Jacobi Elliptic Solutions of $\lambda\phi^4$ Theory in a Finite Domain

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Abstract

The general static solutions of the scalar field equation for the potential $V(\phi) = -\frac{1}{2}M^2\phi^2 + \frac{\lambda}{4}\phi^4$ are determined for a finite domain in $(1 + 1)$ dimensional space-time. A family of real solutions is described in terms of Jacobi Elliptic Functions. We show that the vacuum–vacuum boundary conditions can be reached by elliptic cn-type solutions in a finite domain, such as of the Kink, for which they are imposed at infinity. We proved uniqueness for elliptic sn-type solutions satisfying Dirichlet boundary conditions in a finite interval (box) as well the existence of a minimal mass corresponding to these solutions in a box.

We define expressions for the “topological charge”, “total energy” (or classical mass) and “energy density” for elliptic sn-type solutions in a finite domain. For large length of the box the conserved charge, classical mass and energy density of the Kink are recovered. Also, we have shown that using periodic boundary conditions the results are the same as in the case of Dirichlet boundary conditions. In the case of anti-periodic boundary conditions all elliptic sn-type solutions are allowed.

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1. General Solutions

In this paper we obtain static solutions of the classical equation of motion of a scalar field $\phi$, with potential $V(\phi) = -\frac{1}{2} M^2 \phi^2 + \frac{1}{4} \lambda \phi^4$ in a finite domain. Static solutions with vacuum-vacuum boundary conditions at $x = \pm \infty$ were firstly obtained by Dashen et all (DHN)\(^1\). However, our case may be important since, as far as we know, no general solution satisfying the same boundary conditions for the above potential was found in the literature in a finite domain. Also this has an evident connection with the Casimir effect which attracted much attention for its applications in several topics of physics \(^2\).

Let us consider the Lagrangian density of a scalar field theory in (1 + 1) dimensions given by

$$L(\phi, \partial_\mu \phi) = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\partial_x \phi)^2 - V(\phi),$$

where $V(\phi) = -\frac{1}{2} M^2 \phi^2 + \frac{1}{4} \lambda \phi^4$, $M$ is the mass of the field and the dot means derivative with respect to $t$.

The extrema of the potential $V(\phi)$ are reached for $\phi_e = 0$, $\phi_m = \pm M \sqrt{\lambda}$ (vacua).

We are interested in the static case $\phi = \phi(x)$. So the classical equation of motion is

$$\partial_{xx} \phi - \frac{\partial V}{\partial \phi} = 0. \quad (1)$$

It is well known that the solutions for Eq. (1), satisfying vacuum to vacuum boundary conditions at $\pm \infty$, are given by

$$\phi_0(x) = \pm \frac{M}{\sqrt{\lambda}} tanh \left( \frac{M}{\sqrt{2}} (x - x_0) \right). \quad (2)$$

These solutions are called “Kink” (+ sign) and “Anti-Kink” (− sign). The solution $\phi_0$ approaches to two different values $\pm \frac{M}{\sqrt{\lambda}}$ as $x \to \pm \infty$, which correspond to a degenerate vacuum configuration.

We now turn to the calculation of the static solutions of the classical equation of motion (1) with boundary conditions in a finite domain which, in this paper, is a bounded interval. In order to have well behaved solutions we impose our solutions belonging to class $C^2(\Omega)$, where $\Omega \subset \mathbb{R}$ is an interval.

A first integral of Eq. (1), after a change of variable, is given by

$$x - x_0 = \pm \frac{1}{M} \int \frac{dz}{\sqrt{z^4 - z^2 + \xi^2}}. \quad (3)$$
where \( z = \frac{\sqrt{\lambda}}{\sqrt{2M}} \) is the dimensionless variable and \( x_0, c \) are constants of integration.

Since we are interested to find real solutions of Eq. (3), we must evaluate the integral of right-hand side for \( (z^4 - z^2 + \frac{c}{2}) > 0 \).

