Novel Superposed Kinklike and Pulselike Solutions for Several Nonlocal Nonlinear Equations

Avinash Khare
Physics Department, Savitribai Phule Pune University
Pune 411007, India

Avadh Saxena
Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

Abstract:
We show that a number of nonlocal nonlinear equations including the Ablowitz-Musslimani and the Yang variant of the nonlocal nonlinear Schrödinger (NLS) equation, nonlocal modified Korteweg de Vries (mKdV) equation as well as the nonlocal Hirota equation admit novel kinklike and pulselike superposed periodic solutions. Further, we show that the nonlocal mKdV equation, in addition also admits the superposed (hyperbolic) kink-antikink solution. Besides, we show that while the nonlocal Ablowitz-Musslimani variant of the NLS admits complex (parity-time reversal or) PT-invariant kink and pulse solutions, neither the local NLS nor the Yang variant of the nonlocal NLS admits such solutions. Finally, except for the Yang variant of the nonlocal NLS, we show that the other three nonlocal equations admit both the kink and pulse solutions in the same model.

1 Introduction

Few years ago Ablowitz and Musslimani proposed a nonlocal variant of the nonlinear Schrödinger equation (NLS) \cite{1,2} and showed that it is an integrable system. The interesting point of this nonlocal variant of NLS is that the corresponding nonlinearity-induced potential is in general complex and (parity-time reversal or) PT-invariant. We note here that in recent years it has been realized that optics can provide an ideal ground for testing some
of the consequences of such theories [3]. It has been realized that the PT-symmetric optics can give rise to an entirely new class of optical structures and devices with altogether new properties [3]. Few years later, Yang introduced another variant of the nonlocal NLS equation [4] and showed that it is also an integrable system. He obtained it as a special reduction of the Manakov system [5]. He has suggested that his nonlocal NLS variant could be useful in case the two components of the Manakov system are related by a parity symmetry. In contrast to the Ablowitz-Musslimani nonlocal NLS, the nonlinearity-induced potential in the Yang’s nonlocal NLS is real and symmetric in $x$. In recent years the nonlocal variants of the mKdV [7] and the Hirota equation [8, 9] have also been proposed.

It is clearly of interest to obtain the various exact solutions admitted by these nonlocal equations and contrast them with the exact solutions of the corresponding local equations. Few years back we [10] considered several nonlocal equations including the Ablowitz-Musslimani variant of the nonlocal NLS equation [1] and obtained their exact periodic and hyperbolic solutions. One of the main purposes of this paper is to show that both the nonlocal variants of NLS, the nonlocal mKdV as well as the nonlocal Hirota equation admit periodic superposed solutions in terms of $\text{sn}(x+\Delta, m) \pm \text{sn}(x-\Delta, m)$ as well as $\text{dn}(x+\Delta, m) \pm \text{dn}(x-\Delta, m)$. Here $\text{sn}(x, m)$ and $\text{dn}(x, m)$ are the Jacobi elliptic functions [11] with $m$ being the modulus $0 \leq m \leq 1$. By superposed we mean that a solution can be expressed as a linear combination of two kink solutions or two pulse solutions or the corresponding periodic solutions.

Besides these, we also discuss several other solutions admitted by these equations. Further, we also discuss those solutions of the Ablowitz-Musslimani nonlocal variant of the NLS which we had missed earlier. We show that while the Ablowitz-Musslimani variant of the nonlocal NLS admits complex PT-invariant kink and pulse solutions, neither the Yang variant of the nonlocal NLS nor the local NLS admits such solutions. Further, while the Ablowitz-Musslimani variant of the nonlocal NLS admits both the kink and pulse solutions in the same model, neither the Yang variant of the nonlocal NLS, nor the local NLS admits these solutions in the same model. We also show that while the local mKdV equation is known to admit the complex PT-invariant kink and pulse solutions, the corresponding nonlocal variant does not admit such solutions. Finally, we show that neither the Hirota equation [12] nor the nonlocal Hirota equation (discussed in Sec. 5) admits complex PT-invariant solutions.

The plan of the paper is the following. In Sec. 2 we show that the Yang variant of the nonlocal NLS equation admits several solutions including
novel superposed periodic solutions. In Sec. 3 we consider the Ablowitz-Musslimani variant of the nonlocal NLS equation and show that apart from the several solutions already known [10], this nonlocal equation also admits a few more solutions including the complex PT-invariant solutions as well as periodic superposed solutions. In Sec. 4 we consider the nonlocal mKdV equation and obtain its several new solutions including the periodic as well as the hyperbolic superposed solutions. In Sec. 5 we consider a nonlocal variant of the Hirota equation and obtain its several solutions including the periodic superposed solutions. Finally, in Sec. 6 we summarize the results obtained and point out some of the open problems. In Appendix A we present many of the exact solutions of Yang’s nonlocal NLS equation which are also the solution of the local NLS equation. In Appendix B we similarly present those solutions of the Ablowitz-Musslimani variant of the nonlocal NLS which we had missed in our earlier paper [10] and which are also the solutions of the local NLS equation. Finally, in Appendix C we present those solutions of the nonlocal attractive (repulsive) mKdV which are also the solutions of the corresponding local attractive (repulsive) mKdV equation.

2 Periodic and Hyperbolic Solutions of Yang’s version of Nonlocal NLS

In 2013, Ablowitz and Musslimani introduced a novel nonlocal version of the NLS [1]

\[ i\psi_t(x,t) + \psi_{xx}(x,t) + 2g\psi^2(x,t)\psi^*(-x,t) = 0, \quad g = \pm 1, \]  

where the power defined by

\[ P = \int_{-\infty}^{\infty} dx |\psi(x,t)|^2, \]  

is not conserved but the pseudo-power \( Q \) defined by

\[ Q = \int_{-\infty}^{\infty} dx \psi(x,t)\psi(-x,t), \]  

is conserved. Here \( g = +1 \) (\(-1\)) corresponds to the attractive (repulsive) case. In addition, The Hamiltonian \( H \) defined by

\[ H = \int_{-\infty}^{\infty} dx \left[ \psi_x(x,t)\psi_x^*(-x,t) - \frac{g}{2}\psi^2(x,t)\psi^2(-x,t) \right], \]
is also conserved. Ablowitz-Musslimani showed that like the usual NLS, 
this nonlocal NLS is also integrable. Notice that the nonlinearity induced 
potential \( \psi(x,t)\psi^*(-x,t) \) is in general complex but PT-invariant. 

Few years later, Yang \[5\] introduced another version of the nonlocal NLS 
given by 

\[
\begin{align*}
 i\psi_t(x,t) + \psi_{xx}(x,t) + g[\psi^2(x,t) + |\psi(-x,t)|^2]\psi(x,t) = 0, \quad g = \pm 1. 
\end{align*}
\]

(5)

In this model, not only the power given by Eq. (2) is conserved but also the 
two different pseudo-powers given by 

\[
\begin{align*}
P_1 &= \int_{-\infty}^{\infty} dx \psi^*(-x,t)\psi(x,t), \quad P_2 = \int_{-\infty}^{\infty} dx \psi^*(x,t)\psi(-x,t),
\end{align*}
\]

(6)

are conserved. Besides the Hamiltonian given by 

\[
\begin{align*}
H &= \int_{-\infty}^{\infty} dx [\psi_x(x,t)\psi_x^*(-x,t) - \frac{g}{4}[|\psi(x,t)|^2 + |\psi(-x,t)|^2]^2,
\end{align*}
\]

(7)

is also conserved. Further, Yang \[5\] showed that his nonlocal version of NLS 
is also integrable. Notice that the nonlinearity-induced potential \(|\psi(x,t)|^2 + |\psi(-x,t)|^2\) in Yang’s nonlocal NLS is real and symmetric in \(x\). 

