Uniform Bound of the Highest Energy for the 3D Incompressible Elastodynamics

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Abstract

This article concerns the time growth of Sobolev norms of classical solutions to the 3D incompressible isotropic elastodynamics with small initial displacements. Given initial data in $H^k$ for a fixed big integer $k$, the global well-posedness of this Cauchy problem has been established by Sideris and Thomases in [19] and [20, 21], where the highest-order generalized energy $E_k(t)$ may have a certain growth in time. Alinhac [3] conjectured that such a growth in time may be a true phenomenon, where he proved that $E_k(t)$ is still uniformly bounded in time only for 3D scalar quasilinear wave equation under null condition. In this paper, we show that the highest-order generalized energy $E_k(t)$ is still uniformly bounded for the 3D incompressible isotropic elastodynamics. The equations of incompressible elastodynamics can be viewed as a nonlocal systems of wave type and is inherently linearly degenerate in the isotropic case. There are three ingredients in our proof: The first one is that we still have a decay rate of $t^{-\frac{3}{2}}$ when we do the highest energy estimate away from the light cone even though in this case the Lorentz invariance is not available. The second one is that the $L^\infty$ norm of the good unknowns, in particular, $\nabla(v+G\omega)$, is shown to have a decay rate of $t^{-\frac{3}{2}}$ near the light cone. The third one is that the pressure is estimated in a novel way as a nonlocal nonlinear term with null structure, as has been recently observed in [16]. The proof employs the generalized energy method of Klainerman, enhanced by weighted $L^2$ estimates and the ghost weight introduced by Alinhac.

Keyword: Incompressible elastodynamics, uniform bound, null condition, ghost weight method, generalized energy method.

1 Introduction

This article considers the time growth of Sobolev norms of classical solutions to the Cauchy problem of the 3D incompressible isotropic elastodynamics. The equations of incompressible elastodynamics display a linear degeneracy in the isotropic case; i.e., the equation inherently satisfies a null condition. By virtue of this nature, in a series of seminal works in [19] and [20, 21], Sideris and Thomases proved the global well-posedness of classical solutions to the 3D incompressible isotropic elastodynamics with small initial displacements for initial data in $H^k$ for some fixed big integer $k$ (notations will be introduced at the beginning of Section

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2). In those papers the highest-order generalized energy \( E_k(t) \) are shown to have an upper bound depending on time.

This kind of time growth of the highest Sobolev norms of the generalized energy also appears in the work of Sideris \[17, 18\] and Agemi \[1\] for the 3D compressible nonlinear elastic waves, of Sideris and Tu \[22\] for the 3D nonlinear wave systems and of Alinhac \[2\] for the 2D scalar nonlinear wave equation (see also \[6, 7, 9, 10, 15\] for further results). In \[3\], Alinhac has proved that for 3D scalar quasilinear wave equation with small initial data under null condition, the energy is uniformly bounded, and he also conjectured that by analogy with similar problems where such a growth has been proved, that this time growth of the highest energy is a true phenomenon.

In this paper, we show that the highest-order generalized energy \( E_k(t) \) is still uniformly bounded for the Cauchy problem of the 3D incompressible isotropic elastodynamics with small initial data. As a byproduct, our method presented here also provides a new and simpler proof for global well-posedness of this problem. Our main theorem is as follows:

**Theorem 1.1.** Let \((v_0, G_0) \in H^k_\Lambda\) with \(k \geq 9\). Suppose that \((v_0, F_0) = (v_0, I + G_0)\) satisfy the constraints (2.3) (2.4), and \(|| (v_0, G_0) ||_{H^k_\Lambda} < \epsilon\). Then there exist two positive constants \(M\) and \(\epsilon_0\) which depend only on \(k\) such that, if \(\epsilon \leq \epsilon_0\), then the system of incompressible Hookean elastodynamics (2.5) with initial data \((v_0, F_0) = (v_0, I + G_0)\) has a unique global solution \((v, F) = (v, I + G)\) which satisfies \((v, G) \in H^k_\Gamma\) and \(E_{1/2}^k(t) \leq M\epsilon\) for all \(t \in [0, +\infty)\).

The theorem is also true for general incompressible isotropic elastodynamics, which can be regarded as a higher order nonlinear correction to the Hookean case. See Section 10 for more details.

We recall that the global existence of nonlinear wave and elastic systems in dimension two or three, if it is true, hinges on two basic assumptions: the smallness of the initial data and the null condition of nonlinearities. The omission of either of these assumptions can lead to the break down of solutions in finite time \[5, 23\]. For 3D quasilinear wave equation whose quadratic nonlinearities satisfy the null condition, the global well-posedness theory for small initial data was shown independently by Christodoulou \[4\] and Klainerman \[12\]. The result of Christodoulou \[4\] relied on the conformal method, the null condition implies then that the nonlinear terms of the equations transform into smooth terms, and the problem is reduced to a local problem with small data. While the proof of Klainerman uses a special energy inequality for the wave equation, which is obtained by multiplying by an appropriate vector field with quadratic coefficients, which we called the generalized energy method, see also \[8\] for an account of both aspects. The generalized energy method of Klainerman can be refined to prove the global existence for compressible elastic waves with small initial data under the null condition \[17, 18\], 3D quasilinear wave equations with small initial data under the null condition \[22\], and incompressible elastic fluids with small initial data \[19, 20, 21, 11\]. Alinhac \[2\] can even enhance Klainerman’s generalized energy method to prove the global well-posedness of 2D quasilinear wave equation with small initial data under the null condition via a ghost weighted method. However, in all the aforementioned results, the upper bound of the highest generalized energy depends on time.

The incompressible elasticity equation can be viewed as a nonlocal system of wave type and is inherently linearly degenerate in the isotropic case. Recently, a significant progress has been made in \[16\] where the authors proved the almost global existence of classical solutions for the 2D incompressible isotropic elastodynamics by combining the generalized
energy method of Klainerman, the weighted $L^2$ estimates of Sideris and the ghost weight method of Alinhac. A key point in [16], among other things, is that one can still use the null condition in the highest order energy estimate. Motivated by [16], in this paper, we show a uniform bound for the highest generalized energy of solutions to the Cauchy problem of 3D incompressible isotropic elastodynamics with small initial displacements. Let us mention that the final landmark work is the global wellposedness of the 2d incompressible elastodynamics, which is highly nontrivial and was recently solved by Lei in [13].

There are three ingredients in our proof: The first one is that we still have a decay rate of $t^{-3/2}$ when we do the highest energy estimate away from the light cone even though in this case the Lorentz invariance is not available. This is achieved by a refined Sobolev embedding theorem which enables us to gain one spatial derivative. This one extra derivative allows us to obtain the $L^2$ norm of $\langle t \rangle^{3/2} \nabla^2 U$ away from the light cone. See Lemma 8.2 for details. And we also need to use the weighted generalized energy defined by Sideris in [19]. The second one is that the $L^\infty$ norm of the good unknowns, in particular, $\nabla (v + G \omega)$, is shown to have a decay rate of $t^{-3/2}$ near the light cone. Let us mention that the special quantities $\Gamma^\alpha vw$, $w \Gamma^\alpha G$ are also shown to have $t^{-3/2}$ time decay in the region $r \geq \langle t \rangle^{10}$. The third one is that the pressure is estimated in a novel way as a nonlocal nonlinear term with null structure, as has been recently observed in [16] in the two-dimensional case. The proof also employs the generalized energy method of Klainerman, enhanced by weighted $L^2$ estimates and the ghost weight introduced by Alinhac.

This paper is organized as follows: In Section 2 we introduce some notations and the system of isotropic elastodynamics. In Section 3 we discuss the commutation properties of modified Klainerman’s vector fields with equations and constraints. Then some calculus inequalities are presented in Section 4. In Section 5, we will treat the pressure term. And in Section 6, we show that the weighted generalized energy is controlled by the generalized energy for small solutions. In Section 7, we prove that the good unknown $\partial_r (\Gamma^\alpha v + \Gamma^\alpha G \omega)$ has a better decay estimate [7,1], which gives $t^{-3/2}$ time decay in $L^\infty (r \geq \langle t \rangle^{10})$. And in Section 8, we obtain $t^{-3/2}$ time decay in $L^\infty (r \leq \langle t \rangle^{10})$ for the derivative of the solution. In Section 9 we complete the proof of the main result via the ghost weight method introduced first by Alinhac in [2]. Finally, we treat the general isotropic case by regarding it as a higher order nonlinear correction to the Hookean case (in Section 10).

## 2 Preliminaries

Classically the motion of an elastic body is described as a second-order evolution equation in Lagrangian coordinates. In the incompressible case, the equations are more conveniently written as a first-order system with constraints in Eulerian coordinates. We start with a time-dependent family of orientation-preserving diffeomorphisms $x(t, \cdot)$, $0 \leq t < T$. Material points $X$ in the reference configuration are deformed to the spatial position $x(t, X)$ at time $t$. Let $X(t, x)$ be the corresponding reference map: $X(t, x)$ is the inverse of $x(t, \cdot)$.

**Lemma 2.1.** Given a family of deformations $x(t, X)$, define the velocity and deformation gradient as follows:

$$v(t, x) = \frac{dx(t, X)}{dt} \bigg|_{X = X(t, x)}, \quad F(t, x) = \frac{\partial x(t, X)}{\partial X} \bigg|_{X = X(t, x)}. \quad (2.1)$$
Then for \( 0 \leq t < T, i, j, k \in \{1, 2, 3\} \), we have
\[
\partial_t F + v \cdot \nabla F = \nabla v F, \tag{2.2}
\]
\[
\partial_j G_{ik} - \partial_k G_{ij} = G_{mk} \partial_m G_{ij} - G_{mj} \partial_m G_{ik}, \tag{2.3}
\]
where \( G(t, x) = F(t, x) - I \). If in addition \( x(t, X) \) is incompressible, that is \( \det F(t, x) \equiv 1 \), then
\[
\nabla \cdot F^T = \partial_m F_{mi} = 0 \quad \text{and} \quad \nabla \cdot v = 0. \tag{2.4}
\]

**Proof.** (2.3) is proved in [14], for other identity, see for instance [19, 21].

Here and in what follows, we use the summation convention over repeated indices.

