Model of resistances in systems of Tomonaga-Luttinger liquid wires

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In a recent paper, we combined the technique of bosonization with the concept of a Rayleigh dissipation function to develop a model for resistances in one-dimensional systems of interacting spinless electrons [arXiv:1011.5058]. We also studied the conductance of a system of three wires by using a current splitting matrix $M$ at the junction. In this paper we extend our earlier work in several ways. The power dissipated in a three-wire system is calculated as a function of $M$ and the voltages applied in the leads. By combining two junctions of three wires, we examine a system consisting of two parallel resistances. We study the conductance of this system as a function of the $M$ matrices and the two resistances; we find that the total resistance is generally quite different from what one expects for a classical system of parallel resistances. We will do a sum over paths to compute the conductance of this system when one of the two resistances is taken to be infinitely large. Finally we study the conductance of a three-wire system of interacting spin-1/2 electrons, and show that the charge and spin conductances can generally be different from each other.

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I. INTRODUCTION

For non-interacting electrons, the conductance of a ballistic quantum wire is well-known to be quantized in units of $2e^2/\hbar$ at low temperatures [1,3]. This remains valid when interactions between the electrons in the wire are taken into account, provided that there are no sources of backscattering (such as impurities and junctions) and that the wire is connected to leads where there are no interactions [4–8]. Thus, if the wire is modeled as a Tomonaga-Luttinger liquid (TLL) and the interaction strength is given by the Luttinger parameter $K_W$, the conductance of a clean wire does not depend on $K_W$. This breaks down if there are impurities in a wire with interacting electrons. In that case, the impurity strengths vary with the length scale according to some renormalization group (RG) equations, and the conductance depends on $K_W$ and other parameters like the length of the wire, the distances between the impurities, and the temperature [2,10]. A considerable amount of work has also been done on junctions of several quantum wires both theoretically [13,24] and experimentally [25,26]. In these systems, the conductance matrix again becomes length scale dependent due to the interactions. A junction of three quantum wires with interacting spin-1/2 electrons has been studied in Ref. [27], and it has been found that some of the fixed points of the RG equations exhibit different charge and spin conductances. Thus impurities in a single wire or a junction of three or more wires effectively give rise to a resistance which leads to power dissipation. There have been some studies of power dissipation on the edges of a quantum Hall system [11] and at a junction of quantum wires [12]. However, there has been relatively little discussion in the literature of the effects of an extended region of dissipation (a resistive patch) within the framework of TLL theory or bosonization which is the most efficient way to study the effects of interactions [15].

To remedy this situation, we recently introduced a formalism which can combine the technique of bosonization with the classical notion of resistance; both a single wire and a system of three wires with a junction were studied using this formalism [14]. Our analysis was restricted to spinless electrons and zero temperature. In contrast to this, Ref. [28] considered the effect of an extended region of inhomogeneity in a quantum wire at low temperatures; it was shown that this leads to weak backscattering which gives rise to a resistance which is linear in the temperature.

It is useful to briefly recapitulate our earlier work [14]. We introduced the resistances phenomenologically using a Rayleigh dissipative function [29]. Our treatment was classical in the sense that the resistance was taken to be purely a source of power dissipation, and the microscopic quantum mechanical origins of the resistance were not specified. This is equivalent to treating scattering by the resistance as a phase incoherent process; a consequence of this is that the resistances of different patches add up in series with no effects of interference. We found expressions for the conductance of a single wire and of a three-wire system with a junction (this was described by a current splitting matrix $M$ which is orthogonal) in terms of the resistances in the wires. The conductance can be calculated even when the Luttinger parameter $K_W$, the velocity $v_W$ of the quasiparticles and the resistivity $\rho$ all vary with the spatial coordinate $x$ in an arbitrary way in the different wires. Remarkably, we found that the conductance of the three-wire system is independent of the Luttinger parameter $K_W$ (which is determined by the strength of the interaction between the electrons near the junction) if the matrix $M$ is determined by the strength of the interaction between the electrons near the junction and is a function of the resistances in the wires. The conductance can be calculated even when the Luttinger parameter $K_W$, the velocity $v_W$ of the quasiparticles and the resistivity $\rho$ all vary with the spatial coordinate $x$ in an arbitrary way in the different wires. Remarkably, we found that the conductance of the three-wire system is independent of the Luttinger parameter $K_W$ (which is determined by the strength of the interaction between the electrons near the junction) if the matrix $M$ is determined by the strength of the interaction between the electrons near the junction and is a function of the resistances in the wires.
in Sec. II and a three-wire system in Sec. III. We will then examine the issue of power dissipation in a three-wire system in Sec. IV. The dissipation occurs due to the resistances in the wires and the contact resistance in the leads. We will study the dependence of the power dissipation on the junction matrix $M$ and the relative magnitudes of the voltages applied in the leads of the system. In Sec. V, we will study a system of two resistances in parallel; the system consists of two junctions of three wires. We will make a detailed study of the parallel resistances is recovered only in one special case. In Sec. VI, we will carry out a sum over paths to calculate the conductance of the same system in the special case where the bosonic Lagrangian for the system is given by

$$L = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2\nu K} (\partial_t \phi)^2 - \frac{v}{2K} (\partial_x \phi)^2 \right],$$

