Vertex-minor-closed classes are $\chi$-bounded

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Abstract

We prove a conjecture of Geelen that every proper vertex-minor-closed class of graphs is $\chi$-bounded.

1 Introduction

A class of graphs $\mathcal{G}$ is $\chi$-bounded if graphs in $\mathcal{G}$ with bounded clique number also have bounded chromatic number. Geelen (see [9]) conjectured that every proper vertex-minor-closed class of graphs is $\chi$-bounded (we delay certain definitions such as that of vertex-minors until Section 2). We prove this conjecture.

Theorem 1.1. Every proper vertex-minor-closed class of graphs is $\chi$-bounded.

Scott [23] conjectured that for every graph $H$, the class of graphs containing no induced subdivision of $H$ is $\chi$-bounded. This conjecture was disproved by Pawlik et al [22]. However if a graph contains an induced subdivision of a graph $H$, then it also contains $H$ as a vertex-minor. So Theorem 1.1 recovers one possible weakening of Scott’s conjecture. For further results and conjectures on $\chi$-boundedness, there is an excellent recent survey by Scott and Seymour [25].

Several special cases of Theorem 1.1 have been proved in the past. Most classically, Gyárfás [16] proved that circle graphs are $\chi$-bounded. Another important vertex-minor-closed class of graphs is those with bounded rank-width; Dvořák and Král’ [9] proved that such graphs are $\chi$-bounded. Geelen, Kwon, McCarty and Wollan [14] proved that if $H$ is a circle graph, then the class of graphs with no $H$ vertex-minor has bounded rank-width and so is also $\chi$-bounded. Let $W_n$ denote the wheel graph consisting of an $n$-cycle and a single additional dominating vertex. Two of the three minimal forbidden vertex-minors for circle graphs are wheel graphs [2], and so generalising Gyárfás’s result that circle graphs are $\chi$-bounded, Choi, Kwon, Oum and Wollan [3] proved that the for each $n$, the class of graphs with no $W_n$ vertex-minor are $\chi$-bounded. Kostochka [20] proved that the complements of circle graphs are $\chi$-bounded. This class of graphs is not vertex-minor-closed, however its closure under vertex-minors can be shown to be $\chi$-bounded as an extension of Kostochka’s result [13].

We also prove a natural weakening of $\chi$-boundedness but with a linear bound. For a graph $G$ and a positive integer $\rho$, a $\rho$-ball is an induced subgraph
formed by a vertex $v$ and the vertices at distance at most $\rho$ from $v$. We let $\chi^{(\rho)}(G)$ denote the maximum chromatic number of a $\rho$-ball contained in a graph $G$, and we say that a class of graphs $\mathcal{G}$ is $\rho$-controlled if there exists a function $f$ such that $\chi(G) \leq f(\chi^{(\rho)}(G))$ for all $G \in \mathcal{G}$. A class of graphs $\mathcal{G}$ is linearly $\rho$-controlled if there exists a constant $c$ such that $\chi(G) \leq c\chi^{(\rho)}(G)$ for all $G \in \mathcal{G}$.

With the idea of $\rho$-control in mind, one may naturally split the problem of proving that a class of graphs $\mathcal{G}$ is $\chi$-bounded into subproblems. The first is to show that for some $\rho \geq 2$, $\mathcal{G}$ is $\rho$-controlled. The next is to reduce control and show that $\mathcal{G}$ is 2-controlled. The final subproblem is to make use of the fact that $\mathcal{G}$ is 2-controlled to prove $\chi$-boundedness.

We follow this strategy to prove Theorem 1.1 and along the way prove that proper vertex-minor-closed classes of graphs are in fact linearly 9-controlled.

**Theorem 1.2.** Every proper vertex-minor-closed class of graphs is linearly 9-controlled.

To go from 9-control to 2-control we shall simply apply a theorem of Chudnovsky, Scott and Seymour [5]. Unfortunately this theorem does not preserve linearity if the class of graphs has unbounded clique number. However we conjecture that every proper vertex-minor-closed class of graphs is linearly 2-controlled. Note that in general not all vertex-minor-closed classes of graphs have linear $\chi$-bounding functions, or are even linearly 1-controlled. For instance Kostochka [19,20] showed that no linear $\chi$-bounding function exists for circle graphs, while 1-balls contained in circle graphs are permutation graphs, which are perfect (see [15]). For more on the notion of $\rho$-control, see [25].

Building on Geelen’s conjecture, Kim, Kwon, Oum and Sivaraman [18] further asked if all proper vertex-minor-closed classes of graphs are polynomially $\chi$-bounded. Recently there have also been significant developments on this problem. The author and McCarty [8] proved a quadratic $\chi$-bounding function for circle graphs and Bonamy and Pilipczuk [1] proved that graphs of bounded rank-width are polynomially $\chi$-bounded. As a result, it also follows that if $H$ is a circle graph, then the class of graphs with no $H$ vertex-minor are polynomially $\chi$-bounded [11].

Much less is known for pivot-minor-closed classes of graphs. However again circle graphs [8,16,20] and graphs of bounded rank-width [1,9] are both $\chi$-bounded and closed under pivot-minors. In addition, Choi, Kwon and Oum [4] proved that for each $n$, the class of graphs containing no $n$-cycle as a pivot-minor is $\chi$-bounded. More recently Scott and Seymour [24] proved a significant generalisation of this, that for all integers $k \geq 0$ and $\ell \geq 1$, the class of all graphs with no induced cycle of length $k$ modulo $\ell$ is $\chi$-bounded. This implies the result of Choi, Kwon and Oum as a cycle of length $n + 2$ contains a cycle of length $n$ as a pivot-minor.

We will also make a first step towards proving the conjecture of Choi, Kwon and Oum [4] that proper pivot-minor-closed classes of graphs are $\chi$-bounded. Following the idea of $\rho$-control, the first step we make is the following:

**Theorem 1.3.** Every pivot-minor-closed class of graphs that is 2-controlled is also $\chi$-bounded.
This reduces the problem of proving that a pivot-minor-closed class of graphs is $\chi$-bounded to proving that it is 2-controlled.

2 Preliminaries

Given two sets $A$ and $B$, we let $A - B$ denote the subset of $A$ obtained by removing the elements of $A \cap B$.

Given a vertex $v$ of a graph $G$, we let $N(v)$ denote its neighbourhood, i.e., the set of vertices adjacent to $v$. More generally given a set of vertices $A$ of a graph $G$, we let $N(A)$ be the set of vertices in $V(G) - A$ that are adjacent to a vertex of $A$. If the graph is not clear from context, we use $N_G(v)$ or $N_G(A)$. Given an integer $t \geq 0$, we let $N_t(A)$ be the set of vertices at distance exactly $t$ from $A$, and we let $N_t[A]$ be the set of vertices at distance at most $t$ from $A$. We may denote the closed neighbourhood $N_1[A]$ by $N[A]$.

A clique in a graph $G$ is a set of pairwise adjacent vertices. The clique number $\omega(G)$ of $G$ is equal to the size of the largest clique contained in $G$. A stable set is a set of pairwise non-adjacent vertices. For positive integers $n, m$, we let $R(n, m)$ denote the Ramsey number of $(n, m)$, i.e., all graphs on at least $R(n, m)$ vertices contain either a clique of size $n$ or a stable set of size $m$.

We say that two sets of vertices $A$ and $B$ in a graph $G$ are complete to each other if for all $a \in A$ and $b \in B$, we have $ab \in E(G)$. Similarly $A$ and $B$ are anti-complete if for all $a \in A$ and $b \in B$, we have $ab \notin E(G)$. If for all $b \in B$, there exists a vertex $a \in A$ that is adjacent to $b$, then we say that $A$ dominates $B$. For a simple example observe that if $v$ is a vertex of a graph $G$, then $N_{t-1}(v)$ dominates $N_t(v)$. For a positive integer $n$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Given a set of vertices $C$ of a graph $G$, we denote the induced subgraph of $G$ on vertex set $C$ by $G[C]$. For convenience we often use $\chi(C)$ for $\chi(G[C])$. Given a set $A$ of vertices of a graph $G$, we let $G - A$ be the graph obtained from $G$ by deleting the vertices $A$. Similarly for a set $E$ of edges of $G$, we let $G$ be the graph obtained from $G$ by deleting the edges $E$. For a set $E$ edges in $G$, the graph obtained from $G$ by contracting each edge of $E$ (and then removing any resulting loops or multiple edges) is denoted by $G/E$. Given two disjoint sets $A$ and $B$ of vertices in a graph $G$, we let $E(A, B)$ denote the set of edges between $A$ and $B$. For a set $A$ of vertices in a graph $G$, we let $E(A)$ denote the set of edges between vertices of $A$.

The action of performing local complementation at a vertex $v$ in a graph $G$ replaces the induced subgraph on $N(v)$ by its complement. We denote the resulting graph by $G \ast v$. We say that a graph $H$ is a vertex-minor of a graph $G$ if $H$ can be obtained from $G$ by a sequence of vertex deletions and local complementations.

In a graph $G$ the operation of pivoting an edge $uv$ is: $G \setminus uv = G \ast u \ast v \ast u$. A graph $H$ is a pivot-minor of a graph $G$ if $H$ can be obtained from $G$ by a sequence of vertex deletions and pivots. For an edge $uv$ of a graph $G$, let $V_1 = N(u) - N[v]$, $V_2 = N(v) - N[u]$, and $V_3 = N(u) \cap N(v)$. It is straightforward to show that $G \setminus uv$ is the graph obtained from $G$ by first complementing the
edges between each the three pairs of vertex sets \((V_1, V_2), (V_2, V_3),\) and \((V_1, V_3),\) and then swapping the vertex labels of \(u\) and \(v.\) We will use this often and without explicit reference; for a formal proof, see [21].

In a graph \(G,\) the act of replacing an edge \(uw\) with a vertex \(v\) adjacent to \(u\) and \(w\) only is known as subdividing the edge \(uw.\) A graph \(H\) is a subdivision of a graph \(G\) if \(H\) can be obtained from \(G\) by a sequence of subdivisions. We let \(G^k\) denote the graph obtained from \(G\) by subdividing each edge \(k\) times.

If \(v\) is a vertex of degree two in a graph \(G\) and \(v\) is adjacent to two non-adjacent vertices \(u\) and \(w\) then we say that the graph obtained by removing the vertex \(v\) and adding an edge between \(u\) and \(w\) is the graph obtained from \(G\) by smoothing the vertex \(v.\) Observe that the graph obtained from \(G\) by smoothing a vertex \(v\) is \((G * v) - v,\) and so in particular is a vertex-minor of \(G.\) So more generally, by repeated smoothing of vertices, a graph \(G\) is a vertex-minor of any subdivision of \(G.\)

For positive integers \(n, m,\) we let \(K_{n,m}\) denote the complete bipartite graph whose vertices can be partitioned into two stable sets of size \(n\) and \(m\) that are complete to each other. We prove a motivating lemma.

