Ward identities for charge and heat currents of particle-particle and particle-hole pairs

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Abstract - The Ward identities for the charge and heat currents are derived for particle-particle and particle-hole pairs. They are the exact constraints on the current-vertex functions imposed by conservation laws and should be satisfied by consistent theories. While the Ward identity for the charge current of electrons is well established, that for the heat current is not understood correctly. Thus the correct interpretation is presented. On this firm basis the Ward identities for pairs are discussed. As the application of the identity we criticize some inconsistent results in the studies of the superconducting fluctuation transport and the transport anomaly in the normal state of high-\(T_c\) superconductors.

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Introduction. - The Ward identity plays crucial roles in various aspects of theoretical physics. It is a consequence of a conservation law and a basic relation which should be satisfied by consistent theories. One of the most effective applications of the Ward identity in condensed-matter physics is the gauge-invariant formulation of the Meissner effect in superconductors [1]. At the same time it leads to the discovery of the Nambu-Goldstone mode which appears in the state with spontaneously broken symmetry.

In this letter two kinds of Ward identity are discussed in the context of condensed-matter physics. One is for the charge current and the other is for the heat current. The Ward identity for the charge-current vertex of electrons is well known by the textbook discussion [2]. On the other hand, there is no literature which summarizes the correct understanding of the heat-current vertex. Thus we give a summary on the heat-current vertex including our original finding.

The above description concerns the vertex function for electrons. However, the main purpose of this letter is to establish the constraint on the vertex function, the Ward identity, for particle-particle and particle-hole pairs. Although the Ward identity for the charge current carried by particle-particle pairs has been discussed in the study of superconducting fluctuation transport [3,4], the proof of the identity has not been given. Although it is obvious that particle-hole pairs, which are charge-neutral, do not carry charge, only perturbational results have been reported [5,6] but the rigorous proof of it, which can be achieved on the basis of the Ward identity, is absent. For the heat current there is no literature discussing the Ward identity for pairs. These absent discussions are given in this letter.

By establishing the rigorous constraint on the vertex function we criticize some inconsistent arguments seen in the published results. Several examples are discussed as the applications of the Ward identity.

Detailed calculations are given in refs. [7–12].

Algebraic proof. - First we show the proof [2] of the Ward identity for the charge current of electrons at zero temperature. Let us start from the charge conservation law

\[
\nabla \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial}{\partial t} \rho(\mathbf{r}, t) = 0,
\]

where \(\rho(\mathbf{r}, t)\) and \(\mathbf{j}(\mathbf{r}, t)\) are the charge and current densities at the position \(\mathbf{r}\) and the time \(t\). The three-current \(\mathbf{j}\) is represented as \(\mathbf{j} = (j_1, j_2, j_3)\). In Fourier-transformed variables the conservation law becomes

\[
\mathbf{q} \cdot \mathbf{j}(\mathbf{q}, \omega) - \omega \rho(\mathbf{q}, \omega) = 0.
\]

By introducing the four-current \(j = (j_1, j_2, j_3, j_0)\) with \(j_0 \equiv \rho\) and the four-vectors \(x = (x_1, x_2, x_3, x_0) = (\mathbf{r}, t)\)
and \( q = (q_1, q_2, q_3, q_0) = (q, -\omega) \), the conservation law is expressed as the vanishing four-divergence

\[
\sum_{\mu=0}^{3} \frac{\partial}{\partial x_\mu} j_\mu(x) = 0 = \sum_{\mu=0}^{3} q_\mu j_\mu(q). \tag{3}
\]

Here we introduce the three-point function \( \Lambda_\mu \) defined by

\[
\Lambda_\mu(x, y, z) = \langle T \{ j_\mu(z) \psi_\uparrow(x) \psi_\uparrow(y) \} \rangle, \tag{4}
\]

where \( x, y, \) and \( z \) are the four-vectors in real space, \( \langle A \rangle \) represents the expectation value of \( A \) in the ground state, \( T \) is the time-ordering operator, and \( \psi_\uparrow(x) \) and \( \psi_\uparrow(y) \) are the annihilation and creation operators of the \( \uparrow \)-spin electron. Under the conservation law (3) the four-divergence of \( \Lambda_\mu \) reduces to

\[
\langle T \{ \rho(z), \psi_\uparrow(x) \psi_\uparrow(y) \} \rangle \delta(z_0 - x_0) + \langle T \{ \psi_\uparrow(x) [\rho(z), \psi_\uparrow(y)] \} \rangle \delta(z_0 - y_0). \tag{5}
\]

The commutation relations reduce to the annihilation and creation of electron charge as

\[
[\rho(z), \psi_\uparrow(x)] \delta(z_0 - x_0) = -e \psi_\uparrow(x) \delta^4(z - x), \tag{6}
\]

and

\[
[\rho(z), \psi_\uparrow(y)] \delta(z_0 - y_0) = e \psi_\uparrow(y) \delta^4(z - y), \tag{7}
\]

so that we obtain

\[
\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda_\mu(x, y, z) = -ieG(x, y) \delta^4(z - x) + ieG(x, y) \delta^4(z - y), \tag{8}
\]

where the electron propagator \( G(x, y) \) is introduced as

\[
G(x, y) = -i \langle T \{ \psi_\uparrow(x) \psi_\uparrow(y) \} \rangle. \tag{9}
\]