(1)

where $K$ and $v$ respectively denote the Luttinger parameter and velocity of the bosonic quasiparticles; we will allow $K$ and $v$ to vary with $x$ within a finite region which we will take to be $-L/2 < x < L/2$. (For noninteracting electrons, $K = 1$, while for short-range repulsive interactions like a screened Coulomb interaction, we have $K < 1$.) The limit of noninteracting electrons, with $K = 1$ and $v = v_F$ (the Fermi velocity), is used to model the two- or three-dimensional Fermi liquid leads situated in the regions $|x| > L/2$. In the leads, the frequency and wave number of a plane wave are related by $\omega = v_F|k|$. The electron charge density $n$ and current $j$ can be expressed in terms of the bosonic field by the relations: $n = -e\partial_x \phi/\sqrt{\pi}$ and $j = e\partial_t \phi/\sqrt{\pi}$, where $e$ is the electron charge; these satisfy the equation of continuity $\partial_t n + \partial_x j = 0$. To describe resistances phenomenologically, we introduce a Rayleigh dissipation function

$$F = \frac{1}{2} \int_{-\infty}^{\infty} dx \ r \ j^2,$$

(2)

where the resistivity $r$ can vary with $x$. We then obtain the following equation of motion as described in Ref. [14]

$$\frac{1}{vK} \partial_t^2 \phi - \partial_x \left( \frac{v}{K} \partial_x \phi \right) + \frac{e^2}{\pi} r \ \partial_t \phi = 0.$$  

(3)

Note that we have set $\hbar = 1$, so that $e^2/(2\pi) = e^2/h$.

B. Scattering approach

We will now derive an expression for the DC conductance for a general resistance profile $r(x)$, and will show that only the total resistance $R = \int dx r(x)$ of all the resistive patches appears in the final expression for $\sigma_{dc}$. Also, we will show that a $\delta$-function resistance with the same integrated value of $R$ gives the same value of the conductance.

As mentioned earlier, we set $K = 1$ and $v = v_F$ in the leads. However, $K(x)$ and $v(x)$ can have any profile in the wire region given by $|x| < a$. Similarly, we will assume that the resistivity $r(x) = 0$ for $|x| > a$, but can have any profile in the region $|x| < a$ such that $\int_{-a}^{a} dx \ r(x) = R$ [one such profile is illustrated in Fig.1(ii)]. As described in Ref. [14], in the scattering approach, a plane wave with frequency $\omega$ is incident on the resistive patch, the reflection and transmission amplitudes are calculated as functions of $\omega$, and finally the limit $\omega \to 0^+$ is taken to obtain the expression for $\sigma_{dc}$. For a plane wave incident from left with $k = \omega/v_F$, the spatial part of the solution $\phi(x, t) = f_k(x)e^{-i\omega t}$ outside the resistive patch is given by

$$f_k = e^{ikx} + s_k e^{-ikx} \quad \text{for} \quad x \leq -a,$$

$$= t_k e^{ikx} \quad \text{for} \quad a \leq x.$$  

(4)

Up to zero-th order in $\omega$ (and hence in $k$) the solution of Eq. (3) is $\phi = f_k = c$, where $c$ is a constant. Therefore we can see from Eq. (4) that

$$t_k = 1 + s_k = c \quad \text{up to zero-th order in } \omega.$$  

(5)
Since we are eventually interested in the limit $\omega \to 0^+$, we can work out the solution to Eq. (3) up to first order in $\omega$ and then take the limit $\omega \to 0^+$. We first rewrite Eq. (3) up to first order in $\omega$,

$$-\partial_x \left( \frac{v}{K} \partial_x f_k \right) - i\omega \frac{e^2}{\pi} r \ f_k = 0. \quad (6)$$

In the second term in Eq. (6), we can replace $f_k$ by a constant $c$ (= $t_k$ up to zero-th order in $\omega$) since this term has a factor of $\omega$ already. Integrating this equation from $x = -a - \epsilon$ to $a + \epsilon$, we obtain $i k v_F [t_k - (1 - s_k)] = -i \omega t_k (e^2 R/\pi)$ which is the same as

$$(1 + \frac{e^2 R}{\pi}) t_k + s_k = 1. \quad (7)$$

Solving Eqs. (5) and (7), we obtain $t_k = 1/[1 + \frac{e^2 R}{2 \pi}]$. Note that this expression for $t_k$ is correct only in the limit $\omega \to 0^+$ (or $k \to 0^+$). Hence,

$$\sigma_{dc} = \frac{e^2}{2\pi} \left[ \frac{1}{\tau_{k \to 0^+}} \right] = \frac{e^2}{2\pi} \frac{1}{1 + \frac{e^2 R}{2 \pi}}. \quad (8)$$

One can easily redo the calculations from Eq. (4) to Eq. (8) and see that the result remains unchanged if we choose a $\delta$-function resistivity profile described by $r(x) = R \delta(x - x_0)$, where $|x_0| < a$. We will use this fact later in Sec. III. We thus see that $\sigma_{dc}$ does not depend on the precise functional forms of $K(x)$, $v(x)$, and $r(x)$, as long as $K$ and $v$ are equal to the constants 1 and $v_F$ in the leads, and $R$ is the total resistance of the wire. An implication of this is that the effective resistance of two or more resistive patches in series will be given by the sum of the individual resistances.

In this context, it is worthwhile to look at the phase coherent and phase incoherent transport in the literature (see for instance pp. 125-129 of Ref. 2). In general, the effective resistance of two resistances can have an extra term that depends on some phase factors at the two resistive patches in addition to the sum of the individual resistances. However, the effective resistance reduces to the sum of individual resistances in the incoherent limit. In this sense, our formalism assumes that the system is phase incoherent.

**III. THREE WIRES WITH A JUNCTION**

Let us consider a junction of three TLL wires as shown in Fig. 2. We will assume that each wire has three regions:

(i) $0 \leq x_i \leq L_{11} \neq 0$ — the wire region around the junction where $K(x_i) = K_W$ and $v(x_i) = v_W$; elsewhere $K(x_i) = 1$ and $v(x_i) = v_F$.

(ii) $L_{11} \leq x_i \leq L_{12}$ — the dissipative region where $r(x_i) = r_0$; elsewhere $r(x_i) = 0$, and

(iii) $x_i \geq L_{12}$ — the semi-infinite leads.

