THE RAMANUJAN PROPERTY FOR REGULAR CUBICAL COMPLEXES

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Abstract. We consider cubical complexes which are uniformized by an ordered product of regular trees. For these we define the notion of being Ramanujan, generalizing the one-dimensional definition introduced by Lubotzky, Phillips, and Sarnak [15]. As in [12], we also allow local systems. We discuss the significance of this property, and then we construct explicit arithmetic examples using quaternion algebras over totally real fields. Here we reduce the Ramanujan property to special cases of the Ramanujan-Petersson conjecture, many of which are known. Our examples subsume the constructions of [15], [12], and [17].

Introduction

Ramanujan graphs were defined by Lubotzky, Phillips, and Sarnak in [15] as regular graphs whose adjacency matrices, or their laplacians, have eigenvalues satisfying some “best possible” bounds. Such graphs possess many interesting properties. In this paper we will give a higher dimensional generalization of this theory to regular cubical complexes. By definition, \( \vec{r} = (r_1, \ldots, r_g) \)-regular complexes are cell complexes locally isomorphic to the (ordered) product of \( g \) regular trees, with the \( j \)th tree of regularity \( r_j \geq 3 \). Each cell is an \( i \)-cube (i.e., an \( i \)-dimensional cube) with \( 0 \leq i \leq g \). Throughout each \((g-1)\)-cube exactly one of the tree factors, say the \( j \)th one, is constant, and there are \( r_j \) \( g \)-cubes passing through it. When \( g = 1 \) we simply have an \( r \)-regular graph.

The spaces of \( i \)-cochains \( C^i(X) \) (with real or complex coefficients) of a finite cubical complex \( X \) are inner product vector spaces with an orthonormal basis corresponding to the characteristic functions of the \( i \)-cells. There are partial boundary operators \( \partial_j = \partial_{j,i} : C^i(X) \to C^{i+1}(X) \) for \( 1 \leq j \leq g \). With these we get the adjoint operators \( \partial_j^* = \partial_{j,i}^* : C^{i+1}(X) \to C^i(X) \), and hence the partial laplacians

\[
\Box_j = \Box_{j,i} = \partial_{j,i}^* \partial_{j,i} + \partial_{j,i-1} \partial_{j,i-1}^* : C^i(X) \to C^i(X).
\]

Each \( \Box_{j,i} \) is a self-adjoint nonnegative operator. For \( i \) fixed they all commute, and one gets a combinatorial harmonic theory (compare [21]).

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When $X$ is infinite, these notions extend to $L^2$-cochains. When $X = \Delta$ is an $\vec{r} = (r_1, \ldots, r_g)$-regular product of trees, Kesten’s 1-dimensional results extend and we get that each $\lambda$ in the spectrum of $r_j \text{Id} - \Box_j$ acting on $L^2$-cochains of $\Delta$, satisfies $|\lambda| \leq 2\sqrt{r_j - 1}$. As in the 1-dimensional case we say that a $(r_1, \ldots, r_g)$-regular cubical complex $X$ is Ramanujan if the eigenvalues of $r_j \text{Id} - \Box_j$ on $X$ are $\pm r_j$ or satisfy the same properties for each $j$.

One justification for this definition in the 1-dimensional case is the Alon-Boppana result, which shows that these bounds are essentially the best possible for the trivial local system. We generalize this result under a natural hypothesis. Another parallel with the 1-dimensional case is that when $X$ is finite, connected, and uniformized by a lattice $\Gamma$ in a product $\prod_{1 \leq j \leq g} G_j$ of $p$-adic $\text{SL}_2$’s, the Ramanujan property is equivalent to the following condition. For a $(g - 1)$-cube $\sigma$ in the universal covering complex, which is constant in the $j$th direction, let $\Gamma_\sigma$ be the stabilizer of $\sigma$ in $\Gamma$. Then no nontrivial unramified complementary series representations should appear in $L^2(\Gamma_\sigma \backslash G_j)$ for any $j$ and $\sigma$.

A fundamental problem in the 1-dimensional case is to construct explicitly an infinite family of Ramanujan graphs of a fixed regularity. The only such families known have regularities of the form $k = q + 1$, with $q$ a power of a prime. The standard LPS examples ([15]) depend in fact on the Ramanujan-Petersson conjectures for weight 2 holomorphic cusp forms on $\text{GL}_2(\mathbb{Q})$. These were reduced to the Weil bounds for curves by Eichler, Shimura, and Igusa. They only give examples with $q$ a prime. The case of any prime power was subsequently handled in [16] via function fields. Ramanujan local systems were defined and constructed in [12] with $q$ a prime. This construction requires Deligne’s results on the Ramanujan-Petersson conjecture for cusp forms of higher weight. They depend on the Weil bounds for higher dimensional varieties.

In the higher dimensional case a product of Ramanujan graphs always gives a Ramanujan cubical complex, and the problem is to construct irreducible examples. In this work we construct infinite families of Ramanujan regular cubical complexes, with fixed regularities $(r_1, \ldots, r_g)$, where $r_j = q_j + 1$ and the $q_j$’s are any prime powers. As in the LPS examples, our construction uses quaternion algebras, this time over totally real number fields. It reduces the Ramanujan property to the Ramanujan-Petersson conjecture for certain holomorphic Hilbert modular forms. The Ramanujan-Petersson conjecture seems not to be known for all holomorphic Hilbert modular forms. In particular, the results of [3] do not help us, because they establish the bounds outside a finite set of primes which is inexplicit, and a priori may depend on the form. However, in the literature this conjecture is proved under additional hypotheses, which still permit us to construct infinite families of examples with arbitrary prime powers for regularities. If all the $q_j$’s are powers of the same prime, one could give, as before, an alternate construction using function fields. As in [12], we also construct Ramanujan local systems on our examples. Among the interesting features of our examples is the fact that their cohomology vanishes except in the top dimension $g$ (and 0). In addition, one can bound the girth from below as in the 1-dimensional case ([14], Theorem 7.3.7).
As in [12], this theory is valid for hermitian local systems over cubical complexes. We similarly give examples of Ramanujan local systems on cubical complexes.

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1. The harmonic theory of regular cubical complexes

1.0 Cubical complexes

In this article we will study cell complexes locally isomorphic to a finite product of regular trees, in which the order and the regularities of the factors are globally constant.

Let \( g \) be the dimension of such a complex \( X \), and let \( r_j \) be the regularity of the \( j \)th factor tree \( (1 \leq j \leq g) \). We will call such a complex \( X \) an \((r_1, \ldots, r_g)\)-regular cubical complex. Each cell in \( X \) is a cube of dimension \( \leq g \). The 0-dimensional cells are called vertices and the 1-dimensional cells are called edges.

For each subset \( I \subset \{1, \ldots, g\} \), the \( I \)-cubes, or cubes of direction \( I \) of \( X \), are the products of edges from the factors \( i \in I \) with vertices from the factors not in \( I \). Each such cube has \( 2^{|I|} \) orientations, and we denote the set of oriented \( I \)-cubes by \( \Sigma_I \). There are bottom and top maps \( \text{bot}_j, \text{top}_j : \Sigma_I \to \Sigma_{I\setminus\{j\}} \) and inversion maps \( \text{inv}_j : \Sigma_I \to \Sigma_I \) for any \( j \in I \). These are subject to the following axioms:

1. \( \{\text{inv}_j\}_{j \in I} \) generate a group, isomorphic to the group \((\mathbb{Z}/2\mathbb{Z})^I\) of maps from \( I \) to \( \mathbb{Z}/2\mathbb{Z} \), which acts simply transitively on the orientations of each \( I \)-cube.
2. \( \text{top}_j \text{inv}_{j'} = \text{inv}_{j'} \text{top}_j \) and \( \text{bot}_j \text{inv}_{j'} = \text{inv}_{j'} \text{bot}_j \) for \( j \neq j' \).
3. \( \text{top}_j \text{inv}_j = \text{bot}_j \) (and hence also \( \text{bot}_j \text{inv}_j = \text{top}_j \)) for all \( j \).
4. Any oriented \( I \)-cube is the \( j \)th top of precisely \( r_j \) oriented \( I \cup \{j\} \)-cubes for \( j \notin I \).

Geometrically, these combinatorial conditions mean that \( X \) is locally isomorphic to the ordered product \( \Delta = \prod_j \Delta_j \) of \( g \) regular trees of respective regularities \( r_j \). In particular, when \( X \) is connected these conditions hold if and only if the universal cover \( \tilde{X} \) is isomorphic to \( \Delta \), with the covering transformations preserving the order of the tree factors. The if part is clear, and the only if part holds because \( \Delta \) is simply connected (in fact, explicitly contractible), and locally isomorphic to \( X \). Hence \( \Delta \) is isomorphic to \( \tilde{X} \) by the uniqueness of the universal cover. The directions are preserved under the covering map, and hence the deck transformations form indeed a subgroup of \( \prod_j \text{Aut}(\Delta_j) \).

