Unitary Dilations of Discrete-Time Quantum-Dynamical Semigroups

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We show that the discrete-time evolution of an open quantum system generated by a single quantum channel $T$ can be embedded in the discrete-time evolution of an enlarged closed quantum system, i.e. we construct a unitary dilation of the discrete-time quantum-dynamical semigroup $(T^n)_{n \in \mathbb{N}_0}$. In the case of a cyclic channel $T$, the auxiliary space may be chosen (partially) finite-dimensional. We further investigate discrete-time quantum control systems generated by finitely many commuting quantum channels and prove a similar unitary dilation result as in the case of a single channel.

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I. INTRODUCTION

Stimulated by the seminal work of Arveson\(^1\), Lindblad\(^2\), Gorini, Kossakowski and Sudarshan\(^3\) in the mid 1970s many efforts have been made to obtain dilation results of various degrees of generality for semigroups of completely positive operators.

For instance, Davies\(^4\) (Ch. 9, Thm. 4.3) proved that for any continuous semigroup \((T_t)_{t \in \mathbb{R}_0^+}\) of completely positive, unital operators acting on a finite-dimensional Hilbert spaces \(\mathcal{H}\) (or, more precisely, on the corresponding \(W^*\)-algebra \(\mathcal{B}(\mathcal{H})\) of all bounded linear operators) there exists a Hilbert space \(\mathcal{K}\), a pure state \(\omega\) in \(\mathcal{B}(\mathcal{K})\), and a strongly continuous one-parameter group \((U_t)_{t \in \mathbb{R}}\) of unitaries on \(\mathcal{H} \otimes \mathcal{K}\) such that

\[
T_t(A) = \text{tr}_\omega(U_t^\dagger (A \otimes \text{id}_\mathcal{K}) U_t)
\]

holds for all \(A \in \mathcal{B}(\mathcal{H})\) and \(t \in \mathbb{R}_0^+\). For infinite-dimensional \(\mathcal{H}\), there is a whole zoo of similar results. While Davies\(^5,6\), Evans\(^7\), and Evans \& Lewis\(^8,9\) focused primarily on one-parameter semigroups \((T_t)_{t \in \mathbb{R}_0^+}\) of different continuity type, Kümmerer\(^10\) discussed at great length the discrete-time case \((T^n)_{n \in \mathbb{N}}\). However, to the best of our knowledge, for arbitrary Hilbert spaces a dilation result of the above form \((\boxed{1})\) is still not available.

In the following, we give a short chronological overview on those contributions which are relevant and closely related to our work. Further results and a brief survey over the latest developments can be found in\(^{11-14}\) and\(^{15}\). For the readers’ convenience we collected some standard terminology and basic results on dilations and (completely) positive maps, which are well known to experts in this area, in the glossary of Appendix \[B\].

In\(^7\) (Thm. 1 and Thm. 2), Evans shows that every family \((T_g)_{g \in G}\) of completely positive, unital operators acting on a unital \(C^*\)-algebra \(\mathcal{A}\) and indexed by an arbitrary group \(G\) admits a unitary dilation, i.e.

\[
T_g(A) = E(U_g^\dagger J(A) U_g)
\]

for all \(A \in \mathcal{A}\) and \(g \in G\), where \((U_g)_{g \in G}\) is a unitary representation of \(G\) on some Hilbert space \(\mathcal{K}\) and \(E\) a conditional expectation with corresponding injection \(J\) into \(\mathcal{B}(\mathcal{K})\). Remarkably, he need not assume that \(g \mapsto T_g\) is a group homomorphism. His result can be regarded as \(C^*\)-counterpart to Sz.-Nagy’s\(^{16,17}\) and Stroescu’s\(^{18}\) work on isometric dilations on Hilbert and Banach spaces, respectively. While possible generalizations to \(W^*\)-algebras are addressed by
Evans, continuity issues of the map $g \mapsto U_g$ are disregarded completely. His proof is based on Stinespring’s representation $T_g(x) = V_g^* \pi_g(x) V_g$ which of course exists for all $g \in G$. However, he did not exploit the fact that one can choose a common Hilbert space for all $\pi_g$ which leads to a substantial simplification in our approach. In (Thm.2) Evans & Lewis focus on norm-continuous semigroups $(T_t)_{t \in \mathbb{R}^+_0}$ of ultraweakly continuous, completely positive and unital operators acting on a separable Hilbert space $\mathcal{H}$. They obtain a unitary dilation

$$T_t(A) = E(U_t^\dagger J_t(A)U_t),$$

for all $A \in \mathcal{B}(\mathcal{H})$ and $t \in \mathbb{R}^+_0$, where $(U_t)_{t \in \mathbb{R}}$ is a strongly continuous group of unitary operators acting on some extended Hilbert space $\mathcal{K}$ (and $E$, $J$ as above). Their proof exploits the fact that the explicit form of the infinitesimal generator of $(T_t)_{t \in \mathbb{R}^+_0}$ is well-known due to the work of Lindblad. In Evans & Lewis provide an overview on dilation results known at that time including some minor generalisations of their previous work.

For locally compact groups $G$, Davies (Thm.2.1 and Thm.3.1) obtains the following rather general result: Let $(T_g)_{g \in G}$ be a strongly continuous family of ultraweakly continuous, completely positive and unital operators on $\mathcal{B}(\mathcal{H})$. Then there exists a Hilbert space $\mathcal{K}$, a strongly continuous unitary representation $U$ of $G$ on $\mathcal{H} \otimes \mathcal{K}$ and conditional expectations $E_n : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{B}(\mathcal{H})$ (for all $n \in \mathbb{N}$) such that

$$T_g(A) = \lim_{n \to \infty} E_n(U_g^\dagger (A \otimes \text{id}_\mathcal{K})U_g)$$

holds for all $A \in \mathcal{B}(\mathcal{H})$ and all $g \in G$ in the weak operator topology. Here, $E_n$ is of the form $E_n(A) := V_n^\dagger AV_n$ where $V_n : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ are isometric embeddings. This seems to be the result which is closest to (1) in infinite dimensions, but it is not known whether the limit in (2) is necessary or not (cf. p. 335). For discrete-time systems $T_n := T^n$, $n \in \mathbb{N}$ or, more accurately, for an appropriate extension to $G = \mathbb{Z}$ Davies’ approach and ours are quite similar—in particular due to the fact that in this case, the limit in (2) can be avoided as $G$ is discrete. More precisely, Davies first extends the state space from $\mathcal{H}$ to $L^2(\mathbb{Z}, \mathcal{H}) \cong \ell_2(\mathbb{Z}) \otimes \mathcal{H}$ such that $(T_n)_{n \in \mathbb{Z}}$ can be regarded as one completely positive, unital operator from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(L^2(\mathbb{Z}, \mathcal{H}))$. He then applies Stinespring’s representation theorem to obtain an dilation of $(T_n)_{n \in \mathbb{Z}}$ on a larger state space $L^2(\mathbb{Z}, \mathcal{H}) \otimes \mathcal{K}$. We, however, exploit Stinespring’s result first to guarantee for all $n \in \mathbb{N}$ a dilation of the form

$$T^n(A) = \text{tr}_\omega((U^\dagger)^n (A \otimes \text{id}_\mathcal{K})U^n),$$
where $\omega$ and $K$ are independent of $n \in \mathbb{N}$, to then enlarge the state space to $\ell_2(\mathbb{Z}) \otimes \mathcal{H} \otimes \mathcal{K} \cong L^2(\mathbb{Z}, \mathcal{H} \otimes \mathcal{K})$. Although both approaches differ only in the order of the construction steps the resulting dilations behave quite differently: While Davies’ construction is more “flexible” as one can see, e.g., in Section III B, Remark 12.2, ours yields the desired partial trace structure of (1) which is in general not satisfied for (2) even if the limit can be avoided.

Kümmerer discussed the discrete-time case ${(T^n)}_{n \in \mathbb{N}}$ in detail. However, his setting significantly differs from ours. In his sense, a discrete-time quantum dynamical system consists of a triple $(\mathcal{A}, \varphi, T)$, where $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a $W^*$-algebra, $T$ an ultraweakly continuous, completely positive and unital operator which acts on $\mathcal{A}$ and leaves a faithful normal state $\varphi \in \mathcal{A}^*$ invariant, i.e. $\varphi \circ T = \varphi$. The latter condition can be thermodynamically motivated as $\varphi$ can be interpreted as an equilibrium state which is preserved under composition with $T$ and every power of it. This constraint on the quantum channel $T$ obviously narrows down the possible choices of $T$. Even more restrictive is Kümmerer’s definition of a first order dilation of $(\mathcal{A}, \varphi, T)$. Here, he requires the existence of a reversible quantum dynamical system $(\mathfrak{A}, \hat{\varphi}, \hat{T})$, i.e. $\hat{T}$ is a $*$-automorphism on $\mathfrak{A}$ and $E$ is a conditional expectation with corresponding injection $J$ such that

$$T(A) = E(\hat{T}(J(A))) \quad \text{and} \quad \varphi \circ E = \hat{\varphi}.$$ 

for all $A \in \mathcal{A}$. In doing so, the condition $\varphi \circ E = \hat{\varphi}$ is the delicate part. For instance, the standard Kraus/Stinespring representation which constitutes a (first order) dilation does in general not satisfy this condition—note that, by definition, $\hat{\varphi}$ has to be a faithful normal state—and therefore even first order dilations in Kümmerer’s sense need not exist as the existence of a $\varphi$-adjoint is not guaranteed, cf. (Prop. 2.1.8 ff.). Within his setting, Kümmerer proved (cf. Thm 4.2.1, Cor 4.2.3) that a quantum dynamical system $(\mathcal{B}(\mathcal{H}), \varphi, T)$ has a dilation of first order if and only if it admits a Markovian one of first order which in turn implies that $(\mathcal{B}(\mathcal{H}), \varphi, T)$ also allows a Markovian dilation of arbitrary order. His definition of Markovianity can be regarded as a $W^*$-algebra counterpart of a well-known subspace condition which guarantees for contractions on Hilbert spaces that a first order unitary dilation $T = P_H U|_H$ is already a dilation (of arbitrary order), i.e. $T^n = P_H U^n|_H$ holds for all $n \in \mathbb{N}$. To achieve a Markovian dilation he imbedded the given $W^*$-algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$ in an infinite product/sum of $W^*$-algebras. Our approach considerably deviates from his construction since we use first order Stinespring/Kraus dilations for $T^n$ which of course exist.
for all $n \in \mathbb{N}$ but in general do not satisfy Kümmel’s faithful state condition.

Probably one of the strongest semigroup dilation results so far was presented by Gaebler (Thm. 5.10). Using Sauvageot’s theory he showed that for a norm-continuous semigroup $(T_t)_{t \in \mathbb{R}^+}$ of ultraweakly continuous, completely positive and unital operators acting on a $W^*$-algebra $\mathcal{A}$ with separable pre-dual, there exists a unital dilation $(\mathfrak{A}, (\sigma_t)_{t \in \mathbb{R}^+_0}, J, E)$ of $(T_t)_{t \in \mathbb{R}^+_0}$ (cf. Def. 18) where $\mathfrak{A}$ has separable pre-dual and $((\sigma_t)_{t \in \mathbb{R}^+_0}, J, E)$ satisfies the strong dilation property, i.e. $T_t \circ E = E \circ \sigma_t$ for all $t \in \mathbb{R}^+$. The strength of this result, however, comes at the cost of lacking any partial trace structure of the form (1).

