BICATEGORICAL FIBRATION STRUCTURES AND STACKS

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ABSTRACT. The familiar construction of categories of fractions, due to Gabriel and Zisman, allows one to invert a class \( W \) of arrows in a category in a universal way. Similarly, bicategories of fractions allow one to invert a collection \( W \) of arrows in a bicategory \( B \). In this case the arrows are inverted in the sense that they are made into equivalences. As with categories of fractions, bicategories of fractions suffer from the defect that they need not be locally small even when \( B \) is locally small. Similarly, in the case where \( B \) is a 2-category, the bicategory of fractions will not in general be a 2-category.

In this paper we introduce two notions—systems of fibrant objects and fibration systems—which will allow us to associate to a bicategory \( B \) a homotopy bicategory \( \text{Ho}(B) \) in such a way that \( \text{Ho}(B) \) is the universal way to invert weak equivalences in \( B \). This construction resolves both of the difficulties with bicategories of fractions mentioned above. We also describe a fibration system on the 2-category of prestacks on a site and prove that the resulting homotopy bicategory is the 2-category of stacks. Further examples considered include algebraic, differentiable and topological stacks.

INTRODUCTION

It is widely known that Quillen’s [17] notion of model structure provides an adequate (for many purposes) formal setting for the development of the theory of \((\infty, 1)\)-categories, as has been studied by Joyal [8], Lurie [11] and others. Moreover, a model structure on a category \( C \) provides a technical tool for forming the localization of \( C \) with respect to a class of weak equivalences: weak equivalences are inverted in a universal way in the passage to the homotopy category \( \text{Ho}(C) \) of \( C \). Consequently, it is possible to invert weak equivalences in this setting without having to resort to the Gabriel-Zisman [5] calculus of fractions.

In the bicategorical setting, one might like to be able to invert a collection of weak equivalences in the sense of turning them into equivalences. In [15, 16], the first author gave a bicategorical generalization of the Gabriel-Zisman calculus of fractions which accomplishes this goal:

Theorem (Pronk [16]). Given a collection of arrows \( \mathcal{W} \) in a bicategory \( C \) satisfying certain conditions, there exists a bicategory \( C[\mathcal{W}^{-1}] \) (called the bicategory of fractions for \( \mathcal{W} \)) and a homomorphism \( I : C \to C[\mathcal{W}^{-1}] \) such that I sends arrows in \( \mathcal{W} \) to equivalences in \( C[\mathcal{W}^{-1}] \) and I is universal with this property.

Like the ordinary category of fractions, this construction suffers from the technical defect that \( C[\mathcal{W}^{-1}] \) will not in general have small hom-categories even when \( C \) does. Moreover, \( C[\mathcal{W}^{-1}] \) will be a bicategory even when \( C \) is a 2-category. In the present paper, we introduce the notion of a system of fibrant objects in a bicategory \( C \) and the notion of a
fibration structure on a bicategory which will allow us to form the localization of a bicategory \( \mathcal{C} \) with respect to a class of weak equivalences in such a way that the result will both have small hom-categories when \( \mathcal{C} \) does and will be a 2-category when \( \mathcal{C} \) is.

A system of fibrant objects consists of a collection \( \mathcal{W} \) (weak equivalences) of maps in \( \mathcal{C} \), a pseudofunctor \( Q: \mathcal{C} \to \mathcal{C} \) (fibrant replacement) and a pseudonatural transformation \( \eta: 1_\mathcal{C} \to Q \) (whose components are weak equivalences) satisfying certain factorization conditions. To each bicategory \( \mathcal{C} \) with a system of fibrant objects, there is an associated bicategory \( \text{Ho}(\mathcal{C}) \), called the homotopy bicategory of \( \mathcal{C} \), and a pseudofunctor \( I: \mathcal{C} \to \text{Ho}(\mathcal{C}) \). By construction, \( \text{Ho}(\mathcal{C}) \) has small hom-categories when \( \mathcal{C} \) does and it is a 2-category when \( \mathcal{C} \) is. Our first main result is as follows:

**Theorem** (Theorem 2.2 below). The pseudofunctor \( I \) inverts weak equivalences and is universal with this property.

The remainder of this paper is concerned with investigating specific examples of bicategories with systems of fibrant objects. Our leading example is the 2-category \( \text{St}(\mathcal{C}) \) of stacks on a site \( \mathcal{C} \), which we obtain (Corollary 4) as the homotopy 2-category \( \text{Ho}(\text{PreSt}(\mathcal{C})) \) of the 2-category of prestacks on \( \mathcal{C} \). This result is made possible using a characterization of the fibrations of prestacks which is analogous to an earlier characterization of stacks given, albeit in a different setting from the one considered here, by Joyal and Tierney [9] (cf. also the work of Hollander [7]).

The system of fibrant objects on \( \text{PreSt}(\mathcal{C}) \) is notable in that it exhibits a number of additional features making it more closely resemble the notion of a model structure. These additional features are sufficiently interesting that we introduce the notion of a *fibration structure* on a bicategory to capture them.

A category \( \mathcal{C} \) has a fibration structure when there are stronger lifting and factorization conditions in place which among other things imply that the category has path objects and that the factorization lemma holds, so that one can construct generalized universal bundles.

In [16], the first author gave a number of examples of bicategorical equivalences between well-known 2-categories and bicategories of fractions. These examples include topological, differentiable and algebraic stacks and we show that these examples can also be captured in our setting. Note however that the characterizations given here of these 2-categories differ from those in *ibid*. In *ibid* these 2-categories were characterized as bicategories of fractions of certain categories of groupoids with respect to Morita equivalences. Here we will view them as homotopy categories of certain categories of prestacks with respect to local weak equivalences.

Part of the motivation of this work is the goal of trying to find a formal setting, analogous to the setting of model structures, in which to develop the theory of \((\infty,2)\)- and \((\infty,n)\)-categories. Ultimately we would like to extend the axiomatization given here to the lax setting (we are always working in a “pseudo” setting) and to relate the results presented here to Street’s notion of 2-topos [19]. Intuitively, every 2-topos should arise as a homotopy 2-category by analogy with the way Grothendieck toposes arise as localizations of presheaf categories.

**Summary.** In Section 1 we recall basic definitions and results on bicategories, pseudofunctors, pseudonatural transformations, and so forth. In Section 2 we introduce systems of fibrant objects and fibration structures on bicategories and we prove our main result (Theorem 2.2). In Section 3 we introduce a fibered notion of stack: *local fibrations*. Let \((\mathcal{C},J)\) be a site and let pseudofunctors \( E, B: \mathcal{C}^{\text{op}} \to \text{Cat} \) and a pseudonatural transformation \( p: E \to B \) be given. For each cover \( \mathcal{S} \) of an object \( U \) of \( \mathcal{C} \) we introduce the category
Desc\((p, \mathcal{F})\) of descent data with respect to \(p\) and \(\mathcal{F}\). This category, like the usual category of descent data \(\text{Desc}(E, \mathcal{F})\), can be defined as a pseudolimit (although here we give a direct description) and we define \(p\) to be a local fibration when it satisfies an effective descent condition with respect to \(\text{Desc}(p, \mathcal{F})\) analogous to the usual descent condition for stacks. In Section 4 we describe a fibration structure on the 2-category of prestacks \(\text{PreSt}(\mathcal{C})\) and prove that the resulting homotopy bicategory is the 2-category \(\text{St}(\mathcal{C})\) of stacks. In particular, we introduce the local weak equivalences (which are already known in the literature on stacks) and prove, using the Axiom of Choice, that the local fibrations are exactly those maps having a bicategorical version of the right lifting property with respect to the local weak equivalences. Further examples (algebraic, differentiable and topological prestacks) of systems of fibrant objects are considered in Section 5.

1. Basics and notation

It is worth mentioning that we make free use of the Axiom of Choice. As such, we do not distinguish between strong and weak categorical equivalences (see [3] for more on the differences between strong and weak equivalences). We assume that the reader is familiar with the basic theory of 2-categories and refer the reader to [10] for further details. For more information regarding stacks we refer the reader to [6] and [13].

1.1. Bicategories. We now review the definitions of bicategories, pseudofunctors, pseudonatural transformations and modifications.

**Definition 1.1** (Bénabou [2]). A bicategory \(\mathcal{C}\) consists of a collection of objects \(A, B, \ldots\) together with the following data:

- Categories \(\mathcal{C}(A, B)\) for objects \(A\) and \(B\) of \(\mathcal{C}\). The objects of \(\mathcal{C}(A, B)\) are called arrows and the arrows are called 2-cells. When \(\alpha\) and \(\beta\) are composable 2-cells in \(\mathcal{C}(A, B)\) we denote their composite by \(\beta \cdot \alpha\).
- For objects \(A, B\) and \(C\) of \(\mathcal{C}\), a functor \(c_{A,B,C}: \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)\). We denote \(c_{A,B,C}(f, g)\) by \(g \circ f\), for arrows \(f \in \mathcal{C}(A, B)\) and \(g \in \mathcal{C}(B, C)\), and we denote \(c_{A,B,C}(\alpha, \beta)\) by \(\beta \ast \alpha\), for 2-cells \(\beta \in \mathcal{C}(A, B)\) and \(\alpha \in \mathcal{C}(B, C)\). When no confusion will result we omit the subscripts and write \(c\) instead of \(c_{A,B,C}\).
- For each object \(A\) of \(\mathcal{C}\), an arrow \(1_A \in \mathcal{C}(A, A)\).
- For objects \(A, B, C\) and \(D\) of \(\mathcal{C}\), a natural isomorphism:

\[
\mathcal{C}(A, B) \times \mathcal{C}(B, C) \times \mathcal{C}(C, D) \xrightarrow{c \times \mathcal{C}(C,D)} \mathcal{C}(A, C) \times \mathcal{C}(C, D)
\]

As with the composition functors \(c\), we will omit subscripts and write \(\alpha\) instead of \(\alpha_{A,B,C,D}\).
- For objects \(A\) and \(B\) of \(\mathcal{C}\), natural isomorphisms \(\lambda_{A,B}\) and \(\rho_{A,B}\) as indicated in the following diagrams:

\[
\begin{array}{ccc}
1 \times \mathcal{C}(A, B) & \xrightarrow{\lambda_{A,B}} & \mathcal{C}(A, A) \times \mathcal{C}(A, B) \\
\mathcal{C}(A, B) & \xrightarrow{\rho_{A,B}} & \mathcal{C}(A, B) \times 1 & \mathcal{C}(A, B) \\
\pi_1 & \xrightarrow{c} & \mathcal{C}(A, B) & \pi_0
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C}(A, B) \times \mathcal{C}(A, B) & \xrightarrow{\lambda_{A,B}} & \mathcal{C}(A, B) \\
\mathcal{C}(A, B) & \xrightarrow{\rho_{A,B}} & \mathcal{C}(A, B) \times 1 & \mathcal{C}(A, B)
\end{array}
\]
We again omit subscripts and simply write $\lambda$ and $\rho$.

