A model for eutectic growth in multicomponent alloys

A V Catalina1, P W Voorhees2, R K Huff1 and A L Genau3
1 Caterpillar Inc., 14009 Old Galena Road, Mossville, IL 61552, USA
2 Northwestern University, 2220 Campus Dr, Evanston, IL 60208, USA
3 The University of Alabama at Birmingham, 1075 13th Street South, Birmingham, AL 35294-4440, USA
Email: Catalina_Adrian_V@cat.com
Phone: +1 309 578 6314

Abstract. A model for eutectic growth of multicomponent alloys is proposed. Among several other modifications brought to the classic theory of eutectic growth, the model introduces the phase fraction change as the main mechanism for maintaining an isothermal solid/liquid interface during solidification of eutectic alloys. A comparison between the theoretical results and experimental measurements for the Fe-C based alloys showed a good agreement for alloys of composition not too far removed from the eutectic point. The analysis also showed that an earlier proposed criterion for spacing selection of binary irregular eutectics, based on volume fraction of the faceted phase, can be applied to multicomponent alloys as well. The investigation performed on the effect of Si on the growth kinetics of Fe-C-Si eutectic alloys showed that the growth coefficient $\mu$ decreases with the increase of Si content.

1. Introduction/Background

The Jackson and Hunt (JH) [1] classical treatment of the binary eutectic growth starts with solving the solute diffusion equation in front of a planar solid/liquid (s/l) interface moving at a constant velocity, $V$, in the $z$-direction:

$$\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial z^2} + \frac{V}{D_L} \frac{\partial C}{\partial z} = 0 \quad (1)$$

where $C$ is concentration and $D_L$ is the diffusion coefficient in the liquid. For a eutectic microstructure (Fig. 1a) consisting of alternating lamellae of $\alpha$ and $\beta$ phases the appropriate boundary conditions are:
- periodicity: $C(x+\lambda) = C(x)$
- symmetry: $\partial C/\partial x = 0$ for $x = 0$ and $x = \lambda/2$
- far-field: $C = C_{eut} + \Delta C_{\infty}$ for $z \to \infty$

and therefore the general solution of Eq.(1) is given by:

$$C(x, z) = C_{eut} + \Delta C_{\infty} + B_0 \cdot \exp\left(-\frac{V \cdot z}{D_L}\right) + \sum_{n=1}^{\infty} B_n \cdot \cos\left(\frac{2n\pi \lambda}{\lambda} \cdot x\right) \cdot \exp(-\omega_n \cdot z) \quad (3)$$

where $C_{eut}$ is the eutectic composition, $\Delta C_{\infty}$ is the difference between the eutectic composition and the composition far from the interface, $\lambda$ is the lamellar spacing ($\lambda/2=S_{\alpha}+S_{\beta}$), and the exponent $\omega_n$ can be
approximated as \( \omega_n \equiv 2n\pi/\lambda \) at small growth rates when \( \frac{\lambda V}{2D_L} \ll 2\pi \) [1,2]. The Fourier coefficients \( B_o \) and \( B_n \) have to be determined from the mass balance of solute at the s/l interface which, by ignoring the density differences between phases as in [1], can be expressed as:

\[
\begin{aligned}
-\frac{D_L}{V}\frac{\partial C}{\partial z} &= \begin{cases} 
(1-k_\alpha)\cdot C_\ell(x) & \text{for } \alpha \text{-phase } (0 \leq x \leq S_\alpha) \\
(1-k_\beta)\cdot C_\ell(x) & \text{for } \beta \text{-phase } (S_\alpha \leq x \leq S_\alpha + S_\beta)
\end{cases} \\
&= \beta_\alpha \alpha_\beta S_\alpha S_\beta V \sin(2\pi z/2) 
\end{aligned}
\]

(4)

where \( C_\ell(x) \) is the liquid composition at the s/l interface (i.e., at \( z = 0 \)), and \( k_\alpha \) and \( k_\beta \) are the partition coefficients between the liquid and the \( \alpha \)- and \( \beta \)-phase, respectively.

