Parameter rigid actions of simply connected nilpotent Lie groups

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Abstract. We show that for a locally free $C^\infty$-action of a connected and simply connected nilpotent Lie group on a compact manifold, if every real-valued cocycle is cohomologous to a constant cocycle, then the action is parameter rigid. The converse is true if the action has a dense orbit. Using this, we construct parameter rigid actions of simply connected nilpotent Lie groups whose Lie algebras admit rational structures with graduations. This generalizes the results of dos Santos [Parameter rigid actions of the Heisenberg groups. Ergod. Th. & Dynam. Sys. 27 (2007), 1719–1735] concerning the Heisenberg groups.

1. Introduction

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $M$ a $C^\infty$-manifold without boundary. Let $\rho: M \times G \to M$ be a $C^\infty$ right action. We call $\rho$ locally free if every isotropy subgroup of $\rho$ is discrete in $G$. Assume that $\rho$ is locally free. Then we have the orbit foliation $\mathcal{F}$ of $\rho$ whose tangent bundle $T\mathcal{F}$ is naturally isomorphic to a trivial bundle $M \times \mathfrak{g}$.

The action $\rho$ is parameter rigid if any action $\rho'$ of $G$ on $M$ with the same orbit foliation $\mathcal{F}$ is $C^\infty$-conjugate to $\rho$; more precisely, there exist an automorphism $\Phi$ of $G$ and a $C^\infty$-diffeomorphism $F$ of $M$ which preserves each leaf of $\mathcal{F}$ and is homotopic to the identity through $C^\infty$-maps preserving each leaf of $\mathcal{F}$ such that

$$F(\rho(x, g)) = \rho'(F(x), \Phi(g))$$

for all $x \in M$ and $g \in G$.

Parameter rigidity of actions has been studied by several authors, for instance, Katok and Spatzier [3], Matsumoto and Mitsumatsu [4], Mieczkowski [5], Ramírez [7] and dos Santos [8]. Most of the known examples of parameter rigid actions are those of abelian groups, and non-abelian actions have not been considered so much.

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Parameter rigidity is closely related to cocycles over actions. Now suppose $G$ is contractible and $M$ is compact. Let $H$ be a Lie group. A $C^\infty$-map $c : M \times G \to H$ is called a $H$-valued cocycle over $\rho$ if $c$ satisfies
\[ c(x, gg') = c(x, g)c(\rho(x, g), g') \]
for all $x \in M$ and $g, g' \in G$.

A cocycle $c$ is constant if $c(x, g)$ is independent of $x$. A constant cocycle is just a homomorphism $G \to H$.

$H$-valued cocycles $c, c'$ are cohomologous if there exists a $C^\infty$-map $P : M \to H$ such that
\[ c(x, g) = P(x)^{-1}c'(x, g)P(\rho(x, g)) \]
for all $x \in M$ and $g \in G$.

The action $\rho$ is $H$-valued cocycle rigid if every $H$-valued cocycle over $\rho$ is cohomologous to a constant cocycle.

**Proposition 1.** If $\rho$ is $G$-valued cocycle rigid, then it is parameter rigid.

**Remark.** In [4] Matsumoto and Mitsumatsu assume that $\rho$ has at least one trivial isotropy subgroup, but this assumption is not necessary.

**Proposition 2.** When $G = \mathbb{R}^n$, the following are equivalent:

1. $\rho$ is $\mathbb{R}$-valued cocycle rigid;
2. $\rho$ is $\mathbb{R}^n$-valued cocycle rigid;
3. $\rho$ is parameter rigid.

**Remark.** The equivalence of the first two conditions is obvious.

In this paper we consider actions of simply connected nilpotent Lie groups. In [8], dos Santos proved that, for actions of a Heisenberg group $H_n$, $\mathbb{R}$-valued cocycle rigidity implies $H_n$-valued cocycle rigidity, and, using this, he constructed parameter rigid actions of Heisenberg groups. To the best of my knowledge these are the only known non-trivial parameter rigid actions of non-abelian nilpotent Lie groups. We prove the following.

**Theorem 1.** Let $N$ be a connected and simply connected nilpotent Lie group, $M$ a compact manifold and $\rho$ a locally free $C^\infty$-action of $N$ on $M$. Then, the following are equivalent:

1. $\rho$ is $\mathbb{R}$-valued cocycle rigid;
2. $\rho$ is $N$-valued cocycle rigid;
3. $\rho$ is parameter rigid and every orbitwise constant real-valued $C^\infty$-function of $\rho$ on $M$ is constant on $M$.

This theorem enables us to construct parameter rigid actions of nilpotent Lie groups. The most interesting one is the following.

**Theorem 2.** Let $N$ denote the group of all upper triangular real matrices with 1 on the diagonal, $\Gamma$ a cocompact lattice of $\text{SL}(n, \mathbb{R})$ and $\rho$ the action of $N$ on $\Gamma \backslash \text{SL}(n, \mathbb{R})$ by right multiplication. If $n \geq 4$, $\rho$ is $\mathbb{R}$-valued cocycle rigid.
Remark. In [7], Ramírez proved more general theorems.

**Corollary.** The above action $\rho$ is parameter rigid.

In §4 we construct parameter rigid actions of nilpotent groups using Theorem 1. It is a generalization of dos Santos’ example. Let $N$ be a connected and simply connected nilpotent Lie group and $\Gamma$, $\Lambda$ be lattices in $N$. Consider the action of $\Lambda$ on $\Gamma \backslash N$ by right multiplication and let $\tilde{\rho}$ be its suspended action of $N$.

