The measure in three dimensional Nambu-Goto string theory

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Abstract

We show that the measure of the three dimensional Nambu-Goto string theory has a simple decomposition as a measure on two parameter group of induced area-preserving transformations of the immersed surface and a trivial measure for the area of the surface.

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1. String theory was invented to describe hadronic physics in four dimensions [1]. The action is simple and beautiful: the area of the world sheet of the string [2, 3]. However, canonical quantization of this so called Nambu-Goto string is only consistent in 26 dimensions. The string theory of Polyakov arose as an attempt to perform a consistent quantization in lower dimensions than \( d = 26 \) [4]. His approach can be viewed as two-dimensional gravity coupled to \( d \) scalar fields which are identified with the coordinates of the string in target space. The goal of a consistent quantization in lower dimensional space was achieved, but unfortunately only in dimensions \( d \leq 1 \). A consistent quantization of bosonic string theories in the physical interesting dimensions \( d = 3 \) and \( d = 4 \) still remains to be constructed.

It is unknown if the quantum theories of the Nambu-Goto string and the Polyakov string agree in non-critical dimensions, partly because the path integral approach is not readily available for the Nambu-Goto string. In this article we will investigate the path integral which enters in the quantization of Nambu-Goto string in \( d = 3 \).

The Nambu-Goto string invites us to work directly with the immersed surfaces \( X(\xi) \), rather than the matter field \( X(\xi) \) taken together with an independent metric \( g_{\alpha\beta} \), as in the Polyakovs [4] approach. Our aim will be to formulate the path integral as an integral over the surfaces

\[
X(\xi) \equiv (X_1(\xi_1, \xi_2), X_2(\xi_1, \xi_2), X_3(\xi_1, \xi_2)).
\] (1)

We will show that the generators of infinitesimal deformations of \( X \) which leave the Nambu-Goto Lagrangian invariant form a two-parameter local Lie algebra, which can be considered as an algebra of vector fields on the manifold of maps [3, 4] of the two-dimensional world sheet into the three-dimensional target space \( R^3 \). This "induced" area preserving algebra contains the ordinary area preserving algebra of two-dimensional manifolds as a subalgebra. Further, the measure \( D_X \) of integration over surfaces is equal to the measure for the area of the surface, \( D\ln\sqrt{g} \), times the measure \( D\epsilon_\alpha \), where \( g_{\alpha\beta}(\xi) \) is the metric induced on \( X \) in \( R^3 \) and \( \epsilon_\alpha \) are the parameters of the above mentioned new symmetry group. No Jacobian appears and we will be left with the following expression for the partition function:

\[
Z(\alpha_0) = \int D\!X(\xi) \, e^{-S_{NG}(X)} = \int D\ln\sqrt{g}(\xi) \, e^{-\frac{1}{\alpha_0} \int d^2\xi \sqrt{\hat{g}} D\epsilon_\alpha(\xi)},
\] (2)

where \( 1/\alpha_0 \) is the string tension and

\[
S_{NG}(X) = \frac{1}{\alpha_0} \int d^2\xi \sqrt{\left(\partial_a X^\mu(\xi)\right)^2(\partial_b X^\nu(\xi))^2 - \left(\frac{1}{2} e^{ab} \partial_a X^\mu(\xi) \partial_b X^\mu(\xi)\right)^2}.
\] (3)

2. Let us first consider the analogous problem for the particle in two dimensions. The action of a particle is the length of the world line:

\[
S(X) = m \int_0^1 d\xi \, e(\xi), \quad e(\xi) \equiv \sqrt{\dot{X}_\mu(\xi)^2}
\] (4)
where $\dot{X}(\xi)$ denotes differentiation with respect to the parameter $\xi$, which we assume takes values in $[0, 1]$. Let us consider the transformations of paths

$$
\xi \mapsto X(\xi) \equiv (X_1(\xi), X_2(\xi))
$$

which preserve the induced metric $e(\xi)$. The general form of an infinitesimal transformation is

$$
\delta X(\xi) = \epsilon(\xi) \dot{X}(\xi) + B(\xi) N(\xi),
$$

(5)

where $N(\xi)$ is the normal to the curve $X$ and $\epsilon$ and $B$ are infinitesimal. A length-preserving transformation $\delta g$ satisfies

$$
\delta e(\xi) = 0,
$$

(6)

and one easily checks that it is equivalent to

$$
B_\epsilon(\xi) = \frac{1}{k(\xi)} D_\xi \epsilon,
$$

(7)

where $D_\xi \equiv e^{-1} \dot{e}$ is the covariant derivative with respect to the metric and $k(\xi)$ is the curvature of the trajectory $X(\xi)$, i.e.

