Inversion in a four-terminal superconducting device on the quartet line:
II. Quantum dot and Floquet theory

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In this paper, we provide Keldysh microscopic calculations and physical pictures for the voltage-V dependence of the quartet current in a four-terminal quantum dot Josephson junction containing a loop pierced by reduced flux $\Phi/\Phi_0$. Landau-Zener tunneling at avoided crossings in the Floquet spectrum produces hybridization between the two Floquet states. An inversion $I_{gc}(V,0) < I_{gc}(V,1/2)$ in the critical current $I_{gc}(V,\Phi/\Phi_0)$ on the quartet line is obtained at sufficiently large $V$ if an avoided crossing appears at $(V,\Phi/\Phi_0 = \pi)$ but not at $(V,1/2)$. The inversion originates from the reduction in $I_{gc}$ produced by the dynamical quantum mechanical superposition of the two Andreev bound state manifolds. In addition, $\pi$-0 and 0-$\pi$ crossovers emerge in the current-phase relations as $V$ is further increased. This $\pi$-shift is an effect of the nonequilibrium Floquet populations produced by voltage biasing. Numerical calculations show that the inversions are robust against strong Landau-Zener tunneling and many levels in the quantum dot. Our theory supports that the recent Harvard group experimental observation of an inversion at finite $V$ is a signature of quantum fluctuations in the dc-quartet current.

I. INTRODUCTION

Quantum optics and cold atom experiments have revealed entanglement among two, \( n = 3 \) or four \( n = 4 \) particles. The progress in nanofabrication technology made it possible to consider solid-state analogues since the early 2000s. However, even 20 years after the first theoretical and experimental efforts (see for instance Refs. [2]-[19] for the theory, and Refs. [20]-[27] for the experiments), no proof of entanglement between pairs of electrons has been reported so far in solid-state superconducting nanoscale electronic devices.

A first difficulty is to single out and manipulate pairs, triplets or quartets of electrons in the presence of a collective ground state such as a BCS superconductor or a Fermi sea. The proposed Bell test \([20,21]\) is also challenging from the point of view of practically controlling the device. Rather than a direct proof of entanglement, solid-state experiments \([22,23]\) focused on providing evidence for correlations among pairs of electrons in three-terminal ferromagnet-superconductor-ferromagnet (FSF) or normal-metal-superconductor-normal metal (NSN) devices. For instance, measurements of the nonlocal conductance \( G_{ab} = \partial I_a/\partial V_b \) demonstrated \([24]\) how the current \( I_a \) through lead \( S_a \) depends on the voltage \( V_b \) on lead \( S_b \). In addition, the zero-frequency positive current-current cross-correlations \( S_{ab} \) in FSF or NSN three-terminal devices demonstrated \([25]\) the theoretically predicted [26] quantum fluctuations in the current operators \( I_a \) and \( I_b \). In what follows, we show that the voltage sensitivity of the dc-current in the recent Harvard group experiment \([30]\) on a dc-terminal Josephson junction is a signature of quantum fluctuations in the current operator, due to the quantum mechanical Landau-Zener tunneling in the Andreev Bound State (ABS) dynamics.

Now, we explain that correlations among four fermions (the so-called quartets) can be obtained in three- or four-terminal Josephson devices. The supercurrent flows classically in the bulk of a superconductor but three superconducting leads connected to less than the coherence length \( \xi \) allow for a nonstandard quantum mechanical exchange mechanism known as “the quartets”. Namely, transmitting two pairs from \( (S_a, S_b) \) into the grounded \( S_c \) fulfills energy conservation if the voltage biasing condition of the \( (S_a, S_b, S_c) \) three-terminal junction is such that \( (V_a, V_b, V_c) = (V, -V, 0) \). This resonance is revealed by the predicted \([31,32]\) dc-Josephson-like anomaly along the \( V_a + V_b = 0 \) line in the 2D plane of the bias voltages \( (V_a, V_b) \), with \( V_c = 0 \) for the grounded \( S_c \).

The quartet anomaly was confirmed by three groups: first by the Grenoble group with an Aluminum/Copper metallic structure \([33]\) and next by the Weizmann Institute experiment with semiconducting nanowire \([34]\). The third experiment realized recently by the Harvard group \([35]\) reports evidence for the quartet resonance in the presence of a loop in the grounded \( S_c \), terminated by the contact points \( S_{c,1} \) and \( S_{c,2} \) (see figure 1). Compared to the previous Grenoble \([33]\) and Weizmann \([34]\) group experiments, the very recent Harvard group experiment \([35]\) features the additional control parameter of the reduced flux $\Phi/\Phi_0$ piercing through the loop.

The first theoretical paper I of the series was focused on modeling the Harvard group experiment within lowest-order perturbation theory in the tunneling amplitude between the sheet of graphene and the four superconducting leads. Most of the previous paper I was supported by microscopic theory in the \( V = 0^+ \) adiabatic limit. This following paper II deals with microscopic calculations at finite bias voltage \( V \) on the quartet line. The goal of paper II is to explain the counterintuitive experimental inversion at finite bias voltage \( V \) in the critical current \( I_{gc}(eV/\Delta, \Phi/\Phi_0) \) on the quartet line, between reduced fluxes \( \Phi/\Phi_0 = 0 \) and \( \Phi/\Phi_0 = 1/2 \), i.e., we show that \( I_{gc}(eV/\Delta, 1/2) < I_{gc}(eV/\Delta, 0) \) can be produced in relevant voltage ranges by the quantum fluctuations of the quartet current originating from Landau-Zener tunneling. (The variable $\Delta$ denotes the superconducting gap in these equations.) The goal of the next paper III is to address the emergence of a small voltage scale in the Harvard group experiment \([35]\). This paper III will also provide Floquet theory for a device based
on a 2D metal, thus bridging the gap between the “2D metal quartet beam splitter” of paper I (treated in perturbation in the tunneling amplitudes), and the Floquet theory (treated here in paper II in the limit of a quantum dot).

Current-current cross-correlations in a \((S_a, S_b, S_c)\) three-terminal all-superconducting device provided evidence for quantum fluctuations of the quartet current. The interpretation of the emergent positive cross-correlations \(S_{ab} > 0\) is two-fold: (i) A finite value for \(S_{ab} \neq 0\) is produced by the quantum mechanical Landau-Zener tunneling. (ii) The quartet process implies splitting a supercurrent from \(S_i\) towards \((S_a, S_b)\). The resulting cross-correlations \(S_{ab} > 0\) are generically positive, as for any splitting process such as Cooper pair splitting, see the in-depth investigations presented in Refs. [35, 36] and [38].

As mentioned above, the following paper II provides the theory of the inversion \(I_{qc}(eV/\Delta, 0) < I_{qc}(eV/\Delta, 1/2)\) reported by the Harvard group, in the simplest situation where a quantum dot with a level at zero energy is connected to four superconducting leads. Our numerical calculations consistently point out Landau-Zener tunneling as being at the origin of the inversion \(I_{qc}(eV/\Delta, 0) < I_{qc}(eV/\Delta, 1/2)\). Robustness of the inverted behavior is demonstrated, against including strong Landau-Zener tunneling and a finite number of levels in the gap window.

