NONPERTURBATIVE SOLUTION OF SUPERSYMMETRIC
GAUGE THEORIES

J. R. HILLER

Department of Physics
University of Minnesota-Duluth
Duluth MN 55812 USA
E-mail: jhiller@d.umn.edu

Recent work on the numerical solution of supersymmetric gauge theories is de-
scribed. The method used is SDLCQ (supersymmetric discrete light-cone quan-
tization). An application to $N = 1$ supersymmetric Yang–Mills theory in 2+1
dimensions at large $N_c$ is summarized. The addition of a Chern–Simons term is
also discussed.

1. Introduction

Although much has been learned about supersymmetric gauge theories by
analytic methods, numerical methods can yield much more of the nonper-
turbative structure. In particular, the method known as supersymmet-
ric discrete light-cone quantization (SDLCQ),$^1,2$ an extension of ordinary
DLCQ,$^3,4$ has been quite successful in the analysis of (1+1)-dimensional
supersymmetric theories. This work has recently been extended to 2+1
dimensions$^5,6,7,8$ with consideration of $N = 1$ supersymmetric Yang–Mills
(SYM) theory, including a Chern–Simons (CS) term.$^9$ The mass spectrum,
Fock-state wave functions, and a stress-energy correlator are all computed.
The CS term provides an effective mass that reduces the tendency of SYM
to produce stringy states with many constituents. This work was done at
large-$N_c$, but the method is also applicable to finite $N_c$.

As the name SDLCQ implies, light-cone coordinates$^{10}$ are used. They
are defined by spacetime coordinates

$$x^\pm = (t \pm z)/\sqrt{2}, \ x_\perp = (x, y)$$

$^1$Preprint UMN-D-02-4, to appear in the proceedings of the fifth workshop on Continuous
Advances in QCD (Arkadyfest), Minneapolis, Minnesota, May 17-23, 2002.
and momentum components

\[ p^\pm = (E \pm p_z) / \sqrt{2}, \quad \mathbf{p}_\perp = (p_x, p_y). \]  

The dot product of two such four-vectors then becomes

\[ \mathbf{p} \cdot \mathbf{x} = p^+ x^- + p^- x^+ - \mathbf{p}_\perp \cdot \mathbf{x}_\perp. \]  

The \( x^+ \) direction is treated as the direction of time evolution, which makes the conjugate variable \( p^- \) the light-cone energy. The light-cone three-momentum is \( \mathbf{p} \equiv (p^+, \mathbf{p}_\perp) \). In a frame where the net transverse momentum \( \mathbf{P}_\perp \) is zero, the mass eigenvalue problem becomes

\[ 2 P^+ P^- |P\rangle = M^2 |P\rangle, \]  

where \( |P\rangle \) is also an eigenstate of three-momentum \( \mathbf{P} \). The SDLCQ method provides a means to solve this eigenvalue problem with supersymmetry preserved exactly at any level of the approximation.

One of the advantages of light-cone coordinates is that there exists a well-defined Fock-state expansion for each mass eigenstate. There are no disconnected vacuum pieces, because the longitudinal momentum of each constituent, virtual or real, must be positive. SDLCQ uses a Fock-state expansion for \( |P\rangle \) to obtain a matrix eigenvalue problem for the Fock-state wave functions at discrete values of the momentum. The matrix is then diagonalized by appropriate means. For large matrices the Lanczos diagonalization technique\(^1\) has been used, as discussed in Ref. [6].

The discretization is accomplished by restricting the fields to periodic boundary conditions in a light-cone box\(^3,4\) defined by \(-L_\parallel < x^+ < L_\parallel\) and \(0 < x, y < L_\perp\). This leads to a discrete momentum grid

\[ p^+ \rightarrow \frac{\pi}{L_\parallel} n, \quad \mathbf{p}_\perp \rightarrow \left( \frac{2\pi}{L_\perp} n_x, \frac{2\pi}{L_\perp} n_y \right). \]  

The product \( P^+ P^- \) is independent of \( L_\parallel \), and the limit \( L_\parallel \to \infty \) is exchanged for a limit in terms of an integer \( K \), called the harmonic resolution,\(^3\) defined by

\[ K \equiv \frac{L_\parallel}{\pi} P^+. \]  

Longitudinal momentum fractions \( x = p^+ / P^+ \) then reduce to \( n/K \). The number of particles in a Fock state is limited to \( K \), because negative longitudinal momentum is not allowed and the individual integers \( n \) must sum to \( K \).\(^a\) The eigenvalue equation (4) becomes a coupled set of integral equations