These data are required to satisfy the following conditions:

- Given $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$, $h \in \mathcal{C}(C, D)$ and $k \in \mathcal{C}(D, E)$, the following diagram commutes:

\[
\begin{bmatrix}
((k \circ h) \circ g) \circ f & (k \circ (h \circ g)) \circ f \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
(k \circ h) \circ (g \circ f) & \rho(k \circ (h \circ (g \circ f))) & k \circ ((h \circ g) \circ f).
\end{bmatrix}
\]

- Given $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, the following diagram commutes:

\[
\begin{bmatrix}
(g \circ 1_B) \circ f & g \circ (1_B \circ f) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha \\
\end{bmatrix}
\]

The following definition also involves coherence data which should technically carry subscripts. These are indicated explicitly the first time they appear, but afterwards we adopt a policy of omitting subscripts wherever possible as in Definition 1.1.

**Definition 1.2.** Given bicategories $\mathcal{C}$ and $\mathcal{D}$, a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ is given by the following data:

- An assignment of objects $F C$ of $\mathcal{D}$ to each object $C$ of $\mathcal{C}$.
- For all objects $A$ and $B$ of $\mathcal{C}$, a functor $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$.
- For all objects $A, B$ and $C$ of $\mathcal{C}$, natural isomorphisms as indicated in the following diagrams:

\[
\begin{bmatrix}
\mathcal{C}(A, B) \times \mathcal{C}(B, C) & \mathcal{C}(A, C) \\
F \times F & F \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{D}(FA, FB) \times \mathcal{D}(FB, FC) & \mathcal{D}(FA, FC) \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & \mathcal{C}(A, A) \\
\text{Id}_A & \text{Id}_A \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{D}(FA, FA). \\
\end{bmatrix}
\]

These data are required to be such that the following diagrams commute:

\[
\begin{bmatrix}
(Fh \circ Fg) \circ Ff & F(h \circ g) \circ Ff & F((h \circ g) \circ f) \\
\alpha \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
Fh \circ (Fg \circ Ff) & Fh \circ F(g \circ f) & F(h \circ (g \circ f)). \\
\phi & \phi & \phi \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\end{bmatrix}
\]
Definition 1.3. Given pseudofunctors \( F, G : \mathcal{C} \rightarrow \mathcal{D} \), a **pseudonatural transformation** \( \psi : F \rightarrow G \) consists of the following data:

- For each object \( C \) of \( \mathcal{C} \), an arrow \( \psi_C : FC \rightarrow GC \) in \( \mathcal{D} \).
- For objects \( A \) and \( B \) of \( \mathcal{C} \), a natural isomorphism

\[
\begin{array}{c}
\mathcal{C}(A, B) \xrightarrow{F} \mathcal{D}(FA, FB) \\
G \downarrow \psi_{AB} \downarrow \\
\mathcal{D}(GA, GB) \xrightarrow{\psi_{GB}} \mathcal{D}(FA, GB).
\end{array}
\]

Note that here we are overloading the notation \( \psi \). In practice this should not result in any confusion.

These data are required to be such that the following diagrams commute:

\[
(Gg \circ Gf) \circ \psi \xrightarrow{\alpha} Gg \circ (Gf \circ \psi) \xrightarrow{Gg \circ \psi} Gg \circ (\psi \circ Ff) \xrightarrow{\alpha^{-1}} (Gg \circ \psi) \circ Ff
\]

\[
\begin{array}{c}
G(g \circ f) \circ \psi \xrightarrow{\alpha} \psi \circ (g \circ f) \xrightarrow{\psi \circ \phi} \psi \circ (Fg \circ Ff) \xrightarrow{\alpha} (\psi \circ Fg) \circ Ff
\end{array}
\]

\[
\begin{array}{c}
1_{GA} \circ \psi \xrightarrow{\lambda} \psi \xrightarrow{\rho^{-1}} \psi \circ 1_{FA}
\end{array}
\]

\[
\begin{array}{c}
G1_A \circ \psi \xrightarrow{\psi} \psi \circ F1_A
\end{array}
\]

Definition 1.4. A **modification** \( \omega : \psi \rightarrow \psi' \), for \( \psi \) and \( \psi' \) pseudonatural transformations \( F \rightarrow G \), consists of an assignment of 2-cells \( \omega_C : \psi_C \rightarrow \psi'_C \) to each object \( C \) of \( \mathcal{C} \) such that

\[
Gf \circ \psi \xrightarrow{Gf \circ \omega} Gf \circ \psi'
\]

\[
(\psi \circ Ff) \xrightarrow{\psi \circ Ff} \psi' \circ Ff
\]

commutes, for each \( f : A \rightarrow B \) in \( \mathcal{C} \).

1.2. **Pseudofunctor bicategories.** Given bicategories \( \mathcal{C} \) and \( \mathcal{D} \), we denote by \([\mathcal{C}, \mathcal{D}]\) the bicategory which has as objects pseudofunctors \( \mathcal{C} \rightarrow \mathcal{D} \), as arrows pseudonatural transformations, and as 2-cells modifications. Note that \([\mathcal{C}, \mathcal{D}]\) is a 2-category when \( \mathcal{D} \) is.

Definition 1.5. An arrow \( f : A \rightarrow B \) in a bicategory \( \mathcal{C} \) is an **equivalence** if there exists an arrow \( f' : B \rightarrow A \) together with invertible 2-cells \( f \circ f' \cong 1_B \) and \( 1_A \cong f' \circ f \).
It is a well-known fact that an equivalence of categories can always be altered to give an adjoint equivalence. The same fact holds in an arbitrary bicategory:

**Lemma 1.1.** If \( f : A \to B \) is an equivalence in a bicategory \( \mathcal{C} \), then there exists a \( f' : B \to A \) together with invertible 2-cells \( \eta : 1_A \cong f' \circ f \) and \( \varepsilon : f \circ f' \cong 1_B \) which are the unit and counit of an adjunction \( f \dashv f' \).

**Lemma 1.2.** Given pseudofunctors \( F, G : \mathcal{C} \to \mathcal{D} \) between bicategories \( \mathcal{C} \) and \( \mathcal{D} \), if \( \xi : F \to G \) is a pseudonatural transformation such that, for each \( A \) in \( \mathcal{C} \), \( \xi_A : FA \to GA \) is an equivalence, then there exists a pseudonatural transformation \( \xi' : G \to F \) such that \( \xi' \) is an adjoint pseudoinverse of \( \xi \) in the bicategory \( [\mathcal{C}, \mathcal{D}] \).

**Proof.** By Lemma 1.1, we may choose \( \xi' \) together with \( \eta_A : 1_{FA} \cong \xi_A' \circ \xi_A \) and \( \varepsilon_A : \xi_A \circ \xi_A' \cong 1_{GA} \) making \( \xi_A \dashv \xi_A' \). Then, for \( f : A \to B \) in \( \mathcal{C} \), the isomorphism \( Ff \circ \xi_A' \cong \xi_B' \circ Gf \) is constructed by composing the isomorphisms

\[
Ff \circ \xi_A' \cong \xi_B \circ (\xi_B \circ Ff) \circ \xi_A \cong \xi_B' \circ (Gf \circ \xi_A) \circ \xi_A' \cong \xi_B' \circ Gf
\]

where the first isomorphism is a result of the coherence isomorphisms together with \( \eta_B \), the second isomorphism is by \( \xi_f \) and the third is by coherence and \( \varepsilon_A \). The coherence conditions on pseudonatural transformations follow from pseudonaturality of \( \xi \) and the triangle laws for adjunctions. \( \square \)

**Definition 1.6.** Given bicategories \( \mathcal{C} \) and \( \mathcal{D} \), a pseudofunctor \( F : \mathcal{C} \to \mathcal{D} \) is an equivalence of bicategories if there exists a pseudofunctor \( G : \mathcal{D} \to \mathcal{C} \) together with maps \( \eta : 1_{\mathcal{C}} \to G \circ F \) and \( \varepsilon : F \circ G \to 1_{\mathcal{D}} \) which are equivalences in the bicategories \([\mathcal{C}, \mathcal{C}]\) and \([\mathcal{D}, \mathcal{D}]\), respectively.

**Definition 1.7.** A pseudofunctor \( F : \mathcal{C} \to \mathcal{D} \) is a weak equivalence of bicategories if the following conditions are satisfied:

- For each object \( D \) of \( \mathcal{D} \), there exists an object \( C \) of \( \mathcal{C} \) and an equivalence \( FC \to D \) in \( \mathcal{D} \).
- For all objects \( C \) and \( C' \) of \( \mathcal{C} \), the map \( \mathcal{C}(C, C') \to \mathcal{D}(FC, FC') \) is an equivalence of categories.

Note that, in the presence of the Axiom of Choice, the notions of equivalence and weak equivalence of bicategories coincide.

### 1.3. Arrow bicategories

Given a bicategory \( \mathcal{C} \) we define a new bicategory \( \mathcal{C} \rightarrow \) as follows:

**Objects:** An object is an arrow \( f : A \to B \) in \( \mathcal{C} \).

**Arrows:** Given objects \( f : A \to B \) and \( g : C \to D \), an arrow \( f \to g \) is given by arrows \( h : A \to C \) and \( k : B \to D \) together with an invertible 2-cell \( \gamma \) as indicated in the following diagram:

```
\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow f & & \downarrow \gamma \\
B & \xrightarrow{k} & D
\end{array}
\]
```
2-cells: Given objects \( f \) and \( g \), and arrows \((h,k,\gamma)\) and \((h',k',\gamma')\) from \( f \) to \( g \), a 2-cell \( \varphi: (h,k,\gamma) \to (h',k',\gamma') \) consists of invertible 2-cells \( \varphi_0: h \to h' \) and \( \varphi_1: k \to k' \) in \( \mathcal{C} \) such that

\[
\gamma' \cdot (g \ast \varphi_0) = (\varphi_1 \ast f) \cdot \gamma.
\]

Horizontal composition: Given objects \( f: A \to B \), \( g: C \to D \) and \( i: E \to F \), and arrows \( (h,k,\gamma): f \to g \) and \( (l,m,\delta): g \to i \), we define \( (l,m,\delta) \ast (h,k,\gamma) \) to be the composite

\[
(l \circ h, m \circ i, i \circ (l \circ h)) \xrightarrow{\alpha^{-1}} (i \circ l) \circ h \xrightarrow{\delta \circ h} (m \circ g) \circ h \xrightarrow{\alpha} m \circ (g \circ h) \xrightarrow{m \ast \gamma} m \circ (k \circ f) \xrightarrow{\alpha^{-1}} (m \circ k) \circ f
\]

as in the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{f} & & \downarrow{i} \\
B & \xrightarrow{k} & D \\
\end{array}
\]

Given 2-cells \( (\varphi_0, \varphi_1): (h,k,\gamma) \to (h',k',\gamma') \) and \( (\psi_0, \psi_1): (l,m,\delta) \to (l',m',\delta') \) we define the horizontal composite by

\[
(\psi_0, \psi_1) \ast (\varphi_0, \varphi_1) := (\psi_0 \ast \varphi_0, \psi_1 \ast \varphi_1).
\]

The verification that, with these definitions, \( \mathcal{C} \Rightarrow \mathcal{C} \) forms a bicategory is lengthy, but straightforward, and is left to the reader. Note that when \( \mathcal{C} \) is a 2-category, so is \( \mathcal{C} \Rightarrow \mathcal{C} \).