**Lemma 2.1.** The graph \(K^1_{n,(n/2)}\) contains all \(n\)-vertex graphs as vertex-minors.

**Proof.** First observe that by smoothing and deleting vertices we may obtain \(K^1_n\) as a vertex-minor. Then it can easily be checked for \(n \leq 3\) that \(K^1_n\) contains every \(n\)-vertex graph as a vertex-minor by smoothing and deleting vertices. So we may assume that \(n \geq 4.\) Now given an \(n\)-vertex graph \(G,\) we may associate its vertices with the vertices of degree at least 3 in \(K^1_n.\) Now for each pair of distinct vertices \(u\) and \(v\) of \(G\) we may do one of two things. If \(uv\) is an edge of \(G\) then we may simply smooth the corresponding degree-2 vertex of \(K^1_n.\) If \(uv\) is not an edge then we may just delete the corresponding degree-2 vertex of \(K^1_n.\)

Doing this for each such pair \(u, v\) results in the desired vertex-minor \(G.\)

So \(K^1_{n,n}\) provides a suitable universal graph for vertex-minors. We take Lemma 2.1 a step further. A graph \(G\) with vertex-set \(\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\} \cup \{z_{i,j} : i \in [n], j \in [m]\}\) is an interfered \(K^1_{n,m}\) if

- for each \(i \in [n], j \in [m], x_iz_{i,j} \in E(G)\) and \(z_{i,j}y_j \in E(G),\)
- all other edges of \(G\) are contained in \(\{x_kz_{i,j} : i, k \in [n], j \in [m] \text{ with } k < i\}.\)

An interfered \(K^1_{n,m}\) is completely interfered if \(x_iz_{i,j}\) is an edge for all \(i, k \in [n], j \in [m]\) with \(k < i.\) See Figure 1 for an illustration of a completely interfered \(K^1_{3,4}.

**Lemma 2.2.** Let \(G\) be an interfered \(K^1_{N_1,N_2}\) where \(N_1 = R(n,n)\) and \(N_2 = 2^{\binom{n}{2}} + 1.\) Then \(G\) contains \(K^1_{n,n}\) as a pivot-minor.

**Proof.** For each \(j \in [N_2],\) let \(G_j\) be an auxiliary graph on the vertex set \([N_1]\) such that for each pair \(k < i,\) the vertex \(k\) is adjacent to the vertex \(i\) if and only if \(x_k\) is adjacent to \(z_{i,j}\) in \(G.\) By the pigeonhole principle there exists some \(Y \subseteq [N_2]\) with \(|Y| \geq n + 1\) such that for each \(j, j' \in Y,\) we have that \(G_j = G_{j'}.
Therefore for each $k < i$, the vertex $x_k$ of $G$ is either complete or anti-complete to $\{z_i,j : j \in Y\}$.

Now consider another auxiliary graph $A$, this time on the vertex set $[N_1]$ such that for each pair $k < i$, the vertex $k$ is adjacent to the vertex $i$ if and only if $x_k$ is complete to $\{z_i,j : j \in Y\}$ in $G$. A stable set $I$ of $A$ corresponds to an induced $K^1_{|I|,|Y|}$, while a clique $C$ corresponds to a completely interfered $K^1_{|C|,|Y|}$. Hence we may assume that $G$ contains an induced subgraph $H$ that is a completely interfered $K^1_{n,n+1}$.

Then observe that

$$(H \land x_nz_{n,n+1} \land \cdots \land x_1z_{1,n+1}) - \{y_{n+1}, x_n, \ldots, x_1\}$$

is isomorphic to $K^1_{n,n}$ as we require.

As a consequence of Lemma 2.1 and Lemma 2.2 we get the following.

**Lemma 2.3.** For every graph $G$ there exist integers $q$ and $h$ such that every interfered $K^1_{q,h}$ contains $G$ as a vertex-minor.

So to find a graph $G$ as a vertex-minor it is enough to find an interfered $K^1_{q',h'}$ for some sufficiently large $q'$ and $h'$ that depend only on $G$. This is the tactic we follow in proving that vertex-minor-closed classes of graphs are 9-controlled.

One may naturally partition the edge set of a interfered $K^1_{q,h}$ into two halves, an “non-interfered” half that consists of $h$ disjoint stars and an “interfered” half that consists of $q$ stars with some possible additional edges between them. Sections 3 and 5 deal with finding induced structures that shall contain as vertex-minors stars of the “interfered” and “non-interfered” half of the $K^1_{q,h}$ respectively. Sections 4 and 6 shall deal with showing that vertex-minors can simulate suitable edge-like contraction operations on each of these “interfered” and “non-interfered” induced structures respectively. Then Section 7 shall make use of these results and Lemma 2.3 to prove Theorem 1.2 that proper vertex-minor-closed classes of graphs are (linearly) 9-controlled.
Afterwards in Section 8 we make use of a theorem of Chudnovsky, Scott and Seymour \cite{chudnovsky2006} to very quickly extend the work of Sections 3-7 by proving that proper vertex-minor-closed classes of graphs are 2-controlled. In Section 9 we use this to prove Theorem \ref{thm:chi-bounded} that proper vertex-minor-closed classes of graphs are $\chi$-bounded. Lastly, in Section 10, we extend the work from Section 9 a little further to prove Theorem \ref{thm:rho-bounded}. We remark that Sections 3-7, Section 8 and Sections 9-10 (each of which is dedicated to a different subproblem in the usual $\rho$-control strategy) can be read independently of each other.

3 Large induced bloated trees

This section is devoted to the analysis of large induced tree-like structures. From these structures we shall later obtained the “interfered” half of the $K_{q,h'}^1$ that we seek. Statements proven in this section are tailored to our needs but some of them could possibly be of more general interest.

If $T$ is a tree then we say that a vertex of degree at most 1 is a leaf and that a vertex of degree at least 3 is a branching vertex. The degree sum of an $n$-vertex tree is $2n - 2$, therefore if a tree has $\ell > 1$ leaves, then it has at most $\ell - 2$ branching vertices. Likewise, a tree with $b \geq 1$ branching vertices has at least $b + 2$ leaves. We use these two facts repeatedly without reference.

We call maximal cliques (with respect to vertex inclusion) of size at least three big cliques, or big $k$-cliques when we wish to refer to their size. We say that a graph $G$ is a bloated tree if

- every edge is contained in at most one big clique,
- the vertices of every big clique of size $k \geq 3$ have degree at most $k$, and
- the graph obtained by contracting each big clique is a tree.

Note that a bloated tree is not necessarily a tree, but all trees are bloated trees. An alternative definition for bloated trees is that they are block graphs such that for each $k \geq 3$, the vertices that are contained in the big $k$-cliques have degree at most $k$. We say that a vertex of a bloated tree is a leaf if it has degree at most 1. A vertex of a bloated tree is branching if it has degree at least 3 and is not contained in a triangle.

Erdős, Saks and Sós \cite{erdos1985} proved that for each $r \geq 3$, there exists an increasing function $t_r : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n \to \infty} t_r(n) = \infty$, and every connected $K_r$-free graph with at least $n$ vertices contains an induced tree on at least $t_r(n)$ vertices. Fox, Loh and Sudakov \cite{fox2009} later improved the optimal bounds for $t_r$ up to constant factors.

We require a version for bloated trees, and for convenience we may make this independent of the clique number. We do not attempt to optimize the bounds. A clique is itself a bloated tree, so letting $f(n) = t_n(n)$ we obtain the following version of the theorem of Erdős, Saks and Sós.

**Theorem 3.1** (Erdős, Saks and Sós \cite{erdos1985}). There exists an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n \to \infty} f(n) = \infty$, and if $G$ is a connected graph on
at least \( n \) vertices, then it contains an induced bloated tree \( T \) on at least \( f(n) \) vertices.

We will further require a suitable version of Theorem 3.1 in which we seek an induced bloated tree containing distinguished vertices. First we need a lemma on cut vertices and bridges in maximal induced bloated trees of a graph. A vertex \( v \) of a connected graph \( G \) is a cut vertex if \( G - v \) is disconnected. Similarly, an edge \( e \) of a connected graph \( G \) is a bridge if \( G - e \) is disconnected.

**Lemma 3.2.** Let \( G \) be a connected graph and \( T \) a maximal induced bloated tree of \( G \). If \( u \) and \( v \) are adjacent vertices that have degree two in \( T \), and both \( u \) and \( v \) are cut vertices in \( G \), then \( uv \) is a bridge of \( G \).

**Proof.** As \( u \) is a cut vertex of \( G \) and has degree two in the maximal induced bloated tree \( T \), we observe that, \( G - u \) must have exactly two connected components. In particular the two vertices that \( u \) is adjacent to in \( T \) are in separate connected components of \( G - u \). So \( u \) is not contained in a big clique of \( T \). Similarly for \( v \).

Suppose that \( uv \) is not a bridge. Then there exists an induced cycle \( C \) of \( G \) containing the edge \( uv \). No vertex of \( C - \{u, v\} \) is adjacent to any vertex of \( T - \{u, v\} \) as this would contradict the fact that both \( G - u \) and \( G - v \) have exactly two connected components. Consider the vertex \( w \) of \( C \) which is adjacent to \( u \) and distinct from \( v \). Then as \( u \) and \( v \) are cut vertices, we see that \( u \) and possibly \( v \) are the only vertices of \( T \) that are adjacent to \( w \) in \( G \). This contradicts the maximality of the induced bloated tree \( T \) as we may add the vertex \( w \). We conclude that \( uv \) is a bridge of \( G \).

**Theorem 3.3.** There exists an increasing function \( g : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \lim_{n \to \infty} g(n) = \infty \), and if \( G \) is a connected graph and \( S \) is a non-empty subset of its vertices, then \( G \) contains an induced bloated tree with at least \( g(|S|) \) vertices of \( S \).

**Proof.** Let \( f \) be as in Theorem 3.1 and let \( g(n) = \lceil \frac{1}{12} f(n) \rceil \). We will show that \( g \) satisfies the conclusion of the theorem.

Consider the graph \( G' \) obtained from \( G \) by repeating the following two operations until neither can be done.

- If \( v \) is not a vertex of \( S \) and not a cut vertex, then delete \( v \).
- If \( v \) is not a vertex of \( S \) and \( v \) has degree 2 in \( G \) with both incident edges being bridges, then contract an edge incident to \( v \).

Observe that any induced bloated tree of \( G' \) corresponds to an induced bloated tree of \( G \) that contains the same vertices from \( S \). Hence we just need to find an induced bloated tree of \( G' \) that contains at least \( \frac{1}{12} f(|S|) \) vertices of \( S \). By definition of \( f \), we can find a maximal induced bloated tree \( T' \) of \( G' \) with at least \( f(|V(G')|) \geq f(|S|) \) vertices. We will show that \( |V(T') \cap S| \geq \frac{1}{12} |V(T')| \).

Let:

- \( \ell \) be equal to the number of leaves of \( T' \),


\begin{itemize}
  \item $b$ be equal to the number of branching vertices of $T'$,
  \item $x$ be equal to the number of big cliques $X$ of $T'$ having at least three vertices with a neighbour contained in $V(T') - X$,
  \item $y$ be equal to the number of big cliques $X$ of $T'$ having exactly two vertices with a neighbour contained in $V(T') - X$, and
  \item $z$ be equal to the number of big cliques $X$ of $T'$ having exactly one vertex with a neighbour contained in $V(T') - X$.
\end{itemize}

Every vertex $v$ of $G'$ that does not belong to $S$ is a cut vertex of $G'$. So by the maximality of $T'$, the set $S$ contains every leaf of $T'$ and every vertex contained in a big clique $X$ of $T'$ that has no neighbour in $V(T') - X$. If $T'$ is a clique then $V(T') \subseteq S$, we so may assume that $\ell + z \geq 2$. So the set $W$ of leaves, branching vertices and vertices contained in big cliques of $T'$ must contain at least $\ell + y + 2z$ vertices of $S$.

