Uniformly Flat Semimodules*

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Abstract

We revisit the notion of flatness for semimodules over semirings. In particular, we introduce and study a new notion of uniformly flat semimodules based on the exactness of the tensor functor. We also investigate the relations between this notion and other notions of flatness for semimodules in the literature.

Introduction

The homological classification of monoids, suggested by L. A. Skornjakov [Sko1969a, Sko1969b], is still an ongoing project attracting the attention of many experts in Semigroup Theory and Universal Algebra. Many papers were devoted to study the category $\text{Act}_S$ of right $S$-acts over a monoid $S$ (a right $S$-act is a set $A$ with a map $\mu : A \times S \to A$ such that $a(st) = (as)t$ and $a1_S = a$ for all $a \in A$ and $s, t \in S$); for more information see the encyclopedic manuscript of Kilp et al. [KKM2000]. The philosophy in several of these papers is to model the theory of modules over rings (e.g. [AF1974], [Wis1991]) by studying the interplay between the (categorical) properties of $\text{Act}_S$ and the (algebraic) properties of $S$.

Another approach to study Abelian monoids is to consider them as semimodules over the semiring $\mathbb{N}_0$ of nonnegative integers [Gol1999a]. This provides us with a richer structure, motivates a non-additive version of the theory of modules over rings and opens the door for developing non-Abelian homological algebra [Ina1997]. It is worth mentioning that this approach is supported by the important role that semirings and semimodules play in emerging areas of research like idempotent analysis, tropical geometry and several aspects of theoretical physics [LM2005, KM1997] in addition to many applications in several branches of mathematics and computer science (e.g. [GM2008, Gol1999a, HW1998]).

Although some notions of flatness which are different for $S$-acts (e.g. [KKM2000, Chapter III], [B-F2009] and the papers cited there) coincide for semimodules as shown by Katsov [Kat2004a], several notions of flatness which turn out to be the same for modules are in fact different for semimodules (e.g. flatness and mono-flatness [KN2011]). This results in a rich theory of flatness for semimodules. In this manuscript, we revisit some of these notions and introduce a new notion of uniformly flat semimodules based on the exactness of the tensor product functor simulating the classical notion of flat modules over rings.

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The motivation for introducing a new notion of flatness for semimodules can be understood in light of the following observations: the notions of flat and k-flat semimodules introduced in [Alt2004] use Takahashi’s tensor products of semimodules [Tak1982a] which are not the natural tensor products. Among the main disadvantages of such tensor products is that the category of semimodules over a commutative semiring is not monoidal and that the tensor functor is not left adjoint to the hom functor as one would expect. In fact, Takahashi’s tensor products solve the universal problem related to such structures in the subcategory of cancellative semimodules, but they fail to provide a universal solution in the whole category of semimodules (see Section 2 for more details). Moreover, several results use Takahashi’s notion of exact sequences of semimodules [Tak1981] (see also [Gol1999a]), which we believe is not natural as well; for more details see the recent manuscript [Abu]. On the other hand, while the notion of flat semimodules introduced in [Kat2004a] is quite natural, it does not provide a notion of relative flatness w.r.t. a given family of semimodules which showed to be important in studying several notions related to pure exact sequences of modules over rings (e.g. [Wis1991]). This motivated us to introduce a new notion of flatness, namely that of uniformly flat semimodules, using the natural tensor products of semimodules [Kat1997] and what we believe is a more appropriate notion of exact sequences of semimodules introduced recently in [Abu].

This paper is organized as follows. In Section one, we recall some preliminaries about semirings, semimodules and exact sequences of semimodules. In Section two, we recall the construction of the natural tensor products of semimodules over semirings, clarify their connection with Takahashi’s tensor products and study some of their properties. In Section 3, we introduce the notion of uniformly flat semimodules and investigate its connection with other notions of flatness in the literature. We also generalize several results known for modules over rings to semimodules over semirings.

1 Preliminaries

As pointed out in [KN2011]: “when investigating semirings and their representations, one should undoubtedly use methods and techniques of both ring and lattice theory as well as diverse techniques and methods of categorical and universal algebra.”

For the convenience of the readers who might have different backgrounds, and to make this manuscript as much self-contained as possible, we collect in this section some definitions, remarks and results that will be used in the sequel. For unexplained terminology, our main references are [Mac1998] for Category Theory, [Gra2008] for Universal Algebra and [Wis1991] for Ring and Module Theory.

Semirings and Semimodules

Semirings (semimodules) are roughly speaking, rings (modules) without subtraction. Recall that a semigroup \((S, \cdot)\) is said to be cancellative iff for any \(s_1, s_2, s \in S\) we have

\[
[s_1 \cdot s = s_2 \cdot s \implies s_1 = s_2] \text{ and } [s \cdot s_1 = s \cdot s_2 \implies s_1 = s_2].
\]

**Definition 1.1.** A semiring is an algebraic structure \((S, +, \cdot, 0, 1)\) consisting of a non-empty set \(S\) with two binary operations “+” (addition) and “\(\cdot\)” (multiplication) satisfying the following conditions:

1. \((S, +, 0)\) is an Abelian monoid with neutral element 0;

[98x672]
2. \((S, \cdot, 1)\) is a monoid with neutral element 1;  
3. \(x \cdot (y + z) = x \cdot y + x \cdot z\) and \((y + z) \cdot x = y \cdot x + z \cdot x\) for all \(x, y, z \in S\);  
4. \(0 \cdot s = 0 = s \cdot 0\) for every \(s \in S\) (i.e. 0 is absorbing).

1.2. Let \(S, S'\) be semirings. A map \(f : S \to S'\) is said to be a morphism of semirings iff for all \(s_1, s_2 \in S\) :
\[
f(s_1 + s_2) = f(s_1) + f(s_2), \quad f(s_1 s_2) = f(s_1) f(s_2), \quad f(0_S) = 0_{S'} \text{ and } f(1_S) = 1_{S'}.
\]

1.3. Let \((S, +, \cdot)\) be a semiring. We say that \(S\) is
- cancellative iff the additive semigroup \((S, +)\) is cancellative;
- commutative iff the multiplicative semigroup \((S, \cdot)\) is commutative;
- semifield iff \((S \setminus \{0\}, \cdot, 1)\) is a commutative group.

Examples 1.4. Rings are indeed semirings. The first natural example of a commutative semiring is \((\mathbb{N}_0, +, \cdot)\), the set of nonnegative integers. The semirings \((\mathbb{R}^+_0, +, \cdot)\) and \((\mathbb{Q}^+_0, +, \cdot)\) are indeed semifields. Moreover, for any ring \(R\) we have a semiring structure \((+, \cdot)\) on the set \(\text{Ideal}(R)\) of ideals of \(R\) with the usual addition and multiplication of ideals of \(R\). For more examples, the reader may refer to [Gol1999a].

Definition 1.5. Let \(S\) be a semiring. A right \(S\)-semimodule is an algebraic structure \((M, +, 0_M)\) consisting of a non-empty set \(M\), a binary operation “+” along with a right \(S\)-action
\[
M \times S \to M, \quad (m, s) \mapsto ms,
\]
such that:

1. \((M, +, 0_M)\) is an Abelian monoid with neutral element 0;
2. \((ms)s' = m(ss')\), \((m + m')s = ms + m's\) and \(m(s + s') = ms + ms'\) for all \(s, s' \in S\) and \(m, m' \in M\);
3. \(m1_S = m\) and \(m0_S = 0_M = 0_M s\) for all \(m \in M\) and \(s \in S\).

1.6. 1. Let \(M\) and \(M'\) be right \(S\)-semimodules. A map \(f : M \to M'\) is said to be a morphism of \(S\)-semimodules (or \(S\)-linear) iff for all \(m_1, m_2 \in M\) and \(s \in S\) :
\[
f(m_1 + m_2) = f(m_1) + f(m_2) \quad \text{and} \quad f(ms) = f(m)s.
\]

The set \(\text{Hom}_S(M, M')\) of \(S\)-linear maps from \(M\) to \(M'\) is clearly an Abelian monoid under addition. The category of right \(S\)-semimodules is denoted by \(\mathcal{S}_S\). Analogously, one can define the category \(\mathcal{S}_S\) of left \(S\)-semimodules. A right \(S\)-semimodule \(M_S\) is said to be cancellative iff the semigroup \((M, +)\) is cancellative. With \(\mathcal{C}_S \subseteq \mathcal{S}_S\) (resp. \(\mathcal{S}_S \subseteq \mathcal{S}_S\)) we denote the full subcategory of cancellative right (left) \(S\)-semimodules.

1.7. Let \(S\) be a semiring and \(M\) a right \(S\)-semimodule. A non-empty subset \(L \subseteq M\) is said to be an \(S\)-subsemimodule, and we write \(L \leq_S M\) if \(L\) is closed under “+” and \(ls \in L\) for all \(l \in L\) and \(s \in S\).

Example 1.8. Every Abelian monoid \((M, +, 0_M)\) is an \(\mathbb{N}_0\)-semimodule in the obvious way. Moreover, the categories \textbf{AbMon} of Abelian monoids and the category \(\mathcal{S}_{\mathbb{N}_0}\) of \(\mathbb{N}_0\)-semimodules are isomorphic.
1.9. Let \( S, T \) be semirings, \( M \) a left \( S \)-semimodule and a right \( T \)-semimodule. We say that \( M \) is an \((S,T)\)-bisemimodule iff \((sm)t = s(mt)\) for all \( s \in S, m \in M \) and \( t \in T \). For \((S,T)\)-bisemimodules \( M, M' \), we call an \( S \)-linear \( T \)-linear map \( f : M \rightarrow N' \) a morphism of \((S,T)\)-bisemimodules \( (\text{or } (S,T)\text{-bilinear}) \). The set \( \text{Hom}_{(S,T)}(M, M') \) of \((S,T)\)-bilinear maps from \( M \) to \( M' \) is clearly an Abelian monoid under addition. The category of \((S,T)\)-bisemimodules will be denoted by \( S\mathcal{S} T \).

Throughout, and unless otherwise explicitly specified, \( S \) is an associative semiring with \( 1_S \neq 0_S \). We mean by an \( S \)-semimodule a right \( S \)-semimodule unless something different is mentioned explicitly.

