On the asymptotic expansion of $\Gamma(x)$, Lagrange’s inversion theorem and the Stirling coefficients

R. B. Paris

University of Abertay Dundee, Dundee DD1 1HG, UK
E-Mail: r.paris@abertay.ac.uk

Abstract

We show how the asymptotic expansion for the gamma function $\Gamma(x)$, similar to that obtained by Boyd [Proc. Roy. Soc. London A447 (1994) 609–630], can be obtained by using a form of Lagrange’s inversion theorem with a remainder. A (possibly) new closed-form representation for the Stirling coefficients is given.

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1. Introduction

The gamma function $\Gamma(x)$ has the well-known asymptotic expansion as $x \to \infty$

$$\Gamma(x) = \int_0^\infty e^{-\tau} \tau^{x-1} d\tau \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \sum_{n=0}^\infty \frac{(-)^n \gamma_n}{x^n}, \quad (1.1)$$

where $\gamma_n$ are the so-called Stirling coefficients, the first few being (with $\gamma_0 = 1$)

$$\gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}, \quad \gamma_4 = -\frac{571}{2488320}.$$  

The above expansion holds for large complex $x$ in the sector $|\arg x| \leq \pi - \delta$, $\delta > 0$, although in this note we shall restrict our attention throughout to positive values of $x$. The slowly varying part of $\Gamma(x)$ (when $x$ is large) is given by

$$\Gamma^*(x) = \frac{\Gamma(x)}{\sqrt{2\pi x^{x-\frac{1}{2}} e^{-x}}}, \quad (1.2)$$

and, from (1.1), its asymptotic expansion is

$$\Gamma^*(x) \sim \sum_{n=0}^\infty \frac{(-)^n \gamma_n}{x^n} = 1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \cdots \quad (x \to \infty).$$

Employing the reformulation of the method of steepest descents developed by Berry & Howls [2] (for a summary, see [10, pp. 94–99]), Boyd [3] established the result for positive integer $m$

$$\Gamma^*(x) = \sum_{n=0}^{m-1} \frac{(-)^n \gamma_n}{x^n} + \tilde{R}_m(x), \quad (1.3)$$
The quantity $h(z) = e^z - 1 - z$ and $C'$ (for $m \geq 1$) is a contour that can be taken to be a pair of straight parallel lines situated on either side of the real $z$-axis. By expanding the contour $C'$ to coincide with the other saddle points of the integrand in (1.1), Boyd then obtained the elegant expression
\[
\tilde{R}_m(x) = \frac{i^m x^{-m}}{2\pi i} \int_0^\infty s^{m-1} e^{-2\pi s} \left\{ \frac{\Gamma^*(is)}{1-is/x} - (-)^m \frac{\Gamma^*(-is)}{1+is/x} \right\} ds,
\]
from which he was able to derive a bound on $\tilde{R}_m(x)$ (valid for complex $x$). This bound has been recently improved in [8] by employing more refined bounds on $\Gamma^*(is)$.

The Stirling coefficients appearing in the expansions (1.1) and (1.3) can be generated numerically by means of the following recurrence relation:
\[
\gamma_n = (-2)^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} d_{2n},
\]
\[
d_n = \frac{n+1}{n+2} \left\{ \frac{d_{n-1}}{n} - \sum_{j=1}^{n-1} \frac{d_j d_{n-j}}{j+1} \right\} \quad (n \geq 1),
\]
where $d_0 = 1$ and an empty sum is interpreted as zero. A closed-form representation involving the 3-associated Stirling number $S_3(\ell,k)$ is found in [5] as
\[
\gamma_n = \sum_{j=0}^{2n} \frac{(-)^j S_3(2j+2n,j)}{2^{j+n}(j+n)!},
\]
where
\[
\exp \left[ \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \right] = \sum_{k,\ell \geq 0} S_3(\ell,k) \frac{u^k t^\ell}{\ell!}.
\]
A proof of this result is given in [2]. A different representation has been obtained recently in [7] in the form
\[
\gamma_n = \sum_{m=0}^{2n} \sum_{r=0}^{m} \frac{(\frac{1}{2})_{m+n}}{r!2^{r-m-n}} \sum_{j=0}^{m-r} \frac{(-)^{j+n} S_{2m+2n-2r-j}}{j!(2m+2n-2r-j)!},
\]
where $S_k^{(m)}$ denotes the Stirling number of the first kind [1, p. 824].

