Resistive double-diffusive instability in the dead zones of protostellar discs

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ABSTRACT
We outline a novel linear instability that may arise in the dead zones of protostellar discs, and possibly the fluid interiors of planets and protoplanets. In essence it is an axisymmetric buoyancy instability, but one that would not be present in a purely hydrodynamical gas. The necessary ingredients for growth include a negative radial entropy gradient (of any magnitude), weak magnetic fields and efficient resistive diffusion (in comparison with thermal diffusion). The character of the instability is local, axisymmetric and double-diffusive, and it attacks lengths much shorter than the resistive scale. Like the axisymmetric convective instability, it draws its energy from the negative radial entropy gradient, but by utilizing the diffusing magnetic field, it can negate the stabilizing influence of rotation. Its non-linear saturated state, while not transporting appreciable angular momentum, could drive radial and vertical mixing, which may influence the temperature structure of the disc, dust dynamics and, potentially, planet formation.

Key words: accretion, accretion discs – convection – instabilities – MHD – protoplanetary discs.

1 INTRODUCTION
In this paper, we consider the dynamical behaviour of a gaseous system characterized by rotation (with a strong positive angular momentum gradient), a weak magnetic field and a small negative entropy gradient. As is well known, these ingredients lead to the magnetorotational instability (MRI), possibly in its more general convective form (Balbus & Hawley 1991; Balbus 1995). It might appear that when resistivity is present only the longest wavelengths can remain unstable. If the resistivity is too large, these wave-lengths can be longer than the size of the system, and global stability results. (Note that the small negative entropy gradient is stabilized in the hydrodynamical limit by strong rotation.)

Surprisingly, this is not an accurate description of the dynamical behaviour of this system. Here we show that the presence of resistivity activates unstable buoyancy modes that rely on the arbitrarily small adverse entropy gradient as their free energy source. Moreover, these unstable modes lie on the same branch of the dispersion relation as the MRI, to which they revert at small wavenumbers. What is particularly striking is that at very large wavenumbers, resistivity never damps these modes. Ultimately, when thermal diffusion is considered, there is in fact a critical wavenumber above which damping occurs. But, at least in protostellar discs and planetary interiors, this still allows for a very broad range of unstable large number modes. (The presence of both Ohmic resistivity and a much smaller thermal diffusivity is what gives this instability its ‘double-diffusive’ character.)

In this introductory paper, our interest is in a formal presentation of the properties of the instability in the simplest possible setting. Our fiducial model will be a protostellar disc. Such systems are both cold and dense; excepting the hot regions near the central star and in the surface layers, much of the gas is poorly ionized, and, as a consequence, the electrical conductivity too low to develop classical MRI turbulence (Blaes & Balbus 1994; Igea & Glassgold 1999; Iglewski & Nelson 2006; Wardle 2007; Okuzumi 2009; Turner & Drake 2009). Most protostellar disc models hence consist of an ‘active’ well-ionized envelope, in which the MRI could in principle be present, encasing an extensive body of poorly ionized gas which remains ‘dead’, at least as far as the MRI is concerned. The dead region nevertheless may support other dynamics, such as spiral density waves excited by the turbulence in the surface layers (Fleming & Stone 2003), streaming instability caused by settling dust grains (Youdin & Goodman 2005) and flows driven by torques applied by large-scale magnetic fields (Turner & Sano 2008). Generally, however, dead zones have been considered magnetically ‘deactivated’. Nevertheless, even in the very resistive heart of the dead zone we will see that magnetic fields can trigger local small-scale instability.

The analysis will be explored in a local model, the details of which we present in Section 2. In Section 3, we investigate the dynamics of linear axisymmetric Boussinesq disturbances by solving their fifth-order dispersion relation numerically and by obtaining asymptotic estimates of the growth rates in the double-diffusive limit. The instability is subsequently related to the idea of magnetostrophic balance. In Section 4, our conclusions are drawn and future work outlined.
2 PHYSICAL AND MATHEMATICAL OUTLINE

Consider a weakly ionized region of a protostellar disc, a nominal ‘dead zone’. A negative radial entropy gradient is assumed to be present, but the details of how this gradient is established and maintained do not concern us directly in this calculation: they may plausibly be present due to a combination of outwardly decreasing heating from the central star and an outwardly increasing opacity. The entropy gradient is presumed to be small relative to the angular momentum gradient of the disc. More specifically, the radial Brunt–Väisälä (BV) frequency

\[ N_R^2 = -\frac{1}{\gamma \rho} \frac{\partial \ln \rho}{\partial R} \frac{\partial \rho}{\partial R} \]  

(1)

is assumed to be small compared with the epicyclic frequency

\[ \kappa^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R} . \]

(2)

Here, \( R \) is the radial coordinate in a standard cylindrical system \( (R, \phi, z) \), \( \rho \) is the mass density, \( P \) is the gas pressure, \( \Omega(R) \) is the angular velocity and \( \gamma \) is the adiabatic index (ratio of constant pressure specific heat to constant volume specific heat). Recall that \( \kappa = \Omega \) in a Keplerian disc, so the magnitude of \( N_R \) is assumed much less than an orbital frequency \( (N_R \text{ itself is, of course, imaginary here}) \).