The paper is organized as the following. The model and the Hamiltonians are provided in section [II]. The rate of Landau-Zener tunneling is evaluated in section [III] in connection with the Keldysh numerical calculations of section [IV]. Section [V] presents the robustness of the inversion against changing the coupling parameters for a single level quantum dot, and against including multichannel effects. Concluding remarks are presented in section [VI].

II. MODEL AND HAMILTONIANS

In this section, we present the model and the Hamiltonians. Specifically, the single level quantum dot device Hamiltonian is presented in section [II A]. The infinite gap limit and the gauge-invariant quartet phase variable are presented in section [II B]. The expression of the quartet current is provided in section [II C]. The parameters used in the numerical calculation are given in section [II D]. The multilevel quantum dot is presented in subsection [II E] and inversion in the \(V = 0^+\) adiabatic limit is discussed in subsection [II F].

A. Single-level quantum dot

In this subsection, we provide the Hamiltonian of the four-terminal device in figure [I] in the limit where the quantum dot supports a single level at zero energy.

The Hamiltonian is the sum of the BCS Hamiltonian of the superconducting leads and the tunneling term between the dot and the leads. In absence of voltage biasing, the Hamiltonian of each superconducting lead takes the form

\[
\mathcal{H}_{BCS} = -W \sum_{\langle i,j \rangle} \sum_{\sigma=\uparrow,\downarrow} \left( c^\dagger_{i,\sigma} c_{j,\sigma} + c^\dagger_{j,\sigma} c_{i,\sigma} \right) - |\Delta| \sum_{i} \left( e^{i\phi_{i}} c^\dagger_{i,\uparrow} c_{i,\downarrow} + e^{-i\phi_{i}} c_{i,\downarrow} c^\dagger_{i,\uparrow} \right),
\]

where the summations run over all pairs \((i,j)\) of neighboring tight-binding sites in the kinetic energy given by Eq. (1), and over all the tight-binding site labeled by \(i\) in the pairing term given by Eq. (2). The superconducting phase variable is denoted by \(\phi_{i}\) in Eq. (2) and the gap is denoted by \(|\Delta|\). We assume that no magnetic field penetrates in leads \(S_a, S_b\), therefore \(\phi_{i}\) is constant in each of them, with \(\phi_{a} = \phi_{a} \) in \(S_a\) and \(\phi_{b} = \phi_{b} \) in \(S_b\). We also assume that no magnetic flux penetrates in \(S_c\), but we choose to encode the Aharonov-Bohm flux \(\Phi\) around the loop made by \(S_i\) through a pure gauge vector potential. As a result, \(\phi_{i}\) varies inside \(S_c\), and it takes values \(\phi_{c,1}\) and \(\phi_{c,2}\) at the two extremities of \(S_c\), which are closest to the dot. Minimizing the condensate energy in the presence of the Aharonov-Bohm vector potential in \(S_c\) implies that \(\phi_{c,2} - \phi_{c,1} = \Phi\). Throughout this paper, we shall use the notation \(\phi_{c,1} = \phi_{c}\), and \(\phi_{c,2} = \phi_{c} + \Phi\).

The coupling between the dot \(c\) and each superconductor \(S_p\) takes the form of a usual tunneling Hamiltonian with tunneling amplitude \(J_p\):

\[
\mathcal{H}_{p} = J_p \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} e^{-i\phi_{c} a_{\sigma} c^\dagger_{\sigma,p}(k)d_{\sigma}} + h.c.
\]

Here \(c^\dagger_{\sigma,p}(k)\) and \(c_{\sigma,p}(k)\) are creation and annihilation operators for an electron on reservoir \(p\) with momentum \(k\) and spin.
σ along the quantization axis. The corresponding operators on the dot are denoted by \( d_\sigma \) and \( \sigma_\sigma \). We use the notation \( \omega_0 = eV/h \).

The paper is focused on voltage biasing on the quartet line, according to the experimental result of the Harvard group. This is why we use \( V_j = s_jV \) for the bias voltages. Specifically, the following values \( s_a = 1, s_b = -1, s_{c1} = s_{c2} = 0 \) will are assigned to the parameters \( s_j \), corresponding to voltage biasing at \((V_a, V_b, V_{c1}, V_{c2}) = (V, -V, 0, 0)\).

We neglect quasiparticle tunneling through the loop from \( S_{c1} \) to \( S_{c2} \), i.e. we assume that \( S_{c1} \) and \( S_{c2} \) are solely coupled by the condensate of the grounded \( S_c \). Since most of the current is carried by resonances which are within the gap of \( S_c \), this implies that the perimeter of the loop is large compared to the BCS coherence length. The Harvard group experiment meets this condition, since the loop has a perimeter of 12μm and the coherence length of dirty Aluminum is in between 100nm and 200nm.

### B. Infinite gap limit and gauge-invariant quartet phase

This subsection presents the infinite gap limit and the gauge-invariant phase variable.

In the infinite gap limit, we obtain the following 2 × 2 Hamiltonian in the Nambu representation:

\[
\mathcal{H}_\infty = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}.
\]  

Eq. (4) implies two ABS at opposite energies ±\( E_{ABS} \), with \( E_{ABS} = |z| \).

The expression of \( z \) is the following for a \((S_a, S_b, S_c)\) device which is phase-biased at \((\varphi_a, \varphi_b, \varphi_c)\):

\[
z_{ST} = \Gamma_a \exp(i\varphi_a) + \Gamma_b \exp(i\varphi_b) + \Gamma_c \exp(i\varphi_c).
\]  

The Josephson relations for three terminals \((S_a, S_b, S_c)\) biased at \((V, -V, 0)\) are the following:

\[
\varphi_a(t) = \varphi_a + \frac{2eVt}{\hbar},
\]

\[
\varphi_b(t) = \varphi_b - \frac{2eVt}{\hbar},
\]

\[
\varphi_c(t) = \varphi_c.
\]

The corresponding expression of \( z_{ST} \) for four superconducting leads \((S_a, S_b, S_{c1}, S_{c2})\) which are phase-biased at \((\varphi_a, \varphi_b, \varphi_{c1}, \varphi_{c2})\) is the following:

\[
z_{4T} = \Gamma_a \exp(i\varphi_a) + \Gamma_b \exp(i\varphi_b) + \Gamma_{c1} \exp(i\varphi_{c1}) + \Gamma_{c2} \exp(i\varphi_{c2}),
\]

and we have the following expression for the superconducting phases in the presence of voltage biasing:

\[
\varphi_a(t) = \varphi_a + \frac{2eVt}{\hbar},
\]

\[
\varphi_b(t) = \varphi_b - \frac{2eVt}{\hbar},
\]

\[
\varphi_{c1}(t) = \varphi_{c1},
\]

\[
\varphi_{c2}(t) = \varphi_{c2}.
\]

