Standing waves with large frequency for 4-superlinear Schrödinger-Poisson systems

Huayang Chen \(^a\)  Shibo Liu \(^b\)*

\(^a\) Department of Mathematics, Shantou University, Shantou 515063, China
\(^b\) School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

We consider standing waves with frequency \(\omega\) for 4-superlinear Schrödinger-Poisson system. For large \(\omega\) the problem reduces to a system of elliptic equations in \(\mathbb{R}^3\) with potential indefinite in sign. The variational functional does not satisfy the mountain pass geometry. The nonlinearity considered here satisfies a condition which is much weaker than the classical (AR) condition and the condition (Je) of Jeanjean. We obtain nontrivial solution and, in case of odd nonlinearity an unbounded sequence of solutions via the local linking theorem and the fountain theorem, respectively.

Keywords: Schrödinger-Poisson systems; 4-superlinear; \((PS)\) condition; local linking; fountain theorem.

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1 Introduction

When we are looking for standing wave solutions \(\psi(t,x) = e^{-i\omega t/\hbar}u(x)\) for the nonlinear Schrödinger equation
\[
 i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + U(x) \psi + \phi \psi - \tilde{g}(|\psi|) \psi, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3
\] (1.1)
coupled with the Poisson equation
\[
 -\Delta \phi = |\psi|^2, \quad x \in \mathbb{R}^3,
\]
we are led to a system of elliptic equations in \(\mathbb{R}^3\) of the form
\[
 \begin{cases}
 -\Delta u + V(x)u + \phi u = g(u), & \text{in } \mathbb{R}^3, \\
 -\Delta \phi = u^2, & \text{in } \mathbb{R}^3,
\end{cases}
\] (1.2)
where the potential \(V(x) = U(x) - \omega\) and, without lose of generality we assume \(\hbar^2 = 2m\), so that the coefficient of \(\Delta u\) in the first equation is \(-1\). Due to the physical context, the nonlinearity \(g(t) = \tilde{g}(|t|)t\) satisfies
\[
 \lim_{|t| \to 0} \frac{g(t)}{t} = 0.
\] (1.3)

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†email: liusb@xmu.edu.cn
The problem (1.2) has a variational structure. It is known that there is an energy functional \( \mathcal{J} \) on \( H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \),
\[
\mathcal{J}(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) \, dx - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 \, dx - \int_{\mathbb{R}^3} G(u) \, dx,
\]
such that \((u, \phi)\) solves (1.2) if and only if it is a critical point of \( \mathcal{J} \). However, the functional \( \mathcal{J} \) is strongly indefinite and is difficult to investigate.

For \( u \in H^1(\mathbb{R}^3) \), it is well known that under suitable assumptions, \( \Phi \) is of class \( C^1 \) on some Sobolev space and, if \( u \) is a critical point of \( \Phi \), then \((u, \phi_u)\) is a solution of (1.2); see e.g. [9, pp. 4929–4931] for the details. In other words, finding critical points \((u, \phi)\) of \( \mathcal{J} \) has been reduced to looking for critical points \( u \) of \( \Phi \). The idea of this reduction method is originally due to Benci and his collaborators [7, 8].

In this paper we assume that the potential \( V \) and the nonlinearity \( g \) satisfy the following conditions.

1. \( V \in C(\mathbb{R}^3) \) is bounded from below and, \( \mu(V^{-1}(-\infty, M]) < \infty \) for every \( M > 0 \), where \( \mu \) is the Lebesgue measure on \( \mathbb{R}^3 \).
2. \( g \in C(\mathbb{R}) \) satisfies (1.3) and there exist \( C > 0 \) and \( p \in (4, 6) \) such that
\[
|g(t)| \leq C(1 + |t|^{p-1}).
\]
3. There exists \( b > 0 \) such that \( 4G(t) \leq tg(t) + bt^2 \), and
\[
\lim_{|t| \to \infty} \frac{g(t)}{t^3} = +\infty.
\]

We emphasize that unlike all previous results about the system (1.2), see e.g. [1, 4, 9, 17, 19], we have not assumed that the potential \( V \) is positive. This means that we are looking for standing waves of (1.1) with large frequency \( \omega \).

