Finite-range model potentials for resonant interactions

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Abstract

We show that it is possible to model two-body resonant interactions at low energy with a class of finite-range potentials based on the methods of Jost and Kohn. These potentials are expressed in terms of the effective range $r_0$ and the $s$-wave scattering length $a_s$. We derive continuum solutions of these potentials. By writing $V_\pm(r) = V_0(r) + V_\pm^\epsilon(r)$, where the sign $+$(-) refers to positive(negative) scattering length, $V_0(r)$ is of the form of Pöschl-Teller potential and $V_\pm^\epsilon$ is expressed as a power series of the small parameter $\epsilon = (\sqrt{1 - 2r_0/a_s})^{-1} - 1$ when $a_s$ is large, we derive Green function of $V_0(r)$. Using the Green function, solutions of $V_\pm(r)$ for $|a_s| >> r_0$ can be obtained numerically by treating $V_\pm^\epsilon(r)$ as a perturbation. We describe the threshold behavior of scattering phase shift for $V_0(r)$. This study may be important for developing a better understanding of physics of strongly interacting ultracold atomic gases with tunable interactions.
Physics of interacting many-particle systems is described, to a first approximation, in terms of a mean-field potential. At low energy, there is a well-known mean-field potential of zero range. This is known as Fermi’s contact potential or pseudopotential [1], usually expressed as a delta function with the potential strength being proportional to s-wave scattering length $a_s$ only. For such a delta-type potential, s-wave scattering wave function at zero energy becomes singular at $r = 0$, where $r$ is the separation between two particles. To circumvent this problem, contact interaction is replaced with a suitable boundary condition at $r = 0$ in the Schrödinger equation resulting in the regularized pseudopotential [2–5]

$$V(r) = \frac{4\pi\hbar^2 a_s}{2\mu} \delta(r) \frac{\partial}{\partial r} r$$

where $\mu$ is the reduced mass of the two particles. The zero-range potential method is applicable to low temperature dilute atomic or molecular gases whose two-body interactions obey Wigner’s threshold laws [6]. For two spherically symmetric atoms, Wigner’s threshold laws dictate that $a_s$ becomes an energy-independent parameter. The s-wave scattering amplitude then reduces to the form $f_0(k) \simeq -\frac{a_s}{1+ika_s}$ which is $f_0(k) \simeq -a_s$ for $k|a_s| \ll 1$. However, when a quasibound or bound state exists near the threshold of a potential or a scattering resonance occurs at low energy, $a_s$ becomes large leading to the breakdown of the condition $k|a_s| \ll 1$. Wigner’s threshold laws do not apply when a resonance occurs near zero energy. As a consequence, Fermi’s pseudopotential approach becomes inadequate to describe collision or many-body physics near resonances.

The main purpose of this paper is to seek a finite-range model potential that can describe resonant interactions. Resonances and physical phenomena related to resonant interactions are ubiquitous in physics and chemistry. In ultracold atoms, Feshbach resonances can be induced by external magnetic [7, 8] or optical [9–11] or both magnetic and optical fields [12]. In current literature, many-body phenomena in ultracold atomic gases near a Feshbach resonance are by and large described in terms of contact potential [13]. However, the effective range and energy dependence of many-body phenomena can hardly be ignored when a resonance occurs. It is theoretically shown that for $a_s \gg R_{vdW}$ when $a_s$ is much larger than the characteristic length scale $R_{vdW} = (2\mu C_6/\hbar^2)^{1/4}/2$ of van der Waals potential, the effective range $r_0$ is about three times $R_{vdW}$ [14, 15]. The values of $R_{vdW}$ for different alkali atoms used in cold atom experiments are tabulated in Ref. [8]. For $a_s \gg R_{vdW}$, the effective range $r_0$ of a potential with van der Waals dispersion at long separation will be several tens of Bohr radius ($a_0$). This means $r_0$ is large
compared to typical molecular equilibrium positions. It has been recently shown that the range and energy dependence of magnetic Feshbach resonances of ultracold atoms are quite important \cite{16, 17}. Particularly in case of narrow Feshbach resonances, $r_0$ can be very large, even of the order of hundred or thousand $a_0$. Currently, attempts are being made to construct an improved model potential by resorting to a toy model \cite{18} or contact potentials with energy-dependent scattering length \cite{19}. The fact that a contact interaction can not accurately describe resonant interactions calls for the formulation of a better model potential that can take into account both the finite range and the energy dependence of scattering amplitudes.

