Bloch oscillations of cold atoms in optical lattices

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(Dated: October 22, 2018)

This work is devoted to Bloch oscillations (BO) of cold neutral atoms in optical lattices. After a general introduction to the phenomenon of BO and its realization in optical lattices, we study different extensions of this problem, which account for recent developments in this field. These are two-dimensional BO, decoherence of BO, and BO in correlated systems. Although these problems are discussed in relation to the system of cold atoms in optical lattices, many of the results are of general validity and can be well applied to other systems showing the phenomenon of BO.

I. INTRODUCTION

This review is mainly addressed to researchers, who’s prime scientific interests are far from the topic announced in the title. We assume the reader to have no preliminary knowledge of Bloch oscillations and introduce the problem step by step, beginning from the notion of optical lattices. On the other side, we try to avoid any extended derivations and the theoretical analysis of the considered phenomena is presented in a simplified form. Moreover, in some cases we only explain the main idea of the analytical approach (referring to the original papers) and directly proceed with the results. In this sense, this review serves only as an introduction to Bloch dynamics of cold atoms. For those already familiar with the subject we advise to skip the first sections and move directly to Sec. IV and Sec. V where the most recent developments in the field are discussed.

A. Brief historical review

Originally the problem of Bloch oscillations (BO) was formulated in context of crystalline electrons. Considering the response of the system to a static electric field, Zener came to the conclusion that, instead of the uniform motion which one would naively expects, the electrons in the crystal should oscillate [1]. The period of these oscillations, now known as the Bloch period, is given by \( T_B = 2 \pi \hbar / dF \), where \( d \) is the lattice period and \( F \) is the magnitude of the static electric force. However, for a realistic strength of the electric field, the period \( T_B \) appears to be much smaller than the characteristic relaxation time \( \tau \) in the system. (The main contributors to \( \tau \) are the scattering by impurities or phonons and the electron-electron interactions.) Because of this reason BO have never been observed in the bulk crystal.

The status of BO as a pure theoretical problem changed after fabrication of semiconductor superlattices [2]. Here, due to the essentially larger period \( d \) and the lower density of the carriers, one can satisfy the condition \( T_B < \tau \) and in 1992 the first experimental observation of BO was reported for such systems [3]. It should be stressed, however, that BO in semiconductor superlattices are still dominated by the relaxation process. This difficulty is overcome by optical lattices, where standing laser waves and cold neutral atoms play the role of the crystal lattices and the electrons, respectively. In the latter system, the relaxation processes can be suppressed to any desired level, which has offered unique opportunities for experimental studies of BO [4-8].
We would like also to note that the semiconductor and optical lattices are not the only systems showing BO. As will be shown below, the deep origin of BO lays in the band spectrum of the system. In this sense, any spatially periodic system may show BO. A recent example is the periodic oscillation of a light beam in periodic photonic structures \[10, 11\].

B. List of notations

For the sake of quick reference we list below the notations used throughout the paper.

- \( \lambda \) – the laser wave length, defines the period of the optical lattices \( d = \lambda / 2 \);
- \( p_L = 2 \hbar k_L \) – double recoil momentum \( (k_L = 2\pi / \lambda) \);
- \( E_R = \hbar^2 k_L^2 / 2M \) – recoil energy, defines the characteristic energy scale of the system;
- \( \phi_{\alpha,\kappa}(x) \) – Bloch states, i.e. the eigenstates of the quantum particle in a periodic potential;
- \( E_{\alpha}(\kappa) \) – energy spectrum of Bloch waves, with \( \alpha \) being the band index and \( \kappa \) the quasimomentum \((-\pi/d < \kappa \leq \pi/d)\);
- \( \psi_{\alpha,l}(x) \) – Wannier states, which provide an alternative basis in the Hilbert space of the system, the site index \( l \) labels the wells of the optical lattice;
- \( \Psi_{\alpha,l}(x) \) – Wannier-Stark states, i.e. the eigenstates of the quantum particle in a periodic potential plus a homogeneous field. Rigorously speaking, the Wannier-Stark states are resonance or metastable states.
- \( \varepsilon_{\alpha,l} = E_{\alpha,l} - i\Gamma_{\alpha}/2 \) – the spectrum of the Wannier-Stark states, where \( \hbar / \Gamma_{\alpha} \) defines the lifetime of the states.

II. OPTICAL LATTICES

An optical lattice is a practically perfect periodic potential for atoms, produced by the interference of two or more laser beams. In this section, we explain the origin of optical lattices, their properties and some limitations.

A. Optical potential

The physical origin of the optical lattice is the so-called dipole force, which acts on the atoms in the laser field. Indeed, let us consider a two-level atom in a standing plane wave:

\[
\hat{H} = \begin{pmatrix} E_e & 0 \\ 0 & E_g \end{pmatrix} + \frac{\hbar^2}{2M} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\Omega \cos(\omega_L t) \cos(k_L x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}
\]

In Eq. (1), \( M \) is the atomic mass, \( E_g \) and \( E_e \) are the ground and excited electronic states of the atom, \( \Omega \) Rabi frequency of the dipole transition between these states, \( \omega_L \) the frequency, and \( k_L \) the wave vector of the standing wave. Substituting the wave function \( \Psi(x, t) = \exp(-i\omega_L t)\psi_e(x, t)|e\rangle + \psi_g(x, t)|g\rangle \) into the Schrödinger equation with the Hamiltonian (1) and using the rotating wave approximation, one obtains the following system of coupled equations,

\[
i\hbar \frac{\partial \psi_e(x, t)}{\partial t} = \hbar \delta \psi_e(x, t) + \frac{\hbar^2}{2M} \psi_e(x, t) - \hbar \Omega \cos(k_L x) \psi_g(x, t) \tag{2}
\]

\[
i\hbar \frac{\partial \psi_g(x, t)}{\partial t} = \frac{\hbar^2}{2M} \psi_g(x, t) - \hbar \Omega \cos(k_L x) \psi_e(x, t), \tag{3}
\]
where $\delta = (E_e - E_g)/\hbar - \omega_L$ is the detuning. Let us now assume that the atom is initially in its ground electronic state and that the detuning $\delta$ is much larger than the Rabi frequency $\Omega$. (More precisely, $\delta$ is the largest characteristic frequency of the system.) Then $\psi_e(x,t) \approx (\Omega/\delta) \cos(k_L x) \psi_g(x,t)$ and we end up with the Schrödinger equation

$$i\hbar \frac{\partial \psi_g(x,t)}{\partial t} = \left( \frac{\hbar^2}{2M} + V(x) \right) \psi_g(x,t) ,$$

(4)

$$V(x) = V_0 \cos^2(k_L x) , \quad V_0 = -\hbar \Omega^2 / \delta ,$$

(5)

which describes the motion of the atom along the standing wave. The potential (5), which has a spatial period $d$ equal to one half of the laser wave length, $d = \lambda/2$, is called the optical potential or, simply, the optical lattice. Conveniently, the depth of the optical lattice is measured in units of the recoil energy $E_R = \hbar^2 k_L^2 / 2M$. For example, for sodium atoms in a laser field detuned by 60 GHz from the $D_2$ sodium line (resonant wave length 589 nm), a 4 mW power laser creates an optical potential with an amplitude $V_0$ of about 10 recoil energies [12].

