SOME ACHIEVED WEDGE PRODUCTS OF POSITIVE CURRENTS

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Abstract. In this paper, we study the existence of the current $gT$ for positive plurisubharmonic currents $T$ and unbounded plurisubharmonic functions $g$.

1. Introduction

The wedge product is a very vital subject in the field of currents, and mainly targeted indeed. In general, wedge product of currents cannot be achieved unless further conditions are considered. Throughout this paper we consider $\Omega$ to be an open subset of $\mathbb{C}^n$ and $T$ to be a current of bi-dimension $(p, p), p \geq 1$. We denote by $Psh^-(\Omega)$ the set of all negative plurisubharmonic functions on $\Omega$. For a function $g \in Psh^-(\Omega), put L_g$ to be the set of all locus points of $g$ which consists of the points $z \in \Omega$ where $g$ is unbounded in every neighborhood of $z$. The pole set of $g$ is by definition $P_g = \{g = -\infty\}$. It is obvious that $L_g$ is closed, and $P_g \subset L_g$. Recall also that $T$ is said to be closed if $dT = 0$, and is said to be plurisubharmonic (resp. plurisuperharmonic) if $dd^c T \geq 0$ (resp. $dd^c T \leq 0$).

Our main concern is to give a definition of the current $gT$. Obviously, apart of $L_g$, the current $gT$ is well defined. The real challenge is studying the existence of this product across $L_g$. In such a situation, this product may have no sense due to the behaviors of $g$ and $T$. For example, In $\mathbb{C}$ put

$$T = \left(\frac{-\log|z|^2}{|z|^2}\right) dz \wedge d\bar{z}$$

and $g = \log|z|^2$.

Then we get a positive and integrable current $T$ of bi-dimension $(1, 1)$ on $B(0, 1)$ which is plurisubharmonic outside the origin, and a function $g \in Psh^-(B(0, 1)) \cap C^\infty(B(0, 1) \setminus \{0\})$. Despite the fact that $\mathcal{H}_{2(1)} = 1$, the current $gT$ is of infinite mass near the origin. Now we can feel the motivation behind the paper which is basically about finding sufficient conditions on $T$ and $L_g$ that make $gT$ well defined. The study is also consistent with the evolution of the subject as the case when $T$ is closed was considered before in many works. In fact, Demailly [8] (1993) proved the existence of $gT$ and $dd^c g \wedge T$ as soon as $\mathcal{H}_{2p-2}(L_g \cap \text{Supp} T) = 0$. Fornaess and Sibony [10].

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(1994) generalized the work of Demailly to higher Hausdorff dimension when they succeeded to define the currents $gT$ and $dd^c g \wedge T$ where $H_{2p}(L_g) = 0$. In both studies the closedness property played a main role since it was used to give the relation between $gT$ and $dd^c g \wedge T$. Namely, with such a property, the definition of $dd^c g \wedge T$ is given by $dd^c (gT)$ in the sense of distribution.

Unfortunately, this relation becomes more complicated once we deal with $dd^c$-signed currents as the terms $d g \wedge d^c T$ and $g dd^c T$ have their contribution. This fact causes difficulties to achieve a definition of $gT$, and forced some interested researchers to study the current $dd^c g \wedge T$ separately. In what follows we summarize our main results.

Let $T$ be a positive plurisubharmonic current of bi-dimension $(p, p)$ on $\Omega$ and $g \in \text{Psh}^{-}(\Omega) \cap C^1(\Omega \setminus L_g)$ for some compact subset $A$ of $\Omega$. Assume that $(g_j)$ is a sequence of decreasing smooth plurisubharmonic functions on $\Omega$ converging to $g$ in $C^1(\Omega \setminus L_g)$, then $g_j T$ converges weakly* to a current denoted by $gT$.

1. $L_g$ is compact, $L_g = P_g$ and $p \geq 2$. (Theorem 2.3)
2. $L_g$ is compact and $gd^c T$ is well defined. (Theorem 2.6)
3. $H_{2p-1}(L_g \cap \text{Supp} T) = 0$ and $gd^c T$ is well defined. (Theorem 2.7)
4. $H_{2p-2}(L_g \cap \text{Supp} T)$ is locally finite. (Theorem 2.8)

The precautions taken in the previous results on the thickness of $L_g$ and the properties of $T$ are extremely important to guaranty the existence of $gT$. Actually, the failure to define $gT$ with the choices of (1.1) due to fact that both $gd^c T$ and $gdd^c T$ are of infinite mass near the origin. In a recent work, Al Abdulaali-El Mir [3] obtained the current $g^k T$, $k > 0$ when $g$ is radial and $L_g$ is reduced to a single point.

