Modified gravity in Arnowitt-Deser-Misner formalism

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Motivated by Hořava-Lifshitz gravity theory, we propose and investigate two kinds of modified gravity theories, the \( f(R) \) kind and the K-essence kind, in the Arnowitt-Deser-Misner (ADM) formalism. The \( f(R) \) kind includes one ultraviolet (UV) term and one infrared (IR) term together with the Einstein-Hilbert action. We find that these two terms naturally present the ultraviolet and infrared modifications to the Friedmann equation. The UV and IR modifications can avoid the past Big-Bang singularity and the future Big-Rip singularity, respectively. Furthermore, the IR modification can naturally account for the current acceleration of the Universe. The Lagrangian of K-essence kind modified gravity is made up of the three dimensional Ricc scalar and an arbitrary function of the extrinsic curvature term. We find the cosmic acceleration can also be naturally interpreted without invoking any kind of dark energy. The static, spherically symmetry and vacuum solutions of both theories are Schwarzschild or Schwarzschild-de Sitter solution. Thus these modified gravity theories are viable for solar system tests.

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I. INTRODUCTION

Recently, Hořava [1–4] proposed a four dimensional gravity theory in which the space and time are treated on an unequal footing. The theory is very much interesting because of its power counting renormalizability. There-

Here \( n^i \) is the unit normal vector on the hypersurface and \( R^{(3)} \) is the three dimensional Ricci scalar. Rewrite the Hilbert-Einstein action in the Einstein frame:

\[
S = \int d^4x \sqrt{-g} \frac{R}{16\pi},
\]

in the ADM formalism:

\[
S = \int dt d^3x \sqrt{g^{(3)}} \frac{1}{16\pi} \left[ R^{(3)} + K_{ij}K^{ij} - K^2 + 2\nabla_i (n^i \nabla_j n^j) - 2\nabla_i (n^i \nabla_j n^j) \right],
\]

where \( g^{(3)} \) is the trace of three dimensional metric. One find the last two terms in the integrand would contribute a boundary term which does not enter the equation of motion [24]. Therefore, the action can be written as

\[
S = \int dt d^3x N \sqrt{g^{(3)}} \frac{1}{16\pi} \left( R^{(3)} + K_{ij}K^{ij} - K^2 - 2\nabla_i (n^i \nabla_j n^j) \right).
\]

We stress that for nonlinear terms of Ricci scalar, \( R^n \), the last two terms would enter the equation of motion such that the \( f(R) \) theory in the ADM formalism:

\[
S = \int dt d^3x N \sqrt{g^{(3)}} \frac{1}{16\pi} f(R) + S_m,
\]

is equivalent to that in the Jordan frame:

\[
S = \int d^4x \sqrt{-g} \frac{1}{16\pi} f(R) + S_m.
\]

However, if we neglect the boundary term for nonlinear Ricci scalar terms and take the modified gravity as follows

\[
S = \int dt d^3x N \sqrt{g^{(3)}} \frac{1}{16\pi} f \left( R^{(3)} + K_{ij}K^{ij} - K^2 \right) + S_m.
\]
Then the theory would be different from the $f(R)$ version in the Jordan frame. To make our theory different from the usual $f(R)$ version, we shall neglect the boundary terms in this paper.

Up to the lowest possible orders for IR and UV corrections, the modified gravity in the Einstein frame takes the form of

$$ S = \int d^4x \sqrt{-g} \frac{1}{16\pi} \left( R + \alpha R^2 + \frac{\beta}{\ell^2} \right) + S_m. \quad (10) $$

The theory has been investigated very extensively [23]. Correspondingly, we will explore the first modified gravity in the Einstein frame takes the form of

$$ S = \int dt d^3x N \sqrt{g^{(3)}} \frac{1}{16\pi} \left[ \left( R^{(3)} + K_{ij} K^{ij} - K^2 \right) + \alpha \left( R^{(3)} + K_{ij} K^{ij} - K^2 \right)^2 + \beta \left( R^{(3)} + K_{ij} K^{ij} - K^2 \right)^{-1} \right] + S_m, \quad (11) $$

where $\alpha, \beta$ are two positive constants. We find that, in the ADM formalism, the corresponding Friedmann equation is remarkably simple and very different from that in the Jordan frame [23]. Therefore, the theory will present a different cosmic evolution history.

