Finitary Spacetime Sheaves

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Abstract

The notion of finitary spacetime sheaves is introduced based on locally finite approximations of the continuous topology of a bounded region of a spacetime manifold. Finitary spacetime sheaves are seen to be sound mathematical models of approximations of continuous spacetime observables.

1 Introduction cum Physical Motivation

The present paper associates with a finitary substitute $F_n$ of a bounded region $X$ of a continuous spacetime manifold $M$ (Sorkin, 1991), a collection $S_n$ of appropriately defined continuous functions on $X$, which, as a space on its own, is seen to be (locally) homeomorphic to $F_n$, thus technically speaking, a sheaf over $F_n$ (Bredon, 1967). This finitary spacetime sheaf is denoted by $S_n(F_n)$.

Then, we consider an inverse system $\mathcal{K} = \{ F_n(U_n) \}$ of finitary spacetime substitutes, derived from a net $\mathcal{L} = \{ U_n \}$ of locally finite open covers of $X$ as in (Sorkin, 1991), and the corresponding inverse system $\mathcal{N} = \{ S_n(F_n) \}$ of finitary spacetime sheaves associated with each element of $\mathcal{K}$. We show that as the elements $F_n$ of $\mathcal{K}$ get more refined (in a sense to be defined), their corresponding sheaves $S_n$ in $\mathcal{N}$ ‘converge’ to $S(X)$-the sheaf of (germs of) continuous functions on $X$.

The central physical idea we wish to model by the finitary spacetime sheaves $S_n$ is ‘locally finite approximations of the continuous spacetime observables on $X$’. In more detail, we intuit that as the continuous topology of $X$ can be finitely (or coarsely) approximated by the finitary topologies $F_n$, so

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can the continuous maps on it, that constitute the sheaf $S(X)$, be effectively approximated by the finitary sheaves $S_n$. Since only the continuous (i.e., $C^0$) topological structure of $X$ concerns us here, that is to say, we consider the continuous topology of spacetime to be its sole physically significant property, the physical spacetime observables that are of relevance are precisely the continuous functions on it. Then, the maps in each finitary sheaf $S_n$ over the locally finite substitute $F_n$ of $X$ represent sound coarse approximations of the continuous observables in $S(X)$ in the sense that, ‘in the limit of infinite resolution’ of the $F_n$s into $X$, they effectively reproduce them. The latter may be formally written as $\lim_{n \to \infty} S_n(F_n) \equiv S_\infty(F_\infty) \simeq S(X)$ ($S_n(F_n) \in \mathcal{N}$), with an explanation of this limiting procedure pending.

The present paper is organized as follows: in section 2, a brief reminder of finitary substitutes of a bounded region $X$ of a continuous spacetime manifold $M$ is given. We also allude, without proof, to the ‘inverse limit’ topological space $F_\infty$ to which an appropriately defined ‘inverse system’ $\mathcal{K}$ of such substitutes derived from a net $\mathcal{L} = < U_n >$ of locally finite open covers of $X$, converges ‘at maximum resolution of $X$ by $U_\infty$’. This topological space is essentially homeomorphic to $X$. For detailed proofs the reader is referred to (Sorkin, 1991).

In section 3, a space $S$ that is locally homeomorphic to $X$ is defined. This is the sheaf $S(X)$ of continuous functions on $X$.

In section 4, finitary spacetime sheaves $S_n$ of continuous functions on $X$ associated with the finitary substitutes $F_n$ of section 2, are defined constructively.

Like their finitary ‘domains’ $F_n$, the finitary spacetime sheaves $S_n$ also form an inverse system $\mathcal{N}$ having an inverse limit topological space $S_\infty(F_\infty)$ that is essentially homeomorphic to the sheaf $S(X)$ of section 3. This limiting procedure is briefly presented in section 5 by simulating in $\mathcal{N}$ Sorkin’s proof that the inverse system $\mathcal{K}$ ‘converges’ to $X$. This is evidence of the soundness of the finitary sheaves as models of locally finite approximations of the continuous functions on $X$-the observables of the continuous spacetime topology of $X$.

In the concluding section 6 we discuss the physical significance of the elements of finitary spacetime sheaves as the locally finite approximations of the continuous observables on the region $X$ of the spacetime manifold $M$. In view of a recent definition of quantum causal sets (Raptis, 2000), $\simeq$ denoting ‘homeomorphic to’.
partially motivated by the poset finitary substitutes of continuous spacetime topologies in (Sorkin, 1991), their quantum algebraic analogues in (Raptis and Zapatrin, 2000) and their causal relatives in (Bombelli et al., 1987), we entertain the idea of a finitary spacetime sheaf of quantum causal sets, as well as the possibility that this structure be curved, thus serve as a sound model of some sort of ‘finitary quantum gravity’. However, the analytic development of this possible application of finitary spacetime sheaves will have to be postponed for another paper (Mallios and Raptis, 2000).

2 Finitary Spacetime Substitutes Revisited

In (Sorkin, 1991), with finite open covers \( \mathcal{U}_n \) of a bounded region \( X \) of a continuous spacetime manifold \( M \), finite topological spaces \( F_n \) were associated by the following ‘equivalence algorithm’: two points \( x \) and \( y \) in \( X \) are said to be equivalent with respect to the open cover \( \mathcal{U}_n \) (write \( x \sim_{\mathcal{U}_n} y \)), when \( \forall U \in \mathcal{U}_n: x \in U \Leftrightarrow y \in U \).

