A LEVEQUE-TYPE INEQUALITY ON THE RING OF $p$-ADIC INTEGERS

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Abstract. We derive an inequality on the discrepancy of sequences on the ring of $p$-adic integers $\mathbb{Z}_p$ using techniques from Fourier analysis. The inequality is used to obtain an upper bound on the discrepancy of the sequence $\alpha_n = na + b$, where $a$ and $b$ are elements of $\mathbb{Z}_p$. This is a $p$-adic analogue of the classical LeVeque inequality on the circle group $\mathbb{R}/\mathbb{Z}$.

1. Introduction

The theory of equidistribution of sequences modulo one was initiated by Hermann Weyl in 1916. Since then, it has spurred a lot of interest in many areas of mathematics, including number theory, harmonic analysis, and ergodic theory. The standard reference in this subject is Kuipers and Niederreiter [7].

Equidistribution of sequences on the ring of $p$-adic integers was previously studied in [1, 2, 3]. In particular, Cugiani in [3] defines equidistribution and shows that the sequence $na + b$ is equidistributed if $a$ is a unit. Beer does a quantitative analysis in [1] and [2]. Our aim is to derive a LeVeque-type inequality on the discrepancy of a finite sequence using Fourier analysis.

Let $|\cdot|_p$ denote the $p$-adic absolute value on $\mathbb{Q}_p$, and let

$$\mathbb{Z}_p = \{ x \mid |x|_p \leq 1 \},$$

be the ring of $p$-adic integers. Any element of $\mathbb{Z}_p$ can be given a unique canonical expansion of the form $x = a_0 + a_1p + a_2p^2 + \ldots$, where the $a_i$ are elements of $\{0, 1, 2, \ldots, p-1\}$ (see for example [5, 6]).

For $k \geq 0$, and $a \in \mathbb{Z}_p$, we denote by

$$D(a, 1/p^k) = \{ x \mid |x - a|_p \leq 1/p^k \} = a + p^k\mathbb{Z}_p,$$

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a disc of radius $1/p^k$ centered at $a$. Note that $\mathbb{Z}_p$ can be written as the union of $p^k$ disjoint discs of the form

$$\mathbb{Z}_p = \bigcup_{j=0}^{p^k-1} D(j, 1/p^k).$$

Hence, it is natural to define a notion of equidistribution using such sets.

**Definition 1.** A sequence $\{\alpha_n\}$ is said to be equidistributed in $\mathbb{Z}_p$ if for every $a$ in $\mathbb{Z}_p$ and every $k \in \mathbb{N}$, we have

$$\lim_{N \to \infty} \left| \frac{D(a, 1/p^k) \cap \{\alpha_1, ..., \alpha_N\}}{N} - \frac{1}{p^k} \right| = 0.$$

That is, the proportion of the first $N$ elements of $\{\alpha_n\}$ lying in a disc $D(a, 1/p^k)$ is equal to its measure in the limit of large $N$, and this holds true for all such discs.

This definition of equidistribution in $\mathbb{Z}_p$ was first given by Cugiani in [3], where propositions 4 and 10 were also proved. The details are also given in Kuipers and Niederreiter [7]. One also wants to measure how well a sequence distributes itself. To this end, we define the notion of discrepancy to quantify the idea that some sequences are better equidistributed than others.

**Definition 2.** The discrepancy of a finite sequence $\{\alpha_1, \alpha_2, ..., \alpha_N\}$ in $\mathbb{Z}_p$ is

$$D_N = \sup_{a \in \mathbb{Z}_p, k \in \mathbb{N}} \left| \frac{D(a, 1/p^k) \cap \{\alpha_1, ..., \alpha_N\}}{N} - \frac{1}{p^k} \right|.$$

Some elementary arguments show that

$$\frac{1}{N} \leq D_N \leq 1.$$

The main aim of this paper is to prove a Fourier analytic upper bound on the discrepancy of a set of $N$ elements $\{\alpha_1, \alpha_2, ..., \alpha_N\}$ in $\mathbb{Z}_p$.

Let $\mathbb{Z}(p^\infty)$ denote the Prüfer $p$-group, the group of all $p$-th power roots of unity in $\mathbb{C}$. Suppose that $\zeta \in \mathbb{Z}(p^\infty)$ has order $p^n$, and let $x \in \mathbb{Z}_p$ have the canonical expansion $x = a_0 + a_1 p + a_2 p^2 + ... + a_n p^n + ...$.

