The Local Spin Structure of Large Spin Fermions

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We show that large spin fermions have very rich spin structures. The local spin order of a spin-$f$ Fermi gas is a linear combination of $2f$ (particle-hole) angular momentum states, $L = 1, \ldots, 2f$. $L = 1, 2$ represent ferromagnetic and nematic spin order, while $L \geq 3$ are higher spin orders that have no analog in spin-$1/2$ systems. Each $L$ spin sector is characterized as $L$ pairs of antipodal points on a sphere. Model calculations show that some of these spin-orders have the symmetry of Platonic solid, and many of them have non-abelian line defects.

Prior to the discovery of Bose-Einstein condensates, the only quantum liquids realized experimentally are the electron liquids in solids, and the low temperature phases of liquid $^4$He and $^3$He. All these systems are made up of spin-$1/2$ fermions (like electrons and and $^3$He atoms), or spin-$0$ bosons ($^4$He atoms). Recent advances in cooling atomic gases to quantum degeneracy, however, have created the exciting opportunities of studying the spin analog of quantum fluids. The spins of atomic bosons can range from $f = 1, 2$ ($^{87}$Rb bosons) to $f = 8$ ($^{162}$Dy bosons)\textsuperscript{[1]}, and those of fermions can be as high as $f = 7/2$ ($^{89}$K) and $f = 21/2$ ($^{161}$Dy)\textsuperscript{[2]}. For fundamental phenomena like Bose condensation and fermion pairing, there have been theoretical generalizations to high spin particles\textsuperscript{[3] [4] [5]}. While there are many experiments on large spin Bose condensates (or spinor condensates), there are few experimental studies on large spin fermions.

Realization of the large spin fermion superfluids is difficult due to their low transition temperatures. On the other hand, scatterings in different angular momentum channel and dipolar effects can lead to non-trivial spin structures in the normal state, which can be realized at higher temperatures. Many of these are truly new structures which have no analogs in solid state systems. Here, we show that the spin structures of a spin-$f$ Fermi gas can be decomposed into $2f$ (particle-hole) angular momentum states ($L = 1, \ldots, 2f$), each of which is characterized as $L$ pairs of antipodal points on a sphere. Model calculations show that some of these spin-orders have the symmetry of Platonic solid, and many of them have non-abelian line defects.

The spinor Bose condensate of spin-$f$ bosons is a $(2f+1)$ component vector $\Psi_m(R) = \langle \tilde{\psi}_m(R) \rangle$, $m = f, f-1, \ldots, -f$\textsuperscript{[2]}. In fermion superfluids, two spin-$f$ particles form a pair with total angular momentum $F$, $\hat{O}_M^{(F)}(R) = \sum_{m_1, m_2} \langle F, M | f, m_1; f, m_2 \rangle \tilde{\psi}_{m_1}(R) \tilde{\psi}_{m_2}(R)$, which rotates like a spin-$F$ bosons under spin rotation\textsuperscript{[3]}. The order parameter $\langle \hat{O}_M^{(F)} \rangle = \Psi_M^{(F)}$ then reduces to that of a spin-$F$ spinor condensate. In contrast, local spin order of a spin-$f$ Fermi gas is contained in the density matrix $\rho = \langle \hat{\rho} \rangle$, where $\rho_{m_1, m_2}(R) = \langle \tilde{\psi}_{m_2}(R) \tilde{\psi}_{m_1}(R) \rangle$. The spatial variations of these quantities can be caused by external conditions, or intrinsic interaction effects including dipolar interactions. From now on, we shall suppress the spatial coordinate until they are needed.

Under a spin rotation $\theta$, the field operator and the density matrix transform as

$$\tilde{\psi}_m \to D^{(f)}_{m,m'}(\theta)\tilde{\psi}_{m'}, \quad \rho \to \rho' = D^{(f)}(D^{(f)})^\dagger,$$  \hspace{1cm} (1)

where $D^{(f)}_{m,m'}(\theta) = \langle f, m | e^{-i\vec{\theta} \cdot \vec{F}} | f, m' \rangle$ is the rotational matrix, and $\vec{F}$ is the spin operator for spin-$f$. To sort out the spin structure of $\hat{\rho}$, we decompose it into tensor operators of different angular momenta (made up of a particle-hole pair). This is achieved by noting that the $(2f+1) \times (2f+1)$ matrix which is proportional to the Clebsch-Gordon coefficient

