Abstract. The Lewis Carroll identity expresses the determinant of a matrix in terms of sub-
determinants obtained by deleting one row and column or a pair of rows and columns. Using the
matrix tree theorem, we can convert this into an equivalent identity involving sums over pairs of
forests. Unlike the Lewis Carroll Identity, the Forest Identity involves no minus signs. Using the
Involution Principle, we can pull back Zeilberger’s proof of the Lewis Carroll Identity to a bijective
proof of the Forest Identity. This bijection is implemented by the Red Hot Potato algorithm, so
called because the way edges get tossed back and forth between the two forests is reminiscent of
the children’s game of hot potato.

1 Introduction

This paper presents a combinatorial interpretation of the Lewis Carroll identity [4]. Let
$U$ and $W$ be sets of nodes of the same size, and let $M_{U,W}$ be the matrix $M$ with the rows corresponding to
the elements in $U$ removed and the columns corresponding to the elements in $W$ removed.

Theorem 1.0.1 Lewis Carroll Identity. Let $M$ be a square matrix. Then

$$\det(M) \cdot \det(M_{12,12}) = \det(M_{2,2}) \cdot \det(M_{1,1}) - \det(M_{2,1}) \cdot \det(M_{1,2}).$$

Example 1.0.2 Let $M = \begin{pmatrix} 1' & 2' & 3' & 4' \\ 1 & 3 & 7 & 0 \\ 2 & 8 & 1 & 0 \\ 3 & 0 & 0 & 4 \\ 4 & 0 & 0 & 2 \end{pmatrix}$. Then $\det(M) = -424, \det(M_{12,12}) = 8, \det(M_{2,2}) = 24,$ $\det(M_{1,1}) = 8,$ $\det(M_{2,1}) = 56,$ and $\det(M_{1,2}) = 64,$ so our identity gives us $-424 \cdot 8 = 24 \cdot 8 - 56 \cdot 64$.

We will interpret the Lewis Carroll identity in terms of pairs of directed rooted forests. In our
notion of a tree, every node may have at most one out-edge (it can have any number of in-edges).
A root is a node in a tree such that there is a directed path from all other nodes to the root, and
the root has no out-edges. Thus in a directed rooted forest, the roots are exactly the nodes that
have no out-edges, and the other nodes of a tree have exactly one out-edge.

We call a path from node $i$ to node $j$ a meta-edge $i \rightarrow j$. For instance, in the following
example, the edges $1 \rightarrow 4, 4 \rightarrow 3, 3 \rightarrow 2$ together form the meta-edge $1 \rightarrow 2$, and the edge $2 \rightarrow 1$
forms the meta-edge $2 \rightarrow 1$. 
To give the Forest Identity, we will first introduce some notation. Let $a_F$ be the weight of a forest $F$, where the weight of $F$ is calculated by taking the product of the weights $a_{ij}$ of all edges $i \to j$ in $F$. For a set of nodes $U$, let $R_U$ be the set of forests where the nodes in $U$ (and only those nodes) are the roots. Thus $R_0$ is the set of all trees with root 0. For an added superscript $i \to j$, let $R_{i \to j}^U$ be the set of forests with roots at $U$ and with node $i$ in the tree rooted at $j$ (that is to say there exists a meta-edge $i \to j$). Thus $R_{0,2}^{1 \to 2}$ is the set of two-forests (forests containing two trees) with roots at nodes 0 and 2 and a meta-edge $1 \to 2$. We will let $R^{NF}$ stand for non-forbidden forests, which are the two-forests that are allowed as part of the Forest Identity, as defined below.

**Theorem 1.0.3 Forest Identity.** Let $R^{NF}$ be the set $R_{0,2} \times R_{0,1} \setminus R_{0,2}^{1 \to 2} \times R_{0,1}^{2 \to 1}$. Then

$$\sum_{(F,G) \in R_0 \times R_{0,2}} a_F a_G = \sum_{(F,G) \in R^{NF}} a_F a_G. \quad (1.1)$$

We can also represent this diagramatically. Let a star represent a root and edges represent meta-edges. Then the following diagram represents two-forests on labeled vertices 0, 1, \ldots, $n$ with roots at nodes 0 and 2 and a meta-edge $1 \to 0$ (that could involve other nodes, but doesn’t have to).

