Outline

1. Introduction to Four-Fermion Theories
2. Two Four-Fermion Interactions
3. Fierz Identities for the Thirring Model
4. Fermion Bag Approach
5. Summary
What are Four-Fermion Theories?

QFTs of fermions with 4th power of fermion fields as interaction:

\[
\mathcal{L} = \bar{\psi}_j (\partial + m) \psi_j + \sum_{\alpha} \frac{g_\alpha^2}{2N_f} (\bar{\psi}_j \Gamma_\alpha \psi_j)^2 \quad j = 1, \ldots, N_f
\]

Main Models

- **Thirring** 1958, soluble fermionic theory in 2D
  \[\Gamma_\alpha = \gamma_\mu\]

- **Nambu & Jona-Lassinio** 1961, dynamical mass generation in 4D
  \[\Gamma_1 = \mathbb{1}, \Gamma_2 = i\gamma_5\]

- **Gross & Neveu** 1974, asymptotic freedom, chiral symmetry breaking in 2D
  \[\Gamma = \mathbb{1}\]
What is the Thirring model?

QFT with $N_f$ flavours of massless fermions with \textbf{current} interaction

Euclidean Lagrangian

$$\mathcal{L} = \bar{\psi}_j \partial \psi_j + \frac{g^2}{2N_f} \sum_{\mu=1}^{3} \left( \bar{\psi}_j \gamma^\mu \psi_j \right)^2 \quad j = 1, \ldots, N_f$$

- 3-dimensional euclidean spacetime
- representation of Clifford algebra:
  - irreducible 2-dimensional (later in this talk)
  - reducible 4-dimensional (now)

Motivation

Similarity to QED$_3$ and possible applications in superconductors, graphene, ...
Why is the Thirring model interesting?

Symmetries

- chiral symmetry, generated by \( \{1, \gamma_4, \gamma_5, i\gamma_4\gamma_5\} \)
- flavour symmetry

Result: \( U(N_f, N_f) \), can be spontaneously broken to \( U(N_f) \otimes U(N_f) \)

\[ \Rightarrow \text{chiral condensate } \langle \bar{\psi}\psi \rangle \neq 0 \]

Chiral Symmetry Breaking

“\( N_f = 0.5 \)” irreducible representation for \( N_{f,\text{irr}} = 1 \) corresponds to Gross-Neveu model with chiral symmetry breaking

\( N_f \to \infty \) no chiral symmetry breaking

\[ \Rightarrow \text{There is a } N_{f}^{cr} \text{ where chiral behaviour changes.} \]
What is the value of $N_f^{cr}$?

Results for $N_f^{cr}$ from Schwinger-Dyson equations, $\frac{1}{N_f}$-expansion, functional renormalization group, Lattice simulation with staggered fermions:

- Kim & Kim (1996)
- Del Debbio, Hands et al. (1996-1999)
- Kondo (1995)
- Hands et al. (2007)
- Gomes et al. (1991)
- Itoh et al. (1995)
- Hong & Park (1994)
- Jansen & Gies (2012)
Chiral Symmetry on the Lattice

Nielsen-Ninomiya Theorem

It is not possible to have a chiral, local and translational invariant Dirac operator with correct continuum limit without doublers.

older results: staggered fermions with mass
- mass breaks symmetry explicitly
- still doublers and no full chiral symmetry
- symmetry correct in the continuum limit?

our approach: SLAC derivative [fine for non-gauge theories: Bergner et al. arXiv:0705.2212; Wozar, Wipf arXiv:1107.3324]
- in momentum space: multiplication by $i\gamma^\mu p_\mu$
- exact chiral symmetry
- not local: need to do Fourier transformation
Problems

Technical Problems

- **Chiral condensate always zero** due to exact chiral symmetry and integration over fermions.
- Peak in susceptibility may indicate **lattice artefact phase**.

Coupling to Global Model [PoS(LATTICE 2015)050]

- Can obtain nice histograms of \( \langle \bar{\psi} \psi \rangle \).
- Hard to recover Thirring model.

\[ \Rightarrow \text{No reliable conclusion regarding} \ N_f^{cr} \ \text{, but likely} \ N_f^{cr} \leq 2. \]
Why Coupled Four-Fermion Interactions?

- Easy to study chiral symmetry breaking in Gross-Neveu model. 
  ⇒ Gain new insights into Thirring model and the larger theory space by coupling there two models.
- Test predictions from functional renormalization group.

Interaction with $i\gamma_4\gamma_5$

$(\bar{\psi}\gamma_4\gamma_5\psi)^2$ is interesting because

- same symmetry (and problems) as Thirring, while
- expected to be Gross-Neveu-like, corresponds to irreducible Gross-Neveu model.
\[ \lambda = \frac{1}{2g^2}, \text{ Lattice } 8\times7\times7, \, N_f = 1 \]
Gross-Neveu and $i\gamma_4\gamma_5$

$$\lambda = \frac{1}{2g^2}, \text{ Lattice } 12 \times 11 \times 11, \ N_f = 1$$

Chiral Condensate

Susceptibility
Gross-Neveu and $i\gamma_4\gamma_5$

$$\lambda = \frac{1}{2g^2}, \text{Lattice } 12 \times 11 \times 11, N_f = 2$$

Chiral Condensate

Susceptibility
Gross-Neveu and $i\gamma_4\gamma_5$

$$\lambda = \frac{1}{2g^2}, \text{ Lattice } 12\times11\times11, \ N_f = 3$$

Chiral Condensate

Susceptibility

$\text{pure } i\gamma_4\gamma_5 \text{ limit}$

$\text{pure Gross-Neveu limit}$
$$\lambda = \frac{1}{2g^2}, \text{Lattice } 12 \times 11 \times 11, \, N_f = 4$$