The paper is organized as follows: After some preliminaries on trace-class operators and quantum channels, we present our main results in Section II: (i) For discrete-time quantum-dynamical semigroups on separable Hilbert spaces, a unitary dilation of the form (1) is proved. (ii) If the semigroup in question is generated by a cyclic quantum channel, then the auxiliary Hilbert space can be chosen partially finite-dimensional. (iii) Finally, for discrete-time quantum control systems, the control of which can be switched between a finite number of commuting channels, a unitary dilation of the form (1) is derived.

II. PRELIMINARIES AND NOTATION

In this section, we fix our notation and recall some basic material on Schrödinger and Heisenberg quantum channels. These results should be known to experts in this area.

Henceforth, let $\mathcal{G}, \mathcal{H}$ be infinite-dimensional separable complex Hilbert spaces and $\mathcal{X}, \mathcal{Y}$ real or complex Banach spaces. By convention, all scalar products on complex Hilbert spaces are assumed to be conjugate linear in the first argument and linear in the second. Moreover, let $\mathcal{B}(\mathcal{G}), \mathcal{B}(\mathcal{H})$ denote the set of all bounded operators acting on $\mathcal{G}, \mathcal{H}$ and let $\mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{Y})$ be defined respectively.

Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ on a complex Hilbert space is said to be positive semi-definite, denoted by $A \geq 0$, iff $\langle x, Ax \rangle \geq 0$ for all $x \in \mathcal{H}$. Because we consider complex Hilbert spaces, $A \geq 0$ directly implies that $A$ is self-adjoint via the polarization identity, cf. (Prop. 2.4.6)—else, self-adjointness would have to be required in the definition of $A \geq 0$. 
A. Quantum Channels

Let $\mathcal{B}^1(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ be the subset of all *trace-class operators*, i.e. $\mathcal{B}^1(\mathcal{H})$ is the largest subspace of $\mathcal{B}(\mathcal{H})$ which allows to define the *trace* of an operator $A$ via

$$\text{tr}(A) := \sum_{i \in I} \langle e_i, Ae_i \rangle$$

such that the right-hand side of (3) is finite and independent of the choice of the orthonormal basis $(e_i)_{i \in I}$. More precisely, $\mathcal{B}^1(\mathcal{H})$ can be defined either as the set of all compact operators $A \in \mathcal{B}(\mathcal{H})$ whose singular values $\sigma_n(A)$ are summable, i.e.

$$\nu_1(A) := \sum_{n \in \mathbb{N}} \sigma_n(A) < \infty$$

or, equivalently (Thm.VI.21), as the set of all $A \in \mathcal{B}(\mathcal{H})$ such that

$$\sum_{i \in I} \langle e_i, \sqrt{A^\dagger A}e_i \rangle < \infty$$

is summable for some orthonormal basis $(e_i)_{i \in I}$ of $\mathcal{H}$.

Because of $\sqrt{A^\dagger A} \geq 0$, all summands in (3) are non-negative and therefore (Thm.VI.18), the value of the left-hand side of (3) is independent of the chosen orthonormal basis. Moreover, one has $\text{tr}(\sqrt{A^\dagger A}) = \nu_1(A)$ for all $A \in \mathcal{B}^1(\mathcal{H})$ which readily implies $\text{tr}(A) = \nu_1(A)$ if $A \geq 0$. Finally, we note that for finite-dimensional Hilbert spaces the sets $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}^1(\mathcal{H})$ coincide with the set of all linear operators acting on $\mathcal{H}$ and that for arbitrary Hilbert spaces, $\mathcal{B}^1(\mathcal{H})$ constitutes a Banach space with respect to the *trace norm* $\nu_1$ given by (4). For more on these topics we refer to (Ch.VI.6) and (Ch.16).

An operator $\rho \in \mathcal{B}^1(\mathcal{H})$ which is positive semi-definite and fulfills $\text{tr}(\rho) = 1$ is called a *state* and the set of all states is denoted by

$$\mathbb{D}(\mathcal{H}) := \{ \rho \in \mathcal{B}^1(\mathcal{H}) \mid \rho \text{ is state} \}.$$  

A state $\rho$ is said to be *pure* if it has rank one. Certainly, the corresponding definitions apply to $\mathcal{B}^1(\mathcal{G})$ and $\mathbb{D}(\mathcal{G})$. After these preliminaries, we can introduce the key concepts.

**Definition 1.** (a) A linear map $T : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{G})$ is said to be *positive* if $T(A) \geq 0$ for all positive semi-definite $A \in \mathcal{B}^1(\mathcal{H})$.

(b) A linear map $T : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{G})$ is said to be *completely positive* if for all $n \in \mathbb{N}$ the maps $T \otimes \text{id}_n : \mathcal{B}^1(\mathcal{H} \otimes \mathbb{C}^n) \to \mathcal{B}^1(\mathcal{G} \otimes \mathbb{C}^n)$ are positive.
A Schrödinger quantum channel is a linear, completely positive and trace-preserving map $T : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{G})$. Furthermore, we define

$$Q_S(\mathcal{H}, \mathcal{G}) := \{ T : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{G}) \mid T \text{ is Schrödinger quantum channel} \}$$

and $Q_S(\mathcal{H}) := Q_S(\mathcal{H}, \mathcal{H})$.

Note that Definition 1 (a) and (b) also make sense for maps from $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{G})$ instead of $\mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{G})$.

Clearly, every positive, trace-preserving map and thus every Schrödinger quantum channel maps states to states. Further algebraic and topological properties of $Q_S(\mathcal{H})$ which are crucial in the following are summarized in the following theorem, the proof of which can be found in Appendix A.

**Theorem 1.** The set $Q_S(\mathcal{H})$ is a convex subsemigroup of $\mathcal{B}(\mathcal{B}^1(\mathcal{H}))$ with unity element $\text{id}_{\mathcal{B}^1(\mathcal{H})}$. Moreover, $Q_S(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{B}^1(\mathcal{H}))$ with respect to the weak operator, strong operator and uniform operator topology.

**Remark 2.**

1. Here one should emphasize that it is not necessary to require boundedness of Schrödinger quantum channels. In fact, one can easily prove that any positive linear map is bounded automatically (Ch. 2, Lemma 2.1).

2. In Proposition 2 we will see that $Q_S(\mathcal{H})$ is actually a closed convex subset of the unit sphere of $\mathcal{B}(\mathcal{B}^1(\mathcal{H}))$. Note that the existence of non-trivial convex subsets on the unit sphere of $\mathcal{B}(\mathcal{B}^1(\mathcal{H}))$ is a consequence of the non-strict convexity of the operator norm.

The following beautiful and well-known representation result for Schrödinger quantum channels which can be traced back to Kraus is the starting of our work.

**Theorem 2.** For every $T \in Q_S(\mathcal{H})$ there exists a separable Hilbert space $\mathcal{K}$, a pure $\omega \in \mathbb{D}(\mathcal{K})$ and a unitary $U \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that

$$T(A) = \text{tr}_\mathcal{K}(U(A \otimes \omega)U^\dagger)$$

(6)

for all $A \in \mathcal{B}^1(\mathcal{H})$. 

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Here \( \text{tr}_K : \mathcal{B}^1(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{B}^1(\mathcal{H}) \) is the partial trace with respect to the Hilbert space \( \mathcal{K} \) which is defined via

\[
\text{tr}(B \text{tr}_K(A)) = \text{tr}((B \otimes \text{id}_\mathcal{K})A)
\]

for all \( B \in \mathcal{B}(\mathcal{H}) \) and all \( A \in \mathcal{B}^1(\mathcal{H} \otimes \mathcal{K}) \).

For a complete proof of Theorem 2, see\(^\text{24}\) (second part of Thm.2). Here, we only emphasize that the separable auxiliary space \( \mathcal{K} \) can be chosen independently of \( T \); for instance, \( \mathcal{K} := \ell_2(\mathbb{N}) \) constitutes such a universal auxiliary space. Moreover, once \( \mathcal{K} \) is fixed, \( \omega \in \mathbb{D}(\mathcal{K}) \) can be chosen as any orthogonal rank-1 projection. Thus \( \omega \) is pure and independent of \( T \), too.

**Corollary 1** (General Stinespring Dilation). **For every** \( T \in Q_S(\mathcal{H}, \mathcal{G}) \) **there exists a separable Hilbert space** \( \mathcal{K} \), **pure** \( \omega_G \in \mathbb{D}(\mathcal{G}) \), \( \omega_K \in \mathbb{D}(\mathcal{K}) \) **and a unitary** \( U \in \mathcal{B}(\mathcal{H} \otimes \mathcal{G} \otimes \mathcal{K}) \) **such that**

\[
T(A) = (\text{tr}_\mathcal{H} \circ \text{tr}_\mathcal{K})(U(A \otimes \omega_G \otimes \omega_K)U^\dagger)
\]

**for all** \( A \in \mathcal{B}^1(\mathcal{H}) \).

**Proof.** Consider arbitrary \( \omega_G \in \mathbb{D}(\mathcal{G}) \) and \( \omega_H \in \mathbb{D}(\mathcal{H}) \) of rank one. Applying Theorem 2 to \( X(\cdot) := \omega_H \otimes T(\text{tr}_G(\cdot)) \in Q_S(\mathcal{H} \otimes \mathcal{G}) \) which is obviously a composition of Schrödinger quantum channels yields a separable Hilbert space \( \mathcal{K} \), a pure \( \omega_K \in \mathbb{D}(\mathcal{K}) \) and a unitary \( U \in \mathcal{B}(\mathcal{H} \otimes \mathcal{G} \otimes \mathcal{K}) \) such that \( X \) is of form \( ^6 \). For any \( A \in \mathcal{B}^1(\mathcal{H}) \) one gets

\[
T(A) = \text{tr}_\mathcal{H}(X(A \otimes \omega_G)) = (\text{tr}_\mathcal{H} \circ \text{tr}_\mathcal{K})(U(A \otimes \omega_G \otimes \omega_K)U^\dagger)
\]

with \( \omega_G \otimes \omega_K \in \mathbb{D}(\mathcal{G} \otimes \mathcal{K}) \) rank one. \( \square \)

The last result of this subsection provides a characterization of invertible quantum channels which leads to a nice simplification later on (cf. Remark 9). For finite dimensions, this was essentially shown in\(^\text{25}\) (Coro. 3).

**Proposition 1.** **Let** \( T \in Q_S(\mathcal{H}) \) **be bijective. Then the following statements are equivalent.**

(a) \( T^{-1} \) is positive.

(b) There exists unitary \( U \in \mathcal{B}(\mathcal{H}) \) such that \( T(A) = UAU^\dagger \) for all \( A \in \mathcal{B}^1(\mathcal{H}) \).

