Geometric Drive of the Universe’s Expansion

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Abstract. What if physics is just the way we perceive geometry? That is, what if geometry and physics will one day become one and the same discipline? I believe that will mean we will at last really understand physics, without postulates other than those defining the particular space where the physics play is performed. In this paper I use 5-dimensional spacetime as a point of departure and make a very peculiar assignment between coordinates and physical distances and time. I assume there is an hyperspherical symmetry which is made apparent by assigning the hypersphere radius to proper time and distances on the hypersphere to usual 3-dimensional distances. Time, or Compton time to distinguish from cosmic time is the 0th coordinate and I am able to project everything into 4-dimensions by imposing a null displacement condition.

Surprisingly nothing else is needed to explain Hubble’s expansion law without any appeal to dark matter; an empty Universe will expand naturally at a flat rate in this way. I then discuss the perturbative effects of a small mass density in the expansion rate in a qualitative way; quantitative results call for the solution of equations that sometimes have not even been clearly formulated and so are deferred to later work. A brief outlook of the consequences an hyperspherical symmetry has for galaxy dynamics allows the derivation of constant rotation velocity curves, again without appealing to dark matter.

An appendix explains how electromagnetism is made consistent with this geometric approach and justifies the fact that photons must travel on hypersphere circles, to be normal to proper time.

1. INTRODUCTION

The validity of any theory and its usefulness stem from the correctness of the predictions it allows; this is an unquestionable truth for all physicists and for the public in general. The elegance of a theory, however, is usually associated to a small number of principles or postulates and to a small set of mathematical equations, even if these turn out mathematically intricate and difficult to solve. This has been the case with General Relativity (GR) for many years, a theory which many physicists see as the paradigm of elegance. In spite of the unescapable validity of GR in celestial mechanics and laboratory experiments, the situation is not as clear in cosmology. The frustration of all known attempts to unify GR with Quantum Mechanics and the Standard Model of particle physics is another motivation for many serious people to burn their eyelashes in the search for some alternative way of formulating a new all encompassing theory.

In this work we will discuss geometry under the assumption that a well chosen geometry will allow, one day, the derivation of all the equations of physics from purely geometrical relations. This is, to a great extent, a question of the author’s personal faith without too much evidence to support it at the present time, but enough to motivate his continued search. If the assumption that physics is born out of geometry is true, then what we have to do is start off with the appropriate space, make the correct assignments between coordinates and physical entities and formulate the equations resulting from space symmetries and other space properties; these equations shall be the same as we encounter in physics. In previous work \[1\] it was shown that hyperbolic 5-dimensional space, also known as 5-dimensional spacetime, can generate 4-dimensional space without a metric by the condition of null displacement. This 4D space acquires a metric by promoting one of the coordinates to interval; depending on the choice of coordinate one can obtain either the usual GR space or an Euclidean 4D space designated as 4-Dimensional Optics (4DO) in view of the similarities with standard 3-dimensional optics. Mapping of geodesics between the two spaces can be done for all static metrics, as we will show below; it is not clear at present if the same operation is possible in some cases for non-static metrics, although it seems very likely that it is not. However, many interesting cases in GR are governed by a static metric and we can easily analyse these in 4DO to gain a different perspective. Einstein’s equations cannot be applied in 4DO and a suitable replacement was proposed in the cited paper, which leads to similar results in many cases but not in extreme ones.

The purpose of this paper is to show how 4DO can be used to explain a flat rate expansion of the Universe under
zero mass density. When one of the coordinates of 4DO is associated with the radius of an hypersphere this coordinate takes the physical meaning of proper time and flat rate expansion becomes a direct consequence of geometry. The basic principles involved have been explained in another paper [2], but the formulation is now cleaner than the original one. The usual 3 spatial coordinates are then associated with arc lengths on the hypersphere surface. The metric of Euclidean 4-space in hyperspherical coordinates is dependent on the hypersphere radius (proper time) which precludes its direct mapping into a GR metric; mapping would be possible by resorting to Cartesian coordinates at the expense of a difficult interpretation of their significance. We will also discuss the influence of non-zero mass density to show that small curvature and cosmological constants are expected. This conclusion can be reached independently of the set of equations used to find the metric of space with uniform mass density. Schwarzschild’s metric is PPN equivalent to the exponential metric proposed in both cited papers and consequently it is irrelevant which one is chosen if only first order approximation is envisaged.

Dark matter has been postulated not only to explain the rate of expansion in the Universe but also to account for the incredible orbital velocities found in spiral galaxies. This is a subject which cannot be properly addressed in this short presentation; galaxy dynamics is a difficult subject which the author did not investigate properly, but, also in this case, the postulate of 4DO in connection with an hyperspherical Universe seems to provide a qualitative explanation for the observations. We will give a brief indication of what may become an interesting subject for further work.

2. DYNAMICS IN 5D SPACETIME

In this section we characterize 5-dimensional spacetime and introduce the pertinent geometric algebra, $G_{4,1}$. For a comprehensive introduction to geometric algebra readers are referred to the two excellent books [3, 4]; here we will assume some familiarity with this tool.

This paper is about geometry and its relation to physics, which poses a problem with units right from the start. Geometry only cares about distances and angles, while physics uses a plethora of different units. Any parallel between the two fields must solve the units question right from the start. We note that, at least for the macroscopic world, physical units can all be reduced to four fundamental ones; we can, for instance, choose length, time, mass and electric charge as fundamental, as we could just as well have chosen others. Measurements are then made by comparison with standards; of course we need four standards, one for each fundamental unit. But now note that there are four fundamental constants: Planck constant ($\hbar$), gravitational constant ($G$), speed of light in vacuum ($c$) and proton electric charge ($e$), with which we can build four standards for the fundamental units. Table 1 lists the standards of this units’ system, frequently called Planck units, which the author prefers to designate by non-dimensional units. In this system all the fundamental constants, $\hbar$, $G$, $c$, $e$, become unity, a particle’s Compton frequency, defined by $\nu = mc^2/\hbar$, becomes equal to the particle’s mass and the frequent term $GM/(c^2r)$ is simplified to $M/r$. We can, in fact, take all measures to be non-dimensional, since the standards are defined with recourse to universal constants; this will be our posture. Geometry and physics become relations between pure numbers, vectors, bivectors, etc., but the geometric concept of distance is needed only for graphical representation.

