Whittaker pairs for the Virasoro algebra
and the Gaiotto - BMT states

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Abstract

In this paper we analyze Whittaker modules for two families of Wittaker pairs related to the subalgebras of the Virasoro algebra generated by $L_r, \ldots, L_{2r}$ and $L_1, L_n$. The structure theorems for the corresponding universal Whittaker modules are proved and some of their consequences are derived. All the Gaiotto [17] and the Bonelli-Maruyoshi-Tanzini [34] states in an arbitrary Virasoro algebra Verma module are explicitly constructed.

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1 Introduction

Whittaker modules were first introduced by Arnal and Pinzcon [11] in their study of the \(\mathfrak{sl}_2(\mathbb{C})\) algebra representations. For an arbitrary complex infinite-dimensional semisimple Lie algebra the theory of Whittaker modules was developed by Kostant [3]. The construction was based on the triangular decomposition \(\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+\) and a regular Lie algebra homomorphisms \(\phi : \mathfrak{n}^+ \rightarrow \mathbb{C}\). For non-regular homomorphisms the Kostant construction was analyzed in [4, 5, 6]. More recently the Whittaker modules have been intensively investigated for various infinite-dimensional algebras with a triangular decomposition: the Heisenberg and the affine Lie algebras [7], the generalized Weyl algebras [8], the Virasoro algebra [9, 10], the twisted Heisenberg-Virasoro algebra [11], the Schrödinger-Witt algebra [12], the graded Lie algebras [13], the \(W\)-algebra \(W(2,2)\) [14] and the Lie algebras of Block type [15].

A general categorial framework for Whittaker modules was proposed in [16] where the Kostant construction was generalized to a pair of Lie algebras \(\mathfrak{n} \subset \mathfrak{g}\). In this setting the Whittaker module is defined as a \(\mathfrak{g}\)-module, generated by a one-dimensional \(\mathfrak{n}\)-invariant subspace.

In two-dimensional conformal field theory the interest in Whittaker modules was stimulated by Gaiotto’s paper [17] on a particular versions of the AGT relation [18] where on the CFT theory side so called irregular blocks appear. They are defined as a scalar products of vectors of a Virasoro algebra Verma module determined by the conditions

\[
L_1 |w\rangle = \mu_1 |w\rangle, \quad L_2 |w\rangle = \mu_2 |w\rangle, \quad L_n |w\rangle = 0, \quad n > 2.
\]  

(1)

It has been conjectured in [17] that not only these vectors but also higher order vectors

\[
L_r |w\rangle = \mu_r |w\rangle, \quad \ldots, \quad L_{2r} |w\rangle = \mu_{2r} |w\rangle, \quad L_n |w\rangle = 0, \quad n > 2r, \quad r > 1
\]  

(2)

exist and are uniquely defined by the conditions above. The existence of the first order Gaiotto states has been soon verified by an explicite construction in [21]. Another construction in terms of the Jack symmetric polynomials was given in [22]. The corresponding irregular blocks were the simplest objects for which the AGT relation could be verified [23, 24]. In particular the norm of the first order Gaiotto state with \(\mu_2 = 0\) corresponds to the pure gauge partition function in four dimensions. This relation has been analyzed for various extensions of the AGT relations with the CFT side described by the \(A_n\)-Toda theories [25, 26, 27], the \(N = 1\) super-symmetric Liouville theories [28, 29, 30], the para-\(A_n\)-Toda theories [31, 32] and the general Toda theories [33].

From the point of view of the general approach of [16] the Gaiotto conditions (2) define a Whittaker vector of the Whittaker pair \(\mathcal{V}_r \subset \mathcal{V}\) where \(\mathcal{V}\) is the Virasoro algebra and \(\mathcal{V}_r\) its subalgebra generated by \(L_r, \ldots, L_{2r}\). Recently a new type of coherent states corresponding to the Whittaker pairs \(\mathcal{V}_{1,n} \subset \mathcal{V}\) where \(\mathcal{V}_{1,n}\) is the subalgebra generated by \(L_1, L_n\) has been

\[^4\] In the context of instanton counting the Whittaker vectors of affine algebras were analyzed earlier in [19, 20].
introduced by Bonelli, Maruyoshi and Tanzini in [34]. It was proposed that the corresponding irregular conformal blocks describe partition functions of wild quiver gauge theories.

In the present paper we analyze general algebraic properties of the Gaiotto and the BMT states for the Virasoro algebra. This includes the problem of the structure of Whittaker modules for the pairs $\mathcal{V}_r \subset \mathcal{V}$ and $\mathcal{V}_{1,n} \subset \mathcal{V}$ and the construction of corresponding states. Since $\mathcal{V}_1 = \mathcal{V}_{1,2}$ the lower orders coincide. In this case the structure of Whittaker modules is already known [9, 10] and the Gaiotto states are constructed [21, 22]. Our aim is to extend these results to higher orders.

In Section 2 we use the method of [9] to analyze modules of the higher order Whittaker pairs $\mathcal{V}_r \subset \mathcal{V}$. In view of CFT applications we restrict ourselves to modules with a fixed central charge. Our main result is the structure theorem for the universal Whittaker module of a general type. We say that a Lie algebra homomorphism $\psi_r : \mathcal{V}_r \to \mathcal{V}$ is of the high rank if $\psi_r(L_{2r}) \neq 0$ or $\psi_r(L_{2r-1}) \neq 0$. For the high rank homomorphisms the corresponding universal Whittaker module is simple. In all other cases it has an infinite composition series with a single composition factor uniquely determined by $\psi_r$.

Section 3 is devoted to the Whittaker pairs $\mathcal{V}_{1,n} \subset \mathcal{V}$. Most of the techniques developed in Section 2 can be applied in this case as well. Note that for a non-regular Lie algebra homomorphisms $\psi_{1,n} : \mathcal{V}_{1,n} \to \mathbb{C}$ ($\psi_{1,n}(L_1) = 0$ or $\psi_{1,n}(L_n) = 0$) we are back to the case $\mathcal{V}_r \subset \mathcal{V}$. We give a complete analysis of Whittaker vectors in the universal Whittaker module in the cases $n = 3, 4$ and present some examples for higher orders.

In Section 4 we investigate subspaces of Whittaker vectors of a given pair and type in the Virasoro module of all anti-linear functionals on a Verma module. Whenever the Shapovalov form is non-degenerate the Gaiotto and the BMT states can be recovered by "raising indices" level by level by the inverse to the Gram matrix. For the first order states this is the construction of [21]. One can define irregular blocks directly in terms of forms and the scalar product determined level by level by the inverse Gram matrices. This approach conveniently separates the universal algebraic properties from those specific to the weight dependent Shapovalov form.

It turns out that the first order Gaiotto states are uniquely (up to a scale factor) defined by the conditions (1). This is no longer true for higher orders. We present general representation of the Gaiotto and the BMT states in terms of systems of basic states.

The two types of Whittaker pairs considered in this paper are special cases of a more general pair $\mathcal{V}_{m,n} \subset \mathcal{V}$ where the subalgebra $\mathcal{V}_{m,n}$ is generated by

$$L_m, L_n, L_{n+1}, \ldots, L_{n+m-1}, \quad m < n.$$ 

One can easily invent even more general subalgebras. If they contain the subalgebra $\mathcal{V}_r$ for a certain integer $r$ the method of [9] we have used in the present paper can be applied to analyze the corresponding Whittaker modules. One may expect new classes of simple modules of the Virasoro algebra. It is an interesting question to what extend this general construction might be helpful in solving the problem of classification of all simple modules of
The abundance of higher order Gaiotto states rises the question of their classification. One may expect [17] that at least some of them arise as decoupling limits of \( n \)-point conformal blocks. It would be interesting to find their algebraic characterization. Similar questions arise for the BMT states. Finally, in view of the developments mentioned above the extensions to other algebras would be desirable.

When the present paper was completed we became aware of a series of papers [35, 36, 37]. In [35] irreducible Virasoro algebra modules were studied. In particular all Whittaker modules for higher order pairs \( \mathcal{V}_r \subset \mathcal{V} \) were explicitly constructed by twisting oscillator representation of the twisted Heisenberg-Virasoro algebra. The construction implies that all high rank Whittaker modules of an arbitrary order are simple (Theorem 7 of [35]). In the present paper we obtain the same result using different methods (Corollary 2.2). The other two papers are devoted to the Whittaker modules of the generalized Virasoro algebras [36] and of the Virasoro-like algebras [37].