In particular if one (and thus both) conditions are fulfilled, then \( T \in Q_S(\mathcal{H}) \) is invertible as a channel, i.e. \( T \) is bijective and \( T^{-1} \in Q_S(\mathcal{H}) \).
Proof. (b) ⇒ (a): ✓ (a) ⇒ (b): The proof idea is the same as in \textsuperscript{26} (Prop. 4.31). Consider the restricted channel $T|_{D} : \mathbb{D}(\mathcal{H}) \rightarrow \mathcal{B}^{1}(\mathcal{H})$ which by assumption is convex-linear and injective. As $T$ and $T^{-1}$ are linear, trace-preserving and, by assumption, positive, the restricted channel satisfies

$$T(\mathbb{D}(\mathcal{H})) \subseteq \mathbb{D}(\mathcal{H}) \quad T^{-1}(\mathbb{D}(\mathcal{H})) \subseteq \mathbb{D}(\mathcal{H}),$$

so $T|_{D} : \mathbb{D}(\mathcal{H}) \rightarrow \mathbb{D}(\mathcal{H})$ is surjective and thus a state automorphism, i.e. convex-linear and bijective. Then Corollary 3.2 in \textsuperscript{4} or, more explicitly, Theorem 2.63 in \textsuperscript{26} imply the existence of unitary or anti-unitary $U$ such that $T|_{D}(\cdot) = U(\cdot)U^{\dagger}$. If $U$ were anti-unitary, then $T$ would not be completely positive \textsuperscript{26} (Prop. 4.14) hence $U$ has to be unitary. Due to $\text{span}_{C}(\mathbb{D}(\mathcal{H})) = \mathcal{B}^{1}(\mathcal{H})$, this representation extends linearly to all of $\mathcal{B}^{1}(\mathcal{H})$ which concludes the proof.

\[\square\]

B. Dual Channels

It is well known\textsuperscript{23} (Prop.16.26) that the dual space of $\mathcal{B}^{1}(\mathcal{H})$ is isometrically isomorphic to $\mathcal{B}(\mathcal{H})$ by means of the map $\psi_{\mathcal{H}} : \mathcal{B}(\mathcal{H}) \rightarrow (\mathcal{B}^{1}(\mathcal{H}))'$, $B \mapsto \psi_{\mathcal{H}}(B)$ with

$$(\psi_{\mathcal{H}}(B))(A) := \text{tr}(BA)$$

for all $A \in \mathcal{B}^{1}(\mathcal{H})$. Note that the weak-$*$-topology and the ultraweak topology on $\mathcal{B}(\mathcal{H})$ coincide under the above identification $(\mathcal{B}^{1}(\mathcal{H}))' \cong \mathcal{B}(\mathcal{H})$, cf.\textsuperscript{4} (Section 1.6).

Now, since every positive linear map $T : \mathcal{B}^{1}(\mathcal{H}) \rightarrow \mathcal{B}^{1}(\mathcal{G})$ is bounded (cf. Remark\textsuperscript{2}1) the dual map

$$T' : (\mathcal{B}^{1}(\mathcal{G}))' \rightarrow (\mathcal{B}^{1}(\mathcal{H}))' \quad X \mapsto T'(X) := X \circ T$$

is well defined and this allows us to construct the so called dual channel of $T$

$$T^{*} : \mathcal{B}(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H}) \quad B \mapsto T^{*}(B) := (\psi_{\mathcal{H}}^{-1} \circ T' \circ \psi_{\mathcal{G}})(B)$$

which then satisfies

$$\text{tr}(BT(A)) = \text{tr}(T^{*}(B)A) \quad (8)$$

for all $B \in \mathcal{B}(\mathcal{G})$ and $A \in \mathcal{B}^{1}(\mathcal{H})$. Alternatively, onc can use (8) as defining equation for $T^{*}$. Furthermore, one has $\|T\| = \|T^{*}\|$ by definition of $T^{*}$, because $T$ and $T'$ have the same operator norm and $\psi_{\mathcal{G}}$ and $\psi_{\mathcal{H}}$ are isometric isomorphisms. Some basic properties of $T^{*}$ are:
(a) $T^*$ is positive and ultraweakly continuous.

(b) $T^*$ is completely positive if and only if $T$ is completely positive.

(c) $T^*$ is unital (i.e. $T^*(\text{id}_G) = \text{id}_H$) if and only if $T$ is trace-preserving.

For more details and proofs we refer to [24] (p. 35) or [26] (Ch. 4.1.2).

**Definition 3.** A Heisenberg quantum channel is a linear, ultraweakly continuous, completely positive and unital map $S : \mathcal{B}(G) \to \mathcal{B}(\mathcal{H})$. Furthermore, we define

$$Q_H(G, \mathcal{H}) := \{S : \mathcal{B}(G) \to \mathcal{B}(\mathcal{H}) \mid S \text{ is Heisenberg quantum channel}\}$$

and $Q_H(\mathcal{H}) := Q_H(G, \mathcal{H})$.

By the properties listed above, it is evident that the map $\star : Q_S(\mathcal{H}, G) \to Q_H(G, \mathcal{H})$ which assigns its dual channel is well-defined. Furthermore, it is—as we will see next—bijective.

**Theorem 3.** (a) For every $S : \mathcal{B}(G) \to \mathcal{B}(\mathcal{H})$ linear, ultraweakly continuous and positive there exists unique $T : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(G)$ linear and positive such that $T^* = S$.

(b) For every $S \in Q_H(G, \mathcal{H})$ there exists unique $T \in Q_S(\mathcal{H}, G)$ such that $T^* = S$.

**Proof.** (a) By the above construction of the dual channel it is obvious that the map $\star : Q_S(\mathcal{H}, G) \to Q_H(G, \mathcal{H})$ which assigns its dual channel is well-defined. Furthermore, it is—as we will see next—bijective.

First one shows, similar to [4] (Ch. 1, Lemma 6.1), that for every positive, linear and ultraweakly continuous functional $\lambda : \mathcal{B}(G) \to \mathbb{C}$ there exists a unique positive semi-definite $\rho \in \mathcal{B}^1(G)$ such that $\lambda(\cdot) = \text{tr}(\rho(\cdot))$. Next choose arbitrary positive semi-definite $A \in \mathcal{B}^1(\mathcal{H})$ and consider the linear functional

$$B \mapsto \text{tr}(S(B)A)$$

which by assumption on $S$ is ultraweakly continuous. Our preliminary consideration yields a unique positive semi-definite $\rho_A \in \mathcal{B}^1(G)$ such that $\text{tr}(S(B)A) = \text{tr}(B\rho_A)$ for all $B \in \mathcal{B}(G)$. This allows to define an $\mathbb{R}^+$-linear map $\hat{T}$ on the positive semi-definite elements of $\mathcal{B}^1(\mathcal{H})$ via $\hat{T}(A) := \rho_A$. Finally, $\hat{T}(A)$ can be uniquely extended to a positive, linear map $T : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(G)$ satisfying $T^* = S$.

Now (b) follows from (a) together with the above connections between properties of a positive, linear map and its dual channel.
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Remark 4. 1. Note that, again, boundedness is not required in the definition of a Heisenberg quantum channel because, similar to Schrödinger quantum channels, they are automatically bounded, see Proposition 2 below.

2. In finite dimensions, ultraweak continuity is of course always satisfied and the $\ast$-map is an involution as the sets of trace-class operators and bounded operators coincide.

Proposition 2. Let $T \in Q_S(\mathcal{H}, \mathcal{G})$ and $S \in Q_H(\mathcal{G}, \mathcal{H})$. Then $\|T\| = 1$ and $\|S\| = 1$.

Proof. As each $S \in Q_H(\mathcal{G}, \mathcal{H})$ in particular is linear, positive and unital it has operator norm $\|S\| = 1$ as a consequence of the Russo-Dye Theorem, cf. (Cor. 1) or Rem. 19.1. This directly implies $\|T\| = \|T^\ast\| = 1$. □

Alternatively, one can prove Proposition 2 via the general Stinespring dilation (Corollary 1) because all maps involved in the Stinespring representation have operator norm one. Either way, with this one readily verifies that $Q_H(\mathcal{H})$ forms a convex subsemigroup of the Banach space $B(B(\mathcal{H}))$ with unity element $\text{id}_{B(\mathcal{H})}$.

The partial trace $\text{tr}_\omega : B(\mathcal{H} \otimes \mathcal{K}) \to B(\mathcal{H})$ with respect to a state $\omega \in \mathcal{D}(\mathcal{K})$ is defined via

$$\text{tr}(\text{tr}_\omega(B)A) = \text{tr}(B(A \otimes \omega))$$

for all $B \in B(\mathcal{H} \otimes \mathcal{K})$, $A \in B^1(\mathcal{H})$, cf. (Ch. 9, Lemma 1.1). Be aware that the map $\text{tr}_\mathcal{K}$ from (7) and the extension

$$i_\omega : B^1(\mathcal{H}) \to B^1(\mathcal{H} \otimes \mathcal{K}) \quad A \mapsto A \otimes \omega$$

with some state $\omega \in \mathcal{D}(\mathcal{K})$ are Schrödinger quantum channels so we immediately get their dual channels $i_\omega^\ast = \text{tr}_\omega$ and $\text{tr}_\mathcal{K}^\ast = i_\mathcal{K}$ with

$$i_\mathcal{K} : B(\mathcal{H}) \to B(\mathcal{H} \otimes \mathcal{K}) \quad B \mapsto B \otimes \text{id}_\mathcal{K}.$$

This leads to the following result.

Corollary 2. For every $S \in Q_H(\mathcal{G}, \mathcal{H})$ there exists a separable Hilbert space $\mathcal{K}$, pure states $\omega_G \in \mathcal{D}(\mathcal{G})$ and $\omega_K \in \mathcal{D}(\mathcal{K})$ and a unitary $U \in B(\mathcal{H} \otimes \mathcal{G} \otimes \mathcal{K})$ such that

$$S(B) = (\text{tr}_{\omega_G} \circ \text{tr}_{\omega_K})(U^1(\text{id}_H \otimes B \otimes \text{id}_K)U)$$
for all $B \in \mathcal{B}(\mathcal{G})$. For $\mathcal{G} = \mathcal{H}$ this reduces to

$$S(B) = \text{tr}_{\omega_K}(U^\dagger B \otimes \text{id}_K U)$$

(10)

for all $B \in \mathcal{B}(\mathcal{H})$ where the unitary operator $U$ now acts on $\mathcal{H} \otimes \mathcal{K}$.

Proof. Note that (8) implies $(T_1 \circ T_2)^* = T_2^* \circ T_1^*$ for arbitrary positive, linear maps $T_1$ and $T_2$. Hence this is a simple consequence of Theorem 2, Theorem 3 (b) and Corollary 1.

Remark 5. The result in Corollary 2 is a more structured version of Stinespring’s theorem for Heisenberg quantum channels due to the following: Let $S \in Q_H(\mathcal{H})$ (the same argument works for $S \in Q_H(\mathcal{G}, \mathcal{H})$) and $\omega_K \in \mathbb{D}(\mathcal{K})$ be the state from (10) of rank one, i.e. $\omega_K = \langle y, \cdot \rangle y$ for some $y \in \mathcal{K}$ with $\|y\| = 1$. Defining the isometric embedding $V_y : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$, $x \mapsto x \otimes y$, one readily verifies via (9) that $\text{tr}_{\omega_K}(B) = V_y^\dagger B V_y$ for all $B \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Now (10) becomes

$$S(\cdot) = V^\dagger \pi(\cdot)V$$

with the auxiliary Hilbert space $\mathcal{H} \otimes \mathcal{K}$ being of tensor form, the Stinespring isometry $V = UV_y$ and the unital $*$-homomorphism $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ being $\pi(B) := B \otimes \text{id}_\mathcal{K}$. To the best of our knowledge, the above representation so far only appeared in an unpublished (as of now) book by S. Attal (Thm. 6.15).

The above concept of dual channels will be useful to transfer dilation results from the Schrödinger to the Heisenberg picture and vice versa so one is independent of the used quantum-mechanical framework.