1.10. Let \( M \) be an \( S \)-semimodule. An equivalent relation \( \equiv \) on \( M \) is said to be an \( S \)-congruence iff for any \( m, m', m_1, m_2, m'_2 \in M \) and \( s \in S \) we have
\[
[(m_1 \equiv m'_1 \land m_2 \equiv m'_2) \Rightarrow m_1 + m_2 \equiv m'_1 + m'_2] \quad \text{and} \quad [m \equiv m' \Rightarrow ms \equiv m's].
\]
Every \( S \)-subsemimodule \( L \leq_S M \) induces two \( S \)-congruences on \( M \) given by
\[
m_1 \equiv_L m_2 \iff m_1 + l_1 = m_2 + l_2 \quad \text{for some } l_1, l_2 \in L;
\]
\[
m_2 \equiv_{[L]} m_2 \iff m_1 + l_1 + m'' = m_2 + l_2 + m'' \quad \text{for some } l_1, l_2 \in L \text{ and } m'' \in M.
\]
We call the \( S \)-semimodule \( M/L := M/_{\equiv_L} \) the quotient (factor) semimodule of \( M \) by \( L \). If \( M \) is cancellative, then \( L \) and \( M/L \) are cancellative. On the other hand, \( M/_{\equiv_{[L]}} \) is obviously cancellative.

1.11. Let \( M \) be an \( S \)-semimodule and recall the \( S \)-congruence relation \( \equiv_{[0]} \) on \( M \) defined by
\[
m \equiv_{[0]} m' \iff m + m'' = m' + m'' \quad \text{for some } m'' \in M.
\]
The quotient \( S \)-semimodule \( M/ \sim \) is indeed cancellative and we have a canonical surjection \( \iota_M : M \rightarrowtail c(M) \) with
\[
\text{Ker}(\iota_M) = \{m \in M \mid m + m'' = m'' \text{ for some } m'' \in M\}.
\]
The class of cancellative right \( S \)-semimodules is a reflective subcategory of \( S_S \) in the sense that the functor
\[
c : S_S \rightarrowtail \mathcal{C} S_S, \quad M \mapsto M/ \equiv_{[0]}
\]
is left adjoint to the embedding functor \( \mathcal{C} S_S \rightarrowtail S_S \), i.e. for any \( S \)-semimodule \( M \) and any cancellative \( S \)-semimodule \( N \) we have a natural isomorphism of Abelian monoids \( \text{Hom}_S(\iota(M), N) \simeq \text{Hom}_S(M, N) \) [Tak1981, Page 517].

**Proposition 1.12.** The category \( S_S \) and its full subcategory \( \mathcal{C} S_S \) have kernels and cokernels, where for any morphism of \( S \)-semimodules \( f : M \rightarrow N \) we have
\[
\text{Ker}(f) = \{m \in M \mid f(m) = 0\} \quad \text{and} \quad \text{Coker}(f) = N/f(M).
\]

1.13. We call a subset \( Y \subseteq N \) subtractive iff \( Y = \overline{Y} \), the subtractive closure of \( Y \), where
\[
\overline{Y} = \{n \in N \mid n + y_1 = y_2 \text{ for some } y_1, y_2 \in Y\}.
\]
An \( S \)-semimodule \( M \) is said to be completely subtractive iff every \( S \)-subsemimodule of \( M \) is subtractive.
We call a morphism of $S$-semimodules $\gamma : M \to N$:

- **$k$-uniform** iff for any $m_1, m_2 \in M$:
  \[ \gamma(m_1) = \gamma(m_2) \implies \exists k_1, k_2 \in \text{Ker}(\gamma) \text{ s.t. } m_1 + k_1 = m_2 + k_2; \]  

- **$i$-uniform** iff $\gamma(M) = \gamma(M)$;

- **uniform** iff $\gamma$ is $k$-uniform and $i$-uniform.

**Remark 1.14.** The uniform ($k$-uniform, $i$-uniform) morphisms of semimodules were called regular ($k$-regular, $i$-regular) by Takahashi [Tak1982c]. We think that our terminology avoids confusion since a regular monomorphism (regular epimorphism) has a different well-established meaning in the language of Category Theory.

**1.15.** Let $M$ be an $S$-semimodule, $L \leq_S M$ an $S$-subsemimodule and consider the factor semimodule $M/L$. Then we have a surjective uniform morphism of $S$-semimodules

\[ \pi_L := M \to M/L, \ m \mapsto [m] \]

with

\[ \text{Ker}(\pi_L) = \{ m \in M \mid m + l_1 = l_2 \text{ for some } l_1, l_2 \in L \} = \overline{L}; \]

in particular, $L = \text{Ker}(\pi_L)$ if and only if $L \subseteq M$ is subtractive.

In [Abu] we introduced a new notion of *exact sequences* of semimodules. Takahashi’s exact sequences [Tak1981] shall be called *semi-exact* in the sequel:

**Definition 1.16.** We call a sequence of $S$-semimodules

\[ L \overset{f}{\to} M \overset{g}{\to} N \]  

**exact** iff $f(L) = \text{Ker}(g)$ and $g$ is uniform. An exact sequence

\[ 0 \to L \overset{f}{\to} M \overset{g}{\to} N \to 0 \]  

is called a short exact sequence.

**1.17.** ([Abu]) We call a sequence of $S$-semimodules $L \overset{f}{\to} M \overset{g}{\to} N$:

- **proper-exact** iff $f(L) = \text{Ker}(g)$;

- **semi-exact** iff $\overline{f(L)} = \text{Ker}(g)$;

- **quasi-exact** iff $\overline{f(L)} = \text{Ker}(g)$ and $g$ is uniform.

**1.18.** We call a (possibly infinite) sequence of $S$-semimodules

\[ \cdots \to M_{i-1} \overset{f_{i-1}}{\to} M_i \overset{f_i}{\to} M_{i+1} \overset{f_{i+1}}{\to} M_{i+2} \to \cdots \]  

**chain complex** iff $f_{j+1} \circ f_j = 0$ for every $j$;

**exact** (resp. **proper-exact**, **semi-exact**, **quasi-exact**) iff each partial sequence with three terms $M_j \overset{f_j}{\to} M_{j+1} \overset{f_{j+1}}{\to} M_{j+2}$ is exact (resp. proper-exact, semi-exact, quasi-exact);
1.19. An $S$-semimodule $N$ is said to be a retract of an $S$-semimodule $M$ iff there exist a (surjective) $S$-linear map $\theta : M \to N$ and an (injective) $S$-linear map $\psi : N \to M$ such that $\theta \circ \psi = 1_N$ (equivalently, $N \cong \alpha(M)$ for some idempotent endomorphism $\alpha \in \text{End}(M_S)$). On the other hand, $N$ is a direct summand of $M$ (i.e. $M = N \oplus N'$ for some $S$-subsemimodule $N'$ of $M$) if and only if there exists $\alpha \in \text{Comp(End}(M_S))$ s.t. $\alpha(M) = N$ where for any semiring $T$ we set

$$\text{Comp}(T) = \{ t \in T \mid \exists \tilde{t} \in T \text{ with } t + \tilde{t} = \text{id}_T \text{ and } t\tilde{t} = 0_T = \tilde{t} \}.$$ 

Indeed, every direct summand of $M$ is a retract of $M$; the converse is not true in general (cf. [Gol1999a, Proposition 16.6]).

**Lemma 1.20.** ([Abu, Proposition 3.10, Corollary 3.11]) Let $A, B$ and $C$ be $S$-semimodules.

1. $0 \to A \xrightarrow{f} B$ is exact if and only if $f$ is injective.
2. $B \xrightarrow{g} C \to 0$ is exact if and only if $g$ is surjective.
3. $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is semi-exact and $f$ is uniform if and only if $A = \ker(g)$.
4. $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is semi-exact and $g$ is uniform if and only if $C = \text{coker}(f)$.
5. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if $A = \ker(g)$ and $C = \text{coker}(f)$.

The following technical result follows immediately from the definitions and [Tak1983, Lemmas 1.11, 1.15].

**Lemma 1.21.**

1. Consider a commutative diagram of $S$-semimodules with $\pi \circ i = 1_N$ and $\pi' \circ i' = 1_N'$:

```
\begin{array}{ccc}
N & \xrightarrow{\tilde{\gamma}} & N' \\
\downarrow{\iota} & & \downarrow{\iota'} \\
M & \xrightarrow{\gamma} & M' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
N & \xrightarrow{\tilde{\gamma}} & N' \\
\end{array}
```

If $\gamma$ is uniform (resp. $k$-uniform, $i$-uniform), then $\tilde{\gamma}$ is uniform (resp. $k$-uniform, $i$-uniform).

2. Consider a commutative diagram of $S$-semimodules with $\pi \circ i = 1_N$, $\pi' \circ i' = 1_N'$ and $\pi'' \circ i'' = 1_N''$:

```
\begin{array}{ccc}
N & \xrightarrow{f} & N' \\
\downarrow{\iota} & & \downarrow{\iota'} \\
M & \xrightarrow{f} & M' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
N & \xrightarrow{f} & N' \\
\end{array}
```

```
\begin{array}{ccc}
N' & \xrightarrow{\tilde{g}} & N'' \\
\downarrow{\iota'} & & \downarrow{\iota''} \\
M' & \xrightarrow{g} & M'' \\
\downarrow{\pi'} & & \downarrow{\pi''} \\
N'' & \xrightarrow{\tilde{g}} & N'' \\
\end{array}
```

If $M \xrightarrow{f} M' \xrightarrow{g} M''$ is exact (resp. proper-exact, semi-exact, quasi-exact), then $N \xrightarrow{f} N' \xrightarrow{\tilde{g}} N''$ is exact (resp. proper-exact, semi-exact, quasi-exact).
Some redundant assumptions in [Abu, Lemma 4.5] do not hold in some situations which we will handle in this paper. A slight adjustment of the proof of the above mentioned result yields

**Lemma 1.22.** Consider the following commutative diagram of $S$-semimodules

$$
\begin{array}{ccccc}
L_1 & f_1 & M_1 & g_1 & N_1 \\
\alpha_1 & & \alpha_2 & & \alpha_3 \\
L_2 & f_2 & M_2 & g_2 & N_2
\end{array}
$$

1. Let the second sequence be quasi-exact (i.e. $f_2(L_2) = \text{Ker}(g_2)$ and $g_2$ is $k$-uniform) and $g_1, \alpha_1$ be surjective.
   (a) Let $g_1 \circ f_1 = 0$. If $\alpha_2$ is injective, then $\alpha_3$ is injective.
   (b) If $\alpha_3$ is surjective (and $\alpha_2$ is $i$-uniform), then $\alpha_2$ is a semi-epimorphism (surjective).