We owe special thanks to Volodymyr Mazorchuk and Kaiming Zhao for sending us their work [38] and in particular for pointing out that Theorems 3.5, 3.6 and Corollary 3.7 of Section 3 in the first published version of our paper are valid only in the case of \( n = 3 \). The present version contains necessary corrections of Section 3.

2 Whittaker pairs \( \mathcal{V}_r \subset \mathcal{V} \)

Let \( \mathcal{V} \) be the Virasoro algebra i.e. \( \mathcal{V} = \text{span}_\mathbb{C}\{z, L_n : n \in \mathbb{Z}\} \) with the Lie bracket

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{z}{12}m (m^2 - 1) \delta_{m+n}, \\
[L_m, z] &= 0.
\end{align*}
\]

Let \( r \) be a a positive integer. We introduce the subalgebra \( \mathcal{V}_r \) generated by \( L_r, \ldots, L_{2r} \)

\[ \mathcal{V}_r = \text{span}_\mathbb{C}\{L_r, L_{r+1}, \ldots\} \subset \mathcal{V}. \]

A Lie algebra homomorphism \( \psi_r : \mathcal{V}_r \rightarrow \mathbb{C} \) is uniquely defined by its values on the algebra generators. This justifies the notation

\[ \psi_r = \{\psi_r(L_r), \ldots, \psi_r(L_{2r})\}. \]

\footnote{A complete answer was recently found by Mazorchuk and Zhao in [38].}
We consider only non trivial $\psi_r$ i.e. at least one of $\psi_r(L_i)$ does not vanish. For non trivial homomorphisms we define the rank:

$$\text{rank} \psi_r = \max\{r \leq i \leq 2r : \psi_r(L_i) \neq 0\}.$$ 

From commutation relations one has $\psi_r(L_k) = 0$ for all $k > \text{rank} \psi_r$.

**Definition 2.1.** Let $V$ be a left $\mathcal{V}$-module, $\psi_r : \mathcal{V}_r \to \mathbb{C}$ - a non trivial Lie algebra homomorphism and $c$ - a complex number. A vector $|w\rangle \in V$ is called a Whittaker vector of the Whittaker pair $\mathcal{V}_r \subset \mathcal{V}$, the central charge $c$ and the type $\psi_r$ if

$$z |w\rangle = c |w\rangle \quad \text{and} \quad L_k |w\rangle = \psi_r(L_k) |w\rangle \quad \text{for} \quad k \geq r.$$ 

A $\mathcal{V}$-module $V$ is called a Whittaker module of the Whittaker pair $\mathcal{V}_r \subset \mathcal{V}$, the central charge $c$ and the type $\psi_r$ if it is generated by a Whittaker vector of the same pair, central charge and type.

The order and the rank of the Whittaker vector or module of a type $\psi_r$ are defined as $r$ and $\text{rank} \psi_r$, respectively.

Let us note that the standard notions of the Whittaker vectors and modules for the Virasoro algebra as defined in [9, 10] correspond to the case $r = 1$ with no central charge condition.

We assume that all Whittaker vectors and modules in this section are of the Whittaker pairs $\mathcal{V}_r \subset \mathcal{V}$ and have the same fixed value of the central charge. For a Whittaker module $V$ of a Whittaker pair $\mathcal{V}_r \subset \mathcal{V}$ and a type $\psi_r$ we shall use compact notation $V_{\psi_r}$.

**Definition 2.2.** A Whittaker module $W_{\psi_r}$ generated by $|w\rangle$ is called a universal Whittaker module of a type $\psi_r$, if for any other Whittaker module $V_{\psi_r}$ generated by $|v\rangle$ there exists a surjective module homomorphism $\Phi : W_{\psi_r} \to V_{\psi_r}$ such that $\Phi |w\rangle = |v\rangle$. The Whittaker vector $|w\rangle$ of the type $\psi_r$ generating the universal module $W_{\psi_r}$ is called the universal Whittaker vector of type $\psi_r$.

**Theorem 2.1.** For each Lie algebra homomorphism $\psi_r : \mathcal{V}_r \to \mathbb{C}$ there exists a unique, up to an isomorphism, universal Whittaker module $W_{\psi_r}$.

**Proof.** For each Lie algebra homomorphism $\psi_r : \mathcal{V}_r \to \mathbb{C}$

$$I_r \equiv \sum_{i \geq r} U(\mathcal{V})(L_i - \psi_r(L_i)) + U(\mathcal{V})(z - c)$$

is a left ideal in $U(\mathcal{V})$. $U(\mathcal{V})/I_r$ with the left action

$$u[v] \equiv [uv], \quad u, v \in U(\mathcal{V})$$

is a $U(\mathcal{V})$-module generated by $[1]$. For any $n \geq r$ one has

$$(L_n - \psi_r(L_n))[1] = [L_n - \psi_r(L_n)] = 0 \quad \text{and} \quad (z - c)[1] = 0.$$
Hence $|1\rangle$ is a Whittaker vector and $U(\mathcal{V})/I_r$ is a Whittaker module. Let $V_\psi$ be an arbitrary Whittaker module of a type $\psi_r$ generated by a Whittaker vector $|v\rangle$. The map 

$$\Phi : U(\mathcal{V})/I_r \ni [u] \to u|v\rangle \in V_\psi$$

is a surjective homomorphism such that $\Phi([1]) = |v\rangle$. The uniqueness is a simple consequence of the universality. 

We define a pseudo partition $\lambda$ of order $r$ as a non-decreasing finite sequence of integers smaller than $r$:

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \leq \ldots \leq \lambda_n < r.$$ 

With each $\lambda$ we associate an element of the universal enveloping algebra $U(\mathcal{V})$:

$$L_\lambda \equiv L_{\lambda_1} \ldots L_{\lambda_n}.$$ 

It is convenient to supplement the set $\mathcal{P}^r$ of all pseudo partitions of order $r$ by the empty sequence $\emptyset$ for which

$$L_\emptyset \equiv 1.$$ 

Sometimes it is useful to denote a pseudo partition $\lambda$ of order $r$ as:

$$\lambda = (\lambda(-l), \ldots, \lambda(r-1))$$

where $\lambda(k)$ is the number of times the integer $k$ appears in $\lambda$. In this notation $L_\lambda$ takes the form

$$L_\lambda = L_{\lambda(-l)} \ldots L_{\lambda(r-1)}.$$ 

As a consequence of the PBW theorem one has

**Theorem 2.2.** Let $W_\psi$ be the universal Whittaker module generated by $|w\rangle$. Then the vectors $L_\lambda |w\rangle$ where $\lambda$ runs over the set $\mathcal{P}^r$ of all pseudo partitions of order $r$, form a basis of $W_\psi$. 

Each pseudo partition $\lambda$ can be uniquely decomposed $\lambda = \lambda_- \cup \lambda_+$ into the negative $\lambda_- = (\lambda(-l), \ldots, \lambda(-1))$ and the nonnegative $\lambda_+ = (\lambda(0), \ldots, \lambda(r-1))$ part. We shall introduce the length $\#\lambda$

$$\#\lambda = \sum_{i=0}^{r-1} \lambda(i)$$

and the level $|\lambda|$ 

$$|\lambda| = -\sum_{i<0} i\lambda(i)$$

of a pseudo partition $\lambda$. For an arbitrary vector $|v\rangle = \sum_{\lambda} p_\lambda L_\lambda |w\rangle$ we define the maximal length

$$\max\# |v\rangle = \max\{\#\lambda : p_\lambda \neq 0\}$$

and the maximal level

$$\max |v\rangle = \max\{|\lambda| : p_\lambda \neq 0\}$$
Lemma 2.1. Let $|w\rangle$ be the universal Whittaker vector of a type $\psi_r$ and a rank $s$. Then for any pseudo partition $\lambda \in \mathcal{P}^r$

$$[L_m, L_{\lambda_+}] |w\rangle = 0 \quad \text{for} \quad s < m,$$

$$\max \# [L_m, L_{\lambda_+}] |w\rangle < \# \lambda \quad \text{for} \quad r \leq m \leq s.$$

Proof. For $L_{\lambda_+} = L_{\lambda_{k+1}} \ldots L_{\lambda_n}$ one gets

$$[L_m, L_{\lambda_+}] = \sum_{i=k+1}^{n} (m - \lambda_i)L_{\lambda_{k+1}} \ldots L_{m+\lambda_i} \ldots L_{\lambda_n}.$$

If $m > s$ then $m + \lambda_i > s$ for all $i$ in the sum. Commuting all generators $L_l$ with $l > s$ to the right one gets only terms annihilating $|w\rangle$. Terms with a maximal number of $L$ generators are of the form

$$L_{\lambda_{k+1}} \ldots L_{\lambda_{i-1}}L_{\lambda_{i+1}} \ldots L_{\lambda_n}L_{m+\lambda_i} |w\rangle = \psi_r(L_{m+\lambda_i})L_{\lambda_{k+1}} \ldots L_{\lambda_{i-1}}L_{\lambda_{i+1}} \ldots L_{\lambda_n} |w\rangle.$$

They do not necessarily vanish for $r \leq m \leq s$. Hence the maximal possible number of generators is $\# \lambda - 1$. 

