ON BAND MODULES AND $\tau$-TILTING FINITENESS

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ABSTRACT. In this paper we study general properties of band modules and their endomorphisms in the module category of a finite dimensional algebra. As an application we describe properties of torsion classes containing band modules. Furthermore, we show that a special biserial algebra is $\tau$-tilting finite if and only if no band module is a brick.

1. INTRODUCTION

An important breakthrough in the representation theory of finite-dimensional algebras was the systematic introduction of quivers, a powerful tool bringing linear algebra into the theory. In particular, every finite-dimensional algebra over an algebraically closed field $k$ is Morita equivalent to a quotient $kQ/I$ of a path algebra $kQ$ of a quiver $Q$ quotient by an admissible ideal $I$ [11]. Presentations of algebras in terms of quivers with relations encode much homological and geometric information. As a consequence, quiver representations now play an important role in many areas of mathematics such as, for example, algebraic geometry, mathematical physics, and mirror symmetry.

Since the introduction of quivers to representation theory, much work has been done to show that many families of algebras defined by homological properties can, in fact, be characterised by properties of their quivers and relations.

One important such family of algebras is that of special biserial algebras. This class contains many well-known families of algebras, such as gentle algebras which play a central role in cluster theory and homological mirror symmetry of surfaces [8, 12] and Brauer graph algebras which originate in the modular representation theory of finite groups [9]. For these algebras much of the representation theory is encoded in the combinatorics of their quivers and relations. For example, the isomorphism classes of indecomposable finitely generated modules are given by certain words in the alphabet consisting of the arrows and formal inverses of the arrows of their quivers [7, 20]. This naturally divides the indecomposable modules over these algebras into two classes, the so-called string modules and infinite families of band modules. The morphisms between string and band modules can also be described in terms of word combinatorics in the quiver [8, 13].

One of the motivating observations of this paper is that for any finite-dimensional algebra given by quiver and relations, that is algebras which are not necessarily special biserial, we can still consider the combinatorics of string and band modules. We show that in the general case, these modules still encode significant information on the modules categories of the algebras. Even though, in this case there are (possibly infinitely many) indecomposable modules that cannot be described in terms of string and band modules.

More recently, the theory of cluster algebras has given new impetus to representation theory with the introduction of many new cluster-inspired representation theoretic concepts. An excellent example of this is the introduction of $\tau$-tilting theory, inspired by mutation in cluster algebras [2]. Since its introduction in 2014, $\tau$-tilting theory has been intensively studied and led to the
introduction of many new concepts promising to relate representation theory with other areas of mathematics such as Hall algebras, Donaldson-Thomas invariants and Riemannian geometry.

From a representation theoretic point of view, one of the important results in [2] are explicit bijections between functorially finite torsion classes, support \( \tau \)-tilting modules and 2-term silting complexes in the bounded derived category \( D^b(A) \) of a finite dimensional algebra \( A \). Accordingly an algebra is called \( \tau \)-tilting finite if has finitely many support \( \tau \)-tilting modules. Furthermore, for a \( \tau \)-tilting finite algebra, there are only finitely many torsion classes and all are functorially finite [10]. A further characterisation of \( \tau \)-tilting finite algebras is via bricks in their module category. An object in the module category of an algebra is called a brick if its endomorphism algebra is a division ring. By [10], an algebra \( A \) is \( \tau \)-tilting finite if and only if there are finitely many bricks in its module category. This immediately implies that if the module category of an algebra contains a band modules which is a brick then the algebra is \( \tau \)-tilting infinite (see Proposition 5.1).

The representation theory of a \( \tau \)-tilting finite algebra is considerably easier to understand than that of a \( \tau \)-tilting infinite algebra. For example, the support of the scattering diagram of a \( \tau \)-tilting finite algebra is completely determined by its support \( \tau \)-tilting modules [5, 6] and the stability manifold of a finite dimensional algebra \( A \) is contractible if the algebra is silting-discrete which implies, in particular, that the heart of any bounded \( \tau \)-structure of the derived category of \( A \) is a module category over a \( \tau \)-tilting finite algebra [16].

Therefore, a classification of \( \tau \)-tilting finite algebras is almost as important in today’s representation theory as the determination of representation finite algebras was in the last century.

In this paper we will exploit the interplay between the combinatorics of band modules and bricks.

As an application to the case of special biserial algebras we give necessary and sufficient conditions for the \( \tau \)-tilting finiteness of these algebras.

More precisely, we show the following.

**Theorem 1.1** (Theorem 5.5). Let \( A = kQ/I \) be a finite dimensional algebra such that the module category of \( A \) contains a band module. Then there exists an infinite family of band modules \( M \) such that every non-trivial endomorphism of \( M \) has a semisimple image and factors through a map from the top to the socle of \( M \).

As a consequence of Theorem 1.1, we show the following results on torsion classes based on the band modules they contain. We note that in the first part of Theorem 1.2, the result holds for any finite dimensional algebra, whereas in the second part we only consider the class of special biserial algebras.