2. Let the first row be semi-exact (i.e. $f_1(L_1) = \text{Ker}(g_1)$) and $f_2$ be injective.
   (a) Let $f_1, \alpha_2$ be cancellative and $g_1$ be $k$-uniform. If $\alpha_1, \alpha_3$ are injective, then $\alpha_2$ is injective.
   (b) Let $g_2 \circ f_2 = 0$. If $\text{Ker}(\alpha_3) = 0$, and $\alpha_2$ is surjective, then $\alpha_1$ is a semi-epimorphism. If moreover, $\alpha_1$ or $f_1$ is $i$-uniform, then $\alpha_1$ is surjective.

## 2 Tensor products of semimodules

Tensor products of semimodules were introduced and investigated by Takahashi [Tak1981]. However, they did not provide a solution to the universal problem related to such structures in the whole category of semimodules. On the other hand, Katsov [Kat1997] considered a different tensor product in the category of semimodules (over a commutative semiring) which solved several of the problems that Takahashi’s tensor products had. It is worth mentioning, as Katsov pointed out, that his construction of the tensor product and the elementary results related to it seem to be folklore (e.g. Grillet [Gril1969] gave an explicit construction of a non-associative tensor product in the variety $\text{Sgr}$ of semigroups and suggested that the same construction works for all algebraic varieties of Universal Algebra). Varieties in which the tensor products behave nicely were considered by F. Linton [Lin1966] (see also [Bor1994b, Theorem 3.10.3]).

**Construction of tensor products**

As before, $S$ denotes an associative semiring with $1_S \neq 0_S$. With $S^S$ and $S_S$ we denoted the categories of left and right $S$-semimodule, respectively. For the convention of the reader, we recall the construction of tensor products of semimodules and some of its properties (e.g. [Kat1997], [Kat2004a], [KN2011]):

2.1. Let $M_S$ be a right $S$-semimodule and $S_N$ a left $S$-semimodule. An $S$-balanced map $g : M \times N \to G$, where $G$ is an Abelian monoid, is a bilinear map such that $g(ms, n) = g(m, sn)$ for all $m \in M$, $s \in S$ and $n \in N$. Let $F$ be the free Abelian monoid with basis $M \times N$. 

Every element of \( F \) can be written uniquely as a linear combination of elements of the set \( \{ \delta_{(m,n)} \mid (m,n) \in M \times N \} \) where \( \delta_{(m,n)} \) is the Kronecker delta function. Let \( \sigma \subseteq F \times F \) be the congruence relation generated by the set of all ordered pairs

\[
\{ (\delta_{(m_1+m_2,n)}, \delta_{(m_1,n)} + \delta_{(m_2,n)}), (\delta_{(m,n_1+n_2)}, \delta_{(m,n_1)} + \delta_{(m,n_2)}), (\delta_{(m,s,n)}, \delta_{(m,s,n)}) \},
\]

where \( m_1, m_2, m \in M, n_1, n_2, n \in N, s \in S \) and consider canonical maps

\[
\pi_\sigma : F \to F/\sigma \quad \text{and} \quad \tau := \pi_\sigma \circ \iota : M \times N \to F/\sigma.
\]

Let \( G \) be an Abelian monoid and \( \beta : M \times N \to G \) an \( S \)-balanced map. Since \( F \) is free over \( M \times N \), the map \( \beta \) induces a unique map \( \beta' : F \to G \). Since \( \text{Ker}(\pi_\sigma) = \sigma \subseteq \text{Ker}(g) \), there exists a unique map \( \gamma : F/\sigma \to G \) such that \( \gamma \circ \pi_\sigma = \beta' \) (given by \( \gamma(f) = \beta'(f) \), for every \( f \in F \)) and so \( \gamma \circ \tau = \gamma \circ \pi_\sigma \circ \iota = \beta' \circ \iota = \beta \):

\[
\begin{array}{ccc}
M \times N & \xrightarrow{\pi_\sigma} & F/\sigma & \xrightarrow{\gamma} & G \\
\downarrow{\tau} & & & \downarrow{\beta'} & \\
F/\sigma & \xrightarrow{\gamma} & G \\
\end{array}
\]

So, \( M \otimes_S N := (F/\sigma, \tau) \) is solution for the following universal problem: For every Abelian monoid \( G \) with an \( S \)-balanced map \( \beta : M \times N \to G \), there exists a unique morphism of monoids \( \gamma : M \otimes_S N \to G \) that completes the right triangle in (5) commutatively.

2.2. Let \( M_S \) a right \( S \)-semimodule, \( sN \) a left \( S \)-semimodule and \( F \) the free Abelian monoid with basis \( M \times N \). Let \( N(M) \subseteq F \times F \) be the symmetric \( S \)-subsemimodule generated by the set of elements of the form

\[
(\delta_{(m_1+m_2,n)}, \delta_{(m_1,n)} + \delta_{(m_2,n)}), \quad (\delta_{(m,n_1+n_2)}, \delta_{(m,n_1)} + \delta_{(m,n_2)}), \quad (\delta_{(m,s,n)}, \delta_{(m,s,n)}),
\]

and consider the \( S \)-congruence relation on \( F \) defined by

\[
f \rho g \iff f + h = g + h' \quad \text{for some} \quad (h,h') \in N(M).
\]

Takahashi's tensor product of \( M \) and \( N \) is defined as \( M \boxtimes_S N = F/\rho \). Notice that there is an \( S \)-balanced map

\[
\tilde{\tau} : M \times N \to M \boxtimes_S N, \quad (m,n) \mapsto m \boxtimes_S n := (m,n)/\rho
\]

with the following universal property \[\text{Tak1982a}\]: for every cancellative Abelian monoid \( \tilde{G} \) and every \( S \)-balanced map \( \tilde{\beta} : M \times N \to \tilde{G} \) there exists a unique morphism of monoids \( \tilde{\gamma} : M \boxtimes_S N \to \tilde{G} \) such that \( \tilde{\gamma} \circ \tilde{\tau} = \tilde{G} \).

The above mentioned property means that \(- \boxtimes_S -\) plays the role of a tensor product w.r.t. cancellative semimodules. On the other hand, notice that for every Abelian monoid \( G \), we
have a commutative diagram

\[
\begin{array}{ccc}
M \times N & \xrightarrow{\beta} & G \\
\downarrow{\gamma} & & \downarrow{\epsilon_G} \\
c(M \otimes_S N) & \xrightarrow{\epsilon_{M \otimes_S N}} & c(G)
\end{array}
\]

which suggests that \(c(- \otimes_S -)\) plays the same role.

The above observations motivates the following connection between the bifunctors \(- \otimes_S -\) and \(- \boxtimes_S -\), where \(\text{CAbMon}\) is the category of cancellative Abelian monoids:

**Theorem 2.3.** We have an equivalence of functors

\[- \boxtimes_S - \cong \approx c(-) \circ (- \otimes_S -) : \text{S}_S \times \text{S}_N \longrightarrow \text{CAbMon}.\]

In particular, for every right \(S\)-semimodule \(M_S\) and every left \(S\)-semimodule \(S_N\), we have a natural isomorphism of Abelian monoids

\[M \boxtimes_S N \cong c(M \otimes_S N).\]

**Proof.** Let \(M_S\) be a right \(S\)-semimodule, \(S_N\) a left \(S\)-semimodule and consider the Abelian monoids \((M \otimes_S N; \tau), (M \boxtimes_S N; \overline{\tau})\) along with the canonical morphisms of monoids

\[\epsilon_{M \otimes_S N} : M \otimes_S N \longrightarrow c(M \otimes_S N)\]

and

\[\epsilon_{\gamma} : c(M \otimes_S N) \longrightarrow M \boxtimes_S N.\]

Since \(G = c(M \otimes_S N)\) is cancellative and \(\beta := \epsilon_{M \otimes_S N} \circ \tau : M \times N \longrightarrow c(M \otimes_S N)\) is \(S\)-balanced, there exists a unique morphism of monoids \(\overline{\gamma} : M \boxtimes_S N \longrightarrow c(M \otimes_S N)\) such that \(\overline{\gamma} \circ \overline{\tau} = \beta = \epsilon_{M \otimes_S N} \circ \tau\). On the other hand, for \(G = M \boxtimes_S N\), the map \(\beta := \gamma : M \times N \longrightarrow M \boxtimes_S N\) is \(S\)-balanced and there exists a unique morphism of monoids \(\gamma : M \otimes_S N \longrightarrow M \boxtimes_S N\) such that \(\gamma \circ \tau = \beta = \overline{\tau}\). Consider the morphisms of monoids

\[
\varphi : M \boxtimes_S N \xrightarrow{\gamma} c(M \otimes_S N) \xrightarrow{\epsilon_{\gamma}} M \boxtimes_S N;
\]

\[
\theta : M \otimes_S N \xrightarrow{\epsilon_{M \otimes_S N}} c(M \otimes_S N) \xrightarrow{\epsilon_{\gamma}} M \boxtimes_S N \xrightarrow{\gamma} c(M \otimes_S N).
\]

Notice that

\[
\varphi \circ \overline{\tau} = \epsilon_{\gamma} \circ \gamma \circ \overline{\tau} = \epsilon_{\gamma} \circ \beta = \epsilon_{\gamma} \circ \epsilon_{M \otimes_S N} \circ \tau = \gamma \circ \tau = \overline{\tau}.
\]

Since \(\text{id}_{M \otimes_S N} : M \boxtimes_S N \longrightarrow M \boxtimes_S N\) is the unique morphism of monoids satisfying this property, we conclude that \(\epsilon_{\gamma} \circ \gamma = \text{id}_{M \otimes_S N}\). On the other hand, we have

\[
\theta \circ \tau = \overline{\gamma} \circ \epsilon_{\gamma} \circ c(M \otimes_S N) \circ \tau = \overline{\gamma} \circ \gamma \circ \tau = \overline{\gamma} \circ \overline{\tau} = \epsilon_{M \otimes_S N} \circ \tau.
\]

Although \(\tau\) is not an epimorphism (in general), \(\tau(M \times N)\) is a generating set for \(M \otimes_S N\), whence \(\theta = \epsilon_{M \otimes_S N}\) and so \(\overline{\gamma} \circ \epsilon_{\gamma} \circ c(M \otimes_S N) = \text{id}_{c(M \otimes_S N)} \circ \epsilon_{M \otimes_S N}\). Since \(\epsilon_{M \otimes_S N}\) is an epimorphism, we conclude that \(\gamma \circ \epsilon_{\gamma} = \text{id}_{c(M \otimes_S N)}\). One can easily check that this isomorphism is natural in \(M_S\) and \(S_N\).
Remarks 2.4. Let $S$ and $T$ be semirings.

