A Novel Spatial–Temporal Radial Trefftz Collocation Method for 3D Transient Wave Propagation Analysis with Specified Sound Source Excitation

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Abstract: In this paper, a novel semi-analytical collocation solver, the spatial–temporal radial Trefftz collocation method (STRTCM) is proposed to solve 3D transient wave equations with specified sound source excitations. Unlike the traditional time discretization strategies, the proposed numerical scheme introduces the spatial–temporal radial Trefftz functions (STRTFs) as the basis functions for the spatial and temporal discretization of the transient wave equations. The STRTFs are constructed in the spatial–temporal domain, which is a combination of 3D Euclidean space and time into a 4D manifold. Moreover, since the initial and boundary conditions are imposed on the spatial–temporal domain boundaries, the original transient wave propagation problem can be converted to an inverse boundary value problem. To deal with the specified time-dependent sound source excitations, the composite multiple reciprocity technique is extended from the spatial domain to the spatial–temporal domain, which transforms the original problem with a source term into a high-order problem without a source term. By deriving the related STRTFs for the considered high-order problem, the proposed scheme only requires the node discretization on the spatial–temporal domain boundaries. The efficiency of the proposed method is numerically verified by four benchmark examples under 3D transient wave equations with specified time-dependent sound source excitation.

Keywords: meshless collocation method; semi-analytical; radial Trefftz basis functions; transient wave propagation analysis

MSC: 65M70; 35L05

1. Introduction

As is well known, the phenomenon of wave propagation [1–7] widely exists in various areas of science and engineering, such as acoustics, elastodynamics, electromagnetics, and fluid dynamics. Numerical simulation plays an important role in understanding and mastering the fundamental laws of such wave propagation. Traditional numerical methods [8–10], such as the finite difference method and finite element method (FEM), have been widely used in wave propagation analysis. However, they usually have the problems of low computational efficiency and poor computational accuracy due to the use of universal polynomial functions. To overcome these drawbacks, several basis functions, including wave characteristics, have been introduced, and then a series of semi-analytical numerical methods have been constructed, such as the wave-based method [11], scaled boundary finite element method [12] and boundary element method [13–15], and so on. The aforementioned numerical methods belong to the mesh-based methods, which are sensitive to the mesh quality. To eliminate the effect of mesh generation, a group of meshless methods is developed. Similarly, by introducing the basis functions including the wave characteristics, several boundary-type meshless methods are proposed. They can be divided into two categories: weak-form boundary meshless methods and strong-form boundary meshless methods. The weak-form boundary meshless methods mainly
include the local boundary integral equation method [16], boundary node method [17], hybrid boundary node method [18], boundary face method [19], null-field boundary integral equation method [20], and so on. The strong-form boundary meshless methods mainly include the wave superposition method [21,22], method of fundamental solutions (MFS) [23,24], regularized meshless method [25], boundary distributed source method [26], singular boundary method (SBM) [27–31], collocation Trefftz method (CTM) [32,33], and so on. Due to their simpler form, integral-free and easy-to-use merits, this study focused on the strong-form boundary meshless methods based on the semi-analytical basis functions.

A broader and more challenging problem in wave propagation analysis is the simulation of wave propagation in the time domain. The following three popular approaches have been widely used to treat transient wave propagation problems: (1) Time-stepping methods (TSM) [23,34] transform the transient wave propagation problem into a series of time-independent problems, and the accuracy and stability of this method highly depend on the time-step size. (2) Frequency domain techniques (FDT) [35,36] use the transformation technique to eliminate the time derivative leading to a time-independent equation in the frequency domain, and then employ a numerical inversion scheme to invert the frequency domain solutions back into the time-dependent solutions. The FDT does not require time stepping, and thus avoids the effect of the time step on numerical accuracy. However, for systems with no intrinsic damping and mismatched initial and ending responses, the numerical inversion transformation fails to produce accurate results. This is why in practical calculations often small artificial damping is added to the model. (3) Spatial–temporal semi-analytical basis function methods [37–39] employ the spatial–temporal semi-analytical basis function a priori to satisfy the transient wave equation and then solve it directly. Among these three time-discretization schemes, the first two have been widely used for transient wave propagation analysis; the last one has not been widely used because the time-dependent semi-analytical basis functions are not easy to construct, in particular, the transient wave equation, including the source excitations.

Fortunately, the composite multiple reciprocity method (CMRM) [40] has been proposed and applied to deal with some specified source terms in time-independent nonhomogeneous PDEs. In this study, the CMRM is extended from the spatial domain to the spatial–temporal domain, which transforms the original transient wave propagation problem with a source term into the high-order time-dependent problem without a source term. Then a group of spatial–temporal semi-analytical basis functions and spatial–temporal radial Trefftz functions are derived to satisfy the governing equation of such high-order time-dependent problem in advance. Correspondingly, the so-called spatial–temporal radial Trefftz collocation method (STRTC) is constructed to solve 3D transient wave equations with specified sound source excitations, which only require the node discretization on the spatial–temporal domain boundaries.

