COHERENT STATES, LINE BUNDLES AND DIVISORS

Stefan Berceanu

1 Institute for Physics and Nuclear Engineering
Department of Theoretical Physics
PO BOX MG-6, Bucharest-Magurele,
Romania; E-mail: Berceanu@theor1.ifa.ro

Abstract

For homogeneous simply connected Hodge manifolds it is proved that the set of coherent vectors orthogonal to a given one is the divisor responsible for the homogeneous holomorphic line bundle of the coherent vectors. In particular, for naturally reductive spaces, the divisor is the cut locus.

1. INTRODUCTION

The coherent states [1, 2, 3] are a powerful tool in global differential geometry [4, 5]. For example, the remark Polar divisor = Cut locus, proved on naturally reductive spaces [6], gives a description of the cut locus in terms of coherent states. Let us briefly recall these notions.

Let $X$ be complete Riemannian manifold. The point $q$ is in the cut locus $\mathrm{CL}_p$ of $p \in X$ if $q$ is the nearest point to $p$ on the geodesic emanating from $p$ beyond which the geodesic ceases to minimize his arc length (cf. [7], see also Ref. [6] for more references).

We call polar divisor of $e_0$ the set $\Sigma_0 = \{ e \in e(G)| (e_0, e) = 0 \}$, where $e(G)$ is the family of coherent vectors [3]. This denomination is inspired after Wu [8], who used this term in the case of the complex Grassmann manifold.

In this paper we shall emphasize an aspect of the deep relationship between coherent states and algebraic geometry. Indeed, the notion of polar divisor, introduced in the context of coherent states, is in agreement with the notion of divisor in algebraic geometry [9]. The main result of this paper is the establishment of the relationship between vector coherent state manifold $\tilde{M}$, viewed as a holomorphic homogeneous line bundle over the coherent state manifold $\tilde{M}$, and the polar divisor. The set $\Sigma_0$ is the divisor responsible for the line bundle $\tilde{M}$. The result is proved for homogeneous simply connected Hodge manifolds. In particular, for naturally reductive spaces, the divisor responsible for $\tilde{M}$ is the cut locus $\mathrm{CL}_0$. The enunciation of this theorem was included in Ref. [10]. Using Rawnsley’s definition of coherent states [3] instead of Perelomov’s one, it is possible [11] to drop out the restriction that the manifold $\tilde{M}$ to be a homogeneous one.
The lay out of this paper is as follows: §2 collects some feature needed during the paper – a short remember of the notion of coherent states viewed as homogeneous line bundles, a breviary on divisors, and a brief review of the results established in Ref. [6] on cut locus and coherent states. The main results are proved in §3. The illustration on the complex Grassmann manifold follows the notation of [12].

2. INGREDIENTS

2.1. Homogeneous Line Bundles and Coherent States

1.) Let us consider the principal bundle

\[ K \overset{i}{\rightarrow} G \overset{\lambda}{\rightarrow} \tilde{M}, \]

where \( \tilde{M} \) is diffeomorphic with \( G/K \), \( i \) is the inclusion and \( \lambda \) is the natural projection \( \lambda(g) = gK \). Let \( \chi \) be a continuous representation of the group \( K \) on the Hilbert space \( \mathcal{K} \) and let \( \mathcal{M}_\chi := \tilde{M} \times_\chi \mathcal{K} \), or simply \( \mathcal{M} := \tilde{M} \times K \mathcal{K} \), be the \( G \)-homogeneous vector bundle [13] associated by \( \chi \) to the principal \( K \)-bundle (2.1). Let \( U \subset \tilde{M} \) be open. We introduce the notation

\[ (G)^U = \{ g \in G | go \in U \}, \]

where \( o \) is the base point in \( \tilde{M} \). Then the continuous (holomorphic) sections of \( \mathcal{M}_\chi \) over \( U \) are precisely the continuous (resp. holomorphic) maps \( \sigma : U \rightarrow (G) \times_\chi \mathcal{K} \) of the form

\[ \sigma(go) = [g, e_\sigma(g)], \quad e_\sigma : (G)^U \rightarrow \mathcal{K}, \]

where \( e_\sigma \) satisfies the “functional equation”:

\[ e_\sigma(gp) = \chi(p)^{-1}e_\sigma(g), \quad g \in (G)^U, \quad p \in K. \]