**Theorem 3.** If $\Lambda$ is Diophantine with respect to $\Gamma$, then the action $\tilde{\rho}$ of $N$ is parameter rigid.

For the definition of Diophantine lattices, see §4.

2. Preliminaries

Let $G$ be a contractible Lie group with Lie algebra $\mathfrak{g}$, $M$ a compact manifold and $\rho$ a locally free action of $G$ on $M$ with orbit foliation $\mathcal{F}$. Let $H$ be a Lie group with Lie algebra $\mathfrak{h}$. We denote by $\Omega^p(\mathcal{F}, \mathfrak{h})$ the set of all $C^\infty$-sections of $\text{Hom}(\wedge^p T\mathcal{F}, \mathfrak{h})$. The exterior derivative

$$d_\mathcal{F} : \Omega^p(\mathcal{F}, \mathfrak{h}) \to \Omega^{p+1}(\mathcal{F}, \mathfrak{h})$$

is defined since $T\mathcal{F}$ is integrable.

By differentiating, $H$-valued cocycles over $\rho$ are in one-to-one correspondence with $\mathfrak{h}$-valued leafwise one-forms $\omega \in \Omega^1(\mathcal{F}, \mathfrak{h})$ such that

$$d_\mathcal{F}\omega + [\omega, \omega] = 0.$$

**Proposition 3.** Let $c_1$, $c_2$ be $H$-valued cocycles over $\rho$ and let $\omega_1$, $\omega_2$ be corresponding differential forms. For a $C^\infty$-map $P : M \to H$, the following are equivalent:

1. $c_1(x, g) = P(x)^{-1}c_2(x, g)P(\rho(x, g))$ for all $x \in M$ and $g \in G$;
2. $\omega_1 = \text{Ad}(P^{-1})\omega_2 + P^*\theta$ where $\theta \in \Omega^1(H, \mathfrak{h})$ is the left Maurer–Cartan form on $H$.

**Corollary.** [4] The following are equivalent:

1. $\rho$ is $G$-valued cocycle rigid;
2. for each $\omega \in \Omega^1(\mathcal{F}, \mathfrak{g})$ such that $d_\mathcal{F}\omega + [\omega, \omega] = 0$, there exist an endomorphism $\Phi : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra and a $C^\infty$-map $P : M \to G$ such that

$$\omega = \text{Ad}(P^{-1})\Phi + P^*\theta.$$

Proposition 3 is obtained by examining the proof of [4, Corollary 2]. In this paper, we will identify a cocycle with its corresponding differential form.

Let us consider real-valued cocycles. A real-valued cocycle over $\rho$ is given by $\omega \in \Omega^1(\mathcal{F}, \mathbb{R})$ satisfying $d_\mathcal{F}\omega = 0$. Two real-valued cocycles $\omega_1$, $\omega_2$ are cohomologous if and only if $\omega_1 = \omega_2 + d_\mathcal{F}P$ for some $C^\infty$-function $P : M \to \mathbb{R}$. Leafwise cohomology $H^*(\mathcal{F})$ of $\mathcal{F}$ is the cohomology of the cochain complex $(\Omega^*(\mathcal{F}, \mathbb{R}), d_\mathcal{F})$. Thus $H^1(\mathcal{F})$ is the set of all equivalence classes of real-valued cocycles.

The identification $T\mathcal{F} \simeq M \times \mathfrak{g}$ induces a map $H^*(\mathfrak{g}) \to H^*(\mathcal{F})$ where $H^*(\mathfrak{g})$ is the cohomology of the Lie algebra $\mathfrak{g}$. By the compactness of $M$, this map is injective on $H^1(\mathfrak{g})$. Hence we identify $H^1(\mathfrak{g})$ with its image. Note that $H^1(\mathfrak{g})$ is the set of all equivalence
classes of constant real-valued cocycles. Thus real-valued cocycle rigidity is equivalent to $H^1(\mathcal{F}) = H^1(\mathfrak{g})$.

Notice that $H^0(\mathcal{F})$ is the set of leafwise constant real-valued $C^\infty$-functions of $\mathcal{F}$ on $M$ and $H^0(\mathfrak{g})$ consists of constant functions on $M$. Therefore the equivalence of (1) and (3) in Theorem 1 can be stated as follows: $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ if and only if $\rho$ is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$.

3. Proof of Theorem 1

Let $N$ be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, $M$ a compact manifold and $\rho$ a locally free action of $N$ on $M$ with orbit foliation $\mathcal{F}$.

We first prove that $N$-valued cocycle rigidity implies real-valued cocycle rigidity. There exist closed subgroups $N'$ and $A$ of $N$ such that $N' \triangleleft N$, $N = N' \rtimes A$ and $A \cong \mathbb{R}$. Let $c$ be any real-valued cocycle over $\rho$. We regard $c$ as an $N$-valued cocycle over $\rho$ via the inclusion $\mathbb{R} \cong A \hookrightarrow N$. By $N$-valued cocycle rigidity, there exist an endomorphism $\Phi$ of $N$ and a $C^\infty$-map $P : M \to N$ such that $c(x, g) = P(x)^{-1}\Phi(g)P(\rho(x, g))$ for all $x \in M$ and $g \in N$. Applying the natural projection $\pi : N \to A \cong \mathbb{R}$, we obtain $c(x, g) = (\pi \circ P)(x)^{-1}(\pi \circ \Phi)(\pi \circ P)(\rho(x, g))$. Thus $c$ is cohomologous to a constant cocycle $\pi \circ \Phi$.

Next we assume $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ and prove $N$-valued cocycle rigidity. We need the following two lemmata.