**Theorem 1.2** (Theorem 4.1 and Theorem 4.3). Let \( A \) be an algebra and \( b \) be a band in \( A \).

1. Suppose that the band module \( M(b, \lambda, 1) \) is not a brick for some \( \lambda \in k^* \). If \( M(b, \lambda, 1) \) is in some torsion class \( T \), then \( M(b, \lambda', n) \in T \), for all \( \lambda' \in k^* \) and all \( n \in \mathbb{N} \).
2. Suppose that the band module \( M(b, \lambda, 1) \) is a brick and that \( A \) is special biserial. Then there exists an infinite family of distinct torsion classes \( T_\mu, \mu \in k^* \), such that \( M(b, \lambda, 1) \in T_\mu \) if and only if \( \lambda = \mu \).

As a further application of Theorem 5.5 we obtain the following characterisation of \( \tau \)-tilting finite special biserial algebras.

**Theorem 1.3** (Theorem 5.2). Let \( A = kQ/I \) be a special biserial algebra. Then \( A \) is \( \tau \)-tilting finite if and only if no band module of \( A \) is a brick.

A different classification of \( \tau \)-tilting finite special biserial algebras has recently been obtained in [15] based on the classification of minimal representation infinite algebras in [18]. We also note that gentle algebras are a subclass of special biserial algebras. For these algebras a similar criterion to the above was given in [17].

We finish the paper in Section 6 with an application of our criterion to Brauer graph algebras, obtaining a new proof of the fact that a Brauer graph algebra with Brauer graph \( G \) is \( \tau \)-tilting finite.
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if and only if \( G \) contains at most one cycle and if that cycle is of odd length. This has originally been shown in [11, Theorem 6.7].

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2. Background

In this section we fix some of the notation and definitions which will be used throughout this paper.

We fix an algebraically closed field \( k \), and \( A \) a basic finite dimensional \( k \)-algebra, which is Morita equivalent to \( kQ/I \) for some finite quiver \( Q \) and admissible ideal \( I \) in \( kQ \) (see [11]). Furthermore, unless otherwise stated, an algebra given by quiver and relations \( kQ/I \) is assumed to be finite dimensional and the ideal \( I \) is assumed to be admissible. Note that we use the same notation for elements in \( kQ \) and elements in \( kQ/I \) with the implicit understand that the latter are representatives in their equivalence class.

Given an \( A \)-module \( M \), we call the top of \( M \), denoted \( \text{top}M \), the largest semisimple quotient of \( M \). Similarly, we call the socle of \( M \), denoted \( \text{soc}M \), the largest semisimple submodule of \( M \). Given \( f \in \text{End}_A(M) \), we say that \( f \) is non-trivial if \( f \) is non-zero and is not a scalar multiple of the identity.

For a quiver \( Q \), let \( Q_0 \) be the set of vertices of \( Q \) and \( Q_1 \) the set of arrows. If \( \alpha \) is an arrow of \( Q \), we denote by \( s(\alpha) \in Q_0 \) and by \( t(\alpha) \in Q_0 \) the source and target point of \( \alpha \), respectively.

For every arrow \( \alpha : i \to j \), we define \( \bar{\alpha} : j \to i \) to be its formal inverse. Let \( \overline{Q_1} \) be the set of formal inverses of the elements of \( Q_1 \). We refer to the elements of \( Q_1 \) as direct arrows and to the elements of \( \overline{Q_1} \) as inverse arrows. A walk is a sequence \( \alpha_1 \ldots \alpha_n \) of elements of \( \overline{Q_1} \cup Q_1 \) such that \( t(\alpha_i) = s(\alpha_{i+1}) \) for every \( i = 1, \ldots, n-1 \) and such that \( \alpha_{i+1} \neq \alpha_i^{-1} \).

We recall that a string in \( A \) is by definition a walk \( w \) in \( Q \) avoiding the zero relations and such that neither \( w \) nor \( w^{-1} \) is a summand in a relation. A band \( b \) is defined to be a cyclic string such that every power \( b^n \) is a string, but \( b \) itself is not a proper power of some string \( c \). A string \( w = \alpha_1 \ldots \alpha_n \) is a direct (inverse) if \( \alpha_i \) is a direct arrow (resp. inverse arrow) for every \( i = 1, \ldots, n \).

Given a string \( w \), the string module \( M(w) \) is obtained by replacing each vertex in \( w \) by a copy of the field \( k \) and every arrow in \( w \) by the identity map. In a similar way, given a band \( b = \alpha_1 \ldots \alpha_t \), a non-zero element \( \lambda \) in \( k^* \) and \( n \in \mathbb{N} \), the band module \( M(b, \lambda, n) \) is obtained from the band \( b \) by replacing each vertex by a copy of the \( k \)-vector space \( k^n \) and every arrow \( \alpha_i \) for \( 1 \leq i < t \) by the identity matrix of dimension \( n \), and \( \alpha_t \) by a Jordan block of dimension \( n \) and eigenvalue \( \lambda \) (we refer [7] for the precise definition).

