Switched Server Systems Whose Parameters are Normal Numbers in Base 4

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Abstract
Switched server systems are mathematical models of manufacturing, traffic and queueing systems. Recently, it was proved in (Eur J Appl Math 31(4), 682–708, 2020) that there exist switched server systems with 3 buffers (tanks), a server, filling rates \( \rho_1 = \rho_2 = \rho_3 = \frac{1}{3} \) and parameters \( d_1, d_2, d_3 > 0 \) whose \( \omega \)-limit set is a fractal set. In this article, we give an explicit large subset of parameters for which the corresponding switched server systems have no fractal \( \omega \)-limit set. More precisely, the Poincaré map of each system has a finite \( \omega \)-limit set. The approach we use is to study the topological dynamics of a family of piecewise \( \lambda \)-affine contractions that includes the Poincaré maps of the switched server systems as a particular case.

Keywords
Switched server system · Piecewise contraction · Topological dynamics

Mathematics Subject Classification
Primary 37E05 · 37C25; Secondary 37B10

1 Introduction
This article contains two types of contributions to the area of dynamical systems. The first contribution is in the research topic named one-dimensional dynamics, dynamics of interval maps with gaps or dynamics of piecewise smooth interval maps. In this
regard, we study the topological dynamics of \( n \)-interval piecewise \( \pm \frac{1}{\beta} \)-affine contractions \( f : [0, 1) \to [0, 1) \) considering arbitrary integers \( \beta, n \geq 2 \). Our main contribution here is an explicit condition on the parameters of \( f \) which makes it possible to construct concrete examples with a finite \( \omega \)-limit set \( \omega(f) \). The second contribution is an application of a special case of the first result (with \( \beta = 2 \) and \( n = 4 \)) to understand the dynamics of switched server systems with 3 buffers and 1 server, which are mathematical models of manufacturing, traffic and queueing systems. We provide more details along the specialised forthcoming subsections.

1.1 Piecewise \( \lambda \)-Affine Contractions

Throughout this article, we denote by \( f : I \to I \) a self-map of the interval \( I = [0, 1) \). Given \( -1 < \lambda < 1 \), we say that \( f \) is an \( n \)-interval piecewise \( \lambda \)-affine contraction if there exist points \( 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1 \) and real numbers \( a_1, a_2, \ldots, a_n \) such that

\[
f(x) = \lambda x + a_i \quad \text{for all} \quad x \in [x_{i-1}, x_i) \quad \text{and} \quad 1 \leq i \leq n.
\]

We call the non-empty set

\[
\omega(f) = \bigcup_{x \in I} \omega(x, f), \quad \text{where} \quad \omega(x, f) = \bigcap_{r \geq 0} \bigcup_{k \geq r} \{f^k(x)\},
\]

the \( \omega \)-limit set of \( f \).

An important class of piecewise \( \lambda \)-affine contractions arises from the self-map \( f : \mathbb{T}^1 \to \mathbb{T}^1 \) of the circle \( \mathbb{T}^1 \) defined by \( f(x) = [\lambda x + a] \), where \( 0 < \lambda < 1 \), \( 0 < a < 1 \) and \([\cdot]\) denotes the fractional part of a real number. Writing \( f \) as an interval map leads to the 2-interval piecewise \( \lambda \)-affine contraction \( f : I \to I \) defined by

\[
f(x) = \begin{cases} 
\lambda x + a & \text{if} \quad x \in [0, \frac{1-a}{\lambda}) \\
\lambda x + a - 1 & \text{if} \quad x \in \left[\frac{1-a}{\lambda}, 1\right). 
\end{cases}
\]

The topological dynamics of the 2-interval piecewise \( \lambda \)-affine contractions given by (3) was studied by many authors by means of a rotation number approach and Farey trees (see [1, 4, 14, 16–19]). In particular, it is known that there exists a dichotomy: the \( \omega \)-limit set of \( f \) is either a finite set (generic case) or a Cantor set. The same study was made for another family of 2-interval piecewise \( \lambda \)-affine contractions topologically conjugate to \( f \) (see [10]). The study of the \( \omega \)-limit set of a piecewise (affine or not affine) contraction \( f \) defined on the union \( E_0 \cup E_1 \) of two complete metric spaces and such that \( f|_{E_0} \) and \( f|_{E_1} \) are contractions was carried out in [13].

