One-dimensional transient hygrothermoelastic field in a porous strip considering nonlinear coupling between heat and binary moisture

Masayuki ISHIHARA*, Keita OGASAWARA*, Yoshihiro OOTAO* and Yoshitaka KAMEO**
*Graduate School of Engineering, Osaka Prefecture University,
1-1 Gakuen-cho, Naka-ku, Sakai-shi, Osaka 599-8531, Japan
E-mail: isihara@me.osakafu-u.ac.jp
**Institute for Frontier Medical Sciences, Kyoto University
53 Kawahara-cho, Shogoin, Sakyo-ku, Kyoto 606-8507, Japan

Received 20 June 2016

Abstract
This study aims to develop practical tools for the mechanical design of porous media subjected to a broad gap in a hygrothermal environment. The one-dimensional transient hygrothermoelastic field in a porous strip is investigated considering the nonlinear coupling between heat and binary moisture. The derivation of the system of governing equations is summarized first, by considering the nonlinear relation between temperature, dissolved moisture content, and vapor concentration and diffusivities of both dissolved moisture and vapor. The nonlinearity between the temperature, dissolved moisture content, and vapor concentration is investigated qualitatively and quantitatively. Next, the system of governing equations is applied to the infinite strip subjected to broad gaps of temperature and dissolved moisture content and solved by the finite difference method. The distributions of temperature, dissolved moisture content, vapor concentration, and dissolution rate were illustrated numerically, and the moisture distribution was found to be highly nonlinear. Finally, based on hygrothermoelasticity, the distribution of the resulting in-plane stress in a strip free from mechanical constraints is analyzed theoretically. The occurrence of residual stress, which is quite unlike what it is under the linear theory, is confirmed. Finally, the effect of gaps in a hygrothermal environment on the residual stress is investigated qualitatively and quantitatively.

Key words: Diffusion, Porous medium, Binary moisture, Nonlinear coupling, Hygrothermal stress

1. Introduction

Porous media in engineering applications are usually subjected to a coupled field of heat and moisture, and to the resulting stresses. Therefore, coupled hygrothermal problems in relation to porous media are important issues. The system of governing equations for such problems was derived for two-phase media, i.e., porous media where vapor diffuses through the voids (Hartranft and Sih, 1980). In the derivation, the changes in temperature, dissolved moisture content, and vapor concentration were assumed to be linear with each other, which reduced the system of governing equations to a linear system with respect to temperature and moisture content (or vapor concentration). Moreover, moisture was assumed to be carried through the voids, not through the substantial solid; in other words, the diffusivity of dissolved moisture was not considered. The coupled hygrothermal fields obtained from this system of governing equations (and the resulting elastic fields) were theoretically studied for several cases. The cases studied were—an infinite plate (Sih et al., 1980, Sih and Shih, 1980), a half-space (Sih and Ogawa, 1982), and cylinders in an axisymmetric (Chang et al., 1991, Chang, 1994, Chang and Weng, 1997) and a non-axisymmetric (Sugano and Chuuman, 1993) environment.

The way it is for lumber that is subjected to dried-through, to saturated moist air, porous media are often subjected to a broad range of hygrothermal environmental conditions. In such cases, the assumption of linearity is no longer valid and the system of governing equations would incorrectly estimate the hygrothermal field. Moreover, moisture in typical porous media can be carried through voids and substantial solids. For instance, it is carried essentially through substantial solid and voids in portions with low and high porosities, respectively, and the main channels of moisture in
lumber are the substantial solid and voids in the radial and axial directions, respectively.

The system of governing equations that take into consideration the nonlinear relationship between temperature, dissolved moisture content, and vapor concentration and the diffusivities of both dissolved moisture and vapor (Ishihara et al., 2014) has been derived in the past by the same authors to bridge the gap between assumptions and observed behavior. In the derivation, the above nonlinear relationship was obtained by considering the equilibrium between the chemical potentials of vapor and dissolved moisture. Furthermore, an analysis of the steady hygrothermal field in an infinite strip was carried out as an application of the governing equations. This elucidated the distributions of field quantities that are free from errors owing to the linearization assumption, and the effects of the aforementioned binary diffusivity on the distributions.

The above analysis for an infinite strip (Ishihara et al., 2014) concentrated on a steady hygrothermal field; in other words, it did not consider a transient field or the resulting elastic field. As reported in several older works (Sih et al., 1980, Sih and Shih, 1980, Chang et al., 1991, Chang, 1994, Chang and Weng, 1997, Sugano and Chuuman, 1993), the stresses resulting from hygrothermal loads reach their peak often in a transient state. Because the porous media in engineering applications are usually subjected to a time-varying hygrothermal environment, transient hygrothermoelastic behaviors need to be analyzed in order to ensure the safe use of porous media.

This study therefore extends the older analysis that used a steady hygrothermal field (Ishihara et al., 2014) to a transient hygrothermoelastic one. After a summary of the system of governing equations derived previously (Ishihara et al., 2014), the nonlinearity between the temperature, dissolved moisture content, and vapor concentration is explained. The system of governing equations (Ishihara et al., 2014) is then applied to an infinite strip that is exposed to broad gaps of temperature and dissolved moisture content on both surfaces, and is in a transient state. The resulting system of governing equations is solved using the finite difference method in order to numerically illustrate the distributions of hygrothermal field quantities such as temperature, dissolved moisture content, vapor concentration, and dissolution rate. Next, the effect of diffusive properties is investigated on the distributions. The distribution of the resulting in-plane stress in the strip that is free from mechanical constraints is theoretically analyzed, based on the fundamental equations for hygrothermoelasticity. As a result, the occurrence of the stress that never occurred under the framework of the linear hygrothermal theory is found. Then, the effect of the gaps of hygrothermal environments between both surfaces on that stress is investigated.

