Replacing Mark Bits with Randomness in Fibonacci Heaps

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Abstract

A Fibonacci heap is a deterministic data structure implementing a priority queue with optimal amortized asymptotic operation costs. An unaesthetic aspect of Fibonacci heaps is that they must maintain a “mark bit” which serves only to ensure efficiency of heap operations, not their correctness. Karger proposed a simple randomized variant of Fibonacci heaps in which mark bits are replaced by coin flips. This modified data structure still has expected amortized cost $O(1)$ for insert, decrease-key, and merge. Karger conjectured that this data structure has expected amortized cost $O(\log s)$ for delete-min, where $s$ is the number of heap operations.

In this paper, we give a tight analysis of randomized Fibonacci heaps, resolving Karger’s conjecture. Specifically, we obtain matching upper and lower bounds of $\Theta(\log^2 s / \log \log s)$ for the runtime of delete-min. We also prove a tight lower bound of $\Omega(\sqrt{n})$ on delete-min in terms of the number of heap elements $n$. Finally, we give a simple additional modification to these heaps which yields a tight runtime $O(\log^2 n / \log \log n)$ for delete-min.

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1 Introduction

Often, a data structure stores additional information whose sole purpose is to ensure efficiency rather than correctness. The defining characteristic of what we will refer to as extraneous data is that the data structure still functions correctly—but perhaps more slowly—if the extraneous data is corrupted.

There are numerous examples of extraneous data in data structures. For example, in red-black trees [GS78], the color of a node is extraneous data because even if we adversarially change it, the tree will still answers queries correctly—though perhaps more slowly. The balance factor of nodes in AVL trees and the mark bits of nodes in Fibonacci heaps are also extraneous data [AVL62, FT87].

A natural question to ask about a data structure with extraneous data is whether one can slightly tweak it to eliminate the extraneous data, while still preserving asymptotic performance. Doing so is desirable because—among other reasons—having extraneous information is arguably inelegant and maintaining it makes the implementation more complicated.

In unpublished work in 2000, Karger proposed a tweak to Fibonacci heaps that eliminates the extraneous “mark bit” data using randomization [Kar00]. However, Karger’s analysis of the performance of these heaps—which we’ll refer to as randomized Fibonacci heaps—was not tight. Specifically, Karger proved an upper bound of \(O(\log^2 s)\) on the expected amortized cost of delete-min where \(s\) is the total number of heap operations performed so far. (It is easy to see that the expected amortized cost of all other operations is \(O(1)\).) In terms of lower bounds, none better than the trivial sorting lower bound was known.

Following Karger’s initial work, the problem has a slightly amusing history. Hoping to encourage somebody to obtain a tight analysis of delete-min, Karger added this as a recurring bonus problem in MIT’s annual graduate algorithms course around 2000 [Kar13]. As a result, virtually every graduate student in MIT’s theory group in the past 14 years has at least seen this problem, and many have worked on it.

Despite this attention, relatively little progress was made. We initially thought there had been none at all. After posting this paper, however, we were informed of two unpublished results by Eric Price. These consist of two bounds in terms of \(s\): a lower bound weaker than ours and an upper bound that is essentially the same as ours. See Appendix C for details.

1.1 Our Contributions

We fully resolve Karger’s question, giving a tight analysis of randomized Fibonacci heaps. We give a lower bound of \(\Omega(\log^2 s / \log \log s)\) on the worst-case expected amortized runtime of delete-min. Importantly, our lower bounds assume only a standard oblivious adversary. We also obtain a matching upper bound of \(O(\log^2 s / \log \log s)\).

The above two bounds are in terms of the number of heap operations \(s\). In terms of the heap size \(n\), we give a lower bound of \(\Omega(\sqrt{n})\). (Previous work on pairing heaps implies a matching upper bound [FSST86].)

Finally, we give a simple modification that improves worst-case expected amortized performance to \(\Theta(\log^2 n / \log \log n)\) by periodically rebuilding the heap.

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1 For the purposes of this paper, data which is information theoretically redundant yet does not have the defining characteristic mentioned above isn’t considered “extraneous data.”

2 Fibonacci heaps also store the number of children of each node. Depending on implementation details, this information may or may not be extraneous data. Our interest in this paper is solely mark bits, not this other data.
1.2 Related Work

More broadly, there are several data structures that have been studied which implement priority queues and achieve the same asymptotic as performance Fibonacci heaps. These include [Pet87], [DGST88], [Høy95], [Tak03], [KT08], [Elm10], [HST11], [Cha13]. Additionally, there are many other works that deal with pairing heaps and their variants; e.g., [FSST86], [Pet05], [Elm09]. Pairing heaps offer slightly worse asymptotic performance than Fibonacci heaps but are often faster in practice.

