TIDALLY INDUCED ELONGATION AND ALIGNMENTS OF GALAXY CLUSTERS

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ABSTRACT

We show that tidal interaction among galaxy clusters can account for their observed alignments and very marked elongation and, consequently, that these characteristics of clusters are actually consistent with them being formed in hierarchical clustering. The well-established distribution of projected axial ratios of clusters with richness class $R \geq 0$ is recovered very satisfactorily by means of a simple model with no free parameters. The main perturbers are relatively rich ($R \geq 1$) single clusters and/or groups of clusters (superclusters) of a wider richness class ($R \geq 0$) located within a distance of about 65 $h^{-1}$ Mpc from the perturbed cluster. This makes the proposed scheme be also consistent with all reported alignment effects involving clusters. We find that this tidal interaction is typically in the saturate regime (i.e., the maximum elongation allowed for systems in equilibrium is reached), which explains the very similar intrinsic axial ratio shown by all clusters. Tides would therefore play an important role in the dynamics of large scale structures, in particular, they should be taken into account when estimating the virial mass of clusters.

Subject headings: galaxies: clustering – celestial mechanics, stellar dynamics – cosmology: theory

To be published in The Astrophysical Journal
1. INTRODUCTION

Analyses of large samples of rich galaxy clusters show that their projected galaxy distribution is on the average considerably elongated (e.g., McGillivray et al. 1976; Carter & Metcalfe 1980; Binggelli 1982; Di Fazio & Flin 1988; Plionis, Barrow & Frenk 1991), even more than elliptical galaxies (Binggeli 1982). The elongated shape of clusters does not seem to be supported by rotation (Rood et al. 1972; Gregory & Tifft 1976; Dressler 1981). The distribution of observed (projected in 2D) elongations is consistent with clusters being prolate spheroids (Di Fazio & Flin 1988; Plionis et al. 1991) with intrinsic (in 3D) axial ratio Gaussian-distributed with mean $\sim 0.5$, and standard deviation $\sim 0.15$. It is certainly not consistent with pure oblate spheroids, although triaxial or combined configurations cannot be discarded. This result is, in principle, hard to understand in a hierarchical clustering scenario. Indeed, high density peaks in the primordial density field are triaxial, and collapse is achieved along the short axis leading to oblate-like shapes (Peacock & Heavens 1985; Bardeen et al. 1986).

Prolate spheroids can develop in relaxed spherical systems via bar instability. However, there is so far no observational evidence of such highly elongated velocity tensors in galaxy clusters as to favor this kind of instability. Binggeli’s (1982) finding that cluster projected major axis tends to point towards the nearest neighboring cluster if its separation is less than $\sim 15 h^{-1}$ Mpc seemed to suggest the action of tidal forces. However, this alignment effect has not been confirmed; it is found either marginally significant (Flin 1987; Rhee & Katgert 1987) or simply insignificant (Struble & Peebles 1985; Ulmer, McMillan, & Kowalski 1989; Fong, Stevenson, & Shanks 1990). Instead, some alignment seems to exist between the cluster major axis orientation and the direction of any cluster neighbor within $30 h^{-1}$ Mpc, and possibly as much as $50-60 h^{-1}$ Mpc (Binggeli 1982; West 1989b). Other alignment effects involving clusters concern the excess of galaxies in the vicinity of these structures in the direction of their major axis (Argyres et al. 1986; Lambas, Groth, & Peebles 1988), and the alignment of galaxy groups with their more or less elongated spatial distribution in superclusters (West 1989a). All these alignments would rather point to an intrinsic origin of cluster elongation related to the aspherical shape of large scale structures. This is naturally expected from pancake cosmogonies (Oort 1983), an idea supported by N-body simulations of galaxy formation (Frenk, White, & Davis 1983; Dekel, West, & Aarseth 1984). However, other kinds of observations seem to favor hierarchical cosmogonies. Besides, there is the clear trend for first ranked galaxies in clusters to show the same orientation as their parent structures (Sastry 1968; Carter & Metcalfe 1980; Binggeli 1982), which, given the mobility of galaxies and the extreme fragility of their orientation through merging and capture, is hard to understand in terms of a mere innate effect. So the possibility that cluster elongation is, after all, tidally induced should be investigated.
in detail, particularly if clusters are really prolate spheroids that form via hierarchical clustering.

Binney & Silk (1981) have shown that tidal interaction in the linear regime of density fluctuations should yield prolate shapes with an axial ratio of protoclusters of $\sim 0.5$, i.e., exactly the typical value found in clusters. This might explain the elongation of not yet collapsed large scale structures as cluster haloes. However, it can hardly account for the elongation of clusters themselves because these have evolved through a highly non-linear phase including violent relaxation, during which any preexisting elongation is severely damped out (Aarseth & Binney 1978). A rough estimate of the typical axial ratio that should prevail after relaxation leads to values of about $0.7 - 0.8$ (Binney & Silk 1981), i.e., much higher (or, equivalently, the shape much less elongated) than observed. However, this estimate does not take into account that tidal interaction keeps going on after cluster virialization, which should make the actual elongation be greater.

In the present paper we investigate the elongation induced by tidal interaction among single and grouped virialized clusters embedded in non-steady halos. In § 2 we derive the main equations dealing with such an interaction. In § 3 we determine the distribution of projected axial ratios and compare it with observation. The expected alignment effects and other consequences of the proposed scenario are discussed in § 4.

2. TIDALLY-INDUCED ELONGATION

Let us consider two particle systems, hereafter referred to as perturbed and perturbing, of masses $M$ and $M'$, respectively, with barycenters separated by a distance $s$. Let the perturbed system be in steady state (in particular, with no angular momentum), and its velocity tensor isotropic as a consequence of violent relaxation (Lynden-Bell 1967). The latter assumption guarantees sphericity in the lack of appreciable tidal forces, although this is not mandatory. Indeed, the tidally-induced elongation prior to virialization might yield some anisotropy in the velocity tensor able to support by its own some degree of asphericity. However, this anisotropy is hard to determine. In this sense, the isotropic assumption is a necessary approximation, with the advantage of notably simplifying the calculations.

In cartesian coordinates, with origin at the barycenter of the perturbed system and $x_1$-axis towards the perturbing one, the integral over the perturbed system of the
\( i \) component of the equation of motion of a fluid element in the global potential field times \( x_j \) leads to (Chandrasekhar & Lebovitz 1963; Chandrasekhar 1969)

\[
-\frac{2}{3} \tau \delta_{ij} = U_{ij} + 2\mu \delta_{i1} I_{1j} - \mu \delta_{i2} I_{2j} - \mu \delta_{i3} I_{3j}.
\] (1)

In equation (1) \( \tau \) is the internal kinetic energy, and \( U_{ij} \) and \( I_{ij} \) are the potential energy and inertia tensors, respectively, the former defined as

\[
U_{ij} = \int \rho \frac{\partial \Phi}{\partial x_i} x_j \, dx,
\] (2)

with \( \rho \) the density and \( \Phi \) the potential of the perturbed system alone, and \( \mu \) is

\[
\mu = \frac{GM'}{s^3}.
\] (3)

In deriving relation (1) the potential due to the perturbing system has been approximated by that of a point mass \( M' \) located at its barycenter. Since the external potential of a system is more spherical than the mass distribution causing it, this should be a good approximation even for relatively small separations \( s \). Besides, it neglects third order terms in \( l/s \), with \( l \) the size of the perturbed system, which would give an error of 13% in the most unfavorable case of two interacting systems in physical contact.

