Path integral marginalization for cosmology: scale-dependent galaxy bias and intrinsic alignments

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ABSTRACT
We present a path integral likelihood formalism that extends parametrized likelihood analyses to include continuous functions. The method finds the maximum-likelihood point in function-space, and marginalizes over all possible functions, under the assumption of a Gaussian-distributed function-space. We apply our method to the problem of removing unknown systematic functions in two topical problems for dark energy research: scale-dependent galaxy bias in redshift surveys and galaxy intrinsic alignments in cosmic shear surveys. We find that scale-dependent galaxy bias will degrade information on cosmological parameters unless the fractional variance in the bias function is known to 10 per cent. Measuring and removing intrinsic alignments from cosmic shear surveys with a flat prior can reduce the dark energy figure of merit by 20 per cent, however provided that the scale and redshift dependence is known to better than 10 per cent with a Gaussian prior, the dark energy figure of merit can be enhanced by a factor of 2 with no extra assumptions.

Key words: methods: analytical – methods: data analysis – methods: statistical – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION
The standard cosmological model is a phenomenological model containing two unknown components, dark energy and dark matter, and two untested assumptions, general relativity on cosmic scales and cosmological inflation. To understand this model at a deeper level, by testing the evolution of dark energy, the nature of the dark matter, signatures of inflation and possible deviations from Einstein gravity, we need highly sensitive cosmological probes that must constrain an expanded set of cosmological parameters to (sub-) per cent accuracy. In such a scenario, the correct assessment of systematic effects is of critical importance.

To account for systematic effects, it is common to use a parametrized description of the effect where each nuisance parameter is calibrated to some accuracy and removed. However, there are many cases where characterization of the systematic effect by a parametrized function is not well justified, and in some cases the systematic is effectively an unknown function. The assumption of a given functional form can lead to a bias in measured cosmological parameters, or at least an underestimate of the contribution to errors due to uncertainty in the systematic function.

In Taylor & Kitching (2010), we described a new method which allowed the analysis of large-dimensional parameter spaces composed of cosmological parameters and nuisance parameters describing residual systematics, in both calibrated (Gaussian prior) and self-calibrating (flat prior) regimes. Here we present a new method that can account for systematics in a model-free way such that the impact of every possible function on the cosmological signal is accounted for. To do this we introduce a path integral marginalization approach in which we maximize the likelihood and marginalize over a space of functions rather than parameters. We construct a likelihood functional in parameter space that accounts for the impact of all known systematic functions. The function-space may be either weighted with a flat prior about a fiducial function, where the data themselves are used to measure and remove the systematic effect, or if there is external data weighted with a Gaussian prior, with a variance that is zero when the systematic function is known.

We derive an expression for the Fisher matrix that accounts for functional marginalization, and apply this method to two test cases in cosmology in which the cosmological signal is contaminated by a systematic effect with a poorly understood functional behaviour. The first is scale-dependent galaxy bias (e.g. Peacock & Smith 2000; Conway et al. 2005; Hamann et al. 2008; Cresswell & Percival 2009) in redshift surveys, an unknown function that relates the underlying dark matter distribution to the galaxy distribution. The second is cosmic shear intrinsic alignments (IAs) (e.g. Catelan, Kamionkowski & Blandford 2001; Crittenden et al. 2001; Brown et al. 2002; Heymans et al. 2004; Hirata & Seljak 2004; Bridle & King 2007; Kitching et al. 2008; Joachimi & Schneider 2008, 2010).
2 PATH INTEGRAL MARGINALIZATION

We consider an arbitrary likelihood function, \( L[\theta, \psi(x|\theta)] \), which is a function of a finite number of cosmological parameters of interest, \( \theta \), and a finite number of continuous nuisance functions, \( \psi_a(x|\theta) \), labelled by Greek subscripts. The nuisance functions could depend on position \( x = x \), or wavevector \( x = k \) if we are working in Fourier space. The nuisance functions may themselves also depend on the cosmological parameters. For convenience, we shall work with the log-likelihood functional:

\[
L[\theta, \psi(x|\theta)] = -2 \ln L[\theta, \psi(x|\theta)].
\] (1)

Expanding the log-likelihood functional to second order around an arbitrary nuisance function, we find

\[
\mathcal{L} = \mathcal{L}_0 + \int dx' \delta \psi_a(x') \frac{\delta \mathcal{L}_0}{\delta \psi_a(x')} + \frac{1}{2} \int dx' dx'' \delta \psi_a(x') \frac{\delta^2 \mathcal{L}_0}{\delta \psi_a(x') \delta \psi_a(x'')} \delta \psi_b(x'').
\] (2)

This can be minimized in function-space with the solution

\[
\delta \psi_a(x|\theta) = -\int dx' \frac{\delta \mathcal{L}_0}{\delta \psi_a(x)} \left[ \frac{\delta^2 \mathcal{L}_0}{\delta \psi_a(x') \delta \psi_b(x'')} \right]^{-1} \delta \psi_b(x''),
\] (3)

where \( \delta \psi_a(x|\theta) \) is the displacement in function-space from the fiducial function and the maximum-likelihood solution, and we have assumed the inverse of the second functional derivative of the log-likelihood exists.

2.1 Flat priors

Marginalizing over all paths for the functions, assuming a flat prior, with boundaries sufficiently far away, they do not contribute to the integral, we find

\[
\mathcal{L} = \mathcal{L}_0 + \int dx' dx'' \text{Tr} \left[ \frac{\delta^2 \mathcal{L}_0}{\delta \psi_a(x') \delta \psi_b(x'')} \right] - \frac{1}{2} \int dx' dx'' \text{Tr} \left[ \frac{\delta^2 \mathcal{L}_0}{\delta \psi_a(x') \delta \psi_b(x'')} \right]^{-1} \frac{\delta \mathcal{L}_0}{\delta \psi_b(x')},
\] (4)

where we have omitted an unimportant constant.