In Fourier-transformed variables eq. (8) is expressed as

\[
\sum_{\mu=0}^{3} k_\mu \Lambda_\mu(p, k) = eG(p) - eG(p + k), \tag{10}
\]

where \( k_\mu \) is the momentum. If the interaction is non-local, we can obtain (14) only in the limit of \( k \to 0 \). To prove this [7,8] we should use the Fourier variables as

\[
\sum_{\mu=0}^{3} k_\mu \Lambda_\mu^Q(p, k) = \hat{F} I_{p,k}(x_0, y_0, z_0), \tag{15}
\]

with

\[
\hat{F} \equiv \int \mathrm{d}(x_0 - y_0) e^{-i p_0(x_0 - y_0)} \times \int \mathrm{d}(z_0 - x_0) e^{-i k_0(z_0 - x_0)}, \tag{16}
\]

where the integrand \( I_{p,k}(x_0, y_0, z_0) \) is given by

\[
\langle T \{ [\rho^Q_k(z_0), a_{p-k}(x_0)] a_{p}^\dagger(y_0) \} \rangle \delta(z_0 - x_0) + \langle T \{ a_{p-k}(x_0) [\rho^Q_k(z_0), a_{p}^\dagger(y_0)] \} \rangle \delta(z_0 - y_0). \tag{17}
\]

Here \( \rho^Q_k = \int \mathrm{d}r \exp(-i k \cdot r) S^Q(r) \) and \( a_p \) and \( a_p^\dagger \) are the Fourier transform of \( \psi_\uparrow(r) \) and \( \psi_\uparrow^\dagger(r) \). In the limit of \( k \to 0 \) [7,8] we can replace \( \rho^Q_k \) with \( K \) in (17) and obtain (14).
The above formulation at zero temperature is straightforwardly translated into that at finite temperature [7,10]. The Ward identities, (12) for the charge current and (14) for the heat current, are consistent with the Jonson-Mahan formula [14] which is the exact relation between electric and thermal conductivities as discussed in [7].

**Diagrammatic proof.** – The diagrammatic proof of the Ward identity (12) for the charge current is a subject of a standard textbook [15]. Most of the contributions to the current vertex cancel out and only the “end” contribution remains [12]. The right-hand side of (12) is the “end” contribution. The cancelation in the case of the heat current is too complicated to show here, but the proof is given in [12]. In the proof the Jonson-Mahan transmutation [16] plays a crucial role. It explains the way how the kinetic energy that the heat-current vertex for Cooper pairs reported in [18] transforms of the particle-particle pair is given as

\[ P(R) = \int \Delta_{\chi}(r) \psi_{r}(r_{1}) \psi_{r}(r_{2}), \]  

(22)

where \( r_{1} = R + r/2 \) and \( r_{2} = R - r/2 \). The Fourier transform of the particle-particle pair is given as

\[ P_{\mathbf{q}} = \sum_{\mathbf{q}} \chi(r) b_{-p+q/2} c_{p+q/2}, \]  

(23)

where \( b_{p} \) is the Fourier transform of \( \psi_{r}(r) \). As seen in (17) it is essential for the derivation of the Ward identity to evaluate the equal-time commutation relation, \( [\rho_{k}, \rho_{q}] \) for the heat current and \( [\rho_{k}^{Q}, \rho_{q}^{Q}] \) for the heat current. In the limit of \( k \rightarrow 0 \) we obtain \( [\rho_{k}, \rho_{q}] \rightarrow e^{*} P_{q-k}^{Q} \) and \( [\rho_{k}^{Q}, \rho_{q}^{Q}] \rightarrow [K, P_{q-k}^{Q}] \) so that the same Ward identities as (20) and (21) result.

**Applications.** – The Ward identity imposes a constraint on the vertex function and can be a guide to a consistent theory.

In the study of superconducting fluctuation transport a relatively recent report [20] claims the result which is not consistent with the time-dependent Ginzburg-Landau (TDGL) theory [21,22]. Since the TDGL theory is consistent with our Ward identities [10] and obeys the conservation laws, it is concluded that such a claim violates the conservation laws. Our microscopic theory is consistent with the TDGL theory as the microscopic Fermi-liquid theory [23,24] is consistent with the Boltzmann-transport theory.

In the fluctuation-exchange (FLEX) approximation [25] discussing the transport anomaly in the normal state of high-\( T_{c} \) superconductors, the Aslamazov-Larkin process of the particle-hole pair fluctuation vanishes [9] in accordance with the Ward identity (20). On the other hand, if it vanishes, the FLEX approximation loses the consistency with the Fermi-liquid theory or the Boltzmann-transport theory. The inconsistency arises from the replacement of the renormalized interaction in the Fermi-liquid theory with the particle-hole fluctuation. Such a replacement violates the Pauli principle [26–28] essential for the degenerate Fermi systems. The correct microscopic treatment [23,24] obeying the Pauli principle leads to the expected collision term in the Boltzmann equation.
Concluding remarks. — In the discussion of the Ward identity the equal-time commutation relation plays the central role.

In the case of the charge current it picks up the integrated charge of the object in the limit of vanishing external momentum. In this limit the wavelength of the electromagnetic field exceeds the size of the object so that the object can be treated as a point with its integrated charge in the discussion of the electromagnetic response. Thus it is concluded that charge-neutral pairs do not couple to the electromagnetic field as expected [29]. Namely, charge- and spin-density fluctuations do not carry charge. On the other hand, the particle-particle pair, i.e. the Cooper pair, carrying charge $2e$ couples to the electromagnetic field.

In the case of the heat current it picks up the energy of the object.

Although only Ward identities for charge and heat currents are discussed in this letter, other Ward identities are also actively discussed. As examples of recent developments, the spin current is discussed in [30,31] and the sum rules are discussed in [32,33].

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