We note that the \((S_{c1}, S_{c2})\) contacts can be gathered into a single \( S_{c,eff} \) coupled by \( \Gamma_{c,eff} \) to the dot, and with the phase \( \varphi_{c,eff} \):

\[
\Gamma_{c,eff} \exp(i\varphi_{c,eff}) = \Gamma_{c1} \exp(i\varphi_{c1}) + \Gamma_{c2} \exp(i\varphi_{c2}),
\]

with \( \varphi_{c,eff} = \varphi_c + \alpha(\Phi) \), where \( \alpha(\Phi) \) depends only on \( \Phi \), i.e. it is independent on \( \varphi_c \). Then, all of the currents (which are gauge-invariant) depend on the gauge-invariant quartet phase \( \phi_q \) which is expressed as the following combination of the phase variables \( \varphi_a, \varphi_b \) and \( \varphi_c \):

\[
\phi_q = \varphi_q + \alpha(\Phi),
\]

where the quartet phase is given by \( \varphi_q = \varphi_a + \varphi_b - 2\varphi_c \).

### C. Quartet critical current

The expression of the quartet current is presented in this subsection.

The two-terminal dc-Josephson current is odd in the phase difference. In perturbation theory in the tunnel amplitudes, the lowest-order quartet current is also odd in the superconducting phases, and it is even in voltage. Generalizing to arbitrary values of the contact transparencies, the quartet current \( I_q(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0) \) is defined as the component of

\[
I_{S_a}(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0) = I_{S_{c1}}(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0) + I_{S_{c2}}(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0)
\]

which is odd in \( \phi_q \) and in \( \Phi \):

\[
I_q(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0) = I_{S_a}(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0) - I_{S_{c2}}(eV/\Delta, \phi_q/2\pi, -\Phi/\Phi_0).
\]

Equivalently, \( I_q(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0) \) is the component of Eq. (16) which is even in voltage:

\[
I_q(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0) = I_{S_a}(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0) + I_{S_{c2}}(-eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0).
\]

Eq. (18) is used in the following numerical calculations.

The Harvard group experiment measures the critical current on the quartet line for the device in figure 1 which we call in short as “the critical current”:

\[
I_{q,c}^c(eV/\Delta, \Phi/\Phi_0) = \max_{\phi_q} I_q(eV/\Delta, \phi_q/2\pi, \Phi/\Phi_0),
\]

where the quartet current \( I_q(V, \phi_q) \) is given by Eqs. (17), (18) above. Given Eq. (15), taking the Max over \( \phi_q \) is equivalent to taking the Max over \( \phi_c \). This implies that \( I_{q,c}^c(eV/\Delta) \) is independent on \( \alpha(\Phi) \). Thus, it is only through \( \Gamma_{c,eff}(\Phi) \) that \( I_{q,c}^c(eV/\Delta, \Phi) \) depends on \( \Phi \).

### D. Parameters used in the numerical calculation

In this subsection, we present the parameters which are used in the forthcoming numerical calculations.
Considering first a \((S_a, S_b, S_c)\) three-terminal Josephson junction, the gap closes if the following condition on \((\Gamma_a, \Gamma_b, \Gamma_c)\) is fulfilled\(^{[38]}\)

\[
\Gamma_{c, \text{eff}} \exp(i\varphi_{c, \text{eff}}) = \frac{|\Gamma_a^2 - \Gamma_b^2|}{\sqrt{\Gamma_a^2 + \Gamma_b^2 - 2\Gamma_a\Gamma_b \cos \varphi}}. \tag{20}
\]

Specializing to \(\varphi = 0\) leads to

\[
\Gamma_{c, \text{eff}} = \Gamma_a + \Gamma_b \tag{21}
\]

\[
\varphi_{c, \text{eff}} = 0. \tag{22}
\]

In the following numerical calculations, the four-dimensional \((\Gamma_a, \Gamma_b, \Gamma_{c,1}, \Gamma_{c,2})\) space of the coupling constants between the dot and the superconducting leads will be scanned according to the following 1D subspace:

\[
\frac{\Gamma_a}{\Delta} = 0.4 \tag{23}
\]

\[
\frac{\Gamma_b}{\Delta} = 0.2 \tag{24}
\]

\[
\frac{\Gamma_{c,1}}{\Delta} = \frac{1}{2} \left(0.3 + \frac{\gamma}{\Delta}\right) \tag{25}
\]

\[
\frac{\Gamma_{c,2}}{\Delta} = \frac{1}{2} \left(0.9 + \frac{\gamma}{\Delta}\right). \tag{26}
\]

Eqs. (23)-(26) imply

\[
\Gamma_{c,1} + \Gamma_{c,2} - \Gamma_a - \Gamma_b = \frac{\gamma}{\Delta}. \tag{27}
\]

and thus the ABS gap closes at \(\varphi = 0\) if \(\gamma = 0\).

### E. Multilevel quantum dot

Now, we mention the multilevel quantum dot model describing \(M\) energy levels (see section I of the Supplemental Material\(^{[22]}\)). This multilevel quantum dot can be mapped onto an effective single-level quantum dot with appropriate energy-dependent Green’s functions if a specific condition of factorization is fulfilled. This model is used in section IV in order to demonstrate robustness of the inversion against multichannel effects.

### F. Inversion in the \(V = 0^+\) adiabatic limit

In this subsection, we comment on section II of the Supplemental Material which provides a mechanism for the inversion in the \(V = 0^+\) adiabatic limit (still with biasing on the quartet line). It turns out that this inversion between \(\Phi/\Phi_0 = 0\) and \(\Phi/\Phi_0 = 1/2\) appears in the range of the coupling parameters which fulfills the conditions of convergence of perturbation theory in \(\Gamma_a\) and \(\Gamma_b\) with respect to \(\Gamma_{c,1}\) and \(\Gamma_{c,2}\), assumed to take much larger values. This predicted inversion requires asymmetric couplings \(\Gamma_{c,1}\) and \(\Gamma_{c,2}\).

However, this assumption on the couplings is not directly relevant to the Harvard group experiment\(^{[23]}\) where the couplings \(\Gamma_{c,1}\) and \(\Gamma_{c,2}\) are expected to be symmetric. Now, we consider inversion appearing at finite bias voltage \(V\) in the remaining of the paper, on the basis of the couplings given by Eqs. (23)-(26).

### III. LANDAU-ZENER TUNNELING RATE

This section provides calculations for the Landau-Zener tunneling rate. Subsection III A presents the analytical calculations. Section III B shows the corresponding numerical results for a set of parameters relevant to the forthcoming sections IV and V.

#### A. Analytical results

In this subsection, we present an analytical theory for an indicator of the strength of quantum fluctuations in the quartet current: the rate \(\mathcal{R}\) of Landau-Zener tunneling between the two ABS manifolds.