$$
k(\xi) = \frac{\varepsilon^{\mu\nu} \dot{X}_\mu \dot{X}_\nu}{e^2}.
$$

In addition one can show that the transformations (5) with the imposed constraint (7),

$$
\delta e X = \epsilon \dot{X} + \frac{D_\xi \epsilon}{k} N,
$$

(8)

form an algebra with the commutation relations

$$
\delta_2(\delta_1 X) - \delta_1(\delta_2 X) = \delta_{(1,2)} X,
$$

(9)

where

$$
\epsilon_{(1,2)} = \epsilon_1 \dot{\epsilon}_2 - \epsilon_2 \dot{\epsilon}_1 - \frac{1}{(ke)^2} (\dot{\epsilon}_1 \dot{\epsilon}_2 - \dot{\epsilon}_2 \dot{\epsilon}_1).
$$

(10)

The measure $\mathcal{D}X(\xi)$ can be transformed to a measure $\mathcal{D} \ln e(\xi) \mathcal{D}\epsilon(\xi)$. The metric on the tangent space of maps $X : [0, 1] \mapsto \mathbb{R}^2$ is defined by

$$
\langle \delta X, \delta X \rangle = \int_0^1 e(\xi) d\xi \ \delta X(\xi) \cdot \delta X(\xi).
$$

(11)

It is possible to decompose the deformations $\delta X$ into the deformations $\delta_\epsilon X$ given by (8), which are area preserving and the deformations $\delta_\perp X$ orthogonal to these in the metric (11). From

$$
\langle \delta_\epsilon X, \delta_\perp X \rangle = 0
$$

(12)

one obtains

$$
\delta_\perp X = -\frac{\hat{B}_\perp}{e^2} \dot{X} + B_\perp N,
$$

(13)
where \( B_\perp(\xi) \) is a free parameter. While the deformation \( \delta_\epsilon X \) is characterized by the property \( e(X(\xi)) = e(X(\xi) + \delta_\epsilon X(\xi)) \) the deformations \( \delta_\perp X \) will result in a change in \( e(\xi) \). One finds

\[
\delta \ln e = -\left[k^2 - \left(\frac{1}{e} \frac{d}{d\xi}\right)^2\right] \frac{B_\perp}{k}.
\]

(14)

This implies that the transformation from \( B_\perp/k \) to \( \delta e \) has a Jacobian

\[
\det \left[k^2 - \left(\frac{1}{e} \frac{d}{d\xi}\right)^2\right]^{-1}.
\]

(15)

In addition we have

\[
\langle \delta X, \delta X \rangle = \int_0^1 e \, d\xi \left( \left(\epsilon \dot{X} + \frac{1}{k} D_\xi \epsilon N\right)^2 + \left(-\frac{\dot{B}_\perp}{e^2} \dot{X} + B_\perp N\right)^2 \right)
\]

(16)

\[
= \int_0^1 e \, d\xi \left(e \epsilon \left[1 - \frac{1}{k} \frac{d}{d\xi} \frac{1}{k^2} \frac{d}{d\xi}\right] e \epsilon + \frac{B_\perp}{k} \left[\left(\frac{1}{e} \frac{d}{d\xi}\right)^2 - k^2\right] \frac{B_\perp}{k} \right).
\]

From (14) and (16) it follows that

\[
\mathcal{D}X = \mathcal{D} \ln e \mathcal{D} \left(\epsilon \epsilon/k\right).
\]

(17)

This formula shows that there is no Jacobian related to the transformation

\[
\delta X \rightarrow \left(\frac{\delta e}{e}, \frac{\epsilon \epsilon}{k}\right).
\]

(18)

The geometric interpretation of the variable \( ke \epsilon \) is seen if we use the proper time parametrization of the path \( X(\xi) \). If \( t \) denotes the proper time

\[
e \epsilon = dt = \frac{dt}{d\phi} = \frac{1}{k} d\phi.
\]

and the integration over \( ke \epsilon \) can be viewed as the integration over successive angles \( \phi(t) \) of the tangents of paths. Of course one could have introduced this decomposition of \( \mathcal{D}X \) from the beginning. However, the derivation of (17) presented above has the virtue that it allows generalization to surfaces immersed in \( \mathbb{R}^3 \).

3.