We now show that like the Ablowitz-Musslimani nonlocal NLS \[10\], the 
Yang’s nonlocal NLS Eq. (5) also admits a large number of periodic and 
hyperbolic solutions. Besides, it also admits novel superposed periodic solu-

\section*{Solution I}

It is easy to show that 

\[
\psi(x,t) = \text{Adn}(\beta x, m)e^{i\omega(t+t_0)},
\]

(8)

is an exact solution to Eq. (5) provided 

\[
g = 1, \quad A^2 = \beta^2, \quad \omega = (2 - m)\beta^2.
\]

(9)

Note that for any solution of the above form, translation shift in time, 
i.e. \(t_0 \neq 0\) is allowed. For simplicity, from now onwards we put \(t_0 = 0\) 
while discussing rest of the solutions, but this point should be kept in mind. 
However, unlike the local case, for the nonlocal case, a translation shift in \(x\) 
is not allowed. In particular, while \(\text{Adn}[\beta(x + x_0), m]\) for arbitrary \(x_0\) is a 
solution of the local NLS equation in case constraints (9) are satisfied, it is 
the solution of Eq. (5) only in case \(x_0 = 0\).
It turns out that most of the solutions (including the solution as given by Eq. (8)) of the local NLS and the Yang variant of the nonlocal NLS are valid for the same values of the parameters and we have therefore decided not to present them here but for completeness present them in Appendix A.

As mentioned above, unlike the local NLSE, the solutions of the nonlocal NLSE are not invariant with respect to shifts in $x$. For example, while $A \text{dn} [\beta (x + x_0), m] e^{i \omega t}$ is an exact solution of the local NLSE no matter what $x_0$ is, it is not an exact solution of the nonlocal Eq. (5). However, for special values of $x_0$, $\text{sn}(x, m)$, $\text{cn}(x, m)$ and $\text{dn}(x, m)$ are still the solutions of the nonlocal Eq. (5). In particular, we now show that when $x_0 = K(m)$, where $K(m)$ is the complete elliptic integral of the first kind, there are exact solutions of the nonlocal Eq. (5) in both the focusing ($g > 0$) and the defocusing ($g < 0$) cases. This is because

\[
\begin{align*}
\text{dn}[x + K(m), m] &= \frac{\sqrt{1 - m}}{\text{dn}(x, m)}, \quad \text{sn}[x + K(m), m] = \frac{\text{cn}(x, m)}{\text{dn}(x, m)}, \\
\text{cn}[x + K(m), m] &= -\frac{\sqrt{1 - m} \text{sn}(x, m)}{\text{dn}(x, m)}.
\end{align*}
\]  

(10)

**Solution II**

It is easy to show that

\[
\psi(x, t) = \frac{A}{\text{dn}(\beta x, m)} e^{i \omega t},
\]  

(11)

is an exact solution to Eq. (5) provided

\[
g = 1, \quad A^2 = (1 - m) \beta^2, \quad \omega = (2 - m) \beta^2.
\]  

(12)

**Solution III**

It is easy to show that

\[
\psi(x, t) = \frac{A \sqrt{m} \text{sn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{i \omega t},
\]  

(13)

is an exact solution to Eq. (5) provided

\[
g = 1, \quad A^2 = (1 - m) \beta^2, \quad \omega = (2m - 1) \beta^2.
\]  

(14)

It is worth pointing out that in contrast to the Yang variant of the NLS Eq. (5) or the local NLS, the Ablowitz-Musslimani variant of the nonlocal NLS Eq. (1) admits the solution (13) only if $g = -1$ [10].
Solution IV
It is easy to show that
\[ \psi(x,t) = A \sqrt{m} \text{cn}(\beta x, m) \text{dn}(\beta x, m) e^{i\omega t}, \] (15)
is an exact solution to Eq. (5) provided \( m \neq 1 \) and
\[ g = -1, \ A^2 = \beta^2, \ \omega = -(1 + m)\beta^2. \] (16)

Solution V
Remarkably, it turns out that not only \( \text{dn}[\beta x, m] \) and \( \text{dn}[\beta x + K(m), m] \) but even their superposition is a solution of the nonlocal NLS Eq. (5). In particular, it is easy to check that
\[ \psi(x,t) = e^{i\omega t} \left[ A \text{dn}(\beta x, m) + B \sqrt{1-m} \text{dn}(\beta x, m) \right], \] (17)
is also an exact solution of the nonlocal Eq. (5) provided
\[ g = 1, \ A^2 = \beta^2, \ B = \pm A, \ \omega = [2 - m \pm 6\sqrt{1-m}]\beta^2, \] (18)
where the \( \pm \) sign in \( B = \pm A \) and in \( \omega \) are correlated.

Recently, we [18] have shown that the local NLS Equation admits three novel periodic solutions which can be re-expressed as the superposition of a periodic kink and an antikink or two periodic pulse solutions. Now we show that the nonlocal NLS Eq. (5) also admits the same three novel periodic solutions which can be re-expressed as superposition of a periodic kink and an antikink or two periodic pulse solutions \( \text{sn}(x, m) \) and \( \text{dn}(x, m) \), respectively. Since most of the algebra is the same as in the local case and is discussed in detail in [18], we avoid giving details here.

Solution VI
It is easy to check that the nonlocal NLS Eq. (5) admits the periodic solution
\[ \psi(x,t) = e^{i\omega t} \left[ \frac{A \text{dn}(\beta x, m) \text{cn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)} \right], \ B > 0, \] (19)
provided \( g = -1 \) and further
\[ 0 < m < 1, \ B = \frac{\sqrt{m}}{1-\sqrt{m}}, \]
\[ \omega = -(1 + m + 6\sqrt{m})\beta^2 < 0, \ A^2 = \frac{4\sqrt{m}\beta^2}{(1-\sqrt{m})^2}. \] (20)
Note that this solution is not valid for $m = 1$, i.e. the nonlocal NLS Eq. (5) does not admit a corresponding hyperbolic solution.

At this stage, we recall the well known addition theorem for $\text{sn}(x, m)$ i.e.

$$
\text{sn}(a + b, m) = \frac{\text{sn}(a, m)\text{cn}(b, m)\text{dn}(b, m) + \text{cn}(a, m)\text{dn}(a, m)\text{sn}(b, m)}{1 - m\text{sn}^2(a, m)\text{sn}^2(b, m)}. \quad (21)
$$

From here it is straightforward to derive the identity

$$
\text{sn}(y + \Delta, m) - \text{sn}(y - \Delta, m) = \frac{2\text{cn}(y, m)\text{dn}(y, m)\frac{\text{sn}(\Delta, m)}{\text{dn}^2(\Delta, m)}}{1 + B\text{cn}^2(y, m)} , \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}. \quad (22)
$$

On comparing Eqs. (19) and (22) and using Eq. (20), one can re-express the periodic solution (19) as superposition of a periodic kink and an antikink solution, i.e.

$$
\psi(x, t) = e^{i\omega t} \sqrt{\frac{m}{2}} \beta [\text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m)]. \quad (23)
$$

Here $\Delta$ is defined by $\text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4}$, where use has been made of the identity [11]

$$
\sqrt{m\text{sn}(y, m)} = \text{sn}(\sqrt{my}, 1/m), \quad (24)
$$

**Solution VII**

It is easy to check that the nonlocal NLS Eq. (5) admits another periodic solution

$$
\psi(x, t) = e^{i\omega t} \frac{A\text{sn}(\beta x, m)\text{cn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \quad (25)
$$

provided

$$
0 < m < 1 , \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}} , \quad g = 1 ,
$$

$$
\omega = (2 - m - 6\sqrt{1 - m})\beta^2 , \quad A^2 = \frac{2(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}}. \quad (26)
$$

On using the well known addition theorem for $\text{dn}(x, m)$ i.e.

$$
\text{dn}(a + b, m) = \frac{\text{dn}(a, m)\text{dn}(b, m) - m\text{sn}(a, m)\text{cn}(a, m)\text{sn}(b, m)\text{cn}(b, m)}{1 - m\text{sn}^2(a, m)\text{sn}^2(b, m)}, \quad (27)
$$
one can derive the identity

\[
\text{dn}(x-\Delta, m) - \text{dn}(x+\Delta, m) = \frac{2m\text{sn}(\Delta, m)\text{cn}(\Delta, m)\text{sn}(x, m)\text{cn}(x, m)}{\text{dn}^2(\Delta, m)[1 + \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}\text{cn}^2(x)]}, \tag{28}
\]

On comparing solutions (25) and (28) and using Eq. (26) we find that the solution given in Eq. (25) can be re-expressed as a superposition of two periodic pulse solutions, i.e.

\[
\psi(x, t) = e^{i\omega t} \beta \sqrt{\frac{1}{2}} \left( \text{dn}[\beta(x) - K(m)/2, m] - \text{dn}[\beta(x) + K(m)/2, m] \right). \tag{29}
\]

**Solution VIII**

The nonlocal NLS Eq. (11) also admits another periodic solution

\[
\psi(x, t) = e^{i\omega t} \frac{A\text{dn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \tag{30}
\]

provided

\[
0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad g = 1, \quad \omega = [2 - m + 6\sqrt{1 - m}]\beta^2, \quad A^2 = \frac{4}{\sqrt{1 - m}}\beta^2. \tag{31}
\]

Note that for this solution \( g > 0, \omega > 0 \) and it is not valid for \( m = 1 \).