We set \( \text{Ver} = \Sigma_{\emptyset} \), \( \text{Ed}_o^j = \Sigma_{\{j\}} \), and \( \text{Ed}^o = \sqcup_j \text{Ed}_o^j \). For an oriented edge \( e \) of direction \( \{j\} \) we write \( \text{bot}_j(e) = o(e) \), \( \text{top}_j(e) = t(e) \), and \( \text{inv}_j(e) = \tau \). We will refer to these as the origin, the terminal vertex, and the opposite edge respectively. If necessary we will indicate the dependence of all these objects on \( X \).

Examples 1.1.

1. The unit cube in \( \mathbb{R}^g \) is the product of \( g \) intervals, which are 1-regular trees.
2. The tiling of \( \mathbb{R}^g \) by unit cubes with integer vertices is the product of \( g \) lines, viewed as 2-regular trees.
3. The graphs in the sense of [20] which are $k$-regular are precisely the $k$-regular cubical complexes (of dimension 1).

4. A finite product of regular cubical complexes is a regular cubical complex.

A connected regular cubical complex is called irreducible if it has no finite unramified cover by a product of regular cubical complexes of positive dimension.

In many important cases, the vertices of an $(r_1, \ldots, r_g)$-regular cubical complex $X$ come with parities: these are maps $p_j$ from the vertices to $\{0, 1\}$ which satisfy $p_j(\text{top } e) = p_j(\text{bot } e)$ if and only if $i \neq j$ for any edge $e$ of direction $\{i\}$ of $X$, $1 \leq i, j \leq g$. On a complex with parities we give each cube a canonical orientation by agreeing that its bottom-most vertex has all its parities equal to 0. In the 1-dimensional case we recover the notion of a bipartite graph.

1.1 Local systems

A real/complex local system, or a flat vector bundle, on a regular cubical complex $X$ depends only on the cells of dimension $\leq 2$. For the 1-dimensional case, see, e.g., [12]. In the general case, a local system $\mathcal{L}$ on $X$ consists of a real/complex vector space $\mathcal{L}(v)$ for any vertex $v$ of $X$. In addition, for any oriented edge $e$ one is given a linear isomorphism $\mathcal{L}_e : \mathcal{L}(\text{bottom } e) \rightarrow \mathcal{L}(\text{top } e)$ so that $\mathcal{L}_e = \mathcal{L}_{-1} e$. The $\mathcal{L}_e$’s must also satisfy the flatness condition: for any 2-cell of direction $\{j, j'\}$ in $X$ we require

\[ \mathcal{L}_{\text{top } j} \mathcal{L}_{\text{bot } j} = \mathcal{L}_{\text{top } j'} \mathcal{L}_{\text{bot } j}. \]

The local system is metrized if each fiber $\mathcal{L}(v)$ is a finite dimensional (positive definite) inner product space and all transition maps $\mathcal{L}_e$ are isometries. The (metrized) trivial local system $\mathcal{T}_V$ for a finite dimensional (inner product) vector space $V$ has all fibers $V$ and all transition maps the identity. Over a contractible space all local systems admit a trivialization, i.e., an isomorphism to some $\mathcal{T}_V$. The notions of maps and direct sums make sense for (metrized) local systems over a fixed space (see, e.g., [12]). A local system is irreducible if it is not the direct sum of nonzero local systems. Every metrized local system is the direct sum of irreducible ones and a sub local system has an orthogonal complement.

Let $X_1, X_2$ be regular cubical complexes with (metrized) local systems $\mathcal{L}_i$ on $X_i$. The product $X = X_1 \times X_2$ is naturally a regular cubical complex on which we have a (metrized) local system $\mathcal{L} = \mathcal{L}_1 \boxtimes \mathcal{L}_2$, called the external product of the $\mathcal{L}_i$’s. It is irreducible if and only if both $\mathcal{L}_i$’s are irreducible, and if $X_1$ and $X_2$ are connected, then any irreducible local system on $X$ is an external product.

Let $X$ be a connected regular cubical complex and let $v_0$ be a base vertex. The universal cover $\tilde{X}$ is a product of regular trees $\{\Delta_j\}_{1 \leq j \leq g}$, and the fundamental group $\Pi = \pi_1(X, v_0)$ is discrete in $\prod_j \text{Aut}(\Delta_j)$ which acts properly on $\tilde{X}$ with quotient $X$. (Metrized) local systems $\mathcal{L}$ on $X$ are equivalent to (orthogonal or unitary) representations $\rho_\mathcal{L}$ of $\Pi$ on $\mathcal{L}(v_0)$. We reconstruct $\mathcal{L}$ as $\Pi \backslash (\tilde{X} \times V)$.

A similar construction is possible in the disconnected case: if $\Pi$ acts freely on a regular cubical complex $\tilde{X}$ preserving directions, and $\rho$ is an (orthogonal or unitary) representation of $\Pi$ on $V$, then the quotient $X = \Pi \backslash \tilde{X}$ is again a regular cubical complex and we
get a (metrized) local system \( \mathcal{L} = \Pi \backslash (\tilde{X} \times V) \). One gets this way all local systems on \( X \) whose pullback to \( \tilde{X} \) is trivial.

### 1.2 Cochains

Let \( X \) be an \((r_1, \ldots, r_g)\)-regular cubical complex and let \( \mathcal{L} \) be a local system on \( X \). For a subset \( I \subset \{1, \ldots, g\} \) and \( \sigma \in \Sigma_I \) denote by \( o(\sigma) \) the iterated bottom vertex \( (\prod_{j \in I} \text{bot}_j)(\sigma) \), and for \( j \in I \) let \( e_j(\sigma) \) be the oriented edge of direction \( \{j\} \) defined by \( e_j(\sigma) = (\prod_{j' \neq j} \text{bot}_{j'})(\sigma) \). By the combinatorial conditions these are well defined, i.e., do not depend on the order of the operators. Now set \( \mathcal{L}(\sigma) = \mathcal{L}(o(\sigma)) \) and \( \mathcal{L}_{\sigma,j} = \mathcal{L}_{e_j(\sigma)} \). By definition, the space of \( I \)-cochains \( C^I(X, \mathcal{L}) \) is the space of completely alternating collections

\[
I = \{ s(\sigma) \}_{\sigma \in \Sigma_I} \in \prod_{\sigma \in \Sigma_I} \mathcal{L}(\sigma).
\]

The condition of being completely alternating is vacuous for \( 0 \)-cochains. Otherwise, a collection \( s \) as above is completely alternating if for any \( j \in I \) we have \( s(\text{inv}_j(\sigma)) = -\mathcal{L}_{\sigma,j} s(\sigma) \). For \( \mathcal{L} = \mathcal{T}_V \) we can view the \( I \)-cochains as maps from \( \Sigma_I \) to \( V \).

For an integer \( 0 \leq i \leq g \) put \( C^i(X, \mathcal{L}) = \bigoplus_{|I|=i} C^I(X, \mathcal{L}) \). The partial boundary operators \( \partial_j = \partial_{j,I} : C^I(X, \mathcal{L}) \rightarrow C^{i+1}(X, \mathcal{L}) \) are defined by linearity and the formula

\[
\partial_j s(\sigma) = \begin{cases} 
\mathcal{L}^{-1}_{j,I} s(\text{top}_j(\sigma)) - s(\text{bot}_j(\sigma)) & \text{for } \sigma \in \Sigma_I \text{ with } |I| = i + 1 \text{ and } j \in I \\
0 & \text{for } \sigma \in \Sigma_I \text{ with } |I| = i + 1 \text{ and } j \notin I
\end{cases}
\]

for any \( s \in C^i(X, \mathcal{L}) \). These are well defined and satisfy \( \partial_j \partial_{j'} = \partial_{j} \partial_{j'} \) for all \( j, j' \). Clearly \( \partial_j^2 = 0 \). We will denote by \( \partial_{j,I} \) the restriction of \( \partial_j \) to \( C^I(X, \mathcal{L}) \). If \( j \in I \), then \( \partial_{j,I} = 0 \). As in [19, II.1], the total boundary of \( s \in C^I(X, \mathcal{L}) \) is defined by \( d(s) = d_I(s) = \sum_{j \notin I} (-1)^{\alpha_I(j)} \partial_{j,I} \), where \( \alpha_I(j) \) is the place of \( j \) in \( I \cup \{j\} \). As usual \( d^2 = 0 \). We get the spaces of cohomology with respect to \( \partial_j \) and \( d \) by the usual formulas \( H^j_I(X, \mathcal{L}) = \ker \partial_{j,I} / \text{im} \partial_{j,I-1} \) and \( H^i(X, \mathcal{L}) = \ker d_i / \text{im} d_{i-1} \). We can moreover break \( H^j_I(X, \mathcal{L}) \) as a direct sum of subspaces \( H^j_I(X, \mathcal{L}) \) with \( |I| = i \) defined by restricting \( d_j \) to \( C^j(X, \mathcal{L}) \).

When \( \mathcal{L} \) is the trivial local system \( \mathcal{T}_\mathbb{R} \), it is usually omitted from the notation.