In this note we obtain the expansion of $\Gamma^*(x)$ in the form (1.3) and (1.4) by making use of Lagrange’s inversion theorem with a remainder, so that the inversion is valid on an infinite interval. The derivation of the remainder in Lagrange’s inversion theorem is given in the appendix. The approach we use also provides a (possibly) new closed-form representation for the Stirling coefficients.

2. The expansion for $\Gamma^*(x)$ as $x \to \infty$

We make the change of variable $t = \log(\tau/x)$ in Euler’s integral representation for $\Gamma(x)$ in (1.1) to find
\[
\Gamma(x) = x^x e^{-x} \int_{-\infty}^{\infty} e^{-x h(t)} dt,
\]
where
\[ h(t) = e^t - t - 1. \]
The scaled gamma function defined in (1.2) then becomes
\[ \Gamma^*(x) = \frac{x^{1/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-xh(t)} dt. \] (2.1)

The function \( h(t) \) has saddle points (where \( h'(t) = 0 \)) at \( t = 2\pi ki, \ k = 0, \pm 1, \pm 2, \ldots \). The saddle at \( t = 0 \) is the active saddle and the integration path in (2.1) coincides with the paths of steepest descent from the origin. We now make the quadratic transformation
\[ h(t) = \frac{1}{2}u^2 \] (2.2)
with the assumption that \( \text{sign}(t) = \text{sign}(u) \), to yield
\[ \Gamma^*(x) = \frac{x^{1/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}xu^2} \frac{dt}{du} du. \] (2.3)

To proceed we require the inversion of (2.2) to express \( t \) as a function of the new variable \( u \). Following the argument given in [6, p. 54], it can be seen that the inversion \( t(u) \) is a many-valued function with branch points at \( u = 0 \) and \( u = \pm 2\sqrt{\pi}e^{\pm\pi i/4}, \ k = 1, 2, \ldots \). Since
\[ \frac{dt}{du} = \frac{u}{e^t - 1}, \]
the only singularities of \( t(u) \) are at these branch points, and so the series expansion of \( t(u) \) will converge in \( |u| < 2\sqrt{\pi} \).

2.1 The derivation of the expansion for \( \Gamma^*(x) \)

We employ Lagrange’s inversion theorem with a remainder given in the appendix to obtain the inversion \( t(u) \) valid for \( u \in [0, \infty) \). Writing (2.2) in the form
\[ u = \frac{t}{\phi(t)} \]
and \( \phi(t) = \left( \frac{1}{2} \frac{t^2}{e^t - t - 1} \right)^{1/2} = \left( 1 + 2 \sum_{r=1}^{\infty} \frac{t^r}{(r+2)!} \right)^{-1/2} \), (2.4)
we have from (A.4)
\[ t = \sum_{n=1}^{m-1} \frac{u^n}{n!} D^{n-1} \phi^n(0) - \frac{u^m}{(m-1)!} D^{m-1} \phi^m(0) + Q_m(u) \]
for positive integer \( m \), where \( D^k \phi(0) \equiv (d/dt)^k \phi(t)|_{t=0} \ (k = 0, 1, 2, \ldots) \), and
\[ Q_m(u) = \frac{u^m}{2\pi i} \oint_C \frac{1 - u\phi'(z)}{z - u\phi(z)} \phi^m(z) z^{m-1} dz. \] (2.5)
The contour \( C \) denotes a closed path described in the positive sense surrounding the points \( z = 0 \) and \( z = t \). Making the change of summation index \( m \rightarrow 2m \) and differentiating we find
\[ \frac{dt}{du} = \sum_{n=0}^{m-1} \frac{u^{2n}}{(2n)!} D^{2n} \phi^{2n+1}(0) + \text{odd terms in } u + \frac{d}{du} Q_{2m}(u), \] (2.6)
where we have not specified the terms in the finite sum with odd parity in \( u \) since they make no contribution to the integral in (2.3).