It was shown in section II that the four-terminal device on figure\(^{[1]}\) can be mapped onto three terminals with suitable couplings \(\Gamma_{c, \text{eff}}\) between the dot and the grounded lead \(S_{c, \text{eff}}\) [see Eq. (14)]. Thus, the Landau-Zener tunneling rate \(\mathcal{R}\) is now evaluated for a three-terminal device, without loss of generality with respect to four terminals. We use the following notation \(k\) for the fast combination of the superconducting phases: \(\varphi_a(k) = \varphi_a + k\), \(\varphi_b(k) = \varphi_b + k\), \(\varphi_{c,1}(k) = \varphi_{c,1} + k\), and \(\varphi_{c,2}(k) = \varphi_{c,2} + k\). Eq. (5) leads to the following expression for the ABS energies:

\[
E_{\text{ABS,3T}} \left| \Gamma_{3T} \right| = \left| \Gamma_{a,3T} e^{i\varphi_a(k)} + \Gamma_{b,3T} e^{i\varphi_b(k)} + \Gamma_{c,3T} e^{i\varphi_{c,2}(k)} \right|. \tag{28}
\]

We first evaluate the value \(k_s\) of \(k\) which minimizes \(E_{\text{ABS,3T}}\) in Eq. (28). The corresponding energy at the minimum is denoted by \(\delta_{\text{min}}\):

\[
\delta_{\text{min}} = \text{Inf}_k \left| E_{\text{ABS,3T}}(k) \right|, \tag{29}
\]

which depends on all of the junction parameters. Eq. (29) can be called as “the Andreev gap” if the ABS spectrum is plotted as a function of the fast variable \(k\). We have shown previously\(^{[20]}\) that a single or two local minima can occur in the variations of \(E_{\text{ABS}}\) with \(k\), depending on the values of the device parameters. As a simplifying assumption, the Landau-Zener processes are considered to be dominated by the global minimum in the presence of two local minima. In a second step, \(E_{\text{ABS}}\) given by Eq. (28) is expanded to second order in the vicinity of \(k_s\):

\[
E_{\text{ABS}}^2 = \delta_{\text{min}}^2 + \tilde{\Gamma}_0^2 (k - k_s)^2 + \Theta \left[(k - k_s)^3\right], \tag{30}
\]

where the coefficient \(\tilde{\Gamma}_0\) is the following:

\[
\tilde{\Gamma}_0^2 = -4\Gamma_{a,3T} \Gamma_{b,3T} \cos(2k_s - \varphi_a + \varphi_b) - \Gamma_{a,3T} \Gamma_{c,3T} \cos(k_s - \varphi_a) - \Gamma_{b,3T} \Gamma_{c,3T} \cos(k_s + \varphi_b). \tag{31}
\]
The rate $R$ of Landau-Zener tunneling can be approximated as the following:

$$R = \exp \left( -\frac{\pi \delta_{\min}^2}{4eVT_0} \right).$$

Eq. (32) appeared previously in the literature, see for instance Eq. (20) in a review article on Landau-Zener-Stückelberg interferometry.3

FIG. 2. The figure shows the rate $R$ of Landau-Zener tunneling for $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 0$ (panels a1 and a2), $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 1/2$ (panels b1 and b2) and $\gamma/\Delta = -0.25$ and $\Phi/\Phi_0 = 0$ (panels c1 and c2). Panels a1, b1 and c1 show colorplots of $R$ as a function of $\phi_q/2\pi$ (on $\gamma$-axis) and $\log_{10}(eV/\Delta)$ (on $\gamma$-axis). Panels a2, b2 and c2 show $\delta_{\min}/\Delta$ (magenta lines) as a function of $\phi_q/2\pi$, and $R(\phi_q/2\pi)$ evaluated for $eV/\Delta = 0.01, 0.02, 0.03, 0.04, 0.05$. #3
B. Numerical results

In this subsection, we present on figure 2 our numerical results for the rate $\mathcal{R}$ of Landau-Zener tunneling [see Eq. (22)].

Figures 2-a1, b1 and c1 show colorplots of $\mathcal{R}$ in the plane of the reduced parameters $(\tilde{\phi}_q/2\pi, \log_{10}(eV/\Delta))$. The following parameters are used: $\gamma/\Delta = 0.3$, $\Phi/\Phi_0 = 0$ (panel a1), $\gamma/\Delta = 0.3$, $\Phi/\Phi_0 = 1/2$ (panel b1), and $\gamma/\Delta = -0.25$, $\Phi/\Phi_0 = 0$ (panel c1). The yellow colorcode on figures 2-a1, b1 and c1 corresponds to strong Landau-Zener tunneling with $\mathcal{R} \approx 1$. The black colorcode corresponds to the adiabatic limit with negligibly small Landau-Zener tunneling $\mathcal{R} \approx 0$.

Figures 2-a2, b2 and c2 represent the “Andreev gap” $\delta_{\min}$ as a function of the gauge-invariant quartet phase $\tilde{\phi}_q$ for the same parameters as figures 2-a1, b1 and c1 (see above). In addition, figures 2-a2, b2 and c2 show the variations of $\mathcal{R}$ with $\tilde{\phi}_q/2\pi$, for the following values of voltage: $eV/\Delta = 0.01, 0.02, 0.03, 0.04, 0.05$. These reduced voltage values $eV/\Delta$ are close to those used in the forthcoming sections IV and V.

Considering now interpretation of figure 2, the rate $\mathcal{R}$ of Landau-Zener tunneling given by Eq. (22) has exponential variations with all of the following parameters: the reduced voltage $eV/\Delta$, the reduced flux $\Phi/\Phi_0$, the gauge-invariant quartet phase $\tilde{\phi}_q$, and the parameter $\gamma/\Delta$ used to parameterize between the dot and the superconducting leads [see Eqs. (19–24)]. This exponential behavior of $\mathcal{R}$ is compatible with the narrow cross-over along the $\gamma$-voltage axis on figure 2, between the low-voltage adiabatic and the higher-voltage antadiabatic behaviors, corresponding to the black and yellow color respectively.

Figures 2-a1, b1, c1 correlate with the gauge-invariant quartet phase $\tilde{\phi}_q/2\pi$-sensitivity of the Andreev gap $\delta_{\min}$ on figures 2-a2, b2 and c2 respectively. Namely, closing the Andreev gap $\delta_{\min}$ at $\tilde{\phi}_q/2\pi$ around $\tilde{\phi}_q/2\pi \approx 0.2, 0.8$ (magenta line on panels b2, c2) results in strong nonadiabaticity. The Andreev gap $\delta_{\min}$ does not close at any value of $\tilde{\phi}_q/2\pi$ for weak Landau-Zener (see the magenta line on figure 2-a1). Panel a1 shows $\mathcal{R} \approx 0$ in most of the considered voltage range $-8 \leq \log_{10}(eV/\Delta) \leq -1$ while yellow-colored regions with $\mathcal{R} \approx 1$ clearly develop on panels b1, c1.

