Solutions of Lévy-driven SDEs with unbounded coefficients as Feller processes

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Abstract

Let $(L_t)_{t \geq 0}$ be a $k$-dimensional Lévy process and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$ a continuous function such that the Lévy-driven stochastic differential equation (SDE)

$$dX_t = \sigma(X_{t-}) \, dL_t, \quad X_0 \sim \mu$$

has a unique weak solution. We show that the solution is a Feller process whose domain of the generator contains the smooth functions with compact support if, and only if, the Lévy measure $\nu$ of the driving Lévy process $(L_t)_{t \geq 0}$ satisfies

$$\nu(\{y \in \mathbb{R}^k : |\sigma(x)y + x| < r\}) \xrightarrow{|x| \to \infty} 0.$$ 

This generalizes a result by Schilling & Schnurr [14] which states that the solution to the SDE has this property if $\sigma$ is bounded.

Keywords: Feller process, stochastic differential equation, unbounded coefficients.

MSC 2010: Primary: 60J35. Secondary: 60H10, 60G51, 60J25, 60J75, 60G44.

1 Introduction

Feller processes are a natural generalization of Lévy processes. They behave locally like Lévy processes, but – in contrast to Lévy processes – Feller processes are, in general, not homogeneous in space. Although there are several techniques to prove existence results for Feller processes, many of them are restricted to Feller processes with bounded coefficients, i.e. they assume that the symbol is uniformly bounded with respect to the space variable $x$; see [2, 3] for a survey on known results. In fact, there are only few Feller processes with unbounded coefficients which are well studied, including affine processes and the generalized Ornstein–Uhlenbeck process, cf. [3, Example 1.3f),(i)] and the references therein. In order to get a better understanding of Feller processes with unbounded coefficients, it is important to find further examples.

In the present paper, we investigate under which assumptions the solution to the Lévy-driven stochastic differential equation (SDE)

$$dX_t = \sigma(X_{t-}) \, dL_t, \quad X_0 = x.$$  

is a Feller process whose domain of the generator contains the smooth functions with compact support, i.e. a so-called rich Feller process; here $(L_t)_{t \geq 0}$ is a $k$-dimensional Lévy process and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$ a continuous function such that the SDE has a unique weak solution. If $\sigma$ is bounded and Lipschitz continuous then this follows from a result by Schilling & Schnurr [14].
On the other hand, it is known that \((X_t)_{t \geq 0}\) may fail to be a Feller process if \(\sigma\) is not bounded. For instance, if \((L_t)_{t \geq 0}\) is a Poisson process with jump intensity \(\lambda > 0\), then it is not difficult to see that the solution to the SDE

\[
dX_t = -X_t \, dL_t, \quad X_0 = x,
\]

is not a Feller process, see [14, Remark 3.4] for details. Up to now, it was an open problem under which necessary and sufficient conditions the solution to the SDE (1) is a (rich) Feller process if \(\sigma\) is not necessarily bounded. We resolve this question by showing that \((X_t)_{t \geq 0}\) is a rich Feller process if, and only if, the Lévy measure \(\nu\) of the driving Lévy process \((L_t)_{t \geq 0}\) satisfies

\[
\nu_L \left( \{ y \in \mathbb{R}^d; \sigma(x) y \in B(-x, r) \} \right) \overset{|x| \to \infty}{\to} 0;
\]

here \(B(-x, r)\) denotes the open ball of radius \(r\) centered at \(-x\). The symbol of the Feller process is then given by \(q(x, \xi) = \psi(\sigma(x)^T \xi)\) where \(\psi\) denotes the characteristic exponent of \((L_t)_{t \geq 0}\). This provides us with a new class of Feller processes with unbounded coefficients.

The following result is our main theorem; the required definitions will be explained in Section 2.

1.1 Theorem Let \((L_t)_{t \geq 0}\) be a \(k\)-dimensional Lévy process with Lévy triplet \((b_L, Q_L, \nu_L)\) and characteristic exponent \(\psi\). Let \(\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times k}\) be a continuous function satisfying the linear growth condition

\[
|\sigma(x)| \leq c(1 + |x|), \quad x \in \mathbb{R}^d
\]

for some absolute constant \(c > 0\). Suppose that the SDE

\[
dx_t = \sigma(X_t) \, dL_t, \quad X_0 \sim \mu
\]

admits a unique weak solution for each initial distribution \(\mu\). Then the weak solution \((X_t)_{t \geq 0}\) to (3) is a rich \(d\)-dimensional Feller process if, and only if,

\[
\nu_L \left( \{ y \in \mathbb{R}^d; \sigma(x) y \in B(-x, r) \} \right) \overset{|x| \to \infty}{\to} 0 \quad \text{for all } r > 0.
\]

In this case the symbol \(q\) of the Feller process \((X_t)_{t \geq 0}\) is given by \(q(x, \xi) := \psi(\sigma(x)^T \xi)\) and \(C^2_2(\mathbb{R}^d)\) is contained in the domain of the generator.

For some classes of Lévy processes the existence of a unique weak solution to (3) can be proved under rather weak regularity assumptions on \(\sigma\), typically Hölder continuity.

For Lipschitz continuous functions \(\sigma\) we obtain the following corollary.

1.2 Corollary Let \((L_t)_{t \geq 0}\) be a \(k\)-dimensional Lévy process with Lévy triplet \((b_L, Q_L, \nu_L)\) and characteristic exponent \(\psi\). If \(\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times k}\) is globally Lipschitz continuous, then the (strong) solution to the SDE (3) is a rich \(d\)-dimensional Feller process if, and only if, (4) holds. In this case the symbol \(q\) of the Feller process \((X_t)_{t \geq 0}\) is given by \(q(x, \xi) := \psi(\sigma(x)^T \xi)\) and \(C^2_2(\mathbb{R}^d)\) is contained in the domain of the generator.