\(^a\)Zero modes are ignored.
for the Fock-state wave functions in which the integrals are approximated by discrete sums over the momentum grid

\[
\int dp^+ \int d^2p_L f(p^+, p_L) \simeq \frac{\pi}{L_{\parallel}} \left(\frac{2\pi}{L_{\perp}}\right)^2 \sum_{n_x, n_y} f(nP^+/K, 2n_{\perp}\pi/L_{\perp}).
\] (7)

The harmonic resolution provides a natural cutoff for \( n \). The transverse sums must be truncated explicitly, which is done by limiting \( n_x \) and \( n_y \) to range from \(-T\) to \( T\). The integer \( T \) can be viewed as a transverse cutoff or, at fixed dimensionful cutoff \( \Lambda_{\perp} \equiv 2\pi T/L_{\perp} \), as the transverse resolution.

The distinction between DLCQ and SDLCQ lies in the choice of operator for discretization. In ordinary DLCQ one discretizes the Hamiltonian, \( P^- \); in SDLCQ, one discretizes the supercharge \( Q^- \) and constructs \( P^- \) from the superalgebra relation

\[
\{Q^-, Q^-\} = 2\sqrt{2}P^-,
\] (8)

which guarantees that the discrete eigenvalue problem preserves supersymmetry.\(^1,2\) The \( P^- \) of ordinary DLCQ differs from the supersymmetric \( P^- \) by terms which disappear in the large-\( K \) limit but which break the supersymmetry at finite \( K \).

The remainder of this paper is organized as follows. In Sec. 2, (2+1)-dimensional SYM theory is reviewed and numerical results discussed for the spectrum and for a correlator of the stress-energy tensor. Section 3 extends the study of the spectrum to include the CS term, in both a dimensionally reduced theory and the full (2+1)-dimensional case. A brief summary is given in Sec. 4.

2. SYM\(_{2+1}\) theory

2.1. Formulation

The action for \( N = 1 \) SYM theory in 2+1 dimensions is

\[
S = \int dx^+ dx^- dx_{\perp} \text{tr}(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\bar{\Psi} \gamma^\mu D_\mu \Psi),
\] (9)

with

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \quad D_\mu = \partial_\mu + ig[A_\mu, \_].
\] (10)

The fermion field is separated into chiral projections

\[
\psi = \frac{1 + \gamma^5}{2^{1/4}} \Psi, \quad \chi = \frac{1 - \gamma^5}{2^{1/4}} \Psi,
\] (11)
only one of which is dynamical. In light-cone gauge, $A^+ = 0$, with the transverse component of the gauge field $A_\perp$ written as $\phi$, the action becomes

$$S = \int dx^+ dx^- dx_\perp \text{tr} \left[ \frac{1}{2} (\partial^- A^-)^2 + D_+ \phi \partial^- \phi + i \psi D_+ \psi + i \chi \partial^- \chi + \frac{i}{\sqrt{2}} \psi D_\perp \phi + \frac{i}{\sqrt{2}} \phi D_\perp \psi \right]. \quad (12)$$

The non-dynamical fields $A^-$ and $\chi$ satisfy constraint equations

$$A^- = \frac{g}{\partial^2} (i [\phi, \partial^- \phi] + 2 \psi \psi), \quad \chi = -\frac{1}{\sqrt{2} \partial^-} D_\perp \psi, \quad (13)$$

by which they can be eliminated from the action. The dynamical fields are expanded in terms of creation operators

$$\phi_{ij}(0, x^-, x_\perp) = \frac{1}{\sqrt{2\pi L_\perp}} \sum_{n_\perp = -\infty}^{\infty} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left[ a_{ij}(k^+, n^+) e^{-ik^+ x^- - i \frac{2\pi n^+}{L_\perp} x_\perp} + a_{ji}^\dagger(k^+, n^+) e^{ik^+ x^- + i \frac{2\pi n^+}{L_\perp} x_\perp} \right], \quad (14)$$

$$\psi_{ij}(0, x^-, x_\perp) = \frac{1}{2\sqrt{\pi L_\perp}} \sum_{n_\perp = -\infty}^{\infty} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left[ b_{ij}(k^+, n^+) e^{-ik^+ x^- - i \frac{2\pi n^+}{L_\perp} x_\perp} + b_{ji}^\dagger(k^+, n^+) e^{ik^+ x^- + i \frac{2\pi n^+}{L_\perp} x_\perp} \right], \quad (15)$$

where in (2+1) dimensions $n^+$ is the only transverse momentum index.

