Improvement of the Staggered Fermion Operators

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Abstract

We present a complete and detailed derivation of the finite lattice spacing corrections to staggered fermion matrix elements. Expanding upon arguments of Sharpe, we explicitly implement the Symanzik improvement program demonstrating the absence of order $a$ terms in the Symanzik improved action. We propose a general program to improve fermion operators to remove $O(a)$ corrections from their matrix elements, and demonstrate this program for the examples of matrix elements of fermion bilinears and $B_K$. We find the former does have $O(a)$ corrections while the latter does not.
1 Introduction

With the occurrence of a new generation of teraflop parallel supercomputers, we will be able to simulate lattice QCD with smaller and smaller statistical errors. It is now of increased importance to gain control of various kinds of systematic errors which either affect the numerical results directly or affect the way in which physical quantities are extracted. One of the most important systematic errors comes from the finite lattice spacing \( a \) which generates errors of the order of \( a \Lambda_{QCD} \). For the present lattice computation, this corresponds to the corrections of the order of \( 20\% \sim 30\% \). Another important systematic error comes from the choice of lattice operators. There exist a variety of lattice operators which approach the same continuum operator in the limit \( a \to 0 \). However, many of these operators differ from the continuum limit at \( O(a) \), and a systematic formalism is needed to improve the lattice operators so as to remove these \( O(a) \) corrections.

For the case of Wilson fermions, the standard lattice action differs from the continuum quark action by a term of \( O(a) \). So, both the action and the operators need corrections at order \( a \). Applying the improvement program of Symanzik\[1\] to Wilson fermions, a procedure was proposed in ref.\[2\] \[3\] to reduce the systematic errors due to the finiteness of the lattice spacing, from terms of \( O(a) \) to ones of \( O(g^2_0 a) \), and it was numerically demonstrated in ref.\[4\] that this procedure can reduce the finite \( a \) corrections from 30\% to around 5\%.

The meaning of the statement that there is no term of order \( a \) in the staggered fermion action is not clear. Let us use the free staggered fermion action as an example:

\[
S_F = \sum_{x,\mu} a^4 \bar{\chi}(x)\eta_\mu(x) \frac{1}{2a} [\chi(x + \mu) - \chi(x - \mu)] + m \sum_x a^4 \bar{\chi}(x)\chi(x). \tag{1}
\]

Following Golterman and Smit \[5\], we denote the Fourier components of the fields \( \chi \) and \( \bar{\chi} \) as \( \tilde{\chi} \) and \( \tilde{\bar{\chi}} \), and decompose momentum space as

\[
k = p + \pi_A, \tag{2}
\]

where \( k_\mu \in (0, 2\pi/a), p_\mu \in (0, \pi/a), (\pi_A)_\mu = A_\mu \pi, \) in which \( A_\mu = 0, 1 \). If we define the fermion fields as

\[
\tilde{\psi}(p) = \frac{1}{8} \sum_{A,B} (-1)^{A+B} \gamma_A \tilde{\chi}(p + \pi_B), \tag{3}
\]

\[
\tilde{\bar{\psi}}(p) = \frac{1}{8} \sum_{A,B} (-1)^{A+B} \gamma_A^\dagger \tilde{\bar{\chi}}(p + \pi_B), \tag{4}
\]

where

\[
\gamma_A = \gamma_1 A_1 \gamma_2 A_2 \gamma_3 A_3 \gamma_4 A_4,
\]

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we can write the action as:

\[ S_F = \frac{(2\pi)^4}{\Omega} \sum_p \tilde{\psi}(p) \left( \sum_{\mu} \gamma_\mu \frac{i}{a} \sin p_\mu a + m \right) \tilde{\psi}(p). \quad (6) \]

where \( \Omega \) is the lattice volume. It is clear that there is no order \( a \) term in the action, hence the free staggered action is accurate to \( O(a^2) \). However, the coordinate fields corresponding to \( \tilde{\psi} (\tilde{\psi}) \) are non-local superpositions of the \( \chi \)'s (\( \bar{\chi} \)'s) over all the lattice sites.

On the other hand, if we define the local hypercubic fermion fields as in ref.\[11\]

\[ q(y) = \frac{1}{8} \sum_A \gamma_A \chi(y + A) = \frac{1}{2} \sum_A \gamma_A \chi_A(y), \quad (7) \]

\[ \bar{q}(y) = \frac{1}{8} \sum_A \bar{\chi}(y + A) \gamma_A^\dagger = \frac{1}{2} \sum_A \bar{\chi}_A(y) \gamma_A^\dagger, \quad (8) \]

where

\[ x = y + A, \quad y_\mu = 0, \pm 2, ... \quad (9) \]

then the action in the momentum space can be written as:

\[ S_F = \frac{(2\pi)^4}{\Omega} \sum_p \tilde{q}(p) \left\{ \sum_{\mu} \left( \gamma_\mu \otimes I \right) \frac{i}{2a} \sin p_\mu 2a + a(\gamma_5 \otimes \xi_5)_\mu \left( \frac{1}{a} \sin p_\mu a \right) + m \right\} \tilde{q}(p). \quad (10) \]

