the elements $u$ and $k$ commute with $h$. Moreover, by the second case above $[\tilde{U}_{2(\alpha - \gamma)}, U_\alpha] = 1$ and we also see that $g'$ commutes with $h$. It follows that $[g, h] = 1$ and therefore $[\tilde{U}_{2\alpha}, U_\alpha] = 1$, in other words, $\tilde{U}_{2\alpha} \subset Z(U_\alpha)$.

$$\square$$

### 8.2 Commuting unipotent elements

The following result is well known in the setting of algebraic groups. We give here a geometric proof that works for any spherical building.

**Proposition 8.2.** The product of two commuting unipotent isometries is again unipotent.

**Proof.** Let $g_1, g_2$ be two commuting unipotent isometries. Then $g_i$ stabilizes $Fix(g_{3-i})$. Let $f_i$ be the function on $Fix(g_i)$ given by Theorem 5.1 and let $x_i$ be the corresponding weighted center of $Fix(g_i)$.

Let $\{i, j\} = \{1, 2\}$. Let $A$ be an apartment containing $x \in Fix(g_i)$. Then $f_i(g_jx) = f_{Fix(g_i) \cap g_jA}(g_jx) = f_{Fix(g_i)^{-1} \cap g_jA}(x) = f_{Fix(g_i)^{-1} \cap A}(x) = f_i(x)$. Hence, the function $f_i$ is $g_j$-invariant. Therefore $g_j$ must fix the unique minimum $x_1$ and $x_1x_2 \subset Fix(g_1) \cap Fix(g_2)$. Notice that $x_1$ is an interior point of $Fix(g_1)$ by definition. It follows that the midpoint $x_0$ of the segment $x_1x_2$ is interior in $Fix(g_1)$ and in $Fix(g_2)$. This implies that there is a chamber $\sigma \subset Fix(g_1) \cap Fix(g_2)$ containing $x_0$. Then, by Corollary 2.21, $g_i \in U_\sigma$ and in particular $g_1g_2 \in U_\sigma$.

$$\square$$

### 8.3 Jordan decomposition

In this section we want to consider special examples of spherical buildings that include the buildings associated to algebraic groups and isometries for which there exists a Jordan decomposition. First we will explain the setting that occurs in the algebraic groups and then we state the results in a purely geometric manner forgetting the algebraic group structure.

Let $G$ be a semisimple algebraic group defined over an algebraically closed field $k$. Let $B_{G,k}$ denote the associated spherical building. The faces of $B_{G,k}$ correspond to parabolic subgroups and the chambers to minimal parabolic subgroups, that is, Borel subgroups. An apartment corresponds to a maximal torus, the faces contained in the apartment are the parabolics containing the maximal torus. The group $G(k)$ acts on $B_{G,k}$ by type-preserving isometries. An element $g \in G(k)$ has fixed point set a subbuilding if and only if it is a semisimple element. Let $s \in G(k)$ be semisimple, the fixed point set $Fix(s)$ consists on all apartments corresponding to maximal tori containing $s$. There is a torus $S \subset G$ such that
$C_G(S) = C_G(s)$. The torus $S$ corresponds to a singular sphere $\varsigma \subset B_{G,k}$ and $Fix(s)$ is the union of the apartments containing $\varsigma$. In particular $Fix(s)$ is always top-dimensional.

Let now $G$ be a semisimple algebraic group defined over an arbitrary field $k$. Suppose further that $G$ is $k$-split, that is, $G$ contains a $k$-split (diagonalizable over $k$) maximal torus. Let $\bar{k}$ be the algebraic closure of $k$. We have that $B_{G,k}$ is a subbuilding of $B_{G,\bar{k}}$. Since $G$ is $k$-split, $B_{G,k}$ is top-dimensional in $B_{G,\bar{k}}$. Let $s \in G(k)$ be a semisimple element. In general, $Fix(s) \subset B_{G,k}$ is not top-dimensional anymore. Suppose first that $Fix(s) \subset B_{G,k}$ is top-dimensional. Then by the above discussion, the apartments in the fixed point set of the action $s \circ B_{G,k}$ must contain the singular sphere $\varsigma \subset B_{G,k}$. This implies that the torus $S$ is $k$-defined and $\varsigma \subset B_{G,k}$. The fixed point set $Fix(s)$ is the union of the apartments in $B_{G,k}$ containing $\varsigma$.