The chiral components of the supercharge are

$$Q_{\text{SYM}}^+ = 2^{1/4} \int dx^- dx_\perp \text{tr} \left[ \phi \partial^- \psi - \psi \partial^- \phi \right], \quad (16)$$

$$Q_{\text{SYM}}^- = 2^{3/4} \int dx^- dx_\perp \text{tr} \left[ \partial^+ \phi \psi + g (i [\phi, \partial^- \phi] + 2 \psi \psi) \frac{1}{\partial^-} \psi \right]. \quad (17)$$

They satisfy the supersymmetry algebra

$$\{ Q^+, Q^- \} = 2\sqrt{2} P^+, \quad \{ Q^-, Q^- \} = 2\sqrt{2} P^-, \quad \{ Q^+, Q^+ \} = -4P_\pm. \quad (18)$$

This theory has the additional symmetries of transverse parity, $P$: $a_{ij}(k, n^+) \to -a_{ij}(k, -n^+)$, $b_{ij}(k, n^+) \to b_{ij}(k, -n^+)$ and Kutasov’s transposition$^{12}$ $S$: $a_{ij}(k, n^+) \to -a_{ji}(k, n^+)$, $b_{ij}(k, n^+) \to -b_{ji}(k, n^+)$. These allow the matrix representation to be block diagonalized by an appropriate choice of basis. Eigenstates are labeled by the quantum numbers $\pm 1$ associated with $P$ and $S$. 

2.2. Spectrum and wave functions

The main results for the spectrum of the SYM$_{2+1}$ theory are given in Figs. 1, 11, and 12 of Ref. [6]. They show that the masses squared can be classified according to three main forms of behavior: $1/L^2_\perp$, $g^2 N_c A_\perp$, and $\Lambda^2_\perp$. In particular, the spectrum as a function of $g$ separates into two bands, one of approximately constant $M^2 L^2_\perp$ and the other growing rapidly.

For states in the lower band, the average number of constituents increases rapidly with $g$. At $g \simeq 1.5 \sqrt{4\pi^3/N_c L_\perp}$ the DLCQ limit of $K$ constituents is saturated. Thus in SYM theory the low-mass states are dominated by Fock states with many constituents, in close correspondence with string theory. However, as a practical matter, the saturation means that the SDLCQ approximation breaks down, and numerical studies in this band must be limited to smaller couplings.

Within the coupling limitation, extrapolations to infinite resolution are easily done for low-mass states. One first considers $M^2$ as a function of $1/T$ for a sequence of fixed $K$ values. The extrapolations to $T = \infty$ then yield $M^2$ as a function of $1/K$ alone, to extrapolate to $K = \infty$. The different representatives of continuum eigenstates are disentangled by studying their properties, such as average constituent content and momentum. Of course, the different $P$ and $S$ symmetry sectors are explicitly separated at the start. Typical extrapolations are illustrated in Ref. [6] with plots in Figs. 4-8 and results in Tables II and III.

For the spectrum as a whole there is a curious behavior with respect to the average number of fermion constituents $\langle n_F \rangle$. Calculations for transverse resolution $T = 1$ and longitudinal resolutions $K = 5$ and 6, and for many different coupling strengths, show a gap between $\langle n_F \rangle = 4$ and 6, where no state is found.

Wave functions are also obtained in the diagonalization process. In the analysis of the spectrum they were used to compute various average quantities that helped identify states computed at different resolutions. More of the form of the wave function is revealed in the structure function

$$ g_A(n, n^\perp) = \sum_{q=2}^{K} \sum_{n_1, \ldots, n_q=1}^{K-q} \sum_{n^\perp_1, \ldots, n^\perp_q=-T}^T \delta \left( \sum_{i=1}^{q} n_i - K \right) \delta \left( \sum_{j=1}^{q} n^\perp_j \right) $$

$$ \times \sum_{l=1}^{q} \delta^{n_l} \delta^{n^\perp_l} \delta_{A_l} \langle \psi(n_1, n^\perp_1; \ldots; n_q, n^\perp_q) \rangle^2, \tag{19} $$

where $A$ and $A_l$ represent the statistics (bosonic or fermionic) of the probed type and the $l$-th constituent, respectively, and $\psi$ is a Fock-state wave function. In the lower band the shapes are typically simple and are found to
confirm the identification of states at different resolutions. In the upper band, there are complicated shapes with multiple bumps in transverse momentum, such as in Figs. 13 and 14 of Ref. [6].