1.3 Terminology

We will differentiate between Fibonacci heaps as defined in [FT87] and Karger’s randomized Fibonacci heaps by referring to the former as standard Fibonacci Heaps and the latter as randomized Fibonacci heaps. When the data structure we are referring to is clear from context, we may simply say Fibonacci heaps. (We may also say F-heaps or heaps instead of Fibonacci heaps.)

For variables, \( s \) will always refer to the total number of operations that have been executed on a heap, and \( n \) will refer to the number of elements stored in the heap.

1.4 Roadmap

In Section 2, we review the basic properties of standard Fibonacci heaps and define randomized Fibonacci heaps.

In Section 3, we prove the tight \( O(\log^2 s / \log \log s) \) upper bound on the expected amortized cost of delete-min in randomized Fibonacci heaps, where \( s \) is the number of heap operations.

In Section 4, we give a tight lower bound of \( \Omega(\sqrt{n}) \), where \( n \) is the heap size. This bound serves as a warmup to our more challenging lower bound in the next section.

In Section 5, we give a lower bound of \( \Omega(\log s / \log \log s) \) on the cost of delete-min.

In Section 6, we give a simple modification to Karger’s randomized Fibonacci heaps which improves the performance of delete-min to \( O(\log^2 n / \log \log n) \), replacing the \( s \) in the runtime with an \( n \). We also show how to extend our work in Section 4 to yield a matching lower bound.

In Section 7, we conclude and give possible directions for future work.

In the appendix, we have proofs and large figures that could not be included in the main paper due to page limits.

2 Background

A Fibonacci heap or F-heap is a data structure that implements a priority queue and supports the following operations:

| operation       | amortized runtime |
|-----------------|-------------------|
| insert          | \( O(1) \)        |
| merge (meld)    | \( O(1) \)        |
| decrease-key    | \( O(1) \)        |
| delete-min      | \( O(\log n) \)   |

where \( n \) is the heap size.

We will generally assume that the reader is familiar with the basic design and analysis of Fibonacci heaps. Those wishing to review this information may refer to the original paper [FT87] or to any typical algorithms textbooks.
Recall that each node in a Fibonacci heap allocates one bit of data called a mark bit. The only operation that uses the mark bit is the decrease-key operation. Specifically, the decrease-key operation starts by updating the key of the desired node and promoting it into the root list. Then, it starts from the node’s former parent and walks up the tree, promoting nodes to the root list until it encounters a node with an unset mark bit. It then sets this node’s mark bit and clears the mark bits of all nodes it promoted.

Karger defined randomized Fibonacci heaps as follows. A randomized Fibonacci heap behaves exactly like a Fibonacci heap with one exception: how it decides to stop promoting nodes in the decrease key operation. Recall that standard Fibonacci heaps look at the mark bit to determine whether to stop walking up the tree. In contrast, randomized Fibonacci heaps flip a coin to make this decision.

Equivalently, one can think of a randomized Fibonacci heap as a simulation of a regular Fibonacci heap which intercepts queries to mark bits and responds with a random bit instead.

3 An \(O(\log^2 s / \log \log s)\) Upper Bound

In this section, we upper bound the expected amortized cost of the operations of randomized Fibonacci heaps. After obtaining this result, we were informed that unpublished work by Price gives a similar proof of essentially this same theorem [Pri09].

**Theorem 1.** The expected amortized costs for a randomized Fibonacci heap’s operations are \(O(\log s \log n / \log \log s) \leq O(\log^2 s / \log \log s)\) for delete-min and \(O(1)\) for everything else.

We use a simplified version of the amortization function than the one introduced in [FT87]: if \(F\) is an F-heap, then we let \(\Phi(F)\) be the number of root nodes in \(F\). With this amortization, it is easy to see that insert, merge, and decrease-key all run in expected constant time. Thus it suffices to demonstrate that delete-min runs in expected time \(O(\log^2 s / \log \log s)\). Recall the specification for delete-min: we remove the minimum element from the list of roots, add all of its children to the root list, then perform consolidation by rank.

If \(k\) was the number of roots before the delete-min, \(c\) the number of children of the deleted element, and \(r\) was the maximum rank of any element in the F-heap before the consolidation, then the real work performed is \(O(k + c + r) = O(k + r)\) since \(c \leq r\). The change in potential is \(O(\log n - k)\), so with the correct scaling, the amortized cost of this operation \(O(r + \log n)\). Thus it suffices to show that \(r \leq O(\log^2 s / \log \log s)\) in expectation.

We first upper-bound the probability that a node has lost a lot of its children since the last time it was in the root list.

