Tractable ADMM schemes for computing KKT points and local minimizers for $\ell_0$-minimization problems

Yue Xie1 · Uday V. Shanbhag2

Received: 18 December 2019 / Accepted: 12 September 2020 / Published online: 1 October 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
We consider an $\ell_0$-minimization problem where $f(x) + \gamma \|x\|_0$ is minimized over a polyhedral set and the $\ell_0$-norm regularizer implicitly emphasizes the sparsity of the solution. Such a setting captures a range of problems in image processing and statistical learning. Given the nonconvex and discontinuous nature of this norm, convex regularizers as substitutes are often employed and studied, but less is known about directly solving the $\ell_0$-minimization problem. Inspired by Feng et al. (Pac J Optim 14:273–305, 2018), we consider resolving an equivalent formulation of the $\ell_0$-minimization problem as a mathematical program with complementarity constraints (MPCC) and make the following contributions towards the characterization and computation of its KKT points: (i) First, we show that feasible points of this formulation satisfy the relatively weak Guignard constraint qualification. Furthermore, if $f$ is convex, an equivalence is derived between first-order KKT points and local minimizers of the MPCC formulation. (ii) Next, we apply two alternating direction method of multiplier (ADMM) algorithms, named (ADMM$_{cf}$) and (ADMM$_{cf}$), to exploit the special structure of the MPCC formulation. Both schemes feature tractable subproblems. Specifically, in spite of the overall nonconvexity, it is shown that the first update can be effectively reduced to a closed-form expression by recognizing a hidden convexity property while the second necessitates solving a tractable convex program. In (ADMM$_{cf}$), subsequential convergence to a perturbed KKT point under mild assumptions is proved. Preliminary numerical experiments suggest that the proposed tractable ADMM schemes are more scalable than their standard counterpart while (ADMM$_{cf}$) compares well with its competitors in solving the $\ell_0$-minimization problem.

Keywords Nonconvex sparse recovery · Constraint qualifications and KKT conditions · Nonconvex ADMM · Tractability · Convergence analysis

* Yue Xie
xie86@wisc.edu; xieyue1990@gmail.com

Extended author information available on the last page of the article
1 Introduction

In this paper, we consider the $\ell_0$-minimization problem:

$$\min_x f(x) + \gamma \|x\|_0 \quad \text{subject to } Ax \geq b, \quad (1)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\gamma > 0$. Suppose $f(x) \triangleq f_Q(x) + g(x)$, where $f_Q : \mathbb{R}^n \to \mathbb{R}$ is a quadratic function and $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function. The $\ell_0$-norm of a vector captures the number of nonzero entries while an $\ell_0$-norm regularizer implicitly emphasizes the sparsity of the resulting minimizer. $\ell_0$-minimization problems of the form (1) assume relevance in applications in image processing and statistical learning (cf. [16, 18, 33]). The nonconvexity and discontinuity of the $\ell_0$-norm has prompted the usage of convex $\ell_1$ or $\ell_2$-norm regularizers or other tractable variants [2, 33]. While relatively less is known about directly solving problem (1), a solution of (1) may have better statistical properties. In fact, global solutions of (1) achieve model selection consistency and are known to be sparse under weaker conditions than when utilizing the $\ell_1$-norm (cf. [41]). Therefore, despite the computational challenges in addressing the $\ell_0$-norm penalty, resolution of the $\ell_0$-minimization problem is still desirable. In this work, we focus on the direct resolution of (1).

Related work To solve (1), Feng, Mitchell, Pang, Shen, and Wächter [22] introduced two complementarity-based formulations equivalent with (1) and processed them by standard nonlinear programming solvers. Blumensath and Davis proposed an iterative hard-thresholding (IHT) algorithm, applicable when $f(x)$ is a least-squares metric and the constraint $Ax \geq b$ is absent [9]. Convergence to a local minimizer may be claimed and performance of the scheme can be improved if warm-started from a point computed by matching pursuit.

A problem class closely related to (1) is the $\ell_0$-constrained problem (2).

$$\min_x f(x) \quad \text{subject to } Ax \geq b, \|x\|_0 \leq M. \quad (2)$$

This problem finds application in best subset regression [6, 7], cardinality constrained portfolio optimization [7], and graphical model estimation [21]. To solve (2), Bersekas et al. combined first-order methods and mixed-integer optimization [6] and this approach was seen to be promising. By considering an equivalent complementarity formulation of (2), Burdakov et al. [13] developed a regularization scheme. Moreover, a relatively weak constraint qualification was shown to hold at every feasible point of this reformulation and consequently KKT conditions are necessary at local minima. In [3], problem (2) without the linear inequality constraints is considered. The authors proposed different types of stationary points such as basic feasible points, $L$-stationary points and coordinatewise(CW) minima. Gradient projection and coordinate descent types of methods are proposed to locate different types of stationary points.
In addition to $\ell_0$-norm penalization, related work has examined the usage of the $\ell_p$-norm ($p \in (0, 1)$) [23, 24], the smoothly clipped absolute deviation (SCAD) penalty [20, 29], the minimax concave penalty (MCP) [40], and the capped-$\ell_1$ penalty [42]. More recently, a generalization of the $\ell_0$-norm constraint was considered in the form of an affine sparsity constraint [17].

**Nonconvex ADMM schemes** Since our focus lies in designing an ADMM framework to exploit the structure of an equivalent nonconvex formulation of (1), we provide a brief review of the available convergence statements in the context of ADMM schemes for nonconvex programs.

Table 1 lists some of the main theoretical findings regarding variants of ADMM schemes employed to address nonconvex problems [12, 27, 28, 37]. In the second column, we include the assumptions necessary for proving subsequential convergence or deriving complexity bounds. For some of the findings, it is shown that if the KL property (See Definition 6 in Appendix) is assumed, sequential convergence can be guaranteed [12, 37]. Note that all of the papers in Table 1 assume global resolution of each subproblem, even when the subproblem is nonconvex. Specifically, in [37], it is explained that the proposed ADMM scheme can address MPCC but requires globally resolving an MPCC at each step; this is in sharp contrast with the tractable structure of our scheme in this paper.

Besides the ones in Table 1, relevant studies include extensions of nonconvex ADMM schemes to the linearized regime [30], nonlinear equality constrained settings [36], amongst others [25, 35, 38, 39]. Despite all of these theoretical achievements on nonconvex ADMM, we point out that, to our best knowledge, no scheme introduced above can guarantee the convergence or even the boundedness of the iterates when applied to the following formulation with both blocks constrained and one being nonconvex:

$$\begin{align*}
\min & \quad F(x) + G(y) \\
\text{subject to} & \quad x - y = 0, \\
& \quad x \in X \subseteq \mathbb{R}^N, y \in Y \subseteq \mathbb{R}^N,
\end{align*}$$

where $F$ is a quadratic function, $G$ is smooth and convex, $X$ is a nonconvex set defined by a quadratic equality constraint, while $Y$ is a convex set. Formulation (3) is our focus in this paper because it allows for tractable subproblems, or possibly closed-form solution. Jiang et al. [28] discussed how ADMM schemes may be applied to (3) to promise convergence. Yet it requires modification of problem (3) itself through adding an unconstrained auxiliary block and penalizing 2-norm of the auxiliary variable in the objective function.

**Motivation and contributions** Despite the breadth of prior research, less is known regarding the nature of solutions and tractable convergent schemes for continuous reformulations of (1). Motivated by this gap and inspired by [22], we consider an equivalent MPCC reformulation of (1):

$$\begin{align*}
\min & \quad F(x) + G(y) \\
\text{subject to} & \quad x - y = 0, \\
& \quad x \in X \subseteq \mathbb{R}^N, y \in Y \subseteq \mathbb{R}^N,
\end{align*}$$
Table 1 Main results on convergence of nonconvex ADMM

| Problem | Assumptions (not exhaustive) | Result | Lit. |
|---------|------------------------------|--------|-----|
| $\min \quad f(x_1, x_2, \ldots, x_N)$ | $f$ is differentiable. For any $i$, $\mathcal{X}_i$ is convex. $A_N$ has full row rank | Iter. complexity of $O(1/\epsilon^2)$ to obtain an $\epsilon$-stationary point | [28] |
| $\min \quad \sum_{k=1}^{K} g_k(x_k) + h(x_0)$ | For any $k = 1, \ldots, K$, $\nabla g_k(x)$ is Lipschitz continuous. $h(\cdot)$ is convex. $\mathcal{X}$ is convex and compact | Subsequential convergence to stationary points | [27] |
| $\min \quad \sum_{k=1}^{K} g_k(x_k) + \ell(x_0)$ | $g_k(\cdot)$ is either convex or Lipschitz continuously differentiable. $\nabla l(\cdot)$ is Lipschitz continuous. $\mathcal{X}_k$ is convex and compact. $A_k$ has full column rank | Subsequential convergence to stationary points | [27] |
| $\min \quad F(z) + G(y) + H(x, y)$ | $F$ and $G$ are proper lower semicontinuous. $\nabla H$ is Lipschitz continuous. $A$ is surjective | Subsequential convergence to KKT points | [12] |
In particular, we focus on characterizing stationary points of (4) as well as developing tractable convergent scheme to recover such solutions.

(i) Regularity properties and characterization of KKT points In Sect. 2, we show that a feasible point of the MPCC reformulation satisfies the Guignard constraint qualification (GCQ). Under convexity of $f$, we derive an equivalence between first-order KKT points and local minimizers.

(ii) ADMM schemes with tractable subproblems In Sects. 3 and 4, we propose two ADMM schemes to exploit the special structure of the MPCC: (ADMM$_{cf}^{u}$, $\rho$) and (ADMM$_{cf}$). In particular, we reformulate the MPCC (4) in the form of (3) and apply the proposed ADMM schemes. Both schemes require resolving two subproblems at each iteration where, one is convex and the other, while nonconvex, is shown to possess a hidden convexity property [5], and allows for closed-form solutions. In the perturbed proximal ADMM scheme (ADMM$_{cf}^{u}$, $\rho$), the perturbation technique (inspired by Hajinezhad and Hong [26]) enables proving subsequential convergence. We also show that a limit point of this scheme is a perturbed KKT point where the inexactness depends on the choice of the perturbation parameters of the algorithm.

(iii) Numerics In Sect. 5, we present some preliminary numerical experiments demonstrating that the tractable ADMM schemes are more scalable than their standard counterpart and (ADMM$_{cf}$) competes well with other solution methods for a special case of the $\ell_0$-minimization problem.

Notation We let $e$ denote $(1; \ldots; 1)$ for an appropriate dimension. Given a set $Z$ and a vector $z$, $1_{Z}(z) = 0$ if $z \in Z$ and $\infty$ otherwise. The requirement $a \perp b$ is equivalent to $a_i b_i = 0$ for $i = 1, \ldots, n$. The matrix $I_n$ denotes the $n$–dimensional identity matrix. $[1, n] \triangleq \{1, 2, \ldots, n\}$. $|S|$ denotes the cardinality of set $S$. $(a)_i$ or $[a]_i$ denote the $i$th entry of vector $a$. We may also use $a_i$ to denote $i$th entry of vector $a$, but often $a_i$ may have other connotations such as the $i$th iterate in an algorithm, which will be specified. Let the support set of $x$ be defined as $\text{supp}(x) \triangleq \{i \in \{1, \ldots, n\} \mid x_i \neq 0\}$. For any vector $z \in \mathbb{R}^N$ and positive semidefinite matrix $M \in \mathbb{R}^{N \times N}$, $\|z\|_M^2 \triangleq z^T M z$.
\[
\begin{align*}
\min_{x, \xi} & \quad f(x) + \gamma e^T(e - \xi) \\
\text{subject to} & \quad Ax \geq b, \ x_i \xi_i = 0, \\
& \quad 0 \leq \xi_i \leq 1, \text{ for } i = 1, \ldots, n.
\end{align*}
\]

The term “half-complementarity” arises from noting that the equality constraint may be recast as \( x \perp \xi \geq 0 \).

**(b) Full-complementarity**

\[
\begin{align*}
\min_{x, x^\pm, \xi} & \quad f(x) + \gamma e^T(e - \xi) \\
\text{subject to} & \quad Ax \geq b, \ x^+ - x^- = x, \quad (x^+)T x^- = 0, \quad (x^+ + x^-)T \xi = 0, \\
& \quad x_i^+, x_i^- \geq 0, \ 0 \leq \xi_i \leq 1, \text{ for } i = 1, \ldots, n.
\end{align*}
\]

where \( x^+, x^-, \xi \in \mathbb{R}^n \). (6) may be further simplified by relaxing \((x^+)T x^- = 0\), resulting in (4). It can be formally shown that (4) is a tight relaxation of (6). In particular, a solution of (4) can be appropriately modified to get a minimizer of (6) with the same objective function value.

**Lemma 1** (Tightness of relaxation) Consider the problem (4) and suppose a global minimizer to this problem is denoted by \((x^\pm, \xi)\). Let \((\bar{x}; x^\pm; \bar{\xi})\) be defined as follows:

\[
\bar{x}_i \triangleq \begin{cases} 
  x_i^+ - x_i^-, & \text{if } x_i^+ \geq x_i^- \\
  0, & \text{otherwise}
\end{cases}, \quad \bar{\xi}_i \triangleq \begin{cases} 
  x_i^- - x_i^+, & \text{if } x_i^+ \geq x_i^- \\
  0, & \text{otherwise}
\end{cases},
\]

\( \forall i = 1, \ldots, n.\) \( \bar{x} \triangleq x^+ - x^- \), \( \bar{\xi} \triangleq \xi. \) Then \((\bar{x}; x^\pm; \bar{\xi})\) is a global minimizer of (6).

**Proof** We first prove that the constructed solution \((\bar{x}; x^\pm; \bar{\xi})\) is feasible with respect to (6) and then prove that it is optimal.

**Feasibility of \((\bar{x}; x^\pm; \bar{\xi})\).** By definition (7), we have that \( \bar{x}^\pm \geq 0 \) and \( \min\{\bar{x}_i^+, \bar{x}_i^-\} = 0 \) for \( i = 1, \ldots, n \). Consequently, \( \bar{x} \perp \bar{\xi} \). Furthermore, \( \bar{x} = x^+ - x^- = x^+ - x^- \) and \( \bar{\xi} = \xi \). This implies that \( Ax\bar{x} = Ax^+ - Ax^- = Ax^+ - Ax^- \geq b \). Finally, it suffices to show that \((x^+ + \bar{x}^-)T \bar{\xi} = 0\). By the feasibility of \((x^\pm; \xi)\) with respect to (4), we have that \((x_i^+ + \bar{x}_i^-)\xi_i = 0\), for \( i = 1, \ldots, n \). For any \( i \), if \( \xi_i = 0 \), then \( \bar{\xi}_i = 0 \) by (7), which implies \((\bar{x}_i^+ + \bar{x}_i^-)\bar{\xi}_i = 0\). If \( \xi_i > 0 \), then \( x_i^+ = x_i^- = 0 \), which indicates that \( \bar{x}_i^+ = \bar{x}_i^- = 0 \) by (7), implying that \((x_i^+ + \bar{x}_i^-)\bar{\xi}_i = 0\) also holds. Therefore, \((x_i^+ + \bar{x}_i^-)\bar{\xi}_i = 0\), for \( i = 1, \ldots, n \) and \((\bar{x}^+ + \bar{x}^-)T \bar{\xi} = 0\).

**Optimality of \((\bar{x}; x^\pm; \bar{\xi})\).** We observe that the function \( f(\bar{x}) + \gamma \sum_{i=1}^n (1 - \bar{\xi}_i) = f(x^+ - x^-) + \gamma \sum_{i=1}^n (1 - \xi_i) \) for \( \bar{x} = x^+ - x^- \) and \( \bar{\xi} = \xi \). But since (4) is a relaxation of (6), the optimality of \((\bar{x}; x^\pm; \bar{\xi})\) follows from feasibility of \((\bar{x}; x^\pm; \bar{\xi})\) with respect to the tightened optimization problem with an identical objective value.

Since the equivalence between (6) and (1) has been established \[22\], the tightness of relaxation indicates equivalence between (4) and (1). Moreover, the
The following result shows that local minimizers of (1) can also be recovered by local minimizers of (4).

**Lemma 2** Given \(\hat{x}, \hat{x}^+, \hat{x}^-, \hat{\xi} \in \mathbb{R}^n\) such that \(\hat{x} = \hat{x}^+ - \hat{x}^-\) and \((\hat{x}^+; \hat{x}^-; \hat{\xi})\) is a local minimum of (4). Then \(\hat{x}\) is a local minimum of (1).

**Proof** Suppose \(\mathcal{Z}\) denotes the feasible region of (4). Since \(\hat{\xi} \triangleq (\hat{x}^+; \hat{x}^-; \hat{\xi})\) is a local minimum of (4), \(\hat{\xi} \in \mathcal{Z}\) and there exists an open neighbourhood \(\mathcal{N} \triangleq B(\hat{\xi}, r) \triangleq \{z \in \mathbb{R}^{3n} | \|z - \hat{\xi}\| < r\}\) such that for all \((x^+; x^-; \xi) \in \mathcal{N} \cap \mathcal{Z}\),

\[
f(x^+ - x^-) + \gamma e^T(e - \hat{\xi}) \geq f(\hat{x}^+ - \hat{x}^-) + \gamma e^T(e - \hat{\xi}).
\]

Let \(X \triangleq \{x | Ax \geq b\}\). It suffices to show that (a) \(\hat{x} \in X\) and (b) there exists an open neighbourhood \(U \ni \hat{x}\) such that for all \(x \in U \cap X\),

\[
f(x) + \gamma \|x\|_0 \geq f(\hat{x}) + \gamma \|\hat{x}\|_0.
\]

Moreover, global resolution of convex nonlinear programs in that standard regularity conditions (such as LICQ or MFCQ) may fail to hold at any feasible point [31]. Moreover, global resolution of such problems is generally challenging. We now discuss what constraint qualifications do hold at a feasible point of (4).

\[\square\] Springer
2.2 Constraint qualifications

In this subsection, we analyze whether regularity conditions hold at feasible points for the simplified full complementarity formulation (4). This allows for stating necessary conditions of optimality. Recall that some common CQs are related as follows.

