Chiral corrections to the Kroll–Ruderman theorem

V. Bernard\textsuperscript{a}, N. Kaiser\textsuperscript{b}, Ulf-G. Meißner\textsuperscript{c}

\textsuperscript{a}Laboratoire de Physique Théorique, Institut de Physique
3-5 rue de l’Université, F-67084 Strasbourg Cedex, France
Centre de Recherches Nucléaires, Physique Théorique
BP 28, F-67037 Strasbourg Cedex 2, France
email: bernard@crnhp4.in2p3.fr

\textsuperscript{b}Technische Universität München, Physik Department T39
James-Franck-Straße, D-85747 Garching, Germany
email: nkaiser@physik.tu-muenchen.de

\textsuperscript{c}Universität Bonn, Institut für Theoretische Kernphysik
Nussallee 14-16, D-53115 Bonn, Germany
email: meissner@itkp.uni-bonn.de

Abstract

We calculate the one–loop corrections to the Kroll–Ruderman low–energy theorems for charged pion photoproduction in the framework of heavy baryon chiral perturbation theory. We predict the threshold S–wave multipole $E_{0+}^{\text{thr}}(\gamma p \rightarrow \pi^+ n) = (28.2 \pm 0.6) \cdot 10^{-3}/M_\pi$ and $E_{0+}^{\text{thr}}(\gamma n \rightarrow \pi^- p) = (-32.7 \pm 0.6) \cdot 10^{-3}/M_\pi$, respectively, for a fixed pion–nucleon coupling constant, $g_{\pi N} = 13.4$. A comparison to the existing data is also given.
1. Over the last years, there has been considerable experimental and theoretical activity devoted to the subject of neutral photopion production off protons at threshold, for reviews see e.g. [1] [2]. In that reaction, the S-wave multipole $E_{0+}$ vanishes in the chiral limit of zero pion mass and is therefore very sensitive to the explicit chiral symmetry breaking in QCD due to the finite quark masses. In particular, in Refs. [3] [4] it was stressed that of zero pion mass and is therefore very sensitive to the explicit chiral symmetry breaking

$$E_{0+}^\text{thr}(\pi^+ n) = \frac{e g_{\pi N}}{4\pi \sqrt{2m(1+\mu)^{3/2}}} = 27.6 \cdot 10^{-3} / M_{\pi^+} \ ,$$

$$E_{0+}^\text{thr}(\pi^- p) = -\frac{e g_{\pi N}}{4\pi \sqrt{2m(1+\mu)^{1/2}}} = -31.7 \cdot 10^{-3} / M_{\pi^+} \ ,$$

with $\mu = M_{\pi^+}/m$ and using $g_{\pi N}^2/4\pi = 14.28$, $e^2/4\pi = 1/137.036$, $m = 928.27$ MeV and $M_{\pi^+} = 139.57$ MeV. In the limit $M_{\pi^+} = 0$, this simplifies to

$$E_{0+}^\text{thr}(\pi^+ n) = -E_{0+}^\text{thr}(\pi^- p) = 34 \cdot 10^{-3} / M_{\pi^+} \ .$$

By comparing the numbers in Eq. (1) and Eq. (2) one notices that the kinematical corrections which are suppressed by powers of the small parameter $\mu \simeq 1/7$ are quite substantial for $E_{0+}^\text{thr}(\pi^+ n)$. However, there are other corrections which are related to pion loop diagrams and higher dimension operators. These will be dealt with in a systematic fashion

$$E_{0+}^\text{thr}(\pi^+ n) = (27.9 \pm 0.5) \cdot 10^{-3} / M_{\pi^+} \ [7] \ , \quad (28.8 \pm 0.7) \cdot 10^{-3} / M_{\pi^+} \ [8] \ ,$$

$$E_{0+}^\text{thr}(\pi^- p) = (-31.4 \pm 1.3) \cdot 10^{-3} / M_{\pi^+} \ [7] \ , \quad (-32.2 \pm 1.2) \cdot 10^{-3} / M_{\pi^+} \ [8] \ .$$

