Quantum Cooperative Games

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Abstract

We study two forms of a symmetric cooperative game played by three players, one classical and other quantum. In its classical form making a coalition gives advantage to players and they are motivated to do so. However in its quantum form the advantage is lost and players are left with no motivation to make a coalition.

1 Introduction

Many situations in the recent research in quantum games \[1, 2, 3\] appear to be based on a general idea that is quite interesting as well. It is to take a classical game exhibiting certain features, generalize it to quantum domain, and see how the situation changes in the course of this generalization. In this course noncooperative games have attracted an earlier attention with the ruling solution concept of a Nash equilibrium (NE). This development looks reasonable because in classical game theory as well the earlier research was focused on noncooperative games and interest in coalition formation was revived later. Players in noncooperative games are not able to form binding agreements even if they may communicate. On the other hand the distinguishing feature of cooperative games is a strong incentive to work together to receive the largest total payoff. These games allow players to form coalitions, binding agreements, pay compensations, make side payments etc. In fact, von Neumann and Morgenstern \[4\] in their pioneering work in the theory of games offered models of coalition formation where the strategy of each player consists of choosing the coalition he wishes to join. In coalition games, that are part of cooperative game theory, the possibilities of the players are described by the available resources of different groups (coalitions) of players. Joining a group or remaining outside is part of strategy of a player affecting his/her payoff. Recent work in quantum games \[1, 2, 3\] gives rise to a natural and interesting question: what is the possible quantum mechanical role in cooperative games that are an important part of the classical game theory? In our opinion it may be quite interesting,
and fruitful as well, to investigate coalitions in quantum versions of cooperative games. Our motivation in present paper is to investigate what might happen to the advantage of forming a coalition in a quantum game compared to its classical analogue. We rely on the concepts and ideas of von Neumann's cooperative game theory and consider a three-player coalition game in a quantum form. We then compare it to the classical version of the game and see how the advantage of forming a coalition can be affected.

In usual classical analysis of the coalition games the notion of a strategy disappears; the main features are those of a coalition and the value or worth of the coalition. The underlying assumption is that each coalition can guarantee its members a certain amount called the “value of a coalition". The value of coalition measures the worth the coalition possesses and is characterized as the payoff which the coalition can assure for itself by selecting an appropriate strategy, whereas the ‘odd man’ can prevent the coalition from getting more than this amount. Using this idea we study cooperative games in quantum settings to see how advantages of making coalitions can be influenced in the new settings.

The preferable scheme to us to play a quantum game has been recently proposed by Marinatto and Weber. In this scheme an initial quantum state is prepared by an arbiter and forwarded to the players. Each player possesses two quantum unitary and Hermitian operators i.e. the identity $I$ and the inversion or Pauli spin-flip operator $\sigma$. Players apply the operators with classical probabilities on the initial quantum state and send the quantum state to the 'measuring agent' who decides the payoffs the players should get. Interesting feature in this scheme is that the classical game is reproduced when the initial quantum state becomes unentangled. Classical game is therefore embedded in the quantum version of the game.

In this paper using Marinatto and Weber’s scheme we find a quantum form of a symmetric cooperative game played by three players. In classical form of this game any two players out of three get an advantage when they successfully form a coalition and play the same strategy. We find a quantum form of this game where the advantage for coalition forming is lost and players are left with no motivation to cooperate.

# 2 A three player symmetric cooperative game

## 2.1 Classical form

A classical three person normal form game is given by three non-empty sets $\Sigma_A$, $\Sigma_B$, and $\Sigma_C$, the strategy sets of the players $A$, $B$, and $C$ and three real valued functions $P_A$, $P_B$, and $P_C$ defined on $\Sigma_A \times \Sigma_B \times \Sigma_C$. The product space $\Sigma_A \times \Sigma_B \times \Sigma_C$ is the set of all tuples $(\sigma_A, \sigma_B, \sigma_C)$ with $\sigma_A \in \Sigma_A$, $\sigma_B \in \Sigma_B$ and $\sigma_C \in \Sigma_C$. A strategy is understood as such a tuple $(\sigma_A, \sigma_B, \sigma_C)$ and $P_A$, $P_B$, $P_C$ are payoff functions of the three players. The game is usually denoted as $\Gamma = \{\Sigma_A, \Sigma_B, \Sigma_C; P_A, P_B, P_C\}$. Let $\mathcal{R} = \{A, B, C\}$ be the set of players and $\varphi$ be an arbitrary subset of $\mathcal{R}$. The players in $\varphi$ may form a coalition so that, for
all practical purposes, the coalition \( \emptyset \) appears as a single player. It is expected that players in \( \mathbb{R} - \emptyset \) will form an opposing coalition and the game has two opposing “coalition players” i.e. \( \emptyset \) and \( \mathbb{R} - \emptyset \).