Denoting by \( \Lambda(X) \) the smallest open neighborhood in the subtopology \( \mathcal{T}_n \) of \( X \) generated by the open sets \( U \) in \( \mathcal{U}_n \)\(^2\),

\[
\Lambda(x) := \bigcap \{ U \in \mathcal{U}_n \mid x \in U \},
\]

one may alternatively define \( x \sim_{\mathcal{U}_n} y \) as

\[
(x \rightarrow y) \land (y \rightarrow x),
\]

where \( x \rightarrow y \) stands for \( x \in \Lambda(y) \).

The quotient space \( X / \sim_{\mathcal{U}_n} =: F(\mathcal{U}_n) \equiv F_n = \{ [x] \} \) consisting of equivalence classes \( [x] \) of points \( x \) in \( X \) relative to its finite open covering \( \mathcal{U}_n \), is seen to be a \( T_0 \) topological space having the structure of a poset (Sorkin, 1991). With this \( T_0 \)-quotienting of \( X \) by the equivalence relation \( \sim_{\mathcal{U}_n} \) to \( F_n \), that is to say, the substitution of \( X \) by its finitary approximation \( F_n \) relative to \( \mathcal{U}_n \), a continuous function \( f_n \) from the subtopology \( \mathcal{T}_n \) of \( X \) to \( F_n \), may be

\(^2\)A subset of a topological space is said to be bounded if its closure is compact. Our \( X \) in \( M \) is bounded in this sense.

\(^3\)That is to say, \( \mathcal{T}_n \) consists of arbitrary unions of finite intersections of the open sets \( U \) in \( \mathcal{U}_n \).
associated (Sorkin, 1991). $f_n$ is continuous in the usual sense that open sets in $F_n$ are mapped by $f_n^{-1}$ to open subsets of $X$ in $\mathcal{T}_n$.

In (Sorkin, 1991), a net $\mathcal{L}$ of open covers $\{U_n\}$ is also considered. For every pair $U_i$ and $U_j$ of locally finite open covers of $X$ in $\mathcal{L}$, there is a ‘finer’ open cover $U_k \in \mathcal{L}$ such that $U_i, U_j \subset U_k$. With the net $\mathcal{L}$ and the $f_n$s corresponding its elements to the $F_n$s, the inverse system $\mathcal{K} = \langle F_n \rangle$ of finitary substitutes of $X$ may be derived. Here too, $F_1 \preceq F_k$ may be taken to stand for $F_1 \subset T_k$, and means that the open sets in the poset $F_1$ are contained in the finitary poset topology $T_k$ generated by the open sets in the poset $F_k$.

For the pair $i$ and $k$ of indices above, the ‘finer’ relation may be denoted as $i \preceq k$ and it is seen to be a partial order between the locally finite open covers of $X$ in $\mathcal{L}$, or their associated finitary substitutes in $\mathcal{K}$. Intuitively, the procedure of ‘refinining’ an open cover $U_i$ to a finer $U_j$ ($i \preceq j$), corresponds to adding more and finer or ‘smaller’ open subsets of $X$ to the open cover $U_i$, thus, in a way, it represents the employment of higher power of resolution in operations of determination of the topological structure of $X$.

For every pair $U_i$ and $U_j$ in $\mathcal{L}$ (or the corresponding pair of posets in $\mathcal{K}$), with $U_i \preceq U_j$ (or $F_i \preceq F_j$), one defines a map $f_{ij} : F_j \rightarrow F_i$, which is seen to be a unique continuous surjection.

The central result from (Sorkin, 1991) is that the inverse system $\mathcal{K}$ ‘converges’ to $F_\infty$ at the (inverse) limit of maximum resolution (i.e., formally, as $n \rightarrow \infty$) of the finite open covers of $X$ in $\mathcal{L}$. In particular, $F_\infty$ is essentially $f_\infty$-homeomorphic to $X$, so that, effectively, the maximally refined finitary substitute of $X$ is topologically indistinguishable from (or equivalent to) it. This description of $f_\infty$ as being ‘essentially a homeomorphism’ between $X$ and $F_\infty$, pertains to the fact, shown in (Sorkin, 1991), that $X$ is $f_\infty$-embedded in $F_\infty$ as a dense subset. Then, Sorkin shows how to ‘discard’ from $F_\infty$ the collection of its ‘extra-points’ $y$ that are ‘infinitely close’ to those of $f_\infty(X)$, thus establish $f_\infty$ as a homeomorphism between $\tilde{F}_\infty = F_\infty/\{y\}$ and $X$.

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4 Defined with respect to the partial order relation $\rightarrow$ in $F_n$ as the sets that can be obtained as unions of the following basic open sets: $\forall x \in F_n : \Lambda(x) := \{y \in F_n \mid y \rightarrow x\}$; where we have used the points $x$ instead of the equivalence classes $[x]$ where they belong for simplicity of notation.