$$\langle \gamma_M^{(L)} \rangle_{m_1, m_2} = \sqrt{\frac{2L+1}{2f+1}} \langle f, m_1 | L, M; f, m_2 \rangle,$$  \hspace{1cm} (2)

is a tensor operator (with angular momentum $L$) in the spin-$f$ space, as it transforms under rotation as

$$D^{(f)}(\theta) \gamma_M^{(L)} (\theta) D^{(f)\dagger}(\theta) = \sum_{M'} \gamma_{M'}^{(L)} D^{(L)}_{M'M} (\theta),$$  \hspace{1cm} (3)

and $\gamma_M^{(L)\dagger} = \gamma_M^{(L)\dagger}$. Expanding $\hat{\rho}$ as

$$\rho_{m_1, m_2} = \sum_{L=0}^{2f} \sum_{M=-L}^{L} \Phi_M^{(L)} (\gamma_M^{(L)})_{m_1, m_2},$$  \hspace{1cm} (4)

the transformation properties in Eq.\textsuperscript{[1]} and \textsuperscript{[3]} imply that $\{\Phi_M^{(L)}\}$ transforms as a spin-$L$ vector

$$\Phi_M^{(L)} \to \Phi_M^{(L)} = D^{(L)}_{M,M'} \Phi_M^{(L')}.$$  \hspace{1cm} (5)

From the properties of Clebsch-Gordon coefficients\textsuperscript{[6]},

$$\gamma_M^{(L)} = (-1)^M \gamma_M^{(-L)}, \quad Tr\gamma_M^{(L)} \gamma_M^{(-L)} = \delta_{LL'} \delta_{MM'},$$  \hspace{1cm} (6)

we have

$$\Phi_M^{(L)} = Tr (\hat{\rho} \gamma_M^{(L)\dagger}).$$  \hspace{1cm} (7)
The fact that $\rho$ is hermitian implies that
\[ \Phi_M^{(L)*} = (-1)^M \Phi_{-M}^{(L)}. \]
(8)
Thus there are only $2L + 1$ independent real variables for $\Phi_M^{(L)}$. Eq.(3) also implies that $\text{Tr} Y_M^{(L)} = \sum_{M'} \text{Tr} Y_{M'}^{(L)} D_{M'M}^{(L)}(\theta)$ for all $L \geq 1$ and all $\theta$. This forces $\text{Tr} Y_M^{(L)} = 0$ for $L \geq 1$. Note that $\Phi(0)$ is proportional to the density $\rho$ of the system, $\Phi(0) = n/(2f + 1)$. Non-trivial spin structures are given by the traceless part $\rho_M = \rho_{mn} - \frac{1}{2f+1} \delta_{mn}$.

Although $\Phi_M^{(L)}$ is formally similar to a spin-$L$ spinor, it can have very different meaning. From Wigner-Eckart theorem, we note that $Y_M^{(L)}$ is proportional to a product of $L$ spin operators $\mathbf{F}$ in spin-$f$ space. Thus, $\Phi_M^{(L)}$ and $\Phi_{-M}^{(L)}$ represent ferromagnetic and nematic order respectively in spin-$f$ space. The vectors $\Phi_L$ for $L \geq 1$ will be referred to as the $L$-th spin order of the system, $L = 1, 2, ..., 2f$, which are all contained in the traceless part of $\rho$.