We study quantum version of an example of a classical three player cooperative game discussed in ref. [5]. Each of three players \( A, B \) and \( C \) chooses one of the two strategies 1, 2. If the three players choose the same strategy there is no payoff; otherwise, the two players who have chosen the same strategy receive one unit of money each from the ‘odd man.’ Payoff functions \( P_A, P_B \) and \( P_C \) for players \( A, B \) and \( C \) respectively are given as

\[
P_A(1, 1, 1) = P_A(2, 2, 2) = 0
\]
\[
P_A(1, 1, 2) = P_A(2, 2, 1) = P_A(1, 2, 1) = P_A(2, 1, 2) = 1
\]
\[
P_A(1, 2, 2) = P_A(2, 1, 1) = -2
\]

with similar expressions for \( P_B \) and \( P_C \). Suppose \( \emptyset = \{B, C\} \); hence \( \mathbb{R} - \emptyset = \{A\} \). The coalition game represented by \( \Gamma_\emptyset \) is given by the following payoff matrix

|     | [1] | [2] |
|-----|-----|-----|
| [11] | 0   | 2   |
| [12] | -1  | -1  |
| [21] | -1  | -1  |
| [22] | 2   | 0   |

Here the strategies [12] and [21] are dominated by [11] and [22]. After eliminating these dominated strategies the payoff matrix becomes

|     | [1] | [2] |
|-----|-----|-----|
| [11] | 0   | 2   |
| [22] | 2   | 0   |

It is seen that the mixed strategies

\[
\frac{1}{2} [11] + \frac{1}{2} [22]
\]
\[
\frac{1}{2} [1] + \frac{1}{2} [2]
\]

are optimal for \( \emptyset \) and \( \mathbb{R} - \emptyset \) respectively. With these strategies a payoff 1 for players \( \emptyset \) is assured for all strategies of the opponent; hence, the value of the coalition \( v(\Gamma_\emptyset) \) is 1 i.e. \( v(\{B, C\}) = 1 \). Since \( \Gamma \) is a zero-sum game \( v(\Gamma_\emptyset) \) can also be used to find \( v(\Gamma_{\mathbb{R} - \emptyset}) \) as \( v(\{A\}) = -1 \). The game is also symmetric and one can write
\[ v(\Gamma_\nu) = 1, \quad \text{and} \quad v(\Gamma_{R-\nu}) = -1 \text{ or} \]
\[ v(\{A\}) = v(\{B\}) = v(\{C\}) = -1 \]
\[ v(\{A, B\}) = v(\{B, C\}) = v(\{C, A\}) = 1 \quad (6) \]

2.2 Quantum form

In quantum form of this three player game the players implement their strategies by applying the identity operators in their possession with probabilities \( p, q, \) and \( r \) respectively on the initial quantum state. In Marinatto and Weber’s scheme \(^3\) the Pauli spin-flip or simply the inversion operator \( \sigma \) is then applied with probabilities \((1 - p), (1 - q), \) and \((1 - r)\) by players \( A, B \) and \( C \) respectively. If \( \rho_{in} \) is the density matrix corresponding to initial quantum state the final state after players have played their strategies corresponds to \(^6\)

\[ \rho_{fin} = \sum_{U=I,\sigma} \Pr(U_A) \Pr(U_B) \Pr(U_C) U_A \otimes U_B \otimes U_C \rho_{in} U_A^\dagger \otimes U_B^\dagger \otimes U_C^\dagger \quad (7) \]

where the unitary and Hermitian operator \( U \) can be either \( I \) or \( \sigma \). \( \Pr(U_A), \) \( \Pr(U_B) \) and \( \Pr(U_C) \) are the probabilities with which players \( A, B \) and \( C \) apply the operator \( U \) on the initial state respectively. \( \rho_{fin} \) corresponds to a convex combination of all possible quantum operations. Let the arbiter prepares the following three qubit pure initial quantum state

\[ |\psi_{in}\rangle = \sum_{i,j,k=1,2} c_{ijk} |ijk\rangle, \quad \text{where} \quad \sum_{i,j,k=1,2} |c_{ijk}|^2 = 1 \quad (8) \]

where the eight basis vectors of this quantum state are \( |ijk\rangle \) for \( i, j, k = 1, 2 \). The initial state \(^3\) can be imagined as a global state (in a \( 2 \otimes 2 \otimes 2 \) dimensional Hilbert space) of three two-state quantum systems or ‘qubits’. A player applies the unitary operators \( I \) and \( \sigma \) with classical probabilities on \( \rho_{in} \) during his ‘move’ or ‘strategy’ operation. Fig. 1 shows the scheme to play this three player quantum game where players \( B \) and \( C \) form a coalition and player \( A \) is ‘leftout’.

