Hidden symmetries in a gauge covariant approach, Hamiltonian reduction and oxidation

Mihai Visinescu *

Department of Theoretical Physics,
Horia Hulubei National Institute for Physics and Nuclear Engineering,
P.O.B. MG-6, 077125 Magurele, Romania

Abstract

Hidden symmetries in a covariant Hamiltonian formulation are investigated involving gauge covariant equations of motion. The special role of the Stäckel-Killing tensors is pointed out. A reduction procedure is used to reduce the original phase space to another one in which the symmetries are divided out. The reverse of the reduction procedure is done by stages performing the unfolding of the gauge transformation followed by the Eisenhart lift in connection with scalar potentials.

1 Introduction

The motion of a free point particle in a (pseudo)-Riemannian space is determined by the kinetic energy and the trajectories are geodesics corresponding to the metric tensor of the configuration space. More general than the free mechanical system, in the case of a conservative holonomic dynamical system whose kinetic energy is modified by the addition of a potential, the trajectories are not geodesics of the metric tensor.

*E-mail: mvisin@theory.nipne.ro
In many cases it is preferably to represent the trajectories as geodesics of a related metric. For example the Jacobi metric \[1\], conformally related to the original one, is obtained rescaling the potential and the total energy. The drawback of the conformally related metric is that its geodesics describe the trajectories of a fixed energy.

An attractive alternative is represented by the Eisenhart lift or oxidation \[2\] of a dynamical system which permits to put into correspondence the trajectories of a mechanical system with the geodesics of a configuration space extended in dimension.

It is well known that a geodesic system has a first integral linear in the momenta if the metric admits a Killing vector corresponding to an infinitesimal isometry. In some cases there exist additional nontrivial first integrals quadratic (or more general polynomial) in the momenta provided that the configuration space admits Stäckel-Killing (SK) tensors. Superintegrable systems have been thoroughly studied in connection with hidden symmetries and separability of the Hamilton-Jacobi equations and the corresponding ones in the quantum theory. From this point of view the superintegrable systems, formulated as geodesic systems, offer us examples of manifolds with nontrivial Killing tensors.

In this paper we analyze the hidden symmetries of a dynamical system in the presence of external gauge fields in a covariant approach \[3, 4, 5\]. This approach proves to be more convenient in the study of the conserved quantities involving gauge covariant equations of motion.

In the case of a symplectic manifold on which a group of symmetries acts symplectically, it is possible to reduce the original phase space to another symplectic manifold in which the symmetries are divided out. Such a situation arises when one has a particle moving in an electromagnetic field \[6\].

On the other hand the reverse of the reduction procedure can be used to investigate complicated systems. It is possible to use a sort of unfolding of the initial dynamics by imbedding it in a larger one which is easier to integrate \[7\]. Sometimes the equations of motion in a higher dimensional space are quite transparent, e.g. geodesic motions, but the equations of motion of the reduced system appear more complicated \[8\].

As an illustration of the reduction of a symplectic manifold with symmetries and the opposite procedure of oxidation of a dynamical system we shall consider the principal bundle \(\pi : \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}\) with structure group \(U(1)\). The Hamiltonian function on the cotangent bundle \(T^*(\mathbb{R}^4 - \{0\})\) is in-
variant under the $U(1)$ action and the reduced Hamiltonian system proves to describe the three-dimensional Kepler problem in the presence of a centrifugal potential and Dirac’s monopole field. Moreover this reduction procedure is also relevant for many other problems like the geodesic flows of the generalized Taub-NUT metric, conformal Kepler system, MIC-Kepler system, etc.

Concerning the unfolding of the reduced Hamiltonian system we shall perform it by stages. In a first stage of unfolding we use an opposite procedure to the reduction by an $U(1) \simeq S^1$ action to a four-dimensional generalized Kepler problem. Finally we resort to the method introduced by Eisenhart who added one or two extra dimensions to configuration space to represent trajectories by geodesics. The kinetic energy metric and scalar potential are involved in the construction of the metric of the extended configuration space in such a way that geodesics on the extended space project to trajectories of the initial configuration space.