Gross-Neveu and $i \gamma_4 \gamma_5$ limit

Chiral Condensate

Susceptibility
Fierz Identity (1) for Thirring Interaction

**Goal:** rewrite interaction term, use irreducible representation:

2-component spinors $\chi^a$, $a = 1, \ldots, 2N_f := N_{f,irr}$

\[
(\bar{\chi}^a \sigma_\mu \chi^a) (\bar{\chi}^b \sigma^\mu \chi^b) = - (\bar{\chi}^a \chi^a) (\bar{\chi}^b \chi^b) - 2 (\bar{\chi}^a \chi^b) (\bar{\chi}^b \chi^a) \quad (1)
\]
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\]  (1)

Lagrangian after Hubbard-Stratonovich Transformations

$\phi$ real scalar field, $T^{ab}$ hermitian, traceless matrix field

\[
\mathcal{L}_1 = \bar{\chi}_a \left[ (\phi + \phi) \delta^{ab} + T^{ab} \right] \chi_b + \frac{N_{f,\text{irr}}}{4g^2} T_{ab} T^{ba} + \frac{N^2_{f,\text{irr}}}{2g^2 (2 + N_{f,\text{irr}})} \phi^2
\]
Fierz Identity (1) for Thirring Interaction

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\mathcal{L}_1 = \bar{\chi}_a \left[ (\partial + \phi) \delta^{ab} + T^{ab} \right] \chi_b + \frac{N_{f,irr}}{4g^2} T_{ab} T^{ba} + \frac{N_{f,irr}^2}{2g^2(2 + N_{f,irr})} \phi^2
\]

- $D_1^{ab} = \frac{\delta^{ab}}{4g^2} + \frac{T^{ab}}{2g^2(2 + N_{f,irr})}$
**Fierz Identity (1) for Thirring Interaction**

**Goal:** rewrite interaction term, use irreducible representation:
2-component spinors $\chi^a$, $a = 1, \ldots, 2N_f := N_{f,\text{irr}}$

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**Lagrangian after Hubbard-Stratonovich Transformations**

$\phi$ real scalar field, $T^{ab}$ hermitian, traceless matrix field

\[
\mathcal{L}_1 = \bar{\chi}_a \left[ (\bar{\phi} + \phi) \delta^{ab} + T^{ab} \right] \chi_b + \frac{N_{f,\text{irr}}}{4g^2} T_{ab} T^{ba} + \frac{N_{f,\text{irr}}^2}{2g^2(2 + N_{f,\text{irr}})} \phi^2
\]

$D_1^{ab}$ has no special properties $\Rightarrow$ complex eigenvalues
Sign Problem after Fierz Rearrangement

- **Thirring** has no sign problem
- **Fierz (1)** has a complex phase with \( \langle \psi \rangle := \langle e^{-i \text{Im} S} \rangle \approx 0 \) for \( \lambda_{\text{Th}} \in [0.2, 0.3] \)
- **Fierz (2)** another identity, imaginary eigenvalues, switches sign on nearly every update \( \Rightarrow \langle \psi \rangle \approx 0 \ \forall \lambda_{\text{Th}} \)
Fermion Bag Approach

Current approach to get informations about chiral symmetry breaking and solving the sign problem of Fierz (1):
\[ \Rightarrow \text{Introduce a spin field } k_{\chi i}^{a b} \in \{0, 1\}, \text{ integrate fermions, } T^{a b} \text{ and } \phi: \]

Final Partition Sum

\[ Z(\lambda) \propto \sum_k (-\lambda)^{-\frac{k}{2}} \det(i\mathcal{O}[k])2^{\tilde{n}_2} \prod_{\chi} f(n_{\chi}^1, n_{\chi}^2) \]

- \( \mathcal{O}[k] \) is the SLAC operator matrix with columns and rows deleted corresponding to \( k \).
- \( n_{\chi}^1, n_{\chi}^2 \) and \( \tilde{n}_2 \) count certain entries of \( k_{\chi i}^{a b} \).
- \( f(a, b) \) is a product of gamma- and confluent hypergeometric functions.
Simple Metropolis simulation for $N_{f,\text{irr}} = 1$:

- Agreement with **analytical** calculation and original formulation.
- Deviations for **Fierz(1)** (sign problem!).

\[
\langle k \rangle \propto \left\langle \frac{d \ln Z(\lambda)}{d \lambda} \right\rangle
\]

$N_{f,\text{irr}} = 1, 2\times3\times3$
The Thirring model shows chiral symmetry breaking for $N_f < N_f^{cr}$, for which there are many different predictions.

Simulations with multiple four-fermion couplings may allow new insights.

We get a sign problem after Fierz transformation.

Fermion bag approach may solve the sign problem and allow access to the chiral condensate.

Thank you for your attention!