(I) LICQ ⇒ CRCQ and (II) LICQ ⇒ MFCQ ⇒ ACQ ⇒ GCQ

(8)

The first relation is obvious from the definition of LICQ and CRCQ (See [19, Page 262]) while the proof of the second relation may be found in [15]. In the context of the half-complementarity formulation (5), the constant rank constraint qualification (CRCQ) is proven to hold at points satisfying certain nondegeneracy property while LICQ may fail [22]. In this section, we focus on the simplified full-complementarity formulation (4). It can be shown that GCQ may hold at every feasible point while ACQ may fail.

We begin our discussion with some definitions. Suppose $g : \mathbb{R}^n \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are continuously differentiable functions while $\Omega$ is a set defined as follows.

$\Omega \triangleq \{ x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0 \}$.  

(9)

Then the tangent cone $T_\Omega(x^*)$ and linearized cone $L_\Omega(x^*)$ of $\Omega$ at $x^*$ and the ACQ and the GCQ are defined as follows:

Definition 1  (Abadie and Guignard CQ (ACQ, GCQ)) If $I(x^*) = \{ i : g_i(x^*) = 0 \}$, then $T_\Omega(x^*)$ and $L_\Omega(x^*)$ of $\Omega$ at $x^*$ are defined as follows:

$T_\Omega(x^*) \triangleq \{ d : \exists \{ x_k \} \subseteq \Omega, \{ t_k \} \downarrow 0, \text{ s.t. } x_k \to x^* \text{ and } d = \lim_{k \to \infty} \frac{x_k - x^*}{t_k} \}$.  

(10)

$L_\Omega(x^*) \triangleq \{ d : \nabla g_i(x^*)^T d \leq 0, \forall i \in I(x^*), \nabla h_j(x^*)d = 0, j = 1, \ldots, q \}$.  

(11)

Then $x^*$ satisfies the Abadie Constraint Qualification (ACQ) iff $T_\Omega(x^*) = L_\Omega(x^*)$. Further, $x^*$ satisfies the Guignard Constraint Qualification (GCQ) iff $(T_\Omega(x^*))^* = (L_\Omega(x^*))^*$, where for a cone $C \subseteq \mathbb{R}^n$, $C^* \triangleq \{ v : d^Tv \leq 0, \forall d \in C \}$.

Next, we prove that the GCQ holds at every feasible point of (4).

Lemma 3  (GCQ holds at feasible points) Consider the problem (4) and consider a feasible point $x = (x^+, x^-, \xi)$. Then the GCQ holds at this point.

Proof  For the point $x = (x^+, x^-, \xi)$, define

$A^T \triangleq (a_1, \ldots, a_m)$ and $E(x) \triangleq \{ i : a_i^T(x^+ - x^-) = b_i \}$.  

(12)
\[
S(x) = \{i : x_i^+ = x_i^- = 0\}, \\
S_0(x) = \{i \in S(x) : \xi_i = 0\}, S_1(x) = \{i \in S(x) : \xi_i = 1\}.
\]

In addition, define cones \(C_1(x)\) and \(C_2(x)\) as

\[
C_2(x) \triangleq \left\{ \begin{array}{l}
(d_1)_i = 0, (d_2)_i = 0, \forall i \in S(x) \setminus S_0(x); \\
(d_1)_i \geq 0, \forall i \in S_0(x) \cup (S(x)^c \cap \{i : x_i^+ = 0\}); \\
(d_2)_i \geq 0, \forall i \in S_0(x) \cup (S(x)^c \cap \{i : x_i^- = 0\}); \\
(d_3)_i \geq 0, \forall i \in S(x)^c; a_j^T d_1 - a_j^T d_2 \geq 0, \forall j \in E(x)
\end{array} \right\},
\]

\[
C_1(x) \triangleq C_2(x) \cap \{d = (d_1; d_2; d_3) : (d_1)_i + (d_2)_i)(d_3)_i = 0, \forall i \in S_0(x)\},
\]

respectively where it may be noted that \(C_1(x)\) is characterized by an extra constraint \([(d_1)_i + (d_2)_i)(d_3)_i = 0\) for all \(i \in S_0(x)\). Further, denote

\[
X \triangleq \left\{ (y^+; y^-; \xi) : y^+ \geq 0, y^- \geq 0, 0 \leq \xi \leq 1, \forall i = 1, \ldots, n \right\}.
\]

We proceed to show the following.

(i) \(T_X(x) = C_1(x)\): Suppose \(d \in T_X(x)\). Then there exist sequences \(\{x_k\}\) and \(\{t_k\}\) such that \(\{x_k\} \subseteq X, x_k \to x, \{t_k\} \downarrow 0\) and \(d = \lim_{k \to \infty} \frac{x_k - x}{t_k}\). Denote \(x_k \triangleq (x^+(k); x^-(k); \xi(k))\), where \(x^+(k), x^-(k), \xi(k) \in \mathbb{R}^n\). Suppose that \(d \triangleq (d_1; d_2; d_3), d_1, d_2, d_3 \in \mathbb{R}^n\). Based on feasibility of \(x_k, \forall k \geq 1\) and the fact that \(x_k \to x\), we may claim the following:

\[
\forall i \in S(x) \setminus S_0(x), \exists K_1, \text{ s.t. } \forall k \geq K_1, (x^+(k))_i = (x^-(k))_i = 0 \\
\implies (d_1)_i = (d_2)_i = 0, \forall i \in S(x) \setminus S_0(x)
\]

if \(i \in S_1(x)\), then \(\xi_i = 1\) and \((\xi(k))_i \leq 1, \forall k
\implies (\xi(k))_i - \xi_i \leq 0, \forall k, (d_3)_i \leq 0, \forall i \in S_1(x).
\]

Similarly we may claim the following:

\[
\forall i \in S(x)^c, \exists K_2, \text{ s.t. } \forall k \geq K_2, (\xi(k))_i = 0, (x^+(k))_i \geq 0, (x^-(k))_i \geq 0 \\
\implies (d_1)_i = 0, \forall i \in S(x)^c; (d_1)_i \geq 0, \forall i \in S(x)^c \cap \{i : x_i^+ = 0\};
\]

and \((d_2)_i \geq 0, \forall i \in S(x)^c \cap \{i : x_i^- = 0\}\).

For indices \(i \in S_0(x)\), the following holds:
\[ x_i^+ = x_i^- = 0 \Rightarrow (x^+_{(k)})_i - x_i^+ \geq 0, (x^-_{(k)})_i - x_i^- \geq 0, (\xi_{(k)})_i = 0, \forall k, \]
\[ \implies (d_1)_i \geq 0, (d_2)_i \geq 0, (d_3)_i \geq 0. \]
\[ (x^+_{(k)})_i + (x^-_{(k)})_i = 0, \forall k \]
\[ \implies (x^+_{(k)})_i + (x^-_{(k)})_i = 0, \text{ or } (\xi_{(k)})_i = 0, \text{ infinitely often}; \]
\[ \implies (d_1)_i \geq (d_2)_i = 0, \text{ or } (d_3)_i = 0 \iff [(d_1)_i + (d_2)_i](d_3)_i = 0. \]

Furthermore,
\[ \forall j \in E(x), a_j^T x^+ - a_j^T x^- = b_j, a_j^T x^+_{(k)} - a_j^T x^-_{(k)} \geq 0, \text{ for all } k \geq 1 \]
\[ \implies a_j^T (x^+ - x^-) - a_j^T (x^-_{(k)} - x^-) \geq 0, \forall j \in E(x) \text{ and } k \geq 1 \]
\[ \implies a_j^T d_1 - a_j^T d_2 \geq 0, \forall j \in E(x). \]

Therefore, we may conclude from (14) that \( d \in C_1(x) \) and \( T_X(x) \subseteq C_1(x) \).

We now proceed to show that \( C_1(x) \subseteq T_X(x) \). Choose any \( d \in C_1(x) \). Then based on the property of \( C_1(x) \), it is easy to see that we may choose \( \lambda \) large enough such that \( x + d/(k \lambda) \in X, \forall k \geq 1 \). Let \( x_k \triangleq x + d/(k \lambda), t_k \triangleq 1/(k \lambda) \) for all \( k \geq 1 \), implying that \( \{x_k\} \subseteq X, x_k \rightarrow x, t_k \downarrow 0, d = \lim_{k \to \infty} \frac{\lambda}{k} \) implying that \( d \in T_X(x) \), which further implies \( C_1(x) \subseteq T_X(x) \).

(ii) \( L_X(x) = C_2(x) \): The set \( X \) contains the following active constraints.
\[ -y^+_i \leq 0, \forall i \in S(x) \cup \{i \in S(x)^c : x^+_i = 0\}; \]
\[ -y^-_i \leq 0, \forall i \in S(x) \cup \{i \in S(x)^c : x^-_i = 0\}; \]
\[ -\xi_i \leq 0, \forall i \in S_0(x) \cup S(x)^c; \quad \xi_i \leq 1, \forall i \in S_1(x); \]
\[ a_i^T (y^+ - y^-) \geq b_i, \forall i \in E(x); \quad (y^+ + y^-)^T \xi = 0. \]

This allows for defining the linearized cone \( L_X(x) \) at \( x \in X \).

\[ L_X(x) \triangleq \begin{cases} 
(d_1) \\
(d_2) \\
(d_3) 
\end{cases} : \begin{align*}
(d_1)_i & \leq 0, \forall i \in S(x) \cup \{i \in S(x)^c : x^+_i = 0\}; \\
(d_2)_i & \leq 0, \forall i \in S(x) \cup \{i \in S(x)^c : x^-_i = 0\}; \\
a_i^T (d_1 - d_2) & \geq 0, \forall i \in E(x); \\
\xi^T (d_1 + d_2) + (x^+ + x^-)^T d_3 & = 0.
\end{align*} \]

Suppose \( d \in L_X(x) \). Then the following holds:
\[ \xi^T (d_1 + d_2) + (x^+ + x^-)^T d_3 = 0 \]
\[ \iff \sum_{i \in S(x) \setminus S_0(x)} \xi_i [(d_1)_i + (d_2)_i] + \sum_{i \in S(x)^c} (d_3)_i (x^+_i + x^-_i) = 0 \]
\[ \iff (d_1)_i = (d_2)_i = 0, \forall i \in S(x) \setminus S_0(x); \quad (d_3)_i = 0, \forall i \in S(x)^c, \]
where the first equivalence follows from the definition of \( S_0(x) \) and \( S(x) \) while the second follows from noting that \( (d_1)_i \geq 0, (d_2)_i \geq 0, (d_3)_i \geq 0, \forall i \in S(x) \setminus S_0(x) \).
and \((d_3)_i \geq 0, x^+_i + x^-_i > 0, \forall i \in S(x)^c\). Therefore, by replacing 
\(\xi^T (d_1 + d_2) + (x^+ + x^-)^T d_3 = 0\) with \((16)\) in the representation \((15)\), we observe that 
\(L_X(x) = C_2(x)\).

(iii) We conclude the proof by showing that \(C_2(x) = \text{cl}(\text{conv}(C_1(x)))\). Since \(C_2(x)\) is a polyhedral cone, it is closed and convex. Furthermore, by definition, \(C_2(x) \supseteq C_1(x)\), implying that \(C_2(x) \supseteq \text{cl}(\text{conv}(C_1(x)))\). To prove the reverse direction, choose any vector \(d \triangleq (d_1; d_2; d_3) \in C_2(x)\) where \(d_1, d_2, d_3 \in \mathbb{R}^n\). It is easy to verify that both vectors \(\hat{d} \triangleq (0_{n \times 1}; 0_{n \times 1}; 2d_3)\) and \(\hat{d} \triangleq (2d_1; 2d_2; 0_{n \times 1})\) are in \(C_1(x)\). Note that \(d = \frac{1}{2} \hat{d} + \frac{1}{2} \hat{d} \in \text{cl}(\text{conv}(C_1(x)))\). Therefore, \(C_2(x) \subseteq \text{cl}(\text{conv}(C_1(x)))\).

By (iii) \(L_X(x) = \text{cl}(\text{conv}(T_X(x)))\), implying that \(T_X(x)^* = L_X(x)^*\).  

Remark 1

(i) At a feasible point \(x = (x^+; x^-; \xi)\) such that \([x^+ + x^-]_i = 0\) and \(\xi_i = 0\) for some index \(i\), ACQ may fail to hold. In fact, it is very likely that \(T_X(x) = C_1(x) \nsubseteq C_2(x) = L_X(x)\). On the other hand, at all other points, \(S_0(x) = \emptyset\) and ACQ holds. (ii) KKT conditions are necessary at local minimum.

2.3 KKT conditions and local optimality

In this subsection, we discuss the relation between (first-order) KKT conditions and local optimality. We begin with the definition of KKT conditions.

Definition 2 (KKT conditions) Consider the problem \(\{\min_{x \in \Omega} F(x)\}\), where \(F\) is a continuously differentiable function and \(\Omega\) is defined in \((9)\). Suppose \(x^*\) denotes a feasible solution of \(\Omega\). Then \(x^*\) satisfies the first-order KKT conditions if and only if there exist \(\lambda \in \mathbb{R}^p_+, \mu \in \mathbb{R}^q\) such that

\[
\nabla F(x^*) + \sum_{i=1}^p \lambda_i \nabla g_i(x^*) + \sum_{j=1}^q \mu_j \nabla h_j(x^*) = 0, \\
\lambda_i g_i(x^*) = 0, \quad \forall i = 1, \ldots, p.
\]

Then by Definition 2, a point \(x \triangleq (x^+; x^-; \xi)\) satisfies the first-order KKT conditions of problem (4) if there exist multipliers \((\mu, \beta_1, \beta_2, \beta_3, \beta_4, \pi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m\) such that the following conditions hold:

\[
\begin{aligned}
0 &= \begin{pmatrix} \nabla f(x^+ - x^-) \\ -\nabla f(x^+ - x^-) \\ -\gamma e \\ x^+ + x^- \end{pmatrix} + \mu \begin{pmatrix} \xi \\ -\xi \\ x^+ + x^- \end{pmatrix} + \begin{pmatrix} -\beta_1 - A^T \pi \\ -\beta_2 - A^T \pi \\ -\beta_4 - \beta_3 \end{pmatrix}, \\
0 &\leq \beta_1 \perp x^+ \geq 0, \\
0 &\leq \beta_2 \perp x^- \geq 0,
\end{aligned}
\]

\(\square\) Springer
Before presenting the main result, we point out a non-degeneracy property of KKT points.

**Lemma 4** (Nondegeneracy of first-order KKT points) Consider a point \( x = (x^+, x^-; \xi) \) and a set of multipliers \((\mu, \beta_1, \beta_2, \beta_3, \beta_4, \pi)\) that satisfy the first-order KKT conditions (18) of (4). Then \( x \) satisfies the nondegeneracy property:

\[
[x^+ + x^-]_i = 0 \Rightarrow \xi_i = 1.
\]

**Proof** Suppose that \((x^+, x^-; \xi)\) verifies KKT conditions (18) with multipliers \(\mu, \beta_1, \beta_2, \beta_3, \beta_4, \pi\). Then, by (18a), we have that \((x^+ + x^-)_i = 0 \Rightarrow (\beta_4 - \beta_3)_i = \gamma > 0\). But for a given \(i\), for both \([\beta_4]_i\) and \([\beta_3]_i\), to be positive, we require that both \([\mu]_i = 0\) and \([1 - \xi]_i = 0\) hold, which is impossible. It follows that the only possibility is that \([\beta_4]_i = \gamma\) and \([\beta_3]_i = 0\), implying that \([\xi]_i = 1\). It follows that \((x^+, x^-; \xi)\) satisfies the property (19). \(\square\)

Lemma 4 leads to the rather surprising equivalence between local minimizers and (first-order) KKT points.