The second part of the paper is devoted to discuss the current $dd^c g \wedge T$. As a consequence of the discussions of this part, we show that the quantity

$$
\mu(T, g) = \lim_{r \to -\infty} \int_\{g<r\} T \wedge (dd^c g) \wedge \beta^{p-1}, \beta = dd^c \|z\|^2
$$

exists under the hypotheses of Theorem 2.6. Furthermore, by a counterexample it is shown that the induced results can not be obtained for the case of positive plurisuperharmonic currents without further hypotheses.

2. The Current $gT$

Let us start with a result due to Al Abdulaali [2]. In the following few lines we include the proof in our settings.

**Lemma 2.1.** Let $T$ be a positive plurisubharmonic current of bi-dimension $(p, p)$ on $\Omega$ and $g \in \text{Psh}^{-}(\Omega) \cap C^1(\Omega \setminus A)$ for some compact subset $A$ of $\Omega$. Assume that $(g_j)$ is a sequence of decreasing smooth plurisubharmonic functions on $\Omega$ converging to $g$ in $C^1(\Omega \setminus A)$. Then
Hence by \( \text{the residual current} \) 
\( \text{the trivial extension} \) 
\( \text{complete pluripolar and} \) 
\( \text{is positive and} \) 
\( \text{is considered to be a single point.} \)

**Proof.** Let \( W \) and \( W' \) be neighborhoods of \( A \) such that \( W \Subset W' \Subset \Omega \), and take a positive function \( f \in C_0^\infty(W') \) so that \( f = 1 \) on a neighborhood of \( W \). Then we have

\[
\int_{W'} \dd\!^c(fg_j) \wedge T \wedge \beta^{p-1} = \int_{W'} fg_j \dd\!^c T \wedge \beta^{p-1} \leq 0. \tag{2.1}
\]

This implies that

\[
0 \leq \int_{W'} f \dd\!^c g_j \wedge T \wedge \beta^{p-1} - \int_{W'} fg_j \dd\!^c T \wedge \beta^{p-1} \\
\leq \left| \int_{W'} d\!^c g_j \wedge d\!^c f \wedge T \wedge \beta^{p-1} \right| + \left| \int_{W'} df \wedge d\!^c g_j \wedge T \wedge \beta^{p-1} \right| \tag{2.2}
\]

+ \left| \int_{W'} g_j \dd\!^c f \wedge T \wedge \beta^{p-1} \right|.

Thanks to the properties of \( f \), each term of the first line integrals of (2.2) is uniformly bounded. Therefore, one can infer the existence of both extensions \( \dd\!^c T \) and \( \dd\!^c g \wedge T \). Notice that, the current \( \dd\!^c T \) is well defined by the monotone convergence. And by Banach-Alaoglu the sequence \( (\dd\!^c g_j \wedge T) \) has a subsequence \( (\dd\!^c g_j, T) \) which converges weakly* to a current denoted by \( S \). To show (2), we first note that \( \dd\!^c S \) is a well defined current as well. Hence by [2] the residual current \( R = \dd\!^c S - \dd\!^c (\dd\!^c g \wedge T) \) is positive and supported in \( A \). Now, if we set \( F := S - \dd\!^c g \wedge T \) we find clearly that \( F \) is a positive current where

\[
\dd\!^c F = \dd\!^c S - \dd\!^c (\dd\!^c g \wedge T) \geq \dd\!^c S - \dd\!^c S = 0.
\]

As \( F \) is a compactly supported current with bi-dimension \( (p-1, p-1) \), one can deduce that \( F \equiv 0 \). The third statement comes immediately from the fact that the distribution \( \mu := (S - \dd\!^c g \wedge T) \wedge \beta^{p-1} \) is positive and supported in \( A \). Indeed, \( A \) can be assumed to be the origin, and hence there exists a positive constant \( c \) such that \( \mu = c\delta_0 \) where \( \delta_0 \) is the Dirac measure.