B. Spontaneous emission

In a more sophisticated approach, where the electromagnetic field is treated quantum-mechanically, the dipole force appears due to the stimulated exchange of photons between the modes of the electromagnetic field, associated with two counter-propagating running waves. Namely, the atom absorbs a photon from one of the running waves and ‘immediately’ emits it into the other wave, getting a recoil kick $p_L$ along the standing wave. During this absorption-emission process, the atom may emit a photon in the other modes of the electromagnetic field, getting a recoil kick in an arbitrary direction. The latter process, known as spontaneous emission (which should be opposed to the stimulated emission discussed above) is a kind of relaxation process due to interaction of the system with the environment, i.e. a bath of the electromagnetic modes. The rate $\tilde{\gamma}$ of spontaneous emission is given by the product of the natural width of the excited level $\gamma$ (which is a unique characteristic of the chosen atomic transition) and the population of the upper state,

$$\tilde{\gamma} = \gamma (\Omega/\delta)^2 .$$

(6)

This equation implies that by simultaneously increasing the detuning and the intensity of the laser field, one can keep the depth of the optical potential constant but suppresses the interaction of the system with its environment. For example, in the case of sodium atoms in a laser field detuned by 60 GHz, the rate of spontaneous emission is $\tilde{\gamma} \sim 100 \text{s}^{-1}$, which is actually negligible on the time scale of the laboratory experiment [12].

C. Lattice dimensionality

Up to now, we have considered an idealized situation of plane waves. In practice, however, one deals with beams of a finite width, i.e.,

$$V(r) = V_0 \exp \left[ - \left( \frac{r}{r_0} \right)^2 \right] \cos^2(k_L x) ,$$

(7)

where $r_0 \sim 50 \mu m$ is the $1/e$ diameter of the beam. Note that for a red detuning, the laser field also provides a transverse confinement for the atoms. Besides the optical potential (4), there is an additional harmonic confinement $V_{\text{trap}}(r) \sim \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2$ in the laboratory experiment due to a magnetic time-orbiting potential, which is used to capture the atoms during the cooling procedure. After the sample preparation (the sample preparation includes the cooling of the
atoms and an adiabatically switching on of the optical potential) this harmonic potential can be kept 'switched on' or relaxed towards zero. If not stated otherwise, we assume the second case throughout the paper.

Using two crossed standing laser waves (4 running waves) one can create two-dimensional lattices with approximately \((r_0/\lambda) \sim 100\) wells in each direction. There is a high degree of freedom in choosing a particular form of the 2D potential. For example, by playing with the frequencies of the waves, one can realize separable ‘egg-crate’ or non-separable ‘quantum dot’ potentials (see figure 5 below); changing the angle between the crossing standing waves from 90 to 60 degrees transforms the square lattice into a hexagonal one; and so on. We shall study 2D optical lattices in more detail in Sec. IV. Needless to mention that using three mutually perpendicular standing waves one gets a true 3D lattice – an extension of the results of Sec. IV to the 3D case is straightforward.

Going ahead, we note that, beside providing a richer dynamics of BO, the higher dimensionality of an optical lattice also affects the strength of the atom-atom interaction. Namely, due to a stronger confinement of the atomic wave function, the effective constant of atom-atom interactions in 2D lattices is at least one order of magnitude (two - four orders for 3D lattices) larger than in 1D lattices (see Sec. V B).

D. Bloch waves

As well known, the energy spectrum of a quantum particle in a periodic potential consists of the Bloch bands. An example of the atomic Bloch band spectrum is given in Fig. 1 where zero energy corresponds to the bottom of the potential wells. If the optical lattice is switched on adiabatically, the atoms populate the bottom of the ground band. The characteristic width of the distribution over the quasimomentum \(\kappa\) depends mainly on the frequency of the harmonic trap. For a small frequency (a weak confinement) the atoms may coherently populate hundreds of the wells, which results in a very narrow distribution in the quasimomentum, \(\Delta \kappa \sim 0.01\pi/d\). Thus one may speak
of an atomic Bloch wave,
\begin{equation}
\phi_{\alpha,\kappa}(x) = \exp(i\kappa x)\chi_{\alpha,\kappa}(x), \quad \chi_{\alpha,\kappa}(x + d) = \chi_{\alpha,\kappa}(x).
\end{equation}

One of the facilities provided by optical lattices is that the atomic Bloch waves can be directly measured in the laboratory experiment. Usually, the measurement goes as follows. After preparation of the Bloch wave, the optical potential is abruptly switched off and the atoms move in free space for a given time (the so-called ‘time-of-flight’). Then the atoms are exposed to resonant light and, by ‘taking a picture’ of the atomic cloud, one records the spatial distribution of the atoms. Because the time-of-flight is known, this spatial distribution carries information about the momentum distribution of the atoms in the optical lattice. The latter is given by the squared Fourier transform of the Bloch wave \(\phi_{\alpha,\kappa}\) and consists of a number of peaks, separated by \(p_L = 2\pi\hbar/d\). This peak-like structure of the momentum distribution (see Fig. 2 below), well observed in the laboratory experiments, is a direct indication of the atomic Bloch waves.

We conclude this section by introducing the Wannier states, which we shall use later on. These Wannier states are obtained by integrating the Bloch states \(\phi_{\alpha,\kappa}\) over the quasimomentum,
\begin{equation}
\psi_{\alpha,l}(x) = \int_{-\pi/d}^{\pi/d} d\kappa \exp(-id\kappa l) \phi_{\alpha,\kappa}(x),
\end{equation}
and provide an alternative basis in the Hilbert space. Unlike the Bloch waves, the Wannier states are localized in space. Note that, because of \(\psi_{\alpha,l}(x) = \psi_{\alpha,0}(x - ld)\), it suffices to calculate only one Wannier state \((l = 0\) in what follows) for each energy band.

\section{III. BLOCH OSCILLATIONS IN 1D LATTICES}

This section studies different regimes of BO of cold atoms in 1D optical lattices. It is implicitly assumed in what follows that neither spontaneous emission nor atom-atom interaction are important and, thus, we can use the single-particle Schrödinger equation to analyze the problem. Beside this, we assume that the transverse motion of the atoms is frozen (i.e., we are dealing with a 1D problem). Although we do not discuss the validity of this approximation, the experimental results indicate that this is, indeed, the case realized in quasi 1D lattices.

\subsection{A. Bloch period and Landau-Zener tunneling}

Bloch oscillations are the dynamical response of the system to a static force:
\begin{equation}
\tilde{H} = \tilde{H}_0 + Fx, \quad \tilde{H}_0 = \frac{p^2}{2M} + V_0\cos^2(k_Lx).
\end{equation}

For neutral atoms, the static force \(F\) is usually introduced by accelerating the optical lattice, \(V(x) \rightarrow V(x - at^2/2)\), which can be done by an appropriate chirping of the frequencies of two counter-propagating waves. Then, in the lattice coordinate frame, the atoms experience an inertial force of magnitude \(F = aM\). The other option is to employ the gravitational force for a vertically oriented optical lattice, then \(a = 9.8\ m/s^2\).