III. MAIN RESULTS

A. Unitary Dilation of Discrete-Time Quantum-Dynamical Systems

Consider a discrete-time quantum-dynamical system, the evolution of which is described by

$$\rho_{n+1} = T(\rho_n), \quad \rho_0 \in \mathbb{D}(\mathcal{H})$$

(11)

for arbitrary but fix $T \in Q_S(\mathcal{H})$. Obviously, the explicit solution of (11) is given by

$$\rho_n = T^n(\rho_0)$$
for all $n \in \mathbb{N}_0$ By Theorem 1, one has $T^n \in Q_S(\mathcal{H})$ and thus Theorem 2 yields separable Hilbert spaces $\mathcal{K}_n$, pure states $\omega_n \in \mathcal{D}(\mathcal{K}_n)$ and unitaries $U_n \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}_n)$ such that

$$T^n(A) = \text{tr}_{\mathcal{K}_n}(U_n(A \otimes \omega_n)U_n^\dagger)$$

(12)

for all $A \in \mathcal{B}^1(\mathcal{H})$ and all $n \in \mathbb{N}_0$. Now our goal is to simplify the right-hand side of (12) in the following sense: We want to embed the evolution of $\rho_0$ into an evolution of a closed discrete-time quantum-dynamical system, i.e. we want to replace the r.h.s. of (12) by

$$\text{tr}_{\tilde{\mathcal{K}}}(V^n(A \otimes \tilde{\omega})(V^{\dagger})^n)$$

where $V$ is an appropriate unitary operator and the separable Hilbert space $\tilde{\mathcal{K}}$ as well as the pure state $\tilde{\omega}$ does no longer depend on $n \in \mathbb{N}_0$. Our established result reads as follows

**Theorem 4.** For every $T \in Q_S(\mathcal{H})$ there exists a separable Hilbert space $\mathcal{K}$, a pure state $\omega \in \mathcal{D}(\mathcal{K})$ and a unitary $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that $(\mathcal{H} \otimes \mathcal{K}, (V^n)_{n \in \mathbb{Z}}, \ell_\omega, \text{tr}_\mathcal{K})$ is a unitary dilation of $(T^n)_{n \in \mathbb{N}_0}$ (in the sense of Definition 21.2). In particular, for all $A \in \mathcal{B}^1(\mathcal{H})$ and $n \in \mathbb{N}_0$, one has

$$T^n(A) = \text{tr}_{\mathcal{K}}(V^n(A \otimes \omega)(V^{\dagger})^n) .$$

(13)

**Proof.** First we consider the $n$-dependence of $\mathcal{K}_n$ and $\omega_n$. By construction, cf. Theorem 2 $\mathcal{K}_n$ does not depend on $T^n$ anymore, thus we can choose $\tilde{\mathcal{K}}$ with a countably infinite basis, for example $\tilde{\mathcal{K}} = \ell_2(\mathbb{N})$, and replace every $\mathcal{K}_n$ with $\tilde{\mathcal{K}}$. Moreover, also by construction, the pure state $\omega_n$ is determined via $\mathcal{K}_n$ and thus can be chosen independently of $n$, too. Hence we obtain a joint Hilbert space $\tilde{\mathcal{K}}$ and a pure state $\tilde{\omega}$ such that

$$T^n(A) = \text{tr}_{\tilde{\mathcal{K}}}(U_n(A \otimes \tilde{\omega})U_n^\dagger) .$$

for all $n \in \mathbb{N}_0$. Finally, in order to remove the $n$-dependence of the unitary operators $U_n$ we define $\mathcal{K} := \tilde{\mathcal{K}} \otimes \ell_2(\mathbb{Z})$ and $U_n = \text{id}_{\mathcal{H} \otimes \tilde{\mathcal{K}}}$ for all $n \leq 0$. Furthermore, let $(e_n)_{n \in \mathbb{Z}}$ denote the standard basis of $\ell_2(\mathbb{Z})$ so $\sigma : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ given by $\sigma = \sum_{i \in \mathbb{Z}} e_i e_{i-1}^\dagger$ yields the right shift on $\ell_2(\mathbb{Z})$. With this $U, W : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ are defined by

$$U := \sum_{n \in \mathbb{Z}} U_n U_{n-1}^\dagger \otimes e_n e_n^\dagger \quad \text{and} \quad W := \text{id}_{\mathcal{H} \otimes \tilde{\mathcal{K}}} \otimes \sigma .$$
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Thus $U$ can be visualised as follows:

$$
\begin{pmatrix}
\vdots \\
\id_{\mathcal{H} \otimes \tilde{\mathcal{K}}} \\
U_1 \\
U_2 U_1^\dagger \\
U_3 U_2^\dagger \\
\vdots \\
\end{pmatrix}
\leftarrow
$$

where the arrows indicate the zero-zero entry of this both-sided “infinite matrix”. A simple calculation shows that $U$, $W$ and therefore also $V := UW$ are unitary. Next, using the results from Section II B, one readily verifies that the maps $E := \text{tr}_\mathcal{K}$ and $J := i_\omega$ (where $\omega := \tilde{\omega} \otimes e_0 e_0^\dagger \in \mathbb{D}(\mathcal{K})$ is obviously pure) satisfy the conditions from Definition 21.1. Then, by induction, one shows

$$
V^n (A \otimes \omega) (V^\dagger)^n = U_n (A \otimes \tilde{\omega}) U_n^\dagger \otimes e_n e_n^\dagger
$$

for all $A \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}_0$. Finally, $\id_{\tilde{\mathcal{K}} \otimes \ell_2} = \id_{\mathcal{K}} \otimes \id_{\ell_2}$ implies $\text{tr}_{\tilde{\mathcal{K}} \otimes \ell_2} = \text{tr}_{\mathcal{K}} \circ \text{tr}_{\ell_2}$ so

$$
\text{tr}_{\mathcal{K}} (V^n (A \otimes \omega) (V^\dagger)^n) = \text{tr}_{\mathcal{K}} (\text{tr}_{\ell_2} (U_n (A \otimes \tilde{\omega}) U_n^\dagger \otimes e_n e_n^\dagger)) = \text{tr}_{\mathcal{K}} (U_n (A \otimes \tilde{\omega}) U_n^\dagger) = T^n (A)
$$

for all $A \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}_0$. Hence we constructed a unitary dilation of $(T^n)_{n \in \mathbb{N}_0}$ of the form (13) which concludes the proof.

\[\square\]

Remark 6. Note that $i_\omega$ is trace-preserving because $\omega \in \mathbb{D}(\mathcal{K})$, so the above (tensor type) dilation is trace-preserving.

Now we can easily extend this result to Heisenberg quantum channels.

Corollary 3. For every $S \in Q_H(\mathcal{H})$ there exists a separable Hilbert space $\mathcal{K}$, a pure state $\omega \in \mathbb{D}(\mathcal{K})$ and a unitary $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that $(\mathcal{H} \otimes \mathcal{K}, ((V^\dagger)^n)_{n \in \mathbb{Z}}, i_\mathcal{K}, \text{tr}_\omega)$ is a unitary dilation of $(S^n)_{n \in \mathbb{N}_0}$ (in the sense of Definition 21.2). In particular, for all $B \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}_0$, one has

$$
S^n (B) = \text{tr}_\omega ( (V^\dagger)^n (B \otimes \id_{\mathcal{K}}) V^n ) .
$$
Proof. By Theorem 3 (b) there exists a unique \( T \in Q_S(\mathcal{H}) \) such that \( S = T^* \) and therefore \( S^n = (T^*)^n = (T^n)^* \). Now Theorem 4 yields a separable Hilbert space \( \mathcal{K} \), a pure state \( \omega \in \mathbb{D}(\mathcal{K}) \) and a unitary \( V \) such that (13) holds. By duality we obtain

\[
S^n(B) = \left( \text{tr}_\mathcal{K} (V^n i_\omega (\cdot)(V^\dagger)^n) \right)^*(B) = \text{tr}_\omega ((V^\dagger)^n (B \otimes \text{id}_\mathcal{K}) V^n)
\]

for all \( B \in \mathcal{B}(\mathcal{H}) \) and for all \( n \in \mathbb{N}_0 \).

Remark 7. 1. Due to \( \text{id}_H \otimes \text{id}_\mathcal{K} = \text{id}_{H \otimes \mathcal{K}} \) we even constructed a unital dilation (of tensor type).

2. Recall that a “classical” unitary dilation \( T^n = P_H \circ U^n \circ \text{inc}_H \) of some Hilbert space contraction \( T : \mathcal{H} \to \mathcal{H} \) (cf. Rem. 17.3), where \( P_H \) denotes the orthogonal projection onto \( \mathcal{H} \) and \( \text{inc}_H \) the inclusion map, is called minimal if the domain of \( U \in \mathcal{B}(\mathcal{K}) \) is minimal in the sense of

\[
\mathcal{K} = \bigvee_{n \in \mathbb{Z}} U^n \mathcal{H} .
\]

Here the right-hand side of (14) denotes the smallest closed subspace of \( \mathcal{K} \) which contains all images \( U^n \mathcal{H} , n \in \mathbb{Z} \), cf.\(^{31,32}\). Kümmere\(^{10}\) captures this idea and defines a dilation \( T^n(A) = E(\hat{T}^n(J(A))) \) of an ultraweakly continuous, completely positive and unital map \( T : \mathcal{A} \to \mathcal{A} \) on a \( W^*\)-algebra \( \mathcal{A} \) to be minimal if

\[
\mathcal{A} = \bigvee_{n \in \mathbb{Z}} \hat{T}^n(i(\mathcal{A}))
\]

holds, where the right-hand side of (15) now denotes the smallest closed \( W^*\)-algebra which contains all images \( \hat{T}^n(i(\mathcal{A})) , n \in \mathbb{Z} \), cf.\(^{10}\) (Def. 2.1.5). It is easy to see that our constructions in Theorem 3 / Corollary 3 do in general not lead to a minimal dilation in the above sense. However, one can always restrict a given dilation to the right-hand side of (15) to obtain a minimal one.

3. As seen above in (14) the space \( \mathcal{H}_{-\infty} := \bigvee_{n \in \mathbb{Z}} U^{-n} \mathcal{H} \) and its forward and backward invariant counterparts

\[
\mathcal{H}^\infty := \bigvee_{n \in \mathbb{N}_0} U^n \mathcal{H} , \quad \text{and} \quad \mathcal{H}_{-\infty} := \bigvee_{n \in \mathbb{N}_0} U^{-n} \mathcal{H} ,
\]

play an essential role in the theory of “classical” unitary dilations. In particular, they admit orthogonal decompositions

\[
\mathcal{H}^\infty = \mathcal{H} \oplus \hat{\mathcal{H}}^\infty , \quad \mathcal{H}_\infty = \mathcal{H} \oplus \hat{\mathcal{H}}_{-\infty} \quad \text{and} \quad \mathcal{H}_{-\infty} = \hat{\mathcal{H}}^\infty \oplus \mathcal{H} \oplus \hat{\mathcal{H}}_{-\infty} \quad (16)
\]
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such that \( \hat{\mathcal{H}}_\infty \) and \( \hat{\mathcal{H}}_{-\infty} \) are invariant under \( U \) and \( U^{-1} \), respectively, cf.\(^{32}\) (Lemma VI.3.1) and\(^{33}\). Eventually, (16) establishes the relation to Kümmerer’s notion of Markovianity, cf.\(^{10}\) (Prop. 2.2.3 (b)).

Next we want to improve Theorem 4 for cyclic \( T \), i.e. in the case of \( T^m = T \) for some \( m \in \mathbb{N} \setminus \{1\} \).

**Definition 8.** In doing so, we define a modified modulo function

\[
\nu : \mathbb{N} \setminus \{1\} \times \mathbb{N} \to \mathbb{N}
\]

\[
(m, n) \mapsto (n - 1) \mod (m - 1) + 1
\]

as well as

\[
\mu : \mathbb{N} \setminus \{1\} \times \mathbb{N} \to \mathbb{N}_0
\]

\[
(m, n) \mapsto \frac{n - \nu(m, n)}{m - 1}.
\]

To connect \( \nu(m, n) \) to the above cyclicity condition of \( T \) we represent \( n - 1 \) as

\[
n - 1 = j(m - 1) + r \tag{17}
\]

with unique \( j \in \mathbb{N}_0 \) and \( r \in \{0, \ldots, m - 2\} \). This yields \( \nu(m, n) = r + 1 \) as well as \( \mu(m, n) = j \in \mathbb{N}_0 \) and we obtain the following result.