For any \( c \), the stationary points of the function \( f(z) = \frac{1}{\sqrt{z^4 - z^2 + \frac{c}{2}}} \) are given by the values:

\[
z = 0, \quad \pm \sqrt{2}.
\]

On the other hand, note that for \( c = 0 \) and \( c = \frac{1}{2} \), the integral in Eq. (3) is not defined at these points. So, we discuss the following cases:

1) \( c \leq 0 \)
2) \( 0 < c < \frac{1}{2} \) and
3) \( c \geq \frac{1}{2} \).

It is important to emphasize that we are interested to find static solutions \( \phi(x) \) that satisfy the vacuum-vacuum boundary conditions, like the solution (2), but now in a finite domain and to study some of their properties.

1) **CASE \( c \leq 0 \)**

In this case \( (z^4 - z^2 + \frac{c}{2}) > 0 \) is satisfied for \( |z| > \sqrt{\frac{1 + \sqrt{1 - 2c}}{2}} \).

The integral in Eq. (3) can be performed to give

\[
\int \frac{dz}{\sqrt{z^4 - z^2 + \frac{c}{2}}} = \frac{1}{\sqrt{1 - 2c}} F(\theta, m),
\]

where

\[
F(\theta, m) = \int_0^\theta \frac{dt}{\sqrt{1 - msin^2t}}
\]

is an elliptic integral of the first kind, and

\[
\theta = arccos\left(\frac{1 + \sqrt{1 - 2c}}{\sqrt{2z}}\right), \quad m = \frac{-1 + \sqrt{1 - 2c}}{2\sqrt{1 - 2c}}.
\]

So, the Eq. (3) is equivalent to

\[
F\left(arccos\left(\frac{1 + \sqrt{1 - 2c}}{\sqrt{2z}}\right), \frac{-1 + \sqrt{1 - 2c}}{2\sqrt{1 - 2c}}\right) = \pm M\sqrt{1 - 2c} (x - x_0).
\]

Solving (3) for \( z \) we get (we substitute \( z \) by \( \frac{\sqrt{\lambda}}{\sqrt{2M}} \phi \))

\[
\phi_c(x) = \frac{M\sqrt{1 + \sqrt{1 - 2c}}}{\sqrt{\lambda}} \frac{1}{cn \left(\pm \sqrt{1 - 2c} M (x - x_0), \frac{-1 + \sqrt{1 - 2c}}{2\sqrt{1 - 2c}}\right)},
\]
where \( cn(u,m) \) is a Jacobi Elliptic Function. The solution (6) is unbounded because \( cn \) has zeros. Observe that the functions \( cn^{-1}(u,m) \) are larger than or equal to 1 and since \( \sqrt{1 + \sqrt{1-2c}} \geq \sqrt{2} \), then the value of \( \phi_c(x) \) is always larger than \( \frac{M}{\sqrt{\lambda}} \). So the solutions given by (6) can not satisfy vacuum-vacuum boundary conditions.

2) CASE \( 0 < c < \frac{1}{2} \)

Here \( (z^4 - z^2 + \frac{c}{2}) > 0 \) is satisfied for the following cases

\[
|z| \geq \sqrt{\frac{1 + \sqrt{1 - 2c}}{2}} \quad \text{and} \quad |z| \leq \sqrt{\frac{1 - \sqrt{1 - 2c}}{2}}.
\]

\( a) \ |z| \geq \sqrt{\frac{1 + \sqrt{1 - 2c}}{2}} \)

The integral in Eq. (3) is given by

\[
\int \frac{dz}{\sqrt{z^4 - z^2 + \frac{c}{2}}} = -\frac{\sqrt{2}}{\sqrt{1 + \sqrt{1 - 2c}}} F\left(\frac{\arcsin\left(\sqrt{\frac{1 + \sqrt{1 - 2c}}{2z}}\right)}{1 - 1 + \frac{1 + \sqrt{1 - 2c}}{c}}, \frac{1}{1 + \frac{1 + \sqrt{1 - 2c}}{c}}\right). \tag{7}
\]

