Distributional solutions to the Maxwell-Vlasov equations

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Abstract. The distributional form of the Maxwell-Vlasov equations are formulated. Submanifold distributions are analysed and the general submanifold distributional solutions to the Vlasov equations are given. The properties required so that these solutions can be a distributional source to Maxwell’s equations are analysed and it is shown that a sufficient condition is that spacetime be globally hyperbolic. The cold fluid, multicurrent and water bag models of charge are shown to be particular cases of the distributional Maxwell-Vlasov system.

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1. Introduction

The Maxwell-Vlasov equations give a model for the dynamics of a large collection of charged particles. They are used to analyse the motion of beams of particles in a particle accelerator, in order to address problems such as the effects of coherent synchrotron radiation and space charge. They are also used for low energy particle dynamics as in Klystrons and magnetrons. With several species of particles, the Maxwell-Vlasov equations model the dynamics of plasmas both man made, as in laser plasma wakefield acceleration and fusion reactors, and naturally occurring as in the solar winds and the ionosphere.

In this article we formulate the distributional form of the Maxwell-Vlasov equations, write down the general solution to the Vlasov equation and identify sufficient conditions such that these solutions are valid sources for Maxwell’s equations. We then give some example solutions. This work is relevant in several areas of active research.

- It unifies the Maxwell-Vlasov equations and the cold fluid, multicurrent and water bag model of charge together with the Klimontovich distribution.
• By comparing the statistical limit of the Klimontovich distribution with the regular solutions of the Maxwell-Vlasov equations, it may provide a way of finding a dispersion term for the Boltzmann equation.

• In many scenarios, such as micro-bunching and emittance reduction in accelerators and charge moving on surfaces of constant magnetic flux in a Tokamak, the resulting charge distribution may be better modelled by the use of lower dimensional distributions.

• It enables the description of the ultra-relativistic expansion for the Maxwell-Vlasov and the Klimontovich distribution. Since a single system applies to all the solutions, finding the expansion of this system will enable one to write down the expansion of all the solutions above.

• It enables the use of both the retarded Greens potential and the Liénard-Weichart potential to find the electromagnetic field due to an arbitrary distributional source.

• It enables alternative methods of finding numerical results. The usual method is to use a collection of world-line distributions based on the motion of individual point particles. Alternative methods could use ribbon or higher dimensional distributions. This may avoid some of the regularisation problems associated with point charges.

1.1. Notation

Sections 2 and 3 deal with general properties of distributions and the transport equations (also known as the Vlasov, Liouville and collisionless Boltzmann equation). For generic objects in these sections we use the following symbols:

- Manifolds: $M, N, P, Q$.
- Subsets: $U, V \subset M$.
- Boundary: $\partial M$.
- Tangent bundle: $TM$.
- Bundle of $p$-forms: $\Lambda^p M$.
- Bundle of forms: $\Lambda M$.
- Generic bundle: $\pi : E \to M$.
- Smooth sections of a bundle: $\Gamma E$.
- Space of test forms: $\Gamma_0 \Lambda M$.
- Space of continuous (not differentiable) forms: $\Gamma_{ct} \Lambda M$.
- Space of piecewise continuous forms: $\Gamma_{pc} \Lambda M$.
- Space of test forms: $\Gamma_0 \Lambda M$.
- Space of distributions: $\Gamma_D \Lambda M$.
- Set of submanifold distributions: $\Gamma_S \Lambda M$.
- Vectors and vector fields: $u, v, w$.
- Forms and form fields: $\alpha, \beta$.
- Evaluation of a field at a point: $\alpha|_x, u|_x$.
- Test forms: $\phi, \psi$.
- Distributional forms: $\Psi, \Phi$.
- Regular distributions: $D(\alpha)$.
- Degree of a form or distribution:
  \[ \deg(\alpha), \deg(\Psi). \]
- Sign involution $\alpha^\eta = (-1)^{\deg(\alpha)} \alpha$.
- Smooth map between manifolds:
  \[ a, b, c, \quad a : N \to M. \]
- Image set of the map $a(N) \subset M$.
- Preimage set of the subset $U \subset M$:
  \[ a^{-1}(U) \subset N. \]
- Embedding: $a : N \hookrightarrow M$.
- Composition of maps: $a \circ b$.
- Pushforward for distributions: $a_* \varsigma, b_* \varsigma$.
- Submanifold distribution: $a_{\xi} \alpha$.
- Pullback for distribution: $a^*$. 
Expectation of a distribution $\Psi$ with respect to box $s:S \hookrightarrow M$: $[s^c(\Psi)]$. Initial hypersurface: $\sigma: \Sigma \hookrightarrow M$.

In sections 1 and 4 we look at the Maxwell-Vlasov equation on spacetime. We use the following symbols:

- Spacetime: $\mathcal{M}$.
- Spacetime metric: $g$.
- Upper unit hyperboloid: $\mathcal{E}$.
- Metric dual of a 1-form: $\tilde{\alpha}$.
- Electromagnetic 2-form: $F$.
- Liouville vector field: $W$.

### 1.2. Maxwell-Vlasov equations

Let $(\mathcal{M}, g)$ be spacetime and let

$$\pi: \mathcal{E} \to \mathcal{M} \quad \text{where} \quad \mathcal{E} = \{ u \in T\mathcal{M} \mid g(u, u) = -1, \ u^0 > 0 \}$$

be the unit upper hyperboloid bundle over spacetime. Let the coordinates $(x^0, x^1, x^2, x^3, y^1, y^2, y^3)$, for $\mathcal{E}$ with the embedding $\mathcal{E} \hookrightarrow T\mathcal{M}$ be given by $x^a = x^a$, $\hat{x}^i = y^i$ and let $\hat{x}^0 = y^0 = y^0(x^a, y^i)$ be the solution to $g_{ab}y^ay^b = -1$, $y^0 > 0$, and $y_0 = g_{0a}y^a$. Here the indices $a, b, c = 0, 1, 2, 3$ and $i, j, k = 1, 2, 3$. We choose physical units of time and length so that the speed of light $c = 1$ and the permittivity of free space $\varepsilon_0 = 1$.

Given the electromagnetic 2-form field $F \in \Gamma \Lambda^2 \mathcal{M}$, the standard way of expressing the Maxwell-Vlasov equations is to prescribe the Vlasov vector field $W = W(F) \in \Gamma T\mathcal{E}$, which depends on the electromagnetic field as

$$W = y^a \frac{\partial}{\partial x^a} - \Gamma^i_{bc} y^b y^c \frac{\partial}{\partial y^i} + \frac{q}{m} y^a F_{ab} y^b \frac{\partial}{\partial y^a}$$

for the species of particle with mass $m$ and charge $q$. For most of this article we deal with a single species and therefore choose physical units of mass and charge so that $m = 1$ and $q = 1$.

This vector field is chosen so that it is horizontal, i.e., given $u \in \mathcal{E}$ then $\pi_*(W|_u) = u$ and that if $\gamma: \mathbb{R} \to \mathcal{E}$ is an integral curve of $W$, that is $\gamma'(\tau) = \gamma_*(\partial_\tau) = W|_{\gamma(\tau)}$, then $\gamma = \hat{C} = C_*(\partial_\tau)$ where $C = \pi \circ \gamma: \mathbb{R} \to \mathcal{M}$ satisfies the Lorentz force equation

$$\nabla_c \hat{C} = i_c \hat{F}$$

with $g(\hat{C}, \hat{C}) = -1$, see [1, 2] and lemma 29, section 3.1. Here $\tilde{\gamma}: \Lambda^1 M \to TM$ is the metric dual given by $\beta(\tilde{\gamma}) = g^{-1}(\beta, \alpha)$. A function $f \in \Gamma \Lambda^0 \mathcal{E}$ is called the one particle probability function. The Vlasov equation, which is also know as the Liouville

‡ The use of the word distribution is avoided in this context. This word will be reserved for distribution in the sense of Schwartz or De Rham currents.
equation and the collisionless Boltzmann equation, is now given by
\[ W(f) = 0 \] (4)
Maxwell’s equations for the electromagnetic field \( F \) are given by
\[ dF = 0 \quad \text{and} \quad d\star F = -\mathcal{J} \] (5)
where the source \( \mathcal{J} \in \Gamma^3 \mathcal{M} \) is given by
\[ \mathcal{J} = \left( \int_{\mathbb{R}^3} \frac{f y^a}{y^0} \sqrt{|\det(g)|} dy^{123} \right) i_{\partial/\partial x^a} \star 1 \] (6)
where \( dy^{123} = dy^1 \wedge dy^2 \wedge dy^3 \) etc. There is a natural non-vanishing 7-form on \( \mathcal{E} \) given by
\[ \Omega = -\frac{\det(g)}{y^0} dy^{123} \wedge dx^{0123} \in \Gamma^7 \mathcal{E} \] (7)
Since the Lie derivative \( L_W \Omega = 0 \) then
\[ di_W (f \Omega) = L_W (f \Omega) = L_W (f) \Omega + f L_W \Omega = L_W (f) \Omega = W(f) \Omega \]
Thus (4) is equivalent to \( di_W (f \Omega) = 0 \).

We introduce the charge 6-form given by \( \theta \in \Gamma^6 \mathcal{E} \)
\[ \theta = i_W (f \Omega) \] (8)
Thus we can rewrite (4) as
\[ d\theta = 0 \quad \text{and} \quad i_W \theta = 0 \] (9)
Observe that given \( \theta \) and \( \Omega \) it is easy to construct the probability function \( f \) since the coefficient of \( \theta \) with respect to \( dy^{123} \wedge dx^{123} \) is given by \( f \det(g) \).

Comparing (6) and (8) we see that the integration for the source \( \mathcal{J} \in \Gamma^3 \mathcal{M} \), is in fact a integration along a fibre \([3]\) given by
\[ \int_{\mathcal{E}} \pi^* \phi \wedge \theta = \int_{\mathcal{M}} \phi \wedge \mathcal{J} \] (10)
for all test 1-forms \( \phi \in \Gamma^1 \mathcal{M} \), that is the space of smooth 1-forms with compact support on \( \mathcal{M} \).

The 6-form \( \theta \) has a natural interpretation in terms of probabilities or charge. Given a “box”, that is a compact 6-dimensional region \( S \subset \mathcal{E} \) which is transverse to \( W \) then the probability that the particle passes though \( S \) (or the total charge passing though \( S \)) is given by the integral \( \int_S \theta \). An example of such a box is the set \( S = \{ (x^a, y^i) | x^0 = x^0_c, \ x^i < x^i < x^i_u, \ y^i_l < y^i < y^i_u \} \) for some constants \( \{ x^0_c, x^i, x^i_u, y^i_l, y^i_u \} \). Assuming the correct orientation for \( S \) it is usual to demand \( \int_S \theta \geq 0 \). An initial hypersurface \( \Sigma \subset \mathcal{E} \) is any 6 dimensional hypersurface transverse to \( W \). If \( \theta \) is to be interpreted as probability then we further demand that \( \int_{\Sigma} \theta = 1 \). However if \( \theta \) is the charge distribution then \( \int_{\Sigma} \theta = Q_{\text{total}} \) is the total charge. It is easy to show that \( Q_{\text{total}} \) is a conserved quantity. In order to evaluate a distribution on a box or a hypersurface we require the definition of the pullback of a distribution given in section 2.3.
In order to write the Maxwell-Vlasov system in distributional language we replace the 6-form $\theta$ with the **charge distributional 6-form**

$$\Theta \in \Gamma_D \Lambda^6 \mathcal{E},$$

(11)

This is defined as an element in the dual to the space of test 1-forms $\Gamma_0 \Lambda^1 \mathcal{E}$. Given a test 1-form $\phi \in \Gamma_0 \Lambda^1 \mathcal{E}$ then the action of $\Theta$ on $\phi$ is the real number written $\Theta[\phi]$.

Inspired by (10) we define the pushforward of a distribution $\pi_\varsigma$ via

$$\pi_\varsigma(\Theta)[\phi] = \Theta[\pi^*(\phi)]$$

(12)

and we have the source for Maxwell’s equations as

$$J \in \Gamma_D \Lambda^3 \mathcal{M}; \quad J = \pi_\varsigma \Theta$$

(13)

The details of this are given in section 2.1.

Since $J$ is a distribution, then in general, the solution to Maxwell’s equations (5) will also be a distribution. However in order for $F$ to drive the Liouville equation (2.9) we require that $F$ is continuous, though not necessarily differentiable. Since $F$ is not differentiable, we must convert it into a distribution before inserting it into Maxwell’s equations. Every continuous $q$-form $\alpha \in \Gamma_{cts} \Lambda^q \mathcal{M}$ gives rise to a regular distribution $q$-form $D(\alpha) \in \Gamma_D \Lambda^q \mathcal{M}$. Thus we can define $D(F) \in \Gamma_D \Lambda^2 \mathcal{M}$ given by $D(F)[\phi] = \int_\mathcal{M} \phi \wedge F$ for any $\phi \in \Gamma_0 \Lambda^2 \mathcal{M}$. Maxwell’s equations become

$$d(DF) = 0 \quad \text{and} \quad d(D(*F)) = -J$$

(14)

As is noted in section 2 we can define the exterior derivative and the internal contraction of a distribution, thus the Liouville equation (9) becomes the equivalent for $\Theta$ namely

$$d\Theta = 0$$

(15)

and

$$i_W \Theta = 0$$

(16)

where the **Liouville operator** $W$ is given by (2). These are also called the **transport equations** of $\Theta$ for $W$. Theorem 27 implies that the source $J$ is closed, $dJ = 0$.

We limit ourselves to “submanifold distributions”. These are written $\Theta = a_\varsigma \alpha$ where $a: N \hookrightarrow \mathcal{E}$ is an an embedding with $N$ possibly having a boundary and $\alpha \in \Gamma \Lambda^p N$ is a smooth $p$–form where $p = \dim N - 1$. The action of this distribution on a test form $\phi \in \Gamma_0 \Lambda^1 \mathcal{E}$ is given by

$$a_\varsigma \alpha[\phi] = \int_N a^*(\phi) \wedge \alpha$$

(17)

Most of section 2 details the mathematics needed to manipulate these distributions.

