T-MODEL STRUCTURES ON CHAIN COMPLEXES OF PRESHEAVES

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Abstract. A tensor model structure is constructed on the category of chain complexes of presheaves of $R$-modules for a sheaf of rings $R$ in a Grothendieck topos. If the topos has enough points, then the homotopy category is equivalent to the derived category. Some t-structures, generalizing the perverse t-structures, are constructed on the derived category.

1. INTRODUCTION

There is an injective and a projective model structure on the category of chain complexes of $R$-modules for a unital ring $R$ [15, 2.3]. The injective model structure extends to the category of chain complexes for any Grothendieck abelian category. This model structure fails to interact well with the tensor structure; there are too many cofibrant objects. On the other hand, the projective model structure is well-behaved with respect to the tensor product, but it does not readily generalize to chain complexes of sheaves. For certain ringed spaces, Mark Hovey has constructed a tensor model structure on the category of chain complexes of sheaves of $R$-modules [16]. See also Gillespie [12].

The first goal of this paper is to construct a tensor model structure on the category of chain complexes of sheaves of $R$-modules for any Grothendieck topos. We do this in two steps. First we generalize the projective model structure to the category of chain complexes of presheaves of $R$-modules. Then we localize this model structure with respect to the stalkwise homology isomorphisms.

The second goal of this paper is to study some t-structures on the derived category $D_R$. These t-structures “lift” to the model category of chain complexes of presheaves to form a t-model structure [5].

We now give a summary. In Section 2 we recall the definition of the derived category. In Section 3 we generalize the projective model structure to the category of chain complexes of presheaves of $R$-modules. The weak equivalences are presheaf homology-isomorphisms. In Section 4 we prove
that the projective model structure is tensorial. In Section 5 we define quasi-simplicial model structures and show that the projective model structure is quasi-simplicial.

In Section 6 we localize the projective model category with respect to the stalkwise homology isomorphisms. We call the resulting model structure the stalkwise model structure. It is a tensor model category. If the ringed topos has enough points, then the weak equivalences are exactly the sheaf homology isomorphisms. So under this assumption the homotopy category of the stalkwise model category is equivalent to the derived category $\mathcal{D}_R$.

In Section 7 we construct some families of t-structures on $\mathcal{D}_R$ and show that they lift to form t-model structures on the category of chain complexes. The t-structures we consider are “locally” the standard t-structures. The simplest class of t-structures are given by assigning to each point in the topos a cut-off value in $\mathbb{Z} \cup \{\pm \infty\}$. In particular, we generalize the perverse t-structures on $\mathcal{D}_R$ and show that they lift to t-model structures. In Section 8 we give an explicit description of $(\mathcal{D}_R)_{\geq 0}$ and $(\mathcal{D}_R)_{\leq 0}$ associated to particularly well-behaved t-structures.

In Section 9 we consider various examples and compare our model structures to the flat model structure and to the injective model structure. In an Appendix we recall, and slightly extend, Bousfield’s cardinality argument. This is needed in Section 7.

We assume the reader is familiar with the fundamentals of model category theory. See for example [13, 15].

2. The derived category

In this section some notation and terminology is introduced. Let $C$ be a (skeletally) small Grothendieck site. Let $E$ be the category of sheaves of sets on $C$, and let $P$ be the category of presheaves of sets on $C$. Let $i: E \to P$ be the inclusion functor. It has a left adjoint, the sheafification functor. We denote the sheafification functor by $L^2 = L \circ L$ [1 II.3.0.5]. Assume that $E$ has a set of isomorphism classes of points, and let $\text{pt}(E)$ denote this set.

Let $R$ be a sheaf of rings on $C$. Let $M$ denote the category of left $R$-modules in $E$, and let $N$ denote the category of left $iR$-modules in $P$. Both $M$ and $N$ are abelian closed tensor categories with units $R$ and $iR$, respectively. The functor $i$ induces an inclusion functor $i: M \to N$, and the functor $L^2$ induces a left adjoint sheafification functor $L^2: N \to M$.

For any object $C$ in $C$, let $RC_C$ denote the free $R$-module in $N$ generated by $C$ [1 IV.11.3.3]. There is a natural isomorphism

$$(2.1) \quad N(RC_C, X) \cong X(C).$$

Similarly, $L^2RC_C$ is the free $R$-module in $M$ generated by $C$. Let $\bullet$ be the terminal object in $C$. Then $R$ is isomorphic to $R_\bullet$. 

Definition 2.2. Let \( \text{ch}(\mathcal{N}) \) denote the category of chain complexes of presheaves of \( iR \)-modules on \( C \), and let \( \text{ch}(\mathcal{M}) \) denote the category of chain complexes of sheaves of \( R \)-modules on \( C \).

The categories \( \text{ch}(\mathcal{M}) \) and \( \text{ch}(\mathcal{N}) \) are abelian closed tensor categories.

Definition 2.3. A map \( f: X \to Y \) in \( \text{ch}(\mathcal{N}) \) is a **presheaf homology-isomorphism** if

\[
H_n(f): H_n(X) \to H_n(Y)
\]
is an isomorphism, for each \( n \in \mathbb{Z} \).

A map \( f: X \to Y \) in \( \text{ch}(\mathcal{N}) \) is a **sheaf homology-isomorphism** if the sheafification of the induced map on homology

\[
L^2H_n(f): L^2H_n(X) \to L^2H_n(Y)
\]
is an isomorphism, for each \( n \in \mathbb{Z} \).

Definition 2.4. A map \( f: X \to Y \) in \( \text{ch}(\mathcal{N}) \) is a **stalkwise homology-isomorphism** if \((L^2f)_p\) is a homology-isomorphism of chain complexes of \( R_p \)-modules for all points \( p \) in \( E \).

Let \( i \) also denote the inclusion functor \( i: \text{ch}(\mathcal{M}) \to \text{ch}(\mathcal{N}) \). A map \( f \) in \( \text{ch}(\mathcal{M}) \) is a presheaf homology-isomorphism, sheaf homology-isomorphism, or stalkwise homology-isomorphism if \( i(f) \) has this property.

If \( E \) has enough points, then a map \( f \) in \( \text{ch}(\mathcal{N}) \) is a stalkwise homology-isomorphism if and only if it is a sheaf homology-isomorphism [1, IV.6.4.1]

Definition 2.5. The localization of \( \text{ch}(\mathcal{M}) \) with respect to the class of all sheaf homology-isomorphisms is called the **derived category** of chain complexes of sheaves of \( R \)-modules on \( C \). It is denoted by \( \mathcal{D}_R \).

The unit \( X \to i \circ L^2X \) of the \((L,i)\)-adjunction is a sheaf homology-isomorphism, for all presheaves \( X \). So the derived category is equivalent to the localization of \( \text{ch}(\mathcal{N}) \) with respect to the class of all sheaf homology-isomorphisms. (The injective model structure, for example, gives the existence of the derived categories.)

In Section 3 a tensor model structure on \( \text{ch}(\mathcal{N}) \) is constructed such that the weak equivalences are the stalkwise homology-isomorphisms. Its homotopy category is equivalent to the derived category provided \( E \) has enough points. We first describe a preliminary model structure on \( \text{ch}(\mathcal{N}) \) with weak equivalences the smaller class of presheaf homology-isomorphisms.

3. **The projective model structure**

The projective model structure on the category of chain complexes of \( R \)-modules, for a ring \( R \), is generalized to the category of chain complexes of presheaves of \( R \)-modules, for a sheaf of rings \( R \).

We define the cofibrant generators.
Definition 3.1. Let $i_{C,n}$ be the map of chain complexes
\[
\cdots \rightarrow 0 \rightarrow R_C \rightarrow \cdots
\]
where $C \in C$ and the vertical identity map on $R_C$ is in degree $n$. Let $I$ be the set of all $i_{C,n}$ for $C \in C$ and $n \in \mathbb{Z}$.

Let $j_{C,n}$ be the map of chain complexes
\[
\cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots
\]
where $C \in C$ and the rightmost copy of $R_C$ is in degree $n$. Let $J$ be the set of all $j_{C,n}$ for $C \in C$ and $n \in \mathbb{Z}$.

The following model structure on $\text{ch}(N)$ is called the projective model structure on the category of chain complexes of presheaves of $R$-modules. Relative $I$-cell complexes are defined in [13, 10.5]. Relative $I$-cell complexes are defined in [13, 10.5].

