Hydrodynamic Limit of the Kawasaki Dynamics on the 1d-lattice with Strong, Finite-Range Interaction

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Abstract. We derive the hydrodynamic limit of the Kawasaki dynamics for the one-dimensional conservative system of unbounded real-valued spins with arbitrary strong, quadratic and finite-range interactions. This significantly extends prior results for bounded interaction by Rezakhanlou and complements results obtained by H.T. Yau. The result is obtained by adapting two-scale approach of Grunewald, Otto, Villani and Westdickenberg combined with the authors’ recent approach on conservative systems with strong interactions.

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1. Introduction

The broader scope of this article is the study of the continuum approximations of large discrete systems. Since the fundamental observation of Boltzmann that large particle systems in equilibrium are governed by Gibbs states, understanding the connection between discrete systems and their approximation to the continuum has been one of the main challenges in statistical physics. One of the most actively studied problem in this field is the hydrodynamic limit, which can be thought as a dynamical version of law of large numbers. This means that in a proper time and space macroscopic scales, the random evolution of microscopic system can be macroscopically described by a solution of deterministic partial differential equations.

Since 1980s, the hydrodynamic limit has been established in several settings. An interesting and crucial model is the Kawasaki dynamics, a stochastic dynamics which preserves the mean spin of the system (cf. [8]). Several methods have been developed to establish the hydrodynamic limits of Kawasaki
dynamics. In [10,32], authors developed the martingale approach and entropy method, respectively, to analyze the Kawasaki dynamics of continuous spin systems. In addition, in [31], the similar hydrodynamic limit was established for the discrete spin systems. More recently, Grunewald, Otto, Villani and Westdickenberg presented the two-scale method (cf. [9]). For more details, we refer to [9] or the classical reference on hydrodynamic limits [14].

To understand the difficulties and obstacles when studying the hydrodynamic limit, let us introduce the notion of grand canonical and canonical ensemble. The grand canonical ensemble $\mu_N$ is a probability measure on $\mathbb{R}^N$ given by

$$\mu_N(dx) := \frac{1}{Z} \exp(-H(x)) \, dx.$$ 

Here, $Z$ denotes a generic normalization constant and $H$ is the Hamiltonian of the system. Let us consider the $N-1$-dimensional hyperplane $X_{N,m}$ given by

$$X_{N,m} := \left\{ x \in \mathbb{R}^N : \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\} \subset \mathbb{R}^N.$$ 

The canonical ensemble $\mu_{N,m}$ is the restriction of $\mu_N$ to the $N-1$-dimensional hyperplane $X_{N,m}$:

$$\mu_{N,m}(dx) := \mu_N \left( dx \mid \frac{1}{N} \sum_{i=1}^{N} x_i = m \right) = \frac{1}{Z} \mathbb{1}_{\left\{ \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\}}(x) \exp(-H(x)) \mathcal{L}^{N-1}(dx),$$

where $\mathcal{L}^{N-1}(dx)$ denotes the $N-1$-dimensional Hausdorff measure restricted to the hyperplane $X_{N,m}$. In particular, the canonical ensemble $\mu_{N,m}$ is a stationary distribution of the Kawasaki dynamics.

Recalling that the hydrodynamic limit can be understood as a dynamical version of law of large numbers, it is obvious that the problem becomes relatively much easier if the underlying stationary distribution is a product measure; because this leads to independence of random variables. If the Hamiltonian $H$ is non-interacting, we observe that the grand canonical ensemble is a product measure, whereas the canonical ensemble is not due to the restriction to a hyperplane. This makes deducing hydrodynamic limit for the Kawasaki dynamics non-trivial even in the non-interactive Hamiltonian case. However, given the equivalence of ensembles (cf. [15]) meaning that the canonical ensemble is equivalent to properly modified grand canonical ensemble, it is not surprising that one is able to deduce the hydrodynamic limit of Kawasaki dynamics in the case of a non-interactive Hamiltonian.

The problem becomes a lot more subtle if interactions are present between spins within the Hamiltonian $H$. In this case, even the grand canonical ensemble is not a product measure, deducing hydrodynamic limit for the Kawasaki dynamics becomes even more challenging. Another difficulty is the lack of understanding of the canonical ensemble with interacting Hamiltonian.
Therefore, even though the case of the absence of interactions was tackled in 1980s (e.g., [8,10]), up to authors’ knowledge, there are only few known hydrodynamic results of Kawasaki dynamics for interacting lattice systems. In [29], the hydrodynamic limit was deduced for the translation-invariant and bounded interactions. Also, in [32], the model of quadratic interactions with super-quadratic single-site potentials was considered. It is crucial to note that in both [29] and [32], an important condition on the interaction is that it is strictly dominated by the single-site potentials.

In this article, we first analyze the hydrodynamic limit of Kawasaki dynamics when the interaction and self-potential possess the same growth. Precisely, we consider a (perturbed) quadratic single-site potential with strong quadratic interactions beyond the perturbative regime. We emphasize that our condition on the Hamiltonian makes deducing the hydrodynamic limit much harder, as a standard cutoff approach does not work. In fact, arguments in [29,32] are not applicable, since a strong dominance of the interaction by single-site potentials plays a crucial role to prove a local ergodic theorem. To the best of authors’ knowledge, our result is the first rigorous derivation of the hydrodynamic limit of Kawasaki dynamics for which the interaction is not dominated by the single-site potential but of the same order.

Let us also mention that in [29], the translation invariance of the interaction was crucially used to deduce a large deviation for the empirical measure under the grand canonical ensemble. Our argument does not heavily rely on the assumption of translation invariance. We only assume it for a convenience simplifying some calculations when analyzing the convergence of the free energies.

Another distinguishing characteristic between [10,29,32] and this article is that we work with the canonical ensemble instead of the grand canonical ensemble. This is makes a crucial difference on assumptions of initial configuration of the Kawasaki dynamics. In [10] and [29], the entropy bound of initial profile with respect to the grand canonical ensemble was assumed. In [32], it is assumed that the initial state is close to the local Gibbs state. This excludes initial conditions with a fixed mean spin because a measure defined on a hyperplane is singular to the grand canonical ensemble and any local Gibbs state. In this manuscript, we assume the entropy bound of initial configuration with respect to the canonical ensemble. This allows initial configurations with fixed mean spin. Therefore, in contrast to the approaches of [10,29,32], it might even be possible to extend our method to allow deterministic initial conditions. The reason is that one expects that the relative entropy of the Kawasaki dynamics with respect to the canonical ensemble becomes of order $N$ within finite time (see, e.g., [19,23]). This suggests that it is feasible to relax our assumption on the initial condition allowing deterministic state.

Our approach to the hydrodynamic limit is also different from [29,32] in terms of the mode of convergence. In [29], the dynamics is shown to converge in probability, and in [32], the convergence of relative entropy with respect to local Gibbs state is established. Because we follow the framework of the two-scale approach, we show that the dynamics converge wrt. Wasserstein
distance associated with the $H^{-1}$ norm, which is natural choice considering the underlying geometry of the Kawasaki dynamics.

In all mentioned methods, i.e., the martingale method (cf. [10]), the entropy method (cf. [32]) and the two-scale approach (cf. [9]), a key ingredient to deduce the hydrodynamic limit is the control of the relative entropy. Here, the entropy method has a technical advantage over the two-scale approach. In the entropy method, one only needs to show that the normalized relative entropy stays bounded. The two-scale approach needs a tighter control on the relative entropy. It is controlled by a uniform logarithmic Sobolev inequality (LSI), which yields an exponential decay of the relative entropy.

Deducing the uniform LSI for the canonical ensemble, even in the case of non-interacting Hamiltonian, is a non-trivial problem. This is because there exist long-range interactions due to mean spin conservation. For instance, LSI for conservative systems with bounded discrete spin system was studied in [33], where the LSI—scaling optimal in the systems size—was deduced under the Dobrushin–Shlosman mixing conditions. However, the problem becomes more subtle for the continuous unbounded spin systems due to the technical difficulties arising from lack of compactness. In [22], the LSI for the conservative dynamics with non-interacting Hamiltonian was obtained. A major breakthrough in the case of interacting Hamiltonian was accomplished in [25]. There, the problem of deriving the uniform LSI was solved in the case of weakly interactive system.

While the preparatory study of the canonical ensemble in the case of weak interactions (cf. [25]) would have prepared the ground to derive the hydrodynamic limit in the weakly interacting system, the authors chose to tackle the much harder problem of studying arbitrary strong interactions. The case of weak interactions does not face the problems we encounter in the study of arbitrary strong interactions, because one would expect that everything is close to the case of absent interaction. Indeed, the results were obtained by a perturbation argument, proving that weakly interactive system is close to a perturbed non-interactive system.

In an attempt to provide a better understanding of the Kawasaki dynamics from the aspect of the canonical ensemble, the authors presented a series of articles regarding the canonical ensemble with strong finite-range interactions (see [15–18]). The most important result is the uniform LSI for the canonical ensemble (cf. [18]). With this powerful control of relative entropy, it became possible to apply two-scale approach to deduce the hydrodynamic limit of the Kawasaki dynamics.

In this article, we follow the two-scale approach (cf. [9]) to deduce the hydrodynamic limit of Kawasaki dynamics. One reason is that it seems feasible to relax our assumptions on the initial state allowing deterministic configuration as we expect that the relative entropy with respect to the canonical ensemble becomes of order $N$ in finite time (cf. [19,23]). Another reason is that it is also possible to prove the quantitative hydrodynamic limit with two-scale approach. However, we do not prove the quantitative hydrodynamic limit of Kawasaki dynamics as this approach will result in sub-optimal scaling of convergence and
unnecessarily complicates our argument. Nevertheless, the quantitative hydrodynamic limit would be an important ingredient when studying fluctuations not starting in equilibrium.

One possible way to improve the scaling of the convergence would be to adapt two-scale approach with a more carefully chosen mesoscopic dynamics, as was done in [4] for the non-interactive case. The main difference to [9] is that [4] introduces a mesoscopic dynamics as the Galerkin approximation of the macroscopic dynamics, while [9] uses a projection onto piecewise constant functions to define the mesoscopic scale. This approach using Galerkin approximation has an advantage of gaining regularity of the mesoscopic scale, resulting an optimal error estimate. It would be a challenging problem to extend this approach to the case of strongly interactive Hamiltonian.

We briefly mention a new contribution and importance of our work. A main contribution of our work is a derivation of the hydrodynamic limit of Kawasaki dynamics for the spin system where the interaction is as strong as a single-site potential. We emphasize that the strong interaction is not just a technical difficulty, as a naive cutoff argument is not applicable to establish a local ergodic theorem via [10] methods. To overcome this problem, we do not involve any cutoff argument and instead make use of the two-scale approach. A crucial aspect of the two-scale approach is a strict convexity of the coarse-grained Hamiltonian for sufficiently large mesoscopic blocks. This implies the coercivity of Kawasaki dynamics at the mesoscopic scale, which is a crucial ingredient to deduce the closeness between the microscopic and macroscopic dynamics.

Now, let us mention main challenges when applying the two-scale approach in the case of strong interactions. First of all, the convergence of the one-dimensional coarse-grained Hamiltonian should be handled. In case of non-interacting spin system, the local Cramér theorem implies that the one-dimensional coarse-grained Hamiltonian converges to the Cramér transform of a single-site potential. However, this is no longer true under the existence of strong interactions. Second, a uniform LSI needs to be extended from one block to multi-blocks. That is, we consider the ensemble with conservation laws in each block and deduce the uniform LSI independent of block size, number of blocks and the whole system size. Last, due to the strong finite-range interactions, the neighboring blocks are not independent anymore, resulting that the coarse-grained Hamiltonian is no longer a sum of one-dimensional coarse-grained Hamiltonians, which makes the analysis much more delicate.

To overcome the first difficulty, we recall that the local Cramér theorem implies the uniform convergence of one-dimensional coarse-grained Hamiltonian to a Legendre transform of the free energy of the grand canonical ensemble in the absence of interactions. Motivated by this, we first prove the convergence of the free energy of the grand canonical ensemble under the presence of strong interactions. In fact, we show that the sequence of (non-normalized) free energy is sub-additive up to moment bounds. The moments are then compared with Gaussian moments, resulting bounds uniform on system size and depend only on the mean spin $m$. Then we argue that the coarse-grained Hamiltonian
converges to the Legendre transform of the limit of the free energy of the grand canonical ensemble. We refer to Sect. 3 for more details.

The second difficulty, when deducing the multi-block LSI, is handled by applying a combination of the two-scale approach (cf. [9]) and the Zegarlinski decomposition (cf. [34]). We decompose the lattice into two types of blocks $\Lambda_1$ and $\Lambda_2$ motivated by Zegarlinski’s decomposition (cf. Fig. 3). Then the measure is decomposed into a conditional distribution conditioned on $\Lambda_2$ and marginal distribution. By a careful choice of $\Lambda_1$ and $\Lambda_2$, the conditional distribution factorizes, and thus, the uniform LSI for the conditional distributions follows from a uniform LSI for the canonical ensemble (cf. [18]) and the tensorization principle. For the marginal distribution, we apply Otto–Reznikoff criterion (see [28]) where interactions between blocks are controlled via decay of correlations. Then a usual two-scale argument for LSI combines the LSIs for conditional and marginal distributions and the uniform LSI for the original measure is obtained. For more details, we refer to Sect. 4.

For the last difficulty, we artificially introduce an auxiliary Hamiltonian $H_{\text{aux}}$ where we remove the interactions between neighboring blocks. Removing the interactions makes each block independent, and as a consequence, the corresponding coarse-grained Hamiltonian of $M$ blocks is decomposed into a sum of $M$ coarse-grained Hamiltonians of single blocks. Because we assume finite-range interactions, the number of interactions we remove is relatively small compared to the whole system size. Therefore, as expected, we prove that difference between the coarse-grained Hamiltonians arising from the formal Hamiltonian $H$ and an auxiliary Hamiltonian $H_{\text{aux}}$ goes to 0 as we increase the block size $K$. This is well explained in Sect. 6.

Let us comment on open questions and problems:

- Instead of finite-range interaction, could one deduce similar results for infinite-range, algebraically decaying interactions? More precisely, is it possible to extend the results of [27] from the GCE to the CE? If yes, is the same order of algebraic decay sufficient, i.e., of the order $2+\varepsilon$, or does one need a higher order of decay? For solving this problem one would have to overcome several difficulties. For example, generalizing the equivalence of ensembles (see [16]) would need new work. Also, because we use ideas of the Zegarlinski method, the arguments of this article are restricted to the one-dimensional lattice with finite-range interaction. Applying our method to infinite-range interaction would yield a cyclic dependence of the different parameters. A possible alternative approach to this problem is to generalize the approach of [26–28] from the canonical ensemble to the grand canonical ensemble.

- In this article, we prove the hydrodynamic limit of Kawasaki dynamics in one-dimensional lattice. The main reason for considering one-dimensional lattice is that we use Zegarlinski decomposition to prove the uniform LSI for the canonical ensemble. Another possible approach for deducing the LSI for the canonical ensemble is an adaptation of Otto–Reznikoff criterion. If this is possible, we believe that our approach can be extended
to higher dimensions given sufficient decay of correlations of the grand canonical ensemble.

• Is it possible to consider more general Hamiltonians? For example, our argument is based on the fact that the single-site potentials are perturbed quadratic, especially when we use the results of [16]. In [7], the hydrodynamic limit of Kawasaki dynamics for non-interaction Hamiltonians with a super-quadratic single-site potential was established. It is an interesting further direction to deduce a hydrodynamic limit for interacting Hamiltonians with general class of single-site potentials.

• Is it possible to derive the convergence of the microscopic entropy to the hydrodynamic entropy as in [6]?

• Is it possible to generalize the results to vector-valued spin systems?

• In [23], it was proved that for the Kawasaki dynamics with non-interacting Hamiltonian, the relative entropy with respect to the canonical ensemble decays to order $N$ in a finite time. Is it possible to generalize this to the system with strong interactions? Given this, it seems possible to relax our initial condition assuming the relative entropy of the initial state with respect to the canonical ensemble is of order $N$.

We conclude this section by giving an overview over the article. In Sect. 2, we introduce precise setting and present main results. In Sect. 3, we state key ingredients and prove several auxiliary results. In Sects. 4 and 5, two main ingredients uniform LSI and strict convexity of the coarse-grained Hamiltonian are proved, respectively. In Sect. 6, we give the proof of the main result of this article, namely hydrodynamic limit of Kawasaki dynamics.

Conventions and Notation

• The symbol $T_{(k)}$ denotes the term that is given by the line $(k)$.

• We denote $0 < C < \infty$ by a generic uniform constant. This means that the actual value of $C$ might change from line to line or even within a line.

• Uniform means that a statement holds uniformly in the system size $N$ and the mean spin $m$.

• $a \lesssim b$ denotes that there is a uniform constant $C > 0$ such that $a \leq Cb$.

• $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$.

• $\mathcal{L}^k$ denotes the $k$-dimensional Hausdorff measure. If there is no cause of confusion, we write $\mathcal{L}$.

• $Z$ is a generic normalization constant. It denotes the partition function of a measure.

• For each $N \in \mathbb{N}$, $[N]$ denotes the set $\{1, \ldots, N\}$.

• For a vector $x \in \mathbb{R}^N$ and a set $A \subset [N]$, $x^A \in \mathbb{R}^A$ denotes the vector $(x^A)_i = x_i$ for all $i \in A$.

• For a vector $x \in \mathbb{R}^N$ and a set $A \subset [N]$, $\bar{x}^A = x^{[N] \setminus A} \in \mathbb{R}^{[N] \setminus A}$ denotes the vector $(\bar{x}^A)_i = x_i$ for all $i \in [N] \setminus A$.

• For a function $f : \mathbb{R}^N \to \mathbb{C}$, we denote with $\text{supp} \, f = \{i_1, \ldots, i_k\}$ the minimal subset of $[N]$ such that $f(x) = f(x_{i_1}, \ldots, x_{i_k})$. 
2. Setting and Main Results

2.1. The Model

The Gibbs measure we consider throughout the paper is a canonical ensemble with strong interactions. The simplest case of canonical ensembles, where all of the interactions are removed, is considered in [4, 9]. The interactions we consider are strong in the sense that interactions are beyond the perturbative regime.

Let us describe the precise model. Let $\Lambda$ be the sub-lattice given by $\Lambda = [N] = \{1, \ldots, N\}$. We consider a system of unbounded continuous spins on $\Lambda$. The formal Hamiltonian $H = H_N : \mathbb{R}^N \rightarrow \mathbb{R}$ of the system is defined as

$$
H_N(x) = \sum_{i=1}^{N} \left( \psi(x_i) + \frac{1}{2} \sum_{1 \leq |j-i| \leq R} M_{ij} x_i x_j \right),
$$

where $\psi(z) = \frac{1}{2} z^2 + \psi_b(z)$. For each $i \in [N]$, we define $M_{ii} := 1$ and set $x_j = 0$ for all $j \not\in [N]$. We also make the following assumptions:

- The function $\psi_b : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$
|\psi_b|_{\infty} + |\psi'_b|_{\infty} + |\psi''_b|_{\infty} < \infty.
$$

It is best to imagine $\psi$ as a double-well potential with quadratic growth at infinity (see Figure 1).