Now using the addition theorem for \( \text{dn}(x, m) \) as given by Eq. (27), one can derive another identity, i.e.

\[
\text{dn}(x+\Delta, m) + \text{dn}(x-\Delta, m) = \frac{2\text{dn}(x, m)}{\text{dn}(\Delta, m)(1 + B\text{cn}^2(x, m))}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}. \tag{32}
\]

On comparing Eqs. (30) and (32) and using Eq. (31), the periodic solution (30) can be re-expressed as superposition of two periodic pulse solutions, i.e.

\[
\psi(x, t) = e^{i\omega t} \beta [\text{dn}(\beta x + K(m)/2, m) + \text{dn}(\beta x - K(m)/2, m)], \tag{33}
\]

where \( \Delta = \pm K(m)/2 \)

It is worth noting that for both the superposed periodic pulse solutions VII and VIII, not only the value of \( B \) is the same but also \( g > 0 \) for both the solutions. Further, while the solutions IV and VI are valid if \( g = -1 \), the remaining six solutions are valid provided \( g = 1 \).
3 Superposed Solutions of the Ablowitz-Musslimani version of Nonlocal NLS

Few years ago, we had already obtained \[10\] several exact solutions of the Ablowitz-Musslimani version of the nonlocal NLS as given by Eq. (1). However, it turns out that we had missed a few exact solutions in that paper. We now mention only those missed solutions which are either not the solutions of the local NLS or are not the solutions of the local NLS for the same values of the parameters. However, for the sake of completeness, in Appendix B we have given those solutions of the Ablowitz-Musslimani variant of the nonlocal NLS which are also the solutions of the local NLS as well as the Yang variant of the nonlocal NLS.

We now show that unlike the local NLS or the Yang version of the nonlocal NLS \[5\], the Ablowitz-Musslimani variant of the nonlocal NLS admits complex PT-invariant periodic and hyperbolic superposed solutions.

**Solution I**

It is straightforward to show that the nonlocal Eq. (1) admits the complex PT-invariant periodic solution

\[
\psi(x, t) = e^{i\omega t} \left[ A \text{dn}(\beta x, m) + iB \sqrt{m} \text{sn}(\beta x, m) \right],
\]

provided

\[
B = \pm A, \quad g = 1, \quad A = \frac{\beta}{2}, \quad \omega = -\frac{(2m-1)}{2} \beta^2.
\]

**Solution II**

Another complex PT-invariant periodic solution of the nonlocal Eq. (1) is

\[
\psi(x, t) = e^{i\omega t} \left[ A \sqrt{m} \text{cn}(\beta x, m) + iB \sqrt{m} \text{sn}(\beta x, m) \right],
\]

provided

\[
B = \pm A, \quad g = 1, \quad A = \frac{\beta}{2}, \quad \omega = -\frac{(2-m)}{2} \beta^2.
\]

**Solution III**

In the limit \(m = 1\), both the solutions I and II go over to the complex PT-invariant hyperbolic solution of the nonlocal Eq. (1)

\[
\psi(x, t) = e^{i\omega t} \left[ A \text{sech}(\beta x) + iB \tanh(\beta x) \right],
\]

provided

\[
B = \pm A, \quad g = 1, \quad A = \frac{\beta}{2}, \quad \omega = -\frac{\beta^2}{2}.
\]
Solution IV
Remarkably, the nonlocal Eq. (1) also admits the complex PT-invariant periodic solution
\[
\psi(x,t) = e^{i\omega t} \left[ A\sqrt{m}\text{sn}(\beta x,m) + iB\text{dn}(\beta x,m) \right],
\] (40)
provided the same relations as given in Eq. (35) are satisfied.

Solution V
Another complex PT-invariant periodic solution of the nonlocal Eq. (1) is
\[
\psi(x,t) = e^{i\omega t} \left[ A\sqrt{m}\text{sn}(\beta x,m) + iB\sqrt{m}\text{cn}(\beta x,m) \right],
\] (41)
provided the same relations as in Eq. (37) are satisfied.

Solution VI
In the limit \(m = 1\), both the solutions IV and V go over to the complex PT-invariant hyperbolic solution of the nonlocal Eq. (1)
\[
\psi(x,t) = e^{i\omega t} \left[ A\tanh(\beta x) + iB\text{sech}(\beta x) \right],
\] (42)
provided the same relations as in Eq. (39) are satisfied.

The Ablowitz-Musslimani variant of the nonlocal NLS is rather unusual in the sense that the same model (i.e. with \(g = 1\)) admits not only the kink and pulse solutions but also the complex PT-invariant pulse and kink solutions with PT-eigenvalue +1 and −1, respectively.

We now show that Eq. (1) also satisfies four novel periodic solutions which can be re-expressed either as the superposition of a periodic kink and an antikink or two periodic kink or two periodic pulse solutions, respectively.

Solution VII
It is easy to check that the nonlocal NLS Eq. (1) admits the periodic solution
\[
\psi(x,t) = e^{i\omega t} \left[ A\text{dn}(\beta x,m)\text{cn}(\beta x,m) \right], \quad B > 0,
\] (43)
provided \(g = -1\) and further
\[
0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}},
\]
\[
\omega = -[1 + m + 6\sqrt{m}]\beta^2 < 0, \quad A^2 = \frac{4\sqrt{m}\beta^2}{(1 - \sqrt{m})^2}.
\] (44)
Note that this solution is not valid for \(m = 1\), i.e. the nonlocal NLS Eq. (1) does not admit a corresponding hyperbolic solution.
On comparing Eqs. (43) and the identity (22) and using Eq. (44), one can re-express the periodic solution (43) as superposition of a periodic kink and a periodic antikink solution, i.e.

$$\psi(x,t) = e^{i\omega t} \sqrt{\frac{m}{2}} \beta [\text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m)] .$$  

(45)

Here $\Delta$ is defined by $\text{sn}(\sqrt{m} \Delta, 1/m) = \pm m^{1/4}$, where use has been made of the identity (24).

**Solution VIII**

Remarkably, Eq. (1) also admits another periodic solution

$$\psi(x,t) = e^{i\omega t} \frac{A \text{sn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad B > 0,$$

(46)

provided

$$0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad g = -1, \quad a = [6\sqrt{m} - (1 + m)]/\beta^2, \quad A^2 = 4\sqrt{m}\beta^2.$$  

(47)

Note that this solution too is not valid in the hyperbolic limit of $m = 1$. Now on using the $\text{sn}(x, m)$ addition theorem (21), one can derive another identity

$$\text{sn}(y + \Delta, m) + \text{sn}(y - \Delta, m) = \frac{2\text{sn}(y, m) \text{cn}(\Delta m)}{1 + B \text{cn}^2(y, m)}, \quad B = \frac{m \text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}.$$  

(48)

On comparing Eqs. (46) and (48) and using Eq. (47), the periodic solution VIII given by Eq. (46) can be re-expressed as superposition of two periodic kink solutions

$$\psi(x,t) = ie^{i\omega t} \sqrt{m} \beta \left[\text{sn}(\beta x + \Delta, m) + \text{sn}(\beta x - \Delta, m)\right].$$

(49)

Here $\Delta$ is defined by $\text{sn}(\sqrt{m} \Delta, 1/m) = \pm m^{1/4}$, where use has been made of the identity (24).

It is worth noting that for both the solutions VII and VIII, not only the value of $B$ is the same but even $g < 0$ for both the solutions.