The first and third terms in this expression represent the maximum log-likelihood value, which can be seen by substituting the solution for the nuisance function \( \psi(x|\theta) \) given by equation (3) into equation (2). As we always find the maximum-likelihood solution for the nuisance functions, the marginalized likelihood is independent of the fiducial nuisance functions, \( \psi(x|\theta) \) in the Gaussian approximation. The second term in equation (4) is the width of the likelihood function.

In Taylor & Kitching (2010), we showed that in the case of marginalization over discrete nuisance parameters the Fisher information is preserved about the cosmological parameters of interest. This result holds in the case of continuous functions, \( \psi(x|\theta) \).

2.2 Gaussian priors

If the nuisance functions are constrained by another independent experiment, we can include this constraint as a Gaussian prior on \( \psi(x|\theta) \), with covariance matrix

\[
C_{\psi\psi}(x, x') = \langle \delta \psi_a(x|\theta) \delta \psi_b(x'|\theta) \rangle.
\] (5)

Including this constraint in the marginalization path integral, we find

\[
\mathcal{L} = \mathcal{L}_0 - \frac{1}{2} \int dx' dx'' \frac{\delta \mathcal{L}_0}{\delta \psi_a(x')} M_{\psi\psi}(x', x'') \frac{\delta \mathcal{L}_0}{\delta \psi_b(x'')} + \int dx' dx'' \text{Tr} \left[ \frac{1}{2} C_{\psi\psi}(x', x'') M_{\psi\psi}(x', x'') \right],
\] (6)

where

\[
M_{\psi\psi}(x', x'') = \frac{\delta^2 \mathcal{L}_0}{\delta \psi_a(x') \delta \psi_b(x'')} + 2C_{\psi\psi}(x', x'')
\] (7)

and we assume the matrix \( M_{\psi\psi} \) is invertible. The flat prior, which we have assumed is non-zero over a finite but sufficiently large region, is not a limiting case of the Gaussian prior. When the covariance matrix of the Gaussian prior goes to infinity, the prior distribution becomes arbitrary, close to zero everywhere, and the infinite wings of the prior suppress information.

2.3 Gaussian likelihoods with systematics in the mean

A common distribution for cosmological data sets is that of a multivariate Gaussian where, as discussed in Taylor & Kitching (2010), the effects of cosmological parameters and nuisance parameters can appear in both the mean and/or the covariance. The likelihood Gaussian function is given by

\[
\mathcal{L}_0 = \Delta D \mathbf{C}^{-1} \Delta D' + \text{Tr} \ln \mathbf{C},
\] (8)

where \( \Delta D = D - \mu \) is the variation of the data around its mean value, \( \mu = \mu[\theta, \psi(x|\theta)] \), and \( C[\theta, \psi(x|\theta)] = \langle \Delta D | \Delta D' \rangle \) is the data covariance matrix.

For simplicity, we shall assume the nuisance functions and cosmological parameters are only in the mean term, \( \mu \), however as shown in Taylor & Kitching (2010) results can be generalized for nuisance functions in the covariance matrix. We assume the curvature of the likelihood in function-space can be approximated by its ensemble average, so that

\[
F_{\mu\mu}(x, x') = \frac{1}{2} \left( \frac{\delta^2 \mathcal{L}_0}{\delta \psi_a(x') \delta \psi_b(x'')} \right) \delta \mu_a(x) \delta \mu_b(x')
\] (9)

is the generalized Fisher matrix, and

\[
\frac{\delta \mathcal{L}_0}{\delta \psi(x')} = -2 \Delta D C^{-1} \frac{\delta \mu_a(x)}{\delta \psi_a(x')}
\] (10)
is the gradient of the log-likelihood. The maximum-likelihood solution for the function is
\[
\delta \psi_\mu(x) = \Delta C^{-1} \int dx' dx'' \frac{\delta \mu'_\psi(x)}{\delta \psi_\mu(x')} F^{-1}_{\alpha \beta}(x', x''),
\]
where again Greek indices represent a discrete infinite set of functions.

2.3.1 Flat prior
We can now write the full path integral marginalized log-likelihood for a flat prior as
\[
\mathcal{L}_0 = \Delta D C_M^{-1} \Delta D' + \text{Tr} \ln C_M,
\]
where
\[
C_M = [C^{-1} - C^{-1} P C^{-1}]^{-1}
\]
is the new marginalized covariance matrix,
\[
P = \int dx' dx'' \frac{\delta \mu'_\psi(x)}{\delta \psi_\mu(x')} F_{\alpha \beta}(x', x'') \frac{\delta \mu'_\psi(x)}{\delta \psi_\mu(x')}.
\]
and we assume the functional Fisher matrix, \( F_{\alpha \beta}(x, x') \), is invertible. The marginalized covariance matrix, \( C_M = C_M(\theta) \), is now a function of cosmological parameters through the inclusion of the functional derivatives of the mean, \( \mu(\theta) \).

2.3.2 Gaussian prior
In the presence of a Gaussian prior, the form of the marginalized likelihood and covariance matrix still holds, with the matrix \( P \) now given by
\[
P = \int dx' dx'' \frac{\delta \mu'_\psi(x)}{\delta \psi_\mu(x')} M_{\alpha \beta}(x', x'') \frac{\delta \mu'_\psi(x)}{\delta \psi_\mu(x')}.
\]
Here the \( M_{\alpha \beta} \) is a sum of the path integral Fisher and a prior covariance term
\[
M_{\alpha \beta}(x, x', x'') = F_{\alpha \beta}(x, x') + C_M^{-1}(x, x').
\]
This contains information on the nuisance parameters from the prior \( C_M(x, x') \) and the data itself, \( F_{\alpha \beta}(x, x') \). If we include this prior covariance on the nuisance functions, we find (see Appendix A for details) that the marginalized likelihood is still a Gaussian in the data, with a new covariance matrix:
\[
C_M = C + \int dx' dx'' C_{\alpha \beta}(x', x'') \frac{\delta \mu'_\psi(x)}{\delta \psi_\mu(x')} \frac{\delta \mu'_\psi(x)}{\delta \psi_\mu(x')}.
\]
This form of the marginalized data covariance matrix has the clear advantage that it does not require the inversion of a functional Fisher matrix, or prior covariance matrix.