- The interactions are symmetric, i.e.,

$$
M_{ij} = M_{ji}, \quad \forall i, j \in [N].
$$

- A fixed positive integer $R \in \mathbb{N}$ models the range of interactions between the particles in the system, i.e.,

$$
M_{ij} = 0, \quad \forall i, j \in [N], \; |i - j| > R.
$$

- The matrix $(M_{ij})$ is strictly diagonal dominant, i.e., for some $\delta > 0$, it holds for any $i \in [N]$ that

$$
\sum_{1 \leq |j-i| \leq R} |M_{ij}| + \delta \leq M_{ii} = 1.
$$

- We assume spacial homogeneity of interactions. That is, there exists a function $h : \mathbb{Z} \rightarrow (-1, 1)$ such that

$$
M_{ij} = h(|i-j|), \quad \forall i, j \in \mathbb{N}.
$$

It is crucial to note that the interaction and single-site potential are of the same order (i.e., both are quadratic, see the Hamiltonian (1)). In addition, a crucial aspect of the condition (5) is that the interaction is beyond the perturbative regime. Here, perturbative regime means that the left-hand side of (5) (with $\delta = 0$) is at most $\varepsilon$ for sufficiently small enough $\varepsilon > 0$.

Let us define $X = X_{N,m}$ to be the $(N-1)$-dimensional hyperplane with mean $m$. More precisely, define

$$
X = X_{N,m} := \left\{ x \in \mathbb{R}^N : \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\} \subset \mathbb{R}^N.
$$
We equip the $l^2$ inner product on $X$ as follows:

$$\langle x, \tilde{x} \rangle_X := \sum_{i=1}^{N} x_i \tilde{x}_i.$$  

The grand canonical ensemble (GCE) $\mu_N$ associated with the Hamiltonian $H$ is the probability measure on $\mathbb{R}^N$ given by the Lebesgue density

$$\mu_N(dx) := \frac{1}{Z} \exp(-H(x)) \, dx.$$  

The canonical ensemble (CE) emerges from the GCE by conditioning on the mean spin

$$\frac{1}{N} \sum_{i=1}^{N} x_i = m.$$  

More precisely, the CE $\mu_{N,m}$ is the probability measure on $X$ with density

$$\mu_{N,m}(dx) := \mu_N \left( dx \mid \frac{1}{N} \sum_{i=1}^{N} x_i = m \right) = \frac{1}{Z} \mathbf{1} \left\{ \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\} (x) \exp(-H(x)) \mathcal{L}^{N-1}(dx),$$  

where $\mathcal{L}^{N-1}(dx)$ denotes the $(N-1)$-dimensional Hausdorff measure supported on $X$.

### 2.2. Hydrodynamic Limit of the Kawasaki Dynamics

A natural dynamics for the conservative system is the Kawasaki dynamics, which is defined as follows. Let $A$ denote the second-order difference operator given by the $N \times N$ matrix

$$A_{ij} := N^2 (-\delta_{i,j-1} + 2\delta_{i,j} - \delta_{i,j+1}),$$  

where we define $\delta_{i,0} = \delta_{i,N}$ and $\delta_{i,N+1} = \delta_{i,1}$. The Kawasaki dynamics is a stochastic process $X(t) \in \mathbb{R}^N$ satisfying the following stochastic differential equation:

$$dX(t) = -A \nabla H(X(t)) \, dt + \sqrt{2A} \, dB(t),$$
where $B(t)$ denotes a standard Brownian motion on $\mathbb{R}^N$. The Kawasaki dynamics preserves its mean spin, i.e.,

$$
\frac{1}{N} \sum_{i=1}^{N} X_i(t) = \frac{1}{N} \sum_{i=1}^{N} X_i(0) = m.
$$

This implies that we can restrict the state space $\mathbb{R}^N$ to the hyperplane $X = X_{N,m}$ and consider the corresponding CE $\mu_{N,m}$ as an invariant measure. If the process $X_t$ is distributed according to $f\mu_{N,m}$, then the time dependent probability density $f = f(t, x)$ satisfies

$$
\frac{\partial}{\partial t} (f\mu_{N,m}) = \nabla \cdot (A \nabla f\mu_{N,m}).
$$

(7)

Note that the Dirichlet form of the Kawasaki dynamics is given by

$$
D(f) := \frac{N^2}{2} \int \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}} \right)^2 \mu_{N,m},
$$

where we denote $x_{N+1} := x_1$.

In order to define a continuous counterpart of the configuration space $X_{N,m}$, let us define the space $\bar{X}$ of piecewise constant functions on $T^1 = \mathbb{R} \setminus \mathbb{Z}$ with mean $m$ by

$$
\bar{X} := \left\{ \bar{x} : T^1 \to \mathbb{R}; \bar{x} \text{ is constant on } \left( \frac{j-1}{N}, \frac{j}{N} \right) \text{ for } j = 1, \ldots, N, \text{ and has mean } m \right\}.
$$

We shall identify the space $X = X_{N,m}$ with $\bar{X}$ by the following relation:

- For each $x \in X$, the step function $\bar{x} \in \bar{X}$ associated with $x$ is
  $$
  \bar{x}(\theta) = x_j, \quad \text{if } \theta \in \left( \frac{j-1}{N}, \frac{j}{N} \right).
  $$

- For each step function $\bar{x} \in \bar{X}$, the corresponding vector $x \in X$ is
  $$
  x_j = \bar{x} \left( \frac{j}{N} \right), \quad j = 1, \ldots, N.
  $$

We equip the space of locally integrable functions $f : T^1 \to \mathbb{R}$ having a mean $m$ with $H^{-1}$ norm by

$$
\|f\|_{H^{-1}}^2 = \int_{T^1} \omega^2(\theta) d\theta,
$$

where $\omega$ is a function such that

$$
\omega' = f, \quad \int_{T^1} \omega(\theta) d\theta = 0.
$$

Now we are ready to formulate our main result, namely the hydrodynamic limit of the Kawasaki dynamics for the canonical ensemble (6) having strong interactions. We establish that the evolution along the Kawasaki dynamics gets close to the solution to a certain nonlinear parabolic equation as $N \to \infty$. 
Theorem 2.1. Let $f = f(t, x)$ be a solution of the Kawasaki dynamics (7) with initial condition $f(0, \cdot) = f_0(\cdot)$. Assume that there exists a constant $C > 0$ such that for any $N \geq 1$,
\[
\int f_0(x) \log f_0(x) \mu_{N,m}(dx) \leq CN. \tag{8}
\]
Assume also that there is a $\zeta_0 \in L^2(\mathbb{T}^1)$ such that $\int \zeta_0 d\theta = m$ and
\[
\lim_{N \to \infty} \int \|\bar{x} - \zeta_0\|_{H^{-1}}^2 f_0(x) \mu_{N,m}(dx) = 0.
\]
Let $\zeta = \zeta(t, \theta)$ be the unique weak solution of the nonlinear parabolic equation
\[
\begin{cases}
\frac{\partial \zeta}{\partial t} = \frac{\partial^2}{\partial \theta^2} \varphi'(\zeta), \\
\zeta(0, \cdot) = \zeta_0,
\end{cases}
\tag{9}
\]
where $\varphi$ is defined as
\[
\varphi(m) := \lim_{N \to \infty} -\frac{1}{N} \log \int_{\{\frac{1}{N} \sum_{i=1}^N x_i = m\}} \exp(-H(x)) L_{N-1}^N(dx). \tag{10}
\]
Then, for any $T > 0$,
\[
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \int \|\bar{x} - \zeta(t, \cdot)\|_{H^{-1}}^2 f(t, x) \mu_{N,m}(dx) = 0.
\]
Here, we say that $\zeta = \zeta(t, \theta)$ is a weak solution of (9) on $[0, T] \times \mathbb{T}^1$ if
\[
\zeta \in L^\infty_t(L^2_\theta), \quad \frac{\partial \zeta}{\partial t} \in L^2_t(H^{-1}_\theta), \quad \varphi'(\zeta) \in L^2_t(L^2_\theta),
\]
and
\[
\left\langle \xi, \frac{\partial \zeta}{\partial t} \right\rangle_{H^{-1}} = -\int_{\mathbb{T}^1} \xi \varphi'(\zeta) d\theta, \quad \forall \xi \in L^2, \text{ for almost every } t \in [0, T].
\]

Remark 2.2. There are some issues in Theorem 2.1 to be resolved. First, one has to verify that the pointwise limit of (10) exists and is differentiable. This will be established in Sect. 3. In addition, the existence and uniqueness of a weak solution of (9) for convex $\varphi$ follows from the standard argument in the nonlinear PDE theory (see, for example, [9, Lemma 38]).

Remark 2.3. Although we assumed the initial relative entropy bound (8) in Theorem 2.1, we expect that the condition (8) can be relaxed to cover more general class of initial configurations including deterministic states. In fact, it was proved in [23] that for the Kawasaki dynamics with non-interacting Hamiltonians starting from deterministic configurations, the relative entropy with respect to the canonical ensemble instantaneously becomes of order $N$.

Remark 2.4. The quantity inside the limit of (10), denoted by $\bar{H}_N(m)$, represents the distribution $f_N(m)dm$ of the mean value $(x_1 + \cdots + x_N)/N$ under $\mu_{N,m}$:
\[
f_N(m)dm = \frac{1}{Z_N} e^{-N\bar{H}_N(m)} dm.
\]
In the case of CE without interactions, i.e., $M_{ij} = 0$, as a consequence of local Cramèr theorem (see [9, Proposition 31]), $\varphi$ in (10) is a Legendre transform of the logarithmic generating function of the distribution $\frac{1}{Z} e^{-\psi(x)} dx$. On the other hand, in the presence of interactions, $\varphi$ in (10) can also be expressed in terms of the Legendre transform of the thermodynamic free energy. This point will be discussed in Sect. 3.

3. Two-Scale Decomposition

In this section, we introduce a two-scale decomposition method, originally introduced in [9], which plays a crucial role to study the concentration properties of the CE and their hydrodynamic limit. Then, we state key results on the logarithmic Sobolev inequality and the strict convexity for the coarse-grained Hamiltonian, generalizing the previous results in [16,18], which are crucial to implement a two-scale approach to establish a hydrodynamic limit.

Let us divide $N$ spins into $M$ blocks with size $K$ (see Fig. 2), denoted by

$$B(l) := \{(l - 1)K + 1, \ldots, lK\} \quad \text{for each } l \in [M].$$

We then define the mesoscopic space $Y$ as

$$Y = Y_{M,m} := \left\{ (y_1, \ldots, y_M); \frac{1}{M} \sum_{l=1}^{M} y_l = m \right\}.$$

The $L^2$ inner product on $Y$ is defined as follows:

$$\langle y, \tilde{y} \rangle_Y := \frac{1}{M} \sum_{l=1}^{M} y_l \tilde{y}_l.$$

The projection $P = P_{N,K} : X \to Y$ is defined via

$$P(x_1, \ldots, x_N) := (y_1, \ldots, y_M), \quad y_l = \frac{1}{K} \sum_{i \in B(l)} x_i.$$

We observe that the adjoint operator $P^* : Y \to X$ given by

$$P^*(y_1, \ldots, y_M) = \frac{1}{N} (\underbrace{y_1, \ldots, y_1}_{K \text{ times}}, \ldots, \underbrace{y_M, \ldots, y_M}_{K \text{ times}})$$

satisfies the identity $P N P^* = \text{Id}_Y$, where $\text{Id}_Y$ is the identity operator on $Y$. 
Remark 3.1. For notational simplicity, we assumed that all blocks $B(l)$ have equal size $K$ so that $N = MK$. If $N/K$ is not an integer, we decompose $[N]$ into $M$ blocks with different sizes $K_1, \ldots, K_M$. More precisely, we define

$$Px = (y_1, \ldots, y_M),$$

where

$$y_l = \frac{1}{K_l} \sum_{i \in B(l)} x_i \quad \text{for each } l \in [M].$$

The space $Y$ is defined as

$$Y = \left\{ (y_1, \ldots, y_M); \frac{1}{M} \sum_{l=1}^M \alpha_l y_l = m, \text{ where } \alpha_l = \frac{MK_l}{N} \right\}.$$

Here, we choose block sizes $\{K_l\}_{l=1}^M$ carefully so that $1 \leq \alpha_l \leq 2$ for all $l \in [M]$.

Finally, we disintegrate the CE $\mu_{N,m}$ into the conditional measure $\mu_{N,m}(dx|y) = \mu_{N,m}(dx|Px = y)$ and the marginal measure $\bar{\mu}_{N,m}(y)$ defined on $Y$. This means that for any test function $\xi$,

$$\int_X \xi d\mu_{N,m} = \int_Y \left( \int_{Px=y} \xi(x) \mu_{N,m}(dx|y) \right) \bar{\mu}_{N,m}(dy).$$

3.1. Key Ingredients: Logarithmic Sobolev Inequality and Coarse-Grained Hamiltonian

The two-scale decomposition method has been successfully used to study the hydrodynamic limit of the Kawasaki dynamics of CE without interactions. Key ingredients to establish hydrodynamic limit are the uniform logarithmic Sobolev inequality for the conditional distributions and the strict convexity of the coarse-grained Hamiltonian (see Sect. 6 for details). In this section, we state new results on the logarithmic Sobolev inequality and coarse-grained Hamiltonians in the context of CE with strong interactions.

Let us first introduce the definition of the logarithmic Sobolev inequality (LSI):

**Definition 3.2 (Logarithmic Sobolev Inequality (LSI)).** Let $X$ be a Euclidean space. A Borel probability measure $\mu$ satisfies a logarithmic Sobolev inequality with constant $\rho > 0$ if for any locally Lipschitz functions $f \geq 0$,

$$\int_X f \log f d\mu - \int_X f d\mu \log \left( \int_X f d\mu \right) \leq \frac{1}{2\rho} \int_X \frac{\nabla f|^2}{f} d\mu.$$

When $X = \mathbb{R}^N$, we say $\mu$ satisfies a uniform LSI with constant $\rho > 0$ if $\rho$ is independent of the system size $N$.

There have been numerous works on studying LSI for the conservative spin systems. Important works include [24], where a martingale method was implemented, and [9], where a two-scale method was introduced. Recently, the uniform LSI for $\mu_{N,m}$, the CE with strong interactions, was obtained in [18]:
Lemma 3.3 (Theorem 2 in [18]). The CE $\mu_{N,m}$ given by (6) satisfies a uniform LSI($\rho$), where $\rho > 0$ is independent of the system size $N$ and the mean spin $m \in \mathbb{R}$.

Lemma 3.3 says that for any Lipschitz density $f : X_{N,m} \to \mathbb{R}$,

$$\int_{X_{N,m}} f \log f d\mu_{N,m} - \int_{X_{N,m}} f d\mu_{N,m} \log \left( \int_{X_{N,m}} f d\mu_{N,m} \right) \leq \frac{1}{2\rho} \int_{X_{N,m}} \frac{1}{f} \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 d\mu_{N,m},$$

where $f$ is extended to be constant in a direction normal to the hypersurface $X_{N,m}$.

Recall that the measure $\mu_{N,m}$ conditions on the mean value $m$ of the spins $x_1, \ldots, x_N$. We therefore call the LSI for the measure $\mu_{N,m}$ the one-block LSI.

Remark 3.4. Although Lemma 3.3 states the uniform LSI for the Dirichlet form associated with Glauber dynamics, one immediately obtains LSI for the Kawasaki dynamics. In fact, by discrete Poincaré inequality: For any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ with $\sum_{i=1}^{N} x_i = 0$,

$$\sum_{i=1}^{N} x_i^2 \leq CN^2 \sum_{i=1}^{N} (x_i - x_{i+1})^2$$

($x_{N+1} := x_1$ and $C > 0$ is a constant), we deduce that for any $f$ satisfying $\sum_{i=1}^{N} \frac{\partial f}{\partial x_i} = 0$,

$$\sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 \leq CN^2 \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}} \right)^2. \quad (11)$$

Thus, by Lemma 3.3 and (11),

$$\int_{X_{N,m}} f \log f d\mu_{N,m} - \int_{X_{N,m}} f d\mu_{N,m} \log \left( \int_{X_{N,m}} f d\mu_{N,m} \right) \leq C N^2 \frac{1}{2\rho} \int_{X_{N,m}} \frac{1}{f} \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}} \right)^2 d\mu_{N,m}.$$
We show that this measure also satisfies the uniform LSI, which is called *multi-block LSI*. This is one of the key ingredients to implement the two-scale approach to establish the hydrodynamic limit. It is also highly non-trivial because, due to the interactions between blocks, the measure $\mu_{N,m}(dx|y)$ does not tensorize.

**Theorem 3.5.** For large enough $K$, the conditional measure $\mu_{N,m}(dx|y)$ satisfies a uniform LSI$(\rho)$, where $\rho > 0$ is independent of the system size $N$, the block size $K$, the mean spin $m$ and the macroscopic state $y$.

Theorem 3.5 will be proved in Sect. 4.

Now, we define and study some properties about the coarse-grained Hamiltonian. Recall the disintegration $\mu_{N,m}(dy) = \mu_{N,m}(dx|y)\bar{\mu}_{N,m}(dy)$.

The coarse-grained Hamiltonian $\bar{H}_Y(y)$ is defined to be a Hamiltonian corresponding to $\bar{\mu}_{N,m}(dy)$:

$$\bar{\mu}_{N,m}(dy) = \exp(-N\bar{H}_Y(y))dy.$$ 

In other words, one can define a coarse-grained Hamiltonian $\bar{H}_Y : Y \to \mathbb{R}$ as follows:

$$\bar{H}_Y(y) := -\frac{1}{N} \log \int_{P(x)=y} \exp(-H(x))\mathcal{L}^{N-M}(dx).$$  

(12)

In particular, in the simple case $Y = \mathbb{R}$ and $P(x) = (1/N)\sum_{i=1}^N x_i$, we define $\bar{H}_N : \mathbb{R} \to \mathbb{R}$, called one-dimensional coarse-grained Hamiltonian, by

$$\bar{H}_N(m) := -\frac{1}{N} \log \int_{\{\frac{1}{N} \sum_{i=1}^N x_i = m\}} \exp(-H(x))\mathcal{L}^{N-1}(dx).$$  

(13)

The strict convexity of coarse-grained Hamiltonian of the CE plays a crucial role in [9] to establish a uniform LSI and the hydrodynamic limit without interactions. In the one-block setting, this has been verified for strong interactions $\mu_{N,m}$ (see [16] or [18]).