**Theorem 1** (Equivalence between local minimizers and KKT points) Consider problem (4), and let \( x = (x^+, x^-; \xi) \) denote a feasible point. Assume that \( f \) in (4) is convex. Then the following statements are equivalent:

(a) \( x \) is a local minimizer of (4);
(b) There exist \( \mu \in \mathbb{R}, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}^n \), and \( \pi \in \mathbb{R}^m \) such that the first-order KKT conditions (18) hold:

\[
0 \leq \beta_3 \perp \xi \geq 0, \tag{18d}
\]
\[
0 \leq \beta_4 \perp e - \xi \geq 0, \tag{18e}
\]
\[
0 \leq \pi \perp A(x^+ - x^-) - b \geq 0, \tag{18f}
\]
\[
(x^+ + x^-)^T \xi = 0. \tag{18g}
\]

**Proof** (a)\(\Rightarrow\)(b). This is true because GCQ holds at every feasible point by Lemma 3.

(b)\(\Rightarrow\)(a). Suppose that \( x = (x^+, x^-; \xi) \) satisfies KKT conditions (18) with multipliers \((\mu, \beta_1, \beta_2, \beta_3, \beta_4, \pi)\). Then by the nondegeneracy property of a KKT point (Lemma 4), the set \(\{1, \ldots, n\}\) can be partitioned into the following two sets, as in the same fashion when proving the CQ: \( S(x) \triangleq \{i \in \{1, \ldots, n\} : x^+_i = x^-_i = 0, \xi_i = 1\} \) and \( S^c(x) \triangleq \{i \in \{1, \ldots, n\} : x^+_i + x^-_i > 0, \xi_i = 0\} \). We denote that \( A = (a_1, \ldots, a_n) \) (Note different notation from (12)). Then (18a) implies

\[ x^+_i = x^-_i = 0, \xi_i = 1. \]
(\nabla_x f(x^+ - x^-))_i - a_i^T \pi = (\beta_1)_i \geq 0 \\
-(\nabla_x f(x^+ - x^-))_i + a_i^T \pi = (\beta_2)_i \geq 0 \\
\forall i \in S^c(x).$

because $\xi_j = 0$ for all $i \in S^c(x)$. Consequently, $(\beta_1)_i = -(\beta_2)_i$ where $\beta_1$ and $\beta_2$ are nonnegative. It follows that $(\beta_1)_i = (\beta_2)_i = 0$, and

$$(\nabla_x f(x^+ - x^-))_i = a_i^T \pi, \quad \forall i \in S^c(x).$$

We proceed to prove that $(x^+; x^-)$ is a global minimizer of the following program:

$$\min \tilde{f}(z) \triangleq f(z^+ - z^-), \quad \text{subject to } z = (z^+; z^-) \in \tilde{X}(x),$$

where

$$\tilde{X}(x) \triangleq \{(z^+; z^-) \mid z^+, z^- \in \mathbb{R}^n, A(z^+ - z^-) \geq b, z^+_i = z^-_i = 0, \forall i \in S(x)\}.$$

Consider any feasible point $(\tilde{x}^+; \tilde{x}^-)$ of (21). By applying (20) and noticing $x^+_i, \tilde{x}^-_i = 0 \forall i \in S(x)$ (by def.), $\pi^T A(x^+ - x^-) = \pi^T b, \pi \geq 0$ (by (18)), and $A(\tilde{x}^+ - \tilde{x}^-) - b \geq 0$,

$$\begin{align*}
\begin{bmatrix}
-\nabla_x f(x^+ - x^-) \\
\nabla_x f(x^+ - x^-)
\end{bmatrix}
&= \nabla_x f(x^+ - x^-)[(\tilde{x}^+ - \tilde{x}^-) - (x^+ - x^-)] \\
&= \sum_{i \in S^c(x)} (\nabla_x f(x^+ - x^-))_i[(\tilde{x}^+_i - \tilde{x}^-_i) - (x^+_i - x^-_i)] \\
&= \sum_{i \in S^c(x)} a_i^T \pi(\tilde{x}^+_i - \tilde{x}^-_i) - \sum_{i \in S^c(x)} a_i^T \pi(x^+_i - x^-_i) \\
&= \sum_{i \in S^c(x) \cup S^c(x)} a_i^T \pi(\tilde{x}^+_i - \tilde{x}^-_i) - \sum_{i \in S^c(x) \cup S^c(x)} a_i^T \pi(x^+_i - x^-_i) \\
&= \pi^T [A(\tilde{x}^+ - \tilde{x}^-) - b] \geq 0.
\end{align*}$$

It follows that $(x^+; x^-)$ is a solution of VI$(\tilde{X}(x), \nabla \tilde{f})$. By convexity of $f$ (thus $\tilde{f}$) and $\tilde{X}(x)$, $(x^+; x^-)$ is a global minimizer of (21). Since $\xi_i = 1$ for $i \in S(x)$, by the separability of the objective and the structure of the constraint sets, it follows that $(x^+; x^-; \xi)$ is a minimizer of the tightened (4) as follow:

$$\min f(\bar{x}_1 - \bar{x}_2) + \gamma e^T (e - \bar{x}_3) \quad \text{subject to } (\bar{x}_1; \bar{x}_2; \bar{x}_3) \in X_{\text{tight}}(x),$$

where

$$X_{\text{tight}}(x) \triangleq \left\{(\bar{x}_1; \bar{x}_2; \bar{x}_3) : (\bar{x}_1)_i = (\bar{x}_2)_i = 0, \forall i \in S(x), \quad (\bar{x}_3)_i = 0, \forall i \in S^c(x) \right\}.$$

If $X$ denotes the feasible region in (4), then we can take a sufficiently small neighborhood of $x$, denoted by $N(x)$, such that $X \cap N(x) = X_{\text{tight}}(x) \cap N(x)$. Since
\(x = (x^+; x^-; \xi)\) is a global minimizer of \(f(\hat{x}_1 - \hat{x}_2) + \gamma e^T(e - \hat{x}_3)\) over \(X_{\text{tight}}(x)\), it is a global minimizer of \(f(\hat{x}_1 - \hat{x}_2) + \gamma e^T(e - \hat{x}_3)\) over the smaller set \(\mathcal{N}(x) \cap X_{\text{tight}}(x)\). Since \(\mathcal{N}(x) \cap X_{\text{tight}}(x) = \mathcal{N}(x) \cap X_{\text{tight}}(x)\), it follows that \(x\) is a local minimizer of (4). \square

**Remark 2** Note that while convexity of \(f\) is observed for many loss functions, it does not guarantee the overall convexity of the problem and (4) is still a nonconvex problem.

### 3 Tractable ADMM frameworks

In this section we discuss how to adapt ADMM to efficiently address MPCC (4). In Sect. 3.1, we present a perturbed proximal ADMM framework for obtaining a suitably defined solution of (4) and show in Sect. 3.2 that both of the ADMM subproblems can be solved tractably, of which, one can be recast as a convex program, while the other can be resolved in closed form. In Sect. 3.3, a basic ADMM framework will be presented, along with a discussion regarding why we consider its perturbed proximal variant. A standard ADMM applied to an alternative formulation of (4) is introduced in Sect. 3.4. Note that ADMM applied to this formulation is easier to analyze but does have computational disadvantages arising from the intractability of the subproblem.

#### 3.1 A perturbed proximal ADMM framework

We may reformulate (4) as follows.

\[
\min f(x^+ - x^-) + \gamma \sum_{i=1}^{n} (1 - \xi_i) + 1_{Z_1}(w) + 1_{Z_2}(w) .
\]  

(22)

Recall that \(f(x) = f_0(x) + g(x)\), where \(f_0(x) \triangleq x^T M x + d^T x\), \(g(x)\) is convex and smooth, \(M \in \mathbb{R}^{n \times n}\) is a symmetric matrix, and \(d \in \mathbb{R}^n\). Let \(Z_1, Z_2, \) and \(w\) be defined as

\[
Z_1 \triangleq \{(x^+; x^-; \xi) : (x^+ + x^-)^T \xi = 0\},
\]

\[
Z_2 \triangleq \begin{cases} 
(x^+, x^-) : 0 \leq \xi_i \leq 1, \forall i \\
(x^+ - x^-) \leq b \leq A(x^+ - x^-)
\end{cases},
\]

(23)

and \(w \triangleq (x^+; x^-; \xi)\), respectively. We introduce separability into the objective by adding a variable \(y \triangleq (y^+; y^-; \xi), y^+, y^-, \xi \in \mathbb{R}^n\) and imposing an additional linear constraint.

\[
\min_{w = y} f_0(x^+ - x^-) + \gamma \sum_{i=1}^{n} (1 - \xi_i) + 1_{Z_1}(w) + g(y^+ - y^-) + 1_{Z_2}(y) .
\]

(24)
Note that (24) is in the form of (3). The intuition behind this formulation is that by separating the nonconvex set $Z_1$ from the convex polytope $Z_2$, we may potentially obtain easier subproblems when applying a splitting method. We now define a perturbed augmented Lagrangian function as follows.

$$
\hat{\mathcal{L}}_{\rho, \alpha}(w, y, \lambda) \triangleq f_Q(x^+ - x^-) + \sum_{i=1}^{n} (1 - \xi_i) + g(y^+ - y^-) + (1 - \rho \alpha) \lambda^T (w - y - \alpha \lambda) + \frac{\rho}{2} \|w - y\|^2,
$$

where $w \triangleq (x^+, x^-, \xi)$, $\alpha > 0$, $\rho > 0$. The perturbed proximal ADMM algorithm is presented as Algorithm 1, denoted as $(\text{ADMM}^\mu_{\text{cf}}, \alpha, \rho)$, where “cf” stands for “complementarity formulation”, and $\mu, \alpha, \rho$ are algorithm parameters. The perturbation technique is inspired by Hajinezhad and Hong [26]. Note that $(\text{ADMM}^\mu_{\text{cf}}, \alpha, \rho)$ reduces to a basic ADMM when $\mu = \alpha = 0$, which will be discussed in Sect. 3.3. We refer the readers to Remark 4 and Remark 6 for discussion of the stopping criteria.

We observe that there are indeed some benefits by considering decomposition (24) that separates the nonconvex domain $Z_1$ and the convex polytope $Z_2$. It turns out that this approach reduces the difficulty of both subproblems of the ADMM framework. Next, (Update-1) and (Update-2) are shown to be tractable.

### 3.2 Tractable resolution of ADMM updates

We now show that (Update-1) possesses a hidden convexity property [5], allowing for claiming tractability of (Update-1) and obtaining a solution in closed form.

**Proposition 1** (Tractability of Update-1) Recall that $f_Q(x) = x^T M x + d^T x$ where $M$ may be a symmetric indefinite matrix with real eigenvalues given by

---

1 By saying that an optimization problem is tractable we mean that it either has a closed-form solution or lies in the range of convex programs that are polynomially solvable. We refer the readers to [4] for detailed discussion.
Consider (Update-1) in scheme (ADMM$^a_{cf}$) at iteration $k + 1$. Then $(\rho + \mu)I_n + 4M > 0$ implies $\rho + \mu + 4s_i > 0$, $\forall i = 1, \ldots, n$, and the following hold:

(i) The solution $w_{k+1} \triangleq (x_{k+1}^+; x_{k+1}^-; z_{k+1})$ can be obtained as a solution to a tractable convex program.

(ii) The solution $w_{k+1}$ is available in closed form. In particular, let $h \triangleq (d; -d; -\gamma e) + (1 - \rho a)\lambda_k - \rho y_k - \mu w_k$, and let $V$ be an orthogonal matrix such that $V^T MV = \text{diag}(s_1, \ldots, s_n) \triangleq S$, and let

$$G \triangleq \begin{pmatrix}
\frac{1}{2}I_n & \frac{\sqrt{2}}{2}I_n & \frac{1}{2}I_n \\
\frac{1}{2}I_n & -\frac{\sqrt{2}}{2}I_n & \frac{1}{2}I_n \\
-\frac{\sqrt{2}}{2}I_n & \frac{\sqrt{2}}{2}I_n & -\frac{1}{2}I_n
\end{pmatrix},$$

(25)

Also let $q \triangleq G^Th \triangleq (q_1; q_2; q_3)$, $z \triangleq (z_1; z_2; z_3), q_1, q_2, q_3, z_1, z_2, z_3 \in \mathbb{R}^n$, and

$$z_1 \triangleq \begin{cases}
\frac{-\langle\|q_1\| + \|q_3\|\rangle}{2(\rho + \mu)} & , \|q_1\| > 0 \\
\frac{-\langle\|q_3\|\rangle}{2(\rho + \mu)} & , \|q_1\| = 0
\end{cases}, \quad z_3 \triangleq \begin{cases}
\frac{-\langle\|q_1\| + \|q_3\|\rangle}{\sqrt{2(\rho + \mu)}} & , \|q_3\| > 0 \\
\frac{-\langle\|q_3\|\rangle}{\sqrt{2(\rho + \mu)}} & , \|q_3\| = 0
\end{cases}$$

Then $w_{k+1} \triangleq Gz$ is the solution to (Update-1).

**Proof** (i) The first subproblem in (ADMM$^a_{cf}$) is equivalent to the following:

$$\min_{w \in Z_i} \tilde{L}_{\rho, a}(w, y_k, \lambda_k) + \frac{\mu}{2} \|w - w_k\|^2$$

$$\equiv \min_{(x^+; x^-) \gamma_{\xi} = 0} \left\{ f_Q(x^+ - x^-) + \gamma \sum_{i=1}^n (1 - \xi_i) + (1 - \rho a)\lambda_k^T w + \frac{\rho}{2} \|w - y_k\|^2 \right\}$$

$$+ \frac{\mu}{2} \|w - w_k\|^2 \right\}$$

$$\equiv \min_{w^T Qw = 0} \left\{ w^THw + h^Tw \right\},$$

(27)

where $H \triangleq \begin{pmatrix}
M + \frac{\rho + \mu}{2}I_n & -M \\
-M & M + \frac{\rho + \mu}{2}I_n
\end{pmatrix}$, $\tilde{Q} \triangleq \begin{pmatrix}
I_n \\
I_n
\end{pmatrix}$ and recall that

$h = (d; -d; -\gamma e) + (1 - \rho a)\lambda_k - \rho y_k - \mu w_k$. Through matrix multiplication, we have

$$G^THG = \begin{pmatrix}
\frac{\rho + \mu}{2}I_n \\
2S + \frac{\rho + \mu}{2}I_n \\
\frac{\rho + \mu}{2}I_n
\end{pmatrix} > 0, \quad G^T \tilde{Q} = \begin{pmatrix}
-\sqrt{2I_n} \\
0 \\
\sqrt{2I_n}
\end{pmatrix},$$

(28)
To obtain an optimal solution of (29), we require that the objective value is bounded below. By completing squares, a sufficient condition for boundedness of (29) is $\frac{\rho + \mu}{2} + 2s_i > 0, \forall i = 1, \ldots, n$ because $z_2$ is unconstrained. This is implied by the condition $(\rho + \mu)I_n + 4M > 0$. The result of (ii) follows by noting that all optimal solutions $(z_1^*, z_2^*, z_3^*)$ of (29) can be characterized as follows:

$$z_1^* = \begin{cases} \frac{-\langle q_1, z_1 \rangle + \langle q_2, q_1 \rangle}{\|q_1\|}, & \|q_1\| > 0, \\ \frac{2(\rho + \mu)\|q_1\|}{\|q_1\| + 2(\rho + \mu)n, \|u\| = 1, \|q_1\| = 0}, \end{cases} \quad z_3^* = \begin{cases} \frac{-\langle q_1, z_1 \rangle + \langle q_2, q_1 \rangle}{\|q_1\|}, & \|q_3\| > 0 \\ \frac{2(\rho + \mu)\|q_3\|}{\|q_3\| + 2(\rho + \mu)n, \|v\| = 1, \|q_3\| = 0}, \end{cases}$$

$$(z_2^*)_i = -\langle q_2, q_1 \rangle / (\rho + \mu + 4s_i), \quad \text{for } i = 1, \ldots, n. \quad (30)$$

Next we show that this is true. Note that $(z_2^*)_i = -\langle q_2, q_1 \rangle / (\rho + \mu + 4s_i), \forall i$, because $z_2$ is unconstrained. Since the problem is separable with respect to $z_2$, it may be removed, leading to the problem of

$$\min_{z_1, z_3} \left( \frac{\rho + \mu}{2} \|z_1\|^2 + \frac{\rho + \mu}{2} \|z_3\|^2 + q_1^T z_1 + q_3^T z_3 \right) \quad \text{subject to } \|z_1\|^2 - \|z_3\|^2 = 0.$$ 

Since $z_1$ and $z_3$ have the same magnitude, let $z_1 \triangleq rd_1$ and $z_3 \triangleq rd_3$, where $\|d_1\| = \|d_3\| = 1$. Then the constraint may be removed and the problem is further simplified as

$$\min_{r, d_1, d_3} (\rho + \mu)r^2 + q_1^T d_1 + q_3^T d_3 \quad \text{subject to } r \geq 0, \|d_1\| = 1, \|d_3\| = 1.$$ 

It follows that $r^* = \arg\min_{r \geq 0} \left( (\rho + \mu)r^2 - (\|q_1\| + \|q_3\|)r \right) = (\|q_1\| + \|q_3\|) / (2(\rho + \mu))$.

This leads to concluding that if $\|q_1\| > 0, \|q_3\| > 0$, $z_1^* = -\langle q_1, z_1 \rangle / (2(\rho + \mu)\|q_1\|)$, $z_3^* = -\langle q_1, z_1 \rangle / (2(\rho + \mu)\|q_3\|)$ and $z_2^*$ can take any direction. If $\|q_3\| = 0, \|q_1\| > 0$, then $z_1^* = \langle q_1, q_3 \rangle / (2(\rho + \mu))$ and $z_1^* = -\langle q_1, q_3 \rangle / (2(\rho + \mu))$ and $z_2^*$ can take any direction. If $\|q_1\| = \|q_3\| = 0$, then $z_1^* = z_2^* = z_3^* = 0$. \qed

**Remark 3** Note that in order to compute the closed form solution, we do need the eigenvalue decomposition of $M$ give by $M = VSV^{-1}$. However, this only needs to be done once instead of in every iteration.
Next, (Update-2) is shown to be tractable and its solution may be available in closed-form.

**Proposition 2 (Tractability of Update-2)** Consider (Update-2) at iteration $k + 1$ in $(\text{ADMM}_{\text{cf}})$. Then the following hold: (i) (Update-2) is a convex program and can be computed tractably. (ii) If $g(x) \equiv 0$ and the constraints $Ax \geq b$ are absent, (Update-2) reduces to

$$y_{k+1} = \begin{cases} (y_{k+1,1}, y_{k+1,2}, y_{k+1,3}) \\ (y_{k+1,1})_i := \max \{ (x_{k+1}^+)_i + (1 - \rho \alpha)(\lambda_{k,1})_i / \rho, 0 \} \\ (y_{k+1,2})_i := \max \{ (x_{k+1}^-)_i + (1 - \rho \alpha)(\lambda_{k,2})_i / \rho, 0 \} \\ (y_{k+1,3})_i := \Pi_{[0, 1]}(\xi_{k+1})_i + (1 - \rho \alpha)(\lambda_{k,3})_i / \rho \} \forall i, \quad (31) \end{cases}$$

where $\lambda_k \triangleq (\lambda_{k,1}; \lambda_{k,2}; \lambda_{k,3})$, $y_{k+1,1}, y_{k+1,2}, y_{k+1,3}, \lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3} \in \mathbb{R}^n$, and $\Pi_Z(z)$ denotes the projection of $z$ onto set $Z$.

**Proof** (i) (Update-2) can be cast as a linearly constrained convex smooth program: 

$$\min_{y \in Z_2} \left\{ g(y^+ - y^-) - (1 - \rho \alpha)\lambda_k^T y + \frac{\rho}{2} \| (x_{k+1}^+; x_{k+1}^-; \xi_{k+1}) - y \|^2 \right\},$$

which may be tractably resolved [4].