Then we interpret the notation $\zeta^x$ as

$$\zeta^x = \zeta^{a_0 + a_1 p + a_2 p^2 + ... + a_{n-1} p^{n-1}}.$$

Every element of $\mathbb{Z}(p^\infty)$ has finite order, and we denote the order of $\zeta \in \mathbb{Z}(p^\infty)$ by $\|\zeta\|$. 
Theorem 1 (Main Theorem). The discrepancy of a finite sequence \( \{\alpha_1, ..., \alpha_N\} \) in \( \mathbb{Z}_p \) is bounded by

\[
D_N \leq C(p) \left( \sum_{\zeta \in \mathbb{Z}(p^\infty) \setminus \{1\}} \frac{1}{\|\zeta\|^3} \left| \frac{1}{N} \sum_{n=1}^{N} \zeta^{\alpha_n} \right|^2 \right)^{\frac{1}{4}},
\]

where \( C(p) \) is a constant dependent on \( p \).

As an example application of Theorem 1, we have the following corollary

Corollary 2. The sequence \( na + b \) where \( a \) is a unit in \( \mathbb{Z}_p \) has discrepancy

\[
D_N = O \left( N^{-1/2} \right).
\]

Some quantitative results on the discrepancy of \( p \)-adic sequences were done by Beer in [1] and [2]. In particular, the author proves in [1] that the discrepancy of the sequence \( na + b \) with \( a \) a unit is exactly equal to \( D_N = N^{-1} \), the best possible.

It is not surprising that the LeVeque type inequality gives us a weaker bound, as this is the case in the classical setting on \( \mathbb{R}/\mathbb{Z} \). Montgomery in [8] provides a detailed discussion and considers some examples. In particular, the sequence \( n \theta \) where \( \theta = \frac{1 + \sqrt{5}}{2} \) has discrepancy \( D_N \ll \log(N)/N \), whereas the use of the LeVeque inequality gives only \( D_N \ll N^{-2/3} \).

Our paper is structured as follows. In section 2 we set up the relevant Fourier analysis that is required for our calculations. We prove the main theorem in section 3. We analyze the quantitative behavior of the linear sequence \( \alpha_n = na + b \) in section 4 and prove Corollary 2.

2. Fourier analysis on \( \mathbb{Z}_p \)

If \( G \) is a compact abelian group, then the set of all continuous group homomorphisms (or characters) from \( G \) to the multiplicative unit circle \( \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} \) forms a discrete group under multiplication, the Pontryagin dual group \( \hat{G} \) (see for example [9]). Note that \( \mathbb{Z}_p \) is a compact abelian group. The next lemma states that the dual group \( \hat{\mathbb{Z}}_p \) of \( \mathbb{Z}_p \) is isomorphic to the Prüfer \( p \)-group \( \mathbb{Z}(p^\infty) \). The result is known, but we include a proof due to the lack of a suitable reference.

**Lemma 3.** For each \( \zeta \in \mathbb{Z}(p^\infty) \), the map \( x \mapsto \zeta^x \) is a character of \( \mathbb{Z}_p \). Moreover, the map

\[
\Psi : \mathbb{Z}(p^\infty) \rightarrow \hat{\mathbb{Z}}_p \quad \zeta \mapsto (x \mapsto \zeta^x)
\]
is an isomorphism from the Prüfer $p$-group $\mathbb{Z}(p^{\infty})$ to the Pontryagin dual group of $\mathbb{Z}_p$.

**Proof.** It is easily shown that the map $x \mapsto \zeta^x$ is a character of $\mathbb{Z}_p$. To show the injectivity of $\Psi$, suppose that $\zeta_1^x = \zeta_2^x$ for all $x$ in $\mathbb{Z}_p$. Then picking $x = 1$, we get $\zeta_1 = \zeta_2$.

We need argue that $\Psi$ is surjective. Let $\gamma$ be in the dual group of $\mathbb{Z}_p$. Since $\gamma(0) = 1$ and $\gamma$ is continuous, there exists a disc of radius $1/p^n$ centered at zero $D = p^n\mathbb{Z}_p$, such that $|\gamma(x) - \gamma(0)| < 1$ for all $x$ in $D$, and we can pick a smallest $n$ such that this is true. Moreover, since $D$ is a subgroup of $\mathbb{Z}_p$, we must have that the image $\gamma(D)$ is a subgroup of $\mathbb{T}$.

Note that there does not exist any non-trivial subgroup of $\mathbb{T}$ satisfying the condition $|x - y| < 1$ for all elements $x$ and $y$ in the subgroup. Hence, we conclude that $\gamma(D) = \{1\}$.