It is obvious that there are order \( a \) terms in the action and in the propagator for the fields \( q \) and \( \bar{q} \). In this case, we say that the fields \( q \) and \( \bar{q} \) need to be improved. There exists a set of improved fields

\[ \chi_A^I(y) = (1 - a \sum_\mu A_\mu \partial_\mu^L) \chi_A(y), \quad (11) \]

\[ \bar{\chi}_A^I(y) = \bar{\chi}_A(y) (1 - a \sum_\mu A_\mu \partial_\mu^{L^*}), \quad (12) \]

where

\[ \partial_\mu^L f(y) = \frac{1}{4a} [f(y + 2\mu) - f(y - 2\mu)], \quad (13) \]

such that

\[ S_F = \frac{(2\pi)^4}{\Omega} \sum_p \tilde{q}^I(p) \left( \sum_{\mu} \gamma_\mu \frac{i}{a} \sin p_\mu a + m \right) \tilde{q}^I(p) + O(a^2). \quad (14) \]

Note those improved fields are still local and superior to the nonlocal fields both computationally and theoretically when gauge couplings are included. So,
if we use the improved fields which remove the order $a$ terms from the action to construct a lattice fermion operator, there will be no $O(a)$ corrections to its free field matrix elements. For Landau gauge, Sharpe \cite{9, 10} proposed the following smeared operator:

$$
\chi_A(y)^{\text{smeared}} = \frac{1}{4} \sum_\nu \chi_A(y + 2\nu[1 - 2A_\nu]). \quad (15)
$$

It is easy to show that

$$
\chi_A(y)^{\text{smeared}} = \chi_A(y) + O(a^2). \quad (16)
$$

The full staggered fermion action including gauge couplings is much more complicated. In section 2, we will give a set of improved fermion field variables in terms of which the action has no explicit order $a$ terms at tree level. In section 3, we will expand upon the argument given by Sharpe in ref. \cite{6} to prove that there are no $O(a)$ terms which can be added to the staggered fermion action. Based on these two arguments, we conclude that staggered fermion action is already accurate to $O(a^2)$, and that we should use the improved field variables to construct fermion operators to reduce order $a$ corrections from their matrix elements. We apply this program to the case of $<0|\bar{s}\gamma_5 d|K>$ and $B_K$ as examples. We will also determine the additional operators that must be added to improve the standard staggered fermion currents to define operators whose matrix elements are accurate to $O(a^2)$. We list the lattice symmetry transformation properties of the fermion fields needed in this paper in the Appendix A.

## 2 Improving the fermion fields

Working with the even position variable $y$ of Eq.(9), the lattice gauge-covariant derivative is defined as follows:

$$
D_\mu^L f(y) = \frac{1}{4a} [U_\mu(y)U_\mu(y + \mu)f(y + 2\mu) - U_\mu(y - \mu)U_\mu(y - 2\mu)f(y - 2\mu)]. \quad (17)
$$

Following ref. \cite{11}, we define the gauge covariant fermion fields as

$$
\varphi_A(y) = U_A(y)\chi_A(y), \quad \bar{\varphi}_A(y) = \bar{\chi}_A(y)U_A^\dagger(y), \quad (18)
$$

where $U_A(y)$ is the average of link products along the shortest paths from $y$ to $y + A$. The classical continuum limit of action can be written as:

$$
S_F[\varphi, \bar{\varphi}] \rightarrow \int [\sum y_{AB} \varphi_A(y) \{ \sum_\mu (\gamma_\mu \otimes I)_{AB} D_\mu + m\delta_{AB} - a(\sum_\mu (\gamma_5 \otimes \xi_{5\mu})_{AB} D_\mu^2
$$

3
\[ S_F[\varphi^I, \bar{\varphi}^J] = \int_y \sum_{A} \bar{\varphi}^I_A(y) \sum_{\mu} (\bar{\gamma}_\mu \otimes \bar{I})_{A BA} D_\mu + m\delta_{AB} \varphi^J_B(y) + O(a^2), \]

where \( D_\mu = \partial_\mu + igA_\mu \) is the continuum covariant derivative. At first sight, the action contains order \( a \) terms. Likewise, it is clear that the fermion propagator for the hypercubic fields \( \varphi \) and \( \bar{\varphi} \) deviates from the continuum propagator by terms of order \( a \). However, if we introduce the following improved field variables

\[
\chi^I_A(y) = (1 - a \sum_\nu A_\nu D^L_\nu) \chi_A(y), \tag{20}
\]

\[
\bar{\chi}^I_A(y) = \bar{\chi}_A(y)(1 - a \sum_\nu A_\nu \bar{D}^L_\nu), \tag{21}
\]

and replace \( \chi, \bar{\chi} \) in Eq.\[18] by \( \chi^I \) and \( \bar{\chi}^I \), then the classical continuum limit of the action can be written as

\[
S_F[\varphi^I, \bar{\varphi}^J] = \int_y \sum_{AB} \bar{\varphi}^I_A(y) \sum_{\mu} (\bar{\gamma}_\mu \otimes \bar{I})_{AB} D_\mu + m\delta_{AB} \varphi^J_B(y) + O(a^2), \tag{22}
\]