2. Hygrothermal diffusion equations considering nonlinear coupling between heat and binary moisture

2.1 Theoretical analysis

This subsection presents a summary of the results of the authors’ previous work (Ishihara et al., 2014), namely, the nonlinear relationship between the temperature, dissolved moisture content, and vapor concentration, and the system of governing equations considering that nonlinear relation and the diffusivities of both dissolved moisture and vapor.

From a microscopic viewpoint, a porous medium consists of both voids and substantial solid. Moisture is assumed to be in the dissolved and gaseous forms in the substantial solid and voids, respectively. Absolute temperature $T$ should be defined microscopically. The dissolved moisture content $M$ and the vapor concentration $C$ should also be defined microscopically, respectively, as the mass of dissolved moisture per unit mass of the substantial solid and as the mass of gaseous moisture—namely, vapor per unit volume of the voids. However, the subject has been approached from a macroscopic viewpoint in order to develop a practical tool for mechanical design. More specifically, $T$, $M$, and $C$ are considered to be distributed continuously as functions of the location. This approach defines the mass densities of the dried substantial solid and the porous medium, $\rho_s$ and $\rho$, respectively, and the volume fraction of the voids $f$, all of which are also defined in the macroscopic sense satisfying the relationship $\rho = \rho_s (1-f)$. Then, the total mass of moisture per unit mass of the substantial solid, $m$, is obtained by the sum,

$$ m = \rho M + f C. \quad (1) $$

The dissolved and gaseous moistures are considered, at all times, to be in an equilibrium state that depends on the absolute temperature. Therefore, the relationship between $T$, $M$, and $C$ is described as

$$ C = C(M, T). \quad (2) $$

From Fourier’s law and Fick’s law, the heat flux vector $q_{\theta}$ and the mass flux vectors of the dissolved moisture
\( q_d \) and gaseous moisture \( q_g \) are related to the gradients of the field quantities as

\[
q_h = -kVT, \quad q_d = -D_d \nabla (\rho M), \quad q_g = -D_g \nabla (\rho C).
\]

(3)

In Eq. (3), \( k = k(M, T) \) denotes the thermal conductivity, and \( D_d = D_d(M, T) \) and \( D_g = D_g(M, T) \) denote the diffusivities of the dissolved and gaseous moistures, respectively, which are referred to as the dissolved moisture diffusivity and gas diffusivity, respectively, for brevity. Note that, as a first step, these diffusivities are considered isotropic.

The balance of heat and that of mass are described, respectively, as

\[
cp \frac{\partial T}{\partial t} = -\nabla \cdot q_h + \rho L_{\text{dissolution}} \left( \frac{\partial M}{\partial t} \right)_{\text{dissolution}}, \quad \frac{\partial (\rho M)}{\partial t} = -\nabla \cdot q_d - \nabla \cdot q_g.
\]

(4)

In Eq. (4), \( -\nabla \cdot q_h \), \( -\nabla \cdot q_d \), and \( -\nabla \cdot q_g \) denote the influxes of heat, dissolved moisture, and gaseous moisture, respectively. Moreover, \( c = c(M, T) \) denotes the specific heat, \( L_{\text{dissolution}} \) denotes the heat generated by the dissolution of the unit mass of gaseous moisture into the substantial solid, and \( \left( \partial M / \partial t \right)_{\text{dissolution}} \) denotes the contribution by dissolution for the time rate of dissolved moisture content, which is referred to as the dissolution rate and related to \( \partial M / \partial t \) as

\[
\rho \frac{\partial M}{\partial t} = -\nabla \cdot q_d + \rho \left( \frac{\partial M}{\partial t} \right)_{\text{dissolution}}.
\]

(5)

By substituting Eq. (3) into Eq. (4) combined with Eqs (1), (2), and (5) and arranging the results, the system of nonlinear coupling diffusion equations was obtained (Ishihara et al., 2014) with respect to the absolute temperature \( T \) and dissolved moisture content \( M \) as follows:

\[
D_T(M, T) \nabla^2 T + N_T(M, T) = \frac{\partial T}{\partial t} - v_T(M, T) \frac{\partial M}{\partial t}, \quad D_d(M, T) \nabla^2 M + N_d(M, T) = \frac{\partial M}{\partial t} + \lambda_d(M, T) \frac{\partial T}{\partial t},
\]

(6)

where

\[
\begin{align*}
D_T(M, T) &= \frac{1}{\Delta_T} \left( k + \frac{D_d D_e f \omega L_{\text{dissolution}}}{\rho c} \right), \quad D_d(M, T) = \frac{1}{\Delta_d} \left( D_d + \frac{D_d D_e f \omega L_{\text{dissolution}}}{\rho c} \right), \\
v_T(M, T) &= \frac{1}{\Delta_T} \left( \frac{\nabla k}{c p} + \frac{D_D V D_g f \omega L_{\text{dissolution}}}{\rho c} \right), \quad \lambda_d(M, T) = \frac{1}{\Delta_d} \left( \frac{D_d}{\kappa} + \frac{D_d D_e f \omega L_{\text{dissolution}}}{\rho c} \right), \\
N_T(M, T) &= \frac{1}{\Delta_T} \left( \frac{\nabla k}{c p} + \frac{D_D V D_g f \omega L_{\text{dissolution}}}{\rho c} \right) \cdot \nabla (T) - \frac{D_d V D_d - D_d V D_g f \omega L_{\text{dissolution}}}{\rho c} \cdot \nabla (M), \\
N_d(M, T) &= \frac{1}{\Delta_d} \left( \frac{\nabla k}{k} - \nabla D_g \right) \frac{f \omega L_{\text{dissolution}}}{\rho c} \cdot \nabla (T) + \left( \nabla D_d + \frac{f \sigma}{\rho c} \nabla D_g + \frac{D_D V D_d f \omega L_{\text{dissolution}}}{\rho c} \right) \cdot \nabla (M),
\end{align*}
\]