Thus, Eq. (3) can be rewritten as

\[
F\left(\frac{\arcsin\left(\sqrt{\frac{1 + \sqrt{1 - 2c}}{2z}}\right)}{1 - 1 + \frac{1 + \sqrt{1 - 2c}}{c}}, \frac{1}{1 + \frac{1 + \sqrt{1 - 2c}}{c}}\right) = \mp M \sqrt{\frac{1 + \sqrt{1 - 2c}}{2}} (x - x_0). \tag{8}
\]

As in the previous case the solution of the Eq. (8) is given by

\[
\phi_c(x) = \frac{M \sqrt{1 + \sqrt{1 - 2c}}}{\sqrt{\lambda}} \frac{1}{\text{sn} \left( \pm \sqrt{\frac{1 + \sqrt{1 - 2c}}{2}} \frac{M}{\sqrt{\lambda}} (x - x_0), \frac{1}{1 - 1 + \frac{1 + \sqrt{1 - 2c}}{c}} \right)}, \tag{9}
\]

where \( \text{sn}(u,m) \) is a Jacobi Elliptic Function. As in the previous case, this solution is unbounded since \( \text{sn} \) also has zeros.

It is easy to see that the amplitude of \( \phi_c(x) > \frac{M}{\sqrt{\lambda}} \). Therefore the solutions given by (9) also can not satisfy vacuum-vacuum boundary condition. Observe that, for the limit \( c \rightarrow \frac{1}{2} \), we obtain from (9) the solution

\[
\phi(x) = \pm \frac{M}{\sqrt{\lambda}} \text{coth} \left( \frac{M}{\sqrt{2}} (x - x_0) \right).
\]

Although this solution reaches the vacuum-vacuum boundary conditions (at \( \pm \infty \)) it has a discontinuity at \( x = x_0 \) and therefore must be discarded.

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\*Jacobi Elliptic Functions can be used to describe solutions of other non-linear equations such as KdV equation. \(^5\).
b) \(|z| \leq \sqrt{1-\sqrt{1-\frac{2c}{2}}}

In this case we have \(^3\)

\[
\int \frac{dz}{\sqrt{z^4 - z^2 + \frac{c}{2}}} = \frac{\sqrt{2}}{\sqrt{1 + \sqrt{1-2c}}} F\left(\frac{\arcsin(\frac{\sqrt{2}z}{\sqrt{1 - \sqrt{1-2c}}})}{\sqrt{1 \pm 1 + \frac{1+\sqrt{1-2c}}{c}}}, -1 + \frac{1}{1+\sqrt{1-2c}}\right).
\]

(10)

From (10) and (3) we get

\[
F\left(\frac{\arcsin(\frac{\sqrt{2}z}{\sqrt{1 - \sqrt{1-2c}}})}{\sqrt{1 \pm 1 + \frac{1+\sqrt{1-2c}}{c}}}, -1 + \frac{1}{1+\sqrt{1-2c}}\right) = \pm \frac{M \sqrt{1 + \sqrt{1-2c}}}{\sqrt{2}} (x - x_0).
\]

(11)

The solution of the Eq. (11) is given by

\[
\phi_c(x) = \pm \frac{M}{\sqrt{\lambda}} \frac{\sqrt{2c}}{\sqrt{1 + \sqrt{1-2c}}} \text{sn}\left(\frac{M \sqrt{1 + \sqrt{1-2c}}}{\sqrt{2}} (x - x_0), -1 + \frac{1}{1+\sqrt{1-2c}}\right).
\]

(12)

This is a family of sn-type elliptic functions parametrized by the constant of integration \(c\). Observe that, although we are not considering it by now, the Kink solution (2) is re-obtained as a limiting case \(c \to \frac{1}{2}\).

![Fig. 1. Family of sn elliptic functions for \(c = 0.008, 0.049, 0.09, ..., 0.418, 0.459, 0.5\). The solution for \(c = 0.5\) is the DHN’s Kink. Here and below we have defined \(\psi = \frac{M}{\sqrt{\lambda}} \phi\).](image)

Next, we show that the vacuum-vacuum boundary condition cannot be satisfied by any solution (12), unless \(c = \frac{1}{2}\). For the Jacobi Elliptic Functions we have \(|\text{sn}(u, m)| < 1\) and since
\[ \sqrt{1 - \sqrt{1 - 2c}} < 1, \text{ also } |\phi_c(x)| < \frac{M}{\sqrt{\lambda}}. \] So the general solution (12) can not reach the boundary condition vacuum-vacuum. Therefore we proved that there are no solutions of Kink-type for \( 0 < c < \frac{1}{2} \). Only the kink solution \( (c = \frac{1}{2}) \) satisfies vacuum to vacuum boundary condition at \( \pm \infty \) among the solutions with \( c \leq \frac{1}{2} \).