We see the Feynman rules of the improved fields differ from those of the continuum theory by terms of \( O(a^2) \). In ref.\[4\] it was proven that if the matrix elements of the tree level improved operators differ from the continuum ones only by terms which are of \( O(a^2) \), then in the full theory there are no terms of \( O(a) \) nor terms of \( O(\alpha g_0^2 \ln a) \). We restate their argument here. Since the lowest order correction to the propagator and vertices are of order \( a^2 \), a term proportional to \( \alpha g_0^2 \ln a \) from an n-loop diagram can only occur if one of the loop integrals diverges like \( 1/a \). If this is true, there will be at most \( n-1 \) logarithmically divergent loop integrals which will result in a term of \( O(\alpha g_0^2 \ln a^{n-1}) \) which behaves like \( O(\alpha g_0^2) \) as \( a \to 0 \). So we conclude that there are no terms of \( O(\alpha g_0^2 \ln a^{n-1}) \), and therefore the matrix elements differ from their continuum ones by terms at most of \( O(\alpha g_0^2) \).

Using the new fermion fields, we can construct improved fermion operators. For example, the improved fermion bilinears have the following form:

\[
\bar{\chi}^I_A(y)(\bar{\gamma}_S \otimes \bar{\xi}_F)_{AB} \chi^J_B(y) = \bar{\chi}_A(y)(\bar{\gamma}_S \otimes \bar{\xi}_F)_{AB} \chi_B(y)
- \frac{a}{2} \sum_\nu \partial^L_\nu [\bar{\chi}_A(y)(\bar{\gamma}_S \otimes \bar{\xi}_F)_{AB} - (\bar{\gamma}_{5\nu} \otimes \bar{\xi}_{5\nu} F)_{AB} \chi_B(y)]
- \frac{a}{2} \sum_\nu \bar{\chi}_A(y)(\bar{\gamma}_{5\nu} \otimes \bar{\xi}_{5\nu} F)_{AB} - (\bar{\gamma}_{S5\nu} \otimes \bar{\xi}_{S5\nu} F)_{AB} D^L_\nu \chi_B(y)
+ O(a^2). \tag{23}
\]
3 Improving the staggered fermion action

In contrast to the calculation of matrix elements, the action is already accurate through order $a$ to all orders in $g_0^2$, as we will now discuss. Thus physical quantities that depend only on the form of the action (for example, particle masses determined from correlation functions) will have no corrections of order $g_0^2a$. This can be demonstrated by recognizing that if there were a correction of order $g_0^2a$, we must necessarily be able to add some dimension-5 operator \( a \sum_i c_i g_0^2 O_i^{(5)} \) which must be invariant under the lattice symmetry transformations to cancel this order $g_0^2a$ correction (see ref.[6, 7]). However, following ref.[6], we will now prove that there exists no dimension-5 operator (for the definition of the dimension of lattice operators, see ref.[7]) which is invariant under the lattice symmetry group (rotations, axis reversal, translations, $U(1) \otimes U(1)$, charge conjugation), and therefore, no order $a$ term can be added to the staggered fermion action.

Following standard notation, we rewrite the staggered fermion action as

\[
S_f = (2a)^4 \sum_{y, y'} \sum_{A, B} \overline{\chi}_A(y)\gamma^a(\sum_{\mu} (\gamma_\mu \otimes I)_{AB} D_\mu(y, y')_{BC}) \\
+ m \delta(y - y')\delta_{AC} \chi(y') U(y + A, y' + C)^{ab},
\]

(24)

where

\[
(\overline{\gamma}_S \otimes \xi_F)_{AB} = \frac{1}{4} Tr(\gamma_A^I \gamma_S \gamma_B^I) ;
\]

(25)

\[
D_\mu(y, y')_{AB} = \overline{D}_\mu(y, y')\delta_{AB} + a \overline{\Delta}_\mu(y, y') (\gamma_{\mu 5} \otimes \xi_{5\mu})_{AB},
\]