Say a non-root node \(v\) in a Fibonacci heap is missing a child if the child was removed from the node in the course of a decrease-key operation since the last time \(v\) was in the root list.

**Lemma 1.** Suppose we have an empty randomized Fibonacci heap and we intend to perform \(s\) operations on it which will result in a heap of size \(n\). Then the probability that every non-root node in the resulting heap is missing at most \(k\) children is at least \(1 - ns/2^k\).

The proof is given in the appendix. As a corollary, we get the following:

**Corollary 1.** With probability at least \(1 - 1/n\), no node in the heap described in the above lemma is missing more than \(k = 2 \log n + \log s \leq 3 \log s\) children.

For any integer \(k \geq 2\), let \(f_k(x) = x^k - x^{k-1} - 1\). It is not hard to see that \(f_k(x)\) is increasing for \(x \geq 1\), has a unique positive root \(\lambda_k\), and that \(\lambda_k > 1\). By a slight generalization of the analysis in [FT87], one can obtain the following result:
Lemma 2. Suppose a rooted tree on \( n \) nodes has the property that no non-root node in the heap is missing more than \( k \) children. Then the root has \( O(\log_{\lambda_k} n) \) children.

We now need a technical lemma about the behavior of \( \lambda_k \).

Lemma 3. For \( k \) sufficiently large, \[
\frac{1}{\log \lambda_k} \leq \frac{2k}{\log k}.
\]

Proof. The statement is equivalent to showing that \( \lambda_k \geq k^{1/2k} \). Since both sides are greater than 1, by the monotonicity of \( f_k \) in that interval it suffices to prove that \( f_k(k^{1/2k}) \leq 0 \). Since \( f_k(k^{1/2k}) = k^{1/2} - k^{(k-1)/2k} - 1 \) and \( k^{1/2} - k^{(k-1)/2k} \to 0 \) as \( k \to \infty \), we arrive at the desired conclusion.

Now we can prove the main theorem of this section.

Proof of Theorem 1. Again insert, merge, and decrease-key are obviously \( O(1) \) so we focus our attention to demonstrating the bound for delete-min. The expected amortized cost of delete-min is at most the maximum rank \( r \) of any root node. By Lemma 2, \( r \leq O(\log_{\lambda_k} n) \) where \( k \) is a bound on how many children are missing from any non-root node in the tree. We can break up \( E[r] \) into two terms and bound them separately. We have,

\[
E[r] = E[r \mid k < 3 \log s] + E[r \mid k \geq 3 \log s].
\]

The first term is \( E[r \mid k \geq 3 \log s] \leq n Pr[k \geq 3 \log s] \leq 1 \) by Corollary 1. The second is \( E[r \mid k < 3 \log s] \leq \frac{\log n}{\log \lambda_{3\log s}} = O(\log s \log n / \log \log s) \) by Lemma 3. Thus, the total expected amortized cost of delete-min is \( O(\log s \log n / \log \log s) \). \( \square \)

4 An \( \Omega(\sqrt{n}) \) Lower Bound

The following section is dedicated to the proof of the following lower bound:

Theorem 2. There exists a request sequence for randomized Fibonacci heaps whose expected cost is \( \Omega(\sqrt{n}) \) per operation on average, where \( n \) is the size of the heap.

Note that the analysis used in Section 2 of [FSST86] proves a matching upper bound.

It is worth clarifying what we formally mean when we say “\( \Omega(\sqrt{n}) \) per operation on average” since the heap size can change from operation to operation. Formally, this means the sum of the square roots of the heap sizes before each operation divided by the number of operations.

Notice that the theorem is equivalent to saying that there is a request sequence such that, no matter how one tries to amortize the cost of the operations, there will always be an operation with cost \( \Omega(\sqrt{n}) \). (Under the most natural amortization scheme, this operation would probably be the delete-min operation.)

While it is easy to slightly modify randomized F-heaps to “get around” this lower bound, we include this construction for two reasons. First, it applies to Karger’s randomized Fibonacci heaps as they were originally defined. Second, it is a good warm-up to the more complicated construction in Section 3, which is extended in Section 6 to apply even to these modified heaps—where \( s \) is replaced with \( n \) in the statement of the bound.
Figure 1: The bad state. It has a tree of rank $i$ for all $i$ from $0$ to $\sqrt{n}$. All trees except the first have height $1$. For simplicity, we assume $\sqrt{n}$ is integral. Notice that the total number of heap elements is $\Theta(n)$. Thus, by an appropriate variable substitution, we can equivalently think of this as an $n$ element heap where the highest-rank node has $\Theta(\sqrt{n})$ children.