2.) Let \( \xi : \mathcal{H}^* = \mathcal{H} \setminus \{0\} \rightarrow \mathbb{P}(\mathcal{H}) \), \( \xi(z) = [z] \) be the mapping which associates to the point \( z \) in the punctured Hilbert space the linear subspace \([z]\) generated by \( z \), where \([\lambda z] = [z], \lambda \in \mathbb{C}^* \).

Let us consider the principal bundle (2.1) and let us suppose the existence of a map \( e : G \rightarrow \mathcal{H}^* \) as in eq. (2.3) with the property (2.4) but globally defined, i.e. on the neighbourhood \( (G)^\tilde{M} \). Then \( e(G) \) is called family of coherent vectors [3]. If there is a morphism of principal bundles, i.e. the following diagram is commutative,

\[
\begin{array}{ccc}
G & \xrightarrow{e} & \mathcal{H}^* \\
\lambda \downarrow & & \downarrow \xi \\
\tilde{M} & \xrightarrow{\iota} & \mathbb{P}(\mathcal{H})
\end{array}
\]

then \( \iota(\tilde{M}) \) is called family of coherent states corresponding to the family of coherent vectors \( e(G) \) [3]. The manifold \( \tilde{M} \) is called coherent state manifold and the \( G \)-homogeneous line bundle \( \mathcal{M}_\chi \) is called coherent vector manifold [14].

We impose the following restrictions: a) the mapping \( \iota \) is an embedding in some projective Hilbert space

\[ \iota : \tilde{M} \hookrightarrow \mathbb{P}(\mathcal{H}); \]

b) \( \tilde{M} \) is homogeneous [4]; c) the embedding \( \iota \) is kählerian (cf. Ch. 8 in [15]); d) the line bundle \( \mathcal{M} \) is very ample. If \( \chi \) induces a unitary representation \( \pi_\chi \) on the Hilbert space
of holomorphic sections \( \mathcal{H} = \Gamma_{\text{hol}}(\tilde{M}, M_\chi) = H^0(\tilde{M}, M_\chi) \), then we have an effective realization of Perelomov’s coherent states.

The Perelomov’s coherent vectors are
\[
e_{Z,j} = (\exp \sum_{\phi \in \Delta_\chi^+} (Z_\phi F^+_{\phi})) j, \quad e_{Z,j} = (e_{Z,j}, e_{Z,j})^{-1/2} e_{Z,j}.
\] (2.7)

In eq. (2.7) \( \Delta_\chi^+ \) denotes the positive non-compact roots, \( Z := (Z_\phi) \in \mathbb{C}^n \) are local coordinates in the neighbourhood \( V_0 \subset \tilde{M} \) of the base point \( o \), \( F^+_{\phi} j \neq 0, (F^-_{\phi} j = 0) \), \( \phi \in \Delta_\chi^+ \), and \( j \) is the dominant weight vector of the representation. The system \( \{ e(g) \} \), \( g \in G^c \) is overcomplete [1, 3] and \((e(g), e(g'))\), up to a factor, is a reproducing kernel for the holomorphic line bundle \( M_\chi \rightarrow \tilde{M} \) [7].

2.2. Divisors

Let \( \tilde{M} \) be a complex manifold and \( D \) a locally finite formal combination \( D = \sum a_i V_i \) of irreducible analytic not necessarily smooth hypersurfaces of \( \tilde{M} \). \( D \) is called a Weil divisor of \( \tilde{M} \). Let \( \text{Div}(\tilde{M}) \) denote the abelian group of Weil divisors. If \( a_i \geq 0 \), then \( D \) is an effective (holomorphic) divisor and this is denoted \( D \geq 0 \).

