Black Hole and Dark Matter. Phase Equilibrium

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Abstract. Equilibrium of a gravitating scalar field inside a black hole compressed to the state of boson matter, in balance with a longitudinal vector field (dark matter) from outside is considered. Analytical analysis, confirmed numerically, shows that there are static solutions of Einstein’s equations with arbitrary high total mass of a black hole, where the component of the metric tensor $g^{rr}(r)$ changes its sign twice. The balance of the energy-momentum tensors of the scalar field and the longitudinal vector field at the interface ensures the equilibrium of these phases. Considering a gravitating scalar field as an example, the internal structure of a black hole is revealed. Its phase equilibrium with the longitudinal vector field, describing dark matter on the periphery of a galaxy, determines the dependence of the velocity on the plateau of galaxy rotation curves on the mass of a black hole, located in the center of a galaxy.

1. Introduction

In analyzing the structure of the Universe, due attention is not paid to the interaction of black holes, presumably located in the centers of galaxies, with dark matter on the periphery. Describing the manifestations of dark matter via a longitudinal vector field [1], the covariant divergence $\phi^{0}_k(0)$ of the field $\phi^{i}(r)$ at the center $r = 0$ is the main parameter of the theory. Its numerical value, which determines the speed on the plateau of a galaxy rotation curve, depends on what happens in the center. It is necessary to analyze the internal structure of a black hole in balance with dark matter outside.

A black hole is a process of unlimited compression (collapse) of matter under the action of dominating forces of its own gravitational field. During compression, the pressure increases unlimitedly, and the energy per particle will inevitably reach the binding energy of “elementary particles” in neutrons, and further in their composite components. Note that the time of the existence of galaxies (and hence the black holes in their centers) is of the order of the life time of the Universe. With such a slow evolution of a black hole, locally equilibrium concentrations of particles entering the “chemical reactions” of transforming one into another depend on temperature and pressure and do not depend on the specific reaction channels [2]. With unlimited compression, the stage of domination of Z, W, and/or Higgs scalar bosons is inevitable. The fact, that the lifetime of a galaxy with a collapsar in the center is of the order of the lifetime of the Universe, suggests that the transformation of particles in chemical reactions can slow down the process of compression, and even stop it. This is the mean reason for searching and analyzing static configurations of gravitating objects in the general theory of relativity. After all, if there is a so long living super-heavy object (with the mass up to $4.3 \cdot 10^6 \, M_{\text{Sun}}$ and the radius less than 0.002 light years) in the center of our galaxy [3], then we have to figure out how this can be. It exceeds the critical values of Chandrasekhar [4], Landau [5], Oppenheimer and
Volkov [6] by six orders of magnitude.

2. Regular static solution for a black hole
From my point of view this problem, along with some others, can be solved within the existing General Theory of Relativity. We just have to reject the prejudice that the signature of a metric tensor remains unchanged with arbitrarily strong curvature of space-time. Using the component $g_{rr}$ in the form $g_{rr} = -e^\lambda$, possible solutions with changing signature are excluded in advance. Solutions, in which the sign of $g_{rr}$ remains unchanged, exist, but only if the total mass $M$ does not exceed a critical value $M_{cr}$. For a degenerate neutron Fermi gas $M_{cr}$ is of the order of solar mass. For a gravitating Bose-Einstein condensate $M_{cr} \sim M_{Pl}^2/m$ see [5]. The Planck mass $M_{Pl} = \sqrt{\hbar/c^3} = 2.177 \cdot 10^{-5} \text{g}$, $k = 6.67 \cdot 10^{-8} \text{cm}^3\text{g}^{-1}\text{sec}^{-2}$ is the gravitational constant. For bosons of the Standard Model having the mass $M_{cr}$ the total mass remains finite: $M_{cr} < M < M_{Pl}$.

