From matter to galaxies: General relativistic bias for the one-loop bispectrum.

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Abstract

We write down the Lagrangian bias expansion in general relativity up to 4th order in terms of operators describing the curvature of an early-time hypersurface for comoving observers. They can be easily expanded in synchronous or comoving gauges. This is necessary for the computation of the one-loop halo bispectrum, where relativistic effects can be degenerate with a primordial non-Gaussian signal. Since the bispectrum couples scales, an accurate prediction of the squeezed limit behavior needs to be both non-linear and relativistic. We then evolve the Lagrangian bias operators in time in comoving gauge, obtaining non-local operators analogous to what is known in the Newtonian limit. Finally, we show how to renormalize the bias expansion at an arbitrary time and find that this is crucial in order to cancel unphysical $1/k^2$ divergences in the large-scale power spectrum and bispectrum that could be mistaken for a contamination to the non-Gaussian signal.
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1 Introduction

The study of the Large Scale Structure of the Universe (LSS) is thriving, with next generation experiments (Euclid, LSST, SKA, SPHEREx [1–4]) starting to collect data within the next decade. Of particular relevance for our work, they are expected to be sensitive to a non-Gaussian signal of $f_{NL} = \mathcal{O}(1)$, opening the possibility of constraining fundamental physics with the LSS. More precisely, single field inflation predicts a very specific shape for the three-point correlation function in the squeezed limit [5–7]. Any deviation from this behavior would be a smoking gun for other degrees of freedom active during inflation\(^1\) such as several scalar fields [9], higher spins [10], modified gravity [11], anisotropic inflation [12] and the presence of an electromagnetic field [13].

Very recently, we pointed out that the squeezed limit of the three-point correlation function may be contaminated by non-linear relativistic contributions [14]. On large scales, the universe is linear while Newtonian physics is a good approximation for the dynamics of small scales. But the bispectrum couples scales leading, in the squeezed limit, to a large non-linear relativistic signal [14]. This is particularly relevant since that limit is the most sensitive to the field spectrum during inflation. The main result of that work is the solution for the metric in the weak-field approximation. We now turn to observable quantities.

There exist other motivations to work in a relativistic framework than probing primordial physics with the squeezed limit of the three-point correlation function. Relativistic effects are in effect relevant for instance for neutrinos physics, backreaction and modified gravity, see [15] and references therein.

In this work, we take a further step toward computing the observed bispectrum at one loop by considering biased tracers (such as galaxies). Galaxy clustering is a complex non-linear problem that involves astrophysical processes that are not fully understood. A pragmatic approach is to use an effective expansion: the small scale galaxy density field is smoothed out, in order to focus on the larger scales where the (unknown) physics is parameterized by (unknown) bias coefficients $b_\mathcal{O}$ that multiply gravitational operators $\mathcal{O}$ [16].

Bias was historically developed in a Newtonian framework and was also generalized to GR [17–23]. Taking inspiration from [24] (see also [25]), in this paper we generalize their results to higher orders needed for the one-loop bispectrum.

We structure our paper as follows: in section 2, we review our relativistic results on dark matter. In section 3, we describe the core of our work: we write down a relativistic bias expansion to fourth order by using operators describing the curvature of the initial time hypersurface, and we evolve them in time using the continuity equation. We also enumerate our assumptions and approximations in that section. We then show in section 4 how to extend the renormalization of the bias operators to the relativistic case. Finally, we plot the contributions of some of the operators for the one-loop power spectrum and bispectrum, and conclude in section 5. For clarity, we relegate some of the more technical calculations to the appendices.

\(^1\)See [8] for a review
**Notation** We use Greek letters (e.g. $\mu, \nu$) for space-time indices that run from 0 to 3, and reserve Latin letters (e.g. $i,j$) for spatial indices that run from 1 to 3. Latin indices are written arbitrary up or down as they differ only by powers of $a(\eta)$, which are always written down explicitly. We will indifferently write quantities in position space or in Fourier space. Our convention is

$$f(x) = \int \frac{d^3k}{(2\pi)^3} e^{ix\cdot k} f(k) \equiv \int_k e^{ik\cdot x} f(k),$$

where we also introduced a short hand notation for integrals. An asterisk $*$ denotes a quantity evaluated at a very early time $\eta_* \to 0$.

2 Dark matter perturbations self-gravitating in an expanding universe

We now review our previous results for the dark matter density contrast. For more details on the physical setup, the reader can consult Ref. [14]. In order to simplify calculations, we will assume a matter-dominated Einstein de Sitter universe throughout. The inclusion of dark energy is straightforward though we would loose the simple scaling of all quantities with time that greatly helps with the bookkeeping. Our starting point is the perturbed FLRW metric

$$ds^2 = a(\eta)^2 \left\{ - (1 + 2\phi) d\eta^2 + 2\omega_i dx^i d\eta + [(1 - 2\psi)\delta_{ij} + \gamma_{ij}] dx^i dx^j \right\}, \quad (2.1)$$

where $a(\eta)$ is the background scale factor, $\eta$ is the conformal time, and $x^i$ are Cartesian comoving coordinates. The off-diagonal part of the metric is split into its transverse and longitudinal pieces $\omega_i = \partial_i \omega + w_i$, with $\partial_i w_i = 0$. The dark matter is taken to be a perfect irrotational fluid, with stress-energy

$$T_{\mu\nu} = \bar{\rho}(1 + \delta) u_\mu u_\nu, \quad (2.2)$$

where $\bar{\rho}(\eta)$ is the background density, $\delta(\eta, x)$ is the dark matter density contrast, and $u_\mu = \partial_\mu \varphi/\sqrt{X}$ is the matter 4-velocity, with $X = -\partial_\mu \varphi \partial^\mu \varphi$. Unless stated differently, we work in comoving gauge defined such that $\varphi = \eta$ and $\gamma_{ij}$ is transverse and traceless: $\partial^i \gamma_{ij} = \gamma^i_\ i = 0$. One can show that the lapse can be set to $N = 1$ to all orders [23], though the shift is different from zero $N^i \neq 0$.\footnote{The lapse and the shift are defined, as usual, by writing the ADM decomposition of the metric

$$ds^2 = a^2(\eta) \left[ -N^2 d\eta^2 + h_{ij}(dx^i + N^i d\eta)(dx^j + N^j d\eta) \right].$$}

The continuity and Euler equations that describe the evolution of the matter fluid follow from the conservation of this stress-energy tensor, and are

$$\nabla_\mu (\bar{\rho}(1 + \delta) u^\mu) = 0, \quad (2.3)$$

$$u^\mu \nabla_\mu u^\nu = 0. \quad (2.4)$$

Following the scheme described in section 3 of [14], we adopt a *weak-field approximation* which consists in taking metric fluctuations as small, but spatial derivatives large, which is a good approximation inside the Hubble radius. We then expand in the parameter $\epsilon \equiv H^2/\nabla^2 \ll 1,$
which characterizes the smallness of the relativistic corrections. We use the 4-velocity with an upper index that can be written
\[ u^\mu = (1, u^i) , \quad (2.5) \]
which we split into a longitudinal piece \( \theta \equiv \partial_t u^t \) and a transverse part \( \partial_i u^i = 0 \). In the weak-field approximation, equations (2.3) and (2.4) lead to the generalization of the continuity and Euler equation which we split into a dominant Newtonian part (written with a subscript N), and relativistic corrections (written with a subscript R) sourced by the Newtonian terms:
\[
\begin{align*}
\delta_R + \theta_R = & -\partial_i (\delta_N u_R^i + \delta_R u_R^i) + S_\delta[\delta_N, \theta_N], \quad (2.6) \\
\theta_R + 2H\theta_R + & \frac{3}{2} H^2 \delta_R = \partial_j (u_R^j \partial_j u_N^i + u_R^i \partial_j u_N^j) + S_\theta[\delta_N, \theta_N]. \quad (2.7)
\end{align*}
\]
An expression for the relativistic sources can be found in equations (C.4)-(C.5) of [14]. We perform perturbation theory of these equations in the usual sense, and define the Newtonian perturbation kernels \( F_n^N \) and \( G_n^N \) along with their relativistic counterparts \( F_n^R \) and \( G_n^R \):
\[
\delta(\eta, k) = \sum_{n=1}^{\infty} a^n \int_{k_1 \ldots k_n} (2\pi)^3 \delta_D(k - k_{1 \ldots n}) \left[ F_n^N(k_1, \ldots, k_n) + a^2 H^2 F_n^R(k_1, \ldots, k_n) \right] \delta_i(k_1) \ldots \delta_i(k_n), \quad (2.8)
\]
\[
\theta(\eta, k) = -H \sum_{n=1}^{\infty} a^n \int_{k_1 \ldots k_n} (2\pi)^3 \delta_D(k - k_{1 \ldots n}) \left[ G_n^N(k_1, \ldots, k_n) + a^2 H^2 G_n^R(k_1, \ldots, k_n) \right] \delta_i(k_1) \ldots \delta_i(k_n), \quad (2.9)
\]
\[
\begin{align*}
\mathbf{u}_T(\eta, k) = & \ H^3 a^2 \sum_{n=1}^{\infty} a^n \int_{k_1 \ldots k_n} (2\pi)^3 \delta_D(k - k_{1 \ldots n}) G_n^T(k_1, \ldots, k_n) \delta_i(k_1) \ldots \delta_i(k_n), \quad (2.10) \\
\psi(\eta, k) = & \ H^3 a^2 \sum_{n=1}^{\infty} a^n \int_{k_1 \ldots k_n} (2\pi)^3 \delta_D(k - k_{1 \ldots n}) F_n^\psi(k_1, \ldots, k_n) \delta_i(k_1) \ldots \delta_i(k_n), \quad (2.11)
\end{align*}
\]
where \( k_{1 \ldots n} \equiv \sum_{i=1}^{n} k_i \), and \( \delta_i \) is the linear matter density contrast evaluated at redshift zero.

