Recovery of Full $N = 1$ Supersymmetry in Non(anti-)commutative Superspace

Akifumi Sako† and Toshiya Suzuki∗§

† Department of Mathematics, Faculty of Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

∗ Department of Physics, Faculty of Science, Ochanomizu University
and

§ Institute of Humanities and Sciences, Ochanomizu University
2-1-1 Otsuka, Bunkyo-ku, Tokyo 112-8610, Japan

† sako@math.keio.ac.jp
∗ tsuzuki@phys.ocha.ac.jp

Abstract
We investigate SUSY of Wess-Zumino models in non(anti-)commutative Euclidean superspaces. Non(anti-)commutative deformations break 1/2 SUSY, then non(anti-)commutative Wess-Zumino models do not have full SUSY in general. However, we can recover full SUSY at specific coupling constants satisfying some relations. We give a general way to construct full SUSY non(anti-)commutative Wess-Zumino models. For a some example, we investigate quantum corrections and $\beta$-functions behavior.
1 Introduction

Non(anti-)commutative superspaces have attracted much interest [1] - [60]. One of significant features of non(anti-)commutative field theories is 1/2 SUSY. Non(anti-)commutative theories do not have full SUSY, but one half of SUSY is preserved in many cases. Non(anti-)commutative theories are constructed by several deformations of usual (anti-)commutative SUSY theories. We will treat 2 types of the deformations given by SUSY $\ast$ product and non-SUSY $\ast$ product. (These definitions will be given in the next section.) The SUSY $\ast$ product does not break the SUSY algebra, but SUSY $\ast$ products of chiral superfields are not chiral usually, then 1/2 SUSY is broken (see, for example, [10]). On the other hand, non-SUSY $\ast$ products of chiral superfields are chiral superfields, but the SUSY algebra itself is broken as we saw in [9] and so on.

In this article, we will show that tuning of coupling constants regains full SUSY for not only SUSY $\ast$ deformation cases but also non-SUSY $\ast$ deformation cases where the normal SUSY algebra is broken. In non(anti-)commutative theories, all lagrangians, algebras and transformation laws should be defined by using $\ast$ (or $\ast$ etc.) products. Since the $\ast$ (or $\ast$ etc.) product often breaks SUSY, it is non-trivial question to ask that one can define the SUSY algebra in non(anti-)commutative superspaces or there exist non(anti-)commutative full SUSY lagrangians. At first we will propose a general method to construct the full SUSY Wess-Zumino models in non(anti-)commutative superspaces. Using this procedure, we will understand that full SUSY recovers at specific values of the coupling constants. After that we will observe the quantum effect around the supersymmetry recovering points. The quantum corrections will be calculated and $\beta$-functions will be determined at one-loop order. In some phases, full SUSY is stable in the IR limit.

2 Conventions

We study the Wess-Zumino models on 4-dim Euclidean superspace and use the convention of [62]. Non(anti-)deformation is given by several ways. For example, in [10], we can see the systematic explanation of the deformations. So we use the notation of [10].

The left covariant derivative is identical to the ordinary supersymmetric covariant derivative,

$$\overrightarrow{D}\Phi := D\Phi .$$

(1)

On the other hand, the right covariant derivative is defined through the following relation.

$$\Phi \overleftarrow{D} := (-1)^{p_D (p_A + 1)} \overrightarrow{D}\Phi ,$$

(2)

where $p_A$ is parity of $A$ i.e. $p_A = 0$ for bosonic $A$ and $p_B = 1$ for fermionic $B$. With this
setting, for some superfields Φ and Ψ the SUSY ∗ product is defined by

\[ Φ \ast Ψ := Φ \cdot Ψ + P^{αβ} D_α Φ D_β Ψ + \frac{1}{4} \text{det} P \text{det} D^2 \ast D^2 Ψ \]
\[ = Φ \cdot Ψ - P^{αβ} D_α Φ D_β Ψ - \frac{1}{4} \text{det} P \text{det} D^2 \ast D^2 Ψ. \] (3)

This ∗ product is naively extended to the product of operators. For arbitrary operators (or superfields) \( O_1 \) and \( O_2 \),

\[ O_1 \ast O_2 = O_1 \cdot O_2 + (-)^{pD(p_0+1)} P^{αβ} [D_α, O_1] [D_β, O_2] \]
\[ - \frac{1}{4} \text{det} P e^{βα} [D_α, [D_β, O_1]] e^{δγ} [D_γ, [D_δ, O_2]] , \] (4)

where \( [A, B] := AB - (-)^{pApB} BA \).

We introduce another typical non(anti-)commutative deformation with using not \( D \) but \( Q \). The left SUSY generating operator action is identical to the ordinary action

\[ \overline{Q} Φ := Q Φ . \] (5)

On the other hand, the right action is defined by

\[ Φ \overline{Q} := (-1)^{pQ(p_0+1)} \overline{Q} Φ . \] (6)

With this setting, the non-SUSY ∗ product is defined by

\[ Φ \ast Ψ := Φ \cdot Ψ + P^{αβ} Φ \overline{Q}_α Q_β Ψ + \frac{1}{4} \text{det} P \text{det} Q^2 \overline{Q}^2 Ψ \]
\[ = Φ \cdot Ψ - P^{αβ} Q_α Φ Q_β Ψ - \frac{1}{4} \text{det} PQ^2 \overline{Q}^2 Ψ . \] (7)

### 3 Full SUSY in Non(anti-)commutative Field Theory

In this section, we propose a general procedure to build full SUSY lagrangians in non(anti-)commutative \( \mathbb{R}^4 \) from 1/2 SUSY lagrangians.