In this paper, to best illustrate our methods and ideas, let us first consider the equations of motion for incompressible Hookean elasticity, which corresponds to the Hookean strain energy function \( W(F) = \frac{1}{2} |F|^2 \) and reads
\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \nabla \cdot (FFFF^T), \\
\partial_t F + v \cdot \nabla F &= \nabla v F, \\
\nabla \cdot v &= 0, \quad \nabla \cdot F^T = 0.
\end{aligned} \tag{2.5}
\]

As will be seen in Section 10 where the case of general energy function is discussed, there is no essential loss of generality in considering this simplest case. Throughout this paper we will adopt the notations of
\[
(\nabla \cdot F)_i = \partial_j F_{ij}, \quad (\nabla v)_{ij} = \partial_j v_i.
\]

Most of our norms will be in \( L^2 \) and most integrals will be taken over \( \mathbb{R}^3 \), so we write
\[
\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^3)}, \tag{2.6}
\]
and
\[
\int = \int_{\mathbb{R}^3}. \tag{2.7}
\]

We use the usual derivative vector fields
\[
\partial = (\partial_t, \partial_1, \partial_2, \partial_3) \quad \text{and} \quad \nabla = (\partial_1, \partial_2, \partial_3). \tag{2.8}
\]

The scaling operator is denoted by:
\[
S = t \partial_t + \sum_{j=1}^3 x_j \partial_j = t \partial_t + r \partial_r, \tag{2.9}
\]
and its time independent analogue is
\[
S_0 = r \partial_r, \tag{2.10}
\]
here, the radial derivative is defined by \( \partial_r = \frac{x}{r} \cdot \nabla, \quad r = |x| \). The angular momentum operators are defined by
\[
\Omega = x \wedge \nabla. \tag{2.11}
\]
In this paper, the notation \( U = (v, G) \) is used frequently, where \( v \) are three dimensional real valued vector functions and \( G \) are three by three real valued matrix functions. The arguments of these functions are nearly always suppressed. As in [19, 21] and [11], we define

\[
\tilde{\Omega}_i(U) = (\Omega_i G + [V^i, G], \Omega v + V^i v),
\]

where

\[
\begin{align*}
V^1 &= e_2 \otimes e_3 - e_3 \otimes e_2, \\
V^2 &= e_3 \otimes e_1 - e_1 \otimes e_3, \\
V^3 &= e_1 \otimes e_2 - e_2 \otimes e_1,
\end{align*}
\]

and \([A, B] = AB - BA\) denotes the standard Lie bracket product. Occasionally we will write

\[
\tilde{\Omega} G = \Omega G + [V, G] \quad \text{for} \quad G \in \mathbb{R}^3 \otimes \mathbb{R}^3,
\]

and

\[
\tilde{\Omega} v = \Omega v + V v \quad \text{for} \quad v \in \mathbb{R}^3.
\]

For scalar functions \( f \) we define

\[
\tilde{\Omega} f = \Omega f.
\]

We shall frequently use the decomposition

\[
\nabla = \frac{x}{r} \partial_r - \frac{x}{r^2} \wedge \Omega.
\]

Our vector fields will be written succinctly as \( \Gamma \). We let

\[
\Gamma = (\Gamma_1, \cdots, \Gamma_8) = (\partial, \tilde{\Omega}, S).
\]

Hence by \( \Gamma U \) we mean any one of \( \Gamma_i U \). By \( \Gamma^a, a = (a_1, \cdots, a_k) \), we denote an ordered product of \( k = |a| \) vector fields \( \Gamma_{a_1} \cdots \Gamma_{a_k} \), we note that the commutator of any \( \Gamma^a \) is again a \( \Gamma \).

Define the generalized energy by

\[
E_k(t) = \sum_{|\alpha| \leq k} \left( \|\Gamma^\alpha v(t, \cdot)\|^2 + \|\Gamma^\alpha G(t, \cdot)\|^2 \right).
\]

We also define the weighted energy norm

\[
X_k(t) = \sum_{|\alpha| \leq k-1} \left( \langle t - r \rangle \nabla \Gamma^\alpha v \|^2 + \langle t - r \rangle \nabla \Gamma^\alpha G \|^2 \right),
\]

in which we denote \( \langle \sigma \rangle = \sqrt{1 + \sigma^2} \).

In order to characterize the initial data, we introduce the time independent analogue of \( \Gamma \). The only difference will be in the scaling operator. Set

\[
\Lambda = (\Lambda_1, \cdots, \Lambda_7) = (\nabla, \tilde{\Omega}, S_0).
\]

Then the commutator of any two \( \Lambda \)'s is again a \( \Lambda \). Define

\[
H^k_{\Lambda} = \{ U = (v, G) : \mathbb{R}^3 \to \mathbb{R}^3 \times (\mathbb{R}^3 \otimes \mathbb{R}^3) : \sum_{|\alpha| \leq k} \|\Lambda^\alpha U\| < \infty \}.
\]

Solutions will be constructed in the space

\[
H^k_{\Gamma} = \{ U = (v, G) : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3 \times (\mathbb{R}^3 \otimes \mathbb{R}^3) : \langle v, G \rangle \in \cap_{j=0}^k C^j([0, \infty); H^k_{\Lambda} - j) \}.
\]
3 Commutation

Write $F = I + G$, we obtain the following systems:
\[
\begin{aligned}
&\partial_t v - \nabla \cdot G = -\nabla p - v \cdot \nabla v + \nabla \cdot (GG^T), \\
&\partial_t G - \nabla v = -v \cdot \nabla G + \nabla vG, \\
&\nabla \cdot v = 0, \quad \nabla \cdot G^T = 0,
\end{aligned}
\] (3.1)
where $G$ satisfies the following equation
\[
\partial_k G_{ij} - \partial_j G_{ik} = G_{ij} \partial_t G_{ik} - G_{ik} \partial_t G_{ij}. \tag{3.2}
\]
For any multi-index $\alpha$, apply $\Gamma^\alpha$ to the equation, we have the following commutation properties, see, for instance [16],
\[
\begin{aligned}
&\begin{cases}
\partial_t \Gamma^\alpha v - \nabla \cdot \Gamma^\alpha G = -\nabla \Gamma^\alpha p + f_\alpha, \\
\partial_t \Gamma^\alpha G - \nabla \Gamma^\alpha v = g_\alpha,
\end{cases} \\
&\nabla \cdot \Gamma^\alpha v = 0, \quad \nabla \cdot \Gamma^\alpha G^T = 0,
\end{aligned}
\] (3.3)
where
\[
\begin{aligned}
f_\alpha &= -\sum_{\beta+\gamma=\alpha} \Gamma^\beta v \cdot \nabla \Gamma^\gamma v + \sum_{\beta+\gamma=\alpha} \nabla \cdot (\Gamma^\beta G^T \Gamma^\gamma G^T), \\
g_\alpha &= -\sum_{\beta+\gamma=\alpha} \Gamma^\beta v \cdot \nabla \Gamma^\gamma G + \sum_{\beta+\gamma=\alpha} \nabla \Gamma^\beta v \Gamma^\gamma G,
\end{aligned}
\] (3.4)
we also have
\[
\partial_k \Gamma^\alpha G_{ij} - \partial_j \Gamma^\alpha G_{ik} = h_\alpha, \tag{3.5}
\]
where
\[
h_\alpha = \sum_{|\beta|+|\gamma|=|\alpha|} \Gamma^\beta G_{ij} \partial_t \Gamma^\gamma G_{ik} - \Gamma^\beta G_{ik} \partial_t \Gamma^\gamma G_{ij}.
\]

4 Calculus Inequalities

In this section, a few elementary but useful inequalities will be prepared. Such inequalities capture decay at spatial infinity through the use of the vector field $\Gamma$. The first result appeared in [18], and the second appeared in [19, 21]. We will omit the proofs.

**Lemma 4.1.** For $u \in C_0^\infty(\mathbb{R}^3)$, $r = |x|$, and $\rho = |y|$, 
\[
\begin{aligned}
&\frac{1}{2} |u(x)| \lesssim \sum_{|\alpha| \leq 1} \|\nabla \tilde{\Omega}^\alpha u\|, \\
&|u(x)| \lesssim \sum_{|\alpha| \leq 1} \|\partial_r \tilde{\Omega}^\alpha u\|^{1/2} \cdot \left( \sum_{|\alpha| \leq 2} \|\tilde{\Omega}^\alpha u\|^{1/2} \right) \cdot \left( \sum_{|\alpha| \leq 2} \|\nabla \tilde{\Omega}^\alpha u\|^{1/2} \right).
\end{aligned}
\] (4.1)

In this paper, we write $X \lesssim Y$ to indicate $X \leq CY$ for some constant $C > 0$, and $X \sim Y$ whenever $X \lesssim Y \lesssim X$.

**Lemma 4.2.** Let $U \in H^k_k$, with $X_k[U(t)] < \infty$ and $|U| < \delta$ small. Then we have
\[
\begin{aligned}
&\langle r \rangle |\Gamma^\alpha U(t, x)| \lesssim E_k^{1/2} [U(t)], \quad |\alpha| + 2 \leq k, \\
&\langle r \rangle (t-r) |\nabla \Gamma^\alpha U(t, x)| \lesssim X_k^{1/2} [U(t)], \quad |\alpha| + 3 \leq k.
\end{aligned}
\] (4.2)
Lemma 4.4. Suppose the second one can be carried out in exactly the same way.

Proof. The quantities $\langle t \rangle f$, $\langle t \rangle f$, $\langle t \rangle f$, $\langle t \rangle f$, and $\langle t \rangle f$ provide the right hand side is finite.

Lemma 4.3. For all $f \in H^2(\mathbb{R}^3)$, there holds

$$
\langle t \rangle f_{L^2(r \leq \frac{\alpha}{16})} \lesssim \| f \|_{L^2(r \leq \frac{\alpha}{16})} + \| \langle t - r \rangle \nabla f \|_{L^2(r \leq \frac{\alpha}{16})} + \| \langle t - r \rangle \nabla^2 f \|_{L^2(r \leq \frac{\alpha}{16})},
$$

(4.3)

Remarks 4.1. For any $\alpha$, we have

$$
\langle t \rangle |\Gamma^\alpha f|_{L^2(r \leq \frac{\alpha}{16})} \lesssim E_{|\alpha| + 2}^{1/2} + X_{|\alpha| + 2}^{1/2}.
$$

(4.5)

The following result is a consequence of Corollary 8.1 in [11], which states that the special quantities $w^\alpha G$, $w^\alpha v$ have $(1 + t)^{-3/2}$ time decay outside the light cone.

Lemma 4.4. Suppose $\nabla \cdot v = 0$, $\nabla \cdot G^T = 0$, then we have

$$
r^{3/2} |w \cdot v| \lesssim \sum_{|\alpha| \leq 2} \| \Gamma^\alpha v \|,
$$

$$
r^{3/2} |w_i \cdot G_{ij}| \lesssim \sum_{|\alpha| \leq 2} \| \Gamma^\alpha G \|, \forall j = 1, 2, \ldots, n.
$$

(4.6)

Proof. For completeness, let us give a simpler proof. It suffices to prove the first one, since the second one can be carried out in exactly the same way.

For $r \leq 1$, it is an immediate consequence of the Sobolev embedding

$$
H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3).
$$
While for $r \geq 1$, using Lemma 4.1 and the decomposition of $\nabla$, we have

$$r^{\frac{1}{2}} |rv \cdot w| \lesssim \sum_{|\alpha| \leq 1} \|\nabla \tilde{\Omega}^\alpha (rv \cdot w)\| = \sum_{|\alpha| \leq 1} \|\nabla (r \tilde{\Omega}^\alpha v \cdot w)\|$$

$$\lesssim \sum_{|\alpha| \leq 1} \|\partial_r (r \tilde{\Omega}^\alpha v \cdot w)\| + \sum_{|\alpha| \leq 1} \|w \wedge \Omega \| (r \tilde{\Omega}^\alpha v \cdot w)\|$$

$$\lesssim \sum_{|\alpha| \leq 1} \|\tilde{\Omega}^\alpha v\| + \sum_{|\alpha| \leq 1} \|r \partial_r \tilde{\Omega}^\alpha v \cdot w\|$$

$$\lesssim \sum_{|\alpha| \leq 1} \|\partial_j \Gamma^\alpha v\| + \sum_{|\alpha| \leq 1} \|\frac{w \wedge \Omega}{r}(r \tilde{\Omega}^\alpha v_i)\|$$

$$\lesssim \sum_{|\alpha| \leq 1} \|\Gamma^\alpha v\|.$$ (4.7)

\[\square\]

### 5 Bound for the Pressure Gradient

The following Lemma shows that the pressure gradient may be treated as a nonlinear term. The first estimate appeared in [21], and the second estimate appeared in [16], it is a novel refinement which saves one derivative over the first bound and which allows us to exploit the null structure. This is essential in Section 9 when we estimate the ghost weighted energy.

