$k$-FREE NUMBERS AND INTEGER PARTS OF $\alpha p$

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ABSTRACT. In this note, we obtain asymptotic results on integer parts of $\alpha p$ that are free of $k$th powers of primes, where $p$ is a prime number and $\alpha$ is a positive real number.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\alpha$ and $\beta$ be real numbers such that $\alpha > 0$. Let $[x]$ denote the largest integer not greater than $x$. Sequences of the form $\{[\alpha n + \beta]\}_{n=1}^{\infty}$ are called Beatty sequences. A Beatty sequence is said to be homogeneous if $\beta = 0$. Beatty sequences have been attracting a lot of attention since they can be viewed as analogues of arithmetic progressions, therefore they show up in a broad context. The interested reader is referred to [1,2,4–6,8–11,14–16,19,24].

Let $k \geq 2$ be an integer. An integer is said to be $k$-free if it is not divisible by a $k$th power of a prime. Very recently in [3], an asymptotic formula with an explicit error term is obtained for $k$-free values of homogeneous Beatty sequences at prime arguments (i.e. sequences of the form $\{[\alpha p]\}_{p=2}^{\infty}$) provided that $\alpha$ is of finite type (see Definition 1). This result can be viewed as a natural analogue of the result of Mirsky [20]. In this article, we pursue this result and obtain two asymptotic formulas that are of the same flavour. The results we present here are well applicable to non-homogeneous Beatty sequences.

**Theorem 1.** Let $k \geq 2$ be an integer. Let $\{\alpha_i\}_{i=1}^\ell$ be a finite type subset of irrational numbers each greater than one. Assume that $\{\alpha_i\}_{i=1}^\ell$ satisfies (1) for some $\tau > 0$. Let $\mathbf{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and

$$
\pi(x, k, \mathbf{\alpha}) = \#\{p \leq x : \lfloor \alpha_i p \rfloor \text{ is } k\text{-free for each } i = 1, \ldots, \ell\}.
$$

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Then the following asymptotic is satisfied:
\[ \pi(x, k, \alpha) = \frac{\pi(x)}{\zeta(k)} + O \left( x^{1 - \frac{1}{k-1} + \frac{C\log x}{\log \log x}} \right) \]
for some constant \( C = C(\alpha_1, \ldots, \alpha_\ell) \) and every large \( x \).

A nested version of Theorem 1 is given below.

**Theorem 2.** Let \( k \geq 2 \) be an integer. Let \( \{\alpha_1, \alpha_2\} \) be a finite type subset of irrational numbers each greater than zero. Then the following asymptotic is satisfied:
\[ \#\{p \leq x : [\alpha_1 [\alpha_2 p]] is k-free\} = \frac{\pi(x)}{\zeta(k)} + O(x^{1-\varepsilon}) \]
for some \( \varepsilon > 0 \).

Here, the interested reader is invited to investigate the following problem: Let \( \{\alpha_i\}_{i=1}^n \) be positive real numbers. Define
\[ a_j = \prod_{i=1}^{j} \alpha_{n+1-i} \]
Assuming that \( \{a_1, a_2, \ldots, a_n\} \) is of finite type (see Definition 1), show that
\[ \#\{p \leq x : [a_n [a_{n-1} \cdot [a_1 p]]] is k-free\} = \frac{\pi(x)}{\zeta(k)} + O(x^{1-\varepsilon}) \]
for some \( \varepsilon > 0 \). It might also be fruitful to investigate the possible power saving in the error term above.

### 1.1. Preliminaries and Notation

#### 1.1.1. Notation.**
We recall that for functions \( F \) and \( G \) where \( G \) is real non-negative, the notations \( F \ll G \) and \( F = O(G) \) are equivalent to the statement that the inequality \( |F| \leq \alpha G \) holds for some constant \( \alpha > 0 \). Further we use \( F \sim G \) to indicate \( (F/G)(x) \) tends to 1 as \( x \to \infty \).

Given a real number \( x \), we use the notation \( \{x\} \) for the fractional part of \( x \), the notation \( \lfloor x \rfloor \) for the greatest integer not exceeding \( x \) and \( e(x) = e^{2\pi i x} \).

We use \( \|x\| \) to denote the distance from the real number \( x \) to the nearest integer, \( \Lambda(n) = \log p \) if \( n = p^r \) where \( p \) is a prime number (here and hereafter). Otherwise, \( \Lambda(n) = 0 \). \( \mu(n) \) denotes the Mobius function. \( \phi(n) \) denotes the Euler’s totient function. \( \tau(n) \) denotes the number of positive divisors of \( n \). We also use \( \pi(x) \) to denote the number of primes not more than \( x \).
1.1.2. Preliminaries.

Definition 1. An irrational number \( \alpha \) is called of finite type \( \tau \), if
\[
\tau = \sup \left\{ \beta : \liminf_{q \to \infty} q^{\beta} ||\alpha q|| = 0 \right\} < \infty.
\]

If \( \alpha \) is an irrational number of finite type \( \tau \), then by Dirichlet’s approximation theorem (Lemma 2.1 of [25]) one has \( \tau \geq 1 \). The celebrated theorems of Khinchin [17] and of Roth [21,22] state that \( \tau = 1 \) for almost all (in the sense of the Lebesgue measure) real numbers and for all irrational algebraic numbers respectively.