**Lemma 3.6** (Lemma 1 in [18]). The coarse-grained Hamiltonian $\bar{H}_N : X \to \mathbb{R}$ is uniformly strictly convex. In other words, there exists a constant $C > 0$ such that for any $N \geq 1$ and $m \in \mathbb{R}$,

$$\frac{1}{C} \leq \bar{H}_N''(m) \leq C.$$ 

The second main result of this section is an extension of Lemma 3.6 to the multi-block case:

**Theorem 3.7.** The coarse-grained Hamiltonian $\bar{H}_Y$ is uniformly strictly convex. In other words, there exists a constant $\lambda > 0$ independent of the system size $N$ and the mean spin $m$, such that for any $y \in Y$,

$$\lambda \text{Id}_Y \leq \text{Hess}_Y \bar{H}_Y(y) \leq \frac{1}{\lambda} \text{Id}_Y.$$ 

Theorem 3.7 will be proved in Sect. 5.
Remark 3.8. One should compare Theorems 3.5 and 3.7 with Lemmas 3.3 and 3.6, respectively. In Lemmas 3.3 and 3.6, the authors in [18] considered the case where Gibbs measure has only one constraint

$$\frac{1}{N} \sum_{i=1}^{N} x_i = m.$$ 

Theorems 3.5 and 3.7 are generalization of Lemmas 3.3 and 3.6 in the sense that the measure $\mu_{N,m}(dx|y)$ has multiple constraints, having one conservation law for each block:

$$\frac{1}{K} \sum_{i \in B(l)} x_i = y_l, \quad l = 1, \ldots, M.$$ 

That is, if we let $M = 1$, the statements of Theorems 3.5 and 3.7 reduce to that of Lemmas 3.3 and 3.6, respectively.

Finally, we verify that the pointwise limit of the one-dimensional coarse-grained Hamiltonian $\bar{H}_N(m)$ in (13) exists as the system size goes to infinity. It turns out that this limiting function, denoted by $\varphi$, appears in the nonlinear parabolic Eq. (9). The following lemma provides a quantitative convergence of $\bar{H}_N(m)$ as $N \to \infty$.

**Proposition 3.9.** There exists a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that for each $m \in \mathbb{R}$,

$$\bar{H}_N(m) \to \varphi(m) \quad \text{as } N \to \infty. \quad (14)$$

Moreover, there exist a constant $C > 0$ such that for any $N \geq 1$ and $m \in \mathbb{R}$,

$$\left| \bar{H}_N(m) - \varphi(m) \right| \leq C \frac{m^2 + 1}{N}.$$ 

**Remark 3.10.** In the case of CE without interactions, as mentioned in Remark 2.4, $\varphi$ is a Legendre transform of the logarithmic generating function of the distribution $\frac{1}{Z} e^{-\psi(x)} dx$, and moreover, the convergence in (14) holds in the $C^2$ topology. This is a consequence of the local Cramér theorem obtained in [9, Proposition 31]. Under the presence of strong interactions, we find a candidate $\varphi$ for Theorem 2.1 as a limit of one-dimensional coarse-grained Hamiltonians. It turns out that $\varphi$ can also be represented by a Legendre transform of the thermodynamic free energy of the corresponding GCE.

For the rest of section, we prove Proposition 3.9. As a key ingredient, we first establish a sharp moment estimates with respect to the CE $\mu_{N,m}$. Then, equipped with moment estimates, we establish Proposition 3.9 by showing that $\varphi$ is a Legendre transform of the thermodynamic free energy of GCE.

### 3.2. Moment Estimates

In this section, we obtain sharp moment estimates for the CE and GCE. We first study moments for the GCE and then analyze in the case of CE using the principle of equivalence of ensembles.
3.2.1. Moment Estimates Under Grand Canonical Ensembles. We establish a sharp moment estimate under the following GCE with external fields

\[ \mu^*_N(dx) := \frac{1}{Z} \exp \left( \sigma \sum_{i=1}^{N} x_i - H(x) \right) dx. \]  

Next lemma provides the first moment bound under the GCE (15).

**Lemma 3.11.** For any \( N \geq 1 \) and \( i \in [N] \), we have

\[ \left| E_{\mu^*_N} [X_i] \right| \lesssim |\sigma| + 1. \]

It is delicate to directly estimate the first moment under the measure (15). We overcome this problem by comparing the first moment under the GCE and Gaussian ensemble using the method of interpolation. Since it is straightforward to compute the first moment under the Gaussian measure, one can finally deduce Lemma 3.11.

**Proof of Lemma 3.11.** As mentioned above, proof consists of the following two steps:

- Transfer from the GCE to Gaussian ensembles using interpolation.
- Obtain a sharp estimate on the first moment under the Gaussian ensembles.

**Step 1.** Comparison with Gaussian ensembles.

For \( s \in \mathbb{R} \), define a Hamiltonian

\[ H_s(x) := \sum_{i=1}^{N} \left( \frac{1}{2} x_i^2 + \sum_{j:1 \leq |j-i| \leq R} M_{ij} x_i x_j + s \psi_b(x_i) \right) \]

and the corresponding GCE with the external field

\[ \nu^*_N,s(dx) := \frac{1}{Z} \exp \left( \sigma \sum_{i=1}^{N} x_i - H_s(x) \right) dx. \]

Note that since \( \psi(x) = \frac{1}{2} x^2 + \psi_b(x) \),

\[ \nu^*_{N,1} = \mu^*_N \]

(see (15) for the definition of \( \mu^*_N \)), and \( \nu^*_{N,0} \) is a Gaussian ensemble.

We interpolate between two measures \( \nu^*_{N,0} \) and \( \nu^*_{N,1} \): For \( i \in [N] \),

\[ E_{\nu^*_N,1} [X_i] - E_{\nu^*_N,0} [X_i] = \int_0^1 \frac{d}{ds} E_{\nu^*_{N,s}} [X_i] \, ds \]

\[ = \int_0^1 \text{cov}_{\nu^*_N,s} \left( X_i, \sum_{j=1}^{N} \psi_b(X_j) \right) \, ds \]

\[ = - \int_0^1 \sum_{j=1}^{N} \text{cov}_{\nu^*_N,s} (X_i, \psi_b(X_j)) \, ds. \]
By Proposition A.5, for some constant $C_s > 0$,
\[
\left| \text{cov}_{\nu_{N,s}}(X_i, \psi_b(X_j)) \right| \leq C_s \| \nabla X_i \|_{L^2(\nu_{N,s})} \| \nabla \psi_b(X_j) \|_{L^2(\nu_{N,s})} \exp(-C_s|i-j|) \\
\leq C \| \psi'_b \|_\infty \exp(-C_s|i-j|).
\]
Note that the constant $C_s$ can be chosen to be uniformly bounded for $s \in [0, 1]$. Thus,
\[
\left| E_{\nu_{N,1}}[X_i] - E_{\nu_{N,0}}[X_i] \right| \leq C \sum_{j=1}^{N} \exp(-C|i-j|) \lesssim 1.
\]
This implies that
\[
\left| E_{\mu_N}[X_i] \right| = \left| E_{\nu_{N,1}}[X_i] \right| \lesssim \left| E_{\nu_{N,0}}[X_i] \right| + 1. \tag{16}
\]
**Step 2.** First moment estimate under Gaussian ensembles.

Let us denote a mean vector of the Gaussian measure $\nu_{N,0}$ by $(\eta^1, \ldots, \eta^N)$. Then, we have
\[
(\eta^1, \ldots, \eta^N)^T = M^{-1}(\sigma, \ldots, \sigma)^T.
\]
Since the Hamiltonian $H_0(x)$ is strictly convex, the measure $\nu_{N,0}$ satisfies a uniform LSI independent of $\sigma$, and thus, we have the exponential decay of correlations (see [12], for example). For multivariate Gaussian distribution, the covariance matrix is given by the inverse of the quadratic coefficient matrix $M$, i.e., $\text{cov}_{\nu_{N,0}}(X) = M^{-1}$. In particular, the coefficient of $M^{-1}$ decays exponentially in the sense that there is a constant $C > 0$ such that for any $N \geq 1$,
\[
|(M^{-1})_{ij}| \leq C \exp(-C|i-j|), \quad \forall i, j.
\]
Therefore, there exists a constant $c'_1 > 0$ such that for any $i \in [N],
\[
\left| E_{\nu_{N,0}}[X_i] \right| = |\eta_i^\sigma| \leq c'_1 |\sigma| \tag{17}
\]
Thus, by (16) and (17), one can deduce that there exist constants $c_1, c_2 > 0$ such that for any $\sigma \in \mathbb{R},$
\[
\left| E_{\mu_N}[X_i] \right| \leq c_1 |\sigma| + c_2.
\]
Because the GCE $\mu_N$ satisfies a uniform LSI and hence Poincaré inequality, one can deduce the following corollary as a consequence of Lemma 3.11.

**Corollary 3.12.** For any $N \geq 1$ and $i \in [N],$
\[
\left| E_{\mu_N}^2[X_i^2] \right| \lesssim \sigma^2 + 1.
\]

**Proof of Corollary 3.12.** By Poincaré inequality,
\[
\text{var}(X_i) \lesssim 1.
\]
Combining this with Lemma 3.11, we conclude the proof.
3.2.2. Moment Estimates Under Canonical Ensembles. In the previous section, we obtained moment estimates under GCE. Combining this with the principle of equivalence of observables (Proposition A.8), we finally obtain the moment estimate for the CE.

**Lemma 3.13.** For any \( N \geq 1 \) and \( i \in [N] \),
\[
|\mathbb{E}_{\mu_{N,m}}[X_i]| \lesssim |m| + 1.
\]

**Proof of Lemma 3.13.** Let us define the normalized free energy of Hamiltonian with external field as follows:
\[
A_N(\sigma) := \frac{1}{N} \log \int_{\mathbb{R}^N} \exp \left( \sigma \sum_{i=1}^{N} x_i - H(x) \right) dx.
\]

We first prove that there exist constants \( \gamma_1, \gamma_2 > 0 \) such that for any \( \sigma \) and \( m \) satisfying
\[
\frac{d}{d\sigma} A_N(\sigma) = m, \tag{18}
\]
we have
\[
|\sigma| \leq \gamma_1 |m| + \gamma_2. \tag{19}
\]

By Lemma A.3, there exists a constant \( C > 0 \) such that for any \( \sigma \in \mathbb{R} \) and \( N \geq 1 \),
\[
\frac{1}{C} \leq \frac{d^2}{d\sigma^2} A_N(\sigma) \leq C. \tag{20}
\]

Also, note that
\[
\frac{d}{d\sigma} A_N(0) = \frac{1}{N} \mathbb{E}_{\mu_0^N} \left[ \sum_{i=1}^{N} X_i \right]. \tag{21}
\]

This quantity is uniformly bounded in \( N \) thanks to Lemma 3.11. Thus, by (18), (20) and (21), we have (19) for some constants \( \gamma_1, \gamma_2 > 0 \).

Note that by (18) and the equivalence of observable result (Proposition A.8), we have
\[
|\mathbb{E}_{\mu_{N,m}}[X_i] - \mathbb{E}_{\mu_0^N}[X_i]| = O\left(\frac{1}{N}\right). \tag{22}
\]

Thus, by Lemma 3.11, (19) and (22), proof is concluded.

Because the CE \( \mu_{N,m} \) satisfies a uniform LSI and hence Poincaré inequality, the variance of the CE is well behaved. Therefore, we have the following statement.

**Corollary 3.14.** For any \( N \geq 1 \) and \( i \in [N] \),
\[
|\mathbb{E}_{\mu_{N,m}}[X_i^2]| \lesssim m^2 + 1.
\]
3.3. Convergence of the Coarse-Grained Hamiltonian

In this section, we prove the quantitative convergence result for the coarse-grained Hamiltonian. Let us first define a (non-normalized) free energy of the GCE

\[ a_N(\sigma) := \log \int_{\mathbb{R}^N} \exp \left( \sigma \sum_{i=1}^N x_i - H_N(x) \right) \, dx. \]

First of all, we prove that for each \( \sigma \in \mathbb{R} \), \( a_N(\sigma) \) and \( a'_N(\sigma) \) are sub-additive up to constants.

**Lemma 3.15.** There exists a constant \( C > 0 \) such that for any \( N_1, N_2 \in \mathbb{N} \) and \( \sigma \in \mathbb{R} \),

\[
|a_{N_1+N_2}(\sigma) - a_{N_1}(\sigma) - a_{N_2}(\sigma)| \leq C(\sigma^2 + 1), \tag{23}
\]

\[
|a'_{N_1+N_2}(\sigma) - a'_N(\sigma) - a'_N(\sigma)| \leq C(|\sigma| + 1). \tag{24}
\]

**Proof of (23) in Lemma 3.15.** We write

\[
a_{N_1+N_2}(\sigma) - a_{N_1}(\sigma) - a_{N_2}(\sigma) = \log \frac{\int_{\mathbb{R}^{N_1+N_2}} \exp \left( \sigma \sum_{k=1}^{N_1+N_2} w_k - H_{N_1+N_2}(w) \right) \, dw}{\int_{\mathbb{R}^{N_1}} \exp \left( \sigma \sum_{i=1}^{N_1} u_i - H_{N_1}(u) \right) \, du \cdot \int_{\mathbb{R}^{N_2}} \exp \left( \sigma \sum_{j=1}^{N_2} v_j - H_{N_2}(v) \right) \, dv}
\]

\[
= \log \frac{\int_{\mathbb{R}^{N_1+N_2}} \exp \left( \sigma \sum_{k=1}^{N_1+N_2} w_k - H_{N_1+N_2}(w) \right) \, dw}{\int_{\mathbb{R}^{N_1+N_2}} \exp \left( \sigma \sum_{i=1}^{N_1} u_i + \sigma \sum_{j=1}^{N_2} v_j - H_{N_1}(u) - H_{N_2}(v) \right) \, dw}.
\]

Let us define

\[
I_{N_1,N_2} := \{(i, j) \mid i \in \{1, \ldots, N_1\}, j \in \{N_1 + 1, \ldots, N_1 + N_2\}, |i - j| \leq R\}. \tag{25}
\]

Writing \( w = (u, v) \in \mathbb{R}^N \times \mathbb{R}^M \), we have

\[
H_{N_1+N_2}(w) - H_{N_1}(u) - H_{N_2}(v) = \sum_{(i,j) \in I_{N_1,N_2}} M_{ij} w_i w_j. \tag{26}
\]

Thus, we can write

\[
a_{N_1+N_2}(\sigma) - a_{N_1}(\sigma) - a_{N_2}(\sigma) = \log \left( \mathbb{E}_{\mu_{N_1}^\sigma} \otimes \mu_{N_2}^\sigma \left[ \exp \left( - \sum_{(i,j) \in I_{N_1,N_2}} M_{ij} w_i w_j \right) \right] \right). \tag{27}
\]

We shall only prove that (27) is bounded from above as the proof of lower bound is almost identical to that of upper bound.

To begin with, by Young’s inequality,

\[
T_i(27) \leq \log \left( \mathbb{E}_{\mu_{N_1}^\sigma} \otimes \mu_{N_2}^\sigma \left[ \exp \left( \frac{1}{2} \sum_{(i,j) \in I_{N_1,N_2}} |M_{ij}| (w_i^2 + w_j^2) \right) \right] \right).
\]
Due to the strictly diagonal dominant assumption (5),

\[
\mu_{N_1} = \log \left( \mathbb{E}_{\mu_{N_1}} \left[ \exp \left( \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| u_i^2 \right) \right] \right)
\]

and hence,

\[
T_{(28)} = \log \left( \int_{\mathbb{R}^{N_1}} \exp \left( \sigma \sum_{i=1}^{N_1} u_i - H_{N_1}(u) + \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| u_i^2 \right) du \right)
\]

where application of Corollary 3.12 implies

\[
\mu_{N_1}(s) := \frac{1}{2} \exp \left( \sigma \sum_{i=1}^{N_1} u_i - H_{N_1}(u) + s \cdot \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| u_i^2 \right) du.
\]

We then apply a method of interpolation to obtain

\[
T_{(28)} = \log \left( \int_{\mathbb{R}^{N_1}} \exp \left( \sigma \sum_{i=1}^{N_1} u_i - H_{N_1}(u) + \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| u_i^2 \right) du \right)
\]

\[
= \int_0^1 \frac{d}{ds} \log \left( \int_{\mathbb{R}^{N_1}} \exp \left( \sigma \sum_{i=1}^{N_1} u_i - H_{N_1}(u) + s \cdot \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| u_i^2 \right) du \right) ds
\]

\[
= \int_0^1 \mathbb{E}_{\mu_{N_1}^s} \left[ \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| u_i^2 \right] ds,
\]

where \( \mu_{N_1}^s \) is the probability distribution given by

\[
\mu_{N_1}^s(du) := \frac{1}{Z} \exp \left( \sigma \sum_{i=1}^{N_1} u_i - H_{N_1}(u) + s \cdot \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| u_i^2 \right) du.
\]

We observe that

\[
H_{N_1}(u) - s \cdot \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| u_i^2
\]

\[
= \sum_{i=1}^{N_1 - R} \left( \psi(u_i) + \frac{1}{2} \sum_{1 \leq |j-i| \leq R} M_{ij} u_i u_j \right)
\]

\[
+ \sum_{i=N_1 - R + 1}^{N_1} \left( \psi(u_i) - s \cdot \frac{1}{2} \sum_{j \in \{N_1 + 1, \ldots, N_1 + N_2\}} \sum_{1 \leq |j-i| \leq R} |M_{ij}| u_i^2 + \frac{1}{2} \sum_{j \in \{1, \ldots, N_1\}} \sum_{1 \leq |j-i| \leq R} M_{ij} u_i u_j \right)
\]

Due to the strictly diagonal dominant assumption (5),

\[
\frac{1}{2} - s \cdot \frac{1}{2} \sum_{j \in \{N_1 + 1, \ldots, N_1 + N_2\}} 1 \leq |j-i| \leq R |M_{ij}| \geq \frac{1}{2} \sum_{j \in \{1, \ldots, N_1\}} 1 \leq |j-i| \leq R M_{ij} + \frac{1}{2} \delta.
\]

This means that the interaction terms of \( \mu_N^s \) also satisfy the strictly diagonal dominant assumption (5), and hence, \( \mu_N^s \) is a GCE. Therefore, an application of Corollary 3.12 implies

\[
\left| T_{(28)} \right| \leq \int_0^1 \frac{1}{2} \sum_{(i,j) \in I_{N_1 \cdot N_2}} |M_{ij}| \sup_{s \in [0,1]} \left( \mathbb{E}_{\mu_{N_1}^s} \left[ u_i^2 \right] \right) ds
\]
In particular, recalling (26), it holds that
\[ \frac{1}{2} \sum_{(i,j) \in I_{N_1,N_2}} |M_{ij}|(\sigma^2 + 1) \sim \sigma^2 + 1. \]

Similarly, one gets \(|T_{(29)}| \lesssim \sigma^2 + 1\), and thus,
\[ T_{(27)} \lesssim \sigma^2 + 1. \]