Substitution of the expansion (2.6) into (2.3) then produces

\[
\Gamma^*(x) = \frac{x^\frac{3}{2}}{\sqrt{2\pi}} \sum_{n=0}^{m-1} \frac{D^{2n} \phi^{2n+1}(0)}{(2n)!} \int_{-\infty}^{\infty} u^{2n} e^{-\frac{1}{2} xu^2} du + R_m(x)
\]

\[
= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{m-1} \frac{2^n \Gamma(n + \frac{1}{2})}{(2n)! x^n} D^{2n} \phi^{2n+1}(0) + R_m(x),
\]

(2.7)

where the remainder after \( m \) terms \( R_m(x) \) is given by

\[
R_m(x) = \frac{x^\frac{3}{2}}{\sqrt{2\pi}} \int_0^\infty u e^{-\frac{1}{2} xu^2} Q_{2m}(u) du.
\]

(2.8)

Identification of the coefficients in the finite sum in terms of the Stirling coefficients \( \gamma_n \) (see (1.1)) then yields

\[
\Gamma^*(x) = \sum_{n=0}^{m-1} \frac{(-)^n \gamma_n}{x^n} + R_m(x),
\]

(2.9)

where

\[
\gamma_n = \frac{(-2)^n \Gamma(n + \frac{1}{2})}{\sqrt{\pi} (2n)!} D^{2n} \phi^{2n+1}(0) = \frac{(-)^n}{2^n n!} D^{2n} \phi^{2n+1}(0).
\]

(2.10)

2.2 An integral representation for the remainder \( R_m(x) \)

Substituting the representation of \( Q_{2m}(u) \) in (2.5) into the expression for the remainder \( R_m(x) \) in (2.8) we obtain

\[
R_m(x) = \frac{x^\frac{3}{2}}{\sqrt{2\pi}} \int_0^\infty u^{2m+1} e^{-\frac{1}{2} xu^2} \frac{1}{2\pi i} \oint_C \frac{\phi^{2m}(z)}{z^{2m+1}} \left\{ \frac{1 - u\phi'(z)}{z - u\phi(z)} - \frac{1 + u\phi'(z)}{z + u\phi(z)} \right\} dz du
\]

\[
= \frac{2x^\frac{3}{2}}{\sqrt{2\pi}} \int_0^\infty u^{2m+2} e^{-\frac{1}{2} xu^2} \frac{1}{2\pi i} \oint_C \frac{z(-\phi(z)/z)'}{1 - u^2(\phi(z)/z)^2} (\phi(z)/z)^{2m} dz du.
\]

Since, from (2.4),

\[
\phi(z)/z = (2h(z))^{-1/2}, \quad (\phi(z)/z)'/z = -h'(z)/(2^{1/2}h^{1/2}(z)),
\]

we then find after some straightforward rearrangement, together with the change of variable \( w = \frac{1}{2} xu^2 \), that

\[
R_m(x) = \frac{x^{-m}}{\sqrt{2\pi}} \int_0^\infty e^{-w^{m+\frac{1}{2}}} \frac{1}{2\pi i} \oint_C \frac{zh'(z)}{h(z) - w/x} \{h(z)\}^{-m-\frac{1}{2}} dz du,
\]

(2.11)

where the contour \( C \) denotes a closed path described in the positive sense surrounding the points \( z = 0 \) and the two zeros (one positive and one negative) of \( h(z) = w/x \).

**Remark 1.** As in (1.4), the contour \( C \) in (2.11) can be replaced by \( C' \) which is a pair of parallel lines just above and below the real \( z \)-axis.
Remark 2. Referring to (1.4), we see that Boyd’s expression for the remainder after \(m\) terms is given by

\[
\frac{x^{-m}}{\sqrt{2\pi}} \int_0^\infty e^{-w} w^{-m+\frac{1}{2}} \frac{1}{2\pi i} \oint_C \frac{\{h(z)\}^{-m+\frac{1}{2}}}{h(z) - w} \, dz \, du.
\]

(2.12)

We have been unable to demonstrate the equivalence between this form of the remainder and that in (2.11). We believe, however, that these two expressions are equivalent, a conjecture that is supported by high-precision numerical evaluation of the double integrals using Mathematica. In the particular case \(m = 2, x = 8\) for example, we found agreement between the remainder terms in (2.11) and (2.12) to more than 30dp.