1.4. The 2-categorical case. We later will be concerned with pseudofunctors \( F: \mathcal{C}^{\text{op}} \to \text{Cat} \) where \( \mathcal{C} \) is a category understood as having a trivial 2-category structure. This means precisely that for each object \( U \) of \( \mathcal{C} \) there is a category \( F(U) \) and for each map \( f: V \to U \) there is a functor \( F(f): F(U) \to F(V) \). We will often denote the action of \( F(f) \) on \( x \in F(U) \) by \( x \cdot f \) or, when the map \( f \) is understood, by \( x|_V \). For each object \( U \) we require a distinguished natural isomorphism \( \psi_U: F(1_U) \to 1_{F(U)} \). Finally, for \( f: V \to U \) and \( g: E \to V \), we require a distinguished natural isomorphism \( \varphi_{f,g}: F(g \circ f) \to F(g) \circ F(f) \).

When \( f \) and \( g \) are understood we omit subscripts and simply write \( \varphi \). Similarly, we sometimes write \( \psi \) instead of \( \psi_U \).

Assume given a fixed object \( U \) of \( \mathcal{C} \) together with \( x \) in \( F(U) \) and arrows \( f: U_a \to U \) and \( g: U_B \to U \). In this situation we often denote by \( x|_a \) the object \( x|_{U_a} \) and, similarly, we denote by \( x|_{a \beta} \) the object \( x|_{U_a \times U_B} \). In this situation, we will make use below of the isomorphism from \( x|_a|_{a \beta} \) to \( x|_a|_{a \beta} \) constructed using the coherence maps \( \varphi \) and for which we introduce the notation \( \sigma_{a \beta}(x) \). Explicitly, \( \sigma_{a \beta}(x) \) is defined to be the composite

\[
x|_a|_{a \beta} \xrightarrow{\varphi^{-1}(x)} x|_a|_{a \beta} \xrightarrow{\varphi(x)} x|_a|_{a \beta}.
\]

We also remark that \( \sigma_{a \beta} \) is the inverse of \( \sigma_{a \beta} \). We similarly write \( \sigma_{a \beta \gamma}(x) \) for the map \( x|_{a \gamma}|_{a \beta \gamma} \to x|_{a \beta}|_{a \beta \gamma} \) which is defined in the same way as the composite

\[
x|_{a \gamma}|_{a \beta \gamma} \xrightarrow{\varphi^{-1}(x)} x|_{a \beta \gamma} \xrightarrow{\varphi(x)} x|_{a \beta}|_{a \beta \gamma},
\]

for \( x \) an object of \( F(U_a) \).
2. Fibrant objects and fibration structures

We will now axiomatize two bicategorical notions: bicategories with systems of fibrant objects and bicategories with fibration structures. The former suffices for the construction of the homotopy bicategory. However, the latter concept, which is a refinement of the former, captures additional structure present in certain examples and provides additional structure such as path objects for the homotopy category.

2.1. Systems of fibrant objects. We will now turn to consider the axiomatic structure on a bicategory which will allow us to form the homotopy bicategory and prove that it possesses the correct universal property.

**Definition 2.1.** For arrows \( f : A \to B \) and \( g : C \to D \) in a bicategory \( C \), we write \( f \downarrow g \) to indicate that for any square of the form

\[
\begin{array}{c}
A \quad \xrightarrow{h} \quad C \\
\downarrow f \quad \downarrow \gamma \\
B \quad \xrightarrow{k} \quad D
\end{array}
\]

with \( \gamma \) an invertible 2-cell, there exists a map \( l : B \to C \) together with invertible 2-cells \( \lambda : h \Rightarrow l \circ f \) and \( \rho : g \circ l \Rightarrow k \) such that

\[
g \circ h \xrightarrow{g \circ \lambda} \circ (l \circ f)\]

\[
\gamma \quad \downarrow \quad \alpha^{-1}
\]

\[
k \circ f \xleftarrow{p \circ f} (g \circ l) \circ f
\]

commutes in \( C(A,D) \).

Given a class \( \mathcal{M} \) of maps in \( C \) we write \( \mathcal{M} \downarrow g \) to indicate that \( f \downarrow g \) for all \( f \) in \( \mathcal{M} \). For \( C \) an object of \( C \), we write \( \mathcal{M} \downarrow C \) to indicate that, for all maps \( f : A \to B \) and \( h : A \to C \), if \( f \) is in \( \mathcal{M} \), then there exists a map \( l : B \to C \) and an invertible 2-cell \( h \Rightarrow l \circ f \).

Observe that if a bicategory \( C \) has a terminal object \( 1 \), then \( \mathcal{M} \downarrow C \) if and only if \( \mathcal{M} \downarrow (C \to 1) \).

**Definition 2.2.** A system of fibrant objects on a bicategory \( C \) consists of a collection of maps \( \mathcal{M} \) (weak equivalences) of \( C \) together with a pseudofunctor \( Q : C \to C \) (fibrant replacement) and a pseudonatural transformation \( \eta : 1_C \to Q \) such that the following axioms are satisfied:

**Identities:** All identity arrows \( 1_A : A \to A \) are in \( \mathcal{M} \).

**3-for-2:** Given a diagram

\[
\begin{array}{c}
A \quad \xrightarrow{f} \quad C \\
\downarrow g \quad \downarrow \gamma \\
B \quad \xrightarrow{h}
\end{array}
\]
with $\gamma$ an isomorphism, if any two of $f$, $g$ and $h$ are weak equivalences, then so is the third.

**Fibrant Replacement:** The components of $\eta$ are weak equivalences and $W \cap Q(A)$ for any object $A$ of $\mathcal{C}$.

The notion of a fibration structure on a bicategory $\mathcal{C}$ is a slight refinement of the notion of a system of fibrant objects:

**Definition 2.3.** A fibration structure on a bicategory $\mathcal{C}$ with terminal object $1$ is given by collections of maps $W$ (weak equivalences) and $F$ (fibrations) of $\mathcal{C}$ such that $W$ satisfies the identities and 3-for-2 conditions from Definition 2.2 above and such that the following additional axioms are satisfied:

**Lifting:** $p: E \to B$ is a fibration if and only if $W \cap p$.

**Factorization:** There exists a pseudofunctor $Q: \mathcal{C} \to \mathcal{C}$ together with a pseudo-natural transformation $\eta: 1 \to Q$ such that $\partial_1 \circ Q = \partial_1$, $\partial_1 \circ \eta = \partial_1$, and, for each $f: A \to B$ in $\mathcal{C}$, the arrow part of $\eta_f$ is a weak equivalence and $Q(f)$ is a fibration. Here $\partial_1$ is the pseudofunctor $\mathcal{C} \to \mathcal{C}$ which projects onto the codomain.

Note that every fibration structure on a bicategory $\mathcal{C}$ determines a corresponding system of fibrant objects.

**Remark.** When we apply the factorization condition to the diagonal $\Delta_A: A \to A \times A$, we obtain a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_{\Delta_A}} & \partial_0 Q\Delta_A \\
\Delta_A \downarrow & \cong & \downarrow Q\Delta_A \\
A \times A & \xrightarrow{Q\Delta_A} & A \times A
\end{array}
\]

Here, $\eta_{\Delta_A}$ is a weak equivalence and $Q\Delta_A$ is a fibration. So we find that we can take $A' = \partial_0 Q\Delta_A$ as a path object for $A$ and the classical factorization lemma holds up to an invertible 2-cell. Furthermore, when we take $d_i = \pi_i \circ Q\Delta_A: A' \to A$ we obtain a fibration with the property that $d_i \circ \eta_{\Delta_A} \cong 1_A$.

For the remainder of this section we assume that we are working in a bicategory $\mathcal{C}$ with a system of fibrant objects.

**Definition 2.4.** An object $A$ of $\mathcal{C}$ is **fibrant** when $W \cap A$.

**Lemma 2.1.** If $f: A \to B$ is a weak equivalence between fibrant objects, then $f$ is an equivalence.

**Proof.** First, since $A$ is fibrant there exists a map $f': B \to A$ and an invertible 2-cell

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow f & \Downarrow \cong & \downarrow f' \\
B & \xrightarrow{f'} & A
\end{array}
\]
It follows from the 3-for-2 property that \( f' \) is also a weak equivalence. Therefore, since \( B \) is fibrant, there exists another map \( f'' : A \to B \) and an invertible 2-cell

\[
\begin{array}{c}
B \\
\downarrow^{1_B} \\
A
\end{array}
\]

\[
\begin{array}{c}
f' \\\n\downarrow \\
f''
\end{array}
\]

Now, the 2-cells above, together with the coherence 2-cells of \( \mathcal{C} \), give us an isomorphism \( f \cong f'' \) and therefore \( f' \) is the pseudo-inverse of \( f \), as required.

\[\square\]

**Definition 2.5.** The **homotopy bicategory** \( \text{Ho}(\mathcal{C}) \) of \( \mathcal{C} \) is the full sub-bicategory of fibrant objects of \( \mathcal{C} \).

We denote by \( I : \mathcal{C} \to \text{Ho}(\mathcal{C}) \) the pseudofunctor induced by \( Q : \mathcal{C} \to \mathcal{C} \). It is an immediate consequence of Lemma 2.1 that \( I \) sends weak equivalences to equivalences. For any bicategory \( \mathcal{D} \), let \( [\mathcal{C}, \mathcal{D}]_{\mathfrak{W}} \) denote the sub-bicategory of \([\mathcal{C}, \mathcal{D}]\) consisting of those pseudofunctors which send maps in \( \mathfrak{W} \) to equivalences. Let \( J : \text{Ho}(\mathcal{C}) \to \mathcal{C} \) be the inclusion and observe that \( Q = J \circ I \).

We will now prove that \( I \) is the universal map from \( \mathcal{C} \) to a bicategory which sends weak equivalences to equivalences.

**Theorem 2.2.** For any bicategory \( \mathcal{D} \), \( I \) induces an equivalence of bicategories

\[
[\text{Ho}(\mathcal{C}), \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]_{\mathfrak{W}},
\]

where the subscript \( \mathfrak{W} \) indicates that we are considering only those pseudofunctors which send weak equivalences to equivalences.

**Proof.** Precomposition with \( J \) gives a pseudofunctor \([\mathcal{C}, \mathcal{D}]_{\mathfrak{W}} \to [\text{Ho}(\mathcal{C}), \mathcal{D}]\) which we denote by \([I, \mathcal{D}]\). The pseudonatural transformation \( \eta : 1_{\text{Ho}(\mathcal{C})} \to I \circ J \) (obtained by restricting \( \eta \) to \( \text{Ho}(\mathcal{C}) \)) induces a pseudonatural transformation \([\eta, \mathcal{D}] : 1_{[\text{Ho}(\mathcal{C}), \mathcal{D}]} \to [I, \mathcal{D}] \circ [I, \mathcal{D}]\). Observe that, by Lemmas 1.2 and 2.1, \( \eta : 1_{\text{Ho}(\mathcal{C})} \to I \circ J \) is an equivalence. Therefore the induced \( [\eta, \mathcal{D}] \) is also an equivalence.