**Lemma 1.** Let $V$ be a finite dimensional real vector space. Assume that $\omega \in \Omega^1(\mathcal{F}, V)$ satisfies the equation $d_\mathcal{F} \omega = \varphi$, where $\varphi \in \text{Hom}(\bigwedge^2 \mathfrak{n}, V)$ is a constant leafwise two-form. Then there exists a constant leafwise one-form $\psi \in \text{Hom}(\mathfrak{n}, V)$ with $\varphi = d_\mathcal{F} \psi$.

**Proof.** Since $N$ is amenable, there exists an $N$-invariant Borel probability measure $\mu$ on $M$. Define $\psi \in \text{Hom}(\mathfrak{n}, V)$ by

$$\psi(X) = \int_M \omega(X) \, d\mu,$$

where $X \in \mathfrak{n}$. Since $\varphi(X, Y) = X \omega(Y) - Y \omega(X) - \omega([X, Y])$ for all $X, Y \in \mathfrak{n}$, we obtain

$$\varphi(X, Y) = -\int_M \omega([X, Y]) \, d\mu.$$

Thus

$$d_\mathcal{F} \psi(X, Y) = -\psi([X, Y]) = -\int_M \omega([X, Y]) \, d\mu = \varphi(X, Y),$$

and hence $d_\mathcal{F} \psi = \varphi$. \hfill \Box

Set $\mathfrak{n}^1 = \mathfrak{n}$, $\mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}]$. Then $\mathfrak{n}^s \neq 0$, $\mathfrak{n}^{s+1} = 0$ for some $s$. For each $1 \leq i \leq s$, choose a subspace $V_i$ with $\mathfrak{n}^i = V_i \oplus \mathfrak{n}^{i+1}$, so that $\mathfrak{n} = \bigoplus_{i=1}^s V_i$.

**Lemma 2.** Let $\omega \in \Omega^1(\mathcal{F}, \mathfrak{n})$ be such that $d_\mathcal{F} \omega + [\omega, \omega] = 0$. Decompose $\omega$ as

$$\omega = \xi + \omega_k + \omega_{k+1},$$

where $\xi \in \Omega^1(\mathcal{F}, \bigoplus_{i=1}^{k-1} V_i)$, $\omega_k \in \Omega^1(\mathcal{F}, V_k)$ and $\omega_{k+1} \in \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})$. If $\xi$ is constant, then there exists $\omega' \in \Omega^1(\mathcal{F}, \mathfrak{n})$ with $d_\mathcal{F} \omega' + [\omega', \omega'] = 0$ which is cohomologous to $\omega$ and
Comparing this with the $\omega$ decompose $\eta$ canonical one-form is rigidity.

Proof. By the cocycle equation,

$$0 = d\varphi \xi + d\varphi \omega_k + d\varphi \omega_{k+1} + [\xi, \xi] + \text{an element of } \Omega^2(\mathcal{F}, n^{k+1}).$$

Comparing this with the $V_k$ component, we see that $d\varphi \omega_k$ is constant. Hence, by Lemma 1, $d\varphi \omega_k = d\varphi \psi$ for some $\psi \in \text{Hom}(n, V_k)$. Since we are assuming that $H^1(\mathcal{F}) = H^1(n)$, there exists $\psi' \in \text{Hom}(n, V_k)$ and a $C^\infty$-map $h : M \to V_k$ such that

$$\omega_k = \psi + \psi' + d\varphi h.$$  

Put $P = e^h : M \to N$. Let $x \in M$ and $X \in T_x \mathcal{F}$. Choose a path $x(t)$ such that $X = \left((d/dt)x(t)\right)_{t=0}$. Let $\theta \in \Omega^1(N, n)$ be the left Maurer–Cartan form on $N$. Then

$$P^* \theta(X) = \left. \frac{d}{dt} P(x)^{-1} P(x(t)) \right|_{t=0} = \left. \frac{d}{dt} e^{-h(x)} e^{h(x(t))} \right|_{t=0}
= \left. \frac{d}{dt} \exp(-h(x) + h(x(t))) + \text{an element of } n^{k+1} \right|_{t=0}
= d\varphi h(X) + \text{an element of } n^{k+1}.$$

Thus $P^* \theta = d\varphi h$ + an element of $\Omega^1(\mathcal{F}, n^{k+1})$. Note that $\text{Ad}(P^{-1}) = \exp \text{ad}(-h)$ is the identity on $\bigoplus_{i=1}^k V_i$ and preserves $n^{k+1}$. Hence

$$\omega - P^* \theta = \xi + \psi + \psi' + \text{an element of } \Omega^1(\mathcal{F}, n^{k+1})
= \text{Ad}(P^{-1})(\xi + \psi + \psi' + \text{an element of } \Omega^1(\mathcal{F}, n^{k+1})).$$

Let $\omega$ be any $N$-valued cocycle. Using Lemma 2, we can exchange $\omega$ for a cohomologous cocycle whose $V_1$-component is constant. Applying Lemma 2 repeatedly, we eventually get a constant cocycle cohomologous to $\omega$. This proves $N$-valued cocycle rigidty.

Next we assume that $\rho$ is parameter rigid and $H^0(\mathcal{F}) = H^0(n)$. Let $n'$ and $V_i$ be as above. Note that $n'$ is central in $n$. Fix a non-zero $Z \in n'$.

