Killing symmetry on the Finsler manifold

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Abstract
Symmetry and conservation law are discussed on the Finsler manifold \( M \). We adopt the point Finsler approach, where we consider the geometry on a point manifold \( M \) not on \( TM \). Generalized vector fields are defined on oriented curves on \( M \), and Finsler non-linear connections are considered on \( M \), not on the tangent space \( TM \). Killing vector fields \( K \) are defined as generalized vector fields as \( \mathcal{L}_K F = dB \), and the Killing symmetry is also reformulated simply as \( \mathcal{S} K^\gamma = 0 \) by using the Killing 1-form \( K^\gamma \) and the spray operator \( \mathcal{S} \) defined by using the non-linear connection. \( K^\gamma \) is related to the generalization of Killing tensors on the Finsler manifold, and our ansatz of \( K^\gamma \) and \( \mathcal{S} K^\gamma = 0 \) give an analytical method of finding higher derivative conserved quantities, which may be called hidden conserved quantities. We show two examples: the Carter constant on Kerr spacetime and the Runge–Lentz vectors in Newtonian gravity.

Keywords: Killing symmetry, Killing tensor, Finsler geometry

1. Introduction
Recently, great interest in Finsler geometry has been aroused. Numerous papers have appeared and it has a great variety of applications: mathematical physics, special and general relativity, cosmology, and even in ecology [1–7]. One can look up other earlier applications in [8]. Our group has also applied Finsler geometry to physics in various fields to date: (including related works) classical mechanics [9, 10], path integral [11], constrained systems [12], Lagrangian formulation of field theories and general relativity [13–15], string duality [16].

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thermodynamics [17], fluid dynamics [18] and so on. In this paper, we discuss the symmetry and conservation law on Lagrangian systems which can be described as Finsler manifolds. Our treatment is based on the point Finsler approach [19], which is slightly different from the usual treatments of Finsler geometry that are prevalent in the literature. We only treat the geometry on a point manifold $M$ and have interest in physics defined on $M$, not on its tangent bundle $TM$. Firstly, we explain the point Finsler approach from its general background.

After Riemann presented his metric

$$ (1.1) $$

he proposed, as the next simplest example, a fourth polynomial metric,

$$ (1.2) $$

This is the starting point of the Finsler metric. As in the examples of equations (1.1) and (1.2), the Finsler metric $F$ is defined as the function $F(x, dx)$ of the coordinates $x^i(p)$ of the $(n+1)$-dimensional manifold $M$ and their derivatives $(dx^i) = (dx^0, dx^1, \cdots, dx^n)$, which satisfies the homogeneity condition with respect to $dx^n$:

$$ (1.3) $$

$F$ defines the norm of a tangent vector $v = v^\mu \left( \frac{\partial}{\partial x^\mu} \right)_p \in T_pM$ at the point $p$ on $M$ in the form

$$ (1.4) $$

This $\|v\|_F$ is also dependent upon the direction of $v$ as well as its coordinates $x^\mu(p)$ of the point $p$, and this is the difference from the Riemannian metric in the form (1.1).

By differentiating equation (1.3) with $k$ and then setting $k = 1$, we obtain the Euler’s formula,

$$ (1.5) $$

which is another form of the homogeneity condition of $F$. Applying this form to the unit tangent vector $v$, we obtain

$$ (1.6) $$

which defines a hypersurface called indicatrix on $T_pM$. In Riemannian case (1.1), the indicatrix becomes

$$ (1.7) $$

which is a quadratic hypersurface on the tangent space $T_pM$. In the case of the fourth polynomial metric (1.2), equation (1.6) gives

$$ (1.8) $$

which represents a quartic hypersurface in $T_pM$. Compared with the Riemannian metric (1.7), we see the metric in (1.8) depends on the direction of $v$.