3) **CASE** \( c \geq \frac{1}{2} \)

In this case \( (z^4 - z^2 + \frac{c}{2}) > 0 \) is satisfied for any \( z \). So, we have

\[
\int \frac{dz}{\sqrt{z^4 - z^2 + \frac{c}{2}}} = \frac{1}{2} \sqrt{\frac{2}{c}} F \left( 2 \arctan \left( \frac{\sqrt{2} z}{c} \right), \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2c}} \right) \right). \tag{13}
\]

Substituting (13) in (4) we obtain

\[
F \left( 2 \arctan \left( \sqrt{\frac{2}{c}} z \right), \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2c}} \right) \right) = \pm 2M \sqrt{\frac{c}{2}} (x - x_0). \tag{14}
\]

The solution of the Eq. (14) is

\[
|\phi_c(x)| = \frac{M \sqrt{2c}}{\sqrt{\lambda}} \left| \frac{\text{sn} \left( \sqrt{\frac{c}{2}} M(x - x_0), \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2c}} \right) \right) \text{dn} \left( \sqrt{\frac{c}{2}} M(x - x_0), \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2c}} \right) \right)}{\text{cn} \left( \sqrt{\frac{c}{2}} M(x - x_0), \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2c}} \right) \right)} \right|. \tag{15}
\]

Due to the identity

\[
\frac{\text{sn}x \text{dn}x}{\text{cn}x} = \left( \frac{1 - \text{cn}2x}{1 + \text{cn}2x} \right)^{\frac{1}{2}}
\]

this solution may be called cn-type solution.

Since above solution is periodic, it is easy to see that \( \phi_c(x) \) has an infinite number of branches for each value of \( c \). In Fig(s) 2 and 3 we show, respectively, the cases \( c = \frac{1}{2} \) and \( c = 1 \). Notice that in the case \( c = 1 \) we show just four possibilities for the solutions (15) in the real line. In general, for each period of the function we get four different cases. When we consider all possibilities for each period an infinite number of branches is obtained. Nevertheless, for our purposes only the behaviour of the function inside one period is significant. So we restrict our considerations below to the interval \((-2K(m'), 2K(m'))\), where

\[
K(m') = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - m' \sin^2 t}}
\]

is a Complete Elliptic Integral of the first kind and

\[
m' = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2c}} \right).
\]
For $c = \frac{1}{2}$ the Kink and Anti-Kink solutions are recovered (see Fig. 2). However, for the solutions given by fig(s) 2.1 and 2.2 the situation is different, since they satisfy the condition vacuum-vacuum, but for the same vacuum. Also these solutions have a discontinuity for the first derivative at $x = 0$. This discontinuity in the solutions of this type will appear for any length $L$. So, these solutions does not belong to the class $C^2(\mathbb{R})$ we are considering for solving Eq. (1). Therefore we discard them as acceptable solutions.

![Fig. 2](image1)

Fig. 2. Solutions given by Eq. (15) for $c = \frac{1}{2}$. The Kink and AntiKink solutions are recovered in this case. For the first two functions $\phi'$ is not continuous at $x = 0$, so they are not admissible.