**Lemma 5.1.** Let $(v, F) = (v, I + G)$, $(v, G) \in H^k$, solve the equation (2.5), then we have

$$\|\nabla \Gamma^\alpha p\| \lesssim \|f_\alpha\|, \quad (5.1)$$

$$\|\nabla \Gamma^\alpha p\| \lesssim \sum_{\beta + \gamma = \alpha, |\beta| \leq |\gamma|} \|\partial_j \Gamma^\beta v_i \Gamma^\gamma v_j - \partial_j \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk}\|, \quad (5.2)$$

for all $|\alpha| \leq k - 1$.

**Proof.** Applying the divergence operator to the first equation of (3.3), we have

$$\Delta \Gamma^\alpha p = \nabla \cdot f_\alpha + \nabla \cdot (\nabla \cdot \Gamma^\alpha G) - \partial_t \nabla \cdot \Gamma^\alpha v.$$ (5.3)

Using the last equation in (3.3), one has

$$\Delta \Gamma^\alpha p = \nabla \cdot f_\alpha.$$ (5.4)
By the definition of \( f_\alpha \), and the last equation in (6.3), we have

\[
\nabla \cdot f_\alpha = - \sum_{\beta + \gamma = \alpha} \partial_i \partial_j (\Gamma^\beta v_i \Gamma^\gamma v_j - \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk}) \\
= - \sum_{\beta + \gamma = \alpha, |\beta| \leq |\gamma|} \partial_i \partial_j (\Gamma^\beta v_i \Gamma^\gamma v_j - \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk}) \\
- \sum_{\beta + \gamma = \alpha, |\beta| > |\gamma|} \partial_i \partial_j (\Gamma^\beta v_i \Gamma^\gamma v_j - \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk}) \\
= - \sum_{\beta + \gamma = \alpha, |\beta| \leq |\gamma|} \partial_i (\partial_j \Gamma^\beta v_i \Gamma^\gamma v_j - \partial_j \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk}) \\
- \sum_{\beta + \gamma = \alpha, |\beta| > |\gamma|} \partial_j (\Gamma^\beta v_i \partial_i \Gamma^\gamma v_j - \Gamma^\beta G_{ik} \partial_i \Gamma^\gamma G_{jk}).
\]

The result now follows since

\[
\nabla \Gamma^\alpha p = \Delta^{-1} \nabla (\nabla \cdot f_\alpha),
\]

and since \( \Delta^{-1} \nabla \otimes \nabla \) is bounded in \( L^2 \).

\[\tag{6.6}\]

6 Weighted \( L^2 \) Estimate

In this section, we show that the weighted norm is controlled by the energy, for small solutions.

Lemma 6.1. Suppose that \( (v, F) = (v, I + G), \ (v, G) \in H^k, k \geq 4 \), solves (2.5). Then

\[
\| N_k(t) \| \lesssim E_k(t) + E_k(t)^{1/2} X_k(t)^{1/2},
\]

there

\[
N_k = \sum_{|\alpha| \leq k-1} [t|f_\alpha| + t|g_\alpha| + (t + r)|h_\alpha| + t|\nabla \Gamma^\alpha p|].
\]

Proof. By Lemma 5.1 we have

\[
\| N_k(t) \| \leq \sum_{|\alpha| \leq k-1} [t\| \nabla \Gamma^\alpha p \| + t|f_\alpha| + t|g_\alpha| + (t + r)|h_\alpha|]
\]

\[
\lesssim \sum_{|\alpha| \leq k-1} [t|f_\alpha| + t|g_\alpha| + (t + r)|h_\alpha|].
\]

To estimate these terms, we shall consider the following two cases: \( r \leq \frac{10}{16} \) and \( r \geq \frac{10}{16} \).

**Estimates of nonlinearities for** \( r \leq \frac{(t)}{16} \). Examining the definitions, we find that

\[
\sum_{|\alpha| \leq k-1} [t|f_\alpha|_{L^2(r \leq (t)/16)} + t|g_\alpha|_{L^2(r \leq (t)/16)} + (t + r)|h_\alpha|_{L^2(r \leq (t)/16)}]
\]

\[
\lesssim \sum_{\beta + \gamma = \alpha, |\alpha| \leq k-1} (t)(|\Gamma^\beta v| + |\Gamma^\beta G|)(|\nabla \Gamma^\gamma v| + |\nabla \Gamma^\gamma G|)_{L^2(r \leq (t)/16)}.
\]

\[\tag{6.4}\]
To simplify the notation a bit, we shall write
\[
[(\Gamma^k v, \Gamma^k G)] = \sum_{|\alpha| \leq k} |\Gamma^\alpha v| + |\Gamma^\alpha G|.
\] (6.5)

We make use of the fact that since \(\beta + \gamma = \alpha\), \(|\alpha| \leq k - 1\) and \(k \geq 4\), either \(|\beta| \leq k'\) or \(|\gamma| \leq k'\), with \(k' = \lfloor \frac{k-1}{2} \rfloor\), we have \(k' + 3 \leq k\). Here, \([s]\) stands for the largest integer not exceeding \(s\). Note that \( (t) \leq (t - r) \) in the region \( r \leq \frac{(t)}{16} \), thus, using (6.5), we have
\[
\sum_{k' + 3 \leq k} (\lim_{t \to 0} (t)^{\frac{1}{2}} E_{k+3}^1(t)) \leq E_{k+3}^1(t) + X_{k+3}(t).
\]

**Estimates of nonlinearities for** \( r \geq \frac{(t)}{16} \). By the decomposition of \(\nabla \) (2.16), we have
\[
|f_\alpha| + |g_\alpha| + |h_\alpha| \lesssim \sum_{\beta + \gamma = \alpha, |\alpha| \leq k-1} (|\Gamma^\beta v| + |\Gamma^\beta G|)(|\partial_r \Gamma^\gamma v| + |\partial_r \Gamma^\gamma G|)
\]
\[
+ \sum_{\beta + \gamma = \alpha, |\alpha| \leq k-1} \frac{1}{r}(|\Gamma^\beta v| + |\Gamma^\beta G|)(|\Omega \Gamma^\gamma v| + |\Omega \Gamma^\gamma G|).
\] (6.7)

Thus, using the second inequality of (4.4), we have
\[
\sum_{|\alpha| \leq k-1} [t|f_\alpha| L^2(r \geq (t)/16) + t|g_\alpha| L^2(r \geq (t)/16) + (t + r)|h_\alpha| L^2(r \geq (t)/16)] \lesssim \sum_{\beta + \gamma = \alpha, |\alpha| \leq k-1} ||r(\Gamma^\beta v| + |\Gamma^\beta G|)(|\partial_r \Gamma^\gamma v| + |\partial_r \Gamma^\gamma G|)|| L^2(r \geq (t)/16)
\]
\[
+ \sum_{\beta + \gamma = \alpha, |\alpha| \leq k-1} ||(\Gamma^\beta v| + |\Gamma^\beta G|)(|\Omega \Gamma^\gamma v| + |\Omega \Gamma^\gamma G|)|| L^2(r \geq (t)/16),
\] (6.8)

Then Lemma 6.1 follows by collecting the estimates above.

**Lemma 6.2.** Suppose that \((v, F) = (v, I + G)\), \((v, G) \in H_k^k\), solves (2.5). Then for all \(|\alpha| \leq k - 1\), we have
\[
|| (t - r) \nabla \Gamma^\alpha G ||^2 \lesssim || \Gamma^\alpha G ||^2 + || (t - r) \nabla \cdot \Gamma^\alpha G ||^2 + Q_\alpha,
\] (6.9)

here
\[
Q_\alpha = \sum_{\beta + \gamma = \alpha, |\alpha| \leq k-1} \int (t - r)^2 \partial_j \Gamma^\alpha G_{ik} \Gamma^\beta G_{ik} \partial_i \Gamma^\gamma G_{ij} - \Gamma^\beta G_{ij} \partial_i \Gamma^\gamma G_{ik},
\] (6.10)

if in addition \(k \geq 4\), then we have
\[
Q_\alpha \lesssim X_k E_k^{1/2}.
\] (6.11)
Proof. Using integration by parts and (6.2), combining with Young’s inequality, we have

\[
\|(t-r)\nabla G\|^2 = \int (t-r)^2 \partial_j G_{ik} \partial_j G_{ik} \nonumber
\]
\[
= \int (t-r)^2 \partial_j G_{ik} \partial_k G_{ij} + Q \nonumber
\]
\[
= \int 2(t-r)w_j G_{ik} \partial_k G_{ij} - (t-r)^2 G_{ik} \partial_j G_{ij} + Q \nonumber
\]
\[
= \int 2(t-r)w_j G_{ik} \partial_k G_{ij} - 2(t-r)w_k G_{ik} \partial_j G_{ij} \nonumber
\]
\[
+ (t-r)^2 \partial_k G_{ik} \partial_j G_{ij} + Q \leq 4\|G\|^2 + \frac{3}{2} \|(t-r)\nabla G\|^2 \nonumber
\]
\[
+ \frac{1}{2} \|(t-r)\nabla G\|^2 + Q, \quad (6.12) \nonumber
\]

hence

\[
\|(t-r)\nabla G\|^2 \lesssim \|G\|^2 + \|(t-r)\nabla G\|^2 + Q, \quad (6.13) \nonumber
\]

here

\[
Q = \int (t-r)^2 \partial_j G_{ik} (G_{ik} \partial_j G_{ij} - G_{ij} \partial_j G_{ik}). \quad (6.14) \nonumber
\]

Lemma 6.2 follows by applying this inequality to \(\Gamma^\alpha G\). On the other hand, since \(\beta + \gamma = \alpha\), \(|\alpha| \leq k - 1\), \(\gamma \neq \alpha\) and \(k \geq 4\), either \(|\beta| \leq k'\) or \(|\gamma| \leq k'\), with \(k' = \left[\frac{k-1}{2}\right]\). Hence, by Sobolev embedding \(H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)\), Lemma 4.2 we have

\[
Q_\alpha = \sum_{\beta + \gamma = \alpha, |\alpha| \leq k - 1} \int (t-r)^2 \partial_j \Gamma^\alpha G_{ik}(\Gamma^\beta G_{ik} \partial_j \Gamma^\gamma G_{ij} - \Gamma^\beta G_{ij} \partial_j \Gamma^\gamma G_{ik}) \nonumber
\]
\[
\lesssim \int (t-r)^2 |\nabla \Gamma^k G| (|\Gamma^{k'} G \nabla \Gamma^{k-1} G| + |\Gamma^{k-1} \nabla \Gamma^{k'} G|) \nonumber
\]
\[
\lesssim \|(t-r)\nabla \Gamma^{k-1} G\|^2 \|\Gamma^{k'} G\|_{L^\infty} + \|(t-r)\nabla \Gamma^{k-1} G\| \|\Gamma^{k-1} G\| \|\nabla \Gamma^{k'} G\|_{L^\infty} \nonumber
\]
\[
\lesssim X_k E_{k'+2}^{1/2} + X_k^{1/2} E_k^{1/2} X_{k'-3}^{1/2} \lesssim X_k E_k^{1/2}. \quad (6.15) \nonumber
\]

\[
\square \quad \text{Lemma 6.3.} \quad \text{Suppose that} \quad (v, F) = (v, I + G), \quad (v, G) \in H^k_\Gamma, \quad k \geq 1, \quad \text{solves} \quad (2.5). \quad \text{Define} \nonumber
\]

\[
L_k = \sum_{|\alpha| \leq k} \left[ |\Gamma^\alpha v| + |\Gamma^\alpha G| \right], \quad (6.16) \nonumber
\]

Then for all \(|\alpha| \leq k - 1\),

\[
|\nabla \cdot \Gamma^\alpha G \otimes w - \partial_t \Gamma^\alpha G| \lesssim L_k + N_k, \quad (6.17) \nonumber
\]
\[
|\nabla \Gamma^\alpha v - \partial_t \Gamma^\alpha v \otimes w| \lesssim L_k, \quad (6.18) \nonumber
\]
\[
|t \pm r| \nabla \Gamma^\alpha v \pm \nabla \Gamma^\alpha G \otimes w| \lesssim L_k + N_k. \quad (6.19) \nonumber
\]
Proof. By the decomposition of $\nabla$, we have
\[
(\nabla \cdot \Gamma^\alpha G \otimes w)_{ij} = \partial_k \Gamma^\alpha G_{ik} w_j = w_k \partial_r \Gamma^\alpha G_{ik} w_j - \frac{(w \wedge \Omega)_j}{r} \Gamma^\alpha G_{ik} w_j
\]
\[
= \partial_j \Gamma^\alpha G_{ik} w_k + \frac{(w \wedge \Omega)_j}{r} \Gamma^\alpha G_{ik} w_j - \frac{(w \wedge \Omega)_k}{r} \Gamma^\alpha G_{ik} w_j.
\] (6.20)
Notice that, by (3.5), we have
\[
\partial_j \Gamma^\alpha G_{ik} w_k = \partial_k \Gamma^\alpha G_{ij} w_k - h_\alpha w_k = \partial_r \Gamma^\alpha G_{ij} - h_\alpha w_k,
\] (6.21)
which is (6.17).