Outside the condensate $(1)$ that $\sqrt{g_{rr}}$ can have a maximum when the terms on the right-hand side of (1) become equal. Outside the condensate $g_{rr}^0(r)$ decreases, crosses back the $x$-axis at $x = x_g$, and tends to $-1$ at $r \to \infty$. Thus, there should be solutions of the Einstein equations (1)-(2), in which the metric component $g_{rr}^0$ of the metric tensor does not change sign. One can see from the equation (1) that $g_{rr}^0(r)$ turns to zero twice with increasing $r$: $g_{rr}^0(r_g) = g_{rr}^0(r_h) = 0$, $r_g < r_h$. $r_g$ is a regular gravitational radius within the gravitating matter. The gravitational radius $r_h$ is the event horizon. It is located on the surface of a gravitating body, because we have the Schwarzschild solution [8]

$$g_{rr}^0(r) = -1 + r_h/r, \quad r \geq r_h,$$

provided that $g_{rr}^0(r) = 0$ at $r \geq r_h$. The event horizon radius is proportional to the total mass: $r_h = 2kM/c^2$. The signature of the metric tensor is $(+, +, +, -)$ in the spherical layer $r_g < r < r_h$.

It follows from Einstein equations (1)-(2) that in the solutions with a broken signature the energy density $\varepsilon$ on the surface of a gravitating body vanishes, and the pressure $p$ remains finite:

$$T_0^0(r_h) = 0, \quad T_r^r(r_h) = \frac{1}{k} = \frac{c^8}{32\pi k^3 M^2} \neq 0.$$
According to the equation of state of a degenerate relativistic Fermi gas, both - the energy density and the pressure - vanish at the surface. Therefore, it is more likely that the realization of a static state of a super heavy black hole can take place at the stage of domination of boson matter. At temperatures close to absolute zero, most bosons in equilibrium state are in the ground state, that is, at the quantum level with the lowest energy. The ensemble of bosons, accumulated on the quantum level of the lowest possible energy, is named Bose–Einstein condensate. At absolute zero, the wave function of the ultra quantum condensate of neutral bosons is the classical scalar field [9].

In curved space-time the wave function, being a scalar field, satisfies the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} \left( \sqrt{-g} g^{ik} \psi_{,ik} \right) = - \frac{\partial U}{\partial |\psi|^2} \psi, \quad g = \det g_{ik}. \quad (3)$$

Considering the equilibrium state of matter in its own gravitational field, it is assumed that the number of quanta is large, and all interactions, except the gravitational one, are not significant. The main characteristic, determining gravitational properties of a scalar field, is the mass of a quantum \(m\). In power series of the potential

$$U \left( |\psi|^2 \right) = \left( \frac{mc}{\hbar} \right)^2 |\psi|^2 + \frac{1}{2} \lambda |\psi|^4 + ...$$

the term \( \lambda |\psi|^4 \) and terms of higher orders are corrections for non-gravitational interactions, including collisions of particles.

As applied to laboratory observations of the Bose–Einstein condensate in atoms of rubidium [10] and sodium [11] vapors, the non-ideality of boson gases is small. The nonlinear term \( \lambda |\psi|^2 \) in the Gross-Pitaevskii equation [12], [13] is expressed in terms of the amplitude (length) of scattering \(a\) of atoms in pair collisions: \(\lambda = 4\pi \hbar^2 a/m\). In a gravitating condensate, the boson density is not small, and there is no reason to believe that the term \( \lambda |\psi|^4/2 \) in the expansion of the potential \(U \left( |\psi|^2 \right)\) in powers of \( |\psi|^2 \) reduces exclusively to pair collisions. \( \lambda \) is a macroscopic parameter characterizing the elasticity of a condensate. Without non-gravitational interaction of bosons, the Klein-Gordon equation (3) still remains non-linear due to the non-linearity of Einstein equations (1)-(2).

In a static central field, time is a cyclic variable. The energy of a single quantum \(E = \hbar \omega\) is the integral of motion. In a flat space-time, the Klein-Gordon equation is linear. Its solution is a plane wave \( \psi(x^i) = \psi_0 \exp(i(\mathbf{p} r - E t)/\hbar) \), describing the motion of a particle with a relativistic spectrum \(E^2 = p^2 c^2 + m^2 c^4\). In the curved space-time \(E\) is the conserved energy of the field per one quantum. The scalar field in the state of definite energy \(E\) has the form:

$$\psi_E (x^i) = \psi (r) \exp \left( -i E x^0 / \hbar \right).$$

The radial function \(\psi(r)\) satisfies the equation

$$g^{rr} \psi'' + \frac{1}{\sqrt{-g}} \left( \sqrt{-g} g^{rr} \right) \psi' = \frac{1}{\hbar^2 c^2} \left( g^{00} E^2 - m^2 c^4 \right) \psi. \quad (4)$$

Note that this equation is not defined at the gravitational radius \(r_g\) and at the horizon \(r_h\), where the coefficient at the highest derivative \(g^{rr} = 0\). On hyper-surfaces \(r = r_g, r_h\) space-time is not necessarily locally flat.

