On the scalar consistency relation away from slow roll

V. Sreenath, Dhiraj Kumar Hazra and L. Sriramkumar

$^a$Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India
$^b$Asia Pacific Center for Theoretical Physics, Pohang, Gyeongbuk 790-784, Korea

E-mail: sreenath@physics.iitm.ac.in, dhiraj@apctp.org, sriram@physics.iitm.ac.in

Received October 6, 2014
Revised January 2, 2015
Accepted January 27, 2015
Published February 23, 2015

Abstract. As is well known, the non-Gaussianity parameter $f_{NL}$, which is often used to characterize the amplitude of the scalar bi-spectrum, can be expressed completely in terms of the scalar spectral index $n_s$ in the squeezed limit, a relation that is referred to as the consistency condition. This relation, while it is largely discussed in the context of slow roll inflation, is actually expected to hold in any single field model of inflation, irrespective of the dynamics of the underlying model, provided inflation occurs on the attractor at late times. In this work, we explicitly examine the validity of the consistency relation, analytically as well as numerically, away from slow roll. Analytically, we first arrive at the relation in the simple case of power law inflation. We also consider the non-trivial example of the Starobinsky model involving a linear potential with a sudden change in its slope (which leads to a brief period of fast roll), and establish the condition completely analytically. We then numerically examine the validity of the consistency relation in three inflationary models that lead to the following features in the scalar power spectrum due to departures from slow roll: (i) a sharp cut off at large scales, (ii) a burst of oscillations over an intermediate range of scales, and (iii) small, but repeated, modulations extending over a wide range of scales. It is important to note that it is exactly such spectra that have been found to lead to an improved fit to the CMB data, when compared to the more standard power law primordial spectra, by the Planck team. We evaluate the scalar bi-spectrum for an arbitrary triangular configuration of the wavenumbers in these inflationary models and explicitly illustrate that, in the squeezed limit, the consistency condition is indeed satisfied even in situations consisting of strong deviations from slow roll. We conclude with a brief discussion of the results.

Keywords: inflation, non-gaussianity, cosmological perturbation theory

ArXiv ePrint: 1410.0252
## Contents

1 Introduction ........................................................................................................... 1

2 The scalar bi-spectrum in the squeezed limit ...................................................... 3
   2.1 The scalar power spectrum and bi-spectrum ............................................. 3
   2.2 The consistency relation ....................................................................... 5

3 Analytically examining the validity of the condition away from slow roll ...... 6
   3.1 The simple example of power law inflation ........................................... 6
   3.2 A non-trivial example involving the Starobinsky model ....................... 8

4 Numerical verification of the relation during deviations from slow roll ........... 12
   4.1 \( f_{\text{NL}} \) for an arbitrary triangular configuration of the wavenumbers 13
   4.2 \( f_{\text{NL}} \) in the squeezed limit ......................................................... 14

5 Discussion ............................................................................................................. 18

## 1 Introduction

Over the past two decades, cosmologists have dedicated a considerable amount of attention to hunting down credible models of inflation. The inflationary scenario, which is often invoked to resolve certain puzzles (such as the horizon problem) that plague the hot big bang model, is well known to provide an attractive mechanism for the origin of perturbations in the early universe \([1–28]\). In the modern viewpoint, it is the primordial perturbations generated during inflation that leave their signatures as anisotropies in the Cosmic Microwave Background (CMB) and later lead to the formation of the large scale structure. Ever since the discovery of the CMB anisotropies by COBE \([29–32]\), there has been a constant endeavor to utilize cosmological observations to arrive at stronger and stronger constraints on models of inflation. While the CMB anisotropies have been measured with ever increasing precision by missions such as WMAP \([33–38]\), Planck \([39–41] \) and, very recently, by BICEP2 \([42, 43]\), it would be fair to say that we still seem rather far from converging on a small class of well motivated and viable inflationary models (in this context, see refs. \([44–47]\)).

The difficulty in arriving at a limited set of credible models of inflation seems to lie in the simplicity and efficiency of the inflationary scenario. Inflation can be easily achieved with the aid of one or more scalar fields that are slowly rolling down a relatively flat potential. Due to this reason, a plethora of models of inflation have been proposed, which give rise to the required 60 or so e-folds of accelerated expansion that is necessary to overcome the horizon problem. Moreover, there always seem to exist sufficient room to tweak the potential parameters in such a way so as to result in a nearly scale invariant power spectrum of the scalar perturbations that lead to a good fit to the CMB data. In such a situation, non-Gaussianities in general and the scalar bi-spectrum in particular have been expected to lift the degeneracy prevailing amongst the various inflationary models. For convenience, the extent of non-Gaussianity associated with the scalar bi-spectrum is often expressed in terms of the parameter commonly referred to as \( f_{\text{NL}} \) \([48]\), a quantity which is a dimensionless ratio of the scalar bi-spectrum to the power spectrum. The expectation regarding non-Gaussianities...
alluded to above has been largely corroborated by the strong limits that have been arrived at by the Planck mission on the value of the $f_{NL}$ parameter [49]. These bounds suggest that the observed perturbations are consistent with a Gaussian primordial distribution. Also, the strong constraints imply that exotic models which lead to large levels of non-Gaussianities are ruled out by the data.

Despite the strong bounds that have been arrived at on the amplitude of the scalar bi-spectrum, there exist many models of inflation that remain consistent with the cosmological data at hand. The so-called scalar consistency relation is expected to play a powerful role in this regard, ruling out, for instance, many multi-field models of inflation, if it is confirmed observationally (for early discussion in this context, see, for instance, refs. [50, 51]; for recent discussions, see refs. [52–58]; for similar results that involve the higher order correlation functions, see, for example, refs. [59–64]). According to the consistency condition, in the squeezed limit of the three-point functions wherein one of the wavenumbers associated with the perturbations is much smaller than the other two, the three-point functions can be completely expressed in terms of the two-point functions.\footnote{It should be added here that, in a fashion similar to that of the purely scalar case, one can also arrive at consistency conditions for the other three-point functions which involve tensors [65–73].} In the squeezed limit, for instance, the scalar non-Gaussianity parameter $f_{NL}$ can be expressed completely in terms of the scalar spectral index $n_S$ as $f_{NL} = 5(n_S - 1)/12$ [50, 51]. As we shall briefly outline later, the consistency conditions are expected to hold [73] whenever the amplitude of the perturbations freeze on super-Hubble scales, a behavior which is true in single field models where inflation occurs on the attractor at late times (see refs. [1–8]; in this context, also see refs. [74–76]). While the scalar consistency relation has been established in the slow roll scenario, we find that there has been only a limited effort in explicitly examining the relation in situations consisting of periods of fast roll [77–83]. Moreover, it has been shown that there can be deviations from the consistency relation under certain conditions, particularly when the field is either evolving away from the attractor [84, 85] or when the perturbations are in an excited state above the Bunch-Davies vacuum [86–88]. In this work, our aim is to verify the validity of the scalar consistency relation in inflationary models which exhibit non-trivial dynamics. By considering a few examples, we shall explicitly show, analytically and numerically, that the scalar consistency relation holds even in scenarios involving strong deviations from slow roll.

The remainder of this paper is organized as follows. In the next section, we shall quickly summarize a few essential points and results concerning the scalar power spectrum and the bi-spectrum. We shall also briefly revisit the proof of the scalar consistency relation in the squeezed limit. In the succeeding section, we shall explicitly verify the validity of the consistency condition analytically in the cases of power law inflation and the Starobinsky model which is described by a linear potential with a sudden change in its slope. We shall then evaluate the scalar bi-spectrum numerically for an arbitrary triangular configuration of the wavenumbers in three inflationary models that lead to features in the power spectrum, and examine the consistency condition in the squeezed limit. We conclude with a brief discussion on the results we obtain.

A few remarks on our conventions and notations seem essential at this stage of our discussion. We shall work with natural units wherein $\hbar = c = 1$, and define the Planck mass to be $M_{Pl} = (8\pi G)^{-1/2}$. We shall adopt the signature of the metric to be $(-, +, +, +)$. We shall assume the background to be the spatially flat, Friedmann-Lemaître-Robertson-Walker (FLRW) line element that is described by the scale factor $a$ and the Hubble parameter $H$. As is convenient, we shall switch between various parametrizations of time, viz. the cosmic
time \( t \), the conformal time \( \eta \) or e-folds denoted by \( N \). An overdot and an overprime shall represent differentiation with respect to the cosmic and the conformal time coordinates, respectively. We shall restrict our attention in this work to inflationary models involving the canonical scalar field. Note that, in such a case, the first and second slow roll parameters are defined as \( \epsilon_1 = -\dot{H}/H^2 \) and \( \epsilon_2 = \text{d} \ln \epsilon_1/\text{d}N \).

2 The scalar bi-spectrum in the squeezed limit

In this section, we shall quickly summarize the essential definitions and governing expressions concerning the scalar power spectrum, the bi-spectrum and the corresponding non-Gaussianity parameter \( f_{\text{NL}} \). We shall also sketch a simple proof of the consistency relation obeyed by the non-Gaussianity parameter \( f_{\text{NL}} \) in the squeezed limit of the scalar bi-spectrum.

2.1 The scalar power spectrum and bi-spectrum

Consider the following line-element which describes the spatially flat, FLRW spacetime, when the scalar perturbations, characterized by the curvature perturbation \( R \), have been taken into account:

\[
\text{d}s^2 = -\text{d}t^2 + a^2(t) e^{2R(t,x)} \text{d}x^2. \tag{2.1}
\]

Let \( f_k \) denote the Fourier modes associated with the curvature perturbation at the linear order in the perturbations. In the case of inflation driven by the canonical scalar field that is of our interest here, the modes \( f_k \) satisfy the differential equation

\[
f_k'' + 2\frac{z'}{z} f_k' + k^2 f_k = 0, \tag{2.2}
\]

where \( z = \sqrt{2\epsilon_1} M_p a \). Upon quantization, the curvature perturbation can be decomposed in terms of the Fourier modes \( f_k \) as

\[
\hat{R}(\eta, x) = \int \frac{\text{d}^3k}{(2\pi)^{3/2}} \hat{R}_k(\eta) \ e^{i k \cdot x} = \int \frac{\text{d}^3k}{(2\pi)^{3/2}} \left[ \hat{a}_k f_k(\eta) e^{i k \cdot x} + \hat{a}_k^\dagger f_k^*(\eta) e^{-i k \cdot x} \right], \tag{2.3}
\]

where \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) are the usual creation and annihilation operators that obey the standard commutation relations.