Remark 2.1. If \( A \) is not special biserial and \( b \) is a band then the induced band module \( M(b, \lambda, n) \) might not lie in a homogenous tube of rank 1. For example, this is the case for any band in the 3-Kronecker algebra.

Let \( w = \alpha_1 \ldots \alpha_i \ldots \alpha_j \ldots \alpha_t \) be a (finite or infinite) string and let \( u = \alpha_i \ldots \alpha_j \). Then we say that \( u \) is a submodule substring if \( \alpha_{i-1} \) is direct and \( \alpha_{j+1} \) is inverse. We say that \( u \) is a factor substring if \( \alpha_{i-1} \) is inverse and \( \alpha_{j+1} \) is direct.

Given two strings \( v, w \) such that they have a common substring \( u \) which is a submodule string of \( w \) and a factor string of \( v \), by [8] there is a map from \( M(v) \) to \( M(w) \) and the maps of this form give a basis of \( \text{Hom}_A(M(v), M(w)) \).

Maps between bands are slightly different from maps between strings [13], see also, for example, [14]. For completeness we recall the construction of a basis morphism between bands. We first define \( \approx \) to be the string formed by infinitely many composition of a band \( b \) with itself.

Let \( b \) and \( c \) be two bands, \( \lambda, \mu \in k^* \) and \( n, m \) two positive integers. If \( b \) is different from \( c \) or \( \lambda \) is different from \( \mu \), a basis of \( \text{Hom}_A(M(b, \lambda, n), M(c, \mu, m)) \) is given by maps \( \phi_{(w, g)} \) induced by pairs
(w, g) (detailed in an example below), where w is a string of finite length which is a factor substring of $\mathcal{b}^\infty$ and a submodule substring of $\mathcal{c}^\infty$ and $g \in \text{Hom}_k(k^n, k^m)$.

If $b = c$ and $\lambda = \mu$, a basis of $\text{Hom}_A(M(b, \lambda, n), M(b, \lambda, m))$ is given maps induced by the pairs (w, g) as above and maps $f_h$ induced by $k$-linear maps $h \in \text{Hom}_k(k^n, k^m)$ such that for every vertex $i$ in $Q$, $(f_h)_i : (M(b, \lambda, n))_i \rightarrow (M(b, \lambda, n))_i$ is equal to $h$.

**Example 2.2.** Let $A$ be the algebra given by the quiver

\[
\begin{array}{cccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow \\
\alpha & \rightarrow & \beta & \rightarrow & \gamma & \rightarrow & \delta & \rightarrow \\
\end{array}
\]

and the relations $\alpha \delta = \beta \gamma = 0$. Observe that $b = \gamma \delta$ and $c = \alpha \gamma \delta \gamma \delta \beta$ are bands in $A$ and that $\gamma \delta \gamma \delta$ is a submodule string of $\mathcal{b}^\infty$ and a factor substring of $\mathcal{c}^\infty$. Let $M(b, \lambda, n)$ and $M(c, \mu, m)$ be two band modules associated to $b$ and $c$ respectively, with $\lambda, \mu \in k^*$ and $n, m \in \mathbb{N}$. Then, for any morphism $g \in \text{Hom}_k(k^n, k^m)$, the pair $(\gamma \delta \gamma \delta, g)$ gives rise to a basis element $\phi(\gamma \delta \gamma \delta, g) \in \text{Hom}_A(M(b, \lambda, n), M(c, \mu, m))$ induced by the diagram in Figure 1 with the following notation: $U = k^n$ and $V = k^m$ and $\Psi$ is the $(n \times n)$-Jordan block of eigenvalue $\lambda$ and $\Phi$ is the $(m \times m)$-Jordan block of eigenvalue $\mu$.

![Figure 1. Diagram of $(\gamma \delta \gamma \delta, g)$](image)

Then the morphism $\phi((\alpha \beta, g)) : M(b, \lambda, n) \rightarrow M(c, \mu, m)$ is as follows.

\[
\begin{array}{cccc}
0 & \rightarrow & U & \rightarrow & U \\
A & \rightarrow & B & \rightarrow & C \\
V & \rightarrow & V^3 & \rightarrow & V^2 \\
\end{array}
\]

where $A = [0]$, $B = \begin{bmatrix} g & g \Psi^{-1} \\ g \Psi^{-2} & g \Psi^{-2} \end{bmatrix}$, and $C = \begin{bmatrix} g \Psi^{-1} \\ g \Psi^{-2} \end{bmatrix}$.

**Remark 2.3.** Let $b$ be a band in $A$. Then we have that every non-zero nilpotent endomorphism of the band module $M(b, \lambda, 1)$ is a linear combination of maps that are determined by pairs $(w, 1)$, where $w$ is a string of finite length which is at the same time a factor substring and a submodule substring of $\mathcal{b}^\infty$. 
3. Band modules and their endomorphisms

In this section, we study some general results about bands and band modules in the module category of an algebra $A = kQ/I$. In particular, we focus on the endomorphism algebras of band modules.