Similarly, by the decomposition of $\nabla$, we also have
\[
(\nabla \Gamma^\alpha v)_{ij} = \partial_j \Gamma^\alpha v_i = w_j \partial_r \Gamma^\alpha v_i - \frac{(w \wedge \Omega)_j}{r} \Gamma^\alpha v_i,
\] (6.23)
and using the fact that $\nabla \Gamma^\alpha G = S \Gamma^\alpha G - t f_\alpha + t \nabla \Gamma^\alpha p$, we can write
\[
t \nabla \cdot \Gamma^\alpha G + r \partial_r \Gamma^\alpha v = S \Gamma^\alpha G - t f_\alpha + t \nabla \Gamma^\alpha p,
\]
\[t \nabla \Gamma^\alpha v + r \partial_r \Gamma^\alpha G = S \Gamma^\alpha G - t g_\alpha.
\] (6.24)
This is rearranged as follows:
\[
t \nabla \cdot \Gamma^\alpha G \otimes w + r \nabla \Gamma^\alpha v = r \left[ \nabla \Gamma^\alpha v - \partial_r \Gamma^\alpha v \otimes w \right] + [S \Gamma^\alpha G - t f_\alpha + t \nabla \Gamma^\alpha p] \otimes w,
\]
\[t \nabla \Gamma^\alpha v + r \nabla \cdot \Gamma^\alpha G \otimes w = r \left[ \nabla \cdot \Gamma^\alpha G \otimes w - \partial_r \Gamma^\alpha G \right] + S \Gamma^\alpha G - t g_\alpha.
\] (6.25)
Then, combining with the inequality (6.17) (6.18), we have
\[
(t \pm r) |\nabla \Gamma^\alpha v \pm \nabla \cdot \Gamma^\alpha G \otimes w| \lesssim L_k + N_k,
\] (6.26)
which is inequality (6.19). \)

Lemma 6.4. Suppose that $(v, F) = (v, I + G)$, $(v, G) \in H^k_1$, $k \geq 4$, solves (2.5). If $E_k(t) \ll 1$, then $X_k(t) \lesssim E_k(t)$.

Proof. Starting with the definition of $X_k$, and using the fact that $|t - r| \leq 1 + |t - r|$, we obtain that
\[
X_k(t) = \sum_{|\alpha| \leq k-1} \left( \|t - r\| \nabla \Gamma^\alpha v \|^2 + \|t - r\| \nabla \Gamma^\alpha G \|^2 \right)
\]
\[
\lesssim E_k + \sum_{|\alpha| \leq k-1} \left( \|t - r\| \nabla \Gamma^\alpha v \|^2 + \|t - r\| \nabla \Gamma^\alpha G \|^2 \right).
\] (6.27)
Since
\[
\nabla \Gamma^\alpha v = \frac{1}{2} [\nabla \Gamma^\alpha v + \nabla \cdot \Gamma^\alpha G \otimes w] + \frac{1}{2} [\nabla \Gamma^\alpha v - \nabla \cdot \Gamma^\alpha G \otimes w],
\] (6.28)
\[
\n\nabla \cdot \Gamma^\alpha G = \frac{1}{2} [\nabla \Gamma^\alpha v + \nabla \cdot \Gamma^\alpha G \otimes w] w - \frac{1}{2} [\nabla \Gamma^\alpha v - \nabla \cdot \Gamma^\alpha G \otimes w] w, \tag{6.29}
\]

we see that
\[
|t - r|[|\nabla \Gamma^\alpha v| + |\nabla \cdot \Gamma^\alpha G|] \lesssim |t + r|(|\nabla \Gamma^\alpha v + \nabla \cdot \Gamma^\alpha G \otimes w| + |t - r|(|\nabla \Gamma^\alpha v - \nabla \cdot \Gamma^\alpha G \otimes w|).
\tag{6.30}
\]

It follows from (6.19) that
\[
\| (t - r) \nabla \Gamma^\alpha v \|^2 + \| (t - r) \nabla \cdot \Gamma^\alpha G \|^2 \lesssim E_k + \| N_k \|^2. \tag{6.31}
\]

Then by Lemma 6.2, Lemma 6.1, we obtain
\[
X_k(t) \lesssim E_k + \sum_{|\alpha| \leq k-1} \left( \| t - r |\nabla \Gamma^\alpha v \|^2 + \| t - r |\nabla \Gamma^\alpha G \|^2 \right)
\lesssim E_k + \sum_{|\alpha| \leq k-1} \left( \| t - r |\nabla \Gamma^\alpha v \|^2 + \| t - r |\nabla \cdot \Gamma^\alpha G \|^2 \right) + Q_\alpha \tag{6.32}
\lesssim E_k + \| N_k \|^2 + Q_\alpha
\lesssim E_k + E_k^2 + E_k X_k + X_k E_k^{1/2}.
\]

Under the assumption that \( E_k \ll 1 \), we obtain
\[
X_k \lesssim E_k. \tag{6.33}
\]

7 Estimate for Good Unknown

Lemma 7.1. Suppose that \((v, F) = (v, I + G), (v, G) \in H^k \), solves (2.5). If \( E_k(t) \ll 1 \), then for any \(|\alpha| + 4 \leq k \), we have the following estimate
\[
|\partial_r \Gamma^\alpha v + \partial_r \Gamma^\alpha G w|_{L^\infty(r \geq \frac{t}{16})} \lesssim \langle t \rangle^{-\frac{3}{2}} \left( E_{|\alpha|+3}^{1/2} + E_{|\alpha|+4} \right). \tag{7.1}
\]

Proof. Using (6.24), we have
\[
r \partial_r \Gamma^\alpha v + tw \cdot \nabla \Gamma^\alpha v + t \nabla \cdot \Gamma^\alpha G + r \partial_r \Gamma^\alpha G w = S \Gamma^\alpha v - tf_\alpha + t \nabla \Gamma^\alpha p + (S \Gamma^\alpha G - t g_\alpha) \cdot w, \tag{7.2}
\]

hence, using the decomposition of \( \nabla \) (2.16), we have
\[
(r + t) \partial_r \Gamma^\alpha v + \partial_r \Gamma^\alpha G w
= \frac{t}{r} (w \wedge \Omega) \cdot \Gamma^\alpha G + S \Gamma^\alpha v - tf_\alpha + t \nabla \Gamma^\alpha p + (S \Gamma^\alpha G - t g_\alpha) \cdot w. \tag{7.3}
\]
In the region \( r \geq \frac{(t)}{16} \), we still need to squeeze out an additional decay factor of \( \langle t \rangle^{-1/2} \). Multiplying this inequality with \( \langle t \rangle^{\frac{1}{2}} \), as \( r \geq \frac{(t)}{16} \), using Lemma 4.1

\[
\langle t \rangle^{\frac{1}{2}} (r + t) |\partial_i \Gamma^\alpha v + \partial_i \Gamma^\alpha G w|
\lesssim \langle t \rangle^{\frac{1}{2}} |(w \wedge \Omega) \cdot \Gamma^\alpha G| + \langle t \rangle^{\frac{1}{2}} |\nabla^\alpha v| + \langle t \rangle^{\frac{1}{2}} |tf_\alpha| + \langle t \rangle^{\frac{1}{2}} |\nabla^\alpha p|
\leq \langle t \rangle^{\frac{1}{2}} |\nabla^\alpha p| + \langle t \rangle^{\frac{1}{2}} |L_{\alpha} + 1| + \langle t \rangle^{\frac{1}{2}} N_{\alpha} + 1
\leq E_{\alpha+3} + E_{\alpha+4} + \langle t \rangle^{\frac{1}{2}} |\nabla^\alpha p|.
\]

By Sobolev embedding \( H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3) \), Lemma 5.1 we have

\[
\langle t \rangle^{\frac{3}{2}} |\nabla^\alpha p| \lesssim \langle t \rangle^{\frac{3}{2}} |\nabla^\alpha+2 p| \lesssim \sum_{\beta+\gamma=\alpha+2, |\beta| \leq |\gamma|} \langle t \rangle^{\frac{3}{2}} \| \partial_j \Gamma^\beta v_i \Gamma^\gamma v_j - \partial_j \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk} \|_{L^2}
= \sum_{\beta+\gamma=\alpha+2, |\beta| \leq |\gamma|} \langle t \rangle^{\frac{3}{2}} \| \partial_j \Gamma^\beta v_i \Gamma^\gamma v_j - \partial_j \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk} \|_{L^2(r \leq \frac{(t)}{16})}
+ \sum_{\beta+\gamma=\alpha+2, |\beta| \leq |\gamma|} \langle t \rangle^{\frac{3}{2}} \| \partial_j \Gamma^\beta v_i \Gamma^\gamma v_j - \partial_j \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk} \|_{L^2(r \geq \frac{(t)}{16})}
= P_1 + P_2,
\]

(7.5)

here and in what follows, the notation \( \Gamma^{\alpha+r} \) means \( \Gamma^{\alpha+\beta} \) for \( |\beta| = r \). By (4.5), and Lemma 6.4, \( P_1 \) is estimated as follows

\[
P_1 \lesssim \sum_{\beta+\gamma=\alpha+2, |\beta| \leq |\gamma|} \langle t \rangle^{\frac{3}{2}} \| \Gamma^\gamma U \nabla \Gamma^\beta U \|_{L^2(r \leq \frac{(t)}{16})}
\lesssim \| \langle t \rangle^{\Gamma^\alpha+2} U \|_{L^\infty(r \leq \frac{(t)}{16})} \| \langle t - r \rangle \nabla \Gamma^\alpha U \|
\lesssim (E_{\alpha+2} + X_{\alpha+4})^{\frac{1}{2}} E_{\alpha+4} \lesssim E_{\alpha+4}.
\]

(7.6)

By the decomposition of \( \nabla \) (2.16) and Lemma 4.3, Lemma 4.1

\[
P_2 \lesssim \sum_{\beta+\gamma=\alpha+2, |\beta| \leq |\gamma|} \| \langle t \rangle^{\frac{3}{2}} (w_j \partial_i \Gamma^\beta v_i \Gamma^\gamma v_j - w_j \partial_i \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk}) \|_{L^2(r \geq \frac{(t)}{16})}
+ \sum_{\beta+\gamma=\alpha+2, |\beta| \leq |\gamma|} \| \langle t \rangle^{\frac{3}{2}} ((w \wedge \Omega)_{ij} \Gamma^\beta v_i \Gamma^\gamma v_j - (w \wedge \Omega)_{ij} \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk}) \|_{L^2(r \geq \frac{(t)}{16})}
\lesssim E_{\alpha+4}.
\]

(7.7)

Combining all the estimates above, we obtain the estimate (7.1).