For homogeneous ellipsoids one has (Chandrasekhar 1969)

\[
M = \frac{4}{3} \pi \rho a_1 a_2 a_3,
\] (4)

\[
I_{ij} = \frac{1}{5} M a_i^2 \delta_{ij},
\] (5)

\[
U_{ij} = -2\pi G \rho A_i I_{ij},
\] (6)

with \( a_i \) the semi-axis lengths, and some \( A_i \) geometrical parameters, which, for prolate spheroids \( a_1 > a_2 = a_3 \), write

\[
A_1 = \frac{2\beta^2}{(1-\beta^2)^{3/2}} \left[ \ln\left(\frac{\beta}{1-(1-\beta^2)^{1/2}}\right) - (1-\beta^2)^{1/2} \right],
\] (7a)

\[
A_2 = A_3 = \frac{\beta^2}{(1-\beta^2)^{3/2}} \left[ \frac{(1-\beta^2)^{1/2}}{\beta^2} - \ln\left(\frac{\beta}{1-(1-\beta^2)^{1/2}}\right) \right],
\] (7b)

in terms of the axial ratio

\[
\beta = \frac{a_2}{a_1}.
\] (8)

Taking into account relations (4)–(7b), the non-vanishing components of the tensorial equation (1),

\[
-\frac{2}{3} \tau = U_{11} + 2\mu I_{11} = U_{22} - \mu I_{22} = U_{33} - \mu I_{33},
\] (9)
lead to the relation (Chandrasekhar 1969)

\[
\frac{2\beta^2}{(1 - \beta^2)^{3/2}} \left[ \ln \left( \frac{\beta}{1 - (1 - \beta^2)^{1/2}} \right) - \frac{3 (1 - \beta^2)^{1/2}}{2 + \beta^2} \right] = \frac{\mu}{\pi G \rho},
\]

between the intrinsic axial ratio \(\beta\) of the perturbed system and the mass and distance, through \(\mu\), of the perturbing one. Whereas, taking into account the same relations (4)–(7b) and equation (9), the trace of tensorial equation (1) leads to the virial theorem generalized for the case of non-negligible tidal elongations.

The solution \(\beta(\mu)\) of equation (10) is bivaluate, each pair of values corresponding to one stable and one unstable figure of equilibrium. The two sets of solutions form two different branches of the so-called Jeans sequence, separated at \(\frac{\mu}{\pi G \rho} = 0.1255\) by the degenerate solution \(\beta_{\text{min}} = 0.4693\) giving the minimum stable solution and maximum unstable one. Since we are concerned here only with the stable branch of the Jeans sequence, we can regard expression (10) as a unimodal relation between \(\mu\) and \(\beta\). For values of \(\frac{\mu}{\pi G \rho}\) larger than 0.1255, there is no solution. Note the coincidence between the value \(\beta_{\text{min}}\) and the typical axial ratio of clusters.

Galaxy clusters are not homogeneous ellipsoids but self-similar inhomogeneous ones (Salvador-Solé, Sanromà, & González-Casado 1992). Under these circumstances the density distribution can be expressed as a radial density profile in terms of the equivalent radius \(r = \left(\frac{3V}{4\pi}\right)^{1/3}\), with \(V\) the volume inside each homologous isodensity contour, the total equivalent radius \(R\) being equal to the geometrical mean of the semiaxes \(a_1, a_2,\) and \(a_3\). This leads to (Roberts 1962)

\[
M = \frac{4\pi}{3} \rho(0) a_1 a_2 a_3 f_M, \quad (11)
\]

\[
I_{ij} = \frac{1}{5} M a_i^2 \delta_{ij} f_I, \quad (12)
\]

\[
U_{ij} = -2\pi G \rho(0) A_i I_{ij} f_U, \quad (13)
\]

with \(A_i\) the geometrical coefficients given by expressions (7a) and (7b). So the only difference with the homogeneous case (eqs. [4]–[6]) comes from the shape factors

\[
f_M = 3 \int_0^1 \eta(x) x^2 \, dx, \quad (14a)
\]

\[
f_I = \frac{15}{2 f_M} \int_0^1 F(x) x^2 \, dx, \quad (14b)
\]

\[
f_U = \frac{15}{8 f_I f_M} \int_0^1 F^2(x) \, dx, \quad (14c)
\]
with $\eta(x) = \rho(x)/\rho(0)$, $x = r/R$, and

$$F(\xi) = 2 \int_\xi^1 \eta(x) \, x \, dx.$$  \hfill (14d)

Thus, by substituting $U_{ij}$ and $I_{ij}$ given by relations (12) and (13) into equation (9) and taking into account relations (11), (7a), (7b), we obtain the wanted relation between the axial ratio of the perturbed system, and the mass and distance of the perturbing one for the case of self-similar inhomogeneous systems. This relation turns out to be identical to relation (10) for the homogeneous case, but for the fact that constant density $\rho$ must be replaced by the effective density $\tilde{\rho} = \rho(0) \, f_U$.

Now let us consider the case of tidal interaction between a couple of relaxed clusters with similar shape of their mass density profile, assumed with the modified Hubble law form, $\eta(x) = \left(1 + (xR/r_0)^2\right)^{-\alpha}$, but different central densities, axial ratios, and orientations. Then equation (10) with effective density $\tilde{\rho}$ can be written explicitly as

$$G(\beta) = Q \, \frac{4}{3} \, f_M(R/r_0, \alpha) \left(\frac{R}{s}\right)^3,$$  \hfill (15)

where $G(\beta)$ stands for the left-hand-side member of equation (10) and $Q$ is the ratio of central densities of the perturbing and the perturbed clusters. Actually, equation (15) is not yet well-suited for our purposes. Firstly, galaxy clusters are not isolated steady bodies, but are surrounded by large non-steady halos. Secondly, we are not concerned with the elongation of the whole perturbed system, but only of its galactic component.

Non-steady halos are elongated in the same direction as their respective clusters (Argyres et al. 1986; Lambas, et al. 1988). This is consistent with the elongation of both subsystems being tidally induced by the same perturber. Since cluster halos are in the linear or moderately non-linear regime, their typical axial ratio should be about 0.5, hence, very similar to that found in relaxed clusters. Under these circumstances the potential inside the perturbed system keeps with the same form as above. So by integrating the $i$ component of the equation of motion of a fluid element times $x_j$ over the the steady region alone and neglecting the pressure at the edge of that region (as usually done for estimating the cluster virial mass) we are once again led to equation (1) and, consequently, to equation (15) but for two small differences: 1) due to the non-vanishing potential of the perturbed halo, $f_U(R/r_0, \alpha)$ turns out to be replaced by

$$f_U(R/r_0, \alpha, R_h/r_0, \alpha_h) = f_U(R/r_0, \alpha)$$

$$\times \left[1 + \frac{\left(\frac{R_h}{R}\right)^{2(1-\alpha_h)} - 1}{\frac{2-2\alpha_h}{5} \left(\frac{R}{r_0}\right)^{-2\alpha} f_I(R/r_0, \alpha) f_U(R/r_0, \alpha)}\right],$$  \hfill (16a)
and 2) due to the potential of the perturbing halo, \( f_M(R/r_0, \alpha) \) is also replaced by

\[
\tilde{f}_M(R/r_0, \alpha, R_h/r_0, \alpha_h) = f_M(R/r_0, \alpha) \left[ 1 + \frac{\left( \frac{R_h}{R} \right)^{3-2\alpha_h} - 1}{(3-2\alpha_h) \left( \frac{R}{r_0} \right)^{2\alpha} f_M(R/r_0, \alpha)} \right].
\]