The scalar power spectrum \( P_s(k) \) is defined in terms of the two-point correlation function of the curvature perturbation as follows:

\[
\langle 0 | \hat{R}_k(\eta) \hat{R}_{k'}(\eta) | 0 \rangle = \frac{(2\pi)^2}{2k^3} P_s(k) \delta^{(3)}(k + k'), \tag{2.4}
\]

where \( | 0 \rangle \) denotes the Bunch-Davies vacuum annihilated by the operator \( \hat{a}_k \) [89]. In terms of the modes \( f_k \), the scalar power spectrum \( P_s(k) \) is given by

\[
P_s(k) = \frac{k^3}{2\pi^2} |f_k|^2. \tag{2.5}
\]

The inflationary model governed by a given potential determines the behavior of the quantity \( z \). In order to arrive at the scalar power spectrum using the above expression, one first
solves the differential equation (2.2) for the modes \( f_k \) with the Bunch-Davies initial conditions [89], and then evaluates the amplitude of the modes at sufficiently late times when they are well outside the Hubble radius during inflation. The scalar spectral index is defined as

\[
\eta_s(k) = 1 + \frac{d \ln P_s(k)}{d \ln k} .
\] (2.6)

It should be stressed here that the scalar spectral index proves to be a constant only in simple situations such as power law and slow roll inflation. In general, when the power spectrum contains features, the quantity \( \eta_s \) depends on the wavenumber \( k \).

The scalar bi-spectrum \( B_s(k_1, k_2, k_3) \) evaluated, say, in the vacuum state \(|0\rangle\), is defined in terms of the three-point function of the curvature perturbation as follows [35, 36] :

\[
\langle 0 | \hat{R}_{k_1} \hat{R}_{k_2} \hat{R}_{k_3} | 0 \rangle = (2 \pi)^3 B_s(k_1, k_2, k_3) \delta^{(3)}(k_1 + k_2 + k_3) .
\] (2.7)

Note that the delta function on the right hand side imposes the triangularity condition, viz. that the three wavevectors \( k_1, k_2 \) and \( k_3 \) have to form the edges of a triangle. For the sake of convenience, we shall set

\[
B_s(k_1, k_2, k_3) = (2 \pi)^{-9/2} G(k_1, k_2, k_3) .
\] (2.8)

The non-Gaussianity parameter \( f_{\text{NL}} \) that is often used to characterize the extent of non-Gaussianity indicated by the bi-spectrum is introduced through the relation [48]

\[
R(\eta, x) = R_G(\eta, x) - \frac{3 f_{\text{NL}}}{5} \left[ R_G^2(\eta, x) - \langle R_G^2(\eta, x) \rangle \right] ,
\] (2.9)

where \( R_G \) denotes the Gaussian part of the curvature perturbation. Using the above relation and Wick’s theorem (which applies to Gaussian perturbations), one can arrive at the following expression for the dimensionless non-Gaussianity parameter \( f_{\text{NL}} \) in terms of the bi-spectrum \( G(k_1, k_2, k_3) \) and the scalar power spectrum \( P_s(k) \):

\[
f_{\text{NL}}(k_1, k_2, k_3) = -\frac{10}{3} \frac{1}{(2 \pi)^4} (k_1 k_2 k_3)^3 \frac{1}{8} \frac{1}{(2 \pi)^4} (k_1 k_2 k_3)^3 G(k_1, k_2, k_3)
\]

\[
\times [k_1^2 P_s(k_2) P_s(k_3) + \text{two permutations}]^{-1} .
\] (2.10)

The scalar bi-spectrum generated during inflation can be evaluated using the Maldacena formalism [50]. The approach basically makes use of the third order action governing the curvature perturbation and the standard rules of perturbative quantum field theory to arrive at the scalar three-point function [50, 90–96]. It is found that, in the case of inflation driven by the canonical scalar field, the third order action consists of six terms and the scalar bi-spectrum receives a contribution from each of these ‘vertices’. In fact, there also occurs a seventh term which arises due to a field redefinition, a procedure which is necessary to reduce the action to a simpler form. One can show that the complete contribution to the
2.3 The contribution to the bi-spectrum

Conditions are imposed on the modes until very late times, say, towards the end of inflation

The squeezed limit refers to the case wherein one of the wavenumbers of the triangular configuration vanishes, say, \( k_3 \to 0 \), leading to \( k_2 = -k_1 \). Or, equivalently, one of the modes is assumed to possess a wavelength which is much larger than the other two. The long wavelength mode would be well outside the Hubble radius. In models of inflation driven by a single scalar field, the amplitude of the curvature perturbation freezes on super-Hubble scales, provided the inflaton evolves on the attractor at late times \([74–76, 84, 85]\). As a result, the long wavelength mode simply acts as a background as far as the other two modes are
concerned. If $R^B$ is the amplitude of the curvature perturbation associated with the long wavelength mode, then the unperturbed part of the original FLRW metric will be modified to

$$ds^2 = -dt^2 + a^2(t) e^{2R^B} \, dx^2.$$  

(2.14)

In other words, the effect of the long wavelength mode is to modify the scale factor locally, which is equivalent to a spatial transformation of the form $x' = \Lambda x$, with the components of the matrix $\Lambda$ being given by $\Lambda_{ij} = e^{R^B} \delta_{ij}$. Under such a transformation, the modes of the curvature perturbation transform as $R_k \rightarrow \text{det} (\Lambda^{-1}) R_{\Lambda^{-1}k}$. Further, we have $|\Lambda^{-1}k| = (1 - R^B) k$ and $\delta^{(3)} (\Lambda^{-1} k_1 + \Lambda^{-1} k_2) = \text{det} (\Lambda) \delta^{(3)} (k_1 + k_2)$. Utilizing these relations, the scalar two-point function can be written as

$$\langle \hat{R}_{k_1} \hat{R}_{k_2} \rangle_k = \frac{(2\pi)^2}{2k_3^2} P_s (k_1) \left[ 1 - (n_s - 1) R^B \right] \delta^{(3)} (k_1 + k_2),$$  

(2.15)

where the suffix $k$ on the two-point function indicates that the correlator has been evaluated in the presence of a long wavelength perturbation. Upon using the above expression for the scalar power spectrum, we can write the scalar bi-spectrum in the squeezed limit as [51, 72, 73]

$$\langle \hat{R}_{k_1} \hat{R}_{k_2} \hat{R}_{k_3} \rangle_k \equiv \langle \langle \hat{R}_{k_1} \hat{R}_{k_2} \rangle_{k_3} \hat{R}_{k_3} \rangle = \frac{(2\pi)^{5/2}}{4 k_1^3 k_2^3 k_3^3} (n_s - 1) P_s (k_1) P_s (k_2) \delta^{(3)} (k_1 + k_2).$$  

(2.16)

On making use of this expression for the scalar bi-spectrum in the squeezed limit and the definition of the scalar power spectrum, one can immediately arrive at the consistency relation for $f_{NL}$, viz. that $f_{NL} = 5 (n_s - 1)/12$ [50–58].

3 Analytically examining the validity of the condition away from slow roll

As was outlined in the previous section, the only requirement for the validity of the consistency relation is the existence of a unique clock during inflation. Hence, in principle, this relation should be valid for any single field model of inflation irrespective of the detailed dynamics, if the field is evolving on the attractor at late times. Therefore, it should be valid even away from slow roll. In this section, we shall analytically examine the validity of the consistency condition in scenarios consisting of deviations from slow roll. After establishing the relation first in the simple case of power law inflation, we shall consider the Starobinsky model which involves a brief period of fast roll.

3.1 The simple example of power law inflation

We shall first consider the case of power law inflation with no specific constraints on the power law index, so that the behavior of the scale factor can be far different from that of its behavior in slow roll inflation. In power law inflation, the scale factor can be written as

$$a(\eta) = a_1 \left( \frac{\eta}{\eta_1} \right)^{\gamma + 1},$$  

(3.1)

where $a_1$ and $\eta_1$ are constants, and $\gamma < -2$. In such a background, the Fourier modes $f_k$ associated with the curvature perturbation that satisfy the Bunch-Davies initial conditions are found to be [98, 100–103]

$$f_k (\eta) = \frac{1}{\sqrt{2} \epsilon_1 M_{Pl} a(\eta)} \sqrt{-\frac{\pi \eta}{4}} e^{-i \pi \gamma/2} H_{-(\gamma + 1/2)} (-k \eta),$$  

(3.2)
where the first slow roll parameter $\epsilon_1$ is a constant given by $\epsilon_1 = (\gamma + 2)/(\gamma + 1)$. Note that $H^{(1)}_\nu(x)$ denotes the Hankel function of the first kind [104], while the scale factor $a(\eta)$ is given by eq. (3.1). For real arguments, the Hankel functions of the first and the second kinds, viz. $H^{(1)}_\nu(x)$ and $H^{(2)}_\nu(x)$, are complex conjugates of each other [104]. Moreover, as $x \to 0$, the Hankel function has the following form

$$
\lim_{x \to 0} H^{(1)}_\nu(x) = \frac{i}{\pi \nu} \left[ \Gamma(1 - \nu) e^{-i \pi \nu} \left( \frac{x}{2} \right)^\nu - \Gamma(1 + \nu) \left( \frac{x}{2} \right)^{-\nu} \right]. \quad (3.3)
$$

Upon using this behavior, one can show that the corresponding scalar power spectrum, evaluated at late times, i.e. as $\eta \to -0$, is given by

$$
P_s(k) = \frac{1}{2 \pi^2 M_{Pl}^2 \epsilon_1} \left( \frac{|\eta|}{a} \right)^2 \left( |\Gamma[-(\gamma + 1/2)]| \right)^2 \left( \frac{k}{2} \right)^{2(\gamma + 2)}, \quad (3.4)
$$

where $\Gamma(x)$ represents the Gamma function [104]. The scalar spectral index corresponding to such a power spectrum is evidently a constant and can be easily determined to be $n_s = 2\gamma + 5$. If the consistency condition is true, it would then imply that the scalar non-Gaussianity parameter has the value $f_{NL} = 5(\gamma + 2)/6$ in the squeezed limit.

