

1. Introduction and Preliminaries

Suppose that $\Gamma$ is a connected, finite, and undirected graph. Its vertex set is $V_\Gamma$ and its edge set is $E_\Gamma$. The distance $d(f, g)$ between two vertices $f, g \in V_\Gamma$ in a connected graph is the shortest path connecting those two vertices. If $e \in V_\Gamma$ is a vertex such as $d(e, f) \neq d(e, g)$, then we say $e$ resolves the vertices $f$ and $g$. The representation $r(h, J_\Gamma)$ of a vertex $h$ with respect to an ordered set $J_\Gamma = \{j_v|1 \leq v \leq p\} \subseteq V_\Gamma$ is defined as a $p$-vector $(d(h, j_1), \ldots, d(h, j_p))$, also called the distance vector. If each vertex of the graph $\Gamma$ has a unique distance vector with respect to the set $J_\Gamma$, then $J_\Gamma$ is called the resolving set (or sometimes called locating set) for graph $\Gamma$. The minimum possible cardinality of a resolving set in $\Gamma$ is called the metric dimension (MD), represented as $\text{dim}(\Gamma)$. A resolving set of cardinality $\text{dim}(\Gamma)$ is said to be the metric basis of $\Gamma$.

The notion of MD was first proposed by Slater in [1] and investigated independently by Harary and Melter in [2] because of the problem of locating an intruder in a network. Later, Chartrend and Zang [3] demonstrated various applications in biology, robotics, and chemistry. From a two-dimensional real plane, the MD of graphs can extend the concept of trilateration. For instance, distances are used by the Global Positioning System (GPS) to identify an object’s location on Earth. In the context of MD, Hamming graphs are closely related to the difficulties in weighing of coins discussed in [4, 5], and the comprehensive study of the Mastermind game given in [6].

Resolvability of graphs has become a significant parameter in graph theory as a result of its widespread applicabilities in different areas of mathematics, including facility location problems, network discovery and verification [7], applications in molecular chemistry [8], the positioning of robots in a network [9], routing protocols geographically [10], the problems of sonar or LORAN stations [1], and the optimization problem in combinatorics [11]. After that, we will have a look at some research work on the mathematical significance of this distance-based parameter.

The MD has been used to study a variety of mathematically interesting graph families. We will highlight some of the significant work in this section: Similar to many other graph-theoretic parameters, finding the MD of arbitrary graphs is a computationally tough task [9, 12]. For example, the bounds for the MD of the Petersen graph family was...
studied by Shao et al. [13]. On various distance-regular graphs (such as kayak paddle graphs and chorded cycles), Ahmad et al. investigated the MD [14]. Bailey et al. [15] investigated the MD of Kneser graphs. For wheel graphs, Buczkowski et al. in [16] investigated the MD, whereas Baca et al. in [17] examined the MD of regular bipartite graphs. Chartrand et al. [8] categorized \( n \)-vertex graphs with MD 1, \( n - 2 \) and \( n - 1 \). From the perspective of MD, graphs of relevance in group theory, such as Cayley digraphs [18] and Cayley graphs formed by certain finite groups [19], have also been investigated. [20] provides a response to the question of whether the MD is a finite number or an infinite quantity. The minimum ordered resolving sets of Jahangir graphs (resp. necklace graphs) were studied by Tomescu et al. in [21] (resp. in [22]). Investigations have also been carried out into the MD and the resolving sets of product graphs, such as the categorical product of graphs [23] and the cartesian products [24].

