Field Theoretical Treatment of a Three-Spin Interaction Model on Triangular Lattice

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Abstract. We provide an effective description of a triangular-lattice model with a three-spin interaction. On the basis of a symmetry consideration, we argue that a vector dual sine-Gordon field theory is relevant, and provides us a framework to analyze long-distance behaviors. In particular, we can characterize a criticality of an intermediate phase between the low- and high-temperature phases. Also, in comparison with a model on the Union Jack lattice, we explain that the sublattice symmetry of the triangular lattice plays a crucial role to exhibit a novel phase transition. Theoretical predictions obtained are then confirmed by numerical calculations.

Introduction. — Theoretical models which include two-body interactions have been frequently investigated due to their principal importance in classical and quantum mechanical problems. Those including multi-body interactions such as three- and four-body exchange interactions have been also discussed. But, there only exist a limited number of theoretical models which proffer a clear understanding on multi-body interaction effects, although they can be new sources for emergence of novel phenomena. In this report, we provide our analysis on a three-spin interaction model defined on the triangular lattice ($\Lambda$) introduced by Alcaraz et al [1, 2]. Suppose that $\langle k,l,m \rangle$ denotes three sites at corners of each elementary plaquette of $\Lambda$, then the three-spin interaction model is defined by the Hamiltonian:

$$\beta H = -\frac{J}{k_B T} \sum_{\langle k,l,m \rangle} \cos (\varphi_k + \varphi_l + \varphi_m), \quad \varphi_k \in \left\{ 0, \frac{2\pi}{p}, \ldots, \frac{2\pi(p-1)}{p} \right\}. \quad (1)$$

The symmetry of this model is termed $\mathbb{Z}_p \times \mathbb{Z}_p$, and is expressed by a global spin rotation,

$$\hat{R} : \varphi_k \rightarrow \varphi_k + \sum_{\rho=a,b,c} \sum_{i \in \Lambda_{\rho}} \frac{2\pi n_{\rho}}{p} \delta_{k,i} \quad \text{with} \quad n_a + n_b + n_c = 0 \quad (\text{mod} \; p), \quad (2)$$

where sublattices of $\Lambda$ are denoted as $\Lambda_a$, $\Lambda_b$, and $\Lambda_c$. This possesses two independent phases, say $(n_a, n_b)$ in units of $2\pi/p$, and is broken at a sufficiently low temperature to give a $p^2$-fold degenerate ordered state. In addition, an intermediate critical phase between the ordered and the disordered phases has been expected for the case with $p$ greater than a certain value. Thus, in this report, we shall investigate an effective description of the model (1) to characterize the criticality of the intermediate phase as well as phase transitions. Also, in the last part, we give some numerical evidences which support our argument.

Theory. — Likewise the $\mathbb{Z}_p$ clock model [3], it is plausible that $p^2$ ordered states are expressed as locking points of a phase field. But, unlike the model, its periodicity is not known a priori, and
is determined based on a symmetry consideration. For this, we employ a strategy development by Kondev and Henley [4]. According to them, let us construct the so-called ideal-state graph $\mathcal{I}$ for the present model, whose dimension and structure determine a number of components and a periodicity of field, respectively. By definition, each node of $\mathcal{I}$ represents one of ordered states, and each link exhibits two neighbor nodes connected by minimal spin rotations:

$$R_a : (n_a, n_b, n_c) = (1, p - 1, 0), \quad R_b : (0, 1, p - 1), \quad R_c : (p - 1, 0, 1).$$

(3)

Two of the three are generators of the symmetry (2), so $\mathcal{I}$ is located in a 2D space. Also, due to the sublattice symmetry and some relations among spin rotations, e.g., $R_c R_b R_a = 1$, the shortest loop of links should form a regular triangle. These in turn require $\mathcal{I}$ to be on a triangular lattice. Further, the conditions $\hat{R}^p_i = 1$ give a unit cell of the so-called repeat lattice $\mathcal{R}$. In figure 1 of reference [5], we have exhibited $\mathcal{I}$ for $p = 6$, where 36 ideal states are specified by sublattice dependent angles—"000" at corners represents the spin configuration $\{\phi_i\} = 0$. Consequently, one finds that a two-component vector field is relevant. When we writing it as $\Phi(x)$ ($x$ is the 2D real-space position vector), then the periodicity is expressed as $\Phi = \Phi + 2\pi N$, where $N$ is a lattice vector in $\mathcal{R}$. In figure 2 of [5] the lattice was drawn by solid lines.