Let $[\omega] \in H^1(\mathcal{F})$. Let $\omega_0$ be the $N$-valued cocycle over $\rho$ corresponding to the constant cocycle id : $N \to N$. We call $\omega_0$ the canonical one-form of $\rho$. Fix an $\epsilon > 0$ and put $\eta := \omega_0 + \epsilon \omega Z$. Then $\eta$ is an $N$-valued cocycle over $\rho$, since

$$d\varphi \eta + [\eta, \eta] = d\varphi \omega_0 + \epsilon (d\varphi \omega) Z + [\omega_0, \omega_0] = 0.$$  

Since $M$ is compact, we can assume $\eta_x : T_x \mathcal{F} \to n$ is bijective for all $x \in M$ by choosing $\epsilon > 0$ small. There exists a unique action $\rho'$ of $N$ on $M$ whose orbit foliation is $\mathcal{F}$ and canonical one-form is $\eta$. See [1]. By parameter rigidity, $\rho'$ is conjugate to $\rho$. Thus there exist a $C^\infty$-map $P : M \to N$ and an automorphism $\Phi$ of $N$ satisfying

$$\omega_0 + \epsilon \omega Z = \text{Ad}(P^{-1}) \Phi_\star \omega_0 + P^* \theta.$$  

1. Note that $\log : N \to n$ is defined since $N$ is simply connected and nilpotent. Let us decompose $\omega_0 = \sum_{i=1}^{s_i} \omega_{0i}$, $\Phi_\star \omega_0 = \sum_{i=1}^{s_i} \omega'_{0i}$ and $\log P = \sum_{i=1}^{s_i} P_i$ according to the decomposition $n = \bigoplus_{i=1}^{s_i} V_i.$
Lemma 3. Assume that $P_1 = \cdots = P_{k-1} = 0$; that is, $\log P \in \mathfrak{n}^k$.

1. If $k < s$, then there exist a $C^\infty$-map $Q : M \to N$ and an automorphism $\Psi$ of $N$ such that

$$\omega_0 + \epsilon \omega Z = \text{Ad}(Q^{-1})\Psi_* \omega_0 + Q^* \theta$$

and $Q_1 = \cdots = Q_k = 0$, where $\log Q = \sum_{i=1}^s Q_i$.

2. If $k = s$, then $\omega$ is cohomologous to a constant cocycle.

Proof. For all $X = (d/dt)x(t)\big|_{t=0} \in T_x \mathcal{F}$,

$$P^* \theta(X) = \frac{d}{dt} P(x(t))^{-1} P(x(t)) \bigg|_{t=0} = \frac{d}{dt} \exp \left( - \sum_{i=k}^s P_i(x(t)) \right) \exp \left( \sum_{i=k}^s P_i(x(t)) \right) \bigg|_{t=0}$$

$$= \frac{d}{dt} \exp \left\{ \sum_{i=k}^s (P_i(x(t)) - P_i(x)) + \text{an element of } \mathfrak{n}^{k+1} \right\} \bigg|_{t=0}$$

$$= \frac{d}{dt} \exp(P_k(x(t)) - P_k(x) + \text{an element of } \mathfrak{n}^{k+1}) \bigg|_{t=0}$$

$$= d_{\mathcal{F}} P_k(X) + \text{an element of } \mathfrak{n}^{k+1}.$$ 

We have

$$\text{Ad}(P^{-1})\Phi_* \omega_0 = \exp \left( \text{ad} \left( - \sum_{i=k}^s P_i \right) \right) \sum_{i=1}^s \omega'_0$$

$$= \sum_{i=1}^s \omega'_0 + \text{an element of } \mathfrak{n}^{k+1}.$$

Comparing this with the $V_k$-component of (1) we get

$$\omega_0 + \delta_{k,s} \epsilon \omega Z = \omega'_0 + d_{\mathcal{F}} P_k.$$

When $k = s$, the equation

$$\omega Z = \epsilon^{-1} (\omega'_0 - \omega_0) + d_{\mathcal{F}} (\epsilon^{-1} P_s)$$

shows that $\omega$ is cohomologous to a constant cocycle.

If $k < s$, then $d_{\mathcal{F}} P_k = \phi \circ \omega_0$ for some linear map $\phi : \mathfrak{n} \to V_k$. For any $X \in \mathfrak{n}$, let $\tilde{X}$ denote the vector field on $M$ determined by $X$ via $\rho$. We have $\tilde{X} P_k = \phi(X)$, and by integrating over an integral curve $\gamma$ of $\tilde{X}$ we get $P_k(\gamma(T)) - P_k(\gamma(0)) = \phi(X)T$ for all $T > 0$. Since $M$ is compact, $\phi(X) = 0$. Therefore $d_{\mathcal{F}} P_k = 0$, so $P_k$ is constant on each leaf of $\mathcal{F}$. Thus $P_k$ is constant on $M$ by our assumption. Put $g := \exp(-P_k)$ and $Q := gP = \exp(\sum_{i=k+1}^s P_i + \text{an element of } \mathfrak{n}^{k+1})$. Then

$$\omega_0 + \epsilon \omega Z = \text{Ad}(Q^{-1})g \Phi_* \omega_0 + (L_{g^{-1}} \circ Q)^* \theta$$

$$= \text{Ad}(Q^{-1})\Psi_* \omega_0 + Q^* \theta$$

where $\Psi_* := \text{Ad}(g) \Phi_*$. 

