Quantum tunneling with friction

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Using the phenomenological quantum friction models introduced by Caldirola-Kanai, Kostin, and Albrecht, we study quantum tunneling of a one-dimensional potential in the presence of energy dissipation. To this end, we calculate the tunneling probability using a time-dependent wave packet method. The friction reduces the tunneling probability. We show that the three models provide similar penetrabilities to each other, among which the Caldirola-Kanai model requires the least numerical effort. We also discuss the effect of energy dissipation on quantum tunneling in terms of barrier distributions.

I. INTRODUCTION

In low-energy heavy-ion fusion reactions, it has been known that excitations of the colliding nuclei considerably influence the reaction dynamics, that is, fusion cross sections are largely enhanced as compared to the prediction of a simple potential model [1-4]. In order to take into account the excitations during reactions, the coupled-channels method has been developed. Many experimental data have been successfully accounted for with this method by including a few internal degrees of freedom which are coupled strongly to the ground state [3, 5]. However, when a large number of channels are involved, the coupled-channels calculations become increasingly difficult. This is the case, e.g., fusion reactions in massive systems, in which many non-collective excitations may play an important role [6-9].

To deal with this problem, many phenomenological models based on the classical concept of friction were proposed in connection with deep inelastic heavy-ion collisions [10]. Among them, it has been found that the classical Langevin treatment works well for fusion reactions and deep inelastic collisions when the incident energy is higher than the Coulomb barrier [11-13]. When the incident energy is close to the barrier, however, the fusion reaction takes place by quantum tunneling. Hence, in order to apply these models to low-energy fusion reactions, a quantum mechanical extension of the friction models is essential.

Another important issue is to extend the coupled-channels approach to massive systems by taking into account the dissipation effects and to develop a quantal theory for deep inelastic collision with energy and angular momentum dissipations. Such theory would be able to describe simultaneously dissipative quantum tunneling below the Coulomb barrier and deep inelastic collision above the barrier. In that way, one may resolve a long standing problem of surface diffuseness anomaly in heavy-ion fusion reactions, that is, an anomaly that a significantly large value for the surface diffuseness parameter in an inter-nuclear Woods-Saxon potential has to be used in order to account for above barrier data of fusion cross sections [14-16]. Such theory would also provide a consistent description for deep subbarrier hindrance of fusion cross sections [17], for which the dynamics after the touching of the colliding nuclei play a crucial role [17-19].

Quantum friction has attracted lots of attention as a general problem of open quantum systems [20-27]. To date, many attempts at developing a quantum friction model have been made. They can be mainly categorized into the following two approaches. The first is to consider a system with bath, for which the environmental bath is often simplified as, e.g., a collection of harmonic oscillators [28-30]. The second approach is to treat the couplings to the bath implicitly and introduce a phenomenological Hamiltonian with which the classical equation of motion with a frictional force is reproduced as expectation values. For this approach, Caldirola and Kanai [31, 32], Kostin [33], and Albrecht [34] proposed a hermitian Hamiltonian, whose equation of motion contains a linear frictional force, while Dekker [35] invented an approach with a non-hermitian Hamiltonian.

In this paper, we employ the second approach and investigate quantum tunneling in the presence of friction. Even though the first approach is more microscopic, physical quantities are easier to calculate with the second approach, and thus it is easier to gain physical insight into the effect of friction on quantum tunneling. We particularly consider the three hermitian models for quantum friction, that is, the Caldirola-Kanai, the Kostin, and the Albrecht models, in order to discuss the tunneling problem with quantum friction. We mention that these Hamiltonians have been applied to a tunneling problem [36-41], but a systematic study, including a comparison among the models, has yet to be carried out with respect to tunneling probabilities. In this connection, we notice that Hasse has compared the three models for a free wave packet propagation and for a damped harmonic oscillator. He has shown that the time dependence of the width of a Gaussian wave packet varies significantly from one model to another while all of these three models lead to the same classical equation of motion [42]. It is therefore not obvious whether the three models lead to similar penetrabilities to each other.