The geometric nature of these distributions means that we can define many useful properties such as pullbacks, without having the worry about the convergence and other topological properties. However these distributions are sufficiently general that physically interesting solutions to the Maxwell-Vlasov system can be written using them. In section 4 we show that the cold fluid, the multicurrent and the water bag
models of charge dynamics are all examples of submanifold distributional solutions to the Maxwell-Vlasov equations. Another example of a submanifold distribution, discussed in section 4.1, is the worldline for a single charged particle given by \( \Theta = \dot{C}(1) \) so that \( \Theta[\phi] = \int_{\mathbb{R}} C^\ast(\phi) \) for all \( \phi \in \Gamma_0 \Lambda^1 \mathcal{E} \). In Minkowski space, this is equivalent to the usual expression for a charged particle, written in terms of a Dirac \( \delta \)-function probability function as

\[
f = \left( \prod_{i=1}^{3} \delta(x^i - C^i(\tau)) \right) \left( \prod_{j=1}^{3} \delta(y^j - \dot{C}^j(\tau)) \right)
\]

where \( t = C^0(\tau) \) so that \( \tau = (C^0)^{-1}(t) \). Thus we have

\[
\dot{C}(1) = i_W(f \Omega) \tag{18}
\]

To see this recall that \( \dot{C} : \mathbb{R} \to \mathcal{E} \) is an integral curve of \( W \) so that \( W|_{\dot{C}(\tau)} = \dot{C}(\tau) \) where \( \dot{C} = \dot{C}(\partial_{\tau}) \), thus given a test 1-form \( \phi \in \Gamma_0 \Lambda^1 \mathcal{E} \) we have

\[
\dot{C}^\ast \phi|_\tau = i_{\partial_{\tau}}(\dot{C}^\ast \phi)|_\tau d\tau = \dot{C}^\ast(i_{\partial_{\tau}} \dot{C}(\tau) \phi) d\tau = \dot{C}^\ast(i_W|_{\dot{C}(\tau)} \phi) d\tau = (i_W \phi)|_{\dot{C}(\tau)} d\tau
\]

Hence

\[
\dot{C}(1)[\phi] = \int_{\mathbb{R}} \dot{C}^\ast(\phi) = \int_{\mathbb{R}} (i_W \phi)|_{\dot{C}(\tau)} d\tau
\]

and

\[
i_W(f \Omega)[\phi] = (f \Omega)[i_W \phi] = \int_{\mathbb{R}} \left( \prod_{i=1}^{3} \delta(x^i - C^i(\tau)) \right) \left( \prod_{j=1}^{3} \delta(y^j - \dot{C}^j(\tau)) \right) (i_W \phi) \frac{dx^{123} \wedge dy^{123} \wedge dt}{y^0}
\]

\[
= \int_{\mathbb{R}} (i_W \phi)|_{\dot{C}(\tau)} \frac{1}{C^0(\tau)} dt = \int_{\mathbb{R}} (i_W \phi)|_{\dot{C}(\tau)} \frac{d\tau}{dt} dt = \int_{\mathbb{R}} (i_W \phi)|_{\dot{C}(\tau)} d\tau
\]

Equation (18) is true for general spacetimes, assuming the correct definition of the \( \delta \)-function. One of the advantages of the pushforward approach to distributions is that the definitions do not explicitly contain the coordinates system. They also do not depend on the metric and therefore can be applied to the upper unit hyperboloid bundle \( \mathcal{E} \) without having to choose a metric for this manifold.

The Klimontovich distribution is given by the sum of worldline distributions one for each particle \( \mathbb{R} \). We show that the worldline of a point charge and the Klimontovich distribution satisfy the Liouville equation and have the correct form for the source of Maxwell’s equations. However the solutions to Maxwell’s equations with the worldline as a source, which in free space are given by the Liénard-Weichart potentials, are not continuous and therefore we cannot insert them into the Liouville equation without some form of regularisation, leading, for example, to the Lorentz-Dirac equation. The same is true for the Klimontovich distribution, where the usual regularisation is simply to ignore the field due to a particular particle when looking at the dynamics of that particle.

In section 3 we look at the submanifold distributional solutions to the transport equations for a general vector field \( v \in \Gamma TM \) on a general manifold \( M \). Theorem 21
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Figure 1. A submanifold solution to the transport equation. Observe that the submanifold is tangential to the vector field and that the boundary of submanifold is also tangential to the vector field.

gives the necessary and sufficient conditions for a submanifold solution. A sketch of such a solution is given in figure 1. We look at initial hypersurfaces and boxes and show the conservation of charge, theorem 22, as well as that, given a distribution on an initial hypersurface, there exists a unique solution to the transport equations, theorem 24. This is achieved by extending the initial value along the integral curves of $v$.

Given a solution $\Theta$ to the Liouville equations on $\mathcal{E}$, we can ask if $\mathcal{J} = \pi_* \Theta$ is a valid source for Maxwell’s equations. The problem is that, in general, $\pi_* \Theta$ may not be a distribution. This is because $\pi$ is not a proper map. A map is proper if the preimage of any compact set is compact. This is required so that $\pi^*(\varphi)$ in (12) has compact support. In section 3.1, lemma 28, we show that an appropriately bounded submanifold distribution is a valid source. We say that $\Theta$ is bounded with respect to $\pi$. This is the case in the examples given in section 4. We then pose the following question (70): If a distribution is initially bounded, does it remain bounded? Surprisingly the answer to this question is no and two counter examples are given, one using a drifting source in Galilean spacetime, the other, on Minkowski spacetime, where the initial data is specified on a lightlike surface. However if, as we generally assume, $\mathcal{M}$ is a globally hyperbolic spacetime, and the initial data is given on a Cauchy surface, then theorem 30 shows that the solution will remain bounded. However, as in the case of the worldline for a point charge, the source may be valid distribution but the resulting electromagnetic 2-form need not be continuous.

It is easy to extend (13-16) to model several species of particle with masses $m_\alpha$ and charges $q_\alpha$. Let $\Theta_\alpha \in \Gamma_D \Lambda^6 \mathcal{E}$ be the corresponding one particle 6-form distribution, and $W_\alpha \in \Gamma T\mathcal{M}$ be the Vlasov vector field (2) with appropriate $m$ and $q$. The Vlasov
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Equations are given by
\[ d\Theta_\alpha = 0 \quad \text{and} \quad i_{W_\alpha} \Theta_\alpha = 0 \] (19)
and Maxwell’s equations are given by (14) where the source
\[ \mathcal{J} = \sum_\alpha \pi_\varsigma (\Theta_\alpha) \] (20)

2. Distributions

Let \( M \) be an arbitrary manifold of dimension \( m \). Let \( \Gamma_0 \Lambda^p M \) be the space of test \( p \)-forms.
\[ \Gamma_0 \Lambda^p M = \left\{ \phi \in \Lambda^p M \mid \phi \text{ has compact support} \right\} \] (21)
Let the space of \( p \)-forms distributions be the vector space dual of the space of test functions.
\[ \Gamma_D \Lambda^p M = (\Gamma_0 \Lambda^{m-p} M)^* \] (22)
We use square bracket notation.
\[ \Gamma_D \Lambda^{m-p} M \times \Gamma_0 \Lambda^p M \rightarrow \mathbb{R} , \quad (\Psi, \phi) \mapsto \Psi[\phi] \in \mathbb{R} \] (23)
Observe that the space of piecewise continuous \( p \)-forms is a subspace of the space of \( p \)-form distributions. These are called regular distributions.
\[ D: \Gamma_{pc} \Lambda^p M \hookrightarrow \Gamma_D \Lambda^p M ; \quad D(\alpha)[\phi] = \int_M \phi \wedge \alpha \quad \text{for} \quad \phi \in \Gamma_0 \Lambda^{m-p} M \] (24)
The exterior derivative of a \( p \)-form distribution is given by
\[ d: \Gamma_D \Lambda^p M \rightarrow \Gamma_D \Lambda^{p+1} M ; \quad d\Psi[\phi] = -\Psi[d\phi^n] \quad \text{for} \quad \phi \in \Gamma_0 \Lambda^{m-p+1} M \] (25)
The internal contraction of a vector field \( v \in \Gamma TM \) with a \( p \)-form distribution \( \Psi \in \Gamma_D \Lambda^p M \) is given by
\[ \Gamma TM \times \Gamma_D \Lambda^p M \rightarrow \Gamma_D \Lambda^{p-1} M \, , \quad (v, \Psi) \mapsto i_v \Psi \] (26)
where \( i_v \Psi[\phi] = -\Psi[i_v \phi^n] \) for all \( \phi \in \Gamma_0 \Lambda^{m-p+1} M \)
Given a regular distribution \( D(\beta) \) with \( \beta \in \Gamma_{pc} \Lambda M \) then
\[ i_v (D(\beta)) = D(i_v (\beta)) \] (27)
Given a regular distribution \( D(\beta) \) with \( \beta \in \Gamma_{cts} \Lambda M \) and \( \beta \) has a continuous derivative and where \( M \) has no boundary then
\[ d(D(\beta)) = D(d(\beta)) \] (28)
We define the support of a distribution \( \mathcal{S}(\Psi) \subset M \) as
\[ \mathcal{S}(\Psi) = \bigcap \left\{ U \subset M \mid U \text{ is closed and if} \phi \in \Gamma_0 \Lambda M, \, \mathcal{S}(\phi) \subset M \setminus U \text{ then} \Psi[\phi] = 0 \right\} \] (29)
where \( \mathcal{S}(\phi) \) is the closed support of \( \phi \). That is \( \mathcal{S}(\Psi) \) is the smallest closed subset of \( M \) such that for every test form \( \phi \) with \( \mathcal{S}(\Psi) \cap \mathcal{S}(\phi) = \emptyset \) then \( \Psi[\phi] = 0 \).
Lemma 1 For regular distributions $\mathcal{S}(D(\alpha)) = \mathcal{S}(\alpha)$.

proof: Clearly if $\mathcal{S}(\phi) \cap \mathcal{S}(\alpha) = \emptyset$ then $D(\alpha)[\phi] = \int \phi \wedge \alpha = 0$. Thus $\mathcal{S}(D(\alpha)) \subset \mathcal{S}(\alpha)$. If $\mathcal{S}(D(\alpha)) \neq \mathcal{S}(\alpha)$ then there exists an open set $U \subset \mathcal{S}(\alpha) \setminus \mathcal{S}(D(\alpha))$. It is easy to construct a test form $\phi$ with $\mathcal{S}(\phi) \subset U$ such that $\int \phi \wedge \alpha \neq 0$. So $\mathcal{S}(D(\alpha)) \cap U \neq 0$ leading to the contradiction $\mathcal{S}(D(\alpha)) \cap (\mathcal{S}(\alpha) \setminus \mathcal{S}(D(\alpha))) \neq \emptyset$. Hence result. $\blacksquare$

Lemma 2 For $\Psi \in \Gamma_0 \Lambda \mathcal{M}$ and $v \in \Gamma TM$

$$\mathcal{S}(d\Psi) \subset \mathcal{S}(\Psi) \quad \text{and} \quad \mathcal{S}(i_v \Psi) \subset \mathcal{S}(\Psi)$$

(30)

proof: Given $\phi \in \Gamma_0 \Lambda \mathcal{M}$ such that $\mathcal{S}(\phi) \subset M \setminus \mathcal{S}(\Psi)$ then $\mathcal{S}(d\phi) \subset M \setminus \mathcal{S}(\Psi)$ and $\mathcal{S}(i_v \phi) \subset M \setminus \mathcal{S}(\Psi)$. Thus $0 = -\Psi[d\phi^\eta] = d\Psi[\phi]$ and $0 = -\Psi[i_v \phi^\eta] = i_v \Psi[\phi]$. Hence $M \setminus \mathcal{S}(\Phi) \subset M \setminus \mathcal{S}(d\Phi)$ and $M \setminus \mathcal{S}(\Phi) \subset M \setminus \mathcal{S}(i_v \Phi)$ and (30) follows. $\blacksquare$

2.1. Pushforward of distributions

The **pushforward** [4] of a distribution with respect to a smooth map $a: \mathcal{N} \to \mathcal{M}$ where $\dim \mathcal{M} = m$ and $\dim \mathcal{N} = n$ is given by

$$a_*: \Gamma_0 \Lambda^p \mathcal{N} \to \Gamma_0 \Lambda^{m-n+p} \mathcal{M} ; \quad a_*[\Psi][\phi] = \Psi[a^*(\phi)] \quad \text{for} \quad \phi \in \Gamma_0 \Lambda^{n-p} \mathcal{M}$$

(31)

The pushforward does not preserve the degree of a distribution. Instead we have

$$\deg(a_*[\Psi]) = \deg(\Psi) + \dim \mathcal{M} - \dim \mathcal{N}$$

(32)

We note that the pushforward given in definition (31) is not always defined. The problem is that in general there is no guarantee that $a^*(\phi)$ has compact support. This can lead to the pushforward being infinite as in the following example. Let $a: \mathbb{R} \to \{x\}$ and let $\Psi = D(dt) \in \Gamma_0 \Lambda^1 \{x\}$. For $\phi \in \Gamma_0 \Lambda^0 \{x\}$ we have

$$a_*[\Psi][\phi] = \Psi[a^*(\phi)] = \int \phi(x) \, dt = \phi(x) \int dt$$

which is undefined if $\phi(x) \neq 0$. It can also lead to problems with boundaries.

By contrast, we can define the pushforward of a distribution when the map $a: \mathcal{N} \to \mathcal{M}$ is **proper**, that is, when the preimage of every compact set is compact. However the bundle map $\pi: \mathcal{E} \to \mathcal{M}$ is not proper but from [12] we see we wish to define the pushforward of a distribution with respect to $\pi$. This is not always possible, but it is for certain distribution which we say are bounded with respect to $\pi$.

