Adiabatic Approximation in the Density Matrix Approach: Non-Degenerate Systems

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Abstract

We study the adiabatic limit in the density matrix approach for a quantum system coupled to a weakly dissipative medium. The energy spectrum of the quantum model is supposed to be non-degenerate. In the absence of dissipation, the geometric phases for periodic Hamiltonians obtained previously by M.V. Berry are recovered in the present approach. We determine the necessary condition satisfied by the coefficients of the linear expansion of the non-unitary part of the Liouvillian in order to the imaginary phases acquired by the elements of the density matrix, due to dissipative effects, be geometric. The results derived are model-independent. We apply them to spin $\frac{1}{2}$ model coupled to reservoir at thermodynamic equilibrium.
1 Introduction

Since the fascinating work by Berry in ’84 [1] showing the existence of geometric phases (path-dependent phases) in vector states driven by adiabatic periodic Hamiltonians, authors in the literature have looked for geometric phases in other physical contexts. In particular, Joye et al. [2] and Berry [3] independently showed that the transition probability of instantaneous eigenstates of non-real Hamiltonians in the non-adiabatic regime gets an imaginary phase. This imaginary phase was measured by Zwanziger et al. for a two-level system [4]. The appearance of an imaginary correction to the geometric phase in quantum models coupled to dissipative media has been discussed in the literature [5, 6, 7]. In those references, the non-unitary evolution of the quantum system is implemented by a phenomenological non-hermitian Hamiltonian. This phenomenological approach has been extensively applied to the study of the properties of open quantum systems [8, 9].

Recently we have considered a spin $\frac{1}{2}$ model in the presence of an external magnetic field and coupled to a weakly dissipative medium [10]; this system precesses with constant angular velocity around a fixed axis. We applied two Lindblad operators to represent the non-unitary part of the Liouvillians of the quantum system in contact with two distinct reservoirs. Using the master equations for these models, we concluded that the geometric phases acquired by the spin $\frac{1}{2}$ instantaneous eigenstates of the Hamiltonian were not modified by the presence of the dissipation. The effective result of the interaction of the quantum system with the reservoir is the shrinking of the Bloch vector, which can be used to give a geometric description to the density matrix [11, 12, 13, 14].

Certainly it is still an open question whether our results in reference [10] are of general nature or particular to the dissipative models studied. We remind that they are in opposition to the ones derived by the non-hermitian Hamiltonian approach [8, 9, 10]. In order to prove that the results of [10] are valid in general for reservoirs at equilibrium, we must have a model-independent approach. We should say that master equations are the proper way to study an open quantum system, whereas phenomenological non-hermitian Hamiltonians are supposed to give a “bone fide” description of an open system only when the coherence of states is not destroyed by the interaction of the system with its neighborhood. We point out that the general expression for the imaginary phase acquired by the non-hermitian Hamiltonians is given by eqs. (75) and (76) of reference [8], from which we conclude that this imaginary phase has been geometric for any non-hermitian Hamiltonian, as derived in references [8, 9] and [10] for some specific models. However, by the end of reference [8] Garrison and Wright affirm that their result should be checked out by a
density matrix approach.

In reference [10] the density matrix already appears as our central object. However, even there our explanation is based on the time evolution of the instantaneous eigenstates of Hamiltonian. In the present paper, we rederive the evolution of a quantum system in the adiabatic approximation directly in the density matrix formulation, which is the natural approach in the study of quantum systems coupled to dissipative environment.

The Adiabatic Theorem discussed in references [15, 16, 17] applies to quantum systems driven by unitary evolution. It states that if initially the system is in an eigenstate of the Hamiltonian, its time evolution is fasten, at each time, to an instantaneous eigenstate of Hamiltonian with the same original quantum numbers. Therefore, it is a natural choice to write our density matrix in the basis of the instantaneous eigenstates of the Hamiltonian and derive its adiabatic limit.

The dynamics of the operator density $\rho(t)$ is given by a Liouville-von Neumann equation. In the Liouvillian we add a non-unitary term to take into account the interaction with a dissipative medium, that is

$$
\frac{d\rho(t)}{dt} = -i[H(t), \rho(t)] + \mathcal{L}_D \rho(t),
$$

where $H(t)$ is the time-dependent Hamiltonian of the quantum system and $\mathcal{L}_D$ is a superoperator that acts on $\rho(t)$ and is responsible for the non-unitary evolution of the quantum system.

In the basis of the instantaneous eigenstates of the Hamiltonian $H(t)$ (i.e., $H(t)|u_i; t\rangle = E_i(t)|u_i; t\rangle$), the dynamical equations of the elements of the density matrix are

$$
\frac{d\rho_{ij}^H(t)}{dt} = \sum_l \left( \frac{d\langle u_i; t|}{dt} |u_l; t\rangle \rho_{lj}^H(t) + \rho_{il}^H(t) \langle u_i; t| \frac{d|u_j; t\rangle}{dt} \right) \ - i(E_i(t) - E_j(t))\rho_{ij}^H(t) + \langle u_i; t| \mathcal{L}_D \rho(t) |u_j; t\rangle,
$$

where $\rho_{ij}^H(t) \equiv \langle u_i; t| \rho(t) |u_j; t\rangle$. Not all elements of matrix $\rho_{ij}^H(t)$ are independent, due to the constraints that $\text{Tr}[\rho(t)] = 1$ and $\rho_{ij}^H(t)$ be a hermitian matrix.

The time-dependent Hamiltonian that drives the quantum system takes into account its effective interaction with its neighborhood. This interaction is realized through a set of time-dependent external parameters $\vec{R}(t) = (R_1(t), R_2(t), \cdots, R_m(t))$. By hypothesis, the time-dependence of Hamiltonian comes only from the external parameters $(H(t) =$
For a periodic Hamiltonian, there are in general two distinct time scales: the time scale $T_{ij}$ associated to transitions between two instantaneous eigenstates $i$ and $j$ of the Hamiltonian (being typically of the same order of magnitude) and the time scale (period) $T$, associated to the periodicity of the external parameters ($\mathbf{R}(T) = \mathbf{R}(0)$).

Our aim is to derive the dynamical equations satisfied by the elements $\rho^H_{ij}(t)$ in the limit $T_{ij} \ll T \to \infty$ [13, 16, 17]. We consider a quantum system with non-degenerate energy spectrum. To present the main details of the calculation, in section 2 we take the simplest situation when the system is not coupled to a dissipative medium, in which case we have to recover the known results of literature [1]. In section 3, this non-degenerate quantum system is coupled to a weakly dissipative medium and the density matrix has to be used to describe the quantum behavior of the system. In subsection 3.1, we discuss the conditions under which the quantum system could acquire an imaginary geometric phase due to the presence of dissipation. In section 4 we apply the results of section 3 to the spin $\frac{1}{2}$ model coupled to two particular reservoirs at thermodynamic equilibrium to verify the nature (time-dependent or path-dependent) of their imaginary phases. Finally, in section 5, we present our conclusions.

2 The Quantum Systems in the Absence of Dissipation

In the absence of dissipation, the time evolution of the quantum system in the adiabatic limit is completely described by its vector state. The adiabatic evolution of vector states is well done in references [13, 16, 17]. The appearance of the real geometric phase in the unitary evolution of vector states is beautifully described in reference [1].

Our aim in this section is to consider a simpler physical situation to present the main details of the calculations of getting the adiabatic limit directly from the matrix density approach. This have the advantage of checking the correctness of our results.