On the other hand, $K_{ij} K^{ij} - K^2$ may be understood as a kinetic term of extrinsic curvature. Similar to the K-essence theory [27], we may construct another K-essence kind of modified gravity. To this end, we define

$$ X = K_{ij} K^{ij} - K^2, \quad (12) $$

then the second modified gravity we will explore can be written as:

$$ S = \int dt d^3x N \sqrt{g^{(3)}} \frac{1}{16\pi} \left[ R^{(3)} + X + F(X) \right] + S_m, \quad (13) $$

where $F(X)$ is an arbitrary function of $X$. When $F(X) = \text{const}$, the theory reduces to General Relativity. Similar to the $f(R)$ modified gravity, we expect the non-linear terms of $X$ may arise in the quantum corrections to GR. With this modifications, we find the cosmic acceleration can also be interpreted without invoking any kind of dark energy. It is interesting that this “K-essence” can cross the phantom divide.

The paper is organized as follows. In Section II and Section IV, we investigate the cosmological behavior of the $f(R)$ kind and the K-essence kind of modified gravity, respectively. In Section III and Section V, we look for the static, spherically symmetry and vacuum solutions. In Section VI we make the conclusion and discussion. Throughout the paper, we use the units in which $c = G = h = 1$.

## II. COSMOLOGY-F(R) KIND

Consider the spatially flat Friedmann-Robertson-Walker Universe

$$ ds^2 = -N(t)^2 dt^2 + a(t)^2 \left( dr^2 + r^2 d\Omega^2 \right), \quad (14) $$

So

$$ K_{ij} = \frac{H}{N} g_{ij}, \quad R^{(3)}_{ij} = 0, \quad (15) $$

The action is given by

$$ S = \int dt d^3x \frac{Na^3}{16\pi} \left[ \frac{6H^2}{N^2} + \frac{36\alpha H^4}{N^4} - \frac{\beta N^2}{6H^2} \right] + S_m. \quad (16) $$

Variation of the action with respect to $N$ and then put $N = 1$, we obtain the Friedmann equation

$$ 3H^2 = 8\pi \sum_i \rho_i + 54\alpha H^4 + \frac{\beta}{4H^2}, \quad (17) $$

where $\rho_i$ is the energy density for $i$th component of matter which mainly include dark matter and radiation. We note that here the Friedmann equation is remarkably simple and very different from that in the Einstein frame [23]. Therefore, it will present us a different cosmic evolution history. It is interesting that in many brane world models, the modifications to Friedmann equation effectively corresponds to $H^2$ and $H^{-2}$ [20,28].

On the other hand, variation of the metric with respect to $a(t)$, we obtain the acceleration equation

$$ 2\dot{H} + 3H^2 = -8\pi \sum_i \rho_i + 72\alpha H^2 \dot{H} + 54\alpha H^4 + \frac{\beta}{4}H^{-2} + \frac{\beta}{6}H^{-4}\dot{H}, \quad (18) $$

where $\rho_i$ is the pressure for the $i$th matter. We are able to derive the energy conservation equation from the Friedmann equation and the acceleration equation

$$ \sum_i \left[ \frac{d\rho_i}{dt} + 3H (\rho_i + p_i) \right] = 0. \quad (19) $$

If we assume there is no interaction between dark matter and radiation, we will have

$$ \frac{d\rho_i}{dt} + 3H (\rho_i + p_i) = 0. \quad (20) $$

So for convenience, we can only consider the Friedmann equation and the energy conservation equation. Put

$$ \alpha = \frac{1}{192\pi \rho_U}, \quad \beta = \frac{64\pi^2}{3}\rho_U^2, \quad (21) $$
where $\rho_U$, $\rho_I$ are constant energy densities. We assume $\rho_U$ is on the order of Planck energy density, $\rho_U = \rho_p$. In order to explain the current acceleration of the Universe, we find shortly later $\rho_I$ should on the order of present-day cosmic energy density. Therefore they represent the UV and IR modification of Friedmann equation. With this assumptions, we find the energy density of $\alpha$ term is negligible for the present-day Universe:

$$\frac{9}{32\pi\rho_U \rho_0} H^4 |_{H = H_0} \approx 10^{-123} .$$

(22)

This energy density becomes significantly only when the Hubble radius is on the order of Planck length. Therefore they represent the UV and IR modification of Friedmann equation. With this assumptions, we find the energy density of $\alpha$ term is negligible for the present-day Universe:

$$\frac{16\pi^2 \rho_I^2}{3H^2 \rho_0} |_{H = H_0} \approx O (1) .$$

(23)

This term plays a great role in the present-day Universe. It is negligible at very high redshifts (large $H$) while becomes significant in the future (small $H$). Therefore, it is an IR modification.