**Lemma 1.** Let \( T \in Q_S(\mathcal{H}) \) be cyclic so \( T^m = T \) for \( m \in \mathbb{N} \setminus \{1\} \). Then

\[
T^n = T^{\nu(m, n)}
\]

for all \( n \in \mathbb{N} \).

**Proof.** Via (17) we get \( T^n = T^{j(m - 1) + r + 1} = T^{r + 1 - j(T^m)j} = T^{r + 1 - j} = T^{r + 1} = T^{\nu(m, n)} \). \( \Box \)

Thus \( \mu(m, n) \) indicates how often the cyclicity condition of \( T \) can be applied to reduce the exponent \( n \) to its remaining non-cyclic portion \( \nu(m, n) \). With this we obtain the following simplification of Theorem 4.

**Theorem 5.** Let \( T \in Q_S(\mathcal{H}) \) be cyclic, i.e. \( T^m = T \) for some \( m \in \mathbb{N} \setminus \{1\} \). Then for the unitary dilation \( (\mathcal{H} \otimes \mathcal{K}, (V^n)_{n \in \mathbb{Z}}, i_\omega, \text{tr}_\mathcal{K}) \) of \( (T^n)_{n \in \mathbb{N}_0} \) from Theorem 4, one can choose \( \mathcal{K} = \tilde{\mathcal{K}} \otimes \mathbb{C}^m \) such that (after modifying \( V \) and \( \omega \) accordingly)

\[
T^n(A) = \text{tr}_\mathcal{K} \left( V^{n + \mu(m, n)}(A \otimes \omega)(V^\dagger)^{n + \mu(m, n)} \right)
\]

for all \( A \in \mathcal{B}^1(\mathcal{H}) \) and all \( n \in \mathbb{N}_0 \). Note that \( \omega \in \mathbb{D}(\mathcal{K}) \) still is a pure state.
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Proof. Choose $\tilde{K}$ and $\tilde{\omega} \in \mathcal{D}(\tilde{K})$ as in the proof of Theorem 4. For every $T, \ldots, T^{m-1}$ there again exist unitary $U_1, \ldots, U_{m-1} \in \mathcal{B}(\mathcal{H} \otimes \tilde{K})$ satisfying Theorem 2. This allows to define

$$ U := \sum_{i=1}^{m} U_i U^\dagger_{i-1} \otimes e_i e_i^\dagger \quad \text{and} \quad W := \text{id}_{\mathcal{H} \otimes \tilde{K}} \otimes \sum_{i=1}^{m} e_{i+1} e_i^\dagger $$

where $e_{m+1} := e_1$ and $U_0 := \text{id}_{\mathcal{H} \otimes \tilde{K}} =: U_m$. Then $W$ represents a cyclic shift acting on $\mathbb{C}^m$ and $U$ is of the following form.

$$ U = \begin{pmatrix} U_1 \\ U_2 U_1^\dagger \\ \vdots \\ U_{m-1} U_{m-2}^\dagger \\ U_{m-1}^\dagger \end{pmatrix} $$

Obviously, $U$, $W$ and thus $V := UW$ are unitary. Again choosing $E := \text{tr}_K$ and $J := i\omega$ with pure state $\omega := \tilde{\omega} \otimes e_m e_m^\dagger \in \mathcal{D}(K)$, one readily verifies via induction

$$ V^{n+\mu(m,n)}(A \otimes \omega)(V^\dagger)^{n+\mu(m,n)} = U_{\nu(m,n)}(A \otimes \omega) U_{\nu(m,n)}^\dagger \otimes e_{\nu(m,n)} e_{\nu(m,n)}^\dagger $$

for all $A \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Together with Lemma 1 one gets

$$ \text{tr}_K(V^{n+\mu(m,n)}(A \otimes \omega)(V^\dagger)^{n+\mu(m,n)}) = \text{tr}_K \circ \text{tr}_{\mathbb{C}^m}(U_{\nu(m,n)}(A \otimes \tilde{\omega}) U_{\nu(m,n)}^\dagger \otimes e_{\nu(m,n)} e_{\nu(m,n)}^\dagger) $$

$$ = \text{tr}_K(U_{\nu(m,n)}(A \otimes \tilde{\omega}) U_{\nu(m,n)}^\dagger) = T^{\nu(m,n)}(A) = T^n(A) $$

for all $A \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. \hfill \qed

Remark 9. Note that quantum channels which have an inverse channel (or are “just” bijective with positive inverse) can be written as a unitary conjugation $\text{Ad}_U$, cf. Prop. 1. For such channels, Theorem 4 is trivially fulfilled by choosing $K = \mathbb{C}$, $E = J = \text{id}_{\mathcal{B}(\mathcal{H})}$ and $V = U$. The same holds for cyclic quantum channels which are bijective because cyclicity implies $T^{-1} = T^{m-2} \in Q_S(\mathcal{H})$.

B. Unitary Dilation of Discrete-Time Quantum-Control Systems

Here, we investigate discrete-time quantum-mechanical control systems of the form

$$ \rho_{n+1} = T_n(\rho_n), \quad \rho_0 \in \mathcal{D}(\mathcal{H}) \quad (18) $$
where \( T_n, n \in \mathbb{N}_0 \) is regarded as control input which can be chosen freely from some subset \( \mathcal{C} \subset Q_S(\mathcal{H}) \). We define \( \rho(\cdot,(T_n)_{n\in\mathbb{N}_0},\rho_0) \) to be the unique solution of (18) generated by the control sequence \( (T_n)_{n\in\mathbb{N}_0} \) and the initial value \( \rho_0 \). In the sequel, we are interested in whether the dynamics of (18) can be embedded in the dynamics of a unitary discrete-time quantum control system of the same form.

**Definition 10.** Let \( R_N(\rho_0) \) denote the set of all states which can be reached from \( \rho_0 \) in \( N \) time steps via (18), i.e.

\[
R_N(\rho_0) := \{\rho(N,(T_n)_{n\in\mathbb{N}_0},\rho_0) \mid (T_n)_{n\in\mathbb{N}_0} \text{ arbitrary control sequence}\}.
\]

Moreover, the overall reachable set of \( \rho_0 \) is defined by

\[
R(\rho_0) := \bigcup_{N\in\mathbb{N}_0} R_N(\rho_0).
\]

For the remaining section, we assume \( \mathcal{C} := \{T,S\} \) where \( T \) and \( S \) are commuting but otherwise arbitrary quantum channels over \( \mathcal{H} \). Then the following result is a direct consequence of the fact that \( T \) and \( S \) commute.

**Lemma 2.** For all \( N \in \mathbb{N}_0 \) one has \( R_N(\rho_0) := \{T^kS^{N-k}\rho_0 \mid k = 0,\ldots,N\} \).

Based on this we are interested in dilations of quantum channels of the form \( T^kS^{N-k} \).

**Theorem 6.** Let \( T,S \in Q_S(\mathcal{H}) \) be commuting. Then there exists a separable Hilbert space \( \mathcal{K} \), a pure state \( \omega \in \mathcal{D}(\mathcal{K}) \) and unitary \( U,V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \) such that

\[
T^kS^{N-k}(A) = \text{tr}_\mathcal{K} \left( U^kV^{N-k}(A \otimes \omega)(V^\dagger)^{N-k}(U^\dagger)^k \right).
\]

for all \( A \in \mathcal{B}^1(\mathcal{H}) \), \( N \in \mathbb{N}_0 \) and \( k = 0,\ldots,N \).

**Proof.** For fixed \( N \in \mathbb{N} \) and \( k = 0,\ldots,N \), one has \( T^kS^{N-k} \in Q_S(\mathcal{H}) \) by Theorem 4 and thus Theorem 2 yields a separable Hilbert space \( \mathcal{K}_{N,k} \), a pure state \( \omega_{N,k} \in \mathcal{D}(\mathcal{K}_{N,k}) \) and unitary \( U_{N,k} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \) such that

\[
T^kS^{N-k}(A) = \text{tr}_{\mathcal{K}_{N,k}} \left( U_{N,k}(A \otimes \omega_{N,k})U_{N,k}^\dagger \right).
\]

for all \( A \in \mathcal{B}^1(\mathcal{H}) \). The same line of arguments as in the proof of Theorem 4 show that \( \mathcal{K}_{N,k} \) and \( \omega_{N,k} \) can be chosen independently of \( N \) and \( k \), so there exists some mutual auxiliary space \( \tilde{\mathcal{K}} \) as well as a mutual pure state \( \tilde{\omega} \in \mathcal{D}(\tilde{\mathcal{K}}) \) such that

\[
T^kS^{N-k}(A) = \text{tr}_{\tilde{\mathcal{K}}} \left( U_{N,k}(A \otimes \tilde{\omega})U_{N,k}^\dagger \right). \quad (19)
\]
for all $A \in \mathcal{B}^1(\mathcal{H})$, $N \in \mathbb{N}$ and $k = 0, \ldots, N$. In particular, to every $T^kS^{N-k}$ we can assign some unitary $U_{N,k} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that (19) holds. Now, choose $\mathcal{K} := \mathcal{K} \otimes \ell_2(\mathbb{Z}) \otimes \ell_2(\mathbb{Z})$ and again $E := \text{tr}_\mathcal{K}$ and $J := i_\omega$ with pure state $\omega := \tilde{\omega} \otimes e_0 \tilde{e}_0^\dagger \in \mathbb{D}(\mathcal{K})$. Moreover, by means of the right shift $\sigma$ from the proof of Theorem 4 one defines

$$W_1 := \text{id}_{\mathcal{H} \otimes \tilde{\mathcal{K}}} \otimes \sigma \otimes \sigma, \quad U_1 := \sum_{m,n \in \mathbb{Z}} U_{m,n}U_{m-1,n-1}^\dagger \otimes e_m e_m^\dagger \otimes e_n e_n^\dagger,$$

$$W_2 := \text{id}_{\mathcal{H} \otimes \tilde{\mathcal{K}}} \otimes \sigma \otimes \text{id}_{\ell_2}, \quad U_2 := \sum_{n \in \mathbb{Z}} U_{n,0}U_{n-1,0}^\dagger \otimes e_n e_n^\dagger \otimes \text{id}_{\ell_2},$$

where $U_{m,n} := \text{id}_{\mathcal{H} \otimes \tilde{\mathcal{K}}}$ if $m < 1$ or $n \notin \{0, \ldots, m\}$. Obviously, $W_1$ and $W_2$ are unitary. The unitarity of $U_1$, and $U_2$ is readily verified via the unitarity of $U_{N,k}$ so $U := U_1W_1$ and $V := U_2W_2$ are unitary, too. As before, by induction one shows

$$V^j(A \otimes \omega)(V^\dagger)^j = U_{j,0}(A \otimes \tilde{\omega})U_{j,0}^\dagger \otimes e_j e_j^\dagger \otimes e_0 e_0^\dagger \quad (20)$$

for all $A \in \mathcal{B}^1(\mathcal{H})$ and $j \in \mathbb{N}_0$ and based on this

$$U^kV^{N-k}(A \otimes \omega)(V^\dagger)^{N-k}(U^\dagger)^k = U_{N,k}(A \otimes \tilde{\omega})U_{N,k}^\dagger \otimes e_N e_N^\dagger \otimes e_k e_k^\dagger \quad (21)$$

for all $A \in \mathcal{B}^1(\mathcal{H})$, $N \in \mathbb{N}_0$ and $k = 0, \ldots, N$. Note that the case $k = 0$ reproduces (20) and thus can be omitted. Finally, (19) and (21) imply

$$\text{tr}_\mathcal{K} \left( U^kV^{N-k}(A \otimes \omega)(V^\dagger)^{N-k}(U^\dagger)^k \right) = \text{tr}_\mathcal{K} \left( \text{tr}_{\ell_2 \otimes \ell_2}(U_{N,k}(A \otimes \tilde{\omega})U_{N,k}^\dagger \otimes e_N e_N^\dagger \otimes e_k e_k^\dagger) \right)$$

$$= \text{tr}_\mathcal{K}(U_{N,k}(A \otimes \tilde{\omega})U_{N,k}^\dagger) = T^kS^{N-k}(A)$$

for all $A \in \mathcal{B}^1(\mathcal{H})$, $N \in \mathbb{N}$ and $k = 0, \ldots, N$ which concludes this proof. \hfill \Box

**Remark 11.** The statement of Theorem 4 can be extended to finitely many commuting channels $T_1, \ldots, T_m \in QS(\mathcal{H})$. Obviously, it is natural to choose

$$\mathcal{K} = \mathcal{K} \otimes \underbrace{\ell_2(\mathbb{Z}) \otimes \ldots \otimes \ell_2(\mathbb{Z})}_{\text{m-times}}$$

as common auxiliary space. The rest of the proof is completely analogous.

