Compact calculation of the Perihelion Precession of Mercury in General Relativity, the Cosmological Constant and Jacobi’s Inversion problem.

G. V. Kraniotis *
Humboldt Universität zu Berlin,
Mathematisch-Naturwissenschaftliche Fakultät I
Institute für Physik,
Newtonstraße 15, D-12489 Berlin,
Germany †

S. B. Whitehouse ‡
Royal Holloway, University of London,
Physics Department,
Egham Surrey TW20-0EX, U.K.

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Abstract

The geodesic equations resulting from the Schwarzschild gravitational metric element are solved exactly including the contribution from the Cosmological constant. The exact solution is given by genus 2 Siegelsche modular forms. For zero cosmological constant the hyperelliptic curve degenerates into an elliptic curve and the resulting geodesic is solved by the Weierstraß Jacobi modular form. The solution is applied to the precise calculation of the perihelion precession of the orbit of planet Mercury around the Sun.

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* georgios.kraniotis@physik.hu-berlin.de
† HU-EP-03/20, May 2003
‡ steve.whitehouse@CSIconsulting.co.uk
1 Introduction

1.1 Motivation

Mercury is the inner most of the four terrestrial planets in the Solar system, moving with high velocity in the Sun’s gravitational field. Only comets and asteroids approach the Sun closer at perihelion. This is why Mercury offers unique possibilities for testing General Relativity [1] and exploring the limits of alternative theories of gravitation with an interesting accuracy [6].

As seen from the Earth the precession of Mercury’s orbit is measured to be 5599.7 seconds of arc per century [4]. As observed, in 1859 by Urbain Jean-Joseph Leverrier there is a deviation of Mercury’s orbit from Newtonian’s predictions that could not be due to the presence of other planets. Urbain Jean-Joseph Leverrier had also been the first to attempt to explain this effect.

Perihelion precessions of Mercury and other bodies have been the subject of experimental study from AD 1765 up to the present. In 1882 Simon Newcomb obtained the value 43 seconds per century for the discrepancy for Mercury [5]. According to Pireaux et al [6], the observed advance of the perihelion of Mercury that is unexplained by Newtonian planetary perturbations or solar oblatness is

\[
\Delta \omega_{\text{obs}} = 42.980 \pm 0.002 \text{ arc-seconds per century}
\]

\[
= \frac{2\pi(3.31636 \pm 0.00015) \times 10^{-5} \text{radians}}{415.2019 \text{revolutions}}
\]

\[
= 2\pi(7.98734 \pm 0.00037) \times 10^{-8} \text{radians/revolution} \tag{1}
\]

It is the purpose of this paper to provide the first precise calculation of the orbit and the perihelion precession of Mercury in General Relativity including cosmological constant contributions [2, 11]. We also compare the result with observations (1). We assume that the motion of Mercury is a time-like geodesic in a Schwarzschild space-time surrounding the Sun.

Bounds on the magnitude of the cosmological constant from the observed value of Mercury’s perihelion shift have been estimated using different methods of approximation by Islam [12] and Cardona et al [13] with conflicting results. The upper bounds obtained were respectively, \( |\Lambda| \leq 10^{-42} \text{cm}^{-2} \) [12] and \( |\Lambda| \leq 10^{-55} \text{cm}^{-2} \) [13]. The latter value compares favourably to the values for the magnitude of the cosmological constant obtained from large-scale observations [39] and cosmological considerations [11].

In [11] a direct connection between general exact solutions of general relativity with a cosmological constant in large-scale cosmology and the theory of modular forms and elliptic curves [15, 14] was established.

As we shall see in the main body of the paper, the two-body central orbit problem under consideration is very interesting mathematically and becomes

\footnote{The observations seem to exclude Brans-Dicke theory with \( \omega \sim 5 \) whose post-Newtonian contribution to the perihelion shift would thus have been 39 arc-seconds per century 1) [4, 6].}
more involved with the inclusion of the cosmological constant. In particular, the
integration of the geodesic orbital equations involves the inversion problem of
Abelian hyperelliptic integrals and the resulting solution which was first studied
by outstanding mathematicians such as Jacobi, Abel, Weierstraß, Riemann,
Göpel, Rosenhain and Baker. The exact treatment of the orbital problem and
the techniques developed in this paper are general and should be of interest for
a variety of problems in various fields of cosmology which are discussed in the
main text.

The material of this paper is organized as follows. In the rest of the intro-
duction we review planned satellite experiments that are relevant to the study
of Mercury and the minor planets. In section 2, starting with the Schwarzschild
metric, and including the cosmological constant, we derive the time-like geodesic
equations. In section 3, we solve exactly the geodesic differential equations and
calculate the precession of the perihelion of Mercury as well as the perihelion
and aphelion distances of the planet, with and without the cosmological constant.
Our approach involved the solution of Jacobi’s inversion problem i) for
an elliptic integral (case without cosmological constant) and ii) for a genus-2
hyperelliptic integral (in the general case with \( \Lambda \neq 0 \)). Here we acknowledge
the contribution by Whittaker of the exact solution of time-like geodesics in the
Schwarzschild field with vanishing cosmological constant [40]. However, in his
account no attempt was made in calculating the physical characteristics of the
orbit of planet Mercury. For an alternative method of computation of perihelion
precession in the case of vanishing cosmological constant we refer the reader to
ref.[16]. In order to make the presentation self-contained we include in the same
section a comprehensive presentation of the inversion problem. In section four
we present a discussion and a summary of our main results as well as further
possible applications. Finally, in an appendix we present the definitions of the
genus-2 theta functions that solve Jacobi’s Inversion problem.

1.2 Planned Satellite Experiments

Solar and planetary astronomers and cosmologists are still fascinated with the
precession of the Perihelion of Mercury and the dynamics and properties of near
earth objects such as the minor planets (i.e. Icarus). In fact, there is a growing
interest in this exciting and vibrant field with several satellite missions planned
for the next few years. Listed below are some of the key satellite experiments
(see Pireaux et. al. [6] for an overview):

- **GAIA (ESA)**

  planned launch 2009, key goal to provide observational data to tackle an
  enormous range of important problems related to to the origin, structure
  and evolutionary history of our Galaxy. The data will create a precise
  three - dimensional map of more than one billion stars throughout our
  Galaxy - a stellar census. GAIA will observe and discover several hun-
  dred thousand minor planets, determine their physical properties and or-
  bital characteristics. Also three earth crossing asteroids, Icarus, Talos
and Phaeton, with favourable combination of distance and eccentricity, will provide data to more accurately determine the PPN parameters giving upper limits on the relativistic and solar quadrupole contributions to the perihelion precessions of solar objects enabling a more stringent test of general relativity. The GAIA mission is a hugely ambitious and exciting project with a multitude of objectives (see GAIA Study Report [17] for details).

- BepiColombo (ESA)
  planned launch 2010, Mercury Orbiter Experiment, key objective to study the Mercury environment in order to understand the formation and evolution of of planets within the solar system, especially the four closest to the Sun - Mercury, Venus, Earth and Mars. BepiColombo will measure Mercury’s motion more accurately than ever before, providing data that will allow the determination of the precession of the perihelion, thus providing one of the most rigorous tests of general relativity [18].

- Messenger (NASA)
  planned launch 2004, a Mercury Orbiter experiment whose main goal is to study the characteristics and environment of Mercury in order to better understand the fundamental forces that have formed it and the other terrestrial planets [19]

- Picard (C.N.E.S.)
  planned launch end of 2005, main object is to measure the Sun’s quadrupole moment contribution to the precession of the Perihelion of Mercury, thereby offering another exacting test of general relativity [20]

Also there are several near Earth Satellites presently operational, LAGEOS, LAGEOS II, CHAMP and GRACE. The combined data from these satellites will enable the accurate measurement of the perigee advance in the Earth’s gravitational field (the determination of the PPN parameters $\gamma, \beta$, in the Earth’s gravitational field), providing yet another test of general relativity [7].

