A SHORT PROOF OF CARTAN’S NULLSTELLENSATZ
FOR ENTIRE FUNCTIONS IN $\mathbb{C}^n$

RAYMOND MORTINI

Abstract. Using the fact that the maximal ideals in the polydisk algebra are given by the kernels of point evaluations, we derive a simple formula that gives a solution to the Bézout equation in the space of all entire functions of several complex variables. Thus a short and easy analytic proof of Cartan’s Nullstellensatz is obtained.

1. Introduction

The aim of this note is to give a short and easy proof of Cartan’s Nullstellensatz:

Theorem 1.1. Let $H(\mathbb{C}^n)$ be the space of functions holomorphic in $\mathbb{C}^n$. Given $f_j \in H(\mathbb{C}^n)$, the Bézout equation $\sum_{j=1}^{N} g_j f_j = 1$ admits a solution $(g_1, \ldots, g_N) \in H(\mathbb{C}^n)^N$ if and only if the functions $f_j$ have no common zero in $\mathbb{C}^n$.

The usual proofs use a lot of machinery from sheaf theory, cohomology, (see for example [2]), or are based on the Hörmander-Wolff method by solving higher order $\overline{\partial}$-equations using the Koszul complex, a tool from homological algebra (see [6, p. 128-131]). The one-dimensional case, first done by Wedderburn, is very easy (see for example [4, p. 118-120] for the classical approach or [1, p. 130] for the $\overline{\partial}$-approach). For our proof to work, we shall only use a standard fact from an introductory course to functional analysis, namely Gelfand’s main theorem: the maximal ideals in a commutative unital complex Banach algebra coincide with the kernels of the multiplicative linear functionals (see for instance [5]). The idea is to apply this result to the polydisk algebras on an increasing sequence of polydisks $D_k$ and to glue together the solutions to the Bézout equations $\sum_{j=1}^{N} g_j f_j = 1$ on $D_k$ by using a Mittag-Leffler type trick. The major hurdle to overcome was of course to find suitable summands that guarantee at the end the holomorphy.
2. The general solution to the Bézout equation

Let $\mathbb{D}$ be the unit disk and let $A(\mathbb{D}^n)$ be the polydisk algebra; that is, the algebra of those continuous functions on the closed polydisk

$$\mathbb{D}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| \leq 1\}$$

which are holomorphic in $\mathbb{D}^n$. Endowed with the supremum norm, $A(\mathbb{D}^n)$ becomes a uniform algebra and coincides with the closure on $\mathbb{D}^n$ of the polynomials in $\mathbb{C}[z_1, \ldots, z_n]$. We actually only need that $A(\mathbb{D}^n)$ is the uniform algebra generated by the coordinate functions $Z_j$, $j = 1, \ldots, n$ on $\mathbb{D}^n$. It is now straightforward to show that an ideal $I$ in $A(\mathbb{D}^n)$ is maximal if and only if it coincides with $M(a_0) = \{f \in A(\mathbb{D}^n) : f(a_0) = 0\}$ for some $a_0 \in \mathbb{D}^n$ (just take a character $m$ on $A(\mathbb{D}^n)$ and put $a_0 = (m(Z_1), \ldots, m(Z_n))$). Hence the Bézout equation $\sum_{j=1}^n x_j f_j$ in $A(\mathbb{D}^n)$ has a solution if and only if $\bigcap_{j=1}^N Z_{\mathbb{D}^n}(f_j) = \emptyset$, where $Z_{\mathbb{D}^n}(f)$ is the zero set of $f$ on $\mathbb{D}^n$. We will use the following well-known elementary result. For the reader’s convenience we reproduce the proof here (see [3]), because its understanding is fundamental for our construction.

**Lemma 2.1.** Let $R$ be a commutative unital ring. Suppose that $a = (a_1, \ldots, a_N)$ is an invertible $N$-tuple in $R^N$ and let $x = (x_1, \ldots, x_N)$ satisfy $\sum_{j=1}^N x_j a_j = 1$; that is $xa^t = 1$. Then every other representation $1 = \sum_{j=1}^N y_j a_j$ of 1 can be deduced from the former by letting $y = x + aH$, where $H$ is an antisymmetric $(N \times N)$-matrix over $R$; that is $H = -H^t$, where $H^t$ is the transpose of $H$.

**Proof.** Suppose that $1 = xa^t$ and $1 = ya^t$. For $k = 1, \ldots, N$, multiply the first equation by $y_k$ and the second by $x_k$. Then

$$x_k - y_k = \sum_{j \neq k} a_j (y_j x_k - y_k x_j).$$

Thus $y = x + aH$ for some antisymmetric matrix $H$.