Before we prove the results, let us give some intuition what condition (4) means. For simplicity consider the one-dimensional case, i.e. \(k = d = 1\). Then

\[
\sigma(x) y \in B(-x, r) \iff \exists z \in (-r, r) : y = -\frac{x}{\sigma(x)} + \frac{z}{\sigma(x)}.
\]

Assuming that \(|\sigma(x)| \to \infty\) as \(|x| \to \infty\), we have \(z/\sigma(x) \to 0\) as \(|x| \to \infty\), and therefore, heuristically,

\[
\nu_L \left( \{ y \in \mathbb{R}; \sigma(x) y \in B(-x, r) \} \right) = \nu_L \left( \left\{ -\frac{x}{\sigma(x)} \right\} \right) \quad \text{for all } |x| \gg 1.
\]

There are two cases:
has the Feller property on the domain \( D \) with left-hand limits) sample paths is called a \( q \)\( -\)Feller process. A result due to von Waldenfels and Courrèges, cf. [3, Theorem 2.21], states that

\[
\text{if } \mathbb{P}_x(\tau_x < \infty) \quad \text{then the exit time } \tau_x = \text{inf}\{t \geq 0 : x + \mathbb{E}^{x} f(t) \geq R \} \text{ for } R > 1, \text{ see Example 4.3 and Example 4.4.}
\]

## 2 Preliminaries

We consider the Euclidean space \( \mathbb{R}^d \) endowed with the canonical scalar product \( x \cdot y = \sum_{j=1}^{d} x_j y_j \) and the Borel-\( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d) \). We denote as above the open ball of radius \( r \) centered at \( x \) by \( B(x, r) \). We use \( C_c^2(\mathbb{R}^d) \) to denote the space of twice continuously differentiable functions with compact support and \( C_0(\mathbb{R}^d) \) is the space of continuous bounded functions \( f : \mathbb{R}^d \to \mathbb{R} \).

A \( d \)-dimensional Markov process \((\Omega, \mathcal{A}, \mathbb{P}^x, x \in \mathbb{R}^d, X_t, t \geq 0)\) with càdlàg (right-continuous with left-hand limits) sample paths is called a Feller process if the associated semigroup \((T_t)_{t \geq 0}\) defined by

\[
T_t f(x) := \mathbb{E}^x f(X_t), \quad x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{R} ; f \text{ bounded, Borel measurable} \}
\]

has the Feller property and \((T_t)_{t \geq 0}\) is strongly continuous at \( t = 0 \), i.e. \( T_t f \in C_c(\mathbb{R}^d) \) for all \( C_c(\mathbb{R}^d) \) and \( \| T_t f - f \|_{C_c} \to 0 \) for any \( f \in C_c(\mathbb{R}^d) \). Here, \( C_0(\mathbb{R}^d) \) denotes the space of continuous functions vanishing at infinity. If the smooth functions with compact support \( C_c^\infty(\mathbb{R}^d) \) are contained in the domain of the generator \((L, \mathcal{D}(L))\), then we speak of a rich Feller process. A result due to van Wyk, cf. [3, Theorem 2.21], states that the generator \( L \) of a rich Feller process is, when restricted to \( C_c^\infty(\mathbb{R}^d) \), a pseudo-differential operator with negative definite symbol:

\[
L f(x) = -\int_{\mathbb{R}^d} e^{ix \cdot \xi} q(x, \xi) \hat{f}(\xi) d\xi, \quad f \in C_c^\infty(\mathbb{R}^d), x \in \mathbb{R}^d
\]

where \( \hat{f}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx \) denotes the Fourier transform of \( f \) and

\[
q(x, \xi) = q(x, 0) - ib(x) \cdot \xi + \frac{1}{2} \xi \cdot Q(x) \xi + \int_{\mathbb{R}^{d}\{0\}} \left( 1 - e^{iy \cdot \xi} + iy \cdot \xi 1_{\{0\}}(y) \right) \nu(x, dy).
\]

We call \( q \) the symbol of the rich Feller process \((X_t)_{t \geq 0}\) and \( -q \) the symbol of the pseudo-differential operator. For each fixed \( x \in \mathbb{R}^d \), \( (b(x), Q(x), \nu(x, dy)) \) is a Lévy triplet, i.e. \( b(x) \in \mathbb{R}^d \), \( Q(x) \in \mathbb{R}^{d \times d} \) is a symmetric positive semidefinite matrix and \( \nu(x, dy) \) a \( \sigma \)-finite measure on \( (\mathbb{R}^d\{0\}, \mathcal{B}(\mathbb{R}^d\{0\})) \) satisfying \( \int_{\mathbb{R}^{d}\{0\}} \min\{|y|^2, 1\} \nu(x, dy) < \infty \). A point \( x \in \mathbb{R}^d \) is called absorbing if \( \mathbb{P}^x(X_t = x) = 1 \) for all \( t \in \mathbb{R}^d \). If \((X_t)_{t \geq 0}\) is a Feller process and \( x \in \mathbb{R}^d \) is not absorbing, then the exit time \( \tau_x^\uparrow := \text{inf}\{t \geq 0 : |X_t - x | \geq r \} \) satisfies \( \mathbb{E}^x \tau_x^\uparrow < \infty \) for \( r > 0 \) sufficiently small, cf. [13, Lemma 7.24]. Dynkin’s characteristic operator is the linear operator defined by

\[
\mathcal{L} f(x) := \begin{cases} 
\lim_{t \to 0} \frac{1}{\mathbb{E}^x t} \left( \mathbb{E}^x f(X_{t \uparrow}) - f(x) \right), & \text{if } x \text{ is not absorbing,} \\
0, & \text{if } x \text{ is absorbing,}
\end{cases}
\]

on the domain \( \mathcal{D}(\mathcal{L}) \) consisting of all functions \( f \in \mathcal{B}_b(\mathbb{R}^d) \) such that the limit (7) exists for all \( x \in \mathbb{R}^d \). Our standard reference for Feller processes are the monographs [6, 7, 8, 3].

A Lévy process \((L_t)_{t \geq 0}\) is a rich Feller process whose symbol \( q \) does not depend on \( x \). This is equivalent to saying that \((L_t)_{t \geq 0}\) has stationary and independent increments and càdlàg sample paths. The symbol \( q = q(\xi) \) (also called characteristic exponent) and the Lévy process \((L_t)_{t \geq 0}\) are related through the Lévy–Khintchine formula:

\[
\mathbb{E}^x e^{i\xi (L_t - x)} = e^{-tq(\xi)} \quad \text{for all } t \geq 0, x, \xi \in \mathbb{R}^d.
\]
Weak uniqueness holds for the Lévy-driven stochastic differential equation (SDE, for short)

\[ dX_t = \sigma(X_{t-}) dB_t, \quad X_0 \sim \mu, \]

if any two weak solutions of the SDE have the same finite-dimensional distributions. We speak of pathwise uniqueness if any two strong solutions \((X_t^{(1)})_{t \geq 0}\) and \((X_t^{(2)})_{t \geq 0}\) with \(X_0^{(1)} = X_0^{(2)}\) are indistinguishable, i.e. \(\mathbb{P}(\forall t \geq 0 : X_t^{(1)} = X_t^{(2)}) = 1\). We refer the reader to the Ikeda & Watanabe [5] and Protter [12] for further details.