Lemma 2.2. Let $|w\rangle$ be the universal Whittaker vector of a type $\psi_r$ and a rank $s$. Then for any pseudo partition $\lambda \in \mathcal{P}^r$

$$[L_m, L_{\lambda_-}]L_{\lambda_+} |w\rangle = 0 \quad \text{for} \quad s + |\lambda| < m,$$

$$\max [L_m, L_{\lambda_-}]L_{\lambda_+} |w\rangle \leq |\lambda| + s - m \quad \text{for} \quad s < m \leq s + |\lambda|,$$

$$\max [L_m, L_{\lambda_-}]L_{\lambda_+} |w\rangle < |\lambda| \quad \text{for} \quad r \leq m \leq s.$$

Proof. Let $s + |\lambda| < m$. Ordering all terms in the commutator $[L_m, L_{\lambda_-}]$ one gets a sum of monomials with the generators $L_l$ on the right such that $l > s$. Hence $[L_m, L_{\lambda_-}]L_{\lambda_+} |w\rangle = 0$ by Lemma 2.1.

Let $r \leq m \leq s + |\lambda|$. For $L_{\lambda_-} = L_{\lambda_1} \ldots L_{\lambda_k}$ one has

$$[L_m, L_{\lambda_-}]L_{\lambda_+} |w\rangle = \sum_{m+\lambda_i < 0} (m - \lambda_i)L_{\lambda_1} \ldots L_{m+\lambda_i} \ldots L_{\lambda_k}L_{\lambda_+} |w\rangle + \sum_{m+\lambda_i \geq 0} (m - \lambda_i)L_{\lambda_1} \ldots L_{m+\lambda_i} \ldots L_{\lambda_k}L_{\lambda_+} |w\rangle.$$

After reordering the first sum takes the form

$$\sum_{|\gamma| = |\lambda| - m \atop \# \gamma = 0} p_{\gamma} L_{\gamma} L_{\lambda_+} |w\rangle$$

and if nonzero it is of level $|\lambda| - m$. The second sum can be rewritten as

$$\sum_{|\gamma| = |\lambda| - m \atop \# \gamma = 0} p_{\gamma} L_{\gamma} L_{\lambda_+} |w\rangle + \sum_{l} \sum_{|\gamma| = |\lambda| - l \atop \# \gamma = 0} p_{\gamma} L_{\gamma} L_{m-l} L_{\lambda_+} |w\rangle.$$

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where the sum over \( l \) runs over the partial sums
\[
m > l = -\sum_{j=1}^{l} \lambda_{ij} \geq m - s
\]
corresponding to all possible subseries \( \{\lambda_{i1}, \ldots, \lambda_{it}\} \subset \{\lambda_1, \ldots, \lambda_k\} \). Let \( l' \) be the smallest of such sums. Then the terms \( L_{\lambda_1} L_{m-l'} L_{\lambda_+} |w\rangle \) are of the maximal possible level
\[
\max L_{\lambda_1} L_{m-l'} L_{\lambda_+} |w\rangle = |\lambda| - l' \leq |\lambda| + s - m.
\]
For \( r \leq m \leq s \) one simply has
\[
\max L_{\gamma} L_{m-l'} L_{\lambda_+} |w\rangle = |\lambda| - l' < |\lambda|.
\]
\[ \blacksquare \]

Lemma 2.3. Let \( |w\rangle \) be the universal Whittaker vector of a type \( \psi_r \) and a rank \( s \) and \( \lambda \) a pseudo partition of order \( r \). If \( k > 0 \) is the smallest number for which \( \lambda(-k) \neq 0 \), then
\[
[L_{k+s}, L_{\lambda_-}] L_{\lambda_+} |w\rangle = L(-k)_{\psi_r} (L_s) (2k + s) L_{\lambda+} L_{\lambda-} \sum_{k=1}^{\lambda} L_{\lambda} |w\rangle + |v\rangle + |v'\rangle,
\]
where \( \max |v\rangle < |\lambda| - k \) and \( \max |v'\rangle \leq |\lambda| - k \), \( \max |v'\rangle < \# \lambda \).

Proof. Let \( L_{\lambda'} = L_{\lambda-n} \cdots L_{\lambda-k-1} \), so that \( L_{\lambda_-} = L_{\lambda'} L_{\lambda-k} \). Then
\[
[L_{k+s}, L_{\lambda_-}] L_{\lambda_+} |w\rangle = [L_{k+s}, L_{\lambda'}] L_{\lambda-} L_{\lambda_+} |w\rangle + L_{\lambda'} [L_{k+s}, L_{\lambda-k}] L_{\lambda_+} |w\rangle. \tag{4}
\]

Repeating the reasoning from the proof of Lemma 2.2 one can write
\[
[L_{k+s}, L_{\lambda'}] L_{\lambda-} L_{\lambda_+} |w\rangle = \sum_{|\gamma| = |\lambda'| - k \cdot s}^{p_\gamma} L_{\gamma} L_{\lambda-k} L_{\lambda_+} |w\rangle
\]
\[
+ \sum_{l=k+1}^{k+s} \sum_{|\gamma| = |\lambda'| - l \# \gamma = 0}^{p_\gamma} L_{\gamma} L_{k+s-l} L_{\lambda-k} L_{\lambda_+} |w\rangle
\]
where the range in the sum over \( l \) follows from the assumption that \( L_{\lambda'} \) does not contain generators \( L_{-i} \) with \( i \leq k \). An element of the maximal degree in the second sum takes the form
\[
q_{\gamma} L_{\gamma} L_{\lambda-k} L_{s-1} L_{\lambda_+} |w\rangle,
\]
where \( |\gamma| = |\lambda'| - k - 1 \). Hence
\[
\max [L_{k+s}, L_{\lambda'}] L_{\lambda-} L_{\lambda_+} |w\rangle < |\lambda| - k.
\]
Let us now turn to the second term in (4). A simple algebra yields
\[
L_{\lambda_+}^{(k)}[L_{k+s},L_{-k}]|w\rangle = (2k + s)L_{\lambda_+}^{(k)} \sum_{j=1}^{\lambda(k)} L_{-k}^{j-1} L_{s} L_{-k}^{\lambda(k)-j} |w\rangle = \lambda(-k)\psi_r(L_s)(2k + s)L_{\lambda_+}^{(k-1)} |w\rangle \\
+ (2k + s)L_{\lambda_+}^{(k)} \sum_{j=1}^{\lambda(k)} L_{-k}^{j-1} [L_s, L_{-k}^{\lambda(k)-j}] |w\rangle \\
+ \lambda(-k)(2k + s)L_{\lambda_+}^{(k-1)} |L_s, L_{\lambda_+}| |w\rangle.
\]
By Lemma 2.2 maximal level of the second term on the r.h.s. is strictly smaller than $|\lambda| - k$. If $\lambda_+(0) \neq 0$ the last term does not vanish. Since
\[
[L_s, L_{\lambda_+}]|w\rangle = [L_s, L_0^{k}L_{\lambda_+}]|w\rangle = \psi_r(L_s) \sum_{l=1}^{k} \binom{k}{l} s^l L_0^{k-l} L_{\lambda_+}^{k} |w\rangle
\]
its length is strictly smaller than $\#\lambda_+$. □

Lemma 2.4. Let $W_{\psi_r}$ be the universal Whittaker module of type $\psi_r$ and let $|u\rangle \in W_{\psi_r}$ be an arbitrary vector. If max $|u| > 0$ then $|u\rangle$ is not a Whittaker vector of any pair $V_r' \subset V$.