Next we explain our notation $dx^\mu$. We regard the derivative symbols $dx^\mu$ as the fiber coordinates of tangent bundle $TM$. The Finsler metric $F(x^\mu, dx^\mu)$ can be regarded as a function on $TM$ from this viewpoint. We can also define the Finsler metric $F$ of Finsler manifold $(M, F)$ as a smooth function on a subbundle $D(F)$ of $TM$: 

\[ F(x, dx) = \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu}, \]  

\[ (1.1) \]

\[ F(x, dx) = \sqrt{g_{\mu\nu\rho\sigma}(x)dx^\mu dx^\nu dx^\rho dx^\sigma}. \]  

\[ (1.2) \]

\[ F(x^\mu, kdx^\mu) = kF(x^\mu, dx^\mu), \quad k > 0. \]  

\[ (1.3) \]
In this paper we will apply the Finsler structure to finite dimensional Lagrangian systems, where $F$ is not generally well-defined on the whole region of slit bundle $TM - \{0\}$. Then, we are forced to define a domain $D(F)$ on which $F$ and its derivatives are well-defined. It is usual to suppose two conditions [20–22] that the fundamental metric $(\mu_{\nu} = \frac{\partial^2 F}{\partial y^\mu \partial y^\nu})$ is (1) nondegenerate and (2) positive definite. However, they are too restricted for physical applications. We do not require both conditions in general, while we suppose only the homogeneity condition (1.3) or (1.5).

In this paper we adopt the point Finsler approach [19], where we regard Finsler geometry not as geometry of $TM$ such as line element space, but as one of the point manifold $M$. In the point Finsler approach, the geometry of indicatrix is the main topic of our research. The indicatrix is a distorted hypersurface on a tangent space $T_pM$ of each point $p$, on which vectors $v$ have the same length $\varepsilon$ from the origin of $T_pM$.

\[ F(x(p), dx(v)) = \varepsilon. \]  

The indicatrix is a scalar or a distorted compass on $T_pM$ which gives lengths of oriented curves in $M$. From the indicatrix, we can also define area and angle spanned by two vectors [23], and they are applied directly to physics [11]. Furthermore, the Finsler non-linear connection which preserves $F$ and the shape of the indicatrix can be naturally defined based on this picture [12], as we will explain in the next section.

The purpose of this paper is an application of Finsler geometry to finite dimensional Lagrangian systems and their symmetry discussions, and we consider the point Finsler approach is the most suitable for it, since the Lagrangian formulation is based on second-order differential equations. On the other hand, when we regard Finsler geometry as the geometry on $TM$ such as line element space or sphere bundle, we think it should be handled in the Hamiltonian formalism. As is well known, the projective tangent bundle $PTM$ has a contact form [24] \[ \theta := \frac{\partial F}{\partial y^\mu} (x, y) dx^\mu, \quad \theta \wedge d\theta^\mu = 0, \] which is naturally deduced from $(M, F)$ and we can develop the covariant Hamiltonian formalism [25]. We will discuss the Lagrangian formulation and use the point Finsler approach on a point manifold throughout this paper.

Next, we give comments on our notation: We avoid using $y^\mu$ or $\dot{x}^\mu$ but adopt $dx^\mu$ as the fiber coordinates, since we do not treat geometries of $TM$. Especially, $dx^\mu$ is a representation for a fixed parametrization and is not suitable for our covariant formulation.

Our symbol $dx^\mu$ has two meanings: It is the fiber coordinate of $TM$ as we already explained. At the same time, it can be regarded as a 1-form on the point manifold $M$ since exterior derivatives do not appear in our scheme before pulling various formulas back to the parameter space $T \subset \mathbb{R}$. That is, 2-forms and 3-forms do not appear in almost all stages of our calculations. Instead, we use the notation $d^2 x^\mu$ to represent the second rank derivatives, which are defined, from a parametrization of an oriented curve $c : \tau \in T \subset \mathbb{R} \mapsto c(\tau) \in M$ as

\[ c^* d^2 x^\mu := d(c^* dx^\mu) = d \left( \frac{dx^\mu(\tau)}{d\tau} \right) = \frac{d^2 x^\mu(\tau)}{d\tau^2} (d\tau)^2. \]  

Or, using vector field $v = y^\mu(x) \frac{\partial}{\partial x^\mu}$ on the point manifold $M$, $d^2 x^\mu$ are defined by the formula...
\[ d^2x^\mu(v) := dv^\mu(v) = v^\nu \frac{\partial v^\mu}{\partial x^\nu}. \] (1.13)

which represents the Euler derivative. The \( d^2x^\mu \) notation matches well with the second or higher order differential equations of various Lagrangian systems which are familiar to physicists. The symbol \( d \) corresponds to the total derivative in Olver’s textbook [26].