Theorem 3.2. There is a proper cofibrantly generated model structure on $\text{ch}(N)$. The weak equivalences are presheaf homology-isomorphisms. The cofibrations are retracts of relative $I$-cell complexes. The fibrations are levelwise surjective maps of presheaves. The cofibrant generators, $I$, and the acyclic cofibrant generators, $J$, are small.

A map of presheaves $f : X \rightarrow Y$ is (levelwise) surjective if $f_n(C)$ is surjective as a map of sets, for all $C \in C$ and $n \in \mathbb{Z}$. This is equivalent to $f$ being epic (in the categorical sense) in the category $N$.

Proof. It suffices to check that: (1) inj ($I$) is equal to inj ($J$) $\cap W$, and (2) proj (inj ($J$)) is contained in $W$ [13, 11.3.1].

We first show that inj $J$ is equal to the class of surjective maps, and that inj $I$ is equal to the class of surjective maps that are also homology-isomorphisms. This gives (1) and the description of the fibrant objects.

A map from $j_{C,n-1}$ to $f : X \rightarrow Y$ is specified by $y \in Y_n(C)$, and a lift is given by an element $x \in X_n(C)$ such that $f_n(C)(x) = y$. Hence $f$ has the right lifting property with respect to $j_{C,n-1}$ if and only if $f_n(C)$ is surjective.

We denote the $n$-cycles of $Y$ at $C$, $\ker(Y_n(C) \rightarrow Y_{n-1}(C))$, by $Z(Y(C))_n$. Observe that a map $f : X \rightarrow Y$ has the right lifting property with respect to $i_{C,n}$ if and only if the canonical map from $X_{n+1}(C)$ to $W_n(C)$ in the pullback diagram
\[
\begin{array}{ccc}
W_n(C) & \longrightarrow & Y_{n+1}(C) \\
\downarrow & & \downarrow \\
Z(X(C))_n & \longrightarrow & Z(Y(C))_n
\end{array}
\]
is surjective.

Assume that \( f \) has the right lifting property with respect to \( i_{C,n-1} \). Then

\[
Z(X(C))_n \to Z(Y(C))_n
\]

is surjective since \( 0 \times Z(Y(C))_n \) is contained in \( W_{n-1}(C) \). Hence \( H_n(f)(C) \)

is surjective and \( W_n(C) \to Y_{n+1}(C) \) is surjective. The right lifting property

with respect to both \( i_{C,n-1} \) and \( i_{C,n} \) implies that \( f_{n+1}(C) \): \( X_{n+1}(C) \to Y_{n+1}(C) \)

is surjective. Hence \( \text{im}(X_{n+1}(C) \to X_n(C)) \to \text{im}(Y_{n+1}(C) \to Y_n(C)) \)

is surjective. Hence \( H_n(f)(C) \) is bijective.

We now prove the converse claim. Assume that \( f \) is a homology-isomorphism

of presheaves and \( f_n(C) \) is surjective, for all \( C \in \mathcal{C} \) and \( n \in \mathbb{Z} \). Given an

element \( x \in Z(X(C))_n \) and an element \( y \in Y_{n+1}(C) \) such that \( d(y) = f_n(x) \).

We need to show that the element \( (x, y) \in W_n(C) \) comes from an element

in \( X_{n+1}(C) \). By our assumption there exists an element \( x' \in X_{n+1}(C) \)

such that \( f_{n+1}(C)(x') = y \). Hence \( H_n(f) \) sends the cycle \( d(x') - x \) to zero.

So \( d(x') - x \) is a boundary since \( H_n(f) \) is injective. We conclude that

there exists an element \( x'' \in X_{n+1}(C) \) such that \( dx'' = x \). We get that \( f_{n+1}(C)(x'') = y \) is a cycle. Since \( H_{n+1}(f)(C) \)

is surjective there exists an element \( x'' \in X_{n+1}(C) \) such that \( dx'' = x \) and \( f_{n+1}(C)(x'') = y \). This shows

that the diagram lifts. So \( i_{C,n} \) has the left lifting property with respect to \( f \).

We now verify (2). Assume that \( f: X \to Y \) is a map in \( \text{proj} (\text{inj}(J)) \).

Consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f, \text{id}} & Y \oplus X \\
\downarrow{f} & & \downarrow{\text{id}_Y \oplus 0} \\
Y & = & Y.
\end{array}
\]

The rightmost vertical map is surjective. So by our assumption on \( f \) the

diagram lifts. Hence \( H_n(f) \) is injective for all \( n \).

Now consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{id}, 0} & X \oplus \text{Tot}(Y \oplus Y) \\
\downarrow{f} & & \downarrow{f \oplus g} \\
Y & = & Y
\end{array}
\]

where \( \text{Tot}(Y \oplus Y) \) is the total complex associated to the double complex

\[
\cdots \to Y_{n+1} \to Y_n \to Y_{n-1} \to \cdots \\
\downarrow{(-\text{id})^{n+1}} \quad \Downarrow{(-\text{id})^n} \quad \downarrow{(-\text{id})^{n-1}} \\
\cdots \to Y_{n+1} \to Y_n \to Y_{n-1} \to \cdots
\]
The map $g: \text{Tot}(Y \oplus Y) \to Y$ is given by the identity map on the upper copy of $Y$ in the double complex and by 0 on the lower copy of $Y$. Hence the rightmost map in the diagram is surjective. By our assumption on $f$ there is a lift in the diagram. Since the homology of $\text{Tot}(Y \oplus Y)$ is 0 we get that $H_n(f)$ is surjective for all $n$.

The verification of properness reduces to the category of chain complexes of $R(C)$-modules for a ring $R(C)$, for each $C \in \mathcal{C}$. A tedious verification shows that the pushout of a homology-isomorphism along a levelwise injective map of chain complexes is again a homology-isomorphism. A simpler verification shows that the pullback of a homology-isomorphism along a levelwise surjective map of chain complexes is again a homology-isomorphism. Both pushouts and pullbacks in $\text{ch}(\mathcal{N})$ are formed levelwise. Since the cofibrations are levelwise injective, and the fibrations are levelwise surjective, it follows that the model structure is proper.

The cofibrant generators are small since evaluation of presheaves at an object $C$ of $\mathcal{C}$ commutes with direct sum. □

We refer to Hovey for an alternative description of the cofibrant objects and the cofibrations [15, 2.3.6, 2.3.8-9]. Note that the isomorphism in equation (2.1) shows that the presheaf $R_C$ is a projective object in $\mathcal{N}$, for each object $C \in \mathcal{C}$. In fact all projective objects in $\mathcal{N}$ are retracts of direct sums of object of the form $R_C$, where $C \in \mathcal{C}$.

**Remark 3.3.** The projective model structure on $\text{ch}(\mathcal{N})$ is a stable model category. Hence its homotopy category is a triangulated category by [15, 7.1]. This triangulated category is typically different from the derived category of chain complexes of sheaves of $R$-modules on the site $\mathcal{C}$.

**Definition 3.4.** The unit interval complex, denoted $U$, consists of two copies of $R$ in degree 0, and one copy of $R$ in degree 1, the differential is the identity map on the first copy and minus the identity map on the second copy of $R$.

If $X$ is a cofibrant object, then a cylinder object for $X$ is given by $X \coprod X \to X \otimes U \to X$, where $U$ is the unit interval complex. Hence we get the following.

**Lemma 3.5.** If $X$ is a cofibrant object and $Y$ an arbitrary object in the projective model structure on $\text{ch}(\mathcal{N})$, then $\text{Ho}(\text{ch}(\mathcal{N}))(X,Y)$ is isomorphic to the group of chain homotopy classes of (degree 0) chain maps from $X$ to $Y$.

**Proof.** This follows since all objects are fibrant in the projective model structure on $\text{ch}(\mathcal{N})$. □

**Remark 3.6.** The projective model structure on $\text{ch}(\mathcal{N})$ is certainly well known. We give an alternative way to construct the projective model structure on chain complexes of $R$-modules that mesh better with the language used in other examples of model structures of this kind.
There is a tensor category of presheaves on $\mathcal{C}$ in the category of chain complexes of abelian groups. Let $A$ be a ring object (monoid) in this abelian category. Then the category of $A$-modules in the category of presheaves of chain complexes of abelian groups is equivalent to $A$-ch($\mathcal{N}$) (see Definition 4.7). The projective model structure on $A$-ch($\mathcal{N}$), under this identification, is inherited from the set of right adjoint functors

$$A$-ch($\mathcal{N}$) \to \text{ch}(\mathbb{Z})$$
given by evaluating at objects $C$ in $\mathcal{C}$ and composing with the forgetful functor from $A(C)$-modules to $\mathbb{Z}$-modules (at least one object in each isomorphism class in $\mathcal{C}$).