Proof. For an arbitrary nonzero vector $|u\rangle = \sum p_{\lambda} L_{\lambda}|w\rangle$ we introduce
\[
M = \max |u|, \quad \Lambda_M = \{\lambda : p_{\lambda} \neq 0 \land |\lambda| = M\}.
\]
Since $M > 0$, there exists a smallest positive number $k$ for which there exists a partition $\lambda \in \Lambda_M$ such that $\lambda(-k) \neq 0$. For $s = \operatorname{rank} \psi_r$ one has
\[
L_{k+s} |u\rangle = \sum_{\lambda \notin \Lambda_M} p_{\lambda}[L_{k+s}, L_{-\lambda}] |L_{\lambda_+}| |w\rangle + \sum_{\lambda \in \Lambda_M} p_{\lambda}[L_{k+s}, L_{-\lambda}] |L_{\lambda_+}| |w\rangle.
\]
If $\lambda \notin \Lambda_M$ then $|\lambda| < M$ and Lemma 2.2 implies
\[
\max_{\lambda \notin \Lambda_M} \sum p_{\lambda}[L_{k+s}, L_{-\lambda}] |L_{\lambda_+}| |w\rangle = |\lambda_-| - k < M - k.
\]
Taking this into account and applying Lemma 2.3 to all terms of the second sum one can write
\[
L_{k+s} |u\rangle = \sum_{\lambda \in \Lambda_M} p_{\lambda} \lambda(-k)\psi_r(L_s)(2k + s)L_{\lambda_+}^{'-\lambda} |w\rangle + |v\rangle + |v'|, \quad (5)
\]
where max $|v| < M - k$, max $|v'| < \#\lambda$ and $\lambda_- = (\lambda(-l), \ldots, \lambda(-k - 1), \lambda(-k) - 1)$. All vectors $L_{\lambda_+}^{'} |w\rangle$ in the sum above are linearly independent. Since for all of them $|L_{\lambda_+}^{'} |w\rangle| = M - k$, they form with $|v\rangle$ and $|v'|$ a linearly independent system as well. The decomposition (5) thus imply that $L_{k+s} |u\rangle$ is a nonzero vector not proportional to $|u\rangle$. Hence $|u\rangle$ is not a Whittaker vector of any type of order $k + s$ or lower.
On the other hand, for \( n > k + s \) one has
\[
L_n |u\rangle = \sum_{\lambda} p_{\lambda} L_n L_{\lambda_-} L_{\lambda_+} |w\rangle = \sum_{\lambda} p_{\lambda} [L_n, L_{\lambda_-}] L_{\lambda_+} |w\rangle.
\]
Lemma 2.2 now implies
\[
\max \left( \sum_{\lambda} p_{\lambda} [L_n, L_{\lambda_-}] L_{\lambda_+} |w\rangle \right) < \max |u\rangle
\]
so \( L_n |u\rangle \) cannot be of the form \( \alpha |u\rangle \) for any \( \alpha \neq 0 \). Hence \( |u\rangle \) is not a Whittaker vector of any order higher than \( k + s \).

For an arbitrary element \( L_{\lambda_r} |w\rangle \) of the basis in the universal Whittaker module \( W_{\psi_r} \) with \( \# \lambda \neq 0 \) we denote by \( l_\lambda \) the smallest nonnegative integer \( i \) such that \( \lambda(i) \neq 0 \). It is convenient to assume that \( l_\lambda = r \) if \( \# \lambda = 0 \). One has for instance \( l_\emptyset = r \) for \( |w\rangle \).

Theorem 2.3. Let \( |w\rangle \) be a universal Whittaker vector of a type \( \psi_r \). All Whittaker vectors in \( W_{\psi_r} \) of a given type \( \psi_r' \) form a linear subspace \( Wh_{\psi_r'} \subset W_{\psi_r} \).

1. If \( \text{rank} \psi_r = s \in \{2r, 2r - 1\} \) there are Whittaker vectors in \( W_{\psi_r} \) of the type \( \psi_r \) and of the higher order types
\[
\psi_{r'} = \{ \psi_r(L_{r'}), \ldots, \psi_r(L_s), 0, \ldots, 0 \}, \quad r' = s - r + 2, \ldots, s,
\]
and
\[
Wh_{\psi_r} = \text{span}\{ |w\rangle \},
\]
\[
Wh_{\psi_r'} = \text{span}\{ L_{\lambda} |w\rangle : |\lambda_r| = 0, l_\lambda \geq s - r' + 1 \}.
\]

2. If \( \text{rank} \psi_r = s < 2r - 1 \) there are Whittaker vectors in \( W_{\psi_r} \) of the type \( \psi_r \) and of the higher order types
\[
\psi_{r'} = \{ \psi_r(L_{r'}), \ldots, \psi_r(L_s), 0, \ldots, 0 \}, \quad r' = r + 1, \ldots, s,
\]
and
\[
Wh_{\psi_r} = \text{span}\{ L_{\lambda} |w\rangle : |\lambda_r| = 0, l_\lambda \geq s - r + 1 \},
\]
\[
Wh_{\psi_r'} = \text{span}\{ L_{\lambda} |w\rangle : |\lambda_r| = 0, l_\lambda \geq s - r' + 1 \}.
\]

There are no other Whittaker vectors of any type in the universal Whittaker module \( W_{\psi_r} \).

Proof. By Lemma 2.4 it is enough to consider vectors of the form
\[
|u\rangle = \sum_{\lambda} p_{\lambda} L_{\lambda} |w\rangle
\]
where \( |\lambda| = 0 \) for all \( \lambda \) in the sum. Let \( N = \max \# |u\rangle \) and
\[
\Lambda^N = \{ \lambda : p_{\lambda} \neq 0 \land \# \lambda = N \},
\]
\[
l = \min \{ l_\lambda : \lambda \in \Lambda^N \},
\]
\[
\Lambda^N_l = \{ \lambda \in \Lambda^N : l_\lambda = l \}.
\]
For $k \geq r$ one has

$$\langle L_k - \psi_r(L_k) \rangle |u\rangle = \sum_{\lambda \in \Lambda^N} p_{\lambda L} [L_k, L_{\lambda}] |w\rangle + \sum_{\lambda \in \Lambda^N} p_{\lambda L} [L_k, L_{\lambda}] |w\rangle.$$  

For the first sum Lemma 2.1 implies

$$\max |\langle L_k - \psi_r(L_k) \rangle |w\rangle < N - 1.$$  

If $s - l \geq r$ the second sum for $k = s - l$ takes the form

$$\sum_{\lambda \in \Lambda^N} p_{\lambda L} [L_{s-l}, L_{\lambda}] |w\rangle = \sum_{\lambda \in \Lambda^N} p_{\lambda \psi_r} (L_s) \lambda(l)(s - 2l) \lambda(l-1) \ldots \lambda(r-1) \lambda(s-l) |w\rangle$$  

and one one gets

$$\langle L_{s-l} - \psi_r(L_{s-l}) \rangle |u\rangle = |v\rangle + \sum_{\lambda \in \Lambda^N} p_{\lambda \psi_r} (L_s) \lambda(l)(s - 2l) \lambda(l-1) \ldots \lambda(r-1) \lambda(s-l) |w\rangle$$

where $\max |\langle L_k - \psi_r(L_k) \rangle |w\rangle < N - 1$ so that all the terms on the r.h.s are linearly independent. It follows that $L_{s-l} |u\rangle$ is a nonzero vector not proportional to $|u\rangle$. Hence $|w\rangle$ is not a Whittaker vector of any type of order $s - l$ or lower.

We shall now discuss separate cases.

1. If $s \in \{2r - 1, 2r\}$ the condition $s - l \geq r$ is always satisfied and there are no Whittaker vectors in $W_{\psi_r}$ of any type of the order $r$ or lower except vectors proportional to $|w\rangle$.

   For $r'$ in the range $s - r + 2 \leq r' \leq s$ all vectors of the form

   $$\sum_{|\lambda| = 0 \atop L_\lambda \geq s - r' + 1} p_{\lambda L} |w\rangle$$

   are Whittaker vectors of type $\psi_{r'} = \{\psi_r(L_{r'}), \ldots, \psi_r(L_s), 0, \ldots, 0\}$.