Let the matrix of three player game be given by 24 constants \( \alpha_t, \beta_t, \gamma_t \) with \( 1 \leq t \leq 8 \) \(^3\). We write the payoff operators for players \( A, B, \) and \( C \) as \(^3\)

\[
\begin{align*}
(P_{A,B,C})_{oper} &= \alpha_1, \beta_1, \gamma_1 |111\rangle \langle 111| + \alpha_2, \beta_2, \gamma_2 |211\rangle \langle 211| + \\
&\quad \alpha_3, \beta_3, \gamma_3 |121\rangle \langle 121| + \alpha_4, \beta_4, \gamma_4 |112\rangle \langle 112| + \\
&\quad \alpha_5, \beta_5, \gamma_5 |221\rangle \langle 221| + \alpha_6, \beta_6, \gamma_6 |212\rangle \langle 212| + \\
&\quad \alpha_7, \beta_7, \gamma_7 |222\rangle \langle 222| + \alpha_8, \beta_8, \gamma_8 |222\rangle \langle 222| \quad (9)
\end{align*}
\]
Payoffs to players $A$, $B$, and $C$ are then obtained as mean values of these operators

$$P_{A,B,C}(p,q,r) = \text{Trace} [(P_{A,B,C})_{\text{oper}} \rho_{\text{fin}}]$$

(10)

where, for convenience, we identify the players’ moves only by the numbers $p$, $q$, and $r$. The cooperative game of eq. (10) with the classical payoff functions $P_A$, $P_B$, and $P_C$ for players $A$, $B$, and $C$ respectively, together with the definition of payoff operators for these players in eq. (9), imply that

$$\alpha_1 = \alpha_8 = 0, \quad \alpha_3 = \alpha_4 = \alpha_6 = \alpha_7 = 1 \quad \text{and} \quad \alpha_2 = \alpha_5 = -2$$

(11)

With these constants the payoff to player $A$, for example, can be found as

$$P_A(p,q,r) = \begin{bmatrix}
-4rq - 2p + 2pr + 2pq + r + q \\
-4rq + 2p - 2pr - 2pq + 3r + 3q - 2 \\
4rq + 2pr - 2pq - 3r - q + 1 \\
4rq - 2pr + 2pq - r - 3q + 1
\end{bmatrix}
\begin{bmatrix}
|c_{111}|^2 + |c_{222}|^2 \\
|c_{211}|^2 + |c_{122}|^2 \\
|c_{121}|^2 + |c_{212}|^2 \\
|c_{112}|^2 + |c_{221}|^2
\end{bmatrix}$$

(12)
Similarly payoffs to players $B$ and $C$ can be obtained. Classical mixed strategy payoffs can be recovered from the eq. (12) by taking $|c_{111}|^2 = 1$. The classical game is therefore imbedded in its quantum form.

The classical form of this game is symmetric in the sense that payoff to a player depends on his/her strategy and not on his/her identity. These requirements making symmetric the three-player game are written as

$$P_A(p, q, r) = P_A(p, r, q) = P_B(q, p, r) = P_B(r, p, q) = P_C(r, q, p) = P_A(q, r, p) \quad (13)$$

Now in this quantum form of the game $P_A(p, q, r)$ becomes same as $P_A(p, r, q)$ when

$$|c_{121}|^2 + |c_{212}|^2 = 0, \quad |c_{112}|^2 + |c_{221}|^2 = 0 \quad (14)$$

and then payoff to a $p$ player remains same when other two players interchange their strategies. The symmetry conditions (13) hold if, together with eqs. (14), following relations are also true

$$\alpha_1 = \beta_1 = \gamma_1, \quad \alpha_5 = \beta_6 = \gamma_7$$
$$\alpha_2 = \beta_3 = \gamma_4, \quad \alpha_6 = \beta_5 = \gamma_6$$
$$\alpha_3 = \beta_2 = \gamma_3, \quad \alpha_7 = \beta_7 = \gamma_5$$
$$\alpha_4 = \beta_4 = \gamma_2, \quad \alpha_8 = \beta_8 = \gamma_8 \quad (15)$$