**Proposition 3.1.** Let $A$ be a finite dimensional algebra. Suppose that $w$ is a string that has a repeated direct (resp. inverse) letter $\alpha$ such that there is an inverse (resp. direct) letter $\overline{b}$ in between the two copies of $\alpha$. Then there is a subword $w'$ of $w$ which is a band.

**Proof.** By hypothesis and without loss of generality we can write $w = w_0\omega w_1\overline{b}w_2\alpha w_3$, such that $\alpha$ does not appear in either $w_1$ nor $w_2$. Set $w' = \omega w_1\overline{b}w_2$. Clearly $s(\alpha) = t(w_2)$. Since $w'$ is not a directed string and $w_2\alpha$ is a substring of $w$, any power $(w')^n$ is a string. Moreover, $w'$ is not a proper power of any string $c$, because $\alpha$ appears exactly once in $w'$. This finishes the proof. \[\Box\]

**Proposition 3.2.** Let $A$ be an algebra and $M(b, \lambda, n)$ a band module with $\lambda \in k^*$ and $n \in \mathbb{Z}_{>0}$. If $M(b, \lambda, n)$ is a brick, then $n = 1$.

**Proof.** Let $M(b, \lambda, n)$ be a band module such that $n \geq 2$ and $\lambda \in k^*$. Then there is a morphism $f_\lambda : M(b, \lambda, n) \to M(b, \lambda, n)$ where the linear map $(f_\lambda)_i : (M(b, \lambda, n))_i \to (M(b, \lambda, n))_i$, for each vertex $i$ of $Q$ corresponds to the $n \times n$ Jordan block of eigenvalue $\lambda$. Furthermore, $f_\lambda$ is an isomorphism of $M(b, \lambda, n)$ which is not a scalar multiple of the identity map of $M(b, \lambda, n)$. Hence $M(b, \lambda, n)$ is not a brick, as claimed. \[\Box\]

**Proposition 3.3.** Let $w$ be a string in $A$ containing a substring $\alpha_1 \ldots \alpha_k \alpha_1$ such that $b = \alpha_1 \ldots \alpha_k$ is a band. Then $\text{top}(M(b, \lambda, 1))$ is a direct summand of $\text{top}(M(w))$, and $\text{soc}(M(b, \lambda, 1))$ is a direct summand of $\text{soc}(M(w))$, for every $\lambda \in k^*$.

**Proof.** Up to cyclic permutation $b$ can be written as $b = w_1w_2 \ldots w_{2t-1}w_{2t}$, where $w_{2i-1}$ is a direct string and $w_{2i}$ is an inverse string for all $1 \leq i \leq t$, for some positive integer $t$. Then $\text{soc}(M(b, \lambda, 1))$ is the direct sum of the simple modules $S(t(w_{2i-1})) = S(s(w_{2i}))$ for all $1 \leq i \leq t$. Without loss of generality, $f_\lambda$ is an isomorphism of $M(b, \lambda, 1)$ which is not a scalar multiple of the identity map of $M(b, \lambda, 1)$. Hence $M(b, \lambda, 1)$ is not a brick, as claimed. \[\Box\]

**Proposition 3.4.** Let $b$ be a band such that any repeated letter in $b$ appears in the same direct or inverse substring of $b$. Then there exists a band $c$ with no repeated letters such that $\text{soc}(M(b, \lambda, 1))$ is isomorphic to $\text{soc}(M(c, \lambda, 1))$ and $\text{top}(M(b, \lambda, 1))$ is isomorphic to $\text{top}(M(c, \lambda, 1))$, for every $\lambda \in k^*$.

**Proof.** Suppose $b = w_1w_2 \ldots w_{t-1}w_t$ where without loss of generality we can assume that $w_i$ is a direct string when $i$ is odd and $w_i$ is inverse when $i$ is even.

Assume that $\alpha$ appears twice in some direct string $w_i$. Then the path $l$ going from $\alpha$ to itself is a cycle in $Q$. Now we can construct a band $b_l$ from $b$ by deleting the cycle $l$. Moreover, we have $\text{soc}(M(b, \lambda, 1)) \cong \text{soc}(M(b_l, \lambda, 1))$ and $\text{top}(M(b, \lambda, 1)) \cong \text{top}(M(b_l, \lambda, 1))$.

We repeat this process until there are no more repeated letters, using similar arguments if there are repeated letters in an inverse string $w_j$. In this way we construct a band $c$ as in the statement. \[\Box\]

**Theorem 3.5.** Let $A = kQ/I$ be an algebra and suppose that there exists a band in $A$. Then there is a band $b$ in $A$ such that, for all $\lambda \in k^*$, every non-trivial nilpotent endomorphism $f \in \text{End}_A(M(b, \lambda, 1))$ induces a map

$$\overline{f} : \text{top}(M(b, \lambda, 1)) \to \text{soc}(M(b, \lambda, 1)).$$

In particular, the image of every non-trivial nilpotent endomorphism of $M(b, \lambda, 1)$ is semisimple.
The proof of Theorem 3.3 directly follows from the next Lemma.