Eq. (2) in the absence of a coupling with a dissipative medium becomes

\[
\frac{d\rho^H_{ij}(t)}{dt} = -i \left[ (E_i(t) - E_j(t)) + i(-\langle u_i; t | \frac{d}{dt} | u_i; t \rangle + \langle u_j; t | \frac{d}{dt} | u_j; t \rangle) \right]\rho^H_{ij}(t) -
\sum_{i \neq j} \frac{\langle u_i; t | \frac{dH(t)}{dt} | u_i; t \rangle}{E_i(t) - E_i(t)} \rho^H_{ij}(t) + \sum_{i \neq j} \frac{\langle u_i; t | \frac{dH(t)}{dt} | u_j; t \rangle}{E_j(t) - E_i(t)} \rho^H_{ij}(t).
\]

(3)

Since we intend to get the adiabatic limit of the elements of the density matrix, we
proceed as usual [4, 13, 18] and factorize the dynamical phase in \( \rho^H_{ij}(t) \),

\[
\tilde{\rho}_{ij}(t) \equiv e^{i \int_0^t dt' (E_i(t') - E_j(t'))} \rho^H_{ij}(t).
\] (4)

We distinguish three types of dynamical equations for the elements \( \tilde{\rho}_{ij}(t) \):

i) for the diagonal elements \( \tilde{\rho}_{ii}(t) \), where \( i = 1, 2, \ldots, N - 1 \), namely,

\[
\frac{d\tilde{\rho}_{ii}(t)}{dt} = \sum_{l=1}^{i-1} \frac{1}{E_i(t) - E_l(t)} 2\text{Re} \left[ \langle u_i; t | \frac{dH(t)}{dt} | u_i; t \rangle e^{i \int_0^t dt' (E_i(t') - E_l(t'))} \langle \tilde{\rho}_{ii}(t) \rangle^* \right] + \\
+ \sum_{l=i+1}^{N} \frac{1}{E_i(t) - E_l(t)} 2\text{Re} \left[ \langle u_i; t | \frac{dH(t)}{dt} | u_i; t \rangle e^{-i \int_0^t dt' (E_i(t') - E_l(t'))} \tilde{\rho}_{il}(t) \right];
\] (5a)

ii) for the elements \( \tilde{\rho}_{ij}(t) \), where \( i = 1, 2, \ldots, N - 1 \) and \( j = i + 1, \ldots, N - 1 \), namely,

\[
\frac{d\tilde{\rho}_{ij}(t)}{dt} = \left[ - \langle u_i; t | \frac{d|u_i; t\rangle}{dt} | u_j; t \rangle + \langle u_j; t | \frac{d|u_j; t\rangle}{dt} | u_i; t \rangle \right] \tilde{\rho}_{ij}(t) + \\
+ \sum_{l=1}^{j} \frac{\langle u_i; t | \frac{dH(t)}{dt} | u_l; t \rangle e^{-i \int_0^t dt' (E_i(t') - E_l(t'))} \tilde{\rho}_{lj}(t) +}{E_i(t) - E_l(t)} \\
+ \sum_{l=j+1}^{N} \frac{\langle u_i; t | \frac{dH(t)}{dt} | u_l; t \rangle e^{-i \int_0^t dt' (E_i(t') - E_l(t'))} \langle \tilde{\rho}_{lj}(t) \rangle^* +}{E_i(t) - E_l(t)} \\
+ \sum_{l=1}^{i-1} \frac{\langle u_j; t | \frac{dH(t)}{dt} | u_l; t \rangle e^{i \int_0^t dt' (E_j(t') - E_l(t'))} \langle \tilde{\rho}_{li}(t) \rangle^* +}{E_j(t) - E_l(t)} \\
+ \sum_{l=j+1}^{N} \frac{\langle u_j; t | \frac{dH(t)}{dt} | u_l; t \rangle e^{i \int_0^t dt' (E_j(t') - E_l(t'))} \tilde{\rho}_{li}(t);}{E_j(t) - E_l(t)}
\] (5b)

iii) for the elements \( \tilde{\rho}_{iN}(t) \), where \( i = 1, 2, \ldots, N - 1 \), namely,
whereas eq. (5c) can be written as a Volterra integral equation of the second type.

Eqs. (5a) and (5b) can be written as Volterra integral equations \[19\] of the first type, whereas eq. (5c) can be written as a Volterra integral equation of the second type.

Following references \[14\] and \[18\] we redefine the time scale using the period \( T \) of the external parameters

\[
s = \frac{t}{T}. \tag{6}\]

In this new variable \( s \), the integral equations obtained from eqs. (5) are:

i) for the diagonal elements \( \tilde{\rho}_{ii}(s) \), where \( i = 1, 2, \ldots, N - 1 \),

\[
\begin{align*}
\frac{d\tilde{\rho}_{ii}(t)}{dt} &= -\left[ \langle u_i; t \rangle \frac{d[u_i; t]}{dt} - \langle u_N; t \rangle \frac{d[u_N; t]}{dt} \right]\tilde{\rho}_{ii}(t) + \\
&- \sum_{l=1}^{N-1} \frac{\langle u_i; t \rangle [dH(t)]_N u_N(t)}{E_i(t) - E_N(t)} e^{i \int_0^t dt' (E_i(t') - E_N(t'))} \tilde{\rho}_{ii}(t) + \\
&+ \sum_{l=1}^{N-1} \frac{\langle u_i; t \rangle [dH(t)]_N u_i(t)}{E_i(t) - E_i(t)} e^{-i \int_0^t dt' (E_i(t') - E_i(t'))} \tilde{\rho}_{ii}(t) + \\
&+ \sum_{l=1}^{i-1} \frac{\langle u_i; t \rangle [dH(t)]_N u_N(t)}{E_i(t) - E_i(t)} e^{i \int_0^t dt' (N E_i(t') - E_N(t'))} (\tilde{\rho}_{ii}(t))^* + \\
&+ \sum_{l=1}^{N-1} \frac{\langle u_i; t \rangle [dH(t)]_N u_i(t)}{E_i(t) - E_i(t)} e^{-i \int_0^t dt' (E_i(t') - E_i(t'))} \tilde{\rho}_{ii}(t) + \\
&+ \frac{\langle u_i; t \rangle [dH(t)]_N u_N(t)}{E_i(t) - E_N(t)} e^{i \int_0^t dt' (E_i(t') - E_N(t'))} \tilde{\rho}_{ii}(t). \tag{5c}
\end{align*}
\]

Eqs. (5a) and (5b) can be written as Volterra integral equations \[19\] of the first type, whereas eq. (5c) can be written as a Volterra integral equation of the second type.

Following references \[14\] and \[18\] we redefine the time scale using the period \( T \) of the external parameters

\[
s = \frac{t}{T}. \tag{6}\]