(26)

in which

\[
\overline{D}_\mu(y, y') = \frac{1}{4a^2} [\delta(y + 2\mu - y') - \delta(y - 2\mu - y')],
\]

\[
\overline{\Delta}_\mu(y, y') = \frac{1}{4a^2} [\delta(y + 2\mu - y') + \delta(y - 2\mu - y') - 2\delta(y - y')],
\]

For convenience, we will not write out the $SU(3)$ links explicitly in the remainder of this section unless there would otherwise be confusion. Given an operator, the reader can write out the full form very easily. For example, starting with the operator \( \sum_\mu \overline{\chi}_A(y) (\gamma_5 \otimes \xi_{5\mu}) D_\mu \chi \) we would construct the corresponding gauge invariant operator by the following substitution:

\[
\sum_{y', y''} \sum_\mu \overline{\chi}_A(y) (\gamma_5 \otimes \xi_{5\mu})_{AB} D_\mu(y, y')_{BC} D_\mu(y'y'')_{CD} \chi_D(y'') \rightarrow \\
\sum_{y', y''} \sum_\mu \overline{\chi}_A(y) (\gamma_5 \otimes \xi_{5\mu})_{AB} U(y + A, y + B) D_\mu(y, y')_{BC} U(y' + B, y' + C)
\]

\[
\times D_\mu(y', y'')_{CD} U(y' + C, y'' + D) \chi_D(y'').
\]

(27)

where $U(y + A, y + B)$ is the average of the products of link matrices corresponding to each of the shortest paths from point $y + A$ to $y + B$. 

5
Using the transformation properties of the staggered fermion action (see Appendix A in detail), we try to construct all symmetrical dimension-5 operators which have the general form $\bar{\chi}\gamma_5S \otimes \xi_F f(D, \nabla)\chi$ where $f$ is a homogenous real polynimial of degree 2.

Invariance under $U_A(1)$ requires that $S + F$ is odd, so only the following combinations of $S \otimes F$ are valid:

\[(I, \gamma_5, \gamma_5\gamma_5 \mu \nu) \otimes (\xi_5, \xi_{5\lambda}),\]  
\[(\gamma_5, \gamma_5\gamma_5 \mu) \otimes (I, \xi_5, \xi_{5\lambda\tau}, \xi_{5\lambda\tau}).\]  

Under reflection with respect to a hyperplane normal to the $\rho$ direction, we have the following transformation:

\[\chi \rightarrow I_\rho \chi,\]  
\[\bar{\chi} \rightarrow \bar{\chi} I_{-1\rho},\]

and

\[\nabla_\mu \rightarrow (1 - 2\delta_{\mu\rho})J_\mu \nabla_\rho I_{-1\rho},\]  
\[\nabla_\mu \rightarrow J_\mu \nabla_\rho I_{-1\rho},\]  
\[D_\mu \rightarrow (1 - 2\delta_{\mu\rho})J_\mu D_\rho I_{-1\rho}.\]

Using the transformation formulae of $\gamma_5 \otimes \xi_F$ listed in Eq. (79) of the Appendix, we deduce that axis reversal invariance and $U_A(1)$ invariance allow only the following terms:

\[\gamma_5 \otimes \xi_5 D_\mu^2,\]  
\[\gamma_5 \otimes \xi_5 D_\mu^2,\]  
\[\gamma_5[\gamma_5, \gamma_\nu] \otimes \xi_5(\xi_5 + \xi_\nu)[D_\mu, D_\nu],\]  
\[\gamma_5[\gamma_5, \gamma_\nu] \otimes \xi_5(\xi_5 - \xi_\nu)[D_\mu, D_\nu].\]

where $D_\mu, D_\nu$ can be replaced by $\nabla_\mu, \nabla_\nu$ without affecting these operators up to order $a^2$.

Under a rotation around the center of a hypercube, we have the following:

\[\chi \rightarrow R^{(\sigma\rho)}\chi,\]  
\[\bar{\chi} \rightarrow \bar{\chi} R^{(\sigma\rho)}^{-1},\]  
\[\nabla_\mu \rightarrow R^{(\sigma\rho)} R_{\mu\nu} \nabla_\nu R^{(\sigma\rho)}^{-1}\]  
\[\nabla_\mu \rightarrow R^{(\sigma\rho)} |R_{\mu\nu}| \nabla_\nu R^{(\sigma\rho)}^{-1}\]  
\[D_\mu \rightarrow R^{(\sigma\rho)} R_{\mu\nu} D_\nu R^{(\sigma\rho)}^{-1}\]  
\[D_\mu \rightarrow R^{(\sigma\rho)} R_{\mu\nu} D_\nu R^{(\sigma\rho)}^{-1}\]
Combining the transformation properties listed in Eq. (84) of the Appendix, the rotational invariance will further eliminate the term in Eq. (35) but allows the remaining three terms Eq. (36 - 38).

