On integrals for some class of ordinary difference equations admitting a Lax representation

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Abstract

We consider two infinite classes of ordinary difference equations admitting Lax pair representation. Discrete equations in these classes are parameterized by two integers \( k \geq 0 \) and \( s \geq k + 1 \). We describe the first integrals for these two classes in terms of special discrete polynomials. We show an equivalence between two difference equations belonging to different classes corresponding to the same pair \((k, s)\). We show that solution spaces \( \mathcal{N}_k^s \) of different ordinary difference equations with a fixed value of \( s + k \) are organized in a chain of inclusions.

Keywords: Lax pair, integrable, first integrals

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we consider some infinite classes of \( N \)th order autonomous ordinary difference equations yielded by nonlinear recurrence relations

\[
T(i + N) = R(T(i), \ldots, T(i + N - 1)), \quad i \in \mathbb{Z}
\]

with right-hand sides being some rational functions of their arguments. Every equation of the form (1) generates a special birational map \( \mathbb{R}^N \to \mathbb{R}^N \) for real-valued initial data \( \{y_j := T(i_0 + j) : j = 0, \ldots, N - 1\} \)

\[
(y_0, \ldots, y_{N-1}) \mapsto (y_1, \ldots, y_{N-1}, R(y_0, \ldots, y_{N-1}))
\]
provided that one may rewrite (1) in equivalent form as

\[ T(i - 1) = R(T(i), \ldots, T(i + N - 1)) \]

with corresponding rational function \( R \) of its arguments.

It should be noted that in a sense ordinary difference equations (1) are natural discrete counterparts of their differential analogs and many of the notions encountered in the theory of ordinary differential equations can be extended to them. For instance, it can be said that the function \( J = J(i) = J(T(i), \ldots, T(i + N - 1)) \) is a first integral for (1) if by virtue of this equation one has \( J(i + 1) = J(i) \). It is very important that the notion of Liouville–Arnold integrability can also be extended to symplectic maps with suitable modification of the Liouville theorem [1, 5, 11]. Many examples of integrable autonomous ordinary difference equations were given, for example, in [3, 4, 9, 10], etc. Experience suggests that such equations are usually grouped in infinite classes of their own kind and their first integrals have a similar appearance.

A goal that we set in this paper is to describe the first integrals for some classes of difference equation (1) in terms of their associated discrete polynomials. Specifically, we consider ordinary difference equations that admit a Lax representation and might be integrable in a Liouville–Arnold sense. In addition, the peculiar property of these difference equations and their Lax pairs is that they are presented in terms of special discrete polynomials [8]. This suggests that their first integrals also should be expressed via these polynomials. The article aims to address the problem of constructing the first integrals for whole classes of ordinary difference equations, which are derived from Lax representation. In the following section we first describe discrete \((h, n)\)-polynomials of one field \( T = T(i) \) and associate to them the ordinary difference equations. These classes of polynomials were introduced in [7] and [8]. They are closely related to formal pseudo-difference operators. Also in this section, we show a couple of examples of ordinary difference equations corresponding to the discrete \((1, 2)\)-polynomials and provide the reader with the first integrals derived from the Lax representation. It should be noted here that, in general, there is no problem to construct first integrals for the ordinary difference equations under consideration from the corresponding Lax representation. The problem we set in the article is to construct the first integrals for whole classes of discrete equations and reveal the relationship of these integrals with corresponding discrete polynomials. Unfortunately, only in the case \((h, n) = (1, 1)\) we have advanced enough in addressing this problem. In section 3, we discuss the discrete \((1, 1)\)-polynomials to prepare the ground for further investigation. In section 4, we discuss the Lax representation for two infinite classes of ordinary difference equations in the case \((h, n) = (1, 1)\). The equations in these classes are parametrized by pairs of integers \( k \geq 0 \) and \( s \geq k + 1 \). Based on actual calculations using Lax representation, in section 5, we construct the first integrals starting from in a sense a universal discrete polynomial (rational function) which yields all the first integrals which can be obtained from the Lax representation for all equations in the corresponding class. With some technical conjecture, we show that two \((k, s)\)th discrete equations with any fixed \((k, s)\) belonging to different, at first glance, classes, are related to each other through the interchange of integrals and parameters (see [6]). Also we prove that solution spaces \( N^k_1 \) with fixed values of the sum \( s + k \) are organized in chains of inclusions.
2. Discrete \((h, n)\)-polynomials and ordinary difference equations

To begin with, we fix some notion which will be used in the sequel. Throughout the paper we call any polynomial \(P = P(T(i + a), ..., T(i + b))\) of its arguments with \(a \leq b\) a discrete polynomial of the field \(T = T(i)\). This notion is in analogy with the notion of the differential polynomial. We also use discrete rational functions which can be presented as a ratio of two discrete polynomials.

2.1. Ordinary difference equations

In \([8]\) we have shown a large number of systems of ordinary difference equations which have a Lax pair representation. In particular, two \((h, n)\)-classes of one-field systems in this collection are presented by

\[ T(i + sh + kn)\hat{T}^k_s(i) = T(i)\hat{T}^k_s(i + h + n) \]  
(2)

and

\[ T(i + kn)\hat{S}^k_s(i) = T(i + sh)\hat{S}^k_s(i + h + n). \]  
(3)

\(\hat{T}^k_s(i)\) and \(\hat{S}^k_s(i)\) in (2) and (3), respectively, are non-homogeneous discrete polynomials of the field \(T = T(i)\) to be described.

2.2. Discrete polynomials \(\hat{T}^r_s(i)\)

To begin with, we first describe the polynomials \(\hat{T}^r_s(i)\) in this subsection and then we define polynomials \(\hat{S}^r_s(i)\). Let \((h, n)\) be a pair of positive integers which are supposed to be co-prime. Let us define an infinite class of homogeneous discrete polynomials \(\hat{T}^r_s(i)\) via the explicit formula\(^1\)

\[ \hat{T}^r_s(i) := \sum_{\{\lambda\} \in D_{r,s}} T(i + \lambda_1 h)T(i + \lambda_2 h + n)\cdots T(i + \lambda_r h + (r - 1)n) \]  
(4)

with \(D_{r,s} := \{\lambda : 0 \leq \lambda_1 < \cdots < \lambda_r \leq s - 1\}\), for \(r \geq 0\) and \(s \geq 1\). In particular, we put \(T^0_s(i) \equiv 1\). It is evident that, from the definition, it follows that

\[ \hat{T}^r_s(i) \equiv 0, \quad \forall s \leq r - 1 \]  
(5)

since in this case \(D_{r,s} = \emptyset\). Polynomials (4), by virtue of their definition, satisfy identities \([8]\)

\[ \hat{T}^r_s(i) = T^r_{s-1}(i + h) + T(i)T^r_{s-1}(i + h + n) \]  
(6)

\[ = T^r_{s-1}(i) + T(i + (s - 1)h + (r - 1)n)T^r_{s-1}(i). \]  
(7)

Indeed, it can be seen that identity (6) corresponds to the partition of \(D_{r,s}\) into two non-intersecting sets

\[ D_{r,s}^{(1)} := \{\lambda : \lambda_1 = 0, \quad 1 \leq \lambda_2 < \cdots < \lambda_r \leq s - 1\}\]

and

\[ D_{r,s}^{(2)} := \{\lambda : 1 \leq \lambda_1 < \cdots < \lambda_r \leq s - 1\}. \]

\(^1\) Strictly speaking, we should indicate a dependence for each class of polynomials on \(h\) and \(n\), but we prefer to use simplified notations for these polynomials in the hope that it will not lead to a confusion.
while identity (7) appears as a result of the partition $D_{r,s} = \tilde{D}^{(1)}_{r,s} \sqcup \tilde{D}^{(2)}_{r,s}$ with two parts

$$\tilde{D}^{(1)}_{r,s} := \{ \lambda_j : 0 \leq \lambda_1 < \cdots < \lambda_{r-1} \leq s-2, \ \lambda_r = s-1 \}$$

and

$$\tilde{D}^{(2)}_{r,s} := \{ \lambda_j : 0 \leq \lambda_1 < \cdots < \lambda_r \leq s-2 \}.$$

Taking as initial conditions the data $T^0_s(i) \equiv 1, \ \forall \ s \geq 1$ and $T^1_r(i) \equiv T(i), \ \forall \ r \geq 2$ and using one of the recurrent relations (6) or (7), we ultimately obtain the whole $(h,n)$-class of homogeneous discrete polynomials $\{T^r(i)\}$.

It is a simple observation that discrete polynomials $T^{r-jh}_s(i)$, for any fixed $j = 0, \ldots, \kappa$, satisfy recurrent relations of the same form as (6) and (7), where $r = \kappa h + \bar{r}$, i.e., $\bar{r}$ is supposed to be the remainder of the division of $r$ by $h$. This means that, in general, we can present the common solution of (6) and (7) as a linear combination

$$\tilde{T}^r_s(i) := \sum_{j=0}^{\kappa} c_j T^{r-jh}_s(i), \quad (8)$$

where $c_j$ are arbitrary constants. Note that, due to (5), it makes sense to consider non-homogeneous discrete polynomials (8) and consequently discrete equations of the form (2) only for $s \geq k$. Moreover, since

$$T^k_s(i) = \prod_{j=1}^{k} T(i + (j-1)(h+n))$$

and therefore the relation

$$T(i + k(h+n))T^k_s(i) = T(i)T^k_s(i + h+n)$$

is an identity, then equation (2) in the case $s = k$ is reduced to the following:

$$T(i + k(h+n))\tilde{T}^{k-h}_{k+n}(i) = T(i)\tilde{T}^{k-h}_{k+n}(i + h+n).$$

This means that, without loss of generality, we may consider an $(h,n)$-class of the ordinary difference equations of the form (2) only for $s \geq k + 1$.

### 2.3. Discrete polynomials $\tilde{S}^r_s(i)$

Non-homogeneous discrete polynomials $\tilde{S}^r_s(i)$ are given by linear combinations

$$\tilde{S}^r_s(i) := \sum_{j=0}^{\kappa} (-1)^j H_j \tilde{S}^{r-jh}_s(i + jhn), \quad (9)$$

where $H_j$ are supposed to be arbitrary constants. It is supposed in (9) that $s - jn \geq 1$ and $r - jh \geq 0$. The homogeneous discrete polynomials $\tilde{S}^r_s(i)$ can be defined explicitly as

$$\tilde{S}^r_s(i) := \sum_{\{\lambda_j \in B_{r,s}\}} T\left(i + \lambda_1 h + (r-1)n\right) \cdots T\left(i + \lambda_{r-1} h + (r-1)n\right) T(i + \lambda_r h),$$

where the summation is performed over the set $B_{r,s} := \{ \lambda_j : 0 \leq \lambda_1 \leq \cdots \leq \lambda_r \leq s-1 \}$. These polynomials satisfy identities

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2. It is evident that in equation (2), without loss of generality, we can put $c_0 = 1$.

3. In what follows, we assume that $H_0 = 1$. 

which come from the partitions \( B_{r,s} = B^{(1)}_{r,s} \sqcup B^{(2)}_{r,s} \) and \( \tilde{B}_{r,s} = \tilde{B}^{(1)}_{r,s} \sqcup \tilde{B}^{(2)}_{r,s} \), respectively, with
\[
B^{(1)}_{r,s} := \{ \lambda_j : 0 \leq \lambda_1 \leq \cdots \leq \lambda_r \leq s - 1 \},
\]
\[
B^{(2)}_{r,s} := \{ \lambda_j : 1 \leq \lambda_1 \leq \cdots \leq \lambda_r \leq s - 1 \}
\]
and
\[
\tilde{B}^{(1)}_{r,s} := \{ \lambda_j : 0 \leq \lambda_1 \leq \cdots \leq \lambda_{r-1} \leq s - 1, \lambda_r = s - 1 \},
\]
\[
\tilde{B}^{(2)}_{r,s} := \{ \lambda_j : 0 \leq \lambda_1 \leq \cdots \leq \lambda_r \leq s - 2 \}.
\]

Taking as initial conditions the data \( \mathcal{S}_0^{(i)}(i) \equiv 1, \forall s \geq 1 \) and
\[
\mathcal{S}_0^{(i)}(i) = T(i)T(i + n)\cdots T(i + (r - 1)n), \forall r \geq 1
\]
and using one of the recurrent relations (10) or (11), we obtain the \((h, n)\)-class of homogeneous discrete polynomials \( \mathcal{S}_r^{(h)}(i + jnh) \) and their linear combinations of the form (9) satisfy (10) and (11).