1. For every right $S$-semimodule $M$ and left $S$-semimodule $S N$ we have canonical isomorphisms of Abelian monoids $M \cong M \otimes_S S$ and $N \cong S \otimes_S N$, whence

$$M \boxtimes_S S \simeq \mathbb{c}(M \otimes_S S) \simeq \mathbb{c}(M)$$

and

$$S \boxtimes_S N \simeq \mathbb{c}(S \otimes_S N) \simeq \mathbb{c}(N).$$

2. If $M$ is a right $S$-semimodule, $N$ is an $(S,T)$-bisemimodule and $X$ is a left $T$-semimodule, then we have a canonical isomorphism of Abelian monoids

$$(M \otimes_S N) \otimes_T X \simeq M \otimes_S (N \otimes_T X).$$

Proposition 2.5. (cf. [KN2011]) Let $M$ be a right $S$-semimodule and $N$ a left $S$-semimodule.

1. If $M$ is a $(T,S)$-bisemimodule, then $M \otimes_S - : S S \rightarrow T S$ is left adjoint to $\text{Hom}_T(M, -) : T S \rightarrow S S$, i.e. for every left $S$-semimodule $X$ and every left $T$-semimodule $Y$, we have a canonical isomorphism of Abelian monoids that is natural in $S X$ and $T Y$:

$$\text{Hom}_T(M \otimes_S X, Y) \cong \text{Hom}_S(X, \text{Hom}_T(M, Y)).$$

2. If $N$ is an $(S,T)$-bisemimodule, then $- \otimes_S N : S S \rightarrow S T$ is left adjoint to $\text{Hom}_T(N, -) : S T \rightarrow S S$, i.e. for every right $S$-semimodule $X$ and every right $T$-semimodule $Y$ we have a canonical isomorphism of Abelian monoids that is natural in $S X$ and $T Y$:

$$\text{Hom}_T(X \otimes_S N, Y) \cong \text{Hom}_S(X, \text{Hom}_T(N, Y)).$$

As a consequence of Lemma 2.3 and Proposition 2.5 we recover [Tak1982c, Corollary 4.5]:

Corollary 2.6. Let $M$ be a right $S$-semimodule and $N$ a left $S$-semimodule.

1. If $M$ is a $(T,S)$-bisemimodule, $S X$ a left $S$-semimodule and $Y \in T CS$ a cancellative left $T$-semimodule, then we have a canonical isomorphism

$$\text{Hom}_T(M \boxtimes_S X, Y) \simeq \text{Hom}_S(X, \text{Hom}_T(M, Y)).$$

2. If $N$ is an $(S,T)$-bisemimodule, $X$ is a right $S$-semimodule and $Y \in CS T$ a cancellative right $T$-semimodule, then we have a canonical isomorphism

$$\text{Hom}_T(X \boxtimes_S N, Y) \simeq \text{Hom}_S(X, \text{Hom}_T(N, Y)).$$

Proof. We prove “1”. The proof of “2” is similar. The required isomorphism is given by

$$\text{Hom}_T(M \boxtimes_S X, Y) = \text{Hom}_T(\mathbb{c}(M \otimes_S X), Y)$$

$$\simeq \text{Hom}_T(M \otimes_S X, Y)$$

$$\simeq \text{Hom}_S(X, \text{Hom}_T(M, Y)).$$

Definition 2.7. A category $\mathcal{C}$ is said to be (finitely) complete iff every functor $F : \mathcal{D} \rightarrow \mathcal{C}$, with $\mathcal{D}$ a small (finite) category, has a limit. Dually, $\mathcal{C}$ is said to be (finitely) cocomplete iff every functor $F : \mathcal{D} \rightarrow \mathcal{C}$ with $\mathcal{D}$ a small (finite) category has a colimit.
Taking into account the fact that $S_S$ is a variety (in the sense of Universal Algebra) we have (e.g. [Sch1972, Theorem 21.6.4]):

**Proposition 2.8.** The category $S_S$ of right S-semimodules is complete (has equalizers and products) and cocomplete (has coequalizers and coproducts).

**2.9.** Let $J$ be a directed set. The directed limit (inductive limit, filtered colimit) of a directed system of $S$-semimodules $(M_j, \{f_{jj'} : M_j \to M_{j'} \mid j \leq j'\})_J$ can be constructed as follows: consider the disjoint union $\coprod_{j \in J} M_j = \bigcup_{j \in J} (M_j \times \{j\})$, the embeddings $\iota_j : M_j \to \coprod_{j \in J} M_j$, and the congruence relation on $\coprod_{j \in J} M_j$:

$$(x, j) \sim (x', j') \iff \exists j'' \geq j, j' \text{ s.t. } x \in M_j, x' \in M_{j'} \text{ and } f_{jj''}(x) = f_{j'j''}(x'). \tag{7}$$

We define

$$\lim_{\to} M_j : = \coprod_{j \in J} M_j / \sim \text{ and } \gamma_j : M_j \to \lim_{\to} M_j, m \mapsto [(m, j)].$$

Notice that $\lim M_j$ is an S-semimodule with $[(m, j)]s = [(ms, j)]$ for all $s \in S$ and $m \in M_j$ and $[(m, j)] + [(m', j')] = [(f_{jj''}(m) + f_{j'j''}(m'), j'')]$, where $j'' \geq j, j'$.

**2.10.** Let $J$ be a directed set. The inverse limit (projective limit) of an inverse system of $S$-semimodules $(M_j, \{f_{jj'} : M_j \to M_{j'} \mid j \leq j'\})_J$ is given by:

$$\lim_{\leftarrow} M_j = \{(m_j)_{j \in J} \mid m_j = f_{jj'}(m_{j'}) \text{ whenever } j \leq j'\}.$$ 

The proof of the following important observation is straightforward:

**Proposition 2.11.** Every $S$-semimodule $M$ is a direct limit of its finitely generated $S$-subsemimodules.

**Lemma 2.12.** (Abu-b) Let $J$ be a directed set and $(X_j, \{f_{jj'} : X_j \to X_{j'}\})_J, (Y_j, \{g_{jj'} : Y_j \to Y_{j'}\})_J$ be directed systems of $S$-semimodules. Let $\{h_j : X_j \to Y_j\}_J$ be a class of $S$-linear morphisms satisfying $h_{j'} \circ f_{jj'} = g_{jj'} \circ h_j$ for all $j, j' \in J$ with $j \leq j'$.

1. There exists a unique morphism $h : (\lim_{\to} X_j, f_j) \to (\lim_{\to} Y_j, g_j)$ which satisfies $g_j \circ h_j = h \circ f_j$.

2. If $h_j$ is injective (surjective) for every $j \in J$, then $h$ is injective (surjective).

3. If $h_j$ is uniform (resp. $k$-uniform, $i$-uniform) for every $j \in J$, then $h$ is uniform (resp. $k$-uniform, $i$-uniform).

**Proposition 2.13.** Let $(L_j, \{f_{jj'}\})_J, (M_j, \{g_{jj'}\})_J$ and $(N_j, \{h_{jj'}\})_J$ be directed systems of $S$-semimodules.

1. If $(L_j \xrightarrow{\alpha_j} M_j \xrightarrow{\beta_j} N_j)_J$ is a class of exact (resp. semi-exact, proper-exact, quasi-exact) sequences of $S$-semimodules, with $\alpha_{j'} \circ f_{jj'} = g_{jj'} \circ \alpha_j$ and $\beta_j \circ g_{jj'} = h_{jj'} \circ \beta_j$ for all $j \in J$, then the induced sequence of $S$-semimodules $(\lim_{\to} L_j \xrightarrow{\alpha} \lim_{\to} M_j \xrightarrow{\beta} \lim_{\to} N_j)$ is exact (resp. semi-exact, proper-exact, quasi-exact).
Lemma 2.14. Let \((M_j, \{f_{jj'}\})_J\) be a directed system of left \(S\)-semimodules with associated directed system of \(S\)-linear maps \(f_j : M_j \to M_j\) and let \(X\) be a left \(S\)-semimodule.

1. \((\text{Hom}_S(X, M_j), (X, f_{jj'}))_J\) is a directed system of Abelian monoids. Moreover, \((X, f_j) : \text{Hom}_S(X, M_j) \to \text{Hom}_S(X, \lim M_j)\) is a directed system of morphisms of Abelian monoids and induces a morphism of Abelian monoids
\[
\psi_X = \lim_{\to}(X, f_j) : \lim_{\to} \text{Hom}_S(X, M_j) \to \text{Hom}_S(X, \lim M_j), \quad [(\alpha_j, j)] \mapsto [(f_j \circ \alpha_j, j)].
\]

2. If \(sX\) is finitely generated, then \(\psi_X\) is injective.

Proof. The first statement is obvious. Assume that \(sX\) is finitely generated. Suppose that \(\psi_X([(\alpha_j, j)]) = \psi_X([(\alpha_j', j')])), i.e. \(f_j \circ \alpha_j = f_{jj'} \circ \alpha_j'\) for some \(\alpha_j \in \text{Hom}_S(X, M_j)\), \(\alpha_j' \in \text{Hom}_S(X, M_{j'})\) and \(j, j' \in J\). Since \(sX\) is finitely generated, there exists \(j'' \geq j, j'\) such that \(f_{jj''} \circ \alpha_j = f_{jj'j''} \circ \alpha_{j''}, i.e. (X, f_{jj''})((\alpha_{j''})) = (X, f_{jj''})((\alpha_j)),\) whence \([(\alpha_j, j)] = [(\alpha_{j''}, j'')]\).\)

Proposition 2.15. (cf. [Bor1994a Proposition 3.2.2]) Let \(\mathcal{C}, \mathcal{D}\) be arbitrary categories and \(\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}\) be functors such that \((F, G)\) is an adjoint pair.