We can now transfer the above result to obtain a characterization of the reachable set of the control system (18).
Corollary 4. There exists a separable Hilbert space \( K \), a pure state \( \omega \in \mathbb{D}(K) \) and unitary \( U, V \in \mathcal{B}(\mathcal{H} \otimes K) \) such that

\[
\rho(N, (T_n)_{n \in \mathbb{N}_0}, \rho_0) = \text{tr}_K \left( U^k V^{N-k} (\rho_0 \otimes \omega)(V^\dagger)^{N-k}(U^\dagger)^k \right)
\]

for all controls \((T_n)_{n \in \mathbb{N}_0}\), initial states \( \rho_0 \in \mathbb{D}(\mathcal{H}) \) and \( N \in \mathbb{N}_0 \), where \( k = k(N, (T_n)_{n \in \mathbb{N}_0}) \in \{0, \ldots, N\} \) counts how often \( T \) occurs in the control sequence \((T_n)_{n \in \mathbb{N}_0}\) during the first \( N \) time steps.

Proof. By Definition 10, \( \rho(N, (T_n)_{n \in \mathbb{N}_0}, \rho_0) \in \mathbb{R}^N(\rho_0) \) and hence by Lemma 2 there exists \( k \in \{0, \ldots, N\} \) such that \( \rho(N, (T_n)_{n \in \mathbb{N}_0}, \rho_0) = T^k S^{N-k}(\rho_0) \). Thus the result follows immediately from Theorem 6. \( \square \)

Corollary 5. Let \( K, \omega \in \mathbb{D}(K) \) and \( U, V \in \mathcal{B}(\mathcal{H} \otimes K) \) be as in Corollary 4. Then, for all \( N \in \mathbb{N}_0 \) and \( \rho_0 \in \mathbb{D}(\mathcal{H}) \) one has

\[
\mathbb{R}_N(\rho_0) \subseteq \text{tr}_K(\tilde{\mathbb{R}}_N(\rho_0 \otimes \omega))
\]

and thus \( R(\rho_0) \subseteq \text{tr}_K(\tilde{\mathbb{R}}(\rho_0 \otimes \omega)) \). Here, \( \tilde{\mathbb{R}}(\rho_0) \) and \( \tilde{\mathbb{R}}_N(\rho_0) \) denote the reachable sets of the discrete-time closed quantum control system

\[
\tilde{\rho}_{n+1} = U_n \tilde{\rho}_n U_n^\dagger, \quad \tilde{\rho}_0 \in \mathbb{D}(\mathcal{H} \otimes K)
\]

with \( U_n \in \{U, V\} \) for all \( n \in \mathbb{N}_0 \).

Proof. By Lemma 2 and Theorem 6 one has

\[
\mathbb{R}_N(\rho_0) = \{T^k S^{N-k}(\rho_0) \mid k = 0, \ldots, N\}
\]

\[
= \{\text{tr}_K(U^k V^{N-k} (\rho_0 \otimes \omega)(V^\dagger)^{N-k}(U^\dagger)^k) \mid k = 0, \ldots, N\}
\]

\[
= \text{tr}_K(\{U^k V^{N-k} (\rho_0 \otimes \omega)(V^\dagger)^{N-k}(U^\dagger)^k \mid k = 0, \ldots, N\}) \subseteq \text{tr}_K(\tilde{\mathbb{R}}_N(\rho_0 \otimes \omega)). \quad \square
\]

Remark 12. 1. Note that the unitary channels \( U \) and \( V \) of Corollary 5 do in general not commute, so (22) states a proper inclusion rather than an equality for \( N > 1 \).

2. Consider the dual problem of (18), i.e. let \( T, S \in Q_H(\mathcal{H}) \) be two commuting Heisenberg channels. Of course, one can translate the above results—which we will omit here—into the Heisenberg picture via Corollary 3. However, we want to comment on the result of
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Davies\(^5\) which was already mentioned in the introduction and yields a unitary dilation with \textit{commuting} unitary channels, at the cost of our desired partial trace structure.

Let \(G = (\mathbb{Z} \times \mathbb{Z}, +)\) with subgroup \(S := \{(N, k) \in G \mid N \in \mathbb{N}_0 \text{ and } 0 \leq k \leq N\}\) and define the family \((T_g)_{g \in G}\) of Heisenberg channels via \(T_g := T^k S^{N-k}\) for \(g = (N, k) \in S\) and \(T_g := \text{id}_{\mathcal{B}(\mathcal{H})}\) otherwise. Adjusting the proof of\(^5\) (Thm. 3.1) to discrete groups and using Corollary \(^2\) one gets a Hilbert space \(\mathcal{K}\), a unitary representation \(U\) of \(G\) on \(\mathcal{H} \otimes \mathcal{K}\) and a conditional expectation \(E\) such that \(T_g(B) = E(U_g(B \otimes \text{id}_\mathcal{K})U_g^\dagger)\) for all \(B \in \mathcal{B}(\mathcal{H})\). As \((U_g)_{g \in G}\) is a representation of \(G\) we may consider the commuting unitary operators \(U_{(1,1)} =: U\), \(U_{(1,0)} =: V\) resulting in

\[
T^k S^{N-k}(B) = T_g(B) = E(U^k V^{N-k}(B \otimes \text{id}_\mathcal{K})(V^\dagger)^{N-k}(U^\dagger)^k)
\]

for all \(B \in \mathcal{B}(\mathcal{H})\), \(N \in \mathbb{N}_0\) and \(k = 0, \ldots, N\). Observe that we did not use the fact that \(T, S\) commute so this result even holds for arbitrary channels \(T\) and \(S\) with the drawback of \(E\) lacking any partial trace structure, see also\(^10\).

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Appendix A: Topological Properties of \(Q_S(\mathcal{H})\)

For the following definition, we refer to\(^35\) (Ch.VI.1).

**Definition 13.** Let \(\mathcal{X}\) and \(\mathcal{Y}\) be arbitrary Banach spaces.

(a) The strong operator topology (s.o.t.) on \(\mathcal{B}(\mathcal{X}, \mathcal{Y})\) is the locally convex topology induced by the family of seminorms of the form \(T \rightarrow \|Tx\|\) with \(x \in \mathcal{X}\).

(b) The weak operator topology (w.o.t.) on \(\mathcal{B}(\mathcal{X}, \mathcal{Y})\) is the locally convex topology induced by the family of seminorms of the form \(T \rightarrow |y(Tx)|\) with \((x, y) \in \mathcal{X} \times \mathcal{Y}'\).
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Note that both topologies, the s.o.t. as well as the w.o.t., are Hausdorff so limits are unique.

By the natural isomorphism $(\mathcal{B}^1(\mathcal{H}))' \cong \mathcal{B}(\mathcal{H})$, see Section [1B], one has the following equivalence: A net $(T_{\alpha})_{\alpha \in I}$ in $\mathcal{B}(\mathcal{B}^1(\mathcal{H}))$ converges to $T \in \mathcal{B}(\mathcal{B}^1(\mathcal{H}))$ in w.o.t. if and only if

$$\lim_{\alpha \in I} |\text{tr}(BT_{\alpha}(A)) - \text{tr}(BT(A))| = 0$$

for all $A \in \mathcal{B}^1(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$.

**Remark 14** (Metrizability of s.o.t. and w.o.t. on bounded subsets). At this point one might ask whether the strong or weak operator topology is metrizable. If this is the case, closed and sequentially closed sets do coincide which, of course, is of interest for further investigations. The following is well known in the literature, cf. [36, Thm. 1.2 and 1.13]: If $\mathcal{X}$ is separable, then the s.o.t. is metrizable on bounded subsets of $\mathcal{B}(\mathcal{X})$. If $\mathcal{X}'$ is also separable, then the w.o.t. is metrizable on bounded subsets of $\mathcal{B}(\mathcal{X})$.

Now, recall that $\mathcal{H}$ is assumed to be separable. Therefore it is evident that the subspace of finite-rank operators $\mathcal{F}(\mathcal{H})$ and hence $\mathcal{B}^1(\mathcal{H})$ itself, which is the $\nu_1$-closure of $\mathcal{F}(\mathcal{H})$ (cf. [35, Lemma XI.9.11]), is separable. Moreover, we already know from Proposition [2] that $Q_S(\mathcal{H})$ is a subset of the unit ball in $\mathcal{B}(\mathcal{B}^1(\mathcal{H}))$. This implies that the s.o.t. on $Q_S(\mathcal{H})$ is metrizable and thus convergence, closedness, continuity, etc. can be fully characterized by sequences. On the other hand, it is also well known that $\mathcal{B}(\mathcal{H})$ is not separable with respect to the operator norm topology as the non-separable space $\ell^\infty$ can be isometrically embedded into $\mathcal{B}(\mathcal{H})$. Hence $(\mathcal{B}^1(\mathcal{H}))'$ is not separable and the above metrizability result does not apply to the w.o.t. on $Q_S(\mathcal{H})$.

However, one could make use of the result that for convex sets in $\mathcal{B}(\mathcal{B}^1(\mathcal{H}))$, the closures with respect to the w.o.t. and the s.o.t coincide, cf. [35, Coro. VI.1.6]. Therefore, in the proof of Theorem [1] one could focus on the s.o.t. On the other hand, Lemma [4] ff. show that a direct approach via the w.o.t. is just as simple.

For clarity of the proof of Theorem [1] we first state some auxiliary results.

**Lemma 3.** For every linear map $S : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{G})$ the following statements are equivalent.

(a) $S$ is positive.

(b) For all $A \in \mathcal{B}^1(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{G})$ with $A, B \geq 0$, one has $\text{tr}(BS(A)) \geq 0$. 

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Proof. (a) \(\Rightarrow\) (b): For \(A, B \geq 0\) and \(S\) positive, we obtain \(S(A) \geq 0\) and thus
\[
\text{tr}(BS(A)) = \text{tr}(\sqrt{B}S(A)\sqrt{B}) \geq 0,
\]
where \(\sqrt{B} \geq 0\) denotes the unique square root of \(B\).