The topological dynamics of \( n \)-interval piecewise \( \lambda \)-affine contractions, with \( n \geq 2 \), was considered in [9, 12, 22]. In contrast to the case \( n = 2 \) where a dichotomy is present, when \( n \geq 3 \) the \( \omega \)-limit set can be of three types: a finite set (generic case), finitely many Cantor sets, or the union of a finite set and finitely many Cantor sets (see [5, Theorem 1.1]). Piecewise \( \lambda \)-affine maps with a Cantor \( \omega \)-limit set were constructed in [6, 11] and [23, Corollary 2.5 and Theorem 2.7] [see also [18, Corollary 8] for piecewise contractions given by (3)]. It is also worth mentioning that every injective
piecewise (affine or not affine) contraction is topologically conjugate to a piecewise affine contraction (see [20, Theorem 1.2]).

Although the results provided in [12, 21, 22] are far comprehensive, they are not appropriate for concrete applications since it is not possible to exhibit parameters that lie in the full measure set of generic parameters. In this respect, the advantage of our result is that we can provide explicit (rational and irrational) parameters \( \lambda, a_i \) and \( x_i \) of \( f \) in (1) that make \( \omega(f) \) finite. The approach we use to prove the results connects in a natural way piecewise contractions to the fascinating world of \( \beta \)-transformations.

1.2 Switched Server Systems

Switched server systems are mathematical models of manufacturing, traffic and queueing systems. We consider here the same model studied in [11], which turns out to be a generalisation of the model introduced by Chase et al. in [8, Section II.B, p.72]. It consists of three buffers (tanks) numbered 1, 2, 3, and a server. At each time \( t \geq 0 \), a fluid is delivered to each tank \( i \) at the constant rate \( \rho_i = \frac{1}{3} (i = 1, 2, 3) \) and is removed from a selected tank \( j \) in \( \{1, 2, 3\} \) by the server at the constant rate \( \rho = 1 \). The volume of fluid in the tank \( i \) at the time \( t \) is denoted by \( v_i(t) \). When the tank \( i \) is emptied by the server at the time \( t \), the server changes its location to the tank \( j \neq i \) with the largest scaled volume \( d_{ij}v_j(t) \), where \( \{d_{ij} : 1 \leq i, j \leq 3, i \neq j\} \subset \mathbb{R}^+_* \) are parameters of the system. It was shown in [11] that the dynamics of the switched server system depends only on the proportions of pairs of parameters

\[
\frac{d_{13}}{d_{12}} = d_1, \quad \frac{d_{21}}{d_{23}} = d_2, \quad \frac{d_{32}}{d_{31}} = d_3. \tag{4}
\]

We assume that \( \sum_{i=1}^{3} v_i(0) = 1 \) so that \( \sum_{i=1}^{3} v_i(t) = 1 \) for every \( t \geq 0 \). In this way, the state \( v(t) = (v_1(t), v_2(t), v_3(t)) \) of the system at the time \( t \geq 0 \) is a vector in the phase space

\[
\Delta = \{ v = (v_1, v_2, v_3) : v_1, v_2, v_3 \geq 0 \text{ and } v_1 + v_2 + v_3 = 1 \}.
\]

Let \( l(t) \) denote the position of the server at the time \( t \). We assume that the map \( t \mapsto l(t) \) is right-continuous.

It follows from Nogueira et al. [22, Theorem 1.4] and from [11, Equation (1.2)] that, for Lebesgue almost every vector \( (d_1, d_2, d_3) \) with positive entries, any switched server system with parameters \( d_{ij} \) satisfying (4) is structurally stable and admits finitely many limit cycles that attract all the orbits. However, the set of generic parameters \( (d_1, d_2, d_3) \) is not computable, i.e., it is not possible (or at least not easy) to know if a previously chosen positive vector \( (d_1, d_2, d_2) \) lies in the generic set of parameters. In [11, Theorem 2.2] it was provided a non-generic positive vector \( (d_1, d_2, d_2) \) such that any switched server system with parameters \( d_{ij} \) satisfying (4) admits a fractal \( \omega \)-limit set.
1.3 The Interplay between Switched Server Systems and Interval Dynamics

The dynamics of the switched server system described in Sect. 1.2 with \( \rho_1 = \rho_2 = \rho_3 = \frac{1}{3} \) is governed by an injective piecewise \(-\frac{1}{2}\)-affine contraction. More precisely, the dynamics of the switched server system is completely determined by the Poincaré map \( F : \partial \Delta \rightarrow \partial \Delta \) induced by the flow

\[
t \in [0, \infty) \mapsto v(t) \in \Delta
\]
on the boundary \( \partial \Delta \) of the phase space. The Poincaré map \( F \) is topologically conjugate to the interval map \( f : I \rightarrow I \) defined by \( f = \phi^{-1} \circ F \circ \phi \), where \( \phi : [0, 1) \rightarrow \partial \Delta \) is the anticlockwise arc-length parametrisation of \( \partial \Delta \) with \( \phi(0) = e_2 = (0, 1, 0) \), \( \phi(\frac{1}{3}) = e_3 = (0, 0, 1) \) and \( \phi(\frac{2}{3}) = e_1 = (1, 0, 0) \). The map \( f \), computed in [11, Eq. (1.2)], is given by