The divisors can be defined in terms of sheaf theory. A Cartier divisor \( D \) on an algebraic variety \( \tilde{M} \) is a global section of the quotient sheaf \( D \). We use the following notation: \( A \) – the sheaf of germs of complex \( C^\infty \)-valued functions; \( A^* \) – the sheaf of germs of complex \( C^\infty \)-valued functions nowhere zero; \( O \) – the sheaf of germs of holomorphic functions; \( O^* \) – the sheaf of germs of holomorphic functions vanishing nowhere; \( M \) – the sheaf of germs of local meromorphic functions; \( M^* \) – the sheaf of germs of invertible meromorphic functions; \( \mathcal{D} = M^*/O^* \).

Note that the equivalence classes of \( C^\infty \) line bundles are in 1-1 correspondence with the elements of the cohomology group \( H^1(\tilde{M}, A^*) \) and there is an isomorphism between the group of continuous line bundles and the second cohomology group with integer coefficients: \( H^1(\tilde{M}, A^*) \xrightarrow{\delta} H^2(\tilde{M}, \mathbb{Z}) \). However, for holomorphic line bundles, there is only a homomorphism \( H^1(\tilde{M}, O^*) \xrightarrow{\delta} H^2(\tilde{M}, \mathbb{Z}) \).

On complex manifolds there is an the isomorphism of Weil and Cartier groups of divisors: \( \text{Div}(\tilde{M}) = H^0(\tilde{M}, D) \).

There is a functorial homomorphism \( [ \ ] \) between the group of divisors and the Picard group of equivalence class of holomorphic line bundles \( [ \ ] : \text{Div}(\tilde{M}) \rightarrow H^1(\tilde{M}, O^*) \). \([D]\) is a \( \mathbb{C}^* \)-bundle, but we denote by the same symbol an analytic line bundle determined up to an isomorphism.

The exact sequence \( 1 \rightarrow O^* \xrightarrow{i} M^* \xrightarrow{k} D \rightarrow 1 \) induces the exact cohomology sequence
\[
0 \rightarrow H^0(\tilde{M}, O^*) \xrightarrow{i^*} H^0(\tilde{M}, M^*) \xrightarrow{k^0} H^0(\tilde{M}, D) \xrightarrow{d^0} H^1(\tilde{M}, O^*) \rightarrow \cdots \quad (2.8)
\]
The quotient group \( \text{Cl}(\tilde{M}) := H^0(\tilde{M}, D)/k^0 H^0(\tilde{M}, M^*) \), called the group of divisor classes with respect to linear equivalence, is isomorphic to a subgroup of \( H^1(\tilde{M}, O^*) \). If \( f \) is a meromorphic function on \( \tilde{M} \), then \( (f) \) is its associated divisor. The divisors \( D, D' \) are linearly equivalent if \( D = D' + (f) \), where \( f \in M^*(\tilde{M}) \).

The holomorphic divisor \( D \) is said non-singular if, with respect to some open covering \( U = \{ U_i \} \), it is represented by place functions \( f_i \) with the property: either \( f_i \equiv 1 \) or \( U_i \) admits a system of local complex coordinates for which \( f_i \) is one of the coordinates.
2.3. Cut Locus

Remark 1 codim_{c} CL_{p} \geq 1.

Let \( g \) be the Lie algebra of the group \( G \). Let \( g = \mathfrak{k} \oplus \mathfrak{m} \) be the orthogonal decomposition with respect to the \( B \)-form as explained below, \( \text{Exp}_{p} : T_{p} \tilde{M} \rightarrow \tilde{M} \) the geodesic exponential map from the tangent space to the manifold and \( \exp : g \rightarrow G \) the exponential from the Lie algebra to the group.

