Isomorphisms between curve graphs of infinite-type surfaces are geometric

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Abstract

Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. In this paper we show that in this situation $S$ and $S'$ are homeomorphic and $\phi$ is induced by a homeomorphism $h : S \to S'$.

1 Introduction

This is the last of three papers on which the authors study the natural action of the extended mapping class group $\text{Mod}^*(S)$ of an infinite-type surface $S$ on the curve graph $\mathcal{C}(S)$ and whether any isomorphism between curve graphs actually comes from a homeomorphism, see [4], [3]. Our main result is the following:

Theorem 1. Let $S$ and $S'$ be infinite-type connected orientable surfaces with empty boundary and $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ a simplicial isomorphism. Then $S$ is homeomorphic to $S'$ and $\phi$ is induced by a homeomorphism $h : S \to S'$.

As an immediate consequence of this result and Corollary 1.2 in [3] we obtain an analogue for infinite-type surfaces of a foundational well-known result by Ivanov (see Theorem 1 in [6]):

Theorem 2. Let $S$ be an infinite-type connected orientable surface with empty boundary. Then every automorphism of the curve graph $\mathcal{C}(S)$ is induced by a homeomorphism. More precisely, the natural map:

$$\Psi : \text{Mod}^*(S) \to \text{Aut}(\mathcal{C}(S))$$

is an isomorphism.

It is important to remark that both of these results were known to be true for infinite-type surfaces for which all topological ends carry (infinite) genus [4]. With this in mind we highlight the main contribution of this text: A new and very simple proof to the fact that every automorphism of the curve graph of an infinite-type surface is geometric. The technology we present is based on principal exhaustions, which were introduced in [3].

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1 A surface is of infinite type if its fundamental group is not finitely generated.
Roughly speaking, these are nested sequences of finite-type subsurfaces whose union is $S$ and that allow us to “pull” classical results about simplicial actions of mapping class groups to the realm of infinite-type surfaces. In particular, the proof that we present of Theorem 1 makes no use of Dehn-Thurston coordinates and is hence completely independent (and simpler) from the proofs presented in [4].

Applications. Given that the curve graph of an infinite-type surface has diameter 2, the natural action of $\text{Mod}^*(S)$ on $\mathcal{C}(S)$ gives no large scale information. However, the results that we present in this text have found the following non-trivial applications:

1. Let $S$ be an infinite-type surface whose genus is finite and at least 4 (e.g. a closed surface of genus 5 to which we have removed a Cantor set) and denote by $\text{PMod}(S)$ the subgroup of $\text{Mod}(S)$ consisting of all orientation preserving mapping classes acting trivially on the topological ends of $S$. This group is called the pure mapping class group of $S$. There is a natural monomorphism from $\text{Mod}^*(S)$ to $\text{Aut}(\text{PMod}(S))$ given by the action of $\text{Mod}^*(S)$ on the pure mapping class group by conjugation. The following result, obtained by Patel and Vlamis in [7], is a generalization of a famous result of Ivanov for mapping class groups of surfaces of finite type (see [5]):

**Theorem 3.** If $S$ is an infinite-type surface of genus at least 4, then the natural monomorphism from $\text{Mod}^*(S)$ modulo its center to $\text{Aut}(\text{PMod}(S))$ is an isomorphism.

Indeed, Patel and Vlamis show that any automorphism of $\text{PMod}(S)$ preserves Dehn twists. As they remark, if every automorphism of $\mathcal{C}(S)$ is induced by a homeomorphism of the surface, then Theorem 3 follows by a standard argument that can be found in Ivanov’s original paper [Ibid].

2. Let $\Sigma_g$ be the surface that results from removing a Cantor set from a closed orientable surface of genus $g \geq 0$. In [1] Aramayona and Funar study $B_g$ the asymptotically rigid mapping class group of $\Sigma_g$. This is a finitely presented subgroup of $\text{Mod}^*(\Sigma_g)$ which contains the mapping class group of every surface of genus $g$ with nonempty boundary. Using Theorem 2, they prove that $B_g$ is rigid, that is:

**Theorem 4.** For every $g < \infty$, $\text{Aut}(B_g)$ coincides with the normalizer of $B_g$ within $\text{Mod}^*(\Sigma_g)$.