Thus, equation (15) transforms into

\[
G(\beta) = Q \frac{4 \tilde{f}_M(R/r_0, \alpha, R_h/r_0, \alpha_h)}{3 \tilde{f}_U(R/r_0, \alpha, R_h/r_0, \alpha_h)} \left( \frac{R}{s} \right)^3.
\]

(17)

In equations (16a) and (16b) halos are assumed with total radius \( R_h \) and mass density profile, corrected from the uniform mean background density, of the power law form with index \(-2\alpha_h\) (different from 2 and 3), and amplitude matching the interior mass density run. In calculating the potentials of the whole perturbing system and of the halo of the perturbed system one should correct, indeed, their respective density profiles for the uniform mean density because the potential at any point of the inner perturbed cluster due to an outer uniform mass distribution vanishes. Since performing the same correction to the density profile of the inner perturbed cluster makes a negligible difference because of the very high density contrast there, we can use corrected profiles everywhere, which notably simplifies the modelling.

To deal with the second problem we have followed the same derivation above from the equation of motion of an element of the galactic component. This leads to a tensorial equation similar to (1) but with the inertia tensor \( I^g_{ij} \) written in terms of the mass density of the galactic component instead of the total mass density, and the potential energy tensor \( U^g_{ij} \) in terms of both densities (\( \mu \) is always written in terms of the total mass of the perturber). It can be shown (see Appendix B) that this latter tensor has the same form as \( U_{ij} \) (eq. [13]), with \( I^g_{ij} \) instead of \( I_{ij} \) and the shape factor \( f^g_{ij} \) given by equation (B4). The density profile of the galactic component in the inner relaxed cluster is also assumed with the modified Hubble law form with identical total and core radii as for the total mass distribution (final results are very insensitive to the exact value of the core radius), but a distinct power index \( \alpha^g \). The non-vanishing components of tensorial equation (1) then lead to an analogous relation to (15)

\[
G(\beta) + 4 \left( \frac{1}{2 + (\beta^g)^2} - \frac{1}{2 + \beta^2} \right) = Q \frac{4 f_M(R/r_0, \alpha)}{3 f^g_{ij}(\alpha^g, R/r_0, \alpha)} \left( \frac{R}{s} \right)^3.
\]

(18)

Equation (18) tells us that for different shaped density profiles of the galactic component and the total mass, the axial ratio \( \beta^g \) of the galactic distribution will be different from that of the whole mass distribution. Of course, as for relation (15), relation (18)
presumes the whole system is in steady state. When non-steady cluster haloes are taken into account one must replace, as in the analogous relation for the total mass (eq. [17]), functions $f_M$ and $f_U^g$ by $\tilde{f}_M$ and $\tilde{f}_U^g$, respectively, with $\tilde{f}_U^g$ defined in terms of $f_U^g$ exactly as $\tilde{f}_U$ in terms of $f_U$ (eq. [16a]). And by substituting $G(\beta)$ by its explicit expression (17), the new and rather complicated relation can be put in the practical form

$$
\frac{4}{G(\beta)} \left( \frac{1}{2 + (\beta^g)^2} - \frac{1}{2 + \beta^2} \right) = \frac{\tilde{f}_U(R/r_0, \alpha, R_h/r_0, \alpha_h)}{\tilde{f}_{U}^g(\alpha^g, R/r_0, \alpha, R_h/r_0, \alpha_h)} - 1,
$$

(19)

which will be used in next section in order to obtain the observable distribution of $\beta^g$’s from the much easier to calculate (though hidden to observation) distribution of $\beta$’s. Note that, for $\beta = 1$, $G(\beta)$ is equal to zero and, therefore, that $\beta^g$ is also unity, whereas the value of $\beta^g$ in the saturate regime, $\beta^g_{\text{min}}$, differs from $\beta_{\text{min}}$.

A last comment is in order before inferring the distribution of axial ratios. As mentioned above, equation (10) and, consequently, equations (17) and (19) may have no solution, i.e., there may be no figure of equilibrium, for too strong a tidal interaction or, equivalently, too large an $M'$ and/or small $s$. But “relaxed” clusters are necessarily in equilibrium. So, if this situation is met we shall admit that some assumption is wrong. It might be argued that the lack of solution can be due to the point-mass approximation and/or the truncation at third order in $l/s$, whose validity become increasingly deficient for small values of $s$. However, as pointed out, these approximations are reasonably good even in the most unfavorable case of systems in physical contact. The configuration $(M', s)$ being what it is, the only wrong assumption must concern the assumed shape of the density profile. Indeed, for too strong a tidal interaction, the perturbed system would be tidally truncated or would not have accreted the surplus material, accomodating its structure to the nearest available state of equilibrium. Hence, for consistency with the overall scheme, in this “saturate regime” we shall simply take the minimum allowed axial ratio of equilibrium as the effective solution.

**3. DISTRIBUTION OF PROJECTED AXIAL RATIOS**

The morphological analysis of rich relaxed clusters shows that their “circularized” galaxy number density profiles are consistent with a universal radial profile of the modified Hubble law form, with core radius $r_0$ equal to 0.25 $h^{-1}$ Mpc (Dressler 1978; Binggeli 1982; Salvador-Solé et al. 1992), and index $\alpha^g$ equal to 3/2 (Dressler 1978; West, Dekel, & Oemler 1987; Salvador-Solé et al. 1992). This universal profile refers, however, just to the central parts of clusters (up to about one Abell radius) while we are interested here in the density run up to the edge of the systems. The statistical analysis by means of the cluster-galaxy cross-correlation function, of the angular distribution of galaxies...
around the position of rich Abell clusters (Lilje & Efstathiou 1988) shows that this is consistent with a function of the form (see also Peebles 1980)

\[ \xi_{cg}(r) = \left( \frac{hr}{6} \right)^{-2.3} + \left( \frac{hr}{7} \right)^{-1.7} \]  