Here $i$ labels the wires, and on wire $i$, the coordinate $x_i$ runs from 0 to $\infty$, with $x_i = 0$ denoting the junction point. The regions $x_i \geq L_{12}$ model the two- or three-dimensional leads which are assumed to be Fermi liquids with no interactions between the electrons; we therefore set $K = 1$ and $v = v_F$ in those regions.

![FIG. 2. Schematic diagram of a three-wire junction with interacting regions close to the junction (pink solid region), dissipative regions (green zig-zag region further from the junction), and Fermi liquid leads (brown shaded region furthest from the junction).](image)

**A. Green’s function approach**

We now follow Ref. 4 and write

$$I_i = \sum_{j=1}^{3} \int_{0}^{L_{12}} dx_j' \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{2\pi} \sigma_{ij,\omega}(x_i, x_j') E_\omega(x_j'), \quad (9)$$

in the linear response regime, where $E_\omega(x_j')$ is the Fourier component of the electric field $E(x_j', t)$ on wire $j$, and $\sigma_{ij,\omega}(x_i, x_j')$ is the nonlocal ac conductance matrix. Using the Kubo formula, we then obtain

$$\sigma_{ij,\omega}(x_i, x_j') = -\frac{e^2 \omega}{\pi} G_{ij,\omega}(x_i, x_j'), \quad (10)$$

where $\omega = -i\omega$, and

$$G_{ij,\omega}(x_i, x_j') = \int_{0}^{\infty} \frac{d\tau}{2\pi} (T_{ij}\phi_1(x_i, \tau)\phi_j(x_j', 0)) e^{i\omega\tau} \quad (11)$$

is the Fourier transform of the bosonic field in imaginary time, $\tau = it$. The expressions for $\tau$ and $\bar{\omega}$ in terms of $t$
and \( \omega \) arise as follows. The actions in real (Minkowski) time \( t \) and imaginary (Euclidean) time \( \tau \) are given by

\[
S_M = \int dt dx \left[ \frac{1}{2vK} (\partial_t \phi)^2 - \frac{v}{2K} (\partial_x \phi)^2 \right],
\]

\[
S_E = \int d\tau dx \left[ \frac{1}{2vK} (\partial_\tau \phi)^2 + \frac{v}{2K} (\partial_x \phi)^2 \right]
\]

respectively. The requirement that the exponentials appearing in a path integral formulation be equal to each other, i.e., \( e^{iS_M} = e^{-iS_E} \) implies that \( \tau = it \). Secondly, we want an outgoing plane wave in the lead of wire \( i \) to be given by \( e^{i(\omega x_i/v_F - \omega t)} \) in terms of a real frequency \( \omega \) and \( e^{-i\omega x_i/v_F - \bar{\omega} \tau} \) in terms of an imaginary frequency \( \bar{\omega} \) (this expression for an outgoing wave will be used in the boundary condition (iii) given below). The two planes will be identical if \( \bar{\omega} = -i\omega \).

The Green’s function satisfies the equation

\[
\left[ -\partial_x^2 + K(x) \right] G_{ij}(x, x') = \delta_{ij} \delta(x_i - x_j),
\]

with the following three boundary conditions:

(i) \( G_{ij}(x_i, x_j') \) is continuous at \( x_i = x_j' \) (where \( 0 < x_j' < L_{ij2} \)) and \( -\frac{v(x)}{K(x)} \partial_x G_{ij}(x_i, x_j') \bigg|_{x_j' = x_j} = \delta_{ij} \).

(ii) \( G_{ij}(x_i, x_j') \) and \( -\frac{v(x)}{K(x)} \partial_x G_{ij}(x_i, x_j') \bigg|_{x_j' = x_j} \) are continuous at \( x_i = L_{ij1} \) and \( L_{ij2} \).

(iii) if \( G_{ij}(x_i, x_j') = A_{ij} e^{i \omega x_i/v_F} + B_{ij} e^{-i \omega x_i/v_F} \) for \( 0 < x_i < \min(x_j', L_{ij1}) \) \( \delta_{ij} + L_{ij1}(1 - \delta_{ij}) \), then \( B = -M A \), where \( M \) is the current splitting matrix at the junction \([15][24]\).

The boundary condition in (iii) encodes the fact that the incoming and outgoing currents (and hence the bosonic fields) at the junction are related by the matrix \( M \). Various constraints at the junction such as current conservation and unitarity of the evolution of the system in real time (i.e., no power is dissipated exactly at the junction) imply that each row and column of \( M \) must add up to unity and that \( M \) must be orthogonal. It turns out that for a junction of three wires, the possible \( M \) matrices must belong to one of two classes both of which are parameterized by a single parameter \( \theta \) \([15][24]\): (a) \( \det(M_1) = 1 \) and (b) \( \det(M_2) = -1 \); these can be expressed as:

\[
M_1 = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} b & a & c \\ a & c & b \\ c & b & a \end{pmatrix},
\]

where \( a = (1 + 2 \cos \theta)/3, \ b = (1 - \cos \theta + \sqrt{3} \sin \theta)/3, \) and \( c = (1 - \cos \theta - \sqrt{3} \sin \theta)/3 \). We note that \( (M_2)^2 = 1 \) for any value of \( \theta \); this relation will be used below.

Note that by introducing the orthogonal matrix \( M \), we have made the assumption that there is no dissipation exactly at the junction. The analysis is simpler if the junction, which governs how the incoming currents are distributed among the different wires, is separated from the dissipative regions which lie some distance away from the junction.