Now suppose that $\gamma(1) = \zeta = e^{2\pi i \theta}$ for some $\theta$ in $[0, 1)$. Then, $\gamma(p^n) = \gamma(1)^{p^n} = e^{2\pi i \theta p^n} = 1$ or $p^n \theta \in \mathbb{Z}$. We conclude that $\theta = m/p^n$, where $p \nmid m$ by the minimality of $n$.

For any integer value $k$, we have $\gamma(k) = \gamma(1)^k = \zeta^k$. This completely determines $\gamma$, since we can write $\mathbb{Z}_p$ as the union of $p^n$ disjoint balls $\mathbb{Z}_p = \bigcup_{k=0}^{p^n-1} (k + p^n \mathbb{Z}_p)$ and for any $x \in \mathbb{Z}_p$ we have $x = k + p^n y$, and $\gamma(x) = \gamma(k)$. We conclude that

$$\gamma(x) = \zeta^x,$$

for all $x$ in $\mathbb{Z}_p$ where $\zeta = e^{2\pi i m/p^n}$. \hfill \Box

Using Lemma 3, we shall express the Fourier series of any $f \in L^1(\mathbb{Z}_p)$ in terms of the elements of $\mathbb{Z}(p^{\infty})$. As a compact group there exists a normalized Haar measure $\mu$ on $\mathbb{Z}_p$ (see for example [4]). Let $f \in L^1(\mathbb{Z}_p)$. The Fourier coefficients of $f$ are given by

$$\hat{f}(\zeta) = \int_{\mathbb{Z}_p} f(x) \zeta^{-x} d\mu,$$

and the Fourier inversion formula gives

$$f(x) = \sum_{\zeta \in \mathbb{Z}(p^{\infty})} \hat{f}(\zeta) \zeta^x,$$

whenever $\sum_{\zeta \in \mathbb{Z}(p^{\infty})} |\hat{f}(\zeta)| < \infty$.

A Weyl type criterion holds for equidistribution in $\mathbb{Z}_p$. We state it here in terms of the elements of $\mathbb{Z}(p^{\infty})$, although it holds for a more general class of Riemann integrable functions on $\mathbb{Z}_p$ (see [7]).
Proposition 4 (Weyl’s Criterion). A sequence \( \{ \alpha_n \} \) is equidistributed in \( \mathbb{Z}_p \) if and only if for every non-trivial \( \zeta \) in \( \mathbb{Z}(p^\infty) \) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \zeta^{\alpha_n} = 0.
\]

We denote by \( \mathcal{X}_{D(a,R)}(x) \) the characteristic function of the disc \( D(a, R) \) centered at \( a \) of radius \( R \). We have the following change of variables formula, the proof of which is elementary and we omit.

Proposition 5. Let \( f : \mathbb{Z}_p \to \mathbb{C} \) be an integrable function. Then

\[
\int_{\mathbb{Z}_p} \mathcal{X}_{D(a,1/p^k)}(x) f(x) d\mu(x) = \frac{1}{p^k} \int_{\mathbb{Z}_p} f(a + p^k x) d\mu(x).
\]

We use Proposition 5 to calculate the Fourier coefficients of the characteristic function of a disc.

Lemma 6. The Fourier coefficients of the characteristic function \( \mathcal{X}_{D(a,1/p^k)}(x) \) are

\[
\hat{\mathcal{X}}_{D(a,1/p^k)}(\zeta) = \begin{cases} 
\zeta^{-a} p^{-k} & \text{if } \|\zeta\| \leq p^k, \\
0 & \text{if } \|\zeta\| > p^k.
\end{cases}
\]

Proof of Lemma 6. Suppose that \( \|\zeta\| \leq p^k \), then \( \zeta^{p^k x} = 1 \) for all \( x \) in \( \mathbb{Z}_p \). Therefore, we have

\[
\int_{\mathbb{Z}_p} \mathcal{X}_{D(a,1/p^k)}(x) \zeta^{-x} d\mu(x) = p^{-k} \int_{\mathbb{Z}_p} \zeta^{-(a+p^k x)} d\mu(x) = \zeta^{-a} p^{-k} \int_{\mathbb{Z}_p} \zeta^{-p^k x} d\mu(x) = \zeta^{-a} p^{-k}.
\]