Another problem we have to tackle is one of notation. Since we work in 5 dimensions but need also to consider 4-dimensional and 3-dimensional subspaces, we introduce an indexing convention which allows us to recognize immediately to which space or subspace each index refers. The following diagram shows the index naming convention used in this paper.

### TABLE 1. Standards for non-dimensional units used in the text; $\hbar \rightarrow$ Planck constant divided by $2\pi$, $G \rightarrow$ gravitational constant, $c \rightarrow$ speed of light and $e \rightarrow$ proton charge.

| Length       | Time       | Mass       | Charge    |
|--------------|------------|------------|-----------|
| $\sqrt{\frac{G\hbar}{c^3}}$ | $\sqrt{\frac{G\hbar}{c^5}}$ | $\frac{\hbar c}{G}$ | $e$ |
Indices in the range \{0, 4\} will be denoted with Greek letters \(\iota, \kappa, \lambda\). Indices in the range \{0, 3\} will also receive Greek letters but chosen from \(\mu, \nu, \xi\). For indices in the range \{1, 4\} we will use Latin letters \(i, j, k\) and finally for indices in the range \{1, 3\} we will use also Latin letters chosen from \(m, n, o\). Einstein’s summation convention will be adopted, as well as the compact notation for partial derivatives \(\partial_\iota = \partial / \partial x^\iota\). When convenient we will also make the assignments justified in Almeida [1]: \(x^0 \equiv t\) and \(x^4 \equiv \tau\). The squares of coordinates will be denoted by enclosing in parenthesis, to avoid confusion with superscript indices, but the same procedure will not be needed for \(t, \tau\) and spherical coordinates.

The geometric algebra \(G_{4,1}\) of the hyperbolic 5-dimensional space we want to consider is generated by the frame of orthonormal vectors \(\sigma_\iota\) verifying the relations
\[
(\sigma_0)^2 = -1, \quad (\sigma_\iota)^2 = (\sigma_{0\jmath})^2 = (i\sigma_0)^2 = 1; \tag{1}
\]
\[
\sigma_0 \sigma_\iota + \sigma_\iota \sigma_0 = 0, \quad (\sigma_\iota \sigma_\jmath + \sigma_\jmath \sigma_\iota = 2\delta_{ij}. \tag{2}
\]

We will simplify the notation for basis vector products using multiple indices, i.e. \(\sigma_\iota \sigma_\kappa \equiv \sigma_{\iota\kappa}\). The algebra is 32-dimensional and is spanned by the basis
- 1 scalar, \(1\),
- 5 vectors, \(\sigma_\iota\),
- 10 bivectors (area), \(\sigma_{\iota\kappa}\),
- 10 trivectors (volume), \(\sigma_{\iota\kappa\lambda}\),
- 5 tetravectors (4-volume), \(i\sigma_\iota\),
- 1 pseudoscalar (5-volume), \(i \equiv \sigma_{01234}\).

Several elements of this basis square to unity:
\[
(\sigma_\iota)^2 = (\sigma_0)^2 = (\sigma_{0\jmath})^2 = (i\sigma_0)^2 = 1;
\]
and the remaining square to \(-1:\)
\[
(\sigma_0)^2 = (\sigma_{0\jmath})^2 = (i\sigma_0)^2 = i^2 = -1. \tag{5}
\]

Note that the pseudoscalar \(i\) commutes with all the other basis elements, while being a square root of \(-1\), and plays the role of the scalar imaginary in complex algebra.

The geometric product of any two vectors \(a = a^\iota \sigma_\iota\) and \(b = b^\kappa \sigma_\kappa\) is evaluated making use of the distributive property
\[
ab = \left( -a^0 b^0 + \sum_i a^i b^i \right) + \sum_{i \neq \kappa} a^i b^\kappa \sigma_{\iota\kappa}; \tag{6}
\]
and we notice it can be decomposed into a symmetric part, a scalar called the inner or interior product, and an anti-symmetric part, a bivector called the outer or exterior product.
\[
ab = a \cdot b + a \wedge b, \quad ba = a \cdot b - a \wedge b. \tag{7}
\]

Reversing the definition one can write inner and outer products as
\[
a \cdot b = \frac{1}{2} (ab + ba), \quad a \wedge b = \frac{1}{2} (ab - ba). \tag{8}
\]

When a vector is operated with a multivector the inner product reduces the grade of each element by one unit and the outer product increases the grade by one. By convention the inner product of a vector and a scalar produces a vector.
We will encounter exponentials with multivector exponents; two particular cases of exponentiation are specially important. If \( u \) is such that \( u^2 = -1 \) and \( \theta \) is a scalar

\[
e^{u \theta} = 1 + u \theta - \frac{u^2 \theta^2}{2!} - \frac{u^3 \theta^3}{3!} + \frac{u^4 \theta^4}{4!} + \ldots
\]

\[
= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \{ = \cos \theta \}
+ u \theta - \frac{u^3 \theta^3}{3!} + \ldots \{ = u \sin \theta \}
= \cos \theta + u \sin \theta.
\]

Conversely if \( h \) is such that \( h^2 = 1 \)

\[
e^{h \theta} = 1 + h \theta + \frac{h^2 \theta^2}{2!} + h^3 \frac{\theta^3}{3!} + \frac{h^4 \theta^4}{4!} + \ldots
\]

\[
= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots \{ = \cosh \theta \}
+ h \theta + \frac{h^3 \theta^3}{3!} + \ldots \{ = h \sinh \theta \}
= \cosh \theta + h \sinh \theta.
\]