A similar model structure on the category of presheaves of simplicial sets has been given by Benjamin Blander [5]. See also Sharon Hollander's model structure on the category of stacks [14].

A more elaborate class of examples of this type are the strict model structures on diagram spectra. These examples are most naturally studied in an enriched setting. See [18], [20] and [21].

These examples of model categories of presheaves on a site, as well as ours, first become interesting after we suitably localize them. We consider localizations in Section 6.

4. Tensor structures

The category ch($\mathcal{N}$) is a symmetric closed tensor category. Let $\otimes_R$, or simply $\otimes$, denote the tensor product in ch($\mathcal{N}$). Let $F_R$, or simply $F$, denote the inner hom functor in ch($\mathcal{N}$). Our discussion of tensor model categories follows [21]. The next Lemma says that all cofibrant objects in the projective model structure are flat chain complexes.

**Lemma 4.1.** Let $K$ be a cofibrant object in ch($\mathcal{N}$), and let $f: X \to Y$ be a weak equivalence. Then $K \otimes f$ is also a weak equivalence in ch($\mathcal{N}$).

**Proof.** The complex $K$ is a retract of a relative $I$-cell complex $K'$. Let $f: X \to Y$ be a map of presheaves. There is a natural isomorphism

$$(R_C \otimes_R X)(D) = \oplus_{C(D,C)} X(D)$$

for any two objects $C$ and $D$ in $\mathcal{C}$. Hence $K'(D)$ is a directed colimit of bounded below complex of free $R(D)$-modules. The tensor product of a homology-isomorphism with a bounded below complexes of free modules is again a homology-isomorphism [23, 3.2, 5.8]. Homology commutes with directed colimits. Hence $K' \otimes_R X \to K' \otimes_R Y$ is a homology-isomorphism. A retract of a homology-isomorphism is again a homology-isomorphism, so $K \otimes f$ is a homology-isomorphism. \qed

The unit object for the tensor product on ch($\mathcal{N}$) is the chain complex with $R$ in degree 0. We denote this chain complex by $R$ or $R[0]$. 
Lemma 4.2. The unit object $R$ is cofibrant. More generally, $R_C$ is cofibrant, for $C \in \mathcal{C}$.

Proof. The map of chain complexes $0 \to R_C$ is the pushout of $i_U, \cdot 1$ along the map to the zero chain complex. Hence it is a cofibration. □

Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two maps. Then the pushout-product map is the canonical map

$$M(f_1, f_2) : \text{colim}(Y_1 \otimes X_2 \xrightarrow{f_1 \otimes 1} X_1 \otimes X_2 \xrightarrow{1 \otimes f_2} X_1 \otimes Y_2) \to Y_1 \otimes Y_2.$$

Definition 4.3. A model category with a tensor product satisfies the pushout-product axiom if $M(f_1, f_2)$ is a cofibration whenever $f_1$ and $f_2$ are cofibrations, and $M(f_1, f_2)$ is an acyclic cofibration if $f_1$ or $f_2$ in addition is a weak equivalence.

Lemma 4.4. The projective model structure on $\text{ch}(\mathcal{N})$ satisfies the pushout-product axiom.

Proof. The (acyclic) cofibrations are closed under retracts, transfinite compositions, and pushout. So it suffices to show that if $f_1$ and $f_2$ are maps in $I$, then $M(f_1, f_2)$ is a relative $I$-cell complex, and if $f_1$ is a map in $I$ and $f_2$ is a map in $J$, then $M(f_1, f_2)$ is a relative $J$-cell complex. Note that $R_{C_1} \otimes R_{C_2}$ is isomorphic to $R_{C_1 \times C_2}$, for objects $C_1, C_2 \in \mathcal{C}$. Denote this object by $R_{12}$ for brevity. We have that $M(i_{C_1,0}, i_{C_2,0})$ is the inclusion map

$$(\cdots \to 0 \to 0 \to R_{12} \oplus R_{12} \xrightarrow{a} R_{12} \to 0 \to \cdots) \to$$

$$(\cdots \to 0 \to R_{12} \xrightarrow{b} R_{12} \oplus R_{12} \xrightarrow{a} R_{12} \to 0 \to \cdots)$$

where $a$ is the fold map and $b$ is given by the identity map on the first factor and minus the identity map on the second factor. This is a relative $I$-cell complex. Similarly, the map $M(i_{C_1,0}, j_{C_2,0})$ is a relative $J$-cell complex. □

Definition 4.5. A model category with a tensor product satisfies the monoid axiom if $j \otimes X$ is a weak equivalence for every acyclic cofibration $j$ and any object $X$, and if pushout and transfinite directed composition of such maps is again a weak equivalence [21].

Lemma 4.6. The category $\text{ch}(\mathcal{N})$ with the projective tensor model structure satisfies the monoid axiom.

Proof. It suffices to prove this when $j$ is a map in the set $J$ of generators for the acyclic cofibrations. This follows since there is a natural isomorphism

$$(R_C \otimes_R X)(D) = \oplus_{C(D,C)} X(D)$$

for any two objects $C$ and $D$ in $\mathcal{C}$. □

Definition 4.7. Let $A$ be a monoid in $\text{ch}(\mathcal{N})$, i.e. $A$ is a differential graded $R$-algebra. Denote the category of $A$-modules in $\text{ch}(\mathcal{N})$ by $A \text{-ch}(\mathcal{N})$. 
Lemma 4.8. The category $A$-$ch(N)$ inherits a cofibrantly generated model structure from $ch(N)$ via the forgetful functor $A$-$ch(N) \to ch(N)$. If $A$ is a symmetric monoid, then $A$-$ch(N)$ is a tensor category (with tensor product over $A$) and the model structure on $A$-$ch(N)$ satisfies the pushout-product axiom and the monoid axiom.

Proof. This follows from Lemmas 4.1, 4.4, and 4.6 and [21, 3.1]. □

Lemma 4.9. Let $A$ be a symmetric monoid. Then the category of $A$-algebras inherits a model structure via the forgetful functor from $A$-algebras to $A$-$ch(N)$.

Proof. This follows from Lemmas 4.1, 4.2, 4.4, and 4.6 and [21, 3.1]. □

5. Quasi-simplicial model structures

We model theoretically enrich $ch(N)$ in simplicial sets. We define a weakening of the axioms for a simplicial model category [13, 9.1.6].

Definition 5.1. A quasi-simplicial model category is a model category $K$ that is a simplicial category (see [13, 9.1.2]) satisfying the axioms below. Map denotes the (based) simplicial mapping space, and $X \square S$ and $F_{\square}(S, X)$ are the tensor and cotensor of $X \in K$ by a simplicial set $S$, respectively.

weakM6: Let $S$ be the category of simplicial sets. Let $X, Y$ be objects in $K$ and let $S$ be an object in $S$. There is a natural isomorphism of simplicial sets

$$\text{Map}(X \square S, Y) \cong \text{Map}(X, F_{\square}(S, X)).$$

There is a natural isomorphism of sets

$$S(S, \text{Map}(X, Y)) \cong K(X \square S, Y).$$

M7: Let $i: A \to B$ be a cofibration in $K$ and $f: X \to Y$ a fibration in $K$. Then the map

$$i^* \times f_*: \text{Map}(B, X) \to \text{Map}(A, X) \times_{\text{Map}(A,Y)} \text{Map}(B, Y)$$

is a fibration of simplicial sets. If, in addition, $i$ or $f$ is a weak equivalence, then $i^* \times f_*$ is a weak equivalence.

Lemma 5.2. Let $K$ be a quasi-simplicial model category. Then the natural map $X \square * \to X$ (corresponding to $1_X$ under the adjunction in weakM6) is an isomorphism, for all $X \in K$.