Finally, let’s discuss invariances under translation by one lattice unit:

\[ \chi \rightarrow S^{\rho}(\chi), \]
\[ \bar{\chi} \rightarrow \bar{\chi} S^{\rho -1}, \]
\[ D_{\mu} \rightarrow S^{\rho} D_{\mu} S^{\rho -1}, \]
\[ \Delta_{\mu} \rightarrow S^{\rho} \Delta_{\mu} S^{\rho -1}, \]
\[ D_{\mu} \rightarrow S^{\rho} \Lambda_{\mu}^{\rho -1} D_{\mu} S^{\rho -1}, \]

where

\[ \Lambda_{S\otimes F}(y, y')_{AB} = \epsilon(F) \left\{ (-1)^{F_{\rho}} \delta_{AB} \delta(y - y') + a\left[ (-1)^{F_{\rho}} - (-1)^{S_{\rho}} \right] \right\} \]
\[ \times [a \delta_{AB} \bar{\Delta}_{\rho}(y, y') + \overline{\left( \gamma_{5\rho} \otimes \xi_{5\rho} \right)_{AB} D_{\rho}(y, y')}], \]
\[ \Lambda_{S\otimes F}^{\rho -1}(y, y')_{AB} = \epsilon(F) \left\{ (-1)^{F_{\rho}} \delta_{AB} \delta(y - y') + a\left[ (-1)^{F_{\rho}} - (-1)^{S_{\rho}} \right] \right\} \]
\[ \times [a \delta_{AB} \bar{\Delta}_{\rho}(y, y') - \overline{\left( \gamma_{5\rho} \otimes \xi_{5\rho} \right)_{AB} D_{\rho}(y, y')}], \]

and

\[ S^{\rho -1} \bar{\gamma}_{S} \otimes \xi_{F} S^{\rho} = \bar{\gamma}_{S} \otimes \xi_{F} \Lambda_{S\otimes F}^{\rho}. \]

From these properties, we can see that none of the terms listed in Eq. (36 - 38) are invariant under lattice translation! So, we conclude that there is no dimension-5 fermion operator which is invariant under the lattice symmetry group, and therefore no dimension-5 operator can be added to the staggered fermion action.

4 Applications:

As we argued above, actual numerical simulation should use the improved fermion field variables. However, in most situations, we can use the improvement program proposed in this paper to remove the \( O(a) \) corrections without increasing the computational work. Here, we apply this program to the calculation of the matrix element \( \langle 0| \bar{s} \gamma_{5d} | K^0 \rangle \) which gives \( f_K \) in the continuum, and the calculation of \( B_K \). We will show that the former differs from its continuum counterpart by \( O(m_{\pi} a) \), but \( B_K \) has no \( O(a) \) corrections. We also apply the improvement program to the matrix elements of lattice currents.
4.1 $< 0|s\gamma_{54}d|K^0 >$:

The axial current used in the (Landau gauge) numerical simulation is:

$$A_\mu(y) = \sum_{AB} \bar{\chi}_A(\gamma_5 \otimes \xi_5)_{AB} \chi_B(y). \quad (53)$$

From the continuum expression

$$P(t)^{cont} = <0|A_4(t)^{cont}|K^0 > = \sqrt{2} f_K m_Ke^{-m_k |t|}, \quad (54)$$

we define, on the lattice,

$$P(t) = <0|\sum_{\vec{x}} A_4(\vec{x},t)|K^0 >, \quad (55)$$

and put the wall source that creates the $K^0$ on the time slice at $t = 0$. Then we will have

$$P(t) = \begin{cases} \sqrt{2} f_K^+ m_Ke^{-m_k |t|}, & (t > 0) \\ \sqrt{2} f_K^- m_Ke^{-m_k |t|}, & (t < 0) \end{cases} \quad (56)$$

where

$$f_K^\pm = f_K \pm O(m_ka). \quad (57)$$

If we don’t consider $O(g_0^2a)$ terms, we can take only the term

$$-\frac{a}{2} \sum_{\nu} \partial_\nu [\bar{\chi}_A(\gamma_{54} \otimes \xi_5)_{AB} \chi_B]$$

in Eq. (23) because other terms contribute zero “flavor” trace at the tree level. So, we have

$$P(t)^{imp} = P(t) - \frac{a}{2} \partial_t^\nu P(t) = \begin{cases} \sqrt{2} f_K^{+,imp} m_Ke^{-m_k |t|}, & (t > 0) \\ \sqrt{2} f_K^{-,imp} m_Ke^{-m_k |t|}, & (t < 0) \end{cases} \quad (58)$$

and

$$f_K^{+,imp} = (1 + \frac{1}{2} m_ka) f_K^+ + O(a^2), \quad (59)$$
$$f_K^{-,imp} = (1 - \frac{1}{2} m_ka) f_K^- + O(a^2), \quad (60)$$
$$f_K^{\pm,imp} = f_K \pm O(a^2). \quad (61)$$

So, we get

$$f_K^\pm = (1 \mp \frac{1}{2} m_ka) f_K + O(a^2). \quad (62)$$
Figure 1: The value of $\ln(<0|A_4(t)|K^0>)$ with respect to the time $t$. Where $m_d a = m_s a = 0.01$. Calculated with the cubic wall source method.

