A PROPERTY OF THE BERWALD SCALAR CURVATURE

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Abstract. In this short paper, we prove that a Finsler manifold with vanishing Berwald scalar curvature has zero E-curvature. As a consequence, Landsberg manifolds with vanishing Berwald scalar curvature are Berwald manifolds. This improves a previous result in [5]. For (α, β)-metrics on manifold of dimension greater than 2, if the mean Landsberg curvature and the Berwald scalar curvature both vanish, the Berwald curvature also vanishes.

Introduction

Let (M, F) be an n-dimensional Finsler manifold. In this short paper, we only investigate regular Finsler manifolds, which means F defines on TM and has positive definite fundamental tensor. Let SM = \{F(x, y) = 1\} be the unit sphere bundle (or indicatrix bundle) with the natural projection π : SM → M. Let ω = F_y^i dx^i be the Hilbert form, which defines a contact structure on SM. The dual of ω with respect to the Sasaki-type metric on SM is the spray G. Let D be the contact distribution \{ω = 0\}. The Berwald curvature B as a part of the curvature endomorphism of the Berwald connection is divided into four parts. Along the spray direction B(G) = 0, however the transpose of B along ω gives twice the Landsberg curvature B^t(ω) = 2L. The remain part of B on D is in general not a symmetric endomorphism. B is totally symmetric on D when L = 0. M is called a Berwald manifold if B = 0.

Let E = F · trB be the mean Berwald curvature or the E-curvature. There are examples with vanishing E-curvature but which are not Berwald manifolds. In [5], we proved that L = 0 and E = 0 implies B = 0.

It is natural to consider the scalar e := trE, which will be called the Berwald scalar curvature. In general e is a function on SM, and the symmetric tensor E is not trace free. The main result of this paper is the following property about the Berwald scalar curvature.

Theorem 1. Let (M, F) be an n-dimensional Finsler manifold. If e = trE is a function on M, then E is trace free, i.e.,

$$E = \frac{1}{n-1}eh(\cdot, J\cdot),$$

where h is the angular metric, J the almost complex structure on D. In this case, the Finsler manifold has weak isotropic S-Curvature.

A direct consequence is that vanishing Berwald scalar curvature implies vanishing E-curvature. The following theorem improves the above mentioned result in [5].

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Theorem 2. Let \((M, F)\) be an \(n\)-dimensional Landsberg manifold. If the Berwald scalar curvature vanishes, then \(M\) is a Berwald manifold.

For Finsler manifolds with \((\alpha, \beta)\)-metrics, we obtain the following result.

Theorem 3. Let \(M\) be an \(n \geq 3\) dimensional manifold equipped an \((\alpha, \beta)\)-metric \(F = \alpha \phi(s)\), where \(s = \frac{\beta}{\alpha}\), \(\alpha\) is a Riemannian metric, \(\beta\) an one form on \(M\). If the mean Landsberg curvature \(J = 0\) and the Berwald scalar curvature \(e = 0\), then \(M\) is a Berwald manifold.

By the result in [16], we only need to prove Theorem 3 for the case \(\phi\) not a polynomial. By Theorem 1, \(M\) has zero \(E\)-curvature. For \((\alpha, \beta)\)-metrics on a manifold \(n \geq 3\), by a result in [3], vanishing \(E\)-curvature implies \(\beta\) has constant length with respect to the Riemannian metric \(\alpha\). In [16], it is proved that \((\alpha, \beta)\)-metrics with \(J = 0\) and \(\|\beta\|\alpha\) constant must be Berwald metrics. Therefore Theorem 3 is proved.

This paper contains two parts. In Sect. 1, we introduce some basic results of the Chern connection and the Berwald connection in Finsler geometry. Then we prove the main theorem in Sect. 2.

In this paper we adopt the index range

\[1 \leq i, j, k, \ldots \leq n, \quad 1 \leq \alpha, \beta, \gamma, \ldots \leq n - 1.\]

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1. Preliminary results

In this section, we would like to review some facts in Finsler geometry which will be used later.

Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. Let \(SM = \{F(x, y) = 1\}\) be the unit sphere bundle (or indicatrix bundle) with the natural projection \(\pi : SM \rightarrow M\). Let \(\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n-1}\}\) be a local adapted orthonormal frame with respect to the Sasaki-type Riemannian metric \(g_{T(SM)}\) on \(T(SM)\), where \(e_n = G\) is the spray. Let \(\theta = \{\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^{2n-1}\}\) be the dual frame, then \(\omega^n = F_y dx^i\) is the Hilbert form. \(\omega^n\) defines a contact structure on \(SM\). Let \(I\) be the trivial bundle generated by \(G\). Let \(D\) be the contact distribution \(\{\omega^n = 0\}\). We define an almost complex structure on \(SM\). Let \(\theta = \{\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^{2n-1}\}\) be the dual frame, then \(\omega^n = F_y dx^i\) is the Hilbert form. \(\omega^n\) defines a contact structure on \(SM\). Let \(l\) be the trivial bundle generated by \(G\). Let \(D\) be the contact distribution \(\{\omega^n = 0\}\). We define an almost complex structure on \(D\) by \(J = -\omega^n \otimes e_{n+\gamma} + \omega^{n+\gamma} \otimes e_\alpha\). Let \(F := V(SM)\) be the integrable distribution given by the tangent spaces of the fibers of \(SM\). Let \(p : T(SM) \rightarrow F\) be the natural projection. Then the tangent bundle \(T(SM)\) admits a splitting

\[T(SM) = I \oplus JF \oplus F =: H(SM) \oplus F.\]

The horizontal subbundle \(H(SM) = I \oplus JF\) is spanned by \(\{e_1, \ldots, e_n\}\) on which the Chern connection is defined. And \(\{e_{n+1}, \ldots, e_{2n-1}\}\) gives a local frame of \(F\).

Let \(\nabla^{\text{Ch}}\) be the Chern connection, i.e.,

\[\nabla^{\text{Ch}} : \Omega^*(SM; H(SM)) \rightarrow \Omega^{*+1}(SM; H(SM)).\]
It is proved in [4] that the symmetrization of Chern connection is just the Cartan connection $\nabla^{Ca}$. The difference between $\nabla^{Ca}$ and $\nabla^{Ch}$ will be referred as the Cartan endomorphism,

$$H = \nabla^{Ca} - \nabla^{Ch} \in \Omega^1(SM, \text{End}(H(SM))).$$

Set $H = H_{ij} \omega^j \otimes e_i$. By Lemma 3 and Lemma 4 in [4], $H_{ij} = H_{ji} = H_{ij}^\gamma \omega^{n+\gamma}$ has the following form under natural coordinate systems

$$H_{ij}^\gamma = -A_{ijk} u^p_i u^q_j u^k_\gamma,$$

where $A_{ijk} = \frac{1}{4} F[F^2]_{ij} e^k$ and $u^i_j$ are the transformation matrix from adapted orthonormal frames to natural frames.

Let $\omega = (\omega^i_j)$ be the connection matrix of the Chern connection with respect to the local adapted orthonormal frame field, i.e.,

$$\nabla^{Ch} e_i = \omega^i_j \otimes e_j.$$

**Lemma 1 ([1, 2, 6, 10]).** The connection matrix $\omega = (\omega^i_j)$ of $\nabla^{Ch}$ is determined by the following structure equations,

$$(1.1) \quad \begin{cases} d\vartheta = -\omega \wedge \vartheta, \\
\omega + \omega^t = -2H,
\end{cases}$$

where $\vartheta = (\omega^1, \ldots, \omega^n)^t$. Furthermore,

$$\omega^i_a = -\omega_a^i = \omega^{n+\alpha}, \quad \text{and} \quad \omega^n_a = 0.$$

**Remark 1.** In [4], we proved that the Chern connection is just the Bott connection on $H(SM)$ in the theory of foliation (c.f. [15]).