On the other hand, for \( F \) in \([\mathcal{C}, \mathcal{D}]_{\mathfrak{W}}\), Lemma 1.2 exhibits \( F \eta : F \to FQ \) as an adjoint equivalence. Let \( \vartheta^F \) denote the adjoint pseudoinverse of \( F \eta \). Allowing \( F \) to vary, we have that \( \vartheta \) is an equivalence \([I, \mathcal{D}] \circ [I, \mathcal{D}] \to 1_{[\mathcal{C}, \mathcal{D}]_{\mathfrak{W}}} \).

\[\square\]

### 3. Stacks and local fibrations

We will now begin developing the machinery required to explain our first example of a fibration system in a bicategory (actually, in this case a 2-category): the 2-category of prestacks. In this section we recall some of the basic notions involved and we also introduce a fibered version of the usual category of descent data that will allow us to describe the maps, which we call **local fibrations**, that provide the fibrations in the fibration structure for the 2-category of prestacks.

#### 3.1. Coverings and sites

Throughout we assume given a fixed site \( (\mathcal{C}, I) \) for \( \mathcal{C} \) a category with finite limits. Given an object \( U \) of \( \mathcal{C} \), recall that a **sieve on \( U \)** is a family of maps with codomain \( U \) which is a right ideal for composition. To say that \( (\mathcal{C}, J) \) is a site then means that \( J \) assigns to each object \( U \) of \( \mathcal{C} \) a collection \( J(U) \) of sieves on \( U \) (called **covering sieves**, **covering families** or **covers**) in such a way that the following conditions are satisfied:
we define the category of descent data by making this data vary relative to a fixed morphism.

locally (i.e., on a cover). Note that every stack is a prestack.

order to tell whether and not exists a family which sends sheaf. In particular, in a prestack it is possible to construct arrows in the categories

3.3. Prestacks. Given a pseudofunctor \( F : \mathcal{C}^{\text{op}} \to \text{Cat} \) and objects \( a \) and \( b \) of \( F(U) \), there is an induced functor

\[
\begin{array}{ccc}
(\mathcal{C}/U)^{\text{op}} & \to & \text{Set} \\
F(a,b) & \mapsto & \text{Set}
\end{array}
\]

which sends

\[
f : V \to U \quad \text{Hom}_{F(U)}(a|_V, b|_V).
\]

A pseudofunctor \( F : \mathcal{C}^{\text{op}} \to \text{Cat} \) is a prestack if, for any \( U \) and \( a, b \in F(U) \), \( F(a,b) \) is a sheaf. In particular, in a prestack it is possible to construct arrows in the categories \( F(U) \) locally (i.e., on a cover). Note that every stack is a prestack.

3.4. Descent data. The following definition generalizes the familiar definition of the category of descent data by making this data vary relative to a fixed morphism.

\[ \text{Definition 3.1.} \quad \text{Given } p : E \to B \text{ in } [\mathcal{C}^{\text{op}}, \text{Cat}]_{\text{ps}} \text{ and a cover } \mathcal{J} = (f_\alpha : U_\alpha \to U)_{\alpha} \text{ of some } U \text{ we define the category } \text{Desc}(p, \mathcal{J}) \text{ as follows}
\]

\[ \text{Objects: } \text{An object is a tuple } (b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha \beta})) \text{ where } b \in B(U), \ e_\alpha \in E(U_\alpha), \ \psi_\alpha \text{ is an isomorphism } p(e_\alpha) \to b|_\alpha \text{ and } \vartheta_{\alpha \beta} \text{ is an isomorphism } e_\beta|_{\alpha \beta} \to e_\alpha|_{\alpha \beta}. \]

This data is furthermore required to satisfy the conditions that

\[ \vartheta_{\alpha \alpha} = 1_{e_\alpha} \]
and that the diagrams

\[
\begin{array}{c}
\begin{array}{ccc}
e_\gamma|_{\alpha\beta\gamma} & \xrightarrow{\sigma_{\alpha\beta}(e_\gamma)} & e_\gamma|_{\beta\gamma}\alpha\beta\gamma \\
\downarrow & & \downarrow \\
\varepsilon_\gamma|_{\alpha\beta\gamma} & \xrightarrow{\sigma_{\alpha\beta}(\varepsilon_\gamma)} & \varepsilon_\gamma|_{\alpha\beta\gamma}
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{ccc}
e_\alpha|_{\alpha\beta\gamma} & \xrightarrow{\sigma_{\alpha\beta\gamma}(e_\gamma)} & e_\alpha|_{\alpha\beta\gamma} \\
\downarrow & & \downarrow \\
\varepsilon_\alpha|_{\alpha\beta\gamma} & \xrightarrow{\sigma_{\alpha\beta\gamma}(\varepsilon_\alpha)} & \varepsilon_\alpha|_{\alpha\beta\gamma}
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{ccc}
p(e_\beta|_{\alpha\beta}) & \xrightarrow{\eta(e_\beta)} & p(e_\beta)|_{\alpha\beta} \\
\downarrow & & \downarrow \\
p(\varepsilon_\beta|_{\alpha\beta}) & \xrightarrow{\sigma_{\alpha\beta}(\varepsilon_\beta)} & p(\varepsilon_\beta)|_{\alpha\beta}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
p(e_\alpha|_{\alpha\beta}) & \xrightarrow{\eta(e_\alpha)} & p(e_\alpha)|_{\alpha\beta} \\
\downarrow & & \downarrow \\
p(\varepsilon_\alpha|_{\alpha\beta}) & \xrightarrow{\sigma_{\alpha\beta}(\varepsilon_\alpha)} & p(\varepsilon_\alpha)|_{\alpha\beta}
\end{array}
\end{array}
\]

commute, where the \( \eta \) are the natural isomorphisms associated to \( p \).

**Arrows:** An arrow \( (b,(e_\alpha),(\psi_\alpha),(\varepsilon_\alpha),(\sigma_{\alpha\beta})) \rightarrow (b',(e'_\alpha),(\psi'_\alpha),(\varepsilon'_\alpha),(\sigma'_{\alpha\beta})) \) is given by a pair \( (g,(g_\alpha)) \) such that \( g: b \rightarrow b' \) in \( B(U) \) and \( g_\alpha: e_\alpha \rightarrow e'_\alpha \) in \( E(U_\alpha) \). This data is subject to the requirements that the diagrams

\[
\begin{array}{c}
\begin{array}{ccc}
e_\beta|_{\alpha\beta} & \xrightarrow{\sigma_{\alpha\beta}} & e_\alpha|_{\alpha\beta} \\
\downarrow & & \downarrow \\
\varepsilon_\beta|_{\alpha\beta} & \xrightarrow{\sigma_{\alpha\beta}} & \varepsilon_\alpha|_{\alpha\beta}
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{ccc}
p(e_\alpha) & \xrightarrow{\psi_\alpha} & b|_{\alpha} \\
\downarrow & & \downarrow \\
p(\varepsilon_\alpha) & \xrightarrow{\sigma_\alpha} & g|_{\alpha}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
p(e'_\alpha) & \xrightarrow{\psi'_\alpha} & b'|_{\alpha} \\
\downarrow & & \downarrow \\
p(\varepsilon'_\alpha) & \xrightarrow{\sigma'_{\alpha\beta}} & g'|_{\alpha}
\end{array}
\end{array}
\]

commute.

There is a projection functor \( \pi: \text{Desc}(p,\mathcal{S}) \rightarrow B(U) \). When \( p \) is the canonical map \( E \rightarrow 1 \) into the terminal object we write \( \text{Desc}(E,\mathcal{S}) \) instead of \( \text{Desc}(p,\mathcal{S}) \) and observe that this is the usual category of descent data. There is also an evident functor

\[
E(U) \xrightarrow{\Phi_{\mathcal{S}}} \text{Desc}(p,\mathcal{S}) \xrightarrow{\pi} B(U)
\]
Lemma 3.1. we now turn.

\[ (p(e), (e|\alpha), (1_{p(e\alpha)}), (\sigma_{\alpha\beta}(e))) \]

where \( \sigma_{\alpha\beta} \) is as in Section 1.4.

Given a commutative triangle

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow{p} & & \downarrow{p'} \\
B & \xrightarrow{\pi} & B(U)
\end{array}
\]

in \([\mathcal{C}^{\text{op}}, \mathbf{Cat}]_{ps}\) and a covering family \( \mathcal{S} \) of \( U \), there is a corresponding commutative diagram

\[
\begin{array}{ccc}
\text{Desc}(p, \mathcal{S}) & \xrightarrow{f_*} & \text{Desc}(p', \mathcal{S}) \\
\downarrow{\pi} & & \downarrow{\pi} \\
B(U) & & B(U)
\end{array}
\]

in \( \mathbf{Cat} \). Here the functor \( f_* \) acts as follows:

- **On objects:** \( f_* \) sends \((b, (e_\alpha), (\psi_\alpha), (\theta_{\alpha\beta}))\) in \( \text{Desc}(p, \mathcal{S}) \) to \((b, (f(e_\alpha)), (\psi_\alpha), (\xi_{\alpha\beta}))\) where \( \xi_{\alpha\beta} \) is the composite

\[
\begin{array}{cccc}
& f(e_\beta)|_{\alpha\beta} & \rightarrow & f(e_\beta|_{\alpha\beta}) \\
& \downarrow{f(\theta_{\alpha\beta})} & & \downarrow{f(\theta_{\alpha\beta})} \\
f(e_\alpha|_{\alpha\beta}) & \rightarrow & f(e_\alpha|_{\alpha\beta}) & \rightarrow & f(e_\alpha|_{\alpha\beta})
\end{array}
\]

where the unnamed arrows are from the coherence isomorphisms associated to the pseudonatural transformation \( f \).

- **On arrows:** An arrow \((g, (g_\alpha))\) in \( \text{Desc}(p, \mathcal{S}) \) is sent to \((g, f(f(g_\alpha)))\).

Moreover, this construction is functorial in the sense that \( \text{Desc}(\dash, \mathcal{S}) \) is a functor from \([\mathcal{C}^{\text{op}}, \mathbf{Cat}]_{ps}/B \) to \( \mathbf{Cat}/B(U) \). This fact is a special case of a more general result to which we now turn.

**Lemma 3.1.** For a fixed object \( U \) of \( \mathcal{C} \) and a covering family \( \mathcal{S} \) of \( U \), \( \text{Desc}(\dash, \mathcal{S}) \) is a functor \([\mathcal{C}^{\text{op}}, \mathbf{Cat}]_{ps} \) \( \rightarrow \) \( \mathbf{Cat} \).

**Proof.** Given a square

\[
\begin{array}{ccc}
A & \xrightarrow{h} & E \\
\downarrow{p} & & \downarrow{p} \\
C & \xrightarrow{k} & B
\end{array}
\]

in \([\mathcal{C}^{\text{op}}, \mathbf{Cat}]_{ps} \) with \( \gamma \) invertible, the induced functor \( (h, k, \gamma) : \text{Desc}(i, \mathcal{S}) \rightarrow \text{Desc}(p, \mathcal{S}) \) sends descent data \((c, (a_\alpha), (\psi_\alpha), (\theta_{\alpha\beta}))\) in \( \text{Desc}(i, \mathcal{S}) \) to \((k(c), (h(a_\alpha)), (\psi_\alpha), (\theta_{\alpha\beta}))\) in \( \text{Desc}(p, \mathcal{S}) \) where \( \psi_\alpha \) is the composite

\[
\begin{array}{cccc}
p(h(a_\alpha)) & \xrightarrow{\gamma(a_\alpha)} & k(i(a_\alpha)) & \rightarrow & k(c)|_{\alpha} \\
& & k(\psi_\alpha) & \rightarrow & k(c)|_{\alpha}
\end{array}
\]
and $\hat{\gamma}_{\alpha\beta}$ is the composite

$$h(a_\beta)|_{a_\beta} \xrightarrow{\hat{\gamma}_{\alpha\beta}} h(a_\beta|_{a_\beta}) \xrightarrow{h(a_\beta)} h(a_\alpha|_{a_\beta}) \xrightarrow{\hat{\gamma}_{\alpha\beta}} h(a_\alpha|_{a_\beta}).$$

Here the unnamed arrows are from the coherence isomorphisms associated to the pseudonatural transformations.