We will begin by considering the SUSY algebra determined by the SUSY ∗ product.

\[ \{Q_α, Q_β\}_* = \{Q_α, \overline{Q}_β\}_* = 0 \] (8)
\[ \{Q_α, \overline{Q}_β\}_* = 2σ^α_β P μ = -2iσ^α_β \partial_μ \] (9)
\[ [P_μ, Q_α]_* = [P_μ, \overline{Q}_α]_* = 0 \] (10)
\[ [M_μν, Q_α]_* = -(iσ_μν)^β_α Q_β \] (11)
\[ [M_μν, \overline{Q}^α]_* = -(iσ_μν)^β_α \overline{Q}^β , \] (12)

where \( \{A, B\}_* := A \ast B + B \ast A \) and \( [A, B]_* := A \ast B - B \ast A \). The definition of SUSY ∗ product (3) uses the covariant derivative \( D_α \), and all generators of supersymmetric
In general, we can divide $\tilde{SUSY}$ part where $V_{\text{ucts}}$. From (3),

Note that the super-$\ast$ symmetry is defined in the non(anti-)commutative superspace.

Let us construct full SUSY Wess-Zumino like models in the non(anti-)commutative Euclidean space that is deformed by the SUSY $\ast$ product. Let $\Phi$ be a chiral super field. Let $\int d^4\theta V^0(\Phi; \ast)\bar{\theta}^2$ be a 1/2 SUSY invariant lagrangian, where the $V^0(\Phi; \ast)$ is a some polynomial in $\Phi$ and its derivative $\partial_{\mu} \Phi, \partial_{\alpha} \Phi$ and so on. A general 1/2 SUSY lagrangian is given from the elements of the 1/2 SUSY ring discussed in [55]. In $V^0(\Phi; \ast)$, all products are $\ast$ products, but $V^0(\Phi; \ast)$ is possible to be expressed by usual products by using (3).

Let $V^0(\Phi; \cdot)$ denote the usual product representation of $V^0(\Phi; \ast)$ i.e. $V^0(\Phi; \ast) = \tilde{V}^0(\Phi; \cdot)$. We can regard $\tilde{V}^0(\Phi; \cdot)$ as the sum of supersymmetric lagrangian $\tilde{V}^0_s(\Phi; \cdot)$ and 1/2 SUSY lagrangian $-\tilde{V}^0_{1/2}(\Phi; \cdot)$:

$$
\tilde{V}^0(\Phi; \cdot) = \tilde{V}^0_s(\Phi; \cdot) - \tilde{V}^0_{1/2}(\Phi; \cdot),
$$

where $\tilde{V}^0_s(\Phi; \cdot)$ and $\tilde{V}^0_{1/2}(\Phi; \cdot)$ satisfy

$$
\int d^4x d^4\theta d^2Q_\alpha \tilde{V}^0_s(\Phi; \cdot) = 0, \quad \int d^4x d^4\bar{\theta} d^2\bar{Q}_\alpha \tilde{V}^0_s(\Phi; \cdot) = 0
$$

Next step, we introduce a new lagrangian $\int d^4\theta \tilde{V}^1(\Phi; \ast)$ by

$$
\int d^4x d^4\theta d^2Q_\alpha \tilde{V}^1_s(\Phi; \cdot) = 0, \quad \int d^4x d^4\bar{\theta} d^2\bar{Q}_\alpha \tilde{V}^1_s(\Phi; \cdot) = 0
$$

where $\tilde{V}^1_s(\Phi; \cdot)$ is $\tilde{V}^1_{1/2}(\Phi; \cdot)$ deformed by replacing all usual products with SUSY $\ast$ products. From (3),

$$
V^0_{1/2}(\Phi; \ast) = \tilde{V}^0_{1/2}(\Phi; \cdot) + \text{(higher order of } P^{\alpha\beta})
$$

$$
:= \tilde{V}^0_{1/2}(\Phi; \cdot) + \tilde{V}^1_{\text{def}}(\Phi, P^{\alpha\beta}; \cdot).
$$

In general, we can divide $\tilde{V}^1_{\text{def}}(\Phi, P^{\alpha\beta}; \cdot)$ into a full SUSY part $\tilde{V}^1_s(\Phi, P^{\alpha\beta}; \cdot)$ and a 1/2 SUSY part $\tilde{V}^1_{1/2}(\Phi, P^{\alpha\beta}; \cdot)$:

$$
\tilde{V}^1_{\text{def}}(\Phi, P^{\alpha\beta}; \cdot) = \tilde{V}^1_s(\Phi, P^{\alpha\beta}; \cdot) - \tilde{V}^1_{1/2}(\Phi, P^{\alpha\beta}; \cdot),
$$

where $\tilde{V}^1_s(\Phi, P^{\alpha\beta}; \cdot)$ and $\tilde{V}^1_{1/2}(\Phi, P^{\alpha\beta}; \cdot)$ satisfy

$$
\int d^4x d^4\theta d^2Q_\alpha \tilde{V}^1_s(\Phi, P^{\alpha\beta}; \cdot) = 0, \quad \int d^4x d^4\bar{\theta} d^2\bar{Q}_\alpha \tilde{V}^1_s(\Phi, P^{\alpha\beta}; \cdot) = 0
$$