We say the distribution $\Psi \in \Gamma_0 \Lambda \mathcal{N}$ is **bounded with respect to** $a: \mathcal{N} \to \mathcal{M}$ if for any compact set $U \subset \mathcal{M}$ then $\mathcal{S}(\Psi) \cap a^{-1}(U) \subset \mathcal{N}$ is compact. In this case we define

$$a_*[\Psi][\phi] = \Psi[h \, a^*(\phi)] \quad \text{for} \quad \phi \in \Gamma_0 \Lambda \mathcal{M}$$

(33)

where $h \in \Gamma_0 \Lambda^0 \mathcal{N}$ is any test bump function such that

$$\mathcal{S}(\Psi) \cap \mathcal{S}(1-h) \cap \mathcal{S}(a^* \phi) = \emptyset$$

(34)

Lemma 3 **The pushforward of a distribution bounded with respect to** $a$ **is well defined.**
proof: Since $S(\phi)$ is compact and $S(a^*\phi) \subset a^{-1}(S(\phi))$ then $S(a^*\phi) \cap S(\Psi)$ is compact. There exists a compact set $V_1$ and $V_2$ such that $S(a^*\phi) \cap S(\Psi) \subset \text{interior}(V_1)$ and $V_2 \subset \text{interior}(V_1)$ and a bump function $h$ such that $h(x) = 1$ for $x \in V_1$ and $h(x) = 0$ for $x \notin V_2$.

To see that $\varsigma_{a}(\Psi)[\phi]$ is well defined, consider two functions $h$ and $\hat{h}$, then since $h - \hat{h} = (1 - \hat{h}) - (1 - h)$ we have $S(h - \hat{h}) \subset S(1 - \hat{h}) \cup S(1 - h)$. Thus

$$S(\Psi) \cap S(a^*\phi) \cap S(h - \hat{h}) \subset \left( S(\Psi) \cap S(a^*\phi) \cap S(1 - \hat{h}) \right) \cup \left( S(\Psi) \cap S(a^*\phi) \cap S(1 - h) \right) = \emptyset$$

Thus $\Psi[(h - \hat{h}) a^*(\phi)] = 0$.

To guarantee that the pushforward is well defined we limit ourselves to three cases:

- When $a: N \to M$ is proper.
- When $\Psi$ is bounded with respect to $a$.
- When $\pi: E \to M$ is a fibre bundle and $\alpha \in \Gamma \Lambda E$ has the property that integration along the fibres $\pi^{-1}(x)$, given by $\int_E \pi^* \phi \wedge (\pi_\varsigma(D\alpha)) = \int_M \phi \wedge \alpha$ for all $\phi$ is defined [3]. This last case is simply for regular distributions which are not bounded with respect to $\pi$ but such that integral exists. We will not consider this standard type of pushforward further in the article.

**Lemma 4** Given manifolds $N, M, P$ and smooth proper maps $b: P \to N$ and $a: N \to M$ then the composition of the push forwards is the push forward of the composition:

$$(a \circ b)_\varsigma = a_\varsigma \circ b_\varsigma$$

(proof: The composition of two proper maps is proper. Let $\Psi \in \Gamma_D \Lambda M$ and $\phi \in \Gamma_0 \Lambda P$ then

$$a_\varsigma(b_\varsigma(\Psi))[\phi] = b_\varsigma(\Psi)[a^*\phi] = \Psi[b^*a^*\phi] = \Psi[(a \circ b)^*\phi] = ((a \circ b)_\varsigma \Psi)[\phi]$$

Observe that if $a$ or $b$ is not proper then care must be taken. See lemma 28.)

**Lemma 5** If $a: N \to M$ is proper then the pushforward of distributions commutes with the exterior derivative.

$$d \circ a_\varsigma = a_\varsigma \circ d$$

(proof: $d(a_\varsigma \Psi)[\phi] = -a_\varsigma \Psi[d\phi^\eta] = -\Psi[a^*(d\phi^\eta)] = -\Psi[da^*(\phi^\eta)] = -\Psi[d(a^*\phi^\eta)] = d\Psi[a^*(\phi)]$

$$= a_\varsigma(d\Psi)[\phi]$$

Observe that if $a$ or $b$ is not proper then care must be taken. See lemma 28.)

**Lemma 6** Let $a: N \to M$ be smooth and let $\Psi$ be bounded with respect to $a$, then

$$d(a_\varsigma \Psi) = a_\varsigma(d\Psi)$$
Lemma 8

From \( (30) \), \( d\Psi \) is bounded with respect to \( a \). Given \( h \in \Gamma_0 \Lambda^0 N \) such that \( (34) \) holds, then from \( (30) \) \( S(d\Psi) \cap S(1 - h) \cap S(a^*\phi) = \emptyset \) and since \( S(dh) \subset S(1 - h) \) we have \( S(\Psi) \cap S(dh) \cap S(a^*\phi) = \emptyset \) so \( \Psi[dh \wedge a^*\phi^\eta] = 0 \). Thus

\[
d(a\xi\Psi)[\phi] = -a_\xi \Psi[d\phi^\eta] = -\Psi[h a^*(d\phi^\eta)] = -\Psi[h da^*(\phi^\eta)] = -\Psi[d(h a^*(\phi^\eta))] + \Psi[dh \wedge a^*\phi^\eta] = -\Psi[d(\phi^\eta)] = (d\Psi)[a^*(\phi)] = a_\zeta(d\Psi)[\phi]
\]

\[\blacksquare\]

Given \( a : N \to M \) and a vector field we say that \( v \in \Gamma TM \) is tangential to \( a \) if there exists \( u \in \Gamma TN \) such that for each \( x \in N \), \( a_*(u|_x) = v|_{a(x)} \). We write \( a_*(u) = v|_{a(N)} \) if \( a_*(u|_x) = v|_{a(x)} \) for all \( x \in N \).

**Lemma 7** Let \( a : N \to M \) be proper with \( v \in \Gamma TM \) tangential to \( a \) and let \( u \in \Gamma TN \) satisfy \( a_*(u) = v|_{a(N)} \) then

\[
i_v \circ a_\zeta = a_\zeta \circ i_u \tag{38}
\]

**proof:** Given \( \Psi \in \Gamma_0 \Lambda^0 M, \phi \in \Gamma_0 \Lambda^0 M, a : N \to M \) and \( x \in N \), let \( y = a(x) \) then

\[
(i_u a^*(\phi)|_x = i_{u|_x} a_\zeta^*(\phi||_y) = a_\zeta^*(i_{a_*|_x}(\phi||_y)) = a_\zeta^*((i_v\phi)|_y) = (a^*(i_v\phi))|_x
\]

where \( a_\zeta^* : \Lambda_y N \to \Lambda_x M \). Thus \( i_v a^*\phi = a^*(i_v\phi) \). Hence

\[
i_v a_\zeta(\Psi)[\phi] = -a_\zeta(\Psi)[i_v \phi^\eta] = -\Psi[a^*(i_v\phi^\eta)] = -\Psi[i_v a^*\phi^\eta] = i_u \Psi[a^*\phi] = a_\zeta(i_u \Psi)[\phi]
\]

\[\blacksquare\]

**Lemma 8** Let \( a : N \to M \) be proper and \( \Psi \in \Gamma_0 \Lambda^0 N \) then

\[
S(a_\zeta \Psi) \subset a(S \Psi) \tag{39}
\]

**proof:** Given any \( x \in a^{-1}(M \setminus a(S \Psi)) \) then \( a(x) \in M \setminus a(S \Psi) \) and so \( a(x) \notin a(S \Psi) \) so \( x \notin S(\Psi) \) hence \( x \in N \setminus S(\Psi) \). This gives \( a^{-1}(M \setminus a(S \Psi)) \subset N \setminus S(\Psi) \)

Let \( \phi \in \Gamma_0 \Lambda^0 M \) with \( S(\phi) \subset M \setminus a(S \Psi) \). Then

\[
S(a^*\phi) = a^{-1}(S \phi) \subset a^{-1}(M \setminus a(S \Psi)) \subset N \setminus S(\Psi)
\]

Thus \( a_\zeta \Psi[\phi] = \Psi[a^*\phi] = 0 \).

Hence we have shown \( S(\phi) \subset M \setminus a(S \Psi) \) implies \( a_\zeta \Psi[\phi] = 0 \). Hence \( M \setminus a(S \Psi) \subset M \setminus S(a_\zeta \Psi) \) and hence \( (39) \).

\[\blacksquare\]

To see that in general \( S(a_\zeta \Psi) \neq a(S \Psi) \) consider the following counter example. Let \( M = \mathbb{R} \) and \( N = \mathbb{R} \times \{-1, 1\} \) with \( a : N \to M \) given by \( a(x, i) = x \). Let \( \Psi = D(\alpha) \in \Gamma_0 \Lambda^0 N \) where \( \alpha \in \Gamma_0 \Lambda^0 N \) is given by \( \alpha|_{(x, 1)} = 1 \) and \( \alpha|_{(x, -1)} = -1 \) then given \( \phi \in \Gamma_0 \Lambda^1 M \) we have

\[
a_\zeta a[\phi] = \int_N a^*(\phi) \wedge \alpha = \int_{x \in \mathbb{R}} \phi|_x \wedge \alpha|_{(x, 1)} + \int_{x \in \mathbb{R}} \phi|_x \wedge \alpha|_{(x, -1)} = \int_{\mathbb{R}} \phi - \int_{\mathbb{R}} \phi = 0
\]

Hence \( S(a_\zeta \Psi) = \emptyset \neq \mathbb{R} = a(S \Psi) \).
2.2. Submanifold Distributions

A special kind of distribution is the submanifold distribution. Before we introduce submanifold distributions we observe some basic properties of closed embedding, i.e. embedding $a: N \hookrightarrow M$ where $a(N) \subset M$ is closed. First observe that $a$ is proper.

**Lemma 9** Given an embedding $a: N \hookrightarrow M$ such that $a(N) \subset M$ is closed it follows that $a$ is proper.

**proof:** Given a compact $U \subset M$ then $a(N) \cap U$ is closed and therefore compact. The restriction $a|_{a^{-1}(U)}: a^{-1}(U) \to a(N) \cap U$ is a homeomorphism so $a^{-1}(U)$ is compact. Thus $a$ is proper.

The next three lemmas use the embedding nature of $a$ to relate $a_* (D\alpha)$ and $\alpha$.

**Lemma 10** Let $a: N \hookrightarrow M$ be an embedding. Given an open set $U \subset M$ and $\phi \in \Gamma\Lambda N$ such that $\mathcal{S}(\phi) \subset a^{-1}(U)$ then there exists $\psi \in \Gamma\Lambda M$ such that $a^* \psi = \phi$ and $\mathcal{S}(\psi) \subset U$.

**proof:** Consider $x \in a(N)$. Since $a$ is an embedding there exists an open coordinate patch $V \subset M$, $(x^1, \ldots, x^m)$ about $x$ and a coordinate patch $a^{-1} V \subset N$, $(y^1, \ldots, y^n)$ such that $a(y^1, \ldots, y^n) = (x^1, \ldots, x^m, 0, \ldots, 0)$.

Assume for the moment that $\mathcal{S}(\phi) \subset a^{-1}(V)$ and $\phi = \sum I \phi_I dy^I$ where $I \subset \{1, \ldots n\}$ and $dy^I = dy^{i_1} \wedge \cdots \wedge dy^{i_l}$ refers to multi-index notation, then let

$$\psi = \sum_{I \subset \{1, \ldots n\}} \phi_I dx^I h(x^{n+1}, \ldots, x^m)$$

where $h(x^{n+1}, \ldots, x^m)$ is a smooth function with $h(0, \ldots, 0) = 1$ and so that $\mathcal{S}(\psi) \subset V$. Thus $a^* \psi = \phi$.

Now in general use a partition of unity to partition $U$ into coordinate patches $V_i$. Thus if $\phi = \sum \phi_i$ with $\mathcal{S}(\phi_i) \subset a^{-1} V_i$ then $\psi = \sum \psi_i$ and $a^* \psi = \phi$.

**Lemma 11** Let $a: N \hookrightarrow M$ be an embedding with $a(N) \subset M$ closed then

$$\mathcal{S}(a_* (D\alpha)) = a(\mathcal{S}(\alpha)) \quad (40)$$

**proof:** Lemma 1 implies $\mathcal{S}(D\alpha) = \mathcal{S}(\alpha)$ so $a(\mathcal{S}(D\alpha)) = a(\mathcal{S}(\alpha))$. Lemma 8 implies $\mathcal{S}(a_* (D\alpha)) \subset a(\mathcal{S}(D\alpha))$. Thus $\mathcal{S}(a_* (D\alpha)) \subset a(\mathcal{S}(\alpha))$.

By contrast let $x = a(y) \in a(\mathcal{S}(\alpha))$ so $y \in \mathcal{S}(\alpha)$ as $a$ is injective. Given any neighbourhood $U \subset M$ of $x$ then $a^{-1}(U) \subset N$ is a neighbourhood of $y$. As $y \in \mathcal{S}(\alpha)$ there exists $\psi \in \Gamma_0 \Lambda N$ with $\mathcal{S}(\phi) \subset a^{-1} U$ such that $\int_N \phi \wedge \alpha \neq 0$. From lemma 10 there exists a $\psi \in \Gamma_0 \Lambda M$ such that $\mathcal{S}(\psi) \subset U$ and $a^* \psi = \phi$. Thus $a_* (D\alpha)[\psi] = D\alpha [a^* \psi] = D\alpha [\phi] = \int_N \phi \wedge \alpha \neq 0$. Since this is true for all neighbourhoods $U$ about $x$ then $x \in \mathcal{S}(a_* (D\alpha))$. Thus we have shown $x \in a(\mathcal{S}(\alpha)) \implies x \in \mathcal{S}(a_* (D\alpha))$ hence (40).

**Lemma 12** If $a: N \hookrightarrow M$ is an embedding, $a(N) \subset M$ is closed and $\alpha \in \Gamma\Lambda N$ then $a_* (D\alpha) = 0$ if and only if $\alpha = 0$.

**proof:** Follows trivially from lemma 11.
A submanifold distribution is a distribution of the form $\Psi = a_\varsigma(D\alpha)$ where

- $a: N \hookrightarrow M$ is an embedding. \hfill (41)
- $\alpha \in \Gamma \Lambda N$ with $\mathcal{S}(\alpha) = N$. \hfill (42)
- $\gamma(N) \subset M$ is closed. \hfill (43)
- $a: N \rightarrow a(N)$ is a diffeomorphism. \hfill (44)

The set of all submanifold distributions over $M$ is written $\Gamma_\varsigma \Lambda M$.