The common approach to the problem of BO is to look for the solution as a superposition of Houston functions \(13\),
\begin{equation}
\psi(x,t) = \sum_{\alpha} c_{\alpha}(t)\psi_{\alpha}(x,t),
\end{equation}
\begin{equation}
\psi_{\alpha}(x,t) = \exp\left(-\frac{i}{\hbar} \int_0^t dt' E_{\alpha}(k')\right) \phi_{\alpha,k'}(x),
\end{equation}
FIG. 2: Dynamics of the atomic momentum distribution $P(p)$, induced by a weak static force $F < 0$. The depth of the optical potential is $V_0 = 10E_R$. The weak wiggling of $P(p)$ is an artifact due to the finite size of the lattice ($L = 10$, periodic boundary conditions) used in the numerical simulations.

where $\phi_{\alpha, \kappa'}(x)$ is the Bloch function with quasimomentum $\kappa'$ evolving according to the classical equation of motion $\dot{p} = -F$, i.e $\kappa' = \kappa_0 - Ft/\hbar$. Substituting Eq. (12) into the time-dependent Schrödinger equation with the Hamiltonian $\mathcal{H}$, we obtain

$$\frac{i\hbar}{\hbar} \dot{c}_\alpha = F \sum_\beta X_{\alpha, \beta}(\kappa') \exp \left( -\frac{i}{\hbar} \int_0^t dt' [E_\alpha(\kappa') - E_\beta(\kappa')] \right) c_\beta,$$

(13)

where $X_{\alpha, \beta}(\kappa) = \int dx \chi_{\alpha, \kappa}(x) \partial \chi_{\beta, \kappa}(x) / \partial \kappa$. When neglecting the inter-band coupling, i.e $X_{\alpha, \beta}(\kappa) = 0$ for $\alpha \neq \beta$, we have $i\hbar \dot{c}_\alpha = FX_{\alpha, \alpha}(\kappa')c_\alpha$ and, thus,

$$|c_\alpha(t)| = |c_\alpha(0)|.$$

(14)

This solution is the essence of the so-called single-band approximation.

The correction to the solution (13) is obtained by using the formalism of Landau-Zener tunneling. In fact, when the quasimomentum $\kappa'$ explores the Brillouin zone, an adiabatic transition occurs at the points of ‘avoided’ crossings between the adjacent Bloch bands (see, for example, the avoided crossing between the 1st and 2nd band in Fig. 1(a) at $\kappa = \pi/d$). As a result, the population of the $\alpha$th band decreases exponentially with the decay time

$$\tau = \hbar / \Gamma_\alpha, \quad \Gamma_\alpha = a_\alpha F \exp(-b_\alpha/F),$$

(15)

where $a_\alpha$ and $b_\alpha$ are band-dependent constants. Note, that Eq. (15) provides only an estimate for the mean decay rate and, to find the exact dependence $\Gamma_\alpha(F)$, one has to employ a different approach which we shall briefly discuss later on in Sec. III D.

As follows from the estimate (15) for a weak static force the Landau-Zener tunneling can be neglected and the solution of the problem is essentially given by Eq. (12), i.e., $\psi(x,t) \sim \phi_{\alpha, \kappa_0 - Ft/\hbar}(x)$. The linear change of the quasimomentum of the Bloch wave results in a periodic change of the
atomic momentum distribution \( P(p) = |\psi(p, t)|^2 \) (see Fig. 2) which is the quantity measured in in the laboratory experiments [1, 3]. (Since the measurement is destructive, one has to repeat the experiment several times to record the time-evolution of the momentum distribution.) The period of these oscillations is given by the Bloch period,

\[
T_B = \frac{2\pi \hbar}{dF}
\]

with \( d = \lambda/2 \) and \( F = aM \). Using the single-band approximation, it is also easy to show that the mean momentum evolves as \( \langle p(t) \rangle = M v(Ft) \), where \( v(\kappa) = \partial E_\alpha(\kappa)/\hbar \partial \kappa \) is the group velocity. Note, that the amplitude of oscillations of \( \langle p(t) \rangle \) is proportional to the band width and, thus, can be extremely small (deep optical lattices). The oscillations of the atomic momentum distribution, however, are qualitatively the same independent of the particular choice of the parameters and, in this sense, are a more reliable signature of BO.

B. Wannier-Stark ladder

Bloch oscillations can also be described in terms of the Wannier-Stark states, which provide a useful insight into the physics of this phenomenon. To simplify the analysis, we consider a tight-binding model – an additional (after the single-band) approximation to the original problem. This approximation is not crucial and one obtains similar results for the single-band model. Using the notion of the Wannier states [9], the Hamiltonian of the tight-binding model has the form,

\[
\hat{H}_{TB} = \sum_l (E_\alpha + dFl) |l\rangle \langle l| + \frac{J_\alpha}{2} \sum_l (|l+1\rangle \langle l| + |l-1\rangle \langle l|)
\]

where \( J_\alpha \) is the hopping matrix element, \( E_\alpha = E_\alpha(\kappa) \), and \( \langle x|l\rangle = \psi_{\alpha,l}(x) \). The Hamiltonian (17) can be easily diagonalized, giving the spectrum,

\[
E_\alpha(\kappa) = E_\alpha + J_\alpha \cos(d\kappa), \quad \text{if} \quad F = 0
\]

(note, in passing, that the tight-binding model approximates the Bloch dispersion relation by a cosine function), and

\[
E_{\alpha,l} = E_\alpha + dF l, \quad \text{if} \quad F \neq 0.
\]

The discrete spectrum (19) is known as the Wannier-Stark ladder and the eigenfunctions corresponding to \( E_{\alpha,l} \),

\[
|\Psi_{\alpha,l}\rangle = \sum_m J_{m-l} \left( \frac{J_\alpha}{2dF} \right) |m\rangle,
\]

(here \( J_n(z) \) is the ordinary Bessel function) are known as the Wannier-Stark states. Because the Bessel functions \( J_n(z) \) are exponentially small for \( |n| > |z| \), the Wannier-Stark states are localized in space with a localization length \( l_{WS} = 1 \) (in units of lattice period) for \( dF > J_\alpha \) and \( l_{WS} \approx J_\alpha/dF \) for \( dF < J_\alpha \). One may refer to these two cases as the ‘strong force’ and ‘weak force’ regimes. In this paper, however, we reserve the term ‘strong force’ to static force magnitudes which break the single-band approximation (strong Landau-Zener tunneling).

C. Wave packet dynamics

In Sec. IIIA we have considered the case of a Bloch wave as an initial condition. It is also interesting to study a situation, where only a few wells of the optical potential are populated [14]. Then, along with the oscillations in momentum space, the atoms also oscillate in configuration.
FIG. 3: Spatial oscillations of the localized wave-packet. The amplitude of the oscillations is given by the localization length of the Wannier-Stark states $l_{WS} \sim 1/F$.