Proof. The Yoneda lemma applied to the composition of isomorphisms

$$K(X, Y) \cong \text{Map}(X, Y)_0 \cong S(*, \text{Map}(X, Y)) \cong K(X \square *, Y)$$

gives the result. □
The second axiom has an equivalent formulations in terms of the tensor or cotensor functors instead of the simplicial mapping space \[13, 9.3.6\]. One implication of M7 is that \( X \square S \to X \square T \) is a weak equivalence in \( \mathcal{K} \) whenever \( X \) is cofibrant in \( \mathcal{K} \) and \( S \to T \) is a weak equivalence of simplicial sets. Combined with Lemma \[5.2\] this gives that the suspension of a cofibrant object \( X \) is equivalent to \( X \square S^1 \) and the loop of a fibrant object \( Y \) is equivalent to \( F \square (S^1, Y) \).

In a simplicial structure the last isomorphism in weakM6 is an isomorphism of simplicial sets. In a quasi-simplicial structure, unlike a simplicial structure, repeated applications of the tensor and cotensor functors need not respect the cartesian product (based: smash product) \[13, 9.1.11\].

Lemma 5.3. The projective model structures on \( \text{ch}(\mathcal{N}) \) is quasi-simplicial.

Proof. We make use of the Dold-Kan adjunction \[25, 8.4\]. We note that the inclusion functor, \( k \), from nonnegative chain complexes to unbounded chain complexes is left adjoint to the truncation functor, \( \tau_{\geq 0} \), given by sending

\[
\cdots \to X_2 \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots
\]

to

\[
\cdots \to X_2 \to X_1 \to \ker(X_0 \to X_{-1}) \to 0 \to \cdots .
\]

Let \( D \) denote the right adjoint of the normalized chain complex functor \( N \). Let \( S \) be a simplicial set. Let \( R[S] \) be the free simplicial \( R \)-module, and let \( N R[S] \) be the associated chain complex. We define the simplicial tensor to be \( X \otimes_R kN R[S] \) and the simplicial cotensor to be \( F(kN R[S], X) \), for \( X \in \text{ch}(\mathcal{N}) \). The simplicial hom functor, \( \text{map}(X, Y) \), is defined to be

\[
Dj \tau_{\geq 0} \Gamma(F(X, Y)),
\]

for \( X, Y \in \text{ch}(\mathcal{N}) \), where \( \Gamma \) denotes the global sections functor, \( j \) is the forgetful functor from the category of \( \Gamma \)-chain complexes to the category of chain complexes of abelian groups. Clearly the adjunctions in axiom weakM6 are satisfied.

Note first that the weakened version of axiom M6 still implies that axiom M7 has an alternative formulation as a pushout-product axiom in terms of the tensor \[13, 9.3.7\]. If \( i: S \to S' \) is an inclusion of simplicial sets, then \( kN R[i] : kN R[S] \to kN R[S'] \) is a cofibration in the projective model structure, and if \( i \) is a weak equivalence, then \( kN R[i] \) is a presheaf homology isomorphism in \( \text{ch}(\mathcal{N}) \). Hence axiom M7 follows from Lemma \[4.4\].

6. The stalkwise model structure

Given a (skeletally) small Grothendieck site \( \mathcal{C} \). The following model structure is called the stalkwise model structure on \( \text{ch}(\mathcal{N}) \).

Theorem 6.1. There is a proper quasi-simplicial cofibrantly generated model structure on \( \text{ch}(\mathcal{N}) \). The weak equivalences are stalkwise homology-isomorphisms and the cofibrations are retracts of relative I-cell complexes. With this model
structure the tensor closed structure on \( \text{ch}(\mathcal{N}) \) satisfies the pushout-product axiom and the monoid axiom.

Proof. The stalk functors from \( \text{ch}(\mathcal{N}) \) to the categories of chain complexes of \( R_p \)-modules respect both pushout and pullback squares, levelwise surjective maps, and levelwise injective maps. Furthermore, it takes homology-isomorphism of presheaves to homology-isomorphisms of \( R_p \)-modules.

The functor \((H_n)_p\), for each point \( p \) in \( \mathcal{E} \), is a homology theory in \( \text{ch}(\mathcal{N}) \) which commutes with arbitrary directed colimits. Hence we can Bousfield localize \( \text{ch}(\mathcal{N}) \) with respect to \((H_n)_p\)-equivalences for \( n \in \mathbb{Z} \) and \( p \) in \( \text{pt}(\mathcal{E}) \). This requires a cardinality argument involving the cardinality of \( R(C) \) for all \( C \in \mathcal{C} \). More details are included in Appendix A.

The model structure is proper and satisfies the pushout-product axiom and the monoid axiom. This follows from Theorem 3.2, Lemmas 4.4 and 4.6, and stalkwise verification of weak equivalences. The model structure is quasi-simplicial, with a quasi-simplicial structure given by the quasi-simplicial structure on the projective model structure (Lemma 5.3). This follows because the stalkwise model structure satisfies the pushout-product axiom.

\[ \square \]

Proposition 6.2. Assume that \( L^2 R_C \) is small in \( \mathcal{M} \), for all \( C \in \mathcal{C} \). Then there is a proper quasi-simplicial cofibrantly generated model structure on \( \text{ch}(\mathcal{M}) \), with cofibrant generators \( L^2 I \) and acyclic cofibrant generators \( L^2 J \). The weak equivalences are the stalkwise homology-isomorphisms and the cofibrations are retracts of relative \( L^2 I \)-cell complexes. The model structure satisfies the pushout-product axiom and the monoid axiom.

Proof. By our assumptions the sources of \( L^2 I \) and \( L^2 J \) are small. Relative \( L^2 J \)-cell complexes in \( \text{ch}(\mathcal{M}) \) are stalkwise homology-isomorphisms. Hence the result follows by Theorem 6.1 and [13, 11.3.2]. The model structure is proper, quasi-simplicial, and satisfies the pushout-product axiom and the monoid axiom. This is verified as in the proof of Theorem 6.1 \[ \square \]

If \( X \) is a Noetherian topological space and \( R \) is a ring of sheaves on \( X \), then \( L^2 R_U \) is small for all open subsets \( U \) of \( X \) (since direct sums of sheaves in the category of presheaves are themselves sheaves).

Proposition 6.3. Assume that \( L^2 R_C \) is small in \( \mathcal{M} \), for all \( C \in \mathcal{C} \). Let \( \text{ch}(\mathcal{M}) \) and \( \text{ch}(\mathcal{N}) \) both have the stalkwise model structure. The map of topoi \( i: \mathcal{M} \to \mathcal{N} \) and \( L^2: \mathcal{N} \to \mathcal{M} \) induces a Quillen equivalence of model categories \( i: \text{ch}(\mathcal{M}) \to \text{ch}(\mathcal{N}) \).

Proof. The sheafification functor respects cofibrations and weak equivalences. A map \( L^2 X \to Y \) is a stalkwise homology-isomorphism if and only if the adjoint map \( X \to i(Y) \) is a stalkwise homology-isomorphism for every \( X \in \text{ch}(\mathcal{N}) \) and \( Y \in \text{ch}(\mathcal{M}) \). \[ \square \]

Recall Definition 2.5 of the derived category \( \mathcal{D}_R \).
Proposition 6.4. Assume that the topos $\mathcal{E}$ has enough points. The homotopy category of $\text{ch}(\mathcal{N})$ with the stalkwise model structure is equivalent to $\mathcal{D}_R$ as tensor triangulated categories.

Proof. Since $\mathcal{E}$ has enough points the stalkwise homology-isomorphism and the sheaf homology-isomorphism coincide [1, IV.6.4.1].

Let $\text{ch}(\mathcal{N})_+$ be the full subcategory of $\text{ch}(\mathcal{N})$ consisting of chain complexes $\{X_n\}$ such that $X_n = 0$, whenever $n < 0$. The acyclic cofibrant generators $J'$ is the set of maps $j_{C,n}$, for $n \geq 0$ and $C \in \mathcal{C}$, and the cofibrant generators $I'$ is the set of maps $i_{C,n}$, for $n \geq 0$, together with the maps $0 \to R_C[0]$, for $C \in \mathcal{C}$.

Proposition 6.5. There is a cofibrantly generated model structure on $\text{ch}(\mathcal{N})_+$ such that the weak equivalences are presheaf homology isomorphisms. The cofibrations are retracts of relative $I'$-cell complexes. The fibrations are maps $f : X \to Y$ such that $f_n(C)$ is surjective, for all $C \in \mathcal{C}$ and $n \geq 1$.

