RECOVERING TWO COEFFICIENTS IN AN ELLIPTIC EQUATION VIA PHASELESS INFORMATION

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Abstract. For fixed \( y \in \mathbb{R}^3 \), we consider the equation
\[
Lu + k^2 u = -\delta(x - y), \quad x \in \mathbb{R}^3,
\]
where \( L = \text{div}(n(x)^{-2} \nabla) + q(x) \), \( k > 0 \) is a frequency, \( n(x) \) is a refraction index and \( q(x) \) is a potential. Assuming that the refraction index \( n(x) \) is different from 1 only inside a bounded compact domain \( \Omega \) with a smooth boundary \( S \) and the potential \( q(x) \) vanishes outside of the same domain, we study an inverse problem of finding both coefficients inside \( \Omega \) from some given information on solutions of the elliptic equation. Namely, it is supposed that the point source located at point \( y \in S \) is a variable parameter of the problem. Then for the solution \( u(x, y, k) \) of the above equation satisfying the radiation condition, we assume to be given the following phaseless information
\[
f(x, y, k) = |u(x, y, k)|^2 \text{ for all } x, y \in S \text{ and for all } k \geq k_0 > 0,
\]
where \( k_0 \) is some constant. We prove that this phaseless information uniquely determines both coefficients \( n(x) \) and \( q(x) \) inside \( \Omega \).

1. Introduction. We here study the phaseless inverse problem for the equation
\[
Lu + k^2 u = -\delta(x - y), \quad x \in \mathbb{R}^3,
\]
where \( L = \text{div}(n(x)^{-2} \nabla) + q(x) \), \( k > 0 \) is a frequency, \( n(x) \) is a refraction index and \( q(x) \) is a potential. We assume that these functions are infinitely differentiable and satisfy
\[
n \in C^\infty(\mathbb{R}^3), \quad n(x) \geq 1, \quad \text{supp}(n - 1) \subset \Omega,
\]
\[
q \in C^\infty(\mathbb{R}^3), \quad q(x) \geq 0, \quad \text{supp} q \subset \Omega,
\]
where \( \Omega \) is an open bounded convex domain with a smooth boundary \( S \). The condition \( n(x) \geq 1 \) means that a medium filling \( \Omega \) is more dense than in the remain space. Henceforth we set \( i = \sqrt{-1} \).

Let the function \( u(x, y, k) \) be the solution to equation (1.1) satisfying the radiation conditions
\[
(1.4) \quad u(x, y, k) = O(r^{-1}), \quad \frac{\partial u}{\partial r}(x, y, k) - iku = O(r^{-1}) \quad \text{as } r := |x| \to \infty.
\]
Here condition (1.4) is valid uniformly for every fixed source position $y$ and all directions $x/r$. Assume that $y$ is an arbitrary point of $S$. Below we shall consider an inverse problem of recovering coefficients $n(x)$ and $q(x)$ from given data $|u(x, y, k)|^2$ of the solution $u(x, y, k)$ to problem (1.1)-(1.4) for all $x, y \in S$ with $x \neq y$ and all large $k$. In other words, we here do not use any phase information on the solution.

The phaseless inverse problems has recently attracted more attention. A formulation of the phaseless problem was given in the book [2] by Chadan and Sabatier 40 years ago. The authors noted that the phase of a solution to the Schrödinger equation can not be measured for the large frequencies (energies). Related to this, they suggested to study the inverse problem when only modules of the scattering field are measured. Later Newton [17] discussed this problem as a very important problem in the quantum theory. The first results for a phaseless inverse problem related to recovering potential in Schrödinger equation were obtained by Klibanov and Romanov [3]-[5], [7]-[9] and Novikov [18]-[20]. In the papers [7]-[9] the phaseless inverse problems for incident point sources and incident plane waves were reduced to the well-known X-ray tomography problem. In the papers [18]-[19], it was shown how to recover the phase of the scattering field by some additional observations of the field, when one enters in the space $\mathbb{R}^3$ successively two new known potentials.