When \( V \) is positive, the quadratic part of the functional \( \Phi \) is positively definite, and \( \Phi \) has a mountain pass geometry. Therefore, the mountain pass lemma [2] can be applied. In our case, the quadratic part may posses a nontrivial negative space \( X^- \), so \( \Phi \) no longer possesses the mountain pass geometry. A natural idea is that we may try to apply the linking theorem (also called generalized mountain pass theorem) [18, Theorem 2.12]. Unfortunately, due to the presence of the term involving \( \phi_u \), it turns out that \( \Phi \) does not satisfy the required linking geometry either. To overcome this difficulty, we will employ the idea of local linking [14, 15].

Since the term involving \( \phi_u \) in the expression of \( \Phi \) is homogeneous of degree 4 (see (2.1)), it is natural to consider the case that (1.5) holds. In this case, we say that the nonlinearity \( g(t) \) is 4-superlinear. In the study of such problems, the following Ambrosetti–Rabinowitz condition [2]
(AR) there exists $\theta > 4$ such that $0 < \theta G(t) \leq tg(t)$ for $t \neq 0$

is widely used, see [9, 19]. Another widely used condition is the following condition introduced by Jeanjean [12]

(Je) there exists $\theta \geq 1$ such that $\theta \mathcal{G}(t) \geq \mathcal{G}(st)$ for all $s \in [0, 1]$ and $t \in \mathbb{R}$, where $\mathcal{G}(t) = g(t) t - 4G(t)$.

It is well known that if $t \mapsto |t|^{-3} g(t)$ is nondecreasing in $(-\infty, 0)$ and $(0, \infty)$, then (Je) holds. Obviously, our condition $(g_1)$ is weaker than both (AR) and (Je). Therefore, it is interesting to consider 4-superlinear problems under the condition $(g_1)$.

The condition $(g_1)$ is motivated by Alves et. al [1]. Assuming in addition

$$\alpha = \inf_{\mathbb{R}^3} V > 0$$

and $b \in [0, \alpha)$, they were able to show that all Cerami sequences of $\Phi$ are bounded. In the present paper we will show that with the compact embedding $X \hookrightarrow L^2(\mathbb{R}^3)$ mentioned below, we can get the boundedness of Palais-Smale sequences under much weaker condition $(g_1)$, see Lemma 3.2.

Since $V$ is bounded from below, we may chose $m > 0$ such that

$$\tilde{V}(x) := V(x) + m > 1,$$

for all $x \in \mathbb{R}^3$.

A main difficulty to solve problem (1.2) is that the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is not compact. Thanks to the condition $(V)$, this difficulty can be overcame by the compact embedding $X \hookrightarrow L^2(\mathbb{R}^3)$ of Bartsch and Wang [6], where

$$X = \left\{ u \in H^1(\mathbb{R}^3) \left| \int_{\mathbb{R}^3} V(x) u^2 dx < \infty \right. \right\}$$

is a linear subspace of $H^1(\mathbb{R}^3)$, equipped with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \tilde{V}(x) uv) \, dx$$

and the corresponding norm $\|u\| = \langle u, u \rangle^{1/2}$. With this inner product, $X$ is a Hilbert space. We also note that if $V$ is coercive, namely

$$\lim_{|x| \to \infty} V(x) = +\infty,$$

then $(V)$ is satisfied.

Under our assumptions, the functional $\Phi$ given in (1.4) is of class $C^1$ on $X$ and, to solve (1.2) it suffices to find critical points of $\Phi \in C^1(X)$.

According to the compact embedding $X \hookrightarrow L^2(\mathbb{R}^3)$ and the spectral theory of self-adjoint compact operators, it is easy to see that the eigenvalue problem

$$-\Delta u + V(x) u = \lambda u, \quad u \in X$$

(1.7)

possesses a complete sequence of eigenvalues

$$-\infty < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots, \quad \lambda_k \to +\infty.$$
Each $\lambda_k$ has been repeated in the sequence according to its finite multiplicity. We denote by $\phi_k$ the eigenfunction of $\lambda_k$, with $|\phi_k|_2 = 1$, where $|\cdot|_q$ is the $L^q$ norm. Note that the negative space $X^-$ of the quadratic part of $\Phi$ is nontrivial, if and only if some $\lambda_k$ is negative. Actually, $X^-$ is spanned by the eigenfunctions corresponding to negative eigenvalues.

We are now ready to state our results.