For each big clique $X$ of $T'$, let $P_X$ be a path on the vertices of $X$ that have a neighbour contained in $V(T') - X$. Now let $T''$ be the tree obtained from $T'$ by replacing each big clique $X$ with the path $P_X$. If $X$ is a big clique of $T'$ that has exactly one vertex with a neighbour contained in $V(T') - X$, then this vertex is a leaf of $T''$. Otherwise if $X$ is a big clique of $T'$ that has at least two vertices with a neighbour contained in $V(T') - X$, then two of these vertices have degree 2 in $T''$, while the others are all branching vertices of $T''$. Hence $T''$ has $\ell + z$ leaves and therefore at most $\ell + z - 2$ branching vertices. It further follows that $T'$ has at most $3(\ell + z - b - 2)$ vertices $v$ contained in a big clique $X$ of $T'$ such that $v$ and at least two other vertices of $X$ have a neighbour contained in $V(T') - X$. So the number of vertices that are not contained in $S$, but are contained in a big clique of $T'$ is at most $3(\ell + z - b - 2) + 2y + z$. Therefore $|W - S| \leq b + 3(\ell + z - b - 2) + 2y + z = 3\ell + 2y + 4z - 2b - 6$.

Since $\ell + y + 2z \leq |W \cap S|$, we get that $|W - S| \leq 3|W \cap S|$. Hence $|W| \leq 4|W \cap S|$.

Now let $P$ be the connected components of $T' - W$. Clearly every graph contained in $P$ is a path. Also by considering the tree obtained from $T'$ by contracting each big clique, we observe that $b + x \leq \ell + z - 2$. Therefore $|P| \leq \ell + b + x + y + z - 1 \leq 2\ell + y + 2z - 3$.

If $\sum_{P \in P} |V(P)| \leq 4(2\ell + y + 2z - 3)$, then we have that $|V(T')| \leq 4|W \cap S| + 4(2\ell + y + 2z - 3) < 12|W \cap S| \leq 12|V(T') \cap S|$.

Hence we may assume that $\sum_{P \in P} |V(P)| > 4(2\ell + y + 2z - 3)$.

Consider one such path $P \in P$. Suppose that $P$ contains three consecutive vertices $u, v, w$ that are all not contained in $S$. Then $u, v, w$ would all be cut vertices of $G'$. By Lemma 3.2 both $uw$ and $vw$ must be bridges of $T'$. Hence by the maximality of $T'$, $v$ has degree 2 in $G'$. But now this contradicts the choice of $G'$, so we may conclude that $P$ contains no three consecutive vertices that are all not contained in $S$. Hence $|V(P) \cap S| \geq \left\lceil \frac{|V(P)|}{3} \right\rceil$. 

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Then summing over all $P \in \mathcal{P}$, we get that
\[
\sum_{P \in \mathcal{P}} |V(P) \cap S| \geq \sum_{P \in \mathcal{P}} \frac{|V(P)|}{3} - (2\ell + y + 2z - 3)
\]
\[
> \sum_{P \in \mathcal{P}} \frac{|V(P)|}{3} - \sum_{P \in \mathcal{P}} \frac{|V(P)|}{4}
\]
\[
= \sum_{P \in \mathcal{P}} \frac{|V(P)|}{12}.
\]
So it follows that
\[
|V(T') \cap S| = |W \cap S| + \sum_{P \in \mathcal{P}} |V(P) \cap S| > \frac{|W|}{4} + \sum_{P \in \mathcal{P}} \frac{|V(P)|}{12} \geq \frac{|V(T')|}{12}
\]
as desired. \hfill \square

Next we aim to prune bloated trees with many leaves to obtain a smaller bloated tree, still with many leaves, but without near branching vertices or big cliques.

The following lemma is due to Esperet and de Joannis de Verclos [11].

**Lemma 3.4** (Esperet and de Joannis de Verclos [11]). *Every tree $T$ with at least $\ell$ leaves has a subtree which contains at least $\sqrt{\ell}$ of the leaves of $T$ and has no adjacent branching vertices.*

**Proof.** We will prove a stronger statement on rooted trees. For a rooted tree $T$, let $f_0(T)$ be the largest number of leaves of $T$ in a subtree of $T$ that includes the root vertex and all its children without having adjacent branching vertices. Similarly for a rooted tree $T$, let $f_1(T)$ be equal to the largest number of leaves of $T$ in a subtree of $T$ that contains the root vertex and at most one of its children without having adjacent branching vertices.

We will prove that $f_0(T) \cdot f_1(T) \geq \ell$, which clearly implies the lemma. If $T$ has height either 0 or 1 then the result is clear. So we may assume that $T$ has height at least 2. We proceed by induction on the height of $T$. Let $T_1, \ldots, T_k$ be the subtrees obtained by taking a child of the root of $T$, rooting at this vertex and then taking all its descendants. Then $f_0(T) \geq \sum_{i=1}^{k} f_1(T_i)$ and $f_1(T) \geq \max\{ f_0(T_i) : i \in \{1, \ldots, k\} \}$. Hence
\[
f_0(T) \cdot f_1(T) \geq \sum_{i=1}^{k} (f_1(T_i) \cdot f_0(T_i)) \geq \ell.
\]
as required. \hfill \square

**Lemma 3.5.** Let $T$ be a bloated tree with $\ell$ leaves. Then $T$ contains an induced bloated tree $T'$ that has at least $\ell^4 \ell^4$ leaves and whose branching vertices and big cliques are all at distance at least 4 from every other branching vertex or big clique of $T'$. 

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Proof. First note that if a vertex of a big clique of size \( k \) has degree \( k - 1 \), then we may just delete the vertex. So we may assume that \( T \) has no such vertex.

Next we reduce the problem to trees. We may contract each big clique of \( T \) into a single vertex to obtain a tree \( T^* \). Now by reversing this operation we observe that such a desired subtree of \( T^* \) corresponds to such a desired induced subgraph of \( T \). Hence we may assume that \( T \) is a tree. We need only consider the distance between two branching vertices in \( T \) as there are no big cliques in a tree.

By Lemma 3.4, the tree \( T \) contains an induced subtree \( T_2 \) with at least \( \sqrt{\ell} \) leaves and with no adjacent branching vertices. Now by considering the graph obtained by smoothing degree-2 vertices of \( T_2 \) and applying Lemma 3.4 again, we may find a subtree \( T'' \) of \( T_2 \) and so of \( T \) with at least \( \ell^2 \) leaves and with no pair of branching vertices at a distance of less than 4 from each other.

Next we combine the previous few lemmas so that we may find our desired bloated trees, this is the main result of this section.

**Lemma 3.6.** For every positive integer \( \ell \), there exists a positive integer \( \ell' \) such that every connected graph \( G \) with a set \( S \) of \( \ell' \) distinguished vertices of degree 1 contains an induced bloated tree \( T \) whose branching vertices and big cliques are all at distance at least 4 from each other and with \( \ell \) leaves, all contained in \( S \).

Proof. By Theorem 3.3, there exists some positive integer \( \ell' \) such that every connected graph \( G \) with a set \( S \) of \( \ell' \) distinguished vertices of degree 1 contains an induced bloated tree \( T' \) with \( \ell^4 \) leaves, all contained in \( S \). Now by Lemma 3.5, there is an induced subgraph \( T \) of \( T' \) that is a bloated tree with \( \ell \) leaves all contained in \( S \) and whose big cliques and branching vertices are all at distance at least 4 from each other as required.

### 4 Vertex-minors and induced bloated trees

In this section we will be concerned with simulating an edge contraction-like operation on bloated trees by using vertex-minors. We will require some additional properties on how the induced bloated tree \( T \) interacts with the rest of the graph it lies in than was obtained in Section 3, however this shall be dealt with in Section 7.

The next lemma will allow us to eliminate the big cliques from these bloated trees.

**Lemma 4.1.** Let \( c \) be a degree-\( k \) vertex of a graph \( G \) contained in a big \( k \)-clique \( C \) such that it has a single neighbour \( d \) that is not adjacent to any other vertex of \( C \). Then \( (G - E(C - c))/cd \) is a vertex-minor of \( G \).

Proof. Simply consider \( G * c - c \).

The next lemma will allow us to eliminate undesirable “fanning” that interferes with the bloated tree.
Lemma 4.2. Let $H$ be an induced subgraph of a graph $G$ such that $H$ consists of an induced path $av_0v_1\ldots v_kb$ and an additional vertex $c$ with no neighbours in $\{a,v_0,b\}$ and such that $N_G(\{v_0,v_1,\ldots,v_k\}) \subseteq \{a,b,c\}$. Then $G$ contains either $G/\{v_0v_1,\ldots,v_{k-1}v_k\}$ or $(G-E(c,\{v_1,\ldots,v_k\})\setminus\{v_0v_1,\ldots,v_{k-1}v_k\}$ as a vertex-minor.

Proof. By smoothing vertices we may assume that $\{v_1,\ldots,v_k\} \subseteq N(c)$. The $k=0$ case is trivially true. If $k=1$ then we may obtain the desired vertex-minor by smoothing $v_0$. If $k=2$ then:

$$(G*v_1*v_2) - v_1 - v_2 = G/\{v_0v_1,v_1v_2\}.$$ 

If $k=3$, then:

$$(G*v_1*v_2*v_3) - v_1 - v_2 - v_3 = (G-E(c,\{v_1,v_2,v_3\})\setminus\{v_0v_1,v_1v_2,v_2v_3\}.$$ 

Similarly for $k>3$ we may reduce to the $k-3$ case via the vertex-minor;

$$(G*v_{k-1}*v_{k-2}*v_k) - v_{k-2} - v_{k-1} - v_k = G/\{v_{k-3}v_{k-2},v_{k-2}v_{k-1},v_{k-1}v_k\}.$$ 

This completes the proof. 

Consider an induced bloated tree $T$ of a graph $G$. Let $L$ be the set of leaves of $T$. Let $B$ be the set of branching vertices. Let $Z$ be the set of vertices $z \in V(T)-L$ such that $N_G(z) - V(T)$ is non-empty.

We call $T$ shrinkable if

- for each $z \in Z$, $|N_G(z) - V(T)| = 1$,
- in $T$, the distance between pairs of vertices $z, z' \in Z$ with $N_G(z) - V(T) \neq N_G(z') - V(T)$ is at least 4,
- in $T$, no vertex of $B \cup Z$ is within distance 3 of a big clique,
- in $T$, no vertex of $Z$ is within distance 3 of a vertex in $B$, and
- in $T$, vertices of $L$ are at distance at least 2 from every big clique and distance at least 3 from every vertex of $Z$.

In a similar but much simpler manner, we say that an induced tree $T$ is shrinking if $B \cup Z$ is a stable set. We remark that the sets $B$ and $Z$ need not be disjoint in the definition of a shrinking tree. The next step is to modify shrinkable bloated trees into shrinking trees.

Lemma 4.3. Let $T$ be an induced shrinkable bloated tree of a graph $G$ and let $L$ be the set of leaves of $T$. Then there is a vertex-minor $G'$ of $G$ with an induced shrinking tree $T'$ that has $L$ as its set of leaves such that $G' - (V(T') - L) = G - (V(T) - L)$ and $N_{G'}(V(T') - L) \subseteq N_G(V(T) - L)$.
Proof. Firstly by appropriately removing vertices of $T - L$ we can assume that no big clique of $T$ contains a vertex with no neighbour in $T$ that is outside the big clique.