Then the papers [6, 10, 11] considered the phaseless problems for generalized Helmholtz operator $\Delta + k^2 n^2(x)$ with incident point sources or incident plane waves. These problems were reduced to the inverse kinematic problem of recovering refraction index $n(x)$ in $\Omega$ through given Riemannian distances $\tau(x, y)$ for arbitrary points of $S$. The length element $d\tau$ of the conformal Riemannian metric is defined here via $n(x)$ by the formula: $d\tau = n(x)|dx|$ where $dx = (dx_1, dx_2, dx_3)$. The theory of the inverse kinematic problem was developed in the 1970’s. Therefore the given reduction is useful both for theoretical arguments and numerical implementations. In the paper [12] a special procedure for extracting information of Riemannian distances $\tau(x, y)$ from given data of the phaseless inverse problem related to the Helmholtz operator was developed. Recently the inverse phaseless problem of recovering the permittivity coefficient in the Maxwell equations was considered in [26] for point incident source, and in [27] for the incident plane wave.

In the present paper we consider the problem of recovering coefficients $n(x)$ and $q(x)$ from the given phaseless information. This work develops [7] and [10]. At first we derive an asymptotic formula for scattering field as $k \to \infty$. Then we demonstrate how to extract information on Riemannian distances $\tau(x, y)$ for all $x, y \in S$ from the given data. As a result we reach an inverse kinematic problem. It allows us to uniquely recover the refraction index $n(x)$. After this we show that the given information uniquely determines the integrals along the geodesic lines of the Riemannian metric $\Gamma(x, y)$ of the potential $q(x)$ for all $x, y \in S$. Hence, the problem of recovering the potential is reduced to an integral geometry problem: find $q(x)$ in $\Omega$ from its given integrals along geodesics joining arbitrary points $x$ and $y$ of $S$. The uniqueness and the stability of the latter problem follow from known results.

The paper is organized as follows. In Section 2 we formulate our inverse problem to state the main result Theorem 1 on the uniqueness, and establish an asymptotic formula Proposition 1 for solution to problem (1.1)-(1.4), which is essential for the proof of the main result. In Section 3, we describe a procedure for recovering $n(x)$ and show a stability result for it. In Section 4, we demonstrate how the problem of recovering $q(x)$ is reduced an integral geometry problem, and show the stability.
Recovering two coefficients

2. Formulation of the inverse problem, the main result and an asymptotic formula. Represent the solution $u(x, y, k)$ to problem (1.1)-(1.4) in the form

$$u(x, y, k) = u_0(x, y, k) + u_{sc}(x, y, k),$$

where $u_0(x, y, k)$ is defined by the formula

$$u_0(x, y, k) = A_0(x, y)e^{ik|x-y|},$$

is the fundamental solution of the Helmholtz operator $-\Delta - k^2$ with the conditions (1.4), and $u_{sc}(x, y, k)$ is the scattering field.

Consider the inverse problem of recovering $n(x)$ and $q(x)$ inside $\Omega$ from given functions $f(x, y, k)$ defined by the formula

$$f(x, y, k) = |u_{sc}(x, y, k)|^2, \quad \forall (x, y) \in (S \times S), \forall k \geq k_0,$$

where $k_0$ is a fixed positive number.

Introduce the Riemannian metric by the formula

$$d\tau = n(x)|dx|, \quad |dx| = \left(\sum_{i=1}^{3} dx_i^2\right)^{1/2},$$

where $d\tau$ is the element of length and by $\tau(x, y)$ we denote the Riemannian distance between points $x$ and $y$ belonging to $\mathbb{R}^3$. It is well-known that $\tau(x, y)$ is the solution to the following problem

$$|\nabla_x \tau(x, y)|^2 = n^2(x), \quad \tau(x, y) = O(|x - y|) \text{ as } x \to y.$$

We introduce

**Assumption.** The Riemannian metric is simple, that is, every two points $x, y \in \mathbb{R}^3$ can be connected by a single geodesic line $\Gamma(x, y)$.

Henceforth let $u_{sc}^{(j)}(x, y, k)$, $j = 1, 2$, be the scattering field corresponding to the refraction index $n_j$ and the potential $q_j$.

Now we are ready to state our main result, that is, the uniqueness in determining $n$ and $q$ by the phaseless data $|u_{sc}(x, y, k)|^2$.

**Theorem 1.** Let (1.2), (1.3) and Assumption be satisfied for $n_j, q_j$, $j = 1, 2$. If

$$|u_{sc}^{(1)}(x, y, k)| = |u_{sc}^{(2)}(x, y, k)|, \quad x, y \in S, x \neq y, \quad k \geq k_0,$$

then $n_1 = n_2$ and $q_1 = q_2$ in $\mathbb{R}^3$.