**Lemma 1.** The sets of homogeneous discrete polynomials \( \mathcal{T} := \{ T_r^{(i)}(i) \} \) and \( \mathcal{S} := \{ S_r^{(i)}(i) \} \) are related to each other through identities [8]
\[
(S, T)_r(i) := \sum_{j=0}^{r} (-1)^j S_{r-j}^{-1}(i)T_{r-j}^{(i)}(i + (r - j)n) = 0, \forall r, s \geq 1
\]
and
\[
(T, S)_r(i) := \sum_{j=0}^{r} (-1)^j T_{r-j}^{(i)}(i)S_{r-j}^{(i)}(i + (r - j)n) = 0, \forall r, s \geq 1.
\]

We present the proof of this lemma based on set theoretical reasonings in the appendix. It should be noted that this proof in principle does not depend on \((h, n)\).

For (12) and (13), let us make some remarks. Let us assign to each set of the discrete polynomials \( F = \{ F_r^{(i)}(i) : s \geq 1, r \geq 1 \} \) such that \( F_r^{(i)}(i) \equiv 1 \), semi-infinite lower triangular matrices \( F_r^{(i)} \) with matrix elements
\[
(F_r^{(i)})_{pq} = F_{r-p-q}^{(i)}(i + (q - 1)n), \quad p \leq q.
\]
As can be easily checked, a product \( (F, G)_r(i) := DG_r(i)DG_r(i), \) where \( D := \text{diag}(1, -1, 1, \ldots) \) has matrix elements \( (F, G)_r(i + (q - 1)n) \). This means that we can represent an infinite number of identities (13) in the form of matrix relations:
\[
(T, S)_r(i) = E, \quad \forall s \geq 1,
\]
where \( E \) stands for the unit semi-infinite matrix. Since any semi-infinite lower triangular matrix has its unique inverse, one can easily derive that from (14) it follows that \( (S, T)_r(i) = E \) and vice versa. Equivalently, this means that (12) implies (13) and vice versa. Also, we remark that matrix representation of identities (12) and (13) makes
it possible to express all the discrete polynomials $S'_i(i)$ via $T_i(i)$ in determinant form and vice versa.

2.4. Equivalent forms of discrete equations (2) and (3)

We remark that both equations (2) and (3) are autonomous ordinary difference equations in the form of nonlinear recurrence relation (1) with $N = sh + kn$. Substituting

$$\tilde{T}_i^k(i) = \tilde{T}_i^{k-1}(i + h) + T(i)\tilde{T}_i^{-1}(i + h + n)$$

into the left-hand side of (2) and

$$\tilde{T}_i^k(i + n + h) = \tilde{T}_i^{k-1}(i + n + h) + T(i + kn + sh)\tilde{T}_i^{-1}(i + n + h)$$

into the right-hand side of this equation, we obtain it in an equivalent form

$$T(i + kn + sh)\tilde{T}_i^{k-1}(i + h) = T(i)\tilde{T}_i^{k-1}(i + n + h).$$

By analogy, we find that equation (3) is equivalent to

$$T(i + kn)\tilde{S}_i^{k}(i) = T(i + sh)\tilde{S}_i^{k}(i + n).$$

2.5. Lax pair representations

Equations of the form (2) and (3) have the property of having Lax pair representation. The first linear equation is common to them; namely,

$$z\psi_{i+n} + T(i + n)\psi_i = z\psi_{i+h+n}. \quad (17)$$

The $(k, s)$th equation in the $(h, n)$-class of equations (2) arises as a compatibility condition of (17) and linear equation [8]

$$w\psi_{i+sh+kn} = \sum_{j=0}^{k} z^{k-j}\tilde{T}_s^j(i + (k - j + 1)n)\psi_{i+(k-j)n}. \quad (18)$$

In turn, the $(k, s)$th equation in the $(h, n)$-class of equations (3) appears as a compatibility condition of (17) and

$$w\psi_{i-sh+kn} = \sum_{j=0}^{k} z^{k-j}(-1)^{j}\tilde{S}_s^j(i - (j - k - 1)n - sh)\psi_{i+(k-j)n}. \quad (19)$$

The following remarks are in order. It is quite natural to suppose that ordinary difference equations admitting a Lax pair representation should be integrable in the Liouville–Arnold sense and proving this fact is the general problem. We can derive a number of first integrals for the ordinary difference equations under consideration making use of Lax pairs, but, unfortunately, this does not provide us with full information about the integrability of the equation under consideration since we do not know the Poisson structure. Moreover, in general, we do not know whether a full number of integrals is obtained from the corresponding Lax pair.

To extract the first integrals of the equation under consideration, we rewrite the corresponding two linear equations in equivalent matrix form

$$L_i\Psi = 0, \quad \Psi_{i+1} = A_i\Psi$$

where $\Psi = \begin{pmatrix} \psi_i^{(0)}, \ldots, \psi_i^{(N-1)} \end{pmatrix}^T$, with $\psi_i^{(j)} := \psi_{i+j}$ and $N = sh + kn$ for the linear problem (18) and
\[
\mathcal{N} = \begin{cases} 
sh, & \text{if } sh - kn \geq 0, \\
kn, & \text{if } sh - kn < 0,
\end{cases}
\]

for (19). Clearly, \(L\) in (20) is some \(\mathcal{N} \times \mathcal{N}\) matrix function. The condition \(\det(L) = 0\) gives the polynomial equation \(P(w, z) = 0\), yielding a corresponding spectral curve and first integrals. Below we give some simple examples of ordinary difference equations of the form (2) and (3) together with their first integrals.

### 2.6. Examples

Here we show two examples of ordinary difference equations (2) and (3) in the case \((h, n) = (1, 2)\). The case \((h, n) = (1, 1)\) is investigated in much greater detail in sections 4 and 5.

#### 2.6.1. Equation of the form (2) in the case \((h, n) = (1, 2), k = 1, s = 2\).

The Lax pair in this case is given by

\[
z\psi_{i+3} = z\psi_{i+2} + T(i + 2)\psi_i, \quad w\psi_{i+4} = z\psi_{i+2} + T^2(i + 2)\psi_i.
\]

The ordinary difference equation (2) in this case becomes a fourth-order one:

\[
T(i + 4) = T(i)\frac{T(i + 3) + c_1}{T(i + 1) + c_1}.
\]

Using (21), we obtain the following first integrals for this equation:

\[
\begin{align*}
\mathcal{H}_0 &= \frac{y_0 y_1 y_2 y_3}{(y_1 + c_1)(y_2 + c_1)}, \\
\mathcal{H}_1 &= \frac{y_0 + y_1 + y_2 + y_3 + \frac{y_0 y_3(y_1 + y_2 + c_1)}{(y_1 + c_1)(y_2 + c_1)}}{y_1 + c_1 + y_2 + c_1}, \\
G &= \frac{y_0 y_2 + y_1 y_3}{y_1 + c_1 + y_2 + c_1}.
\end{align*}
\]

Calculations show that a spectral curve in this case is given by

\[
\mathcal{H}_0 w^3 + z^3(z + \mathcal{H}_1)w - z^4(z + \tilde{\mathcal{H}}) - Gz^2(w - z)^2 = 0
\]

with \(\tilde{\mathcal{H}} \equiv \mathcal{H} + c_1\). One can find out that this algebraic curve is in fact elliptic.

#### 2.6.2. Equation of the form (3) in the case \((h, n) = (1, 2), k = 1, s = 3\).

The Lax pair in this case is given by

\[
z\psi_{i+3} = z\psi_{i+2} + T(i + 2)\psi_i, \quad w\psi_{i+4} = z\psi_{i+2} - T^3(i - 1)\psi_i.
\]

The ordinary difference equation (3) in this case looks like the fifth-order one:

\[
\begin{align*}
T(i + 2)(T(i) + T(i + 1) + T(i + 2) - \mathcal{H}_1) \\
&= T(i + 3)(T(i + 3) + T(i + 4) + T(i + 5) - \mathcal{H}_1).
\end{align*}
\]

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It is more convenient to present the first integrals in terms of initial data \(\{y_i\}\).
The Lax matrix $L$, in terms of the initial data, looks, in this case, like

$$ L = \begin{pmatrix} \frac{z}{w} \left( n + \frac{y_2 + y_3 - H_i}{w} \right) & \frac{z}{w} & \frac{z}{w} \\ y_2 - w & -y_0 - y_1 - y_2 + H_1 & -y_1 - y_2 - y_3 + H_1 \\ y_0 + y_1 + y_2 - H_1 - w & -y_1 - y_2 - y_3 + H_1 & -y_0 - y_1 - y_2 + H_1 \end{pmatrix}. $$

From the latter, we obtain the following first integrals for this equation

\begin{align*}
\mathcal{H}_0 &= y_2 (y_0 + y_1 + y_2 + y_3 - H_1)(y_1 + y_2 + y_3 + y_4 - H_1), \\
\mathcal{H}_1 &= -y_2 (y_0 + y_1 + y_2) - y_3 (y_1 + y_2) - y_4 y_2 + y_2 H_1, \\
\mathcal{H}_0 \bar{H}_1 &= -y_2 (y_0 + y_1 + y_2 - H_1)(y_1 + y_2 + y_3 - H_1)(y_2 + y_3 + y_4 - H_1).
\end{align*}

A spectral curve in this case is given by

$$ zw^3 - z(z + H_1)w^2 - H_2 zw - \mathcal{H}_0(z + \bar{H}_1) = 0. $$

This curve is hyper-elliptic of genus 2. Performing the birational transformation

$$ w = Z, \quad z = \frac{1}{2} \frac{Z^3 - H_1 Z^2 - H_2 Z - \mathcal{H}_0 - W}{Z^2}, $$

we get

$$ W^2 = \left( Z^3 - H_1 Z^2 - H_2 Z - \mathcal{H}_0 \right)^2 - 4 \mathcal{H}_0 \bar{H}_1 Z^2. $$

3. Discrete polynomials in the case $(h, n) = (1, 1)$

In this section we list a number of identities for discrete polynomials $T_i^r(i)$ and $S_i^r(i)$ in the case $(h, n) = (1, 1)$ that may be needed in the following. It should be noted that polynomials $T_i^r(i)$ in this particular case first appeared in [3].

In addition, we introduce two associated classes of discrete polynomials $\{P_i^r(i)\}$ and $\{Q_i^r(i)\}$ which are necessary for the construction of the first integrals for corresponding ordinary difference equations.

Clearly, identities (6) and (7) are specified in this case as

$$ T_i^r(i) = T_i^{r+1}(i + 1) + T(i) T_i^{r-1}(i + 2) $$

$$ = T_i^{r-1}(i) + T(i + s + r - 2) T_i^{r-1}(i), \quad \text{(23)} $$

while (10) and (11) become

$$ S_i^r(i) = S_i^{r+1}(i + 1) + T(i + r - 1) S_i^{r-1}(i) $$

$$ = S_i^{r-1}(i) + T(i + s - 1) S_i^{r-1}(i + 1). \quad \text{(25)} $$

In what follows we will use identity (12) which in the case $n = 1$ is specified as

$$ (S, T_i^r(i) := \sum_{j=0}^{r} (-1)^j S_i^{r-j}(i) T_i^{j}(i + r - j) = 0. \quad \text{(27)} $$
3.1. Two associated classes of discrete polynomials \( \{ P'_s(i) \} \) and \( \{ Q'_i(i) \} \)

Here we define two classes of discrete polynomials which as it turns out are necessary ingredients for the construction of the first integrals of ordinary difference equations under consideration.