We consider the stability of axisymmetric disturbances. If we were to treat this highly resistive system as purely hydrodynamic, we would conclude that it is stable: there is no instability if \( N_R^2 + \kappa^2 > 0 \) (the Hoiland–Solberg criterion), which for small \( |N_R| < \Omega \) is certainly true. On the other hand, magnetohydrodynamical (MHD) processes are generally also stabilized when the resistivity is high. The classical MRI, for example, is banished to unfeasibly long wavelengths in the presence of strong Ohmic dissipation. Consequently, as noted in Section 1, one might be tempted to conclude that adding a very small negative entropy gradient to a magnetized resistive disc would make very little difference to the axisymmetric stability. In fact, it makes a dramatic difference.

Consider two fluid elements at different vertical locations embedded in a negative radial entropy gradient. A weak magnetic field tethers these two elements to one another. When these elements are radially displaced, they will try to bring the magnetic field with them, but the field diffuses because of resistivity (which would quell the classical MRI). Nevertheless, because of the transient magnetic torque, some finite amount of angular momentum will have been exchanged between the two elements in the process. As a consequence, each element will settle into a new orbit at a new radial location – and, most critically, in a new entropy environment. If thermal diffusion is too slow to equilibrate the temperature of the displaced fluid with its surroundings, the elements will continue to move radially, because they are now buoyantly unstable. Subsequently, the diffusing field will ensure that these elements will magnetically connect to new fluid parcels, and further angular momentum will be exchanged. Thus, the cycle continues and an instability is at hand: an instability that depends upon the transfer of angular momentum between magnetized fluid elements, yet whose seat of free energy is not the shear, but rather the unstable thermal stratification. Note that this instability is double-diffusive, relying on the speedy diffusion of angular momentum (accomplished by a diffusing magnetic field) and the slow (or negligible) diffusion of heat. This field diffusion counters the stabilizing tendency of an angular momentum gradient by breaking the constraint of specific angular momentum conservation.

Though buoyantly driven, the instability clearly has features in common with the MRI. On sufficiently small vertical wavenumbers (sufficiently spaced fluid elements), the influence of resistive diffusion will be minor and we recover the MRI. In fact, we will show that the double-diffusive instability and the MRI are, in effect, two sides of the same instability. In the presence of a negative radial entropy gradient, one of the modes that emerges is, at small wavenumbers, the classical MRI, and, at large wavenumbers, the double-diffusive instability. The instability is suppressed only at extremely large wavenumbers, when thermal conduction becomes important. Thus, the MRI and the resistive double-diffusive instability are intimately linked.

In the following sections this idea is explored using a number of approaches, mainly within the framework of the linearized Boussinesq equations of resistive MHD. (An important omission, which we will examine in a later study, is the electromotive force associated with the Hall term in the induction equation.) The local modes under consideration satisfy a fifth-order dispersion relation, which we will study in some detail. It is possible to obtain a number of clean numerical and analytical results from which further physical insights emerge.

2.1 Model equations

Our model employs the resistive MHD equations, which comprise the continuity, momentum and entropy equations, and the magnetic induction equation with Ohmic diffusion. As noted, the Hall effect is also of practical importance, but it is omitted in this first analysis. Ohmic resistivity will exceed viscosity and thermal diffusion by several orders of magnitude; hence, the viscous stress is dropped. At least initially, however, it is helpful to retain the thermal diffusion term, which we model as the divergence of a radiative energy flux. The equations are

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = -\rho \nabla \cdot \mathbf{v} , \]

(3)

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla (P + \frac{B^2}{8\pi}) - \nabla \Phi + \left( \frac{B \cdot \nabla}{4\pi \rho} \right) B \]

(4)

\[ \frac{\partial B}{\partial t} + \mathbf{v} \cdot \nabla B = B \cdot \nabla \mathbf{v} - B \nabla \cdot \mathbf{v} + \eta \nabla^2 B , \]

(5)

\[ E \left( \frac{\partial \sigma}{\partial t} + \mathbf{v} \cdot \nabla \sigma \right) = -\nabla \cdot \mathbf{F} + \eta |\nabla \times B|^2 . \]

(6)

Here \( \mathbf{v} \) is the velocity, \( E \) the internal energy density, \( \Phi \) the gravitational potential, \( B \) the magnetic field and \( \sigma = \ln P \rho^{-\gamma} \) the entropy function. The resistivity is represented by \( \eta \) and the radiative energy flux density by \( \mathbf{F} \) which is defined through

\[ F = -\frac{16}{3} \frac{\sigma_B T^3}{\rho \kappa_{\text{op}}} \nabla T , \]

(7)

where \( T \) is the temperature, \( \kappa_{\text{op}} \) is the opacity and \( \sigma_B \) is the Stefan–Boltzmann constant (the latter two are not to be confused with the epicyclic frequency \( \kappa \) and the entropy function \( \sigma \)). Finally, the equation of state is of an ideal gas which gives

\[ E = \frac{1}{\gamma - 1} P . \]

(8)

In this study, we restrict the analysis to current-free initial equilibria, i.e. \( \nabla \times B = 0 \), and the last term of equation (6) will not be used.
2.2 Linearized equations

It is assumed that the disc is in near-Keplerian equilibrium, with a small pressure gradient supplementing the star’s gravity in the radial force balance. The equilibrium is \( v_0 = \Omega(R)e_\theta, \rho = \rho(R), P = P(R), \) and \( S = S(R). \) It is also assumed from the outset that the radial gradients of \( P \) and \( S \) are negative and relatively small. In addition, weak magnetic fields thread the disc, but they are too small to influence the equilibrium balances. For simplicity, we treat only a vertical and uniform configuration: \( \mathbf{B} = B_0 \mathbf{e}_z. \)

In order to examine local disturbances, we adopt the shearing sheet model (Goldreich & Lynden-Bell 1965), which represents a small block of disc as a Cartesian box with \((x, y, z)\) denoting the radial, azimuthal and vertical coordinates, respectively. Inside the box \( \rho, \frac{dP}{dR}, \) and \( \frac{dS}{dR} \) are assumed to be constant.