3. A representation for the Stirling coefficients \(\gamma_n\)

Our representation for the Stirling coefficients is given in the following theorem.

**Theorem 1.** The Stirling coefficients \(\gamma_n\) \((n \geq 1)\) are given by

\[
\gamma_n = 2^n \sum (-2)^m (\frac{1}{2})_{m+n} \prod_{k=1}^{2n} (m_k!((k+2)!))^{m_k},
\]

where \((a)_n = \Gamma(a + n)/\Gamma(a)\) is Pochhammer’s symbol,

\[
m = m_1 + m_2 + \cdots + m_{2n}
\]

and the summation is taken over all nonnegative integer solutions \((m_1, \ldots, m_{2n})\) of the partition

\[
P_{2n} = \{(m_1, m_2, \ldots, m_{2n}) : \sum_{k=1}^{2n} km_k = 2n\}.
\]

**Proof.** From (2.10), the Stirling coefficients \(\gamma_n\) are given by

\[
\gamma_n = \frac{(-2)^n \Gamma(n + \frac{1}{2})}{\sqrt{\pi}} D^{2n} \phi^{2n+1}(0),
\]

(3.3)

where \(\phi(t)\) is defined in (2.4). To evaluate the derivatives \(D^{2n} \phi^{2n+1}(0)\) we make use of Faà di Bruno’s formula [1, p. 823], [9, p. 5]

\[
\frac{d^n}{dt^n}[f(g(t))] = n! \sum f^{(m)}(g(t)) \prod_{k=1}^{n} \frac{1}{m_k!} \left( \frac{g^{(k)}(t)}{k!} \right)^{m_k},
\]

(3.4)

where

\[
m = m_1 + m_2 + \cdots + m_n
\]

and the summation is taken over all nonnegative integer solutions \((m_1, \ldots, m_n)\) of the partition

\[
m_1 + 2m_2 + \cdots + nm_n = n.
\]

From (2.4), we set \(f(u) = u^{-n-1/2}\) and \(g(t) = 1 + 2 \sum_{r=1}^{\infty} t^r/(r+2)!\). Then a simple calculation shows that

\[
f^{(k)}(1) = (-)^k \frac{\Gamma(n + k + \frac{1}{2})}{\Gamma(n + \frac{1}{2})}, \quad g^{(k)}(0) = \frac{2}{(k+1)(k+2)}
\]

(3.5)
for \( k = 1, 2, \ldots \). From (3.4) we then obtain

\[
D^{2n} \phi^{2n+1}(0) = \frac{(2n)! \sqrt{\pi}}{\Gamma(n + \frac{1}{2})} \sum \frac{(-2)^m (\frac{1}{2})_{m+n}}{\prod_{k=1}^{2n} m_k! ((k + 2)!)^{m_k}}.
\]

Substitution of these values into (3.3) then yields the result in (3.1).

An alternative version of (3.1) is

\[
\gamma_n = \frac{2^n}{(2n)!} \sum \frac{(-2)^m (\frac{1}{2})_{m+n} C_{\vec{m}}}{\prod_{k=1}^{2n} ((k + 1)(k + 2))^{m_k}},
\]

where the coefficients \( C_{\vec{m}} \) are given by

\[
C_{\vec{m}} = \prod_{k=1}^{2n} \frac{(2n)!}{m_k! (k!)^{m_k}}.
\]

Values of these coefficients for \( n \leq 5 \) are tabulated in [1, p. 831].

### 4. Concluding remarks

In (2.7) we have obtained the expansion of the scaled gamma function \( \Gamma^* (x) \) as a finite sum involving inverse powers of \( x \) together with a remainder \( R_m (x) \) using Lagrange’s inversion theorem. This result is similar to that found by Boyd [3] who employed the Berry-Howls reformulation of the treatment of Laplace-type integrals. From this we derived an expression for the Stirling coefficients \( \gamma_n \) given in (2.10) and in Theorem 1.