Solving Eq. (13) with the above boundary conditions and taking the limit \( \bar{\omega} \to 0^+ \), we obtain the following expression for the dc conductance matrix:

\[
G = -\frac{e^2 K_W}{\pi} \left[ 1 + M + K_W(1 - M)(1 + \frac{e^2}{\pi} R) \right]^{-1} \times [1 - M],
\]

where \( R \) is a \( 3 \times 3 \) diagonal matrix with \( R_{ii} = R_i = r_0(L_{ij2} - L_{ij1}) \); we note that \( R_0 \) is simply the total resistance in wire \( i \). The conductance matrix relates the outgoing current \( I_i \) to the potential \( V_i \) applied in lead \( i \) as \( I_i = \sum_j G_{ij} V_j \). One can show in general that each row and column of \( G \) must add up to zero; the columns adding up to zero is a consequence of current conservation \( \sum_i I_i \) must be zero), while the rows must add up to zero because each of the \( I_i \) must vanish if the \( V_j \)'s have the same values in all the wires.

Eq. (15) can be understood as a combination of the conductance of a system with no resistances \( (R_0 = 0) \) and resistances \( R_i \) on the three wires. Let us denote the conductance with no resistances by

\[
G_0 = -\frac{e^2 K_W}{\pi} \left[ 1 + M + K_W(1 - M) \right]^{-1} [1 - M]. \tag{16}
\]

If \( V_i \) denote the potentials at the points \( x_i = L_{ij} \) (i.e., the points which lie after the interacting regions but before the dissipative regions), we have \( I_i = \sum_j G_{ij} V_j \). Further, \( I_i = (V_i - V_i)/R_i \). Combining these equations with \( I_i = \sum_j G_{ij} V_j \), we obtain

\[
G = [1 - G_0 R]^{-1} G_0. \tag{17}
\]

This relation will be used in Sec. VII below.

### B. Scattering approach

Eq. (15) can be derived in general using the equation of motion approach in the limit \( \omega \to 0^+ \) in the same way as described above for the single wire case. Let \( \phi_i(x, t) = f_i(x) e^{-i\omega t} \) denote the bosonic field in real time which satisfies Eq. (10) in wire \( i \). Current conservation at the junction implies that \( \sum_i \partial_t \phi_i(x_i \rightarrow 0 + \epsilon, t) = 0 \) which implies that \( \sum_i f_i(x_i \rightarrow 0 + \epsilon) = 0 \). [Note that this is a different condition than the one used in the single wire case where \( f(x) \) was assumed to be continuous everywhere; for the case of more than two wires, it is no more convenient to assume \( \sum_i f_i(x_i \rightarrow 0 + \epsilon) = 0 \) rather than the continuity of \( f_i(x_i \rightarrow 0 + \epsilon) \) between different values of \( i \).] We now assume that

\[
f_i(x_i \rightarrow 0 + \epsilon) = a_i e^{-ik' x_i} + b_i e^{ik' x_i},
\]

\[
f_i(x_i \rightarrow \infty) = a_i e^{-ik x_i} + b_i e^{ik x_i}, \tag{18}
\]

where \( a_i, \ b_i \) denote the incoming fields and \( b_i, \ b_i \) denote the outgoing fields. Assuming that the Luttinger

parameter and the velocity are given by $K_W$ and $v_W$ as $x_i \to 0 + \epsilon$ and by 1 and $v_F$ as $x_i \to \infty$, we must have $v_W k' = v_F k = \omega$. In the limit $\omega \to 0^+$, the field on wire $i$ is given by $a_i + b_i$ at $x_i \to 0 + \epsilon$ and by $a_i + \beta_i$ as $x_i \to \infty$ at zero-th order in $\omega$, $k$, and $k'$. Since $f_i(x_i)$ equal to a constant is a solution of Eq. (3) for $\omega = 0$, we must have $a_i + b_i = a_i + \beta_i$. Next, the coefficients $a_i$ and $b_i$ must be related by the current splitting matrix at the junction, $b = -M a$. The coefficients $a_i$ and $\beta_i$ must be related by the conductance matrix in the leads, $\beta = -(2\pi/e^2)G + 1\alpha$; this follows from the statement that $\sum_i (a_i + \beta_i) = \sum_i (a_i + b_i) = 0$. Finally, we integrate Eq. (3) from $x_i = 0 + \epsilon$ to $\infty$, ignoring the first term which is of order $\omega^2$ and setting the third term equal to $-i\omega(e^2/\pi)(a_i + \beta_i)R_{ii}$, where $R_{ii} = \int_{L_{ii}} dx_i r(x_i)$. This gives the equation $\beta_i - a_i - (1/K_W)(b_i - a_i) = -(e^2/\pi)(a_i + \beta_i)R_{ii}$, where we have used the fact that $v_W k' = v_F k = \omega$ and taken the limit $\omega \to 0^+$. Using all these equations, we recover Eq. (15). We thus see that the precise profiles of $K(x_i), v(x_i)$, and $r(x_i)$ in the different wires are not important; all that matters is that the values of $K$ and $v$ are given by $K_W$ and $v_W$ as $x_i \to 0 + \epsilon$ and by 1 and $v_F$ as $x_i \to \infty$, and that $R_{ii} = \int dx_i r(x_i)$.

C. Conductance for the $M_1$ and $M_2$ classes

Within the $M_1$ class, the case $\theta = 0$ is trivial because $M_1(0) = \mathbb{1}$ and $G = 0$. We now consider all other values of $\theta$. We find that in general $G$ depends on $K_W$, $\theta$, and the resistances $R_i = r_{ii}(L_{i2} - L_{ii})$. An exception arises for the case $\theta = \pi$ where we find that $G$ is independent of $K_W$ and depends only on the $R_i$. As shown below, this occurs whenever $M^2 = \mathbb{1}$ which is true for $M_1(\pi)$ and also for the $M_2$ class for any $\theta$. The dependence of $G$ on $K_W$ for the $M_1$ class is to be contrasted to the case of a single wire where the conductance is independent of $K_W$ [18].