The plan of the paper is as follows. Section 2 is concerned with the covariant formulation of the dynamics of particles in the presence of external gauge fields and scalar potentials. In Section 3 we make a brief review of the Hamiltonian reduction of symplectic manifolds with symmetries pointing out the Hamiltonian systems defined on the cotangent bundle $T^*(\mathbb{R}^4 - \{0\})$ with standard symplectic form. Next Section is devoted to the reverse of the reduction procedure associated with an $S^1$ action to a four dimensional generalized Kepler problem. In Section 5 we discuss the Eisenhart procedure for the oxidation limiting ourselves to dynamical systems and scalar potentials which do not involve time. Conclusions and open problems are discussed in the last Section.

2 Symmetries and conserved quantities

The geodesic flow for an $n$-dimensional manifold $M$ equipped with a (pseudo)-Riemmanian metric $g$ is generated by the quadratic Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j. \tag{1}$$

In terms of the phase-space variables $(x^i, p_i)$ the canonical symplectic structure $\omega$ of $T^*M$ is $\omega = dp_i \wedge dx^i$ and the corresponding Poisson bracket of two
observables $P, Q$ is
\[
\{P, Q\} = \frac{\partial P}{\partial x^i} \frac{\partial Q}{\partial p_i} - \frac{\partial P}{\partial p_i} \frac{\partial Q}{\partial x^i}.
\]

Let us consider a conserved quantity of motion expanded as a power series in momenta:
\[
K = \sum_{i=0}^{s} K^{(i)} = K_0 + \sum_{k=1}^{s} \frac{1}{k!} K^{i_1 \cdots i_k}(x)p_{i_1} \cdots p_{i_k},
\]
with $K^{(i)}$ homogeneous polynomial of degree $i$ in momenta. It has vanishing Poisson bracket with the Hamiltonian, $\{K, H\} = 0$, which implies
\[
K^{(i_1 \cdots i_k;i)} = 0,
\]
where a semicolon denotes the covariant differentiation corresponding to the Levi-Civita connection $\nabla$ and round brackets indicate full symmetrization over the indices enclosed. A symmetric tensor $K^{i_1 \cdots i_k}$ satisfying (4) is called a SK tensor of rank $k$. The SK tensors represent a generalization of the Killing vectors and are responsible for the hidden symmetries of the motions, connected with conserved quantities of the form (3) polynomials in momenta.

The traditional means to deal with the coupling to a gauge field $F_{ij}$ expressed (locally) in terms of the potential 1-form $A_i$
\[
F = dA,
\]
is to replace the Hamiltonian by
\[
H = \frac{1}{2} g^{ij}(p_i - A_i)(p_j - A_j) + V(x),
\]
work with the Poisson bracket (2) and consider the polynomials (3) in the variables $(p_i - A_i)$ for $i = 1, \cdots, n$ [9]. For completeness, in (6) we included a scalar potential $V(x)$.

The disadvantage of this approach is that the canonical momenta $p_i$ and implicitly the Hamilton equations of motion are not manifestly gauge covariant. This inconvenience can be removed using van Holten’s receipt [3] by introducing the gauge invariant momenta:
\[
\Pi_i = p_i - A_i.
\]
The Hamiltonian (6) becomes
\[
H = \frac{1}{2} g^{i j} \Pi_i \Pi_j + V(x),
\] (8)
and the equations of motion are derived using the modified Poisson bracket [3, 4, 10]
\[
\{P, Q\} = \frac{\partial P}{\partial x^i} \frac{\partial Q}{\partial \Pi_i} - \frac{\partial P}{\partial \Pi_i} \frac{\partial Q}{\partial x^i} + q F_{ij} \frac{\partial P}{\partial \Pi_i} \frac{\partial Q}{\partial \Pi_j}.
\] (9)

Searching for conserved quantities [3] expanded rather into powers of the gauge invariant momenta Π_i, the vanishing of the Poisson bracket \{K, H\} yields a series of constraints in the form of a system of coupled differential equations [5]. Only the equation for the leading order term \(K^{i_1 \cdots i_s}\) defines a \(SK\) tensor of rank \(s\). The rest of equations mixes the derivatives of the terms \(K^{i_1 \cdots i_k}, (k < s)\) and potential \(V\) with the gauge field strength \(F_{ij}\).

Several applications using van Holten’s covariant framework [3] are given in [4, 5, 11, 12, 13].

3 Hamiltonian reduction

It is simplest to work with the Hamiltonian formulation in order to see how the reduction and the oxidation of a dynamical system affect constants of motion.