Under the weak field approximation, it is possible to obtain expressions for the relativistic kernels, which we reproduce in appendix A.1.

## 3 Relativistic Bias Expansion

### 3.1 Geometric approach to bias expansion

Since halo formation is a local process, it should be described in the frame of reference of an observer moving with the halo’s center of mass [21]. The bias expansion should only depend on the quantities that such an observer would measure such as the local curvature (corresponding to second derivatives of the gravitational potential in Newtonian physics). This description can
be carried out by expanding around such an observer by working in Fermi coordinates [21, 26]. However, in practice this can be cumbersome since one would like to easily compute correlations between far away points.

A different approach was adopted recently in [25], who write down the bias expansion in synchronous gauge, defined by the lapse and the shift being $N = 1$ and $N^i = 0$. This gauge is Lagrangian in the sense that the 4-velocity of fluid elements is equal to $u^\mu = (1, 0)$, and is suitable for a Lagrangian bias expansion. They also pointed out (as remarked previously in [27]) that the condition of making the bias expansion depend only on locally measurable quantities can be satisfied by requiring that there exist a local coordinate system where short scale physics is independent of the value and first gradient of a long wavelength gravitational potential. In synchronous gauge, the coordinate transformation that guarantees this takes a particularly simple form, given by dilations and special conformal transformations.

However, in the synchronous gauge the metric does not stay close to an unperturbed FLRW. This is because the spatial coordinate position of a fluid element is constant, and its physical displacement is contained in the metric, which always happens in a Lagrangian approach to perturbation theory, even in the Newtonian limit. Since the metric is greatly distorted with the gravitational evolution, this makes a weak field approximation impossible in this gauge.

It would be interesting to define a "post-Lagrangian" expansion around Newtonian Lagrangian perturbation theory, but we take a different approach. In order to describe biasing we use the comoving gauge, for which the comoving time of observers is equal to the coordinate time in matter domination, given that one can choose $N = 1$ (see [14, 23]). Therefore, the constant-time hypersurfaces of the comoving gauge are the same as those in the synchronous gauge (they have the same slicing, the difference being in the threading). Furthermore, both gauges coincide at early times, that is $N^i = 0$ at $\eta_\ast \to 0$. We then follow a Lagrangian biasing prescription, and write down the bias expansion in terms of geometrical quantities describing those hypersurfaces of constant comoving time of observers at a very early time $\eta_\ast \to 0$. These quantities will be by definition gauge-independent once one chooses the hypersurface. Moreover, the coordinate transformation that eliminates the long-wavelength mode is also well known in this gauge [28].

A similar approach was used to write down the effective theory of inflation [29], and we follow their analysis with a few differences. Our first building blocks are the extrinsic curvature $K^\ell_\nu$ of the constant-time hypersurfaces, and the matter density contrast $\delta$. This second quantity is geometrical in the sense that it can be written as proportional to the Einstein tensor contracted with the 4-velocity $G^\mu\nu u_\mu u_\nu$ using the Einstein equation. Since the galaxy number density transforms as a scalar under spatial diffeomorphisms, each term in the bias expansion is guaranteed to transform appropriately under such transformations. We ignore stochastic terms, which we expect behave as in the Newtonian case.

We thus write the Lagrangian galaxy number over-density, up to third order, as

$$
\delta_g(\eta_\ast, \mathbf{x}) = \frac{b_1}{a_\ast} \delta + \frac{b_2}{a_\ast^2} \delta^2 + \frac{b_3}{a_\ast^2} S^i_j S^j_i + \frac{b_4}{a_\ast^3} \delta S^i_j S^j_i + \frac{b_5}{a_\ast^3} S^i_j S^k_j S^i_k,
$$

where $S^i_j \equiv (K^\ell_\ell \delta^i_j/3 - K^i_j)/H^2$ and all quantities are evaluated at $\eta_\ast \to 0$. The extension of this
expression to fourth order is straightforward, and is given in equation (3.5). Finally, since the coordinate transformation that eliminates the long mode involves only a spatial diffeomorphism, both sides of this expression transform in the same way, thus guaranteeing that the consistency relation is automatically satisfied, which we check in Appendix B.

There are many other operators that can in principle be included, we discuss them in order:

- One can form spatial derivatives by projecting covariant derivatives orthogonal to the constant-time hypersurface. Higher spatial derivatives of the operators considered will be relevant for large enough halos. We expect them to work as in the Newtonian case, where each spatial derivative is accompanied by the Lagrangian size of the halo $R \partial_i$, which is small for scales larger than $R$. It is straightforward to include this in our expansion, but we stick to the lowest derivative operators for simplicity.

- One can apply a time derivative of any operator by acting on it with $u^\mu \nabla_\mu$. In synchronous gauge, or in comoving gauge at $\eta_i$, this is simply a time derivative $u^\mu \nabla_\mu = \partial_\eta$. Since in matter domination each operator scales with powers of the growth factor, at each order one has that $u^\mu \nabla_\mu O^{(n)} = \partial_\eta O^{(n)}$ is simply proportional to $O^{(n)}$ at the initial time (see section 2.5.2 of [16] for further details).

- Other contractions of the Ricci tensor will be different from the extrinsic curvature and $u^\mu u^\nu G_{\mu\nu}$. In general they will be independent operators, but at the lowest order in perturbation theory each operator will be expanded in terms of second derivatives of the metric fluctuations. At a given order $n$ the operators we chose $O^{(n)}$ give all possible combinations of second derivatives of the curvature fluctuation $(\partial_i \partial_j \zeta)^n$. Additional operators will be different from ours at higher orders in perturbation theory, but the different combinations that can appear are quite constrained. For example, at order $n + 1$ one can have additional combinations of second derivatives $(\partial_i \partial_j \zeta)^{n+1}$, but they will be degenerate with the operators starting at that order $O^{(n+1)}$. Additionally, they can have relativistic corrections at a higher order, which to subleading order in the relativistic expansion are $\zeta(\partial_i \partial_j \zeta)^n$ or $\nabla \zeta, \nabla \zeta(\partial_i \partial_j \zeta)^n$, but they are fixed by the requirement of satisfying the consistency relation: A very specific combination of $(\partial_i \partial_j \zeta)^n$, $\zeta(\partial_i \partial_j \zeta)^n$ and $\nabla \zeta, \nabla \zeta(\partial_i \partial_j \zeta)^n$ is needed such that the unphysical correlation of the operator with a constant $\zeta$ or a constant gradient $\nabla \zeta$ disappears in the local frame (see Appendix B).\(^3\)

- Continuing with the previous point, one can in general have second derivatives of the tensor fluctuations $\partial_i \partial_j \gamma_{k\ell}$, but we take them to be sourced only by the scalar fluctuations such that the considerations of the previous point apply.

### 3.2 Explicit Lagrangian expansion and dynamical evolution

We follow a Lagrangian framework for the bias expansion as detailed in the previous section. That is, we write the galaxy number density contrast in terms of geometrical quantities describing an

\(^3\)The dynamical evolution will however generate non-local terms that cannot be written as simple expansions in derivatives, as is remarked in [16, 30] and computed explicitly in section 3.2.
early time hypersurface. These can be explicitly written in synchronous or comoving gauge to obtain the relativistic Lagrangian bias prescription. In order to obtain the later time Eulerian density contrast, we work in comoving gauge and evolve the galaxy number density as follows:

- We assume conservation of the fraction of Lagrangian volume that will eventually collapse into halos by the time of observation,

\[ \nabla_\mu (u^\mu u^\nu \rho_g) = 0, \tag{3.2} \]

where we have also assumed that the 4-velocity of galaxies equals that of the dark matter fluid, i.e. we ignore velocity bias. The solution to this equation in the Newtonian limit recovers the standard relation between Lagrangian and Eulerian bias coefficients (see section 2.3 of [16]).