Symmetries of differential equations and Lagrangian systems were discussed in [26] where he developed his theory in jet bundle formulation. In this paper we will give a formulation of metrical geometry treating the same topic based on Finsler geometry. The conservation law will be derived from a general background, Killing symmetry, which is the aim of the present paper and will be discussed in chapter 3. The Killing vector field on the Finsler manifold is defined [27]. Furthermore, the Killing symmetry will be given in another form. We will define the Killing 1-form \( K \) which can directly generalize the concept of ‘Killing tensor’ to Finsler manifold in chapter 4. The conserved current is given directly from \( K \). We will treat two examples: the Carter constant on Kerr spacetime and the Runge–Lentz vector in classical mechanics. Before that, in the next chapter, we will review the Finsler–Lagrangian formulation. The Euler–Lagrange equation is formulated in covariant, that is, reparametrization invariant form. They are rewritten in a geometrical form by using Finsler non-linear connection, which will be also reviewed briefly.

2. Review of Finsler–Langrangian formulation

2.1. Finsler–Lagrangian formulation

Finite dimensional Lagrangian system is described by an \( n \)-dimensional configuration space \( Q \) and a Lagrangian \( L(t, q^i, \dot{q}^i) \) \( (i = 1, 2, \cdots, n) \). Then, on the \( (n + 1) \)-dimensional extended configuration space \( M := \mathbb{R} \times Q \) including time \( \tau^0 := t \), Finsler metric \( F \) can be defined by

\[ F(x^\mu, dx^\mu) := L\left(x^0, x^i, \frac{dx^i}{d\tau^0}\right) d\tau^0. \] (2.1)

We use Greek indices such as \( \mu, \nu, \rho, \cdots \) take values \( (0, 1, 2, \cdots, n) \), while Latin indices \( a, b, \cdots, i, j, \cdots \) take \( (1, 2, \cdots, n) \) throughout this paper. \((M, F)\) can be regarded as a Finsler manifold. Time evolutions of the system are represented by oriented curves \( c \subset M \) on the extended configuration space \( M \). The greatest advantage of using Finsler manifold is that the time evolution of the system is described independently of choices of time parameters, that is, it is independent of various parametrizations of the oriented curve \( c \).

\( F \) gives a geometric action of the Lagrangian system defined by the Finsler length of an oriented curve \( c \) as

\[ A[c] = \int_c F(x, dx) = \int_{\tau_0}^{\tau_1} F\left(x^\mu(\tau), \frac{dx^\mu(\tau)}{d\tau}\right) d\tau, \] (2.2)

where \( \tau \) is an arbitrary time parameter of \( c : [\tau_0, \tau_1] \subset \mathbb{R} \to M \). The integrand is the pull-back of \( F \) by a parametrization \( c \),

\[ c^*F := F(c^*x^\mu, c^*dx^\mu) = F\left(x^\mu(\tau), \frac{dx^\mu(\tau)}{d\tau}\right) d\tau, \] (2.3)
which is a usual 1-form on the parameter space. Because of the homogeneity condition (1.3),
the action integral $\mathcal{A}[e]$ is defined geometrically as Finsler length of $e$, independently of choices
of parametrizations.

We defined a variational method of Finsler–Lagrangian formulation in our previous work
[15]. This method should be extended and more general variations are necessary for the applications
in the next chapter.

Covariant Euler–Lagrange equation is derived from stationary condition of $\mathcal{A}$ under variations
of oriented curve $c$. $\varphi$ is a differentiable map $\varphi : [-\varepsilon_0, \varepsilon_1] \times c \to M$ with both ends
of $c$, $p_0$ and $p_1$ being fixed. Denoting $\varphi(c, \tau) : c \to M$ as $\varphi_\tau$ and setting $\varphi_\tau(0) = id$, it follows
$\varphi_\tau(p_0) = p_0$, $\varphi_\tau(p_1) = p_1$. This map $\varphi$ is generated by a generalized vector field.

**Definition 2.1.** A vector field $X = X^a \frac{\partial}{\partial x^a}$ on Finsler manifold $M$, of which coefficients
$X^a = X^a(x, dx)$ are homogeneity zero with respect to $dx$, $X^a(x, kdx) = X^a(x, dx)$, $k > 0$, is
called a generalized vector field.