Let \((A, \mathcal{D})\) be a linear operator with domain \(\mathcal{D} \subseteq \mathcal{B}_b(\mathbb{R}^d)\) and \(\mu\) a probability measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\). A \(d\)-dimensional stochastic process \((X_t)_{t \geq 0}\) with càdlàg sample paths is a solution to the \((A, \mathcal{D})\)-martingale problem with initial distribution \(\mu\) if \(X_0 \sim \mu\) and

\[ M_t^f := f(X_t) - f(X_0) - \int_0^t Af(X_s) \, ds, \quad t \geq 0, \]

is a martingale with respect to the canonical filtration of \((X_t)_{t \geq 0}\) for any \(f \in \mathcal{D}\). The \((A, \mathcal{D})\)-martingale problem is well-posed if for any initial distribution \(\mu\) there exists a unique (in the sense of finite-dimensional distributions) solution to the \((A, \mathcal{D})\)-martingale problem with initial distribution \(\mu\). For a comprehensive study of martingale problems see Ethier & Kurtz [4, Chapter 4].

3 Proofs

Let us recall the connection between Dynkin’s characteristic operator, cf. (7), and the infinitesimal generator of a Feller process \((X_t)_{t \geq 0}\), cf. [13, Theorem 7.35].

3.1 Theorem Let \((X_t)_{t \geq 0}\) be a Feller process with infinitesimal generator \((L, \mathcal{D}(L))\) and characteristic operator \((\mathcal{L}, \mathcal{D}(\mathcal{L}))\). If \(f \in \mathcal{D}(\mathcal{L}) \cap C_0(\mathbb{R}^d)\) is such that \(\mathcal{L}f \in C_0(\mathbb{R}^d)\), then \(f \in \mathcal{D}(L)\) and \(Lf = \mathcal{L}f\).

For the proof of Theorem 1.1 we need some auxiliary statements.

3.2 Lemma Let \((q(x, \cdot))_{x \in \mathbb{R}^d}\) be a family of continuous negative definite functions (i.e. functions of the form (6)) with \(q(x, 0) = 0\) and denote by \((b(x), Q(x), \nu(x, dy))_{x \in \mathbb{R}^d}\) the associated family of Lévy triplets. Assume that \(x \mapsto q(x, \xi)\) is continuous for all \(\xi \in \mathbb{R}^d\) and that \(q\) is locally bounded in \(x\), i.e., for any compact set \(K \subseteq \mathbb{R}^d\) there exists \(C > 0\) such that \(|q(x, \xi)| \leq C(1 + |\xi|^2)\) for all \(x \in K, \xi \in \mathbb{R}^d\).

For the pseudo-differential operator \(A\) with symbol \(-q\) the following statements are equivalent:

(i). \(A\) maps \(C_0^\infty(\mathbb{R}^d)\) into \(C_0(\mathbb{R}^d)\).

(ii). \(\nu(x, B(-x, r)) \xrightarrow{|r| \to \infty} 0\) for any \(r > 0\).

Both conditions are, in particular, satisfied if

\[
\lim_{r \to \infty} \sup_{|x| \leq r, |\xi| \leq 1} |q(x, \xi)| = 0. \tag{8}
\]

For a proof of Lemma 3.2 see [10, Theorem 1.27] or [3, Lemma 3.26].

3.3 Lemma Let \(\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times k}\) be a measurable locally bounded mapping and let \((L_t)_{t \geq 0}\) be a Lévy process with characteristic exponent \(\psi\). If the pseudo-differential operator \(A\) with symbol \(-q(x, \xi) := -\psi(\sigma(x)^T \xi)\) maps \(C_0^\infty(\mathbb{R}^d)\) into \(C_0(\mathbb{R}^d)\), then the following statements are equivalent.

(i). \((X_t)_{t \geq 0}\) is a weak solution to the SDE

\[ dX_t = \sigma(X_{t-}) \, dB_t, \quad X_0 \sim \mu \tag{9} \]
(ii). \((X_t)_{t \geq 0}\) is a solution to the \((A, C_c^\infty(\mathbb{R}^d))\)-martingale problem with initial distribution \(\mu\).

Proof. Let \((X_t)_{t \geq 0}\) be a solution to the SDE (9). For \(r > 0\) denote by \(\tau_r := \inf\{t \geq 0; |X_t| \geq r\}\) the exit time from the ball \(B(0,r)\). Since \((X_t)_{t \geq 0}\) is non-explosive, we have \(\tau_r \to \infty\) as \(r \to \infty\) almost surely. It follows easily from Itô’s formula that

\[
M_{t}^{f,r} := f(X_{t \wedge \tau_r}) - f(X_0) - \int_{0}^{t \wedge \tau_r} Af(X_s) \, ds
\]

is a \(\mathbb{P}^\mu\)-martingale for any \(f \in C_c^\infty(\mathbb{R}^d)\). As \(Af \in C_0(\mathbb{R}^d)\) and \(f \in C_c^\infty(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)\), the dominated convergence theorem gives

\[
M_{t}^{f,r} \xrightarrow{\mathbb{P}^\mu} M_{t}^f := f(X_t) - f(X_0) - \int_{0}^{t} Af(X_s) \, ds \quad \text{in } L^1(\mathbb{P}^\mu).
\]

This implies that \((M_t^f)_{t \geq 0}\) is a martingale for each \(f \in C_c^\infty(\mathbb{R}^d)\). For the proof of the implication (ii) \(\implies\) (i) we refer to Kurtz [9].

The next result allows us to remove the large jumps from the driving Lévy process \((L_t)_{t \geq 0}\).

**3.4 Theorem** Let \((L_t)_{t \geq 0}\) be a \(k\)-dimensional Lévy process with Lévy triplet \((b_L, Q_L, \nu_L)\) and characteristic exponent \(\psi\). Let \(\sigma: \mathbb{R}^d \to \mathbb{R}^{d \times k}\) be a continuous function satisfying the linear growth condition \(|\sigma(x)| \leq c(1 + |x|)\).