The paper is organized as follows. In Sec. II we briefly introduce the three quantum friction models which we employ. In Sec. III we present our results for penetra-
bility of a one-dimensional barrier. In order to compare among the three models, we first carry out a detailed study on numerical accuracy of the calculations for a free wave packet propagation. We then discuss the energy dependence of the penetrability obtained with each of these three models. We also discuss the results in terms of barrier distribution. We finally summarize the paper in Sec. IIIC.

II. QUANTUM FRICTION MODELS

A. Classical equation of motion

We consider a system with a particle whose mass is \( m \), moving in a one-dimensional space \( q \) with a potential \( V(q) \) and a linear frictional force. We here consider potential scattering, and regard \( q \) as the distance between the particle and the center of the potential. The classical equation of motion for the particle reads

\[
\frac{dp}{dt} + \gamma_0 p + \frac{\partial V}{\partial q}(q) = 0,
\]

where \( p = m\dot{q} \) is the kinetic momentum, the dot denoting the time derivative, and \( \gamma_0 \) is a friction coefficient. We have assumed that the potential depends only on \( q \). From the classical equation of motion, Eq. (1), the time derivative of the energy \( E = p^2/2m + V(q) \) reads

\[
\frac{dE}{dt} = -\frac{\gamma_0}{m} p^2. \tag{2}
\]

In constructing the phenomenological quantum friction models, Eqs. (1) and (2) have been used as a guiding principle, that is, it is demanded that the time dependence of the expectation values obeys the same equations as Eqs. (1) and (2) [42].

B. The Caldirola-Kanai model

In the Caldirola-Kanai model, the Hamiltonian depends explicitly on time as [31, 32]

\[
H = \frac{\pi^2}{2m} e^{-\gamma_0 t} + V(q)e^{\gamma_0 t}, \tag{3}
\]

where \( \pi \) is a canonical momentum conjugate to \( q \). The canonical quantization \( [q, \pi] = i\hbar \) with \( p = \pi e^{-\gamma_0 t} \) leads to the desired equations,

\[
\frac{d}{dt} \langle p \rangle + \gamma_0 \langle p \rangle + \left\langle \frac{\partial V}{\partial q}(q) \right\rangle = 0, \tag{4}
\]

\[
\frac{d}{dt} \langle E \rangle = -\frac{\gamma_0}{m} \langle p^2 \rangle. \tag{5}
\]

Here, the expectation value of an operator \( O \) is denoted as \( \langle O \rangle = \int dq \psi^* O\psi \) with a wave function \( \psi = \psi(q,t) \). \( p \) can be regarded as the kinetic momentum operator, since the relation \( \langle p \rangle = m \langle d\langle q \rangle/dt \rangle \) holds. Notice that the kinetic momentum operator depends explicitly on time in this model.

Since the Hamiltonian (3) is hermitian, the probability is conserved with the continuity equation of

\[
\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial q} e^{-\gamma_0 t} = 0, \tag{6}
\]

where \( \rho = |\psi|^2 \) and \( J = (\hbar/m) \Im (\psi^* \partial \psi/\partial q) \) are the probability density and the current, respectively, \( \Im \) denoting the imaginary part.

Since the kinetic momentum operator in this model depends explicitly on time, the commutation relation between the coordinate and the physical momentum is of the form

\[
[q, p(t)] = i\hbar e^{-\gamma_0 t}. \tag{7}
\]

Hence, the quantum fluctuation disappears as \( t \gg 1/\gamma_0 \).

One may consider that this unphysical feature can be neglected if one considers only a short time behavior. However, the friction is not active in that time regime, since the factor \( e^{-\gamma_0 t} \) determines how much the momentum is damped, and thus the dynamics may be rather trivial there.

C. The Kostin and the Albrecht models

In the Kostin and the Albrecht models, the momentum operator is kept time-independent, but a nonlinear potential \( W \) is introduced in the Hamiltonian:

\[
H = \frac{p^2}{2m} + V(q) + \gamma_0 W. \tag{8}
\]

In the Kostin model, the nonlinear potential is taken to be [33]

\[
W_{\text{Ko}} = \frac{\hbar}{2i} \left( \ln \frac{\psi^*}{\psi} - \left\langle \frac{\psi^*}{\psi} \right\rangle \right), \tag{9}
\]

while in the Albrecht model it is taken as [34],

\[
W_{\text{Al}} = \langle p \rangle (\langle q \rangle - \langle q \rangle). \tag{10}
\]

With the canonical quantization, one obtains Eq. (4) together with

\[
\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial q} = 0, \tag{12}
\]

for both Hamiltonians.