Clearly, the constant \( c \) is independent from the choice of \( j_s \) since

\[
c = \mu(f) = \lim_{j_s \to \infty} \int_{W'} f \dd\!^c g_{j_s} \wedge T \wedge \beta^{p-1} - \int_{W'} f \dd\!^c g \wedge T \wedge \beta^{p-1} \\
= \int_{W'} (\dd\!^c f + 2dg \wedge d\!^c f) \wedge T \wedge \beta^{p-1} \\
+ \int_{W'} fg \dd\!^c T \wedge \beta^{p-1} \tag{2.2}
\]

- 0.
In other words, \( dd^c g \wedge T \) is well defined. \[ \square \]

As a consequence of the previous result, the wedge products in [4] and [1] can be generalized to case when \( \mathcal{H}_{2p-2}(A) \) is locally finite. (See [2])

**Lemma 2.2.** Let \( T \) be a positive plurisubharmonic current of bi-dimension \((p, p)\), \( p \geq 1 \) on \( \Omega \) and \( g \in Psh^{-}(\Omega) \cap \mathcal{C}^{1}(\Omega \setminus L_{g}) \). If \( L_{g} \) is compact, then for every compact subset \( K \) of \( \Omega \) we have

\[
\left| \frac{dg \wedge d^c g}{(-g)^{1+\epsilon}} \right|_{K \setminus L_{g}} < \infty, \quad 0 < \epsilon < 1.
\]

**Proof.** Notice first that for every \( 0 < \epsilon < 1 \) the function \( -(g)^{1-\epsilon} \) is plurisubharmonic. Hence, by Lemma 2.1 the current \( -(dd^c g)^{1-\epsilon} \wedge T \) is of locally finite mass across \( L_{g} \). But, by simple computation one has

\[
-dd^c(g)^{1-\epsilon} = \frac{dg \wedge d^c g}{(-g)^{1+\epsilon}} + \frac{dd^c g}{(-g)^{\epsilon}}.
\] (2.3)

This shows the result. \[ \square \]

**Theorem 2.3.** Let \( T \) be a positive plurisubharmonic current of bi-dimension \((p, p)\), \( p \geq 1 \) on \( \Omega \) and \( g \in Psh^{-}(\Omega) \cap \mathcal{C}^{1}(\Omega \setminus L_{g}) \). If \( L_{g} \) is compact, then for all \( 0 < \epsilon < 1 \) the current \( |g|^{1-\epsilon} T \) is well defined. Moreover, if \( L_{g} = P_{g} \) and \( p \geq 2 \), then \( gT \) is well defined.

**Proof.** Take \( W, W', f, g_{j} \) as in the proof of Lemma 2.1 and for \( 0 < \epsilon < 1 \) set \( u_{j} = -(g_{j})^{1-\epsilon}, \ j \in \mathbb{N} \). Observe that \( (u_{j})_{j} \) is a sequence of negative plurisubharmonic functions where

\[
du_{j} = (1 - \epsilon) \frac{dg_{j}}{(-g_{j})^{\epsilon}}.\] (2.4)

Clearly we have

\[
\int_{W'} dd^c (-u_{j} f |z|^2) \wedge T \wedge \beta^{p-1} = \int_{W'} -u_{j} f |z|^2 dd^c T \wedge \beta^{p-1} \] (2.5)

Therefore,

\[
I_{j} := \int_{W'} -u_{j} f T \wedge \beta^{p} \leq \int_{W'} -u_{j} f |z|^2 dd^c T \wedge \beta^{p-1} + \int_{W'} f |z|^2 dd^c u_{j} \wedge T \wedge \beta^{p-1}
\] (2.6)

\[
+ 2 \left| \int_{W'} f du_{j} \wedge d^c |z|^2 \wedge T \wedge \beta^{p-1} \right| + |O'(f)|,
\]