Since the combination of a pushforward of a regular distribution, for example $a_\varsigma(D(\alpha))$ is so common, we introduce the notation (the bold subscript $\varsigma$) to represent the pushforward of a regular distribution so that $a_\varsigma \alpha = a_\varsigma(D(\alpha))$.

Observe that if $a: N \hookrightarrow M$ and $a_\varsigma \alpha \in \Gamma_\varsigma \Lambda M$ is a submanifold distribution then from lemma [9] $a$ is proper and from lemma [11]

$$\mathcal{S}(a_\varsigma \alpha) = a(N)$$ \hfill (45)

Given $\Psi \in \Gamma_\varsigma \Lambda M$, the following lemma establishes the essential uniqueness of the embedding and the form on the domain of the embedding.

**Lemma 13** Given two submanifold distributions $a_\varsigma \alpha \in \Gamma_\varsigma \Lambda M$ and $b_\varsigma \beta \in \Gamma_\varsigma \Lambda M$ with $a: N \hookrightarrow M$ and $b: P \hookrightarrow M$ then $a_\varsigma \alpha = b_\varsigma \beta$ if and only if there exist a diffeomorphism $c: N \rightarrow P$ with $\alpha = c^* \beta$ and the following diagram commutes

$$\begin{array}{ccc}
N & \xrightarrow{c} & P \\
ap & \downarrow & \downarrow b \\
M & \end{array}$$

**proof:** Since $\mathcal{S}(a_\varsigma \alpha) = \mathcal{S}(b_\varsigma \beta)$ then $a(N) = b(P)$. Furthermore $a: N \rightarrow a(N)$ and $b: P \rightarrow b(P)$ are diffeomorphisms. Thus we can let $c = (b|_{b(P)})^{-1} \circ a: N \rightarrow P$, so $c$ is a diffeomorphism and $a = b \circ c$.

Given $\phi \in \Gamma_0 \Lambda P$, from lemma [10] there exists $\psi \in \Gamma_0 \Lambda M$ such that $b^* \psi = \phi$. Now

$$\int_P \phi \wedge \beta = \int_P b^* \psi \wedge \beta = b_\varsigma(\beta)[\psi] = a_\varsigma(\alpha)[\psi] = \int_N a^* \psi \wedge \alpha = \int_N (b \circ c)^* \psi \wedge \alpha$$

$$= \int_N (c^* b^* \psi) \wedge \alpha = \int_N c^* \phi \wedge \alpha = \int_P c^{-1\star}(c^* \phi \wedge \alpha) = \int_P \phi \wedge (c^{-1\star} \alpha)$$

Since this is true for all $\phi$ then $\beta = c^{-1\star} \alpha$. \hfill $\blacksquare$

The following lemmas relate embeddings with internal contraction.

**Lemma 14** Let $a: N \hookrightarrow M$ and $a_\varsigma \alpha \in \Gamma_\varsigma \Lambda M$. Let $x \in a(N)$ and let $v \in \Gamma TM$ such that $v|_x \not\in T_x(a(N))$, that is $v$ is transverse to $a$ at $x$. Then $i_v(a_\varsigma \alpha) \neq 0$.

**proof:** There exists an open neighbourhood $U \subset M$ of $x$ such that $v|_y$ is transverse to $a_\varsigma \alpha$ for all $y \in a(N) \cap U$. Shrinking $U$ if necessary we can assume that $U \cap a(\partial N) = \emptyset$. Further shrinking $U$ we can make it contain a coordinate chart about $x$ adapted to $v$. Thus there exists a $t: U \rightarrow \mathbb{R}$ such that $t(y) = 0$ for all $y \in a(N)$ and $v(t) = 1$. 


Lemma 15 Let \( \alpha \in \Gamma_S \Lambda^p M \) with \( \mathcal{S} (\psi) \subset U \) then
\[
a_{\varsigma} \alpha [dt \wedge \psi] = \int_N a^* (dt \wedge \psi) \wedge \alpha = \int_N da^* (t) \wedge a^* (\psi) \wedge \alpha
\]
\[
= \int_{\partial N} a^* (t) \wedge a^* (\psi) \wedge \alpha - \int_N a^* (t) \ w (a^* (\psi) \wedge \alpha) = 0
\]
since \( a^* t (z) = t (a (z)) = 0 \) as \( a (z) \in a (N) \), and since also \( a^* \psi |_{\partial N} = 0 \).

There exists \( \phi \in \Gamma_0 \Lambda M \) such that \( \mathcal{S} (\phi) \subset U \) and \( a_{\varsigma} \alpha [\phi] \neq 0 \). To see this consider the contrary that \( \mathcal{S} (\phi) \subset U \) implied \( a_{\varsigma} \alpha [\phi] = 0 \). This would imply that \( U \subset M \setminus \mathcal{S} (a_{\varsigma} \alpha) \), a contradiction.

Now consider \( i_v (a_{\varsigma} \alpha) [\phi \wedge dt] \)
\[
i_v (a_{\varsigma} \alpha) [\phi \wedge dt] = -i_v (a_{\varsigma} \alpha) [(dt \wedge \phi) \eta] = a_{\varsigma} \alpha [i_v (dt \wedge \phi)] = a_{\varsigma} \alpha [\phi] - a_{\varsigma} \alpha [dt \wedge i_v \phi]
\]
\[
= a_{\varsigma} \alpha [\phi] \neq 0
\]
Hence \( i_v (a_{\varsigma} \alpha) \neq 0 \).

### Lemma 15

Let \( a_{\varsigma} (\alpha) \in \Gamma_S \Lambda^p M \) with \( a: N \hookrightarrow M \). Then
\[
i_v a_{\varsigma} \alpha = 0 \tag{46}
\]
if and only if \( v \) is tangential to \( a \) and
\[
i_u \alpha = 0 \tag{47}
\]
where \( u \in \Gamma \tau TN \) is the unique vector field satisfying \( a_* (u) = v |_{a (N)} \).

**Proof:** That (47) implies (46) follows trivially from lemma 7.

If (46) is true and \( x \in a (N) \) then from lemma 14 \( v |_x \) must be tangential to \( a \). Then lemma 7 gives \( a_{\varsigma} (i_u \alpha) = 0 \). Finally (47) follows from lemma 12.

The set of \( p \)-form distributions \( \Gamma_D \Lambda^p M \) forms a vector space in that it is closed under addition and multiplication by a scalar. Also the exterior derivative \( d: \Gamma_D \Lambda^p M \to \Gamma_D \Lambda^{p+1} M \). By contrast the set of \( p \)-form submanifold distributions \( \Gamma_S \Lambda^p M \) does not in general form a vector space. For example if \( a_{\varsigma} \alpha \in \Gamma_S \Lambda^p M \) and \( b_{\varsigma} \beta \in \Gamma_S \Lambda^p M \) with \( a: N \hookrightarrow M \), \( b: P \hookrightarrow M \) and \( \dim N \neq \dim P \) then \( a_{\varsigma} \alpha + b_{\varsigma} \beta \notin \Gamma_S \Lambda^p M \). Also if \( \dim N = \dim P \) and \( a (N) \cap b (P) \neq \emptyset \) then in general \( a_{\varsigma} \alpha + b_{\varsigma} \beta \notin \Gamma_S \Lambda^p M \).

Likewise in section 2.4 we see that the exterior derivative does not map \( \Gamma_S \Lambda^p M \) to \( \Gamma_S \Lambda^{p+1} M \). This is a due to the possible additional boundary terms.

### 2.3. Pullback of Distributions

We first define the pullback for a general “pushforward” distribution. This is a distribution of the form \( a_{\varsigma} \alpha \in \Gamma_D \Lambda M \). This is required for the proof of lemma 35. However in most cases the map \( a \) is an embedding, and this simplifies the concept. This definition of a pullback of a distribution can be compared to the definition using the weak limit 6.
Let $a: N \to M$ and $b: P \to M$ be proper maps and let $\alpha \in \Gamma \Lambda N$. Thus $a_\varsigma \alpha \in \Gamma \Lambda M$ is a distribution and we wish to define the pullback $b^*(a_\varsigma \alpha) \in \Gamma \Lambda P$ with respect to $b$. Let
\[ Q = \{(y, p) \in N \times P | a(y) = b(p)\} \tag{48} \]
be the induced manifold (also known as the pullback manifold), such that the following diagram commutes
\[ \begin{array}{ccc}
Q & \xrightarrow{\hat{b}} & N \\
\downarrow \hat{a} & & \downarrow a \\
P & \xrightarrow{b} & M
\end{array} \quad \text{where } \hat{b}(y, p) = y \text{ and } \hat{a}(y, p) = p. \tag{49} \]
We say that $a_\varsigma \alpha \in \Gamma \Lambda M$ is transverse to $b$ if
\[ \dim Q = \dim N + \dim P - \dim M \tag{50} \]
Given $a_\varsigma \alpha \in \Gamma \Lambda M$ transverse to $b$ then we can define the pullback of $a_\varsigma \alpha$ by $b$
\[ b^*(a_\varsigma \alpha) = \hat{a}_\varsigma (\hat{b}^* \alpha) \in \Gamma \Lambda P \tag{51} \]

**Lemma 16** The pullback preserves the degree.

**Proof:** From (32) we have
\[ \deg(b^*(a_\varsigma \alpha)) = \deg(\hat{a}_\varsigma (\hat{b}^* \alpha)) = \deg(\hat{b}^* \alpha) + \dim P - \dim Q = \deg(\alpha) + \dim M - \dim N = \deg(a_\varsigma \alpha) \]

**Lemma 17** For regular distributions we observe
\[ a^\varsigma (D\alpha) = D(a^* \alpha) \tag{52} \]

**Proof:** For regular distributions (49) becomes
\[ \begin{array}{ccc}
P & \xrightarrow{b} & M \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
P & \xrightarrow{b} & M
\end{array} \]
Hence for $\alpha \in \Gamma \Lambda M$, $a^\varsigma (D\alpha) = \text{Id}_\varsigma (a^* \alpha) = D(a^* \alpha)$. \hfill \blacksquare

**Lemma 18** Composition of pullbacks: Let $b: P \to M$ and $c: Q \to P$, where $b^*$ and $c^*$ are defined, then
\[ (b \circ c)^\varsigma = c^\varsigma \circ b^\varsigma \tag{53} \]

**Proof:** Let $a_\varsigma \alpha \in \Gamma \Lambda M$ with $a: N \to M$ and $\alpha \in \Gamma \Lambda N$. Define $\hat{P}, \hat{Q}, \hat{a}, \hat{b}, \hat{c}, \hat{a}$ using (48, 49) so that the following commutes
\[ \begin{array}{ccc}
\hat{Q} & \xrightarrow{\hat{c}} & \hat{P} & \xrightarrow{\hat{b}} & N \\
\downarrow \hat{a} & & \downarrow \hat{a} & & \downarrow a \\
Q & \xrightarrow{c} & P & \xrightarrow{b} & M
\end{array} \]
then \( \dim Q - \dim P = \dim N - \dim M \), so that the pullbacks are defined.

\[
(b \circ c)^*(a_\xi \alpha) = \tilde a_\xi((\hat b \circ \hat c)^* \alpha) = \tilde a_\xi(\hat c^*(\hat b^* \alpha)) = c^*(a_\xi(\hat b^* \alpha)) = c^*(b^*(a_\xi \alpha))
\]

hence (53).}

It is often the case that both \( a: N \hookrightarrow M \) and \( b: P \hookrightarrow M \) are embeddings. Observe that in this case the intersection \( Q \simeq a(N) \cap b(P) \) since

\[
a(\hat b(Q)) = b(\hat a(Q)) = a(N) \cap b(P)
\]

and all these maps are embeddings.

### 2.4. Boundaries

**Lemma 20** Let \( N \) be a manifold with \( \dim N = n \) and have boundary \( B = \partial N \) with \( \iota: B \to N \) and let \( \alpha \in \Gamma \Lambda^p N \) (\( \alpha \) is smooth). Then

\[
d(D\alpha) = D(d\alpha) + (-1)^{n-p}\iota_\ast(\iota^*\alpha) \tag{54}
\]

**proof:** Let \( \phi \in \Gamma_0 \Lambda^{n-p-1}N \) then

\[
d(D\alpha)[\phi] = -D\alpha[d\phi^p] = -\int_N d\phi^p \wedge \alpha = -\int_N d(\phi^p \wedge \alpha) + \int_N \phi \wedge d\alpha
\]

\[
= -\int_B \iota^*(\phi^p \wedge \alpha) + D(d\alpha)[\phi] = D(d\alpha)[\phi] + (-1)^{n-p} \int_B \iota^*(\phi) \wedge \iota^*(\alpha)
\]

\[
= D(d\alpha)[\phi] + (-1)^{n-p}D(\iota^*\alpha)[\iota^*(\phi)]
\]

\[
= D(d\alpha)[\phi] + (-1)^{n-p}\iota_\ast(\iota^*\alpha)[\phi]
\]

We observe that in general the right hand side of (54) is not a submanifold distribution, since the domain of \( D(d\alpha) \) is \( N \) whereas the domain of \( \iota_\ast(\iota^*\alpha) \) is \( B \) and \( \dim N \neq \dim B \).

Also in general \( \iota_\ast(\iota^*\alpha) \) is not a submanifold distribution. This is because in general \( \iota: B \to N \) is not an embedding. For example, if \( N \) is a 3-dimensional solid bounded cylinder then \( B \) consists of two discs and a cylinder \( S^1 \times I \) where \( I \) is a closed interval, and \( \iota : B \to N \) is not injective.

Although this doesn’t matter for smooth forms since the set where \( \iota \) is not injective has measure zero, one has to be careful when dealing with distributional forms.

**Lemma 20** Let \( \alpha \in \Gamma \Lambda N \), so that \( \alpha \) is smooth then

\[
d(D\alpha) = 0 \implies d\alpha = 0 \tag{55}
\]

**proof:** Let \( B = \partial N \) with \( \iota: B \to N \), deg(\( \alpha \)) = \( p \) and \( \dim N = n \). Given \( \phi \in \Gamma_0 \Lambda M \) with \( \mathcal{S}(\phi) \subset N \setminus B \) then

\[
0 = d(D\alpha)[\phi] = D(d\alpha)[\phi] + (-1)^{n-p}\iota_\ast(\iota^*\alpha)[\phi] = D(d\alpha)[\phi] + (-1)^{n-p}D(\iota^*\alpha)[\iota^*\phi]
\]

\[
= D(d\alpha)[\phi]
\]
since \( \tau^\phi = 0 \). Thus \( \mathcal{S}(D(d\alpha)) \subset B \). From lemma 1, \( \mathcal{S}(d\alpha) \subset B \). So \( d\alpha = 0 \) on \( N \setminus B \), but since \( d\alpha \) is continuous and \( B \) contains no open sets, \( d\alpha = 0 \).

