Local Quantum Measurement Demands Type-Sensitive Information Principles for Global Correlations

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Physical theories with local structure similar to quantum theory can allow beyond-quantum global states that are in agreement with unentangled Gleason’s theorem. In a standard Bell experiment any such bipartite state produces correlations that are always quantum simulable. In this limited classical-input-classical-output Bell scenario, we show that there exist bipartite beyond-quantum states that produce correlations all of which are in-fact classically simulable. However, if the type of Bell scenario is generalized to consider quantum states as inputs, we then show that any such bipartite beyond-quantum state yields beyond-quantum input-output correlations. We also analyze the implication of this quantum input scenario while studying generic multipartite correlations obtained from local quantum theory but potentially allowing different global structure. Our study suggests the requirement of type sensitive information principles for isolating the quantum correlations from the beyond-quantum ones.

Introduction.— Correlations among distant events established through the violation of Bell type inequalities confirm nonlocal behavior of the physical world \([1–4]\). Nonseparable multipartite quantum states yielding such correlations, in Schrödinger’s words, are “...the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought” \([5]\). The advent of quantum information science identifies the power of such nonlocal correlations in numerous device independent protocols – cryptographic key distribution \([6]\), randomness certification \([7]\) and amplification \([8]\), dimension witness \([9]\) are few canonical examples. Cirel’son’s result \([10]\), however, establishes that the nonlocal strength of quantum correlations is limited compared to the general no-signaling (NS) ones \([11]\) as depicted in the celebrated Clauser-Horne-Shimony-Holt (CHSH) inequality violation \([12]\).

To comprehend the limited nonlocal behavior of quantum theory and to obtain a better understanding of the theory itself, researchers have proposed several approaches to compare and contrast quantum theory with other conceivable physical theories constructed within more general mathematical frameworks \([13]\). Here, we consider a class of theories wherein local measurements are described quantum mechanically, but they allow global structure more generic than quantum theory \([14–21]\). Gleason’s celebrated result in quantum foundations proves that any map from generalized measurements to probability distributions can be written as the trace rule with the appropriate quantum state \([22]\) (see also \([23, 24]\) for simpler proof). This theorem when appraised to the case of local observables acting on bipartite or general multipartite systems, hence called the unentangled Gleason’s theorem, endorses the joint NS probability distributions to be obtained from some Hermitian operator called the positive over all pure tensors (POPT) state \([14–17]\). Although the set of POPT states is strictly larger than the set of quantum states (density operators), in a recent work, Barnum et al. have shown that the set of bipartite correlations attainable from the POPT states is precisely the set of quantum correlations \([18]\). Consequently, their result provokes a far-reaching conclusion “…that if nonlocal correlations beyond quantum mechanics are obtained in any experiment then quantum theory would be invalidated even locally.”

In this work we analyze the correlations of multipartite POPT states obtained from local measurements performed on their constituent parts by considering a generalized Bell scenario as introduced in \([25]\). While in the standard Bell scenario spatially separated parties receive some classical inputs and accordingly generate some classical outputs by performing local measurements on their respective parts of some composite system, recently Buscemi has generalized the scenario where the parties receive quantum inputs instead of classical variables \([25]\). In this generalized scenario he has shown that all entangled states exhibit nonlocality, whereas some of them allow local-hidden-variable (LHV) model in classical input scenario \([26–28]\). Considering this generalized scenario, here we show that not all correlations obtained from bipartite POPT states are quantum simulable. In fact, every beyond quantum POPT state produces some beyond quantum correlations in some quantum input game. On the other hand, to illustrate the limitations of the standard Bell scenario, we show that there are POPT states which produce classical-input-classical-output correlations that are not only quantum simulable, rather simulable classically. Our result shows that the strong claim made by the authors in \([18]\) will not be correct anymore in this generalized Bell scenario which is allowed within the framework of local quantum theory. From a foundational perspective our study welcomes new information principles incorporating this generalized Bell type scenario to isolate quantum correlation from beyond-quantum ones. We also analyze the implication of this generalized scenario for multipartite
correlations and answer an open question raised in [19].

Gleason’s theorem.– We investigate the class of locally quantum theories studied in a series of works in the recent past [14–21]. In accordance with these works, by locally quantum we mean an elementary system possessed by some party (say Alice) is described by some Hilbert space $\mathcal{H}_A$ with dimension $d_A$. A generic quantum measurement $M_A$, also called positive-operator-valued measurement (POVM) [29], is given by a collection of effect operators $\{\pi_A^a\}$ satisfying the constraint $\sum_a \pi_A^a = 1_A$; where $\forall a$, $\pi_A^a \in \mathcal{E}(\mathcal{H}_A) \subset \mathcal{L}(\mathcal{H}_A)$, with $\mathcal{E}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_A)$ respectively denoting the set of all positive operators and bounded linear operators acting on $\mathcal{H}_A$; and $1_A$ is the identity operator on $\mathcal{H}_A$. The probability $p(a|\pi_A^a)$ that Alice obtains an outcome $a$ for the measurement $m_A \equiv \{\pi_A^a\}$ is given by a generalized probability measure $\mu: \mathcal{E}(\mathcal{H}_A) \rightarrow [0,1]$, satisfying the properties (i) $\forall \pi_A \in \mathcal{E}(\mathcal{H}_A)$, $0 \leq \mu(\pi_A) \leq 1$, (ii) $\mu(1_A) = 1$, and (iii) $\mu(\sum_a \pi_A^a) = \sum_a \mu(\pi_A^a)$ for any sequence $\pi_A^a, \pi_A^b, \cdots$ with $\sum_a \pi_A^a \leq 1_A$. According to Gleason’s theorem for POVM any such generalized probability measure is of the form $\mu(\pi_A^a) = \text{Tr}(\rho_A \pi_A^a)$, for some density operator $\rho_A \in D(\mathcal{H}_A)$ [23]; $D(\mathcal{H}_A)$ denotes the set of positive operators with unit-trace on $\mathcal{H}_A$. Importantly, Gleason’s theorem is general enough to hold true for multipartite systems associated with the tensor product Hilbert space $\bigotimes_i \mathcal{H}_A$, i.e. a generalized probability measure on POVM, $\mu: \bigotimes_i \mathcal{E}(\mathcal{H}_A) \rightarrow [0,1]$, satisfying (i)-(iii), is of the form $\mu(\pi) = \text{Tr}(\rho \pi)$ for some density operator $\rho \in D(\bigotimes_i \mathcal{H}_A)$ for every effect $\pi \in \mathcal{E}(\bigotimes_i \mathcal{H}_A)$. Interesting situations arise when the probability measure is restricted to local observables acting on bipartite and general multipartite systems [14–17]. Like the original theorem, in this case the goal is to characterize the probability measure on product effects, i.e. to specify $\mu: \bigotimes_i \mathcal{E}(\mathcal{H}_A) \rightarrow [0,1]$ satisfying the conditions (i)-(iii). It has been shown that any such measure is of the form $\mu(\bigotimes_i \pi_i^a) = \text{Tr}[W(\bigotimes_i \pi_i^a)]$ for some Hermitian operator $W$ acting on $\bigotimes_i \mathcal{H}_A$. Clearly, $W$ is positive over all pure tensors (POPT). Furthermore, condition (ii) implies $\text{Tr}(W) = 1$. However, positivity of $W$ over entangled states is not assured and such a negative $W$ can act as an entanglement witness operator [43]. We will denote the set of POPT states as $\mathcal{W}(\bigotimes_i \mathcal{H}_A)$ which includes $D(\bigotimes_i \mathcal{H}_A)$ as a proper subset. A $W$ will be called ‘beyond quantum state’ (BQS) whenever $W \in \mathcal{W}(\bigotimes_i \mathcal{H}_A)$ but $W \notin D(\bigotimes_i \mathcal{H}_A)$. In this work, our aim is to study the correlations obtained from BQSs. But before going to our main result we briefly recall the standard input-output scenario for the Bell correlation experiment.