We should emphasize that generalized vector fields are not on tangent bundle $TM$, but ones
only on oriented curves on manifold $M$. Thus, $X$ does not include $\frac{\partial}{\partial \varepsilon}$ or $\frac{\partial}{\partial \varepsilon\varepsilon}$ components.
From a generalized vector field $X$, we can define a vector field along an oriented curve $c$
as
$X(p) := X^a(x(p), dx(v)) \left( \frac{\partial}{\partial x^a} \right)_p, \quad p \in c, \quad v = \frac{dc}{d\tau} \in T_pM. \quad (2.4)$

The homogeneity zero condition for $X^a$ is assumed so that the vector field $\dot{X}$ does not depend on its parameter choices. If we take a parametrization $c : [\tau_0, \tau_1] \to c \subset M$, the vector field $\dot{X} := \pi \circ c : [\tau_0, \tau_1] \to TM$ is also defined by

$$\dot{X}(\tau) := \pi(c(\tau)) = X^a (x(c(\tau)), dx(\frac{dc}{d\tau})) \left( \frac{\partial}{\partial x^a} \right)_{c(\tau)}, \quad (2.5)$$

which satisfies $\pi \circ (\dot{X} \circ c) = \dot{c}$ for the projection $\pi : TM \to M$. Thus, it is nothing but a vector
field on the map $c$ in accordance with the textbook by O’Neill [28]. Furthermore, if we take time parameter $\tau = t$,
we have $X = X^a(x, dx) \frac{\partial}{\partial x^a} = X_0(t, x^i, \dot{x}^j) \frac{\partial}{\partial t} + X_1(t, x^i, \dot{x}^j) \frac{\partial}{\partial \dot{x}^a}$,
which is the same form given in [26] and is also called the generalized vector field. Though
we can assume that $X^a$ depends on arbitrary higher order derivatives $d^{(a)} x, (a = 2, 3, \cdots)$,
only first order dependence is enough for our discussion. This is because we consider the
generalized vector field $X$ only on a solution curve $c$ of the Euler–Lagrange equation, which
is an ordinary differential equation of at most second order derived from the Finsler metric
$F(x, dx)$ of first order.

For small $\varepsilon$, an infinitesimal deformation map $\varphi_\varepsilon$ of an oriented curve $c$ is defined from this
$\dot{X}$ as

$$\varphi_\varepsilon = \text{Exp} (\varepsilon \cdot \dot{X}). \quad (2.6)$$

Then, the variational principle is represented by

$$0 = \delta \mathcal{A}[e] := \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \mathcal{A}[\varphi_\varepsilon(e)] = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{c(e)} F = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{c} \varphi_\varepsilon^* F. \quad (2.7)$$

Taking a parametrization of $e, c : [\tau_0, \tau_1] \to M$, the RHS is calculated as
\[
\frac{d}{de} \bigg|_{e=0} \int_{c} \varphi^{\tau} F = \frac{d}{de} \bigg|_{e=0} \int_{\gamma_{0}}^{\gamma_{1}} c^{\tau} \varphi^{\tau} F = \int_{\gamma_{0}}^{\gamma_{1}} F(x^{\mu}(\varphi_{e}(c(\tau))), dx^{\mu}(\varphi_{e}(c(\tau))))
\]

\[
= \int_{\gamma_{0}}^{\gamma_{1}} \left[ \delta x^{\mu} c^{\mu} \left( \frac{\partial F}{\partial x^{\mu}} \right) + d \delta x^{\mu} c^{\mu} \left( \frac{\partial F}{\partial dx^{\mu}} \right) \right]
\]

\[
= \left[ \delta x^{\mu} c^{\mu} \left( \frac{\partial F}{\partial x^{\mu}} \right) \right]_{\gamma_{0}}^{\gamma_{1}} + \int_{\gamma_{0}}^{\gamma_{1}} \delta x^{\mu} \left[ c^{\mu} \left( \frac{\partial F}{\partial x^{\mu}} \right) - d \left\{ c^{\mu} \left( \frac{\partial F}{\partial dx^{\mu}} \right) \right\} \right],
\]  
\text{(2.8)}

where \( \delta x^{\mu} \) is defined by

\[
\delta x^{\mu} := \frac{d}{de} \bigg|_{e=0} x^{\mu}(\varphi_{e}(c(\tau))) = c^{\tau} \mathcal{L}_{c} x^{\mu} = c^{\tau} X^{\mu},
\]  
\text{(2.9)}

\[
\frac{d}{de} \bigg|_{e=0} dx^{\mu}(\varphi_{e}(c(\tau))) \delta x^{\mu} = d \delta x^{\mu} = d(c^{\tau} X^{\mu}).
\]  
\text{(10.10)}