If \( a: N \to M \) and neither \( M \) nor \( N \) has a boundary then we have the trivial result following from (28) lemma 5:

\[
d(a_\zeta \alpha) = a_\zeta (d\alpha)
\]

### 3. Distributional solutions to transport equations

Given a manifold \( M \), with \( \dim M = n \) and a nowhere zero vector field \( v \in \Gamma TM \) then we say that the distribution \( \Psi \in \Gamma_D \Lambda^{n-1}M \) is a solution to the transport equations if

\[
d\Psi = 0 \quad \text{and} \quad i_v \Psi = 0
\]

We use the name transport equations to refer to any vector field \( v \in \Gamma TM \). The transport equations with respect to the vector field \( W \in \Gamma T\mathcal{E} \) where \( \pi: \mathcal{E} \to \mathcal{M} \), \( \mathcal{E} \subset TM \) is a bundle (usually the upper unit hyperboloid over spacetime) and \( W \) is horizontal, that is \( \pi_\star(W|_u) = u \) for \( u \in \mathcal{E} \), are also known as the Liouville equation, the collisionless Boltzmann equation and the Vlasov equation (15-16).

In general we will only be interested in solutions to the transport equations which are submanifold distributions and which, if they possess a boundary then the boundary is an embedding. See figure 1. These are given by

**Theorem 21** Let \( N \) be a manifold which is either without boundary or if it possess a boundary \( B \) then let \( i: B \hookrightarrow N \) be a closed embedding and let \( M \) be without boundary. Let \( a: N \hookrightarrow M \) and \( \alpha \in \Gamma \Lambda^{\dim N-1}M \). Then the submanifold distribution \( \Psi = a_\zeta \alpha \in \Gamma_S \Lambda^M \) satisfies the transport equations (57) if and only if the following three conditions hold

- \( d\alpha = 0 \)
- There exists \( u \in \Gamma TN \) such that \( a_\star(u) = v|_{a(N)} \) and \( i_u \alpha = 0 \).
- If \( N \) has a boundary then \( u \) is tangential to \( i \).

**proof:** First show that the three conditions (58-60) imply that \( \Psi \) is a solution to the transport equations (57). Let \( w \in \Gamma TB \) satisfy \( \iota_\star w = u|_{\iota(B)} \) then

\[
i_w \tau^\star \alpha = \tau^\star(i_w \alpha) = \tau^\star(i_u \alpha) = \tau^\star 0 = 0
\]

Now since \( \deg(\tau^\star \alpha) = \dim N - 1 = \dim(B) \) and \( w \neq 0 \) then \( \tau^\star \alpha = 0 \).

From lemmas 5 and 12, \( d\Psi = da_\zeta \alpha = a_\zeta(d(D\alpha)) = a_\zeta(d\alpha) - a_\zeta(t_\zeta(\tau^\star \alpha)) = 0 \)

and from lemma 7,

\[
i_v \Psi = i_v a_\zeta \alpha = a_\zeta(i_v(D\alpha)) = a_\zeta(D(i_v \alpha)) = 0
\]

In order to show that the transport equations (57) imply (58-60): From (57) we have \( 0 = i_v \Psi = i_v a_\zeta \alpha \). Hence from lemma 13, \( v \) is tangential to \( a \) and we have (59).

Let \( \iota: B \hookrightarrow N \) be the boundary of \( N \). Then \( 0 = d(a_\zeta \alpha) = a_\zeta(d(D\alpha)) \). From lemma 12, \( 0 = d(D\alpha) \) hence from lemma 20, \( d\alpha = 0 \) and hence \( a_\zeta(t_\zeta(\tau^\star \alpha)) = 0 \). Since \( a_\zeta \iota: B \hookrightarrow M \)
is a closed embedding and $0 = a_{\varsigma}(\iota^*\alpha) = (a \circ \iota)_\varsigma(\iota^*\alpha)$, it follows from lemma 12 that $\iota^*\alpha = 0$.

Now $\alpha$ satisfies the transport equations with respect to $u$, i.e. $d\alpha = 0$ and $i_\alpha = 0$. If $u$ is transverse to $\iota$ at any point in $B$ then $u$ is transverse to $\iota$ in an open subset $V \subset B$. Furthermore since $\iota^*\alpha = 0$, $i_\alpha = 0$ and $u$ is transverse to $\iota$ then $\alpha|_y = 0$ for all $y \in V$. Using the transport equations for $\alpha$ this implies there exists an open subset $U \subset N$ such that $\alpha|_U = 0$. This contradicts the statement that $S(\alpha) = N$. Hence $u$ is tangential to $\iota$.

An initial hypersurface associated with $v$ is a submanifold $\sigma: \Sigma \hookrightarrow M$ such that each integral curve of $v$ intersects $\Sigma$ exactly once, and $v$ contains no closed curves. For the following we further demand that each integral curve of $v$ has domain $\mathbb{R}$. If $\tau$ is the parameter along an integral curve $\gamma_x: \mathbb{R} \rightarrow M$, with $\gamma_x(0) = x \in \Sigma$, then $v|_{\gamma_x(\tau)} = \gamma_x(\tau)$. These two conditions imply that there is a diffeomorphism $c: \mathbb{R} \times \Sigma \rightarrow M$ with $c(\tau, x) = \gamma_x(\tau)$.

For $M = E$ and $v = W$, the Liouville vector field $\mathbf{2}$, then since $W$ is horizontal the requirement that the domain of $\gamma_x$ is $\mathbb{R}$ is equivalent to demanding that the proper time for each curve $C_x$ where $C_x = \gamma_x$ is from $-\infty$ to $+\infty$.

A box $s: \Sigma \hookrightarrow M$ is any compact submanifold (usually with boundary) of an initial hypersurface and of the same dimension as the hypersurface. That is $v$ is transverse to $s$ and $\dim S = \dim M - 1$. Given a box $s: \Sigma \hookrightarrow M$ we can define the expectation of $\Psi$ as

$$[s^\varsigma(\Psi)] = (s^\varsigma\Psi)[1]$$

which is valid since $1 \in \Gamma\Lambda S$ has compact support on $S$.

The future of a box or an initial hypersurface is the set of points which lie on the integral curves of $v$ in the direction of $v$ away from $S$ or $\Sigma$.

The total charge on an initial hypersurface $\sigma: \Sigma \hookrightarrow M$ associated with $\Psi$ is given by

$$Q_{\text{total}} = [\sigma^\varsigma(\Psi)]$$

However in order to define charge we require that given a series of boxes $S_i \subset S_{i+1}$ with $\bigcup S_i = \Sigma$ then $\lim_{i \rightarrow \infty}[s_i^\varsigma(\Psi)]$ is well defined and independent of the choice of $\{S_1, S_2, \ldots\}$.

**Theorem 22** Given a solution $a_{\varsigma}\alpha \in \Gamma S\Lambda M$ to the transport equations where $a: N \hookrightarrow M$ is an embedding. Let $s_1: S_1 \hookrightarrow M$ and $s_2: S_2 \hookrightarrow M$ be two boxes with the property that any integral curve of $v$ passing through one also passes through the other. Then

$$[s_1^\varsigma(a_{\varsigma}\alpha)] = [s_2^\varsigma(a_{\varsigma}\alpha)]$$

**proof:** Assume first that $S_2$ lies to the future of $S_1$. Let $\iota: U \hookrightarrow M$ be the closed submanifold with $\dim U = \dim M$, given by the union of all the integral curves of $v$
between $S_1$ and $S_2$. The pullback manifold with respect to $i$ and $a$ is given by $a^{-1}(U)$ and (49) becomes (defining the embeddings $i, \hat{a}, \hat{a}_i$ and $\hat{s}_i$.)

\[
\begin{array}{c}
\text{S}_1 \xrightarrow{\hat{s}_i} \text{U} \xrightarrow{\hat{a}_i} \text{N} \xrightarrow{a} \text{M} \\
\text{S}_2 \xrightarrow{\hat{s}_i} \text{U} \xrightarrow{\hat{a}_i} \text{N} \xrightarrow{a} \text{M}
\end{array}
\]

By definition $s_i^\varsigma(a_\varsigma \alpha) = (\hat{a}_i)_\varsigma(\hat{s}_i^* \alpha)$ so

\[
[s_i^\varsigma(a_\varsigma \alpha)] = s_i^\varsigma(a_\varsigma \alpha)[1] = (\hat{a}_i)_\varsigma(\hat{s}_i^* \alpha)[1] = D(\hat{s}_i^* \alpha)[1] = \int_{a^{-1}(S_i)} \hat{s}_i^* \alpha
\]

Now $\alpha \in \Gamma \Lambda N$ so $i^* \alpha \in \Gamma \Lambda (a^{-1}U)$ is a regular form. Therefore we can perform the standard analysis. From the transport equations then (58) implies $d\alpha = 0$ and (59) implies $i_u \alpha = 0$ where $u \in \Gamma \tau N$ is the unique vector field satisfying $a_* (u) = v|_{a(N)}$.

The boundary of $U$ is given by

\[
\partial(U) = S_2 - S_1 + V
\]

where the $-$ sign refers to the orientation of $S_1$ and $u$ is tangential to $V$. Since $a$ is an embedding the boundary of $a^{-1}(U)$ is given by

\[
\partial(a^{-1}(U)) = a^{-1}(S_2) - a^{-1}(S_1) + a^{-1}(V)
\]

where $u$ is tangential to $a^{-1}(V)$. Therefore $\int_{a^{-1}(V)} \alpha = 0$. Thus

\[
0 = \int_{a^{-1}(U)} d\alpha = \int_{a^{-1}(S_2)} \alpha - \int_{a^{-1}(S_1)} \alpha + \int_{a^{-1}(V)} \alpha = [s_1^\varsigma(a_\varsigma \alpha)] - [s_2^\varsigma(a_\varsigma \alpha)]
\]

For two general boxes $S_1$ and $S_2$ simply take another box $S_3$ lying to the future of both $\Sigma_1$ and $\Sigma_2$.

**Corollary 23** Conservation of charge. Given two initial hypersurfaces $\sigma: \Sigma \hookrightarrow M$ and $\hat{\sigma}: \hat{\Sigma} \hookrightarrow M$ and given that the charge (62) is well defined with respect to $\Sigma$ then the charge is well defined with respect to $\hat{\Sigma}$ and

\[
[\hat{\sigma}^\varsigma(\Psi)] = [\sigma^\varsigma(\Psi)]
\]

**proof:** Any box $s: S \hookrightarrow \Sigma$ corresponds to a box $\hat{s}: \hat{S} \hookrightarrow \hat{\Sigma}$. Therefore from theorem 22 $[s^\varsigma(\Psi)] = [\hat{s}^\varsigma(\Psi)]$. Taking the limit of boxes shows that $[\sigma^\varsigma(\Psi)]$ is well defined and that (64) holds.

Given a submanifold distribution on an initial hypersurface we can generate a unique solution to the transport equations.
Theorem 24 Let $\sigma: \Sigma \hookrightarrow M$ be an initial hypersurface of $v \in \Gamma TM$ and let $c: \mathbb{R} \times \Sigma \to M$ be the corresponding diffeomorphism. Let $b_\zeta \alpha \in \Gamma S\Lambda \Sigma$ be a submanifold distribution, with $b: N \hookrightarrow \Sigma$ and $\alpha \in \Gamma^{\dim N}M$. Let

$$\Psi = a_\varsigma(\pi_2^*\alpha) \in \Gamma S\Lambda^{\dim M-1}M$$

(65)

where

$$a: \mathbb{R} \times N \to M; \quad a(\tau, y) = c(\tau, b(y))$$

(66)

and $\pi_2: \mathbb{R} \times N \to N$ is the natural projection. Then $\Psi$ satisfies the transport equations (57) and the initial conditions (41-44). The following commutes

$$\begin{array}{ccc}
\mathbb{R} \times N & \xrightarrow{\text{Id} \times b} & N \\
\sigma & \xrightarrow{a} & M
\end{array}$$

Thus $a$ is an embedding. Since $\mathbb{R} \times b(N) \subset \mathbb{R} \times \Sigma$ is a closed submanifold. Then $a(N \times \mathbb{R}) \subset M$ is closed. Clearly $S(\pi_2^*\alpha) = \mathbb{R} \times N$. Also since $\text{Id} \times b: \mathbb{R} \times N \to \mathbb{R} \times b(N)$ is a diffeomorphism so is $a$.

To establish that $d\Psi = 0$ it is necessary to realise that $\mathbb{R} \times N$ may have a boundary. Let $\iota: B \hookrightarrow N$ be the boundary of $N$. Then the boundary of $\mathbb{R} \times N$ is given by $\hat{i}: \mathbb{R} \times B \to \mathbb{R} \times N$ where $\hat{i}(\tau, y) = (\tau, \iota(y))$.

$$d\Psi = da_\varsigma(\pi_2^*\alpha) = da_\varsigma D(\pi_2^*\alpha) = a_\varsigma dD(\pi_2^*\alpha) = a_\varsigma (d\pi_2^*\alpha) - a_\varsigma \iota^*_\varsigma(i^*\pi_2^*\alpha)$$

Now $d\pi_2^*\alpha = \pi_2^*d\alpha = 0$ since $\deg(\alpha) = \dim N$.

Let $\pi_2: \mathbb{R} \times B \to B$ be the second projection so that $\pi_2 \circ \hat{i} = \iota \circ \pi_2$. Then

$$\iota^*\pi_2^*\alpha = (\pi_2 \circ \hat{i})^*\alpha = (\iota \circ \pi_2)^*\alpha = \pi_2^* (\iota^*\alpha) = 0$$

since $\deg \alpha = \dim N > \dim B$. Thus $d\Psi = 0$.