2. If $s < 2r - 1$ there are no Whittaker vectors of any type of order $s$ or lower with nonzero components along the vectors $L_{\lambda} |w\rangle$ with $s - l_\lambda \geq r$ ($|\lambda| = 0$). The only possibility left over are vectors of the form

   $$\sum_{|\lambda| = 0 \atop L_\lambda \geq s - r} p_{\lambda L} |w\rangle.$$  

   One easily checks that all of them are Whittaker vectors of the same type as $|w\rangle$. Thus there are no lower order Whittaker vectors of any type and all Whittaker vectors of order $r$ are of the $\psi_r$ type.

   The higher order vectors can be constructed as in the other two cases. For $r'$ in the range $r + 1 \leq r' \leq s$ all vectors of the form

   $$\sum_{|\lambda| = 0 \atop L_\lambda \geq s - r' + 1} p_{\lambda L} |w\rangle$$

   are Whittaker vectors of the type $\psi_{r'} = \{\psi_r(L_{r'}), \ldots, \psi_r(L_s), 0, \ldots, 0\}$.
Since all vectors $|u\rangle$ of the maximal level zero were considered there are no other Whittaker vectors.

**Definition 2.3.** Let $V_{\psi}$ be a Whittaker module generated by a Whittaker vector $|w\rangle$ of a type $\psi_r$. The dot-action $\cdot$ of the subalgebra $V_r$ on $V_{\psi}$ is defined by

$$L_n \cdot |v\rangle \equiv (L_n - \psi_r(L_n)) |v\rangle, \quad n \geq r, \quad |v\rangle \in V_{\psi}.$$

For an arbitrary vector $|v\rangle = \sum p_{\lambda, r} L_{\lambda, r} |w\rangle$ one has

$$L_n \cdot |v\rangle = \sum p_{\lambda, r} [L_n, L_{\lambda, r}] |w\rangle.$$

**Lemma 2.5.** All generators of $V_r$ are locally nilpotent on $V_{\psi}$ with respect to the dot-action, i.e. for each $n \geq r$ and $|v\rangle \in V_{\psi}$ there exists an integer $k_{n, |v\rangle}$ such that

$$L_{k_{n, |v\rangle}}^n |v\rangle = 0.$$

**Proof.** It is enough to consider vectors of the form $L_{\lambda} |w\rangle$. For any $n \geq r$ and $\lambda \in P^r$ one has, up to numerical coefficients,

$$\left[ L_n, [L_n, [L_n, [L_n, L_{\lambda, -} L_{\lambda, +}]]]] \right] |w\rangle \sim \sum_{l=0}^{k} \left[ L_n, [L_n, [L_n, [L_n, \ldots, L_n, L_{\lambda, -} L_{\lambda, +}]]]] \right] |w\rangle.$$

By Lemmas 2.1 and 2.2 the level of the expression $[L_n, L_{\lambda, -} L_{\lambda, +} |w\rangle$ is smaller than $|\lambda|$ and the length of the expression $L_{\lambda, -} L_{\lambda, +} |w\rangle$ is smaller than $\# \lambda$. Let $k_+$ and $k_-$ be the biggest numbers for which

$$\max_{k_-} \left[ L_n, [L_n, [L_n, [L_n, \ldots, L_n, L_{\lambda, -}]]]] \right] |w\rangle > 0,$$

$$\max_{k_+} \left[ L_n, [L_n, [L_n, [L_n, \ldots, L_n, L_{\lambda, +}]]]] \right] |w\rangle > 0.$$

To ensure vanishing of (6) one can choose $k > 2 \max\{k_+ + 1, k_- + 1\}$.

**Lemma 2.6.** Let $|v\rangle \in V_{\psi}$ be an arbitrary vector. $U(V_r) \cdot |v\rangle$ is a finite dimensional $V_r$ submodule of $V_{\psi}$ with respect to the dot action.

**Proof.** The PBW basis of $U(V_r)$ consists of all monomials of the form $L_{r}^{\alpha(r)} \ldots L_{n}^{\alpha(n)}$. By Lemmas 2.1 and 2.2 there exists $N$ such that $L_n \cdot |v\rangle = 0$ for all $n > N$. Using Lemma 2.5 one can then show that there are only finitely many pseudo partitions

$$\lambda^N = (\lambda^N(r), \ldots, \lambda^N(N - 1))$$

such that $L_{\lambda^N} \cdot |v\rangle \neq 0$. 

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**Theorem 2.4.** Any submodule of a Whittaker module of a type $\psi_r$ contains a Whittaker vector of the same type.

**Proof.** Let $S$ be a submodule of a Whittaker module $V_{\psi_r}$ and $0 \neq |v\rangle \in S$. By Lemma 2.6 $F = U(V_r) \cdot |v\rangle$ is a finite $V_r$ submodule with respect to the dot action. By Lemmas 2.1 and 2.2 there exists $N$ such that $L_n \cdot F = 0$ for all $n > N$. The quotient $U = V_r / V_N$ is a finite dimensional Lie algebra and $F$ is a $U$ module with respect to the induced dot action

$$[L_n] \cdot |u\rangle = L_n \cdot |u\rangle, \quad n = r, \ldots, N - 1, \quad |u\rangle \in F.$$

It follows from Lemma 2.5 that this action is locally nilpotent. Thus by the Engel theorem there exists a vector $|w'\rangle \in F$ such that

$$[L_n] \cdot |w'\rangle = 0, \quad n = r, \ldots, N - 1.$$

But this is a Whittaker vector of the type $\psi_r$ by construction of the induced dot action.

**Theorem 2.5.** Let $W_{\psi_r}$ be the universal Whittaker module generated by $|w\rangle$ and $s = \text{rank } \psi$.

1. If $s \in \{2r, 2r - 1\}$ then $W_{\psi_r}$ is simple.

2. If $s < 2r - 1$ then there exists an infinite composition series

$$\ldots W^{(n)}_{\psi_r} \subset \ldots \subset W^{(1)}_{\psi_r} \subset W^{(0)}_{\psi_r} = W_{\psi_r},$$

such that all composition factors $W^{(n-1)}_{\psi_r} / W^{(n)}_{\psi_r}$ are Whittaker modules of the type

$$\psi_{\frac{s}{2}} = \{0, \ldots, 0, \psi_r(L_r), \ldots, \psi_r(L_s)\} \quad \text{for } s \in 2\mathbb{N},$$

$$\psi_{\frac{s-1}{2}} = \{0, \ldots, 0, \psi_r(L_r), \ldots, \psi_r(L_s), 0\} \quad \text{for } s \in 2\mathbb{N} + 1.$$

**Proof.** If $S \subset W_{\psi_r}$ is a submodule then Theorem 2.4 implies that $S$ contains a Whittaker vector $|w'\rangle$ and also the submodule $U(V_r) |w'\rangle$ generated by $|w'\rangle$. By Theorem 2.3 if $s \in \{2r, 2r - 1\}$ then any Whittaker vector of the type $\psi_r$ in $W_{\psi_r}$ is proportional to $|w\rangle$ hence $U(V_r) |w'\rangle = W_{\psi_r}$. But this is a Whittaker vector of the type $\psi_r$ by construction of the induced dot action.

For the proof of the second part we construct a strictly decreasing series of submodules:

$$W^{(n)}_{\psi_r} = \text{span}_{U(V_r)} \{L_{\frac{s}{2}} |w\rangle, \ldots, L_{r-2} |w\rangle, L_n^r |w\rangle\}$$

For all $n \in \mathbb{N}$ the quotient $W^{(n-1)}_{\psi_r} / W^{(n)}_{\psi_r}$ is generated by the vector $[L_{r-1}^n |w\rangle]$ which by construction is a Whittaker vector of one of the types stated above. By the first part of the theorem the quotient module is simple. The construction is not unique. It works for subsequent powers of an arbitrary linear combinations of the form

$$\alpha^r L_{\frac{s}{2}} + \cdots + \alpha_{r-1} L_{r-1}.$$
Corollary 2.1. Any Whittaker module of order $r$ and rank $2r$ or $2r - 1$ is isomorphic to the universal Whittaker module of the same type.

Proof. Let $V_{\psi}$ be a Whittaker module of a type $\psi_r$ and rank $2r$ or $2r - 1$ generated by a vector $|v\rangle$. By the universal property of $W_{\psi}$ there exists a surjective homomorphism $\Phi : W_{\psi} \rightarrow V_{\psi}$ such that $\Phi(|w\rangle) = |v\rangle$. The kernel of $\Phi$ is a submodule of $W_{\psi}$. By Theorem 2.5 $W_{\psi}$ is simple and $\Phi(|w\rangle) = |v\rangle \neq 0$ hence $\ker \Phi = \{0\}$. Thus $\Phi$ is an isomorphism of $V$ modules.