**Lemma 3.6.** Let \( b \) be a band in \( A \) with no repeated letters. Then for all \( \lambda \in k^\ast \), every non-trivial nilpotent endomorphism \( f \in \text{End}_A(M(b, \lambda, 1)) \) induces a map

\[
\overline{f} : \text{top}(M(b, \lambda, 1)) \to \text{soc}(M(b, \lambda, 1)).
\]

In particular, the image of every non-trivial nilpotent endomorphism \( f \) is semisimple.

**Proof.** Let \( f \in \text{End}_A(M(b, \lambda, 1)) \) be a non zero endomorphism. By Remark 2.3 \( f \) is given by a linear combination of morphisms, which are given by submodule strings of \( b^\infty \) which are at the same time factor strings of \( \infty b^\infty \). Since \( b \) has no repeated letters, every summand of \( f \) is induced by a simple module which is a direct summand of both the top and socle of \( M(b, \lambda, 1) \). \( \square \)

**Proof of Theorem 3.3** By Propositions 3.1 and 3.3 we can assume without loss of generality that there exists a band in \( A \) with no repeated letters. The result then follows from Lemma 3.6. \( \square \)

**Corollary 3.7.** Let \( b \) be a band in \( A \) and suppose that \( M(b, \lambda, 1) \) is not a brick. Then there exists a simple module \( S \) which is at the same time a direct summand of \( \text{top}(M(b, \lambda, 1)) \) and \( \text{soc}(M(b, \lambda, 1)) \).

**Proof.** Given that \( M(b, \lambda, 1) \) is not a brick, by Lemma 3.6 there exists a non-trivial endomorphism \( f \in \text{End}_A(M(b, \lambda, 1)) \), whose image is semisimple. Hence, \( \text{Im} f \) is a direct summand of both \( \text{top} M(b, \lambda, 1) \) and \( \text{soc} M(b, \lambda, 1) \), which are semisimple. \( \square \)

## 4. Bands and Torsion Classes

In this section we study torsion classes containing band modules. We show that if a torsion class contains a band module which is not a brick then the torsion class contains all band modules in the same infinite family. In the case of special biserial algebras, we show that if a band module \( M(b, \lambda, 1) \) is a brick then, for any \( \mu \in k^\ast \) with \( \mu \neq \lambda \), the minimal torsion class containing \( M(b, \lambda, 1) \) is distinct from the minimal torsion class containing \( M(b, \mu, 1) \).

**Theorem 4.1.** Let \( A \) be an algebra and \( \mathcal{T} \) be a torsion class in \( \text{mod} A \). Suppose that \( b \) is a band in \( A \) such that \( M(b, \lambda, 1) \in \mathcal{T} \), for some \( \lambda \in k^\ast \). If \( M(b, \lambda, 1) \) is not a brick then \( M(b, \lambda', n) \in \mathcal{T} \), for all \( \lambda' \in k^\ast \) and all \( n \in \mathbb{N} \).

**Proof.** Let \( M(b, \lambda, 1) \in \mathcal{T} \) be a band module which is not a brick. Corollary 3.7 implies the existence of a simple module \( S \) which is at the same time a direct summand of \( \text{top}(M(b, \lambda, 1)) \) and \( \text{soc}(M(b, \lambda, 1)) \). Then there is a short exact sequence

\[
0 \rightarrow S \rightarrow M(b, \lambda, 1) \rightarrow N \rightarrow 0
\]

where \( N \) is the string module \( M(b, \lambda, 1)/S \). Both \( S \) and \( N \) are quotients of \( M(b, \lambda, 1) \) and \( S \in \mathcal{T} \) and \( N \in \mathcal{T} \) because torsion classes are closed under quotients.

Since \( M(b, \lambda, 1) \), we have that \( M(b, \lambda', n) \) is not a brick for all \( \lambda' \in k^\ast \) and all \( n \in \mathbb{N} \). Moreover, we have that \( S \) and the string module \( N \) are quotients of \( M(b, \lambda', 1) \) because the top and the socle of a band module are independent of the parameter \( \lambda' \in k^\ast \). Hence, for every \( \lambda' \in k^\ast \) we have a short exact sequence

\[
0 \rightarrow S \rightarrow M(b, \lambda', 1) \rightarrow N \rightarrow 0
\]

and since \( \mathcal{T} \) is closed under extensions, \( M(b, \lambda', 1) \in \mathcal{T} \) for all \( \lambda' \in k^\ast \).

Now \( M(b, \lambda', n) \) is an extension of \( M(b, \lambda', 1) \) by \( M(b, \lambda', n - 1) \) for all \( n \in \mathbb{N} \) and \( M(b, \lambda', n) \in \mathcal{T} \) because \( \mathcal{T} \) is closed under extensions. \( \square \)

**Remark 4.2.** The result in Theorem 4.1 serves as an indication of why \( \tau \)-tilting finite algebras of wild representation type should exist.