Let us consider the conditions:

A) \( \text{Exp}|_{o} = \lambda \circ \exp |_{m} \).

B) On the Lie algebra \( g \) of \( G \) there exists and \( \text{Ad}(G) \)-invariant, symmetric, non-degenerate bilinear form \( B \) such that the restriction of \( B \) to the Lie algebra \( \mathfrak{k} \) of \( K \) is likewise non-degenerate.

Note that for the homogeneous space \( \tilde{M} \simeq G/K \) property B), implies A). The condition A) is verified by the symmetric spaces, but also by the naturally reductive spaces because they verify the condition B).

Theorem 1 Let \( \tilde{M} \) be a homogeneous manifold \( \tilde{M} \simeq G/K \). Suppose that there exists a unitary irreducible representation \( \pi_{j} \) of \( G \) such that in a neighbourhood \( \mathcal{V}_{0} \) around \( Z = 0 \) the coherent states are parametrized as in eq. (2.7). Then the manifold \( \tilde{M} \) can be represented as the disjoint union \( \tilde{M} = \mathcal{V}_{0} \cup \Sigma_{0} \). Moreover, if the condition B) is true, then \( \Sigma_{0} = CL_{0} \).

Corollary 1 Suppose that \( \tilde{M} \) verifies B) and admits the embedding \( ([2.6]) \). Let \( 0, Z \in \tilde{M} \). Then \( Z \in CL_{0} \) iff \( d_{c}(\iota(0), \iota(Z)) = \pi/2 \), where \( d_{c}([\omega'], [\omega]) = \arccos \frac{\|\omega' \cdot \omega\|}{\|\omega\| \cdot \|\omega'\|} \).

We remember the explicit expression of the cut locus on the complex projective space and Grassmannian.

Remark 2 On \( \mathbb{C}P^{n} \), \( CL_{0} = \Sigma_{0} = H_{1} = \mathbb{C}P^{n-1} \).

Proof. Let the notation \( \mathcal{V}_{i} = \{ z \in \mathcal{H}^{*} | z_{i} \neq 0 \} \), \( \mathcal{U}_{i} = \xi(\mathcal{V}_{i}) \), \( H_{1} = \mathbf{P}(\mathcal{H}) \setminus \mathcal{U}_{i} \). The point \( p_{0} = [(1, 0, 0, \ldots)] \in \mathbf{P}(\mathcal{H}) \) corresponds to the point \( 0 \) in the Remark. Then the solution of the equation \( ([p_{0}], [z]) = 0 \) is \( [z] = [(0, x, x, \ldots)] = H_{1} = \mathbb{C}P^{n-1} \subset \mathbb{C}P^{n} \) for \( \mathcal{H} = \mathbb{C}^{n+1} \).

The complex Grassmann manifold \( G_{n}(\mathbb{C}^{m+n}) \) consists of the \( n \)-planes passing through the origin of \( \mathbb{C}^{n+m} \). The Plücker embedding \( \iota : G_{n}(\mathbb{C}^{m+n}) \hookrightarrow \mathbb{C}P^{N(n)-1} \) is given by \( \iota(Z) = [Z_{i_{1},\ldots,i_{n}}] \), where

\[
Z = z_{1} \wedge \ldots \wedge z_{n} = \sum_{1 \leq i_{1} < \ldots < i_{n} \leq N} Z^{i_{1},\ldots,i_{n}} e_{i_{1}} \wedge \ldots \wedge e_{i_{n}}, \quad z_{i} = \sum_{a=1}^{N} \hat{Z}_{ia} e_{a}. \tag{2.9}
\]