This is illustrated in Fig. 3 which shows the dynamics of $P(x) = |\psi(x,t)|^2$ for a ‘minimum uncertainty’ wave packet. The amplitude of the oscillations is given by the localization length of the Wannier-Stark states. Indeed, since the general solution of the Schrödinger equation can be written as a sum over the Wannier-Stark states,

$$\psi(x,t) = \sum_l c_l \exp(-iE\alpha,t/\hbar) \Psi_{\alpha,l}(x),$$

the localization length $l_{SW}$ defines the maximum distance where the wave-packet can move to.

For the purpose of future use, we also display the solution (21) in the momentum representation. Expanding the initial state in terms of Bloch states, $\psi(x,0) = \int d\kappa g_{\alpha}(\hbar\kappa)\phi_{\alpha,\kappa}(x)$, we introduce the envelope functions $g_{\alpha}(p) = g_{\alpha}(\hbar\kappa)$. Then, using the extended Brillouin zone representation, the solution $\psi(p,t)$ can be represented as

$$\psi(p,t) = \sum_{\alpha} \exp(-iE_\alpha t/\hbar) \sum_{n=-\infty}^{\infty} g_{\alpha}(p + Ft + np_L) \Psi_{\alpha,0}(p)$$

(in comparison with Eq. (21) here we also included the sum of $\alpha$). Equation (22) is well suited for a numerical simulation of Bloch dynamics and has actually been used in Sec. III A to illustrate the oscillations of the momentum distribution.

D. Bloch oscillations for strong static forcing

Above we have considered only the case of a weak static force, where the Landau-Zener tunneling is negligible and one can use the single-band approximation to study the system dynamics. This
section is devoted to the regime of strong forcing, which we shall analyze by using the formalism of metastable (or resonance) Wannier-Stark states. The latter are defined as non-hermitian eigenstates of the Hamiltonian $H_0$, corresponding to the complex energies

$$E_{\alpha,l} = E_{\alpha} + dF \ell - i \Gamma_{\alpha}/2.$$  

The spectrum (23) generalizes the notion of the Wannier-Stark ladder (19) and is given by the poles of the scattering matrix of the system. For the details of the scattering matrix approach to the Wannier-Stark problem we refer the reader to the review [15]. Here we only note that this approach does not involve any approximation and, hence, the results presented below are rigorous.

The decay constant $\Gamma_{\alpha}$ in Eq. (23), which defines the lifetime of the Wannier-Stark states, has a rather nontrivial dependence on the static force. An example is given in Fig. 4, where the six different curves correspond to $\Gamma_{\alpha}$ of the Wannier-Stark ladder, originating from the six lowest bands of the Hamiltonian $H_0$. Strong fluctuations of the decay rate, superimposed on the Landau-Zener dependence (15), are noticed. These fluctuations are due to resonance tunneling, occurring when the positions of the Wannier-Stark levels in different wells of the optical potential coincide, i.e., when $\text{Re}[E_{\alpha,l}(F)] = \text{Re}[E_{\beta,m}(F)]$.

Having the scattering problem solved, one generalizes Eq. (22) for the system dynamics simply by substituting the stationary Wannier-Stark states (20) by the metastable Wannier-Stark states, the real energy $E_{\alpha}$ by the complex energy $E_{\alpha} = E_{\alpha} - i\Gamma_{\alpha}/2$, and multiplying the whole expression by the Heaviside step-function $\Theta(p + Ft)$ which truncates the momentum distribution at $p < -Ft$ [15]:

$$\psi(p, t) = \Theta(p + Ft) \sum_{\alpha} \exp(-iE_{\alpha}t/\hbar) \sum_{n=-\infty}^{\infty} g_{\alpha}(p + Ft + np_L) \Psi_{\alpha,0}(p).$$  

The characteristic feature of the resonance Wannier-Stark states $\Psi_{\alpha,0}$ is an exponentially growing tail for negative momentum, $\Psi_{\alpha,l}(p) \sim \exp(\Gamma_{\alpha}p/\hbar F)$. Then, as follows from Eq. (24), the solution

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Imaginary part of the complex energies (23) as the function of the inverse scaled static force $1/F$ ($F \rightarrow dF/2\pi V_0$). The depth of the optical potential is $V_0 = 8E_R$.}
\end{figure}
ψ(p,t) is essentially a sequence of wave packets, separated by a distance \( p_L \). In the coordinate representation, these equidistant sequence of the ‘momentum’ wave packets transforms into a train of ‘coordinate’ wave packets, distributed in space according to a square law. Thus, in the strong field regime, the atomic array acts as a matter laser, emitting one pulse of matter per Bloch period. 

E. Related problems

In this subsection, we briefly discuss two important modifications of the problem of atomic BO. The first one deals with BO in the presence of harmonic confinement, i.e. \( V(x) = V_0 \cos^2(k_L x) + M \omega_x^2 x^2/2 \). (The characteristic value of the frequency \( \omega_x \) is a few Hz, which should be compared with the frequency of small atomic oscillations \( \sim \omega_R(V_0/E_R)^{1/2} \) of few kHz.) Let us consider the situation where the atoms are located far from the trap origin. In practice, this initial condition is realized by a sudden shifting of the center of the trap to a distance \( x_0 \gg d \). Then the atoms locally feel a static force \( F = M \omega_x^2 x_0 \), which is one of the preconditions for BO. It should be noted, however, that the analogy with BO should be drawn here with some precautions. Indeed, using the tight-binding approximation, the Hamiltonian of the atom in the combined potential reads as

\[
\hat{H}_{\text{eff}} = \sum_l \nu l^2 |l\rangle\langle l| + \frac{J_a}{2} \sum_l (|l+1\rangle\langle l| + |l-1\rangle\langle l|) 
\]

with \( \nu = M \omega_x^2 d^2 \). This Hamiltonian formally corresponds to the Hamiltonian of the quantum pendulum,

\[
\hat{H}_{\text{eff}} = -\frac{\nu}{2} \frac{d^2}{d\theta^2} + J_a \cos \theta ,
\]

and, hence, the characteristic frequency of atomic oscillations is given by the frequency of the pendulum. The latter is known to have a rather nontrivial dependence on pendulum energy and can be approximated by a linear law only in the asymptotic region of large energies. Thus, to mimic BO (in a sense that the frequency of oscillations is inversely proportional to the local static field) one should satisfy the condition that the pendulum is well above its separatrix.

The second problem we would like to mention deals with BO in a resonant or near-resonant laser field. In this case, the external degree of freedom of the atom (the motion of the center of mass) cannot be decoupled from its internal (electronic) degree of freedom and, thus, we have to solve the system of partial differential equations (with a static term added) exactly, without adiabatic elimination of the upper state. It was found, in particular, that BO of the atoms in a resonant field enforce a kind of Rabi oscillations, where up to 90 percent of the atoms may appear in the excited electronic state. We note, however, that an experimental realization of this interesting regime of BO requires a very narrow width of the optical transition \( h\gamma < E_R \).