(ii) When $g \equiv 0$ and the constraints $Ax \geq b$ are absent, then (Update-2) can be viewed as a projection of $(x_{k+1}^+; x_{k+1}^-; \xi_{k+1}) + (1 - \rho \alpha)\lambda_k / \rho$ onto a Cartesian set: 

$$\hat{Z}_2 \triangleq \{ (y_1, y_2, y_3) | y_1, y_2, y_3 \in \mathbb{R}^n, y_1 \geq 0, y_2 \geq 0, 0 \leq (y_3)_i \leq 1, \forall i \}. $$

Consequently, the projection onto this set reduces to the update given by (31). $\square$

### 3.3 A basic ADMM framework with tractable subproblems

In Algorithm 1, a perturbation parameter and a proximal term are introduced. In fact, as we see in Sect. 4, an explicit bound can be derived for the multiplier sequence generated by Algorithm 1. Therefore, we may show subsequential convergence and estimate a norm of the limit point. In this subsection, we present the vanilla ADMM framework (denoted by $(\text{ADMM}_{\text{cf}})$) and defined in Algorithm 2) applied to the tractable decomposition (24). In $(\text{ADMM}_{\text{cf}})$, the augmented Lagrangian function $\mathcal{L}_\rho$ is defined as follows.
Algorithm 2 (ADMM_{cf})

(0) Given $y_0, \lambda_0, \epsilon > 0$, $k := 0$; Choose $\rho_0, \text{s.t. } \rho_0 I_n + 4M \succ 0$;
(1) Let $x_{k+1}^+, x_{k+1}^-, y_{k+1}, \kappa_{k+1}$ be given by the following:

\[
(x_{k+1}^+, x_{k+1}^-, \kappa_{k+1}) \in \arg\min_{(x, y, \kappa)} \mathcal{L}_{\rho_k}(x^+, x^-, \kappa, y, \kappa_k),
\]

(Update-1)

\[y_{k+1} := \arg\min_{y} \mathcal{L}_{\rho_k}(x_{k+1}^+, x_{k+1}^-, \kappa_{k+1}, y, \kappa_k),\]

(Update-2)

\[\kappa_{k+1} := \kappa_k + \rho_k(x_{k+1}^+; x_{k+1}^-; \kappa_{k+1} - y_{k+1}).\]

(Update-3)

(2) Update $\rho_k$ and let $\rho_{k+1} \leftarrow \rho_k$;
(3) If $\max\{|(x_{k+1}^+; x_{k+1}^-; \kappa_{k+1}) - y_{k+1}|, \rho_k|y_{k+1} - y_k|\} < \epsilon$, STOP; else $k := k + 1$, return to (1).

\[
\mathcal{L}_{\rho}(x^+, x^-, \kappa, y, \lambda) \triangleq f_Q(x^+ - x^-) + \gamma \sum_{i=1}^n (1 - \xi_i) + g(y^+ - y^-)
\]

\[+ \lambda^T (w - y) + \frac{\rho}{2} ||w - y||^2 + 1l_{Z_1}((x^+; x^-; \xi)) + 1l_{Z_2}(y).\]

We will specify in Sect. 5 the update rule for $\rho_k$ in Step 2. Note that if we let $\mu = 0$, $\alpha = 0$ and replace $\rho$ by $\rho_k$, then (ADMM_{cf}^{\mu, \alpha, \rho}) reduces to (ADMM_{cf}). The similarity between these two algorithms allows for (ADMM_{cf}) to maintain the property of tractability of the subproblems (A special case of Propositions 1 and 2 when $\alpha = 0$, $\mu = 0$). However, convergence analysis of (ADMM_{cf}) is by no means straightforward. Since (24) is in the form of (3), as discussed in Sect. 1, we believe that no existing convergence theory for ADMM schemes in nonconvex regimes is currently available. It turns out that even showing boundedness of the multiplier sequence is challenging. We leave convergence analysis of (ADMM_{cf}) under mild and checkable assumptions for future work.

3.4 A standard ADMM framework on an alternative formulation

In Sect. 3.1, we consider reformulation (24) of (4), which allows for efficient resolution of the subproblem; Note that an alternative formulation of (4) exists as specified next.

\[
\min \ 1l_{Z}(w) + f(y^+ - y^-) + \gamma e^T(e - \zeta) \quad \text{subject to } w - y = 0,
\]

(32)

where $w = (x^+; x^-; \xi), y = (y^+; y^-; \zeta), Z \triangleq Z_1 \cap Z_2$, $Z_1$ and $Z_2$ are defined as in (23). Note that in (32), the $y$ block is unconstrained and has a smooth objective function. Such a reformulation of optimization over complementarity constraints is considered in [37] and an ADMM scheme can be applied (See Algorithm 3 below). We referred to this framework as a standard ADMM framework (or (ADMMo)), since this type of ADMM scheme is favored and most studied in literature due to clear convergence guarantee. As indicated in [37, Corollary 3], if $\rho_k \equiv \rho$ is large enough and the augmented Lagrangian function has the KL property, (ADMMo) generates a
sequence convergent to a stationary point. However, such a framework is potentially slow because (Update-1) requires globally resolving an MPCC and may render the scheme impractical. This will be further demonstrated with numerical experiments in Sect. 5.

### 4 Convergence analysis

In the prior section, we consider the formulation (24) to resolve (4) and present a perturbed proximal ADMM framework (ADMM\textsubscript{cf}) reliant on tractable updates at each iteration. In this section, we analyze the convergence properties of this framework. Specifically, we show that under mild assumptions, the sequence \( \{w_k, y_k, \lambda_k\} \) is bounded and a subsequence of \( \{w_k, y_k, \lambda_k\} \) converges to a perturbed KKT point of (24). The main results are Theorem 2, Corollary 1 and Theorem 3. First we present some definitions used in this section. We refer interested readers to [1] and [32] for more details.

#### Definition 3 ((Limiting) subdifferential and critical point)

Let \( F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a proper lower semicontinuous function and let \( \tilde{\partial}F(x) \) denote the Fréchet subdifferential of \( F \) at \( x \), i.e.,

\[
\tilde{\partial}F(x) \triangleq \left\{ d : \liminf_{z \to x, z \neq x} \frac{1}{\|z - x\|} [F(z) - F(x) - d^T(z - x)] \geq 0 \right\}
\]

for \( x \in \text{dom}F = \{x : F(x) < +\infty\} \) and \( \tilde{\partial}F(x) = \emptyset \) if \( x \notin \text{dom}F \). Then,

1. The (limiting) subdifferential of \( F \) at \( x \in \text{dom}F \) is defined as follows.
Tractable ADMM schemes for computing KKT points and local…

\[ \partial F(x) \triangleq \{ d \in \mathbb{R}^n : \exists \{ x_k \}_{k \geq 1}, \text{ s.t. } x_k \to x, F(x_k) \to F(x), d_k \to d, d_k \in \partial F(x_k) \}. \]

(ii) \( x \) is a critical point of \( F \) if and only if \( 0 \in \partial F(x) \).

**Definition 4** ((Limiting) normal cone) Suppose \( Z \) is a nonempty closed subset of \( \mathbb{R}^n \), and \( \tilde{N}_Z(x) \) denotes the Fréchet normal cone to \( Z \) at \( x \), defined as

\[ \tilde{N}_Z(x) \triangleq \{ v \in \mathbb{R}^n : v^T (z - x) \leq o(\|z - x\|), \forall z \in Z \} \]

if \( x \in Z \) and \( \tilde{N}_Z(x) \triangleq \emptyset \) if \( x \not\in Z \). Then the (limiting) normal cone to \( Z \) at \( x \in Z \), denoted as \( N_Z(x) \), is defined as follows.

\[ N_Z(x) \triangleq \{ v \in \mathbb{R}^n : \exists \{ x_k \}_{k \geq 1}, \text{ s.t. } x_k \to x, x_k \in Z, v_k \to v, v_k \in \tilde{N}_Z(x_k) \}. \]

These concepts have the following properties:

(i) (Closedness of \( \partial F \)) If \( d_k \to d \), \( x_k \to x \in \text{dom} F \) and \( d_k \in \partial F(x_k), F(x_k) \to F(x), \) then \( d \in \partial F(x) \).

(ii) Let \( F = F_0 + F_1 \). If \( F_0 \) is finite at \( x \) and \( F_1 \) is smooth in a neighborhood of \( x \), then at \( x \), we have \( \partial F(x) = \partial F_0(x) + \nabla F_1(x) \).

(iii) If \( x^* \in \text{argmin} \ F \), then \( x^* \) is a critical point of \( F \).

(iv) \( \partial \Pi_Z(z) = N_Z(z), \forall z \in Z \).

To simplify the notation, we rewrite (24) as the following structured program:

\[
\min_{w \in Z_1, y \in Z_2} h(w) + p(y) \quad \text{subject to } w - y = 0, \quad (33)
\]

where \( Z_1, Z_2 \) are defined by (23),

\[
h(w) \triangleq f_Q(x^+ - x^-) + \gamma \sum_{i=1}^n (1 - \xi_i), \quad f_Q(x^+ - x^-) \triangleq (x^+ - x^-)M(x^+ - x^-) + d^T(x^+ - x^-), \quad p(y) \triangleq g(y^+ - y^-).
\]

In addition, the perturbed augmented Lagrangian function is rewritten as follows.

\[
\tilde{L}_{\rho,\alpha}(w, y, \lambda) \triangleq h(w) + p(y) + (1 - \rho \alpha)\lambda^T(w - y - \alpha \lambda) + \frac{\rho}{2}\|w - y\|^2.
\]

Let \( r_k \triangleq w_k - y_k \) and \( \Delta \lambda_{k+1} \triangleq \lambda_{k+1} - \lambda_k \) for all \( k \geq 0 \). We define a Lyapunov function \( P^k_\tau \) for any \( \tau > 0 \) and \( k \geq 1 \).

\[
P^k_\tau \triangleq \tilde{L}_{\rho,\alpha}(w_k, y_k, \lambda_k) + \frac{(1 - \rho \alpha)\alpha}{2}\|\lambda_k\|^2 + \tau \left( \frac{1 - \rho \alpha}{2\rho} \right)\|\lambda_k - \lambda_{k-1}\|^2. \quad (34)
\]

We intend to show that the sequence \( \{ P^k_\tau \}_{k \geq 1} \) is nonincreasing and the following two lemmas are needed.

**Lemma 5** Consider the sequence \( \{ w_k, y_k, \lambda_k \} \) generated by \( \text{ADMM}^{\mu,\alpha,\rho}_c \).

Then the following holds for any \( \nu > 0 \), and any \( k \geq 1 \),

\[ \text{property} \]
\[
\frac{1 - \rho \alpha}{2 \rho} \left( \| \lambda_{k+1} - \lambda_k \|^2 - \| \lambda_k - \lambda_{k-1} \|^2 \right) \\
\leq - \left( \alpha - \frac{v}{2} \right) \| \lambda_{k+1} - \lambda_k \|^2 + \frac{1}{2v} \| w_{k+1} - w_k \|^2.
\]

(35)

**Proof** Let \( G_{k+1} \triangleq \nabla_y p(y_{k+1}) \). By (Update-2), for all \( y \in \mathbb{Z}_2 \) and \( k \geq 0, \)

\[
0 \geq \left( G_{k+1} - (1 - \rho \alpha) \lambda_k - \rho r_{k+1} \right)^T (y_{k+1} - y) \\
= \left( G_{k+1} - \lambda_{k+1} \right)^T (y_{k+1} - y).
\]

(36)

Consequently, we have that \( \forall k \geq 1, \)

\[
(G_k - \lambda_k)^T (y - y_k) \leq 0, \quad \forall y \in \mathbb{Z}_2.
\]

(37)

By choosing \( y = y_k \) in (36), \( y = y_{k+1} \) in (37), then adding (36) and (37), we have that for any \( k \geq 1, \)

\[
(G_{k+1} - G_k - \lambda_{k+1} + \lambda_k)^T (y_{k+1} - y_k) \leq 0, \\
\implies (G_{k+1} - G_k)^T (y_{k+1} - y_k) - (\lambda_{k+1} - \lambda_k)^T (y_{k+1} - y_k) \leq 0, \\
\implies - (\lambda_{k+1} - \lambda_k)^T (y_{k+1} - y_k) \leq 0,
\]

(38)

where the last step follows from convexity of \( p(y) \). Recall \( \Delta \lambda_k \triangleq \lambda_k - \lambda_{k-1}, \forall k \geq 1. \)

Then by adding \( \Delta \lambda_{k+1}^T (w_{k+1} - w_k) \) on both sides, (38) can be rewritten as follows for \( \forall k \geq 1, \)

\[
\begin{align*}
\Delta \lambda_{k+1}^T (w_{k+1} - w_k) \\
\geq \Delta \lambda_{k+1}^T (w_{k+1} - y_{k+1} - w_k + y_k) \\
= \Delta \lambda_{k+1}^T (r_{k+1} - r_k) \\
= \Delta \lambda_{k+1}^T (r_{k+1} - \alpha \lambda_k - r_k + \alpha \lambda_{k-1}) + \Delta \lambda_{k+1}^T (\alpha \lambda_k - \alpha \lambda_{k-1}) \\
= \Delta \lambda_{k+1}^T \left( \frac{\Delta \lambda_{k+1}}{\rho} - \frac{\Delta \lambda_k}{\rho} \right) + \alpha \Delta \lambda_{k+1}^T \Delta \lambda_k \\
= \frac{1 - \rho \alpha}{\rho} \Delta \lambda_{k+1}^T (\Delta \lambda_{k+1} - \Delta \lambda_k) + \alpha \| \Delta \lambda_{k+1} \|^2.
\end{align*}
\]

(39)

Note that \( \Delta \lambda_{k+1}^T (\Delta \lambda_{k+1} - \Delta \lambda_k) = \frac{1}{2} (\| \Delta \lambda_{k+1} \|^2 - \| \Delta \lambda_k \|^2 + \| \Delta \lambda_{k+1} - \Delta \lambda_k \|^2). \) Let \( \Delta w_{k+1} \triangleq w_{k+1} - w_k. \) Then by (39),
\[
\frac{1 - \rho \alpha}{2\rho} \cdot (\|\Delta \lambda_{k+1}\|^2 - \|\Delta \lambda_k\|^2 + \|\Delta \lambda_{k+1} - \Delta \lambda_k\|^2) + \alpha \|\Delta \lambda_{k+1}\|^2 \\
\leq \Delta \lambda_{k+1}^T \Delta w_{k+1}
\]

\[\Rightarrow \quad \frac{1 - \rho \alpha}{2\rho} \cdot (\|\Delta \lambda_{k+1}\|^2 - \|\Delta \lambda_k\|^2) \leq -\alpha \|\Delta \lambda_{k+1}\|^2 + \Delta \lambda_{k+1}^T \Delta w_{k+1}
\]
\[\leq -\alpha \|\Delta \lambda_{k+1}\|^2 + \frac{v}{2} \|\Delta \lambda_{k+1}\|^2 + \frac{\|\Delta w_{k+1}\|^2}{2v}
\]
\[= \frac{\|\Delta w_{k+1}\|^2}{2v} - \left(\alpha - \frac{v}{2}\right)\|\Delta \lambda_{k+1}\|^2.
\]

Then the proof is complete. \qed

**Lemma 6.** Consider \(\{w_k, y_k, \lambda_k\}\) generated by (ADMM\(^{\mu,\alpha,\rho}\)). Then

\[
\begin{aligned}
\left(\tilde{L}_{\rho,\alpha}(w_{k+1}, y_{k+1}, \lambda_{k+1}) + \frac{(1 - \rho \alpha)\alpha}{2}\|\lambda_{k+1}\|^2\right)
\end{aligned}
\]

\[-\left(\tilde{L}_{\rho,\alpha}(w_k, y_k, \lambda_k) + \frac{(1 - \rho \alpha)\alpha}{2}\|\lambda_k\|^2\right) \\
\leq -\frac{\mu}{2}\|w_{k+1} - w_k\|^2 - \frac{\rho}{2}\|y_{k+1} - y_k\|^2 + \frac{(1 - \rho \alpha)(2 - \rho \alpha)}{2\rho}\|\Delta \lambda_{k+1}\|^2.
\]

**Proof.** From (Update-1),

\[
\begin{aligned}
\tilde{L}_{\rho,\alpha}(w_{k+1}, y_k, \lambda_k) + \frac{\mu}{2}\|w_{k+1} - w_k\|^2 - \tilde{L}_{\rho,\alpha}(w_k, y_k, \lambda_k) \leq 0
\end{aligned}
\]

\[
\Rightarrow \quad \tilde{L}_{\rho,\alpha}(w_{k+1}, y_k, \lambda_k) - \tilde{L}_{\rho,\alpha}(w_k, y_k, \lambda_k) \leq -\frac{\mu}{2}\|w_{k+1} - w_k\|^2.
\]

Also, by the optimality condition of (Update-2), if \(\tilde{G}_{k+1} \triangleq \nabla_y \tilde{L}_{\rho,\alpha}(w_{k+1}, y_{k+1}, \lambda_k)\), then \(\tilde{G}_{k+1}^T(y - y_{k+1}) \geq 0, \forall y \in Z_2\). Using this fact and the strong convexity of \(\tilde{L}_{\rho,\alpha}\) in terms of \(y\) with constant \(\rho\),

\[
\begin{aligned}
\tilde{L}_{\rho,\alpha}(w_{k+1}, y_{k+1}, \lambda_k) - \tilde{L}_{\rho,\alpha}(w_{k+1}, y_k, \lambda_k) \\
\leq -\tilde{G}_{k+1}^T(y_{k+1} - y_k) - \frac{\rho}{2}\|y_{k+1} - y_k\|^2 \leq -\frac{\rho}{2}\|y_{k+1} - y_k\|^2.
\end{aligned}
\]

The fact that \(\Delta \lambda_{k+1} = \rho r_{k+1} - \rho \alpha \lambda_k\) and \(\lambda_{k+1}^T \Delta \lambda_{k+1} = \frac{1}{2}(\|\lambda_{k+1}\|^2 - \|\lambda_k\|^2 + \|\Delta \lambda_{k+1}\|^2)\) imply:
Finally, by adding (41), (42) and (43), the following holds ∀k ≥ 0,

\[
\begin{align*}
\mathcal{L}_{\rho,a}(w_{k+1}, y_{k+1}, \lambda_{k+1}) &= \mathcal{L}_{\rho,a}(w_k, y_k, \lambda_k) \\
&= \mathcal{L}_{\rho,a}(w_{k+1}, y_{k+1}, \lambda_{k+1}) - \mathcal{L}_{\rho,a}(w_{k+1}, y_{k+1}, \lambda_k) + \mathcal{L}_{\rho,a}(w_{k+1}, y_{k+1}, \lambda_k) \\
&- \mathcal{L}_{\rho,a}(w_{k+1}, y_{k+1}, \lambda_k) + \mathcal{L}_{\rho,a}(w_{k+1}, y_{k+1}, \lambda_k) - \mathcal{L}_{\rho,a}(w_{k+1}, y_{k+1}, \lambda_k) \\
&\leq -\frac{\mu}{2} ||w_{k+1} - w_k||^2 - \frac{\rho}{2} ||y_{k+1} - y_k||^2 \\
&\quad + \frac{(1 - \rho\alpha)(2 - \rho\alpha)}{2\rho} ||\Delta \lambda_{k+1}||^2 - \frac{(1 - \rho\alpha)\alpha}{2}(||\lambda_{k+1}||^2 - ||\lambda_k||^2).
\end{align*}
\]

Then the result follows. \qed

We now impose a requirement on \( h(w) + p(y) + \frac{\rho}{2} ||w - y||^2 \) and define several constants to be used later.