Proof. This is essentially contained in the proof of Theorem 3.2. Note that the class inj $(0 \to R_C[0])$ in $\text{ch}(\mathcal{N})_+$ consists of maps that are surjective in degree 0 when evaluated at $C$.

Proposition 6.6. There is a cofibrantly generated model structure on $\text{ch}(\mathcal{N})_+$ such that the weak equivalences are stalkwise homology isomorphisms, and the cofibrations are retracts of relative $I'$-cell complexes.

Proof. This follows by localizing the model structure in Proposition 6.5 with respect to stalkwise equivalences. Note that $\text{ch}(\mathcal{N})_+$ is closed under directed colimits.

7. SOME T-STRUCTURES ON DERIVED CATEGORIES

We construct t-structures on the homotopy category of $\text{ch}(\mathcal{N})$ with the stalkwise model structures. These t-structures interact well with the model structure on $\text{ch}(\mathcal{N})$. More precisely, they all arise from a t-model structure on $\text{ch}(\mathcal{N})$. Homological grading of t-structures is used. So $\mathcal{D}_{\geq n}, \mathcal{D}_{\leq n-1}$ corresponds to $\mathcal{D}^{\leq -n}, \mathcal{D}^{\geq -n+1}$ in cohomological notation. Let $\mathcal{K}$ be a stable model category, with a t-structure on its triangulated homotopy category $\text{Ho}(\mathcal{K})$ [15, 7.1].

Definition 7.1. The class of (co-)n-equivalences in $\mathcal{K}$ is the class of maps $f$ in $\mathcal{K}$ such that the homotopy type of $\text{hofib}(f)$ is in $\text{Ho}(\mathcal{K})_{\geq n} \cap (\text{Ho}(\mathcal{K})_{\leq n-1})$ [8, 3.1].

We now make precise what we mean by lifting a t-structure on $\text{Ho}(\mathcal{K})$ to $\mathcal{K}$.

Definition 7.2. A (weak) t-model category is a proper quasi-simplicial stable model category $\mathcal{K}$ with functorial factorizations equipped with a t-structure on its homotopy category together with a functorial factorization
of maps in $\mathcal{K}$ into $n$-equivalences followed by co-$n$-equivalences, for each $n \in \mathbb{Z}$.

This is a weakening of the definition of a t-model structure in [8, 4.1]. We require the model structure to be quasi-simplicial instead of simplicial. This is a harmless weakening and all the results of [8] are still valid (with simplicial replaced by quasi-simplicial).

We are mainly interested in t-structures on $\text{ch}(\mathcal{N})$ with the stalkwise model structure, but we consider a more general framework. Let $\mathcal{K}$ be a proper quasi-simplicial cofibrantly generated stable cellular model category together with a t-structure on its homotopy category. Let $I$ be a set of cofibrant generators of $\mathcal{K}$ [12, 12]. We make the following assumptions:

1. the maps in $I$ have small sources;
2. the heart of the t-structure is the category of sheaves, $\mathcal{M}$, (or of presheaves, $\mathcal{N}$,) of $R$-modules for a ring of sheaves $R$ on a Grothendieck site $\mathcal{C}$; and
3. the heart functor, $\mathcal{H}: \mathcal{K} \to \mathcal{M}$ is $\sigma$-uniform, for a cardinal number $\sigma$ (see Definition A.3), respects sums, and directed colimits of relative $I$-cell complexes.

The topos $\mathcal{E}$ is assumed to have a set, $\text{pt}(\mathcal{E})$, of isomorphism classes of points. Let $\mathcal{D}$ denote the homotopy category of $\mathcal{K}$. Let $d: \text{pt}(\mathcal{E}) \to \mathbb{Z} \cup \{\pm \infty\}$ be a function. We construct t-model structures on $\mathcal{K}$ by shifting the original t-structure such that at each point of $\mathcal{E}$ in the isomorphism class $p \in \text{pt}(\mathcal{E})$ the shift is given by $\Sigma^{d(p)}$.

**Proposition 7.3.** There is a t-model structure (with simplicial relaxed to quasi-simplicial) on $\mathcal{K}$ such that

$$\mathcal{D}_{\geq 0} = \{X \mid (\mathcal{H}_{n_p}(X))_p = 0 \text{ for all } n_p < d(p)\}.$$

**Proof.** The associated class of $n$-equivalences, $W_n$, consists of all maps $f$ such that $(\mathcal{H}_{n_p}(f))_p$ is an isomorphism for all $n_p < d(p) + n$ and $(\mathcal{H}_{d(p)+n}(f))_p$ is a surjection if $|d(p)| < \infty$.

The pushout of a $W_n$-map along a cofibration is again a $W_n$-map. For each point $p$ in the topos $\mathcal{E}$ the functor $(\mathcal{H})_p$ from $\mathcal{K}$ to abelian groups respects sums and directed colimits of relative $I$-cell complexes. We can localize the category $\mathcal{K}$ with respect to the $\mathbb{Z}$-indexed homology theory whose $n$-th functor is

$$X \mapsto \bigoplus_p(\mathcal{H}_{d(p)+n}(X))_p$$

using Proposition A.9. See also [8, 7.5].

The full subcategory $\mathcal{D}_{\leq -1}$ is given by

$$\{Y \in \mathcal{D} \mid \mathcal{D}(X,Y) = 0 \text{ for } X \in \mathcal{D}_{\geq 0}\}.$$

We describe $\mathcal{D}_{\leq -1}$ more explicitly in Section 8 when $\mathcal{K}$ is $\text{ch}(\mathcal{N})$. 

□
Corollary 7.4. Let $Z$ be a subset of $pt(\mathcal{E})$. Then there is a proper quasi-simplicial model structure on $\mathcal{K}$ such that the weak equivalences are $\oplus_{p \notin Z}(H_*)^p$-isomorphisms and the cofibrations are retracts of relative $I$-cell complexes.

Proof. Let $d_Z: pt(\mathcal{E}) \to \mathbb{Z} \cup \{\pm \infty\}$ be the function defined by letting $d_Z(z) = \infty$, for $z \notin Z$, and $d_Z(z) = -\infty$, for $z \in Z$. The result follows from Proposition 7.3 applied to the function $d_Z$. □

We refine the t-structure in Proposition 7.3 by taking the structure of the ring $R$ into account. Let $d: \coprod_{p \in pt(\mathcal{E})} \text{spec } R_p \to \mathbb{Z} \cup \{\pm \infty\}$ be a function. We can localize the category $\mathcal{K}$ with respect to the $Z$-indexed homology theory whose $n$-th functor is $X \mapsto \oplus_{p, p}(H_{d(p, p)+n}(X))^p$.

Proposition 7.5. There is a t-model structure (with simplicial relaxed to quasi-simplicial) on $\mathcal{K}$ such that

$$\mathcal{D}_{\geq 0} = \{X \mid (H_{n_{p, p}}(f))^p = 0 \text{ for all } n_{p, p} < d(p, p)\}.$$

Proof. The corresponding class of $n$-equivalences is the class of maps $f$ in $\mathcal{K}$ such that

$$(H_{n_{p, p}}(f))^p$$

is an isomorphism for all $n_{p, p} < d(p, p) + n$, and a surjection for $n_{p, p} = d(p, p) + n$ if $|d(p, p)| < \infty$. The result follows from Proposition A.9 since the class of $n$-equivalences is closed under pushouts along cofibrations and the sources of the cofibrant generators are small. □

If $d$ is constant on each spec $R_p$, then Proposition 7.5 reduces to Proposition 7.3. We consider another t-model structure on $\mathcal{K}$. Let

$$d: \coprod_{p \in pt(\mathcal{E})} \text{spec } R_p \to \mathbb{Z} \cup \{\pm \infty\}$$

be a function.