Suppose now that the field $k$ is perfect. Let $g \in G(k)$, then $g$ has a unique Jordan decomposition $g = su$ with $s \in G(k)$ semisimple, $u \in G(k)$ unipotent and $s, u$ commute with each other. In the case of algebraically closed fields, this implies that $Fix(g)$ is always top-dimensional and therefore it is a subbuilding or has circumradius $\leq \frac{3}{2}$ by Theorem 6.1. We generalize this conclusion to perfect fields and non-split groups in Propositions 8.5 and 8.6.

Motivated by this discussion, we make the following definition. We say that an isometry $g \in Isom(B)$ of a spherical building $B$ is split if $Fix(s)$ is the union of all apartments containing a singular sphere $\varsigma \subset B$. In particular, the fixed point set of a split isometry factorizes as a spherical join $Fix(k) \cong \Sigma \varsigma B \circ \varsigma$.

In the following results we consider isometries $g$ of spherical buildings that admit a kind of Jordan decomposition, that is, can be written as $g = uk$ with $u$ unipotent, $Fix(k)$ is a subbuilding and $u$ and $k$ commute.

**Lemma 8.3.** Let $B$ be an irreducible spherical building. Let $k$ be an isometry, whose fixed point set is a subbuilding, and let $u$ be a unipotent isometry. Suppose that $u$ and $k$ commute. Then $Fix(u) \cap Fix(k)$ is a top-dimensional subcomplex of the building $B' = Fix(k)$ with its thick structure.

**Proof.** Let $x_0$ be the weighted incenter of $Fix(u)$ given by Theorem 5.1. Then $k$ fixes $x_0$ because it commutes with $u$. By definition $x_0$ must be an interior point of $Fix(u)$, therefore by Corollary 2.23 \textit{St}$_{x_0} B \subset Fix(u)$. Since $Fix(k)$ is a subbuilding, it follows that $x_0$ is an interior point of $Fix(u) \cap Fix(k)$ and $Fix(u) \cap Fix(k)$ is of full dimension in $Fix(k)$.

Let $\tau \subset \partial (Fix(u) \cap Fix(k))$ be a face of full dimension and let $\sigma, \sigma' \subset Fix(k)$ be faces of full dimension containing $\tau$ and such that $\sigma \subset Fix(u)$, $\sigma' \not\subset Fix(u)$. Observe that $u$ stabilizes $Fix(k)$ because $u$ and $k$ commute. Then $\sigma, \sigma', u\sigma'$ are three pairwise distinct faces in $Fix(k)$ containing $\tau$. It follows that $\tau$ is contained in a wall with respect to the thick structure of $Fix(k)$. \hfill $\Box$

Notice that if $k$ is a split isometry with $Fix(k) \cong \Sigma \varsigma B \circ \varsigma$, then a top-dimensional subcomplex $K \subset Fix(k) \subset B$ is a subcomplex with respect to the thick structure of $Fix(k)$ if and only if $K$ contains the singular sphere $\varsigma$. In the next result we see that the converse of Lemma 8.3 is also true if the isometry $k$ is split.

**Lemma 8.4.** Let $B$ be an irreducible spherical building. Let $k$ be a split isometry with $Fix(k) \cong \Sigma \varsigma B \circ \varsigma$. If the fixed point set of a unipotent isometry $u$ contains the singular
sphere \( \varsigma \), then \( u \) and \( k \) commute.

**Proof.** Let \( A \subset B \) be an apartment containing \( \varsigma \) and such that \( K = \text{Fix}(u) \cap A \) is top-dimensional. Then by Proposition \[223\] \( g \) is a product of root elements of roots in \( \Lambda_k = \{ \alpha \subset A \mid K \subset A \} \subset \{ \alpha \subset A \mid \varsigma \subset A \} \). Hence, we may assume that \( u \in U_\alpha \) for some \( \alpha \subset A \) containing \( \varsigma \). Then \( kuk^{-1} \in U_\alpha \) because \( k \) fixes \( \alpha \). Further, the action of \( kuk^{-1} \) and \( u \) on \( \Sigma_k B \) is the same because \( k \) acts as the identity on \( \Sigma_k B \). Since \( B \) is irreducible, this implies that \( kuk^{-1} = u \).

The following proposition applies in particular to the buildings associated to algebraic groups \( G \) defined over algebraically closed fields \( k \) and isometries \( g \in G(k) \).