The general setting for reduction of symplectic manifolds with symmetries is presented in [1, 6]. Here we confine ourselves to the \(U(1)\) reduction of a Hamiltonian system defined on the cotangent bundle \(T^* (\mathbb{R}^4 - \{0\})\) with standard symplectic form. The reduced phase space is not symplectomorphic to the cotangent bundle \(T^* (\mathbb{R}^3 - \{0\})\) with standard symplectic form. It proves that the reduced symplectic form on \(T^* (\mathbb{R}^3 - \{0\})\) contains a two-form describing Dirac’s monopole field beside the standard symplectic form.

Let us start to consider the principal fiber bundle \(\pi: \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}\) with structure group \(U(1)\) whose action is given by [14]
\[
x \mapsto T(t) x, \quad x \in \mathbb{R}^4, \quad t \in \mathbb{R},
\] (10)
where
\[
T(t) = \begin{pmatrix} R(t) & 0 \\ 0 & R(t) \end{pmatrix}, \quad R(t) = \begin{pmatrix} \cos \frac{t}{2} & -\sin \frac{t}{2} \\ \sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix}.
\] (11)
The $U(1)$ action is lifted to a symplectic action on $T^* (\mathbb{R}^4 - \{0\})$

$$ (x, y) \rightarrow (T(t)x, T(t)y), \quad (x, y) \in (\mathbb{R}^4 - \{0\}) \times \mathbb{R}^4. \tag{12} $$

Let $\Psi : T^* (\mathbb{R}^4 - \{0\}) \rightarrow \mathbb{R}$ be the moment map associated with the $U(1)$ action (12)

$$ \Psi(x, y) = \frac{1}{2}(-x_2 y_1 + x_1 y_2 - x_4 y_3 + x_3 y_4). \tag{13} $$

The reduced phase-space $P_\mu$ is defined through

$$ \pi_\mu : \Psi^{-1}(\mu) \rightarrow P_\mu := \Psi^{-1}(\mu)/U(1), \quad \tag{14} $$

which is diffeomorphic with $T^* (\mathbb{R}^3 - \{0\}) \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$.

The coordinates $(q_k, p_k) \in (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ are given by the Kustaanheimo-Stiefel transformation

$$ \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ 0 \end{pmatrix} = \frac{1}{r} \begin{pmatrix} x_3 & x_4 & x_1 & x_2 \\ -x_4 & x_3 & -x_1 & x_2 \\ x_1 & x_2 & -x_3 & -x_4 \\ -x_2 & x_1 & -x_4 & x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{15} $$

$$ \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \Psi/r \end{pmatrix} = \frac{1}{2r} \begin{pmatrix} x_3 & x_4 & x_1 & x_2 \\ -x_4 & x_3 & x_2 & -x_1 \\ x_1 & x_2 & -x_3 & -x_4 \\ -x_2 & x_1 & -x_4 & x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}. \tag{16} $$

where $r = \sum_{j=1}^4 x_j^2 = \sqrt{\sum_{k=1}^3 q_k^2}$.

The phase-space $T^* (\mathbb{R}^4 - \{0\})$ is equipped with the standard symplectic form

$$ d\Theta = \sum_{j=1}^4 dy_j \wedge dx_j, \quad \Theta = \sum_{j=1}^4 y_j \wedge dx_j. \tag{17} $$

Let $\iota_\mu : \Psi^{-1}(\mu) \rightarrow T^* (\mathbb{R}^4 - \{0\})$ be the inclusion map. The reduced symplectic form $\omega_\mu$ is determined on $P_\mu$ by

$$ \pi_\mu^* \omega_\mu = \iota_\mu^* d\Theta, \tag{18} $$

namely

$$ \omega_\mu = \sum_{k=1}^3 dp_k \wedge dq_k - \frac{\mu}{r^3} (q_1 dq_2 \wedge dq_3 + q_2 dq_3 \wedge dq_1 + q_3 dq_1 \wedge dq_2). \tag{19} $$
ω_µ consists of the standard symplectic form on T^*(\mathbb{R}^3 - \{0\}) and in addition a term corresponding to the Dirac’s monopole field

\[ \vec{B} = -\mu \frac{\vec{q}}{r^3}, \]  

of strength \(-\mu\).