A more recent measurement of the inverse reaction $\pi^- p \to \gamma n$ (pion radiative capture) from TRIUMF (experiment E643) for energies slightly above threshold lead to the preliminary value of $E_{0+}^\text{thr}(\pi^- p) = (-34.6 \pm 1.0) \cdot 10^{-3} / M_{\pi^+} \ [10]$. Here, the error is only statistical and the final result of this experiment has not yet been reported. If it holds up, it would amount to a rather sizeable deviation from the previously reported numbers.

2. The tool to systematically calculate all corrections to a given order in the pion mass is chiral perturbation theory (CHPT). It amounts to a systematic expansion around the chiral limit in terms of two small parameters related to the quark masses and the external momenta. Threshold pion photoproduction is particularly suited since these expansion parameters are given by one single number, $M_{\pi}/4\pi F_{\pi} = 0.12$ (the pion energy at threshold is nothing but the pion mass). Here, $F_{\pi} = 92.4$ MeV is the pion decay constant. We remark that due to the presence of nucleons this small parameter does not only appear
squared as it is the case for purely mesonic processes. Chiral corrections for charged pion photoproduction have already been considered in Ref.\[3\] within the one–loop approximation. However, in that paper relativistic nucleon CHPT was used and thus it could not be proven that the calculated terms of order $O(\mu^2 \ln \mu, \mu^2)$ are not touched by higher loop corrections. This is due to the fact that the presence of the additional mass scale related to the nucleon mass (in the chiral limit) complicates the power counting. This difficulty can be overcome by treating the nucleons as very heavy (static) sources \[11\]. In what follows, we will use the systematic SU(2) approach developed in Ref.\[12\]. The calculations are most easily done in the isospin basis,

$$E_{0+}(\pi^+ n) = \sqrt{2} (E_{0+}^{(0)} + E_{0+}^{(\pi^-)}), \quad E_{0+}(\pi^- p) = \sqrt{2} (E_{0+}^{(0)} - E_{0+}^{(\pi^0)}) , \quad (4)$$

and the amplitude $E_{0+}^{(0)}$ is already known from the calculation of the processes $\gamma N \rightarrow \pi^0 N$ since $E_{0+}(\pi^0 p/\pi^0 n) = \pm E_{0+}^{(0)} + E_{0+}^{(+)}$ \[4\]. In the framework of heavy baryon CHPT, we have to consider pion loop diagrams and local contact terms accompanied with a priori unknown coefficients, the so–called low-energy constants (LECs). These we are estimating by resonance exchange since not enough precise data exist yet to pin them all down. However, previous calculations have already shown that this approach of treating the LECs is fairly accurate as long as no big cancellations appear (for details, see \[2\]).

Consider first $E_{0+}^{(0)}$. To order $M^3_\pi$, i.e. for the first three terms in the chiral expansion, we find

$$E_{0+}^{(0)} = \frac{C \mu}{2} \left\{ -1 + \frac{M_\pi}{2m} (3 + \kappa_s) - \frac{3M^2_\pi}{8m^2} (5 + 2\kappa_s) + \frac{M^2_\pi}{3\pi^2 F^2_\pi} \left( \ln \frac{M_\pi}{\lambda} - \frac{5}{6} \right) \right. $$

$$+ \frac{mM^2_\pi}{24\pi F^2_\pi M_K} + \frac{(1 + \kappa_\rho)mM^2_\rho}{16\pi^2 g_{\pi N} F^3_\pi} \right\} \quad (5)$$