Proposition 7.6. There is a t-model structure (with simplicial relaxed to quasi-simplicial) on $\mathcal{K}$ such that

$$\mathcal{D}_{\geq 0} = \{X \mid (H_{n_{d, p}}(X))^p \otimes_{R_p} R_p/p = 0 \text{ for all } n_{d, p} < d(p, p)\}.$$

Proof. Note that $- \otimes_{R_p} R_p/p$ is exact on the category of $R_p$-modules since $R_p/p$ is a field. The corresponding class of $n$-equivalences is the class of maps $f: X \to Y$ such that

$$(H_{n_{d, p}}(f))^p \otimes_{R_p} R_p/p$$

is an isomorphism, for $n_{d, p} < d(p, p)$, and an surjection for $n_{d, p} = d(p, p)$ if $|d(p, p)| < \infty$. The t-model structure is obtained from Proposition A.9 since the sources of the cofibrant generators are small and the class of $n$-equivalences is closed under pushouts along cofibrations. □
Example 7.7. Propositions 7.5 and 7.6 apply to the category \( \text{ch}(\mathcal{N}) \) with the stalkwise model structure and the standard \( t \)-structure on its homotopy category. The heart valued homology functor associated to the standard \( t \)-structure on the derived category \( \mathcal{D} \) is the usual homology of a chain complex. It respects sums and directed colimits of relative \( I \)-cell complexes.

Let \( A \) be a monoid in \( \text{ch}(\mathcal{N}) \) such that the sheaf valued homology is zero in negative degrees. The assumption on \( A \) gives that there is a standard \( t \)-structure on the homotopy category of \( A \)-\( \text{ch}(\mathcal{N}) \). Propositions 7.5 and 7.6 apply to the category of \( A \)-modules in \( \text{ch}(\mathcal{N}) \) with the standard \( t \)-structure. See Definition 4.7.

A class of \( t \)-structures on \( \text{ch}(\mathcal{N}) \) with the projective model structure can be constructed using another technique. Let \( I' \) be a subset of \( I \) such that whenever \( i_{C,m} \in I' \), then \( i_{C,m} \in I' \) for all \( m \leq n \). We associate to \( I' \) a class of maps closed under (nonnegative) suspensions:

\[
W(I') = \{ f \mid H_{m-1}(f)(C) \text{ is an isomorphism in } \mathcal{N} \text{ and } H_m(f)(C) \text{ is a surjection in } \mathcal{N}, \text{ whenever } i_{C,m} \in I' \}.
\]

Lemma 7.8. There is a \( t \)-model structure on \( \text{ch}(\mathcal{N}) \) with the projective model structure such that the associated class of \( 0 \)-equivalences is \( W(I') \).

Proof. This follows from the proof of Theorem 3.2 and [8, 4.13]. \( \Box \)

8. A DESCRIPTION OF \( \mathcal{D}_{\leq 0} \) FOR THE DERIVED CATEGORY

Recall that if \( \mathcal{E} \) has enough points, then the homotopy category of \( \text{ch}(\mathcal{N}) \) with the stalkwise model structure is equivalent to the derived category, \( \mathcal{D} \).

We describe the full subcategory, \( \mathcal{D}_{\leq 0} \), of \( \mathcal{D} \) for \( t \)-structures obtained from functions \( d: \text{pt}(\mathcal{E}) \to \mathbb{Z} \cup \{\pm \infty\} \) such that \( d^{-1}([n, \infty]) \) is an open subset of \( \text{pt}(\mathcal{E}) \), for every \( n \in \mathbb{Z} \cup \{\pm \infty\} \). We first recall some terminology and a Lemma.

The support of an object \( X \) in \( \mathcal{E} \) is defined to be

\[
\text{sup}(X) = \{ p \in \text{pt}(\mathcal{E}) \mid X_p \neq \emptyset \}.
\]

Note that if \( X \to Y \) is a map in \( \mathcal{E} \), then \( \text{sup}(X) \subset \text{sup}(Y) \). Recall that the topology on \( \text{pt}(\mathcal{E}) \) is generated by a basis of open sets consisting of \( \text{sup}(S) \), for all subobjects \( S \) of the terminal object \( \bullet \) of \( \mathcal{E} \) [1 IV.7.1.7]. Neighborhoods of points are defined in [1 IV.6.8].

Lemma 8.1. Let \( X \) be an object in \( \mathcal{E} \) and let \( p \) be a point in \( \mathcal{E} \). Suppose given an element \( x \in X_p \) and an open subset \( U \) of \( \text{pt}(\mathcal{E}) \) containing \( p \). Then there exists an object \( C \in \mathcal{C} \) with support contained in \( U \) and a map \( C \to X \) in \( \mathcal{E} \) such that \( x \) is in the image of \( C_p \to X_p \).

In other words, \( p \) has a neighborhood with support in \( U \).

Proof. By definition of the topology on \( \text{pt}(\mathcal{E}) \) there is a subobject \( S \) of \( \bullet \) such that \( p \in \text{sup}(S) \subset U \). The stalk, \( S_p = \bullet \), is the colimit of \( S(C) \)
for neighborhoods \((C, c \in C_p)\) of \(p\). Hence there exists a neighborhood \((C, c \in C_p)\) of \(p\) such that \(x\) in the image of \(C_p \to X_p\) and \(S(C) \neq \emptyset\) (so there is a map \(C \to S\) in \(E\)). Since \(\text{sup}(C) \subset \text{sup}(S)\) the claim follows. \(\square\)

**Proposition 8.2.** Assume that \(E\) has enough points. Let \(d: pt(E) \to \mathbb{Z} \cup \{\pm \infty\}\) be a function such that \(d^{-1}([n, \infty])\) is an open subset of \(pt(E)\), for every \(n \in \mathbb{Z} \cup \{\pm \infty\}\). Then there is a t-model structure on \(\text{ch}(N)\) (with simplicial relaxed by quasi-simplicial) such that

\[
\mathcal{D}_{\geq 0} = \{X \mid (\mathcal{H}_{n_p}(X))_p = 0 \text{ for all } n_p < d(p)\}
\]

and

\[
\mathcal{D}_{\leq 0} = \{X \mid (\mathcal{H}_{n_p}(X))_p = 0 \text{ for all } n_p > d(p)\}.
\]

**Proof.** Except for the description of \(\mathcal{D}_{\leq 0}\) the result follows from Proposition 7.3.

We construct an explicit truncation functor. For each object \(C \in C\) let \(n(C) \in \mathbb{Z} \cup \{\pm \infty\}\) be the largest number such that \(n(C) \leq d(p)\), for all \(p \in \text{sup}(C)\). Define \(X_{\geq 0}\) to be the complex such that \((X_{\geq 0})_k(C) = X_k(C)\) for \(k > n(C)\), \(\ker(X_{n(C)}(C) \to X_{n(C)-1}(C))\) for \(k = n(C)\), and \(0\) for \(k < n(C)\). There is a canonical inclusion map \(X_{\geq 0} \to X\) and we denote the cokernel by \(X_{\leq -1}\). The truncation functors give well defined functors in the homotopy category \(\mathcal{D}\). The quotient map \(X \to X_{\leq -1}\) is a fibration (between fibrant objects) since it is levelwise surjective. Since \(X_{\geq 0}\) is its fiber

\[
X_{\geq 0} \to X \to X_{\leq -1} \to \Sigma X_{\geq 0}
\]
gives a triangle in the homotopy category [15 6.2.6, 6.3, 7.1]. The map \(X_{\geq 0} \to X\) is a stalkwise equivalence if and only if \(X \in \text{ch}(N)_{\geq 0}\), and \(X \to X_{\leq -1}\) is a stalkwise homology isomorphism if and only if \(X \in \mathcal{D}_{\leq -1}\) by Lemma 8.1. Both \(X \mapsto X_{\geq 0}\) and \(X \mapsto X_{\leq -1}\) are idempotent functors, and \((X_{\geq 0})_{\leq -1} = (X_{\leq -1})_{\geq 0} = 0\). Hence \(\mathcal{D}(X_{\geq 0}, Y_{\leq 1}) = 0\), for all \(X, Y \in \mathcal{D}\). \(\square\)

**Corollary 8.3.** The heart of the t-structure in Proposition 8.2 is given by

\[
\{X \mid (\mathcal{H}_{n_p}(X))_p = 0 \text{ for all } n_p \neq d(p)\}.
\]

**Remark 8.4.** The assumption on \(d\) in Proposition 8.2 is not optimal. There might be more general functions \(d\) that have the same description of \(\mathcal{D}_{\leq 0}\). For example if \(d^{-1}([n, \infty])\) is closed (instead of open), for all \(n \in \mathbb{Z} \cup \{\pm \infty\}\), then Proposition 8.2 is still valid (make an explicit construction of the truncation \(X \to X_{\leq -1}\)).

