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New Results in Sasaki-Einstein Geometry

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Abstract

This article is a summary of some of the author’s work on Sasaki-Einstein geometry. A rather general conjecture in string theory known as the AdS/CFT correspondence relates Sasaki-Einstein geometry, in low dimensions, to superconformal field theory; properties of the latter are therefore reflected in the former, and vice versa. Despite this physical motivation, many recent results are of independent geometrical interest, and are described here in purely mathematical terms: explicit constructions of infinite families of both quasi-regular and irregular Sasaki-Einstein metrics; toric Sasakian geometry; an extremal problem that determines the Reeb vector field for, and hence also the volume of, a Sasaki-Einstein manifold; and finally, obstructions to the existence of Sasaki-Einstein metrics. Some of these results also provide new insights into Kähler geometry, and in particular new obstructions to the existence of Kähler-Einstein metrics on Fano orbifolds.

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1 Introduction

Sasaki-Einstein geometry is the odd-dimensional cousin of Kähler-Einstein geometry. In fact the latter, for positive Ricci curvature, is strictly contained in the former; Sasaki-Einstein geometry is thus a generalisation of Kähler-Einstein geometry. The author’s initial interest in this subject stemmed from a rather general conjecture in string theory known as the AdS/CFT correspondence [31]. This is probably the most important conceptual development in theoretical physics in recent years. AdS/CFT conjecturally relates quantum gravity, in certain backgrounds, to ordinary quantum field theory without gravity. Moreover, the relation between the two theories is holographic: the quantum field theory resides on the boundary of the region in which gravity propagates.

In a particular setting the AdS/CFT correspondence relates Sasaki-Einstein geometry, in dimensions five and seven, to superconformal field theory, in dimensions four and three, respectively. Superconformal field theories are very special types of quantum field theories: they possess superconformal symmetry, and hence in particular conformal symmetry. The five-dimensional case of this correspondence is currently understood best. One considers a ten-dimensional Riemannian product $(B \times L, g_B + g_L)$, where $(L, g_L)$ is a Sasaki-Einstein five-manifold and $(B, g_B)$ is five-dimensional hyperbolic space. We may present $B$ as the open unit ball $B = \{x \in \mathbb{R}^5 \mid \|x\| < 1\}$ with metric

$$g_B = \frac{4 \sum_{i=1}^5 dx_i \otimes dx_i}{(1 - \|x\|^2)^2}. \quad (1.1)$$
Here $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^5$. We may naturally compactify $(B, g_B)$, in the sense of Penrose, by adding a conformal boundary at infinity. One thus considers the closed unit ball $\bar{B} = \{ x \in \mathbb{R}^5 \mid \|x\| \leq 1 \}$ equipped with the metric

$$g_{\bar{B}} = f^2 g_B$$

where $f$ is a smooth function on $\bar{B}$ which is positive on $B$ and has a simple zero on $\partial \bar{B} = S^4$. For example, $f = 1 - \|x\|^2$ induces the standard metric on $S^4$, but there is no natural choice for $f$. Thus $S^4$ inherits only a conformal structure. The isometric action of $SO(1,5)$ on $(B, g_B)$ extends to the action of the conformal group $SO(1,5)$ on the four-sphere. The AdS/CFT correspondence conjectures that type IIB string theory, which is supposed to be a theory of quantum gravity, propagating on $(B \times L, g_B + g_L)$ is equivalent to a four-dimensional superconformal field theory that resides on the boundary four-sphere. Indeed, the ten-dimensional manifold is a supersymmetric solution to type IIB supergravity; in differentio-geometric terms, this means that there exists a solution to a certain Killing spinor equation. The AdS/CFT correspondence thus in particular implies a correspondence between Sasaki-Einstein manifolds in dimension five and superconformal field theories in four dimensions: for each Sasaki-Einstein five-manifold $(L, g_L)$ we obtain a different superconformal field theory. The AdS/CFT correspondence then naturally maps geometric properties of $(L, g_L)$ to properties of the dual superconformal field theory.

I should immediately emphasize, however, that this article is aimed at geometers, rather than theoretical physicists. Unfortunately, explaining AdS/CFT to a mathematical audience is beyond the scope of the present paper. Instead I shall focus mainly on the new geometrical results obtained by the author. The paper is based on a talk given at the conference “Riemannian Topology: Geometric Structures on Manifolds,” in Albuquerque, New Mexico.

The outline of the paper is as follows. Section 2 contains a brief review of Sasakian geometry, in the language of Kähler cones. Section 3 summarises the properties of several infinite families of Sasaki-Einstein manifolds that were constructed in [21, 22, 23, 13, 14, 33], focusing on the (most physically interesting) case of dimension five. Section 4 reviews toric Sasakian geometry, as developed in [34]. Section 5 is a brief account of an extremal problem that determines the Reeb vector field for a Sasaki-Einstein metric [34, 35]. This is understood completely for toric Sasaki-Einstein manifolds.

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1The Lorentzian version of hyperbolic space is known as anti-de Sitter spacetime (AdS). The acronym CFT stands for conformal field theory.
whereas the general case currently contains some technical gaps. Finally, section 6 reviews some obstructions to the existence of Sasaki-Einstein metrics [25]. A far more detailed account of Sasakian geometry, together with many other beautiful results in Sasaki-Einstein geometry not described here, may be found in the book [7].

2 Sasakian geometry

While Kähler geometry [27] has been studied intensively for more than seventy years, Sasakian geometry [39] has, in contrast, received relatively little attention. Sasakian geometry was originally defined in terms of metric-contact geometry, but this does not really emphasize its relation to Kähler geometry. The following is a good

**Definition** A compact Riemannian manifold $(L, g_L)$ is Sasakian if and only if its metric cone $(X_0 = \mathbb{R}_+ \times L, g = dr^2 + r^2 g_L)$ is a Kähler cone.

Here $r \in (0, \infty)$ may be regarded as a coordinate on the positive real line $\mathbb{R}_+$. The reason for the subscript on $X_0$ will become apparent later. Note that $(L, g_L)$ is isometrically embedded $\iota : L \to X_0$ into the cone with image $\{r = 1\}$. By definition, $(X_0, g)$ is a non-compact Kähler manifold, with Kähler form

$$\omega = \frac{1}{4} dd^c r^2 . \quad (2.1)$$

Here $d^c = J \circ d = i(\bar{\partial} - \partial)$, as usual, with $J$ the complex structure tensor on $X_0$. The square of the radial function $r^2$ thus serves as a global Kähler potential on the cone. It is not difficult to verify that the homothetic vector field $r \partial/\partial r$ is holomorphic, and that

$$\xi = J \left( r \frac{\partial}{\partial r} \right) \quad (2.2)$$

is holomorphic and also a Killing vector field: $\mathcal{L}_\xi g = 0$. $\xi$ is known as the Reeb vector field. It is tangent to the surfaces of constant $r$, and thus defines a vector field on $L$, which, in a standard abuse of notation, we also denote by $\xi$. Another important object is the one-form

$$\eta = d^c \log r . \quad (2.3)$$

This is homogeneous degree zero under $r \partial/\partial r$ and pulls back via $\iota^*$ to a one-form, which we also denote by $\eta$, on $L$. In fact $\eta$ is precisely a contact one-form on $L$; that is,

\[ \mathcal{L}_{r \partial/\partial r} \eta = 0. \]
\( \eta \wedge (d\eta)^{n-1} \) is a volume form on \( L \). Here \( n = \text{dim}_\mathbb{C} X_0 \), or equivalently \( 2n - 1 = \text{dim}_\mathbb{R} L \).

The pair \( \{\eta, \xi\} \) then satisfy, either on the cone or on \( L \), the relations

\[
\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0.
\] (2.4)

This is the usual definition of the Reeb vector field in contact geometry. Indeed, the open cone \((X_0, \omega)\) is the symplectization of the contact manifold \((L, \eta)\), where one regards \( \omega = \frac{1}{2}d(r^2\eta) \) as a symplectic form on \( X_0 \).

The square norm of \( \xi \), in the cone metric \( g \), is \( \|\xi\|^2_g = r^2 \), and thus in particular \( \xi \) is nowhere zero on \( X_0 \). It follows that the orbits of \( \xi \) define a foliation of \( L \). It turns out that the metric transverse to these orbits \( g_T \) is also a Kähler metric. Thus Sasakian structures are sandwiched between two Kähler structures: the Kähler cone of complex dimension \( n \), and the transverse Kähler structure of complex dimension \( n - 1 \).

Consideration of the orbits of \( \xi \) leads to a global classification of Sasakian structures. Suppose that all the orbits of \( \xi \) close. This means that \( \xi \) generates an isometric \( U(1) \) action on \((L, g_L)\). Such Sasakian structures are called quasi-regular. Since \( \xi \) is nowhere zero, this action must be locally free: the isotropy subgroup at any point must be finite, and therefore isomorphic to a cyclic group \( \mathbb{Z}_m \subset U(1) \). If the isotropy subgroups for all points are trivial then the \( U(1) \) action is free, and the Sasakian structure is called regular. We use the term strictly quasi-regular for a quasi-regular Sasakian structure that is not regular. In either case there is a quotient \( V = L/U(1) \), which is generally an orbifold. The isotropy subgroups descend to the local orbifold structure groups in the quotient space \( V = L/U(1) \); thus \( V \) is a manifold when the Sasakian structure is regular. The transverse Kähler metric descends to a Kähler metric on the quotient, so that \((V, g_V)\) is a Kähler manifold or orbifold. Indeed, \((V, g_V)\) may be regarded as the Kähler reduction of the Kähler cone \((X_0, \omega)\) with respect to the \( U(1) \) action, which is Hamiltonian with Hamiltonian function \( \frac{1}{2}r^2 \).