with

$$C = \frac{eg_{\pi N}}{8\pi m} = 24.01 \cdot 10^{-3}/M_\pi^{+} , \quad (6)$$

and $\kappa_s = \kappa_p + \kappa_n = -0.12$ is the isoscalar anomalous magnetic moment of the nucleon. Let us briefly discuss the various contributions appearing in Eq.(5). The first three terms come from the expansion of the Born graphs (i.e. tree diagrams with photon absorption including the anomalous magnetic moment coupling followed by pion emission). The fourth term is the pion loop contribution. Here, $\lambda$ is the scale of dimensional regularization. We note that the contribution $\sim M^3_\pi \ln M_\pi$ agrees with the result of the relativistic calculation (as it should) \[3\]. The fifth term in Eq.(4) is the contribution from frozen kaon loops, with $M_K = 493.65$ MeV the kaon mass. Finally, the last term stems from $\rho$–meson exchange with $\kappa_\rho \simeq 6$ and we use some symmetry relations for the $\rho$–meson mass and couplings \[4\]. These last two terms constitute the counter term contribution. At this order, there are no other contributions to $E_{0+}^{(0)}$. Higher mass resonances play no role within the accuracy of the calculation, see below.

We turn now to the calculation of $E_{0+}^{(-)}$ to $O(M^3_\pi)$, i.e. the first four terms in the chiral
expansion. These read

\[
E_{0^+}^{(-)} = C \left\{ 1 - \frac{M_\pi}{m} + \frac{9M_\pi^2}{8m^2} + \frac{M_\pi^2}{8\pi^2 F_\pi^2} \left( \frac{\pi^2}{8} - \ln \frac{M_\pi}{\lambda} \right) - \frac{M_\pi^2}{16\pi^2 F_\pi^2} \left( \frac{1}{2} + \ln \frac{M_K}{\lambda} \right) \right.
\]

\[
- \frac{M_\pi^2}{6} < r_A^2 > + \frac{M_\pi^2}{3\sqrt{2}m_\Delta} \left[ g_1 \left( 2Y(2Z - 1) + \frac{m_\Delta}{m} (1 - 2Y - 2Z - 8YZ) \right) \right.
\]

\[
+ g_2 \left( (X + \frac{1}{2})(Z - \frac{1}{2}) - \frac{m_\Delta}{2m} (1 + X + Z + 4XZ) \right) \right] - \frac{5M_\pi^3}{4m^3} + \frac{\pi}{4} M_\pi a^+
\]

\[
+ \frac{M_\pi^3}{8\pi^2 m F_\pi^2} \left( \ln \frac{M_\pi}{\lambda} - \frac{\pi^2}{4} - \frac{\pi}{2} \right) + \frac{g_{\pi N}^2 M_\pi^2}{16\pi^2 m^3} \left( \frac{\pi^2}{8} + 1 - \ln \frac{M_\pi}{\lambda} \right) \right\}, \quad (7)
\]

which consists of the expanded Born terms, pion loop contributions and a variety of counter terms. These are estimated in part by \( \Delta (1232) \) excitation. The parameters related to the \( N\Delta\pi\gamma \) system \( (g_1, g_2, X, Y, Z) \) have been previously determined in chiral corrections to Compton scattering, pion–nucleon scattering and neutral pion photoproduction \[13\]. In addition, there are potentially large corrections related to one loop graphs with one insertion of the dimension two operators which come with the LECs \( c_1, c_2, c_3 \) and \( c_4 \). There are also relativistic corrections with fixed coefficients of the type \( 1/2m \) and the term proportional to the isovector/isoscalar anomalous magnetic moments \( \kappa_{v,s} \) with \[14\]

\[
\kappa_v \gg 1, \quad c_1, c_2, c_3, c_4 \gg \frac{1}{2m} \frac{g_A^2}{8m} . \quad (8)
\]