(9)

The exponential of bivectors is useful for defining rotations; a rotation of vector \( a \) by angle \( \theta \) on the \( \sigma_{12} \) plane is performed by

\[
d' = e^{\sigma_{12} \theta / 2} a e^{\sigma_{12} \theta / 2} = \tilde{R} a R;
\]

the tilde denotes reversion and reverses the order of all products. As a check we make \( a = \sigma_1 \)

\[
e^{-\sigma_{12} \theta / 2} \sigma_1 e^{\sigma_{12} \theta / 2} = \left( \cos \frac{\theta}{2} - \sigma_{12} \sin \frac{\theta}{2} \right) \sigma_1 \left( \cos \frac{\theta}{2} + \sigma_{12} \sin \frac{\theta}{2} \right)
= \cos \theta \sigma_1 + \sin \theta \sigma_2.
\]

(11)

Similarly, if we had made \( a = \sigma_2 \), the result would have been \(-\sin \theta \sigma_1 + \cos \theta \sigma_2 \).

If we use \( B \) to represent a bivector belonging to Euclidean 4-space and define its norm by \( |B| = (B \tilde{B})^{1/2} \), a general rotation is represented by the rotor

\[
R \equiv e^{-B / 2} = \cos \left( \frac{|B|}{2} \right) - B \frac{|B|}{2} \sin \left( \frac{|B|}{2} \right).
\]

(12)

The rotation angle is \( |B| \) and the rotation plane is defined by \( B \). A rotor is defined as a unitary even multivector (a multivector with even grade components only) which squares to unity: we are particularly interested in rotors with bivector components. It is more general to define a rotation by a plane (bivector) then by an axis (vector) because the latter only works in 3D while the former is applicable in any dimension.

The space spanned by frame vectors \( \sigma_i \) is flat; its geodesics are straight lines and we can define an elementary displacement on a geodesic by the vector

\[
\text{d}x = \sigma_i \text{d}x^i = \sigma_0 \text{d}x^0 + \sigma_i \text{d}x^i.
\]

(13)

Collapsing 5-dimensional space into 4 dimensions can be achieved by a projection; we choose to make this transition by imposing a null displacement condition, that is, the norm of the displacement vector must be null;

\[
(\text{d} x)^2 = \text{d} x \cdot \text{d} x = 0.
\]

(14)

Introducing (13) above we verify immediately that

\[
(\text{d} x^0)^2 - \sum (\text{d} x^i)^2 = 0;
\]

(15)
and this is equivalent to either of the relations

\[
(dx^0)^2 = \sum (dx^i)^2; \tag{16}
\]

\[
(dx^4)^2 = (dx^0)^2 - \sum (dx^m)^2. \tag{17}
\]

The former of these relations defines an Euclidean 4-space where \((dx^0)^2\) is taken as interval and the latter defines Minkowski spacetime with \((dx^4)^2\) as interval. We see by this construction that Euclidean and Minkowski 4-spaces can be taken as belonging to the null subspace of 5D spacetime; we can obtain one or the other, depending on the coordinate that we choose for interval. Remember though we are only considering displacements along geodesics, i.e. straight lines. A very different approach to the same subject was used in \([5]\) and the first author to notice this equivalence was probably Montanus \([6, 7]\).

Everything that was said above is true in geometry and has no implications for physics until we decide to assign some of the coordinates to physical entities. Some of those assignments are carried over from previous work; for instance we have already established that coordinate \(x^4\) is to be taken as time and coordinate \(x^4\) as proper time \([1]\); accordingly we will frequently represent \(x^0\) with the letter \(t\) and \(x^4\) with the letter \(\tau\). We will also simplify the notation for time and proper time derivatives by writing \(df/dt \equiv \dot{f}; df/d\tau \equiv \dot{f}\).

Dividing both members of (13) by \(dt\) one defines a 4-dimensional velocity vector \(v\);

\[
\dot{x} = \sigma_0 + \sigma_i \dot{x}^i = \sigma_0 + v. \tag{18}
\]

If we are in the null displacement subspace \((\dot{x})^2\) is necessarily null and we recognize that \(v\) is unitary

\[
v \cdot v = \sum (\dot{x}^i)^2 = 1. \tag{19}
\]

The velocity vector can then be obtained by rotation of any unitary vector and it is particularly interesting to note that it can be expressed as a rotation of the \(\sigma_4\) frame vector.

\[
v = \tilde{R}\sigma_4 R. \tag{20}
\]

The rotation angle is a measure of the 3-dimensional, physical, velocity. A null angle corresponds to a tangent vector; \(\pi/2\) is the plane normal to both \(\dot{x}^0\) and \(\dot{x}^4\). The idea that physical velocity can be seen as the 3D component of a unitary 4D vector has been explored in several papers but see \([8]\).

Instead of dividing (13) by \(dt\) we can divide by \(d\tau\), obtaining

\[
\dot{x} = \sigma_0 \dot{x}^0 + \sigma_m \dot{x}^m + \sigma_4; \tag{21}
\]

squaring the second member and noting that it must be null we obtain

\[
(\dot{x}^0)^2 - \sum (\dot{x}^m)^2 = 1. \tag{22}
\]

We then define a bivector, called relativistic 4-velocity, by

\[
v = \sigma_0 \dot{x}^0 + \sigma_m \dot{x}^m, \tag{23}
\]

such that \(v^2 = v \cdot v = 1\). The relativistic 4-velocity is a bivector in this space and not a vector as in special relativity but it represents the same physical concept; in particular we note that any 4-velocity can be obtained by a Lorentz transformation of bivector \(\sigma_{04}\).

\[
v = T \sigma_{04} T, \tag{24}
\]

where \(T\) is of the form \(T = \exp(B)\) and \(B\) is a bivector whose plane is normal to \(\sigma_4\). Note that \(T\) is a pure rotation when the bivector plane is normal to both \(\sigma_0\) and \(\sigma_4\).