Since \( \delta x^{\mu}(\gamma_{0}) = \delta x^{\mu}(\gamma_{1}) = 0, \)

\[
0 = \delta A[e] = \int_{\gamma_{0}}^{\gamma_{1}} \delta x^{\mu} \left[ c^{\mu} \left( \frac{\partial F}{\partial x^{\mu}} \right) - d \left\{ c^{\mu} \left( \frac{\partial F}{\partial dx^{\mu}} \right) \right\} \right].
\]  
\text{(11.11)}

Thus, the variational principle derives the equation

\[
0 = c^{\mu} \left( \frac{\partial F}{\partial x^{\mu}} - d \left\{ \frac{\partial F}{\partial dx^{\mu}} \right\} \right), \quad (\mu = 0, 1, 2, \ldots, n),
\]  
\text{(2.12)}

as a critical condition of \( A[e] \) under variations of an oriented curve \( e \). We call equation (2.12) the covariant Euler–Lagrange equation, which is reparametrization invariant and defines a solution curve \( e \).

**Definition 2.2.** Covariant total derivative operator \( d \) is given by

\[
d : x^{\mu} \mapsto dx^{\mu}, \quad d : dx^{\mu} \mapsto d^{2}x^{\mu}, \quad (d : d^{k}x^{\mu} \mapsto d^{k+1}x^{\mu}, \quad k = 0, 1, 2, \ldots),
\]  
\text{(2.13)}

which are defined as the reparametrization invariant extension of the total derivative operator \( D : q^{i} \mapsto dq^{i}, \quad D : q^{i} \mapsto dq^{i}, \ldots \) in jet bundle formulation [26], where parameters of time and configuration space are fixed as \((x^{\mu}) = (t, q^{i})\). The pull-back of \( d^{k}x^{\mu} \) by a parametrization of an oriented curve \( e \) is given in equation (1.12). The pull-back of \( d^{k}x^{\mu} \) is defined in a similar manner.

For zeroth homogeneity function with respect to \( dx, f = f(x, dx), \)

\[
f(x, kdx) = f(x, dx), \quad (k > 0) \quad \Rightarrow \quad \frac{\partial f}{\partial dx^{\mu}} dx^{\mu} = 0,
\]  
\text{(2.14)}

\( df \) is defined by

\[
df = \frac{\partial f}{\partial x^{\mu}} dx^{\mu} + \frac{\partial f}{\partial dx^{\mu}} d^{2}x^{\mu}.
\]  
\text{(2.15)}
Our $d$ is not the exterior derivative, but a covariant extension of ‘total derivative’ $D$ in Olver [26] as stated above. Our $d'$ given above corresponds to $Df = f(q, 1, \dot{q})$ given above corresponds to $\tilde{D}(q, 1, \dot{q})$. The homogeneity transformation $dx \rightarrow kdx$ is extended to $(dx, d^2x)$ as

$$dx^\mu \rightarrow kdx^\mu, \quad d^2x^\mu \rightarrow k^2d^2x^\mu + ldx^\mu, \quad k > 0, \quad l \in \mathbb{R}, \quad (2.16)$$

with arbitrary $\ell$ and arbitrary positive $k$. Then, $df = df(x, dx, d^2x)$ in equation (2.15) satisfies the homogeneity relation of first degree under this homogeneity transformation (2.16):

$$\frac{df}{dx^\mu}(x, kdx)kdx^\mu + \frac{df}{d^2x^\mu}(x, kdx)(k^2d^2x^\mu + ldx^\mu)$$

$$= \frac{df}{dx^\mu}(x, dx)kdx^\mu + \frac{1}{k} \frac{df}{d^2x^\mu}(x, dx)k^2d^2x^\mu + \frac{l}{k} \frac{df}{d^2x^\mu}(x, dx)dx^\mu$$

$$= k \left\{ \frac{df}{dx^\mu}(x, dx)dx^\mu + \frac{df}{d^2x^\mu}(x, dx)d^2x^\mu \right\}$$

$$= kdf(x, dx, d^2x), \quad (2.17)$$

where $\ell$ term vanishes because of the zeroth homogeneity of $f$.