\[
f(x) = f_{d_1,d_2,d_3}(x) = \begin{cases} 
  -\frac{1}{2}x + \frac{1}{2} & \text{if } x \in [x_0, x_1) \\
  -\frac{1}{2}x + 1 & \text{if } x \in [x_1, x_2) \\
  -\frac{1}{2}x + 1 & \text{if } x \in [x_2, x_3) \\
  -\frac{1}{2}x + \frac{1}{2} & \text{if } x \in [x_3, x_4) \\
  -\frac{1}{2}x + 1 & \text{if } x \in [x_4, x_5),
\end{cases} \tag{5}
\]

where

\[
x_0 = 0, \quad x_1 = \frac{d_2}{3(d_2 + d_3)}, \quad x_2 = \frac{d_3}{3(d_1 + d_3)} + \frac{1}{3}, \quad x_3 = \frac{d_1}{3(d_1 + d_2)} + \frac{2}{3}, \quad x_4 = 1. \tag{6}
\]

Figure 1 shows a graphical representation of the interplay between the switched server system, the Poincaré map \( F : \partial \Delta \rightarrow \partial \Delta \) and the piecewise \(-\frac{1}{2}\)-affine contraction \( f : I \rightarrow I \) considering the parameter values \( d_1 = d_2 = d_3 = 1 \) and \( d_{ij} = 1 \) for all \( i \neq j \).

2 Statement of the Results

We will need some terminology from Symbolic Dynamics and Number Theory. Let \( \beta \geq 2 \) be an integer and set \( A = \{0, 1, \ldots, \beta - 1\} \). Given an infinite word \( w = w_0w_1 \ldots \) over the alphabet \( A \) and a positive integer \( k \), we denote by \( L_k(w) = \{w_iw_{i+1} \ldots w_{i+k-1} : i \geq 0\} \) the set of factors of \( w \) of length \( k \). The factor complexity of \( w \) is the map \( p_w : \mathbb{N} \rightarrow \mathbb{N} \) defined by \( p_w(k) = \# L_k(w) \), where \# stands for cardinality. To each infinite word \( w = w_0w_1 \ldots \) over the alphabet \( A \), we can associate the real number \( x \) in \([0, 1]\) whose \( \beta \)-ary expansion is \( w \). More precisely,
We say that \( x = \varphi(w) \) is a rich number to base \( \beta \) or a disjunctive number to base \( \beta \) if \( p_w(k) = \beta^k \) for all \( k \geq 1 \), i.e., if all finite words over \( \mathcal{A} \) occur in the \( \beta \)-ary expansion of \( x \) (see [2, §4.4, p. 91]). Concerning numbers in the interval \([0, 1]\), the set \( \mathcal{R}_\beta \) made up of rich numbers to base \( \beta \) contains the proper subset \( \mathcal{N}_\beta \) of normal numbers in base \( \beta \). In particular, \( \mathcal{R}_\beta \) is a residual \( G_\delta \)-set of Lebesgue measure 1 (see [24, Lemma 19]). The property of being a rich number depends on the base considered: see for instance the examples constructed in [15]. Furthermore, there exist numbers that are rich in every base, see the explicit examples provided in [3]. Anyway, it is much easier to construct a rich number to base \( \beta \) than a normal number in the same base. We will also consider a third class of numbers of \([0, 1]\) defined as follows. Given integers \( \beta, n \geq 2 \), set

\[
\ell = \min\{k \in \mathbb{N} : 2\beta^{-k} < (1 - \beta^{-1})/(n + 1)\} = 1 + \left\lceil \frac{\log(2(n + 1)/(\beta - 1))}{\log \beta} \right\rceil,
\]

where \( \lceil x \rceil \) denotes the smallest integer strictly greater than \( x \), and let \( S_{\beta,n} \subset [0, 1] \) be the set defined by

\[
S_{\beta,n} = \{\varphi(w) : w \in \mathcal{A}^\mathbb{N} \text{ and } p_w(\ell) = \beta^\ell\}. \tag{7}
\]

The following nested sequence of inclusions holds for each integer \( n \geq 2 \):

\[
\mathcal{N}_\beta \subsetneq \mathcal{R}_\beta \subsetneq S_{\beta,n}.
\]

Moreover, \( S_{\beta,n} \) is a neighbourhood of \( \mathcal{R}_\beta \) and \( I \setminus S_{\beta,n} \) has Hausdorff dimension less than 1.
Now we state our results concerning the topological dynamics of piecewise \( \lambda \)-affine maps. We denote by \( \mathbb{Q} \) the set of rational numbers in \( I \).