On the other hand suppose \( \|\zeta\| > p^k \), and let \( \omega = \zeta^{p^k} \). Then \( \|\omega\| = \|\zeta\|/p^k > 1 \) and hence

\[
\int_{\mathbb{Z}_p} \mathcal{X}_{D(a,1/p^k)}(x) \zeta^{-x} d\mu(x) = \zeta^{-a} p^{-k} \int_{\mathbb{Z}_p} \zeta^{-p^k x} d\mu(x) = \zeta^{-a} p^{-k} \int_{\mathbb{Z}_p} \omega^{-x} d\mu(x) = 0.
\]

\[\square\]
3. Proof of the main theorem

Let \( \{\alpha_1, \alpha_2, ..., \alpha_N\} \) be a finite sequence in \( \mathbb{Z}_p \). Define the function
\[
f(x, y) = \left\lfloor \frac{\{\alpha_1, \alpha_2, ..., \alpha_N\} \cap D(x, |y|_p)}{N} \right\rfloor - |y|_p,
\]
where \( D(x, |y|_p) \) is a disc of radius \( |y|_p \) centered at \( x \). The discrepancy of the points \( \{\alpha_1, ..., \alpha_N\} \) is then
\[
D_N = \sup_{x, y \in \mathbb{Z}_p} |f(x, y)|.
\]
We suppress the \( p \) in \( | \cdot |_p \) as it would be clear from the context. We can also write
\[
f(x, y) = \frac{1}{N} \left( \sum_{n=1}^{N} \mathcal{X}_{D(x, |y|)}(\alpha_n) \right) - |y|
= \frac{1}{N} \left( \sum_{n=1}^{N} \mathcal{X}_{D(\alpha_n, |y|)}(x) \right) - |y|.
\]

Our proof of Theorem 1 proceeds as follows. We shall bound the \( L^2 \) norm \( \|f\|_2^2 = \int \int_{\mathbb{Z}_p^2} |f(x, y)|^2 \mu(x) d\mu(y) \) from below by \( D_N^4 \) using geometrical arguments, and from above by using Parseval’s theorem. The two steps are given below as lemmas

**Lemma 7.** The discrepancy \( D_N \) is bounded by
\[
D_N^4 \leq C_1(p) \|f\|_2^2,
\]
where \( C_1(p) \) is a constant dependent on \( p \).

**Lemma 8.** The \( L^2 \) norm of the function \( f \) is bounded by
\[
\|f\|_2^2 \leq C_2(p) \sum_{\zeta \in \mathbb{Z}(p^\infty) \setminus \{1\}} \frac{1}{\|\zeta\|^3} \left| \frac{1}{N} \sum_{n=1}^{N} \zeta^{x_n} \right|^2.
\]
where \( C_2(p) \) is a constant dependent on \( p \).

The proof of Theorem 1 then follows by combining Lemmas 7 and 8.

**Remark 1.** For \( x > 0 \), we use the notation \( \lfloor x \rfloor \) and \( \lceil x \rceil \) to denote
\[
\lfloor x \rfloor = \max \{ p^k \mid k \in \mathbb{Z}, p^k \leq x \}
\]
\[
\lceil x \rceil = \min \{ p^k \mid k \in \mathbb{Z}, x \leq p^k \}.
\]

Note that \( \lfloor x \rfloor \leq x < p \lfloor x \rfloor \) and \( \frac{1}{p} \lfloor x \rfloor < x \leq \lceil x \rceil \).
**Proof of Lemma 7**. Pick a point \((x_0, y_0)\) for which \(f(x_0, y_0)\) is not zero. We consider each of the two possibilities \(f(x_0, y_0) > 0\) and \(f(x_0, y_0) < 0\) separately. Our strategy in each case is to find a small neighborhood around the point \((x_0, y_0)\) where \(|f(x, y)|\) is bounded away from zero. Using this fact and integrating over this neighborhood, we produce a bound of the form \(\|f\|_2^2 \geq C(p)|f(x_0, y_0)|^4\), where \(C(p)\) is a constant depending only on \(p\).

**Case 1.** Suppose that \(\Delta = f(x_0, y_0) > 0\). Let \(R = \lfloor \Delta + |y_0| \rfloor\). Since, \(|y_0| < |y_0| + \Delta\) and \(|y_0|\) is in the value group of \(\mathbb{Q}_p\), we have \(|y_0| \leq R\). We consider the two cases \(|y_0| < R\) and \(|y_0| = R\).