More precisely, their result uses the following lemma, which is proven using Theorem 2:

**Lemma 5.** If $g < \infty$ then $\text{Aut}(\text{PMod}_c(\Sigma_g)) = \text{Mod}^*(\Sigma_g)$.

Here $\text{PMod}_c(\Sigma_g)$ denotes the subgroup of $\text{Mod}^*(\Sigma_g)$ formed by compactly supported pure mapping classes. For details we refer the reader to [Ibid].

It is in the light of these facts that we conjecture that further applications of Theorems 1 and 2 to the study of big mapping class groups should exist. In particular, it is a natural to
wonder which of the classical applications of Ivanov’s theorem have an analog in the realm of infinite-type surfaces.

**Reader’s guide.** In §2 we make a short discussion on the general aspects of pants decompositions. We also recall the notion of principal exhaustion. In §3 we prove several topological properties that are preserved under isomorphisms of curve graphs. Finally, in §4 we prove that these isomorphisms are geometric.

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## 2 Preliminaries

The main tools that we use in this text to prove our theorem are pants decompositions and a special kind of exhaustion for infinite-type surfaces called *principal exhaustions*. In what follows we recall the definitions of these objects and the main properties that we need.

Abusing language and notation, we call *curve*, a topological embedding $S^1 \hookrightarrow S$, the isotopy class of this embedding and its image on $S$. A curve is said to be *essential* if it is neither homotopic to a point nor to the boundary of a neighbourhood of a puncture. Hereafter all curves considered are essential unless otherwise stated.

A collection of essential curves $L$ in $S$ is *locally finite* if for every $x \in S$ there exists a neighbourhood $U_x$ of $x$ which intersects finitely many elements of $L$. A locally finite collection of pairwise disjoint non-isotopic essential curves is called a *multicurve*.

**Definition 6 (The Curve Graph).** The curve graph of $S$, denoted by $C(S)$, is the abstract graph whose vertices are isotopy classes of essential curves in $S$, and two vertices $\alpha$ and $\beta$ span an edge if the corresponding curves are disjoint modulo homotopy. We denote the set of vertices of $C(S)$ by $C^0(S)$.

**Remark 7.** The curve graph is the 1–skeleton of the curve complex, that is, the abstract simplicial complex whose simplices are multicurves of finite cardinality. The curve complex is a flag complex, in particular it is completely determined by its 1-skeleton and for this reason in this text we restrain our discussion to the curve graph.

**Definition 8 (Pants Decomposition).** A maximal (w.r.t. inclusion) multicurve is called a pants decomposition.

In this text we call both a maximal multicurve $P = \{\alpha_k\}_{k \in K}$ and its image $\{[\alpha_k]\}_{k \in K}$ in $C^0(S)$, a pants decomposition. The following lemma gives a simplicial characterization of pants decompositions.

**Lemma 9.** Let $S$ be a surface and $P = \{a_k\}_{k \in \mathbb{N}} \subset C^0(S)$. Then $P = \{a_k\}_{k \in \mathbb{N}}$ is pants decomposition for $S$ if and only if this collection satisfies:

1. $i(a_k,a_l) = 0$ for all $k, l \in \mathbb{N}$,
2. for each \( a \in C^0(S) \setminus P \) we have:

(a) \( i(a, a_k) \neq 0 \) for some \( k \in \mathbb{N} \), and

(b) \( |\{ k \in \mathbb{N} : i(a, a_k) \neq 0 \}| < \infty \).