All of these satellite experiments are expected, to various degrees, to provide data that will provide stringent tests for general relativity and to more accurately determine solar and planetary geophysical characteristics.

Our precise results that follow should be of interest to all of the above experiments.

### 2 Schwarzschild solution and time-like geodesics

Einstein’s equations with the cosmological constant $\Lambda$ are as follows

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = (8\pi G/c^4)T_{\mu \nu} + \Lambda g_{\mu \nu}$$  \hspace{1cm} (2)
where $R_{\mu\nu}, R$ denotes the Ricci tensor and scalar respectively. Also $G$ denotes
Newton’s gravitational constant and $c$ the velocity of light. For vanishing stress-
energy momentum tensor $T_{\mu\nu}$ the field equations reduced to

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad (3)$$

The motion of a planet according to General Relativity is a time-like geodesic
in a Schwarzschild space-time [26] surrounding the Sun. The Schwarzschild
solution for the metric taking into account the cosmological constant (see e.g.,
ref.[42, 12]) is

$$ds^2 = c^2 (1 - \frac{2GM_\odot}{c^2r} + \frac{1}{3}\Lambda r^2)dt^2$$
$$- (1 - \frac{2GM_\odot}{c^2r} + \frac{1}{3}\Lambda r^2)^{-1}dr^2$$
$$- r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (4)$$

where $M_\odot$ denotes the mass of the Sun. For zero cosmological constant Eq.(4)
reduces to the original Schwarzschild metric element [26]. We note at this
point, that by taking the time-like geodesic equation as a starting point
for the subsequent calculus represents an approximation to the real situation: it is
strictly true for point-like bodies with negligible mass. However, since $m_M \ll
M_\odot$ where $m_M$ denotes the mass of Mercury the approximation is adequate for
all practical purposes.

The geodesic equation has the general form:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \quad (5)$$

It is easily shown that the motion is confined to the equatorial plane $\theta = \frac{\pi}{2}$,
so that (5) with $\mu = 2$ is trivially satisfied [42]. Furthermore, we can ignore (5)
with $\mu = 1$ in favour of (4), which is a first integral of the geodesic equations.

The resulting equations can be written as

$$1 = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$
$$= c^2 \left(1 - \frac{2GM_\odot}{c^2r} + \frac{1}{3}\Lambda r^2\right) \left(\frac{dt}{ds}\right)^2$$
$$- \left(1 - \frac{2GM_\odot}{c^2r} + \frac{1}{3}\Lambda r^2\right)^{-1} \left(\frac{dr}{ds}\right)^2 - r^2 \left(\frac{d\phi}{ds}\right)^2 \quad (6)$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\phi}{ds} = 0 \quad (7)$$

$$\frac{d^2t}{ds^2} + \frac{2}{r(r - \frac{2GM_\odot}{c^2} + \frac{1}{3}\Lambda r^3)} \frac{dr}{ds} \frac{dt}{ds} = 0 \quad (8)$$

5
Next, define \( v := dt/ds \) and \( w := d\phi/ds \). Then eqs(7),(8) can be written as

\[
\frac{dw}{dr} + \frac{2}{r} w = 0 \quad (9)
\]

\[
\frac{dv}{dr} + \left[ -\frac{1}{r} + \frac{1 + \Lambda r^2}{r - 2GMc^2 + \frac{1}{3} \Lambda r^3} \right] v = 0 \quad (10)
\]

which can be integrated to yield [42]

\[
w = \frac{d\phi}{ds} = \frac{L}{r^2} \quad (11)
\]

\[
v = \frac{dt}{ds} = \frac{\mathcal{E}}{1 - \frac{2GMc^2}{c^2 r} + \frac{1}{3} \Lambda r^2} \quad (12)
\]

where \( L, \mathcal{E} \) are arbitrary constants. Substituting in (6) we get the following equation after defining a new variable \( u := r^{-1} \)

\[
\left( \frac{du}{d\phi} \right)^2 = \frac{2GMc^2}{c^2} u^3 - u^2 + \frac{2GMc}{c^2 L^2} u - \frac{1}{3} \left( \frac{1}{u^2 L^2} + 1 \right) \Lambda + \left( \frac{E^2 - 1}{L^2} \right) \quad (13)
\]

3 Exact solution of the time-like geodesic and the perihelion precession

3.1 Precise calculation of the perihelion advance assuming zero Cosmological Constant.

In the previous section we derived the geodesic equation for the orbital motion of the planet Mercury around the Schwarzschild gravitational field of the Sun given by eq.(13). In this section we will solve Eq.(13) with a zero Cosmological Constant, the non-zero, more general case, will be considered in section 3.3. Making the following definitions,

\[
\alpha_S := \frac{2GMc^2}{c^2}, \beta := \frac{2GMc}{c^2 L^2}, \gamma := -\frac{1 - E^2}{L^2} \quad (14)
\]

our geodesic equation reduces to:

\[
\left( \frac{du}{d\phi} \right)^2 = \alpha_S u^3 - u^2 + \frac{\alpha S}{L^2} u + \gamma \quad (15)
\]

Differentiating Eq.(15) with respect to \( \phi \) and comparing with the Newtonian term we get \( cL = L_M \), where \( L_M \) is the angular momentum per unit mass of the planet [12]. Equation (15) is a cubic equation which can be reduced to the Weierstraß form using the substitution:

\[
u = \frac{4}{\alpha_S} U + \frac{1}{3\alpha_S} \quad (16)
\]
Then the Weierstraß representation of the geodesic equation becomes
\[ \left( \frac{dU}{d\phi} \right)^2 = 4 U^3 + \left\{ -\frac{1}{12} + \frac{1}{4L^2 \alpha_S^2} \right\} U - \frac{1}{216} + \frac{\alpha_S^2}{48L^2} + \frac{\alpha_S^2}{16} \gamma \] (17)

where the Weierstraß cubic invariants are given by:
\[ g_2 = \frac{1}{12} - \frac{(2GM_{\odot}/c^2)^2}{4L_M^4}, \] 
\[ g_3 = \frac{1}{216} - \frac{(2GM_{\odot}/c^2)^2}{48L_M^2} - \frac{(2GM_{\odot}/c^2)^2}{16} \left( \frac{E^2 - c^2}{L_M^2} \right) \] (18)

where \( \mathcal{L} := \frac{\alpha_S^2}{L_M^2} \) and \( E = cE' \). Then the solution of Eq.(17) is given by:
\[ U = \wp(\phi + \epsilon) \] (19)

and \( \epsilon \) is a constant of integration. The solution in terms of the original variables is given by:
\[ u = \frac{1}{r} = \frac{4}{\alpha_S} \wp(\phi + \epsilon) + \frac{1}{3\alpha_S} \] (20)
or
\[ r = \frac{\alpha_S}{4\wp(\phi + \epsilon) + \frac{1}{3}} \] (21)

For the calculation of the perihelion precession and the orbital characteristics of Mercury we use the following values for the physical constants:
\[ c = 299 792 458 \text{ m s}^{-1}, \quad \alpha_S = 2.953 250 08 \text{ Km} \] (22)

The data is taken from [8] and [9], respectively. The value of the speed of light in vacuum, given in Eq.(22), is exact, since the meter is the length of the path travelled by light in vacuum during a time interval of \( 1/299 792 458 \) of a second [8]. As free parameters we may use \( \mathcal{L} \) and \( E \). Then \( L_M^2 = \frac{c^2 L^2}{\mathcal{L} \alpha_S^2} \). Our free parameters are mixed through the Weierstraß invariants \( g_2, g_3 \) with exact numbers.