However, the terms proportional \( c_{1,2,3} \) only appear in a combination that can be expressed in terms of the small isoscalar \( S \)-wave pion–nucleon scattering length \( a^+ \),

\[
a^+ = \frac{M_\pi^2}{2\pi F_\pi^2} \left( -2c_1 + c_2 + c_3 - \frac{g_A^2}{8m} \right), \quad (9)
\]

and the other large combination \( 2c_4 - c_3 + 1/2m \) which appears is fully absorbed in the renormalization of the pion–nucleon vertex,

\[
\hat{g}_A \to g_{\pi N}/m \quad , \quad (10)
\]

where \( \hat{g}_A \) \( (F) \) denotes the axial–vector (pion decay) constant in the chiral limit. Furthermore, the low–energy constant \( b_{11} \) related to the Goldberger–Treiman discrepancy \[14\] does only enter via the strong coupling constant renormalization. Also, terms proportional to \( \kappa_{v,s} \) appear only in such graphs which vanish at threshold \[4\]. Again, some of the terms appearing in Eq.\( (7) \) agree with the ones of the expanded relativistic calculation of Ref.\[3\] (like e.g. the term \( \sim M_\pi^2 \ln M_\pi \)). We also note that compared to that reference, we now have a much better understanding of the \( \Delta(1232) \) contribution to certain LECs, i.e. it is much more constrained since a variety of different processes have been calculated in the mean time. We do not take into account isospin breaking via the difference of the neutral and charged pion masses. Cusp effects play no role here since the secondary thresholds \( (\pi^0p/\pi^0n) \) lie below the physical ones.
3. We are now in the position to analyse the chiral corrections to the Kroll–Ruderman LETs. The numerical values of the various parameters not yet given are \( X = 2.75 \), \( Y = 0.1 \), \( Z = -0.2 \) and \( g_1 = g_2 = 5 \) for the \( N\Delta\pi\gamma \) system [13]. We will vary these within their bounds determined from the fit to the LEC \( a_1 + a_2 \) in neutral pion photoproduction and from the contribution to the \( \pi N \) scattering volume \( a_{33} \). For the axial mean square radius, we use the dipole relation \( < r_A^2 > = 12/M_A^2 \) with \( M_A = 1.032 \text{GeV} \). As a variation of \( M_A \), we also consider \( M_A = 0.96 \text{GeV} \) and \( 1.15 \text{ GeV} \) (see e.g. [15]). In the case of resonance saturation for the LECs, there remains a spurious mild scale–dependence since we have to let \( \lambda \) run in the interval \( M_\rho \leq \lambda \leq m_\Delta \).

Consider first \( E_{0+}^{(0)} \). Keeping the pion–nucleon coupling constant fixed, the Born terms are well determined. We remark that at present there is not a generally accepted uncertainty for \( g_{\pi N} \) and we thus refrain from varying it. Note, however, that \( E_{0+}^{(0)} \) essentially scales with \( g_{\pi N} \) and thus a different value than the one used here can easily be accounted for. Letting \( \lambda \) vary as described, the pion loop contribution changes from \( -0.35 \cdot 10^{-3}/M_\pi \) to \( -0.41 \cdot 10^{-3}/M_\pi \). Further uncertainties can be estimated as follows. For the \( \rho \)-contribution we change \( \kappa_\rho \) from 6 to 6.6 and in the term from the frozen kaon loops, we use \( F_K = 1.21 F_\pi \) instead of \( F_\pi \). Adding all these uncertainties in quadrature, we have (\( \lambda = m \))

\[
E_{0+}^{(0)} = (-1.6 \pm 0.1) \cdot 10^{-3}/M_\pi \quad .
\]  

The chiral expansion is rapidly converging,

\[
E_{0+}^{(0)} = (-1.79 + 0.38 - 0.07 - 0.38 + 0.10 + 0.14) \cdot 10^{-3}/M_\pi ,
\]

\[
= (-1.79 + 0.38 - 0.21) \cdot 10^{-3}/M_\pi ,
\]  

where the first three terms are the Born contributions of \( \mathcal{O}(M_\pi^n) \) (\( n = 1, 2, 3 \)), while the fourth, fifth and sixth term refer to the pion loop, the frozen K–loop and the \( \rho \)-exchange contributions, in order. In the second line of Eq.(12), we have collected the various contributions to \( E_{0+}^{(0)} \) of order \( M_\pi, M_\pi^2 \) and \( M_\pi^3 \).