(7)

\[
\begin{align*}
\kappa &= \frac{k}{c p}, \quad D_D = D_d + D_g \frac{f \sigma}{\rho c}, \quad \Delta_T = 1 + \frac{D_d}{D_D} \frac{f \omega L_{\text{dissolution}}}{\rho c}, \quad \Delta_d = 1 + \frac{f \sigma}{\rho c} + \frac{D_d}{D_D} \frac{f \omega L_{\text{dissolution}}}{\rho c}, \\
\sigma &= \frac{\partial C}{\partial M}, \quad \omega = \frac{\partial C}{\partial T}, \quad \sigma_T = \frac{\partial C}{\partial T}, \quad \omega_T = \frac{\partial C}{\partial T},
\end{align*}
\]

where \( N_T(M, T) \) and \( N_d(M, T) \) are referred to as the steady nonlinear sources for the temperature and dissolved moisture content, respectively, in that they serve as the nonlinear sources even in a steady state.
Aiming at the formula to be employed in Eq. (7), the concrete example of Eq. (2) was also presented previously (Ishihara et al., 2014), by considering the equilibrium between the dissolved and gaseous moistures, as
\[
C(M, T) = f_0(T_0) \frac{p_s(T_0)}{\rho_0} \exp \left[ -\frac{L_{\text{dissolution}}}{R_g} \left( \frac{1}{T} - \frac{1}{T_0} \right) \right] \frac{\omega_j M}{1 + \omega_j M},
\]  
where \( T_0, f_0(T), p_s(T), R_g, \) and \( \omega_j \) denote a reference absolute temperature that can be chosen arbitrarily, the activity coefficient, the saturated vapor pressure, the gas constant of the moisture, and the constant dependent on the combination of the moisture and substantial solid.

2.2 Numerical results

This subsection numerically illustrates the nonlinear relation between temperature, dissolved moisture content, and vapor concentration described by Eq. (8). As a first step, material properties \( c, k, D_d, \) and \( D_g \) are assumed to be independent of the field quantities \( M \) and \( T \).

The following non-dimensional quantities were introduced in order to extract the governing parameters,
\[
\tilde{T} = \frac{T - T_0}{T_0}, \quad \tilde{C}(M, \tilde{T}) = \frac{fC(M, T)}{\rho}, \quad \tilde{D}_d = \frac{(D_d, D_g)}{\kappa}, \quad \tilde{C}_{m0} = \frac{f_0 p_s(T_0)}{\rho R_g T_0}, \quad \tilde{L}_{\text{dissolution}} = \frac{L_{\text{liquefaction}}}{R_g T_0}, \quad \tilde{c} = \frac{c}{R_g}.
\]  
The reference absolute temperature was chosen to be \( T_0 = 298.15 \, [\text{K}] = 25 \, [\text{C}] \). Sitka spruce (a species of wood) is chosen as the porous medium and water to play the role of moisture. The required parameters are given as \( \tilde{L}_{\text{dissolution}} = 18.347, \quad \tilde{C}_{m0} = 35.834 \times 10^{-6}, \quad \omega_s = 3.333, \quad f_0(T_0) \approx 2.1111, \quad \tilde{c} = 2.7214, \) and \( D_d = 10^{-7} \) (Ishihara et al., 2014). The value for \( \tilde{D}_d \) is chosen because a typical value of the thermal diffusivity for wood is found to be of the order of \( 10^{-7} \, [\text{m}^2/\text{s}] \) (Glass and Zelinka, 2010) and a typical value of the dissolved moisture diffusivity for Sitka spruce is found to be of the order of \( 10^{-9} \, [\text{m}^2/\text{s}] \) (Nakao, 1998). Because the connectivity of voids is diverse, the value for \( \tilde{D}_g \) can be considered to take various values, and is given where necessary.

Figure 1 shows the variation of vapor concentration with dissolved moisture content and temperature as illustrated by Eq. (8). Note that if the relationship in Eq. (2) is linear, a differential form \( dC = \sigma_0 M + \omega_0 T \) obtained from Eqs (2) and (7) has constant coefficients \( \sigma_0 \) and \( \omega_0 \), and therefore the contours in Fig. 1 become straight and equidistant ones. From Fig. 1, it is found that the relationship is highly nonlinear and the vapor concentration increases with the dissolved moisture content and temperature in order to maintain the equilibrium between the dissolved and gaseous moistures.

Fig. 1 Variation of vapor concentration with temperature and dissolved moisture content.

3. Transient hygrothermal field in a strip

This section applies the system of nonlinear coupling diffusion equations presented in Eq. (6) to an infinite strip which is initially under uniform distributions of temperature and dissolved moisture content, and then exposed to broad gaps of temperature and dissolved moisture content on both surfaces, in a one-dimensional transient state. As stated in
Subsection 2.2, the material properties $c$, $k$, $D_d$, and $D_g$ are assumed to be independent on the field quantities $M$ and $T$.