We will show in the next section that there is a small parameter space for \( \mathcal{L} \) and \( E \) that reproduces the observed values for the characteristics of Mercury’s

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\(^2\)The differential equation that the Weierstraß function satisfies is the equation of an elliptic curve and is given by:
\[(\wp'(z))^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3, \] \( \wp'(z) = \frac{\partial \wp(z)}{\partial z} \), see [11] for further details. The inversion of the elliptic integral \( \int_{U}^{U} \frac{dU}{\sqrt{4U^3 - g_2 U - g_3}} = \phi \), by the Weierstraß function, i.e.
\( U = \wp(\phi + \epsilon) \), is a simple case of the problem of inversion for elliptic integrals.

\(^3\)The integration constant \( L \) has dimensions of length and thus scales with \( \alpha_S \).
orbit. Non-physical solutions that predict complex, negative or too small orbital radii and precession values are disregarded. The sign of the discriminant \( \Delta = g_3^2 - 27g_2^2 \) determines the roots \( e_i, i = 1, 2, 3 \) of the elliptic curve: \( \Delta > 0 \), corresponds to three real roots \( e_i \) while for \( \Delta < 0 \) two roots are complex conjugates and the third is real. In the degenerate case \( \Delta = 0 \), (where at least two roots coincide) the elliptic curve becomes singular and the solution is not given by modular functions.

We find that the physically acceptable solutions that reproduce the orbital data of Mercury correspond to the case where \( \Delta > 0 \), \( e_3 \to -\frac{1}{12}, e_2 \to -\frac{1}{12} \) and \( e_1 \to 1/6 \). The analytic expressions for the roots of the elliptic curve may be obtained by using the algorithm of Tartaglia and Cardano [43]. Their expressions are given in appendix B. The Weierstraß function, \( \wp(z) \), is an even meromorphic elliptic function of periods 2\( \omega \), 2\( \omega' \) (i.e., \( \wp(z + 2\omega) = \wp(z) = \wp(z + 2\omega') \), for all complex numbers \( z \)). The two half-periods \( \omega \) and \( \omega' \) are given by the following Abelian integrals (for \( \Delta > 0 \)) [40]:

\[
\omega = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \quad \omega' = i \int_{-\infty}^{e_3} \frac{dt}{\sqrt{-4t^3 + g_2 t + g_3}}
\]  

(23)

The value of the Weierstraß function at the half-periods are the three roots of the cubic. For positive discriminant \( \Delta \) one half-period is real while the second is imaginary. The exact expression for the precession of the perihelion of planet Mercury, is given by

\[
\Delta_{\omega}^{\text{GTR}} = 2(\omega - \pi)
\]  

(24)

which is proportional to the deviation of the real half-period \( \omega \) of the Weierstraß modular form, from the transcendental number \( \pi \).

The exact expressions for the minimum distance of planet Mercury from the Sun (Perihelion) and its maximum distance (Aphelion) are given by \(^4\):

\[
\begin{align*}
    r_P &\equiv r_{\text{Perihelion}} = \frac{\alpha_s}{4e_2 + \frac{1}{3}} \quad (25) \\
    r_A &\equiv r_{\text{Aphelion}} = \frac{\alpha_s}{4e_3 + \frac{1}{3}} \quad (26)
\end{align*}
\]

For such orbits the radius vector oscillates between \( r_A \) and \( r_P \), as the argument of the Weierstraß function travels along the straight line from \( \omega' \) to \( \omega' + \omega \) and then to \( \omega' + 2\omega \) in the fundamental period region [21].

We also note the following. Although by construction the roots of the cubic are calculated with arbitrary precision, our output for the perihelion and aphelion distances can only be displayed with nine significant figures given the nine digit accuracy of the Schwarzschild length \( \alpha_S \). However, their ratio which is given by:

\[
\frac{r_P}{r_A} = \frac{4e_3 + 1/3}{4e_2 + 1/3}
\]  

(27)

\(^4\)We organize the roots as: \( e_1 > e_2 > e_3 \)
constitutes a genuine and precise prediction. For a given choice of values for the free parameters, equations (24)-(27) are the output of the precise theory assuming zero cosmological constant, for the corresponding physical quantities, that should be tested against observations.

3.2 Theoretical results

The exact solution of the geodesic equations of the General Theory of Relativity outlined in the previous section predicts a set of coupled results,

\[
\begin{align*}
\Delta \omega^{GTR} &\quad \text{- Perihelion Precession} \\
r_P &\quad \text{- distance of closest approach} \\
r_A &\quad \text{- distance of farthest approach}
\end{align*}
\]

based solely upon the values chosen for the \( L \) and \( E \). As we mentioned not all of input values for the free parameters provide us with physical solutions, or produce realistic data for the orbit of Mercury. In appendix B, we provide some examples of solutions that correspond to specific choices of the free parameters that do not describe the orbit of Mercury. More specifically, we describe solutions for the motion of the test particle that are described by special elliptic curves for which one of the Weierstraß invariants, \( g_2, g_3 \), vanishes. In addition, we describe choices of the initial conditions that lead to a vanishing discriminant \( \Delta \) (singular elliptic curves).

For \( L = 1.18554647 \times 10^{-28} \text{ cm}^{-2} \text{ s}^{-2} \), \( (L_M^2 = 7.35668 \times 10^{38} \text{ cm}^4 \text{ s}^{-2}) \) and \( E = 0.029979245176 \times 10^{12} \text{ cm s}^{-1} \) we obtain

\[
\begin{align*}
\epsilon_2 &= -0.0833333172749111339, \quad \epsilon_3 = -0.083333322753852543388 \\
\Delta \omega^{GTR} &= 43.0017 \text{arcsec century} \\
r_P &= 4.59766539 \times 10^{12} \text{ cm}, \\
r_A &= 6.97872168 \times 10^{12} \text{ cm}
\end{align*}
\]

while the two periods are: \( \omega = 3.141592904646, \omega' = 20.40864976 \text{ i} \) and the period ratio \( \tau = 6.496i \), and the radius vector oscillates between \( \frac{a}{\sqrt{4e^2 + 1}} \leq r \leq \frac{a}{\sqrt{4e^2 + 1}} \). For \( L = 1.1848820116975453 \times 10^{-28} \text{ cm}^{-2} \text{ s}^{-2} \), \( (L_M^2 = 7.36008550 \times 10^{38} \text{ cm}^4 \text{ s}^{-2}) \) and \( E = 0.0299792454178 \times 10^{12} \text{ cm s}^{-1} \) we obtain, \( \Delta \omega^{GTR} = 42.9776 \text{arcsec century} \) with the half-periods, \( \omega = 3.141592904505435, \omega' = 20.409634 \text{ i} \) and the period ratio \( \tau = 6.497i \). Also \( r_P = 4.60057668 \times 10^{12} \text{ cm}, \quad r_A = 6.98186816 \times 10^{12} \text{ cm} \).

