THE GENERALIZED NEWTON–KANTOROVICH METHOD FOR EQUATIONS WITH NONDIFFERENTIABLE OPERATORS

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Abstract: The article deals with the generalized Newton–Kantorovich method for solving operator equations with nondifferentiable operators in Banach spaces. The convergence theorem is proved by means of majorant scalar equations.

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1 Introduction

Let $X$ and $Y$ be Banach spaces, $D$ is a convex subset of $X$, $f$ and $g$ are nonlinear operators, defined on $D$ and taking values in $Y$, where $f$ is differentiable at every interior point of $D$, $g$ is nondifferentiable. One of the most effective iterative methods for solving operator equation of the form

$$f(x) + g(x) = 0$$

(1)

is the generalized Newton–Kantorovich method with successive approximations

$$x_{n+1} = x_n - [f'(x_n)]^{-1}(f(x_n) + g(x_n)) \quad (n = 0, 1, \ldots),$$

(2)

where $x_0 \in D$ is given.

A thorough convergence analysis of the sequence (2) was carried in [1] by means of the approach based on the application of majorant scalar equations and originating from Kantorovich’s investigations ([2], chapter XVIII). However, the hypotheses given there are tediously formulated and difficult to verify. For this reason in [3] was proposed a more flexible approach for solving the equation (1) under the following hypotheses on the operators $f$ and $g$:

$$\|f'(x'') - f'(x')\| \leq \varphi(t)\|x'' - x'\|, \quad \forall x', x'' \in \overline{B(x_0,t)} \subseteq D,$$

(3)

$$\|g(x'') - g(x')\| \leq \psi(t)\|x'' - x'\|, \quad \forall x', x'' \in \overline{B(x_0,t)} \subseteq D,$$

(4)

where $\varphi(t)$ and $\psi(t)$ are nondecreasing functions of the nonnegative argument. If $\varphi(t)$ and $\psi(t)$ are constants, the conditions (3) and (4) are reduced to the classic Lipschitz conditions.

In the case when $g = 0$ the most precise error estimates for the process (2) were obtained in [4, 5] under a new smoothness assumption imposed on the operator $f$ called regular smoothness. In this paper we generalize the main result from [5] to equations of the form (1) under the hypotheses that the operator $f$ is regularly smooth on $D$ and the operator $g$ satisfies (4). The convergence theorem for the process (2) is proved by means of majorant equations.
2 Regular smoothness

Let $\mathcal{N}$ denote the class of continuous strictly increasing functions $\omega : [0, \infty) \to [0, \infty)$ that are concave and vanishing at zero: $\omega(0) = 0$. Assume without loss of generality that $f'(x_0) = I$.

Denote by $h(f)$ the quantity $\inf_{x \in D} \|f'(x)\|$. Given an $\omega \in \mathcal{N}$, we say in accordance with \[5\] that $f$ is $\omega$-regularly smooth on $D$ (or, equivalently, that $\omega$ is a regular smoothness modulus of $f$ on $D$), if there exists $h \in [0, h(f)]$ such that the inequality

$$\omega^{-1}(h_f(x', x'') + \|f'(x'') - f'(x')\|) - \omega^{-1}(h_f(x', x'')) \leq \|x'' - x'\|,$$

where

$$h_f(x', x'') = \min\{\|f'(x')\|, \|f'(x'')\|\} - h,$$

holds for all $x', x'' \in D$.

The operator $f$ is called regularly smooth on $D$, if it is $\omega$-regularly smooth on $D$ for some $\omega \in \mathcal{N}$.

The condition (5) may be written in the form

$$\|f'(x'') - f'(x')\| \leq \omega \left( \omega^{-1}(h_f(x', x'')) + \|x'' - x'\| \right) - h_f(x', x''),$$

or

$$\|f'(x'') - f'(x')\| \leq \omega(\xi(x', x'') + \|x'' - x'\|) - \omega(\xi(x', x'')),$$

where $\xi(x', x'') = \omega^{-1}(h_f(x', x''))$.

It should be remarked that in [4] a more restrictive definition of regular smoothness was used, which coincides with the definition in [5] when $h = 0$. In fact, if for some $h = h_0$ and some $x', x'' \in D$ the inequality (6) holds, then it will be true for all $h > h_0$ with the same $x', x'' \in D$ because of the difference $\omega(t + \tau) - \omega(t)$ does not increase in $t$ for each fixed $\tau > 0$.