In this paper, a novel spatial–temporal radial Trefftz collocation method is proposed without differential approximation for the temporal derivatives, which may cause the accumulated error to solve the 3D transient wave equations, and the composite multiple reciprocity method is extended from the space domain to the spatial–temporal domain to treat the time-dependent source term. Due to the use of the related spatial–temporal radial Trefftz functions, the proposed STRTC requires fewer node discretizations to obtain more accurate results. A brief outline of this paper is as follows. In Section 2, the numerical procedure of the spatial–temporal radial Trefftz collocation method for solving 3D transient wave equations with specified sound source excitations is introduced. The efficiency of the proposed method is numerically verified by four benchmark examples in Section 3. In Section 4, several conclusions are drawn based on the present study.
2. Methodology

Considering a transient wave equation in 3D finite domain \( \Omega \) bounded by \( \Gamma \), the governing equation of 3D transient wave propagation problem with sound source excitations is stated as follows:

\[
\left( \frac{\partial^2}{\partial t^2} - \nu^2 \Delta \right) u(x, t) = f(x, t), \quad x \in \Omega, \quad 0 < t \leq T
\]

(1)

subjected to the initial conditions

\[
u(x, t) \big|_{t=0} = u_0, \quad x \in \Omega
\]

(2)

\[
\frac{\partial u(x, t)}{\partial t} \big|_{t=0} = u_1, \quad x \in \Omega
\]

(3)

and the Dirichlet boundary condition

\[
u(x, t) \big|_{\Gamma} = \bar{\nu}, \quad x \in \Gamma, \quad 0 < t \leq T.
\]

(4)

where \( \Delta \) is the Laplace operator, \( \nu \) denotes the wave speed, \( T \) represents the final time instant, \( u_0, u_1 \) and \( \bar{\nu} \) are the known functions, and \( f(x, t) \) is the known sound source function.

If the sound source function \( f(x, t) = 0 \), then the homogeneous type of Equation (1) is obtained as

\[
\left( \frac{\partial^2}{\partial t^2} - \nu^2 \Delta \right) u(x, t) = 0, \quad x \in \Omega, \quad 0 < t \leq T.
\]

(5)

By using the derived spatial–temporal radial Trefftz function \([41]\),

\[
G_0(x, t; s, \tau) = \left[ \cos(\nu(t - \tau)) + \frac{\sin(\nu(t - \tau))}{\nu} \right] \frac{\sin(r(x, s))}{r(x, s)}
\]

(6)

the approximate solution of Equation (5) by using the spatial–temporal radial Trefftz collocation method can be represented as follows

\[
u^0(x, t) = \sum_{j=1}^{N_S} \alpha_{0j} G_0(x, t; s_j, \tau_j)
\]

(7)

where \( r(x, s) = \|x - s\|_2 \) denotes the Euclidean distance between collocation nodes \( x_i \) and source nodes \( s_j \), \( t \) and \( \tau \) are the time variables corresponding to the collocation nodes \( x_i \) and source nodes \( s_j \), respectively. \( \{\alpha_{0j}\} \) are the unknown coefficients and \( N_S \) represents the number of the source node pair \( (s_j, \tau_j) \in \partial(\Omega \times (0, T)) \), in which \( \partial(\Omega \times (0, T)) = [\Gamma \times (0, T)] \cup [\Omega \times \{0, T\}] \) stands for the boundaries of the considered spatial–temporal domain \( \Omega \times (0, T) \). Substituting expression (7) into the initial conditions (2) (3) and boundary conditions (4), one may have

\[
\sum_{j=1}^{N_S} \alpha_{0j} G_0(x, 0; s_j, \tau_j) = u_0, \quad x \in \Omega, \quad t = 0
\]

(8)

\[
\sum_{j=1}^{N_S} \alpha_{0j} \frac{\partial G_0(x, 0; s_j, \tau_j)}{\partial t} = u_1, \quad x \in \Omega, \quad t = 0
\]

(9)

\[
\sum_{j=1}^{N_S} \alpha_{0j} G_0(x, t; s_j, \tau_j) = \bar{\nu}, \quad x \in \Gamma, \quad 0 < t \leq T
\]

(10)
To determine the unknown coefficient \( \{ \alpha_0 \} \), \( N \) collocation node pairs \((x_i, t_i) \) in \([\Gamma \times (0,T)] \cup [\Omega \times \{0\}]\) are placed on the boundaries of the considered spatial–temporal domain, and then the discretized formulation can be represented as follows

\[
\sum_{j=1}^{N_5} \alpha_{0j} G_0(x_i, t_i; s_j, \tau_j) = u_0, \quad x_i \in \Omega, \quad t_i = 0
\]  
\[
\sum_{j=1}^{N_5} \frac{\partial G_0(x_i, t_i; s_j, \tau_j)}{\partial t} = u_1, \quad x_i \in \Omega, \quad t_i = 0
\]  
\[
\sum_{j=1}^{N_5} \alpha_{0j} G_0(x_i, t_i; s_j, \tau_j) = \bar{u}_0, \quad x_i \in \Gamma, \quad 0 < t_i \leq T.
\]