Let $R^{Ch} = (\nabla^{Ch})^2$ be the curvature of $\nabla^{Ch}$. Let $\Omega = (\Omega^i_j)$ be the curvature forms of $R^{Ch}$. From the torsion freeness, the curvature form has no pure vertical differential form

$$\Omega^i_j = \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l + P^i_{j\gamma} \omega^k \wedge \omega^{n+\gamma}.$$

The Landsberg curvature is defined as

$$\mathbf{L} := L^i_j \gamma \omega^{n+\gamma} \otimes \omega^j \otimes e_i = -P^i_{j\gamma} \omega^{n+\gamma} \otimes \omega^j \otimes e_i,$$

the mean Landsberg curvature is defined by

$$\mathbf{J} = \text{tr} \mathbf{L} = J^i_\gamma \omega^{n+\gamma} = -P^i_{j\gamma} \omega^{n+\gamma}.$$

If a Finsler manifold satisfies $P = 0$, $\mathbf{L} = 0$ or $\mathbf{J} = 0$, then it is called a Berwald, Landsberg or weak Landsberg manifold, respectively.

Another important linear connection in Finsler geometry is the Berwald connection. We will review some relations between the curvatures of the Chern connection and the Berwald curvatures. The Berwald connection is defined by $\nabla^{B} = \nabla^{Ch} + J^* \mathbf{L}$, where $J^*$ is dual of the almost complex structure $J$. Let $\tilde{\omega} = (\tilde{\omega}^i_j)$ denote the Berwald connection form, then

$$\tilde{\omega}^i_j = \omega^i_j - L^i_j_\alpha \omega^\alpha.$$
From the torsion freeness the Chern connection and the following well known formula (c.f. [1, 2, 6, 10])

\[ P^i_{\alpha k\gamma} = -H_{k\gamma|i} = -L_{ik\gamma}, \]

the Berwald connection is torsion free,

\[ d\omega^i = \omega^j \wedge \tilde{\omega}^i_j. \]

Let \( \tilde{\Omega}^i_{jk} \) be the curvature forms of the Berwald connection. By the torsion freeness of the Berwald connection, we have

\[ \tilde{\Omega}^i_{jk} = \frac{1}{2} \tilde{R}^i_{jkl} \omega^k \wedge \omega^l + \tilde{P}^i_{jkl} \tilde{\omega}^k \wedge \omega^l, \]

where \( \tilde{R}^i_{jkl} = -\tilde{R}^j_{ikl} \). By using Lemma 1, we have the formulae of the curvatures of the Berwald connection,

\[ \tilde{R}^\alpha_{\beta \gamma \mu} = R^\alpha_{\beta \gamma \mu} - (L^\alpha_{\beta \gamma} L^\mu_{\nu \mu} - L^\alpha_{\beta \mu} L^\mu_{\nu \gamma}) + (L^\alpha_{\beta \gamma} L^\mu_{\nu \mu} - L^\alpha_{\beta \mu} L^\mu_{\nu \gamma}), \]

\[ \tilde{P}^\alpha_{\beta \gamma \mu} = P^\alpha_{\beta \gamma \mu} + L^\alpha_{\beta \gamma \mu}, \quad \tilde{P}^\alpha_{\beta \gamma \mu} = 2L^\alpha_{\beta \gamma \mu}, \quad \tilde{P}^\alpha_{\beta \gamma \mu} = 0, \quad \tilde{P}^\alpha_{\beta \gamma \mu} = 0. \]

Using the explicit formulae of the connections and the curvature tensors under natural coordinate systems (cf. [1]. pp. 27-67), one finds that

\[ \tilde{P}^i_{jkl} = F \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} =: FB^i_{jkl} \]

is the usual Berwald curvature \( B \) of the spray \( G \). It is clear that the curvature \( P = 0 \) of the Chern connection if and only if \( B = 0 \). The mean Berwald curvature or the \( E \)-curvature is defined by

\[ E = E_{\gamma \mu} \omega^\gamma \otimes \omega^{n+\mu} = \text{tr} \tilde{P}. \]

Under the local adapted frame, the coefficients of the \( E \)-curvature is given by

\[ E_{\gamma \mu} = \tilde{P}^i_{j \gamma \mu} = \tilde{P}^\alpha_{\beta \gamma \mu} = P^\alpha_{\beta \gamma \mu} + L^\alpha_{\beta \gamma \mu} = P^\alpha_{\beta \gamma \mu} + J^\alpha_{\gamma \mu}. \]

In the following we mention some facts of the geometry of the unit sphere bundle of a Finsler manifold.