**Case 1.1:** Suppose that \(|y_0| < R\). We must then have \(|y_0| \leq \frac{1}{p}R\). If we fix \(|y| = \frac{1}{p}R\) and \(|x - x_0| \leq \frac{1}{p}R\), then \(D(x, |y|) \subseteq D(x, |y|)\). We get a nonnegative lower bound on \(f(x, y)\) as follows

\[
f(x, y) = \frac{1}{N} \sum_{n=1}^{N} X_{D(x, |y|)}(\alpha_n) - |y| \\
\geq \frac{1}{N} \sum_{n=1}^{N} X_{D(x_0, |y_0|)}(\alpha_n) - |y| \\
= |y_0| + f(x_0, y_0) - |y| \\
= |y_0| + \Delta - |y| \\
\geq \left(1 - \frac{1}{p}\right) R.
\]

We can bound the \(L^2\) norm of \(f\) from below by evaluating the required integral only on the set \(|y| = \frac{1}{p}R, |x - x_0| \leq \frac{1}{p}R\)

\[
\|f\|_2^2 = \iint_{\mathbb{Z}_p^2} |f(x, y)|^2 d\mu(x) d\mu(y) \\
\geq \iint_{|y|=\frac{1}{p}R, |x-x_0| \leq \frac{1}{p}R} |f(x, y)|^2 d\mu(x) d\mu(y) \\
\geq \iint_{|y|=\frac{1}{p}R, |x-x_0| \leq \frac{1}{p}R} \left(1 - \frac{1}{p}\right)^2 R^2 d\mu(x) d\mu(y) \\
= \left(1 - \frac{1}{p}\right)^3 \frac{1}{p^2} R^4 \\
\geq \left(1 - \frac{1}{p}\right)^3 \frac{1}{p^6} \Delta^4 \\
= \frac{(p-1)^3}{p^9} \Delta^4,
\]
using \( R = |\Delta + |y_0|| \geq \frac{1}{p} (|y_0| + \Delta) \geq \frac{1}{p} \Delta. \)

**Case 1.2:** Suppose that \(|y_0| = R\). If we let \(|y| = R\) and \(|x - x_0| \leq R\), then \(D(x_0, |y_0|) = D(x, |y|)\). From this, we get

\[
f(x, y) = \frac{1}{N} \sum_{n=1}^{N} \chi_{D(x, |y|)}(\alpha_n) - |y|
= \frac{1}{N} \sum_{n=1}^{N} \chi_{D(x_0, |y_0|)}(\alpha_n) - |y_0|
= |y_0| + f(x_0, y_0) - |y|
= f(x_0, y_0) = \Delta.
\]

Therefore,

\[
\|f\|_2^2 = \int \int_{\mathbb{R}^2} |f(x, y)|^2 \, d\mu(x) \, d\mu(y)
\geq \int \int_{\{y = R\} \cap |x - x_0| \leq R} \Delta^2 \, d\mu(x) \, d\mu(y)
= \left(1 - \frac{1}{p}\right) R^2 \Delta^2
\geq \left(1 - \frac{1}{p}\right) \frac{1}{(p-1)^2} \Delta^4
= \frac{1}{p(p-1)} \Delta^4,
\]

using \(R + \Delta = |y_0| + \Delta < p(|y_0| + \Delta) = pR\) and therefore \(\Delta < (p-1)R\).

Finally, since \(\frac{(p-1)^3}{p^3} < \frac{1}{p(p-1)}\) we conclude

\[
\|f\|^2 \geq \frac{(p-1)^3}{p^9} \Delta^4
\]
holds in both cases 1.1 and 1.2, so it holds in general for case 1.

**Case 2:** Suppose that \(f(x_0, y_0) < 0\) and \(\Delta = |f(x_0, y_0)| = -f(x_0, y_0)\). In other words, the disc \(D(x_0, |y_0|)\) contains fewer than the expected number of points \(\alpha_n\).

Now let \(R = |y_0|\). Then if \(|y| = R\) and \(|x - x_0| \leq R\), by the strong triangle inequality \(D(x, |y|) = D(x_0, |y_0|)\) and we have

\[
f(x, y) = \frac{1}{N} \sum_{n=1}^{N} \chi_{D(x, |y|)}(\alpha_n) - |y|
\]
\[ \frac{1}{N} \sum_{n=1}^{N} \mathcal{X}_{D(x_0, |y_0|)}(\alpha_n) - |y_0| = f(x_0, y_0). \]