Let us generalize the results of the last section to surfaces immersed in \( \mathbb{R}^3 \). Let \( M_2 \) be a two-dimensional manifold and \( X : M_2 \rightarrow \mathbb{R}^3 \) the map to \( \mathbb{R}^3 \). We use the notation

\[
g_{ab} = \partial_a X^\mu \partial_b X^\nu, \quad \det g_{ab} = g,
\]

(19)

for the induced metric and its determinant, respectively.

A general deformation of the surface \( X(\xi) \) can be written as

\[
\delta X = \epsilon^a \partial_a X + BN
\]

(20)
where \( \partial_a X \) denote the two tangents and \( N \) the normal to \( X \). The deformation \( \delta_g X \) is area preserving if \( \delta g = 0 \), which implies

\[
B_g = \frac{1}{h} D_a e^a,
\]

where \( D_a \) is the covariant derivative with respect to the induced metric \( g_{ab} \), \( e^a \) are free parameters and

\[
h_{ab} = N \cdot D_a \partial_b X, \quad h = h_{ab} g^{ab}
\]

are the second fundamental form and the mean curvature of \( X \), respectively.

As in the particle case one can check that deformations (20) with the constraint (21), i.e. deformations

\[
\delta_g X = e^a \partial_a X + \frac{1}{h} D_a e^a N
\]

form a Lie algebra:

\[
\delta_2 (\delta_1 X) - \delta_1 (\delta_2 X) = \delta_{(1,2)} X,
\]

where

\[
e^{a}_{(1,2)} = e^b D_b e^a_2 - e^b_1 D_b e^a_1 + \left( \frac{D_c e^c_2}{h} \right) \partial^a \left( \frac{D_b e^b_1}{h} \right) - \left( \frac{D_c e^c_1}{h} \right) \partial^a \left( \frac{D_b e^b_2}{h} \right).
\]

The first two terms correspond to the usual terms present in the Lie algebra for vector fields on the tangent space of \( X \) and if \( D_a e^a = 0 \) we have according to (23) the standard area preserving diffeomorphism algebra.

The measure \( DX \) is defined on the tangent space of the maps \( X : M^2 \to R^3 \) by

\[
\langle \delta X, \delta X \rangle = \int_{M^2} \sqrt{g} d^2 \xi \, \delta X(\xi) \cdot \delta X(\xi),
\]

as in the case of the particle. Again we can split the deformations (20) in two classes: \( \delta X_g \) which leaves \( \sqrt{g(\xi)} \) invariant and where \( B_g \) satisfy (21), and \( \delta X_\perp \), orthogonal to \( \delta X_g \) with respect to the scalar product (26). In analogy with the particle case (13) one finds from

\[
\langle \delta_g X, \delta_\perp X \rangle = 0
\]

that

\[
e^a_\perp = \partial^a (B_\perp / h),
\]

for the deformation \( \delta_\perp X = e^a_\perp \partial_a X + B_\perp N \). With this decomposition the norm of an arbitrary tangent vector on the space of maps \( X : M^2 \to R^3 \) is

\[
\langle \delta X, \delta X \rangle = \int_{M^2} \sqrt{g} d^2 \xi \left( \delta_g X \cdot \delta_g X + \delta_\perp X \cdot \delta_\perp X \right)
\]

\[
= \int_{M^2} \sqrt{g} d^2 \xi \, e^a [\delta_{ab} - D_a \left( \frac{1}{h^2} D_b \right)] e^b + B_\perp \left[ 1 - \frac{1}{h} D_a \partial^a \frac{1}{h} \right] B_\perp.
\]

An arbitrary two-dimensional vector \( e^a \) can be represented as

\[
e^a = \partial^a \varphi + \frac{1}{\sqrt{g}} \varepsilon^{ab} \partial_b \varphi.
\]
and we have
\[ \int_{M_2} \sqrt{g} d^2 \xi \ e^a e_a = - \int_{M_2} \sqrt{g} d^2 \xi \ (\varphi \nabla^2 \varphi + \phi \nabla^2 \phi) \] (31)
from which we deduce
\[ \mathcal{D} e^a = \det(-\nabla^2) \mathcal{D} \varphi \mathcal{D} \phi. \] (32)

From (29) we now obtain
\[ \langle \delta X, \delta X \rangle = \int_{M_2} \sqrt{g} d^2 \xi (\varphi (-\nabla^2 + \nabla^2 \frac{1}{h^2} \nabla^2) \phi + \phi (-\nabla^2) \phi + B_\perp (1 - \frac{1}{h^2} \nabla^2 \frac{1}{h} B_\perp)). \] (33)