The operators \( \partial^*_j = \partial^*_{j,I} : C^{i+1}(X, \mathcal{L}) \rightarrow C^i(X, \mathcal{L}) \) are defined by

\[
\partial^*_j s(\sigma) = \sum_{\text{top}_j(\tau) = \sigma} \mathcal{L}_{r,j}^{-1} s(\tau).
\]

These operators satisfy the analogous relations \( \partial_j^* \partial^*_j = \partial^*_j \partial_j^* \) and \( (\partial^*_j)^2 = 0 \). We let \( \partial^*_{j,I} \) denote the restriction of \( \partial^*_j \) to \( C^{I\cup\{j\}}(X, \mathcal{L}) \) for \( j \notin I \). Finally, for \( t \in C^I(X, \mathcal{L}) \) we define \( d^*(t) = \sum_{j \in I} (-1)^{\alpha_I(j)} \partial^*_{j,I-\{j\}} \). Again \( (d^*)^2 = 0 \).

### 1.3 Hodge Theory

From now on we consider only metrized local systems. Let \( C^*_I(X, \mathcal{L}) \) denote the Hilbert space completion of the space of \( I \)-cochains with coefficients in \( \mathcal{L} \), with respect to the pre-Hilbert norm

\[
\|s\|^2 = 2^{-|I|} \sum_{\sigma \in \Sigma_I} \|s(\sigma)\|^2.
\]
We let $C^2_2(X, \mathcal{L})$ be the orthogonal sum of the corresponding $C^I$-s. It is clear that the operators previously defined (e.g., $\partial_j$, $d_j^I$, etc.) induce bounded operators, denoted by the same letters, on the corresponding spaces $C^2(X, \mathcal{L})$. In particular we have $\|\partial_j\| \leq r_j$ and $\|\partial_{j,i}\| \leq r_j$. To see this, notice that for $|I| = i$ and $s \in C^I(X, \mathcal{L})$ we have $\partial_j s = 0$ if $j$ is in $I$. Otherwise set $I' = I \cup \{j\}$. Then
\[
\|\partial_j s\|^2 = 2^{-i-1} \sum_{\sigma \in \Sigma_I} \|L_{j,\sigma}^{-1} s(\text{top}_j(\sigma)) - s(\text{bot}_j(\sigma))\|^2 \\
\leq 2^{-i+1} \sum_{\sigma \in \Sigma_I} \|s(\text{bot}_j(\sigma))\|^2 = r_j 2^{-i} \sum_{\tau \in \Sigma_I} \|s(\tau)\|^2 \\
= r_j \|s\|^2
\]
(see \([12], \text{Lemma 1.2}\)). This implies the bound $\|\partial_j\| \leq r_j$. Since the $C^I(X, \mathcal{L})$'s are mutually orthogonal, the same bound holds for the direct sum operator $\partial_j$. The case of $\partial_{j,i}$ is similar and we omit it.

It is routine and easy to verify that $\partial^*_{j,i}$ is the adjoint of $\partial_{j,i}$.

The laplacians $\Box_{j,i}: C^i(X, \mathcal{L}) \to C^i(X, \mathcal{L})$ are defined as usual by $\Box_{j,i} = \partial_{j,i} \partial^*_{j,i} \partial_{j,i} + \partial^*_{j,i} \partial_{j,i}$. On each $C^I(X, \mathcal{L})$ this simplifies to $\partial_{j,I-(j)} \partial^*_{j,I-(j)}$ if $j \in I'$ and to $\partial^*_{j,i} \partial_{j,i}$ if $j \notin I$. These are commuting bounded self-adjoint nonnegative operators. The Laplacian $\Box_{\text{tot},i}$ is defined by
\[
\Box_{\text{tot},i} = \Box_{1,i} + \Box_{2,i} + \cdots + \Box_{r,i} = d_{i-1}^I d_i^I + d_i^I d_i^I.
\]
The restriction of $\Box_{j,i}$ ($\Box_{\text{tot},i}$) to $C^I(X, \mathcal{L})$ will be denoted by $\Box_{j,I}$ ($\Box_{\text{tot},I}$). All these are also bounded, self-adjoint, and nonnegative. By definition, the space of $\bullet$-harmonic forms for $\bullet = i$ or $I$, $\mathcal{H}^\bullet(X, \mathcal{L}) \subset C^\bullet(X, \mathcal{L})$, is $\text{Ker} \Box_{\text{tot},\bullet}$. We likewise set $\mathcal{H}^I_j(X, \mathcal{L}) = \text{Ker} \Box_{j,I}$. We now have the following routine

**Proposition 1.2.** Let $X$ be a finite regular cubical complex, and let $\mathcal{L}$ be a metrized local system on $X$. Then

1. $\mathcal{H}^i(X, \mathcal{L}) = \text{Ker} d_i \cap \text{Ker} d_{i-1}^I$ and $\mathcal{H}^i_j(X, \mathcal{L}) = \text{Ker} \partial_{j,i} \cap \text{Ker} \partial^*_{j,i-1}$. Moreover $\mathcal{H}^I_j(X, \mathcal{L}) = \text{Ker} \partial_{j,I}$ if $j \notin I$ and $\mathcal{H}^I_j(X, \mathcal{L}) = \text{Ker} \partial^*_{j,I-(j)}$ if $j \in I$.

2. We have orthogonal sum decompositions (the Hodge Decomposition)
\[
\begin{align*}
C^2_2(X, \mathcal{L}) &= \mathcal{H}^i(X, \mathcal{L}) \oplus \text{Im} d_{i-1} \oplus \text{Im} d_i^I, \\
C^2(X, \mathcal{L}) &= \mathcal{H}^I_j(X, \mathcal{L}) \oplus \text{Im} \partial_{j,i-1} \oplus \text{Im} \partial^*_{j,i}, \quad \text{and} \\
C^I_2(X, \mathcal{L}) &= \mathcal{H}^I_j(X, \mathcal{L}) \oplus \text{Im} \partial^*_{j,I} \quad \text{for } j \notin I \\
&= \mathcal{H}^I_j(X, \mathcal{L}) \oplus \text{Im} \partial_{j,I-(j)} \quad \text{for } j \in I.
\end{align*}
\]

2. **The Spectrum of the Laplacian**

Our main interest will be in the eigenvalues of the $\Box_{j,i}$'s. First, we have the following
Lemma 2.1. For every subset $I \subset \{1, \ldots, g\}$ the $I$-cochains $C^I(X, \mathcal{L})$ are preserved by $\square_{j,i}$ for all $j$, with $i = |I|$. Moreover, for $j \notin I$ the maps $\partial_j$ and $\partial^*_j$ induce isomorphisms between the $\lambda$-eigenspace of $\square_j$ on $C^I(X, \mathcal{L})$ and on $C^{I \cup \{j\}}(X, \mathcal{L})$ for any $\lambda \neq 0$.

Proof: Suppose $\square_j c = \lambda c$ and for $\square_j c' = \lambda c'$ for $c \in C^I(X, \mathcal{L})$ and $c' \in C^{I \cup \{j\}}(X, \mathcal{L})$, with $j \notin I$. Then $\square_j \partial_j c = \partial_j \partial^*_j \partial_j c = \lambda \partial_j c$ and likewise $\square_j \partial_j^* c' = \lambda \partial_j^* c'$. Since $\partial_j \partial_j^*$ and $\partial_j^* \partial_j$ are both multiplication by $\lambda \neq 0$ on all such $c$, $c'$ respectively, the lemma follows.

It follows that the study of the nonzero eigenvalues and eigenspaces of $\square_j$ on any $C^I(X, \mathcal{L})$ can be reduced to the case $j \notin I$. In fact, it is also possible to reduce the study of the full diagram

$$C^I(X, \mathcal{L}) \xrightarrow{\partial_j} C^{I \cup \{j\}}(X, \mathcal{L}),$$

to the case when $X$ is an $r_j$-regular graph and $I$ is empty. To do this, we make for convenience the following hypothesis.

PAR: $X$ has parities $p_j$ for all $1 \leq j \leq g$.

Under hypothesis PAR we define, for any $j \notin I$, a graph $\text{Gr}_j(X) = \text{Gr}_{j,I}(X)$ and a (metrized) local system $\mathcal{L} = \mathcal{L}_{j,I}(X)$ as follows. The vertices of $\text{Gr}_j(X)$ are the cubes of type $I$ of $X$; its (oriented) edges are the (oriented) cubes of type $I \cup \{j\}$, (with the $I$ part of the orientation all 0’s). For an $I$-cube $\sigma$ of $X$ define $\mathcal{L}_{j,I}(X)(\sigma) = \mathcal{L}(v_\sigma)$, where $v_\sigma$ is the bottom-most vertex of $\sigma$. The transition maps are then those induced from $\mathcal{L}$. The compatibilities of the corresponding operators $\partial_j, \partial^*_j, \square_j$ with $\partial$, $\partial^*$, and $\square$ are obvious from the definitions. In any mention of these graphs hypothesis PAR will be assumed.