Now we apply Lemma 4.3 to a vertex of each big clique of $T$ to obtain a vertex-minor $G^*$ of $G$ and an induced tree $T^*$ of $G^*$ with its set of leaves being $L$, its set $B^*$ of branching vertices, and $Z^* = \{ z \in V(T) - L : N_{G^*}(z) - V(T^*) \neq \emptyset \}$ such that:

- for each $z \in Z^*$, $|N_{G^*}(z) - V(T^*)| = 1$,
- in $T^*$, the distance between pairs of vertices $z, z' \in Z$ with $N_{G^*}(z) - V(T^*) \neq N_{G^*}(z') - V(T^*)$ is at least 4,
- in $T^*$, all vertices of $B^*$ are at distance at least 3 from vertices of $Z^*$ and at distance at least 2 from other vertices of $B^*$, and
- in $T^*$, vertices of $L$ are at distance at least 3 from vertices of $Z^*$.

In particular, this eliminated the big cliques of $T$. Next we must eliminate the undesirable “fanning”. Now we may partition the vertices of $T^* - B^* - L$ into sets $X_1, \ldots, X_h$ such that for each $i \in [h]$, we have $|N_{G^*}(X_i) - V(T^*)| \leq 1$, and $G^*[X_i]$ is a path with endpoints $a$ and $b$ and a vertex $v_0$ adjacent to $a$ with $\{a, v_0, b\} \notin Z^*$. Then to obtain our desired vertex-minor $G'$ we just apply Lemma 4.2 to each set of the induced paths $G^*[X_i]$ such that $X_i \cap Z^* \neq \emptyset$.

With this we may now simulate another contraction operation on shrinkable bloated trees.

Lemma 4.4. Let $T$ be an induced shrinkable bloated tree of a graph $G$, with leaves $L$. Then there is a vertex-minor $G'$ of $G$ with an induced star $T'$ that has $L$ as its set of leaves such that $G' - (V(T') - L) = G - (V(T) - L)$ and $N_{G'}(V(T') - L) \subseteq N_G(V(T) - L)$. Or equivalently there exists some subset $E^*$ of $E(V(T) - L, V(G) - V(T))$ such that $G' = (G - E^*) / E(T - L)$ is a vertex-minor of $G$.

Proof. Firstly by Lemma 4.3 we may instead assume that $T$ is a shrinking tree of $G$.

If $|V(T) - L| = 1$ then the result is trivial. If $|V(T) - L| = 2$ then we may simply smooth some vertex. Suppose for sake of contradiction that $T$ is a counter-example with $|V(T) - L|$ minimum, we may assume that $|V(T) - L| \geq 3$.

Suppose first that there exist two adjacent vertices $u$ and $v$ of $T$ such that $u, v \notin L \cup B \cup Z$. Then both $u$ and $v$ have degree 2 in $G$ and we may smooth one, but this contradicts $|V(T) - L|$ being minimum.

Similarly suppose that there exists a vertex $v$ of $T$ such that $v \notin L \cup B \cup Z$ and $v$ is adjacent to a leaf of $T$. Then we may again smooth $v$ contradicting $|V(T) - L|$ being minimum.

So there must exist a vertex $v \in V(T) - (L \cup B \cup Z)$ of degree 2 with neighbours $u$ and $w$ such that $u, w \in (B \cup Z) - L$. Let $E'$ be the set of edges between either $u$ or $w$ and the set $(N_G(u) \cap N_G(w)) - v \subseteq N_G(V(T) - L)$. Then $(G \cap uv) - u - v$ is the graph obtained by deleting the set of edges $E'$, and
Lemma 5.1 there exist a lollipop \((P, C)\) such that \(\chi(G) \geq c + k\kappa\). Then there is a lollipop \((P, C')\) contained in \(C\) with \(\chi(C') \geq c + k\kappa\) and a \(k\)-stripe of \((P, C')\).

Proof. First we handle the case that \(k = 1\). Let \(s_1\) be a vertex of \(C\), and let \(C'\) be the vertices of a connected component of \(G[C - N_7(s_1)]\) with chromatic number at least \(\chi(C) - \kappa \geq c\). Then let \(P\) be a shortest path in \(G[C]\) between the vertex \(s_1\) and \(N(C') \cap C\). Let \(t\) be the other end vertex of \(P\). Then \((P, C')\) provides the desired lollipop with \(\{s_1\}\) being a 1-stripe of \((P, C')\).

So now we may proceed inductively. Let \((P^*, C^*)\) be a lollipop with \(k - 1\) stripes contained in \(C\) with \(\chi(C^*) \geq \chi(G) - (k - 1)\kappa \geq c + k\kappa\). Let \(t^*\) be the end vertex of \(P^*\) neighbouring a vertex of \(C^*\). Let \(\{s_1, \ldots, s_{k-1}\}\) be a \((k - 1)\)-stripe of \((P^*, C^*)\). Let \(C'\) be the vertex set of a connected component of \(G[C^* - N_8(t^*)]\) with chromatic number at least \(\chi(C^*) - \kappa \geq c\). Let \(P'\) be a shortest path in \(G[C^* \cup \{t^*\}]\) between \(t^*\) and \(N(C') \cap C^*\) and let \(t\) be the other end vertex of \(P'\). Let \(P = P^* \cup P'\) and let \(s_k\) be the vertex of \(P\) adjacent to \(t^*\) and contained in \(C^*\). Then \((P, C')\) is a lollipop contained in \(C\), and \(\{s_1, \ldots, s_k\}\) is a \(k\)-stripe of \((P, C')\) as required.

The purpose of the lollipop structure was to aid us in finding the paths we seek. We may now be more precise with what we need from this section.

Lemma 5.2. Let \(q, h, \kappa\) be non-negative integers with \(\kappa \geq 1\). Let \(G\) be a graph such that \(\chi(G) \leq \kappa\) and let \(C \subseteq V(G)\) be such that \(\chi(C) \geq \text{ghr}(G)\). Then there exist induced paths \(P_1, \ldots, P_h\) contained in \(C\) that are pairwise at distance at least 3 from each other in \(G\) and such that, for each \(j \in [h]\), the path \(P_j\) contains a set \(\{s_{i,j}, \ldots, s_{q,j}\}\) of \(q\) vertices and the vertices of \(\{s_{i,j} : i \in [q], j \in [h]\}\) are pairwise at distance at least 8 from each other.

Proof. The result is vacuously true if \(h = 0\), so may proceed inductively. By Lemma 5.1 there exist a lollipop \((P, C')\) contained in \(C\) and a \(q\)-stripe of \((P, C')\).
We remark that no condition on the chromatic number of $G[C']$ is necessary here and that we may take $C' = \emptyset$.

Next we choose a set $\{s_1, h, \ldots, s_q, h\}$ of vertices contained in $V(P)$ that are pairwise at distance at least 8 from each other in $G$, and subject to this so that the distance in $P$ between the first and last vertex of $\{s_1, h, \ldots, s_q, h\}$ as they appear on $P$ is minimised. Such a set exists since there is a $q$-stripe of $(P, C')$.

Let $P_h$ be the subpath of $P$ between the first and last vertex of $\{s_1, h, \ldots, s_q, h\}$ as they appear on $P$. Then by the choice of $\{s_1, h, \ldots, s_q, h\}$, we have that $\{s_1, h, \ldots, s_q, h\} \subseteq V(P_h)$ and $V(P_h) \subseteq N[\{s_1, h, \ldots, s_q, h\}]$.

Now let $C^*$ be the vertex set of a connected component of the induced subgraph $G[C - N_9([s_1, h, \ldots, s_q, h])]$ such that $\chi(C^*) \geq \chi(C) - q\kappa \geq q(h - 1)\kappa$. Notice that $C^*$ is at distance at least 3 from $V(P_h)$ and at distance at least 10 from $\{s_1, h, \ldots, s_q, h\}$ in $G$.

Then by the inductive hypothesis we may find paths $P_1, \ldots, P_{h-1}$ contained $C^*$ satisfying the conclusion of the lemma. Then $P_1, \ldots, P_h$ provide the desired paths contained in $C$.

\begin{proof}
This is trivially true if $|X'| = 0$, so we proceed inductively. If $X'$ is non-empty, then the path $P$ has at least two vertices. Let $x \in X'$, and let $y$ be a vertex of $V(P)$ that is adjacent to $x$.

Suppose first that $y$ is an end vertex of $P$. Then let $G'' = G - y$, and let $X'' = X' - x$. Then in the graph $G''$, the path $P'' = P - y$ dangles from the set $X$, and $X''$ is exactly the subset of $X$ consisting of the vertices with an even number of neighbours in $V(P)$. If the distance between two vertices in $X'$ is at least 3, then there is a vertex-minor $H$ of $G$ such that $G[V(G) - V(P)] = H[V(G) - V(P)]$ and $H[V(H) - (V(G) - V(P))]$ is an induced path $P'$ that dangles oddly from the set $X$.

Let $G$ be a graph, we say that an induced path $P$ dangles from a set $X \subset V(G)$ if $N(V(P)) = X$. If for each $x \in X$, there is an odd number of edges between $x$ and $V(P)$, then we say that the path $P$ dangles oddly.

\begin{lemma}
Let $P$ be a path dangling from a stable set $X$ in a graph $G$ and let $X'$ be the subset of $X$ consisting of all the vertices that have an even number of neighbours in $V(P)$. If the distance between two vertices in $X'$ is at least 3, then there is a vertex-minor $H$ of $G$ such that $G[V(G) - V(P)] = H[V(G) - V(P)]$ and $H[V(H) - (V(G) - V(P))]$ is an induced path $P'$ that dangles oddly from the set $X$.
\end{lemma}

\begin{proof}
This is trivially true if $|X'| = 0$, we so proceed inductively. If $X'$ is non-empty, then the path $P$ has at least two vertices. Let $x \in X'$, and let $y$ be a vertex of $V(P)$ that is adjacent to $x$.

Suppose first that $y$ is an end vertex of $P$. Then let $G'' = G - y$, and let $X'' = X' - x$. Then in the graph $G''$, the path $P'' = P - y$ dangles from the set $X$, and $X''$ is exactly the subset of $X$ consisting of the vertices with an even number of neighbours in $V(P'')$. Clearly in $G'' = G - y$ the distance between vertices in $X''$ is at least 3 and $X$ remains a stable set. So by induction $G''$ (and thus $G$) contains a vertex-minor $H$ as in the conclusion of the lemma.

So we may assume now in the second case that $y$ is not an end vertex of $P$. Let $a, b$ be the two vertices of $P$ adjacent to $y$. Then let $G'' = G - y - x$, and let $X'' = X' - x$. The graph $G''$ is identical to $G$, except that $y$ is deleted, the edge $ab$ is added and the number of edges between $x$ and $\{a, b\}$ is still equal modulo

\end{proof}
2. In particular $X$ is a stable set in $G''$. Also $P'' = G''[V(P) - y]$ is an induced path dangling from $X$ in $G$ so that $X''$ is exactly the subset of $X$ consisting of the vertices with an even number of neighbours in $V(P'')$. The neighbourhood of a vertex in $X - x$ is the same in both $G$ and $G''$. So in $G''$ the distance between vertices of $X''$ is at least 3. Then as in the first case, by induction we get that $G$ contains a vertex-minor $H$ as in the conclusion of the lemma.

Next we show that vertex-minors can be used to simulate another edge contraction-like operation, this time on paths dangling oddly.

