Bubbles and Wormholes: Analytic Models

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Abstract

The first junction conditions of spherically symmetric bubbles are
solved for some cases, and whereby analytic models to the Einstein
field equations are constructed. The effects of bubbles on the space-
time structure are studied and it is found that in some cases bubbles
can close the spatial sector of the spacetime and turn it into a com-
 pact one, while in other cases they can give rise to wormholes. One of
the most remarkable features of these wormholes is that they do not
necessarily violate the weak and dominant energy condition even at
the classical level.

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I. INTRODUCTION

Spherically symmetric thin shells or bubbles have been the focus of interest since the early days of Einstein’s General Relativity, for example, see [1] and references therein, and studied intensively in the past decade [2 - 13], mainly because of their notable implications to the inflationary Universe scenario [14]. Most of the investigations [2 - 11] have been centered in the dynamics of bubbles by using Israel’s method [1]. The advantage of this method is that the four-dimensional coordinates can be chosen independently in each side of the bubble. Because of this advantage the relations between the two coordinate systems have been frequently ignored [4, 8]. The neglect of these conditions is partially because of their irrelevance to the study of the dynamics of bubbles and partially because of the complexity of the problem concerned.

Moreover, the effects of bubbles on the spacetime properties, specially the global ones, have been hardly studied. However, it has been shown in the plane-wall case [15] that the existence of these defects could dramatically change the spacetime geometry.

In the present paper, we shall stress the above issues by considering some exact solutions to the Einstein field equations, starting from the first junction conditions. Whereby we are enabled to study the global structure of the resulting spacetimes. Specifically, the paper is organized as follows: In Sec.
II, using an algorithm [13], we first find the corresponding first junction conditions. After solving them, we construct exact solutions which represent spacetimes of bubbles, and then study the spacetime structure. In Sec. III, our main results are summarized.

II. EXACT SOLUTIONS OF BUBBLES AND WORMHOLES

To study solutions that represent static or non-static bubbles and wormholes, let us begin with the following solution

\[ ds^2 = Fdt^2 + 2Gdtdr - Hdr^2 - R^2d\Omega^2, \tag{1} \]

where \( d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2 \), and

\[
F \equiv \frac{1}{f}(f^2T_{,t}^2 - R_{,r}^2), \quad G \equiv \frac{1}{f}(f^2T_{,t}T_{,r} - R_{,t}R_{,r}), \\
H \equiv \frac{1}{f}(R_{,r}^2 - f^2T_{,r}^2), \quad f \equiv 1 - \frac{2m}{r}, \tag{2}
\]

and \( T, R \) are functions of \( t \) and \( r \) only. The coordinates will be numbered as \( \{x^\mu\} = \{t, r, \theta, \phi\} \), \( (\mu = 0, 1, 2, 3) \) with \(-\infty < t, r < +\infty, 0 \leq \theta \leq \pi, \) and \( 0 \leq \phi \leq 2\pi \). Note that, provided that the functions \( T \) and \( R \) are well defined in terms of \( t \) and \( r \), the above solution is essentially the Schwarzschild solution but written in a different coordinate system. Actually, in terms of \( T \) and \( R \), one can see that it will take the form commonly used. The reason to
adopt the \((t, r)\) coordinates will be seen clearly in the following discussions. Following Ref. 13, we make the ansatz

\[
T = M(\xi - |\psi|) + N(\xi + |\psi|), \quad R = U(\xi - |\psi|) + V(\xi + |\psi|),
\]

where \(M, N, U\), and \(V\) are at least \(C^4\) functions of their indicated arguments in the sense defined in [16], and \(\xi, \psi\) are smooth functions of \(t\) and \(r\). Note that the notations used here are slightly different from the ones used in Ref. 13. It is easy to see that Eq. (3) is well defined, respectively, in the regions \(\Omega^+\) and \(\Omega^-\), where \(\Omega^+ \equiv \{x^\mu : \psi \geq 0\}\) and \(\Omega^- \equiv \{x^\mu : \psi \leq 0\}\). Therefore, the resulting solutions in \(\Omega^+\) and \(\Omega^-\) are locally isometric to the Schwarzschild solution. However, across the hypersurface \(\psi = 0\) the functions \(T\) and \(R\) are only \(C^0\) with respect to \(t\) and \(r\). As a result, the metric coefficients are only \(C^{-1}\). It is these “pathological” coordinate transformations that will lead to new solutions. As a matter of fact, Eq. (3) is not simply coordinate transformations. Technically, it is equivalent to cut and then glue two identical parts of the Schwarzschild solutions together along the hypersurface \(\psi = 0\). Of course, such a “surgery” does not always give us physically meaningful solutions unless some additional conditions are imposed. One of our purposes in the following is to find out these conditions.