From Fig(s) 3.3 and 3.4 an interesting case can be studied. If we impose vacuum-vacuum boundary conditions $\phi\left(\frac{L}{2}\right) = \phi\left(-\frac{L}{2}\right) = M \sqrt{\frac{\lambda}{2}}$ for a fixed length $L$ we obtain from (15) (for $c = 1$, for example), the following relation

$$\frac{\text{sn}\left(\frac{ML}{2\sqrt{2}}, 0.85\right)\text{dn}\left(\frac{ML}{2\sqrt{2}}, 0.85\right)}{\text{cn}\left(\frac{ML}{2\sqrt{2}}, 0.85\right)} = 0.84.$$  

This result can be generalized for $c$ arbitrary, that is,

$$\frac{\text{sn}\left(\frac{\sqrt{c}ML}{2\sqrt{2}}, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}})\right)\text{dn}\left(\frac{\sqrt{c}ML}{2\sqrt{2}}, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}})\right)}{\text{cn}\left(\frac{\sqrt{c}ML}{2\sqrt{2}}, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}})\right)} = \frac{1}{\sqrt{2c}}.$$  

(16)

Therefore, this case (Fig(s) 3.3 and 3.4) shows that the vacuum-vacuum boundary condition can be satisfied on a finite domain in the spirit of Casimir Effect, but here in the classical level.
For the case in the Figs 3.1 and 3.2 the condition vacuum-vacuum is satisfied for the same vacuum. Moreover, as in the previous case \( c = \frac{1}{2} \) these solutions have a discontinuity for the first derivative at \( x = 0 \). So, also here we discard them as acceptable solutions.

![Fig. 3. Solutions given by Eq. (15) for \( c = 1 \). Here we took \( x_0 = 0 \). For a given length \( L \) there exists just one solution satisfying vacuum-vacuum condition at \( \pm \frac{L}{2} \).](image)

2. Properties of sn-Type Solutions in an One-Dimensional Box

Now, we study the bounded sn-type solutions of class \( C^2(\Omega) \) (Eq. (12)) that satisfy the Dirichlet boundary conditions.

First of all, it is clear that translation invariance is not valid any more due to Dirichlet boundary conditions on a finite domain, that is, a translated sn in general does not satisfy the same boundary conditions (in our case Dirichlet’s) in the same interval. Only the Kink \( c = \frac{1}{2} \) has this property in the infinite interval \( (-\infty, +\infty) \).

Now we are going to show that in a given interval \( [-\frac{L}{2}, +\frac{L}{2}] \) there exists only one sn satisfying Dirichlet boundary conditions at \( x = \pm \frac{L}{2} \). The proof is as follows. We note that, in order a solution of Eq. (12) satisfies Dirichlet boundary conditions, that is \( \phi_c(\pm \frac{L}{2}) = 0 \), we must have

\[
\text{sn} \left( \frac{ML}{2\sqrt{2}} \sqrt{1 + \sqrt{1 - 2c}}, \frac{1}{-1 + \frac{1}{c\sqrt{1 - 2c}}} \right) = 0. \quad (17)
\]
(we take, without loss of generality, \(x_0 = 0\)).

We have from the theory of Jacobi Elliptic Functions

\[
\frac{ML}{2\sqrt{2}} \sqrt{1 + \sqrt{1 - 2c}} = 2K_n(m),
\]

where

\[
K_n(m) = \frac{K(m)}{n}, \quad n = 1, 2, 3 \ldots, \quad \text{and} \quad m = \frac{1}{-1 + \frac{1 + \sqrt{1 - 2c}}{c}}.
\]

Here \(4K(m)\) is a period of the Jacobi Elliptic Functions \(sn(u, m)\) \(^4\).

On the other hand, since our solutions are Jacobi Elliptic Functions we must have \(K_n(m) \geq \frac{\pi}{2}\). This is true only for \(n = 1\). So, there is no solution of the \(sn\)-type with semi-period less than \(2K(m)\).

Also we observe from (12) that

\[
\lim_{c \to 0} |\phi_c(x)| = 0 \quad \forall x \in \mathbb{R}.
\]

For fixed \(x \in \mathbb{R}\), but large, the convergence of the limit above is very slow.