1. \(F\) preserves all colimits which turn out to exist in \(\mathcal{C}\).
2. \(G\) preserves all limits which turn out to exist in \(\mathcal{D}\).

The following results can be obtained as a direct consequence of Propositions 2.15 and 2.16.

Corollary 2.16. Let \(S, T\) be semirings and \(TF_S\) a \((T, S)\)-bisemimodule.

1. \(F \otimes_S : s\mathcal{S} \to t\mathcal{S}\) preserves all colimits.
   
   (a) For every family of left \(S\)-semimodules \(\{X_{\lambda}\}_\Lambda\), we have a canonical isomorphism of left \(T\)-semimodules
   \[
   F \otimes_S \bigoplus_{\lambda \in \Lambda} X_{\lambda} \simeq \bigoplus_{\lambda \in \Lambda} (F \otimes_S X_{\lambda}).
   \]

   (b) For any directed system of left \(S\)-semimodules \((X_j, \{f_{jj'}\})_J\), we have an isomorphism of left \(T\)-semimodules
   \[
   F \otimes_S \lim_{\to} X_j \simeq \lim_{\to} (F \otimes_S X_j).
   \]

   (c) \(F \otimes_S\) preserves coequalizers.
(d) \( F \otimes_S - \) preserves cokernels (uniform quotients).

2. \( \text{Hom}_T(F, -) : T \mathcal{S} \to S \mathcal{S} \) preserves all limits.

   (a) For every family of left \( T \)-semimodules \( \{Y_\lambda\}_{\lambda \in \Lambda} \), we have a canonical isomorphism of left \( S \)-semimodules
   \[
   \text{Hom}_T(F, \prod_{\lambda \in \Lambda} Y_\lambda) \cong \prod_{\lambda \in \Lambda} \text{Hom}_T(F, Y_\lambda).
   \]

   (b) For any inverse system of left \( T \)-semimodules \( (X_j, \{f_{jj'}\})_J \), we have an isomorphism of left \( S \)-semimodules
   \[
   \text{Hom}_T(F, \lim_{\leftarrow} X_j) \cong \lim_{\leftarrow} \text{Hom}_T(F, X_j).
   \]

   (c) \( \text{Hom}_T(F, -) \) preserves equalizers;

   (d) \( \text{Hom}_T(F, -) \) preserves kernels (uniform subsemimodules).

3. \( \text{Hom}_T(-, F) : T \mathcal{S} \to S \mathcal{S} \) preserves all limits.

   (a) For every family of left \( T \)-semimodules \( \{Y_\lambda\}_{\lambda \in \Lambda} \), we have a canonical isomorphism of right \( S \)-semimodules
   \[
   \text{Hom}_T(\bigoplus_{\lambda \in \Lambda} Y_\lambda, F) \cong \prod_{\lambda \in \Lambda} \text{Hom}_T(Y_\lambda, F).
   \]

   (b) For any directed system of left \( T \)-semimodules \( (X_j, \{f_{jj'}\})_J \), we have an isomorphism of right \( S \)-semimodules
   \[
   \text{Hom}_T(\lim_{\rightarrow} X_j, F) \cong \lim_{\rightarrow} \text{Hom}_T(X_j, F).
   \]

   (c) \( \text{Hom}_T(-, F) \) converts coequalizers into equalizers;

   (d) \( \text{Hom}_T(F, -) \) converts cokernels into kernels (uniform quotients into uniform subsemimodules).

Corollary 2.16 allows us to improve [Tak1982a, Theorem 2.6].

**Proposition 2.17.** Let \( T \mathcal{G} \mathcal{S} \) an \( S \)-bisemimodule and consider the functor \( \text{Hom}_T(G, -) : T \mathcal{S} \to S \mathcal{S} \). Let
\[
0 \to L \overset{f}{\to} M \overset{g}{\to} N
\]
be a sequence of left \( T \)-semimodules and consider the following sequence of left \( S \)-semimodules
\[
0 \to \text{Hom}_T(G, L) \overset{(G,f)}{\to} \text{Hom}_T(G, M) \overset{(G,g)}{\to} \text{Hom}_T(G, N).
\]

1. If \( 0 \to L \overset{f}{\to} M \) is exact and \( f \) is uniform, then \( 0 \to \text{Hom}_T(G, L) \overset{(G,f)}{\to} \text{Hom}_T(G, M) \) is exact and \( (G, f) \) is uniform.

2. If \( (9) \) is semi-exact and \( f \) is uniform, then \( (10) \) is semi-exact (proper exact) and \( (G, f) \) is uniform.
3. If $\mathbf{[9]}$ is exact and $\text{Hom}_T(G, -)$ preserves $k$-uniform morphisms, then $\mathbf{[14]}$ is exact.

**Proof.** 1. The following implications are obvious: $0 \rightarrow L \xrightarrow{f} M$ is exact $\implies f$ is injective $\implies (G, f)$ is injective $\implies 0 \rightarrow \text{Hom}_T(G, L) \xrightarrow{(G,f)} \text{Hom}_T(G, M)$ is exact. Assume that $f$ is uniform and consider the exact sequence of $S$-semimodules

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{\pi_L} M/L \rightarrow 0.$$

Notice that $L = \text{Ker}(\pi_L)$ by Lemma $\mathbf{[120]}(5)$. By Corollary $\mathbf{2.16}$ $\text{Hom}_T(G, -)$ preserves kernels and so $(G, f) = \ker(G, \pi_L)$ whence uniform.

2. Apply Lemma $\mathbf{1.20}$ (3): The semi-exactness of $\mathbf{[9]}$ and the uniformity of $f$ are equivalent to $L \simeq \text{Ker}(g)$. Since $\text{Hom}_T(G, -)$ preserves kernels, we deduce that $\text{Hom}_T(G, L) = \text{Ker}((G, g))$ which is equivalent to the semi-exactness of $\mathbf{[10]}$ and the uniformity of $(G, f)$. Notice that $(G, f)(\text{Hom}_T(G, L)) = (G, f)(\text{Hom}_T(G, L)) = \text{Ker}(G, g)$, i.e. $\mathbf{[10]}$ is proper exact.

3. The statement follows directly from “2” and the assumption on $\text{Hom}_T(G, -)$.

**Proposition 2.18.** Let $T S$ be a $(T, S)$-bisemimodule and consider the functor $\text{Hom}_T(-, G) : T S \rightarrow S$. Let

$$L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0 \quad (11)$$

be a sequence of left $T$-semimodules and consider the sequence of right $S$-semimodules

$$0 \rightarrow \text{Hom}_T(N, G) \xrightarrow{(g,G)} \text{Hom}_T(M, G) \xrightarrow{(f,G)} \text{Hom}_T(L, G). \quad (12)$$

1. If $M \xrightarrow{g} N \rightarrow 0$ is exact and $g$ is uniform, then $0 \rightarrow \text{Hom}_T(N, G) \xrightarrow{(g,G)} \text{Hom}_T(M, G)$ is exact and $(g, G)$ is uniform.

2. If $\mathbf{[11]}$ is semi-exact and $g$ is uniform, then $\mathbf{[12]}$ is semi-exact (proper-exact) and $(g, G)$ is uniform.

3. If $\mathbf{[11]}$ is exact and $\text{Hom}_T(-, G)$ converts $i$-uniform morphisms into $k$-uniform ones, then $\mathbf{[12]}$ is exact.

**Proof.** 1. The following implications are clear: $M \xrightarrow{g} N \rightarrow 0$ is exact $\implies g$ is surjective $\implies (g, G)$ is injective $\implies 0 \rightarrow \text{Hom}_T(N, G) \xrightarrow{(g,G)} \text{Hom}_T(M, G)$ is exact. Assume that $g$ is uniform and consider the exact sequence of $S$-semimodules

$$0 \rightarrow \text{Ker}(g) \xrightarrow{\iota} M \xrightarrow{g} N \rightarrow 0.$$

Notice that $N \simeq \text{Coker}(\iota)$. By Corollary $\mathbf{2.16}$ $\text{Hom}_T(-, G)$ converts cokernels into kernels, we conclude that $(g, G) = \ker((f, G))$ whence uniform.

2. Apply Lemma $\mathbf{1.20}$: $L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is semi-exact and $f$ is uniform $\iff M \simeq \text{Coker}(f)$. Since the contravariant functor $\text{Hom}_T(-, G)$ converts cokernels into kernels, it follows that $\text{Hom}_T(N, G) = \text{Ker}((f, G))$ which is in turn equivalent to $\mathbf{[12]}$ being semi-exact and $(g, G)$ being uniform. Notice that $(g, G)(\text{Hom}_S(N, G)) = (g, G)(\text{Hom}_S(N, G)) = \text{Ker}((f, G))$, i.e. $\mathbf{[12]}$ is proper-exact.
3. This follows immediately from “2” and the assumption on $\text{Hom}_T(\cdot, G)$.

**Proposition 2.19.** Let $TGS$ be a $(T, S)$-bisemimodule and consider the functor $G \otimes_S - : sS \to T$. Let
\[ L \to M \to N \to 0 \] (13)
be a sequence of left $S$-semimodules and consider the sequence of left $T$-semimodules
\[ G \otimes_S L \xrightarrow{id_G \otimes_S f} G \otimes_S M \xrightarrow{id_G \otimes_S g} G \otimes_S N \to 0 \] (14)

1. If $M \to N \to 0$ is exact and $g$ is uniform, then $G \otimes_S M \xrightarrow{id_G \otimes_S g} G \otimes_S N \to 0$ is exact and $id_G \otimes_S g$ is uniform.