If the orbits of \( \xi \) do not all close, the Sasakian structure is said to be irregular. The generic orbit is \( \mathbb{R} \), and in this case one cannot take a meaningful quotient. The closure of the orbits of \( \xi \) defines an abelian subgroup of the isometry group of \((L, g_L)\). Since \( L \) is compact, the isometry group of \((L, g_L)\) is compact, and the closure of the orbits of \( \xi \) therefore defines a torus \( \mathbb{T}^s \), \( s > 1 \), which acts isometrically on \((X_0, g)\) or \((L, g_L)\). Thus irregular Sasakian manifolds have at least a \( \mathbb{T}^2 \) isometry group.
The main focus of this article will be Sasaki-Einstein manifolds. A simple calculation shows that

\[ \text{Ric}(g) = \text{Ric}(g_L) - 2(n-1)g_L = \text{Ric}(g_T) - 2ng_T \]  

(2.5)

where \( \text{Ric}(\cdot) \) denotes the Ricci tensor of a given metric. Thus the Kähler cone \((X_0, g)\) is Ricci-flat if and only if \((L, g_L)\) is Einstein with positive scalar curvature \(2(n-1)(2n-1)\), if and only if the transverse metric is Einstein with positive scalar curvature \(4n(n-1)\). \((L, g_L)\) is then said to be a Sasaki-Einstein manifold. Notice that \((X_0, g)\) is then a Calabi-Yau cone, and in the quasi-regular case the circle quotient of \((L, g_L)\) is a Kähler-Einstein manifold or orbifold of positive Ricci curvature. The converse is also true: give a positively curved Kähler-Einstein manifold or orbifold \((V, g_V)\), there exists a Sasaki-Einstein metric on the total space of a \(U(1)\) principal (orbi-)bundle over \(V\). This was proven in general by Boyer and Galicki in \[4\].

3 Explicit constructions of Sasaki-Einstein manifolds

Explicit examples of Sasaki-Einstein manifolds were, until recently, quite rare. In dimension five, the only simply-connected examples that were known in explicit form were the round sphere and a certain homogeneous metric on \(S^2 \times S^3\) \[42\], known as \(T^{1,1}\) in the physics literature. These are both regular, being circle bundles over \(\mathbb{C}P^2\) and \(\mathbb{C}P^2 \times \mathbb{C}P^1\) with their standard Kähler-Einstein metrics, respectively. In fact regular Sasaki-Einstein manifolds are classified \[18\]. This follows since smooth Kähler-Einstein surfaces with positive Ricci curvature have been classified by Tian and Yau \[43, 44\]. The result is that the base may be taken to be a del Pezzo surface obtained by blowing up \(\mathbb{C}P^2\) at \(k\) generic points with \(3 \leq k \leq 8\); although proven to exist, the Kähler-Einstein metrics on these del Pezzo surfaces are not known explicitly. More recently, Boyer, Galicki and collaborators have produced vast numbers of quasi-regular Sasaki-Einstein metrics using existence results of Kollár for Kähler-Einstein metrics on Fano

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3 Notice the slight abuse of notation here: we are regarding all tensors in this equation as tensors on \(X_0\).

4 In the orbifold case this lifting may or may not be an orbifold. If \(\{\Gamma_\alpha\}\) denote the local orbifold structure groups of \(V\), then the data that defines an orbibundle over \(V\) with structure group \(G\) includes elements of \(\text{Hom}(\Gamma_\alpha, G)\) for each \(\alpha\), subject to certain gluing conditions. A moment’s thought shows that the total space of a \(G\) principal orbibundle over \(V\) is smooth if and only if all these maps are injective.

5 Positively curved Einstein manifolds have finite fundamental group \[37\].
orbifolds, together with the $U(1)$ lifting mentioned at the end of the previous section. For a review of their work, see [6, 7].

Until 2004 no explicit examples of non-trivial strictly quasi-regular Sasaki-Einstein manifolds were known, and it was not known whether or not irregular Sasaki-Einstein manifolds even existed. In fact Cheeger and Tian conjectured in [10] that they did not exist. The following theorem disproves this conjecture:

**Theorem 3.1** ([22]) There exist a countably infinite number of Sasaki-Einstein metrics $Y^{p,q}$ on $S^2 \times S^3$, labelled naturally by $p, q \in \mathbb{N}$ where $\gcd(p, q) = 1$, $q < p$. $Y^{p,q}$ is quasi-regular if and only if $4p^2 - 3q^2$ is the square of a natural number, otherwise it is irregular. In particular, there are infinitely many strictly quasi-regular and irregular Sasaki-Einstein metrics on $S^2 \times S^3$.

These metrics were constructed explicitly in [22], based on supergravity constructions by the same authors in [21]. The metrics are cohomogeneity one, meaning that the generic orbit under the action of the isometry group has real codimension one. The Lie algebra of this group is $su(2) \times u(1) \times u(1)$. The volumes of the metrics are given by the formula

$$\frac{\text{vol}[Y^{p,q}]}{\pi^3} = \frac{q^2(2p + \sqrt{4p^2 - 3q^2})}{3p^2(3q^2 - 2p^2 + \sqrt{4p^2 - 3q^2})}. \quad (3.1)$$

The result that $Y^{p,q}$ is diffeomorphic to $S^2 \times S^3$ follows from Smale’s classification of 5-manifolds [41]. Interestingly, the cone $X_0$ corresponding to $Y^{2,1}$ is the open complex cone over the first del Pezzo surface [32]. Note that the first del Pezzo surface was missing from the list of Tian and Yau; it cannot admit a Kähler-Einstein metric since its Futaki invariant is non-zero. The Ricci-flat Kähler cone metric is in fact irregular for $Y^{2,1}$. We shall return to this in section 6. Recently, Conti [12] has classified cohomogeneity one Sasaki-Einstein five-manifolds: they are precisely the set $\{Y^{p,q}\}$. The construction of the above metrics also easily extends to higher dimensions [23, 24, 11]. This leads to the following

**Corollary 3.2** There exist countably infinitely many strictly quasi-regular and irregular Sasaki-Einstein structures in every (odd) dimension greater than 3.

This corollary should be contrasted with several other results. It is known that for fixed dimension there are finitely many (deformation classes of) Fano manifolds [28]; thus there are only finitely many positively curved Kähler-Einstein structures in each
dimension, and hence finitely many regular Sasaki-Einstein structures in each odd dimension. On the other hand, these may occur in continuous families. This is already true for del Pezzo surfaces with \( k \geq 5 \) blow-ups, which have a complex structure moduli space of complex dimension \( 2(k - 4) \). The quasi-regular existence results of Boyer and Galicki also produce examples with, sometimes quite large, moduli spaces \([6]\). It is currently unknown whether or not there exist continuous families of irregular Sasaki-Einstein structures.

Perhaps surprisingly, there also exist explicit cohomogeneity two Sasaki-Einstein five-manifolds. The following subsumes Theorem 3.1:

**Theorem 3.3 (13), (14), (33)** There exist a countably infinite number of Sasaki-Einstein metrics \( L^{a,b,c} \) on \( S^2 \times S^3 \), labelled naturally by \( a, b, c \in \mathbb{N} \) where \( a \leq b, c \leq b, d = a + b - c, \gcd(a, b, c, d) = 1, \gcd\{a, b\}, \{c, d\} = 1 \). Here the latter means that each of the pair \( \{a, b\} \) must be coprime to each of \( \{c, d\} \). Moreover, \( L^{p-q,p+q,p} = Y^{p,q} \).

The metrics are generically cohomogeneity two, generically irregular, and generically have volumes that are the product of quartic irrational numbers with \( \pi^3 \).

The condition under which the metrics are (strictly) quasi-regular is not simple to determine in general. The quartic equation with integer coefficients that is satisfied by \( \text{vol}[L^{a,b,c}]/\pi^3 \) is written down explicitly in [13]. For integers \( (a, b, c) \) not satisfying some of the coprime conditions one obtains Sasaki-Einstein orbifolds.

We conclude this section with a comment on how the metrics in Theorem 3.3 were constructed. In [13, 14] the local form of the metrics was found by writing down the Riemannian forms of known black hole metrics, and then taking a certain “BPS” limit. The initial family of metrics are local Einstein metrics, and in the limit one obtains a local family of Sasaki-Einstein metrics. One then determines when these local metrics extend to complete metrics on compact manifolds, and this is where the integers \( (a, b, c) \) enter. The same metrics were independently discovered in a slightly different manner in [33]. In the latter reference the local Kähler-Einstein metrics in dimension four are constructed first. It turns out that these are precisely the orthotoric Kähler-Einstein metrics in [1]. The construction again easily extends to higher dimensions [13, 14]: the metrics are generically cohomogeneity \( n - 1 \).
4 Toric Sasakian geometry

In this section we summarise some of the results in [34] on toric Sasakian geometry. This probably warrants a

Definition A Sasakian manifold \((L, g_L)\) is said to be toric if there exists an effective, holomorphic and Hamiltonian action of the torus \(\mathbb{T}^n\) on the corresponding Kähler cone \((X_0, g)\). The Reeb vector field \(\xi\) is assumed to lie in the Lie algebra of the torus \(\xi \in \mathfrak{t}_n\).