The Lagrangian of a scalar field \(L = g^{IK} \psi_{,I} \psi^*_K - U \left( |\psi|^2 \right)\) does not depend on the derivatives of the metric tensor. The energy-momentum tensor is simply \(T_{ik} = \frac{2}{\sqrt{-g}} \frac{\partial (\sqrt{-g} L)}{\partial g^{ik}}\). The set of Einstein and Klein-Gordon equations (1)-(2) and (4) are reduced to a normal form in my paper [14]. It is convenient for the analysis of spherically symmetric gravitating boson condensate. With account of non-gravitational interactions, in dimensionless variables

$$x = \frac{m c}{\hbar} r, \quad u(x) = \sqrt{x} \psi (r), \quad w(x) = \frac{h \sqrt{x}}{m c} g^{rr} (r) \psi' (r), \quad h(x) = \frac{E^2}{m c^2} g^{00} (r), \quad g(x) = g^{rr} (r),$$

$$h r, u$$
the set of equations
\[ u' = \frac{w}{g}, \quad w' = (h - 1 - \Lambda u^2) u - \frac{2}{x} w + \frac{x}{g} \left( u^2 h - \frac{u^2}{g} \right) w, \]
\[ g' = x \left( u^2 h + u^2 - \frac{w^2}{g} + \frac{1}{2} \Lambda u^4 \right) - \frac{1 + g}{x}, \]
\[ h' = \frac{h}{x} \left( 1 + \frac{1}{g} \right) - \frac{h}{g} \left( -u^2 h + u^2 + \frac{w^2}{g} + \frac{1}{2} \Lambda u^4 \right) \]
ccontain one parameter \( \Lambda = \frac{\hbar^2}{m c^2 r_s} \), characterizing the elasticity of a condensate. For the case \( M < M_{cr} \) (when the metric component \( g^{rr} \) does not change sign) the properties of solutions of these equations with \( \Lambda = 0 \) are described in detail in the survey [14].

Analytical analysis, confirmed numerically, proves the existence of solutions with no limit of the mass of a condensate. In these solutions \( g^{rr} \) crosses the x-axis twice. Excluding non-gravitational interactions of bosons (\( \Lambda = 0 \)), the condensate has no elasticity, and its density diverges logarithmically in the center. With account of elasticity (\( \Lambda < 0 \)) there are regular solutions, in which the wave function of the condensate is finite, and there are no restrictions on the total mass.

We face a nonlinear eigenvalue problem. Trivial solution \( u = 0, \quad w = 0, \quad g = -1, \quad h = h_0 \) is a flat space-time in the absence of a scalar field. The constant \( h_0 \) is the freedom for the choice of time units.

The above set of equations has a simple non-trivial regular solution:
\[ u = u_0, \quad w = 0, \quad g(x) = -1 + u_0^2 x^2 / 3, \quad h = 1/3, \quad \Lambda u_0^2 = -2/3. \] (5)

It is as if the Universe was uniformly filled by a scalar field. The relation \( \Lambda u_0^2 = -2/3 \) reflexes the balance of density and elasticity of the condensate. The less is the elasticity, the denser condensate is compressed by its own gravitational field. Though this solution is unrealistic, it has a physical meaning. Its energy-momentum tensor \( T_0^0 = \frac{\hbar^2}{m c^2 r_s} u_0^2, \quad T_{rr} = \frac{\hbar^2}{m c^2} u_0^2 \) corresponds to ultra relativistic equation of state \( p = \varepsilon / 3 \). Naturally, it is not achievable. However, by a slightest deviation \( \Lambda u_0^2 = -2/3 + \alpha, \quad \alpha \ll 1 \) (with a proper \( h_0 = 1/3 + \beta (\alpha) \)) we get regular solutions with finite total masses, and with \( g(x) \) transforming from the parabola \(-1 + u_0^2 x^2 / 3\) into Schwarzschild’s asymptotic \(-1 + x_h / x\) as \( x \) grows from zero to infinity. Solution (5) is the upper boundary of regular solutions with no restriction on the mass of a black hole. As an example, a solution with the parameters \( u_0 = \sqrt{2}, \quad \alpha = 0.00025, \quad h_0 = 0.33358(3), \quad \Lambda = -0.333208(3), \quad x_g = 1.2245, \quad x_h = 8.245 \cdot 10^7 \) is presented in Figures 1, 2, and 3.