The main idea is that by using a very large number of requests, we can force the heap into a very bad configuration with high probability. In particular, we exhibit a configuration which we call the bad state shown in Figure 1 shown below.

The bad state has the following two key properties, which we encapsulate in the following two lemmas:

**Lemma 4.** There exists a constant length sequence of operations which—when applied to a heap in the bad state—returns the F-heap to the bad state and takes $\Omega(\sqrt{n})$ time to execute.

**Lemma 5.** There exists a request sequence which, when applied to an empty heap, results in a heap in the bad state at the end with probability at least $1/2$.

Together, these properties imply Theorem 2.

**Proof of Theorem 2 assuming Lemma 4 and Lemma 5.** Fix an $n$. Construct a request sequence as follows: use Lemma 5 to construct the first part of the request sequence and follow it with a sufficiently large large number of copies of the request sequence guaranteed by Lemma 4. Then the expected average cost of executing this request sequence is $\Omega(\sqrt{n})$ per step on average.

Thus, all that is left is to prove Lemma 4 and Lemma 5 which we do in the following two subsections, respectively.

### 4.1 Proof of Lemma 4

From the bad state, it is straightforward to force the F-heap to spend $O(\sqrt{n})$ time on delete-min. In this subsection, we prove this fact.

**Proof of Lemma 4.** Consider the following request sequence:

1. Add two elements $t_1, t_2$ smaller than every element in the heap with $t_1 < t_2$.
2. Delete-min twice.

If our earlier steps give us the heap shown earlier, applying this procedure yields the cycle of states shown in Figure 5 deterministically. Notice that the state of the F-heap after applying these operations is unchanged. Moreover, it is clear that the last delete-min operation in the procedure takes $\Theta(\sqrt{n})$ time.
4.2 Proof of Lemma 5

This subsection proves Lemma 5.

Call a tree of height 1 where the root has $c$ children the $c$-star (see Figure 2), so that the bad state consists of one $c$-star, for each $0 \leq c \leq \sqrt{n}$.

![Figure 2: A c-star](image)

We will show how to force the heap to construct the bad state by forcing it to construct each $c$-star in the bad state in order from large $c$ to small. Specifically, we will use the following lemma

**Lemma 6.** For every $c$ and $\epsilon > 0$, there exists a sequence of operations which (starting from an empty heap) results in an F-heap which is a $c$-star with probability at least $1 - \epsilon$, and which at no point ever constructs a node with rank $> c$.

We first explain why Lemma 6 implies Lemma 5. Proof of Lemma 6 is in the appendix due to its length.

**Proof of Lemma 5 assuming Lemma 6.** Our request sequence is obtained by taking the sequences obtained from Lemma 6 for each $c$ with $\epsilon$ sufficiently small, then concatenating them in order from largest $c$ to smallest $c$. It is easy to see that this sequence results in the desired heap.

5 The $\Omega(\log^2 s / \log \log s)$ lower bound

This section is devoted to proving the following

**Theorem 3.** There exists a request sequence for randomized Fibonacci heaps whose expected cost is $\Omega(\log^2 s / \log \log s)$ per operation on average, where $s$ is the number of heap operations.

Our approach to this bound has the similar structure to Theorem 2: get the heap into a “bad” state then have it perform a costly operation repeatedly. However, to prove the previous bound, we constructed an exponentially long request sequence. The challenge in proving the present bound is that we now need a subexponential length request sequence.

For this bound, the “bad” state we will force the heap into is defined as a generalized bad state of rank $m$ and is shown in Figure 3. Formally, a F-heap is in a generalized bad state of rank $m$ if it has $m$ root nodes, where the $i$th root node has rank $i$, for $1 \leq i \leq m$. Once we get the heap into a state of this form, we will use an analog of Lemma 4 to make the heap perform costly operations, just as in the proof of Theorem 2.

The analogs of Lemma 4 and Lemma 5 we will use are the following:

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3More specifically, we will force the heap into a specific known state which is of the form shown in the figure.
Lemma 7. There exists a constant length sequence of operations which—when applied to a heap in a generalized bad state of rank $m$—returns the F-heap to a generalized bad state of rank $m$ and takes $O(m)$ time to execute.

Proof. The sequence of operations and proof is exactly the same as Lemma 4.

Lemma 8. There exists a request sequence of length $2^{O(\sqrt{m\log m})}$ which, when applied to an empty heap, results in a heap in a generalized bad state with probability at least $1/2$.