Then the Forest Identity deals with pairs of forests as represented below.
Example 1.0.4 Below is the case $n = 2$:

The rest of the paper will be organized as follows: In section 2, we explain the connection between the Lewis Carroll identity and the Forest Identity. In sections 3 and 4, we give the Red Hot Potato algorithm and provide an example. In section 5, we prove that the Red Hot Potato algorithm works. In section 6, we use the algorithm to prove the Forest Identity and the Lewis Carroll identity.

2 Matrix Tree Theorem connection between Forest and Lewis Carroll Identities

We can use the All Minors Matrix Tree Theorem \[1\] to derive the forest identity from the Lewis Carroll identity. Let $a_{ij}$ be the weight of the edge $i \rightarrow j$. Define the Laplacian $A$ by
A_{ij} = \begin{cases} -a_{ij} & i \neq j \\ \sum_{k \neq i} a_{ik} & i = j \end{cases}

Replacing M in the Lewis Carroll identity with A_{0,0}, where A is the matrix corresponding to a complete graph on nodes \{0, 1, \ldots, n\} and A_{0,0} is A with the zeroth row and column removed, we obtain the following:

$$\det(A_{0,0}) \cdot \det(A_{012,012}) = \det(A_{01,01}) \cdot \det(A_{02,02}) - \det(A_{02,01}) \cdot \det(A_{01,02}). \quad (2.1)$$

Now we can use the matrix tree theorem [1] to see that when \( U = W \) in \( \det(A_{U,W}) \), we have the sum of the weights of the forests rooted at the nodes in set \( U \). When \( U \neq W \), we have that \( \det(A_{02,01}) \cdot \det(A_{01,02}) \) is pairs of forests of the form

Since we are subtracting these from the total number of pairs of forests rooted at 01 and 02, we will call these forbidden forests. The meta-edges 1 \( \rightarrow \) 2 and 2 \( \rightarrow \) 1 that cause a forbidden forest will together be called a forbidden meta-cycle. Recall from our definition of the Forest Identity that \( R^{NF} \) represents the non-forbidden forests since it removes the set of forbidden forests \( R^{\rightarrow 2}_{0,2} \times R^{\rightarrow 1}_{0,1} \) from the total set of pairs two-forests rooted at 02 and 01.

Taken together, we have that the left hand side of the Forest Identity (the left hand side of 2.1) is equal to the right hand side of the Forest Identity (the right hand side of 2.1). The Lewis Carroll identity thus proves the Forest Identity. Our goal for the rest of the paper is to prove the Forest Identity directly, and then use it to prove the Lewis Carroll identity, showing that the two identities are equivalent.

### 3 The Red Hot Potato algorithm

Our algorithm is based on a consequence of the Involution Principle [2], [3]:

**Theorem 3.0.1** Given any sequence of signed sets \( S_0, S_1, \ldots, S_{k+1} \) where \( S_0 \) and \( S_{k+1} \) contain only positive elements, and sign-reversing involutions \( \phi_0, \ldots, \phi_k \) where \( \phi_i : S_i - S_{i+1} \rightarrow S_i - S_{i+1} \), there is a constructible bijection between \( S_0 \) and \( S_{k+1} \).

Our goal, then, will be to find a sequence of signed sets and sign-reversing involutions satisfying the assumptions of Theorem 3.0.1 such that \( S_0 \) is the set of pairs of trees and three-forests, and \( S_{k+1} \) is the set of non-forbidden two-forests.
3.1 Sets

Our signed sets will each involve pairs of graphs with edges colored either black or red (dashed).