Proof. The necessity condition is easily verified. We prove the sufficiency. Since \( S \) is of infinite type, it is uniformized by the Poincaré disc and hence we can choose a complete Riemannian metric on \( S \) with negative constant curvature. Since the metric is complete, we can choose for each \( a_k \) the only geodesic representative \( \alpha_k \) in its class. Conditions 1 and 2.a above assure that \( \{ \alpha_k \}_{k \in \mathbb{N}} \) is a maximal collection of non-isotopic curves. To see that this collection is locally finite we proceed as follows. Let \( s \in S \) be a point and \( N \subset S \) a compact finite-type subsurface containing \( s \) such that each connected component of \( \partial N \) is a closed geodesic that is essential in \( S \). Suppose first that there exists finitely many curves in \( \{ \alpha_k \}_{k \in \mathbb{N}} \) which intersect \( \partial N \). In this case one can easily find a neighbourhood \( U_s \) of \( s \) in \( N \) which intersects only finitely many elements of \( \{ \alpha_k \}_{k \in \mathbb{N}} \). On the other hand, if infinitely many elements of \( \{ \alpha_k \}_{k \in \mathbb{N}} \) intersect \( \partial N \) then we get a contradiction with condition 2.b. \( \square \)

Corollary 10. Let \( \phi : C(S) \to C(S') \) be a simplicial isomorphism. Then \( \phi \) sends pants decompositions in \( S \) to pants decompositions in \( S' \).

Let \( \mathcal{L} \) be a multicurve. We say that \( \mathcal{L} \) bounds a subsurface \( \Sigma \) of \( S \), if the elements of \( \mathcal{L} \) are exactly all the boundary curves of the closure of \( \Sigma \) on \( S \). Also, we say that \( \Sigma \) is induced by \( \mathcal{L} \) if there exists a subset \( \mathcal{M} \subset \mathcal{L} \) that bounds \( \Sigma \) and there are no elements of \( \mathcal{L} \setminus \mathcal{M} \) in its interior.

Remark 11. Given that every pants decomposition \( P \) is locally finite, the index set \( K \) in \( P = \{ \alpha_k \}_{k \in K} \) is at most countable. Moreover, every connected component of \( S \setminus P \) is homeomorphic to the thrice-punctured sphere. In particular, every closed subsurface \( \Sigma \) induced by \( P \) is homeomorphic to the compact surface of genus zero and three boundary components. This topological surface is called a pair of pants.

Recall that an essential curve \( \alpha \) is called separating if \( S \setminus \alpha \) is disconnected. A separating curve \( \alpha \) is called outer if it bounds a twice-punctured disc.

Let \( P \) be a pants decomposition, and let \( \alpha, \beta \in P \). We say \( \alpha \) and \( \beta \) are adjacent w.r.t. \( P \) if there exists a subsurface \( \Sigma \) induced by \( P \) such that \( \alpha \) and \( \beta \) are two of its boundary curves.

In all the proofs of our main theorems we use the following graph associated to a pants decomposition.

Definition 12 (Adjacency Graph). Let \( S \) be a surface and \( P \) a pants decomposition of \( S \). We define the adjacency graph of \( P \), denoted by \( A(P) \), as the abstract simplicial graph whose set of vertices is \( P \) and where two curves \( \alpha \) and \( \beta \) span an edge if they are adjacent w.r.t. \( P \).

Given a pants decomposition \( P \) and a subset \( Y \subseteq P \), we denote by \( V(Y) \) the set of vertices in \( A(P) \) defined by elements in \( Y \).

Finally, we recall a particular way to exhaust infinite-type surfaces that is used to prove that every automorphism of the curve graph is geometric.
Definition 13 (Principal exhaustion). Let \( \{S_i\}_{i \in \mathbb{N}} \) be an (set-theoretical) increasing sequence of open connected subsurfaces of \( S \). We say \( \{S_i\}_{i \in \mathbb{N}} \) is a principal exhaustion of \( S \) if \( S = \bigcup_{i \geq 1} S_i \) and for all \( i \geq 1 \) it satisfies the following conditions:

1. \( S_i \) is a surface of finite topological type,
2. \( S_i \) is contained in the interior of \( S_{i+1} \),
3. \( \partial S_i \) is the finite union of pairwise disjoint essential separating curves on \( S \),
4. each connected component of \( S_{i+1} \setminus \overline{S_i} \) has complexity at least 4, and
5. each connected component of \( S \setminus \overline{S_i} \) is of infinite topological type.