Let us now evaluate the scalar bi-spectrum in the squeezed limit using the Maldacena formalism and illustrate that it indeed leads to the above consistency condition for $f_{NL}$. It should be clear that, in order to arrive at the complete scalar bi-spectrum, we first need to carry out the integrals (2.12) associated with the six vertices, calculate the corresponding contributions $G_C(k_1, k_2, k_3)$ for $C = (1,6)$, and lastly add the contribution $G_7(k_1, k_2, k_3)$ [cf. eq. (2.13)] that arises due to the field redefinition. However, since $\epsilon_1$ is a constant in power law inflation, the second slow roll parameter $\epsilon_2$ vanishes identically. As a result, the contribution corresponding to the fourth term that is determined by the integral (2.12d) as well as the seventh term $G_7(k_1, k_2, k_3)$ prove to be zero. Moreover, in the squeezed limit of our interest, i.e. as $k_3 \to 0$, the amplitude of the mode $f_{k_3}$ freezes and hence its time derivative goes to zero. Therefore, terms that are either multiplied by the wavenumber corresponding to the long wavelength mode or explicitly involve the time derivative of the long wavelength mode do not contribute, as both vanish in the squeezed limit. Due to these reasons, one finds that it is only the first and the second terms, determined by the integrals (2.12a) and (2.12b), that contribute in power law inflation. After an integration by parts, we find that, in the squeezed limit, these two integrals can be combined to be expressed as

$$
\lim_{k_3 \to 0} [G_1(k, -k, k_3) + G_2(k, -k, k_3)] = \lim_{k_3 \to 0} 2 i \epsilon_1^2 f'_{k_3} \left[ (a^2 f^*_k f^*_k)^{0}_{-\infty} + 2k^2 \int_{-\infty}^{0} d\eta a^2 f^*_k \right], \quad (3.5)
$$

where we have set $\eta_i = -\infty$ and $\eta_c = 0$. One can show that the derivative $f'_{k_3}$ can be written as

$$
f'_{k_3}(\eta) = \frac{-k}{\sqrt{2 \epsilon_1 M_{Pl} a(\eta)}} \sqrt{-\frac{\pi \eta}{4}} e^{-i \pi \gamma/2} H^{(1)}_{-(\gamma + 3/2)}(-k \eta). \quad (3.6)
$$

Therefore, upon using this expression for the derivative $f'_{k}$, the behavior (3.3), the following asymptotic form of the Hankel function

$$
\lim_{x \to \infty} H^{(1)}_\nu(x) = \sqrt{\frac{2}{\pi x}} e^{i(x - \pi \nu/2 - \pi/4)}, \quad (3.7)
$$
and the integral [104]

$$\int dx \left[ H^{(1)}_{\nu}(x) \right]^2 \approx \frac{x^2}{2} \left\{ \left[ H^{(1)}_{\nu}(x) \right]^2 - H^{(1)}_{\nu-1}(x) H^{(1)}_{\nu+1}(x) \right\},$$

(3.8)

we find that the bi-spectrum in the squeezed limit can be written as

$$\lim_{k_3 \to 0} k_3^3 k_3^3 G(k, -k, k_3) = -8 \pi^4 (\gamma + 2) P_s(k) P_s(k_3).$$

(3.9)

This expression and the definition (2.10) for the scalar non-Gaussianity parameter then leads to

$$f_{NL} = \frac{5(\gamma + 2)}{6},$$

which is the result suggested by the consistency relation. We should add here that such a result has been arrived at earlier using a slightly different approach (see the third reference in refs. [52–58]).

### 3.2 A non-trivial example involving the Starobinsky model

The second example that we shall consider is the Starobinsky model. In the Starobinsky model, the inflaton rolls down a linear potential which changes its slope suddenly at a particular value of the scalar field [105]. The governing potential is given by

$$V(\phi) = \begin{cases} V_0 + A_+ (\phi - \phi_0) & \text{for } \phi > \phi_0, \\ V_0 + A_- (\phi - \phi_0) & \text{for } \phi < \phi_0, \end{cases}$$

(3.10)

where $V_0, A_+, A_-$ and $\phi_0$ are constants. An important aspect of the Starobinsky model is the assumption that it is the constant $V_0$ which dominates the value of the potential around $\phi_0$. Due to this reason, the scale factor always remains rather close to that of de Sitter. This in turn implies that the first slow roll parameter $\epsilon_1$ remains small throughout the domain of interest. However, the discontinuity in the slope of the potential at $\phi_0$ causes a transition to a brief period of fast roll before slow roll is restored at late times. One finds that the transition leads to large values for the second slow roll parameter $\epsilon_2$ and, importantly, the quantity $\dot{\epsilon}_2$ grows to be even larger, in fact, behaving as a Dirac delta function at the transition. As we shall discuss, it is this behavior that leads to the most important contribution to the scalar bi-spectrum in the model [97, 106–108].

Clearly, it would be convenient to divide the evolution of the background quantities and the perturbation variables into two phases, before and after the transition at $\phi_0$. In what follows, we shall represent the various quantities corresponding to the epochs before and after the transition by a plus sign and a minus sign (in the super-script or sub-script, as is convenient), while the values of the quantities at the transition will be denoted by a zero. Let us quickly list out the behavior of the different quantities which we shall require to establish the consistency relation.

The first slow roll parameter before and after the transition is found to be [97, 105–108]

$$\epsilon_{1+}(\eta) \simeq \frac{A_+^2}{18 M^2_{\text{Pl}} H_0^2},$$

(3.11a)

$$\epsilon_{1-}(\eta) \simeq \frac{A_-^2}{18 M^2_{\text{Pl}} H_0^2} \left[ 1 - \frac{\Delta A}{A_-} \left( \frac{\eta}{\eta_0} \right) \right]^2,$$

(3.11b)

where $\Delta A = A_- - A_+$, $H_0$ is the Hubble parameter determined by the relation $H_0^2 \simeq V_0/ (3 M^2_{\text{Pl}})$, and $\eta_0$ denotes the conformal time when the transition takes place. The second
slow roll parameter is given by

\[ \varepsilon_2^+(\eta) = 4 \varepsilon_{1+}, \quad (3.12a) \]
\[ \varepsilon_2^- (\eta) = \frac{6 \Delta A}{A_-} \left( \frac{\eta}{\eta_0} \right)^3 - \frac{(\eta/\eta_0)^3}{1 - (\Delta A/A_-) (\eta/\eta_0)^3} + 4 \varepsilon_{1-}. \quad (3.12b) \]

In fact, to determine the modes associated with the scalar perturbations and to evaluate the dominant contribution to the scalar bi-spectrum, we shall also require the behavior of the quantity \( \dot{\varepsilon}_2 \). One can show that \( \dot{\varepsilon}_2 \) can be expressed as

\[ \dot{\varepsilon}_2 = -\frac{2 V_{\phi\phi}}{H^2} + 12 H \varepsilon_1 - 3 H \varepsilon_2 - 4 H \varepsilon_1^2 + 5 H \varepsilon_1 \varepsilon_2 - \frac{H}{2} \varepsilon_2^2, \quad (3.13) \]

where \( V_{\phi\phi} = d^2V/d\phi^2 \), and it should be stressed that this is an exact relation. It should be clear that the first term in the above expression involving \( V_{\phi\phi} \) will lead to a Dirac delta function due to the discontinuity in the first derivative of the potential in the case of the Starobinsky model. Hence, the dominant contribution to \( \dot{\varepsilon}_2 \) at the transition can be written as [106–108]

\[ \dot{\varepsilon}_2^0 \simeq \frac{2 \Delta A}{H_0} \delta^{(1)} (\phi - \phi_0) = \frac{6 \Delta A}{A_+ a_0} \delta^{(1)} (\eta - \eta_0), \quad (3.14) \]

where \( a_0 \) denotes the value of the scale factor when \( \eta = \eta_0 \). Post transition, the dominant contribution to \( \dot{\varepsilon}_2 \) is found to be [97]

\[ \dot{\varepsilon}_2^0 \simeq -3 H \varepsilon_2 - \frac{H}{2} \varepsilon_2^2 \simeq -\frac{18 H_0 \Delta A}{A_-} \left( \frac{\eta}{\eta_0} \right)^3 \left[ 1 - (\Delta A/A_-) (\eta/\eta_0)^3 \right]^2. \quad (3.15) \]

Due to the fact that the potential is linear and also since the first slow roll parameter remains small, the modes \( f_k \) governing the curvature perturbation can be described by the conventional de Sitter modes to a good approximation before the transition. For the same reasons, one finds that the scalar modes can be described by the de Sitter modes soon after the transition as well. However, due to the transition, the modes after the transition are related by the Bogoliubov transformations to the modes before the transition. Therefore, the scalar mode and its time derivative before the transition can be written as [97, 105–111]:

\[ f^+_k (\eta) = \frac{i H_0}{2 M_{pl} \sqrt{k^3} \varepsilon_{1+}} \left( 1 + i k \eta \right) e^{-i k \eta}, \quad (3.16a) \]
\[ f^{+\prime}_k (\eta) = \frac{i H_0}{2 M_{pl} \sqrt{k^3} \varepsilon_{1+}} \left[ \frac{3 \varepsilon_{1+}}{\eta} \left( 1 + i k \eta + k^2 \eta \right) \right] e^{-i k \eta}. \quad (3.16b) \]

Whereas, the mode and its derivative after the transition can be expressed as follows:

\[ f^-_k (\eta) = \frac{i H_0 \alpha_k}{2 M_{pl} \sqrt{k^3} \varepsilon_{1-}} \left( 1 + i k \eta \right) e^{-i k \eta} - \frac{i H_0 \beta_k}{2 M_{pl} \sqrt{k^3} \varepsilon_{1-}} \left( 1 - i k \eta \right) e^{i k \eta}, \quad (3.17a) \]
\[ f^{-\prime}_k (\eta) = \frac{i H_0 \alpha_k}{2 M_{pl} \sqrt{k^3} \varepsilon_{1-}} \left[ \left( \varepsilon_{1-} + \frac{\varepsilon_{2-}}{2} \right) \frac{1}{\eta} \left( 1 + i k \eta + k^2 \eta \right) \right] e^{-i k \eta} \]
\[ - \frac{i H_0 \beta_k}{2 M_{pl} \sqrt{k^3} \varepsilon_{1-}} \left[ \left( \varepsilon_{1-} + \frac{\varepsilon_{2-}}{2} \right) \frac{1}{\eta} \left( 1 - i k \eta + k^2 \eta \right) \right] e^{i k \eta}, \quad (3.17b) \]
with \( \alpha_k \) and \( \beta_k \) denoting the Bogoliubov coefficients. Upon matching the above modes and their time derivatives at the transition, the Bogoliubov coefficients can be determined to be

\[
\alpha_k = 1 + \frac{3 i \Delta A}{2 A_+} \frac{k_0}{k} \left( 1 + \frac{k_0^2}{k^2} \right), \tag{3.18a}
\]
\[
\beta_k = -\frac{3 i \Delta A}{2 A_+} \frac{k_0}{k} \left( 1 + \frac{i k_0}{k} \right) e^{2i k/k_0}, \tag{3.18b}
\]