5 We will return to this issue at the end of section 5 and in section 6.

6 For the purposes of the present paper, we need not explain Sorkin’s proof of the inverse limit in more detail. A brief outline of the basic steps of the proof is given in section 5 where we argue that $S_n(F_n)$ in $\mathcal{N}$ converge to $S(X)$. 

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The last result from (Sorkin, 1991) of interest to us here is that the tiling of $F_\infty$ by the open sets in a particular covering $U_i$ of $X$, symbolized as $f_{i\infty}^{-1}(x)$ ($\forall x \in F_\infty$), becomes arbitrarily fine as $i \to \infty$. In other words, the open sets get ‘smaller’ and ‘smaller’ as the ‘experimenter’ employs higher power to resolve $X$ into its points (ie, ideally, to determine or localize individual spacetime events in $X$).

In the next section we present a space $S$ that, similarly to $F_\infty$, is (locally) homeomorphic to $X$.

3 S(X)-The Sheaf of Continuous Functions over $X$

We want to organize the observables on the bounded spacetime region $X$ of the continuous manifold $M$ into a space $S$ that, for all practical purposes, is topologically indistinguishable from (ie, homeomorphic to) $X$. If we succeed in this, then the information about the topology of spacetime encoded in the points of $X$, which serve as the ‘carriers of its topology’ (Sorkin, 1991), will be the same as that of our observations of them, at least locally and our description of the topological relations between spacetime events will be the same as that of the topological relations between our observations of them. Thus, the problem of localization of spacetime events will be effectively translated to the more operationally sound problem of localization of our observations of them.

Now, there is such an organization of the continuous functions on $X$ called a sheaf. Below, we first introduce the notion of a presheaf $P(X)$ of functions defined on the open sets of $X$, then we endow this presheaf with a topology that is locally equivalent (ie, homeomorphic) to that of $X$. This process converts the presheaf $P(X)$ into the sheaf $S(X)$ of (germs of) continuous functions on $X$ and is called ‘sheafification’. Only the basic definitions from sheaf theory, that will help us define finitary spacetime sheaves in the next section, are given below. For a more detailed treatment of sheaves, the reader is referred to (Bredon, 1967).

A presheaf $P$ on our region $X$ of the continuous spacetime manifold $M$ is an assignment to each open subset $U$ of $X$ of a set $P(U)$ and to each pair

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7Where $f_{i\infty} : F_\infty \to F_i$ can be thought of as the continuous finite approximation of $F_\infty$ by $F_i$.

8See section 6 for the discussion of this physical role of the points of $X$.

9That is to say, ‘about every point-event $x$ in $X$’.

10Again, see the concluding section 6 for more discussion on this.
U, V of open subsets of X of a ‘restriction map’ \( \rho_{U,V} : P(V) \to P(U) \), so that: \( \rho_{U,U} = id \), and \( \rho_{U,V} \rho_{V,W} = \rho_{U,W} \), with \( id \) the identity map and \( U \subset V \subset W \subset X \). One may think of the presheaf \( P(X) \) as collections of functions defined on open subsets of X which \( \rho \)-reduce to one another when their respective domains of definition are nested by inclusion.

To sheafify or ‘topologize’ the presheaf \( P(X) \) to the sheaf \( S(X) \), we embed each \( P(U) \) in the presheaf to the collection \( \Gamma(U, S) \) of sections of continuous functions on \( \mathsf{U} \subset \mathsf{V} \subset \mathsf{W} \subset X \) by the map \( \sigma_U : P(U) \to \Gamma(U, S) \), that is to say, in some sense we select from \( P(U) \) the continuous maps on \( \mathsf{U} \subset \mathsf{X} \).

With the definition of \( \tilde{\sim} \), one may define the stalk of the sheaf \( S \) over \( x \) as the following equivalence class of continuous functions at \( x \)

\[
S_x := \bigcup \{ P(U) \mid x \in U \} / \tilde{\sim} \equiv \lim \{ P(U) \mid x \in U \},
\]

with \( \lim \) denoting ‘direct limit’\(^{12}\). As a non-topologized set, the sheaf \( S(X) \) may be expressed as a disjoint union (or direct sum) of its stalks \( S(X) = \bigcup_{x \in X} S_X \).

We may now endow \( S \) with the following topology \( T \): let \( f \) be a member of the presheaf \( P(U) \) and \( x \) a point in \( U \), then the germ of \( f \) at \( x \), \( [f]_x \), is the \( \tilde{\sim} \)-equivalence class of \( f \). A basis for the topology of \( S \) consists of open sets of the following sort: \( (x, [f]_x) \) \( (x \in U) \).

Now, \( \Gamma(U, S) \) above is the set of continuous sections of the sheaf \( S(X) \) over its open subset \( U \), that is to say, the set of continuous maps \( s : U \to S \).

\(^{11}\)This nesting by inclusion of the open subsets of \( X \) is a partial order on the collection of all open subsets of \( X \).

\(^{12}\)This limit effectively yields the ‘smallest’ class of functions in the presheaf \( P \) over the ‘finest’ neighborhood of \( x \in X \) (that is also included in both \( U \) and \( V \)). This direct limit effectively corresponds to maximum localization/resolution of \( S(X) \) into its stalks \( S_x \) over the points \( x \) of \( X \), and it is dual to the inverse limit procedure employed to resolve \( X \) into its points \( x \) in (Sorkin, 1991)-read the physical significance of finitary spacetime sheaves in section 6.