In this new variable \( s \), the integral equations obtained from eqs. (5) are:

i) for the diagonal elements \( \tilde{\rho}_{ii}(s) \), where \( i = 1, 2, \ldots, N - 1 \),

\[
\begin{align*}
\tilde{\rho}_{ii}(s) &= \tilde{\rho}_{ii}(0) + \\
&+ \sum_{l=1}^{i-1} \int_0^s ds' \frac{1}{E_i(s') - E_i(s')} 2 \text{Re} \left[ \langle u_i; s' \rangle [dH(s')]_i u_i; s' \rangle e^{iT \int_0^s ds''(E_i(s'') - E_i(s'))} (\tilde{\rho}_{ii}(s'))^* \right] + \\
&+ \sum_{l=i+1}^{N} \int_0^s ds' \frac{1}{E_i(s') - E_i(s')} 2 \text{Re} \left[ \langle u_i; s' \rangle [dH(s')]_i u_i; s' \rangle e^{-iT \int_0^s ds''(E_i(s'') - E_i(s'))} \tilde{\rho}_{ii}(s') \right]; \tag{7a}
\end{align*}
\]

ii) for the elements \( \tilde{\rho}_{ij}(s) \), where \( i = 1, 2, \ldots, N - 1 \) and \( j = i + 1, \ldots, N - 1 \),
\[ \tilde{\rho}_{ij}(s) = \tilde{\rho}_{ij}(0) + \int_{0}^{s} ds' \left[ -\langle u_i; s'| \frac{d|u_j; s'}{ds'} \rangle + \langle u_j; s'| \frac{d|u_i; s'}{ds'} \rangle \right] \tilde{\rho}_{ij}(s') + \int_{0}^{s} ds' \sum_{l=1 \atop l \neq i}^{i} \int_{0}^{s} ds'' \langle u_i; s'| \frac{dH(s')}{ds'} \rangle \frac{u_j; s''}{E_i(s') - E_i(s'')} e^{-iT \int_{0}^{s} ds''(E_i(s'') - E_i(s''))} \tilde{\rho}_{ij}(s'') + \int_{0}^{s} ds' \sum_{l=1 \atop l \neq j}^{j} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_j; s''}{E_j(s') - E_j(s'')} e^{iT \int_{0}^{s} ds''(E_i(s'') - E_i(s''))} \tilde{\rho}_{ij}(s'') + \int_{0}^{s} ds' \sum_{l=1 \atop l \neq j}^{i} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_j; s''}{E_j(s') - E_j(s'')} e^{-iT \int_{0}^{s} ds''(E_i(s'') - E_i(s''))} \tilde{\rho}_{ij}(s'') + \int_{0}^{s} ds' \sum_{l=1 \atop l \neq i}^{i} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_j; s''}{E_i(s') - E_i(s'')} e^{iT \int_{0}^{s} ds''(E_i(s'') - E_i(s''))} \tilde{\rho}_{ij}(s''); \] (7b)

iii) for the elements \( \tilde{\rho}_{iN}(s) \), where \( i = 1, 2, \ldots, N - 1 \),

\[ \tilde{\rho}_{iN}(s) = \tilde{\rho}_{iN}(0) - \int_{0}^{s} ds' \left[ \langle u_i; s'| \frac{d|u_N; s'}{ds'} \rangle - \langle u_N; s'| \frac{d|u_i; s'}{ds'} \rangle \right] \tilde{\rho}_{iN}(s') - \int_{0}^{s} ds' \sum_{l=1}^{N-1} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_N; s''}{E_i(s') - E_N(s'')} e^{iT \int_{0}^{s} ds''(E_i(s'') - E_N(s''))} \tilde{\rho}_{iN}(s'') + \int_{0}^{s} ds' \sum_{l=1}^{N-1} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_N; s''}{E_i(s') - E_N(s'')} e^{iT \int_{0}^{s} ds''(E_i(s'') - E_N(s''))} \tilde{\rho}_{iN}(s'') + \int_{0}^{s} ds' \sum_{l=1}^{i-1} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_N; s''}{E_i(s') - E_i(s'')} e^{iT \int_{0}^{s} ds''(E_i(s'') - E_i(s''))} \tilde{\rho}_{iN}(s'') + \int_{0}^{s} ds' \sum_{l=1}^{i-1} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_N; s''}{E_i(s') - E_i(s'')} e^{iT \int_{0}^{s} ds''(E_i(s'') - E_i(s''))} \tilde{\rho}_{iN}(s'') + \int_{0}^{s} ds' \sum_{l=1}^{i-1} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_N; s''}{E_i(s') - E_i(s'')} e^{iT \int_{0}^{s} ds''(E_i(s'') - E_i(s''))} \tilde{\rho}_{iN}(s'') + \int_{0}^{s} ds' \sum_{l=1}^{i-1} \frac{\langle u_i; s'| [dH(s')/ds'] \rangle u_N; s''}{E_N(s') - E_i(s'')} e^{iT \int_{0}^{s} ds''(E_N(s'') - E_i(s''))} \tilde{\rho}_{iN}(s''). \] (7c)

Some of the integrals on the r.h.s. of eqs. (7) are of the following type
where \( i \neq l \) and \( \tilde{\rho}(s') \) representing one element of the density matrix.

By defining

\[
    g_{il}(s) = \int_0^s ds' (E_i(s') - E_l(s'))
\]

we recognize \( I_{il}(s) \) as the Stieltjes integral \([20]\)

\[
    I_{il}(s) = \int_0^s ds' F_{il}(s') e^{iT g_{il}(s')} \quad \text{so that} \quad F_{il}(s') = \frac{\langle u_i; s' | [\frac{dH(s')}{ds}] | u_l; s' \rangle}{\dot{g}_{il}(s')} \tilde{\rho}(s'),
\]

and \( \dot{g}_{il}(s) = \frac{dg_{il}(s)}{ds} \) (we are not summing over the indices \( i \) and \( l \)).

Assuming that \( F_{il}(s') \) is a piece-wise continuous function in the interval \([0, s]\), then the Riemann-Lebesgue Theorem \([21]\) gives the value of \( I_{il}(s) \) in the adiabatic limit \((T \to \infty)\)

\[
    \lim_{T \to \infty} \int_0^s ds' F_{il}(s') e^{iT g_{il}(s')} dg_{il}(s') = 0,
\]

which means that for large but finite values of \( T \), the r.h.s. of eq. \((10)\) can be written in terms of the inverse powers of \( T \). Integrating eq. \((10)\) by parts, we get

\[
    I_{il}(s) = \frac{1}{iT} \left[ \frac{F_{il}(s)}{\dot{g}_{il}(s)} e^{iT g_{il}(s)} \right]_s^0 - \frac{1}{iT} \int_0^s ds' e^{iT g_{il}(s')} \frac{d(F_{il}(s')/\dot{g}_{il}(s'))}{ds'}
\]

\[
= \frac{1}{iT} \left[ \frac{F_{il}(s)}{\dot{g}_{il}(s)} e^{iT g_{il}(s)} \right]Id_{s=0} + o\left(\frac{1}{iT}\right).
\]

From the result \((12)\) we obtain that the elements \( \tilde{\rho}_{ij}(s) \) can be expanded in powers of \((\frac{1}{T})\) \([18]\),

\[
    \tilde{\rho}_{ij}(s) = \sum_{n=0}^{\infty} \left(\frac{i}{T}\right)^n \tilde{\rho}_{ij}^{(n)}(s),
\]

where \( \tilde{\rho}_{ij}^{(n)}(s) \) is the coefficient of order \((\frac{1}{T})^n\) in the expansion.
We are interested in the adiabatic limit \((T \to \infty)\). This correspond to substitute expansion \((T)\) in eqs. \((7)\) and keeping only terms of order \((\frac{1}{T})^0\) in the differential equations. At this order, the equations become:

i) for the diagonal elements \(\tilde{\rho}_{ii}(s), i = 1, 2, \cdots, N - 1\).

\[
\frac{d\tilde{\rho}_{ii}(s)}{ds} = 0 \quad \Rightarrow \quad \tilde{\rho}_{ii}^H(s) = \tilde{\rho}_{ii}^H(0). \tag{14}
\]

This result gives us the meaning of the Adiabatic Theorem in the density matrix approach: the population of an instantaneous eigenstate of the Hamiltonian does not change in an adiabatic process.

ii) for the elements \(\tilde{\rho}_{ij}(s), \) where \(i = 1, 2, \cdots, N - 1\) and \(j = i + 1, \cdots, N\).

\[
\frac{d\tilde{\rho}_{ij}^{(0)}(s)}{ds} = \left[ -\langle u_i; s| \frac{d|u_i; s\rangle}{ds} \rangle + \langle u_j; s| \frac{d|u_j; s\rangle}{ds} \rangle \right] \tilde{\rho}_{ij}^{(0)}(s) \tag{15}
\]

whose solution in the variable \(t\) is

\[
\rho_{ij}^H(t) = e^{\int_0^t dt' \left[ -\langle u_i; t'| \frac{d|u_i; t'\rangle}{dt'} \rangle + \langle u_j; t'| \frac{d|u_j; t'\rangle}{dt'} \rangle \right]} e^{-i \int_0^t dt' (E_i(t') - E_j(t'))} \rho_{ij}^H(0). \tag{16}
\]

Global phases do not contribute to the density matrix. Eq. \((16)\) is compatible with eq. \((14)\) since the element \(\rho_{ij}^H(t)\) is only different from zero if in the initial state we already have a superposition of the \(i^{th}\) and \(j^{th}\) instantaneous eigenstates of \(H(t)\). Finally, the difference of phases, either the geometrical and the dynamical phases, appears naturally in this formalism.