For a representation of an effective theory in terms of $\Phi$, field transformations $\Phi \to \Phi + \delta \Phi$ corresponding to spin rotations (3) play a crucial role: Writing the normalized non-orthogonal primitive vectors of $\mathcal{R}$ as $(e_1, e_2)$, then we obtain

$$\hat{R}_a : \delta \Phi = (2\pi/p) e_1, \quad \hat{R}_b : (2\pi/p) (-e_1 + e_2), \quad \hat{R}_c : -(2\pi/p) e_2.$$

(4)

To derive the form of the phase-locking potential, we need to address following issues: 1) The potential should be invariant under the field transformations (4). 2) The point group symmetry of $\mathcal{I}$ should be taken into account. 3) In a renormalization-group (RG) sense, it is sufficient to keep the most relevant terms. These can be realized as $V_{p,0}(\Phi) = h_{p,0} \sum_{||M||=pa^*} e^{iM \cdot \Phi}$, where $M$ is a lattice vector in a reciprocal lattice of $\mathcal{R}$, $\mathcal{R}^*$, and $a^* = (2/\sqrt{3})$ is its lattice constant. We also write the primitive vectors of $\mathcal{R}^*$ as $\{e^1, e^2\}$ (see figure 2 of [5]). The summation is taken over all vectors with their norm $||M|| = pa^*$: Explicitly, it is done over six vectors $\pm pe^1, \pm pe^2, \pm p(e^1 + e^2)$. In figure 1 of [5], the contour plot of $V_{6,0}(\Phi)$ with $h_{6,0} = 1/2$ has been given; one can see that the points with a minimum value form a triangular lattice and reproduce $\mathcal{I}$.

While $V_{p,0}$ succeeds in reproducing the ordered state, a spatial fluctuation of field becomes important with increase of temperature. For this issue, it has been expected that a critical intermediate phase is stabilized for $p$ greater than a certain value, and it corresponds to a roughening phase of an interface model. To account for the phase, it is natural to introduce a free-boson term as $L_0 = (K/4\pi) \sum_{i=x,y} ||\partial_i \Phi(x)||^2$, where the Gaussian coupling $K$ controls the stiffness of an interface in a roughening phase. The summation is over $i = (x, y)$ specifying the Cartesian component of $x$.

At this stage, we can describe the ordered and the critical phases. However, to describe the disordered phase, we should next consider discontinuity of $\Phi$ by an amount of $2\pi N$ ($N \in \mathcal{R}$), which becomes frequent with increase of temperature. This topological defect is created by the vertex operator $e^{iN \cdot \Theta}$, where $\Theta$ is the dual field of $\Phi$ and is defined as $iK\partial_i \Phi = \epsilon_{ij} \partial_j \Theta$ ($\epsilon_{ij}$ is the antisymmetric symbol). To obtain an explicit form, we can repeat the similar argument to the above, and arrive at the following expression: $V_{0,1}(\Theta) = h_{0,1} \sum_{||N||=1} e^{iN \cdot \Theta}$. The summation is taken over six vectors: $\pm e_1, \pm e_2, \pm (e_1 - e_2)$ (see the left panel of figure 1).

Consequently, we obtain a Lagrangian density [5]:

$$L_0 = L_0 + L_1 + L_2$$

with

$$L_0 = \frac{K}{4\pi} \sum_{i=x,y} ||\partial_i \Phi(x)||^2, \quad L_1 = \frac{yp_{0}}{2\pi\alpha} \sum_{||M||=pa^*} e^{iM \cdot \Phi}, \quad L_2 = \frac{yp_{0}}{2\pi\alpha} \sum_{||N||=1} e^{iN \cdot \Theta},$$

(5)
We consider a system on \( \mathbb{Z}^d \) with \( \mathbf{M} \) electric and \( \mathbf{N} \) magnetic vector charges and its scaling dimension in the critical region parameterized by \( K \) are respectively given by

\[
O_{\mathbf{M},\mathbf{N}} \equiv \exp \left[ i (\mathbf{M} \cdot \Phi + \mathbf{N} \cdot \Theta) \right] \quad \text{and} \quad x_{\mathbf{M},\mathbf{N}} \equiv \frac{1}{2} \left( K^{-1} ||\mathbf{M}||^2 + K ||\mathbf{N}||^2 \right). \tag{6}
\]

This formula supports our treatment that vertex operators with the shortest vector charges were kept in the effective theory, and also tells us that \( \mathcal{L}_0 \) is perturbed by \( \mathcal{L}_{1,2} \), but these are both irrelevant for \( p^2/3 \geq K \geq 4 \). Therefore, we conclude that our model may possess the intermediate critical phase for \( p \geq 4 \), and it exhibits a transition to the low-temperature (high-temperature) ordered (disordered) phase at a certain point, say \( T_L \left( T_H \right) \) \cite{5, 6, 7}.