Using the perturbed metric (2.1) and the background equations, we find

\[ \delta_g + \theta = -\partial_i (\delta_g u^i) + S_\delta [\delta_g, \theta_N], \tag{3.3} \]

where \( S_\delta \) is the same source as in equation (2.6) but evaluated for the biased tracer. We reproduce it here for completeness

\[ S_\delta [\delta_g, \theta_N] = 3(1 + \delta_g) \dot{\psi} + 3(1 + \delta_g) u^i \partial_i \psi, \tag{3.4} \]

- We set adiabatic initial conditions at a very early time \( \eta_* \rightarrow 0 \). Explicitly, up to fourth order we write

\[ \delta_g(\eta_*) = \sum_{n=1}^{4} \frac{b_n}{n! a_n^2} \delta^n + \sum_{n=2}^{4} \frac{b_n^*}{a_n^3} [S^n] + \frac{b_2^*}{a_2^3} [S^2] + \frac{b_3^*}{a_3^4} [S^3] + \frac{b_4^*}{a_4^4} [S^2]^2, \tag{3.5} \]

where all quantities are evaluated at the initial time \( \eta_* \). Here, \([S^n]\) stands for the trace of \(n\) operators, e.g., \([S^n] = \text{tr}[S \cdots S]\), and \(S^i_j\) is the traceless part of the extrinsic curvature, which reduces to the usual expression in the Newtonian limit \((S^i_j) = (k_i k_j/k^2 - \delta_{ij}/3) \delta_\hat{L}(k)\).

In the perturbative expression for the bias expansion, we keep only those terms which don’t vanish in the limit \( \eta \rightarrow 0 \).

For future reference, after some work the operator \(S^i_j\) can be explicitly written as

\[ S^i_j = \frac{1}{2} (\partial_i u^j + \partial_j u^i) - \frac{1}{3} \partial_k u^k \delta^i_j + O(\epsilon^2). \tag{3.6} \]

Following this prescription allows us to consistently write solutions order by order. The explicit solution for the relativistic contribution to the galaxy number density contrast will be written in terms of bias Kernels given in Appendices A.2 and A.3.

### 3.2.1 First order

The integration of (3.3) gives:

\[ \delta_g^{(1)} = a(\eta) \delta_\hat{L}(k) + C_1(k), \tag{3.7} \]

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Following this prescription allows us to consistently write solutions order by order. The explicit solution for the relativistic contribution to the galaxy number density contrast will be written in terms of bias Kernels given in Appendices A.2 and A.3.
At second order, the Lagrangian bias expansion (3.5) is given by

$$C_1 = b_i^* \delta_\ell(k),$$

(3.8)

to get

$$\delta_g^{(1)}(k) = a(\eta) \delta_\ell(k) \left(1 + \frac{b_1^*}{a(\eta)}\right).$$

(3.9)

### 3.2.2 Second order

At second order, the Lagrangian bias expansion (3.5) is given by

$$\delta_g^{(2)}(k, \eta) = \int k_1 k_2 (2\pi)^3 \delta_D(k - k_{12}) \left[b_i^* a^3 H^2 F_2^R(k_1, k_2) + \frac{1}{2} b_2^* + b_{s2}^* s^2(k_1, k_2)\right] \delta_\ell(k_1) \delta_\ell(k_2),$$

(3.10)

where we have taken the limit $a_\eta \to 0$ such that the Newtonian quadratic kernel vanishes. This sets the initial conditions for the evolution of $\delta_g$, and the first term reflects the fact that initial conditions for $\delta$ must be set at second order. We have also defined

$$s^2(k_1, k_2) = \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} - \frac{1}{3}.$$  

(3.11)

The integration of equation (3.3) gives

$$\delta_g^{(2)}(k, \eta) = \int k_1 k_2 (2\pi)^3 \delta_D(k - k_{12}) \left\{ \left[a^2 F_2^N(k_1, k_2) + a b_i^* a(k_1, k_2)\right] \delta_\ell(k_1) \delta_\ell(k_2) + C_2(k_1, k_2) \right\},$$

(3.12)

where the second term is usually rewritten as $(F_2^N - (2/7)s^2 - 2/31)\delta_\ell^2$. Fixing the integration constant then gives

$$\delta_g^{(2)}(k, \eta) = a^2 \int k_1 k_2 (2\pi)^3 \delta_D(k - k_{12}) \left[ \left(1 + \frac{b_1^*}{a}\right) F_2(k_1, k_2) + \frac{1}{2} \left(\frac{b_2^*}{a^2} - \frac{4}{21} \frac{b_1^*}{a}\right) + \left(\frac{b_{s2}^*}{a^2} - \frac{2}{7} \frac{b_1^*}{a}\right) s^2(k_1, k_2)\right] \delta_\ell(k_1) \delta_\ell(k_2),$$

(3.13)

where we denote $F_2 = F_2^N + a^2 H^2 F_2^R$ for brevity.

### 3.2.3 Third order

At third and higher orders, it won’t be possible to neatly regroup the different terms into the geometric operators at a given time. This may be a gauge issue, but it doesn’t spoil the result, merely complicating the bookkeeping.

At third order, the Lagrangian bias expansion is given by

$$\delta_g^{(3)}(k, \eta) = \int k_1 k_2 k_3 (2\pi)^3 \delta_D(k - k_{123}) \left[b_i^*\frac{a^5}{2} H^2 F_2^R(k_1, k_2) + \frac{b_{s3}^*}{a^2} a^5 H^2 M_3^2 R(k_1, k_2, k_3)\right] \delta_\ell(k_1) \delta_\ell(k_2) \delta_\ell(k_3),$$

(3.14)
where all the terms are constant in time, we define

\[
s^3(k_1, k_2, k_3) = \frac{k_1 k_2 k_3 k_2 k_3}{k_1 k_2 k_3} - \frac{(k_1 k_2)^2}{3k_1 k_2 k_3} + 2 \text{ perms.} + \frac{2}{9},
\]

and \(M^{2,R}_\epsilon\) is the cubic piece of the relativistic correction to the operator \(S^\epsilon_j S^\epsilon_i\), given in equation (A.34) and obtained from equation (3.6). Note that there is no term proportional to \(b^*_1\) since its cubic piece goes to zero as \(\eta \to 0\). Integration of equation (3.3) then gives

\[
\delta^{(3)}_g(k, \eta) = \int_{k_1, k_2, k_3} (2\pi)^3 \delta_D(k - k_1) \left\{ a^3 F_3(k_1, k_2, k_3) + \frac{1}{2} a^2 b_1^* \alpha(k_1 + k_2, k_3) G_2^N(k_1, k_2) + \alpha(k_3, k_1 + k_2) \left[ \frac{1}{2} a^2 b_1^* F_2^N(k_1, k_2) + \frac{1}{2} \left( a b_2^* - \frac{2}{21} a^2 b_1^* \right) s^2(k_1, k_2) \right] + a^4 H^2 b_1^* \left[ \alpha(k_3, k_1 + k_2) F_2^R(k_1, k_2) + \alpha(k_1 + k_2, k_3) G_2^R(k_1, k_2) + 3 F_2^\psi(k_1, k_2) \right] - i k_3 \cdot G_2^T(k_1, k_2) + \frac{15}{2} \frac{k_1 k_2}{k_1 k_2} \right\} \delta_\ell(k_1) \delta_\ell(k_2) \delta_\ell(k_3) + \delta^{(3)}_g(k_1, k_2, k_3, \eta^*)\right). \tag{3.15}
\]

We don’t attempt to rewrite this expression in terms of simple quantities since not much insight is gained by doing so. Even in the Newtonian limit, non-local terms appear in the evolved galaxy density contrast (see [16, 30]), and we expect the same to hold for the relativistic terms. This means that it won’t be possible to neatly write them in terms of geometrical quantities of the hypersurface of constant time \(\eta\).

### 3.2.4 Fourth order

At fourth order, the calculation proceeds in an analogous fashion, and we obtain schematically

\[
\delta^{(4)}_g = a^4 \left( F_4(k_1, k_2, k_3, k_4) + \sum b_\epsilon^2 M_4^O(k_1, k_2, k_3, k_4) \right) \delta_\ell(k_1) \delta_\ell(k_2) \delta_\ell(k_3) \delta_\ell(k_4), \tag{3.16}
\]

where we used the notation \(b_\epsilon^2 = b_\epsilon^n/a^n\) with \(n\) the order at which a given operator starts, e.g., \(b_2^2 = b_2^2/a^2\) and \(b_{s_2}^2 = b_{s_2}^2/a^3\). The kernels are given by \(M_4^O = M_4^{O,N} + a^2 H^2 M_4^{O,R}\), and \(M_4^{O,N}\) and \(M_4^{O,R}\) can be found in Appendixes A.2 and A.3.