In Tables 1-3 we list the predicted results for \( \Delta \omega^{GTR}, r_P, r_A \) keeping \( E \) fixed at the indicated value and varying \( L \). We see that by increasing the input value for \( L \) one can obtain a slightly larger value for the precession of perihelion. The

\[\text{In Newtonian theory the orbit of a planet is described by an ellipse with eccentricity } e \text{ and semi-major axis } a. \text{ For an ellipse, the perihelion } (r_P^N) \text{ and aphelion } (r_A^N) \text{ distances, are } a(1 - e), a(1 + e), \text{ respectively, and then } e = \frac{1 - r_P^N/a}{1 + r_A^N/a}.\]

\[\text{These elliptic curves have the property of complex multiplication, see [11] for the definition.}\]
Table 1: Predictions for $\Delta_{GTR}^{\omega}, r_P, r_A$ for the indicated choice for $\mathcal{L}, E$. The two half-periods are: $\omega = 3.14159290452929, \omega' = 20.409059 i$ and the period ratio $\tau = 6.496i$. Also $L_M^2 = 7.36010550 \times 10^{38} \text{cm}^4 \text{s}^{-2}$.

| parameters | roots | predicted results |
|------------|-------|-------------------|
| $\mathcal{L} = 1.1849947026647969 \times 10^{-28} \text{cm}^2 \text{s}^2$ | $0.16666664004166$ | $\Delta_{GTR}^{\omega} = 42.9817 \text{arc-sec/century}$ |
| $E = 0.02997924541779875 \times 10^{12} \text{cm s}^{-1}$ | $-0.083333317282230892$ | $r_P = 4.59976206 \times 10^{12} \text{cm}$ |
| | $-0.083333322758930472$ | $r_A = 6.98207293 \times 10^{12} \text{cm}$ |

Table 2: Predictions for $\Delta_{GTR}^{\omega}, r_P, r_A$ for the indicated choice for $\mathcal{L}, E$. The two half-periods are: $\omega = 3.14159290452929, \omega' = 20.409059 i$ and the period ratio $\tau = 6.4965i$. Also $L_M^2 = 7.36030420 \times 10^{38} \text{cm}^4 \text{s}^{-2}$.

| parameters | roots | predicted results |
|------------|-------|-------------------|
| $\mathcal{L} = 1.1849627128268641 \times 10^{-28} \text{cm}^2 \text{s}^2$ | $0.166666664004188$ | $\Delta_{GTR}^{\omega} = 42.9805 \text{arc-sec/century}$ |
| $E = 0.02997924541779875 \times 10^{12} \text{cm s}^{-1}$ | $-0.083333331728350096$ | $r_P = 4.60012605 \times 10^{12} \text{cm}$ |
| | $-0.08333332275837917$ | $r_A = 6.98170894 \times 10^{12} \text{cm}$ |

Parameter $E$ fixes the sum of the radii $r_P + r_A$. In fact, in tables 1-3 the sum $\frac{r_P + r_A}{2}$ is equal to $5.7909175 \times 10^{12} \text{cm}$. The latter value is reported as the data for the semi-major axis $a$ in ref. [10]. In table 2 the value for $L_M$ that corresponds to the choice for $\mathcal{L}$ for fixed $\alpha_S$ is equal to the Newtonian value that would be obtained by assuming the orbit is an ellipse with eccentricity $e$ [42, 12].

The choice of initial conditions presented in Tables 1-3, lead to a set of predictions for the perihelion advance in agreement with observations in Eq.(1). In addition, they lead to a set of predictions for the perihelion and aphelion distances of the planet, whose half-sum, $\frac{r_P + r_A}{2}$, is in agreement with the data for the semi-major axis $a$ in ref. [10]. The calculated theoretical values for the perihelion and aphelion distances compare favourably with best current values for the orbital data for Mercury [27]. More precise measurements of the orbit of Mercury as described for instance in the ESA mission BeriColombo, will determine the perihelion and aphelion distances as well as the precession of the perihelion more accurately and will further restrict the choice of initial conditions.

### 3.3 Precise calculation of the perihelion advance with the contribution of the Cosmological Constant.

As we saw in section 3.1, the integration of the geodesic equation that describes the orbit of Mercury in the central gravitational field of the Sun (assuming
Table 3: Predictions for $\Delta^{GTR}_{\omega, r_P, r_A}$ for the indicated choice for $L, E$. The two half-periods are: $\omega = 3.141592904505435, \omega' = 20.410231661 i$ and the period ratio $\tau = 6.497 i$. Also $L^2_M = 7.36080550 \times 10^{38} \text{ cm}^4 \text{s}^{-2}$.

vanishing cosmological constant) involved an elliptic integral, whose inversion by the Weierstraß elliptic function, provided the exact solution in closed analytic form for the orbit of the planet eq.(19).

Elliptic integrals are special cases of the so called Abelian integrals. According to Jacobi an Abelian integral is an integral of the form $\int R(x,y)dx$, where $R(x,y)$ is a rational function in $x$ and $y$ and $x,y$ are connected through an equation $f(x,y) = 0$, where $f$ is an irreducible polynomial. In the special case $y^2 = P(x)$, where $P(x)$ is a polynomial of $n^{th}$ degree, with no multiple roots, the Abelian integral is called elliptic when $n = 3$ or $n = 4$, and hyperelliptic when $n \geq 5$.

Including cosmological constant effects we need to calculate the integral:

$$\int^u \frac{du}{\sqrt{\frac{2GM_0}{c^2}u^3 - u^2 + \frac{2GM_0}{c^4L^2}u - \frac{1-E^2}{L^2}u - \frac{\Lambda}{3L^2}u - \frac{\Lambda}{3}u^2}} = \phi$$

or

$$\int^u \frac{u \, du}{\sqrt{\frac{2GM_0}{c^2}u^5 - u^4 + \frac{2GM_0}{c^4L^2}u^3 - \frac{1-E^2}{L^2}u^2 - \frac{\Lambda}{3L^2}u - \frac{\Lambda}{3}u^2}} = \phi$$

Eq.(29) defines a hyperelliptic integral whose inversion involves genus 2 Abelian-Siegelche modular functions. The problem of inversion (whose solution was culminated in the Jacobisches Umkehrtheorem) of hyperelliptic integrals were first investigated by Abel [28], Jacobi, Göpel and Rosenhain [29], [30], [31]. The explicit solution of Jacobi’s inversion problem in terms of higher genus theta functions was provided by Göpel and Rosenhain for the case $n = 5$ or 6, and the general solution for the hyperelliptic case (i.e. $\forall n \geq 5$) was provided by Weierstraß [32]. Riemann introduced the idea of a Riemann surface to study algebraic singularities. He also introduced the Riemann theta function which served as a useful tool for solving the Jacobi’s inversion problem [33]. For full details and extended bibliography to the original literature we refer the reader to the book of Baker [34].

In what follows we discuss first the Jacobi’s inversion problem and then we proceed to determine the effect of the cosmological constant on the perihelion advance of Mercury as well as its effect on the radii $r_P, r_A$. 

| parameters | roots | predicted results |
|------------|-------|-------------------|
| $L = 1.184882011697545 \times 10^{-28} \text{ cm}^{-2} \text{s}^2$ | $0.166666640043693405$ | $\Delta^{GTR}_{\omega} = 42.9776 \text{arc-sec}$ |
| $E = 0.0297924541779875 \times 10^{12} \text{ cm s}^{-1}$ | $-0.083333317286706146$ | $r_P = 4.60104489 \times 10^{12} \text{cm}$ |
| | $-0.083333322756987258$ | $r_A = 6.98079010 \times 10^{12} \text{cm}$ |
3.4 Abel’s theorem and Jacobi’s inversion problem

Let the genus \( g \) Riemann hyperelliptic surface be described by the equation:

\[ y^2 = 4(x - a_1) \cdots (x - a_g)(x - c)(x - c_1) \cdots (x - c_g) \]  

For \( g = 2 \) the above hyperelliptic Riemann algebraic equation reduces to:

\[ y^2 = 4(x - a_1)(x - a_2)(x - c)(x - c_1)(x - c_2) \]  

where \( a_1, a_2, c, c_1, c_2 \) denote the finite branch points of the surface.

The Jacobi’s inversion problem involves finding the solutions, for \( x_i \) in terms of \( u_i \), for the following system of equations of Abelian integrals \[34\]:

\[ u_1^{x_1,a_1} + \cdots + u_1^{x_g,a_g} \equiv u_1 \]

\[ \vdots + \cdots + \vdots \]

\[ u_2^{x_1,a_1} + \cdots + u_2^{x_g,a_g} \equiv u_g \]

where \( u_1^{x,a} = \int_y^x \frac{dx}{y}, u_2^{x,a} = \int_y^x \frac{dx}{y}, \cdots, u_g^{x,a} = \int_y^x \frac{dx}{y} \).