To gain further insight, we express $\Phi_M^{(L)}$ in Majorana representation as a set of $2L$ points (referred to as Majorana points) on the unit sphere $S^2$. This follows from the fact that any state in the spin-$L$ manifold made up of spin-$1/2$ (Schwinger) bosons $(a, b)$, can be factorized as
\[ |\Phi_L^{(M)}\rangle = \sum_{M=-L}^{L} \Phi_M^{(L)} |L, M\rangle = \lambda^L (2L-1) \prod_{i=1}^{2L} (a_i^0 + v_i^0) |0\rangle, \]
(9)
where $|L, M\rangle = \frac{a_i^L b_i^{-M} - a_i^{-L} b_i^M}{\sqrt{(L+M)!(L-M)!}}$ is the angular momentum basis, $\lambda$ is a constant, and $\zeta_i = (u_i, v_i)^T$ is a normalized spinor. Both $\lambda$ and $\zeta_i$ should also be labelled by an index “$i$”. We suppress it for simplicity, and will reinstate it later when needed. The rotational properties of the Schwinger bosons $(a, b)$ imply that $\zeta_i$ rotates like a spin-1/2 particle. Using the standard representation $(u_i, v_i)^T = (\cos \frac{\theta}{2} e^{-i\phi}/\sqrt{2}, \sin \frac{\theta}{2} e^{i\phi}/\sqrt{2})$, $\zeta_i$ can be represented as a point on the unit sphere $S_2$ in the direction $\mathbf{n}$, with polar angle $(\theta_i, \phi_i)$. The phase $\chi_i$ can be absorbed into $\lambda$. Eq.(8), however, implies
\[ |\Phi_L^{(M)}\rangle = \sum_{\zeta^{(i)}} \frac{\Phi_M^{(L)*}}{\sqrt{(L+M)!(L-M)!}} b_i^{L+M} (-a_i^0)^{L-M} |0\rangle = \lambda^{(L)} \prod_{i=1}^{2L} (u_i^* b_i^1 - v_i^* a_i^1) |0\rangle. \]
(10)
Eq.(10) shows that any $\zeta^{(i)}$ in Eq.(9) must be accompanied by its time-reversed partner $-\sigma_2 \zeta^{(i)*} = (-v_i^*, u_i^*)$. Therefore, the 2L points must appear in antipodal pairs $(\mathbf{n}_i, -\mathbf{n}_i) \equiv (\mathbf{n}_i)$, and it is sufficient to represent each pair by only one of its members. The presence of antipodal pairs implies Eq.(9) is of the form
\[ |\Phi_L^{(M)}\rangle = \lambda^L \prod_{i=1}^{L} (-u_i v_i^* a_i^1 + (|u_i|^2 - |v_i|^2) a_i^1 b_i^1 + u_i^* v_i b_i^2) |0\rangle, \]
(11) with $\lambda^{(L)} = \lambda^{(L)}$ because of Eq.(8). The $(2L + 1)$ variables of $\Phi_M^{(L)}$ is now represented by the $L$ unit vectors $\mathbf{n}_i$ and real number $\lambda^{(L)}$. In equilibrium, the signs of different $\lambda^{(L)}$’s are correlated to minimize the energy.

Configurations of Majorana points: For $L = 1$, the ferromagnetic order, there is only one Majorana pair $[\mathbf{n}]$. (See Fig.1(i)). Since $\mathbf{n}$ can be in any direction, the configurations space is the unit sphere $S_2$. Note that $[\mathbf{n}]$ and $[-\mathbf{n}]$ are distinct because $|\Phi^{(1)}\rangle$ becomes $-|\Phi^{(1)}\rangle$, (hence $\Phi_{1}^{(1)} \rightarrow -\Phi_{1}^{(1)}$), as $\mathbf{n}$ changes continuously to $-\mathbf{n}$.[10]. Since the first homotopy group of $S_2$ is trivial, $(\pi_1(S_2) = 0)$, the vector field $\{ \Phi_m^{(1)}(\mathbf{r}) \}$ has no topologically stable line defects[11].

For $L = 2$, the nematic order, there are two Majorana pairs $[\mathbf{n}] [\mathbf{m}]$. If $\mathbf{n} = \pm \mathbf{m}$, the system is uniaxial nematics characterized by a single antipodal pair on the unit sphere with each pole doubly occupied. (See Fig.1(ii)). Unlike the $L = 1$, where $[\mathbf{n}]$ and $[-\mathbf{n}]$ are distinct, the states $[\mathbf{n}] [\mathbf{m}]$ and $[-\mathbf{n}] [-\mathbf{m}]$ are identical, as they correspond to the same state $|\Phi\rangle = \lambda(ua^1 + vb^1)^2 (-u^* a + v^* b)^2 |0\rangle$. (Note that $(|\mathbf{n}] [-\mathbf{m}] = -[\mathbf{n}] [\mathbf{m}]$). The configurational space is therefore $S_2$ with antipodal points identified, which is the projected space $P_2$. Since $\pi_1(P_2) = Z_2$, there is only one type of nontrivial line defect. If $\mathbf{n} \neq \pm \mathbf{m}$, it is straightforward to show that $|\Phi^{(2)}\rangle$ is unchanged only under a $\pi$ rotation along the orthogonal axes $\mathbf{n} \times \mathbf{m}$, $\mathbf{n} + \mathbf{m}$, and $\mathbf{n} - \mathbf{m}$. (See Fig.1(iii)). The system is therefore a bi-axial nematics, and has nonabelian line defects[11].

The discussions above also show that for even and odd $L$, the configuration space for the state with all Majorana points collapsing into a single antipodal pair is $P_2$ and $S_2$ respectively. Another consequence of antipodal pairs is that there are no distribution of Majorana points with tetrahedral symmetry, as it is inconsistent with the condition of antipodal pairs. (See Fig.1).