which can be also written as the following matrix form

\[
\begin{bmatrix}
G_0(x_i, t_i; s_j, \tau_j) \\
\frac{\partial G_0(x_i, t_i; s_j, \tau_j)}{\partial t} \\
G_0(x_i, t_i; s_j, \tau_j)
\end{bmatrix}_{N_5 \times N_5}
\begin{bmatrix}
\alpha_{0j}
\end{bmatrix}_{N_5 \times 1} =
\begin{bmatrix}
[u_0]_{N_5 \times 1} \\
[u_1]_{N_5 \times 1} \\
[\bar{u}_0]_{N_5 \times 1}
\end{bmatrix}
\]  

(14)

where \( N = 2N_t + N_pN_t \), in which \( N_t \) and \( N_p \) represent the number of the collocation nodes inside the spatial domain \( \Omega \) and on the boundary \( \Gamma \) of spatial domain \( \Omega \), respectively. The total node number in the spatial domain \( \Omega \) is \( N_{\text{total}} = N_t + N_p \), and \( N_t \) represents the number of the collocation nodes along the time axis. If the same set of nodes to the collocation node discretization is used in the source node discretization inside the spatial domain \( \Omega \) and on the boundary \( \Gamma \) of spatial domain \( \Omega \), the resultant matrix in Equation (14) is square due to \( N_5 = 2N_t + N_pN_t \).

Next, consider 3D transient wave propagation problems (1–4) with the non-zero sound source function \( f(x,t) \). The approximate solution can be first divided into two parts, homogeneous solution \( u_h(x,t) \) and particular solution \( u_p(x,t) \), i.e.,

\[
u(x,t) = u_h(x,t) + u_p(x,t)
\]  

(15)

where the particular solution \( u_p(x,t) \) is constructed to satisfy the following equation

\[
\left( \frac{\partial^2}{\partial t^2} - v^2 \Delta \right) u_p(x,t) = f(x,t)
\]  

(16)

and then the following updated homogeneous problem can be represented by substituting Equations (15) and (16) into the original transient wave propagation problems (1–4),

\[
\begin{cases}
\left( \frac{\partial^2}{\partial t^2} - v^2 \Delta \right) u_h(x,t) = 0, \quad x \in \Omega, \quad 0 < t \leq T \\
u_b(x,t) |_{\Gamma} = \bar{u} - u_p \\
u_b(x,t) |_{t=0} = u_0 - u_p \\
\frac{\partial u_b(x,t)}{\partial t} |_{t=0} = u_1 - \frac{\partial u_p}{\partial t}
\end{cases}
\]  

(17)

where the homogeneous solution \( u_b(x,t) \) of Equation (17) can be obtained by using the spatial–temporal radial Trefftz collocation method with node discretization on the boundaries of the considered spatial–temporal domain.

To evaluate the particular solution \( u_p(x,t) \), the composite multiple reciprocity method (CMRM) is extended from the spatial domain to the spatial–temporal domain. The key issue...
is to introduce the different spatial/spatial–temporal differential operators $L_1, L_2, \cdots L_M$ to eliminate the related non-zero sound source function $f(x,t)$ in Equation (16), namely,

$$L_M \cdots L_2 L_1 f(x,t) \equiv 0 \quad (18)$$

It should be pointed out that the commonly used differential operators (Laplace, Helmholtz, modified Helmholtz, diffusion equation and wave equation operators) can be chosen as $L_1, L_2, \cdots L_M$ according to the form of $f(x,t)$, which could be the polynomial, exponential and trigonometric functions, or a combination of these functions. For complex functions, e.g., non-smooth functions, a set of discrete measured data or large-gradient functions in the source term $f(x,t)$, we can express the complex functions or a set of discrete measured data by a series representation of polynomial or trigonometric functions, and then Laplace and Helmholtz operators can be chosen as $L_1, L_2, \cdots L_M$ to satisfy Equation (18). In the numerical implementation, the order $M$ is usually finite, or can be determined by a specified truncation error.

Then the particular solution $u_p(x,t)$ can be obtained by solving the following $m$-order homogeneous equation with $m - 1$ constraint conditions

$$\begin{cases}
L_m \cdots L_2 L_1 \mathcal{R} u_p(x,t) = 0 & (x,t) \in (\Omega \times (0,T]) \\
\vdots \\
L_2 L_1 \mathcal{R} u_p(x,t) = L_2 L_1 f(x,t) & (x,t) \in \partial(\Omega \times (0,T]) \\
L_1 \mathcal{R} u_p(x,t) = L_1 f(x,t) & (x,t) \in \partial(\Omega \times (0,T]) \\
\mathcal{R} u_p(x,t) = f(x,t) & (x,t) \in \partial(\Omega \times (0,T])
\end{cases} \quad (19)$$