The numerical data (from the full QCD simulation on a $16^3 \times 40$ lattice with the cubic wall source, at the quark mass $m_s a = m_d a = 0.01$, see ref. [12]) for the unimproved and improved matrix elements are shown in Figure 1, from which we see that $(f^- - f^+)/f_K \approx m_k a \sim 25\%$ and $(f^{Imp}_K - f^{Imp+}_K)/f_K \sim 5\%$ is much smaller.

From this simple example, we can see that if we do not consider $O(g^2 a)$ corrections, the improved operator is equivalent to the extrapolation:

$$P(t) = \sqrt{2} f_k^{Imp} m_k e^{-m_k |t+1|/2},$$

(63)

4.2 $B_K$

The formula for calculating $B_K$ is:

$$B_K = \frac{\mathcal{M}_K}{\hat{s}^2 \mathcal{M}_K^K},$$

(64)

where

$$\mathcal{M}_K = <K^0|\bar{s}\gamma_\mu(1 + \gamma_5)d\bar{s}\gamma_\mu(1 + \gamma_5)d|K^0 >.$$

(65)
\[ \mathcal{M}_K^V = \langle K^0 | \bar{s} \gamma_4 \gamma_5 d | 0 \rangle \langle 0 | \bar{s} \gamma_4 \gamma_5 d | K^0 \rangle. \] (66)

The improved numerator is (omitting the \(O(g_0^2 a)\) terms):
\[ \mathcal{M}_K^{imp} = \mathcal{M}_K - \frac{a}{2} \partial_\mu \mathcal{M}_K + O(a^2). \] (67)

Since \(\mathcal{M}_K(t)\) is a plateau (i.e. time independent) within the statistical error, there is no \(O(a)\) corrections to the numerator.

Since \(f_K^+ f_K^- = f_K^2 + O(a^2)\),
\[ (68) \]
the denominator \(\mathcal{M}_K^V\) also has no \(O(a)\) corrections even if no attention is paid to an accurate definition of \(f_K\).

Hence, we showed that there is neither \(O(a)\) nor \(O(g_0^2 a \log n)\) corrections to \(B_K\). Sharpe [6] has examined this question in greater detail and argued that in fact there are no corrections of \(O(g_0^2 a)\) also.

However, if we calculated the denominator only in one time direction and took the square of \(f_K^+\) (or \(f_K^-\)), there would be an error of order of \(O(m_K a)\).

4.3 Renormalization of lattice currents:

The lattice currents can be written as
\[ J_{\text{lat}}^F = \bar{\chi} \gamma_j \otimes \xi^F \chi, \] (69)
and according to ref [13], their renormalized continuum forms can be written as
\[ J_{\text{cont}}^F = Z_j \kappa_j^F J_{\text{lat}}^F, \] (70)
where \(Z_j\) is the usual (divergent) renormalization constant and \(\kappa_j^F\) is a finite lattice renormalization constant. Using the method developed in this paper, we can explicitly determine the improved currents accurate to \(O(a^2)\). For example, the conserved vector current and axial vector current corresponding to the \(U_V(1) \otimes U_A(1)\) lattice symmetry can be written as follows:

\[ V_\mu^I(y) = V_\mu(y) - \frac{a}{2} \sum_\nu \partial_\nu [\bar{\chi}_A (\gamma_\mu \otimes T)_{AB} \chi_B] \]
\[ - \frac{a}{2} \partial_\mu [\bar{\chi}_A (\gamma_5 \otimes T)_{AB} \chi_B] \]
\[ - \frac{a}{4} \sum_\nu \partial_\nu [\bar{\chi}_A (\gamma_5 [\mu, \nu] \otimes 5_{A B} \chi_B] + O(a^2), \] (71)

\[ A_{\mu, F}^I(y) = A_{\mu, F}^I(y) - \frac{a}{2} \sum_\nu \partial_\nu [\bar{\chi}_A (\gamma_5 \otimes 5)_{AB} \chi_B] \]
\[ - \frac{a}{2} \partial_\mu [\bar{\chi}_A (\gamma_5 \otimes 5)_{AB} \chi_B] \]
\[ + \frac{a}{4} \sum_\nu \partial_\nu [\bar{\chi}_A (\gamma_5 [\mu, \nu] \otimes 5)_{AB} \chi_B] + O(a^2). \] (72)
The effect of the second term on the right hand side is to shift the position of the current, from the corner to the center of the hypercube. The third term whose effect is to shift in the $\mu$’s direction occurs here because the currents are non-local operators which involve an overlap between two nearest hypercubes. The forth term is a mixing of a different spin-flavor operator and is necessary to remove all order $a$ effects from a general matrix element.