Consider now \( E_{0+}^{(-)} \). The pion loop contribution (the fourth and the last two terms in Eq.(10)) is \((1.80, 1.94, 2.12) \cdot 10^{-3}/M_\pi \) for \( \lambda = (M_\rho, m, m_\Delta) \) and similar for the frozen K–loop we have \((-0.02, 0.05, 0.14) \cdot 10^{-3}/M_\pi \). Setting furthermore \( F_K = 1.21 F_\pi \), we assign a total uncertainty of \( \pm 0.1 \cdot 10^{-3}/M_\pi \) to this contribution. Varying the parameters \( g_2 \) and \( X \) under the constraints given from \( \pi^0 \) photoproduction [12], we have for the \( \Delta \)-contribution \((-0.57 \pm 0.10) \cdot 10^{-3}/M_\pi \). Similarly, the variation in \( M_A \) leads to \((-0.88 \pm 0.17) \cdot 10^{-3}/M_\pi \) from the axial radius term. Furthermore, the term proportional to the \( \pi N \) scattering length \( a^+ \) induces some uncertainty. For the Karlsruhe–Helsinki value, \( a^+ = (-0.83 \pm 0.38) \cdot 10^{-2}/M_\pi \) [13], we get a contribution \( \delta E_{0+}^{(-)} = (-0.16 \pm 0.07) \cdot 10^{-3}/M_\pi \). The new value from the ETH group based on level shifts in pionic atoms is \( a^+ = (+0.25 \pm 0.18) \cdot 10^{-2}/M_\pi \) [17], leading to \( \delta E_{0+}^{(-)} = (0.05 \pm 0.03) \cdot 10^{-3}/M_\pi \). Adding all these uncertainties in quadrature, we have

\[
E_{0+}^{(-)} = (21.5 \pm 0.4) \cdot 10^{-3}/M_\pi \quad .
\]
Higher resonance contributions are well within the given uncertainty. It is again instructive to dissect the various terms in the chiral expansion,

$$E_{0+}^{(-)} = (24.01 - 3.57 + 1.38 - 0.29) \cdot 10^{-3}/M_\pi,$$

which are the terms of $O(M_n^3)$ with $n = 0, 1, 2, 3$, respectively. Again, we find a quick convergence.

Finally, we can translate the results Eqs.(11,13) into the physical channels,

$$E_{0+}^{thr}(\gamma p \rightarrow \pi^+ n) = (28.2 \pm 0.6) \cdot 10^{-3}/M_\pi,$$
$$E_{0+}^{thr}(\gamma n \rightarrow \pi^- p) = (-32.7 \pm 0.6) \cdot 10^{-3}/M_\pi,$$

for a fixed pion–nucleon coupling constant, $g_{\pi N}^2/4\pi = 14.28$. These compare favorably with the existing data Eq.(3). We note, however, that the large (in magnitude) preliminary value from TRIUMF for $E_{0+}^{thr}(\gamma n \rightarrow \pi^- p)$ would be difficult to understand. For comparison, in the calculation based on relativistic CHPT we had found $E_{0+}^{thr}(\gamma p \rightarrow \pi^+ n) = 28.4 \cdot 10^{-3}/M_\pi$ and $E_{0+}^{thr}(\gamma n \rightarrow \pi^- p) = -31.1 \cdot 10^{-3}/M_\pi$. The differences stem mostly from a better treatment of the $\Delta$–contribution and the fact that in the heavy fermion approach given here, all terms of order $M_\pi^3$ could be given. However, the statement made in [3] that the loop corrections are fairly small in the case of charged pion photoproduction remains valid.