Here we show that one can deduce, based on the methods of Jost and Kohn \cite{20, 21}, a class of finite-range model potentials that can account accurately for $s$-wave resonances at low energy. We demonstrate that, for large $a_s$, the model potentials reduce to Pöschl-Teller form \cite{22} that can admit analytical solutions. Using these analytical solutions, we construct Green function that can be used for solving the full potential numerically. The primary physical motivations behind this work is to establish the connections of strongly or unity-limited interactions with the exactly solvable Pöschl-Teller potentials. This study showing that the unitarity-limited interactions can be treated with exact analytical solutions may be important for gaining new insight into the physics of strongly interacting ultracold atomic gases. There is another motivation behind undertaking this work. In a seminal paper Butsch \textit{et al.} \cite{23} have presented exact solutions of two identical cold atoms interacting via regularized contact potential in one, two and three dimensional harmonic traps. However, as expected, these exact solutions will not apply to strongly interacting atoms in traps due to finite-range effects. It is therefore worth seeking exact solutions of two atoms interacting via the finite-range model potentials with large scattering length in a harmonic trap. Such solutions will lead to new insight into physics of strongly interacting systems in confined or low-dimensional space. In this paper we do not make any attempt to find solutions of two trapped atoms interacting with finite-range potentials. However, the finite-range potentials and their exact solutions presented in this paper may serve as a precursor towards generalizing the results of Butsch \textit{et al.} \cite{23} for finite-range potentials.

This paper is organized in the following way. In the next section, we present our proposed finite-range potentials for resonant interactions and discuss how they are obtained. In Sec. 3, we present continuum and bound state solutions of the potentials. Analytical and numerical results are discussed in Sec. 4. The paper is concluded in Sec. 5.
II. FINITE-RANGE MODEL POTENTIALS

The model potentials we consider are some variants of Bargmann potentials [24] and derived using effective range expansion of phase shift. The connections between phase shifts and potentials were first rigorously established by Bargmann [25]. The early works on the method derivations of finite-range potential from the phase shift data were carried out by Gel’fand and Levitan [26], Jost and Kohn [20, 21], and many others. Here we follow the method of Jost and Kohn. A particularly simple model potential results in when the phase shift \( \delta_0(k) \) is given by effective range expansion

\[
\cot \delta_0(k) = -\frac{1}{ka_s} + \frac{1}{2} r_0 k + \cdots \tag{2}
\]

at low energy under the conditions \( kr_0 < 1 \) and \( r_0 << |a_s| \). Jost and Kohn [20] showed that, for negative \( a_s \), the model potential takes the form

\[
V_-(r) = -\frac{4\hbar^2}{\mu r_0^2} \frac{\alpha \beta^2 \exp(-2\beta r/r_0)}{[\alpha + \exp(-2\beta r/r_0)]^2} \tag{3}
\]

where \( \alpha = \sqrt{1 - 2r_0/a_s} \), \( \beta = 1 + \alpha \) and \( \mu \) is the reduced mass. This is valid for \( |\delta_0(E)| < \pi/2 \) in the limit \( E \to 0 \). When a scattering resonance occurs, \( k|a_s| >> 1 \) such that if one neglects the effects of effective range then \( \delta_0 \simeq \pi/2 \). This means that the \( S \)-matrix element \( \exp[2i\delta_0] \simeq -1 \) leading to unitarity-limited interactions. We define unitarity regime by \(-1 << (ka_s)^{-1} << 1\).

When \( a_s \) is positive and large, the potential can support one near-zero energy bound state. Therefore, to obtain a model potential for positive \( a_s \) from effective range expansion, one needs to incorporate the binding energy of the bound state. Then the potential becomes a three-parameter potential. This potential is not unique unless an additional parameter corresponding to the bound state is taken into account. As shown by Jost and Kohn [21], the \( s \)-wave binding energy \( E_b = -\hbar^2\kappa_0^2/2\mu \) (where \( \kappa_0 > 0 \)) can be parametrized, under effective range expansion, by introducing a parameter \( \Lambda \) to express \( \kappa_0 \) in the following form

\[
\kappa_0 = \frac{1}{r_0} \left[ 1 + \sqrt{1 - 2r_0/a_s} \right] \frac{1 + \Lambda}{1 - \Lambda} \tag{4}
\]