**Standard Bell scenario.–** A multipartite Bell scenario considers $n$ distant parties, $A_1, A_2, \cdots, A_n$, with each party $A_i$ receiving an independent and random classical input $s_i \in S_i$ from a Referee. Each party then produces a classical output $a_i \in O_i$, based on which the Referee yields some payoff $\mathcal{P} : \times_{i=1}^n (S_i \times O_i) \rightarrow \mathbb{R}$. An implicit rule is that the parties cannot communicate with one another once the game starts, although they can agree upon some pre-shared strategy. At the end of the game the players generate a joint input-output probability distribution $P = \{p(a_1 \cdots a_n|s_1 \cdots s_n)\}_{a, s \in O}^n$, which is also called the input-output correlation. For a NS correlation communication between any disjoint subgroups of the parties is prohibited. The collection of all NS correlations forms a convex polytope $\mathcal{N}S$. A correlation is called classical if and only if it is of the form $p(a_1 \cdots a_n|s_1 \cdots s_n) = \int_A p(\lambda) \prod_i p(a_i|s_i, \lambda) d\lambda$, where $\lambda \in \Lambda$ is some classical variable shared among the parties. In Bell’s terminology such correlations are called locally causal and the collection of such correlations also forms a convex polytope $\mathcal{L}$. On the other hand, a correlation is called quantum if it is obtained from some quantum state through local measurements, i.e. $p_Q(a_1 \cdots a_n|s_1 \cdots s_n) = \text{Tr}(p(\rho, \pi_i^a))$ for some $\pi_i^a \in \mathcal{E}(\mathcal{H}_A)$ and $\rho \in D(\bigotimes_i \mathcal{H}_A)$. The set of all quantum correlations $\mathcal{Q}$ forms a convex set but not a polytope. The framework of locally quantum theories allows us to define the correlation set obtained form the POPT states. Following the terminology of Ref. [19] we call such a correlation ‘Gleason correlation’ and denote the set as $\mathcal{G}L$. The following set inclusion relations are immediate: $\mathcal{L} \subseteq \mathcal{Q} \subseteq \mathcal{G}L \subseteq \mathcal{N}S$. In the simplest bipartite scenario, with each party having two inputs and each input with two outputs, Bell’s seminal work establishes the proper subset inclusion $\mathcal{L} \subseteq \mathcal{Q}$ [1]. In the same setup, Cirac’s result and Popescu-Rohlich’s result establish the proper subset inclusion $\mathcal{Q} \subseteq \mathcal{N}S$ [10, 11]. With a similar spirit, recently Barnum et al. have proved an interesting result in the generic bipartite scenario by establishing the equality $\mathcal{Q} = \mathcal{G}L$ [18]. More precisely, they have shown that correlations obtained from any BQS $W_{AB} \in \mathcal{W}(\mathcal{H}_A \otimes \mathcal{H}_B)$, are in-fact quantum simulable, i.e. for every $W_{AB}$ and for every local measurements $M_A = \{\pi_A^a\}$ and $M_B = \{\pi_B^b\}$, there always exists a quantum state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ and measurements $\tilde{M}_A = \{\tilde{\pi}_A^a\}$, $\tilde{M}_B = \{\tilde{\pi}_B^b\}$ such that, $\text{Tr}[W_{AB}(\pi_A^a \otimes \pi_B^b)] = \text{Tr}[\rho_{AB}(\tilde{\pi}_A^a \otimes \tilde{\pi}_B^b)]$. In this classical input-output scenario we are now in a position to prove our first result that in some sense can be considered stronger than the result of Barnum et al.

**Proposition 1.** There exist beyond quantum bipartite states yielding correlations that are classically simulable.

**Proof.** Consider the one parameter family of operators $W_p := p \Gamma[|\phi^+\rangle \langle \phi^+|] + (1-p) I/4$ acting on the bipartite Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$, where $|\phi^+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ and $\Gamma$ denotes partial transposition. $W_p$ is a beyond-quantum state for $1/3 < p \leq 1$. The POPTness directly follows from the expression $\text{Tr}[W_p(\mathcal{P}_p \otimes \mathcal{P}_m)] = 1/4(1 + p \hat{n} \hat{m})$, where $\mathcal{P}_p := 1/2(1 + \hat{x} \sigma)$; and beyond
quantumness follows from the explicit eigenvalue calculation of the operator \( W_p \). The correlations obtained from \( W_p \) are classically simulable, i.e. allows a local hidden variable (LHV) description for certain values of \( p \). If we consider projective measurements only then LHV description is possible whenever \( p \leq 1/2 \), whereas for generic POVMs one can have such a description for \( p \leq 5/12 \). The LHV models are motivated from the well known constructions of Werner \([26]\) and Barrett \([27]\). The explicit construction we defer to the Appendix.

The result of Barnum et al. \([18]\) and our Proposition 1 depicts the limitation of classical-input classical-output Bell scenario to reveal the full correlation strength of BQSs. At this point a more general Bell scenario turns out to be advantageous.

**Semiquantum Bell scenario.**— The scenario was introduced by Buscemi to establish the nonlocal behaviour of all entangled quantum states \([25]\). While in the standard Bell scenario the distant parties are given classical inputs, here they are given some quantum states as the inputs. In general, the input quantum states can be chosen from a nonorthogonal set. Whenever the states are mutually orthogonal the scenario boils down to the standard one. In fact, it is the indistinguishability of nonorthogonal input states that makes the scenario powerful enough to reveal the nonlocal behaviour of entangled quantum states that otherwise allow LHV in the classical input scenario \([26, 27]\). Subsequently, Buscemi’s work has generated a plethora of research interest both from foundational and practical perspectives \([31–36]\).

For a generic \( n \)-partite case, a semiquantum non-local game \( G_{SQ} \) is specified by the index sets \( S_i = \{ s_i \}, \{ a_i \} \) on \( n \) quantum systems \( A_i \) respectively, and a payoff function \( \beta : \times^n_{i=1}(S_i \times O_i) \rightarrow \mathbb{R} \). A referee picks the indices \( \{ s_i \}_{i=1}^n \) with probability \( \{ p_i(s_i) \}_{i=1}^n \) and sends the corresponding state \( \psi_{s_i}^{A_i} \) to the \( i \)th player (without revealing the actual index). Each of the players, without communicating with others, computes some classical index \( a_i \in O_i \) and sends it to the referee, who then gives some payoff \( \beta(s_1, \ldots, s_n, a_1, \ldots, a_n) \). The players can share some quantum state or BQS (say, \( Z_{A_1\ldots A_n} \)) and by making appropriate joint measurements on their respective parts of \( Z_{A_1\ldots A_n} \) and on their input states they can generate a correlation \( P_{Z_{A_1\ldots A_n}} := \{ p(a_1, \ldots, a_n|\psi_{s_1}^{A_1}, \ldots, \psi_{s_n}^{A_n}) \} \equiv \{ \text{Tr}[ (\otimes_i \pi^{A_i}_{s_i}) (\otimes_i \psi_{s_i}^{A_i} \otimes Z_{A_1\ldots A_n}) ] \} \) and thus obtain the expected payoff \( \mathcal{I}_{G_{SQ}}(Z_{A_1\ldots A_n}) := \sum_{s_1, a_1, \ldots, s_n, a_n} \beta(s_1, a_1, \ldots, s_n, a_n) \times p(a_1, \ldots, a_n|\psi_{s_1}^{A_1}, \ldots, \psi_{s_n}^{A_n}) \) (see Fig. 1). It is worth mentioning that an effect \( e \in \mathcal{E}(\otimes_i \mathcal{H}_{A_i}) \) on some BQS might yield negative probability. However, this does not affect our scenario as we consider that each party performs local measurements on her respective part of the BQS and the quantum input(s) received from the referee. Furthermore the input quantum states are assumed to be generated from independent sources, i.e. they are not entangled with each other. If it turns out that for some BQS \( W_{A_1\ldots A_n} \) we have \( \mathcal{I}_{G_{SQ}}(W_{A_1\ldots A_n}) < 0 \), while \( \mathcal{I}_{G_{SQ}}(\rho_{A_1\ldots A_n}) \geq 0, \forall \rho_{A_1\ldots A_n} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \), then the game \( G_{SQ} \) establishes the correlation strength of \( W_{A_1\ldots A_n} \) over quantum states. This motivates us to the following generic result for bipartite BQS.