A superficially similar procedure (but not equivalent) has been described by Brassesco and Méndez [4]. They started with the result (for \( x > 0 \))

\[
\Gamma(x) = x^x \int_0^\infty e^{-xt} t^x dt
\]

and made the linear transformation \( t \to 1 + w, \ w = ux^{-1/2} \) to obtain

\[
\Gamma(x) = x^x e^{-x} \int_{-1}^{\infty} e^{x(\log(1+w)-w)} dw = x^x e^{-x} \int_{-\sqrt{x}}^{\infty} e^{-u^2/2} e^{u^2 \lambda(u)} du,
\]

where

\[
\lambda(z) = z^{-2} \{ \log(1 + z) - z + \frac{1}{2} z^2 \}.
\]

Substituting the Maclaurin expansion

\[
e^{u^2 \lambda(z)} = \sum_{j \geq 0} \frac{z^j}{j!} D^j e^{u^2 \lambda(z)}|_{z=0}, \quad D \equiv \frac{d}{dz}
\]

\[1\] In [1, p. 831] these quantities are called \( M_3 = (2n; m_1, m_2, \ldots, m_{2n})' \).

\[2\] This follows from the Euler integral for \( \Gamma(x+1) \), followed by the change of variable \( \tau \to xt \) and use of the result \( \Gamma(x+1) = x\Gamma(x) \).
with \( z \) replaced by \( ux^{-1/2} \) into the above integral, they found upon reversal of the order of summation and integration

\[
\Gamma(x) \sim x^{\frac{1}{2}} e^{-x} \sum_{j \geq 0} \frac{x^{-j/2}}{j!} D^j \left( \int_{-\sqrt{x}}^{\infty} u^j e^{-u^2} \Lambda(uz)^{2j/2} du \right) \quad \Lambda(z) = 1 - 2\lambda(z). \tag{4.2}
\]

The above integral is then extended over \((-\infty, \infty)\), so that the terms with odd index \( j \) vanish, to yield the representation for the Stirling coefficients

\[
\gamma_n = \frac{(-1)^n}{2^nn!} D^{2n} \Lambda - n - \frac{1}{2} (0). \tag{4.3}
\]

This representation is equivalent to that in (2.10).

The implication here is that the evaluation of the \( \gamma_n \) by this means has resulted in the neglect of exponentially small terms produced by extending the above integral to include the interval \((-\infty, -\sqrt{x})\). In addition, Brassesco and Méndez [4, Eq. (2.26)] incorrectly write (4.2) as an equality when this cannot be the case since the expansion (4.1) is convergent in \(|z| < 1\). This fact results in integration of the series on \([0, \infty)\) beyond its interval of convergence. In our treatment, we make the quadratic transformation in (2.2) to obtain the Stirling coefficients expressed exactly in terms of an integral over the interval \((-\infty, \infty)\). This results in no exponentially small terms being neglected. Also the use of the Lagrange inversion theorem with a remainder circumvents the problem of integration beyond the interval of convergence (which in the case of \( t(u) \) in (2.4) is \(|u| < 2\sqrt{\pi} \)) and leads to an expression for the remainder term in the expansion.

The closed-form expression for the Stirling coefficients \( \gamma_n \) in (3.1), and its alternative form (3.5), involves the partition \( P_{2n} \). The cardinality of this set is equal to the partition function \( p(n) \), where \( p(n) \) represents the number of partitions of the positive integer \( n \). To illustrate the use of (3.5) we take the case \( n = 2 \), so that \( p(4) = 5 \) and [1, p. 831]

\[
P_4 = \{ (0, 0, 0, 1), (1, 0, 1, 0), (0, 2, 0, 0), (2, 1, 0, 0), (4, 0, 0, 0) \},
\]

\[
C_m = \{ 1, 4, 3, 6, 1 \}.
\]

Then

\[
\gamma_2 = \frac{2^2}{4!} \left\{ \frac{2 \cdot 1(\frac{1}{2})_3}{5 \cdot 6} + 4(\frac{1}{2})_4 \left( \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{3}{(3 \cdot 4)^2} \right) - \frac{8 \cdot 6(\frac{1}{2})_5}{(2 \cdot 3)^2 2 \cdot 3 \cdot 4} + \frac{16 \cdot 1(\frac{1}{2})_6}{(2 \cdot 3)^4} \right\} = \frac{1}{288}.
\]

It is clear that \( p(n) \) grows rapidly with \( n \). Consequently, (3.5) is not a practical means for the computation of these coefficients for large values of \( n \).