2.3. Stress-energy correlator

Consider the following correlator of the stress-energy component $T^{++}$:

$$F(x^+,x^-,x^+) \equiv \langle 0 | T^{++}(x^+,x^-,x^+) T^{++}(0,0,0) | 0 \rangle ,$$

(20)

at strong coupling, as an example of what one might compare with a supergravity approximation to string theory for small curvature.

In the discrete approximation, $F$ can be written as

$$F(x^+,x^-,0) = \sum_{n,m,s,t} \left( \frac{\pi}{2L_\parallel L_\perp} \right)^2 \times \langle 0 | \frac{L_\parallel}{\pi} T(n,m) e^{-iP_\parallel x^+ - iP_\perp x^-} \frac{L_\parallel}{\pi} T(s,t) | 0 \rangle ,$$

(21)

where

$$\frac{L_\parallel}{\pi} T^{++}(n,m) | 0 \rangle = \sqrt{nm} e^{iP_\parallel x^+ - iP_\perp x^-} \frac{L_\parallel}{\pi} T^{++}(n,m) | 0 \rangle .$$

(22)

Insertion of a complete set of bound states $|\alpha\rangle$ with light-cone energies $P_\alpha = (M_\alpha^2 + P_\perp^2)/P_\parallel$ at resolution K (and therefore $P_\parallel = \pi K/L_\parallel$) and with total transverse momentum $P_\perp = 2T\pi/L_\perp$ yields

$$\frac{1}{\sqrt{-i}} \left( \frac{x^-}{x^+} \right)^2 F(x^+,x^-,0) = \sum_{\alpha} \frac{1}{2(2\pi)^{3/2}} \frac{M_\alpha^{9/2}}{\sqrt{r} K_{9/2}(M_\alpha r)} \frac{|\langle u|\alpha\rangle|^2}{L_\parallel K_3|N_u|^2} ,$$

(23)

with $r^2 = x^+x^-, x_\perp = 0$, and

$$|u\rangle = N_u \frac{L_\parallel}{\pi} \sum_{n,m} \delta_{n+m,K} \delta_{n_\perp+m_\perp,N_\perp} T(n,m) | 0 \rangle .$$

(24)

Here $N_u$ is a normalization factor such that $\langle u|u\rangle = 1$. The sum over the full set of eigenvalues can be avoided by a Lanczos-based technique.

For free particles, $(x^+/x^-)^2 F$ has a $1/r^6$ behavior. In the interacting case, this behavior should be recovered for small $r$, where the bound states behave as free particles. Because this behavior depends on having a mass spectrum that extends to infinity, the finite resolution of the numerical
calculation yields only $1/r^5$; however, the $1/r^6$ behavior is recovered in a careful limiting process.

The behavior for large $r$ is determined by the massless states. Because this theory has zero central charge, there are exactly massless Bogomol’nyi–Prasad–Sommerfield (BPS) states at any coupling. However, their wave functions remain sensitive to the coupling, and, at a particular (resolution dependent) value of $g$, the correlator is exactly zero in the large-$r$ limit. The associated ‘critical’ value of $g$ increases in proportion to the square root of the transverse resolution $T$.

3. SYM-CS theory

The following CS term can be added to the Lagrangian:

$$L_{CS} = \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} \left( A_\mu \partial_\nu A_\lambda + \frac{2i}{3} g A_\mu A_\nu A_\lambda \right) + \kappa \bar{\Psi} \Psi.$$  \hspace{1cm} (25)

This induces an additional term in the supercharge

$$\kappa Q_{CS}^{-} \equiv -2^{3/4} \kappa \int dx^- \partial^- \phi \frac{1}{\partial^-} \psi,$$  \hspace{1cm} (26)

which generates in $P^-$ terms proportional to $\kappa^2$ that act like a constituent mass squared. The presence of an effective mass reduces the tendency for low-mass states to be composed of high-multiplicity Fock states. This creates a theory in which the eigenstates are more likely to be QCD-like, i.e. valence dominated, and improves the applicability of the SDLCQ approximation to a greater range of couplings.$^{15,16,8}$

The dominance of the valence state is most prominent in the dimensionally reduced theory.$^{15}$ This (1+1)-dimensional theory is obtained by requiring the fields to be constant in the transverse direction and replacing $\partial_\perp$ by zero in the full supercharge $Q_{SYM} + \kappa Q_{CS}^{-}$. The SYM contribution is then proportional to $g$. Figure 3 of Ref. [15] illustrates the dramatic reduction in the average number of constituents as the ratio $\kappa/g$ is increased. Also, there are states for which the mass is nearly independent of $g$ at fixed $\kappa$. These are identified as approximate BPS states and are the reflection of the massless BPS states in the underlying SYM theory.$^{16}$ This behavior can be seen in Fig. 1 of Ref. [16].