### 3.1 Theoretical analysis

The analytical model is an infinite strip with thickness $L$ that occupies the region $\{(x,y,z)| 0 \leq x \leq L\}$ in the Cartesian coordinate system $(x,y,z)$ as shown in Fig. 2.

Equation (6) is simplified as

$$D_f(M,T)\frac{\partial^2 T}{\partial x^2} + N_f(M,T) = \frac{\partial T}{\partial t} - v_u(M,T)\frac{\partial M}{\partial t} + D_u(M,T)\frac{\partial^2 M}{\partial x^2} + N_u(M,T) = \frac{\partial M}{\partial t} + \lambda_r(M,T)\frac{\partial T}{\partial t}, \quad (10)$$

where the formulas for $D_f(M,T)$, $D_u(M,T)$, $v_u(M,T)$, and $\lambda_r(M,T)$ are given in Eq. (7) and those for $N_f(M,T)$ and $N_u(M,T)$ are reduced to

$$N_f(M,T) = \frac{1}{\Delta_x} D_f \frac{D_e}{D_d} \frac{L_{\text{dissolution}}}{c} \frac{f}{\rho} N_{\psi_c}(M,T), \quad N_u(M,T) = \frac{1}{\Delta_x} D_g \frac{f}{\rho} N_{\psi_c}(M,T).$$

$$N_{\psi_c}(M,T) = \sigma_x \left( \frac{\partial M}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial x} \right)^2 + \frac{\omega_t}{\omega_x} \frac{\partial T}{\partial x}. \quad (11)$$

The initial and boundary conditions are given by

$$t = 0: T = T_0, M = M_0; \quad x = 0: T = T_0, M = M_0; \quad x = L: T = T_s, M = M_s. \quad (12)$$

The solution to Eq. (10) subjected to Eq. (12) can be obtained by a finite difference method. More specifically, the domain $0 \leq x \leq L$ is divided into $n_{\text{diss}}$ equal parts, and Eq. (10) is regarded as the time evolution equation for $2(n_{\text{diss}}+1)$ degree of freedom field quantities, namely, the temperature and dissolved moisture content at thus generated $(n_{\text{diss}}+1)$ discrete points. The spatial derivatives required on the left hand sides of Eq. (10) are evaluated by the corresponding finite differences with truncation errors of the order of the segment length squared. Time evolution is evaluated by the Adams method, in which the predictor-corrector method by the Adams-Bashforth and Adams-Moulton methods is employed (LeVeque, 2007).

If the relationship in Eq. (2) is linear, the functions $\sigma$ and $\omega$ are constant with respect to $M$ and $T$. In that case, from Eq. (11), the steady nonlinear sources for the temperature and dissolved moisture content, $N_f(M,T)$ and $N_u(M,T)$, are found to be absent, and the system of diffusion equations for a steady case is reduced to a linear and uncoupled system that can be depicted as

$$\frac{\partial^2 T}{\partial x^2} = 0, \quad \frac{\partial^2 M}{\partial x^2} = 0. \quad (13)$$
The solution to Eq. (13) subjected to the boundary conditions in Eq. (12) can then be obtained in a closed form as

$$T = T_a + \left( T_b - T_a \right) \frac{x}{L}, \quad M = M_a + \left( M_b - M_a \right) \frac{x}{L},$$

(14)

which is linear with respect to $x$.

### 3.2 Numerical results

This subsection numerically illustrates the distributions of hygrothermal field quantities such as temperature, dissolved moisture content, vapor concentration, and dissolution rate. In addition to Eq. (9), the non-dimensional quantities are introduced as

$$\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{k_t}{L^2}, \quad \tilde{T}_t = \frac{T - T_0}{T_0}, \quad \tilde{T}_a = \frac{T_a}{T_0}, \quad \tilde{T}_b = \frac{T_b}{T_0}, \quad \dot{M}_{\text{dissolution}} = \frac{L^2}{K} \left( \frac{\partial M}{\partial t} \right)_{\text{dissolution}}.$$

(15)

As in Subsection 2.2, Sitka spruce and water are chosen as the porous medium and the moisture, respectively, and numerical parameters in that subsection are employed unless otherwise stated. The initial and boundary values for the absolute temperature and dissolved moisture content are taken, unless otherwise stated, as

$$T_i = T_0, \quad T_a = T_0, \quad T_b = 1.2T_0, \quad M_i = 0.12, \quad M_a = 0.12, \quad M_b = 0.2,$$

(16)

which are rewritten in non-dimensional forms as

$$\tilde{T}_i = 0, \quad \tilde{T}_a = 0, \quad \tilde{T}_b = 0.2, \quad M_i = 0.12, \quad M_a = 0.12, \quad M_b = 0.2.$$

(17)

Note that $T_b = 357.78\text{[K]} = 84.63\text{[°C]}$ from Eq. (16), the value 0.12 for $M_i$ and $M_a$ is a representative value for the dissolved moisture content under the air-dried condition (Bergman, 2010), and the value 0.2 for $M_b$ is intended to represent a moderate value as compared to the fiber saturation point, namely around 0.3 (Glass and Zelinka, 2010). As for the finite difference method, the division number is taken as $n_{\text{fin}} = 20$, and the time increment and interpolation order for each evolutiona step are determined such that the maximum absolute error of the non-dimensional field quantities, $\tilde{T}$ and $\dot{M}$, is less than $10^{-6}$.