As formulated in Eq.(4) the mass balance equation cannot be easily applied to calculating the Fourier coefficients because of different values of \( k_\alpha \) and \( k_\beta \). Consequently, it was suggested in JH treatment that \( C_\ell(x) \) can be replaced by the eutectic composition \( (C_{\text{eut}}) \) thus allowing \( B_o \) and \( B_n \) be calculated as:

\[
B_o = \frac{C_\alpha^\ell \cdot S_\alpha - C_\beta^\ell \cdot S_\beta}{S_\alpha + S_\beta} 
\]

(5)

\[
B_n = \frac{2}{(n\pi)^2} \left( S_\alpha + S_\beta \right) V D_L C_\ell \sin \left( \frac{n \cdot \pi \cdot S_\alpha}{S_\alpha + S_\beta} \right) 
\]

(6)

where \( C_\alpha^\ell \) and \( C_\beta^\ell \) are the differences between \( C_{\text{eut}} \) and the maximum solute solubility in the solid \( \alpha \)- and \( \beta \)-phase, respectively, and \( C_\ell = C_\alpha^\ell + C_\beta^\ell \). Also \( S_a \) and \( S_\beta \) are half the width of the \( \alpha \)- and \( \beta \)-lamellae, respectively. With the composition thus defined and by accounting for the effect of s/l interface curvature, the average interface undercooling for the \( \alpha \)- and \( \beta \)-phase can be calculated as [1]:

\[
\Delta T_\alpha = m_\alpha \left[ \Delta C_\ell + B_o + \frac{2V}{D_L} C_\ell \left( \frac{S_\alpha + S_\beta}{S_\alpha} \right)^2 P \right] + \frac{\Gamma_\alpha \sin(\theta_\alpha)}{S_\alpha} 
\]

(7a)

\[
\Delta T_\beta = m_\beta \left[ -\Delta C_\ell - B_o + \frac{2V}{D_L} C_\ell \left( \frac{S_\alpha + S_\beta}{S_\beta} \right)^2 P \right] + \frac{\Gamma_\beta \sin(\theta_\beta)}{S_\beta} 
\]

(7b)

where \( m_\alpha \) and \( m_\beta \) are the liquidus slopes, \( \Gamma_\alpha \) and \( \Gamma_\beta \) are the Gibbs-Thomson coefficients of the \( \alpha \)- and \( \beta \)-phase, respectively, \( \theta_\alpha \) and \( \theta_\beta \) are the contact angles at the triple point junction (see Fig.1b), and \( P \) has the expression

\[
P = \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} \sin^2 \left( \frac{n\pi S_\alpha}{S_\alpha + S_\beta} \right).
\]
An isothermal s/l interface would require \( \Delta T_\alpha = \Delta T_\beta \) but this equality is not necessarily obeyed by Eq.(7a) and Eq.(7b) because no adjustment mechanism has been considered in the JH model. It was in fact pointed out by JH that treating the ratio \( S_\beta / S_\alpha \), as a constant (i.e., constant phase fractions) was a drawback of the model and that a minor change of the phase fractions would have a considerable effect on the magnitude of \( B_\alpha \) while negligible for the other terms involving \( S_\alpha \) and \( S_\beta \). Therefore, it was considered that the isothermal condition can still be imposed by eliminating \( B_\alpha \) between Eq.(7a) and Eq.(7b). The resulting undercooling was of the form:

\[
\Delta T = K_1 \cdot \lambda \cdot V + K_2 / \lambda
\]

where \( K_1 \) and \( K_2 \) are material constants of which expressions are defined in ref.[1]. Eq.(8) contains three variables (i.e., \( \Delta T \), \( V \), and \( \lambda \)) and therefore requires and additional condition for the direct correlation of pairs of variables. Consequently, JH made use of the extremum condition according to which the system selects the particular spacing, \( \lambda_{ex} \), which minimizes the undercooling and maximizes the growth velocity. The result of this assumption is the following group of equations:

\[
\begin{align*}
\lambda_{ex}^2 V &= K_2 / K_1 \\
\lambda_{ex} \Delta T &= 2 K_2 \\
\Delta T^2 / V &= 4 K_1 K_2
\end{align*}
\]

(9)

While the drawback of constant phase fraction treatment could be easily circumvented in the case of binary eutectic alloys through the elimination of the sensitive term \( B_\alpha \), the complexity increases dramatically when a third element is added to the system [3]. Donaghey and Tiller (DG) [4] relaxed the simplifying assumption of JH model related to constant phase fraction and mass balance at the s/l interface but the inconvenience of their treatment is that the Fourier coefficients must be determined numerically from an infinite linear system of equations. The actual phase fraction can then be calculated from the extremum condition. The possibility of adding a third element to the system is also illustrated in [4]. A more recent investigation on DG solution was published by Ludwig and Leibbrandt [5] showing that the extremum condition leads to relationships similar to those of JH (Eq.(9)).