**Theorem 1.** For every beyond quantum state \( W_{AB} \in W(\mathcal{H}_A \otimes \mathcal{H}_B) \) there exists a semiquantum game \( G_{SQ} \) such that \( \mathcal{I}_{G_{SQ}}(W_{AB}) < 0 \), while \( \mathcal{I}_{G_{SQ}}(\rho_{AB}) \geq 0, \forall \rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \).

*Proof.** At the core of our proof lies the classic Hahn-Banach separation theorem of convex analysis and the fact that for every beyond quantum states \( W_{AB} \in W(\mathcal{H}_A \otimes \mathcal{H}_B) \) there exists an entangled state \( \chi_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \) such that \( \text{Tr}[W_{AB}\chi_{AB}] < 0 \), whereas \( \text{Tr}[\sigma_{AB}\chi_{AB}] \geq 0, \forall \sigma_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \). Also note that, there exits (non-unique) choices of states \( \psi_{A_1}^{\delta_1} \in D(\mathcal{H}_A) \) and \( \psi_{B}^{\delta_2} \in D(\mathcal{H}_B) \) and some real coefficients \( \{ \delta_{s,t} \} \) such that \( \chi_{AB} = \sum_{s,t} \delta_{s,t} \psi_{A_1}^{\delta_1} \otimes \psi_{B}^{\delta_2} \), where \( T \) represents the transposition with respect to the computational basis. This leads us to the required game \( G_{SQ}^{1} \) where Alice and Bob receive quantum inputs \( \psi_{A_1}^{\delta_1} \) and \( \psi_{B}^{\delta_2} \) respectively, and are asked to produce binary outputs \( \in \{0,1\} \). Their average payoff will be calculated as \( \mathcal{I} := \sum_{s,t} \beta_{s,t} \rho(11|\psi_{A_1}^{\delta_1} \otimes \psi_{B}^{\delta_2}) \). Alice performs the measurement \( \{ P_{AA_1}^{+}, P_{AA_1}^{-} - P_{AA_1}^{+} \} \) on her part of the shared BQS and the quantum input \( \psi_{A_1}^{\delta_1} \). Here \( P_{AA_1}^{+} := |\phi^{+}\rangle_{AA_1} \langle \phi^{+}| \) with \( |\phi^{+}\rangle_{AA_1} := \frac{1}{\sqrt{d_A}} \sum_{i=0}^{d_A-1} |ii\rangle \) and \( P_{AA_1}^{-} \) corresponds to the outcome 1. Bob does the same measurement on his part of the shared BQS and the quantum input \( \psi_{B}^{\delta_2} \).
We therefore have,
\[
\mathcal{I}_{G_{sq}}^X(W_{AB}) = \sum_{s,t} \beta_{s,t} \text{Tr} \left[ D_{AA'}^+ \otimes D_{BB'}^+ \left( \psi_A^s \otimes \psi_B^t \right) \right]
\]
\[
= \sum_{s,t} \beta_{s,t} \text{Tr} \left[ (R_A \otimes R_B) W_{AB} \right];
\]
where \(R_A\) and \(R_B\) are the effective POVMs acting on the parts of Alice’s and Bob’s shares of the BQS respectively and are given by \(R_u := \text{Tr}_{\text{in}} \left[ \rho_u^\text{in} (I_u \otimes \psi_u^T) \right] = \frac{1}{d} \psi_u^T; \ u \in \{A, B\}\). Therefore, we have
\[
\mathcal{I}_{G_{sq}}^X(W_{AB}) = \frac{1}{d_{BA}} \sum_{s,t} \beta_{s,t} \text{Tr} \left[ \left( \psi_A^s \otimes \psi_B^t \right) W_{AB} \right]
\]
\[
= \frac{1}{d_{BA}} \text{Tr} \left[ \left( \sum_{s,t} \beta_{s,t} \psi_A^s \otimes \psi_B^t \right) W_{AB} \right]
\]
\[
= \frac{1}{d_{BA}} \text{Tr} \left[ \chi_{AB} W_{AB} \right] < 0.
\]
Given a quantum state \(\rho_{AB}\) let Alice and Bob perform some measurements \(M_{AA'} = \{ \pi_{AA'}^a \}\) and \(N_{BB'} = \{ \pi_{BB'}^b \}\) on their respective joint systems, where \(a, b \in \{0, 1\}\). The average payoff turns out to be
\[
\mathcal{I}_{G_{sq}}^X(\rho_{AB}) = \sum_{s,t} \beta_{s,t} \text{Tr} \left[ \pi_{AA'}^1 \otimes \pi_{BB'}^1 \left( \psi_A^s \otimes \rho_{AB} \otimes \psi_B^t \right) \right]
\]
\[
= \sum_{s,t} \beta_{s,t} \text{Tr} \left[ R_{A'B'} ( \psi_A^s \otimes \psi_B^t ) \right],
\]
where \(R_{A'B'} := \text{Tr}_{AB} [(\pi_{AA'}^1 \otimes \pi_{BB'}^1)(I_{A'B'} \otimes \rho_{AB})]\) is a positive operator. Using linearity of trace we get,
\[
\mathcal{I}_{G_{sq}}^X(\rho_{AB}) = \text{Tr} \left[ R_{A'B'} \left( \sum_{s,t} \beta_{s,t} \psi_A^s \otimes \psi_B^t \right) \right]
\]
\[
= \text{Tr} \left[ R_{A'B'} \chi_{A'B'}^T \right] \geq 0.
\]
The last inequality follows due to the fact that \(\chi_{A'B'}^T\) is also a valid density operator, and this completes the proof.

The semi quantum scenario also has important implication while studying correlations in multipartite (involving more than two parties) scenarios. Acín et al. have already pointed out that the result of Barnum et al. does not generalize to the multipartite scenario even in the classical-input classical-output paradigm [19]. They have provided examples of multipartite BQSs producing beyond quantum correlations within the standard Bell scenario. They have also pointed out that a BQS of the form
\[
W_{A_1 \cdots A_N} = \sum_k \psi_k (A_{A_1}^k \otimes \cdots \otimes A_{A_N}^k),
\]
will not generate any classical-input classical-output correlation that lies outside the set of correlations generated by quantum states. Here, \(\{\psi_k\}\) is a probability distribution, \(\rho_{A_1 \cdots A_N} \in \mathcal{D}(\bigotimes_{i=1}^N \mathcal{H}_{A_i})\), and \(\Lambda^k\) are positive but not completely positive trace preserving maps on \(\mathcal{L}(\mathcal{H}_{A_i})\) [39]. The authors in [19] have left the question open to identify the additional requirements to close the gap in their result. Our next result provides a solution to close this gap.

**Theorem 2.** For every BQS \(W_{A_1 \cdots A_N} \in \mathcal{W}(\bigotimes_{i=1}^N \mathcal{H}_{A_i})\) there exists a semiquantum game \(G_{sq}\) such that \(\mathcal{I}_{G_{sq}}(W_{A_1 \cdots A_N}) < 0\), whereas \(\mathcal{I}_{G_{sq}}(\rho_{A_1 \cdots A_N}) \geq 0, \ \forall \rho_{A_1 \cdots A_N} \in \mathcal{D}(\bigotimes_{i=1}^N \mathcal{H}_{A_i})\).