We first define a class of homogeneous discrete polynomials \( \{ P_i(i) \} \). To this aim, we consider the partition \( D_{r,s} = D_r \sqcup D_s \) with

\[
D_{r,s} := \left\{ \lambda_j : \lambda_1 = 0, \ 1 \leq \lambda_2 < \cdots < \lambda_{r-1} \leq s - 2, \ \lambda_r = s - 1 \right\}
\]

and

\[
D_{r,s} := \left\{ \lambda_j : 1 \leq \lambda_1 < \cdots < \lambda_{r-1} \leq s - 2, \ \lambda_r = s - 1 \right\}
\]

(28)

Let us define a class of discrete polynomials \( \{ P'_s(i) \} \) by

\[
P'_s(i) = \sum_{\{\lambda_j\} \in D_{r,s}} T(i + \lambda_1)T(i + \lambda_2 + 1) \cdots T(i + \lambda_r + r - 1).
\]

It is important to notice the following. Since \( D_{r,s} = \emptyset \), for \( r < s \), hence

\[
P'_s(i) = 0, \ \text{for } s \leq r.
\]

(30)

We can easily find out the relationship between polynomials \( P'_s(i) \) and \( T'_s(i) \). Since

\[
\sum_{\{\lambda_j\} \in D_{r,s}} T(i + \lambda_1)T(i + \lambda_2 + 1) \cdots T(i + \lambda_r + r - 1)
\]

\[
= T(i)T(i + s + r - 2) \sum_{1 \leq \lambda_2 < \cdots < \lambda_{r-1} \leq s - 2} T(i + \lambda_2 + 1) \cdots T(i + \lambda_{r-1} + r - 2)
\]

\[
= T(i)T(i + s + r - 2)T^{r-2}_{s-2}(i + 2)
\]

then

\[
P'_s(i) = T'_s(i) - T(i)T(i + s + r - 2)T^{r-2}_{s-2}(i + 2).
\]

(31)

In turn, partitions (28) and (29) give the identities

\[
P'_s(i) = T(i + s + r - 2)T^{r-1}_{s-2}(i + 1) + T'_{r-1}(i)
\]

(32)

\[
= T(i)T^{r-1}_{s-2}(i + 1) + T'_{r-1}(i + 1),
\]

(33)

respectively. Suppose now that \( T(i + s + r - 1) = T(i) \). Taking into account (32) and (33), we obtain that by virtue of this periodicity condition

\[
P'_s(i + 1) = T(i + s + r - 1)T^{r-1}_{s-2}(i + 2) + T'_{r-1}(i + 1)
\]

(34)

\[
= T(i)T^{r-1}_{s-2}(i + 2) + T'_{r-1}(i + 1)
\]

(35)

\[
= P'_s(i).
\]

The latter shows the meaning of the discrete polynomials \( P'_s(i) \). They are the well-known first integrals of the periodicity equation \( T(i+N) = T(i) \) (see, for example, [12]) with \( N = s + r - 1 \), while (32) and (33) give two expressions of these integrals via polynomials \( T'_s(i) \).

Now we define a class of discrete polynomials \( \{ Q'_i(i) \} \), but first let us prove the following lemma.
Lemma 2. Two set theoretic equalities

\[ \mathcal{B}_{r,s+1} \sqcup \{ \lambda_j : \lambda_j = -1, \ -1 \leq \lambda_2 + \cdots + \lambda_{r-1} \leq s, \ \lambda_r = s \} = \mathcal{B}_{r,s} \sqcup \{ \lambda_j : -1 \leq \lambda_1 \leq \cdots \leq \lambda_{r-1} \leq s, \ \lambda_r = s \} \]  

(36)

and

\[ \mathcal{B}_{r,s+1}^- \sqcup \{ \lambda_j : \lambda_j = -1, \ -1 \leq \lambda_2 \leq \cdots \leq \lambda_{r-1} \leq s, \ \lambda_r = s \} = \mathcal{B}_{r,s}^- \sqcup \{ \lambda_j : -1 \leq \lambda_2 \leq \cdots \leq \lambda_r \leq s \} \]  

(37)

are valid.

Let us notice that we denote \( \mathcal{B}_{r,s+1} = \{ \lambda_j : -1 \leq \lambda_1 \leq \cdots \leq \lambda_r \leq s-1 \} \) in (37).

From (36) it follows that

\[ S_{r+1}'(i) + T(i + r - 2)T(i + s) \sum_{-1 \leq \lambda_2 \leq \cdots \leq \lambda_{r-1} \leq r} T(i + \lambda_2 + r - 2) \cdots T(i + \lambda_{r-1} + 1) = S_r'(i) + T(i + s) \sum_{\lambda_j \in \mathcal{B}_{r,s+1}^-} T(i + \lambda_1 + r - 1) \cdots T(i + \lambda_{r-1} + 1). \]

Observe that the latter can be rewritten as the identity

\[ S_{r+1}'(i) = Q_r'(i) + T(i + s)S_{r+2}'^{-1}(i), \]  

(38)

where, by definition,

\[ Q_r'(i) = S_r'(i) - T(i + r - 2)T(i + s)S_{r+2}'^{-2}(i). \]  

(39)

In a similar way we derive from (37) the identity

\[ S_{r+1}'(i) = Q_r'(i + 1) + T(i + r - 1)S_{r+2}'^{-1}(i). \]  

(40)

Some remarks are in order. Relations (31) and (39) yield two mappings \( \psi : \{T_r'(i)\} \rightarrow \{P_r'(i)\} \) and \( \varphi : \{S_r'(i)\} \rightarrow \{Q_r'(i)\} \). It can be checked that \( \psi \circ \varphi = \varphi \circ \psi \) and inverse mappings exist. It should be also noted that, in general, there is no problem to determine the \((h, n)\)-classes of polynomials \( \{P_r'(i)\} \) and \( \{Q_r'(i)\} \) together with identities similar to (38), (39) and (40), but at the moment we do not see the need for this.

3.2. Non-homogeneous polynomials

Clearly, linear combinations (8) and (9) in the case \((h, n) = (1, 1)\) become

\[ T_r'(i) = \sum_{j=0}^{r} c_j T_{r+j}'(i) \]

(41)

and

\[ S_r'(i) = \begin{cases} \sum_{j=0}^{r} (-1)^j H_j S_{r+j}'(i), & \text{for } s > r \\ \sum_{j=0}^{s-1} (-1)^j H_j S_{r+j}'(i), & \text{for } s \leq r. \end{cases} \]  

(42)

As for non-homogeneous polynomials \( P_r'(i) \) and \( Q_r'(i) \), they are defined by (41) and (42), respectively, replacing \( T \rightarrow P \) and \( S \rightarrow Q \).
3.3. Factorization property for polynomials $S^r_s(i)$

Polynomials $S^r_s(i)$ also have the following property which will be used later on.

**Lemma 3.** Polynomials $S^r_s(i)$ satisfy the factorization identity

$$S^r_s(i) = S^r_{s+1}(i) \prod_{q=s-1}^{r-1} T(i + q)$$

for all $s \leq r$.

Observe the following. For $s \leq r$, taking into account lemma 3, we get

$$S^r_s(i) = \sum_{j=0}^{s-1} (-1)^j H_j S^r_{s-j}(i + j)$$

$$= \left( \sum_{j=0}^{s-1} (-1)^j H_j S^r_{s-j}(i + j) \right) \prod_{q=s-1}^{r-1} T(i + q)$$

$$= S^r_{s+1}(i) \prod_{q=s-1}^{r-1} T(i + q).$$

\[ (43) \]

3.4. New binary operation on the sets of discrete polynomials

In addition to the operation $(\cdot, \cdot)$ given by (27), we introduce a new one which will be needed in the following. We let $\{F^r_s(i)\}$ and $\{G^r_s(i)\}$ be two arbitrary sets of discrete polynomials, such that $F^0_s(i) = G^0_s(i) \equiv 1$. Then, by definition,

$$\{F, G\}_r^s(i) = \sum_{j=0}^{r} (-1)^j F^r_{s+j}(i) G^r_{s-r-j}(i + r - j).$$

Note that in this formula we put $F^r_s(i) \equiv 0$ for $s \leq 0$. For example, $\{F, G\}_r^r(i) = F^r_r(i) + (-1)^r G^r_r(i)$. Let

$$F^r_s(i) = \sum_{j=0}^{r} (-1)^j H_j F^r_{s+j}(i + j) \quad \text{and} \quad G^r_s(i) = \sum_{j=0}^{r} c_j G^r_{s+j}(i).$$

As can be checked by direct inspection, one has the following properties of this binary operation:

$$\{F, G\}_r^r(i) = \sum_{j=0}^{r-1} (-1)^j c_j \{F, G\}_{s+j}^r(i) + (-1)^r c_r$$

\[ (45) \]

and

$$\{F, G\}_r^r(i) = \sum_{j=0}^{r-1} (-1)^j H_j \{F, G\}_{s+j}^r(i + j) + (-1)^r H_r.$$  

We also observe that by virtue of (5)

$$\{F, T\}_r^s(i) = F^r_s(i), \quad \forall s \leq r - 1$$

\[ (47) \]
Conjecture 1. The relations

\[ \langle S, T^i_s \rangle (i) = \sum_{j=0}^{r-2} (-1)^j T(i + r - j - 2)T(i + s + j) \]
\[ \times S_{r+j+s}^{r-j-2} (i) T^j_s (i + r - j), \quad \forall \, s \geq r - 1 \]  

are identities.

Unfortunately, at the moment we can prove this conjecture only for \( r = 1, 2, 3 \). These proofs can be found in the appendix.

It is remarkable that by virtue of (31) and (39), the identities of the form (48) are equivalent to the following:

\[ \langle S, P^i_s \rangle (i) = 0, \quad \forall \, s \geq r - 1 \] 

and

\[ \langle Q, T^i_s \rangle (i) = 0, \quad \forall \, s \geq r - 1. \] 

Moreover, from (45), (46), (49) and (50), it follows that

\[ \langle S, P^i_s \rangle (i) = (-1)^r H_r, \quad \forall \, s \geq r - 1 \] 

and

\[ \langle Q, T^i_s \rangle (i) = (-1)^r c_r, \quad \forall \, s \geq r - 1. \]

4. Ordinary difference equations in the case \((h, n) = (1, 1)\)

Clearly, in the case \((h, n) = (1, 1)\), equations of the form (2) and (3) are specified as

\[ T(i + k + s) \tilde{T}^k_s (i) = T(i) \tilde{T}^k_s (i + 2) \] 

and

\[ T(i + k) \tilde{S}^k_s (i) = T(i + s) \tilde{S}^k_s (i + 2), \] 

respectively, while the equivalents (15) and (16) become

\[ T(i + k + s) \tilde{T}^k_{r-1} (i + 1) = T(i) \tilde{T}^k_{r-1} (i + 2) \] 

and

\[ T(i + k) \tilde{S}^k_{r+1} (i) = T(i + s) \tilde{S}^k_{r+1} (i + 1), \]

respectively. These equations are the main objects to be studied throughout the rest of the paper.

We observe the following. Due to the factorization property (44), for \( s \leq k \), equation (54) can be rewritten as

\[ T(i + s - 1) \tilde{S}^{s-1}_{k+1} (i) = T(i + k + 1) \tilde{S}^{s-1}_{k+1} (i + 2). \]

This means that the \( s \leq k (k, s) \)th equation (54) is equivalent to the \((s - 1, k + 1)\)th equation (54) and therefore, without loss of generality, it is enough to consider this equation only for \( s \geq k + 1 \).
4.1. Lax pairs and integrals

Here we discuss equations (53) and (54) from the point of view of their Lax pair representation [8].

4.1.1. Lax pair for equation (54). A Lax pair for nonlinear difference equation (54) is given by two linear equations:

\[ z\psi_{i+2} = z\psi_{i+1} + T(i + 1)\psi_i \]  
(57)

and

\[ w\psi_{i+k-j} = \sum_{j=0}^{k} (-1)^j z^{k-j} S^j_i (i + k - s - j + 1)\psi_{i+k-j} \]  
(58)

which are a specification of (17) and (19) in the case \((h, n) = (1, 1)\). In what follows, we distinguish two cases: \(s + k = 2g + 1\) and \(s + k = 2g + 2\), where \(g \geq k\).

Let \(\Psi := (\psi_i^{(0)}, \ldots, \psi_i^{(s-1)})^T\). One sees that the pair of linear equations (57) and (58) is equivalent to two linear systems

\[ L_i \Psi_i = 0, \quad \Psi_{i+1} = A_i \Psi_i, \]  
(59)

the first one of which is given exactly by

\[ T(i)\psi_{i+1} + z\psi_i^{(0)} - z\psi_i^{(1)} = 0, \]
\[ T(i + 1)\psi_i^{(0)} + z\psi_i^{(1)} - z\psi_i^{(2)} = 0, \ldots \]
\[ T(i + s - 2)\psi_i^{(s-3)} + z\psi_i^{(s-2)} - z\psi_i^{(s-1)} = 0, \]
\[ T(i + s - 1)\psi_i^{(s-2)} + z\psi_i^{(s-1)} - z\psi_{i+s} = 0, \]

while the second one is given by

\[ \psi_{i+1}^{(1)} = \psi_i^{(1)}, \ldots, \psi_i^{(s-2)} = \psi_i^{(s-1)}, \psi_i^{(s-1)} = \psi_i^{(s-1)}, \]

where

\[ \psi_{i-1} = \frac{1}{w} \sum_{j=0}^{k} (-1)^j z^{k-j} S^j_i (i - j)\psi_i^{(s-j-1)} \]

and

\[ \psi_{i+s} = \frac{w}{z^k} \psi_i^{(0)} - \sum_{j=1}^{k} (-1)^j z^{-j} S^j_i (i - j + 1)\psi_i^{(s-j)}. \]

Checking the condition \(\det(L) = 0\) gives the following. In the case \(k + s = 2g + 1\), we get

\[ z^{2g-2k+4}w^2 - z^{s-k+1}R_k(z)w - H_0\tilde{R}_k(z) = 0, \]  
(60)

while in the case \(k + s = 2g + 2\), we obtain

\[ z^{2g-2k+6}w^2 - z^{s-k+1}R_k+1(z)w + H_0\tilde{R}_k(z) = 0, \]  
(61)

where

\[ R_k(z) := z^k + \sum_{j=1}^{k} H_{j} z^{s-j}, \quad \tilde{R}_k(z) := z^k + \sum_{j=1}^{k} \tilde{c}_j z^{k-j} \]  
(62)

and \(\tilde{c}_r := \sum_{j=0}^{r} c_{j} H_{r-j}\). Here, \(\{H_1, \ldots, H_k\}\) is the set of parameters entering equation (54), while \(H_0^{(k,s)} := H_0\) and \(c_r^{(k,s)} := c_r\) for \(r = 1, \ldots, k\) and \(H_r^{(k,s)} := H_r\), for \(r \geq k + 1\), are...
supposed to be nontrivial first integrals of the \((k, s)\)th equation \((54)\). In particular,
\[
\mathcal{H}^{(k,s)}_0(i) = (-1)^k S_{k+1}^{k}(i) \prod_{q=k}^{s-1} T(i + q).
\]
In section 5, we describe these integrals regardless of Lax representation \((59)\).