The Hamiltonian condition means that there exists a \(\mathbb{T}^n\)-invariant moment map

\[ \mu : X_0 \rightarrow \mathfrak{t}_n^* \quad (4.1) \]

The condition on the Reeb vector field implies that the image is a strictly convex rational polyhedral cone [17, 29]. Symplectic toric cones with Reeb vector fields not satisfying this condition form a short list and have been classified in [29]. The main result of this section is Proposition 4.1 which describes, in a rather explicit form, the space of toric Sasakian metrics on the link of a fixed affine toric variety \(X\). Any toric Sasakian manifold is of this form, with the open Kähler cone \(X_0 = X \setminus \{p\}\) being the smooth part of \(X\), with \(p\) an isolated singular point.

We begin by fixing a strictly convex rational polyhedral cone \(C^*\) in \(\mathbb{R}^n\), where the latter is regarded as the dual Lie algebra of a torus \(t_n^* \cong \mathbb{R}^n\) with a particular choice of basis:

\[ C^* = \{ y \in t_n^* \mid \langle y, v_a \rangle \geq 0, \forall a = 1, \ldots, d \} \quad (4.2) \]

The strictly convex condition means that \(C^*\) is a cone over a convex polytope of dimension \(n - 1\). It follows that necessarily \(n \leq d \in \mathbb{N}\). The rational condition on \(C^*\) means that the vectors \(v_a \in t_n \cong \mathbb{R}^n\) are rational. In particular, one can normalise the \(v_a\) so that they are primitive vectors in \(\mathbb{Z}^n \cong \ker\{\exp : t_n \rightarrow \mathbb{T}^n\}\). The \(v_a\) are thus the inward-pointing primitive normal vectors to the bounding hyperplanes of the polyhedral cone \(C^*\). We may alternatively define \(C^*\) in terms of its generating vectors \(\{u_\alpha \in \mathbb{Z}^n\}:\)

\[ C^* = \left\{ \sum_\alpha \lambda_\alpha u_\alpha \mid \lambda_\alpha \geq 0 \right\} \quad (4.3) \]

The primitive vectors \(u_\alpha\) generate the one-dimensional faces, or rays, of the polyhedral cone \(C^*\).
Define the linear map

\[ A : \mathbb{R}^d \to \mathbb{R}^n \]

\[ e_a \mapsto v_a \quad (4.4) \]

where \( \{e_a\} \) denotes the standard orthonormal basis of \( \mathbb{R}^d \). Let \( \Lambda \subset \mathbb{Z}^n \) denote the lattice spanned by \( \{v_a\} \) over \( \mathbb{Z} \). This is of maximal rank, since \( C^* \) is strictly convex. There is an induced map of tori

\[ T^d \cong \mathbb{R}^d/2\pi \mathbb{Z}^d \to \mathbb{R}^n/2\pi \mathbb{Z}^n \cong T^n \quad (4.5) \]

where the kernel is a compact abelian group \( A \), with \( \pi_0(A) \cong \Gamma \cong \mathbb{Z}^n/\Lambda \).

Using this data we may construct the following Kähler quotient:

\[ X = \mathbb{C}^d//A \quad (4.6) \]

Here we equip \( \mathbb{C}^d \) with its standard flat Kähler structure \( \omega_{\text{flat}} \). \( A \subset T^d \) acts holomorphically and Hamiltonianly on \((\mathbb{C}^d, \omega_{\text{flat}})\). We then take the Kähler quotient (4.6) at level zero. The origin of \( \mathbb{C}^d \) projects to a singular point in \( X \), and the induced Kähler metric \( g_{\text{can}} \) on its complement \( X_0 \) is a cone. Moreover, the quotient torus \( T^d/A \cong T^n \) acts holomorphically and Hamiltonianly on \((X_0, \omega_{\text{can}})\), with moment map

\[ \mu : X_0 \to \mathfrak{t}_n^*; \quad \mu(X) = C^* \quad (4.7) \]

The quotient (4.6) may be written explicitly as follows. One computes a primitive basis for the kernel of \( A \) over \( \mathbb{Z} \) by finding all solutions to

\[ \sum_a Q^a_I v_a = 0 \quad (4.8) \]

with \( Q^a_I \in \mathbb{Z} \), and such that for each \( I \) the \( \{Q^a_I | a = 1, \ldots, d\} \) have no common factor. The number of solutions, which are indexed by \( I \), is \( d - n \) since \( A \) is surjective; this latter fact again follows since \( C^* \) is strictly convex. One then has

\[ X = \mathcal{K}/A \equiv \mathbb{C}^d//A \quad (4.9) \]

with

\[ \mathcal{K} \equiv \left\{ (Z_1, \ldots, Z_d) \in \mathbb{C}^d \mid \sum_a Q^a_I |Z_a|^2 = 0 \right\} \subset \mathbb{C}^d \quad (4.10) \]
where $Z_a$ denote standard complex coordinates on $\mathbb{C}^d$ and the charge matrix $Q^a_I$ specifies the torus embedding $\mathbb{T}^{d-n} \subset \mathbb{T}^d$.

It is a standard fact that the space $X$ is an affine toric variety; that is, $X$ is an affine variety equipped with an effective holomorphic action of the complex torus $\mathbb{T}^n_\mathbb{C} \cong (\mathbb{C}^*)^n$ which has a dense open orbit. Let

$$C = \{\xi \in \mathfrak{t}_n \mid \langle \xi, y \rangle \geq 0, \forall y \in \mathbb{C}^* \}.$$  (4.11)

This is the dual cone to $\mathbb{C}^*$, which is also a convex rational polyhedral cone by Farkas’ Theorem. In the algebro-geometric language, the cone $C$ is precisely the fan for the affine toric variety $X$. We have $X_0 = \mathbb{R}_+ \times L$ with $L$ compact. If one begins with a general strictly convex rational polyhedral cone $C^*$, the link $L$ will be an orbifold; in order that $L$ be a smooth manifold one requires the moment polyhedral cone $C^*$ to be good [29]. This puts certain additional constraints on the vectors $v_a$; the reader is referred to [34] for the details. Note that $L$ inherits a canonical Sasakian metric from the Kähler quotient metric $g_{\text{can}}$ on $X_0$.

Let $\partial/\partial \phi_i, i = 1, \ldots, n$, be a basis for $\mathfrak{t}_n$, where $\phi_i \in [0, 2\pi)$ are coordinates on the real torus $\mathbb{T}^n$. Then we have the following

**Proposition 4.1 ([34])** The space of toric Kähler cone metrics on the smooth part of an affine toric variety $X_0$ is a product

$$C_{\text{int}} \times H^1(C^*)$$

where $\xi \in C_{\text{int}} \subset \mathfrak{t}_n$ labels the Reeb vector field, with $C_{\text{int}}$ the open interior of $C$, and $H^1(C^*)$ denotes the space of homogeneous degree one functions on $C^*$ that are smooth up to the boundary (together with the convexity condition below).

Explicitly, on the dense open image of $\mathbb{T}^n_\mathbb{C}$ we have

$$g = G_{ij} dy^i dy^j + G^{ij} d\phi_i d\phi_j$$  (4.12)

where

$$G_{ij} = \frac{\partial^2 G}{\partial y^i \partial y^j}$$  (4.13)

with matrix inverse $G^{ij}$, and the function

$$G(y) = G_{\text{can}}(y) + G_\xi(y) + h(y)$$  (4.14)

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6By a standard abuse of notation we identify vector fields on $X_0$ with corresponding elements of the Lie algebra.
is required to be strictly convex with \( h(y) \in \mathcal{H}^1(\mathcal{C}^*) \) and
\[
G_{\text{can}}(y) = \frac{1}{2} \sum_{a=1}^{d} \langle y, v_a \rangle \log \langle y, v_a \rangle
\]
\[
G_{\xi}(y) = \frac{1}{2} \langle \xi, y \rangle \log \langle \xi, y \rangle - \frac{1}{2} \left( \sum_{a=1}^{d} \langle v_a, y \rangle \right) \log \left( \sum_{a=1}^{d} \langle v_a, y \rangle \right).
\]

In particular, the canonical metric on \( X_0 \), induced from Kähler reduction of the flat metric on \( \mathbb{C}^d \), is given by setting \( G(y) = G_{\text{can}}(y) \). This function has a certain singular behaviour at the boundary \( \partial \mathcal{C}^* \) of the polyhedral cone; this is required precisely so that the metric compactifies to a smooth metric on \( X_0 \).

The space of Reeb vector fields is the interior of the cone \( \mathcal{C} \). One can show [35] that for \( \xi \in \partial \mathcal{C} \) the vector field \( \xi \) must vanish somewhere on \( X_0 \). Specifically, the bounding facets of \( \mathcal{C} \) correspond to the generating rays of \( \mathcal{C}^* \) under the duality map between cones; \( \xi \) being in a bounding facet of \( \mathcal{C} \) implies that the corresponding vector field then vanishes on the inverse image, under the moment map, of the dual generating ray of \( \mathcal{C}^* \).

However, since the Reeb vector field is nowhere vanishing, we see that the boundary of \( \mathcal{C} \) is a singular limit of Sasakian metrics on \( X_0 \).

Fixing a particular choice of Kähler cone metric on \( X_0 \), the image of \( L = \{ r = 1 \} \) under the moment map is
\[
\mu(L) = \{ y \in \mathcal{C}^* \mid \langle y, \xi \rangle = \frac{1}{2} \} \equiv H(\xi).
\]

The hyperplane \( \langle y, \xi \rangle = \frac{1}{2} \) is called the characteristic hyperplane [5]. This intersects the moment cone \( \mathcal{C}^* \) to form a compact \( n \)-dimensional polytope \( \Delta(\xi) = \mu(\{ r \leq 1 \}) \), bounded by \( \partial \mathcal{C}^* \) and the compact \( (n-1) \)-dimensional polytope \( H(\xi) \). In particular, the image \( H(\xi) \) of \( L \) under the moment map depends only on the Reeb vector field \( \xi \), and not on the choice of homogeneous degree one function \( h \) in Proposition 4.1. Moreover, the volume of a toric Sasakian manifold \((L, g_L)\) is [34]
\[
\text{vol}[g_L] = 2n(2\pi)^n \text{vol}[\Delta(\xi)]
\]
where \( \text{vol}[\Delta(\xi)] \) is the Euclidean volume of \( \Delta(\xi) \).