The reduced Hamiltonian is determined by

\[ H \circ \iota_\mu = H_\mu \circ \pi_\mu. \]  

For the purpose of the present paper, we shall be concerned with the reduction of the dynamical system associated with the geodesic flows of the generalized Taub-NUT metric on \(\mathbb{R}^4 - \{0\}\). This metric is relevant for (conformal) Coulomb problem [14], MIC-Zwanziger system [15, 16], Euclidean Taub-NUT [17, 18, 19] and its extensions [20, 21], etc. The generalized Taub-NUT metric is

\[ ds_4^2 = f(r)(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) + g(r)(d\psi + \cos \theta d\phi)^2, \]  

where the curvilinear coordinates \((r, \theta, \phi, \psi)\) are

\[ x_1 = \sqrt{r} \cos \frac{\theta}{2} \cos \frac{\psi + \phi}{2}, \quad x_2 = \sqrt{r} \cos \frac{\theta}{2} \sin \frac{\psi + \phi}{2}, \]
\[ x_3 = \sqrt{r} \sin \frac{\theta}{2} \cos \frac{\psi - \phi}{2}, \quad x_4 = \sqrt{r} \sin \frac{\theta}{2} \sin \frac{\psi - \phi}{2}. \]  

In what follows we consider the Hamiltonian on the cotangent bundle \(T^*(\mathbb{R}^4 - \{0\})\)

\[ H = \frac{1}{2f(r)} p_r^2 + \frac{1}{2r^2 f(r)} p_\theta^2 + \frac{(p_\phi - p_\psi \cos \theta)^2}{2r^2 f(r) \sin^2 \theta} + \frac{p_\psi^2}{2g(r)} + V(r), \]  

where, to make things more specific, we consider a potential \(V(r)\) function of the radial coordinate \(r\).

The Hamiltonian function is invariant under the \(U(1)\) action with the infinitesimal generator \(\frac{\partial}{\partial \psi}\) so that the conserved momentum is

\[ \mu = p_\psi = \Theta \left( \frac{\partial}{\partial \psi} \right), \]  

7
where the canonical one-form $\Theta$ could be expressed in curvilinear coordinates

$$
\Theta = p_r dr + p_\theta d\theta + p_\phi d\phi + p_\psi d\psi ,
$$

on the cotangent bundle $T^*(\mathbb{R}^4 - \{0\})$.

The reduced Hamiltonian (21) has the form

$$
H_\mu = \frac{1}{2f(r)} \sum_{k=1}^{3} p_k^2 + \frac{\mu^2}{2g(r)} + V(r) .
$$

Now the search of conserved quantities of motion in the 3-dimensional curved space in the presence of the potential $V(r)$ plus the contribution of the monopole field proceeds in standard way. First of all we remark that the reduced Hamiltonian is still spherical symmetric and one can easily show that the angular momentum vector

$$
\vec{J} = \vec{q} \times \vec{p} + \frac{\mu}{r} \vec{q} ,
$$

is conserved.

In some cases the system admits additional constants of motion polynomial in momenta. Here are some notable cases:

1) For

$$
f(r) = 1 , \quad g(r) = r^2 , \quad V(r) = -\frac{\kappa}{r}
$$

we recognize the MIC-Kepler problem with the Runge-Lenz type conserved vector

$$
\vec{A} = \vec{p} \times \vec{J} - \kappa \frac{\vec{q}}{r} .
$$

1a) Moreover, for $\mu = 0$, $H_\mu$ becomes the Hamiltonian for the Coulomb - Kepler problem.

2) For

$$
f(r) = \frac{a + br}{r} , \quad g(r) = \frac{ar + br^2}{1 + cr + dr^2} , \quad V(r) = 0 ,
$$

with $a, b, c, d$ real constants we recover the extended Taub-NUT space which still admit a Runge-Lenz type vector

$$
\vec{A} = \vec{p} \times \vec{J} - (aE - \frac{1}{2} c\mu^2) \frac{\vec{q}}{r} ,
$$

8
where $E$ is the conserved energy.

2a) In the particular case, if the constants $a, b, c, d$ are subject to the constraints

$$c = \frac{2b}{a}, \quad d = \frac{b^2}{a^2},$$

the extended metric coincides, up to a constant factor, with the original Taub-NUT metric.

Other examples could be found in [4, 20, 22].