2.4 Marginalized Fisher matrices
The Fisher matrix for the marginalized likelihood is given by (Tegmark, Taylor & Heavens 1997)
\[
F_{\alpha \beta} = \frac{1}{2} \text{Tr} \left[ \mu'_\alpha \mu'_\beta + \mu'_\alpha \mu'_\beta \right] C_M^{-1},
\]
where Roman indices, \( (a, b) \), denote cosmological parameters, and for path integral marginalized covariance \( C_M \) is given by equation (13) for a flat prior and equation (17) for a Gaussian prior. We have implicitly taken account of the cosmological parameter information contained in the mean of the data via \( \mu(\theta) \). However, even though the marginalized covariance matrix, \( C_M \), now depends on cosmological parameter through the functional derivatives of the mean, \( \mu \), at the level of the Fisher matrix \( C_M \) is a constant in parameter space.

Throughout Section 2 we have made the assumption of a Gaussian-distributed function-space. If this assumption is dropped then the functional integration can be done numerically, one way to do this integration is using Monte Carlo integration techniques or functional form-filling (Kitcch et al. 2009). We discuss these alternatives in Appendix B.

Having developed the formal aspects of path integral marginalization to cosmological likelihoods and Fisher matrices, we now apply it to the case of marginalization over two cases of systematic effect in cosmology. The first is scale-dependent galaxy bias, and the second is the case of IAs in weak gravitational lensing.

3 SCALE-DEPENDENT GALAXY BIAS
As an example, let us consider the case when the mean signal is an estimate of the galaxy power spectrum measured from a galaxy redshift survey:
\[
\mu(k, r) = P_g(k) + N(r),
\]
where \( N(r) = 1/n(r) \) is the distance-dependent galaxy shot noise and \( n(r) \) the mean galaxy number density. The galaxy power spectrum is a biased representation of the underlying matter power spectrum
\[
P_M(k) = b^2(k) P_M(k),
\]
where \( P_M(k) \) is the matter power spectrum and \( b(k) \) is a dimensionless, scale-dependent galaxy bias function. This scale-dependent galaxy bias only depends on scale and not on phase, so is a function of \( k = |k| \) only. While the bias function is, in this model, deterministic, it is in general completely degenerate with the matter power spectrum. Hence we need to introduce a Gaussian prior from the start, where the prior knowledge could come from weak-lensing measurements (e.g. van Waerbeke 1998; Bernstein 2009) or a semi-analytic galaxy bias model. In either case, there will be some level of uncertainty associated with the bias model, \( \delta \psi(k) = \delta b(k) \). We can, of course, extend this model to include stochastic bias effects (e.g. Dekel & Lahav 1999).

In the following, we include the effect of linear redshift-space distortions by the transformation (Kaiser 1987):
\[
P_{\delta\delta}(k) = \left[ b(k) + f \mu_\delta \right]^2 P_M(k),
\]
where \( f = d \ln \delta_m / d \ln \alpha \) is the growth rate of matter perturbations, and \( \mu_\delta = \cos \alpha \), the cosine of the angle between the wavevector \( k \) and the line of sight. We shall assume we are working on scales large enough to ignore non-linear finger-of-god effects. If we assume the power is a Gaussian random variable, the covariance of the galaxy power is given by
\[
C_k = 2 \left[ P_{\delta\delta}(k) + N(r) \right]^2.
\]
Applying equation (17), assuming a Gaussian prior, the functionally marginalized covariance is
\[
C_M(k) = 2 \left[ P_{\delta\delta}(k) + N(r) \right]^2 + B(k),
\]
where
\[
B(k) = 4 \left( \frac{P_M(k)}{b^2(k)} \right) \left| P_{\delta\delta}(k) \right|^2
\]
is the increase in the covariance matrix after we have marginalized over all bias functions. As the galaxy bias is independent of phase, and both the galaxy and matter distributions are statistically homogeneous, we only need to know the power, or functional variance, in the bias:

$$P_b(k) = \langle |\delta b(k)|^2 \rangle,$$

(25)

where the averaging is taken over all functions of $b(k)$.

If we substitute the redshifted galaxy power spectrum into the Fisher matrix formalism, (see e.g. Taylor & Watts 2001; Seo & Eisenstein 2003; Burkey & Taylor 2004), we find the Fisher matrix including path integral marginalization over all possible bias functions is

$$F_{ab} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\partial \ln P_{\ell s}(k)}{\partial \theta_a} \frac{\partial \ln P_{\ell s}(k)}{\partial \theta_b} V_{\text{eff}}(k),$$

(26)

where

$$V_{\text{eff}}(k) = \int d^3r \left( \frac{[P_{\ell s}(k)]^2}{2[P_{\ell s}(k) + N(r)]^2 + B(k)} \right),$$

(27)

is the effective volume of the survey. As $B(k)$ also depends on $P_{\ell s}^2(k)$, the effective volume fully specifies our survey. In the limit of no systematic uncertainty in the bias, $P_b(k) = 0$, or $B(k) = 0$, this reduces to the usual Fisher matrix for the redshifted power spectrum.