Observe that, given a square \(3\) in \([G^{op}, \textbf{Cat}]_{ps}\) and a cover \(\mathcal{S}\) of some \(U\), the following diagram commutes:

$$\text{Desc}(i, \mathcal{S}) \xrightarrow{(h, k, \gamma)} \text{Desc}(p, \mathcal{S})$$

$$\pi \downarrow \bigg\downarrow \phantom{(h, k, \gamma)} \pi \downarrow \bigg\downarrow$$

$$C(U) \xrightarrow{k} B(U).$$

On the other hand, we merely have a natural isomorphism \(\hat{\gamma}\) as indicated in the following diagram:

$$A(U) \xrightarrow{h} E(U)$$

$$\Phi_{\mathcal{S}} \downarrow \bigg\downarrow \Phi_{\mathcal{S}} \downarrow \bigg\downarrow$$

$$\text{Desc}(i, \mathcal{S}) \xrightarrow{(h, k, \gamma)} \text{Desc}(p, \mathcal{S})$$

$$\pi \downarrow \bigg\downarrow \phantom{(h, k, \gamma)} \pi \downarrow \bigg\downarrow$$

$$C(U) \xrightarrow{k} B(U)$$

which, for \(a\) an object of \(A(U)\), is the map of descent data

\((\gamma(a): ph(a) \rightarrow ki(a), (h(a)|_{a} \rightarrow h(a|_{a})): \Phi_{\mathcal{S}}(h(a)) \rightarrow (h, k, \gamma), (\Phi_{\mathcal{S}}(a)).\)

This has the property that

$$A(U) \xrightarrow{h} E(U)$$

$$\Phi_{\mathcal{S}} \downarrow \bigg\downarrow \Phi_{\mathcal{S}} \downarrow \bigg\downarrow$$

$$\text{Desc}(i, \mathcal{S}) \xrightarrow{(h, k, \gamma)} \text{Desc}(p, \mathcal{S})$$

$$\pi \downarrow \bigg\downarrow \phantom{(h, k, \gamma)} \pi \downarrow \bigg\downarrow$$

$$C(U) \xrightarrow{k} B(U)$$

The construction of the category of descent data is also functorial in the second argument in the sense that if \(\mathcal{S}\) and \(\mathcal{R}\) are both covers of some \(U\) with \(\mathcal{R} \subseteq \mathcal{S}\), then there exists an associated restriction functor \(\cdot|_{\mathcal{R}}: \text{Desc}(p, \mathcal{S}) \rightarrow \text{Desc}(p, \mathcal{R})\) which acts by restricting descent data to those maps in \(\mathcal{R}\). These restrictions satisfy the functoriality condition \((\cdot|_{\mathcal{R}}) \circ (\cdot|_{\mathcal{S}}) = (\cdot|_{\mathcal{R}})\) and are well-behaved with respect to the associated maps \(\Phi_{\mathcal{R}}: E(U) \rightarrow \text{Desc}(p, \mathcal{S})\), in the sense that the diagram

$$E(U) \xrightarrow{\Phi_{\mathcal{R}}} \text{Desc}(p, \mathcal{S})$$

$$\text{Desc}(p, \mathcal{R})$$

$$\Phi_{\mathcal{R}} \downarrow \bigg\downarrow |_{\mathcal{R}}$$
commutes for any \( S \subseteq T \).

In addition to the functorial behavior of \( \text{Desc}(\_ , \_ ) \) described above, if we are given a fixed \( p : E \to B \), a cover \( \mathcal{U} \) of \( U \) and a map \( g : V \to U \) in the site, we obtain a further restriction functor \( g^* : \text{Desc}(p, \mathcal{U}) \to \text{Desc}(p, g^*(\mathcal{U})) \) which sends descent data \( (b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta})) \) to the descent data given by:

- the object \( b|_V \) of \( B(V) \);
- the family of objects \( (e_\alpha) \) (this makes sense by virtue of the definition of \( g^*(\mathcal{U}) \));
- the family of maps given by the composites

\[
p(e_\alpha) \xrightarrow{\psi_\alpha} b|_\alpha \longrightarrow (b|_V)|_\alpha,
\]

which we denote by \( g^*(\psi)_\alpha \) when no confusion will result; and
- the family of maps given by the composites

\[
e_\beta|_{U_\alpha \times_V U_\beta} \xrightarrow{e_\beta|_{U_\alpha \times_V U_\beta}} e_\beta|_{U_\alpha \times_V U_\beta} \xrightarrow{\vartheta_{\alpha\beta}|_{U_\alpha \times_V U_\beta}} e_\alpha|_{U_\alpha \times_V U_\beta}
\]

where \( \vartheta_{\alpha\beta} \) is here restricted along the induced map \( U_\alpha \times_V U_\beta \to U_\alpha \times U_\beta \) and the unlabeled maps are the structural isomorphisms associated with pseudofunctoriality of \( E \).

and which acts on arrows by sending \((g, (g_\alpha))\) to \((g|_V, (g_\alpha))\).

### 3.5. Local fibrations

We are now in a position to describe the maps which will be the fibrations in our fibration structure.

**Definition 3.2.** A map \( p : E \to B \) is a **local fibration** if and only if, for every \( U \) and cover \( \mathcal{U} \) of \( U \), the map

\[
\Phi_{\mathcal{U}} : E(U) \to \text{Desc}(p, \mathcal{U})
\]
described in Section 3.4 above is a weak equivalence.

**Example 3.3.** When \( p \) is the canonical map \( F \to 1 \), \( \text{Desc}(p, (U_\alpha)) \) is the pseudolimit from (2) and this map is a local fibration if and only if \( F \) is a stack.

**Example 3.4.** Let \( 2 \) be the category with two objects, one non-identity arrow and one connected component. Then, for \( A : \mathcal{C}^{\text{op}} \to \textbf{Cat} \) a pseudofunctor, \([2, A] \) denotes the cotensor with \( 2 \). I.e., \([2, A](U) = A(U)^2\). \( A \) is a prestack if and only if the induced map \( \langle \partial_0, \partial_1 \rangle : [2, A] \to A \times A \) is a local fibration.

Notice that the map \( \Phi_{\mathcal{U}} \) is always faithful and that we have the following characterization of local fibrations between prestacks:

**Lemma 3.2.** If \( E \) and \( B \) are prestacks, then \( p : E \to B \) is a local fibration if and only if, for each \( U \) and cover \( \mathcal{U} \), \( \Phi_{\mathcal{U}} \) is essentially surjective on objects.

**Proof.** Given a map \( (f, f_\alpha) : \Phi_{\mathcal{U}}(e) \to \Phi_{\mathcal{U}}(e') \) in \( \text{Desc}(p, \mathcal{U}) \) it follows from the fact that \( E \) is a prestack that the \( f_\alpha \) possess a unique amalgamation \( g : e \to e' \). Since \( B \) is a prestack we may test locally to see that \( p(g) = f \). \( \square \)
4. The Fibration Structure on Prestacks

We will now describe the fibration structure on $\text{PreSt}(\mathcal{C})$ for a site $(\mathcal{C}, J)$ such that the topology $J$ is precanonical. We begin by defining what we will call local weak equivalences (this definition can be found in [14] and similar definitions appear throughout the literature on stacks and homotopy theory).

**Definition 4.1.** A map $h: A \to B$ in $\text{PreSt}(\mathcal{C})$ is said to be **locally essentially surjective on objects** if and only if for any $U$ and $b \in B(U)$ there exists a cover $\mathcal{U} = (f_\alpha: U_\alpha \to U)$ of $U$ together with, for each $\alpha$, an element $\tilde{b}_\alpha \in A(U_\alpha)$ and an isomorphism $\psi_\alpha: h(\tilde{b}_\alpha) \to b|_\alpha$.

**Definition 4.2.** A map $h: A \to B$ in $\text{PreSt}(\mathcal{C})$ is a **local weak equivalence** if it is full, faithful and locally essentially surjective on objects.

Here being full and faithful means being pointwise full and faithful.

The remainder of this section is devoted to giving a proof of the following result:

**Theorem 4.1.** There is a fibration structure on $\text{PreSt}(\mathcal{C})$ with fibrations the local fibrations and weak equivalences the local weak equivalences.

Consequently, the fibrant objects in this case are precisely the stacks.

**Corollary.** There is an equivalence of 2-categories $\text{St}(\mathcal{C}) \simeq \text{Ho}(\text{PreSt}(\mathcal{C}))$.

Throughout the remainder of this section we denote by $\mathcal{W}$ the class of local weak equivalences and by $\mathfrak{S}$ the class of maps $p$ such that $\mathcal{W} \cap p$.

4.1. Three-for-two. We will now show that the local weak equivalences satisfy the three-for-two condition:

**Proposition 4.2.** Given a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow{g} & & \downarrow{h} \\
B & \xleftarrow{h} & \text{C} \\
\end{array}
$$

with $\gamma$ an isomorphism, if any two of $f, g$ and $h$ are local weak equivalences, then so is the third.

**Proof.** If $h$ and $g$ are local weak equivalences, then it is trivial to verify that $f$ is also a local weak equivalence.

When $f$ and $g$ are local weak equivalences it is easily seen that $h$ is locally essentially surjective since $f$ is. To see that $h$ is full, suppose given a map $j: h(x) \to h(y)$ in $C(U)$. Because $g$ is locally essentially surjective on objects we can find a cover $\mathcal{U}$ of $U$ and isomorphisms $\varphi_\alpha: g(a_\alpha) \to x|_\alpha$ and $\psi_\alpha: g(b_\alpha) \to y|_\alpha$ in $B(U_\alpha)$ for each $U_\alpha \to U$ in the cover. We can then construct composites

$$
\begin{array}{ccc}
& f(b_\alpha) \\
\downarrow{\psi^{-1}_\alpha} & & \downarrow{\psi^{-1}_\alpha} \\
\varphi_\alpha & h(g(a_\alpha)) & h(g(b_\alpha)) \\
\downarrow{h(\varphi_\alpha)} & \downarrow{h(x|_\alpha)} & \downarrow{h(\gamma)} \\
h(x) & h(y) & h(\gamma) \\
\end{array}
$$

where the unlabelled arrows are the coherence isomorphisms associated to $h$. Since $f$ is full and faithful there exists a canonical lift $u_\alpha: a_\alpha \to b_\alpha$ in $A(U_\alpha)$ for each $U_\alpha \to U$.
in the cover \( \mathcal{S} \). Using these lifts we similarly obtain maps \( v_\alpha : x|\alpha \to y|\alpha \) defined as \( \psi_\alpha \circ g(u_\alpha) \circ \phi^{-1}_\alpha \). These constitute a matching family for the presheaf \( B(x,y) \). To see this it suffices to show that, for each \( U_\alpha \to U \) and \( U_\beta \to U \) in \( \mathcal{S} \), the diagram

\[
\begin{array}{ccc}
g(a_\alpha)|_{\alpha\beta} & \xrightarrow{\phi_\alpha^{-1}|_{\alpha\beta}} & x|_{\alpha\beta} \\
g(a_\alpha) & \downarrow & \downarrow \\
g(a_\alpha)|_{\alpha\beta} & \xrightarrow{\phi_\beta^{-1}|_{\alpha\beta}} & g(a_\beta)|_{\alpha\beta}
\end{array}
\]

commutes, where the unnamed arrows are the evident coherence isomorphisms. Since \( g \) is full and faithful both ways around this diagram induces canonical lifts \( \xi, \zeta : a_\alpha|_{\alpha\beta} \to b_\beta|_{\alpha\beta} \).