8 Local Energy Decay

In this section, we will prove that \( \langle t \rangle^{3/2} |\nabla \Gamma^\alpha U|_{L^\infty(r \leq \frac{(t)}{16})} \) is bounded by the energy, hence, we obtain \( (1+t)^{-3/2} \) time decay in \( L^\infty(r \leq \frac{(t)}{16}) \). This is important for us to prove our main theorem.
Lemma 8.1. Suppose that \((v, G)\) is a solution of

\[
\begin{aligned}
\partial_t v - \nabla \cdot G &= f, \\
\partial_t G - \nabla v &= g, \\
\nabla \cdot v &= 0, \quad \nabla \cdot G = 0,
\end{aligned}
\]  

(8.1)

then inside the cone \(r \leq \frac{t}{\delta},\) we have the following

\[
\|t^{3/2} \nabla^2 G\|_{L^2(r \leq \frac{t}{\delta})}^2 + \|t^{3/2} \nabla^2 v\|^2_{L^2(r \leq \frac{t}{\delta})} \lesssim \|\eta t^{3/2} \nabla^2 G\|^2 + \|\eta t^{3/2} \nabla^2 v\|^2 \\
\lesssim \|\eta t^{3/2} \nabla f\|^2 + \|\eta t^{1/2} \nabla S v\|^2 + \|\eta t^{1/2} \nabla^2 v\|^2 + \|\eta t^{1/2} \nabla g\|^2 \\
+ \|\eta t^{1/2} \nabla S G\|^2 + \|\eta t^{1/2} \nabla^2 G\|^2 + \|t^{1/2} \nabla G\|_{L^2(r \leq \frac{t}{\delta})}^2 + \|\eta t^{3/2} \nabla h_0\|^2,
\]  

(8.2)

here \(h_0 = G_{ij} \partial_i G_{ik} - G_{ik} \partial_i G_{ij}\) and \(\eta(t, x) = \varphi(r/t)\), \(\varphi\) is as before (Lemma 4.3).

Proof. Multiplying the first and second equation of (8.1) by \(t\), moving the \(t \partial_t\) terms on the right and rewriting them using the scaling operator gives

\[
\begin{aligned}
-t \nabla \cdot G &= tf - Sv + r \partial_r v = tf - Sv + x \cdot \nabla v, \\
-t \nabla v &= tg - SG + r \partial_r G = tg - SG + x \cdot \nabla G.
\end{aligned}
\]

(8.3)

Taking one more derivative on both sides of equation (8.3), we obtain

\[
\begin{aligned}
-t \nabla \cdot \nabla G &= tf \nabla f - \nabla (S v) + (x \cdot \nabla v) \nabla v, \\
-t \nabla^2 v &= tf \nabla g - \nabla (S G) + (x \cdot \nabla G) \nabla v.
\end{aligned}
\]

(8.4)

Equation (8.4) will be our starting point for the derivation of the estimates (8.5).

To begin we multiply equation (8.4) by \(\eta t^{1/2}\), and use the triangle inequality to obtain

\[
\begin{aligned}
\|\eta^{3/2} \nabla \cdot G\|^2 + \|\eta t^{3/2} \nabla^2 v\|^2 \\
\leq 4 \|\eta t^{3/2} \nabla f\|^2 + 4 \|\eta t^{1/2} \nabla S v\|^2 + 4 \|\eta t^{1/2} \nabla v\|^2 + 4 \|\eta t^{1/2} \nabla^2 v\|^2 \\
+ 4 \|\eta t^{3/2} \nabla g\|^2 + 4 \|\eta t^{1/2} \nabla S G\|^2 + 4 \|\eta t^{1/2} \nabla G\|^2 + 4 \|\eta t^{3/2} \nabla h_0\|^2.
\end{aligned}
\]

(8.5)

Notice that \(r \leq \frac{t}{\delta}\) on \(\text{supp}\eta\), hence we have

\[
\begin{aligned}
\|\eta^{3/2} \nabla \cdot G\|^2 + \|\eta t^{3/2} \nabla^2 v\|^2 \\
\leq 4 \|\eta t^{3/2} \nabla f\|^2 + 4 \|\eta t^{1/2} \nabla S v\|^2 + 4 \|\eta t^{1/2} \nabla v\|^2 + \frac{1}{4} \|\eta t^{1/2} \nabla^2 v\|^2 \\
+ 4 \|\eta t^{3/2} \nabla g\|^2 + 4 \|\eta t^{1/2} \nabla S G\|^2 + 4 \|\eta t^{1/2} \nabla G\|^2 + \frac{1}{4} \|\eta t^{3/2} \nabla^2 G\|^2 \\
+ \frac{1}{4} \|\eta t^{3/2} \nabla^2 v\|^2 + \frac{1}{4} \|\eta t^{3/2} \nabla^2 G\|^2.
\end{aligned}
\]

(8.6)
The last equality follows from (3.2), hence Lemma 8.2.

Suppose that (l for any Proof. Combining Lemma 8.1 with equations (3.3) give us
we obtain
Combining with (8.6), we obtain
Then Lemma 8.1 follows.

Let

The last equality follows from (3.2), hence

\[
\|\eta^{3/2} \nabla G\|^2 \leq 16 \max|\eta| \|\eta^{3/2} \nabla G\|^2 \left( t^{1/2} \nabla G \right)_{L^2(r \leq \frac{\alpha}{4})} + 16 \max|\eta| \|\eta^{3/2} \nabla^2 G\|^2 \left( t^{1/2} \nabla G \right)_{L^2(r \leq \frac{\alpha}{4})} + \|\eta^{3/2} \nabla^2 G\|^2 \left( \nabla G \right)_{L^2(r \leq \frac{\alpha}{4})} + \|\eta^{3/2} \nabla h_0\|^2.
\]

(8.9)

Combining with (8.6), we obtain

\[
\|\eta^{3/2} \nabla^2 G\|^2 + \|\eta^{3/2} \nabla v\|^2 
\leq \|\eta^{3/2} \nabla f\|^2 + \|\eta t^{1/2} \nabla S v\|^2 + \|\eta t^{1/2} \nabla^2 v\|^2 + \|\eta t^{1/2} \nabla v\|^2 + \|\eta t^{3/2} \nabla g\|^2 
+ \|\eta t^{1/2} \nabla S G\|^2 + \|\eta t^{1/2} \nabla^2 G\|^2 + \|t^{1/2} \nabla G\|^2 \left( \eta^{3/2} \nabla h_0\right)_{L^2(r \leq \frac{\alpha}{4})}.
\]

(8.10)

Then Lemma 8.1 follows.

\[
\Psi_l(U) = \sum_{|\alpha| \leq l} \|\eta(t)^{3/2} \nabla^2 \Gamma^\alpha U\|.
\]

(8.11)

**Lemma 8.2.** Suppose that \((v, F) = (v, I + G)\), \((v, G) \in H^k_1\), solves (2.5). If \(E_k(t) \ll 1\), then for any \(l + 4 \leq k\), we have \(\Psi_l(U) \lesssim E_{l+4}^\frac{1}{4}\).

**Proof.** Combining Lemma 8.1 with equations (3.3) give us

\[
\Psi_l^2(U) \lesssim \sum_{|\alpha| \leq l} \|\eta(t)^{3/2} \nabla^2 \Gamma^\alpha U\|^2 
\lesssim E_{l+2} + \|t^{3/2} \nabla \Gamma^\alpha f_\alpha\|^2 + \|\eta^{3/2} \nabla f_\alpha\|^2 + \|\eta t^{1/2} \nabla G\|^2 
+ \|t^{1/2} \nabla \Gamma^\alpha G\|^2 + \|\eta^{3/2} \nabla S G\|^2 + \|\eta t^{1/2} \nabla S G\|^2 
+ \|t^{1/2} \nabla \Gamma^\alpha G\|^2 + \|t^{1/2} \nabla G\|^2 \left( \eta^{3/2} \nabla h_0\right)_{L^2(r \leq \frac{\alpha}{4})} + \|\eta^{3/2} \nabla h_0\|^2.
\]

(8.12)
In a similar fashion as we estimate (7.5) - (7.7) earlier, we have
\[ t^{3/2} \| \nabla \Gamma^{\alpha+1} p \| \lesssim E_{l+4}. \] (8.13)

On the other hand, by the definition of \( f_{\alpha}, g_{\alpha}, h_{\alpha} \), inequality (1.3) and Lemma 6.4, we have
\[ \| \eta t^{3/2} \nabla f_{\alpha} \|^2 + \| \eta t^{3/2} \nabla g_{\alpha} \|^2 + \| \eta t^{3/2} \nabla h_{\alpha} \|^2 \lesssim E_{l+4}^2, \] (8.14)
hence, we have
\[ \Psi_l^2(U) \lesssim E_{l+2} + E_{l+4}^2 + X_{l+2} \lesssim E_{l+4}. \] (8.15)

\[ \square \]

Remark 8.1. By Lemma [4.3] replacing \( f \) by \( \langle t \rangle^{1/2} \nabla f \), we have
\[ \langle t \rangle^{3/2} \| \nabla f \|_{L^\infty(r \leq \frac{\omega}{16})} \lesssim \langle t \rangle^{1/2} \| \nabla f \|_{L^2(r \leq \frac{\omega}{4})} + \langle t \rangle^{3/2} \| \nabla^2 f \|_{L^2(r \leq \frac{\omega}{4})} + \langle t \rangle^{3/2} \| \nabla^3 f \|_{L^2(r \leq \frac{\omega}{4})}, \] (8.16)
then, for any \( \alpha \), we have
\[ \langle t \rangle^{3/2} \| \nabla \Gamma^\alpha f \|_{L^\infty(r \leq \frac{\omega}{16})} \lesssim X_{|\alpha|+1}^\frac{3}{2} + \langle t \rangle^{3/2} \| \nabla^2 \Gamma^\alpha f \|_{L^2(r \leq \frac{\omega}{4})} + \langle t \rangle^{3/2} \| \nabla^3 \Gamma^\alpha f \|_{L^2(r \leq \frac{\omega}{4})}; \] (8.17)
Hence, inside the cone, under the condition of Lemma 8.2 for any \( |\alpha| + 5 \leq k \), we have
\[ \langle t \rangle^{3/2} \| \nabla \Gamma^\alpha U \|_{L^\infty(r \leq \frac{\omega}{16})} \lesssim E_{|\alpha|+5}^\frac{3}{2}. \] (8.18)