Therefore,
\[ \|f\|_2^2 = \iint_{\mathbb{Z}_p^2} |f(x, y)|^2 d\mu(x) d\mu(y) \]
\[ \geq \iint_{|y|=R, |x-x_0| \leq R} |f(x, y)|^2 d\mu(x) d\mu(y) \]
\[ = \iint_{|y|=R, |x-x_0| \leq R} \Delta^2 d\mu(x) d\mu(y) \]
\[ = \left(1 - \frac{1}{p}\right) R^2 \Delta^2 \]
\[ \geq \left(1 - \frac{1}{p}\right) \Delta^4, \]

where the last line follows because \( \Delta \leq R \). To see this, note that
\[ \Delta = -f(x_0, y_0) \]
\[ = |y_0| - \frac{1}{N} \sum_{n=1}^{N} \mathcal{X}_{D(x_0, |y_0|)}(\alpha_n) \]
\[ \leq |y_0| = R. \]

\[ \square \]

Next, we need to prove Lemma 8. Our goal is to find an upper bound on the \( L^2 \)-norm of \( f(x, y) \) using Parseval’s theorem. Suppose \( f(x, y) \) has a Fourier series
\[ f(x, y) = \sum_{\zeta, \omega \in \mathbb{Z}(p^{\infty})} \hat{f}(\zeta, \omega) \zeta^x \omega^y. \]

Then by Parseval’s theorem we would get
\[ \|f\|_2^2 = \sum_{\zeta, \omega \in \mathbb{Z}(p^{\infty})} |\hat{f}(\zeta, \omega)|^2. \]

Therefore, we need to bound the Fourier coefficients of \( f(x, y) \). The Fourier coefficients are
\[ \hat{f}(\zeta, \omega) = \iint_{\mathbb{Z}_p^2} f(x, y) \zeta^{-x} \omega^{-y} d\mu(x) d\mu(y) \]
\[ = \frac{1}{N} \sum_{n=1}^{N} \iint_{\mathbb{Z}_p^2} \mathcal{X}_{D(\alpha_n, |y|)}(x) \zeta^{-x} \omega^{-y} d\mu(x) d\mu(y) \]
(1) \[- \iint_{\mathbb{Z}_p^2} |y| \zeta^{-x} \omega^{-y} \, d\mu(x) \, d\mu(y).\]

Note that if \(\zeta = 1\) we get
\[
\hat{f}(1, \omega) = \frac{1}{N} \sum_{n=1}^{N} \iint_{\mathbb{Z}_p^2} \mathcal{X}_{D(\alpha_n, |y|)}(x) \omega^{-y} \, d\mu(x) \, d\mu(y) - \int_{\mathbb{Z}_p} |y| \omega^{-y} \, d\mu(y)
\]
\[
= \frac{1}{N} \sum_{n=1}^{N} \int_{\mathbb{Z}_p} |y| \omega^{-y} \, d\mu(y) - \int_{\mathbb{Z}_p} |y| \omega^{-y} \, d\mu(y) = 0.
\]

When \(\zeta \neq 1\), the second integral in line 2 of Equation (1) is zero
\[
\iint_{\mathbb{Z}_p^2} |y| \zeta^{-x} \omega^{-y} \, d\mu(x) \, d\mu(y) = \int_{\mathbb{Z}_p} |y| \omega^{-y} \left( \int_{\mathbb{Z}_p} \zeta^{-x} \, d\mu(x) \right) \, d\mu(y) = 0.
\]

Therefore,
\[
\hat{f}(\zeta, \omega) = \frac{1}{N} \sum_{n=1}^{N} \iint_{\mathbb{Z}_p^2} \mathcal{X}_{D(\alpha_n, |y|)}(x) \zeta^{-x} \omega^{-y} \, d\mu(x) \, d\mu(y).
\]

Using Lemma 6, we have
\[
\int_{\mathbb{Z}_p} \mathcal{X}_{D(\alpha_n, |y|)}(x) \zeta^{-x} \, d\mu(x) = \begin{cases} 
\zeta^{-\alpha_n} |y| & \text{if } \|\zeta\| \leq 1/|y|, \\
0 & \text{else.}
\end{cases}
\]

Hence, for \(\zeta \neq 1\),
\[
\hat{f}(\zeta, \omega) = \frac{1}{N} \sum_{n=1}^{N} \zeta^{-\alpha_n} \int_{|y| \leq 1/\|\zeta\|} |y| \omega^{-y} \, d\mu(y).
\]

The following lemma makes some estimates that are useful in our succeeding calculations