**Theorem 2.1** Let \( \beta, n \geq 2 \) be integers, \( x_0 = 0, x_n = 1, 0 < x_1 < \cdots < x_{n-1} < 1 \) be elements of the set \( \mathbb{Q} \cup S_{\beta,n} \) defined by (7) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be elements of the set \( \{1, 2, \ldots, \beta\} \). Then \( \omega \)-limit set of the \( n \)-interval piecewise \( \frac{1}{\beta} \)-affine contraction \( f : I \to I \) defined by

\[
    f(x) = \frac{1}{\beta} x + \alpha_i - 1 \beta 
    \text{ for all } x \in [x_{i-1}, x_i) \text{ and } 1 \leq i \leq n,
\]

is finite.

Given integers \( \beta, n \geq 2 \), set

\[
    \ell' = \min\{ k \in \mathbb{N} : 2\beta^{-2k} < (1 - \beta^{-1})/(n + 1) \} = \left\lceil \frac{1}{2} + \frac{\log (2(n + 1)/(\beta - 1))}{2 \log \beta} \right\rceil
\]

and let \( S'_{\beta,n} \subset [0, 1] \) be the set defined by

\[
    S'_{\beta,n} = \left\{ \varphi(w) : w \in \{0, 1, \ldots, \beta^2 - 1\}^{\mathbb{N}} \text{ and } p_w(\ell') = \beta^{2\ell'} \right\}.
\]

The following nested sequence of inclusions holds for each integer \( n \geq 2 \) :

\[
    \mathcal{N}_{\beta^2} \subset \mathcal{R}_{\beta^2} \subset S'_{\beta,n}.
\]

**Theorem 2.2** Let \( \beta, n \geq 2 \) be integers, \( x_0 = 0, x_n = 1, 0 < x_1 < \cdots < x_{n-1} < 1 \) be elements of the set \( \mathbb{Q} \cup S'_{\beta,n} \) defined by (8) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be elements of the set \( \{1, 2, \ldots, \beta\} \) with \( \alpha_1 \neq \beta \). Then the \( \omega \)-limit set of the \( n \)-interval piecewise \( -\frac{1}{\beta} \)-affine contraction \( f : I \to I \) defined by

\[
    f(x) = -\frac{1}{\beta} x + \frac{\alpha_i}{\beta} \text{ for all } x \in [x_{i-1}, x_i) \text{ and } 1 \leq i \leq n,
\]

is finite.

Notice that taking \( \beta = 2, n = 4, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 2 \) in Theorem 2.2 leads to the map \( f_{d_1,d_2,d_3} \) given by (5). Moreover, the result in Theorem 2.2 is optimal because in [11, Definition 6.2, Proof of Proposition 6.4 and Figure 4] it was constructed an example with \( \beta = 2, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 2 \) such that the interval map in Theorem 2.2 has a Cantor \( \omega \)-limit set. In this case, it follows from Theorem 2.2 that at least one of the numbers \( x_i \) (denoted by \( z_i \) in [11]), \( 1 \leq i \leq 3 \), is an irrational number that is not rich to base 4 (in particular, such \( x_i \) is not a normal number in base 4). More precisely, since the integer \( \ell' \) corresponding to these parameters is equal to 3, there exists at least one word of length 3 over \( \{0, 1, 2, 3\} \) which does not occur in the 4-ary expansion of (at least) one of the \( x_i \)'s.
Now we state a consequence of Theorem 2.2 in the dynamics of switched server systems.

**Corollary 2.3** Let \( x_1 \) in \((0, \frac{1}{4})\), \( x_2 \) in \((\frac{1}{4}, \frac{3}{4})\) and \( x_3 \) in \((\frac{3}{4}, 1)\) be rational numbers or irrational numbers whose 4-ary expansions contain all words of length 3 over \(\{0, 1, 2, 3\}\) and let \((d_1, d_2, d_3)\) be the vector with positive entries satisfying

\[
\begin{align*}
    d_1 &= \frac{1}{3x_1} - 1, \\
    d_2 &= \frac{2 - 3x_2}{3x_2 - 1}, \\
    d_3 &= \frac{3 - 3x_3}{3x_3 - 2}.
\end{align*}
\]

Then any switched server system with parameters \(d_{ij}\) satisfying (4) has no fractal \(\omega\)-limit set or, more precisely, the \(\omega\)-limit set of its Poincaré map \(F : \partial \Delta \to \partial \Delta\) is finite.