4. To summarize, we have calculated the corrections to the Kroll–Ruderman low–energy theorem up–to–and–including all terms of order $O(M_\pi^3)$. The chiral expansion of the S–wave multipoles $E_{0+}^{(0)}$ and $E_{0+}^{(-)}$ shows a rapid convergence and thus one is able to give a rather accurate prediction for $E_{0+}^{thr}(\gamma p \rightarrow \pi^+ n)$ and $E_{0+}^{thr}(\gamma n \rightarrow \pi^- p)$, compare Eq.(15). It would be important to have these observables determined with high precision for a couple of reasons. First, an accurate determination of these multipoles gives a stringent constraint on the much discussed value of the pion–nucleon coupling constant $g_{\pi N}$ via the Goldberger–Miyazawa–Oehme sum rule [18] combined with the Panofsky ratio. Second, together with a precise determination of the two neutral pion production amplitudes, one would have an excellent testing ground for the investigation of isospin symmetry violation beyond leading order in the electromagnetic coupling $e$. For such a test, it is mandatory to determine the elementary neutron amplitude $\gamma n \rightarrow \pi^0 n$ (as was stressed already in Refs.[3, 4]). We point out that the $O(q^4)$ CHPT calculation leads us to expect that $E_{0+}^{thr}(\gamma n \rightarrow \pi^0 n) \simeq -2 E_{0+}^{thr}(\gamma p \rightarrow \pi^0 p)$ [4, 13] and thus a determination using e.g. the deuteron does appear feasible.
References

[1] D. Drechsel and L. Tiator, J. Phys. G: Nucl. Part. Phys. 18, 449 (1992)

[2] V. Bernard, N. Kaiser and Ulf-G. Meißner, Int. J. Mod. Phys. E4, 193 (1995)

[3] V. Bernard, N. Kaiser and Ulf-G. Meißner, Nucl. Phys. B383, 442 (1992)

[4] V. Bernard, N. Kaiser and Ulf-G. Meißner, "Neutral Pion Photoproduction off Nucleons Revisited", Z.Phys. C70 (1996) in press.

[5] N.M. Kroll and M.A. Ruderman, Phys. Rev. 93, 233 (1954)

[6] G. Ecker and Ulf–G. Meißner, Comm. Nucl. Part. Phys. 21, 347 (1995)

[7] J.P. Burg, Ann. Phys. (Paris) 10, 363 (1965)

[8] M.J. Adamovitch et al., Sov. J. Nucl. Phys. 2, 95 (1966)

[9] E.L. Goldwasser et al., Proc. XII Int. Conf. on High–Energy Physics, Dubna, 1964, ed. Y.–A. Smorodinsky (Atomzidat, Moscow, 1966)

[10] M.A. Kovash, talk given at the workshop “Chiral Dynamics: Theory and Experiment”, A.M. Bernstein and B.R. Holstein (eds), Springer Lecture Notes in Physics, Heidelberg, 1995;

[11] E. Jenkins and A.V. Manohar, Phys. Lett. B255, 558 (1991)

[12] V. Bernard, N. Kaiser, J. Kambor and Ulf-G. Meißner, Nucl. Phys. B388, 315 (1992)

[13] V. Bernard, N. Kaiser and Ulf-G. Meißner, “Chiral symmetry and the reaction $\gamma p \to \pi^0 p$”, preprint CRN 95-38 and TK 95 30, [hep-ph/9512234]

[14] V. Bernard, N. Kaiser and Ulf-G. Meißner, Nucl. Phys. B457, 147 (1995)

[15] S. Choi et al., Phys. Rev. Lett. 71, 3927 (1993)

[16] R. Koch, Nucl. Phys. A448, 707 (1986)

[17] D. Chatellard et al., Phys. Rev. Lett. 74, 4157 (1995);
   D. Sigg et al., Phys. Rev. Lett. 75, 3245 (1995);
   J. Leisi, private communication

[18] M.L. Goldberger, H. Miyazawa and R. Oehme, Phys. Rev. 99, 986 (1955)