As an immediate consequence of Corollary 2.1 and Theorem 2.5 one has

Corollary 2.2. Any Whittaker module of order $r$ and rank $2r$ or $2r - 1$ is simple.

3 Whittaker pairs $\mathcal{V}_{1,n} \subset \mathcal{V}$

Let $n$ be a positive integer. We introduce the subalgebra $\mathcal{V}_{1,n}$ generated by $L_1, L_n$:

$$
\mathcal{V}_{1,n} = \text{span}_C\{L_1, L_n, L_{n+1}, \ldots\} \subset \mathcal{V}.
$$

A Lie algebra homomorphism $\psi_{1,n} : \mathcal{V}_{1,n} \rightarrow \mathbb{C}$ is uniquely defined by its values on the algebra generators. As it was mentioned in the introduction the cases when $\psi_{1,n}(L_1) = 0$ or $\psi_{1,n}(L_n) = 0$ are already described in the previous section. So is the case $n = 2$. We shall assume therefore that homomorphism $\psi_{1,n}$ is regular i.e. $\psi_{1,n}(L_1) \neq 0$, $\psi_{1,n}(L_n) \neq 0$ and $n > 2$. From commutation relations (3) one has $\psi_{1,n}(L_k) = 0$ for all $k > n$.

Definition 3.1. Let $V$ be a left $\mathcal{V}$-module, $\psi_{1,n} : \mathcal{V}_{1,n} \rightarrow \mathbb{C}$ - a non trivial Lie algebra homomorphism and $c$ - a complex number. A vector $|w\rangle \in V$ is called a Whittaker vector of the Whittaker pair $\mathcal{V}_{1,n} \subset \mathcal{V}$, the central charge $c$ and the type $\psi_{1,n}$ if

$$
z |w\rangle = c |w\rangle, \quad L_1 |w\rangle = \psi_{1,n}(L_1) |w\rangle \quad \text{and} \quad L_k |w\rangle = \psi_{1,n}(L_k) |w\rangle \quad \text{for} \quad k \geq n.
$$

A $\mathcal{V}$-module $V$ is called a Whittaker module of the Whittaker pair $\mathcal{V}_{1,n} \subset \mathcal{V}$, the central charge $c$ and the type $\psi_{1,n}$ if it is generated by a Whittaker vector of the same central charge and type.

We say that the Whittaker pair $\mathcal{V}_{1,n} \subset \mathcal{V}$ and its Whittaker vectors are of order $n$.

For a Whittaker module $V$ of a Whittaker pair $\mathcal{V}_{1,n} \subset \mathcal{V}$ and a type $\psi_{1,n}$ we shall use compact notation $V_{\psi_{1,n}}$. As before we assume that all Whittaker vectors and modules in this section are of Whittaker pairs $\mathcal{V}_{1,n} \subset \mathcal{V}$ and have the same fixed value of the central charge.

Both the definition of the universal Whittaker module for the pair $\mathcal{V}_{1,n} \subset \mathcal{V}$ and proofs of the theorems below are obvious modifications of the considerations of the previous section.

Theorem 3.1. For each Lie algebra homomorphism $\psi_{1,n} : \mathcal{V}_{1,n} \rightarrow \mathbb{C}$ there exists a unique, up to an isomorphism, universal Whittaker module $W_{\psi_{1,n}}$. 

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Theorem 3.2. Let $W_{\psi_{1,n}}$ be the universal Whittaker module generated by $|w\rangle$. Then the vectors $L_{\lambda}|w\rangle$ where $\lambda$ runs over the set
$$\mathcal{P}^{1,n} = \{ \lambda \in \mathcal{P}^n : \lambda(1) = 0 \},$$
form a basis of $W_{\psi_{1,n}}$.

The counterpart of Lemma 2.1 takes the form

Lemma 3.1. Let $|w\rangle$ be the universal Whittaker vector of a type $\psi_{1,n}$. Then for any pseudo partition $\lambda \in \mathcal{P}^{1,n}$
1. $[L_m, L_{\lambda_+}] |w\rangle = 0$ for $m > n$,
2. $\max^\# [L_n, L_{\lambda_+}] |w\rangle < \#\lambda$,
3. there exists a positive integer $m_\lambda$ such that
   $$\max^\# [L_1, [L_1, \ldots [L_1, L_{\lambda_+}] \ldots ]] |w\rangle < \#\lambda.$$

By straightforward extensions of Lemmas 2.2, 2.3 one gets

Lemma 3.2. Let $W_{\psi_{1,n}}$ be the universal Whittaker module of type $\psi_{1,n}$ and let $|u\rangle \in W_{\psi_{1,n}}$ be an arbitrary vector. If $\max |u\rangle > 0$ then $|u\rangle$ is not a Whittaker vector of any type.

Lemma 3.3. Let $W_{\psi_{1,n}}$ be the universal Whittaker module $W_{\psi_{1,n}}$ of type $\psi_{1,n}$ and let

$$|u\rangle = \sum p_\lambda L_\lambda |w\rangle$$

be an arbitrary vector in $W_{\psi_{1,n}}$. If $p_\lambda \neq 0$ for pseudo-partitions $\lambda \in \mathcal{P}^{1,n}$ with $\lambda(0) \neq 0$ then $|u\rangle$ is not a Whittaker vector of any type.

Proof. By Lemma 3.2 it is enough to consider vectors of the form

$$|u\rangle = \sum p_\lambda L_\lambda |w\rangle$$

where $|\lambda\rangle = 0$ for all $\lambda$ in the sum. Let $\Lambda^0$ be the set of all partitions such that $p_\lambda \neq 0$ and $\lambda(0) \neq 0$. Let

$$\lambda_{\max}^0 = \max \{ \lambda(0) : \lambda \in \Lambda^0 \}.$$

One can write

$$(L_n - \psi_{1,n}(L_n)) |u\rangle = \sum_{\lambda \in \Lambda^0 \atop \lambda(0) = \lambda_{\max}^0} p_\lambda [L_n, L_\lambda] |w\rangle + \sum_{\lambda \in \Lambda^0 \atop \lambda(0) < \lambda_{\max}^0} p_\lambda [L_n, L_\lambda] |w\rangle$$

$$= n\lambda_{\max}^0 \psi_{1,n}(L_n) \sum_{\lambda \in \Lambda^0 \atop \lambda(0) = \lambda_{\max}^0} p_\lambda L_0^{\lambda(0) - 1} L_1^{\lambda(1)} \ldots L_{n-1}^{\lambda(n-1)} |w\rangle$$

$$+ \sum_{\lambda(0) < \lambda_{\max}^0 - 1} p_\lambda' L_\lambda |w\rangle.$$ 

All terms of the first sum with the whole second sum form a set of linearly independent vectors. Hence, as each term in the first sum does not vanish neither the whole sum does. $\square$
Let us now turn to the analysis of Whittaker vectors in the universal Whittaker modules $W_{\psi_1,n}$ of a type $\psi_1,n$. By Lemma 3.2 and 3.3 the only possible Whittaker vectors in $W_{\psi_1,n}$ are of the form

$$|u\rangle = \sum_{\lambda:\lambda(0)=\lambda(1)=0} p_{\lambda} L_{\lambda} |w\rangle.$$ 

All vectors of this form satisfy

$$L_m |u\rangle = \psi_{1,n}(L_n)\delta_{n,m} |u\rangle, \quad m \geq n$$

and are therefore Whittaker vectors of the type

$$\psi_n = \{\psi_{1,n}(L_n), 0, \ldots, 0\}.$$ 

We shall call them trivial.

**Lemma 3.4.** Let $W_{\psi_1,n}$ be the universal Whittaker module $W_{\psi_1,n}$ of a type $\psi_1,n$. The only possible nontrivial Whittaker vectors $|u\rangle \in W_{\psi_1,n}$ are of the type $\psi_{1,n}$.