Similar anomalously light states appear in the full (2+1)-dimensional theory.$^{8}$ The bulk of the spectrum is driven to large $M^2$ values as $g$ is increased, but one or more states remain at low values. The presence of the transverse degree of freedom makes this more difficult to disentangle numerically, because the matrices are larger and because one must consider the
transverse resolution limit. Also, the eigenstates are less valence-dominated
at stronger YM coupling, which makes the SDLCQ approximation less use-
ful. However, structure functions have been extracted at intermediate cou-
pling to show that the approximate BPS states have a distinctly flat depen-
dence in longitudinal momentum. Figure 3b of Ref. [8] gives an example
of this behavior.

4. Summary
This work shows that one can compute spectra, wave functions, and matrix
elements nonperturbatively in supersymmetric theories. The introduction
of a Chern–Simons term brings an effective constituent mass which has the
effect of reducing the tendency of SYM theory to form stringy, low-mass
states with many constituents. Instead, the lowest-mass states tend to be
dominated by their valence Fock state. The massless BPS states of SYM
theory survive in SYM-CS theory as states with masses nearly independent
of the YM coupling.
A number of interesting issues remain to be explored. They include
theories in 3+1 dimensions, matter in the fundamental representa-
tion,17 and supersymmetry breaking. All of these are important for making contact
with QCD.

Acknowledgments
This talk was based on work done in collaboration with S.S. Pinsky and U.
Trittmann and supported in part by the U.S. Department of Energy and by
grants of computing time from the Minnesota Supercomputing Institute.

References
1. Y. Matsumura, N. Sakai, and T. Sakai, Phys. Rev. D52, 2446 (1995).
2. O. Lunin and S. Pinsky, AIP Conf. Proc. 494, 140 (1999), hep-th/9910222.
3. H.-C. Pauli and S.J. Brodsky, Phys. Rev. D32, 1993 (1985); D32, 2001 (1985).
4. S.J. Brodsky, H.-C. Pauli, and S.S. Pinsky, Phys. Rep. 301, 299 (1997), hep-
ph/9705477.
5. P. Haney, J.R. Hiller, O. Lunin, S. Pinsky, and U. Trittmann, Phys. Rev. D62,
075002 (2000), hep-th/9911243.
6. J.R. Hiller, S. Pinsky, and U. Trittmann, Phys. Rev. D64, 105027 (2001),
hep-th/0106193.
7. J.R. Hiller, S. Pinsky, and U. Trittmann, Phys. Rev. D63, 105017 (2001),
hep-th/0101120.
8. J.R. Hiller, S.S. Pinsky, and U. Trittmann, to appear in Phys. Lett. B, hep-
th/0206197.
9. G.V. Dunne, “Aspects of Chern–Simons Theory,” Lectures at the 1998 Les Houches NATO Advanced Studies Institute, Session LXIX, Topological Aspects of Low Dimensional Systems, edited by A. Comtet et al., pp. 177-263, (Springer–Verlag, Berlin, 2000), hep-th/9902115.
10. P.A.M. Dirac, Rev. Mod. Phys. 21, 392 (1949).
11. C. Lanczos, J. Res. Nat. Bur. Stand. 45, 255 (1950); J. Cullum and R.A. Willoughby, Lanczos Algorithms for Large Symmetric Eigenvalue Computations (Birkhauser, Boston, 1985), Vol. I and II.
12. D. Kutasov, Nucl. Phys. B414, 33 (1994).
13. J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
14. J.R. Hiller, O. Lunin, S. Pinsky, and U. Tittmann, Phys. Lett. B482, 409 (2000), hep-th/0003249.
15. J.R. Hiller, S.S. Pinsky, and U. Tittmann, Phys. Rev. D65, 085046 (2002), hep-th/0112151.
16. J.R. Hiller, S.S. Pinsky, and U. Tittmann, submitted for publication, hep-th/0203162.
17. Some work on fundamental matter has been done in 1+1 dimensions; see O. Lunin and S. Pinsky, Phys. Rev. D63, 045019 (2001).