Before proceeding, let us first note that the gluing of two Schwarzschild solutions through a bubble has been considered by several authors, see, for
instance, Refs. 7 and 9. However, our solutions are different from theirs at least in two points: First, the first junction conditions are worked out in our case. As a result, the global structure of the spacetime can be studied easily. Second, our bubbles do not necessarily satisfy the “barotropic” equation of state as do in Refs. 7, 9 and others.

Inserting Eq.(3) into Eq.(2), we find that all the metric coefficients can be written in the form

$$Y = Y^+ H(\psi) + Y^- [1 - H(\psi)],$$

(4)

where $H(\psi)$ is the Heaviside function, which is one for $\psi \geq 0$ and zero for $\psi < 0$, and $Y^\pm$ are the quantities defined in $\Omega^\pm$ respectively. Specifically, we have

$$F^+ = \frac{1}{f^+} \{ [f^+ + 2] (\dot{M}_+ + \dot{N}_-) - (\dot{U}_+ + \dot{V}_-) )^2 \} \xi_{,t}^2$$

$$-2 [ f^+ + 2 ( \dot{M}_+^2 - \dot{N}_-^2 ) - (\dot{U}_+^2 - \dot{V}_-^2) ] \xi_{,t} \psi_{,t}$$

$$+ [ f^+ + 2 ( \dot{M}_+^2 - \dot{N}_-^2 ) - (\dot{U}_+^2 - \dot{V}_-^2) ] \psi_{,t}^2 \},$$

$$G^+ = \frac{1}{f^+} \{ [f^+ + 2] (\dot{M}_+ + \dot{N}_-) - (\dot{U}_+ + \dot{V}_-) )^2 \} \xi_{,r} \xi_{,r}$$

$$- [ f^+ + 2 ( \dot{M}_+^2 - \dot{N}_-^2 ) - (\dot{U}_+^2 - \dot{V}_-^2) ] (\xi_{,t} \psi_{,t} + \xi_{,r} \psi_{,r} )$$

$$+ [ f^+ + 2 ( \dot{M}_+^2 - \dot{N}_-^2 ) - (\dot{U}_+^2 - \dot{V}_-^2) ] \psi_{,t} \psi_{,r} \},$$

$$H^+ = \frac{1}{f^+} \{ [ (\dot{U}_+ + \dot{V}_-) )^2 - f^+ + 2 (\dot{M}_+ + \dot{N}_-) )^2 \} \xi_{,r}^2$$

$$-2 [ (\dot{U}_+^2 - \dot{V}_-^2 ) - f^+ + 2 (\dot{M}_+^2 - \dot{N}_-^2 ) ] \xi_{,r} \psi_{,r}$$

5
\[ +[(\dot{U}_+ - \dot{V}_-)² - f^+²(\dot{M}_+ - \dot{N}_-)²)\psi_r²] \}, \]

\[ R^+ = \dot{U}_+ + \dot{V}_-, \tag{5} \]

and the quantities \( F^-, G^-, H^- \) and \( R^- \) can be obtained from the above equations by the following replacements

\[ M_+ \rightarrow M_-, \ N_- \rightarrow N_+, \ U_+ \rightarrow U_-, \ V_- \rightarrow V_+, \ f_+ \rightarrow f_-, \ \psi \rightarrow -\psi, \tag{6} \]

where

\[ M_\pm \equiv M(\xi \mp \psi), \ f^\pm \equiv 1 - \frac{2m}{\tilde{R}^\pm}. \tag{7} \]

An overdot denotes the ordinary differentiation with respect to the indicated argument.