9 Energy Estimate with a Ghost Weight

Choosing \( q = q(t-r) \), with \( q(\sigma) = \int_0^\sigma \frac{1}{1+\sigma} d\sigma \) so that \( q'(\sigma) = \frac{1}{1+\sigma^2} \) and \( |q(\sigma)| \leq \frac{\sigma}{2} \). Let \( |\alpha| \leq k \), taking the inner product of the first and second equation in (3.3) with \( e^{-q} \Gamma^\alpha v \) and \( e^{-q} \Gamma^\alpha G \) respectively, and then adding them up, one has
\[ \int \left( e^{-q} \partial_t (|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2) - 2e^{-q} (\Gamma^\alpha v_i \partial_j \Gamma^\alpha G_{ij} + \Gamma^\alpha G_{ij} \partial_j \Gamma^\alpha v_i) \right) dx \]
\[ = -2 \int e^{-q} \Gamma^\alpha v \cdot \nabla \Gamma^\alpha v dx + 2 \int e^{-q} (f_{\alpha} \cdot \Gamma^\alpha v + (g_{\alpha})_{ij} \Gamma^\alpha G_{ij}) dx. \] (9.1)
Integration by parts gives that
\[
\frac{d}{dt} \int e^{-q(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)}dx \\
= - \int e^{-q}[\partial_t q(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2) - 2 \partial_j q \Gamma^\alpha v_i \Gamma^\alpha G_{ij}]dx \\
- 2 \int e^{-q} \Gamma^\alpha v \cdot \nabla \Gamma^\alpha p dx + 2 \int e^{-q} (f_\alpha \cdot \Gamma^\alpha v + (g_\alpha)_{ij} \Gamma^\alpha G_{ij})dx \\
= - \int \frac{e^{-q}}{1 + (t - r)^2} \left[ (|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2) + 2w_j \Gamma^\alpha v_i \Gamma^\alpha G_{ij} \right]dx \\
- 2 \int e^{-q} \Gamma^\alpha v \cdot \nabla \Gamma^\alpha p dx + 2 \int e^{-q} (f_\alpha \cdot \Gamma^\alpha v + (g_\alpha)_{ij} \Gamma^\alpha G_{ij})dx \\
= - \int \frac{e^{-q}}{1 + (t - r)^2} \left[ (|\Gamma^\alpha v + \Gamma^\alpha G\omega|^2 + |\Gamma^\alpha G - \Gamma^\alpha G w \otimes w|^2) - 2 \int e^{-q} \Gamma^\alpha v \cdot \nabla \Gamma^\alpha p dx. \right.
\] (9.2)

That is
\[
\frac{d}{dt} \int e^{-q(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)}dx \\
+ \int \frac{e^{-q}}{1 + (t - r)^2} \left[ (|\Gamma^\alpha v + \Gamma^\alpha G\omega|^2 + |\Gamma^\alpha G - \Gamma^\alpha G w \otimes w|^2) \right]dx \right.
\] (9.3)

We emphasize that here we do not use integration by parts in the term involving pressure. We also point out that we cannot use the approach of Lemma 6.1 to estimate the nonlinear terms because we now have that $|\alpha| \leq k$ rather than $|\alpha| \leq k - 1$ as we had earlier. To simplify the notation a bit, we shall write
\[
z = \sum_{|\alpha| \leq k} \left( |\Gamma^\alpha v + \Gamma^\alpha G\omega|^2 + |\Gamma^\alpha G - \Gamma^\alpha G w \otimes w|^2 \right), \quad (9.4)
\]
and
\[
Z = \int \frac{e^{-q}}{1 + (t - r)^2} z dx \\
= \sum_{|\alpha| \leq k} \int \frac{e^{-q}}{1 + (t - r)^2} \left( |\Gamma^\alpha v + \Gamma^\alpha G\omega|^2 + |\Gamma^\alpha G - \Gamma^\alpha G w \otimes w|^2 \right) dx. \quad (9.5)
\]

Let us first treat the first term in (9.3). Recall that $f_\alpha$ and $g_\alpha$ are given by (3.4). By
To estimate the last term in this equality, by integration by parts, we have

\[
\int e^{-q} [f_\alpha \cdot \Gamma^\alpha v + (g_\alpha)_{ij} \Gamma^\alpha G_{ij}] \, dx
= \sum_{\beta + \gamma = \alpha, \gamma \neq \alpha} \int e^{-q} \Gamma^\alpha v_1 [\partial_j \Gamma^\gamma G_{ik} \Gamma^\beta G_{jk} - \Gamma^\beta v_j \partial_j \Gamma^\gamma v_1] \, dx
+ \sum_{\beta + \gamma = \alpha, \gamma \neq \alpha} \int e^{-q} \Gamma^\alpha G_{ik} [\partial_j \Gamma^\gamma v_1 \Gamma^\beta G_{jk} - \Gamma^\beta v_j \partial_j \Gamma^\gamma G_{ik}] \, dx
+ \frac{1}{2} \int e^{-q} \partial_j [2 \Gamma^\alpha v_1 \Gamma^\alpha G_{ik} G_{jk} - v_j (|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)] \, dx
= I + II + III.
\]

To estimate the last term in this equality, by integration by parts, we have

\[
2III = \int e^{-q} \partial_j [2 \Gamma^\alpha v_1 \Gamma^\alpha G_{ik} G_{jk} - v_j (|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)] \, dx \\
= - \int \frac{e^{-q}}{1 + (t - r)^2} [2 \Gamma^\alpha v_1 \Gamma^\alpha G_{ik} G_{jk} - v_j (|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)] \, dx.
\]

Inside the cone ($|x| \leq \frac{(t)}{16}$), this term is bounded by

\[
2III \lesssim \frac{1}{(1 + t)^2} E_k^{3/2}.
\]

Away from the origin ($|x| \geq \frac{(t)}{16}$), by Lemma 4.3, this term is bounded by

\[
2III \lesssim \frac{1}{(1 + t)^{3/2}} E_k E_\gamma^{1/2} \lesssim \frac{1}{(1 + t)^{3/2}} E_k^{3/2}.
\]

Next, we are going to estimate the terms I and II.

### 9.1 Inside the cone

On the region: $|x| \leq \frac{(t)}{16}$. The terms $I$ and $II$ are bounded by

\[
\sum_{\beta + \gamma = \alpha, |\alpha| \leq k', r \leq \frac{(t)}{16}} e^{-q} \Gamma^\alpha v_1 [\partial_j \Gamma^\gamma G_{ik} \Gamma^\beta G_{jk} - \Gamma^\beta v_j \partial_j \Gamma^\gamma v_1] \, dx
+ \sum_{\beta + \gamma = \alpha, |\alpha| \leq k', r \leq \frac{(t)}{16}} e^{-q} \Gamma^\alpha G_{ik} [\partial_j \Gamma^\gamma v_1 \Gamma^\beta G_{jk} - \Gamma^\beta v_j \partial_j \Gamma^\gamma G_{ik}] \, dx
\lesssim \sum_{\beta + \gamma = \alpha, |\alpha| \leq k', r \leq \frac{(t)}{16}} |\Gamma^\alpha U \Gamma^\beta U \nabla \Gamma^\gamma U|.
\]

We make use of the fact that since $\beta + \gamma = \alpha$, $|\alpha| \leq k$, $\gamma \neq \alpha$, either $|\beta| \leq k'$ or $|\gamma| \leq k'$, with $k' = \frac{k}{2}$. Note that $k \geq 9$, hence, we have $k' + 5 \leq k$. 

In the first case, using (4.5) we have:

$$
\sum_{\beta + \gamma = \alpha, |\alpha| \leq k} \int_{r \leq \frac{\langle t \rangle}{16}} |\Gamma^{\alpha} U \Gamma^{\beta} U \nabla \Gamma^{\gamma} U| \lesssim \frac{1}{\langle t \rangle^2} \|\Gamma^{k} U\|_{L^2(r \leq \frac{\langle t \rangle}{16})} \|\langle t \rangle^{\frac{3}{2}} \nabla \Gamma^{k'} U\|_{L^\infty(r \leq \frac{\langle t \rangle}{16})}
$$

(9.11)

In the second case, using (8.18):

$$
\sum_{\beta + \gamma = \alpha, |\alpha| \leq k} \int_{r \leq \frac{\langle t \rangle}{16}} |\Gamma^{\alpha} U \Gamma^{\beta} U \partial_t \Gamma^{\gamma} U| \lesssim \frac{1}{\langle t \rangle^{3/2}} \|\Gamma^{k} U\|_{L^2(r \leq \frac{\langle t \rangle}{16})} \|\langle t \rangle^{3/2} \nabla \Gamma^{k'} U\|_{L^\infty(r \leq \frac{\langle t \rangle}{16})}
$$

(9.12)

9.2 Away from the origin

On the region: $|x| \geq \frac{\langle t \rangle}{16}$ By the decomposition of $\nabla$ (2.16), the terms I and II can be written as

$$
\sum_{\beta + \gamma = \alpha, |\alpha| \leq k} \int_{r \geq \frac{\langle t \rangle}{16}} e^{-q \Gamma^{\alpha} v_i} \left[ \partial_j \Gamma^{\gamma} G_{ik} \Gamma^{\beta} G_{jk} - \Gamma^{\beta} v_j \partial_j \Gamma^{\gamma} v_i \right] dx
$$

$$
+ \int_{r \geq \frac{\langle t \rangle}{16}} e^{-q \Gamma^{\alpha} G_{ik} \left[ \partial_j \Gamma^{\gamma} v_i \Gamma^{\beta} G_{jk} - \Gamma^{\beta} v_j \partial_j \Gamma^{\gamma} G_{ik} \right]} dx
$$

$$
= \sum_{\beta + \gamma = \alpha, |\alpha| \leq k} \int_{r \geq \frac{\langle t \rangle}{16}} e^{-q \Gamma^{\alpha} v_i} \left[ w_j \partial_j \Gamma^{\gamma} G_{ik} \Gamma^{\beta} G_{jk} - \Gamma^{\beta} v_j w_j \partial_j \Gamma^{\gamma} v_i \right] dx
$$

(9.13)

$$
+ \int_{r \geq \frac{\langle t \rangle}{16}} e^{-q \Gamma^{\alpha} G_{ik} \left[ w_j \partial_j \Gamma^{\gamma} v_i \Gamma^{\beta} G_{jk} - \Gamma^{\beta} v_j w_j \partial_j \Gamma^{\gamma} G_{ik} \right]} dx
$$

$$
+ \int_{r \geq \frac{\langle t \rangle}{16}} R_\alpha dx,
$$

here

$$
|R_\alpha| \lesssim \frac{1}{r} \sum_{\beta + \gamma = \alpha, |\alpha| \leq k} |(\Gamma^{\alpha} v, \Gamma^{\alpha} G)||(|\Gamma^{\beta} v, \Gamma^{\beta} G)||(|\Omega^{\gamma} v, \Omega^{\gamma} G)|,
$$

(9.14)

since $\beta + \gamma = \alpha, \gamma \neq \alpha, |\alpha| \leq k,$ using Lemma 4.1 the last term is bounded by

$$
\int_{r \geq \frac{\langle t \rangle}{16}} R_\alpha dx \lesssim \frac{1}{\langle t \rangle^2} \|\Gamma^{k} U\|^2_{L^2} |r \Gamma^{\frac{3}{2}} + 1 U|_{L^\infty} \lesssim \frac{1}{\langle t \rangle^2} E^3_k,
$$

(9.15)
On the other hand, we have the following simply calculation,

\[
\sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha v_i} [w_j\partial_r \Gamma^\gamma G_{ik} \Gamma^\beta G_{jk} - \Gamma^\beta v_j w_j \partial_r \Gamma^\gamma v_i] \\
= - \sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha v_i} (\Gamma^\beta v_j + \Gamma^\beta G_{jl} w_l) w_j \partial_r \Gamma^\gamma v_i \\
+ \sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha v_i} \Gamma^\beta G_{jl} w_l (w_j \partial_r \Gamma^\gamma v_i + w_j \partial_r \Gamma^\gamma G_{ik} w_k) \\
+ \sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha v_i} (\partial_r \Gamma^\gamma G(I - w \otimes w))_{il} w_j (\Gamma^\beta G(I - w \otimes w))_{jl} \\
= I_1 + I_2 + I_3.
\]