### 4 Renormalization of bias parameters

In order for the bias expansion to be well defined, the operators should be appropriately renormalized such that the expectation value of \(\delta_\eta\) vanishes, and its correlation with long-wavelength perturbations behaves as expected. We do this at the initial time \(\eta_*\) take these as renormalized initial conditions for the dynamical evolution. Since this evolution is non-linear, one then needs to renormalize the result at each time \(\eta\).
To be more explicit, let’s study how this works for the operator proportional to $b^2$. At initial time, the bare operator is given by

$$
\frac{1}{2a_+^2} b^2 \delta^2(k, \eta) = \frac{1}{2} b^2 \int_{q_1, q_2} \delta_D(k - q_1 - q_2) \delta_\ell(q_1) \delta_\ell(q_2)
$$

\[ + b^2 a^3 H^2 \int_{q_1, q_2, q_3} \delta_D(k - q_1 - q_2 - q_3) F^R_2(q_1, q_2) \delta_\ell(q_1) \delta_\ell(q_2) \delta_\ell(q_3). \tag{4.1} \]

Following [31], we compute the expectation value of this operator, which is

$$
\frac{1}{2a_+^2} b^2 \langle \delta^2 \rangle = \frac{1}{2} b^2 \int_q P(q) = \frac{1}{2} b^2 \sigma^2(\Lambda). \tag{4.2} \]

where $\Lambda$ is the UV cutoff of the integral, or equivalently the scale over which the density contrast is smoothed. This needs to be subtracted from the bare operator in order to guarantee that the full operator have zero expectation value. Next, we correlate with a long-wavelength perturbation, which gives

$$
\frac{1}{2a_+^2} b^2 \lim_{k \to 0} \langle \delta_\ell(k) \delta^2(-k) \rangle = b^2 a^3 H^2 \lim_{k \to 0} P(k) \int_q \left( F^R_2(k, q) + F^R_2(k, -q) \right) P(q)
$$

\[ = \left( -\frac{5}{k^2} \sigma^2(\Lambda) - 5\sigma^2_{-2}(\Lambda) \right) b^2 a^3 H^2 P(k). \tag{4.3} \]

where $\sigma^2_{-2} = \int_q P(q)/q^2$, and the prime denotes the fact that we factor out the Dirac delta of momentum conservation. In order to cancel the unphysical cutoff dependence coming from this result, we need to write

$$
\left[ \frac{1}{2a_+^2} b^2 \delta^2 \right]_R = \frac{1}{2a_+^2} b^2 \delta^2 - \frac{1}{2} b^2 \sigma^2 + 5b^2 a^3 H^2 \sigma^2_{-2} \delta_\ell - b^2 \sigma^2 \zeta. \tag{4.4} \]

where $\zeta$ is the curvature perturbation in comoving gauge. The last term of this expression would seem to suggest that we need to include an unphysical “non-Gaussian scale-dependent bias” term in our expansion. This is not the case as remarked in [32], as it merely represents the fact that we are working in a set of coordinates that don’t represent what a local observer measures. Specifically, the cutoff scale has been fixed to be some coordinate value $\Lambda$, while it should correspond, physically, to a scale defined by a local observer $\Lambda_{\text{ph}}$. It was shown by [32] that $\sigma^2(\Lambda_{\text{ph}}) = (1 + 4\zeta) \sigma^2(\Lambda)$, such that the problematic last term combines with the second term to form $-\frac{1}{2} b^2 \sigma^2(\Lambda_{\text{ph}})$. Finally, the third term proportional to $\sigma^2_{-2}$ represents a relativistic correction to the standard renormalization.

We now want to take this renormalized operator as initial conditions, and evolve it using equation (3.3). For this, let us write the evolution equation for a given operator, which is obtained by defining $\delta_g = \delta + O$ in equation (3.3), such that

$$
\dot{O}(k) = \int_{q_1, q_2} (2\pi)^3 \delta_D(k - q_{12}) \left[ -\alpha(q_1, q_2) \theta(q_1) O(q_2) + 3O(q_1) \dot{\psi}(q_2) - iq_2 \cdot u_T(q_1) O(q_2) \right]
$$

\[ + 3 \int_{q_1, q_2, q_3} (2\pi)^3 \delta_D(k - q_{123}) \frac{q_1 \cdot q_3}{q_1^2} \theta(q_1) O(q_2) \psi(q_3). \tag{4.5} \]
Since this equation is linear in $\mathcal{O}$, the evolution of the initial renormalized operator is given by the superposition of the solution for the bare operator presented in the previous section, and the solution for which the initial conditions are only given by the counter-terms in equation (4.4). Since this evolution is non-linear, it will induce further cutoff dependences that need to be subtracted at each order. In particular,

$$
\lim_{k_1 \to 0} \lim_{k_2 \to 0} \left\langle \delta_\ell(k_1) \delta_\ell(k_2) \left[ \frac{1}{2a^2} b^2_s \delta^2 \right](k, \eta) \right\rangle' = -\frac{115}{6} a^4 H^2 b_2^s \sigma^2 P(k_1) P(k_2)
$$

(4.6)

where $[b^2_s \delta^2 / 2a^2]$ is the operator evolved from expression (4.4) using equation (4.5). We add an additional counter-term to the evolved operator in order to cancel this cutoff dependence. In this way, we obtain for this second piece

$$
M^{s^2.c.t.}_0(k) = -\frac{1}{2} b^s_0 a^2 \sigma^2 (2\pi)^3 \delta_D(k),
$$

(4.7)

$$
M^{s^2.c.t.}_1(k) = -\frac{1}{2} b^s_0 a^2 \sigma^2 + 5 b^s_0 a^5 H^2 \sigma^2 + \frac{5}{2} b^s_0 a^5 H^2 \sigma^2 \frac{1}{k^2},
$$

(4.8)

$$
M^{s^2.c.t.}_2(k_1, k_2) = -\frac{1}{4} b^s_2 a^4 \sigma^2 \left( G^N_2(k_1, k_2) + \alpha(k_1, k_2) \right) + \frac{b^s_2}{a^2} a^6 H^2 \sigma^2 \left( 5 \alpha(k_1, k_2) + \frac{115}{12} \right)
$$

$$
-\frac{5}{2} b^s_2 a^6 H^2 \sigma^2 \left( G^R_2(k_1, k_2) + 3 F^N_2(k_1, k_2) + \frac{15}{2} k_1 k_2 \right) - \frac{10}{k^2} \alpha(k_1, k_2),
$$

(4.9)

One can check that, after a reshuffling of terms, this procedure recovers the standard renormalized bias expansion in the Newtonian limit. The renormalization of the other operators used proceeds in a similar fashion, and is performed in Appendix C.

5 Results and conclusions

In order to have a better understanding of the interplay between the bias operators, dark matter, and their corresponding counter-terms, we now explicitly compute the power spectrum and bispectrum generated by the second-order operators proportional to $b^s_2$ and $b^s_2$. For each plot, we set all bias parameters to zero except for the one being studied.

The one-loop power spectrum is plotted in figure 1, where the tree-level, loop contributions and counter-terms are plotted separately. We find that, as expected, the integrals involved in the loop are sensitive to the UV cutoff chosen, but this dependence is cancelled by the counter-terms. Interestingly, the relativistic terms in the loop induce a $1/k^2$ contribution that could be interpreted as a relativistic signal degenerate with local-type non-Gaussianity. However, this contribution is unphysical and is exactly cancelled by the counter-terms. This works as remarked in [32], who pointed out that such a $1/k^2$ behavior is absent when a physical cutoff must be chosen for the integrals instead of a coordinate cutoff. As pointed out in section 4, this naturally appears in our counter-terms.

In figure 2, we plot the behavior of the bispectrum as it approaches the squeezed limit. Since the bispectrum is generated by both the dark matter non-linearities and the bias expansion, we
Figure 1. The one-loop power spectrum, computed from the operators proportional to $b_2^*$ and $b_s^2$. For each plot, we set all bias parameters to zero except for the one being studied. Each contribution is plotted separately. Notice that the unphysical $1/k^2$ behavior of the loop is cancelled by the counter-terms. All quantities are evaluated at redshift $z = 0$.

Figure 2. The one-loop bispectrum in the squeezed limit, computed from the operators proportional to $b_2^*$ and $b_s^2$. For each plot, we set all bias parameters to zero except for the one being studied. Each contribution is plotted separately. For these plots $k_1 = 0.1\,\text{Mpc}^{-1}h$ is held fixed and $k$ is varied. The squeezed limit is approached towards the left as $k$ becomes smaller. All quantities are evaluated at redshift $z = 0$.

We find that there is a $1/k^2$ behavior as it approaches that limit, similar to what happens for the dark matter case. As remarked in [14, 33], this is expected from the consistency relation since we use a quantity which is not what a local observer would measure. This $1/k^2$ behavior must be present such that it can be cancelled by the coordinate transformation that goes to the local frame, and in that sense it is a geometric projection effect. We also show the full bispectrum in figure 3.