Note that the metric coefficients given by Eqs.(5) and (6) are all continuous across the hypersurface \( \psi = 0 \), except for the ones that are proportional to \( \psi_t, \psi_r \) that change signs when crossing \( \psi = 0 \). Therefore, to have the first junction conditions hold, that is, the metric coefficients must be at least \( C^0 \) [17], these terms need vanish on the surface, i.e.,

\[ \dot{M}² - \dot{N}² = \frac{\dot{U}² - \dot{V}²}{f²}, \quad (\psi = 0). \tag{8} \]

On the other hand, since we are concerned with physical bubbles, we require that the hypersurface \( \psi = 0 \) be time-like, \( \psi,\mu \psi,\nu g^{\mu\nu} > 0 \), which now reads

\[ E(t, r) \equiv F\psi_r² - 2G\psi_r \psi_t - H\psi_t² > 0, \quad (\psi = 0). \tag{9} \]
From the results obtained in Ref. 12, we find that corresponding to the solutions of Eqs. (5) - (9) the energy-momentum tensor (EMT) is given by

\[ T_{\mu\nu} = \tau_{\mu\nu} \delta(\psi), \quad \text{(10)} \]

where \( \delta(\psi) \) is the Dirac delta function, and \( \tau_{\mu\nu} \) the surface EMT of the bubble located on the hypersurface \( \psi = 0 \), and given by

\[
\tau_{\mu\nu} = \sigma u_\mu u_\nu - \tau (\theta_\mu \theta_\nu + \phi_\mu \phi_\nu),
\]

with

\[
\sigma = -\frac{2E}{FH + G^2} \frac{[R_{,\psi}]^-}{R_0(\xi)},
\]

\[
\tau = -\frac{1}{2(FH + G^2)} \left[ \frac{2E}{R_0(\xi)} [R_{,\psi}]^- \right. \\
\left. + (\psi_r^2 [F_{,\psi}]^- - 2\psi_{,r} \psi_r [G_{,\psi}]^- - \psi_r^2 [H_{,\psi}]^-) \right],
\]

\[
u_\mu = E^{-1/2} [(F_{,\psi} - G_{,\psi}) \delta_\mu^t + (G_{,\psi} + H_{,\psi}) \delta_\mu^r],
\]

\[
\theta_\mu = R_0(\xi) \delta_\mu^\theta, \quad \phi_\mu = \sin \theta R_0(\xi) \delta_\mu^\phi,
\]

\[
u_\lambda \nu^\lambda = - \theta_\lambda \theta^\lambda - \phi_\lambda \phi^\lambda = 1,
\]

and

\[
R_0(\xi) \equiv R(\xi, \psi = 0) = U(\xi) + V(\xi). \quad \text{(14)}
\]

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\(^3\) The surface EMT of the wall is usually defined as \([1-11] \) \( S_{\mu\nu} \equiv \int T_{\mu\nu} dn \), where \( n \) is the proper distance in the direction perpendicular to the wall. When \( \xi = t, \psi = \psi(r) \), we have \( dn = [H^{1/2}/\psi_r] d\psi \). Then, we find \( S_{\mu\nu} = [H^{1/2}/\psi_r]_{\psi=0} \tau_{\mu\nu} \). Thus, the surface energy density and tensions of the wall defined here are different from the ones in \([1-11]\) by a factor \([H^{1/2}/\psi_r]_{\psi=0}\).
From Eqs. (11) and (13) we can see that the quantity $\sigma$ represents the surface energy density of the bubble and $\tau$ its tensions in the tangential directions$^4$.

To study the above solutions as a whole is too complicated. Thus, in the following we shall restrict ourselves to the cases where

$$\xi = t, \quad \psi = \psi(r).$$

(15)

Now we need to solve the restraint equation (8). Since the functions $M, N, U$ and $V$ are arbitrary, we can always first choose any functions for three of them and then integrate Eq.(8) to get the other. This freedom is partially due to the arbitrary choice of the coordinates and partially due to the fact that different choice of these functions implies different matching of the two parts of the spacetime in each side of the bubble. In the following, we shall consider the cases with

$$\dot{M}^2 - \dot{N}^2 = \frac{\dot{U}^2 - \dot{V}^2}{f^2} = \mu, \quad (\psi = 0),$$

(16)

where $\mu$ is an arbitrary real constant and must not be confused with the tensor index. Since for $\mu = 0$ and $\mu \neq 0$ we will have physically different solutions, we shall consider them separately.