And

\[
\sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha G_{ik}} [w_j\partial_r \Gamma^\gamma v_i \Gamma^\beta G_{jk} - \Gamma^\beta v_j w_j \partial_r \Gamma^\gamma G_{ik}] \\
= \sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha G_{ik}} (w_j \partial_r \Gamma^\gamma v_i + w_j \partial_r \Gamma^\gamma G_{il} w_l) \Gamma^\beta G_{jk} \\
- \sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha G_{ik}} (\Gamma^\beta v_j + \Gamma^\beta G_{jl} w_l) w_j \partial_r \Gamma^\gamma G_{ik} \\
+ \sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha G_{ik}} w_j \Gamma^\beta G_{jl} w_l (\partial_r \Gamma^\gamma G(I - w \otimes w))_{ik} \\
- \sum_{\beta+\gamma=\alpha,|\alpha|\leq k \atop \gamma \neq \alpha} \int_{r \geq \frac{(t)}{16}} e^{-q\Gamma^\alpha G_{ik}} w_j \partial_r \Gamma^\gamma G_{il} w_l (\Gamma^\beta G(I - w \otimes w))_{jk} \\
= II_1 + II_2 + II_3 + II_4.
\]

In the following, we will estimate the above terms in the region \( r \geq \frac{(t)}{16} \) one by one. We make use of the fact that since \( \beta + \gamma = \alpha \), \(|\alpha| \leq k\) and \( k \geq 9\), either \(|\beta| \leq k'\) or \(|\gamma| \leq k'\), with \( k' = \left\lceil \frac{k}{2} \right\rceil \). Hence, we have
\[ I_1 = - \sum_{\beta + \gamma = a, |\alpha| \leq k} \int_{r \geq \langle t \rangle^{(k)}} e^{-q\Gamma^\alpha v_i (\Gamma^\beta v_j + \Gamma^\gamma G_{j_i} w_i)} w_j \partial_r \Gamma^\gamma v_i \]

\[ \lesssim \sum_{\beta + \gamma = a, |\alpha| \leq k} \int_{r \geq \langle t \rangle^{(k)}} |\Gamma^\alpha v||\Gamma^\beta v w + w \Gamma^\beta G w||\partial_r \Gamma^\gamma v| \]

\[ \lesssim \int_{r \geq \langle t \rangle^{(k)}} \frac{1}{r^{3/2}} |\Gamma^k v|r^{3/2}|\Gamma^k v w + w \Gamma^k G w||\partial_r \Gamma^{k-1} U| \]

\[ + \int_{r \geq \langle t \rangle^{(k)}} |\Gamma^k v| \frac{1}{\langle r \rangle \langle t - r \rangle} |\Gamma^k v + \Gamma^k G w| \langle r \rangle \langle t - r \rangle |\partial_r \Gamma^k U| , \]

the first term of this inequality is estimated by Lemma 4.4 and the second term is estimated by Lemma 4.2.

\[ I_1 \lesssim \frac{1}{(1 + t)^{3/2}} E_k^2 E_k^{1/2} E_k^{1/2} + \frac{1}{1 + t} E_k^{1/2} Z_k^{1/2} X_k^{1/2} + 3 \]

\[ \lesssim \frac{1}{(1 + t)^{3/2}} E_k^2 + \frac{1}{1 + t} E_k Z_k^{1/2} . \] (9.19)

Similarly, the terms \( I_2 \) can be bounded as follows

\[ I_2 = \sum_{\beta + \gamma = a, |\alpha| \leq k} \int_{r \geq \langle t \rangle^{(k)}} e^{-q\Gamma^\alpha v_i (\Gamma^\beta G_{j_i} w_i + w_j \partial_r \Gamma^\gamma G_{i_k} w_k)} \]

\[ \lesssim \sum_{\beta + \gamma = a, |\alpha| \leq k} \int_{r \geq \langle t \rangle^{(k)}} |\Gamma^\alpha v||w \Gamma^\beta G w||\partial_r \Gamma^\gamma G w| \]

\[ \lesssim \int_{r \geq \langle t \rangle^{(k)}} |\Gamma^k v||w \Gamma^k G w||\partial_r \Gamma^{k-1} v + \partial_r \Gamma^k U| \]

\[ + \int_{r \geq \langle t \rangle^{(k)}} |\Gamma^k v||w \Gamma^k G w||\partial_r \Gamma^k v + \partial_r \Gamma^k G w| . \]

The first term is handled using Lemma 4.4, the second term is handled using (7.1), thus we have

\[ II_1 \lesssim \frac{1}{(1 + t)^{3/2}} E_k^2 E_k^{1/2} E_k^{1/2} + \frac{1}{(1 + t)^{3/2}} E_k (E_k^{1/2} + E_k^{1/2}) + 3 \]

\[ \lesssim \frac{1}{(1 + t)^{3/2}} E_k^2 . \] (9.21)
The terms $I_3$ are handled in a similar fashion as $I_1$, $I_2$

$$I_3 = \sum_{\beta+\gamma=\alpha, |\alpha| \leq k, \gamma \neq \alpha} \int r \geq (t)^{16} e^{-q \Gamma^\alpha v_i (\partial_t \Gamma^\gamma G(I - w \otimes w))_{ij} w_j (\Gamma^\beta G(I - w \otimes w))_{jl}}$$

$$\lesssim \sum_{\beta+\gamma=\alpha, |\alpha| \leq k, \gamma \neq \alpha} \int r \geq (t)^{16} |\Gamma^\alpha v| |\partial_t \Gamma^\gamma G(I - w \otimes w)||w\Gamma^\beta G(I - w \otimes w)|$$

$$\lesssim \int \frac{r}{r \geq (t)^{16}} |\Gamma^k v||\partial_t \Gamma^{k-1} G(I - w \otimes w)||w\Gamma^{k'} G(I - w \otimes w)| (9.22)$$

$$+ \int \frac{r}{r \geq (t)^{16}} |\Gamma^k v||\partial_t \Gamma^{k'} G(I - w \otimes w)||w\Gamma^k G(I - w \otimes w)|$$

$$\lesssim \frac{1}{(1 + t)^3/2} E_k^3 E_{k'}^3 E_{k'+2}^3 + \frac{1}{1 + t} E_k^3 Y_{k'+3}^3 Z_{k'+3}^3$$

$$\lesssim \frac{1}{(1 + t)^3/2} E_k^3 + \frac{1}{1 + t} E_k Z_{k'}^3. (9.23)$$

It remains to estimate the four terms of $II$. In a similar fashion, the first term of $II$ is bounded by

$$II_1 = \sum_{\beta+\gamma=\alpha, |\alpha| \leq k'} \int r \geq (t)^{16} e^{-q \Gamma^\alpha G_{ik}(w_j \partial_t \Gamma^\gamma v_i + w_j \partial_t \Gamma^\gamma G_{il} w_l)\Gamma^\beta G_{jk}}$$

$$\lesssim \sum_{\beta+\gamma=\alpha, |\alpha| \leq k'} \int r \geq (t)^{16} |\Gamma^\alpha G||\partial_t \Gamma^\gamma v + \partial_t \Gamma^\gamma G w||w\Gamma^\beta G| (9.23)$$

$$\lesssim \int \frac{r}{r \geq (t)^{16}} |\Gamma^k G| r^{-3/2} (\partial_t \Gamma^{k-1} v + \partial_t \Gamma^{k-1} G w) (r^{3/2} w\Gamma^{k'} G)$$

$$+ \int \frac{r}{r \geq (t)^{16}} |\Gamma^k G|(\partial_t \Gamma^{k'} v + \partial_t \Gamma^{k'} G w) w\Gamma^k G.$$
The second term of $II$ is handled in a similar fashion as $I_1$:

$$II_2 = - \sum_{\beta+\gamma=\alpha, |\alpha| \leq k} \int_{r \geq (\frac{\Omega}{10})} e^{-q\Gamma^{\alpha}} G_{ik}(\Gamma^\beta v_j + \Gamma^\beta G_{jl} w_l) w_j \partial_r \Gamma^\gamma G_{ik} \nonumber$$

$$\leq \sum_{\beta+\gamma=\alpha, |\alpha| \leq k} \int_{r \geq (\frac{\Omega}{10})} |\Gamma^{\alpha} G| |\Gamma^\beta v + \Gamma^\beta G w| |\partial_r \Gamma^\gamma G| \nonumber$$

$$\leq \int_{r \geq (\frac{\Omega}{10})} |\Gamma^{k} G||w \Gamma^{k'} G| |\nabla \Gamma^{k-1} G| \quad (9.25)$$

$$+ \int_{r \geq (\frac{\Omega}{10})} |\Gamma^{k} G| |\frac{1}{|r|} \frac{1}{|l-r|} |\Gamma^{k} v w + \Gamma^{k} G w| |\langle r \rangle \langle t-r \rangle |\partial_r \Gamma^{k'} G| \nonumber$$

$$\leq \frac{1}{(1+t)^{3/2}} E_k^\frac{3}{2} E_{k'}^{\frac{3}{2}} + \frac{1}{1+t} E_k Z_k^{\frac{3}{2}} X_{k'}^{\frac{3}{2}} + 3$$

$$\leq \frac{1}{(1+t)^{3/2}} E_k^\frac{3}{2} + \frac{1}{1+t} E_k Z_k^{\frac{3}{2}}. \nonumber$$

The third term is bounded by

$$II_3 = \sum_{\beta+\gamma=\alpha, |\alpha| \leq k} \int_{r \geq (\frac{\Omega}{10})} e^{-q\Gamma^{\alpha}} w_j \Gamma^\beta G_{jl} w_l (\partial_r \Gamma^\gamma G(1 - w \otimes w))_{ik} \nonumber$$

$$\leq \int_{r \geq (\frac{\Omega}{10})} |\Gamma^{k} G||w \Gamma^{k'} G| |\nabla \Gamma^{k-1} G| \quad (9.26)$$

$$+ \sum_{\beta+\gamma=\alpha, |\alpha| \leq k} \int_{r \geq (\frac{\Omega}{10})} |\Gamma^{k} G_{ik} w_j \Gamma^\beta G_{jl} w_l (\partial_r \Gamma^{k'} G(1 - w \otimes w))_{ik}|, \nonumber$$

using Lemma 4.4 we have

$$\int_{r \geq (\frac{\Omega}{10})} |\Gamma^{k} G||w \Gamma^{k'} G| |\nabla \Gamma^{k-1} G| \nonumber$$

$$\leq \frac{1}{(1+t)^{3/2}} E_k^\frac{3}{2}, \quad (9.27)$$

on the other hand, in order to estimate the second term of $II_3$, we use the decomposition of $\nabla \quad (2.16)$ and the constraint of $G \quad (3.5)$,

$$\Gamma^{k} G_{ik} w_j \Gamma^\beta G_{jl} w_l (\partial_r \Gamma^{k'} G(1 - w \otimes w))_{ik} \nonumber$$

$$= \Gamma^{k} G_{ik} \Gamma^{k} G_{jl} w_l (w_j \partial_r \Gamma^{k'} G_{ik} - w_j \partial_r \Gamma^{k'} G_{ih} w_h w_k) \nonumber$$

$$= \Gamma^{k} G_{ik} \Gamma^{k} G_{jl} w_l (w_j \partial_r \Gamma^{k'} G_{ik} - \partial_j \Gamma^{k'} G_{ih} w_h w_k - \frac{(w \otimes \Omega)}{r} \Gamma^{k'} G_{ih} w_h w_k) \quad (9.28)$$