Let $\text{Gr}$ be a a locally finite graph and $\mathcal{L}$ be a local system on $\text{Gr}$. The star operator $S(\text{Gr}, \mathcal{L}) : C^0(\text{Gr}, \mathcal{L}) \to C^0(\text{Gr}, \mathcal{L})$ is defined to be the map sending a cochain $s \in C^0(\text{Gr}, \mathcal{L})$ to the cochain $S(\text{Gr}, \mathcal{L})(s)$ given by

$$S(\text{Gr}, \mathcal{L})(s)(v) = \sum_{\{e | I(e) = v\}} \mathcal{L}^{-1}(o(e)).$$

In case the graph $\text{Gr} = \text{Gr}_{j,I}(X)$ we shall call $S_{j,I} = S(\text{Gr}_{j,I}(X), \mathcal{L})$ the $j$th star operator and put $S_{j,i} = \oplus_{|I|=i} S_{j,I}$. The operator $S_{j,I}$ then acts on the space $C^I(X, \mathcal{L}) = C^0(\text{Gr}_{j,I}(X), \mathcal{L})$ and accordingly $S_{j,i}$ acts on the space $C^I(\text{Gr}_j(X), \mathcal{L})$. Clearly the norm of $S_{j,I}$ is bounded by $r_j$. Since $\square_{j,I}$ we get the following

Proposition 2.2. The norm of $\square_{j,I}$ restricted to $C^I(X, \mathcal{L})$ is bounded by $2r_j$ and its spectrum is contained in $[0, 2r_j]$.

As a corollary, the same bound is valid for the norm of $\square_{j,i}$ (use the orthogonal decomposition of $C^I(X, \mathcal{L})$ to the $C^I(X, \mathcal{L})$’s, treating separately the cases $j \in I$ and $j \notin I$).

In dimension 1, the classical result of Alon and Boppana (see, e.g., [14, Proposition 4.5.4]) shows that the nontrivial eigenvalues of the star operator for the trivial local
system are essentially bounded in terms of the norm of the star operator on the universal covering regular tree (see also [13] or [4]). This result is the following:

**Proposition 2.3.** (Alon-Bopanna) Let $\text{Gr}_n$ be a family of finite $r$-regular connected graphs whose number of vertices goes to $\infty$ with $n$. Let $L_{2,0}(\text{Gr}_n)$ denote the space of zero-sum $L_2$ zero cochains. Then

$$\liminf_{n \to \infty} \|S(\text{Gr}_n)|_{L_{2,0}(\text{Gr}_n)}\| \geq 2\sqrt{r - 1}.$$ 

In the higher dimensional case set $\bullet = i$ or $I$, and define for $j \notin I$

$$\mu_{j,\bullet}(X, L) = \max\{|\mu| \mid \mu \text{ is an eigenvalue of } S_{j,\bullet} \text{ different from } \pm r_j\}.$$ 

Here $L$ is any metrized local system on $X$. Put also

$$\mu_{j,\bullet}(X) = \mu_{j,\bullet}(X, T_j).$$

In particular, $\mu_{j,\bullet}(X, L)$ lies in the interval $[-r_j, r_j]$. The Alon-Boppana result now implies the following $g$-dimensional version:

**Proposition 2.4.** Let $X_n$ be a sequence of $(r_1, \ldots, r_g)$-regular connected cubical complexes. If the number of vertices of each connected component of $\text{Gr}_{j,I}(X_n)$ tend to $\infty$ with $n$, then $\liminf_n \mu_{j,I}(X_n) \geq 2\sqrt{r_j - 1}$.

In analogy with the 1-dimensional case ([12, Definition 2.25]) we make the following

**Definition 2.5.** A local system $L$ on a regular cubical complex $X$ is Ramanujan if $\mu_{j,I}(X, L) \leq 2\sqrt{r_j - 1}$ for all $j$ and $I$ as above.

We say that $X$ is Ramanujan if the trivial local system $T_C$ on $X$ is Ramanujan.

It is of course possible to define the notion of an $(j, I)$-Ramanujan system, so that $L$ is Ramanujan if and only if it is $(j, I)$-Ramanujan for all $(j, I)$ ($j \notin I$).

Many examples are known in the 1-dimensional case (see, e.g., [14]). Since the external tensor product of Ramanujan local systems is Ramanujan, we get many examples in the higher-dimensional case as products, or more generally by pulling back a product system from a finite unramified cover. We will say that a connected regular cubical complex is reducible if a finite unramified cover of it is a product of cubical complexes of lower dimensions. Otherwise we will say that $X$ is irreducible.

When $X$ be connected, we can write $X$ as a quotient $X = \Gamma \backslash \Delta$ of a product $\Delta = \prod_j \Delta_j$ of trees. Then $X$ is reducible if and only if the following two conditions are satisfied: a) $\Delta$ is a product $\Delta = \Delta' \times \Delta''$, with $\Delta', \Delta''$ products over complementary subsets of the $j$’s; and b) A subgroup of finite index of $\Gamma$ is compatibly a product $\Gamma' \times \Gamma''$, so that the corresponding finite unramified cover of $X$ is $\Gamma' \backslash \Delta' \times \Gamma'' \backslash \Delta''$.

A metrized local system is always an orthogonal sum of irreducible local systems. The irreducible (metrized) local systems over a product (connected) complex are precisely the external tensor products of irreducible ones. The challenge is then to find Ramanujan local systems over irreducible cubical complexes. Even when a graph or a cubical complex is Ramanujan, there are local systems on it which are not: a generic deformation of the transition maps of the trivial local system is an example.
Let $F_1, \ldots, F_g$ be nonarchimedean local fields, and set $G_j = \text{PGL}_2(F_j)$ and $G = \prod_j G_j$. Let $G_\pm$ be the elements of $G$ whose $j$th components have determinants with even valuations for each $j$. The standard trees $\Delta_j$ associated to the $G_j$’s are $r_j = (q_j + 1)$-regular, with $q_j$ the cardinality of the residue field of $F_j$. Set $\Delta = \prod_j \Delta_j$. For a discrete subgroup $\Gamma$ of $G_+$ the quotient complex $\Gamma \backslash \Delta$ satisfies hypothesis PAR. For any $I \subset \{1, \ldots, g\}$, $G$ acts on $\Delta_I = \prod_{j \in I} \Delta_j$ and on $\Delta^I = \prod_{j \notin I} \Delta_j$, and $\Delta = \Delta_I \times \Delta^I$. For an $I$-cube $\sigma \in \Sigma^I(\Delta_I)$ let $\Gamma_\sigma$ denote the projection to $G^I = \prod_{j \notin I} G_j$ of the stabilizer of $\sigma$. We now have the following

**Proposition 2.6.** 1. Let $\Gamma$ be a discrete, cocompact, torsion-free subgroup of $G$. Let $\mathcal{L}$ be the (metrized) local system $\mathcal{L}$ on $X = \Gamma \backslash \Delta$ corresponding to a unitary representation $\rho$ of $\Gamma$ on a (finite-dimensional) space $V$. Then $\mathcal{L}$ is $(j, I)$-Ramanujan for $j \notin I$ if and only if the following condition holds. For any $I$-cube $\sigma$ of $\Delta^{\{j\}}$, no (nontrivial) representations of $G$ of the unramified complementary series appear in $L_2(\Gamma_\sigma \backslash G_j \times V)$, where $G_j$ acts through its right action on the $G_j$-factor.

2. The local system $\mathcal{L}$ is Ramanujan on $X$ if and only if no nontrivial representations of $G$ of the unramified complementary series appear in $L_2(\Gamma_\sigma \backslash G_j \times V)$, for any cube $\sigma$ of $\Delta^{\{j\}}$ of direction $I^j = \{1, \ldots, g\} - \{j\}$.

**Proof:** 1. After the choice of a maximal unramified compact subgroup of $G$, the unramified vectors in $L_2(\Gamma_\sigma \backslash G_j \times V)$ can be identified with $L_2(\Gamma_\sigma \backslash \text{Ver} \Delta_j \times V)$, and the action of the $j$th factor Hecke operator $T_{\nu_j}$ corresponds to the action of the $j$th star operator (compare Step 1 in the proof of [12, Theorem 3.4]). Our claim now follows from Satake’s reformulation of the Ramanujan-Petersson conjecture ([13]).

2. For $j \notin I \subset \{1, \ldots, g\}$, an $I^j$-cube has $2^{g-1-|I|}$ faces of direction $I$. Hence the diagram

$$C^I(X, \mathcal{L}) \xrightarrow{\partial^j} C^{I \cup \{j\}}(X, \mathcal{L}),$$

embeds (in $2^{g-1-|I|}$ ways) into the corresponding diagram with $I$ replaced by $I^j$. Hence to verify the Ramanujan property it suffices to verify the $(j, I^j)$-Ramanujan property for all $1 \leq j \leq g$. For any $I$ as above we have the decomposition into connected components

$$\text{Gr}_{j,I}(X, \mathcal{L}) = \sqcup \Gamma_\sigma \backslash \Delta^{I \cup \{j\}},$$

the union taken over representatives $\sigma$ of $\Gamma \backslash \Sigma^I(\Delta^{\{j\}})$. It suffices therefore to verify the Ramanujan property for $\mathcal{L}$ over each of these components for $I = I^j$, and our claim follows from the first part.