A. Solutions with $\mu = 0$

$^4$ See Footnote 3.
When $\mu = 0$, Eq.(16) has the solution

$$M = \epsilon_1 N + M_0, \quad U = \epsilon_2 V + R_0, \quad (\epsilon_{1,2} = \pm 1), \quad (17)$$

where $M_0$ and $R_0$ are two integration constants, and $N$ and $V$ are arbitrary functions. It can be shown that in the present case only does the choice $\epsilon_1 = -\epsilon_2 = 1$ give physically meaningful solutions. For this choice, Eq.(12) becomes

$$\sigma = \gamma^{-1} = -\frac{2(R_0 - 2m)}{R_0^2 V(t)}, \quad \gamma \equiv \frac{R_0 - m}{2(R_0 - 2m)}. \quad (18)$$

That is, in the present case the bubbles satisfy the “barotropic” equation of state. Thus, they must belong to the solutions studied in Refs. 7 and 9. On the other hand, Eq.(9) now is equivalent to

$$R_0 > 2m, \quad (19)$$

which means that the bubbles must be greater than the Schwarzschild sphere.

Combining Eqs.(18) and (19) we find

$$\gamma = \frac{1}{2} + \frac{m}{2(R_0 - 2m)} > \frac{1}{2}. \quad (20)$$

Thus, in the present case all the bubbles are gravitationally repulsive [7].

To study further this class of solutions, let us consider the cases with

$$V = -\alpha t^\beta, \quad \psi = \prod_{k=1}^{n} (r - a_k), \quad (21)$$
where \( a_k \geq 0, \alpha, \beta \) are constants. Then, we find that

\[
\sigma \delta(\psi) = \gamma^{-1} \tau \delta(\psi) = \sum_{k=1}^{n} \sigma_k \delta(r - a_k),
\]

where

\[
\sigma = \frac{\sigma_0}{2\alpha \beta} t^{1-\beta}, \quad \sigma_k = \sigma \left\{ \prod_{i \neq k} |a_i - a_k| \right\}^{-1},
\]

and \( \sigma_0 \equiv 4(2\gamma - 1)/[m(4\gamma - 1)^2] \). Therefore, the corresponding solutions actually represent \( n \) bubbles that connect \((n+1)\) regions, each of which is locally isometric to the Schwarzschild solution. Since on each of the bubbles, we have \( R = R_0 \), all the bubbles have the same physical radius.

**Case \( \alpha \):** \( \alpha = \frac{1}{2}, \beta = 1, a_1 = R_0, \) and \( n = 1 \). In this case, we find that the metric is given by

\[
ds^2 = \begin{cases} 
(1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2d\Omega^2, & r \leq R_0, \\
(1 - \frac{2m}{2R_0-r})dt^2 - (1 - \frac{2m}{2R_0-r})^{-1}dr^2 - (2R_0 - r)^2d\Omega^2, & r \geq R_0,
\end{cases}
\]

and that Eqs. (22) and (23) simply yield \( \sigma = \sigma_0 \). Therefore, now the solution represents a spherical static bubble with constant surface energy density and tensions. The solution has two horizons at \( r = 2m, 2(R_0 - m) \) and is singular respectively at \( r = 0, 2R_0 \), which seal off the spacetime and turn it into a compact one (A different interpretation is given in [18]). While its cosmological interest is not clear, the above solution does show that the existence of bubbles can dramatically change the spacetime geometry.
Case $\beta$: $\alpha = \frac{1}{2}, \beta = 1, a_2 > a_1$, and $n = 2$. Then, the solution represents two bubbles that divide the spacetime into three regions, in each of which the metric takes the form

$$ds^2 = (1 - \frac{2m}{R})dt^2 - 4(r - r_m)^2(1 - \frac{2m}{R})^{-1}dr^2 - R^2d\Omega^2,$$

(25)

where

$$R = \begin{cases} R_0 - (a_1 - r)(a_2 - r), & r \leq a_1, \\ R_0 - (r - a_1)(a_2 - r), & a_1 \leq r \leq a_2, \\ R_0 - (r - a_1)(r - a_2), & r \geq a_2, \end{cases}$$

(26)

and $r_m \equiv (a_1 + a_2)/2$. The corresponding surface energy densities are given by

$$\sigma_1 = \sigma_2 = \frac{\sigma_0}{a_2 - a_1}.$$

(27)

It can be shown that in general the spacetime is singular at

$$r_{1,2} \equiv r_m \pm \sqrt{(a_2 - a_1)^2 + 4R_0}, \quad r_{3,4} \equiv r_m \pm \sqrt{(a_2 - a_1)^2 - 4R_0},$$