Applying Lemma 3 repeatedly, we see that $\omega$ is cohomologous to a constant cocycle.
Finally we assume $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ and prove that $\rho$ is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$. Parameter rigidity of $\rho$ follows from Proposition 1. Let $f \in H^0(\mathcal{F})$. Fix a non-zero $\phi \in H^1(\mathfrak{n})$ and denote the corresponding leafwise one-form on $M$ by $\tilde{\phi}$. Then $f \tilde{\phi} \in H^1(\mathcal{F}) = H^1(\mathfrak{n})$. Thus there exist $\psi \in H^1(\mathfrak{n})$ and a $C^\infty$-function $g : M \rightarrow \mathbb{R}$ such that $f \tilde{\phi} - \tilde{\psi} = dxg$ where $\tilde{\psi}$ is the leafwise one-form corresponding to $\psi$. Choose $X \in \mathfrak{n}$ satisfying $\phi(X) \neq 0$. Let $x(t)$ be an integral curve of $\tilde{X}$ where $\tilde{X}$ is the vector field corresponding to $X$. We have

$$f(x(t))\phi(X) = \tilde{x}_X g = \frac{d}{dt}g(x(t)).$$

By integrating over $[0, T]$, we get $T(\psi(x(0)))\phi(X) - \psi(X)) = g(x(T)) - g(x(0))$. Since $g$ is bounded, $f(x(0))\phi(X) - \psi(X)$ must be zero. Then $f(x(0)) = \psi(X)/\phi(X)$ and $f$ is constant on $M$.

This completes the proof of Theorem 1. \hfill $\square$

4. A construction of parameter rigid actions

Let us now construct real-valued cocycle rigid actions of nilpotent groups. For the structure theory of nilpotent Lie groups, see [2]. A basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ is called a strong Malcev basis if $\text{span}_{\mathbb{R}} \{X_1, \ldots, X_i \}$ is an ideal of $\mathfrak{n}$ for each $i$. If $\Gamma$ is a lattice in $N$, there exists a strong Malcev basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ such that $\Gamma = e^{\mathbb{Z}X_1} \ldots e^{\mathbb{Z}X_n}$. Such a basis is called a strong Malcev basis strongly based on $\Gamma$. Let $\Gamma$ and $\Lambda$ be lattices in $N$.

Definition 1. $\Lambda$ is Diophantine with respect to $\Gamma$ if there exists a strong Malcev basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ strongly based on $\Gamma$ and a strong Malcev basis $Y_1, \ldots, Y_n$ of $\mathfrak{n}$ strongly based on $\Lambda$ such that $Y_i = \sum_{j=1}^{i} a_{ij}X_j$ for every $1 \leq i \leq n$, where $a_{ii}$ is Diophantine.

Let $\rho$ be the action of $\Lambda$ on $\Gamma \backslash N$ by right multiplication. First we will prove the following.

Theorem 4. If $\Lambda$ is Diophantine with respect to $\Gamma$, then every real-valued $C^\infty$ cocycle $c : \Gamma \backslash N \times \Lambda \rightarrow \mathbb{R}$ over $\rho$ is cohomologous to a constant cocycle.

Proof. Note that $X_1$ is in the center of $\mathfrak{n}$. Let $\pi : N \rightarrow \tilde{N} := e^{\mathbb{R}X_1} \backslash N$ be the projection. Since $\Gamma \cap e^{\mathbb{R}X_1} = e^{\mathbb{Z}X_1}$ is a cocompact lattice in $e^{\mathbb{R}X_1}$, $\tilde{\Gamma} := \pi(\Gamma) = e^{\mathbb{R}X_1} \backslash e^{\mathbb{R}X_1}$ is a cocompact lattice in $\tilde{N}$. Let $\tilde{n} = \mathbb{R}X_1 \backslash \mathfrak{n}$. Then $\tilde{X}_2, \ldots, \tilde{X}_n$ is a strong Malcev basis of $\tilde{n}$ strongly based on $\tilde{\Gamma}$.

We will see that the naturally induced map $\tilde{\pi} : \Gamma \backslash N \rightarrow \tilde{\Gamma} \backslash \tilde{N}$ is a principal $S^1$-bundle. Indeed,

$$\Gamma \backslash e^{\mathbb{R}X_1} \hookrightarrow \Gamma \backslash N \rightarrow e^{\mathbb{R}X_1} \backslash N$$

is a principal $\Gamma \backslash e^{\mathbb{R}X_1}$-bundle, and we have

$$\Gamma \backslash e^{\mathbb{R}X_1} \simeq \Gamma \cap e^{\mathbb{R}X_1} \backslash e^{\mathbb{R}X_1} = e^{\mathbb{Z}X_1} \backslash e^{\mathbb{R}X_1} \simeq \mathbb{Z} \backslash \mathbb{R}$$

and the following diagram.

$$
\begin{array}{ccc}
ge^{\mathbb{R}X_1} \backslash \Gamma & \longrightarrow & e^{\mathbb{R}X_1} \backslash N \\
\longrightarrow & \Gamma \backslash N
\end{array}
$$
We prove this by induction on $n$. When $n$ is constant, by (2), $g$ assume that $c$ to a constant cocycle $\int$ the action of $e \in R$.

Proof. We use induction on $n$. When restricted to $e \in R$, preserves fibers of $\tilde{\pi}$. Let $z \in \tilde{\Gamma} \setminus \tilde{N}$. Choose a point $x$ in $\tilde{\pi}^{-1}(z)$. Then we have a trivialization $t_{\Gamma z} : \mathbb{Z} \setminus \mathbb{R} \simeq \tilde{\pi}^{-1}(z)$ of $\tilde{\pi}^{-1}(z)$ given by $t_{\Gamma z}(x) = \Gamma e^{sX1}x$. Note that if we take another point $\Gamma y \in \tilde{\pi}^{-1}(z)$, $t_{\Gamma y} \circ t_{\Gamma x} : \mathbb{Z} \setminus \mathbb{R} \rightarrow \mathbb{Z} \setminus \mathbb{R}$ is a rotation.