The energy dissipation for the Kostin model is given by

\[
\frac{d}{dt} \langle E \rangle = -\frac{\gamma_0}{m} \left\langle \left( \frac{mJ}{\rho} \right)^2 \right\rangle. \tag{13}
\]

However, the friction is not active in that time regime, since the factor \( e^{-\gamma_0 t} \) determines how much the momentum is damped, and thus the dynamics may be rather trivial there.
Hamiltonians with $\gamma$ form factor dissipation occurs only during interaction. That is, the potential, $V$, introduce a friction form factor in the collision problem, however, we have to consider only the Albrecht model in this paper. A damped harmonic oscillator. For simplicity, however, we consider only the Albrecht model in this paper.

D. Generalization for a collision problem

In the original models for quantum friction, the friction constant $\gamma_0$ is treated to be a constant. When considering friction in a collision problem, however, we have to introduce a friction form factor $f(q)$, since the energy dissipation occurs only during interaction. That is, the form factor $f(q)$ vanishes outside the range of the potential, $V(q)$. A naive replacement of $\gamma_0$ in the model Hamiltonians with $\gamma_0 f(q)$ does not work due to the $q$ dependence in the form factor. Alternatively, in this paper we consider a time dependent friction coefficient $\gamma(t)$ which vanishes after the interaction. In the simple form of $\gamma(t) = \gamma_0 f(\langle q \rangle_t)$, the dissipation continuously occurs even after the interaction if an incident wave is equally bifurcated into transmitted and reflected waves. To avoid this undesired behavior, we choose the form,

$$\gamma(t) = \gamma_0 \langle f(q) \rangle_t.$$  

In Ref. [40], Hahn and Hasse discussed a generalization of the Albrecht model and suggested a better nonlinear potential, $W$, which reproduces the classical reduced frequency for a damped harmonic oscillator. For simplicity, however, we consider only the Albrecht model in this paper.

III. RESULTS

To calculate the tunneling probability with the three models discussed in the previous section, we integrate the time dependent nonlinear Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi.$$  

In what follows, we employ the same potential as in Ref. [40], that is,

$$V(q) = V_0 e^{-\frac{q^2}{2s^2}},$$  

with $V_0 = 100$ MeV and $s = 3$ fm. This potential somehow simulates the $^{58}$Ni$^{+^{58}}$Ni reaction, and thus we take $me^2 = 29 \times 938$ MeV.

A. Wave packet tunneling without friction

Before we introduce the friction, we first discuss the time-dependent approach to quantum tunneling. For the calculation of the tunneling probability for the time-dependent nonlinear Hamiltonians, the usual time-independent approach, which imposes the asymptotic plane wave boundary condition, would not be applicable. An alternative method is to make a wave packet propagate, then observe how it bifurcates after it passes the potential region.

A wave packet is a superposition of various waves, each of which has a different energy. Hence, in order to obtain the tunneling probability for a certain energy, one needs either to perform the energy projection [47, 48] or to broaden the spatial distribution of the wave packet so that the energy distribution becomes narrow [49]. In the former method, the tunneling probability is calculated as the ratio of the energy distribution of a transmitted wave packet to that of the incident one at a fixed energy. This method, however, is not applicable in our case, since we do not know a priori how much energy is lost during a collision at each energy. Therefore, we shall employ the latter approach here. To this end, it is necessary to know how narrow the energy distribution should be in the wave packet in order to obtain meaningful results.