It is bounded by \(-1 < \Lambda < 1 \) for \( a_s > 2r_0 \) and \( r_0 > 0 \). Using these three parameters \( a_s, r_0 \) and \( \Lambda \), an expression for the model potential is given in Eq. (2.29) of Ref. [21], where \( \alpha \) is bounded by \( 0 < \alpha \leq 1 \). Now, if we make a choice \( \Lambda = -\sqrt{1 - 2r_0/a_s} \) the Eq. (2.29) of Ref. [21] reduces to a two-parameter potential of the form

\[
V_+(r) = -\frac{4\hbar^2}{\mu r_0^2} \frac{\alpha \beta^2 \exp(-2\beta r/r_0)}{[1 + \alpha \exp(-2\beta r/r_0)]^2}. \tag{5}
\]
This choice of $\Lambda$ corresponds to the binding energy $E_{\text{bin}} \simeq \hbar^2/(2\mu a_s^2)$ for $2r_0/a_s \ll 1$.

It is easy to notice that in the limit $a_s \to \pm \infty$, both the potential of Eqs. (3) (5) reduces to the form

$$V_\infty = -4\hbar^2 \frac{\kappa^2}{\mu} \frac{\alpha^{-1}}{\cosh^2 (2\kappa r)}$$

This form of the potential has been employed by Carlson et al. [27] for quantum Monte Carlo simulation of a homogeneous superfluid Fermi gas with infinite negative scattering length. Shea et al. [28] have calculated the energy spectrum of two harmonically trapped atoms interacting via the potential of Eq. (6), showing that bound state spectrum remains almost the same as for zero-range pseudopotential [23] if $r_0$ is much less than the characteristic length scale $l_{ho} = \sqrt{\hbar/\mu \omega_{ho}}$ of the harmonic oscillator trap with frequency $\omega_{ho}$.

The potentials of Eqs. (3) and (5) can be written in the forms $V_\pm (r) = V_0(r) + V^{(e)}_\pm (r)$ where

$$V_0(r) = -\frac{\hbar^2 \kappa^2}{\mu} \frac{\alpha^{-1}}{\cosh^2 (\kappa r)}$$

and

$$V^{(e)}_\pm = V_0 \sum_{n=1}^{\infty} (-1)^n (n+1) \left[ \frac{\epsilon}{1 + \exp(\pm 2\kappa r)} \right]^n.$$  

$V_0(r)$ is in the form of Pöschl-Teller potential of second kind. Pöschl-Teller potentials and their different variants are well-known in quantum mechanics [29, 30] as exactly solvable potentials in one dimension. In three dimension (3D), s-wave bound and continuum (scattering) solutions of Pöschl-Teller potentials are obtained by a group theoretic algebraic approach [31, 32] as well as by non-algebraic methods [30, 33]. The Schroedinger equation of relative motion can be expressed in the form

$$\mathcal{L}_r \psi_\pm (r) = -V^{(e)}_\pm \psi_\pm (r)$$

where

$$\mathcal{L}_r = -\frac{\hbar^2}{2\mu} \left[ \frac{d^2}{dr^2} + k^2 \right] + V_0$$

By treating the right hand side of Eq. (9) as a source term, we seek solutions of the homogeneous equation

$$\mathcal{L}_r \psi_\pm (r) = 0$$

In the following section we present exact solution of $V_0(r)$. 

5
III. SOLUTIONS OF $V_0(r)$

To obtain solutions of Eq. (11), we first convert this equation into the standard equation for associated Legendre functions. We then write down the desired solutions as a superposition of two linearly independent associated Legendre functions by imposing the boundary conditions for scattering states. Next, we rewrite the solutions in terms of hypergeometric functions for asymptotic analysis. We then construct Green function of Eq. (11). In the following subsection we briefly describe our method of solution. In passing it is worth noting that as the range $r_0$ goes to zero, Pöschl-Teller potential in 1D reduces to a delta well potential [34]. So, it is expected that for $s$-wave interactions three dimensional Pöschl-Teller potential will behave as a contact potential.