**Assumption 1** \( h(w) + p(y) + \frac{\rho}{2} ||w - y||^2 \geq \bar{L} \) for all \( w \in Z_1, y \in Z_2 \).

**Definition 5** Recall that \( \alpha, \tau, \mu, \rho \) are nonnegative parameters from Algorithm 1 and the definition (34). Let \( \nu \) and \( R \) be nonnegative constants. Then \( c_1(\nu) \triangleq \frac{\mu}{2} - \frac{\tau}{2\nu}, c_2 \triangleq \frac{\rho}{2}, c_3(\nu) \triangleq \nu \left( \alpha - \frac{\nu}{2} \right) - \frac{(1 - \rho\alpha)(2 - \rho\alpha)}{2\rho}, \)
\[ c_4(R) \triangleq \frac{(1 - \rho\alpha)(R + 1)\rho - 1}{2\rho R}, c_5(R) \triangleq \frac{1 - \rho\alpha}{2\rho} [\tau - (1 - \rho\alpha)R]. \]

**Assumption 2** \( \exists \nu > 0, R > 0 \) such that \( c_1(\nu), c_3(\nu), c_4(R), c_5(R) > 0 \).

Assumption 2 imposes some restrictions on the choice of parameters \( \alpha, \tau, \mu, \rho \). In the next lemma we discuss the conditions for these parameters in order to guarantee Assumption 2.

**Lemma 7** Let \( \alpha, \tau, \mu, \rho \) be parameters from Algorithm 1 and the definition (34). Then Assumption 2 holds if \( \alpha, \tau, \mu, \rho \) are positive and satisfy the following inequalities:
\[
\tau \rho / \mu < 2\rho - (1 - \rho\alpha)(2 - \rho\alpha) / \tau, \quad (1 - \rho\alpha)^2 < \rho \alpha, \quad 0 < \rho\alpha < 1. \quad (44)
\]
Proof Let \( \alpha, \tau, \mu, \rho \) be positive. Then \( c_1(v) > 0, c_3(v) > 0 \iff v > \tau / \mu \) and \( v < 2\alpha - (1 - \rho \alpha)(2 - \rho \alpha) / (\tau \rho) \). Therefore, such \( v > 0 \) exists if \( \tau / \mu < 2\alpha - (1 - \rho \alpha)(2 - \rho \alpha) / (\tau \rho) \), which is exactly the first inequality in (44). \( c_4(R) > 0, c_3(R) > 0 \) if \( R > 1 / (\rho \alpha) - 1 \), \( R < \tau / (1 - \rho \alpha) \) and \( 0 < \rho \alpha < 1 \). Therefore existence of \( R > 0 \) is implied by \( 1 / (\rho \alpha) - 1 < \tau / (1 - \rho \alpha) \) and \( 0 < \rho \alpha < 1 \), which represent the last two inequalities in (44). \( \square \)

Note that solution of (44) does exist. We may let \( \rho \alpha = 1/2, \tau = 2, \mu = 4\rho \). Then (44) holds. Moreover, \( \rho \) can be chosen large enough such that \( (\rho + \mu)I_n + 4M > 0 \), as required by Algorithm 1.

In the next lemma, we prove that \( \{P^k_r\}_{k \geq 1} \) is a nonincreasing sequence.

Lemma 8 Consider \( \{w^k_r, y^k_r, \lambda^k_r\} \) generated by (ADMM\( ^{\mu, \alpha, \rho} \)). Then the following hold.

(i) \( P^{k+1}_r - P^k_r \leq -c_1(v)\|w^k_r - w^k\|^2 - c_2\|y^k_r - y^k\|^2 - c_3(v)\|\lambda^k_{r+1} - \lambda^k_r\|^2 \), for any \( k \geq 1 \).

(ii) Suppose that Assumption 2 holds. Then \( \{P^k_r\}_{k \geq 1} \) is non-increasing. If Assumption 1 also holds, then \( P^k_r \) is bounded from below.

(iii) If Assumption 1 and 2 hold, then

\[
\lim_{k \to \infty} (w^k_r - w^k) = \lim_{k \to \infty} (y^k_r - y^k) = \lim_{k \to \infty} (\lambda^k_{r+1} - \lambda^k_r) = 0.
\]

Proof (i) Take \( \tau \times (35) + (40) \) and the result follows.

(ii) When \( c_1(v), c_2, c_3(v) > 0 \) for certain \( v \), we conclude from (i) that \( P^{k+1}_r \leq P^k_r \) for all \( k \geq 1 \). Further,

\[
(1 - \rho \alpha)\lambda^T_k (r_k - \alpha \lambda_k) = (1 - \rho \alpha)\lambda^T_k (r_k - \alpha \lambda_{k-1} - \alpha \Delta \lambda_k) = (1 - \rho \alpha)\lambda^T_k \left[ \frac{\Delta \lambda_k}{\rho} - \alpha \frac{\Delta \lambda_k}{\rho} \right] = \frac{(1 - \rho \alpha)^2}{\rho} \lambda_k (\lambda_k - \lambda_{k-1}) = \frac{(1 - \rho \alpha)^2}{2 \rho} \left( \|\lambda_k\|^2 - \|\lambda_{k-1}\|^2 + \|\lambda_k - \lambda_{k-1}\|^2 \right) \]

\[
\geq [(1 - \rho \alpha)^2 / (2 \rho)](\|\lambda_k\|^2 - \|\lambda_{k-1}\|^2), \quad k \geq 1.
\]

Then,
\[ P_k^\varepsilon = h(w_k) + p(y_k) + \frac{\rho}{2} ||w_k - y_k||^2 \]
\[ + (1 - \rho\alpha)\lambda_k^T (r_k - \alpha \lambda_k) + \frac{(1 - \rho\alpha)}{2} ||\lambda_k||^2 + \tau \left( \frac{1 - \rho\alpha}{2\rho} \right) ||\Delta \lambda_k||^2 \]
\[ \geq \tilde{L} + (1 - \rho\alpha)\lambda_k^T (r_k - \alpha \lambda_k) + \frac{(1 - \rho\alpha)}{2} ||\lambda_k||^2 + \tau \left( \frac{1 - \rho\alpha}{2\rho} \right) ||\Delta \lambda_k||^2 \]
\[ \geq \tilde{L} + (1 - \rho\alpha)\lambda_k^T (r_k - \alpha \lambda_k) \geq \tilde{L} + \frac{(1 - \rho\alpha)}{2\rho} (||\lambda_k||^2 - ||\lambda_{k-1}||^2). \]  

Then, \[ \sum_{k=1}^{K} (P_k^\varepsilon - \tilde{L}) \geq \frac{(1 - \rho\alpha)^2}{2\rho} \sum_{k=1}^{K} (||\lambda_k||^2 - ||\lambda_{k-1}||^2) \geq -\frac{(1 - \rho\alpha)^2}{2\rho} ||\lambda_0||^2, \] for any \( K \geq 1. \) Since \( \{P_k^\varepsilon - \tilde{L}\}_{k\geq1} \) is a non-increasing sequence and the above inequality holds, \( \{P_k^\varepsilon - \tilde{L}\}_{k\geq1} \) is nonnegative. Thus \( \{P_k^\varepsilon\}_{k\geq1} \) is bounded from below.

(iii) This may be concluded based on (i) and (ii). \( \square \)

**Remark 4** By Lemma 8(iii) and the stopping criterion, Algorithm (ADMM\(_{cf}\)) may terminate in finite time.

The following lemma provides an inequality related to \( ||\lambda_k|| \), which helps in showing boundedness of \( ||\lambda_k|| \).

**Lemma 9** Consider \( \{w_k, y_k, \lambda_k\} \) generated by (ADMM\(_{cf}\)). Suppose Assumption 1 holds. Then \( P_k^\varepsilon \geq \tilde{L} + c_4(R) ||\lambda_k||^2 + c_5(R) ||\lambda_k - \lambda_{k-1}||^2 \) for all \( R > 0 \) and \( k \geq 1. \) Therefore, if Assumption 2 holds, \( ||\lambda_k||^2 \leq c_6(R) \sum_{k=1}^{K} (P_k^\varepsilon - \tilde{L}) \) for all \( k \geq 1. \)

**Proof** We may use the following result for any \( R > 0 \):

\[ ||\lambda_{k-1}||^2 = ||\lambda_{k-1} - \lambda_k + \lambda_k||^2 \]
\[ \leq (1 + R)||\lambda_{k-1} - \lambda_k||^2 + (1 + 1/R)||\lambda_k||^2. \]  

From the definition of \( P_k^\varepsilon \), we have that:

\[ P_k^\varepsilon \geq \tilde{L} + \frac{(1 - \rho\alpha)^2}{2}\lambda_k^T \lambda_k + \tau \left( \frac{1 - \rho\alpha}{2\rho} \right) ||\lambda_k - \lambda_{k-1}||^2 \]
\[ + \frac{(1 - \rho\alpha)^2}{2\rho} (||\lambda_k||^2 - ||\lambda_{k-1}||^2 + ||\lambda_k - \lambda_{k-1}||^2) \]
\[ \geq \tilde{L} + \frac{(1 - \rho\alpha)^2}{2} ||\lambda_k||^2 \]
\[ + \frac{(1 - \rho\alpha)^2}{2\rho} \left( \frac{||\lambda_k||^2}{R} + R||\lambda_k - \lambda_{k-1}||^2 \right) \]
\[ - \frac{2\rho}{(1 - \rho\alpha)(R + 1)\rho\alpha - 1} \left[ ||\lambda_k||^2 + \frac{1 - \rho\alpha}{2\rho} [\tau - (1 - \rho\alpha)R] ||\lambda_k - \lambda_{k-1}||^2. \right] \]
Then the result follows from definitions of $c_3(R)$ and $c_5(R)$. \hfill \Box

Boundedness of $\|\lambda_k\|$ and subsequential convergence are proved in the next theorem.

**Theorem 2** Suppose \{(w_k,y_k;\lambda_k)\}_{k\geq 0} is generated by (ADMM$_{\mu,a,\rho}$). Assume that the sequences \{w_k\} and \{y_k\} are bounded. Suppose Assumptions 1 and 2 hold. Then the sequence \{\lambda_k\}_{k\geq 1} is bounded and a subsequence of \{(w_k,y_k;\lambda_k)\} converges to \((w^*;y^*;\lambda^*)\) such that

$$0 \in \partial(h + 11Z_1)(w^*) + \lambda^*, \quad 0 \in \partial(p + 11Z_2)(y^*) - \lambda^*, \quad w^* - y^* = a\lambda^*.$$  \hfill (49)

**Proof** Since $c_1(v) > 0$ and $c_3(v) > 0$, then by Lemma 8, \{ $P^k$ \} is a non-increasing sequence. Futhermore, since $c_4(R) > 0$, $c_5(R) > 0$, Lemma 9 indicates that

$$\|\lambda_k\|^2 \leq \frac{1}{c_4(R)}(P^k - \bar{L}) \leq \frac{1}{c_4(R)}(P^1 - \bar{L}) < +\infty.$$  \hfill (50)

Therefore, the sequence \{\lambda_k\} is bounded, implying that \{(w_k,y_k;\lambda_k)\} is bounded. Suppose \{(w_{n_k},y_{n_k};\lambda_{n_k})\} denotes a convergent subsequence of \{(w_k,y_k;\lambda_k)\} such that \((w_{n_k},y_{n_k};\lambda_{n_k}) \rightarrow (w^*;y^*;\lambda^*)\) as $k \rightarrow \infty$. Based on the optimality conditions of (Update-1), (Update-2) translated using the property of critical points, and the multiplier update, the following hold:

$$0 \in \partial(h + 11Z_1)(w_{n_k}) + \lambda_{n_k} + \rho(y_{n_k} - y_{n_k-1}) + \mu(w_{n_k} - w_{n_k-1}),$$
$$0 \in \partial(p + 11Z_2)(y_{n_k}) - \lambda_{n_k}, \quad w_{n_k} - y_{n_k} - \alpha \lambda_{n_k-1} = (\lambda_{n_k} - \lambda_{n_k-1})/\rho.$$  \hfill (51)

By Lemma 8, $w_{n_k} - w_{n_k-1} \rightarrow 0$, $y_{n_k} - y_{n_k-1} \rightarrow 0$, $\lambda_{n_k} - \lambda_{n_k-1} \rightarrow 0$, $k \rightarrow +\infty$, so we also have $\lambda_{n_k-1} \rightarrow \lambda^*$, $k \rightarrow +\infty$. Since $Z_1$ and $Z_2$ are closed, we have that $w^* \in Z_1$, $y^* \in Z_2$. Therefore, by taking limits and the closedness of a subdifferential map, we may conclude the result. \hfill \Box

**Remark 5**

(i) Boundedness of \{y_k\} and \{w_k\} is a mild assumption. First, boundedness of \{y_k\} can be obtained from adding constraints such as $x^+ \leq ub^+$, $x^- \leq ub^-$ to $Z_2$. If $ub^+$ and $ub^-$ are large enough, $Z_2$ will still include the optimal solution. Second, since $r_k = w_k - y_k = \frac{1}{\rho} \lambda_k - \frac{1-\alpha}{\rho} \lambda_{k-1}$, for any $k \geq 1$ and \{\lambda_k\} is bounded, $r_k$ is also a bounded sequence. Thus, boundedness of \{w_k\} is implied by boundedness of \{y_k\}.

(ii) It can be shown that the conditions (49) are equivalent to KKT conditions with a feasibility error (See Theorem 3 below).

(iii) Denote $\mathcal{H}_x(w,y,\lambda)$ as
\[ \mathcal{H}_\tau(w, y, \lambda) \triangleq \mathcal{L}_{\rho, \alpha}(w, y, \lambda) + 11Z_1(w) + 11Z_2(y) + \frac{(1 - \rho \alpha)\alpha}{2} \| \lambda \|^2 + \rho\| w - y - \alpha \lambda \|^2 \frac{2}{2(1 - \rho \alpha)/\tau}. \]

Then \( \mathcal{H}_\tau(w_k, y_k, \lambda_k) = P_k^\tau, \forall k \geq 1, \tau > 0. \) If the assumptions in Theorem 2 hold, and in addition, \( \mathcal{H}_\tau(w, y, \lambda) \) satisfies the KL property at \( (w^*; y^*; \lambda^*) \), then \( \{ (w_k, y_k, \lambda_k) \} \) converges to \( (w^*; y^*; \lambda^*) \). See Lemma 10 in the Appendix.

(iv) Although in the context of this paper we focus on problem (24), it should be noted that Theorem 2 may be generalized. Specifically, Theorem 2 may hold if we apply \( \text{ADMM}_{\text{cf}}^{\rho, \alpha, \varphi} \) to tackle a more general class of problem:

\[ \min f(x) + g(y) \quad \text{subject to } Ax + By = b, x \in X, y \in Y, \]

where \( g \) is smooth and convex, \( Y \) is convex, but \( f \) could be nonsmooth and nonconvex, \( X \) may be nonconvex, \( A, B \) and \( b \) are matrices and vector with appropriate dimensions and do not need to be \( I, -I \) and \( 0 \), respectively. The analysis will basically remain the same.

Note that (49) are not the precise conditions for \( (w^*; y^*; \lambda^*) \) to be a critical point of the Lagrangian \( \mathcal{L}(w, y, \lambda) \triangleq h(w) + 11Z_1(w) + p(y) + 11Z_2(y) + \lambda^T (w - y) \), i.e. \( 0 \in \partial \mathcal{L}(w^*; y^*; \lambda^*) \). There exists an infeasibility error \( \alpha \lambda^* \) and the following corollary discusses how to choose the parameters such that this error can be made arbitrarily small.