The covariant Euler–Lagrange equation (2.12) can be represented by using this $d$ operator. We define the Euler–Lagrange derivative as

$$\mathcal{E}_{\mu}(F) = \frac{\partial F}{\partial x^\alpha} - d \left( \frac{\partial F}{\partial x^\mu} \right) - \frac{\partial^2 F}{\partial x^\alpha \partial x^\mu} dx^\nu - \frac{\partial F}{\partial x^\nu \partial x^\mu} d^2x^\mu. \quad (2.18)$$

Finsler metric $F = F(x, dx)$ is a function of $(x, dx)$ and it is homogeneity one with respect to $dx$, and thus, $\frac{df}{dx^\mu}(x, dx)$ is the zeroth homogeneity function. From equation (2.17), $\mathcal{E}_{\mu}(F)$ are homogeneity functions of first degree under the transformation (2.16). This means that the Euler–Lagrange equation $0 = c^* \mathcal{E}_{\mu}(F)$ is covariant, since

$$c^* \mathcal{E}_{\mu}(F) = 0 \quad \Leftrightarrow \quad c^* \mathcal{E}_{\mu}(F) = 0, \quad (2.19)$$

where $c$ and $\tilde{c}$ are two different parametrizations of the same solution curve $e$.

2.2. Finsler non-linear connection

Based on the point Finsler approach, we consider a connection in parallel translation between two points on $M$, not on $TM$. This corresponds to Berwald’s non-linear connection. We reconsider the Berwald’s non-linear connection in terms of point Finsler picture [12].

**Definition 2.3.** Finsler non-linear connection $\nabla$ is defined as a connection on point manifold $M$, satisfying following three conditions:

$$\nabla dx^\mu = -dx^\alpha \otimes N^\mu_{\alpha}(x, dx), \quad (2.20)$$

$$\frac{\partial N^\mu_{\alpha}}{\partial x^\beta} - \frac{\partial N^\mu_{\beta}}{\partial x^\alpha} = 0, \quad (2.21)$$

$$\frac{\partial F}{\partial x^\alpha} - \frac{\partial F}{\partial x^\mu} N^\mu_{\alpha} = 0, \quad (2.22)$$

where the coefficients $N^\mu_{\alpha}(x, dx)$ of $\nabla$ are functions of $(x, dx)$, which are the homogeneity one.
with respect to $dx^\mu$ of degree 1,

$$N^\mu_\alpha(x, k dx) = k N^\mu_\alpha(x, dx), \quad k > 0.$$  

(2.23)

Then, we can prove the following proposition [12].

**Proposition 2.1.** In the case of

$$\text{rank}\left(\frac{\partial F}{\partial dx^\mu \partial dx^\nu}\right) = n \quad \Leftrightarrow \quad \text{det}(g_{\mu\nu}(x, dx)) \neq 0, \quad g_{\mu\nu}(x, dx) := \frac{1}{2} \frac{\partial^2 F^2}{\partial dx^\mu \partial dx^\nu},$$

(2.24)

the three conditions in definition 2.3 uniquely determine $N^\mu_\alpha(x, dx)$ as

$$N^\mu_\alpha = \frac{\partial G^\mu}{\partial dx^\alpha}, \quad G^\mu = \frac{1}{2} \left( dx^\beta \frac{\partial F}{\partial x^\beta} + F^{ab} \varepsilon^\mu_{ab} \left( \frac{\partial F}{\partial x^b} + dx^\alpha \frac{\partial^2 F}{\partial dx^b \partial x^\alpha} \right) \right),$$

(2.25)

where

$$\ell^\mu_a := F \frac{\partial}{\partial dx^a} \left( \frac{dx^\mu}{F} \right) = \delta^\mu_a - p_a dx^\mu, \quad p_a := \frac{\partial F}{\partial dx^a}, \quad F_{\mu\nu} := \frac{\partial^2 F}{\partial dx^\mu \partial dx^\nu}.$$ 

(2.26)

$$\text{det}(F_{\mu\nu}) = 0, \quad F^{ab} \delta^\mu_a (\mu, \nu = 0, 1, 2, \ldots, n), \quad (a, b, c = 1, 2, 3, \ldots, n).$$

(2.27)

Our formulation is covariant, that is, reparametrization invariant. The relations $\text{det}(F_{\mu\nu}) = 0$ and $\text{rank}(F_{\mu\nu}) = n$ show that there is a gauge freedom of reparametrization invariance. The second equation corresponds to a system with gauge freedoms in the usual sense of physics.