**Theorem 1.1.** Suppose $(V)$, $(g_0)$ and $(g_1)$ are satisfied. If 0 is not an eigenvalue of (1.7), then the Schrödinger–Poisson system (1.2) has at least one nontrivial solution $(u, \phi) \in X \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.

As far as we know, this is the first existence result for (1.2) in the case that the Schrödinger operator $S = -\Delta + V$ is not necessary positively definite. In case that $g$ is odd, we can obtain an unbounded sequences of solutions.

**Theorem 1.2.** If $(V)$, $(g_0)$, $(g_1)$ are satisfied, and $g$ is odd, then the Schrödinger–Poisson system (1.2) has a sequence of solutions $(u_n, \phi_n) \in X \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that the energy $\mathcal{J}(u_n, \phi_n) \to +\infty$.

**Remark 1.3.** (i) Note that if $u$ is a critical point of $\Phi$ then $\Phi(u) = \mathcal{J}(u, \phi_u)$. Therefore, to prove Theorem 1.2 it suffices to find a sequence of critical points $\{u_n\}$ of $\Phi$ such that $\Phi(u_n) \to +\infty$.

(ii) Theorem 1.2 improves the recent results in [9] and [13]. In these two papers the authors assumed in addition (1.6), and (AR) or (Je) respectively.

Schrödinger–Poisson systems of the form (1.2) has been extensively studied in recent years. To overcome the difficulty that the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is not compact, many authors restrict their study to the case that $V$ is radially symmetric, or even a positive constant, see e.g. [3, 10, 16]. Obviously, in this case replacing $X$ with the radial Sobolev space $H^1_{\text{rad}(\mathbb{R}^3)}$, the conclusions of Theorems 1.1 and 1.2 remain valid.

## 2 Tools from critical point theory

Evidently, the properties of $\phi_u$ play an important role in the study of $\Phi$. According to [11, Theorem 2.2.1] we know that

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy. \tag{2.1}$$

Using this expression, a more complete list of properties can be found in [1, Lemma 1.1]. Here, we only recall the ones that will be used in our argument.

**Proposition 2.1.** There is a positive constant $a_1 > 0$ such that for all $u \in X$ we have

$$0 \leq \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx \leq a_1 \|u\|^4. \tag{2.2}$$

For any $q \in [2,6]$ we have a continuous embedding $X \hookrightarrow L^q(\mathbb{R}^3)$. Consequently there is a constant $\kappa_q > 0$ such that

$$|u|_q \leq \kappa_q \|u\|, \quad \text{for all } u \in X. \tag{2.3}$$

If $0 < \lambda_1$, it is easy to see that $\Phi$ has the mountain pass geometry. This case is simple and will be omitted here. For the proof of Theorem 1.1, since 0 is not an eigenvalue of (1.7), we may assume that $0 \in (\lambda_{\ell}, \lambda_{\ell+1})$ for some $\ell \geq 1$. Let

$$X^- = \text{span} \{\phi_1, \cdots, \phi_\ell\}, \quad X^+ = (X^-)^\perp. \tag{2.4}$$
Then $X^-$ and $X^+$ are the negative space and positive space of the quadratic form
\[ Q(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) \, dx \]
respectively. Note that $\dim X^- = \ell < \infty$. Moreover, there is a positive constant $\kappa$ such that
\[ \pm Q(u) \geq \kappa \|u\|^2, \quad u \in X^\pm. \tag{2.5} \]

As we have mentioned, because of the term involving $\phi u$ in (1.4), our functional $\Phi$ does not satisfy the geometric assumption of the linking theorem. In fact, choose $\phi \in X^+$ with $\|\phi\| = 1$. For $R > r > 0$ set
\[ N = \{ u \in X^+ | \|u\| = r \}, \quad M = \{ u \in X^- \oplus R^+ \phi | \|u\| \leq R \}, \]
then $M$ is a submanifold of $X^- \oplus R^+ \phi$ with boundary $\partial M$. We do have
\[ b = \inf_{N} \Phi > 0, \quad \sup_{u \in \partial M, \|u\| = R} \Phi < 0 \]
provided $R$ is large enough. However, for $u \in X^-$ we have
\[ \Phi(u) = Q(u) + \frac{1}{4} \int_{\mathbb{R}^3} \phi u^2 \, dx - \int_{\mathbb{R}^3} G(u) \, dx. \]
Because $\phi u \geq 0$, the term involving $\phi u$ may be very large and for some point $u \in \partial M \cap X^-$ we may have $\Phi(u) > b$. Therefore the following geometric assumption of the linking theorem
\[ b = \inf_{N} \Phi > \sup_{\partial M} \Phi \]
can not be satisfied. Fortunately, we can apply the local linking theorem of Luan and Mao [15] (see also Li and Willem [14]) to overcome this difficulty and find critical points of $\Phi$.