**Theorem 4.3.** Let \( A \) be a special biserial algebra and let \( b \) be a band such that \( M(b, \lambda, 1) \) is a brick for some \( \lambda \in k^\ast \). Then there exists an infinite family of distinct torsion classes \( \mathcal{T}_\mu, \mu \in k^\ast \) such that \( M(b, \lambda, n) \in \mathcal{T}_\mu \) for any \( n \in \mathbb{N} \) if and only if \( \lambda = \mu \).
Proof. Recall that given a band $b$ such that $M(b, \lambda, 1)$ is a brick for some $\lambda \in k^*$, we have $\text{Hom}_A(M(b, \lambda, 1), M(b, \mu, 1)) = 0$ for all $\mu \neq \lambda$.

Define $T_\mu$ to be $\text{Filt}(\text{Fac}(M(b, \mu, 1)))$, the class of all modules in $\text{mod}A$ filtered by quotients of an element in $\text{add}(M(b, \mu, 1))$. We claim that $M(b, \lambda, 1)$ is not in $T_\mu$ if $\mu \neq \lambda$. Suppose to the contrary that $M(b, \lambda, 1) \in T_\mu$. Then, there is a submodule $0 \neq L$ of $M(b, \lambda, 1)$ which is a quotient of $M(b, \mu, 1)$. Hence there is a map from $M(b, \mu, 1) \rightarrow M(b, \lambda, 1)$ whose image is isomorphic to $L$, contradicting the fact that $\text{Hom}_A(M(b, \lambda, 1), M(b, \mu, 1)) = 0$. \hfill $\square$

5. $\tau$-tilting finiteness for special biserial algebras

In this section we apply the results of the Section 3 to construct a necessary and sufficient criterion to determine the $\tau$-tilting finiteness of special biserial algebras. We note a criterion for $\tau$-tilting finiteness of biserial algebras has been determined in [15] together with a description of the minimal $\tau$-tilting infinite special biserial algebras.

For gentle algebras, a subclass of special biserial algebras, $\tau$-tilting finiteness has been determined in [17] and has been shown to coincide with the gentle algebras of finite representation type.

We start by stating a direct consequence of [10, Theorem 1.4].

Proposition 5.1. Let $A = kQ/I$ be a finite dimensional algebra. If there exists a band module $M$ which is a brick, then $A$ is $\tau$-tilting infinite.

Proof. Since $k$ is algebraically closed and hence infinite, for any band $b$ there is an infinite family $\{M(b, \lambda, 1) : \lambda \in k^*, n \in \mathbb{N}\}$ of non-isomorphic indecomposable modules.

By hypothesis, there exists a band $b$, $\lambda \in k^*$, and $n \in \mathbb{N}$ such that $M(b, \lambda, 1)$ is a brick. Since $\text{End}_A(M(b, \lambda, 1)) \cong \text{End}_A(M(b, \lambda', 1))$, for all $\lambda, \lambda' \in k^*$, we have that $M(b, \lambda, 1)$ is a brick for all $\lambda \in k^*$. In particular, this implies that there is an infinite number of bricks in $\text{mod}A$ and by [10] $A$ is $\tau$-tilting infinite. \hfill $\square$

We will see now that for special biserial algebras the converse of the above also holds.

Theorem 5.2. Let $A = kQ/I$ be a special biserial algebra. Then $A$ is $\tau$-tilting finite if and only if no band module of $A$ is a brick.

Proof. Suppose that no band module in $\text{mod}A$ is a brick. Then we have that any brick in $\text{mod}A$ is a string. We claim that there are only finitely many string modules which are bricks.

Given that $A$ is finite dimensional, we have in particular that $Q$ has finitely many arrows. Since all objects in $\text{mod}A$ are finite dimensional as $k$-vector spaces, by the pigeon hole principle all but finitely many strings are as in the statement of Proposition 3.1. Thus all but finitely many strings $w$ have a substring $b$ which is a band. By Proposition 3.1 we can assume without loss of generality that every repeated letter in $b$ appears in the same direct or inverse string.

By Proposition 3.4 there exists a band $b'$ with no repeated letters such that $\text{top}(M(b', \lambda, 1))$ is isomorphic to $\text{top}(M(b', \lambda, 1))$ and $\text{soc}(M(b', \lambda, 1))$ is isomorphic to $\text{soc}(M(b', \lambda, 1))$. By hypothesis, $M(b', \lambda, 1)$ is not a brick and Lemma 3.6 implies that there exists a non-trivial nilpotent endomorphism $f'' : M(b', \lambda, 1) \rightarrow M(b', \lambda, 1)$ that factors through a morphism $\overline{f''} : \text{top}(M(b', \lambda, 1)) \rightarrow \text{soc}(M(b', \lambda, 1))$. By Proposition 3.4 and Proposition 3.3 we have that $\text{top}(M(b', \lambda, 1))$ is a direct summand of $\text{top}(M(w))$ and $\text{soc}(M(b', \lambda, 1))$ is a direct summand of $\text{soc}(M(w))$. Hence, there is a non-trivial nilpotent endomorphism $f : M(w) \rightarrow M(w)$ that factors through $\overline{f} : \text{top}(M(w)) \rightarrow \text{soc}(M(w))$. As a consequence, at most finitely many strings can be bricks, proving our claim. The converse is given by Proposition 5.1. \hfill $\square$

6. Characterisation of $\tau$-tilting finite Brauer Graph Algebras

An algebra $A$ is said to be symmetric if $A \cong \text{Hom}_A(A, k)$ as $A$-$A$-bimodule. It was shown in [19] that every symmetric special biserial algebra is a Brauer graph algebra. A Brauer graph is a finite undirected connected graph, possibly with multiple edges and loops, in which every vertex is
equipped with a cyclic ordering of the edges incident with it and with a strictly positive integer, its multiplicity. The construction of a symmetric special biserial algebra from a Brauer graph can be found, for example, in [1].