Figures 3 (a)–(d) show the distributions of the temperature, dissolved moisture content, vapor concentration, and dissolution rate for $D_g = 1$. From Fig. 3 (a), it is found that the temperature increases from the heated side, $\tilde{x} = 1$, and finally exhibits an approximately linear distribution at time $\tilde{t} = 1$. From Fig. 3 (b), it is found that the dissolved moisture content increases from the humid side, $\tilde{x} = 1$, but reaches a steady state at $\tilde{t} = 50$ even slower than the temperature does. Significantly, Fig. 3 (b) shows that the steady distribution is highly nonlinear, unlike the result of the linear theory described by Eq. (14) or described in older works (Hartranft and Sih, 1980, Sih et al., 1980, Sih and Shih, 1980, Sih and Ogawa, 1982, Chang et al., 1991, Chang, 1994, Chang and Weng, 1997, Sugano and Chuuman, 1993). Moreover, the steady distribution shown in Fig. 3 (b) is found to exhibit a negative curvature, i.e., $\frac{\partial^2 M}{\partial \tilde{x}^2} < 0$, while the distributions of vapor concentration are found to exhibit positive curvatures, i.e., $\frac{\partial^2 C}{\partial \tilde{x}^2} > 0$ as shown in Fig. 3 (c). From Eq. (3), the mass of dissolved moisture and vapor flowing into a unit volume of the porous medium per unit time is expressed, respectively, by

$$-\nabla \cdot \mathbf{q}_d = D_g \rho \frac{\partial^2 M}{\partial \tilde{x}^2}, \quad -\nabla \cdot \mathbf{q}_d = D_g f \frac{\partial^2 C}{\partial \tilde{x}^2}.$$

(18)

Therefore, the profile of $\frac{\partial^2 M}{\partial \tilde{x}^2} < 0$ at the steady state, shown in Fig. 3 (b), means that a local portion of the body is subjected to the outflow of dissolved moisture. Meanwhile, the mass of dissolved moisture in a local portion must be kept constant for the steady state. Therefore, the dissolved moisture must be supplied by the dissolution of gaseous moisture into the substantial solid, which is indicated by Eq. (5) for the steady state:

$$\rho \left( \frac{\partial M}{\partial t} \right)_{\text{dissolution}} = -\nabla \cdot \mathbf{q}_d = -D_g \rho \frac{\partial^2 M}{\partial \tilde{x}^2} > 0.$$

(19)
Moreover, as found in Eq. (4) for the steady state, the mass of total moisture flowing into a local portion must be zero, which leads to \(-\nabla \cdot \mathbf{q}_d = \nabla \cdot \mathbf{q}_g > 0\) from Eq. (19) and, in turn, leads to \(\partial^2 C / \partial x^2 > 0\) from Eq. (18), as illustrated in Fig. 3 (c). The distributions of dissolution rate, for the steady state and for transient states, are shown in Fig. 3 (d). The structure of the hygrothermal field for the steady aforementioned state is integrated into Fig. 4. In Fig. 4, the \(x\) axis lies horizontally, and the heights of the upper and lower rectangles appear in a ratio of 0.3 to 0.7 reflecting the volume fraction of voids \(f = 0.7\) that was used in a previous work by the authors (Ishihara et al., 2014) to construct the parameters in Subsection 2.2. Moreover, the depths of blue and red denote the values of \(M\) and \(\dot{C}\) shown in Figs 3 (b) and (c), respectively; the horizontal arrows in the upper and lower rectangles correspond to the mass flux vectors of dissolved moisture \(\mathbf{q}_d\) and gaseous moisture \(\mathbf{q}_g\), respectively, given by Eq. (3). The lengths of the vertical arrows at the interfaces of the rectangles are proportional to the value of \(\dot{M}_{\text{dissolution}}\) shown in Fig. 3 (d). Figure 4 shows approximately that the moisture supplied at the humid surface \((\hat{x} = 1)\) diffuses toward the air-dried surface \((\hat{x} = 0)\) in the gaseous form at first, transforms into the dissolved form, and diffuses in the dissolved form.

The effect of diffusive properties on the nonlinearity of moisture distribution is studied next. At first, \(m = M + \dot{C}\) is obtained from Eqs (1) and (9). In other words, the total mass of moisture per unit mass of the substantial solid is the sum of the dissolved moisture content and the non-dimensional vapor concentration. Meanwhile, by comparing Figs 3

![Figure 3](image1.png)

![Figure 4](image2.png)
(b) and (c), \( M \) is found to be much greater than \( \hat{C} \). Therefore, the total mass of moisture can be evaluated substantially by the mass of dissolved moisture. From this viewpoint, a measure of the total mass of moisture per unit surface area of the strip is introduced, and is defined by

\[
W = \int_0^1 \! M \, d\hat{x},
\]

which corresponds to the area that is enclosed by each curve in Fig. 3 (b) and the axes described by \( M = 0, \, \hat{x} = 0, \) and \( \hat{x} = 1 \). The measure \( W \) for the linear and steady case given by Eq. (14), denoted by \( W_{\text{linear}} \), is calculated as \( W_{\text{linear}} = (M_0 + M_1)/2 \). Therefore, the ratio of the difference \( W - W_{\text{linear}} \) to \( W_{\text{linear}} \), namely,

\[
R_{\text{nonlinear}} = \frac{W - W_{\text{linear}}}{W_{\text{linear}}},
\]

can be regarded as one of the measures for the nonlinearity of moisture distribution. Figure 5 shows the variation of the nonlinearity with the dissolved moisture diffusivity \( D_j \) and gas diffusivity \( D_g \). From Fig. 5, it is found that the nonlinearity increases with the increase in gas diffusivity and the decrease in dissolved moisture diffusivity. Moreover, the contours in Fig. 5 exhibit a relation \( \log_{10} D_j \equiv \log_{10} D_g + (\text{constant}) \), i.e., \( D_j/D_g \equiv (\text{constant}) \), which means the magnitude of the nonlinearity is governed roughly by the ratio of the gas diffusivity to the dissolved moisture diffusivity.