4.1.2. Lax pair for equation \((53)\). Now let us consider a Lax pair completed by linear equations \((57)\) and
\[
w_i \psi_{i+k+s} = \sum_{j=0}^{k} z^{k-j} T_{s}^{j} (i + k - j + 1) \psi_{i+k-j} \quad (63)
\]
which is a specification of \((18)\). Let \(\Psi_i = (\psi_i^{(0)}, \ldots, \psi_i^{(k+s-1)})^T\). We rewrite the two linear equations \((57)\) and \((63)\) in the form \((59)\) or more exactly as a system
\[
T(i) \psi_{i-1} + z \psi_{i}^{(0)} - z \psi_{i}^{(1)} = 0,
T(i + 1) \psi_{i}^{(0)} + z \psi_{i}^{(1)} - z \psi_{i}^{(2)} = 0, \ldots
T(i + k + s - 2) \psi_{i}^{(k+s-3)} + z \psi_{i}^{(k+s-2)} - z \psi_{i}^{(k+s-1)} = 0,
T(i + k + s - 1) \psi_{i}^{(k+s-2)} + z \psi_{i}^{(k+s-1)} - z \psi_{i+k+s} = 0,
\]
and
\[
\psi_{i+1}^{(0)} = \psi_{i}^{(1)}, \ldots, \psi_{i+1}^{(k+s-2)} = \psi_{i}^{(k+s-1)} \Rightarrow \psi_{i+1}^{(k+s-1)} = \psi_{i+k+s}^{(0)}
\]
with
\[
\psi_{i-1} = \frac{1}{T_{k}^{s}(i)} \left( w \psi_{i}^{(k+s-1)} - \sum_{j=0}^{k-1} z^{k-j} T_{s}^{j} (i + k - j) \psi_{i}^{(k-j)} \right)
\]
and
\[
\psi_{i+k+s} = \frac{1}{w} \sum_{j=0}^{k} z^{k-j} T_{s}^{j} (i + k - j + 1) \psi_{i}^{(k-j)}.
\]
Actual calculations give the following. In the case \(k + s = 2g + 1\), we get
\[
\mathcal{H}_0 w^2 + z^{g+1} R_g(z) w - z^{2g+1} \tilde{R}_k(z) = 0, \quad (64)
\]
while in the case \(k + s = 2g + 2\), we obtain
\[
\mathcal{H}_0 w^2 - z^{g+1} R_{g+1}(z) w + z^{2g+2} \tilde{R}_k(z) = 0, \quad (65)
\]
where \(R_g(z)\) and \(\tilde{R}_k(z)\) are as in \((62)\), but \(\{\mathcal{H}_0, H_0, \ldots, H_g\}\) in this case is the set of the first integrals depending on \(c_j\) being the parameters contained in difference equation \((53)\). The curves given by equations \((64)\) and \((65)\) are hyper-elliptic ones. Indeed, applying the birational transformation
\[
z = \frac{1}{Z}, \quad w = \frac{W - r_{g}(Z)}{2 \mathcal{H}_0 Z^{2g+1}}
\]
to equation \((64)\), we obtain
\[
W^2 = r_{g}^2(Z) + 4 \mathcal{H}_0 Z^{2g-k+1} \tilde{R}_k(Z),
\]
while transforming (65) with the help of
\[ z = Z, \quad w = Z^{g+1}W + R_{g+1}(Z) \]
yields
\[ W^2 = R^2_{g+1}(Z) - 4\mathcal{H}_0\mathcal{R}_k(Z). \]

Also, it is important to notice that using the transformation
\[ w \mapsto z^{g+1} \left( z^{-k}w - R_k(z) \right) \]
for (64) gives (60), while the transformation
\[ w \mapsto z^{g+1} \left( R_{g+1}(z) - z^{-k+1}w \right) \]
converts (65) to (61). This suggests that associated difference equations have to be closely related to each other and we discuss this question in the following.

5. First integrals

5.1. Integrals for equation (54)

Let us first consider an infinite class of equations of the form (54) for \( s \geq k + 1 \). We denote the solution space of (54) by \( \mathcal{N}_s^k \). Every solution of this equation is defined by a set of initial data \( \{y_j = T(i_0 + j) : j = 0, \ldots, k + s - 1\} \) and constants \( \{H_1, \ldots, H_k\} \). Thus, the dimension of \( \mathcal{N}_s^k \) is \( 2k + s \).

From the equivalent form of equation (54) given by (56), it is evident that the polynomial\(^5\)
\[ G_0^{(k, 1)}(i) = \tilde{S}_{k+1}(i) \prod_{q=k}^{s-1} T(i + q) \]
is its first integral. To construct some number of integrals for this equation, let us consider discrete polynomials of the form
\[ G_r^{(k)}(i) = \sum_{j=0}^{r} Q_{r-k-j-1}(i + k + 1)G_0^{(k+j-r-j)}(i). \]

Let us remember that \( Q_r(i) \) are the discrete polynomials defined by (39). Clearly, these polynomials satisfy recurrent relations
\[ G_r^{(k)}(i) = G_r^{(k)}(i) + Q_{r-k-r-2}(i + k + 1)G_0^{(k+r+1,s-r-1)}(i). \]

**Proposition 1.** The polynomial (66), for any fixed \( k, r \geq 0 \) and \( s \geq k + 2r + 1 \) is the first integral of the \((k + r, s - r)\)th equation (54). Moreover, the relation

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\(^5\) Equations of the form \( G_0 = \tilde{S}_{k+1}(i) \prod_{q=1}^{r-1} T(i + q) \) with arbitrary constant \( G_0 \) and \( H_1 = 0 \) have appeared and are studied in [2].
\[ G_{r}^{(k,s)}(i+1) - G_{r}^{(k,s)}(i) = \lambda_{r}^{(k,s)}(i) \left( T(i + s - r)S_{r-r+1}^{k+r}(i) - T(i + k + r)S_{r-r+1}^{k+r}(i) \right) \]

(68)

\[ = \lambda_{r}^{(k,s)}(i) \left( S_{r-r+1}^{k+r+1}(i) + 1 - S_{r-r}^{k+r+1}(i) \right) \]

(69)

with

\[ \lambda_{r}^{(k,s)}(i) := S_{r-k-r}(i + k + 1) \prod_{q=k+r+1}^{x-r-1} T(i + q) \]

being a corresponding integrating factor, is valid.

Therefore \{\( G_{r}^{(k,s)}(i) \)\} is a multitude of the first integrals for the \((k, s)\)th equation (54). We easily deduce the system

\[ G^{(k,s)}(i) := Q_{s-r}(i + k)G^{(k,s)}(i) \]

(70)

with

\[ G^{(k,s)}(i) := \left( G_{0}^{(k,s)}(i), G_{1}^{(k,s)}(i), \ldots, G_{k}^{(k,s)}(i) \right)^{T} \]

and

\[ G^{(k,s)}(i) := \left( G_{0}^{(k,s)}(i), G_{0}^{(k-1,s+1)}(i), \ldots, G_{0}^{(0,s+k)}(i) \right)^{T} \]

which follows from (66). The matrix \( Q(i) \) in (70) is supposed to be a \((k + 1) \times (k + 1)\) matrix with elements \((Q_{i}(i))_{pq} = Q_{i+p+q-1}(i - p + 2)\).

From (68) and (69), it is obvious that

\[ G_{r}^{(k,s)}(i+1) - G_{r}^{(k,s)}(i) = \lambda_{r}^{(k-r,s+r)}(i) \left( T(i + s)S_{r+1}^{k}(i) + 1 - T(i + k)S_{r+1}^{k}(i) \right) \]

\[ = \lambda_{r}^{(k-r,s+r)}(i) \left( S_{r+1}^{k+1}(i) + 1 - S_{r}^{k+1}(i) \right). \]

In what follows, it is also useful to consider the following linear combinations of integrals:

\[ \tilde{G}_{r}^{(k,s)}(i) := \sum_{q=0}^{r} (-1)^{q}H_{q}G_{r-q}^{(k,s)}(i), \quad r = 0, \ldots, k. \]

(71)

One can see that \( \tilde{G}_{r}^{(k,s)}(i) = G_{r}^{(k-r,s+r)}(i) \) with

\[ \tilde{G}_{r}^{(k,s)}(i) := \sum_{j=0}^{r} \tilde{G}_{r-k-j+1}(i + k + 1)G_{0}^{(j,s+j)}(i) \]

and

\[ \tilde{G}_{r}^{(k,s)}(i+1) - \tilde{G}_{r}^{(k,s)}(i) = \lambda_{r}^{(k,s)}(i) \left( T(i + s)S_{r+1}^{k}(i) + 1 - T(i + k)S_{r+1}^{k}(i) \right) \]

\[ = \lambda_{r}^{(k,s)}(i) \left( S_{r+1}^{k+1}(i) + 1 - S_{r}^{k+1}(i) \right) \]

with \( \lambda_{r}^{(k,s)}(i) = G_{0}^{(r,s-k+r-1)}(i + k - r + 1) \).
Let us consider
\[ G^{(0,1)}(i) = \sum_{j=0}^{r} \tilde{Q}_j(i) \tilde{S}_j(i+1) \prod_{q=j}^{r-j-1} T(i+q). \] (72)
Using the identity of the form (25), namely,
\[ \tilde{S}_j(i+1) = \tilde{S}_{j-1}(i+1) + T(i+j) \tilde{S}_j(i), \]
let us present (72) as
\[ G^{(0,1)}(i) = \sum_{j=0}^{r} g_j(i) + \sum_{j=1}^{r} f_j(i) \]
with
\[ g_j(i) = \tilde{Q}_j(i) \tilde{S}_j(i+1) \prod_{q=j}^{r-j-1} T(i+q) \]
and
\[ f_j(i) = \tilde{Q}_j(i) \tilde{S}_{j-1}(i+1) \prod_{q=j-1}^{r-j-2} T(i+q). \]
In what follows, we need the following lemma.

**Lemma 4.** The relations
\[ \sum_{j=0}^{r} g_j(i) + \sum_{j=1}^{r+1} f_j(i) = \tilde{S}_{r+1}(i) \prod_{q=r}^{s-r-1} T(i+q) \] (73)
and
\[ \sum_{q=0}^{s} g_q(i) + \sum_{q=1}^{s+1} f_q(i) = \tilde{S}_{s+1}(i) \prod_{q=s}^{r-s-1} T(i+q) \] (74)
are identities.

Therefore by virtue of this lemma,
\[ G^{(k,s)}(i) = G^{(0,r+k)}(i) = \tilde{S}_r(i) \tilde{S}_s(i+1) \prod_{q=k}^{s-1} T(i+q). \] (75)
Here we have replaced \( r \rightarrow k \) and \( s - r \rightarrow s \). It should be noted that we in fact obtained the expected result since (75) is an obvious first integral of the \((k, s)\)th equation (54).