To clarify the effect of finite width in the energy distribution, we take the initial wave function in the energy space $\psi_0(E; E_i)$ with the Gaussian form,

$$|\tilde{\psi}_0(E; E_i)|^2 = \frac{1}{\sqrt{2\pi \sigma_E}} e^{-\frac{(E-E_i)^2}{2\sigma_E^2}},$$  

where $E_i$ and $\sigma_E$ are the mean energy and the width of the energy distribution, respectively. Multiplying

Kan and Griffin rederived the Kostin Hamiltonian from a fluid dynamics point of view [43, 44]. In that context, $mJ/\rho$ is the kinetic momentum, and hence Eq. [13] is similar to Eq. [2]. For the Albrecht model, on the other hand, one obtains

$$\frac{d}{dt} \langle E \rangle = -\frac{\gamma_0}{m} \langle p \rangle^2,$$  

as is desired. Notice that the energy dissipation is proportional to $\langle p^2 \rangle$ in the Caldirola-Kanai and the Kostin models (see Eqs. [5] and [13]), while it is $\langle p \rangle^2$ in the Albrecht model. In the classical limit, these quantities are the same to each other, but they may differ in quantum mechanics.
\( \psi_0(E; E_1) \) by \( e^{ik(q-\phi_0)} \) with \( E = \hbar^2 k^2 / (2m) \) and making its Fourier transform into the coordinate space, one obtains the initial wave function in the coordinate space, \( \psi(q, t = 0; E_1) \), which is consistent with the energy distribution given by Eq. (20). Such wave function has the mean position of \( q_0 \). Notice that the energy alone does not determine the direction of propagation of the wave packet. It is determined by the interval of the integration with respect to \( k \). We consider a propagation of the wave packet from \( q_0 < 0 \) towards the positive \( q \) direction, and thus we make the integration from \( k = 0 \) to \( \infty \). Even though this initial wave function is somewhat different from the one used in Refs. [38, 39], we find that this form is more convenient in order to discuss a correspondence to the time-independent solutions (see Eq. (22) below).

By integrating the time dependent Schrödinger equation, (13), from \( t = 0 \) to \( t = t_f \), by which time the bifurcation of the wave packet has been completed, we calculate the tunneling probability \( T_{wp}(E_i) \) as

\[
T_{wp}(E_i) = \int_0^\infty dq |\psi(q, t_f; E_i)|^2.
\] (21)

In implementing the time integration, it is helpful to introduce the dimensionless time \( \tau \equiv t/t_0 \), by measuring the time in units of a typical time scale of the problem, \( t_0 \). For this, we take \( t_0 \) as the time taken by a free classical particle to travel some distance \( L \), that is, \( t_0 = L/\sqrt{2E_i/m} \). We choose \( L \) so that the final mean position of the transmitted wave packet at \( t = t_0 \) is almost independent of \( E_i \) (and the friction coefficient, \( \gamma_0 \)) for each parameter set.

When \( \sigma_E \) is small enough, it is expected that \( T_{wp}(E_i) \) is nearly the same as the tunneling probability obtained for a certain energy \( E_i \), \( T_{ex}(E_i) \). More generally, the following relation between \( T_{wp} \) and \( T_{ex} \) should hold:

\[
T_{wp}(E_i) = \int_0^\infty dE |\tilde{\psi}_0(E; E_i)|^2 T_{ex}(E).
\] (22)

Without friction, \( T_{ex}(E) \) can be calculated with the time-independent approach. The upper panel of Fig. 1 shows the result for \( \sigma_E = 1 \) MeV. To solve the time dependent Schrödinger equation with a wave packet, we employ the Crank-Nicholson method together with the tridiagonal matrix algorithm [50] with grid sizes of \( \Delta_x = 0.00025 \) and \( \Delta q = 0.01 \) fm. We take a space of \(-100 \text{ fm} < q < 100 \text{ fm} \), and set \( q_0 = -50 \text{ fm} \) and \( L = 165 \text{ fm} \). The solid line shows the penetrability obtained with the time-independent method, while the dots are obtained with the wave packet method. The dashed line denotes the average penetrability according to Eq. (22). One can find that Eq. (22) is valid until the tunneling probability falls below \( 10^{-7} \).

In order to improve the agreement between \( T_{wp} \) and \( T_{ex} \), one needs a smaller value of \( \sigma_E \). The lower panel of Fig. 1 shows the result for \( \sigma_E = 0.5 \) MeV. In this case, we enlarge the space to \(-150 \text{ fm} < q < 150 \text{ fm} \) to accommodate a spatially wider wave packet. As one can see, the penetrability with the time-dependent wave packet method is now in a good agreement with \( T_{ex} \) for the tunneling probability higher than \( 10^{-4} \). We therefore use \( \sigma_E = 0.5 \) MeV for all the calculations shown below.