A more concrete example can be constructed by considering the base-4 Champernowne constant

\[
c = \varphi(\zeta), \quad \text{where} \quad \zeta = 1 2 3 10 11 12 13 20 21 22 23 30 31 32 33 100 \ldots
\]

is the infinite word over \(\{0, 1, 2, 3\}\) obtained by concatenation of the finite 4-ary expansions of the positive integers. Clearly, by construction, \(c\) is rich to base 4. Notice that \(c\) is the real number whose decimal expansion is

\[
0.4261111111111106576455657142016198509554623896723 \ldots
\]

**Corollary 2.4** Let \( x_1 = c - \frac{1}{4}, \ x_2 = c \) and \( x_3 = c + \frac{1}{2}, \) where \(c\) is the base-4 Champernowne constant. Let \((d_1, d_2, d_3)\) be as in Corollary 2.3. Then any switched server system with parameters \(d_{ij}\) satisfying (4) has no fractal \(\omega\)-limit set or, more precisely, the \(\omega\)-limit set of its Poincaré map \(F : \partial \Delta \to \partial \Delta\) is finite.

In Fig. 2 we display the result of some computational simulations of the switched server system considering the parameter values of Corollary 2.4. The results are in agreement with the claim of Corollary 2.4 since the orbits \(t \in [0, \infty) \mapsto v(t)\) with \(v(0) = (0.11, 0, 0.89)\) and \(v(0) = (0.8, 0, 0.2)\), drawn in red, converge to a periodic (therefore non-fractal) \(\omega\)-limit set, drawn in blue. More precisely, the \(\omega\)-limit set of any \(v(0)\) in \([(0.11, 0, 0.89), (0.8, 0, 0.2)]\) under the Poincaré map \(F : \partial \Delta \to \partial \Delta\) is the finite set

\[
\omega(v(0), F) = \{(0.4, 0.6, 0), (0, 0.8, 0.2), (0.4, 0, 0.6), (0, 0.2, 0.8)\}.
\]

## 3 Proofs of the Results

**Definition 3.1** \((f\text{-invariant quasi-partition})\) Let \(f : I \to I\) be a self-map of \(I\) with a finite set of discontinuities \(D \subset I\). We say that a finite collection of pairwise disjoint open subintervals \(J_1, J_2, \ldots, J_m\) of \(I \setminus D\) is an \(f\text{-invariant quasi-partition}\

if \(I \setminus \bigcup_{i=1}^{m} J_i\) is a finite set and there exists a self-map \(\tau\) of \(\{1, 2, \ldots, m\}\) such that \(f(J_i) \subseteq J_{\tau(i)}\) for all \(1 \leq i \leq m\).
The next lemma is a variation of [21, Lemma 2.2, p. 1605] and [22, Proof of Theorem 2.1, p. 7], see also [7, Proof of Theorem 3.1] and [12, Proof of Theorem 1].

**Lemma 3.2** Assume that there exists an $f$-invariant quasi-partition for a piecewise $\lambda$-affine contraction $f : I \to I$. Then the set $\omega(f)$, the $\omega$-limit set of $f$, is finite.

**Proof** Let $J_i = (a_{i-1}, a_i)$, $1 \leq i \leq m$, be an $f$-invariant quasi-partition of $I$ and $\tau$ be the self-map of $\{1, 2, \ldots, m\}$ as in Definition 3.1. Let $P \subseteq \{1, 2, \ldots, m\}$ denote the set of periodic points of $\tau$ and let $f^{(0)}$ denote the identity map of $I$. Let

$$F = \bigcap_{k \geq 0} \bigcup_{i \in P} f^k(J_i), \quad G = [0, 1] \setminus \bigcup_{i=1}^m J_i. \quad (9)$$

We will split the proof in three claims:

**Claim A.** The set $\omega(f)$ is contained in $F \cup G$.

In fact, let $x$ be in $I$. Then, either $\omega(x, f) \subseteq G$ or there exist $i$ in $P$ and a non-negative integer $k$ such that $f^k(x)$ is in $J_i$. Then, by (2) and (9), we have that $\omega(x, f) = \omega \left( f^k(x), f \right) \subseteq F$. Since

$$\bigcup_{i \in P} f^{k+1}(J_i) \subseteq \bigcup_{i \in P} f^k(J_{\tau(i)}) = \bigcup_{i \in P} f^k(J_i),$$

we have that the sequence of sets $\left\{ \bigcup_{i \in P} f^k(J_i) \right\}_{k \geq 0}$ in (9) is a nested sequence of compact sets, thus $F$ is a compact non-empty set. Therefore, $\bar{F} = F$, implying that $\omega(x, f)$ is in $F$. In either case, $\omega(x, f) \in F \cup G$.

**Claim B.** We have $F = \bigcap_{j \geq 0} \bigcup_{i \in P} f^{jq}(J_i)$, where the positive integer $q$ is such that $\tau^q$ is the identity map on $P$.