**Proof of (24) in Lemma 3.15.** Following the notations used in the proof of (23) in Lemma 3.15, recalling \( w = (u,v) \in \mathbb{R}^N \times \mathbb{R}^M \), we have
\[
a'_{N_1+N_2}(\sigma) - a'_{N_1}(\sigma) - a'_{N_2}(\sigma) = \mathbb{E}_{\mu_{N_1+N_2}^{\sigma}} \left[ \sum_{k=1}^{N_1+N_2} w_k \right] - \mathbb{E}_{\mu_{N_1}^{\sigma}} \left[ \sum_{i=1}^{N_1} u_i \right] - \mathbb{E}_{\mu_{N_2}^{\sigma}} \left[ \sum_{j=1}^{N_2} v_j \right]
\]
\[
= \mathbb{E}_{\mu_{N_1+N_2}^{\sigma}} \left[ \sum_{k=1}^{N_1+N_2} w_k \right] - \mathbb{E}_{\mu_{N_1}^{\sigma}} \otimes \mu_{N_2}^{\sigma} \left[ \sum_{k=1}^{N_1+N_2} w_k \right].
\]

(31)

Let us recall the definition (25) of \( I_{N_1,N_2} \). For \( s \in [0,1] \), define
\[
\nu_{N_1+N_2}^{\sigma}(s)(dw) := \frac{1}{Z} \exp \left( \sigma \sum_{k=1}^{N_1+N_2} w_k - H_{N_1+N_2}(w) + s \sum_{(i,j) \in I_{N_1,N_2}} M_{ij} w_i w_j \right).
\]
In particular, recalling (26), it holds that
\[
\nu_{N_1+N_2}^{\sigma}(0) = \mu_{N_1+N_2}^{\sigma}, \quad \nu_{N_1+N_2}^{\sigma}(1) = \mu_{N_1}^{\sigma} \otimes \mu_{N_2}^{\sigma}.
\]
Therefore, by the method of interpolation,
\[
\left| T_{(31)} \right| = \left| - \int_0^1 \frac{d}{ds} \mathbb{E}_{\nu_{N_1+N_2}^{\sigma}(s)} \left[ \sum_{k=1}^{N_1+N_2} w_k \right] dx \right|
\]
\[
\leq \int_0^1 \left| \text{cov}_{\nu_{N_1+N_2}^{\sigma}(s)} \left( \sum_{k=1}^{N_1+N_2} w_k, \sum_{(i,j) \in I_{N_1,N_2}} M_{ij} w_i w_j \right) \right| ds
\]
\[
\leq \int_0^1 \sum_{k=1}^{N_1+N_2} \left| \text{cov}_{\nu_{N_1+N_2}^{\sigma}(s)} \left( w_k, \sum_{(i,j) \in I_{N_1,N_2}} M_{ij} w_i w_j \right) \right| ds.
\]

(32)

Since \( \nu_{N_1+N_2}^{\sigma}(s) \) is a GCE and \(|I_{N_1,N_2}| \leq 2R^2\), by the decay of correlation for GCE (Proposition A.5) and the second moment estimate (Corollary 3.12),
\[
\left| \text{cov}_{\nu_{N_1+N_2}^{\sigma}(s)} \left( w_k, \sum_{(i,j) \in I_{N_1,N_2}} M_{ij} w_i w_j \right) \right| \\
\leq C_s \left( 2R^2(\sigma^2 + 1) \right)^{\frac{s}{2}} \exp \left( -C_s \text{dist}(k, I_{N_1,N_2}) \right) \\
\leq C(\|\sigma\| + 1) \exp \left( -C \text{dist}(k, I_{N_1,N_2}) \right)
\]

(33)
for some \( C > 0 \). Plugging (33) into (32) yields

\[
\left| T_{(31)} \right| \leq C(|\sigma| + 1) \sum_{k=1}^{N_1+N_2} \exp\left(-C \text{dist}(k, I_{N_1, N_2})\right) = C(|\sigma| + 1).
\]

As an application of Fekete’s sub-additive lemma to Lemma 3.15, we have the following corollary. Note that in the limit \( N \to \infty \), the error term above vanishes after a normalization.

**Corollary 3.16.** The normalized free energy

\[
A_N(\sigma) := \frac{1}{N} a_N(\sigma) = \frac{1}{N} \log \int_{\mathbb{R}^N} \exp \left( \sigma \sum_{i=1}^{N} x_i - H_N(x) \right) dx
\]

and its derivative \( A'_N(\sigma) \) converge pointwisely to some functions \( A(\sigma), B(\sigma) : \mathbb{R} \to \mathbb{R} \) as \( N \to \infty \), respectively.

Next, we provide quantitative bounds of the convergences of \( A_N \) and \( A'_N \) as \( N \to \infty \).

**Lemma 3.17.** There exists a constant \( C > 0 \) such that

\[
|A_N(\sigma) - A(\sigma)| \leq C \frac{\sigma^2 + 1}{N}, \quad \forall \sigma \in \mathbb{R}, \quad (34)
\]

\[
|A'_N(\sigma) - B(\sigma)| \leq C \frac{|\sigma| + 1}{N}, \quad \forall \sigma \in \mathbb{R}. \quad (35)
\]

**Proof of Lemma 3.17.** We shall only provide the proof of (34) as there is only a cosmetic difference between the proof of (34) and that of (35).

Let us fix \( N \in \mathbb{N} \). We first claim that for each \( k \in \mathbb{N} \),

\[
|A_{kN}(\sigma) - A_N(\sigma)| \leq C \frac{k-1}{k} \cdot \frac{\sigma^2 + 1}{N}, \quad (36)
\]

where \( C > 0 \) is a constant from Lemma 3.15.

First of all, (36) is obviously true for \( k = 1 \). The case \( k = 2 \) also holds by putting \( M = N \) in (23) and dividing it by 2\( N \). Let us assume that (36) holds for some \( k = p \in \mathbb{N} \). That is,

\[
|A_{pN}(\sigma) - A_N(\sigma)| \leq C \frac{p-1}{p} \cdot \frac{\sigma^2 + 1}{N}. \quad (37)
\]

Then

\[
|A_{(p+1)N}(\sigma) - A_N(\sigma)| = \left| \frac{a_{(p+1)N}(\sigma)}{(p+1)N} - \frac{a_N(\sigma)}{N} \right| \leq \left| \frac{a_{(p+1)N}(\sigma) - a_{pN}(\sigma) - a_{pN}(\sigma)}{(p+1)N} \right| + \left| \frac{a_{pN}(\sigma) - pa_N(\sigma)}{(p+1)N} \right|.
\]

\[
\leq C \frac{1}{p+1} \cdot \frac{\sigma^2 + 1}{N} + \frac{p}{p+1} \cdot \left| A_{pN}(\sigma) - A_N(\sigma) \right| \leq \left( \frac{p}{p+1} \cdot \frac{\sigma^2 + 1}{N} \right) + \frac{p}{p+1} \cdot \left| A_{pN}(\sigma) - A_N(\sigma) \right|.
\]
Therefore, (36) holds for $k = p + 1$ as well, and thus, it holds for all $k \in \mathbb{N}$.

We now take $k \to \infty$ in (36) to conclude that

$$|A(\sigma) - A_N(\sigma)| \leq C \frac{\sigma^2 + 1}{N}.$$  

\[ \square \]

Lemma 3.17 implies that $A_N$ and $A'_N$ uniformly converge to $A$ and $B$ on each bounded interval $[a, b]$, respectively. Since $B$ is continuous, we have the following statement:

**Corollary 3.18.** $A$ is a $C^1$ function, and $A' = B$. In other words,

$$A'(\sigma) = \lim_{N \to \infty} A'_N(\sigma), \quad \forall \sigma \in \mathbb{R}.$$  

Let $H_N$ and $\varphi$ be the Legendre transforms of $A_N$ and $A$, respectively:

$$H_N(m) := \sup_{\sigma \in \mathbb{R}} (\sigma m - A_N(\sigma)),$$

$$\varphi(m) := \sup_{\sigma \in \mathbb{R}} (\sigma m - A(\sigma)).$$  \hspace{1cm} \text{(38)}

We recall that

$$A'_N(\sigma) = \frac{1}{N} \mathbb{E}_{\mu_N} \left[ \sum_{i=1}^N X_i \right] \quad \text{and} \quad A''_N(\sigma) = \frac{1}{N} \operatorname{var}_{\mu_N} \left( \sum_{i=1}^N X_i \right).$$

Then Lemmas 3.11 and A.2 imply that there exists a constant $C > 0$ such that

$$-C \leq A'_N(0) \leq C \quad \text{and} \quad \frac{1}{C} \leq A''_N(\sigma) \leq C.$$  \hspace{1cm} \text{(39)}

Since the bounds (39) are uniform in $N$, it also holds that

$$-C \leq A'(0) \leq C.$$  \hspace{1cm} \text{(40)}

and $A$ is strictly convex in the sense that

$$\frac{1}{C} |x - y| \leq |A'(x) - A'(y)| \leq C |x - y|.$$  \hspace{1cm} \text{(41)}

The strict convexity of $A_N$ implies that for each $N$, there exists a unique real number $\sigma_N \in \mathbb{R}$ such that

$$H_N(m) = \sup_{\sigma \in \mathbb{R}} (\sigma m - A_N(\sigma)) = \sigma_N m - A_N(\sigma_N).$$  \hspace{1cm} \text{(42)}

We also denote $\sigma_{\infty}$ by a unique real number satisfying

$$\varphi(m) = \sup_{\sigma \in \mathbb{R}} (\sigma m - A(\sigma)) = \sigma_{\infty} m - A(\sigma_{\infty}).$$  \hspace{1cm} \text{(43)}

Next, we prove that $H_N$ converges pointwisely to $\varphi$ as $N \to \infty$.

**Lemma 3.19.** There exists a constant $C > 0$ such that

$$|H_N(m) - \varphi(m)| \leq C \frac{m^2 + 1}{N}, \quad \forall m \in \mathbb{R}.$$
Proof of Lemma 3.19. By definition (42) and (43) of $\sigma_N$ and $\sigma_\infty$, it holds that

$$A'_N(\sigma_N) = A'(\sigma_\infty) = m.$$  \hspace{1cm} (44)

By (39), (40), (41) and (44), there exist constants $\gamma_1, \gamma_2 > 0$ such that for any $N \geq 1$,

$$|\sigma_N|, |\sigma_\infty| \leq \gamma_1 |m| + \gamma_2. \hspace{1cm} (45)$$

Note that by (42) and (43),

$$|H_N(m) - \varphi(m)| \leq |\sigma_\infty - \sigma_N| |m| + |A_N(\sigma_N) - A(\sigma_\infty)|.$$  \hspace{1cm} (46)

Let us begin with the estimation of the first term in the right-hand side of (46). By (41),

$$\frac{1}{C} |\sigma_\infty - \sigma_N| \leq |A'(\sigma_\infty) - A'(\sigma_N)| \leq C |\sigma_\infty - \sigma_N|. \hspace{1cm} (47)$$

Therefore, we have

$$|\sigma_\infty - \sigma_N| \leq C |A'(\sigma_\infty) - A'(\sigma_N)| \leq C |A_N(\sigma_N) - A'(\sigma_N)| \leq C |\sigma_\infty - \sigma_N|. \hspace{1cm} (48)$$

Let us turn to the estimation of the second term in the right-hand side of (46). It holds that

$$|A_N(\sigma_N) - A(\sigma_\infty)| \leq |A_N(\sigma_N) - A(\sigma_N)| + |A(\sigma_N) - A(\sigma_\infty)| \leq C \frac{\sigma_N^2 + 1}{N}.$$

Hence, applying (40), (41), (45) and (47) to (48),

$$|A_N(\sigma_N) - A(\sigma_\infty)| \leq C \frac{m^2 + 1}{N}. \hspace{1cm} (49)$$

Plugging the estimates (47) and (49) into (46) gives the desired estimate

$$|H_N(m) - \varphi(m)| \leq C \frac{m^2 + 1}{N}. \hspace{1cm} \square$$

The last ingredient for proving Lemma 3.9 is the local Cramér theorem.

Lemma 3.20 (Theorem 2.6 in [16]). There exists a constant $C > 0$ such that for $N$ large enough,

$$|H_N(m) - \mathcal{H}_N(m)| \leq C \frac{1}{N} \quad \text{for all } m \in \mathbb{R}. \hspace{1cm} \square$$

We can now conclude the Proof of Proposition 3.9.
Proof of Proposition 3.9. Let $\varphi$ be the function defined by (38). A combination of Lemmas 3.19 and 3.20 implies that
\[
|\bar{H}_N(m) - \varphi(m)| \leq |\bar{H}_N(m) - H_N(m)| + |H_N(m) - \varphi(m)| \\
\leq C \frac{1}{N} + C' \frac{m^2 + 1}{N} = C \frac{m^2 + 1}{N}.
\]
The differentiability of $\varphi$ is obvious. In fact, the Legendre transform of a $C^1$ strictly convex function with super-linear growth is also differentiable. \qed

4. Logarithmic Sobolev Inequality: Proof of Theorem 3.5

The Proof of Theorem 3.5 is motivated by Zegarlinski’s decomposition which was used to prove the uniform LSI for the GCE $\mu_N$ (cf. [34]). In [18], the authors used this idea combined with the two-scale approach (cf. [9]) to prove that the CE $\mu_{N,m}$ satisfies a uniform LSI on the one-dimensional lattice. In this proof, we adapt this idea using Zegarlinski’s decomposition and two-scale approach to deduce the uniform LSI for the measure $\mu_{N,m}(dx|y)$.

Let us begin with decomposing the lattice with two types of blocks (cf. Fig. 3):
\[
\Lambda := [N] = \{1, 2, \ldots, N\}, \\
\Lambda_1 := \bigcup_{l=1}^{M} \Lambda \cap ([1, K-R] + (l-1)K) = \bigcup_{l=1}^{M} \Lambda_1^{(l)}, \\
\Lambda_2 := \bigcup_{l=1}^{M} \Lambda \cap ([K-R+1, K] + (l-1)K) = \bigcup_{l=1}^{M} \Lambda_2^{(l)},
\]
where $R$ is the interaction range of the particles (cf. (4)). We disintegrate the measure $\mu_{N,m}(dx|y)$ as follows:
\[
\mu_{N,m}(dx|y) = \mu_{N,m}(dx^{A_1}|x^{A_2}, y)\tilde{\mu}_{N,m}(dx^{A_2}|y).
\]
In other words, for any test function $\xi$,
\[
\int \xi(x)\mu_{N,m}(dx|y) = \int \left( \int \xi(x^{A_1}, x^{A_2})\mu_{N,m}(dx^{A_1}|x^{A_2}, y) \right) \tilde{\mu}_{N,m}(dx^{A_2}|y).
\]
We prove the uniform LSI for the conditional measure $\mu_{N,m}(dx^{A_1}|x^{A_2}, y)$ and the marginal measure $\tilde{\mu}_{N,m}(dx^{A_2}|y)$ separately. Then uniform LSI for the
full measure $\mu_{N,m}$ is deduced via the two-scale criterion for the LSI (cf. [9, Theorem 3] or [18, Proposition 6]). More precisely, we have

**Lemma 4.1.** There is a constant $K_1$ such that for any $K \geq K_1$, the conditional measure $\mu_{N,m}(dx^{A_1}|x^{A_2}, y)$ satisfies LSI($\rho_1$), where $\rho_1 > 0$ is a constant independent of the system size $N$, the block size $K$, the mean spin $m$, conditioned spins $x^{A_2}$, and the macroscopic state $y$.

**Lemma 4.2.** There is a constant $K_2$ such that for any $K \geq K_2$, the marginal measure $\bar{\mu}_{N,m}(dx^{A_2}|y)$ satisfies LSI($\rho_2$), where $\rho_2 > 0$ is a constant independent of the system size $N$, the block size $K$, the mean spin $m$ and the macroscopic state $y$.

**Lemma 4.3.** Assume that
- The conditional measure $\mu_{N,m}(dx^{A_1}|x^{A_2}, y)$ satisfies LSI($\rho_1$), where $\rho_1 > 0$ is a constant independent of $N$, $K$, $m$, $x^{A_2}$ and $y$.
- The marginal measure $\bar{\mu}_{N,m}(dx^{A_2}|y)$ satisfies LSI($\rho_2$), where $\rho_2 > 0$ is a constant independent of $N$, $K$, $m$ and $y$.

Then the CE $\mu_{N,m}(dx|y)$ satisfies LSI($\rho$), where $\rho > 0$ is a constant independent of the system size $N$, the block size $K$, the mean spin $m$ and the macroscopic state $y$.

Let us briefly summarize the main ideas to prove those lemmas:
- Lemma 4.1 is a consequence of Lemma 3.3 and tensorization principle (cf. Theorem C.1). Indeed, by conditioning on the spins $x^{A_2}$, the blocks $\Lambda_j^{(i)}$ do not interact within the Hamiltonian. Therefore, the conditional measure $\mu_{N,m}(dx^{A_1}|x^{A_2}, y)$ tensorizes on $\otimes_{l=1}^M X_{K-R,\tilde{y}_l}$, where
  \[ \tilde{y}_l := \frac{Ky_l - \sum_{j \in \Lambda_j^{(i)}} x_j}{K-R}. \]
  Because each tensorized measure on $X_{K-R,\tilde{y}_l}$ has the same structure as one-dimensional CE $\mu_{N,m}$, it satisfies a uniform LSI by Lemma 3.3. Then an application of Theorem C.1 yields Lemma 4.1, and we omit the details of the argument.

- The Proof of Lemma 4.2 utilizes the Otto–Reznikoff criterion (cf. Theorem C.4). The details are provided in Section 4.1.

- The Proof of Lemma 4.3 is almost identical to that of [18, Proposition 6]. We refer to [18] or [9] for more details.

With the help of the lemmas above, we establish Theorem 3.5:

**Proof of Theorem 3.5.** A combination of Lemmas 4.1, 4.2 and 4.3 immediately yields the proof. \(\square\)
4.1. Proof of Lemma 4.2

The main idea of the Proof of Lemma 4.2 is to apply the Otto–Reznikoff criterion (Theorem C.4).