**Proposition 8.5.** Let \( B \) be an irreducible spherical building. Let \( k \) be a split isometry and let \( u \) be a unipotent isometry such that \( u \) and \( k \) commute. Let \( g = uk \). Then \( \text{Fix}(g) = \text{Fix}(u) \cap \text{Fix}(k) \).

**Proof.** Clearly, \( \text{Fix}(g) \supset \text{Fix}(u) \cap \text{Fix}(k) \). Let \( \sigma \subset B \) be a chamber such that \( \sigma \cap (\text{Fix}(u) \cap \text{Fix}(k)) \) is a panel \( \tau \subset \partial(\text{Fix}(u) \cap \text{Fix}(k)) \). By Lemma \[8.3\] \( \tau \) is contained in a wall of the thick structure of \( \text{Fix}(k) \cong \Sigma_k B \circ \varsigma \). This implies that there is a panel \( \nu \subset \Sigma_k B \) such that \( \tau \subset \nu \circ \varsigma \subset \text{Fix}(k) \cong \Sigma_k B \circ \varsigma \). In particular, \( St_\nu B \subset St_\nu(\Sigma_k B) \circ \varsigma \subset \text{Fix}(k) \). This in turn implies that \( \sigma \subset \text{Fix}(k) \). Since, \( \sigma \not\subset \text{Fix}(u) \cap \text{Fix}(k) \), it follows that \( \sigma \not\subset \text{Fix}(g) \). Therefore, we conclude that \( \text{Fix}(g) = \text{Fix}(u) \cap \text{Fix}(k) \).

Let again \( G \) be a semisimple algebraic group defined over a field \( k \), but we do not assume that \( G \) is \( k \)-split. Let \( s \in G(k) \) be a semisimple element. Even if \( \text{Fix}(s) \subset B_{G,k} \) is top-dimensional, it may not be top-dimensional in \( B_{G,k} \). Thus, the isometry \( s \) of \( B_{G,k} \) must not be split. Nevertheless, we can use Proposition \[8.5\] applied to \( B_{G,k} \) to conclude that for a perfect field \( k \) (that is, when we have a Jordan decomposition), the fixed point set \( \text{Fix}(g) \) is either a subbuilding or it has circumradius \( \leq \frac{4}{7} \).

**Proposition 8.6.** Let \( \hat{B} \) be an irreducible spherical building Let \( \hat{k} \) be a split isometry and let \( \hat{u} \) be a unipotent isometry such that \( \hat{u} \) and \( \hat{k} \) commute. Suppose further that \( \hat{u}, \hat{k} \) stabilize a subbuilding \( B \subset \hat{B} \), in particular, their restrictions to \( B \) induce isometries \( u, k \in \text{Isom}(B) \). Assume also that \( u \) is unipotent. Let \( g = uk \). Then \( \text{Fix}(g) = \text{Fix}(k) \cap \text{Fix}(u) \subset B \) is either a subbuilding or it has circumradius \( \leq \frac{4}{7} \).

**Proof.** By Proposition \[8.5\] \( \text{Fix}(g) = B \cap \text{Fix}(u\hat{k}) = B \cap \text{Fix}(\hat{k}) \cap \text{Fix}(\hat{u}) = \text{Fix}(k) \cap \text{Fix}(u) \). Theorem \[5.1\] implies that \( \text{Fix}(u) \) has a unique weighted incenter \( x_0 \). Since \( u \) and \( k \) commute, \( k \) fixes \( x_0 \) and therefore \( x_0 \in \text{Fix}(g) \). It follows that \( \text{rad}(\text{Fix}(g)) \leq \text{rad}(\text{Fix}(g), x_0) \leq \text{rad}(\text{Fix}(u), x_0) \leq \frac{4}{7} \) again by Theorem \[5.1\]

**Remark 8.7.** Proposition \[8.6\] in the case of semisimple Lie groups gives another proof of [Ebe96 Proposition 4.1.5], see also [Mos73 Lemma 12.3].

