Status of the Standard Model Prediction of the Muon g-2

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The current status of the Standard Model prediction for the anomalous magnetic moment of the muon is briefly reviewed and compared with the present experimental value.

1. Introduction

Schwinger’s 1948 calculation [1] of the leading QED contribution to the anomalous magnetic moment of the muon \( a_\mu = (g_\mu - 2)/2 \), equal to the one of the electron, was one of the very first results of this theory, and one of its early confirmations. During the last few years, in a sequence of increasingly precise measurements, the E821 Collaboration at Brookhaven has determined \( a_\mu \) with a fabulous relative precision of 0.5 parts per million (ppm) [2,3,4], serving as an invaluable tool to test all sectors of the Standard Model (SM) and to scrutinize viable alternatives to this theory [5]. This note provides a brief summary of the present status of the three contributions into which the SM prediction \( a_\mu^{SM} \) is usually split – QED, electroweak and hadronic – and a comparison with the current experimental value.

2. QED and Electroweak Contributions

The QED contribution to \( a_\mu \) arises from the subset of SM diagrams containing only leptons (\( e, \mu, \tau \)) and photons. The lowest-order contribution is \( a_\mu^{QED}(1 \text{ loop}) = \alpha/(2\pi) \) [4]. Also the two- and three-loop QED terms are known analytically – see [6] for an update and a review of these contributions. The four-loop term has been evaluated numerically, a formidable task first accomplished by Kinoshita and his collaborators in the early 1980s [4]. The latest analysis appeared in [8]. Note that this four-loop contribution is about six times larger than the present experimental uncertainty of \( a_\mu \) [1]. The evaluation of the five-loop QED contribution is in progress [9].

Adding up these terms, using the latest CODATA [10] recommended value for the fine-structure constant \( \alpha^{-1} = 137.03599911(46) \), known to 3.3 ppb, one obtains [6] \( a_\mu^{QED} = 116584718.8(0.3)(0.4) \times 10^{-11} \). The first error is mainly due to the uncertainty of the \( O(\alpha^5) \) term, while the second one is caused by the 3.3 ppb uncertainty of the fine-structure constant.

The electroweak (EW) contribution to \( a_\mu \) is suppressed by a factor \( (m_\mu/M_W)^2 \) with respect to the QED effects. The one-loop part was computed in 1972 by several authors [11]: \( a_\mu^{EW}(1 \text{ loop}) = \frac{5G_\mu m_\mu^2}{24\sqrt{2}\pi^3} \left[ 1 + \frac{1}{3} (1 - 4\sin^2\theta_W)^2 \right] + O(m_\mu^2/M_W^2) \), where \( G_\mu = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2} \). Employing the on-shell definition \( \sin^2\theta_W = 1 - M_Z^2/M_W^2 \) [12], where \( M_Z = 91.1875(21) \text{ GeV} \) and \( M_W \) is the SM prediction of the \( W \) mass (which can be derived, for example, from the simple formulae of [13] leading to \( M_W = 80.383 \text{ GeV} \) for the Higgs mass \( M_H = 150 \text{ GeV} \}}, we obtain \( a_\mu^{EW}(1 \text{ loop}) = 194.8 \times 10^{-11} \).

The two-loop EW contribution to \( a_\mu \) is not negligible because of large factors of \( \ln(M_{Z,W}/m_f) \), where \( m_f \) is a fermion mass scale much smaller than \( M_W \) [14]. It was computed in 1995 [15]. The proper treatment of the contribution of the light quarks was addressed in [16,17]. These refinements significantly improved the reliability of the fermionic part (that containing closed fermion loops) of \( a_\mu^{EW}(\text{two loop}) \) leading, for \( M_H = 150 \text{ GeV} \), to \( a_\mu^{EW} = 154(1)(2) \times 10^{-11} \) [17]. The first error is due to hadronic loop uncertainties, while the second one corresponds to an allowed range of \( M_H \) in [114, 250] GeV, to the current top mass uncertainty, and to unknown three-loop ef-
fects. The leading-logarithm three-loop contribution to $a_\mu^{\text{EW}}$ is extremely small \cite{17,18}. The results of \cite{19} for the two-loop bosonic part of $a_\mu^{\text{EW}}$, performed without the large $M_\mu$ approximation previously employed, agree with the previous evaluation \cite{14} in the large Higgs mass limit. Work is also in progress for an independent recalculation based on the numerical methods of \cite{20}.

3. The Hadronic Contribution

The evaluation of the hadronic leading-order contribution $a_\mu^{\text{HLO}}$, due to the hadronic vacuum polarization correction to the one-loop diagram, involves long-distance QCD for which perturbation theory cannot be employed. However, using analyticity and unitarity, it was shown long ago that this term can be computed from hadronic analyticity and unitarity, it was shown long ago that this term can be computed from hadronic $e^+e^-$ annihilation data via the dispersion integral \[ a_\mu^{\text{HLO}} = \frac{1}{2\pi^2} \int_{4m_e^2}^{\infty} ds K(s)\sigma^{(0)}(s) \] \cite{21}, where $\sigma^{(0)}(s)$ is the total cross section for $e^+e^-$ annihilation into any hadronic state, with extraneous QED corrections subtracted off. The kernel function $K(s)$ decreases monotonically for increasing $s$.

