Classification and Unification of the Microscopic Deterministic Traffic Models with Identical Drivers

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We show that all existing deterministic microscopic traffic models with identical drivers (including both two-phase and three-phase models) can be understood as special cases from a master model by expansion around well-defined ground states. This allows two traffic models to be compared in a well-defined way. The three-phase models are characterized by the vanishing of leading orders of expansion within a certain density range, and as an example the popular intelligent driver models (IDM) is shown to be equivalent to a generalized optimal velocity (OV) model. We also explore the diverse solutions of the generalized OV model that can be important both for understanding human driving behaviors and algorithms for autonomous driverless vehicles.

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It is of great interest, both theoretically and practically, to understand via simple models the emergent behaviors of the complex systems containing a large number of interacting components. Examples like crowd dynamics[1, 2], highway traffic systems[3, 4], and the more recent urban traffic flows[5] have attracted physicists for many years. Despite the observed complex patterns from these systems, some of the essential characteristics can be universal and well-defined, leading to the possibility of them being captured by simple mathematical models[5-6]. In contrast to the traditional many-body physical systems (e.g. involving identical particles like electrons), the crowd or traffic systems lack almost any symmetry at the microscopic level: even individual components are different from one another, and stochasticity is intrinsic. This poses great challenges in finding the simplest model with enough predictive power to adequately characterize these complex systems.

The lack of symmetries at the microscopic level leads to a certain arbitrariness in the model construction. The modeling of the highway traffic system has led to a plethora of traffic models[3-6]. Analysis of the empirical data has led to profound understanding of the traffic dynamics[5-6]; however since the interaction between vehicles is non-linear, even simple interactions can lead to very complex spatio-temporal patterns. It is thus difficult to decide which detailed driving behavior or specific functional form needs to be used in the model. This is the primary concern behind the controversy between the two-phase and three-phase traffic theories[10-11], which leads to two classes of traffic models distinguished by the presence or absence of a fundamental diagram in the flow-density plot[6]. More importantly, the lack of a fundamental principle in understanding various different traffic models makes it difficult to decide if in addition to the non-linear interactions between the vehicles, factors like stochasticity or diversity of driving behaviors are also essential to certain observed empirical features.

To tackle the dilemma of the apparent (over)abundance of traffic models, we need answers to the following questions: a). How do we properly characterize the differences between two traffic models? b). Is there a standard way of extending an existing traffic model or construction of a new traffic model? c). Is there a standard way in selecting the best traffic model based on the experimental data? In this Letter, we answer in details the first two questions for deterministic microscopic models with identical drivers, by proposing a general framework of obtaining all such existing traffic models from a master model. The controlled expansion around properly defined “ground states” of the master model shows that the two-phase and three-phase traffic models are both just special cases. The general framework also allows us to classify all existing as well as new traffic model, and show explicitly various types of approximations in a well-defined way. Qualitative features of various possible solutions to the master model are also explored.

The answer to the third question lies in the fact that in principle the master model can be obtained experimentally, via a renormalization-like procedure by averaging over unimportant factors influencing the reaction of individual drivers. The details can be found in[12] and will also be discussed elsewhere. We will now develop the general framework by assuming the following general form of the master model

\[ \tilde{a}_n = \kappa^2 \tilde{f} \left( \tilde{h}_n, \tilde{v}_n, \Delta \tilde{v}_n \right) \]  

(1)

Here the assumption we make is that the headway of the n-th vehicle, \( \tilde{h}_n \), its velocity \( \tilde{v}_n \), and the velocity difference between the preceding vehicle \( \Delta \tilde{v}_n = \tilde{v}_{n+1} - \tilde{v}_n \) are the important factors affecting the acceleration of the n-th vehicle, and all other factors are averaged over. One can choose more (or different) dynamic variables in the function \( \tilde{f} \), but for the purpose of illustration we choose this simple yet sufficient case. The parameter \( \kappa \) has the dimension of inverse time.