**5.2. Chains of inclusions of \( N^k_s \)**

Suppose that, for some \( k \geq 1 \) and \( s \geq k + 1 \), the condition
\[ \tilde{S}_k(i) = 0 \] (76)
is fulfilled. It can easily be seen that
\[ G^{(k,s)}(i) = G^{(k-1, s+1)}(i), \quad r = 1, \ldots, k. \]
This means that under this condition the set of integrals \( \{G^{(k,i)}_r : r = 0, \ldots, k\} \) for the \((k, s)\)th equation \((56)\) is reduced to the set \( \{G^{(k-1,i+1)}_r : r = 0, \ldots, k - 1\} \). This analysis of the first integrals suggests that it is possible that \( \mathcal{N}^k_{s+1} \subset \mathcal{N}^k_s \). To clarify this question, let us first consider the simplest case \( k = 1 \). One can rewrite the \((1, s)\)th equation \((56)\) as

\[
T(i + s + 1) = \frac{T(i + 1)}{T(i + s)} \left( \sum_{j=1}^{s+1} T(i + j) - H_i \right) - \sum_{j=1}^s T(i + j) + H_i. \tag{77}
\]

Suppose now that

\[
\bar{S}^1_{s+1}(i) := \sum_{j=1}^{s+1} T(i + j) - H_i = 0. \tag{78}
\]

By direct inspection, one can check that this constraint is compatible with \((77)\). Indeed, by virtue of \((77)\), one has

\[
\bar{S}^1_{s+1}(i + 1) = \frac{T(i + 1)}{T(i + s)} \bar{S}^1_{s+1}(i).
\]

In turn, equation \((78)\) is equivalent to the \((0, s + 1)\)th equation \((56)\), which is a periodicity one: \(T(i + s + 1) = T(i)\).

Now let us turn to the general situation. Rewriting the \((k, s)\)th equation \((56)\) as

\[
\bar{S}^k_{s+1}(i + 1) = \frac{T(i + k)}{T(i + s)} \bar{S}^k_{s+1}(i) = 0 \tag{79}
\]

we immediately see that condition \((76)\) is evidently compatible with this equation. Moreover we observe that, taking into account identities \((25)\) and \((26)\), we can rewrite this restriction as the \((k - 1, s + 1)\)th equation \((56)\); that is,

\[
T(i + k - 1) \bar{S}^{k-1}_{s+2}(i) = T(i + s + 1) \bar{S}^{k-1}_{s+2}(i + 1). \tag{80}
\]

For any solution of this equation corresponding to some fixed initial data \( \{y_j : j = 0, \ldots, k + s - 1\} \) and a set of the parameters \( \{H_1, \ldots, H_{k-1}\} \) we calculate the constant \( H_k \) through the condition \((76)\), where one substitute \( T(i + j) = y_j \) for \( j = 0, \ldots, k + s - 1 \). For example, in the case \( k = 1 \) we have to put \( H_1 = \sum_{j=0} y_j \). Then we can assert that this solution of \((80)\) also solves the \((k, s)\)th equation \((56)\) with the same initial data \( \{y_j : j = 0, \ldots, k + s - 1\} \) and a set of parameters \( \{H_1, \ldots, H_k\} \). Therefore, we conclude that \( \mathcal{N}^{k+1}_{s+1} \subset \mathcal{N}^k_s \).

Given some \( g \geq k \), there are two different cases to consider: \( s + k = 2g + 1 \) and \( s + k = 2g + 2 \). For these two cases we have two chains of inclusions

\[
\mathcal{N}^g_{2g+1} \subset \mathcal{N}^g_{2g} \subset \cdots \subset \mathcal{N}^g_{g+1}
\]

and

\[
\mathcal{N}^g_{2g+2} \subset \mathcal{N}^g_{2g+1} \subset \cdots \subset \mathcal{N}^g_{g+2},
\]

respectively.

Consider now the solution space \( \mathcal{N}^g_{g+2} \) corresponding to the \((g, g + 2)\)th equation \((54)\); that is,

\[
T(i + g) \bar{S}^g_{g+2}(i) = T(i + g + 2) \bar{S}^g_{g+2}(i + 2)
\]
and then observe that this equation is a consequence of the relation
\[ T(i + g)\tilde{S}_{g+2}^g(i) = T(i + g + 1)\tilde{S}_{g+2}^g(i + 1) \]  
which is the \((g, g + 1)\)th equation (56). This means that \(\mathcal{N}_g^{g+1} \subset \mathcal{N}_g^{g+2}\).

Now we are in a position to formulate the following theorem.

**Theorem 1.** All the solution spaces \(\mathcal{N}^k_g\) are organized in the following diagram of inclusions:

\[
\begin{align*}
\mathcal{N}_1^0 & \subset \mathcal{N}_2^0 \mathcal{N}_3^0 \quad \mathcal{N}_4^0 \mathcal{N}_5^0 \quad \mathcal{N}_6^0 \mathcal{N}_7^0 \quad \mathcal{N}_8^0 \cdots \\
\mathcal{N}_1^1 & \subset \mathcal{N}_2^1 \mathcal{N}_3^1 \quad \mathcal{N}_4^1 \mathcal{N}_5^1 \quad \mathcal{N}_6^1 \mathcal{N}_7^1 \cdots \\
\mathcal{N}_1^2 & \subset \mathcal{N}_2^2 \mathcal{N}_3^2 \quad \mathcal{N}_4^2 \mathcal{N}_5^2 \quad \mathcal{N}_6^2 \mathcal{N}_7^2 \cdots \\
\mathcal{N}_1^3 & \subset \mathcal{N}_2^3 \cdots
\end{align*}
\]

Let us remark that from this theorem it follows that every equation (54) admits periodic solutions.

Let us consider now the set of integrals \(\{\mathcal{G}_{j}^{(g, g+2)} : j = 1, \ldots, g\}\) and investigate how they behave when restricting them on \(\mathcal{N}_1^g\). To this aim, the following technical lemma is useful.

**Lemma 5.** The relations
\[
\mathcal{G}^{(k-1,s)}_r(i) - \mathcal{G}^{(k-1,s+1)}_{r+1}(i) = S^k_{s-k-r}(i + k)\tilde{S}^{k+r}_{s-r}(i + 1) \prod_{q=k+r}^{s-r-1} T(i + q) - S^k_{s-k-r-1}(i + k)\tilde{S}^{k+r}_{s-r}(i) \prod_{q=k+r}^{s-r-2} T(i + q)
\]  
are identities.

Putting \(k = g - r\) and \(s = g + r + 2\) into (5) gives
\[
\mathcal{G}^{(g, g+2)}_r(i) - \mathcal{G}^{(g, g+1)}_{r+1}(i) = \mathcal{G}^{(g-r, g+r+2)}_r(i) - \mathcal{G}^{(g-r-1, g+r+2)}_{r+1}(i) = S^g_{r+2}(i + g - r)\tilde{S}^{g}_{g+2}(i + 1)T(i + g)T(i + g + 1) - S^g_{r+1}(i + g - r)\tilde{S}^{g}_{g+2}(i)T(i + g).
\]

Therefore, we obtain
\[
\mathcal{G}^{(g, g+2)}_r(i) \mid_{(\mathfrak{g}1)} = \mathcal{G}^{(g, g+1)}_{r+1}(i) = T(i + g)\tilde{S}^{g}_{g+2}(i)\left(T(i + g)S^g_{r+2}(i + g - r) - S^g_{r+1}(i + g - r)\right).
\]

In turn, by virtue of the factorization property (43), the latter is identically zero. Therefore we can conclude that when restricting \(\mathcal{N}^k_{g+2}\) on \(\mathcal{N}^k_{g+1}\) one has
\[
\mathcal{G}^{(g, g+2)}_r \mid_{(\mathfrak{g}1)} = \mathcal{G}^{(g, g+1)}_{r+1} \quad r = 0, \ldots, g - 1.
\]
Let us remark that this set of relations can also be written as
\[ G_r^{(g,g+2)} |_{(81)} = G_r^{(g+1)} + (-1)^r H_r G_0^{(g,g+1)}. \]

Finally, we want to find out what happens with the integral \( G_g^{(g,g+2)} \) under this restriction. It turns out more convenient to consider \( G_g^{(g,g+2)} \) instead of \( G_0^{(g,g+2)} \). Putting \( k = g \) and \( s = g + 2 \) into (75) yields the identity
\[ G_g^{(g,g+2)} = \tilde{S}^g_{g+2}(i) \tilde{S}^g_{g+2}(i + 1) T(i + g) T(i + g + 1). \]

Therefore we know that restricting of the integral \( G_g^{(g,g+2)} \) on \( G_0^{(g,g+2)} \) yields
\[ G_g^{(g,g+2)}(i) |_{(81)} = \left( \tilde{S}^g_{g+2}(i) T(i + g) \right)^2 = \left( G_0^{(g,g+2)}(i) \right)^2. \]

### 5.3. Equivalence of equations (53) and (54)

Consider a set of the first integrals
\[ c_r^{(k,s)}(i) = (-1)^r \frac{G_r^{(k,s)}(i)}{G_0^{(k,s)}(i)} \]
\[ = (-1)^r \sum_{j=0}^r Q_{i-k+2r-j-1}^{(j)}(i + k - r + 1) \xi_{r-j}^{(k,s)}(i), \quad r = 1, \ldots, k \] (83)

for equation (54). Here
\[ \xi_{r}^{(k,s)}(i) = \frac{G_0^{(k-r,s+r)}(i)}{G_0^{(k,s)}(i)} \]
\[ = \frac{\tilde{S}^{k-r}_{s+r}(i)}{\tilde{S}^{k}_{s+1}(i)} \prod_{q=1}^{r} T(i + k - q + 1) \prod_{q=1}^{r} T(i + s + q - 1). \]

Let \( c_r^{(k,s)} \) be values of the integrals given by (83) corresponding to the initial data \( \{y_0, \ldots, y_{s+k-1}\} \) and the set of parameters \( \{H_1, \ldots, H_k\} \). In what follows it makes sense to use simplified notation \( c_r^{(k,s)} \equiv c_r \). The system equations
\[ c_r = (-1)^r \sum_{j=0}^r Q_{i-k+2r-j-1}^{(j)}(i + k - r + 1) \xi_{r-j}^{(k,s)}(i), \quad r = 1, \ldots, k \] (84)
yield a relationship between the set of parameters \( \{H_r\} \) of the \((k, s)\)th equation (56) and its integrals \( \{c_r\} \).

The following question arises: what would happen if we interchange \( \{H_r\} \) and \( \{c_r\} \)? We claim the following: as a result, we obtain the \((k, s)\)th equation (55). Before discussing the general situation, let us first illustrate this in the simple example \( k = 1.6 \). In this case, system (84) is specified as a single equation
\[ c_1 = -Q_{i-1}^{(i+1)} - \frac{T(i)T(i + s)}{S_{i+1}^{1}}. \] (85)

6 It is obvious that in the case \( k = 0 \) both the equations (55) and (56) are periodicity equations: \( T(i + s) = T(i) \).
Since $Q_{i-1}^1(i) = T_{i-1}^1(i) := \sum_{j=1}^{s} T(i + j - 1)$, we can rewrite (85) as

$$S_{s+1}^1(i) = -\frac{T(i)T(i + s)}{T_{s-1}^1(i + 1)}$$

or

$$\xi_1^{(1,s)}(i) = -\tilde{T}_{s-1}^1(i + 1).$$

One can check that substituting the latter into the $(1, s)$th equation (56) gives the $(1, s)$th equation (55). Therefore we proved that in the case $k = 1$, equations (53) and (54), for any fixed $s \geq 2$, yield the same dynamical system. Given some value of $H_1$ and initial data $\{y_0, \ldots, y_s\}$ we calculate the corresponding parameter $c_1$ with the help of

$$c_1 = -\sum_{j=1}^{s-1} y_j - \frac{y_0 y_j}{\sum_{j=1}^{s} y_j + c_1} - H_1.$$

In turn, resolving the latter in favor of $H_1$ gives

$$H_1 = \sum_{j=0}^{s} y_j + \frac{y_0 y_j}{\sum_{j=1}^{s} y_j + c_1}.$$

Therefore in this case $c_1$ and $H_1$ are related to each other by birational transformation. Clearly, requiring $H_1 = \sum_{j=0}^{s} y_j$ yields $c_1 = \infty$.

Now let us turn to the general situation. We would like to show that the system of equations (84) is equivalent to the following:

$$\xi_r^{(k,s)}(i) = (-1)^r T_{s-k+r-1}^{r}(i + k - r + 1), \quad r = 1, \ldots, k. \tag{86}$$

Substituting (86) into (84) gives

$$c_r = \sum_{j=0}^{k} (-1)^j Q_{r-k+2r-j-1}^j(i + k - r + 1) T_{r-k+r-j-1}^{r}(i + k - r + j + 1)$$

$$= (-1)^r \langle Q, \tilde{T} \rangle_{s-k+r-1}^r(i + k - r + 1).$$

The latter is an identity provided that (52) holds.