5 Summary

In this paper, based on the demonstration of that there is no dimension-5 fermion operator which is invariant under all lattice symmetry transformations and that there exists a set of improved fermion fields with respect to which the tree-level action has no order $a$ terms, we concluded that the staggered fermion action is already in fact improved to $O(a^2)$. We argued that to remove order $a$ corrections from the matrix elements, the first step is to use the proposed improved fermion field variables to construct fermion operators so that they differ from the continuum ones by order of $a^2$ at the tree level. Then we showed that to all orders of perturbation theory treating $g_0^2 \ln a \sim O(1)$, the correction is at most of $O(g_0^2 a)$. Furthermore, for the on-shell quantities, they are accurate to $O(a^2)$. We applied our program to the matrix element $<0|\bar{s}_5\gamma_5 d|K^0>$ and found that the unimproved one differs from the continuum by a factor of $O(m_ka)$. At the same time, we showed that there is no $O(a)$ corrections to $B_K$, which is consistent with the result of Sharpe [6]. We also discussed the matrix elements of the lattice currents, and obtained the explicit terms which should be added to the original current operators to define improved operators accurate through $O(a)$.

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A Symmetry Properties of Staggered Fermion

For completeness, we collect some formulae connected with the transformation properties of the staggered fermion under the lattice symmetry group from
A.1 \( U(1)_A \):

\[
\chi_A(y) \to e^{i\alpha \varepsilon(A)} \chi_A(y) \\
\bar{\chi}_A(y) \to e^{i\alpha \varepsilon(A)} \bar{\chi}_A(y)
\]

where

\[
\varepsilon(A) = (-1)\sum_\mu A_\mu
\]

A.2 Reflection with respect to a Hyperplane:

\[
I_\rho^\mu : \begin{cases} 
  x'_\rho = -x_\rho + 1, \\
  x'_\mu = x_\mu \quad (\mu \neq \rho)
\end{cases}
\]

The transformation of the fermion fields are:

\[
\chi_A(y) \to \sum_B \sum_{y'} \mathcal{I}_\rho(y, y')_{AB} \chi_B(y'),
\]

\[
\bar{\chi}_A(y) \to \sum_B \sum_{y'} \bar{\chi}_B(y') \mathcal{I}_\rho^{-1}(y', y)_BA,
\]

where

\[
\mathcal{I}_\rho(y, y')_{AB} = \overline{(\gamma_\rho \otimes \xi_5)_{AB}} \delta(I_\rho y - y'),
\]

\[
\mathcal{I}_\rho^{-1}(y, y')_{AB} = (\gamma_\rho \otimes \xi_5)_{AB} \delta(I_\rho y - y'),
\]

and

\[
(I_\rho y)_\mu = \begin{cases} 
  y_\mu, \quad (\mu \neq \rho) \\
  -y_\rho, \quad (\mu = \rho)
\end{cases}
\]

The spin-flavor matrices transform as:

\[
(\gamma_S \otimes \xi_F) \to (\gamma_\rho \otimes \xi_5)(\gamma_S \otimes \xi_F)(\gamma_5 \otimes \xi_5),
\]

which are explicitly listed as following:

| \( S \) | \( F \) |
| --- | --- |
| \( I \to I \) | \( I \to I \) |
| \( \gamma_5 \to -\gamma_5 \) | \( \xi_5 \to \xi_5 \) |
| \( \gamma_\mu \to (1 - 2\delta_{\rho\mu})\gamma_\mu \) | \( \xi_\mu \to -\xi_\mu \) |
| \( \gamma_{5\mu} \to -(1 - 2\delta_{\rho\mu})\gamma_{5\mu} \) | \( \xi_{5\mu} \to -\xi_{5\mu} \) |
| \( \gamma_{\mu\nu} \to (1 - 2\delta_{\rho\mu})(1 - 2\delta_{\rho\nu})\gamma_{\mu\nu} \) | \( \xi_{\mu\nu} \to \xi_{\mu\nu} \) |
| \( \gamma_{5\mu\nu} \to -(1 - 2\delta_{\rho\mu})(1 - 2\delta_{\rho\nu})\gamma_{5\mu\nu} \) | \( \xi_{5\mu\nu} \to \xi_{5\mu\nu} \) |
A.3 Rotations by $\pi/2$ arround the center of a hyperplane:

\[
R_{H}^{(\rho\sigma)} : \begin{cases} 
  x'_{\rho} = x_{\sigma}, \\
  x'_{\sigma} = -x_{\rho} + 1, \\
  x'_{\mu} = x_{\mu}, \quad (\mu \neq \rho, \sigma)
\end{cases}
\]