For each \( l \in [M] \), let

\[ x^{B(l)} := (x_i)_{i \in [N] \setminus B(l)}. \]

Now, we define (with a slight abuse of notation) a conditional version of Hamiltonian

\[ H(x^{B(l)}|\bar{x}^{B(l)}) := \sum_{i \in B(l)} \psi(x_i) + \frac{1}{2} \sum_{i,j \in B(l)} M_{ij}x_ix_j + \sum_{i \in B(l)} \sum_{j \notin B(l)} M_{ij}x_ix_j \]

and

\[ H(\bar{x}^{B(l)}) := H(x) - H(x^{B(l)}|\bar{x}^{B(l)}) = \sum_{i \notin B(l)} \psi(x_i) + \frac{1}{2} \sum_{i,j \notin B(l)} M_{ij}x_ix_j. \]

Note that the Hamiltonian \( Q(x^A_2|y) \) associated with the marginal measure \( \bar{\mu}_{N,m}(dx^{A_2}|y) \) is

\[ Q(x^{A_2}|y) = -\log \int_{\frac{1}{M}} \sum_{i \in \Lambda^{(k)}_{A_2}} x_i = \bar{y}_k \exp(-H(x)) \mathcal{L}(dx^{A_1}), \]

where \( \bar{y}_l \) is given by (50). Denoting

\[ \bar{x}^{A_1(l)} := (\bar{x}^{(k)}_{A_1})_{k \in [M] \setminus \{l\}}, \quad \bar{x}^{A_2(l)} := (\bar{x}^{(k)}_{A_2})_{k \in [M] \setminus \{l\}}, \]

the Hamiltonian \( Q(x^{A_2(l)}|\bar{x}^{A_2(l)}, y) \) associated with the (conditional) marginal measure \( \bar{\mu}_{N,m}(dx^{A_2(l)}|\bar{x}^{A_2(l)}, y) \) is

\[ Q(x^{A_2(l)}|\bar{x}^{A_2(l)}, y) = -\log \int_{\frac{1}{M}} \sum_{i \in \Lambda^{(k)}_{A_2}} x_i = \bar{y}_k \exp(-H(\bar{x}^{B(l)})) \exp(-Q_l(x^{A_2(l)}|\bar{x}^{B(l)}, y)) \mathcal{L}(dx^{A_1(l)}), \]

where \( Q_l \) is the block Hamiltonian defined by

\[ Q_l(x^{A_2(l)}|\bar{x}^{B(l)}, y) := -\log \int_{\frac{1}{M}} \sum_{i \in \Lambda^{(k)}_{A_2}} x_i = \bar{y}_l \exp(-H(x^{B(l)}|\bar{x}^{B(l)})) \mathcal{L}(dx^{A_1(l)}). \]

It was deduced in [18] (with \( M = 1 \)) that the block Hamiltonian \( Q_l \) can be decomposed into a sum of strictly convex function \( \bar{\Psi}_l^c \) and bounded perturbation \( \bar{\Psi}_l^b \). Note that although our Hamiltonian \( H(x^{B(l)}|\bar{x}^{B(l)}) \) is a conditional version, the proof of [18] works in the same way since the conditioning \( \bar{x}^{B(l)} \) only affects linearly in \( x^{B(l)} \) (see (51)).

Lemma 4.4 ((7.2) and (7.3) in [18]). There exist functions \( \bar{\Psi}_l^c \) and \( \bar{\Psi}_l^b \) such that

- \( Q_l = \bar{\Psi}_l^c + \bar{\Psi}_l^b \).
• For block size $K$ large enough, $\bar{\Psi}_i^c$ is strictly convex.
• $\bar{\Psi}_i^b$ is uniformly bounded.

Moreover, the strict convexity of $\bar{\Psi}_i^c$ and boundedness of $\bar{\Psi}_i^b$ is independent of the system size $N$, the block size $K$ and conditioned spins $\bar{x}^{B(l)}$.

The crucial step toward the Proof of Lemma 4.2 is to prove that each block (conditional) marginal measure $\bar{\mu}_{N,m}(dx^{A_2^{(l)}}|x^{A_2^{(l)}}_2,y)$ satisfies a uniform LSI.

**Lemma 4.5.** There is a constant $K_0$ such that for any $K \geq K_0$, each block (conditional) marginal measure $\bar{\mu}_{N,m}(dx^{A_2^{(l)}}|x^{A_2^{(l)}}_2,y)$ satisfies a uniform LSI.

**Proof of Lemma 4.5.** Let us fix $l \in [M]$ and decompose a block Hamiltonian $Q_l(x^{A_2^{(l)}}|x^{B(l)},y)$ into the strictly convex part $\bar{\Phi}_i^c$ and bounded perturbation part $\bar{\Phi}_i^b$ using Lemma 4.4. Our aim is to decompose the Hamiltonian $Q(x^{A_2^{(l)}}|x^{A_2^{(l)}}_2,y)$ into two parts $\bar{\Phi}_i^c$ and $\bar{\Phi}_i^b$ such that $\bar{\Phi}_i^c$ is strictly convex and $\bar{\Phi}_i^b$ is bounded. Then the desired statement follows from the application of Bakry–Émery criterion (Theorem C.3) and Holley–Stroock perturbation Principle (Theorem C.2).

To see this, let us decompose $Q(x^{A_2^{(l)}}|x^{A_2^{(l)}}_2,y)$ as follows:

$$Q(x^{A_2^{(l)}}|x^{A_2^{(l)}}_2,y) = -\log \int_{\kappa^{-1}} \sum_{\ell \in \Lambda^{(l)}} \frac{1}{\exp(-H(x^{B(l)}))} \exp(-\bar{\Psi}_i^c - \bar{\Psi}_i^b) L(d\bar{x}^{A_2^{(l)}})$$

$$= -\log \int_{\kappa^{-1}} \sum_{k \in [M] \setminus \{l\}} \exp(-H(x^{B(l)})) \exp(-\bar{\Psi}_i^c) L(d\bar{x}^{A_2^{(l)}})$$

$$+ \left( \log \int_{\kappa^{-1}} \sum_{k \in [M] \setminus \{l\}} \exp(-H(x^{B(l)})) \exp(-\bar{\Psi}_i^c) L(d\bar{x}^{A_2^{(l)}}) \right)$$

$$- \log \int_{\kappa^{-1}} \sum_{k \in [M] \setminus \{l\}} \exp(-H(x^{B(l)})) \exp(-\bar{\Psi}_i^c) L(d\bar{x}^{A_2^{(l)}})$$

$$= \bar{\Phi}_i^c + \bar{\Phi}_i^b.$$

**Step 1.** Strict convexity of $\bar{\Phi}_i^c$.

We note that $H(x^{B(l)})$ is independent of the spins $x^{A_2^{(l)}}$, and thus,

$$\frac{d}{dx_i} \bar{\Psi}_i^c = \frac{d}{dx_i} \left( H(x^{B(l)}) + \bar{\Psi}_i^c \right) \quad \text{for } i \in \Lambda_2^{(l)}.$$

In particular, $\bar{\Psi}_i^c$ is strictly convex. Therefore, by Brascamp–Lieb inequality (cf. [3,5]), $\bar{\Phi}_i^c$ is also uniformly strictly convex on $\mathbb{R}^{\Lambda_2^{(l)}}$.

**Step 2.** Boundedness of $\bar{\Phi}_i^b$.
We write
\[
\tilde{\Phi}_l^b = -\log \frac{\int \sum_{i \in \Lambda(l)} x_i = \tilde{y}_k \exp \left( -H(\tilde{x}^B(l)) \right) \exp \left( -\tilde{\Psi}_l^c \right) \mathcal{L}(d\tilde{x}^{A(l)})}{\int \sum_{i \in \Lambda(l)} x_i = \tilde{y}_k \exp \left( -H(\tilde{x}^B(l)) \right) \exp \left( -\tilde{\Psi}_l^c \right) \mathcal{L}(d\tilde{x}^{A(l)})}
\]
\[
= -\log \mathbb{E}_{\tilde{\mu}_l} \left[ \exp \left( -\tilde{\Psi}_l^b \right) \right].
\]
Therefore, the boundedness of \( \tilde{\Phi}_l^b \) follows from the boundedness of \( \tilde{\Psi}_l^b \). \( \square \)

Next, we shall prove that the strength of interactions between different blocks \( \Lambda_{(l)}^2 \) and \( \Lambda_{(n)}^2 \) becomes arbitrary small for large enough block size \( K \). We first introduce a formula for the second derivative of the Hamiltonian \( Q(x^{A_2}|y) \).

**Lemma 4.6** (Lemma 16 in [18]). For any \( i \in \Lambda_{(n)}^2 \) and \( j \in \Lambda_{(l)}^2 \) with \( n \neq l \),
\[
\frac{d^2}{dx_idx_j} Q(x^{A_2}|y) = - \text{cov}_{\mu_{N,m}(dx^{A_1}|x^{A_2},y)} \left( \frac{\partial}{\partial x_i} H(x) - \frac{\partial}{\partial x_l} H(x), \frac{\partial}{\partial x_j} H(x) - \frac{\partial}{\partial x_l} H(x) \right).
\]

The following statement is the second main ingredient for proving Lemma 4.2.

**Lemma 4.7.** For any \( n, l \in [M] \) with \( n \neq l \), it holds that
\[
\left| \frac{d^2}{dx_idx_j} Q(x^{A_2}|y) \right| \lesssim \frac{R}{K} + R \exp \left( -CK|n-l| \right) \quad \text{for all } i \in \Lambda_{(n)}^2, j \in \Lambda_{(l)}^2.
\]

**Proof of Lemma 4.7.** Let \( f \) and \( g \) be functions supported on \( \Lambda_{(l)}^1 \) and \( \Lambda_{(n)}^1 \), respectively. Since the measure \( \mu_{N,m}(dx^{A_1}|x^{A_2},y) \) tensorizes on \( \bigotimes_{l=1}^M X_{K-R,\tilde{y}_l} \), for \( l \neq n \),
\[
\text{cov}_{\mu_{N,m}(dx^{A_1}|x^{A_2},y)}(f,g) = 0. \tag{52}
\]
Then a combination of Lemma 4.6, (52) and Proposition A.9 yields
\[
\left| \frac{d^2}{dx_idx_j} Q(x^{A_2}|y) \right| \lesssim \frac{R}{K} + R \exp \left( -CK|n-l| \right).
\]
\( \square \)

Now we are ready to present the Proof of Lemma 4.2.

**Proof of Lemma 4.2.** By choosing the block size \( K \) large enough, the Otto–Reznikoff criterion (Theorem C.4) applied with Lemmas 4.5 and 4.7 concludes the Proof of Lemma 4.2. \( \square \)
5. Strict Convexity of Coarse-Grained Hamiltonian: Proof of Theorem 3.7

In this section, we establish the strict convexity of coarse-grained Hamiltonian, Theorem 3.7. The proof consists of two ingredients. The first one is a uniform estimate on the diagonal elements of Hess $\bar{H}(y)$ and the second one is a control on off-diagonal elements of Hess $\bar{H}(y)$. First, we state the uniform positivity of the diagonal terms of Hess $\bar{H}(y)$, obtained by the strict convexity of the one-dimensional coarse-grained Hamiltonian.

**Lemma 5.1.** There is a constant $\tau > 0$ such that for sufficiently large $K$ and each $l \in [M]$,

$$\tau \leq (\text{Hess } \bar{H}(y))_{ll} \leq \frac{1}{\tau}. $$

Next, using the decay of correlations result (cf. Proposition A.9), we establish the smallness of off-diagonal terms of Hess $\bar{H}(y)$.

**Lemma 5.2.** For each $l \neq n \in [M]$,

$$|((\text{Hess } \bar{H}(y))_{ln}| \lesssim \frac{1}{K}. $$

The Proof of Lemmas 5.1 and 5.2 are given in Sects. 5.1 and 5.2, respectively. Theorem 3.7 is a direct consequence of these two lemmas.

**Proof of Theorem 3.7.** It follows directly from Lemmas 5.1 and 5.2 by choosing $K$ large enough. \[ \square \]

### 5.1. Proof of Lemma 5.1

Recall the definition (51) of the conditional Hamiltonian $H(x_{B(l)}^{B(l)}|\bar{x}_{B(l)}^{B(l)})$. The coarse-grained Hamiltonian associated with $H(x_{B(l)}^{B(l)}|\bar{x}_{B(l)}^{B(l)})$ is given as follows: for $y_l \in \mathbb{R}$,

$$\bar{H}(y_l|x_{B(l)}^{B(l)}) := -\frac{1}{K} \log \int \exp \left( -H(x_{B(l)}^{B(l)}|\bar{x}_{B(l)}^{B(l)}) \right) \mathcal{L}^{K-1}(dx_{B(l)}^{B(l)}).$$

In addition, we disintegrate the measure $\mu_{N,m}(dx|y)$ into $\mu_{N,m}(dx_{B(l)}^{B(l)}|\bar{x}_{B(l)}^{B(l)},y)$ and $\hat{\mu}_{N,m}(d\hat{x}_{B(l)}^{B(l)}|y)$.

Observing that the Hamiltonian $H(x_{B(l)}^{B(l)}|\bar{x}_{B(l)}^{B(l)})$ has the same structure as the Hamiltonian given by (1), a straightforward calculation yields the following statement:

**Lemma 5.3** (cf. [25]). For any $l \in [M]$,

\[ (\text{Hess } \bar{H}(y))_{ll} \]

\[= \int \frac{d^2}{dy_l^2} \bar{H}(y_l|x_{B(l)}^{B(l)}) \hat{\mu}_{N,m}(d\hat{x}_{B(l)}^{B(l)}|y) \]

\[= -\frac{1}{K} \text{var}_{\hat{\mu}_{N,m}(d\hat{x}_{B(l)}^{B(l)}|y)} \left( \int \sum_{x_j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi(x_j) \right) \mu_{N,m}(dx_{B(l)}^{B(l)}|\bar{x}_{B(l)}^{B(l)},y). \]

(53)
In [16], the authors proved that under the assumption of lower bound of variance of the mean spin of the modified GCE $\mu_N^x$, the one-dimensional coarse-grained Hamiltonian is uniformly strictly convex (cf. [16, Corollary 2]). Combined with Lemma A.2, we get

**Lemma 5.4** (Extension of Corollary 2 in [16]). There is a constant $\lambda > 0$ independent of the system size $N$, mean spin $m$, block $B(l)$ and conditioning $\bar{x}^{B(l)}$ such that

$$
\lambda \leq \frac{d^2}{dy^2} H(y|\bar{x}^{B(l)}) \leq \frac{1}{\lambda}, \quad \forall y \in Y.
$$

The next statement implies that the right-hand side of (53) can be arbitrary small for small enough $K$.

**Lemma 5.5.** It holds that

$$
\frac{1}{K} \text{var}_{\bar{\mu}_{N,m}(d\bar{x}^{B(l)})|y} \left( \int \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j) \right) \mu_{N,m}(d\bar{x}^{B(l)}|\bar{x}^{B(l)}, y) \right) \lesssim \frac{1}{K}.
$$

**Proof of Lemma 5.5.** Note that the conditional measure $\mu_{N,m}(d\bar{x}^{B(l)}|\bar{x}^{B(l)}, y)$ is given by

$$
\mu_{N,m}(d\bar{x}^{B(l)}|\bar{x}^{B(l)}, y) = \frac{1}{Z} \mathbb{1}_{\{ \frac{1}{N} \sum_{i \in B(l)} x_i = y \}} (\bar{x}^{B(l)}) \exp \left( -H(\bar{x}^{B(l)}|\bar{x}^{B(l)}) \right) \mathcal{L}^{K-1}(d\bar{x}^{B(l)}).
$$

For each $l \in [M]$, define $E_l := \{ k \notin B(l) : \exists i \in B(l) \text{ such that } |i-k| \leq R \}$. Note that

$$
\int \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j) \right) \mu_{N,m}(d\bar{x}^{B(l)}|\bar{x}^{B(l)}, y) \tag{54}
$$

depends only on the spins $x_k$ with $k \in E_l$. In particular, (54) is a function of $\bar{x}^{B(l)}$. Thus,

$$
\text{var}_{\bar{\mu}_{N,m}(d\bar{x}^{B(l)})|y} \left( \int \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j) \right) \mu_{N,m}(d\bar{x}^{B(l)}|\bar{x}^{B(l)}, y) \right)

= \text{var}_{\mu_{N,m}(d\bar{x}|y)} \left( \int \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j) \right) \mu_{N,m}(d\bar{x}^{B(l)}|\bar{x}^{B(l)}, y) \right).
$$

Then an application of Poincaré inequality for $\mu_{N,m}(d\bar{x}|y)$ (cf. Theorem 3.5) yields

$$
\text{var}_{\mu_{N,m}(d\bar{x}|y)} \left( \int \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j) \right) \mu_{N,m}(d\bar{x}^{B(l)}|\bar{x}^{B(l)}, y) \right)

\lesssim \int \left( \mu_{N,m}(d\bar{x}^{B(l)}|\bar{x}^{B(l)}, y) \right)^2 \mu_{N,m}(d\bar{x}|y).
$$
Because (54) depends only on the spins $x_k, k \in E_l$, it holds that

$$\left| \nabla \left( \int \left( \sum_{j \in B(l)} \left( \sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right) \right|$$

$$= \sum_{k \in E_l} \left( \frac{\partial}{\partial x_k} \int \left( \sum_{j \in B(l)} \left( \sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right) \mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y) \right)^2$$

$$= \sum_{k \in E_l} \left( \sum_{j \in B(l)} M_{kj} - \text{cov}_{\mu_{N,m}(dx^{B(l)}|\bar{x}^{B(l)}, y)} \left( \sum_{j \in B(l)} \left( \sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j), \sum_{j \in B(n)} \left( \sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right) \right)^2. \quad (55)$$

Lastly, an application of Proposition A.9 combined with the fact that $|E_l| \leq 2R$ implies

$$T_{(55)} \lesssim 1.$$  

This finishes the Proof of Lemma 5.5. \qed

Proof of Lemma 5.1. Lemma 5.1 is a direct consequence of Lemmas 5.3, 5.4 and 5.5. Indeed, by choosing $K$ large enough, we can find a constant $\tau > 0$ such that

$$\tau \leq (\text{Hess}_Y \bar{H}(y))_{ll} \leq \frac{1}{\tau}. \quad \square$$

5.2. Proof of Lemma 5.2

The main ingredient for the Proof of Lemma 5.2 is the following representation of the Hessian of $\bar{H}$ from Lemma B.3.

Lemma 5.6. For any $l \neq n \in [M],

$$\left( \text{Hess}_Y \bar{H}(y) \right)_{ln} = \frac{1}{K} \sum_{i \in B(l), j \in B(n)} M_{ij}$$

$$- \frac{1}{K} \text{cov}_{\mu_{N,m}(dx|y)} \left( \sum_{j \in B(l)} \left( \sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j), \sum_{j \in B(n)} \left( \sum_{i=1}^N M_{ij} x_i \right) + \psi_b'(x_j) \right). \quad (56)$$

Proof of Lemma 5.2. We observe that for $l \neq n$, there are at most $R^2$ many pairs $(i, j)$ with $i \in B(l), j \in B(n)$ and $|i - j| \leq R$. For such $(i, j)$, we know $|M_{ij}|$ is uniformly bounded by 1, and hence,

$$|T_{(56)}| \lesssim R^2 \cdot \frac{1}{K} \lesssim \frac{1}{K}.$$
Let us turn to the estimation of (57). The law of total variance yields
\[
\text{cov}_{\mu_{N,m}}(dx|y) \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j), \sum_{j \in B(n)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j) \right) \\
= \text{cov}_{\hat{\mu}_{N,m}}(dx^n|y) \left( \int \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j) \right) \mu_{N,m}(dx^B(l)|\hat{x}^B(l), y), \right) \\
+ \int \text{cov}_{\mu_{N,m}}(dx^B(l)|\hat{x}^B(l), y) \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j), \right) \\
+ \sum_{j \in B(n)} \left( \sum_{i=1}^{N} M_{ij} x_i \right) + \psi'_b(x_j) \right) \hat{\mu}_{N,m}(dx^B(l)|y) \\
\right)
\]

Then an application of Proposition A.9 as in Lemma 5.1 yields the desired estimate
\[
\left| T(57) \right| \lesssim \frac{1}{K}.
\]

Hence, we conclude
\[
\left| (\text{Hess}_Y \hat{H}(y))_{lm} \right| \leq \left| T(56) \right| + \left| T(57) \right| \lesssim \frac{1}{K}.
\]

\[\Box\]

6. Hydrodynamic Limit: Proof of Theorem 2.1

In this section, we provide the proof of our main theorem, a hydrodynamic limit of Kawasaki dynamics (Theorem 2.1). As mentioned in the introduction, the main idea of proof is a two-scale approach (cf. [9, Theorem 8]). In [9], the hydrodynamic limit of the Kawasaki dynamics was deduced via two-scale approach where there are no interactions within the Hamiltonian (see [9, Theorem 17]). The problem becomes a lot more subtle when we add strong finite-range interactions within the Hamiltonian. For example, because neighboring blocks interact with each other, the coarse-grained Hamiltonian $\hat{H}_Y$ (cf. (12)) cannot be decomposed into a sum of one-dimensional coarse-grained Hamiltonian of the form (13).