In the $M_2$ class, we find that although $G$ depends on $\theta$ and the $R_i$, it is completely independent of $K_W$. In Eq. (15), we write $\mathbb{1} + M + K_W(\mathbb{1} - M)[\mathbb{1} + (e^2/\pi)\mathbb{R}] = A + B$, where $A = \mathbb{1} + M + K_W(\mathbb{1} - M)$ and $B = (e^2/\pi)K_W(\mathbb{1} - M)\mathbb{R}$. We can then use the relations that $M^2 = \mathbb{1}$, $A^{-1} = [K_W(\mathbb{1} + M) + (\mathbb{1} - M)]/(4KW)$, and $(A + B)^{-1} = A^{-1} - A^{-1}BA^{-1} - A^{-1}BA^{-1} - \cdots$ to show that $G$ does not depend on $K_W$ for any choice of $\theta$ and $R_i$. The exact expression for $G$ turns out to be

$$ G = -\frac{e^2}{\pi} \frac{3(1 - M_2)}{D}, $$

where $D = 2(g_1 + g_2 + g_3) + \cos\theta(\sqrt{g_1 + g_2 - 2g_3}) - \sqrt{3}\sin\theta(\sqrt{g_1 - g_2})$, (19)

where $g_1 = 1 + (e^2/\pi)R_i$. We thus see that $G$ does not depend on $K_W$, the Luttinger parameter in the wire regions.

IV. POWER DISSIPATION

In our model, there is no power dissipation exactly at the junction since the current splitting matrices $M_1$ and $M_2$ are orthogonal. Power dissipation occurs only at the resistive patches and in the leads due to the contact resistance. The power dissipation at the contact resistance occurs due to the energy relaxation of the electrons in the leads (reservoirs) which are maintained at some particular chemical potentials. Classically, if a current $I_0$ passes through a resistance $R_0$, the power dissipated $P_0$ is given by $P_0 = I_0^2 R_0$. For a three-wire junction, we can define the power dissipated in two equivalent ways as follows

$$ P = -\sum_{i=1}^3 V_i I_i, \quad (20a) $$

and

$$ P = \sum_{i=1}^3 I_i^2 \left( R_i + \frac{\hbar}{2e^2} \right). \quad (20b) $$

We have verified analytically that these two definitions give the same result. (A minus sign appears in Eq. (20a) because we have defined the $I_i$ to be outgoing currents.)

We know that when the voltages applied in all the three leads are equal to each other, there should be no current in any of the three wires, and hence the power dissipated should be zero. To incorporate this fact, we choose a new coordinate system for the $V_i$’s known as the Jacobi coordinates

$$ \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \quad (21) $$

In this coordinate system, the power dissipated $P$ does not depend on the voltage $V_c$. Further, it turns out that $V_b$ and $V_c$ can be parameterized in such a way that $P$ depends only on one of two parameters: this parametrization is different for junctions described by $M_1$ and $M_2$ as we will see below. Since we know from Eq. (20b) that the power dissipated at the resistance $R_i$ is $I_i^2 R_i$, for simplicity we will only consider the case of $R_i = 0$ (for $i = 1, 2, 3$) for further analysis.

A. Power dissipated for the $M_1$ class

In this case, the $M$-matrix is invariant under a cyclic permutation of the wires 1, 2 and 3, and the dissipated power $P$ turns out to be invariant under rotations in the $V_b - V_c$ plane. If we write $V_b = V_0 \cos \phi$ and $V_c = V_0 \sin \phi$, the power dissipated is proportional to $V_0^2$ and does not depend on $\phi$. However, $P$ depends on the parameters $K_W$ and $\theta$ since the conductance matrix $G$ depends on those two parameters. The dependence of $P$ on $K_W$ and $\theta$ is shown as a contour plot in Fig. 3. The choice $\theta = 0$ decouples the three wires at the junction, making the currents in the three wires zero; hence, the power dissipated
is zero for this choice. The power dissipated is maximum on the contour $\theta = \pm \pi$.

FIG. 3. Power dissipated for a $Y$-junction (see Fig. 2) shown as a contour plot in the $\theta - K_W$ plane for the $M_1$ class. The numbers shown indicate the power dissipated along the nearest contour in units of $e^2/h$; we have set $V_0 = 1$ and $R_i = 0$.

B. Power dissipated for the $M_2$ class

Junctions described by $M_2$-class are time reversal invariant (the $M$-matrix is symmetric). In this case, the power dissipated is a constant over straight lines in the $V_b - V_c$ plane. If we define the variables $V_\pm = V_b \cos(\theta/2) \pm V_c \sin(\theta/2)$, then the power dissipated $P$ is found to be independent of $V_+$ and is given by $P = (e^2/h) V_+^2$. Further, $P$ does not depend on $K_W$ in this case.

V. RESISTANCES IN PARALLEL

At the end of Sec. III we saw that the effective resistance of two or more resistances in series is the sum of the individual resistances in agreement with the classical result. By the word ‘classical’ we mean the result obtained for the effective resistance by using Kirchoff’s circuit laws. In this spirit, it is interesting to study whether the effective resistance of two resistances in parallel will agree with the classical result for a similar system.

To begin with, let us consider a general model illustrated in Fig. 3. The three wires at each junction are labeled by the index $i = 1, 2, 3$, and the two junctions are described by current splitting matrices $M_L$ and $M_R$. The coordinate $x_i$ runs from

(i) $-\infty$ to $-l$ on the wire $i = 1$ on the left,
(ii) $-l$ to $l$ on the wires $i = 2, 3$,
(iii) $l$ to $\infty$ on the wire $i = 1$ on the right.