**Lemma 6.2.** Let $q$ be a positive integer and let $P$ be a path dangling oddly from a set $X$ of vertices in a graph $G$ with $|X| \geq R(q,q)$. Then there exists $Y \subseteq X$ with $|Y| \geq q$ such that the graph $(G - (X - Y) - E(Y))/E(P)$ is a vertex-minor of $G$.

**Proof.** First we suppose that the path $P$ consists of a single vertex $p$. If there exists a stable set of $I$ in $X$ of size $q$, then it is enough to take $Y = I$, so we may assume not. Then by Ramsey theorem there exists a clique $C$ in $X$ of size $q$. In this case we may take $Y = C$ and $(G - (X - Y)) \ast p$.

We shall now proceed inductively. Let the path $P$ be $p_1p_2 \ldots p_m$, we may now assume that $m \geq 2$. Observe that in the graph $G' = G \ast p_m - p_m$, the path $P' = p_1p_2 \ldots p_{m-1}$ dangles oddly from the set $X$. Furthermore $G' - V(P')$ and $G - V(P)$ may differ only on the adjacencies between vertices of $X$.

So by induction there exists $Y \subseteq X$ with $|Y| \geq q$ such that the graph $(G' - (X - Y) - E_G(Y))/E(P') = (G - (X - Y) - E_G(Y))/E(P)$ is a vertex-minor of $G'$, and thus of $G$ as required.

In a graph $G$, we say that a path dangling from a set $X$ *dangles spaciously* if the distance between vertices in $X$ is at least 6. By the previous two lemmas we obtain the main result of this section.

**Lemma 6.3.** Let $q$ be a positive integer and let $P$ be a path dangling spaciously from a vertex set $X$ with $|X| \geq R(q,q)$ in a graph $G$. Then there exists $Y \subseteq X$ with $|Y| \geq q$ such that the graph $(G - (X - Y))/E(P)$ is a vertex-minor of $G$.

Of course this lemma still holds if we just required that vertices of $X$ are at distance at least 3 from each other, rather than at least 6 as in the definition of a path dangling spaciously. But for our purposes a minimum distance of at least 6 is enough.

### 7 Linear 9-control

In this section we shall see the fruits of our labour in Sections 3-6 and prove that vertex-minor-closed classes of graphs are linearly 9-controlled. We remark that with some more care one may certainly argue directly that such graphs are linearly $\rho$-controlled for some slightly smaller $\rho$. However showing 9-control shall be sufficient for later extending to 2-control in Section 8.

We call a collection $\mathcal{L} = (L^0, L^1, L^2, L^3)$ a *long cover* of a set $C \subseteq V(G)$ if:
the subsets \( L^0, L^1, L^2, L^3, C \subseteq V(G) \) are pairwise disjoint,

- \( G[L^0] \) is connected,
- for each \( i \in \{0, 1, 2\} \), \( L^i \) dominates \( L^{i+1} \), and \( L^3 \) dominates \( C \),
- for each \( i \in \{0, 1, 2\} \), \( C \) is anti-complete to \( L^i \), and
- for each \( i, j \in \{0, 1, 2\} \), \( L_i \) is anti-complete to \( L_j \) if \(|i - j| > 1| \).

We say that two long covers \( \mathcal{L}_i = (L^0_i, L^1_i, L^2_i, L^3_i) \) and \( \mathcal{L}_j = (L^0_j, L^1_j, L^2_j, L^3_j) \) are disjoint if the two sets \( L^0_i \cup L^1_i \cup L^2_i \cup L^3_i \) and \( L^0_j \cup L^1_j \cup L^2_j \cup L^3_j \) are disjoint. We say that a collection of pairwise disjoint long covers \( (\mathcal{L}_i : i \in [q]) \) of a set \( C \) is a long \( q \)-cover of a set \( C \subseteq V(G) \) if for each \( i, j \in [q] \), with \( i < j \), the set of vertices \( L^0_i \cup L^1_i \cup L^2_i \cup L^3_i \) is anti-complete to \( L^0_i \cup L^1_i \cup L^2_i \cup L^3_i \).

We start by showing that for large \( q \), we may find long \( q \)-covers of sets with large chromatic number. This is just a straightforward levelling argument.

**Lemma 7.1.** Let \( q, c, \kappa \) be non-negative integers with \( \kappa \geq 1 \). Then every graph \( G \) satisfying \( \chi(G) > 2^q \max\{c, \kappa\} \) and \( \chi^{(3)}(G) \leq \kappa \) contains a long \( q \)-cover \( (\mathcal{L}_i : i \in [q]) \) of a set \( C \), with \( \chi(C) > c \).

**Proof.** For \( q = 0 \), the result is trivial. We proceed inductively on \( q \). Let \( G \) be such a graph and let \( v \) be a vertex of \( G \) in a component with largest chromatic number. Let \( t \) be the smallest positive integer such that \( G[N_t(v)] \) has chromatic number more than \( 2^{q-1} \max\{c, \kappa\} \) (such a \( t \) exists as otherwise for each \( i \), dependent on if \( i \) is odd or even, we could colour the vertices of \( N_i(v) \) from one of two sets of \( 2^{q-1} \max\{c, \kappa\} \) colours, thus yielding a colouring of \( G \) with at most \( 2^q \max\{c, \kappa\} \) colours). Note that \( t \geq 4 \) as \( \chi(N_3[v]) \leq \kappa \). Now \( G[N_t(v)] \) contains a long \((q - 1)\)-cover \( (\mathcal{L}_i : i \in [q - 1]) \) of a set \( C \subseteq N_t(v) \), with \( \chi(C) > c \). Now let \( L^0_q = N_{t-4}(v), L^1_q = N_{t-3}(v), L^2_q = N_{t-2}(v), L^3_q = N_{t-1}(v) \), and let \( \mathcal{L}_q = (L^0_q, L^1_q, L^2_q, L^3_q) \). Then \( (\mathcal{L}_i : i \in [q]) \) is a long \( q \)-cover of \( C \) as required.

We plan to find a large interfered \( K^1_{q', h'} \) as a vertex-minor. We shall find the “interfered” half of our \( K^1_{q', h'} \), within the vertices of some long \( q \)-cover for sufficiently large \( q \). The stars of the “non-interfered” half of our \( K^1_{q', h'} \) shall be found within certain paths disjoint from the long \( q \)-cover. For now we must focus on the “non-interfered” half of our \( K^1_{q', h'} \).

Let \( G \) be a graph containing a long \( q \)-cover \( (\mathcal{L}_i : i \in [q]) \). We say that a collection of induced paths \( P_1, \ldots, P_h \) that are disjoint from \( (\mathcal{L}_i : i \in [q]) \) dangle spaciously from the long \( q \)-cover \( (\mathcal{L}_i : i \in [q]) \) if they are pairwise disjoint and anti-complete to each other, and there exists a set \( M = \{m_{i,j} : i \in [q], j \in [h]\} \) of vertices such that

- the vertices of \( M \) are at distance at least 6 from each other,
- for each \( i \in [q] \), the vertices \( \{m_{i,1}, \ldots, m_{i,h}\} \) are contained in \( L^3_i \), and
• for each $j \in [h]$, the path $P_j$ dangles spaciously from the set $\{m_{1,j}, \ldots, m_{q,j}\}$ of vertices in the subgraph of $G$ induced by the vertices of the long $q$-cover $(L_i : i \in [q])$ and the paths $P_1, \ldots, P_h$.

Next we use the main result of Section 5 to find such paths dangling spaciously from a long $q$-cover.

**Lemma 7.2.** Let $q, h, \kappa$ be positive integers and let $G$ be a graph such that $\chi^{(9)}(G) \leq \kappa$ and $\chi(G) > 2^q q h \kappa$. Then $G$ contains a long $q$-cover $(L_i : i \in [q])$ with $h$ paths $P_1, \ldots, P_h$ dangling spaciously.

**Proof.** By Lemma 7.1 there exists a set $C^* \subseteq V(G)$ with $\chi(C^*) > q h \kappa$ and a long $q$-cover $(L^*_i : i \in [q]) = ((L^*_i^0, L^*_i^1, L^*_i^2, L^*_i^3) : i \in [q])$ of $C^*$. Now by Lemma 5.2 there exist induced paths $P_1, \ldots, P_h$ contained in $C^*$ that are pairwise at distance at least 3 from each other in $G$ and such that for each $j \in [h]$, the path $P_j$ contains a set of $q$ vertices $\{s_{1,j}, \ldots, s_{q,j}\}$ such that the vertices of $\{s_{1,j} : i \in [q], j \in [h]\}$ are pairwise at distance at least 8 from each other.

Now for each $i \in [q], j \in [h]$, let $m_{i,j}$ be a vertex of $L^*_i^3$ that is adjacent to $s_{i,j}$. Let $M = \{m_{i,j} : i \in [q], j \in [h]\}$. Then as the vertices of $\{s_{i,j} : i \in [q], j \in [h]\}$ are pairwise at distance at least 8 from each other, we see that the vertices of $M$ must be pairwise at distance at least 6 from each other. Then let $(L_i : i \in [q])$ be the long $q$-cover (of an empty set) obtained from the long $q$-cover $(L^*_i : i \in [q])$ by removing the vertices of $N(V(P_1) \cup \cdots \cup V(P_h)) - M$ that are contained in $(L^*_i : i \in [q])$.

Notice that in Lemma 7.2 it is not important what set $C$ of vertices that $(L_i : i \in [q])$ is a long $q$-cover of, and indeed we may as well assume that the set $C$ is empty. The next step will be to apply the main result of Section 6 to simulate an edge contraction-like operation on these paths dangling spaciously.

A graph $F$ is a $(q, h)$-frame if there exists a partition of the vertices into sets $A_1, \ldots, A_q, M = \{m_{i,j} : i \in [q], j \in [h]\}, S = \{s_1, \ldots, s_h\}$ such that

• for each $i \in [q], F[A_i]$ is connected,

• the vertex sets $A_1, \ldots, A_q$ are pairwise anti-complete to each other,

• for each $i \in [q], j \in [h]$, the vertex $m_{i,j}$ has a single neighbour $y_{i,j}$ contained in $A_i$, $y_{i,j}$ has degree 2, and all other neighbours of $m_{i,j}$ are contained in $A_1 \cup \cdots \cup A_{i-1} \cup \{s_j\}$,

• for each $j \in [h]$, we have $N(s_j) = \{m_{1,j}, \ldots, m_{q,j}\}$, and

• the vertices of $M$ are pairwise at distance at least 6 from each other in $F - S$.

Next we will find as vertex-minors large frames within long $q'$-covers with many dangling paths.
Lemma 7.3. For all pairs of positive integers \( q \) and \( h \), there exists a positive integer \( q' \) with the following property. Let \( G \) be a graph containing a long \( q' \)-cover \( \{ \mathcal{L}_i : i \in [q'] \} \) with \( h \) paths \( P_1, \ldots, P_h \) dangling spaciously. Then \( G \) contains as a vertex-minor a \((q,h)\)-frame \( F \).

Proof. Fix \( q \) and \( h \). Let \( q_0 = q \) and for each \( j \in [h] \) in order, let \( q_j = R(q_{j-1}, q_{j-1}) \). Let \( q'_0 = q_0 \).