**Solution IX**

It is easy to check that the nonlocal NLS Eq. (1) admits another periodic solution

$$\psi(x,t) = e^{i(\omega t)} \frac{A \text{sn}(\beta x, m) \text{cn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)},$$

(50)
provided

\[ 0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad g = -1, \]

\[ \omega = (2 - m - 6\sqrt{1 - m})\beta^2, \quad A^2 = \frac{2(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}}. \]  \hspace{1em} (51)

Note that whereas this solution is valid if \( g = -1 \), the same solution in the Yang’s nonlocal case, as well as in the local NLS case is valid only if \( g = +1 \).

On comparing the solution (50) with the identity (28) and using Eq. (51) we find that the solution as given by Eq. (50) can be re-expressed as a superposition of two periodic pulse solutions, i.e.

\[ \psi(x, t) = e^{i\omega t} \beta \sqrt{\frac{1}{2}} (\text{dn}[\beta(x) - K(m)/2, m] - \text{dn}[\beta(x) + K(m)/2, m]) \]. \hspace{1em} (52)

**Solution X**

Remarkably, the nonlocal NLS Eq. (1) also admits another periodic solution

\[ \psi(x, t) = e^{i\omega t} \frac{A\text{dn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \]  \hspace{1em} (53)

provided

\[ 0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad g = 1, \]

\[ \omega = [2 - m + 6\sqrt{1 - m}]\beta^2, \quad A^2 = \frac{4\sqrt{1 - m}\beta^2}{\sqrt{1 - m}}. \]  \hspace{1em} (54)

Note that this solution is also not valid for \( m = 1 \). Thus for this solution \( g > 0, \omega > 0 \).

On comparing the solution (53) and the identity (32) and using Eq. (54), the periodic solution (53) can be re-expressed as superposition of two periodic pulse solutions, i.e.

\[ \psi(x, t) = e^{i\omega t} \beta [\text{dn}(\beta x + K(m)/2, m) + \text{dn}(\beta x - K(m)/2, m)], \]  \hspace{1em} (55)

where \( \Delta = \pm K(m)/2 \)

It is worth noting that for both the superposed periodic pulse solutions IX and X, while the value of \( B \) is the same but the value of \( g \) is opposite for the two solutions.

Note that out of the 10 solutions, the solutions VII to X are only valid if \( m \neq 1 \). Further, while the solutions VII, VIII and IX are valid if \( g < 0 \), the remaining seven solutions are valid if \( g > 0 \).
4 Periodic and Hyperbolic Solutions of Nonlocal mKdV Equation

Recently, a nonlocal variant of the mKdV equation (both attractive and repulsive) has been proposed [7] and is given by

$$\psi_t(x, t) + \psi_{xxx}(x, t) + 6g\psi(x, t)\psi(-x, -t)\psi_x(x, t) = 0,$$

where $g = 1 \ (−1)$ corresponds to attractive (repulsive) nonlocal mKdV. Let us first note that all those solutions of the attractive (repulsive) local MKdV equation

$$\psi_t(x, t) + \psi_{xxx}(x, t) + 6g\psi^2(x, t)\psi_x(x, t) = 0,$$

for which $\psi(-x, -t) = \psi(x, t)$ are obviously also the solutions of the corresponding nonlocal attractive (repulsive) mKdV Eq. (56). However, because of the translational invariance, unlike the local mKdV Eq. (57), the nonlocal mKdV Eq. (56) does not admit arbitrary translational shift in $x$ or $t$.

Solution I

For example, one of the exact solutions of the nonlocal mKdV Eq. (56) as well as the local mKdV Eq. (57) is

$$\psi(x, t) = \text{Adn}(\xi, m), \ \xi = \beta(x - vt),$$

provided

$$g = 1, \ A^2 = \beta^2, \ v = (2 - m)\beta^2.$$

Hence we do not present those solutions of the nonlocal mKdV Eq. (56) for which $\psi(-x, -t) = \psi(x, t)$ here. However, for the sake of completeness, we have given those solutions in Appendix C.

One comment is in order here. Because of the translational invariance, even $\psi(x, t) = \text{Adn}(\xi + \xi_0, m)$ is an exact solution of the corresponding local attractive mKdV Eq. (57) under the same conditions (59). Here $\xi_0$ is an arbitrary constant. The major difference between the local and the nonlocal case is that in the nonlocal case neither the translation in $x$ nor $t$ and hence $\xi = \beta(x - vt)$ is allowed and the solution can only exist provided $\xi = 0$. This remark is valid for all the solutions mentioned below, for both the attractive as well as the repulsive mKdV Eq. (56) and hence we will not repeat this comment now onwards.

Solution II

Remarkably, unlike the local mKdV, for the nonlocal case even

$$\psi(x, t) = A\sqrt{m}\text{sn}(\xi, m), \ \xi = \beta(x - vt)$$

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is an exact solution to the attractive mKdV, i.e. of Eq. (56) provided
\[ g = 1, \quad A^2 = \beta^2, \quad v = -(1 + m)\beta^2. \] (61)

Note that the local mKdV Eq. (57) admits this solution only if \( g = -1 \).

**Solution III**

In the limit \( m = 1 \), the solution II goes over to the hyperbolic solution
\[ \psi(x, t) = A \tanh(\xi), \quad \xi = \beta(x - vt), \] (62)
provided
\[ g = 1, \quad A^2 = \beta^2, \quad v = -2\beta^2. \] (63)

Thus unlike the local mKdV, the same nonlocal attractive mKdV Eq. (56) (i.e. with \( g = +1 \)) admits both the kink and pulse solutions. This is because for both the solutions II and III, \( \psi(-x, -t) = -\psi(x, t) \).

As mentioned above, unlike the local case, the solutions of the nonlocal mKdV Eq. (56) are not invariant with respect to shifts in \( x \) and \( t \). However, for special values of \( \xi = \beta(x - vt) \), \( sn(x, m) \), \( cn(x, m) \) and \( dn(x, m) \) are still the solutions of the nonlocal Eq. (56). In particular, we now show that when the shift is by \( K(m) \), where \( K(m) \) is the complete elliptic integral of the first kind, there are exact solutions of the nonlocal Eq. (56) in both the focusing (\( g > 0 \)) and the defocusing (\( g < 0 \)) cases. This is because of the relations (10).

**Solution IV**

It is easy to show that
\[ \psi(x, t) = A \frac{dn(\xi, m)}{\sqrt{m}sn(\xi, m)}, \quad \xi = \beta(x - vt), \] (64)
is an exact solution to Eq. (56) (as well as to the local mKdV Eq. (57)) provided
\[ g = 1, \quad A^2 = (1 - m)\beta^2, \quad v = (2 - m)\beta^2. \] (65)

**Solution V**

It is easy to show that
\[ \psi(x, t) = A \frac{\sqrt{m}sn(\xi, m)}{dn(\xi, m)}, \quad \xi = \beta(x - vt), \] (66)
is an exact solution to the nonlocal mKdV Eq. (56) provided
\[ g = -1, \quad A^2 = (1 - m)\beta^2, \quad v = (2m - 1)\beta^2. \] (67)
Solution VI
It is easy to show that
\[
\psi(x, t) = \frac{A\sqrt{mc}\text{cn}[\beta(x - vt), m]}{\text{dn}[\beta(x - vt), m]},
\]
is an exact solution to Eq. (56) provided
\[
g = -1, \quad A^2 = \beta^2, \quad v = -(1 + m)\beta^2, \quad m \neq 1.
\]

Solution VII
Remarkably, it turns out that not only \(\text{dn}[\beta(x - vt), m]\) and \(\text{dn}[\beta(x - vt) + K[m], m]\) but even their superposition is a solution of the nonlocal mKdV Eq. (56). In particular, it is easy to check that
\[
\psi(x, t) = A\text{dn}[\beta(x - vt), m] + B\sqrt{1 - m}\text{dn}[\beta(x - vt), m],
\]
is also an exact solution of the nonlocal Eq. (56) provided
\[
g = 1, \quad A^2 = \beta^2, \quad B = \pm A, \quad v = [2 - m \pm 6\sqrt{1 - m}]\beta^2,
\]
where the \(\pm\) sign in \(B = \pm A\) and in \(v\) are correlated.