IV. BLOCH OSCILLATIONS IN 2D LATTICES

In this section, we study Bloch oscillations in 2D optical lattices. We restrict ourselves to square lattices, created by laser beams of equal intensities. If the frequencies of the crossing standing waves also coincide, the optical potential is given by

\[
V(x,y) = V_0[\cos(k_L x) + \cos(k_L y)]^2 ,
\]

as can be easily shown by repeating the derivation of Sec. 2.1. Note that the potential is not separable. If, however, the frequencies of the waves are slightly mismatched, we obtain a separable potential

\[
V(x,y) = V_0[ \cos^2(k_L x) + \cos^2(k_L y) ] .
\]
Obviously, the property of separability can be attributed only to 2D or 3D potentials. The consequences of this property for the dynamics of BO is one of the main questions we address in this section.

FIG. 5: Contour plot of the non-separable optical potential (27) (left panel), and the separable potential (28) (right panel). Note that the primitive translation vectors for the lattice are rotated by 45 degrees with respect to the laboratory coordinate system.

A. Band spectrum

We begin with the analysis of the Bloch band spectrum for the specified 2D lattices. To simplify the equations, we shall use a scaling where the coordinate is measured in units of the lattice period \( d \) and the time in periods of the recoil frequency \( t \to \omega_R t \). This scaling (which also involves a rotation of the coordinate system) leads to the dimensionless Hamiltonian

\[
\hat{H}_0 = -\frac{\hbar^2}{2} \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) + \cos x + \cos y - \varepsilon \cos x \cos y
\]

where the only independent parameter is the scaled Planck’s constant \( \hbar \sim (E_R/V_0)^{1/2} \) and the constant \( \varepsilon \) equal to zero or ±1 for a separable and non-separable potential, respectively. (Although \( \varepsilon \) can take only the specified integer values, for theoretical purposes it might be useful to consider the whole interval \( 0 \leq |\varepsilon| \leq 1 \).)

To characterize the two-dimensional dispersion relation \( E_\alpha(\kappa_x, \kappa_y) \), we consider the cross-section of the spectrum along the lines \( \kappa_y = 0 \) for \( \kappa_x < 0 \) and \( \kappa_y = \kappa_x \) for \( \kappa_x > 0 \). The result is depicted in Fig. 6 where the left and right panels refer to the case \( \varepsilon = 0 \) and \( \varepsilon = 1 \), respectively. It is seen in the figure that a nonzero value of \( \varepsilon \) strongly modifies the central part of the spectrum. (In particular, it removes the degeneracy between the bands along the lines \( \kappa_y = \pm \kappa_x \).) As concerning the ground Bloch band, the difference shows up in the coefficients of the Fourier expansion of the dispersion relation

\[
E_0(\kappa_x, \kappa_y) = \sum_{m,n} J_{m,n} \exp(i2\pi m \kappa_x) \exp(i2\pi n \kappa_y).
\]
In the separable case, only the coefficients $J_{m,n}$ with $n = 0$ or $m = 0$ differ from zero while in the non-separable case all elements have non-zero values. (Also note the symmetry $J_{n,m} = J_{m,n}$.)

At the same time, even the largest non-trivial coefficient $J_{1,1}$ is only $1/20$ of the coefficient $J_{0,1}$, which practically alone determines the width of the ground Bloch band.

### B. Fractional Wannier-Stark ladder

Let us briefly discuss the energy spectrum of the 2D Wannier-Stark system,

$$\hat{H} = \hat{H}_0 + F_x x + F_y y,$$

where $\hat{H}_0$ is given in Eq. (29). First we consider the special case when the vector $\mathbf{F}$ of the static force is parallel to one of the crystallographic axes of the lattice (the $x$-axis, to be certain). In this case, the Hamiltonian possesses the ‘ladder symmetry’ along the field direction and a translational symmetry in the direction perpendicular to the field. Thus, the spectrum of the system consists of replica of the Bloch band $E_{\alpha}(\kappa_y)$, shifted relative to each other by the Stark energy $2\pi F$ (here $2\pi$ stands for the lattice period).

The above result can be extended to the case of arbitrary ‘rational’ directions of the field,

$$\frac{F_x}{F_y} = \frac{q}{r},$$

where $q, r$ are co-prime integers. In this case, one uses the transformation of the coordinates

$$x' = \frac{q x + r y}{(r^2 + q^2)^{1/2}}, \quad y' = \frac{q y - r x}{(r^2 + q^2)^{1/2}}.$$
which introduces a new lattice with the period \( d' = 2\pi(r^2 + q^2)^{1/2} \) and matches the vector \( \mathbf{F} \) to the primitive vector of this new lattice,

\[
\hat{H}' = \frac{p'_x^2}{2} + \frac{p'_y^2}{2} + V(x', y') + F x'.
\]  

(34)

It is also easy to see that the transformation \( \mathcal{T} \) actually introduces \( s = r^2 + q^2 \) different (sub)lattices, whose Hamiltonians differs from \( \mathcal{H}_0 \) by an additive term \((d'F/s)j, j = 1, \ldots, s-1\). Thus, for rational directions of \( \mathbf{F} \), the spectrum of the original Hamiltonian \( \mathcal{H}_0 \) is a fractional Wannier-Stark ladder along the direction of the field, constructed from the Bloch bands with \( \sqrt{s} \) times reduced Brillouin zone in the direction perpendicular to the static force, i.e.,

\[
\mathcal{E}_{\alpha,f}(\kappa_\perp) = \mathcal{E}_\alpha(\kappa_\perp) + \frac{2\pi F l}{(r^2 + q^2)^{1/2}}, \quad \mathcal{E}_\alpha(\kappa_\perp) = E_\alpha(\kappa_\perp) - \frac{\Gamma_\alpha(\kappa_\perp)}{2}.
\]  

(35)

Note that for a separable potential the sub-bands \( \mathcal{E}_\alpha(\kappa_\perp) \), \(-s^{-1/2} \leq \kappa_\perp < s^{-1/2}\), have zero width for any direction of the fields \( \theta = \arctan(r/q) \), except \( \theta = 0, \pi/2 \). (For the imaginary part of the dispersion relation one obviously has \( \Gamma_\alpha(F, \theta) = \tilde{\Gamma}_\alpha(F \cos \theta) + \tilde{\Gamma}_\alpha(F \sin \theta) \), where \( \tilde{\Gamma}_\alpha(F) \) is obtained by solving the 1D problem.) As \( \varepsilon \) deviates from zero, the sub-bands gain a small but finite width. A more dramatic consequence of the non-separability, however, is the strong dependence of the decay rate \( \Gamma_\alpha(\kappa_\perp) \) on the quasimomentum \( \kappa_\perp \), where the decay rate may vary by several orders of magnitude \( [20] \).

C. Wave packet dynamics

It is interesting to compare the wave packet dynamics for separable and non-separable potentials. Restricting ourselves to the ground Bloch band, the results of these studies \( [21, 22] \) can be summarized as follows.