where \( O'(f) \) consists of all terms involving \( df \), \( d^c f \) and \( dd^c f \). The first two line integrals of the right hand side of (2.6) are uniformly bounded, thanks to Lemma 2.1. Furthermore, Cauchy-Schwartz inequality shows that
Proof. In virtue of the precedent argument, the currents
\[ \int_{W'} f \delta u_j \wedge d^c|z|^2 \wedge T \wedge \beta^{p-1} \]
\[ \leq \left| \int_{W'} \frac{f}{\delta u_j} \delta u_j \wedge d^c u_j \wedge T \wedge \beta^{p-1} \right|^\frac{1}{2} \times \left| \int_{W'} (-\delta u_j) f d|z|^2 \wedge d^c|z|^2 \wedge T \wedge \beta^{p-1} \right|^\frac{1}{2} \]
\[ \leq \left| \int_{W'} \frac{f}{\delta(-g_j)^{1+\varepsilon}} \delta(-g_j)^{1+\varepsilon} d^c g_j \wedge d^c g_j \wedge T \wedge \beta^{p-1} \right|^\frac{1}{2} \times \left| \int_{W'} -f u_j T \wedge \beta^p \right|^\frac{1}{2}, \]
where \( \delta \) is a positive constant chosen so that \( \delta d|z|^2 \wedge d^c|z|^2 \leq dd^c|z|^2 \). By
complying the last two inequalities, we have
\[ I_j \leq M_j + A_j \times I_j^p. \]
And as \( M_j \) together with \( A_j \) are uniformly bounded, one can conclude the
definition of \( |g|^{1-\varepsilon} T \). Suppose now that \( p \geq 2 \) and \( L_g = P_g \). The
current \(-T \wedge dd^c(-g)^{1-\varepsilon} \) is positive and plurisubharmonic on \( \Omega \). Hence, the
precedent part guarantees the existence of the trivial extension of \(-|g|^{1-\varepsilon} T \wedge dd^c(-g)^{1-\varepsilon}, \) and obviously the current \( \frac{dg \wedge d^c g}{(-g)^{2\varepsilon}} \wedge T \) is of locally finite mass.
Notice also that
\[ \lim_{j \to \infty} \left| \int_{W'} f \delta g_j \wedge d^c|z|^2 \wedge T \wedge \beta^{p-1} \right| \]
\[ \leq \lim_{j \to \infty} \left| \int_{W'} \frac{f}{\delta(-g_j)^{1+\varepsilon}} \delta(-g_j)^{1+\varepsilon} d^c g_j \wedge d^c g_j \wedge T \wedge \beta^{p-1} \right|^\frac{1}{2} \times \left| \int_{W'} -f u_j T \wedge \beta^p \right|^\frac{1}{2} \]
\[ < \infty. \]
Therefore, by similar argument as above, one can replace \( u_j \) by \( g_j \) in (2.6)
and deduce that \( g_j T \) converges to a current denoted by \( gT \).

\textbf{Theorem 2.4.} Let \( T \) be a positive pluriharmonic current of bi-dimension
\( (p,p), \ p \geq 2 \) on \( \Omega \) and \( g \in Psh^{-}(\Omega) \cap C^2(\Omega \setminus L_g) \). If \( L_g \) is compact and
\( L_g = P_g \), then for all \( 0 < \varepsilon < 1 \) the current \( |g|^{1+\varepsilon} T \) is well defined.

\textbf{Proof.} In virtue of the precedent argument, the currents \( (-g)^{1+\varepsilon} \wedge T \) and
\( \frac{dg \wedge d^c g}{(-g)^{1+\varepsilon}} \wedge T \) are of locally finite mass across \( L_g \). On the other hand
\[ dd^c(-g)^{1+\varepsilon} \wedge T = \epsilon (1 + \varepsilon) \frac{dg \wedge d^c g}{(-g)^{1+\varepsilon}} \wedge T - (1 + \varepsilon)(-g)^{1+\varepsilon} \wedge T. \]
This implies that the trivial extension \( dd^c(-g)^{1+\varepsilon} \wedge T \) exists. Set \( v_j = (-g_j)^{1+\varepsilon}, \ j \in \mathbb{N} \). By the features of \( T \) we have
\[ \int_{W'} dd^c(v_j f |z|^2) T \wedge \beta^{p-1} = 0 \]
Now, by analogous discussion as in the previous proof, we infer the definition of \((-g)^{1+T}\) since
\[
\int_{W'} v_j f T \wedge \beta^p \leq \left| \int_{W'} f |z|^2 d\bar{c}v_j \wedge T \wedge \beta^{p-1} \right| + |\mathcal{O}'(f)| \quad (2.12)
\]
\[
+ 2 \left| \int_{W'} \frac{f}{\delta v_j} dv_j \wedge d\bar{c}v_j \wedge T \wedge \beta^{p-1} \right|^{\frac{1}{2}} \times \left| \int_{W'} v_j f T \wedge \beta^p \right|^{\frac{1}{2}}.
\]
\]

\[\square\]

Lemma 2.5. Let \(T\) be a positive plurisubharmonic current of bi-dimension \((p, p), p \geq 1\) on \(\Omega\) and \(g \in Psh^-(\Omega) \cap C^1(\Omega \setminus L_g)\). If \(L_g\) is a single point, then \(gT\) is well defined.