The resistive regions on the wires $i = 2, 3$ lie in the range $|x_i| < a$, the total resistances being $R_i$. In the leads, $K = 1$ and $v = v_p$ as usual, while $K = K_W$ and $v = v_W$ in the regions $a < |x_i| < L/2$ shown in pink (solid lines). In the resistive regions, $K(x)$ and $v(x)$ can have any profiles and these do not affect the final result. The incoming and outgoing files at the two junctions (at $x_i = \pm l$) are related by the matrices $M_R$ and $M_L$. A plane wave incident from the left has the solution $\phi_{i,k}(x_i,t) = f_{i,k} e^{-i\omega t}$













FIG. 4. Schematic diagram of resistances $R_2$ and $R_3$ in parallel attached to Fermi liquid leads through junctions $L$ and $R$.

The relations (23) and (24) along with the
continuity of $\phi_{1,k}$ and $\frac{1}{h} \partial_x \phi_{1,k}$ at $x = \pm L/2$ give us enough conditions to solve for all the unknowns, namely, $s_k, t_{ij,p,k}, s_{ij,p,k}$, and $t_k$.

Once we know the final expression for the DC conductance $\sigma_{dc}$ ($= \frac{e^2}{2\pi} t_k$ in the limit $k \to 0^+$), we can obtain the effective resistance $R_{||}$ by subtracting out the contact resistance from the total resistance,

$$R_{||} = \frac{1}{\sigma_{dc}} - \frac{\hbar}{e^2}.$$

We have listed the results obtained for different choices of $M_L$ and $M_R$ in Table I.

| $M_L(\theta_L)$ | $M_R(\theta_R)$ | Expression for $R_{||}$ |
|-----------------|-----------------|-------------------------|
| $M_1(\theta)$   | $M_1(-\theta)$  | $R_2 R_3/(R_2 + R_3)$   |
| $\theta \neq 0$ | $M_1(\theta)$   | Depends on $\theta_L, E_R$ and $K_W$ as shown in Figs. 3 and 5 |
| $\theta_L \neq \theta_R$ | $M_2(\theta_R)$ | Depends only on $\theta_R = \theta$ as shown in Fig. 4 |
| $M_1(\theta_L) = 0$ | $M_2(\theta_R)$ | $\infty$ |
| $M_1(\theta_L) = 0$ | $M_2(\theta_R)$ | $\infty$ |
| $M_2(\theta)$ | $M_2(\theta)$ | Depends only on $\theta$ as shown in Fig. 4 |
| $M_2(\theta_L \neq \theta_R)$ | $M_2(\theta_R)$ | $\infty$ |

TABLE I. The behavior of the effective resistance $R_{||}$ for different choices of $M_L$ and $M_R$.

In an earlier paper, we observed that in the case of a three-wire junction, if time reversal invariance is broken at the junction (i.e., if $M$ is not a symmetric matrix), then the conductance matrix depends on $K_W$ [14]. In the model studied here, we find that the final DC conductance depends on $K_W$ only when time reversal invariance is broken at both the junctions (i.e., if both $M_L$ and $M_R$ belong to the $M_1$ class) and $\theta_L \neq \theta_R$. The dependence of the conductance $\sigma_{dc}$ on $K_W (\theta_L / \theta_R)$ is shown in Fig. 5(b). The contour $\theta_L = -\theta_R = \theta \neq 0$ corresponds to the maximum value of $\sigma_{dc}$. On this contour, $\sigma_{dc}$ does not depend on either $K_W$ or $\theta$; moreover we get an expression for the effective resistance $R_{||}$ which agrees with the classical result for $R_{||}$.

The choice $\theta = 0$ at a junction described by $M_1$-matrix decouples all the three wires at the junction. Hence, the conductance of the system is zero (or equivalently $R_{||} = \infty$) if either of the two junctions have $\theta = 0$. We can see this in both Table I and Fig. 6. Another interesting case that results in infinite effective resistance arises when both $M_L$ and $M_R$ lie in the $M_2$ class and $\theta_L \neq \theta_R$. The choices, (i) $M_L$ and $M_R$ lie in the $M_2$ class with $\theta_L = \theta_R = \theta$ and (ii) $M_L$ lies in the $M_1$ class and $M_R$ lies in the $M_2$ class, with $0 < \theta_L < 2\pi$ and $\theta_R = \theta$, yield the same $\theta$-dependent expression for the conductance, as shown in Fig. 6 for one particular choice of the parameters. The case $M_L(\theta_L) = M_2(\theta_2)$ and $M_L(\theta_R) = M_1(\theta_1)$ is redundant since we have already analyzed the case $M_L(\theta_L) = M_1(\theta_1)$ and $M_L(\theta_R) = M_2(\theta_2)$, and these two cases are equivalent. The equivalence of these two cases follows from the parity transformation $x_i \to -x_i$, $i = 1, 2, 3$, since the conductance is parity invariant for this system.

Finally, as a special case, we look at a symmetric situation treating the wires 2 and 3 on the same footing, with both junctions being described by the same $M$-matrix. The case $M = M_1$ requires $\theta = \pi$ for symmetry between the wires 2 and 3. As $\theta_L = -\theta_R$ for $\theta_L = \pi$, we conclude
from Table II that \( R_1 = R_2 R_3 / (R_2 + R_3) \). Turning to the case \( M = M_2 \), the symmetry requirement imposes \( \theta = -\pi / 3 \). The effective resistance in this case takes the form \( R_1 = (R_2 + R_3) / 4 \). An interesting implication of this is that when one of the resistances (say \( R_3 \)) is taken to infinity keeping the other (\( R_2 \)) finite, the effective resistance \( R_1 \) also goes to infinity. This is unlike the classical case where, if one of the resistances \( (R_3) \) in a parallel geometry is taken to infinity, the effective resistance approaches the value of the other resistance \( (R_2) \).