Let $Y_1 = aX_1$, where $a$ is Diophantine. If we identify $\tilde{\pi}^{-1}(z)$ with $\mathbb{Z} \setminus \mathbb{R}$ by $t_{\Gamma x}$, then the action of $e^{Y_1} \in \mathbb{Z} \setminus \mathbb{R}$ is $s \mapsto s + a$.

Let $\mu$, be the normalized Haar measure naturally defined on $\tilde{\pi}^{-1}(z)$, $\mu$ the $N$-invariant probability measure on $\Gamma \setminus N$ and $\nu$ the $\tilde{N}$-invariant probability measure on $\tilde{\Gamma} \setminus \tilde{N}$. For any $f \in C(\Gamma \setminus N)$,

$$\int_{\Gamma \setminus N} f \, d\mu = \int_{\tilde{\Gamma} \setminus \tilde{N}} \int_{\tilde{\pi}^{-1}(z)} f \, d\mu_z \, d\nu. \quad (2)$$

Lemma 4. $\rho$ is ergodic with respect to $\mu$.

Proof. We use induction on $n$. If $n = 1$, $\rho$ is an irrational rotation on $\mathbb{Z} \setminus \mathbb{R}$, and hence the result is well known. In general, let $f : \tilde{\Gamma} \setminus \tilde{N} \rightarrow \mathbb{C}$ be a $\Lambda$-invariant $L^2$-function with $\int_{\Gamma \setminus N} f \, d\mu = 0$. Since the action of $e^{Y_1} \in \tilde{\pi}^{-1}(z)$ is ergodic, $f|_{\tilde{\pi}^{-1}(z)}$ is constant $\mu_z$-almost everywhere. We denote this constant by $g(z)$. Then $g : \tilde{\Gamma} \setminus \tilde{N} \rightarrow \mathbb{C}$ is a $\Lambda$-invariant measurable function. By induction, $g$ is constant $\nu$-almost everywhere. By (2), this constant must be zero. Therefore $f$ is zero $\mu$-almost everywhere.

Let $c : \Gamma \setminus N \times \Lambda \rightarrow \mathbb{R}$ be a $C^\infty$-cocycle over $\rho$. We must show that $c$ is cohomologous to a constant cocycle $\bar{c} : \Lambda \rightarrow \mathbb{R}$ where $\bar{c}(\lambda) := \int_{\Gamma \setminus N} c(x, \lambda) \, d\mu(x)$. Therefore we may assume that $\int_{\Gamma \setminus N} c(x, \lambda) \, d\mu(x) = 0$ for all $\lambda \in \Lambda$, and we will show that $c$ is a coboundary. We prove this by induction on $n$. When $n = 1$, $\rho$ is a Diophantine rotation on $\mathbb{Z} \setminus \mathbb{R}$, and hence the result is well known.

Lemma 5. For all $m \in \mathbb{Z}$,

$$\int_{\tilde{\pi}^{-1}(z)} c(s, e^{mY_1}) \, d\mu_z(s) = 0.$$

Proof. Fix $m$ and put $g(z) = \int_{\tilde{\pi}^{-1}(z)} c(s, e^{mY_1}) \, d\mu_z(s)$. For any $\lambda \in \Lambda$, the cocycle equation gives $c(x, \lambda) + c(x, e^{mY_1}) = c(x, e^{mY_1}) + c(x e^{mY_1}, \lambda)$. By integrating this equation on $\tilde{\pi}^{-1}(z)$, we get $g(z \pi(\lambda)) = g(z)$. Since the action of $\tilde{\Lambda}$ on $\tilde{\Gamma} \setminus \tilde{N}$ is ergodic, $g$ is constant. By (2), $g$ must be zero.

Let $f : \mathbb{Z} \setminus \mathbb{R} \xrightarrow{t_{\Gamma z}} \tilde{\pi}^{-1}(z) \xrightarrow{c(\cdot e^{Y_1})} \mathbb{R}$. We define

$$h_z(t_{\Gamma z}(s)) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{1 + e^{2\pi i k a}} e^{2\pi i k s}.$$
Then \( h_z : \tilde{\pi}^{-1}(z) \to \mathbb{R} \) is \( C^\infty \), since \( f \) is \( C^\infty \) and \( a \) is Diophantine. By Lemma 5, we have

\[
c(t_{\Gamma x}(s), e^Y) = -h_z(t_{\Gamma x}(s)) + h_z(t_{\Gamma x}e^Y).\]

If we choose another point \( \Gamma e^{\sigma X_1}x \in \pi^{-1}(z) \) to define \( h_z \), then

\[
h_z(t_{\Gamma x}(s)) = h_z(\Gamma e^{sX_1}x) = h_z(t_{\Gamma e^{\sigma X_1}x}(s-s_0))
= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{(1 + e^{2\pi i ka})} \int_0^1 c(\Gamma e^{(u+s_0)X_1}x, e^Y)e^{-2\pi i ku} du e^{2\pi ik(s-s_0)}
= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{(1 + e^{2\pi i ka})} \int_0^1 f(u+s_0)e^{-2\pi i ku} du e^{2\pi ik(s-s_0)}
= \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(k) e^{2\pi ik s},
\]

so \( h_z \) is determined only by \( z \). Define \( h : \Gamma \setminus N \to \mathbb{R} \) by \( h|_{\pi^{-1}(z)} = h_z \). Then for all \( x \in \Gamma \setminus N \) and \( m \in \mathbb{Z} \), \( c(x, e^{mY}) = -h(x) + h(xe^{mY}) \).