Let \( S_0 \) be pairs of trees and three-forests, the trees with roots at node 0 and the three-forests with roots at nodes 0, 1, and 2 (the pairs involved in the left hand side of the Forest Identity). In this set, all edges will be black and all elements will be positive.

Let \( S_1 \) be pairs of graphs, one with no edge out of node 0 and one edge out of the rest of the nodes, and one with no edge out of nodes 0, 1, or 2 and one edge out of the rest of the nodes. Each cycle can either be colored red or black, and edges not involved in cycles must be colored black. If an even number of cycles in the two graphs is red, then the pair is positive. If an odd number of cycles is red, the pair is negative.

Let \( S_2 \) be pairs of graphs, one with no edge out of nodes 0 and 2 and one edge out of the rest of the nodes, and one with no edge out of nodes 0 and 1 and one edge out of the rest of the nodes. If there is a forbidden meta-cycle in the pair, we count it as a single cycle. Each cycle can either be colored red or black. The rest of the edges must be colored black. If an even number of cycles is colored red, then the pair is positive. If an odd number of cycles is colored red, the pair is negative.

Let \( S_3 \) be pairs of two-forests, one with roots at nodes 0 and 2 and the other with roots at nodes 0 and 1, that are not forbidden (the pairs involved in the right hand side of the Forest Identity). In this set, all edges will be black and all elements will be positive.

3.2 Involutions

We define involutions \( \phi_0 \), \( \phi_1 \), and \( \phi_2 \) on differences of sets and prove they are sign-reversing. For clarity, we will call the graph with an edge coming out of node 1 \( A \) and we will call the other \( B \).

We define \( \phi_0 : S_0 - S_1 \to S_0 - S_1 \) and \( \phi_2 : S_2 - S_3 \to S_2 - S_3 \) in the same way. Below, we will define \( \phi_0 \). We define \( \phi_2 \) in a similar way by replacing \( S_0 \) with \( S_3 \) and \( S_1 \) with \( S_2 \).

- Notice that \( S_0 \subset S_1 \). If \( t \in S_0 \), then \( \phi_0(t) = t \in S_1 \). Similarly, if \( t \in S_1 \) and \( t \in S_0 \), then the involution sends the \( t \) in \( S_1 \) to itself in \( S_0 \). This is clearly an involution. It is sign-reversing since \( t \) is positive in both \( S_0 \) and \( S_1 \), so it is negative in \(-S_1\).

- If \( t \in S_1 \) and \( t \notin S_0 \), then there must be at least one cycle in the graphs. Then the involution changes the color of one cycle. If there is a cycle in \( A \), we change the color of the cycle in \( A \) containing the largest node. If there are no cycles in \( A \), we change the color of the cycle in \( B \) containing the largest node. If there is a forbidden meta-cycle, we change its color if there are no other cycles in the two graphs. This is clearly an involution. It is also sign-reversing since changing the color of one cycle changes the parity of the number of red cycles.

The involution \( \phi_1 : S_1 - S_2 \to S_1 - S_2 \) is the involution that actually moves edges back and forth between the graphs. When we talk about moving a red meta-edge \( i \to a_1 \to \cdots \to a_k \to j \) from \( A \) to \( B \), we mean that we move all of the red edges in the \( i \to j \) meta-edge from \( A \) to \( B \), and move all of the black edges in \( B \) coming out of the nodes \( a_1, a_2, \ldots, a_k \) to \( A \). Then \( \phi_1(\{A, B\}) = \{C, D\} \) where \( C \) and \( D \) are defined as follows:

- If the edge coming out of node 1 is black, move that edge from \( A \) to \( B \) to form a new pair \( \{C, D\} \) (where now \( D \) has the edge out of node 1).