where $\mathcal{R} = (\frac{\partial^2}{\partial t^2} - \nabla^2) \Omega$ denotes the governing differential operator. Then the particular solution $u_p(x,t)$ can be represented by a linear combination of high-order spatial–temporal radial Trefftz functions $G_1(x,t;s, \tau), \cdots, G_m(x,t;s, \tau)$, namely,

$$u_p(x,t) = \sum_{k=1}^{m} \sum_{j=1}^{N_k} \alpha_{kj} G_k(x,t;s_j, \tau_j) \quad (20)$$

where the high-order spatial–temporal radial Trefftz functions are derived by satisfying the following equations:

$$\begin{cases}
L_1 \mathcal{R} G_1(x,t;s_j, \tau_j) = 0 \\
\vdots \\
L_m \cdots L_2 L_1 \mathcal{R} G_m(x,t;s_j, \tau_j) = 0
\end{cases} \quad (21)$$

Table 1 lists the related radial Trefftz functions for several commonly used spatial/spatial–temporal differential operators.

| $L_k$ | 3D |
|-------|----|
| $\Delta + k^2$ | $\sin(kr)/(4\pi r)$ |
| $\Delta - k^2$ | $\sinh(kr)/(4\pi r)$ |
| $\frac{\partial}{\partial t} - k\Delta$ | $e^{t(\pi - \tau)}\sin(r)$ |
| $\frac{\partial^2}{\partial t^2} - \nu^2 \Delta$ | $\left[\cos(\nu_k(t - \tau)) + \frac{\sin(\nu_k(t - \tau))}{\nu_k}\right]\sin(r)$ |
By combining Equations (7), (15) and (20), the approximate solution of 3D transient wave propagation problems (1–4) can be expressed as follows:

\[ u(x, t) = \sum_{k=0}^{m} \sum_{j=1}^{N_{S}} a_{kj} G_{k}(x, t; s_{j}, \tau_{j}), \]  

(22)

By employing Equation (22) to discretize Equations (17) and (19), the unknown coefficients \( \{a_{kj}\}_{k=0,1,\cdots,m} \) can be obtained. After that, the numerical solution at any node pair \((x, t) \in [\Omega \times (0, T)]\) can be calculated by using Equation (22).

3. Numerical Results and Discussions

This section presents four benchmark examples of 3D transient wave propagation problems with specified sound source excitations to verify the efficiency of the proposed spatial–temporal radial Trefftz collocation method (STRTC). To assess the performance of the proposed solver, the following \( L_{2} \) relative error \( Lerr \), relative error \( Rerr \) and maximum relative error \( MRE \) are adopted as follows:

\[ Lerr = \sqrt{\frac{1}{NN} \sum_{n=1}^{NN} \left( \frac{u_{num}(x_{n}, t) - u_{ana}(x_{n}, t)}{u_{ana}(x_{n}, t)} \right)^{2}} \]  

(23)

\[ Rerr = \frac{u_{num}(x_{n}, t) - u_{ana}(x_{n}, t)}{u_{ana}(x_{n}, t)} \]  

(24)

\[ MRE = \max_{1 \leq n \leq NN} \left| \frac{u_{num}(x_{n}, t) - u_{ana}(x_{n}, t)}{u_{ana}(x_{n}, t)} \right| \]  

(25)

where \( u_{ana}(x_{n}, t) \) and \( u_{num}(x_{n}, t) \) stand for the analytical solution and the numerical solution on the test nodes \( x_{n} \in \Omega, n = 1, \cdots, NN \) at time instant \( t \), respectively. \( NN \) denotes the number of test nodes \( \{x_{n}\} \). Unless otherwise specified, the test nodes \( \{x_{n}\} \) are chosen as the same set of the collocation nodes inside 3D spatial domain \( \{x_{i}\} \in \Omega \setminus \Gamma \) and \( NN = N_{i} \) in this study.

Example 1. Transient wave equation with specified sound source excitation under a unit cube.

In this example, the efficiency and accuracy of the proposed spatial–temporal radial Trefftz collocation method (STRTC) in the solution of transient wave equations with the sound source \( f(x, t) = -(\sin(x_1) + \cos(x_2) + \sin(x_3)) \cos(\sqrt{2}t) \) under the unit cubic domain \( \Omega_{1} = \{(x_{1},x_{2},x_{3}|0 \leq x_{1},x_{2},x_{3} \leq 1}\} \) are investigated. The geometry and node distribution are depicted in Figure 1. The governing equation is represented as

\[ \left( \frac{\partial^{2}}{\partial t^{2}} - \Delta \right) u(x, t) = - (\sin(x_1) + \cos(x_2) + \sin(x_3)) \cos(\sqrt{2}t), \ x \in \Omega, \ 0 < t \leq T, \]  

(26)

subjected to the initial conditions

\[ u(x, t)|_{t=0} = (\sin(x_1) + \cos(x_2) + \sin(x_3)), \ x \in \Omega, \]  