3 Topological properties

In this section we prove several topological properties preserved under an isomorphism \( \phi : \mathcal{C}(S) \to \mathcal{C}(S') \). All surfaces in this section are of infinite type, unless otherwise stated.

The following two propositions and lemma can be deduced from the work of Shackleton [8]. More precisely, Propositions 14 and 16 below are Lemmas 8 and 12 in [Ibid.]; on the other hand, Lemma 15 below follows from Lemmas 9 and 10 in [Ibid.] and the fact that \( \phi \) is an isomorphism.

As a matter of fact, Shackelton does not work on the context of infinite-type surfaces but the arguments that he uses to prove these results are of local nature and hence can be immediately extrapolated to all infinite-type surfaces. For the sake of completeness, we include a sketch of proof in each case.

**Proposition 14.** Let \( \phi : \mathcal{C}(S) \to \mathcal{C}(S') \) be a simplicial isomorphism between curve graphs of infinite-type surfaces. Then \( \phi \) induces a graph isomorphism

\[
\tilde{\phi} : \mathcal{A}(P) \to \mathcal{A}(\phi(P))
\]

for any pants decomposition \( P \) of \( S \).

**Sketch of proof.** Since pants decompositions are maximal multicurves, \( \tilde{\phi} \) is a biyective correpondence between the set of vertices of \( \mathcal{A}(P) \) and the vertices of \( \mathcal{A}(\phi(P)) \). Then we only need to check that \( \tilde{\phi} \) and \( \tilde{\phi}^{-1} \) preserve edges, but this follows from the fact that any two vertices \( \alpha \) and \( \beta \) are adyacent in \( \mathcal{A}(P) \) if and only if there exist a curve \( \gamma \) in \( S \) that intersects \( \alpha \) and \( \beta \) but does not intersect any other element in \( P \setminus \{\alpha, \beta\} \).

**Lemma 15.** Let \( \phi : \mathcal{C}(S) \to \mathcal{C}(S') \) be a simplicial isomorphism between curve graphs of infinite-type surfaces \( S \) and \( S' \). Then \( \phi \) maps nonouter separating curves to nonouter separating curves, nonseparating curves to nonseparating curves and hence outer curves to outer curves.
Sketch of proof. Nonouter separating curves are cut vertices of the graph $\mathcal{A}(P)$, for any pants decomposition $P$ and vice versa. Since $\tilde{\phi}$ is an isomorphism, cut vertices must go to cut vertices. On the other hand, outer curves are vertices of degree at most 2 in $\mathcal{A}(P)$, for any pants decomposition $P$. Given that $\phi$ is an isomorphism, nonseparating curves cannot be mapped to nonouter separating curves. If for a nonseparating curve $\alpha$, $\phi(\alpha)$ were an outer curve, then we could find a pants decomposition $P$ containing $\alpha$ for which the vertex corresponding to $\alpha$ has degree four. This contradicts the fact that $\phi$ is an isomorphism.

Proposition 16. Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. Then $S$ and $S'$ have the same genus.

Sketch of proof. Let $\mathcal{L}$ be a multicurve in $S$ such that each curve in $\mathcal{L}$ bounds a once-punctured torus in $S$ and $S\setminus \mathcal{L}$ has only one connected component of infinite type and genus zero. In other words, $\mathcal{L}$ is the multicurve that “captures” all genus in $S$, see Figure 1. Hence the genus of $S$ is equal to the cardinality of $\mathcal{L}$. By Lemma 12 in [Ibid.], for each $\alpha \in \mathcal{L}$ the curve $\phi(\alpha)$ bounds a once-punctured torus in $S'$ induced by $\phi(\mathcal{L})$. Hence $\text{genus}(S) \leq \text{genus}(S')$. As $\phi$ is an isomorphism we obtain the equality.