- Suppose the edge coming out of node 1 is red. We move the edges designated by the crabwalk, defined as follows. Create a graph with the same node set as \( A \), and with edge set the set of
all red edges from $A$ and $B$. Color the edges coming from $A$ dark red and the edges coming from $B$ light red (dashed). This is the crabwalk graph. The crabwalk begins by moving along the dark red edge coming out of 1. We continue along that meta-edge until we reach a node that has a light red edge going into it. We travel backwards along the light red meta-edge until we reach a node that has a dark red edge coming out of it. Then we travel forward along the dark red meta-edge until we reach a node that has a light red edge going into it. We continue in this manner until we have reached either node 2 or node 1. We then change the color of each of the edges we traveled along, so the dark edges become light and the light ones become dark. Returning to graphs $A$ and $B$, we move the red meta-edges that changed color in the crabwalk. Figure 1 gives an example of the crabwalk.

We will prove that $\phi_1$ is a sign-reversing involution in section 5.

Assuming that $\phi_1$ is in fact a sign-reversing involution, then these sets and involutions satisfy the hypotheses of Theorem 3.0.1 so we have proved the Forest Identity. Following the algorithm for finding the bijection that the Involution Principle guarantees (see [3]), we can construct the bijection as follows:

**Red Hot Potato algorithm:**

We begin with a pair of forests in $S_0$. We apply $\phi_0$ and then apply $\phi_1$. Once finished with $\phi_1$, we change the color of the appropriate cycle using $\phi_2$ and do $\phi_1$ again. Then we change the color of the appropriate cycle and so on. We finish when, upon performing $\phi_2$, there are no colors to be changed, i.e. when $\phi_2$ yields a pair of graphs in $S_3$. 
Figure 1: An illustration of the crabwalk. All of the edges shown here are red meta-edges. The smaller arrows along the edges indicate the order in which each meta-edge is moved from one graph to another.
4 Example

Applying $\phi_0$ simply returns the same pair in $S_1$.

The left is the result of $\phi_1$ in $S_2$, moving the single black edge out of 1 from $A$ to $B$. The right is the result of $\phi_2$, which changes the color of the forbidden meta-cycle.

The left is the result of $\phi_1$ in $S_1$, where we moved the meta-edge $1 \to 2$. The right is the result of $\phi_2$, which changes the color of the cycle in $A$.

The left is the result of $\phi_1$ in $S_2$, moving the single black edge out of node 1. The right is the result of $\phi_2$, which changes the color of the cycle in $A$ (which in this case is the right-hand graph).

The left is the result of $\phi_1$ in $S_1$, moving the single black edge out of node 1. The right is the result of $\phi_2$, which changes the color of the cycle in $A$.

The left is the result of $\phi_1$ in $S_2$. We began in $A$, moving the edge $1 \to 4$. This intersected a red cycle in $B$, so we then moved $3 \to 4$ followed by $4 \to 3$ (since we move backwards in $B$). Upon returning to 4, we intersected the red cycle in $A$, so we moved $4 \to 2$ and, on reaching 2, finished the crabwalk. The right is the result of $\phi_2$, which changes the color of the cycle in $B$. 


The left is the result of $\phi_1$ in $S_1$, moving the meta-edge $1 \to 2$. The right is the result of $\phi_2$, which changes the color of the cycle in $A$.

The result of $\phi_1$ in $S_2$, moving the single black edge out of node 1. When we apply $\phi_2$ to this graph, no colors change, so we end up in $S_3$. Thus we are done!

5 Proof that $\phi_1$ is a sign-reversing involution

Lemma 5.0.1 Let $\{A,B\} \in S_1 - S_2$. Then $\phi_1(\{A,B\}) \in S_1 - S_2$.

Proof. To prove that $\phi_1(\{A,B\}) = \{C,D\} \in S_1 - S_2$, we need to show that $\{C,D\}$ has the following two defining characteristics:

1. There are two edges out of each node, one in each graph, except for nodes 0, 1, and 2. Node 0 has no out-edges in either graph, and nodes 1 and 2 each have a total of one out-edge between the two graphs (if the out-edges for nodes 1 and 2 are in the same graph, we are in $S_1$, and if they are in different graphs, we are in $S_2$).