3. **Ramanujan local systems arising from quaternion algebras**

Let $F$ be a totally real field of degree $d$ over $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$. Let $\infty_i, i = 1, \ldots, d$, be the infinite places of $F$. For an algebraic group $\mathbf{H}$ over $F$ we let $H_F$ be the $F$-rational points of $\mathbf{H}$, let $H_\mathbb{A}$ be the adelic points of $\mathbf{H}$, and let $H^f$ denote the
finite-adelic points of $H$. For any finite set of finite places $v_1, \ldots, v_n$ of $F$ let $H^{t,v_1,\ldots,v_n}$ denote the finite-adelic points of $H$ without the $v_1, \ldots, v_n$ component.

Let $B$ be a totally definite quaternion algebra over $F$ with reduced norm $\text{Nm} = \text{Nm}_{B/F}$. (A general reference for quaternion algebras over totally real fields is [23].) Let $G$ be the algebraic group over $F$ associated with the multiplicative group $B^\times$. Let $q_v$ denote the cardinality of the residue field of the completion $F_v$ of $F$ at a nonarchimedean place $v$.

With $\mathbb{H}$ denoting the Hamilton quaternions, let $\text{Symm}^k$ denote the $k$th symmetric power of the 2-dimensional representation of $\mathbb{H}^\times$ obtained by identifying $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ with $\text{Mat}_{2 \times 2}(\mathbb{C})$. We have $G_\infty := \prod_i G(F_\infty) \simeq (\mathbb{H}^\times)^d$. The irreducible representations of $G_\infty$ are then $\text{Symm}^{\vec{k},\vec{s}} = \otimes_i (\text{Nm}_{B_\infty/F})^{s_i} \text{Symm}^{k_i}$, with $\vec{k} = (k_1, \ldots, k_d)$ and $\vec{s} = (s_1, \ldots, s_d)$, the $s_i$ being any complex numbers. We denote by $V^{\vec{k},\vec{s}}$ the space on which $\text{Symm}^{\vec{k},\vec{s}}$ acts. They can be unitarized if and only if the center of $G_\infty$ acts through a unitary character, and this happens if and only if $2s_i + k_i$ is imaginary for each $i$. In this event there is a $G_\infty$-invariant positive definite hermitian inner product on $V^{\vec{k},\vec{s}}$ which is unique up to a scalar. In what follows we shall also assume that $\vec{s} = -\vec{k}/2$. Then an element $(z_1, \ldots, z_d)$ in the center of $G_\infty$ acts via $\prod_i \text{sign}(z_i)^{k_i}$. In particular, an element $x$ of the center $F^\times$ of $G_F \subset G_\infty$ acts via $\text{sign}(\text{Nm}_F/Q(x)^k)$ if all the $k_i$’s have the same parity $k$.

Let $Z$ be the center of $G$, viewed as an algebraic group over $F$. Let $v_j, j = 1, \ldots, g$, be distinct nonarchimedean places of $F$ where $B$ is unramified, and fix identifications

$$B_{v_j} \simeq \text{Mat}_{2 \times 2}(F_{v_j}).$$

Set $G_j = \text{GL}_2(F_{v_j})$, $G = \prod_j G_j$, and $K_j = Z_{v_j} \text{GL}_2(O_{F,v_j})$. We view the vertices of the $(r_j = q_{v_j} + 1$-regular) tree $\Delta_j$ of $G_j$ as $\text{Ver} \Delta_j = G_j/K_j$, and set $\Delta = \prod_j \Delta_j$. For a compact open subgroup $K^0$ of $G^{t,v_1,\ldots,v_g}$ set $\Gamma = \Gamma(K^0) = G_F \cap K^0$. Then $\Gamma$ divided by its center acts discretely on $\Delta$, and if $K^0$ is sufficiently small the action is free. We can therefore use the notation and results of the previous section. Recall that the Ramanujan-Petersson conjecture asserts that the eigenvalues of the Hecke operator $T_{v_j}$ are bounded by $2\sqrt{q_{v_j}}$ on the automorphic forms on $\text{GL}_2(F)$ which are unramified at $v_j$. We now have the following

**Theorem 3.1.** 1. The cubical complex $X = \Gamma \setminus \Delta$ is irreducible, and the graph $\text{Gr}_{j,I}(X)$ is connected for any $j \not\in I \subseteq \{1, \ldots, g\}$.

2. Suppose that every element in $\Gamma$ which fixes a point on $\Delta$ acts trivially on $V^{\vec{k},-\vec{k}/2}$. Then the formula $\mathcal{L} = \mathcal{L}^{\vec{k},-\vec{k}/2} = \Gamma \setminus (\Delta \times V^{\vec{k},-\vec{k}/2})$ defines a local system on $X = \Gamma \setminus \Delta$, which is metrized, irreducible, and nontrivial unless all the $k_j$ are 0.

3. The cohomology groups $H^*(X, \mathcal{L})$ vanish except in dimensions 0 and $g$.

4. When $\mathcal{L}$ defines a local system on $X$, it is Ramanujan provided the Ramanujan-Petersson conjecture holds for holomorphic automorphic forms on $\text{GL}_2(F)$ of weight $(k_1 + 2, \ldots, k_g + 2)$.

**Proof:** 1. By the property of strong approximation the projection of $\Gamma$ to each strictly partial $G_j$ is dense, and the $G$-action on the (unoriented) $I$-cubes of $\Delta^{(I)}$ is transitive. The first fact implies that $X$ is irreducible, and the second that $\text{Gr}_{j,I}(X)$ is connected.
2. The given condition clearly implies that $L$ as defined is a local system on $X$. (In our case it simply means that if $z$ is a central element in $\Gamma$, then $\prod_{1 \leq i \leq g}(\text{sign}_\infty, x)^{k_i} = 1$.) In our case since $s = -\frac{k}{2}$, $\rho_{B, k/2}$ is unitary, and hence $L$ is metrized. The Eichler-Kneser strong approximation theorem implies that the image of $\Gamma$ in $G_\infty$ contains a Zariski dense subgroup of the group $(G_1)_\infty \simeq SU(2)^d$ of the norm $1$ elements of $G_\infty$, implying the irreducibility of $L_{k, -k/2}$. The nontriviality statement is clear.

3. This is a consequence of representation-theoretic results of Garland and Casselman, see [4, Chapter 13, Proposition 3.6(i)].

4. Hypothesis PAR holds since the elements of $\Gamma$ have norm $1$. We will deduce the Ramanujan property from bounds on Hecke eigenvalues as in [12, Theorem 3.4.2]. Define a finite $\vec{r}$-regular cubical complex $Y = Y(K^0)$ and a local system $L_y$ on $Y$ by

$$Y = G_F/(\Delta \times (G_{F, v_1, \ldots, v_g}/K^0)) \quad \text{and} \quad L_y = G_F/(\Delta \times (G_{F, v_1, \ldots, v_g}/K^0) \times V_{k, -k/2}),$$

with the diagonal $G_F$ action. The connected components of $Y$ are in bijection with the idèle class group $F^\times \backslash A_F^\times /\Nm_B/F(K/K^0)$ by the Eichler-Kneser strong approximation theorem. Moreover, the natural inclusion of $\Delta$ into $\Delta \times G_{F, v_1, \ldots, v_g}$ exhibits $X$ as a connected component of $Y$, with $\mathcal{L}$ the restriction of $L_y$ to $X$. By Proposition 2.6(2), the Ramanujan property is equivalent to bounds for the “interesting” eigenvalues of $S_{j, U}$ on each $C^0(Gr := Gr_{j, U}(Y), L_y), 1 \leq j \leq g$.

First observe that the stabilizer $U_j$ of a fixed $I^j$-cube $\sigma$ of $\Delta^{(j)}$ in $G$ is the product $\prod_{i \in I^j} U_i$ of the stabilizers $U_i$ of each $i$th edge factor of $\sigma$. Set $K = K^0 K_j U_j$. Then we are reduced to studying the eigenvalues of the star operator $S_{j, U}$ on

$$C^0(Gr, \mathcal{L}_y) = G_F/(\text{Ver}(\Delta_j) \times (G_{F, v_j}/K^0 U_j) \times V_{k, -k/2}) \quad = \quad G_F/((G_{F}/K^0 K_j U_j) \times V_{k, -k/2}).$$

Let $B_{k, -k/2}(G)$ be the space of continuous maps $\phi : G \rightarrow V_{k, -k/2}$ satisfying $\phi(g^f x) = \rho_{B, -k/2}(g_{\infty}) \phi(x)$, for any $g = g_{\infty} g^f \in (G_{\infty} \times G_{\text{f}}) \cap G_F$, with $G_{\text{f}}$ acting on it through right translations. We may view $C^0(Gr, \mathcal{L}_y)$ as the $K$-invariants of $B_{k, -k/2}(G)$, with $S_{j, U}$ corresponding to the Hecke operator $T_{U_j}$. We shall express this space as a space of automorphic forms for $B^\times$. First we decompose under the action of $Z_f$:

$$B_{k, -k/2}(G)^K = \oplus_{\omega \in \Omega(G, K^0)} B_{k, -k/2}(G, \omega)^K,$$

with the sum over the set $\Omega(G, K^0)$ characters of $Z_{\hbar}$ trivial on $Z_{\text{f}}$, on the connected component of the the identity of $Z_{\infty}$, and on $K \cap Z_{\text{f}}$. In particular, it is a finite set.