(i). For any initial distribution \(\mu\) the SDE

\[
dY_t = \sigma(Y_t) \, dL_t, \quad Y_0 \sim \mu,
\]

has a weak solution (which does not explode in finite time).

(ii). For fixed \(r > 0\) denote by \((L_t^{(r)})_{t \geq 0}\) the Lévy process which is obtained by removing all jumps of modulus larger than \(r\) from \((L_t)_{t \geq 0}\), i.e. a Lévy process with Lévy triplet \((b_L, Q_L, \nu_L)_{\mathbb{R}^d \times [0,r]}\). If (10) has a unique weak solution for any initial distribution, then the SDE

\[
dX_t = \sigma(X_t) \, dL_t^{(r)}, \quad X_0 \sim \mu,
\]

has a unique weak solution for any initial distribution \(\mu\).

Proof. Denote by \(A\) the pseudo-differential operator with symbol \(-\psi(\sigma(x)^T \xi)\), and fix an initial distribution \(\mu\). By [4, Theorem 4.5.4] there exists a process \((Y_t)_{t \geq 0}\) which takes values in the one-point compactification of \(\mathbb{R}^d\) and which is a solution to the \((A, C_c^\infty(\mathbb{R}^d))\)-martingale problem. It follows from the linear growth condition on \(\sigma\) and [11, Corollary 3.2] that \((Y_t)_{t \geq 0}\) is conservative, i.e. it does not explode in finite time. Applying Lemma 3.3 shows that \((Y_t)_{t \geq 0}\) is a weak solution to (10), and this proves (i).

It remains to prove (ii). Let \((X_t)_{t \geq 0}\) be a solution to (11), and let \((N_j)_{j \geq 1}\) be sequence of independent random variables such that \((N_j)_{j \geq 1}\) and \((X_t)_{t \geq 0}\) are independent and \(N_j \sim \text{Exp}(\lambda)\) is exponentially distributed with intensity \(\lambda := \nu_L(\{y; |y| > r\})\) for \(j \geq 1\). We define

\[
\tilde{L}_t^{(r)} := L_t^{(r)} + C_1 \mathbb{1}_{\{t \leq N_1\}} \quad \tilde{X}_t := X_t + C_1 X_{N_1} - \mathbb{1}_{\{t \geq N_1\}}
\]

for a random variable \(C_1 \sim \lambda^{-1} \nu_L(\mathbb{R}^d \setminus [0,r])\), which is independent from \((X_t)_{t \geq 0}\) and \((N_j)_{j \geq 1}\), then \((\tilde{X}_t)_{t \geq 0}\) is a solution to the SDE

\[
Z_{t \wedge N_1} - Z_0 = \int_{0}^{t \wedge N_1} \sigma(Z_s) \, d\tilde{L}_s^{(r)}, \quad Z_0 \sim \mu.
\]

It follows from Itô’s formula and [4, Lemma 4.5.16] that there exist a solution \((Y_t)_{t \geq 0}\) to the \((A, C_c^\infty(\mathbb{R}^d))\)-martingale problem and a random variable \(\tau_1\) taking values in \([0, \infty)\) such that \((\tilde{X}_{t \wedge N_1})_{t \geq 0}\) and \((Y_{t \wedge \tau_1})_{t \geq 0}\) have the same distribution. By Lemma 3.3 \((Y_t)_{t \geq 0}\) is a weak solution to (10). Set

\[
\theta_j(\omega) := k2^{-j}; \quad \text{for } \omega \in \{(k - 1)2^{-j} \leq \tau_1 < k2^{-j}), k \geq 1,
\]
then $g_j \downarrow \tau_1$, and it follows from the dominated convergence theorem that
\[
E^\mu(f(Y_{t_1 \wedge \tau_1}, \ldots, Y_{t_n \wedge \tau_1}) \mid \tau_1) = \lim_{j \to \infty} \sum_{k=1}^\infty I_{(k-1)2^{-j},k2^{-j})}(\tau_1)E^\mu(f(Y_{t_1 \wedge k2^{-j}}, \ldots, Y_{t_n \wedge k2^{-j}}))
\]
for any bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and any $0 < t_1 < \ldots < t_n$. This shows that the finite-dimensional distributions of $(Y_{t_1 \wedge \tau_1})_{t \geq 0}$ conditioned on $\tau_1$ are uniquely determined by the finite-dimensional distributions of $(Y_t)_{t \geq 0}$. Consequently,
\[
P^\mu(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n, t_n < N_1) = P^\mu(\hat{X}_{t_1} \in B_1, \ldots, \hat{X}_{t_n} \in B_n, t_n < N_1) = E^\mu(P^\mu(Y_{t_1 \wedge \tau_1} \in B_1, \ldots, Y_{t_n \wedge \tau_1} \in B_n \mid \tau_1)I_{(\tau_1 < t_n)})
\]
is uniquely determined by the finite-dimensional distributions of $(Y_t)_{t \geq 0}$ and the distribution of $\tau_1 \sim \text{Exp}(\lambda)$ for any Borel sets $B_i$ and $0 < t_1 < \ldots < t_n$, $n \in \mathbb{N}$. We iterate the procedure. Since the shifted process $Z_t := X_{t_1 + t_2 + \ldots + t_n}$ is a weak solution to the SDE
\[
dZ_t = \sigma(Z_t) dL_t^{(r)}, \quad Z_0 \sim \mu
\]
we can construct a weak solution $(Y_t)_{t \geq 0}$ to (10), $Y_0 \sim \mu$, and a random variable $\tau_j$ such that $(Z_{t \wedge N_j}, N_j)_{t \geq 0} \equiv (Y_{t \wedge \tau_j}, \tau_j)_{t \geq 0}$ in distribution, and now we can use the same reasoning as in the first part of the proof. (Note that $\mu_j$ is uniquely determined by $X_t$, $t < N_1 + \ldots + N_{j-1}$, since $X_{N_1 + \ldots + N_{j-1}} = X_{(N_1 + \ldots + N_{j-1})}$ almost surely.) We conclude that
\[
P^\mu(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n, t_n < N_1 + \ldots + N_j)
\]
is uniquely determined by finite-dimensional distributions of the unique weak solutions to (17) started at $Y_0 \sim \mu_i$, $i \leq j$. As $N_1 + \ldots + N_j \to \infty$ as $j \to \infty$, this proves the weak uniqueness. \(\Box\)

The next result is the key step to prove Theorem 1.1.