For \( g = 2 \) the above system of equations takes the form:

\[ \int_{x_1}^{x_2} \frac{dx}{y} \equiv u_1 \]

\[ \int_{x_1}^{x_2} \frac{x \, dx}{y} \equiv u_2 \]

where \( u_1, u_2 \) are arbitrary. The solution \[7\] is given by the five equations \[34\]

\[ \frac{\theta^2(u|u^{b,a})}{\theta^2(u)} = A(b - x_1)(b - x_2) \cdots (b - x_g) \]

\[ = A(b - x_1)(b - x_2) \]

\[ = \pm \frac{(b - x_1)(b - x_2)}{\sqrt{e^{\pi iPP'} f(b)}} \]  

(35)

where \( f(x) = (x - a_1)(x - a_2)(x - c)(x - c_1)(x - c_2) \), and \( e^{\pi iPP'} = \pm 1 \) according as \( u^{b,a} \) is an odd or even half-period. Also \( b \) denotes a finite branch point and the branch place \( a \) being at infinity \[34\]. The symbol \( \theta(u|u^{b,a}) \) denotes a genus 2 theta function with characteristics: \( \theta(u; q, q') \) \[34\], where \( u = (u_1, u_2) \), denotes two independent variables, see appendix A for further details. From any 2 of these equations, \( \text{eq.(35)} \), the upper integration bounds \( x_1, x_2 \) of the system of differential equations \( \text{eq.(34)} \) can be expressed as single valued functions of the arbitrary arguments \( u_1, u_2 \). For instance,

\[ x_1 = a_1 + \frac{1}{A_1(x_2 - a_1)} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)} \]  

(36)

\[ \text{The definitions of the genus two theta functions appearing in the solution of Jacobi’s inversion problem are given in the appendix A.} \]
\[
x_2 = \pm \sqrt{\left( (a_2 - a_1)(a_2 + a_1) + \frac{\theta_1^2(u|u^{a_1,+})}{\theta_1^2(u)} - \frac{\theta_2^2(u|u^{a_2,+})}{\theta_2^2(u)} \right)^2 - 4(a_1 - a_2)\eta} \frac{2(a_1 - a_2)}{2(a_1 - a_2)} - \frac{1}{a_1} \] 

(37)

where
\[
\eta := \frac{a_2}{A_1} \frac{a_1}{a_1 - a_2} - a_2 \frac{\theta_1^2(u|u^{a_1,+})}{\theta_1^2(u)} + a_1 \frac{\theta_2^2(u|u^{a_2,+})}{\theta_2^2(u)} \] 

(38)

Also, \( A_i = \pm \sqrt{\frac{e^{\pi i P' P}}{p'(u)}} \).

The solution can be reexpressed in terms of generalized Weierstraß functions:
\[
x_1^{(1,2)} \left( \frac{\omega_{2,2}(u) \pm \sqrt{\omega_{2,2}(u) + 4\omega_{2,1}(u)}}{2} \right), \quad k = 1, 2 \] 

(39)

where
\[
\omega_{2,2}(u) = \frac{(a_1 - a_2)(a_2 + a_1) - \frac{\theta_1^2(u|u^{a_1,+})}{\theta_1^2(u)} + \theta_2^2(u|u^{a_2,+})}{a_1 - a_2} \] 

(40)

and
\[
\omega_{2,1}(u) = \frac{-a_1 a_2 (a_1 - a_2) - \frac{\theta_1^2(u|u^{a_2,+})}{\theta_1^2(u)} + \theta_2^2(u|u^{a_1,+})}{a_1 - a_2} \] 

(41)

Thus, \( x_1, x_2 \), that solve Jacobi’s inversion problem Eq.(34), are solutions of a quadratic equation [29, 34]
\[
Ux^2 - U'x + U'' = 0 \] 

(42)

where \( U, U', U'' \) are functions of \( u_1, u_2 \). In the particular case that the coefficient of \( x^2 \) in the quintic polynomial is equal to 4, \( U = 1, U' = \omega_{2,2}(u), U'' = \omega_{2,1}(u) \).

### 3.5 Inversion of the hyperelliptic integral.

Let us apply the solution of the Jacobi’s inversion problem given by eqs.(35)-(37) to the hyperelliptic integral (30). We have the following correspondence between the variables of the hyperelliptic integral and the variables in (34): the upper integration bounds in (34) \( x_i \), correspond to \( u_i \) and the periods \( u_i \) correspond to \( \phi_i \). Then for:
\[
\phi_1 = \phi_1^{u,a_2} \quad \phi_2 = \phi_2^{u,a_2} \] 

(43)
Thus we have that:

\[
\frac{1}{A_1} \frac{\theta^2(\Phi|\Phi^{a_1,a})}{\theta^2(\Phi)} = (u_1 - a_1)(u_2 - a_1) = 0
\]

\[
\frac{1}{A_2} \frac{\theta^2(\Phi|\Phi^{a_2,a})}{\theta^2(\Phi)} = (a_1 - a_2)(u - a_2)
\]

and the solution for \( u \) in the hyperelliptic integral \( \phi_2 = \phi_2^{u,a_2} = \int_{a_2}^u \frac{u \, du}{y} \) (where \( y \) is the radical of the quintic polynomial \( ^8 \)) is given by

\[
u_2,1(\Phi) = -a_1a_2 - \frac{a_1}{A_2(a_1 - a_2)} \frac{\theta^2(\Phi|\Phi^{a_2,a})}{\theta^2(\Phi)} \quad \text{(47)}
\]

\[
u_2,2(\Phi) = a_1 + a_2 + \frac{1}{A_2(a_1 - a_2)} \frac{\theta^2(\Phi|\Phi^{a_2,a})}{\theta^2(\Phi)} \quad \text{(48)}
\]

**3.6 Roots of the quintic, periods of the hyperelliptic surface and the cosmological constant effect**

In principle, the sign of the cosmological constant is an additional parameter, besides its magnitude. Recent observations of large-scale cosmology indicate a positive cosmological constant of magnitude \( \sim 10^{-56} \text{cm}^{-2} \) [39].

In the presence of the cosmological term there are five branch points for the hyperelliptic surface, eq.(32), which are obtained by solving the quintic polynomial that appears in the time-like geodesics, Eq.( 29).

For negative cosmological constant all the roots are real. For positive cosmological constant and depending on its magnitude and the values of the parameters \( L, E \) three roots are real and two complex conjugates. For some particular values all the roots are real.

Let us start our discussion with the case of the negative sign. Let us arrange the roots in ascending order of magnitude and denote them by \( e^2, e^3, \ldots, e_0 \), \( g = 2 \), so that \( e_2, e_2, -1 \), are respectively \( c_2 - 1, a_2 - 1 \) and \( c_0 \) is \( c \). We then define linearly independent Abelian integrals of the first kind [34], denoted by \( U_r^{x,a}, i = 1, \ldots, g \), whose periods we want to calculate. These integrals are such that \( dU_r^{x,a}/dx = \psi_r/x, \) where \( \psi_r \) is an integral polynomial in \( x \), of degree \( g - 1 = 1 \) at most, with only real coefficients. Then the half-periods, \( U_r^{x_1,x_2} \) and \( U_r^{x_2,x_1} \), are real, while the half-periods \( U_r^{x_3,x_2} \) and \( U_r^{x_1,x_0} \) are purely imaginary.
For clarity, by $U_{c_4}^{c_4, c_3}$ we denote $\int_{e_3}^{e_4} \frac{\alpha}{\beta} \, d\alpha$, $U_{c_1}^{c_4, c_3}$ denotes $\int_{e_3}^{e_4} \frac{\alpha}{\beta} \, d\alpha$ and similarly for the rest of the periods.