**Proposition 2.20.** If (13) is semi-exact and $g$ is uniform, then (13) is semi-exact and $id_G \otimes_S g$ is uniform.

**Proposition 2.21.** If (13) is exact and $G \otimes_S -$ preserves $i$-uniform morphisms, then (13) is exact.

**Proof.** The following implications are obvious: $M \to N \to 0$ is exact $\implies g$ is surjective $\implies id_G \otimes_S g$ is surjective $\implies G \otimes_S M \xrightarrow{id_G \otimes_S g} G \otimes_S N \to 0$ is exact. Assume that $g$ is uniform and consider the exact sequence of $S$-semimodules
\[ 0 \to \text{Ker}(g) \xrightarrow{i} M \xrightarrow{g} N \to 0. \]
Then $N \simeq \text{Coker}(i)$. By Corollary 2.16 $G \otimes_S -$ preserves cokernels and so $id_G \otimes_S g = \text{coker}(id_G \otimes_S i)$ whence uniform.

**Proof.** Apply Lemma 1.20. The assumptions on (13) are equivalent to $N = \text{Coker}(f)$ by Lemma 2.20. Since $G \otimes_S -$ preserves cokernels, we conclude that $G \otimes_S N = \text{Coker}(id_G \otimes_S f)$, i.e. (14) is semi-exact and $id_G \otimes_S g$ is uniform.

**Proof.** This follows directly from “2” and the assumption on $G \otimes_S -$.

We say that an $S$-semimodule $P$ is *projective* iff for every surjective morphism of $S$-semimodules $M \to N \to 0$, the induced morphism of Abelian monoids $\text{Hom}_S(P, M) \to \text{Hom}_S(P, N)$ is surjective. It is well-known that $sP$ is projective if and only if $P$ is a retract of a free $S$-semimodule (e.g. [Tak1983, Theorem 1.9], [Gol1999a, Proposition 17.16]).

The proof of the following lemma is straightforward:

**Lemma 2.22.**

1. Let $\{f_\lambda : L_\lambda \to M_\lambda\}_\Lambda$ be a family of left $S$-semimodule morphisms and consider the induced $S$-linear map $f : \bigoplus_{\lambda \in \Lambda} L_\lambda \to \bigoplus_{\lambda \in \Lambda} M_\lambda$. Then $f$ is uniform (resp. $k$-uniform, $i$-uniform) if and only if $f_\lambda$ is uniform (resp. $k$-uniform, $i$-uniform) for every $\lambda \in \Lambda$. In particular, $\bigoplus_{\lambda \in \Lambda} L_\lambda \lesseqqg s \bigoplus_{\lambda \in \Lambda} M_\lambda$ if and only if $L_\lambda \lesseqqg s M_\lambda$ for every $\lambda \in \Lambda$.

2. A morphism $\varphi : L \to M$ of left $S$-semimodules is uniform (resp. $k$-uniform, $i$-uniform) if and only if $id_F \otimes_S \varphi : F \otimes_S L \to F \otimes_S M$ is uniform (resp. $k$-uniform, $i$-uniform) for every non-zero free right $S$-semimodule $F \neq 0$.

3. If $P_S$ is projective and $\varphi : L \to M$ is a uniform (resp. $k$-uniform, $i$-uniform) morphism of left $S$-semimodules, then $id_F \otimes_S \varphi : P \otimes_S L \to P \otimes_S M$ is uniform (resp. $k$-uniform, $i$-uniform).
It is well-known, that for every (finitely generated) \( S \)-semimodule \( X \), there is a free \( S \)-semimodule \( S^{(J)} \), for some (finite) index set \( J \), and a surjective \( S \)-linear map \( S^{(J)} \rightarrow X \rightarrow 0 \).

**Definition 2.23.** We call a left \( S \)-semimodule \( X \):

- uniformly finitely generated iff there exists a uniform surjective \( S \)-linear map \( S^n \rightarrow X \rightarrow 0 \) for some \( n \in \mathbb{N} \);
- uniformly finitely presented iff \( sX \) is uniformly finitely generated and for any exact sequence of \( S \)-semimodules
  
  \[ 0 \rightarrow K \xrightarrow{f} S^n \xrightarrow{g} X \rightarrow 0, \]

the \( S \)-semimodule \( K \) is finitely generated.

**Remark 2.24.** Takahashi [Tak1983] defined an \( S \)-semimodule \( X \) to be normal iff there exists a projective \( S \)-semimodule \( P \) and a uniform surjective \( S \)-linear map \( P \xrightarrow{\varepsilon} X \xrightarrow{0} \) (called a projective presentation of \( X \)). Indeed, every uniformly finitely generated \( S \)-semimodule is normal.

**Proposition 2.25.** If \( sX \) is uniformly finitely presented, then there exist \( m,n \in \mathbb{N} \) and an exact sequence of \( S \)-semimodules

\[ S^m \xrightarrow{f} S^n \xrightarrow{g} X \rightarrow 0. \]

**Proof.** Since \( sX \) is uniformly finitely generated, there exists a uniform surjective \( S \)-linear map \( g : S^n \rightarrow X \). Let \( K = \text{Ker}(g) \) and consider the exact sequence of left \( S \)-semimodules

\[ 0 \rightarrow K \xrightarrow{\ker(g)} S^n \xrightarrow{g} X \rightarrow 0. \]

By assumption, \( sK \) is finitely generated and so there exists a surjective \( S \)-linear map \( \pi : S^m \rightarrow K \) for some \( m \in \mathbb{N} \). Notice that \( f := \ker(g) \circ \pi \) is \( i \)-uniform by [Abu Lemma 3.8 “1-c”] and \( g \) is uniform by assumption. Indeed, \( f(S^m) = \text{Ker}(g) \) and so the following sequence is exact

\[ S^m \xrightarrow{f} S^n \xrightarrow{g} X \rightarrow 0. \]

**Definition 2.26.** ([Abr-3]) We say that a right \( S \)-semimodule \( Q \) is (uniformly) \( M \)-injective, where \( M \) is a class of right \( S \)-semimodules, iff for every (uniform) injective morphism \( 0 \rightarrow L \xrightarrow{f} M \) with \( M \in M \), the induced morphism of Abelian monoids \( \text{Hom}_S(M,Q) \xrightarrow{f} \text{Hom}_S(L,Q) \) is surjective (and uniform). If \( sQ \) is (uniformly) \( M \)-injective for every \( M \in sS \), then we say that \( sQ \) is (uniformly) injective. In fact, \( sQ \) is uniformly injective if and only if \( \text{Hom}_S(\_,Q) \) preserves exact sequences.

## 3 Flat Semimodules

As before, \( S \) is a semiring with \( 1_S \neq 0_S \). If \( M \) is a left \( S \)-semimodule, then we write \( U \leq_S M \) to indicate that \( U \) is a uniform (subtractive) \( S \)-subsemimodule of \( M \) (i.e. the embedding map \( U \hookrightarrow M \) is uniform).

The following definition applies to any variety in the sense of Universal Algebra (e.g. [BR2004]):

**Definition 3.1.** We say that a right \( S \)-semimodule \( F \) is flat iff \( F = \varinjlim F_t \), a directed limit (filtered colimit) of finitely presented projective right \( S \)-semimodules.
Lemma 3.2. (cf. [Kat2004a]) The following are equivalent for a right $S$-semimodule $F_S$:

1. $F \otimes_S -$ is left exact (i.e. preserves finite limits);
2. $F \otimes_S -$ preserves pullbacks and equalizers;
3. $F_S$ is pullback-flat, i.e. $F \otimes_S -$ preserves pullbacks;
4. $F_S$ is $L$-flat, i.e. $F \simeq \lim_{\to} F_\lambda$, a filtered (directed) colimit of finitely generated free $S$-semimodules;
5. $F_S$ is flat.

Although the above definition is quite natural, a notion of flatness w.r.t. to a family of semimodules is important. This motivates introducing the following notion.

Definition 3.3. Let $F$ be a right $S$-semimodule and $\mathcal{M}$ a class of left $S$-semimodules. We say that $F$ is uniformly flat w.r.t. $\mathcal{M}$ (or uniformly $\mathcal{M}$-flat) iff for every exact sequence of left $S$-semimodules

$$0 \to L \overset{f}{\to} M \overset{g}{\to} N \to 0,$$

with $M \in \mathcal{M}$, the following sequence of Abelian monoids is exact

$$0 \to F \otimes S L \overset{id_F \otimes_S f}{\to} F \otimes S M \overset{id_F \otimes_S g}{\to} F \otimes S N \to 0. \quad (15)$$

If $F_S$ is uniformly $\mathcal{M}$-flat for every left $S$-semimodule $S \mathcal{M}$, then we say that $F$ is uniformly flat.

Theorem 3.4. Let $F$ be a right $S$-semimodule.

1. Let $S \mathcal{M}$ be a left $S$-semimodule. Then $F_S$ is uniformly $\mathcal{M}$-flat if and only if for every $U \leq_{S}S \mathcal{M}$ we have $F \otimes_S U \leq_{S}F \otimes_S S \mathcal{M}$.
2. $F_S$ is uniformly flat if and only if $F \otimes_S -$ preserves uniform subsemimodule.

Proof. We need only to prove “1”.

$(\Longrightarrow)$ Assume that $F_S$ is uniformly $\mathcal{M}$-flat. Let $U \leq_{S}S \mathcal{M}$ and consider the exact sequence of $S$-semimodules $0 \to U \overset{\iota}{\to} M \overset{\pi}{\to} M/U \to 0$, where $\iota$ is the canonical embedding and $\pi$ is the canonical uniform surjection. By assumption, the sequence $0 \to F \otimes_S U \overset{id_F \otimes_S \iota}{\to} F \otimes_S M \overset{id_F \otimes_S \pi}{\to} F \otimes_S M/U \to 0$ is exact; in particular, $F \otimes_S U \leq_{S} F \otimes_S S \mathcal{M}$ is a uniform submonoid.