For a weak static force (negligible Landau-Zener tunneling) and \( \varepsilon = 0 \), the two-dimensional BO are given by a superposition of the one-dimensional BO. In other words, BO is a (quasi)periodic process with two periods defined by the projections of the static force to the crystallographic axes of the lattice, \( T_{x,y} = 2\pi \hbar / dF_{x,y} \). In coordinate space, this is reflected in Lissajous-like trajectories of a localized wave packet in the \( xy \)-plane. Note that for \( \varepsilon = 0 \), the motion of the wave packet is generally non-dispersive. (Exclusions are \( \theta = 0, \pi/2 \), where BO along one axis are accompanied by a dispersive spreading of the wave packet along the other axis.) A nonzero \( |\varepsilon| \leq 1 \) only slightly modifies this dynamics which is actually not surprising, because the dispersion relation \( \mathcal{T} \) for the ground Bloch band can be well approximated by a ‘separable’ dispersive relation, \( E(\kappa_x, \kappa_y) \approx (J_{1,0}/2) \cos(2\pi \kappa_x) + (J_{0,1}/2) \cos(2\pi \kappa_y) \). In terms of the energy spectrum \( \mathcal{E}_\alpha(\kappa_\perp) \) this approximation amounts to neglecting the width of the bands \( E_\alpha(\kappa_\perp) \).

The case of a strong static force is essentially more complicated and here the difference between the separable and non-separable potentials appears on the \textit{qualitative} level. Indeed, in the strong force regime, the atoms may escape out of the potential wells through a sequence of Landau-Zener tunneling transitions to higher bands. The details of this process crucially depend on the particular structure of the upper bands which, as seen in Fig. \( \text{[4]} \) is quite different for \( \varepsilon = 0 \) and \( \varepsilon \neq 0 \). Similar to the 1D case, one can analyze the tunneling of the atoms by using the formalism of metastable (now 2D) Wannier-Stark states. This analysis leads to the prediction that in the separable case the atoms may escape out of the potential wells only along the \( x \) - and \( y \)-axis of the lattice, while for the non-separable potential additional escape channels appear \( [21] \). This is illustrated in Fig. \( \text{[4]} \) which shows ‘snapshots’ of two-dimensional BO in the strong force regime for \( \varepsilon = 0 \) (left panel) and \( \varepsilon = 1 \) (right panel). The additional channel along the \( (1,1) \)-crystallographic direction is clearly seen in the right panel. (Also notice a rich interference structure between the channels, which is absent in the separable case.)
FIG. 7: A fragment of the wave function of the atom in the separable (left panel) and non-separable (right panel) potential in the case of a strong static force. The system parameters are $\hbar = 2$, $F = 0.2$, and $F_x/F_y = 1$.

D. Related problems

The 2D optical lattice also offers an opportunity for studying a number of related problems like, for example, ‘two-bands’ BO. Indeed, superimposing the potentials shown in Fig. 5 one can easily realize the case, where the two lowest Bloch bands of the combined potential are separated by an arbitrary small gap (located along the edges of the Brillouin zone). Then, performing a BO, the atom tunnels between these two bands at each crossing of the Brillouin zone, which results in a very non-trivial dynamics of BO [21, 22].

The other problem we would like to mention is the scattering of an atomic beam by a 2D optical potential [23]. In classical dynamics, and for a non-separable potential, this would be a chaotic scattering process, with fractal basins for the scattered channels. The classical and quantum scattering of the atoms by 2D lattices is studied in some details in the paper cited above.

V. DECOHERENCE OF BLOCH OSCILLATIONS

Since BO is a coherent quantum phenomenon, it is important to study the processes which cause a decoherence of the system. In this section we consider two of these processes – decoherence due to spontaneous emission and due to atom-atom interactions. We would like to stress that decoherence or relaxation, usually considered as ‘unwelcome’ phenomena, are also of interest on their own. Indeed, as was already noticed by Esaki [24], the conventional conductivity is an interplay between BO and relaxation processes. Thus, the analysis of decoherence of BO is a necessary step for developing a theory of atomic conductivity.

A. Decoherence by spontaneous emission

As mentioned above in Sec. 1, spontaneous emission is a particular case of the systems interaction with an environment. For this kind of problem, the approach based on the single-
FIG. 8: Dispersion $\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$ of the atomic wave packet as a function of time. Time is measured in the units of $T_J = 2\pi \hbar / J$, the scaled rate of spontaneous emission $\hbar \gamma / J = 0.05$, $d = \pi$, and the scaled magnitude of the static force ($F \rightarrow dF / J$) is indicated in the figure. The slopes of the dashed lines are given by the values of the diffusion coefficient $D$. The particle Schrödinger equation is not applicable, and the dynamics of the system should be described in terms of the density matrix $\rho(x, x'; t)$, which is defined as the trace of the total wave function (system plus environment) over irrelevant variables of the environment. For a specified environment (the photon bath), the density matrix obeys the master equation (36),

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \frac{\tilde{\gamma}}{2} \int du P(u) \left( \hat{L}_u \hat{L}_u \hat{\rho} - 2 \hat{L}_u \hat{\rho} \hat{L}_u^\dagger + \hat{\rho} \hat{L}_u^\dagger \hat{L}_u \right),$$

where $\hat{H}$ is the Hamiltonian (10) (we consider a quasi one-dimensional lattice), $\tilde{\gamma}$ the rate of spontaneous emission (6), and $\hat{L}_u$ the projection of the recoil operator on the $x$ axis,

$$\hat{L}_u = \cos(k_Lx) \exp(iu_k L x), \quad |u| \leq 1.$$  

Note that Eq. (36) has the Lindblad form and, thus, $\text{Tr}[\rho(t)] = \int dx \rho(x, x; t) = 1$. The distribution $P(u)$ of the random variable $u$ in Eq. (36) is defined by the angle distribution for the momentum of the spontaneously emitted photons (26), and $\hat{L}_u$ the projection of the recoil operator on the $x$ axis, is, in the case of linearly polarized light considered here, approximately given by $P(u) = 1/2$.

Using the tight-binding approximation, i.e. substituting the Hamiltonian $\hat{H}$ by the tight-binding Hamiltonian (17), and the recoil operator (37) in its tight-binding version $\hat{L}_u = \sum_i (-1)^i \exp(i\pi u L_i) |l\rangle \langle l|$, the master equation (36) can be solved analytically, with the following main results (27): The spontaneous emission leads to decoherence of the system, i.e. the density matrix tends to a diagonal one in the basis of the Wannier states. As a consequence, BO decay as

$$\langle p(t) \rangle = p_0 \exp(-\nu t) \sin(\omega_B t)$$

where the decay rate $\nu$ appears to coincide with the rate $\tilde{\gamma}$ of the spontaneous emission.
of BO is accompanied by (asymptotically) diffusive spreading of the atoms (see Fig. 8),
\[
\langle x^2(t) \rangle \sim D t, \quad D = \left( \frac{p_0}{M} \right)^2 \frac{\tilde{\gamma}}{\omega_B^2 + \tilde{\gamma}^2}
\] (39)
(here \( p_0 \) is the amplitude of BO in the absence of the relaxation process). Note that (since \( \omega_B \sim F \)) the static force actually suppresses the diffusion. Considering the diffusion as a ‘generalized conductivity’, this result agrees with the prediction of Esaki and Tsu that the conductivity of the system tends to zero when the frequency of BO becomes much larger than the characteristic frequency of the relaxation processes.