Let \( \pi : SM \to M \) be the unit sphere bundle. The tensor

\[ g^{T(SM)} = \sum_{i=1}^{n} \omega^i \otimes \omega^i + \sum_{\alpha=1}^{n-1} \omega^{n+\alpha} \otimes \omega^{n+\alpha} =: g + \dot{g} \]

gives raise a Sasaki-type Riemannian metric on \( SM \). Using the adapted frame and the Chern connection, the connection matrix of the Levi-Civita connection \( \nabla^{T(SM)} \) of \( g^{T(SM)} \) is given by (c.f. [6])

\[ \Theta = \begin{bmatrix} \omega^i + (H_{ij\gamma} + \frac{1}{2} R^\gamma_{n ij}) \omega^{n+\gamma} - \frac{1}{2} R^\alpha_{n ij} H_{ij\alpha} \omega^j - P^\alpha_{n ij} \omega^{n+\beta} + H_{\alpha\beta\gamma} \omega^{n+\gamma} \end{bmatrix}. \]
As the restriction the Levi-Civita connection $\nabla_{T(SM)}^p$ on $F$, $\nabla^F := p\nabla_{T(SM)}^p$ is the Euclidean connection of the bundle $(F, \hat{g})$. It is clear that along each fiber of $S_xM$, $x \in M$, $\nabla^F$ is just the Levi-Civita connection of the Riemannian manifold $(S_xM, \hat{g}_x)$.

For any $x \in M$, the fiber of $S_xM$ is the indicatrix of the Minkowski space $(T_xM, F_x)$, where $F_x := F|_{T_xM}$ for simplicity. By the discussion in our previous paper [5], $\hat{g}_x$ is just the centro-affine metric of $S_xM$, when it is considered as an affine hypersurface in $T_xM$. We define a connection on $F$ by $\nabla^{Ch} := \nabla^F + J \circ H \circ J$. Therefore, the connection forms of $\nabla^{Ch}$ are precisely given by

$$\nabla^{Ch}e_{n+\beta} = \omega_{\beta}^\alpha \otimes e_{n+\alpha}.$$ 

In fact $\nabla^{Ch}$ gives the affine connection along the fibers of $SM$.

2. A VERTICAL ELLIPTIC EQUATION OF THE S-CURVATURE

On a local coordinate chart $(U; x^i)$, let $dV_M = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ be any volume form on $M$. The following important function on $SM$ is well defined,

$$\tau = \ln \sqrt{\det g_{ij}} \sigma(x).$$

$\tau$ is called the distortion of $(M, F)$. $\tau$ is a very important invariant of the Finsler manifold, which is first introduced by Zhongmin Shen. One refers to [2, 10] for more discussion about the distortion $\tau$. The restriction of $\tau$ on each fiber of $SM$ is also critical for the investigation of the centro-affine differential geometry of the fiber. One consults [14] for more details.

Set $\eta = \text{tr}[H] \in \Omega^1(SM)$. It is referred as the Cartan-type form in [3]. The Cartan-type form has the following local formula

$$\eta = \sum_{i=1}^n H_{ii\gamma}^\gamma \omega^{n+\gamma} =: H_{\gamma}^{\gamma} \omega^{n+\gamma}.$$ 

By definition $\eta$ is the vertical differential of the distortion.

Lemma 2. The Cartan-type one form is the vertical differential of the distortion $\tau$.

$$d^V \tau = \eta,$$

where $d^V := \omega^{n+\alpha} \wedge \nabla^{Ch}_{e_{n+\alpha}}$, $\nabla^{Ch}$ is the dual connection of the Levi-Civita connection $\nabla^{T(SM)}$.