MD has also been generalised and extended by offering more mathematically rigorous general ideas, such as the double metric dimension (DMD). Caceres et al. [24] proposed and defined the notion of doubly resolving sets (DRSs) of graph \( \Gamma \) as follows: a pair of vertices \( g \) and \( h \) of graph \( \Gamma \) is said to double resolve vertices \( g' \) and \( h' \) if the following equation holds: \( d(g', g) - d(g', h) \neq d(h', g) - d(h', h) \). A DRS of \( \Gamma \) is a subset \( N_\Gamma = \{k_1, k_2, \ldots, k_s\} \) of \( V_\Gamma \), where every two different vertices of \( \Gamma \) are resolved by some two vertices of \( N_\Gamma \). The minimal doubly resolving set (MDRS) problem is to find a DRS of \( \Gamma \) with the smallest cardinality, which is called the DMD of \( \Gamma \) indicated by \( \psi(\Gamma) \). If the vertices \( g' \) and \( h' \) can be doubly resolved by the vertices \( g \) and \( h \) then either \( d(g', g) - d(h', g) \neq 0 \) or \( d(g', h) - d(h', h) \neq 0 \) which follows that either \( g \) or \( h \) resolve the vertices \( g' \) and \( h' \).

Because a DRS is also a resolving set, we have \( \psi(\Gamma) \geq \dim(\Gamma) \), so we can use the MDRSs to get an upper bound on the MD of the graph under discussion. The idea of establishing upper boundaries in the cartesian product encouraged us to explore on DRSs of different graph classes. In general graphs, the MDRS problem has been shown to be NP-hard [25]. Many families of graphs, such as cocktail graphs, prisms, and jellyfish graphs, have been studied for the problem of finding the MDRSs (for details see: [26, 27]). The MDRSs for convex polytope structures and Hamming graphs have been derived and may be found in [28], and [29], respectively. The DMD and minimal order resolving sets of Harary and circulant graphs were studied in [30, 31]. It was Chen et al. that provided the first approximated upper bounds for the MDRS problem [32]. The authors in [33, 34] demonstrated that the DMD of some convex polytope structures is finite and constant. The line graphs of chorded cycles [35], kayak paddle graphs [36], \( n \)-Sunlet, and prism graphs [37] were discovered to have the MD and DRSs. In [38], layersun graphs and associated line graphs were studied for the MDRSs. Liu et al. gave the results for the minimum order resolving sets and DMD of the line graph of the Necklace graph in [39].

Using resolving sets is a natural way to find the origins of a network spread. For example, determining where a disease originated as it spreads through a population might be helpful in a variety of scenarios. A direct solution can be found if the time at which the spread began and the internode distances are reliable and known. Resolvability, however, needs to be expanded to account for things like an unknown start time and irregular transmission delays among nodes. The former problem can be solved by employing DRSs. The variation involved with arbitrary internode distances is critical in successfully determining the source of a spread [40].

Detecting virus sources in starlike networks are more complicated than in pathlike networks. While the DMD is \( n - 1 \) for a star-type structure with \( n \) nodes, and for a path-type structure with the same number of nodes it is 2 (see [40]). In addition, this demonstrates that the DMD is always reliant on the topology of the network being used.

The purpose of this research is to find the MDRSs and DMD for a class of chordal ring networks, which are helpful in the designing of local area networks. The following result regarding the MD of chordal ring networks \( CR_n \) is helpful in determining its DMD:

**Theorem 1.** [41] Let \( CR_n(1, 3, 5) \) be the chordal ring network. Then, for any even integer \( n \geq 6 \), we have \( \dim(CR_n(1, 3, 5)) = \begin{cases} 3, & \text{if } n \equiv 0 \text{ (mod 4)} \\ 4, & \text{if } n \equiv 2 \text{ (mod 4)} \end{cases} \)

The rest of the article is arranged in this following way:

(i) In Section 2, we discussed the construction of chordal ring networks and investigated the MDRSs of \( CR_n(1, 3, 5) \)

(ii) Finally, in Section 3, we summarize the findings of the article

**2. Double Metric Dimension of Chordal Ring Networks ** \( CR_n(1, 3, 5) \)

In this section, we calculated the MDRSs and DMD for chordal ring network \( CR_n(1, 3, 5) \).

If every closed path of length 4 or more in an undirected network contains a chord, it is referred to as chordal. Arden and Lee [42] first proposed chordal ring networks of degree 3. An undirected cycle of even order can be used to form a chordal ring network, in which all newly added chords connect an even labelled node to an oddly labelled node.