We wrote a generalization of the Lagrangian bias prescription in the relativistic case (3.5), written in terms of operators describing the curvature of the initial time hypersurface. We explicitly wrote these operators in comoving gauge (which coincides with the synchronous gauge...
Figure 3. Logarithm of the ratio between the relativistic one-loop bispectrum to its Newtonian counterpart, computed from the operators proportional to $b_2^*$ and $b_2^s$. For each plot, we set all bias parameters to zero except for the one being studied. Here, one of the momenta was fixed to $k_1 = 0.1 \, h \text{Mpc}^{-1}$. All quantities are evaluated at redshift $z = 0$.

at the initial time hypersurface) and evolved them in time to obtain an Eulerian result, see section 3. It would be interesting to write this expansion in Poisson gauge for a couple of reasons: First it is a popular gauge chosen for instance to implement the N-body code gevolution [34]. Second, the UV behavior of the non-linear loop integrals is better modeled in the Poisson gauge with EFT techniques. Indeed, in synchronous and comoving gauges the inverse of the smoothed large scale metric is not the same as the smoothed inverse metric, see Appendix B of [35]. However, the Poisson gauge does not coincide with the synchronous gauge at the initial time hypersurface, making this task an arduous one to fourth order since it requires a gauge transformation. It would also be interesting to write down the bias expansion at formation time instead of the far past. Our bias expansion can be straightforwardly used in that case, but writing the geometric operators explicitly in comoving gauge is again difficult due to the fact that this gauge does not coincide with the synchronous one at an arbitrary time. Finally, we are computing the photon geodesics in the weak-field approximation in order to write down the observed galaxy power spectrum and bispectrum to one loop.

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A. Explicit expressions of the large scale structure kernels

A.1 For dark matter

The relativistic kernels for dark matter were computed in [14], we list them here since we use them to write the galaxy bias kernels.

\[ F_1^R(k) = 0, \] 
\[ F_2^R(k_1, k_2) = \frac{5}{2} \left( \frac{k_1^2 + k_2^2}{k_1^2 k_2^2} \right) + \frac{5}{4} \frac{k_1 \cdot k_2}{k_1^2 k_2^2}, \]
\[ F_2^\psi(k_1, k_2) = \frac{1}{4k_2^2} \left[ 1 - 6F_2^N(k_1, k_2) - 4G_2^N(k_1, k_2) - \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} \right], \]
\[ G_2^R(k_1, k_2) = F_2^R(k_1, k_2) - 3F_2^\psi(k_1, k_2) - \frac{15}{2} \frac{k_1 \cdot k_2}{k_1^2 k_2^2}, \]
\[ G_2^T(k_1, k_2) = i \left[ 4 \frac{k_{12}}{k_{12}^2} F_2^\psi(k_1, k_2) + \frac{5}{k_2} \left( -\frac{k_1}{k_1^2} + \frac{k_{12} k_1}{k_{12}^2} + \frac{k_2}{k_1^2} + \frac{k_1 \cdot k_2}{k_1^2 k_{12}^2} \right) \right], \]
\[ F_3^R(k_1, k_2, k_3) = \frac{5}{7} \alpha(k_1, k_23) F_2^R(k_2, k_3) + G_2^R(k_2, k_3) \left( \frac{5}{7} \alpha(k_23, k_1) + \frac{4}{7} \beta(k_1, k_23) \right) \]
\[ + F_2^\psi(k_2, k_3) \left( \frac{19}{7} + \frac{6}{7} \frac{k_1 \cdot k_{23}}{k_1^2} \right) + \frac{95}{14} \frac{(k_1 \cdot k_3)}{k_1^2 k_3^2} \]
\[ - \frac{i}{7} \frac{1}{2} G_2^T(k_1, k_3) \cdot k_2 \left( 7 + 4 \frac{k_{13} \cdot k_2}{k_2^2} \right) + \frac{15}{7} \frac{(k_1 \cdot k_2)(k_2 \cdot k_3)}{k_1^2 k_2^2 k_3^2}, \]
\[ F_3^\psi(k_1, k_2, k_3) = \frac{1}{2k_{123}^2} \left[ G_2^N(k_1, k_3) \left( 1 - \frac{(k_{13} \cdot k_2)^2}{k_{13}^2 k_2^2} \right) - 3F_3^N(k_1, k_23) - 2G_3^N(k_1, k_2 k_3) \right], \]
\[ G_3^R(k_1, k_2, k_3) = 2F_3^R(k_1, k_2, k_3) - 6F_3^\psi(k_1, k_2, k_3) - \alpha(k_1, k_23) F_2^R(k_2, k_3) \]
\[ - \alpha(k_{13}, k_2) G_2^R(k_1, k_3) - \frac{15}{2} \frac{k_1 \cdot k_3}{k_1^2 k_3^2} - \frac{15}{2} \frac{G_2^N(k_1, k_3) k_{13} \cdot k_2}{k_{13}^2 k_2^2} \]
\[ - 3F_2^\psi(k_2, k_3) \left( 1 - \frac{k_1 \cdot k_{23}}{k_1^2} \right) + i k_2 \cdot G_2^T, \]
\[ G_3^T(k_1, k_2, k_3) = i \left[ \frac{8k_{123}}{k_{123}^2} F_3^\psi(k_1, k_2, k_3) \right. \]
\[ + 5 \frac{1}{2} G_2^N(k_1, k_3) \left( -\frac{k_{13}}{k_{13}^2} + \frac{k_{123} k_1}{k_{123}^2} + \frac{k_{13} \cdot k_2}{k_{123}^2} + \frac{k_{13} \cdot k_3}{k_{123}^2} \right) \]
\[ - 2F_2^\psi(k_2, k_3) \left( -\frac{k_1}{k_1^2} + \frac{k_{123} k_{123} \cdot k_1}{k_{123}^2} + \frac{k_2}{k_1^2} + \frac{k_1 \cdot k_23}{k_{123}^2} + \frac{k_1 \cdot k_{23} k_{23}}{k_{123}^2} \right) \right], \]
\[ F_4^R(k_1, k_2, k_3, k_4) = \frac{7}{18} \alpha(k_1, k_{234}) F_3^R(k_2, k_3, k_4) + \frac{7}{18} \alpha(k_{12}, k_{34}) G_2^N(k_1, k_2) F_2^R(k_3, k_4) \]
\[ + \frac{7}{18} \alpha(k_{14}, k_2) G_3^R(k_1, k_3, k_4) + \frac{7}{18} \alpha(k_{12}, k_{34}) G_2^R(k_1, k_2) F_2^N(k_3, k_4) \]
\[ + \frac{2}{9} \beta(k_{14}, k_2) G_3^R(k_1, k_3, k_4) + \frac{2}{9} \beta(k_{12}, k_{34}) G_2^N(k_1, k_2) G_2^R(k_3, k_4) \]
\[ M_2^{\delta,N} (k_1, k_2) = \frac{F_2^N (k_1, k_2) + 4}{21} - \frac{2}{7} s^2 (k_1, k_2), \quad (A.11) \]
\[ M_2^{\delta^2,N} (k_1, k_2) = \frac{1}{2}, \quad (A.12) \]
\[ M_2^{s^2,N} (k_1, k_2) = s^2 (k_1, k_2), \quad (A.13) \]
\[ M_3^{\delta,N} (k_1, k_2, k_3) = \frac{1}{2} \left[ G_2^N (k_1, k_2) \alpha (k_1, k_3) + F_2^N (k_1, k_2) \alpha (k_3, k_1) + \frac{4}{21} - \frac{2}{7} s^2 (k_1, k_2) \right] \alpha (k_3, k_12), \quad (A.14) \]
\[ M_3^{\delta^2,N} (k_1, k_2, k_3) = \frac{1}{2} \alpha (k_1, k_23), \quad (A.15) \]
\[ M_3^{s^2,N} (k_1, k_2, k_3) = s^2 (k_2, k_3) \alpha (k_1, k_23), \quad (A.16) \]
\[ M_3^{s^2,N} (k_1, k_2, k_3) = \frac{1}{6}, \quad (A.17) \]
\[ M_3^{s^2,N} (k_1, k_2, k_3) = s^2 (k_1, k_23), \quad (A.18) \]
\[ M_3^{\delta s^2,N} (k_1, k_2, k_3) = \frac{1}{3} \left[ s^2 (k_1, k_2) + s^2 (k_1, k_3) + s^2 (k_2, k_3) \right], \quad (A.19) \]
\[ M_4^{\delta,N} (k_1, k_2, k_3, k_4) = \frac{1}{3} M_3^{\delta,N} (k_2, k_3, k_4) \alpha (k_1, k_234) + \frac{1}{3} M_2^{\delta,N} (k_2, k_3) G_2^N (k_1, k_3) \alpha (k_1, k_24), \quad (A.20) \]
\[ M_4^{\delta^2,N} (k_1, k_2, k_3, k_4) = \frac{1}{2} M_3^{\delta^2,N} (k_2, k_3, k_4) \alpha (k_1, k_234) + \frac{1}{3} M_2^{\delta^2,N} (k_2, k_3) G_2^N (k_1, k_3) \alpha (k_1, k_24), \quad (A.21) \]
\[ M_4^{s^2,N} (k_1, k_2, k_3, k_4) = \frac{1}{2} M_3^{s^2,N} (k_2, k_3, k_4) \alpha (k_1, k_234) + \frac{1}{3} M_2^{s^2,N} (k_2, k_3) G_2^N (k_1, k_3) \alpha (k_1, k_24), \quad (A.22) \]
\[ M_4^{\delta^3,N} (k_1, k_2, k_3, k_4) = M_3^{\delta^3,N} (k_2, k_3, k_4) \alpha (k_1, k_234), \quad (A.23) \]
\[ M_4^{s^3,N} (k_1, k_2, k_3, k_4) = M_3^{s^3,N} (k_2, k_3, k_4) \alpha (k_1, k_234), \quad (A.24) \]
\[ M_4^{\delta s^2,N} (k_1, k_2, k_3, k_4) = M_3^{\delta s^2,N} (k_2, k_3, k_4) \alpha (k_1, k_234), \quad (A.25) \]
\[ M_4^\delta, N(k_1, k_2, k_3, k_4) = \frac{1}{24} \cdot (A.26) \]
\[ M_4^{\delta^2, N}(k_1, k_2, k_3, k_4) = s^2(k_1, k_2) \cdot (A.27) \]
\[ M_4^{\delta s, N}(k_1, k_2, k_3, k_4) = s^3(k_1, k_2, k_3) \cdot (A.28) \]
\[ M_4^{(s^2)^2,N}(k_1, k_2, k_3, k_4) = s^2(k_1, k_2)s^2(k_3, k_4) \cdot (A.29) \]
\[ M_4^{s^3,N}(k_1, k_2, k_3, k_4) = \frac{k_1k_2k_3k_4}{k_1^2k_2^2k_3k_4} - \frac{4}{3} \frac{k_2k_4}{k_2^2k_3k_4} + \frac{2}{3} \frac{(k_3k_4)}{k_3^2k_4} - \frac{1}{9} \cdot (A.30) \]