We overcome this difficulty by introducing an auxiliary Hamiltonian $H_{aux}$ obtained by removing the interactions between different blocks in the Hamiltonian $H$ (see (62) later). Due to the nature of finite-range interactions of the Hamiltonian $H$, the amount of such removed interactions is negligible compared to the original Hamiltonian $H$. Therefore, one can expect that $H$ and $H_{aux}$ are close, which will be quantified explicitly later. This allows us to take advantage of nice structure of $H_{aux}$ such as block decomposition and tensorization property.
Recall the definition of the mesoscopic space $Y$ in Sect. 3, and let us introduce a mesoscopic version of Kawasaki dynamics. First of all, define the coarse-grained operator $\bar{A} : Y \to Y$ by

$$(\bar{A})^{-1} = PA^{-1}NP^*.$$  

For given $\eta_0 \in Y$, consider the mesoscopic analog of Kawasaki dynamics:

$$\left\{ \begin{array}{l}
\frac{d\eta}{dt} = -\bar{A}\nabla Y \bar{H}_Y(\eta), \\
\eta(0) = \eta_0.
\end{array} \right. \quad (58)$$

Recalling the identification of $X$ and $\bar{X}$ (see Sect. 2.2), we identify $Y$ with the space $\bar{Y}$ of piecewise constant functions on $T_1 = \mathbb{R} \setminus \mathbb{Z}$ with mean $m$:

$$\bar{Y} := \{ \bar{y} : T_1 \to \mathbb{R}; \bar{y} \text{ is constant on } \left( \frac{l-1}{M}, \frac{l}{M} \right) \text{ for } l \in [M], \text{ and has mean } m \}.$$  

The main idea of the two-scale approach is to prove the closeness of microscopic–mesoscopic solutions and mesoscopic–macroscopic solutions.

Consider a sequence $\{M_\nu, N_\nu, K_\nu\}_{\nu=1}^{\infty}$ such that $M_\nu \to \infty$, $N_\nu \to \infty$, $K_\nu = \frac{N_\nu}{M_\nu} \to \infty$. This means that the size of each block and the number of blocks are simultaneously increasing to the infinity.

**Convention.** Following the convention of [9], we write $M,N,K$ for $M_\nu,N_\nu,K_\nu$. We also denote $X = X_{N_\nu \cdot m}$, $Y = Y_{M_\nu \cdot m}$, and so on in the remaining sections.

For given $\zeta_0$, choose a sequence of step functions $\{\bar{\eta}_0^\nu\}_{\nu=1}^{\infty}$ in $\bar{Y}$ that converges to $\zeta_0$ in $L^2$:

$$\|\bar{\eta}_0^\nu - \zeta_0\|_{L^2} \to 0 \quad \text{as } \nu \to \infty. \quad (59)$$

For each $\nu$, let $\eta_0^\nu \in Y$ be the vector that corresponds to the step function $\bar{\eta}_0^\nu$, and denote $\eta^\nu$ by a solution of the mesoscopic parabolic equation (58) with the initial data $\eta(0) = \eta_0^\nu$.

The first main ingredient is the closeness of microscopic–mesoscopic solutions.

**Proposition 6.1.** For any $T > 0$,

$$\lim_{\nu \to \infty} \sup_{0 \leq t \leq T} \int \|\bar{x} - \bar{\eta}^\nu(t,\cdot)\|_{H^{-1}} f(t,x) \mu_{N,m}(dx) = 0.$$  

The second ingredient is the closeness of mesoscopic–macroscopic solutions.

**Proposition 6.2.** The step functions $\bar{\eta}^\nu$ converge in $L^\infty(H^{-1})$ to the unique weak solution $\zeta$ of (9). In particular, for any $T > 0$,

$$\lim_{\nu \to \infty} \sup_{0 \leq t \leq T} \|\bar{\eta}^\nu(t,\cdot) - \zeta(t,\cdot)\|_{H^{-1}}^2 = 0.$$  

We provide the proof of Propositions 6.1 and 6.2 in Sect. 6.1 and 6.2, respectively. Assuming Propositions 6.1 and 6.2, one can conclude the Proof of Theorem 2.1.
**Proof of Theorem 2.1.** Following the notations from above, Propositions 6.1 and 6.2 imply

\[
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \int \| \bar{x} - \zeta(t, \cdot) \|^2_{N^{-1}} f(t, x) \mu(\text{d}x) \\
\leq 2 \lim_{N \to \infty} \sup_{0 \leq t \leq T} \left( \int \| \bar{x} - \bar{\eta}^v(t, \cdot) \|^2_{N^{-1}} f(t, x) \mu(\text{d}x) + \int \| \bar{\eta}^v - \zeta(t, \cdot) \|^2_{N^{-1}} f(t, x) \mu(\text{d}x) \right) \\
= 2 \lim_{N \to \infty} \sup_{0 \leq t \leq T} \left( \int \| \bar{x} - \bar{\eta}^v(t, \cdot) \|^2_{N^{-1}} f(t, x) \mu(\text{d}x) + \| \bar{\eta}^v - \zeta(t, \cdot) \|^2_{N^{-1}} \right) \\
= 0.
\]

\[ \square \]

**6.1. Proof of Proposition 6.1**

The key ingredient of the proof of Proposition 6.1 is a two-scale criterion for the hydrodynamic limit which was originally obtained in [9, Theorem 8]. This was successfully used to establish a hydrodynamic limit of Kawasaki dynamics where there are no interactions within the Hamiltonian. We first introduce a general two-scale criterion for the hydrodynamic limit developed in [9] and then apply this to the general CE using the results established so far.

**Theorem 6.3** (Two-scale criterion for the hydrodynamic limit [9]). Let

\[
\mu(\text{d}x) = \frac{1}{Z} \exp(-H(x)) \text{d}x
\]

be a probability measure on \( X \). Assume a linear operator \( P : X \to Y \) satisfies \( PNP^* = \text{Id}_Y \) for some large \( N \in \mathbb{N} \). Assume further the following:

(i) It holds that

\[
\kappa := \max \{ \langle \text{Hess} H(x) \cdot u, v \rangle : u \in \text{Ran}(NP^*P), v \in \text{Ran}(\text{Id}_X - NP^*P), |u| = |v| = 1 \} < \infty;
\]

(ii) There is \( \rho > 0 \) such that \( \mu(\text{d}x|y) \) satisfies LSI(\( \rho \)) for all \( y \);

(iii) There is \( \lambda > 0 \) such that \( \langle \bar{y}, \text{Hess} \bar{H}(y)\bar{y} \rangle_Y \geq \lambda \langle \bar{y}, \bar{y} \rangle_Y \);

(iv) There is \( \alpha > 0 \) such that \( \int |x|^2 \mu(\text{d}x) \leq \alpha N \);

(v) There is \( \beta > 0 \) such that \( \inf_{y \in Y} \bar{H}(y) \geq -\beta \);

Define \( M := \dim Y \) and let \( A : X \to X \) be a symmetric linear operator such that:

(vi) There is \( \gamma > 0 \) such that for all \( x \in X \), \( |(\text{Id}_X - NP^*P)x|^2 \leq \gamma M^{-2} \langle x, Ax \rangle_X \).

Let \( f(t, x) \) and \( \eta(t) \) be the solution of (7) and

\[
\frac{\text{d} \eta(t)}{\text{d}t} = -\bar{A} \nabla_Y \bar{H}(\eta),
\]

with initial data \( f(0, \cdot) \) and \( \eta_0 \), respectively, where \( (\bar{A})^{-1} = PA^{-1}NP^* \).

Assume

(vii) \( \int f(0, x) \log f(0, x) \mu(\text{d}x) \leq C_1 N, \bar{H}(\eta_0) \leq C_2 \).

Define

\[
\Theta(t) := \frac{1}{2N} \int \langle (x - NP^*\eta(t)), A^{-1}(x - NP^*\eta(t)) \rangle f(t, x) \mu(\text{d}x).
\]
Then for any \( T > 0 \),
\[
\max \left\{ \sup_{0 < t \leq T} \Theta(t), \frac{\lambda}{2} \int_0^T \left( \int_Y |y - \eta(t)|^2 \tilde{f}(t, y) \tilde{\mu}(dy) \right) dt \right\}
\leq \Theta(0) + T \left( \frac{M}{N} \right) + \left( \frac{C_1 \gamma \kappa^2}{2 \lambda \rho^2 M^2} \right)
+ \left[ \sqrt{2T \gamma} \left( \alpha + \frac{2C_1}{\rho} \gamma \right)^\frac{1}{2} \left( C_1^2 + (C_2 + \beta)^2 \right) \right] \frac{1}{M},
\]
where
\[
\hat{\rho} := \frac{1}{2} \left( \rho + \lambda + \frac{\kappa^2}{\rho} - \sqrt{\left( \rho + \lambda + \frac{\kappa^2}{\rho} \right)^2 - 4 \rho \lambda} \right) > 0.
\]

As a corollary, we have both microscopic and mesoscopic closeness between the microscopic Kawasaki dynamics and the evolution (58).

**Corollary 6.4** (Propagation of hydrodynamic behavior [9]). Consider a sequence \( \{X_\nu, Y_\nu, P_\nu, A_\nu, \mu_\nu, f_{0,\nu}, \eta_{0,\nu} \}_{\nu = 1}^\infty \) of data satisfying the assumptions of Theorem 6.3 for every \( \nu \) with uniform constants \( \lambda, \rho, \kappa, \alpha, \beta, \gamma, C_1, C_2 \). Suppose that
\[
M_\nu \to \infty, \quad N_\nu \to \infty, \quad \frac{N_\nu}{M_\nu} \to \infty,
\]
and the initial data satisfies
\[
\lim_{\nu \to \infty} \frac{1}{N_\nu} \int (x - N_\nu P_\nu t \eta_{0,\nu}) \cdot A_\nu^{-1} (x - N_\nu P_\nu t \eta_{0,\nu}) f_{0,\nu}(x) \mu_\nu(dx) = 0.
\]
Then for any \( T > 0 \),
\[
\lim_{\nu \to \infty} \sup_{0 \leq t \leq T} \frac{1}{N_\nu} \int (x - N_\nu P_\nu t \eta) \cdot A_\nu^{-1} (x - N_\nu P_\nu t \eta) f(t, x) \mu(dx) = 0,
\]
and
\[
\lim_{\nu \to \infty} \int_0^T \int_Y |y - \eta(t)|^2 \tilde{f}(y) \tilde{\mu}(dy) dt = 0.
\]

The convergence (60) implies the closeness of microscopic variables in the weak norm induced by \( A^{-1} \), and the convergence (61) implies the closeness of macroscopic variables in the strong \( L^2(Y) \) norm.

**6.1.1. Auxiliary Hamiltonian.** As mentioned at the beginning of Sect. 6, we introduce the auxiliary Hamiltonian and related notions. First of all, define the auxiliary Hamiltonian \( H_{N,\text{aux}} = H_{\text{aux}} \) as
\[
H_{\text{aux}}(x) := H(x) - \frac{1}{2} \sum_{n,l \in [M]} \sum_{i \in B(l)} \sum_{j \in B(n)} M_{ij} x_i x_j.
\]
Because the interactions between different blocks are removed, $H_{\text{aux}}$ is decomposed as follows:

$$H_{\text{aux}}(x) = \sum_{l=1}^{M} \left( \sum_{i \in B(l)} \left( \psi(x_i) + \frac{1}{2} \sum_{j \in B(l), \ 1 \leq |j-i| \leq R} M_{ij} x_i x_j \right) \right)$$

$$=: \sum_{l=1}^{M} H_K(x^{B(l)}).$$

Here, we note that there are at most $2R^2 M$ many pairs of $(i, j, l, n)$ such that

- $l, n \in [M]$ and $l \neq n$,
- $i \in B(l)$, $j \in B(n)$ and $|i-j| \leq R$.

Next, define the corresponding canonical ensemble $\mu_{N,m,\text{aux}}$ by

$$\mu_{N,m,\text{aux}}(dx) := \frac{1}{Z} \mathbb{P}\left\{ \sum_{i=1}^{N} x_i = m \right\}(x) \exp\left(-H_{\text{aux}}(x)\right) \mathcal{L}^{N-1}(dx).$$

Now, let us decompose the CE $\mu_{N,m,\text{aux}}$ into the conditional measure $\mu_{N,m,\text{aux}}(dx|y)$ and the marginal measure $\bar{\mu}_{N,m,\text{aux}}(y)$, and then define the corresponding coarse-grained Hamiltonian $\bar{H}_{Y,\text{aux}}$ by

$$\bar{H}_{Y,\text{aux}}(y) := -\frac{1}{N} \log \int_{P_x=y} \exp\left(-H_{\text{aux}}(x)\right) \mathcal{L}^{N-M}(dx)$$

$$= \frac{1}{M} \sum_{l=1}^{M} \left( -\frac{1}{K} \log \int_{\sum_{i \in B(l)} x_i = y} \exp\left(-H_K(x^{B(l)})\mathcal{L}^{K-1}(dx^{B(l)})\right) \right)$$

$$= \frac{1}{M} \sum_{l=1}^{M} \bar{H}_K(y). \quad (63)$$

**Convention.** In Sects. 6.1 and 6.2, we write $\mu = \mu_{N,m}$, $\mu_{\text{aux}} = \mu_{N,m,\text{aux}}$, $\bar{H} = \bar{H}_Y$, and $\bar{H}_{\text{aux}} = \bar{H}_{Y,\text{aux}}$ to reduce our notational burden.

**6.1.2. Auxiliary Lemmas.** In this section, we provide auxiliary statements that are needed in the Proof of Proposition 6.1. We first obtain the quantitative closeness between $\bar{H}$ and $\bar{H}_{\text{aux}}$.

**Lemma 6.5.** There exists a constant $C > 0$ such that for any $y \in Y$,

$$\left| \bar{H}(y) - \bar{H}_{\text{aux}}(y) \right| \leq \frac{C}{K} \left( 1 + \|y\|_{L^2(Y)}^2 \right).$$

**Proof of Lemma 6.5.** As there is only cosmetic difference between the proof of Lemmas 6.5 and 3.15, we briefly outline the proof.

Recalling the definition (62) of $H_{\text{aux}}$, we have

$$\bar{H}_{\text{aux}}(y) - \bar{H}(y) = \frac{1}{N} \log \frac{\int_{P_x=y} \exp\left(-H(x)\right) \mathcal{L}^{N-M}(dx)}{\int_{P_x=y} \exp\left(-H_{\text{aux}}(x)\right) \mathcal{L}^{N-M}(dx)}$$

$$= \frac{1}{N} \log \left( \mathbb{E}_{\mu_{\text{aux}}(dx|y)} \left[ \exp\left(-\frac{1}{2} \sum_{n,l \in [M]} \sum_{i \in B(l), n \neq l, j \in B(n)} M_{ij} x_i x_j \right) \right] \right).$$
\begin{align*}
&\leq \frac{1}{N} \log \left( \mathbb{E}_{\mu_{\text{aux}}(dx|y)} \left[ \exp \left( \frac{1}{4} \sum_{n,l \in [M]} \sum_{i \in B(l)} \sum_{n \neq l \in B(n)} |M_{ij}|(x_i^2 + x_j^2) \right) \right] \right) \\
&= \frac{1}{N} \log \left( \mathbb{E}_{\mu_{\text{aux}}(dx|y)} \left[ \exp \left( \frac{1}{2} \sum_{n,l \in [M]} \sum_{i \in B(l)} \sum_{n \neq l \in B(n)} |M_{ij}|x_i^2 \right) \right] \right). \quad (64)
\end{align*}

Because the conditional measure \( \mu_{\text{aux}}(dx|y) \) tensorizes, i.e.,
\[
\mu_{\text{aux}}(dx|y) = \bigotimes_{l=1}^{M} \frac{1}{Z} \{ x \in \mathbb{R}^{|B(l)|} : x_i = y_i \} \left( x^{B(l)} \right) \exp \left( -H_K(x^{B(l)}) \right) \mathcal{L}^{K-1}(dx^{B(l)})
\]
\[=: \bigotimes_{l=1}^{M} \mu_K(dx^{B(l)}|y_l), \]
we have
\[
T(64) = \frac{1}{N} \sum_{l=1}^{M} \log \left( \mathbb{E}_{\mu_K(dx^{B(l)}|y_l)} \left[ \exp \left( \frac{1}{2} \sum_{n,l \in [M]} \sum_{i \in B(l)} \sum_{n \neq l \in B(n)} |M_{ij}|x_i^2 \right) \right] \right).
\]

Since \( \mu_K(dx^{B(l)}|y_l) \) is a canonical ensemble with a single constraint for each \( l \in [M] \), a similar argument from the Proof of Lemma 3.15 using Corollary 3.14 yields
\[
T(64) \lesssim \frac{1}{N} \sum_{l=1}^{M} (1 + y_l^2) = \frac{1}{K} \left( 1 + \|y\|_{L^2(Y)}^2 \right).
\]
The lower bound of \( T(64) \) is similarly deduced. \( \square \)

Next, we bound the coarse-grained Hamiltonian \( \bar{H} \) by quadratic functions.

**Lemma 6.6.** There exists a constant \( C > 0 \) such that
\[
-C + \frac{1}{C} \|y\|_{L^2(Y)}^2 \leq \bar{H}(y) \leq C \left( 1 + \|y\|_{L^2(Y)}^2 \right).
\]

**Proof of Lemma 6.6.** By Lemma 6.5, it suffices to prove
\[
-C + \frac{1}{C} \|y\|_{L^2(Y)}^2 \leq \bar{H}_{\text{aux}}(y) \leq C \left( 1 + \|y\|_{L^2(Y)}^2 \right). \quad (65)
\]

First of all, by Proposition 3.9,
\[
\bar{H}_{\text{aux}}(0) = \frac{1}{M} \sum_{l=1}^{M} \bar{H}_K(0) = \bar{H}_K(0) \to \varphi(0) \quad \text{as } \nu \to \infty.
\]

Thus, \( \bar{H}_{\text{aux}}(0) \) is uniformly bounded. Next, Lemma B.2 implies
\[
\left| \frac{\partial}{\partial y_l} \bar{H}_{\text{aux}}(0) \right| = \frac{1}{N} \mathbb{E}_{\mu_{\text{aux}}(dx|0)} \left[ \sum_{i,j \in B(l)} M_{ij} X_i + \sum_{i \in B(l)} \psi'_b(X_i) \right]
\]
Lemma \text{3.13} \quad \frac{1}{N} (2KR^2 + K) \sim \frac{1}{M}.