(27)

\[ \frac{\partial u(x, t)}{\partial t} |_{t=0} = 0, \ x \in \Omega, \]  

(28)

and the Dirichlet boundary condition

\[ u(x, t)|_{\Gamma} = (\sin(x_1) + \cos(x_2) + \sin(x_3)) \cos(\sqrt{2}t), \ x \in \Gamma, \ 0 < t \leq T \]  

(29)
chosen as the same set of the collocation nodes inside 3D spatial–temporal domain. The number of total nodes distributed on the boundaries of spatial–temporal domain is 4892. Four-node tetrahedral elements are used in the COMSOL simulation. In addition, some other nodal arrangements are investigated. The geometry and node distribution are depicted in Figure 1. The governing equation is represented as

\[
\partial \Delta = -\partial - \Theta \in \Omega < \leq 2
\]

For ease of comparison, the same set of discretized nodes are used in both the proposed STRTCM and COMSOL (FEM). Figure 2 shows the numerical errors along with the time evolution by using the proposed STRTCM in Example 1. It can be observed that the numerical errors may slightly increase with the time evolution. Table 3 presents relative errors obtained by the proposed STRTCM with different total node numbers, it can be found that with the increasing total node number, the \(L_2\) relative error decreases rapidly and then remains at the same level.

**Table 2.** Relative errors obtained by the proposed STRTCM and COMSOL at different time instants in Example 1.

|          | \(T = 0.1\) s | \(t = 0.5\) s | \(T = 1.0\) s | CPU Time |
|----------|--------------|--------------|--------------|----------|
| STRTCM   | \(1.92 \times 10^{-7}\) | \(6.21 \times 10^{-7}\) | \(1.20 \times 10^{-6}\) | \(1.76\) s |
| COMSOL   | \(2.14 \times 10^{-3}\) | \(2.33 \times 10^{-2}\) | \(5.98 \times 10^{-2}\) | \(1.00\) s |

\[u(x, t) = (\sin(x_1) + \cos(x_2) + \sin(x_3)) \cos(\sqrt{2}t)\]  

(30)

Figure 1. Schematic configurations of the unit cube model. (a) Geometry, (b) Node distribution.

Its analytical solution of Example 1 is

\[u(x, t) = (\sin(x_1) + \cos(x_2) + \sin(x_3)) \cos(\sqrt{2}t)\]

(30)

Table 2 presents \(L_2\) relative errors obtained by the proposed STRTCM and COMSOL at different time instants in Example 1. It can be found that under the same node discretization, the proposed STRTCM produces more accurate results with a slight longer computational time than the COMSOL (FEM). Figure 2 shows the numerical errors along with the time evolution by using the proposed STRTCM in Example 1. From Table 2 and Figure 2, it can be observed that the numerical errors may slightly increase with the time evolution. Table 3 presents \(L_2\) relative errors obtained by the proposed STRTCM with different total node numbers, it can be found that with the increasing total node number, the \(L_2\) relative error decreases rapidly and then remains at the same level.

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Figure 2. Numerical errors along with the time evolution by using the proposed STRTCM in Example 1.
Table 3. Relative errors obtained by the proposed STRTCM with different total node numbers in Example 1.

| $N_{\text{Total}}$ | 32   | 81   | 432  | 896  | 1600 | 4725 |
|---------------------|------|------|------|------|------|------|
| $L_{\text{err}}$    | $1.01 \times 10^{-1}$ | $4.04 \times 10^{-6}$ | $1.30 \times 10^{-6}$ | $1.31 \times 10^{-6}$ | $1.06 \times 10^{-6}$ | $1.34 \times 10^{-6}$ |

For ease of comparison, the same set of discretized nodes are used in both the proposed STRTCM and COMSOL, in which the number of collocation nodes is $N_{\text{Total}} = 1023$, the boundary nodes number is $N_b = 374$ and the interior nodes number is $N_i = 649$, and the number of total nodes distributed on the boundaries of spatial–temporal domain is $N = 5421$. It should be mentioned that, based on these collocation nodes, 4892 four-node tetrahedral elements are used in the COMSOL simulation. In addition, some other parameters are set as follows: the wave speed $v = 1.0$ m/s, the final time $T = 1.0$ s, the time interval $dt = 0.1$ s. The annihilation spatial–temporal differential operator $L_1 = \left( \frac{\partial^2}{\partial t^2} - v^2 \Delta \right)$ is employed to vanish the specified sound source excitation $f(x,t)$ in Equation (26) by using the extended CMRM.

Example 2. Transient wave equation with specified sound source excitation under a circular tube.