Solution VIII
It has been shown in [20] that
\[
\psi(x, t) = -2\frac{d}{dx}\text{tanh}^{-1}\left[\alpha\text{sn}(ax + bt, k)\text{sn}(cx + dt, m)\right],
\]
is an exact solution of the local mKdV Eq. (57) in case \(g = -1\). Since for this solution \(\psi(-x, -t) = -\psi(x, t)\), hence it is clear that Eq. (72) is an exact solution of the nonlocal mKdV Eq. (56) provided
\[
g = 1, \quad ka^4 = mc^4, \quad \alpha^2 = \sqrt{km},
\]
\[
b = a[a^2(1 + k) + 3c^2(1 + m)], \quad d = c[3a^2(1 + k) + (1 + m)c^2].
\]

Solution IX
It has been shown in [20] that there is a periodic solution of the local mKdV Equation (57)
\[
\psi(x, t) = -2\frac{d}{dx}\tan^{-1}\left[\alpha\text{cn}(ax + bt + a_0, k)\text{cn}(cx + dt + c_0, m)\right],
\]

provided

\[ g = 1, \ k(1 - k)a^4 = m(1 - m)c^4, \ \alpha^2 = \frac{km}{(1-k)(1-m)}, \]

\[ b = a[a^2(1 - 2k) + 3c^2(1 - 2m)], \ d = c[3a^2(1 - 2k) + (1 - 2m)c^2]. \]

Now observe that for this solution \( \psi(-x, -t) = -\psi(x, t) \) provided \( a_0 = c_0 = 0 \) and hence

\[ \psi(x, t) = -2\frac{d}{dx}\tan^{-1}\left[\text{cn}(ax + bt, k)\text{cn}(cx + dt, m)\right], \]

is in fact a solution of the nonlocal mKdV Eq. (56) provided the constraints (75) are satisfied except that \( g = -1 \).

Recently, we [18] have obtained four novel periodic and one hyperbolic solutions of the local mKdV Eq. (57) and shown that they can be re-expressed as a superposed kink or pulse solution. We now show that the nonlocal mKdV Eq. (56) also admits these five superposed solutions.

**Solution X**

Following [18] it is easy to see that

\[ \psi(x, t) = \frac{A\text{dn}(\xi, m)\text{cn}(\xi, m)}{1 + B\text{cn}^2(\xi, m)}, \ B > 0, \ \xi = \beta(x - vt), \]

is an exact solution of the nonlocal mKdV Eq. (56) provided

\[ g = -1, \ 0 < m < 1, \ B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \]

\[ v = [-1 + m + 6\sqrt{m}]\beta^2 < 0, \ A^2 = \frac{4\sqrt{m}\beta^2}{(1 - \sqrt{m})^2}. \]

Note that this solution is not valid for \( m = 1 \), i.e. the MKdV Eq. (56) does not admit a corresponding hyperbolic solution. Notice that for this solution \( v < 0 \).

On using the identity (22) one can then rewrite the periodic pulse solution (77) as the superposition of a periodic kink and a periodic antikink solution, i.e.

\[ \psi(x, t) = \sqrt{2m}\beta[\text{sn}(\xi + \Delta, m) - \text{sn}(\xi - \Delta, m)], \ \xi = \beta(x - vt). \]

Here \( \Delta \) is defined by \( \text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4} \), where use has been made of the identity (23).
Solution XI
Following [18] it is easy to show that the nonlocal mKdV Eq. (56) admits the periodic kink solution
\[\psi(x,t) = \frac{A \operatorname{sn}(\xi,m)}{1 + B \operatorname{cn}^2(\xi,m)}, \quad B > 0, \quad (80)\]
provided
\[g = -1, \quad 0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad (81)\]
\[v = [6\sqrt{m} - (1 + m)]\beta^2, \quad A^2 = 4\sqrt{m}\beta^2. \quad (82)\]

Note that this solution does not exist for \(m = 1\), i.e. the corresponding hyperbolic solution does not exist. Notice that for this solution \(\psi(-x,-t) = -\psi(x,t)\) and hence unlike the local mKdV, the nonlocal mKdV Eq. (56) admits such a solution in the repulsive case \((g = -1)\).

On using the identity (48), the periodic solution (80) can be re-expressed as
\[\psi(x,t) = i\sqrt{m}\beta[\operatorname{sn}(\xi + \Delta, m) + \operatorname{sn}(\xi - \Delta, m)], \quad \xi = \beta(x - vt). \quad (83)\]
Here \(\Delta\) is defined by \(\operatorname{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4}\), where use has been made of the identity (24).

Solution XII
Following [18] it is easy to show that another periodic solution to the nonlocal mKdV Eq. (56) is
\[\psi(x,t) = \frac{A \operatorname{sn}(\xi,m)\operatorname{cn}(\xi,m)}{1 + B \operatorname{cn}^2(\xi,m)}, \quad (84)\]
provided
\[0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad v = (2 - m - 6\sqrt{1 - m})\beta^2, \quad (85)\]
\[g = -1, \quad A^2 = \frac{4(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}}. \quad (86)\]

Notice that for this solution \(\psi(-x,-t) = -\psi(x,t)\) and hence unlike the local mKdV, the nonlocal mKdV Eq. (56) admits such a solution in the repulsive case \((g = -1)\).
On using the identity (28), the periodic solution (83) can be re-expressed as a superposition of two periodic pulse solutions, i.e.
\[ \psi(x, t) = \beta(\text{dn}[\xi - K(m)/2, m] - \text{dn}[\xi + K(m)/2, m]), \quad \xi = \beta(x - vt). \] (85)

**Solution XIII**

Following [18], yet another periodic solution to the nonlocal mKdV Eq. (56) is
\[ \psi(x, t) = A \text{dn}(\xi, m) + B \text{cn}^2(\xi, m), \] (86)
provided
\[ 0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad g = 1, \]
\[ v = [2 - m + 6\sqrt{1 - m}]\beta^2, \quad A^2 = \frac{4}{\sqrt{1 - m}}\beta^2. \] (87)
Thus for this solution \( v > 0. \)

On using the identity (32) the periodic solution (86) can be re-expressed as superposition of two periodic pulse solutions, i.e.
\[ \psi(x, t) = \beta[\text{dn}(\beta x + K(m)/2, m) + \text{dn}(\beta x - K(m)/2, m)]. \] (88)

Summarizing, while the solutions X and XII are satisfied for the same values of the parameters in both the local and nonlocal mKdV Eqs. (57) and (56) respectively, the solutions XI and XII are satisfied for the opposite values of \( g \) in the two models.

**Superposed Solution XIV**

Following [18, 21] it is easy to check that the repulsive mKdV Eq. (56) admits a hyperbolic pulse solution
\[ \psi(x, t) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad B > 0, \quad \xi = \beta(x - vt), \] (89)
provided
\[ g = -1, \quad A = 2\sqrt{B(B + 1)}\beta, \quad \beta^2 = \frac{4(B + 1)}{(B + 2)^2} < 1, \quad v = 4\beta^2 - 6. \] (90)
Now by starting from the identity (22) and taking \( m = 1, \) we obtain the corresponding hyperbolic identity
\[ \tanh(y + \Delta) - \tanh(y - \Delta) = \frac{\sinh(2\Delta)}{B + \cosh^2(y)}, \quad B = \sinh^2(\Delta). \] (91)
On using the identity (91), the solution (89) can be re-expressed as the superposition of a kink and an antikink solution

\[ \psi(x, t) = 1 - \beta [\tanh(\xi + \Delta) - \tanh(\xi - \Delta)] , \tag{92} \]

where \( \xi = \beta(x - vt) \) while \( \sinh(\Delta) = \sqrt{B} \).

5 Exact Solutions of Nonlocal Hirota Equation

Recently it has been proposed [8, 9] that the nonlocal Hirota equation is given by

\[ iu_t(x, t) + \alpha [u_{xx}(x, t) + 2gu^2(x, t)u(-x, -t)] + i\beta [u_{xxx}(x, t) + 6gu(x, t)u(-x, -t)u_x(x, t)] = 0 . \tag{93} \]

Note that in case \( \beta = 0 \) we have an attractive or repulsive nonlocal NLS depending on whether \( g = 1 \) or \( g = -1 \), respectively. Similarly, when \( \alpha = 0 \), and \( u \) is real we have an attractive or repulsive nonlocal mKdV depending on whether \( g = 1 \) or \( g = -1 \), respectively. Note that in case \( u(-x, -t) = u(x, t) \), then all the solutions of the local Hirota Equation

\[ iu_t + \alpha [u_{xx} + 2g|u|^2u] + i\beta [u_{xxx} + 6g|u|^2u_x] = 0 , \tag{94} \]

are also the solutions of the nonlocal Hirota Eq. (93). On the other hand, in case \( u(-x, -t) = -u(x, t) \), then all the solutions of \( g = 1 \) \((-1) \) Hirota Eq. (94) are also the solutions of the nonlocal Hirota Eq. (93) in case \( g = -1 \) \((+1) \).