**Corollary 1** Suppose that sequences \( \{ w_k \} \) and \( \{ y_k \} \) are bounded. In addition, assume that \( \exists \rho_+ > 0 \) and \( l \in \mathbb{R}^n \) such that \( h(w) + p(y) + \rho_+ \| w - y \|^2 \geq l \) for all \( w \in Z_1, y \in Z_2 \). Then for any \( \epsilon > 0 \) such that \( \epsilon \leq 1/(4\rho_+ + 2) \), if the parameters in Algorithm 1 satisfy \( \alpha = \epsilon, \rho = \frac{1}{2\epsilon}, \mu > \frac{\epsilon}{\rho}, w_0 = y_0, (\rho + \mu)I_n + M > 0 \), then a subsequence of \( \{ (w_k; y_k; \lambda_k) \}_{k \geq 1} \) converges to \( (w^*; y^*; \lambda^*) \) such that

\[ 0 \in \partial (h + 11Z_1)(w^*) + \lambda^*, \quad 0 \in \partial (p + 11Z_2)(y^*) - \lambda^*, \]

\[ \| w^* - y^* \|^2 \leq \alpha^2 \| \lambda^* \|^2 \leq (64\| h(w_0) + p(y_0) \| - 64l + (14 + 5\epsilon))\| \lambda_0 \|^2 \epsilon. \]

**Proof** Let \( \nu = \alpha = \epsilon, R = 2 \) and \( \tau = 2 \), then

\[ c_1(\nu) = \frac{\mu}{2} - \frac{\tau}{2\nu} = \frac{\mu}{2} - \frac{1}{\epsilon} > 0, \quad c_2 = \frac{\rho}{2} = \frac{1}{4\epsilon}, \]

\[ c_3(\nu) = \nu \left( \frac{\alpha - \nu}{\tau} \right) - \frac{(1 - \rho \alpha)(2 - \rho \alpha)}{2\rho} = 2 \left( \frac{\epsilon}{2} - \frac{\epsilon}{2} \right) - \frac{(1 - \frac{1}{2})(2 - \frac{1}{2})}{1/\epsilon} = \frac{\epsilon}{4}, \]

\[ c_4(R) = \frac{(1 - \rho \alpha)[(R + 1)\rho \alpha - 1]}{2\rho R} = \frac{(1 - 1/2)(2 + 1/2 - 1)}{2/\epsilon} = \frac{\epsilon}{8}, \]

\[ c_5(R) = \frac{1 - \rho \alpha}{2\rho} \left[ \tau - (1 - \rho \alpha)R \right] = \frac{1}{1/\epsilon} \left[ 2 - (1 - 1/2) \cdot 2 \right] = \frac{\epsilon}{2}. \]

Therefore, Assumption 2 holds. Since \( \rho = \frac{1}{2\epsilon} \geq 2\rho_+ + 1 \), we have that
Tractable ADMM schemes for computing KKT points and local...

\[ h(w) + p(y) + \frac{\rho}{2} \|w - y\|^2 \geq h(w) + p(y) + \frac{\rho}{2} \|w - y\|^2 \geq l, \forall w \in Z_1, y \in Z_2. \]

Thus Assumption 1 holds. Based on Theorem 2, it suffices to show that \( \alpha^2 \| \lambda^* \|^2 \leq (64(h(w_0) + p(y_0)) - 64l + (14 + 5e)\| \lambda_0 \|^2) e. \) By (50) in Theorem 2, for \( k \geq 1, \)

\[ \| \lambda_k \|^2 \leq \frac{1}{c_4(R)} (P^1_k - l) = \left( \frac{2}{1 - \rho \alpha} \right) \left( \frac{\rho R}{(R + 1) \rho \alpha - 1} \right) (P^1_k - l) \]

\[ \Rightarrow \alpha^2 \| \lambda_k \|^2 \leq \left( \frac{2}{1 - \rho \alpha} \right) \left( \frac{\alpha R}{R + 1 - 1/(\rho \alpha)} \right) (P^1_k - l). \]

Since \( P^1_k = \tilde{L}_{\rho,\alpha}(w_1, y_1, \lambda_1) + \frac{(1 - \rho \alpha)\alpha}{2} \| \lambda_1 \|^2 + \tau \left( \frac{1 - \rho \alpha}{2 \rho} \right) \| \lambda_1 - \lambda_0 \|^2, \)

\[ P^1_k \leq \tilde{L}_{\rho,\alpha}(w_0, y_0, \lambda_0) + \frac{(1 - \rho \alpha)\alpha}{2} \| \lambda_0 \|^2 + \tau \left( \frac{1 - \rho \alpha}{2 \rho} \right) \| \lambda_1 - \lambda_0 \|^2 \]

\[ - \frac{\mu}{2} \| w_1 - w_0 \|^2 - \frac{\rho}{2} \| y_1 - y_0 \|^2 + \frac{(1 - \rho \alpha)(2 - \rho \alpha)}{2 \rho} \| \lambda_1 - \lambda_0 \|^2 \]

\[ \leq h(w_0) + p(y_0) + \frac{\rho}{2} \| r_0 \|^2 + (1 - \rho \alpha)\lambda_0^T(r_0 - \alpha \lambda_0) \]

\[ + \frac{(1 - \rho \alpha)\alpha}{2} \| \lambda_0 \|^2 + \frac{(1 - \rho \alpha)(2 + \tau - \rho \alpha)}{2 \rho} \| \lambda_1 - \lambda_0 \|^2 \]

\[ = h(w_0) + p(y_0) + 0 - (1 - 1/2)\epsilon \| \lambda_0 \|^2 + \frac{(1 - 1/2)\epsilon}{2} \| \lambda_0 \|^2 \]

\[ + \frac{(1 - 1/2)(2 + 2 - 1/2)}{1/\epsilon} \| (1 - 1/2) \lambda_0 + \rho (w_1 - y_1) - \lambda_0 \|^2 \]

\[ \leq h(w_0) + p(y_0) - \frac{\epsilon}{4} \| \lambda_0 \|^2 + \frac{7\epsilon}{4} \left( \frac{1}{2} \| \lambda_0 \|^2 + 2 \rho^2 \| w_1 - y_1 \|^2 \right) \]

\[ \leq h(w_0) + p(y_0) + \frac{5\epsilon}{8} \| \lambda_0 \|^2 + \frac{7}{8\epsilon} \| w_1 - y_1 \|^2. \]

Adding (41) and (42) and letting \( k = 0, \) we obtain the following.

\[ \tilde{L}_{\rho,\alpha}(w_1, y_1, \lambda_0) - \tilde{L}_{\rho,\alpha}(w_0, y_0, \lambda_0) \leq - \frac{\mu}{2} \| w_1 - w_0 \|^2 - \frac{\rho}{2} \| y_1 - y_0 \|^2, \]

which indicates that
\[ h(w_1) + p(y_1) + (1 - \rho \alpha)\lambda_0^T(w_1 - y_1 - \alpha \lambda_0) + \frac{\rho}{2}\|w_1 - y_1\|^2 \leq h(w_0) + p(y_0) + (1 - \rho \alpha)\lambda_0^T(w_0 - y_0 - \alpha \lambda_0) + \frac{\rho}{2}\|w_0 - y_0\|^2 \]
\[ = h(w_0) + p(y_0) - (1 - \rho \alpha)\alpha\|\lambda_0\|^2 \]
\[ \implies (1 - \rho \alpha)\lambda_0^T(w_1 - y_1) + \frac{\rho - \rho_+}{2}\|w_1 - y_1\|^2 \leq h(w_0) + p(y_0) - h(w_1) - p(y_1) - \frac{\rho - \rho_+}{2}\|w_1 - y_1\|^2 \]
\[ \implies \frac{\rho - \rho_+ - (1 - \rho \alpha)}{2}\|w_1 - y_1\|^2 \leq h(w_0) + p(y_0) - l + (1 - \rho \alpha)\|\lambda_0\|^2/2 + (1 - \rho \alpha)\|w_1 - y_1\|^2/2 \]
\[ \implies \frac{1/\epsilon - 2\rho_- - 1}{4}\|w_1 - y_1\|^2 \leq h(w_0) + p(y_0) - l + \|\lambda_0\|^2/4. \]
\[ \implies \|w_1 - y_1\|^2 \leq 8\epsilon(h(w_0) + p(y_0) - l + \|\lambda_0\|^2/4), \] (54)

where the last inequality holds because \( \epsilon \leq 1/(4\rho_+ + 2). \) By (53) and (54),
\[ P_1^1 \leq 8(h(w_0) + p(y_0)) - 7l + \left(\frac{7}{4} + \frac{5\epsilon}{8}\right)\|\lambda_0\|^2 \] (55)

By combining (52) and (55), we have for any \( k \geq 1, \)
\[ \alpha^2\|\lambda_k\|^2 \leq \frac{2}{1 - 1/2} \cdot \frac{2}{2 + 1 - 2} \left(8(h(w_0) + p(y_0)) - 7l + \left(\frac{7}{4} + \frac{5\epsilon}{8}\right)\|\lambda_0\|^2 - l\right)\epsilon \] (56)
\[ = (64(h(w_0) + p(y_0)) - 64l + (14 + 5\epsilon)\|\lambda_0\|^2)\epsilon. \]

This implies that \( \alpha^2\|\lambda^*\|^2 \leq (64(h(w_0) + p(y_0)) - 64l + (14 + 5\epsilon)\|\lambda_0\|^2)\epsilon. \)

**Remark 6** Based on the optimality conditions of (Update-1), (Update-2), and the multiplier update, the following holds for any \( k \geq 0: \)

\[ 0 \in \partial(h + 11Z_1)(w_{k+1}) + \lambda_{k+1} + \rho(y_{k+1} - y_k) + \mu(w_{k+1} - w_k), \]
\[ 0 \in \partial(p + 11Z_2)(y_{k+1}) - \lambda_{k+1}, \]
\[ w_{k+1} - y_{k+1} - \alpha \lambda_k = (\lambda_{k+1} - \lambda_k)/\rho. \]

According to (56), if we choose the parameters as in Corollary 1, the stopping criteria in Algorithm 1 indicates that:
\[ \text{dist}(0, \partial(h + 11Z_1)(w_{k+1}) + \lambda_{k+1}) < \epsilon_0, \ 0 \in \partial(p + 11Z_2)(y_{k+1}) - \lambda_{k+1}, \]
\[ \|w_{k+1} - y_{k+1}\| < \sqrt{(64(h(w_0) + p(y_0)) - 64l + (14 + 5\epsilon)\|\lambda_0\|^2)\epsilon + \epsilon_0}. \]

Finally we will show that the conditions (49) in Theorem 2 are equivalent to KKT conditions of (24) with a feasibility error \( \alpha \lambda^*. \)
Denote $w^* \triangleq (x^*_1; x^*_4; \xi^*)$, $w \triangleq (x^*: x^-; \xi)$, $y^* \triangleq (y^*_1; y^*_2; y^*_3)$, $\gamma \triangleq (y_1; y_2; y_3)$. By Definition 2, $(w^*; y^*)$ satisfies first-order KKT conditions of (24) if there exist $\mu \in \mathbb{R}$, $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}^n$, $\pi \in \mathbb{R}^m$ such that

\[
\begin{pmatrix}
\nabla f_Q(x^*_1 - x^*_2) + \nabla g(y^*_1 - y^*_2) \\
\nabla f_Q(x^*_4 - x^*_2) - \nabla g(y^*_1 - y^*_2) \\
-\gamma e
\end{pmatrix}
\begin{pmatrix}
\mu \xi^* - \beta_1 - A^T \pi \\
\mu \xi^* - \beta_2 + A^T \pi \\
\mu(x^*_1 + x^*_4) + \beta_4 - \beta_3
\end{pmatrix} = 0,
\]

\[0 \leq \beta_1 \perp y_1^* \geq 0,
\]

\[0 \leq \beta_2 \perp y_2^* \geq 0,
\]

\[0 \leq \beta_3 \perp y_3^* \geq 0,
\]

\[0 \leq \beta_4 \perp e - y_3^* \geq 0,
\]

\[0 \leq \pi \perp A(y^*_1 - y^*_2) - b \geq 0,
\]

\[(x^*_1 + x^*_4)\xi^* = 0,
\]

\[w^* - y^* = 0.
\]

It can be easily seen that (57) is equivalent to (18) by merging $w^*$ and $y^*$. Recall that according to discussions in Sect. 2, point satisfying (18) is exactly the local minimum of (4), thus the local minimum of $\ell_0$-minimization (1).

**Theorem 3** Suppose that $(w^*; y^*; \lambda^*)$ satisfies (49), and recall that $h(w) = f_Q(x^+ - x^-) + \gamma e^T(e - \xi)$, $p(y) = g(y_1 - y_2)$ are smooth functions. Assume that vector $(\xi^*: \xi^*; x^*_1 + x^*_4) \neq 0$. Then $\exists \mu \in \mathbb{R}$, $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}^n$, $\pi \in \mathbb{R}^m$ such that (57a) - (57g) hold and $w^* - y^* = \alpha \lambda^*$.

**Proof** We know that $\partial 11_{Z_1}(w) = N_{Z_1}(w)$, $\partial 11_{Z_2}(y) = N_{Z_2}(y)$. Due to (49) and the smoothness of function $h$, $0 \in \partial(h + 11_{Z_1})(w^*) + \lambda^* \Leftrightarrow 0 \in \nabla_w h(w^*) + \lambda^* + \partial 11_{Z_1}(w^*) \Rightarrow -\nabla_w h(w^*) - \lambda^* \in N_{Z_1}(w^*)$. Recall $Z_1 = \{(x^+; x^-; \xi) \in \mathbb{R}^n \mid \xi^T(x^+ - x^-) = 0\}$. Then by Lemma 11 and the assumption $(\xi^*: \xi^*; x^*_1 + x^*_4) \neq 0$, we have $N_{Z_1}(w^*) = \{\mu(\xi^*: \xi^*; x^*_1 + x^*_4) \mid \mu \in \mathbb{R}\}$. Therefore, $\exists \mu \in \mathbb{R}$ s.t.
\[ \nabla_w h(w^*) + \lambda^* + \mu (\xi^*; x_+^* + x_-^*) = 0. \tag{58} \]

On the other hand, (49) and smoothness of function \( p \) imply \( 0 \in \partial (p + 1)Z_2)(y^*) - \lambda^* \Rightarrow 0 \in \nabla_y p(y^*) - \lambda^* + \partial 1(1)Z_2(y^*) \Rightarrow -\nabla_y p(y^*) + \lambda^* \in N_{Z_2}(y^*). \]

Since \( Z_2 \) is a convex set, \( N_{Z_2}(y^*) = \{ v | v^T (y - y^*) \leq 0, \forall y \in Z_2 \}. \) Therefore, \((\nabla_y p(y^*) - \lambda^*)^T (y - y^*) \geq 0, \forall y \in Z_2 \). This indicates that \( y^* \) is the optimal solution of the linear program: \( \min_{y \in Z_2} [(\nabla_y p(y^*) - \lambda^*)^T y]. \) Thus the KKT conditions are satisfied at \( y^* \), i.e. \( \exists \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}^n, \pi \in \mathbb{R}^m \) s.t.

\[
\begin{align*}
\nabla_y p(y^*) - \lambda^* + (-\beta_1 - AT \pi; -\beta_2 + AT \pi; -\beta_3) &= 0, \\
0 \leq \beta_1 \perp y_1^* \geq 0, 0 \leq \beta_2 \perp y_2^* \geq 0, 0 \leq \beta_3 \perp y_3^* \geq 0, \\
0 \leq \beta_4 \perp e - y_3^* \geq 0, 0 \leq \pi \perp A(y_1^* - y_2^*) - b \geq 0.
\end{align*}
\tag{59}
\]

From (58) and (59), utilizing the def. of \( h \) and \( p \), and adding the feasibility constraints \((x_+^* + x_-^*)^T \xi^* = 0, w^* - y^* = a \lambda^* \), we obtain the perturbed KKT conditions.

\[ \square \]

5 Preliminary numerics

In Sect. 5.1, we describe the test problem of interest while in Sect. 5.2, we study the impact of tractability by comparing tractable ADMM frameworks with their standard counterpart. In Sect. 5.3, performance of (ADMM\text{cf}) is examined by comparing it with other methods.\(^2\)

5.1 Least squares regression with \( \ell_0 \)-norm

Suppose \( f_Q(x) = \|Cx - d\|^2, C \in \mathbb{R}^{p \times n}, g(x) \equiv 0 \), and there is no linear constraint \( Ax \geq b \) in (1), leading to the following \( \ell_0 \)-regularized least-squares regression:

\[ \min \|Cx - d\|^2 + \gamma \|x\|_0. \quad (\ell_0\text{-LSR}) \]

This special case finds application in signal recovery and regression problems. The rows of \( C \) are generated from a multivariate normal \( N(0, I_n) \) while the true coefficients \( x^\text{true} \) are created as follows: (1) Generate \( x_i^\text{true} \) for \( i = 1, \ldots, n \) from uniform distribution \( U(-60, 60) \). (2) If \( |x_i^\text{true}| \geq \frac{50k}{n} \), then \( x_i^\text{true} \leftarrow 0 \) for \( i = 1, \ldots, n \). Then \( x^\text{true} \) is approximately \( \kappa \)-sparse (or \( \|x^\text{true}\|_0 \approx \kappa \)). The observation vector \( d = Cx^\text{true} + \epsilon \), where \( \epsilon_i \sim N(0, \sigma^2) \) and \( \sigma^2 = \|x^\text{true}\|^2 / 10 \).

\(^2\) All experiments are conducted on Matlab and the code is uploaded to https://github.com/yue-xie/l0-minimization.
5.2 Impact of tractable subproblems

In this subsection, we will compare the ADMM based algorithms (Algorithms 1-3) proposed in Sect. 3 to resolve (ℓ₀-LSR).

Algorithm descriptions and settings We start all algorithms from an initial point \(w_0 = y_0 = (e_{1\times n}; \lambda_0)\). The maximum runtime allowed is 200s. Experiments are run on CPU of 1.3GHz Intel Core i5 with 8GB memory. Other settings are as follows.

(ADMM cf): Please refer to Algorithm 2 for details. Specifically, we use the following rule for updating \(\rho_k\):

\[
\text{If } (\rho_k - \delta) \cdot ||y_{k+1} - y_k|| < \sqrt{2} \cdot \beta_{k+1} - \beta_k \text{ and } \rho_k \leq \rho_{\text{max}}, \text{ then } \rho_{k+1} := \delta \cdot \rho_k; \\
\text{else } \rho_{k+1} := \rho_k.
\]

In addition, \(\rho_0 = 1, \rho_{\text{max}} = 2000, \delta = \rho_0/2, \delta_{\rho} = 1.01, \epsilon = 10^{-4}\).