The proof is a straightforward generalization of the proof of Theorem 1. For the sake of completeness, however, we provide the proof in the Appendix. While Theorem 1 & 2 are just existence theorems, it is not hard to see that given an arbitrary BQS there is an efficient algorithm to construct a semiquantum game (the procedure is discussed in the Appendix). It is important to note that nonorthogonal quantum inputs are necessary to reveal the beyond quantum signature of correlation for any BQS of the form of Eq.(1). This implicitly follows from the results of Barnum et al. [18] and Acín et al. [19]. However, for the sake of better clarity we discuss this point further in the Appendix. It is worth mentioning that this semi quantum scenario is different from local tomography as it establishes beyond quantum nature of POPT states in a measurement device independent manner [31].

**Discussion:** One of the earnest research endeavours in quantum theory is to understand the limited nonlocal behaviour of quantum correlations. Apart from the foundational appeal, this question also has practical relevance as nonlocal correlations have been established as useful resources for several tasks. In the bipartite scenario the result of Barnum et al. [18] ensures that the correlations obtained from bipartite BQSs are attainable in quantum theory, Theorem 1 reports beyond quantum correlation from all such BQSs in quantum-input scenario (see Fig. 1). Naturally, it welcome new principle(s) to isolate the quantum correlations from beyond-quantum ones in this generalized scenario.
which is legitimate within the structure of local quantum description, the quantum correlations are not singled out naturally. Our Theorem 1 shows that all bipartite beyond quantum states compatible with unentangled Gleason’s theorem can yield beyond quantum correlations in this quantum input scenario, and accordingly divulges a more complex picture within the correlations zoo. Our study therefore welcomes new principles to identify the correlations in the physical world and importantly it also tells us that such a principle should take into consideration the type of the input-output scenario.

Our Theorem 2 establishes that within quantum input paradigm all multipartite BQSs yield beyond quantum correlation which was known only for a particular class of such state [19]. After the work of [19], Torre et al. have shown that when the local systems are identical qubits, then any theory admitting at least one continuous reversible interaction must be identical to quantum theory [20]. However, the result in [20] has also been obtained within the classical-input classical-output paradigm. It might be provoking to see what additional structures are required there to single out the quantum correlations in quantum-input scenario. On the other hand, within the classical-input-classical-output paradigm the authors in [41] and the present authors with other collaborators in [42] have studied beyond quantum correlations in time-like domain. Similar studies with quantum inputs might also provide new insight there, which is one of the questions present authors are trying to explore further.

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I. PROOF OF PROPOSITION 1

Proof. Our aim is to show that for certain range of the parameter $p$, the classical-input classical-output correlations obtained from the class of BQSs $W_p := p\Gamma[|\phi^+\rangle \langle \phi^+|] + (1-p)\mathbb{I}/4$ can be classically simulated. Given the classical inputs, the parties Alice and Bob perform some local measurement on their part of the BQS to obtain some classical outputs. The joint input-output probabilities are calculated using Born rule as the local systems are assumed to be quantum. By classically simulable we mean that the obtained correlations allow a local hidden variable (LHV) model, i.e. if Alice and Bob perform some measurements $A \equiv \{ A_i \mid A_i \geq 0 \& \sum_i A_i = \mathbb{I} \}$ and $B \equiv \{ B_j \mid B_j \geq 0 \& \sum_j B_j = \mathbb{I} \}$ respectively, then the joint probability distributions are factorizable.

$$P(A_i, B_j | A, B, W_p) = \int_{\Lambda} \omega(\lambda | W_p) P(A_i | A, \lambda) P(B_j | B, \lambda) d\lambda,$$

where $\lambda \in \Lambda$ is some shared variable (also called common cause/HV) and $\omega(\lambda | W_p)$ is a probability distribution on the HV space $\Lambda$.

Let us first consider the particular case, where measurement effects of Alice’s and Bob’s measurements are proportional to some rank one projection operator, i.e. $A_i = x_i P_i$ & $B_j = y_j Q_j$, with $0 < x_i, y_j \leq 1$. Note that, $\text{Tr} \left[ \Gamma[|\phi^+\rangle \langle \phi^+|] (P_i \otimes Q_j) \right] = 1/4(1 + \hat{m}_i \cdot \hat{n}_j)$, where $P_i := 1/2(\mathbb{I} + \hat{m}_i \cdot \sigma)$ and $Q_j := 1/2(\mathbb{I} + \hat{n}_j \cdot \sigma)$. This expression differs from $\text{Tr} \left[ |\psi^-\rangle \langle \psi^-| (P_i \otimes Q_j) \right] = 1/4(1 - \hat{m}_i \cdot \hat{n}_j)$ just by a negative sign, which motivates us to construct the LHV for $W_p$ by simply modifying the LHV model known for the noisy singlet state [27].

Let the hidden variables $\lambda \in \Lambda$ (which we shall now denote by $|\lambda\rangle$) be the unit vectors of a 2-dimensional complex Hilbert space. The local responses are given by,

$$P(A_i | A, \lambda) := \langle \lambda | A_i | \lambda \rangle \Theta \left( \langle \lambda | P_i | \lambda \rangle - \frac{1}{2} \right) + \frac{x_i}{2} \left( 1 - \sum_k \langle \lambda | A_k | \lambda \rangle \Theta \left( \langle \lambda | P_i | \lambda \rangle - \frac{1}{2} \right) \right);$$

$$P(B_j | B, \lambda) := y_j \left( 1 - \langle \lambda^\perp | Q_j | \lambda^\perp \rangle \right);$$

where $\Theta(x)$ is the Heaviside step function and $|\lambda^\perp\rangle$ is the state perpendicular to $|\lambda\rangle$. It is noteworthy that the response on Alice’s side is contextual, since the response of the effect $A_i$ depends on the other effects $A_k$’s constituting the measurement $A$. Substituting Eqs.(3) and (4) in Eq.(2) we obtain

$$P \left( A_i, B_j | A, B, W_p \right) = \int_{\Lambda} \omega(\lambda | W_p) \left[ \langle \lambda | A_i | \lambda \rangle \Theta \left( \langle \lambda | P_i | \lambda \rangle - \frac{1}{2} \right) + \frac{x_i}{2} \left( 1 - \sum_k \langle \lambda | A_k | \lambda \rangle \Theta \left( \langle \lambda | P_i | \lambda \rangle - \frac{1}{2} \right) \right) \right] $$

$$\times \left[ y_j \left( 1 - \langle \lambda^\perp | Q_j | \lambda^\perp \rangle \right) \right] d\lambda.$$ 

To check the reproducibility condition let us define the following quantity:

$$I_{ij} := x_i y_j \int d\lambda \omega(\lambda | W_p) \Theta \left( \langle \lambda | P_i | \lambda \rangle - \frac{1}{2} \right) \langle \lambda | P_i | \lambda \rangle \langle \lambda^\perp | Q_j | \lambda^\perp \rangle .$$

Using Eq.(6), Eq.(5) can be written as

$$P \left( A_i, B_j | A, B, W_p \right) = \left( -I_{ij} - \frac{1}{2} x_i y_j \int d\lambda \omega(\lambda | W_p) \langle \lambda^\perp | Q_j | \lambda^\perp \rangle \right) + \left( \frac{x_i}{2} \sum_k I_{kij} \right)$$

$$+ \left( y_j \sum_l I_{ijl} \right) + \left( \frac{x_i y_j}{2} \int d\lambda \omega(\lambda | W_p) \right) - \left( \frac{x_i y_j}{2} \sum_{kl} I_{ikl} \right)$$

$$= \frac{x_i y_j}{2} \left( \int d\lambda \omega(\lambda | W_p) - \int d\lambda \omega(\lambda | W_p) \langle \lambda^\perp | Q_j | \lambda^\perp \rangle \right)$$

$$+ \left( -I_{ij} - \frac{1}{2} x_i y_j \sum_k I_{kij} \right) + \left( y_j \sum_l I_{ijl} - \frac{x_i y_j}{2} \sum_{kl} I_{ikl} \right).$$