It suffices by faithfulness of \( E \) has the property that \( x \in B \).

To see that \( f \) is faithful one uses roughly the same kind of approach. Given \( j,k : x \to y \in B(U) \) with \( h(j) = k(j) \) we use local essential surjectivity of \( g \) to obtain a cover \( \mathcal{S} \) and isomorphisms \( g(a_\alpha) \cong x|\alpha \) and \( g(b_\alpha) \cong y|\alpha \). Conjugation of \( j|\alpha \) and \( k|\alpha \) by these isomorphisms gives two families of maps \( g(a_\alpha) \to g(b_\alpha) \) and since \( g \) is full and faithful these induce canonical lifts \( u_\alpha, v_\alpha : a_\alpha \to b_\alpha \) in \( A(U_\alpha) \). Using the fact that \( h(j) = k(j) \) we can then show that \( f(u_\alpha) = f(v_\alpha) \) so that \( u_\alpha = v_\alpha \). It then follows by the fact that \( B \) is a prestack that \( j = k \).

The proof that \( g \) is a local weak equivalence when \( f \) and \( h \) are is similar and is left to the reader.

4.2. Characterization of the fibrations. We now turn to providing a characterization of the fibrations \( \mathcal{S} \). This result is analogous to an earlier result of Joyal and Tierney [9] in which they characterize stacks as weakly fibrant objects. The differences between our result and theirs are as follows. First, they consider a Grothendieck topos \( \mathcal{E} \) with the canonical topology and they characterize those groupoids \( G \) in \( \mathcal{E} \) such that the externalization \( \mathcal{E}(\mathcal{E}(\cdot,G)) \) is a stack. In our case, the site is an arbitrary precanonical site and our prestacks are fibered in categories rather than groupoids. In the setting of ibid it is not necessary to consider prestacks and it is not necessary to make use of the axiom of choice. Because we work in a more general setting we must restrict first to prestacks and we also appeal to the axiom of choice. Finally, the characterization we give is of local fibrations in general and not just stacks.

**Lemma 4.3.** For \( i : A \to C \) in \( \mathcal{M} \) and \( U \) in \( \mathcal{E} \), every object \( c \) of \( C(U) \) determines a cover \( \mathcal{S} \) and an object of \( \text{Desc}(i, \mathcal{S}) \) which projects via \( \pi : \text{Desc}(i, \mathcal{S}) \to C(U) \) onto \( c \).

**Proof.** Let an object \( c \) of \( C(U) \) be given. Because \( i \) is locally essentially surjective on objects there exists a family of isomorphisms \( \psi_\alpha : i(\tilde{c}_\alpha) \to c|\alpha \). We may form the composites

\[
\begin{array}{cccccccc}
l(\tilde{c}_\beta)|_{\alpha\beta} & \xrightarrow{\psi_\beta|_{\alpha\beta}} & c|\beta|_{\alpha\beta} & \xrightarrow{\phi_{\alpha\beta}(c)} & c|\alpha|_{\alpha\beta} & \xrightarrow{\psi_\alpha^{-1}|_{\alpha\beta}} & l(\tilde{c}_\alpha)|_{\alpha\beta} & \xrightarrow{\phi_{\alpha\beta}} & l(\tilde{c}_\alpha)|_{\alpha\beta}
\end{array}
\]
where the unlabelled arrows are induced by the coherence 2-cell associated to the pseudo-natural transformation $i$. Since $i$ is full and faithful these possess canonical invertible lifts $\theta_{a\beta} : c_\beta|_a \to c_a|_a$ in $A(U_{a\beta})$. It is routine to verify that $(c, (c_\alpha), (\psi_\alpha), (\theta_{a\beta}))$ is an object of $\text{Desc}(i, \mathcal{S})$.

**Lemma 4.4.** If $p : E \to B$ is a local fibration, then $\mathbb{W} \cap p$.

**Proof.** Suppose $p : E \to B$ is a local fibration and let a diagram of the form

$$
\begin{array}{ccc}
A & \xrightarrow{h} & E \\
\downarrow{i} & \swarrow{Y} & \downarrow{p} \\
C & \xrightarrow{k} & B
\end{array}
$$

be given with $i : A \to C$ in $\mathbb{W}$. Given an object $c$ of $C(U)$ we may choose, by Lemma 4.3, a cover $\mathcal{S}$ together with descent data $(c, (a_\alpha), (\psi_\alpha), (\theta_{a\beta}))$ in $\text{Desc}(i, \mathcal{S})$. By Lemma 3.1 this gives descent data $\hat{c} := (k(c), (h(a_\alpha)), (\hat{\psi}_\alpha), (\hat{\theta}_{a\beta}))$ in $\text{Desc}(p, \mathcal{S})$. Thus, we choose $l(c)$ to be an amalgamation of this descent data.

Given $f : c \to d$ in $C(U)$ assume that $\mathcal{S}$ and $\mathcal{R}$ are the covers chosen in the definition of $l(c)$ and $l(d)$ and assume that $(c, (a_\alpha), (\psi_\alpha), (\theta_{a\beta}))$ and $(d, (b_\beta), (\varphi_\alpha), (\omega_{a\beta}))$ are the descent data chosen in the definition of $l(c)$ and $l(d)$, respectively. Let $\mathcal{W}$ be the common refinement $\mathcal{S} \cap \mathcal{R}$ of $\mathcal{S}$ and $\mathcal{R}$ and observe that, for $U_a \to U$ in $\mathcal{W}$, we have isomorphisms $\chi : \text{Desc}(l(c)) \cong \hat{c}$ and $\mu : \text{Desc}(l(d)) \cong \hat{d}$. We also have $i(a_\alpha) \xrightarrow{\psi_\alpha} c|_\alpha \xrightarrow{f|_\alpha} d|_\alpha \xrightarrow{\varphi_\alpha^{-1}} i(b_\beta)$.

As such, since $i : A \to C$ is full and faithful, there exists a canonical map $f_\alpha : a_\alpha \to b_\alpha$ which is mapped by $i$ onto this composite. This gives a map of descent data

$$(f, f_\alpha) : (c, (a_\alpha), (\psi_\alpha), (\theta_{a\beta}))|_\mathcal{W} \to (d, (b_\beta), (\varphi_\alpha), (\omega_{a\beta}))|_\mathcal{W}$$

and by Lemma 3.1, we have that $(k(f), (h(f_\alpha))) : (\hat{c})|_\mathcal{W} \to (\hat{d})|_\mathcal{W}$ in $\text{Desc}(p, \mathcal{W})$. Therefore we may form the composite

$$
\begin{array}{ccc}
\Phi_{\mathcal{S}}(l(c))|_\mathcal{W} & \xrightarrow{\chi|_\mathcal{W}} & (k(f), (h(f_\alpha)))|_\mathcal{W} \\
& \xrightarrow{\mu^{-1}|_\mathcal{W}} & \Phi_{\mathcal{R}}(l(d))|_\mathcal{W}
\end{array}
$$

which gives us a family of maps $l(c)|_\alpha \to l(d)|_\alpha$ for $U_a \to U$ in $\mathcal{W}$. This family constitutes a matching family for $E(l(c), l(d))$ and since $E$ is a prestack there exists a canonical amalgamation $l(f) : l(c) \to l(d)$. Functoriality follows from the uniqueness of amalgamations.

We now construct the natural isomorphisms $\lambda : h \cong l \circ i$ and $\rho : p \circ l \cong k$. First, for $\lambda$, assume given an object $u$ of $A(U)$. Assume that $\mathcal{S}$ is the cover of $U$ and $(iu, (a_\alpha), (\psi_\alpha), (\theta_{a\beta}))$ is the descent data chosen in the construction of $l(iu)$. Then $(1_{iu}, (\psi_\alpha^{-1}))$ is an isomorphism in $\text{Desc}(i, \mathcal{S})$ from $\Phi_{\mathcal{S}}(u)$ to $(iu, (a_\alpha), (\psi_\alpha), (\theta_{a\beta}))$. As such, we may form the following composite

$$
\begin{array}{ccc}
\Phi_{\mathcal{S}}(ha) & \xrightarrow{\gamma(a)} & (h, k, \hat{\gamma})_*\Phi_{\mathcal{R}}(a) \\
& \xrightarrow{(h, k, \hat{\gamma})_*(1_{iu}, (\psi_\alpha^{-1}))} & \Phi_{\mathcal{R}}(l(iu))
\end{array}
$$

in $\text{Desc}(\mathcal{S}, p)$, where $\hat{\gamma}$ is as in the discussion of $(h, k, \gamma)_*$ from Section 3.4 and the unnamed map is the isomorphism associated to the definition of $l(iu)$. Because $\Phi_U$ is full and faithful this gives a canonical isomorphism $\lambda(a) : h(a) \to l(iu)$ with $\Phi_{\mathcal{S}}(\lambda(a))$ the
composite above. Naturality of \( \lambda \) follows from faithfulness of the \( \Phi_\mathcal{S} \) together with the definition of the action of \( l \) on arrows. Next, we define \( p(c) : p(l(c)) \to k(c) \) to be the first component of the isomorphism \( \Phi_\mathcal{S}(l(c)) \cong \hat{c} \) of descent data associated to the definition of \( l(c) \). This is natural by definition of \( l \). Finally, it is immediate from the definitions that \( \gamma \) can be recovered by composing the pasting diagram obtained from \( \lambda \) and \( p \).

\[ \square \]

**Theorem 4.5.** For a map \( p : E \to B \) the following are equivalent:

1. \( p \) is a local fibration.
2. \( p \) is in \( \bar{\mathcal{S}} \).

**Proof.** By Lemma 4.4 it suffices to prove that if \( p : E \to B \) is in \( \bar{\mathcal{S}} \), then it is a local fibration. To this end, let \( U \) together with a cover \( \mathcal{S} \) be given. Assume given descent data \((b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta})) \) in \( \text{Desc}(p, \mathcal{S}) \). Then we have a square

\[
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{e} & E \\
\downarrow i & \cong & \downarrow p \\
\gamma U & \xrightarrow{b} & B
\end{array}
\]

where \( \gamma U \) is the representable functor and \( \hat{\mathcal{S}} \) is the subfunctor of \( \gamma U \) induced by the cover \( \mathcal{S} \) (note that both of these are prestacks). Also, \( e \) is the pseudonatural transformation representing the family \((e_\alpha)\) with coherence isomorphisms constructed using the \( \vartheta_{\alpha\beta} \). Similarly, \( b \) is the pseudonatural transformation representing \( b \). Finally, \( \psi \) is the modification with component at \( U_\alpha \to U \) in \( \mathcal{S} \) given by \( \psi_\alpha \).