To prove the converse, let $1 = xa^t$. Since $H$ is antisymmetric we have (due to the transitivity of matrix multiplication and $xy^t = yx^t$)

$$aH a^t = a(Ha^t) = a(aH^t)^t = a(-aH)^t = (aH)a^t.$$

Thus $(aH)a^t = 0$. Hence

$$ya^t = (x + aH)a^t = xa^t + (aH)a^t = 1 + 0 = 1.$$

$\square$

3. Proof of Theorem 1.1

**Proof.** Let $f_j \in H(\mathbb{C}^n)$ and put $f = (f_1, \ldots, f_N)$. Suppose that $\bigcap_{j=1}^N Z(f_j) = \emptyset$. For $k \in \mathbb{N}^*$, let $\mathbb{D}_k = (k \mathbb{D})^n$ be the closed polydisk

$$\mathbb{D}_k = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| \leq k\}.$$
Note that $D_k \subseteq D_{k+1}$. Let $a_k \in A(D_{k+1})^N$ be a solution to the Bézout equation $a_k \cdot \mathbf{f}^t = 1$ on $D_{k+1}$. Using Tietze's extension theorem $^1$, we may assume that the tuples $a_k$ have been continuously extended to $C^n$. By Lemma 2.1, there is an antisymmetric matrix $H_k$ over $A(D_{k+1})$ such that

$$a_{k+1} = a_k + \mathbf{f} \cdot H_k.$$ 

Put $a_0 = 0$ and $H_0 = 0$. For $k = 0, 1, \ldots$, let $P_k$ be an antisymmetric $N \times N$-matrix of polynomials in $C[z_1, \ldots, z_n]$ such that

$$\max_{D_{k+1}} ||\mathbf{f} \cdot H_k - \mathbf{f} \cdot P_k||_N < 2^{-k}.$$ 

We claim that the $N$-tuple

$$\mathbf{g} := \sum_{k=0}^{\infty} (a_{k+1} - a_k - \mathbf{f} \cdot P_k)$$

belongs to $H(C^n)^N$ and is a solution to the Bézout equation $\mathbf{g} \cdot \mathbf{f}^t = 1$ in $H(C^n)$. In fact, let $D_m$ be fixed. Then the series defining $\mathbf{g}$ is uniformly convergent on $D_m$ since

$$\mathbf{g} = \sum_{k=0}^{m} (a_{k+1} - a_k - \mathbf{f} \cdot P_k) + \sum_{k=m+1}^{\infty} (a_{k+1} - a_k - \mathbf{f} \cdot P_k)$$

$$= a_{m+1} - \mathbf{f} \cdot (\sum_{k=0}^{m} P_k) + \sum_{k=m+1}^{\infty} (a_{k+1} - a_k - \mathbf{f} \cdot P_k)$$

and the tail can be majorated on $D_m$ by

$$\sum_{k=m+1}^{\infty} ||\mathbf{f} \cdot H_k - \mathbf{f} \cdot P_k||_N < 2^{-m}.$$ 

Moreover, on $D_m$, $a_{m+1}$ and all the summands in the series $\sum_{k=m+1}^{\infty} \mathbf{g}$ are holomorphic. Since $m$ was arbitrarily chosen, we conclude that $\mathbf{g} \in H(C^n)^N$. Moving again to $D_m$ we see that, due to the antisymmetry of the matrices $H_k$ and $P_k$,

$$\mathbf{g} \cdot \mathbf{f}^t = a_{m+1} \cdot \mathbf{f}^t - \mathbf{f} \cdot (\sum_{k=0}^{m} P_k) \cdot \mathbf{f}^t + \sum_{k=m+1}^{\infty} \mathbf{f} \cdot (H_k - P_k) \cdot \mathbf{f}^t$$

$$= 1 - 0 + 0 = 1.$$ 

$^1$ Since we will consider a telescoping series $\sum T_j$, where the domains of definition of the summands $T_j$ are strictly increasing, even an application of Tietze’s theorem is not necessary.

$^2$ Of course, outside $D_k$ the Bézout equation is not necessarily satisfied.
Acknowledgements
I thank the referee of the journal “American Math. Monthly” for some useful comments concerning the introductory section of a previous version of the paper.

References
[1] M. Andersson. Topics in Complex Analysis, Springer, New York 1997.
[2] S. Krantz. Function theory of several complex variables, Reprint of the 1992 edition. AMS Chelsea Publishing, Providence, RI, 2001.
[3] R. Mortini, B. Wick, Simultaneous stabilization in $A_R(\mathbb{D})$, Studia Math. 191 (2009), 223–235.
[4] R. Remmert. Funktionentheorie II, Springer, Berlin, 1991.
[5] W. Rudin, Real and Complex Analysis, third edition, McGraw-Hill, New York, 1986.
[6] E. Sawyer, Function Theory: Interpolation and Corona Problems, Fields Institute Monographs, Amer. Math. Soc. 2009.

Université de Lorraine, Département de Mathématiques et Institut Élie Cartan de Lorraine, UMR 7502, Ile du Saulcy, F-57045 Metz, France
E-mail address: raymond.mortini@univ-lorraine.fr