4. Transient hygrothermoelastic field in a strip

This section investigates the resulting stress in the strip shown in Fig. 2 that is subjected to the hygrothermal field treated in Section 3. The elastic properties are assumed to be isotropic.

4.1 Theoretical analysis

The components of the displacement, strain, and stress in the Cartesian coordinate system shown in Fig. 2 are denoted by \( u_x, \, e_y, \) and \( \sigma_{ij} \) \((i, j = x, y, z)\), respectively. The strip, subjected to the hygrothermal field treated in Section 3, free from stresses on the surfaces \( x = 0 \) and \( x = L \), experiences the following stress distribution,

\[
\sigma_{yy}(x,t) = \sigma_{xx}(x,t) = \sigma(x,t), \quad \sigma_{xy}(x,t) = 0, \quad \sigma_{yx}(x,t) = 0, \quad \sigma_{xx} = \sigma_{yy} = 0, \quad \sigma_{zz} = \sigma_{xy} = \sigma_{yx} = 0,
\]

that is one-dimensional in the \( x \)-direction and isotropic in \( yz \)-plane. Assuming a quasi-static process for the resulting elastic field, the equilibrium equation of stresses under Eq. (22) is obtained as \( \partial \sigma_{xx}/\partial x = 0 \) (Timoshenko and Goodier, 1955). \( \sigma_{xx} = 0 \) on the surfaces \( x = 0 \) and \( x = L \) gives

\[
\sigma_{xx}(x,t) = 0: 0 \leq x \leq L.
\]
The constitutive equations are given by (Sih et al., 1980)

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{E} \left[ \sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz}) \right] + \alpha \Delta T + \beta \Delta M, \\
\varepsilon_{yy} &= \frac{1}{E} \left[ \sigma_{yy} - \nu (\sigma_{xx} + \sigma_{zz}) \right] + \alpha \Delta T + \beta \Delta M, \\
\varepsilon_{zz} &= \frac{1}{E} \left[ \sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy}) \right] + \alpha \Delta T + \beta \Delta M, \\
\varepsilon_{xy} &= \frac{1}{E} \sigma_{xy}, \\
\varepsilon_{yz} &= \frac{1}{E} \sigma_{yz}, \\
\varepsilon_{xz} &= \frac{1}{E} \sigma_{xz}, \\
\varepsilon_{yy} &= \frac{1}{E} \sigma_{yy}, \\
\varepsilon_{zz} &= \frac{1}{E} \sigma_{zz}, \quad \varepsilon_{yx} = \frac{1}{E} \sigma_{yx}, \quad \varepsilon_{zx} = \frac{1}{E} \sigma_{zx}
\end{align*}
\]

(24)

where

\[\Delta T = T - T_{\text{free}}, \quad \Delta M = M - M_{\text{free}}.\]

(25)

\(T_{\text{free}}\) and \(M_{\text{free}}\) denote the absolute temperature and dissolved moisture content, respectively, that give a natural state, i.e., \(\varepsilon_0 = 0\) and \(\sigma_0 = 0\). \(E\) and \(\nu\) denote the Young's modulus and Poisson's ratio, respectively. \(\alpha\) and \(\beta\) denote the coefficients of thermal and moisture expansion, respectively. Substituting Eqs (22) and (23) into Eq. (24) produces

\[
\begin{align*}
\varepsilon_{yy}(x,t) &= \varepsilon_0(x,t) + \frac{1}{E} \sigma(x,t) + \alpha \Delta T + \beta \Delta M \quad (= \varepsilon(x,t)), \\
\varepsilon_{xx}(x,t) &= -\frac{\nu}{E} (2\sigma(x,t) + \alpha \Delta T + \beta \Delta M), \\
\varepsilon_{xy}(x,t) &= \frac{1}{E} \sigma_{xy}, \\
\varepsilon_{yz}(x,t) &= \frac{1}{E} \sigma_{yz}, \\
\varepsilon_{xz}(x,t) &= \frac{1}{E} \sigma_{xz}, \quad \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0
\end{align*}
\]

(26)

For Eq. (26), the conditions of compatibility (Timoshenko and Goodier, 1955) are reduced to \(\partial^2 \varepsilon(x,t)/\partial x^2 = 0\), producing the following equation.

\[
\varepsilon(x,t) = \varepsilon_0(t) + \kappa_0 \left( x - \frac{L}{2} \right),
\]

(27)

where the integration constants \(\varepsilon_0(t)\) and \(\kappa_0(t)\) denote the in-plane strain and anti-plane curvature of the midplane \(x = L/2\). Equations (26) and (27) together produce

\[
\sigma = \frac{E}{1 - \nu} \left[ \varepsilon_0 + \kappa_0 \left( x - \frac{L}{2} \right) - (\alpha \Delta T + \beta \Delta M) \right].
\]