\(^{13}\)Thus, in some sense, the stalks \( S_x \) of \( S(X) \) are its ‘points’, like the \( x \) in \( X \)-see last footnote and read the physical significance of finitary spacetime sheaves in section 6.
such that, locally (ie, point-wise in $X$), they map each point $x$ in $X$ to (an element $[f]_x$ of) the stalk $S_x$ over it. Also, the selection $\sigma_U$ in $\Gamma(U, S)$ of the map $f$ in $P(U)$ above reduces locally to the germ of $f$ at $x$, that is: $\sigma_U(f)(x) = [f]_x \in S_x$ ($x \in U$). It follows that, as a topological space $T(S)$, the sheaf $S(X)$ is generated by the (germs of) sections of the continuous maps on $X$, so that one can easily show that the ‘projection map’ $\pi : S(X) \to X$, given locally by $\pi(x, [f]_x) = x$, is a local homeomorphism; hence, the slogan that ‘a sheaf is a local homeomorphism’. Equivalently, one can verify that every (germ of a) section of a continuous map in $\Gamma(U, S)$ is also such a local homeomorphism of $X$ to $S(X)\vert_U$ so that for such a section $s \in \Gamma(U, S)$, the composition of $s$ with $\pi$ corresponds to the identity map $id : S(X)\vert_U \to S(X)\vert_U$ (ie, $s \circ \pi \equiv id_{S(X)\vert_U} \iff s \equiv \pi^{-1}$), where $S(X)\vert_U$ is the restriction of the sheaf $S(X)$ to the open subset $U$ of $X$.

In the next section we define finitary spacetime sheaves $S_n(F_n)$ as the finitary substitutes of the continuous sheaf $S(X)$ in a way analogous to how the $F_n$s were seen to be the locally finite poset substitutes of the continuous topological space $X$ in (Sorkin, 1991) and section 2.

4 Constructive Definition of $S_n(F_n)$

A point in $X$ (ie, a spacetime event) corresponds to an ideal determination of location in spacetime—an ideal measurement of the locus of an event. A more pragmatic and operationally sound model of spacetime measurements, one taking into account their actual ‘roughness’ or ‘approximate character’, or even the ‘fuzziness’ due to the uncontrollable perturbations that such realistic acts of measurement inflict on spacetime, is that they, at least, determine open sets in $X$ (Sorkin, 1995, Raptis and Zapatrin, 2000). Since

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14Hence the equivalent slogan that ‘a sheaf is its sections’.

15We may add here that usually the stalks $S_x \in S(X)$ are assumed to have some kind of algebraic structure, so that the algebraic operations that define it respect the ‘horizontal continuity’ of the base space $X$ (ie, they satisfy ‘compatibility conditions’ with the topology of the underlying space $X$). It must be noted that the continuous functions on $X$ form indeed an algebra, the prototype being $C^0(X, \mathbb{C})$-the algebra of complex valued continuous functions on $X$. It is tacitly assumed that the operations in $C^0(X, \mathbb{C})$ respect the base topology in a sheaf of such algebras over $X$. In the present paper we are not interested in the algebraic structure of the stalks of the (finitary) sheaves. In section 6 however, we mention finitary spacetime sheaves whose stalks have some specific algebraic structure of special importance to the physical situation that they are employed to model.
points in $X$ serve as the ‘carriers of the spacetime topology’ (Sorkin, 1991)\textsuperscript{16}, the aforementioned pragmatic determinations of events by open sets of $X$ effectively correspond to ‘approximations of its continuous topology’. Thus, the latter may be thought of as the ideal limit-topology an experimenter determines at his (also ideal !) maximum power of resolution of spacetime into its finest/’smallest’ open sets containing its points. This was essentially the moral of section 2.

Now our main physical motivation for defining finitary spacetime sheaves $S_n$ for every finitary approximation $F_n$ of the continuous topology of $X$, is to similarly approximate the sheaf $S$ of continuous functions on $X$, and effectively recover it at the limit of maximum resolution into its ‘ultra-local elements’ that in the previous section were seen to be its stalks $S_x$ over the points $x$ in its base space $X$. As we saw in the last section, the stalks of $S$ over the points of $X$ consist of the germs of continuous functions on $X$. Thus, the main idea is to define our finitary sheaves $S_n(F_n)$ in such a way that their stalks consist of ‘gross observables’ that are continuous over the ‘rough’ open subsets of a finite cover $U_n$ of $X$ and that, in the limit of maximum resolution of the $U_n$s to $F_\infty$ as in section 2, they reduce to the $S_x$ of $S(X)$. This (inverse) limit will be the subject of the next section.