Recall that by definition, $\Phi$ has a local linking at 0 with respect to the direct sum decomposition $X = X^- \oplus X^+$, if there is $\rho > 0$ such that
\[ \begin{cases} 
\Phi(u) \leq 0, & \text{for } u \in X^-, \|u\| \leq \rho, \\
\Phi(u) \geq 0, & \text{for } u \in X^+, \|u\| \leq \rho. 
\end{cases} \tag{2.6} \]
It is then clear that 0 is a (trivial) critical point of $\Phi$. Next, we consider two sequences of finite dimensional subspaces
\[ X_0^\pm \subset X_1^\pm \subset \cdots \subset X^\pm \]
such that
\[ X^\pm = \bigcup_{n \in \mathbb{N}} X_n^\pm. \]
For multi-index $\alpha = (\alpha^-, \alpha^+) \in \mathbb{N}^2$ we set $X_\alpha = X_{\alpha^-} \oplus X_{\alpha^+}$ and denote by $\Phi_\alpha$ the restriction of $\Phi$ on $X_\alpha$. A sequence $\{\alpha_n\} \subset \mathbb{N}^2$ is admissible if, for any $\alpha \in \mathbb{N}^2$, there is $m \in \mathbb{N}$ such that $\alpha \leq \alpha_n$ for $n \geq m$; where for $\alpha, \beta \in \mathbb{N}^2$, $\alpha \leq \beta$ means $\alpha^\pm \leq \beta^\pm$. Obviously, if $\{\alpha_n\}$ is admissible, then any subsequence of $\{\alpha_n\}$ is also admissible.
Definition 2.2 ([15, Definition 2.2]). We say that $\Phi \in C^1(X)$ satisfies the Cerami type condition $(C)^\ast$, if whenever $\{\alpha_n\} \subset \mathbb{N}^2$ is admissible, any sequence $\{u_n\} \subset X$ such that

$$u_n \in X_{\alpha_n}, \quad \sup_n \Phi(u_n) < \infty, \quad (1 + \|u_n\|) \|\Phi'_{\alpha_n}(u_n)\|_{X_{\alpha_n}} \to 0 \quad (2.7)$$

contains a subsequence which converges to a critical point of $\Phi$.

Theorem 2.3 (Local Linking Theorem, [15, Theorem 2.2]). Suppose that $\Phi \in C^1(X)$ has a local linking at $0$, $\Phi$ satisfies $(C)^\ast$, $\Phi$ maps bounded sets into bounded sets and, for every $m \in \mathbb{N}$,

$$\Phi(u) \to -\infty, \quad \text{as} \quad \|u\| \to \infty, \quad u \in X^+ \oplus X_m^+. \quad (2.8)$$

Then $\Phi$ has a nontrivial critical point.

Remark 2.4. Theorem 2.3 is a generalization of the well known local linking theorem of Li and Willem [14, Theorem 2], where instead of $(C)^\ast$, the stronger Palais-Smale type condition $(PS)^\ast$ is assumed.

For the proof of Theorem 1.2 we will use the fountain theorem of Bartsch [5], see also [18, Theorem 3.6]. For $k = 1, 2, \cdots$, let

$$Y_k = \text{span} \{\phi_1, \cdots, \phi_k\}, \quad Z_k = \text{span} \{\phi_k, \phi_{k+1}, \cdots\}. \quad (2.9)$$

Theorem 2.5 (Fountain Theorem). Assume that the even functional $\Phi \in C^1(X)$ satisfies the $(PS)$ condition. If there exists $k_0 > 0$ such that for $k \geq k_0$ there exist $\rho_k > r_k > 0$ such that

(i) $b_k = \inf_{u \in Z_k, \|u\|=r_k} \Phi(u) \to +\infty$, as $k \to \infty$,

(ii) $a_k = \max_{u \in Y_k, \|u\|=\rho_k} \Phi(u) \leq 0$,

then $\Phi$ has a sequence of critical points $\{u_k\}$ such that $\Phi(u_k) \to +\infty$.