(28)

and has horizons at

$$r_{5,6} \equiv r_m \pm \sqrt{(a_2 - a_1)^2 + 4(R_0 - 2m)}, \quad r_{7,8} \equiv r_m \pm \sqrt{(a_2 - a_1)^2 - 4(R_0 - 2m)}.$$

(29)

Note that in general the maximum number of singular points or event horizons is $n(n + 1)$, where $n$ is the number of bubbles.
Case $\gamma$): $\alpha = -\frac{1}{2}, \beta = 1, a_1 = R_0$, and $n = 1$. Then, we find that
\[
 ds^2 = \begin{cases} 
 (1 - \frac{2m}{2R_0 - r})dt^2 - (1 - \frac{2m}{2R_0 - r})^{-1}dr^2 - (2R_0 - r)^2d\Omega^2, & r \leq R_0, \\
 (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2d\Omega^2, & r \geq R_0.
\end{cases}
\] (30)

Comparing Eq.(24) with the above equation we find that these two cases are related each other by exchanging the form of the metric inside and outside of the bubble. Because of this exchanging, one can show that in the present case the spacetime has no singularities and horizons. Also, as $r \to \pm \infty$, the spacetime becomes asymptotically flat. Thus, a remote observer who moves along a time-like radial geodesic toward the bubble will pass through it within finite time and soon finds himself in another asymptotically flat region. Therefore, in the present case, the bubble acts like the throat of a wormhole [19]. As shown by Morris and Thorne in [19], the price to construct such a static wormhole is to violate the weak energy condition (WEC). In the present model this particular feature is manifested by the fact that the surface energy density of the bubble is negative, $\sigma = -\sigma_0 < 0$. It should be noted that the above solution was first studied in [20] in a different manner.

Case $\delta$): $\alpha = -\frac{1}{2}, \beta = 1, a_2 > a_1$, and $n = 2$. Clearly, this corresponds to Case $\beta$), in which there are two bubbles with radii $a_1$ and $a_2$, respectively.
The metric takes the same form as Eq.(25) but with

\[
R = \begin{cases} 
R_0 + (a_1 - r)(a_2 - r), & r \leq a_1, \\
R_0 + (r - a_1)(a_2 - r), & a_1 \leq r \leq a_2, \\
R_0 + (r - a_1)(r - a_2), & r \geq a_2.
\end{cases}
\]

(31)

The above equation together with Eq.(19) show that the whole spacetime is free of any singularities and horizons. The region between the two bubbles is isometric to a compact region of the Schwarzschild spacetime, and the ones in outside of the two bubbles are asymptotically flat. Thus, this solution also represents a wormhole with a finite thickness of throat, \( \Delta l = a_2 - a_1 \), and the “exotic” matter is concentrated at the two mouths of the throat, \( r = a_1 \) and \( r = a_2 \), with

\[
\sigma_1 = \sigma_2 = -\frac{\sigma_0}{a_2 - a_1}.
\]

(32)

B. Solutions with \( \mu \neq 0 \)

Part of the work to be presented in this subsection has been briefly reported in [21, 22]. In the following, we shall provide a systematic study. When \( \mu \neq 0 \), let us first assume

\[
M = AN + B, \quad U = aV + b,
\]

(33)
where $A, B, a$ and $b$ are arbitrary constants. Then, the integration of Eq. (16) yields

$$N(t) = \varepsilon_1 \mu_1 t + N_0,$$

$$\varepsilon_2 \mu_2 t = V(t) + \frac{2m}{1 + a} \ln[R_0(t) - 2m] + V_0,$$  \hspace{1cm} (34)

with $N_0$ and $V_0$ being integration constants,

$$\mu_1 \equiv \left(\frac{\mu}{A^2 - 1}\right)^{1/2}, \quad \mu_2 \equiv \left(\frac{\mu}{a^2 - 1}\right)^{1/2}, \quad \varepsilon_1, \varepsilon_2 = \pm 1, \hspace{1cm} (35)$$

and

$$R_0(t) \equiv R(t, \psi = 0) = (1 + a)V(t) + b. \hspace{1cm} (36)$$

Once Eq. (4) is solved, the metric coefficients of (2) are in turn fixed in terms of $t$ and $\psi$. Inserting Eqs.(33) and (34) into Eq.(12) we find