The properties of the master model can be studied by
The key assumption of the three-phase traffic theory is that the scattering of the flow-density plot in the “synchronized phase” corresponds to a multitude of equilibrium states with non-unique flow-density relations.\textsuperscript{[9]} One should note, however, it is empirically difficult to verify if those states are indeed equilibrium states. The macroscopic quantities like flow and density are empirically obtained from the raw data of the double loop sensors and averaged quantities\textsuperscript{[11]}. A transient state with qualitative or quantitative features that evolves very slowly will be captured by the sensor as the “equilibrium state”. Thus for most practical purposes we cannot distinguish such long lasting quasisteady states from the equilibrium states.

We thus argue that instead of dealing with a non-analytic function as shown in Fig.(1), we can for most practical purposes replace it with an analytic function with an inflection point at the x-axis (i.e. Fig.(1d)). Such a model still has a fundamental diagram, but for any state not too far away from the inflection point, it evolve very slowly due to the smallness of the accelerations. Numerical simulation of such analytic models show very similar scattering of the flow-density plot and the spatiotemporal patterns as the proposed three-phase models within any physically reasonable time scale. Therefore fundamentally there is no strict boundary between the two-phase and three-phase models, and we will proceed to show the way to differentiate them is no different from the way of differentiating between different two-phase models.

We will now only deal with master models that are analytic. The ground states of this master model are defined as the solutions with all vehicles equally spaced apart with headway \( h \) and travelling at the same time-independent velocity \( \bar{v} \). In Eq.(1) we use dimensionless quantities with headway and velocity measured in the units of \( \rho_{op} \), the maximum density of the traffic, and \( V_{max} \), the maximum velocity of the vehicles: \( \partial^{2} \bar{h}_{n}/\partial t^{2} = \bar{a}_{n+1} - \bar{a}_{n} \) and the only scale in the model is \( \kappa = \rho_{op} V_{max} \). Clearly the ground states are indexed by the average density \( h^{-1} \), and at some density there may be more than one ground states (i.e. Fig.(1b)). We can thus expand the master model around a chosen ground state:

\[
\bar{a}_{n} = \sum_{p,q} \kappa_{p,q} \left( \bar{h}_{n} \right) \left( \bar{v}_{n} - V_{op} \left( \bar{h}_{n} \right) \right)^{p} \Delta \bar{v}_{n}^{q} \quad (2)
\]

\[
\kappa_{p,q} \left( \bar{h}_{n} \right) = \left. \frac{\partial^{p+q} \tilde{f}}{\partial^{p} \bar{v}_{n} \partial^{q} \Delta \bar{v}_{n}} \right|_{\bar{v}_{n}=V_{op} \left( \bar{h}_{n} \right), \Delta \bar{v}_{n}=0} \quad (3)
\]

where \( V_{op} \left( h \right) \) satisfies \( \tilde{f} \left( h, V_{op} \left( h \right) \right, 0 \) = 0. Clearly any existing two-phase models can be expanded this way, and they can be quantitatively compared by the set of coefficients \( \kappa_{p,q} \). The optimal velocity (OV) model\textsuperscript{[16]} is the special case where \( \kappa_{1,0} \) is a negative constant while all other \( \kappa_{p,q} \) vanishes. The full velocity difference model (FVD) has an additional constant \( \kappa_{0,1} \textsuperscript{[17]} \).
while the asymmetric FVD [18] can be understood as having additional non-linear corrections with non-vanishing $\kappa_{1,q}, q > 1$.

Within this framework the three-phase models and the phase transition can be understood as follows: there is a certain range of the headway (typically when the headway is smaller than some “synchronization gap” but larger than that within a wide moving jam), $\kappa_{p,0}$ vanishes for $p < p_0$ so that the leading term involving $v_n$ are of higher orders. The larger the value of $p_0$, the more long lasting the transient states are around the inflection points. Since the exact form of $\tilde{f}$ can be obtained experimentally, the framework offers a way to justify the assumptions of the three-phase traffic theory empirically.