Let \( \alpha, \beta, \gamma \in \{1, 2, \ldots, n-1\} \) be three distinct odd integers and \( n \geq 3 \) be an even natural number. A chordal ring network having order \( n \) with chords \( \alpha, \beta, \gamma \) represented by \( CR_n(\alpha, \beta, \gamma) \), has the nodes set \( Z_n \) and links are given by \( \delta \sim \delta + \alpha, \delta \sim \delta + \beta \) and \( \delta \sim \delta + \gamma \) for every even node \( \delta \in Z_n \). By definition, every chordal ring network \( CR_n(\alpha, \beta, \gamma) \) is bipartite and 3-regular. Further, every odd node \( \theta \in Z_{n-1} \) forms a link with \( \theta - \alpha, \theta - \beta, \theta - \gamma \) in \( CR_n(\alpha, \beta, \gamma) \). There are no specific rules for selecting three different odd integers \( \alpha, \beta, \gamma \) from the set \( \{1, 2, \ldots, n-1\} \). Actually, we can choose any three different odd integers and, as a result, we obtain different chordal ring networks each time, with the possibility that few of them are isomorphic. To make a \( CR_n(\alpha, \beta, \gamma) \), select 3 different even integers from the set \( \{1, 2, \ldots, n-1\} \) and connect odd numbered nodes to them, as explained earlier. In addition,
\(\alpha, \beta, \) and \(\gamma\) are stated to be chords because they help to connect nodes in the \(\mathcal{CR}_n(\alpha, \beta, \gamma)\), and each obtained link represents a chord for at least one path in \(\mathcal{CR}_n(\alpha, \beta, \gamma)\) as demonstrated in Figure 1.

Here, the DMD. \(\psi(CR_n, (1, 3, 5)) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod{4} \\ 4, & \text{if } n \equiv 2 \pmod{4} \end{cases}\) for any even integer \(n \geq 6\) by applying Theorem 1. Also, we are going to prove the DMD \(\psi(CR_n, (1, 3, 5)) = 4\), for any even integer \(n \geq 6\).

Define \(S_t(h_0) = \{h \in V_{CR,(1,3,5)} : d(h_0, h) = v\}\) to be a vertex set in \(V_{CR,(1,3,5)}\) at distance \(v\) from \(h_0\). Table 1 is simply constructed for \(S_t(h_0)\) and employed to compute the distance between any pair of vertices in \(V_{CR,(1,3,5)}\).

The symmetry of \(CR_n(1, 3, 5)\) for any even integer \(n \geq 6\), shows the following fact that: \(d(h_0, h_1) = d(h_0, h_{t-v})\) if \(0 \leq |t-v| \leq n-1\).

As a conclusion, by knowing the distance \(d(h_0, h)\) for all \(h \in V_{CR,(1,3,5)}\), we can reconstruct the distances between every two vertices in \(V_{CR,(1,3,5)}\).

**Lemma 1.** For \(n \geq 8\) and \(n \equiv 0 \pmod{4}\), we have \(\psi(CR_n, (1, 3, 5)) > 3\).

**Proof.** We know that \(\psi(CR_n, (1, 3, 5)) \geq 3\). Therefore, we need to explain that every set \(D_{CR,(1,3,5)} \subseteq V_{CR,(1,3,5)}\) of order 2 is not a DRS for \(CR_n(1, 3, 5)\). There are some different types of set \(D_{CR,(1,3,5)}\) as well as their nondoubly resolved pair of vertices from \(V_{CR,(1,3,5)}\) listed in Table 2. Let us show that the vertices \(h_0, h_{n-3}\) are not doubly resolved by any two vertices of the set \(h_0, h_{n-3}, h_1, 1 \leq v \leq 2l - 2\) and \(\tau = n - 1\):

\[
\begin{align*}
d(h_0, h_0) &= 0, \\
d(h_0, h_{n-3}) &= 1, \\
d(h_0, h_v) &= v, \\
d(h_v, h_{n-3}) &= d(h_0, h_{n-3}) = v + 1, \\
d(h_0, h_1) &= 1, \\
d(h_v, h_{n-3}) &= d(h_0, h_{n-3}) = 2.
\end{align*}
\]