This equilibrium is perturbed by a planar axisymmetric Boussinesq disturbance, \( \rho', \mathbf{v}', \mathbf{B}', \mathbf{B}'_s, \mathbf{B}'_z, \) proportional to \( e^{i \xi z + \epsilon t} \), the linearized equations of which read as

\[
\begin{align*}
\dot{s}' &= 2\Omega_0 u'_s e_z - \frac{k^2}{2\Omega_0} v'_s e_y + \frac{\nu'_s}{\eta} \left( \frac{d\rho}{dR} \right)_0 e_z + \frac{i k B_0}{2\eta \rho_0} \mathbf{B}' \\
\dot{B}' &= i k B_0 \mathbf{v}' + \left( \frac{d\Omega}{dR} \right)_0 \mathbf{B}'_s e_y - \eta k^2 \mathbf{B}'_z, \\
\dot{\rho}' &= \frac{\rho_0}{\gamma} \left( \frac{dS}{dR} \right)_0 v'_s - \xi k^2 \rho',
\end{align*}
\]

(9)–(11)

where the subscript 0 indicates evaluation at the point on which the shearing sheet is anchored. Hereafter, the subscript 0 will be dropped. The thermal diffusivity \( \xi \) has been introduced which is defined by

\[
\xi = \frac{16 \sigma (\gamma - 1) T^4}{3 \gamma \kappa_\text{ep} \rho P}.
\]

(12)

Notice that in the entropy equation (11), we have used the relation

\[
T' = -(T/\rho)\rho',
\]

(13)

which comes from the perturbed ideal gas law. The pressure perturbation is identically zero for our planar disturbances. But Boussinesq perturbations are in near pressure equilibrium with their surroundings, and so the fractional pressure perturbation is negligible in the ideal gas law, in any case.

3 ANALYSIS

Equations (9)–(11) present an algebraic eigenvalue problem for \( s \), which yields the following dispersion relation:

\[
s^4 + a_2 s^3 + a_1 s^2 + a_0 s + a_0 = 0,
\]

(14)

with the coefficients

\[
a_2 = 2 \eta k^2 + \xi k^2,
\]

(15)

\[
a_3 = N^2_k + \xi k^2 + 2 v^2_\Lambda k^2 + \eta^2 k^4 + 2 \eta \xi k^4,
\]

(16)

\[
a_2 = 2 N^2_k \eta k^2 + 2(\eta + \xi)v^2_\Lambda k^4 + (2 \eta + \xi)k^2 k^2 + \eta^2 \xi k^6,
\]

(17)

\[
a_1 = v^2_\Lambda k^2 \left( \frac{\rho}{\text{cm}^2 \text{g}^{-1}} \right) T^{1/2} + v^2_\Lambda k^4 + \left( \eta^2 k^2 + v^2_\Lambda \right) k^2 N^2_k
\]

\[
+ (2 \eta \xi + \eta^2) k^4 k^2 + 2 \eta \xi v^2_\Lambda k^6,
\]

(18)

\[
\eta = 2.34 \times 10^{17} \left( \frac{T}{100 \text{ K}} \right)^{1/2} \left( \frac{10^{-14}}{x_e} \right) \text{ cm}^2 \text{ s}^{-1},
\]

(21)

\[
\xi = 2.39 \times 10^{12} \left( \frac{\rho}{10^{-9} \text{ g cm}^{-3}} \right)^{-2} \left( \frac{T}{100 \text{ K}} \right)^3
\]

\[
\times \left( \frac{\kappa_\text{ep}}{\text{ cm}^2 \text{ g}^{-1}} \right)^{-1} \text{ cm}^2 \text{ s}^{-1},
\]

(22)

where \( x_e \) denotes the electron fraction (Blaes & Balbus 1994). In the latter expression, we have used \( \gamma = 7/5 \) and endowed the gas with

\[
a_0 = N^2_k \eta v^2_\Lambda k^4 + \xi v^2_\Lambda k^6 + \eta^2 \xi k^8 \kappa^2 + \xi v^2_\Lambda k^4 \left( \frac{d\Omega^2}{dR} \right),
\]

(19)

where the Alfvén speed is defined through \( v_\Lambda = B_0/\sqrt{\rho \eta}. \) Equation (14) is a special case of the dispersion relation calculated by Menou, Balbus & Spruit (2004), which also accounts for viscosity; vertical variation in \( \Omega, P, \) and \( S; \) and radial wavenumbers. On the other hand, by taking \( \eta = \xi = 0 \) we recover the ideal MHD relation of Balbus (1995), which yields the two convective modes and the two convective-MRI modes. The dispersion relation in this case is of fourth order, which means that the inclusion of magnetic diffusion (and thermal diffusion) leads to the emergence of a new fifth mode.