Density-Matrix texture: An example of the Majorana representation of the spin order for a spin $f = 7/2$ Fermi gas is given in Figure 1. We shall consider a general non-equilibrium situation where $\{ \Phi^{(L)}(\mathbf{R}) \}$ can be arbitrary functions. The figure displays the spin order $\Phi_M^{(L)}$ along a loop $C$ in real space, which can be represented as a straight line along $x$ with end points identified. The entire set of spin order $\{ \Phi_M^{(L)}(x), L = 1, 2, ..., 2f \}$ is represented as an array of $2f$ unit spheres $\{S^{(L)}\}$ with radius $\lambda^{(L)}(x)$ and $L$ pairs of antipodal Majorana points. As an illustration, we have shown local spin orders with high symmetry. As we discuss later, many of these symmetries are found in model calculations. As long as $\lambda^{(L)}(x) \neq 0$ and different Majorana pairs do not merge as on traverses the loop $C$, each $L$ sector can have its own line defects in general.
Energy considerations and Platonic solid inert states: Next, we discuss how interaction effects give rise to spin order. We shall consider a general short range spin-exchange interaction between fermions [3]. Many high spin systems have dipolar interactions which will lead to non-uniform spin ordering in the normal state. As a first step, we ignore dipolar interactions. In practice, dipolar interactions can be averaged out to zero through a sequence of magnetic pulses [12]. The Hamiltonian for only local spin-exchange interaction is $H = T + V$, where

$$
T = \int d\mathbf{r} \hat{\psi}^\dagger_m (-\nabla^2 / 2\hbar^2 - \mu) \hat{\psi}_m, \quad V = \frac{1}{2} \int \hat{\psi}^\dagger_1 \hat{\psi}^\dagger_2 V_{12;34} \hat{\psi}_4 \hat{\psi}_3,
$$

$$
V_{12;34} = \sum_{F=0}^{2F} g_F \sum_{M=-F}^{F} \langle 1, 2 | FM \rangle \langle F, M | 3, 4 \rangle \quad (12)
$$

where the integers (1, 2, ..) stand for spin indices $(m_1, m_2, ..)$ and repeated indices are summed over; and $\hat{m}$ is the fermion mass. $(1, 2|FM)$ means $<f, m_1; f, m_2|FM>$. $g_F \equiv 4\pi \hbar^2 a_F / \hat{m}$ is the interaction constant in the scattering channel with total spin $F$ and $a_L$ is the corresponding scattering length. For
fermions with half integer spin $f$, Fermi statistics requires $F = 0, 2, 4, \ldots 2f - 1$.

We shall study uniform spin order using mean field approximation. The mean field Hamiltonian is

$$H_{MF} = T - \int \hat{\psi}_4^\dagger B_{14} \hat{\psi}_4 = \sum_k c_{k,1}^\dagger H_{14}(k) c_{k,4}$$  \hspace{1cm} (13)

$$H_{14}(k) = (\epsilon_k - \mu)\delta_{14} - B_{14}, \quad B_{14} = 2\mathcal{V}_{12,34}\rho_{32}. \hspace{1cm} (14)$$

where $c_{k,m}$ is the Fourier transform of $\psi_m(r)$, and $B_{mn}$ is the mean field that will be determined self consistently. Our mean field calculation does not include superfluid order. That means that we stay at temperature regimes above superfluid condensation. Still, it is useful to extend the calculation to zero temperature to find out the spin order in the absence of superfluid order, as the latter may only appear at extremely low temperatures.

We have solved the self consistency equation Eq. (13) numerically for the case of $f = 21/2$ with some specified value of gas parameters $\{k_0, a_F\}$'s, where $k_0 = (6\pi^2n)^{1/3}$, and $n$ is the total number density. Since $a_F$'s are unknown at present, we have tried various parameter sets, (labelled (I) to (III)) in Footnote[13]. Their mean field states are:

(I): Only $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \Phi^{(4)}$ are nonzero. The Majorana points of each one of them collapse into the antipodal pair like (a) and (b) in Fig. 1. The pairs of different $L$ orient differently.

(II): Only $\Phi^{(4)}, \Phi^{(6)}, \Phi^{(8)}$ are non-zero. $\Phi^{(4)}$ and $\Phi^{(8)}$ form a cube ((d) in Fig 1). $\Phi^{(6)}$ form an octahedron ((c) in Fig 1). For $\Phi^{(8)}$ and $\Phi^{(6)}$, the vertices of the cube and octahedron are doubly occupied.