Lemma 1 [5] If the operator $f$ is $\omega$-regularly smooth on $D$ with some $h$, then

$$\left| \omega^{-1}(\|f'(x'')\| - h) - \omega^{-1}(\|f'(x')\| - h) \right| \leq \|x'' - x'\|$$

for all $x', x'' \in D$.

It follows from the definition of $\xi$ and Lemma [4] that

$$\xi(x', x'') \geq \omega^{-1}(\|f'(x')\| - h) - \|x'' - x'\|$$

for all $x', x'' \in D$.

3 Some preliminary results

The proof of the main theorem is based on several preliminary propositions.

Let $\omega \in \mathcal{N}$, $\Omega(t) = \int_0^t \omega(\tau) \, d\tau$, $\Psi(t) = \int_0^t \psi(\tau) \, d\tau$, $\chi = \omega^{-1}(1 - h)$, $a$ is a positive number such that

$$\|f(x_0) + g(x_0)\| \leq a$$

and

$$\Phi_h(t) = a - \Omega(\chi) + \Omega(\chi - t) - th, \quad t \in [0, \chi].$$
Let us define the numerical sequence \( \{t_n\} \) by the following recurrence formula:

\[
t_{n+1} = t_n + \frac{a - \Omega(\chi) + \Omega(\chi - t_n) - t_nh + \Psi(t_n)}{h + \omega(\chi - t_n)}, \tag{8}
\]

\( n = 0, 1, \ldots; \ t_0 = 0. \)

In terms of the function

\[
W(t) = \Phi_h(t) + \Psi(t)
\]

the relation (8) may be rewritten as follows:

\[
t_{n+1} = t_n - \frac{W(t_n)}{\Phi'_h(t_n)},
\]

\( n = 0, 1, \ldots; \ t_0 = 0. \)

**Lemma 2** Suppose that the function (9) has a unique zero \( t_* \) in the interval \([0, \chi]\) and

\[
a < \Omega(\chi) + h \cdot \chi - \Psi(\chi). \tag{10}
\]

Then the sequence (8) is defined for all \( n \), monotonically increases and converges to \( t_* \).

**Proof.** The function \( W \) is positive on the interval \([0, t_*]\), since \( t_* \) is a unique zero of the equation \( W(t) = 0 \), \( W(0) = a > 0 \) and \( W \) is continuous on \([0, \chi]\). Moreover, the function \( \Phi'_h(t) = -\omega(\chi - t) - h \) is negative on the interval \([0, t_*]\). Hence the function

\[
u(t) = -\frac{W(t)}{\Phi'_h(t)}
\]

is positive on \([0, t_*]\).

Let us show that the function \( t + u(t) \) is nondecreasing on \([0, t_*]\). In fact,

\[
(t + u(t))' = 1 + u'(t) = 1 + \left( \frac{\Phi_h(t) + \Psi(t)}{\omega(\chi - t) + h} \right)' = \\
= 1 + \frac{\Phi'_h(t) + \Psi'(t) \cdot (\omega(\chi - t) + h) + (\Phi_h(t) + \Psi(t)) \cdot \omega'(\chi - t)}{(\omega(\chi - t) + h)^2} = \\
= \frac{\Psi'(t) \cdot (\omega(\chi - t) + h) + (\Phi_h(t) + \Psi(t)) \cdot \omega'(\chi - t)}{(\omega(\chi - t) + h)^2} \geq 0
\]

on \([0, t_*]\). This implies that the sequence \( \{t_n\} \) monotonically increases and \( t_{n+1} = t_n + u(t_n) \leq t_* + u(t_*) = t_* \) for \( t_n \leq t_* \). Consequently, the sequence \( \{t_n\} \) converges to \( t_* \in [0, t_*] \) and \( t_{n+1} = t_* + u(t_n) \), hence \( W(t_n) = 0 \). Since \( t_* \) is a unique zero of \( W \) in the interval \([0, \chi]\), it follows that \( t_{**} = t_* \).

The sequence \( \{t_n\} \) is defined for all \( n \). In fact, it is clear from (10) that \( W(\chi) < 0 < a = W(0) \) and hence there exists \( \theta \in (0, \chi) \) such that \( W(\theta) = 0 \). Consequently, \( \theta = t_* = \lim_{n \to \infty} t_n \) and \( t_n \leq \theta < \chi \) for all \( n = 0, 1, \ldots \). Because of the monotonicity of \( \omega \) the inequality \( \omega(\chi - t_n) > 0 \) is true for all \( n = 0, 1, \ldots \). This completes the proof of Lemma 2.