Definition 2. A finite subset of real numbers \( \{\beta_1, \beta_2, \ldots, \beta_\ell\} \) is said to be of finite type if there is \( \tau > 0 \) such that the inequality
\[
||h_1 \beta_1 + h_2 \beta_2 + \cdots + h_\ell \beta_\ell|| < \left( \max\{1, |h_1|, \ldots, |h_\ell|\} \right)^{-\tau}
\]
has only finitely many solutions for \( h_i \in \mathbb{Z} \).

If \( \{\beta_i\}_{i=1}^\ell \) satisfies (1) for some \( \tau > 0 \), then it follows from Dirichlet’s theorem on rational approximations that \( \tau \geq 1 \). Furthermore, such a set is linearly independent over \( \mathbb{Q} \).

Throughout this paper, we shall mostly use the weak form of the prime number theorem, that is
\[
\pi(x) \sim \frac{x}{\log x}.
\]

Lemma 1. For every positive integer \( n \geq 1 \),
\[
\tau(n) < e^{C \log \log n}
\]
for some constant \( C > 0 \).

Proof. Follows from [23, Theorem 2.11]. \( \square \)

Lemma 2. If
\[
|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}
\]
for some integers \( a \) and \( q \) such that \( (a,q) = 1 \), then
\[
\sum_{p \leq x} e(\alpha p) \ll x \log^3 x \left( q^{-\frac{1}{2}} + x^{-\frac{1}{2}} + q^\frac{1}{2} x^{-\frac{1}{2}} \right).
\]

Proof. This follows in a standard way using the main result of [12, §25]. \( \square \)

Lemma 3 (Erdős-Turán-Koksma Inequality). If \( \{x_i\}_{i=1}^N \) is a finite sequence in \( \mathbb{R}^\ell \), then for any \( J \subseteq [0,1)^\ell \) that is a Cartesian product of subintervals of \( [0,1) \) and any \( H \geq 1 \), we have
\[
\#\{1 \leq i \leq N : x_i \equiv J \mod 1\} - |J|N \ll \frac{N}{H} + \sum_{0 < ||h|| \leq H} \frac{1}{r(h)} \sum_{1 \leq i \leq N} e((h, x_i)).
\]
Here \(|J|\) denotes the \(\ell\)-dimensional Lebesgue measure of \(J\), \(\langle \cdot, \cdot \rangle\) denotes the standard inner product in \(\mathbb{R}^\ell\) and we set \(|h| = \max_{1 \leq i \leq \ell} |h_i|\) and

\[
r(h) = \prod_{i=1}^\ell \max\{|h_i|, 1\}
\]

for all \(h = (h_1, h_2, \ldots, h_\ell) \in \mathbb{Z}^\ell\). Moreover, the implied constant depends only on \(\ell\).

**Proof.** For the proof see [18]. \(\square\)

The following lemma is a classical result due to Vinogradov [26, Lemma 12].

**Lemma 4.** Let \(\alpha, \beta\) and \(\Delta\) be real numbers such that

\[
0 < \Delta < \frac{1}{2} \quad \text{and} \quad \Delta \leq \beta - \alpha \leq 1 - \Delta.
\]

Then there exists a periodic function \(\Psi(x)\), with period 1, satisfying

(i) \(\Psi(x) = 1\) in the interval \(\alpha + \frac{1}{2}\Delta \leq x \leq \beta - \frac{1}{2}\Delta\),

(ii) \(\Psi(x) = 0\) in the interval \(\beta + \frac{1}{2}\Delta \leq x \leq 1 + \alpha - \frac{1}{2}\Delta\),

(iii) \(0 \leq \Psi(x) \leq 1\) in the remainder of the interval \(\alpha - \frac{1}{2}\Delta \leq x \leq 1 + \alpha - \frac{1}{2}\Delta\),

(iv) \(\Psi(x)\) has a Fourier expansion of the form

\[
\Psi(x) = \sum_{h=-\infty}^{\infty} a_h e(hx),
\]

where

\[
|a_h| \leq c \cdot \min\left\{|h|^{-1}, |h|^{-2}\Delta^{-1}\right\}
\]

for every \(|h| \geq 1\) and some \(c\) fixed. Furthermore, \(a_0 = \beta - \alpha\).

**2. Proof of The Main Results**

**2.1. Proof of Theorem 1.** Let \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)\) and

\[
\pi(x, k, \alpha) = \# \{p \leq x : [\alpha_i p] \text{ is } k\text{-free for each } i = 1, \ldots, \ell\}.
\]