(ADMM\(\mu,\lambda,\rho\)): Algorithm 1. \(\alpha = 10^{-3}, \rho = 1/(2\alpha), \mu = 3/\alpha, \epsilon_0 = 10^{-2}\).

(ADMM\(\mu\)): Algorithm 3 with Update-1 solved by Baron. The maximum runtime for Baron is set to 200s. Note that we do not fix the penalty parameter at a value suggested by theory, which more often than not is impractical and involves problem parameter estimation. Instead, we update it adaptively. The update rule for \(\rho_k\), inspired by augmented Lagrangian schemes [8], is as follows:

\[
\text{If } k = 0 \text{ or } (h_{k+1} \geq 10^{-2} \text{ and } h_{k+1} > 0.9 h_k \text{ and } \rho_k < 500), \rho_{k+1} = 1.01 \rho_k; \\
\text{otherwise } \rho_{k+1} = \rho_k;
\]

where \(h_k = ||w_k - y_k|| \text{ for all } k \geq 0, \rho_0 = 1, \epsilon = 10^{-4}\.)

Table 2 Comparison of methods on (ℓ₀-LSR), \(p = 10\)

| (\(\alpha, ||x^r||_0, \gamma\)) | \(\text{(ADMM cf)}\) | \(\text{(ADMM\(\mu,\lambda,\rho\)}\) | \(\text{(ADMM\(\mu\))}\) |
|-----------------|-----------------|-----------------|-----------------|
| \(t(s)\) | \(\text{res.}\) | \(\text{iter.}\) | \(t(s)\) | \(\text{res.}\) | \(\text{iter.}\) | \(t(s)\) | \(\text{res.}\) | \(\text{iter.}\) |
| (20, 1, 1) | 0.38 | 9.4e–5 | 587 | 2.2e0 | 1.0e–2 | 4.90e3 | 201 | 3.0e–3 | 135 |
| (20, 1, 10) | 0.46 | 7.9e–5 | 515 | 2.0e2 | 3.5e–1 | 2.05e5 | 200 | 1.0e–2 | 158 |
| (20, 4, 1) | 0.26 | 7.2e–5 | 214 | 2.0e2 | 7.6e–2 | 2.67e5 | 200 | 8.7e–3 | 197 |
| (20, 4, 10) | 0.70 | 9.9e–5 | 798 | 2.0e2 | 1.8e–1 | 2.18e5 | 200 | 2.1e–2 | 150 |
| (50, 10, 1) | 1.10 | 1.0e–4 | 781 | 6.3e–1 | 9.9e–3 | 551 | 209 | 7.2e0 | 2 |
| (50, 10, 10) | 2.10 | 9.9e–5 | 1093 | 2.0e0 | 1.6e–1 | 1.18e5 | 201 | 2.5e–1 | 101 |
| (50, 18, 1) | 0.62 | 7.7e–5 | 283 | 2.0e0 | 1.8e–1 | 1.42e5 | 215 | 1.3e1 | 2 |
| (50, 18, 10) | 4.20 | 9.9e–5 | 1802 | 2.0e0 | 6.9e–1 | 1.29e5 | 201 | 4.8e–1 | 47 |
| (100, 6, 1) | 0.58 | 9.3e–5 | 617 | 2.5e–1 | 9.6e–3 | 200 | 209 | 9.8e0 | 2 |
| (100, 6, 10) | 0.66 | 9.8e–5 | 593 | 2.0e2 | 1.4e–1 | 2.10e5 | 399 | 4.8e0 | 14 |
| (100, 19, 1) | 0.76 | 9.2e–5 | 483 | 4.0e–1 | 9.9e–3 | 376 | 211 | 1.0e1 | 2 |
| (100, 19, 10) | 1.80 | 9.0e–5 | 535 | 2.0e2 | 1.3e–1 | 1.72e5 | 289 | 5.5e0 | 5 |
Stopping criteria. The stopping criteria for (ADMM cf) and (ADMM 0) guarantee that the KKT residual is below \( \epsilon \) if terminated within time limit. For (ADMM cf), it is guaranteed that the KKT residual is below \( \epsilon_0 + O(\sqrt{\epsilon}) \) (See Remark 6). Thus, stopping criteria of the three algorithms are related to the optimality conditions.

Metric In Table 2, KKT residual for (ADMM cf), (ADMM cf,\( \mu,\alpha,\rho \)) and (ADMM 0) are \( \max \{ \rho_{K-1} \|y_K - y_{K-1}\|, \|x_K^+ - x_{K-1}^+ + \mu z_K - y_K\| \} \) and \( \max \{ \rho_{K-1} \|y_K - y_{K-1}\|, \|w_K - y_K\| \} \), respectively. \( K \) is the last iteration.

Results In Table 2, we provide a comparison of (ADMM cf), (ADMM cf,\( \mu,\alpha,\rho \)) and (ADMM 0) to address (\( \ell_0 \)-LSR). Note that (ADMM cf), (ADMM cf,\( \mu,\alpha,\rho \)) are designed for formulation (24), which renders tractable subproblems, while (ADMM 0) is for formulation (32), which requires global resolution of an MPCC as the subproblem. Therefore, even though (ADMM 0) can be efficient when the dimension is relatively low, but becomes less so when \( n \) is larger. This is because the subproblem solver
does not scale well with problem size and requires significant time for larger dimensions. This is supported by the drastically reduced number of outer loop iterations and large KKT residual upon termination when \( n \) is larger. Meanwhile, (ADMM\(_{cf}^\mu\)) and (ADMM\(_{cf}^{\mu,a,\rho}^\)) appear to be scale far better with \( n \) due to tractability of the sub-problem. Furthermore, it can be seen that (ADMM\(_{cf}^\)) is far more efficient than the other two methods. It spends far less time to find solutions with lower KKT residual. In fact, it also provides better objective function value than the other two methods as observed during the experiment. Further exploration of (ADMM\(_{cf}^\)) through comparison with other \( \ell_0 \)-minimization solvers is presented in the next subsection.

### 5.3 Comparison between (ADMM\(_{cf}^\)) and other methods

In this set of experiments, we test (ADMM\(_{cf}^\)) on (\( \ell_0 \)-LSR) with higher dimensions \( (p = 256, n = 1024) \) and compare it with other known methods directly addressing \( \ell_0 \)-minimization: iterative hard thresholding (IHT) and iterative hard thresholding with warm start (IHTWS) \cite{9}. We again test the schemes on (\( \ell_0 \)-LSR) and choose almost the same settings as in Sect. 5.1, the only difference being that \( e \in \mathbb{R}^p, e_i \sim N(0, \sigma^2) \), i.i.d., \( \sigma^2 = \frac{(y_{\text{true}}) y_{\text{true}}^T}{\text{SNR}} \), where SNR refers to the signal-to-noise ratio. All experiments are conducted on CPU of 3.4GHz Intel Core i7 with 16GB memory.

#### Algorithm descriptions and settings

**(IHT)** and **(IHTWS)**: (IHT) is implemented with 50 initial points (including the origin and points drawn from normal distribution \( N(0, I_n) \)), and the best solution is chosen. (IHTWS) is warm-started from a point computed by matching pursuit. The termination condition for both (IHT) and (IHTWS) is \( \|x_{k+1} - x_k\| < 1 \times 10^{-6} \).

**(ADMM\(_{cf}^\))**: Implementation of (ADMM\(_{cf}^\)) is almost the same with the last experiment in Sect. 5.2: Initial point is selected as \( y_0 = (e_n, 0_n, 0_n) \), \( \lambda_0 = 0_{3n} \), and the parameters are chosen as \( \rho_0 = \gamma \), \( e = 10^{-4} \), \( \delta_\rho = 1.01 \), \( \delta = \rho_0 / 2 \), \( \rho_{\max} = 2000 \), \( \text{time}_{\max} = 300 \) for all cases.

#### Metric

In Table 3, RDF \( \triangleq \frac{f_{\text{method}} - f_{\text{ADMM}_{cf}^\mu}}{f_{\text{ADMM}_{cf}^\mu}} \), where \( f_{\text{ADMM}_{cf}^\mu} \) is calculated as follows. Suppose Algorithm 2 terminates when \( k = T \). Let \( (\tilde{x}^+, \tilde{x}^-, \tilde{y}) = y_{T+1} \). Then the solution given by (ADMM\(_{cf}^\)) is \( \tilde{x}^+ - \tilde{x}^- \) and

\[
f_{\text{ADMM}_{cf}^\mu} = \begin{cases} 
\|C(\tilde{x}^+ - \tilde{x}^-) - d\|^2 + \gamma(n - e^T \tilde{y}), & \text{if } \max(\|w_{T+1}^T - y_{T+1} \|, \rho_T \|y_{T+1} - y_T\|) \leq \epsilon; \\
\|C(\tilde{x}^+ - \tilde{x}^-) - d\|^2 + \gamma \|\tilde{x}^+ - \tilde{x}^-\|_0, & \text{if } \max(\|w_{T+1}^T - y_{T+1} \|, \rho_T \|y_{T+1} - y_T\|) > \epsilon.
\end{cases}
\]

#### Results

From Table 3, we conclude the following:

1. Although (ADMM\(_{cf}^\)) takes more time, it generally produces solutions that are superior to (IHT) in objective function value and provides better values than (IHTWS) in most cases. Note that (ADMM\(_{cf}^\)) is cold started.
(2) \( \text{ADMM}_{cf} \) generally produces sparser solution than \( \text{IHTWS} \) and \( \text{IHT} \), which indicates that \( \text{ADMM}_{cf} \) scheme is potentially more favorable from a statistical standpoint.

6 Concluding remarks and future work

We consider a full complementarity reformulation of a general class of \( \ell_0 \)-norm minimization problems. The focus of this paper lies on the characterization and efficient computation of KKT points for this formulation. In particular, we show that a suitable (Guignard) constraint qualification holds at every feasible point. Moreover, when \( f \) is a convex function, a point satisfies the first-order KKT conditions if and only if it is a local minimizer. Next, two tractable ADMM schemes are presented for resolution. In these schemes, a hidden convexity property is leveraged to allow for tractable resolution of ADMM subproblems. For the perturbed proximal ADMM framework, subsequential convergence to KKT points with arbitrarily small error under mild assumptions can be shown. Preliminary empirical studies show the significance of having tractable subproblems in ADMM schemes and that the tractable ADMM framework compares well with its competitors. In future work, we may consider characterization and computation of KKT points of problems complicated by cardinality constraints (2) and affine sparsity constraints [17].

Acknowledgements The authors would like to acknowledge an early discussion with Dr. Ankur Kulkarni of IIT, Mumbai, as well as the inspiration provided by Dr. J. S. Pang during his visit to Penn. State University, and suggestion by Dr. Mingyi Hong in INFORMS 2018, Denver.

Appendix

KŁ property and global convergence

In this subsection we present the missing proof of global convergence of the sequence generated by \( \text{ADMM}_{cf}^{\mu, \nu, \rho} \) under the assumption of KŁ property. In the end, we will discuss the cases when KŁ does hold for the Lyapunov function. First we introduce several concepts necessary for the discussion. More details of the math background could be found in [1, 11, 34].

Definition 6 (Kurdyka–Łojasiewicz (KŁ) property [1]) A proper lower semi-continuous function \( \mathcal{L}: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\} \) has the KŁ property at \( \bar{x} \in \text{dom} (\partial \mathcal{L}) \), if there exists \( \eta \in (0, +\infty) \), a neighborhood \( U \) of \( \bar{x} \), and a continuous concave function \( \phi: [0, \eta) \rightarrow \mathbb{R}^+ \) such that the following hold: (i) \( \phi(0) = 0 \), and \( \phi \) is continuously differentiable on \( (0, \eta) \). For all \( s \in (0, \eta) \), \( \phi'(s) > 0 \); (ii) For all \( x \) in \( U \cap \{ x \in \mathbb{R}^N : \mathcal{L}(\bar{x}) < \mathcal{L}(x) < \mathcal{L}(\bar{x}) + \eta \} \), the Kurdyka–Łojasiewicz (KŁ) inequality holds: \( \phi'(\mathcal{L}(x) - \mathcal{L}(\bar{x})) \text{dist}(0, \partial \mathcal{L}(x)) \geq 1 \).
Definition 7 (Semialgebraic function) A semialgebraic set $S \subseteq \mathbb{R}^n$ can be written as finite union of sets of the following form:

$$S \triangleq \{ x \in \mathbb{R}^n : p_i(x) = 0, q_i(x) < 0, i = 1, \ldots, m \},$$

where $p_i$ and $q_i$ are real polynomial functions. A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a semialgebraic function if and only if its graph $\{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y = F(x)\}$ is a semialgebraic subset in $\mathbb{R}^{n+1}$.

Remark 7 A semialgebraic function has the following properties: (i) If it is proper lower semi-continuous, then it satisfies the KŁ property with $\phi(s) = cs^{1-\theta}$ for some $\theta \in [0, 1) \cap \mathbb{Q}$ and $c > 0$. (ii) Finite sums and products of semialgebraic functions are semialgebraic. See [1, Section 4.3] for more details.

Definition 8 (o-minimal structure [34]) An o-minimal structure on the real field $(\mathbb{R}, +, \cdot)$ is a sequence $G = (G_n)_{n \in \mathbb{N}}$ such that:

1. $G_n$ is a boolean algebra of subsets in $\mathbb{R}^n$, i.e., $\mathbb{R}^n \in G_n$ and if $A,B \in G_n$, then $A \cap B, A \cup B, \mathbb{R}^n \setminus A$ are in $G_n$.
2. If $A \in G_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are in $G_{n+1}$.
3. If $A \in G_{n+1}$, then $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | (x_1, \ldots, x_n, x_{n+1}) \in A\}$ is in $G_n$.
4. For $i, j$ such that $1 \leq i < j \leq n$, $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_i = x_j\}$ is in $G_n$.
5. The graphs of addition and multiplication are in $G_3$.
6. $G_1$ consists exactly finite unions of intervals and singletons.

Remark 8 Given $G$, if the graph of function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ belongs to $G_{n+1}$, then $f$ is called definable. Note that summation of two definable functions is definable, and composition of definable functions is definable.

Theorem 4 (Theorem 14 [1]) Any proper lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ which is definable in an o-minimal structure $G$ has the Kurdyka–Lojasiewicz property at each point of $\text{dom} f$.

Next we prove the statement we make in Remark 5 (iii).

Lemma 10 Suppose that assumptions in Theorem 2 hold. $(w_k; y_k; \lambda_k)$ is generated by (ADMM$_{\text{cf}}^{\rho, \alpha, \rho}$) and denote $(w^*, y^*, \lambda^*)$ as the limit point. Let

$$\mathcal{H}_c(w, y, \lambda) \triangleq \tilde{L}_{\rho, \alpha}(w, y, \lambda) + 1L_1(w) + 1L_2(y) + \frac{(1 - \rho \alpha)a}{2}\|\lambda\|^2 + \frac{\rho\|w - y - \alpha \lambda\|^2}{2(1 - \rho \alpha)}.$$ 

Suppose that $\mathcal{H}_c$ satisfies the KL property at $(w^*, y^*, \lambda^*)$. Then $\{(w_k; y_k; \lambda_k)\}$ converges to $(w^*; y^*; \lambda^*)$ globally.
Proof. Denote $\mathcal{H}^k \triangleq \mathcal{H}_r(w_k, y_k, \lambda_k)$. Then it can be verified that $P_r^k = \mathcal{H}^k$, $\forall k \geq 1$ ($P_r$ defined in (34)). Then by Lemma 8, for any $k \geq 1$,

$$\mathcal{H}^k - \mathcal{H}^{k+1} \geq c_1(v)\|w_{k+1} - w_k\|^2 + c_2\|y_{k+1} - y_k\|^2 + c_3(v)\|\lambda_{k+1} - \lambda_k\|^2. \quad (60)$$

By Theorem 2, we know that there exists a subsequence $\{(w_{n_k}, y_{n_k}, \lambda_{n_k})\}$ that converges to $(w^*; y^*; \lambda^*)$ as $k \to \infty$. Therefore, $\mathcal{H}^{n_k} \to \mathcal{H}^* = \mathcal{H}_r(w^*, y^*, \lambda^*)$ as $k \to \infty$. Then by Assumption 2 and (60), we know that $\mathcal{H}^k \to \mathcal{H}^*$ as $k \to \infty$. Therefore, by the monotonicity of $\mathcal{H}^k$, we have that $\mathcal{H}^k \downarrow \mathcal{H}^*$.

Denote $z_k \triangleq (w_k; y_k; \lambda_k)$ and $z^* \triangleq (w^*; y^*; \lambda^*)$. By KL property, there exist neighbourhood $U \supset B(z^*, r) \triangleq \{z \in \mathbb{R}^{3n} | \|z - z^*\| < r\}$, $\eta > 0$ and concave continuous function $\phi : [0, \eta) \to \mathbb{R}_+$ such that $\phi(0) = 0$, $\phi$ is continuously differentiable on $(0, \eta)$ and $\phi'(s) > 0$ on $(0, \eta)$. Moreover, for any $z \in U \cap \{\mathcal{H}^* < \mathcal{H}_r(z) < \mathcal{H}^* + \eta\}$,

$$\phi'(\mathcal{H}_r(z) - \mathcal{H}^*)\text{dist}(0, \partial \mathcal{H}_r(z)) \geq 1. \quad (61)$$

By subsequential convergence to $z^*$, $\|z_k - z_{k+1}\| \to 0$ (Lemma 8(iii)), monotonicity of $\phi$ and the fact that $\mathcal{H}^k \downarrow \mathcal{H}^*$, there exists $K_0$ large enough such that (let $\Delta z_{k+1} \triangleq z_{k+1} - z_k$)

$$\|z_{K_0} - z^*\| + \|\Delta z_{K_0+1}\| < r/4, \mathcal{H}_{K_0+1}^* - \mathcal{H}^* < \eta, \quad (62)$$

where $C_{\min} \triangleq \min\{c_1(v), c_2, c_3(v)\}$, $C_{\max} \triangleq \max\{C(\rho, \alpha, \tau), \rho, \mu/2\}$, $C(\rho, \alpha, \tau) \triangleq 2(1 - \rho \alpha + \tau) + |(1 - \rho \alpha)^2/\rho - \tau \alpha|$. WLOG let $K_0 = 0$. Then $z_0, z_1 \in B(z^*, r)$ and $\|\Delta z_i\| < r$. Suppose that for any $k = 1, \ldots, K_1, K_1 \geq 1, z_k \in B(z^*, r)$, and $\sum_{k=1}^{K_1} \|\Delta z_k\| < r$. We want to show that the same is true when $k = K_1 + 1$.