2. Red edges are only in cycles (this includes the forbidden meta-cycle).

We begin with requirement 1. Since $\phi_1$ simply moves edges around, and does not add or delete any edges, then since we have started with no edges out of node 0, one edge out of nodes 1 and 2, and two edges out of the rest, we will end with that as well. Thus we just need to show that for the two edges out of nodes $3, \ldots, n$ in $\{C,D\}$, one is in each graph. Since we begin with one edge out of a node $i$ in each graph, we want to show that, if we move one edge out of $i$ from $A$ to $B$, then we must move the other edge out of $i$ from $B$ to $A$. Notice that if both edges out of node $i$ are black, then neither will get moved, so they end as they started with one in each graph. If one edge is red and the other is black then, by how we defined moving red meta-edges, if the red edge gets moved, then the black one will as well. Finally, suppose that both edges out of $i$ are red. Suppose we move the red edge out of $i$ from $B$ to $A$. Then we intersect with a red cycle in $A$, so the next step in the crabwalk is to move the red edge out of $i$ from $A$ to $B$, and thus both edges have switched graphs. This is the only way to have moved edges out of $i$ when both of the edges are red.

We now prove requirement 2, that the red edges in $\{C,D\}$ will only be involved in cycles. For contradiction, suppose that $j \to k$ is a red meta-edge that is not part of a cycle in $\{C,D\}$. That is to say, there is no red edge pointing into $j$ and no red edge pointing out of $k$.

First let us suppose that $j$ is not 1 or 2. Then in $\{A,B\}$, since $j$ has a red edge pointing out of it, it must also have a red edge pointing into it. In order for the red edge into $j$ and the red edge out of $j$ to be separated, one must have moved and one must not have, so the crabwalk must have switched between dark and light red at $j$. Thus $j$ must have a red edge into and out of it in both
A and B. Then in order for there to be no red edge into \( j \) at the end, both of the red edges into \( j \) must end up in the other graph. However, by the definition of the crabwalk, either both red edges into a node move or both do not, i.e. they always end up in different graphs. Thus \( j \) will end up having a red edge pointing into it.

Now let us suppose that \( k \) is not 1 or 2. Then similarly to \( j \), there must be a red edge into and out of \( k \) in both \( A \) and \( B \). In order for there to be no red edge out of \( k \) at the end, both of the red edges out of \( k \) must end up in the other graph. By requirement 1, that is impossible. Thus \( k \) will end up having a red edge pointing out of it.

We have just proved that if there is a red meta-edge \( j \rightarrow k \) that is not part of a cycle in \( \{C, D\} \), then \( j \) and \( k \) must be 1 or 2, that is to say \( j \rightarrow k \) is either \( 1 \rightarrow 2 \) or \( 2 \rightarrow 1 \). Notice that in \( \{A, B\} \), if there is a red edge out of node 1 and node 2, then there is exactly one red edge into 1 and one red edge into 2. Thus if \( j \rightarrow k \) is \( 1 \rightarrow 2 \), then the other graph must have a red edge out of 2. However, we’ve already use the single red edge into 2 as part of our \( 1 \rightarrow 2 \) meta-edge, so the other graph must have a meta-edge \( 2 \rightarrow k \) where \( k \neq 2 \). But then \( k \) must be 1, so we have a forbidden meta-cycle \( 1 \rightarrow 2 \) and \( 2 \rightarrow 1 \). If \( j \rightarrow k \) is \( 2 \rightarrow 1 \), then by the same argument, the other graph must have a red meta-edge \( 1 \rightarrow 2 \). Thus red edges only end up in cycles or in the forbidden meta-cycle.

\( \square \)

**Lemma 5.0.2** If the crabwalk ends in graph \( A \), then the parity of cycles remains the same after applying \( \phi_1 \). If the crabwalk ends in graph \( B \), then the parity of cycles changes after applying \( \phi_1 \).