Every other period can be expressed as a linear combination of the above Abelian integrals. For instance, the period $U_{c_1}^{c_4, c_3}$ is given by the equation:

$$U_{c_1}^{c_4, c_3} = -U_{c_2}^{c_1} - U_{c_3}^{c_4, c_3} + U_{c_0}^{c_1, c_0}$$

while the period $U_{c_0, c_\infty}^{c_4, c_3}$ is given by

$$U_{c_0, c_\infty}^{c_4, c_3} = -U_{c_2}^{c_1} - U_{c_3}^{c_4, c_3}$$

(49)

For $\Lambda = 10^{-55} \text{cm}^{-2}, \mathcal{L} = 1.1848820116975453 \times 10^{-28} \text{ cm}^{-2} s^2 (L_M^2 = 7.3608055 \times 10^{38} \text{ cm}^4 s^{-2}), E = 0.0299792454178 \times 10^{12} \text{ cm s}^{-1}$ the roots are:

$$e_4 = -1.143376827586711663025 \times 10^{-24}$$
$$e_3 = 1.1433768275867116230255 \times 10^{-24}$$
$$e_2 = 1.43228141354887622443346 \times 10^{-13}$$
$$e_1 = 2.1736405449702531058427 \times 10^{-13}$$
$$e_0 = 0.000003386099606939$$

(50)

The half-periods are calculated to be:

$$U_{c_3, c_2}^{c_3, c_2} = 2.264048243121482 i, \ U_{c_1, c_0}^{c_1, c_0} = 20.4094269 i$$
$$U_{c_1, c_2}^{c_1, c_2} = 3.14159290450544$$

(52)

the real half period: $\int_{c_2}^{c_1} \frac{\alpha d\alpha}{\beta} = 3.14159290450544$, leads to $\Delta_{\omega}^{GTR} = 42.9776 \text{ arc-sec century}$ in agreement with observations (1).

For $\Lambda = 10^{-42} \text{ cm}^{-2}, \mathcal{L} = 1.1848820116975453 \times 10^{-28} \text{ cm}^{-2} \ s^2, E = 0.0299792454178 \times 10^{12} \text{ cm s}^{-1}$ the roots are:

$$e_4 = -3.61559923152843540392 \times 10^{-18}$$
$$e_3 = 3.615750711682713946388 \times 10^{-18}$$
$$e_2 = 1.4322814108727319240989 \times 10^{-13}$$
$$e_1 = 2.173640464132212312989 \times 10^{-13}$$
$$e_0 = 0.000003386099606939$$

(53)

The periods are calculated to be:

$$U_{c_3, c_2}^{c_3, c_2} = 2.26404824460 i, \ U_{c_1, c_0}^{c_1, c_0} = 20.4094269 i$$
$$U_{c_1, c_2}^{c_1, c_2} = 3.14159290255, \ U_{c_2, c_3}^{c_2, c_3} \in \mathbb{R}$$

(54)

Here there is substantial effect on the perihelion advance due to the cosmological constant with $\Delta_{\omega}^{GTR} = 42.6427 \text{ arc-sec century}$ in conflict with observations (1).

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Table 4: Theoretical predictions for perihelion precession for $\Lambda \neq 0$, and $L = 1.1848820116975453 \times 10^{-28} \text{cm}^{-2} \text{s}^2, E = 0.0299792454178 \times 10^{12} \text{cm} \text{s}^{-1}$. For comparison we list the $\Lambda = 0$ case for the same set of values for $L, E$. For this choice of $L, L^2_M = 7.36080550 \times 10^{38} \text{cm}^4 \text{s}^{-2}$.

| $\omega$ | $\Lambda = 0$ | $\Lambda = -10^{-42} \text{cm}^{-2}$ | $\Lambda = 10^{-42} \text{cm}^{-2}$ |
|----------|---------------|---------------------------------|-------------------------------|
| $\Delta_{\omega}^{\text{GTR}}$ | $42.9776 \text{arcsec/century}$ | $42.6427 \text{arcsec/century}$ | $42.8680 \text{arcsec/century}$ |
| $r_P$ | $4.60057668 \times 10^{12} \text{cm}$ | $4.60057668 \times 10^{12} \text{cm}$ | $4.60057668 \times 10^{12} \text{cm}$ |
| $r_A$ | $6.98186816 \times 10^{12} \text{cm}$ | $6.98186817 \times 10^{12} \text{cm}$ | $6.98186815 \times 10^{12} \text{cm}$ |

For $\Lambda = 10^{-42} \text{cm}^{-2}, L = 1.1848820116975453 \times 10^{-28} \text{cm}^{-2} \text{s}^2, E = 0.0299792454178 \times 10^{12} \text{cm} \text{s}^{-1}$ the roots are:

$$e_1 = 0.00000010574713017622 \times \frac{4}{\alpha_s},$$

$$e_2 = 0.000000160482602747 \times \frac{4}{\alpha_s},$$

$$e_3 = 0.249999733770 \times \frac{4}{\alpha_s},$$

$$e_4 = (-5.58970979303625 \times 10^{-18} - 2.6694981178048 \times 10^{-13} i) \times \frac{4}{\alpha_s},$$

$$e_5 = \bar{e}_4 \quad (55)$$

with the periods $\int_{e_2}^{e_1} \frac{ud\nu}{y} = 3.1415929045713017622$, $\int_{e_3}^{e_2} \frac{ud\nu}{y} = 0.0299792454178 \times 10^{12} \text{cm} \text{s}^{-1}$ the roots are:

$$e_1 = 0.00000010574713017622 \times \frac{4}{\alpha_s},$$

$$e_2 = 0.000000160482602833 \times \frac{4}{\alpha_s},$$

$$e_3 = 0.2499997337770 \times \frac{4}{\alpha_s},$$

$$e_4 = (-5.5897098056 \times 10^{-32} - 2.669498120170 \times 10^{-20} i) \times \frac{4}{\alpha_s},$$

$$e_5 = \bar{e}_4 \quad (56)$$

with the real half period: $\int_{e_2}^{e_1} \frac{ud\nu}{y} = 3.141592904505435$, which leads to $\Delta_{\omega}^{\text{GTR}} = 42.9776 \text{arcsec/century}$ in agreement with observations (1).

We summarise our results in Table 4 and 5. We also repeated our calculations for the values for $L, E$ as in Tables 1 and 2, and we found: For the values for $L, E$ as in table 1 for $\Lambda = -10^{-55} \text{cm}^{-2}$, $\int_{e_2}^{e_1} \frac{ud\nu}{y} = 3.1415929045292983,$
\[ \Lambda = -10^{-55} \text{cm}^{-2} \quad \Lambda = 10^{-56} \text{cm}^{-2} \quad \Lambda = 0 \]

| \( \omega \) | \( \Delta \) | \( r_P \) | \( r_A \) |
|----------------|----------------|----------------|----------------|
| 3.14159290450544 | 42.9776 arcsec century | 4.60057668 \times 10^{14} \text{cm} | 6.98186816 \times 10^{12} \text{cm} |
| 3.141592904505435 | 42.9776 arcsec century | 4.60057668 \times 10^{14} \text{cm} | 6.98186816 \times 10^{12} \text{cm} |
| 3.141592904505435 | 42.9776 arcsec century | 4.60057668 \times 10^{14} \text{cm} | 6.98186816 \times 10^{12} \text{cm} |

Table 5: Theoretical predictions for perihelion precession for \( \Lambda \neq 0 \), and \( L = 1.1848820116975453 \times 10^{-28} \text{ cm}^{-2} \text{s}^{-2} \), \( E = 0.0299792454178 \times 10^{12} \text{cm} \text{s}^{-1} \). For comparison we list the corresponding results for the \( \Lambda = 0 \) case. For this choice of \( L \), \( L_M^2 = 7.36080550 \times 10^{38} \text{cm}^4 \text{s}^{-2} \).