(III): Only $\Phi^{(6)}$ and $\Phi^{(10)}$ are non-zero. $\Phi^{(6)}$ form an isochadron (like (f) in Fig 1) and $\Phi^{(10)}$ form a dodacahedron ((g) in Fig 1).

The states found in (II) and (III) are the Platonic solids. All the states in (I) to (III) are the so-called inert states as the distances between Majorana points in these states are independent of interactions. All these states are found in a region containing the parameter set in Footnote[13]. There are also non-inert state in other regions of parameter space.

Mean field phase boundary: Additional insight can be gained by exploring interaction effects near the transition temperature $T_c$ of the spin order. Within mean field approximation, the grand potential is

$$\Omega = -\text{Th}_\text{Tr} \text{e}^{-i(H_{MF} - \mu)/T} + \langle H - H_{MF}\rangle_{MF}, \hspace{1cm} (15)$$

$$\langle H - H_{MF}\rangle_{MF} = \frac{1}{2} \text{Tr} B = \rho_{41} \mathcal{V}_{12,34}\rho_{32}. \hspace{1cm} (16)$$

These expressions can be simplified using the identity

$$\langle f, m_f; f, m_b | F, M | \gamma_{M}^{(L)} \rangle_{m_{1}, m_{2}} \langle F, M | f, m_{1} ; f, m_{a} \rangle = (-1)^{2F - F}(2F + 1)W(f f f f; F, L)\gamma_{M}^{(L)}_{m_{b}, m_{a}} (17)$$

where $W(f f f f; F, L)$ is the Racah coefficient. For simplicity, we shall simply denote it as $W(F, L)$. Eq. (14) then becomes

$$B_{14} = \sum_{L=0}^{2f} \alpha_{L} \sum_{M=-L}^{L} \Phi_{M}^{(L)}(\gamma_{M}^{(L)})_{14}, \hspace{1cm} (18)$$

$$\alpha_{L} = 2 \sum_{F=0,2,4,6,8,10} g_{F}(2F + 1)W(F, L). \hspace{1cm} (19)$$

Since $\gamma_{M}^{(L)}$ is traceless for $L \geq 1$, the mean field for spin order is given by the traceless part of $B$, denoted as $\tilde{B}$.

Near $T_c$, we can expand the thermodynamic potential in powers of the mean field $\tilde{B}$. To the lowest order in $\tilde{B}$, and retaining only the contribution of the traceless part, we have

$$\Delta \Omega = \frac{1}{2} \left( -\chi \text{Tr} \tilde{B}^2 + \text{Tr} \tilde{B}^2 \right), \hspace{1cm} (20)$$

where $\chi(T, \mu) = \frac{1}{T} \sum_{k} n_k (1 - n_k) = -\int \frac{\text{d}v D(e)}{\text{d}e} \frac{\partial f(e)}{\partial e}$, $D(e)$ is the density of state, and $f(e) = (e^{\epsilon - \mu}/T + 1)^{-1}$ is the Fermi distribution function. $\chi(T, \mu)$ increases as temperature is lowered. Evaluating the traces in Eq. (20) explicitly, we have

$$\Delta \Omega = \sum_{L=0}^{2f} \sum_{M=-L}^{L} \left[ \frac{1}{2} \frac{2F + 1}{2L + 1} \alpha_{L}(1 - \chi(T, \mu)\alpha_{L}) \right] |\Phi_{M}^{(L)}|^2. \hspace{1cm} (21)$$

Eq. (21) shows $\Phi_{M}^{(L)}$ will emerge if

(i) $\alpha_{L} > 0$, and (ii) $\chi_{AL} \geq 1$, \hspace{1cm} (22)

where the equal sign gives the phase boundary. Up to the quadratic order in $\Phi_{M}^{(L)}$, all $M$ components are degenerate. Higher order terms in $\Phi_{M}^{(L)}$ will lift the degeneracy and mix different $L$ components.