**Proof.** We claim that each time we switch from dark to light red (or vice versa) in the crabwalk, the parity of the cycles changes. Then if we end in \( A \), we have switched shades an even number of times, so the parity is the same after applying \( \phi_1 \). If we end in \( B \), we have switched shades an odd number of times, so the parity has changed after applying \( \phi_1 \).

To prove our claim, we take advantage of a topological fact: suppose we have a graph made entirely of cycles, and each node can have at most two cycles through it. We break each node with two cycles into two separate nodes, which we will call a pair of split nodes. To distinguish the two split nodes, we will label them \( d \) and \( l \) based on which edge comes out of each node, that is to say, the edges coming out of the nodes are fixed. Then we have two choices for how to split the cycles since there are two nodes in which to place the two edges coming in. Having fixed a choice for each of the split nodes, we now have a graph made of disjoint cycles. Now suppose that for a single pair of split nodes, we were to switch our choice of which in-edge is assigned to which of the split nodes. Then the parity of disjoint graphs changes: if the edges at that node were part of two separate cycles, then switching our choice combines those two cycles into one. If instead the edges at that node were part of one large cycle, then switching our choice separates that cycle into two different cycles. See Figure 2. Either way, the parity changes.

Switching a choice at a split node is equivalent to switching between dark and light at that node in the crabwalk: The crabwalk graph is made entirely of cycles, and each node can have at most two cycles (a dark one and a light one) through it. Then each point that the dark and light cycles pass through can be thought of as a split node. Let us label the split node with a dark edge coming out of it \( d \) and the split node with the light edge coming out of it \( l \). Then our original configuration has the dark edge coming into \( d \) and the light edge coming into \( l \). When we travel along the dark edge coming out of node 1, we stop when we reach the first split node. When we reach the first split node, we change the meta-edge we were traveling along to light, so there is now a light edge coming into \( d \). Then we move backwards along the light edge going into \( l \), changing it to dark. Thus the edge coming into \( l \) is now dark. Then in changing from shade, we have switched the choice that
we made at that node. The same is true when we change from light to dark. We have thus proved
our claim that each time we switch between dark and light in the crabwalk, the parity of the cycles
changes.

\begin{center}
\begin{tabular}{c}
\includegraphics[width=0.5\textwidth]{diagram1.png}
\end{tabular}
\end{center}

Figure 2: Switching a choice at a split node

**Theorem 5.0.3** The function $\phi_1$ is a sign-reversing involution on $S_1 - S_2$.

**Proof.**

Lemma 5.0.1 shows that $\phi_1$ is indeed a function into the correct range.

If the edge coming out of node 1 in $A$ is black, then we simply move that edge over, which
means if we apply $\phi_1$ again, we just move that edge back again. Therefore in this case, $\phi_1$ is an
involution. It is sign-reversing because if \{A, B\} $\in S_1$, then moving the single black edge over puts
\{C, D\} $\in S_2$ and vice versa. The parity of cycles does not change, which means that in $S_1 - S_2$, $\phi_1$
is sign-reversing.

Now suppose that the edge coming out of node 1 in $A$ is red. Then we perform the crabwalk.
By definition, if we apply $\phi_1$ twice, each time the sequence of edges traveled along will be the same.
Since we are simply changing shades in that sequence of edges, it is clear that $\phi_1$ is an involution.

We break our proof that $\phi_1$ is sign-reversing into two cases:

**Case 1** Let \{A, B\} $\in S_1$. Since B has no edges out of nodes 1 or 2, there can be no red edges in
B involving either of these nodes, so the crabwalk must end in A. Then by Lemma 5.0.2 the parity
of cycles has not changed. Since we have moved over the edge out of 1, but not the edge out of 2,
then \{C, D\} $\in S_2$, and since the parity of cycles has not changed, \{C, D\} has the same sign in $S_2$
as \{A, B\} $\in S_1$. Thus the sign has changed in $S_1 - S_2$.

\begin{center}
\begin{tabular}{c}
\includegraphics[width=0.5\textwidth]{diagram2.png}
\end{tabular}
\end{center}

Figure 3: An example of Case 1. Here, we have switched colors twice at the bottom node
Case 2 Let \( \{A, B\} \in S_2 \).