Proof. Without loss of generality, one can assume that \(\Omega\) is the unit ball and \(L_g\) is the origin. Take \(\chi \in C_0^\infty(B(0, \frac{1}{2}))\) so that \(\chi = 1\) on a neighborhood of \(B(0, \frac{1}{4})\). First notice that for all \(0 < t < 1\) we have
\[
\lim_{j \to \infty} \left| \int_{\{z \leq t\}} d(-\chi g_j) \wedge d^c T \wedge \beta^{p-1} \right| = - \int_{\{z \leq t\}} \chi gd^c T \wedge \beta^{p-1}. \quad (2.13)
\]

But
\[
\lim_{j \to \infty} \left| \int_{\{z \leq \frac{1}{2}\}} -\chi g_j d|z|^2 \wedge d^c T \wedge \beta^{p-1} \right|
= \lim_{j \to \infty} \left| \frac{1}{t} \int_0^t \left| \int_{\{z \leq t\}} -\chi g_j d^c T \wedge \beta^{p-1} \right| dt \right|
\]
\[
= \lim_{j \to \infty} \left| \int_0^\infty dt \int_{\{z \leq t\}} |d(-\chi g_j) \wedge d^c T - \chi g_j d^c T \wedge \beta^{p-1}| < \infty. \quad (2.14)
\]

Therefore,
\[
\sup_j \left| \int_{\{z \leq \frac{1}{2}\}} -g_j d(\chi |z|^2) \wedge d^c T \wedge \beta^{p-1} \right| < \infty. \quad (2.15)
\]

By a simple computation, one finds that
\[
\chi |z|^2 d g_j \wedge d^c T = g_j d(\chi |z|^2) \wedge d^c T + g_j \chi |z|^2 d\bar{c} T. \quad (2.16)
\]

Hence,
\[
\sup_j \left| \int_{\{z \leq \frac{1}{2}\}} -\chi |z|^2 d g_j \wedge d^c T \wedge \beta^{p-1} \right| < \infty. \quad (2.17)
\]

Now, once again Stokes’ formula shows that
\[
\int_{\{z \leq \frac{1}{2}\}} -d\bar{c}(\chi |z|^2) \wedge g_j T \wedge \beta^{p-1} = \int_{\{z \leq \frac{1}{2}\}} -\chi |z|^2 d\bar{c}(g_j T) \wedge \beta^{p-1}. \quad (2.18)
\]
But
\[
\int_{\{|z| \leq \frac{1}{2}\}} -\chi |z|^2 dd^c(g_j T) \wedge \beta^{p-1} = \int_{\{|z| \leq \frac{1}{2}\}} -\chi |z|^2 dd^c g_j \wedge T \wedge \beta^{p-1}
+ \int_{\{|z| \leq \frac{1}{2}\}} -\chi |z|^2 g_j dd^c T \wedge \beta^{p-1}
+ 2 \int_{\{|z| \leq \frac{1}{2}\}} -\chi |z|^2 dg_j \wedge d^c T \wedge \beta^{p-1}.
\]

Thus, we infer that
\[
\sup_j \left| \int_{\{|z| \leq \frac{1}{2}\}} -\chi |z|^2 dd^c(g_j T) \wedge \beta^{p-1} \right| < \infty,
\]
thanks to Lemma 2.1 and (2.17). On the other hand, the left hand side of (2.18) involves \(O(\chi)\) which consists of the terms where \(d\chi\), \(d^c\chi\) and \(dd^c\chi\) appear. Notice that \(O(\chi)\) is under control since the origin is isolated. Hence,
\[
-\int_{\{|z| \leq \frac{1}{4}\}} g_j T \wedge \beta^p \leq - \int_{\{|z| \leq \frac{1}{4}\}} \chi g_j T \wedge \beta^p
\leq |O(\chi)| + \left| \int_{\{|z| \leq \frac{1}{4}\}} -\chi |z|^2 dd^c(g_j T) \wedge \beta^{p-1} \right| < \infty.
\]

Clearly, the current \(gT\) is obtained by the monotone convergence of \(g_j T\). □

**Theorem 2.6.** Let \(T\) be a positive plurisubharmonic current of bi-dimension \((p, p)\), \(p \geq 1\) on \(\Omega\) and \(g \in Psh^-(\Omega) \cap C^1(\Omega \setminus L_g)\). If the current \(gd^cT\) is well defined and \(L_g\) is compact, then \(gT\) is a well defined current on \(\Omega\).