**Remark 1.** One interesting question arise: what will happen if we assume that the function $f(x, y, k)$ is given not for all $k \geq k_0$, but only for a discrete wave numbers set, i.e., for $k = k_j$ with $k_1 < k_2 < \ldots$ and $k_j \to \infty$? To our regret, our method does not give an answer to this question because in the used expansion (3.1), the leading periodic term contains $\cos(k\gamma)$ and it vanishes if $k_j$ are choosing as $k_j\gamma = \pi/2 + \pi j$, $j = 1, 2, \ldots$.

The rest part of this paper is devoted to the proof of Theorem 1. Indeed our proof yields some stability results in determining $n_j$ and $q_j$, which are stated as Theorems 2 and 3 respectively, and Theorem 1 directly follows from these theorems.
Our proof is based on

**Proposition 1.** Let conditions (1.2), (1.3) and Assumption be fulfilled. Then for \( x \neq y \) the following asymptotic formula holds

\[
\begin{align*}
    u_{sc}(x, y, k) &= e^{ik\tau(x, y)} \left[ A(x, y) - \frac{\alpha_0(x, y)}{ik} \right] \\
    - A_0(x, y)e^{ik|x-y|} + O\left(\frac{1}{k^2}\right) \text{ as } k \to \infty,
\end{align*}
\]

where \( A(x, y) > 0 \) and \( \alpha_0(x, y) \) are infinitely differentiable functions.

**Proof.** Consider the auxiliary Cauchy problem

\[
\begin{align*}
    \frac{\partial^2 v}{\partial t^2} - Lv &= \delta(x - y, t), \quad x \in \mathbb{R}^3; \quad v|_{t<0} = 0,
\end{align*}
\]

Here \( \delta(x - y, t) \) is the Dirac delta-function located at the point \((y, 0)\) of the 4-D space. It means that we should consider the solution of the problem (2.6) as a distribution. Needless to say, the problem (2.6) is equivalent to the Cauchy problem for the homogeneous equation with the initial data \( v(x, 0) = 0 \) and \( v_t(x, 0) = \delta(x-y) \) and the right-hand side in (2.6) which is equal to zero, but we prefer to use the present form of this problem, taking into account that the solution is a distribution. Notice that the formal Fourier transform with respect to \( t \) of the hyperbolic equation leads to (1.1). We shall study some properties of the solution to problem (2.6) in order to use them for deriving the asymptotic expansion (2.5).

Introduce the following functions:

\[
\begin{align*}
    \theta_0(t) &:= \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases} \\
    \theta_m(t) &:= \frac{t^m}{m!} \theta_0(t), \quad m = 1, 2, \ldots.
\end{align*}
\]

Then the following lemma holds (see Lemma 2.2.1 in [25]).

**Lemma 1.** Let \( n = n(x) \) and \( q = q(x) \) be in \( C^\infty(\mathbb{R}^3) \) and Assumption hold. Then the solution to problem (2.6) can be represented in the form of the asymptotic series

\[
v(x, t; y) = \theta_0(t) \left[ \alpha_{-1}(x, y)\delta(t^2 - \tau^2(x, y)) + \sum_{m=0}^\infty \alpha_m(x, y)\theta_m(t^2 - \tau^2(x, y)) \right],
\]

where \( \tau^2(x, y), \alpha_m(x, y), m = -1, 0, 1, \ldots \), are infinitely differentiable functions of \( x, y \) and, \( \alpha_{-1}(x, y) > 0 \).

Let \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \) be the Riemannian coordinates of a point \( x \) with respect to a fixed point \( y \). They can be calculated through the function \( \tau^2(x, y) \) by the formula (see formula (2.2.28) in [25]):

\[
(2.7) \quad \zeta = -\frac{1}{2} \left( \nabla_y \tau^2(x, y) \right) n^{-2}(y).
\]

Denote by \( J(x, y) \) the Jacobian of the transformation of the Riemannian coordinates into the Cartesian ones, i.e.,

\[
J = \det \left( \frac{\partial \zeta}{\partial x} \right).
\]
Then coefficients of the expansion (2.7) are defined by the formulae (see (2.2.44), (2.2.45) in [25])

\[(2.8)\quad \alpha_{-1}(x, y) = \frac{1}{2\pi} n^3(y) \sqrt{J(x, y)},\]

\[(2.9)\quad \alpha_m(x, y) = \frac{\alpha_{-1}(x, y)}{4\pi m+1(x, y)} \int_{\Gamma(x, y)} \tau^m(\xi, y) \left[ \text{div} \left( \frac{n(\xi)^{-2}\nabla}{\alpha_m(\xi, y)} \right) \right] d\tau_\xi,\]

\[m = 0, 1, 2, \ldots,\]

where \(\Gamma(x, y)\) is the geodesic line connecting \(x\) and \(y\), and \(d\tau = d\tau_\xi\) is the element of the Riemannian length and \(\xi \in \Gamma(x, y)\) is a variable point.