As in [11, Section 1.1], each $B_{k, -k/2}(G, \omega)$ is closely related to the space $\mathcal{A}(G, \omega)$ of automorphic forms on $B^\times$ in the sense of Jacquet-Langlands [10, Chapter 14]. (By the compactness of $G_{\infty}^1$ these can be viewed as the complex-valued continuous functions $f : G_F \rightarrow \mathbb{C}$ which are right $G_{\hbar}$-finite and which satisfy $f(gz) = \omega(z)f(g)$ for any $g \in G_{\hbar}$ and $z \in Z_{\hbar}$.) For an irreducible representation $\rho$ of $G_{\infty}^1$ which is isomorphic to
\( \rho_{G_\infty}^{\vec{k},-\vec{k}/2} \), the Peter-Weyl theory furnishes an isomorphism

\[
B^{\vec{k},-\vec{k}/2}(G, \omega) \simeq \text{Hom}_{G_\infty}(V^{\vec{k},-\vec{k}/2}, \mathcal{A}(G, \omega)) \simeq \text{Hom}_{G_\infty}(V^{\vec{k},-\vec{k}/2}, \mathcal{A}(G, \omega)).
\]

The second isomorphism holds because both \( \omega \) and \( \rho_{G_\infty}^{\vec{k},-\vec{k}/2} \) are trivial on the connected component of \( Z_\infty \simeq \prod_i F_{\infty_i}^\times \). The spaces involved are nonzero if and only if the central character of \( \rho^{k,s} \) agrees with \( \omega \) on \( Z_\infty \simeq \prod_i F_{\infty_i}^\times \). In the opposite direction, we can identify the \( \rho \)-isotypical part \( \mathcal{A}(G, \omega)^\rho \) of \( \mathcal{A}(G, \omega) \) with \( V^{\vec{k},-\vec{k}/2} \otimes_\mathbb{C} B^{\vec{k},-\vec{k}/2}(G, \omega) \); see, e.g., [11, Section 1.1] for explicit formulas for these isomorphisms (in the case \( F = \mathbb{Q} \), but the generalization is immediate). These isomorphisms are \( G^\mathbb{A} \)-equivariant. We need therefore to study the eigenvalues of \( T_{v_j} \) on \( \mathcal{A}(G, \omega)^K \) for each \( \omega \in \Omega(G, K^0) \).

Let \( \mathcal{A}(G, \omega)^K \) be the space of functions in \( \mathcal{A}(G, \omega)^K \) factorizing through the norm, and \( \mathcal{A}(G, \omega)^K \) its orthogonal complement. Then

\[
\mathcal{A}(G, \omega)^K = \mathcal{A}(G, \omega)^K \cap \mathcal{A}(G, \omega)^K.
\]

Moreover, \( \mathcal{A}(G, \omega)^K \) is the sum of character spaces \( V(\chi) \), each 1-dimensional, with \( \chi \) going over the idèle class characters trivial on \( \text{Nm}_{B/F}(K) \) and satisfying \( \chi^2 = \omega \). On the corresponding space \( T_{v_j} \) acts as the scalar \( \lambda_\chi = (q_{v_j} + 1)\chi(\pi_{v_j}) \), where \( \pi_{v_j} \) is a uniformizer for \( v_j \). Hence \( \lambda_\chi^2 = r_{v_j}^2 \omega(\pi_{v_j}) \), and since \( \pi_{v_j} \) is in \( Z_{v_j} \) we get \( \lambda_\chi^2 = \pm r_{v_j} \).

The elements of \( \mathcal{A}(G, \omega)^K \) lift Hecke-equivariantly to cusp forms on \( \text{GL}_2(F) \) by the Eichler-Shimizu-Jacquet-Langlands theory. The lift is injective, and its image is characterized by square integrability at each prime \( v \) which is ramified for \( B \). When \( v \) is infinite, the square integrability is essentially the same as holomorphy. Moreover, the weights correspond as indicated: to \( \text{Symm}^k \) at an infinite place correspond forms of weight \( k + 2 \) at that place on \( \text{GL}_2(F) \). Hence we have reduced the Ramanujan property for \( X \) to the Ramanujan-Petersson conjecture for holomorphic forms on \( \text{GL}_2(F) \) of the type claimed.

**Remarks 3.2.** 1. We need only special cases of the holomorphic Ramanujan-Petersson conjecture. Let \( B/F; v_1, \ldots, v_g; K^0 \); and \( \vec{k} \) be as above. To prove that \( \mathcal{L}^{\vec{k},-\vec{k}/2} \) is a Ramanujan local system on \( X \) only requires that the eigenvalues of \( T_{v_1}, \ldots, T_{v_g} \) on those automorphic forms on \( \text{GL}_2(F) \) which are lifts of \( \rho_{G_\infty}^{\vec{k},-\vec{k}/2} \)-isotypical forms on \( \mathcal{A}(G, \omega)^K \) with \( \omega \in \Omega(G, K^0) \) satisfy the Ramanujan-Petersson bound. These are the automorphic forms on \( \text{GL}_2(F) \), of level dividing \( K^0 \), of \( \infty \)-type as indicated, and of central character in \( \Omega(G, K^0) \) which are moreover square integrable at all places where \( B \) ramifies. However, the Ramanujan-Petersson conjecture seems not to be known in this generality.

2. Fortunately, when \( B \) is ramified at some finite place \( v \) (which always happens if \( [F : \mathbb{Q}] \) is odd), then the Ramanujan-Petersson conjecture is known. If \( F = \mathbb{Q} \) the automorphic form can be lifted to \( \text{GL}_2(\mathbb{Q}) \), where the Ramanujan-Petersson conjecture was proved by Deligne to follow from the Weil conjectures ([4]), which he subsequently proved ([5]). Else, let \( B'/F \) be a quaternion algebra ramified precisely at the places where \( B \) is except for \( \infty_1 \) and \( v \). Then our form can be lifted to \( B' \), and Carayol [5] showed that they occur
in the cohomology of a local system on a Shimura curve with good reduction at \( v \). The Ramanujan-Petersson conjecture then follows from the Weil-Deligne bounds \([1]\).

3. More cases of the Ramanujan-Petersson conjecture can undoubtedly be proved by the techniques of \([2]\) or of \([4]\). The scope of results that this might yield is unclear.

As a result, we get the following

**Theorem 3.3.** In the notation of Theorem 3.1 assume in addition that \( B \) is ramified in at least one finite place, and that all the \( k_i \)'s are of the same parity. Then \( L^{\vec{k}, -\vec{k}/2} \) is Ramanujan on \( X \).

4. **Explicit arithmetic examples**

Let \( p_j^{f_j} \) be \( g \) powers of rational primes \( p_j \), not necessarily distinct, and put \( r_j = p_j^{f_j} + 1 \). Our results suffice to give the following

**Theorem 4.1.** There exist infinitely many irreducible Ramanujan local systems over infinitely many irreducible \((r_1, \ldots, r_g)\)-regular complexes.

**Proof:** Let \( \Delta_j \) be an \( r_j \)-regular tree and put \( \Delta = \prod_j \Delta_j \). Let \( d \) be an integer \( \geq \max \left\{ \frac{p}{\text{a rational prime}} : \frac{\prod f_j}{\{ j \mid p_j = p \}} \right\} \).

There exists a totally real number field \( F \) of degree \( d \) over \( \mathbb{Q} \), and pairwise distinct finite places \( v_j \) of \( F \) whose residue fields have \( p_j^{f_j} \) elements. There exists a totally definite quaternion algebra \( B/F \) which is unramified at all the \( v_j \)'s and ramified over at least one finite prime (again, this is automatic if \( d \) is odd). Viewing each \( \Delta_j \) as the tree associated to \( B \times v_j \cong \text{GL}_2(F_{v_j}) \), we get an action of \( B^{\times} \) modulo its center \( F^{\times} \) on \( \Delta \).

Choose an order \( \mathcal{O} \) in \( B \), which for simplicity we take to contain \( \mathcal{O}_F \), and set \( S = \{ v_1, \ldots, v_g \} \). Let \( \mathcal{O}_{F,S} \) and \( \mathcal{M}_S \) be the localizations at \( S \), namely the elements of \( F \) and \( B \) that are integral outside of \( S \). For an ideal \( N \) of \( \mathcal{O}_F \) prime to \( S \) let \( \Gamma(N) \) be the principal congruence subgroup in \( \mathcal{M}_S \), namely the kernel of the (surjective) reduction \( \mathcal{M}_S^{\times} \to (\mathcal{M}/N\mathcal{M})^{\times} \) modulo \( N \). Suppose that \( N \) is sufficiently small. Then \( \Gamma(N) \) divided by its center acts freely on \( \Delta \). Now take \( \vec{k} = (k_1, \ldots, k_g) \), where the \( k_j \)'s are nonnegative integers of the same parity. Then \( \mathcal{L} = \Gamma(N) \setminus (\Delta \times V^{\vec{k}/2}) \) is a local system on \( X(N) = \Gamma(N) \setminus \Delta \) if the condition in Theorem 3.1(2) is satisfied. In our case it means that if the parity of the \( k_i \)'s is odd, then there are no elements in \( \mathcal{O}_{F,S}^{\times} \) congruent to 1 modulo \( N \) whose norm to \( \mathbb{Q} \) is negative. For example, this holds if either the \( k_i \)'s are even, or if \( -1 \) is not in the subgroup of \( (\mathbb{Z}/(N \cap \mathbb{Z}))^{\times} \) generated by the \( p_j^{f_j} \)'s.