The result generalizes the case when \(dT = 0\). One can also implement it for the currents \(T\) where \(dT\) has \(L^q\) coefficients, \(q > 1\).

**Proof.** We keep the notation of the proof of Lemma 2.1 taking into consideration that \(L_g \subset W\). It is obvious that \(dg_j \wedge d^c T = d(g_j d^c T) - g_j dd^c T\).

Hence by Lemma 2.1 the current \(dg \wedge d^c T\) is well defined. Now, by applying Stokes’ formula we have
\[
\int_{W'} -dd^c(f|z|^2) \wedge g_j T \wedge \beta^{p-1} = \int_{W'} -f|z|^2 dd^c(g_j T) \wedge \beta^{p-1}.
\]

But
\[
\int_{W'} -f|z|^2 dd^c(g_j T) \wedge \beta^{p-1} = \int_{W'} f|z|^2 dd^c g_j \wedge T \wedge \beta^{p-1}
+ \int_{W'} f|z|^2 g_j dd^c T \wedge \beta^{p-1}
+ 2 \int_{W'} f|z|^2 dg_j \wedge d^c T \wedge \beta^{p-1}.
\]
Which means that
\[ \sup_j \left| \int_{W'} -f|z|^2dd^c(g_jT) \wedge \beta^{p-1} \right| < \infty \] (2.24)
because of Lemma 2.3. Meanwhile, the left hand side of (2.22) involves terms \( \mathcal{O}(f) \) where \( df, d'f \) and \( dd^c f \) appear. And the properties of \( f \) make these terms defined and uniformly bounded as the locus points of \( g \) are avoided. Hence,
\[
- \int_{W' } g_jT \wedge \beta^p \leq - \int_{W' } fg_jT \wedge \beta^p
\]
\[
\leq |\mathcal{O}(f)| + \left| \int_{W'} -f|z|^2dd^c(g_jT) \wedge \beta^{p-1} \right| < \infty.
\] (2.25)
This yields to the current \( gT \).

Next we give conditions on the locus points of \( g \) that allow the existence of \( gT \) regardless the compactness property.

**Theorem 2.7.** Let \( T \) be a positive plurisubharmonic current of bi-dimension \((p,p)\) on \( \Omega \) and \( g \in Psh^{-}(\Omega) \cap C^1(\Omega \setminus L_g) \). If the current \( gd^cT \) is well defined and \( \mathcal{H}_{2p-1}(L_g \cap \text{Supp} \ T) = 0 \), then the current \( gT \) is well defined.

**Proof.** Let us assume that \( 0 \in \text{Supp} \ T \cap L_g \). Since \( \mathcal{H}_{2p-1}(L_g \cap \text{Supp} \ T) = 0 \), then by [6] and [11], there exist a system of coordinates \((z',z'')\) in \( \mathbb{C}^s \times \mathbb{C}^{n-s} \), \( s = p - 1 \) and a polydisk \( \Delta^n = \Delta' \times \Delta'' \) such that \( \overline{\Delta'} \times \partial \Delta'' \cap (\text{Supp} \ T \cap L_g) = \emptyset \). Now, take \( 0 < t < 1 \) so that \( \Delta' \times \{z'' \mid |z''| < 1\} \cap (\text{Supp} \ T \cap L_g) = \emptyset \). As \( \overline{\Delta''} \cap L_g \) is compact set, one can find a neighborhood \( \omega \) of \( \overline{\Delta''} \cap L_g \) such that \( \omega \cap (\Delta' \times \{z'' \mid |z''| < 1\}) = \emptyset \). Let \( a \in (t,1) \). Notice that the slice \( \langle T,\pi,z' \rangle \) exists for a.e. \( z' \), and is a positive plurisubharmonic current of bi-dimension \((1,1)\) on \( \Omega \), supported in \( \{z'\} \times \Delta^{n-p+1} \). Hence, by Theorem 2.3, the sequence \( \{g_jT,\pi,z'\} \) is weakly* convergent since \( \pi^{-1}(z') \cap L_g \) is a compact subset of \( \{z'\} \times \Delta^{n-p+1} \). Thus by applying the slice formula we have
\[
\lim_{j \to \infty} \int_{\Delta' \times \Delta''} g_jT \wedge \pi^* \beta^{p-1} \wedge \beta'' = \lim_{j \to \infty} \int_{z'} \langle g_jT,\pi,z' \rangle \beta^{p-1} \wedge \beta''
\]
\[= \int_{z'} gT \wedge \pi^* \beta^{p-1} \wedge \beta''
\]
\[= \int_{\Delta' \times \Delta''} gT \wedge \pi^* \beta^{p-1} \wedge \beta'' \].
This completes the proof.