The finitary spacetime sheaf $S_n$, associated with the locally finite open cover $U_n$ of $X$, is defined as follows: take a finitary open cover $U_n$ of $X$. Recall from section 2 that $U_n \sim$ denotes the following equivalence relation between points $x$ and $y$ of $X$

$$x \sim_y \iff (x \rightarrow \Lambda(y)) \land (y \rightarrow \Lambda(x)); \quad \Lambda(X) := \bigcap\{U \in U_n \mid x \in U\},$$

according to which, the poset finitary substitute $F_n$ of $X$ with respect to its locally finite open cover $U_n$, is defined by the quotient $F_n := X/\sim$ and consists of equivalence classes $[x]$ of points $x \in X$. Then, force the following ‘collapse’ equivalence relation between the stalks $S_x = \{[f]_x\}$ of $S(X)$ over the $x$ in $X$, induced by the $\sim$ equivalence relation between them, as

$$x \sim_y \Rightarrow [f]_x \equiv [f]_y,$$

where we have effectively identified the germs of continuous functions in the

\textsuperscript{16}See section 6.
stalks $S_x$ and $S_y$ of $S(X)$ over the $\sim_{\mathcal{U}}$-equivalent points $x$ and $y$ in $X$. This $\sim_{\mathcal{U}}$-identification (equivalence relation) of the stalks of $S(X)$ has the following physical interpretation: for point-events in $X$ that are $\sim_{\mathcal{U}}$-indistinguishable at the power of resolution employed to analyze it by $\mathcal{U}$, one may use any of the germs of the continuous observables from $S(X)$, residing in the stalks over them, in order to describe their ‘local topological relations’.

Then, like the $F_n$s, we define the corresponding finitary sheaf $S_n(F_n)$, as the following equivalence class of stalks in $S(X)$

$$S_n(F_n) := S(X)/\sim_{\mathcal{U}} = \bigcup_{[x] \in F_n} S_{[x]}.$$

It is plain to see that the finitary sheaves $S_n(F_n)$ inherit the poset $T_0$-topology of their corresponding finitary substitutes $F_n$ as follows

$$[x] \to [y] \in F_n \Rightarrow [f]_{[x]} \to [f]_{[y]} \in S_n(F_n),$$

with the partial order in $S_n(F_n)$ pending a bit of further explanation. This explanation can be drawn straightforwardly by considering the following ‘commutative diagram’

$$\begin{array}{ccc}
X & \xrightarrow{f_n} & F_n \\
\pi^{-1} \downarrow & & \downarrow \\
S & \xrightarrow{s_n} & S_n
\end{array}$$

where $s_n \equiv \pi_n^{-1}$ is the ‘local homeomorphism’ from $F_n$ to $S_n$ that we are searching for. The diagram shows that, as a map: $s_n = f_n \circ \pi^{-1} \circ f_n^{-1}$, where $f_n^{-1}$ is the inverse of the bijective correspondence between the smallest open subsets $\{\Lambda(x)\}$ containing the points of $X$ with respect to its locally finite open covering $\mathcal{U}$ and the $\sim_{\mathcal{U}}$-equivalence classes of points of $X$ in $F_n$. $\pi^{-1} \equiv s$ is the local homeomorphism from $X$ to $S$ as defined in the previous

\footnote{Note that all our definitions are implemented ‘point-wise in $X$. As noted earlier, we hold to the primitive intuition that the points of $X$ are ‘the carriers of its topology’-the basic tenet of point set topology (read the physical motivation in section 6 for a ‘physical justification’ of this primitive intuition).}

\footnote{Again, read section 6 for more on this.}

\footnote{Defined in section 2 as the map $f_n : T_n \to F_n$. $f_n$ as a map from $X$ to $F_n$ is a continuous surjection, but as a map from $T_n$ to $F_n$ it is a homeomorphism (i.e., a bijective continuous map whose inverse is also continuous).}
section which, in terms of the corresponding smallest open neighborhoods \( \Lambda(x) \) of points \( x \) in \( X \) with respect to \( U_n \), reads: \( s : \Lambda(x) \to \Gamma(\Lambda(x), S) \), and \( \hat{f}_n \) is the one-to-one map defined by the equivalence relation \( \equiv \) between the stalks of \( S \) with respect to \( U_n \). Thus, as the partial order \( x \to y \) (or \([x] \to [y]\)) in \( F_n \) stands (by definition) for \( x \in \Lambda(y) \Rightarrow [f]_x \to [f]_y \) in \( S_n \) is simply interpreted as the following set-theoretic inclusion in \( S \): \( [f]_x \in \{ [f]_{\Lambda(y)} \} = \pi^{-1}(\Lambda(y)) = S_{\Lambda(y)} = \bigcup_{x \in \Lambda(y)} S_x \)—the restriction of \( S(X) \) at \( \Lambda(y) \subset X \). All in all, \( s_n : F_n \to S_n \) is a local homeomorphism, because, by construction, it is one-to-one and (locally) preserves the \( T_0 \) order-topology of both \( F_n \) and \( S_n \).

This completes the definition of the finitary sheaf \( S_n \) over the locally finite poset substitute \( F_n \) of \( X \). In the next section we argue that, as the inverse system \( K = \langle F_n \rangle \) converges to \( X \), so its derivative \( N = \langle S_n(F_n) \rangle \) converges to \( S(X) \).