It follows from Lemma 1 that the solution to problem (2.6) for \(x \neq y\) can be represented in the form

\[(2.10)\quad v(x, t; y) = A(x, y) \delta(t - \tau(x, y)) + \tilde{v}(x, t; y)H(t - \tau(x, y)),\]

where \(A(x, y) = \alpha_{-1}(x, y)/(2\tau(x, y))\), \(\tilde{v}(x, t; y)\) is a \(C^\infty(\mathbb{R}^3)\)-function in \(x\) and

\[(2.11)\quad \tilde{v}(x, \tau(x, y) + 0; y) = \alpha_0(x, y).\]

Below we shall use a result in Vainberg [28] related to the decay of the solution to problem (2.6) as \(t \to \infty\). The result says that for any \(R > 0\) and the domain \(D(y, R) = \{x \in \mathbb{R}^3; |x - y| < R\}\), there exist positive numbers \(M, \ell, t_0\), depending only on \(n(x), q(x)\) and \(R, y\), such that

\[|\partial_t^m v(x, y, t)|, |\partial_x v(x, y, t)|, |\partial_{x,x} v(x, y, t)| \leq Me^{-\ell t}, \quad m = 0, 1, 2\]

for all \(t \geq t_0\) and all \(x \in D(y, R)\).

Applying this result, we obtain

\[u(x, y, k) = \int_{-\infty}^{\infty} e^{ikt} v(x, t; y) dt.\]

Taking into account the representation (2.10) for \(x \neq y\), we find

\[u(x, y, k) = e^{ikt(x,y)} A(x, y) + \int_{\tau(x,y)}^{\infty} e^{ikt} \tilde{v}(x, t; y) dt\]

\[= e^{ikt(x,y)} \left[ A(x, y) - \frac{\tilde{v}(x, \tau(x, y) + 0; y)}{ik} \right.\]

\[\left. + \frac{1}{(ik)^2} \int_{\tau(x,y)}^{\infty} e^{ikt} \tilde{v}_{t\ell}(x, t; y) dt \right].\]

From here, using formula (2.11), we obtain

\[(2.12)\quad u(x, y, k) = e^{ikt(x,y)} \left[ A(x, y) - \frac{\alpha_0(x, y)}{ik} \right] + O \left( \frac{1}{k^2} \right) \quad \text{as } k \to \infty.\]

Hence, for \(x \neq y\) we reach (2.5), and the proof of the proposition is complete.
3. Recovering the refraction index. We can represent (2.5) as
\[ u_{sc}(x, y, k) = A(x, y) \cos(k\gamma(x, y)) - A_0(x, y, k) \cos(k|x - y|) \]
\[ + \frac{1}{k} \alpha_0(x, y) \sin(k\gamma(x, y)) + i \left[ A(x, y) \sin(k\gamma(x, y)) \right. \]
\[ \left. - A_0(x, y) \sin(k|x - y|) + \frac{1}{k} \alpha_0(x, y) \cos(k\gamma(x, y)) \right] \]
\[ + O \left( \frac{1}{k^2} \right). \]