We now establish that $i_v\Psi = 0$.

$$i_v a_\varsigma(\pi_2^*\alpha) = i_v a_\varsigma D(\pi_2^*\alpha) = a_\varsigma i_v D(\pi_2^*\alpha) = a_\varsigma (i_v \pi_2^*\alpha) = a_\varsigma (\pi_2^* i_v \pi_2, \partial_\tau, \alpha) = 0$$

since $a_\varsigma(\partial_\tau) = v$ and $\pi_2^* \partial_\tau = 0$.

We now establish that $\Psi$ satisfies the initial conditions. Let $I_0: N \hookrightarrow \mathbb{R} \times N$, $I_0(y) = (0, y)$ then the following commutes and is the pullback manifold

$$\begin{array}{ccc}
\mathbb{R} \times N & \xrightarrow{a} & M \\
\Sigma & \xrightarrow{\sigma} & M
\end{array}$$
and \( I_0^*(\pi_2^*(\alpha)) = \alpha \) so \( \sigma^*(\Psi) = b_*\alpha \).

We now establish that \( \Psi \) is unique. We first show that \( \mathcal{S}(\Psi) = \mathcal{S}(\Phi) \). Let \( s: S \hookrightarrow M \) be a box so that \( x \in S \) and \( S \cap \Sigma = \emptyset \). Now there lies a box \( \hat{s}: \hat{S} \hookrightarrow \Sigma \) which lies to the future or past of \( S \) hence from lemmas 22 and 18 we have

\[
[s^*(\Phi)] = [(\sigma \circ \hat{s})^*(\Phi)] = [\hat{s}^*(\sigma^*\Phi)] = [(\sigma \circ \hat{s})^*(\Psi)] = [s^*(\Psi)]
\]

Thus given \( x \in \mathcal{S}(\Phi) \), then for any box \( s: S \hookrightarrow M \) with \( x \in s(S) \) then \( [s^*(\Psi)] = [s^*(\Phi)] \neq 0 \). Thus \( x \in \mathcal{S}(\Psi) \) and visa versa.

Since \( \mathcal{S}(\Psi) = \mathcal{S}(\Phi) \) we may consider up to diffeomorphism that \( \Phi = a_*\beta \) for some \( \beta \in \Gamma\Lambda(\mathbb{R} \times N) \). From theorem 21 both \( \pi_2^*\alpha \) and \( \beta \) satisfy the transport equations with respect to \( \partial_\tau \) on \( \mathbb{R} \times N \). Furthermore \( b_* (I_0^* \pi_2^*\alpha) = \sigma^*\Psi = \sigma^*\Phi = b_* (I_0^* \beta) \) hence \( I_0^* \beta = I_0^* \pi_2^*\alpha \), i.e. they agree on the initial hypersurface \( I_0: N \hookrightarrow (\mathbb{R} \times N) \). Hence \( \beta = \pi_2^*\alpha \) and so \( \Phi = \Psi \).

There are two alternative formulations of \( \Psi \), one simply using the pullback, a second using an integral (65) which can easily be generalised for general distributions.

**Corollary 25** Let \( \sigma: \Sigma \hookrightarrow M \), \( v \in \Gamma TM \), \( c: \mathbb{R} \times \Sigma \rightarrow M \), \( b_*\alpha \in \Gamma_S\Lambda_\Sigma \), \( b: N \hookrightarrow \Sigma \), \( \alpha \in \Gamma_L^{\dim N} \) and \( \Psi \in \Gamma_S\Lambda^{\dim M-1} M \) be as in theorem 24. Then

\[
\Psi = (\hat{\pi}_2 \circ (c^{-1})^*)(b_*\alpha)
\]

where \( \hat{\pi}_2: \mathbb{R} \times \Sigma \rightarrow \Sigma \) is the second projection.

**proof:** Since \( c: \mathbb{R} \times \Sigma \rightarrow M \) is a diffeomorphism then \( (\hat{\pi}_2 \circ (c^{-1})^*): M \rightarrow \Sigma \). Thus to define the pullback \( (\hat{\pi}_2 \circ c^{-1})^* \) we have the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{R} \times N & \xrightarrow{\pi_2} & N \\
\downarrow a & & \downarrow b \\
M & \xrightarrow{\hat{\pi}_2 \circ c^{-1}} & \Sigma \\
\end{array}
\]

Clearly \( \dim(\mathbb{R} \times N) + \dim \Sigma = (\dim N + 1) + (\dim M - 1) = \dim N + \dim M \) so \( \hat{\pi}_2 \circ c^{-1} \) is transverse to \( b \), and (67) follows from (65) and (51).

**Corollary 26** Let \( \sigma: \Sigma \hookrightarrow M \), \( v \in \Gamma TM \), \( c: \mathbb{R} \times \Sigma \rightarrow M \), \( b_*\alpha \in \Gamma_S\Lambda_\Sigma \), \( b: N \hookrightarrow \Sigma \), \( \alpha \in \Gamma_L^{\dim N} \) and \( \Psi \in \Gamma_S\Lambda^{\dim M-1} M \) be as in theorem 24. Then

\[
\Psi[\phi] = \int_{\tau \in \mathbb{R}} \Phi[\zeta^*_\tau \iota_v \phi] d\tau 
\]

for all \( \phi \in \Gamma_0 L^1 M \) where \( \Phi = b_*\alpha \) and \( \zeta_\tau: \Sigma \rightarrow M \), \( \zeta_\tau(y) = c(\tau, y) \).

**proof:** The pushforward of the vector \( \partial_\tau|_{(\tau,y)} \in T_{(\tau, y)}(\mathbb{R} \times N) \) under \( \mathbb{R} \times N \xrightarrow{\text{Id} \times b} \mathbb{R} \times \Sigma \xrightarrow{c} M \) is given by \( (\text{Id} \times b)_*(\partial_\tau|_{(\tau, y)}) = \partial_\tau|_{(\tau, b(y))} \) and \( c_*\partial_\tau|_{(\tau, b(y))} = v|_{c(\tau, b(y))} \).

Let \( I_\tau: N \rightarrow \mathbb{R} \times N \), \( I_\tau(y) = (\tau, y) \) so that \( (c \circ (\text{Id} \times b) \circ I_\tau)(y) = c((\text{Id} \times b)(\tau, y)) = c(\tau, b(y)) = \zeta_\tau(b(y)) \). Thus

\[
\Psi[\phi] = a_*(\pi_2^*\alpha)[\phi] = D(\pi_2^*\alpha)[a^*\phi] = \int_{\mathbb{R} \times N} a^*\phi \wedge \pi_2^*\alpha = \int_{\mathbb{R} \times N} (\text{Id} \times b)^*(c^*\phi) \wedge \pi_2^*\alpha
\]
\[
\begin{align*}
&= \int_{\tau \in \mathbb{R}} d\tau \int_{N} I_\tau^* (\text{Id} \times b)^* (c^\ast \phi) \wedge \pi_2^* \alpha = \int_{\tau \in \mathbb{R}} d\tau \int_{N} I_\tau^* (\text{Id} \times b)^* (c^\ast \phi) \wedge \pi_2^* \alpha \\
&= \int_{\tau \in \mathbb{R}} d\tau \int_{N} I_\tau^* (\text{Id} \times b)^* (c^\ast i_{\partial \tau} (c^\ast \phi) \wedge \pi_2^* \alpha = \int_{\tau \in \mathbb{R}} d\tau \int_{N} (c \circ (\text{Id} \times b) \circ I_\tau)^* (i_{\partial \tau} (c^\ast \phi) \wedge \alpha \\
&= \int_{\tau \in \mathbb{R}} d\tau \int_{N} b^\ast (\zeta_\tau^* (i_{\partial \tau} \phi)) \wedge \alpha = \int_{\tau \in \mathbb{R}} d\tau b_\zeta (\alpha) [\zeta_\tau^* (i_{\partial \tau} \phi)] = \int_{\tau \in \mathbb{R}} d\tau \Phi [\zeta_\tau^* (i_{\partial \tau} \phi)]
\end{align*}
\]

3.1. Distributional solutions to the Liouville equations as a source for Maxwell’s equations.

The following shows that the source for Maxwell’s equations is closed.

**Theorem 27** The source for Maxwell’s equations (5) is closed i.e. \(d\mathcal{J} = 0\).

**proof:** If \(\Theta\) is bounded with respect to \(\pi\) then lemma 6 implies \(d(\pi_\zeta \Theta) = \pi_\zeta (d\Theta)\). If \(\Theta\) is regular and unbounded but \(\pi_\zeta \Theta\) is defined then in 34 it is also shown that \(d \circ \pi_\zeta = \pi_\zeta \circ d\). Thus

\[
d\mathcal{J} = d(\pi_\zeta \Theta) = \pi_\zeta (d\Theta) = 0
\]

We have established that a submanifold distributional solution to the transport equations can be constructed from a distribution on an initial hypersurface. From now on we consider distributional solutions for the transport equations for distributions \(\Psi \in \Gamma S \Lambda E\) where \(\pi : E \rightarrow M\) is a fibre bundle. If this solution is to be the source for an electromagnetic field then we need to be able to take its pushforward with respect to the projection map \(\pi\). However this projection map is not proper. Therefore we must establish under what condition we can guarantee that \(\Psi\) is bounded with respect to \(\pi\).

**Lemma 28** Let \(a : N \hookrightarrow E\) and \(a_\zeta (\alpha) \in \Gamma S \Lambda E\) be bounded with respect to \(\pi : E \rightarrow M\) then \(D(\alpha)\) is bounded with respect to \((\pi \circ a) : N \rightarrow M\) and

\[
\pi_\zeta (a_\zeta \alpha) = (\pi \circ a)_\zeta \alpha \tag{69}
\]

**proof:** Let \(U \subset M\) be compact. Since \(a_\zeta (\alpha)\) is bounded with respect to \(\pi\), it follows that \(\pi^{-1} U \cap S(a_\zeta \alpha)\) is compact. As \(a\) is proper \(a^{-1} (\pi^{-1} U \cap a(N)) \subset N\) is compact. Now

\[
a^{-1} (\pi^{-1} U \cap a(N)) = (a^{-1} \pi^{-1} U) \cap (a^{-1} a(N)) = (\pi \circ a)^{-1} U \cap N = (\pi \circ a)^{-1} U \cap S(\pi D\alpha)
\]

Thus \(D(\alpha)\) is bounded with respect to \((\pi \circ a)\). Given \(\phi \in \Gamma_0 \Lambda M\) and \(h \in \Gamma_0 \Lambda E\) satisfies 34 i.e. \(S(1 - h) \cap S(a^\ast \phi) \cap S(a_\zeta \alpha) = \emptyset\) then we wish to show that \(a^\ast h \in \Gamma_0 \Lambda N\) satisfies 34 for \(\Psi = D\alpha\) and \(\phi\) replaced by \(\pi^\ast \phi\):

\[
S(1 - a^\ast h) \cap S(a^\ast \pi^\ast \phi) \cap S(D\alpha)
\]

\[
= S(1 - a^\ast h) \cap S(a^\ast \pi^\ast \phi) \cap N = S(a^\ast (1 - h)) \cap S(a^\ast \pi^\ast \phi)
\]

\[
\subset a^{-1} (S(1 - h)) \cap a^{-1} (S(\pi^\ast \phi)) = a^{-1} (S(1 - h) \cap S(\pi^\ast \phi))
\]

\[
= a^{-1} (S(1 - h) \cap S(\pi^\ast \phi) \cap a(N)) = a^{-1} (S(1 - h) \cap S(a^\ast \phi) \cap S(a_\zeta \alpha)) = \emptyset
\]
Figure 2. Distribution of charge for a Galilean spacetime for two different values of $x^0$.

Hence we have

$$\pi_\varsigma (a_\varsigma \alpha)[\phi] = a_\varsigma \alpha [h \pi^* \phi] = D \alpha [a^*(h \pi^* \phi)] = D \alpha [a^*(h)(a^* \pi^* \phi)]$$

$$= D \alpha [a^*(h)(\pi \circ a)^*(\phi)] = (\pi \circ a)_\varsigma \alpha [\phi]$$

giving (69). \hfill \blacksquare

Recall that a vector field $W \in \Gamma E$ with $\pi: E \to M$ and $E \subset TM$ is **horizontal** if $\pi_*(W|_u) = u$ for all $u \in E$. The following lemma relates the integral curves of horizontal vector fields with curves on the base space $M$.

**Lemma 29** Let $\pi: E \to M$ with $E \subset TM$ and $W \in \Gamma E$ be a horizontal vector field and let $\gamma: \mathbb{R} \to E$ be an integral curve of $W$ then $\gamma = \dot{C}$ where $C = \pi \circ \gamma: \mathbb{R} \to M$.

**proof:** Let $\tau \in \mathbb{R}$. Recall that $\partial_\tau \in T_\tau \mathbb{R}$ is the natural vector field and $\partial_\tau|_\tau \in \Gamma T_\tau \mathbb{R}$ is a vector at the point $\tau \in \mathbb{R}$. The lifts of the curves $\dot{\gamma} = \gamma_*(\partial_\gamma): \mathbb{R} \to TE$ and $\dot{C} = C_*(\partial_\tau): \mathbb{R} \to TM$.

$$\dot{C}(\tau) = C_*(\partial_\tau|_\tau) = (\pi \circ \gamma)_*(\partial_\gamma|_\tau) = \pi_*(\gamma_*(\partial_\gamma|_\tau)) = \pi_*(\dot{\gamma}(\tau)) = \pi_*(W_{\gamma(\tau)}) = \gamma(\tau)$$

We can ask the question:

If given an initial hypersurface $\sigma: \Sigma \hookrightarrow E$, with respect to a horizontal vector field, such that $\sigma^* \Psi$ is bounded with respect to $\pi \circ \sigma$ then is $\Psi$ bounded with respect to $\pi$? \hfill (70)

It turns out that even for simple cases this is not the case. For example consider the Vlasov operator corresponding to a simple force free drift in one dimensional Galilean spacetime $M = \mathbb{R}^2$ coordinates $(t,x)$ with $E = \mathbb{R}^3$ coordinated by $(t,x,\dot{x})$ and $\pi(t,x,\dot{x}) = (t,x)$.

$$W = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x}$$

A solution to the transport equations for this vector field is given by $\Psi = a_\varsigma \alpha \in \Gamma_S \Lambda^2 E$ where $a: \mathbb{R}^2 \to E$, $a(\tau,y) = (\tau, \tau y, y)$ and $\alpha = \hat{\alpha}(y)dy \in \Gamma \Lambda^1 \mathbb{R}^2$.  

\[\text{Figure 2. Distribution of charge for a Galilean spacetime for two different values of } x^0.\]
Now to see that $\Psi$ is a solution to the transport equations we use theorem 21. Clearly $da = 0$ and
\[ a_\ast \left( \frac{\partial}{\partial \tau} \right)_{(\tau, y)} = \frac{\partial}{\partial t} + y \frac{\partial}{\partial x} = W |_{a(\tau, y)} \]
Hence it is a solution.