A. Scattering solutions of $V_0$

Let $\kappa = \beta/r_0$, $\kappa r = \rho$, $\cos \theta = \tanh \rho = z$, then we have

$$\mathcal{L}_\rho = -\frac{\hbar^2}{2\mu} \left[ \frac{d^2}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \frac{d}{d\theta} - \frac{q^2}{\sin^2 \theta} \right] \sin^2 \theta + V_0$$

where

$q = i k/\kappa = \sqrt{-2\mu E / \hbar^2 / \kappa}$.  

Furthermore, substituting $V_0$ given by Eq. (7) in Eq. (11), we have

$$\left[ (1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + \lambda(\lambda + 1) - \frac{q^2}{1 - z^2} \right] \psi(z) = 0.$$  

where the parameter $\lambda$ is given by $\lambda(\lambda + 1) = 2\alpha^{-1}$. This is the familiar equation of associated Legendre functions. Two linearly independent basic solutions of this equation are $P^q_{\lambda}(z)$ and $Q^q_{\lambda}(z)$.

By doing asymptotic analysis of the two basic solutions as given in the appendix, one can write down the regular scattering solution of the potential $V_0$ in the form

$$\psi(z) = \mathcal{N}_{\lambda,q} \left[ \Gamma \left( \frac{1}{2} + \frac{\lambda}{2} + \frac{q}{2} \right) P^q_{\lambda}(z) - 2e^{-izq - iz(q-\lambda-1)} Q^q_{\lambda}(z) \frac{1}{\Gamma \left( \frac{1}{2} - \frac{\lambda}{2} - \frac{q}{2} \right)} \right]$$

with $\mathcal{N}_{\lambda,q} = A_{q,\lambda}^{-1}$ where $A_{q,\lambda}$ is given in the appendix. One can notice that $\alpha$ is bounded by $1 \leq \alpha < 2$ or $0 < \alpha \leq 1$ depending on whether $a_s$ is negative or positive, respectively. In the limits $a_s \to \pm\infty$, $\alpha \to 1$ and therefore $\lambda$ assumes integer values of either -2 or 1.
B. Green function

Although Green functions for one-dimensional Pöschl-Teller potentials are studied before [35], to the best of our knowledge the green functions of Pöschl-Teller potentials in three dimension (3D) is not considered before. We first note down the Wronskian [36]

\[ W \{ Pq_{\lambda}(z), Qq_{\lambda}(z) \} = e^{i\pi q/2} \frac{\Gamma(1 + \frac{\lambda + q}{2}) \Gamma(\frac{1}{2} + \frac{\lambda + q}{2})}{(1 - z^2) \Gamma(1 + \frac{\lambda - q}{2}) \Gamma(\frac{1}{2} + \frac{\lambda - q}{2})} \]  

(16)

between \( Pq_{\lambda}(z) \) and \( Qq_{\lambda}(z) \). To construct the Green function, we prepare another state \( \phi(z) \) by linear superposition between \( Pq_{\lambda}(z) \) and \( Qq_{\lambda}(z) \) such that this state satisfies the outgoing boundary condition \( \phi(r \to \infty) \sim -e^{i(kr + \delta_0)} \). This leads to

\[ \phi(z) = -\frac{\pi e^{-i\pi(\lambda + q/2)}}{\Gamma(q) [a_{\lambda}(q) - a_{\lambda}(-q)]} e^{i\delta_0} Pq_{\lambda} \]  

(17)

where

\[ a_{\lambda}(q) = \cos[\pi(\lambda + q)/2] \sin[\pi(q - \lambda)/2] \]  

(18)

The Green function can be written in the form

\[ G_{E}(\rho, \rho') \equiv G_{E}(z, z') = -\pi \psi(z_<) \phi(z_> \) 

(19)

That the Green function \( G_{E}(\rho, \rho') \) is the solution of the Green equation

\[ \mathcal{L}_\rho G_{E}(\rho, \rho') = -\frac{\hbar^2}{2\mu} (1 - z^2) \delta(z - z') = -\frac{\hbar^2}{2\mu} \delta(\rho - \rho') \]  

(20)

can be ascertained from the relation

\[ \int_{\rho' = 0^+}^{\rho' = 0^+} d\rho \mathcal{L}_\rho G_{E}(\rho, \rho') = 1 \]  

(21)