We demonstrate the effect of path integration over all possible galaxy scale-dependent bias functions with a Gaussian prior in Fig. 1, where we consider a Euclid spectroscopic survey (Laureijs et al. 2009) with a survey volume of $19.7 \, h^{-3} \, \text{Gpc}^3$ and a maximum wavenumber of $0.5 \, h \, \text{Mpc}^{-1}$. We shall assume that the signal-to-noise ratio of the survey is unity, $\bar{n} P_{\ell s} = 1$ (Seo & Eisenstein 2003). We use a cosmological parameter set that allows for curved cosmologies with parameters $\Omega_m$, $\Omega_b$, $\Omega_r$, $h$, $\sigma_8$, and $n_s$, given by (0.25, 0.7, 0.75, 0.8, 0.95), and parametrize the dark energy equation-of-state using a first-order Taylor expansion, $w(z) = w_0 + (1 - a)w_a$ (Linder 2003; Chevallier & Polarski 2001) with $(w_0, w_a) = (-0.95, 0.0)$. We assume a fiducial bias function of $b(k) = 1$. In Fig. 1 we show the marginal errors on $\Omega_m$, $w_0$ and $w_a$, respectively, as a function of the functional variance in the bias. In the limit that the bias variance is small, the accuracy converges towards the systematic-free case. However, as the bias variance increases beyond $P_b(k)/b^2(k) \approx 10^{-3}$, the uncertainty on cosmological parameters grows large. Using this result and from equation (27), we define an approximate scaling relation

$$P_b(k) \lesssim 0.22 \frac{1 + \frac{1}{\bar{n} P_{\ell s}}}{1},$$

(28)

for cosmological parameter constraints to be unaffected by uncertainty in the scale-dependent bias.

Constraints on the bias function of $P_b(k)/b^2(k) \approx 10^{-1}$ can be achieved using large area surveys and techniques such as shear-galaxy cross-correlations (e.g. van Waerbeke 1998; Bernstein 2009) or bispectrum statistics (e.g. Matarrese, Verde & Heavens 1997; Verde et al. 1998; Guo & Jing 2009). A more complete analysis should include the redshift-dependence of galaxy bias, as well as scale-dependence, and also consider stochastic bias effects. However, consideration of equation (27) suggests that all of these effects will only become important if they become larger than the sampling and shot-noise terms.

We note here that we have assumed that the galaxy bias is scale but not redshift dependent. In general, we may expect the bias (and the shot noise) to be both redshift- and scale-dependent $b(k, r)$ [$N(k, r)$], and other effects such as galaxy-type dependence may also introduce additional complexity. For a given scale, we would expect additional freedom of the bias as a function of redshift to degrade cosmological parameter constraints further. We leave an exploration of this additional flexibility for future work.

### 4 COSMIC SHEAR INTRINSIC ALIGNMENTS

We now consider the effect of IA on weak lensing. Cosmic shear uses the weak-lensing distortion caused by large-scale structure along the line of sight as a probe of cosmology; this assumes that galaxies are randomly aligned, so that the mean observed ellipticity is zero. However, there are two effects which act to align galaxies by adding `intrinsic’ ellipticity which is not part of the cosmic shear signal. Intrinsic–intrinsic alignment is the small-scale effect of galaxies aligning due, for example, to local tidal forces. We will ignore this effect in our example since this can be removed by neglecting close galaxy pairs in angle and redshift from the cosmic shear signal (e.g. Heymans & Heavens 2003). The Galaxy, or shear, Intrinsic (GI) alignment (Hirata & Seljak 2004) is the correlation between the foreground galaxy ellipticity and the induced shear in a background galaxy and this is more difficult to correct for.

The tomographic cosmic shear power spectra $C_{ij}^{\text{GI}}(\ell)$, where the two-point correlation of the shear field is binned in redshift, can be written as

$$C_{ij}^{\text{GI}}(\ell) = \int_0^{\ell_\text{H}} dr \, W_{ij}^{\text{GI}}(r) \, P_{\ell s} \left( \frac{\ell}{S_i(r)} \right),$$

(29)

where lensing weight can be expressed as

$$W_{ij}^{\text{GI}}(r) = \frac{q_i(r) q_j(r)}{S_i^2(r)},$$

(30)

and the kernel is

$$q_i(r) = \frac{3H_0^2 \Omega_m S_i(r)}{2a(r)} \int_0^{\ell_\text{H}} dr' \, p_i(r') \, S_i(r' - r).$$

(31)

We follow the notation of Joachimi & Bridle (2009). The comoving distance is $r$. $H_0$ is the horizon distance, while $S_i(r) = \sin(r)$. $r$, $\sinh(r)$ for curvatures $k = -1, 0, +1$, $a$ is the scale factor and $P_{\ell s}(k; r)$ is the 3D density–density matter power spectrum. The comoving
4.1 Intrinsic alignment removal

In this section we investigate two distinct cases. The first case is marginalization over the uncertainty in the density–ellipticity relation itself. In this first case, we assume both a flat prior (the self-calibration case), and a Gaussian prior. A Gaussian prior may use information on the density–ellipticity relation gained from, for example, galaxy–galaxy lensing. In the second case, we make some assumptions about how the density–ellipticity power spectrum is related to the matter power spectrum via a set of IA bias functions, and assume only a Gaussian prior.

4.1.1 Intrinsic alignment removal: flat prior

The functional variation of the IA signal with the density–ellipticity cross-spectrum is given by (see Appendix C)

\[
\frac{\delta C_{ij}^{\text{GI}}(\ell)}{\delta F_{\mu}\overline{C}(\ell, r, r')} = W_{ij}^{\text{GI}}(\pi) \frac{2 \pi \delta D(\ell - \ell')}{\ell}.
\] (36)

For a flat prior, where we use the data itself to fit and marginalize over the IA term, we use equations (13) and (14) to find the marginalized covariance

\[
C_{\mu\nu}^{\text{GI}}(\ell) = \left[ C_{\mu\nu}^{\text{GI}}(\ell) - C_{\mu\nu}^{\text{IA}}(\ell) P_{\mu\nu}(\ell) C_{\rho\varsigma}^{-1}(\ell) \right]^{-1},
\] (37)

where

\[
P_{\mu\nu}(\ell) = \int_0^\infty dr dr' W_{\mu\nu}(r)[F(\ell; r, r') - 1] W_{\mu\nu}(r'),
\] (38)

and the functional Fisher matrix for the IA power spectrum is

\[
F(\ell; r, r') = \sum_{\mu\nu} C_{\mu\nu}^{-1}(\ell) W_{\mu\nu}(r) W_{\mu\nu}(r').
\] (39)

To invert the functional Fisher matrix, we bin the radial GI weights over the IA term, we use equations (13) and (37) using 200 radial bins, while the angular terms are always diagonal. The Fisher matrix for cosmological parameters is given by equation (18), this is more explicitly shown for the IA case in Appendix C.