From equations (1) to (6), we have \(d(h_0, h_0) - d(h_0, h_{n-3}) = d(h_0, h_v) - d(h_v, h_{n-3}) = d(h_0, h_1) = 1\), that is, the set \(\{h_0, h_1, 1 \leq v \leq 2l - 2\, \text{and} \, \tau = n - 1\} \neq \{h_0, h_{n-3}\}\) is not a DRS of \(CR_n(1, 3, 5)\). All other forms of the set \(D_{CR,(1,3,5)}\) in Table 2 can be examined and confirmed to have nondoubly resolved vertices, as well.

**Lemma 2.** Let \(n \equiv 0 \pmod{4}\) for \(n \geq 8\), we have \(\psi(CR_n, (1, 3, 5)) = 4\).

**Proof.** Consider the case where \(n \equiv 2 \pmod{4}\) for \(n \geq 6\). To demonstrate that \(\psi(CR_n, (1, 3, 5)) = 4\), a DRS of order 4 is all that is required. Now, by using the sets \(S_t(h_0)\) from Table 1, Tables 3 and 4 demonstrate the metric coordinate vectors for each vertex of \(CR_n(1, 3, 5)\) in relation to the set \(D_{CR,(1,3,5)} = \{h_0, h_1, h_2, h_{2l-1}\}\), where \(n = 4l\) and \(l \geq 2\) be an integer.

Using Tables 3 and 4, it is possible to verify directly that, for any given integer \(v \in \{1, 2, \ldots, l + 1\}\) if two vertices \(a_1, a_2 \in S_t(h_0)\), then the difference between their first coordinates is zero, but the difference between their overall coordinates is not zero that is the representation \(r(a_1, D_{CR,(1,3,5)}) - r(a_2, D_{CR,(1,3,5)}) \neq 0\). In the same way, for the vertices \(a_1 \in S_t(h_0)\) and \(a_2 \in S_t(h_0)\) for any \(v \neq \tau \in \{1, 2, \ldots, l + 1\}\), the difference between their first coordinates is \(v - \tau\) but the difference between all coordinates is not \(v - \tau\) at the same time, that is the representation \(r(a_1, D_{CR,(1,3,5)}) - r(a_2, D_{CR,(1,3,5)}) \neq v - \tau\). Therefore, the set \(D_{CR,(1,3,5)} = \{h_0, h_1, h_2, h_{2l+1}\}\) is the MDRS. Thus, Lemma 3 holds.

**Lemma 3.** Let \(n \equiv 2 \pmod{4}\) for \(n \geq 6\), we have \(\psi(CR_n, (1, 3, 5)) = 4\).

**Proof.** Consider the case where \(n \equiv 2 \pmod{4}\) for \(n \geq 6\). To demonstrate that \(\psi(CR_n, (1, 3, 5)) = 4\), a DRS of order 4 is all that is required. Now, by using the sets \(S_t(h_0)\) from Table 1, Tables 3 and 6 demonstrate the metric coordinate vectors for each vertex of \(CR_n(1, 3, 5)\) in relation to the sets \(D_{CR,(1,3,5)} = \{h_0, h_1, h_2, h_{2l+1}\}\) and \(D_{CR,(1,3,5)} = \{h_0, h_1, h_2, h_{2l+2}\}\) respectively, where \(n = 4l + 2\) and \(l \geq 1\) be an integer.