(with \( r \) in Megaparsecs) for \( 1 \text{ Mpc} \leq hr \leq 20 \text{ Mpc} \). According to the statistical meaning of \( \xi_{cg} \) expression (20) should be proportional to the galaxy number density profile in cluster/halos corrected from the mean cosmological number density as needed. At small radii, expression (20) is consistent with a corrected galaxy distribution in the relaxed part of clusters following a modified Hubble law with index \( \alpha^g = 1.15 \), whereas, at large radii, it is consistent with a power law of logarithmic slope \(-1.7\) or, equivalently, with index \( \alpha^h_{g} \) for the galaxy component in the outer halo about equal to 0.85. It is important to remark that the shape of \( \xi_{cg} \) at large radii may be influenced by the clustering of galaxy clusters, the observed behavior being the geometrical mean of both contributions. However, since this latter contribution is necessarily close to a power law of logarithmic slope \(-1.8\) (Bahcall 1988), if there is a significant contribution of cluster halos, this must also have the same form. (The only halo whose morphology has been studied in detail, the Virgo Supercluster, shows this kind of profile, with \( \alpha^h \approx 1 \); Yahil 1974; Yahil, Sandage, & Tammann 1980.) On the other hand, the roughly uniform l.o.s. velocity dispersion \( \sigma_{los} \) of galaxies in clusters implies a value of the power index \( \alpha \) for the total mass density profile of clusters roughly equal to 1, in agreement with what is expected from violent relaxation (Lynden-Bell 1967). This is readily seen from the hydrostatic equation for the galactic component in spherical clusters and the assumed universality of the equivalent radial profile. To estimate the value of index \( \alpha_h \) for the total mass in the halo we must make some assumption on the radial run of the \( M/L \) ratio there. Since the infalling material has the same infall velocity irregardless of its nature and well mixed material (with uniform \( M/L \)) seems a reasonable guess for the initially unperturbed medium around protoclusters, we shall assume that light traces mass in cluster halos and take \( \alpha_h = \alpha_{g}^h \approx 0.85 \). (Results are, anyhow, quite insensitive to the value of this latter parameter, as in the case of \( R_h \).)

A rough estimate of the value of \( R \) is given by the velocity dispersion of galaxies times the age of the universe. Such a radius probably embraces the region of rebounding although not yet fully virialized layers. However, we do not need matter to be strictly virialized but only with negligible streaming velocity. Poor clusters show a clear correlation between \( \sigma_{los} \) and Abell (1958) richness \( N \), while rich clusters show a much larger scatter around a roughly fixed value of \( \sigma_{los} \) (Bahcall 1981). This behavior leads to \( R \approx 0.14 \ N \ h^{-1} \ \text{Mpc} \) (for \( \Omega = 1 \)) for \( N \leq 50 \), and \( R \approx 7 \ h^{-1} \ \text{Mpc} \) for \( N \geq 50 \). (Note that these values are fully consistent with the radius at which there is the change
between the two extreme power-law regimes of $\xi_{cg}$, this latter function having been obtained from rich clusters.) A similar situation probably holds for $R_h$. According to the observed cluster-galaxy cross-correlation this latter radius would be, for rich clusters, of at least $20 \, h^{-1}$ Mpc and possibly as large as $\sim 40 \, h^{-1}$ Mpc (Seldner and Peebles 1977; Peebles 1980), while for poor and moderately rich clusters it is certainly much smaller. For this reason we shall simply take $R_h = 30 \left( \frac{N}{50} \right) h^{-1}$ Mpc for $N \leq 50$, and $R_h = 30 \, h^{-1}$ Mpc for $N \geq 50$, in the case that the perturbing system is more distant than twice the corresponding value of $R_h$, and half the separation $s$ for closer systems. (Results are also quite insensitive to this upper bound value of $R_h$.) Note that, at separations smaller than twice the theoretical value of $R$, the total radius $R_h = s/2$ will get smaller than $R$, so not only will there be no halo, but the relaxed part of clusters will then be truncated at $R_h$.

The dependence of $Q$ on cluster richness can be inferred from the generalized version of the virial theorem (§ 2) for the galactic component of clusters. By dividing it by the mass $M^g$ we obtain

$$3 \sigma_{los}^2 = G \rho(0) R^2 f_I^g \tilde{f}_U^g J(\beta, \beta^g),$$

(21)

From the definitions of $f_I^g$ and $\tilde{f}_U^g$ (eq. [14b] and [16a] for the galactic component, respectively) we have that factor $\rho(0) R^2 f_I^g \tilde{f}_U^g$ is proportional to the total mass $M$ over some characteristic length of the cluster. So one recovers the usual expression of the virial theorem for isolated spherical systems except for the function $J(\beta, \beta^g)$. The product $f_I^g \tilde{f}_U^g$ is essentially proportional to $R^{-2\alpha}$. Thus, given the proportionality between $R$ and $\sigma_{los}$, equation (21) leads to the fact that $\rho(0) R^{-2\alpha} J(\beta, \beta^g)$ should not depend on cluster richness. Since the function $J(\beta, \beta^g)$ will typically take the value corresponding to the saturate regime (see below) it will not depend on cluster richness neither. So the central density $\rho(0)$ must be proportional to $N^{2\alpha} \sim N^2$ or, equivalently, $Q$ must be proportional to the square of the richness ratio. (This dependence of $\rho(0)$ with $N$ leads to a $M/L$ ratio, up to a fixed radius, essentially proportional to $N$.) Notice that, since the richer the cluster, the harder it is that its axial ratio can systematically reach the saturate value, equation (21) implies that, for rich clusters, the empirical correlation $\sigma_{los}$ vs $N$ must show a larger scatter than for poorer ones. Besides, since $J(\beta, \beta^g)$ will then begin to depend, on the average, on cluster richness as $N^\gamma$ with $\gamma < 0$, the correlation $\sigma_{los}$ vs $N$ must become flatter (disregarding of the actual dependence of $R$ on $N$ for these clusters). Both trends are in full agreement with observation.

With these values and/or functionalities of the parameters entering in the model let us now derive the predicted distribution of tidally-induced elongations. For a cluster of given richness $N$ in the range for which the empirical distribution has been obtained (i.e., $N \geq 30$), the probability $P(N, \beta) \, d\beta$ of it having intrinsic axial ratio between $\beta$
and $\beta + d\beta$ due to the tidal action of neighboring clusters with richness above some threshold $N_t$ is

$$P(N, \beta) \, d\beta = P_{\text{neig}}(N, \beta) \, d\beta \, \mathcal{P}_{\text{neig}}(N, \leq \beta), \quad (22)$$

where $P_{\text{neig}}(N, \beta) \, d\beta$ is the probability of finding one of such neighbors able to produce the wanted axial ratio, and $\mathcal{P}_{\text{neig}}(N, \leq \beta)$ is the probability that its tidal interaction be dominant, which coincides with the probability that there is no other neighbor of the same kind able to yield a smaller axial ratio.