From simply inspecting the sign of the last coefficient \( a_0, \) it is already clear that the system possesses novel stability behaviour when \( \xi \) is small. If we consider scales upon which the thermal diffusion has little influence, the stability discriminant is the first term on the right-hand side of (19). It follows that stability is linked to the sign of \( N^2_k, \) which means that the Schwarzschild criterion is recovered — the stabilizing effect of rotation has vanished. Notice also that the discriminant combines the three elements essential to the new instability: resistivity, entropy stratification and magnetic fields. If any of these is missing \( a_0 \) vanishes and the system’s stability properties revert to the familiar situation of Balbus (1995), which is governed by the coefficient \( a_1. \)

In the following subsections, we adopt a number of approaches in order to describe the mathematics and physics of this double-diffusive mode more fully. First, we briefly discuss the main parameters that appear in the model and estimate realistic values they may take in protostellar discs. Next, we solve the dispersion relation (14) numerically and obtain an analytical asymptotic result in the regime of interest. Following that, we show how the mode relies on a generalization of ‘magnetostrophic balance’ in order to function.

Finally, we give a more physical explanation of the mechanism of instability.

3.1 Fiducial parameters and length-scales

Time and wavenumber units are chosen so that \( \Omega = 1 \) and the Alfvén wavenumber, defined through \( k_\Lambda = \Omega/v_\Lambda, \) is also 1. The Alfvén length is denoted by \( l_\Lambda \) and is defined by \( l_\Lambda = 2\pi/k_\Lambda. \) Consequently, the governing parameters of the dispersion relation comprise the scaled squared BV frequency \( N^2_k/\Omega^2, \) the scaled squared epicyclic frequency \( \kappa^2/\Omega^2, \) the Roberts number \( q \) and the Elsasser number \( \Lambda. \) The latter two are defined by

\[
q = \frac{\xi}{\eta}, \quad \Lambda = \frac{v^2_\Lambda}{\eta \Omega}.
\]

(20)

In order to estimate \( q, \) we use the following formulae for the resistivity and thermal diffusivity:

\[
\eta = 2.34 \times 10^{17} \left( \frac{T}{100 \text{ K}} \right)^{1/2} \left( \frac{10^{-14}}{x_e} \right) \text{ cm}^2 \text{ s}^{-1},
\]

(21)

\[
\xi = 2.39 \times 10^{12} \left( \frac{\rho}{10^{-9} \text{ g cm}^{-3}} \right)^{-2} \left( \frac{T}{100 \text{ K}} \right)^3
\]

\[
\times \left( \frac{\kappa_\text{ep}}{\text{ cm}^2 \text{ g}^{-1}} \right)^{-1} \text{ cm}^2 \text{ s}^{-1},
\]

(22)

where \( x_e \) denotes the electron fraction (Blaes & Balbus 1994). In the latter expression, we have used \( \gamma = 7/5 \) and endowed the gas with...
a mean molecular weight of 2.3. According to the commonly used minimum mass model, at a few au the disc may be characterized by \( T \sim 100 \text{K} \) and \( \rho \sim 10^{-9} \text{g cm}^{-3} \) (Hayashi, Nakazawa & Nakagawa 1985), while at these low temperatures \( \kappa_{\text{eq}} \sim 1 \text{cm}^2 \text{g}^{-1} \) (Henning & Stognienko 1996). The ionization fraction \( x_e \), on the other hand, is poorly constrained and extremely sensitive to dust grain size and abundance, the sources of ionization and vertical location. Various models show it ranging over many orders of magnitude: from \( 10^{-13} \) to values as low as \( 10^{-19} \) (Sano et al. 2000; Ilgner & Nelson 2006; Wardle 2007; Turner & Drake 2009). Given this uncertainty, we estimate the Roberts number by \( q \sim x_e/10^{-9} \). This suggests an upper limit for \( q \) of \( 10^{-3} \) and an extreme lower limit of \( 10^{-10} \).

The Elsasser number requires knowledge of the strength of the latent magnetic field, which we parametrize by the plasma \( \beta \):

\[
\beta = \frac{2 c^2}{\mathcal{v}^2},
\]

where \( c \) is the gas sound speed. This gives

\[
\eta \sim \kappa H k/k_1 \quad \text{and} \quad q \gtrsim 2 \eta/\Omega 1 \quad \text{of} \quad 10^{\sim 3} \quad \text{and} \quad \lambda \text{extreme lower limit of} \quad 10^{-10},
\]

The Alfvén length scales as

\[
l_\perp \sim \beta^{-1/2} H.
\]

Finally, it is necessary to estimate the squared BV frequency \( N_\perp^2 \), which represents the entropy gradient from which the double-diffusive modes derive their energy. Most \( \alpha \)-disc models in fact yield a positive BV frequency (Balbus & Hawley 1998), but this is because they omit the central star’s radiation field which will tend to drive a negative radial temperature gradient. However, it is difficult to quantify how this central heating is mediated by the turbulent inner radii of the disc, and the dead zone itself, and so it is unclear which estimates one should use for the magnitude of the constant coefficient \( \kappa_1 \), which we parametrize by the plasma \( \beta > \Lambda > \omega_1 \). Hence, the Elsasser number \( \Lambda \) may be interpreted as a surrogate for magnetic field strength. The results of this subsection are limited to the MRI/double-diffusive mode, which is the only mode that grows, and we neglect the other four decaying modes.