Figure 1: Curves capturing the genus of the surface.

Recall that two curves $\{\alpha, \beta\}$ form a peripheral pair if they bound a once-punctured annulus.

Proposition 17. Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. Then $\phi$ maps peripheral pairs to peripheral pairs.

Proof. Remark that if $\{\alpha, \beta\}$ is a peripheral pair then both curves forming it have to be either separating or nonseparating. Therefore we only consider the following three cases: (1) both $\alpha$ and $\beta$ are separating curves with $\alpha$ an outer curve, (2) both $\alpha$ and $\beta$ are nonouter separating curves, and (3) both $\alpha$ and $\beta$ are nonseparating curves.

Case 1: Let $P$ be a pants decomposition for $S$ containing both $\alpha$ and $\beta$. Then $\alpha$ is a vertex of degree 1 in $\mathcal{A}(P)$ which is adjacent only to $\beta$, these properties are preserved by simplicial isomorphism. Therefore $\phi(\alpha)$ is an outer curve adjacent only to $\phi(\beta)$ and hence these curves must form a peripheral pair.
**Case 2**: This is an immediate result of the fact that $\phi$ is an isomorphism and the following technical lemma, which gives a simplicial characterisation of peripheral pairs formed by nonouter separating curves. Recall that the *link* of a vertex $\alpha \in \mathcal{C}^0(S)$ is the complete subgraph of $\mathcal{C}(S)$ induced by all vertices adjacent to $\alpha$ in $\mathcal{C}(S)$. We denote it by $L(\alpha)$. Remark that $L(\alpha)$ is naturally isomorphic to $\mathcal{C}(S \setminus \alpha)$. For any subgraph $\Gamma \subset \mathcal{C}(S)$, we define $\Gamma^*$ as the graph whose vertices are $V(\Gamma)$ and two vertices span an edge if they do not span an edge in $\Gamma$.

**Lemma 18.** Let $\alpha$ and $\beta$ be disjoint nonouter separating curves. Then $(L(\alpha) \cap L(\beta))^*$ has 2 connected components if and only if $\{\alpha, \beta\}$ forms a peripheral pair.

**Proof.** The necessity of the statement is evident. To prove the sufficiency, remark that $S_{(\alpha, \beta)} = S_1 \sqcup S_2 \sqcup S_3$. Since $(L(\alpha) \cap L(\beta))^*$ has 2 connected components there exists $j$ such that $S_j$ has nonpositive topological complexity. Moreover, given that $\alpha$ and $\beta$ are nonouter separating curves we have that $\alpha \cup \beta = \partial S_j$. A straightforward calculation of the possible topological types for $S_j$ gives us the desired result.

**Case 3**: Up to homeomorphism we can find a separating curve $\gamma$ such that $\{\alpha, \beta, \gamma\}$ bound a pair of pants as in Figure 2. Let $P$ be a pants decomposition containing $\{\alpha, \beta, \gamma\}$. By construction and the fact that $\tilde{\phi}$ is a graph isomorphism, we have that $S'_\gamma \setminus \phi(\gamma) = S_1 \sqcup S_2$ and w.l.o.g. we can suppose that $S_1$ has topological complexity equal to 2 and contains $\phi(\alpha) \cup \phi(\beta)$. Since both $\phi(\alpha)$ and $\phi(\beta)$ are nonseparating curves, $S_1$ has positive genus. Therefore $S_1$ is homeomorphic to a torus with one boundary component (the boundary curve $\phi(\gamma)$) and one puncture, and the result follows.

![Figure 2: Nonseparating curves $\alpha$ and $\beta$ forming a peripheral pair.](image)
4 Proof of Theorem 1

In this section we use the topological results from the previous section to prove Theorem 1.