**Lemma 3** Let the operator \( f \) be \( \omega \)-regularly smooth on \( D \) with some \( h \), the operator \( g \) satisfies (4), the function (9) has a unique zero \( t_* \) in the interval \([0, \chi]\) and the closed ball \( B(x_0, t_*^*) \) is contained in \( D \). Then the equation (11) has a unique solution \( x_* \) in the ball \( B(x_0, t_*^*) \).
Proof. Let us prove the existence of a solution in the ball $B(x_0, t_*)$. Consider the sequence

$$u_{n+1} = Du_n \quad (n = 0, 1, \ldots; u_0 = x_0),$$

where $D = I - \left[f'(x_0)\right]^{-1}(f + g) = I - (f + g)$, and the numerical sequence

$$\rho_{n+1} = d(\rho_n) \quad (n = 0, 1, \ldots; \rho_0 = 0),$$

where $d(t) = t + W(t)$. Since

$$d'(t) = 1 + W'(t) = 1 + \Phi'_h(t) + \Psi'(t) =
= 1 - h - \omega(\chi - t) + \psi(t) = \omega(\chi) - \omega(\chi - t) + \psi(t) \geq 0$$

on the interval $[0, \chi]$, the function $d$ is monotonically increasing on $[0, \chi]$. Consequently, by the induction hypothesis the inequality $\rho_n \leq t_*$

holds. In fact, for $n = 0$ the inequality (11) is obvious: $\rho_0 = 0 \leq t_*$. Suppose that (11) holds for all $n \leq k$. Then from $\rho_k \leq t_*$ because of the monotonicity of $d$ we obtain $d(\rho_k) \leq d(t_*)$, that is $\rho_{k+1} \leq t_*$. Consequently, the induction hypothesis the inequality (11) is true for all $n$.

Let us prove by induction that the sequence $\{\rho_n\}$ is monotone. Clearly $0 = \rho_0 \leq \rho_1 = a$. Suppose that $\rho_k \leq \rho_{k+1}$. Then $\rho_{k+1} = d(\rho_k) \leq d(\rho_{k+1}) = \rho_{k+2}$.

Thus the sequence $\{\rho_n\}$ is monotonically increasing and bounded from above. Consequently, it converges to some $\hat{\rho} \in [0, t_*]$. By letting $n \to \infty$ in $\rho_{n+1} = \rho_n + W(\rho_n)$ we obtain $W(\hat{\rho}) = 0$ and $\hat{\rho} = t_*$.

Let us show that for all $n = 0, 1, \ldots$ the inequality

$$\|u_{n+1} - u_n\| \leq \rho_{n+1} - \rho_n \quad (12)$$

holds.

For $n = 0$ the inequality (12) is obvious:

$$\|u_1 - u_0\| = \|x_0 - (f(x_0) + g(x_0)) - x_0\| = \|f(x_0) + g(x_0)\| \leq a = W(0) = \rho_1 - \rho_0.$$

Suppose that (12) holds for all $n < k$. Then

$$\|u_{k+1} - u_k\| = \|Du_k - Du_{k-1}\| = \|u_k - u_{k-1} - (f(u_k) - f(u_{k-1})) - (g(u_k) - g(u_{k-1}))\| \leq$$

$$\leq \|u_k - u_{k-1} - (f(u_k) - f(u_{k-1}))\| + \|g(u_k) - g(u_{k-1})\| \leq$$

$$\leq \int_0^1 \|f'(u_t) - f'(x_0)\| \|u_k - u_{k-1}\| \, dt + \|g(u_k) - g(u_{k-1})\| \leq$$

$$\leq \int_0^1 (\omega(\xi(x_0, u_t) + \|u_t - x_0\|) - \omega(\xi(x_0, u_t))) \|u_k - u_{k-1}\| \, dt + \|g(u_k) - g(u_{k-1})\|,$$

where $u_t = u_{k-1} + t(u_k - u_{k-1})$, $0 \leq t \leq 1$. 