where \( k_0 = -1/\eta_0 = a_0 H_0 \) denotes the mode that leaves the Hubble radius at the transition. At late times, the scalar mode behaves as

\[
f_k(\eta) = \frac{i H_0}{2 M_{Pl} \sqrt{\kappa} \epsilon_1(\eta)} (\alpha_k - \beta_k), \tag{3.19}
\]

where \( \epsilon_1(\eta) = A^2/(18 M_{Pl}^2 H_0^4) \). Therefore, the scalar power spectrum, evaluated as \( \eta \to 0 \), can be expressed as

\[
P_s(k) = \left( \frac{H_0}{2 \pi} \right)^2 \left( \frac{3 H_0^2}{A_+} \right)^2 |\alpha_k - \beta_k|^2
\]
\[
= \left( \frac{H_0}{2 \pi} \right)^2 \left( \frac{3 H_0^2}{A_+} \right)^2 \left[ (I(k) + I_c(k)) \cos \left( \frac{2 k}{k_0} \right) + I_s(k) \sin \left( \frac{2 k}{k_0} \right) \right], \tag{3.20}
\]

where the quantities \( I(k), I_c(k) \) and \( I_s(k) \) are given by

\[
I(k) = 1 + \frac{9}{2} \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^2 + 9 \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^4 + \frac{9}{2} \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^6, \tag{3.21a}
\]
\[
I_c(k) = \frac{3 \Delta A}{2 A_+} \left( \frac{k_0}{k} \right)^2 \left[ \left( \frac{3 A_-}{A_+} - 7 \right) - \frac{3 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^4 \right], \tag{3.21b}
\]
\[
I_s(k) = -\frac{3 \Delta A}{A_+} \frac{k_0}{k} \left[ 1 + \left( \frac{3 A_-}{A_+} - 4 \right) \left( \frac{k_0}{k} \right)^2 + \frac{3 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^4 \right]. \tag{3.21c}
\]

Note that, because of the features in the power spectrum, the corresponding scalar spectral index \( n_s \) depends on the wavenumber \( k \), and is found to be

\[
n_s(k) = \frac{1}{2} \left[ I(k) + I_c(k) \cos \left( \frac{2 k}{k_0} \right) + I_s(k) \sin \left( \frac{2 k}{k_0} \right) \right]^{-1}
\]
\[
\times \left[ J(k) + J_c(k) \cos \left( \frac{2 k}{k_0} \right) + J_s(k) \sin \left( \frac{2 k}{k_0} \right) \right], \tag{3.22}
\]

where \( J(k), J_c(k) \) and \( J_s(k) \) are given by

\[
J(k) = 2 - 9 \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^2 - 54 \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^4 - 45 \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^6, \tag{3.23a}
\]
\[
J_c(k) = -\frac{3 \Delta A}{A_+} \left[ 4 + \left( \frac{15 A_-}{A_+} - 23 \right) \left( \frac{k_0}{k} \right)^2 + \frac{12 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^4 - \frac{15 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^6 \right], \tag{3.23b}
\]
\[
J_s(k) = -\frac{6 \Delta A}{A_+} \frac{k_0}{k} \left[ \left( \frac{3 A_-}{A_+} - 7 \right) - \left( \frac{3 A_-}{A_+} - 4 \right) \left( \frac{k_0}{k} \right)^2 + \frac{15 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^4 \right]. \tag{3.23c}
\]
If the consistency condition is indeed satisfied, then the non-Gaussianity parameter, as predicted by the relation, would prove to be

\[ f_{\text{NL}}(k) = \frac{5}{12} \left[ n_s(k) - 1 \right] \]

\[ = \frac{5}{24} \left[ \mathcal{I}(k) + \mathcal{I}_c(k) \cos \left( \frac{2k}{k_0} \right) + \mathcal{I}_s(k) \sin \left( \frac{2k}{k_0} \right) \right]^{-1} \]

\[ \times \left\{ \left[ \mathcal{J}(k) - 2\mathcal{I}(k) \right] + \left[ \mathcal{J}_c(k) - 2\mathcal{I}_c(k) \right] \cos \left( \frac{2k}{k_0} \right) + \left[ \mathcal{J}_s(k) - 2\mathcal{I}_s(k) \right] \sin \left( \frac{2k}{k_0} \right) \right\}. \]

Let us now examine whether we do arrive at the same result upon using the Maldacena formalism to compute the scalar bi-spectrum. It is known that, when there exist deviations from slow roll, it is the fourth vertex that leads to the most dominant contribution to the bi-spectrum. In other words, we need to focus on the contribution from slow roll, it is the fourth vertex that leads to the most dominant contribution to the formalism to compute the scalar bi-spectrum. It is known that, when there exist deviations, the bi-spectrum in the squeezed limit can be written as

\[ G_{4\rightarrow4}(k_1, k_2, k_3) \]

The corresponding contribution to the bi-spectrum can be easily evaluated using the late time behavior (3.19) of the mode \( f_k \). The contribution after the transition is governed by the integral

\[ \lim_{k_3 \rightarrow 0} G^0_{4\rightarrow4}(k, -k, k_3) = \lim_{k_3 \rightarrow 0} \frac{12 i a_0^2 \epsilon_1 + \Delta A}{A_+} \left[ f_k^{+\ast}(\eta_0) f_k^{+\ast}\eta_0 f_k^{\ast}(\eta_0) \right]. \] (3.25)

We find that the resulting integral, arrived at upon making use of the behavior (3.11b) and (3.15) of the slow roll parameters and the modes (3.17), can be easily evaluated. On adding the above two contributions at the transition and post-transition, one can show that the bi-spectrum in the squeezed limit can be written as

\[ \lim_{k_3 \rightarrow 0} G(k, -k, k_3) = -\frac{81}{8} H_0^2 \left( k \right) \left[ \mathcal{J}(k) - 2\mathcal{I}(k) \right] + \left[ \mathcal{J}_c(k) - 2\mathcal{I}_c(k) \right] \cos \left( \frac{2k}{k_0} \right) \]

\[ + \left[ \mathcal{J}_s(k) - 2\mathcal{I}_s(k) \right] \sin \left( \frac{2k}{k_0} \right) \]. \]

There are a few points concerning this result that require emphasis. The above bi-spectrum goes to a constant value at large scales, while it is found to oscillate with a constant amplitude in the small scale limit. In the equilateral limit, the contribution at the transition is known to lead to a term that grows linearly with \( k \) at large wavenumbers [106–108]. This essentially arises due to the infinitely sharp transition in the Starobinsky model.
the squeezed limit, one does not encounter such a growing term, but the sharpness of the transition is reflected in the oscillations of a fixed amplitude that persist indefinitely at small scales. Clearly, one can expect these oscillations to die down at suitably large wavenumbers if one smoothens the transition [108]. As far as our primary concern here, viz. the validity of the consistency condition, we find that, upon making use of the expression (3.27) for the bi-spectrum and the power spectrum (3.20), we indeed recover the $f_{\text{NL}}$ as given by eq. (3.24), implying that the consistency relation does hold even in the case of the infinitely sharp Starobinsky model. Moreover, it is important to appreciate the point that, while it is the contribution at the transition that dominates the amplitude of the non-Gaussianity parameter at large wavenumbers, the contribution after the transition proves to be essential for establishing the consistency relation at small wavenumbers. This suggests that the contributions after the transition are essential in order to arrive at the complete bi-spectrum in the Starobinsky model [97, 108].

4 Numerical verification of the relation during deviations from slow roll

In this section, we shall numerically examine the validity of the consistency relation in three models that lead to features in the scalar power spectrum due to deviations from slow roll. We shall consider models that result in features of the following types: (i) a sharp drop in power at large scales, roughly associated with the Hubble scale today (see refs. [112, 113]; for recent discussions, see refs. [114–116]), (ii) a burst of oscillations around scales corresponding to the multipoles of $\ell \simeq 20–40$ [117–128], and (iii) small and repeated modulations extending over a wide range of scales [129–139]. Such features are known to result in a better fit to the cosmological data than the more simple and conventional, nearly scale invariant, spectra. It should be highlighted that it is essentially these three types of spectra that have been considered by the Planck team while examining the possibility of features in the primordial spectrum [41]. We should also clarify that, though the fit to the data improves in the presence of features, the Bayesian evidence does not necessarily alter significantly, as the improvement in the fit is typically achieved at the cost of a few extra parameters [41, 44–46, 108, 140–143]. Nevertheless, we believe that the possibility of features require to be explored further since repeated exercises towards model independent reconstruction of the primordial power spectrum seem to point to their presence [144–146].

The different types of power spectra mentioned above can be generated by three inflationary models which we shall now briefly describe. Power spectra with a sharp drop in power on large scales can be generated in scenarios dubbed punctuated inflation [112, 113], which is a situation wherein a short period of departure from inflation is sandwiched between two epochs of slow roll inflation. Such a punctuated inflationary scenario can be produced, for example, by the following potential which contains a point of inflection:

$$V(\phi) = \frac{m^2}{2} \phi^2 - \frac{\sqrt{2} \lambda (n-1) m}{n} \phi^n + \frac{\lambda}{4} \phi^{2(n-1)}, \quad (4.1)$$

where $n \geq 3$. The point of inflection proves to be crucial to recover the second stage of slow roll inflation, after inflation has been interrupted briefly. (We should clarify that we shall restrict ourselves to the $n = 3$ case in this work.) The second class of features wherein there arises a burst of oscillations over an intermediate range of scales can be generated by introducing a step in a potential that otherwise leads to slow roll inflation. The step results in a brief period of fast roll, which leads to the oscillations in the scalar power spectrum.
For instance, if a step is introduced in the conventional quadratic potential, the complete potential can be written as

\[ V(\phi) = \frac{m^2}{2} \phi^2 \left[ 1 + \alpha \tanh \left( \frac{\phi - \phi_0}{\Delta \phi} \right) \right], \]

where, evidently, \( \phi_0, \alpha \) and \( \Delta \phi \) represent the location, the height and the width of the step, respectively [117–124, 126–128]. Spectra with repeated modulations extending over a wide range of scales can be generated by potentials which contain oscillatory terms such as in the axion monodromy model [133–136]. The potential in such a case is given by

\[ V(\phi) = \lambda \left[ \phi + \alpha \cos \left( \frac{\phi}{\beta} + \delta \right) \right], \]

where \( 1/\beta \) represents the frequency of oscillations in the potential, while \( \delta \) is a phase. (For the best fit values of the potential parameters, arrived at upon comparison with the CMB data, as well as for an illustration of the scalar power spectra that arise in the above three models, we would refer the reader to ref. [99].) Let us now turn to the numerical evaluation of the scalar bi-spectrum in these models and the verification of the consistency relation.