Using Tables 5 and 6, it is possible to verify directly that, for any given integer \(v \in \{1, 2, \ldots, l + 1\}\) if two vertices \(b_1, b_2 \in S_t(h_0)\), then the difference between their first coordinates is zero, but the difference between their overall coordinates is not zero, that is the representation \(r(b_1, D_{CR,(1,3,5)}) - r(b_2, D_{CR,(1,3,5)}) \neq 0\). In the same way, for the vertices \(b_1 \in S_t(h_0)\) and \(b_2 \in S_t(h_0)\) for any \(v \neq \tau \in \{1, 2, \ldots, l + 1\}\), the difference between their first coordinates is \(v - \tau\) but the difference between all coordinates is not \(v - \tau\) at the same time, that is the representation \(r(b_1, D_{CR,(1,3,5)}) - r(b_2, D_{CR,(1,3,5)}) \neq v - \tau\). Therefore, the sets \(D_{CR,(1,3,5)} = \{h_0, h_1, h_2, h_{2l+1}\}\) and \(D_{CR,(1,3,5)} = \{h_0, h_1, h_2, h_{2l+2}\}\) are the MDRSs. Thus, Lemma 3 holds.

Using Lemmas 2 and 3, the main theorem is stated below:
Table 1: $S_v(h_0)$ for $CR_n(1, 3, 5)$.

| $n$  | $v$  | $S_v(h_0)$ |
|------|------|------------|
| 1    | $2 \leq v \leq l$ | $\{h_{l-1}, h_{n-3}, h_{n-1}\}$, if $v$ is even $\{h_{2v-2}, h_{2v}, h_{2v+2}, h_{2v+4}, h_{2v+6}, h_{2v+8}, h_{2v+10}, h_{2v+12}\}$, if $v$ is odd |
| $4l$, ($l \geq 2$) | $l + 1$ | $\{h_{2v-2}\}$, if $v$ is even $\{h_{2v-3}\}$, if $v$ is odd |
| $4l + 2$, ($l \geq 1$) | $2 \leq v \leq l + 1$ | $\{h_{2v-2}, h_{2v}, h_{2v+2}, h_{2v+4}, h_{2v+6}, h_{2v+8}, h_{2v+10}, h_{2v+12}\}$, if $v$ is even $\{h_{2v-3}, h_{2v-1}, h_{2v-2}, h_{2v+1}, h_{2v+2}, h_{2v+3}, h_{2v+4}, h_{2v+5}\}$, if $v$ is odd |

Table 2: Nondoubly resolved pairs of vertices for $CR_n(1, 3, 5)$, for $n \geq 8$ and $n \equiv 0 \text{ (mod 4)}$.

| $D_{CR,(1,3,5)}$ | Nondoubly resolved pairs |
|-----------------|--------------------------|
| $[h_0, h_n, h_1]$, $v = 1, n - 4 \leq t \leq n - 3$ | $[h_0, h_n, 1]$ |
| $[h_0, h_n, h_1]$, $2 \leq v \leq 3, n - 4 \leq l \leq n - 3$ | $[h_1, h_n, 1]$ |
| $[h_0, h_n, h_1]$, $4 \leq v \leq 5, n - 4 \leq l \leq n - 3$ | $[h_0, h_n, 1]$ |
| $[h_0, h_n, h_1]$, $1 \leq v \leq 2l - 2$ |
| $[h_0, h_n, h_1]$, $2l + 1 \leq v \leq n - 1$ |
| $[h_0, h_n, h_1]$, $1 \leq v \leq 2l - 2, \tau = n - 1$ |
| $[h_0, h_n, h_1]$, $2l - 1 \leq v \leq 2l, \tau = n - 1$ |

Table 3: Metric coordinate vectors for $CR_n(1, 3, 5)$, where $n = 4l$, $l \geq 2$ is even.

| $v$  | $S_v(h_0)$ |
|------|------------|
| 0    | $h_0$ $(0, 1, 2, l + 1)$ |
| 1    | $h_1$ $(1, 0, 1, l)$ |
| 2 \leq v \leq l, when $v$ is even | $h_{2v}$ $(v, v - 1, v - 2, l - v + 1)$ |
| 2 \leq v \leq l, when $v$ is odd | $h_{2v+1}$ $(v, v - 1, v, l - v + 1)$ |
| $v = l + 1$, when $v$ is odd | $h_{2v+2}$ $(v, v - 1, v - 2, 0)$ |

Table 4: Metric coordinate vectors for $CR_n(1, 3, 5)$, where $n = 4l$, $l \geq 2$ is odd.