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By the inequality (7) we have

\[ \xi(x_0, u_t) \geq \omega^{-1}(\|f'(x_0)\| - h) - \|u_t - x_0\| = \chi - \|u_t - x_0\|. \]

By the induction hypothesis

\[ \|u_k - x_0\| = \|u_k - u_0\| \leq \sum_{j=1}^{k} \|u_j - u_{j-1}\| \leq \sum_{j=1}^{k} (\rho_j - \rho_{j-1}) = \rho_k. \]

Consequently,

\[ \|u_t - x_0\| = \|(1 - t)(u_{k-1} - u_0) + t(u_k - u_0)\| \leq (1 - t)\|u_{k-1} - u_0\| + t\|u_k - u_0\| \leq \]

\[ \leq (1 - t)\rho_{k-1} + t\rho_k. \]

From (14) and Proposition 1 in [3] it follows that

\[ \|g(x'') - g(x')\| \leq \Psi(t + \|x'' - x'\|) - \Psi(t) \quad \forall x', x'' \in \overline{B(x_0, t)} \subseteq D. \quad (13) \]

Because of concavity of \( \omega \) and (13) we have

\[ \|u_{k+1} - u_k\| \leq \int_0^1 (\omega(\chi - \|u_t - x_0\| + \|u_t - x_0\|) - \omega(\chi - \|u_t - x_0\|))\|u_k - u_{k-1}\| \ dt + \]

\[ + \Psi(\rho_{k-1} + \|u_k - u_{k-1}\|) - \Psi(\rho_{k-1}) \leq \]

\[ \leq \int_0^1 (\omega(\chi - \|u_t - x_0\|)) (\rho_k - \rho_{k-1}) \ dt + \Psi(\rho_k) - \Psi(\rho_{k-1}) \leq \]

\[ \leq \int_0^1 (\omega(\chi - \omega(\chi - ((1 - t)\rho_{k-1} + t\rho_k)))(\rho_k - \rho_{k-1}) \ dt + \Psi(\rho_k) - \Psi(\rho_{k-1}) = \]

\[ = \int_0^1 (1 + \Phi_h'((1 - t)\rho_{k-1} + t\rho_k))(\rho_k - \rho_{k-1}) \ dt + \Psi(\rho_k) - \Psi(\rho_{k-1}) = \]

\[ = \int_{\rho_{k-1}}^{\rho_k} (1 + \Phi_h'(\theta)) \ d\theta + \Psi(\rho_k) - \Psi(\rho_{k-1}) = \]

\[ = \rho_k - \rho_{k-1} + \Phi_h(\rho_k) - \Phi_h(\rho_{k-1}) + \Psi(\rho_k) - \Psi(\rho_{k-1}) = d(\rho_k) - d(\rho_{k-1}) = \rho_{k+1} - \rho_k. \]

Thus the inequality (12) holds for \( n = k \).

It follows from (12) that for \( m > n \)

\[ \|u_m - u_n\| \leq \|u_m - u_{m-1}\| + \ldots + \|u_{n+1} - u_n\| \leq \rho_m - \rho_{m-1} + \ldots + \rho_{n+1} - \rho_n = \rho_m - \rho_n. \]

Hence for all \( m \) and \( n \)

\[ \|u_m - u_n\| \leq |\rho_m - \rho_n|. \quad (14) \]
Since the sequence \( \{\rho_n\} \) converges to \( t_* \), it follows from (14) that the sequence \( \{u_n\} \) also converges to some \( x_* \). Further,

\[
\|u_n - u_0\| \leq \rho_n \leq t_* \quad (n = 0, 1, \ldots)
\]

and, consequently, all \( u_n \) with \( x_* \) belong to the ball \( \overline{B(x_0, t_*)} \). By letting \( n \to \infty \) in \( u_{n+1} = Du_n \) we obtain that \( x_* = D(x_*) \) or \( f(x_*) + g(x_*) = 0 \). Thus \( x_* \) is a solution of the equation (1) in the ball \( \overline{B(x_0, t_*)} \).

To prove the uniqueness of the solution \( x_* \) in the ball \( \overline{B(x_0, t_*)} \) consider the second solution \( x_{**} \in B(x_0, t_*) \) of (1) and show that for all \( n = 0, 1, \ldots \) the inequality

\[
\|x_{**} - u_n\| \leq t_* - \rho_n
\]

holds.