In particular the partial derivatives of $\bar{H}_{\text{aux}}(0)$ are also bounded.

Since $\bar{H}_{\text{aux}}$ is uniformly strictly convex (Theorem 3.7), by Taylor’s theorem, we obtain (65). \hfill \Box

As a special case of Lemma 6.6, we have the following corollary.

\textbf{Corollary 6.7.} The one-dimensional coarse-grained Hamiltonian $\bar{H}_K$ and its limit $\varphi$ are strictly convex. In particular, there exists a constant $C > 0$ such that

$$-C + \frac{1}{C} m^2 \leq \bar{H}_K(m), \varphi(m) \leq C(1 + m^2).$$

We now state lemmas which are additional ingredients for deducing Proposition 6.1.

\textbf{Lemma 6.8.} Define $\kappa$ by

$$\kappa := \max \{ \langle \text{Hess} H(x) \cdot u, v \rangle : u \in \text{Ran}(NP^*P), v \in \text{Ran}(\text{Id}_X - NP^*P), |u| = |v| = 1 \}. $$

Then, we have $\kappa < \infty$ (uniformly in $N$).

\textbf{Proof of Lemma 6.8.} Since $\text{Hess} H(x)$ is symmetric, for any $|u| = |v| = 1,$

$$|\langle \text{Hess} H(x) \cdot u, v \rangle| \leq \| \text{Hess} H(x) \|_{L^2} \leq \| \text{Hess} H(x) \|_{L^1}.$$ 

Due to the conditions (2) and (5), we have $\max_x \| \text{Hess} H(x) \|_{L^1} < \infty$ uniformly in $N.$ This concludes the proof. \hfill \Box

\textbf{Lemma 6.9 (92), (93) in [9].} There exists a constant $C > 0$ such that

$$\frac{1}{C} \langle \bar{x}, \bar{x} \rangle_{H^{-1}} \leq \frac{1}{N} \langle x, A^{-1}x \rangle_X \leq C \langle \bar{x}, \bar{x} \rangle_{H^{-1}}. \quad (66)$$

If $\bar{x}$ is bounded in $L^2,$ then

$$\left| \langle \bar{x}, \bar{x} \rangle_{H^{-1}} - \frac{1}{N} \langle x, A^{-1}x \rangle_X \right| \leq \frac{C}{N}.$$ 

Since the statement of Lemma 6.9 is same as (92) and (93) in [9], we omit the proof.

\textbf{6.1.3. Proof of Proposition 6.1.} In this section, we prove Proposition 6.1 using Corollary 6.4.

\textit{Proof of Proposition 6.1.} Let us begin with verifying the assumptions of Theorem 6.3.

- (i) is a consequence of Lemma 6.8.
- (ii) is a consequence of Theorem 3.5.
- (iii) is a consequence of Theorem 3.7.
• (iv) follows from Lemma 3.14.
• (v) is a consequence of Lemma 6.6.
• (vi) is the same as (66) in Lemma 6.9.

Lastly, we verify the condition (vii). The first assumption of (vii) is the same as (8). To verify the second condition, let us recall (59). Because \( \zeta_0 \in L^2(\mathbb{T}^1) \), there is a positive constant \( C \) such that
\[
\| \bar{\eta}_0^{\nu} \|_{L^2} \leq C. \tag{67}
\]
Thus, Lemma 6.6 implies the second condition of (vii) as follows:
\[
\bar{H}(\eta_0^{\nu}) \leq C(1 + \| \eta_0^{\nu} \|_{L^2(Y)}^2) = C(1 + \| \bar{\eta}_0^{\nu} \|_{L^2}^2) \leq C.
\]

Therefore, all assumptions in Theorem 6.3 are satisfied, and hence, one can apply Corollary 6.4, once the closeness of initial data is verified. In fact,
\[
\lim_{\nu \to \infty} \sup_{0 \leq t \leq T} \frac{1}{N} \int (x - NP^t_\nu \eta_0^{\nu}) \cdot A^{-1} (x - NP^t_\nu \eta_0^{\nu}) f_0(x) \mu(dx) = 0.
\]
In particular, (66) implies
\[
\lim_{\nu \to \infty} \sup_{0 \leq t \leq T} \int \| \bar{x} - \bar{\eta}^{\nu}(t, \cdot) \|_{H^{-1}} f(t, x) \mu_N(dx) = 0.
\]

\section*{6.2. Proof of Proposition 6.2}

\subsection*{6.2.1. Auxiliary Lemmas}
In this section, we provide auxiliary statements that will be needed in the Proof of Proposition 6.2. The statements are extensions of [9], where the Hamiltonian with no interactions is considered, to the general CE with strong interactions. Recall that \( \{ \eta^{\nu} \}^\infty_{\nu=1} \) are solutions to the mesoscopic parabolic Eq. (58).

\textbf{Lemma 6.10} (Analogue of Lemma 34 in [9]). Let \( \{ \eta^{\nu} \}^\infty_{\nu=1} \) be solutions to the mesoscopic parabolic Eq. (58) satisfying (67). Then, there is a constant \( C > 0 \) such that
\[
\sup_{0 \leq t \leq T} \langle \eta^{\nu}(t), \eta^{\nu}(t) \rangle_Y \leq C,
\]
\[
\int_0^T \left\langle \frac{d\eta^{\nu}}{dt}(t), A^{-1}(A) \frac{d\eta^{\nu}}{dt}(t) \right\rangle_Y dt \leq C. \tag{68}
\]
In particular, (68) implies, up to a subsequence, the associated step functions \( \bar{\eta}_\nu \) converges to \( \eta_* \) weak-* in \( L_t^\infty(L_\theta^2) = (L_t^1(L_\theta^2))^* \). Next lemma provides some properties of the function \( \eta_* \).

**Lemma 6.11** (Analogue of Lemma 35 in \([9]\)). Let \( \{\eta^\nu\}_{\nu=1}^\infty \) be solutions to the mesoscopic parabolic Eq. (58) satisfying (67). Assume that the associated step functions of any subsequence of \( \{\eta^\nu\}_{\nu=1}^\infty \) weak-* converges to \( \eta_* \) in \( L_t^\infty(L_\theta^2) = (L_t^1(L_\theta^2))^* \). Then \( \eta_* \) satisfies

\[
\eta_* \in L_t^\infty(L_\theta^2), \quad \frac{\partial \eta_*}{\partial t} \in L_t^2(H_{-1}^{-1}), \quad \varphi'(\eta_*) \in L_t^2(L_\theta^2).
\]

The following lemma provides a integral criteria to ensure a function to be a weak solution to the nonlinear parabolic equation.

**Lemma 6.12** (Analogue of Lemma 36 in \([9]\)). Assume \( \bar{H} \) is convex. Then \( \eta \) satisfies

\[
\frac{d\eta^\nu}{dt} = -A\nabla_Y \bar{H}(\eta^\nu)
\]

if and only if for all \( \xi \in Y \) and smooth \( \beta : [0,T] \to [0,\infty) \),

\[
\int_0^T \bar{H}(\eta(t))\beta(t)dt \leq \int_0^T \bar{H}(\eta + \xi(t,\theta))\beta(t)dt - \int_0^T \langle \xi, (A)^{-1}\eta \rangle_Y \dot{\beta}(t)dt.
\]

Similarly, if \( \varphi \) is convex, then \( \zeta \) is a weak solution of

\[
\frac{\partial \zeta}{\partial t} = \frac{\partial^2}{\partial \theta^2} \varphi'(\zeta)
\]

if and only if for all \( \xi \in L^2(T^1) \) and smooth \( \beta : [0,T] \to [0,\infty) \),

\[
\int_0^T \int_{T^1} \varphi(\zeta(t,\theta))\beta(t)d\theta dt \leq \int_0^T \int_{T^1} \varphi(\zeta(t,\theta) + \xi(\theta))\beta(t)d\theta dt - \int_0^T \langle \xi, \zeta(t,\cdot) \rangle_{H^{-1}} \dot{\beta}(t)dt.
\]

In addition, thanks to the convexity of \( \varphi \) (see (38)), we have a uniqueness of the weak solution to the nonlinear parabolic Eq. (9).

**Lemma 6.13** (Analogue Lemma 38 in \([9]\)). There is at most one weak solution to the Eq. (9).

We shall not provide the Proof of Lemma 6.10, 6.11, 6.12 and 6.13 as there are only cosmetic differences to that of Lemma 34, 35, 36 and 38 in \([9]\), respectively. The last ingredient of the proof of Proposition 6.2 is the following lemma.

**Lemma 6.14** (Lemma 37 in \([9]\)). Let \( \{\eta^\nu\}_{\nu=1}^\infty \) and \( \eta_* \) as in Lemma 6.11. Define \( \xi^\nu := \pi(\xi + \eta_* - \eta^\nu) \), where \( \pi \) is the \( L^2 \) projection onto \( Y \). Then it holds that

\[
\liminf_{\nu \to \infty} \int_0^T \bar{H}(\eta^\nu(t))\beta(t)dt \geq \int_0^T \int_{T^1} \varphi(\eta_*(t,\theta))\beta(t)d\theta dt,
\]
\[
\lim_{\nu \to \infty} \int_0^T \tilde{H}(\eta^\nu(t) + \xi^\nu(t))\beta(t)dt = \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_*(t, \theta) + \xi(\theta))\beta(t)d\theta dt, \quad (70)
\]
\[
\lim_{\nu \to \infty} \int_0^T \langle \xi^\nu(t), (A)^{-1}\eta^\nu(t) \rangle_{\mathcal{Y}} \beta(t)dt = \int_0^T \langle \xi(\theta), \eta_*(t, \theta) \rangle_{\mathcal{H}^-1}\beta(t)dt. \quad (71)
\]

Since the proof of a statement (108) in [9] can be adapted to prove (71) without any changes, we only present the proof of (69) and (70). Compared to [9], the main difficulty one encounters when deducing (69) and (70) is the lack of uniform convergence of the coarse-grained Hamiltonian \( \tilde{H}_K \) toward \( \varphi \). The key ingredient to solve this problem is Proposition 3.9, which gives a quantitative convergence of \( \bar{\eta} \).

**Proof of (69) in Lemma 6.14.** Let us write
\[
\int_0^T \tilde{H}(\eta^\nu)\beta(t)dt = \int_0^T (\tilde{H}(\eta^\nu) - \tilde{H}_{aux}(\eta^\nu))\beta(t)dt \quad (72)
\]
\[
+ \int_0^T \tilde{H}_{aux}(\eta^\nu)\beta(t)dt. \quad (73)
\]
To begin with, by Lemma 6.5 and (68),
\[
\left| T_{(72)} \right| \leq \frac{C}{K} \int_0^T (1 + \|\eta^\nu\|^2_{L^2})\beta(t)dt \leq \frac{C}{K} \int_0^T \beta(t)dt. \quad (74)
\]
Next, we have
\[
T_{(73)} = \int_0^T \int_{\mathbb{T}^1} \tilde{H}_K(\bar{\eta}^\nu)\beta(t)d\theta dt
\]
\[
= \int_0^T \int_{\mathbb{T}^1} \varphi(\bar{\eta}^\nu)\beta(t)d\theta dt + \int_0^T \int_{\mathbb{T}^1} (\tilde{H}_K(\bar{\eta}^\nu) - \varphi(\bar{\eta}^\nu))\beta(t)d\theta dt.
\]
Since \( \varphi \) is convex, the functional \( f \mapsto \int \int \varphi(f)\beta(t)d\theta dt \) is weakly lower semi-continuous with respect to the weak-* \( L^\infty_\text{aux}(L^2_\theta) \) topology. Thus, we have
\[
\liminf_{\nu \to \infty} \int_0^T \int_{\mathbb{T}^1} \varphi(\bar{\eta}^\nu)\beta(t)d\theta dt \geq \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_*)\beta(t)d\theta dt. \quad (75)
\]
Next, by Proposition 3.9,
\[
\left| \int_0^T \int_{\mathbb{T}^1} (\tilde{H}_K(\bar{\eta}^\nu) - \varphi(\bar{\eta}^\nu))\beta(t)d\theta dt \right|
\]
\[
\leq \frac{C}{K} \int_0^T (1 + \|\eta^\nu\|^2_{L^2})\beta(t)dt \leq \frac{C}{K} \int_0^T \beta(t)dt. \quad (76)
\]
Therefore, plugging the estimations (74), (75) and (76) into (72), (73) and then taking \( \nu \to \infty \), we conclude the proof.

**Proof of (70) in Lemma 6.14.** Let us write
\[
\int_0^T \tilde{H}(\eta^\nu + \xi^\nu)\beta(t)dt = \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_* + \xi)\beta(t)d\theta dt \quad (63)
\]
\[
= \int_0^T \int_{\mathbb{T}^1} \tilde{H}_K(\bar{\eta}^\nu + \bar{\xi}^\nu)\beta(t)d\theta dt - \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_* + \xi)\beta(t)d\theta dt
\]
\[
\begin{align*}
&= \int_0^T \int_{\mathbb{T}^1} \left( \bar{H}_K(\bar{\eta}^\nu + \bar{\xi}^\nu) - \varphi(\bar{\eta}^\nu + \bar{\xi}^\nu) \right) \beta(t) d\theta dt \\
&\quad + \int_0^T \int_{\mathbb{T}^1} \left( \varphi(\bar{\eta}^\nu + \bar{\xi}^\nu) - \varphi(\eta_* + \xi) \right) \beta(t) d\theta dt. \quad (77)
\end{align*}
\]

Let us begin with the estimation of (77). By Proposition 3.9,
\[
\left| T_{(77)} \right| \leq \frac{C}{K} \int_0^T \int_{\mathbb{T}^1} \left( 1 + \|\bar{\eta}^\nu + \bar{\xi}^\nu\|^2_{L^2} \right) \beta(t) d\theta dt \\
\leq \frac{C}{K} \int_0^T \int_{\mathbb{T}^1} \left( 1 + \|\eta_* + \xi\|^2 \right) \beta(t) d\theta dt \\
\leq \frac{C}{K} \int_0^T \beta(t) dt. \quad (79)
\]

Let us turn to the estimation of (78). Recalling that \( \xi^\nu \) is defined by \( \eta^\nu + \xi^\nu = \pi(\xi + \eta_*) \), we have
\[
\bar{\eta}^\nu + \bar{\xi}^\nu \to \eta_* + \xi \quad \text{in } L^2 \quad \text{for a.e. } t. \quad (80)
\]

A combination of Corollary 6.7 and (80) yields
\[
\int_{\mathbb{T}^1} \varphi(\bar{\eta}^\nu + \bar{\xi}^\nu) d\theta \to \int_{\mathbb{T}^1} \varphi(\eta_* + \xi) d\theta \quad \text{for a.e. } t. \quad (81)
\]

In addition, we have
\[
\left| \int_{\mathbb{T}^1} \varphi(\bar{\eta}^\nu + \bar{\xi}^\nu) d\theta \right| \leq \int_{\mathbb{T}^1} C \left( 1 + |\bar{\eta}^\nu + \bar{\xi}^\nu|^2 \right) d\theta \leq \int_{\mathbb{T}^1} C \left( 1 + |\eta_* + \xi|^2 \right) d\theta \leq C. \quad (82)
\]

Thus, the dominated convergence theorem applied with (81) and (82) gives
\[
\lim_{\nu \to \infty} T_{(78)} = \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_* + \xi) \beta(t) d\theta dt. \quad (83)
\]

Now letting \( \nu \to \infty \) in (79) and plugging this with (83) into (77) and (78) yields
\[
\lim_{\nu \to \infty} \int_0^T \bar{H}(\eta^\nu(t) + \xi^\nu(t)) \beta(t) dt = \int_0^T \int_{\mathbb{T}^1} \varphi(\eta_*(t, \theta) + \xi(\theta)) \beta(t) d\theta dt. \quad \square
\]

6.2.2. Proof of Proposition 6.2. In this section, we prove Proposition 6.2.

Proof of Proposition 6.2. Since \( \zeta_0 \in L^2(\mathbb{T}) \) and \( \bar{\eta}_0^\nu \) converges to \( \zeta_0 \) in \( L^2 \), \( \|\bar{\eta}_0^\nu\|_{L^2} \) is uniformly bounded. Thus, by Lemma 6.10, up to a subsequence,
\[
\bar{\eta}^\nu \rightharpoonup \eta_* \quad \text{weak * in } L^\infty_t(L^2_\theta) = (L^1_t(L^2_\theta))^*, \quad \text{strongly in } L^\infty_t(H^{-1}_\theta).
\]

Lemma 6.11 implies that \( \eta_* \) satisfies
\[
\eta_* \in L^\infty_t(L^2_\theta), \quad \frac{\partial \eta_*}{\partial t} \in L^2_t(H^{-1}_\theta), \quad \varphi'(\eta_*) \in L^2_t(L^2_\theta).
\]
Next, by Lemma 6.12, for any smooth $\beta : [0, T] \to [0, \infty)$,
\[
\int_0^T \bar{H}(\eta')\beta(t)dt \leq \int_0^T \bar{H}(\eta' + \xi')\beta(t)dt - \int_0^T \langle \xi', (\bar{A}^{-1}\eta')_Y \rangle Y \beta(t)dt,
\] (84)
where $\xi' := \pi(\xi + \eta_*) - \eta'$. By taking the limit in (84) and applying Lemma 6.14, one gets
\[
\int_0^T \int_{T^1} \varphi(\zeta(t, \theta))\beta(t)d\theta dt \leq \int_0^T \int_{T^1} \varphi(\zeta(t, \theta) + \xi(\theta))\beta(t)d\theta dt
- \int_0^T \langle \xi(\cdot), \zeta(t, \cdot) \rangle_{H^{-1}H} \beta(t)dt.
\]
Therefore, Lemma 6.12 implies that $\eta_*$ is a weak solution of
\[
\begin{aligned}
\frac{\partial \zeta}{\partial t} &= \frac{\partial^2}{\partial \theta^2} \varphi'(\zeta), \\
\zeta(0, \cdot) &= \zeta_0,
\end{aligned}
\]
and by uniqueness (Lemma 6.13), one can conclude that the sequence $\{\bar{\eta}'\}_{\nu=1}^\infty$ converges to $\eta_*$ in $L_t^\infty(H^{-1}_\theta)$.