**B. Free wave packet evolution with friction**

In order to discuss the value of a friction coefficient as well as numerical accuracy of the calculations, we next consider free wave packet in this subsection. As discussed in Sec. 11 the potential and the corresponding friction form factor should have a similar range. In this paper we simply employ the same form for the form factor as that for the potential, Eq. (19).

\[
f(q) = \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{q^2}{2s^2}}.
\] (23)

Here \( f(q) \) is normalized so that \( \gamma_0 \) is interpreted as the strength of friction. Note that the dimension of \( \gamma_0 \) is altered from inverse time to velocity.

The friction strength \( \gamma_0 \) is determined based on the amount of energy loss. Since the energy loss depends on energy, we choose the barrier top energy, \( E_f = 100 \) MeV, as a reference. With the mean energy of the transmitted wave, \( E_f \), the energy loss \( E_{loss} \) is given by \( E_{loss} = E_i - E_f \).
FIG. 2: Panels (a)-(c): The numerical accuracy defined by Eq. (27) in the case of the strong friction. The dashed and the straight lines show the results without and with friction, respectively. The dotted lines are the expectation value of the form factor, $\langle f(q) \rangle$. Panel (d): the time dependence of $\langle q \rangle$ for the Caldirola-Kanai model. All the calculations are performed with $L = 160$ fm.

$E_f$. We here consider weak and strong friction cases, for which $E_{\text{loss}}$ is 5 MeV and 30 MeV, respectively. These are realized when $\gamma_0$ is chosen as listed in Table I.

| strength  | Caldirola - Kanai | Kostin | Albrecht |
|-----------|-------------------|--------|----------|
| weak ($E_{\text{loss}} = 5$ MeV) | 2.14  | 2.16   | 2.17     |
| strong ($E_{\text{loss}} = 30$ MeV) | 13.8  | 14.0   | 14.0     |

TABLE I: The dimensionless friction coefficient $\gamma_0/c$, $c$ being the speed of light, for the weak and strong friction cases. All the values are given in units of $10^{-3}$.

Using these friction coefficients, the accuracy of integration of the time dependent nonlinear Schrödinger equation is tested by checking how well the equation of motion, Eq. (4), is reproduced. The equation cannot be solved in the same way as in Sec. III A, since the matrix is no longer in a tridiagonal form due to the nonlinearity. Instead, we carry out the numerical integration in the following way.

The discretized Schrödinger equation may be given by

$$i\hbar \frac{\psi^{n+1} - \psi^n}{\Delta t} = \frac{H_{n+1}^2 + H_n^2}{2}$$

at the $n$-th time grid. Here $H$ is the Hamiltonian which depends on $\psi$. In our calculation, we simply neglect the time dependence of the Hamiltonian and obtain

$$i\hbar \frac{\psi^{n+1} - \psi^n}{\Delta t} = \frac{H_{n+1} \psi^{n+1} + H_n \psi^n}{2}.$$  (24)

We integrate this equation with grid sizes of $\Delta \tau = 0.00025$, $\Delta q = 0.01$ fm for the Caldirola-Kanai and the Kostin models, and $\Delta \tau = 0.00015$, $\Delta q = 0.01$ fm for the Albrecht model.

A care must be taken in integrating the equation for the Kostin model. The nonlinear potential Eq. (10) is nothing but the phase of a wave function, and hence a naive estimation leads to discontinuities in the potential. However, one can estimate the continuous phase by the
following definition \[44\] (see also Ref. \[23\]):

\[
\text{arg } \psi(q, t) = \Im \psi(q, t) + 2\pi (n_+ - n_-),
\]

(26)

where \(n_+\) and \(n_-\) are the number of crossing the discontinuous points from \(\pi\) to \(-\pi\) and from \(-\pi\) to \(\pi\), respectively. We take \(q = 0\) as a reference and compute \(n_+\) (\(n_-\)) by counting the point where adjacent phase differs by less than \(-4\) (more than \(4\)).