It follows from the fact that the sequence of sets $\left\{ \bigcup_{i \in P} f^k(J_i) \right\}_{k \geq 0}$ in (9) is nested.
The set $G$ is finite because $J_1, J_2, \ldots, J_m$ is an $f$-invariant quasi-partition. Let us prove now that $F$ is finite. By Definition 3.1, $J_i \cap D$ is empty for all $i$, where $D$ is the set of discontinuities of $f$. Yet, since $f^k(J_i) \subseteq J_{\tau^k(i)}$, we have that $f^k(J_i) \cap D$ is empty for all $i$. Hence, because $f$ is a piecewise $\lambda$-affine map continuous on $\mathcal{I}/D$ we conclude that $f^k(J_i)$ is an interval of length less than or equal to $\lambda^k$ for all $k \geq 0$. Since

$$f^{(j+1)q}(J_i) \subseteq f^{jq}(J_{\tau^q(i)}) = f^{jq}(J_i),$$

we have that $\{f^{jq}(J_i)\}_{j \geq 0}$ is a nested sequence of compact intervals. Therefore, there exists a point $c_i$ in $[0, 1]$ such that $c_i = \bigcap_{j \geq 0} f^{jq}(J_i)$. Moreover, as already pointed out, $f^{jq}(J_i)$ is an interval of length less or equal to $\lambda^j$ containing $c_i$. By Claim B, we conclude that $F = \{c_1, \ldots, c_m\}$, where it may occur that $c_i = c_j$ for some $i \neq j$. In this way, $F$ is a finite set.

**Proof of Theorem 2.1** We assume all the hypotheses and notation in the statement of Theorem 2.1. We begin by observing that the map $f$ is injective. In fact, suppose that $f(x) = f(y)$. Then $|x - y| = |\alpha_j - \alpha_i|$. In particular, $|x - y|$ is an integer and $0 \leq |x - y| < 1$, that is, $x = y$. In this way, if $x$ is in $[0, 1)$ and $k$ is an integer, then, by the injectivity of $f$, the set $f^{-k}([x]) = \{y \in \mathcal{I} : f^k(y) = x\}$ is either a one-point-set or empty. We affirm that:

**Claim D.** For each $1 \leq i \leq n - 1$, there exists a positive integer $k_i$ such that

$$\bigcup_{k=0}^{\infty} f^{-k}([x_i]) = \bigcup_{k=0}^{k_i} f^{-k}([x_i]).$$

First assume that Claim D holds. Without any restriction, we assume that $k_i$ is minimal for the property of the claim. Then, for each $0 \leq k \leq k_i$, the set $f^{-k}([x_i])$ has a unique element, which we denote by $f^{-k}(x_i)$, thus $\{f^{-k}(x_i)\} = f^{-k}([x_i])$. Let

$$H = \left\{f^{-k}(x_i) : 0 \leq k \leq k_i \text{ and } 1 \leq i \leq n - 1 \right\}.$$

Let $J_1, J_2, \ldots, J_m$ be the pairwise disjoint connected components (open intervals) of $(0, 1) \setminus H$. It is clear that no point of $\bigcup_{s=1}^{m} J_s$ can be mapped into a point of $H$. In particular, $f(J_s)$ is an interval that contains no point of $H$, i.e. $f(J_s) \subseteq J_{\tau(s)}$ for some $\tau(s)$ in $\{1, 2, \ldots, m\}$. In this way, $J_1, J_2, \ldots, J_m$ is an $f$-invariant quasi-partition. Applying Lemma 3.2 we conclude that Theorem 2.1 holds provided we prove Claim D.

Now it remains to prove Claim D. Let $i$ be in $\{1, 2, \ldots, n - 1\}$. If $f^{-k}([x_i])$ is the empty-set for some positive integer $k$, then Claim D holds for such $i$. Otherwise, by the injectivity of $f$, we have that $f^{-k}([x_i])$ is a one-point-set for all positive integers $k$. As before, for each positive $k$, let $f^{-k}(x_i)$ denote the unique element of the set $f^{-k}([x_i])$. Since $f^{-1}(f^{-k}(x_i)) = f^{-(k+1)}(x_i)$, we have that $f(f^{-(k+1)}(x_i)) = f^{-k}(x_i)$, that is, $f^{-k}(x_i)$ is in $f(I)$ for all integers $k \geq 0$. No
In the sequel, let \( k \geq 0 \) be fixed. Since \( f^{-k}(x_i) \) is in \( f(I) \), we know that there exist \( j \) in \( \{1, \ldots, n\} \) and \( y_i \) in \([x_j-1, x_j)\) such that

\[
f^{-k}(x_i) = f(y_i) = \frac{1}{\beta} y_i + \frac{\alpha_j - 1}{\beta}.
\] (10)