We start from the local Hirota Eq. (94) and choose the ansatz

\[ u(x, t) = e^{i\omega t} \phi(\xi) , \ \xi = x - vt , \tag{95} \]

where \( \phi \) is real. On substituting the ansatz (95) in Eq. (94) we obtain

\[ \alpha [\phi_{\xi\xi} + 2g\phi^3 - \frac{\omega}{\alpha} \phi] + i\beta \frac{d}{d\xi} [\phi_{\xi\xi} + 2g\phi^3 - \frac{v\phi}{\beta}] = 0 . \tag{96} \]

It then follows that all the solutions of the real part of Eq. (96), i.e.

\[ \phi_{\xi\xi} = -\frac{\omega}{\alpha} \phi - 2g\phi^3 , \tag{97} \]

are automatically also the solutions of the imaginary part of Eq. (96) provided

\[ \omega\beta = v\alpha . \tag{98} \]
It is then straightforward to obtain the solutions of Eq. (97) and hence the solutions of the nonlocal Eq. (93) which we now mention one by one.

**Solution I**
One of the periodic pulse-like solutions of the nonlocal (as well as the local) Hirota equation (93) is

\[ u(x,t) = Ae^{i\omega t} \text{cn}(\xi, m), \quad \xi = \delta(x - vt), \]  

provided the relation (98) is satisfied and further

\[ g = 1, \quad A^2 = m\delta^2, \quad \omega = (2m - 1)\alpha\delta^2. \]  

**Solution II**
Another periodic pulse-like solution of the nonlocal (as well as the local) Hirota Eq. (93) is

\[ u(x,t) = Ae^{i\omega t} \text{dn}(\xi, m), \]  

provided the relation (98) is satisfied and further if

\[ g = 1, \quad A^2 = \beta^2, \quad \omega = (2 - m)\alpha\delta^2. \]  

**Solution III**
In the limit \( m = 1 \), both the solutions I and II go over to the hyperbolic pulse-like solution

\[ u(x,t) = Ae^{i\omega t} \text{sech}(\xi), \]  

provided the relation (98) is satisfied and further if

\[ g = 1, \quad A^2 = \beta^2, \quad \omega = \alpha\delta^2. \]  

**Solution IV**
The nonlocal Hirota Eq. (93) also admits the periodic kink-like solution

\[ u(x,t) = Ae^{i\omega t} \text{sn}(\xi, m), \]  

provided the relation (98) is satisfied and further if

\[ g = 1, \quad A^2 = m\beta^2, \quad \omega = -(1 + m)\alpha\delta^2. \]  

Note that in contrast, for the local Hirota Eq. (94), (105) is a solution only if \( g = -1 \).

**Solution V**
In the limit $m = 1$, the periodic kink-like solution \(105\) goes over to the celebrated single kink-like solution
\[
\begin{align*}
  u(x, t) &= Ae^{i\omega t} \tanh(\xi),
  \end{align*}
\]
provided the relation \(98\) is satisfied and if further
\[
  g = 1, \quad A^2 = \beta^2, \quad \omega = -2\alpha\delta^2.
\]
Note that unlike the Hirota Eq. \(94\), the nonlocal Hirota Eq. \(93\) admits both the kink and pulse solutions in the same model (i.e. with $g = 1$).

**Solution VI**

Remarkably, the nonlocal (as well as the local) Hirota Eq. \(93\) also admits superposition of the two periodic pulse-like solutions, i.e.
\[
\begin{align*}
  u(x, t) &= e^{i\omega t}[A\text{dn}(\xi, m) + B\sqrt{m}\text{cn}(\xi, m)],
  \end{align*}
\]
provided the relation \(98\) is satisfied and further if
\[
  g = 1, \quad B^2 = A^2, \quad 4A^2 = \beta^2, \quad \omega = (1 + m)/2\alpha\delta^2.
\]
Note that in the limit $m = 1$ and $B = A$, the solution VI also goes over to the pulse solution III while the solution with $B = -A$ goes to the vacuum solution $u = 0$.

As mentioned above for the nonlocal NLS and nonlocal mKdV equations, even for the Hirota equation, unlike the local case, the solutions of the nonlocal Hirota Eq. \(93\) are not invariant with respect to shifts in $x$, $t$ and hence $\xi$. For example, while $A\text{dn}(\xi + x_0, m)e^{i\omega t}$ is an exact solution of the local Hirota Eq. \(94\) no matter what $x_0$ is, it is not an exact solution of the nonlocal Eq. \(93\). However, for special values of $x_0$, $\text{sn}(x, m)$, $\text{cn}(x, m)$ and $\text{dn}(x, m)$ are still the solutions of the nonlocal Hirota Eq. \(93\). In particular, we now show that when $x_0 = K(m)$, where $K(m)$ is the complete elliptic integral of the first kind, there are exact solutions of the nonlocal Hirota Eq. \(93\) in both the focusing ($g > 0$) and defocusing ($g < 0$) cases. This is because \(11\) of the relations \(10\).

**Solution VII**

Yet another periodic solution to the nonlocal Hirota Eq. \(93\) is
\[
\begin{align*}
  u(x, t) &= e^{i\omega t} \frac{A}{\text{dn}(\xi, m)},
  \end{align*}
\]
provided the relation \(98\) is satisfied and further if
\[
  g = 1, \quad A^2 = (1 - m)\beta^2, \quad \omega = (2 - m)\alpha\delta^2.
\]
Solution VIII
Yet another periodic solution to the nonlocal Hirota Eq. (93) is
\[ u(x, t) = e^{i\omega t} A \sqrt{msn(\xi, m)} \frac{dn(\xi, m)}{dn(\xi, m)}, \]  
(113)
provided the relation (98) is satisfied and further if
\[ g = -1, \quad A^2 = (1 - m)\beta^2, \quad \omega = (2 - m)\alpha\delta^2. \]  
(114)

Solution IX
Yet another periodic solution to the nonlocal Hirota Eq. (93) is
\[ u(x, t) = e^{i\omega t} A \sqrt{mcn(\xi, m)} \frac{dn(\xi, m)}{dn(\xi, m)}, \]  
(115)
provided the relation (98) is satisfied and further if
\[ g = -1, \quad 0 < m < 1, \quad A^2 = \beta^2, \quad \omega = -(1 + m)/2\alpha\delta^2. \]  
(116)

Solution X
Remarkably, it turns out that not only \( dn(\xi, m) \) and \( dn(\xi + K(m)), m \) but even their superposition is a solution of the nonlocal Hirota Eq. (93). In particular, it is easy to check that
\[ \psi(x, t) = e^{i\omega t} \left[ A dn(\xi, m) + \frac{B\sqrt{1 - m}}{dn(\xi, m)} \right], \]  
(117)
is also an exact solution of the nonlocal Eq. (93) provided
\[ g = 1, \quad A^2 = \beta^2, \quad B = \pm A, \quad \omega = [2 - m \pm 6\sqrt{1 - m}]\beta^2, \]  
(118)
where the ± sign in \( B = \pm A \) and in \( \omega \) are correlated.

We now mention three periodic solutions of the Hirota Eq. (1) which can be written as the superposition of either the periodic kink or pulse solutions \( sn(x, m) \) or \( dn(x, m) \), respectively.