Note that for any $k \geq 1$,

$$\partial \mathcal{H}_r(w_k, y_k, \lambda_k) = \partial (11Z_1(w_k) + 11Z_2(y_k))$$

\[
\begin{align*}
&\nabla h(w_k) + (1 - \rho \alpha)\lambda_k + \rho(w_k - y_k) + \frac{\tau \rho}{1 - \rho \alpha}(w_k - y_k - \alpha \lambda_k) \\
&\nabla p(y_k) - (1 - \rho \alpha)\lambda_k - \rho(w_k - y_k) - \frac{\tau \rho}{1 - \rho \alpha}(w_k - y_k - \alpha \lambda_k) \\
&\quad + (1 - \rho \alpha)(w_k - y_k - 2\alpha \lambda_k) + (1 - \rho \alpha)\alpha \lambda_k + \frac{\tau \rho}{1 - \rho \alpha}(w_k - y_k - \alpha \lambda_k)(\alpha)
\end{align*}
\]

$$= \frac{\partial 11Z_1(w_k)}{\partial 11Z_2(y_k)} + \frac{\partial 11Z_2(y_k)}{\partial 11Z_1(w_k)} + (1 - \rho \alpha)\lambda_k + \rho(w_k - y_k) + \frac{\tau \rho}{1 - \rho \alpha}(w_k - y_k - \alpha \lambda_k)$$

\[
\begin{align*}
&\quad + (1 - \rho \alpha - \frac{\tau \rho}{1 - \rho \alpha})(w_k - y_k - \alpha \lambda_k)
\end{align*}
\]  

(63)

where the first equation holds because of differentiability of the smooth part of $\mathcal{H}_r$ and property (ii) after Definition 4. The second equation is implied by the subdifferential calculus for separable functions [32, Proposition 10.5, p. 426].

By the optimality conditions of Update-1 and Update-2 of (ADMM$^\mu_{\alpha, \rho}$), for any $k \geq 1$, there exist $u_k \in \partial 11Z_1(w_k), v_k \in \partial 11Z_2(y_k)$ such that
\[-u_k = \nabla h(w_k) + (1 - \rho \alpha)\lambda_{k-1} + \rho (w_k - y_{k-1}) + \frac{\mu}{2} (w_k - w_{k-1}) \]

\[-v_k = \nabla p(y_k) - (1 - \rho \alpha)\lambda_{k-1} - \rho (w_k - y_k) \]  

\[ (64) \]

Denote $\Delta w_k \triangleq w_k - w_{k-1}$, $\Delta y_k \triangleq y_k - y_{k-1}$, $\Delta \lambda_k \triangleq \lambda_k - \lambda_{k-1}$. Then for any $k \geq 1$,

\[
\text{dist}(0, \partial \mathcal{H}_c(z_k)) \leq \left\| \begin{bmatrix} u_k + \nabla h(w_k) + (1 - \rho \alpha)\lambda_k + \rho (w_k - y_k) + \frac{\tau \rho}{1 - \rho \alpha} (w_k - y_k - \alpha \lambda_k) \\ v_k + \nabla p(y_k) - (1 - \rho \alpha)\lambda_k - \rho (w_k - y_k) - \frac{\tau \rho}{1 - \rho \alpha} (w_k - y_k - \alpha \lambda_k) \end{bmatrix} \right\| 
\]

\[ (65) \]

For any $k = 1, \ldots, K$, suppose that $\mathcal{H}^k \succ \mathcal{H}^\ast$. Otherwise there exists $\bar{k}$ such that $\mathcal{H}^{\bar{k}} = \mathcal{H}^\ast$. Together with (60) and $c_1(v), c_2, c_3(v) > 0$, this implies that $z_{k+1} = z_k = z^\ast$, \( \forall k \geq \bar{k} \), i.e., $z_k$ converges to $z^\ast$ already. Then by $\mathcal{H}^k \leq \mathcal{H}^1 < \mathcal{H}^\ast + \eta$ from (62) and the hypothesis $z_k \in B(z^\ast, r)$, (61) holds at $z = z_k$.

Also, by concavity of $\phi$ and the fact that $\mathcal{H}^\ast < \mathcal{H}^k \leq \mathcal{H}^1 < \eta$, we have

\[ 0 \leq \phi'(\mathcal{H}^k - \mathcal{H}^\ast)(\mathcal{H}^k - \mathcal{H}^{k+1}) \leq \phi(\mathcal{H}^k - \mathcal{H}^\ast) - \phi(\mathcal{H}^{k+1} - \mathcal{H}^\ast). \]  

\[ (66) \]

Therefore, by (65), (66) and KL inequality, we have the following:
\[(\phi(\mathcal{H}^k - \mathcal{H}^*) - \phi(\mathcal{H}^{k+1} - \mathcal{H}^*)) \left( \frac{\mu}{2} \| \Delta w_k \| + \rho \| \Delta y_k \| + C(\rho, \alpha, \tau) \| \Delta \lambda_k \| \right) \]
\[\geq \mathcal{H}^k - \mathcal{H}^{k+1} \stackrel{(60)}{\geq} c_1(\nu) \| \Delta w_{k+1} \|^2 + c_2(\nu) \| \Delta y_{k+1} \|^2 + c_3(\nu) \| \Delta \lambda_{k+1} \|^2 \]
\[\Rightarrow \sqrt{c_1(\nu) \| \Delta w_{k+1} \|^2 + \frac{\rho}{2} \| \Delta y_{k+1} \|^2 + c_3(\nu) \| \Delta \lambda_{k+1} \|^2} \]
\[\leq \sqrt{\phi(\mathcal{H}^k - \mathcal{H}^*) - \phi(\mathcal{H}^{k+1} - \mathcal{H}^*)} \cdot \sqrt{\frac{\mu}{2} \| \Delta w_k \| + \rho \| \Delta y_k \| + C(\rho, \alpha, \tau) \| \Delta \lambda_k \|} \]
\[\forall M > 0 \Rightarrow \sqrt{C_{\min}} \sum_{k=1}^{K} \| \Delta z_{k+1} \| \leq M \left( \frac{\mu}{2} \| \Delta w_k \| + \rho \| \Delta y_k \| + C(\rho, \alpha, \tau) \| \Delta \lambda_k \| \right) \]
\[+ \frac{1}{2M} \left( \frac{\mu}{2} \| \Delta w_k \| + \rho \| \Delta y_k \| + C(\rho, \alpha, \tau) \| \Delta \lambda_k \| \right) \]
\[\leq \frac{M}{2} (\phi(\mathcal{H}^k - \mathcal{H}^*) - \phi(\mathcal{H}^{k+1} - \mathcal{H}^*)) + \frac{C_{\max}}{2M} \left( \| \Delta w_k \| + \| \Delta y_k \| + \| \Delta \lambda_k \| \right) \]
\[\leq \frac{M}{2} (\phi(\mathcal{H}^k - \mathcal{H}^*) - \phi(\mathcal{H}^{k+1} - \mathcal{H}^*)) + \frac{\sqrt{3}C_{\max}}{2M} \| \Delta z_k \| \]

(67) The last inequality holds because \((\| \Delta w_k \| + \| \Delta y_k \| + \| \Delta \lambda_k \|)^2 \leq 3(\| \Delta w_k \|^2 + \| \Delta y_k \|^2 + \| \Delta \lambda_k \|^2) = 3\| \Delta z_k \|^2\). Sum up (67) from \(k = 1\) to \(K\) and we have:

\[\sqrt{C_{\min}} \sum_{k=1}^{K} \| \Delta z_{k+1} \| \]
\[\leq \frac{M}{2} (\phi(\mathcal{H}^1 - \mathcal{H}^*) - \phi(\mathcal{H}^{K+1} - \mathcal{H}^*)) + \frac{\sqrt{3}C_{\max}}{2M} \sum_{k=1}^{K} \| \Delta z_k \| \]
\[\leq \frac{M}{2} (\phi(\mathcal{H}^1 - \mathcal{H}^*) + \frac{\sqrt{3}C_{\max}}{2M} \sum_{k=1}^{K} \| \Delta z_k \| \]

(68)

\[\Rightarrow \sum_{k=0}^{K} \| \Delta z_{k+1} \| \leq \frac{M}{2\sqrt{C_{\min}}} (\phi(\mathcal{H}^1 - \mathcal{H}^*) + \frac{\sqrt{3}C_{\max}}{2M\sqrt{C_{\min}}} \sum_{k=1}^{K} \| \Delta z_k \| + \| \Delta z_1 \| \]

Let \(M = \frac{\sqrt{3}C_{\max}}{\sqrt{C_{\min}}} \) in (68) and use (62) and the hypothesis \(\sum_{k=1}^{K} \| \Delta z_k \| < r\), we have that

\[\sum_{k=1}^{K+1} \| \Delta z_k \| < \frac{r}{4} + \frac{r}{2} + \frac{r}{4} = r, \| \Delta z_{k+1} + \Delta z_k \| \leq \sum_{k=0}^{K} \| \Delta z_{k+1} \| + \| \Delta z_0 - \Delta z_k \| < r. \]

Therefore, the hypothesis is verified at \(k = K + 1\). By induction, \(z_k \in B(z^*, r), \sum_{i=1}^{k} \| \Delta z_i \| < r, \forall k \geq 1\). Therefore sequence \(\{z_k\}\) is Cauchy and converges. \(\square\)

**Remark 9** We introduce two general cases when \(\mathcal{H}_r\) satisfies the KL property:

\(\copyright\) Springer
(i) \( p(y) \) is a polynomial function. In this case, \( p(y) \) is semialgebraic (Definition 7). Therefore, \( \mathcal{H}_\tau \) is a sum of semialgebraic functions so itself is semialgebraic. Then the result follows from the fact that a semialgebraic function satisfies the KL property at every point in its domain \([1]\). Note that if we reformulate (\( \ell_0 \)-LSR) in Sect. 5.1 as the structured program (33), then \( p(y) \equiv 0 \), which belongs to this case.

(ii) \( \mathcal{H}_\tau \) is in \( \mathcal{G}(\mathbb{R}_{\text{an,exp}}) \). \( \mathcal{G}(\mathbb{R}_{\text{an,exp}}) \) is a type of o-minimal structure that contains the graphs of many function classes including semialgebraic functions, restricted analytic functions (an analytic function \( f : \mathbb{R}^n \to \mathbb{R} \) restricted to \([-1, 1]^n\)), \( \exp : \mathbb{R} \to \mathbb{R} \) and \( \log : (0, +\infty) \to \mathbb{R} \) \([34]\). In particular, when \( g(x) \) in (1) is a logistic loss function, i.e.,

\[
g(x) = \frac{1}{N} \sum_{i=1}^N \log(1 + \exp(-l_i x^T s_i)),
\]

\( p(y) \) is definable w.r.t. \( \mathcal{G}(\mathbb{R}_{\text{an,exp}}) \) since the composition and summation of definable function is definable. Therefore, \( \mathcal{H}_\tau \) is also definable since other summands of \( \mathcal{H}_\tau \) are semialgebraic functions.

(iii) Other types of functions such as uniformly convex functions, convex function that satisfies a growth condition and convex subanalytic functions may also satisfies the KL property, which is beyond of the scope of this paper. We refer the interested reader to \([1, 10]\) for more details.

**Miscellaneous**

**Lemma 11** (Theorem 10 \([14]\) ) In \( \mathbb{R}^{n_1} \), let \( C = \{x \in X \mid F(x) \in D\} \), for closed convex sets \( X \subset \mathbb{R}^{n_1}, D \subset \mathbb{R}^{n_2} \), and a \( \mathcal{C}^1 \) mapping \( F : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2} \), written component-wise as \( F(x) = (f_1(x); \ldots; f_{n_2}(x)) \). Suppose the following constraint qualification is satisfied at a point \( \bar{x} \in C \):

\[
\sum_{j=1}^{n_2} y_j \nabla f_j(\bar{x}) + z = 0, y = (y_1; \ldots; y_{n_2}) \in \mathcal{N}_D(F(\bar{x})), z \in \mathcal{N}_X(\bar{x})
\]

\[\implies y = 0, z = 0.\]

Then the normal cone \( \mathcal{N}_C(\bar{x}) \) consists of all vectors \( v \) of the form

\[
v = y_1 \nabla f_1(\bar{x}) + \ldots + y_{n_2} \nabla f_{n_2}(\bar{x}) + z \] with \( y = (y_1; \ldots; y_{n_2}) \in \mathcal{N}_D(F(\bar{x})), z \in \mathcal{N}_X(\bar{x}). \)

**Note:** When \( X = \mathbb{R}^{n_1} \), the normal cone \( \mathcal{N}_X(\bar{x}) = \{0\} \), so the \( z \) terms here drop out. When \( D \) is a singleton, \( \mathcal{N}_D(F(\bar{x})) = \mathbb{R}^{n_2} \).
References

1. Attouch, H., Bolte, J., Redont, P., Soubeyran, A.: Proximal alternating minimization and projection methods for nonconvex problems: an approach based on the Kurdyka–Łojasiewicz inequality. Math. Oper. Res. 35, 438–457 (2010)
2. Bach, F., Jenatton, R., Mairal, J., Obozinski, G.: Optimization with sparsity-inducing penalties. Found. Trends Mach. Learn. 4, 1–106 (2012)
3. Beck, A., Eldar, Y.C.: Sparsity constrained nonlinear optimization: optimality conditions and algorithms. SIAM J. Optim. 23, 1480–1509 (2013)
4. Ben-Tal, A., Nemirovski, A.: Computational tractability of convex programs. Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications, vol. 2. SIAM, Philadelphia (2001)
5. Ben-Tal, A., Teboulle, M.: Hidden convexity in some nonconvex quadratically constrained quadratic programming. Math. Program. 72, 51–63 (1996)
6. Bertsimas, D., King, A., Mazumder, R.: Best subset selection via a modern optimization lens. Ann. Stat. 44, 813–852 (2016)
7. Bertsimas, D., Shi, D.: Algorithm for cardinality-constrained quadratic optimization. Comput. Optim. Appl. 43, 1–22 (2009)
8. Birgin, E.G., Floudas, C.A., Martínez, J.M.: Global minimization using an augmented Lagrangian method with variable lower-level constraints. Math. Program. 125, 139–162 (2010)
9. Blumensath, T., Davies, M.E.: Iterative thresholding for sparse approximations. J. Fourier Anal. Appl. 14, 629–654 (2008)
10. Bolte, J., Daniilidis, A., Lewis, A.: The Łojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems. SIAM J. Optim. 17, 1205–1223 (2007)
11. Bolte, J., Daniilidis, A., Lewis, A., Shiota, M.: Clarke subgradients of stratifiable functions. SIAM J. Optim. 18, 556–572 (2007)
12. Boţ, R., Csetnek, E., Nguyen, D.: A proximal minimization algorithm for structured nonconvex and nonsmooth problems. SIAM J. Optim. 29, 1300–1328 (2019)
13. Burdakov, O.P., Kanzow, C., Schwartz, A.: Mathematical programs with cardinality constraints: reformulation by complementarity-type conditions and a regularization method. SIAM J. Optim. 26, 397–425 (2016)
14. Burke, J.: Fundamentals of optimization, Chapter 5, Langrange multipliers. Course Notes, AMath/Math 515, University of Washington
15. Burke, J.: Numerical optimization. Course Notes, AMath/Math 516, University of Washington, Spring Term (2012)
16. Candès, E.J., Wakin, M.B.: An introduction to compressive sampling. IEEE Signal Process. Mag. 25, 21–30 (2008)
17. Dong, H., Ahn, M., Pang, J.-S.: Structural properties of affine sparsity constraints. Math. Program. 176, 95–135 (2019)
18. Donoho, D.L.: Compressed sensing. IEEE Trans. Inf. Theory 52, 1289–1306 (2006)
19. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I. Springer, Berlin (2007)
20. Fan, J., Li, R.: Variable selection via nonconcave penalized likelihood and its oracle properties. J. Am. Stat. Assoc. 96, 1348–1360 (2001)
21. Fang, E.X., Liu, H., Wang, M.: Blessing of massive scale: spatial graphical model estimation with a total cardinality constraint approach. Math. Program. 176, 175–205 (2019)
22. Feng, M., Mitchell, J.E., Pang, J.-S., Shen, X., Wächter, A.: Complementarity formulations of $\ell_0$-norm optimization problems. Pac. J. Optim. 14, 273–305 (2018)
23. Fung, G., Mangasarian, O.: Equivalence of minimal $\ell_0$ and $\ell_p$ norm solutions of linear equalities, inequalities and linear programs for sufficiently small p. J. Optim. Theory Appl. 151, 1–10 (2011)
24. Ge, D., Jiang, X., Ye, Y.: A note on the complexity of $L_p$ minimization. Math. Program. 129, 285–299 (2011)
25. Gonçalves, M.L., Melo, J.G., Monteiro, R.D.: Convergence rate bounds for a proximal ADMM with over-relaxation stepsize parameter for solving nonconvex linearly constrained problems (2017). arXiv:1702.01850
26. Hajinezhad, D., Hong, M.: Perturbed proximal primal-dual algorithm for nonconvex nonsmooth optimization. Math. Program. 176, 207–245 (2019)
Tractable ADMM schemes for computing KKT points and local…

Affiliations

Yue Xie
Uday V. Shanbhag

Affiliations

Yue Xie1 · Uday V. Shanbhag2

Uday V. Shanbhag
vvs3@psu.edu

1 Wisconsin Institute for Discovery, Madison, USA
2 Pennsylvania State University, State College, USA