3.5 Theorem Let $(L_t)_{t \geq 0}$ and $\sigma$ be as in Theorem 1.1. If (4) holds, then the solution $(X_t)_{t \geq 0}$ to the SDE (3) is a rich Feller process with symbol $q(x, \xi) = \psi(\sigma(x)^T \xi)$, $x, \xi \in \mathbb{R}^d$.

Note that $(X_t)_{t \geq 0}$ may fail to be a Feller process if (4) is not satisfied; consider for instance the SDE
\[
dx_t = -x_t \, dN_t, \quad X_0 = x
\]
for a Poisson process $(N_t)_{t \geq 0}$.

Proof. To keep notation simple, we consider only the case $k = d = 1$. Since $\sigma$ is at most of linear growth, we can choose $r \in (0, 1)$ such that
\[
|\sigma(x)y| \leq \frac{1}{2}(|x|+1) \quad \text{for all } x \in \mathbb{R}, |y| \leq r.
\]
Denote by $(L_t^{(r)})_{t \geq 0}$ the Lévy process which is obtained from $(L_t)_{t \geq 0}$ by removing all jumps of modulus larger than $r$. By Theorem 3.4, there exists a unique weak solution to the SDE
\[
dY_t = \sigma(Y_t) dL_t^{(r)}, \quad Y_0 \sim \mu
\]
for any initial distribution $\mu$. We split the proof into several steps.

Step 1: $(Y_t)_{t \geq 0}$ is a Feller process. It is well-known that the unique weak solution $(Y_t)_{t \geq 0}$ is a Markov process (the proof works exactly as in the diffusion case, see e.g. [13, Section 19.7]), see also [4, Theorem 4.4.2]. By [3, Lemma 1.4], it suffices to prove that $P_t f(x) := E^\mu f(Y_t)$ satisfies the following three properties.

(i). $|P_t f(x) - f(x)| \xrightarrow{t \to 0} 0$ for all $x \in \mathbb{R}$ and $f \in C_0(\mathbb{R})$

(ii). $|P_t f(x)| \xrightarrow{|x| \to \infty} 0$ for all $t \geq 0$, $f \in C_0(\mathbb{R})$
(iii). \( x \mapsto P_t f(x) \) is continuous for all \( t \geq 0, f \in C^\infty_c(\mathbb{R}) \)

Since \( (Y_t)_{t \geq 0} \) has càdlàg sample paths, it follows from the dominated convergence theorem that (i) holds. To prove (ii), we are going to show that

\[
\lim_{|x| \to \infty} \mathbb{P}^x(|Y_t| \leq R) = 0 \quad \text{for all } R > 0.
\]  

(14)

If we define a function \( f \) by \( f(x) := (x^2 + 1)^{-1} \), then

\[
|f'(x)| \leq 2|x|f(x)^2 \quad \text{and} \quad |f''(x)| \leq 6f(x)^2, \quad x \in \mathbb{R};
\]

combining this with (12) and the fact that \( \sigma \) grows at most linearly, we find that there exists an absolute constant \( C > 0 \) such that

\[
Bf(x) := \left( \int_{|y| < 1} y \nu(dy) + b_L \right) \sigma(x) f'(x) + \frac{1}{2} \sigma(x)^2 Q_L f''(x) \\
+ \int_{|y| \geq 1} (f(x + \sigma(x)y) - f(x) - f'(x)\sigma(x)y 1_{|y| \geq 1}) \nu_L(dy)
\]

satisfies \(|Bf(x)| \leq Cf(x), x \in \mathbb{R} \), for some absolute constant \( C > 0 \). For fixed \( R > 0 \) set \( g_R := \inf \{ t \geq 0 ; |Y_t| \leq R \} \) and \( Y_t^R := Y_{t\wedge g_R} \). An application of Itô’s formula gives

\[
\mathbb{E}^x f(Y_{t \wedge 
abla^R}) - f(x) = \mathbb{E}^x \left( \int_{[0, t \wedge \nabla^R]} Bf(Y_s) \, ds \right);
\]

here \( \nabla_t^x := \inf \{ t \geq 0 ; |Y_t - x| \geq g \} \) denotes the exit time from the ball \( B(x, g) \). Consequently,

\[
\mathbb{E}^x f(Y_{t \wedge 
abla^R}) \leq f(x) + C \int_0^t \mathbb{E}^x(f(Y_{s \wedge 
abla^R})) \, ds.
\]

Applying Gronwall’s lemma we get

\[
\mathbb{E}^x f(Y_{t \wedge 
abla^R}) \leq f(x)e^{Ct} \quad \text{for all } t \geq 0, x \in \mathbb{R}.
\]

Since the constant \( C \) does not depend on \( g \), we obtain from Fatou’s lemma

\[
\mathbb{E}^x f(Y_t^R) \leq f(x)e^{Ct} \quad \text{for all } t \geq 0, x \in \mathbb{R}.
\]

This implies by the Markov inequality

\[
\mathbb{P}^x(|Y_t| \leq R) \leq \frac{1}{(f(R))^t} \mathbb{E}^x f(Y_t^R) \leq \frac{1}{(f(R))^t} f(x)e^{Ct}.
\]