The (numerical) analysis of the system dynamics beyond the tight-binding approximation leads to qualitatively the same results, although the quantitative deviation can be larger than 50 percent [27]. It should also be mentioned that the validity of Eq. (36) still assumes a low population of the excited electronic state of the atom and, hence, the case of resonant driving (briefly analyzed in Sec. IIIE for \( \gamma = 0 \)) is excluded from this consideration.

### B. Interacting atoms and the Bose-Hubbard model

Up to now, we have studied BO using single-particle quantum mechanics. This is justified only for a very dilute gas of atoms, where the atom-atom interactions can be neglected. If this is not the case, the problem of BO becomes very diverse and, first of all, one should distinguish between Bose and Fermi statistics. In this review we restrict ourselves to bosonic atoms. Moreover, we assume in what follows that the initial state of the system is a Bose-Einstein condensate, where all atoms occupy the zero quasimomentum state of the ground Bloch band. To study the time evolution of this state, one usually uses the Gross-Pitaevskii (or nonlinear Schrödinger) equation. It is understood, however, that this equation has a limited applicability – in particular, the Gross-Pitaevskii equation is unable to describe the decoherence of the system. Because of this, we employ here a more general approach, based on the Bose-Hubbard model.

The Bose-Hubbard model (with a static term added) generalizes the tight-binding Hamiltonian (17) to the multi-particle case. To simplify the analysis we shall consider only the 1D Bose-Hubbard model, even when discussing the 3D lattices. (This approximation is not crucial for the phenomena discussed below.) Then the Hamiltonian of the system has the form
\[
\hat{H} = -\frac{J}{2} \left( \sum_l \hat{a}^\dagger_{l+1} \hat{a}_l + \text{h.c.} \right) + dF \sum_l \hat{n}_l + \frac{W}{2} \sum_l (\hat{n}_l - 1)
\] (40)
(the band index \( \alpha = 0 \) is omitted, \( J > 0 \)). In the Hamiltonian [10], the creation operator \( \hat{a}^\dagger_l \) and the annihilation operator \( \hat{a}_l \) ‘creates’ or ‘annihilates’ an atom in the \( l \)th well of the optical potential in the Wannier state \( \psi_l(x) \). Hence, the first term on the right hand side of Eq. (40) is responsible for the tunneling (hopping) of the atoms between the wells of the optical potential. Note that the hopping matrix elements \( J \) (defined by the overlap integral of the Wannier states in neighboring wells) exponentially depends on the depth of the optical potential and, hence, can be easily varied by several order of magnitude (0.30 \( \lesssim J/E_R \lesssim 0.0035 \) for \( 2 \leq V_0/E_R \leq 22 \)). The second term in the Hamiltonian [10] is the Stark energy of the atoms in the homogeneous field. The third term is the interaction energy of the atoms sharing one and the same well, where the interaction constant \( W \) is mainly defined by the \( s \)-wave scattering length \( a_{sc} \) of the atoms and by the geometry of the lattice,
\[
W = \frac{4\pi a_{sc} \hbar^2}{M} \int \psi^4_l(r) dr.
\] (41)
As an estimate, one can use \( W = 0.28 E_R \) – the experimental value for \( ^{87} \text{Rb} \) atoms (\( a_{sc} = 5.8 \text{ nm} \)) in 3D separable potential of the depth \( V_0 = 22 E_R \) [28]. Interpolating this result to \( V_0 = 2 E_R \) would give \( W = 0.027 E_R \).
A remark about the phase diagram of the Bose-Hubbard model (the Hamiltonian \( H \) without the static term) is appropriate here. As it is known, the Bose-Hubbard model shows the superfluid/Mott-insulator quantum phase transition when the ratio between the parameters \( J \) and \( W \) exceeds some critical value \( 22 \). (In the laboratory experiments one varies the ratio \( W/J \) by changing the depth of the optical potential – the critical value for \( ^{87}\text{Rb} \) atoms is \( V_0 \approx 15E_R \) and \( V_0 \approx 44E_R \) for the 3D and 1D lattices, respectively \( 30, 31 \).) Obviously, the dynamical response of the system to a static force also depends on whether the system is in the super-fluid or Mott-insulator regime. In the next section, we study the Bloch dynamics in the super-fluid regime, where the ground state of the Bose-Hubbard system (which we take as the initial state in our numerical simulations) can be well approximated by the product of Bloch waves with zero quasimomentum. The response of the system to a static force in the Mott-insulator regime will be briefly discussed in Sec. \( \text{V} \).

C. Decoherence due to the atom-atom interactions

Numerically, the problem of BO of interacting bosonic atoms consists of solving the Schrödinger equation for the multi-particle wave function \( \Psi(t) = \sum_n c_n(t)|n\rangle \), where \( |n\rangle = |\ldots, n_{i-1}, n_i, n_{i+1}, \ldots\rangle \) are the Fock (number) states, given by the symmetrized product of Wannier states \( \psi_l(x) \). Note that the translation symmetry of the system, obviously broken by the static term, can be actually recovered by using the gauge transformation,

\[
\hat{H} \rightarrow \tilde{H}(t) = \frac{J}{2} \left( \sum_t e^{-i\omega_B t} \hat{a}_{t+1} \hat{a}_t + h.c. \right) + \frac{W}{2} \sum_t \hat{n}_t(\hat{n}_t - 1). \tag{42}
\]

This allows us to impose periodic boundary conditions, which greatly facilitate the convergence in the thermodynamic limit \( N, L \to \infty, N/L = \bar{n} \). Here \( L \) is the lattice size and \( N \) the number of atoms. For given \( L \) and \( N \), the dimension of the Hilbert space (the total number of Fock states)
is $N = (N + L - 1)!/N!(L - 1)!$. Note that the dimension of the Hilbert space but not the size of the system controls the convergence in the thermodynamic limit.

Having a (numerical) solution of the Schrödinger equation, $i\hbar \partial |\Psi(t)\rangle/\partial t = \hat{H}(t)|\Psi(t)\rangle$, we then calculate the single-particle density matrix,

$$\rho(x, x'; t) = \sum_{l,m} \psi_l(x)\psi_m(x') \rho_{l,m}(t) , \quad \rho_{l,m}(t) = \langle \Psi(t)|\hat{a}_l^\dagger \hat{a}_m|\Psi(t)\rangle ,$$

which carries essential (although not complete) information about the system. In particular, the diagonal elements of the density matrix (43) in the momentum representation define the momentum distribution of the atoms $P(p)$ which, as discussed above in Sec. II D, is the quantity most easily measured in the laboratory experiments.