(7)
The quantity $c \equiv \int d\lambda \, \omega (\lambda|W_p) \langle \lambda^+|Q_j|\lambda^+ \rangle$ is invariant under the change of $Q_j$. Now, $\Sigma_j y_j c = \Sigma_j y_j \int d\lambda \, \omega (\lambda|W_p) \langle \lambda^+|Q_j|\lambda^+ \rangle = \int d\lambda \, \omega (\lambda|W_p) = 1$. Thus $c = 1/2$; which yields,

$$P(A_i, B_j|A, B, W_p) = \frac{x_i y_j}{4} + \left( -I_{ij} + \frac{x_i}{2} \sum_k I_{kj} \right) + \left( y_j \sum_l I_{li} - \frac{x_j y_j}{2} \sum_{kl} I_{kl} \right). \quad (8)$$

In order to evaluate $I_{ij}$ we write $|\lambda\rangle = z_0 |0\rangle + z_1 |1\rangle$ where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis for $C^2$. Let $z_\nu = r_\nu e^{i\theta_\nu}$ for $\nu \in \{0, 1\}$. We choose $|0\rangle$ to be such that $|0\rangle \langle 0| = P_i$. Writing $u_\nu = r_\nu^2$ and $Q_{ij} = |q_{ij}\rangle \langle q_{ij}|$ and using the fact $\langle \lambda|q_{ij}\rangle \langle q_{ij}|\lambda\rangle = \langle \lambda|q_{ij}\rangle \langle q_{ij}|\lambda\rangle$, we get

$$I_{ij} = x_i y_j \int d\lambda \, \omega (\lambda|W_p) \Theta(\langle \lambda|P_i|\lambda\rangle - 1/2) \langle \lambda|P_i|\lambda\rangle \langle \lambda|q_{ij}\rangle \langle q_{ij}|\lambda\rangle$$

$$= \frac{1}{N} x_i y_j \sum_{\nu=0}^{1} |\langle q_{ij}|\nu\rangle|^2 \int_{1/2}^{1} du_0 \int_{0}^{1} du_1 \delta(u_0 + u_1 - 1) u_0 u_\nu = x_i y_j \sum_{\nu=0}^{1} |\langle q_{ij}|\nu\rangle|^2 J_\nu, \quad (9)$$

where we have assumed $\omega (\lambda|W_p)$ to be a uniform distribution over $\Lambda$ and

$$N := \int_{0}^{1} du_0 \int_{0}^{1} du_1 \delta(u_0 + u_1 - 1), \quad (10)$$

$$J_\nu := \frac{1}{N} \int_{1/2}^{1} du_0 \int_{0}^{1} du_1 \delta(u_0 + u_1 - 1) u_0 u_\nu. \quad (11)$$

Defining $\bar{J} \equiv \frac{1}{N} \int_{1/2}^{1} du_0 \int_{0}^{1} du_1 \delta(u_0 + u_1 - 1) u_0$ and using $u_0 + u_1 = 1$, we can write $J_1 = \bar{J} - J_0$. Using normalization condition for $|q_{ij}\rangle$ and the fact $|p_i\rangle = |0\rangle$ we get $|\langle q_{ij}|p_i\rangle|^2 = 1 - |\langle q_{ij}|p_i\rangle|^2$, which thus yields

$$I_{ij} = x_i y_j \left[ (\bar{J} - J_0) + (2J_0 - J) |\langle q_{ij}|p_i\rangle|^2 \right]. \quad (12)$$

Substituting Eq.(12) in Eq.(8) we get,

$$P(A_i, B_j|A, B, W_p) = x_i y_j \left( 1 + \frac{4J_0 - 2\bar{J}}{4} - (2J_0 - J) |\langle q_{ij}|p_i\rangle|^2 \right) \quad (13)$$

Straightforward integration yields $J_0 = \frac{7}{24}$ and $\bar{J} = \frac{3}{8}$, which further implies

$$P(A_i, B_j|A, B, W_p) = \frac{x_i y_j}{48} \left( 17 - 10 |\langle q_{ij}|p_i\rangle|^2 \right) = \frac{x_i y_j}{48} \left( 17 - 10 \text{Tr} (P_iQ_{ij}^+) \right) = x_i y_j \left( 1 + \frac{5}{12} \hat{m}_i \cdot \hat{n}_j \right). \quad (14)$$

Again, from the Born rule we have

$$P(A_i, B_j|A, B, W_p) = \text{Tr} \left[ (A_i \otimes B_j)(W_p) \right] = x_i y_j \left( 1 + p \hat{m}_i \cdot \hat{n}_j \right). \quad (15)$$

Therefore, for $p = \frac{5}{12}$ the BQS $W_p$ allows a LHV model when all the effects constituting Alice’s and Bob’s measurements are proportional to rank one projectors. We are now left to extend this model for more general measurements (consisting more than rank one effects). This can be argued by noticing that any POVM element $A$ is a Hermitian operator with $0 < A \leq 1$, and hence allows spectral decomposition of the form $A = \sum A_i$, where $A_i = x_i P_i$ are operators proportional to rank-one projectors like the ones considered above with $0 < x_i \leq 1$ and $P_i P_j = \delta_{ij} P_i$. Thus any general POVM measurement can be regarded as a coarse-grained measurement of the special scenario considered above. We associate the outcome $A_i$ of the finer measurement with the outcome $A$ of the coarse-grained measurement for all values of $i$. Thus we have a LHV model for $W_{5/12}$.

Once the LHV model is defined for a particular state, it can be extended for a large class of states. Suppose we have a LHV model for the state $\varphi_1$. It is then possible to construct a LHV model for a state $\varphi_2$ if it can be written in the form $\varphi_2 = \sum_j M_j \otimes N_j \varphi_1 M_j^\dagger \otimes N_j^\dagger$, such that $\sum_j M_j^\dagger M_j = \mathbb{I}$, $\sum_j N_j^\dagger N_j = \mathbb{I}$. For describing the LHV model of $\varphi_2$ we just need to modify the responses in the following way

$$P_{\varphi_2}(A_i|A, \lambda) := P_{\varphi_1}(A_i'|A', \lambda) \quad \text{where} \quad A_i' := \sum_k M_k^\dagger A_i M_k, \quad (16)$$

$$P_{\varphi_2}(B_j|B, \lambda) := P_{\varphi_1}(B_j'|B', \lambda) \quad \text{where} \quad B_j' := \sum_l N_l^\dagger B_j N_l. \quad (17)$$
If we now take $\omega(\lambda|\sigma_2) = \omega(\lambda|\sigma_1)$, the above construction will give

$$
\int d\lambda \; \omega(\lambda|\sigma_2) P_{\gamma}(A_i|A,\lambda)P_{\gamma}(B_j|B,\lambda) = \text{Tr}\left[ (A_i \otimes B_j^\dagger) \sigma_1 \right]
= \sum_{kl} \text{Tr}\left[ (M_k^i A_i M_k \otimes N_j B_j^\dagger N_j^\dagger) \sigma_1 \right]
= \sum_{kl} \text{Tr}\left[ (A_i \otimes B_j) \left( M_k^i \otimes N_j^\dagger \sigma_1 M_k^\dagger \otimes N_j^\dagger \right) \right]
= \text{Tr}\left[ (A_i \otimes B_j) \sigma_2 \right].
$$

(18)

Thus the above construction is successful in defining a valid LHV model for $\sigma_2$. It is obvious that any $W_{p'}$ can be created from $W_p$ just by using local operations if $p' \leq p$. This implies existence of a LHV model for $W_{p^{1/2}}$. Therefore, any classical-input classical-output correlation obtained from $W_p$ is classically simulable whenever $p \leq 5/12$. On the other hand, as discussed in the manuscript, $W_p$ is BQS for $p > 1/3$. This proves the claim of Proposition 1.