The last relationship in Eq.(9) is mostly used in solidification modeling and by commercial casting simulation software packages in the form:

\[
V = \mu \cdot \Delta T^2
\]

with \( \mu = (4 \cdot K_1 \cdot K_2)^{-1} \) being defined as the eutectic growth constant (or coefficient) of the alloy. While this relationship along with the others in Eq.(9) closely match the experimental measurements for regular eutectics, such as Al-Cu or Pb-Sn for instance, they do not readily apply to alloys of commercial importance such as Fe-C or Al-Si (i.e., irregular eutectics) which exhibit both lamellar spacing and undercooling much higher than the theoretical predictions [6,7,8,9,10]. For this latter type of eutectics the average spacing, \( \bar{\lambda} \), follows a relationship of the type:

\[
\bar{\lambda} = \phi \cdot \lambda_{ex}
\]

(11)

and the equivalent of Eq.(9) is:

\[
\bar{\lambda}^2 V = \phi^2 K_2 / K_1 \\
\bar{\lambda} \Delta T = (\phi^2 + 1) K_2 \\
\Delta T^2 / V = (\phi + 1 / \phi)^2 K_1 K_2
\]

(12)

According to the treatment of Magnin and Trivedi (MT) [2] the coefficient \( \phi \) takes the values of 5.4, 1.8, and 3.2 for binary Fe-C_graphite, Fe-Fe,C, and Al-Si system, respectively. A more recent treatment of Catalina and Stefanescu (CS) [11] showed that for all the above binary systems the value of \( \phi \) can be related to the volume fraction of the faceted phase \( f_\beta \) in the microstructure:

\[
\phi = f_\beta \frac{1}{2} = (1 - f_\alpha) \frac{1}{2}
\]

(13)

In both MT and CS treatments the value of \( \phi \) was determined by comparing the theoretical predictions against the experimental measurements. The advantage of Eq.(13), however, is that it relates the eutectic spacing to a microstructure feature and the validity of this relationship in a multicomponent system will be investigated in this paper.

In this paper we propose a theoretical model to describe the eutectic growth in multicomponent alloys. While the mathematical derivation follows the general line of JH treatment, several
modifications are made in order to account for the density difference between the phases, precipitation of faceted phases with fixed chemical composition, as well as the change of the volume fractions of the eutectic phases with respect to chemistry and solidification conditions. It will be shown that, according to the proposed model, the phase fraction adjustment affects the composition of the boundary layer ahead the solid/liquid interface and is also the main factor responsible for maintaining an isothermal interface. The presentation of theoretical results will also focus on the influence of alloying elements on the spacing selection and growth velocity of eutectic microstructures. Available experimental measurements will also be presented for comparison. Particular attention will be given to the Fe-C_{graphite} based alloys, i.e., commercial gray cast iron.

2. Model description

2.1. The concentration field

We consider directional solidification of a multicomponent alloy that solidifies with a eutectic microstructure of which one of the phases (α) is non-faceted while the other (β) is a faceted phase of fixed chemical composition. The problem set-up is identical to that of JH (Fig.1) except the number of alloying elements in the system. The diffusion equation therefore is:

\[
\frac{\partial^2 C_j}{\partial x^2} + \frac{\partial^2 C_j}{\partial z^2} + \frac{V}{D_{L,j}} \frac{\partial C_j}{\partial z} = 0
\]

where \(C_j\) is the concentration and \(D_{L,j}\) is the liquid diffusion coefficient of the alloying element \(j\). The same boundary conditions as in Eq.(2) are considered for each element and therefore, for solidification at low Peclet number, the general solution is:

\[
C_j(x, z) = C_{ij}^0 + B_{o,j} \cdot \exp \left( -\frac{V \cdot z}{D_{L,j}} \right) + \sum_{n=1}^{\infty} B_{n,j} \cdot \cos \left( \frac{2n\pi}{\lambda} x \right) \exp \left( -\frac{2n\pi}{\lambda} \cdot z \right)
\]

where \(C_{ij}^0\) is the initial concentration of element \(j\) (i.e., far field) and \(\lambda\) is the lamellar spacing. Because the solid β-phase is considered a faceted phase of fixed chemical composition, the mass balance equation at the s/l interface that also accounts for the density differences can be formulated as:

\[
- \frac{D_{L,j}}{V} \frac{\partial C_j}{\partial z} \bigg|_{z=0} = \begin{cases} 
C_{1,j}(x) - \eta_{\alpha} \cdot k_{\alpha,j} \cdot C_{1,j}(x) & \text{for } \alpha - \text{phase } (0 \leq x \leq f_{\alpha} \lambda/2) \\
C_{1,j}(x) - \eta_{\beta} \cdot C_{\beta,j} & \text{for } \beta - \text{phase } (f_{\alpha} \lambda/2 \leq x \leq \lambda/2) 
\end{cases}
\]

where \(C_{1,j}(x)\) is the liquid concentration of element \(j\) at the s/l interface (i.e., at \(z = 0\)), \(k_{\alpha,j}\) is the partition coefficient of \(j\) between the liquid and the \(\alpha\)-phase, \(C_{\beta,j}\) is the concentration of \(j\) in the \(\beta\)-phase, \(f_{\alpha}\) is the volume fraction of the \(\alpha\)-phase in the solid microstructure, \(\eta_{\alpha}\) and \(\eta_{\beta}\) are quantities accounting for the density differences between the phases [11]:

\[
\eta_i = \rho_i / \rho_{iq}, \quad i = \alpha, \beta
\]

where \(\rho_{iq}\) is the density of the liquid phase and \(\rho_i\) is the density of the solid phase \(i\) (\(i = \alpha, \beta\)).

As already pointed out earlier, Fourier coefficients cannot be immediately calculated based on Eq.(17) because of the discontinuity of the \(C_{1,j}(x)\) multipliers at \(x = f_{\alpha} \lambda/2\). Therefore we consider a simplified formulation of Eq.(16) in the form:

\[
- \frac{D_{L,j}}{V} \frac{\partial C_j}{\partial z} \bigg|_{z=0} = \begin{cases} 
C_{1,j}(x) - \eta_{\alpha} \cdot k_{\alpha,j} \cdot \overline{C}_{\alpha,j} & \text{for } \alpha - \text{phase } (0 \leq x \leq f_{\alpha} \lambda/2) \\
C_{1,j}(x) - \eta_{\beta} \cdot C_{\beta,j} & \text{for } \beta - \text{phase } (f_{\alpha} \lambda/2 \leq x \leq \lambda/2) 
\end{cases}
\]

where \(\overline{C}_{\alpha,j}\) is the average liquid concentration of element \(j\) at the s/l interface. Note that, as opposed to JH treatment, the approximation made in Eq.(18) is only for the concentration in the solid \(\alpha\)-phase (i.e., \(k_{\alpha,j} \cdot \overline{C}_{\alpha,j}\)). At this time \(\overline{C}_{\alpha,j}\) is not known but it can be calculated by making use of Eq.(15) and Eq.(16). From Eq.(15) one obtains:
\[
\overline{C}_{\alpha,j} = C_j^\infty + B_{\alpha,j} + S_n / f_{\alpha} \\
\overline{C}_{\beta,j} = C_j^\infty + B_{\alpha,j} - S_n / (1 - f_{\alpha})
\]

where \( \overline{C}_{\beta,j} \) is the interface concentration of \( j \) in front of the \( \beta \)-phase and the quantity \( S_n \) is given by:

\[
S_n = \sum_{n=1}^{\infty} B_n \sin (n \pi \sigma_{\alpha}) / n \pi
\]

Also, the integration of Eq.(16) on the range \( x = [0, \lambda / 2] \) gives:

\[
\eta_{\alpha} \cdot k_{\alpha,j} \cdot (f_{\alpha} \cdot B_{\alpha,j} + S_n) = (1 - f_{\alpha} \cdot \eta_{\alpha} \cdot k_{\alpha,j}) \cdot C_j^\infty - (1 - f_{\alpha}) \cdot \eta_{\beta} \cdot C_{\beta,j}^\infty
\]

and therefore the concentrations \( \overline{C}_{\alpha,j} \) and \( \overline{C}_{\beta,j} \) can be calculated by making use of Eq.(19), Eq.(20), and Eq.(22). The result is:

\[
\overline{C}_{\alpha,j} = C_j^\infty - (1 - f_{\alpha}) \cdot \eta_{\beta} \cdot C_{\beta,j}^\infty / (1 - f_{\alpha} \cdot \eta_{\alpha} \cdot k_{\alpha,j})
\]

\[
\overline{C}_{\beta,j} = \left( \eta_{\alpha} \cdot k_{\alpha,j} - 1 \right) \cdot C_j^\infty + (1 - f_{\alpha}) \cdot \eta_{\beta} \cdot C_{\beta,j}^\infty + \eta_{\alpha} \cdot k_{\alpha,j} \cdot B_{\alpha,j} / (1 - f_{\alpha}) \cdot \eta_{\alpha} \cdot k_{\alpha,j}
\]