Probability $P_{\text{neig}}(N, \beta) \, d\beta$ for $\beta > \beta_{\text{min}}$ is simply given by

$$P_{\text{neig}}(N, \beta) \, d\beta = 4\pi n_c \int_0^\infty (1 + \xi_{cc}(s)) \, s^2 \, N_c(N_{\text{neig}}(N, \beta, s)) \left| \frac{\partial N_{\text{neig}}(N, \beta, s)}{\partial \beta} \right| d\beta \, ds. \quad (23)$$

In equation (23) $n_c$ and $\xi_{cc}$ are the mean number density and correlation function of clusters with $N \geq N_t$, $N_c(N)$ is the normalized cluster richness function, and $N_{\text{neig}}(N, \beta, s)$ is the richness necessary for a neighbor at $s$ be able to yield the wanted axial ratio. This latter value is obtained from equation (17) where, for simplicity, radii $R$ and $R_h$ are taken equal to those corresponding to a cluster of median richness $N_{\text{med}}$ in the allowed range for perturbers. Note also that in equation (23) we have neglected any spatial segregation among clusters with different richnesses within this range. These two practical approximations have noticeable consequences discussed below. The cluster richness function $N_c(N)$ is approximated by two power laws of indexes $-2$ at the poor end and $-5$ at the rich one, matching at $N^* \simeq 65$ (Bahcall 1979). This allows us to calculate the value $N_{\text{med}}$ and the mean density $n_c$ for any threshold $N_t$ from some given value of reference; here we use $n_c = 1.0 \times 10^{-5} \, h^3$ clusters per Megaparsec for $N_t = 50$ (Bahcall 1988). Finally, the correlation function $\xi_{cc}(s)$ for threshold $N_t$ is taken equal to $A \, s^{-1.8}$ for $s \leq 50 \, h^{-1}$ Mpc and null beyond that distance, with the correlation amplitude $A$ depending on cluster richness according to the empirical relation $A \propto N_{\text{med}}$ (Bahcall & West 1992, and references therein). Notice that factor $d\beta$ in the right hand side of equation (23) allows us to calculate the probability of finding one neighbor with the appropriate richness by simply taking the integral over the volume of the density probability of finding it. Although the integral extends to the whole space, one can take a finite though sufficiently large piece of universe warranting the convergence of the result. It may also be more realistic to exclude some small volume around the perturbed system, but this makes no appreciable difference in the final results.

On the other hand, probability $\mathcal{P}_{\text{neig}}(N, \leq \beta)$ is given by

$$\mathcal{P}_{\text{neig}}(N, \leq \beta) = \exp \left[ -4\pi n_c \int_0^\infty (1 + \xi_{cc}(s)) \, s^2 \, N_c(\geq N_{\text{neig}}(N, \beta, s)) \, ds \right], \quad (24)$$
with the exponent of the right-hand-side member equal to minus the expected number of neighbors able to yield an axial ratio smaller than or equal to $\beta$, which coincides with the number of neighbors with richness larger than or equal to that yielding the wanted axial ratio. Taking into account that expression (23) is equal to the partial derivative of the exponent in expression (24), probability $P(N, \beta) \, d\beta$ can be written as minus the partial derivative with respect to $\beta$ of $\mathcal{P}_{neig}(N, \leq \beta)$ or, equivalently,

$$P(N, \geq \beta) = \mathcal{P}_{neig}(N, \leq \beta).$$

(25)

From equation (25) it can be readily seen that the probability $P(N, \beta) \, d\beta$ is correctly normalized to unity for each richness $N$ of the perturbed system.

The distribution of intrinsic axial ratios $\Psi(\beta)$ for $\beta > \beta_{\text{min}}$ of clusters with $N \geq 30$ can be thus obtained by deriving with respect to $\beta$ the integral over $N$ of probability $P(N, \geq \beta)$, given by equation (25), weighted by $N_c(N)$. And the value of the distribution function at $\beta = \beta_{\text{min}}$ can be readily calculated by taking into account that the whole integral of $\Psi$ over $\beta$ must be unity. This leads to the following expression

$$\Psi(\beta) = \frac{\partial}{\partial \beta} \int_{\beta_{\text{min}}}^{\infty} \mathcal{P}_{neig}(N, \leq \beta) \, N_c(N) \, dN$$

$$+ \delta(\beta - \beta_{\text{min}}) \left( 1 - \int_{\beta_{\text{min}}}^{\infty} \mathcal{P}_{neig}(N, \leq \beta_{\text{min}}) \, N_c(N) \, dN \right).$$

(26)

Finally, in order to obtain the distribution of axial ratios for the galactic component we must simply take into account that each axial ratio $\beta$ gives rise to an “observable” axial ratio $\beta^g$, related to the former through equation (19). So the wanted distribution of axial ratios $\Psi(\beta^g)$ is

$$\Psi(\beta^g) = \Psi(\beta) \frac{d\beta}{d\beta^g}$$

(27)

in terms of the distribution $\Psi(\beta)$ given by equation (26) and the function $\beta(\beta^g)$ given by equation (19) (with $R$ and $R_h$ in $\tilde{f}_U$ and $\tilde{f}^g_U$ taking the values for perturbers with richness equal to $N_{\text{med}}$ as above). Notice that the distribution $\Psi(\beta^g)$ also has a Dirac delta as high as that of $\Psi(\beta)$ but shifted to $\beta_{\text{min}}^g = \beta^g(\beta_{\text{min}})$. The resulting distribution of intrinsic axial ratios $\beta^g$ (for a richness threshold of perturbers equal to 45; see discussion below) is plotted in Figure 1a. The solution shows a very sharp peak at $\beta_{\text{min}}^g \approx 0.5$ in agreement with what is required by observation (Plionis et al. 1991). However, it notably differs from a Gaussian function centered on that value and standard deviation equal to 0.15. The disagreement seems to be twofold: the peak at $\beta_{\text{min}}^g$ is too sharp, and there is an important bump at large axial ratios. But before drawing any definite conclusion we should first look at the distribution of projected axial ratios.
By using the relation between the distributions of intrinsic and projected axial ratios, $\beta$ (or $\beta^g$) and $\beta_p$ ($\beta_p^g$) for prolate spheroids (Hubble 1926)

$$
\Phi(\beta_p) = \frac{1}{\beta_p^2} \int_0^{\beta_p} \frac{\beta^2 \Psi(\beta) \, d\beta}{(1 - \beta^2)^{1/2} (\beta_p^2 - \beta^2)^{1/2}}.
$$

we can obtain the really compelling distribution, $\Phi(\beta_p^g)$, from the intrinsic one, $\Psi(\beta^g)$, given by expression (27) and directly compare it with observation. In Figure 1b we plot the function $\Phi(\beta_p^g)$ corresponding to the distribution in 3D shown in Figure 1a, together with the empirical distribution obtained by Plionis et al. (1991). In fact to properly compare both distributions we must take into account measuring errors. According to Plionis et al. (1991) the standard deviation in the axial ratio obtained by his method, calculated from Monte Carlo clusters, is equal to 0.05–0.07, independently of the specific value of the intrinsic axial ratio simulated. The scatter shown by their data on real clusters is somewhat larger, with average standard deviation equal to 0.10 (see the data quoted by Plionis et al. 1991 for different thresholds of cluster density), such an increase being probably caused by the varying center location and background contamination which are both absent in the simulations. Thus, in order to mimic the effects of measuring errors we have convolved our theoretical distribution by a Gaussian function with standard deviation equal to 0.10. As shown in Figure 1c, the maximum at $\sim 0.6$ of the convolved theoretical distribution of projected axial ratios is now sufficiently smooth. So the sharpness of the Dirac delta at 3D is actually not any problem. Projection and convolution has also somewhat mitigated the bump at large axial ratios, but there is still a very marked secondary maximum at $\beta_p \sim 0.90$, fully absent in the empirical histogram. This would only disappear for a high enough Dirac delta in 3D.