In this way, we have that

\[
f^{-k}(x_i) \in \left[ \frac{\alpha_j - 1}{\beta}, \frac{\alpha_j}{\beta} \right]
\] and

\[
f^{-(k+1)}(x_i) = y_i = \beta f^{-k}(x_i) + 1 - \alpha_j.
\] (11)

We may rewrite (11) as

\[
f^{-(k+1)}(x_i) = T_\beta\left(f^{-k}(x_i)\right),
\] (12)

where \( T_\beta : I \rightarrow I \) is the \( \beta \)-transformation \( x \mapsto \{\beta x\} \) or, equivalently,

\[
T_\beta(x) = \beta x + 1 - r \quad \text{for all } x \in \left[ \frac{r - 1}{\beta}, \frac{r}{\beta} \right] \quad \text{and } r \in \{1, 2, \ldots, \beta\}.
\] (13)

Since \( k \geq 0 \) is arbitrary in (12), by induction on \( k \geq 0 \), we obtain that

\[
\{T_\beta^k(x_i) : k \geq 0\} = \left\{f^{-k}(x_i) : k \geq 0\right\} \subset f(I).
\] (14)

Now we have two cases to consider.

Case (a): \( x_i \in \mathbb{Q} \).

In this case, the \( \beta \)-ary expansion \( w \) of \( x_i \) is ultimately periodic meaning that the \( T_\beta \)-orbit of \( x_i \) is finite. Then, by (14), Claim D holds for such \( i \).

Case (b): \( x_i \in S_{\beta,n} \).

In this case, since \( f \) is a piecewise \( \frac{1}{\beta} \)-affine contraction, we have that \( I \setminus f(I) \) contains an open interval \( J \) of length \( |J| = (1 - \beta^{-1})/(n + 1) \). By the definition of the integer \( \ell \) in Section 2, we have that \( \beta^{-\ell} < \frac{1}{2} |J| \). Hence, there exists \( 0 \leq p \leq \beta^\ell - 1 \) such that \( \left[ \frac{p}{\beta^\ell}, \frac{p + 1}{\beta^\ell} \right] \subset J \). Since \( x_i \) is in \( S_{\beta,n} \), the \( T_\beta \)-orbit of \( x_i \) visits the intervals \( \left[ \frac{r}{\beta^\ell}, \frac{r + 1}{\beta^\ell} \right] \) for all \( 0 \leq r \leq \beta^\ell - 1 \). In particular, there exists a positive integer \( k' \) such that \( T_\beta^{k'}(x_i) \) is in \( J \subset I \setminus f(I) \), which contradicts (14). \( \square \)

**Definition 3.3** The fractional part of \( x \in \mathbb{R} \), denoted by \( \{x\} \), equals \( x - \lfloor x \rfloor \), where \( \lfloor x \rfloor \) denotes the largest integer \( n \) such that \( n \leq x \).

**Lemma 3.4** Let \( I = [0, 1) \), \( \alpha \in \mathbb{Z} \) and \( T_\alpha : I \rightarrow I \) be the transformation defined by \( T_\alpha(x) = \{\alpha x\} \). If \( \beta \geq 2 \) is an integer, then \( (T_\beta^-)^2 = T_{\beta^2} \).
Proof  Let $x \in I$, then

\[
(T_{-\beta})^2(x) = \{-\beta(-\beta x)\} = \{-\beta(-\beta x - [-\beta x])\} = \{\beta^2 x + \beta[-\beta x]\}
\]

\[
= \beta^2 x + \beta[-\beta x] - \left[ \beta^2 x + \beta[-\beta x] \right] \\
= \beta^2 x - \left[ \beta^2 x + \beta[-\beta x] - \beta[-\beta x] \right] \in [0, 1).
\]

Thus, by the uniqueness of the integral part of a real number, we conclude that

\[
\left[ \beta^2 x + \beta[-\beta x] \right] - \beta[-\beta x] = \beta^2 x.
\]

In this way,

\[
(T_{-\beta})^2(x) = \beta^2 x - \beta[-\beta x] - \left[ \beta^2 x + \beta[-\beta x] - \beta[-\beta x] \right] \\
= \beta^2 x - \left( \beta^2 x + \beta[-\beta x] - \beta[-\beta x] \right)
\]

\[
= \beta^2 x.
\]

This concludes the proof. \(\square\)

Proof of Theorem 2.2  The proof is exactly the same as the proof of Theorem 2.1 until before Eq. (10). From that point on, the proof should be modified as follows.