Firstly by removing vertices we may assume that all vertices of \( G \) belong to either the long \( q' \)-cover \( \{ \mathcal{L}_i : i \in [q'] \} \) or one of its paths dangling spaciously. Let \( M' = \{ m'_{i,j} : i \in [q'], j \in [h] \} \) be the vertices of the long \( q' \)-cover that the paths \( P_1, \ldots, P_h \) dangle spaciously from. Now for each \( m'_{i,j} \in M' \), let \( y'_{i,j} \) be a vertex of \( L_2^q \) adjacent to \( m'_{i,j} \) and let \( z'_{i,j} \) be a vertex of \( L_1^q \) adjacent to \( y'_{i,j} \). Let \( Y' = \{ y'_{i,j} : i \in [q'], j \in [h] \} \) and \( Z' = \{ z'_{i,j} : i \in [q'], j \in [h] \} \). Let \( G' = G[M' \cup Y' \cup Z' \cup (\bigcup_{i=1}^{h} L_0^q) \cup (\bigcup_{j=1}^{h} V(P_j))]. \)

Now in \( G' \), for each \( j \in [h] \), the path \( P_j \) dangles spaciously from the set \( \{ m'_{i,j}, \ldots, m'_{q',j} \} \). Notice that \( G' \setminus \bigcup_{i=1}^{h} E(P_i) \) is a \((q',h)\)-frame (where each path \( P_j \) is contracted to a single vertex \( s_j \), and for each \( i \in [q'] \) we have \( A_i = L_0^q \cup \{ z'_{i,j} : j \in [h] \} \)). However of course \( G' \setminus \bigcup_{i=1}^{h} E(P_i) \) need not be a vertex-minor of \( G^* \). But with \( h \) applications of Lemma 3.6 we see that \( G' \) does at least contain a \((q,h)\)-frame \( F \) as a vertex-minor as we require. \( \square \)

We say that a \((q,h)\)-frame \( F \) is trimmed if for each \( i \in [q] \), the induced subgraph \( F[A_i \cup \{ m_{i,1}, \ldots, m_{i,h} \}] \) is a bloated tree \( T_i \) such that

- the leaves of \( T_i \) are \( \{ m_{i,1}, \ldots, m_{i,h} \} \), and
- in \( T_i \) all big cliques and branching vertices are at distance at least 4 from each other.

Given a \((q,h')\)-frame with \( h' \) sufficiently large we can apply Lemma 3.6 to each of the connected induced subgraphs \( F[A_1], \ldots, F[A_q] \) in order (with the distinguished vertices being contained in \( \{ m_{i,1}, \ldots, m_{i,h} \} \) for each \( F[A_i] \)) to obtain a trimmed \((q,h)\)-frame as an induced subgraph.

To consolidate our position, by Lemma 7.3 and repeated application of Lemma 3.6 as discussed, we obtain the following lemma.

Lemma 7.4. For all pairs of positive integers \( q \) and \( h \), there exists a pair of positive integers \( q' \) and \( h' \) with the following property. Let \( G \) be a graph containing a long \( q' \)-cover \( \{ \mathcal{L}_i : i \in [q'] \} \) with \( h' \) paths \( P_1, \ldots, P_{h'} \) dangling spaciously. Then \( G \) contains as a vertex-minor a trimmed \((q,h)\)-frame \( F \).

We call a \((q,h)\)-frame pure if it is trimmed and in addition, for each \( i \in [q] \), the induced bloated tree \( T_i = F[A_i \cup \{ m_{i,1}, \ldots, m_{i,h} \}] \) is shrinkable (as defined in Section 4).

Notice that the bloated trees of a trimmed \((q,h)\)-frame are rather close to being shrinkable. For example if \( T_i \) is a bloated tree of a trimmed \((q,h)\)-frame \( F \), then the leaves \( \{ m_{i,1}, \ldots, m_{i,h} \} \) of \( T_i \) are at distance at least 2 in \( T_i \) from big cliques of \( T_i \) since by the definition of a \((q,h)\)-frame, the unique neighbour of
each vertex of \( \{ m_{i,1}, \ldots, m_{i,h} \} \) in \( T_i \) has degree 2. Additionally if \( Z_i \) is the set of non-leaf vertices of \( T_i \) that have a neighbour outside the bloated tree \( T_i \), then each vertex of \( Z_i \) has a neighbour in \( M \) and all their neighbours outside the bloated tree are contained in \( M \). Since the distance in \( F - S \) between vertices of \( M \) is at least 6, each vertex in \( Z_i \) has just a single neighbour outside \( T_i \), and similarly if \( z, z' \in Z_i \) have different neighbours outside \( T_i \), then the distance between \( z \) and \( z' \) in \( T_i \) is at least 4. Also in \( T_i \), the vertices in \( Z_i \) are at distance at least 5 from leaf vertices of \( T_i \) since the leaves are contained in \( M \). So a bloated tree \( T_i \) of a trimmed \( (q,h) \)-frame is shrinkable if in \( T_i \), the vertices in \( Z_i \) are at distance at least 4 from the branching vertices and big cliques of \( T_i \).

Therefore an equivalent definition of a pure \((q,h)\)-frame is that it is a trimmed \((q,h)\)-frame such that for each \( i \in [q] \) the vertices in \( Z_i \) in the bloated tree \( T_i \) are at distance at least 4 in \( T_i \) from the branching vertices and big cliques of \( T_i \).

The next step is to make trimmed frames pure.

**Lemma 7.5.** Let \( F \) be a trimmed \((q(3h - 5), h)\)-frame with \( h \geq 2 \). Then \( F \) contains as an induced subgraph a pure \((q,h)\)-frame \( F' \).

**Proof.** First of all we may assume that no bloated tree \( T_j \) of \( F \) contains a vertex of degree \( k - 1 \) which belongs to a big \( k \)-clique, as otherwise we may simply delete this vertex and any new leaves this creates as necessary.

We consider a \((q(3h-5))\)-vertex auxiliary graph \( H \) with vertex set \([q(3h-5)]\) corresponding to the bloated trees \( T_{1}, \ldots, T_{q(3h-5)} \). For each \( i < j \), \( ij \) is an edge of \( H \) if and only if in \( F \) there is an edge between some leaf of \( T_j \) and some vertex \( w \) of \( T_i \) that is at distance at most 3 in \( T_i \) from a big clique or a branching vertex of \( T_i \).

Fix \( i \in [q(3h-5)] \) and consider a set \( C \) which is either a big clique or a single branching vertex of \( T_i \). Then take \( k = |N(C) \cap V(T_i)| \). Let \( J \) be the set of all \( j \in [q(3h-5)] - [i] \), such that some leaf of \( T_j \) has a neighbour in \( T_i \) at distance at most 3 in \( T_i \) from \( C \). Note that the vertices at distance at most 3 from \( C \) in \( T_i \) form a bloated tree with \( C \) being the only big clique or branching vertex. So observe that, as the vertices of \( M \) are at distance at least 6 from each other in \( F - S \), we must have that \( |J| \leq k \).

In a tree \( T \) with \( \ell \) leaves, the sum of the degrees of the branching vertices is at most \( 3(\ell - 2) = 3\ell - 6 \). Hence in \( H \), \( i \) is adjacent to at most \( 3h - 6 \) vertices \( j \) with \( j > i \). Hence \( H \) is \((3h - 6)\)-degenerate and so \((3h - 5)\)-colourable. Hence \( H \) has a stable set \( I \) of size \( \frac{q(3h-5)}{3h-5} = q \).

Finally observe that we may obtain a pure \((q,h)\)-frame \( F \) by deleting the bloated trees \( T_j \) such that \( j \) is not contained in the stable set \( I \).

Each bloated tree of a pure frame is shrinkable and so Lemma 4.4 can be applied. We now prove the last lemma of this section.

**Lemma 7.6.** Let \( H \) be a graph, then there exists positive integers \( q' \) and \( h' \) such that every graph \( G \) containing a long \( q'\)-cover \((L_i : i \in [q'])\) with \( h' \) paths \( P_1, \ldots, P_{h'} \), dangling spaciously contains the graph \( H \) as a vertex-minor.
Proof. Let $q^*, h^*$ be such that every interfered $K_{q^*, h^*}^1$ contains $H$ as a vertex-minor (as in Lemma 7.3). By Lemma 7.4 and Lemma 7.5 there exist positive integers $q'$ and $h'$ such that every graph $G$ containing a long $q'$-cover ($L_i : i \in [q']$) with $h'$ paths $P_1, \ldots, P_{h'}$ dangling spaciously contains a pure $(q^*, h^*)$-frame $F$ as a vertex-minor. Next we shall show that $F$ contains an interfered $K_{q^*, h^*}^1$ as a vertex-minor.

For each $i, j \in [q']$ with $i < j$, let $E_{i,j}'$ be the set of edges between the two sets of vertices $A_i$ and $\{m_{j,1}, \ldots, m_{j,h^*}\}$ in $F$. Let $E'$ be the union of all such sets $E_{i,j}'$. Then by an application of Lemma 4.4 to each of the bloated trees $T_1, \ldots, T_{q'}$, there exists a set $E^*$ of edges contained in $E'$ such that $F$ contains $(F - E^*)/\bigcup_{i=1}^{q'} E(A_i)$ as a vertex-minor. This graph is an interfered $K_{q^*, h^*}^1$. So we conclude that $G$ contains $H$ as a vertex-minor as required.

We may now prove the main result of this section, Theorem 1.2 that proper vertex-minor-closed classes of graphs are linearly 9-controlled.

Proof of Theorem 1.2. Let $G$ be a proper vertex-minor-closed class of graphs and let $H$ be a graph not contained in $G$. Let $q', h'$ be as in Lemma 7.6. We will show that for each $G \in G$, with $\chi(G) \leq \kappa$, we have $\chi(G) \leq 2^{q'h'}\kappa$.

Suppose that $\chi(G) > 2^{q'h'}\kappa$. Then by Lemma 7.2, $G$ contains a long $q'$-cover ($L_i : i \in [q']$) with $h'$ paths $P_1, \ldots, P_{h'}$ dangling spaciously. But then by Lemma 7.6 the graph $G$ would contain $H$ as a vertex-minor, a contradiction. Hence $\chi(G) \leq 2^{q'h'}\kappa$ as required.

8 From 9-control to 2-control

In this section we quickly extend the fact that proper vertex-minor closed classes are 9-controlled to prove that they are in fact 2-controlled. We make use of a theorem of Chudnovsky, Scott and Seymour [5].

Theorem 8.1 (Chudnovsky, Scott and Seymour [5, 1.10]). Let $\mu \geq 0$, and let $\rho \geq 2$. Let $G$ be a $\rho$-controlled class of graphs that is closed under taking induced subgraphs. Then the class of all graphs in $G$ that do not contain any of $K_{\mu, \mu}^1, \ldots, K_{\mu, \mu}^{\rho+1}$ as an induced subgraph is 2-controlled.

Theorem 8.2. Every proper vertex-minor-closed class of graphs is 2-controlled.

Proof. Let $G$ be a proper vertex-minor class of graphs. Let $H$ be a graph not contained in $G$ and let $\mu = (|V(H)|)$. Then by smoothing vertices and applying Lemma 2.1 we see that each of the graphs $K_{\mu, \mu}^1, \ldots, K_{\mu, \mu}^{\rho+1}$ are not contained in $G$. Hence by Theorem 1.2 and Theorem 8.1 it follows that $G$ is 2-controlled.