**Lemma 9.** Let \(R = p^k, k \in \mathbb{Z}\) satisfy \(0 < R < 1\), and let \(\omega \in \mathbb{Z}(p^{\infty})\).

Then,
\[
(3) \quad \left| \int_{|y| \leq R} |y| \omega^{-y} \, d\mu(y) \right| \leq \frac{p}{\max(1/R, \|\omega\|)^2}.
\]

Moreover,
\[
(4) \quad \sum_{\omega \in \mathbb{Z}(p^{\infty})} \left| \int_{|y| \leq R} |y| \omega^{-y} \, d\mu(y) \right|^2 \leq 2p^2 R^3.
\]
Proof of Lemma 9. Let $R = 1/p^k$ and let $||\omega|| = p^l$. We have

$$\int_{|y| \leq R} |y| \omega^{-y} d\mu(y) = \sum_{j \geq k} \frac{1}{p^j} \int_{|y| = 1/p^j} \omega^{-y} d\mu(y)$$

(5)

$$= \sum_{j \geq k} \frac{1}{p^j} \int_{\mathbb{Z}_p} (\chi_{D(0,1/p^j)}(y) - \chi_{D(0,1/p^{j+1})}(y)) \omega^{-y} d\mu(y).$$

When $||\omega|| \leq 1/R$, that is when $l \leq k$, using Lemma 6 and (5) we have

$$\int_{|y| \leq R} |y| \omega^{-y} d\mu(y) = \sum_{j \geq k} \frac{1}{p^j} \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right) = \frac{p}{(p+1)(1/R)^2}.$$ 

Thus (3) holds in this case. If $||\omega|| > 1/R$, that is when $l \geq k+1$, again using Lemma 6 and (5) we have

$$\int_{|y| \leq R} |y| \omega^{-y} d\mu(y) = \sum_{j \geq l} \frac{1}{p^j} \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right) - \frac{1}{p^l-1/p^l} = -\frac{p^2}{(p+1)||\omega||^2},$$

and thus (3) holds in this case as well.

To check (4), we use the fact that for each $j \geq 1$, the group $\mathbb{Z}(p^\infty)$ contains $p^j$ elements of order at most $p^j$, and $p^j - p^{j-1}$ elements of order exactly $p^j$. We then have

$$\sum_{\omega \in \mathbb{Z}(p^\infty)} \left| \int_{|y| \leq R} |y| \omega^{-y} d\mu(y) \right|^2 \leq p^2 \sum_{\omega \in \mathbb{Z}(p^\infty)} \frac{1}{1/(R, ||\omega||)^4} \leq p^2 \left( \sum_{||\omega|| \leq 1/R} R^4 + \sum_{||\omega|| > 1/R} \frac{1}{||\omega||^4} \right) = p^2 \left( R^3 + \frac{p-1}{p(p^3-1)} R^3 \right) < 2p^2 R^3.$$ 

□

Finally, we prove Lemma 8.

Proof of Lemma 8. Applying Parseval’s theorem to $f(x, y)$ and using Equations (2) and (4), we conclude

$$\|f\|^2 = \sum_{\zeta, \omega \in \mathbb{Z}(p^\infty)} |\hat{f}(\zeta, \omega)|^2$$
\[
\sum_{\zeta \neq 1} \left( \sum_{\omega \in \mathbb{Z}(p^\infty)} \left| \int_{|y| \leq 1/\|\zeta\|} |y| \omega^{-y} d\mu(y) \right|^2 \right) \left( \frac{1}{N} \sum_{n=1}^{N} \zeta^\alpha_n \right)^2 \leq 2p^2 \sum_{\zeta \in \mathbb{Z}(p^\infty)} \frac{1}{\|\zeta\|^3} \left( \frac{1}{N} \sum_{n=1}^{N} \zeta^\alpha_n \right)^2.
\]

4. THE LINEAR SEQUENCE \( na + b \) IN \( \mathbb{Z}_p \)

Consider the sequence \( \alpha_n = na + b \). We have the following proposition, a proof of which is given in [7] using elementary number theory. We present an alternate proof using Fourier analysis.

**Proposition 10.** The sequence \( \alpha_n = na + b \) is equidistributed in \( \mathbb{Z}_p \) if and only if \( a \) is a unit in \( \mathbb{Z}_p \).

**Proof.** The forward implication follows from Weyls criterion (Proposition 4). For suppose, \( a \) was not a unit. Then \( a = p^k c \), where \( k > 0 \) and \( c \) is a unit. Now let \( \zeta = e^{2\pi i/p^k} \). Then \( \zeta^a = 1 \), and Weyls criterion will not hold.