**Theorem 2.8.** Let \( T \) be a positive plurisubharmonic current of bi-dimension \((p,p)\) on \( \Omega \) and \( g \in Psh^{-}(\Omega) \cap C^1(\Omega \setminus L_g) \). If \( \mathcal{H}_{2p-2}(L_g \cap \text{Supp} \ T) \) is locally finite, then the current \( gT \) is well defined.

**Proof.** For each \( z' \) we set \( L_g(z') = (\text{Supp} \ T \cap L_g) \cap (\{z'\} \times \Delta'') \). Since \( \mathcal{H}_{2p-2}(L_g \cap \text{Supp} \ T) \) is locally finite, then by [11] the set \( L_g(z') \) is a discrete
subset for a.e. $z'$. Without loss of generality, we may assume that $L_g(z')$ is reduced to a single point $(z',0)$. On the other hand, $T$ is $C$-flat on $\Omega$. Thus, the slice $(T, \pi, z')$ exists for a.e. $z'$, and is a positive plurisubharmonic current of bi-dimension $(1,1)$ on $\Omega$, supported in $\{z'\} \times \triangle^{n-p+1}$. Now, by Theorem 2.5, the sequence $(g_j T, \pi, z')$ is weakly $^\ast$ convergent since $L_g(z')$ is a single point. Hence the slice formula implies that

$$\lim_{j \to \infty} \int_{\triangle' \times \triangle''} g_j T \wedge \pi^* \beta^{p-1} \wedge \beta'' = \lim_{j \to \infty} \int_{z'} (g_j T, \pi, z') \beta^{p-1} \wedge \beta''$$

$$= \int_{z'} (gT, \pi, z') \beta^{p-1} \wedge \beta''$$

(2.26)

And our desired current is achieved. $\square$

3. The Current $dd^c g \wedge T$

As mentioned earlier in the introduction, for the case under investigation the current $dd^c g \wedge T$ stole the show from the current $gT$. Actually, Alessandrini-Bassanilli [4] and Al Abdulaali [1] studied the definition of $dd^c g \wedge T$ for pluriharmonic current $T$ and $g$ of class $C^2$ apart of its locus points. Al Abdulaali [2] generalized the latter works to the more general case when $T$ is plurisubharmonic and $g$ of class $C^1$ where $H_{2p-2}(L_g)$ is locally finite. In [3], Dihn and Sibony discussed the case when $\Omega$ is a compact Kähler manifold. They obtained the desired current when $T$ is pluriharmonic and $g$ is continuous on $\Omega$.

Theorem 3.1. Under the same hypotheses of Theorem 2.6, the current $dd^c g \wedge T$ is well defined.

Proof. For any $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_{\Omega} \varphi dd^c g_j T \wedge \beta^{p-1} = \int_{\Omega} g_j dd^c \varphi \wedge T \wedge \beta^{p-1}$$

(3.1)

This means that

$$\int_{\Omega} \varphi dd^c g_j \wedge T \wedge \beta^{p-1} = \int_{\Omega} g_j dd^c \varphi \wedge T \wedge \beta^{p-1}$$

$$- \int_{\Omega} \varphi g_j dd^c T \wedge \beta^{p-1}$$

$$- 2 \int_{\Omega} \varphi dg_j \wedge d^c T \wedge \beta^{p-1}$$

(3.2)

In virtue of the previous results, the sequence $dd^c g_j \wedge T$ converges to a current denoted by $dd^c g \wedge T$. $\square$

By analogous discussion as in the proof of Theorem 2.7 one can apply the slice formula to imply the following assertion.
Theorem 3.2. Under the same hypotheses of Theorem 2.7, the current $dd^c g \wedge T$ is well defined.

Theorem 3.1 allows us to define the number

$$\mu(T, g) := \lim_{r \to -\infty} \int_{\{g < r\}} T \wedge dd^c g \wedge \beta^{p-1}.$$ 

For such number one can obtain the following comparison result.