5 The Inverse Limit \( S(X) \) of \( < S_n(F_n) > \)

In this section we present briefly the basic definitions and follow in little detail the main steps in Sorkin’s proof of the ‘convergence’ of the inverse system \( K \) of finitary substitutes \( F_n \) and the maps \( f_{ij} \) between them to a space \( S_\infty \) that is essentially homeomorphic to \( X \), and apply it for the proof of a similar ‘convergence’ of the inverse system \( N \) of the finitary sheaves of the previous section and certain continuous surjections \( \tilde{f}_{ij} \) between them, to a limit space \( S_\infty(F_\infty) \) that is effectively homeomorphic to \( S(X) \). The essential point to mention is that, since our constructive definition of the finitary sheaves \( S_n \) in the previous section followed precisely, point-wise in \( X \), the steps of the constructive definition of their respective domains \( F_n \) in section 2, so that both \( F_n \) and \( S_n \) have the same poset-topology, one expects the proof of the convergence of \( N \) to \( S(X) \) to be effectively the same as that of \( K \) to \( X \) given in (Sorkin, 1991). Thus, our proof of the latter convergence only highlights the important definitions and proof-steps given by Sorkin (1991), referring the reader to it for more details.

The first thing from (Sorkin, 1991) that we mention is that ‘convergence’ of \( K \) does not pertain to the usual notion of a ‘limit’ for its terms, because for that, a topology on the set of all topologies on \( X \) would have to exist, which is not the case. Rather, \( K \) is defined as an inverse system possessing

\(^{20}\)And it literally stands for the convergence of the constant sequence \( x \) to \( y \) (Sorkin, 1991).
an inverse limit.

The terms in \( \mathcal{K} \) are the finitary substitutes of \( X \) corresponding to the net \( \mathcal{L} \) of locally finite open covers of \( X \), together with unique continuous surjections \( f_{ij} : F_j \to F_i, \ (i \preceq j) \) between them. Thus, the inverse system \( \mathcal{N} \) is defined in the same way as \( \mathcal{K} \), though the continuous surjections between the finitary sheaves in it are now denoted by \( \tilde{f}_{ij} (i \preceq j) \).

As the inverse system \( \mathcal{K} = \langle F_j, f_{ij} \rangle \) is seen to possess an inverse limit \((F_\infty, f_{i\infty})\) as \( j \to \infty \), with \( F_\infty \) a \( T_0 \) topological space like all the finite terms in \( \mathcal{K} \) (Sorkin, 1991), so does the inverse system \( \mathcal{N} = \langle S_j(F_j), \tilde{f}_{ij} \rangle \). The limit sheaf space \( S_\infty(F_\infty) \) is a \( T_0 \) topological space, since all the finite terms \( S_n(F_n) \) in \( \mathcal{N} \) are \( T_0 \) posets.

The reader is referred to a series of lemmas in (Sorkin, 1991) that establish that the inverse limit space \( F_\infty \) of the inverse system \( \mathcal{K} \) is a non-Hausdorff space that contains \( X \) as a dense subset. Since, as we noted earlier, the definitions and constructions for the terms in the inverse system \( \mathcal{N} \) are identical, being implemented point-wise in \( X \), with those in \( \mathcal{K} \) of (Sorkin, 1991), we directly infer that the lemmas mentioned above also hold in our scheme, so that we may quote directly their result: the inverse limit \( S_\infty(F_\infty) \) of \( \mathcal{N} \) is a non-Hausdorff space that contains \( S(X) \) as a dense subset. The latter means essentially that for every stalk \( S_y \) over \( y \in F_\infty \) in the limit sheaf \( S_\infty \) over \( F_\infty \), there is a stalk \( S_x \) in \( f_\infty(S(X)) \) ‘infinitely close’ to it.

Next, we mention a further lemma in (Sorkin, 1991) that establishes that if \( X \) is \( T_1 \) and the previous lemmas for the denseness of \( X \) in \( F_\infty \) also hold, then \( f_\infty(X) \) constitute the points of \( F_\infty \) that are closed in its topology. We apply it to our situation and state that the image set \( f_\infty(S(X)) \) in \( S_\infty(F_\infty) \) consists of the closed stalks in the latter’s topology. We refer again the reader to (Sorkin, 1991) to verify that \( F_\infty \), and in extenso our \( S_\infty(F_\infty) \), is non-Hausdorff; moreover, one can ‘delete’ its extra-points, thus render \( f_\infty \) a

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\(^1\)The uniqueness of the continuous \( f_{ij} \)'s follows from the universal mapping theorem for \( T_0 \) topological spaces and the assumption that \( i \preceq j \) (Sorkin, 1991).

\(^2\)The ‘finer’ relation \( S_i \preceq S_j \) means in this case that the poset topology \( T(S_i) \) is a subtopology of \( T(S_j) \), like for their corresponding finitary substitutes.

\(^3\)Again, the proof is again via the universal mapping property of the \( f_{ij} \).s.

\(^4\)A topological space \( X \) is said to be Hausdorff, or satisfying the \( T_2 \) axiom of separation of point-set topology, if for every pair of distinct points \( x \) and \( y \) in it, there exist open neighborhoods \( N(x) \) and \( N(y) \) about them, such that \( N(x) \cap N(y) = \emptyset \).

\(^5\)One should read the lemmas in (Sorkin, 1991) that establish this ‘infinite closeness’ relation between points in \( F_\infty \) and \( f_\infty(X) \), and convince oneself that they also hold, point-wise, in our sheaf-theoretic scheme as well.
homeomorphism between $S(X)$ and $S_\infty(F_\infty)$.