For the reverse implication, let \( \zeta \in \mathbb{Z}(p^\infty) \) with \( \|\zeta\| = p^k \) for \( k \geq 1 \). There exists an \( m \) such that \( 1 \leq m < p^k \), with \( p \nmid m \) and \( \zeta = e^{2\pi im/p^k} \). Suppose that \( a \) is a unit in \( \mathbb{Z}_p \). Let \( a = t_0 + t_1 p + t_2 p^2 + \ldots \) be the canonical expansion of \( a \), with \( t_0 \neq 0 \). Then we let \( a_k = t_0 + t_1 p + \ldots + t_{k-1} p^{k-1} \) be the truncation of this expansion to the first \( k \) terms. We have

\[
\frac{1}{N} \left| \sum_{n=1}^{N} \zeta^na \right| = \frac{1}{N} \left| \sum_{n=1}^{N} \zeta^na \right| = \frac{1}{N} \left| \frac{1 - \zeta^{(N+1)a_k}}{1 - \zeta^a_k} \right| \leq \frac{1}{N} \left| \frac{1 - \zeta^a_k}{\sin(\pi ma_k/p^k)} \right|.
\]

Since \( p \nmid m \) and \( p \nmid a_k \), \( \sin(\pi ma_k/p^k) \neq 0 \) and hence \( \frac{1}{N} \sum_{n=1}^{N} \zeta^na \to 0 \) as \( N \to \infty \); the proof of equidistribution now follows from Weyls criterion. \(\square\)
Proof of Corollary 2. Applying the bound given by Theorem 1 we get

\[ D_N^4 \ll \sum_{\zeta \in \mathbb{Z}(p^{\infty}) \setminus \{1\}} \frac{1}{\|\zeta\|^3} \left| \frac{1}{N} \sum_{n=1}^{N} e^{\pi n a + b} \right|^2 \]

\[ \leq \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{p^{3k}} \sum_{1 \leq m < p\ell/m} \frac{1}{|\sin(\pi m a_k / p^k)|^2} \]

\[ \leq \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{p^{3k}} \sum_{1 \leq m < p_k} \frac{1}{|\sin(\pi m a_k / p^k)|^2} \]

\[ \leq \frac{1}{N^2} \sum_{k=1}^{\infty} \frac{1}{p^{3k}} \sum_{1 \leq l < p^k / 2} \frac{1}{|\sin(\pi l / p^k)|^2} \]

\[ \leq \frac{2}{N^2} \sum_{k=1}^{\infty} \frac{1}{p^{3k}} \sum_{1 \leq l < p^k / 2} \frac{1}{|\sin(\pi l / p^k)|^2} . \]

(7)

Note that the second inequality in Equation (7) comes from the last inequality in (6). For the fourth inequality, note that since \(a\) is a unit we have \(p \nmid a_k\). Hence, \(\gcd(a_k, p^k) = 1\) and so \(a_k\) generates \(\mathbb{Z}/p^k\mathbb{Z}\). That is, \(\mathbb{Z}/p^k\mathbb{Z} = \{ma_k | m = 0, ..., p^k - 1\}\). The final inequality follows from the identities \(|\sin(\theta)| = |\sin(-\theta)| = |\sin(\pi - \theta)|\), so that for \(p^k / 2 \leq l < p^k\) we have \(|\sin(\pi l / p^k)| = |\sin(\pi (p^k / l - l / p^k))|\). This allows us to double the sum over the first half of the interval.

Note that in the interval \([0, \pi / 2]\), \(\sin(\theta)\) is bounded from below by \(2\theta / \pi\), so that

\[ \frac{1}{|\sin(\theta)|} \leq \frac{\pi}{2\theta} . \]

This gives us

\[ \sum_{1 \leq l \leq p^k / 2} \frac{1}{|\sin(\pi l / p^k)|^2} \leq \sum_{1 \leq l \leq p^k / 2} \frac{p^{2k}}{4l^2} \]

\[ \leq \frac{p^{2k}}{4} \sum_{1 \leq l < \infty} \frac{1}{l^2} \]

\[ \leq \frac{p^{2k} \pi^2}{24} . \]

(8)

Finally, applying the bound from (8) to (7) we get

\[ D_N^4 \ll \frac{\pi^2}{12 N^2} \sum_{k=1}^{\infty} \frac{1}{p^k} . \]
We conclude that $D_N = O \left( \frac{1}{\sqrt{N}} \right)$.

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