Proof of Theorem 3 assuming Lemma 7 and Lemma 8. Fix an $m$. Construct a request sequence as follows: use Lemma 8 to construct the first part of the request sequence with length $\ell(m) = 2^{O(\sqrt{m\log m})}$. Follow it with $\ell(m)$ copies of the constant length request sequence given by Lemma 7. Then the expected average cost of this request sequence is $\Omega(\ell^{-1}(m)) = \Omega(\log^2 m/\log\log m)$ per step on average.

The rest of this section is devoted to proving Lemma 8.

5.1 Proof of Lemma 8

We construct our request sequence by stringing together other request sequences which we will call subroutines. We will be very careful to control the exact internal state of the heap throughout this process. To this end, we will construct our subroutines so that they have a completely predictable effect on the heap, even though these subroutines may invoke heap operations that employ randomization. In other words, we require that if we know the exact starting state of the heap, apply a subroutine to the heap, and the subroutine succeeds, we should be able to infer the exact ending state of the heap. From this point forward, when we say “subroutine,” we mean “subroutine with predictable effect.”

Say a subroutine is an evil subroutine for rank $m$ with failure probability $p_{\text{evil}}$ if, when given a heap where every root node has rank $> m$, the subroutine produces a heap which is the union of the heap it was given and the generalized state of rank $m$, and it fails with probability at least $p_{\text{evil}}$. Our strategy is to construct an evil subroutine by making multiple calls to what we call a shifting subroutine which we define below. Our strategy for constructing a shifting subroutine will be to make multiple calls to evil subroutines, resulting in mutual recursion.

Intuitively, a shifting subroutine of rank $k$ takes a heap in a generalized bad state of rank $(k-1)$, looks at the node of rank $(k-1)$ in the root list, and increments its rank. Unfortunately, this subroutine requires a rather long formal definition as there are several technical conditions it must satisfy.
More formally, a shifting subroutine for rank $k$ with failure probability $p_{\text{shift}}$ will give the result heap specified below with probability at least $(1 - p_{\text{shift}})$ assuming the preconditions given below are satisfied.

The preconditions are that the heap have precisely the following nodes in its root list:

1. exactly one node of rank $i$ in the root list for all $0 \leq i \leq (k - 1)$,
2. no nodes of rank $k$, and
3. zero or more nodes of rank $(k + 1)$ or greater.

Let $S$ be the set of trees whose roots have rank $(k + 1)$ or greater prior to the shifting subroutine being applied. The result of the shifting subroutine is a heap which satisfies three conditions. First, it has precisely the following nodes in its root list:

1. exactly one node of rank $i$ in the root list for all $0 \leq i \leq (k - 2)$,
2. no nodes of rank $k - 1$,
3. exactly one node of rank $k$, and
4. zero or more nodes of rank $(k + 1)$ or greater.

Second, the trees from $S$ have not been modified in any way by the shifting subroutine. Third, there are no trees whose root nodes have rank $(k + 1)$ or greater besides those in $S$.

Note that this formal definition does not require the nodes in the root list of rank $< k$ be same before and after the subroutine is applied.

We now prove several lemmas which are necessary to prove Lemma 8.

**Lemma 9.** Suppose for all $k \leq m$ and any probability $p_{\text{shift}}$ that there exists a shifting subroutine for rank $k$ with failure probability at most $p_{\text{shift}}$ and length $f(m, p_{\text{shift}})$. Then for any probability $p_{\text{evil}}$, there exists an evil subroutine for rank $m$ with failure probability at most $p_{\text{evil}}$ and length at most $m^2 f(m, p_{\text{evil}}/m^2)$.

**Proof.** Insert a lone node into the heap. Then, for each $i$ from 1 to $(m - 1)$ in order, do the following two things:

1. Iterate over all $j$ from $i$ to 1 in order and append a shifting subroutine for rank $j$ and failure probability $p_{\text{evil}}/m^2$.
2. Insert a node into the heap.

It is easy to see that the resulting sequence has the desired length and failure probability.

We have established that given a shifting subroutine, we can construct an evil subroutine. We now give a construction in the reverse direction.

**Lemma 10.** Let $m$ be sufficiently large. Suppose for all $k < m$ and any probability $p_{\text{evil}}$ that there exists an evil subroutine for rank $k$ with failure probability at most $p_{\text{evil}}$ and length $g(k, p_{\text{evil}})$. Then for any probability $p_{\text{shift}}$ and any $w$, there exists a shifting subroutine for rank $m$ with failure probability at most $p_{\text{shift}}$ and length at most $2^w \cdot \ln \frac{2}{p_{\text{shift}}} + g(m - w - 1, p_{\text{shift}}/2)$.
Proof. Let \( u \) be the element in the generalized bad state with \( k \) children. To start off, decrease-key on \( u \) so that it is the smallest element in the heap. Then add an element \( t_2 \) such that \( u < t_2 < \) everything else.