9 Vertex-minor $\chi$-boundedness

In this section we prove that if a vertex-minor-closed class of graphs $G$ is 2-controlled then $G$ is $\chi$-bounded. As all proper vertex-minor-closed classes of graphs are 2-controlled by Theorem 8.2, this implies our main result, Theorem 1.1 that proper vertex-minor-closed classes of graphs are $\chi$-bounded.
Let $G$ be a graph, and $X, C \subseteq V(G)$ such that $X$ has a total ordering $\preceq$. For each $x \in X$, let $N_x \subseteq N(x)$. We say that $(N_x : x \in X)$ is a multicover of $C$ if:

- the sets $X, C, (N_x : x \in X)$ are disjoint,
- the set $X$ is stable,
- the set $X$ is anti-complete to $C$,
- for each $x \in X$, $N_x$ dominates $C$, and
- for distinct $x, y \in X$, with $x \prec y$, the vertex $y$ is anti-complete to $N_x$.

If additionally for all distinct $x, y \in X$, the vertex $y$ is anti-complete to $N_x$ then $(N_x : x \in X)$ is a pure multicover of $C$.

We say that $(N_x : x \in X)$ is an impure multicover if it is a multicover and for each distinct $x, y \in X$, with $x \prec y$, the vertex $x$ is complete to $N_y$. A multicover is stable if for each $x \in X$, the set $N_x$ is stable. The length of a multicover is equal to $|X|$.

We begin now by showing that every graph of sufficiently large chromatic number and small clique number in a 2-controlled class of graphs contains a large stable multicover of a set with large chromatic number.

**Lemma 9.1.** Let $c, \ell, \tau, \omega$ be non-negative integers and let $\mathcal{G}$ be a 2-controlled class of graphs that is closed under taking induced subgraphs such that $\chi(G) \leq \tau$ for all $G \in \mathcal{G}$ with $\omega(G) < \omega$. Then there exists a positive integer $c'$ such that every graph $G \in \mathcal{G}$ with $\chi(G) \geq c'$ and $\omega(G) \leq \omega$ contains a length-$\ell$ stable multicover of a set $C \subseteq V(G)$ with $\chi(C) \geq c$.

**Proof.** We fix $c, \tau, \omega$. The result is trivial for $\ell = 0$, so we proceed inductively assuming the result holds for $\ell - 1$. Let $c_0'$ be an integer such that every graph $G \in \mathcal{G}$ with $\chi(G) \geq c_0'$ and $\omega(G) \leq \omega$ contains a length-$(\ell - 1)$ stable multicover of a set $C'$ with chromatic number at least $c$. Now let $c'$ be such that every graph $G \in \mathcal{G}$ with $\chi(G) \geq c'$ contains a 2-ball with chromatic number at least $\tau + \tau c_0'$. It remains to show that $c'$ satisfies the conclusion of the lemma.

Let $G$ be a graph in $\mathcal{G}$ with $\chi(G) \geq c'$ and $\omega(G) \leq \omega$. Let $y$ be a vertex of $G$ such that $N_2[y]$ has chromatic number at least $\tau + \tau c_0'$. Then $\chi(N(y)) \leq \tau$, so $\chi(N_2(y)) \geq \tau c_0'$. Let $N_y$ be a stable subset of $N(y)$ such that $\chi(N(N_y) \cap N_2(y)) \geq c_0'$. So by the induction hypothesis, there exists a set $C$ contained in $N(N_y) \cap N_2(y)$ with $\chi(C) \geq c$ and a stable multicover $(N_{x'} : x' \in X')$ of length $\ell - 1$ in $G[N(N_y) \cap N_2(y)]$ of the set $C$.

Let $X = X' \cup \{y\}$ and let $\preceq$ be the total ordering on $X$ where $y$ is the largest vertex and the restriction to $X'$ is $\preceq'$. Then $(N_x : x \in X)$ provides the desired stable multicover of $C$. \hfill $\Box$

Next we wish to obtain either a pure or an impure multicover.

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¹Our concept of a multicover is slightly different from that of [6] and [24]. Their definition of a multicover corresponds to our definition of a pure multicover.
Lemma 9.2. Let $c$ and $m$ be positive integers, and let $m' = R(m,m)$. If a graph $G$ has a length-$m'$ stable multicover $(N_x : x \in X')$ of a set $C$ with $\chi(C) \geq c2(\binom{m}{2})$, then it has a length-$m$ stable multicover $(N_x : x \in X)$ of a set $C' \subseteq C$ with $\chi(C') \geq c$ that is either pure or impure.

Proof. For each $v \in C$ and $x \in X'$, let $p_v^x \in N_x$ be some vertex adjacent to $v$. For each $v \in C$, let $f(v)$ be the auxiliary graph on vertex set $X'$ such that for each pair $x, y \in X'$ with $x < y$, $x$ is adjacent to $y$ in $f(v)$ if $p_v^x$ is adjacent to $x$ in $G$. By the pigeonhole principle there exists some graph $H$ on vertex set $X'$ such that $f^{-1}(H) \subseteq C$ has chromatic number at least $c$. Let $C' = f^{-1}(H)$.

Now for each pair $x, y \in X$ with $x < y$, the set $P_y = \{p_v^x : v \in V(C')\}$ is either complete or anti-complete to $x$ depending on whether or not $x$ is adjacent to $y$ in $H$. Since $|X'| = m' = R(m,m)$, there exists a set $X \subseteq X'$ with $|X| = m$ that is either stable in $H$ or a clique in $H$. For each $x \in X$, let $N_x = P_x$, then $(N_x : x \in X)$ is a stable multicover of $C'$. Furthermore, if $X'$ is a stable set in $H$, then $(N_x : x \in X)$ is a pure multicover of $C'$ and if $X'$ is a clique in $H$, then $(N_x : x \in X)$ is an impure multicover of $C'$ as required. \qed

We wish to find a stable pure multicover rather than a stable impure multicover. To this end we next show that large stable impure multicovers contain large stable pure multicovers as pivot-minors (and therefore as vertex-minors).

Lemma 9.3. For all positive integers $c, \ell, \tau, \omega, \omega^*$ such that $\omega^* \leq \omega$, there exists a pair of positive integers $c', \ell'$ with the following property. Let $G$ be a pivot-minor-closed class of graphs such that all $H \in G$ with $\omega(H) < \omega$ have $\chi(H) \leq \tau$. Let $G \subseteq G$ be a graph with $\omega(G) \leq \omega$ that contains a length-$\ell$ stable impure multicover $(N_x : x \in X)$ of a set $C$ such that $\chi(C) \geq c'$ and $\omega(\bigcup_{x \in X} N_x) \leq \omega^*$. Then $G$ contains as a pivot-minor a graph $G'$ with $\omega(G') \leq \omega$ that contains a length-$\ell$ stable pure multicover of a set $C'$ with $\chi(C') \geq c$.

Proof. The case $\ell = 1$ is trivial, so we may assume that $\ell \geq 2$. We fix $c, \ell, \tau, \omega$ and will now proceed by induction on $\omega^*$.

First we handle the case that $\omega^* = 1$, or in other words the case that $\bigcup_{x \in X} N_x$ is a stable set. In this case let $c' = c + \ell\tau$ and $\ell' = \ell$. Let $X = \{x_1, \ldots, x_{\ell'}\}$ where $x_1 < \cdots < x_{\ell'}$. For each $x \in X$, let $n_x$ be a vertex of $N_x$ and let $C_x = N(n_x) \cap C$. By deleting vertices we may assume that $V(G) = C \cup X \cup \bigcup_{x \in X} N_x$. Then let $G' = (G \setminus x_{\ell'} n_{x_{\ell'}} \cdots x_1 n_{x_1}) - (X \cup \bigcup_{x \in X} C_x)$. Equivalently, $G'$ is the graph obtained from $G$ by removing the edges between $x$ and $N_y - \{n_y\}$ for each pair of distinct vertices $x, y \in X$, then deleting the vertices $\bigcup_{x \in X} (\{n_x\} \cup C_x)$, and then for each $x \in X$, relabelling the vertex $x$ as $n_x$. So $\omega(G') \leq \omega(G) \leq \omega$. Let $C' = C \setminus \bigcup_{x \in X} C_x$, then $\chi(C') \geq \chi(C) - \ell\tau \geq c$.

Let $X' = \{n_x : x \in X\}$ and for each $x \in X$, let $N_{n_x} = N_x - \{n_x\}$. Then $(N_{n_x} : x' \in X')$ provides the desired stable pure multicover of $C'$.

Now let us assume that $\omega^* > 1$ and the inductive hypothesis holds for $\omega^* - 1$. Let $c'_0, \ell'_0$ be as obtained by the inductive hypothesis for $c, \ell, \tau, \omega, \omega^* - 1$. Let $\ell' = \ell'_0 R(\omega^* + 1, \ell)$ and $c' = 4\binom{\ell}{2} c'_0 + \ell'\tau$.

As before, for each $x \in X$, let $n_x$ be a vertex of $N_x$. Then for each $x \in X$, let $C_x = N(n_x) \cap C$. Let $C'_0 = C - \bigcup_{x \in X} C_x$, then $\chi(C'_0) \geq c' - \ell'\tau = 4\binom{\ell}{2} c'_0$. \qed
Now for each \( v \in C'_0 \) and each \( x \in X \), let \( p^x_v \in N_x \) be a vertex adjacent to \( v \). For each \( v \in C'_0 \), let \( f_1(v) \) be the auxiliary graph on vertex set \( X \) such that for each pair of distinct vertices \( x, y \in X \) with \( x < y \), \( y \) is adjacent to \( x \) in \( f_1(v) \) if \( p^y_v \) is adjacent to \( n_x \). Similarly for each \( v \in C'_0 \), let \( f_2(v) \) be the auxiliary graph on vertex set \( X \) such that for each pair of distinct vertices \( x, y \in X \) with \( x > y \), \( y \) is adjacent to \( x \) in \( f_2(v) \) if \( p^y_v \) is adjacent to \( n_x \).

By pigeonhole principle there exists some pair of graphs \( H_1 \) and \( H_2 \) on the vertex set \( X \) such that \( f_1^{-1}(H_1) \cap f_2^{-1}(H_2) \subseteq C'_0 \) has chromatic number at least \( \chi(C'_0)/4(V(X)) = \chi(C'_0)/4(\ell^2) \geq \varepsilon_0 \). Let \( C'_1 = f_1^{-1}(H_1) \cap f_2^{-1}(H_2) \). Now for each distinct pair \( x, y \in X \), the set \( P_y = \{ p^y_v : v \in C'_1 \} \) is either complete or anti-complete to \( n_x \).

Suppose that \( H_1 \) contains a vertex \( x \in X \) such that \( x \) has at least \( \ell'_0 \) neighbours \( y \) in \( H_1 \) with \( x < y \). Let \( Y \) be the set of neighbours \( y \) of \( x \) in \( H_1 \) with \( x < y \). Then \( |Y| \geq \ell'_0 \) and \( x \) is complete to \( \bigcup_{y \in Y} P_y \) in \( G \). Now \( \omega(\bigcup_{y \in Y} P_y) < \omega^* \), so by the inductive hypothesis, there exists a pivot-minor \( G^* \) of \( G \) containing a length-\( \ell \) stable pure multicover of a set \( C' \) with \( \chi(C') \geq c \) as required. Hence we may assume that \( H_1 \) contains no vertex \( x \in X \) such that \( x \) has at least \( \ell'_0 \) neighbours \( y \) in \( H_1 \) with \( x < y \). So \( H_1 \) is \((\ell'_0 - 1)\)-degenerate, and so \( \ell'_0 \)-colourable. Similarly \( H_2 \) is also \( \ell'_0 \)-colourable.