Fig. 1 illustrates the remarkable stability behaviour unlocked by the combination of resistivity and a negative entropy gradient. The purple dot–dashed curve represents the MRI in ideal MHD (\( \eta = 0 \) but with \( N_\perp^2 \neq 0 \)) and the red dashed curve represents the MRI in resistive MHD but with no radial stratification (\( \eta 
eq 0 \) and \( N_\perp^2 = 0 \)). In both cases, instability is extinguished on relatively long scales: near an Alfvén length in the former case and near the resistive length in the latter case. (Because here \( \Lambda = 1 \) the two scales are comparable.) However, when resistivity and stratification are present (\( \eta \neq 0 \) and \( N_\perp^2 \neq 0 \)) instability extends to much shorter scales, as witnessed by the blue solid curve in Fig. 1. We identify this ‘extension’ of the MRI as the double-diffusive instability. The MRI transforms smoothly into the new instability upon a transitional scale comparable to \( l_\perp \); both instabilities occur on the same branch of the dispersion relation. When the MRI mechanism is quenched (because of resistivity), the double-diffusive mechanism takes its

3.2 Numerical solutions

In this section, the dispersion relation (14) is solved numerically. Once the parameters \( \Lambda, \kappa^2/\Omega^2, N_\perp^2/\Omega^2 \) and \( q \) are stipulated, we may compute the growth rate \( s \) as a function of vertical wavenumber \( k \). Throughout we set \( q \) to be a constant, \( q = 10^{-6} \), and \( \kappa/\Omega = 1 \), while varying \( \Lambda \) and \( N_\perp^2/\Omega^2 \). Hence, the Elsasser number \( \Lambda \) may be interpreted as a surrogate for magnetic field strength. The results of this subsection are limited to the MRI/double-diffusive mode, which is the only mode that grows, and we neglect the other four decaying modes.

Figure 1. Growth rate \( s \) of the unstable MRI/double-diffusive mode as a function of \( k \) for three illustrative cases. The dash–dotted line represents the ideal MHD configuration \( \Lambda = \infty \) with \( N_\perp^2/\Omega^2 = -0.1 \). Here, the unstable mode is the convective MRI and it is extinguished when \( k/k_1 \gtrsim 3 \). The dashed line represents resistive MHD in the absence of a radial entropy gradient; here \( \Lambda = 1 \) and \( N_\perp^2 = 0 \). Instability is extinguished on wavelengths roughly below the resistive scale. The solid line represents the case with both resistivity and the negative entropy gradient: \( \Lambda = 1 \) and \( N_\perp^2/\Omega^2 = -0.1 \). As is plain, instability continues into short scales; this is the double-diffusive instability. In the non-ideal case, the ratio of thermal to resistive diffusivity is set to \( q = 10^{-6} \).
The range of unstable wavenumbers associated with the double-diffusive instability is governed by the Roberts number \( q \) and \( N_R^2 \). It extends from, roughly, the resistive scale \( k_\eta \sim \Lambda \) down to \( k_c \sim |N_R/\Omega|\Lambda q^{-1/2} \) (in units of \( k_\Lambda \)).

### 3.3 Asymptotic analysis

In this section, we derive an analytic expression for the growth rates in the double-diffusive regime: when wavelengths are much less than the resistive scale \( l_\eta \) but much longer than the cut-off scale \( l_c \). These modes are thus heavily influenced by electrical diffusion but are relatively insensitive to radiative diffusion. An asymptotic analysis of very long wavelength modes, which connects this paper to the local stability results of Balbus (1995), is presented in Appendix A.

In order to calculate modes in the double-diffusive regime, we set their length-scale to a value of the order of the Alfvén length, i.e. \( k \sim k_\Lambda \). In addition, a small scaling parameter \( \epsilon \) is introduced so that \( 0 < \epsilon \ll 1 \). From the outset, we assume that both \( \Lambda \) and \( N_R^2/\Omega^2 \) are equally small and of order \( \epsilon \). Subsequently, the growth rate \( s \) is expanded in powers of \( \epsilon \), substituted into the dispersion relation and then terms of equal order are collected. This procedure permits the calculation of the leading order terms of all five modes.

We find that the two hydrodynamic convective modes oscillate at the epicyclic frequency at leading (zeroth) order and decay at a rate \( -\Lambda \Omega \). There exist two other modes, on the other hand, which both decay at the rate \( -\Lambda^{-1} (k/k_\Lambda)^2 \Omega \).

The fifth mode is the double-diffusive mode and it possesses a growth rate that scales like \( \epsilon^2 \). It can be determined by balancing the last two terms in the dispersion relation (14). We find, to leading order,

\[
s = -\frac{N_R^2}{k^2} \Lambda \Omega. \tag{27}
\]

The stability of the mode is governed by the classical Schwarzschild criterion. So when \( N_R^2 < 0 \), the mode grows. Note also that the growth rate is independent of the wavenumber \( k \) in this regime. Expression (27) is, in fact, quite general and holds throughout most of the double-diffusive range of \( k \), as confirmed by the numerical solution (Figs 2 and 3). When \( k \) becomes small, however, the analysis breaks down and the growth rate connects to the classical-MRI curve. When \( k \) is sufficiently large, thermal diffusion stabilises the mode.