Let \( U \subset \Gamma \setminus \tilde{N} \) be open and \( \sigma : U \to \tilde{\pi}^{-1}(U) \) a section of \( \tilde{\pi} \). Then we have a trivialization \( \mathbb{Z} \setminus \mathbb{R} \times U \simeq \tilde{\pi}^{-1}(U) \) which sends \((s, z)\) to \( t_{\sigma(z)}(s) = \Gamma e^{sX_1}\sigma(z) \). Hence

\[
h(t_{\sigma(z)}(s)) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{(1 + e^{2\pi i ka})} \int_0^1 c(t_{\sigma(z)}(u), e^Y)e^{-2\pi i ku} du e^{2\pi ik s}
\]
on \( \tilde{\pi}^{-1}(U) \). The following lemma shows \( h \) is \( C^\infty \) on \( \Gamma \setminus N \).

**Lemma 6.** Let \( U \subset \mathbb{R}^n \) be open and let \( f : \mathbb{Z} \setminus \mathbb{R} \times U \to \mathbb{R} \) be a \( C^\infty \)-function. Define

\[
h(s, z) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{(1 + e^{2\pi i ka})} \hat{f_z}(k) e^{2\pi ik s},
\]

where \( f_z(u) = f(u, z) \). Then \( \mathbb{Z} \setminus \mathbb{R} \times U \to \mathbb{R} \) is \( C^\infty \).

**Proof.** Let \( V \) be open such that \( \tilde{V} \subset U \) and \( \tilde{V} \) is compact. We will show that \( h \) is \( C^\infty \) on \( \mathbb{Z} \setminus \mathbb{R} \times V \). Choose constants \( C, \alpha > 0 \) such that \(|-1 + e^{2\pi i ka}| \geq C|k|^{-\alpha} \) for all \( k \in \mathbb{Z} \setminus \{0\} \).

We will first prove that \( h \) is continuous. Since for any \( m \in \mathbb{Z}_{>0} \),

\[
\frac{\partial^m f_z}{\partial s^m}(s) = \sum_{k \in \mathbb{Z}} (2\pi i k)^m \hat{f_z}(k) e^{2\pi ik s}
\]
in \( L^2(\mathbb{Z} \setminus \mathbb{R}) \),

\[
\left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2^2 = \sum_{k \in \mathbb{Z}} |(2\pi i k)^m \hat{f_z}(k)|^2 
\geq (2\pi)^{2m} |k|^{2m} |\hat{f_z}(k)|^2 
\geq |k|^{2m} |\hat{f_z}(k)|^2.
\]

Since

\[
\left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2 = \left( \int_0^1 \left| \frac{\partial^m f_z}{\partial s^m}(s, z) \right|^2 ds \right)^{1/2}
\]
is continuous in $z$, there exists $M > 0$ such that $\| \partial^m f \zeta / \partial z^m \|_2 < M$ for every $z \in \tilde{V}$. Hence, for all $k \in \mathbb{Z}$ and $z \in \tilde{V}$, $|k|^m |\hat{f}_\zeta (k)| \leq M$. Therefore, for any $z \in \tilde{V}$,

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \left| \frac{1}{1 + e^{2\pi ika}} \hat{f}_\zeta (k) e^{2\pi iks} \right| \leq C^{-1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k|^2} |k|^a |\hat{f}_\zeta (k)| \leq C^{-1} M \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k|^2} < \infty.$$  

This implies continuity of $h$ on $\mathbb{Z} \setminus \mathbb{R} \times \tilde{V}$.

We have

$$\frac{\partial h}{\partial s}(s, z) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{2\pi ik}{-1 + e^{2\pi ika}} \hat{f}_\zeta (k) e^{2\pi iks}.$$  

Thus a similar argument shows that $\partial h / \partial s$ is continuous.

Let $z = (z_1, \ldots, z_n)$. For any $z \in \tilde{V}$,

$$\left| \frac{\partial}{\partial z_j} \left( \frac{1}{-1 + e^{2\pi ika}} \hat{f}_\zeta (k) e^{2\pi iks} \right) \right| \leq \left| \frac{1}{-1 + e^{2\pi ika}} \frac{\partial \hat{f}_\zeta}{\partial z_j} (\cdot, z)(k) e^{2\pi iks} \right| \leq C^{-1} \frac{1}{|k|^2} |k|^a |\hat{f}_\zeta (k)| \leq C^{-1} M \frac{1}{|k|^2} \in L^1(\mathbb{Z} \setminus \{0\}).$$

Thus

$$\frac{\partial h}{\partial z_j}(s, z) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi ika}} \frac{\partial \hat{f}_\zeta}{\partial z_j} (\cdot, z)(k) e^{2\pi iks}.$$  

Hence $\partial h / \partial z_j$ is continuous by an argument similar to those above. For higher derivatives of $h$, continue this procedure.

Set $c_1(x, \lambda) = c(x, \lambda) + h(x) - h(x\lambda)$. Then $c_1 : \Gamma \setminus \mathbb{N} \times \Lambda \to \mathbb{R}$ is a $C^\infty$-cocycle and $c_1(x, e^{my_1}) = 0$. Thus for any $\lambda \in \Lambda$, the cocycle equation implies $c_1(x, \lambda) = c_1(xe^{y_1}, \lambda)$. Since the action of $e^{y_1}$ on $\tilde{\pi}^{-1}(z)$ is ergodic, $c_1(x, \lambda)$ is constant on $\tilde{\pi}^{-1}(z)$. Therefore we can define a cocycle $\tilde{c} : \Gamma \setminus \tilde{N} \times \tilde{\Lambda} \to \mathbb{R}$ by $\tilde{c}(\tilde{\pi}(x), \pi(\lambda)) = c_1(x, \lambda)$. Indeed, if $\tilde{\pi}(x) = \tilde{\pi}(y)$ and $\pi(\lambda) = \pi(\lambda')$, then there exists an $m \in \mathbb{Z}$ with $\lambda = e^{my_1} \lambda'$, so that

$$c_1(x, \lambda) = c_1(x, e^{my_1} \lambda') = c_1(xe^{my_1}, \lambda') = c_1(y, \lambda').$$  