$$
\int d^4x d^4\theta d^2Q_\alpha \tilde{V}^1_{1/2}(\Phi, P^{\alpha\beta}; \cdot) = 0, \quad \int d^4x d^4\bar{\theta} d^2\bar{Q}_\alpha \tilde{V}^1_{1/2}(\Phi, P^{\alpha\beta}; \cdot) \neq 0.
$$
From (17), (18) and (19), the lagrangian is
\[
\int d^4\theta \{ V^1(\Phi; *) \} \bar{\theta}^2 = \int d^4\theta \bar{\theta}^2 \{ [\tilde{V}_s^0(\Phi; \cdot) + \tilde{V}_s^1(\Phi, P^{\alpha\beta}; \cdot)] - \tilde{V}_{1/2}^1(\Phi, P^{\alpha\beta}; \cdot) \}. \tag{22}
\]

Note that \( \tilde{V}_s^0(\Phi; \cdot) + \tilde{V}_s^1(\Phi, P^{\alpha\beta}; \cdot) \) in the right hand side is full supersymmetric. We can duplicate the process from (17) to (22) to eliminate the 1/2 SUSY terms. The \( n \)-th process is as follows. The \( n \)-th lagrangian is
\[
\int d^4\theta \{ V^n(\Phi; *) \} \bar{\theta}^2 := \int d^4\theta \{ V^{n-1}(\Phi; *) + V_{1/2}^{n-1}(\Phi; *) \} \bar{\theta}^2, \tag{23}
\]
where \( V_{1/2}^{n-1}(\Phi; *) \) is \( \tilde{V}_{1/2}^{n-1}(\Phi, P^{\alpha\beta}; \cdot) \) deformed by replacing usual products by SUSY * products. From (23),
\[
V_{1/2}^{n-1}(\Phi; *) = \tilde{V}_{1/2}^{n-1}(\Phi, P^{\alpha\beta}; \cdot) + \text{(higher order of } P^{\alpha\beta})
:= \tilde{V}_{1/2}^{n-1}(\Phi; \cdot) + \tilde{V}_{\text{def}}^n(\Phi, P^{\alpha\beta}; \cdot). \tag{24}
\]

We divide \( \tilde{V}_{\text{def}}^n(\Phi, P^{\alpha\beta}; \cdot) \) into a full SUSY part \( \tilde{V}_s^n(\Phi, P^{\alpha\beta}; \cdot) \) and a 1/2 SUSY part \( \tilde{V}_{1/2}^n(\Phi, P^{\alpha\beta}; \cdot) \). Using this, the \( n \)-th lagrangian is given as
\[
\int d^4\theta \{ V^n(\Phi; *) \} \bar{\theta}^2 = \int d^4\theta \bar{\theta}^2 \{ \tilde{V}_s^n(\Phi, P^{\alpha\beta}; \cdot) + \tilde{V}_s^1(\Phi, P^{\alpha\beta}; \cdot) + \cdots + \tilde{V}_s^1(\Phi, P^{\alpha\beta}; \cdot) \}.
\]

The key point is these processes get over at finite rotation. Because the deformation of (24) makes \( \tilde{V}_{1/2}^n(\Phi, P^{\alpha\beta}; \cdot) \) be higher order terms of \( P^{\alpha\beta} \). In proportion to the square root of the power of \( P^{\alpha\beta} \), the number of \( D_\alpha \) in \( \tilde{V}_{1/2}^n(\Phi, P^{\alpha\beta}; \cdot) \) increases. Since \( D_\alpha D_\beta D_\gamma = 0 \), there is a finite number \( N \) such that \( \tilde{V}_{1/2}^N(\Phi, P^{\alpha\beta}; \cdot) = 0 \).

Then we get the full SUSY (super-* symmetric) lagrangian
\[
\int d^4\theta \{ V^N(\Phi；* ) \} \bar{\theta}^2 = \int d^4\theta \bar{\theta}^2 \{ [\tilde{V}_s^0(\Phi; \cdot) + \tilde{V}_s^1(\Phi, P^{\alpha\beta}; \cdot) + \cdots + \tilde{V}_s^N(\Phi, P^{\alpha\beta}; \cdot)]
\]
\[
= \int d^4\theta \bar{\theta}^2 \{ \tilde{V}_s^0(\Phi; \cdot) + \tilde{V}_s^1(\Phi, P^{\alpha\beta}; \cdot) + \cdots + \tilde{V}_s^N(\Phi, P^{\alpha\beta}; \cdot) \}. \tag{25}
\]

We will take examples to illustrate the above method to construct full SUSY lagrangians.

