VEST is $W[2]$-hard *

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Abstract

In this short note, we show that the problem of VEST is $W[2]$-hard for parameter $k$. This strengthens a result of Matoušek, who showed $W[1]$-hardness of that problem. The consequence of this result is that computing the $k$-th homotopy group of a $d$-dimensional space for $d > 3$ is $W[2]$-hard for parameter $k$.

1 Introduction

The homotopy groups $\pi_k$, for $k = 1, 2, \ldots$ are important invariants of topological spaces. The most intuitive of them is the group $\pi_1$ which is often called fundamental group.

Many topological spaces can be described by finite structures, e.g. by abstract simplicial complexes. Such structure can be used as an input for a computer and therefore, it is natural to ask how hard is to compute these homotopy groups of a given topological space represented by an abstract simplicial complex.

Novikov in 1955 (see [Nov55]) and independently Boone in 1959 (see [Boo59]) showed undecidability of the word problem for groups. Their result also implies undecidability of computing the fundamental group. (Even determining whether the fundamental group of a given topological space is trivial is undecidable.)

On the other hand, it is known that for greater $k$, the corresponding homotopy group $\pi_k$ is a finitely generated abelian group which is always isomorphic to a group of the form

$$\mathbb{Z}^n \oplus \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_m}$$

where $p_1, \ldots, p_m$ are powers of prime numbers. An algorithm for computing $\pi_k$, where $k > 1$, was first introduced by Brown in 1957 (see [Bro57]).

In 1989, Annick (see [Ani89]) proved that computing rank of $\pi_k$, that is the number of direct summands isomorphic to $\mathbb{Z}$ (represented by $n$ in the expression above) is $\#P$-hard for 4-dimensional 1-connected spaces. Another computational problem called VEST, which we define below, was used in Annick’s proof as an intermediate step. Briefly said, $\#P$-hardness of the problem of VEST implies $\#P$-hardness of computing rank of $\pi_k$.

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1Note that $\mathbb{Z}^n$ is a direct sum of $n$ copies of $\mathbb{Z}$ while $\mathbb{Z}_{p_i}$ is a finite cyclic group of order $p_i$.
Vector evaluated after a sequence of transformations (VEST). The input of this problem defined by Anick in [Ani89] is a vector \( v \in \mathbb{Q}^d \), a list of \((T_1,\ldots,T_m)\) of rational \( d \times d \) matrices and a rational matrix \( S \in \mathbb{Q}^{h \times d} \) for \( d,m,h \in \mathbb{N} \).

For an instance of a VEST let \( M\text{-sequence} \) be a sequence of integers \( M_1,M_2,M_3,\ldots \), where

\[ M_k := \{ (i_1,\ldots,i_k) ; ST_i \cdots T_i v = 0 \} \]

Given an instance of a VEST and \( k \in \mathbb{N} \), the goal is to compute \( M_k \). Note, that instead of rational setting we can assume integral setting.

From an instance of a VEST, it is possible to construct a corresponding algebraic structure called 123\( H \)-algebra in polynomial time whose Tor-sequence is equal to the \( M \)-sequence of the original instance of a VEST. This is stated in [Ani89] Theorem 3.4 and it follows from [Ani85] Theorem 1.3 and [Ani87] Theorem 7.6.

Given a presentation of a 123\( H \)-algebra, one can construct a corresponding 4-dimensional simplicial complex in polynomial time whose sequence of ranks \((\text{rk}_2,\text{rk}_3,\ldots)\) is related to the Tor-sequence of the 123\( H \)-algebra. In particular, it is possible to compute that Tor-sequence from the sequence of ranks using an \( \text{FPT} \) algorithm. (Which is defined in the next paragraph).

Parameterized complexity and \( W \) hierarchy  It is also possible to look at the problem of computing \( \pi_k \) from the viewpoint of parameterized complexity which classifies decision computational problems with respect to a given parameter(s). For instance, one can ask if there exists an independent set of size \( k \) in a given graph, where \( k \) is the parameter.

In our case, the number \( k \) of the homotopy group \( \pi_k \) plays the role of such parameter. Since we assume only decision problems, we only ask whether the rank of \( \pi_k(X) \) of a space \( X \) is nonzero (or equal to a particular number). In 2014 Čadek et al. (see [ˇCKM+14b]) proved that this problem is in \( \text{XP} \) in the parameter \( k \). In other words, there is an algorithm solving this problem in time \( cn^{f(k)} \), where \( c \) is a constant, \( n \) is the size of input and \( f(k) \) is a computable function of the parameter \( k \).

A lower bound for the complexity from the parameterized viewpoint was obtained by Matoušek in 2013 (see [Mat13]). He proved that computing \( M_k \) of a VEST is \( W[1] \)-hard. This also implies \( W[1] \)-hardness for the original problem of computing higher homotopy groups \( \pi_k \) (for 4-dimensional 1-connected spaces) for parameter \( k \). Matoušek’s proof also works as a proof for \#P-hardness and it is shorter and much more easier than the original proof of Annick in [Ani89].