With \( \overline{C}_{\alpha,j} \), thus given by Eq.(23) the Fourier coefficients \( B_{n,j} \) can now be calculated from Eq.(18) as:

\[
B_{n,j} = \frac{\lambda \cdot V}{D_{L,j}} \left( \eta_{\beta} \cdot C_{\beta,j}^\infty - C_j^\infty \right) \cdot \frac{1}{f_{\alpha}} \cdot \frac{\sin (n \pi \sigma_{\alpha})}{(n \pi)^2}
\]

Also, by replacing Eq.(25) in Eq.(21) and then in Eq.(19) the coefficient \( B_{n,j} \) can be obtained as:

\[
B_{n,j} = \frac{(1 - \eta_{\alpha} \cdot k_{\alpha,j} \cdot f_{\alpha}) \cdot C_j^\infty - (1 - f_{\alpha}) \cdot \eta_{\beta} \cdot C_{\beta,j}^\infty - P_{\alpha} \cdot A_j \cdot P_{\alpha}}{f_{\alpha}}
\]

where \( P_{\alpha} \), \( A_j \), and \( P_{\alpha} \) are defined as:

\[
P_{\alpha} = \frac{\lambda \cdot V}{D_{L,j}}
\]

\[
A_j = \eta_{\beta} \cdot C_{\beta,j}^\infty - C_j^\infty
\]

\[
P_{\alpha} = \sum_{n=1}^{\infty} \frac{\sin^2 (n \pi \cdot f_{\alpha})}{(n \pi)^3}
\]

With the Fourier coefficients \( B_{n,j} \) and \( B_{n,j} \) given in Eq.(25) and Eq.(26) the concentration field given by Eq.(15) is now completely defined. It must be pointed out that the coefficients \( B_{n,j} \) and \( B_{n,j} \) determined in the present treatment, especially \( B_{n,j} \), are different from those obtained by JH. Eq.(26) shows that \( B_{n,j} \) also depends on \( P_{\alpha} \) and the initial concentration of the element \( j \). The dependency on the phase fraction change is discussed in the next section.

### 2.2. Calculation of the growth undercooling and phase fraction change during solidification

The average undercooling \( \Delta T_{\alpha} \) and \( \Delta T_{\beta} \) in front of the \( \alpha \)-phase and \( \beta \)-phase, respectively, can be expressed as:

\[
\Delta T_{\alpha} = m_{\alpha,1} \cdot (C_{\text{eut}} - \overline{C}_{\alpha,1}) + \sum_{j=2}^{N_{\text{elem}}} m_{\alpha,j} \cdot (C_j^\infty - \overline{C}_{\alpha,j}) + \frac{K_{\alpha}}{\lambda}
\]

\[
\Delta T_{\beta} = m_{\beta,1} \cdot (C_{\text{eut}} - \overline{C}_{\beta,1}) + \sum_{j=2}^{N_{\text{elem}}} m_{\beta,j} \cdot (C_j^\infty - \overline{C}_{\beta,j}) + \frac{K_{\beta}}{\lambda}
\]
where $N_{\text{elems}}$ is the number of the alloying elements in the composition of the alloy. It was considered in Eq.(30) and Eq.(31) that $j=1$ represents the main element in the composition, such as Carbon in Fe-C based alloys for instance. $C_{\text{eut}}$ in the above equations is therefore the concentration of the main element in the multicomponent alloy at the eutectic composition. Also, the quantities $K_{r}^{\alpha}$ and $K_{r}^{\beta}$ represent, for a certain spacing $\lambda$, the contribution of the interface curvature to the total undercooling and are defined as [1]:

$$K_{r}^{i} = \frac{2 \cdot \Gamma_{i}^{\gamma} \cos(\theta)}{f_{i}}; \quad (i = \alpha, \beta; \quad f_{\beta} = 1 - f_{\alpha})$$

(32)

where $\Gamma_{i}^{\gamma}$ is the Gibbs-Thomson coefficient of the phase $i$, and $\theta$ is the contact angle of the phase $i$ at the three-phase junction (see Fig. 1b).