We notice that eq. \((16)\) includes the elements \(\rho_{iN}^H(t), i = 1, 2, \cdots, N - 1\), as well as the solution \((14)\) for the diagonal terms. The inhomogeneous term in eq. \((7c)\) gives a contribution of order \((\frac{1}{T})\); this is why it does not contribute to the dynamics of the elements \(\tilde{\rho}_{iN}(s)\) in the adiabatic limit.

The argument of the first exponential on the r.h.s. of eq. \((14)\) is equal to the difference of the geometric phases acquired by the instantaneous eigenstates \(i\) and \(j\) of the Hamiltonian \(H(t)\). For any closed path in the \(\mathbf{R}\)-space, each of those phases depends only on the path followed by the \(\mathbf{R}\)-parameters. As should be, in the density matrix formalism we recover the known results derived for the unitary evolution of instantaneous eigenstates of Hamiltonian in the adiabatic limit\([1]\).
3 The Quantum System in the Presence of Weak Dissipation

In general we are interested in studying a quantum system that is part of a whole system whose sub-systems interact with one another. This interaction allows the sub-systems to have exchanges among themselves. The traditional way to study a part of the whole system is taking a partial trace over all degrees of freedom of the complementary sub-system. These complementary degrees of freedom are called environment. In this approach we have an effective Hamiltonian that drives the dynamical evolution of the quantum system under study and at the same time the non-unitary part of the Liouvillian takes into account its interaction with the environment. In the general case the effective Hamiltonian depends on a set of time-dependent classical parameters.

In reference [10] we questioned whether the presence of dissipation could introduce an imaginary geometric phase in systems driven by periodic Hamiltonians. We considered a two-level model in the presence of two distinct Lindblad operators representing reservoirs, and concluded that in those cases the imaginary phases are not geometric and, based on eq. (19) of reference [10], we affirmed that the nature of the imaginary phase depends on its origin. Certainly, this last affirmation has to be confirmed by a model-independent approach.

The last term on the r.h.s. of eq. (2) introduces the effects of dissipation in the dynamics of the quantum system. We assume that the dissipation is weakly coupled to the quantum system. In general, the non-unitary part of the Liouvillian is written in a time-independent basis. Let \( \{|v_l\rangle\} \) be this time-independent basis and, for weakly dissipative interaction, we have [22, 23]

\[
\langle v_i | \mathcal{L}_D \rho(t) | v_j \rangle = \sum_{l,m} \tilde{c}_{ij}^{lm}(t) \rho_{lm}(t),
\]

(17)

where \( \rho_{lm}(t) \equiv \langle v_l | \rho(t) | v_m \rangle \). The coefficients \( \tilde{c}_{ij}^{lm}(t) \) take into account the characteristics of the environment (in the case of a quantum system at thermodynamic equilibrium with a reservoir composed of an infinity set of harmonic oscillators, the coefficients \( \tilde{c}_{ij}^{lm}(t) \) take into account the distribution of frequencies, etc, but are time-independent). In the most general case of coupling with a dissipative medium we can be time dependent and consequently those coefficients can vary in time.

From the beginning we chose to write the density matrix in the basis of the instantaneous eigenstates of Hamiltonian \( H(t) \). In this basis, the non-unitary part of Liouvillian
is written as

\[ \langle u_i; t | \mathcal{L}_D \rho(t) | u_j; t \rangle = \sum_{l,m} c_{im}^{ij}(t) \rho_{lm}(t). \]  

(18)

The coefficients \( c_{im}^{ij}(t) \) on the r.h.s. of eq. (18) are obtained from \( \bar{c}_{lm}^{ij}(t) \) by making a similarity transformation in each of its indices, that is

\[ c_{im}^{ij}(t) = \langle u_i; t | v_{l_1} \rangle \langle u_i; t | v_{l_3} \rangle \bar{c}_{l_1 l_2 l_3 l_4}(t) \langle v_{l_2} | u_j; t \rangle \langle v_{l_4} | u_m; t \rangle, \]

(19)

where implicit sum over the indices \( l_1, l_2, l_3 \) and \( l_4 \) is meant. We point out that even in the case when the coefficients \( \bar{c}_{l_1 l_2 l_3 l_4}(t) \) are time independent, their analogous in the instantaneous basis can acquire a time dependence through the transformation (19). From the scalar products in the transformation (19) the coefficients \( c_{im}^{ij}(t) \) do not get a dependence on the variation of Hamiltonian or of any other external classical parameter.

Following the same steps as we did in section 2, we obtain the dynamical equations for the elements \( \tilde{\rho}_{ij}(s) \) (see their definition in eq. (4)) in the limit \( T \to \infty \):

i) for the diagonal elements \( \tilde{\rho}_{ii}(s) \), \( i = 1, 2, \ldots, N - 1 \).

\[ \frac{d \tilde{\rho}_{ii}(t)}{dt} = \sum_{l=1}^{N-1} \left( c_{il}^{ii}(k(t)) - c_{NN}^{ii}(k(t)) \right) \tilde{\rho}_{ll}(t) + c_{NN}^{ii}(k(t)). \]

(20)

In writing the r.h.s. of eq. (20) we have already implemented the condition \( \text{Tr}(\rho(t)) = 1 \).

In the most general case the dissipation couples the dynamics of the diagonal elements of the density matrix. From eq. (20) we recover the adiabatic behavior of the quantum system in the presence of the dissipative medium when the non-unitary part of the Liouvillian has null diagonal terms in the instantaneous basis \( (c_{il}^{ii} = 0) \). In this situation the population of an instantaneous eigenstate of Hamiltonian does not vary along the adiabatic process. In this case the quantum system does not transfer to the environment energy due to electronic transitions.

The elements of matrix \( C(t) \) are defined as

\[ C_{il}(t) \equiv c_{il}^{ii}(t) - c_{NN}^{ii}(t), \]

(21)
The general solution of eq. (20) is
\[
\hat{\rho}_{ii}^H(t) = \left[ T \left( e^{\int_0^t \! dt' C(t')} \right) \right]_{ii} \left\{ \int_0^t \! dt' \left[ T \left( e^{\int_0^{t'} \! dt'' C(t'')} \right) \right]_{jk} c_{kk}^{NN}(t') + \hat{\rho}_{jj}^H(0) \right\}. \tag{22}
\]
The operator \( T \) means a time-ordering operator \[24\]. From eq. (19) we obtain that the coefficients \( c_{ii}^{ll}(t), i, l = 1, 2, \ldots, N \), are the same for any basis of the instantaneous eigenstates of Hamiltonian. This fact avoids any ambiguity in the imaginary phases in the time-ordering terms. In the general case the imaginary phases in eq. (22) are time-dependent. We postpone the discussion under what conditions the time-ordering integrals in eq. (22) can be rewritten as a path integrals in a suitable parameter space.