(28)

In order to investigate the stress that occurs only due to the hygrothermal field and excludes the effects of mechanical constraints, the strip is considered free from the in-plane extensional force and anti-plane bending moment. In that case, as these force and moment (per unit width of the strip) must be zero, the condition

\[
\int_0^L \sigma \, dx = 0, \quad \int_0^L \sigma \left( x - \frac{L}{2} \right) \, dx = 0
\]

(29)

must hold. By substituting Eq. (28) into Eq. (29), \(\varepsilon_0\) and \(\kappa_0\) are obtained as

\[
\varepsilon_0 = \frac{1}{L} \int_0^L (\alpha \Delta T + \beta \Delta M) \, dx, \quad \kappa_0 = \frac{12}{L^2} \int_0^L (\alpha \Delta T + \beta \Delta M) \left( x - \frac{L}{2} \right) \, dx,
\]

(30)

both of which are known from the hygrothermal field treated in Section 3. Therefore, by substituting Eq. (30) into Eq. (28), the stress resulting from the hygrothermal field can be obtained.

As stated in Subsection 3.1, if the relation in Eq. (2) is linear, the hygrothermal field for the steady case is given by Eq. (14). By substituting Eq. (14) into Eqs (28) and (30), and performing the integrations, it is found that the stress due to the linear distribution of the hygrothermal field vanishes at all positions in the strip that are worth mentioning for discussion in the following subsection.

[DOI: 10.1299/jtst.2016jtst0035] © 2016 The Japan Society of Mechanical Engineers
4.2 Numerical results

This subsection numerically illustrates the distributions of hygrothermoelastic stress given by Eq. (28) with Eqs (25) and (30). In addition to Eqs (9) and (15), the non-dimensional quantities are

\[
\tilde{\sigma} = \frac{1-v}{aET_0} \sigma, \quad \tilde{\beta} = \frac{\beta}{aT_0}, \quad \tilde{T}_{\text{free}} = \frac{T_{\text{free}}-T_0}{T_0}.
\]  

(31)

By applying Eqs (9), (15), and (31) to Eq. (28) with Eqs (25) and (30) substituted, the non-dimensional stress is formulated as

\[
\tilde{\sigma} = \int_0^1 \left( \tilde{T} + \tilde{\beta} \Delta M \right) \tilde{\alpha} \tilde{\beta} \tilde{T} \left( \tilde{T} - \frac{1}{2} \right) d\tilde{T} - \left( \Delta \tilde{T} + \tilde{\beta} \Delta M \right),
\]  

(32)

where \( \Delta \tilde{T} = \tilde{T} - \tilde{T}_{\text{free}} \). Therefore, the value for \( \tilde{\beta} \) needs to be evaluated. As in previous sections, Sitka spruce was chosen. The coefficients of thermal expansion as an anisotropic material are found to be \( \alpha_r = 23.8 \times 10^{-6} [1/K] \) and \( \alpha_t = 32.3 \times 10^{-6} [1/K] \) in the radial and tangential directions, respectively (Weathermax and Stamm, 1946). The coefficient of thermal expansion is assumed to be the average of the coefficients in the radial and tangential directions in order to obtain the properties as an isotropic body. This produces the equation \( \alpha = 28.05 \times 10^{-6} [1/K] \). As for the coefficient of moisture expansion, the values in the radial and tangential directions are found to be \( \beta_r = 0.148 \) and \( \beta_t = 0.263 \), respectively (Bergman, 2010). Similarly, the coefficient of moisture expansion gives the value \( \beta = 0.2055 \). Therefore, Eq. (31) and the value of \( T_0 \) described in Subsection 2.2 produce the following value,

\[
\tilde{\beta} = 24.572.
\]  

(33)

From Eqs (32) and (33), it is found that the effect of moisture expansion on the stress is considerably predominant over that of thermal expansion, for unit changes in non-dimensional temperature and dissolved moisture content. The absolute temperature and dissolved moisture content for a natural state are chosen as \( T_{\text{free}} = T_i \) and \( M_{\text{free}} = M_i \), respectively.

Figure 6 shows the distributions of the resulting stress in the strip subjected to the hygrothermal field. Because of the predominant effect of the aforementioned moisture expansion, the distributions at the same stages as in Fig. 3 (b) are chosen in Fig. 6. From Fig. 6, it is found that, for relatively early stages \( (\tilde{t} < 10) \), the stress is compressive and tensile in the regions close to and distant from, respectively, both surfaces. In addition, the profiles of distribution are reversed for relatively late stages \( (\tilde{t} > 20) \). From Figs 3 (a) and (b), it is supposed that, at a certain time of transition \( (10 < \tilde{t} < 20) \), the hygrothermal field exhibits a quasi-linear distribution and, as explained in the last paragraph in Subsection 4.1, the resulting stress almost vanishes. This behavior is consistent with the transition of the profile found in Fig. 6. The most important aspect found in Fig. 6 is that, under the nonlinear hygrothermal theory treated in this paper, the stress at the steady stage, e.g., \( \tilde{t} = 50 \) persists in the strip. This feature is quite unlike the characteristic of a linear framework, which causes no stress at the steady stage as explained in Subsection 4.1.
The stress that persists at the steady stage, shortly, the residual stress, is caused by the nonlinear hygrothermal field as explained above, and therefore can be regarded as a measure of the nonlinearity. In this regard, the effects of the hygrothermal gap between two surfaces of the strip are investigated on the residual stress. The initial and boundary values of the field quantities are taken as indicated by Eq. (17), but $T_b$ and $M_b$ are considered to be variable. When there is no such gap, that is, $\hat{T}_b = \hat{T}_a$ and $M_b = M_a$, the system of nonlinear coupling diffusion equations (10) subjected to the steady condition ($\partial e / \partial t = 0$) is found to give the uniform and therefore linear distributions of the temperature and dissolved moisture content, and to cause no residual stress as explained in Subsection 4.1. On the other hand, when such a gap exists, the residual stress does occur as illustrated in Fig. 6, at $\tau = 50$. Therefore, the variations of residual stress with the gaps in the boundary values, namely $\hat{T}_b - \hat{T}_a$ and $M_b - M_a$ are shown in Fig. 7. Figure 7 shows that at various portions of the strip, the magnitude of residual stress increases with the gaps in boundary values of the temperature and dissolved moisture content.