A prominent role among all $e^+e^-$ annihilation measurements is played by the precise data collected in 1994-95 by the CMD-2 detector at the VEPP-2M collider in Novosibirsk for the $e^+e^- \to \pi^+\pi^-$ cross section at values of $\sqrt{s}$ between 0.61 and 0.96 GeV \cite{22}. The quoted systematic error of these data is 0.6%, dominated by the uncertainties in the radiative corrections (0.4%). Recently \cite{23} the CMD-2 Collaboration released its 1996-98 measurements for the same cross section in the full energy range $\sqrt{s} \in [0.37,1.39]$ GeV. The part of these data for $\sqrt{s} \in [0.61,0.96]$ GeV (quoted systematic error 0.8%) agrees with their earlier result published in \cite{22}. Also the SND Collaboration (at the VEPP-2M collider) recently presented its analysis of the $e^+e^- \to \pi^+\pi^-$ process for $\sqrt{s}$ between 0.39 and 0.98 GeV, with a systematic uncertainty of 1.3% (3.2%) for $\sqrt{s}$ larger (smaller) than 0.42 GeV \cite{24}. A hint of discrepancy, at the level of the combined systematic error, occurs between the CMD-2 and SND measurements (the contribution to $a_\mu^{\text{HLO}}$ of the SND data is a bit higher than the corresponding one from CMD-2) \cite{23}. Further significant progress is expected from the new $e^+e^-$ collider VEPP-2000 under construction in Novosibirsk \cite{23}.

In 2004 the KLOE experiment at the DAΦNE collider in Frascati presented a precise measurement of $\sigma(e^+e^- \to \pi^+\pi^-)$ via the initial-state radiation (ISR) method at the $\phi$ resonance \cite{25}. This cross section was extracted for $\sqrt{s}$ between 0.59 and 0.97 GeV with a systematic error of 1.3% and a negligible statistical one. There are some discrepancies between the KLOE and CMD-2 results (KLOE's data lying higher than the CMD-2 fit below the $\rho$ peak, and lower on the peak and above it), although their integrated contributions to $a_\mu^{\text{HLO}}$ are similar \cite{24}. The data of KLOE and SND disagree above the $\rho$ peak, where the latter are significantly higher. The study of the $e^+e^- \to \pi^+\pi^-$ process via the ISR method is also in progress at BABAR \cite{26} and BELLE \cite{27}.

The evaluations of the dispersive integral based on the analysis \cite{22} of the 1994-95 CMD-2 data are in good agreement:\footnote{The evaluation of \cite{29} is not included as its result is being revised \cite{30}.}

\begin{align*}
\alpha_\mu^{\text{HLO}} &= 6934 (53)_{\text{exp}}^{(35)}_{\text{rad}} \times 10^{-11}, \\
\alpha_\mu^{\text{HLO}} &= 6948 (86) \times 10^{-11}, \\
\alpha_\mu^{\text{HLO}} &= 6924 (59)_{\text{exp}}^{(24)}_{\text{rad}} \times 10^{-11}, \\
\alpha_\mu^{\text{HLO}} &= 6944 (48)_{\text{exp}}^{(10)}_{\text{rad}} \times 10^{-11}.
\end{align*}

Reference \cite{31} updates \cite{35} and already includes KLOE's results. The recently released data of CMD-2 \cite{23} and SND \cite{21} are not yet included.

The authors of \cite{30} pioneered the idea of using vector spectral functions derived from the study of hadronic $\tau$ decays (see \cite{37,38} for recent reviews) to improve the evaluation of the dispersive integral. Indeed, assuming isospin invariance to hold, the isovector part of the cross section for $e^+e^- \to$ hadrons can be calculated via the Conserved Vector Current relations from $\tau$-decay spectra. The latest analysis with ALEPH \cite{39}, CLEO \cite{40}, and OPAL \cite{41} data yields $\alpha_\mu^{\text{HLO}} = 7110 (50)_{\text{exp}}^{(8)}_{\text{rad}} (28)_{\text{SU}(2)} \times 10^{-11}$. Isospin-breaking corrections were applied \cite{42}. Information from $\tau$ decays was also...
included in one of the analyses of \cite{34}, leading to $a_\mu^{HLO} = 7027 (47) \times 10^{-11}$.

Although the precise CMD-2 $e^+e^- \to \pi^+\pi^-$ data \cite{22} are consistent with the corresponding $\tau$ ones for energies below $0.85$ GeV, they are significantly lower for larger energies. KLOE’s $\pi^+\pi^-$ spectral function confirms this discrepancy with the $\tau$ data; on the contrary, the recent SND results are compatible with them \cite{24}. This discrepancy could be caused by inconsistencies in the $e^+e^-$ or $\tau$ data, or in the isospin-breaking corrections which must be applied to the latter. Indeed, the mentioned disagreements between the $e^+e^-$ data sets need careful consideration. On the other hand, in spite of the agreement of the $\tau$ data sets \cite{35}, the question remains whether all possible isospin-breaking effects have been properly taken into account \cite{31,32}.