In diminishing the threshold allowed for perturbers one should in principle obtain more marked elongations and, consequently, a higher Dirac delta. Indeed, all rich perturbers causing the previous elongations are always included, and one is just adding new perturbers, which should tend to increase the resulting elongations. In practice, however, such an expected trend is only followed at large thresholds; at small ones the solution shows just the opposite behavior. This can be seen in Figure 2 where we have plotted the height of the Dirac delta at $\beta^g_{\text{min}}$ of the distribution in 3D as a function of the threshold $N_t$. The wrong behavior shown by the solution at small thresholds is just a consequence of the use of average characteristics for clusters above some given threshold (see comments on eq. [22]). In lowering the threshold we are taking smaller $R$ and $R_h$ and a smaller correlation amplitude. The new values are well suited on the average for clusters in the new range of richesses, but worse suited for the richest ones in this range, which rather partake of the preceding average characteristics. This introduces an error which gets more and more severe for increasingly lower thresholds. Only for large
enough thresholds (above $N^* = 65$) can the effect not balance the dramatic increase in the number of perturbers when diminishing it, so that the good behavior of the distribution function there is guaranteed. From this discussion it is clear that: (1) the lower the threshold, the more underestimated the effects of tidal interaction in the derived distributions of axial ratios, (2) correction of this flaw should lead to, at least, the distribution with the most marked axial ratios obtained, (3) very rich clusters are not numerous enough and, hence, they are placed typically at too large distances from the perturbed system to systematically yield very marked elongations, and (4) poor clusters are, on the contrary, quite numerous and can be found nearer to the perturbed system, but they are not rich enough to yield very marked elongations neither. From Figure 2 we see that the most marked elongations correspond to a threshold $N_t$ of about 45; this was the value used in Figure 1. Note that, since the values of $R$ and $R_h$ keep fixed for $N_t \geq 35$ ($N_{med} = 50$), the distribution plotted in that figure is only affected by the neglect of the segregation in cluster richness, which is not a very important effect.

There is, however, another simplifying assumption in the model that may have important consequences in the final result. In the derivation above we have neglected the tidal action of non-dominant neighbors. The reason for this is that the total number of neighbors located within the distance necessary for their tidal action be relevant (see § 4), is relatively small (about 14 with $N \geq 45$ and $s \leq 55 \, h^{-1} \, \text{Mpc}$). This guarantees that, in any actual realization, there are important gaps in the distributions of neighbor richesses and separations. Consequently, the tidal force of any non-dominant neighbor will usually be negligible compared to that of the dominant one. However, clusters tend to be clustered and non-dominant neighbors will be preferentially located near to the dominant one, boosting its tidal action. Besides, we have also neglected the tidal effects of voids (Ftaclas 1983) which should also boost the tidal action of groups of clusters. So the previous simple approach has been rather underestimating the true tidally-induced elongation of clusters. An accurate prediction of the distribution of cluster axial ratios through the vectorial composition of the tidal force caused by each individual neighbor and void is hopeless because this demands a full statistical knowledge of the spatial distribution of clusters, while only the very first $N$-point correlation functions are available. Nonetheless, it would be interesting to have an idea of the effects of clustering. In fact, a straightforward extension of the previous model can be developed which accounts very approximately for the tidal action of groups of clusters, or superclusters, and voids amidst them making only use of the cluster two- and three-point correlation functions.

Taking advantage of the point mass approximation, we can consider all neighboring clusters within spheres centered on one of them and with radius equal to the distance of this center to the perturbed cluster as contributing with the sum of their masses to the potential of the central perturbing system. One must correct, of course, this
composite mass from the contribution inside the sphere of the mean uniform density of clusters. So we are only concerned with the number of clusters “in excess” inside these spheres. (Note that we should strictly add, on the contrary, the previously subtracted contribution of the mean density in individual clusters, but this is a small correction that can be neglected.) This procedure would accurately account for the tidal effects of any expected anisotropy in the distribution of neighboring clusters around a given one if the density distribution around the center of these spheres were really a continuous spherically symmetric function. Actually, the probability of finding clusters inside the spheres does not only depend on the radial distance from their center but also, to a smaller extent, on the distance to the perturbed cluster. (For simplicity this latter dependence is neglected in the calculations; taking it into account should result in slightly more marked elongations.) Besides, the number of clusters in excess contained in these spheres is not very large. So the associated mass density distribution will notably deviate from a continuous function. (We take it as simply proportional to the probability of finding clusters). However, we are interested in estimating the effects of anisotropies in the spatial distribution of clusters in a statistical manner rather than trying to compute the accurate tidal force exerted on a cluster in any given realization. So our approximate procedure is justified and should actually lead to quite a good estimate of the real distribution of axial ratios.

In Appendix A we derive the equations leading to the distribution of axial ratios according to this much more accurate version of the model. Because of the effects mentioned above, we have tried different thresholds, the most marked elongations being now obtained for $N_t \simeq 35$. The solution is shown in Figure 3. (As in the preceding version the maximum length of positive correlation has been taken equal to $50 \, h^{-1} \text{Mpc}$. It is worthwhile mentioning that in the present version, the larger this length, the more marked the resulting elongations. In fact, for values greater than $\sim 60h^{-1} \text{Mpc}$ they would even be excessively marked, but this effect should not be taken too seriously because it might be balanced by a cluster richness function steeper than assumed at very large richnesses.) For comparison with Figure 1 we plot the distribution of intrinsic and projected axial ratios, the latter in the real as well as degraded versions. As expected, the inclusion of the tidal action of groups of clusters (superclusters) has notably increased the height of the Dirac delta at $\beta_{\text{min}}$ in the distribution in 3D and erased the second maximum at about 0.9 in the distribution in 2D. The result is in very good agreement with observation, which is especially remarkable given the simplifying assumptions involved in the model and the fact that it has no free parameters. (Note that even the small inflexion observed at an axial ratio of 0.9 in the predicted distribution and absent in the empirical one would tend to disappear if we could accurately correct the effects of measuring errors at the upper bound value of one. Indeed, the
4. DISCUSSION

Dominant perturbers are, therefore, typically rich ($N \geq 45$) single clusters and groups of clusters with a slightly wider range of richnesses ($N \geq 35$). In any case, the nearest neighboring cluster with $N \geq 30$ is typically not the dominant perturbing system, which explains the lack of significant correlation between cluster orientation and angular position of the nearest neighboring cluster. Our results also explain the existence, on the contrary, of a correlation between cluster orientation and angular position of any neighbor within some given distance. The predicted value of this distance can be easily obtained from the theoretical distribution of axial ratios by imposing increasingly large minimum separations to the possible perturbers. In doing so, the typical elongation of clusters, measured from the height of the Dirac delta at $\beta_{\min}^g$ in 3D, gets smaller and smaller, and, for a large enough minimum separation, it becomes so small that no significant alignment would be found between the perturbed cluster and the perturbing ones. This upper value of the minimum separation determines, therefore, the maximum distance for significant alignments. By taking the fraction of typically elongated clusters equal to 10\% (5\%) we are led to a value of that distance equal to 55 $h^{-1}$ Mpc (75 $h^{-1}$ Mpc), in very good agreement with observation, too. (The predicted values may seem slightly larger than observed, but this is not important because by using the height of the maximum at 2D instead of the height of the Dirac delta at 3D we would have obtained smaller values.)