Furthermore,

$$\int_{\Gamma \setminus \mathbb{N}} \tilde{c}(x, \pi(\lambda)) \, dv(z) = \int_{\Gamma \setminus \tilde{N}} \int_{\tilde{\pi}^{-1}(z)} c_1(s, \lambda) \, d\mu_z(s) \, dv(z)$$

$$= \int_{\Gamma \setminus \tilde{N}} c_1(x, \lambda) \, d\mu(x) = 0.$$  

By induction, there exists a $C^\infty$-function $P : \Gamma \setminus \tilde{N} \to \mathbb{R}$ such that $\tilde{c}(z, \pi(\lambda)) = -P(z) + P(z\pi(\lambda))$. Put $Q = P \circ \tilde{\pi}$. Then $c_1(x, \lambda) = \tilde{c}(\tilde{\pi}(x), \pi(\lambda)) = -Q(x) + Q(x\lambda)$. This proves Theorem 4.  

$\square$
**Proof of Theorem 3.** Let \( \tilde{\rho} : M \times N \to M \) be the suspension of \( \rho : \Gamma \backslash N \times \Lambda \to \Gamma \backslash N \) where \( M = \Gamma \backslash N \times_{\Lambda} N \) is a compact manifold. Then \( \tilde{\rho} \) is locally free and let \( F \) be its orbit foliation. We have

\[
H^1(F) \cong H^1(\Lambda, C^\infty(\Gamma \backslash N))
\]

by [6], where the right hand side is the first cohomology of the \( \Lambda \)-module \( C^\infty(\Gamma \backslash N) \) obtained by \( \rho \). It is easy to prove that \( \text{Hom}(\Lambda, \mathbb{R}) \to H^1(\Lambda, C^\infty(\Gamma \backslash N)) \) is injective. By Theorem 4,

\[
H^1(\Lambda, C^\infty(\Gamma \backslash N)) = \text{Hom}(\Lambda, \mathbb{R}).
\]

**Lemma 7.** \( \dim \text{Hom}(\Lambda, \mathbb{R}) = \dim H^1(n) \).

**Proof.** Recall that \([N, N] \backslash \Lambda[N, N] \) is a cocompact lattice in \([N, N] \backslash N \) and that \([\Lambda, \Lambda][N, N] \) is finite. Since

\[
0 \to [\Lambda, \Lambda][N, N] \to [\Lambda, \Lambda] \to [N, N][N, N] \to 0
\]

is exact, we have

\[
\text{rank } [\Lambda, \Lambda] = \text{rank } [N, N][N, N] = \dim [N, N] \frac{N}{N}.
\]

Thus

\[
\dim \text{Hom}(\Lambda, \mathbb{R}) = \dim \text{Hom}([\Lambda, \Lambda] \backslash \Lambda, \mathbb{R})
\]

\[
= \text{rank } [\Lambda, \Lambda] \backslash \Lambda
\]

\[
= \dim [N, N] \frac{N}{N}
\]

\[
= \dim \text{Hom}_\mathbb{R}([n, n] \backslash n, \mathbb{R})
\]

\[
= \dim H^1(n). \quad \square
\]

Therefore we obtain

\[
H^1(F) = H^1(n).
\]

This proves Theorem 3. \( \square \)

5. **Existence of Diophantine lattices**

Let \( n_\mathbb{Q} \) be a rational structure of \( n \). We construct Diophantine lattices when \( n_\mathbb{Q} \) admits a graduation. Namely, we assume that \( n_\mathbb{Q} \) has a sequence \( V_i \) of \( \mathbb{Q} \)-subspaces such that \( n_\mathbb{Q} = \bigoplus_{i=1}^k V_i \) and \([V_i, V_j] \subset V_{i+j} \). Let \( X_1, \ldots, X_n \) be a \( \mathbb{Q} \)-basis of \( n_\mathbb{Q} \) such that \( X_1, \ldots, X_{i_1} \in V_k, X_{i_1+1}, \ldots, X_{i_2} \in V_{k-1}, \ldots, X_{i_{k-1}+1}, \ldots, X_n \in V_1 \). Then \( X_1, \ldots, X_n \) is a strong Malcev basis of \( n \) with rational structure constants. Multiplying \( X_1, \ldots, X_n \) by an integer if necessary, we may assume that \( \Gamma := e^{Z X_1} \cdots e^{Z X_n} \) is a cocompact lattice in \( N \). Let \( \alpha \) be a root of an irreducible polynomial of degree \( k + 1 \) over \( \mathbb{Q} \). Since \( \alpha, \alpha^2, \ldots, \alpha^k \) are irrational algebraic numbers, they are Diophantine. If we define a linear map \( \varphi : n \to n \) by \( \varphi(X) = \alpha^i X \) for \( X \in V_i \otimes \mathbb{R} \), then \( \varphi \) is an automorphism of Lie algebra \( n \). Put \( Y_i = \varphi(X_i) \). Then \( Y_1, \ldots, Y_n \) is a strong Malcev basis of \( n \) strongly based on \( \Lambda := e^{Z Y_1} \cdots e^{Z Y_n} \). Thus \( \Lambda \) is Diophantine with respect to \( \Gamma \).
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