Then we find that
\[ f(x, y, k) = A^2(x, y) + A_0^2(x, y) - 2A(x, y)A_0(x, y, k) \cos(k\gamma) \]
\[ + \frac{2}{k} \alpha_0(x, y)A_0(x, y) \sin(k\gamma) + O \left( \frac{1}{k^2} \right) \text{ as } k \to \infty, \]
where \( \gamma(x, y) = \tau(x, y) - |x - y| \) and \( f(x, y, k) = |u_{sc}(x, y, k)|^2 \) is a given function for all \( x \in S, y \in S \) and \( k \geq k_0 \). Fix \( x \in S \) and \( y \in S \) with \( x \neq y \). Then \( f(x, y, k) \) is a function of \( k \) only. Consider the limit of \( f(x, y, k) \) as \( k \to \infty \). It exists if and only if \( \tau(x, y) = |x - y| \). Hence, if the limit of \( f(x, y, k) \) as \( k \to \infty \) exists, then \( \tau(x, y) = |x - y| \). Consider now the case when \( \tau(x, y) \neq |x - y| \). Then \( \tau(x, y) > |x - y| \) because \( n(x) \geq 1 \) in \( \Omega \). Recall that
\[ A_0(x, y) = \frac{1}{4\pi|x - y|}. \]
From formula (3.1), it follows that
\[ \limsup_{k \to \infty} f(x, y, k) = (A(x, y) + A_0(x, y))^2. \]
Hence we find
\[ A(x, y) = \sqrt{\limsup_{k \to \infty} f(x, y, k) - \frac{1}{4\pi|x - y|}}, \quad (x, y) \in S \times S, \quad x \neq y. \]
We set
\[ g(x, y, k) = -\frac{f^2(x, y, k) - A^2(x, y) - A_0^2(x, y)}{2A(x, y)A_0(x, y)}. \]
Now we can determine \( \tau(x, y) \) for \( (x, y) \in S \times S \). Fix again \( x \) and \( y \) on \( S \). Then we can rewrite (3.1) as
\[ g(x, y, k) = \cos(k\gamma) + O \left( \frac{1}{k} \right) \text{ as } k \to \infty. \]
In the paper [12], it was shown that the equation \( g(x, y, k) = 0 \) has one and only one root \( k_m = k_m(x, y) \) on each segment \((m - 1)\pi, m\pi)/\gamma \) for large \( m \). These roots have the following asymptotic formula
\[ \gamma(x, y)k_m(x, y) = -\frac{\pi}{2} + m\pi + O \left( \frac{1}{m} \right) \text{ as } m \to \infty. \]
Moreover, \( \gamma = \gamma(x, y) \) is defined by the formula
\[ \gamma(x, y) = \lim_{m \to \infty} \frac{\pi}{k_{m+1}(x, y) - k_m(x, y)}. \]
Thus, we can uniquely find $\gamma(x, y)$ and then determine

$$\tau(x, y) = \gamma(x, y) + |x - y|. \quad (3.6)$$

Hence the function $\tau(x, y)$ becomes known for all $(x, y) \in S \times S$. Thus we reach an inverse kinematic problem. Solving the problem, we recover $n(x)$ inside $\Omega$.

The inverse kinematic problem is nonlinear one. For the first time this problem was solved by Herglotz and Wiehert in 1905 under the assumption that $n = n(r)$, $r = |x|$. The multidimensional case was considered in a linear approximations in $[13, 21, 22]$. The nonlinear problem in multidimensions was studied under the assumption in papers $[1, 15, 16, 24]$, where the uniqueness and the stability theorems were shown. In view of Romanov [24] (Theorem 3.4 of Chapter 3, p.94), we can establish the stability as well as the uniqueness in determining $n(x)$ by $f(x, y, k) := |u_{sc}(x, y, k)|^2$ for $(x, y) \in S \times S$ with $x \neq y$ and $k \geq k_0$.

**Theorem 2.** Let $\Upsilon(d)$ be the set of functions $n(x)$ such that

1) the conditions (1.2) hold,
2) there exists a number $d > 1$ such that $1 \leq \|n\|_{C^2(\Omega)} \leq d$,
3) Assumption holds.

Then for any $n_1, n_2 \in \Upsilon(d)$ and the Riemannian distances $\tau_1(x, y)$ and $\tau_2(x, y)$ generated by them respectively, there exists a positive constant $C = C(d)$ such that

$$\|n_1 - n_2\|_{L^2(\Omega)} \leq C\|\tau_1 - \tau_2\|_{H^2(S \times S)}.$$ 

Here $\tau_j$, $j = 1, 2$, are determined by $|u_{sc}^j(x, y, k)|^2$ for $(x, y) \in S \times S$ with $x \neq y$ and $k \geq k_0$ via the procedure (3.2) - (3.6) given above successively.

Theorem 2 implies that if $|u_{sc}^1(x, y, k)|^2 = |u_{sc}^2(x, y, k)|^2$ for $(x, y) \in S \times S$ with $x \neq y$ and $k \geq k_0$, then $n_1 = n_2$ in $\mathbb{R}^3$.

4. Recovering the potential. Consider now the problem of recovering a potential $q(x)$. In this case the functions $A(x, y)$ and $\gamma(x, y)$ are given for all $x, y \in S$. First of all, we prove that the function $f(x, y, k)$ allows to find $\alpha_0(x, y)$ for all $x, y \in S$.

**Lemma 2.** Let the conditions of Theorem 2 and (1.3) hold. Then the data of the inverse problem uniquely determine the integrals of the function $q(x)$ along the geodesics $\Gamma(x, y)$ for all $x, y \in S$.