$$\sigma = \frac{\sigma_0}{R_0(t)}, \quad \tau = \frac{\sigma_0[R_0(t) - m]}{2R_0(t)[R_0(t) - 2m]}, \hspace{1cm} (37)$$

where

$$\sigma_0 \equiv \frac{2\varepsilon_2(1 + A)}{\mu_2(a - A)} \hspace{1cm} (38)$$

is a constant. In the following, we shall choose the free parameters such that $\sigma_0 > 0$. On the other hand, Eq.(9) now becomes

$$\mu(a - A)(1 + a)(1 + A) > 0. \hspace{1cm} (39)$$

Thus, provided Eq.(39) holds, the above solutions represent a spherical bubble connecting two regions, each of which is locally isometric to part of
the Schwarzschild spacetime. Since the radius of the bubble $R_0(t)$ is time-
dependent, the bubble in the present case is not static. On the other hand, 
Eqs. (33) and (34) imply that

$$R_0(t) > 2m.$$  \hspace{1cm} (40)

That is, the bubble is always greater than the Schwarzschild sphere.

From Eq. (37) it is easy to show that the bubbles in this case do not satisfy 
the “barotropic” equation of state. Thus, they do not fall into the solutions 
studied in Refs. 7 and 9. From the same equation, we also find that

$$\sigma - \tau = \frac{\sigma_0 [R_0(t) - 3m]}{2[R_0(t) - 2m]} \geq 0, \quad \text{for} \quad R_0(t) \geq 3m.$$  \hspace{1cm} (41)

Therefore, as long as the radius of the bubble is greater than or equal to 
3$m$, it will satisfy both of the weak and dominant energy conditions [16], 
although not the strong one, since for any $R_0(t)$ we always have

$$\sigma - 2\tau = -\frac{\sigma_0 m}{R_0(t) - 2m} < 0.$$  \hspace{1cm} (42)

That is, the “Newtonian” mass of the bubble is negative, and the bubble is 
always gravitational repulsive [7].

The dynamics of the bubble can be studied using the kinematical quanti-
ties

$$\frac{dR_0(t)}{ds} = \beta \left[ \frac{R_0(t) - 2m}{R_0(t)} \right]^{1/2}, \quad \frac{d^2 R_0(t)}{ds^2} = \frac{m\beta^2}{R_0(t)},$$  \hspace{1cm} (43)
where $s$ denotes the proper time measured by observers who are at rest relative to the bubble, and

$$\beta \equiv \varepsilon_2 \mu_2 (1 + a) \left[ \frac{(1 - a)(1 - A)}{2 \mu(a - A)} \right]^{1/2}. \tag{44}$$

The above equations show that in the present case the bubble either expands ($dR_0(t)/ds > 0$) or collapses ($dR_0(t)/ds < 0$), depending on the choice of the free parameters. From Eqs. (5) and (34), we also have

$$\frac{\partial R(t, |\psi|)}{\partial |\psi|} \bigg|_{\psi=0} = \varepsilon_2 (1 - a) \mu_2 f(R_0). \tag{45}$$

That is, the topology of the spacetime in the neighborhood of the bubble depends on the choice of the parameters $\varepsilon_2$ and $a$.

In parallel to the last subsection, now let us turn to consider the following representative cases:

(a) $\varepsilon_2 = -1$, $0 < a < 1$, $0 < A < 1$, $A > a$, $b = 2m$, $V_0 = 0$, and $\psi = r$. Then, we find