Theorem 3.3. In addition to the hypotheses of Theorem 2.6, if $dd^c T = 0$ and $u \in Psh^-(\Omega) \cap C^1(\Omega \setminus L u)$ such that $u d c$ exists and $l := \limsup_{z \to g(z) \to -\infty} u(z) < \infty$, then

$$\mu(T, u) \leq l \mu(T, g).$$

Proof. We follow a similar technique as in [8]. Since $\lambda \mu(T, g) = \mu(T, \lambda g)$ for all $\lambda \geq 0$, it is enough to show the result for $l = 1$. Set $u_c = \max \epsilon(u - c, g)$ where $c$ is a positive constant and $\max \epsilon(x_1, x_2)$ is by definition

$$\max \epsilon(x_1, x_2) = \max(x_1, x_2) \ast \alpha_\epsilon,$$

where $\alpha_\epsilon$ is a regularization kernel on $\mathbb{R}^2$ depending only on $\| (x_1, x_2) \|$. Now take $r' < b < r < 0$. Notice that for $c$ large enough we have $u_c = g$ on $\{b < g \leq r\}$. Therefore,

$$\int_{\{g < r\}} T \wedge dd^c (u_c - g) \wedge \beta^{p-1} = \int_{\{g < r\}} (u_c - g) dd^c T \wedge \beta^{p-1} = 0. \quad (3.5)$$

But the properties of $u$ imply that $\mu(T, u_c) = \mu(T, u - c) = \mu(T, u)$. Hence,

$$\int_{\{g < r\}} T \wedge dd^c g \wedge \beta^{p-1} \geq \int_{\{g < r\}} T \wedge dd^c u_c \wedge \beta^{p-1}$$

$$\geq \int_{\{u_c < r\}} T \wedge dd^c u_c \wedge \beta^{p-1}$$

We finish the proof by letting first $r' \to -\infty$, and secondly $r \to -\infty$. □

In Theorem 3.1 if we consider $T$ to be positive plurisuperharmonic, then the statement fails to remain true. The next example illustrates this fact. Notice that, based on [2], such wedge product exists when the obstacle is assumed to be of zero $(2p - 2)$-Hausdorff measure.

Example 3.4. In $\mathbb{C}$, set $T = g = \log |z|^2$. Then $T$ is negative and plurisubharmonic on $\{|z| < 1\}$ where $gd^c T$ is well defined. But despite the fact that $L g = \{0\}$ is of locally finite 0-Hausdorff measure, the mass of $dd^c g \wedge T$ explodes across $\{0\}$. 
However, local potential currents can be very useful to our settings. Remember that, by [5], if $T$ is positive and closed, then locally there exist a negative plurisubharmonic current $U$ of bi-dimension $(p + 1, p + 1)$ and a smooth form $R$ such that $T = dd^c U + R$. The current $U$ is called the local potential of $T$.

**Corollary 3.5.** Let $T$ be a positive plurisubharmonic current of bi-dimension $(p, p)$, $p \geq 1$ on $\Omega$ and $g \in Psh^-(\Omega) \cap C^1(\Omega \setminus L_g)$. If $L_g$ is a single point, then $dd^c g \wedge S$ is a well defined current on $\Omega$ where $S$ is the potential of $dd^c T$.

**Proof.** As our problem is local, one can assume that $dd^c T = dd^c S$. Now, if we set $F = T - S$, then we get a positive pluriharmonic current. Hence by Theorem 2.5, both currents $dd^c g \wedge F$ and $dd^c g \wedge T$ are well defined. Therefore, one can define $dd^c g \wedge S$ by $dd^c g \wedge T - dd^c g \wedge F$. 

We end this paper by showing a case where the $dd^c g \wedge T$ can be defined without paying any attention to the derivatives of $g$.

**Corollary 3.6.** Let $T$ be a positive or negative plurisubharmonic current of bi-dimension $(p, p)$, $p \geq 1$ on $\Omega$ and $g \in Psh^-(\Omega) \cap L^\infty_{loc}(\Omega)$. If $dT$ is of order zero, then $dd^c g \wedge T$ is a well defined current on $\Omega$.

**Proof.** It is so obvious that the currents $gT$, $gd^c T$ and $gdd^c T$ are well defined. Therefore, one can define

$$dd^c g \wedge T = dd^c (gT) - 2dg \wedge d^c T - gdd^c T.$$  \hfill (3.7)

\[\square\]

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