3 Proofs of Theorems 1.1 and 1.2

To study the functional $\Phi$, it will be convenient to write it in a form in which the quadratic part is $\|u\|^2$. Let $f(t) = g(t) + mt$. Then, by a simple computation, we have

$$F(t) := \int_0^t f(\tau) d\tau \leq \frac{t}{4} f(t) + \frac{\bar{b}}{4} t^2, \quad \text{where} \quad \bar{b} = b + m > 0. \quad (3.1)$$

Note that by (1.5) we easily have

$$\lim_{|t| \to \infty} \frac{f(t)}{t^3} = +\infty. \quad (3.2)$$

Moreover, using (1.3) we get

$$\lim_{|t| \to 0} \frac{f(t)t}{t^4} = \lim_{|t| \to 0} \left( \frac{t^2}{t^4} \cdot \frac{g(t) t + mt^2}{t^2} \right) = +\infty.$$

Therefore, there is $\Lambda > 0$ such that

$$f(t)t \geq -\Lambda t^4, \quad \text{all} \quad t \in \mathbb{R}. \quad (3.3)$$
With the modified nonlinearity $f$, our functional $\Phi : X \to \mathbb{R}$ can be rewritten as follows:

$$\Phi(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_n u^2 dx - \int_{\mathbb{R}^3} F(u) dx.$$ \hspace{1cm} (3.4)

Note that this does not imply that $\Phi$ has the mountain pass geometry, because unlike in (1.4), as $\|u\| \to 0$ the last term in (3.4) is not $o(\|u\|^2)$ anymore. The derivative of $\Phi$ is given below:

$$\langle \Phi'(u), v \rangle = \langle u, v \rangle + \int_{\mathbb{R}^3} \phi_n uv dx - \int_{\mathbb{R}^3} f(u)v dx.$$

**Lemma 3.1.** Suppose $(V)$, $(g_0)$ and $(g_1)$ are satisfied, then $\Phi$ satisfies the $(C)^*$ condition.

**Proof.** Suppose \( \{u_n\} \) is a sequence satisfying (2.7), where \( \{\alpha_n\} \subset \mathbb{N}^2 \) is admissible. We must prove that \( \{u_n\} \) is bounded.

We may assume $\|u_n\| \to \infty$ for a contradiction. By (2.7) and noting that

$$\langle \Phi'_{\alpha_n}(u_n), u_n \rangle = \langle \Phi'(u_n), u_n \rangle \hspace{1cm} (3.5)$$

since $u_n \in X_{\alpha_n}$, for large $n$, using (3.1) we have

$$4 \cdot \sup_n \Phi(u_n) + \|u_n\| \geq 4\Phi(u_n) - \langle \Phi'_{\alpha_n}(u_n), u_n \rangle$$

$$= \|u_n\|^2 + \int_{\mathbb{R}^3} (f(u_n)u_n - 4F(u_n)) dx$$

$$\geq \|u_n\|^2 - b \int_{\mathbb{R}^3} u_n^2 dx. \hspace{1cm} (3.6)$$

Let $v_n = \|u_n\|^{-1} u_n$. Up to a subsequence, by the compact embedding $X \hookrightarrow L^2(\mathbb{R}^3)$ we deduce

$$v_n \rightharpoonup v \text{ in } X, \quad v_n \to v \text{ in } L^2(\mathbb{R}^3).$$

Multiplying by $\|u_n\|^{-2}$ to both sides of (3.6) and letting $n \to \infty$, we obtain

$$b \int_{\mathbb{R}^3} v^2 dx \geq 1.$$

Consequently, $v \neq 0$.

Using (3.3) and (2.3) with $q = 4$, we have

$$\int_{v = 0} \frac{f(u_n)u_n}{\|u_n\|^4} dx = \int_{v = 0} \frac{f(u_n)u_n}{u_n^4} v_n^4 dx$$

$$\geq -\Lambda \int_{v = 0} v_n^4 dx$$

$$\geq -\Lambda \int_{\mathbb{R}^3} v_n^4 dx = -\Lambda |v_n|^4 \geq -\Lambda k_4^4 > -\infty. \hspace{1cm} (3.7)$$