The same tidal action causing the elongation of relaxed clusters should also tend to elongate individual galaxies. Because of their much higher concentration, the resulting elongation should be, however, much less marked than for clusters (eq. [19]). In other words, the shape of galaxies would not be supported, in general, by the tidal action of clusters. However, this might easily cause an appreciable elongation to giant D and cD galaxies. So the proposed scenario would also explain the observed alignment between clusters and their first ranked galaxies. Note that, for the same reason, some alignment should also be found between clusters and any stable substructure inside them such as binary systems (if any). This might explain the recent finding by Trevese, Cirimele, & Flin (1992). In summary, the proposed scenario is in agreement or, at least, consistent with all reported alignment effects involving clusters.

The model of tidal interaction developed here assumes the velocity tensor of galaxies in clusters isotropic. As pointed out, some anisotropy is foreseeable (although difficult
to characterize) coming from the aspherical configuration of the protocluster. Since this would also be tidally-induced by the same dominant perturber as, later on, once the cluster has virialized, such an anisotropy should not prevent the cluster or, more exactly, its anisotropic galactic component to reach the very marked elongation found under the isotropic condition. On the contrary, it should rather favor a more marked elongation. Therefore, the axial ratio in the saturate regime of tidal interaction for such anisotropic systems should not be too different from that found in the isotropic approximation. Otherwise, the predicted typical tidally-induced elongation of clusters might turn out to be unacceptably high. This should be possible to check by means of N-body simulations.

Since the isotropic condition is very well suited for the intracluster medium, the model developed here could be readily applied, without any limitation of the previous kind, to this gaseous component. Unfortunately, there is so far no well-established distribution of projected axial ratios of clusters drawn from their X-ray images. Work in progress in this line seems to point at substantially less marked typical elongations (Jones & Forman 1991). However, this might simply be due to the fact that clusters studied in X-rays tend to be very rich. So it would be of major interest to see whether or not the observed elongation of the gaseous component of clusters is also in agreement with that predicted by the present model for the appropriate range of cluster richnesses.

As an important byproduct of the proposed scenario we have that tidal interaction between clusters is typically in the saturate regime, which explains in a very natural manner the “universal” value of their intrinsic axial ratio. Therefore, the growth and dynamical state of clusters should be notably influenced by this interaction, and in neglecting it one might be committing non-negligible systematic errors. This is apparently the case for the usual estimate of cluster masses by means of the virial theorem in its version for spherically symmetric isolated systems.

Acknowledgements

We thank P.J.E. Peebles for a fruitful discussion, and M. Castillo and J. Carvalho for their aid in a preliminary study. This work has been supported by the Direccin General de Investigacin Científica y Técnica, under contract PB89-0246, and PB90-0448.
APPENDIX A

Multiple Tidal Interaction

The probability $P(N, \beta)\ d\beta$ for a cluster of given richness $N$ to have axial ratio between $\beta$ and $\beta + d\beta$ is equal to the sum of probabilities of this axial ratio being caused by one single neighbor in excess relative to the mean cluster density, $P_1(N, \beta)\ d\beta$, by a group of two neighbors in excess, $P_2(N, \beta)\ d\beta$, by a group of three neighbors in excess, $P_3(N, \beta)\ d\beta$, etc... This multiplicity of galaxies in excess refers to inside spheres $\delta V$ centered on one neighboring cluster (which guarantees some net excess of clusters in the sphere) and with radius equal to $s$.

The probability $P_1(N, \beta)\ d\beta$ is given by equation (22), but for the fact that the probability $P(N, \beta)$ includes the condition that the perturbing neighbor be really isolated in the sphere once it has been corrected from the mean cosmological density of clusters, and the probability $P_{neig}(N, \leq \beta)$ is replaced by the probability that there is no other isolated nor grouped neighbors in excess able to produce a smaller value of $\beta$. On the other hand, the probability $P_2(N, \beta)\ d\beta$ is given by an equation similar to (22) with the product in the right-hand-side member involving the probability of finding a group of two systems in excess able to jointly produce the wanted axial ratio or, equivalently, the corresponding additive value of $G(\beta)$, and the probability of this tidal action be dominant, which coincides with the corresponding probability for the isolated case; we shall write this probability as $P(N, \leq \beta)$. And so on. Therefore, we have

$$P(N, \beta)\ d\beta = \left[ P_1(N, \beta) + P_2(N, \beta) + \ldots \right] P(N, \leq \beta)\ d\beta.$$  \hspace{1cm} (A1)

The probability $P(N, \leq \beta)$ is given by

$$P(N, \leq \beta) = \exp\left\{-4\pi n_c \int_0^\infty ds\ s^2\ (1 + \xi_{cc}(s))\ e^{-\nu(s)} \right\} \times \left[ N_1(N, \geq G(\beta), s) + \frac{1}{2} \nu(s) N_2(N, \geq G(\beta), s) + \frac{1}{3} \frac{\nu^2(s)}{2!} N_3(N, \geq G(\beta), s) + \ldots \right\},$$  \hspace{1cm} (A2)

with

$$\nu(s) = n_c \int_{\delta V} \frac{\xi_{cc}(s) + \xi_{cc}(r) + \xi_{cc}^2(s) + 2 \xi_{cc}(s) \xi_{cc}(r)}{1 + \xi_{cc}(s)} \ dV$$  \hspace{1cm} (A3)

giving the mean number of neighbors in excess, apart from the central one, within a sphere $\delta V$ centered on a galaxy at $s$ from the perturbed one. In equation (A3) we have
taken the reduced three-point correlation function in the simple form of the Kirkwood superposition consistent with observation (Jing & Valdarnini 1991 and references therein), and we have approximated the two-point correlation of all neighbors in each sphere with respect to the perturbed cluster by that of the central neighbor (warranting in this way the necessary spherical symmetry of the density distribution inside the spheres). Note that the different terms in the exponent of equation (A2) contain a factor (e.g., \(\frac{1}{2}, \frac{1}{3}, \text{etc...}\)) which corrects from repeating the configurations when integrating over the spatial location of the central neighbor. Functions \(N_i(N, G(\beta), s)\) in equation (A2) give the probability that a group of \(i\) neighbors located at \(s\) be able to yield an axial ratio smaller than or equal to \(\beta\) (or, equivalently, a value of the additive quantity \(G\) greater than or equal to \(G(\beta)\)). In terms of the cluster richness function they write

\[
\mathcal{N}_i(N, \geq G(\beta), s) = \mathcal{N}_c(\geq N_{\text{neig}}(N, \beta, s)),
\]  

(A4a)

and the iterative relation for \(i \geq 2\)

\[
\mathcal{N}_i(N, \geq G(\beta), s) = \mathcal{N}_c(\geq N_t) \mathcal{N}_{i-1}(N, \geq (G(\beta) - G_t), s) \int_{N_t}^{\sqrt{N_{\text{neig}}^2(N, G(\beta), s) - N_t^2}} \mathcal{N}_c(\tilde{N}) \mathcal{N}_{i-1}(N, \geq (G(\beta) - \tilde{G}), s) d\tilde{N} \quad (A4b)
\]