Before we start, we need to fix some notation. A cycle $C$ in a Brauer graph $G$ is a set of vertices $\{v_1, \ldots, v_n\}$ and a set of edges $\{e_1, \ldots, e_n\}$ such that $e_i$ is incident with $v_i$ and $v_{i+1}$ and $e_i$ is incident with $v_{i-1}$ and $v_i$ for all $1 \leq i \leq n-1$. A cycle $C$ is minimal if all its vertices are distinct. We say that a cycle $C$ is odd (resp. even) if it is a minimal cycle with an odd (resp. even) number of vertices.

As an application of Theorem 5.2, we give a new proof of the characterisation in [1] of the $\tau$-tilting finiteness of Brauer graph algebras in terms of their Brauer graph. More precisely, we show the following.

**Theorem 6.1.** [1,Theorem 6.7] Let $A = kQ/I$ be a Brauer graph algebra with Brauer graph $G$. Then $A$ is $\tau$-tilting finite if and only if $G$ has no even cycles and at most one odd cycle.

In order to show Theorem 6.1 we first show the following lemmas.

**Lemma 6.2.** Let $A = kQ/I$ be a Brauer graph algebra with Brauer graph $G$. If $G$ has an even cycle, then there is a band $b$ such that $M(b, \lambda, 1)$ is a brick.

**Proof.** By hypothesis $G$ has an even cycle $C$ with pairwise distinct vertices $v_1, \ldots, v_2t$ and edges $e_1, \ldots, e_2t$ in $G$ such that $e_i$ is incident with $v_i$ and $v_{i+1}$ for all $i$ and $e_2t$ is incident with $v_{2t}$ and $v_1$. Now, define $w_i$ to be the shortest direct path from $e_i$ to $e_{i+1}$ if $i$ is even and the shortest inverse path from $e_i$ to $e_{i+1}$ if $i$ is odd. Then it is easy to see that the word $w = w_1w_2\ldots w_{2t}$ is a band in $Q$.

We claim that $M(w, \lambda, 1)$ is a brick. Indeed, by construction $w$ has no repeated letters. Then, Lemma 3.3 implies that every non-trivial nilpotent endomorphism $f$ of $M(w, \lambda, 1)$ factors through a map $\bar{f} : \text{top}(M(w, \lambda, 1)) \to \text{soc}(M(w, \lambda, 1))$. By construction, we have that

$$\text{top}(M(w, \lambda, 1)) \cong \bigoplus_{j=1}^{t} S(e_{2j}) \quad \text{and} \quad \text{soc}(M(w, \lambda, 1)) \cong \bigoplus_{j=1}^{t} S(e_{2j-1})$$

where all the $S(e_i)$ are distinct. Then $M(w, \lambda, 1)$ has no non-trivial nilpotent endomorphisms and thus it is a brick. \(\square\)

**Remark 6.3.** If a Brauer graph $G$ is not simply laced then it has a cycle of length 2 and it follows from the previous lemma that it contains a band module which is a brick.

**Lemma 6.4.** Let $A = kQ/I$ be a Brauer graph algebra with Brauer graph $G$. If $G$ has two odd cycles, then there is a band $b$ such that $M(b, \lambda, 1)$ is a brick.

**Proof.** Suppose that $G$ has two odd cycles. Then there exist two sets of vertices, $v_1, \ldots, v_{2t+1}$ and $v_1', \ldots, v_{2r+1}'$, and two set of edges $e_1, \ldots, e_{2t+1}$ and $e_1', \ldots, e_{2r+1}'$ in $G$ such that $e_i$ is incident with $v_i$ and $v_{i+1}$ for all $i$ and $e_i'$ is incident with $v_i'$ and $v_{i+1}'$. Since $G$ is connected, there exists a set of vertices $v''_1, \ldots, v''_s$ and edges $e''_1, \ldots, e''_{s+1}$ such that $e''_i$ is incident with $v_i$ and $v_{i+1}$, $e''_s$ is incident with $v_s''$ and $v_1''$ and $e''_k$ is incident with $v_k''$ and $v_k''$ for all $2 \leq k \leq s$. Note that if $v_1 = v'', then $k = 0$. This proof consist of two cases: when $k$ is odd and when $k$ is even. We prove the case of $k$ odd, the case of $k$ even being very similar.

Similarly to the proof of Lemma 6.2, we construct a suitable band in $Q$ and we will do so in several steps.