In order to study dynamics we must introduce bent space by allowing for non-orthonormed frame vectors;

\[
g_4 = n^k \sigma_k, \tag{25}
\]

\(g_4\) is called the \textit{refractive index frame} and \(n^k\) the \textit{refractive index tensor} or simply the \textit{refractive index}. The designation is borrowed from 3D optics and the refractive index tensor can be seen as the 5D generalization of a dielectric refractive index. The definition of frame vectors with recourse to the orthonormed frame can only be applied to bent spaces and
not to general curved ones but we believe this is sufficient for expressing all dynamics; most of the derivations that follow, however, would apply equally well to spaces of general curvature. We introduce now the reciprocal frame $g^i$ such that

$$g^i \cdot g_k = \delta^i_k.$$

(26)

In non-orthonormed frames we define the elementary optical displacement vector

$$ds = g_i dx^i.$$

(27)

The designation is again borrowed from 3D optics and calls for the optical path length. For simplicity we will consider only those cases where $g_0 = \sigma_0, \ g_i = n^i_j \sigma_j$;

(28)

to get $ds = \sigma_0 dt + g_i dx^i$. That is, we are considering only spaces where the refractive index is a 4-rank tensor and does not modify the zeroth frame vector. A further simplification results from imposing that the refractive index depends only on the 3 spatial $x^m$ coordinates. We now replace the null displacement condition by a similar condition applied to optical displacement. From $ds^2 = 0$ we write immediately

$$dt^2 = g_{ij} dx^i dx^j;$$

(29)

Multiplying the two equations member by member, the first member becomes $g^{i4} ds^2$ and must therefore be null. We have then

$$0 = g^{i4} \left( -dt^2 + g_{mn} dx^m dx^n + g^{44} d\tau^2 \right).$$

(32)

Note that non-scalar terms in the second member cancel out necessarily, so that the first member can be null. Rearranging the equation we can write

$$d\tau^2 = \frac{1}{g^{44} \left( -dr^2 + g_{mn} dx^m dx^n + g^{44} d\tau^2 \right)}.$$

(33)

This is clearly a GR metric if the second member depends only on the $x^m$ coordinates, that is, if the metric is static. We have thus established a metric conversion method between GR and 4DO applicable to static metrics. The importance of this conclusion cannot be overstressed; we have concluded that geodesics of 4DO and GR spaces can be mapped to each other when the refractive index is a function only of the 3 spatial coordinates. When this happens any dynamics which can be studied as free fall in GR can also be studied as free fall in 4DO, providing a different angle of approach to the same problem.

The geodesics of 4DO space can be found, as in any other space, by consideration of the Lagrangian

$$L = \frac{g_{ij} \dot{x}^i \dot{x}^j}{2} = \frac{1}{2};$$

(34)

from the Lagrangian one derives immediately the conjugate momenta

$$v_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij} \dot{x}^j.$$

(35)

Note the use of the lower index ($v_i$) to represent conjugate momenta while velocity components have an upper index ($v^i$). The conjugate momenta are the components of the conjugate momentum vector $v = g^i v_i$, which can be written in two alternative forms

$$v = g^i v_i = g^i g_{ij} \dot{x}^j = g_{ij} \dot{x}^j.$$

(36)

1 This restriction is not applied in the appendix.
We conclude that conjugate momentum and velocity are the same vector but their components are referred to the reciprocal and refractive index frames, respectively. The geodesic equations can now be written in the form of Euler-Lagrange equations

$$\dot{v}_i = \partial_i L; \quad (37)$$

Space is isotropic if the refractive index does not depend on direction, and so the three $g_m$ vectors must be related to $\sigma_m$ by a common scale factor, which may be a function of position. The scale coefficient for $g_4$ does not need to be the same as for the other frame vectors and hence we will characterize an isotropic space by the refractive index frame

$$g_m = n_r\sigma_m, \quad g_4 = n_4\sigma_4. \quad (38)$$

In problems with spherical symmetry we use spherical coordinates and it must be

$$g_r = n_r\sigma_r, \quad (39)$$
$$g_\theta = n_r r \sigma_\theta, \quad (40)$$
$$g_\phi = n_r r \sin \theta \sigma_\phi, \quad (41)$$
$$g_4 = n_4\sigma_\tau; \quad (42)$$

with both $n_r$ and $n_4$ functions of $r$.

We will now look at Schwarzschild’s metric to see how it can be transposed to 4D optics. The usual form of the metric is

$$d\tau^2 = \left(1 - \frac{2m}{\chi}\right) dt^2 - \left(1 - \frac{2m}{\chi}\right)^{-1} d\chi^2 - \chi^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right); \quad (43)$$

where $m$ is a spherical mass and $\chi$ is a radial coordinate, not the distance to the spherical mass’ centre. This form is non-isotropic but a change of coordinates can be made that returns an isotropic form, see D’Inverno [10, section 14.7].

$$r = \left(\chi - m + \sqrt{\chi^2 - 2m\chi}\right)/2; \quad (44)$$

and the new form of the metric is

$$d\tau^2 = \left(1 - \frac{m}{2r}\right)^2 dr^2 - \left(1 + \frac{m}{2r}\right)^4 \left[dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right]. \quad (45)$$

This corresponds to the refractive index coefficients

$$n_4 = 1 + \frac{m}{2r} \frac{1}{1 - \frac{2m}{r}}, \quad n_r = \frac{\left(1 + \frac{m}{2r}\right)^3}{1 - \frac{2m}{r}}, \quad (46)$$

which can then be used in 4DO Euclidean space.