As defined in Eq.(30) and Eq.(31) the undercooling $\Delta T_{\alpha}$ and $\Delta T_{\beta}$ are not necessarily equal (i.e., isothermal s/l interface) for any growth velocity and eutectic spacing. As pointed out by JH [1], it is the phase fraction change (PFC) that insures an isothermal interface and therefore an explicit dependency on PFC must be considered in the model. In the present treatment this dependency is developed based on the first order expansion in the vicinity of the eutectic point. Therefore, if $f_{\alpha}^{o}$ is the volume fraction of the $\alpha$-phase at the equilibrium eutectic point and $\Delta f_{\alpha} = -\Delta f_{\beta}$ is the PFC at the imposed solidification conditions, the corresponding changes of any quantity $q=q(f_{\alpha})$ can be expressed as:

$$q(f_{\alpha}) \equiv q^{o} + q'\Delta f_{\alpha}$$

(33)

where $q^{o} = q(f_{\alpha}^{o})$ and $q' = q'(f_{\alpha}^{o})$ is the first derivative of $q$ with respect to $f_{\alpha}$ at $f_{\alpha}^{o}$. If the expansion defined in Eq.(33) is applied to all the appropriate quantities in Eq.(30) and Eq.(31) then $\Delta f_{\alpha}$ can be determined by imposing the condition of isothermal s/l interface (i.e., $\Delta T_{\alpha} = \Delta T_{\beta}$). The result is:

$$\Delta f_{\alpha} = \frac{m_{\alpha,1}(C_{\text{eut}} - C_{\alpha,1}^{\gamma}) - m_{\beta,1}(C_{\text{eut}} - C_{\beta,1}^{\gamma}) + \sum_{j=2}^{N_{\text{elems}}} m_{\alpha,j}(C_{\alpha,j}^{\gamma} - C_{\alpha,1}^{\gamma}) - m_{\beta,j}(C_{\beta,j}^{\gamma} - C_{\beta,1}^{\gamma}) + K_{r}^{\alpha o} - K_{r}^{\beta o}}{\lambda}$$

(34)

Eq.(34), along with the previous equations defining the composition field, shows that the $\Delta f_{\alpha}$ that ensures an isothermal s/l interface depends on the material parameters (e.g., liquidus slopes, partition, diffusion and Gibbs-Thomson coefficients), initial composition of the liquid alloy, interphase spacing $\lambda$, as well as the imposed growth velocity $V$. Eq.(34) also shows that $\Delta f_{\alpha}$ can be calculated for any pair of $\lambda$ and $V$ and is not restricted to the extremum condition as in the DG model [4].

2.3. Application to multicomponent Fe-C_{\text{graphite}} based alloys

For this alloy system carbon is the main alloying element (i.e., $j=1$). Also, the solid solution $\alpha$ is the austenite while the faceted $\beta$-phase is graphite. The partition coefficient of carbon between the liquid and austenite is defined as $k_{\alpha,1} = C_{E} / C_{\text{eut}}$, where $C_{E}$ is the maximum solubility of carbon at the eutectic temperature ($T_{\text{eut}}$) and $C_{\text{eut}}$ is the concentration of carbon in the multicomponent alloys at the eutectic composition. Both $C_{E}$ and $C_{\text{eut}}$ as well as $T_{\text{eut}}$ depend on the concentration of the other elements in the composition of the alloy and can be calculated by using the relationships given in ref.[12]. Because at the eutectic temperature the quantity $k_{\alpha,1} \cdot C_{\alpha,1}^{\gamma}$ appearing in Eq.(18) is in fact $C_{E}$, it follows that $f_{\alpha}^{c}$ can be calculated from Eq.(23) for $C_{\alpha,1}^{\gamma} = C_{\text{eut}}$. Furthermore, it is considered that the $\beta$-phase is made entirely out of carbon (i.e., $C_{\beta,1}^{\gamma} = 100\%$) and therefore $C_{\beta,1}^{\gamma}(j=1) = 0$. Consequently, it can be shown that the change $\Delta f_{\alpha}$ (see Eq.(34)) for this system can be calculated as:
\[
\Delta f_\alpha = \frac{\sum_{j=2}^{\text{Nelems}} m_{\beta,j} A_j Pe_j - \sum_{j=1}^{\text{Nelems}} (m_{\alpha,j} - m_{\beta,j}) \left(1 - \frac{1}{\eta_\alpha \cdot k_{\alpha,j} \cdot f_\alpha^o} \right) \cdot \left(\frac{K_r^{\alpha,o} - K_r^{\beta,o}}{\lambda} \right) \cdot c_j}{f_\alpha^o \cdot (1 - f_\alpha^o) - \sum_{j=1}^{\text{Nelems}} (m_{\alpha,j} - m_{\beta,j}) H_{o,j} - \sum_{j=1}^{\text{Nelems}} m_{\beta,j} A_j Pe_j + \frac{K_r^{\alpha,o} - K_r^{\beta,o}}{\lambda}} \]  

(35)

with  \( H_{o,j} = \eta_\beta C_{\beta,j}^s - C_{\beta,j}^\infty \)  and  \( U_o = \frac{f_\alpha^o \left(1 - f_\alpha^o\right) \sum_{n=1}^{\infty} \sin \left(2\pi n \frac{x}{\lambda}\right) + \left(3 f_\alpha^o - 2\right) f_\alpha^o}{f_\alpha^o \left(1 - f_\alpha^o\right)} \).