\( Z^{i_{1},\ldots,i_{n}} \) are the Plücker coordinates, \( \hat{Z} = (\hat{Z}_{ia})_{1 \leq i \leq n; 1 \leq a \leq N}, \quad N(n) = \binom{N}{n} = \frac{N!}{n!(N-n)!} \) and \( \{ e_{1}, ..., e_{N} \} \) is a basis of \( \mathbb{C}^{N} \), \( N = n + m \). Let the vectors \( z_{i}^{a} \) be such that \( \hat{Z}^{a} \in \mathcal{V}_{a} \), where \( a \) is a Schubert symbol. Then \( Z_{\sigma(i)\sigma(a)} \), \( i = 1, \ldots, n, a = n + 1, ..., N \), are the Pontrjagain coordinates

\[
z_{i}^{a} = e_{\sigma(i)} + \sum_{\alpha=n+1}^{N} Z_{\sigma(i)\sigma(a)} e_{\sigma(a)}, \quad i = 1, \ldots, n. \tag{2.10}
\]
Remark 3 (Wong[18]) The cut locus of the point \( O \in \mathbb{G}_n(\mathbb{C}^{m+n}) \) is given by

\[
\begin{align*}
\text{CL}_0 &= \Sigma_0 = V_1^m = Z(\omega_1^m) = Z(\omega_1) = Z(m-1, m, \ldots, m) \\
&= \left\{ X \in G_n(\mathbb{C}^{m+n}) \mid \dim(X \cap O^\perp) \geq 1 \right\}.
\end{align*}
\]

(2.11)

\[
V_1^m = \begin{cases} 
\mathbb{C}^{p_{n-1}}, & \text{for } n = 1, \\
W_1^m \cup W_2^m \cup \ldots \cup W_{n-1}^m \cup W_n^m, & 1 < n,
\end{cases}
\]

(2.12)

\[
W_n^m = \begin{cases} 
G_r(\mathbb{C}^{\max(m,n)}), & n \neq m, \\
O^\perp, & n = m.
\end{cases}
\]

(2.13)

The following notation is used:

\[
V_l^p = \left\{ Z \in G_n(\mathbb{C}^{m+n}) \mid \dim(Z \cap \mathbb{C}^p) \geq l \right\},
\]

\[
\begin{align*}
\omega_l^p &= (p-l, \ldots, p-l, m, \ldots, m) \\
W_l^p &= V_l^p - V_{l+1}^p; \ V_l^p = Z(\omega_l^p); \ W_l^p = Z'(\omega_l^p)
\end{align*}
\]

\[
\omega = \{ 0 \leq \omega(1) \leq \ldots \leq \omega(n) \leq m \}; \ \sigma(i) = \omega(i) + i, \ i = 1, \ldots, n
\]

\[
\begin{align*}
Z(\omega) &= \left\{ X \in G_n(\mathbb{C}^{m+n}) \mid \dim(X \cap \mathbb{C}^{\sigma(i)}) \geq i \right\} \\
I_\omega &= \{ 0 = i_0 < i_1 < \ldots < i_l-1 < i_l = n \}
\end{align*}
\]

\[
\begin{align*}
\omega(i_h) < \omega(i_{h+1}), \ \omega(i) = \omega(i_{h-1}), \ i_{h-1} < i \leq i_h, \ h = 1, \ldots, l
\end{align*}
\]

\[
Z'(\omega) = \left\{ X \in G_n(\mathbb{C}^{m+n}) \mid \dim(X \cap \mathbb{C}^{\sigma(i_h)}) = i_h, \ i_h \in I_\omega \right\}.
\]

Remark 4 The cut locus for flag manifolds \( G^c/P \) has a stratified structure consisting of \( r \) \( P \)-orbits. (\( r \)=rank).

3. RESULTS

Proposition 1 If \( \widetilde{M} \) is an homogeneous algebraic manifold embedded in a projective Hilbert space \( \mathbb{H} \) then the polar divisor \( \Sigma_0 \) can be expressed as \( \Sigma_0 = \nu^*H_1 \), and \( \Sigma_0 \) is a divisor.