With initial hypersurface $\sigma: \Sigma \to E$, $\sigma(x_0, \dot{x}_0) = (t_0, x_0, \dot{x}_0)$ an initial distribution is $\sigma^\ast \Psi = b \hat{\alpha}$ where $b: \mathbb{R} \to \Sigma$, $b(y) = (t_0 y, y)$, so that $\sigma(b(y)) = (t_0, t_0 y, y)$. Given compact $U \subset \mathbb{R}$ we have
\[ \mathcal{S}(\sigma^\ast \Psi) \cap \pi^{-1} U = \mathcal{S}(b \hat{\alpha}) \cap \pi^{-1} U = \{(t_0, t_0 y, y) | y \in \mathbb{R}\} \cap \{(t, x, \dot{x}) | (t, x) \in U\} \]
\[ = \{(t_0, t_0 y, y) | (t_0, t_0 y) \in U\} \]
is compact as long as $t_0 \neq 0$. Thus $\sigma^\ast \Psi$ is bounded with respect to $\pi \circ \sigma$, see figure 2. However consider $(0, 0) \in \mathbb{R}$ then
\[ \pi^{-1}\{(0, 0)\} = \{(0, 0, \dot{x}) | \dot{x} \in \mathbb{R}\} = \{(0, 0, y) | y \in \mathbb{R}\} = \{a(0, y) | y \in \mathbb{R}\} \subset \mathcal{S}(\Psi) \]
Thus $\Psi$ is not bounded with respect to $\pi$.

It turns out that for the relativistic Maxwell-Vlasov equations, as long as the initial hypersurface $\Sigma \subset E$ is such that $\pi(\Sigma) \subset \mathbb{R}$ is a Cauchy surface, then we can guarantee that if $\sigma^\ast \Psi$ is bounded with respect to $\pi \circ \sigma$ then $\Psi$ is bounded with respect to $\pi$.

**Theorem 30** Let $\pi: \mathcal{E} \to \mathcal{M}$ be the upper unit hyperboloid bundle over a globally hyperbolic spacetime. Let $W \in \Gamma TE$ be a horizontal vector field, i.e. such that $\pi_\ast(W|_u) = u$. Let $\sigma: \Sigma \hookrightarrow \mathcal{E}$ be an initial hypersurface such that $\pi(\sigma(\Sigma)) \subset \mathcal{M}$ is a Cauchy surface. Let $\Psi \in \Gamma_S \Lambda \mathcal{E}$ be a solution to the transport equations with respect to $W$. Let $\sigma^\ast(\Psi)$ be bounded with respect to $\pi \circ \sigma$ then $\Psi$ is bounded with respect to $\pi$.

*proof:* Let $\hat{\Sigma} = \pi(\sigma(\Sigma))$, $\hat{\sigma}: \hat{\Sigma} \hookrightarrow \mathcal{M}$ and $\hat{\pi} = \pi \circ \sigma : \Sigma \to \hat{\Sigma}$. Given a compact subset $Y \subset \mathcal{M}$, since $\hat{\Sigma}$ is a Cauchy surface, it is shown in [7] that for each $y \in Y$ the light cone of $y$ intersecting with $\hat{\Sigma}$ is compact. Define $X \subset \Sigma$ as
\[ X = \left\{ x \in \hat{\Sigma} \mid \text{there exists a timelike or lightlike curve $C$ passing though $x$ and $Y$} \right\} \]
then $X$ is compact. Furthermore since $X$ and $Y$ are compact the proper time it takes to go from $X$ to $Y$ has a maximum and minimum.
\[ T = \left\{ \tau \in \mathbb{R} \mid \text{there exists a timelike normalised curve $C: \mathbb{R} \hookrightarrow \mathcal{M}$ such that $C(0) \in \hat{\sigma}(X)$ and $C(\tau) \in Y$} \right\} \]
Then $T \subset \mathbb{R}$ is compact. We have the following commutative diagram
\[
\begin{array}{ccc}
N \xrightarrow{\iota} \mathbb{R} \times \mathbb{R} & \xrightarrow{\pi} & N \\
\downarrow{b} & & \downarrow{a} \\
\Sigma \xrightarrow{\sigma} \mathcal{E} & \xrightarrow{\hat{\pi}} & \hat{\Sigma} \\
\downarrow{\hat{\sigma}} & & \downarrow{\pi} \\
\Sigma \xrightarrow{\sigma} \mathcal{M} & & \end{array}
\]
Since $\sigma^*\Psi = b_\mathcal{K}\alpha$ is bounded with respect to $\hat{\pi}$, and $X \subset \hat{\Sigma}$ is compact then $\hat{\pi}^{-1}(X) \cap S(b_\mathcal{K}\alpha) \subset \Sigma$ is compact. Now $\hat{\pi}^{-1}(X) \cap S(b_\mathcal{K}\alpha) = \hat{\pi}^{-1}(X) \cap b(N)$ and since $b: N \to \Sigma$ is proper

$$b^{-1}(\hat{\pi}^{-1}(X) \cap b(N)) = b^{-1}\hat{\pi}^{-1}(X) \cap N = b^{-1}\hat{\pi}^{-1}(X) \subset N$$  is compact

We need to show that $\pi^{-1}(Y) \cap S(\Psi)$ is compact. Now from set theory and the fact that $a$ is injective and $S(\Psi) = a(N)$ then $\pi^{-1}(Y) \cap S(\Psi) = a^{-1}\pi^{-1}(Y))$. Now since $a$ is continuous, it is sufficient to show that $a^{-1}\pi^{-1}(Y) \subset \mathbb{R} \times N$ is compact.

Let $(\tau_0, y) \in a^{-1}\pi^{-1}(Y)$, so that $\pi(a(\tau_0, y)) \in Y$. Let $\gamma: \mathbb{R} \to \mathcal{E}$ be given by $\gamma(\tau) = a(\tau, y)$, then from (66) $\gamma$ is an integral curve of $W$. From lemma 29 $\gamma = \hat{C}$ where $\hat{C} = \pi \circ \gamma$. Since $\hat{C}: \mathbb{R} \to \mathcal{E}$, $g(\hat{C}, \hat{C}) = -1$ so $C$ is a timelike normalised curve. Now

$$C(0) = \pi(\gamma(0)) = \pi(a(0, y)) = \pi(I_0(y)) = \pi(\sigma(b(y))) = \hat{\sigma}(\hat{\pi}(b(y)))$$

thus $\hat{\pi}(b(y)) \in \hat{\Sigma}$. Also $C(\tau_0) = \pi(\gamma(\tau_0)) = \pi(a(\tau_0, y)) \in Y$. From the definitions of $X$ and $T$, $C(0) \in \hat{\sigma}(X)$ and $\tau_0 \in T$. Since $C(0) = \hat{\sigma}(\hat{\pi}(b(y))) \in \sigma(X)$ then $\hat{\pi}(b(y)) \in X$ so $y \in b^{-1}\hat{\pi}^{-1}(X)$. Hence $(\tau_0, y) \in T \times b^{-1}\hat{\pi}^{-1}(X)$. Thus we have shown that

$$a^{-1}\pi^{-1}(Y) \subset T \times b^{-1}\hat{\pi}^{-1}(X)$$

Being the product of two compact sets $T \times b^{-1}\hat{\pi}^{-1}(X)$ is compact and since $a^{-1}\pi^{-1}(Y)$ is closed it is therefore compact. Hence result.

The condition that $\pi(\sigma(\Sigma)) = \hat{\Sigma} \subset \mathcal{M}$ is a Cauchy surface is required. Here we give an example where $\mathcal{M}$ is globally hyperbolic, $\sigma^*(\Psi)$ is bounded with respect to $\pi: \mathcal{E} \to \mathcal{M}$ but $\Psi$ is not bounded with respect to $\pi$.

Consider $\mathcal{M}$ is two dimensional Minkowski spacetime, with coordinates $(x^0, x^1)$ and $\mathcal{E}$ is the upper unit hyperboloid coordinates $(x^0, x^1, y)$. Let $W$ be the Liouville vector field for a zero electric force:

$$W = \sqrt{1 + y^2} \frac{\partial}{\partial x^0} + y \frac{\partial}{\partial x^1}$$
Let
\[ \sigma: \Sigma \hookrightarrow E, \quad \sigma(z,p) = (x^0 = z, x^1 = z, y = p) \]
so that \( \pi \circ \sigma: \Sigma \to M, \pi \circ \sigma(z,p) = (z,z) \) is a lightlike curve so \( \pi(\Sigma) \) must be transverse to \( W \). Also every integral curve of \( W \) must intersect \( \sigma(\Sigma) \) so \( \Sigma \) is an initial hypersurface of \( W \). Clearly \( \pi(\sigma(\Sigma)) \) is not a Cauchy surface. Let
\[ N = \mathbb{R} \times \{ \hat{z} \in \mathbb{R}, \hat{z} \leq 0 \} \]
and
\[ a: N \hookrightarrow E, \quad a(\tau, \hat{z}) = \left( x^0 = \frac{1 - \hat{z}}{\sqrt{1 - 2\hat{z}}}, x^1 = \frac{-\hat{z}}{\sqrt{1 - 2\hat{z}}}, y = \frac{-\hat{z}}{\sqrt{1 - 2\hat{z}}} \right) \]
so that
\[ a_*(\frac{\partial}{\partial \tau}) = \left( \frac{1 - \hat{z}}{\sqrt{1 - 2\hat{z}}}, \frac{-\hat{z}}{\sqrt{1 - 2\hat{z}}}, 0 \right) = W|_{a(\tau, \hat{z})} \]
Hence \( a \) is tangential to \( W \). Let \( \Psi = a_1 \). Now \( \sigma^i(\Psi) = b_1 \) where
\[ b: \{ \hat{z} \in \mathbb{R}, \hat{z} \leq 0 \} \to \Sigma, \quad b(\hat{z}) = (\hat{z}, \hat{z}, -\hat{z}(1 - 2\hat{z})^{-1/2}) \]
This map is proper and \( \sigma^i(\Psi) \) is bounded with respect to \( \pi \). However this example has been contrived so that every integral curve passes through \((x^0, x^1) = (1, 0) \in M \). See figure [3]
\[ a(\sqrt{1 - 2\hat{z}}, \hat{z}) = (1, 0, -\hat{z}(1 - 2\hat{z})^{-1/2}) \]
So
\[ \pi^{-1}(\{(1, 0)\}) \cap S(\Psi) = \{(1, 0, y)|y \geq 0\} \]
is not compact so \( \Psi \) is not bounded with respect to \( \pi \).

4. Known solutions to the Maxwell-Vlasov equations.

4.1. Worldline of a point charge and the Klimontovich distribution

For the worldline of a point charge we can show that the Liouville equation \((15,16)\) implies that the point charge undergoes the Lorentz force equation \((3)\) and that the electromagnetic source due to a point charge is given by \( J = C^D \in \Gamma_S \Lambda^3 \mathcal{M} \) where \( C^D = C_\xi(1) \). However due to the divergence of the electromagnetic field \( F \) due to point sources we cannot demand that there exist solutions to the complete Maxwell-Vlasov equation \((13-16)\). It is therefore necessary to perform a regularisation of \( F \) before substituting into \((2)\). For a single point charge the usual regularisation is the Dirac regularisation which leads to the Lorentz-Dirac equation.

Let \( C: \mathbb{R} \to \mathcal{M} \) be a regular curve with \( g(\dot{C}, \dot{C}) = -1 \) and let \( \dot{C}: \mathbb{R} \to \mathcal{E} \) be the corresponding lift. Let
\[ \Theta \in \Gamma_D \Lambda^6 \mathcal{E}; \quad \Theta = \dot{C}_\xi 1 = \dot{C}^D \quad \text{i.e.} \quad \Theta[\phi] = \int_{\mathbb{R}} \dot{C}^*(\phi) \quad \text{for all} \quad \phi \in \Gamma_0 \Lambda^1 \mathcal{E} \quad (72) \]
Clearly \( d\Theta = 0 \).

**Lemma 31** Equations \((13)\) and \((72)\) give the point source.
\[ J = C^D = C_\xi 1 \quad (73) \]
proof: $\dot{C}$ is bounded with respect to $\pi$ thus from lemma 28 we have
\[ J = \pi_{\dot{\varsigma}}(\Theta) = \pi_{\dot{\varsigma}}(C_{\dot{\varsigma}}1) = (\pi \circ \dot{C})_{\dot{\varsigma}}1 = C_{\dot{\varsigma}}1 = C^D \]

Lemma 32 Equations (2), (16) and (72) imply that the point charge undergoes the Lorentz force equation:
\[ \nabla_{\dot{C}} \dot{C} = i_{\dot{C}}F \tag{74} \]

proof: From (16) and (72) we have $i_W \dot{C}^D = 0$ hence from lemma 15 we have $\dot{C}$ is tangential to $W$ hence $\dot{C}$ is an integral curve of $W$ with a scaling. I.e. $W|_{\dot{C}(\tau)} = \kappa(\tau) \dot{C}(\tau)$. By hitting both sides with $\pi_*$ we see
\[ \dot{C}(\tau) = \pi_* W|_{\dot{C}(\tau)} = \pi_*(\kappa \dot{C}(\tau)) = \kappa \dot{C}(\tau) \]
hence $\kappa = 1$. Thus $W|_{\dot{C}(\tau)} = \dot{C}(\tau)$. So $\dot{C}$ is an integral curve of $W$ and thus (3) gives (74).