### 4.1 \( f_{NL} \) for an arbitrary triangular configuration of the wavenumbers

We shall make use of the code BI-spectra and Non-Gaussianity Operator, or simply, BINGO, which we had developed earlier, to calculate the scalar bi-spectrum [98]. BINGO is a Fortran 90 code that evaluates the scalar bi-spectrum in single field inflationary models involving the canonical scalar field. It is based on the Maldacena formalism, and it efficiently computes all the various contributions to the bi-spectrum. It should be clear from the Maldacena formalism that, in order to arrive at the scalar bi-spectrum, one first requires the behavior of the background quantities (such as, say, the scale factor and the slow roll parameters) and the scalar modes \( f_k \). Then, it is a matter of computing the various integrals that govern the scalar bi-spectrum. The evolution of the background quantities is arrived at by solving the equation describing the scalar field. Once we have the solution to the background, the scalar modes are obtained by solving the corresponding differential equation, viz. eq. (2.2), with the standard Bunch-Davies initial conditions. With these at hand, the integrals involved [cf. eqs. (2.12)] can be carried out from a sufficiently early time to a suitably late time. In the context of power spectrum, it is well known that it is sufficient to evolve the modes from a time when they are sufficiently inside the Hubble radius, say, from \( k/(a H) = 10^2 \), till they are well outside, say, when \( k/(a H) = 10^{-5} \) [147, 148]. One finds that, in order to arrive at the bi-spectrum, it suffices to carry out the integrals involved over roughly the same domain in time [71, 99, 149–156]. However, two points need to be emphasized in this regard. Firstly, in the case of the bi-spectrum, while evaluating for an arbitrary triangular configuration, one needs to make sure that the integrals are carried out from a time when the largest of the three modes (in terms of wavelength) is well inside the Hubble radius to a time when the smallest of the three is sufficiently outside. To achieve the accuracy we desire (say, of the order of 2–3% or better), we perform the integrals from the time when the largest mode satisfies the condition \( k/(a H) = 10^2 \) until a time when the smallest mode satisfies the condition \( k/(a H) = 10^{-5} \). (This is so barring the case of the axion monodromy model wherein we have to integrate from deeper inside the Hubble radius — actually, from \( k/(a H) = 250 \) for the values of the parameters that we work with — to
take into account the resonances that occur in the model [133].) Secondly, due to continued oscillations in the sub-Hubble domain, it is well known that the integrals require a cut-off in order for them to converge. We have introduced a cut-off of the form $\exp \left[ -\kappa k/(a H) \right]$ and have worked with $\kappa = 0.1$, which is known to lead to consistent results [71, 99]. We should mention here that we have made the latest version of BINGO publicly available at the URL: http://www.physics.iitm.ac.in/~sriram/bingo/bingo.html. The earlier public version of the code was limited to the evaluation of the bi-spectrum in the equilateral limit. The current version can compute the bi-spectrum for an arbitrary triangular configuration of the wavenumbers, including the squeezed limit of our interest here.\(^2\)

Before we go on to consider the consistency relation in the squeezed limit, let us make use of BINGO to understand the shape and structure of the bi-spectrum or, equivalently, the non-Gaussianity parameter $f_{NL}$, for an arbitrary triangular configuration of the wavenumbers.\(^3\) Usually, the scalar bi-spectrum and the parameter $f_{NL}$ are illustrated as density plots, plotted as a function of the ratios $k_3/k_1$ and $k_2/k_1$, for a fixed value of $k_1$ (in this context, see, for instance, ref. [99]). While the actual value of $k_1$ will not play a significant role in simple slow roll scenarios, the structure of the bi-spectrum revealed in such density plots will depend on choice of $k_1$ in models which lead to features. In figure 1, we have plotted the scalar non-Gaussianity parameter arising in the three inflationary models of our interest, for suitable values of the quantity $k_1$. We find that, in the cases of punctuated inflation and the quadratic potential with a step, since the features are localized over a small range of scales, the structure of the plot changes to a certain extent with the choice of $k_1$. However, in the case of axion monodromy model, because of the reason that the oscillations extend over a wide range of scales, the choice of $k_1$ does not alter the structure of the plots significantly.

In figure 2, we have attempted to capture the complete structure and shape of the bi-spectrum using a three-dimensional contour plot. We have made use of Mayavi and Python to create the three-dimensional plot [157]. We have plotted the parameter $f_{NL}$ for a wide range of the wavenumbers $k_1$, $k_2$ and $k_3$, over the allowed domain wherein the corresponding wavevectors satisfy the triangularity condition. It is known that the triangularity condition restricts the wavenumbers to a ‘tetrapyd’, as is evident from the figure. In the figure, we have presented two projections of the three-dimensional plot. One of the views clearly shows the fact that the bi-spectrum is symmetric along the three axes, as is expected in an isotropic background. The second illustrates the fact the non-Gaussianity parameter peaks in the equilateral limit, i.e. when $k_1 = k_2 = k_3$.

4.2 $f_{NL}$ in the squeezed limit

Let us now turn to examine the consistency relation in the three models of our interest. Towards this end, we have made use of BINGO to evaluate the non-Gaussianity parameter $f_{NL}$ in the squeezed limit, using the Maldacena formalism. As we had pointed out, BINGO can be made use of to evaluate the power spectrum as well. Using the expression (2.6) and the scalar power spectrum, we arrive at the scalar spectral index $n_s$, which we then utilize to verify the consistency condition $f_{NL}(k) = 5 \left[ n_s(k) - 1 \right]/12$. Before we go on to illustrate

\(^2\)We should add that we have independently reproduced the results being presented here using a different code as well. The latter code was originally used to calculate scalar-tensor three-point functions and the tensor bi-spectrum [71], and it has been modified suitably to calculate the scalar bi-spectrum and the corresponding non-Gaussianity parameter $f_{NL}$.

\(^3\)We should mention here that, apart from the scalar bi-spectrum, we shall also require the scalar power spectrum to arrive at the non-Gaussianity parameter $f_{NL}$. BINGO, as it computes the scalar modes, can easily be made use of to obtain the power spectrum too.
Figure 1. Density plots of the scalar non-Gaussianity parameter $f_{\text{NL}}$, plotted as function of $k_3/k_1$ and $k_2/k_1$, with fixed values of $k_1$, in the three models of our interest, viz. punctuated inflation (on top), quadratic potential with a step (in the middle) and axion monodromy model (at the bottom). We have chosen the wavenumber $k_1$ to be $10^{-3}\,\text{Mpc}^{-1}$ in the case of punctuated inflation, while we have set it to be $2 \times 10^{-3}\,\text{Mpc}^{-1}$ for the other two models. We find that, when the features are localized, as in the cases of punctuated inflation and the quadratic potential with a step, the structure of $f_{\text{NL}}$ varies considerably with the choice of $k_1$. However, in the case of the axion monodromy model, wherein there arises continued oscillations, the shape of $f_{\text{NL}}$ is more or less independent of the choice of $k_1$.

The results for the three models that we are focusing on, a couple of points concerning the squeezed limit needs to be made. We should stress that we take the wavenumber of the squeezed mode to be smallest wavenumber that is numerically tenable in the sense that the mode is sufficiently inside the Hubble radius at a time close to when the integration of the background begins. Moreover, it should be noted that, since the squeezed mode has a finite and non-zero wavenumber, in the squeezed limit of our interest, the numerically evaluated bi-spectrum is expected to be more accurate at larger wavenumbers than the smaller ones.
Figure 2. Three-dimensional contour plots of the parameter $f_{NL}$, plotted against the three wavenumbers $k_1$, $k_2$ and $k_3$ in the three models of interest, organized in the same order as in the last figure. We have shown two different projections of the plots in the figure. The projections in the left column clearly indicate the symmetry along the three different axes, as is expected in an isotropic background. The figures on the right column illustrate the fact that the allowed domain of wavenumbers are confined to a ‘tetrapyd’ and that the bi-spectrum peaks in the equilateral limit, i.e. along the line $k_1 = k_2 = k_3$. 

---

JCAP02(2015)029

---
Figure 3. The behavior of the non-Gaussianity parameter $f_{NL}$ in the squeezed limit has been plotted as a function of $k$ for the three models, organized as in the previous two figures. The blue curve represents the $f_{NL}$ calculated using Maldacena formalism, while the red dashed line corresponds to the same quantity arrived at from the consistency relation. The excellent match between the two curves indicate that the consistency relation is valid even in non-trivial scenarios involving brief departures from inflation.

In figure 3, we have plotted the quantity $f_{NL}$ obtained from the Maldacena formalism as well as the quantity arrived at from the consistency relation. It is clear that the two quantities match very well (they match at the level of a few percent) thus confirming the validity of consistency relation even in scenarios displaying highly non-trivial dynamics.
5 Discussion

At the level of the three-point function, the consistency condition relates the scalar bispectrum to the power spectrum in the squeezed limit wherein the wavelength of one of the three modes is much longer than the other two. As we had discussed, the consistency condition applies to any situation wherein the amplitude of the long wavelength mode freezes. Since the amplitude of the curvature perturbation settles down to a constant value on super-Hubble scales in most single field models of inflation, the consistency relation is expected to be valid in such models. It is easy to analytically establish the consistency relation in slow roll scenarios. In contrast, one needs to often resort to numerical methods to analyze situations involving departures from slow roll. In this work, we had examined the validity of the consistency condition, analytically as well as numerically, in a class of models permitting deviations from slow roll. We find that the condition is indeed satisfied even in situations consisting of strong departures from slow roll, such as the punctuated inflationary scenario.

With the emergence of increasingly precise cosmological data, it has been recognized that correlation functions beyond the power spectrum can act as powerful probes of the early universe. However, as we had discussed in the introductory section, despite the relatively strong bounds that have been arrived at on the scalar non-Gaussianity parameter $f_{NL}$, there exist a wide range of models that remain consistent with the data. The consistency relations can obviously play an important role to alleviate the situation. For instance, if the consistency relations can be observationally confirmed, it can rule out many multi-field models of inflation and even, possibly, alternate scenarios such as the bouncing models (in this context, see, for instance, refs. [158, 159] and references therein). It seems worthwhile to closely investigate the conditions under which the consistency relation holds in, say, two-field models (see, for example, ref. [160]) and, in particular, examine in some detail the role the iso-curvature perturbations may play in this regard. We are currently studying such issues.