**Proof.** We first show that a vector of the form

$$|u\rangle = \sum_{\lambda:\lambda(0)=\lambda(1)=0} p_{\lambda} L_{\lambda} |w\rangle.$$ 

(7)

is not an eigenvector of any $L_k, k = 2, \ldots, L_{n-1}$. To this end let us introduce the lexicographic order in the set of partitions of the form $\lambda = (\lambda(2), \ldots, \lambda(n-1))$:

$$\lambda < \lambda' \iff \begin{cases} \lambda(2) < \lambda'(2) \lor \\ (\lambda(2) = \lambda'(2) \land \lambda(3) < \lambda'(3)) \lor \\ \vdots \\ (\lambda(2) = \lambda'(2) \land \cdots \land \lambda(n-2) = \lambda'(n-2) \land \lambda(n-1) < \lambda'(n-1)) \end{cases}.$$ 

For vectors (7) we define

$$\lambda_{\max}(|u\rangle) = \max\{\lambda : p_{\lambda} \neq 0\}.$$ 

One easily checks that for $k = 2, \ldots, n-1$

$$\lambda_{\max}(L_k|u\rangle) > \lambda_{\max}(|u\rangle)$$

hence $|u\rangle$ is not an eigenvector of $L_k$.

Let

$$\lambda_{\min}(|u\rangle) = \min\{\lambda : p_{\lambda} \neq 0\}.$$ 

For each partition $\lambda = (\lambda(2), \ldots, \lambda(n-1))$

$$\lambda_{\min}([L_1, L_{\lambda}]|w\rangle) < \lambda$$
\[ \lambda_{\text{min}} (L_1 |u\rangle - \psi_{1,n}(L_1) |u\rangle) \leq \lambda_{\text{min}}([L_1, L_{\lambda_{\text{min}}(|u\rangle)}] |w\rangle) < \lambda_{\text{min}}(|u\rangle). \]

It follows that if \( L_1 |u\rangle - \psi_{1,n}(L_1) |u\rangle \neq 0 \) it is not proportional to \(|u\rangle\). Thus if \( L_1 |u\rangle = \lambda |u\rangle \) then \( \lambda = \psi_{1,n}(L_1) \).

**Theorem 3.3.** Let \(|w\rangle\) be a universal Whittaker vector of a type \( \psi_{1,n} \).

1. For \( n = 3 \) there are no nontrivial Whittaker vectors in \( W_{\psi_{1,n}} \) of any type.

2. For \( n = 4 \) the subspace of all nontrivial Whittaker vectors of the type \( \psi_{1,n} \) in \( W_{\psi_{1,n}} \) is span by the family of vectors

\[
|w_{2}'\rangle = \sum_{k=0}^{l} \alpha_k L_2^{l-k} L_3^{2k} |w\rangle, \quad l \in \mathbb{N},
\]

\[
\alpha_k = -\frac{l+1-k}{4k} \psi_{1,4}(L_4) \alpha_{k-1}, \quad \alpha_0 \neq 0.
\]

There are no other nontrivial Whittaker vectors in \( W_{\psi_{1,n}} \) of any type.

**Proof.** By Lemma 3.4 it is enough to look for nontrivial solutions of the equation

\[ L_1 |u\rangle = \psi_{1,3}(L_1) |u\rangle. \]

For \( n = 3 \) the only possibility is \(|u\rangle = \sum p_n L_3^n |w\rangle\) for which

\[ (L_1 - \psi_{1,3}(L_1)) |u\rangle = -\sum n p_n \psi_{1,3}(L_3) L_2^{n-1} |w\rangle. \]

As the sum on the r.h.s is finite and non vanishing there are no nontrivial Whittaker vectors in \( W_{\psi_{1,3}} \) of any type.

For \( n = 4 \) one checks by explicit calculations that vectors (8) are Whittaker vectors of the type \( \psi_{1,4} \). We shall show that for any Whittaker vector of the type \( \psi_{1,4} \) the decomposition

\[ |u\rangle = \sum_{\lambda=(\lambda(2),\lambda(3))} p_\lambda L_\lambda |w\rangle. \]

contains a nonzero term with \( \lambda = (k, 0) \). By assumption

\[ (L_1 - \psi_{1,4}(L_1)) |u\rangle = \sum_{\lambda=(\lambda(2),\lambda(3))} p_\lambda [L_1, L_\lambda] |w\rangle = 0. \]

For each term one has

\[ [L_1, L_2^{\lambda(2)} L_3^{\lambda(3)}] |w\rangle = -\lambda(2) L_2^{\lambda(2)-1} L_3^{\lambda(3)+1} - 2\lambda(3) \psi_{1,n}(L_4)L_2^{\lambda(2)} L_3^{\lambda(3)-1} |w\rangle. \]

In order to achieve cancelation of terms on the r.h.s of (9) if the sum contains nonzero term with \( \lambda = (k, l) \) it has to contain non vanishing terms with \( \lambda' = (k-1, l+2) \) and \( \lambda'' = (k+1, l-2) \). It follows that the following terms must have nonzero coefficients

\[ L_2^{\lambda'} |w\rangle, \quad L_2^{\lambda''} L_3 |w\rangle. \]
The second case however cannot be realized as a non-vanishing term in the decomposition of a Whittaker vector of the type \( \psi_{1,4} \) since

\[
[L_1, L_2'' L_3] |w\rangle = -k'' L_2'' L_3 - 2\psi_{1,4}(L_4) L_2'' |w\rangle
\]

and the second term cannot be canceled on the r.h.s. of (9). (This implies in particular that in decomposition (9) \( \lambda(3) \) assumes only even values). It follows that \( |u\rangle \) contains at least one term with \( \lambda = (k, 0) \). Let \( \lambda_{\text{min}} = (k_{\text{min}}, 0) \) be the partition of this type with smallest \( k \). Then the vector

\[
|u'\rangle = |u\rangle - p_{\lambda_{\text{min}}} |w_k^2\rangle
\]

is a new Whittaker vector of the type \( \psi_{1,n} \) with \( k'_{\text{min}} > k_{\text{min}} \). Hence repeating the subtraction above a finite number of times one has to get a zero vector. Thus \( |u\rangle \) is a linear combination of vectors (8).

Construction (8) of Whittaker vectors can be generalized for \( n > 4 \) as follows

\[
|w_{2,n}^l\rangle = \sum_{k=0}^{l} \alpha_k L_{n-2}^{l-k} L_{n-1}^{2k} |w\rangle, \quad l \in \mathbb{N},
\]

\[
\alpha_0 \neq 0,
\]

\[
\alpha_{k+1} = -\frac{(n-3)(l-k)}{2(n-2)(k+1) \psi_{1,n}(L_n)} \alpha_k.
\]

Another generalization is given by

\[
|w_{l,n}^1\rangle = \sum_{k=l}^{n-1} \alpha_k L_k L_{n-1}^{k-l} |w\rangle, \quad 2 \leq l \leq n-2,
\]

\[
\alpha_l \neq 0,
\]

\[
\alpha_{k+1} = -\frac{k-1}{(n-2)(k+1-l) \psi_{1,n}(L_n)} \alpha_k, \quad l \leq k \leq n-3,
\]

\[
\alpha_{n-1} = -\frac{n-3}{(n-2)(n-l) \psi_{1,n}(L_n)} \alpha_{n-2}.
\]

Constructions above do not exhaust all possibilities. As an illustration we give two more examples for \( n = 5 \)

\[
|w_{2,3;5}^{1,1}\rangle = \left( L_2 L_3 - \frac{1}{3\mu} L_2 L_4 - \frac{1}{3\mu} L_3 L_4 + \frac{5}{3^3\mu^2} L_3 L_4^3 - \frac{2}{3^4\mu^3} L_4^5 \right) |w\rangle
\]

\[
|w_{2;5}^2\rangle = \left( L_2^2 - \frac{2}{3\mu} L_2 L_4 + \frac{2^2}{3^3\mu^2} L_2 L_4^2 + \frac{1}{3^2\mu^2} L_3 L_4^2 - \frac{4}{3^4\mu^3} L_3 L_4^4 - \frac{1}{3} L_4 + \frac{4}{3^6\mu^4} L_4^6 \right) |w\rangle
\]

where \( \mu = \psi_{1,5}(L_5) \). The dimension of the subspace of Whittaker vectors of the type \( \psi_{1,n} \) grows very fast with \( n \). A general discussion is rather involved and goes beyond the scope of this paper.

We close this section with the theorem characterizing \( V_{1,n} \) submodules. Its derivation parallels that of Theorem 2.4 of the previous section.
Note that by construction $zf$ where $f \in V$.