(with \(G_t\) and \(\tilde{G}\) standing for the values of \(G\) yielded by a neighbor of richnesses \(N_t\) and \(\tilde{N}\), respectively) for \(N_{\text{neig}}(N, \beta, s) \geq \sqrt{2} N_t\), and \(\mathcal{N}_i(N, \geq G(\beta), s) = 1\) for \(N_{\text{neig}}(N, \beta, s) > \sqrt{2} N_t\) (notice that in relation [A4b] \(\mathcal{N}_c(\geq N_t)\) is actually equal to one). Given that these expressions make it hard to calculate the exponent of the right-hand-side of equation (A2) it is better to take, for \(i \geq 3\), some approximate but more practical expressions. The very steep cluster richness function allows us to write

\[
\int_{N_t}^{\sqrt{N_{\text{neig}}^2(N, G(\beta), s) - N_t^2}} \mathcal{N}_c(\tilde{N}) \mathcal{N}_{i-1}(N, \geq (G(\beta) - \tilde{G}), s) d\tilde{N} \approx \\
\mathcal{N}_{i-1}(N, \geq G(\beta) - G_t, s) \mathcal{N}_c(\geq N_t) - \mathcal{N}_{i-1}(N, \geq (G(\beta) - G_t), s),
\]

(A5)

so that the iterative relation (A4b) takes the simple form

\[
\mathcal{N}_i(N, \geq G(\beta), s) \approx \mathcal{N}_1(N, \geq (G(\beta) - G_t), s) \left[ 2 - \mathcal{N}_1(N, \geq (G(\beta) - G_t), s) \right]^{i-1}. \quad (A6)
\]

(Note that for \(N_{\text{neig}}(N, \beta, s) > \sqrt{2} N_t\) one has \(G(\beta) - G_t \leq G_t\), and functions \(\mathcal{N}_i(N, \geq G(\beta), s)\) given by equation [A6] reduce to unity as required.) Finally, by substituting functions \(\mathcal{N}_i(N, \geq G(\beta), s)\) given by the approximate relations (A6) into equation (A2) we are led to

\[
\mathcal{P}(N, \leq \beta) = \exp \left\{ -4\pi n_c \int_0^\infty ds \, s^2 \left( 1 + \xi_{cc}(s) \right) e^{-\nu(s)} \left[ \mathcal{N}_1(N, \geq G(\beta), s) \right] \right\}
\]
\[-N_1(N, \geq (G(\beta) - G_t), s) + \frac{N_1(N, \geq (G(\beta) - G_t), s)}{\nu(s)[2 - N_1(N, \geq (G(\beta) - G_t), s)]} \times \left( e^{\nu(s)[2-N_1(N, \geq (G(\beta) - G_t), s)]} - 1 \right) \}\right]. \tag{A7}

It is worthwhile pointing out that approximation (A5) slightly underestimates the elongations of the exact solution.

Like in the one-single-neighbor version (see § 3), the sum \(P_1(N, \beta) + P_2(N, \beta) + \ldots\) in equation (A1) is equal to the partial derivative with respect to \(\beta\) of the exponent in the right-hand-side member of equation (A2). So we always have the practical relation

\[P(N, \geq \beta) = \overline{P}(N, \leq \beta), \tag{A8}\]

whose integration with respect to \(N\), appropriately weighted by \(N_c(N)\), and subsequent derivation with respect to \(\beta\) readily leads to the wanted distribution of intrinsic axial ratios.

The volume of spheres \(\delta V\) increases indefinitely with increasing \(s\). But the number of neighbors in excess from the uniform mean density inside them stabilizes when a distance is reached for which the radius of the corresponding sphere becomes larger than the maximum length of positive correlation of clusters, \(i.e.,\) the typical size of super-clusters. Thus, the probability of finding a group of \(i\) neighbors in excess within these spheres with appropriate richnesses to yield the wanted axial ratio will diminish very rapidly beyond that distance, which limits the distance of significant tidal interaction (see § 4).
APPENDIX B

Potential Energy of the Galactic Component

The potential energy tensor of a self-gravitating system is equal to

\[ U_{ij} = -\frac{1}{2} G \int \int \rho(x) \rho(x') \frac{(x_i - x'_i) (x_j - x'_j)}{|x - x'|} \, dx \, dx'. \]  

(B1)

Taking advantage of factor 1/2, in the case of self-similar ellipsoids (i.e., \( \rho = \rho(m) \), with \( m^2 = \frac{x^2}{a_1^2} + \frac{x^2}{a_2^2} + \frac{x^2}{a_3^2} \)) equation (B1) can be written in the practical form (Roberts 1962)

\[ U_{ij} = -\int \int \rho(m') \Phi_{ij}(m) \, dm \, dm', \]  

(B2)

with \( \Phi_{ij}(m) \) the tensor potential produced by the homoeoid \((m, m + dm)\). But the tensor potential of an homoeoid is constant inside it, so that expression (B2) allows one to readily calculate the potential energy tensor, the result being equation (13) (Roberts 1962). This development is, of course, only possible if the component for which we want to calculate the potential energy tensor is self-gravitating (otherwise there would not be factor 1/2). Since the density distribution of the galactic component of a galaxy cluster \( \rho^g \) does not cause by itself the whole tensor potential \( \Phi_{ij} \) but only contributes marginally to it, we cannot apply this method.

To calculate the potential energy tensor of the galactic component let us decompose the total density \( \rho \) into the sum of two densities, that of galaxies, \( \rho^g \), and the remaining one \( \rho^r = \rho - \rho^g \). Then the integral on the right-hand-side of equation (B1) decomposes in a sum of three integrals involving, respectively, half the product \( \rho^r \rho^r \), half the product \( \rho^g \rho^g \), and product \( \rho^r \rho^g \). The sum of the latter two integrals exactly gives the wanted potential energy tensor \( U_{ij}^g \). Therefore, this can be calculated as the total potential energy tensor \( U_{ij} \) given by equation (B1) minus the potential energy tensor \( U_{ij}^r \) associated to a self-gravitating fluid with density \( \rho^r \) for which the method above also applies. By doing so we are led to

\[ U_{ij}^g = -2\pi G \rho(0) A_i(\beta) I_{ij}^g f_U^g, \]  

(B3)

where we have approximated the axial ratio of component \( r \) to that of the whole mass distribution (this is a very good approximation because the shape of the density profile of the former is essentially equal to that of the whole mass), with \( I_{ij}^g \) defined as \( I_{ij} \) (eq. [12]) in terms of the density profile \( \rho^g \), and \( f_U^g \) given by

\[ f_U^g = \frac{f_{ij} f_U - f_{ij} f'_U}{f'_i}. \]  

(B4)
For simplicity, the present proof has been carried out for the potential energy tensor of the galactic component in the whole system and not in a limited part of it. The changes that should be introduced in this latter case are given in § 3.
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Figure Captions

**Figure 1.** Theoretical distributions of intrinsic (a) and projected (b) axial ratios predicted by the model of tidal interaction in the simplest one-single-neighbor version. In panel (c) we plot the same solution as in panel (b), but conveniently degraded in order to simulate the effects of measuring errors. The histograms in panels (b) and (c) give the empirical distribution of projected axial ratios obtained by Plionis et al. (1991).

**Figure 2.** Height of the Dirac delta at $\beta = 0.4963$ in the predicted distribution of intrinsic axial ratios of Figure 1a as a function of the threshold richness $N_t$ of the perturbing clusters.

**Figure 3.** Same as Figure 1, but for the more accurate version of the model dealing with the tidal action of single and grouped clusters.