Finally in this section we introduce the notion of a Gorenstein singularity:

**Definition** An analytic space \( X \) with isolated singular point \( p \) and smooth part \( X \setminus \{ p \} = X_0 \) is said to be Gorenstein if there exists a smooth nowhere zero holomorphic \((n,0)\)-form \( \Omega \) on \( X_0 \).
We shall refer to $\Omega$ as a \textit{holomorphic volume form}. $X$ being Gorenstein is a necessary condition for $X_0$ to admit a Ricci-flat Kähler metric, and hence for the link $L$ to admit a Sasaki-Einstein metric. Indeed, the Ricci-form $\rho = \text{Ric}(\mathcal{J} \cdot, \cdot)$ is a curvature two-form for the holomorphic line bundle $\Lambda^{n,0}$. The Ricci-flat Kähler condition implies

$$\frac{i^n}{2^n} (-1)^{n(n-1)/2} \Omega \wedge \bar{\Omega} = \frac{1}{n!} \omega^n.$$  \hfill (4.17)

For affine toric varieties, it is again well-known that $X$ being Gorenstein is equivalent to the existence of a basis for the torus $\mathbb{T}^n$ for which $v_a = (1, w_a)$ for each $a = 1, \ldots, d$, and $w_a \in \mathbb{Z}^{n-1}$.

\textbf{Example} \([32, 33]\) From Theorems \([3.1, 3.3]\) one sees that $L^{a,b,c}$, which contain $Y^{p,q}$ as a subset, have a holomorphic Hamiltonian action of $\mathbb{T}^3$ on the corresponding Kähler cones and are thus toric Sasaki-Einstein manifolds. One finds that the image of the cone under the moment map is always a four-sided polyhedral cone ($d = 4$) in $\mathbb{R}^3$. The charge matrix $Q$ is

$$Q = (a, b, -c, -a - b + c).$$ \hfill (4.18)

The Gorenstein condition is reflected by the fact that the sum of the components of $Q$ is zero. In particular, for $Y^{p,q}$ (which is $a = p - q$, $b = p + q$, $c = p$) we have

$$v_1 = [1, 0, 0], \quad v_2 = [1, 1, 0], \quad v_3 = [1, p, p], \quad v_4 = [1, p - q - 1, p - q].$$ \hfill (4.19)

It is relatively straightforward to see that the affine toric Gorenstein singularities for $L^{a,b,c}$ are the most general such that are generated by four rays.

\section{A variational problem for the Reeb vector field}

In this section we consider the following problem: given a Gorenstein singularity $(X, \Omega)$, with isolated singular point and smooth set $X_0 = \mathbb{R}_+ \times L$, what is the Reeb vector field for a Ricci-flat Kähler cone metric on $X_0$, assuming it exists? We shall go quite a long way in answering this question, and give a complete solution for affine toric varieties; in general more work still remains to be done.

The strategy is to set up a variational problem on a space of Sasakian metrics on $L$, or equivalently a space of Kähler cone metrics on $X_0$. To this end, we suppose that $X_0$ is equipped with an effective holomorphic action of the torus $\mathbb{T}^s$ for some $s$; this is clearly necessary. Indeed, for irregular Sasakian metrics one requires $s > 1$, as
commented in section \[2\]. We then assume we are given a space of Kähler cone metrics $S(X_0)$ on $X_0$ such that:

- The torus $\mathbb{T}^s$ acts Hamiltonianly on each metric $g \in S(X_0)$.
- The Reeb vector field for each metric lies in the Lie algebra $\mathfrak{t}_s$ of $\mathbb{T}^s$.

We shall continue to denote Kähler cone metrics by $g$, the corresponding Sasakian metric by $g_L$, and regard either as elements of $S(X_0)$. The second condition above ensures that the torus action is of Reeb type [17, 5]. We then have the following

**Proposition 5.1** [55] The volume of the link $(L, g_L)$, as a functional on the space $S(X_0)$, depends only on the Reeb vector field $\xi$ for the Sasakian metric $g_L \in S(X_0)$.

It follows that vol may be regarded as a function on the space of Reeb vector fields:

$$
\text{vol} : \mathcal{R}(X_0) \rightarrow \mathbb{R}_+
$$

where

$$
\mathcal{R}(X_0) = \{\xi \in \mathfrak{t}_s \mid \xi = \text{Reeb vector field for some } g_L \in S(X_0)\}.
$$

The first and second derivatives are given by

**Proposition 5.2**

$$
d\text{vol}(Y) = -n \int_L \eta(Y) d\mu
$$

$$
d^2\text{vol}(Y, Z) = n(n+1) \int_L \eta(Y)\eta(Z) d\mu.
$$

Here $Y, Z$ are holomorphic Killing vector fields in $\mathfrak{t}_s$, $\eta$ is the contact one-form for the Sasakian metric, and $d\mu$ is the Riemannian measure on $(L, g_L)$.

Note that for toric Sasakian metrics (for which the torus $\mathbb{T}^s$ has maximal dimension: $s = n$) we already noted Proposition 5.1 in the previous section – see equation (4.16). Note also that (5.4) shows that vol is a strictly convex function of $\xi$.

Proposition 5.1 is proven roughly as follows. Suppose one has two Kähler cone metrics on $X_0$ with the same homothetic vector field $r\partial/\partial r$. It is then straightforward to show that the Kähler potentials differ by a multiplicative factor $\exp \varphi$, where $\varphi$ is a basic homogeneous degree zero function: $\mathcal{L}_\xi \varphi = 0 = \mathcal{L}_{r\partial/\partial r} \varphi$, where recall that $\xi = \mathcal{J}(r\partial/\partial r)$. One then shows that the volume is independent of $\varphi$.  

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This is precisely analogous to the situation in Kähler geometry where one fixes a Kähler class. Suppose that \((M, \omega)\) is a compact Kähler manifold with Kähler class \([\omega] \in H^{1,1}(M)\). Then any other Kähler metric on \(M\) in the same Kähler class is given by \(\omega + i \partial \bar{\partial} \varphi\) for some smooth real function \(\varphi\). The volume of \((M, \omega)\) clearly depends only on \([\omega]\).

Indeed, one can push the analogy further. A choice of Reeb vector field on \(X_0\) should be regarded as a choice of polarisation\(^7\). The space of quasi-regular Reeb vector fields is dense in the space of all Reeb vector fields: quasi-regular Reeb vector fields correspond to rational vectors in the Lie algebra \(t_s \cong \mathbb{R}^s\), and these are dense since the rationals are dense in the reals. For \(\xi\) quasi-regular, the \(U(1)\) quotient is a Kähler orbifold \((V, \omega_V)\). Changing the polarisation \(\xi\) thus changes the quotient \(V\), in contrast to the Kähler setting in the last paragraph where \(M\) is fixed and the Kähler class changes. It is also straightforward to show that the space of Reeb vector fields \(\xi\) forms a cone: if \(\xi\) is a Reeb vector field, then \(c\xi\) is also a Reeb vector field for another Kähler cone metric on \(X_0\), for any constant \(c > 0\). Thus the space \(\mathcal{R}(X_0)\) of Reeb polarisations forms a cone, analogous to the Kähler cone in Kähler geometry: we saw this explicitly in the previous section on toric Sasakian manifolds, where the space of Reeb vector fields is the interior \(\mathcal{C}_{\text{int}}\) of the polyhedral cone \(\mathcal{C}\).

We now suppose that \(X\) is also Gorenstein. This is necessary for the existence of a Ricci-flat Kähler metric on \(X_0\). We then introduce the subspace \(\mathcal{S}(X_0, \Omega)\) as the space of metrics in \(\mathcal{S}(X_0)\) for which the Reeb vector field \(\xi\) satisfies

\[
\mathcal{L}_\xi \Omega = i n \Omega .
\]  

(5.5)

Equivalently, \(\Omega\) should be homogeneous degree \(n\) under \(r \partial / \partial r\). This is again clearly a necessary condition for a Ricci-flat Kähler cone metric, cf (4.17).

Recall now that Einstein metrics on \(L\) are critical points of the Einstein-Hilbert action:

\[
\mathcal{I} : \text{Metrics}(L) \rightarrow \mathbb{R}
\]

\[
g_L \mapsto \int_L [s(g_L) + 2(n - 1)(3 - 2n)] d\mu
\]

(5.6)

where \(s(g_L)\) is the scalar curvature of \(g_L\). We then have the following proposition:

**Proposition 5.3** \(\text{[35]}\) The Einstein-Hilbert action, as a functional on \(\mathcal{S}(X_0)\), depends only on the Reeb vector field \(\xi\). It may thus be regarded as a function of \(\xi\).