Now, with \(n = 1\), we obtain from (18)

\[
ML = \frac{4\sqrt{2}}{\sqrt{1 + \sqrt{1 - 2c}}} K(m).
\]

Taking the derivative with respect to parameter \(c\) in (19), it is easy to show that \(\frac{d(ML)}{dc} > 0\). In other words, “\(ML\)” it is an increasing function of the parameter \(c\), and so there exists one to one correspondence between \(ML\) and \(c\) (see Fig. 4 below). Therefore, given an arbitrary \(L\), there exists only one correspondent \(c \in (0, \frac{1}{2})\) as is shown in Fig. 4, and so there exists only one classical solution \(\phi_L(x)\) satisfying Dirichlet boundary conditions.

Taking the limit \(c \to 0\) we have from (19)

\[
ML = 2\pi.
\]

This relation shows that there exists a minimal value of “\(ML\)” for the solution \((12)\) satisfying Dirichlet boundary conditions. This is an interesting aspect of these solutions: once it is “placed” in a box of size \(L\), its mass must be greater than \(\frac{2\pi L}{ML}\). On the other hand if we fix \(M\) as the mass of our field, it can not exist inside a box with size smaller than \(\frac{2\pi}{ML}\). Of course if \(L \to \infty\), the minimal mass \(M \to 0\), and, in this case, all positive mass values are allowed as it should be in the DHN’s model.
On the other hand, we can also define an expression that will be seen as “topological charge” for the solution (12). In the literature we find that the topological charge is defined as \( Q = \frac{1}{2} \int_{-\infty}^{\infty} dx J^0, \) where
\[
J^0 = \frac{\sqrt{\lambda}}{M} \varepsilon^{01} \frac{d\phi}{dx^1}, \quad \text{and} \quad \varepsilon^{ik} \text{ is the antisymmetric tensor with } \varepsilon^{01} = 1.
\]

In our case the “topological charge” will be given by
\[
Q(L) = \frac{1}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\sqrt{\lambda}}{M} d\phi = \frac{1}{2} \frac{\sqrt{\lambda}}{M} \left( \phi\left(\frac{L}{2}\right) - \phi\left(-\frac{L}{2}\right) \right).\]

Using the solution (12), it is easy to verify that
\[
Q(L) = \pm \frac{\sqrt{2c}}{\sqrt{1 + \sqrt{1 - 2c}}} \text{sn} \left( \frac{ML}{2\sqrt{2}} \sqrt{1 + \sqrt{1 - 2c}}, \frac{1}{-1 + \frac{1 + \sqrt{1 - 2c}}{c}} \right). \tag{21}
\]

These are the “topological charges” for an arbitrary length \( L \) of the interval. It is seen from (17) that with the Dirichlet boundary conditions they are equal to zero. For large lengths
(L = ∞ or c = 1/2) we recover the usual conserved charge $Q$ of the Kink (AntiKink) solutions, that is

$$Q = \pm 1.$$  

In a similar way we now define an expression for the “total energy” (the so called classical mass in the case of the Kink \(^7\)), associated to the solution (12), that is

$$\mathcal{M}(L) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \epsilon_c(x),$$  

where $\epsilon_c(x)$ is the “energy density” given by

$$\epsilon_c(x) = \frac{1}{2} (\partial_x \phi_c)^2 + V(\phi_c) - V(\phi_m),$$

and $\phi_m$ are the points of minimum of the potential $V(\phi)$.

So, the “energy density” $\epsilon_c(x)$ is found to be

$$\epsilon_c(x) = \frac{M^4}{\lambda} \left( c \frac{1}{2} \text{cn}^2\left(\frac{Mx}{\sqrt{2}} \sqrt{1 + \sqrt{1 - 2c}}, \frac{1}{1 + \frac{1}{c} \sqrt{1 - 2c}}\right) \text{dn}^2\left(\frac{Mx}{\sqrt{2}} \sqrt{1 + \sqrt{1 - 2c}}, \frac{1}{1 + \frac{1}{c} \sqrt{1 - 2c}}\right) \right)$$

$$- \frac{c}{(1 + \sqrt{1 - 2c})} \text{sn}^2\left(\frac{Mx}{\sqrt{2}} \sqrt{1 + \sqrt{1 - 2c}}, \frac{1}{1 + \frac{1}{c} \sqrt{1 - 2c}}\right) + \frac{c^2}{(1 + \sqrt{1 - 2c})^2} \times$$

$$\text{sn}^4\left(\frac{Mx}{\sqrt{2}} \sqrt{1 + \sqrt{1 - 2c}}, \frac{1}{1 + \frac{1}{c} \sqrt{1 - 2c}}\right) + \frac{1}{4}. \right) \right).$$  