Let us write down the last equation in system (86) as

$$(-1)^k T_{s-1}^k(i + 1) S_{s+1}^k(i) = \prod_{q=1}^{k} T(i + k - q) \prod_{q=1}^{k} T(i + s + q - 1) \tag{87}$$

and observe that replacing in (56) the polynomial $S_{s+1}^k(i)$ by $T_{s-1}^k(i)$ by virtue of (87) yields (55). Therefore we prove the equivalence of two discrete equations (55) and (56) or their equivalents (53) and (54) for any fixed $k \geq 0$ and $s \geq k + 1$ and show how parameters $c_r$ and $H_r$ are related to each other. Of course, it is valid provided that conjecture 1 holds.

Finally, we would like to resolve system (86) in favor of $H_r$. To this aim, we rewrite this system as

$$(-1)^r T_{s-k+r-1}^{r}(i + k - r + 1) S_{s+1}^k(i)$$

$$= \tilde{S}_{s+r+1}^r(i) \prod_{q=1}^{r} T(i + k - q) \prod_{q=1}^{r} T(i + s + q - 1), \quad r = 1, \ldots, k$$

and
and then observe that it is equivalent to the following system:

\[
(-1)^r\tilde{S}^r_{x+k-r+1}(i)\tilde{T}^k_{s-r-1}(i) = \tilde{T}^k_{s-r-1}(i) \prod_{q=1}^{r} T(i + q - 1) \prod_{q=1}^{r} T(i + s + k - q), \quad r = 1, \ldots, k
\]

which in turn we may rewrite as

\[
\eta_r^{(k,s)}(i) = (-1)^r\tilde{S}^r_{x+k-r+1}(i), \quad r = 1, \ldots, k
\]  

(88)

with

\[
\eta_r^{(k,s)}(i) := \frac{\tilde{T}^k_{s-r-1}(i + r + 1)}{\tilde{T}^k_{s-1}(i + 1)} \prod_{q=1}^{r} T(i + q - 1) \prod_{q=1}^{r} T(i + s + k - q).
\]  

(89)

We can solve (88) as

\[
H_r = \sum_{j=0}^{r} P^j_{s+k-r+1}(i + r - j)\eta_r^{(k,s)}(i).
\]  

(90)

Indeed, substituting (88) into (90) gives \(H_r = (-1)^r(\tilde{S}, P)^r_{x+k-r+1}(i)\). By virtue of (51), the latter is an identity. Note that relation (90) gives a number of integrals for the \((k, s)\)th equation (55).

5.4. Integrals for equation (55)

Inspired by (90) we consider the rational discrete function

\[
F_r^{(k,s)}(i) = \sum_{j=0}^{r} P^j_{s+k+j+1}(i - j)F_0^{(k+j,s+j)}(i - j),
\]  

(91)

where

\[
F_0^{(k,s)}(i) := \frac{\tilde{T}^k_{s+1}(i + 1)}{\prod_{q=0}^{k-1} T(i + q)}
\]

is an obvious integral of the \((k, s)\)th equation (55). It is important to notice that \(k\) in (91) is allowed to take negative values. In this case we put \(F_0^{(k,s)}(i) \equiv 0, \quad \forall r < 0\). Clearly, this rational function satisfies the recurrent relation

\[
F_{r+1}^{(k,s)}(i) = F_r^{(k,s)}(i) + P^{r+1}_{s+r+2}(i - r - 1)F_0^{(k+r+1,s+r+1)}(i - r - 1).
\]  

(92)

**Proposition 2.** The rational function (91) is a first integral of the \((k + r, s + r)\)th equation (55). Moreover the relation

\[
F_r^{(k,s)}(i + 1) - F_r^{(k,s)}(i) = \Delta_r^{(k,s)}(i) \left( T(i - r)\tilde{T}^{k+r}_{s+r-1}(i - r + 2) - T(i + k + s + r)\tilde{T}^{k+r}_{s+r-1}(i - r + 1) \right)
\]  

(93)

\[
= \Delta_r^{(k,s)}(i) \left( \tilde{T}^{k+r+1}_{s+r}(i - r) - \tilde{T}^{k+r+1}_{s+r}(i - r + 1) \right).
\]  

(94)
with an integrating factor

\[ \Delta_r^{(k,s)}(i) = \frac{T_{r+k+i}(i-r+1)}{\Pi_{s+k-r} T(i+q)} \]

is valid.

Therefore \( \{ F_r^{(k,s)}(i) = F_r^{(k-r,s-r)}(i+r) \} \) presents a multitude of first integrals for the \((k, s)\)th equation (55). Moreover, from (93) and (94), it follows that

\[ F_r^{(k,s)}(i+1) - F_r^{(k,s)}(i) = \Delta_r^{(k-r,s-r)}(i+r) \left( T(i) T_{s-r-1}^{-1}(i+2) - T(i+k+s) T_{s-r-1}^{k+r-1}(i+1) \right) \]

\[ = \Delta_r^{(k-r,s-r)}(i+r) \left( \tilde{T}_{s-r}^{k+r-1}(i) - \tilde{T}_{s-r}^{k+r-1}(i+1) \right). \]

Observe that due to (30),

\[ F_r^{(k,s)}(i) \equiv 0, \quad \forall 2r \geq s + k + 1. \quad (95) \]

So, taking into account proposition 2 and (95), in the case \( s + k = 2g + 1, \ g \geq k, \) we can describe the first integrals for the \((k, s)\)th equation (55) in the following way. Let \( F^{(k,s)}(i) := (F_0^{(k,s)}, \ldots, F_g^{(k,s)})^T \) and

\[ F^{(k,s)}(i) := \left( F_0^{(k,s)}(i), F_0^{(k-1,s-1)}(i+1), \ldots, F_0^{(0,s)}(i+k), 0, \ldots, 0 \right)^T \quad (96) \]

be \((g+1)\)-dimensional vectors and

\[ \mathcal{P}_{s+k}(i) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ P_{s+k-1}^{1}(i) & 1 & 0 & \cdots & 0 \\ P_{s+k-2}^{1}(i) & P_{s+k-2}^{1}(i+1) & 1 & \cdots & 0 \\ P_{s+k-3}^{2}(i) & P_{s+k-3}^{2}(i+1) & P_{s+k-3}^{1}(i+2) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\ P_{s+k-g-1}^{g}(i) & P_{s+k-g-1}^{g-1}(i+1) & P_{s+k-g-1}^{g-2}(i+2) & \cdots & 1 \end{pmatrix} \]

Then the first integrals are given by the system

\[ F^{(k,s)}(i) = \mathcal{P}_{s+k}(i) F^{(k,s)}(i). \quad (97) \]

Respectively, in the case \( s + k = 2g + 2, \ g \geq k, \) we take \((g+2)\)-dimensional vectors:

\[ F^{(k,s)}(i) := (F_0^{(k,s)}, \ldots, F_{g+1}^{(k,s)})^T \quad \text{and} \quad F^{(k,s)}(i) \text{ given by } (96). \]

Then the first integrals are given by system (97) with the corresponding matrix

\[ \mathcal{P}_{s+k}(i) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ P_{s+k-1}^{1}(i) & 1 & 0 & \cdots & 0 \\ P_{s+k-2}^{2}(i) & P_{s+k-2}^{2}(i+1) & 1 & \cdots & 0 \\ P_{s+k-3}^{3}(i) & P_{s+k-3}^{3}(i+1) & P_{s+k-3}^{1}(i+2) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\ P_{s+k-g}^{g+1}(i) & P_{s+k-g}^{g}(i+1) & P_{s+k-g}^{g-1}(i+2) & \cdots & 1 \end{pmatrix} \quad (98) \]
Let us define now the first integrals $H^{(k,s)}_i(i) = F^{(k,s)}_r(i)/F^{(k,s)}_0(i)$. Clearly, these integrals are described by the system $H^{(k,s)}_r(i) = H^{(k,s)}(i) = \Theta_{i+k}(i)$, where $\Theta^{(k,s)}(i) = F_0^{(k-r,s-r)}(i + r)/F_0^{(k,s)}(i)$ is given in more explicit form by (89). Notice that making use of relationship (88) we construct a number of the first integrals for the $(k, s)$th equation (56). As mentioned above, substituting (88) into (90), for $r = 1, \ldots, k$, produces identities; that is, $H^{(k,s)}_r(i) = H_r, \quad r = 1, \ldots, k$, while doing that for $r \geq k + 1$ gives nontrivial polynomial integrals $H^{(k,s)}_{r+1}(i)$ in the case $s + k = 2g + 1$ and $H^{(k,s)}_{r+1}(i)$ in the case $s + k = 2g + 2$, respectively. It should be noted that we can represent these integrals through the relation

$$R_k(z) = z^g + \sum_{j=1}^{k} H^{(k,s)}_j(i)z^{s-j}$$

$$= \sum_{q=0}^{g+1} (-1)^q S^{g+1}_{k+q-1}(i) \left( z^{s-q} + \sum_{j=1}^{g+1} P_{s+k-2q-j+1}(i + q)z^{s-q-j} \right)$$

in the case $s + k = 2g + 1$ and

$$R_{g+1}(z) = z^{g+1} + \sum_{j=1}^{g+1} H^{(k,s)}_j(i)z^{s-j+1}$$

$$= \sum_{q=0}^{g+1} (-1)^q S^{g+1}_{k+q-1}(i) \left( z^{s-q+1} + \sum_{j=1}^{g+1} P_{s+k-2q-j+1}(i + q)z^{s-q-j+1} \right)$$

in the case $s + k = 2g + 2$, respectively. This representation makes our construction of the first integrals closer to the approach using Lax representation which we have expounded in section 4.

5.5. Examples

5.5.1. Equation (56) in the case $k = 1$ and $s = 3$. In this case equation (56) is

$$T(i + 1)(T(i) + T(i + 1) + T(i + 2) - H)$$

$$= T(i + 3)(T(i + 2) + T(i + 3) + T(i + 4) - H)$$

which generates a map

$$(y_0, y_1, y_2, y_3) \mapsto \left( y_1, y_2, y_3, \frac{y_1}{y_3}(y_0 + y_1 + y_2 - H) - y_2 - y_3 + H \right).$$

Polynomial integrals for this map are

$$G^{(1,3)}_0 = S^{1}_{4}y_1y_2 = \left( y_0 + y_1 + y_2 + y_3 - H \right)y_1y_2,$$

$$G^{(1,3)}_1 = G^{(0,4)}_0 + Q^{1,1}_2C^{(1,3)}_0 = y_0y_1y_2y_3 + (y_1 + y_2)(y_0 + y_1 + y_2 + y_3 - H)y_1y_2,$$

$$H^{(1,3)}_2 = P_3^{1,1}S^{1}_{4} = y_0y_2 + y_1y_3 - (y_1 + y_2)(y_0 + y_1 + y_2 + y_3 - H).$$

Here discrete polynomials are replaced by multi-variate ones via the replacement $T(i + k) \rightarrow y_i$. The notation like $Q^{i,k}_r$ means $k$-shifted polynomials $Q^i_r$ corresponding to a shift $j \rightarrow i + 1$. 

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5.5.2. Equation (55) in the case $k = 1$ and $s = 3$. Equation (55) in this case appears as
\[
T(i + 4) = T(i) \frac{T(i + 2) + T(i + 3) + c_1}{T(i + 1) + T(i + 2) + c_1},
\]
which generates a map
\[
(y_0, y_1, y_2, y_3) \mapsto \left( y_1, y_2, y_3, \frac{y_2 + y_3 + c_1}{y_1 + y_2 + c_1} \right).
\]
Using (97) with matrix (98) gives us the following rational integrals for this map:
\[
F_0^{(1,3)} = \frac{\tilde{T}_2^{1,1}}{y_0 y_1 y_2 y_3} = \frac{y_1 + y_2 + c_1}{y_0 y_1 y_2 y_3},
\]
\[
\mathcal{F}_1^{(1,3)} = P_4^1 F_0^{(1,3)} + F_0^{(0,2),1} = \left( y_0 + y_1 + y_2 + y_3 \right) \frac{y_1 + y_2 + c_1}{y_0 y_1 y_2 y_3} + \frac{1}{y_1 y_2},
\]
\[
\mathcal{F}_2^{(1,3)} = P_3^2 F_0^{(1,3)} + P_2^1 F_0^{(0,2),1} = \left( y_0 y_2 + y_1 y_3 \right) \frac{y_1 + y_2 + c_1}{y_0 y_1 y_2 y_3} + \frac{y_1 + y_2}{y_1 y_2}.
\]

6. Discussion

We have presented in this paper the way to construct the first integrals for two $(1, 1)$-classes of ordinary difference equations (53) and (54) which possess Lax pair representation. These equations are presented in terms of special classes of discrete polynomials and it is natural that their first integrals are also expressed in terms of these polynomials. Based on conjecture 1, we have shown the equivalency of dynamical systems generated by two different at first glance $(k, s)$th equations (53) and (54) with fixed $(k, s)$. We do not discuss in the paper the Liouville–Arnold integrability for these classes of equations and leave this problem for further investigation. We also need to address the problem of expanding the results presented in this paper to general $(h, n)$-classes of difference equations (2) and (3).