To summarize, we calculated the variations of the rate $\mathcal{R}$ of Landau-Zener tunneling for the three sets of parameters which will be used in the next sections IV and V. One of those is representative of “weak Landau-Zener tunneling” characterized by a finite Andreev gap in the entire $\tilde{\phi}_q/2\pi$-parameter range, i.e. $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 0$ on figures 2-a1-a2. The two others correspond to “strong Landau-Zener tunneling” characterized by the closing of the “Andreev gap” at specific values of $\tilde{\phi}_q/2\pi$, i.e. $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 1/2$ on figures 2-b1-b2 and $\gamma/\Delta = -0.25$ and $\Phi/\Phi_0 = 0$ on figures 2-c1-c2.

IV. INVERSION AT FINITE BIAS VOLTAGE $V \neq 0$

Now, we present the main results and discuss how Landau-Zener tunneling can produce an inversion between $\Phi/\Phi_0 = 0$ and $\Phi/\Phi_0 = 1/2$ [i.e. $I_{q,1}(eV/\Delta, 0) < I_{q,2}(eV/\Delta, 1/2)$], in connection with the Harvard group experimental results [40].

The algorithms are presented in section IV A. The quartet critical current is defined in section IV B. Section IV C presents the numerical data. They are next discussed physically in section IV D. A connection to the Harvard group experiment [40] is presented in section IV E.

A. Algorithms

The principle of the codes is summarized in this subsection.

The dC-currents are evaluated from an integral of the spectral current over the energy $\omega$. The spectral quartet current is calculated from Keldysh Green’s functions, see for instance Ref. [54]. The spectral current shows sharp peaks at the energies $\omega = E_n$ of the Floquet levels [40]. An adaptive algorithm is used to integrate over $\omega$, and matrix multiplications are optimized by using sparse matrix algorithms.

Figures 3-a2-d2 show how the peaks in the spectral current evolve as the voltage is increased on figures 3-a1-d1 is scanned in $eV/\Delta$ through a dip in $I_{q,1}(eV/\Delta, \Phi/\Phi_0)$. Further comments about this figure will be presented in section IV D 2 in connection with populations of the two Floquet states.

B. Definition of the quartet critical current as a function of voltage

Figure 4 shows a comparison between i) The Floquet energies $E_n$ as a function of $\log_{10}(eV/\Delta)$, and ii) The critical current $I_{q,c}$. The values $\Phi/\Phi_0 = 0$ and $\Phi/\Phi_0 = 1/2$ of the reduced flux are used on panels a-b and c-d respectively, and the contact transmissions are such that $\gamma/\Delta = 0.3$ in Eqs. (19–24).

We present now a central quantity: the quartet critical current as a function of reduced voltage $eV/\Delta$.

The value of the gauge-invariant quartet phase $\tilde{\phi}_q$ is calculated in such a way as to maximize the current $I_{q,c} = I_{s_{c,1}} + I_{s_{c,2}}$ transmitted into the ground loop $S_1$ at the contacts points $S_{c,1}$ and $S_{c,2}$, as a function of the gauge invariant quartet phase $\tilde{\phi}_q$. This value of $\tilde{\phi}_q$ which maximizes the current is denoted by $\tilde{\phi}_q^*$. In the spirit of Eq. (19), the value of the current at the maximum is denoted by

$$I_{q,c}^* = I_{s_{c,1}}(eV/\Delta, \Phi/\Phi_0) + I_{s_{c,2}}(eV/\Delta, \Phi/\Phi_0)$$

The quantity $I_{q,c}^*(eV/\Delta, \Phi/\Phi_0)$ is called in short as “the critical current”.

Now, we present the currents $I_{q,c,1}^*(eV/\Delta, \Phi/\Phi_0)$ and $I_{q,c,2}^*(eV/\Delta, \Phi/\Phi_0)$ carried by each Floquet state.
Specifically, the spectral current $\tilde{I}(\omega)$ is “folded” into the first Brillouin zone $[0, 2eV]$

$$I_{folded}(\tilde{\omega}) = \sum_n \tilde{I}(\tilde{\omega} + 2neV) \quad (34)$$

where $0 < \tilde{\omega} < 2eV$ in Eq. (34). The currents $\tilde{I}_1$ and $\tilde{I}_2$ carried by each Floquet state are the contributions of the $0 < \tilde{\omega} < eV$ and the $eV < \tilde{\omega} < 2eV$ spectral windows:

$$\tilde{I}_1 = \int_0^{eV} I_{folded}(\tilde{\omega}) d\tilde{\omega} \quad (35)$$

$$\tilde{I}_2 = \int_{eV}^{2eV} I_{folded}(\tilde{\omega}) d\tilde{\omega}. \quad (36)$$

The values of $\tilde{I}_1$ and $\tilde{I}_2$ at $\tilde{\omega}_q = \tilde{\phi}_q^*$ are denoted by $\tilde{I}_{q,c,1}$ and $\tilde{I}_{q,c,2}$ respectively. The contributions $I_{q,c,1}$ and $I_{q,c,2}$ of the Floquet states 1 and 2 are calculated solely from maximizing the total current $I = \tilde{I}_1 + \tilde{I}_2$ with respect to $\tilde{\phi}_q$, not from separately maximizing $\tilde{I}_1$ and $\tilde{I}_2$.

Concerning the choice of the parameters, this section [IV] discusses solely “weak Landau-Zener tunneling” for $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 0$ (corresponding to figures 2-a1, a2 in the preceding section [III]). The discussion of strong Landau-Zener tunneling (such as for $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 1/2$) is postponed for section [V].

C. Presentation of the numerical results

Now, we present our numerical data. The Floquet spectra are discussed in section [IVC]. The critical current is presented in section [IVC2]. The connection between the Floquet spectra and the critical current is presented in section [IVC3].

I. Numerical results for the Floquet spectra

We start with the Floquet spectra in this subsection, focusing on emergence of anticrossings (see figure [IV-1]).

Figure 4a shows the normalized Floquet energies $E_n/eV$ as a function of the reduced voltage $eV/\Delta$. The dynamics is periodic in time with period $\hbar/2eV$ and the Floquet spectrum is thus also periodic in energy with period $2eV$. The
FIG. 4. The figure shows the Floquet spectra (panels a and c) and the critical quartet current $I_{q,c}$ (panels b and d) as a function of $\log_{10}(eV/\Delta)$ on x-axis, for $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 0$ (panels a and b, being representative of “weak Landau-Zener”) and for $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 1/2$ (panels c and d being representative of “strong Landau-Zener”). Panels b and d also show the currents $I_{q,c,1}$ and $I_{q,c,2}$ carried by each Floquet state [see Eqs. (35) and (36)]. The vertical bars show the extrema in the Floquet spectra (on panels a and b) and the minima in $I_{q,c}$ (on panels b and d), with the following colorcode: (i) “Black vertical bars” are used for coincidence between the extrema in the Floquet spectra and the state [see Eqs. (35) and (36)]. The vertical bars show the extrema in the Floquet spectra (on panels a and b) and the minima in $I_{q,c}$, (ii) “Orange vertical bars” on panels a and c are used for the extrema in the Floquet spectra which have no counterpart as an extremum in the Floquet spectrum, (iii) “Green vertical bars” on panels b and d are used for the minima in $I_{q,c}$, which have no counterpart as an extremum in the Floquet spectrum.

shaded green region on panels a and c show the “first Brilloin zone” $-1 < E_{n}/eV < 1$. The other Floquet levels can be obtained by translation along the y-axis of energy according to $\{E_{-1} + 2peV, E_1 + 2qeV\}$ with $p$ and $q$ two integers, where $-eV < E_{-1} < 0$ and $0 < E_1 < eV$.