Let \( \{S_i\} \) be a fixed principal exhaustion of \( S \). For each \( 1 \leq i \) we denote by \( B_i \) the set of boundary curves of \( S_i \) and \( B := \bigcup_{1 \leq i} B_i \).

**Theorem 19.** Let \( S \) and \( S' \) be infinite-type connected orientable with surfaces with empty boundary and \( \phi : \mathcal{C}(S) \to \mathcal{C}(S') \) a simplicial isomorphism. Then \( S \) is homeomorphic to \( S' \). Moreover, we can construct a homeomorphism \( f : S \to S' \) such that \( \phi(\beta) = f(\beta) \) for all \( \beta \in B \).

**Proof.** Let \( S \setminus B = \bigcup_{1 \leq j} \text{int}(\Sigma_j) \), where the collection \( \{\Sigma_j\}_{1 \leq j} \) is formed by closed subsurfaces of \( S \) of finite type whose topological complexity is at least 4 and such that for any \( j \neq k \), \( \Sigma_j \cap \Sigma_k \) is either empty or formed by boundary curves of \( S_i \), for some \( i \in \mathbb{N} \). See Figure 4.

For each \( 1 \leq j \), let \( P_j \) be a pants decomposition of \( \Sigma_j \) which contains a multicurve analogous to the one used in Proposition 16 in other words we choose a pants decomposition \( P_j \) that captures the genus of \( \Sigma_j \). Then \( P = (\bigcup_{1 \leq j} P_j) \cup (\bigcup_{1 \leq i} B_i) \) is a pants decomposition for \( S \).

Let \( A(P) \) be the adjacency graph of \( P \) and \( \tilde{\phi} : A(P) \to A(\phi(P)) \) be the corresponding graph isomorphism. Curves in \( B \) are by definition nonouter and separating, hence every element \( v \in V(B) \) is a cut vertex (i.e. \( A(P) \setminus \{v\} \) is disconnected). Then \( A(P) \setminus V(B) = \bigcup_{1 \leq j} \Gamma_j \), where each \( \Gamma_j \) is a finite subgraph whose vertex set \( V(\Gamma_j) \) is precisely the pants decomposition \( P_j \) of \( \Sigma_j \). By defining \( \Sigma_j' \) as the closed subsurface of \( S' \) bounded by \( \phi(\partial \Sigma_j) \) and recalling that \( \phi \) sends nonouter separating curves to nonouter separating curves (see lemma 15) we have that

\[
\tilde{\phi}(P) = \bigcup_{1 \leq j} \tilde{\phi}(P_j) \cup \bigcup_{1 \leq i} \tilde{\phi}(B_i)
\]

is such that \( \tilde{\phi}(P_j) \) is a pants decomposition for \( \Sigma_j' \subset S' \). Since \( P_j \) captures the genus of \( \Sigma_j \), \( \phi \) is an isomorphism and by construction both surfaces have the same number of boundary components, a direct calculation of the topological complexity of \( \Sigma_j \) and \( \Sigma_j' \) shows that they must be homeomorphic. Moreover, by adjacency w.r.t. \( P \), we have that \( \partial \Sigma_j' = \{\phi(\alpha) : \alpha \subset \partial \Sigma_j\} \). Hence we can find a collection of orientation preserving homeomorphisms \( \{f_j : \Sigma_j \to \Sigma_j'\} \) such that each \( f_j \) maps a boundary curve \( \alpha \subset \partial \Sigma_j \) to \( \phi(\alpha) \). These homeomorphisms can be glued together to define a global homeomorphism \( f : S \to S' \) which coincides with \( \phi \) on \( B \).

\[\square\]

With this result we have proved the topological rigidity, and we only need to prove that isomorphisms between curve complexes are geometric.

Hereafter, \( f : S \to S' \) denotes the homeomorphism obtained from Theorem 19. Remark that every homeomorphism \( h \) of the form \( f \circ g \) with \( g \in \text{stab}_{pt}(B) \), where

\[
\text{stab}_{pt}(B) := \{g \in \text{Homeo}(S) : g \text{ fixes } B \text{ pointwise}\},
\]

also coincides with \( \phi \) on \( B \).
Figure 3: The collection of subsurfaces \( \{\Sigma_j\}_{j \geq 1} \).