For \( n = 0 \) the inequality (15) is obvious:

\[
\|x_{**} - x_0\| \leq t_* - \rho_0 = t_*.
\]

Suppose that (15) holds for all \( n \leq k \). Then

\[
\|x_{**} - u_{k+1}\| = \|x_{**} - Du_k\| = \|x_{**} - u_k + f(u_k) + g(u_k)\| =
\]

\[
= \|f(u_k) - f(x_{**}) - (u_k - x_{**}) + g(u_k) - g(x_{**})\| \leq
\]

\[
\leq \|f(u_k) - f(x_{**}) - f'(x_0)(u_k - x_{**})\| + \|g(u_k) - g(x_{**})\| \leq
\]

\[
\leq \int_0^1 \|f'(\tilde{u}_t) - f'(x_0)\| \|u_k - x_{**}\| dt + \|g(u_k) - g(x_{**})\| \leq
\]

\[
\leq \int_0^1 (\omega(\xi(x_0, \tilde{u}_t) + \|\tilde{u}_t - x_0\|) - \omega(\xi(x_0, \tilde{u}_t))) \|u_k - x_{**}\| dt + \|g(u_k) - g(x_{**})\|
\]

where \( \tilde{u}_t = x_{**} + t(u_k - x_{**}), \ 0 \leq t \leq 1 \).

By the inequality (7) we have

\[
\xi(x_0, \tilde{u}_t) \geq \omega^{-1}(\|f'(x_0)\| - h) - \|\tilde{u}_t - x_0\| = \chi - \|\tilde{u}_t - x_0\|.
\]

Further,

\[
\|\tilde{u}_t - x_0\| = \|(1 - t)(x_{**} - x_0) + t(u_k - x_0)\| \leq (1 - t)\|x_{**} - x_0\| + t\|u_k - x_0\| \leq
\]

\[
\leq (1 - t)t_* + t\rho_k.
\]

Because of concavity of \( \omega \), the inequality (13) and the induction hypothesis we have

\[
\|x_{**} - u_{k+1}\| \leq
\]

\[
\leq \int_0^1 (\omega(\chi - \|\tilde{u}_t - x_0\| + \|\tilde{u}_t - x_0\|) - \omega(\chi - \|\tilde{u}_t - x_0\|)) \|u_k - x_{**}\| dt + \|g(u_k) - g(x_{**})\| \leq
\]

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\[ \leq \int_{0}^{1} (\omega(\chi) - \omega(\chi - \|\bar{a}_{n} - x_{0}\|))(t_{*} - \rho_{k}) \, dt + \Psi(\rho_{k} + \|u_{k} - x_{**}\|) - \Psi(\rho_{k}) \leq \]

\[ \leq \int_{0}^{1} (\omega(\chi) - \omega(\chi - ((1-t)t_{*} + t\rho_{k}))(t_{*} - \rho_{k}) \, dt + \Psi(t_{*}) - \Psi(\rho_{k}) = \]

\[ = \int_{0}^{1} (1 + \Phi_{h}((1-t)t_{*} + t\rho_{k}))(t_{*} - \rho_{k}) \, dt + \Psi(t_{*}) - \Psi(\rho_{k}) = \]

\[ = \int_{\rho_{k}}^{t_{*}} (1 + \Phi_{h}^{'}(\theta)) \, d\theta + \Psi(t_{*}) - \Psi(\rho_{k}) = \]

\[ = t_{*} - \rho_{k} + \Phi_{h}(t_{*}) - \Phi_{h}(\rho_{k}) + \Psi(t_{*}) - \Psi(\rho_{k}) = d(t_{*}) - d(\rho_{k}) = t_{*} - \rho_{k+1}. \]

Hence (15) holds for \( n = k + 1. \)

By letting \( n \to \infty \) in (15) we obtain that

\[ \|x_{**} - x_{*}\| \leq t_{*} - t_{*} = 0 \]

and hence \( x_{**} = x_{*} \). This completes the proof of Lemma 3.

Let us denote for all \( n = 1, 2, \ldots \)

\[ r(x_{n-1}, x_{n}) = \|f(x_{n}) - f(x_{n-1}) - f'(x_{n-1})(x_{n} - x_{n-1})\|. \]

**Lemma 4** Let the operator \( f \) be \( \omega \)-regularly smooth on \( D \) with some \( h \), the operator \( g \) satisfies (11), the sequence \( \{t_{n}\} \) is defined by the recurrence formula (8) and the condition (10) holds. If for all \( 1 \leq k \leq n \) successive approximations \( x_{k} \) are defined and satisfy the inequality

\[ \|x_{k} - x_{k-1}\| \leq t_{k} - t_{k-1}, \tag{16} \]

then

\[ r(x_{n-1}, x_{n}) \leq a - \Omega(\chi) + \Omega(\chi - t_{n}) - t_{n}h + \Psi(t_{n-1}). \tag{17} \]