Assuming this, \( \mathcal{L} \) is Ramanujan local system on \( X(N) \) by Theorem 3.3. If we fix the quaternion algebra and vary the level \( N \), we thus get an infinite family of complexes of the same regularities with growing number of vertices, proving the theorem.

In certain cases the construction takes a particularly simple form, in which, among other things, the \( \text{Gr}_j \)'s are all Cayley graphs on the same group. (The general case is
not much worse — see [12] Section 2.6 for the prototypical examples.) Namely, let us assume there is an ideal \( N_0 \neq 0 \) of \( \mathcal{O}_F \), prime to the \( v_j \)'s (we allow \( N_0 = \mathcal{O}_F \)), such that the following holds:

**Conditions 4.2.**

1. Every ideal of \( F \) has a totally positive generator \( \equiv 1 \mod N_0 \).
2. The class number of \( B \) is 1.
3. The units \( M^\times \) of a maximal order \( \mathcal{M} \) of \( B \) surject onto \( (\mathcal{M}/N_0\mathcal{M})^\times \), with the kernel being contained in the center \( \mathcal{O}_F^\times \) of \( M^\times \).

We then have the following two propositions:

**Proposition 4.3.** 1. For each \( 1 \leq j \leq g \) there are exactly \( r_j \) (principal) ideals \( \mathcal{P}_{j,i} \), \( 1 \leq i \leq r_j \), of \( \mathcal{M} \) whose norm to \( F \) is the prime ideal \( v_j \). This ideal has a totally positive generator, say \( \pi_j \), which is \( \equiv 1 \mod N_0 \). We can next choose generators \( \varepsilon_{j,i} \equiv 1 \mod N_0 \mathcal{M} \) for \( \mathcal{P}_{j,i} \), whose norm is \( \pi_j \), whose image \( \varepsilon_{j,i}^* \) under the main involution \( x \mapsto x^* \) of \( B \) is some \( \varepsilon_{j,i'} \) for some \( 1 \leq i' \leq r_j \) (\( i' = i \) may happen).
2. For every permutation \( \sigma \) of \( \{1, \ldots, g\} \) and any sequence of indices \( i_1, \ldots, i_g \), with \( 1 \leq i_j \leq r_j \), there is a (unique) sequence \( i'_1, \ldots, i'_g \), with \( 1 \leq i'_j \leq r_j \), and a (unique) unit \( u \in \mathcal{O}_F^\times \), satisfying \( u \equiv 1 \mod N_0 \), so that

\[
\varepsilon_{\sigma(1),i_1} \cdots \varepsilon_{\sigma(g),i_g} = u \varepsilon_{1,i'_1} \cdots \varepsilon_{g,i'_g}.
\]

**Proof:** We shall use the notations of Section 3. Fix generators \( \pi_j \) of \( v_j \) as required. Let \( \mathcal{M} \) be a maximal order in \( B \). For each finite prime \( v \) of \( F \) let \( \mathcal{M}_v \) be the completion of \( \mathcal{M} \) at \( v \), and set \( K_v = \mathcal{M}_v^\times \) and \( K^0 = \prod K_v \), the product taken over the finite \( v \)'s not among the \( v_j \)'s. Fix the identifications in (3) so that \( \mathcal{M}_{v_j} = \text{Mat}_{2 \times 2}(\mathcal{O}_F,v_j) \). The assumption that \( F \) and \( B \) have class number one implies, using the Eichler-Kneser strong approximation theorem, that the complex \( Y(K^0) \) coincides with the complex \( X = \Gamma(K^0) \backslash \Delta \) and that it has one vertex. Therefore there are elements \( \varepsilon_{j,i} \in \Gamma(K^0) \), for \( 1 \leq j \leq g \) and \( 1 \leq i \leq r_j \), mapping the vertex of \( \Delta \) fixed by \( K \) to its \( r_j \) neighbours of direction \( \{j\} \). Multiplying by an element in the center, we may assume that these elements are in \( \mathcal{M} \), and not divisible by any of the \( v_j \). Then the norm of each \( \varepsilon_{j,i} \) must be a generator of \( v_j \).

Multiplying by a unit in \( \mathcal{M} \) we may assume that \( \varepsilon_{j,i} \equiv 1 \mod N_0 \mathcal{M} \). Multiplying by a unit in the center \( \mathcal{O}_F \) we may further assume that, in addition, the norm is \( \pi_j \). This is because there are units in \( \mathcal{O}_F \), which have arbitrary signs at the infinite places of \( F \) and which are \( \equiv 1 \mod N_0 \). The result gives the required \( \varepsilon_{j,i} \)'s. For the second part, we see from the action on the complex that \( \varepsilon_{\sigma(1),i_1} \cdots \varepsilon_{\sigma(g),i_g} = \varepsilon_{1,i'_1} \cdots \varepsilon_{g,i'_g} u \) for some unit \( u \) of \( \mathcal{M} \). Then \( u \) must be \( \equiv 1 \mod N_0 \), and therefore in the center.

To formulate the second proposition we need to set some notation first. For a subset \( J \) of \( \{1, \ldots, g\} \) we will denote by \( \prod_{j \in J} g_j \) the product \( g_{j_1} \cdots g_{j_n} \), where the \( g_j \)'s are elements in any semigroup and the \( j_i \)'s are the elements of \( J \) in increasing order.

Let \( N_1 \) be a prime ideal of \( \mathcal{O}_F \) prime to the \( v_j \)'s and to \( N_0 \), and set \( N = N_0 N_1 \). Let \( A \) be the subgroup of \( (\mathcal{O}_F/N_1)^\times \) generated by the images modulo \( N_1 \) of the \( \pi_j \)'s. Let \( B \) be the subgroup of scalars in \( (\mathcal{M}/N_1\mathcal{M})^\times \) generated by the images modulo \( N_1 \mathcal{M} \) of the \( \pi_j \)'s and by those units of \( \mathcal{O}_F \) which are congruent to 1 modulo \( N_0 \). Let \( B' \)
be the subgroup of scalars in $(\mathcal{M}/N_1\mathcal{M})^\times$ generated the images modulo $N_1\mathcal{M}$ of the $\pi_j$'s and by those units of $O_F$ which are congruent to 1 modulo $N_0$ and whose norm to $\mathbb{Q}$ is positive (namely 1). The groups $H = \{ g \in (\mathcal{M}/N_1\mathcal{M})^\times \mid \text{Nm}(g) \in A \}/B$ and $H' = \{ g \in (\mathcal{M}/N_1\mathcal{M})^\times \mid \text{Nm}(g) \in A \}/B'$ are isomorphic to the groups $\text{SL}_2(O_F/N_1)$, $\text{PSL}_2(O_F/N_1)$, or $\text{PGL}_2(O_F/N_1)$. The latter case occurs if and only if at least one of the $\pi_j$'s is a square modulo $N_1$. Moreover, $H$ is a quotient of $H'$, and the kernel subgroup has order 1 if $-1$ is in the subgroup of $(\mathbb{Z}/(N_0 \cap \mathbb{Z}))^\times$ generated by the norms $\text{Nm}_{F/\mathbb{Q}} \pi_j \mod N_0 \cap \mathbb{Z}$, and has order 2 otherwise. If the kernel is 1 assume that the parity of the $k_i$'s is even. For $x \in \mathcal{M}_S$ let $\overline{x}$ and $\overline{x}'$ denote its respective reductions into $H$ and $H'$. These make sense because $\mathcal{M}_S/N_0\mathcal{M}_S$ is isomorphic to $\mathcal{M}/N_0\mathcal{M}$. Choose a section $s : H \to H'$. With this notation we have the following