The first example is
\[
S = \int d^4x d^2\theta d^2\bar{\theta} \bar{\phi} * \phi + \int d^4x d^2\theta \bar{\phi} \frac{m}{2} \phi * \bar{\phi} + \int d^4x d^2\theta \{ \frac{m}{2} \phi * \phi + g_0 \phi * \phi * \phi \}. \tag{27}
\]
The quadratic terms are not deformed by the SUSY \(*\) product. Then the only \(\Phi \ast \Phi \ast \Phi\) should be modified to get the full SUSY action. Let us follow the above instruction in the condition \(\Phi \ast \Phi \ast \Phi = V^0(\Phi; \ast)\). Rewriting this as
\[
\Phi \ast \Phi \ast \Phi = \Phi^3 - \frac{1}{4} \det P \Phi D^2 \Phi D^2 \Phi = \tilde{V}^0(\Phi; \cdot),
\] (28)
then \(\tilde{V}^0_s(\Phi, \cdot) = \Phi^3\) and \(\tilde{V}^0_{1/2}(\Phi, \cdot) = \frac{1}{4} \det P \Phi D^2 \Phi D^2 \Phi\). Thus,
\[
V^0_s(\Phi) = \frac{1}{4} \det P \Phi \ast D^2 \Phi \ast D^2 \Phi, 
\] (29)
where \(D^2 := D^\alpha D_\alpha\). In the following, we often use that \(D^2 = D \ast D\), \(D \ast \Phi = D \Phi\) etc.

Using (29), a new lagrangian is introduced by
\[
\int d^4 \theta \bar{\theta}^2 \{V^1(\Phi; \ast)\} = \int d^4 \theta \bar{\theta}^2 \{\Phi \ast \Phi \ast \Phi + \frac{1}{4} \det P \Phi \ast D^2 \Phi \ast D^2 \Phi\}
= \int d^4 \theta \bar{\theta}^2 \Phi^3.
\] (30)
This is the super-\(*\) symmetric (full SUSY) lagrangian. This lagrangian is regarded as a special case of 1/2 SUSY lagrangian \(\int d^4 \theta \bar{\theta}^2 \{g_0 \ast \Phi \ast \Phi \ast \Phi + g_1 \ast \Phi \ast D^2 \Phi \ast D^2 \Phi\}\), where \(g_0\) and \(g_1\) are coupling constants. From this point of view, we can observe that tuning coupling constants,
\[
g_1 \ast / g_0 \ast \rightarrow \frac{1}{4} \det P,
\]
realizes full SUSY.

The second example is the case of \(\Phi^4_s = V^0(\Phi; \ast)\), where \(\Phi^s = \Phi \ast \cdots \ast \Phi\). From
\[
\Phi^4_s = \Phi^4 - \frac{1}{4} \det P \Phi^2 D^2 \Phi D^2 \Phi - \frac{1}{4} \det P \Phi D^2 \Phi D^2 \Phi - \frac{1}{16} (\det P)^2 (D^2 \Phi)^4,
\] (31)
\(\tilde{V}^0_s\) and \(\tilde{V}^0_{1/2}\) are given by
\[
\tilde{V}^0_s(\Phi, \cdot) = \Phi^4
\]
\[
\tilde{V}^0_{1/2}(\Phi, \cdot) = \frac{1}{4} \det P \Phi^2 D^2 \Phi D^2 \Phi + \frac{1}{4} \det P \Phi D^2 \Phi D^2 \Phi - \frac{1}{16} (\det P)^2 (D^2 \Phi)^4.
\]
Therefore, the modified lagrangian is given by
\[
\int d^4 \theta \bar{\theta}^2 V^1(\Phi; \ast) = \int d^4 \theta \bar{\theta}^2 \{V^0(\Phi; \ast) + V^0_{1/2}(\Phi; \ast)\}
= \int d^4 \theta \bar{\theta}^2 \{\Phi^4_s + \left[\frac{1}{4} \det P \Phi^2 D^2 \Phi D^2 \Phi + \frac{1}{4} \det P \Phi D^2 \Phi D^2 \Phi - \frac{1}{16} (\det P)^2 (D^2 \Phi)^4\]\}
= \int d^4 \theta \bar{\theta}^2 \left\{\Phi^4_s - \frac{1}{8} (\det P)^2 (D^2 \Phi)^4\right\}
= \int d^4 \theta \bar{\theta}^2 \Phi^4.
\] (32)
This is what we want. Again, we can regard this lagrangian is given by tuning of coupling constants of the 1/2 SUSY lagrangian.

This method is extended easily to the case of non-SUSY ★ deformation. However, the SUSY algebra for non-SUSY ★ product has to be modified, because the ★ definition uses Q_µ and it does not commute with ̄Q_µ. So we replace ̄Q_µ by ̄Q_µ defined by

$$Q_\alpha := Q_\alpha + P^{\alpha\beta} \{ Q_\alpha, Q_\beta \} = Q_\alpha - 2P^{\alpha\beta}(i\sigma_\alpha^\mu \partial_\mu)Q_\beta.$$  \hspace{1cm} (33)

From this definition, ̄Q_µ satisfies

$$\bar{Q}_\dot{\alpha} \star \phi = \bar{Q}_\dot{\alpha} \phi,$$  \hspace{1cm} (34)

for arbitrary field \( \phi = \phi(x, \theta, \bar{\theta}) \). Using ̄Q_µ, the super-★ symmetry algebra is defined by

$$\{ Q_\alpha, Q_\beta \}_\star = \{ \bar{Q}_\dot{\alpha}, \bar{Q}_\dot{\beta} \}_\star = 0 \quad \text{(35)}$$
$$\{ Q_\alpha, \bar{Q}_\dot{\beta} \}_\star = 2\sigma_\alpha^{\mu} P_\mu = -2i\sigma_\alpha^{\mu} \partial_\mu \quad \text{(36)}$$
$$[P_\mu, Q_\alpha]_\star = [P_\mu, \bar{Q}_\dot{\alpha}]_\star = 0 \quad \text{(37)}$$
$$[M_{\mu\nu}, Q_\alpha]_\star = -(i\sigma_{\mu\nu})_\alpha^\beta Q_\beta \quad \text{(38)}$$
$$[M_{\mu\nu}, \bar{Q}_\dot{\alpha}]_\star = -(i\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^\dot{\beta} \bar{Q}_\dot{\beta} \quad \text{(39)}$$