Solution XI
One of the periodic solutions of the nonlocal Hirota Eq. (93) is
\[ u(x, t) = e^{i\omega t} A \frac{dn(\xi, m)cn(\xi, m)}{1 + Bcn^2(\xi, m)}, \quad B > 0, \]  
(119)
provided the relation (98) is satisfied and further if

\[ B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad g = -1, \quad \omega = -[1 + m + 6\sqrt{m}]{\alpha}{\delta}^2 < 0, \]
\[ 0 < m < 1, \quad A^2 = \frac{4\sqrt{m}{\beta}^2}{(1 - \sqrt{m})^2}. \]  

(120)

On using the identity (22), the solution (119) can be re-expressed as a superposition of a periodic kink and a periodic antikink solution, i.e.

\[ u(x, t) = e^{i\omega t} \sqrt{m}{\beta}[\operatorname{sn}(\xi + \Delta, m) - \operatorname{sn}(\xi - \Delta, m)]. \]  

(121)

Here \( \Delta \) is defined by \( \operatorname{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4} \), where use has been made of the identity (24).

**Solution XII**

Another periodic solution to the nonlocal Hirota Eq. (93) is

\[ u(x, t) = e^{i\omega t} \frac{A\operatorname{sn}(\xi, m)\operatorname{cn}(\xi, m)}{1 + B\operatorname{cn}^2(\xi, m)}, \quad B > 0, \]  

(122)

provided the relation (98) is satisfied and further if

\[ g = -1, \quad B = \frac{1-\sqrt{1-m}}{\sqrt{1-m}}, \quad 0 < m < 1, \]
\[ \omega = (2 - m - 6\sqrt{1-m}){\alpha}{\delta}^2, \quad A^2 = \frac{4(1-\sqrt{1-m})^2{\beta}^2}{\sqrt{1-m}}. \]  

(123)

On using the identity (28), one can re-express the solution (122) as a superposition of two periodic pulse solutions, i.e.

\[ u(x, t) = e^{i\omega t} \beta(\operatorname{dn}[\xi - K(m)/2, m] - \operatorname{dn}[\xi + K(m)/2, m]). \]  

(124)

**Solution XIII**

Remarkably, the nonlocal Hirota Eq. (93) also admits another periodic solution

\[ u(x, t) = e^{i\omega t} \frac{A\operatorname{dn}(\xi, m)}{1 + B\operatorname{cn}^2(\xi, m)}, \quad B > 0, \]  

(125)

provided the relation (98) is satisfied and further if

\[ g = 1, \quad B = \frac{1-\sqrt{1-m}}{\sqrt{1-m}}, \quad 0 < m < 1, \]
\[ \omega = [2 - m + 6\sqrt{1-m}]{\alpha}{\delta}^2, \quad A^2 = \frac{4}{\sqrt{1-m}}{\beta}^2. \]  

(126)
On using the identity (32), one can re-express the solution (125) as a superposition of two periodic pulse solutions, i.e.

\[ u(x, t) = e^{i\omega t} \beta (dn[\xi + K(m)/2, m] + dn[\xi - K(m)/2, m]) . \]  

(127)

It is worth pointing out that while the solutions XI and XIII are also the solutions of the local Hirota Eq. (94), solution XII is only a solution of the local Hirota Eq. (94) provided \( g = -1 \).

6 Conclusion and Open Problems

In this paper we have shown that a number of nonlocal nonlinear equations such as the Ablowitz-Musslimani variant of the nonlocal NLS [1, 2], Yang variant of the nonlocal NLS [5], nonlocal mKdV equation [7] as well as the nonlocal Hirota equation [8, 9] admit novel superposed periodic kink and pulse solutions. Besides, nonlocal mKdV also admits hyperbolic superposed kink solution. Further, we have shown that except for the Yang variant of the nonlocal NLS, other three nonlocal equations admit both the kink and the pulse solutions in the same model. Besides, unlike the local nonlinear equations the nonlocal nonlinear equations do not admit solutions with arbitrary translation. However, we have shown that all of them support solutions with definite translation. Finally, we have also shown that amongst these nonlocal equations, only the Ablowitz-Musslimani variant of the nonlocal NLS admits complex PT-invariant kink and pulse solutions. Remarkably, all these solutions exist in the same model in which the real kink and pulse solutions are also admitted.

This paper raises several questions which are still not understood. We list some of them below.

1. In this paper, in several different models we have obtained a number of solutions which can be re-expressed as either the sum or the difference of two \( \text{sn}(x, m) \) or two \( \text{dn}(x, m) \) Jacobi elliptic functions. However, we have not been able to obtain similar superposed solutions in the Jacobi elliptic \( \text{cn}(x, m) \) case. It is not clear what is the underlying reason. Clearly it would be worthwhile finding \( \text{cn}(x, m) \) superposed solutions in some nonlocal nonlinear models.

2. So far only for the nonlocal mKdV equation we have been able to obtain a (hyperbolic) solution which can be re-expressed as the sum of a kink and an antikink solution. However, so far we have not been
able to obtain hyperbolic solutions which can be re-expressed either as a sum of two kink or a sum or difference of two pulse solutions. It is clearly of interest to look for such solutions.

3. It is not clear what is the physical interpretation of such superposed periodic or hyperbolic solutions. Do they correspond to a bound state of a kink and an antikink or of two periodic kinks or two periodic pulse solutions? Or do they merely correspond to some excitation of a kink and an antikink, or of two kink or of two pulse solutions? It is worthwhile finding the interpretation of such superposed solutions vis a vis a single kink or pulse solution.

4. It is clearly of interest to discover other nonlocal nonlinear equations which also admit such or even more unusual superposed solutions.

Hopefully one can find answers to some of the questions raised above.

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Appendix A: Solutions of Yang’s nonlocal variant of NLS Eq. (5)

We now present those solutions of the local NLS which are also the solutions of the Yang’s variant of the nonlocal NLS for the same value of the parameters.

Solution IX
Another solution to the Yang variant of the nonlocal NLS Eq. (5) is

\[ \psi(x, t) = A \sqrt{m} \text{cn}(\beta x, m) e^{i\omega t}, \]  

provided

\[ g = 1, \quad A^2 = \beta^2, \quad \omega = (2m - 1)\beta^2. \]  

Solution X
Remarkably, even a linear superposition of solutions I and II is also a solution of Eq. (5), i.e.

\[ \psi(x, t) = [\text{Adn}(\beta x, m) + B \sqrt{m} \text{cn}(\beta x, m)] e^{i\omega t}, \]  

provided

\[ g = 1, \quad 4A^2 = \beta^2, \quad B = \pm A, \quad \omega = \frac{(1 + m)}{2} \beta^2. \]
Solution XI
In the limit \( m = 1 \), the solutions I, II and III (with \( B = A \)) go over to the hyperbolic solution
\[
\psi_1(x, t) = \text{Asech}(\beta x)e^{i\omega t},
\]
provided
\[
g = 1, \quad A^2 = \beta^2, \quad \omega = \beta^2,
\]
while solution III with \( B = -A \) goes over to the vacuum solution \( \psi_1 = \psi_2 = 0 \).

Solution XII
Yet another solution to the nonlocal Eq. (5) is
\[
\psi(x, t) = A\sqrt{m}\text{sn}(\beta x, m)e^{i\omega t},
\]
provided
\[
g = -1, \quad A^2 = -\beta^2, \quad \omega = -(1 + m)\beta^2.
\]

Solution XIII
In the limit \( m = 1 \), solution XII goes over to the hyperbolic solution
\[
\psi(x, t) = Atanh(\beta x)e^{i\omega t},
\]
provided
\[
g = -1, \quad A^2 = -\beta^2, \quad \omega = -2\beta^2.
\]

Thus as in the local NLS case, in this nonlocal NLS model, while pulse solutions are admitted in case \( g > 0 \), the kink solutions exist only if \( g < 0 \). In contrast, in the Ablowitz-Musslimani variant of the nonlocal NLS, both the kink and pulse solutions exist in the same model, i.e. in case \( g > 0 \).

Solution XIV: Peregrine Soliton
Remarkably, the celebrated Peregrine soliton solution \([13, 14]\) of the local NLS is also a solution of the nonlocal Eq. (5). In particular, it is easy to check that
\[
\psi(x, t) = \frac{1}{\sqrt{2}}\left[1 - \frac{4(1 + 2it)}{(1 + 2x^2 + 4t^2)}\right]e^{it}, \quad g = 1,
\]
is an exact solution of Eq. (5).