$$= \Gamma^{k} G_{ik} \Gamma^{k} G_{jl} w_l (w_j \partial_r \Gamma^{k'} G_{ik} - \partial_i \Gamma^{k'} G_{ij} w_k w_k - h_k w_h w_k - \frac{(w \otimes \Omega)}{r} \Gamma^{k'} G_{ih} w_h w_k) \nonumber$$

$$= \Gamma^{k} G_{ik} \Gamma^{k} G_{jl} w_l (w_j \partial_r \Gamma^{k'} G_{ik} - \partial_j \Gamma^{k'} G_{ij} w_k - h_k w_h w_k - \frac{(w \otimes \Omega)}{r} \Gamma^{k'} G_{ih} w_h w_k),$$
while
\[
w_j \partial_t \Gamma^{kl} G_{ik} - \partial_k \Gamma^{kl} G_{ij} w_k
\]
\[
= \partial_j \Gamma^{kl} G_{ik} - \partial_k \Gamma^{kl} G_{ij} + \frac{(w \wedge \Omega)_k^j}{r} \Gamma^{kl} G_{ik} - \frac{(w \wedge \Omega)_k^j}{r} \Gamma^{kl} G_{ij}
\]
\[
= h_{kl}^j + \frac{(w \wedge \Omega)_k^j}{r} \Gamma^{kl} G_{ik} - \frac{(w \wedge \Omega)_k^j}{r} \Gamma^{kl} G_{ij},
\]
hence
\[
\sum_{\beta + \gamma = \alpha, |\alpha| \leq k} \int_{r \geq \frac{1}{16}} |\Gamma^k G_{ik} w_j \Gamma^k G_{jl} (\partial_r \Gamma^k G(I - w \otimes w))_{ik}|
\]
\[
\lesssim \int_{r \geq \frac{1}{16}} |\Gamma^k G||\Gamma^k G|(h_{kl}^j + \frac{\Gamma^k G}{r}) \lesssim \frac{1}{(1+t)^2} E_0^\frac{4}{3}.
\]

The last term of \( II \) is treated as follows
\[
\begin{align*}
II_4 &= - \sum_{\beta + \gamma = \alpha, |\alpha| \leq k} \int_{r \geq \frac{1}{16}} e^{-q} \Gamma^\alpha G_{ik} w_j \partial_r \Gamma^\gamma G_{jl} w_l (\Gamma^\beta G(I - w \otimes w))_{jk} \\
&\lesssim \sum_{\beta + \gamma = \alpha, |\alpha| \leq k} \int_{r \geq \frac{1}{16}} |\Gamma^\alpha G||\partial_r \Gamma^\gamma G w||w \Gamma^\beta G(I - w \otimes w)| \\
&\lesssim \int_{r \geq \frac{1}{16}} |\Gamma^k G||\partial_r \Gamma^{k-1} G w||w \Gamma^{k'} G(I - w \otimes w)| \\
&\quad + \int_{r \geq \frac{1}{16}} |\Gamma^k G||\partial_r \Gamma^{k'} G w||w \Gamma^{k} G(I - w \otimes w)| \\
&\lesssim \frac{1}{(1+t)^{3/2}} E_0^\frac{3}{2} E_0^\frac{3}{2} E_0^{k+2} + \frac{1}{1+t} E_0^\frac{1}{2} X_0^{\frac{3}{2}} Z_0^{\frac{1}{2}} \\
&\lesssim \frac{1}{(1+t)^{3/2}} E_0^\frac{3}{2} + \frac{1}{1+t} E_0 Z_0^{\frac{1}{2}}.
\end{align*}
\]

It remains to treat the pressure term in \((9.3)\). However, thanks to Lemma \[5.1\] this term is handled exactly as the preceding ones.

Finally, we gather our estimate for \((9.3)\) to get
\[
\tilde{E}_k'(t) + Z \leq \mu Z + C_\mu \frac{1}{(1+t)^{3/2}} E_0^{3/2},
\]
with
\[
\tilde{E}_k = \sum_{|\alpha| \leq k} \int e^{-q}(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2) dx.
\]

Notice that \( E_k \sim \tilde{E}_k \), we obtain, for \( \mu \) small
\[
\tilde{E}_k'(t) \leq C_\mu \frac{1}{(1+t)^{3/2}} \tilde{E}_0^{3/2},
\]
This implies that \( E_k(t) \) remains bounded by \( Me^2 \) for all time, here \( M \) is a positive constant which depends only on \( k \). Hence we complete the proof of Theorem \[14\].
10 General Isotropic Elastodynamics

For general isotropic elastodynamics, the energy functional has the form

\[ W(F) = W(QF) = W(FQ), \]  

(10.1)

for all rotation matrices: \( Q = Q^T, \det Q = 1 \). The first relation is due to frame indifference, while the second one expresses the isotropy of materials. This implies that \( W \) depends on \( F \) through the principle invariants of \( FF^T \), namely \( \text{tr}(FF^T) \), \( \frac{1}{4}[(\text{tr}(FF^T))^2 - \text{tr}((FF^T)^2)] \) and \( \det FF^T \) in 3D. Setting \( \tau = \frac{1}{2}\text{tr}(FF^T), \gamma = \frac{1}{4}[(\text{tr}(FF^T))^2 - \text{tr}((FF^T)^2)] \) and \( \delta = \det F = (\det FF^T)^{\frac{1}{3}} \), we may assume that \( W(F) = \bar{W}(\tau, \gamma, \delta) \), for some smooth function \( \bar{W} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \). Since

\[ \frac{\partial \tau}{\partial F} = F, \quad \frac{\partial \gamma}{\partial F} = \text{tr}(FF^T)F - FF^T F \quad \text{and} \quad \frac{\partial \delta}{\partial F} = \delta F^T, \]  

(10.2)

the Piola-Kirchhoff stress has the form

\[ S(F) \equiv \frac{\partial W(F)}{\partial F} = \bar{W}_\tau(\tau, \gamma, \delta)F + \bar{W}_\gamma(\tau, \gamma, \delta)(\text{tr}(FF^T)F - FF^T F) \]

\[ + \bar{W}_\delta(\tau, \gamma, \delta)\delta F^T, \]  

(10.3)

We assume that the reference configuration is stress free, \( S(I) = 0 \), so that

\[ \bar{W}_\tau\left(\frac{3}{2}, \frac{3}{2}, 1\right) + 2\bar{W}_\gamma\left(\frac{3}{2}, \frac{3}{2}, 1\right) + \bar{W}_\delta\left(\frac{3}{2}, \frac{3}{2}, 1\right) = 0. \]  

(10.4)

The Cauchy stress tensor is

\[ T(F) \equiv \delta^{-1}S(F)F^T \]

\[ = \delta^{-1}\bar{W}_\tau(\tau, \gamma, \delta)FF^T + \delta^{-1}\bar{W}_\gamma(\tau, \gamma, \delta)(\text{tr}(FF^T)FF^T - FF^T F) \]

\[ + \bar{W}_\delta(\tau, \gamma, \delta)I, \]  

(10.5)

and the term \( \nabla \cdot FF^T \) in (2.5) is replaced by \( \nabla \cdot T(F) \). Let us now proceed to examine this term.

Write

\[ T(F) = [\bar{W}_\tau\left(\frac{3}{2}, \frac{3}{2}, 1\right) + 2\bar{W}_\gamma\left(\frac{3}{2}, \frac{3}{2}, 1\right)]FF^T \]

\[ + [\delta^{-1}\bar{W}_\tau(\tau, \gamma, \delta) - \bar{W}_\tau\left(\frac{3}{2}, \frac{3}{2}, 1\right)][FF^T - I] \]

\[ + [\delta^{-1}\bar{W}_\gamma(\tau, \gamma, \delta) - \bar{W}_\gamma\left(\frac{3}{2}, \frac{3}{2}, 1\right) + \bar{W}_\delta(\tau, \gamma, \delta)]I \]

\[ + \delta^{-1}\bar{W}_\gamma(\tau, \gamma, \delta)(\text{tr}(FF^T)FF^T - FF^T F) - 2\bar{W}_\gamma\left(\frac{3}{2}, \frac{3}{2}, 1\right)FF^T \]

\[ \equiv \sum_{a=1}^{4} T_a(F). \]

(10.6)

Assume that

\[ \bar{W}_\tau\left(\frac{3}{2}, \frac{3}{2}, 1\right) + 2\bar{W}_\gamma\left(\frac{3}{2}, \frac{3}{2}, 1\right) > 0. \]  

(10.7)
Then $T_1(F)$ gives rise to a Hookean term. Notice that assumption (10.7) rules out the hydrodynamical case $\bar{W}_\delta = 0$. The principal invariants can be expanded about the identity as follows:

\[
\tau = \frac{1}{2} \text{tr} FF^T = \frac{1}{2} \text{tr}(I + G)(I + G^T) = \frac{3}{2} + \text{tr}G + \frac{1}{2} \text{tr}GG^T,
\]

\[
\gamma = \frac{1}{4}[\text{tr}(FF^T)^2 - \text{tr}((FF^T)^2)]
= \frac{1}{4}(2\tau)^2 - \frac{1}{4}\text{tr}(I + G + G^T + GG^T)^2
= \frac{3}{2} + \mathcal{O}(|G|^2),
\]

and

\[
\delta = \det F = \det(I + G) = 1 + \text{tr}G + \frac{1}{2}|(\text{tr}G)^2 - \text{tr}G|^2 + \det G.
\]

In the case of incompressible motion, we have $\delta = 1$, so from (10.10), we get

\[
\text{tr}G + \frac{1}{2}|(\text{tr}G)^2 - \text{tr}G|^2 + \det G = 0,
\]

and hence from (10.8)

\[
\tau - \frac{3}{2} = \mathcal{O}(|G|^2).
\]

Thus, we see that for $|G| \ll 1$

\[
T_2(F) = \mathcal{O}[(\tau - \frac{3}{2})|G| + (\gamma - \frac{3}{2})|G|] = \mathcal{O}(|G|^3),
\]

which produces nonlinearities which are of cubic order or higher, while $T_3(F)$ leads to a gradient term which can be included in the pressure. Similarly, the term $T_4(F)$ can be written as a term $\mathcal{O}(|G|^3)$ plus a term which can be included in the pressure. The conclusion is that the general incompressible isotropic case differs from the Hookean case by a nonlinear perturbation which is cubic in the displacement gradient $G$. Such terms present no further obstacles in the proof of our theorem in 3D. Hence, for general isotropic elastodynamics, we have the following theorem.

**Theorem 10.1.** Let $(v_0, G_0) \in H^k_{A}$, with $k \geq 9$. Suppose that $(v_0, F_0) = (v_0, I + G_0)$ satisfy the constraints (2.3), (2.4), and $\|(v_0, G_0)\|_{H^k_{A}} < \epsilon$. Assume that the smooth strain energy function $W(F)$ is isotropic, frame indifferent, and satisfies (10.4) (10.7). Then there exist two positive constants $M$ and $\epsilon_0$ which depend only on $k$ such that, if $\epsilon \leq \epsilon_0$, then the system of incompressible isotropic elastodynamics

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \nabla \cdot T(F), \\
\partial_t F + v \cdot \nabla F &= \nabla v F, \\
\nabla \cdot v &= 0, \\
\nabla \cdot F^T &= 0.
\end{aligned}
\]

with initial data $(v_0, F_0) = (v_0, I + G_0)$ has a unique global solution $(v, F) = (v, I + G)$, which satisfies $(v, G) \in H^k_{A}$ and $E^1_k(t) \leq M\epsilon$ for all $t \in [0, +\infty)$. 

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