\textbf{Remark:} Motivated from the LHV model constructed in [44], it can be further shown that for $W_p$ a classical model exists for $p \leq 1/2$ whenever Alice’s and Bob measurements are limited to projective measurement only. In this case also $\lambda$’s are given by unit vectors of 2-dimensional complex Hilbert space and $\omega(\lambda|W_p)$ is taken to be uniform distribution. Using spherical polar coordinates we can denote the HVs as $\hat{\lambda} = \sin(\theta)\cos(\phi)\hat{i} + \sin(\theta)\sin(\phi)\hat{j} + \cos(\theta)\hat{k}$ and $\omega(\lambda|W_p) = 2\pi^2 \sin(\theta) d\theta d\phi$. Alice’s and Bob’s response are given by

$$
P(P_i|A,\lambda) = \cos^2(a_1/2) = \frac{1 + \cos(a_1)}{2};
$$

(19)

$$
P(Q_j|B,\lambda) = 1 \text{ if } 2\cos^2(a_2/2) < 1;
= 0 \text{ if } 2\cos^2(a_2/2) > 1.
$$

(20)

(21)

Here $a_1$ is the angle between the block vector of $P_i$ and $-\hat{\lambda}$, and $a_2$ is the angle between the block vector of $Q_j$ and $\hat{\lambda}$. Without any loss of generality we can consider $P_i = 1/2(\mathbb{I} + \hat{M}_i \sigma)$ and $Q_j = 1/2(\mathbb{I} + \hat{N}_j \sigma)$ with $\hat{M}_i = (\cos x, 0, \cos x)$ and $\hat{N}_j = (0, 0, 1)$. This implies $P(Q_j|B,\lambda) = 1$ for $a_2 < 3\pi/2$ and accordingly we have non zero contribution in the integral for $\theta < \pi$. Also we get $\cos(a_1) = -\sin(x)\sin(\theta)\cos(\phi) - \cos(x)\cos(\theta)$. Therefore, we have

$$
P(P_i, Q_j|A, B, W_p) = \int^{\theta=\pi}_{\theta=\pi} \int^{\phi=2\pi}_{\phi=0} \frac{1}{4\pi} \sin(\theta) \sin(\theta) \left[ 1 - \sin(x)\sin(\theta)\cos(\phi) - \cos(x)\cos(\theta) \right] d\theta d\phi
= \frac{1}{4} + \frac{\cos x}{8} = \frac{1}{4} (1 + \frac{1}{2} \hat{M}_i \cdot \hat{N}_j),
$$

(22)

which is same as the Born probability obtained from the state $W_p$ for $p = 1/2$. It is not difficult to see that the model can be extended for any values of $p \leq 1/2$.

\section{II. \textbf{EXPLICIT CONSTRUCTION OF SEMIQUANTUM GAME}}

\textbf{Special case:} Here we first construct a semiquantum game for the BQSs of form $W_p := p\Gamma[|\phi^+\rangle \langle \phi^+|] + (1 - p)\mathbb{I}/4$. Clearly $W_p$ corresponds to a BQS if and only if $1/3 < p \leq 1$. The entangled state $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \in \mathbb{C}^2 \otimes \mathbb{C}^2$ acts as a (beyond quantum) witness for this class of states. This evidently follows from the expression:

$$
W_p = \frac{P}{2} \left[ |0\rangle \langle 0| \otimes |0\rangle \langle 0| + |1\rangle \langle 1| \otimes |1\rangle \langle 1| + |\psi^+\rangle \langle \psi^+| + |\psi^+\rangle \langle \psi^+| - |\psi^-\rangle \langle \psi^-| + \frac{1 - p}{4} \mathbb{I}. \right.
$$

(23)

Now the state $|\psi^-\rangle$ allows the following decomposition:

$$
|\psi^-\rangle = \frac{1}{2} \left[ P_x^T P_x^T + P_y^T P_y^T - P_x^T P_y^T - P_y^T P_x^T + P_z^T P_x^T + P_x^T P_z^T - P_y^T P_z^T - P_z^T P_y^T + P_y^T P_z^T \right];
$$

(24)

where, $P_i P_j := P_i \otimes P_j$ with $P_i$ being the projector onto the up eigenstate of $\sigma_i$ for $i \in \{x, y, z\}$ and it is the projector onto the down eigenstate for $i \in \{\hat{x}, \hat{y}, \hat{z}\}$. This immediately leads us to the required semiquantum game $G_{sq}$. In
each run of the game, referee randomly choose the states $\psi^s_A = P_s$ and $\psi^t_B = P_t$ and respectively sends them to Alice and Bob without revealing the indices $s$ and $t$, where $s, t \in \{x, y, z, \bar{x}, \bar{y}, \bar{z}\}$. Alice and Bob needs to return classical output 1 to the referee and the average payoff will be calculated as $I_{GQL} := \sum_{s, t} p(s, t)I(\psi^s_A, \psi^t_B)$, where

$$\beta_{s,t} = \beta_{s,x} = \beta_{s,y} = \beta_{s,z} = 0 \quad \beta_{s,t} = 1/4.$$  

The winning condition demands Alice and Bob to generate a negative payoff. If Alice performs the measurement $\{P^{s^*}_A \otimes I_B, I_A \otimes P^{t^*}_B\}$ on her part of the shared state $W_{p > 1/3}$ and the quantum input $\psi^s_A$ received from the referee and if Bob also performs the same measurement $\{P^{s^*}_B, I_B \otimes P^{t^*}_B\}$ and send the outcome 1 when the projector $P^{s^*}_A \otimes P^{t^*}_B$ click, then we have

$$I_{GQL}(W_p) = \frac{1}{d_Bd_A} \text{Tr}[\chi_{AB}W_p] = \frac{1}{4} \text{Tr}[\psi^-(\psi^-)|W_p|] = \frac{1 - 3p}{16} < 0 \quad \text{whenever} \quad p > 1/3.$$

On the other hand, for every quantum strategy $I_{GQL}(\rho) \geq 0$. It should be noted that the decomposition in Eq.(24) is not a unique. Considering a different decomposition it is possible to come up with a different semiquantum game. For instance, one has $|\psi^-\rangle\langle\psi^-| = \sum_{a,b=1}^4 \beta_{ab} \psi^a \otimes \psi^b$ where $\{\beta_{ab}\}$ written in a matrix form are given by

$$[\beta_{ab}] := \begin{bmatrix}
-15/64 & 17/64 & 1/2 & -1/32 \\
17/64 & -15/64 & 1/2 & -1/32 \\
1/2 & 1/2 & -1 & 0 \\
-1/32 & -1/32 & 0 & 1/16
\end{bmatrix},$$

and $\psi^1 := P_x$, $\psi^2 := P_y$, $\psi^3 := P_z$ and $\psi^4 := P_t$.