Our numerical test is carried out for \(E_i = 100\) MeV. We compare the following quantity with that of the no friction:

\[
\text{Accu} \equiv \frac{1}{\mathcal{N}} \left| \frac{d}{d\tau} \langle \hat{p} \rangle + \gamma \tau \langle \hat{p} \rangle \right|.
\]

(27)

Since we do not consider the potential in this subsection, this quantity vanishes if the equation of motion is fully satisfied. The accuracy for the strong friction case is shown in Figs. 4 (a)-(c) for the three friction models. The expectation value of the form factor, \(\langle \hat{q} \rangle\), is also shown to illustrate the effect of nonlinearity on numerical accuracy. The corresponding \(\langle \hat{q} \rangle\) as a function of \(\tau\) is also shown in Fig. 2 (d) for the Caldirola-Kanai model (the results for the other two models are almost the same and are not shown in the figure). We have verified that the probability is conserved within a numerical accuracy for all the calculations. It is found that the Caldirola-Kanai model can be integrated as accurately as the no friction case. In contrast, the reproduction of the equation of motion is less satisfactorily with the Kostin and the Albrecht models due to the nonlinearity of the equations. This is expected if the nonlinearity due to the form factor plays a much less important role as compared to the nonlinearity of the equation itself, since the nonlinearity of the Caldirola-Kanai model is caused only by the friction form factor, Eq. (23). Actually, we have verified that the accuracy remains almost the same as Fig. 2 even without the form factor for all the models.

Notice that the increase of Accu at large \(\tau\) is due to the finiteness of our space, that is, \(q\) is limited in the range of \(-150\) fm \(\leq q \leq 150\) fm. An accumulation of numerical errors is rather small, as no increase is observed when \(L\) is taken to be small enough so that the tail of the wave packet does not reach the edge of the box at \(\tau = 1\) while keeping the number of step in the \(\tau\) integration to be the same.

The accuracy of the nonlinear potential models does not improve even for the weak friction, as shown in Figs. 3 (a)-(c). Even though the absolute value of Accu is slightly reduced in this case, the \(\tau\)-dependence remains almost the same.

We should note that the damping from a bound excited state to the ground state can successfully be described with the Kostin model \[44, 51\]. Actually we also have verified it for a harmonic oscillator with the same grid sizes. An application to scattering problems seems more difficult with our present numerical method.

C. Quantum tunneling with friction

Let us now discuss dissipative quantum tunneling, i.e., quantum tunneling in the presence of friction. For an illustration, Fig. 3 shows the time-evolution of the wave packet for the Caldirola-Kanai model. The initial mean energy is chosen so that the transmitted wave packet has an appreciable amount. The calculations are performed with \(E_i = 100\) MeV and \(L = 165\) fm for the no friction, \(E_i = 103\) MeV and \(L = 165\) fm for the weak friction, and \(E_i = 120\) MeV and \(L = 185\) fm for the strong friction. The behavior is quite similar also for the nonlinear potential models. Notice that the reflected and the transmitted wave packets at \(\tau = 1\) largely deviate from a symmetric Gaussian shape in the presence of friction.

Figs. 4 and 5 compare the tunneling probability as a function of energy obtained with the three friction models for the weak and the strong friction cases, respectively. We plot only the tunneling probability larger than \(10^{-4}\), according to the discussion in Sec. III A. One can see that the tunneling probability of the three models is nearly the same, even though there might be a possibility that the results of the Kostin and the Albrecht models suffer
FIG. 5: The energy dependence of the tunneling probability for the Caldirola-Kanai (the dot-dashed lines), the Kostin (the filled circles), and the Albrecht (the dotted lines with filled diamonds) models with the weak friction. The upper panel is in the linear scale, while the lower panel is in the logarithmic scale. The result without friction is also shown by the solid lines for a comparison.

FIG. 6: Same as Fig. 5 but with the strong friction.

FIG. 7: Upper panel: the penetrabilities for the Caldirola-Kanai model as a function of incident energy. The solid line corresponds to the no friction case. The dashed and the dotted lines are for the weak and strong friction cases, respectively. Lower panel: the corresponding barrier distribution defined as the first energy derivative of the penetrability.

from numerical errors with the present setup of numerical calculations (see the discussion in Sec. 11). It is interesting to notice that the Caldirola-Kanai and the Kostin models lead to almost the same results to each other, while the result of the Albrecht model slightly deviates from the other two models. Even though the exact cause of this different behavior is not known, a possible origin may be the fact that the energy dissipation is slightly different between the Albrecht model and the Caldirola-Kanai/Kostin models (see the discussion below Eq. (14)).