Finally, we mention the usefulness for the physical interpretation of the finitary spacetime sheaves $S_n(F_n)$ as finite approximations of $S(X)$ in the next section, (the sheaf-theoretic analogue of Sorkin’s proof) that the open subsets $f_{i\to\infty}(S_{[x]})$ ($S_{[x]} \in S_i(F_i)$) tiling $S_\infty(F_\infty)$ become arbitrarily fine, or ‘small’ as $j \to \infty$ (Sorkin, 1991). Thus, refining the finitary sheaves in $\mathcal{N}$ effectively amounts to better localizations or approximations of the observables residing in $S_x \in S(X)$.

6 Physical Significance and a Future Application of the $S_n$s

If we consider a net $\mathcal{L} = (\{\mathcal{U}_n\}, \prec)$ of locally finite open covers of $X$, with $\mathcal{U}_i \prec \mathcal{U}_j$ denoting the relation ‘finer’ between its elements as defined in section 2, then as shown in (Sorkin, 1991) and briefly discussed above, ‘in the limit of infinite resolution of $X$ into its ‘smallest’ open neighborhoods (in $\mathcal{U}_\infty$) about its points, $X$ is recovered up to homeomorphism’, that is to say, formally: $F_\infty(\mathcal{U}_\infty) := \lim_{n \to \infty} F_n(\mathcal{U}_n)$ is homeomorphic to $X$. Thus, finitary substitutes are regarded as sound finite approximations of continuous topologies (Sorkin, 1991), whereby a ‘rough’ determination of a point in $X$ is modelled after an open neighborhood about it (Sorkin, 1991, 1995).

Now, the transcription of the problem ‘$F_n$ as approximations of $X$’ to ‘$S_n(F_n)$ as approximations of $S(X)$’ that is the essence of the present paper, changes focus from ‘approximate localization/local determination (measurement) of points in $X$', to ‘approximate localization/local determination (measurement) of continuous functions over $X$', thus from a physical point of view, when $X$ is taken to be a bounded region of the spacetime manifold, from ‘localization of events’, to ‘localization of observables of events’. In this paper, as it was mentioned earlier, the continuous topology of $X$ is regarded as its sole physically significant property ‘carried by its points’, thus the continuous functions on it adequately qualify as ‘spacetime observables of events’. This attribute of points as ‘carrying the topology of $X$’ can be realized by requiring that every physical space $X$ is a $T_0$ topological space (Sorkin, 1991). The relation $\mathcal{U}_i \sim \mathcal{U}_j$ between events in $X$ is physically interpreted

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26 See end of section 2.
27 Read next section.
28 Roughly, $\mathcal{U}_j$ has more and ‘smaller’ open subsets of $X$ than $\mathcal{U}_i$.
29 The epithets ‘fuzzy’, ‘blurry’, or ‘foamy’ may be regarded as alternative synonyms to ‘rough’.
as ‘indistinguishability of events at the finite power of resolution of $X$ corresponding to $U_n$’, while $U_n \equiv$ between the observables residing in the stalks of $S(X)$, as ‘indistinguishability of the observables of events at the finite power of resolution of $S(X)$ corresponding to $U_n$’.

The posets corresponding to the locally finite substitutes $F_n$ of $X$ are known to have an equivalent (ie, functorial) representation as simplicial complexes obtained from the nerves of the covering $U_n$ (Alexandrov, 1956, Raptis and Zapatrin, 2000) which, in turn, are categorically equivalent to incidence Rota algebras associated with them (Raptis and Zapatrin, 2000, Zapatrin, 2000)\(^{30}\). Observables too are in a certain sense dual to events\(^{31}\). In (Raptis and Zapatrin, 2000) it was found that the Rota incidence algebra $R_n$ associated with a particular finitary substitute $F_n$ is also a discrete differential manifold in the sense of (Dimakis and Müller-Hoissen, 1999). Thus, not only the reticular analogues of the continuous ($C^0$) functions on $X$ are encoded in a finitary sheaf $R_n$ of Rota algebras over the $F_n$, but also a discrete version of the smooth ($C^\infty$) ones\(^{32}\). Hence, the conjecture is that at the maximum resolution $U_\infty$ of a bounded region $X$ of a smooth spacetime manifold $M$, the Rota algebras $R_n$ associated with the locally finite posets $F_n$, $R_n(F_n)$, are expected to ‘yield’ $(X, \partial, \Omega)$\(^{33}\)—the flat sheaf of sections of smooth differential

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\(^{30}\)That is to say, the category of finitary posets/order morphisms, is functorial to that of simplicial complexes/simplicial mappings, which, in turn, is ‘anti-functorial’ to that of Rota algebras/Rota homomorphisms. The latter means that, since the Rota incidence algebras associated with finitary posets are objects dual to them (Raptis and Zapatrin, 2000), there is a contravariant functor between their respective categories. Thus, while finitary posets constitute the inverse system $K$ in (Sorkin, 1991) and above, their associated incidence algebras may be organized into a ‘direct system’ having a direct limit, much like the finitary sheaves $S_n(F_n)$ above were seen to have a direct limit space isomorphic to the stalk $S_x$ of $S(X)$.