C. Bound state solutions of \( V_0 \)

A bound state occurs for energy \( E = -|E| \) or \( k = i|k| \), and so the parameter \( q \) defined in Eq. \[13\] becomes \( q = -\bar{k} = -|k|/\kappa \). For the solution \( \psi_0^0(r) \) to behave as a bound state, asymptotically \( \psi(r \to \infty) \sim e^{-|k|r} \). For negative energy, the asymptotic form of \( \psi_0^0(r) \) takes the form

\[ \psi(r \to \infty) \sim [G_{\lambda}(q) e^{-|k|r} + G_{\lambda}(-q) e^{|k|r}] \]  

(22)
FIG. 1: Absolute square of energy-normalized continuum wave function $\psi(r)$ in unit of $E_0^{-1}r_0^{-1}$ is plotted against $r$ (in unit of $r_0$) for $a_s = 50r_0$ (solid line), $a_s = 100r_0$ (dashed lines), $a_s = -50r_0$ (dashed-dotted) and $a_s = -100r_0$ (dotted) for $k = 0.01r_0^{-1}$.

Now, for $\psi$ to qualify as a bound state, the coefficient of $e^{ikr}$ must vanish. Thus, the bound state condition is given by

$$G_\lambda(\bar{k}) = \frac{2\bar{k}\Gamma(\bar{k})}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\bar{k}}{2}\right) \Gamma\left(1 + \frac{\lambda}{2} + \frac{\bar{k}}{2}\right)} = 0$$

(23)

Now, assuming $x = -(\frac{\lambda}{2} + \frac{\bar{k}}{2})$, and using the identity $\Gamma(1 - x)\Gamma(x)\pi/\sin[\pi x]$, we can rewrite the above equation in the form

$$G_\lambda(\bar{k}) = \frac{2\bar{k}}{\pi} \frac{\Gamma\left(-\frac{\lambda}{2} - \frac{\bar{k}}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\bar{k}}{2}\right)} \sin\pi\left(-\frac{\lambda}{2} - \frac{\bar{k}}{2}\right)$$

(24)

This shows that $G_\lambda(\bar{k}) = 0$ when

$$-\lambda - \bar{k} = 2n$$

(25)

with $n$ being an integer, or the argument of the gamma function in the denominator on the right hand side of the Eq. (24) is a negative integer. As $a_s$ decreases from $+\infty$ to $2r_0$, $\lambda$ decreases from -2 to $-\infty$. This means that the argument of this gamma function can not be negative since $\bar{k} \geq 0$. Therefore, the bound state condition is given by Eq. (25) which, for low energy, will be satisfied for $n = 1$. On the other hand, when $a_s < 0$ and as $a_s$ changes from $-\infty$ to small negative value $\lambda$ decreases from 1 ranging between $0 < \lambda \leq 1$. This means the above equation can not be fulfilled
FIG. 2: The phase shift $\delta_0(k)$ in unit of $\pi$ radian is plotted against dimensionless momentum $kr_0$ (scaled by a factor of $10^4$) for three positive and three negative values of scattering length $a_s$ as indicated in the legend. The lower dotted, dashed and solid curves correspond to $a_s = -10^3$, $a_s = -10^4$ and $a_s = -10^5$, respectively; while the upper dotted, dashed and solid curves are plotted for positive $a_s$ of same magnitudes, respectively.

for negative scattering length. These findings are consistent with the conditions for bound states obtained earlier by group theoretic approach [32].

IV. RESULTS AND DISCUSSIONS

In figure 1, we show the square of energy-normalized scattering or continuum wave function $|\psi(r)|^2$ as a function of $r$ for four values of relatively large scattering lengths at a fixed low energy. The contrasting behavior of the wave functions for positive and negative scattering lengths for $r \geq r_0$ is noteworthy. The intercepts of the slope of the wave function at $r = r_0$ on the $x$-axis are positive and negative for positive and negative scattering lengths, respectively. The intercepts are nearly equal to the respective scattering lengths. Thus the wave functions behave exactly like those of a 3D square well at low energy. At $r = r_0$, the wave functions for positive and negative scattering lengths bend towards and away from $x$-axis, respectively.