9. **Examples and comparisons**

**Example 9.1** (Rings). Let \(C\) be the one morphism site. Then a ringed topos is the category of sets together with a ring \(R\), and \(\mathcal{M} = \mathcal{N}\) is the category of \(R\)-modules. The projective and the stalkwise model structures on \(\text{ch}(\mathcal{M})\) coincide. The localization in Corollary 7.4 has been constructed by Neeman for the one morphism site [19 3.3].
Let $R$ be a Noetherian ring. Stanley has constructed the t-structures in Proposition 7.5 on the full subcategory of $D_R$ consisting of complexes whose homology groups are finitely generated in each degree and bounded above and below [24]. He also shows that there are no other t-structures on this full subcategory of $D$ [24, 5.3].

**Example 9.2** (Perverse t-structures). We consider the category of $R$-modules for a ringed space $(S, R)$. A space is said to be sober if all closed irreducible sets have a generic point. Let $k: S \to S_{\text{sob}}$ be the universal inclusion into a sober space. The points of $S_{\text{sob}}$ correspond to closed irreducible subsets of $S$, and the map $k$ is given by sending a point $s \in S$ to the closure of $s$ in $S$. The set of (isomorphism classes of) points of $(S, R)$ is the space $S_{\text{sob}}$ [1, IV.7.1.6].

We now assume that $S = S_{\text{sob}}$ and compare our t-structures to the perverse t-structures introduced by Beilinson, Bernstein, Deligne, and Gabber [4]. They consider a nonempty finite partition $\{S_a\}_{a \in A}$ of $S$ into locally closed sets, together with a function $p: A \to \mathbb{Z}$, called the perversity function. A locally closed set is an intersection of an open and a closed set. The t-structure associated to a perversity function $p$ is given by $D_{\geq 0} = \{X \mid H_n(i^*_{S_a}X) = 0 \text{ for } n < p(a), a \in A\}$, for the locally closed sets $S_a$ in $S$ [4, 2.2.1]. This agrees with $\{X \mid H_n(X) = 0 \text{ for } n < p(a), q \in S_a, a \in A\}$.

Given a perversity function $p$ there is associated a corresponding function $d_p: S \to \mathbb{Z} \cup \{\pm \infty\}$, defined by letting

$$d_p^{-1}([n, \infty)) - d_p^{-1}([n + 1, \infty]) = \cup_{a \in p^{-1}(n)} S_a.$$

The perverse t-structure associated to $p$ agrees with the t-structure in Proposition 7.3 for the associated function $d_p$. In particular, the perverse t-structures on $D_R$ lift to t-model structures on $\text{ch}(\mathcal{M})$.

**Example 9.3** (Flat model structure). Let $(S, R)$ be a ringed space, and assume that $(S, R)$ has finite hereditary global dimension [16, 3.1]. Under these assumptions Mark Hovey has constructed a tensorial model structure on $\text{ch}(\mathcal{M})$, called the flat model structure [16, 3.2]. The assumptions are satisfied if $S$ is a finite-dimensional Noetherian space [16, 3.3]. Hovey constructs a cofibrantly generated model structure on $\text{ch}(\mathcal{M})$ with weak equivalences the stalkwise homology-isomorphisms. The fibrations are maps that are levelwise surjective, as presheaves, and whose kernels are complexes of flasque sheaves [16, 3.2].

We compare our model structure to his. Assume that $S = S_{\text{sob}}$ and that $(S, R)$ has finite hereditary global dimension. Let $\text{ch}(\mathcal{N})$ have the stalkwise model structure and $\text{ch}(\mathcal{M})$ the flat model structure. We claim that $L^2: \text{ch}(\mathcal{N}) \to \text{ch}(\mathcal{M})$ is a Quillen left adjoint to the forgetful functor $i: \text{ch}(\mathcal{M}) \to \text{ch}(\mathcal{N})$. The functor $L^2$ respects cofibrations since $L^2$ respects colimits and $L^2$ of the cofibrant generators of the stalkwise model structure
are among the cofibrant generators of the flat model structure \[16, 1.1, 3.2\]. In addition \(L^2\) respects weak equivalences. So \((L^2, j)\) is a Quillen adjunction. This is a Quillen equivalence since \(L^2X \to Y\) is a sheaf homology-isomorphism in \(\text{ch}(\mathcal{M})\) if and only if the adjoint map \(X \to i(Y)\) is a sheaf homology-isomorphism in \(\text{ch}(\mathcal{N})\), for \(X \in \text{ch}(\mathcal{N})\) and \(Y \in \text{ch}(\mathcal{M})\).

**Example 9.4** (Injective model structure). The injective model structure on \(\text{ch}(\mathcal{M})\) (or \(\text{ch}(\mathcal{N})\)) has sheaf homology isomorphisms as weak equivalences and levelwise injections as cofibrations. For a discussion of this model structure see \[15, 2.3.13\]. The fibrant objects are chain complexes of injective sheaves that are \(K\)-injective in the sense of Spaltenstein \[23, 1.1\]. Assume that \(\mathcal{E}\) has enough points. Then the adjoint pair \((L^2, i)\) gives a Quillen equivalence between \(\text{ch}(\mathcal{N})\) with the stalkwise model structure and \(\text{ch}(\mathcal{M})\) with the injective model structure. This follows as in the previous example because \(L^2\) respects injections of presheaves. In particular, the class of fibrations for the stalkwise model structure on \(\text{ch}(\mathcal{N})\) contains the class of fibrations for the injective model structure on \(\text{ch}(\mathcal{M})\) (and \(\text{ch}(\mathcal{N})\)).

**Example 9.5** (Stable homotopy category). We give an application of Proposition \[7.3\] to a category which is not a derived category. The heart of the stable homotopy category \(\mathcal{D}\) with the t-structure given by Postnikov sections is equivalent to the category of abelian groups. Hence Proposition \[7.3\] gives a twisted variant of the Postnikov t-structure. For each rational prime \(p\) let \(N_p \in \mathbb{Z} \cup \{\pm\infty\}\) and let \(N_0 \in \mathbb{Z} \cup \{\pm\infty\}\) be greater or equal to \(N_p\) for all primes \(p\). Then the associated full subcategory, \(\mathcal{D}_{\geq 0}\) (the connective spectra), of \(\mathcal{D}\) consists of spectra \(X\) such that

\[
(H_{n_p}(X))_p = 0
\]

whenever \(n_p < N_p\), for all primes \(p\), and \(H_{n_0}(X) \otimes \mathbb{Q} = 0\) whenever \(n_0 < N_0\).

**Example 9.6** (Quasicoherent sheaves). Let \((S, \mathcal{O}_S)\) be a scheme. There is a full abelian subcategory of \(\mathcal{M}\) consisting of quasi-coherent \(\mathcal{O}_S\)-modules. Typically, \(\mathcal{O}_U\) is not quasi-coherent for an open subset \(U\) of \(S\). So we cannot follow Chapter \[8\] and give a projective model structure on the category of chain complexes of quasi-coherent \(\mathcal{O}_S\)-modules.

We can construct t-model structures on the category of chain complexes of quasi-coherent presheaves using the techniques of Chapter \[7\] and a proper cofibrantly generated model structure given by Mark Hovey for certain schemes \[16, 2.4, 2.5\]. The proof of Lemma \[5.3\] shows that this model structure is quasi-simplicial (the complex associated to a simplicial set are chain complexes of free \(\mathcal{O}_S\)-modules).

With some assumptions on \(S\) we can inherit a t-structure on the derived category of quasi-coherent \(\mathcal{O}_S\)-modules from a t-structure on \(\mathcal{D}\mathcal{O}_S\). Assume that \(S\) is a finite dimensional Noetherian scheme. Since \(S\) is quasi-compact and quasi-separated the derived category of chain complexes of quasi-coherent \(\mathcal{O}_S\)-modules is a full subcategory of the derived category of
T-MODEL STRUCTURES ON CHAIN COMPLEXES OF PRESHEAVES

Moreover, our assumptions guarantee that the objects of this full subcategory are exactly complexes with quasi-coherent homology [2, p.191]. Hence the t-structures on \( D_{\mathcal{O}_S} \) constructed in Proposition 7.5 restrict to give t-model structures on the derived category of chain complexes of quasi-coherent \( \mathcal{O}_S \)-modules. See also the preprint by Roman Bezrukavnikov [3].