Wave function \( u(x)/u_0 \) and metric function \( h(x) \) are shown in Figure 1, using the logarithmic scale \( t = \ln x \) along the horizontal axis. In Figure 2 and Figure 3 the solid red curve is \( g(x) \).
in linear and double logarithmic scales, respectively. The increasing green dashed lines show $g(x) = -1 + u_0^2 x^2/3$, and the decreasing red dashed lines are the Schwarzschild asymptotic $g(x) = -1 + x h/x$. Dashed horizontal line in Figure 3 is the level ln 2 where $g(x) = 0$. $g(x)$ crosses the dashed horizontal line at $x = x_g$ and at $x = x_h$.

The main factor, making possible the existence of a static black hole with no restriction on its mass, is the inside layer of violated signature of the metric tensor.

3. Black hole and dark matter
The wave function $\psi(x^i)$ of a Bose-Einstein condensate inside a black hole is a scalar. The derivative of a scalar field $\psi_{,i}$ is a covariant longitudinal vector. The exactly opposite situation takes place outside a black hole. The wave function $\phi_{,i}$, describing dark matter, is a longitudinal vector. And its covariant divergence $\phi_{;k}$, being a scalar, satisfies the Klein-Gordon equation (3). However, there is a huge difference in characteristic length scales of Bose-Einstein condensate inside and dark matter outside a black hole. The de Broglie wavelength of bosons with the mass $m \sim 100 \text{GeV}/c^2$ is about $10^{-16}$ cm, while the wavelength of dark matter quanta $\lambda/c \approx 10$ kiloparsec. At the horizon $r = r_h$, (it is the interface between black hole and dark matter), both scalar functions $\psi(r_h)$ and $\phi_{;k}(r_h)$ are non-zeros. Their equality provides the balance of pressures at the level of $T^r_r(r_h) = (\kappa r_h^2)^{-1}$ inside and outside.

Having determined $\phi_{;k}(r_h)$, we find the dependence of the speed $V(r)$ of a rotating star on its distance from the center of a galaxy:

$$V(r) = V_{pl} \sqrt{1 - \frac{\hbar}{2 \mu c r} \sin \left( \frac{2 \mu c \hbar}{\hbar} r \right)}, \quad V_{pl} = c \frac{M_{Pl}^2}{\sqrt{\mu m}},$$

$V_{pl}$ - speed on the plateau of a rotation curve, $M$ - black hole mass, $M_{Pl} = \sqrt{c \hbar / k} = 2.177 \cdot 10^{-5}$ g is the Planck mass, $m$ - mass of a quantum of a scalar field, $\mu$ - mass of a quantum of a longitudinal vector field (dark matter).

Rotation curve of the galaxy NGC 3769 in Ursa Major cluster, shown in Figure 4, has a plateau speed $V_{pl} \approx 120$ km/s. The period of damping oscillations is $\lambda \approx 13$ kiloparsec. It corresponds to de Broglie wavelength of a particle with the mass $\mu = \hbar/c\lambda \approx 0.7 \cdot 10^{-60}$ g. The rest energy of massive bosons of the Standard Model is about 100 GeV. The rest mass is $m \approx 1.8 \cdot 10^{-22}$ g. If the black hole in the galaxy NGC 3769 is really compressed to the state of boson condensate, then its mass

$$M = \frac{c}{4V_{pl}} \frac{M_{Pl}^2}{\sqrt{\mu m}} \approx 6.5 \cdot 10^{33} \text{ g}$$
is three times the mass of the Sun.

If we reject the prejudice that the signature of metric tensor remains unchanged with an arbitrarily strong curvature of space-time, then it becomes possible to clarify a number of long-standing questions. In the considered case, the main factor, stopping the gravitational collapse, is the presence of a spherical layer of broken signature inside a black hole. Spherically symmetric static solutions of Einstein equations are found that do not require a limitation of mass of a gravitating body.

Super massive black holes in the centers of galaxies, being in phase equilibrium with surrounding dark matter, form the skeleton of modern structure of the Universe.

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