Obviously, the right-hand side converges to 0 as \(|x| \to \infty \), and this gives (14). It remains to prove (iii), i.e. that \( x \mapsto P_t f(x) \) is continuous. Set \( g(x) := x^2 + 1 \). Using a very similar reasoning as above, we find \(|B \nu f(x)| \leq Cg(x) \) for some absolute constant \( C > 0 \) which implies by Itô’s formula, the optional stopping theorem and Gronwall’s lemma that

\[
\mathbb{E}^x g(Y_{t \wedge 
abla^R}) \leq g(x)e^{CT} \quad \text{for all } T \geq 0, x \in \mathbb{R}
\]

where \( \nabla := \inf \{ t \geq 0 ; |Y_t| \geq R \} \). Hence, by the Markov inequality,

\[
\mathbb{P}^x \left( \sup_{t \leq \nabla} |Y_t| \geq R \right) \leq \mathbb{P}^x \left( g(Y_{t \wedge 
abla^R}) \geq g(R) \right) \leq \frac{1}{g(R)} \mathbb{E}^x g(Y_{t \wedge 
abla^R}) \leq \frac{g(x)}{g(R)} e^{CT}
\]

and the right-hand side converges uniformly (in \( x \)) on compact sets to 0 as \( R \to \infty \). On the other hand, it follows from the existence of a unique weak solution to (13), Lemma 3.1 and [9, Corollary 2.5] that the \((B, C^\infty_c(\mathbb{R}))\)-martingale problem for the operator \( B \) defined in (15) is well-posed. Now we can apply [8, Theorem 4.1.16] to get tightness and then use exactly the same reasoning as in [8, Corollary 4.6.4] to conclude that \( x \mapsto \mathbb{E}^x f(Y_t) \) is continuous.
Combining the above considerations, we conclude that \((Y_t)_{t \geq 0}\) is a Feller process. From now on we denote by \((M, \mathcal{D}(M))\) its generator.

**Step 2:** \(C^\infty_c(R) \subseteq \mathcal{D}(M)\) and \(Mu = Bu\) for any \(u \in C^\infty_c(R)\) with \(B\) defined in (15).

Fix \(u \in C^\infty_c(R)\) and denote as usual by \(\tau^g_\varphi := \inf\{t > 0; |Y_t - x| \geq g\}\) the exit time from the ball \(B(x, \varphi)\). Recall that by Itô’s formula

\[
E^x u(Y_{t_\varphi}) = E^x \left(\int_{(0,t_\varphi]} Bu(Y_s) \, ds\right)
\]

for all \(x \in R\). Fix a non-absorbing point \(x \in R\). By Lemma 3.2 and (4), we have \(Bu \in C_\infty(R)\). Since \(E^x \tau^g_\varphi < \infty\) for \(g > 0\) sufficiently small, it follows from the dominated convergence theorem that

\[
E^x u(Y_{\tau^g_\varphi}) - u(x) = E^x \left(\int_{[0,\tau^g_\varphi]} Bu(Y_s) \, ds\right).
\]

As \(Bf\) is continuous, we get

\[
\left|\frac{E^x u(Y_{\tau^g_\varphi}) - u(x) - Bu(x)}{E^x \tau^g_\varphi}\right| \leq \frac{1}{E^x \tau^g_\varphi} E^x \left(\int_{[0,\tau^g_\varphi]} |Bu(Y_s) - Bu(x)| \, ds\right) \leq \sup_{|y-x|\leq g} |Bu(y) - Bu(x)| \xrightarrow{g \to 0} 0.
\]

If \(x \in R\) is absorbing, then (16) gives \(Bu(x) = 0\). This shows that \(u\) is the domain of Dynkin’s characteristic operator \(\mathfrak{M}\) and \(\mathfrak{M}u = Bu\). As \(Bu \in C_\infty(R)\) this implies \(u \in \mathcal{D}(M)\) and \(Mu = Bu\), cf. Theorem 3.1.

**Step 3:** \((X_t)_{t \geq 0}\) is a rich Feller process with symbol \(\varphi\). We use a perturbation theorem to prove the assertion. Define an operator \(N\) by

\[
Nu(x) := \int_{|y|>r} (u(x + \sigma(x)y) - u(x)) \nu_L(dy), \quad x \in R, u \in C_\infty(R).
\]

We claim that \(N\) is a bounded linear operator which is dissipative and maps \(C_\infty(R)\) into \(C_\infty(R)\). Indeed: Since \(\nu_L(B(0,r)^c) < \infty\), the boundedness of \(N\) is obvious. Moreover, it is clear from the definition that \(N\) is dissipative. Fix \(u \in C_\infty(R)\). Then

\[
|Nu(x)| \leq \left|\int_{|y|>r} (u(x) + \sup_{|y| \leq R} |u(y)|) \nu_L(dy)\right| + \|u\|_{\infty} \nu_L(\{y \in R; |x + \sigma(x)y| \leq R, |y| > r\})
\]

for any \(R > 0\). Letting first \(|x| \to \infty\) and then \(R \to \infty\), it follows from (4) that \(\lim_{|x| \to \infty} |Nu(x)| = 0\). Since a straightforward application of the dominated convergence theorem gives the continuity of \(Nu\), we conclude \(Nu \in C_\infty(R)\).

Applying [4, Corollary 1.7.2], we find that there exists a rich Feller process \((Z_t)_{t \geq 0}\) with generator \(L := M + N\). Note that the generator \(L\) restricted to \(C^\infty_c(R)\) equals the pseudo-differential operator \(A\) with symbol \(-q(x, \xi) = -\psi(\sigma(x)^T \xi)\). Since \((Z_t)_{t \geq 0}\) is a solution to the \((A, C^\infty_c(R))\)-martingale problem and therefore also a weak solution to (3), cf. [9, Corollary 2.5], it follows from the weak uniqueness of the solution that \(X_t \overset{d}{=} Z_t\) for all \(t \geq 0\) (for a given initial distribution). Hence, \((X_t)_{t \geq 0}\) and \((Z_t)_{t \geq 0}\) have the same semigroup, and this implies that \((X_t)_{t \geq 0}\) is a rich Feller process with symbol \(q(x, \xi) = \psi(\sigma(x)^T \xi)\).

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Denote by \(A\) the pseudo-differential operator with symbol \(-q\). Note that the Lévy-Khintchine formula shows that the family of Lévy triplets associated with \(q\) is given by

\[
b(x) = \sigma(x)b_L + \int_{|y|<1} \sigma(x)y(1_{(0,1)}(|\sigma(x)y|) - 1_{(0,1)}(|y|)) \nu_L(dy)
\]

\[
Q(x) = \sigma(x)Q_L\sigma(x)^T
\]

\[
\nu(x, B) = \int 1_{B}(\sigma(x)y) \nu_L(dy), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\]

\[8\]
Since the domain of the generator of any rich Feller process contains $C^2_c(\mathbb{R}^d)$, cf. [3, Theorem 2.37c]), it suffices to show that the unique weak solution $(X_t)_{t \geq 0}$ is a rich Feller process if, and only if, (4) holds, and to identify the symbol of the process.