Now, it is worthy to compare the present model to those defined on other 2D lattices. The Union Jack lattice \( \Lambda_{UJ} \) consists of isosceles right triangles, and each of them includes three sites in different three sublattices, so the three-spin interaction model (1) can be also defined on \( \Lambda_{UJ} \). According to Alcaraz and Cardy \cite{8}, for \( p \geq 5 \), a phase transition between an intermediate critical and the high-temperature disordered phases occurs due to unbindings of vortices. The fundamental vector charges (corresponding to \( \mathbf{N} \) with small norms) were clarified to take a conformation given in the right panel of figure 1, which, in comparison with the left, exhibits a deformation reflecting the lack of the sublattice symmetry. Then, based on the RG analysis they predicted that the usual Berezinskii-Kosterlitz-Thouless (BKT) transition occurs despite of the three-spin interaction. Also from our viewpoint, their conclusion is plausible; here we shall briefly explain its reason. Since, in a RG sense, the defects corresponding to \( \pm \mathbf{a} \) can be neglected due to their higher dimensionality \cite{5, 6, 7}, the rest of four vectors, \( \pm \mathbf{b} \) and \( \pm (\mathbf{a} - \mathbf{b}) \), are responsible for the transition. Then, we can see that any three-point functions among vertex operators with these four vector charges should vanish in the critical phase because they cannot satisfy the vector-charge neutrality condition \cite{5, 7}. This implies that the unbinding of \( \pm \mathbf{b} \) vortices occurs almost independently of \( \pm (\mathbf{a} - \mathbf{b}) \) vortices, and vice versa. Consequently, four vectors are decoupled into two pairs; the charges in each pair are then located in a unidimensional space which is identical to the BKT transition case. In contrast, for the case of the triangular lattice with the sublattice symmetry, such a decoupling/pairing does not occur due to the symmetric conformation of vectors as depicted in the left panel of figure 1. In fact, based on the effective theory, we have predicted that the phase transition to the disordered phase is not the BKT type, but a novel continuous one; details have been discussed analytically and numerically in our recent researches \cite{6, 7}.

\textbf{Numerical Calculations.}— We explain our numerical transfer-matrix calculations and results. We consider a system on \( \Lambda \) with \( M (\to \infty) \) rows of \( L \) (a multiple of 3) sites wrapped on a cylinder, and define the transfer matrix connecting the next-nearest-neighbor rows. We denote its eigenvalues as \( \lambda_q(L) \) or their logarithms as \( E_q(L) = -\frac{1}{2} \ln |\lambda_q(L)| \) \( (q \) specifies a level). Then, the conformal invariance provides expressions of the central charge \( c \) and the scaling dimension \( x_q \) in critical phases as \( E_q(L) \sim L f - \pi c/6L \zeta \) and \( E_q(L) - E_g(L) \sim 2\pi x_q/L \zeta \). Here, \( E_g(L) \), \( \zeta \left( = 2/\sqrt{3} \right) \), and \( f \) correspond to the logarithm of the largest eigenvalue, the geometric factor,
and a free energy density, respectively [9, 10, 11]. When performing diagonalization calculations, we employ two of three spin rotations (3) [e.g., \((R_a, R_b)\)] as well as the lattice translation and space inversion. This is because discrete symmetries can specify a level to be analyzed: For instance, we can find \(E_q(L)\) corresponding to the spin operator \(S_a(x) = e^{i x} + e^{i x}\), in the sector with indexes \((e^{i2\pi/p}, 1)\). Now, we fix \(p = 6\) and treat systems with the size up to \(L = 9\). In the left panel of figure 2, we provide the \(T\) dependence (in units of \(J/k_B\)) of \(c\), where the region with \(c' \approx 2\) can be recognized. From this plot, we can roughly estimate phase transition temperatures as \(1/T_L(1/3)\) and \(1/T_H(1/3)\). Theoretically, \(c\) is expected to go quickly to zero outside of the region, but it is smeared out by finite-size effects, and takes finite values. Next, we plot the \(T\) dependence of the exponent \(\eta = 2x e^{i} + e^{i}\) along its second axis, which shows that \(\eta\) is the increasing function of \(T\) and approaches a close value to \(1/9\) (1/3) around \(T_L(T_H)\) (see dotted lines). In the right panel of figure 2, we give the scaling dimension \(x_{M,0}\) as a function of \(M\) along the path depicted in the right inset. As a representative for the critical region, we numerically pick up the self-dual point of \(K = 4\sqrt{3}\) via a crossing of two levels corresponding to \(L_1\) and \(L_2\) (see the left inset), and then we perform calculations at \(1/T_{SD} \approx 1.252\). From the figure, we can verify that the dimension depends only on the norm of the vector charge. Further, despite the smallness of \(L\), the results (open circles) agree well with the theoretical formula (6) (dotted curves), which quantitatively supports our argument.

**Summary.** — Based on the ideal-state graph concept, we discussed the effective description on the three-spin interaction model. We found that it is given by the vector dual sine-Gordon field theory, where two kinds of vector charges are expressed by the reciprocal and the repeat lattice vectors. Then, we provided numerical evidences to support our predictions.

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