### A. UV modification

In this subsection, we investigate the UV modification. We find that the Big-Bang singularity can be safely avoided. In the presence of only UV modification, the Friedmann equation is given by

$$3H^2 = 8\pi \rho + \frac{9}{32\pi \rho_U} H^4 .$$

(24)

It is a quadratic equation of $H^2$. Mathematically, we would have two roots for $H^2$. But physically, only one root could reduce to the standard Friedmann equation in the limit of smaller $\rho$. We find the root takes the form of

$$H^2 = \frac{16\pi^2 \rho_U}{3} \left( 1 - \sqrt{1 - \frac{\rho}{\rho_U}} \right) .$$

(25)

Here $\rho$ is total energy of dark matter and radiation. Then we obtain the Friedmann equation in GR to zero order of $\rho/\rho_U$,

$$3H^2 = 8\pi \rho ,$$

(26)

and the Friedmann equation in Randall-Sundrum model to the first order of $\rho/\rho_U$ [30],

$$3H^2 = 8\pi \left( \rho + \frac{\rho^2}{2\rho_U} \right) .$$

(27)

It is easy to find that, at very high energy densities, the Big bang singularity is avoided according to Eq. (25). The maximum of cosmic energy density is of the order of Planck energy density and the Universe has the minimum Hubble radius on the order of Planck length. Thus the Universe is created from a de Sitter phase. We note that if $\rho_U$ is negative, the above equation recovers to the loop quantum gravity (or extra time dimension) case [30,32].

### B. IR modification

In this subsection, we investigate the IR modification. We find that the IR modification can account for the acceleration of the Universe. Although the dark energy density contributed by this modification behaves as phantom [34], the Big-Rip singularity can be avoided. For the IR modification, the Friedmann equation is given by

$$3H^2 = 8\pi \rho + \frac{16\pi^2 \rho_I^2}{3H^2} .$$

(28)

It is a quadratic equation of $H^2$. The physical solution is given by

$$3H^2 = 4\pi \rho \left( 1 + \sqrt{1 + \frac{\rho^2}{4\rho^2}} \right) .$$

(29)

Then we obtain the Friedmann equation in GR to zero order of $\rho_I/\rho$,

$$3H^2 = 8\pi \rho ,$$

(30)

and one Friedmann equation in “Cardassian models” [35] to the first order of $\rho_I/\rho$

$$3H^2 = 8\pi \left( \rho + \frac{\rho^2}{4\rho} \right) .$$

(31)

In “Cardassian models” [33], the Friedmann equation is modified as

$$3H^2 = 8\pi \left( \rho + B\rho^3 \right) ,$$

(32)

with $B$ a constant. Supernova and CMB suggest $\eta \leq 0.4$ [35]. It is easy to find that the Big Rip or Big Collapse singularity is avoided according to Eq. (29). With the diluting of cosmic matter, the Universe ends in a de Sitter phase. The minimum of cosmic energy density is $\rho_I/2$ and the Universe has the maximum but finite Hubble radius.

In the next, let’s show the IR modification can account for the acceleration of the Universe. For the present-day Universe, we have

$$3H_0^2 = 8\pi \rho_0 ,$$

(33)

where $H_0$ and $\rho_0$ are the present-day Hubble parameter and the present-day total energy density. Divided Eq. (29) by Eq. (33) and put

$$h = \frac{H}{H_0} , \quad \Omega_{m0} = \frac{\rho_{m0}}{\rho_0} , \quad \varepsilon = \frac{\rho_I}{\rho_{m0}} ,$$

(34)

where $\Omega_{m0}$ is the relative density of the dark matter (For the matter dominated Universe, we can safely neglect...
radiation matter). The Friedmann equation is reduced to
\[ h^2 = \frac{1}{2} \Omega_{m0} a^{-3} \left( 1 + \sqrt{1 + \varepsilon^2 a^6} \right), \]  
(35)

Apply above equation on the present-day Universe \((a = 1, h = 1)\), we have
\[ \varepsilon = 2\Omega_{m0}^{-1} \sqrt{1 - \Omega_{m0}}. \]  
(36)

The present-day matter density parameter \(\Omega_{m0}\) has been obtained by Komatsu et al. \[36\] from a combination of baryon acoustic oscillation, type Ia supernovae and WMAP5 data at a 95% confidence limit, \(\Omega_{m0} = 0.25\). So in the following discussions, we will put \(\Omega_{m0} = 0.25\).