- Suppose the crabwalk ends in \( A \). Then by Lemma 5.0.2 the parity of the cycles remains the same. Since we are not ending at node 2 in \( B \), the edge out of 2 remains in \( D \), and the edge out of 1 switches to \( D \). Thus \( \{C, D\} \in S_1 \) and, since the sign of \( \{A, B\} \in S_2 \) is the same as that of \( \{C, D\} \in S_1 \), the sign is reversed in \( S_1 - S_2 \).

- Suppose the crabwalk ends in \( B \). Then by Lemma 5.0.2 the parity of the cycles has switched. Since we move backwards along light red edges, then the last light edge moved is an edge coming out of a node. Since there is no edge out of 1 in \( B \), then we must end with the edge coming out of node 2. Thus the edge out of 1 has been moved to \( D \) and the edge out of node 2 has been moved to \( C \), so \( \{C, D\} \in S_2 \). Since the parity of the cycles has been switched, the sign is reversed in \( S_2 \), so it is reversed in \( S_1 - S_2 \).

(a) An example of the crabwalk ending in \( A \). The colors have switched once at \( m \) and \( i \) and twice at \( k \), for a total of four switches.

(b) An example of the crabwalk ending in \( B \). The colors have switched once at the bottom node.

Figure 4: Examples of Case 2.

6 Proofs of Forest and Lewis Carroll Identities

We will restate and prove our two identities using the Red Hot Potato algorithm.

Theorem 6.0.1 Forest Identity. Let \( R^{NF} \) be the set \( R_{0,2} \times R_{0,1} \setminus R_{0,2}^{1 \rightarrow 2} \times R_{0,1}^{2 \rightarrow 1} \). Then

\[
\sum_{(F,G) \in R_0 \times R_{0,1,2}} a_F a_G = \sum_{(F,G) \in R^{NF}} a_F a_G.
\]
Proof. We have proven that the signed sets $S_0, S_1, S_2,$ and $S_3,$ and the sign-reversing involutions $\phi_0, \phi_1,$ and $\phi_2$ satisfy the hypotheses of Theorem 3.0.1. Thus there exists a bijection between $S_0$ and $S_3.$ Suppose that the edges in our graphs are unweighted. Then the left hand side of the identity merely counts the number of pairs $(F,G) \in R_0 \times R_{0,1,2},$ i.e. the left hand side counts the number of elements in $S_0.$ Similarly the right hand side counts the number of pairs $(F,G) \in R^N F,$ i.e. the number of elements in $S_3.$ Since there is a bijection between $S_0$ and $S_3,$ then $|S_0| = |S_3|,$ so the identity holds. Now suppose that the graphs are weighted. We modify our bijection between $S_0$ and $S_3$ by first taking away the weights on the edges, performing the bijection, and then putting the weights back on the respective edges. In this manner, we are still matching each pair of forests in $S_0$ with one in $S_3,$ and the total weight $a_F a_G$ remains the same. Thus our identity holds. \qed

Theorem 6.0.2 Lewis Carroll Identity. Let $M$ be a square matrix. Then
\[
\det(M) \cdot \det(M_{12,12}) = \det(M_{22}) \cdot \det(M_{11}) - \det(M_{21}) \cdot \det(M_{12}).
\]

Proof. The Forest Identity proves the Lewis Carroll Identity provided that $M = A_{0,0}$ where $A$ is the Laplacian for some graph, that is to say, the row sums must be zero for every row in $A.$ But given any square matrix $M,$ we can turn it into a Laplacian by adding a zeroth row and column where the zeroth row can be anything that sums to zero, and the $i$th entry of the zeroth column is the opposite of the sum of the $i$th row. Thus any square matrix is of the form $A_{0,0}$ where $A$ is the Laplacian for some graph. \qed

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