| $v$  | $S_v(h_0)$ |
|------|------------|
| 0    | $h_0$ $(0, 1, 2, l)$ |
| 1    | $h_1$ $(1, 0, 1, l - 1)$ |
| 2 \leq v \leq l, when $v$ is even | $h_{2v}$ $(v, v - 1, v - 2, l - v + 2)$ |
| 2 \leq v \leq l, when $v$ is odd | $h_{2v+1}$ $(v, v - 1, v, l - v)$ |
| $v = l + 1$, when $v$ is odd | $h_{2v+2}$ $(v, v - 1, v - 2, 2l - v + 2)$ |
Table 5: Metric coordinate vectors for $CR_n(1,3,5)$, where $n = 4l + 2$, $l \geq 1$ is odd.

| $v$ | $S_v(h_v)$ | $D_{CR_n(1,3,5)} = \{h_0, h_1, h_2, h_{2l+1}\}$ |
|-----|------------|-----------------------------------------------|
| 0   | $h_0$     | $(0, 1, 2, l)$                              |
|     | $h_1$     | $(1, 0, 1, l + 1)$                          |
| 1   | $h_{n-3}$ | $(1, 2, 1, 0)$, if $n = 6$                 |
|     | $h_{n-1}$ | $(1, 2, 3, l - 1)$, if $n \neq 6$          |
| $2 \leq v \leq l + 1$, | $h_{2v}$ | $(v, v - 1, v - 2, l - v + 2)$ |
|     | $h_{2v+1}$ | $(v, v - 1, v - l + v + 2)$ |
| when $v$ is even | $h_{2v+2}$ | $(v, v + 1, v, l - v + 2)$, if $v < l$ |
|     | $h_{2v+3}$ | $(v, v - 1, v, l - v + 2)$, if $v = l$ |
| when $v$ is odd | $h_{2v+1}$ | $(v, v + 1, v, l - v + 2)$, if $v < l$ |
|     | $h_{2v+2}$ | $(v, v + 1, v, l - v + 2)$, if $v = l$ |

Table 6: Metric coordinate vectors for $CR_n(1,3,5)$, where $n = 4l + 2$, $l \geq 1$ is even.

| $v$ | $S_v(h_v)$ | $D_{CR_n(1,3,5)} = \{h_0, h_1, h_2, h_{2l+1}\}$ |
|-----|------------|-----------------------------------------------|
| 0   | $h_0$     | $(0, 1, 2, l)$                              |
|     | $h_1$     | $(1, 0, 1, l + 1)$                          |
| 1   | $h_{n-3}$ | $(1, 2, 3, l - 1)$, if $n \neq 6$          |
|     | $h_{n-1}$ | $(1, 2, 1, l + 1)$                          |
| $2 \leq v \leq l + 1$, | $h_{2v}$ | $(v, v - 1, v - 2, l - v + 2)$ |
|     | $h_{2v+1}$ | $(v, v - 1, v - l + v + 2)$ |
| when $v$ is even | $h_{2v+2}$ | $(v, v + 1, v, l - v + 2)$, if $v < l$ |
|     | $h_{2v+3}$ | $(v, v - 1, v, l - v + 2)$, if $v = l$ |
| when $v$ is odd | $h_{2v+1}$ | $(v, v + 1, v, l - v + 2)$, if $v < l$ |
|     | $h_{2v+2}$ | $(v, v + 1, v, l - v + 2)$, if $v = l$ |

Theorem 2. Let $CR_n(1,3,5)$ be the chordal ring network. Then for any even integer $n \geq 6$, $\psi(CR_n(1,3,5)) = 4$.

3. Conclusion

This study is concerned with the concept of calculating MDRSs of graphs that has been proposed earlier in the literature. The DMD of chordal ring networks $CR_n(1,3,5)$ is computed by describing their MDRSs. In this study, we found that the number of vertices in the chordal ring networks does not affect its DMD.

Data Availability

The whole data are included within this article. However, the reader may contact the corresponding author for more details on the data.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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