Acknowledgments

The authors wish to thank Jérôme Martin for collaborations and discussions as well as detailed comments on the manuscript. DKH wishes to acknowledge support from the Korea Ministry of Education, Science and Technology, Gyeongsangbuk-Do and Pohang City for Independent Junior Research Groups at the Asia Pacific Center for Theoretical Physics, Pohang, Korea. The authors wish to acknowledge the use of Physics PC cluster at Pohang University of Science and Technology, Pohang, Korea and the high performance computing facility at the Indian Institute of Technology Madras, Chennai, India.

References

[1] A.A. Starobinsky, Spectrum of relict gravitational radiation and the early state of the universe, *JETP Lett.* 30 (1979) 682 [nSPIRE].

[2] V.F. Mukhanov and G.V. Chibisov, Quantum Fluctuation and Nonsingular Universe (in Russian), *JETP Lett.* 33 (1981) 532 [nSPIRE].

[3] S.W. Hawking, The Development of Irregularities in a Single Bubble Inflationary Universe, *Phys. Lett. B* 115 (1982) 295 [nSPIRE].

[4] A.A. Starobinsky, Dynamics of Phase Transition in the New Inflationary Universe Scenario and Generation of Perturbations, *Phys. Lett. B* 117 (1982) 175 [nSPIRE].
[5] A.H. Guth and S.Y. Pi, *Fluctuations in the New Inflationary Universe*, *Phys. Rev. Lett.* **49** (1982) 1110 [SPIRE].

[6] A.A. Starobinsky, *The Perturbation Spectrum Evolving from a Nonsingular Initially De-Sitter Cosmology and the Microwave Background Anisotropy*, *Sov. Astron. Lett.* **9** (1983) 302.

[7] V.N. Lukash, *Production of phonons in an isotropic universe*, *Sov. Phys. JETP* **52** (1980) 807 [SPIRE].

[8] D.H. Lyth, *Large Scale Energy Density Perturbations and Inflation*, *Phys. Rev. D* **31** (1985) 1792 [SPIRE].

[9] E.W. Kolb and M.S. Turner, *The Early Universe*, Addison-Wesley, Redwood City, California, U.S.A. (1990).

[10] S. Dodelson, *Modern Cosmology*, Academic Press, San Diego, U.S.A. (2003).

[11] V.F. Mukhanov, *Physical Foundations of Cosmology*, Cambridge University Press, Cambridge, England (2005).

[12] S. Weinberg, *Cosmology*, Oxford University Press, Oxford, England (2008).

[13] R. Durrer, *The Cosmic Microwave Background*, Cambridge University Press, Cambridge, England (2008).

[14] D.H. Lyth and A.R. Liddle, *The Primordial Density Perturbation*, Cambridge University Press, Cambridge, England (2009).

[15] P. Peter and J.-P. Uzan, *Primordial Cosmology*, Oxford University Press, Oxford, England (2009).

[16] H. Mo, F. van den Bosch and S. White, *Galaxy Formation and Evolution*, Cambridge University Press, Cambridge, England (2010).

[17] H. Kodama and M. Sasaki, *Cosmological Perturbation Theory*, *Prog. Theor. Phys. Suppl.* **78** (1984) 1 [SPIRE].

[18] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, *Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions*, *Phys. Rept.* **215** (1992) 203 [SPIRE].

[19] J.E. Lidsey, A.R. Liddle, E.W. Kolb, E.J. Copeland, T. Barreiro and M. Abney, *Reconstructing the inflation potential: An overview*, *Rev. Mod. Phys.* **69** (1997) 373 [astro-ph/9508078] [SPIRE].

[20] A. Riotto, *Inflation and the theory of cosmological perturbations*, *hep-ph/0210162* [SPIRE].

[21] W.H. Kinney, *Cosmology, inflation and the physics of nothing*, *NATO Sci. Ser. II* **123** (2003) 189 [astro-ph/0301448] [SPIRE].

[22] J. Martin, *Inflationary perturbations: The Cosmological Schwinger effect*, *Lect. Notes Phys.* **738** (2008) 193 [arXiv:0704.3540] [SPIRE].

[23] J. Martin, *Inflationary cosmological perturbations of quantum-mechanical origin*, *Lect. Notes Phys.* **669** (2005) 199 [hep-th/0406011] [SPIRE].

[24] J. Martin, *Inflation and precision cosmology*, *Braz. J. Phys.* **34** (2004) 1307 [astro-ph/0312492] [SPIRE].

[25] B.A. Bassett, S. Tsujikawa and D. Wands, *Inflation dynamics and reheating*, *Rev. Mod. Phys.* **78** (2006) 537 [astro-ph/0507632] [SPIRE].

[26] W.H. Kinney, *TASI Lectures on Inflation*, arXiv:0902.1529 [SPIRE].

[27] L. Sriramkumar, *An introduction to inflation and cosmological perturbation theory*, *Carr. Sci.* **97** (2009) 868 [arXiv:0904.4584] [SPIRE].
[28] D. Baumann, *TASI Lectures on Inflation*, arXiv:0907.5424 [inSPIRE].

[29] C.L. Bennett et al., *Cosmic temperature fluctuations from two years of COBE differential microwave radiometers observations*, *Astrophys. J.* **436** (1994) 423 [astro-ph/9401012] [inSPIRE].

[30] E.L. Wright, G.F. Smoot, C.L. Bennett and P.M. Lubin, *Angular Power Spectrum of the Microwave Background Anisotropy seen by the COBE Differential Microwave Radiometer*, *Astrophys. J.* **436** (1994) 443 [astro-ph/9401015] [inSPIRE].

[31] K.M. Gorski, *On Determining the spectrum of primordial inhomogeneity from the COBE DMR sky maps. 1. Method*, *Astrophys. J.* **436** (1994) L85 [astro-ph/9403066] [inSPIRE].

[32] K.M. Gorski et al., *On determining the spectrum of primordial inhomogeneity from the COBE dmr sky maps. 2. Results of two year data analysis*, *Astrophys. J.* **436** (1994) L89 [astro-ph/9403067] [inSPIRE].

[33] WMAP collaboration, J. Dunkley et al., *Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Likelihoods and Parameters from the WMAP data*, *Astrophys. J. Suppl.* **180** (2009) 306 [arXiv:0803.0586] [inSPIRE].

[34] WMAP collaboration, E. Komatsu et al., *Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation*, *Astrophys. J. Suppl.* **180** (2009) 330 [arXiv:0803.0547] [inSPIRE].

[35] D. Larson et al., *Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Power Spectra and WMAP-Derived Parameters*, *Astrophys. J. Suppl.* **192** (2011) 16 [arXiv:1001.4635] [inSPIRE].

[36] WMAP collaboration, E. Komatsu et al., *Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation*, *Astrophys. J. Suppl.* **192** (2011) 18 [arXiv:1001.4538] [inSPIRE].

[37] WMAP collaboration, C.L. Bennett et al., *Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Final Maps and Results*, *Astrophys. J. Suppl.* **208** (2013) 20 [arXiv:1212.5225] [inSPIRE].

[38] WMAP collaboration, G. Hinshaw et al., *Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results*, *Astrophys. J. Suppl.* **208** (2013) 19 [arXiv:1212.5226] [inSPIRE].

[39] PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 results. XV. CMB power spectra and likelihood*, *Astron. Astrophys.* **571** (2014) A15 [arXiv:1303.5075] [inSPIRE].

[40] PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 results. XVI. Cosmological parameters*, *Astron. Astrophys.* **571** (2014) A16 [arXiv:1303.5076] [inSPIRE].

[41] PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 results. XXII. Constraints on inflation*, *Astron. Astrophys.* **571** (2014) A22 [arXiv:1303.5082] [inSPIRE].

[42] BICEP2 collaboration, P.A.R. Ade et al., *BICEP2 II: Experiment and Three-Year Data Set*, *Astrophys. J.* **792** (2014) 62 [arXiv:1403.4302] [inSPIRE].

[43] BICEP2 collaboration, P.A.R. Ade et al., *Detection of B-Mode Polarization at Degree Angular Scales by BICEP2*, *Phys. Rev. Lett.* **112** (2014) 241101 [arXiv:1403.3985] [inSPIRE].

[44] J. Martin, C. Ringeval and V. Vennin, *Encyclopaedia Inflationaris*, *Phys. Dark Univ.* (2014) [arXiv:1303.3787] [inSPIRE].

[45] J. Martin, C. Ringeval, R. Trotta and V. Vennin, *The Best Inflationary Models After Planck*, *JCAP* **03** (2014) 039 [arXiv:1312.3529] [inSPIRE].

[46] J. Martin, C. Ringeval, R. Trotta and V. Vennin, *Compatibility of Planck and BICEP2 in the Light of Inflation*, *Phys. Rev. D* **90** (2014) 063501 [arXiv:1405.7272] [inSPIRE].
[47] J. Martin, C. Ringeval and V. Vennin, How Well Can Future CMB Missions Constrain Cosmic Inflation?, JCAP 10 (2014) 038 [arXiv:1407.4034] [inSPIRE].

[48] E. Komatsu and D.N. Spergel, Acoustic signatures in the primary microwave background bispectrum, Phys. Rev. D 63 (2001) 063002 [astro-ph/0005036] [inSPIRE].

[49] PLANCK collaboration, P.A.R. Ade et al., Planck 2013 Results. XXIV. Constraints on primordial non-Gaussianity, Astron. Astrophys. 571 (2014) A24 [arXiv:1303.5084] [inSPIRE].

[50] J.M. Maldacena, Non-Gaussian features of primordial fluctuations in single field inflationary models, JHEP 05 (2003) 013 [astro-ph/0210603] [inSPIRE].

[51] P. Creminelli and M. Zaldarriaga, Single field consistency relation for the 3-point function, JCAP 10 (2004) 006 [astro-ph/0407059] [inSPIRE].

[52] C. Cheung, A.L. Fitzpatrick, J. Kaplan and L. Senatore, On the consistency relation of the 3-point function in single field inflation, JCAP 02 (2008) 021 [arXiv:0709.0295] [inSPIRE].

[53] S. Renaux-Petel, On the squeezed limit of the bispectrum in general single field inflation, JCAP 10 (2010) 020 [arXiv:1008.0260] [inSPIRE].

[54] J. Ganc and E. Komatsu, A new method for calculating the primordial bispectrum in the squeezed limit, JCAP 12 (2010) 009 [arXiv:1006.5457] [inSPIRE].