$(\Longleftarrow)$ Let $0 \to L \overset{f}{\to} M \overset{g}{\to} N \to 0$ be an exact sequence of left $S$-semimodules, i.e. $L \simeq \text{Ker}(g)$ and $N \simeq \text{Coker}(f)$. By Proposition 2.19 “2”, the sequence $F \otimes_S L \overset{id_F \otimes_S f}{\to} F \otimes_S M \overset{id_F \otimes_S g}{\to} F \otimes_S N \to 0$ is proper exact and $id_F \otimes_S g$ is uniform. By assumption, $id_F \otimes_S f$ is injective and uniform, whence (15) is exact. ■

Corollary 3.5. 1. Let $M$ be a left $S$-semimodule. Any retract of a uniformly $\mathcal{M}$-flat right $S$-semimodule is uniformly $\mathcal{M}$-flat.
2. Any retract of a uniformly flat right $S$-semimodule is uniformly flat.
Proof. We need only to prove “1”. Let $M$ be a left $S$-semimodule and $U \leq_S^n M$. Let $F_S$ be a uniformly $M$-flat right $S$-semimodule and $\widetilde{F}$ a retract of $F$. Then there exist $S$-linear maps $\widetilde{F} \xrightarrow{\psi} F \xrightarrow{\theta} \widetilde{F}$ such that $\theta \circ \psi = \text{id}_{\widetilde{F}}$. Consider the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\widetilde{F} \otimes_S U \\
\downarrow \psi \otimes S \text{id}_U
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\widetilde{F} \otimes_S M \\
\downarrow \psi \otimes S \text{id}_M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\widetilde{F} \otimes_S U \\
\downarrow \theta \otimes S \text{id}_U
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\widetilde{F} \otimes_S M \\
\downarrow \theta \otimes S \text{id}_M
\end{array}
\end{array}
\end{array}
\]

Indeed, $(\theta \otimes S \text{id}_U) \circ (\psi \otimes S \text{id}_U) = \text{id}_{\widetilde{F} \otimes S U}$ and $(\theta \otimes S \text{id}_M) \circ (\psi \otimes S \text{id}_M) = \text{id}_{\widetilde{F} \otimes S M}$, i.e. $\widetilde{F} \otimes_S U$ is a retract of $F \otimes_S U$ and $\widetilde{F} \otimes_S M$ is a retract of $F \otimes_S M$. Since $F_S$ is flat, $\text{id}_{\widetilde{F} \otimes S} \nu : F \otimes_S U \longrightarrow F \otimes_S M$ is injective and uniform. It follows that $\text{id}_{\widetilde{F} \otimes S} \nu$ is injective and indeed uniform by Lemma [1.21] “1”, i.e. $\widetilde{F} \otimes_S U \leq_S^n \widetilde{F} \otimes_S M$. Consequently, $\widetilde{F}$ is uniformly $M$-flat. ■

Proposition 3.6. Let $\{F_\lambda\}_{\Lambda}$ be a family of right $S$-semimodules.

1. Let $M$ be a left $S$-semimodule. Then $\oplus_{\lambda \in \Lambda} F_\lambda$ is uniformly $M$-flat if and only if $F_\lambda$ is uniformly $M$-flat for every $\lambda \in \Lambda$.

2. $\oplus_{\lambda \in \Lambda} F_\lambda$ is uniformly flat if and only if $F_\lambda$ is uniformly flat for every $\lambda \in \Lambda$.

Proof. We need only to prove “1”. Let $F := \oplus_{\lambda \in \Lambda} F_\lambda$ and consider the projections $\pi_\lambda : F \longrightarrow F_\lambda$, $(f_\lambda)_\Lambda \mapsto f_\lambda$ for $\lambda \in \Lambda$. Let $U \leq_S^n M$ be a uniform $S$-subsemimodule. Assume that $F_\lambda$ is $M$-flat for every $\lambda \in \Lambda$. Then $F_\lambda \otimes_S U \leq_S^n F_\lambda \otimes_S M$ for every $\lambda \in \Lambda$, whence $\bigoplus_{\lambda \in \Lambda} (F_\lambda \otimes_S U) \leq_S^n \bigoplus_{\lambda \in \Lambda} (F_\lambda \otimes_S M)$ by Lemma 2.22. Since $\bigoplus_{\lambda \in \Lambda} (F_\lambda \otimes_S U) \simeq \bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S U$ and $\bigoplus_{\lambda \in \Lambda} (F_\lambda \otimes_S M) \simeq \bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S M$, we conclude that $\bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S U \leq_S^n \bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S M$. It follows that $\bigoplus_{\lambda \in \Lambda} F_\lambda \otimes$ is uniformly $M$-flat. On the other hand, assume that $\bigoplus_{\lambda \in \Lambda} F_\lambda$ is uniformly $M$-flat.

Every $F_\lambda, \lambda \in \Lambda$, is a retract of $\bigoplus_{\lambda \in \Lambda} F_\lambda$ whence uniformly $M$-flat by Corollary 3.5. ■

Lemma 3.7. Let $S X$ be a left $S$-semimodule, $S Y_T$ an $(S,T)$-bisemimodule, $T Z$ a uniformly flat left $T$-module and consider the following map of Abelian monoids

$$\nu_{X,Y,Z} : \text{Hom}_S(X,Y) \otimes_T Z \longrightarrow \text{Hom}_S(X,Y \otimes_T Z), \ f \otimes_T z \mapsto f(-) \otimes_S z.$$

1. If $S X$ is uniformly finitely generated, then $\nu_{X,Y,Z}$ is injective and uniform.

2. If $S X$ is uniformly finitely presented, then $\nu_{X,Y,Z}$ is an isomorphism.

Proof. 1. Since $S X$ is uniformly finitely generated, there exists a uniform surjective $S$-linear map

$$S^n \xrightarrow{\tilde{g}} X \longrightarrow 0.$$
By Proposition 2.18, $\text{Hom}_S(X,Y) \leq^S \text{Hom}_S(S^n,Y)$, whence $\text{Hom}_S(X,Y) \otimes_T Z \xrightarrow{(\tilde{g},Y) \otimes I_{TZ}} \text{Hom}_S(S^n,Y) \otimes_T Z$ since $T Z$ is uniformly flat and we have a commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_S(X,Y) \otimes_T Z \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_S(X,Y) \otimes_T Z
\end{array}
$$

Notice that $\nu_{S^n,Y,Z}$ is an isomorphism, whence $\nu_{X,Y,Z}$ is injective. Moreover, it follows by [Abm, Lemma 3.8 (1)] that $\nu_{X,Y,Z}$ is uniform.

2. Since $S X$ is finitely presented, there exists by Proposition 2.25 an exact sequence of $S$-semimodules $S^m \xrightarrow{f} S^n \xrightarrow{\tilde{g}} X \rightarrow 0$ for some $m,n \in \mathbb{N}$. By Proposition 2.18 and the uniform flatness of $T Z$ we obtain the following commutative diagram with proper-exact rows

$$
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_S(X,Y) \otimes_T Z \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_S(X,Y) \otimes_T Z
\end{array}
$$

Notice that $\nu_{S^n,Y,Z}$ and $\nu_{S^n,Y,Z}$ are isomorphisms and so it follows by Lemma 1.22 “3” that $\nu_{X,Y,Z}$ is surjective. Notice that $\nu_{X,Y,Z}$ is injective by “1” whence an isomorphism.

Applying Lemma 3.7 to $S = T$ and $Y = S$, considered as a bisemimodule over itself in the canonical way, we obtain with $X^* = \text{Hom}_S(X,S)$:

**Proposition 3.8.** Let $S X$ be a uniformly finitely presented $S$-semimodule, $S Z$ a uniformly flat left $S$-semimodule and consider the following morphism of Abelian monoids

$$
\nu_{X,Z} : X^* \otimes_S Z \rightarrow \text{Hom}_S(X,Z), \ f \otimes_S z \mapsto f(-)z.
$$

If $S X$ is uniformly finitely generated (uniformly finitely presented), then $\nu_{X,Z}$ is injective and uniform (an isomorphism).

**Definition 3.9.** Let $M$ be a left $S$-semimodule. We say that a right $S$-semimodule $F_S$ is $M$-mono-flat [Kat2004a] (or $M$-k-flat [Alt2004]) iff $F \otimes_S L \leq_S M$ for every $S$-subsemimodule $L \leq_S M$. If $F_S$ is $M$-mono-flat for every left $S$-semimodule $M$, then we call $F_S$ mono-flat (or $k$-flat).

**Notation.** For every left $S$-semimodule $M$, we set

$$
\mathcal{I}_S(M) := \{ G \in S_S \mid G \otimes_S U \xrightarrow{id_G \otimes_S} G \otimes_S M \text{ is } i\text{-uniform } \forall U \leq^u_S M \};
$$

**Remark 3.10.** Let $M$ be a left $S$-semimodule. If $F_S \in \mathcal{I}_S(M)$ and $M$-mono-flat, then $F$ is uniformly $M$-flat.

The following result is straightforward (cf. [Alt2004, Proposition 4.1]):

**Proposition 3.11.** Let $M$ be a left $S$-semimodule and $F_S \in \mathcal{I}_S(M)$. Then $F$ is uniformly $M$-flat if and only if $F \otimes_S L \leq_S M$ for every finitely generated $S$-subsemimodule $L \leq_S M$. 

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Proposition 3.12. Let \( 0 \rightarrow M_1 \xrightarrow{\gamma} M \xrightarrow{\delta} M_2 \rightarrow 0 \) be an exact sequence of left \( S \)-semimodules and assume that \( F_S \) is uniformly \( M \)-flat.

1. \( F_S \) is uniformly \( M_1 \)-flat.

2. If \( F_S \in \mathcal{I}_S(M_2) \), then \( F \) is uniformly \( M_2 \)-flat.

Proof. Assume that \( F_S \) is uniformly \( M \)-flat.

1. Let \( U \leq_S M_1 \). Since \( M_1 \leq_S M \), we have \( U \leq_S M \), whence \( F \otimes_S U \leq_S F \otimes_S M \) and so \( F \otimes_S U \leq_S F \otimes_S M_1 \) (e.g. [Abu, Lemma 3.8 (1-b)]). Consequently, \( F_S \) is uniformly \( M_1 \)-flat.