It is clear from Theorem 3.5 that (4) is sufficient; it remains to prove the necessity of (4). Suppose that the weak solution $(X_t)_{t \geq 0}$ is a rich Feller process. We claim that $q$ is the symbol of the Feller process $(X_t)_{t \geq 0}$, i.e. that the generator $L$ satisfies $Af = Lf$ for any $f \in C^\infty_c(\mathbb{R}^d)$. Denote by $\tau^*_r := \inf\{t \geq 0; |X_t - x| \geq r\}$ the exit time from the ball $B(x, r)$ and fix a non-absorbing point $x \in \mathbb{R}^d$. Since $x \mapsto q(x, \xi) = \psi(\sigma(x)^T \xi)$ is continuous for all $\xi \in \mathbb{R}^d$ and $q$ is locally bounded, it follows from the dominated convergence theorem that the pseudo-differential operator $A$ maps $C^\infty_c(\mathbb{R}^d)$ into $C(\mathbb{R}^d)$. Therefore an application of Itô’s formula and the fundamental theorem of calculus yields

$$\lim_{r \to 0} \frac{1}{E^{x,r}} (E^x f(X_{\tau^*_r}) - f(x)) = Af(x) \quad \text{for all } f \in C^\infty_c(\mathbb{R}^d).$$

On the other hand, $C^\infty_c(\mathbb{R}^d)$ is contained in the domain of the generator $(L, \mathcal{D}(L))$ of the Feller process $(X_t)_{t \geq 0}$, and therefore by Dynkin’s formula

$$\lim_{r \to 0} \frac{1}{E^{x,r}} (E^x f(X_{\tau^*_r}) - f(x)) = Lf(x) \quad \text{for all } f \in C^\infty_c(\mathbb{R}^d).$$

Hence, $Lf(x) = Af(x)$ for any non-absorbing point $x \in \mathbb{R}^d$. If $x \in \mathbb{R}^d$ is absorbing, then it is not difficult to see that $Af(x) = 0 = Lf(x)$. Hence, $A = L$ for any $f \in C^\infty_c(\mathbb{R}^d)$. This shows that $q$ is the symbol of the Feller process $(X_t)_{t \geq 0}$. By Lemma 3.2, $Af = Lf \in C_\infty(\mathbb{R}^d)$ implies (4).

\[\square\]

**Proof of Corollary 1.2.** It is well known (see e.g. [5, Theorem IV.9.1]) that there exists a (pathwise) unique strong solution to the SDE (3) if $\sigma$ is Lipschitz continuous. Since any strong solution is also a weak solution and pathwise uniqueness implies uniqueness in law (see e.g. [15, Corollary 140]), there exists a unique (in law) weak solution to (3) for any $r \in (0, \infty]$. Now the claim follows from Theorem 1.1. \(\square\)

### 4 Examples

In this section we present some illustrating examples.

#### 4.1 Example (Sublinear growth)

Let $(L_t)_{t \geq 0}$ be a $k$-dimensional Lévy process with Lévy triplet $(b_L, Q_L, \nu_L)$ and characteristic exponent $\psi$ and let $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times k}$ be a Lipschitz continuous function. If $\sigma$ is of sublinear growth, i.e.

$$\lim_{|x| \to \infty} \frac{|\sigma(x)|}{|x|} = 0,$$

then the strong solution to the SDE

$$dX_t = \sigma(X_{t-}) \, dL_t, \quad X_0 = x,$$

is a rich $d$-dimensional Feller process with symbol $q(x, \xi) = \psi(\sigma(x)^T \xi)$.

Let us mention that this statement can be also deduced from a result by Böttcher [2].

**Proof of Example 4.1.** We show that any function $\sigma$ of sublinear growth satisfies (4). By the triangle inequality, we have

$$\nu_L \left( \{ y \in \mathbb{R}^k : \sigma(x) y \in B(-x, r) \} \right) = \nu_L \left( \{ y \in \mathbb{R}^k : \frac{|\sigma(x) y + x|}{|x|} \leq \frac{r}{|x|} \} \right) \\ \leq \nu_L \left( \{ y \in \mathbb{R}^k : 1 - \frac{|\sigma(x)| |y|}{|x|} \leq \frac{r}{|x|} \} \right).$$
For any fixed $r > 0$ and $\varepsilon > 0$ we can choose $R > 0$ sufficiently large such that $r/|x| \leq \varepsilon$ and $|\sigma(x)|/|x| \leq \varepsilon$ for all $|x| \geq R$. Hence,
\[
\nu_L((y \in \mathbb{R}^k; \sigma(x)y \in B(-x, r))) \leq \nu_L \left( \left\{ y \in \mathbb{R}^k; 1 - \varepsilon|y| \leq \varepsilon \right\} \right) = \nu_L \left( \mathbb{R}^k \setminus B\left(0, \varepsilon^{-1} - 1\right) \right) \xrightarrow{\varepsilon \to 0} 0.
\]

If the driving Lévy process $(L_t)_{t \geq 0}$ is a one-dimensional symmetric $\alpha$-stable process, we obtain the following stronger result.

4.2 Example Let $(L_t)_{t \geq 0}$ be a one-dimensional symmetric $\alpha$-stable Lévy process for some $\alpha \in (0, 2]$, i.e. a Lévy process with characteristic exponent $\psi(\xi) = |\xi|^\alpha$, $\xi \in \mathbb{R}$. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous mapping such that the SDE (17) has a unique weak solution (taking values in $\mathbb{R}^d$) for any initial distribution $\mu$. Then the unique weak solution $(X_t)_{t \geq 0}$ is a rich Feller process if, and only if,
\[
\lim_{|x| \to \infty} \frac{|\sigma(x)|^\alpha}{|x|^{1+\alpha}} = 0. \tag{18}
\]
In this case, the symbol $q$ of the Feller process $(X_t)_{t \geq 0}$ is given by $q(x, \xi) = |\sigma(x)|^\alpha |\xi|^\alpha$.

Clearly, the growth condition (18) is, in particular, satisfied if $\sigma$ is at most of linear growth.