Let \( H \) be the graph on vertex set \( X \) with edge set \( E(H_1) \cup E(H_2) \). Then \( H \) is \( \ell'_0 \)-colourable. So \( H \) contain a stable set \( Y_0 \) of size at least \( \ell' / \ell'_0 = R(\omega^* + 1, \ell) \).

Since \( \omega(\bigcup_{y \in Y_0} P_y) \leq \omega^* \), there exists a subset \( Y \) of \( Y_0 \) of size \( \ell \) such that the set \( \{ n_y : y \in Y \} \) is stable in \( G \). Let \( Y = \{ y_1, \ldots, y_\ell \} \) with \( y_1 < \cdots < y_\ell \). Then let \( G'_0 \) be the subgraph of \( G \) induced on \( C'' \cup Y \cup \bigcup_{y \in Y} (P_y \cup \{ n_y \}) \). Let \( G^* = G'_0 \setminus \{ y_1 n_{y_1} \setminus \cdots \setminus y_\ell n_{y_\ell} \} \). Similarly to before, \( G^* \) is equivalently the graph obtained from \( G'_0 \) by removing the edges between \( x \) and \( N_y \setminus \{ n_y \} \) for each pair of distinct vertices \( x, y \in Y \), and then for each \( y \in Y \), swapping the labels of \( y \\mbox{and } n_y \). So \( \omega(G^*) \leq \omega(G'_0) \leq \omega(G) \leq \omega \). Let \( Y' = \{ n_y : y \in Y \} \) and for each \( y \in Y \), let \( P'_{y} = P_y \setminus \{ n_y \} \). Finally \( (P'_{y} : y' \in Y') \) provides our desired stable pure multicover of \( C'' \).

We consolidate our position, and by Lemmas 9.1, 9.2 and 9.3, we obtain the following:

**Lemma 9.4.** Let \( c, \ell, \tau, \omega \) be positive integers and let \( G \) be a 2-controlled class of graphs closed under pivot-minors such that \( \chi(H) \leq \tau \) for all \( H \in G \) with \( \omega(H) < \omega \). Then there exists a positive integer \( c' \) such that every graph \( G \in G \) with \( \omega(G) \leq \omega \) and \( \chi(G) \geq c' \) contains as a pivot-minor a graph \( G^* \) with \( \omega(G^*) \leq \omega \) such that \( G^* \) contains a length-\( \ell \) stable pure multicover of a set \( C \) with \( \chi(C) \geq c \).

Let \( G \) be a graph and let \( (N_x : x \in X) \) be a pure multicover of a set \( C \subseteq V(G) \). We say that the pure multicover is stably \( k \)-crested if \( V(G) \setminus X - C \setminus (\bigcup_{x \in X} N_x) \) has distinct elements \( a_1, \ldots, a_k \) and \( a_{i, x} \) for all \( i \in [k] \) and \( x \in X \) such that:

- for each \( i \in [k] \), \( N(a_i) \cap \{ a_{j, x} : j \in [k], x \in X \} = \{ a_{i, x} : x \in X \} \).
for each \( i \in [k] \) and \( x \in X \), \( N(a_{i,x}) \cap (C \cup X \cup \bigcup_{y \in X} N_y) = \{x\} \),

- \( \{a_1, \ldots, a_k\} \) is anti-complete to \( C \cup X \cup \bigcup_{y \in X} N_y \), and
- both \( \{a_1, \ldots, a_k\} \) and \( \{a_{i,x} : i \in [k], x \in X\} \) are stable.

We call the vertices of \( \{a_1, \ldots, a_k\} \), centres of a stably \( k \)-crested pure multico\-ver. Chudnovsky, Scott, Seymour and Spirkl [6] proved that:

**Theorem 9.5** (Chudnovsky, Scott, Seymour and Spirkl [6, 4.4]). For all \( c, \ell, k, \tau, \omega \) positive integers \( c, \ell, k, \tau, \omega \), there exists a pair of positive integers \( \ell', c' \) with the following property. Let \( G \) be a graph with \( \omega(G) \leq \omega \), such that \( \chi(H) \leq \tau \) for every induced subgraph \( H \) of \( G \) with \( \omega(H) < \omega \). Let \( (N_x' : x \in X') \) be a pure multico\-ver of some set \( C' \) such that \( |X'| \geq \ell' \) and \( \chi(C) \geq c' \). Then there exists a stably \( k \)-crested length-\( \ell \) stable pure multico\-ver of a set \( C \) with \( \chi(C) \geq c \).

By combining Lemma 9.6 and Theorem 9.5 we obtain the following (which is also where we will pickup from in Section 10 to prove Theorem 1.3):

**Lemma 9.6.** Let \( c, \ell, k, \tau, \omega \) be positive integers and let \( G \) be a 2-controlled class of graphs closed under pivot-minors such that \( \chi(H) \leq \tau \) for all \( H \in G \) with \( \omega(H) < \omega \). Then there exists a positive integer \( c' \) such that every graph \( G \in \mathcal{G} \) with \( \omega(G) \leq \omega \) and \( \chi(G) \geq c' \) contains a pivot-minor a graph \( G^* \) with \( \omega(G^*) \leq \omega \) such that \( G^* \) contains a stably \( k \)-crested length-\( \ell \) stable pure multico\-ver of a set \( C \) with \( \chi(C) \geq c \).

Notice in particular that if a graph \( G \) contains a stably \( k \)-crested stable pure multico\-ver \( (N_x : x \in X) \), with \( |X| = \ell \), then \( G \) contains an induced \( K_{\ell,k}^1 \).

**Theorem 9.7.** Every vertex-minor-closed class of graphs that is 2-controlled is also \( \chi \)-bounded.

**Proof.** Let \( G \) be a 2-controlled vertex-minor closed class of graphs and suppose for sake of contradiction that \( G \) is not \( \chi \)-bounded. Then there exists a minimum integer \( \omega \geq 2 \) such that graphs \( G \in \mathcal{G} \) with \( \omega(G) \leq \omega \) have unbounded chromatic number. Let \( \tau \geq 0 \) be such that \( \chi(G) \leq \tau \) for all \( G \in \mathcal{G} \) with \( \omega(G) < \omega \). Let \( H \) be some graph not contained in \( \mathcal{G} \) (\( H \) exists as the class of all graphs is not 2-controlled). Let \( \ell = |V(H)| \) and \( k = \binom{|V(H)|}{2} \). Let \( c = 0 \). Then let \( c' \) be as in the conclusion of Lemma 9.6 for \( c, \ell, k, \tau, \omega \). Then there must exist a graph \( G \in \mathcal{G} \) with \( \omega(G) \leq \omega \) and \( \chi(G) \geq c' \). But by Lemma 9.6 \( G \) must contain \( K_{\ell,k}^1 \) as a pivot-minor, and so by Lemma 2.1 \( G \) must contain \( H \) as a vertex-minor, a contradiction. \( \square \)

Now proving the main result, Theorem 1.1 (which states that proper vertex-minor-closed classes of graphs are \( \chi \)-bounded) is straightforward.

**Proof of Theorem 1.1** Let \( G \) be a proper vertex-minor-closed class of graphs. By Theorem 8.2 \( G \) is 2-controlled. So then by Theorem 9.7 \( G \) is \( \chi \)-bounded. \( \square \)
10 Pivot-minors and a step towards $\chi$-boundedness

In this section we make a first step towards proving that proper pivot-minor-closed classes of graphs are $\chi$-bounded. In particular we will prove Theorem 10.3, the pivot-minor analogue of Theorem 9.7. We may continue from where Lemma 9.6 left off in Section 9.

Let $G$ be a graph and let $(N_x : x \in X)$ be a pure multicover of a set $C \subseteq V(G)$. We say that an induced path $P$ is an oddity for the pure multicover if:

- $P$ has length 3 or 5,
- the ends of $P$ are in $X$,
- no vertex of $X$ that is not an end of $P$ has a neighbour or is contained in $V(P)$, and
- $V(P) \subseteq C \cup X \cup \bigcup_{x \in X} N_x$.

Scott and Seymour [24] proved the following:

Theorem 10.1 (Scott and Seymour [24, 2.2]). For all positive integers $n, \tau, \omega$, there exist positive integers $\ell, c$ with the following property. Let $G$ be a graph such that $\omega(G) \leq \omega$, for every induced subgraph $H$ of $G$ with $\omega(H) < \omega$, we have $\chi(H) \leq \tau$, and $G$ contains a stable pure multicover $(N_x : x \in X)$ of length $\ell$ of a set $C$, where $\chi(C) \geq c$. Then the multicover contains $n$ vertex-disjoint oddities $P_1, \ldots, P_n$, where $V(P_1), \ldots, V(P_n)$ are pairwise anti-complete.

A graph $H$ is a proper odd subdivision of a graph $H'$ if $H$ can be obtained from $H'$ by replacing each edge with a path of odd length at least 3. Notice that if a graph $G$ contains a stably $n$-crested stable pure multicover $(N_x : x \in X)$ of a set $C$, such that the multicover contains $\binom{n}{2}$ vertex-disjoint oddities $P_1, \ldots, P_{\binom{n}{2}}$, where $V(P_1), \ldots, V(P_{\binom{n}{2}})$ are pairwise anti-complete, then $G$ contains a proper odd subdivision of $K_n$ (where the vertices of $K_n$ are the centre vertices of the stably $n$-crested stable pure multicover $(N_x : x \in X)$). If $u, v, w, x$ are vertices of a graph $G$ such that $N(v) = \{u, w\}$ and $N(w) = \{v, x\}$, then $(G \wedge vw) - v - w$ is isomorphic to the graph $G/\{vw, wx\}$. Hence a proper odd subdivision of $K_n$ contains any $n$-vertex graph as a pivot-minor. So by Lemma 9.6 and Theorem 10.1, we may obtain the following:

Lemma 10.2. Let $\tau, \omega$ be positive integers, let $J$ be a graph, and let $\mathcal{G}$ be a 2-controlled class of graphs closed under pivot-minors such that $\chi(H) \leq \tau$ for all $H \in \mathcal{G}$ with $\omega(H) < \omega$. Then there exists a positive integer $c$ such that every graph $G \in \mathcal{G}$ with $\omega(G) \leq \omega$ and $\chi(G) \geq c$ contains $J$ as a pivot-minor.

Then a simple induction on $\omega$, making use of Lemma 10.2, proves Theorem 10.3 that 2-controlled pivot-minor-closed classes are $\chi$-bounded:

Proof of Theorem 10.3. Let $\mathcal{G}$ be 2-controlled class of graphs that is closed under pivot-minors. Suppose for the sake of contradiction that $\mathcal{G}$ is not $\chi$-bounded.
Then there exists a minimum integer $\omega \geq 2$ such that graphs $G \in \mathcal{G}$ with $\omega(G) = \omega$ have unbounded chromatic number. Let $\tau \geq 0$ be such that $\chi(G) \leq \tau$ for all $G \in \mathcal{G}$ with $\omega(G) < \omega$. Let $J$ be some graph not contained in $\mathcal{G}$ ($J$ exists because the class of all graphs is not 2-controlled). Then let $c$ be as in the conclusion of Lemma [10.2] for $\tau, \omega$ and $J$. Then there must exist a graph $G \in \mathcal{G}$ with $\omega(G) \leq \omega$ and $\chi(G) \geq c$. But then $G$ must contain $J$ as a pivot-minor, contradicting the fact that $J \not\in \mathcal{G}$.

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