First, define $\alpha_i$ to be the shortest direct path from $e_i$ to $e_{i+1}$ if $i$ is even and the shortest inverse path from $e_i$ to $e_{i+1}$ if $i$ is odd for all $1 \leq i \leq 2t$. Now let $\alpha_{2t+1}$ be the shortest direct path from $e_{2t+1}$ to $e_1'$. Denote by $\beta_i$ the shortest direct path from $e_i''$ to $e_{i+1}''$ if $i$ is odd and the shortest inverse path from $e_i''$ to $e_{i+1}''$ if $i$ is even for all $i$ between 0 and $s-1$. Define $\gamma_0$ as the shortest direct path
from $e''_{s+1}$ to $e'_1$. For all $1 \leq j \leq 2r$ define $\gamma_j$ to be the shortest direct path from $e'_j$ to $e'_{j+1}$ if $j$ is even and the shortest inverse path from $e'_j$ to $e'_{j+1}$ if $j$ is odd. Set $\gamma_{2r+1}$ as the shortest inverse path from $e''_{2r+1}$ to $e''_1$. Finally let $\alpha_0$ be the shortest direct path from $e''_1$ to $e_1$.

By construction $w = \alpha_1 \cdots \alpha_{2r+1} \beta_0 \cdots \beta_{s-1} \gamma_{2r+1} \cdots \beta_0 \alpha_0$ is a band and $w$ has no repeated letters. Then, Lemma 3.6 implies that every non-trivial nilpotent endomorphism $f$ of $M(w, \lambda, 1)$ factors through a map $\tau : \text{top}(M(w, \lambda, 1)) \to \text{soc}(M(w, \lambda, 1))$. Furthermore, by construction, we have that top$(M(w, \lambda, 1))$ and soc$(M(w, \lambda, 1))$ have no common direct summand, thus implying that $M(w, \lambda, 1)$ has no non-trivial nilpotent endomorphisms. In other words, $M(w, \lambda, 1)$ is a brick in mod$A$.

We now prove Theorem 6.1.

Proof of Theorem 6.1. Let $A$ be a Brauer Graph algebra with Brauer graph $G$. Recall that every indecomposable non-projective $A$-module $M$ comes from a string or a band in the Brauer graph. Furthermore, if $G$ contains an even cycle or if $G$ contains two odd cycles then by Lemmas 6.2 and 6.4 the algebra $A$ is $\tau$-tilting infinite. Thus suppose that $G$ contains at most one odd cycle.

If $G$ is a tree and all but at most one multiplicity is equal to one then $A$ is of finite representation type and, in particular, $A$ is $\tau$-tilting finite.

Now suppose that $G$ is a tree and that there are at least two vertices of $G$ with multiplicity strictly greater than one and let $b$ be a band in $A$. Given that $G$ is a tree, there exists a vertex $v$ in $G$ with multiplicity strictly greater than one such that $b = wb'$, where $w$ is a direct or inverse path maximal in $b$ starting and ending at the same edge $x$ of $G$ which is incident with $v$. By maximality of $w$, we have that the simple module $S(x)$ associated to $x$ is a direct summand of both soc$(M(b, \lambda, 1))$ and top$(M(b, \lambda, 1))$, for any $\lambda \in k^*$. Hence no band module in mod$A$ is a brick. So, $A$ is $\tau$-tilting finite by Theorem 5.2.

The last case to consider is when $G$ has exactly one cycle of odd length $2t+1$. Then there exists a set of vertices $v_1, \ldots, v_{2t+1}$ and edges $e_1, \ldots, e_{2t+1}$ in $G$ such that each $e_i$ is incident with $v_i$ and $v_{i+1}$ for all $i$ and $e_{2t+1}$ is incident with $v_{2t+1}$ and $v_1$. Since $G$ has no even cycle, it is simply-laced and $e_i$ is the unique edge incident with $v_i$ and $v_{i+1}$.

Now consider a band $b$ in $A$. If $b$ is such that there exists a vertex $v$ such that $b = wb'$ as in the case of the tree with at least two vertices of of higher multiplicities, then $M(b, \lambda, 1)$ is not a brick.

Otherwise $b$ is of the form $b = \alpha_1 \cdots \alpha_{2t+1} \beta_1 \cdots \beta_{2t+1}$, where $\alpha_i$ is a direct path from $e_i$ to $e_{i+1}$ if $i$ is even and the inverse path from $e_i$ to $e_{i+1}$ if $i$ is odd for all $1 \leq i \leq 2t + 1$ and $\beta_i$ is the inverse path from $e_i$ to $e_{i+1}$ if $i$ is even and the direct path from $e_i$ to $e_{i+1}$ if $i$ is odd for all $1 \leq i \leq 2t + 1$. Then the simple module $S(e_i)$ is a direct summand of both top$(M(w, \lambda, 1))$ and soc$(M(w, \lambda, 1))$. So, $M(w, \lambda, 1)$ is not a brick and by the same argument as above, $A$ is $\tau$-tilting finite.

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