4 Unfolding

It is interesting to analyze the reverse of the reduction procedure which can be used to investigate difficult problems [7]. For example the equations of motion for the dynamical system (19), (27) look quite complicated. Using a sort of unfolding of the 3-dimensional dynamics imbedding it in a higher dimensional space the conserved quantities are related to the symmetries of this manifold.

To exemplify let us start with the reduced Hamiltonian (27) written in curvilinear coordinates

$$H_\mu = \frac{1}{2f(r)} \left[ p_r^2 + \frac{1}{r^2} \left( p_\theta^2 + \frac{(p_\phi - \mu \cos \theta)^2}{\sin^2 \theta} \right) \right] + \frac{\mu^2}{2g(r)} + V(r),$$

on the 3-dimensional space with the metric

$$ds^2_3 = f(r)(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)),\quad (33)$$

and the canonical symplectic form

$$d\Theta_\mu = dp_r \wedge dr + dp_\theta \wedge d\theta + dp_\phi \wedge d\phi,$$

$\mu$ being the strength of the Dirac’s monopole field.

In the specific case of the Dirac’s monopole field (20) the gauge invariant momenta (7) in spherical coordinates are

$$\Pi_r = p_r, \quad \Pi_\theta = p_\theta, \quad \Pi_\phi = p_\phi - \mu \cos \theta.$$

The reduced Hamiltonian (27) has the form

$$H_\mu = \frac{1}{2f(r)} \left[ \Pi_r^2 + \frac{1}{r^2} \left( \Pi_\theta^2 + \frac{\Pi_\phi^2}{\sin^2 \theta} \right) \right] + \frac{\mu^2}{2g(r)} + V(r),$$

(36)
and
\[ d\Theta_\mu = d\Pi_r \wedge dr + d\Pi_\theta \wedge d\theta + d\Pi_\phi \wedge d\phi - \mu \sin \theta d\theta \wedge d\phi , \]  

in agreement with (19) and Poisson bracket (9).

At each point of \( T^*(\mathbb{R}^3 - \{0\}) \) we define the fiber \( S^1 \), the group space of the gauge group \( U(1) \). On the fiber we consider the motion whose equation is
\[ \frac{d\psi}{dt} = \frac{\mu}{g(r)} - \frac{\cos \theta}{r^2 f(r) \sin^2 \theta} (p_\phi - \mu \cos \theta) . \]  

The metric on \( \mathbb{R}^4 \) defines horizontal spaces orthogonal to the orbits of the circle - this is a connection on the principal bundle [23]. Using the above trivialization, we have the coordinates \((r, \theta, \phi, \psi)\) with the horizontal spaces annihilated by the connection
\[ d\psi + \cos \theta d\phi . \]

The metric on \( \mathbb{R}^4 \), which admits a circle action leaving invariant the symplectic form (37), can be written in the form
\[ ds_4^2 = f(r)(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) + h(r)(d\psi + \cos \theta d\phi)^2 \]
\[ = \sum_{i,j=1}^4 g_{ij} dq^i dq^j . \]  

The natural symplectic form on \( T^*(\mathbb{R}^4 - \{0\}) \) is (26)
\[ d\Theta = d\Theta_\mu + dp_\psi \wedge d\psi . \]

Considering the geodesic flow of \( ds_4^2 \) and taking into account that \( \psi \) is a cycle variable
\[ p_\psi = h(r)(\dot{\psi} + \cos \theta \dot{\phi}) , \]  

is a conserved quantity. To make contact with the Hamiltonian dynamics on \( T^*(\mathbb{R}^3 - \{0\}) \) we must identify
\[ h(r) = g(r) . \]

Otherwise the resulting Hamiltonian dynamics projected onto \( T^*(\mathbb{R}^3 - \{0\}) \) is that from the Hamiltonian \( H_\mu \) choosing (43).
The reverse of the reduction proves to be useful since the equations of motion for the Hamiltonian
\[ H = \frac{1}{2}g^{ij}p_ip_j + V , \] (44)
are quite simple and transparent, but the equations of the quotient system (32) appear more complicated [21]. The corresponding differential equations of the trajectories are
\[ g^{ij}\dddot{q}_j + [jk,i]\dddot{q}_j\dot{q}_k + \frac{\partial V}{\partial q^i} = 0 , \] (45)
where \([jk,i]\) is the Christoffel symbol. These equations admit the first integral of motion
\[ \frac{1}{2}g^{ij}\dot{q}_i\dot{q}_j + V = T + V = E , \] (46)
where \(E\) is the conserved energy.