Using the previous relation we obtain from (22)

$$\mathcal{M}(L) = \frac{M^3 \sqrt{2}}{\lambda \sqrt{1 + \sqrt{1 - 2c}}} \left\{ \frac{1 + \sqrt{1 - 2c}}{3} \left[ E \left( \text{am}\left(\frac{ML \sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}}, \frac{1}{1 + \frac{1}{c} \sqrt{1 - 2c}}\right) \right) - \right. \right.$$  

$$\left. \frac{1 - 2c + \sqrt{1 - 2c}}{1 - c + \sqrt{1 - 2c}} \frac{ML \sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}} \right] + \frac{2c}{3} \left( \frac{1}{2} + \frac{1 + \sqrt{1 - 2c} - c}{(1 + \sqrt{1 - 2c})^2} \right) \times$$

$$\left. \text{sn}\left(\frac{ML \sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}}, \frac{1}{1 + \frac{1}{c} \sqrt{1 - 2c}}\right) \text{cn}\left(\frac{ML \sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}}, \frac{1}{1 + \frac{1}{c} \sqrt{1 - 2c}}\right) \times \right.$$  

$$\left. \text{dn}\left(\frac{ML \sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}}, \frac{1}{1 + \frac{1}{c} \sqrt{1 - 2c}}\right) \right. \left. - \frac{2}{3} \left( \frac{1}{1 + \sqrt{1 - 2c}} \right) \right. \times$$

$$\left. \frac{ML \sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}} \right] - \frac{2}{3} \left( \frac{1}{1 + \sqrt{1 - 2c}} \right) \left[ \frac{ML \sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}} - \right.$$
\[
E \left( \text{am} \left( \frac{ML\sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}}, \frac{1}{-1 + \frac{1 + \sqrt{1 - 2c}}{c}} \right) \right) + \left( \frac{1 - 2c + \sqrt{1 - 2c}c}{3(-c + 1 + \sqrt{1 - 2c})} - \frac{2c(1 + \sqrt{1 - 2c} - c)}{3(1 + \sqrt{1 - 2c})^2} + \frac{1}{2} \right) \frac{ML\sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}} \right) \),
\]

where \( E(\gamma, \frac{1}{-1 + \frac{1 + \sqrt{1 - 2c}}{c}}) \) is an elliptic integral of the second kind, and \( \gamma = \text{am} \left( \frac{ML\sqrt{1 + \sqrt{1 - 2c}}}{2\sqrt{2}} \right) \) is the so called amplitude of this integral \(^4\). Here the “total energy” \((24)\) depends on the geometrical parameter \(L\).

The “energy density” \((23)\) is shown in Fig. 5 below for the several values of \(c\).

![Energy density curve](image)

Fig. 5. Energy density for a family of sn-type solutions \((c = 0.008, 0.049, 0.09, ..., 0.418, 0.459, 0.5)\). The curve with \(c = 0.5\) corresponds to the energy density for DHN’s Kink.

For \(c = \frac{1}{2}\) in \((23)\) we recover the energy density of the Kink, i.e.

\[
\epsilon(x) = \frac{M^4}{2\lambda} \text{sech}^4 \left( \frac{Mx}{\sqrt{2}} \right),
\]

\((25)\)

and from \((24)\) we recover the classical mass of the Kink

\[
M = \frac{2\sqrt{3}M^3}{3\lambda}.
\]

\((26)\)
3. Properties of cn-Type Solutions in an One-Dimensional Box

In this case for the different values of $c$ in (15) there exist different values of the “topological charge”. As in the last section we must discard the solutions shown in Figs. 3.1 and 3.2 since they have also a discontinuity in the first derivative.