These form the extra restrictions on the constants of payoff matrix and, together with the conditions (14), give a three player symmetric game in a quantum form. No subscript in a payoff expression is then needed and $P(p, q, r)$ represents the payoff to a $p$ player against two other players playing $q$ and $r$. The payoff $P(p, q, r)$ is found as

$$P(p, q, r) = (|c_{111}|^2 + |c_{222}|^2)(-4rq + 2p - 2qr + 2pq + r + q) +$$
$$((|c_{211}|^2 + |c_{122}|^2)(-4rq + 2p - 2qr - 2pq + 3r + 3q - 2) \quad (16)$$

The term ‘mixed strategy’ in the quantum form of this game is defined as being a convex combination of quantum strategies with classical probabilities. For this assume that the pure strategies [1] and [2] correspond to $p = 0$ and $p = 1$ respectively. The mixed strategy $n [1] + (1 - n) [2]$, where $0 \leq n \leq 1$, means that the strategy [1] is played with probability $n$ and [2] with probability $(1 - n)$. Now suppose the coalition $\wp$ plays the following mixed strategy

$$l[11] + (1 - l)[22] \quad (17)$$
where the strategy [11] means that both players in the coalition \( \varphi \) apply the identity operator \( I \) with zero probability. Similarly the strategy [22] can be defined. The strategy of the coalition in eq. (17) means that the coalition \( \varphi \) plays [11] with probability \( l \) and [22] with probability \( 1 - l \). Similarly we suppose the player in \( \mathbb{R} - \varphi \) plays following mixed strategy

\[
m[1] + (1 - m)[2]
\]

In this case the payoff to the coalition \( \varphi \) is obtained as

\[
P_\varphi = (lm)P_{\varphi[111]} + l(1 - m)P_{\varphi[112]} + (1 - l)mP_{\varphi[221]} + (1 - l)(1 - m)P_{\varphi[222]}
\]

where \( P_{\varphi[111]} \) is the payoff to \( \varphi \) when all three players play \( p = 0 \) i.e. the strategy [1]. Similarly \( P_{\varphi[221]} \) is coalition payoff when coalition players play \( p = 1 \) and the player in \( \mathbb{R} - \varphi \) plays \( p = 0 \). Now from eq. (16) we get

\[
P_{\varphi[111]} = 2P(0, 0, 0) = -4(|c_{111}|^2 + |c_{122}|^2)
\]

\[
P_{\varphi[112]} = 2P(0, 0, 1) = 2(|c_{111}|^2 + |c_{222}|^2 + |c_{211}|^2 + |c_{122}|^2)
\]

\[
P_{\varphi[221]} = 2P(1, 1, 0) = 2(|c_{111}|^2 + |c_{222}|^2 + |c_{211}|^2 + |c_{122}|^2)
\]

\[
P_{\varphi[222]} = 2P(1, 1, 1) = -4(|c_{111}|^2 + |c_{122}|^2)
\]

Therefore from eq. (19)

\[
P_\varphi = -4(|c_{211}|^2 + |c_{122}|^2) \{ lm + (1 - l)(1 - m) \} + 2(|c_{111}|^2 + |c_{222}|^2 + |c_{211}|^2 + |c_{122}|^2) \{ l(1 - m) + (1 - l)m \}
\]

To find the value of coalition \( \nu(\Gamma_\varphi) \) in the quantum game we find \( \frac{\partial P_\varphi}{\partial m} \) and equate it to zero i.e. \( P_\varphi \) is such a payoff to \( \varphi \) that the player in \( \mathbb{R} - \varphi \) cannot change it by changing his/her strategy given in eq. (18). It gives, interestingly, \( l = \frac{1}{2} \) and the classical optimal strategy of the coalition \( \frac{1}{2} [11] + \frac{1}{2} [22] \) becomes optimal in the quantum game as well. In the quantum game the coalition then secures following payoff, also termed as the value of the coalition

\[
\nu(\Gamma_\varphi) = (|c_{111}|^2 + |c_{222}|^2) - (|c_{211}|^2 + |c_{122}|^2)
\]