The following property of the Cartan-type form is critical for us.

Proposition 1 ([5]). The exterior differentiation of $\eta$ is given by

$$d\eta = -\text{tr}[R^{Ch}].$$

Then $d\eta$ has the local formula

$$(2.1) \quad d\eta = d(H_{\gamma}^{\gamma} \omega^{n+\gamma}) = -\frac{1}{2} R_{ijkl}^i \omega^j \wedge \omega^k - P_{i}^{i} \omega^k \wedge \omega^{n+\gamma}.$$ 

From Lemma 2, one has the following corollary.
Corollary 1. The differential of $\tau$ is given by
\begin{equation}
(2.2) \quad d\tau = \tau_i \omega^i + \eta = \tau_\alpha \omega^\alpha + \tilde{S} \omega^n + \eta,
\end{equation}
where we denote $\tau_i := e_i(\tau)$, and $\tilde{S} := \tau_n = e_n(\tau)$.

It is clear that $\tau_i := \tau_\alpha \omega^\alpha$ defines a section of the dual bundle of $J\mathcal{F}$. Therefore
\\
\begin{align*}
\tilde{\tau} := J^* \tau_i = \tau_n \omega^{n+\alpha}
\end{align*}

is a well defined section of the dual bundle of the vertical bundle $\mathcal{F}$.

In literature, $S := F\tilde{S} = G(\tau)$ defined on $TM_0$ is called the $S$-curvature of the Finsler manifold $(M, F)$. The $S$-curvature is also introduced by Zhongmin Shen. For more results related to the $S$-curvature, one refers to [2, 10, 11, 12, 13] and the references in them.

Taking the exterior differentiation of $(2.2)$, we obtain
\begin{equation}
(2.3) \quad 0 = d^2 \tau = d(d\tau_i \wedge \omega^i) + d\eta = d\tau_i \wedge \omega^i + \tau_i d\omega^i + d\eta
\end{equation}

where $\tau_i$ and $\tau_n$ denote the coefficients of the covariant differential of $\tau$ with respect to the Chern connection.

One easily verifies the following identities
\begin{equation}
(2.4) \quad \tau_{\n\alpha} := e_{\alpha, \mu} d\tau_n - \tau_{\n, \alpha} \omega^\mu = e_{\alpha, \mu} (d\tilde{S} - \tau_{\beta} \omega^\beta) = e_{\beta} (\tilde{S}) = \tilde{S}_{\alpha},
\end{equation}

\begin{equation}
\tau_{\n\alpha} := e_{\alpha, \mu} d\tau_n - \tau_{\n, \alpha} \omega^\mu = e_{\alpha, \mu} (\tilde{S}) + \tau_{\gamma} \omega^\gamma = \tilde{S}_{\gamma} = \tau_{\gamma} \omega^\gamma + \tau_{\gamma} \omega^\gamma
\end{equation}

where $\tau_{\alpha, \beta}$ denote the vertical coefficients of $\nabla_{\overline{C}} \tilde{\tau}$.

Plugging $(2.1)$ and $(2.4)$ into $(2.3)$, we have
\begin{equation}
(2.5) \quad 0 = -\frac{1}{2} (\tau_{\alpha\beta} - \tau_{\beta\alpha}) \omega^\alpha \wedge \omega^\beta + \tilde{S}_{\alpha} \omega^\alpha \wedge \omega^n + \tau_{\alpha n} \omega^n \wedge \omega^\alpha
\end{equation}

The following identities are derived from $(2.5)$
\begin{equation}
(2.6) \quad \tau_{\gamma, \mu} - \tilde{S}_{\gamma} \omega^\mu + P_{i \, \gamma}^{\mu} = 0,
\end{equation}

\begin{equation}
(2.7) \quad \tilde{S}_{\mu} + \tau_{\mu} + P_{i \, \mu}^{\mu} = 0.
\end{equation}

Substituting the formula of $\tau_{\mu}$ from $(2.7)$ into $(2.6)$, we obtain by using $(1.3)$
\begin{equation}
(2.8) \quad \tilde{S}_{\gamma, \mu} + \tilde{S}_{\gamma} \omega^\mu = P_{i \, \mu}^{\mu} - P_{i \, \gamma\mu}^{\mu} = E_{\gamma\mu}.
\end{equation}

Using natural frames, the formula $(2.8)$ is known as the relation between the $S$-curvature and the $E$-curvature as below
\begin{equation}
(2.9) \quad F \cdot S_{\nu \gamma} = E_{\nu \gamma}.
\end{equation}
Now we present a proof of Theorem 1.