For every subsurface \( \Sigma \) of \( S \) with topological complexity at least 2, we have that the natural inclusion \( \iota : \Sigma \to S \) induces a simplicial map \( \iota_* : C(\Sigma) \to C(S) \) that is an isomorphism on its image. Abusing notation, we denote by \( C(\Sigma) \) the image of \( \iota_* \) on \( C(S) \). Analogously, we do the same for subsurfaces of \( S' \).

Lemma 20. For all \( 1 \leq i \), and for all curves \( \alpha \in C(S_i) < C(S) \), we have that \( \phi(\alpha) \in C(f(S_i)) < C(S') \). In particular, for each \( 1 \leq i \), the restriction of \( \phi \) to \( C(S_i) \) defines an injective simplicial map \( \phi_i : C(S_i) \to C(f(S_i)) \).

Proof. Let \( 1 \leq i \) be fixed, \( P_1 \) be a pants decomposition of \( S_i \), and \( P_2 \) be a pants decomposition of \( S\setminus S_i \). Then \( P = P_1 \cup B_i \cup P_2 \) is a pants decomposition of \( S \).

Recall that \( f \) coincides with \( \phi \) on \( B \), and that the curves in \( \partial(f(S_i)) \) are all separating curves. Then, by the same argument as in Theorem 19 we have that \( \phi(P_1) \) is a pants decomposition of \( f(S_i) \), and is contained in the interior of \( f(S_i) \). Analogously, the curves in \( \phi(P_2) \) are contained in the interior of \( f(S\setminus S_i) \).

Now, let \( \alpha \) be a curve contained in \( S_i \). If \( \alpha \in P_1 \), then \( \phi(\alpha) \in C(f(S_i)) \) as above. If \( \alpha \notin P_1 \), then there exists \( \beta \in P_1 \) such that \( i(\alpha, \beta) \neq 0 \). Since we have that:

- \( \phi(\alpha) \) is disjoint from every element in \( \phi(B_i) = f(B_i) \),
- \( \phi(\beta) \) is contained in \( f(S_i) \), and
- \( i(\phi(\alpha), \phi(\beta)) \neq 0 \),

we can conclude that \( \phi(\alpha) \) is contained in \( f(S_i) \). □
With this lemma and Shackleton’s result on combinatorial rigidity (see Theorem 1 in [8]), we obtain for each $1 \leq i$ a homeomorphism $g_i : S_i \to f(S_i)$ that induces $\phi_i$, that is, such that for all $\alpha \in C(S_i)$ we have that $\phi(\alpha) = g_i(\alpha)$.

We affirm that each $g_i$ can be extended to a homeomorphism $\overline{g}_i : \overline{S_i} \to \overline{f(S_i)}$ between the closure on $S$ and $S'$ of the respective subsurfaces. To show this we describe first the only possible obstruction to this extension and then why this obstruction never happens.

Each $S_i$ is an open subsurface of $S$ and its punctures can be classified into two categories: those that persist when we take the closure $\overline{S_i}$ of $S_i$ in $S$ (which are precisely those punctures of $S_i$ which are also punctures of $S$) and those that do not (these “become” curves contained in $B_i = \partial S_i$ when taking the closure of $S_i$ in $S$). The obstruction could be that $g_i$ exchanges a puncture of $S_i$ that persist in $\overline{S_i}$ with one that does not. We suppose this is the case and we derive a contradiction.