Similarly we get the value of coalition for \( \mathbb{R} - \varphi \) as

\[
\nu(\Gamma_{\mathbb{R} - \varphi}) = - \left\{ |c_{111}|^2 + |c_{222}|^2 + |c_{211}|^2 + |c_{122}|^2 \right\}
\]
Note that these values reduce to their classical counterparts of eq. (6) when the initial quantum state becomes unentangled and is given by $|\psi_{in}\rangle = |111\rangle$. Classical form of the coalition game is, therefore, a subset of its quantum version. Suppose the arbiter now has at his disposal a quantum state $|\psi_{in}\rangle = c_{111}|111\rangle + c_{222}|222\rangle + c_{211}|211\rangle + c_{122}|122\rangle$ such that $(|c_{211}|^2 + |c_{122}|^2) > (|c_{111}|^2 + |c_{222}|^2)$. In this case $v(\Gamma_\psi)$ becomes a negative quantity and $v(\Gamma_{R-\psi}) = -1$ because of the normalization given in eq. (6). A more interesting case is when the arbiter has the state $|\psi_{in}\rangle = c_{211}|211\rangle + c_{122}|122\rangle$ at his disposal. Because now both $v(\Gamma_\psi)$ and $v(\Gamma_{R-\psi})$ are $-1$ and the players are left with no motivation to form a coalition. A quantum version of this cooperative game, therefore, exists in which players are deprived of motivation to form a coalition.

The payoff to a $p$ player against $q$, $r$ players in classical mixed strategy game can be obtained from eq. (16) by taking $|c_{111}|^2 = 1$. It gives

$$P(p, q, r) = -4rq - 2p + 2pr + 2pq + r + q$$

(24)

Note that $P(p, q, r) + P(q, p, r) + P(r, p, q) = 0$ and classical mixed strategy game is zero-sum and $P(\psi) + P(\mathcal{R} - \psi) = 0$. The quantum version of this game is not zero-sum always because from eq. (16) we have

$$P(p, q, r) + P(q, p, r) + P(r, p, q) = 8(p + q + r - pq - qr - rp) - 6$$

(25)

and the quantum game becomes zero-sum only when $p + q + r - pq - qr - rp = \frac{3}{4}$.

3 Discussion and conclusion

There may appear several guises in which the players can cooperate in a game. One possibility is that they are able to communicate and, hence, able to correlate their strategies. In certain situations players can make binding commitments before or during the play of a game. Even in the post-play behavior the commitments can make players to redistribute their final payoffs. The two-player games are different from multi-player games in an important aspect. In two-player games the question before the players is whether to cooperate or not. In multi-player case the players are faced with a more difficult task. Each player has to decide which coalition to join. There is also certain uncertainty that the player faces about the extent to which players outside his coalition may coordinate their actions. Analysis of cooperative games isolating coalition considerations instead of studying elaborate strategic structures has drawn more attention. Recent exciting developments in quantum game theory provide a motivation to see how forming a coalition and its associated advantages can be influenced in already proposed quantum versions of these cooperative games. To study this we selected an interesting but simple cooperative game as well as a recently proposed scheme telling how to play a quantum game. We allowed the players in the quantum version of the game to form a coalition similar to the classical game. The underlying assumption in this approach is that because
the arbiter, responsible for providing three qubit pure quantum initial states to be later unitarily manipulated by the players, can forward a quantum state that correspond to the classical game, therefore, other games corresponding to different initial pure quantum states are quantum forms of the classical game. This assumption, for example, reduces the problem of finding a quantum version of the classical coalition game we considered, with an interesting property that the advantage of making a coalition is lost, to finding some pure initial quantum states. We showed that such quantum states can be found and, therefore, there are quantum versions of the three-player coalition game where the motivation for coalition formation is lost.

In conclusion, we considered a symmetric cooperative game played by three players in classical and quantum forms. In classical form of this game, which is also embedded in the quantum form, forming a coalition gives advantage to players and players are motivated to do so. In quantum form of the game, however, an initial quantum state can be prepared by the arbiter such that coalition forming is of no advantage. The interesting function in these situations i.e. ‘value of coalition’ is greater for coalition then for player outside; when the game is played classically. These values become same in a quantum form of the game and motivation to form a coalition is lost. There is, nevertheless, an essential difference between the two forms of the game i.e. classical game is zero-sum but its quantum version is not.

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\[ |\psi_{\text{ini}}\rangle = \sum_{i,j,k=1,2} C_{ijk} |i j k\rangle \]

Player A is 'leftout'

Player B

Player C

Players B and C form a coalition

Measurement and payoffs