To make a qualitative assessment of the impact of IA removal, we consider a Euclid weak-lensing survey (Refregier et al. 2010) of 20 000 square degree, with median redshift of \(z_m = 1.0\) and 35 galaxies arcmin\(^{-2}\) with a photometric redshift uncertainty of 0.03(1 + z). We assume the galaxy number density is given by \(n(z) \propto z^2 \exp(-1.4z/z_m)^{1/2}\), and use the same cosmological model as in Section 3.

To quantify the effect of path integral IA removal on dark energy surveys, we use the dark energy figure of merit (FoM; Albrecht et al. 2006), defined as the area constrained in the uncorrelated \((w_0, w_a)\) plane with FoM = \(1/[\sqrt{F_{\mu}^{-1} F_{\mu}^{-1} - (F_{\mu}^{-1})^2}]\). We assume the fiducial model for the GI term is given by the linear/non-linear alignment model of Bridle & King (2007; see also Section 4.1.3).\(^1\)

For comparison we find that, for a default model with fixed IA and no cosmological dependence of the IA term (i.e. cosmic shear only, treating IA as additional known noise term), the FoM = 130. If we allow the ellipticity–density relation to be fixed by the data, with a flat prior and allow a cosmological dependence in the weak-lensing weight of the GI term, we find the FoM is degraded to FoM \(\approx 100\) (see Fig. 2). In this case, information is gained because of the extra cosmological dependence of the GI weights, but also lost as the data now has to measure the density–ellipticity relation.

It is interesting to compare the path integral IA removal approach to the nulling method of Joachimi & Schneider (2008, 2009), which uses the radial information in the lensing kernel to null the GI term from the shear data vector. Nulling makes a number of assumptions, about the lens and source planes being thin sheets, and fixes the cosmology in the lensing kernel. An effect of nulling is to introduce a bias in cosmological parameters due to its effect on the shear power. Nulling finds a factor \(\sim 2\) reduction in a global FoM, combining errors from all cosmological parameters, which seems comparable to path integral marginalization. We leave a full comparison of methods for future work.

\(^1\) Our code is an extension of the icosmo package (Refregier et al. 2008) and is available on request.
Figure 2. The dark energy FoM for a Euclid-like tomographic weak-lensing survey (Refregier et al. 2010) marginalized over all possible non-linear alignment bias functions. We show results as a function of the fractional scatter in the \(P_{\delta f}(k)\) function-space, assuming a Gaussian prior. The lower red line (GG) shows the FoM from cosmic shear alone, assuming the GI term does not depend on cosmological parameters. The upper red line (GG+GI) shows the FoM when we also allow the GI redshift dependence to vary with cosmological parameters. The horizontal (blue) line shows the FoM for the self-calibration case in which flat prior in function-space is assumed (equation 37) – in this case the shear and GI power spectra are dependent on cosmology (similar to the upper red line).

4.1.2 Intrinsic alignment removal: Gaussian prior

If, instead of self-calibration, we use external data to constrain IA with a Gaussian prior, we find the path integral marginalized covariance, equation (17), can be written as (see Appendix C)

\[
C_{\mu \nu}^{\mu}(\ell) = C_{\nu \nu}(\ell) + \int_0^{\infty} dr \sigma_\mu^2(\ell/r, r) W_{ij}^{\mu}(r) W_{lm}^{\mu}(r),
\]

where \(\sigma_\mu^2(\ell/r, r) = \langle |\delta P_{\delta f}(\ell/r, r)\rangle^2\) is the prior uncertainty in the shear–density relation. We again make no assumption about the relationship between matter and ellipticity. The path integral Fisher matrix, marginalizing over all possible shear-density cross-power, is again given by equation (18) and in Appendix C.

In Fig. 2, we show how the dark energy FoM changes as the fractional scatter in the ellipticity–density power spectrum: \(\sigma[P_{\delta f}(\ell/r, r)]/P_{\delta f}(\ell/r, r)\). The lower solid line extends our default model from Section 4.1.1, by treating the IA term as additional noise where the GI term is independent of cosmological parameters, but the ellipticity–shear relation is allowed to vary [constrained by the prior on \(P_{\delta f}(\ell/r, r)\)]. In this case the FoM is rapidly damped from FoM = 130, as the scatter is increased above \(\sigma[P_{\delta f}]/P_{\delta f} \approx 10^{-3}\). As anticipated in Section 2.2, the scatter is allowed to increase this adds greater uncertainty to the model, with ever larger fluctuations in the ellipticity–density relation. As a result the information is actually damped below that of the flat prior.

The upper curve in Fig. 2 allows the redshift-dependence of the GI weights to vary with cosmology and act as a source of information. This increases the FoM, in the limit of no uncertainty in the IA term, to FoM = 300, showing that the known lensing dependence of the GI term contains useful cosmological information. Even though the FoM drops off rapidly, as the scatter is increased, it still remains a factor of a few greater than ignoring the cosmological information in the GI term.