**Appendix: The Lagrange expansion theorem with a remainder**

Let \( f(t) \) and \( \phi(t) \) be analytic on and inside a simple closed contour \( C \) in the complex \( t \)-plane surrounding the point \( t = a \). Suppose further that the function \( \psi(z) = z - a - u\phi(z) \) has only one root \( z = t \) inside \( C \) given by

\[
t - a = u\phi(t), \tag{A.1}
\]

where \( u \) is the expansion variable. The procedure we adopt is a modification of that presented in [11, p. 17].

Our starting point is the identity

\[
f(t) = \frac{1}{2\pi i} \oint_C f(z) \frac{\psi'(z)}{\psi(z)} \, dz = \frac{1}{2\pi i} \oint_C f(z) \frac{1 - u\phi'(z)}{z - a - u\phi(z)} \, dz.
\]
Upon expansion of the factor \((z - a - u\phi(z))^{-1}\) as a finite geometric progression of \(m\) terms with a remainder, we find

\[
f(t) = \frac{1}{2\pi i} \oint_C f(z)(1 - u\phi'(z)) \left\{ \sum_{n=0}^{m-1} \frac{u^n\phi^n(z)}{(z - a)^{n+1}} + \frac{u^m\phi^m(z)}{(z - a)^m(z - a - u\phi(z))} \right\} \, dz.
\]

Making use of the Cauchy formula

\[
F^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\zeta = z} \frac{F(\zeta)}{(\zeta - z)^{n+1}} d\zeta,
\]

we obtain

\[
f(t) = \sum_{n=0}^{m-1} \frac{u^n}{n!} D^n[f(a)\phi^n(a)(1 - u\phi'(a))] + Q_m(u)
\]

\[
= \sum_{n=0}^{m-1} \frac{u^n}{n!} D^n[f(a)\phi^n(a)] - \frac{u}{n+1} f(a) D\phi^{n+1}(a)] + Q_m(u),
\]

where \(D \equiv d/da\), the remainder \(Q_m(u)\) is given by

\[
Q_m(u) = \frac{u^m}{2\pi i} \oint_C f(z) \frac{1 - u\phi'(z)}{z - a - u\phi(z)} \frac{\phi^m(z)}{(z - a)^m} \, dz,
\]

and the points \(z = a\) and \(z = t\) are enclosed by the contour \(C\).

Straightforward rearrangement of the sum over \(n\) then yields Lagrange’s expansion with a remainder in the form\(^3\)

\[
f(t) = f(a) + \sum_{n=1}^{m-1} \frac{u^n}{n!} D^{n-1}[f'(a)\phi^n(a)] - \frac{u^m}{m!} D^{m-1}[f(a)D\phi^m(a)] + Q_m(u) \tag{A.3}
\]

for positive integer \(m\), where \(\phi(t)\) is specified by (A.1).

In the special case \(f(t) = t\) and \(a = 0\), we have from (A.2) and (A.3) the expansion for positive integer \(m\)

\[
t = \sum_{n=1}^{m-1} \frac{u^n}{n!} D^{n-1}\phi^n(0) - \frac{u^m}{(m-1)!} D^{m-1}\phi^m(0) + \frac{u^m}{2\pi i} \oint_C \frac{1 - u\phi'(z)}{z - u\phi(z)} \frac{\phi^m(z)}{z^{m-1}} \, dz, \tag{A.4}
\]

where \(\phi(t)\) is specified in (A.1) and we have used the fact that

\[D^{m-1}[aD\phi^m(a)]_{a=0} = (m-1)D^{m-1}\phi^m(0)\]

and the contour \(C\) encloses the poles at \(z = 0\) and \(z = t\).

\(^3\)We note that the usual form of this theorem [11, p. 17], [12, p. 133] has the additional requirement \(|u\phi(z)| < |z - a|\) for points on \(C\), so that the arbitrary function \(f(t)\) then has the expansion

\[
f(t) = f(a) + \sum_{n=1}^{\infty} \frac{u^n}{n!} D^{n-1}[f'(a)\phi^n(a)].
\]

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