Hence,
\[ \mathcal{D} X = \det(-\nabla^2) \det(-\nabla^2 \frac{1}{h^2} \nabla^2) \det h^{-1} \det(-\nabla^2 + h^2 \mathcal{D} \varphi \mathcal{D} \phi \mathcal{D} \left( \frac{B_\perp}{h} \right), \] (34)
and by the use of (32) we can write \( \mathcal{D} X \) as
\[ \mathcal{D} X = \det(h^2 - \nabla^2) \mathcal{D} \left( \frac{e^a}{h} \right) \mathcal{D} \left( \frac{B_\perp}{h} \right). \] (35)
As in the particle case we can use \( \delta g \) instead of \( B_\perp \). One has
\[ \delta \ln \sqrt{g} = (D_a D^a - h^2) \left( \frac{B_\perp}{h} \right), \] (36)
i.e.
\[ \mathcal{D} \ln \sqrt{g} = \det(h^2 - D_a D^a) \mathcal{D} \left( \frac{B_\perp}{h} \right). \] (37)
Using (36) and (37) we can finally write \( \mathcal{D} X \) as
\[ \mathcal{D} X = \mathcal{D} \ln \sqrt{g} \mathcal{D} e^a, \] (38)
and the partition function for the Nambu-Goto string theory reads
\[ Z(\alpha_0) = \int \mathcal{D} \ln \sqrt{g} e^{-\frac{\alpha_0}{2} \int_{M_2} \sqrt{g} d^2 \xi} \int \mathcal{D} e^a \] (39)

4.

The immediate question which arises is in what kind of mathematical framework one should view the Lie algebra defined by eqs. (23)-(25). We here draw the attention to the work in \[5\] (see \[7\]) for a review) which is a generalization of the classical work of Lie on transformation groups.

The main idea is to introduce the so called the prolongation of the ordinary Lie-Bäcklund operator
\[ L_0 = e^a \frac{\partial}{\partial \xi^a} + B N^\mu \frac{\partial}{\partial x^\mu} \] (40)
by derivatives of the map \( X^\mu(\xi) \) considering the derivatives as independent variables
\[ L = e^a \frac{\partial}{\partial \xi^a} + B N^\mu \frac{\partial}{\partial x^\mu} + \Xi^a \frac{\partial}{\partial x(\partial_a x^\mu)}; \] (41)
where
\[ \Xi^\mu_a = \partial_a (BN^\mu) - \partial_a (\epsilon^b \partial_b X^\mu) + B \partial_a \partial_b X^\nu \frac{\partial N^\mu}{\partial (\partial_b X^\nu)} \] (42)

The first two terms in eq.(41) appeared due to definition of \( \delta X^\mu(\xi) \) in eq.(23) whereas
the next term constitutes the prolongation. The invariance condition

\[ L \sqrt{g} = 0 \] (43)

is called the determining equation for the Lie-Bäcklund operator and it fixes
the constraint on B to be of the form (21). It is easily checked that Lie bracket algebra
defined by prolonged operator (41) forms the algebra we found in eq.(25).

5.
We have shown that the measure \( \mathcal{D}X \) can be written as \( \mathcal{D} \ln \sqrt{g} \mathcal{D} e^a \), without any
determinant. The parameters \( e^a \), as introduced here, were related to what we call
“induced” area preserving deformations of the surface \( X(\xi) \). These deformations
form a continuous group and we showed by explicit calculation that the generators of
deformations form a Lie algebra. The group contains as a subgroup the usual group
of area preserving diffeomorphisms, but in addition it contains the deformations of
\( X \) in \( R^3 \) which preserve the area density \( \sqrt{g} \) and which describe the fluctuations of
the surface viewed as an incompressible membrane in \( R^3 \). The additional part of
the measure, including the area action, is a simple product measure.

The Nambu-Goto action, as well as all our transformations (23) are reparametriza-
tion invariant and therefore the physical space of states should be defined in the fac-
tor space of maps \( \{ e^a, \sqrt{g} \} \) over diffeomorphisms. The physics will be encoded in this
factor space, and it is natural to expect that the fluctuations of the incompressible
membrane in \( R^3 \) will play an important role in describing this physics. In the case
of the particle this is manifest in the sense that one can first perform the integration
over all positions of the world-line with two endpoints fixed and a given length and
only afterwards the integration over world-lines of different length. In this way one
recovers the free massive propagator of a particle \( [8] \). Since the factorization of the
path integral of the three-dimensional string is very similar to the factorization of
the path integral for the two-dimensional particle, it is natural to conjecture that a
similar decomposition will take place for the string and that it should be possible
to perform first the integration over induced area preserving deformations when we
calculate for instance two-point correlators of the bosonic string.

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