$$Exp(-\mu_2 t) = [(1 + a)V(t)]^{2m} \exp[V(t)],$$

$$R(t, |r|) = aV(t - |r|) + V(t + |r|) + 2m,$$

$$R_0(t) = (1 + a)V(t) + 2m. \tag{46}$$

From the above equation, it is easy to show that

$$R(t, |r|) = \begin{cases} +\infty, & \text{as } r \to \pm\infty \text{ at a moment } t = t_1, \\ \geq 2m, & \text{for any } t \text{ and } r. \end{cases} \tag{47}$$
where $R(t, |r|) = 2m$ only when $t = +\infty$. Thus, in the present case the bubble acts as the throat of a wormhole that connects two asymptotically flat Schwarzschild universes. However, this wormhole is distinguishable from all the known ones in the sense: First, it satisfies both of the weak and dominant energy conditions. Note that dynamic wormholes that satisfy the WEC have been recently studied in [23, 24] in the framework of Einstein theory, and in [25] in the framework of Brans-Dicke theory. Second, the physical radial coordinate $R$ is initially decreasing when away from the bubble, as we can see from Eq.(45), which yields $[\partial R(t, |r|)/\partial |r|]_{r=0} = -(1-a)\mu_2 f < 0$. But, as it decreases to a minimum, say, $R_{\text{min}}$, which is always greater than or equal to $2m$, it starts to increase. And as $|r| \to +\infty$, we have $R(t, |r|) \to +\infty$. Dynamic wormholes made of two asymptotically flat Schwarzschild spacetimes were also studied by Visser in [20]. Assuming that $R$ is always an increasing function of $|r|$, i.e., $[\partial R(t, |r|)/\partial |r|]_{r=0} > 0$ for any $r$, Visser found that all the wormholes necessarily violate the WEC.

It should be noted that in most of the previous studies of bubbles, the spacetime topology was classified by the signs of the angular component, $K_\theta^\theta$, of the extrinsic curvature tensor of the bubble, where $K_\theta^\theta$ is given by [26]

$$K_\theta^\theta = \frac{1}{R_0(t)} \frac{\partial R}{\partial N} \bigg|_{r=0},$$

(48)

$N$ denotes the Gaussian normal coordinate to the bubble. Clearly, this is
correct only for the static case. When the spacetime is time-dependent, the situation is different. It determines the spacetime topology only in the neighborhood of the bubble, and the global topology of the spacetime could be quite different from the local one. This fact has been noticed quite recently in [26] and the above solutions provide another example. Regarding to the latter, one can see that all the statements concerning the global structure of the wall spacetimes given in the previous literature should be taken with some cautions.

On the other hand, from Eq.(46) we find that

\[ R_0(t) = \begin{cases} +\infty, & \text{as } t \to -\infty, \\ 2m, & \text{as } t \to +\infty. \end{cases} \]  

(49)

That is, the corresponding solutions represent a collapsing wormhole. The wormhole throat starts to collapse at \( R_0(-\infty) = \infty \) and ends at \( R_0(+\infty) = 2m \). On the other hand, from Eq. (43) we find that

\[ \Delta s = \frac{1}{|\beta|} \int_{2m}^{\infty} \frac{xdx}{x-2m} = \infty. \]  

(50)

Thus, to complete the process of collapse, the throat will take an infinitely long proper time. Consequently, a space adventurer will have enough time to pass through the throat of the wormhole from one asymptotically flat region to the other before the radius of the throat shrinks to 2m, where the event horizon usually appearing in the Schwarzschild solution is developed.
To further study the above solutions, let us consider the embedding of them in the three dimensional Euclidean space

\[ dl^2 = dZ^2 + dR^2 + R^2 d\phi. \]  

(51)

Because of the reflection symmetry of the spacetime, it is sufficient to consider the problem only in the region where \( r \geq 0 \). Then, following Ref. [19], we find that

\[
\left( \frac{dZ}{dr} \right)^2 = (a\dot{V}_+ - \dot{V}_-)^2 \left\{ (a\dot{V}_+ - \dot{V}_-) \left[ 1 - f(R) \right] - \mu_1^2(1 - A)^2 f(R) \right\}. 
\]

(52)

One can show that the right side of the above equation changes signs from point to point. That is, in the present case our solutions cannot be embedded in a 3-dimensional Euclidean space and pictured as an ordinary Euclidean curved surface. Recall that not any two dimensional metric can be embedded into a three dimensional Euclidean space. Classical examples are the Moebius strip and the Gauss-Bólyai-Lobachevski metric

\[ ds^2 = (1 + r^2)^{-1} dr^2 + r^2 d\phi^2 \]

[27].