Next we analyze phase shifts at low energy. From the asymptotic analysis of the solution for
\[ V_0 \text{ as given in the appendix, we have } \psi(r \to \infty) \sim \sin[kr + \delta_0] \text{ where} \]

\[ \delta_0(k) = -\tan^{-1} \left[ \frac{\cot \frac{-\pi \lambda}{2} \tanh \left( \frac{\pi k}{2} \right)}{\sqrt{1 + 8 \alpha^{-1}}} \right] + \arctan \left( \frac{k}{\lambda \kappa} \right) + \sum_{n=1}^{\infty} \left[ \arctan \left( \frac{k}{-(\lambda + n) \kappa} \right) - \arctan \left( \frac{k}{n \kappa} \right) \right] \]  

(26)

From the expression for \( \delta_0 \) given in Eq. (26) one can notice that, for \( k \neq 0 \), as \( \lambda \to 1 \) we get \( \delta_0 \to \pi/2 - 0_+ \) while as \( \lambda \to 0 \) approaches to -2, \( \delta_0 \) approaches to \( \pi/2 + 0_+ \). From the relation \( a_s = -\lim_{k \to 0} \tan[\delta_0(k)]/k \), one finds that, for negative \( a_s \) the resonance \( (a_s \to -\infty) \) will be characterized by \( \delta_0 \to \pi/2 - 0_+ \), that is, by \( \lambda = 1 \) while for positive \( a_s \) the resonance will occur when \( \lambda = -2 \). The Eq. (26) suggests that in order for \( \delta_0(k) \leq \pi/2 \) so that \( a_s < 0 \), one needs to set \( \lambda = \lambda_1 = -1/2 + \sqrt{1 + 8 \alpha^{-1}}/2 \). Similarly, for \( a_s > 0 \) one has to fix \( \lambda = \lambda_2 = -1/2 - \sqrt{1 + 8 \alpha^{-1}}/2 \). This analysis shows that we have 0 < \( \delta_0 < \pi/2 \) and \( \pi/2 < \delta_0 < \pi \) for \( a_s < 0 \) and \( a_s > 0 \), respectively. For \( |\lambda| \ll 1 \) the phase shift \( \delta_0(k) \propto k \) in accordance with Wigner’s threshold laws. In the limit \( \alpha \to 1 \), \( \lambda \) can assume a value of either -2 or 1. In both limits \( a_s \to -\infty \) and \( a_s \to +\infty \), the parameter \( \lambda \) is given by \( \lambda(\lambda + 1) = 2 \).

Figure 2 displays the variation of \( \delta_0(k) \) as a function of \( kr_0 \) for different values of \( a_s \). To remain within the validity regime of the effective range expansion, in this plot we have restricted the variation of \( kr_0 \) at low values below \( 10^{-3} \). From this figure we notice that for large scattering length \( \delta_0(k) \) approaches the resonant value \( \pi/2 \) at finite momentum while in the zero momentum limit \( \delta_0(k) \) significantly deviates from \( \pi/2 \) even for large scattering length. This implies that the momentum- and range-dependence of interactions become particularly important for resonant interactions.

We next discuss how these potentials will be useful to model tunable MFR in ultracold atoms. The \( s \)-wave phase shift near MFR is \( \delta_0(k) = \delta_{bg} + \delta_r \) where the background phase shift \( \delta_{bg} \) can be approximated as \( \delta_{bg} \simeq -ka_{bg} \) with \( a_{bg} \) being the background scattering length. Here \( \delta_r \) is the resonance phase shift. At low energy, the Feshbach resonance width \( \Gamma_f \) is given by \( \Gamma_f/2 \simeq ka_{bg}\Gamma_0 \) where \( \Gamma_0 \) is a parameter related to the width of zero crossing. Now, assuming \( |ka_{bg}| \ll 1 \) one can write

\[ \frac{1}{a_s} = \frac{B_0 - B}{\Delta a_{bg}} \]  

(27)

and

\[ r_0 = 2a_{bg} - \frac{\hbar}{\mu a_{bg}\Gamma_0} \]  