Thus same as ΛCDM model, the IR model is also one parameter model. Then ratio of dark energy density is given by
\[ \Omega_X = \frac{1}{2} \Omega_{m0} a^{-3} h^{-2} \left( 1 + \sqrt{1 + \varepsilon^2 a^6} \right). \]  
(37)

In Fig. 1 and Fig. 2, we plot the evolution of density ratios for dark energy, dark matter and the equation of state of dark energy. We see this dark energy model behaves as phantom matter. The dark energy density is negligible at the redshifts greater than 2. Therefore the theories of structure formation and nucleosynthesis would not be modified. Actually, we can understand this point from Eq. (28). At higher redshift (large H), dark energy is negligible. At late times (small H), dark energy becomes significant and dominant. In Fig. 3, we plot the Hubble parameter and redshift relations for ΛCDM model and the IR model with the same parameters \(\Omega_{m0}\). Both models are very well consistent with observation data. In order to show the IR model can account for the acceleration of the Universe, we plot the deceleration parameter \(q\)
\[ q = \frac{1}{2} + \frac{3p_X}{2\rho_X + 2\rho_m}, \]  
(38)
for ΛCDM model and IR model. We find the two models predict the same transition redshift of the Universe from deceleration to acceleration at \(z_T \simeq 0.8\).

We note that by assuming the dark energy is proportional to \(H^3\), Dvali and Turner \[28\] have constrained \(\eta \leq 1\) with observations. Therefore, our IR modifications is observationally viable.

III. STATIC SPHERICALLY VACUUM SOLUTION-F(R) KIND

In this section, we shall present the static, spherically symmetric and vacuum solutions to verify whether it meets the solar system tests. The metric takes the form of
\[ ds^2 = -N(r)^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2. \]  
(39)

We find the two models predict the same transition redshift of the Universe from deceleration to acceleration at \(z_T \simeq 0.8\).

FIG. 1: The ratio of densities for dark matter (circled line) and dark energy (solid line). The cosmic coincidence problem is relaxed. Here we put \(\Omega_{m0} = 0.25\).

FIG. 2: The evolution of the equation of state for dark energy. This is a phantom dark energy. Here we put \(\Omega_{m0} = 0.25\).

FIG. 3: The Hubble-redshift relations for ΛCDM model (pointed line) and the IR model (solid line). Both models are consistent with the observational data. Here we put \(\Omega_{m0} = 0.25\).
FIG. 4: The evolution of decelerating parameters for ΛCDM model (pointed line) and the IR model (solid line). Both models predict the transition redshift of the Universe from deceleration to acceleration at $z_T \simeq 0.8$. Here we put $\Omega_m = 0.25$. We find

$$K_{ij} = 0, \quad R^{(3)} = \frac{2}{r^2} \left( rf' - 1 + f \right),$$

where prime denotes the derivative with respect to $r$. So the action for the gravitational sector can be written as

$$S = \int dt d^3 x \frac{N}{16\pi\sqrt{f}} \left[ R^{(3)} + \alpha \left( R^{(3)} \right)^2 + \beta \left( R^{(3)} \right)^{-1} \right].$$

In the first place, let’s look for the solution for UV modification. In this case, we should put $\beta = 0$. Variation of action with respect to $N$ yields

$$R^{(3)} + \alpha \left( R^{(3)} \right)^2 = 0.$$  

Solving the equation, we obtain two solutions

$$f = 1 - \frac{2M}{r}, \quad \beta = 0,$$

where $M$ also stands for an integration constant. On the other hand, variation of action with respect to $f$ yields

$$(6r^2 - 12Mr) N' + 6NM = 0,$$

from which we obtain

$$N^2 = 1 - \frac{2M}{r}.$$  

Therefore, the static, spherically symmetric and vacuum solution to IR modification is exactly the Schwarzschild solution. Since the solar system tests mainly base on the Schwarzschild solution, we conclude the theory is viable for solar system tests.