Fig. 7  Effect of hygrothermal gap on residual stress $\{\tau = 50, \hat{D}_x = 1\}$.  

5. Concluding remarks

A one-dimensional transient hygrothermoelastic field in a porous strip is investigated, taking into consideration the nonlinear coupling between heat and binary moisture. The nonlinear relationship between the temperature, dissolved moisture content, and vapor concentration is investigated qualitatively and quantitatively. A system of governing equations is presented, that accounts for the nonlinear relationship and the diffusivities of both dissolved moisture and vapor. The governing equations are then applied to an infinite strip subjected to a broad gap in the hygrothermal environment, and solved using a finite difference method. The structure of the hygrothermal field is investigated in detail, and the moisture distribution is found to be highly nonlinear at the steady stage. Finally, based on hygrothermoelasticity, the distribution of the resulting in-plane stress in the strip free from mechanical constraints is analyzed theoretically. The occurrence of residual stress is confirmed, and the effect of the gaps in hygrothermal environment on the residual stress is investigated qualitatively and quantitatively.
References

Bergman, R., Wood Handbook, Chapter 13: Drying and control of moisture content and dimensional changes. In: Ross, R.J. (ed.), Wood Handbook, Wood as an Engineering Material (General Technical Report FPL-GTR-190) (2010), pp. "13-1"–"13-20", Department of Agriculture, Forest Service, Forest Products Laboratory.

Chang, W. -J., Transient hygrothermal responses in a solid cylinder by linear theory of coupled heat and moisture, Applied Mathematical Modelling, Vol. 18, No. 8 (1994), pp. 467–473.

Chang, W. -J., Chen, T. -C., and Weng, C. -I., Transient hygrothermal stresses in an infinitely long annular cylinder: coupling of heat and moisture, Journal of Thermal Stresses, Vol. 14, No. 4 (1991), pp. 439–454.

Chang, W. -J. and Weng, C. -I., An analytical solution of a transient hygrothermal problem in an axisymmetric double-layer annular cylinder by linear theory of coupled heat and moisture, Applied Mathematical Modelling, Vol. 21, No. 11 (1997), pp. 721–734.

Glass, S. V. and Zelinka, S. L., Wood Handbook, Chapter 4: Moisture relations and physical properties of wood. In: Ross, R.J. (ed.), Wood Handbook, Wood as an Engineering Material (General Technical Report FPL-GTR-190) (2010), pp. "4-1"–"4-19", Department of Agriculture, Forest Service, Forest Products Laboratory.

Hartranft, R. J. and Sih, G. C., The influence of the Soret and Dufour effects on the diffusion of heat and moisture in solids, International Journal of Engineering Science, Vol. 18, No. 12 (1980), pp. 1375–1383.

Ishihara, M., Ootao, Y., and Kameo, Y., Hygrothermal field considering nonlinear coupling between heat and binary moisture diffusion in porous media, Journal of Thermal Stresses, Vol. 37, No. 10 (2014), pp 1173–1200.

LeVeque, R. J., Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems (2007), Society for Industrial and Applied Mathematics.

Nakao, T., Description of moisture dependence of diffusion coefficient by moisture diffusion equation considering Coulomb friction, Holz als Roh und Werkstoff, Vol. 56, No. 4 (1998), p. 266.

Sih, G. C. and Ogawa, A., Transient thermal change on a solid surface: coupled diffusion of heat and moisture, Journal of Thermal Stresses, Vol. 5, Nos 3–4 (1982), pp. 265–282.

Sih, G. C. and Shih, M. T., Hygrothermal stress in a plate subjected to antisymmetric time-dependent moisture and temperature boundary conditions, Journal of Thermal Stresses, Vol. 3, No. 3 (1980), pp. 321–340.

Sih, G. C., Shih, M. T., and Chou, S. C., Transient hygrothermal stresses in composites: coupling of moisture and heat with temperature varying diffusivity, International Journal of Engineering Science, Vol. 18, No. 1 (1980), pp. 19–42.

Sugano, Y. and Chuuman, Y., Analytical solution of transient hygrothermoelastic problem due to coupled heat and moisture diffusion in a hollow cylinder, Transactions of the Japan Society of Mechanical Engineers Series A, Vol. 59, No. 564 (1993), pp. 1956–1963 (in Japanese).

Timoshenko, S. P. and Goodier, J. N., Theory of Elasticity (3rd ed) (1955), McGraw-Hill.

Weathermax, R. C. and Stamm, A. J., The coefficients of thermal expansion of wood and wood products, Report No. 1487 (1946), U.S. Department of Agriculture, Forest Service, Forest Products Laboratory.