A new exactly solvable Eckart-type potential

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Abstract

A new exact analytically solvable Eckart-type potential is presented, a generalisation of the Hulthén potential. The study through Supersymmetric Quantum Mechanics is presented together with the hierarchy of Hamiltonians and the shape invariance property.

I. Introduction

Based on the study of SUSY breaking mechanism of higher dimensional quantum field theories [1], Supersymmetric Quantum Mechanics (SQM) appeared 20 years ago and has so far been considered as a new field of research, providing not only a supersymmetric interpretation of the Schrödinger equation, but important results to a variety of non-relativistic quantum mechanical problems, [2]. Particular examples to be mentioned include the better understanding of the exactly solvable, [3]-[7], the partially solvable, [8]-[10], the isospectral, [11] and the periodic potentials, [12]. The association of the variational method with SQM formalism has been introduced to obtain the approximate energy spectra of non-exactly solvable potentials, [13]-[15]. In the work of reference [15] a new methodology, based on an ansatz for the superpotential which is related to the trial wave function, has been proposed. Using physical arguments it is possible to make an ansatz for the superpotential which satisfies the Riccati equation by an effective potential. The superalgebra enables us to take this superpotential and evaluate the trial wavefunctions containing the variational parameter, i.e., the parameter that minimises the energy expectation value. This new scheme has been successfully applied to obtain the spectra of 3-dimensional atomic systems well fit by the Hulthén, the Morse and the screened Coulomb potentials, [15]-[18].

In particular, when applying the approach to the Hulthén potential, [15], it was found that its effective potential was linked to a new exactly solvable potential, that
presents, unlike the Hulthén potential in one-dimension the property of shape invariance. It is a two parameters potential, apart from the screening parameter $\delta$ and it has already appeared in the literature, in a similar form. Written in terms of hyperbolic functions, this potential is known as the exactly solvable Eckart potential.

In this letter a study of such new exactly solvable potential through SQM is presented. We show its hierarchy and its shape invariance property. For particular values of its constants the Hulthén potential hierarchy is recovered. This material is preceded by a brief review of SQM, in order to fix the notation.

II. Supersymmetric Quantum Mechanics

In SQM for $N = 2$ we have two nilpotent operators, $Q$ and $Q^+$, satisfying the algebra

$$\{Q, Q^+\} = H_{SS}, \quad Q^2 = Q'^2 = 0,$$

where $H_{SS}$ is the supersymmetric Hamiltonian. This algebra can be realized as

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix},$$

where $A^\pm$ are bosonic operators. With this realization the supersymmetric Hamiltonian $H_{SS}$ is given by

$$H_{SS} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} = \begin{pmatrix} H^- & 0 \\ 0 & H^+ \end{pmatrix}.$$  

$H^\pm$ are called supersymmetric partner Hamiltonians and share the same spectra, apart from the nondegenerate ground state, (see for a review),

$$E_n^{(+)} = E_{n+1}^{(-)}.$$  

For the non-spontaneously broken supersymmetry this lowest level is of zero energy, $E_1^{(-)} = 0$. In $\hbar = c = 1$ units, we have

$$H^\pm = -\frac{1}{2} \frac{d^2}{dx^2} + V_\pm(x) = A^\pm A^\pm$$

where $V_\pm(x)$ are called partner potentials. The operators $A^\pm$ are defined in terms of the superpotential $W(x)$,

$$A^\pm = \frac{1}{\sqrt{2}} \left( \mp \frac{d}{dx} + W(x) \right)$$

which satisfies the Riccati equation

$$W^2 \pm W' = 2V_\pm(x)$$

as a consequence of the factorization of the Hamiltonians $H^\pm$. 
By definition, two partner potentials are called shape invariant if they have the same functional form, differing only by change of parameters, including an additive constant. In this case the partner potentials satisfy
\[
V_+(x, a_1) = V_-(x, a_2) + R(a_2),
\] (8)
where \(a_1\) and \(a_2\) denote a set of parameters, with \(a_2\) being a function of \(a_1\),
\[
a_2 = f(a_1)
\] (9)
and \(R(a_2)\) is independent of \(x\).