**Proof of Theorem 1.** Using the Levi-Civita connection along each fiber of $SM$, the equation (2.8) is reformulated as

\[(2.10)\quad \text{Hess}_{\dot{g}} \tilde{S}_\gamma \delta_{\mu} + \sum_{\nu} H_{\gamma \nu \delta \mu} \tilde{S}_{\nu} + \tilde{S} \delta_{\gamma \mu} = E_{\gamma \mu},\]

where $\text{Hess}_{\dot{g}}$ denotes the Hessian operator on each fiber of $SM$ with respect to the metric $\dot{g}$. By taking trace, we obtain from (2.10) the following linear elliptic equation

\[(2.11)\quad \Delta \dot{g} \tilde{S} + \dot{g}(\eta, d^V \tilde{S}) + (n - 1) \tilde{S} = e.\]

In [5], we elaborated that $\eta = -(n - 1) \hat{T}$, where $\hat{T}$ is the Tchebychev form in centro-affine differential geometry. Therefore (2.11) along each fiber is

\[(2.12)\quad \Delta \dot{g} \tilde{S} - (n - 1) \dot{g}(\hat{T}, d^V \tilde{S}) + (n - 1) \tilde{S} = e.\]

Assume that $e$ is a function on $M$, then $f = \tilde{S} - \frac{1}{n-1} e$ solves the following equation along each fiber of $SM$

\[(2.13)\quad \Delta \dot{g} f - (n - 1) \dot{g}(\hat{T}, d^V f) + (n - 1) f = 0.\]

Let $S_x M$ and $S^*_x M$ be the indicatrices of the Minkowski spaces $(T_x M, F_x)$ and $(T^*_x M, F^*_x)$, respectively, for any $x \in M$. It is known that the Legendre transformation $\mathcal{L}$ gives a diffeomorphism between $S_x M$ and $S^*_x M$, which preserves the affine metrics and changes the sign of the Tchebychev forms (cf. [1]). By Satz 3.1 in [8], due to Blaschke and Schneider, the solutions of the equation (2.13) on $S^*_x M$ are of the form

\[(2.14)\quad \langle \xi_x, d^T^* M F^*_x \rangle |_{S^*_x M},\]

where $\xi_x = \xi_i(x) \frac{\partial}{\partial p_i} \in T^*_x M$ are constant co-vectors. The differential of the norm $F^*_x$ is simply given by

\[d^T^* M F^*_x = F^*_{p_i}(x, p) dp_i = \frac{1}{F^*_x(p)} g^{ij}(p) p_j dp^i.\]

Thus the solutions of (2.13) on $S_x M$ are given by

\[f = \mathcal{L}^* \langle \xi_x, d^T^* M F^*_x \rangle = \mathcal{L}^* \left[ \frac{1}{F^*_x(p)} \xi_i(x) g^{ij}(p) p_j \right] = \frac{1}{F^*_x(\xi_x y)} \xi_i(x) g^{ij}(\xi_x y)(\xi_x y)_j = \frac{1}{F^*_x(y)} \xi_i(x) y^j.\]

Therefore the S-curvature is weakly isotropic

\[(2.15)\quad S = F \tilde{S} = F \left( f + \frac{1}{n-1} e \right) = \frac{1}{n-1} Fe + \xi_i y^i.\]

By differentiation

\[(2.16)\quad S_{y^i y^j} = \frac{1}{n-1} e F^{-1} h_{ij}.\]

Recall (2.9), we complete the proof. □
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