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Appendix A. Basic Properties of the Grand Canonical Ensemble and the Canonical Ensemble

In this section, we provide useful auxiliary results for the GCE and CE, previously obtained in [15–18]. The results stated in this section hold under the
conditions (2), (3), (4) and (5). Recall that a generalized GCE $\mu_\sigma^\sigma$ with external field $\sigma$ is defined by

$$\mu_\sigma^\sigma(dx) := \frac{1}{Z} \exp \left( \sigma \sum_{i=1}^{N} x_i - H(x) \right) dx.$$ 

Remark A.1. Again, the CE $\mu_{N,m}$ can be thought as a conditional probability distribution which emerges from the GCE $\mu_\sigma^\sigma$ conditioned on the mean spin

$$\frac{1}{N} \sum_{i=1}^{N} x_i = m.$$ 

More precisely, we have

$$\mu_\sigma^\sigma \left( dx \mid \frac{1}{N} \sum_{i=1}^{N} x_i = m \right) = \frac{1}{Z} \mathbb{I} \left\{ \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\} (x) \exp (\sigma m N - H(x)) \mathcal{L}^{N-1}(dx)$$

$$= \frac{1}{Z} \mathbb{I} \left\{ \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\} (x) \exp (-H(x)) \mathcal{L}^{N-1}(dx)$$

$$= \mu_{N,m}(dx).$$

The following statement tells that the variance of the mean spin of the modified GCE $\mu_\sigma^\sigma$ is well behaved.

Lemma A.2 (Lemma 1 in [18]). There exists a constant $C > 0$, uniform in $N$ and $\sigma$, such that

$$\frac{1}{C} \leq \frac{1}{N} \text{var}_{\mu_\sigma^\sigma} \left( \sum_{k=1}^{N} X_k \right) \leq C.$$ 

Define the (normalized) free energy $A_N : \mathbb{R} \to \mathbb{R}$ by

$$A_N(\sigma) := \frac{1}{N} \log \int_{\mathbb{R}^N} \exp \left( \sigma \sum_{i=1}^{N} x_i - H(x) \right) dx.$$ 

The following lemma states that the free energy $A_N$ is uniformly strictly convex.

Lemma A.3 (Lemma 2 in [18]). There is a constant $C > 0$, uniform in $N$ and $\sigma$, such that

$$\frac{1}{C} \leq \frac{d^2}{d\sigma^2} A_N(\sigma) \leq C.$$ 

Let us now introduce the definition of local, intensive and extensive functions.

Definition A.4 (Local, intensive and extensive functions/observables). For a function $f : \mathbb{R}^Z \to \mathbb{C}$, denote $\text{supp } f$ by the minimal subset of $Z$ with $f(x) = f(x^{\text{supp } f})$. We call $f$ a local function if it has a finite support independent of $N$. A function $f$ is called intensive if there is a positive constant $\varepsilon$ such that $|\text{supp } f| \lesssim N^{1-\varepsilon}$. A function $f$ is called extensive if it is not intensive.
The following proposition claims the exponential decay of correlations for the GCE.

**Proposition A.5** (Lemma 6 in [16]). There exists a constant $C > 0$ such that for any intensive functions $f, g : \mathbb{R}^N \to \mathbb{R},$

$$|\text{cov}_{\mu_N^\sigma} (f, g)| \lesssim \|\nabla f\|_{L^2(\mu_N^\sigma)} \|\nabla g\|_{L^2(\mu_N^\sigma)} \exp (-C \text{dist}(\text{supp } f, \text{supp } g)).$$

The following moment estimate is a consequence of Proposition A.5.

**Lemma A.6** (Lemma 3.2 in [17]). For each $k \geq 1$, there exists a constant $C = C(k) > 0$ such that for any smooth function $f : \mathbb{R}^\Lambda \to \mathbb{R},$

$$\text{E}_{\mu_N^\sigma} \left[ |f(X) - \text{E}_{\mu_N^\sigma}[f(X)]|^k \right] \leq C(k) \|\nabla f\|_{L^\infty(\mu_N^\sigma)}^k.$$

We now introduce the principle of equivalence of observables. We first relate the external field $\sigma$ of $\mu_N^\sigma$ and the mean spin $m$ of $\mu_{N,m}$ as follows:

**Definition A.7.** For each $m \in \mathbb{R}$, we choose $\sigma = \sigma_N^\sigma(m) \in \mathbb{R}$ such that

$$d \sigma A_N^\sigma(\sigma) = m.$$  \hfill (85)

Denoting $m_i := \int x_i \mu_N^\sigma(dx)$ for each $i \in [N]$, we equivalently get

$$m = \frac{d}{d\sigma} A_N^\sigma(\sigma) = \frac{1}{N} \int_{\mathbb{R}^N} \sum_{i=1}^N x_i \exp \left( \sigma \sum_{i=1}^N x_i - H(x) \right) dx = \frac{1}{N} \sum_{i=1}^N m_i.$$

The following proposition states the equivalence of ensembles results for GCE $\mu_N^\sigma$ and CE $\mu_{N,m}$ with $\sigma$ and $m$ related by (85).

**Proposition A.8** (Theorem 2.7 in [15]). There exist constants $C, N_0 > 0$ such that for any intensive function $f : \mathbb{R}^N \to \mathbb{R}$ and $N \geq N_0,$

$$|\text{E}_{\mu_N^\sigma}[f] - \text{E}_{\mu_{N,m}}[f]| \leq C \frac{|\text{supp } f|}{N} \|\nabla f\|_{L^\infty(\mu_N^\sigma)}.$$

Finally, we state the exponential decay of correlations for the CE.

**Proposition A.9** (Theorem 2.10 in [15]). There exist constants $C, N_0 > 0$ such that for any intensive functions $f, g : \mathbb{R}^N \to \mathbb{R}$ and $N \geq N_0,$

$$|\text{cov}_{\mu_{N,m}} (f, g)| \leq C \|\nabla f\|_{L^\infty(\mu_N^\sigma)} \|\nabla g\|_{L^\infty(\mu_N^\sigma)} \frac{\left( |\text{supp } f| + |\text{supp } g| \right)}{N} \exp (-C \text{dist}(\text{supp } f, \text{supp } g)).$$

Here, $\sigma$ and $m$ are related by (85).

Appendix B. Derivatives of Coarse-Grained Hamiltonian

In this section, we provide the first and second derivative formula for the coarse-grained Hamiltonian. First, we state an explicit formula for $\bar{H}$, obtained in [25].
Lemma B.1 (Lemma 1 in [25]). For $z \in X$ with $Pz = 0$ and $y \in Y$, define $H_M(z, y)$ by

$$H_M(z, y) := \frac{1}{2} \langle z, (\text{Id} + M)z \rangle + \langle z, MNP^*y \rangle + \sum_{i=1}^{N} \psi_b(z_i + (NP^*y)_i),$$  \hspace{1cm} (86)$$

where $M = (M_{ij})_{1 \leq i, j \leq n}$ is an interaction matrix in the Hamiltonian (cf. (1)).

Then

$$\bar{H}(y) = \frac{1}{2} \langle y, (\text{Id} + PMNP^*)y \rangle_Y - \frac{1}{N} \log \int_{Px = 0} \exp \left( -H_M(x, y) \right) \mathcal{L}(dx).$$

Proof of Lemma B.1. For $x \in X$ with $Px = y$ and $y \in Y$, let $z = x - NP^*y$. Recalling the identity $PNP^* = \text{Id}_Y$, we have $Pz = 0$. We then write the Hamiltonian $H$ as

$$H(x) = \sum_{i=1}^{N} \left( \psi(x_i) + \frac{1}{2} \sum_{j: 1 \leq |j - i| \leq R} M_{ij}x_i x_j \right)$$

$$= \frac{1}{2} \langle x, (\text{Id} + Mx) \rangle + \sum_{i=1}^{N} \psi_b(x_i)$$

$$= \frac{1}{2} \langle z + NP^*y, (\text{Id} + M)(z + NP^*y) \rangle + \sum_{i=1}^{N} \psi_b(z_i + (NP^*y)_i)$$

$$= \frac{1}{2} \langle z, (\text{Id} + M)z \rangle + \langle z, NP^*y \rangle + \langle z, MNP^*y \rangle + \frac{1}{2} \langle NP^*y, (\text{Id} + M)NP^*y \rangle$$

$$+ \sum_{i=1}^{N} \psi_b(z_i + (NP^*y)_i).$$  \hspace{1cm} (87)$$

Because $Pz = 0$, we have

$$\langle z, NP^*y \rangle = N \langle Pz, y \rangle_Y = 0. \hspace{1cm} (88)$$

It also holds by $PNP^* = \text{Id}_Y$ that

$$\frac{1}{2} \langle NP^*y, (\text{Id} + M)NP^*y \rangle = \frac{N}{2} \langle y, PNP^*y + PMNP^*y \rangle_Y$$

$$= \frac{N}{2} \langle y, (\text{Id} + PMNP^*)y \rangle_Y. \hspace{1cm} (89)$$

Plugging (88) and (89) into (87) yields

$$H(x) = \frac{N}{2} \langle y, (\text{Id} + PMNP^*)y \rangle_Y + H_M(z, y),$$

and hence,

$$\bar{H}(y) = -\frac{1}{N} \log \int_{Px = y} \exp \left( -H(x) \right) \mathcal{L}(dx)$$

$$= \frac{1}{2} \langle y, (\text{Id} + PMNP^*)y \rangle_Y - \frac{1}{N} \log \int_{Pz = 0} \exp \left( -H_M(z, y) \right) \mathcal{L}(dz). \hspace{1cm} \square$$
Next, we compute the first derivative of the coarse-grained Hamiltonian $\bar{H}$ using Lemma B.1.

**Lemma B.2.** For each $l \in [M]$, it holds that

\[
\frac{\partial}{\partial y_l} \bar{H}(y) = \frac{1}{M} y_l + \frac{1}{N} \mathbb{E}_{\mu_{N,m}}(dx|y) \left[ \sum_{i=1}^{N} \sum_{j \in B(l)} M_{ij} X_i + \sum_{i \in B(l)} \psi'_b(X_i) \right].
\]

**Proof of Lemma B.2.** Recall that the inner product $\langle \cdot, \cdot \rangle_Y$ is given by

\[
\langle x, y \rangle_Y = \frac{1}{M} \sum_{l=1}^{M} x_l y_l.
\]

First of all, noting that

\[
\langle y, P M N P^* y \rangle_Y = \frac{1}{N} \sum_{l,n=1}^{M} \sum_{i \in B(l), j \in B(n)} M_{ij} y_l y_n,
\]

we have

\[
\frac{\partial}{\partial y_l} \left( \frac{1}{2} \langle y, (\text{Id} + P M N P^*) y \rangle_Y \right) = \frac{1}{M} y_l + \frac{1}{N} \sum_{n=1}^{M} \sum_{i \in B(l), j \in B(n)} M_{ij} y_n. \tag{90}
\]

In addition, differentiating (86) yields

\[
\frac{\partial}{\partial y_l} (H_M(x, y))
\]

\[
= \frac{\partial}{\partial y_l} (\langle x, M N P^* y \rangle) + \frac{\partial}{\partial y_l} \left( \sum_{i=1}^{N} \psi_b(x_i + (NP^* y)_i) \right)
\]

\[
= \sum_{i=1}^{N} \sum_{j \in B(l)} M_{ij} x_i + \sum_{i \in B(l)} \psi'_b(x_i + (NP^* y)_i)
\]

\[
= \sum_{i=1}^{N} \sum_{j \in B(l)} M_{ij} (x_i + (NP^* y)_i)
\]

\[
- \sum_{i=1}^{N} \sum_{j \in B(l)} M_{ij} (NP^* y)_i + \sum_{i \in B(l)} \psi'_b(x_i + (NP^* y)_i)
\]

\[
= \sum_{i=1}^{N} \sum_{j \in B(l)} M_{ij} (x_i + (NP^* y)_i) + \sum_{i \in B(l)} \psi'_b(x_i + (NP^* y)_i)
\]

\[
- \sum_{n=1}^{M} \sum_{i \in B(n), j \in B(l)} M_{ij} y_n.
\]

As a consequence, we obtain

\[
\frac{\partial}{\partial y_l} \left( -\frac{1}{N} \log \int_{P_x=0} \exp (-H_M(x, y)) \mathcal{L}(dx) \right)
\]
\begin{align*}
&= \frac{1}{N} \int_{p_x = 0} \frac{\partial}{\partial p_y} (H_M(x, y)) \exp (-H_M(x, y)) \mathcal{L}(dx) \\
&= \frac{1}{N} \mathbb{E}_{\mu,N,m(dx|y)} \left[ \sum_{i=1}^{N} \sum_{j \in B(l)} M_{ij} X_i + \sum_{i \in B(l)} \psi_i'(X_i) - \frac{1}{N} \sum_{n=1}^{M} \sum_{i \in B(n), j \in B(l)} M_{ij} y_n \right]. \quad (91)
\end{align*}

Combining (90) and (91) with symmetry of $M_{ij}$, i.e., $M_{ij} = M_{ji}$, we have the desired equation

\begin{align*}
\frac{\partial}{\partial y_l} \bar{H}(y) = \frac{1}{M} y_l + \frac{1}{N} \mathbb{E}_{\mu,N,m(dx|y)} \left[ \sum_{i=1}^{N} \sum_{j \in B(l)} M_{ij} X_i + \sum_{i \in B(l)} \psi_i'(X_i) \right].
\end{align*}

\square

The second derivatives of the coarse-grained Hamiltonian $\bar{H}$ follow from a similar calculation.

**Lemma B.3** (Lemma 2 in [25]). For $l, n \in [M]$, we have

\begin{align*}
\text{(Hessy } \bar{H}(y) \text{)}_{ln} &= \delta_{ln} + \delta_{ln} \frac{1}{K} \int \sum_{i \in B(l)} \psi''_b(x_i) \mu_{N,m}(dx|y) + \frac{1}{K} \sum_{i \in B(l), j \in B(n)} M_{ij} \\
&- \frac{1}{K} \text{cov}_{\mu,N,m(dx|y)} \left( \sum_{j \in B(l)} \left( \sum_{i=1}^{N} M_{ij} X_i + \psi_b'(X_j) \right) \right), \\
&\sum_{j \in B(n)} \left( \sum_{i=1}^{N} M_{ij} X_i + \psi_b'(X_j) \right).
\end{align*}

**Appendix C. Criteria for the Logarithmic Sobolev Inequality**

In this section, we state several standard criteria for deducing a LSI. For proofs we refer to the literature. For a general introduction and more comments on the LSI, we refer the reader to [2,20,21,30].

**Theorem C.1** (Tensorization Principle [11]). Let $\mu_1$ and $\mu_2$ be probability measures on Euclidean spaces $X_1$ and $X_2$, respectively. Suppose that $\mu_1$ and $\mu_2$ satisfy $\text{LSI}(\rho_1)$ and $\text{LSI}(\rho_2)$, respectively. Then the product measure $\mu_1 \otimes \mu_2$ satisfies $\text{LSI}(\rho)$, where $\rho = \min\{\rho_1, \rho_2\}$.

**Theorem C.2** (Holley–Stroock Perturbation Principle [13]). Let $\mu_1$ be a probability measure on Euclidean space $X$ and $\psi_b : X \to \mathbb{R}$ be a bounded function. Define a probability measure $\mu_2$ on $X$ by

\begin{align*}
\mu_2(dx) := \frac{1}{Z} \exp (-\psi_b(x)) \mu_1(dx).
\end{align*}

Suppose that $\mu_1$ satisfies $\text{LSI}(\rho_1)$. Then $\mu_2$ also satisfies $\text{LSI}$ with constant

\begin{align*}
\rho_2 = \rho_1 \exp (-\text{osc } \psi_b),
\end{align*}

where $\text{osc } \psi_b := \sup \psi_b - \inf \psi_b$. 

\begin{align*}
\text{□}
\end{align*}
**Theorem C.3** (Bakry–Émery criterion [1]) Let $X$ be a $N$-dimensional Euclidean space and $H \in C^2(X)$. Define a probability measure $\mu$ on $X$ by

$$
\mu(dx) := \frac{1}{Z} \exp(-H(x)) \, dx.
$$

Suppose that there is a constant $\rho > 0$ such that $\operatorname{Hess} H \geq \rho \cdot |v|^2$, \ \forall v \in T_x X$. Then $\mu$ satisfies LSI$(\rho)$.

**Theorem C.4** (Otto-Reznikoff Criterion [28]) Let $X = X_1 \times \cdots \times X_N$ be a direct product of Euclidean spaces and $H \in C^2(X)$. Define a probability measure $\mu$ on $X$ by

$$
\mu(dx) := \frac{1}{Z} \exp(-H(x)) \, dx.
$$

Assume that

- For each $i \in [N]$, the conditional measures $\mu(dx_i|\bar{x}_i)$ satisfy LSI($\rho_i$) for any $\bar{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$.
- For each $i \neq j \in [N]$ there is a constant $\kappa_{ij} > 0$ such that

  $$
  |\nabla_i \nabla_j H(x)| \leq \kappa_{ij}, \quad \forall x \in X.
  $$

Here, $| \cdot |$ denotes the operator norm of a bilinear form.

- Define a symmetric matrix $A = (A_{ij})_{1 \leq i,j \leq N}$ by

  $$
  A_{ij} = \begin{cases} 
  \rho_i, & i = j, \\
  -\kappa_{ij}, & i \neq j.
  \end{cases}
  $$

Assume that there is a constant $\rho > 0$ with

$$
A \geq \rho \, \operatorname{Id},
$$

in the sense of quadratic forms.

Then $\mu$ satisfies LSI$(\rho)$.

**Theorem C.5** (Two-Scale Criterion for LSI [9]). Let $X$ and $Y$ be Euclidean spaces. Consider a probability measure $\mu$ on $X$ defined by

$$
\mu(dx) := \frac{1}{Z} \exp(-H(x)) \, dx.
$$

Let $P : X \to Y$ be a linear operator such that for some $N \in \mathbb{N}$,

$$
P N P^* = \operatorname{Id}_Y.
$$

Define

$$
\kappa := \max \{ \langle \operatorname{Hess} H(x) \cdot u, v \rangle : u \in \operatorname{Ran}(NP^*P), v \in \operatorname{Ran}(\operatorname{Id}_X - NP^*P), |u| = |v| = 1 \}.
$$

Assume that

- $\kappa < \infty$.
- There is $\rho_1 > 0$ such that the conditional measure $\mu(dx|Px = y)$ satisfies LSI($\rho_1$) for all $y \in Y$. 


• There is $\rho_2 > 0$ such that the marginal measure $\bar{\mu} = P_\# \mu$ satisfies $\text{LSI}(\rho_2 N)$.

Then $\mu$ satisfies $\text{LSI}(\rho)$, where

$$
\rho := \frac{1}{2} \left( \rho_1 + \rho_2 + \frac{\kappa^2}{\rho_1} - \sqrt{\left( \rho_1 + \rho_2 + \frac{\kappa^2}{\rho_1} \right)^2 - 4 \rho_1 \rho_2} \right) > 0.
$$

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