**Proof.** Let \( x_{t} = x_{n-1} + t(x_{n} - x_{n-1}), \quad 0 \leq t \leq 1. \) Then

\[ r(x_{n-1}, x_{n}) \leq \int_{0}^{1} \|f'(x_{t}) - f'(x_{n-1})\| \|x_{n} - x_{n-1}\| \, dt \leq \]

\[ \leq \int_{0}^{1} (\omega(\xi(x_{n-1}, x_{t}) + \|x_{t} - x_{n-1}\|) - \omega(\xi(x_{n-1}, x_{t}))) \|x_{n} - x_{n-1}\| \, dt. \]

By the inequality (7) we have

\[ \xi(x_{n-1}, x_{t}) \geq \omega^{-1}(\|f'(x_{n-1})\| - h) - \|x_{t} - x_{n-1}\| = \]

\[ = \omega^{-1}(\|f'(x_{n-1})\| - h) - t\|x_{n} - x_{n-1}\|. \]

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Since for all $1 \leq k \leq n$ the inequality \((16)\) holds, it follows that
\[
\|x_n - x_0\| \leq \sum_{k=1}^{n} \|x_k - x_{k-1}\| \leq \sum_{k=1}^{n} (t_k - t_{k-1}) = t_n.
\]

By Lemma \([1]\)
\[
\omega^{-1}(\|f'(x_n)\| - h) \geq \omega^{-1}(\|f'(x_0)\| - h) - \|x_n - x_0\|.
\]

Since $\omega^{-1}(\|f'(x_n)\| - h) \geq 0$, we have
\[
\omega^{-1}(\|f'(x_n)\| - h) \geq (\omega^{-1}(\|f'(x_0)\| - h) - \|x_n - x_0\|)^+ \geq (\omega^{-1}(\|f'(x_0)\| - h) - t_n)^+, \tag{18}
\]

where $\lambda^+ = \max\{\lambda, 0\}$.

Further $(\omega^{-1}(\|f'(x_0)\| - h) - t_n)^+ = (\omega^{-1}(1 - h) - t_n)^+ = (\chi - t_n)^+$. Let $\alpha_n = (\chi - t_n)^+$. By the condition (10) we obtain that $t_n < \chi$ and $\alpha_n = \chi - t_n > 0$ for all $n = 0, 1, \ldots$. Hence the inequality (15) may be rewritten in the form
\[
\omega^{-1}(\|f'(x_n)\| - h) \geq \alpha_n.
\]

Analogously we obtain
\[
\omega^{-1}(\|f'(x_{n-1})\| - h) \geq \alpha_{n-1}.
\]

Using $\|x_n - x_{n-1}\| \leq t_n - t_{n-1}$ we get
\[
\omega^{-1}(\|f'(x_{n-1})\| - h) - t\|x_n - x_{n-1}\| \geq \alpha_{n-1} - t(t_n - t_{n-1}),
\]

which implies that
\[
\xi(x_{n-1}, x_t) \geq \alpha_{n-1} - t(t_n - t_{n-1}) = \alpha_{n-1} - t\delta_{n-1},
\]

where $\delta_{n-1} = t_n - t_{n-1}$.

Because of concavity and monotonicity of $\omega$ we have
\[
r(x_{n-1}, x_n) \leq \int_{0}^{1} (\omega(\alpha_{n-1} - t\delta_{n-1} + t\delta_{n-1}) - \omega(\alpha_{n-1} - t\delta_{n-1}))\delta_{n-1} \, dt =
\]
\[
= \int_{0}^{1} (\omega(\alpha_{n-1}) - \omega(\alpha_{n-1} - t\delta_{n-1}))\delta_{n-1} \, dt.
\]