Let \(I_k\) denote the characteristic function of \(k\)-free integers. Since

\[
I_k(n) = \sum_{d^k|n} \mu(d),
\]

we have
\[
\pi(x, k, \alpha) = \sum_{p \leq x} \mathcal{I}_k([\alpha_1p]) \cdots \mathcal{I}_k([\alpha_\ell p])
\]

\[
= \sum_{p \leq x} \left( \sum_{d_1 \mid [\alpha_1p]} \mu(d_1) \right) \cdots \left( \sum_{d_\ell \mid [\alpha_\ell p]} \mu(d_\ell) \right)
\]

\[
= \sum_{p \leq x} \sum_{(d_1, \ldots, d_\ell) \atop d_i \geq z} \mu(d_1) \cdots \mu(d_\ell) \sum_{p \leq x \atop d_i \mid [\alpha_i p]} 1
\]

\[
= \sum_{(d_1, \ldots, d_\ell) \atop d_i \geq z} \mu(d_1) \cdots \mu(d_\ell) \sum_{p \leq x \atop d_i \mid [\alpha_i p]} 1 + \sum_{i=1, \ldots, \ell} \mu(d_1) \cdots \mu(d_\ell) \sum_{p \leq x \atop d_i \mid [\alpha_i p]} 1,
\]

where \(z \leq x^{1/k}\) will be chosen later. It follows from Lemma 1 that for all \(i = 1, 2, \ldots, \ell\) there exists a positive constant \(c_i = c_i(\alpha_i)\) depending on \(\alpha_i\) such that

\[
\tau([\alpha_i p]) \ll e^{c_i \log x / \log \log x}
\]

for every \(p \leq x\). Then, for all \(i = 1, 2, \ldots, \ell\) and \(p \leq x\)

\[
\tau([\alpha_i p]) \ll e^{c \log x / \log \log x},
\]

where \(c = \max\{c_1, \ldots, c_\ell\}\). Set \(C = c(\ell - 1)\). Then, by [4] and using partial summation in the last step, we get

\[
\sum_{(d_1, \ldots, d_\ell) \atop d_i \geq z} \mu(d_1) \cdots \mu(d_\ell) \sum_{p \leq x \atop d_i \mid [\alpha_i p]} 1 < \sum_{(d_1, \ldots, d_\ell) \atop d_i \geq z} \sum_{p \leq x \atop d_i \mid [\alpha_i p]} 1 + \sum_{(d_1, \ldots, d_\ell) \atop d_i > z} \sum_{p \leq x \atop d_i \mid [\alpha_i p]} 1
\]

\[
= \sum_{p \leq x} \left( \sum_{d_1 \mid [\alpha_1 p]} 1 \right) \cdots \left( \sum_{d_\ell \mid [\alpha_\ell p]} 1 \right) + \sum_{p \leq x} \left( \sum_{d_1 \mid [\alpha_1 p]} 1 \right) \cdots \left( \sum_{d_\ell \mid [\alpha_\ell p]} 1 \right)
\]
Therefore,

\[
\pi(x, k, \alpha) = \sum_{(d_1, \ldots, d_\ell) \in \mathcal{D}} \mu(d_1) \cdots \mu(d_\ell) \sum_{p \leq x} \frac{1}{d_1^{\alpha_p}} \cdots \frac{1}{d_\ell^{\alpha_p}} + O \left( \frac{\log x}{z^{k-1}} \right).
\]  

(5)

Next, we will study the sum above appearing in (5) which runs over all tuples \((d_1, \ldots, d_\ell)\) of positive integers where \(d_i \leq z\) for all \(i = 1, \ldots, \ell\).

So, let \(d = (d_1, \ldots, d_\ell)\) be such a tuple and set

\[
D = \prod_{j=1}^{\ell} d_j^k, \quad D_i = \prod_{j=1, j \neq i}^{\ell} d_j^k \quad \text{and} \quad \mathcal{I}_d = \left[ 0, \frac{1}{d_1^k} \right) \times \cdots \times \left[ 0, \frac{1}{d_\ell^k} \right).
\]

(6)

for all \(i = 1, \ldots, \ell\). For a positive integer \(i\), let \(p_i\) denote the \(i\)th prime. Observing that

\[
|\alpha_p| \equiv 0 \pmod{d} \text{ if and only if } \left\{ \frac{\alpha p}{d} \right\} < \frac{1}{d},
\]

(7)
we have
\[
\sum_{p \leq x} 1 = \sum_{p \leq x} \sum_{i=1, \ldots, \ell} d_i^k |\alpha_i p|_{\mathbb{Z}} \left( \frac{\alpha_i p}{d_i} \right) \leq (\frac{x}{d_1})^{\frac{1}{d_1}} \left( \frac{\alpha_1 p}{d_1}, \ldots, \frac{\alpha_\ell p}{d_\ell} \right) \in \mathcal{I}_d
\]
where
\[
t_i = \left( \left\{ \frac{\alpha_1 p_i}{d_1^k} \right\}, \ldots, \left\{ \frac{\alpha_\ell p_i}{d_\ell^k} \right\} \right).
\]
It follows from Erdős-Turán-Koksma Inequality that for all \( H \geq 1, \)
\[
\#\{i \leq \pi(x) : t_i \in \mathcal{I}_d \} - \frac{\pi(x)}{d_1 \cdots d_\ell}
\ll \frac{\pi(x)}{H} + \sum_{0 < |h| \leq H} \frac{1}{r(h)} \left| \sum_{i \leq \pi(x)} e\left(\langle h, t_i \rangle\right) \right| \]  \hspace{1cm}(8)
\[
\ll \frac{\pi(x)}{H} + \sum_{0 < |h| \leq H} \frac{1}{r(h)} \left| \sum_{p \leq x} e\left(\frac{h_1 D_1 \alpha_1 + \cdots + h_\ell D_\ell \alpha_\ell}{D_1 \cdots D_\ell} \cdot p\right) \right| . \]  \hspace{1cm}(9)
Next, we shall prove the following lemma.