From eq. (22) we recover solution (14) in the absence of dissipation \( c_{ii}^{ll} = 0, i, l = 1, \ldots, N \).

\( ii \) for the elements \( \hat{\rho}_{ij}(s), i = 1, 2, \ldots, N - 1 \) and \( j = i + 1, \ldots, N \).

\[
\frac{d\hat{\rho}_{ij}(t)}{dt} = \left[ -\langle u_i; t | d|u_i; t \rangle dt + \langle u_j; t | d|u_j; t \rangle dt + c_{ij}^{ij}(k(t)) \right] \hat{\rho}_{ij}(t) + \\
+ \sum_{\{l,m\} \neq \{i,j\}} c_{ml}^{ij}(k(t)) \hat{\rho}_{lm}(t), \tag{23}
\]
where the set of pair of indices \( \{l, m\} \) are those which satisfy: \( E_i(t) - E_m(t) = E_i(t) - E_j(t) \).

To simplify our discussion, we order the instantaneous eigenenergies such that: if \( E_i(t) < E_j(t) \) then \( i < j \). Since the indices \( (i, j) \) of the elements of the density matrix in eq. (23) are chosen such that \( i < j \), we get that the elements \( \hat{\rho}_{lm}(t) \) that contribute to the r.h.s. of this equation are such that \( l < m \). Once the energy spectrum of the quantum system is non-degenerate, if the elements \( \hat{\rho}_{l_1m_1}(t) \) and \( \hat{\rho}_{l_2m_2}(t) \) that contribute to r.h.s. of eq. (23) are distinct then we must necessarily have \( l_1 \neq l_2 \) and \( m_1 \neq m_2 \).

Let us suppose that the dynamics of \( M \) elements \( \hat{\rho}_{ij}(t) \) are coupled by the presence of dissipation and their dynamics are given by eq. (23). Due to the fact that each pair \( (l, m) \) is unique we may relabel them by using only one index: \( (l_i, m_i), i = 1, 2, \ldots, M \).

Eq. (23) is rewritten as:
\[
\frac{d\hat{\rho}_{l_im_i}(t)}{dt} = \left[ -\langle u_{l_i}; t | d|u_{l_i}; t \rangle dt + \langle u_{m_i}; t | d|u_{m_i}; t \rangle dt \right] \hat{\rho}_{l_im_i}(t) + \sum_{k=1}^M c_{m_kl_k}^{l_im_i} \hat{\rho}_{l_km_k}(t), \tag{24}
\]
\( i = 1, 2, \ldots, M \). We distinguish two possible situations:

1) the dynamics of the elements \( \tilde{\rho}_{ij}(t) \) are not coupled by the coupling to a weakly dissipative medium: \( M = 1 \).

In this case, eq. (24) reduces to

\[
\frac{d\tilde{\rho}_{ij}(t)}{dt} = \left[ -\langle u_i; t | \frac{d}{dt}|u_i; t \rangle + \langle u_j; t | \frac{d}{dt}|u_j; t \rangle \right] \tilde{\rho}_{ij}(t) + c_{ij}^{ji} \tilde{\rho}_{ij}(t),
\]

(25)

whose solution is (see eq. (4))

\[
\rho_{ij}^{H}(t) = e^{-i \int_{0}^{t} dt' (E_i(t')-E_j(t'))} - \int_{0}^{t} dt' \left[ \langle u_i; t' | \frac{d}{dt} | u_i; t' \rangle - \langle u_j; t' | \frac{d}{dt} | u_j; t' \rangle \right] \rho_{ij}^{H}(0).
\]

(26)

Here again the coefficients \( c_{ij}^{ji}(t) \) that appear in the last phase on the r.h.s. of eq. (26) are independent of the chosen basis of the instantaneous eigenstates of Hamiltonian. In the general case this phase is time-dependent. In next sub-section we discuss the conditions satisfied by the coefficients \( c_{ij}^{ji}(t) \) in order to this integral becomes path-dependent.

Eq. (26) reduces to eq. (16) in the absence of interaction with a dissipative medium.

2) the dynamics of \( M \) elements \( \tilde{\rho}_{ij}(t) \) are coupled due to the presence of dissipation: \( M > 1 \).

Making the change of variables

\[
\tilde{\rho}_{lm}(t) = e^{-i \int_{0}^{t} dt' (E_i(t')-E_j(t'))} - \int_{0}^{t} dt' \left[ \langle u_i; t' | \frac{d}{dt} | u_i; t' \rangle - \langle u_l; t' | \frac{d}{dt} | u_l; t' \rangle \right] \tilde{\rho}_{lm}(t),
\]

(27)

where \( i = 1, 2, \ldots, M \), eq. (24) becomes

\[
\frac{d\tilde{\rho}_{lm}(t)}{dt} = A_{ik}(t) \tilde{\rho}_{lm}(t).
\]

(28)

The elements \( A_{ik}(t) \) of matrix \( A(t) \) are defined as

\[
A_{ik}(t) = e^{-i \int_{0}^{t} dt' \left[ \langle u_i; t' | \frac{d}{dt} | u_i; t' \rangle - \langle u_l; t' | \frac{d}{dt} | u_l; t' \rangle - \langle u_m; t' | \frac{d}{dt} | u_m; t' \rangle + \langle u_m; t' | \frac{d}{dt} | u_m; t' \rangle \right] + \langle u_i; t' | \frac{d}{dt} | u_i; t' \rangle - \langle u_l; t' | \frac{d}{dt} | u_l; t' \rangle - \langle u_m; t' | \frac{d}{dt} | u_m; t' \rangle + \langle u_m; t' | \frac{d}{dt} | u_m; t' \rangle}. \]

(29)

The solution of eq. (28) is
\[ \rho_{l_{i}m_{i}}(t) = \left[ \mathcal{T}\left( e^{\int_{t_{0}}^{t} A(t') dt'} \right) \right]_{ij} \rho_{j_{m}r_{j}}(0), \]  
\( \mathcal{T} \) being the time-ordering operator \[24\].

From eqs. (4) and (27) we finally have

\[ \rho_{H_{l_i}m_i}(t) = e^{-i \int_{t_{0}}^{t} dt' (E_{l_{i}}(t') - E_{m_{i}}(t'))} e^{-\int_{t_{0}}^{t} dt' \left[ \langle u_{l_{i}} | t' \frac{d}{dt} | u_{l_{i}} \rangle + \langle u_{m_{i}} | t' \frac{d}{dt} | u_{m_{i}} \rangle \right]} \times \]

\[ \times \left[ \mathcal{T}\left( e^{\int_{t_{0}}^{t} A(t') dt'} \right) \right]_{ij} \rho_{H_{j}m_r}(0). \]  

By choosing a new basis of eigenstates of Hamiltonian the elements \( A_{ik}(t) \) get an irrelevant real phase that does not contribute to the average value of any physical operator. As in the two previous discussion, the time-ordering integral that appears on the r.h.s. of eq. (31) due to the presence of dissipation is time-dependent. In the next sub-section we give the conditions necessary for this imaginary phase to be path-dependent of a suitable set of time dependent parameters.

Eq. (16) is recovered from eq. (31) in the absence of a coupling with a dissipative medium.

### 3.1 Conditions to Obtain an Imaginary Geometric Phase

In the most general case, eqs. (22), (26) and (31) do not give an imaginary geometric phase correction to the real Berry phase\[1\]. In this subsection we discuss the mathematical requirements that the coefficients \( c_{m_{k}}^{ij}(k(t)) \) have to satisfy in order to those imaginary phases are geometric (path-dependent).