We perform the following procedure \( y = 2^w \cdot \ln \frac{2}{p_{\text{shift}}} \) times:

1. Insert an element \( t_1 \) such that \( t_1 < u < t_2 < \) everything else. Then perform a delete-min operation, which removes \( t_1 \) and forces a consolidation.

2. For the \( w \) children of \( t_2 \) having the largest number of children of their own, decrease-key on them to send them to the root list.

The results of this operation are shown in Figure [B.5]. Notably, from the starting state, a single iteration of this operation has one of two effects: with probability \( 1 - 2^{-w} \), it enters into a state \( S_1 \) where \( t_2 \) is made a root in one of the decrease-key operations performed in step (2), and with probability \( 2^{-w} \), it enters a state \( S_2 \) where \( t_2 \) remains a child of \( u \), which now has \( t+1 \) children. By inspection, if we go to state \( S_1 \), then another iteration of this loop has the same effect, and if it goes to state \( S_2 \), then another iteration of the loop will have no effect on the state. Thus, if we repeat this procedure \( y \) times, the probability we do not end up at state \( S_2 \) is at most \((1 - 2^{-w})^y \leq p_{\text{shift}}/2 \).

Moreover, in \( S_2 \), as seen in the figure, in addition to \( u \), the root list exactly consists of elements with \( j \) children, where \( j \in \{k - w, k - w + 1, \ldots, k - 1 \} \). Thus, after exiting the loop, applying an evil subroutine for rank \( k - w - 1 \) with failure probability \( p_{\text{shift}}/2 \) yields a shifting subroutine with the desired runtime and failure probability.

We now prove the following lemma which immediately implies Lemma 8.

**Lemma 11.** For sufficiently large \( m \), all \( 1 \leq j \leq m \), and any probability \( p_{\text{evil}} \), there exists an evil subroutine for rank \( m \) with failure probability \( p_{\text{evil}} \) and length at most \( \ell(m) = m^{3j}2^{m/j} \ln \frac{1}{p_{\text{evil}}} \). Setting \( j = \sqrt{m/\log m} \) and \( p_{\text{evil}} = 1/2 \) gives \( \ell(m) = 2^{O(\sqrt{m \log m})} \).

**Proof.** We proceed by induction on \( j \).

Base Case: For \( j = 1 \), a sequence of length \( 2^m \) suffices. If we add \( 2^{m+1} \) nodes smaller than any other nodes in the heap and delete-min, we get a heap in a generalized bad state. However, if we only did this, we wouldn’t be able to keep track of where all the nodes go, so the request sequence wouldn’t have predictable effect. We can get around this pretty easily, however.

Here is a sketch. When inserting nodes, we frequently stop the insertion process to force a consolidation by inserting a dummy and doing a delete-min operation. If we do this frequently enough (say, after every insertion of a real node), then there will be at most one pair of nodes that can consolidate, so we can predict exactly what will happen.

Induction Hypothesis: Assume there result holds for some \( i = j \). Then for any probability \( p_{\text{evil}} \), there exists an evil subroutine with failure probability \( p_{\text{evil}} \) and length at most \( \ell(m) = m^{3j}2^{m/j} \ln \frac{1}{p_{\text{evil}}} \). Now consider \( i = j + 1 \).

By our induction hypothesis and Lemma 10 with \( w = m/(j+1) \), we have that for any probability \( p_{\text{shift}} \), there exists a shifting subroutine for rank \( m \) with failure probability at most \( p_{\text{shift}} \) and length at most

\[
2^{m/(j+1)} \cdot \ln \frac{2}{p_{\text{shift}}} + m^{3j}2^{m/(j+1)} \ln \frac{2}{p_{\text{shift}}} \leq 4m^{3j}2^{m/(j+1)} \ln \frac{1}{p_{\text{shift}}}.
\]

By this and Lemma 9 there exists evil subroutine for rank \( m \) with failure probability \( p_{\text{evil}} \) and length at most

\[2^{m/(j+1)} \cdot \ln \frac{2}{p_{\text{shift}}} + m^{3j}2^{m/(j+1)} \ln \frac{2}{p_{\text{shift}}} \leq 4m^{3j}2^{m/(j+1)} \ln \frac{1}{p_{\text{shift}}}.
\]

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\[ m^24m^{3j}2^{m/(j+1)}\ln \frac{m^2}{p_{\text{evil}}} \leq m^{3(j+1)}2^{m/(j+1)}\ln \frac{1}{p_{\text{evil}}}. \]

6 Going from \( \Theta(\log^2 s / \log \log s) \) to \( \Theta(\log^2 n / \log \log n) \)

In this section, we eliminate the dependence on \( s \) in the runtime of randomized F-heaps via a simple change. Specifically, after every operation, we rebuild the heap with probability \( 1/n \). Rebuilding is done as follows: Create a new randomized F-heap, and insert all elements from the old heap into the new heap. (We do not allow a new rebuild to happen during an existing rebuild.) We refer to these self-rebuilding heaps as augmented randomized F-heap. It is easy to see the following:

**Theorem 4.** The augmented randomized Fibonacci heap has worst-case expected amortized runtime \( O(\log^2 n / \log \log n) \) for delete-min and \( O(1) \) for everything else.