\(^7\)This terminology was introduced in [8].
Moreover, for Sasakian metrics $g_L \in S(X_0, \Omega)$ we have

$$I(g_L) = 4(n - 1)\text{vol}[g_L].$$

(5.7)

Thus the Einstein-Hilbert action restricted to the space $S(X_0, \Omega)$ is simply the volume functional, and depends only on the Reeb vector field $\xi$ of the metric. This suggests we introduce

$$R(X_0, \Omega) = \{\xi \in R(X_0) \mid L_\xi \Omega = i n \Omega\}.$$  

(5.8)

Since Sasaki-Einstein metrics are critical points of $I$, we see that the Reeb vector field for a Sasaki-Einstein metric is determined by a finite-dimensional extremal problem, namely $dI = 0$, where $I$ is interpreted as a function on $R(X_0, \Omega)$. For toric varieties this is particularly simple:

**Theorem 5.4** Let $X$ be an affine toric Gorenstein variety with fan (or Reeb polytope) $\mathcal{C} \subset \mathfrak{t}_n$, and generating vectors of the form $v_a = (1, w_a)$. Then the Reeb vector field $\xi$ for a Ricci-flat Kähler cone metric on $X_0$ is uniquely determined as the critical point of the Euclidean volume of the polytope $\Delta(\xi)$

$$\text{vol}[\Delta] : N_{\text{int}} \to \mathbb{R}_+$$

(5.9)

where $N$ is the $(n - 1)$-dimensional polytope $N = \{\xi \in \mathcal{C} \mid \langle (1, 0, \ldots, 0), \xi \rangle = n\}$.

Here we may take $R(X_0) = C_{\text{int}}$ and $R(X_0, \Omega) = N_{\text{int}}$, following the classification of toric Sasakian metrics in section 1. The first and second derivatives $\frac{\partial \text{vol}[\Delta]}{\partial \xi_i}, \frac{\partial^2 \text{vol}[\Delta]}{\partial \xi_i \partial \xi_j}$ in Proposition 5.2 may be written

$$\frac{\partial \text{vol}[\Delta]}{\partial \xi_i} = \frac{1}{2 \xi_k \xi_k} \int_{H(\xi)} y^i \, d\sigma$$

(5.10)

$$\frac{\partial^2 \text{vol}[\Delta]}{\partial \xi_i \partial \xi_j} = \frac{2(n + 1)}{\xi_k \xi_k} \int_{H(\xi)} y^i y^j \, d\sigma.$$  

(5.11)

Here $d\sigma$ is the standard measure induced on the $(n - 1)$-polytope $H(\xi) \subset \mathcal{C}^*$. Uniqueness and existence of the critical point follows from a standard convexity argument: $\text{vol}[\Delta]$ is a strictly convex (by (5.11)) positive function on the interior of a compact convex polytope $N$. Moreover, $\text{vol}[\Delta]$ diverges to $+\infty$ at $\partial N$. It follows that $\text{vol}[\Delta]$ must have precisely one critical point in the interior of $N$.  

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Given these results, it is natural to consider the general case, with $s \leq n$. We begin by recalling a classical result. Let $-\nabla^2_L$ be the scalar Laplacian on $(L, g_L)$, with spectrum $\{E_\nu\}_{\nu=0}^\infty$. Then we may define the heat kernel trace

$$\Theta(t) = \sum_{\nu=0}^\infty \exp(-tE_\nu)$$

where $t \in (0, \infty)$. There is a holomorphic analogue of this. Let $f$ be a holomorphic function on $X_0$ with

$$\mathcal{L}_\xi f = \lambda f$$

where $\mathbb{R} \ni \lambda > 0$, and we refer to $\lambda$ as the charge of $f$ under $\xi$. Since $f$ is holomorphic, this immediately implies that

$$f = r^\lambda \tilde{f}$$

where $\tilde{f}$ is homogeneous degree zero under $r\partial/\partial r$; that is, $\tilde{f}$ is the pull–back to $X$ of a function on the link $L$. Moreover, since $(X_0, g)$ is Kähler one can show that $f$ is harmonic, and that

$$-\nabla^2_L \tilde{f} = E \tilde{f}$$

where

$$E = \lambda[\lambda + (2n - 2)] .$$

Thus any holomorphic function $f$ of definite charge under $\xi$, or equivalently degree under $r\partial/\partial r$, corresponds to an eigenfunction of the Laplacian $-\nabla^2_L$ on the link. The charge $\lambda$ is then related simply to the eigenvalue $E$ by the above formula \[5.16\]. We may thus in particular define the holomorphic spectral invariant

$$Z(t) = \sum_{i=0}^\infty \exp(-t\lambda_i)$$

where $\{\lambda_i\}_{i=0}^\infty$ is the holomorphic spectrum, in the above sense. This is also the trace of a kernel, namely the Szegő kernel.

Recall the following classical result:

**Theorem 5.5** \[5.18\] Let $(L, g_L)$ be a compact Riemannian manifold with heat kernel trace $\Theta(t)$ given by \[5.12\]. Then

$$\text{vol}[g_L] = \lim_{t \searrow 0} (4\pi t)^{n-\frac{1}{2}} \Theta(t) .$$
In the holomorphic setting we have:

**Theorem 5.6** ([35]) For $g_L \in \mathcal{S}(X_0, \Omega)$ with Reeb vector field $\xi$ we have

$$\frac{\text{vol}[g_L]}{\text{vol}[S^{2n-1}, g_{\text{can}}]} = \lim_{t \to 0} t^n Z(t).$$  \hspace{1cm} (5.19)

This result first appeared for regular Sasaki-Einstein manifolds in [2]. The proof of Theorem 5.6 is essentially the Riemann-Roch Theorem. Suppose that $g_L \in \mathcal{S}(X_0, \Omega)$ is quasi-regular\footnote{The reader who is uneasy with orbifolds may take a regular Sasakian manifold in what follows. However, the point here is that quasi-regular Reeb vector fields are dense in the space of Reeb vector fields, since the rationals are dense in the reals; regular Sasakian structures are considerably more special.}, so that $\xi$ generates a $U(1)$ action on $L$. The quotient is a Fano orbifold $(V, \omega_V)$:

**Definition** A compact Kähler orbifold $(V, \omega_V)$ is Fano if the cohomology class of the Ricci-form in $H^{1,1}(V)$ is represented by a positive $(1,1)$-form.

Holomorphic functions on $X_0$ that are eigenstates under $\mathcal{L}_\xi$ correspond to holomorphic sections of a holomorphic orbifold line bundle $L^{-k} \to V$ for some $k$; here $\mathcal{L}$ is the associated holomorphic line orbibundle to the $U(1)$ principal orbibundle $U(1) \hookrightarrow L \to V$. This is holomorphic since the curvature is proportional to the Kähler form on $V$, which is of Hodge type $(1,1)$. The number of holomorphic sections is given by an orbifold version of the Riemann-Roch theorem, involving characteristic classes on $V$. In general this is rather more complicated than the smooth Riemann-Roch theorem, but the limit in Theorem 5.6 simplifies the formula considerably: only a leading term contributes. The essential point now is that the volume may also be written in terms of Chern classes:

**Proposition 5.7** For a quasi-regular Sasakian metric $g_L \in \mathcal{S}(X_0, \Omega)$ one has

$$\frac{\text{vol}[g_L]}{\text{vol}[S^{2n-1}, g_{\text{can}}]} = \frac{\beta}{n^n} \int_V c_1(V)^{n-1}. \hspace{1cm} (5.20)$$

Here $(S^{2n-1}, g_{\text{can}})$ is the round sphere metric; $\beta \in \mathbb{Q}$ is defined by

$$c_1(\mathcal{L}) = -\frac{c_1(V)}{\beta} \in H^2_{\text{orb}}(V; \mathbb{Z}) \hspace{1cm} (5.21)$$

where $H^2_{\text{orb}}(V; \mathbb{Z})$ is the orbifold cohomology of Haefliger\footnote{One defines $H^*_{\text{orb}}(V; \mathbb{Z}) = H^*(BV; \mathbb{Z})$ where $BV$ is the classifying space for $V$. For details, see for example [4].} [26], $c_1(V)$ is the first Chern class of the holomorphic tangent bundle of $V$, and $\mathcal{L}$ is the orbifold line bundle associated to the $U(1)$ principal orbibundle $U(1) \hookrightarrow L \to V$.\footnote{One defines $H^*_{\text{orb}}(V; \mathbb{Z}) = H^*(BV; \mathbb{Z})$ where $BV$ is the classifying space for $V$. For details, see for example [4].}
The cohomology group $H^2_{orb}(V; \mathbb{Z})$ classifies orbifold line bundles over $V$, in exactly the same way that $H^2(V; \mathbb{Z})$ classifies line bundles when $V$ is smooth. The Proposition relates the volume of $g_L \in S(X_0, \Omega)$, for quasi-regular $\xi$, to characteristic classes of $V$. Theorem 5.6 then follows from the fact that quasi-regular Reeb vector fields are dense in the space of all Reeb vector fields, and vol is continuous.

We end this section with a localisation formula for the volume. We first require a

**Definition** Let $(X, \Omega)$ be a Gorenstein singularity with isolated singular point $p$, $X_0 = \mathbb{R}_+ \times L$, and let $S(X_0, \Omega)$ be as above, with respect to an effective holomorphic action of $\mathbb{T}^s$. We say that an orbifold $\hat{X}$ is a *partial resolution* of $X$ if

$$\pi : \hat{X} \to X$$

is a $\mathbb{T}^s$-equivariant map with $\pi : \hat{X} \setminus E \to X_0$ a $\mathbb{T}^s$-equivariant biholomorphism for some exceptional set $E$. If $\hat{X}$ is smooth we say that it is a resolution of $X$.