**General Case:** We now provide an explicit procedure to construct a semiquantum game for any BQS $W_{A_1 \cdots A_n} \in W (\otimes_i H_{A_i}).$

First note that for a $d$-dimensional Hilbert space $C^d$ given an orthonormal basis $\{|a\rangle\}_{a=0}^{d-1} \subset C^d$, one can construct a non-orthogonal operator basis ($B^{proj}$) of Projectors from the orthogonal operator (computational) basis ($B^{comp}$) as follows:

$$\mathcal{L}(C^d) \supset B^{comp} := \{|a\rangle\langle b|\}_{a,b=0}^{d-1};$$

$$\mathcal{L}(C^d) \supset B^{proj} := \{|a\rangle\langle a|\}_{a=0}^{d-1} \cup \{P^a_1, P^a_2\}_{a=0}^{d-1}, \quad a < b;$$

where $P^a_1 := \frac{1}{2}(|a\rangle\langle a| + |b\rangle\langle b| + |a\rangle\langle b| + |b\rangle\langle a|), \quad P^a_2 := \frac{1}{2}(|a\rangle\langle a| + |b\rangle\langle b| + i|a\rangle\langle b| - i|b\rangle\langle a|)$. Notice that $B^{proj}$ has $d^2$ linearly independent projectors with $d$ number of them common to $B^{comp}$. If an operator is known in the $B^{comp}$ basis then it can be easily written in the $B^{proj}$ basis by making the following substitution:

$$|a\rangle\langle b| = \begin{cases}
P^a_1 - iP^b_2 - \frac{1 - i}{2}|a\rangle\langle a| - \frac{1 + i}{2}|b\rangle\langle b|, & a < b; \\
P^{a}_1 + iP^b_2 - \frac{1 + i}{2}|a\rangle\langle a| - \frac{1 - i}{2}|b\rangle\langle b|, & b < a.
\end{cases}$$

Now, given an arbitrary beyond quantum state $W_{A_1 \cdots A_n}$, a semi quantum game can be constructed by mimicking the following steps:

**S1:** Write down the spectral decomposition of $W_{A_1 \cdots A_n}$. Hermiticity of $W_{A_1 \cdots A_n}$ guarantees that the eigenvalues are real. Since $W_{A_1 \cdots A_n}$ is a BQS, it has least one negative eigenvalue with entangled eigen-projector. Let the eigen-projector corresponding to a negative eigenvalue ($\lambda < 0$) be $\chi_{A_1 \cdots A_n}$. Clearly,

$$\text{Tr}[W_{A_1 \cdots A_n} \chi_{A_1 \cdots A_n}] = \lambda < 0,$$

$$\text{Tr}[\sigma_{A_1 \cdots A_n} \chi_{A_1 \cdots A_n}] \geq 0, \quad \forall \sigma_{A_1 \cdots A_n} \in D(\bigotimes_{i=1}^n H_{A_i}).$$
S2: Expand $\chi_{A_1\ldots A_n}$ in the computational basis:

$$\chi_{A_1\ldots A_n} = \sum_{a_1^1, \ldots, a_n^1, a_1^n, \ldots, a_n^n} a_{a_1^1, \ldots, a_n^1} a_{a_1^n, \ldots, a_n^n} |a_1^1\rangle A_1 \otimes \cdots \otimes |a_n^n\rangle A_n.$$  

Using 25 we can write this as,

$$\chi_{A_1\ldots A_n} = \sum_{s_1, \ldots, s_n} \beta_{s_1, \ldots, s_n} \otimes \phi_A^{s_j},$$

where, $\phi_A^{s_j} \in B_{A_i}^{proj}$. Since $\chi_{A_1\ldots A_n}$ is Hermitian and $\phi_A^{s_j}$ are linearly independent, all the $\beta_{s_1, \ldots, s_n}$ are real. Let $\psi_{A_i}^{s_j} = \phi_A^{s_j}^T$, where the transpose is taken in the computational basis.

$$\chi_{A_1\ldots A_n} = \sum_{s_1, \ldots, s_n} \beta_{s_1, \ldots, s_n} \otimes \psi_{A_i}^{s_j}$$

In the semiquantum game, the referee gives one of the pure states $\psi_{A_i}^{s_j} \in B_{A_i}^{proj} \otimes B_{A_i}^{proj}$ to the $i^{th}$ party in each run (See the proof of Theorem 2).

### III. PROOF OF THEOREM 2

**Proof.** This proof is a straightforward generalization of the proof of Theorem 1. For every BQS $W_{A_1\ldots A_n} \in W(\bigotimes_{i=1}^n \mathcal{H}_i)$ there exists an entangled state $\chi_{A_1\ldots A_n} \in D(\bigotimes_{i=1}^n \mathcal{H}_i)$ such that $\text{Tr}[W_{A_1\ldots A_n} \chi_{A_1\ldots A_n}] < 0$, whereas $\text{Tr}[\sigma_{A_1\ldots A_n} \chi_{A_1\ldots A_n}] \geq 0$, $\forall \sigma_{A_1\ldots A_n} \in D(\bigotimes_{i=1}^n \mathcal{H}_i)$ [43]. The state allows non-unique decomposition of the form

$$\chi_{A_1\ldots A_n} = \sum_{s_1, \ldots, s_n} \beta_{s_1, \ldots, s_n} \otimes \phi_A^{s_j}, \text{ where } \phi_A^{s_j} \in D(\mathcal{H}_i) \text{ & } \beta_{s_1, \ldots, s_n} \in \mathbb{R}. \quad (26)$$

In the semiquantum game, referee sends the quantum inputs $\psi_{A_i}^{s_j}$ to the $i^{th}$ party who has to produce binary outputs $\in \{0, 1\}$. Their average payoff will be calculated as

$$I_{G_0} = \sum_{s_1, \ldots, s_n} \beta_{s_1, \ldots, s_n} \times \text{p}(1\cdots1 | \psi_{A_1}^{s_1} \cdots \psi_{A_n}^{s_n}). \quad (27)$$

Given the BQS $W_{A_1\ldots A_n}$, the $i^{th}$ party performs the measurement $\left\{ P_{A_1A_i}^{+}, \mathbb{I}_{A_i} - P_{A_1A_i}^{+} \right\}$ on her part of the shared BQS and the quantum input $\psi_{A_i}^{s_j}$ received from the referee; $P_{A_1A_i}^{+}$ corresponds to the output 1. The average payoff turns out to be,

$$I_{G_0} (W_{A_1\ldots A_n}) = \sum_{s_1, \ldots, s_n} \beta_{s_1, \ldots, s_n} \times \text{Tr} \left( P_{A_1A_i}^{+} \otimes \cdots \otimes P_{A_1A_n}^{+} \right) \left( W_{A_1\ldots A_n} \otimes \psi_{A_i}^{s_j} \otimes \cdots \otimes \psi_{A_n}^{s_n} \right)$$

$$= \sum_{s_1, \ldots, s_n} \beta_{s_1, \ldots, s_n} \times \text{Tr} \left( \left( R_{A_1} \otimes \cdots \otimes R_{A_n} \right) W_{A_1\ldots A_n} \right), \quad (28)$$

where $R_{A_i}$ is the effective POVM acting on the $i^{th}$ party’s part of $W_{A_1\ldots A_n}$ and is given by $R_{A_i} := \text{Tr}_{A_i} \left[ P_{A_iA_i}^{+} \left( \mathbb{I}_{A_i} \otimes \psi_{A_i}^{s_j} \right) \right] = \frac{1}{d_{A_i}} \psi_{A_i}^{s_j}$. Therefore, we have,

$$I_{G_0} (W_{A_1\ldots A_n}) = \prod_{i=1}^n d_{A_i}^{-1} \sum_{s_1, \ldots, s_n} \beta_{s_1, \ldots, s_n} \times \text{Tr} \left( \left( \bigotimes_{i=1}^n \psi_{A_i}^{s_j} \right) W_{A_1\ldots A_n} \right)$$

$$= \prod_{i=1}^n d_{A_i}^{-1} \text{Tr} \left[ \left( \sum_{s_1, \ldots, s_n} \psi_{A_i}^{s_j} \right) W_{A_1\ldots A_n} \right] = \prod_{i=1}^n d_{A_i}^{-1} \text{Tr} \left[ \chi_{A_1\ldots A_n} W_{A_1\ldots A_n} \right] < 0. \quad (29)$$
We will now calculate the payoff for an arbitrary quantum strategy. Given a quantum state \( \rho_{A_1 \cdots A_N} \), let the \( i \)-th party perform the measurement \( M_{A_i} = \{ \pi_{A_i}^a \} \) on her respective joint system, where \( a_i \in \{0, 1\} \). The average payoff turns out to be

\[
I_{G_{sq}}^X (\rho_{A_1 \cdots A_N}) = \sum_{s_1, \cdots, s_n} \beta_{s_1 \cdots s_n} \times \text{Tr} \left[ \left( \prod_{i=1}^{n} \pi_{A_i}^1 \otimes \cdots \otimes \pi_{A_n}^1 \right) \left( \rho_{A_1 \cdots A_N} \otimes \psi_{A_1}^{s_1} \otimes \cdots \otimes \psi_{A_n}^{s_n} \right) \right]
\]

\[
= \sum_{s_1, \cdots, s_n} \beta_{s_1 \cdots s_n} \times \text{Tr} \left[ R_{A_1^o \cdots A_n^o} \left( \bigotimes_{i=1}^{n} \psi_{A_i}^{s_i} \right) \right]
\]