### 3.4 Magnetostrophic balances

In this subsection we characterize the instability in a physically illuminating way, by showing how it relies on certain steady balances. While the resistive MRI (or magnetostrophic MRI) works by balancing the Lorentz and inertial forces in the momentum equation (as discussed in Petitdemange, Dormy & Balbus 2008), the double-diffusive instability requires in addition the balancing of advection and diffusion in the induction equation. Recognizing this fact helps simplify some of the algebra of the previous section and adds support to the previous asymptotic estimate.

#### 3.4.1 Balancing the momentum equation

Consider the linearized momentum equation (9). Assume the left-hand side is subdominant. This means that to leading order, we have a steady balance between the Lorentz and inertial forces (magnetostrophic balance), also relevant to the Earth’s core (Petitdemange...
et al. 2008). By keeping the growth rate explicit in the induction and entropy equation, but setting $\xi = 0$, we may derive a (reduced) cubic dispersion relation:

$$k^2 s^3 + 2k^2 \eta k^2 s^2 + [\left(N^2_k + \tilde{\Omega}^2\right) v^4_A k^2 + \left(v^4_A + k^2 \eta^2\right) k^4] s + N_k^2 \eta v^4_A k^4 = 0,$$

(28)

with $\tilde{\Omega}^2 = \langle d\Omega^2/d\ln R \rangle$. While an approximation, this equation captures the physical essence of the two MRI modes and the double-diffusive mode. We may obtain an analytic expression by solving for $k^2$, which, after inversion, yields the growth rates of these modes. For purely real $s$,

$$k^2 = \frac{2k^2 \eta s^2 + \left(N^2_k + \tilde{\Omega}^2\right) v^4_A s}{2 \left(v^4_A s + \eta N^2_k v^4_A + k^2 \eta^2 s^2\right)} \pm \frac{\sqrt{4k^2 \eta^2 s^2 + \left(N^2_k + \tilde{\Omega}^2\right)^2 v^4_A - 4k^2 v^4_A s^2}}{2 \left(v^4_A s + \eta N^2_k v^4_A + k^2 \eta^2 s^2\right)}.$$

(29)

There are thus two branches, which we plot in Fig. 4 for representative parameters. The blue solid line indicates the ‘negative’ branch and the red dashed line the ‘positive’ branch. In addition, we plot the full solution to the fifth-order dispersion relation (14) (the black dotted line). As is clear, the two branches of the reduced magnetostrophic solution present a reasonable approximation to the actual MRI and double-diffusive modes for all $k$. This strengthens the claim that magnetostrophic balance is central to the mechanism of instability in both cases.

### 3.4.2 Balancing the induction equation

It is also instructive to consider the limit in which the time derivatives are negligible in the induction equation as well. We thus set the left-hand side of the linearized equation (10) to zero. This means that in addition to the steady balance between the Lorentz force and the inertial forces in the momentum equation, there is a steady balance between magnetic diffusion and magnetic advection/distortion. This leaves only one time-derivative in the linearized equations (9)–(11).

As a result, the growth rate is straightforward to calculate:

$$s = \frac{-N^2_k \eta v^4_A k^2}{\Omega^2 v^4_A + \left(v^4_A + \eta k^2\right) k^4}.$$

(30)

This expression yields the (scaled) asymptotic growth rate of the double-diffusive instability (27) in the limit of large resistivity and for $k \gtrsim v_A / \Omega$. On the other hand, for sufficiently small $k$ we obtain the growth rate of a decaying purely resistive mode described in Appendix A by equation (A4). These two modes, however, occur on different solution branches and the transition between them, in the above, is discontinuous.

The two steady balances upon which the double-diffusive mode exploits may be understood as a cancelling of the forces associated with the magnetic field and differential rotation. Differential rotation will attempt to stretch out a field line and magnetic diffusion will relax the ensuing magnetic tension. In the meantime, however, what magnetic tension is generated can counter-balance the inertial forces in the momentum equation. This is important because these inertial forces would otherwise stabilize the mode.

### 3.4.3 Angular momentum and heat transport

The solution obtained in the previous subsection presents a convenient framework to compute the angular momentum and heat fluxes aroused by a single linear mode. These may offer an insight into the direction of these fluxes in the saturated non-linear state.

We define the thermal heat flux associated with a single Fourier mode of the double-diffusive instability as

$$F_h = 2 \Re(u'_s \tau) = -2 \Re(u'_s \mathbf{\tau})$$

(31)

where the overline indicates the complex conjugate. From the linearized equations, with $s$ negligible in the momentum and induction equations, we obtain

$$F_h = 2 |u'_s|^2 \left\langle \frac{-\nu \partial_x S}{\gamma s} \right\rangle.$$

(32)

With $s > 0$ and positive, from equation (30), and a negative radial entropy gradient, the double-diffusive mode always transports heat outwards, at a rate proportional to the gradient itself. This is as expected: the mode will act against the destabilizing condition from which it arose.