Define $R_{\mu
u}$ such that

\[
(Ry)_{\mu} = R_{\mu
u}y_{\nu} = \begin{cases} 
  y_{\sigma}, & (\mu = \rho) \\
  -y_{\rho}, & (\mu = \sigma) \\
  y_{\mu}, & (\mu \neq \rho, \sigma)
\end{cases}
\]

Then, we have the following transformation:

\[
\chi_{A}(y) \rightarrow \sum_{B} \sum_{y'} R_{AB}^{(\rho\sigma)}(y, y') \chi_{B}(y'), \quad (80)
\]

\[
\bar{\chi}_{A}(y) \rightarrow \sum_{B} \sum_{y'} \bar{\chi}_{B}(y') R_{AB}^{(\rho\sigma)}(y, y')^{-1}, \quad (81)
\]

where

\[
R_{AB}^{(\rho\sigma)}(y, y') = \frac{1}{2} \left( (1 - \gamma_{\rho\sigma}) \otimes (\xi_{\sigma} - \xi_{\rho}) \right)_{AB} \delta(R^{-1}y - y'), \quad (82)
\]

\[
R_{AB}^{(\rho\sigma)}(y, y')^{-1} = \frac{1}{2} \left( (1 + \gamma_{\rho\sigma}) \otimes (\xi_{\sigma} - \xi_{\rho}) \right)_{AB} \delta(Ry - y'), \quad (83)
\]

The spin-flavor matrices transform as:

\[
(\gamma_{S} \otimes \xi_{F}) \rightarrow \frac{1}{2} \left( (1 - \gamma_{\rho\sigma}) \otimes (\xi_{\sigma} - \xi_{\rho}) \right) (\gamma_{S} \otimes \xi_{F}) \frac{1}{2} \left( (1 + \gamma_{\rho\sigma}) \otimes (\xi_{\sigma} - \xi_{\rho}) \right). \quad (84)
\]

which are explicitly listed as following:

| $S$ | $F$ |
|-----|-----|
| $I \rightarrow I$ | $I \rightarrow I$ |
| $\gamma_{5} \rightarrow \gamma_{5}$ | $\xi_{5} \rightarrow -\xi_{5}$ |
| $\gamma_{\mu} \rightarrow R_{\mu\lambda}\gamma_{\lambda}$ | $\xi_{\mu} \rightarrow -|R_{\mu\lambda}|\xi_{\lambda}$ |
| $\gamma_{5\mu} \rightarrow R_{\mu\lambda}\gamma_{5\lambda}$ | $\xi_{5\mu} \rightarrow |R_{\mu\lambda}|\xi_{5\lambda}$ |
| $\gamma_{\mu\nu} \rightarrow R_{\mu\lambda}R_{\nu\tau}\gamma_{\lambda\tau}$ | $\xi_{\mu\nu} \rightarrow |R_{\mu\lambda}|R_{\nu\tau}\xi_{\lambda\tau}$ |
| $\gamma_{5\mu\nu} \rightarrow R_{\mu\lambda}R_{\nu\tau}\gamma_{5\lambda\tau}$ | $\xi_{5\mu\nu} \rightarrow -|R_{\mu\lambda}|R_{\nu\tau}|\xi_{5\lambda\tau}$ |

A.4 Translations by one lattice unit:

\[
T_{\rho} : \begin{cases} 
  x'_{\rho} = x_{\rho} + 1, \\
  x'_{\mu} = x_{\mu}, \quad (\mu \neq \rho)
\end{cases}
\]
lead to the following transformation on the $\chi_A$, $\bar{\chi}_A$’s:

$$\chi_A(y) \to \sum_B \sum_{y'} S^{(\rho)}(y, y')_{AB} \chi_B(y'), \quad (85)$$

$$\bar{\chi}_A(y) \to \sum_B \sum_{y'} \bar{\chi}_B(y') S^{(\rho)}(y, y')^{-1}_{AB}, \quad (86)$$

where

$$S^{(\rho)}(y, y')_{AB} = \frac{1}{2}([I \otimes \xi_{\rho} - \gamma_{\rho 5} \otimes \xi_5]_{AB} \delta(y - y')$$

$$+ (I \otimes \xi_{\rho} + \gamma_{\rho 5} \otimes \xi_5)_{AB} \delta(y + 2\rho - y')], \quad (87)$$

$$S^{(\rho)}(y, y')^{-1}_{AB} = \frac{1}{2}([I \otimes \xi_{\rho} - \gamma_{\rho 5} \otimes \xi_5]_{AB} \delta(y - 2\rho - y')$$

$$+ (I \otimes \xi_{\rho} + \gamma_{\rho 5} \otimes \xi_5)_{AB} \delta(y - y')], \quad (88)$$

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