Differently from the real phases acquired by the evolution of the instantaneous eigenstates of Hamiltonian in the adiabatic limit\[13, 16, 17\], the imaginary phases in eqs. (22), (26) and (31) have no ambiguity due to the arbitrariness of the basis of the instantaneous eigenstates of Hamiltonian. Consequently in the case of the imaginary phases acquired by the elements of the density matrix \( \rho^H(t) \) due to the presence of dissipation, they can be written as an integral over a suitable time-dependent parameter space if we have an integration over \( t \) of a function \( f(t) \) that has the general form

\[ f(t) = \sum_{i} \varphi_i(t) \frac{d\psi_i(t)}{dt}, \]  

\( \psi(t) \)}}
with the functions \( \Psi_i(t) \) satisfying the following conditions:

1) the functions \( \Psi_i(t) \) are not explicitly time-dependent;

2) the time-dependence of functions \( \Psi_i(t) \) come only from their dependence on the set of parameters \( \vec{k}(t) \equiv (k_1(t), k_2(t), \cdots, k_i(t)) \).

We point out that we do no restrict the regime of the time variation of the set of parameters \( \vec{k}(t) \) and it has not to be a periodic function in time.

We begin by discussing the imaginary phase on the r.h.s. of eq. (22). On the r.h.s. of eq. (22) we have a time-ordering integrals of a matrix \( C(t) \) whose elements are (see eq. (21)):

\[
C_{il}(t) = c_{ii}^{ll}(t) - c_{iN}(t), \quad i, l = 1, 2, \cdots, N - 1.
\]

(33)

The relation between \( c_{ii}^{ll}(t) \) and the coefficients of the non-unitary part of Liouvillian written in a time-independent basis is given by eq. (19)

\[
c_{ii}^{ll}(t) = \langle u_i; t | v_l^1 | u_l; t \rangle \langle v_{l3} | u_i; t \rangle \langle v_{l4} | u_l; t \rangle \bar{c}_{l1l2l3l4}(t),
\]

(34)

where \( i, l = 1, 2, \cdots, N \). In order to be able to write the time-ordering integrals on the r.h.s. of eq. (22) as a path-dependent integral the coefficients \( \bar{c}_{l1l2l3l4}(t) \) must have the form

\[
\bar{c}_{l1l2l3l4}(t) = \frac{d}{dt} \left( \Psi_{l1l2l3l4}(t) \right),
\]

(35)

and the functions \( \Psi_{l1l2l3l4}(t) \) have to satisfy the two previous conditions mentioned. We stress out that the path in the \( \vec{k} \)-parameter space has not to be closed.

Due to the hermicity property of the operator \( L_D \rho(t) \) the elements of matrix \( C \) (see eq.(21)) are real. If condition (35) is satisfied the geometric phase introduced by the presence of dissipation is purely imaginary.

Once eq. (35) is valid for the coefficients \( \bar{c}_{l1l2l3l4}(t) \), the non-diagonal elements \( \rho_{lm}^{H}(t) \) have two distinct situations in the limit \( T \rightarrow \infty \):

i) for \( M = 1 \), the elements \( \rho_{lm}^{H}(t) \) get an imaginary geometric phase. From the hermicity property of operator \( L_D \rho(t) \) we obtain that \( c_{ji}^{ij}(t) = \left( c_{ij}^{ji}(t) \right)^* \) which leaves open the
possibility that the coefficient $c_{ij}^j(t)$ has an imaginary part. For master equations where $c_{ji}^j(t)$ has an imaginary part, the dissipation gives a real correction to the Berry’s phase\[1\]. This correction to the Berry’s phase has not to be a closed-loop in the $\vec{k}$-parameter space.

\textit{ii)} for $M > 1$, the elements $\rho_{lm}^H(t)$ get an imaginary geometric phase due to the interaction with a weakly dissipative medium. The presence of the dissipation couples the dynamics of different elements of the density matrix and the integral over the $\vec{k}$-path involves a matrix that in general does not commute with itself at different instants. This $\vec{k}$-path has not to be a closed loop. As in case \textit{i} if the coefficients $c_{lm}^j(t)$ have an imaginary part they give a correction to the real geometric phase\[1\].

From eq. (35) we see that the presence of the dissipative media gives an imaginary geometric phase only if the coupling between the quantum system and its environment is externally driven by a varying external parameter. In both situations the time-variation of the set of the parameters $\vec{k}(t)$ has not to be slow.

\section{Adiabatic Limit of the Density Matriz of the Spin $\frac{1}{2}$ Model Coupled to a Reservoir}

For systems in contact with reservoirs at thermal equilibrium whose the weak coupling constant does not vary in time, the effect of the presence of the dissipation is to destroy the coherence in the quantum system in a time-dependent process. That is the case of two distinct reservoirs coupled to a spin $\frac{1}{2}$ model studied in detail in reference \[10\].

By using directly eq. (35) we want to verify in the next sub-sections if the conclusions about the nature of the imaginary phases in reference \[10\] are corrected or not.

\subsection{Dephasing Process in a Two-Level System}

An interesting process well studied in the standard textbooks\[23,25\] is the phase destroying process which might appear due to elastic collisions. We consider a spin $\frac{1}{2}$ variable (two level model) coupled to a time dependent magnetic field precessing around the z-axis with $\omega$ constant precession frequency. The external magnetic field $\vec{B}(t)$ has norm $B$ and makes a $\theta$ angle with the z-axis.

For the sake of later calculations, it is convenient to define two unitary transformations: the first one, $R(\omega, t)$, takes us to the rotating frame where the Hamiltonian is no longer time dependent; the second one, $D(B, \theta, \omega)$, diagonalizes the effective Hamiltonian (time independent) that drives the dynamics of the final matrix representation of the density.
operator. We call this the diagonal frame \((10)\).

The master equation to this process written in the diagonal frame is

\[
\frac{d}{dt} \rho_D(t) = -i \left[ \lambda_1 \sigma_z, \rho_D(t) \right] + \frac{k}{2} \left( \sigma_z \rho_D(t) \sigma_z - \rho_D(t) \right),
\]

(36)

where \(k\) is the dissipation constant at zero temperature. The weak coupling regime is characterized by the condition \(\frac{k}{\lambda_1} \ll 1\) with \(\lambda_1 = \sqrt{\left( \mu B \cos(\theta) - \frac{\omega}{2} \right)^2 + \mu^2 B^2 \sin^2(\theta)}\).

We define \(\rho^H(t)\) to be the density matrix in a basis of the instantaneous eigenvectors of Hamiltonian. The relation between \(\rho^H(t)\) and \(\rho_D(t)\) is \((10)\)

\[
\rho^H(t) = V^\dagger(t) D \rho_D(t) D V(t),
\]

(37a)

where the matrix \(V(t)\) is equal to

\[
V(t) = \begin{pmatrix} \cos(\frac{\theta}{2}) e^{-i\omega t \over 2} & -\sin(\frac{\theta}{2}) e^{-i\omega t \over 2} \\ \sin(\frac{\theta}{2}) e^{i\omega t \over 2} & \cos(\frac{\theta}{2}) e^{i\omega t \over 2} \end{pmatrix}
\]

(37b)

and

\[
D = \frac{1}{\sqrt{2}} \sqrt{1 + \Lambda} \sigma_z + \frac{1}{\sqrt{2}} \sqrt{1 - \Lambda} \sigma_x
\]

(37c)

with \(\Lambda \equiv \frac{1}{\lambda_1} \left( \mu B \cos(\theta) - \frac{\omega}{2} \right)\) and \(\pm \lambda_1\) are the eigenvalues of the effective Hamiltonian.