As stated in the introduction the Klimontovich distribution is simply a finite number of point worldlines. Thus we can set
\[ \Theta = \sum_{k=1}^{N} \dot{C}_k^D \]
Which gives the source for Maxwell’s equation as
\[ J = \sum_{k=1}^{N} C_k^D \]
Again, this does not give rise to a continuous electromagnetic field so some method of regularisation is necessary. For the Klimontovich distribution, for example, we can ignore the contribution of $C_j$ to $J$ when calculating the motion of the worldline $\dot{C}_j$. That is we solve $d(DF_j) = 0$ and $d(D(\star F_j)) = \sum_{k \neq j} C_k^D$. We then construct the Liouville vector field $W_j = W(F_j)$ using (2) and solve the transport equations for $\dot{C}_j$ using $W_j$.

4.2. Cold charged fluid model

Here we see that the cold charged fluid model is simply an example of a distributional solution to the Maxwell-Vlasov equations. Let $v \in \Gamma T\mathcal{M}$ with $g(v, v) = -1$ and $\rho \in \Gamma^0\mathcal{M}$. We may consider $v : \mathcal{M} \hookrightarrow \mathcal{E}$ as a closed embedding. Let
\[ \Theta \in \Gamma_{\mathcal{D}}\Lambda^6\mathcal{E} ; \quad \Theta = \nu_{\varsigma}(\rho \star \bar{v}) \tag{75} \]
i.e. $\Theta[\phi] = \int_{x \in \mathcal{M}} \phi|_{\nu|_x} \wedge \rho|_{x} \star \bar{v}|_x$ for all $\phi \in \Gamma_{\mathcal{D}}\Lambda^1\mathcal{E}$
where $\bar{v} \in \Gamma^1\mathcal{M}$ is the metric dual of $v$ satisfying $\bar{v}(u) = g(v, u)$ for all $u \in \Gamma T\mathcal{M}$.

Lemma 33 Equations (13) and (75) give the source for the cold charged fluid:
\[ J = D(\rho \star \bar{v}) \tag{76} \]
proof: $\Theta$ is bounded with respect to $\pi$ thus from lemma 28 we have
$$J = \pi_\xi \Theta = \pi_\xi (v_\xi (\rho \ast \tilde{v})) = (\pi \circ v)_\xi (\rho \ast \tilde{v}) = D(\rho \ast \tilde{v})$$

Also since $dJ = 0$ we have the continuity equation
$$d(\rho \ast \tilde{v}) = 0 \quad (77)$$

Lemma 34 Equations (2), (16) and (75) imply the Lorentz force equation
$$\nabla \cdot v = i \vec{F} \quad (78)$$

proof: From (16) and (75) we have $i_W(v_\xi(\rho \ast \tilde{v})) = 0$. Thus from lemma 15 there exists $u \in \Gamma T M$ such that $v_\ast(u | x) = W | v | x$. Hence $v_\ast(v | x) = W | v | x$.

Given an integral curve $C: \mathbb{R} \to M$ of $v$ then $v | C(\tau) = \dot{C}(\tau)$, i.e. $v \circ C = \dot{C}$. Thus
$$\dot{C}(\tau_0) = \dot{C}_\ast (\partial_{\tau} | \tau_0) = (v \circ C)_\ast (\partial_{\tau} | \tau_0) = v_\ast(C_\ast(\partial_{\tau} | \tau_0)) = v_\ast(\dot{C}(\tau_0))$$
$$= v_\ast(v | C(\tau_0)) = W | v | C(\tau_0) = W | \dot{C}(\tau_0)$$

Hence $\dot{C}$ is an integral curve of $W$ and hence from (3) $\nabla \cdot \dot{C} = \vec{F}$. Since this is true for all integral curves we have (78).

4.3 Multicurrent model

Here we see that the multicurrent model is simply an example of a distributional solution to the Maxwell-Vlasov equations. The conclusions (81,82,84) are in [9,10], with slightly modified notation. An application to wake breaking in plasmas is given in [11].

Let $B = \mathbb{R} \times B$ be a body-time manifold with points $(\tau,y) \in B$ and a measure $d\tau \wedge \mathcal{K} \in \Gamma \Lambda^4 B$ where $\mathcal{K} = \pi_2^*(\mathcal{K}) \in \Gamma \Lambda^3 B$ and $\mathcal{K} \in \Gamma \Lambda^3 B$. Let $C:B \to M$ with $\dot{C} = C_\ast(\partial_{\tau}) \in \mathcal{E}$. Let
$$\Theta \in \Gamma_{D \Lambda^6 \mathcal{E}} ; \quad \Theta = \dot{C}_\ast(\mathcal{K}) \quad (79)$$

i.e. $\Theta[\phi] = \int_{(\tau,y) \in B} \phi | \dot{C}(\tau,y) \wedge \mathcal{K}$ for all $\phi \in \Gamma_0 \Lambda^1 \mathcal{E}$

Lemma 35 Equations (13) and (79) give the source $J$ due to a multicurrent fluid. These may be written in a variety of ways. Using distributional push forwards:
$$J = C_\ast(\mathcal{K}) \quad (80)$$

Furthermore if $J$ is regular in the sense that it is piecewise continuous, then we can write $J = D(\tilde{J})$ where $\tilde{J} \in \Gamma_{pc} \Lambda^3 M$ is piecewise continuous.

In terms of the inverse pull back on generic open sets
$$\tilde{J} = \sum_{i=1}^{N(U,M)} \text{sign}(\det(C_{[i]*})) C^{-1}_{[i]}(\mathcal{K}) \quad (81)$$
where $U^M \subset M$ is an open set where $C$ is generic and therefore the number of preimages $U^B_{[i]}$, $i = 1, \ldots, N(U^M)$ is constant, and the maps $C_{[i]} : U^B_{[i]} \rightarrow U^M$ are invertible. In terms of integrals over spacelike hypersurfaces of $M$.

$$\int_S \mathcal{J} = \int_{C^{-1}(S)} \mathcal{K} \quad \text{for all bounded spacelike hypersurfaces} \quad S \subset M \quad (82)$$

**proof:** Since $\dot{C}(\tau, y) = C_*(\partial_{\tau}|_{(\tau, y)})$,

$$\pi(\dot{C}(\tau, y)) = \pi C_*(\partial_{\tau}|_{(\tau, y)}) = C(\tau, y)$$

i.e. $\pi \circ \dot{C} = C$. Hence from lemma 28

$$\mathcal{J} = \pi_*(\dot{C}_*\mathcal{K}) = (\pi \circ \dot{C})_*(\mathcal{K}) = C_*(\mathcal{K})$$

giving (80).

Let $U^M \subset M$ be a generic subset and $\phi \in \Gamma_0 \Lambda^1 U^M$ then since $C(U^B_{[i]}) = \pm U^M$ depending on the sign($\det(C_{[i]*})$) we have

$$\int_M \phi \wedge \mathcal{J} = D(\mathcal{J})[\phi] = \mathcal{J}[\phi] = C_*(\mathcal{K}) = \int_B C^* \phi \wedge \mathcal{K}$$

$$= \sum_{i=1}^{N(U^M)} \int_{U^B_{[i]}} C_{[i]}^* \phi \wedge \mathcal{K} = \sum_{i=1}^{N(U^M)} \int_{U^M} \text{sign} \det(C_{[i]*}) C_{[i]}^{-1*}(C_{[i]}^* \phi \wedge \mathcal{K})$$

$$= \sum_{i=1}^{N(U^M)} \int_{U^M} \text{sign} \det(C_{[i]*}) \phi \wedge C_{[i]}^{-1*}(\mathcal{K})$$

since this is true for all $\phi \in \Gamma_0 \Lambda^1 U^M$ then we have (81).

Putting $s : S \rightarrow M$ and $C : B \rightarrow M$ into (48) we see that $\{(y, x) \in B \times S \mid C(y) = s(x)\} = C^{-1}(S)$ thus (49) becomes

$$C^{-1}(S) \xrightarrow{s^*} B \xrightarrow[C]{\mathcal{J}} B \xrightarrow[C]{\mathcal{K}} \mathcal{M}$$

where for $y \in C^{-1}(S)$ we have $\dot{C}(y) = C(y)$ and $s(y) = y$. Hence from (80) and since $S$ is compact so $1 \in \Gamma_0 \Lambda S$ we have

$$\int_S s^* \mathcal{J} = D(s^* \mathcal{J})[1] = s^*(D\mathcal{J})[1] = s^* \mathcal{J}[1] = s^*(C_*\mathcal{K})[1] = \dot{C}_*(s^*\mathcal{K})[1] = \int_{C^{-1}(S)} \dot{s}^*\mathcal{K}$$

I.e. (82).

**Lemma 36** Equation (2), (16) and (79) imply

$$\nabla_C \dot{C} = i_{\dot{C}} F \quad (84)$$

**proof:** From (16) and (79) we have $i_W \dot{C}_*(\mathcal{K}) = 0$. Thus from lemma 15 there exists $u \in \Gamma TB$ such that $\dot{C}_*(u|_{(\tau, y)}) = W|_{\dot{C}(\tau, y)}$.

$$C_*(\partial_{\tau}|_{(\tau, y)}) = \dot{C}(\tau, y) = \pi_* W|_{\dot{C}(\tau, y)} = \pi_* \dot{C}_*(u|_{(\tau, y)}) = (\pi \circ \dot{C})_*(u|_{(\tau, y)}) = C_*(u|_{(\tau, y)})$$

Hence $u = \partial_{\tau}$. Thus $\dot{C}_*(\partial_{\tau}|_{(\tau, y)}) = W|_{\dot{C}(\tau, y)}$. Keeping $y$ as a constant then $\dot{C}(\tau, y)$ is an integral curve of $W$ and hence form (3) we have (84).
4.4. The water bag model

Let \( N \subset \mathbb{R}^3 \) be a bounded 3 dimensional manifold with boundary \( \nu: B \hookrightarrow N \). Let \( a: N \times \mathcal{M} \hookrightarrow \mathcal{E} \) be an embedding such that \( \pi \circ a = \pi_2: N \times \mathcal{M} \rightarrow \mathcal{M} \) is the second projection. This induces the map \( v = (a \circ \nu): B \times \mathcal{M} \hookrightarrow \mathcal{E} \) so \( \pi \circ v = \pi_2 \). Let \( \alpha = a^*(i_{W\Omega}) \in \Gamma \Lambda^6(N \times \mathcal{M}) \). We call this the water bag model [5].

The source for Maxwell’s equations for the water bag model is given by

\[
\mathcal{J} = \pi_\nu a_\nu \alpha
\]

Comparing with (6), this is a regular distribution \( \mathcal{J} = D(\ast \tilde{J}) \) where \( \tilde{J} \in \Gamma \Lambda^3 \mathcal{M} \)

\[
\tilde{J} = \left( \int_{\pi^{-1}(x) \cap a(N)} \frac{g_{\nu} \sqrt{|\det(g)|}}{y_0} dy^{123} \right) i_{\partial/\partial x^a} \ast 1
\]

Let \( v = a \circ \nu: B \times \mathcal{M} \hookrightarrow \mathcal{E} \) and \( v(\xi, x) = v_\xi(x) \) so for each \( \xi \in B \), \( v_\xi \in \Gamma \mathcal{E} \). There is no requirement that \( v_\xi \) satisfies the cold charged fluid equations (78). However we can find a reparametrisation \( w_\zeta(x) = w_{\zeta(\xi,x)}(x) = v_\xi(x) \) such that \( w_\zeta \) does satisfy (78).

To see this, let \( \Sigma \in \mathcal{M} \) be a Cauchy hypersurface so that \( \pi^{-1}\Sigma \subset \mathcal{E} \) is an initial hypersurface with respect to \( W \). Thus we have the submanifold \( \sigma: B \times \Sigma \hookrightarrow \mathcal{E} \). Given any \( (\zeta, y) \in B \times \Sigma \) let \( \dot{C}_{\sigma(\zeta,y)}: \mathbb{R} \rightarrow \mathcal{E} \) be the integral curve of \( W \) passing though \( \sigma(\zeta, y) \), which therefore satisfies (74). Given \( \xi \in B \) and \( x \in \mathcal{M} \), since \( v_\xi(x) \in v(B \times \mathcal{M}) \) then \( v_\xi(x) \) lies on some integral curve \( \dot{C}_{\sigma(\zeta,y)} \) and hence there exists unique \( \tau \in \mathbb{R}, \zeta \in B \) and \( y \in \Sigma \) such that \( v_\xi(x) = \dot{C}_{\sigma(\zeta,y)}(\tau) \). Let \( \zeta(\xi, x) \) be the corresponding \( \zeta \) so that \( v_\xi(x) = \dot{C}_{\sigma(\zeta(\xi, x), y)}(\tau) \). Now set \( w_{\zeta(\xi, x)}(x) = v_\xi(x) \) so \( w_\zeta \) obeys (78).

5. Conclusion and Discussion

We have presented the distributional Maxwell-Vlasov equations and analysed the consequences of pushforward distributional solutions. We have paid particular attention to the question of when a pushforward distributional solution can be the source for the Maxwell equations.

There are a number of directions that this research may continue:

It is possible to extend the idea of a submanifold distribution to include distribution with support on submanifolds with different dimension or intersecting components. One can ask how to extend the notion of a pullback to these distributions. From equation (68) we can see it is possible to take any distribution on an initial hypersurface and create a distributional solution to the transport equations.

Submanifold solutions with dimension of 5 or 6 may, for example, correspond to Vlasov one particle probability functions which at every point in spacetime have a range of possible values of one component of velocity, but fixed values for the other components.

As stated in the introduction, it may be possible to solve the Maxwell-Vlasov equation numerically, for distributional solutions of arbitrary dimension. For low dimension it will be necessary to perform some form of regularisation of the
electromagnetic field $F$, however this should be easier for the solutions of dimension 2 or 3. These numerical solutions can either be interpreted in their own right as, for example, the dynamics of strings or disks, or as approximations to regular solutions.

The stress energy tensor for a distributional source on the upper unit hyperboloid is known. This enables one to couple the Maxwell-Vlasov equations with Einstein’s equations to give the Einstein-Maxwell-Vlasov system [12]. Since the equations in this article are already covariant no further modification would be required.

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