IV. COSMOLOGY-K-ESSENCE KIND

In this section, let’s investigate the cosmic behavior of the modified gravity for K-essence kind. The corresponding action is then given by

$$S = \int dt d^3 x \frac{N a^3}{16\pi} \left[ X + F(X) \right] + S_m.$$  

Variation of the action with respect to $N$ and then put $N = 1$, we obtain the Friedmann equation

$$3H^2 = 8\pi \sum_i \rho_i - \frac{F}{2} - 6H^2 F'.$$

On the other hand, variation of the metric with respect to $a(t)$ and then put $N = 1$, we obtain the acceleration equation

$$2\dot{H} + 3H^2 = -8\pi \sum_i \rho_i - \frac{F}{2} - \left( 2\dot{H} + 6H^2 \right) F' + 24H^2 \dot{H} F''.$$  

Naively, the static, spherically symmetric and vacuum solution to UV modification is the Schwarzschild or Schwarzschild-de Sitter solution. However, it is easy to find that in the limit of $\alpha \rightarrow 0$ and $\beta \rightarrow 0$, the action of Eq. (6) would smoothly match GR. But this Schwarzschild-de Sitter solution would be divergent when $\alpha \rightarrow 0$. Therefore, the physical solution is uniquely left with the Schwarzschild solution.

Secondly, let’s look for the solution for IR modification. In this case, we should put $\alpha = 0$. Variation of action with respect to $N$ yields

$$R^{(3)} + \beta \left( R^{(3)} \right)^{-1} = 0.$$  

Solving the equation, we obtain

$$f = 1 - \frac{2M}{r}, \quad \beta = 0,$$

where $M$ also stands for an integration constant. On the other hand, variation of action with respect to $f$ yields

$$\left( 6r^2 - 12Mr \right) N' + 6NM = 0,$$

from which we obtain

$$N^2 = 1 - \frac{2M}{r}.$$  

Therefore, the static, spherically symmetric and vacuum solution to IR modification is exactly the Schwarzschild solution. Since the solar system tests mainly base on the Schwarzschild solution, we conclude the theory is viable for solar system tests.
Here $\rho_0, p_0$ are as defined before. The prime denotes the derivative with respect to $X$. We assume there is no interaction between dark matter and radiation. So the energy conservation equation
\[ \frac{dp_i}{dt} + 3H(\rho_i + p_i) = 0 , \]
still holds. For convenience we shall investigate the exponential function for $F$:
\[ F = F_0 e^{\xi X} , \]
where $F_0, \xi$ are two constants. Then the Friedmann equation is given by
\[ 3H^2 = 8\pi \sum_i \rho_i - F_0 e^{-6\xi H^2} \left( \frac{1}{2} + 6\xi H^2 \right) . \]
With the usual definitions
\[ \Omega_i = \frac{\rho_i}{\rho_0} , \quad h = \frac{H}{H_0} , \]
the Friedmann equation becomes
\[ h^2 = \frac{\Omega_0}{a^3} + \frac{\Omega_{r0}}{a^4} + f_0 e^{-\xi h^2} \left( \frac{1}{2} + \xi h^2 \right) , \]
where $\rho_0, H_0$ are the present-day total cosmic energy density and the present-day Hubble parameter. $\Omega_0, \Omega_{r0}$ are the relative density of dark matter and radiation in present-day Universe. We have defined:
\[ f_0 = -\frac{F_0}{3H_0^2} \quad \xi = 6\xi H_0^2 . \]
Apply above equation on the present-day Universe ($a = 1, h = 1$), we have
\[ f_0 = \frac{2(1 - \Omega_0 - \Omega_{r0})}{e^{-\xi}(1 + 2\xi)} . \]
The ratio of dark energy density is given by
\[ \Omega_X = f_0 e^{-\xi h^2} \left( \frac{1}{2h^2} + \xi \right) . \]
In Fig. 5 and Fig. 6, we plot the equation of state of dark energy for different parameters, $\xi = 0.36, 0.66, 1.26$ and $\xi = 0.1$, respectively. We find that when $\xi < 0.66$, the dark energy model behaves as quintom matter which can crosse phantom divide smoothly. On the other hand, when $\xi \geq 0.66$, the dark energy behaves as phantom matter which always have the equation of state $w < -1$. When $\xi = 0$, it reduces to the cosmological constant. We see this dark energy is negligible at the high redshifts. Therefore the theories of structure formation and nucleosynthesis would not be modified. In order to mimic AC$DM$ model at most, in the following, we will consider $\xi = 0.01$.

In Fig. 7, we plot the relative densities for radiation, dark matter and dark energy. We see this dark energy is negligible at the high redshifts. It is dominant only at very late time. To show the model can account for the acceleration of the Universe, we plot the deceleration parameter $q$
\[ q \equiv \frac{1}{2} \left( 1 + \frac{3\rho}{\rho_{tot}} \right) , \]
for our model and AC$DM$ model in Fig. 8. $\rho_{tot}, \rho_{tot}$ denote the total cosmic density and total pressure. We find the two models predict nearly the same behavior of the Universe from deceleration to acceleration. This is because the equation of state for dark energy is $w \approx -1$ at the redshifts $0 - 2$ (see Fig. 6)). Therefore, the energy density of this dark energy is nearly a constant at the redshifts $0 - 2$. 