The master equation \((36)\) in the instantaneous basis of the Hamiltonian for arbitrary value of \(\omega\) is

\[
\frac{d}{dt} \rho^H(t) = -i \left[ \left( \mu B + \frac{\omega}{2} \right) \sigma_z - \frac{\omega}{2} \sigma_n(t), \rho^H(t) \right] + \frac{k}{2} \mathcal{L}_D^H \rho^H(t)
\]

(38a)

where

\[
\mathcal{L}_D^H \rho^H(t) = \Lambda^2 \sigma_n(t) \rho^H(t) \sigma_n(t) + \Lambda \sqrt{1 - \Lambda^2} \left[ e^{-i\omega t} \left( \sigma_n(t) \rho^H(t) \sigma_+(t) + \sigma_+(t) \rho^H(t) \sigma_n(t) \right) + e^{i\omega t} \left( \sigma_n(t) \rho^H(t) \sigma_-(t) + \sigma_-(t) \rho^H(t) \sigma_n(t) \right) \right] + (1 - \Lambda^2) \left[ e^{-2i\omega t} \sigma_+(t) \rho^H(t) \sigma_+(t) + e^{2i\omega t} \sigma_-(t) \rho^H(t) \sigma_-(t) + \sigma_+(t) \rho^H(t) \sigma_+(t) + \sigma_-(t) \rho^H(t) \sigma_-(t) \right] - \rho^H(t).
\]

(38b)
In eqs. (38), we use the definitions:

\[
\sigma_n(t) \equiv \begin{pmatrix} \cos(\theta) & -\sin(\theta)e^{-i\omega t} \\ -\sin(\theta)e^{i\omega t} & -\cos(\theta) \end{pmatrix},
\]

(39a)

\[
\sigma_+(t) \equiv e^{i\omega t} \begin{pmatrix} \frac{\sin(\theta)}{2} & \cos^2\left(\frac{\theta}{2}\right)e^{-i\omega t} \\ -\sin^2\left(\frac{\theta}{2}\right)e^{i\omega t} & \frac{\sin(\theta)}{2} \end{pmatrix},
\]

(39b)

and \(\sigma_-(t) \equiv (\sigma_+(t))^\dagger\).

From eqs. (22) and (26) we see that to obtain the nature of the imaginary phase (if they are time or path-dependent) we need the coefficients \(c_{1m}^{11}\) and \(c_{1m}^{12}\) gotten from \(L_D^H \rho_H(t)\) in the adiabatic limit \((\omega \to 0)\). However, the weakness condition on the dissipation constant \(k\left(\frac{k}{\lambda}\right)\) does not impose any constraint to the ratio \(\frac{k}{\omega}\). The interesting case happens when the adiabatic evolution and the dissipation effect are of the same order: \(\frac{k}{\omega} \sim 1\).

In the adiabatic limit, weak dissipation regime and for \(\frac{k}{\omega} \sim 1\), eq. (38b) give us

\[
c_{11}^{11} = k \mathcal{O}\left(\frac{\omega}{\mu B}\right) \quad \text{and} \quad c_{22}^{11} = k \mathcal{O}\left(\frac{\omega}{\mu B}\right)
\]

(40a)

and

\[
c_{21}^{12} = -k \left[1 + \mathcal{O}\left(\frac{\omega}{\mu B}\right)\right].
\]

(40b)

We neglect the contribution of terms of order \(k \mathcal{O}\left(\frac{\omega}{\mu B}\right)\) to eqs. (22) and (26) since they are of the same magnitude as the terms of higher order in \(\left(\frac{1}{T}\right)\). The constant \(k\) cannot be written as eq. (35) and consequently the imaginary phase due to the weak coupling to the dissipative medium is not geometric. For completeness we substitute the values of the coefficients in eqs. (22) and (26), and obtain [20]

\[
\rho_{11}^H(t) = \rho_{11}^H(0)
\]

(41a)

\[
\rho_{12}^H(t) = e^{-2i\mu B t} e^{-i\omega(1-\cos(\theta))t} e^{-kt} \rho_{12}^H(0).
\]

(41b)

From eq. (41a) we get that the process continues to be adiabatic even in the presence of dissipation. On the r.h.s. of eq. (41b) the first phase gives the difference of the dynamical phases of the eigenstates of Hamiltonian, the second phase gives the difference of the geometrical phases acquired by the instantaneous eigenstates of \(H(t)\) and the last phase is the time-dependent phase due to the coupling of the quantum system with the dissipative medium. Its effect is to destroy the quantum coherence in the system.
4.2 Adiabatic Limit of a Two-Level Model in Thermal Equilibrium

In our next application, we consider the spin $\frac{1}{2}$ model under the influence of an external magnetic field as described in section 4.1 but now coupled to a reservoir of electromagnetic fields at thermal equilibrium [10]. The master equation of this model in thermodynamic magnetic field as described in section 4.1 but now coupled to a reservoir of electromagnetic fields is [23, 25]

$$\frac{d}{dt} \rho_D (t) = -i \left[ \lambda_1 \sigma_z, \rho_D (t) \right] + k (\overline{n} + 1) (2 \sigma_- \rho_D (t) \sigma_+ \sigma_- - \sigma_+ \sigma_- \rho_D (t)) + k \overline{n} (2 \sigma_+ \rho_D (t) \sigma_- \sigma_+ - \sigma_- \sigma_+ \rho_D (t)),$$

(42)

where $k$ is the dissipation constant at zero temperature and $\overline{n}$ is the average number of excitations of the weakly coupled thermal oscillators at inverse temperature $\beta$.

By doing the transformation [37], the master equation (42) in the instantaneous basis of the Hamiltonian for arbitrary value of $\omega$ becomes

$$\frac{d}{dt} \rho^H (t) = -i \left[ \left( \mu B + \frac{\omega}{2} \right) \sigma_z - \frac{\omega}{2} \sigma_n (t), \rho^H (t) \right] + k L^H_D \rho^H (t)$$

(43)

where

$$L^H_D \rho^H (t) = \frac{2\overline{n} + 1}{2} \left\{ -2 \rho^H (t) + (1 - \Lambda^2) \left[ \sigma_n (t) \rho^H (t) \sigma_n (t) - e^{-i\omega t} \sigma_+ (t) \rho^H (t) \sigma_+ (t) - e^{i\omega t} \sigma_- (t) \rho^H (t) \sigma_- (t) + \right. \right.$$

$$- \left. \left[ (1 + \Lambda^2) \left[ \sigma_+ (t) \rho^H (t) \sigma_- (t) + \sigma_- (t) \rho^H (t) \sigma_+ (t) \right] - \Lambda \sqrt{1 - \Lambda^2} e^{2i\omega t} \left( \sigma_+ (t) \rho^H (t) \sigma_n (t) + \sigma_n (t) \rho^H (t) \sigma_+ (t) \right) \right) \right. \right.$$

$$- \left. \left[ \sigma_+ (t) \rho^H (t) \sigma_n (t) + \sigma_n (t) \rho^H (t) \sigma_+ (t) \right] \right\} -$$

$$\frac{1}{2} \left\{ \{ \rho^H (t), \Lambda \sigma_n (t) + \sqrt{1 - \Lambda^2} (e^{-i\omega t} \sigma_+ (t) + e^{i\omega t} \sigma_- (t)) \} + 2 \Lambda \left[ \sigma_+ (t) \rho^H (t) \sigma_- (t) - \sigma_- (t) \rho^H (t) \sigma_+ (t) \right] +$$

$$\sqrt{1 - \Lambda^2} \left[ e^{-i\omega t} \left( \sigma_n (t) \rho^H (t) \sigma_+ (t) - \sigma_+ (t) \rho^H (t) \sigma_n (t) \right) + e^{i\omega t} \left( \sigma_- (t) \rho^H (t) \sigma_n (t) - \sigma_n (t) \rho^H (t) \sigma_- (t) \right) \right] \right\},$$

(44)

$\sigma_n (t)$ and $\sigma_+ (t)$ are given by eqs. (39a) and (39b) respectively.