Proof of Example 4.2. Fix $r > 0$. Since
\[
\int_{|y| \leq r} 1_{B(x, r)}(\sigma(x)y) \frac{1}{|y|^{1+\alpha}} \, dy = |\sigma(x)|^\alpha \int_{|z| \leq r} 1_{B(x, r)}(z) \frac{1}{|z|^{1+\alpha}} \, dz
\]
for any $|x| > r$, it follows easily from the fact that
\[
\int_{|z| \leq r} 1_{B(x, r)}(z) \frac{1}{|z|^{1+\alpha}} \, dz \leq |x|^{-\alpha-1}, \quad |x| \gg 1
\]
that (4) is equivalent to (18). Now the assertion follows from Theorem 1.1.\[\square\]

It is known that the SDE (17) driven by a one-dimensional symmetric $\alpha$-stable Lévy process has a unique weak solution if $\sigma$ is continuous and one of the following two conditions is satisfied.

(i). (cf. Zanzotto [19]) $\alpha \in (1, 2]$ and
\[
\{ x \in \mathbb{R}; \sigma(x) = 0 \} = \left\{ x \in \mathbb{R}; \forall \delta \in (0, 1) : \int_{|z| \leq \delta} |\sigma(y)|^{-\alpha} \, dy = \infty \right\}. \tag{19}
\]

(ii). (cf. Kühn [11]) $\alpha \in (0, 2]$, $\inf_x \sigma(x) > 0$.

Note that (19) is satisfied for any continuous function $\sigma$ such that $\inf_{x \in \mathbb{R}} \sigma(x) > 0$, and therefore the first condition is for $\alpha \in (1, 2]$ more general than the second one.

Example 4.2 shows, in particular, that for a one-dimensional symmetric $\alpha$-stable Lévy process $(L_t)_{t \geq 0}$, $\alpha \in (1, 2]$, the (weak) solution to the SDE
\[
dX_t = |X_{t-}|^\beta \, dL_t, \quad X_0 = x,
\]
is a rich Feller process with symbol $q(x, \xi) = |x|^{\beta \alpha} |\xi|^\alpha$ for all $\beta \in [1/\alpha, 1)$.

The next example discusses generalized Ornstein–Uhlenbeck processes which have been studied by Behne & Lindner [1].

4.3 Example (Generalized Ornstein–Uhlenbeck process) Let $L_t = (U_t, V_t)$ be a two-dimensional Lévy process with characteristic exponent $\psi$ and Lévy triplet $(b_L, Q_L, \nu_L)$. The general-ized Ornstein–Uhlenbeck process
\[
dX_t = X_{t-} \, dU_t + dV_t, \quad X_0 = x,
\]
is a rich Feller process (with symbol $q(x, \xi) = \psi((x, 1)^T \xi)$) if, and only if,
\[
\nu_L(\{-1\} \times \mathbb{R}) = 0. \tag{20}
\]
Behme & Lindner [1] proved that the generalized Ornstein–Uhlenbeck process is a rich Feller process if (20) holds. Our proof is not only considerably shorter, but also shows that (20) is, in fact, a necessary and sufficient condition.

Proof of Example 4.3. Set \( \sigma(x) := (x, 1) \) for \( x \in \mathbb{R} \) and

\[
A(x) := \{ y \in \mathbb{R}^2; |\sigma(x)y + x| \leq r \} = \{ y \in \mathbb{R}^2; |xy_1 + y_2 + x| \leq r \} 
= \left\{ y \in \mathbb{R}^2; \left| y_1 + \frac{y_2}{x} + 1 \right| \leq \frac{r}{|x|} \right\}.
\]

Since \( \nu_L \) is a \( \sigma \)-finite measure on \( \mathbb{R}^2 \setminus \{0\} \) and \( I_{A(x)}(y) \xrightarrow{|x|\to\infty} I_{\{y_1 = -1\}} \), the dominated convergence theorem yields \( \nu_L(A(x)) \xrightarrow{|x|\to\infty} \nu_L(\{-1\} \times \mathbb{R}) \). Consequently, (4) holds if, and only if, \( \nu_L(\{-1\} \times \mathbb{R}) = 0 \).

Using a very similar reasoning, we obtain the following result on solutions of linear SDEs.

4.4 Example (Linear SDE) Let \( (L_t)_{t \geq 0} \) be a \( d \)-dimensional Lévy process with characteristic exponent \( \psi \) and Lévy triplet \((b_L, Q_L, \nu_L)\). Then for any fixed \( c \in \mathbb{R}^d \) the solution to the linear SDE

\[
dX_t = cX_t \, dL_t, \quad X_0 = x,
\]

is a rich Feller process (with symbol \( q(x, \xi) := \psi(x \xi c), \ x, \xi \in \mathbb{R} \)), if, and only if,

\[
\nu_L \left( \{ y \in \mathbb{R}^d; c \cdot y = -1 \} \right) = 0.
\]

Example 4.5 shows that Theorem 1.1 may fail to hold if the coefficient \( \sigma \) does not satisfy the linear growth condition (2).

4.5 Example The ordinary differential equation

\[
dx_t = -X_t^3 \, dt, \quad X_0 = x
\]

has a unique solution which is given by

\[
X_t = \frac{x}{\sqrt{1 + 2tx^2}}, \quad t \geq 0, x \in \mathbb{R}.
\]

Although (4) is trivially satisfied, the process \( (X_t)_{t \geq 0} \) is not a Feller process since the associated semigroup \((T_t)_{t \geq 0} \) does not satisfy the Feller property:

\[
\lim_{|x| \to \infty} T_t f(x) = f \left( \frac{1}{\sqrt{2t}} \right),
\]

cf. [3, Example 2.26b)] and [17, Remark 2.12].

Acknowledgements I would like to thank Björn Böttcher, Alexei Kulik and René Schilling for their helpful comments and suggestions. An earlier version of this paper relied on a result [18, Theorem 2.5] by van Casteren; many thanks to Tristan Haugomat for pointing out that this result is wrong and that a counterexample can be found in [17, Remark 2.12] (see Example 4.5). The error in van Casteren’s [18] proof is in Proposition 2.2 where it is claimed that \( x \mapsto \exp(tu(X_x)) \) vanishes at \( \infty \) for any \( u \in C_\infty(\mathbb{R}^d) \); as Example 4.5 shows this is, in general, not correct. However, a close look at his proof reveals that a weaker statement than the Feller property holds true: For any \( u \in C_\infty(\mathbb{R}^d) \) and \( t \geq 0 \) there exists a constant \( c \in \mathbb{R} \) such that \( \lim_{|x| \to \infty} \exp(tu(x)) = c \). The current version of this paper does not rely on van Casterens results.
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