We analyse now the constraints on the refractive index so that experimental data on light bending and perihelion advance in closed orbits can be predicted; this will allow us to propose another set of refractive indices which will be more convenient than those just obtained. Light rays are characterized by $d\tau = 0$ both in 4DO and in general relativity; the effective refractive index for light is then

$$\sqrt{\sum (\chi_m)^2} = n_r. \quad (47)$$

For compatibility with experimental observations $n_r$ must be expanded in series as (see [11])

$$n_r = 1 + \frac{2m}{r} + O(1/r)^2. \quad (48)$$

This is the bending predicted by Schwarzschild’s metric and has been confirmed by observations.
For the analysis of orbits its best to rewrite (34) for spherical coordinates. Since we know that orbits are flat we can make $\theta = \pi/2$

$$n_t^2 t^2 + n_r^2 (r^2 + r^2 \phi^2) = 1.$$  

(49)

The metric depends only on $r$ and we get two conservation equations

$$n_t^2 \tau = \frac{1}{\gamma}, \quad n_r^2 r^2 \phi = J_\phi.$$  

(50)

Replacing in (49)

$$\frac{1}{\gamma^2 n_3^2} + n_r^2 r^2 + \frac{J_\phi^2}{n_3^2} = 1.$$  

(51)

The solution of this equation calls for a change of variable $r = 1/b$; as a result it is also $\dot{r} = \phi dr/d\phi$; replacing in the equation and rearranging

$$\left(\frac{db}{d\phi}\right)^2 = \frac{n_r^2}{J_\phi^4} - \frac{n_r^2}{J_\phi^4 \gamma^2 n_3^2} - b^2.$$  

(52)

To account for light bending we have established that $n_r \approx 1 + 2mb$. For $n_4$ we need 2nd order approximation [11], so we make $n_4 \approx 1 + \alpha mb + \beta m^2 b^2$. We can also assume that velocities are low, so $\gamma \approx 1$

$$\left(\frac{db}{d\phi}\right)^2 \approx \frac{2\alpha m^2}{J_\phi^4} b + \left(-1 + \frac{8\alpha m^2}{J_\phi^4} - \frac{3\alpha^2 m^2}{J_\phi^4} + \frac{2\beta m^2}{J_\phi^4}\right) b^2.$$  

(53)

For compatibility with Kepler’s 1st order predictions $\alpha = 1$; then, for compatibility with observed planet orbits, $\beta = 1/2$. Together with the constraint for $n_0$, these are the conditions that must be verified by the refractive indices to be in agreement with experimental data; any refractive indices verifying such conditions are then perfectly legitimate in terms of predictions for those two observations.

We know, of course, that the refractive indices corresponding to Schwarzschild’s metric verify the constraints above, however that is not the only possibility. Schwarzschild’s metric is a consequence of Einstein’s equations when one postulates that vacuum is empty of mass and energy, but the same does not necessarily apply in 4DO. In [1] we proposed a counterpart to Einstein equations in 4DO whose solutions are in full agreement with observations; the resulting refractive index is

$$n_r = e^{2m/r} \approx 1 + \frac{2m}{r};$$

(54)

$$n_4 = e^{m/r} \approx 1 + \frac{m}{r} + \frac{m^2}{2r^2}.$$  

(55)

Montanus [7] arrives at the same solutions with a different reasoning; the same metric is also due to Yilmaz [12, 13, 14, 15].

These refractive index coefficients are as effective as those derived from Schwarzschild’s metric for light bending and perihelium advance prediction for small $m/r$; there is one singularity for $r = 0$ which is not a physical difficulty since before that stage quantum phenomena have to be considered and the metric ceases to be applicable; in other words, we must change from geometric to wave optics approach.

### 3. HYPSHERSPHERICAL COORDINATES

Deriving physical equations and predictions from purely geometrical equations is an exercise whose success depends on the correct assignment of coordinates to physical entities; the same space will produce different predictions if different options are taken for coordinate assignment. Since the birth of special relativity it has been usual to assign three coordinates to orthogonal directions in physical space and a zeroth coordinate to time. This is a totally arbitrary assignment, which has gained acceptance by the correct predictions it originates in many circumstances. We discussed above that it is also perfectly legitimate to replace the assignment of coordinate zero to time by an assignment of coordinate four with proper time. Geodesic (straight line) movement can be predicted equally well in both cases. In terms of curvature, flat space is usually associated with absolute emptiness in a physical sense.
We are now going to experiment with a different assignment of flat space coordinates, which will explore the possibility that physics and the Universe have an inbuilt hyperspherical symmetry. The exercise consists on assigning coordinate \( x^4 = \tau \) to the radius of an hypersphere and the three \( x^m \) coordinates to distances measured on the hypersphere surface. If the hypersphere radius is very large we will not be able to notice the curvature on everyday phenomena, in the same way as everyday displacements on Earth don’t seem curved to us; but the Universe as a whole will manifest the consequences of its hyperspherical symmetry. Using the Earth as a 3-dimensional analogue of an hyperspherical Universe, although our everyday life is greatly unaffected by Earth’s curvature the atmosphere senses this curvature the same way as everyday displacements on Earth don’t seem curved to us; but the Universe as a whole will manifest manifestations of it in winds and climate. What we propose here is an exercise; it is an arbitrary assignment between coordinates and physical entities; the validity of such exercise can only be judged by the predictions it allows and how well they conform with observations.