**Lemma 5.**
\[
\sum_{p \leq x} e\left(\frac{h_1 D_1 \alpha_1 + \cdots + h_\ell D_\ell \alpha_\ell}{D_1 \cdots D_\ell} \cdot p\right)
\ll x \log^3 x \left( x^{-\frac{1}{D_1 \cdots D_\ell}} \max\{|h_1| D_1, \ldots, |h_\ell| D_\ell\} \right)^{\frac{1}{D_1 \cdots D_\ell}} + x^{-\frac{1}{D_1 \cdots D_\ell}}
\]
uniformly for all \( h = (h_1, \ldots, h_\ell) \in \mathbb{Z}^\ell \) such that \( \|h\| > 0, \) where \( D_i \) and \( D \) are defined in \( \mathbf{6}. \)

**Proof.** Since \( \{\alpha_i\} \) satisfies \( \mathbf{1} \) for some \( \tau > 0, \) there exists a positive constant \( A \geq 1 \) such that
\[
\max\{|h_1|, \ldots, |h_\ell|\}^{-\tau} \leq A \|h_1 \alpha_1 + h_2 \alpha_2 + \cdots + h_\ell \alpha_\ell\|
\ll (x^{-\frac{1}{D_1 \cdots D_\ell}} \max\{|h_1| D_1, \ldots, |h_\ell| D_\ell\})^{\frac{1}{D_1 \cdots D_\ell}} + x^{-\frac{1}{D_1 \cdots D_\ell}}
\]
for all \( h = (h_1, \ldots, h_\ell) \in \mathbb{Z}^\ell \) such that \( \max_{1 \leq i \leq \ell} |h_i| > 0. \) Let \( h = (h_1, \ldots, h_\ell) \in \mathbb{Z}^\ell \) be such a tuple and set
\[
m_h = \frac{h_1 D_1 \alpha_1 + \cdots + h_\ell D_\ell \alpha_\ell}{D_1 \cdots D_\ell}.
\]
Let \( 1 \leq Q < x/2 \) to be determined later. By Dirichlet’s rational approximation theorem, there exists \( \frac{r}{q} \in \mathbb{Q} \) such that \( 1 \leq q \leq \frac{x}{Q} \) and
\[
\left| m_h - \frac{r}{q} \right| < \frac{Q}{qx}.
\]
So,
\[ \|q(h_1D_1\alpha_1 + \cdots + h_\ell D_\ell\alpha_\ell)\| < \frac{QD}{x}. \]  
(11)

On the other hand, it follows from (10) that
\[ \|q(h_1D_1\alpha_1 + \cdots + h_\ell D_\ell\alpha_\ell)\| \geq A^{-1}q^{-\tau}(\max\{|h_1D_1|, \ldots, |h_\ell D_\ell|\})^{-\tau}. \]
(12)

Combining (11) and (12), we get
\[ q \geq x^{\frac{1}{2}} \max\{|h_1D_1|, \ldots, |h_\ell D_\ell|\} A^{\frac{1}{2}} D^{\frac{1}{2}} Q^{\frac{1}{2}}. \]
(13)

Then it follows from Lemma 2 that
\[ X_p \leq x \epsilon((m_\ell \cdot \rho) \leq x \log^3 x \left( x^{-\frac{1}{2}} M^{\frac{1}{2}} D^{\frac{1}{2}} Q^{\frac{1}{2}} + x^{-\frac{1}{2}} + Q^{-\frac{1}{2}} \right)). \]
(14)

where for the sake of brevity we set
\[ M = \max\{|h_1D_1|, \ldots, |h_\ell D_\ell|\}. \]

By [13, Lemma 2.4], there exists 1 \( \leq Q < x/2 \) such that the left hand side of (14) is
\[ \ll x \log^3 x \left( x^{-\frac{1}{2}} M^{\frac{1}{2}} D^{\frac{1}{2}} Q^{\frac{1}{2}} + x^{-\frac{1}{2}} + Q^{-\frac{1}{2}} \right). \]

At this point, we can assume that \( x^{-\frac{1}{2}} M^{\frac{1}{2}} D^{\frac{1}{2}} Q^{\frac{1}{2}} \ll 1 \), because otherwise the required upper bound holds trivially. Therefore, the second term is beaten by the first term giving the proof of Lemma 5.