However, the solutions shown in Figs. 3.3 and 3.4 are allowed since for a given $L$ we can find such a $c$, that the solution exists in $(-\frac{L}{2}, \frac{L}{2})$ and it has continuous derivative at all points of the interval, including $x = 0$. The charge is given by

$$Q(L) = \pm \sqrt{2c} \frac{\text{sn}(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}})) \text{dn}(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))}{\text{cn}(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))}.$$ 

When the condition (16) of vacuum-vacuum transition in a finite domain is satisfied we obtain

$$Q = \pm 1.$$ 

So, we recover the conserved charge $Q$ of the Kink (AntiKink) solutions but now for a finite domain.

In the case of the “energy density” we have

$$\epsilon_c(x) = \frac{M^4}{\lambda} \left[ c \frac{\text{sn}^2(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))}{\text{cn}^2(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))} + c \frac{\text{sn}^4(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}})) \text{dn}^4(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))}{\text{cn}^4(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))} - \frac{c}{2} (1 + \frac{1}{\sqrt{2c}}) \frac{\text{sn}^2(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}})) \text{dn}^2(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))}{\text{cn}^2(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))} - \right] \frac{\sqrt{2c}}{2} \times$$

$$\frac{c}{2} (1 + \frac{1}{\sqrt{2c}}) \frac{\text{sn}^2(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}})) \text{dn}^2(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))}{\text{cn}^2(\sqrt{\frac{c}{2}} Mx, \frac{1}{2}(1 + \frac{1}{\sqrt{2c}}))} + \frac{1}{4}.$$
For \( c = \frac{1}{2} \) we recover once more the energy density of the Kink (25).

On the other hand it is not difficult to obtain an expression for the “total energy” associated to these solutions using the previous relation in (22). Again for \( c = \frac{1}{2} \) we recover the classical mass of the Kink, i.e., (26).

A similar calculation can be made considering periodic or anti-periodic boundary conditions for the solution (12). Imposing periodic boundary conditions at \( x = \pm \frac{L}{2} \), namely

\[ \phi_c(-\frac{L}{2}) = \phi_c(\frac{L}{2}), \]

we obtain (both for the positive and negative signs in (12)), the same relation (17) and, therefore, all results of the Dirichlet case are valid.

The anti-periodic boundary conditions \( \phi_c(-\frac{L}{2}) = -\phi_c(\frac{L}{2}) \) are satisfied automatically for \( \forall c \in (0, \frac{1}{2}] \) and \( \forall L \in (0, \infty) \) since the solutions (12) are odd functions (we took \( x_0 = 0 \)). So, this boundary condition does not furnish any constraint like those which we have for a Kink.

4. Conclusion

In this work we found the general solution of the scalar wave equation for the potential \( V(\phi) = -\frac{1}{2}M^2\phi^2 + \frac{1}{4}\phi^4 \) in a box in a two dimensional space-time. These solutions are the well known Jacobi Elliptic Functions. We showed that solutions given by Eqs. (6), (9) and (12) can not satisfy the vacuum-vacuum boundary conditions in a finite domain, but those solution given by Eq. (15) can satisfy these conditions. Finally, we showed that there exists one classical real solution \( \phi_L(x) \) in an interval with size \( L \) satisfying Dirichlet’s boundary conditions. In this case the product “\( ML \)” has a minimal value \( 2\pi \). This implies that a solution corresponding to a mass \( M \) can not exist in a cavity with length smaller than \( \frac{2\pi}{M} \). As in the Casimir Effect these solutions show that physical properties of some systems can change drastically when placed in cavities.

We have defined expressions for the “topological charge” (21), “total energy” (24) and “energy density” (23) associated to the elliptic sn-type solutions in a finite domain and shown that for large lengths \( (L = \infty \text{ or } c = \frac{1}{2}) \) we recover the conserved charge, classical mass and energy density of the Kink. Also, for periodic boundary conditions all results are valid obtained with Dirichlet boundary conditions. For anti-periodic boundary conditions it is easy to see that all solutions of the class defined by Eq. (12) are allowed. These solutions, however, do not satisfy the asymptotic vacuum-vacuum boundary conditions, unless \( c = \frac{1}{2} \) and \( L = \infty \).
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