We now state the matching lower bound.

**Theorem 5.** There exists a request sequence for augmented randomized Fibonacci heaps whose expected cost is \( \Omega(\log^2 n / \log \log n) \) per operation on average, where \( n \) is the number of heap elements.

A proof is given in the appendix, but we give the main idea now. Note that the only thing that prevents our request sequence given in **Theorem 3** from directly applying to augmented randomized F-heaps is that the augmented heaps periodically rebuild themselves, messing up the heap state. Our strategy is to simply prevent the heap from rebuilding itself. Specifically, we add a very large number of nodes to the heap so the rebuild probability will be low.

7 Conclusion

This work gave the first tight analysis of randomized Fibonacci heaps, resolving a 14 year old question of Karger. It is somewhat surprising that the simple modification of replacing mark bits with random bits in Fibonacci heaps requires such complicated analysis for the lower bound. Does replacing extraneous data with randomness in other data structures yield similar results?

In terms of directly extending the results in the paper, the most straightforward direction would be to try and obtain the same performance as standard Fibonacci heaps by further modifying the heaps described in **Section 6**. Here are two ideas along these lines: (1) When doing a decrease-key, walk both down and up the tree promoting nodes to the root list. (2) Simply have the heap split up a node whenever it gets too many children.

8 Acknowledgments

We would like to thank David Karger for making us aware of this problem and for pointing out that our analysis in **Section 3** actually gave us something tighter than we originally thought.
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A Omitted Proofs

A.1 Proof of Lemma 1

Proof. The proof idea is as follows. We can think of the request sequence as being generated by an adversary who wishes to make a bunch of non-root nodes that are each missing at least \( k \) of their children. In order to do this, the adversary will inevitably be required to cascade decrease-key operations up to the children it wants to remove. However, any decrease-key operation that gets rid of a child also has probability \( 1/2 \) of sending the parent to the root list as well. However, the adversary gets many tries to remove \( k \) children from a node, potentially up to one per delete-min operation. The probability of any try succeeding is \( 2^{-k} \) and the adversary gets at most \( s \) tries on each of the \( n \) nodes. A union bound over all tries on all nodes gives a probability of success for the adversary of no more than \( ns2^{-k} \).

More formally, for any node \( v \) in the heap, let \( p_{v,t} \) denote the probability that \( v \) just became a non-root node during operation \( t \), \( v \) never returns to the root list after operation \( t \), and is eventually missing at least \( k \) of its children after operation \( t \). A necessary condition for this event is that \( k \) of \( v \)'s children get promoted to the root after time \( t \), but the cascade does not continue on to \( v \). The probability of this is at most \( 2^{-k} \). So, \( p_{v,t} \leq 2^{-k} \).

Then the probability \( p_v \) that node \( v \) is missing at least \( k \) children after \( s \) operations is \( \sum_{i=1}^{s} p_{v,i} \leq s2^{-k} \). Taking a union bound over all nodes gives an upper bound of adversary success of \( ns2^{-k} \).

A.2 Proof of Lemma 6

Proof. We proceed by induction on \( c \). For \( c = 0 \), simply start with an empty heap and insert \( u \). This results in the desired heap with probability 1. Inductively, suppose the statement is true for \( c = k \). Fix \( \epsilon > 0 \). By induction, there is an request sequence which produces a \( k \)-star with probability at least \( \sqrt{1 - \epsilon} \). Below, we describe an request sequence which constructs a \( (k + 1) \)-star from a \( k \)-star with probability at least \( \sqrt{1 - \epsilon} \). Then by concatenating this request sequence to the prior one we produce the desired request sequence.

Assume the F-heap is a \( k \)-star. Now insert a node \( v \) with \( v > u \). This results in the heap shown below.

Consider the following procedure which we will apply a large number of times.

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\( ^4 \)IE., \( v \) is a non-root node after operation \( t \) and either (1) there was no operation before operation \( t \) or (2) \( v \) was a root node after operation \( t - 1 \).
1. Add $2^k - 1$ nodes $s_1, \ldots, s_{2^k-1}$ such that $u < v < s_1 < s_2 < \ldots < s_{2^k-1}$.