**Proposition 2.2.** By using the quantity $G^\mu(x, dx)$ defined in Proposition 2.1, the covariant Euler–Lagrange equation (2.12) is rewritten in the form

$$0 = c^2 \left\{ \frac{\partial F}{\partial x^\mu} - \frac{1}{2} \left( \frac{\partial F}{\partial dx^\mu} \right) \right\} \quad \Leftrightarrow \quad 0 = c^2 \left[ d^2 x^\mu + 2G^\mu(x, dx) - \lambda dx^\mu \right],$$

(2.28)

where the second equation is an auto-parallel equation. The Lagrange multiplier $\lambda = \lambda(x, dx, d^2x)$ is an arbitrary function of $(x, dx, d^2x)$ and satisfies the homogeneity relation of degree 1 under a special transformation of (2.16) with $l = 0, \lambda(x, k dx, k^2 d^2x) = k\lambda(x, dx, d^2x)$.

**Proof.** Differentiation of (2.22) with respect to $dx^\beta$ gives

$$0 = \frac{\partial^2 F}{\partial dx^\beta \partial dx^\alpha} - \frac{\partial^2 F}{\partial dx^\alpha \partial dx^\beta} N^\mu_\alpha - \frac{\partial F}{\partial dx^\mu} \frac{\partial N^\mu_\alpha}{\partial dx^\beta}.$$ 

(2.29)

By multiplying $dx^\alpha$, contracting $\alpha$ indices, and taking the relation (2.21) into account, we obtain

$$0 = \frac{\partial^2 F}{\partial dx^\beta \partial dx^\alpha} dx^\alpha - \frac{\partial^2 F}{\partial dx^\beta \partial dx^\alpha} N^\mu_\alpha dx^\alpha - \frac{\partial F}{\partial dx^\mu} \frac{\partial N^\mu_\alpha}{\partial dx^\alpha} dx^\alpha,$$

(2.30)

$$= \frac{\partial^2 F}{\partial dx^\beta \partial dx^\alpha} dx^\alpha - \frac{\partial^2 F}{\partial dx^\beta \partial dx^\alpha} N^\mu_\alpha dx^\alpha - \frac{\partial F}{\partial dx^\mu} \frac{\partial N^\mu_\alpha}{\partial dx^\alpha} dx^\alpha.$$ 

(2.31)
where $2G^\mu = N^\mu \alpha dx^\alpha$ and we have used the first-order homogeneity relation $\frac{\partial N^\mu}{\partial x^\alpha} dx^\alpha = N^\mu \beta$ which is obtained from (2.23). Using (2.22) again, we have an identity

$$0 = \frac{\partial^2 F}{\partial x^\mu \partial x^\nu} dx^\mu - \frac{\partial^2 F}{\partial x^\mu \partial x^\alpha} 2G^\mu - \frac{\partial F}{\partial x^\beta}.$$  

From this, we can rewrite the covariant Euler–Lagrange equation into the form

$$0 = c_s \left\{ \frac{\partial F}{\partial x^\alpha} - d \left( \frac{\partial F}{\partial x^\mu} \right) \right\} = c_s \left\{ \frac{\partial F}{\partial x^\alpha} - \frac{\partial^2 F}{\partial x^\mu \partial x^\alpha} dx^\mu - \frac{\partial^2 F}{\partial x^\mu \partial x^\alpha} d^2 x^\mu \right\}$$

$$=-c_s \left\{ \frac{\partial^2 F}{\partial x^\beta \partial x^\alpha} \left( d^2 x^\mu + 2G^\mu \right) \right\}.  

