CLASSIFICATION OF CONFORMALLY FLAT ISOPARAMETRIC SUBMANIFOLDS OF EUCLIDEAN SPACE

CHRISTOS-RAENT ONTI

ABSTRACT. In this note we provide a complete (local) classification of conformally flat isoparametric submanifolds of Euclidean space.

1. Introduction

A submanifold $f : M^n \rightarrow \mathbb{R}^m$ is said to be isoparametric if it has flat normal bundle and the shape operator in any parallel normal direction has constant eigenvalues. This class of Euclidean submanifolds has been investigated by several authors throughout the years; see, for example, [1, 2, 4, 6–11, 13, 14, 16–22]. The aim of this short note is to provide a complete (local) classification of conformally flat isoparametric submanifolds of Euclidean space. Recall that a Riemannian manifold $M^n$ is said to be conformally flat if each point lies in an open neighborhood conformal to an open subset of Euclidean space $\mathbb{R}^n$.

The following is the main result.

Theorem 1.1. Let $f : M^n \rightarrow \mathbb{R}^m$ be an isometric immersion of a conformally flat manifold into Euclidean space. If $f$ is isoparametric and proper, then $f$ is locally an open subset of one of the following:

(i) $S^n$, (ii) $\mathbb{R}^n$, (iii) $\mathbb{R}^{n-k} \times S^1 \times \cdots \times S^1 \subset \mathbb{R}^{n+k}$, (iv) $S^1 \times \cdots \times S^1 \subset S^{2n-1} \subset \mathbb{R}^n$,

(v) $\mathbb{R} \times S^{n-1} \subset \mathbb{R}^{n+1}$, (vi) $S^1 \times S^{n-1} \subset \mathbb{R}^{n+2}$.

Remark 1.2. Theorem 1.1 can also be extended to non-flat space forms.

2. Preliminaries

In this section we recall some basic facts. Let $f : M^n \rightarrow \mathbb{R}^m$ be an isometric immersion of a Riemannian manifold into the Euclidean space. The second fundamental form $\alpha$ of $f$ is a symmetric section of the vector bundle $\text{Hom}(TM \times TM, N_f M)$, where $N_f M$ is the normal bundle of $f$. We say that $f$ is totally umbilical if

$$\alpha(X, Y) = \langle X, Y \rangle H,$$

where $H$ is the mean curvature vector field.

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If $f$ has flat normal bundle, that is, at any point the curvature tensor of the metric induced from the ambient space on the normal bundle of the submanifold vanishes, then it is a standard fact (see [15]) that at any point $x \in M^n$ there exists a set of unique pairwise distinct normal vectors $\eta_i(x) \in N_f M(x)$, $1 \leq i \leq s(x)$, called the principal normals of $f$ at $x$. Moreover, there is an associated orthogonal splitting of the tangent space as

$$T_x M = E_1(x) \oplus \cdots \oplus E_{s(x)}(x),$$

where

$$E_i(x) = \{ X \in T_x M : \alpha(X, Y) = \langle X, Y \rangle \eta_i(x) \text{ for all } Y \in T_x M \}. \quad (2.1)$$

If $s(x) = k$ is constant on $M^n$, then $f$ is said to be proper. In this case, the maps $x \in M^n \mapsto \eta_i(x)$, $1 \leq i \leq k$, are smooth vector fields, called the principal normal vector fields of $f$, and the distributions $x \in M^n \mapsto E_i(x)$, $1 \leq i \leq k$, are also smooth. If $\nabla$ denotes the Levi-Civita connection of $M^n$, then the Codazzi equation is easily seen to yield

$$\langle \nabla_X Y, Z \rangle (\eta_i - \eta_j) = \langle X, Y \rangle \nabla_{\nabla_Z} \eta_i \quad (2.2)$$

and

$$\langle \nabla_X V, Z \rangle (\eta_j - \eta_\ell) = \langle \nabla_V X, Z \rangle (\eta_i - \eta_\ell) \quad (2.3)$$

for all $X, Y \in E_i, Z \in E_j$, and $V \in E_\ell$, where $1 \leq i \neq j \neq \ell \neq i \leq k$.

The following is contained in [5].

**Proposition 2.1.** Let $M^n$ be a conformally flat manifold and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with flat normal bundle. If at some point of $M^n$ we have $k \geq 3$, then the vectors $\eta_j - \eta_i$ and $\eta_j - \eta_\ell$ are linearly independent for $1 \leq i \neq j \neq \ell \neq i \leq k$.

The following is also contained in [5].

**Theorem 2.2.** Let $f : M^n \to \mathbb{R}^m, n \geq 4$, be an isometric immersion with flat normal bundle and proper of a conformally flat manifold. Then $f$ carries at most one principal normal vector field of multiplicity larger than one.