\( \Delta_{GTR} = 42.9817 \text{arc-sec century} \) and for \( \Lambda = 10^{-55} \text{cm}^{-2} \), \( \int_{e_1}^{e_2} \frac{u \, du}{y} = 3.14159290452999 \) and \( \Delta_{GTR} = 42.9817 \text{arc-sec century} \).

For the values for \( L \), \( E \) chosen in table 2, for \( \Lambda = -10^{-55} \text{cm}^{-2} \), \( \int_{e_1}^{e_2} \frac{u \, du}{y} = 3.14159290452253 \), \( \Delta_{GTR} = 42.9805 \text{arc-sec century} \), while for \( \Lambda = 10^{-56} \text{cm}^{-2} \), \( \int_{e_1}^{e_2} \frac{u \, du}{y} = 3.141592904524534 \) and \( \Delta_{GTR} = 42.9809 \text{arc-sec century} \). Thus we see that the cosmological constant depending on its magnitude has an effect on the perihelion precession and in fact is in conflict with observations for magnitudes \( |\Lambda| \sim 10^{-42} \text{cm}^{-2} \) and larger while current favoured values of the cosmological constant \([39, 36, 37]\) are compatible with observations of the Mercury’s perihelion advance and do not have significant effect either on the perihelion precession or the radii \( r_P, r_A \). Evidently, more precise observations can put a rigorous upper bound on the magnitude of the cosmological constant.

### 4 Discussion and further applications

In this paper we dealt with the motion of a test particle in a central gravitational field. In particular we investigated the orbit of planet Mercury around the gravitational Schwarzschild field of the Sun by solving exactly the geodesic differential equations with and without the cosmological constant and determined the perihelion advance. As we saw in the main body of the paper, the analytic solution of the orbital problem is provided by the solution of the inversion problem first enunciated by Abel and Jacobi in their study of elliptic and hyperelliptic modular functions.

The perihelion precession of the planet in the exact solutions obtained depends on fundamental properties of the modular functions and in particular their periods.

We have also calculated the precise perihelion and aphelion distances of the planet. The calculated theoretical values compare favourably with best current values for the orbital data for Mercury [27].

For zero cosmological constant the exact solution for the orbit of Mercury is expressed by the Weierstraß Jacobi modular form. The orbital data of the
planet are reproduced for positive discriminant ($\Delta > 0$) and when all roots of the cubic are real. For the particular choice of the free parameters given in Tables 1-3 the theoretical predictions for the perihelion advance are in agreement with observations Eq.(1). The complex structure of the torus that describe the particular orbital solutions is $\tau = 6.497i$. We note that the modular properties of the exact solutions in large-scale cosmology including cosmological constant effects allow for both signs of the discriminant $\Delta$ and in some interesting cases the complex structure of the corresponding tori is a fixed point of the modular group [11].

For non-zero cosmological constant the exact solution for the orbit is provided by quotients of genus two theta functions and therefore is described by a hyperelliptic curve of genus two. Alternatively, the solution can be reexpressed in terms of generalized Weierstraß functions. We investigated the effect of various values for the cosmological constant on the perihelion precession. Magnitudes of $\Lambda \sim 10^{-42}cm^{-2}$ are in conflict with observations (1), while current favoured values from large scale cosmology are compatible with observations. More accurate measurements will place a rigorous upper bound on the magnitude of the cosmological constant.

The results of this paper, are in full agreement with the method suggested in [11] namely: the properties of modular theta functions associated with Riemann surfaces and in particular the non-linear differential equations they obey, provide a new way to deal with Einstein’ equations. In this way, the non-linearity of the gravitational field equations can be tackled effectively and general exact solutions be generated.

Our exact treatment of the geodesic equations can be applied to a variety of problems in cosmology where the two body central orbit problem in general relativity is encountered. More specifically our techniques can be applied to address problems such as : the determination of orbital parameters of stars around massive gravitational objects such as Neutron Stars [22] or candidate Black Holes [23], galactic dynamics including the missing mass problem [3] which will be a subject of separate publication , or other scales in cosmology [24, 38]. It would also be interesting to generalise the techniques developed in this paper to the gravitational field of a rotating mass. Thus effects, like the Lense-Thirring [25] precession associated with such gravitational fields could be investigated. We note that the Schwarzschild solution because of its static character does not describe rotation of the mass distribution. Such generalisation is of additional interest for the following reason: taking into account very fine corrections due to a non-zero cosmological constant calculated in this paper, one should look carefully if they are not comparable with other subtle relativistic effects, like the rotation of the Sun and the associated Lense-Thirring effect. Other subtle effects include the constant loss of mass during cosmological times.

In addition, the precise general relativistic predictions obtained can be compared with the predictions of scalar-tensor type gravity theories such as the Brans-Dicke theory. We already mentioned that the latter theory for small values of its additional parameter $\omega$ ($\omega \sim 5$) is in conflict with the observations for the perihelion precession of Mercury Eq.(1) and current estimations of the
Sun’s quadrupole moment [6, 4].

Non-commutative field theories are also strongly constrained from the precession of the perihelion of Mercury [35].

We are entering a new era in cosmology where precise theoretical results can be compared with the ever increasing accuracy of observations.

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A Definitions of genus-2 theta functions that solve Jacobi’s inversion problem

Riemann’s theta function [33] for genus $g$ is defined as follows:

$$\Theta(u) := \sum_{n_1, \ldots, n_g} e^{2\pi i un + i\pi \Omega n^2}$$

where $\Omega n^2 := \Omega_{11} n_1^2 + \cdots + 2\Omega_{12} n_1 n_2 + \cdots$ and $un := u_1 n_1 + \cdots + u_g n_g$. The symmetric $g \times g$ complex matrix $\Omega$ whose imaginary part is positive definite is a member of the set called Siegel upper-half-space denoted as $\mathcal{L}_{S_g}$. It is clearly the generalization of the ratio of half-periods $\tau$ in the genus $g = 1$ case. For genus $g = 2$ the Riemann theta function can be written in matrix form:

$$\Theta(u, \Omega) = \sum_{n \in \mathbb{Z}^2} e^{|\pi i n^T \Omega n + 2\pi i n u|}$$

$$= \sum_{n_1, n_2} e^{\pi i \left( n_1 \ n_2 \right) \left( \begin{array}{cc} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{array} \right) \left( \begin{array}{c} n_1 \\ n_2 \end{array} \right) + 2\pi i \left( n_1 \ n_2 \right) \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right)}$$

(58)

Riemann’s theta function with characteristics is defined by:

$$\Theta(u; q, q') := \sum_{n_1, \ldots, n_g} e^{2\pi i u(n+q') + i\pi \Omega(n+q')^2 + 2\pi i q(n+q')}$$

(59)

herein $q$ denotes the set of $g$ quantities $q_1, \cdots, q_g$ and $q'$ denotes the set of $g$ quantities $q'_1, \cdots, q'_g$. Eq.(59) can be rewritten in a suggestive matrix form:

$$\Theta\left[\begin{array}{c} q' \\ q \end{array}\right](u, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i f(n+q')\Omega(n+q') + 2\pi i f(n+q')(u+q)}, \quad q, q' \in Q^g$$

(60)
The theta functions whose quotients provide a solution to Abel-Jacobi’s inversion problem are defined as follows [34]:

\[ \theta(u; q, q') := \sum e^{au^2 + 2hu(n+q') + b(n+q')^2 + 2i\pi q(n+q')} \]  