First note that such a partial resolution $\hat{X}$ always exists: one can take any quasi-regular Reeb vector field $\xi \in \mathfrak{t}_s$ and blow up the corresponding orbifold $V$. In this case the exceptional set $E = V$ and the partial resolution is clearly equivariant. We then have the following

**Theorem 5.8** Let $g_L \in S(X_0, \Omega)$ and pick a partial resolution $\hat{X}$ of $X$. Suppose that the Kähler form of $X_0$ extends to a suitable smooth family of Kähler forms on $\hat{X}$ (see [35] for details). Then

$$\frac{\text{vol}[g_L]}{\text{vol}[S^{2n-1}, g_{can}]} = \sum_{\{F\}} \frac{1}{d_F} \int_F \prod_{m=1}^{R} \frac{1}{\langle \xi, u_m \rangle^{n_m}} \left[ \sum_{a \geq 0} c_a(E_m) \right]^{-1}$$

(5.23)

where

- $E \supset \{F\} = \text{set of connected components of the fixed point set, where } \xi \text{ is a generic vector } \xi \in \mathfrak{t}_s; \text{ that is, the orbits of } \xi \text{ are dense in the torus } \mathbb{T}^s$.

- **For fixed connected component $F$, the linearised $\mathbb{T}^s$ action on the normal bundle $\mathcal{E}$ of $F$ in $\hat{X}$ is determined by a set of weights $u_1, \ldots, u_R \in \mathbb{Q}^s \subset \mathfrak{t}_s^*$. $\mathcal{E}$ then splits $\mathcal{E} = \bigoplus_{m=1}^{R} \mathcal{E}_m$ where rank$_\mathbb{C} \mathcal{E}_m = n_m$ and $\sum_{m=1}^{R} n_m = \text{rank}_\mathbb{C}(\mathcal{E})$.

- $c_a(E_m)$ are the Chern classes of $\mathcal{E}_m$. 

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When $\hat{X}$ has orbifold singularities, the normal fibre to a generic point on $F$ is not a complex vector space, but rather an orbifold $\mathbb{C}^l/\Gamma$. Then $E$ is more generally an orbibundle and $d_F = |\Gamma|$ denotes the order of $\Gamma$.

This theorem is proven as follows. One notes that the volume $\text{vol}[g_L]$ may be written as

$$\text{vol}[g_L] = \frac{1}{2^{n-1}(n-1)!} \int_{X_0} e^{-r^2/2} \omega^n / n!.$$  \hspace{1cm} (5.24)

The function $r^2/2$ is precisely the Hamiltonian function for the Reeb vector field $\xi$. One may then naively apply the theorem of Duistermaat-Heckman [15, 16], which easily extends to non-compact manifolds and orbifolds. This theorem localises such an integral to the fixed point set of $\xi$. However, since $\|\xi\|^2_g = r^2$, the integral formerly localises at the singular point $p = \{r = 0\}$ of $X$. To obtain a sensible answer, one must first resolve the singularity, as in Theorem 5.8. Such a proof requires that the Kähler form of $X_0$ extends to a suitable smooth family of Kähler forms on $\hat{X}$. One then applies the Duistermaat-Heckman theorem to this family, and takes the cone limit $(\hat{X}, g)$. The limit is independent of the choice of resolving family of Kähler metrics on $\hat{X}$.

**Remark** The author believes that the technical condition requiring the ability to extend the Kähler form $\omega$ on $X_0$ to a smooth family of Kähler forms on $\hat{X}$ is probably redundant. In fact, one can also formerly apply the equivariant Riemann-Roch theorem to $\hat{X}$, with respect to the holomorphic action of $T^*$, and Proposition 5.7 to obtain the same result. This has the advantage of not requiring existence of any Kähler metrics.

Note the theorem guarantees that $\text{vol} : \mathcal{R}(X_0) \to \mathbb{R}_+$, relative to the volume of the round sphere, is a rational function of $\xi$ with rational coefficients. Since $\text{vol}$ is strictly convex, its critical points are isolated. By Theorem 5.8 one thus sees that the volume of a Sasaki-Einstein manifold, relative to that of the round sphere, is an algebraic number.

Since the expression in Theorem 5.8 is rather formidable, we end with an

**Example** Let $X$ be an affine toric variety with moment polytope $\mathcal{C}^*$. Let $\pi : \hat{X} \to X$ be a toric resolution of $X$, with some choice of Kähler metric, and denote the moment polytope by $\hat{\mu}(\hat{X}) = P \subset \mathfrak{t}_n^*$. It is standard that such a resolution and Kähler metric always exist. Let $\text{Vert}(P)$ denote the set of vertices of $P$. For each vertex $A \in \text{Vert}(P)$
there are precisely \( n \) outward-pointing edge vectors; these may be taken to be primitive vectors \( u_A^i \in \mathbb{Z}^n \subset \mathfrak{t}_n^* \), \( i = 1, \ldots, n \). This follows since the resolution is smooth. Then

\[
\frac{\text{vol}[g_L]}{\text{vol}[S^{2n-1}, g_{\text{can}}]} = \sum_{A \in \text{Vert}(P)} \prod_{i=1}^n \frac{1}{\langle \xi, u_A^i \rangle}.
\] (5.25)

Note that this gives an unusual way of computing the Euclidean volume of the polytope \( \Delta(\xi) \), which, by (4.16), is also essentially the left-hand side of (5.25).

### 6 Obstructions to the existence of Sasaki-Einstein metrics

In this final section we examine the following

**Problem 6.1** Let \((X, \Omega)\) be a Gorenstein singularity with isolated singular point \( p \), \( X \setminus \{ p \} = X_0 = \mathbb{R}_+ \times L \), and let \( g_L \in \mathcal{S}(X_0, \Omega) \) be a Sasakian metric with Reeb vector field satisfying (5.5). When does \( X_0 \) admit a Ricci-flat Kähler cone metric with this Reeb vector field?

We shall describe three obstructions. The first is a natural corollary of the previous section, whereas the remaining two obstructions are based on classical theorems in differential geometry, and in particular lead to new obstructions to the existence of Kähler-Einstein orbifold metrics.

The previous section implies that a Sasaki-Einstein metric is a critical point of the volume functional, thought of as a function on the space \( \mathcal{R}(X_0, \Omega) \) of Reeb vector fields satisfying (5.5). Thus the Reeb vector field in Problem 6.1 must be a critical point of this function, or equivalently of the Einstein-Hilbert action on \( L \). We have already given this condition, in Proposition 5.2:

\[
d\text{vol}(Y) = -n \int_L \eta(Y).
\] (6.1)

Here \( Y \in \mathfrak{t}_s \) and recall that \( \eta \) is the contact one-form on \( L \). For a Sasaki-Einstein metric (6.1) must be zero for all holomorphic vector fields \( Y \in \mathfrak{t}_s \) satisfying \( L_Y \Omega = 0 \).

Suppose that the Sasakian metric \( g_L \in \mathcal{S}(X_0, \Omega) \) is quasi-regular. Then there is a quotient \( V = L/U(1) \) which is a Fano orbifold. In this case we have the following

**Theorem 6.2** Let \( g_L \in \mathcal{S}(X_0, \Omega) \) be quasi-regular and let \( Y \in \mathfrak{t}_s \) with \( L_Y \Omega = 0 \). Then

\[
d\text{vol}(Y) = -\frac{\ell}{2} F(J_V(Y_V)).
\] (6.2)
Here $\ell = 2\pi \beta/n$ is the length of the generic Reeb $S^1$ fibre, $\mathcal{J}_V$ is the complex structure tensor on $V$, $Y_V$ is the push-forward of $Y$ to $V$, and $F : \text{aut}(V) \to \mathbb{R}$ is the Futaki invariant of $V$.

The Futaki invariant is a well-known obstruction to the existence of a Kähler-Einstein metric on $V$ \[19\]. It is conventionally defined as follows. Let $(V, \omega_V)$ be a compact Kähler orbifold with Kähler form satisfying $[\rho_V] = 2n[\omega_V] \in H^{1,1}(V)$, where $\rho_V = \text{Ric}(\mathcal{J}_V\cdot, \cdot)$ is the Ricci-form of $(V, \omega_V)$. Then there exists a real function $f$ on $V$, unique up to an additive constant, satisfying

$$\rho_V - 2n\omega_V = i\partial\bar{\partial}f.$$ \hspace{1cm}(6.3)

Given any real holomorphic vector field $\zeta \in \text{aut}(V)$ we define

$$F(\zeta) = \int_V \mathcal{L}_\zeta f \frac{\omega_V^{n-1}}{(n-1)!}.$$ \hspace{1cm}(6.4)

Clearly, if $(V, \omega_V)$ is Kähler-Einstein of scalar curvature $4n(n-1)$ then $f$ is constant, and the function $F : \text{aut}(V) \to \mathbb{R}$ vanishes. However, more generally $F$ also satisfies the following rather remarkable properties \[19, 9\]:

- $F$ is independent of the choice of Kähler metric representing $[\omega_V]$.
- The complexification $F_C$ is a Lie algebra homomorphism $F_C : \text{aut}_C(V) \to \mathbb{C}$.

Note that the first point is implied by Proposition 5.1 and Theorem 6.2. The second is due to Calabi \[9\]. The Sasakian setting thus gives a \textit{dynamical} interpretation of the Futaki invariant.

We now turn to two further simple obstructions to the existence of solutions to Problem 6.1. These are based on the classical theorems of Lichnerowicz \[30\] and Bishop \[3\], respectively. We begin with the latter. Recall from Proposition 5.1 that the volume of a Sasakian metric $g_L \in \mathcal{S}(X_0, \Omega)$ is determined by its Reeb vector field $\xi$. In particular, one can compute this volume $\text{vol}(\xi)$ using Theorem 5.6. In some cases one can compute the trace over holomorphic functions on $X$ directly. Now, Bishop’s theorem \[3\] implies that for any $(2n-1)$-dimensional Einstein manifold $(L, g_L)$ with $\text{Ric}(g_L) = 2(n-1)g_L$ we have

$$\text{vol}(L, g_L) \leq \text{vol}(S^{2n-1}, g_{\text{can}}).$$ \hspace{1cm}(6.5)

Combining these two results we have
Theorem 6.3 [Bishop obstruction] ([25]) Let \((X, \Omega)\) be as in Problem 6.1 with Reeb vector field \(\xi\). If \(\text{vol}(\xi) > \text{vol}(S^{2n-1}, g_{\text{can}})\) then \(X_0\) admits no Ricci–flat Kähler cone metric with Reeb vector field \(\xi\).