5 Eisenhart lift

In many concrete problems, after the unfolding of the gauge symmetry, one ends up with a dynamical system on an extended phase space and an Hamiltonian (44) with a "residual" scalar potential.

In the final stage of the oxidation of the dynamical system described by the Hamiltonian (44) we shall apply the Eisenhart’s lift [2] (see also [24, 25]). In the general case of the Eisenhart’s lift when the time enters in the constraints and in the potential function, the dynamics of a mechanical system with an \(n\) dimensional configuration space is related to a system of geodesics in an \((n+2)\) spacetime [2, 26, 27]. In order to simplify the problem, we shall assume that the constrains of the dynamical system and the potential \(V\) do not involve time. In this simplified case it is adequately to consider a Riemannian space with \(n+1\) (in our particular case 4 + 1) dimensions with the metric
\[ ds^2 = \sum_{i,j=1}^{4} g_{ij}dq^idq^j + Adu^2 , \] (47)
where it is assumed that \(A\) does not involve \(u\).
In contrast with equations (45), now the trajectories of motion are given by
\[ g_{ij} \frac{d^2 q^i}{ds^2} + [j k, i] \frac{dq^j}{ds} \frac{dq^k}{ds} - \frac{1}{2} \frac{\partial A}{\partial q^i} \left( \frac{du}{ds} \right)^2 = 0, \]
\[ A \frac{du}{ds} = a, \] (48)
where \( a \) is a constant.

For a non-vanishing constant \( a \) it is possible to choose a parameter \( t \) for each non-minimal geodesic as
\[ t = as, \] (49)
identified with the time. Equations (45) are the same as (48) if \( A \) is defined by
\[ \frac{1}{2A} = V + b, \] (50)
where \( b \) is another constant which should be chosen consistently with
\[ \frac{1}{a^2} = 2(E + b). \] (51)

At last, the coordinate \( u \) is related to the action by
\[ u = -2 \int Tdt + 2(E + b)t. \] (52)

The Hamiltonian on the enlarge phase space (47) is
\[ H_5 = \frac{1}{2} \sum_{i,j=1}^{4} g^{ij} p_i p_j + \frac{1}{2} \frac{1}{A} p_u^2, \] (53)
where \( A \) is given by (50), \( p_i, p_u \) are the conjugate momenta and the new symplectic form is
\[ \omega' = dp_i \wedge dq^i + dp_u \wedge du. \] (54)

Let us assume that the Hamiltonian (32) on \( T^*(\mathbb{R}^4 - \{0\}) \) has a constant of motion polynomial in momenta of the form (3). We lift \( K \) to the extended space
\[ K = \sum_{i=0}^{s} p_u^{s-i} K^{(i)}. \] (55)
It could be easily verified that \( K \) is a constant along geodesics on the enlarge phase space \([17]\) iff \( K \) is a constant of motion for the original system \([27]\). In fact \( K \) is a homogeneous polynomial in momenta corresponding to a Killing tensor of the metric \([17]\).

### 6 Concluding remarks

The aim of this paper is to use the covariant Hamiltonian formulation of the dynamics of particles in external gauge fields and scalar potential.

In general the explicit and hidden symmetries of a spacetime are encoded in the multitude of Killing vectors and higher order SK tensors respectively. The inclusion of gauge fields and scalar potentials affects the geodesic conserved quantities in a nontrivial way.

When we have a symplectic manifold with symmetries, it is possible to reduce the phase space to another symplectic manifold in which the symmetries are divided out. Such a situation arises when one has a particle moving in a gauge field \( F \). If the group of symmetries acts on the manifold leaving the two-form \( F \) invariant, it is possible to find a Hamiltonian system canonically induced on a reduced phase space.

In the usual applications, applying the method of reduction simplifies the equations of motion. However, in some cases, the reverse of the reduction might be useful, namely the equations of motion on the extended phase space are quite transparent, but the equations of motion of the quotient system appear more complicated. Applying an oxidation of a dynamical system with constants of motion polynomial in momenta, one may obtain spacetimes admitting SK tensors of higher rank.