### A.3 Relativistic galaxy bias kernels

The non-symmetrized relativistic galaxy bias kernels are

\[ M_2^{\delta, R}(k_1, k_2) = F_2^R(k_1, k_2) \cdot (A.31) \]
\[ M_2^{\delta^2, R}(k_1, k_2, k_3) = \frac{15}{2} \frac{k_1 \cdot k_3}{k_1^2k_3^2} - ik_2 \cdot G_2^T(k_1, k_3) + 3F_2^\psi(k_2, k_3) + \frac{F_2^R(k_2, k_3)\alpha(k_1, k_23) + G_2^R(k_2, k_3)\alpha(k_23, k_1)}{2} \cdot (A.32) \]
\[ M_3^{\delta^2, R}(k_1, k_2, k_3, k_4) = \frac{1}{6} \left[ F_2^R(k_1, k_2) + F_2^R(k_1, k_3) + F_2^R(k_2, k_3) \right] \cdot (A.33) \]
\[ M_3^{s^2, R}(k_1, k_2, k_3, k_4) = 2 \left[ \frac{G_2^R(k_2, k_3)s^2(k_1, k_23) - ik_2 \cdot G_2^T(k_1, k_3)}{k_3} \right] \cdot (A.34) \]
\[ M_4^{\delta, R}(k_1, k_2, k_3, k_4) = \frac{1}{2} \left[ -i \frac{4}{21} k_24 \cdot G_4^T(k_1, k_3) + k_2 \cdot G_4^N(k_1, k_3) + \frac{10}{7} \frac{k_1 \cdot k_3}{k_1^2k_3^2} - \frac{k_24 \cdot G_4^T(k_1, k_3)F_2^N(k_2, k_4)}{k_1^2k_3^2} + \frac{4}{7} F_2^\psi(k_2, k_4) + 3F_2^N(k_1, k_3)F_2^\psi(k_2, k_4) - 3F_2^\psi(k_3, k_4) \frac{k_1 \cdot k_34}{k_1^2k_3^2} \right] \cdot (A.35) \]

\[ M_4^{\delta^2, R}(k_1, k_2, k_3, k_4) = -ik_24 \cdot G_4^T(k_1, k_3) + 15 \frac{k_1 \cdot k_3}{k_1^2k_3^2} + \frac{3}{2} F_2^\psi(k_2, k_4) + 2 \frac{G_2^N(k_1, k_4)G_2^N(k_2, k_3)}{k_1^2k_3^2k_4^2} - \frac{6}{7} F_2^\psi(k_2, k_4)s^2(k_1, k_3) + \frac{2}{7} F_2^\psi(k_2, k_4)G_2^N(k_1, k_3) + \frac{2}{7} F_2^\psi(k_2, k_4)s^2(k_1, k_3) \cdot (A.36) \]

\[ M_4^{s^2, R}(k_1, k_2, k_3, k_4) = 3F_2^\psi(k_2, k_4)s^2(k_1, k_3) - ik_24 \cdot G_4^T(k_1, k_3) \cdot (A.37) \]
$\frac{15}{2} s^2(k_2, k_4) \frac{\mathbf{k}_1 \cdot \mathbf{k}_3}{k_1^2 k_3^2} + G_2^R(k_2, k_4) s^2(k_1, k_3)^2 \alpha(k_{24}, k_{13}) + M_3 s^2 R(k_2, k_3, k_4) \alpha(k_1, k_{234})$, (A.37)

$M_4 s^3 R(k_1, k_2, k_3, k_4) = \frac{1}{24} \left[ F_2^R(k_1, k_2) + F_2^R(k_1, k_3) + F_2^R(k_2, k_3) + F_2^R(k_1, k_4) + F_2^R(k_2, k_4) + F_2^R(k_3, k_4) \right]$, (A.38)

$M_4 s^3 R(k_1, k_2, k_3, k_4) = 3 s^3 R(k_2, k_3, k_{14}) G_2^R(k_1, k_4) - i 6 \left( \frac{-2 k_2 \cdot k_{14}}{3 k_4} - \frac{k_3 \cdot k_{14}}{3 k_4} + \frac{k_2 \cdot k_3 k_{14} \cdot k_2}{k_2^2 k_3^2} \right) k_2 \cdot G_2^R(k_1, k_4)$, (A.39)

$M_4 s^3 R(k_1, k_2, k_3, k_4) = i k_4 \cdot G_2^R(k_2, k_3) \frac{2 k_4 \cdot k_{23}}{k_1 \cdot k_4} + F_2 R(k_1, k_4) s^2(k_2, k_3) + 2 G_2^R(k_2, k_3) s^2(k_{23}, k_4)$, (A.40)

### B Consistency relation for bias evolution

We check our calculation of the bias evolution kernels by verifying that they satisfy the consistency relations. Consistency relations are based on the fact that the effect of a long mode on the short modes is equivalent to a linear coordinate transformation [36]. Under a coordinate transformation $N$ point correlation functions are related to $N - 1$ correlation function, which is the consistency check that all the kernels must satisfy. We correlate de galaxy density with matter density perturbations.

The coordinate transformation in comoving gauge is given by

$$\tilde{\eta} = \eta,$$  

$$\tilde{x}^i = (1 + \zeta) x^i + \frac{1}{2} \left( 2 x^i x^j \partial_j \zeta \eta - x^2 \partial_i \zeta \right) - \frac{1}{5} \eta^2 \partial_i \zeta,$$  

where $\zeta$ is the curvature perturbation, set at the initial conditions (e. g. by inflation).

Since time does not transform, the galaxy bias solution satisfies the same relation as the dark matter density solution. Under the coordinate transformation, the N-point function in Fourier space is

$$\langle \delta(\eta, \mathbf{q}) \delta_g(\eta_1, \mathbf{k}_1) \cdots \delta(\eta_n, \mathbf{k}_n) \rangle'_{\eta \to 0} = - \frac{5 H_0^2}{2 q^2} a(\eta) P(q) \left[ 3(n - 1) + \sum_a k_a \cdot \partial_{k_a} \right] + \frac{1}{2} q^i D_i - \frac{1}{5} \sum_a q \cdot k_a \eta_a^2 \langle \delta_g(\eta_1, \mathbf{k}_1) \cdots \delta(\eta_n, \mathbf{k}_n) \rangle',$$  

where we used the relation $\zeta(k) = - \frac{5}{2 \pi^2} \delta_l(k)$, and

$$q^i D_i \equiv \sum_{a=1}^n \left[ 6 q \cdot \partial_{k_a} - q \cdot k_a \partial_{k_a}^2 + 2 k_a \cdot \partial_{k_a} (q \cdot \partial_{k_a}) \right].$$
For simplicity we compute the consistency relation at equal times. In this case the last term in equation B.3 is zero at all orders. Since the relativistic solution for the bias evolution starts to contribute to second order, we begin by checking the consistency relation for $n = 2$. Thus, the l.h.s of equation B.3 at equal times for $n = 2$ is

\[ \langle \delta(q)\delta(k_1)\delta(k_2) \rangle'_{q \to 0} = \sum_O 2H_0^3b_C^O \left[ M_2^{O,R}(-q,k_1+q)P(|k_1+q|) + M_2^{O,R}(-q,-k_1)P(k_1) \right] P(q), \]

\[ = \sum_O 2H_0^3b_C^O \left[ M_2^{O,R}(-q,k_1) \left( 1 + \frac{q \cdot k_1}{k_1^2} \right) + M_2^{O,R}(-q,-k_1) \right] P(q)P(k_1) \tag{B.5} \]

and for r.h.s it gives

\[ \langle \delta(q)\delta_g(k_1)\delta(k_2) \rangle'_{q \to 0} = -\frac{5}{2q^2} \left[ 3 + \sum_a k_a \cdot \partial_{k_a} + \frac{1}{2}q^i D_i \right] b_C^O M_1^{hI}(k_1)P(k_1). \tag{B.6} \]