Through the super-algebra, for a given Hamiltonian \(H_1\), factorized in terms of the bosonic operators, it is possible to construct its hierarchy of Hamiltonians. For the general spontaneously broken supersymmetric case we have
\[
H_1 = -\frac{1}{2} \frac{d^2}{dx^2} + V_1(x) = A_1^+ A_1^- + E_0^{(1)}
\] (10)
where \(E_0^{(1)}\) is the lowest eigenvalue.

The bosonic operators are defined by (6) whereas the superpotential \(W_1(r)\) satisfies the Riccati equation
\[
W_1^2 - W_1' = 2V_1(x) - 2E_0^{(1)}.
\] (11)
The eigenfunction for the lowest state is related to the superpotential \(W_1\) by
\[
\Psi_0^{(1)}(x) = N \exp\left(-\int_0^x W_1(\bar{x})d\bar{x}\right).
\] (12)
The supersymmetric partner Hamiltonian is given by
\[
H_2 = A_1^- A_1^+ + E_0^{(1)} = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2}(W_1^2 + W_1') + E_0^{(1)}.
\] (13)
Thus, factorizing \(H_2\) in terms of a new pair of bosonic operators, \(A_2^\pm\) we get,
\[
H_2 = A_2^+ A_2^- + E_0^{(2)} = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2}(W_2^2 - W_2') + E_0^{(2)}
\] (14)
where \(E_0^{(2)}\) is the lowest eigenvalue of \(H_2\) and \(W_2\) satisfy the Riccati equation,
\[
W_2^2 - W_2' = 2V_2(x) - 2E_0^{(2)}.
\] (15)
Thus a whole hierarchy of Hamiltonians can be constructed, with simple relations connecting the eigenvalues and eigenfunctions of the \(n\)-members, \(\mathbf{[2, 3]}\),
\[
H_n = A_n^+ A_n^- + E_0^{(n)}
\] (16)
\[
A_n^\pm = \frac{1}{\sqrt{2}} \left( \mp \frac{d}{dx} + W_n(x) \right)
\] (17)
\[
\Psi_n^{(1)} = A_1^+ A_2^+ ... A_n^+ \psi_0^{(n+1)}, \quad E_n^{(1)} = E_0^{(n+1)}
\]

where \(\psi_0^{(1)}(x)\) is given by (12).

III. The Eckart-type potential

Consider the following one-dimensional potential, a generalization of the Hulthén potential,

\[
V_1(x) = a_1(a_1 - \delta) \frac{e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - a_1(2b_1 + \delta) \frac{e^{-\delta x}}{2(1 - e^{-\delta x})}.
\]

Through the superalgebra the following superpotential

\[
W_1 = -a_1 \frac{e^{-\delta x}}{1 - e^{-\delta x}} + b_1
\]

satisfies the associated Riccati equation (11),

\[
W_1''(x) - W_1'(x) = 2V_1 - 2E_0^{(1)}
\]

Thus, substituting (19) into (21) we find that the lowest energy-eigenvalue is given by

\[
E_0^{(1)} = -\frac{b_1^2}{2}
\]

and from equations (12) and (20) the associated eigenfunction is given by

\[
\Psi_0^{(1)} = (1 - e^{-\delta x})^{a_1} e^{-b_1 x}.
\]

The condition we must impose on the above wave-function is to vanish at infinity and at the origin, i.e.,

\[
\frac{a_1}{\delta} > 0, \quad b_1 > 0
\]

and since \(\delta > 0\) it implies that

\[
a_1 > 0, \quad b_1 > 0.
\]

Thus we can evaluate the whole hierarchy of this new potential. The supersymmetric partner of \(V_1(x)\) comes from equation (13)

\[
W_1^2(x) + W_1'(x) = 2V_2 - 2E_0^{(1)}
\]

and is given by

\[
V_2(x) = a_1(a_1 + \delta) \frac{e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - a_1(2b_1 - \delta) \frac{e^{-\delta x}}{2(1 - e^{-\delta x})}.
\]
Considering now the factorization of the Hamiltonian associated to $V_2(x)$, it depends on the superpotential $W_2$ which is of the following form

$$W_2(x) = -a_2 \frac{e^{-\delta x}}{1 - e^{-\delta x}} + b_2. \quad (28)$$

The Riccati equation it satisfies is given by

$$W''_2(x) - W'_2(x) = 2V_2 - 2E_0^{(2)} \quad (29)$$

$$= a_2(a_2 - \delta) \frac{e^{-2\delta x}}{(1 - e^{-\delta x})^2} - a_2(2b_2 + \delta) \frac{e^{-\delta x}}{(1 - e^{-\delta x})} + b_2^2.$$ 