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Appendix A. Proof of lemma 1

Let us prove, for example, (12) for any odd $r$. To this aim, it is more convenient to present polynomials $S^r_j$ in equivalent form as
where $B_{r,s} := \{ \lambda_j : 0 \leq \lambda_j \leq \cdots \leq \lambda_1 \leq s - 1 \}$, and rewrite (12) as

$$
\sum_{j=0}^{(r-1)/2} S_{r-2j}^2(i)T_s^{2j}(i + (r - 2j)n)
$$

$$
= \sum_{j=0}^{(r-1)/2} S_s^{r-2j-1}(i)T_s^{2j+1}(i + (r - 2j - 1)n).
$$

(A.1)

Now we observe the following. It is evident that the product $S_r(i)T_s(i + ln)$ is given by the summation

$$
\sum_{\{\lambda\}} T(i + \lambda_1 h)T(i + \lambda_2 h + n) \cdots T(i + \lambda_{r+q} h + (l + q - 1)n)
$$

over the set

$B_{r,s} \ast D_{q,s} := \{ \lambda_j : 0 \leq \lambda_1 \leq \cdots \leq \lambda_1 \leq s - 1, \ 0 \leq \lambda_{r+1} < \cdots < \lambda_{r+q} \leq s - 1 \}.$

One can easily check that

$$
B_{r,s} \ast D_{q,s} = K_{l+q,l,s} \sqcup K_{l+q,l-1,s},
$$

(A.3)

where

$K_{l+q,l,s} := \{ \lambda_j : 0 \leq \lambda_1 \leq \cdots \leq \lambda_1 \leq s - 1, \ 0 \leq \lambda_{r+1} < \cdots < \lambda_{l} \leq s - 1 \}.$

Therefore the left-hand side of relation (A.1) presents a summation (A.2) over the set

$B_{r,s} \sqcup (B_{r-2,s} \ast D_{2,s}) \sqcup \cdots \sqcup (B_{1,s} \ast D_{r-1,s}).$

In turn, the right-hand side of this relation is given by a summation over the set

$(B_{r-1,s} \ast D_{1,s}) \sqcup (B_{r-3,s} \ast D_{3,s}) \sqcup \cdots \sqcup D_{r,s}.$

To prove (A.1), it remains only to notice that, by virtue of (A.3), these two sets coincide and appear as $\bigcup_{j=0}^{r-1} K_{r,s,j}.$ For all other cases, the scheme of proof is about the same.

\[ \square \]

Appendix B. Proof of lemma 2

We can easily prove (36) by adding to the left and right sides of this set theoretic equality the following set:

$$
\{ \lambda_j : \lambda_1 = -1, \ -1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq s - 1 \}.
$$

(B.1)

On the one hand, it is obvious that

$$
\{ \lambda_j : \lambda_1 = -1, \ -1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq s \}
$$

$$
\sqcup \{ \lambda_j : \lambda_1 = -1, \ -1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq s - 1 \}
$$

$$
= \{ \lambda_j : \lambda_1 = -1, \ -1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq s \}
$$

and

$$
\{ \lambda_j : \lambda_1 = -1, \ -1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq s \} \sqcup B_{k,r+1} = B_{k,r+2}^{-1}.
$$
that is, as a result of adding \((B.1)\) to the left-hand side of (36), we obtain \(B_{k,s+2}^-\). On the other hand, we get

\[
\{ \lambda_j : \lambda_1 = -1, \ -1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq s - 1 \} \sqcup B_{k,s} = B_{k,s+1}^-
\]

and

\[
B_{k,s+1}^- \sqcup \{ \lambda_j : -1 \leq \lambda_1 \leq \cdots \leq \lambda_{k-1} \leq s, \ \lambda_k = s \} = B_{k,s+2}^-
\]

Since we obtain on the right-hand side of (36) the same set as on the left-hand side, this means that (36) is indeed valid. Similar reasonings are used to prove (37). Namely, we first add the set

\[
\{ \lambda_j : 0 \leq \lambda_1 \leq \cdots \leq \lambda_{k-1} \leq s, \ \lambda_k = s \}
\]

to the left and right side of equality (37) and then we prove that as a result on both sides of this equality we obtain the same set \(B_{k,s+2}^-\). Therefore we proved this lemma. \(\square\)

Appendix C. Proof of lemma 3

To begin with, we notice that relation (43) for \(s = 1\) takes the form

\[
S_k^1(i) = \prod_{q=0}^{k-1} T(i + q), \ \forall k \geq 1.
\]  

(C.1)

It is obvious that the latter is valid simply by virtue of the definition of the polynomials \(S_k^1\). Suppose now we have already proved (43) for some value of \(s \geq 1\) and all \(k \geq s\). Then using identities of the form (25) and (26), we get

\[
S_{k+1}^{s+1}(i) = S_{k+1}^{s+1}(i + 1) + T(i + s)S_{k+1}^s(i)
\]

\[
= S_{k+2}^s(i + 1)T(i + s + 1) + T(i + s)S_{k+1}^s(i)
\]

\[
= (S_{k+2}^s(i + 1)T(i + s + 1) + S_{k+1}^s(i))T(i + s)
\]

\[
= S_{k+2}^s(i)T(i + s),
\]

(C.2)

for this value of \(s\) and

\[
S_{k+1}^{s+1}(i) = S_k^{s+1}(i + 1) + T(i + k)S_k^s(i)
\]

\[
= S_{k+2}^s(i + 1) \prod_{q=i}^{k+1} T(i + q) + T(i + k)S_{k+1}^s(i) \prod_{q=i}^{k-1} T(i + q)
\]

\[
= \left( T(i + k + 1)S_{k+2}^s(i + 1) + S_{k+1}^s(i) \right) \prod_{q=i}^{k} T(i + q)
\]

\[
= S_{k+2}^s(i) \prod_{q=i}^{k} T(i + q)
\]

(C.3)

for \(k \geq s + 1\).

So supposing that the relation (43) is valid for some value of \(s\) and all values of \(k \geq s\), we prove therefore that it is fulfilled for \(s + 1\) and all \(k \geq s + 1\). Now we are in a position, using (C.1), (C.2) and (C.3), to prove (43) by induction on \(k\) for all values of \(s \geq 1\) and \(k \geq s\). Therefore this lemma is proved. \(\square\)
Appendix D. Proof of lemma 4

We first observe that

$$g^s_i(i) = \prod_{q=0}^{s-1} T(i + q) \quad (D.1)$$

and

$$g^s_i(i) + f^s_i(i) = \prod_{q=0}^{s-1} T(i + q) + \tilde{Q}^{s-2}_s(i + 1) \prod_{q=0}^{s-2} T(i + q)$$

$$= \left(T(i + s - 1) + \tilde{Q}^{s-2}_s(i + 1)\right) \prod_{q=0}^{s-2} T(i + q)$$

$$= \tilde{S}^{s-1}_s(i + 1) \prod_{q=0}^{s-2} T(i + q). \quad (D.2)$$

Therefore (73) and (74) are fulfilled for \( r = 0 \). Let us prove this lemma by induction on \( r \). We first suppose that (73) is valid for some \( r \), then, taking into account an identity of the form (40), we calculate

$$\left(\sum_{j=0}^{r} g^j_j(i) + \sum_{j=1}^{r+1} f^j_j(i)\right) + g^r_r(i) = \tilde{S}^{r-1}_r(i) \tilde{S}^{r+1}_r(i + 1) \prod_{q=r}^{s-r-2} T(i + q)$$

$$+ \tilde{Q}^{r-2}_{s-r-2}(i + 1) \tilde{S}^{r+1}_r(i + 1) \prod_{q=r+1}^{s-r-2} T(i + q)$$

$$= \left(T(i + r) \tilde{S}^{r-1}_r(i) + \tilde{Q}^{r+1}_{s-r-2}(i + 1)\right)$$

$$\times \tilde{S}^{r+1}_r(i + 1) \prod_{q=r+1}^{s-r-2} T(i + q)$$

$$= \tilde{S}^{r+1}_r(i) \tilde{S}^{r+1}_r(i + 1) \prod_{q=r+1}^{s-r-2} T(i + q).$$

Therefore we prove that if (73) is valid for some \( r \), then (74) is valid for \( r + 1 \). Now suppose that (74) is valid for some \( r \), then, using an identity of the form (38), we get

$$\left(\sum_{q=0}^{r} g^q_q(i) + \sum_{q=1}^{r+1} f^q_q(i)\right) + f^r_{r+1}(i) = \tilde{S}^{r-1}_r(i) \tilde{S}^{r+1}_r(i + 1) \prod_{q=r}^{s-r-1} T(i + q)$$

$$+ \tilde{Q}^{r-1}_{s-r-2}(i + 1) \tilde{S}^{r+1}_r(i + 1) \prod_{q=r}^{s-r-2} T(i + q)$$

$$= \left(\tilde{Q}^{r+1}_{s-r-2}(i + 1) + T(i + s - r - 1) \tilde{S}^{r-1}_r(i + 1)\right)$$

$$\times \tilde{S}^{r+1}_r(i + 1) \prod_{q=r}^{s-r-2} T(i + q)$$

$$= \tilde{S}^{r+1}_r(i + 1) \tilde{S}^{r+1}_r(i + 1) \prod_{q=r}^{s-r-2} T(i + q).$$
Therefore we prove that if (74) is valid for some \( r \), then (73) is valid for the same value of \( r \). Now, to prove this lemma by induction, it remains to use (D.1) and (D.2). Therefore this lemma is proved.

**Appendix E. Proof of lemma 5**

For further convenience, let us denote

\[
D_r^{(k,s)}(i) = G_{r+1}^{(k,s)}(i) - G_r^{(k,s-1)}(i).
\]

We first remark that, by definition,

\[
D_r^{(k,s)}(i) - D_r^{(k,s)}(i) = Q_r^{r+1}(i + k + 1)S_{r-1}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-2} T(i + q)
- Q_s^{r+2}(i + k + 1)S_{r-1}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-3} T(i + q).
\] (E.1)

Let us prove this lemma by induction on \( r \). Suppose that we have already proved the validity of (82) for some \( r \geq 0 \). Then, taking into account (E.1), we calculate

\[
D_r^{(k,s)}(i) = S_r^{r+1}(i + k)S_{r-1}^{k+r+1}(i + 1) \prod_{q=k+r}^{s-r-1} T(i + q)
- S_{r-1}^{r+1}(i + k)S_{r-1}^{k+r+1}(i) \prod_{q=k+r}^{s-r-2} T(i + q)
+ Q_s^{r+1}(i + k)S_{r-1}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-2} T(i + q)
- Q_s^{r+2}(i + k)S_{r-1}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-3} T(i + q).
\]