Classically, the Floquet spectra consist of two families at energies

$$ E_{+,p} = \langle E_{ABS}\rangle_k + 2peV $$

$$ E_{-,q} = -\langle E_{ABS}\rangle_k + 2qeV. $$

with $n = q - p$.

Now, we note that the quantum mechanical Landau-Zener tunneling opens gaps in the Floquet spectrum in figure 4, instead of the classically nonavoided level crossings discussed above at $\{eV_{cross,n}\}$. The Floquet spectrum can well be approximated by the “classical spectrum” if the voltage is in between two values of $V_{cross,n}$, with the corresponding normalized Floquet level energies $E_{\pm,n}/eV$ given by Eqs. (37) and (38). The gap between the Floquet levels shows a tendency towards taking smaller values as the voltage ratio $eV/\Delta$ is reduced (see figure 4). This is in a qualitative agreement with the reduction of the Landau-Zener tunneling rate $R$ at low voltage [see Eq. (32) and figures 2-a1-a2].
2. Numerical results for the critical current

Now, we comment on the critical current \( I_{q.c}(eV/\Delta, \Phi/\Phi_0) \) defined in the previous section [IV.B]. The variations of \( I_{q.c}(eV/\Delta, \Phi/\Phi_0) \) with \( \log_{10}(eV/\Delta) \) are shown by the blue lines in figure 4b. Figure 2b reveals a regular sequence of “dips towards zero” in the reduced voltage-eV/\( \Delta \) dependence of \( I_{q.c}(eV/\Delta, \Phi/\Phi_0) \). The discussion of the contributions \( I_{q.c,1}(eV/\Delta, \Phi/\Phi_0) \) and \( I_{q.c,2}(eV/\Delta, \Phi/\Phi_0) \) of each Floquet state (green and orange lines on figure 4b) is postponed for section [IV.D] below.

3. Numerical evidence for a connection between the Floquet spectra and the current

Now, we present a connection between the Floquet spectra and the quartet current i.e. we discuss the vertical bars in figures 4a-b:

(i) The extrema in the Floquet spectra are shown by the vertical bars on figure 4a. They are such that \( \partial E_n(V_{FI,\lambda})/\partial V = 0 \) (where the integer \( \lambda \) labels the extrema).

(ii) The minima in \( I_{q.c,\mu}(V) \) are shown by the vertical bars on figure 4b. They are such that \( \partial^2 I_{q.c}(V_{q.c,\mu})/\partial V^2 = 0 \) and \( \partial^2 I_{q.c}(V_{q.c,\mu})/\partial V^2 > 0 \) (where the integer \( \mu \) labels the minima).

The following colorcode is used for these vertical bars:

(i) The black vertical bars on figure 4a-b show the voltage-V values such that \( V_{FI,\lambda} \approx V_{q.c,\mu} \), are coinciding within a small tolerance.

(ii) The thinner vertical orange bars on figure 4b show the values of \( V_{FI,\lambda} \) which are noncoinciding with any of the \( \{V_{q.c,\mu}\} \).

(iii) The thinner vertical magenta bars on figure 4b show the values of \( V_{q.c,\mu} \) which are noncoinciding with any of the \( \{V_{FI,\lambda}\} \).

D. Physical picture

Now, we interpret our “numerical experiments” for \( \gamma/\Delta = 0.3 \), presented above in subsection [IV.C].

Three regimes are obtained upon increasing voltage \( V \) from the \( V = 0^+ \) adiabatic limit, i.e. upon increasing the strength of Landau-Zener tunneling:

(i) At low voltage, Landau-Zener tunneling implies hybridization between the Floquet states at the avoided crossings in the Floquet spectrum (see section [IV.D.1]).

(ii) Increasing voltage has the effect of enhancing Landau-Zener tunneling and populating both Floquet states.

(iii) At higher voltage, the nontrivial populations of the Floquet states produce \( \pi \)-shifted current-phase relations (see section [IV.D.2]).

1. Hybridization between the two Floquet states at very low voltage

Connection between the Floquet spectra and the quartet current: We provide in this subsection a physical picture for the avoided crossings in the Floquet spectrum in the regime of low bias voltage \( V \). Namely, we discuss the coincidences between \( V_{FI,\lambda} = V_{q.c,\mu} \) reported in the preceding section [IV.C] 3.

(i) Landau-Zener tunneling produces quantum mechanical coupling between the two Floquet states. The two ABS at opposite energies contribute for exactly opposite values to the currents \( I_{S,1}(eV/\Delta, \Phi/\Phi_0) \) and \( I_{S,2}(eV/\Delta, \Phi/\Phi_0) \) at the \( S_{1,1} \) and \( S_{2,2} \) contacts. Thus, Landau-Zener tunneling reduces the critical current \( I_{q.c} \).

(ii) Weak Landau-Zener tunneling produces avoided crossings in the Floquet spectra, like any generic quantum-mechanical perturbation.

As a consequence of these items (i) and (ii), the dips in the voltage dependence of \( I_{q.c}(eV/\Delta, \Phi/\Phi_0) \) and the avoided crossings in the Floquet spectrum appear simultaneously at the same voltage values, because they have a common origin, i.e. production of quantum superpositions of the positive and negative-energy ABS manifolds, as a result of Landau-Zener tunneling between them.

Current carried by each Floquet state: Now, we present the voltage dependence of the currents \( I_1 \) and \( I_2 \) carried by each Floquet state [see Eqs. (35) and (36) in the preceding section [IV.B]].

The reduced voltage-eV/\( \Delta \) dependence of \( I_{q.c,1}(eV/\Delta, \Phi/\Phi_0) \) and \( I_{q.c,2}(eV/\Delta, \Phi/\Phi_0) \) is shown in figure 4b.