(b) \(\Rightarrow\) (a): Choosing \(B := \langle x, \cdot \rangle x\) for arbitrary \(x \in \mathcal{G}\) yields \(B \geq 0\) and
\[
\langle x, S(A)x \rangle = \text{tr}(BS(A)) \geq 0,
\]
for all \(A \geq 0\). Hence it follows \(S(A) \geq 0\) so \(S\) is positive. \(\square\)

Lemma 4. Let \((T_\alpha)_{\alpha \in I}\) be a net in \(\mathcal{B}(\mathcal{B}^1(\mathcal{H}))\) which converges to \(T \in \mathcal{B}(\mathcal{B}^1(\mathcal{H}))\) in w.o.t. Then the following statements hold.

(a) If \(T_\alpha\) is trace-preserving for all \(\alpha \in I\) then \(T\) is trace-preserving.

(b) If \(T_\alpha\) is positive for all \(\alpha \in I\) then \(T\) is positive.

Proof. Both statements follow from (A1): (a) by choosing \(B = \text{id}_\mathcal{H}\) and (b) by applying Lemma 3 and taking into account that \([0, \infty)\) is a closed subset of \(\mathbb{R}\). \(\square\)

For the proof of our next result we recall that \(\mathcal{B}^1(\mathcal{H} \otimes \mathbb{C}^m)\) and \(\mathcal{B}^1(\mathcal{H}) \otimes \mathbb{C}^{m \times m}\) can be identified as follows. Any \(A \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^m)\) can be represented as \(A = \sum_{i,j=1}^m A_{ij} \otimes E_{ij}\) with the standard basis \((E_{ij})_{i,j=1}^m\) of \(\mathbb{C}^{m \times m}\) and appropriate \(A_{ij} \in \mathcal{B}(\mathcal{H})\). Then, the following statements are equivalent\(^{24}\) (p. 33-34).

(a) \(A \in \mathcal{B}^1(\mathcal{H} \otimes \mathbb{C}^m)\)

(b) \(A_{ij} \in \mathcal{B}^1(\mathcal{H})\) for all \(i, j \in \{1, \ldots, m\}\)

Lemma 5. Let \((T_\alpha)_{\alpha \in I}\) be a net in \(\mathcal{B}(\mathcal{B}^1(\mathcal{H}))\) converging to \(T \in \mathcal{B}(\mathcal{B}^1(\mathcal{H}))\) in w.o.t. Then, for all \(m \in \mathbb{N}\), the net \((T_\alpha \otimes \text{id}_m)_{\alpha \in I}\) converges to \(T \otimes \text{id}_m \in \mathcal{B}(\mathcal{B}^1(\mathcal{H} \otimes \mathbb{C}^m))\) in w.o.t.

Proof. According to (A1) we have to show
\[
\lim_{\alpha \in I} |\text{tr}(B(T_\alpha \otimes \text{id}_m - T \otimes \text{id}_m)A)| = 0 \quad \text{(A2)}
\]
for all \(A \in \mathcal{B}^1(\mathcal{H} \otimes \mathbb{C}^m)\) and \(B \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^m)\). As seen above every \(A \in \mathcal{B}^1(\mathcal{H} \otimes \mathbb{C}^m)\) and \(B \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^m)\) can be represented as finite linear combinations of elements \(A_{ij} \otimes E_{ij} \in \mathcal{B}^1(\mathcal{H}) \otimes \mathbb{C}^{m \times m}\) and \(B_{ij} \otimes E_{ij} \in \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^{m \times m}\), respectively, with \(i, j = 1, \ldots, m\). Hence
\[
\text{tr}(B(T_\alpha \otimes \text{id}_m - T \otimes \text{id}_m)A) = \text{tr}(B((T_\alpha - T) \otimes \text{id}_m)A) = \sum_{i,j=1}^m \text{tr}(B_{ij}(T_\alpha - T)(A_{ji}))
\]
so convergence of \((T_\alpha \otimes \text{id}_m)_{\alpha \in I}\) can easily be related to the convergence of \((T_\alpha)_{\alpha \in I}\). \(\square\)
Proof of Theorem 1. Since every linear and positive operator on $B^1(\mathcal{H})$ is naturally norm bounded as a simple consequence of (Ch. 2, Lemma 2.1), the set $Q_S(\mathcal{H})$ of all Schrödinger channels is a bounded subset of $B(B^1(\mathcal{H}))$. Now it is readily verified that $Q_S(\mathcal{H})$ is a convex subsemigroup of $B(B^1(\mathcal{H}))$, cf. (Ch. 4.3). Next consider a net $(T_\alpha)_{\alpha \in I}$ in $Q_S(\mathcal{H})$ converging to some $T \in B(B^1(\mathcal{H}))$ in w.o.t. By Lemma 4 (a), the map $T$ is trace-preserving and by Lemma 5 $(T_\alpha \otimes \text{id}_m)_{\alpha \in I}$ converges to $T \otimes \text{id}_m$ with respect to the w.o.t. Then applying Lemma 4 (b) to the net $(T_\alpha \otimes \text{id}_m)_{\alpha \in I}$ yields that $T$ is also $m$-positive for all $m \in \mathbb{N}$. Hence $T$ is a Schrödinger quantum channel and $Q_S(\mathcal{H})$ is closed in $B(B^1(\mathcal{H}))$ with respect to the w.o.t. The well-known fact that the w.o.t. is weaker than the s.o.t. and the uniform operator topology concludes the proof.

Remark 15. Note that in the above proof we did not explicitly use the fact that domain and range of the operator $T$ coincides. Therefore, the convexity and closedness results trivially extend to $Q_S(\mathcal{H}, \mathcal{G})$.

Appendix B: Glossary on Dilations

For the sake of self-containedness, we recall some basic terminology concerning different types of dilations of linear contractions. Let us start with the Banach space case.

Definition 16. Let $\mathcal{X}$ be an arbitrary Banach space.

1. Let $T : \mathcal{X} \to \mathcal{X}$ be a linear contraction, i.e. $\|T\| \leq 1$. A dilation $(\mathcal{Y}, \hat{T}, J, E)$ of $T$ consists of a Banach space $\mathcal{Y}$ and a triple of maps $(\hat{T}, J, E)$ with

$$ T = E \circ \hat{T} \circ J \quad \text{and} \quad E \circ J = \text{id}_\mathcal{X}, \quad \text{(B1)} $$

where the linear maps $\hat{T}$, $J$ and $E$ satisfy:

(a) $\hat{T} : \mathcal{Y} \to \mathcal{Y}$ is a bi-isometry (i.e. $\hat{T}$ is bijective and $\hat{T}, \hat{T}^{-1}$ are isometries)

(b) $J : \mathcal{X} \to \mathcal{Y}$ is an isometric embedding of $\mathcal{X}$ in $\mathcal{Y}$.

(c) $E : \mathcal{Y} \to \mathcal{X}$ has operator norm $\|E\| = 1$. 

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2. Let $S \subset G$ be a semigroup of a group $G$ and $(T_g)_{g \in S}$ be a representation of $S$ with values in the contraction semigroup of $X$. A dilation $(\mathcal{Y}, (\hat{T}_g)_{g \in G_0}, J, E)$ of $(T_g)_{g \in S}$ consists of a Banach space $\mathcal{Y}$, a subgroup $G_0 \subset G$ and a triple $((\hat{T}_g)_{g \in G_0}, J, E)$ with

$$T_g = E \circ \hat{T}_g \circ J \quad \text{and} \quad E \circ J = \text{id}_X$$

for all $g \in S \subset G_0$, where $(\hat{T}_g)_{g \in G_0}$ is a linear representation of $G_0$ with values in the isometry group of $\mathcal{Y}$ and $J, E$ as before.

Remark 17. 1. Note that $E \circ J = \text{id}_X$ implies that $E$ is onto and $J$ is injective. Furthermore, $J \circ E : \mathcal{Y} \to \mathcal{Y}$ is a projection of norm 1 from $\mathcal{Y}$ onto the range of $J$.

2. If $S$ is assumed to be abelian and there exists a “dilation” of $(T_g)_{g \in S}$ such that $\hat{T}_g$ is well-defined for all $g \in S$ then $(\hat{T}_g)_{g \in S}$ obviously extends to a proper dilation in the above sense, where $G_0$ can be chosen to be the subgroup generated by $S$. For non-abelian $S$, however, this extension property is not obvious.

3. Choosing $S := \mathbb{N}_0$ and $T_n := T^n$ in Def. 16.2, where $T : \mathcal{H} \to \mathcal{H}$ is a linear contraction, we recover the “classical” concept of a linear dilation (see also Rem. 17.2). To distinguish such a dilation of $T$—which in our sense is actually a dilation of $(T^n)_{n \in \mathbb{N}_0}$—from a dilation of $T$ in the sense of Def. 16.1 one sometimes calls the latter a “dilation of first order”, cf. 10.

4. Once continuity comes into play, things become more subtle as one can either require that the continuity properties of $g \mapsto T_g$ are preserved by $g \mapsto \hat{T}_g$ (which in some cases is unfeasible) or allow that the continuity is relaxed, cf., e.g. 9 (Rem. 17.5). For our applications, however, this is not an issue as we are only concerned with the case $S = \mathbb{N}_0$.

In the context of quantum channels (or, more generally, completely positive maps) various specializations of the above definitions to

- abstract $C^*$- or $W^*$-algebras
- Heisenberg quantum channels
- and Schrödinger quantum channels
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are available in the literature. For more details we refer to\textsuperscript{14}.

**Definition 18.** Let $\mathcal{A}$ be a unital $C^*$-algebra.

1. Let $T : \mathcal{A} \to \mathcal{A}$ be linear, completely positive and unital (i.e. identity preserving).
   A dilation $(\mathfrak{A}, \hat{T}, J, E)$ of $T$ consists of a unital $C^*$-algebra $\mathfrak{A}$ and a triple of maps $(\hat{T}, J, E)$ with
   \[
   T = E \circ \hat{T} \circ J \quad \text{and} \quad E \circ J = \text{id}_A,
   \]  
   (B2)
   where $\hat{T}$, $J$ and $E$ satisfy:
   \begin{enumerate}
   \item[(a)] $\hat{T} : \mathfrak{A} \to \mathfrak{A}$ is a $*$-automorphism.
   \item[(b)] $J : \mathcal{A} \to \mathcal{A}$ is a $*$-homomorphism of $\mathcal{A}$ into $\mathfrak{A}$.
   \item[(c)] $E : \mathfrak{A} \to \mathcal{A}$ is linear and completely positive with operator norm $\|E\| = 1$.
   \end{enumerate}

2. Let $S \subset G$ be a semigroup of a group $G$ and let $(T_g)_{g \in S}$ be a semigroup representation of $S$ with values in the set of completely positive, unital maps on $\mathcal{A}$. A dilation $(\mathfrak{A}, (\hat{T}_g)_{g \in G_0}, J, E)$ of $(T_g)_{g \in S}$ consists of a unital $C^*$-algebra $\mathfrak{A}$, a subgroup $G_0 \subset G$ and a triple $((\hat{T}_g)_{g \in G_0}, J, E)$ with
   \[
   T_g = E \circ \hat{T}_g \circ J \quad \text{and} \quad E \circ J = \text{id}_A,
   \]
   for all $g \in S \subset G_0$, where $(\hat{T}_g)_{g \in G_0}$ is a representation of $G_0$ with values in the $*$-automorphism group of $\mathfrak{A}$ and $J$, $E$ as before.

If, in addition, $J(\text{id}_A) = \text{id}_A$ then the dilation is said to be unital. On the other hand, if $\mathcal{A}$ is even a $W^*$-algebra, then all involved maps are in general assumed to be ultraweakly continuous.