This example considers the transient wave equations with the sound source $f(x,t) = -\sin(\frac{x_1 + x_2 + x_3}{\sqrt{3}}) \cos(340\sqrt{2}t)$ under the circular tube domain as shown in Figure 3a. The distribution of boundary nodes and interior nodes is depicted in Figure 3b. The governing equation is represented as

$$ \left( \frac{\partial^2}{\partial t^2} - v^2 \Delta \right) u(x,t) = -\sin(\frac{x_1 + x_2 + x_3}{\sqrt{3}}) \cos(340\sqrt{2}t), \quad x \in \Omega, \quad 0 < t \leq T, \quad (31) $$

subjected to the initial conditions

$$ u(x,t)|_{t=0} = -\sin(\frac{x_1 + x_2 + x_3}{\sqrt{3}}), \quad x \in \Omega \quad (32) $$

and the Dirichlet boundary condition

$$ u(x,t)|_{\Gamma} = \sin(\frac{x_1 + x_2 + x_3}{\sqrt{3}}) \cos(340\sqrt{2}t), \quad x \in \Gamma, \quad 0 < t \leq T \quad (34) $$

Figure 3. Schematic configurations of the circular tube model. (a) Geometry, (b) Node distribution.
Its analytical solution of Example 2 is

\[ u(x, t) = \sin\left(\frac{x_1 + x_2 + x_3}{\sqrt{3}}\right) \cos(340\sqrt{2}t) \]  

(35)

In the proposed STRTCM implementation, the number of collocation nodes is \( N_{\text{total}} = 780 \), the boundary nodes number is \( N_b = 499 \), the interior nodes number is \( N_l = 281 \), the wave speed is \( v = 340 \text{ m/s} \), the final time instant is \( T = 10 \text{ s} \), and the annihilation spatial–temporal differential operator \( L_1 = \left(\frac{\partial^2}{\partial t^2} - 231200\Delta\right) \) is employed to vanish the specified sound source excitation \( f(x, t) \) in Equation (31) by using the extended CMRM.

Table 4 presents numerical errors at final time instant \( T = 10 \text{ s} \) obtained by using the proposed STRTCM with different time intervals \( dt \) in Example 2. From Table 3, it can be found that the proposed STRTCM with different time intervals \( dt \) can provide equally accurate results, which reveals that the time interval \( dt \) has a slight influence on the numerical accuracy. Figure 4 plots the absolute and relative error distributions at two time instants (\( t = 5 \text{ s} \) and \( 10 \text{ s} \)) by using the proposed STRTCM with a large time interval \( dt = 5.0 \text{ s} \). Numerical results given in Figure 4 show that the proposed STRTCM performs very accurate results, even with large time interval \( dt = 5.0 \text{ s} \).

**Table 4.** Numerical errors at final time instant \( T = 10 \text{ s} \) obtained by using the proposed STRTCM with different time intervals \( dt \) in Example 2.

| \( dt \) | 0.5 s | 1.0 s | 2.0 s | 2.5 s | 5.0 s | 10 s |
|---|---|---|---|---|---|---|
| **Lerr** | \( 1.55 \times 10^{-5} \) | \( 3.84 \times 10^{-6} \) | \( 3.76 \times 10^{-6} \) | \( 2.34 \times 10^{-5} \) | \( 2.54 \times 10^{-6} \) | \( 3.23 \times 10^{-5} \) |
| **MRE** | \( 3.14 \times 10^{-5} \) | \( 9.94 \times 10^{-6} \) | \( 9.96 \times 10^{-6} \) | \( 8.98 \times 10^{-5} \) | \( 9.77 \times 10^{-6} \) | \( 9.86 \times 10^{-5} \) |

Figure 4. Error distributions at two time instants (\( t = 5 \text{ s} \) and \( 10 \text{ s} \)) by using the proposed STRTCM with large time interval \( dt = 5.0 \text{ s} \): (a) absolute error distribution and (b) relative error distribution at \( t = 5 \text{ s} \); (c) absolute error distribution and (d) relative error distribution at \( t = 10 \text{ s} \).
Example 3. Transient wave equation with specified sound source excitation under a room model.

This example considers the transient wave equations with the sound source
\[ f(x, t) = -(\cos(x_1) + \sin(x_2) + \cos(x_3)) \sin(\sqrt{2}t) \]
under the room model [42] with principal dimensions being 5.0 m in length, 4.0 m in width and 3.0 m in height (see Figure 5a). The governing equation is represented as
\[
\left( \frac{\partial^2}{\partial t^2} - \sigma^2 \Delta \right) u(x, t) = -(\cos(x_1) + \sin(x_2) + \cos(x_3)) \sin(\sqrt{2}t), \ x \in \Omega, \ 0 < t \leq T \quad (36)
\]
subjected to the initial conditions
\[
u(x, t)|_{t=0} = 0, \ x \in \Omega \quad (37)
\]
\[
\frac{\partial u(x, t)}{\partial t}|_{t=0} = \sqrt{2}(\cos(x_1) + \sin(x_2) + \cos(x_3)) \cos(\sqrt{2}t), \ x \in \Omega \quad (38)
\]
and the Dirichlet boundary condition
\[
u(x, t)|_{\Gamma} = (\cos(x_1) + \sin(x_2) + \cos(x_3)) \sin(\sqrt{2}t), \ x \in \Gamma, \ 0 < t \leq T \quad (39)
\]