Here \( \{ A, B \}_\star := A \star B + B \star A \) and \( \{ A, B \}_\star := A \star B - B \star A \). Note that ̄Q_µ is not derivative because it includes second derivative and does not satisfy the Leibniz rule. So, we can not make a group from the above super-★ symmetry algebra. However, it is possible to make full super-★ symmetric field theories, where the super-★ symmetry is defined by invariance under

$$\Phi \rightarrow \Phi + \zeta^\alpha Q_\alpha \star \Phi + \bar{\zeta}_{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \star \Phi = \Phi + \zeta^\alpha Q_\alpha \Phi + \bar{\zeta}_{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \Phi,$$
$$\bar{\Phi} \rightarrow \bar{\Phi} + \zeta^\alpha Q_\alpha \star \bar{\Phi} + \bar{\zeta}_{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \star \bar{\Phi} = \bar{\Phi} + \zeta^\alpha Q_\alpha \bar{\Phi} + \bar{\zeta}_{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \bar{\Phi},$$

where \( \zeta \) and \( \bar{\zeta} \) are fermionic spinor parameters. This super-★ symmetry is defined as a symmetry of non(anti-)commutative field theory. As mentioned above, this transformation does not mean the super Poincare group transformation of the superspace, because ̄Q_µ is not derivative and it does not make a group action, in the sense of non(anti-)commutative theory. Therefore, this symmetry is not a superspace symmetry but a symmetry defined by an infinitesimal field transformation. However, arbitrary super-★ symmetric lagrangians satisfy

$$\int d^4x \ Q_\alpha \star L = 0 \ , \ \int d^4x \ \bar{Q}_{\dot{\alpha}} \star L = 0.$$  \hspace{1cm} (40)

We can construct super-★ symmetric lagrangians by the same way of above super-★ symmetric lagrangian construction.

\(^{1}\)This ̄Q_µ first appeared in [9].
There are some comments. The method in this section is formally extended to other 1/2 SUSY theories in non(anti-)commutative superspaces. We expect, for example, 1/2 SUSY gauge theories are deformed to full SUSY theories by this method. However, there may be some problems on eliminating SUSY breaking terms. There are no assurance that this process stop at finite rotation, for some kind of theories like non-abelian gauge theories. Also, it is not clear that the (deformed) gauge invariance is maintained in this procedure. Therefore, more detailed analyses are needed.

The super-*(*) symmetric action in non(anti-)commutative superspace is equivalent to the normal SUSY action given by $V^0(\Phi; *(*)$) replacing *(*) with the usual multiplication, in above two examples. It may be that there are equivalent non(anti-)commutative theories to arbitrary SUSY theories in usual space.

In this article, we treat the * and * deformed theories. There are many kinds of non(anti-)commutative deformations, and it is known that some deformations do not break any SUSY (see, for example, [42]). In any case, when deformed products are expressed by a polynomial with finite terms, the above method is valid to construct full SUSY non(anti-)commutative lagrangians.

4 1-loop Calculations

In the previous section, we gave the prescription to obtain full SUSY actions in non(anti-)commutative superspaces. In this section, we will investigate quantum effects [11]-[22], concentrating on the $\Phi^3$ model for simplicity.

From the results of the previous section, we know that

$$S = \int d^4x d^2\theta d^2\bar{\theta} \Phi \Phi + \int d^4x d^2\theta d^2\bar{\theta} \frac{\bar{m}}{2} \Phi^2 + \int d^4x d^2\theta d^2\bar{\theta} \{ \frac{m}{2} \Phi^2 + g_0 \Phi^3 + g_1 \Phi D^2 \Phi \Phi \}, \tag{41}$$

and

$$S_* = \int d^4x d^2\theta d^2\bar{\theta} \Phi* \Phi + \int d^4x d^2\theta d^2\bar{\theta} \frac{\bar{m}}{2} \Phi_*^2 + \int d^4x d^2\theta d^2\bar{\theta} \{ \frac{m}{2} \Phi_*^2 + g_0* \Phi_*^3 + g_1* \Phi* D^2 \Phi* \Phi* \}, \tag{42}$$

are equivalent, if

$$g_0 = g_0* , \quad g_1 = g_1* - \frac{1}{4} \text{det}P \ g_0*. \tag{43}$$

We will calculate 1-loop graphs and $\beta$-functions based on (41),^2 and interpret the results in terms of (42).