(28)
where $B_0$ is the magnetic field at which MFR occurs (or equivalently, $a_s$ diverges) at zero energy. Since $a_{bg} \Gamma_0 > 0$, the above equation shows that $r_0$ will be positive if $a_{bg}$ is positive and $a_{bg} > \sqrt{h/(2\mu \Gamma_0)}$. If $a_{bg} < 0$ then $r_0$ is negative. Since the model potentials of Eqs. (3) and (5) are derived assuming $r_0 > 0$, these potentials will be useful to model those MFR for which $r_0$ is positive. For example, magnetic Feshbach Resonances observed in $^{133}$Cs near $B_0 = -11.7, 547$ and 800 G fulfill the conditions for $r_0 > 0$. In general, for a relatively broad Feshbach resonance or an open-channel dominated Feshbach resonance [8] with positive $a_{bg}$ for which binding energy of closed channel bound state has the universal form $E_b \sim h^2/(2\mu a_s^2)$, the resonance can be described effectively by a single-channel potential using our proposed model potentials. The treatment of MFR with our proposed finite-range potentials will give new insight into resonant phenomena in ultracold atoms by providing effective range dependence of scattering phase shifts and near-zero energy bound states. The analytical solutions found in this paper will be useful to calculate photoassociation in the presence of an MFR and thereby to develop quantitative understanding of the Fano effect in photoassociation [39]. Furthermore, in case of two-component Fermi gas of atoms, the modeling of MFR with the finite-range potentials will facilitate to investigate the hither-to-unexplored nontrivial effects of large effective range on the BCS-BEC crossover from Bardeen-Cooper-Schrieffer (BCS) state of Cooper-pairs to Bose-Einstein condensate (BEC) of dimers.

V. CONCLUSIONS AND OUTLOOK

In conclusion we have demonstrated that the effective range dependence of resonant two-body interactions can be taken into account within a class of model potentials constructed under effective range expansion using the methods of Jost and Kohn. We have shown that these potentials reduce to the form of Pöschl-Teller potential when scattering length is large. We have presented analytical scattering and bound state solutions of the potentials for large scattering lengths and established the connections of the nature of the solutions with the sign and strength of the $\lambda$-parameter of the Pöschl-Teller potential. These finite-range model potentials will permit us to explore the finite-range effects of interactions between atoms in low-dimensional traps. In a recent paper [40], using these potentials we have numerically studied bound-state properties of two atoms in a quasi-two dimensional trap and found significant effects of the range on the bound states. For studies of collisional properties of cold atoms near resonances, one can employ more accurate multichan-
nel scattering-based computational methods where one can use molecular potentials. However, to gain insight into many-body physics near resonances, one prefers a single-channel simplified model potential that can be Fourier transformed so that one can conveniently develop many-body treatment of a homogeneous system in momentum space. Keeping this in mind, we have shown that there exists a class of simplified finite-range potentials that are well suited for the purpose of describing resonant interactions more accurately with both finite-range and energy-dependence of the scattering processes being taken into account.

Appendix A: Asymptotic analysis

The regular scattering solution is required to fulfill the boundary conditions \( \psi(r = 0) = 0 \) and \( \psi(r \to \infty) \sim \sin[kr + \delta_0(k)] \) or equivalently, \( \psi(z = 0) = 0 \) and \( \psi(z \to 1) \sim \sin[k\rho + \delta_0(k)] \). This means \( \psi(z) \) will be given by the superposition of the two basic solutions fulfilling these two boundary conditions. Thus we have

\[
\psi(z) \sim \Gamma \left( \frac{1}{2} + \frac{\lambda}{2} + \frac{q}{2} \right) P^q_\lambda(z) - 2e^{-i\pi q - i\frac{\pi}{2}(q - \frac{1}{2})} Q_q^q(z) \frac{1}{\Gamma(\frac{1}{2} - \frac{\lambda}{2} - \frac{q}{2})}
\]  

(A1)

Expressing the two associated Legendre functions in terms of hypergeometric functions, and after a lengthy algebra one obtains

\[
\psi(z) \sim 2i2^q e^{i\pi(\lambda+q)} \pi^{1/2} \frac{\Gamma \left( \frac{1}{2} + \frac{\lambda}{2} + \frac{q}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{\lambda}{2} - \frac{q}{2} \right) \Gamma \left( -\frac{\lambda}{2} - \frac{q}{2} \right) \sin \left[ \pi \left( \frac{\lambda}{2} + \frac{q}{2} \right) \right]} \times z(1 - z^2)^{-1/2}(z^2 - 1)^{\frac{3}{2}} \left( \frac{1}{2} - \frac{\lambda}{2} - \frac{q}{2} \right) \left( \frac{1}{2} - \frac{\lambda}{2} + \frac{q}{2} \right) \left( \frac{1}{2} - \frac{\lambda}{2} + \frac{q}{2} \right) \left( \frac{3}{2} \right) \left( \frac{z^2}{z^2 - 1} \right)\right)
\]  

(A2)

From this expression it is easy to verify that at \( r = 0 \) or \( z = 0 \), \( \psi = 0 \) and therefore \( \psi(z) \) is a regular solution.