Proof. Use is made of the Cauchy formula \([4, 21]\)

\[
(\langle e_Z^*, e_{\nu^*Z} \rangle)_{\widetilde{M}} = (\langle e_{\nu^*Z}^*, e_{\nu^*Z} \rangle)_{\mathbb{P}(\mathbb{H})},
\]

(3.1)

where \( \nu(Z) = [e_Z] \). We equate with 0 both sides of eq. (3.1). The pull-back \( \nu^*(H_1) \) of the divisor \( H_1 \) is itself a divisor \([4]\), because the mapping \( \nu \) is an embedding, i.e. biholomorphic on his image. \( \Box \)

Theorem 2 Let \( \widetilde{M} \) be a homogeneous simply connected Hodge manifold admitting the embedding \( [2, 4] \). Let \( M = \nu^*[1] \) be the unique, up to equivalence, projectively induced line bundle with a given admissible connection. Then \( M = [\Sigma_0] \). Moreover, if the homogeneous manifold \( \widetilde{M} \) verifies condition B), then \( M = [\text{CL}_0] \). In particular, the first relation is true for Kählerian C-spaces, while the second one for hermitian symmetric spaces.
Proof. The main part of the proof is based on the following theorem of Kodaira and Spencer: For an algebraic manifold there is an isomorphism of the group \( \text{Cl}(\tilde{M}) \) of divisor classes with respect to linear equivalence with the Picard group \( \text{Pic}(\tilde{M}) \), i.e. for every complex line bundle \( M \) over an algebraic manifold \( \tilde{M} \) there exists a divisor \( D \) such that \([D] = M\). The next ingredient is the following theorem due to Kostant: Let \( \tilde{M} \) be a simply connected Hodge manifold. Then, up to equivalence, there exists a unique line bundle with a given curvature matrix of the hermitian connection, or, equivalently, with a given admissible connection (Thm. 2.2.1 in [19] p. 135). Farther the theorem [1] is used. The information on Kählerian \( C^- \) spaces is extracted from [20, 21].

We remember also in this context Bertini theorem: Let \( M \) be a projectively induced line bundle over an algebraic manifold \( \tilde{M} \). Then there is a non-singular divisor \( D \) of \( \tilde{M} \) with \( M = [D] \). Another formulation reads as follows: A general hyperplane section \( S \) of a connected non-singular algebraic manifold \( \tilde{M} \) in \( \mathbb{CP}^N \) is itself non-singular and for \( n \geq 2 \), connected [3].

Comment 1 Generally, the divisor \( \Sigma_0 \) is singular because it does not correspond to a general section in Bertini’s theorem.

Proof. We illustrate the assertion on the pedagogical example furnished by the Grassmannian \( G_2(\mathbb{C}^4) \). The Plücker embedding is \( G_2(\mathbb{C}^4) \hookrightarrow \mathbb{CP}^5 \). The coordinate neighbourhoods \( V_1 - V_6 \) are presented in Table 1, while Table II presents the patches.

In \( V_1 \) the Plücker coordinates are

\[
(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) = (1, a_3, a_4, -a_1, -a_2, a_1a_4 - a_2a_3).
\]

They verify the constrained: \( p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23} = 0 \).

Let \( p_0 = (1, 0, 0, 0, 0, 0) \in V_1 \). We want to calculate \( \Sigma_0 \). Firstly note that \( V_1 \cap \Sigma_0 = \emptyset \). Then observe that \( \Sigma_0 \cap V_2 = \{ b_3 = 0 \} \), i.e. an open subset of codimension 1. We proceed similarly on other coordinate neighbourhoods \( V_3 - V_5 \). On \( V_6 \)

\[
(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) = (f_1 f_4 - f_3 f_2, -f_3, f_1, -f_4, f_2, 1).
\]