At low voltage, the current is almost entirely carried by a single Floquet state, if the voltage value is in between two avoided crossings [typically \( \log_{10}(eV/\Delta) \approx -0.9 \) in figure 4b]. The “+” and the “−” Floquet states defined by Eqs. (37) and (38) anticross at the \( \{V_{cross,n}\} \) above, yielding alternation between “current carried mostly by Floquet state 1”, followed by “current carried mostly by Floquet state 2”, ... as voltage increases, see figure 4b. It is seen on figures 4b and b that the “switching voltages” between \( I_{q.c,1} \approx 0 \) and \( I_{q.c,2} \approx 0 \) match perfectly well the anticrossings in the Floquet spectra, which coincide with the most pronounced minima in \( I_{q.c}(eV/\Delta) \), see the discussion above.

Generalization to the full current-phase relations: Our previous discussion was based on taking the maximum of the current with respect to the gauge invariant quartet phase. Now, we focus on the behavior of the Floquet spectrum \( E_n(eV/\Delta, \Phi/\Phi_0) \) and the current \( I_{S,1}(eV/\Delta, \Phi/\Phi_0) \) as a function of the gauge-invariant quartet phase variable \( \Phi_n \). Figures 3a-d1 show the critical current \( I_{q.c}(eV/\Delta, \Phi/\Phi_0) \) as a function of the reduced voltage eV/\( \Delta \), the gauge-invariant quartet phase \( \Phi_n \) taking the value \( \Phi_n = \Phi_n^* \). The \( \Phi_n^* \)-sensitivity of the Floquet spectra and the current-phase relations are shown on panels a2-d2 and a3-d3 respectively, at the values of the reduced voltage eV/\( \Delta \) which are selected on panels a1-d1. Going from panel a1 to panel d1, we scan voltage through one of the dips appearing at low voltage in \( I_{q.c}(eV/\Delta, \Phi/\Phi_0) \).
Fig. 5. The figure shows the evolution of the reduced Floquet level energies $E_n/eV$ (see panels a2-d2) and the current $I_q$ (see panels a3-d3) as a function of the reduced quartet phase $\tilde{\phi}_q/2\pi$. Panels a1-d1 show the voltage values which are selected while scanning through a dip in $I_{q,c}^*(eV/\Delta)$ plotted as a function of $eV/\Delta$.

Figures 5-a2-d2 reveal that the dip in $I_{q,c}(eV/\Delta, \Phi/\Phi_0)$ plotted as a function of $eV/\Delta$ correspond to collisions between the Floquet levels plotted as a function of $\tilde{\phi}_q/2\pi$. Avoided crossings appear in $E_n(eV/\Delta, \tilde{\phi}_q/2\pi, \Phi/\Phi_0)$ plotted as a function of $\tilde{\phi}_q/2\pi$. Part of figure 5 is already presented in the Supplementary Information of the Harvard group paper. But here, panels a3-d3 show in addition the $\tilde{\phi}_q/2\pi$-dependence of the currents $I_1(eV/\Delta, \tilde{\phi}_q/2\pi, \Phi/\Phi_0)$ and $I_2(eV/\Delta, \tilde{\phi}_q/2\pi, \Phi/\Phi_0)$ carried by each Floquet state, see Eqs. (35) and (36) above.

The following is deduced from figure 5:

(i) The current $I_{q,c}(eV/\Delta, \tilde{\phi}_q/2\pi, \Phi/\Phi_0)$ is carried by a single Floquet state for most of the values of $\tilde{\phi}_q/2\pi$, except in the immediate neighborhood of an avoided crossing where both $I_1(eV/\Delta, \tilde{\phi}_q/2\pi, \Phi/\Phi_0)$ and $I_2(eV/\Delta, \tilde{\phi}_q/2\pi, \Phi/\Phi_0)$ have a small contribution to $I_{q,c}(eV/\Delta, \tilde{\phi}_q/2\pi, \Phi/\Phi_0)$.

(ii) We find $I_{q,c}(eV/\Delta, \tilde{\phi}_q/2\pi, \Phi/\Phi_0) \simeq 0$ if the reduced gauge-invariant quartet phase $\tilde{\phi}_q/2\pi$ is tuned at an avoided crossing according to the spectra on figure 5-a2-d2.

2. Populating both Floquet states and the $\pi$-shift

Now, we present in this subsection how a $\pi$-shifted current-phase relation can emerge as the result of nonequilibrium populations of the ABS.

Coming back to figure 3, we see that the evolution from panel a2 to panel d2 across a dip in $I_{q,c}^*(eV/\Delta, \Phi/\Phi_0)$ as a function of $eV/\Delta$ involves spectral current carried by both Floquet states if the voltage is tuned at a minimum in $I_{q,c}^*(eV/\Delta, \Phi/\Phi_0)$, see figure 5-c2.

Populating both Floquet states can be realized by increasing voltage in the considered situation of weak Landau-Zener tunneling (i.e. $\gamma/\Delta = 0.3$ and $\Phi/\Phi_0 = 0$). On figure 6, we scan the reduced voltage $eV/\Delta$ through a dip in $I_{q,c}^*(eV/\Delta, \Phi/\Phi_0)$, but now at higher $eV/\Delta$ values than on figure 5. The current-phase relations are shown on figures 6-a2-f2. A cross-over from $\pi$-shifted current-phase relation (see figure 6-a2) to 0-shift (see figure 6-d2) and back to $\pi$-shift (see figure 6-f2) is obtained as $eV/\Delta$ is increased.

The origin of the low-voltage $\pi$-shifted current phase relation was already discussed in section VII A of paper I of the
The proposed interpretation of the $\pi$-0 and 0-\pi cross-overs obtained in a narrow voltage window around $eV/\Delta \approx 0.2$ (see figure 9) is the following: A $\pi$-shifted Josephson relation was predicted in a superconductor-normal metal-superconductor (SNS) Josephson weak link, originating from injection of nonequilibrium quasiparticle populations from two attached normal leads. This $\pi$-shifted current-phase relation can be understood by noting that the two ABS at opposite energies carry opposite currents. A change of sign in the current-phase relation is obtained in the limit where only the positive-energy ABS is populated, which is an extreme limit of nonequilibrium quasiparticle populations. This is why we relate the $\pi$-0 and the 0-$\pi$ shifts of $I_{q,c}$ to the nonequilibrium Floquet populations produced for these relatively large values of the reduced voltage $eV/\Delta$.

The $\sim -\sin(2q\phi_q)$ current-phase relation appearing at the $\pi$-0 cross-over on panel c2 meets physical expectations regarding emergence of a second-order harmonics of the current-phase relation once the first-order harmonics changes sign.

E. Conclusion on the Harvard group experiment

To summarize, the inversion obtained in the Harvard group experiment emerges also in our quantum dot model, and the mechanism is simple in the regime of weak quantum correlations:

- The Floquet levels plotted as a function of bias voltage $V$ on the quartet line show nonavoided crossings in the classical limit, at the voltages $\{V^*_{\pi}(\Phi/\Phi_0)\}$. The $\Gamma_{c,eff}$ in Eq. (14) is $\Phi/\Phi_0$-dependent and thus, the avoided crossings $\{V^*_{\pi}(0)\}$ at $\Phi/\Phi_0 = 0$ do not generally coincide with $\{V^*_{\pi}(1/2)\}$ at $\Phi/\Phi_0 = 1/2$.