Also, the undercooling at the s/l interface is given by:

\[
\Delta T = \sum_{j=2}^{\text{Nelems}} m_{\alpha,j} \left(1 - \frac{1}{\eta_\alpha \cdot k_{\alpha,j} \cdot f_\alpha^o} \right) \cdot C_j^\infty \cdot \frac{K_r^{\alpha,o}}{\lambda} \cdot \left(1 - \sum_{j=1}^{\text{Nelems}} m_{\beta,j} \cdot H_{o,j}\right) \cdot \Delta f_\alpha \]  

(36)

Eq.(36) shows that the undercooling depends on  \( \Delta f_\alpha \)  and the initial composition of the multicomponent alloy (  \( C_j^\infty \) ) in addition to all the other material parameters considered in the previous models.

A graphical representation of  \( \Delta T = f(\lambda) \)  and  \( \Delta f_\alpha = f(\lambda) \)  shows that both  \( \Delta T \)  and  \( \Delta f_\alpha \)  go through their minimum values for about the same value of  \( \lambda = \lambda_{ex} \)  (see Fig. 2). Therefore, in order to calculate  \( \lambda_{ex} \)  one may find convenient to apply the extremum condition to  \( \Delta f_\alpha \)  instead to  \( \Delta T \). By using the pertinent data for the Fe-C based systems and considering only the terms with relevant order of magnitude in Eq.(35) it can be shown that  \( \lambda_{ex} \)  can be approximated as:

\[
\lambda_{ex} = \frac{\sum_{j=2}^{\text{Nelems}} m_{\alpha,j} \cdot \frac{A_j}{D_{L,j}}}{P_\alpha \cdot \left(1 - \frac{1}{\sum_{j=2}^{\text{Nelems}} m_{\beta,j} \cdot A_j / D_{L,j}} \right)^2} \cdot \lambda \]  

(37)

Also, by accounting for Eq.(37), Eq.(11) and Eq.(13) the approximate average spacing would be:

\[
\bar{\lambda} = \frac{f_\alpha^o \cdot (K_r^{\beta,o} - K_r^{\alpha,o}) \cdot \left(1 - \frac{1}{\sum_{j=2}^{\text{Nelems}} m_{\beta,j} \cdot A_j / D_{L,j}} \right)}{P_\alpha \cdot \left(1 - \frac{1}{\sum_{j=2}^{\text{Nelems}} m_{\beta,j} \cdot A_j / D_{L,j}} \right)^2} \cdot \sqrt{V} \]  

(2)

thus showing that the well-known relationship  \( \bar{\lambda}^2 \cdot V = \text{const.} \) of the eutectic growth (see Eq.(9) and Eq.(12)) is also recovered by the proposed model. shows that the relationship  \( V = \mu \cdot \Delta T^2 \) is also recovered by the model.

3. Theoretical predictions vs. experimental measurements for Fe-C based systems

![Fig. 2](image-url)  Calculated variation with spacing of undercooling and change of austenite fraction for Fe-C-2.0wt%Si eutectic alloy solidifying at a velocity of 0.5 µm/s.

![Fig. 3](image-url)  Growth velocity as function of  \( \Delta T^2 \) for Fe-C-2.0wt%Si eutectic alloy (theoretical measurements).
A comparison between the calculated and measured lamellar spacing for binary Fe-C_{graphite} eutectic alloy is presented in Fig. 4. The figure shows that a very good agreement between model predictions and measurements is achieved up to a velocity of about 6.5-7.0 µm/s which corresponds to the limit of eutectic coupled growth [13] after which the microstructure becomes degenerate [8] and exhibits higher spacing [14]. A good agreement between theoretical calculations and measured undercooling was also reported in ref. [11] for a similar but simpler model which did not account for the phase fraction change. That is not shown here due to the economy of space. Experimental measurements for small additions of other elements to pure Fe-C eutectic are also reported in ref. [14]. The comparison with theoretical results is shown in Fig. 5 for eutectic alloys containing 0.1wt%Si and 0.5 wt%Si. Similar to the binary Fe-C system, the agreement is good for solidification velocity smaller than 10 µm/s.