For $x \in \{x \in \mathbb{R}^3 \mid v \neq 0\}$, we have $|u_n(x)| \to +\infty$. By (3.2) we get

$$\frac{f(u_n(x))u_n(x)}{\|u_n\|^4} = \frac{f(u_n(x))u_n(x)}{u_n^4(x)}v_n^4(x) \to +\infty. \hspace{1cm} (3.8)$$
Consequently, using (3.7), (3.8) and the Fatou lemma we obtain
\[
\int_{\mathbb{R}^3} \frac{f(u_n)u_n}{\|u_n\|^4} \, dx \geq \int_{\nu \neq 0} \frac{f(u_n)u_n}{\|u_n\|^4} \, dx - \Lambda \kappa_4^4 \to +\infty.
\] (3.9)

Since \( \{u_n\} \) is a sequence satisfying (2.7), using (2.2) and (3.9), for large \( n \) we obtain
\[
4a_1 + 1 \geq \frac{1}{\|u_n\|^4} \left( \|u_n\|^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx - \langle \Phi'(u_n), u_n \rangle \right)
= \int_{\mathbb{R}^3} \frac{f(u_n)u_n}{\|u_n\|^4} \, dx \to +\infty,
\] (3.10)
a contradiction.

Therefore, \( \{u_n\} \) is bounded in \( X \). Now, by the argument of [9, pp. 4933] and using (3.5), the compact embedding \( X \hookrightarrow L^2(\mathbb{R}^3) \) and
\[
X = \bigcup_{n \in \mathbb{N}} X_{\alpha_n},
\]
we can easily prove that \( \{u_n\} \) has a subsequence converging to a critical point of \( \Phi \).

**Lemma 3.2.** Suppose \((V), (g_0)\) and \((g_1)\) are satisfied, then \( \Phi \) satisfies the (PS) condition.

**Proof.** Under the assumption there exists \( \tilde{\Lambda} > 0 \) such that
\[
F(t) \geq -\tilde{\Lambda} t^4, \quad \lim_{|t| \to \infty} \frac{F(t)}{t^4} = +\infty.
\] (3.11)

Let \( \{u_n\} \) be a (PS) sequence, that is \( \sup_n |\Phi(u_n)| < \infty, \Phi'(u_n) \to 0 \). We only need to show that \( \{u_n\} \) is bounded. If \( \{u_n\} \) is not bounded, similar to the first part in the proof of Lemma 3.1, we may assume that for some \( \nu \neq 0 \),
\[
\|u_n\|^{-1} u_n \to \nu \quad \text{in} \ X.
\]
Since \( \nu \neq 0 \), similar to (3.9), using (3.11) we deduce
\[
\int_{\mathbb{R}^3} \frac{F(u_n)}{\|u_n\|^4} \, dx \to +\infty.
\]

Therefore
\[
a_1 + 1 \geq \frac{1}{\|u_n\|^4} \left( \frac{1}{2} \|u_n\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx - \Phi(u_n) \right)
= \int_{\mathbb{R}^3} \frac{F(u_n)}{\|u_n\|^4} \, dx \to +\infty,
\]
a contradiction.

**Lemma 3.3.** Under the assumptions \((V), (g_0)\), the functional \( \Phi \) has a local linking at 0 with respect to the decomposition \( X = X^- \oplus X^+ \).
Proof. By \((g_0)\), there exists \(C > 0\) such that
\[
|G(u)| \leq \frac{K}{2\kappa^2} |u|^2 + C \kappa |u|^p.
\]  
(3.12)

If \(u \in X^-\), then using \((2.2)\) and \((2.5)\) we deduce
\[
\Phi(u) = Q(u) + \frac{1}{4} \int_{R^3} \phi_u u^2 dx - \int_{R^3} G(u) dx
\leq -\kappa |u|^2 + a_1 |u|^4 + \frac{K}{2\kappa^2} |u|^2 + C \kappa |u|^p
\leq -\frac{K}{2} |u|^2 + a_1 |u|^4 + C_1 |u|^p,
\]  
(3.13)

where \(C_1 = C \kappa \kappa^p\). Similarly, for \(u \in X^+\) we have
\[
\Phi(u) \geq \frac{K}{2} |u|^2 - C_1 |u|^p.
\]  
(3.14)

Since \(p > 4\), the desired result \((2.6)\) follows from \((3.13)\) and \((3.14)\).