Let $\alpha, \beta, \gamma$ bound a pair of pants in $S$ such that $\alpha \subset \partial S_i$ and $\beta, \gamma \in C(S_i)$. Note this implies that $\{\beta, \gamma\}$ is a peripheral pair in $S_i$. If $g_i$ exchanges the puncture of $S_i$ defined by $S_i \setminus \alpha$ with a puncture of $S$, then $\{\phi(\beta), \phi(\gamma)\}$ would be a peripheral pair in $S$. By the Proposition 2, $\{\beta, \gamma\}$ is also a peripheral pair in $S$. This situation is depicted in Figure 4. It is clear that $S \setminus \alpha$ has one connected component whose topological complexity is strictly less than 3. This is a contradiction, for both connected components of $S \setminus \alpha$ have topological complexity at least 3.

![Figure 4: {β, γ} is a peripheral pair.](image)

Thus for each $1 \leq i$, we have a homeomorphism $\overline{g}_i : \overline{S_i} \to \overline{f(S_i)}$ that induces $\phi_i$. Using the following lemma we can assert that for each $1 \leq i$, $\overline{g}_i$ coincides with $f$ on $B_i$.

**Lemma 21.** Let $\alpha, \beta$ and $\gamma$ be curves on $S$ such that $\alpha$ is a separating curve, and $\alpha, \beta$ and $\gamma$ bound a pair of pants on $S$. Then $\phi(\alpha), \phi(\beta)$ and $\phi(\gamma)$ also bound a pair of pants on $S'$.

**Proof.** Let $P$ be a pants decomposition of $S$ with $\alpha, \beta, \gamma \in P$. Then, with respect to $P$, $\alpha$ is adjacent to $\beta$, $\beta$ is adjacent to $\gamma$, and $\alpha$ is adjacent to $\gamma$. By Proposition 14, we know that adjacency is preserved under $\phi$.
The only possibility for this to happen and having that $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ do not bound a pair of pants on $S'$, is (up to homeomorphism) illustrated in Figure 5. However, if this were to happen we could find a curve $\delta$ on $S'$ that would intersect $\phi(\alpha)$ exactly once, which is impossible since $\phi(\alpha)$ is a separating curve due to Proposition 14. Therefore, $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ bound a pair of pants on $S'$.

Figure 5: the curves $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ do not bound a pair of pants.

For each $1 \leq i$ we can pick an adequate element $l_i$ of the form $f \circ \eta$ with $\eta \in \text{stab}_{pt}(B)$ and define:

$$h_i(x) = \begin{cases} g_i(x) & \text{if } x \in S_i \\ l_i(x) & \text{if } x \in S \setminus S_i \end{cases}$$

We obtain this way a family of homeomorphisms $h_i : S \to S'$ which by construction satisfy that $h_i(\alpha) = \phi(\alpha) = h_j(\alpha)$ for all $i < j$ and $\alpha \in \mathcal{C}(S_i) \subset \mathcal{C}(S_j) \subset \mathcal{C}(S)$. As a consequence of the Alexander method (see [2], chapter 2.3) we have for each $i < j$ that $h_i|_{S_i} = h_j|_{S_j} \circ M_i$, where $M_i \in \text{Homeo}(S_i)$ is a multitwist whose support is contained in a neighbourhood in $S_i$ of $\partial S_i$. In other words, for each $1 \leq i$ there exists a subsurface $\tilde{S}_i \subset S_i \subset S$ isotopic within $S_i$ to $S_i$ such that the support of the multitwist $M_i$ is contained in $S \setminus \tilde{S}_i = \bigsqcup_{k=1}^{n} A_k$, where each $A_k$ is an annulus. In particular $M_i|_{\tilde{S}_i} = \text{Id}_{\tilde{S}_i}$ and hence for each $i < j$ we have that $h_i|_{\tilde{S}_i} = h_j|_{\tilde{S}_i}$. This way we can define the following map:

$$h : S \rightarrow S'$$

$$s \in \tilde{S}_i \rightarrow h_i(s)$$

Since for all $1 \leq i < j$ we have that $h_i|_{\tilde{S}_i} = h_j|_{\tilde{S}_i}$, this map is well-defined. Moreover it is a homeomorphism and by construction it coincides with $\phi$ on the whole curve graph $\mathcal{C}(S)$, as desired.

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