4.1.3 The intrinsic alignment model

In this section we will assume that the shear-density power spectrum, \(P_{\gamma \delta}\), is related to the matter power spectrum by some unknown ellipticity bias which is a function of redshift and scale (e.g. Bernstein 2009)

\[
P_{\gamma \delta}(\ell/r, r) = b_\gamma(\ell) b_\delta(\ell) P_{\delta f}(\ell/r, r),
\]

and that the dimensionless bias function is separable in radial distance \(r\) and angular wavenumber \(\ell\). In this case, the GI effect can be written in the same way as the tomographic shear power spectrum:

\[
C_{ij}^{\gamma \delta}(\ell) = \int_0^\infty dr W_{ij}^{\gamma \delta}(r) b_\gamma(\ell) b_\delta(\ell) P_{\delta f}(\ell/r, r).
\]

This is similar to the (linear/non-linear) alignment model introduced in Hirata & Seljak (2004) and generalized by Bridle & King (2007) and Joachimi & Bridle (2009). The IA bias relation, equation (41), assumes that the IA is a non-local convolution of the angular density field which depends locally on distance.

Given we are assuming a model for which we should have some motivation to trust, we shall only assume a Gaussian prior on the IA bias functions. In Appendix D, we show that the path integral marginalized covariance for this model is given by

\[
C_{\mu \nu}(\ell) = C_{\nu \nu}(\ell) + B_{\mu \nu}(\ell),
\]

where

\[
B_{\mu \nu}(\ell) = b_\gamma^2(\ell) \int_0^\infty dr \sigma_\mu^2(\ell/r, r) W_{ij}^{\gamma \delta}(r) W_{lm}^{\gamma \delta}(r) |P_{\delta f}(\ell/r, r)|^2
\]

\[
+ \left(\frac{\sigma_\mu^2(\ell)}{b_\gamma^2(\ell)}\right) C_{ij}^{\gamma \delta}(\ell) C_{lm}^{\gamma \delta}(\ell).
\]

We assume \(b_\gamma(r[z]) = C_1 \bar{\rho}(z)/D(z)(1+z)\) and \(b_\delta(\ell) = 1\) as the fiducial functions for the biases (see Bridle & King 2007; Joachimi & Bridle 2009 for these definitions) and that the scatter in these functions is given by \(\sigma_\mu(\ell)\) and \(\sigma_\delta(\ell)\). Equation (18) is again used to derive a Fisher matrix for the non-linear alignment bias model (also given in Appendix C).

Fig. 3 shows the dark energy FoM as a function of the radial (left-hand plot) and angular scatter (right-hand plot) in the IA bias, for our fiducial survey. The lower set of lines assumes that the GI terms are independent of cosmology, while the upper set of lines includes the cosmological dependence of the GI terms. We see that including the GI cosmology dependence increases the FoM by a factor of around three. The angular bias reduces the FoM when \(\sigma_\delta(\ell)/P_{\delta f} \gtrsim 10^{-1}\), while the radial bias does not affect the FoM until \(\sigma_\delta(\ell)/P_{\delta f} \gtrsim 10^0\). This is because the dominant effect of the dark energy equation of state is to change the angular part of the lensing power spectrum. While the growth rate is also affected by changes in the dark energy equation of state, the extra geometric constraints from the lensing kernel (in both the shear and GI terms) help to break any degeneracies introduced. As a result the dependence of the FoM on the radial functional scatter is weak (and even more so when both shear and GI contribute to the geometric constraints). This is also in agreement with Joachimi & Bridle (2009), where a large but finite-dimensional marginalization has found a similar behaviour.

Fig. 4 shows a 2D contour plot of the dark energy FoM as a function of the fractional scatter in radial and angular IA bias. Again it is clear that marginalizing over angular, \(\ell\)-dependent bias has a larger impact than the radial bias on the FoM. A constraint in the functional scatter of \(\sigma_\delta/P_{\delta f} \sim 10^{-1}\), required such that the FoM is unaffected, is achievable using current techniques applied to future
functions are marginalized over. We have applied this to the case of a scale-dependent galaxy bias function, relating the galaxy power spectrum to the underlying matter power spectrum. As a scale-dependent bias is completely degenerate with the matter power spectrum, we assume a Gaussian prior with variable scatter. We show that the marginal errors of $\Omega_m$, $w_0$ and $w_a$ rapidly increase as the uncertainty in the fractional power of a scale-dependent bias increases above $P_s(k)/b^2(k) \gtrsim 10^{-1}$, implying we need to constrain any scale-dependent bias to a few per cent accuracy. We did not explore redshift-dependent bias models here, but the marginalized Fisher matrix implies a similar accuracy is needed. In addition, we did not explore stochastic bias models, which will add a further requirement. There are clear ways to constrain scale-dependent bias from redshift-space distortions, which also probe the velocity field and so are bias-free: by the galaxy bispectrum or by combining galaxy redshift surveys with weak-lensing probes of the dark matter (e.g. Matarrese et al. 1997; van Waerbeke 1998; Verde et al. 1998; Bernstein 2009; Guo & Jing 2009).

Our second example is in the removal of IAs effects, in particular the non-local GI effect, on cosmic shear measurements. In addition to nulling (Joachimi & Schneider 2008, 2009) and modelling (Bridle & King 2007; Kitching et al. 2008; Joachimi & Bridle 2009), path integral marginalization represents a third way to remove and assess the impact of this effect. Path integral marginalization has the advantage that it is model-free, and introduces no biases in cosmological parameter estimates. For a flat prior on the ellipticity–density relation, the dark energy FoM drops from a fiducial FoM $= 130$ to FoM $\approx 100$, when we allow the GI term to be completely free in both angular and radial dependence and measured by the data.

If we assume the GI ellipticity–density relation is accurately constrained from external data (e.g. Mandelbaum et al. 2009) or complimentary information (e.g. shear-position information; Bernstein 2009) with a Gaussian prior, and allow the weak-lensing effect in the GI term to depend on cosmology, we find that the dark energy constraint is enhanced from FoM $= 130$ to FoM $= 300$. The GI is degraded as the accuracy drops below $\sigma(P_{\gamma\delta})/P_{\gamma\delta} \approx 10^{-3}$. The assumption of a non-linear alignment model (e.g Hirata & Seljak 2004; Bridle & King 2007) relaxes this requirement to $\approx 10^{-1}$.