**Proof.** For fixed $x \in S$ and $y \in S$, $x \neq y$, consider two cases:

1) $\gamma(x, y) \neq 0$,
2) $\gamma(x, y) = 0$.

In the first case where $\gamma(x, y) \neq 0$, introduce the function

$$h(x, y, k) = \frac{k}{2A_0(x, y)}\left[f(x, y, k) - A^2(x, y) - A_0^2(x, y)ight] + 2A(x, y)A_0(x, y)\cos(k\gamma(x, y))]. \quad (4.1)$$

Then equation (3.1) takes the form

$$h(x, y, k) = \alpha_0(x, y)\sin(k\gamma(x, y)) + O\left(\frac{1}{k}\right) \text{ as } k \to \infty, \quad (4.2)$$

where $h(x, y, k)$ is a given function for all $k \geq k_0$. Putting here $k = k_m' =: (\pi/2 + 2m\pi)/\gamma$, where $m \in \mathbb{N}$. Then we have

$$h(x, y, k_m') = \alpha_0(x, y) + O\left(\frac{1}{m}\right) \text{ as } m \to \infty. \quad (4.3)$$
Hence,  
\begin{equation}
\alpha_0(x,y) = \lim_{m \to \infty} h(x,y,k_m).
\end{equation}

Now use formulae (2.9) with \(m = 0\). Then we obtain  
\begin{equation}
\alpha_0(x,y) = \frac{\alpha^{-1}(x,y)}{4\tau(x,y)} \int_{\Gamma(x,y)} \left[ \text{div}(n^{-2} \nabla \alpha^{-1}(\xi,y)) + q(\xi) \right] d\tau.
\end{equation}

Hence we can find  
\begin{equation}
\int_{\Gamma(x,y)} q(\xi) d\tau = r_1(x,y), \quad x, y \in S,
\end{equation}

where
\begin{equation}
r_1(x,y) = \frac{4\tau(x,y)\alpha_0(x,y)}{\alpha^{-1}(x,y)} - \int_{\Gamma(x,y)} \frac{\text{div}(n^{-2}(\xi) \nabla \alpha^{-1}(\xi,y))}{\alpha^{-1}(\xi,y)} d\tau.
\end{equation}

is a given function because \(\alpha^{-1}(x,y)\) is defined by formula (2.8) with already determined \(n(x)\). The geodesic lines \(\Gamma(x,y)\) are also known, and are defined by the Riemannian metric.

In the second case where \(\gamma(x,y) = 0\) for \(x, y \in S\), we needs more detailed consideration. First we note that the equality \(\gamma(x,y) = 0\) implies that \(\tau(x,y) = |x - y|\). It means that the geodesic line \(\Gamma(x,y)\) does not intersect the set where \(n(x) > 1\), and therefore \(\Gamma(x,y)\) is the straight line \(L(x,y)\) connecting the points \(x\) and \(y\). Consider a small \(\varepsilon\)-neighborhood of \(x\) on \(S\): \(S(x, \varepsilon) = \{x' \in S; |x' - x| < \varepsilon, x' \neq x\}\). Let \(\varepsilon_m \to 0\). Then there are two possibilities:

1) for any \(\varepsilon_m > 0\) there exists \(x' \in S(x, \varepsilon_m)\) such that \(\gamma(x', y) \neq 0\),

2) one can find \(\varepsilon_m > 0\) such that \(\gamma(x', y) = 0\) for all \(x' \in S(x, \varepsilon_m)\).

In the first case we define the relation (4.6) and the function \(r_1(x,y)\) by the continuity \(r_1(x,y) = \lim_{x' \to x} r_1(x', y)\). In the second case we conclude that \(\Gamma(x', y)\) coincides with the straight line \(L(x', y)\) for all \(x' \in S(x, \varepsilon_m)\) and \(K(x,y) := \bigcup_{x' \in S(x,\varepsilon)} L(x', y)\) has no intersection with \(\text{supp} (n - 1)\). Therefore \(n(x) \equiv 1\) in \(K(x,y)\). Moreover, for all \(x' \in K(x,y)\) the following equalities hold:

\[\tau(x', y) = |x' - y|, \quad \zeta(x', y) = x' - y, \quad A(x,y) = A_0(x,y), \quad \alpha_{-1}(x', y) = 1/(2\pi).\]

From these equalities it follows that in the latter case we have the formula

\[\alpha_0(x,y) = \frac{1}{8|x - y|} \int_{L(x,y)} q(\xi) d\tau\]

instead of formula (4.5). Since \(q(x) \geq 0\) by the condition (1.3), we conclude that \(\alpha_0(x,y) \geq 0\).