Hyperspherical coordinates are characterized by one distance coordinate, \( \tau \) and three angles \( \rho, \theta, \phi \); following the usual procedure we will associate with these coordinates the frame vectors \( \{ \sigma_\tau, \sigma_\rho, \sigma_\theta, \sigma_\phi \} \). The position vector for one point in 5D space is quite simply

\[
x = \tau \sigma_0 + \tau \sigma_4. \tag{56}
\]

In order to write an elementary displacement \( dx \) we must consider the rotation of frame vectors, but we don’t need to think hard about it because we can extend what is known from ordinary spherical coordinates \[16\].

\[
dx = \sigma_0 d\tau + \sigma_4 d\tau + \tau \sigma_\rho d\rho + \tau \sin \rho \sigma_\theta d\theta + \tau \sin \rho \sin \theta \sigma_\phi d\phi. \tag{57}
\]

Just as before, we consider only null displacements to obtain time intervals;

\[
d\tau^2 = d\tau^2 + \tau^2 [d\rho^2 + \sin^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)]. \tag{58}
\]

The velocity vector, \( v = \dot{x} - \sigma_0 \), can be immediately obtained from the displacement vector dividing by \( d\tau \)

\[
v = \sigma_0 \dot{\tau} + \tau \sigma_\rho \dot{\rho} + \tau \sin \rho \sigma_\theta \dot{\theta} + \tau \sin \rho \sin \theta \sigma_\phi \dot{\phi}. \tag{59}
\]

Geodesics of flat space are naturally straight lines, no matter which coordinate system we use, however it is useful to derive geodesic equations from a Lagrangian of the form \[24\]; in hyperspherical coordinates the Lagrangian becomes

\[
2L = v^2 = \tau^2 + \tau^2 [\rho^2 + \sin^2 \rho (\theta^2 + \sin^2 \theta \phi^2)]. \tag{60}
\]

Because de Lagrangian is independent of \( \phi \) we can establish a conserved quantity

\[
J_\phi = \tau^2 \sin^2 \rho \sin^2 \theta \dot{\phi}. \tag{61}
\]

It may seem strange that any physically meaningful relation can be derived from the simple coordinate assignment that we have made, that is, proper time is associated with hypersphere radius and the three usual space coordinates are assigned to distances on the hypersphere radius. This unexpected fact results from the possibility offered by hyperspherical coordinates to explore a symmetry in the Universe that becomes hidden when we use Cartesian coordinates. In the real world we measure distances between objects, namely cosmological objects, rather than angles; we have therefore to define a distance coordinate, which is obviously \( r = \tau \rho \). It does not matter where in the Universe we place the origin for \( r \) and we find it convenient to place ourselves on the origin.

Radial velocities \( \dot{r} \) measure movement in a radial direction from our observation point; we are particularly interested in this type of movement in order to find a link to the Hubble relation. Applying the chain rule and then replacing \( \rho \)

\[
\dot{r} = \rho \dot{\tau} + \rho \tau = \frac{\dot{\tau}}{\tau} r + \rho \tau. \tag{62}
\]

We expect objects that have not suffered any interaction to move along \( \sigma_\tau \); from \[59\] we see that this implies \( \dot{\rho} = \dot{\theta} = \dot{\phi} = 0 \) and then \( \dot{\tau} \) becomes unity. Replacing in the equation above and rearranging

\[
\frac{\dot{r}}{r} = \frac{1}{\tau}. \tag{63}
\]

What this equation tells us is exactly what is expressed by the Hubble relation. The value of \( \tau \) can be taken as constant for any given observation because the distance information is carried by photons and these preserve proper time\[2\]. The

\[2\] In order to preserve proper time photons must travel on the hypersphere surface and thus don’t follow geodesics; the way in which this is made compatible with electromagnetism is briefly discussed in the appendix.
first member of the equation is the definition of the Hubble parameter and we can then write \( H = 1/\tau \). In this way we find the physical meaning of coordinate \( \tau \) as being the Universe’s age.

How does the use of hyperspherical coordinates affect dynamics in our laboratory experiments? We would like to know if these coordinates need only be considered in problems of cosmological scale or, on the contrary, there are implications for everyday experiments. The answer implies rewriting \( (67) \) with distance rather than angle coordinates; replacing \( \rho \),

\[
dx = \sigma_0 dt + \left( \sigma_4 - \frac{r}{\tau} \sigma_0 \right) d\tau + \sigma_1 dr + r(\sigma_0 d\theta + \sin \theta \sigma_0 d\phi).
\]

Evaluating time intervals from the null displacement condition, as before

\[
dr^2 = \left[ 1 + \left( \frac{r}{\tau} \right)^2 \right] d\tau^2 - 2 \frac{r}{\tau} d\tau dr + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

This would be a version of \( (16) \) in spherical coordinates, were it not for the extra terms with powers of \( r/\tau \) in the second member. The coefficient \( r/\tau \) implies a comparison between the distance from the object to the observer and the size of the Universe; remember that \( \tau \) is both time and distance in non-dimensional units. We can say that ordinary special relativity will apply for objects which are near us, but distant objects will show in their movement an effect of the Universe’s hyperspherical nature.

We have established the refractive indices \( n_r \) \( (54) \) and \( n_4 \) \( (55) \) to account for the dynamics near a massive sphere using Cartesian coordinates; since this is frequently applied on a cosmological scale, we must find out how the dynamics is modified by the use of hyperspherical coordinates. Using the refractive indices and hyperspherical coordinates, noting that \( n_r = n_4 \), the optical displacement \( (27) \) becomes

\[
ds = \sigma_0 dt + n_4 \sigma_4 d\tau + n_4^2 (\tau \sigma_0 d\rho + \tau \sin \rho \sigma_0 d\theta + \tau \sin \theta \sigma_0 d\phi).
\]

In radial displacements we can set \( \theta = \phi = 0 \); introducing this and dividing by \( dt \)

\[
\dot{s} = \sigma_0 + n_4 \sigma_4 \dot{\tau} + n_4^2 \tau \sigma_0 \dot{\rho}.
\]

Squaring \( \dot{s} \) and invoking null displacement condition

\[
n_4^2 \dot{\tau}^2 + n_4^4 \tau^2 \dot{\rho}^2 = 1.
\]

and replacing \( \tau \rho \) by \( \dot{r} - r \dot{\tau}/\tau \)

\[
n_4^2 \dot{\tau}^2 + n_4^4 \left[ \dot{r}^2 + \left( \frac{\dot{\tau}}{\tau} \right)^2 \left( r^2 - 2 \tau r \right) \right] = 1.
\]