**Example 9.7 (Pro-chain complexes).** A t-model structure on \( \text{ch}(\mathcal{N}) \) is well suited to give a model structure on the category of pro-chain complexes of presheaves of \( R \)-modules. For example there is a proper stable tensor model structure on \( \text{pro-ch}(\mathcal{N}) \) such that the levelwise t-structure on the triangulated homotopy category of pro-ch(\( \mathcal{N} \)) has the property that the intersection

\[
\bigcap_{n \in \mathbb{Z}} \text{Ho}(\text{pro-ch}(\mathcal{N}))_{\geq n}
\]

consists of objects isomorphic to the 0-object [7]. Recent work by the author on a general local cohomology theory makes use of model structures on categories of pro-chain complexes of presheaves of \( R \)-modules [10].

**Appendix A. Bousfield’s cardinality argument**

Bousfield’s cardinality argument is used to localize model categories [6]. We give an extension of this result. Let \( K \) be a cofibrantly generated model category. Let \( I \) be a set of cofibrant generators. We assume that the sources of the maps in \( I \) are small.

**Definition A.1.** Let \( X \) be an \( I \)-cell complex. The cardinality of the set of cells in \( X \) is denoted \( \sharp X \). Let \( i: A \to X \) be a relative \( I \)-cell complex. The cardinality of the set of relative \( I \)-cells in the relative cell complex \( i \) is denoted \( \sharp(X, A) \).

**Definition A.2.** Let \( h \) be a functor from \( K \) to the category of sets. The functor \( h \) is said to satisfy the **colimit axiom** if for all relative \( I \)-cell complexes \( A \to X \)

\[
\operatorname{colim}_a h(X_a) \to h(X)
\]

is a bijection where the colimit is over all relative sub \( I \)-cell complexes \( i_a: A \to X_a \) of \( i \) such that \( \sharp(X_a, A) \) is finite.

The assumption that \( \sharp(X_a, A) \) is finite can be relaxed. We restrict to the less general version as that suffices for our examples.

**Definition A.3.** Let \( \sigma \) be a cardinal number. We say that a functor \( h \) from relative \( I \)-cell complexes to sets is **\( \sigma \)-uniform** if the cardinality of \( h(X) \) is less or equal to \( \sigma \times \sharp X \) for all \( I \)-cell complexes \( X \).

Given two sets of functors \( \{i_a\} \) and \( \{s_b\} \) from \( K \) to the category of sets, we define a class of maps in \( K \) depending on \( \{i_a\} \) and \( \{s_b\} \).

**Definition A.4.** Let \( E \) denote the class of all maps \( f: X \to Y \) such that \( i_a(f) \) is injective and \( s_b(f) \) is surjective, for all \( a \) and \( b \).
Note that the class $E$ is closed under composition and retract but that it need not satisfy the two-out-of-three property. A typical example of a class $E$ is the class of $n$-equivalences associated to a $t$-structure on the homotopy category of $\mathcal{K}$ (when $\mathcal{K}$ is a stable model category).

We want to produce a (functorial) factorization of a map in $\mathcal{K}$ as a map in $C \cap E$ followed by a map that has the right lifting property with respect to all maps in $C \cap E$.

We say that $f: A \to X$ is an I-cell complex pair if $A$ is an I-cell complex and $f$ is a relative I-cell complex.

**Definition A.5.** Let $E_\sigma$ denote the class of all I-cell complex pairs $X \to Y$ with $i_Y \leq \sigma$ such that $i_a(X) \to i_y(Y)$ is injective and $s_b(X) \to s_b(Y)$ is surjective, for all $a$ and $b$. Let $E'$ denote the class of all I-cell complex pairs $X \to Y$ such that $i_a(X) \to i_y(Y)$ is injective and $s_b(X) \to s_b(Y)$ is surjective, for all $a$ and $b$. Let $E''$ denote the class of all relative I-cell complexes $X \to Y$ such that $i_a(X) \to i_y(Y)$ is injective and $s_b(X) \to s_b(Y)$ is surjective, for all $a$ and $b$.

The class $E_\sigma$ is skeletally small for each cardinal $\sigma$, but $E'$ and $E''$ need not be skeletally small. In the next Lemma one ought to be careful about the meaning of intersection sub-cell complexes. We follow Hirschhorn and assume that the model category is cellular [13, 12].

**Lemma A.6.** Let $\mathcal{K}$ be a cellular model category. Let $\sigma$ be an infinite cardinal number, and let $\{i_a\}_{a \in A}$ and $\{s_b\}_{b \in B}$ be $\sigma$-uniform functors from $\mathcal{K}$ to the category of sets. Assume that $A$ and $B$ have cardinality not greater than $\sigma$.

Let $f: A \to X$ be an I-cell complex pair with $A \neq X$. Assume that $f$ is in $E'$. Then there is an I-cell subcomplex $B$ of $X$ such that $\exists B \leq \sigma$, $B \not\subset A$, and $B \cap A \to B$ is in $E'$.

**Proof.** We construct an increasing sequence

$$B_0 \subset B_1 \subset B_2 \subset \cdots$$

of I-cell subcomplexes of $X$ such that:

- $B_0 \not\subset A$.
- whenever $i_1, i_2 \in i_a(B_n \cap A)$ map to the same element in $i_a(B_n)$, then they map to the same element in $i_a(B_{n+1} \cap A)$
- the set $s_b(B_n)$ maps to the image of $s_b(B_{n+1} \cap A)$ in $s_b(B_{n+1})$.

We choose some finite subcomplex $B_0$ of $X$ that is not contained in $A$. We can do this since $A \neq X$ and the gluing map from any I-cell to $A$ factors through a finite I-cell subcomplex of $A$ since the sources of the maps in $I$ are small.

Assume that $B_n$ has been constructed. We construct $B_{n+1}$. Let $i_1, i_2 \in i_a(B_n \cap A)$ be two elements that map to the same element in $i_a(B_n)$. By our assumption on $f$ the two elements are sent to the same element under $i_a(B_n \cap A) \to i_a(A)$. The colimit axiom for $i_a$ implies that there is a finite
relative $I$-cell complex $B_n \to I_{x_1, x_2}$ in $X$ such that the two elements map to the same element in $i_a(I_{x_1, x_2} \cap A)$.

Similarly, for every element $y \in s_b(B_n)$ there is a finite relative $I$-cell complex $B_n \to S_y$ in $X$ such that the image of $y$ in $s_b(S_y)$ is in the image of $s_b(S_y \cap A)$. Now let

\[ B_{n+1} = B_n \cup_{i_1, i_2} I_{i_1, i_2} \cup_y S_y \]

where the sum is over $i_1, i_2 \in i(B_n \cap A)$ and $y \in s_b(B_n)$, for all $a$ and $b$. This complex satisfies the conditions in the list above.

Now let $B$ be the union of all the $B_n$. The colimit axiom gives that $B \cap A \to B$ is in $E_\sigma$. The assumption that $i_a, s_b$ are $\sigma$-uniform, and that the cardinality of $A$ and $B$ are not greater than $\sigma$ give that $\#B \leq \sigma$. □

Lemma A.7. The class $\text{inj} E'$ is equal to the class $\text{inj} E_\sigma$.

Proof. Let $f : X \to Y$ be a map in $E'$. The trick is to use Lemma A.6 to write $f$ as a transfinite composition of maps in $E_\sigma$. There is a transfinite sequence $X_\lambda$ such that:

- $X_\lambda \to X_{\lambda+1}$ is in $E_\sigma$
- if $\lambda$ is a limit ordinal, then $X_\lambda = \cup_{\ell < \lambda} X_\ell$
- if $X_\lambda$ is strictly contained in $X$, then $X_{\lambda+1}$ is strictly larger than $X_\lambda$.

Lemma A.6 implies that we must have that $X_\lambda = X$ for some $\lambda$. □

Lemma A.8. If $E''$ is closed under pushout along cofibrations in $K$, then $\text{inj} E' = \text{inj} E''$.

Proof. This is a consequence of [13, Prop. 13.2.1]. □

Proposition A.9. There is a functorial factorization of the maps in $K$ with an $I$-cell complex source as a map in $E'$ followed by a map in $\text{inj} E'$. Moreover, if $E'$ is closed under pushout along cofibrations in $K$, then there is a functorial factorization of any map in $K$ as a map in $E''$ followed by a map in $\text{inj} E''$.

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