**Remark 19.** 1. Let $\mathcal{A}$, $\mathcal{B}$ be unital $C^*$-algebras and let $T : \mathcal{A} \to \mathcal{B}$ be unital. Then positivity of $T$ is equivalent to the norm condition $\|T\| = 1$, cf.\textsuperscript{23,37}. In particular, one has $\|T\| = \|T(\text{id}_A)\|$ for every positive map $T : \mathcal{A} \to \mathcal{B}$ so unitality of $T$ implies that $T$ is a contraction.

2. As every $*$-homomorphism is trivially completely positive and every injective $*$-homomorphism is always isometric, (B2) yields a dilation in the sense of (B1). Moreover, if a dilation is unital, then $E$ is unital as well because (B2) implies $\text{id}_A = (E \circ J)(\text{id}_A) = E(\text{id}_A)$. 

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3. Every \( \ast \)-automorphism \( \hat{T} : \mathcal{A} \to \mathcal{A} \) on a unital \( C^* \)-algebra \( \mathcal{A} \) is unital itself because of
\[
\hat{T}(\text{id}_\mathcal{A}) = \hat{T}(\text{id}_\mathcal{A}) \text{id}_\mathcal{A} = \hat{T}(\text{id}_\mathcal{A}) \hat{T}^{-1}(\text{id}_\mathcal{A}) = \text{id}_\mathcal{A}.
\]

4. Let \( \mathcal{A} \) be a \( C^* \)-subalgebra of a unital \( C^* \)-algebra \( \mathcal{A} \), i.e. \( \text{id}_\mathcal{A} \in \mathcal{A} \subseteq \mathcal{A} \). Then a linear map \( E : \mathcal{A} \to \mathcal{A} \) is said to be a conditional expectation (of \( \mathcal{A} \) onto \( \mathcal{A} \)) if it is completely positive with norm \( \| E \| = 1 \) and satisfies
\[
E(AB) = AE(B) \quad \text{for all } A \in \mathcal{A} \text{ and } B \in \mathcal{A}.
\]

Obviously, \((B3)\) implies that \( E \) is a unital (cf. Rem. 19.1) projection onto \( \mathcal{A} \), that is \( E(A) = A \) for all \( A \in \mathcal{A} \). The converse is also true, i.e. every projection \( E : \mathcal{A} \to \mathcal{A} \) of norm \( \| E \| = 1 \) is a conditional expectation, cf. \( \text{Thm. II.6.10.2} \) and \( \text{38} \). Moreover, exploiting that \( E(B^*) = E(B)^* \) for all \( B \in \mathcal{A} \), which results from the (complete) positivity of \( E \), one can easily show that \((B3)\) is equivalent to
\[
E(BA) = E(B)A \quad \text{for all } A \in \mathcal{A} \text{ and } B \in \mathcal{A}
\]
and, since \( \mathcal{A} \) is unital, also to
\[
E(A_1BA_2) = A_1E(B)A_2 \quad \text{for all } A_1, A_2 \in \mathcal{A} \text{ and } B \in \mathcal{A}.
\]

In the literature, \((B3)\) is often replaced by the “more symmetric” condition \((B4)\). Now if \( \mathcal{A} \not\subset \mathcal{A} \), but \( \mathcal{A} \) can be embedded into \( \mathcal{A} \) via some unital, injective \( \ast \)-homomorphism \( J : \mathcal{A} \to \mathcal{A} \), then \( E : \mathcal{A} \to \mathcal{A} \) is said to be a conditional expectation with corresponding injection \( J \), if \( E \) is completely positive and \( E \circ J = \text{id}_\mathcal{A} \). Note that in this case \( E \) is also unital (because \( J \) is unital) and thus of norm one. Hence the composed map \( J \circ E : \mathcal{A} \to J(\mathcal{A}) \subset \mathcal{A} \) is a projection of norm one and thus a conditional expectation in the above sense. Thus every unital dilation gives rise to a conditional expectation \( E \) with corresponding injection \( J \).

Now Definition 18 directly applies to Heisenberg channels. Taking into account that the only invertible channels are the unitary ones (cf. Prop. 1) we obtain the following concept.

**Definition 20.** 1. Let \( T \in Q_H(\mathcal{H}) \) be a Heisenberg quantum channel, i.e. \( T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) is linear, ultraweakly continuous, completely positive and unital. A unitary dilation \((\mathcal{K}, U, J, E)\) of \( T \) consists of a Hilbert space \( \mathcal{K} \) and a triple of maps \((U, J, E)\) with
\[
T = E \circ \text{Ad}_U \circ J \quad \text{and} \quad E \circ J = \text{id}_{\mathcal{B}(\mathcal{H})},
\]
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where $U$, $J$ and $E$ satisfy

(a) $U \in \mathcal{B}(\mathcal{K})$ is unitary.

(b) $J : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is an ultraweakly continuous $*$-homomorphism of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$.

(c) $E : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ is linear, ultraweakly continuous and completely positive with operator norm $\|E\| = 1$.

2. Let $S \subset G$ be a semigroup of a group $G$ and let $(T_g)_{g \in S}$ be a semigroup representation of $S$ with values in the set of Heisenberg quantum channels $Q_H(\mathcal{H})$. A unitary dilation $(\mathcal{K}, (U_g)_{g \in G_0}, J, E)$ of $(T_g)_{g \in S}$ consists of a Hilbert space $\mathcal{K}$, a subgroup $G_0 \subset G$ and a triple $((U_g)_{g \in G_0}, J, E)$ with

$$T_g = E \circ \text{Ad}_{U_g} \circ J \quad \text{and} \quad E \circ J = \text{id}_{\mathcal{B}(\mathcal{H})},$$

for all $g \in S \subset G_0$, where $(U_g)_{g \in G_0}$ is a representation of $G_0$ with values in the unitary group on $\mathcal{K}$ and $J, E$ as before.

If, in addition, $J(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{K}}$ then the dilation is said to be unital.

If the dilation is unital, then $J \in Q_H(\mathcal{H}, \mathcal{K})$ and $E \in Q_H(\mathcal{K}, \mathcal{H})$ are Heisenberg channels (cf. Rem. 19.2). Finally, this concept can be transferred to the Schrödinger quantum channels via duality (cf. Section II B).

**Definition 21.** 1. Let $T \in Q_S(\mathcal{H})$ be a Schrödinger quantum channel, i.e. $T : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{H})$ is linear, completely positive and trace-preserving. A unitary dilation $(\mathcal{K}, U, J, E)$ of $T$ consists of a Hilbert space $\mathcal{K}$ and a triple of maps $(U, J, E)$ with

$$T = E \circ \text{Ad}_U \circ J \quad \text{and} \quad E \circ J = \text{id}_{\mathcal{B}^1(\mathcal{H})},$$

where $U$, $J$ and $E$ satisfy

(a) $U \in \mathcal{B}(\mathcal{K})$ is unitary.

(b) $J : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{K})$ is linear and completely positive with operator norm $\|J\| = 1$.

(c) $E : \mathcal{B}^1(\mathcal{K}) \to \mathcal{B}^1(\mathcal{H})$ is linear, completely positive and satisfies

$$E(E^*(B)A) = BE(A) \quad \text{for all } B \in \mathcal{B}(\mathcal{H}) \text{ and } A \in \mathcal{B}^1(\mathcal{K}),$$

(B5)

where $E^*$ is the dual channel of $E$. 

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Let $S \subset G$ be a semigroup of a group $G$ and let $(T_g)_{g \in S}$ be a semigroup representation of $S$ with values in the set of Schrödinger quantum channels $Q_S(\mathcal{H})$. A unitary dilation $(\mathcal{K}, (U_g)_{g \in G}, J, E)$ of $(T_g)_{g \in S}$ consists of a Hilbert space $\mathcal{K}$, a subgroup $G_0 \subset G$ and a triple $((U_g)_{g \in G_0}, J, E)$ with

$$T_g = E \circ \text{Ad}_{U_g} \circ J \quad \text{and} \quad E \circ J = \text{id}_{B^1(\mathcal{H})},$$

for all $g \in S \subset G_0$, where $(U_g)_{g \in G_0}$ is a representation of $G_0$ with values in the unitary group on $\mathcal{K}$ and $J, E$ as before.

If, in addition, $E$ is trace-preserving then the dilation is said to be trace-preserving.

**Remark 22.**

1. Property (B5) which looks quite similar to (B3) implies (by direct computation) that the dual channel $E^*$ is a $\ast$-homomorphism. Moreover, $E^*$ is ultraweakly continuous as this holds for every dual channel. Conversely, for any ultraweakly continuous $\ast$-homomorphism $J$ from Definition 20 one can show that together with its pre-dual channel, it satisfies (B5). In this sense, the dilation definitions 20 and 21 are dual to each other. Similar as for (B3), one can conclude that (B5) is equivalent to

$$E(\text{AE}^*(B)) = E(A)B \quad \text{for all } B \in B(\mathcal{H}) \text{ and } A \in B^1(\mathcal{K}).$$

2. If a dilation is trace-preserving, then $J$ is trace-preserving as well (cf. Remark 19.2.) so in particular, $J \in Q_S(\mathcal{H}, \mathcal{K})$ and $E \in Q_S(\mathcal{K}, \mathcal{H})$ are Schrödinger channels.

3. Corollary 2 shows that for every Heisenberg channel $T \in Q_H(\mathcal{H})$ there exists a unitary (and even unital) dilation of $T$ of the following type $(\mathcal{H} \otimes \mathcal{K}, \text{Ad}_U, i_\mathcal{K}, \text{tr}_\omega)$, where $\mathcal{K}$ is a separable Hilbert space, $\omega \in \mathbb{D}(\mathcal{K})$ a pure state and $U \in B(\mathcal{H} \otimes \mathcal{K})$ a unitary operator. Such a dilation is also said to be of tensor type. This result holds analogously for every $T \in Q_S(\mathcal{H})$ by Theorem 2.

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19 For $G = \mathbb{Z}$ one can show, e.g., via Corollary 2 that Davies’ construction leads to a dilation of the form $T^n(A) = \text{tr}_{\omega \otimes \omega'}(U_n^\dagger U_n^\dagger (A \otimes \text{id}_{L^2(\mathbb{Z}, \mathcal{H}) \otimes \mathcal{K}})U_n U)$. On the one hand, $(U_n U)_{n \in \mathbb{Z}}$ does obviously not yield a unitary representation of $G = \mathbb{Z}$ although $(U_n)_{n \in \mathbb{Z}}$ does. On the other hand, the map $X \mapsto \text{tr}_\omega \text{tr}_{\omega'}(U_n^\dagger X U) = \text{tr}_{\omega \otimes \omega'}(U_n^\dagger X U)$ can in general not be replaced by $X \mapsto \text{tr}_\rho(X)$ so (2) does not have the desired structure.

20 For unitary dilations on Hilbert spaces the condition is necessary and sufficient (cf. Remark 7.2) and for $W^*$-algebras Kümmerer proved sufficiency (Prop. 2.2.7).

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33More precisely, said Lemma shows the existence of the decompositions $\mathcal{H}^\infty = \mathcal{H} \oplus \hat{\mathcal{H}}^\infty$ and $\mathcal{H}^\infty = \mathcal{H} \oplus \hat{\mathcal{H}}_{-\infty}$. Then it is easy to see that $\hat{\mathcal{H}}^\infty$ is orthogonal to $\hat{\mathcal{H}}_{-\infty}$ because $\langle \hat{v}, U^{-n}h \rangle = \langle U^n\hat{v}, h \rangle = 0$ for $\hat{v} \in \hat{\mathcal{H}}^\infty$, $h \in \mathcal{H}$ and $n \in \mathbb{N}_0$.

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