Our first aim is to find in $V$ representations of a form $f$. Let us denote by $V$ to the action defined by $V(f)$.

We consider the space $V$ of all anti-linear forms on $V$. It is a left $V$ module with respect to the action defined by $(L_n f) |u\rangle = f(L_{-n} |u\rangle)$, $(zf) |u\rangle = f(z |u\rangle)$.

Let us denote by $V^n_{c,\Delta}$ the space of all anti-linear forms on $V^n_{c,\Delta}$. There are two bases in $V^n_{c,\Delta}$ dual to the standard bases in $V^n_{c,\Delta}$:

\[ f^{[k_1,\ldots,k_n]}(L_{-k_1} \cdots L_{-k_n} |\Delta\rangle) = \delta^{n_1}_{m_1} \cdots \delta^{n_k}_{m_k}, \]
\[ f^{[1,\ldots,k_n]}(L_{-1} \cdots L_{-k} |\Delta\rangle) = \delta^{n_1}_{m_1} \cdots \delta^{n_k}_{m_k}. \]

Each form $f \in V^n_{c,\Delta}$ is uniquely determined by an infinite sequence $\{f_n\}_{n=0}^{\infty}$ of forms $f_n \in V^n_{c,\Delta}$ which can in order be decomposed in one of the dual bases. This yields two different representations of a form $f \in V^n_{c,\Delta}$ as the infinite series

\[ f = \sum_{n=0}^{\infty} \sum_{\sum i = n} C_{n_1,\ldots,n_k} f^{[k_1,\ldots,k_n]} |\Delta\rangle, \]
\[ f = \sum_{n=0}^{\infty} \sum_{\sum i = n} D_{n_1,\ldots,n_k} f^{[1,\ldots,k_n]} |\Delta\rangle. \]

Our first aim is to find in $V^n_{c,\Delta}$ all Whittaker vectors of the pair $\mathcal{V}_r \subset \mathcal{V}$ and the type $\psi_r = (\mu_r, \ldots, \mu_s, \ldots)$, $r \leq s \leq 2r$, where $s = \text{rank} \psi_r$. These are the forms $f_{\psi_r} \in V^n_{c,\Delta}$ satisfying

\[ L_k f_{\psi_r} = 0 \quad \text{for} \quad s \leq k, \]
\[ L_k f_{\psi_r} = \mu_k f_{\psi_r} \quad \text{for} \quad r \leq k \leq s. \]

Note that by construction $zf = cf$ for all $f \in V^n_{c,\Delta}$.
Theorem 4.1. A form \( f_{\psi_r} \in V^*_c,\Delta \) is a Whittaker vector of the pair \( V_r \subset V \) and the type \( \psi_r = (\mu_r, \ldots, \mu_s, \ldots) \) if and only if it is of the form

\[
f_{\psi_r} = \sum_{n_r-1, \ldots, n_1=0}^{\infty} A_{n_r-1, \ldots, n_1} f_{\psi_r}^{n_r-1, \ldots, n_1},
\]

where

\[
f_{\psi_r}^{n_r-1, \ldots, n_1} = \sum_{n_s, \ldots, n_r=0}^{\infty} \mu^{n_s}_s \ldots \mu^{n_r}_r f^{[ns_s, \ldots, nr_r, (r-1)n_{r-1}, \ldots, 1n_1]}
\]

and \( A_{n_r-1, \ldots, n_1} \) are arbitrary coefficients.

Proof. Let \( f_{\psi_r} \in V^*_c,\Delta \) be a form satisfying conditions (14), (15). Equations (14) imply that \( f_{\psi_r} \) vanishes on all basis vectors \( L_{-i_1} \ldots L_{-i_n} |\Delta\) involving at least one generator \( L_{-n} \) with \( n > s \). Hence \( f_{\psi_r} \in V^*_c,\Delta \) is of the following general form

\[
f_{\psi_r} = \sum_{n=0}^{\infty} \sum_{i_1=0}^{\infty} C_{n_s, \ldots, n_1} f^{[ns_s, \ldots, 1n_1]} = \sum_{n_s, \ldots, n_1=0}^{\infty} C_{n_s, \ldots, n_1} f^{[ns_s, \ldots, 1n_1]}.
\]

Calculating the left hand sides of equations (15) on the basis vectors \( L^{-m_1} \ldots L^{-m_1} |\Delta\) one gets

\[
L_i f_{\psi_r} (L^{-m_1} \ldots L^{-m_1} |\Delta) = f_{\psi_r} (L^{-i} L^{-m_1} \ldots L^{-m_1} |\Delta)
\]

\[
= \sum_{n_s, \ldots, n_1=0}^{\infty} C_{n_s, \ldots, n_1} f^{[ns_s, \ldots, 1n_1]} (L^{-n_s} \ldots L^{-n_1} |\Delta)
\]

\[
= C_{n_s, \ldots, n_1+1, \ldots, n_1}.
\]

Comparing with the right hand sides one gets

\[
C_{n_s, \ldots, n_{i+1}, \ldots, n_1} = \mu_i C_{n_s, \ldots, n_i, \ldots, n_1}
\]

for all \( n_1, \ldots, n_s \) and \( 0 < i < r \). The most general solution of the conditions above takes the form

\[
C_{n_s, \ldots, n_r, n_{r-1}, \ldots, n_1} = A_{n_r-1, \ldots, n_1} \mu^{n_s}_s \ldots \mu^{n_r}_r
\]

with arbitrary coefficients \( A_{n_r-1, \ldots, n_1} \).

In the case of BMT states it is more convenient to work in basis (11). By similar calculations one gets

Theorem 4.2. A form \( f_{\psi_{1,n}} \in V^*_c,\Delta \) is a Whittaker vector of the pair \( V_{1,n} \subset V \) and the type \( \psi_{1,n} \) if and only if it is of the form

\[
f_{\psi_{1,n}} = \sum_{m_2, \ldots, m_{n-1}=0}^{\infty} B_{m_2, \ldots, m_{n-1}} f_{\psi_{1,n}}^{m_2, \ldots, m_{n-1}},
\]

where

\[
f_{\psi_{1,n}}^{m_2, \ldots, m_{n-1}} = \sum_{m_1, m_n=0}^{\infty} \nu^{m_1}_1 \nu^{m_n}_n f^{[m_1, \ldots, m_{n-1} |m_n}_{mn}]
\]

\( \nu_1 = \psi_{1,n}(L_1) \), \( \nu_n = \psi_{1,n}(L_n) \) and \( B_{m_2, \ldots, m_{n-1}} \) are arbitrary coefficients.
Theorems 4.1 and 4.2 provide a general construction of all Gaiotto and BMT states in terms of finite or infinite combinations of the basic forms $f_{\psi_{r-1},n}^{m_2,\ldots,m_n-1}$ and $f_{\psi_{1,n}}^{m_2,\ldots,m_n-1}$, respectively. By the results of the Section 2 in the case of the high rank Gaiotto states all Whittaker modules of a given type are isomorphic. The states can be in principle further characterized by the transformation properties with respect to the lower generators. Our basic states are not convenient from this point of view. For the simplest Gaiotto states one has
\[
L_0 f_{\psi_r}^{0,\ldots,0} = \left( \Delta + \sum_{l=r}^s l \mu_l \frac{\partial}{\partial \mu_l} \right) f_{\psi_r}^{0,\ldots,0},
\]
\[
L_i f_{\psi_r}^{0,\ldots,0} = \sum_{l=r}^{s-i} (l - i) \mu_{i+l} \frac{\partial}{\partial \mu_l} f_{\psi_r}^{0,\ldots,0}, \quad 1 \leq i \leq r - 1.
\]
The corresponding formulae for generic basic forms contain several complicated terms and are not especially illuminating. For the basic BMT states the transformation rules are much less transparent even in the simplest case.

It is not clear which parts of the enormous spaces of states found in this section are relevant for the AGT relation and the CFT itself. We are not aware of any explicit construction of higher order Gaiotto states in this context. For the pair $\mathcal{V}_{1,n} \subset \mathcal{V}$ the only examples are the recently discovered BMT states [29] which in our notation are of the form
\[
\sum_{m_2,\ldots,m_{n-1}=0}^{\infty} \lambda_{m_2}^{n_2} \cdots \lambda_{n-1}^{m_{n-1}} f_{\psi_{1,n}}^{m_2,\ldots,m_{n-1}}.
\]

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