Notice that \( i \) is a local weak equivalence so that, since \( \mathcal{M} \pitchfork p \), it follows that there exists a lift \( l : \gamma U \to E \) together with isomorphisms \( \lambda : e \cong l \circ i \) and \( p : p \circ l \cong b \) such that the square above can be recovered from these. I.e., we have \( l \) an object of \( EU \) together with \( p_V : p(l)|_V \cong b|_V \) for every \( V \to U \) and \( \lambda_\alpha : e_\alpha \cong l|_\alpha \) for each \( U_\alpha \to U \) in the cover. It is then routine to verify that \( l \) is an amalgamation of our descent data.

\[ \square \]

**Corollary.** For any \( F \), the canonical map \( F \to 1 \) is in \( \bar{\mathcal{S}} \) if and only if \( F \) is a stack.

**Corollary.** If \( p : E \to B \) is in \( \bar{\mathcal{S}} \cap \mathcal{M} \), then \( p \) is an equivalence (i.e., there exists a \( p' : B \to E \) together with invertible \( \eta : 1_B \to p \circ p' \) and \( \epsilon : p' \circ p \to 1_E \)).

**Corollary.** If \( p : E \to B \) is in \( \bar{\mathcal{S}} \cap \mathcal{M} \) and \( i : A \to C \) is any map, then \( i \pitchfork p \).

**Corollary.** Theorem 4.5 is equivalent to the Axiom of Choice.

**Proof.** Consider the case where our site consists of the lattice \( \mathcal{O}(\emptyset) \) of open subsets of the empty set with its canonical topology and the notion of covering family is given by the usual topological notion of covering family. In this case we are working directly in \( \text{Cat} \) and we can easily prove that every object is locally fibrant. Using this it is possible to construct pseudo-inverses of weak categorical equivalences. Therefore the Axiom of Choice holds.

\[ \square \]

4.3. **Factorization and isocomma objects.** We will now describe the factorizations in \( \text{PreSt}(\mathcal{C}) \). To a map \( f : A \to B \) we associate a prestack \( \text{Path}(f) \) by letting \( \text{Path}(f)(U) \) be the category with

- **Objects:** Tuples consisting of a cover \( \mathcal{S} \) and an object \((b, (e_\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta})) \) of \( \text{Desc}(f, \mathcal{S}) \).
Arrows: An arrow \((\mathcal{S}, b, (e_a), (\psi_a), (\vartheta_{a\beta})) \to (\mathcal{S}', b', (e'_a), (\psi'_a), (\vartheta'_{a\beta}))\) is an equivalence class of data given by a common refinement \(\mathcal{W}\) of \(\mathcal{S}\) and \(\mathcal{S}'\) together with a map
\[
(b, (e_a), (\psi_a), (\vartheta_{a\beta}))|_{\mathcal{W}} \to (b', (e'_a), (\psi'_a), (\vartheta'_{a\beta}))|_{\mathcal{W}}
\]
in \(\text{Desc}(f, \mathcal{W})\). We identify \((\mathcal{W}, g, (g_a))\) and \((\mathcal{W}', g', (g'_a))\) when there exists a common refinement of \(\mathcal{W}\) and \(\mathcal{W}'\) on which the maps of descent data agree.

Note that \(g = g'\) when \((\mathcal{W}, g, (g_a))\) and \((\mathcal{W}', g', (g'_a))\) are identified in \(\text{Path}(f(U))\).

There is, for \(g: V \to U\), an obvious restriction map \(\text{Path}(f)(U) \to \text{Path}(f)(V)\) which acts by pullback on both covers and descent data. This makes \(\text{Path}(f)\) into a pseudofunctor \(\mathcal{C}^{\text{op}} \to \text{Cat}\). We observe that we have the following lemma, the proof of which is straightforward:

**Lemma 4.6.** If \(A\) and \(B\) are prestacks and \(f: A \to B\), then \(\text{Path}(f)\) is a prestack.

There is a projection map \(\text{Path}(f) \to B\) which sends \((\mathcal{S}, b, (e_a), (\psi_a), (\vartheta_{a\beta}))\) to \(b\) and sends an arrow \([\mathcal{W}, g, (g_a)]\) to \(g\). We define \(Q: \text{PreSt}(\mathcal{C}) \to \text{PreSt}(\mathcal{C}) \to \text{PreSt}(\mathcal{C})\) by letting \(Q(f)\), for \(f: A \to B\) in \(\text{PreSt}(\mathcal{C})\), be the projection \(\text{Path}(f) \to B\).

For the pseudonatural transformation \(\eta: 1_{\text{PreSt}(\mathcal{C})} \to Q\), note that there is a map \(A \to \text{Path}(f)\), which we denote by \(\eta_f\), that sends an \(a\) in \(A(U)\) to \((M_U, (f(a), (a|_U), (\psi_a), (\vartheta_{a\beta})))\) where \(M_U\) denotes the maximal sieve on \(U\), the \(\psi_a\) are the coherence isomorphisms associated to \(f\), and the \(\vartheta_{a\beta}\) are the coherence isomorphisms obtained from the structure of \(A\) as a pseudofunctor. It is straightforward to verify that \(Q(f) \circ \eta_f = f\) and this equation determines the rest of the data of the pseudonatural transformation \(\eta\).

**Lemma 4.7.** For \(f: A \to B\), in the factorization

![Diagram](attachment:image.png)

\(Q(f)\) is a local fibration and \(\eta_f\) is in \(\mathcal{M}\).

**Proof.** It is trivial that \(\eta_f\) is in \(\mathcal{M}\). To see that \(Q(f)\) is a local fibration let a cover \(\mathcal{W} = (h^\gamma: U^\gamma \to U)_\gamma\) of \(U\) be given together with an object
\[
(b, (\mathcal{S}^\gamma, b^\gamma, (e_a^\gamma), (\psi_a^\gamma), (\vartheta_{a\beta}^\gamma)), (\vartheta_{\gamma\delta}^\gamma), (\Theta_{\gamma\delta}^\gamma))
\]
of \(\text{Desc}(Q(f), \mathcal{W})\) where \(\mathcal{S}^\gamma = (h^\gamma a: U^\gamma a \to U^\gamma)_a\). Here \(\Theta_{\gamma\delta}^\gamma = (h_{\gamma\delta}^\gamma, (h_{\gamma\delta}^\gamma a))\) is an isomorphism
\[
(b_{\delta}, (e_a^\delta, (\psi_a^\delta), (\vartheta_{a\beta}^\delta))|_{\gamma\delta} \xrightarrow{\vartheta_{\gamma\delta}^\gamma} (b_{\gamma\delta}, (e_a^\gamma, (\psi_a^\gamma), (\vartheta_{a\beta}^\gamma))|_{\gamma\delta})
\]
of descent data in \(\text{Desc}(f, \mathcal{S}^\gamma|_{\gamma\delta})\). Define a new cover \(\bar{\mathcal{W}}\) of \(U\) as the cover consisting of the maps of the form \(h^\gamma \circ h^\gamma a: U^\gamma a \to U^\gamma \to U\) for \(h^\gamma\) in \(\mathcal{W}\) and \(h^\gamma a\) in \(\mathcal{W}^\gamma\). We then have an object
\[
(\bar{\mathcal{W}}, b, (e_a), (\vartheta_{a\beta}^\delta), (\chi_{a\beta}^\delta))
\]
of $\text{Desc}(f, \mathcal{S})$ where $\varphi^\gamma_\alpha$ is the composite

$$f(e^\gamma_\alpha) \xrightarrow{\psi^\gamma_\alpha} b^\gamma_{|U^\gamma_\alpha} \xrightarrow{\Phi^\gamma_{|U^\gamma_\alpha}} b_{|U^\gamma_\alpha}$$

and $\chi^\delta_{\alpha\beta}$ is the composite

$$e^\delta_{|U^\delta_{\alpha\beta}} \xrightarrow{h^\delta_{\alpha\beta}} e^\gamma_{|U^\gamma_{\alpha\beta}} \xrightarrow{\theta^\gamma_{|U^\gamma_{\alpha\beta}}} e^\gamma_{|U^\gamma_{\alpha\beta}}.$$

With these definitions, it is a (quite) lengthy but straightforward verification that we have described the amalgamation of the descent data. □

**Example 4.3.** When $f$ is the canonical map $A \to 1$ we see that $\text{Path}(f)$ is the associated stack $a(A)$ of $A$ (cf. [13, 14] for more on the associated stack).

We note that when $A$ is a stack it is possible to factor $f$ in a more straightforward way using isocomma objects.

**Definition 4.4.** Given maps $f : A \to B$ and $g : C \to B$ in $\text{PreSt}(\mathcal{C})$, the isocomma object $(f, g)$ is the pseudofunctor given at $U$ by the category $(f, g)(U)$ with

- **Objects:** Tuples consisting of objects $a$ and $c$ of $A(U)$ and $C(U)$, respectively, and an isomorphism $\xi : f(a) \cong g(c)$.
- **Arrows:** An arrow $(a, c, \xi) \to (b, d, \zeta)$ is given by maps $i : a \to b$ and $j : c \to d$ such that the diagram

$$
\begin{array}{ccc}
    f(a) & \xrightarrow{f(i)} & f(b) \\
    \downarrow{\xi} & & \downarrow{\zeta} \\
    g(c) & \xrightarrow{g(j)} & g(d)
\end{array}
$$

commutes.

The action of $(f, g)$ on arrows is simply by restriction of all of the aforementioned data.

There is an invertible 2-cell $\chi$ as indicated in the following diagram:

$$
\begin{array}{ccc}
    (f, g) & \xrightarrow{\chi} & C \\
    \downarrow{f} & & \downarrow{g} \\
    A & \xrightarrow{f} & B
\end{array}
$$

where the unnamed arrows are the obvious projections. Here $\chi$ projects $(a, c, \xi) \mapsto \xi$. The universal property of $(f, g)$ is that for any other diagram

$$
\begin{array}{ccc}
    Z & \xrightarrow{\chi'} & C \\
    \downarrow{f'} & & \downarrow{g} \\
    A & \xrightarrow{f} & B
\end{array}
$$
with \( \chi' \) invertible, there exists a canonical map \( z : Z \to (f, g) \) such that the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{z} & C \\
\downarrow & & \downarrow \\
A & \xrightarrow{(f, g)} & C
\end{array}
\]

commutes and such that \( \chi' = \chi \circ z \). It is straightforward to show that \( (f, g) \) is a prestack when \( A \) and \( C \) are.

Now, the universal property gives us a map \( i : A \to (f, 1_B) \) such that

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow_i & & \downarrow_p \\
(f, 1_B) & &
\end{array}
\]

commutes, where \( p \) is the projection. Here it is clear that this gives a factorization \( f = p \circ i \). In particular, \( i(a) \) is \( (a, f(a), 1_{f(a)}) \) and it is straightforward to verify that \( i \) is in \( \mathfrak{M} \).