^2In [13], $\beta$-functions based on slightly different action were calculated at 2-loop level. Also, in [20], $\beta$-functions for the non(anti-)commutative gauge theory were calculated at 1-loop level.
Before calculating 1-loop corrections, it is convenient to integrate out $\bar{\Phi}$ in (41) \cite{11, 61}.

$$S = \int d^4x d^2\theta d^2\bar{\theta} \{ \frac{1}{2} \Phi(m - \frac{\Box}{\bar{m}})\Phi + g_0\Phi^3 + g_1\Phi D^2\Phi D^2\Phi \}. \quad (44)$$

Also we have to add a term proportional to $\Phi D^2\Phi$ to renormalize a 1-loop divergent graph (See Fig.1) \cite{11} \cite{12}. So we take the following action as the starting point:

$$S = \int d^4x d^2\theta d^2\bar{\theta} \{ \frac{1}{2}\Phi_b(\frac{1}{4}\lambda_b D^2 + m_b - \frac{\Box}{\bar{m}_b})\Phi_b + g_{0b}\Phi_b^3 + g_{1b}\Phi_b D^2\Phi_b D^2\Phi_b \}, \quad (45)$$

where the subscription $b$ denotes $\Phi_b$ etc. are bare quantities. In \cite{16} \cite{17} \cite{18} \cite{19}, the renormalizability of (45) was proved.

To renormalize 1-loop corrections, we define the following renormalized quantities:

$$\Phi_b = \sqrt{Z_\Phi}(\Phi_r + \delta\Phi) , \quad m_b = m_r + \delta m , \quad \bar{m}_b = Z_{\bar{m}} Z_\Phi \bar{m}_r ,$$

$$\lambda_b = Z_\lambda Z_\Phi \lambda_r , \quad g_{0b} = Z_0 \sqrt{Z_\Phi}^{-3} g_{0r} , \quad g_{1b} = Z_1 \sqrt{Z_\Phi}^{-3} g_{1r} , \quad (46)$$

where

$$Z_\Phi = 1 + Z_\Phi^{(1)} + ... , \quad \delta\Phi = 0 + \delta\Phi^{(1)} + ... ,$$

$$\delta m = 0 + \delta m^{(1)} + ... , \quad Z_{\bar{m}} = 1 + Z_{\bar{m}}^{(1)} + ... ,$$

$$Z_\lambda = 1 + Z_\lambda^{(1)} + ... , \quad Z_0 = 1 + Z_0^{(1)} + ... , \quad Z_1 = 1 + Z_1^{(1)} + ... . \quad (47)$$

1-loop calculations tell us

$$\delta m = 0 , \quad Z_{\bar{m}}^{(1)} = 0 ,$$

$$Z_0^{(1)} = 0 , \quad Z_1^{(1)} = 0 , \quad (48)$$

and

$$\frac{1}{2} Z_\Phi^{(1)} m_r + 3g_{0r}\delta\Phi^{(1)} = 0 , \quad (50)$$

so we find $Z_\lambda$ and $\delta\Phi$ are independent. They are determined by the renormalization for 1-loop divergent graphs, Fig.1 and Fig.2.
It is worthwhile to comment on the role of $\frac{1}{4\lambda} \Phi D^2 \Phi$ in (45). Firstly, as we mentioned above, this is necessary to renormalize the 1-loop divergent graph, Fig.1. Secondly, this term changes the propagator:

\[
\frac{\tilde{m}}{m\tilde{m} + p^2} \Rightarrow \frac{\tilde{m}}{m\tilde{m} + p^2 + \frac{\tilde{m}}{4\lambda} \kappa^2},
\]

\[
= \frac{\tilde{m}}{m\tilde{m} + p^2} - \frac{\tilde{m}}{m\tilde{m} + p^2} \frac{\kappa^2}{4\lambda m\tilde{m} + p^2}, \tag{51}
\]

where $\kappa$ is the conjugate momentum of the fermionic coordinate $\theta$. This change yields the divergent tadpole graph, Fig.2, at the 1-loop level. This non-vanishing tadpole causes the field shift $\delta \Phi$, and the wave function renormalization through the relation (50). It is shown that the above field shift does not lift the vacuum energy, then it does not break the (1/2) SUSY [11, 12].

We adopt the dimensional regularization, that is, we replace the dimension 4 by $n \in \mathbb{C}$, and the minimal subtraction renormalization. So we introduce a scale parameter $\mu$ whose mass dimension is 1 and redefine the renormalized quantities by

\[
m_b = \mu \tilde{m}_r, \tag{52}
\]

\[
\tilde{m}_b = Z_\Phi \mu \tilde{m}_r, \tag{53}
\]

\[
\lambda_b = Z_\lambda Z_\Phi \lambda_r, \tag{54}
\]

\[
g_{0b} = Z_\Phi \frac{\mu}{4\pi} Z_\Phi \lambda_r, \tag{55}
\]

\[
g_{1b} = Z_\Phi \frac{\mu}{4\pi} Z_\Phi \lambda_r. \tag{56}
\]

Here, $\tilde{m}_r$, $\tilde{m}_r$, $\lambda_r$, $g_{0r}$ and $g_{1r}$ are dimensionless.