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**FIG. 5:** Evolution of the equation of state for dark energy for $\xi = 0.36, 0.66, 1.26$ up down. When $\xi < 0.66$, the dark energy model behaves as quintom matter which can crosse phantom divide smoothly. When $\xi \geq 0.66$, the dark energy behaves as phantom matter which always have the equation of state $w < -1$. When $\xi = 0$, it reduces to the cosmological constant. Here we put $\Omega_0 = 0.25, \Omega_{r0} = 8.1 \times 10^{-5}$.

**FIG. 6:** The equation of state for a quintom dark energy model. Here we put $\Omega_0 = 0.25, \xi = 0.01, \Omega_{r0} = 8.1 \times 10^{-5}$.
from deceleration to acceleration at $z_T = 0.8$. Here we put $\Omega_{m0} = 0.25$, $\xi = 0.01$, $\Omega_{v0} = 8.1 \cdot 10^{-5}$.

Variation of action with respect to $N$ yields

$$ \frac{2}{r^2} \left( r f' - 1 + f \right) - F_0 = 0. $$

Solving the equation, we obtain

$$ f = 1 - \frac{2M}{r} + \frac{F_0}{6} r^2, $$

where $M$ is an integration constant which has the meaning of the mass of gravitational source. On the other hand, variation of action with respect to $f$ yields

$$ (-F_0 r^4 + 12Mr - 6r^2) N' + (6M + F_0 r^3) N = 0, $$

from which we obtain

$$ N^2 = f = 1 - \frac{2M}{r} + \frac{F_0}{6} r^2 = 1 - \frac{2M}{r} - \frac{f_0}{2} H_0^2 r^2. $$

Equation (61) tells us the dimensionless constant $f_0 \simeq 1.48$ for $\Omega_{m0} = 0.25$, $\xi = 0.01$, $\Omega_{v0} = 8.1 \cdot 10^{-5}$. So the static, spherically symmetric and vacuum solution is the Schwarzschild-de Sitter solution. The solar system tests constrain the Schwarzschild-de Sitter metric that $H_0^2 < 10^{-41} \text{m}^{-2}$ (see, e.g. [37]). Take the present-day Hubble parameter as $H_0 = 71 \text{km sec}^{-1} \text{Mpc}^{-1}$, we then obtain $H_0^2 = 6.6 \cdot 10^{-57} \text{m}^{-2}$. Therefore, the theory is not conflict with solar system tests.

V. STATIC, SPHERICALLY AND VACUUM SOLUTION-K-ESSENCE KIND

In this section, we shall present the static, spherically symmetric and vacuum solution. The general form for a metric describing the static, spherically symmetric space-time is given by

$$ ds^2 = -N(r)^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2. $$

Using the metric, we find the extrinsic curvature and the three dimensional Ricci scalar are

$$ K_{ij} = 0, \quad R = -\frac{2}{r^2} \left( rf' - 1 + f \right), $$

where prime denotes the derivative with respect to $r$. So the action for the gravitational sector can be written as

$$ S = \int dtd^3x \frac{N}{16\pi \sqrt{f}} \left[ -\frac{2}{r^2} \left( rf' - 1 + f \right) + F_0 \right]. $$

VI. CONCLUSION AND DISCUSSION

In conclusion, we have investigated two kinds of modified gravity theories in ADM formalism. The Friedmann equation of $f(R)$ kind is remarkably simple and very different from that in the Jordan frame. The UV modification can avoid the Big-Bang singularity and the IR modification can avoid the Big-Rip singularity, respectively. In this version, the Universe starts from a de Sitter phase and ends in another de Sitter phase. For the K-essence modified gravity, the Universe starts from Big-Bang but ends in de Sitter phase. It is interesting that the corresponding dark energy behaves as quintom matter. We find both theories can account for the current acceleration of the Universe without invoking any dark energy.

We also find the static, spherically symmetry and vacuum solutions to both theories. The solutions are the Schwarzschild or Schwarzschild-de Sitter solution. We verify that the solutions are viable for solar system tests. In view of above simple and interesting results, the modified gravities in the ADM formalism merit further detailed study.
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