\(^{31}\)Intuitively they are dual, for the pairing of an observable $f$ with a point-event $x$ produces a measurable number $f(x)$ (ie, the value of the ‘field’ $f$ at the ‘test-event’ $x$). Thus, in some sense, $x$ is like a state of $X$, while the action of $f$ on it, $f(x)$, is some sort of measurement of the property $f$ of $X$ at $x$ (here, its continuous topology). Evidence for this duality is the mathematical duality of the notions of inverse and direct limit by which the local (point-like) elements of $X$ and $S(X)$ (ie, $x$ and $S_x$) were defined above, respectively. See also previous footnote.

\(^{32}\)Although, our study in the present paper concentrates solely on the continuous (ie, $C^0$) structure of spacetime, not its differential.

\(^{33}\)Ω being the module of differential forms over the algebra of $C^\infty$-smooth functions on $X$, ie, $\Omega := \Omega^0(\equiv C^\infty(X)) \oplus \Omega^1 \oplus \Omega^2 \oplus \ldots$
forms over $X$ (i.e., the smooth spacetime observables). Also, a quantum interpretation has been given to the Rota incidence algebras $R_n$ associated with the finitary substitutes $F_n$ of $X$ (Raptis and Zapatrin, 2000). Accordingly, the limit space of a system of $R_n(F_n)$s is $(X, \partial, \Omega)$—the classical smooth spacetime manifold with differential forms attached, and it was interpreted there as Bohr’s correspondence limit structure of a quantal substratum of finitary incidence algebras.

In (Raptis, 2000), a causal interpretation to the quantal incidence algebras $R_n(F_n)$ of (Raptis and Zapatrin, 2000) was given. The resulting structures were called ‘quantum causal sets’—a quantal version of the causal sets of (Bombelli et al., 1987). It follows from the discussion above that a finitary sheaf of quantum causal sets may be studied as a quantal and locally finite substitute of the causal relations between events in a bounded region of a smooth Lorentzian spacetime manifold. As in (Dimakis and Müller-Hoissen, 1999) a Riemannian metric connection was studied on a discrete differential manifold, so we should be able to define a non-flat pseudo-Riemannian connection on the finitary sheaf of quantum causal sets by using powerful sheaf-theoretic results from (Mallios, 1998). It is expected that in the Bohr correspondence limit of an inverse system of finitary sheaves of quantum causal sets, the smooth (region of a) Lorentzian manifold, together with the smooth fields and a non-flat pseudo-Riemannian metric connection $D$ on it, will emerge. Then, the underlying finitary sheaves of quantum causal sets may be regarded as sound models of a reticular and quantal version of grav-

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34 In (Mallios, 1998), this structure is called the ‘smooth and flat differential triad’. The stalks of this sheaf are isomorphs of $\Omega$ and the Kähler-Cartan differential $\partial$ effects (stalk-wise) vector sheaf morphisms of the following sort: $\partial : (X, \Omega^n) \to (X, \Omega^{n+1})$; where $(X, \Omega^n)$ is the vector subsheaf of $(X, \partial, \Omega)$ having as stalks isomorphs of the vector space $\Omega^n$ of $n$-forms, which is a vector subspace of $\Omega$.

35 These results were obtained for a paracompact and Hausdorff topological base space $X$ of the vector sheaves considered there. A space that is paracompact is akin to one that is finitary in our and Sorkin’s (1991) sense of the latter denomination (i.e., that it admits a locally finite open cover). Our $X$, which is such a finitary topological space, was assumed to be bounded (i.e., having compact closure), and in the usual topological parlance it is called ‘relatively compact’. $X$ was also seen to be $T_1$, but not $T_2$ (i.e., not Hausdorff). If we relax ‘paracompactness’ to ‘relative compactness’ and $T_2$ to $T_1$ for $X$, the essential results of Abstract Differential Geometry via Vector Sheaves still apply to our case (Mallios, private communication).

36 In the sense of (Raptis and Zapatrin, 2000).

37 Interpreted as the classical gravitational potential.
This project however, is still under development (Mallios and Raptis, 2000).

We conclude this paper by discussing its ‘general physico-philosophical moral’. By considering the finitary sheaves approximating $S(X)$ rather than directly the finitary spaces approximating $X$, we regard our observations of spacetime events as being more fundamental than the events themselves. This is the main lesson for physics to learn by applying differential geometry in the inherently algebraic language of sheaf theory, namely, that the physically significant concepts are less those about the ‘geometrical’ background spacetime $X$ and more those about our observations of this background which are organized into sheaves (of algebras) over $X$ (Mallios, 1998). This general principle that underlies the Abstract Differential Geometry via Vector Sheaves theory developed in (Mallios, 1998), is well in accord with the general philosophy of quantum theory holding that inert, background, geometrical ‘state spaces’, such as spacetime, ‘dissolve away’, so that what remains and is of physical significance, the ‘physically real’ so to speak, is (the algebraic mechanism of) our own actions of observing ‘it’ (Finkelstein, 1996).

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38 A ‘finitary quantum gravity’ so to speak.
39 To parallel Saunders Mac Lane’s mathematical motto ‘Every good function is a section of a sheaf’ (Mac Lane, 1986) in physics: ‘Every physically significant action is a section of the sheaf of our operations of observation of the system in focus’.
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