[55] P. Creminelli, G. D’Amico, M. Musso and J. Norena, The (not so) squeezed limit of the primordial 3-point function, JCAP 11 (2011) 038 [arXiv:1106.1462] [inSPIRE].

[56] D. Chialva, Signatures of very high energy physics in the squeezed limit of the bispectrum (violation of Maldacena’s condition), JCAP 10 (2012) 037 [arXiv:1108.4203] [inSPIRE].

[57] K. Schalm, G. Shiu and T. van der Aalst, Consistency condition for inflation from (broken) conformal symmetry, JCAP 03 (2013) 005 [arXiv:1211.2157] [inSPIRE].

[58] E. Pajer, F. Schmidt and M. Zaldarriaga, The Observed Squeezed Limit of Cosmological Three-Point Functions, Phys. Rev. D 88 (2013) 083502 [arXiv:1305.0824] [inSPIRE].

[59] L. Senatore and M. Zaldarriaga, A Note on the Consistency Condition of Primordial Fluctuations, JCAP 08 (2012) 001 [arXiv:1203.6884] [inSPIRE].

[60] P. Creminelli, J. Norena and M. Simonovic, Conformal consistency relations for single-field inflation, JCAP 07 (2012) 052 [arXiv:1203.4595] [inSPIRE].

[61] P. Creminelli, A. Perko, L. Senatore, M. Simonović and G. Trevisan, The Physical Squeezed Limit: Consistency Relations at Order $q^2$, JCAP 11 (2013) 015 [arXiv:1307.0503] [inSPIRE].

[62] L. Berezhiani and J. Khoury, Slavnov-Taylor Identities for Primordial Perturbations, JCAP 02 (2014) 003 [arXiv:1309.4461] [inSPIRE].

[63] L. Berezhiani, J. Khoury and J. Wang, Non-Trivial Checks of Novel Consistency Relations, JCAP 06 (2014) 056 [arXiv:1401.7991] [inSPIRE].

[64] H. Collins, R. Holman and T. Vardanyan, A Cosmological Slavnov-Taylor Identity, JCAP 12 (2014) 007 [arXiv:1405.0017] [inSPIRE].

[65] J.M. Maldacena and G.L. Pimentel, On graviton non-Gaussianities during inflation, JHEP 09 (2011) 045 [arXiv:1104.2846] [inSPIRE].

[66] X. Gao, T. Kobayashi, M. Yamaguchi and J. Yokoyama, Primordial non-Gaussianities of gravitational waves in the most general single-field inflation model, Phys. Rev. Lett. 107 (2011) 211301 [arXiv:1108.3513] [inSPIRE].

[67] X. Gao, T. Kobayashi, M. Shiraishi, M. Yamaguchi, J. Yokoyama and S. Yokoyama, Full bispectra from primordial scalar and tensor perturbations in the most general single-field inflation model, PTEP 2013 (2013) 053E03 [arXiv:1207.0588] [inSPIRE].
[68] D. Jeong and M. Kamionkowski, *Clustering Fossils from the Early Universe*, Phys. Rev. Lett. **108** (2012) 251301 [arXiv:1203.0302] [INSPIRE].

[69] L. Dai, D. Jeong and M. Kamionkowski, *Seeking Inflation Fossils in the Cosmic Microwave Background*, Phys. Rev. D **87** (2013) 103006 [arXiv:1302.1868] [INSPIRE].

[70] L. Dai, D. Jeong and M. Kamionkowski, *Anisotropic imprint of long-wavelength tensor perturbations on cosmic structure*, Phys. Rev. D **88** (2013) 043507 [arXiv:1306.3985] [INSPIRE].

[71] V. Sreenath, R. Tibrewala and L. Sriramkumar, *Numerical evaluation of the three-point scalar-tensor cross-correlations and the tensor bi-spectrum*, JCAP **12** (2013) 037 [arXiv:1309.7169] [INSPIRE].

[72] S. Kundu, *Non-Gaussianity Consistency Relations, Initial States and Back-reaction*, JCAP **04** (2014) 016 [arXiv:1311.1575] [INSPIRE].

[73] V. Sreenath and L. Sriramkumar, *Examining the consistency relations describing the three-point functions involving tensors*, JCAP **10** (2014) 021 [arXiv:1406.1609] [INSPIRE].

[74] S.M. Leach and A.R. Liddle, *Inflationary perturbations near horizon crossing*, Phys. Rev. D **63** (2001) 043508 [astro-ph/0010082] [INSPIRE].

[75] S.M. Leach, M. Sasaki, D. Wands and A.R. Liddle, *Enhancement of superhorizon scale inflationary curvature perturbations*, Phys. Rev. D **64** (2001) 023512 [astro-ph/0101406] [INSPIRE].

[76] R.K. Jain, P. Chingangbam and L. Sriramkumar, *On the evolution of tachyonic perturbations at super-Hubble scales*, JCAP **10** (2007) 003 [astro-ph/0703762] [INSPIRE].

[77] R.H. Ribeiro, *Inflationary signatures of single-field models beyond slow-roll*, JCAP **05** (2012) 037 [arXiv:1202.4453] [INSPIRE].

[78] J. Martin, H. Motohashi and T. Suyama, *Ultra Slow-Roll Inflation and the non-Gaussianity Consistency Relation*, Phys. Rev. D **87** (2013) 023514 [arXiv:1211.0083] [INSPIRE].

[79] M.G. Jackson and G. Shiu, *Study of the consistency relation for single-field inflation with power spectrum oscillations*, Phys. Rev. D **88** (2013) 123511 [arXiv:1303.4973] [INSPIRE].

[80] R. Flauger, D. Green and R.A. Porto, *On squeezed limits in single-field inflation. Part I*, JCAP **08** (2013) 032 [arXiv:1303.1430] [INSPIRE].

[81] J.-O. Gong, K. Schalm and G. Shiu, *Correlating correlation functions of primordial perturbations*, Phys. Rev. D **89** (2014) 063540 [arXiv:1401.4402] [INSPIRE].

[82] P. Adshead, W. Hu, C. Dvorkin and H.V. Peiris, *Fast Computation of Bispectrum Features with Generalized Slow Roll*, Phys. Rev. D **84** (2011) 043519 [arXiv:1102.3435] [INSPIRE].

[83] A. Achúcarro, J.-O. Gong, G.A. Palma and S.P. Patil, *Correlating features in the primordial spectra*, Phys. Rev. D **87** (2013) 121301 [arXiv:1211.5619] [INSPIRE].

[84] M.H. Namjoo, H. Firouzjahi and M. Sasaki, *Violation of non-Gaussianity consistency relation in a single field inflationary model*, Europhys. Lett. **101** (2013) 39001 [arXiv:1210.3692] [INSPIRE].

[85] X. Chen, H. Firouzjahi and M. Sasaki, *A Single Field Inflation Model with Large Local Non-Gaussianity*, Europhys. Lett. **102** (2013) 59001 [arXiv:1301.5699] [INSPIRE].

[86] J. Ganc, *Calculating the local-type fNL for slow-roll inflation with a non-vacuum initial state*, Phys. Rev. D **84** (2011) 063514 [arXiv:1104.0244] [INSPIRE].

[87] I. Agullo and L. Parker, *Non-Gaussianities and the Stimulated creation of quanta in the inflationary universe*, Phys. Rev. D **83** (2011) 063526 [arXiv:1010.5766] [INSPIRE].
I. Agullo and L. Parker, *Stimulated creation of quanta during inflation and the observable universe*, Gen. Rel. Grav. 43 (2011) 2541 [arXiv:1106.4240] [SPIRE].

T.S. Bunch and P.C.W. Davies, *Quantum Field Theory in de Sitter Space: Renormalization by Point Splitting*, Proc. Roy. Soc. Lond. A 360 (1978) 117 [SPIRE].

D. Seery and J.E. Lidsey, *Primordial non-Gaussianities in single field inflation*, JCAP 06 (2005) 003 [astro-ph/0503692] [SPIRE].

X. Chen, *Running non-Gaussianities in DBI inflation*, Phys. Rev. D 72 (2005) 123518 [astro-ph/0507053] [SPIRE].

X. Chen, M.-x. Huang, S. Kachru and G. Shiu, *Observational signatures and non-Gaussianities of general single field inflation*, JCAP 01 (2007) 002 [astro-ph/0605045] [SPIRE].

X. Chen, *Primordial Non-Gaussianities from Inflation Models*, Adv. Astron. 2010 (2010) 638979 [arXiv:1002.1416] [SPIRE].

Y. Wang, *Inflation, Cosmic Perturbations and Non-Gaussianities*, Commun. Theor. Phys. 62 (2014) 109 [arXiv:1303.1523] [SPIRE].

J. Martin and L. Sriramkumar, *The scalar bi-spectrum in the Starobinsky model: The equilateral case*, JCAP 01 (2012) 008 [arXiv:1109.5838] [SPIRE].

D.K. Hazra, J. Martin and L. Sriramkumar, *The scalar bi-spectrum during preheating in single field inflationary models*, Phys. Rev. D 86 (2012) 063523 [arXiv:1206.0442] [SPIRE].

L. Sriramkumar and T. Padmanabhan, *Initial state of matter fields and trans-Planckian physics: Can CMB observations disentangle the two?*, Phys. Rev. D 71 (2005) 103512 [gr-qc/0408034] [SPIRE].

I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, seventh edition, Academic Press, New York, U.S.A. (2007).

JCAP02 (2015) 029
[108] J. Martin, L. Sriramkumar and D.K. Hazra, Sharp inflaton potentials and bi-spectra: Effects of smoothening the discontinuity, JCAP 09 (2014) 039 [arXiv:1404.6093] [inSPIRE].

[109] E.D. Stewart, The spectrum of density perturbations produced during inflation to leading order in a general slow roll approximation, Phys. Rev. D 65 (2002) 103508 [astro-ph/0110322] [inSPIRE].

[110] J. Choe, J.-O. Gong and E.D. Stewart, Second order general slow-roll power spectrum, JCAP 07 (2004) 012 [hep-ph/0405155] [inSPIRE].

[111] J.-O. Gong, Breaking scale invariance from a singular inflaton potential, JCAP 07 (2005) 015 [astro-ph/0504383] [inSPIRE].

[112] R.K. Jain, P. Chingangbam, J.-O. Gong, L. Sriramkumar and T. Souradeep, Punctuated inflation and the low CMB multipoles, JCAP 10 (2009) 008 [arXiv:1005.2175] [inSPIRE].