**Lemma 4.8.** When \( A \) is a stack, \( p : (f, 1_B) \to B \) is a local fibration.

**Proof.** Given descent data \( (b,(e_\alpha : f(a^\alpha) \cong b^\alpha), (\psi_\alpha), (\vartheta_{\alpha\beta})) \) in \( \text{Desc} (\mathcal{S}, p) \), we have that \( \vartheta_{\alpha\beta} \) is a commutative square

\[
\begin{array}{ccc}
f(a^\beta)|_{\alpha\beta} \xrightarrow{f(\chi_{\alpha\beta})} f(a^\alpha)|_{\alpha\beta} \\
\uparrow e_\beta & & \uparrow e_\alpha \\
\downarrow & & \downarrow \\
b^\beta|_{\alpha\beta} \xrightarrow{\vartheta_{\alpha\beta}} b^\alpha|_{\alpha\beta}
\end{array}
\]

of isomorphisms. This gives us descent data \( ((a^\alpha), (\chi_{\alpha\beta})) \) for \( A \) and \( \mathcal{S} \) and since \( A \) is a stack there is an amalgamating object \( a \) of \( A(U) \). For each \( \alpha \), we have the isomorphism

\[
f(a)|_{\alpha} \xrightarrow{e_\alpha} f(a)|_{\alpha} \xrightarrow{e_\alpha} f(a^\alpha) \xrightarrow{\psi_\alpha} b^\alpha \xrightarrow{\vartheta_{\alpha\beta}} b|_{\alpha}
\]

and these are easily seen to constitute a matching family for \( B(f(a), b) \). Therefore, since \( B \) is a prestack there is a canonical amalgam \( e : f(a) \cong b \). We define this isomorphism to be the object of \( (f, B)(U) \) to our descent data. It is routine to verify that this constitutes a pseudo-inverse to the map \( (f, B)(U) \to \text{Desc} (\mathcal{S}, p) \) satisfying the coherence conditions from the definition of local fibrations. \( \blacksquare \)

This completes the proof of Theorem 4.1.

5. **Topological, Differentiable and Algebraic Stacks**

We will now show that the results of Section 4 can be used to give analogous characterizations of the 2-categories of topological, differentiable and algebraic stacks. These three cases are formal analogues. The categories of topological spaces, differentiable manifolds and schemes all have in common that quotients in them are not well-behaved. This gives rise to the situation, familiar from the theory of étendues from [1], in which one would like to form a “generalized quotient” of a space, manifold or scheme. (Indeed, there is an important connection with the theory of étendues as described in [16], but we do not describe it here.) Topological, differentiable and algebraic stacks are the appropriate “generalized quotients” of suitable equivalence relations in each of these situations. These three cases are formally analogous in the sense that topological, differentiable and algebraic stacks
are by definition stacks $X$ which appear in a suitable sense as “quotients” of topological spaces, differentiable manifolds, or schemes, respectively. This formal analogy permits us to give a single argument (here described in detail for topological stacks) which will show that each of these 2-categories can be described as the homotopy 2-category of the corresponding 2-category of prestacks.

5.1. **Topological stacks.** We will briefly recall the definition of topological stacks, which are essentially the topological version of the algebraic stacks of Deligne and Mumford [4]. Throughout this section we will be working with the topological site which consists of a small category $\text{Top}$ of sober topological spaces $U, V, \ldots$ equipped with the étale Grothendieck topology. The étale topology is generated by families $(f_i : U_i \rightarrow U)$, which are said to cover when the map $\sum_i U_i \rightarrow U$ is an étale surjection.

**Definition 5.1.** A map $f : A \rightarrow B$ of prestacks is **representable** if, for any space $U$ in $\text{Top}$ and map $g : yU \rightarrow B$, the isocomma object $(f, g)$ is representable.

We now consider pseudofunctors $[\text{Top}, \text{Gpd}]_{ps}$ valued in groupoids. Throughout this section “prestack” means prestack valued in groupoids and similarly for “stack”. Roughly, topological prestacks are those prestacks which arise as quotients of spaces.

**Definition 5.2.** A topological prestack is a prestack $A$ such that the following conditions are satisfied:

1. The diagonal $\Delta : A \rightarrow A \times A$ is representable.
2. There exists a space $U$ in $\text{Top}$ and a map $q : yU \rightarrow A$ such that, for all spaces $V$ in $\text{Top}$ and maps $f : yV \rightarrow A$, the map $(f, q) \rightarrow V$ is an étale surjection.

Notice that it makes sense in condition (2) to say that $(f, q) \rightarrow V$ is an étale surjection since the domain of this map is, by condition (1), representable. We will often refer to the map $q : yU \rightarrow A$ from condition (2) as a **chart for** $A$. Observe that representables are trivially topological prestacks. We will henceforth omit explicit mention of the Yoneda embedding $y$ when no confusion will result.

We denote by $\text{TopPreSt}$ the 2-category of topological prestacks and we observe that it is an immediate consequence of Lemma 4.4 that if $p : E \rightarrow B$ is a local fibration between topological prestacks, then $\mathcal{W} \cap p$ where $\mathcal{W}$ denotes the class of local weak equivalences in $\text{TopPreSt}$. We will now consider to what extent the additional structure of $\text{PreSt}(\text{Top})$ restricts to $\text{TopPreSt}$.

**Lemma 5.1.** If $f : A \rightarrow B$ is an equivalence between prestacks and $B$ has a representable diagonal, then so does $A$.

**Proof.** Let maps $v : V \rightarrow A$ and $w : W \rightarrow A$ be given. Because $B$ has a representable diagonal the isocomma object $(f \circ v, f \circ w)$ is a representable $U$. This gives us the following diagram of invertible 2-cells:
where $f'$ is a pseudoinverse of $f$. This is easily seen to exhibit $U$ as $(v,w)$. □

**Lemma 5.2.** If $f : A \to B$ is an equivalence between prestacks and $B$ is a topological prestack, then $A$ is also a topological prestack.

**Proof.** By Lemma 5.1 it suffices to prove that there exists a space $U$ and an étale map $U \to A$. Because $B$ is a topological prestack there exists an étale map $e : U \to B$. We claim that the map $f' \circ e : U \to A$ is étale, where $f'$ is a pseudoinverse of $f$. Let another map $v : V \to A$ be given. Then the isocomma object $(f \circ v, e)$ is a representable $W$. We then obtain the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\sim} & U \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B \\
\end{array}
\]

where the vertical map $W \to V$ is an étale surjection. It is straightforward to show that the diagram above exhibits $U$ as the isocomma object $(v, f' \circ e)$ so that $f' \circ e$ is étale. □

Modifying a construction of [16], we associate to each topological prestack $A$ and chart $e : U \to A$ the étale groupoid $G^e$ with space of objects $U$ and space of arrows the space representing the isocomma object $(e, e)$. In *ibid* it is assumed that $A$ is a topological stack, but it is in fact sufficient for $A$ to be a topological prestack. Also in *ibid* it is shown how to associate to any étale groupoid $G$ a topological stack $R(G)$. Combining these two procedures, we obtain, for each topological prestack $A$ and chart $e : U \to A$, a topological stack $Q(A, e)$ given by

\[
Q(A, e)_V := \text{GeomMorph}(\text{Sh}(V), \text{Sh}(G^e))
\]

for $V$ a space. Here the objects are geometric morphisms, arrows are invertible natural transformations, $\text{Sh}(V)$ is the ordinary category of sheaves on the space $V$ and $\text{Sh}(G^e)$ is the category of equivariant sheaves on the groupoid $G^e$. Note that it is shown in *ibid* that there is a map $i : A \to Q(A, e)$ which is a weak equivalence.

**Lemma 5.3.** The associated stack $a(A)$ of a topological prestack is a topological stack.

**Proof.** It suffices by Lemma 5.2, and the fact that both $a(A)$ and $Q(A, e)$ are both stacks, to construct a local weak equivalence $a(A) \to Q(A, e)$. Because the map $\eta : A \to a(A)$ is a local weak equivalence and $Q(A, e)$ is a stack there exists a map $a(A) \to Q(A, e)$ and an invertible 2-cell as indicated in the diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{i} & Q(A, e) \\
\downarrow & \sim & \downarrow \\
a(A) & &
\end{array}
\]

By the three-for-two property for local weak equivalences it then follows that $a(A) \to Q(A, e)$ is also a local weak equivalence. □

Putting these lemmas together with Theorem 4.1 we have proved the following:
Theorem 5.4. There is a system of fibrant objects on \( \text{TopPreSt} \) given by taking the local weak equivalences and with fibrant replacement given by the associated stack.

Corollary. There is an equivalence of 2-categories \( \text{TopSt} \simeq \text{Ho}(\text{TopPreSt}) \).

5.2. Differentiable stacks. We will now turn to differentiable stacks. As mentioned above, this case is proved in precisely the same manner as the topological case. In this case, we work with the site \( \text{Diff} \) of small differentiable manifolds with the étale topology.

Definition 5.3. A differentiable prestack is a prestack \( A \) such that there exists a manifold \( U \) in \( \text{Diff} \) and a map \( q: U \to A \) such that, for all manifolds \( V \) in \( \text{Diff} \) and maps \( f: V \to A \), the isocomma object \( (f, q) \) is representable and the map \( (f, q) \to V \) is an étale surjection.

As in the topological case, we may associate to each differentiable prestack \( A \) and chart \( e: U \to A \) a differentiable groupoid \( G^e \). To such a differentiable groupoid we then have an associated differentiable stack \( Q(A, e) \) given by

\[
Q(A, e)_V := \text{Ringed}((\text{Sh}(V), C^\infty(V)), (\text{Sh}(G^e), C^\infty(U)))
\]

where the objects are morphisms of ringed toposes and the arrows are natural isomorphisms thereof.

Theorem 5.5. There is a system of fibrant objects on \( \text{DiffPreSt} \) given by taking the local weak equivalences and with fibrant replacement given by the associated stack.

Proof. By the differentiable analogues of Lemma 5.2 and the argument given in the proof of Theorem 5.4, it suffices to construct a local weak equivalence \( A \to Q(A, e) \) for any differentiable prestack \( A \) with chart \( e: U \to A \). This was done in ibid. \( \square \)

Corollary. There is an equivalence of 2-categories \( \text{DiffSt} \simeq \text{Ho}(\text{DiffPreSt}) \).

5.3. Algebraic stacks. The case of algebraic stacks is even closer to the topological case. In this case we work with the site \( \text{Sch} \) of small schemes with the étale topology.

Definition 5.4. An algebraic prestack is a prestack \( A \) such that the following conditions are satisfied:

1. The diagonal \( \Delta: A \to A \times A \) is representable and proper.
2. There exists a scheme \( U \) in \( \text{Sch} \) and a map \( q: U \to A \) such that, for all schemes \( V \) in \( \text{Sch} \) and maps \( f: V \to A \), the map \( (f, q) \to V \) is an étale surjection.

Theorem 5.6. There is a system of fibrant objects on \( \text{AlgPreSt} \) given by taking the local weak equivalences and with fibrant replacement given by the associated stack.

Proof. By the algebraic analogues of Lemmas 5.1 and 5.2, and the argument given in the proof of Theorem 5.4, it suffices to construct a local weak equivalence \( A \to Q(A, e) \) for any algebraic prestack \( A \) with chart \( e: U \to A \). This was done in ibid. \( \square \)

Corollary. There is an equivalence of 2-categories \( \text{AlgSt} \simeq \text{Ho}(\text{AlgPreSt}) \).

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