The class \( W[1] \) is a member of the following \( W \) hierarchy, which we briefly define.

\[ \text{FPT} \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq W[P] \subseteq \text{XP} \]

We have already defined the class \( \text{XP} \) above. The class \( \text{FPT} \) consists of decision problems solvable in time \( f(k)n^{O(1)} \), where \( f(k) \) is a computable function of the parameter \( k \) and \( n \) is the size of input. It is only known that \( \text{FPT} \subseteq \text{XP} \) (see [FG04]). The class \( W[1] \) then consists of all problems which can be reduced by an \( \text{FPT} \) algorithm to a boolean circuit of a constant depth with AND, OR and NOT gates such that there is at most 1 gate of higher input size than 2 on each path from the input gate to the final output gate (this number of larger gates is called \( \text{weft} \)) such that the parameter \( k \) from the original problem is translated to setting \( g(k) \) input gates to TRUE. See Figure [1]. It is strongly believed that \( \text{FPT} \subseteq W[1] \). Therefore, one cannot expect existence of an algorithm solving a \( W[1] \)-hard problem in time \( f(k)n^c \) where \( f(k) \) is a computable function of \( k \) and \( c \) is a constant.

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The class $W[i]$ consists of problems FPT-reducible to a boolean circuit of a constant depth and weft at most $i$.

The class $W[P]$ can be defined as a class of problems which can be solved by non-deterministic Turing machine which can make at most $O(g(k) \log n)$ non-deterministic choices and which works in time $f(k) n^{O(1)}$. See [FG04].

According to this definition, it is easy to see that the problem of VEST is in $W[P]$.

**Observation 1.** Computing $M_k$ of a VEST for parameter $k$ is in $W[P]$.

**Proof.** Let $n$ be the size of the input and $m$ the number of the matrices in the collection. In particular, $m \leq n$.

We can guess which $k$ matrices we choose from the collection. Each matrix can be represented by an integer $\leq m$ which can be described by $\log m$ bits. Therefore, we need at most $k \log m \leq k \log n$ non-deterministic choices.

Then, we need to multiply $k + 1$ matrices together with 1 vector. This can be easily done in time $(k+2)n^3$. \qed

In this note, we strengthen the result of Matoušek and show that the problem of VEST is $W[2]$-hard. Our proof is even simpler than the proof of $W[1]$-hardness.

**Theorem 2.** Computing $M_k$ of a VEST is $W[2]$-hard for parameter $k$.

Theorem 2 together with the result of Anick (see [Ani89]) implies the following.

**Corollary 3.** Computing $k$-th homotopy groups of $d$-dimensional space for $d > 3$ is $W[2]$-hard in the parameter $k$.

## 2 The proof

Note that the current complexity of the problem of VEST is a self-contained problem. Our reduction will use only 0,1 matrices and the initial vector $v$. Moreover, each matrix will have at most one 1 in each line. Therefore, such construction also shows $W[2]$-hardness of a VEST for $\mathbb{Z}_2$ setting.
Figure 2: The submatrix realizing the procedure which assures that no matrix can repeat. The first line corresponds to the coordinate \( u_2 \), the second to \( u_3 \) and the third to \( c \).

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

**W[2]-complete problem.** Our reduction is from well-known problem of existence of a dominating set of size \( k \) which is known to be W[2]-complete and which we define in this paragraph. See [FG04].

For a graph \( G(V,E) \) and its vertex \( v \in V \) let \( N[v] \) denote the closed neighborhood of a vertex \( v \). That is, \( N[v] := \{u \in V; \{u, v\} \in E\} \cup \{v\} \).

A **dominating set** of a graph \( G(V,E) \) is a set \( U \subseteq V \) such that for each \( v \) there is \( u \in U \) such that \( v \in N[u] \).

**Proof of Theorem 2.** We show an FPT reduction from the problem of existence of a dominating set of size \( k \) to a VEST.

Let \( G(V,E) \) be a graph and let \( n = |V| \). We start with a description of our vector space. It is of dimension \( 3n + 1 \). For each \( u \in V \) we have 3 dimensions \( u_1, u_2, u_3 \). Then there is one extra dimension \( c \). In the beginning, the corresponding coordinates of the vector \( v \) are set as follows: \( u_1 = 1 \) and \( u_2 = u_3 = 0 \). The coordinate corresponding to \( c \) will be set to 1 during the whole computations. The described coordinates will simulate a data structure during the computation which will correspond to matrix multiplication.

For each \( u \in V \) we create a matrix \( M_u \) as follows. This matrix nullifies the coordinate \( w_1 \) for each \( w \in N[u] \) which corresponds to a domination of vertices in \( N[u] \) by the vertex \( u \). The matrix \( M_u \) also set \( u_2 \) to \( u_3 \) and \( u_3 \) to 1. This can be done by the coordinate \( c \). See Figure 2.

It is similar to the procedure from [Mat13] and it assures that each matrix can be chosen only once, but comparing to that procedure used by Matoušek it also works in \( \mathbb{Z}_2 \) setting; note that in the beginning or after one multiplication by matrix \( M_u \) the coordinate \( u_2 = 0 \) while after two or more multiplications it is 1.

The matrix \( S \) then chooses the coordinates \( u_1 \) and \( u_2 \) for each vector \( u \). The coordinate \( u_1 = 1 \) if and only if corresponding vertex is not dominated. As it was discussed in the previous paragraph \( u_2 = 0 \) if and only if the corresponding matrix was not chosen or was chosen only once.

Therefore, \( M_k = k!D_k \) where \( D_k \) is the number of dominating sets of size \( k \) of graph \( G \).

Note that the reduction is FPT. Indeed, we do not use the parameter \( k \) during it and it is polynomial in the size of input. \( \square \)

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