Again the comparison of the two equation above with the substitution of (27) gives

$$E_0^{(2)} = -\frac{b_2^2}{2} \quad (30)$$

where the coefficients are such that

$$a_2 = a_1 + \delta, \quad b_2 = \frac{a_1(2b_1 - \delta)}{2(a_1 + \delta)} - \frac{\delta}{2}. \quad (31)$$

Written in terms of $a_2$ and $b_2$ the potential (27) is given by

$$V_2(x) = a_2(a_2 - \delta) \frac{e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - a_2(2b_2 + \delta) \frac{e^{-\delta x}}{2(1 - e^{-\delta x})} \quad (32)$$

which is analogous to the potential $V_1$ given by equation (19).

Thus, this process can be repeated sistematically $n$ times in order to derive the whole hierarchy. We arrive at the $n$-th term, given by

$$V_n(x) = a_n(a_n - \delta) \frac{e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - a_n(2b_n + \delta) \frac{e^{-\delta x}}{2(1 - e^{-\delta x})} \quad (33)$$

$$W_n(x) = -a_n \frac{e^{-\delta x}}{1 - e^{-\delta x}} + b_n \quad (34)$$

$$E_0^{(n)} = -\frac{b_n^2}{2} \quad (35)$$

and the ground state wave function of each member of the hierarchy given by

$$\Psi_0^{(n)} = (1 - e^{-\delta x}) e^{\frac{a_n}{2}} e^{-b_n x} \quad (36)$$

where the constants satisfy

$$a_n = a_{n-1} + \delta \quad (37)$$
and
\[ b_n = \frac{a_{n-1}(2b_{n-1} - \delta)}{2a_n} - \frac{\delta}{2}. \]  (38)

After some algebraic manipulations and using induction arguments we arrive at the following recurrence relations for the aʼs and bʼs
\[ b_{n+1} = \frac{1}{2a_{n+1}} \left( a_1(2b_1 + \delta) - 2\delta(na_1 + \frac{n(n-1)\delta}{2}) \right) - \frac{\delta}{2}. \]  (39)

For future analysis it is convenient to rewrite the above coefficients like
\[ b_{n+1} = \frac{1}{2a_{n+1}} a_1(a_1 + 2b_1) - \frac{1}{2}a_{n+1} \]  (40)
where
\[ a_{n+1} = a_1 + n\delta. \]  (41)

Thus, the full hierarchy for this new potential is completely determined in terms of the parameters \(a_1\), \(b_1\) and \(\delta\).

At this point we remark that the new potential is shape invariant, since all the potentials preserve the shape in the hierarchy eq. (33). To prove it we consider the potential \(V_-(a_1, b_1, x) = V_1 - E_0^{(1)}\), given by
\[ V_-(a_1, b_1, x) = a_1(a_1 - \delta) \frac{e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - a_1(2b_1 + \delta) \frac{e^{-\delta x}}{2(1 - e^{-\delta x})} + \frac{b_1^2}{2}. \]  (42)

Its supersymmetric partner is \(V_+(a_1, b_1, x) = V_2(a_1, b_1, x) - E_0^{(1)}\), i.e.,
\[ V_+(a_1, b_1, x) = a_1(a_1 + \delta) \frac{e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - a_1(2b_1 - \delta) \frac{e^{-\delta x}}{2(1 - e^{-\delta x})} + \frac{b_2^2}{2}. \]  (43)

Thus, if the parameters are transformed as
\[ a_1 \rightarrow a_2 = a_1 + \delta, \quad b_1 \rightarrow b_2 = \frac{a_1(2b_1 - \delta)}{2(a_1 + \delta)} - \frac{\delta}{2} \]  (44)
we get
\[ V_-(a_2, b_2, x) = a_2(a_2 - \delta) \frac{e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - a_2(2b_2 + \delta) \frac{e^{-\delta x}}{2(1 - e^{-\delta x})} + \frac{b_2^2}{2} \]
\[ = a_1(a_1 + \delta) \frac{e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - a_1(2b_1 - \delta) \frac{e^{-\delta x}}{2(1 - e^{-\delta x})} + \]
\[ + \frac{1}{2} \frac{a_1(2b_1 - \delta)}{2(a_1 + \delta)} - \frac{\delta}{2}. \]  (45)
Therefore we have the identification, in the usual notation

\[ V_+(a_1, b_1, x) = V_-(a_2, b_2, x) + R(a_2, b_2, x) \]  \hspace{1cm} (46)

where \( a_2 \) and \( b_2 \) are defined in (44). This is precisely the shape invariant condition, [3].