Let us determine $Z_\lambda$. The contribution of Fig.1 is

\[
\Gamma^{(2)}_{D^2} = -\frac{9g_{0r} \tilde{g}_{1r} \tilde{m}_r^2}{\pi^2} \kappa^2 \left( \frac{1}{\epsilon} + \ldots \right), \tag{57}
\]

where $\epsilon = \frac{4-n}{2}$. The divergent part of (57) is canceled by the counter term $\frac{Z^{(1)}_{\lambda}}{4\lambda_r} \Phi_r D^2 \Phi_r$, so

\[
Z^{(1)}_{\lambda} = \frac{1}{\epsilon} \left( -\frac{36g_{0r} \tilde{g}_{1r} \tilde{m}_r^2 \lambda_r}{\pi^2} \right). \tag{58}
\]

Now we turn to $\delta \Phi$ and $Z_\Phi$. The contribution of Fig.2 is

\[
\Gamma^{(1)} = \frac{3g_{0r} \mu^2 \tilde{m}_r^2}{16\pi^2 \lambda_r} \left( \frac{1}{\epsilon} + \ldots \right). \tag{59}
\]

The divergent part of (59) is canceled by the counter term $\mu \tilde{m}_r \delta \Phi^{(1)}$. From (50), we obtain

\[
Z^{(1)}_{\phi} = \frac{1}{\epsilon} \left( \frac{9g_{0r} \tilde{g}_{1r} \tilde{m}_r^2}{8\pi^2 \tilde{m}_r^2 \lambda_r} \right). \tag{60}
\]
Using (58) and (60), we calculate the $\beta$-functions defined by

$$
\beta_0 = \frac{\partial g_0}{\partial \log \mu}, \quad \beta_1 = \frac{\partial g_1}{\partial \log \mu}, \quad \beta_{\lambda^{-1}} = \frac{\partial \lambda^{-1}}{\partial \log \mu},
$$

$$
\beta_{\tilde{m}} = \frac{\partial \tilde{m}}{\partial \log \mu}, \quad \beta_{\tilde{\tilde{m}}} = \frac{\partial \tilde{\tilde{m}}}{\partial \log \mu}.
$$

(61)

The results are

$$
\beta_0 = \frac{-27 g_0^3 \lambda^{-1} \tilde{m}_r^2}{8\pi^2 \tilde{m}_r^2}, \quad \beta_1 = 2 \tilde{g}_1 r - \frac{27 g_0^2 \tilde{g}_1 \lambda^{-1} \tilde{m}_r^2}{8\pi^2 \tilde{m}_r^2},
$$

$$
\beta_{\lambda^{-1}} = -\frac{9g_0^2 \lambda^{-2} \tilde{m}_r^2}{4\pi^2 \tilde{m}_r^2} + \frac{72g_0 \tilde{g}_1 \lambda^{-1} \tilde{m}_r^2}{\pi^2},
$$

$$
\beta_{\tilde{m}} = -\tilde{m}_r, \quad \beta_{\tilde{\tilde{m}}} = -\tilde{\tilde{m}}_r - \frac{9g_0^2 \lambda^{-1} \tilde{m}_r^3}{4\pi^2 \tilde{m}_r^2}.
$$

(62)

5 Discussions

In this section, we argue the RG behavior of $\Phi^3$ model, particularly, around points realizing full SUSY. Notice that full SUSY is recovered when $g_1 = 0$ and $\lambda^{-1} = 0$. Naively, since $g_1$ has mass dimension $-2$, we expect full SUSY is repaired in the IR limit. However, as we will see in the following, the RG behavior of $\lambda^{-1}$ modifies this speculation.

It makes things clear to classify situations according to the sign of $\lambda^{-1}$. From (62),

(i) $\lambda^{-1} > 0$ (Fig.3a-3c)

For $g_0$, $g_0 \to 0$ in the UV limit. For $\lambda^{-1}$,

- $\lambda^{-1} \to 0$ in IR, if $\{g_0 > 0, g_1 > \frac{\lambda^{-2}}{32\pi^2 \tilde{m}_r^2}g_0\}$ or $\{g_0 < 0, g_1 < \frac{\lambda^{-2}}{32\pi^2 \tilde{m}_r^2}g_0\}$,
- $\lambda^{-1} \to 0$ in UV, if $\{g_0 > 0, g_1 < \frac{\lambda^{-2}}{32\pi^2 \tilde{m}_r^2}g_0\}$ or $\{g_0 < 0, g_1 > \frac{\lambda^{-2}}{32\pi^2 \tilde{m}_r^2}g_0\}$.

(ii) $\lambda^{-1} = 0$ (Fig.4a-4c)

$\beta_0 = 0$. For $\lambda^{-1}$,

- $\beta_{\lambda^{-1}} > 0$, if $\{g_0 > 0, g_1 > 0\}$ or $\{g_0 < 0, g_1 < 0\}$,
- $\beta_{\lambda^{-1}} < 0$, if $\{g_0 > 0, g_1 < 0\}$ or $\{g_0 < 0, g_1 > 0\}$,
- $\beta_{\lambda^{-1}} = 0$, if $\{g_0 = 0\}$ or $\{g_1 = 0\}$.
Fig. 4a: $\beta_{\lambda^{-1}} > 0$ in the grayed region.

Fig. 4b: $\beta_1 > 0$ in the grayed region.

Fig. 4c: $\beta_0 = 0$ in the whole (perturbative) region.

(iii) $\lambda^{-1} < 0$ (Fig. 5a-5c)

For $g_0$, $g_0 \to 0$ in the IR limit. For $\lambda^{-1}$,
- $\lambda^{-1} \to 0$ in UV, if $\{g_0 > 0, \ g_1 > \frac{\lambda^{-2}}{32m^2}g_0\}$ or $\{g_0 < 0, \ g_1 < \frac{\lambda^{-2}}{32m^2}g_0\}$,
- $\lambda^{-1} \to 0$ in IR, if $\{g_0 > 0, \ g_1 < \frac{\lambda^{-2}}{32m^2}g_0\}$ or $\{g_0 < 0, \ g_1 > \frac{\lambda^{-2}}{32m^2}g_0\}$.