We will provide an understanding of this surprising result in Sec. VII by a physically intuitive argument.

VI. TRANSMISSION OF A PULSE

Inspired by the scattering method [6, 7, 14], we illustrate a ‘sum over paths’ method to calculate the DC conductance of a system. We consider a special case of the model studied in Sec. V by setting \( K = 1 \) everywhere and \( R_3 = \infty \). We now allow a \( \delta \)-pulse to be incident from the left lead and calculate the transmission amplitude for the pulse to exit at the right lead. (In the bosonic language, the conductance is just proportional to the transmission amplitude). We know that the pulse broadens when reflected from a resistive region of finite width and that the width of the reflected pulse is twice the width of the resistive region [14]. For simplicity, let us choose the pulse to be a \( \delta \)-function. In order to maintain the pulse as a \( \delta \)-function even after scattering from the resistive region, we choose the resistivity profile for \( R_2 \) to also be a \( \delta \)-function given by \( \sigma_2(x_2) = R_2 \delta(x_2) \). We have seen earlier in Sec. III that doing this does not alter the final expression for the conductance as long as the total resistance of the patch remains the same.

Taking the limit \( R_3 \to \infty \) is like attaching stubs on either sides of the resistance \( R_2 \) and connecting it to Fermi liquid leads as shown in Fig. 8. To illustrate the ‘sum over paths’ method, let us first consider a simpler model where we calculate the transmission amplitude at a \( Y \)-junction formed by attaching a single stub to a wire as shown in Fig. 9. Let \( M_p \) be the \( M \)-matrix that relates the outgoing bosonic fields to the incoming bosonic fields at the junction. Of the pulse approaching the \( Y \)-junction through the wire \( i \), a fraction \( M_{pj} \) goes into the wire \( j \) after scattering. So the transmission amplitude for the path \( 1 \to 2 \) is \( M_{p21} \). Since the pulse gets completely reflected at the free end of wire 3, the transmission amplitude for the path \( 1 \to 3 \to 2 \) is \( M_{p23} M_{p31} \).

The total transmission amplitude for this system is a sum over transmission amplitudes for all possible paths. In addition to the two paths shown in Fig. 9 (i), there are infinitely many paths in which the pulse starting in wire 1 ends in wire 2. These paths are characterized by multiple reflections of the pulse between the junction-\( P \) and the free end of the wire 3. One such path is shown in Fig. 9 (ii). The path \( 1 \to (3 \to 3)^n \to 2 \) from wire 1 to wire 2 with \( n \) reflections between junction-\( P \) and the free end of the wire 3, has a transmission amplitude of \( M_{p23} (M_{p3})^n M_{p31} \) (this is non-zero only when \( \theta_P \neq 0 \)). Hence the total transmission amplitude \( t_P(1 \to 2) \) (when \( \theta_P \neq 0 \)) turns out to be

\[
t_{P}(1 \to 2) = M_{p21} + \frac{M_{p23} M_{p31}}{1 - M_{p33}}. \tag{26}
\]

The DC conductance of such a system connected to leads on the two sides, is given by \( \sigma_{dc} = (e^2 / h) t_{P}(1 \to 2) \). Using the fact that the matrix \( M_{p} \) describing the junction is parameterized by a single variable \( \theta_P \), one can show that (i) \( \sigma_{dc} = (e^2 / h) \) for \( M_{P} \) lying in the \( M_1 \) class, and (ii) \( \sigma_{dc} = 0 \) for \( M_{P} \) lying in the \( M_2 \) class, irrespective of the angle \( \theta_P \) as long as \( \theta_P \neq 0 \). For the \( M_1 \) class, \( \theta_P = 0 \) means that the three wires are disconnected.
and it is easy to see that $\sigma_{dc} = 0$, while for the $M_2$ class, $\theta_P = 0$ means that the stub is disconnected from the wire and the calculation gives the expected result, $\sigma_{dc} = e^2/h$. It is easy to see by this method that the effective resistance of a system having two or more scatterers in series is the sum of the individual resistances. In other words, if $t_1, t_2, \ldots$ are the transmission probabilities of the different scatterers in series, the quantities $(1/t - 1)$’s add up to give $1/t - 1$, where $t$ is the transmission probability for the system (see pp. 63-64 of Ref. [2]). Now, the model in Fig. 5 can be looked upon as a system with three scattering centers, i.e., a resistance $R_2$ in the middle and two $Y$-junctions on the two sides of $R_2$. When either of the two junctions ($P$) is described by an $M_2$-matrix (with $\theta \neq 0$), we get $R_{||} = \infty$ since the resistance of the scatterer at $P$ is infinite. When both the junctions are described by $M_1$-matrix (with the constraints that $\theta_L \neq 0$ and $\theta_R \neq 0$), we get $R_{||} = R_2$.

VII. SPIN-1/2 ELECTRONS

Let us briefly discuss an extension of our results to the realistic case of spin-1/2 electrons. In one dimension, it is known that interactions between electrons lead to the phenomenon of spin-charge separation [13]. A bosonic description of the system begins with fields for spin-up and spin-down electrons denoted by $\phi_{\uparrow}$ and $\phi_{\downarrow}$ respectively. The fields for the charge and spin modes are then given by

$$\phi_c = \frac{1}{\sqrt{2}} (\phi_{\uparrow} + \phi_{\downarrow}),$$
$$\phi_s = \frac{1}{\sqrt{2}} (\phi_{\uparrow} - \phi_{\downarrow}).$$

(27)

The system decouples in terms of these fields even when density-density interactions are introduced between the electrons. The Lagrangian is similar in form to the one in Eq. (11) except that there are two sets of parameters denoted by $(K_c, v_c)$ and $(K_s, v_s)$ for the charge and spin fields respectively. Namely,

$$L = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2v_cK_c} (\partial_x \phi_c)^2 - \frac{v_c}{2K_c} (\partial_x \phi_c)^2 \right. + \left. \frac{1}{2v_sK_s} (\partial_x \phi_s)^2 - \frac{v_s}{2K_s} (\partial_x \phi_s)^2 \right],$$

(28)

where we have ignored a term involving the cosine of the field $\phi_s$, arising from fourth-order fermionic terms like $(\psi_{\downarrow}^T \psi_{\downarrow})^2$ [13]. For an inhomogeneous system, the parameters $K_a$ and $v_a$ are generally functions of $x$. However, for a system with $SU(2)$ rotational invariance, $K_s = 1$.