Let $\tau = t\delta_{n-1}$. Then
\[
r(x_{n-1}, x_n) \leq \int_{0}^{\delta_{n-1}} (\omega(\alpha_{n-1}) - \omega(\alpha_{n-1} - \tau)) \, d\tau = \int_{0}^{\delta_{n-1}} \omega(\alpha_{n-1}) \, d\tau - \int_{0}^{\delta_{n-1}} \omega(\alpha_{n-1} - \tau) \, d\tau =
\]
\[
= \omega(\alpha_{n-1})\delta_{n-1} + \int_{\alpha_{n-1}}^{\alpha_{n-1} - \delta_{n-1}} \omega(\theta) \, d\theta = \omega(\alpha_{n-1})\delta_{n-1} + \int_{0}^{\alpha_{n-1} - \delta_{n-1}} \omega(\theta) \, d\theta - \int_{0}^{\alpha_{n-1}} \omega(\theta) \, d\theta =
\]
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It follows from the first of these equalities that

\[
\omega(\alpha_n)\delta_n - \Omega(\alpha_n) = \omega(\alpha_n)\delta_n - \Omega(\alpha_n) = \\
= \omega(\alpha_n)\delta_n - \Omega(\alpha_n) + \Omega(\alpha_n) - \Omega(\alpha_n) = \\
= \omega(\alpha_n)\delta_n - \Omega(\alpha_n).
\]

Let us show that for all \( n = 0, 1, \ldots \) the equality

\[
\omega(\alpha_n - t_n) \cdot (t_{n+1} - t_n) - \Omega(\alpha_n + t_n) + \Psi(t_n) = a - \Omega(\chi)
\]

holds. In fact, by the definition of the sequence \( \{t_n\} \)

\[
(t_{n+1} - t_n)(h + \omega(\chi - t_n)) = a - \Omega(\chi) + \Omega(\chi - t_n) - t_n h + \Psi(t_n)
\]

and

\[
(t_n - t_{n-1})(h + \omega(\chi - t_n)) = a - \Omega(\chi) + \Omega(\chi - t_n) - t_n h + \Psi(t_n).
\]

It follows from the first of these equalities that

\[
a - \Omega(\chi) = t_{n+1}h + \omega(\chi - t_n) \cdot (t_{n+1} - t_n) - \Omega(\chi - t_n) - \Psi(t_n)
\]

and from the second that

\[
a - \Omega(\chi) = t_n h + \omega(\chi - t_{n-1}) \cdot (t_n - t_{n-1}) - \Omega(\chi - t_{n-1}) - \Psi(t_{n-1}).
\]

Consequently,

\[
\omega(\chi - t_n) \cdot (t_{n+1} - t_n) - \Omega(\chi - t_n) + t_{n+1} h - \Psi(t_n) = \\
= \omega(\chi - t_{n-1}) \cdot (t_n - t_{n-1}) - \Omega(\chi - t_{n-1}) + t_n h - \Psi(t_{n-1})
\]

for all \( n = 1, 2, \ldots \) and

\[
\omega(\chi - t_n) \cdot (t_{n+1} - t_n) - \Omega(\chi - t_n) + t_{n+1} h - \Psi(t_n) = \\
= \omega(\chi - t_0) \cdot (t_1 - t_0) - \Omega(\chi - t_0) + t_1 h - \Psi(t_0) = \\
= \omega(\chi) \cdot a - \Omega(\chi) + ah = (1 - h)a - \Omega(\chi) + ah = a - \Omega(\chi).
\]

Thus the equality (19) holds for all \( n = 0, 1, \ldots \) and the estimate for \( \tau(x_{n-1}, x_n) \) may be rewritten in the form (17). This completes the proof of Lemma 4.

4 Convergence Theorem

Let the operator \( f \) be \( \omega \)-regularly smooth on \( D \) with some \( h \), the operator \( g \) satisfies (4), the function (9) has a unique zero \( t_* \) in the interval \([0, \chi]\), the closed ball \( B(x_0, t_*) \) is contained in \( D \) and the condition (10) holds. Then

1) the equation (1) has a unique root \( x_* \) in the ball \( B(x_0, t_*) \);

2) the successive approximations (2) are defined for all \( n = 0, 1, \ldots \), belong to \( B(x_0, t_*) \) and converge to \( x_* \);
3) for all \( n = 0, 1, \ldots \) the inequalities

\[
\|x_{n+1} - x_n\| \leq t_{n+1} - t_n,  \tag{20}
\]

\[
\|x_s - x_n\| \leq t_s - t_n,  \tag{21}
\]

hold, where the sequence \( \{t_n\} \) is defined by the recurrence formula (8), monotonically increases and converges to \( t_s \).

**Proof.** In order to prove the theorem it suffices to show that the successive approximations (2) are defined for all \( n = 0, 1, \ldots \), belong to the ball \( B(x_0, t_s) \) and satisfy the inequalities (20) and (21). Other assertions of the theorem follow from Lemma 2 and Lemma 3.