As a result of the these considerations, we conclude that in the case where \(\tau(x,y) = |x - y|\), the formula (2.5) takes the form

\begin{equation}
u_{xc}(x,y,k) = -e^{ik\tau(x,y)} \frac{\alpha_0(x,y)}{ik} + O \left( \frac{1}{k^2} \right) \text{ as } k \to \infty.
\end{equation}

Hence,

\begin{equation}f(x,y,k) = \frac{\alpha_0^2(x,y)}{k^2} + O \left( \frac{1}{k^3} \right) \text{ as } k \to \infty,
\end{equation}

\[Hence,\]

\begin{equation}a_0(x,y) = \lim_{m \to \infty} h(x,y,k_m).\end{equation}
and so

\begin{equation}
\alpha_0(x, y) = \lim_{k \to \infty} k \sqrt{f(x, y, k)}.
\end{equation}

Then

\[
\int_{L(x, y)} q(\xi) d\tau_\xi = r_2(x, y),
\]

where

\begin{equation}
r_2(x, y) = 8\pi|x - y|/\alpha_0(x, y)
\end{equation}
is a given function.

Finally we obtain

\[
\int_{\Gamma(x, y)} q(\xi) d\tau_\xi = r(x, y), \quad \forall x, y \in S,
\]

where

\begin{equation}
r(x, y) = \begin{cases}
r_1(x, y), & \text{if } \gamma(x, y) \neq 0, \\
r_2(x, y), & \text{if } \gamma(x, y) = 0.
\end{cases}
\end{equation}

Hence, using the given data of the inverse problem, we can find the integrals of \(q(x)\) along the geodesics \(\Gamma(x, y)\) for all \(x, y \in S\).

Henceforth by \(\alpha_{-1}^j, \alpha_0^j, r_1^j, r_2^j, j = 1, 2\), we denote \(\alpha_{-1}, \alpha_0, r_1, r_2\) corresponding to \(n_j, q_j, j = 1, 2\). Moreover let \(r^j, j = 1, 2\), be defined by (4.13) with \(r_1^j\) and \(r_2^j\).

Now we reach the integral geometry problem: given \(r(x, y)\), find \(q(x)\) in \(\Omega\). As for this inverse problem, see for example, [13, 14, 21, 22, 24, 25]. It was intensively studied in the 1970’s, and in particular in [1, 15, 23], under Assumption the uniqueness and the stability theorems were established. For our inverse problem of determining potentials, the following stability estimate follows.

**Theorem 3.** Let conditions of Theorem 2 and (1.3) be fulfilled. Then Theorem 2 asserts that \(n_j\) and \(\tau_j\) are determined uniquely by \(|u_{sc}^j(x, y, k)|^2\) for \(x, y \in S\) with \(x \neq y\) and \(k \geq k_0\) and so in terms of \(|u_{sc}(x, y, k)|^2\), we uniquely determine \(\alpha_{-1}^j\) and \(\alpha_0^j\) respectively by (2.8) and (4.10), so that \(r_1^j\) and \(r_2^j\), accordingly \(r^j, j = 1, 2\) are determined respectively by (4.7) and (4.11), (4.13). Moreover there exists a positive constant \(C = C(d)\) such that

\[
\|q_1 - q_2\|_{L^2(\Omega)} \leq C\|r^1 - r^2\|_{H^1(S \times S)}.
\]

**Remark 2.** Instead of the \(C^\infty(\mathbb{R}^3)\)-regularity of unknown \(n(x)\) and \(q(x)\), one can use only some finite-order smoothness of these coefficients. For example, the \(C^{15}(\mathbb{R}^3)\)-regularity which is used in [10]-[12], is quite enough for our study. For simplicity, we omit further discussions.

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REFERENCES

[1] J. N. Bernstein and M. L. Gerver, On the problem of integral geometry for a family of geodesics and the inverse kinematics seismic problem, Dokl. Akad. Nauk SSSR, 243 (1978), 302–305 (in Russian).

[2] K. Chadan and P. S. Sabatier, Inverse Problems in Quantum Scattering Theory, Texts and Monographs in Physics, Springer-Verlag, New York - Berlin, 1977.

[3] M. V. Klibanov, Phaseless inverse scattering problems in three dimensions, SIAM J. Appl. Math., 74 (2014), 392–410.