(30)

where, \( R_{A_1^o \cdots A_n^o} := \text{Tr}_{A_1 \cdots A_n} \left[ \left( \prod_{i=1}^{n} \pi_{A_i}^1 \otimes \cdots \otimes \pi_{A_n}^1 \right) \left( \rho_{A_1 \cdots A_n} \otimes I_{A_1^o \cdots A_n^o} \right) \right] \) is a positive semidefinite operator, i.e. \( R_{A_1^o \cdots A_n^o} \in \mathcal{E} \left( \bigotimes_{i=1}^{n} \mathcal{H}_{A_i^o} \right) \). Linearity of trace further yields,

\[
I_{G_{sq}}^X (\rho_{A_1 \cdots A_N}) = \text{Tr} \left[ R_{A_1^o \cdots A_n^o} \left( \sum_{s_1, \cdots, s_n} \beta_{s_1 \cdots s_n} \bigotimes_{i=1}^{n} \psi_{A_i}^{s_i} \right) \right] = \text{Tr} \left[ R_{A_1^o \cdots A_n^o} \chi_{A_1^o \cdots A_n^o}^{T} \right] \geq 0.
\]

(31)

The last inequality follows from the fact that \( \chi_{A_1^o \cdots A_n^o}^{T} \in \mathcal{D} \left( \bigotimes_{i=1}^{n} \mathcal{H}_{A_i^o} \right) \), and this completes the proof. ■

IV. NECESSITY OF NON-ORTHOGONAL INPUTS IN THEOREM 1 AND THEOREM 2

In this section we will discuss the necessity of the non-orthogonal quantum inputs in the game \( G_{sq} \) used in Theorem 1 and Theorem 2. While playing the game \( G_{sq} \), let the \( i \)-th party get the quantum input \( \psi_{A_i}^{s_i} \in \mathcal{D} (\mathcal{H}_{A_i^o}) \) and perform some joint measurement \( M_{A_i} = \{ \pi_{A_i}^a \} \), where \( a_i \) is the outcome corresponding to the POVM element \( \pi_{A_i}^a \). For the BQS \( W_{A_1 \cdots A_n} \), the joint probabilities are given by

\[
p (a_1, \cdots, a_n| \psi_{A_1}^{s_1}, \cdots, \psi_{A_n}^{s_n}) = \text{Tr} \left[ \bigotimes_{i} \pi_{A_i}^{a_i} \otimes \psi_{A_i}^{s_i} \otimes W_{A_1 \cdots A_n} \right] = \text{Tr} \left[ \left( \bigotimes_{i} Q_{A_i}^{a_i} [s_i] \right) W_{A_1 \cdots A_n} \right],
\]

(32)

where, \( Q_{A_i}^{a_i} [s_i] := \text{Tr}_{A_i} \left[ \pi_{A_i}^{a_i} (I_{A_i} \otimes \psi_{A_i}^{s_i}) \right] \in \mathcal{E} (\mathcal{H}_{A_i}) \) effectively acts on \( A_i \) subsystem of the shared state \( W_{A_1 \cdots A_n} \) when the quantum input \( \psi_{A_i}^{s_i} \) is given by the referee. Since \( \sum_{a_i} \pi_{A_i}^{a_i} = I_{A_i A_i^o} \), we thus have,

\[
\sum_{a_i} Q_{A_i}^{a_i} [s_i] = \text{Tr}_{A_i} \left[ \sum_{a_i} \pi_{A_i}^{a_i} (I_{A_i} \otimes \psi_{A_i}^{s_i}) \right] = \text{Tr}_{A_i} \left[ I_{A_i} \otimes \psi_{A_i}^{s_i} \right] = I_{A_i}.
\]

(33)

Therefore, \( \{ Q_{A_i}^{a_i} [s_i] \}_{a_i} \) is the effective POVM on \( A_i \) subsystem of the shared state \( W_{A_1 \cdots A_n} \) when the quantum input \( \psi_{A_i}^{s_i} \) is received by the \( i \)-th party. This calculation tells that, whenever the given inputs states are orthogonal, the joint correlation \( \{ p (a_1, \cdots, a_n| \psi_{A_1}^{s_1}, \cdots, \psi_{A_n}^{s_n}) \} \) can be produced by following two procedures.

P-1: The \( i \)-th party performs some fixed joint measurement \( M_{A_i} = \{ \pi_{A_i}^a \} \) on her part of the shared BQS \( W_{A_1 \cdots A_n} \) and the given input state \( \psi_{A_i}^{s_i} \).

P-2: The \( i \)-th party first performs some measurement to identify the index ‘\( s_i \)’ of the given quantum state \( \psi_{A_i}^{s_i} \), and then accordingly she performs the measurement \( M_{A_i} = \{ Q_{A_i}^{a_i} [s_i] \} \) on her part of the shared BQS \( W_{A_1 \cdots A_n} \).

Clearly, the procedure P-1 is the only option to generate the desired correlations when non-orthogonal inputs states are involved in the game \( G_{sq} \).

Let us now assume that the BQS is of the following form

\[
W_{A_1 A_2 \cdots A_n} = \sum_k p_k \left( \Lambda_{A_1}^k \otimes \cdots \otimes \Lambda_{A_n}^k \right) \rho^k
\]

(34)
where, $\rho^k \in D(\otimes_i \mathcal{H}_{A_i})$, $\Lambda_i^k$ are the positive trace preserving map, and $\{p_k\}$ is some probability distribution. In this case we have

$$p (a_1, \cdots, a_n | \psi_{s_1}^{A_1}, \cdots, \psi_{s_n}^{A_n}) = \text{Tr} \left[ \left( \otimes_i Q_{A_i}^{a_i} [s_i] \right) \left( \otimes_i \Lambda_i^k \right) \rho^k \right] = \sum_k p_k \text{Tr} \left[ \left( \otimes_i Q_{A_i}^{a_i} [s_i] \right) \left( \otimes_i \Lambda_i^k \right) \rho^k \right] = \sum_k p_k \text{Tr} \left[ \left( \otimes_i \tilde{Q}_{A_i}^{a_i} [s_i] \right) \rho^k \right],$$

(35)

where $\Lambda^*$ is the dual of $\Lambda$, i.e. $\text{Tr}[U \Lambda(V)] = \text{Tr}[\Lambda^*(U)V]$ for all $U$ and $V$. Clearly $\tilde{M}_i^{a_i} \equiv \{\tilde{Q}_{A_i}^{a_i} [s_i]\}$ is a valid quantum measurement since dual of a positive trace-preserving map is positive and unital. Therefore, for this class of BQSs, whenever the input states are orthogonal, the resulting correlations can be simulable quantum mechanically by performing the local measurements $\tilde{M}_i^{a_i}$ on some multipartite quantum state. This reproduces the results in [18, 19].