- The quantum mechanical Landau-Zener tunneling makes the crossings between the Floquet levels become avoided. At these avoided crossings, the dynamical state of the device is the result of a quantum mechanical superposition between the two ABS.

- The quartet current is reduced at the avoided crossings, because the two ABS in the superposition carry opposite currents. Considering that voltage $V = V_{\pi}(0)$ is tuned at one of the avoided crossings in zero flux $\Phi/\Phi_0 = 0$, we obtain the following possibility of an inversion between $\Phi/\Phi_0 = 0$ and $\Phi/\Phi_0 = 1/2$.
the connection between the extrema in the Floquet spectrum and the minima in the quartet critical current (both being plotted as a function of reduced voltage \(eV/\Delta\)) holds also for strong Landau-Zener tunneling with \(\gamma/\Delta = -0.25\) (see the previous section III). Next, section III B of the Supplemental Material presents a scan from \(\gamma/\Delta = -0.25\) to \(\gamma/\Delta = 0.3\), and provides evidence for inversion in this range of \(\gamma/\Delta\).

Now, we show that inversion \(I_{q,c}(eV/\Delta,0) < I_{q,c}(eV/\Delta,1/2)\) appears generically in the multilevel quantum dot model presented in the previous section IV specialized to the equally spaced energy levels:

\[
\varepsilon_n = n\delta + \delta_0, \quad (40)
\]

with \(n\) an integer. An estimate for the number of energy levels within the gap window is \(2\Delta/\delta\).

Figures 7a1-i1 and figures 7a2-i2 correspond to \(\delta_0/\Delta = 0\) and \(\delta_0/\Delta = 0.6\) respectively, with \(\delta/\Delta\) ranging from \(\delta/\Delta = 1.8\) (panels a1 and a2) to \(\delta/\Delta = 0.2\) (panels i1 and i2). Panels i1 and i2 coincide with each other, because \((\delta_0/\Delta, \delta/\Delta) = (0, 0.2)\) and \((\delta_0/\Delta, \delta/\Delta) = (0.6, 0.2)\) produce the same spectrum of the quantum dot energy levels.

V. ROBUSTNESS OF THE INVERSION

Now, we investigate robustness against strong Landau-Zener tunneling and many levels in the quantum dot.

In section III A of the Supplemental Material, we show that the connection between the extrema in the Floquet spectrum and the magenta shaded region corresponds to the inversion. The parameter \(\delta_0/\Delta = 0\) is used on panels a1-i1, with \(\delta/\Delta\) ranging from 1.8 (panel a1) to 0.2 (panel i1). The parameter \(\delta_0/\Delta = 0.6\) is used on panels a2-i2, and \(\delta/\Delta\) is from 1.8 (panel a2) to 0.2 (panel i2).

\[
\Phi/\Phi_0 = 1/2
\]

\(I_{q,c}(V_{q0}^\ast(0), \Phi/\Phi_0 = 0) < I_{q,c}(V_{q0}^\ast(0), \Phi/\Phi_0 = 1/2), \quad (39)\)

because no reason is seen why \(V_{q0}^\ast(0)\) at \(\Phi/\Phi_0 = 0\) should in general be close to one of the \(V_{q0}^\ast(1/2)\) at \(\Phi/\Phi_0 = 1/2\).

The windows of the reduced voltage \(eV/\Delta\) at which the inversions appear are shown by the magenta shading in figure [4], and they fit well within the proposed mechanism.

In addition, we obtained evidence for a 0-\(\pi\) and \(\pi-0\) crossovers upon increasing the reduced voltage \(eV/\Delta\). This numerical result was interpreted as being a consequence of nontrivial Floquet populations.
It is concluded from figures a1-i1 and figures a2-i2 that crossing-over from $\delta/\Delta = 1.8$ larger than unity on figures a1-a2 (typically with zero of a single energy level in the gap window) to $\delta/\Delta = 0.2$ on figures i1-i2 (thus with about $\sim 10$ energy levels in the gap window) implies emergence of inversion over a broad interval of the reduced voltage $eV/\Delta$. It is concluded from figure a2 that the inversion is favored upon increasing the number of levels on the quantum dot, in comparison with a single level quantum dot.

VI. CONCLUSIONS

In this paper, we presented interpretation of “numerical experiments” for the four-terminal device in figure 1. The critical current $I_{q,c}(eV/\Delta, \Phi/\Phi_0)$ is parameterized by the reduced voltage $eV/\Delta$ and reduced flux $\Phi/\Phi_0$ piercing through the loop.

The Harvard group experiment reports a window of the reduced voltage $eV/\Delta$ such that $I_{q,c}(eV/\Delta,0) < I_{q,c}(eV/\Delta,1/2)$, where $I_{q,c}(eV/\Delta,0)$ and $I_{q,c}(eV/\Delta,1/2)$ are the critical currents at the reduced flux values $\Phi/\Phi_0 = 0$ and $\Phi/\Phi_0 = 1/2$ respectively. Naive expectations about destructive interferences at $\Phi/\Phi_0 = 1/2$ would lead to “noninverted behavior” instead of the inversion observed in the Harvard group experiment.

The “Floquet mechanism” for the inversion is especially simple in the limit of weak Landau-Zener tunneling. First, in absence of Landau-Zener tunneling between the two ABS manifolds, the classical Floquet spectrum shows nonavoided crossings as a function of reduced voltage $eV/\Delta$. The rate of Landau-Zener tunneling increases from zero as $eV/\Delta$ is increased. This yields opening of gaps in the Floquet spectrum, which makes the crossings between the Floquet levels become avoided. The quantum mechanical effects of weak Landau-Zener tunneling are important only if the voltage is close to one of the avoided crossings. Landau-Zener tunneling produces hybridization between the two Floquet states and a reduction of the quartet critical current $I_{q,c}(eV/\Delta, \Phi/\Phi_0)$, due to the time-dependent dynamical quantum superpositions of the two ABS which carry opposite currents. In certain voltage windows, the reduction in $I_{q,c}(eV/\Delta, \Phi/\Phi_0)$ at $\Phi/\Phi_0 = 0$ is such as to produce an inversion with $\Phi/\Phi_0 = 1/2$. In addition, we have also shown that nontrivial populations of the two Floquet states upon increasing voltage eventually result in a change of sign in the relation between the quartet current and the gauge-invariant phase variable.

Finally, we have shown that the phenomenon of the inversion is generic since it holds also for strong Landau-Zener tunneling and for a multilevel quantum dot.

This is why our quantum dot model provides an explanation to the inversion in the recent Harvard group experiment. In the forthcoming paper III of the series, we will start from the “2D metal beam splitter” of the previous paper I (instead of the 0D quantum dot of this paper II), and develop Floquet theory in connection with the emergence of a small energy scale.

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