Our results imply the GI effect in weak lensing can be removed by path integral marginalization and that cosmic shear can still
APPENDIX A: MARGINALIZATION OF SYSTEMATICS IN THE MEAN

We start with the likelihood function where the cosmological parameters, \( \theta \), and nuisance parameters, \( \psi \), both appear in the mean, \( \mu(\theta, \psi) \). Expanding the nuisance parameter to first order and explicitly marginalizing over variations in the nuisance parameters, \( \psi \), we find,

\[ L(D|\theta) = \int d[\delta \psi] \frac{e^{-1/2 \Delta D - \delta \psi} \delta \mu \psi^{-1}(\Delta D - \delta \psi) \delta \mu \psi}{\sqrt{(2\pi)^N \det C}} \frac{e^{-\frac{1}{2}(\psi - \hat{\psi})^2}}{\sqrt{(2\pi)^N \det C_\psi}}. \tag{A1} \]

First, we expand in a Fourier series

\[ f(x, \mu, C) = \int d^n s \int d^n k e^{-\frac{1}{2} \Delta C k^2} e^{i k \cdot (x - \mu)} f \left( d^{n+1} k' e^{-\frac{1}{2} \Delta C k'^2} e^{i k' \cdot s} \right) \]

\[ = \int d^n k e^{-\frac{1}{2} \Delta C k^2} e^{-i k \cdot x} \int d^n k' e^{-\frac{1}{2} \Delta C k'^2} e^{i k' \cdot s} d^n s e^{-i(\mu - k' \cdot s) \cdot \mu} \]

\[ = \int d^n k e^{-\frac{1}{2} \Delta C k^2} e^{i k \cdot (\mu - k' \cdot \mu)} e^{i k \cdot x} \frac{1}{\sqrt{(2\pi)^N \det (C + i(\theta, \mu)C'(\theta, \mu)' - 1) \cdot x}} \]

\[ = \int d^n k e^{-\frac{1}{2} \Delta C k^2} e^{i k \cdot (\mu - k' \cdot \mu)} e^{i k \cdot x} \frac{1}{\sqrt{(2\pi)^N \det (C + (\partial_\psi \mu)(\partial_\psi \mu)')}} \]

\[ \left( 2\pi)^N \det (C + C_\psi(\partial_\psi \mu)(\partial_\psi \mu)') \right)^{\frac{1}{2}} \exp \left( -\frac{1}{2} \Delta D + C_\psi(\partial_\psi \mu)(\partial_\psi \mu)' \Delta D' \right). \tag{A3} \]
We note that this can be encapsulated by a modification of the covariance
\[ C_M = C + C_\phi^\psi (\partial_\phi \mu)(\partial_\psi \mu'), \]  
(A4)
where \( C_{ij}^\psi \) is the covariance of the systematic, which acts as an effective prior. This is a generalization of the results from Bridle et al., (2002) and Taylor & Kitching (2010), where \( \hat{C} = C + \sigma_\alpha^2 \mu \mu' \). After carrying out a path integral marginalization over the bias functions, the new data covariance matrix is given by
\[ C_m = C + \int dx' dx'' \langle \psi(x) \psi'(x') \rangle \frac{\delta \mu[\psi(x)]}{\delta \psi(x)} \frac{\delta \mu'[\psi(x')]}{\delta \psi'(x')}. \]  
(A5)
where the derivatives \( \frac{\delta \mu[\psi(x)]}{\delta \psi(x)} \) are functional derivatives.

This can also be derived using the Woodbury matrix identity (e.g. Press et al. 1989). Using equations (6) and (10), we can write the path integral marginalized likelihood as
\[ \mathcal{L} = \Delta D \left( C^{-1} - C^{-1} \int d\xi \frac{\delta \mu[\psi(x)]}{\delta \psi(x)} \frac{\delta \mu'[\psi(x')]}{\delta \psi'(x')} \right) C^{-1} \Delta D, \]  
(A6)
where we have remove the constant term, and
\[ M_{ij} = F_{ij} + (C_{ij}^\psi)^{-1}, \]  
(A7)
where \( F_{ij} \) is the path integral Fisher matrix given by equation (9). Then using the Woodbury identity, the covariance term in equation (A6) can be directly written as equation (A5).

APPENDIX B: NUMERICAL PATH INTEGRATION

In this paper, we make use of the assumption of Gaussianity in function-space. However, in some cases this assumption will be a poor approximation to the actual functional distribution. In this case, there are two alternative approaches which we review here: numerical path integration and functional form-filling.

Numerical path integration decomposes the functional integration into discrete steps at position \( x_i \), and integrates over all possible field values at each position:
\[ L(D|\theta) \propto \prod_i \int D[\psi(x_i)] p[D|\theta, \psi(x_i)] p[\psi(x_i)] = \prod_i \prod_j \int d\psi(x_i,j) p[D|\theta, \psi(x_i,j)] p[\psi(x_i,j)]. \]  
(B1)
Each integral can be evaluated by Monte Carlo Markov Chain methods (e.g. Lewis & Bridle 2002).

Functional form-filling was developed in Kitching et al., (2009) and uses a Monte Carlo approach where the bias in the measured parameters is evaluated instead of marginalization. In the appendices of Kitching et al. (2009), it is shown that the bias and marginalization procedures are equivalent. Functional form-filling is also a slight generalization of Monte Carlo path integration, in that arbitrary basis sets are used and the coefficients of the expansion sampled randomly.