In what follows, we focus only on the result of the Caldirola-Kanai model. As can be seen in the upper panel of Fig. 7, the stronger the friction is, the lower the tunneling probability results in. This behavior is consistent with the results of Ref. 36 for a rectangular barrier unless the tunneling probability is extremely small. Ref. 36 showed that the tunneling probability is not affected by friction at energies well below the barrier. Whereas the numerical accuracy has yet to be estimated in order to draw a conclusive conclusion concerning the role of friction in quantum tunneling at deep subbarrier energies, we simply could not confirm the result of Ref. 36 because a finite width in the wave packet prevents us to go into the deep subbarrier energy region (see the lower panel of Fig. 1). In Ref. 39, McCoy and Carbonell argued that the tunneling probability is either increased or decreased by friction depending on the magnitude of the barrier height and width. We do not confirm their results either, partly because we do not include the fluctuation term in the Hamiltonian.

In order to gain a deeper insight into the role of friction in quantum tunneling, we next discuss a barrier distribution. In the field of heavy-ion subbarrier fusion reactions, the so called fusion barrier distribution has been widely used in analyses of experimental data 2, 52. This quantity is defined as the second energy derivative of the product of the incident energy $E$ and fusion cross sections $\sigma_{\text{fus}}$, that is, $d^2(E\sigma_{\text{fus}})/dE^2$ 52, and has provided a con-
venient representation in order to study the underlying dynamics of subbarrier fusion reactions. For the tunneling problem, this quantity corresponds to the first energy derivative of the penetrability, $dI/dE$. The barrier distribution for the Caldirola-Kanai model is shown in the lower panel of Fig. 7. Whereas the barrier distribution shows a symmetric peak in the case of no friction, some strength is shifted towards higher energies as the strength of the friction increases and the barrier distribution becomes structured. It is interesting to notice that a similar behavior has been obtained in coupled-channels calculations for fusion in relatively heavy systems, such as $^{100}$Mo+$^{100}$Mo.

The barrier distribution indicates that the energy damping during tunneling results in a increased effective barrier, whose height is thus energy dependent and is determined by the strength of friction. This leads us to two different points of view for dissipative quantum tunneling. From one view point, the incident energy is damped by friction while a wave packet traverses towards a fixed barrier. This can be interpreted in a different way as that the effective barrier increases dynamically due to the friction for a fixed value of incident energy. The barrier distribution shown in Fig. 7 well represents this dynamical point of view of friction.

IV. SUMMARY

We have investigated the effects of friction on quantum tunneling by applying the three friction models, the Caldirola-Kanai, the Kostin, and the Albrecht models, to a one-dimensional tunneling problem. We have studied the energy dependence of the tunneling probability obtained as the barrier penetration rate of a wave packet, whose initial energy variance is set to be small enough. In order to limit a region where the dissipation is active, we have introduced the time dependent friction coefficient. We have shown that the friction tends to prevent the wave packet from penetrating the barrier, and thus the penetrability decreases as a function of the strength of friction. We have found that the three models lead to similar penetrabilities to each other. We have also discussed the effect of friction on quantum tunneling in terms of barrier distribution and have shown that the barrier distribution becomes structured due to friction by shifting effective barriers towards higher energies. Among the three models which we considered in this paper, we have found that the numerical accuracy can be most easily handled with the Caldirola-Kanai model.

Very recently, it has been found experimentally that heavy-ion multi-nucleon transfer processes in $^{16,18}$O, $^{19}$F + $^{208}$Pb reactions populate highly excited states in the target-like nuclei. One may be able to describe such processes by extending the friction models considered in this paper to multi-channel cases. We are now working towards this direction, and we will report our results in a separate paper. Another interesting future direction is to include the random force term to the quantum friction Hamiltonians and investigate its effect on quantum tunneling. For this purpose, a proper quantization of the fluctuation term will be needed.

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