Fig. 5a: $\beta_{\lambda^{-1}} > 0$ in the grayed region. The oblique line represents $g_1 = \frac{\lambda^{-2}}{32m^2}g_0$.

Fig. 5b: $\beta_1 > 0$ in the grayed region. The oblique line represents $g_1 = \frac{\lambda^{-2}}{32m^2}g_0$.

Fig. 5c: $\beta_0 > 0$ in the grayed region. The oblique line represents $g_1 = \frac{\lambda^{-2}}{32m^2}g_0$.

From the above results, full SUSY is realized in the IR limit if the RG flow starts from

- $\{\lambda^{-1} > 0, \ g_0 > 0, \ g_1 > \frac{\lambda^{-2}}{32m^2}g_0\}$, (63)
- $\{\lambda^{-1} > 0, \ g_0 < 0, \ g_1 < \frac{\lambda^{-2}}{32m^2}g_0\}$, (64)
- $\{\lambda^{-1} < 0, \ g_0 > 0, \ g_1 < \frac{\lambda^{-2}}{32m^2}g_0\}$, (65)
- $\{\lambda^{-1} < 0, \ g_0 > 0, \ g_1 > \frac{\lambda^{-2}}{32m^2}g_0\}$. (66)

By using the relations (43) and $\lambda_* = \lambda$, the above argument can be rewritten in terms of the non(anti-)commutative field theory and the super-* symmetry.
(i) $\lambda_*^{-1} > 0$ (Fig.6a-6c)

For $g_{0*}$, $g_{0*} \to 0$ in the UV limit. For $\lambda_*^{-1}$,

- $\lambda_*^{-1} \to 0$ in IR,
  - if $\{g_{0*} > 0, g_{1*} > (\frac{1}{4} det P + \frac{\lambda_*^{-2}}{32m^2}) g_{0*}\}$ or $\{g_{0*} < 0, g_{1*} < (\frac{1}{4} det P + \frac{\lambda_*^{-2}}{32m^2}) g_{0*}\}$,

- $\lambda_*^{-1} \to 0$ in UV,
  - if $\{g_{0*} > 0, g_{1*} < (\frac{1}{4} det P + \frac{\lambda_*^{-2}}{32m^2}) g_{0*}\}$ or $\{g_{0*} < 0, g_{1*} > (\frac{1}{4} det P + \frac{\lambda_*^{-2}}{32m^2}) g_{0*}\}$.

(ii) $\lambda_*^{-1} = 0$ (Fig.7a-7c)

$\beta_0 = 0$. For $\lambda_*^{-1}$,

- $\beta_*^{-1} > 0$, if $\{g_{0*} > 0, g_{1*} > (\frac{1}{4} det P g_{0*})\}$ or $\{g_{0*} < 0, g_{1*} < (\frac{1}{4} det P g_{0*})\}$,

- $\beta_*^{-1} < 0$, if $\{g_{0*} > 0, g_{1*} < (\frac{1}{4} det P g_{0*})\}$ or $\{g_{0*} < 0, g_{1*} > (\frac{1}{4} det P g_{0*})\}$,

- $\beta_*^{-1} = 0$, if $g_{0*} = 0$ or $g_{1*} = (\frac{1}{4} det P g_{0*})$.

(iii) $\lambda_*^{-1} < 0$ (Fig.8a-8c)

For $g_{0*}$, $g_{0*} \to 0$ in the IR limit. For $\lambda_*^{-1}$,
\[ \frac{\lambda^*_1}{\lambda^*} \to 0 \quad \text{in UV,} \]

if \( \{ g_0^* > 0 \quad , \quad g_1^* > \left( \frac{1}{4} \det P + \frac{\lambda^*_1}{32 m^2} \right) g_0^* \} \) or \( \{ g_0^* < 0 \quad , \quad g_1^* > \left( \frac{1}{4} \det P + \frac{\lambda^*_1}{32 m^2} \right) g_0^* \} , \)

\[ \lambda^*_1 \to 0 \quad \text{in IR,} \]

if \( \{ g_0^* > 0 \quad , \quad g_1^* < \left( \frac{1}{4} \det P + \frac{\lambda^*_1}{32 m^2} \right) g_0^* \} \) or \( \{ g_0^* < 0 \quad , \quad g_1^* > \left( \frac{1}{4} \det P + \frac{\lambda^*_1}{32 m^2} \right) g_0^* \} . \)

6 Conclusions

In this article we gave the general prescription to construct full SUSY lagrangians in non(anti-)commutative superspaces, which is relevant for both the SUSY \( \ast \)-deformed superspace and the non-SUSY \( \ast \)-deformed superspace.

We investigated quantum effects at the 1-loop level for the deformed \( \Phi^3 \) Wess-Zumino model with the \( \Phi D^2 \Phi \) proportional term. We found that this term yields a divergent tadpole graph and the renormalization of this causes the wave function renormalization. The wave function renormalization gave the non-trivial \( \beta \)-functions.
From the obtained $\beta$-functions, we found the conditions on the parameters $(g_0, g_1, \lambda^{-1})$ to recover full SUSY in the IR limit. We also rewrote the conditions into ones on $(g_0^*, g_1^*, \lambda_*^{-1})$, the parameters of the non(anti-)commutative field theories.

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