This implies that \( \Sigma_0 \cap V_6 = \{ f_1 f_4 - f_3 f_2 = 0 \} \), i.e. disjoint union of the open subset of codimension 1 already found in \( V_2 - V_3 \) and the point \( (0, 0, 0, 0, 0, 1) \). So, in \( V_6 \): \( p_{12} = 0; p_{14} p_{23} - p_{13} p_{24} = 0 \). This is a cone over a quadric surface whose vertex is the point \( (0, 0, 0, 0, 0, 1) \). The hyperplane \( p_{12} = 0 \) is the embedded tangent hyperplane of \( G_2(\mathbb{C}^4) \) of the line \( x_1 = x_2 = 0 \) in \( \mathbb{CP}^5 \). A general hyperplane section of \( G_2(\mathbb{C}^4) \) is not of the form \( p_{12} = 0 \), since by Bertini’s theorem it has to be smooth.

In fact, we have also proved the Remark [3] in the particular case of \( G_2(\mathbb{C}^4) \), i.e. \( \Sigma_0 = \text{CL}_0 = V_1^2 = W_1^2 \cup [(0, 0, 0, 0, 0, 1)] \), where \( W_1^2 \) is a quasiprojective variety of codimension one, while the point is the singular set. See details in [22].□

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Table 1: The Pontrjagin coordinates in the neighbourhoods $\mathcal{V}_1 - \mathcal{V}_6$ on $G_2(\mathbb{C}^4)$.

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mathcal{V}_1$ | 1 | 0 | $a_1$ | $a_2$ |
|   | 0 | 1 | $a_3$ | $a_4$ |
| $\mathcal{V}_2$ | 1 | $b_1$ | 0 | $b_2$ |
|   | 0 | $b_3$ | 1 | $b_4$ |
| $\mathcal{V}_3$ | 1 | $c_1$ | $c_2$ | 0 |
|   | 0 | $c_3$ | $c_4$ | 1 |
| $\mathcal{V}_4$ | $d_1$ | 1 | 0 | $d_2$ |
|   | $d_3$ | 0 | 1 | $d_4$ |
| $\mathcal{V}_5$ | $e_1$ | 1 | $e_2$ | 0 |
|   | $e_3$ | 0 | $e_4$ | 1 |
| $\mathcal{V}_6$ | $f_1$ | $f_2$ | 1 | 0 |
|   | $f_3$ | $f_4$ | 0 | 1 |

Table 2: The change of coordinates on $G_2(\mathbb{C}^4)$

|   | $\mathcal{V}_1$ | $\mathcal{V}_2$ | $\mathcal{V}_3$ | $\mathcal{V}_4$ | $\mathcal{V}_5$ | $\mathcal{V}_6$ |
|---|----------------|----------------|----------------|----------------|----------------|----------------|
| $\mathcal{V}_1$ | × | $a_3 \neq 0$ | $a_4 \neq 0$ | $a_1 \neq 0$ | $a_2 \neq 0$ | $a_1a_4-a_2a_3 \neq 0$ |
| $\mathcal{V}_2$ | $b_3 \neq 0$ | × | $b_1 \neq 0$ | $b_4 \neq 0$ | $b_1b_4-b_2b_3 \neq 0$ | $b_2 \neq 0$ |
| $\mathcal{V}_3$ | $c_3 \neq 0$ | $c_4 \neq 0$ | × | $c_1c_4-c_2c_3 \neq 0$ | $c_1 \neq 0$ | $c_2 \neq 0$ |
| $\mathcal{V}_4$ | $d_3 \neq 0$ | $d_1 \neq 0$ | $d_1d_4-d_2d_3 \neq 0$ | × | $d_4 \neq 0$ | $d_2 \neq 0$ |
| $\mathcal{V}_5$ | $e_3 \neq 0$ | $e_1e_4-e_2e_3 \neq 0$ | $e_1 \neq 0$ | $e_4 \neq 0$ | × | $e_2 \neq 0$ |
| $\mathcal{V}_6$ | $f_1f_4-f_2f_3 \neq 0$ | $f_3 \neq 0$ | $f_1 \neq 0$ | $f_4 \neq 0$ | $f_2 \neq 0$ | × |
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