The hadronic higher-order contribution can be divided into two parts: $a_\mu^{HHC} = a_\mu^{HHC}(vp) + a_\mu^{HHC}(lbl)$. The first term is the $O(\alpha^3)$ contribution of diagrams containing hadronic vacuum polarization insertions \cite{43}. Its first value is $a_\mu^{HHC}(vp) = -97.9 (0.9) \times 10^{-11}$ \cite{43}, obtained using $e^+e^-$ annihilation data; it changes by $\sim -3 \times 10^{-11}$ if hadronic $\tau$-decay data are used instead \cite{45}. The second term, also of $O(\alpha^3)$, is the hadronic light-by-light contribution; it cannot be determined from data, its evaluation relies on specific models. In 2001 the authors of \cite{49} uncovered a sign error in earlier evaluations of its dominating pion-pole part. Their estimate, based also on previous results for the quark and charged-pions loop parts \cite{47}, is $a_\mu^{HHC}(lbl) = 80 (40) \times 10^{-11}$ \cite{49}. A higher value was obtained in 2003 including short-distance QCD constraints: $a_\mu^{HHC}(lbl) = 136 (25) \times 10^{-11}$ \cite{49}. Further independent calculations would provide an important check of this result for $a_\mu^{HHC}(lbl)$, a contribution whose uncertainty may become the ultimate limitation of the SM prediction of the muon $g-2$.

4. **Standard Model vs. Measurement**

The first column of Table 1 shows $a_\mu^{SM} = a_\mu^{QED} + a_\mu^{EM} + a_\mu^{HLO} + a_\mu^{HHC}$. The values employed for $a_\mu^{HLO}$ are indicated by the reference on the left (\cite{34} quotes two values, see Sec. 3); all $a_\mu^{SM}$ values were derived with $a_\mu^{HHC}(lbl) = 136 (25) \times 10^{-11}$ \cite{49}. Errors were added in quadrature. The present world average experimental value for the muon $g-2$ is $a_\mu^{EXP} = 116 592 080 (60) \times 10^{-11}$ (0.5 ppm) \cite{49}. The differences $\Delta = a_\mu^{EXP} - a_\mu^{SM}$ are listed in the second column of Table 1, while the numbers of “standard deviations” ($\sigma$) appear in the third one. Higher discrepancies, shown in angle brackets, are obtained if $a_\mu^{HHC}(lbl) = 80 (40) \times 10^{-11}$ \cite{49} is used instead of 136 (25) \times 10^{-11} \cite{49}.

5. **Conclusions**

The discrepancies between recent SM predictions of $a_\mu$ and the current experimental value vary in a very wide range, from 0.7 to 3.2 $\sigma$, according to the values chosen for the hadronic contributions. In particular, the leading-order hadronic contribution depends on which of the two sets of data, $e^+e^-$ collisions or $\tau$ decays, are employed. The puzzling discrepancy between the $\pi^+\pi^-$ spectral functions from $e^+e^-$ and isospin-breaking-corrected $\tau$ data could be caused by inconsistencies in the $e^+e^-$ or $\tau$ data, or in the isospin-breaking corrections applied to the latter. Indeed, disagreements occur between $e^+e^-$ data sets, requiring further detailed investigations. On the other hand, $\tau$ data sets are in agreement, but their connection with the leading hadronic contribution to $a_\mu$ is less direct, and one wonders whether all possible isospin-breaking effects have been properly taken into account. Using $e^+e^-$ data, the SM prediction of the muon $g-2$ deviates from the present experimental value by 2–3 $\sigma$.

The impressive results of the E821 experiment are still limited by statistical errors. A new experiment, E969, has been approved (but not yet funded) at Brookhaven in 2004 \cite{31,40}. Its goal is to reduce the present experimental uncertainty by a factor of 2.5 to about 0.2 ppm. A letter of

| $a_\mu^{SM} \times 10^{11}$ | $\Delta \times 10^{11}$ | $\sigma$ |
|--------------------------|-------------------|---------|
| 116 591 845 (69)         | 235 (91)          | 2.6 (3.0) |
| 116 591 859 (90)         | 221 (108)         | 2.1 (2.5) |
| 116 591 835 (69)         | 245 (91)          | 2.7 (3.1) |
| 116 591 855 (55)         | 225 (81)          | 2.8 (3.2) |
| 116 592 018 (63)         | 62 (87)           | 0.7 (1.3) |
| 116 591 938 (54)         | 142 (81)          | 1.8 (2.3) |

Table 1: Standard Model vs. measurement.
intent for an even more precise $g-2$ experiment was submitted to J-PARC with the proposal to reach a precision below $0.1$ ppm [51]. While the QED and EW contributions appear to be ready to rival these precisions, much effort will be needed to reduce the hadronic uncertainty by a factor of two. This effort is challenging but possible, and certainly well motivated by the excellent opportunity the muon $g-2$ is providing us to unveil (or constrain) “new physics” effects.

Acknowledgments. I would like to thank the organizers for their kind invitation and excellent organization of this workshop, and S. Eidelman and B.L. Roberts for very useful communications.

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