Lemma 3.4. Let \(Y\) be a finite dimensional subspace of \(X\), then \(\Phi\) is anti-coercive on \(Y\), that is
\[
\Phi(u) \to -\infty, \quad \text{as} \quad \|u\| \to \infty, \quad u \in Y.
\]

Proof. If the conclusion is not true, we can choose \(\{u_n\} \subset Y\) and \(\beta \in \mathbb{R}\) such that
\[
\|u_n\| \to \infty, \quad \Phi(u_n) \geq \beta.
\]  
(3.15)

Let \(v_n = \|u_n\|^{-1} u_n\). Since \(\dim Y < \infty\), up to a subsequence we have
\[
\|v_n - v\| \to 0, \quad v_n(x) \to v(x) \text{ a.e. } \mathbb{R}^3
\]
for some \(v \in Y\), with \(\|v\| = 1\). Since \(v \neq 0\), similar to \((3.9)\), using \((3.11)\) we have
\[
\int_{\mathbb{R}^3} \frac{F(u_n)}{\|u_n\|^4} dx \to +\infty.
\]  
(3.16)

Using \((2.2)\) and \((3.16)\) we deduce
\[
\Phi(u_n) = \|u_n\|^4 \left( \frac{1}{2 \|u_n\|^2} + \frac{1}{4 \|u_n\|^4} \int_{\mathbb{R}^3} \phi_n u_n^2 dx - \int_{\mathbb{R}^3} \frac{F(u_n)}{\|u_n\|^4} dx \right) \to -\infty,
\]
a contradiction with \((3.15)\).

Now, we are ready to prove our main results.

Proof (Proof of Theorem 1.1). We will find a nontrivial critical point of \(\Phi\) via Theorem 2.3. Using Proposition 2.1, it is easy to see that \(\Phi\) maps bounded sets into bounded sets. In Lemmas 3.1 and 3.3 we see that \(\Phi\) satisfies \((C)^*\) and \(\Phi\) has a local linking at 0. It suffices to verify \((2.8)\). Since \(\dim (X^- \oplus X^+_m) < \infty\), this is a consequence of Lemma 3.4.
Proof (Proof of Theorem 1.2). By Remark 1.3 (i), it suffices to find a sequence of critical points \( \{u_n\} \) of \( \Phi \) such that \( \Phi(u_n) \to +\infty \).

We define subspaces \( Y_k \) and \( Z_k \) of \( X \) as in (2.9). Since \( g \) is odd, \( \Phi \) is an even functional. By Lemma 3.2 we know that \( \Phi \) satisfies \((PS)\). It suffices to verify (i) and (ii) of Theorem 2.5.

Verification of (i). We assume that \( 0 \in [\lambda_\ell, \lambda_{\ell+1}) \). Then if \( k > \ell \) we have \( Z_k \subset X^+ \), where \( X^+ \) is defined in (2.4). Now, by (2.5), we have

\[
Q(u) \geq \kappa \|u\|^2, \quad u \in Z_k.
\]

We recall that by [9, Lemma 2.5],

\[
\beta_k = \sup_{u \in Z_k, \|u\|=1} |u| \to 0, \quad \text{as} \quad k \to \infty.
\]

Let \( r_k = (Cp\beta_k^p)^{1/(2-p)} \), where \( C \) is chosen in (3.12). For \( u \in Z_k \) with \( \|u\| = r_k \), using (3.12) and (3.17) we deduce

\[
\Phi(u) = Q(u) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} G(u) dx \\
\geq \kappa \|u\|^2 - \frac{\kappa}{2\kappa_2} \|u\|^2 - C\kappa \|u\|^p \\
\geq \kappa \left( \frac{1}{2} \|u\|^2 - C\beta_k^p \|u\|^p \right) = \kappa \left( \frac{1}{2} - \frac{1}{p} \right) (Cp\beta_k^p)^{2/(2-p)}.
\]

Since \( \beta_k \to 0 \) and \( p > 2 \), it follows that

\[
b_k = \inf_{u \in Z_k, \|u\|=r_k} \Phi(u) \to +\infty.
\]

Verification of (ii). Since \( \dim Y_k < \infty \), this is a consequence of Lemma 3.4.

Remark 3.5. From the proof of Lemmas 3.2 and 3.4, it is easy to see that if in \((g_1)\) we replace (1.5) by the following weaker condition

\[
\lim_{|t| \to \infty} \frac{F(t)}{t^4} = +\infty,
\]

then Theorem 1.2 remains valid.

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