2. Add a node $t$ smaller than all other nodes in the heap and perform a delete-min. (This results in $t$ being removed and the rest of the nodes being consolidated.)

3. For all $1 \leq i \leq 2^k - 1$, decrease the key of $s_i$ to be minimum in the heap and delete-min, removing it. The order is arbitrary.

Given a heap $H$ as shown in Figure 2, if we apply this procedure over and over again, the state of $H$ after any particular application of the procedure is given by the Markov process shown by the flowchart in Figure 4.

![Flowchart](image)

Figure 4: High-Level description of how one iteration of our procedure works. Each box represents a state of the heap and each arrow represents the probability of going from one state to the other after applying steps 1–3 once. After a large number of applications, we will get stuck in the state on the right with high probability.

A more detailed step-by-step version of the flowchart is given in Figure 6.

Notice that $H$ always has a positive probability of gaining a single extra child (and no extra descendants), resulting in the heap we are trying to create. Furthermore, once $H$ enters this state, it will never leave. Notice additionally that in none of these possible transitions do we ever produce a tree with rank greater than $k + 1$. As such, if we apply the procedure a sufficiently large number of times—and provided $H$ had the structure shown in Figure 2— we can construct a sequence of steps that gives the desired resulting $H$ with probability arbitrarily high. By repeating this request sequence sufficiently many times such that this probability is at least $\sqrt{1 - \epsilon}$, we are done.

A.3 Proof of Theorem 5

Proof. We obtain the desired request sequence by modifying the sequence constructed in the proof of Theorem 3. Specifically, our new request sequence is as follows: insert \(2^{\lceil 100 \cdot \sqrt{m \log n} \rceil} + 1\) entries into the heap, then perform a delete-min. Provided that this final delete-min does not trigger a rebuild, the result of these operations—regardless of any rebuilding that occurs prior to the delete-min operation—is a heap with a single node in the root list which has $\Theta(2^{\lceil 100 \cdot \sqrt{m \log n} \rceil})$ children.

After performing the requests described above, we simply perform the requests given by Theorem 3 as usual. Recall that the large number of nodes we added prior to performing the requests from Theorem 3 are all consolidated under a single node of large rank. Notice that the rank of this node is actually larger than any possible rank of any other node in the heap for the entirety of
the request sequence. Thus, the presence of these extra nodes will not affect any operations from Theorem 3.

So, provided no rebuild happens after the delete-min mentioned above and no rebuild happens during any subsequent operations, this whole request sequence has expected cost $\Theta(2^{c\sqrt{m \log m}})$ for some constant $c$. Furthermore, the probability of such a rebuild happening is so small that the overall expected cost is still $\Theta(2^{d\sqrt{m \log m}})$ for some constant $d$. The desired result immediately follows.

\section*{B Large Figures}

Each subsection in Appendix B contains figures originally from the same-numbered section in the main paper.

B.4

Figure 5: Creating many expensive delete-min operations.
Figure 6: Detailed description of how one iteration of our procedure works. If only one arrow is shown, the probability of going from the state it starts at to the state it ends at is 1. Triangles denote trees/subtrees.
C Discussion of Price’s Unpublished Work

After posting this paper, we were informed of two unpublished bounds by Price. Price proves an upper bound of $O(\log s / \log \log s)$—the same as us.

He also gives a lower bound in the following sense: suppose we have an adversary making requests of the heap. Price allows this adversary to “cheat” by inspecting the internal heap state before deciding on each request. Price shows that such an adversary can force the heap to use $\Omega(\log^2 s / \log \log s)$ expected amortized time per delete-min. An adversary which can cheat by looking at the internal state of a data structure is called adaptive. Otherwise, the adversary is
called oblivious. We distinguish between lower bounds that assume an adaptive adversary and those that assume an oblivious one.

While adaptive adversaries are used in a variety of areas in theory, they are not typically used in the analysis of randomized data structures. There are two good reasons why: (1) If a particular use of a data structure can be modeled as oblivious, then adaptive lower bounds tell us nothing and (2) almost all uses of data structures can be modeled as oblivious. As an example, consider your favorite data structure and try to think of all the algorithms you could use it in. How many of those algorithms needed to examine the internal state of the data structure?

The lower bounds proven in our paper, in contrast, do not cheat by looking at the heap’s internal state; i.e., they are standard oblivious lower bounds. For the reasons outline above, this makes them more desirable. An additional feature of our work is that we prove bounds both in terms of $s$ and $n$, while Price’s work only obtains bounds in terms of $s$. 