Dividing both members by \( n_4^4 r^2 \) and rearranging results in the equation

\[
\left( \frac{\dot{r}}{r} \right)^2 = \left( \frac{1}{n_4^2} - \frac{\dot{\tau}^2}{n_4^4} \right) \frac{1}{r^2} - \left( \frac{\dot{\tau}}{\tau} \right)^2 + 2 \frac{\dot{\tau}}{\tau r}.
\]

As a further step we expand the second member in series of \( m \) and take the two first terms, in order to get an equation that allows comparison to those used in cosmology.

\[
\left( \frac{\dot{r}}{r} \right)^2 \approx \frac{1 - \dot{\tau}^2}{r^2} + \frac{(2 \dot{\tau}^2 - 4) m}{r^3} - \left( \frac{\dot{\tau}}{\tau} \right)^2 + 2 \frac{\dot{\tau}}{\tau r}.
\]

The previous equation applies to bodies moving radially under the influence of mass \( m \) located at the origin which is, remember, the observer’s position. For comparison we derive the corresponding equation in Cartesian coordinates; starting with \( (68) \) it is now

\[
n_4^2 \dot{\tau}^2 + n_4^4 \dot{r}^2 = 1;
\]

dividing by \( n_4^4 r^2 \) and rearranging

\[
\left( \frac{\dot{r}}{r} \right)^2 = \left( \frac{1}{n_4^2} - \frac{\dot{\tau}^2}{n_4^4} \right) \frac{1}{r^2} \approx 1 - \frac{\dot{\tau}^2}{r^2} + \frac{(2 \dot{\tau}^2 - 4) m}{r^3}.
\]
If we want to apply these equations to cosmology it is easiest to follow the approach of Newtonian cosmology, which produces basically the same results as the relativistic approach but presumes that the observer is at the centre of the Universe [14, 17]. In order to adopt a relativistic approach we need equations that replace Einstein’s in 4DO. A set of such equations has been proposed [1] but their application in cosmology has not yet been tested, so we will have to defer this more correct approach to a forthcoming paper. The strategy is to consider a general object at distance \( r \) from the observer, moving away from the latter under the gravitational influence of the mass included in a sphere of radius \( r \). If we designate by \( \mu \) the average mass density in the Universe, then mass \( m \) in (71) is \( 4\pi \mu r^3/3 \); this will have to be considered further down.

Friedman equation governs standard cosmology and can be derived both from Newtonian and relativistic dynamics, with different consequences in terms of the overall size of the Universe and the observer’s privileged position. From the cited references we write Friedman equation as

\[
\left( \frac{\dot{r}}{r} \right)^2 = \frac{8\pi}{3} \mu + \frac{\Lambda}{3} - \frac{k}{r^2};
\]

with \( \Lambda \) a cosmological constant and \( k \) the curvature constant; the gravitational constant was not included because it is unity in non-dimensional units and the equation is written in real, not comoving, coordinates. In order to compare (71) with Friedman equation there is a problem with the last term because the Hubble parameter \( \dot{r}/r \) does not appear isolated in the first member; we will find a way to circumvent the problem later on but first let us look at what (71) tells us when the mass density is zeroed. In this case \( n_4 = 1 \) and we find from (63) that \( \dot{\tau} \) is unity, unless \( \dot{\rho} \) is non-zero, for which we can find no reasonable explanation. Replacing \( n_4 \) and \( \dot{\tau} \) with unity in (71) we find that \( \dot{r}/r = 1/\tau \), confirming what had already been found in (65). Comparing with Friedman equation, this corresponds to a flat Universe with a critical mass density \( \mu = \mu_c \); it is immediately obvious that \( \mu_c = 3/(8\pi \tau^2) \). Let us not overlook the importance of this conclusion because it completely removes the need for a critical density if the Universe is flat; remember this is one of the main reasons to invoke dark matter in standard cosmology. Notice also that this conclusion does not depend on a privileged observer, because it is just a consequence of space symmetry and not of dynamics.

Let us now see what happens when we consider a small mass density; here we are talking about matter that is observed or measured in some way but not postulated matter. The matter density that we will consider is of the order of 1% of the presently accepted value. It is therefore just a perturbation of the flat solution that we described above and the fact that we are presuming a privileged observer has to be taken just for this perturbation. The first thing we note when we consider matter density is that \( \dot{\tau} < 1 \), because there is now a component of the velocity vector along \( \sigma_\rho \). Ideally we should solve the Euler-Lagrange equations resulting from (63) in order to find \( \tau \) and \( \rho \) but this is a difficult process and we shall carry on with just a qualitative discussion. Considering that we are discussing a perturbation it is legitimate to make \( \dot{r}/r \approx \dot{\tau}/\tau \) and the two last terms in the second member of (71) can be combined into one single term \( (\dot{\tau}/\tau)^2 \), the same as we encountered for the flat solution, albeit with a numerator slightly smaller than unity. The first term has now become slightly positive and we can see from Friedman equation that this corresponds to a negative curvature constant, \( k \), and to an open Universe. Lastly the second term includes the mass \( m \) of a sphere with radius \( r \) and can be simplified to \( 8\pi \mu (\dot{\tau}^2 - 2)/3 \); this has the effect of a negative cosmological constant; the combined effect of the two terms is expected to close the Universe [17, 14]. The previous discussion was done in qualitative terms, making use of several approximations, for which reason we must question some of the findings and expect that after more detailed examination they may not be quite as anticipated; in particular there is concern about the refractive indices used, which were derived in Cartesian coordinates both by the author and those that preceded him in using an exponential metric; it may happen that the transposition to hyperspherical coordinates has not been properly made, with consequences in the perturbative analysis that was superimposed on the flat solution. The latter, however, is totally independent of such concerns and allows us to state that the assumption of hyperspherical symmetry for the Universe dispenses with dark matter in accounting for the gross of observed expansion.