Thus the spectrum of \( V_-(a_1, b_1, x) \) is

\[ E_{n+1}^{(1)} = -\frac{b_{n+1}^2}{2} + \frac{j^2}{2} = -\frac{(a_1^2 - a_{n+1}^2 + 2a_1b_1)^2}{8a_{n+1}^2} + \frac{b_1^2}{2}, \quad n = 0, 1, \ldots . \]  \hspace{1cm} (47)

IV. The hyperbolic functions: the Eckart potential

The Eckart potential has the following form, [2]

\[ V_E(x) = a^2 + \frac{b^2}{a^2} + a(a - \alpha)cosech^2 \alpha x - 2b \coth \alpha x. \]  \hspace{1cm} (48)

As it is well known it is exactly solvable and shape-invariant. The superpotential associated through the super-algebra is given by

\[ W_E = -a \coth \alpha x + \frac{b}{a}, \quad b > a^2. \]  \hspace{1cm} (49)

Substituting the hyperbolic functions by exponentials we arrive at

\[ V_E(x) = (a - \frac{b}{a})^2 + 4a(a - \alpha) \frac{e^{-2\alpha x}}{2(1 - e^{-2\alpha x})^2} - 4b \frac{e^{-2\alpha x}}{2(1 - e^{-2\alpha x})} \]  \hspace{1cm} (50)

and

\[ W_E = -\frac{b}{a} + a - \frac{2a}{1 - e^{-2\alpha x}}. \]  \hspace{1cm} (51)

The energy eigenstates of the Eckart potential are given by

\[ E_{n+1}^{(1)} = \frac{1}{2} \left( a^2 - (a + n\alpha)^2 + \frac{b^2}{a^2} - \frac{b^2}{(a + n\alpha)^2} \right), \quad n = 0, 1, \ldots . \]  \hspace{1cm} (52)

Thus, setting \( a = \frac{a_1}{2}, 4b = a_1(a_1 + 2b_1) \) and \( \alpha = \frac{\delta}{2} \) we recover our starting potential (42), the superpotential (20) and the energy spectrum (47).

V. A particular case: the Hulthén potential

The particular case of setting \( a_1 = \delta \) and \( 2b_1 + \delta = 2 \) in (19) reduces \( V_1(x) \) to the known Hulthén potential

\[ V_1(x) = V_H = -\delta \frac{e^{-\delta x}}{1 - e^{-\delta x}}. \]  \hspace{1cm} (53)
In this case the parameters $a_n$ and $b_n$ reduce to

$$a_n = n\delta, \quad b_n = \frac{1}{n} - \frac{n\delta}{2}. \tag{54}$$

Substituting them in the hierarchy of potentials, equation (33), we arrive at

$$V_n(x) = \frac{n(n-1)\delta^2 e^{-2\delta x}}{2(1 - e^{-\delta x})^2} - \frac{\delta(2 + n(1-n)\delta)e^{-\delta x}}{2(1 - e^{-\delta x})}. \tag{55}$$

The lowest energy levels of the spectrum of energy of the potential (53) are given by

$$E_n^{(1)} = E_0^{(n)} = -\frac{1}{2} \left(\frac{1}{n} - \frac{n\delta}{2}\right)^2, \quad n = 1, \ldots. \tag{56}$$

These results are exactly the same obtained to the hierarchy of the Hulthén potential in [15]. It is not shape invariant as we can see from the functional form of $V_1$ and $V_2$ in equation (55).

VI. Conclusions

When dealing with the variational method associated with SQM we found a new potential that depends on tree parameters ($a$, $b$, $\delta$). It is a generalisation of the Hulthén potential. Written in terms of hyperbolic functions it is the known exactly solvable and shape invariant Eckart potential.

In conclusion, we remark that the results presented here allowed us to get an unified description of the Hulthén and the Eckart potentials. Through the superalgebra the spectrum and the hierarchy of both associated Hamiltonians were put together in the same formalism based on SQM.

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