Suppose we are in a setting where a Jordan decomposition always exists. It is a natural question to ask for a geometric way of finding this decomposition. This is what [Ebe96 Problem 2.19.11] is about in the case of symmetric spaces of noncompact type. Let us rephrase the statement of the [Ebe96 Conjecture 2.19.11] in our notation. Let \( G \) be a noncompact semisimple Lie group and let \( B \) be its associated spherical building. \( B \) is the
Tits boundary of the symmetric space \( X = G/K \), where \( K \) is a maximal compact subgroup. Let \( g \in G \) be a parabolic isometry. Then the fixed point set \( Fix(g) \) of \( g \) in \( B \) has circumradius \( \leq \frac{2}{3} \) (cf. Section \[I\]). Let \( x_0 \) be the unique circumcenter of the set of circumcenters of \( Fix(g) \) and let \( \tau \subset B \) be the face containing \( x_0 \) in its interior. Let \( A \subset B \) be an apartment containing \( \tau \). Let \( g = uk \) be the decomposition given by Proposition \[2.21\] with respect to the apartment \( A \), where \( u \in U_\tau \) and \( k \in L_\tau H \). Then the conjecture asks if \( k \) is semisimple and \( g = uk \) is the Jordan decomposition of \( g \). As stated the conjecture cannot be true as we can see in the following example.

**Example 8.8.** Let \( g = \begin{pmatrix} a & -2 \\ 1 & a \end{pmatrix} \in SL(3, \mathbb{R}) \) with \( a \neq 1 \). Then the fixed point set \( Fix(g) \) in \( B = \partial_\tau(SL(3, \mathbb{R})/SO(3)) \) is a root \( \alpha \subset B \). The unipotent part \( u \) in the Jordan decomposition of \( g = us \) is an element in the root group \( U_\alpha \). The center of \( \alpha \) is the center of a chamber \( \sigma \). Let \( A \) be an apartment such that \( A \cap \alpha = \sigma \). Then the decomposition \( g = u'k' \) with respect to this apartment cannot be the Jordan decomposition because \( Fix(u') \cap A = \sigma \) and \( u' \) cannot be a root element. On the other hand, the decomposition with respect to any apartment containing \( \alpha \) is the Jordan decomposition.

In the example above, the key to obtain the Jordan decomposition was to choose an apartment supporting the fixed point set \( Fix(g) \). This works in general for a split algebraic group \( G \) over a perfect field and \( g \in G \) with top-dimensional fixed point set. This is the assertion of the following proposition, thus giving a solution of [Ebe96, Problem 2.19.11] in this case. Notice that it is actually not important whether the apartment contains the circumcenter of the fixed point set or not.

**Proposition 8.9.** Let \( B \) be an irreducible spherical building. Let \( g \) be an isometry admitting a decomposition \( g = uk \) with \( u \) unipotent, \( k \) split and such that \( u \) and \( k \) commute. Then \( g = uk \) is the decomposition as in Proposition \[2.21\] with respect to any apartment supporting \( Fix(g) \).

**Proof.** By Lemma \[8.3\], \( Fix(g) \) is a subcomplex of \( Fix(k) \cong \Sigma B \circ \varsigma \) with respect to its thick structure. It follows that any apartment \( A \) supporting \( Fix(g) \) must contain the singular sphere \( \varsigma \). This in turn implies that \( A \subset Fix(k) \). Let \( g = u'k' \) be the decomposition with respect to \( A \). Then \( u^{-1}u' = kk'^{-1} \) fixes \( A \). By Proposition \[8.5\] we have \( Fix(g) = Fix(u) \cap Fix(k) \), hence, \( Fix(u) \cap A = Fix(g) \cap A = Fix(u') \cap A \) and it follows that \( u^{-1}u' = kk'^{-1} \) is a unipotent isometry by Corollary \[2.21\]. Then \( u^{-1}u' = kk'^{-1} \) must be the identity. \( \square \)

Taking an apartment supporting \( Fix(g) \) does not work anymore in the general case. Actually, it is not possible to extract the Jordan decomposition of \( g \) just by considering its fixed point set as the following example shows.

**Example 8.10.** Let \( g = \begin{pmatrix} R & Id_2 \\ 0 & R \end{pmatrix} \in SL(4, \mathbb{R}) \), where \( Id_2 \in SL(2, \mathbb{R}) \) is the identity matrix and \( \pm Id_2 \neq R \in SO(2) \) is a rotation. The fixed point set of \( g \) consists of only one point. It is the vertex of \( B = \partial_\tau(SL(4, \mathbb{R})/SO(4)) \) corresponding to the plane \( \langle e_1, e_2 \rangle \subset \mathbb{R}^4 \).
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