We define the angular momentum flux as

$$F_h = 2 \left[ \Re(u'_s \mathbf{\pi}) - \frac{1}{4 \pi \rho_0} \Re(B'_s \mathbf{B}) \right].$$

(33)

Using the eigenfunction of the double-diffusive instability, this can be re-expressed as

$$F_h = -|u'_s|^2 \left[ \frac{1}{\Lambda} \left( \frac{k^2}{\Omega^2} - 4 \Lambda \frac{\Omega^2}{v^4_A k^4} \right) \right].$$

(34)

The double-diffusive mode works on scales $k \gtrsim v_A / \Omega$, and we take $\Lambda$ to be small. Thus the first term (associated with the Reynolds stress) in the square brackets will dominate the second, which ensures that angular momentum is transported inwards. However, for very small $k$, i.e. for the decaying resistive mode, the situation is reversed.

### 3.5 Physical description of the unstable mode

Fig. 5 presents a schematic diagram of how the double-diffusive instability works in a physically intuitive way, developing the ideas put forward at the beginning of Section 2. Panel (a) shows two fluid
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Figure 5. Four panels showing schematically the physical mechanism of the double-diffusive instability. Panel (a) presents two fluid blobs, embedded in a negative entropy gradient: $S_{+2} > S_{+1} > S_0 > S_{-1} > S_{-2}$. The fluid elements are separated vertically but connected by a magnetic tether. The following panels, in order, show the development of a horizontal perturbation in time, under the influence of speedy angular momentum transport (by the diffusing magnetic field) and the radial entropy gradient. See Section 3.5.

blobs at different vertical locations but at the same radial location (the violet circles). They are connected by a vertical magnetic field line. The two blobs are embedded in a negative radial entropy gradient, but they possess the same entropy $S_0$ as each other, because they inhabit the same radius. Fluid at smaller radii (to their left) possesses greater entropy, $S_{+1}$ and $S_{+2}$, and fluid at greater radii (to their right) possesses lesser entropy, $S_{-1}$ and $S_{-2}$. In panel (b) the blobs are perturbed horizontally, initially deforming their magnetic tether. Because of the large resistivity, the blobs will slip through the magnetic field, and so the MRI does not set in. Nevertheless, the field will permit some amount of angular momentum to be exchanged between the two blobs and they adopt new orbits at their new radii. But the displaced fluid blobs now possess a different entropy to their surroundings: too little entropy in the case of the upper blob ($S_0 < S_{+1}$) and too much in the case of the lower ($S_0 > S_{-1}$). Unless thermal diffusion acts rapidly to equilibrate them with their ambient gas, they cannot sustain their current positions. Instead, they will continue moving radially (pink arrows). This motion will bring them into contact with new diffusing magnetic field lines (panel c), which will tether them to new fluid partners (the grey circles). Now the new magnetic field lines will facilitate another transient torque, which will transfer more angular momentum. As earlier, the violet blobs adopt new radii but the disparity between their entropy and their environments’ entropy is even larger than before (panel d), and they are compelled to continue drifting. The cycle continues, leading to instability. Note that the cycle incorporates the $B$ field, resistivity and $dS/dr < 0$, the three ingredients which are represented in the stability discriminant in $a_0$ (equation 19).

The essential features of the instability are the suppression of stabilizing rotation by the diffusing magnetic field and slow (or negligible) thermal diffusion. This justifies calling the instability double-diffusive: angular momentum is diffused rapidly, eliminating the stable angular momentum stratification. But heat is diffused slowly, allowing the unstable entropy stratification to do its work unhindered.

It can be shown that a sufficiently large viscosity can do the same job as the magnetic field here, and, in fact, a hydrodynamical disc in which $\xi/\nu \ll 1$ exhibits an analogous instability (where $\nu$ is the kinematic viscosity). Consequently, we may interpret the instability as the ‘dual’ of the Goldreich and Schubert instability (Goldreich & Schubert 1967), which relies on the negation of a stabilizing entropy gradient, by rapid thermal diffusion, and on the tapping of an unstable angular momentum gradient (unimpeded by the weak viscosity). The dead-zone double-diffusion instability is also related to salt-fingering in the ocean, and, in fact, the linear stage of a single unstable mode will be characterized by a vertical
sequence of horizontal ‘fingers’ or channel flows, much as in the MRI. However, these flows, unlike classical MRI, are not non-linear solutions, and will quickly break down into some form of disordered mixing. Another important difference is that nearly all the unstable modes possess comparable growth rates, so no mode will dominate. Noticeable channel flows, or fingers, are therefore unlikely to be observed in simulations.

4 CONCLUSION
This paper introduces a novel linear instability which occurs on short length-scales and which could be at work within the dead zones of protostellar discs. The instability is double-diffusive and relies on (i) the energy provided by a weak negative entropy gradient, (ii) a diffusing magnetic field to relax angular momentum conservation and (iii) very slow thermal diffusion. It grows on a range of length-scales extending from the resistive length down to a length related to the thermal scale. Across this potentially large range, the growth rate scales like $s \sim \Lambda (N_R^2/\Omega^2)\Omega$. Though the unstable mode will almost always physically fit into the dead zone (unlike the MRI), it may grow quite slowly. Nevertheless, it may give rise to dynamically interesting velocity amplitudes after some $10^6$ orbits (depending on the parameters). In principle, the instability’s non-linear saturation may play a significant role in the dynamics of dead zones, leading, in particular, to radial and vertical mixing, which in turn may influence dust grain dynamics (settling and clumping), on one hand, and the global temperature structure of the disc, on the other.