For $n = 3$ the l.h.s is

\[ \langle \delta(q)\delta_g(k_1)\delta(k_2)\delta(k_3) \rangle'_{q \to 0} = \langle \delta^{(1)}(q)\delta^{(3)}(k_1)\delta^{(1)}(k_2)\delta^{(1)}(k_3) \rangle + 2 \text{ perm} \]

\[ \langle \delta^{(1)}(q)\delta^{(2)}(k_1)\delta^{(2)}(k_2)\delta^{(1)}(k_3) \rangle + 2 \text{ perm} \tag{B.7} \]

As we did in [14], it is sufficient to compare only a combination of the momenta. We will compare the terms on both sides which are proportional to $P(k_2)P(k_3)$. The relation that we use is given by

\[ \langle \delta(q)\delta_g(k_1)\delta(k_2)\delta_g(k_3) \rangle'_{q \to 0} \supset P(k_2)P(k_3) \sum_i b_C^O \left[ 6M_3^{O,R}(-q,-k_2,-k_3) \right. \]

\[ + 4F_2^N(-q,k_2)M_3^{O,N}(-q,-k_2,-k_3) \left( 1 + \frac{Q \cdot k_2}{k_2^2} \right) + k_2 \leftrightarrow k_3 \]

\[ + 4F_2^R(-q,k_2)M_3^{O,R}(-q,-k_2,-k_3) \left( 1 + \frac{Q \cdot k_2}{k_2^2} \right) + k_2 \leftrightarrow k_3 \],

\[ \tag{B.8} \]

and the r.h.s is

\[ \langle \delta(q)\delta_g(k_1)\delta(k_2)\delta_g(k_3) \rangle'_{q \to 0} = -\frac{5}{2q^2} \left[ 6 + \sum_a k_a \cdot \partial_{k_a} + \frac{1}{2}q^i D_i \right] \sum_O b_C^O M_2^{O,N}(k_2,k_3)P(k_2)P(k_3). \tag{B.9} \]

For $n = 4$ we take the combination proportional to $P(k_2)P(k_3)P(k_4)$ on both sides of B.3. The l.h.s reads

\[ \langle \delta(q)\delta_g(k_1)\delta(k_2)\delta_g(k_3)\delta(k_4) \rangle'_{q \to 0} = 24 \sum_i b_C^O M_4^{O,R}(-q,-k_2,-k_3,-k_4) \]

\[ + 12 \sum_O b_C^O \left[ F_2^R(-q,k_2+q)M_3^{O,N}(-k_2,-q,-k_3,-k_4) \right. \]

\[ + F_2^R(-q,k_3+q)M_3^{O,N}(-k_3,-q,-k_2,-k_4) \]

\[ + F_2^R(-q,k_4+q)M_3^{O,N}(-k_4,-q,-k_3,-k_2) \], \tag{B.10} \]
and the r.h.s is
\[
\langle \delta(q) \delta_\alpha(k_1) \delta(k_2) \delta(k_3) \delta(k_4) \rangle'_{q \to 0} = 
- \frac{5}{2q^3} \left[ 9 + \sum_a k_a \cdot \partial_{k_a} + \frac{1}{2} q^i D_i \right] \sum_O b^5 S_{\alpha}^O \mathcal{F}_3^{O,N}(k_2, k_3, k_4) P(k_2) P(k_3) P(k_4). \tag{B.11}
\]

As an example, we show here that the operator \( S^2 \) satisfies the consistency relation. We take all the bias parameters equal to zero but \( b^{s_2}_s \) and we check the relativistic evolution for the operator \( S^2 \) which starts to contribute to third order. For \( n=3 \), the limit \( q \to 0 \) in equation B.8, and the corresponding derivatives in equation B.9 give the same result, therefore the consitency relation is satisfied at this order. The result is given by

\[
\langle \delta(q) \delta_\alpha(k_1) \delta(k_2) \delta(k_3) \rangle' = - \frac{1}{q^2} \left[ 40 s^2(k_2, k_3) + \frac{25}{3} \left( \frac{k_2 \cdot q}{k^2} + \frac{k_3 \cdot q}{k^2} \right) - 30 \frac{k_2 \cdot k_3}{k^2 k^2} q \cdot (k_{23}) \right. 
+ \left. 5 \left( \frac{k_2 \cdot k_3}{k^2 k^2} \right)^2 \left( \frac{k_2 \cdot q}{k^2} + \frac{k_3 \cdot q}{k^2} \right) \right] P(k_2) P(k_3). \tag{B.12}
\]

For \( n = 4 \) we only check the leading contribution to the consistency relation, given by the dilation transformation. The result for equations B.10 and B.11 in this case is

\[
\langle \delta(q) \delta_\alpha(k_1) \delta(k_2) \delta(k_3) \delta(k_4) \rangle' = - \frac{1}{q^2} \left[ 60 s^2(k_2, k_3) + 20 k_2 \cdot k_3 \left( \frac{1}{k^2} + \frac{1}{k^2} \right) \right. 
- \left. 60 \frac{(k_2 \cdot k_3)^2}{k^2 k^2 k^2 (k_{23})} + k_3 \to k_4 + k_2 \to k_4 \right] P(k_2) P(k_3) P(k_4). \tag{B.13}
\]

All the other relativistic galaxy bias solutions also satisfy the consistency relation. For \( n = 4 \) we only check the dilation part of the constency relation.

**C Renormalization of operators**

We continue the work of section 4, where we presented the renormalization of the terms proportional to \( b^{s_2}_s \). We now turn to all the other operators.

**Operator proportional to \( b^{s_2}_s \)**

For simplicity of notation, we call this operator \( (S^2) = b^{s_2}_s \text{tr}[S^2]/a^2 \). Its initial conditions are given by

\[
(S^2)^0(k) = b^{s_2}_s \int_{q_1, q_2} (2\pi)^3 \delta_D(k - q_{12}) \left( \frac{(q_1 \cdot q_2)^2}{q_1^2 q_2^2} - \frac{1}{3} \right) \delta_\ell(q_1) \delta_\ell(q_2) 
+ b^{s_2}_s a^3 \mathcal{H}^2 \int_{q_1, q_2, q_3} (2\pi)^3 \delta_D(k - q_{123}) M_{3}^{s_2 R}(q_1, q_2, q_3) \delta_\ell(q_1) \delta_\ell(q_2) \delta_\ell(q_3). \tag{C.1}
\]
where \(M_s^2,R\) is given in equation (A.34), and is obtained by using equation (3.6). The expectation value of the bare operator is
\[
\langle (S_2^2) \rangle = \frac{2}{3} b_s^2 \sigma^2 . \tag{C.2}
\]
This leads to the renormalization \([S_2^2]_R = (S_2^2) - \frac{2}{3} b_s^2 \sigma^2 + \ldots\). Next, we correlate with a long-wavelength perturbation
\[
\lim_{k \rightarrow 0} \langle \delta_t(k)(S_2^2)(k) \rangle' = \lim_{k \rightarrow 0} 3 \int_q M_3^{s2,R}(k, q, -q) P(q) P(k) = -\frac{20}{3k^2} a^3 H^2 b_s^2 P(k) , \tag{C.3}
\]
which is cancelled by writing
\[
[S_2^2]_R = (S_2^2) - \frac{2}{3} b_s^2 \sigma^2 - \frac{8}{3} b_s^2 a^2 H^2 \sigma^2 \zeta , \tag{C.4}
\]
where the term proportional to \(\zeta\) is the same as discussed in section 4, and combines with the first counter-term to form the same evaluated at the physical cutoff. We then use these as initial conditions for the solution of equation (4.5) to obtain an evolved operator \([S^2]\). One can check that it satisfies
\[
\langle [S^2] \rangle = 0 , \quad \lim_{k \rightarrow 0} \langle \delta_t(k)[S^2](-k) \rangle = 0 .
\]
However, there is still a cutoff dependence in the correlator
\[
\lim_{k_1 \rightarrow 0} \lim_{k_2 \rightarrow 0} \langle \delta_t(k_1) \delta_t(k_2)[S^2](k, \eta) \rangle' = -\frac{25}{3} a^4 H^2 b_s^2 \sigma^2 \zeta \left( \frac{(k_1 k_2)^2}{k_1^2 k_2^2} - \frac{1}{3} \right) P(k_1) P(k_2) , \tag{C.5}
\]
which has to be cancelled with an additional counter-term. The resulting counter-terms are
\[
M_0^{s2,c.t.}(k) = -\frac{2}{3} b_s^2 a^2 \sigma^2 (2\pi)^3 \delta_D(k) , \tag{C.6}
\]
\[
M_1^{s2,c.t.}(k) = -\frac{2}{3} b_s^2 a^2 \sigma^2 \left( \frac{20}{3k^2} a^2 H^2 \sigma^2 \right) , \tag{C.7}
\]
\[
M_2^{s2,c.t.}(k_1, k_2) = -\frac{b_s^2}{3} a^4 \sigma^2 \left( \sigma_D^N(k_1, k_2) + \alpha(k_1, k_2) \right) + \frac{25}{6} a^4 H^2 b_s^2 \sigma^2 \left( \sigma_D^N(k_1, k_2) + \alpha(k_1, k_2) \right) + 3 F_2^D(k_1, k_2) + \frac{15}{2} \frac{k_1 k_2}{k_1^2 k_2^2} \frac{\alpha(k_1, k_2)}{k_1^2 + k_2^2} . \tag{C.8}
\]