We next proceed by plugging this upper bound into (9). We also use the upper bound \( |h_i| \ll H \) together with the upper bounds \( D \ll z^{\ell} \) and \( D_i \ll z^{k(\ell - 1)} \). Then the difference in the first line of (9) is
\[ \ll \frac{\pi(x)}{H} + \left( x^{1-\frac{1}{2\ell+1}} H^{\frac{1}{2\ell+1}} z^{\frac{k(\ell - 1)}{2\ell+1}} \log^3 x + x^{\frac{3}{2}} \log^3 x \right) \left( \sum_{0 \leq \|h\| \leq H} \frac{1}{r(h)} \right). \]
(15)

Now, by (2)
\[ \sum_{0 \leq \|h\| \leq H} \frac{1}{r(h)} \ll \sum_{0 \leq \|h\| \leq H} \frac{1}{\Pi_{i=1}^\ell (\max\{|h_i|, 1\})} \ll \left( 1 + 2 \sum_{1 \leq h \leq H} \frac{1}{h} \right) \ll \log^\ell H, \]
(16)

where in the last step we use integral test. Here we note that the implied constant depends on \( \ell \). Coupling (8), (9), (15) and (16), we arrive at
\[ \left( \sum_{p \leq x \atop (d_i^{\ell}) \mid \alpha, p} \frac{1}{d_1^\ell \cdots d_\ell^\ell} \right) - \frac{\pi(x)}{d_1^\ell \cdots d_\ell^\ell} \]
for every $H \geq 1$ and every $(d_1, \ldots, d_\ell)$ such that $d_i \leq z \leq x^{1/k}$ for each $i$. Noting $\pi(x) \ll x$ and choosing $1 \leq H \leq x$ by [13, Lemma 2.4], the left hand side of (17) is

$$
\ll \log^{\ell+3} x \left( x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}+\ell} + x^{1-\frac{1}{3\tau+1}} z^{\frac{k(\ell-1)\tau+k\ell}{2\tau+1}+\ell} + x^4 z^\ell \right).
$$

On summing this over all tuples $(d_1, \ldots, d_\ell)$ of positive integers where $d_i \leq z$ for all $i = 1, \ldots, \ell$, we observe from (5) that for all $1 \leq z \leq x^{1/k}$,

$$
\pi(x, k, \alpha) - \pi(x) \sum_{(d_1, \ldots, d_\ell) \atop d_i \leq z, i=1,\ldots,\ell} \frac{\mu(d_1) \cdots \mu(d_\ell)}{d_1^k \cdots d_\ell^k}
$$

is

$$
\ll \log^{\ell+3} x \left( x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}+\ell} + x^{1-\frac{1}{3\tau+1}} z^{\frac{k(\ell-1)\tau+k\ell}{2\tau+1}+\ell} + x^4 z^\ell \right) + \frac{C \log x}{z^{k-1}}.
$$

Here,

$$
\sum_{(d_1, \ldots, d_\ell) \atop d_i \leq z, i=1,\ldots,\ell} \frac{\mu(d_1) \cdots \mu(d_\ell)}{d_1^k \cdots d_\ell^k} = \left( \sum_{d \leq z} \frac{\mu(d)}{d^k} \right)^\ell
$$

and using the following inequality

$$
\left| \sum_{d \leq z} \frac{\mu(d)}{d^k} - \sum_{d=1}^\infty \frac{\mu(d)}{d^k} \right| \ll \sum_{d \geq z} \frac{1}{d^k} \ll \frac{1}{z^{k-1}},
$$

it follows by the mean value theorem that

$$
\left( \sum_{d \leq z} \frac{\mu(d)}{d^k} \right)^\ell - \left( \sum_{d=1}^\infty \frac{\mu(d)}{d^k} \right)^\ell \ll \frac{1}{z^{k-1}}.
$$

Therefore, the contribution of the sums running over $d_i \leq z$ for all $i = 1, \ldots, \ell$ is

$$
\frac{\pi(x)}{\zeta^\ell(k)} + O \left( \frac{\pi(x)}{z^{k-1}} \right)
$$

yielding for all $1 \leq z \leq x^{1/k}

$$
\pi(x, k, \alpha) - \pi(x) \frac{\zeta^\ell}{\zeta(k)}
$$

is

$$
\ll \log^{\ell+3} x \left( x^{1-\frac{1}{3\tau+2}} z^{\frac{k(\ell-1)\tau+k\ell}{3\tau+2}+\ell} + x^{1-\frac{1}{3\tau+1}} z^{\frac{k(\ell-1)\tau+k\ell}{2\tau+1}+\ell} + x^4 z^\ell \right) + \frac{C \log x}{z^{k-1}}.
$$

(18)
where $C = C(\ell, \alpha)$ is positive. On the right hand side of (18), the first term beats the third term as $\tau \geq 1$ and the second term whenever

$$z \leq x^{\frac{1}{3} + \frac{k}{3}}$$

which one can assume since otherwise (18) holds trivially. Using now [13, Lemma 2.4] to choose optimal $z \leq x^{1/k}$, the left hand side of (18) is

$$\ll e^{C' \log x} \left( x^{1 - \frac{1}{3} + \frac{k}{3}} + x^{\frac{1}{3} + \frac{k}{3} + \frac{k - 1}{3} + \frac{1}{3} + \frac{1}{k} + \frac{1}{3} + \frac{2}{3}} \right)$$

$$\ll x^{1 - \frac{1}{3} + \frac{k}{3} + \frac{k - 1}{3} + \frac{1}{3} + \frac{1}{k} + \frac{1}{3} + \frac{2}{3} + C' \log x}$$

for some constant $C'$ depending on $\ell$ and $\alpha$, therefore the claim follows.