An example of the time evolution of the momentum distribution is depicted in Fig. 9. Comparing this figure with Fig. 2 for BO of non-interacting atoms, a rapid decay of BO is noticed. The reason for this decay is the decoherence of the density matrix, similar to that considered in Sec. V A, but with the fundamental difference that here the system itself plays the role of a ‘bath’. To get a qualitative understanding of this phenomenon, it is useful to consider the instantaneous spectrum of the time-dependent Hamiltonian (42) (see Fig. 10). The thin line in Fig. 10 is the mean-field solution (diabatic continuation of the ground state), where all atoms would oscillate coherently. However, due to Landau-Zener transitions at avoided crossings, the other Fock states become populated when the static force drives the system along the mean-field solution. This leads to a thermalization of the atoms and, as a consequence, to the decay of BO.

In principle, one can try to describe the thermalization process by thoroughly analyzing the Landau-Zener tunneling at the avoided crossings. An alternative (and, actually, more constructive) approach is to study the properties of the Floquet-Bloch operator [32], which we define as the evolution operator over one Bloch period:

$$\hat{U} = \exp \left[ -\frac{i}{\hbar} \int_0^{T_B} \hat{H}(t)dt \right] .$$
It has been found that for the parameter region of a typical experiment with cold atoms in the 3D lattices \((W \sim J, \bar{n} \sim 1, \text{and } dF < J)\) the matrix of the Floquet-Bloch operator \((44)\) can be well identified with a random matrix of the circular orthogonal ensemble and, thus, the system \((40)\) is a quantum chaotic system \([33, 34]\). In fact, this is precisely the Quantum Chaos, which justifies the use of the terms ‘bath’ and ‘thermalization’ for the system of interacting atoms.

D. Different regimes of BO of interacting atoms

As already mentioned in Sec. V B, BO of interacting atoms is a rather diverse problem and the irreversible decay of BO discussed above is in no way the only possible regime of Bloch dynamics. In particular, a static force with magnitude \(dF \gg J\) suppresses the Quantum Chaos and BO become quasiperiodic (see Fig. 11). This regime of BO can be treated analytically and, for example, for the mean atomic momentum we have \([35]\),

\[
\langle p(t) \rangle = p_0 \sin(\omega_B t) \exp\{-2\bar{n}[1 - \cos(\omega_W t)]\} , \quad \omega_W = W/\hbar .
\]

Note, in passing, that the same interaction frequency \(\omega_W = W/\hbar\) (but for a different problem) has recently been observed in a laboratory experiment \([28]\).

The other limiting case of BO is the case of a weak atom-atom interactions, which is typically realized in the quasi 1D lattices. Indeed, because of a weak transverse confinement \(r_0 \gg \lambda\), the integral \(\int \psi_j^*(r)dr\) in Eq. \((11)\) is orders of magnitude smaller than for 3D lattices. An increase of the filling factor from \(\bar{n} \sim 1\) in the 3D case to \(\bar{n} \sim 100\) in the 1D case (present days situation) may not be able to compensate this decrease of the interaction constant \(W\) and, therefore, a regular dynamics of BO can be expected. Let us also remind that a large filling factor and a small interaction constant are usually considered as a validity condition for the Gross-Pitaevskii equation. We shall discuss this issue in more detail in Sec. VII.

It is also interesting to study ‘Bloch oscillations’ in the Mott-insulator regime \((J \ll W)\). In this case, a response of the system to a static force has a resonant character \([30, 36]\), with the main
resonance corresponding to the condition \( dF \approx W \). Providing resonant (or near resonant) forcing, the particle-hole excitations of the Mott-insulator states are created dynamically. This excitation process is reflected in the (almost) periodic dynamics of the atomic momentum distribution \[37\]. It should be stressed, however, that in spite of the formal analogy with BO, the origin of these oscillations as well as the characteristic period \( T_J \sim \hbar/J \) are fundamentally different.

VI. CONCLUSION

We considered the phenomenon of BO for cold neutral atoms in optical lattices. In Sec. II and Sec. III of the review, which mainly serve as tutorials, we focused on a dilute gas of atoms in quasi 1D lattices. In addition to the theoretical analysis, we also discussed the scheme of a typical laboratory experiment on BO and indicated the characteristic values of the parameters.

It can be safely stated that in the above case of a dilute gas in 1D lattices the BO of cold atoms are well understood. The further progress in the field is related to the problems of BO in lattices of higher dimensionality, BO in the presence of relaxation processes, and BO of interacting atoms.

A dilute gas of cold atoms in 2D lattices was discussed in Sec. IV. Naively, one may expect the two-dimensional BO to be a superposition of one-dimensional BO, where the values of the static force \( F_{x,y} \) are given by the projection of the vector \( F \) to the crystallographic axis of the lattice. However, this is true only in the separable case. For a non-separable optical potential, the two-dimensional BO generally cannot be expressed in terms of the 1D problem. This is especially the case for a strong static force (strong Landau-Zener tunneling), where non-separability of the potential manifests itself in a number of effects.

The ultimate goal for studying BO in the presence of relaxation processes is to obtain a directed diffusive current of atoms, similar to that of electrons in a conductor. In Sec. IV A we studied the relaxation (decoherence) of BO due to spontaneous emission. Note, that the rate of spontaneous emission can be increased to any desired level by simply tuning the laser frequency closer to the atomic resonance. The effect of spontaneous emission is shown to lead only to ‘non-directed’ diffusion. The theoretical analysis of the other relaxation mechanisms, like scattering on ‘impurities’ is strongly in need. In this connection we would like to mention the recent experiment \[8\], which studies the dynamics of the fermionic \(^{40}\)K atoms with an admixture of bosonic \(^{87}\)Rb atoms in inhomogeneous optical lattices.

Finally, we comment on the problem of BO for a Bose-Einstein condensate. This field of research opens unlimited perspectives for studying the different phenomena of correlated systems. In particular, by loading a BEC of cold atoms into a quasi 1D lattice, one can realize the case of a large filling factor (mean number of atoms per lattice cite) and a small interaction constant. This is believed to be the realm of the Gross-Pitaevskii equation, where BO becomes a macroscopic quantum phenomenon. We intentionally did not discuss this case because even the mentioning of all publications on this subject would draw this review out of the length limit. Instead, we analyzed BO of interacting atoms in the 3D lattice. Loading BEC in the 3D lattice increases the interaction constant by several orders of magnitude and simultaneously decreases the filling factor to \( \bar{n} \sim 1 \). In this regime the Gross-Pitaevskii equation fails to describe the dynamics of the atoms and a fully microscopic treatment of the system is required. Such a microscopic approach is provided by the Bose-Hubbard model. It was shown that, depending on the value of the static force, BO of interacting atoms may vary from quasiperiodic to irreversibly decaying behavior. Moreover, in the latter case, the system appears to be chaotic in the sense of Quantum Chaos. This intriguing regime of BO is waiting for experimental studies.

Financial support by Deutsche Forschungsgemeinschaft (Grant No. 436 RUS 113/658/0-1) is acknowledged. We also express our gratitude to A. Buchleitner, who is our co-author in Sec. IV of
the present review.

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