Now, we will show that \( \psi(z) \) satisfies the proper asymptotic boundary condition \( \psi(r \to \infty) \sim \sin[kr + \delta_0(k)] \). Since \( z^2 = \tanh^2 \rho \), \( \frac{z^2}{z^2 - 1} = -\sinh^2 \rho \), we can write \( z(1 - z^2)^{-1/2}(z^2 - 1)^{\frac{3}{2}} = \)
$e^{i\pi \lambda/2} \sinh \rho \left[ \cosh^2 \rho \right]^{-\lambda/2}$ and

\[
F \left( \frac{1}{2} - \frac{\lambda}{2}, \frac{1}{2} - \frac{\lambda}{2} + \frac{q}{2}; \frac{3}{2}, z^2 - q^2 \right) = \frac{\Gamma \left( \frac{3}{2} \right) \Gamma(q)}{\Gamma \left( \frac{1}{2} - \frac{\lambda}{2} + \frac{q}{2} \right) \Gamma \left( 1 + \frac{\lambda}{2} + \frac{q}{2} \right)}
\]

\[
\times (\sinh^2 \rho)^\frac{\lambda}{2} \left( \frac{1}{2} - \frac{\lambda}{2} + \frac{q}{2}; 1 - q; -1 + \frac{1}{\sinh^2 \rho} \right)
\]

\[
\times (\sinh^2 \rho)^\frac{\lambda}{2} \left( \frac{1}{2} - \frac{\lambda}{2} + \frac{q}{2}; 1 - q; -1 + \frac{1}{\sinh^2 \rho} \right)
\]

(A3)

Thus we have

\[
\psi(r \to \infty) \sim A_{q, \lambda} \left[ G_{\lambda}(q)e^{ikr} + G_{\lambda}(-q)e^{-ikr} \right]
\]

(A4)

where

\[
A_{q, \lambda} = i^{21+q} e^{i\pi q} e^{i\pi \lambda/2} \pi^{1/2} \Gamma(3/2) \Gamma(1 + \lambda/2 + q/2)
\]

\[
\sin \left[ \pi \left( \frac{\lambda}{2} + \frac{q}{2} \right) \Gamma \left( \frac{1}{2} + \frac{\lambda}{2} - \frac{q}{2} \right) \Gamma \left( -\frac{\lambda}{2} - \frac{q}{2} \right) \right]
\]

(A5)

and

\[
G_{\lambda}(q) = \frac{2^{-q}\Gamma(q)}{\Gamma \left( \frac{1}{2} - \frac{\lambda}{2} + \frac{q}{2} \right) \Gamma \left( 1 + \frac{\lambda}{2} + \frac{q}{2} \right)}
\]

(A6)

The quantity within the third bracket on the right hand side of Eq. (A4) can be expressed in the form $|G_{\lambda}(q)| \sin [kr + \delta_0]$ with $\delta_0 = \frac{\pi}{2} + \phi$ where

\[
\phi = \arg \left[ \frac{2^{-q}\Gamma(q)}{\Gamma \left( \frac{1}{2} - \frac{\lambda}{2} + \frac{q}{2} \right) \Gamma \left( 1 + \frac{\lambda}{2} + \frac{q}{2} \right)} \right]
\]

(A7)

Substituting $q = ik/\kappa$ and making use of the standard phase relationship involving gamma functions we obtain

\[
\phi = \arg \Gamma(ik/\kappa) - \arg \Gamma(-\lambda + ik/\kappa) - \tan^{-1} \left[ \cot \frac{-\pi \lambda}{2} \tanh \left( \frac{\pi k}{2} \right) \right]
\]

(A8)

Using the formulas [6.1.27], [6.3.16], [6.3.7] of Ref. [36], one can obtain Eq.(26).

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