2.2. **Proof of Theorem 2.** The proof will be similar to that of Theorem 1. We shall therefore be brief. Let $\alpha = (\alpha_1, \alpha_2)$ and define

$$\pi_\alpha(x, k) = \# \{ p \leq x : \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \text{ is } k\text{-free} \}.$$

Let $1 \leq z \leq x^{1/k}$ be a number to be determined. Using (3), it follows that

$$\pi_\alpha(x, k) = \sum_{p \leq x} \sum_{d \mid \lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor} \mu(d) = \sum_{p \leq x} \sum_{d \leq z} \mu(d) + \sum_{p \leq x} \sum_{d > z} \mu(d).$$

As we did before, we have

$$\sum_{p \leq x} \sum_{d \leq z} \mu(d) \ll \frac{x}{z^{k-1}},$$

where the implied constant depends only on $\alpha_1$ and $\alpha_2$. This yields

$$\pi_\alpha(x, k) = \sum_{p \leq x} \sum_{d \leq z} \mu(d) + O\left( \frac{x}{z^{k-1}} \right).$$

We now proceed to derive the main term. Writing

$$\sum_{p \leq x} \sum_{d \leq z} \sum_{d \leq z} \mu(d) = \sum_{d \leq z} \mu(d) \left( \sum_{\lfloor \alpha_1 \lfloor \alpha_2 p \rfloor \rfloor \equiv 0 \pmod{d^k}} 1 - \frac{\pi(x)}{d^k} \right) + \pi(x) \sum_{d \leq z} \frac{\mu(d)}{d^k},$$

and using partial summation to get

$$\sum_{d \leq z} \frac{\mu(d)}{d^k} = \frac{1}{\zeta(k)} + O\left( \frac{1}{z^{k-1}} \right),$$
one arrives at
\[
\pi_\alpha(x, k) = \pi(x) \frac{x}{\zeta(k)} + O \left( \frac{x}{z^{k-1}} + \sum_{d \leq z} \left| \sum_{p \leq x, [\alpha_1 \alpha_2 p] \equiv 0 \pmod{d^k}} 1 - \pi(x) \right| d^k \right) \quad (19)
\]
for any \(1 \leq z \leq x^{1/k}\). Let us now concentrate on the error term and proceed to show that it is \(\ll x^{1-\varepsilon}\) for some \(\varepsilon > 0\). Using observation (7), together with Lemma 3 one ends up with
\[
\left( \sum_{p \leq x, [\alpha_1 \alpha_2 p] \equiv 0 \pmod{d^k}} 1 \right) - \frac{\pi(x)}{H_1} \ll \frac{\pi(x)}{H_1} + \sum_{1 \leq |h_1| \leq H_1} \frac{1}{|h_1|} \left| \sum_{p \leq x} e \left( \frac{\alpha_1 \alpha_2 h_1 p}{d^k} - \alpha_1 h_1 \{\alpha_2 p\} \right) \right|, \quad (20)
\]
where \(H_1\) is a positive number to be determined. So, it boils down to estimate the exponential sum above. To do this, we let \(K\) be a sufficiently large number and we write
\[
[\alpha_2 p] = \alpha_2 - \{\alpha_2 p\},
\]
yielding
\[
\sum_{p \leq x} e \left( \frac{\alpha_1 h_1 [\alpha_2 p]}{d^k} \right) = \sum_{0 \leq i \leq K-1} \sum_{p \leq I_i(x)} e \left( \frac{\alpha_1 \alpha_2 h_1 p}{d^k} - \frac{\alpha_1 h_1 \{\alpha_2 p\}}{d^k} \right), \quad (21)
\]
where \(I_i(x) = \{p \leq x : i/K \leq \{\alpha_2 p\} < i+1/K\}\). Since
\[
e(t) = 1 + O(|t|)
\]
uniformly for all \(t \in \mathbb{R}\), we have
\[
e \left( \frac{\alpha_1 \alpha_2 h_1 p}{d^k} - \frac{\alpha_1 h_1 \{\alpha_2 p\}}{d^k} \right) = e \left( - \frac{\alpha_1 h_1 i}{K d^k} \right) e \left( \frac{\alpha_1 \alpha_2 h_1 p}{d^k} \right) + O \left( \frac{|h_1|}{K d^k} \right)
\]
if \(p \in I_i(x)\). Therefore, the left hand side of (21) is
\[
\ll \frac{|h_1| \pi(x)}{K d^k} \sum_{0 \leq i \leq K-1} \sum_{p \in I_i(x)} e \left( \frac{\alpha_1 \alpha_2 h_1 p}{d^k} \right) \quad (22)
\]
Given \(0 \leq i \leq K-1\), let \(\beta_i = i/K\), \(\gamma_i = (i+1)/K\) and \(0 < \Delta < 1/K\) be a number to be chosen. By Lemma 4, there exists a periodic function \(\Psi_i(x)\), with period 1, satisfying
(i) \(\Psi_i(x) = 1\) in the interval \(\beta_i + \frac{1}{2} \Delta \leq x \leq \gamma_i - \frac{1}{2} \Delta\),
(ii) \(\Psi_i(x) = 0\) in the interval \(\gamma_i + \frac{1}{2} \Delta \leq x \leq 1 + \beta_i - \frac{1}{2} \Delta\),
(iii) $0 \leq \Psi_i(x) \leq 1$ in the remainder of the interval $\beta_i - \frac{1}{2} \Delta \leq x \leq 1 + \beta_i - \frac{1}{2} \Delta$, where $\Delta$ is a fixed integer.