(61)

where the summation extends to all positive and negative integer values of the \( g \) integers \( n_1, \cdots, n_g \), \( a \) is any symmetrical matrix whatever of \( g \) rows and columns, \( h \) is any matrix whatever of \( g \) rows and columns, in general not symmetrical, \( b \) is any symmetrical matrix whatever of \( g \) rows and columns, such that the real part of the quadratic form \( bm^2 \) is necessarily negative for all real values of the quantities \( m_1, \cdots, m_g \), other than zero, and \( q, q' \) constitute the characteristics of the function. The matrix \( b \) depends on \( \frac{1}{2}g(g+1) \) independent constants; if we put \( i\pi\Omega = b \) and denote the \( g \)-quantities \( hu \) by \( i\pi U \), we obtain the relation with Riemann’s theta function:

\[ \theta(u; q, q') = e^{au^2}\Theta(U; q, q') \]  

(62)

The dependence of genus-2 theta functions on two complex variables is denoted by: \( \theta(u_1, u_2; q, q') = \theta(u_1, u_2; q, q') \), the dependence on the Siegel moduli matrix \( \Omega \) by: \( \theta(u_1, u_2; \Omega, q, q') \). To every half-period one can associate a set of characteristics. For instance, the period \( u^{a_1,a_1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \) while \( \theta(u) \) is a theta function of two variables with zero characteristics, i.e. \( \theta(u) = \theta(u; 0, 0) = \theta \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} (u, \Omega) \). Also, Weierstraß had associated a symbol for each of the six odd theta functions with characteristics and the ten even theta functions of genus two. For example, \( \theta(u) \) is associated with the Weierstraß symbol 5 or occasionally the number appears as a subscript, i.e. \( \theta(u)_5 \).

The matrix elements \( h_{ij}, \Omega_{ij} \) can be explicitly written in terms of the half-periods \( U^a_r \) defined in section 3.6. For instance, the matrix element \( h_{11} = \frac{1}{2}U^r_1U^r_2 - U^r_3U^r_4 \) \( \times i\pi \), while \( \Omega_{11} = \frac{1}{2}U^r_1U^r_2 - U^r_3U^r_4 \).

Let us perform some consistency checks for particular cases of the inversion problem:

\[ \int_{a_1}^{u_1} \frac{du}{y} + \int_{a_2}^{u_2} \frac{du}{y} \equiv \phi_1 \]

\[ \int_{a_1}^{u_1} \frac{y}{u} \frac{du}{y} + \int_{a_2}^{u_2} \frac{y}{u} \frac{du}{y} \equiv \phi_2 \]  

(63)

For instance for \( \phi_1 = \phi_1^{a_2,a_1}, \phi_2 = \phi_2^{a_2,a_1} \), \( u_2 = a_2 = e_1, u_1 = c_2 = e_2 \). Let us check if the formulas eqs.(35)-(37) reproduce the chosen upper limits of the period integrals in the inversion problem.

In this case,

\[ \frac{1}{A_1} \frac{\theta^2(\phi)\phi_1^{a_1,a}}{\theta^2(\phi)} = (u_1 - a_1)(u_2 - a_1) = (c_2 - a_1)(a_2 - a_1) \]

\[ \frac{1}{A_2} \frac{\theta^2(\phi)\phi_2^{a_2,a}}{\theta^2(\phi)} = 0 \]  

(64)
and therefore using (37) we obtain $u_2 = a_2$. Then from

$$u_1 = a_1 + \frac{1}{A_3(u_2 - a_1)} \frac{\theta^2(q^a_1, a)}{\theta^2(q)} = a_1 + \frac{1}{a_2 - a_1} (c_2 - a_1)(a_2 - a_1) = c_2. \quad (65)$$

### B Roots of the cubic and special orbit cases for vanishing cosmological constant

The roots of the elliptic curve can be calculated analytically using the algorithm developed by Tartaglia and Cardano [43]. Their general expressions are given by:

$$r_1 = \frac{2^{1/3}\left\{2 + 18Lc^2 - 27LE^2 + 3\sqrt{3}\sqrt{\delta}\right\}^{2/3} + 2 - 6Le^2}{12 \left\{2 + 18Lc^2 - 27LE^2 + 3\sqrt{3}\sqrt{\delta}\right\}^{1/3}}$$

$$r_2 = \frac{\rho^{2^{1/3}}\left\{2 + 18Lc^2 - 27LE^2 + 3\sqrt{3}\sqrt{\delta}\right\}^{2/3} + \rho^2(2 - 6Le^2)}{12 \left\{2 + 18Lc^2 - 27LE^2 + 3\sqrt{3}\sqrt{\delta}\right\}^{1/3}}$$

$$r_3 = \frac{\rho^{2^{1/3}}\left\{2 + 18Lc^2 - 27LE^2 + 3\sqrt{3}\sqrt{\delta}\right\}^{2/3} + \rho(2 - 6Le^2)}{12 \left\{2 + 18Lc^2 - 27LE^2 + 3\sqrt{3}\sqrt{\delta}\right\}^{1/3}} \quad (66)$$

where

$$\delta := L\left(8c^4L + 4L^2c^6 + e^2(4 - 36E^2L) + E^2(27E^2L - 4)\right) \quad (67)$$

and $\rho = e^{2\pi i/3}$.

Equation (21), represents the exact solution in closed analytic form for the orbit of a test particle in the central gravitational field of the Sun assuming zero cosmological constant. As it was mentioned in section 3.1, not all choices of initial conditions will lead to planetary orbits.

Below we list the orbits that correspond to specific choices of initial conditions and in particular to the two elliptic curves for which one of their Weierstraß invariants vanishes.

i) Case of negative discriminant and $g_2 = 0, g_3 \neq 0$.

In this case for $L = \frac{1}{3\alpha}$ or equivalently, $L^2_M = 3c^2a_S^2$, we obtain: $g_2 = 0, g_3 = \frac{1}{16} - \frac{1}{16} - \frac{1}{16}(E^2 - c^2)$. The discriminant $\Delta = -27g_2^2 < 0$ and two roots are complex conjugates and one is real. The Weierstraß function $\wp(\phi)$ is real along the diagonal of the fundamental period parallelogram (FPP), and the real root $e_2$ is located at the point $\omega + \omega'$. As the argument of the Weierstraß function changes from $\omega + \omega'$ to 0 along the diagonal of FPP, its value changes from $e_2$ to $\infty$.

For $E = \sqrt{\frac{8}{3}c}$, also $g_3 = 0$ and then there is no elliptic curve since the discriminant vanishes and all three roots are zero.
ii) Case of $g_3 = 0, g_2 > 0$.

In this case there are three real roots, one of them $e_2 = 0$ and the other two are equal in magnitude and opposite in sign, $e_1 = \sqrt{\frac{-c^2}{4\sqrt{3}}}$, $e_3 = -e_1$. Here, as the argument of Weierstraß function $\wp$, travels along the line from $\omega + \omega'$ to $\omega$ in the FPP, its value changes from 0 to $e_1 > 0$. Thus, the radius vector varies from $r_{e_2} = 3\alpha_S$ to $r_{e_1} = \frac{\alpha_S}{4e_1 + \frac{1}{3}}$. On the other hand in the region, of the FPP along the line from $\omega + \omega'$ to $\omega' + 2\omega$ the Weierstraß function is negative $e_3 \leq \wp(\phi) \leq 0$. The radius vector increases from $3\alpha_S$ and tends asymptotically to infinity when $4\wp(\phi) + \frac{1}{3} \to 0$.

iii) Case of vanishing discriminant.

The discriminant $\Delta = g_2^3 - 27g_3^2$ can be written as follows:

$$\Delta = \frac{L( -4L^2c^6 + L(-27E^4 + 36c^2E^2 - 8c^4) + 4(E^2 - c^2))}{256}$$ (68)

It vanishes, besides the case we mentioned above for which $g_2 = g_3 = 0$, also for $L = 0$ and for

$$L = -\frac{1}{8c^6} \left(8c^4 - 36c^2E^2 + 27E^4 \pm \sqrt{-512c^6E^2 + 1728c^4E^4 - 1944c^2E^6 + 729E^8}\right)$$

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