However, a number of issues need to be settled, and these form the basis for future work. First, the governing parameters of the system need to be better constrained, so that one can be assured that the instability grows on dynamically meaningful time-scales. This requires realistic estimates for the magnetic field strength, the radial entropy gradient and the resistivity. Secondly, the linear analysis should be extended to the peculiar non-ideal MHD conditions present in the interior of protostellar discs, where the instability will be altered by Hall effects and ionization and dust grain chemistry. Thirdly, the interaction between the instability and much faster processes in the active zones needs to be clarified. For instance, the turbulent motions in these regions may transport the large-scale magnetic field on a similar time to the growth time (Fromang & Stone 2009; Lesur & Longaretti 2009). On the other hand, rapid density waves, stimulated by the turbulence, may also interact with the instability (Fleming & Stone 2003; Heinemann & Papaloizou 2009a,b). Fourthly, the instability’s non-linear saturation should be simulated numerically in order to establish whether it leads to sustained disordered flows.

Finally, this instability is not restricted to protostellar discs, as the necessary ingredients for instability are rather generic. In particular, double-diffusive modes may be excited within the fluid cores of rapidly rotating planets and protoplanets, possibly modulating pre-existing dynamo activity or perhaps directly instigating convection itself in bodies that spin so rapidly that regular buoyancy instabilities are quenched. We address these issues in future work.

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APPENDIX A: ASYMPTOTIC DESCRIPTION OF LONG WAVELENGTH MODES
In this appendix we calculate the growth rates of the five modes of equation (14) in the limit of a long wavelength, i.e. on scales longer than the resistive scale $l_R$. As earlier, we adopt $\Omega^{-1}$ for the unit of time and the Alfvén wavenumber ($\Omega/v_A$) for the unit of wavenumber. In addition, we drop the thermal diffusion terms, assuming that $k$ and $\xi$ are so small that these effects will be negligible. The dispersion relation (14) in these units is

$$s^5 + b_1 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0 = 0,$$

with coefficients

$$b_4 = 2\Lambda^{-1} k^2,$$

$$b_3 = \kappa^2 + N_R^2 + 2\kappa^2 + \Lambda^{-1} k^4,$$

$$b_2 = 2\Lambda^{-1} k^2 (k^2 + N_R^2 + \kappa^2),$$

$$b_1 = k^2 (k^2 - 4) + k^2 (1 + \Lambda^{-2} k^2) (N_R^2 + \kappa^2),$$

$$b_0 = \Lambda^{-1} N_R^2 k^4,$$

and in which both $\kappa^2$ and $N_R^2$ have been scaled by $\Omega^2$.

A small scaling parameter $\epsilon$ is introduced so that $0 < \epsilon \ll 1$. From the outset we assume that both $\Lambda$ and $N_R^2$ are equally small, and let $\Lambda \sim N_R^2 \sim \epsilon$. In order to pick out the longest wavelength modes we set $k \sim \epsilon^2$, which means that we examine length-scales $\Lambda^{-1}$ times the resistive scale. Next the growth rate $s$ is expanded in small $\epsilon$, which allows the computation of each mode analytically to leading order.

The two hydrodynamical convective modes possess an expansion like $s = s_0 + s_1 \epsilon + s_5 \epsilon^2 + \cdots$, with the leading order term given by $s_0 = 0$. Thus, the modes oscillate at the epicyclic frequency at leading order. The real part of the growth rate comes in only at
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Seventh order in $\epsilon$:

$$\text{Re}(s) = -\frac{4k^4}{\kappa^4 \Lambda} + \mathcal{O}(\epsilon^8),$$

and thus decays at a rate proportional to $\eta$. Note that if a different scaling had been adopted in which $N_2^2 \sim 1$, then the leading order term would be $s_0 = \pm \sqrt{-\kappa^2 - N_2^2}$ and stability would follow the Hoiland–Solberg criterion, $\kappa^2 + N_2^2 > 0$.

The leading order term of the two MRI (or convective-MRI) modes scales as $\epsilon^2$ and is determined by balancing the second and fourth terms in equation (A1). We find

$$s = \pm \frac{k}{\kappa} \sqrt{-\frac{d\Omega^2}{d \ln R}} + \mathcal{O}(\epsilon^4),$$

which returns the MRI stability condition $d\Omega^2/d \ln R > 0$. Thus, these very long scales will be MRI unstable. Note that the entropy gradient has not entered at leading order, it being too small to determine the stability properties of the gas. But if we had assumed that $N_2^2 \sim 1$ the growth rate to leading order would be proportional to $\sqrt{-\kappa^2 - N_2^2 + 4}$, and the instability criterion would reproduce that for convective-MRI modes (Balbus 1995; see also Papaloizou & Szuszkiewicz 1992).

The last mode is a simple resistive mode whose growth rate scales like $\epsilon^4$ and which can be determined from the last two terms of the dispersion relation. We find

$$s = -\frac{k^2 N_2^2}{\Lambda (d\Omega^2/d \ln R)} + \mathcal{O}(\epsilon^6),$$

and thus the mode decays at a rate proportional to $\eta$. Note that this mode is not the double-diffusive mode computed in the main body of the paper. It exists on a separate branch of the dispersion relation and will always decay for all $k$. However, just like the double-diffusive instability, it establishes magnetostrophic balances in both the induction and momentum equations (see Section 3.4.2).

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