![Figure 5. Schematic configurations of the room model. (a) Geometry, (b) Node distribution.](image)

Its analytical solution of Example 3 is
\[
u(x, t) = (\cos(x_1) + \sin(x_2) + \cos(x_3)) \sin(\sqrt{2}t) \quad (40)
\]

In the proposed STRTCM implementation, the number of collocation nodes is \( N_{\text{total}} = 3588 \), the boundary nodes number is \( N_b = 1447 \), the interior nodes number is \( N_i = 2141 \), the number of total nodes distributed on the boundaries of the spatial-temporal domain is \( N = 17,305 \), the wave speed is \( v = 1.0 \) m/s, the final time instant is \( T = 1000 \) s, the time interval is \( dt = 125 \) s, and the annihilation spatial-temporal differential operator \( L_1 = \left( \frac{\partial^2}{\partial t^2} - 2\Delta \right) \) is employed to vanish the specified sound source excitation \( f(x, t) \) in Equation (36) by using the extended CMRM.

Figure 6 shows the relative error distributions on the plane \( x_3 = 1.5 \) at different time instants \( (t = 250, 500, 750, 1000 \) s\) by using the proposed STRTCM with time interval \( dt = 125 \) s in Example 3. It can be observed that the proposed STRTCM with a large
time interval can still obtain very accurate results in the solution of the transient wave problem under a complicated geometry domain (room model). This is because the semi-analytical spatial–temporal radial Trefftz functions are employed as the basis functions in the proposed STRTCM, which allows the few temporal discretizations to simulate the long-term evolution of the wave propagation.

![Figure 6.](image)

**Figure 6.** Relative error distributions on the plane $x_3 = 1.5$ at different time instants ((a) $t = 250$, (b) $t = 500$, (c) $t = 750$, (d) $t = 1000$ s) by using the proposed STRTCM with time interval $dt = 125$ s in Example 3.

**Example 4. Transient wave equation with specified sound source excitation under a submarine model.**

The final example considers the transient wave equations with the sound source $f(x, t) = -(\cos(x_1) + \sin(x_2) + \cos(x_3))\sin(\sqrt{2}t)$ under the submarine model with the principal dimensions being $15.0$ m in length, $4.0$ m in width and $6.0$ m in height (see Figure 7a). The distribution of boundary nodes and interior nodes are depicted in Figure 7b.

The governing equation is represented as

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)u(x, t) = -(\cos(x_1) + \sin(x_2) + \cos(x_3))\sin(\sqrt{2}t), \ x \in \Omega, \ 0 < t \leq T \quad (41)$$

subjected to the initial conditions

$$u(x, t)|_{t=0} = 0, \ x \in \Omega \quad (42)$$

$$\frac{\partial u(x, t)}{\partial t}|_{t=0} = \sqrt{2}(\cos(x_1) + \sin(x_2) + \cos(x_3))\cos(\sqrt{2}t), \ x \in \Omega \quad (43)$$
and the Dirichlet boundary condition

\[ u(x, t) \big|_{\Gamma} = (\cos(x_1) + \sin(x_2) + \cos(x_3)) \sin(\sqrt{2}t), \quad x \in \Gamma, \; 0 < t \leq T \]  

(44)

Its analytical solution of Example 4 is

\[ u(x, t) = (\cos(x_1) + \sin(x_2) + \cos(x_3)) \sin(\sqrt{2}t) \]  

(45)

Figure 7. Schematic configurations of the submarine model. (a) Geometry, (b) Node distribution.

In the proposed STRTCM implementation, the number of collocation nodes is \( N_{\text{Total}} = 2804 \), the boundary nodes number is \( N_b = 1312 \), and the interior nodes number is \( N_i = 1492 \), the wave speed is \( v = 1.0 \text{ m/s} \), the time interval is \( dt = 1 \text{ s} \), the final time is \( T = 10 \text{ s} \), and the annihilation spatial–temporal differential operator \( L_1 = \left( \frac{\partial^2}{\partial t^2} - 2\Delta \right) \) is employed to vanish the specified sound source excitation \( f(x, t) \) in Equation (41) by using the extended CMRM.

By using the proposed STRTCM for Example 4, very accurate results with \( Lerr = 4.40 \times 10^{-6} \) can be obtained in 1 min. However, it requires about 9 GB memory storage for getting the results in a large time instant \( T = 10 \text{ s} \). To enhance the ability of the proposed STRTCM for long-time evolution simulation, the entire time interval \([0, 10] \) is divided into \( NP \) sub-time intervals \(([0, 10/np], \cdots, [10(NP − 1)/NP, 10])\), and the STRTCM is used to solve the problems (41–44) in each sub-time interval in sequence. For each problem in the considered sub-time interval \( np \), the initial conditions are updated by using the final solution at the previous sub-time interval \( np - 1 \), \( np = 1, \cdots, NP \). Table 5 gives the numerical results of Example 4 by using the proposed STRTCM with different numbers of sub-time intervals. It can be found from Table 5 that with the increasing \( NP \), the proposed STRTCM can perform enough accurate results with less computational cost (CPU time and memory storage).