Since (21) is a direct consequence of (20), it suffices to prove (20). For \( n = 0 \) the inequality (20) is obvious:

\[
\|x_1 - x_0\| = \|[f'(x_0)]^{-1}(f(x_0) + g(x_0))\| \leq a = t_1 - t_0.
\]

Suppose that (20) holds for all \( n < k \). We first show that the operator \( f'(x_k) \) is invertible. In fact,

\[
\|[f'(x_0)]^{-1}(f'(x_k) - f'(x_0))\| = \|f'(x_k) - f'(x_0)\| \leq \leq \omega(\xi(x_0, x_k) + \|x_k - x_0\|) - \omega(\xi(x_0, x_k)).
\]

By the inequality (7)

\[
\xi(x_0, x_k) \geq \omega^{-1}(\|f'(x_0)\| - h) - \|x_k - x_0\| = \chi - \|x_k - x_0\|
\]

By the induction hypothesis

\[
\|x_k - x_0\| \leq \sum_{j=1}^{k} \|x_j - x_{j-1}\| \leq \sum_{j=1}^{k} (t_j - t_{j-1}) = t_k
\]

and hence \( \xi(x_0, x_k) \geq \chi - t_k > 0 \) (\( t_k < \chi \) for all \( k = 0, 1, \ldots \) due to (10)). Because of concavity of \( \omega \) we have

\[
\omega(\xi(x_0, x_k) + \|x_k - x_0\|) - \omega(\xi(x_0, x_k)) \leq \omega(\chi - t_k + \|x_k - x_0\|) - \omega(\chi - t_k) \leq \leq \omega(\chi - t_k + t_k) - \omega(\chi - t_k) < \omega(\chi) - \omega(0) = \omega(\chi) = 1 - h \leq 1.
\]

Thus \( \|[f'(x_0)]^{-1}(f'(x_k) - f'(x_0))\| < 1 \) and, consequently, the operator

\[
T = I + [f'(x_0)]^{-1}(f'(x_k) - f'(x_0))
\]

is invertible. Since \( f'(x_k) = f'(x_0)T = T \), the operator \( f'(x_k) \) is also invertible and

\[
\|[f'(x_k)]^{-1}\| = \|T^{-1}\| \leq \frac{1}{1 - \|T - I\|} \leq \frac{1}{1 - [\omega(\chi) - \omega(\chi - t_k)]}.
\]

Further, using the estimate for \( r(x_{k-1}, x_k) \) from Lemma 4 and the inequality (13) we get

\[
\|x_{k+1} - x_k\| = \|[f'(x_k)]^{-1}(f(x_k) + g(x_k))\| =
\]
\[ \| [f'(x_k)]^{-1}(f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1}) + g(x_k) - g(x_{k-1})\| \leq \ \leq \| [f'(x_k)]^{-1}\| \| f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1}) + g(x_k) - g(x_{k-1})\| + \| [f'(x_k)]^{-1}\| \| g(x_k) - g(x_{k-1})\| \leq \frac{r(x_{k-1}, x_k) + \Psi(t_{k-1} + \| x_k - x_{k-1}\|) - \Psi(t_{k-1})}{1 - \omega(\chi) - \omega(\chi - t_k)} \leq \frac{r(x_{k-1}, x_k) + \Psi(t_k) - \Psi(t_{k-1})}{1 - \omega(\chi) - \omega(\chi - t_k)} \leq \frac{a - \Omega(\chi) + \Omega(\chi - t_k) - t_k h + \Psi(t_{k-1}) + \Psi(t_k) - \Psi(t_{k-1})}{h + \omega(\chi - t_k)} = \frac{a - \Omega(\chi) + \Omega(\chi - t_k) - t_k h + \Psi(t_k)}{h + \omega(\chi - t_k)} = t_{k+1} - t_k. \]

Consequently, (20) holds for \( n = k \).

Since for all \( n = 0, 1, \ldots \) the operator \( f'(x_n) \) is invertible and \( \| x_n - x_0 \| \leq t_n \leq t_* \), the successive approximations (2) are defined for all \( n = 0, 1, \ldots \) and belong to the ball \( B(x_0, t_*) \). The convergence of successive approximations to \( x_* \) follows from (21). This proves the theorem.

It is to be noted that each Lipschitz smooth operator is also regularly smooth, but the converse is not true. So the theorem proved is applicable to more wide class of nonlinear operator equations of the form (1) than the corresponding convergence theorems from [1, 3].

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