Seemingly different traffic models can be shown to be qualitatively equivalent under this scheme of controlled expansion. As an example we look at the IDM model which has the following form:

$$a_n = a \left(1 - \frac{v_n}{v_0}\right) - \left(\frac{h^* (v_n, \Delta v_n)}{h_n}\right)^2$$  \hspace{2cm} (4)

$$h^* (v, \Delta v) = s_0 + s_1 \frac{v}{v_0} + T v + \frac{v \Delta v}{2 \sqrt{a b}}$$  \hspace{2cm} (5)

Here we use the dimensionful variables for headways and velocities and pick the standard set of parameters $a = 0.73 m s^{-2}, b = 1.67 m s^{-2}, v_0 = 33 m s^{-1}, s_0 = 2 m, s_1 = 0 m, T = 1.6 s, \delta = 4$. The ground state is unique for any headway and the relationship between $h$ and $V_{op}$ is given by

$$h = (s_0 + V_{op} T) \left(1 - \frac{V_{op}}{v_0}\right)^{\frac{4}{2}}$$  \hspace{2cm} (6)

The Taylor expansion around the ground state configuration has a finite number of terms as follows

$$a_n = \sum_{p=1,q=0}^{p=4,q=2} \lambda_{p,q} (v_n - V_{op} (h_n))^p \Delta v_n^q$$  \hspace{2cm} (7)

The non-vanishing coefficients of expansion are plotted in Fig.2 as functions of the headway. One can clearly see at high traffic density (small $h$) $\lambda_{10}$ dominates, so the acceleration of the vehicle mostly depends on the headway only. $\lambda_{p0}$ decreases with increasing $h$, indicating the driver’s sensitivity to the headway decreases (i.e. drivers become more relaxed) as the traffic is less crowded. The driver is only sensitive to $\Delta v_n$ at intermediate traffic density, this is also intuitive since in the congested traffic it is more important to keep a safe headway than to synchronize the velocity.

Thus from Fig.2 it is probably enough to just keep $\lambda_{10}, \lambda_{11}, \lambda_{11}$ for the qualitative features of the traffic dynamics to be preserved. Numerical simulation shows even ignoring $\lambda_{11}$ may be good enough for the qualitative features, while $\lambda_{10}$ and $\lambda_{01}$ can be tuned by step-like functions for quantitative agreement with the original IDM. Truncation of higher orders and simplification of the expansion coefficients can be applied to other models like Shamoto’s [19]. However it is not a priori true that the original IDM or Shamoto’s models are more realistic than their simplified counterparts. The tuning of the expansion coefficients should be based on the experimental measurement of the master model.

The original OV models [16] [18] are the simplest case of Eq.2 where the coefficients of expansion are constants and only the term linear in $(v_n - V_{op} (h_n))$ is kept. The phase diagram of such models are well-known [21] [25]. To understand the more realistic traffic models as shown above, and for tuning and construction of new traffic models, it is also important to understand the emergent characteristics of the generalized OV models with general coefficients of expansion and higher orders of expansion. This is in general difficult due to the non-linear interaction, and we will just explore some simple examples, leaving a more detailed discussion elsewhere.

We start by generalizing the simplest artificial OV model as follows:

$$a_n = \lambda (h_n) (\tanh (h_n) - v_n)$$  \hspace{2cm} (8)

It is useful to plot $\lambda$ together with the coexistence curve as well as the neutral stability (NS) line of the original OV model. The coexistence curve at $h < 0$ is given by the minimum headway of the cluster solution, which we call the $h_{\text{min}}$-branch; naturally the other half is called the $h_{\text{max}}$-branch [26]. The linear stability condition at average density $h_{\text{av}}$ is given by $\lambda (h_0) > 2 \text{sech}^2 h_0$.