The following is well-known (cf. [3]).

**Proposition 2.3.** A Riemannian product is conformally flat if and only if one of the following possibilities holds:

(i) One of the factors is one-dimensional and the other one has constant sectional curvature.

(ii) Both factors have dimension greater than one and are either both flat or have opposite constant sectional curvatures.
A map \( f: M^n \to \mathbb{R}^m \) from a product manifold \( M^n = \prod_{i=1}^{k} M_i \) is called the **extrinsic product of immersions** \( f_i: M_i \to \mathbb{R}^{m_i} \), \( 1 \leq i \leq k \), if there exist an orthogonal decomposition \( \mathbb{R}^m = \prod_{i=0}^{k} \mathbb{R}^{m_i} \), with \( \mathbb{R}^{m_0} \) possibly trivial, such that \( f \) is given by

\[
f(x) = (v, f_1(x_1), \ldots, f_k(x_k))
\]

for all \( x = (x_1, \ldots, x_k) \in M^n \) and \( v \in \mathbb{R}^{m_0} \).

Let \( f: M^n \to \mathbb{R}^m \) be an isometric immersion of a Riemannian manifold. If \( M^n = \prod_{i=1}^{k} M_i \) is a product manifold then the second fundamental form \( \alpha \) is said to be **adapted** to the product structure of \( M^n \) if

\[
\alpha(X_i, X_j) = 0 \quad \text{for all} \quad X_i \in T M_i, \ X_j \in T M_j \quad \text{with} \quad 1 \leq i \neq j \leq k,
\]

where the tangent bundles \( T M_i \) are identified with the corresponding tangent distributions to \( M^n \). The next result, due to Moore [12], shows that extrinsic products of isometric immersions are characterized by this property among isometric immersions of Riemannian products.

**Theorem 2.4.** Let \( f: M^n \to \mathbb{R}^m \) be an isometric immersion of a Riemannian product manifold \( M^n = \prod_{i=1}^{k} M_i \) with adapted second fundamental form. Then \( f \) is an extrinsic product of isometric immersions.

### 3. Proof of Theorem 1.1

Assume that \( k \geq 3 \), otherwise the result is immediate. We claim that each distribution \( E_i, 1 \leq i \leq k \), is parallel, that is

\[
\nabla_X Y \in E_i \quad \text{for all} \quad X \in T M, \ Y \in E_i \quad \text{and} \quad 1 \leq i \leq k.
\]

Since \( f \) is isoparametric we have that the principal normal vector fields are parallel in the normal connection. Therefore, it follows from the Codazzi equation (2.2) that each distribution \( E_i, 1 \leq i \leq k \), is totally geodesic. Thus, we only need to show that

\[
\nabla_X Y \in E_i \quad \text{for all} \quad X \in E_j, \ Y \in E_i
\]

with \( j \neq i \). Indeed, we consider \( Y \in E_i, \ X \in E_j, \ Z \in E_\ell \subset E_i^\perp \) and distinguish the following two cases.

If \( \ell = j \), then we get

\[
\langle \nabla_X Y, Z \rangle = -\langle Y, \nabla_X Z \rangle = 0, \quad (3.1)
\]

where we have used the fact that \( E_\ell \) is totally geodesic.

If \( \ell \neq j \), then from (2.3) we obtain

\[
\langle \nabla_X Y, Z \rangle (\eta_\ell - \eta_i) = \langle \nabla_Y X, Z \rangle (\eta_\ell - \eta_j).
\]

Using Proposition 2.1 we get

\[
\langle \nabla_X Y, Z \rangle = 0 \quad \text{for all} \quad X \in E_j, Y \in E_i, Z \in E_\ell, \quad (3.2)
\]
with \( \ell \neq i \neq j \). Therefore, from (3.1) and (3.2), we obtain that \( \nabla_X Y \in E_i \) for all \( X \in E_j \) and \( Y \in E_i \) with \( j \neq i \). This completes the proof of the claim.

Now, de Rham’s theorem implies that around every point \( x \in M^n \) there is a neighborhood \( U \) that is the Riemannian product of the integral manifolds \( M_1, \ldots, M_k \) of the distributions \( E_1, \ldots, E_k \) respectively, through a point \( y \in U \). Therefore, since the second fundamental form of \( f \) is adapted, Theorem 2.4 implies that \( f|_U \) is an extrinsic product of isometric immersions \( f_i : M_i \to \mathbb{R}^{m_i}, \ 1 \leq i \leq k \), which due to (2.1) and our hypothesis have to be totally umbilical and with mean curvatures of constant length. Now, the classification follows easily by using Theorem 2.2 and Proposition 2.3. This completes the proof. \( \square \)

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