Solution XV: Akhmediev-Eleonskii-Kulagin Breather Solution
Remarkably, even the celebrated Akhmediev-Eleonskii-Kulagin breather solution [14, 15] of the local NLS is also a solution of the nonlocal Eq. (5). In particular, it is easy to check that

$$\psi(x,t) = \sqrt{\frac{a^2}{2}} e^{ia^2 t} \left[ \frac{b^2 \cosh(\theta) + ib\sqrt{2 - b^2}}{\sqrt{2} \cosh(\theta) - \sqrt{2 - b^2} \cos(abx)} \right], \quad g = 1,$$

where $\theta = a^2 b \sqrt{2 - b^2} t$, is an exact solution of the nonlocal Eq. (5).

Solution XVI: Kuznetsov-Ma Soliton

The celebrated Kuznetsov-Ma soliton solution [14, 16] of the local NLS is also a solution of the nonlocal Eq. (5). In particular, it is easy to check that

$$\psi(x,t) = a \sqrt{\frac{a^2}{2}} e^{ia^2 t} \left[ 1 + 2m(m \cos(\theta) + i n \sin(\theta)) \right] \frac{n \cosh(\sqrt{2}m \alpha)}{n \cosh(\sqrt{2} m \alpha + \cos(\theta))}, \quad g = 1,$$

where $n^2 = 1 + m^2$, $\theta = 2mna^2 t$, is an exact solution of the nonlocal Eq. (5).

Appendix B: Exact Solutions of Ablowitz-Musslimani Variant of Nonlocal NLS Eq. (1)

Solution XI

In [10] we had shown that \( \text{dn} (\beta x, m) \) as well as \( \text{dn} [\beta(x + K[m]), m] \) are the exact solutions of the nonlocal Eq. (1). Remarkably, it turns out that not only \( \text{dn} (\beta x, m) \) and \( \text{dn} [\beta(x + K[m]), m] \) but even their superposition is a solution of the nonlocal NLS Eq. (1). In particular, it is easy to check that

$$\psi(x,t) = e^{i\omega t} \left[ A \text{dn} (\beta x, m) + B \sqrt{1-m} \text{dn}(\beta x, m) \right], \quad g = 1,$$

is also an exact solution of the nonlocal Eq. (5) provided

$$g = 1, \quad A^2 = \beta^2, \quad B = \pm A, \quad \omega = \left[ 2 - m \pm 6\sqrt{1-m} \beta^2 \right],$$

where the $\pm$ sign in $B = \pm A$ and in $\omega$ are correlated.

Solution XII: Peregrine Soliton

It is easy to check that the celebrated Peregrine soliton solution [13, 14] of the local NLS is also a solution of the nonlocal Eq. (1), i.e. in particular, it is easy to check that

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left[ 1 - \frac{4(1+2it)}{(1+2x^2+4t^2)} \right] e^{i t}, \quad g = 1,$$

is an exact solution of Eq. (5).
Solution XIII: Akhmediev-Eleonskii-Kulagin Breather Solution

It is easy to check that even the celebrated Akhmediev-Eleonskii-Kulagin breather solution [14, 15] of the local NLS is also a solution of the nonlocal Eq. (1)

\[
\psi(x, t) = \sqrt{\frac{a^2}{2}} e^{i a^2 t} \left[ \frac{b^2 \cosh(\theta) + ib \sqrt{2 - b^2}}{\sqrt{2} \cosh(\theta) - \sqrt{2 - b^2} \cos(abx)} \right], \quad g = 1,
\]  

where \( \theta = a^2 b \sqrt{2 - b^2} t \).

Solution XIV: Kuznetsov-Ma Soliton

Remarkably, even the celebrated Kuznetsov-Ma soliton solution [14, 16] of the local NLS is also a solution of the nonlocal Eq. (1). In particular, it is easy to check that

\[
\psi(x, t) = \frac{a}{\sqrt{2}} e^{i a^2 t} \left[ 1 + 2m \left( m \cos(\theta) + i n \sin(\theta) \right) \cosh(\sqrt{2} \max) + \cos(\theta) \right], \quad g = 1,
\]

where \( n^2 = 1 + m^2 \), \( \theta = 2mna^2 t \), is an exact solution of the nonlocal Eq. (1).

Appendix C: Exact Solutions of the Nonlocal mKdV Eq. (56)

We present here those solutions of the nonlocal mKdV Eq. (56) which are also the solutions of the local mKdV Eq. (57) for the same set of parameters.

Solution XV

Similarly,

\[
\psi(x, t) = A \sqrt{m} \cosh[\beta(x - vt), m],
\]

is an exact solution to Eq. (56) provided

\[
g = 1, \quad A^2 = \beta^2, \quad v = (2m - 1) \beta^2.
\]

Solution XVI

Remarkably, even a linear superposition of solutions I and II is also a solution of Eq. (56), i.e.

\[
\psi(x, t) = A \sinh[\beta(x - vt), m] + B \sqrt{m} \cosh[\beta(x - vt), m],
\]

provided

\[
g = 1, \quad 4A^2 = \beta^2, \quad B = \pm A, \quad v = \frac{(1 + m)}{2} \beta^2.
\]

Solution XVII

In the limit \( m = 1 \), solutions I, II and III (with \( B = A \)) go over to the hyperbolic solution

\[
\psi(x, t) = A \text{sech}[\beta(x - vt)],
\]

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provided

\[ A^2 = \beta^2, \quad v = \beta^2, \tag{151} \]

while solution III with \( B = -A \) goes over to the vacuum solution \( \psi = 0 \).

**Solution XVIII: Rational Solution**

It is easy to check that the rational solution of the nonlocal mKdV Eq. (57) is \((g = 1)\)

\[ \psi(x, t) = c - \frac{4c}{4c(x - 6c^2 t)^2 + 1}. \tag{152} \]

**Solution XIX: The Bion Solution**

The well known breather (also called bion) solution of the attractive local mKdV Eq. (57) is \([14]\)

\[ \psi(x, t) = \frac{-2}{d} \frac{d}{dx} \tan^{-1} \left[ \frac{c \sin(ax + bt + a_0)}{a \cosh(cx + dt + c_0)} \right], \tag{153} \]

provided \( g = 1, \quad b = a(a^2 - 3c^2), \quad d = c(3a^2 - c^2). \tag{154} \)

Here \( a_0, c_0 \) are arbitrary constants. It is then clear that the bion solution as given by Eq. (153) is also the bion solution of the nonlocal mKdV Eq. (56) provided \( a_0 = c_0 = 0 \).

**Solution XX: Periodic Generalization of the Bion Solution**

It has been shown \([19]\) that the periodic generalization of the bion solution of the local mKdV Eq. (57) is

\[ \psi(x, t) = \frac{-2}{d} \frac{d}{dx} \tan^{-1} \left[ \frac{\alpha \sin(ax + bt + a_0, k) dn(cx + dt + c_0, m)}{a \cosh(cx + dt + c_0)} \right], \tag{155} \]

provided \( a^4k = c^4(1 - m), \quad \alpha = \frac{c}{a}, \quad b = a[a^2(1 + k) - 3c^2(2 - m)], \quad d = c[3a^2(1 + k) - (2 - m)c^2]. \tag{156} \)

As expected, in the limit \( m \to 1, k \to 0, \) the periodic bion solution (155) goes over to the bion solution (153) and the relations between \( c \) and \( d \) as well as between \( a \) and \( b \) as given by Eq. (156) go over to the one given in Eq. (154).

It is then clear that the periodic bion solution as given by Eq. (155) is also the periodic bion solution of the nonlocal mKdV Eq. (56) provided \( a_0 = c_0 = 0 \).

**Solution XXI**
It has been shown [20] that there is another periodic solution of the attractive local mKdV Eq. (57) and it is easy to check that it is also the solution of the nonlocal mKdV Eq. (56), and is given by

\[ \psi(x, t) = -2 \frac{d}{dx} \tan^{-1} \left[ \alpha(ax + bt, k) \text{dn}(cx + dt, m) \right], \quad (157) \]

provided

\[ g = 1, \quad (1 - k)a^4 = (1 - m)c^4, \quad \alpha^2 = -\frac{c}{a}, \]
\[ b = -a[a^2(2 - k) + 3c^2(2 - m)], \quad d = -c[3a^2(2 - k) + (2 - m)c^2]. \quad (158) \]

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