**Operator proportional to** \(b_3^s\)

For simplicity, we call this operator \((\delta^3) = b_3^s \delta^3/6a^3\). The expression for this operator at initial time is
\[
(\delta^3) = \frac{1}{6} b_3^s \int_{q_1, q_2, q_3} (2\pi)^3 \delta_D(k - q_{123}) \delta_t(q_1) \delta_t(q_2) \delta_t(q_3)
+ \frac{1}{6} b_3^s \int_{q_1, q_2, q_3, q_4} (2\pi)^3 \delta_D(k - q_{1234}) M_4^{s3,R}(q_1, q_2, q_3, q_4) \delta_t(q_1) \delta_t(q_2) \delta_t(q_3) \delta_t(q_4) . \tag{C.9}
\]
where $M_4^{\delta^3, R}$ is given in equation (A.38). The correlators give
\[ \lim_{k \to 0} \langle \delta_1(k) (\delta_2^3)(k) \rangle' = \frac{1}{2} \sigma^2 b_3^* P(k), \tag{C.10} \]

\[ \lim_{k_1 \to 0} \lim_{k_2 \to 0} \langle \delta_1(k_1) \delta_1(k_2)(\delta_2^3)(k) \rangle' = \lim_{k_1 \to 0} \lim_{k_2 \to 0} 12 \int_q M_4^{\delta^3, R}(k_1, k_2, q, -q) P(q) P(k_1) P(k_2) = \frac{1}{2} b_3^* a^3 H^2 \left( 2 \sigma^2 F_2^R(k_1, k_2) - 10 \sigma^2 \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) - 20 \sigma_{-2}^2 \right) P(k_1) P(k_2), \tag{C.11} \]

which can be eliminated at second order by
\[ [\delta_2^3]_R = (\delta_2^3) - \frac{1}{2} b_3^* a^3 \sigma^2 \delta_s - \frac{1}{2} b_3^* a^3 \sigma^2 \delta_s \zeta + 10 b_3^* a^3 H^2 \sigma_{-2}^2 \delta_s^2, \tag{C.12} \]

where $\delta_*$ is the dark matter solution at initial time, containing its linear and quadratic piece. We see that the counter-terms reshuffle into geometric operators of the initial time hypersurface as they should, and again we find the term proportional to $\zeta$ which corresponds to a rescaling of the cutoff. In this case there are no additional counter-terms needed for the evolved operator at the order at which we are working, and we find
\[ M_1^{\delta^3, c.t.}(k) = -\frac{1}{2} b_3^* a^3 \sigma^2, \tag{C.13} \]
\[ M_2^{\delta^3, c.t.}(k_1, k_2) = -\frac{1}{2} b_3^* a^3 \sigma^2 \alpha(k_1, k_2) - \frac{1}{2} b_3^* a^6 H^2 \left( \sigma^2 F_2^R(k_1, k_2) - 5 \sigma^2 \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) - 10 \sigma_{-2}^2 \right). \tag{C.14} \]

**Operator proportional to $b_{ss}^*$**

For simplicity, we call this operator $(\delta S)^2 = b_{ss}^* \text{tr}[S^2]/a^3$. The expression for this operator at initial time is
\[ (\delta_*^2 S_*^2) = b_{ss}^* \int_{q_1, q_2, q_3} (2\pi)^3 \delta_D(k - q_1^{123}) \left( \frac{(q_1 q_2)^2}{q_1 q_2} - \frac{1}{3} \right) \delta_1(q_1) \delta_1(q_2) \delta_1(q_3) + b_{ss}^* \int_{q_1, q_2, q_3, q_4} (2\pi)^3 \delta_D(k - q_1^{1234}) M_4^{\delta^2, R}(q_1, q_2, q_3, q_4) \delta_1(q_1) \delta_1(q_2) \delta_1(q_3) \delta_1(q_4), \]

where $M_4^{\delta^2, R}$ is given in equation (A.40). The correlators are
\[ \lim_{k \to 0} \langle \delta_1(k) (\delta_*^2 S_*^2)(k) \rangle' = \frac{2}{3} \sigma^2 b_{ss}^* P(k), \tag{C.15} \]

\[ \lim_{k_1 \to 0} \lim_{k_2 \to 0} \langle \delta_1(k_1) \delta_1(k_2)(\delta_*^2 S_*^2)(k) \rangle' = \lim_{k_1 \to 0} \lim_{k_2 \to 0} 12 \int_q M_4^{\delta^2, R}(k_1, k_2, q, -q) P(q) P(k_1) P(k_2) = \frac{2}{3} b_{ss}^* a^3 H^2 \left( 2 \sigma^2 F_2^R(k_1, k_2) - 10 \sigma^2 \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) - 18 \left( \frac{(k_1 k_2)^2}{k_1^2 k_2^2} - \frac{1}{3} \right) \sigma_{-2}^2 \right) P(k_1) P(k_2), \tag{C.16} \]
which can be eliminated at second order by

\[ [\delta_s S_s^2]_R = (\delta_s S_s^2) - \frac{2 b_s^2}{3 a_s^3} a_s^2 \sigma^2 \delta_s - \frac{8 b_s^2}{3 a_s^5} a_s^2 \sigma^2 \delta_s \zeta + 12 b_s^2 a_s^3 H_s^2 \sigma_2 \text{tr}[S^2], \]  

(C.17)

Evolving the operator renormalized at the initial hypersurface gives a new combination of terms that we call \([\delta S^2]\) which, however, has a non-zero correlation of the form

\[ \lim_{k_1 \to 0} \lim_{k_2 \to 0} \langle \delta \ell(k_1) \delta \ell(k_2) [\delta S^2](k) \rangle' = \frac{2 b_s^2}{a^3} a^4 \sigma^2 \frac{(k_1 k_2)^2}{k_1^2 k_2^2} P(k_1) P(k_2), \]  

(C.18)

that must be cancelled with additional counter-terms. After reordering the terms, we finally obtain

\[ M_1^{\delta s^2, c.t.}(k) = -\frac{2 b_s^2}{3 a^3} a^3 \sigma^2, \]  

(C.19)

\[ M_2^{\delta s^2, c.t.}(k_1, k_2) = -\frac{b_s^2}{a^3} a^4 \sigma^2 s^2(k_1, k_2) - \frac{2 b_s^2}{3 a^3} a^6 H^2 \left( \sigma^2 P_2^R(k_1, k_2) - 5 \sigma^2 \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) - 18 \sigma^2 s^2(k_1, k_2) \right). \]  

(C.20)

Operator proportional to \( b_s^3 \)

The calculation is analogous to the previous ones, and we find

\[ M_1^{s^3, c.t.}(k) = 0, \quad M_2^{s^3, c.t.}(k_1, k_2) = -\frac{4 b_s^3}{3 a^3} a^6 H^2 \sigma^2 s^2(k_1, k_2). \]

Quartic terms

The renormalization of quartic terms is straightforward at the order needed to compute the one-loop bispectrum. We need only require that the correlation with two long-wavelength perturbations at initial time cancel. We find

\[ M_1^{s^2, c.t.}(k_1, k_2) = -\frac{b_s^4}{4 a^4} a^4 \sigma^2, \quad M_2^{s^2, c.t.}(k_1, k_2) = -\frac{b_s^4}{a^4} a^4 \sigma^2 \left( s^2(k_1, k_2) + \frac{2}{3} \right), \]

\[ M_2^{s^3, c.t.}(k_1, k_2) = 0, \quad M_2^{(s^2)^2, c.t.}(k_1, k_2) = -\frac{28 b_s^4}{15 a^4} a^4 \sigma^2 s^2(k_1, k_2), \]

\[ M_2^{s^4, c.t.}(k_1, k_2) = -\frac{14 b_s^4}{15 a^4} a^4 \sigma^2 s^2(k_1, k_2). \]

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