The qualitative features of Eq.8 can be roughly classified by the way $\lambda$ intersects with the coexistence curve and the NS line. If $\lambda$ intersect with the $h_{\text{min}}$- and the $h_{\text{max}}$-branch at most once separately (see Fig.3), the constant headway ($h_n = \text{const}$) solution of Eq.8 can be stable or unstable against the development of cluster solution with minimum and maximum headways $h_{\text{min}}$ and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{(Color Online.)The coefficients of expansion of the IDM in Eq.(4) as the function of the headway $h$, and $\lambda_{pq}$ is defined in Eq.(7).}
\end{figure}
\( \tilde{h}_{\text{max}} \), qualitatively the same as the original OV model; both \( \tilde{h}_{\text{min}} \) and \( \tilde{h}_{\text{max}} \) are independent of the density of the traffic. Clearly only \( \lambda \in \{ \tilde{h}_{\text{min}}, \lambda (\tilde{h}_{\text{max}}) \} \) contributes to the cluster structure, and quantitatively we have the relationship:

\[
\tilde{h}_{\text{max/min}} \in \left[ \tilde{h}_{\text{max/min}} (\lambda(\tilde{h}_{\text{min}}), \lambda (\tilde{h}_{\text{max}})) \right]
\]

If \( \lambda (\tilde{h}_{\text{min}}) < 2 \text{sech}(\tilde{h}_{\text{min}}) \) or \( \lambda (\tilde{h}_{\text{max}}) < 2 \text{sech}(\tilde{h}_{\text{max}}) \), the vehicles in the clusters (or anti-clusters) are also linearly unstable and their headways will fluctuate (see \( \lambda(3) \) in Fig.(3b) and the inset), in contrast to the cluster structure of the original OV model.

If \( \lambda \) intersects with the \( \tilde{h}_{\text{min}} \)-branch or the \( \tilde{h}_{\text{max}} \)-branch more than once as shown in Fig.(3b), the traffic dynamics can be different at different vehicle density. For \( h_0 \) within the thickened part of \( \lambda(1) \) and \( \lambda(2) \), the stability of the \( h_n = \text{const} \) solution and the development of the cluster solution are similar as the cases in Fig.(3a). For other part of \( h_0 \), if \( \lambda (h_0) < 2 \text{sech}(h_0) \), the \( h_n = \text{const} \) solution is linearly unstable but clearly the cluster solution cannot be fully developed due to the conservation of the vehicle density. Partially developed cluster solutions with strong oscillations are obtained as shown in the inset of Fig.(3b).

Higher orders of \( (V_{op} - v_n) \) generally shift the values of \( \tilde{h}_{\text{min}} \) and \( \tilde{h}_{\text{max}} \). Here we look at the simple cases where the coefficients of expansion are constants. For situations like the ones in Fig.(4)), the stable phase can have multiple equilibrium headways with vehicles all traveling at the same velocity (see Fig.(4a)). For three-phase models where the low order terms (at least the linear order) vanishes, the NS line is no longer applicable. Since the accelerations are very small even when the vehicles deviates from their equilibrium headway, it is very difficult for the clusters to form, as shown in Fig.(4b) and Fig.(4c).

One should note that our method can be applied to more general cases, in particular with more realistic optimal velocity functions. This is because the cluster solutions, the coexistence curve and the NS line are universal features for the more complicated traffic models. Though step functions of \( \lambda \) are used in many of the arguments above for its simplicity, they can be deformed moderately or smoothened without altering the qualitative features. Since only \( \lambda \in \{ \lambda (h_{\text{min}}), \lambda (h_{\text{max}}) \} \) determines the instability of the model, one can also alter the shape of \( \lambda \) outside of that range arbitrarily without altering the cluster structure or the phase diagram. For example, the inset of Fig.(3b) will be realized by a monotonically decreasing \( \lambda \) intersecting the \( h_{\text{max}} \)-branch of the coexistence curve twice, as is the case for the IDM model.

In conclusion, we propose that microscopic traffic model should start with identifying the proper optimal velocity function, from which the family of ground states are defined. Simple traffic models are constructed as special cases of the expansion around those ground states, with coefficients of expansion in general dependent on the vehicle density. We thus have a standard way of comparing different traffic models, and in particular the “synchronized phase” can be formally understood as the result of the models, where the leading order expansion about the optimal velocity vanishes within some range of vehicle density. The theoretical framework also proposes standard ways of both model construction, as well.
as validation by the empirical data. The understanding of various possible solutions to a more general class of models introduced here could also be very useful for designing the algorithms of autonomous driverless vehicles for large scale highway transportation.

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