In the sequel, let $k \geq 0$ be fixed. Since $f^{-k}(x_i)$ is in $f(I)$, we know that there exist $j$ in $\{1, \ldots, n\}$ and $y_i$ in $[x_{j-1}, x_j)$ such that

\[
f^{-k}(x_i) = f(y_i) = -\frac{1}{\beta} y_i + \frac{\alpha_j}{\beta}.
\]  \hfill (15)

We affirm that $y_i \neq 0$. In fact, if $y_i = 0$, then $0 = y_i = f^{-k+1}(x_i) \in f(I)$, which contradicts the fact that $0 \notin f(I)$. Hence, $0 < y_i < 1$. By (15), we have that

\[
f^{-k}(x_i) \in \left( \frac{\alpha_j - 1}{\beta}, \frac{\alpha_j}{\beta} \right) \quad \text{and} \quad f^{-(k+1)}(x_i) = y_i = -\beta f^{-k}(x_i) + \alpha_j.
\]  \hfill (16)

We may rewrite (16) as

\[
f^{-(k+1)}(x_i) = T_{-\beta}(f^{-k}(x_i)),
\]  \hfill (17)

where $T_{-\beta} : I \rightarrow I$ is the transformation $x \mapsto \{-\beta x\}$, or equivalently, $T_{-\beta}(0) = 0$ and

\[
T_{-\beta}(x) = -\beta x + r \quad \text{for all} \ x \in \left( \frac{r - 1}{\beta}, \frac{r}{\beta} \right) \cap I \quad \text{and} \quad r \in \{1, 2, \ldots, \beta\}.
\]

Since $k \geq 0$ is arbitrary in (17), by induction on $k \geq 0$, we obtain that

\[
\{T_{-\beta}^k(x_i) : k \geq 0\} = \left\{ f^{-k}(x_i) : k \geq 0 \right\} \subset f(I).
\]  \hfill (18)
By Lemma 3.4 and by (18), we reach
\[
\left\{(T\beta_2)^k(x_i) : k \geq 0\right\} = \left\{(T_{-\beta})^{2k}(x_i) : k \geq 0\right\} \subset f(I). \tag{19}
\]

Now we have two cases to consider.

Case (a): \(x_i \in \mathbb{Q}\).
In this case, the \(\beta_2\)-ary expansion \(w\) of \(x_i\) is ultimately periodic meaning that the \(T\beta_2\)-orbit of \(x_i\) is finite. Then, by (19), Claim D holds for such \(i\).

Case (b): \(x_i \in S'_{\beta,n}\)
In this case, since \(f\) is a piecewise \(-1\)-affine contraction, we have that \(I \setminus f(I)\) contains an open interval \(J\) of length \(|J| = (1 - \beta^{-1})/(n + 1)\). By the definition of the integer \(\ell'\) in Section 2, we have that \(\beta^{-2\ell'} < \frac{1}{2}|J|\). Hence, there exists \(0 \leq p \leq \beta^{2\ell'} - 1\) such that \(\left[\frac{p}{\beta^{2\ell'}}, \frac{p + 1}{\beta^{2\ell'}}\right] \subset J\). Since \(x_i\) is in \(S'_{\beta,n}\), the \(T\beta_2\)-orbit of \(x_i\) visits the intervals \(\left[\frac{r}{\beta^{2\ell'}}, \frac{r + 1}{\beta^{2\ell'}}\right]\) for all \(0 \leq r \leq \beta^{2\ell'} - 1\). In particular, there exists a positive integer \(k'\) such that \(T_k\beta_2(x_i)\) is in \(J \subset I \setminus f(I)\), which contradicts (19). \(\square\)

Proof of Corollary 2.3 Let \((d_1, d_2, d_3)\) be as in the statement of Corollary 2.3. The Poincaré map \(F : \partial \Delta \to \partial \Delta\) of any switched server system with parameters \(d_{ij}\) satisfying (4) is topologically conjugate to the map \(f\) defined by (5). Moreover, given \(1 \leq i \leq 3\), either \(x_i \in \mathbb{Q}\) or the 4-ary expansion of \(x_i\) contains all words of length 3 over \(\{0, 1, 2, 3\}\) and thus \(x_i \in S'_{2,4}\). By Theorem 2.2, the map \(f\) (and therefore \(F\)) has a finite \(\omega\)-limit set, which concludes the proof. \(\square\)

Proof of Corollary 2.4 The base-4 Champernowne constant \(c\) is a normal number in base 4. Since the 4-ary expansions of \(x_1, x_2\) and \(x_3\) equal the 4-ary expansion of \(c\), up to finitely many digits, we have that \(x_1, x_2\) and \(x_3\) are normal numbers in base 4. Now the result follows from Corollary 2.3. \(\square\)

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