[4] M. V. Klibanov, On the first solution of a long standing problem: Uniqueness of the phaseless quantum inverse scattering problem in 3-d, Applied Mathematics Letters, 37 (2014), 82–85.

[5] M. V. Klibanov, Uniqueness of two phaseless non-overdetermined inverse acoustics problems in 3-d, Applicable Analysis, 93 (2014), 1135–1149.

[6] M. V. Klibanov, A phaseless inverse scattering problem for the 3-D Helmholtz equation, Inverse Problems and Imaging, 11 (2017), 263–276.

[7] M. V. Klibanov and V. G. Romanov, The first solution of a long standing problem: Reconstruction formula for a 3-d phaseless inverse scattering problem for the Schrödinger equation, J. Inverse and Ill-Posed Problems, 23 (2015), 415–428.

[8] M. V. Klibanov and V. G. Romanov, Explicit solution of 3-D phaseless inverse scattering problems for the Schrödinger equation: The plane wave case, Eurasian J. of Math. and Comp. Appl., 3 (2015), 48–63.

[9] M. V. Klibanov and V. G. Romanov, Explicit formula for the solution of the phaseless inverse scattering problem of imaging of nano structures, J. of Inverse and Ill-Posed Problems, 23 (2015), 187–193.

[10] M. V. Klibanov and V. G. Romanov, Two reconstruction procedures for a 3-d phaseless inverse scattering problem for the generalized Helmholtz equation, Inverse Problems, 32 (2016), 015005 (16pp).

[11] M. V. Klibanov and V. G. Romanov, Reconstruction procedures for two inverse scattering problem without the phase information, SIAM J. Appl. Math., 76 (2016), 178–196.

[12] M. V. Klibanov and V. G. Romanov, Uniqueness of a 3-D coefficient inverse scattering problem without the phase information, Inverse Problems, 33 (2017), 095007 (10 pp), https://doi.org/10.1088/1361-6420/aa7a18.

[13] M. M. Lavrent’ev and V. G. Romanov, On three linearized inverse problems for hyperbolic equations, Soviet Math. Dokl., 7 (1966), 1650–1652.

[14] M. M. Lavrent’ev, V. G. Romanov and S. P. Shishat-skii, Ill-Posed Problems of Mathematical Physics and Analysis, Transl. of Math. Monographs, Vol. 64, AMS, Providence, Rhode Island, 1986.

[15] R. G. Mukhometov, The reconstruction problem of a two-dimensional Riemannian metric and integral geometry, Dokl. Akad. Nauk SSSR, 232 (1977), 32–35.

[16] R. G. Mukhometov and V. G. Romanov, On the problem of determining an isotropic Riemannian metric in n-dimensional space, Dokl. Akad. Nauk SSSR, 243 (1978), 41–44.

[17] R. G. Newton, Inverse Schrödinger Scattering in Three Dimensions, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1989.

[18] R. G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, Bulletin des Sciences Mathématiques, 139 (2015), 923–936.

[19] R. G. Novikov, Phaseless inverse scattering in the one-dimensional case, Eurasian J. of Math. and Comp. Appl., 3 (2015), 64–70.

[20] R. G. Novikov, Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions, J. Geometrical Analysis, 26 (2016), 346–359.

[21] V. G. Romanov, Reconstructing a function by means of integrals along a family of curves, Siberian Math. J., 8 (1967), 923–925.

[22] V. G. Romanov, Integral Geometry and Inverse Problems for Hyperbolic Equations, Springer-Verlag, Springer Tracts in Natural Philosophy, Vol. 26, Berlin, 1974.

[23] V. G. Romanov, Integral geometry on geodesics of the isotropic Riemannian metric, Soviet Math. Dokl., 19 (1978), 847–851.

[24] V. G. Romanov, Inverse Problems of Mathematical Physics, VNU Science Press, Utrecht, 1987.

[25] V. G. Romanov, Investigation Methods for Inverse Problems, VSP, Utrecht, 2002.
[26] V. G. Romanov, The problem of recovering the permittivity coefficient from the modulus of the scattered electromagnetic field, *Siberian Math. J.*, 58 (2017), 711–717.

[27] V. G. Romanov, Problem of determining the permittivity in the stationary system of Maxwell equations, *Doklady Math.*, 95 (2017), 230–234.

[28] B. R. Vainberg *Asymptotic Methods in Equations of Mathematical Physics*, Gordon and Breach Science Publishers, New York, 1989.

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