**Proposition 4.4.** 1. The vertices of $X(N)$ are the elements of $H$.
2. The (oriented) cubes of direction $J \subset \{1, \ldots, g\}$ of $X(N)$ are the pairs $(v, I)$, where $v \in H = \text{Ver} X(N)$ and $I$ is a $|J|$-tuple $(i_j)_{j \in J}$, with $1 \leq i_j \leq r_j$.
3. For $j \in J \subset \{1, \ldots, g\}$, the $j$th bottom of the $J$-cube $(v, I)$ is the $J'$-cube $(\overline{v}_{J',i_j}, I')$, where $J' = J - \{ j \}$, and $I' = (i_j')_{j \in J'}$ is the $|J'| = (|J| - 1)$-tuple characterized by
   \[
   \prod_{k \in J} \overline{\omega}_{k,i_k} = u \overline{\omega}_{j,i_j} \prod_{k \in J'} \overline{\omega}_{k,i_k'},
   \]
   with $u$ a unit $u \in O_F^\times$ as in Proposition 4.3(2).
   With the same notation, $\text{inv}_j(v, I)$ is the $J$-cube $(\overline{v}_{J,i_j'}, I'')$, with $I''$ characterized by
   \[
   \overline{\omega}_{j,i_j} \prod_{k \in J'} \overline{\omega}_{k,i_k'} = u' \prod_{k \in J} \overline{\omega}_{k,i_k''}
   \]
   for an appropriate $u' \in O_F^\times$.
4. The fiber of $L$ over each $v \in H$ is $L(v) = V^{k_i - k_j/2}$. To describe the transition maps, let $e$ be an edge of direction $\{j\}$, say $e = (v, i) = (v, \{i\})$, with $1 \leq i \leq r_j$. Then $\text{bot}_je = v$ and $\text{top}_je = v \overline{\omega}_{j,i}$. Set $\epsilon = 1$ if $s(\text{top}_je) = s(v) \overline{\omega}_{j,i}$ and $\epsilon = -1$ otherwise. Then $L(e) : L(\text{bot}_je) \to L(\text{top}_je)$ is given by
   \[
   \epsilon^{k_i} \rho^{k_j - k_j/2}(\overline{\omega}_{j,i}).
   \]

**Proof:** Let $K^0(N)$ be the principal congruence subgroup of level $N$ is $K^0$. Using the Eichler-Kneser strong approximation theorem we see that the complex $Y(K^0(N))$ is described as $\Gamma(K^0) \backslash (\Delta \times K^0/CK^0(N))$, where $C$ is the center of $\Gamma(K^0)$. Then $X(N)$ is the connected component of $Y(K^0(N))$ lying under $\Delta \times \{1\}$. On the other hand, $K^0/CK^0(N) \simeq (\mathcal{M}/N\mathcal{M})^\times / B = H$; and since $\Gamma(K^0)$ acts transitively on the set $\text{Ver} \Delta \times (\mathcal{M}/N_0\mathcal{M})^\times$, part 1) follows readily. The rest is routine. We will show directly (without using $\Delta$) that the transition maps $L(e)$ as in 4) satisfy the flatness condition (ï). First notice that $v' = \text{top}_{j_2} \text{top}_{j_1} \sigma$ indeed coincides with $\text{top}_{j_1} \text{top}_{j_2} \sigma$ for any 2-cell $\sigma = (v, i_{j_1}, i_{j_2})$, with $1 \leq i_j \leq r_j$ for $j = j_1, j_2$. Hence the lifts of $v$ and of $v'$ are related
by

\[ s(v') = e' \varepsilon \omega_{j_1,i_1} \varepsilon \omega_{j_2,i_2} s(v) = e'' \varepsilon \omega_{j_2,i_2} \varepsilon \omega_{j_1,i_1} \tilde{v}, \]

where the product of the signs corresponding to \( e' \) and \( e'' \) is the image in \( H \) of \( u \) from equation (4). Condition (3) is an immediate consequence.

The *girth* of a cubical complex is the length, namely the number of edges, of the shortest homotopically nontrivial closed path. Set \( q = \max_j p_j^j \) and \( n = Nm_{F/Q} N_1 \). We now have the following generalization of [14, Theorem 7.3.7]

**Proposition 4.5.** Assume that the group \( H \) of \( X(N) \) is \( SL_2(O_{F}/N_1) \). Then the girth of \( X(N) \) is at least \( 2\log_q(N^2/4^d) \). Put differently,

\[ \text{girth } X(N) \geq \frac{4}{3} \log_q \# \text{Ver } X(N) - \text{ constant}. \]

**Proof:** As in the 1-dimensional case, the girth is \( \min \text{dist}(x, \gamma x) \), the minimum taken over all vertices \( x \) of the universal cover \( \Delta \) and noncentral elements \( \gamma \) of \( \Gamma = \Gamma(K^0) \). Fix such \( \gamma \) and \( x \) where the minimum occurs. Then the distance is, by definition, the sum of the distances \( \text{dist}_j \) of the projections to the \( j \)th tree factor for \( 1 \leq j \leq g \). Moreover, \( \text{dist}_j \geq -2 \text{val}_{v_j} \text{Tr}_{B/F} \gamma \), exactly as in loc. cit., Lemma 7.3.2. We have \( Nm_{B/F}(\gamma - 1) = 2 - \text{Tr}_{B/F} \gamma \). Also, since \( G_{\infty i} \simeq SU(2) \) we have \( |\text{Tr}_{B/F} \gamma|_{\infty i} \leq 2 \) for all \( 1 \leq i \leq d \). Taking norms to \( Q \) gives that \( Nm_{F/Q}(2 - \text{Tr}_{B/F} \gamma) \) is of the form \( n^2m/\prod_j p_j^{l_j} \) for some \( m \), necessarily \( \geq 1 \), where \( l_j = -\text{val}_{v_j}(2 - \text{Tr}_{B/F} \gamma) \). Hence

\[ |Nm_{F/Q}(2 - \text{Tr}_{B/F} \gamma)| \leq \prod_{1 \leq i \leq d} (2 + |\text{Tr}_{B/F} \gamma|_{\infty i}) \leq 4^d, \]

so that \( n^2/4 \leq \prod_j p_j^{l_j} \leq q^{\sum_j l_j} \). Therefore,

\[
\begin{align*}
\text{girth} &= \sum_j \text{dist}_j \geq -2 \sum_j \min(0, \text{val}_{v_j} \text{Tr}_{F/Q} \gamma) = 2 \sum_{l_j > 0} l_j \geq 2 \sum_j l_j \\
&\geq 2 \log_q \left( n^2/4^d \right)
\end{align*}
\]

On the other hand, the cardinality of \( \text{Ver } X(N) \) is at least that of \( \text{PSL}_2(O_{F}/N_1) \), which is at least a positive constant \( c \) times \( n^3 \). This implies the last assertion.

**Examples 4.6.** In [22, Chapter 5] one finds examples satisfying Conditions 4.2. These suffice to give many types of examples. However the regularities are generally assumed to satisfy some congruence conditions besides of being primes or prime powers of a specific type. To get rid of such restrictions it appears necessary to consider the general cases, whose finite description is somewhat messier and hence not discussed here.

**A.** Take \( F = Q \) and let \( B_2 \) have discriminant 2. When \( S \) consists of \( g = 1 \) prime \( p \equiv 1 \text{ mod } 4 \), one gets the Lubotzky-Phillips-Sarnak graphs [15]. The nontrivial local systems over them are handled in [12]. Notice that the sign \( \epsilon \) in (3) is mistakenly missing there. The case of \( g = 2 \) distinct primes \( p \equiv q \equiv 1 \text{ mod } 4 \), without local systems, is studied for different reasons in [17]. The resulting complexes are \((p + 1, q + 1)-regular\).

**B.** The case \( F = Q \), with \( B_3 \) of discriminant 3 and \( N_0 = 2 \), satisfies the Conditions 4.2.
C. The case $F = \mathbb{Q}$, with $B_{13}$ of discriminant $13$ and $N_0 = 1$, satisfies the Conditions 4.2. It is used to get $3$-regular Ramanujan graphs in [6].

D. To get regularities other than $p + 1$ one must use fields other than $\mathbb{Q}$. The simplest case occurs when $F = \mathbb{Q}(\sqrt{5})$ and $B_F = B_2 \otimes F$, with $B_2$ as in A, and $N_1 = 2O_F$. Then $B_F$ is totally definite but unramified at all the finite places. If we assume the Ramanujan-Petersson conjecture for the relevant automorphic forms as in Theorem 3.1, then this gives irreducible Ramanujan local systems over infinitely many $p^2 + 1$-regular graphs for all primes $p \equiv \pm 1 \mod 5$. It also gives irreducible Ramanujan local systems over infinitely many $(p + 1, p + 1)$-regular complexes for all primes $p \equiv \pm 1 \mod 5$. Of course one can combine regularities to get higher dimensional examples. Unfortunately, Theorem 3.3 does not apply and we have not found an adequate reference in the literature to prove the Ramanujan-Petersson conjecture in this case.

E. For our last example take $F = \mathbb{Q}(\cos 2\pi/7)$, and set $B_F = B_2 \otimes F$, with $B_2$ as before. Then $B_F$ is ramified precisely at the prime $2$ of $F$ and at the $3$ infinite places of $F$. Here Theorem 3.3 applies. This example allows therefore to get infinitely many Ramanujan graphs of regularity $p^3 + 1$ (and Ramanujan local systems on them) for any $p$ which is not a square modulo $7$; see [10] for more general examples coming from function fields (with the trivial local system). Our present example also gives infinitely many $(p + 1, p + 1)$-regular and $(p + 1, p + 1, p + 1)$-regular Ramanujan complexes and Ramanujan local systems on them for any $p$ which is a square modulo $7$. Again, different such regularities may be combined to form higher dimensional examples.

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