The charge and spin currents are given by

$$j_c = \frac{e}{\sqrt{\pi}} \sqrt{2} \partial_x \phi_c,$$
$$j_s = \frac{e}{\sqrt{\pi}} \sqrt{2} \partial_x \phi_s$$

(29)

respectively. We note that $j_c$ is invariant under $SU(2)$ rotations, while $j_s$ is only invariant under $U(1)$ rotations about the $z$-axis. The simplest way of introducing resistance in this theory would be to postulate a Rayleigh dissipation function for the charge current given by

$$F = \frac{1}{2} \int_{-\infty}^{\infty} dx \, r \, j_c^2 = \frac{e^2}{2\pi} \int_{-\infty}^{\infty} dx \, 2r \, (\partial_x \phi_c)^2.$$  

(30)

We can now use this dissipative function to calculate the charge conductance in a variety of systems as in the spinless case. In principle, one can also introduce a dissipative function for the spin current; however such a function would have both a term quadratic in the field $\phi_s$ as well as a cosine of $\phi_s$ [28]. This makes it difficult to analyze the corresponding equations of motion.

In analogy with Eqs. (10,11), we can compute the two-point correlation function of the charge current $j_c$ to find the charge conductance $G_c$ for a three-wire junction. The presence of the factors of $\sqrt{2}$ in Eq. (28) and 2 in Eq. (30) as compared to the corresponding expressions for the spinless case implies that $G_c$ will be given by an expression similar to Eq. (15), except for some factors of 2. Namely, we will have

$$G_c = -\frac{2e^2K_{cW}}{\pi} \left[ (1 + M + K_{cW}(1 - M)(1 + 2e^2/\pi)]^{-1} \times [1 - M],$$

(31)

where the current splitting matrix $M$ and the resistance matrix $R$ are defined as before, and $K_{cW}$ denotes the Luttinger parameter for the charge field in the wire regions given by $0 < x_i < L_{11}$.

One can actually define two conductances, $G_c$ and $G_s$, which govern the amounts of charge and spin currents which flow when the corresponding voltage biases or chemical potential differences are applied between different leads, namely, $eDV_c = \Delta \mu_{\uparrow} = \Delta \mu_c$ and $eDV_s = \Delta \mu_{\downarrow} = \Delta \mu_s$ for driving charge and spin currents respectively. In the absence of resistances, it has been shown in Ref. [27] that for a junction of three quantum wires, the RG flows resulting from the interactions between the electrons generally take the system to a fixed point at which the charge and spin conductances, $G_{0c}$ and $G_{0s}$, are different from each other. [We note here that the RG flows occur entirely within the interacting regions $0 \leq x_i < L_{11}$ in Fig. 2 and that no renormalization occurs in the dissipative regions since there is no interaction in those regions].

Let us now introduce resistances in the system. If we then use a relation like Eq. (17) to find $G_a$ from $G_{0a}$ and $R_a$, where $a = c,s$, we expect to find that $G_c$ and $G_s$ would also not be equal to each other since $G_{0c}$ and $G_{0s}$ differ from each other. Note that this is purely an effect of interactions between the electrons; for non-interacting electrons, we would have $G_{0c} = G_{0s}$ and therefore $G_c = G_s$. 

9
VIII. DISCUSSION

We have developed a formalism which allows us to study the effect of resistances in a quantum wire using the technique of bosonization. The analysis can be extended to a system of three wires by using a current splitting matrix $M$ to describe the junction. It is known that there are two classes of such matrices which are called $M_1$ and $M_2$. We have calculated the conductance of a three-wire system using both Green’s function and scattering approaches. We have then examined the power dissipated by the system as a function of the matrix $M$ and the voltages applied in the three leads. For the $M_1$ class, both the conductance and the dissipated power depend on the value of the interaction parameter $K$ near the junction, while in the $M_2$ class, the conductance and power are independent of $K$. Next, by putting together two junctions of three wires, we have studied the effective conductance of a system of two wires in parallel. This is found to depend in a highly non-trivial way on the $M$ matrices at the two junctions, the parameter $K$, and the resistances in the two wires. In some cases, the effective resistance is infinite, while in other cases, it is finite but depends on $K$. In only one special case do we obtain the classical result for two parallel resistances. For the case in which one of the two resistances is infinitely large, we have provided an intuitive way of calculating the effective conductance by summing over all the paths that an electron can take in going from one lead to the other. This method also shows that three-wire junctions with matrices $M_1$ and $M_2$ behave quite differently from each other when the resistance in one of the wires is taken to be infinitely large. Finally, we have generalized our results to the case of interacting spin-$1/2$ electrons. We have argued that the charge and spin conductances will generally be different from each other due to RG flows induced by interactions between the electrons.

The formalism discussed in this paper is well suited for studying systems with interacting electrons in which the resistances are phase incoherent. It would be useful to develop a more general method which can deal with partially coherent resistances. It would also be useful to have a more complete treatment of systems with spin-$1/2$ electrons in which the charge and spin resistivities are position dependent and different from each other, in the spirit of Ref. [28].

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