A priori, it is not clear this condition can ever obstruct existence. We shall provide examples below. However, in [25] we conjectured that for regular Reeb vector fields \(\xi\) Theorem 6.3 never obstructs. This is equivalent to the following conjecture about smooth Fano manifolds:

Conjecture 6.4 ([25]) Let \(V\) be a smooth Fano manifold of complex dimension \(n - 1\) with Fano index \(I(V) \in \mathbb{N}\). Then

\[
I(V) \int_V c_1(V)^{n-1} \leq n \int_{\mathbb{P}^{n-1}} c_1(\mathbb{P}^{n-1})^{n-1} = n^n
\]

with equality if and only if \(V = \mathbb{P}^{n-1}\).

This is related to, although slightly different from, a standard conjecture about Fano manifolds. For further details, see [25].

We turn now to the Lichnerowicz obstruction. Suppose that \((L, g_L)\) is Einstein with \(\text{Ric}(g_L) = 2(n-1)g_L\). The first non–zero eigenvalue \(E_1 > 0\) of \(-\nabla^2_L\) is bounded from below:

\[
E_1 \geq 2n - 1 .
\]

This is Lichnerowicz’s theorem [30]. Moreover, equality holds if and only if \((L, g_L)\) is isometric to the round sphere \((S^{2n-1}, g_{\text{can}})\) [38]. From (5.16), we immediately see that for holomorphic functions \(f\) on \(X_0\) of charge \(\lambda\) under \(\xi\), Lichnerowicz’s bound becomes \(\lambda \geq 1\). This leads to a potential holomorphic obstruction to the existence of Sasaki–Einstein metrics:

Theorem 6.5 [Lichnerowicz obstruction] ([25]) Let \((X, \Omega)\) be as in Problem 6.1 with Reeb vector field \(\xi\). Suppose that \(f\) is a holomorphic function on \(X\) of positive charge \(\lambda < 1\) under \(\xi\). Then \(X_0\) admits no Ricci–flat Kähler cone metric with Reeb vector field \(\xi\).

Again, it is not immediately clear that this can ever obstruct existence. Indeed, for regular \(\xi\) one can prove [25] that this never obstructs. This follows from the fact that \(I(V) \leq n\) for any smooth Fano \(V\) of complex dimension \(n - 1\). However, there exist plenty of obstructed quasi-regular examples.
Notice then that the Lichnerowicz obstruction involves holomorphic functions on $X$ of small charge with respect to $\xi$, whereas the Bishop obstruction is a statement about the volume, which is determined by the asymptotic growth of holomorphic functions on $X$, analogously to Weyl’s asymptotic formula.

**Example** Our main set of examples of Theorems 6.3 and 6.5 is provided by isolated quasi-homogeneous hypersurface singularities. Let $w \in \mathbb{N}^{n+1}$ be a vector of positive weights. This defines an action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ via

$$ (z_1, \ldots, z_{n+1}) \mapsto (q^{w_1}z_1, \ldots, q^{w_{n+1}}z_{n+1}) $$ (6.8)

where $q \in \mathbb{C}^*$. Without loss of generality one can take the set $\{w_i\}$ of components of $w$ to have no common factor. This ensures that the above $\mathbb{C}^*$ action is effective. Let

$$ F : \mathbb{C}^{n+1} \to \mathbb{C} $$ (6.9)

be a quasi–homogeneous polynomial on $\mathbb{C}^{n+1}$ with respect to $w$. This means that $F$ has definite degree $d$ under the above $\mathbb{C}^*$ action:

$$ F(q^{w_1}z_1, \ldots, q^{w_{n+1}}z_{n+1}) = q^d F(z_1, \ldots, z_{n+1}). $$ (6.10)

Moreover, we assume that the affine algebraic variety

$$ X = \{F = 0\} \subset \mathbb{C}^{n+1} $$ (6.11)

is smooth everywhere except at the origin $(0, 0, \ldots, 0)$. For obvious reasons, such $X$ are called isolated quasi–homogeneous hypersurface singularities. The corresponding link $L$ is the intersection of $X$ with the unit sphere in $\mathbb{C}^{n+1}$:

$$ \sum_{i=1}^{n+1} |z_i|^2 = 1. $$ (6.12)

We define a nowhere zero holomorphic $(n, 0)$-form $\Omega$ on the smooth part of $X$ by

$$ \Omega = \frac{dz_1 \wedge \cdots \wedge dz_n}{\partial F/\partial z_{n+1}}. $$ (6.13)

This defines $\Omega$ on the patch where $\partial F/\partial z_{n+1} \neq 0$. One has similar expressions on patches where $\partial F/\partial z_i \neq 0$ for each $i$, and it is simple to check that these glue together into a nowhere zero form $\Omega$ on $X_0$. Thus all such $X$ are Gorenstein, and moreover they come equipped with a holomorphic $\mathbb{C}^*$ action by construction. The orbit space of
this $\mathbb{C}^*$ action, or equivalently the orbit space of $U(1) \subset \mathbb{C}^*$ on the link, is a complex orbifold $V$. In fact, $V$ is the weighted variety defined by $\{ F = 0 \}$ in the weighted projective space $\mathbb{W}/\mathbb{C}P^n_{[w_1,w_2,\ldots,w_{n+1}]}$. It is not difficult to show that $V$ is a Fano orbifold if and only if
\[ |w| - d > 0 \]  
(6.14)
where $|w| = \sum_{i=1}^{n+1} w_i$. To see this, first notice that $|w| - d$ is the charge of $\Omega$ under $U(1) \subset \mathbb{C}^*$. To be precise, if $\zeta$ denotes the holomorphic vector field on $X$ with
\[ \mathcal{L}_\zeta z_j = w_j iz_j \]  
(6.15)
for each $j = 1, \ldots, n + 1$, then
\[ \mathcal{L}_\zeta \Omega = (|w| - d)i\Omega . \]  
(6.16)
Positivity of this charge then implies [35] that the cohomology class of the natural Ricci–form induced on $V$ is represented by a positive $(1, 1)$–form, which is the definition that $V$ is Fano. If there exists a Ricci–flat Kähler metric on $X_0$ which is a cone under $\mathbb{R}_+ \subset \mathbb{C}^*$, then the correctly normalised Reeb vector field $\xi \in R(X_0, \Omega)$ is thus
\[ \xi = \frac{n}{|w| - d} \zeta . \]  
(6.17)
Bishop’s theorem then requires, for existence of a Sasaki–Einstein metric on $L$ with Reeb vector field $\xi$,
\[ d (|w| - d)^n \leq wn^n \]  
(6.18)
where $w = \prod_{i=1}^{n+1} w_i$ is the product of the weights. The computation of the volume that gives this inequality may be found in [25]. It is simple to write down infinitely many examples of isolated quasi-homogeneous hypersurface singularities that violate this inequality, and are thus obstructed by Theorem 6.3.

Lichnerowicz’s theorem requires, on the other hand, that
\[ |w| - d \leq nw_{\text{min}} \]  
(6.19)
where $w_{\text{min}}$ is the smallest weight. Moreover, this bound can be saturated if and only if $(X_0, g)$ is $\mathbb{C}^n \setminus \{0\}$ with its flat metric. It is again clearly trivial to construct many examples of isolated hypersurface singularities that violate this bound, and are hence obstructed by Theorem 6.5.
We conclude by making some remarks on the possible solution to Problem 6.1. Let us first comment on the case of toric varieties. The Reeb vector field for a critical point of the Einstein-Hilbert action, considered as a function on $\mathcal{R}(X_0, \Omega)$, exists and is unique, by Theorem 5.4. The remaining condition for a Ricci-flat Kähler cone metric may be written as a real Monge-Ampère equation on the polytope $\mathcal{C}^*$. This has recently been shown to always admit a solution in [20]. We state this as

**Theorem 6.6** ([20]) Let $(X, \Omega)$ be an affine toric Gorenstein singularity. Then $X_0$ admits a $\mathbb{T}^n$-invariant Ricci-flat Kähler cone metric, with Reeb vector field determined by Theorem 5.4.

Thus toric varieties are unobstructed. However, this still leaves us with the general non-toric case. Problem 6.1 is in fact closely related to a major open conjecture in the Kähler category. If $\xi$ is regular, then existence of a Ricci-flat Kähler cone metric on $X_0$ with this Reeb vector field is equivalent to existence of a Kähler-Einstein metric on the Fano $V$, and we then have the following conjecture due to Yau:

**Conjecture 6.7** ([45]) A Fano manifold $V$ admits a Kähler-Einstein metric if and only if it is stable in the sense of Geometric Invariant Theory.

A considerable amount of progress has been made on this conjecture, notably by Tian and Donaldson. However, it is still open in general. The conjecture is closely related to Problem 6.1 and it is clearly of interest to extend the work in the Kähler setting to the Sasakian setting.