Condition (i) is necessary for self consistency. This is because for a single $\Phi_{M}^{(L)}$, $B = \alpha_{L} \Phi_{M}^{(L)} \gamma_{M}^{(L)}$, Eq. (18). In addition, the mean field term causes a linear response near $T_c$, $\Phi_{M}^{(L)} \gamma_{M}^{(L)} = \bar{B}$. Since $\chi > 0$, these two relations are consistent only when $\alpha_{L} > 0$. Condition (ii) simply means that the spin-ordered state $\Phi_{M}^{(L)}$ is of lower energy. It is useful to write (ii) in a different way. Since $\alpha_{L}$ is a sum of scattering lengths $\{a_F\}$, Eq. (19), we can define an effective scattering length $A_L$ as

$$\alpha_{L} = \frac{8\pi\hbar^2 A_L}{m}, \quad A_L = \sum_{F} a_{F}(2F + 1)W(F, L). \hspace{1cm} (23)$$

Next, we note that $\chi(T, \mu)$ has the dimension of density of state. It scales as $\sqrt{\mu}$ and hence represents a momentum scale. We can then define a wavevector $k(T, \mu)$ as $\chi(T, \mu) = (\bar{m}/2\pi^2\hbar^2)k(T, \mu)$. (Note that $\bar{m}(0, \mu)$ is the Fermi wavevector of a single component Fermi gas with chemical potential $\mu$). Condition (ii) then becomes

$$k(T, \mu)A_L \geq \frac{\pi}{4}, \quad \text{or} \quad k_{n}A_L \geq \frac{\pi}{4} k_{n}(T, \mu). \hspace{1cm} (24)$$
where the equal sign gives the phase boundary. Approaching the phase boundary from the normal state, we have \( \mu = \mathcal{E}_n(2f + 1)^{-2/3}, \mathcal{E}_n = \hbar^2 k_n^2/2m_n. \)

![Figure 2](image-url) (Color online) The transition temperature for spin order \( \Phi^{(L)} \) for \( L = 1, 2, 6, 3, 5, 4, 7 \) (from bottom to top), for a spin \( f = 21/2 \) Fermi gas.

Certainly, the larger the \( A_L \), the easier for the \( L \)-th spin order to emerge. However, for small gas parameters \( k_n a_F < 1 \), it is not clear whether Eq. (24) can be satisfied. On the other hand, one sees from Eq. (21) that \( A_L \) will be maximized if the sign of \( a_F \)'s matches that of the Racah coefficient \( W(F, L) \). To demonstrate this effect, we consider a set of \( a_F \)'s with the same magnitude \( \bar{a} \) with a sign matching that of \( W(F, L) \). Eq. (24) then becomes

\[
k_n \bar{a} \geq \frac{\pi/4}{\sum_{F=0,2} (2F + 1)|W(F, L)| k(T, \mu)} \frac{k_n}{k(T, \mu)}.
\]

This condition is plotted in Figure 2 for a spin \( f = 21/2 \) Fermi gas. It shows spin orders as high as \( \bar{a} \) can emerge at the phase boundary for \( k_n \bar{a} < 1 \). While Eq. (25) is sufficient for the appearance of \( \Phi^{(L)} \), it is not necessary. Once a low order \( \Phi^{(L)} \) is present, say, \( L = 1 \), higher \( L \)-spin order can emerge through non-linear coupling as temperature is lowered. Finally, we note from Eq. (25) that the larger the spin \( f \) of the fermions, the larger the sum in Eq. (25), and the smaller the gas parameter \( k_n \bar{a} \) needed to activate the spin order.

Concluding Remarks: Large spin quantum gases are fertile grounds for new quantum matter. Here, we have pointed out the very rich spin order possible in large spin fermions, most of which have no analog in electron matters. We show that the spin order in different sector can be conveniently described as Majorana antipodal points on a sequence of spheres representing the spin order of different particle-hole angular momenta. Our model calculations show that some of these orders can take the form of Platonic solids, which are structures that exist in certain region of the parameter space but are independent of the exact values of the interaction parameters. These structures are therefore robust and will have good chance to be realized.

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\( |\Phi^{(I)}\rangle \rightarrow -|\Phi^{(I)}\rangle \) as \( \theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi \).
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\( |\Phi^{(I)}\rangle \rightarrow -|\Phi^{(I)}\rangle \) as \( \theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi \).
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[13] (I)=(0.1, 0.3, 0.4, 0.45, 0.5, 0.5, 0.45, 0.45, -0.3, -0.5, -0.5), (II)=(-0.1, -0.6, -0.6, 0.55, 0.75, 0.75, 0.45, -0.75, -0.8, 0.8, 0.8), (III)=(-0.6, -0.74, 0.87, 0.79, 0.84, -0.8, -0.83, 0.82, 0.82, -0.85, 0.85). The row vector stands for \( (k_n a_0, k_n a_2, k_n a_4, ..., k_n a_{20}) \).