(iv) $\Psi_i(x)$ has a Fourier expansion of the form

$$
\Psi_i(x) = \sum_{h=-\infty}^{\infty} a_h e(hx),
$$

where $a_0 = 1/K$ and

$$
|a_h| \leq c \cdot \min \left\{ |h|^{-1}, |h|^{-2} \Delta^{-1} \right\}
$$

for every $|h| \geq 1$ and some $c$ fixed.

Let $\psi_i(x)$ be 1 if $\beta_i \leq \{x\} \leq \gamma_i$ and $\psi_i(x) = 0$ otherwise. It follows that $\Psi_i(x)$ and $\psi_i(x)$ agree on $[0, 1]$ except possibly for two subintervals of $[0, 1]$ of length $\leq \Delta$.

Therefore, we have

$$
\sum_{p \in I_i(x)} e\left(\frac{\alpha_1 \alpha_2 h_1 p}{d^k}\right) = \sum_{p \leq x} \Psi_i(\alpha_2 p) e\left(\frac{\alpha_1 \alpha_2 h_1 p}{d^k}\right) + O\left( \sum_{p \leq x} \frac{1}{(\alpha_2 p) \in I} \right),
$$

where $I$ is a union of two intervals and is of length $\Delta$. Since $\alpha_2$ is of finite type, following the proof of Theorem 5.1 in [8] together with a partial summation argument, it follows that for some $0 < \epsilon'' < 1/5$, one has

$$
\sum_{p \leq x, \{\alpha_2 p\} \in I} 1 = \Delta \pi(x) + O \left( x^{1-\epsilon''} \right),
$$

uniformly for all $0 < \Delta < 1/K$. Therefore, we see that the left hand side of (23) is

$$
= \frac{1}{K} \sum_{p \leq x} e\left(\frac{\alpha_1 \alpha_2 h_1 p}{d^k}\right)
+ O\left( \sum_{|h_2| > 0} a_{h_2} \left| \sum_{p \leq x} e\left(\frac{\alpha_1 \alpha_2 h_1 + \alpha_2 h_2 d^k p}{d^k}\right) \right| + \Delta \pi(x) + x^{1-\epsilon''} \right).
$$

Letting $H_2$ be a positive integer to be determined, we split the sum running over $h_2$ at $H_2$. For $|h_2| > H_2$, estimating the innermost exponential sum by $\pi(x)$, and using the upper bounds $a_h \ll 1/(\Delta h^2)$ and $a_h \ll 1/|h|$, we obtain that the left hand side of (23) is

$$
= \frac{1}{K} \sum_{p \leq x} e\left(\frac{\alpha_1 \alpha_2 h_1 p}{d^k}\right)
+ O\left( \sum_{0 < |h_2| \leq H_2} \frac{1}{|h_2|} \left| \sum_{p \leq x} e\left(\frac{\alpha_1 \alpha_2 h_1 + \alpha_2 h_2 d^k p}{d^k}\right) \right| \right)
+ O\left( \frac{\pi(x)}{\Delta H_2} + \Delta \pi(x) + x^{1-\epsilon''} \right).
$$
Plugging this upper bound into (22) yields that
\[
\sum_{p \leq x} e \left( \frac{\alpha_1 h_1 \lfloor \alpha_2 p \rfloor}{d^k} \right) \ll \sum_{p \leq x} e \left( \frac{\alpha_1 \alpha_2 h_1 p}{d^k} \right) + \sum_{i \leq K, 0 < |h_2| \leq H_2} \sum_{p \leq x} \frac{1}{|h_2|} e \left( \frac{\alpha_1 \alpha_2 h_1 + \alpha_2 h_2 d^k}{d^k} p \right) + \frac{\pi(x) K}{\Delta H_2} + \Delta K \pi(x) + K x^{1-\varepsilon'} + \frac{|h_1| \pi(x)}{K d^k}.
\]  
(25)

We are therefore left with the estimation of
\[
\sum_{p \leq x} e \left( \frac{(\alpha_1 \alpha_2 h_1 + \alpha_2 h_2 d^k)}{d^k} p \right),
\]  
(26)
whenever \(\max\{|h_1|, |h_2|\} > 0\). To estimate the exponential sum, by Dirichlet’s theorem we pick up a rational number \(a/q\) satisfying
\[
\left| \frac{(\alpha_1 \alpha_2 h_1 + \alpha_2 h_2 d^k)}{d^k} \right| - \frac{a}{q} < \frac{1}{q x^{1-\kappa}}
\]  
with \(1 \leq q \leq x^{1-\kappa}\), where \(0 < \kappa < 1\) is to be determined. Since \(\{\alpha_1 \alpha_2, \alpha_2\}\) is of finite type, similar to how we obtain (13) \(x^{1-\kappa} \tau d^k \max\{|h_1|, |h_2|d^k|\} \ll q \ll x^{1-\kappa}\) for some \(\tau \geq 1\). Then by Lemma 2, the exponential sum (26) is
\[
\ll x \log^3 x \left( \max\{|h_1|, |h_2|d^k|\} \right)^{\frac{1}{2}} d^k x^{-\frac{1}{2\tau'}} + x^{-\frac{1}{2}} + x^{-\frac{3}{2}}.
\]