The systems considered in this paper present hidden symmetries described by SK tensors of rank 2. However there are several examples of integrable systems admitting integrals of motion of higher order in momenta. Recently it has been introduced \([28]\) a new superintegrable Hamiltonian as a generalization of the Keplerian one with three terms preventing the particle crossing the principal planes. A generalization of the Runge-Lenz vector is found and also independent isolating integrals quartic in the momenta are identified. An investigation of the Kepler problem on \( N \)-dimensional Riemannian spaces of non constant curvature was done \([29]\) in order to obtain maximally superintegrable classical systems.

Another natural generalization of the Killing vectors is represented by
totally antisymmetric Killing-Yano (KY) tensors. KY tensors generate supercharges in the dynamics of pseudo-classical spinning particles and non standard Dirac operators which commute with the standard one. Given a KY tensor one can construct a rank 2 SK tensor as a symmetric product of KY tensors. It would be interesting to investigate relations between KY tensors and hidden symmetries in the context of Hamilton reduction and oxidation.

Acknowledgments

Support through CNCSIS program IDEI-571/2008 is acknowledged.

References

[1] R. Abraham and J. E. Marsden, *Foundations of mechanics* (Benjamin/Cummings, New York, N. Y., 1978).

[2] L. P. Eisenhart, *Annals Math.* **30**, 591 (1928).

[3] J. W. van Holten, *Phys. Rev. D* **75**, 025027 (2007).

[4] J.-P. Ngome, *J. Math. Phys.* **50**, 122901 (2009).

[5] M. Visinescu, *Mod. Phys. Lett. A* **25**, 341 (2010).

[6] J. Marsden and A. Weinstein, *Rep. Math. Phys.* **5**, 121 (1974).

[7] G. Marmo, E. J. Saletan and A. Simoni, *J. Math. Phys.* **20**, 856 (1979).

[8] D. Kazdan, B. Kostant and S. Sternberg, *Comm. Pure Appl. Math.* **31**, 481 (1978).

[9] C. Duval and G. Valent, *J. Math. Phys.* **46**, 053516 (2005).

[10] J.-M. Souriau, *Structures des Systèmes Dynamiques* (Dunod, Paris, 1970).

[11] P.A. Horváthy and J.-P. Ngome, *Phys. Rev. D* **79**, 127701 (2009).

[12] T. Igata, T. Koike and H. Isihara, *Phys. Rev. D* **83**, 065027 (2011).
[13] M. Visinescu, *SIGMA* **7**, 037 (2011).
[14] T. Iwai and Y. Uvano, *J. Math. Phys.* **27**, 1523 (1986).
[15] D. Zwanziger, *Phys. Rev.* **176**, 1480 (1968).
[16] H. V. McIntosch and A. Cisneros, *J. Math. Phys.* **11**, 896 (1970).
[17] S. W. Hawking, *Phys. Lett. A* **60**, 81 (1977).
[18] N. S. Manton, *Phys. Lett. B* **110**, 54 (1982).
[19] G. R. Gibbons and R. J. Ruback, *Commun. Math. Phys.* **115**, 267 (1988).
[20] T. Iwai and N. Katayama, *J. Math. Phys.* **36**, 1790 (1995).
[21] T. Iwai and N. Katayama, *J. Geom. Phys.* **12**, 55 (1993).
[22] G. W. Gibbons and C. M. Warnick, *J. Geom. Phys.* **57**, 2286 (2007).
[23] N. Hitchin, *Monopoles, Minimal Surfaces and Algebraic Curves* (Séminaire de Mathématiques Supérieures 105, Les Presses de L’Université de Montréal, Montréal, Canada, 1987).
[24] M. Szydlowski, *Gen. Rel. Grav.* **30**, 887 (1998).
[25] I. M. Benn, *J. Math. Phys.* **47**, 022903 (2006).
[26] E. Minguzzi, *Class. Quantum Grav.* **24**, 2781 (2007).
[27] G. W. Gibbons, T. Houri, D. Kubiznak and C. M. Warnick, *Phys. Lett. B* **700**, 68 (2011).
[28] P. E. Verrier and N. W. Ewans, *J. Math. Phys.* **49**, 022902 (2008).
[29] A. Ballesteros, A. Enciso, F. J. Herranz, O. Ragnisco and D. Riglioni, *SIGMA* **7**, 048 (2011).