Dark matter is also called in cosmology to account for the extremely high rotation velocities found in spiral galaxies [18, 19] and we will now take a brief look at how hyperspherical symmetry can help explain this phenomenon. Galaxy dynamics is an extremely complex subject, which we do not intend to explore here due to lack of space but most of all due to lack of author’s competence to approach it with any rigour; we will just have a very brief outlook at the equation for flat orbits, to notice that an effect similar to the familiar Coriolis effect on Earth can arise in an expanding hyperspherical Universe and this could explain most of the observed velocities on the periphery of galaxies. Let us recall (62), divide by \( d\tau \) and invoke null displacement to obtain the velocity

\[
v = (\sigma_\rho - \frac{r}{\tau} \sigma_\tau) \dot{\tau} + \sigma_\rho \dot{r} + r(\sigma_\theta \dot{\theta} + \sin \theta \sigma_\phi \dot{\phi}).
\]
If orbits are flat we can make $\theta = \pi/2$ and the equation simplifies to

$$v = \tau\sigma_4 + \left(\frac{\dot{r}}{\tau} - \frac{r \tau}{\dot{\tau}}\right)\sigma_\rho + r\phi\sigma_\varphi.$$  \hspace{1cm} (76)

Suppose now that something in the galaxy is pushing outwards slightly, so that the parenthesis is zero; this happens if $\dot{r}/r = \tau/\dot{\tau}$ and can be caused by a pressure gradient, for instance. The result is that (76) now accepts solutions with constant $\phi$, which is exactly what is observed in many cases; swirls will be maintained by a radial expansion rate which exactly matches the quotient $\dot{\tau}/\tau$. In any practical situation $\dot{\tau}$ will be very near unity and the quotient will be virtually equal to the Hubble parameter; thus the expansion rate for sustained rotation is $\dot{r}/r \approx H$. If applied to our neighbour galaxy Andromeda, with a radial extent of 30 kpc, using the Hubble parameter value of 81 km s$^{-1}$/Mpc, the expansion velocity is about 2.43 km s$^{-1}$; this is to be compared with the orbital velocity of near 300 km s$^{-1}$ and probably within the error margins. An expansion of this sort could be present in many galaxies and go undetected because it needs only be of the order of 1% the orbital velocity.

4. CONCLUSION

The approach to the equations that govern the Universe examined in this paper can be compared to the revolution brought to 15th century geography and navigation by the consideration of a spherical Earth, a concept as old as Pythagoras and Aristotle but not widely accepted until then. It is unimaginable today to explain any world scale phenomenon without recourse to spherical coordinates, because these make full exploitation of Earth’s spherical symmetry and render equations enormously simpler than they would be if expressed in Cartesian coordinates.

Making an hypothesis that the Universe as a whole has an inbuilt hyperspherical symmetry we were able to derive Hubble’s law as a direct consequence of geometry in a Universe completely devoid of any matter. The existence of a minute mass density can then be seen to introduce a perturbation in the main picture, being responsible for a slight curvature and a small cosmological constant. Similarly to what happens in the Earth’s atmosphere, we were also able to demonstrate the existence of constant rotation velocity swirls that can be the basis for understanding galaxy dynamics.

Mathematically the argument was set on purely geometrical grounds, with 5-dimensional spacetime as a point of departure. This space was shown to produce both GR spacetime and an Euclidean metric 4-dimensional space, by an imposition of null displacement. Euclidean metric 4D space was then used to formulate the hyperspherical symmetry hypothesis and derive its consequences.

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A. ELECTROMAGNETISM IN 5D SPACETIME

We will treat electromagnetism as a local phenomenon, avoiding the need to use hyperspherical coordinates. The easiest way to include electromagnetism in the geometry of 5-dimensional spacetime is to consider a non-orthonormed frame; in terms of the reciprocal frame we make

\[ g^\mu = \sigma_\mu, \quad g^A = \frac{q}{m} A_\mu \sigma^\mu + \sigma^4. \]  

(77)

This is used for the definition of a covariant derivative

\[ \mathcal{D} = g^\iota \partial_\iota = \sigma^\mu \partial_\mu + (\sigma_4 + \frac{q}{m} A_\mu \sigma^\mu) \partial_4. \]  

(78)

A covariant Laplacian is defined as \( \mathcal{D}^2 = \mathcal{D} \cdot \mathcal{D} \) and being the square of a vector it is a scalar. It follows that the Laplacian of a vector must always be a vector and we have by necessity

\[ \mathcal{D}^2 g^4 = \frac{J}{m}. \]  

(79)

The covariant derivative of \( g^4 \) can have scalar and bivector parts but by choice of \( A_\mu \) we can zero the scalar part and thus define the Faraday bivector

\[ F = m \mathcal{D} \wedge g^4, \]  

(80)

so that \( \mathcal{D} F = J \), our version of Maxwell’s equations [3]. In the absence of currents we look for solutions with the second member zero, that is \( \mathcal{D}^2 g^4 = 0 \), which reduces to

\[ \mathcal{D}^2 A = 0, \]  

(81)

with \( A = A_\mu \sigma^\mu = A^\mu \sigma_\mu \). If \( A \) does not depend on \( x^4 \), then the Laplacian reduces to \(-\partial^2 / \partial t^2 + \sum \partial^2 / \partial (x^m)^2\) and the equation is a straightforward wave equation with plane wave solutions normal to \( \sigma_4 \).

We use the argument above to sustain that electromagnetic waves must follow lines normal to \( \sigma_4 \) in any circumstance, even when \( x^4 \) is the radius of an hypersphere. Electromagnetic waves will follow geodesics in usual flat space but they will follow great circles on the hypersphere if hyperspherical symmetry is assumed.