Physics might also provide a different viewpoint. Suppose that $X$ admits a crepant resolution $\pi: \hat{X} \to X$. This means that the holomorphic volume form $\Omega$ on $X_0$ extends smoothly as a holomorphic volume form onto the resolution $\hat{X}$. Then one expects the derived category of coherent sheaves $\mathcal{D}^b(\text{coh}(\hat{X}))$ on $\hat{X}$ to be equivalent to the derived category of representations $\mathcal{D}^b(\text{Reps} (Q, R))$ of a quiver $Q$ with relations $R$. In fact this could be stated formerly as a conjecture; it is known to be true for various sets of examples. A quiver $Q$ is simply a directed graph. $R$ is a set of relations on the path algebra $\mathbb{C}Q$ of the quiver. The above correspondence between derived categories, if correct, would allow for a more precise mathematical statement of what the AdS/CFT map is. Physically, one is placing D3-branes at the singular point $p$ of $X$; mathematically, a D3-brane at a point on $X$ corresponds to the structure sheaf of that

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10This is certainly an extra constraint. For example, the singularities $w^{2k+1} + x^2 + y^2 + z^2 = 0$ admit no crepant resolution by an argument in [25].
point. The corresponding representation of \( \mathcal{Q} \) then defines an \( \mathcal{N} = 1 \) supersymmetric quantum field theory in four dimensions; this is precisely the quantum field theory on the D3-brane. The quiver representation determines the gauge group and matter content of the quantum field theory, while the relations specify the superpotential, which determines the interactions. The choice of \( (\mathcal{Q}, \mathcal{R}) \) is known to be non-unique, and this non-uniqueness is related to a duality known as Seiberg duality \[^4\]. The AdS/CFT correspondence then implies that \( X \) admits a Ricci-flat Kähler cone metric (in dimension \( n = 3 \)) if and only if this supersymmetric quantum field theory flows to a dual infra-red fixed point under renormalisation group flow; this infra-red fixed point is precisely the superconformal field theory in the AdS/CFT correspondence. Thus the solution to Problem 6.1, in complex dimension \( n = 3 \), is related to the low-energy behaviour of certain supersymmetric quiver gauge theories in four dimensions.

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References

[1] V. Apostolov, D. M. Calderbank, P. Gauduchon, “The geometry of weakly selfdual Kähler surfaces,” Compositio Math. 135 (2003), 279-322.

[2] A. Bergman, C. P. Herzog, “The volume of some non-spherical horizons and the AdS/CFT correspondence,” JHEP 0201, 030 (2002) [arXiv:hep-th/0108020].

[3] R. L. Bishop, R. J. Crittenden, “Geometry of manifolds,” Academic Press, New York, 1964.

[4] C. P. Boyer, K. Galicki, “On Sasakian–Einstein Geometry,” Internat. J. Math. 11 (2000), no. 7, 873–909 [arXiv:math.DG/9811098].

[5] C. P. Boyer, K. Galicki, “A Note on Toric Contact Geometry,” J. Geom. Phys. 35 No. 4 (2000), 288-298, [arXiv:math.DG/9907043].
[6] C. P. Boyer, K. Galicki, “Sasakian Geometry, Hypersurface Singularities, and Einstein Metrics,” Supplemento ai Rendiconti del Circolo Matematico di Palermo Serie II. Suppl 75 (2005), 57-87 [arXiv:math.DG/0405256].

[7] C. P. Boyer, K. Galicki, ”Sasakian Geometry,” Oxford Mathematical Monographs, Oxford University Press (2008), ISBN-13: 978-0-19-856495-9.

[8] C. P. Boyer, K. Galicki, S. Simanca, ”Canonical Sasakian Metrics,” Commun. Math. Phys. 279 (2008), 705-733 [arXiv:math.DG/0604325].

[9] E. Calabi, “Extremal Kähler Metrics II,” in “Differential Geometry and Complex Analysis” (ed. I. Chavel and H. M. Farkas), Springer–Verlag, 1985.

[10] J. Cheeger, G. Tian, “On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay,” Invent. Math. 118 (1994), no. 3, 493-571.

[11] W. Chen, H. Lu, C. N. Pope, J. F. Vazquez-Poritz, “A Note on Einstein–Sasaki Metrics in $D \geq 7$,” Class.Quant.Grav. 22 (2005), 3421-3430 [arXiv:hep-th/0411218].

[12] D. Conti, “Cohomogeneity one Einstein-Sasaki 5-manifolds,” Commun. Math. Phys. 274 (2007), N. 3, 751-774 [arXiv:math.DG/0606323].

[13] M. Cvetic, H. Lu, D. N. Page, C. N. Pope, “New Einstein-Sasaki spaces in five and higher dimensions,” Phys. Rev. Lett. 95, 071101 (2005) [arXiv:hep-th/0504225].

[14] M. Cvetic, H. Lu, D. N. Page, C. N. Pope, “New Einstein-Sasaki and Einstein Spaces from Kerr-de Sitter,” [arXiv:hep-th/0505223]

[15] J. J. Duistermaat, G. Heckman, “On the variation in the cohomology of the symplectic form of the reduced space,” Inv. Math. 69, 259-268 (1982).

[16] J. J. Duistermaat, G. Heckman, Addendum, Inv. Math. 72, 153-158 (1983).

[17] S. Falcao de Moraes, C. Tomei, “Moment maps on symplectic cones,” Pacific J. Math. 181 (2) (1997), 357-375.

[18] Th. Friedrich, I. Kath, “Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator,” J. Differential Geom. 29 (1989), 263-279.
[19] A. Futaki, “An obstruction to the existence of Einstein Kähler metrics,” Invent. Math., 73 (1983), 437-443.

[20] A. Futaki, H. Ono, G. Wang, “Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds,” arXiv:math.DG/0607586.

[21] J. P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, “Supersymmetric AdS 5 solutions of M-theory,” Class. Quant. Grav. 21, 4335 (2004) [arXiv:hep-th/0402153].

[22] J. P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, “Sasaki-Einstein metrics on $S^2 \times S^3$,” Adv. Theor. Math. Phys. 8, 711 (2004) [arXiv:hep-th/0403002].

[23] J. P. Gauntlett, D. Martelli, J. F. Sparks, D. Waldram, “A new infinite class of Sasaki-Einstein manifolds,” Adv. Theor. Math. Phys. 8, 987 (2004) [arXiv:hep-th/0403038].

[24] J. P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, “Supersymmetric AdS Backgrounds in String and M-theory,” IRMA Lectures in Mathematics and Theoretical Physics, volume 8, published by the European Mathematical Society [arXiv: hep-th/0411194].

[25] J. P. Gauntlett, D. Martelli, J. Sparks, S.-T. Yau, “Obstructions to the existence of Sasaki-Einstein metrics,” Commun. Math. Phys. 273, 803-827 (2007) [arXiv:hep-th/0607080].

[26] A. Haefliger, “Groupoides d’holonomie et classifiants,” Astérisque 116 (1984), 70-97.

[27] E. Kähler, “Über eine bemerkenswerte Hermitesche Metrik,” Abh. Math. Sem. Hamburg Univ. 9 (1933), 173-186.

[28] J. Kollár, Y. Miyaoka, S. Mori, “Rational connectedness and boundedness of Fano manifolds,” J. Diff. Geom. 36, 765-769 (1992).

[29] E. Lerman, “Contact toric manifolds,” J. Symplectic Geom. 1 (2003), no. 4, 785–828 [arXiv:math.SG/0107201].

[30] A. Lichnerowicz, “Géométrie des groupes de transformations,” Travaux et Recherches Mathematiques III, Dunod, Paris, 1958.
[31] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[32] D. Martelli, J. Sparks, “Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals,” Commun. Math. Phys. 262, 51 (2006) [arXiv:hep-th/0411238].

[33] D. Martelli, J. Sparks, “Toric Sasaki-Einstein metrics on $S^2 \times S^3$,” Phys. Lett. B 621, 208 (2005) [arXiv:hep-th/0505027].

[34] D. Martelli, J. Sparks, S.-T. Yau, “The geometric dual of $a$-maximisation for toric Sasaki-Einstein manifolds,” Commun. Math. Phys. 268, 39-65 (2006) [arXiv:hep-th/0503183].

[35] D. Martelli, J. Sparks, S.-T. Yau, “Sasaki-Einstein Manifolds and Volume Minimisation,” Commun. Math. Phys. 280 (2008) 611-673 [arXiv:hep-th/0603021].

[36] S. Minakshisundaram, A. Pleijel, “Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds,” Canadian J. Math. 1, (1949), 242-256.

[37] S. B. Myers, “Riemannian Manifolds with Positive Mean Curvature,” Duke Math. J. 8 (1941), 401-404.

[38] M. Obata, “Certain conditions for a Riemannian manifold to be isometric to a sphere,” J. Math. Soc. Japan 14 (1962), 333-340.

[39] S. Sasaki, “On differentiable manifolds with certain structures which are closely related to almost contact structure,” Tôhoku Math. J. 2 (1960), 459-476.

[40] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. B 435, 129 (1995) [arXiv:hep-th/9411149].

[41] S. Smale, “On the structure of 5-manifolds,” Ann. Math. 75 (1962), 38-46.

[42] S. Tanno, “Geodesic flows on $C_L$-manifolds and Einstein metrics on $S^3 \times S^2$,” in “Minimal submanifolds and geodesics” (Proc. Japan-United States Sem., Tokyo, 1977), pp. 283-292, North Holland, Amsterdam-New York, 1979.
[43] G. Tian, “On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$,” Invent. Math. 89 (1987) 225-246.

[44] G. Tian, S.-T. Yau, “On Kähler–Einstein metrics on complex surfaces with $C_1 > 0$,” Commun. Math. Phys. 112 (1987) 175-203.

[45] S.-T. Yau, “Open problems in Geometry,” Proc. Symp. Pure Math. 54 (1993), 1-28.