Making use of an identity of the form (25), namely,

\[
S_{r-1}^{k+r+1}(i) = S_{r-1}^{k+r+1}(i + 1) + T_{i+k+r}S_{r-1}^{k+r}(i),
\]

we obtain

\[
D_r^{(k,s)}(i) = S_r^{r+1}(i + k)S_{r-1}^{k+r+1}(i + 1) \prod_{q=k+r}^{s-r-1} T(i + q)
+ \left(Q_s^{r+1}(i + k) - S_{r-1}^{r+1}(i + k)S_{r-1}^{k+r+1}(i) \prod_{q=k+r}^{s-r-2} T(i + q)\right)
+ Q_s^{r+1}(i + k)S_{r-1}^{k+r+1}(i + 1) \prod_{q=k+r+1}^{s-r-2} T(i + q)
- Q_s^{r+2}(i + k)S_{r-1}^{k+r+1}(i + 1) \prod_{q=k+r+1}^{s-r-3} T(i + q).
\]
In turn, using an identity of the form (40), we calculate to get

\[
D_{r+1}^{(k,s)}(i) = S_{i-k-r}^k(i + k) \left( T_{i-r-i} S_{i-r}^{k+r}(i + 1) - T_{i+k+r} S_{i-r}^{k+r}(i) \right) \prod_{q=k+r}^{s-r-2} T(i + q)
\]

\[+ Q_{r-k-r-2}^{r+1}(i + k) S_{s-r-1}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-2} T(i + q)\]

\[+ Q_{r-k-r-2}^{r+2}(i + k) S_{s-r-1}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-3} T(i + q)\]

\[+ Q_{r-k-r-2}^{r+1}(i + k + 1) S_{s-r-1}^{k+r+1}(i + 1) \prod_{q=k+r+1}^{s-r-2} T(i + q)\]

\[+ Q_{r-k-r-2}^{r+2}(i + k + 1) S_{s-r-1}^{k+r+1}(i + 1) \prod_{q=k+r+1}^{s-r-3} T(i + q)\]

\[+ (Q_{r-k-r-2}^{r+1}(i + k + 1) + T_{i+k+r} S_{i-k-r}^{r}(i + k)) S_{s-r-1}^{k+r+1}(i + 1) \prod_{q=k+r+1}^{s-r-2} T(i + q)\]

\[+ (Q_{r-k-r-2}^{r+2}(i + k) + T_{i+k+r} T_{r-r-2} S_{i-k-r}^{r}(i + k)) \prod_{q=k+r+1}^{s-r-3} T(i + q).\]

Finally, using identities (39) and (40), we get

\[
D_{r+1}^{(k,s)}(i) = S_{i-k-r}^k(i + k) S_{i-r-1}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-2} T(i + q)
\]

\[+ Q_{r-k-r-2}^{r+2}(i + k) S_{s-r-1}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-3} T(i + q)\]

Therefore we prove that if (82) is valid for some \(r \geq 0\) then it is valid for \(r + 1\). To prove the lemma, it remains to prove (82) for \(r = 0\). We obtain

\[
D_0^{(k,s)}(i) = S_{i-k-1}^k(i) T(i + q) - S_{s+1}^{k-1}(i) \prod_{q=k-1}^{s-1} T(i + q)
\]

\[+ Q_{r-k-1}^{1}(i + k) S_{i-k}^k(i) \prod_{q=k}^{s-2} T(i + q)\]

\[+ (S_{i-k}^k(i + 1) + T_{i+k-1} S_{i+1}^{k-1}(i)) \prod_{q=k}^{s-1} T(i + q)\]

\[+ S_{i+1}^{k-1}(i) \prod_{q=k-1}^{s-1} T(i + q) - Q_{r-k-1}^{1}(i + k) S_{i}^k(i) \prod_{q=k}^{s-2} T(i + q)\]

\[= S_{i-k-1}^k(i + k) \prod_{q=k+i}^{s-1} T(i + q).\]

Therefore this lemma is proved. \(\square\)
Appendix F. Proof of proposition 1

Firstly, remark that the relation \((68) = (69)\) is valid by virtue of identities for these polynomials. For \(r = 0\), relations \((68)\) and \((69)\) are obvious. To prove the validity of them by induction on \(r\), it remains to show the following. Suppose that we have already proved \((69)\) for some value of \(r\), then making use of recurrent relation \((67)\) and identities of the form \((40)\) and \((38)\), we obtain

\[
G_{r+1}^{(k,s)}(i + 1) - G_{r+1}^{(k,s)}(i) = \Lambda_{r+1}^{(k,s)}(i)\left(\tilde{S}_{s-r}^{k+r+1}(i + 1) - \tilde{S}_{s-r}^{k+r+1}(i)\right) \\
+ Q_{r-k-r-2}^{r+1}(i + k + 2)\tilde{S}_{s-r}^{k+r+1}(i + 1) \prod_{q=k+r+2}^{s-r-1} T(i + q) \\
- Q_{r-k-r-2}^{r+1}(i + k + 1)\tilde{S}_{s-r}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-2} T(i + q) \\
= \left(Q_{r-k-r-2}^{r+1}(i + k + 2) + T(i + k + r + 1)\right) \\
\times S_{r-k-r}(i + k + 1)\tilde{S}_{s-r}^{k+r+1}(i + 1) \prod_{q=k+r+2}^{s-r-1} T(i + q) \\
- \left(Q_{r-k-r-2}^{r+1}(i + k + 1) + T(i + s - r - 1)\right) \\
\times S_{r-k-r}(i + k + 1)\tilde{S}_{s-r}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-2} T(i + q) \\
= S_{r-k-r-1}^{r+1}(i + k + 1)\tilde{S}_{s-r}^{k+r+1}(i + 1) \prod_{q=k+r+2}^{s-r-1} T(i + q) \\
- S_{r-k-r-1}^{r+1}(i + k + 1)\tilde{S}_{s-r}^{k+r+1}(i) \prod_{q=k+r+1}^{s-r-2} T(i + q) \\
= \Lambda_{r+1}^{(k,s)}(i)\left(T(i + s - r - 1)\tilde{S}_{s-r}^{k+r+1}(i + 1) \\
- T(i + k + r + 1)\tilde{S}_{s-r}^{k+r+1}(i)\right).
\]

Therefore this proposition is proved. \(\square\)

Appendix G. Proof of proposition 2

Firstly, remark that the relation \((93) = (94)\) is valid by virtue of identities for these polynomials. For \(r = 0\), relations \((93)\) and \((94)\) are obvious. To prove the validity of them by induction on \(r\), it remains to show the following. Suppose that we have already proved \((94)\) for some value of \(r\), then making use of recurrent relation \((92)\), we obtain
By virtue of identities (32) and (33), we get

\[
F_{r+1}^{(k,s)}(i+1) - F_{r+1}^{(k,s)}(i) = \sum_{n=0}^{i-r} T_{r+1}^{(k,s)}(i) \left( \sum_{j=0}^{i-r} \frac{T_{r+1}^{(k,s)}(i-j) - T_{r+1}^{(k,s)}(i-j+1)}{T(i+j)} \right)
\]

Therefore this proposition is proved. \(\square\)

**Appendix H. Proof of conjecture 1 for \(r = 2\) and \(r = 3\)**

Unfortunately, at the moment, we cannot prove identity (48) for all \(r\). Clearly, in the case \(r = 1\), it is evident. Below we give a proof of this conjecture for \(r = 2, 3\).

**H.1. The case \(r = 2\)**

We start from the proven identity of the form (27); namely,

\[
(S, T_{s}^{2})(i) := S_{s}^{2}(i) - S_{s}^{1}(i)T_{s}^{1}(i+1) + T_{s}^{2}(i) = 0 \quad \forall s \geq 1 \quad (H.1)
\]

We first transform (H.1) into

\[
S_{s}^{2}(i) - \left( S_{s+1}^{2}(i) - T(i+s) \right)T_{s}^{1}(i+1) + T_{s}^{2}(i) = 0
\]

or

\[
S_{s}^{2}(i) - S_{s+1}^{1}(i)T_{s}^{1}(i+1) + T_{s}^{2}(i) = T(i+s)T_{s}^{1}(i+1)
\]

and then into

\[
S_{s}^{2}(i) - S_{s+1}^{1}(i)\left( T_{s-1}^{1}(i+1) + T(i+s) \right) + T_{s}^{2}(i) = -T(i+s)T_{s}^{1}(i+1)
\]
H.2. The case $r = 3$

We start from the proven identity
\[
(S, T^3_{ij}(i)) := S^3_{i}(i) - S^2_{i}(i)T^3_{i}(i + 2) + S^3_{i}(i)T^2_{i}(i + 1) - T^3_{i}(i)
\]
\[
= 0, \quad \forall s \geq 1.
\] (H.2)

Making use of the identities of the form (26) we rewrite (H.2) in equivalent form
\[
S^3_{i}(i) - S^2_{i}(i)T^3_{i}(i + 2) + S^3_{i}(i)T^2_{i}(i + 1) - T^3_{i}(i)
\]
\[
= S^3_{i}(i) - \left( S^2_{i}(i) - T(i + s)S^1_{i+1}(i + 1)\right)T^3_{i}(i + 2)
\]
\[
+ \left( S^3_{i+2}(i) - T(i + s) - T(i + s + 1)\right)T^2_{i}(i + 1) - T^3_{i}(i)
\]

or
\[
S^3_{i}(i) - S^2_{i}(i)T^3_{i}(i + 2) + S^3_{i+1}(i + 1)T^2_{i}(i + 1) - T^3_{i}(i)
\]
\[
= T(i + s)\left( T^3_{i}(i + 1) - T^3_{i}(i + 2)S^1_{i+1}(i + 1)\right) + T(i + s + 1)T^2_{i}(i + 1).
\]

Next, using identities of the form (24), we transform the latter into\(^8\)
\[
S^3_{i}(i) - S^2_{i}(i)\left( T^3_{i-2}(i + 2) + T(i + s) + T(i + s + 1)\right)
\]
\[
+ S^3_{i+2}(i)\left( T^2_{i-1}(i + 1) + T(i + s + 1)T^1_{i-1}(i + 1)\right) - T^3_{i}(i)
\]

or
\[
\langle S, T^3_{ij}(i) \rangle = T(i + s)\xi_{2,0}(i) + T(i + s + 1)\xi_{2,1}(i)
\]

with
\[
\xi_{2,0}(i) := T^2_{i}(i + 1) - T^3_{i}(i + 2)S^1_{i+1}(i + 1) + S^3_{i}(i)
\]

and
\[
\xi_{2,1}(i) := T^3_{i}(i + 1) - T^3_{i-1}(i + 1)S^1_{i+2}(i) + S^3_{i}(i).
\]

It remains for us to convert $\xi_{2,0}$ and $\xi_{2,1}$ into the desired form. Using the identities (25) and (26), we get
\[
\xi_{2,0}(i) = T^2_{i}(i + 1) - T^3_{i}(i + 2)\left( S^3_{i}(i + 1) + T(i + s + 1)\right)
\]
\[
+ S^3_{i}(i + 1) + T(i + s + 1)S^1_{i+1}(i)
\]
\[
= T(i + 1)S^1_{i+1}(i) - T(i + s + 1)T^3_{i}(i + 2)
\]
\[
= T(i + 1)\left( S^1_{i+2}(i) - T(i + s + 1)\right) - T(i + s + 1)\left( T^3_{i+1}(i + 1) - T(i + 1)\right)
\]
\[
= T(i + 1)S^1_{i+2}(i) - T(i + s + 1)T^3_{i+1}(i + 1)
\]

\(^8\) It is evident that the identity $T^3_{i}(i + 2) = T^3_{i-2}(i + 2) + T(i + s) + T(i + s + 1)$ is valid only for $s \geq 2$. 

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and
\[ \xi_{2,1}(i) = T_{s}^{2}(i + 1) - T_{s-1}^{1}(i + 1)\left(S_{i}^{1}(i + 2) + T(i) + T(i + 1)\right) \\
+ S_{i}^{2}(i + 1) + T(i + 1)S_{i+1}^{1}(i) \\
= T_{s}^{2}(i + 1) - \left(T_{s}^{1}(i + 1) - T(i + s)\right)S_{i}^{1}(i + 2) + S_{i}^{2}(i + 1) \\
+ T(i + 1)T(i + T(i + s)) - T(i)T_{s-1}^{1}(i + 1) \\
= -T(i)\left(T_{s-1}^{1}(i + 1) - T(i + 1)\right) + T(i + s)\left(S_{i}^{1}(i + 2) + T(i + 1)\right) \\
= -T(i)T_{s-2}^{1}(i + 2) + T(i + s)S_{i+1}^{1}(i + 1). \]

Therefore
\[ \langle S, T_{s}^{1}(i) \rangle = \langle T(i + s)\left(T(i + 1)S_{i+2}^{1}(i) - T(i + s + 1)T_{i+1}^{1}(i + 1)\right) \\
+ T(i + s + 1)\left(-T(i)T_{s-2}^{1}(i + 2) + T(i + s)S_{i+1}^{1}(i + 1)\right) \rangle \\
= T(i + s)T(i + s + 1)\left(S_{i+1}^{1}(i + 1) - T_{i+1}^{1}(i + 1)\right) \\
+ T(i + 1)T(i + s)S_{i+2}^{1}(i) - T(i)T(i + s + 1)T_{s-2}^{1}(i + 2) \\
= T(i + 1)T(i + s)S_{i+2}^{1}(i) - T(i)T(i + s + 1)T_{s-2}^{1}(i + 2). \]

Therefore we have proved (48) for \( k = 3 \) and all \( s \geq 2 \). Moreover, by virtue of (47), we have
\[ \langle S, T_{s}^{1}(i) \rangle = S_{i}^{1}(i) = T(i)T(i + 1)T(i + 2). \]

\[ \square \]

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