| \( NP \) | 1 | 2 | 5 | 10 |
|---------|---|---|---|----|
| CPU time | 53.0 s | 20.0 s | 15.9 s | 15.7 s |
| \( Lerr \) | \( 4.40 \times 10^{-6} \) | \( 6.92 \times 10^{-6} \) | \( 1.39 \times 10^{-5} \) | \( 5.17 \times 10^{-5} \) |
| Memory requirement | 9973 MB | 5671 MB | 4794 MB | 4647 MB |
4. Conclusions

In this paper, the spatial–temporal radial Trefftz collocation method (STRTCM) is proposed to solve transient wave propagation problems with specified sound source excitations. To deal with the specified sound source excitations, the extended composite multiple reciprocity method (CMRM) is presented from the spatial domain to the spatial–temporal domain for constructing the high-order homogeneous equation with the related constraint conditions. Then, the particular solution can be obtained by using a linear combination of the related high-order spatial–temporal radial Trefftz functions. Therefore, the proposed STRTCM only requires the node discretization on the boundaries of the spatial–temporal domain in 3D transient wave propagation analysis.

Numerical investigation shows that the proposed STRTCM produces more accurate results with a slight longer computational time than the COMSOL (FEM) under the same node discretization. The time interval $dt$ has a slight influence on the numerical accuracy of the proposed STRTCM. The iterative strategy is feasible to reduce the storage and CPU time. Due to the use of the spatial–temporal radial Trefftz functions, the proposed STRTCM only requires few temporal discretizations to accurately simulate the long-term evolution of the wave propagation.

Overall, it is concluded that the proposed STRTCM could be considered as a competitive alternative for the transient wave problems with specified sound source excitations under 3D complicated structures after further theoretical and numerical investigations. Moreover, it should be mentioned that as the first step, only 3D transient wave propagation with specified sound source excitations is considered in this paper, whose analytical solutions with regular form can be easily derived. The STRTCM for 3D transient wave problems with general sound source excitations is still under our intensive investigation. In addition, the present STRTCM scheme cannot handle the problems with heterogeneous materials because it is a nontrivial task to derive the corresponding semi-analytical basis solutions. For this, combining with the localized collocation scheme [43] and the extended multiple reciprocity method—generalized reciprocity method [44]—may be a good way. These topics are under study and will be reported in a subsequent paper.

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Nomenclature

\( u(x, t) \) \hspace{1cm} \text{acoustic pressure}  \\
\( v \) \hspace{1cm} \text{wave speed}  \\
\( \Delta \) \hspace{1cm} \text{Laplace operator}  \\
\( f(x, t) \) \hspace{1cm} \text{spatial-temporal source function}  \\
\( x \) \hspace{1cm} \text{collocation node}  \\
\( s \) \hspace{1cm} \text{source node}  \\
\( T \) \hspace{1cm} \text{final time instant}  \\
\( t \) \hspace{1cm} \text{time variable corresponding to collocation node}  \\
\( \tau \) \hspace{1cm} \text{time variable corresponding to source node}  \\
\( u_0, u_1 \) \hspace{1cm} \text{known functions on boundary}  \\
\( \Omega \) \hspace{1cm} \text{computational domain}  \\
\( \Gamma \) \hspace{1cm} \text{boundary of computational domain}  \\
\( \mathcal{R} \) \hspace{1cm} \text{partial differential operator matrix}  \\
\( N \) \hspace{1cm} \text{number of collocation node pairs}  \\
\( N_S \) \hspace{1cm} \text{number of source node pairs}  \\
\( r(x, s) \) \hspace{1cm} \text{distance between source node and collocation node}  \\
\( \{a\} \) \hspace{1cm} \text{unknown coefficients}  \\
\( N_i \) \hspace{1cm} \text{number of interior nodes on spatial domain}  \\
\( N_b \) \hspace{1cm} \text{number of boundary nodes on spatial domain}  \\
\( N_p \) \hspace{1cm} \text{number of collocation nodes along with time axis}  \\
\( N_{\text{Total}} \) \hspace{1cm} \text{number of total nodes on spatial domain}  \\
\( G_{0}(x,t,s,\tau) \) \hspace{1cm} \text{spatial–temporal radial Trefftz function}  \\
\( u_h(x,t) \) \hspace{1cm} \text{homogeneous solution}  \\
\( u_p(x,t) \) \hspace{1cm} \text{particular solution}  \\
\( L_{M} \cdots L_{2}L_{1} \) \hspace{1cm} \text{differential operators}  \\
\( MRE \) \hspace{1cm} \text{maximum absolute error}  \\
\( L_{err} \) \hspace{1cm} \text{relative error}  \\
\( R_{err} \) \hspace{1cm} \text{relative error}  \\
\( \text{NP} \) \hspace{1cm} \text{numbers of evolution steps}  

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