At this point, we assume that \(0 < \max\{|h_1|, |h_2|\} \leq x^{\varepsilon'}\) where \(\varepsilon'\) is a sufficiently small number to be determined in terms of \(\kappa\). Then,
\[
\sum_{p \leq x} e \left( \frac{(\alpha_1 \alpha_2 h_1 + \alpha_2 h_2 d^k)}{d^k} p \right) \ll \left( d^{\frac{\kappa+1}{2}} x^{1-\frac{1-\varepsilon}{2\tau'} + \frac{\varepsilon'}{2}} + x^\frac{1}{4} + x^{\frac{1}{2}} \right) \log^3 x,
\]  
(27)
uniformly for
\[
0 < \max\{|h_1|, |h_2|\} \leq x^{\varepsilon'}.
\]
Plugging the upper bound (27) into (25), we arrive at
\[
\sum_{p \leq x} e \left( \frac{\alpha_1 h_1 \lfloor \alpha_2 p \rfloor}{d^k} \right) \ll K \left( d^{\frac{\kappa+1}{2}} x^{1-\frac{1-\varepsilon}{2\tau'} + \frac{\varepsilon'}{2}} + x^\frac{1}{4} + x^{\frac{1}{2}} \right) \log^4 x
\]
\[
+ \frac{\pi(x) K}{\Delta H_2} + \Delta K \pi(x) + K x^{1-\varepsilon''} + \frac{H_1 \pi(x)}{K d^k},
\]
uniformly for $|h_1| \leq x^\varepsilon'$, provided that $H_2 \leq x^\varepsilon'$, $0 < \kappa < 1$, $0 < \Delta < 1/K$ and $K$ is sufficiently large. Plugging this upper bound into (20) and summing over $d \leq z$, we see that the error term in (19) is
\[
\ll \frac{xz}{H_1} + K \left( x^{1 - \frac{\kappa + k + \varepsilon'}{2\tau}} x^{1 - \frac{\varepsilon'}{2\tau}} + z x^\frac{4}{7} + z x^{1 - \frac{2}{7}} \right) \log^5 x
\]
\[
+ \left( \frac{z x K}{\Delta H_2} + z \Delta K x + z K x^{1 - \varepsilon''} + \frac{H_1 x}{K} \right) \log x + \frac{x}{z^{k-1}}
\] (28)
provided that $0 < H_1, H_2 \leq x^\varepsilon'$, $0 < \kappa < 1$, $0 < \Delta < 1/K$ and $K$ is sufficiently large. We now make all unspecified constants explicit. For $0 < \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 < 1$ to be determined, we set
\[
K = x^{\varepsilon_1}, H_1 = x^{\varepsilon_2}, H_2 = x^{\varepsilon_3}, \Delta = x^{-\varepsilon_4} \text{ and } z = x^{\varepsilon_5},
\]
where $0 < \varepsilon_5 \leq 1/k$ (this assumption is from the beginning of the proof). Examining each term in (28), the right hand side of (28) is $\ll x^{1-\varepsilon}$ for some $\varepsilon > 0$, if the following inequalities are satisfied:
(1) $\varepsilon_5 < 1/k$,
(2) $\varepsilon_2, \varepsilon_3 < \varepsilon'$,
(3) $\varepsilon_5 < \varepsilon_2 < \varepsilon_1$,
(4) $\varepsilon_1 + \varepsilon_5 < \min\{\varepsilon_4, \varepsilon'', \kappa/2\}$,
(5) $\varepsilon_1 + \varepsilon_4 + \varepsilon_5 < \varepsilon_3$,
(6) $\varepsilon_1 + \varepsilon_5 (1 + \frac{k+1+k}{2}) \varepsilon' < 1 - \frac{\varepsilon}{2\tau}$,
where $\varepsilon'' < 1/5$ is a fixed positive number defined in (24), $\tau \geq 1$ is a fixed number and $0 < \kappa < 1$ and $0 < \varepsilon' < 1$ are to be chosen. We choose $\kappa = 2/5$ and $\varepsilon' = (1 - \kappa)/(4\tau)$. Then since $\varepsilon'' < 1/5$, we assume that $\varepsilon_4 < \varepsilon''$ so that the fourth inequality becomes equivalent to $\varepsilon_1 + \varepsilon_5 < \varepsilon_4$. We next choose $\varepsilon_3 < \varepsilon'$ and $\varepsilon_4 < \min\{\varepsilon_3, \varepsilon''\}$ and $\varepsilon_1 < \min\{\varepsilon_4, \varepsilon_3 - \varepsilon_4, (1 - \kappa)/(4\tau)\}$. Finally, we choose $\varepsilon_2 < \min\{\varepsilon_1, \varepsilon'\}$ and
\[
\varepsilon_5 < \min\left\{\frac{\varepsilon_2 - \varepsilon_4 - \varepsilon_1 - \varepsilon_3 - \varepsilon_1 - \varepsilon_4}{k}, \frac{1}{k}, \frac{2\tau}{(k+2)\tau + k} \left( 1 - \frac{\kappa}{4\tau} - \varepsilon_1 \right) \right\},
\]
completing the proof.

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