2. Let \( U \leq_S M_2 \) and consider \( \tilde{U} := \{ m \in M \mid \delta(m) \in U \} \). Then \( \tilde{U} \leq_S M \) and we have a commutative diagram of left \( S \)-semimodules with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M_1 & \xrightarrow{\gamma} & \tilde{U} & \xrightarrow{\delta} & U & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M_1 & \xrightarrow{\gamma} & M & \xrightarrow{\delta} & M_2 & \rightarrow & 0
\end{array}
\]

Tensoring with \( F_S \), we obtain a commutative diagram of Abelian monoids

\[
\begin{array}{ccccccccc}
0 & \rightarrow & F \otimes_S M_1 & \xrightarrow{id_F \otimes S \gamma} & F \otimes_S \tilde{U} & \xrightarrow{id_F \otimes S \delta} & F \otimes_S U & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & F \otimes_S M_1 & \xrightarrow{id_F \otimes S \gamma} & F \otimes_S M & \xrightarrow{id_F \otimes S \delta} & F \otimes_S M_2 & \rightarrow & 0
\end{array}
\]

Since \( F_S \) is uniformly flat, the second row is exact. By Proposition 2.19, the first row is semi-exact and \( id_F \otimes_S \delta \) is uniform. It follows by Lemma 1.22 “1-a” that \( id_F \otimes_S \iota \) is injective. Since \( F \in \mathcal{I}_S(M) \), we have \( F \otimes_S U \leq_S F \otimes_S M_2 \). Consequently, \( F_S \) is uniformly \( M_2 \)-flat. \( \blacksquare \)

Let

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

be a sequence of left \( S \)-semimodules. The proof of the following result is straightforward:

Proposition 3.13. 1. If \( F_S \) is a free right \( S \)-semimodule and \((16)\) is exact (resp. semi-exact, quasi-exact, proper-exact), then the sequence

\[
F \otimes_S A \xrightarrow{id_F \otimes S f} F \otimes_S B \xrightarrow{id_F \otimes S g} F \otimes_S C
\]

of Abelian monoids is exact (resp. semi-exact, proper-exact, quasi-exact).

2. Every free \( S \)-semimodule is uniformly flat.
Corollary 3.14. 1. If $P_S$ is a projective right $S$-semimodule and (16) is exact (resp. semi-exact, quasi-exact, proper-exact), then the sequence

$$P \otimes_S A \xrightarrow{id \otimes_S f} P \otimes_S B \xrightarrow{id \otimes_S g} P \otimes_S C$$  

(18)

of Abelian monoids is exact (resp. semi-exact, proper-exact, quasi-exact).

2. Every projective $S$-semimodule is uniformly flat.

Proof. 1. This follows directly from Proposition 3.13 and Lemma 1.21 “2”.

2. This follows directly from the definition and “1”.

Definition 3.15. Let $Q$ be a right $S$-semimodule. We say that $Q_S$ (uniformly) cogenerates a class $M$ of right semimodules iff the following holds: for every morphisms $\iota : U \rightarrow M$ with $M \in M$, if $(\iota, Q) : \text{Hom}_S(M, Q) \rightarrow \text{Hom}_S(U, Q)$ is surjective (and uniform), then $0 \rightarrow U \xrightarrow{\iota} M$ is injective (and uniform). If $Q$ (uniformly) cogenerates all left $S$-semimodules, then we say that $Q$ is a (uniform) cogenerator in $S_S$.

Example 3.16. The assumption that $S$ has an injective cogenerator might be empty. For example the semiring $\mathbb{N}_0$ has no injective cogenerators.

Proposition 3.17. Let $T_S F$ be a $(T,S)$-bisemimodule, $M$ a left $S$-semimodule and $X$ a left $T$-semimodule.

1. Let $T_X$ be uniformly $F \otimes_S M$-injective. If $F_S$ is uniformly $M$-flat, then $\text{Hom}_T(F, X)$ is uniformly $M$-injective.

2. Let $M$ be uniformly $X$-cogenerated. If $\text{Hom}_T(F, X)$ is uniformly $M$-injective, then $F_S$ is uniformly $M$-flat.

Proof. Let $M$ be a left $S$-semimodule, $U \leq^*_S M$ and consider the following commutative diagram

$$\begin{array}{ccc}
\text{Hom}_S(M, \text{Hom}_T(F, X)) & \xrightarrow{=} & \text{Hom}_T(F \otimes_S M, X) \\
\downarrow & & \downarrow \\
\text{Hom}_S(U, \text{Hom}_T(F, X)) & \xrightarrow{=}(\text{id}_F \otimes_S U, X) & \text{Hom}_T(F \otimes_S U, X)
\end{array}$$

1. Let $T_X$ be uniformly $F \otimes_S M$-injective. If $F_S$ is uniformly $M$-flat, then $F \otimes_S U \leq^*_F F \otimes_S M$, whence $(\text{id}_F \otimes_S U, X)$ is surjective and uniform. Consequently, $(-, \text{Hom}_T(F, X))$ is surjective and uniform. This means that $\text{Hom}_T(F, X)$ is uniformly $M$-injective.

2. Let $M$ be uniformly $X$-cogenerated. If $\text{Hom}_T(F, X)$ is uniformly $M$-injective, then $(-, \text{Hom}_T(F, X))$ is surjective and uniform, whence $(\text{id}_F \otimes_S U, X)$ is surjective and uniform. So, $\text{id}_F \otimes_S U$ is injective and uniform. This means that $F_S$ is uniformly $M$-flat.

Theorem 3.18. Let $T_S F$ be a $(T,S)$-bisemimodule and assume that $T_S$ has a uniformly injective-cogenerator $Q$. Then $F_S$ is uniformly flat if and only if $\text{Hom}_T(F, Q)$ is uniformly injective.
The analogous of Baer’s criterion for injective modules over rings “$M$ is $R$-injective $\implies M$ is injective” might fail for semimodules over semirings.

**Example 3.19.** ([Hi2008]) The semifield $(\mathbb{Q}^+, +, \cdot)$ has only two ideals $\{0\}$ and $\mathbb{Q}^+$ whence every semimodule is $\mathbb{Q}^+$-injective. However, $\{0\}$ is the only injective $\mathbb{Q}^+$-semimodule (e.g. by [Hi2008]).

The above example motivates the following definitions:

**Definition 3.20.** We say that the semiring $S$ is a left (uniformly) Baer’s semiring iff every (uniformly) injective left $S$-semimodule is (uniformly) injective. The right (uniformly) Baer-injective semirings can be defined analogously.

**Proposition 3.21.** Let $\tau F_S$ be a $(T, S)$-bisemimodule and assume that $\tau S$ has a uniformly injective cogenerator $Q$. If $S$ is a left uniformly Baer semiring, then the following are equivalent:

1. $F_S$ is uniformly flat;
2. For every uniform left ideal $sI \leq u S$, we have $F \otimes_S I \leq u S F \otimes_S S$.

**Proof.** We need only to prove “2” $\implies “1”$. Let $sI \leq u S$ be a left uniform ideal. By assumption, $0 \to F \otimes_S I \overset{\text{id}_F \otimes s \iota}{\to} F \otimes_S S$ is exact and $\text{id}_F \otimes s \iota$ is a uniform morphism of left $T$-semimodule, whence $\text{Hom}_T(F \otimes_S I, Q) \overset{\text{(id}_F \otimes s \iota, Q)}{\to} \text{Hom}_S(F \otimes_S S, Q) \to 0$ is exact and $(\text{id}_F \otimes s \iota, Q)$ is uniform. Notice that $\text{Hom}_T(F \otimes_S I, Q) \simeq \text{Hom}_S(I, \text{Hom}_T(F, Q))$ and $\text{Hom}_S(F \otimes_S S, Q) \simeq \text{Hom}_S(S, \text{Hom}_T(F, Q))$, whence $\text{Hom}_S(I, \text{Hom}_T(F, Q)) \overset{(s \iota, \text{Hom}_T(F, Q))}{\to} \text{Hom}_S(S, \text{Hom}_T(F, Q)) \to 0$ is exact and $(s \iota, \text{Hom}_T(F, Q))$ is uniform, i.e. $\text{Hom}_T(F, Q)$ is uniformly $S$-injective. Since $S$ is a left uniformly Baer semiring, we conclude that $\text{Hom}_T(F, Q)$ is uniformly injective as a left $S$-semimodule, whence $F_S$ is uniformly flat by Theorem 3.18.

**Theorem 3.22.** Let $(F_j, \{f_{jj'}\})_J$ be a directed system of right $S$-semimodules.

1. If each $F_j$ is uniformly $M$-flat, for some left $S$-semimodule $M$, then $\lim_{\to} F_j$ is uniformly $M$-flat.
2. If each $F_j$ is uniformly flat, then $\lim_{\to} F_j$ is uniformly flat.

**Proof.** We need only to prove “1”. Assume that $F_j$ is uniformly $M$-flat for every $j \in J$. Let $U \leq u S M$. Then $F_j \otimes_S U \leq u S F_j \otimes_S M$ for each $j \in J$. It follows by Corollary 2.16 that $\lim_{\to} (F_j \otimes_S U) \leq u S \lim_{\to} (F_j \otimes_S M)$ and so we are done (note that $\lim_{\to} F_j \otimes_S U \simeq \lim_{\to} (F_j \otimes_S U)$ and $\lim_{\to} F_j \otimes_S M \simeq \lim_{\to} (F_j \otimes_S M)$).

**Corollary 3.23.** If every finitely generated subsemimodule of an $S$-semimodule $F$ is uniformly flat, then $F$ is uniformly flat.

**Proof.** This follows directly from Theorem 3.22 and the fact that every semimodule is the direct limit of its finitely generated subsemimodules (cf. Proposition 2.11).

As a direct consequence of Theorem 3.22 we obtain:

**Corollary 3.24.** Every flat $S$-semimodule is uniformly flat.
We finish this manuscript with the following open question:

**Question:** When is every uniformly flat \( S \)-semimodule flat?

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