APPENDIX C: PATH INTEGRAL MARGINALIZATION FOR POWER SPECTRUM INTRINSIC ALIGNMENT MODEL

Starting from Section 4 (equation 34), we can write the mean signal is the cosmic shear and IA power spectra as
\[ \mu_i(\ell) = C_{ij}^{\text{GG}}(\ell) + C_{ij,j>1}^{\text{GI}}(\ell) + N_{ij}, \]  
(C1)
where \( i, j \) refer to the background source redshifts, \( z_i, z_j \), and we have IAs for galaxies in the \( i \)-th redshift bin. The covariance of this is
\[ \text{Cov} \left( \left[ C_{ij}^{\text{GG}}(\ell) + C_{ij,j>1}^{\text{GI}}(\ell) \right] \left[ C_{im}^{\text{GG}}(\ell) + C_{im,m>1}^{\text{GI}}(\ell) \right] \right) = C_{jm}^{\text{GG}}(\ell) \left[ C_{ij}^{\text{GG}}(\ell) + C_{ij,j>1}^{\text{GI}}(\ell) \right] + C_{jm,j>1}^{\text{GG}}(\ell) \left[ C_{ij}^{\text{GG}}(\ell) + C_{ij,j>1}^{\text{GI}}(\ell) \right]. \]  
(C2)
This includes all covariances between redshift bins pairs.

Starting from Section 4, we can treat the ellipticity-density cross-power, \( P_{\gamma\delta}(\ell/r, r) \), itself as an unknown function. Taking the functional derivative with respect to the perturbation we find that
\[ \frac{\delta C_{ij}^{\text{GG}}(\ell)}{\delta P_{\gamma\delta}(\ell/r, r)} = W_{ij}^{\text{GI}}(\ell) \frac{\delta \rho(\ell - \ell')}{\ell}. \]  
(C3)
In a similar way to Appendix B, we can now write the marginalized covariance as
\[ C_{mij}^{\text{M}}(\ell) = C_{ij}^{\text{M}}(\ell) + \int_0^{\Omega} d\ell' \left\langle \delta \sigma_{ij}^2(\ell/r, r) W_{ij}^{\text{GL}}(\ell) W_{ij}^{\text{GL}}(\ell') \right\rangle, \]  
(C4)
where \( \sigma_{ij}^2(\ell/r, r) \) is the functional scatter in \( P_{\gamma\delta} \), and we have assumed the functional covariance is diagonal in \( \ell \) and \( r \).
By taking the derivative of the mean with respect to a set of cosmological parameters $\theta$, we can now write a general expression for the Fisher matrix

$$F_{ab} = \frac{1}{2} \sum_{\mu} \int \frac{d\ell}{2\pi} \left( C_{\mu\mu}^{\mu}(\ell)^{-1} \left[ \frac{\partial a_{\mu}(\ell)}{\partial \theta_{\mu}} \frac{\partial a_{\nu}(\ell)}{\partial \theta_{\nu}} + \frac{\partial a_{\mu}(\ell)}{\partial \theta_{\mu}} \frac{\partial b_{\mu}(\ell)}{\partial \theta_{\mu}} \right] \right).$$  \hspace{1cm} (C5)

where $\mu = (i, j)$ and $\nu = (l, m)$ denote pairs of background redshifts. Equations (D3) and (C5) provide a Fisher matrix for weak-lensing tomography that include marginalization over all possible GI IA bias functions.

APPENDIX D: PATH INTEGRAL MARGINALIZATION FOR THE NON-LINEAR ALIGNMENT INTRINSIC ALIGNMENT MODEL

In the following, we will use the following notation (similar to Hirata & Seljak 2004; Bridle & King 2007)

$$C_{ij}^{GG}(\ell) = \int_0^{\infty} dr W_{ij}^{GG}(r) P_{\delta\delta}(\ell/r; r)$$ and $$C_{ij}^{GI}(\ell) = b_I(\ell) \int_0^{\infty} dr W_{ij}^{GI}(r) P_{\delta\delta}(\ell/r; r)$$ \hspace{1cm} (D1)

which can be related to equations (29) and (42). We have here introduced two bias functions $b_I(\ell)$ and $b_I(r)$. We have assumed that the scale and redshift-dependent part of a more general bias $b_I(\ell, r)$ can be separated into two independent functions in this case; this will allow us to comment on the relative merit of constraining the bias function in these directions.

In a similar way to Appendix C, let us consider the bias functions as unknown systematics such that we can write them as some fiducial function plus some stochastic Gaussian distributed unknown $\psi_I(r)$ and $\psi_I(\ell)$. We can write down the functional derivatives associated with these perturbations as

$$\frac{\delta C_{ij}^{GI}(\ell)}{\delta b_I(r)} = b_I(\ell) W_{ij}^{GI}(r) P_{\delta\delta}(\ell/r; r)$$ and $$\frac{\delta C_{ij}^{GI}(\ell)}{\delta b_I(\ell)} = \frac{C_{ij}^{GI}(\ell)}{b_I(\ell)} \frac{\delta b_I(\ell - \ell')}{\ell - \ell'}.$$  \hspace{1cm} (D2)

Using equation (D1) and assuming that the cross terms in the nuisance function covariance matrix are zero, $\langle \delta b_I(r) \delta b_I(\ell') \rangle = 0$, and that the auto-covariance are diagonal in $r$ and $\ell$ the marginalized covariance becomes

$$C_{\mu\nu}(\ell) = C_{\mu\nu}(\ell) + b_I(\ell) \int_0^{\infty} dr \sigma_I^2(r) W_{ij}^{GI}(r) W_{lm}^{GI}(r) [P_{\delta\delta}(\ell/r; r)]^2 + \left( \frac{\sigma_I^2(\ell)}{b_I(\ell)} \right) C_{ij}^{GI}(\ell) C_{lm}^{GI}(\ell),$$ \hspace{1cm} (D3)

where $\langle |\delta b_I(r)|^2 \rangle = \sigma_I^2(r)$ and similarly for $\ell$.

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