[113] R.K. Jain, P. Chingangbam, L. Sriramkumar and T. Souradeep, The tensor-to-scalar ratio in punctuated inflation, Phys. Rev. D 82 (2010) 023509 [arXiv:0904.2518] [inSPIRE].

[114] L. Lello, D. Boyanovsky and R. Holman, Preslow roll initial conditions: large scale power suppression and infrared aspects during inflation, Phys. Rev. D 89 (2014) 063533 [arXiv:1307.4066] [inSPIRE].

[115] M. Cicoli, S. Downes and B. Dutta, Power Suppression at Large Scales in String Inflation, JCAP 12 (2013) 007 [arXiv:1309.3412] [inSPIRE].

[116] F.G. Pedro and A. Westphal, Low-ℓ CMB power loss in string inflation, JHEP 04 (2014) 034 [arXiv:1309.3413] [inSPIRE].

[117] J.A. Adams, B. Cresswell and R. Easther, Inflationary perturbations from a potential with a step, Phys. Rev. D 64 (2001) 123514 [astro-ph/0102236] [inSPIRE].

[118] L. Covi, J. Hamann, A. Melchiorri, A. Slosar and I. Sorbera, Inflation and WMAP three year data: Features have a Future!, Phys. Rev. D 74 (2006) 083509 [astro-ph/0606452] [inSPIRE].

[119] J. Hamann, L. Covi, A. Melchiorri and A. Slosar, New Constraints on Oscillations in the Primordial Spectrum of Inflationary Perturbations, Phys. Rev. D 76 (2007) 023503 [astro-ph/0701380] [inSPIRE].

[120] M. Joy, V. Sahni and A.A. Starobinsky, A New Universal Local Feature in the Inflationary Perturbation Spectrum, Phys. Rev. D 77 (2008) 023514 [arXiv:0711.1585] [inSPIRE].

[121] M. Joy, A. Shafieloo, V. Sahni and A.A. Starobinsky, Is a step in the primordial spectral index favored by CMB data?, JCAP 06 (2009) 028 [arXiv:0807.3334] [inSPIRE].

[122] M.J. Mortonson, C. Dvorkin, H.V. Peiris and W. Hu, CMB polarization features from inflation versus reionization, Phys. Rev. D 79 (2009) 103519 [arXiv:0903.4920] [inSPIRE].

[123] C. Dvorkin and W. Hu, Generalized Slow Roll for Large Power Spectrum Features, Phys. Rev. D 81 (2010) 023518 [arXiv:0910.2237] [inSPIRE].

[124] W. Hu, Generalized Slow Roll for Non-Canonical Kinetic Terms, Phys. Rev. D 84 (2011) 027303 [arXiv:1104.4500] [inSPIRE].

[125] A. Ashoorioon and A. Krause, Power Spectrum and Signatures for Cascade Inflation, hep-th/0607001 [inSPIRE].

[126] D.K. Hazra, M. Aich, R.K. Jain, L. Sriramkumar and T. Souradeep, Primordial features due to a step in the inflaton potential, JCAP 10 (2010) 008 [arXiv:1005.2175] [inSPIRE].

[127] M. Benetti, M. Lattanzi, E. Calabrese and A. Melchiorri, Features in the primordial spectrum: new constraints from WMAP7+ACT data and prospects for Planck, Phys. Rev. D 84 (2011) 063509 [arXiv:1107.4992] [inSPIRE].
[128] M. Benetti, *Updating constraints on inflationary features in the primordial power spectrum with the Planck data*, Phys. Rev. D 88 (2013) 087302 [arXiv:1308.6406] [SPIRE].

[129] J. Martin and C. Ringeval, *Superimposed oscillations in the WMAP data?*, Phys. Rev. D 69 (2004) 083515 [astro-ph/0310382] [SPIRE].

[130] J. Martin and C. Ringeval, *Exploring the superimposed oscillations parameter space*, JCAP 01 (2005) 007 [hep-ph/0405249] [SPIRE].

[131] M. Zarei, *Short Distance Physics and Initial State Effects on the CMB Power Spectrum and Cosmological Constant*, Phys. Rev. D 78 (2008) 083502 [arXiv:0809.4312] [SPIRE].

[132] C. Pahud, M. Kamionkowski and A.R. Liddle, *Oscillations in the inflaton potential?*, Phys. Rev. D 79 (2009) 083503 [arXiv:0807.0322] [SPIRE].

[133] R. Flauger, L. McAllister, E. Pajer, A. Westphal and G. Xu, *Oscillations in the CMB from Axion Monodromy Inflation*, JCAP 06 (2010) 009 [arXiv:0907.2916] [SPIRE].

[134] M. Aich, D.K. Hazra, L. Sriramkumar and T. Souradeep, *Oscillations in the inflaton potential: Complete numerical treatment and comparison with the recent and forthcoming CMB datasets*, Phys. Rev. D 87 (2013) 083526 [arXiv:1106.2798] [SPIRE].

[135] R. Easther and R. Flauger, *Planck Constraints on Monodromy Inflation*, JCAP 02 (2014) 037 [arXiv:1308.3705] [SPIRE].

[136] P.D. Meerburg and D.N. Spergel, *Searching for oscillations in the primordial power spectrum. I. Perturbative approach*, Phys. Rev. D 89 (2014) 063536 [arXiv:1308.3704] [SPIRE].

[137] P.D. Meerburg, D.N. Spergel and B.D. Wandelt, *Searching for oscillations in the primordial power spectrum. II. Constraints from Planck data*, Phys. Rev. D 89 (2014) 063537 [arXiv:1308.3705] [SPIRE].

[138] P.D. Meerburg, *Alleviating the tension at low ℓ through axion monodromy*, Phys. Rev. D 90 (2014) 063529 [arXiv:1406.3243] [SPIRE].

[139] H. Peiris, R. Easther and R. Flauger, *Constraining Monodromy Inflation*, JCAP 09 (2013) 018 [arXiv:1303.2616] [SPIRE].

[140] R. Easther and R. Flauger, *Planck Constraints on Monodromy Inflation*, JCAP 02 (2014) 037 [arXiv:1308.3736] [SPIRE].

[141] M.J. Mortonson, H.V. Peiris and R. Easther, *Bayesian Analysis of Inflation: Parameter Estimation for Single Field Models*, Phys. Rev. D 85 (2012) 103533 [arXiv:1112.0326] [SPIRE].

[142] R. Easther and H.V. Peiris, *Bayesian Analysis of Inflation II: Model Selection and Constraints on Reheating*, Phys. Rev. D 85 (2012) 103533 [arXiv:1112.0326] [SPIRE].

[143] J. Martin, C. Ringeval and R. Trotta, *Hunting Down the Best Model of Inflation with Bayesian Evidence*, Phys. Rev. D 83 (2011) 063524 [arXiv:1009.4157] [SPIRE].

[144] J. Norena, C. Wagner, L. Verde, H.V. Peiris and R. Easther, *Bayesian Analysis of Inflation III: Slow Roll Reconstruction Using Model Selection*, Phys. Rev. D 86 (2012) 023505 [arXiv:1202.0304] [SPIRE].

[145] D.K. Hazra, A. Shafieloo and T. Souradeep, *Primordial power spectrum: a complete analysis with the WMAP nine-year data*, JCAP 07 (2013) 031 [arXiv:1303.4143] [SPIRE].

[146] D.K. Hazra, A. Shafieloo and T. Souradeep, *Primordial power spectrum from Planck*, JCAP 11 (2014) 011 [arXiv:1406.4827] [SPIRE].

[147] P. Hunt and S. Sarkar, *Reconstruction of the primordial power spectrum of curvature perturbations using multiple data sets*, JCAP 01 (2014) 025 [arXiv:1308.2317] [SPIRE].

[148] D.S. Salopek, J.R. Bond and J.M. Bardeen, *Designing Density Fluctuation Spectra in Inflation*, Phys. Rev. D 40 (1989) 1753 [SPIRE].
[148] C. Ringeval, *The exact numerical treatment of inflationary models*, Lect. Notes Phys. 738 (2008) 243 [astro-ph/0703486] [inSPIRE].

[149] X. Chen, R. Easther and E.A. Lim, *Large Non-Gaussianities in Single Field Inflation*, JCAP 06 (2007) 023 [astro-ph/0611645] [inSPIRE].

[150] X. Chen, R. Easther and E.A. Lim, *Generation and Characterization of Large Non-Gaussianities in Single Field Inflation*, JCAP 04 (2008) 010 [arXiv:0801.3295] [inSPIRE].

[151] S. Hotchkiss and S. Sarkar, *Non-Gaussianity from violation of slow-roll in multiple inflation*, JCAP 05 (2010) 024 [arXiv:0910.3373] [inSPIRE].

[152] S. Hannestad, T. Haugbolle, P.R. Jarnhus and M.S. Sloth, *Non-Gaussianity from Axion Monodromy Inflation*, JCAP 06 (2010) 001 [arXiv:0912.3527] [inSPIRE].

[153] R. Flauger and E. Pajer, *Resonant Non-Gaussianity*, JCAP 01 (2011) 017 [arXiv:1002.0833] [inSPIRE].

[154] P. Adshead, W. Hu, C. Dvorkin and H.V. Peiris, *Fast Computation of Bispectrum Features with Generalized Slow Roll*, Phys. Rev. D 84 (2011) 043519 [arXiv:1102.3435] [inSPIRE].

[155] X. Chen, *Primordial Features as Evidence for Inflation*, JCAP 01 (2012) 038 [arXiv:1104.1323] [inSPIRE].

[156] P. Adshead, W. Hu and V. Miranda, *Bispectrum in Single-Field Inflation Beyond Slow-Roll*, Phys. Rev. D 88 (2013) 023507 [arXiv:1303.7004] [inSPIRE].

[157] P. Ramachandran and G. Varoquaux, *Mayavi: 3D Visualization of Scientific Data*, Comput. Sci. Eng. 13 (2011) 40. [arXiv:1010.4891].

[158] D. Battefeld and P. Peter, *A Critical Review of Classical Bouncing Cosmologies*, arXiv:1406.2790 [inSPIRE].

[159] X. Gao, M. Lilley and P. Peter, *Non-Gaussianity excess problem in classical bouncing cosmologies*, Phys. Rev. D 91 (2015) 023516 [arXiv:1406.4119] [inSPIRE].

[160] V. Assassi, D. Baumann and D. Green, *On Soft Limits of Inflationary Correlation Functions*, JCAP 11 (2012) 047 [arXiv:1204.4207] [inSPIRE].