As in sub-section 4.1, we consider the adiabatic limit and the weak dissipative regime. In those regimes we consider the case $\frac{\omega}{\omega_o} \sim 1$ and the coefficients $c_{lm}^{11}$ and $c_{lm}^{12}$ gotten from $L^H_D \rho^H (t)$ (see eq. (14)) are:

19
\[ c_{11}^{11} = k \left[ -2(1 + \pi) + \mathcal{O} \left( \frac{\omega}{\mu B} \right) \right] \quad \text{and} \quad c_{22}^{11} = k \left[ 2\pi + \mathcal{O} \left( \frac{\omega}{\mu B} \right) \right] \] (45a)

and

\[ c_{21}^{12} = k \left[ -(1 + 2\pi) + \mathcal{O} \left( \frac{\omega}{\mu B} \right) \right]. \] (45b)

By the reason discussed in sub-section 4.1 the terms \( k \mathcal{O} \left( \frac{\omega}{\mu B} \right) \) do not contribute to the adiabatic limit. For the present model we see that neither the coefficients \( c_{1m}^{11} \) and \( c_{1m}^{12} \) can be written as eq. (35) and consequently the phases due to the coupling of the spin \( \frac{1}{2} \) model with a reservoir of electromagnetic fields at thermal equilibrium are not geometric.

The solutions of eqs. (22) and (26) for this model are

\[ \rho_{11}^H(t) = \left[ \rho_{11}^H(0) - \frac{\pi}{1 + 2\pi} \right] e^{-2k(1+2\pi)t} + \frac{\pi}{1 + 2\pi} \] (46a)

and

\[ \rho_{12}^H(t) = e^{-2i\mu B t} e^{-i\omega(1-\cos(\theta))t} e^{-k(1+2\pi)t} \rho_{12}^H(0). \] (46b)

From eq. (46a) we get that due to the weak interaction with the dissipative medium, we have a very slow phenomena but the adiabatic character of the quantum process is lost. As in the previous sub-section, the constants \( k \) and \( \pi \) can not be written as eq. (35) and consequently the imaginary phase due to the weak coupling to the dissipative medium is not geometric. From eq. (46b) we obtain that the interference effects between the instantaneous eigenstates of Hamiltonian is destroyed by the coupling with the dissipative medium. This effect depends on the dissipative constant \( k \) and the time elapsed.

5 Conclusions

We discuss the behavior of the density matrix of non-degenerate quantum systems whose dynamics are driven by a master equation in which the unitary part is a periodic Hamiltonian with period \( T (T \to \infty) \). Our discussion is model-independent. We recover the known results of the literature [1] for the quantum systems that are not coupled to any external dissipative medium. The difference of geometric phases necessary for measuring a
physical effect associated to the existence of these phases appears naturally in the density matrix approach. The obtainment of the adiabatic limit for the density matrix in a closed system is used as a simple situation where we present the details of the calculations.

Next we consider the quantum system coupled to a general weak dissipative medium. We take the case $k_0 \sim 1$ when the effects on the dynamics due to the slow evolution and the dissipative attenuation are of the same order. From eq. (22) we obtain that the adiabatic nature of the phenomena disappears if the non-unitary part of the Liouvillian has non-null diagonal terms in the instantaneous basis of Hamiltonian.

In the general case the loss of coherence due to the coupling to a weak dissipative medium is a time-dependent phenomena (see eqs. (26) and (31)). In eq. (35) we have our main result where we obtain the condition that the coefficients $c_{ijkl}(t)$ have to satisfy in order to the imaginary phase be geometric. This only happens when the interaction between the quantum system and its environment is time-dependent and is a consequence of the time variation of a set of parameters. As in the case of imaginary geometric phases appeared in the transition probabilities of non-real Hamiltonians in the non-adiabatic regime\cite{2,3,4} here also the integral over the parameter space $\vec{k}$ does not have to be a close loop. The condition (35) allows us to say directly from the expression of $L^H_D$ if the dissipative effect in the master equation gives an imaginary geometric correction to the Berry phase\cite{1}. In the case where condition (35) is fulfilled and the coefficients $c_{lm}(t)$ has an imaginary phase, the presence of dissipation gives a correction to the real geometric phase.

The condition (35) is not satisfied by the coefficients $c_{ijkl}(t)$ in the linear expansion of the non-unitary part of the Liouvillian (see eq. (18)) that represents the weak interaction of a non-degenerate quantum system with a reservoir at thermodynamic equilibrium. This implies that for those type of couplings the imaginary phases are time-dependent. In the present work we extend the validity of the results derived for two particular dissipation mechanisms discussed in reference \cite{10} that are rediscussed in section 4 using the approach presented in this work.

The non-hermitian parts in the Hamiltonians in references \cite{5,6,7} take into account the losses of a quantum system to its environment (a suitable reservoir of degrees of freedom at equilibrium). In these models, the couplings between quantum systems and their neighborhoods in the “ab initio” Hamiltonians do not depend on the time variation of a set of classical parameters. This fact, togheter with result (35) (the condition to the existence of an imaginary geometric phase) put in check the correctness of the imaginary geometric phases in the literature due to dissipative effects \cite{5,6,7}. In order to verify if
the imaginary phase for a quantum model described by a phenomenological non-hermitian hamiltonian truly exists — being of true geometric origin, and not a fake one due to eqs. (75) and (76) — one must apply the approach presented here. The presence of imaginary phases of geometric and time-dependent natures in a quantum system yield distinct rates of loss of coherence, and such distinction is experimentally detectable.

Finally we point out that we have applied the results derived in here to quantum systems in contact with reservoirs at thermodynamic equilibrium, but they apply equally well to models whose interaction with the environment varies in time. Only in this situation the imaginary phase acquired by loss of coherence can be eventually geometric (if and only if the condition (35) is fulfilled).

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[24] We define the time-ordering operator as

\[
\mathcal{T} \left( e^{\int_t^\tau A(t')dt'} \right) \equiv 1 + \int_t^\tau A(t_1)dt_1 + \int_t^{t_1} A(t_1)dt_1 \int_t^{t_2} A(t_2)dt_2 + \\
+ \int_t^{t_1} A(t_1)dt_1 \int_t^{t_2} A(t_2)dt_2 \int_t^{t_3} A(t_3)dt_3 + \cdots
\]

where \( \tau \) can be smaller or bigger than \( t \).

[25] W.H. Louisell, *Quantum Statistical Properties of Radiation*, (Wiley, New York, 1973).
[26] Like the case of no dissipation, the off-diagonal elements of the density matrix are not uniquely defined unless we impose a condition on $\langle u_i; t | \left( \frac{d}{dt} | u_i; t \right) \rangle$, $i = 1$ and 2.

In order to get eqs. (41b) and (46b), we impose the conditions: $\langle u_1; t | \left( \frac{d}{dt} | u_1; t \right) \rangle = i\omega \sin^2 \left( \frac{\theta}{2} \right)$ and $\langle u_2; t | \left( \frac{d}{dt} | u_2; t \right) \rangle = -i\omega \sin^2 \left( \frac{\theta}{2} \right)$. 