(b) \( \varepsilon_2 = -1, \ 0 < a < 1, \ 0 < A < 1, \ A > a, \ b = 2m, \ V_0 = 0, \) and \( \psi = (r - a_2)(r - a_1), \) where \( a_2 > a_1 > 0. \) Clearly, in this case we have two bubbles, located respectively on the hypersurfaces \( r = a_1 \) and \( r = a_2. \) The physical radii of these two bubbles are the same and are give by Eqs. (34) and (36). Each of the two bubbles collapses in the same way as the single bubble given in Case (a). In particular, they collapse from \( R_0(-\infty) = \infty \) to
$R_0(+\infty) = 2m$ by taking an infinitely long proper time. The two bubbles connect three regions: $-\infty < r < a_1, a_1 < r < a_2$, and $a_2 < r < +\infty$, each of which is locally isometric to the Schwarzschild spacetime. In the region in between the two bubbles ($a_1 < r < a_2$), the spacetime is compact with the physical radius always being greater than or equal to $2m$, where equality holds only when $t \to +\infty$. The two regions outside of the bubbles are reflection symmetric with respect to the hypersurface $r = (a_2 + a_1)/2$. They are asymptotically flat and free of any singularities and horizons at any finite time. Therefore, they also represent wormholes that satisfy the weak and dominant energy conditions. The only difference between solutions given in this case and the ones given in the last is that the thickness of the throat of the wormhole now is different from zero and equal to $\triangle l = a_2 - a_1$, and that the two bubbles act as two mouths of the throat.

(c) $\varepsilon_2 = 1, \ a > 1, \ A > 1, \ a > A, \ b = 2m, V_0 = 0$, and $\psi = r$. In this case, we have

\[
\begin{align*}
Exp(\mu_2 t) &= [(1 + a)V(t)]^{\frac{2m}{\mu_2}} Exp[V(t)], \\
R(t, |r|) &= aV(t - |r|) + V(t + |r|) + 2m, \\
R_0(t) &= (1 + a)V(t) + 2m. \quad (53)
\end{align*}
\]

Similar to Case (a), one can show that these solutions represent an expanding bubble, which connects two asymptotically flat universes and satisfy the weak
and dominant energy conditions as long as $R_0(t) \geq 3m$. The bubble expands from $R_0(-\infty) = 2m$ to $R_0(+\infty) = +\infty$ by taking an infinite long proper time.

From Eq. (37) we can see that when $R_0(t)$ is approaching $2m$, the tensions in the tangent directions of the bubble tend to infinite. Thus, in the course of the collapse of the bubble, as described in Case (a), it is not difficult to imagine that the bubble will explode due to the enormous tensions before its radius really shrinks to $2m$. By properly arranging the parameters, the explosion could happen before the trapped surface is developed. After the explosion, the material may recompose and form another wormhole, the later evolution of which follows more or less the same process as described by the solutions given in the present case.

(d) $\varepsilon_2 = 1, \ a > 1, \ A > 1, \ a > A, \ b = 2m, V_0 = 0$, and $\psi = (r-a_2)(r-a_1)$, where $a_2 > a_1 > 0$. Clearly, this is the time-reversed process of Case (b), and corresponds to two expanding bubbles. All the properties of this class of solutions can be obtained by the replacement $t$ by $-t$ in the solutions of Case (b).

III. CONCLUSIONS AND DISCUSSIONS

In this paper, by solving the first junction conditions, we have constructed
some analytic solutions to the Einstein field equations, which represent multiple bubbles connecting regions that are locally isometric to the Schwarzschild solution. The obtained solutions can be classified into two categories, one represents static bubbles and the other dynamic bubbles. For the static bubbles, provided that their surface energy densities are positive, the spacetimes are always compact and closed by curvature singularities. For the dynamic bubbles, the resulted spacetimes represent wormholes. However, these wormholes are distinguishable from the known ones in the sense that they satisfy both the weak and dominant energy conditions, but violate the strong one. Therefore, in contrast to the static ones [19], dynamic wormholes can be built without violating the WEC. Note that the violation of the strong energy condition nowadays does not seem to be a very serious drawback. Recall that cosmic bubbles and domain walls formed in the early Universe do not satisfy this condition either.

The recent studies of wormholes usually fall into two different directions. One is concerned with the energy conditions [28], and the other is concerned with the vacuum polarization due to the quantum effects [29 - 31]. To the first, one can see that even it can be shown that the WEC is preserved at the quantum level for the generic cases, the existence of wormholes can not be ruled out. As shown in this paper, they can exist even in the classical level without violating the WEC. To the second, Hawking [29, 30] argued
that, when the vacuum polarization effects are taken into account, one might finally show that such a building of a traversable wormhole is impossible, although Thorne and others [31] seem to defend the opposite opinion. The considerations of the latter now are under investigation, and the results will be discussed somewhere else.

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