Experimental measurements on directionally solidified specimens containing levels of Si and Mn comparable to those in commercial cast iron were very recently reported by Rivera et al. [15]. The composition of the specimens is presented in Table 1 while the measured and theoretical spacing is shown in Table 2.

Of the specimens presented in Table 1, #2, #3, and #4 are hypo-eutectic, #1 is slightly hyper-eutectic, and Sorel Pig is highly hyper-eutectic. Table 2 shows that for the hypo-eutectic specimens the theoretical calculations of the spacing are in good agreement with the measurements.

The agreement is also good for specimen #1 for the lower experimental spacing. However, for the Sorel Pig sample which is quite removed from the eutectic composition the theoretical spacing is only half the value of the measured one. This may be an indication that the proposed model is not valid for alloys of composition too far removed from the eutectic point.

An important application of the model would be the calculation of the eutectic growth coefficient, µ, shown in Eq.(10). The calculated influence of Si content on the value of µ is shown in Fig.6. It can be observed from Fig.6 that, according to the model, µ decreases linearly with the increase of Si content in the eutectic alloy.

**Fig. 4** Average lamellar spacing in Fe-C_{graphite} system (experimental data from refs.[8], [14]).

**Fig. 5** Average spacing and growth temperature for eutectic Fe-C-0.1%Si and Fe-C-0.5%Si (exp. data from ref.[14]).
4. Conclusions
A model for eutectic growth of multicomponent alloys has been proposed. As presented, the model applies to alloy systems of which one of the eutectic phases exhibits a fixed chemical composition, such as Fe-C or Al-Si based type of alloys, but a more general formulation is also possible. The model introduces the phase fraction change as the main mechanism responsible for maintaining an isothermal s/l interface during solidification. A comparison between the theoretical results and available experimental data for the Fe-C based system showed good agreement for alloys of chemistry not too far from that of the eutectic point. An earlier proposed criterion for spacing selection of binary irregular eutectics was also successfully tested for the case of multicomponent alloys. Calculations of the eutectic growth coefficient $\mu$ for the ternary Fe-C-Si alloy showed that Si content decreases the value of $\mu$. The influence on $\mu$ of all the other elements can be analysed based on the proposed model. The model is simple and can accommodate any number of alloying elements as long as they do not form additional precipitates.

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Table 1. Chemistry of the specimens used in ref.[15].

| Specimen ID | Sorel Pig | #1 | #2 | #3 | #4 |
|-------------|-----------|----|----|----|----|
| C           | 4.52      | 3.83 | 3.42 | 3.37 | 3.45 |
| Si          | 0.21      | 1.67 | 2.07 | 2.01 | 2.08 |
| Mn          | 0.025     | 0.075 | 0.095 | 0.593 | 0.669 |
| P           | 0.008     | 0.017 | 0.015 | 0.012 | 0.016 |
| S           | 0.009     | 0.007 | 0.01 | 0.009 | 0.011 |
| Cr          | 0.045     | 0.049 | 0.05 | 0.545 | 0.054 |
| Mo          | 0.057     | 0.06 | 0.06 | 0.06 | 0.06 |
| Ni          | 0.149     | 0.141 | 0.135 | 0.136 | 0.138 |
| Cu          | 0.065     | 0.08 | 0.086 | 0.095 | 0.09 |
| C_equivalent| 4.59      | 4.39 | 4.11 | 4.23 | 4.13 |

Table 2. Experimental and theoretical spacing for the specimens presented in Table 1.

| Specimen ID | Exp. average spacing ($\mu$m) | Exp. std. dev. ($\mu$m) | Calc. average spacing ($\mu$m) |
|-------------|-------------------------------|------------------------|-------------------------------|
| Sorel Pig   | 0.5                           | 75.6                   | 15.4                          |
| #1          | 1                             | 32.3                   | 6.5                           |
| #2          | 0.5                           | 47.2                   | 6.7                           |
| #3          | 0.5                           | 45                     | 7.7                           |
| #4          | 0.5                           | 43.8                   | 44.4                          |

*) a bimodal spacing was reported in [15] due to the presence of austenite dendrites

Fig.6 Calculated influence of Si content on the magnitude of the eutectic growth coefficient, $\mu$
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