(2.34)$$

Since $\left( \frac{\partial F}{\partial x^\mu} \right) = (F_{\mu\nu})$ has only one eigenvector $(dx^\mu)$ with zero eigenvalue, the above relation is equivalent to

$$0 = c_s \left\{ d^2 x^\mu + 2G^\mu(x, dx) - \lambda dx^\mu \right\},  

(2.35)$$

with an arbitrary homogeneity function $\lambda = \lambda(x, dx, d^2 x)$ of degree 1.

In (2.28) the $\lambda$ function is the gauge freedom of reparametrization invariance.

2.3. Examples

We show two examples: 1. free particle on a Lorentzian manifold and 2. potential system on Euclidean space, which will be discussed further in later sections.

2.3.1. Free particle on a Lorentzian manifold. We consider free motions of a particle with mass $m$ in $(n+1)$-dimensional Lorentzian spacetime manifold $(M, g)$. The Finsler–Lagrangian $F$ with the coordinates $(x^0, x^1, x^2, \ldots, x^n)$ describing this particle is given by

$$F(x, dx) = \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu}, \quad g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu,  

(2.36)$$

where $F(x, dx)(v) = F(x(p), dx(v))$ and $dx^\mu$ in $F$ may be regarded as coordinates of $TM$. For simplicity, we set $mc = 1$. Then, the extended configuration space is regarded as $(n+1)$-dimensional Finsler manifold $(M, F)$. The covariant Euler–Lagrange equation is given by

$$0 = \frac{\partial F}{\partial x^\mu} - d \left( \frac{\partial F}{\partial x^\nu} \right) = \frac{1}{2F} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\alpha dx^\alpha - d \left( \frac{F_{\mu\nu}}{2F} dx^\nu \right)$$

$$= \frac{1}{2F} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\alpha + \frac{1}{2F} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\alpha dx^\gamma g_{\nu\alpha} dx^\alpha + \frac{1}{F} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\alpha dx^\nu dx^\alpha - \frac{1}{1} \frac{\partial g_{\mu\nu}}{F} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\alpha dx^\nu - \frac{1}{F} \frac{\partial g_{\mu\nu}}{F} dx^\nu$$

$$= \frac{1}{2F} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\alpha \Gamma_{\mu\nu} dx^\alpha dx^\nu - \frac{1}{F} \left( \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\alpha dx^\nu + \frac{1}{2F} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} dx^\alpha dx^\mu dx^\nu \right)$$

$$= -\frac{g_{\mu\nu}}{F} \left( d^2 x^\nu + \Gamma_{\mu\nu} dx^\alpha dx^\alpha - \lambda dx^\nu \right).  

(2.37)$$
where $\lambda(x, dx, d^2 x)$ is defined by

$$\Gamma^\sigma_{\alpha \beta} = \frac{1}{2} g^{\sigma \gamma} \left( \frac{\partial g_{\beta \gamma}}{\partial x^\alpha} + \frac{\partial g_{\alpha \gamma}}{\partial x^\beta} - \frac{\partial g_{\alpha \beta}}{\partial x^\gamma} \right),$$

$$\lambda = \frac{g_{\nu \beta}}{F^2} dx^\nu d^2 x^\nu + \frac{1}{2F^2} \frac{\partial g_{\nu \beta}}{\partial x^\gamma} dx^\alpha dx^\beta dx^\gamma. \tag{2.39}$$

The pull-back of the above equation by a parametrization $c : [\tau_0, \tau_1] \subset \mathbb{R} \rightarrow c \subset M$ is given by

$$0 = c^* \left\{ \frac{\partial F}{\partial x^\nu} - \frac{\partial F}{\partial x^\mu} \right\} = -\frac{g_{\alpha \beta}}{F^2} \left( \frac{d^2 x^\sigma}{dT^2} + \Gamma^\sigma_{\alpha \beta} \frac{dx^\alpha}{dT} \frac{dx^\beta}{dT} - \lambda^* \frac{dx^\sigma}{dT} \right), \tag{2.40}$$

where

$$F^* = c^* F, \quad \lambda^* = c^* \lambda. \tag{2.41}$$

We consider the special homogeneity transformation

$$dx^\nu \rightarrow k dx^\nu, \quad d^2 x^\nu \rightarrow k^2 d^2 x^\nu + l dx^\nu, \quad (k > 0, \ l \in \mathbb{R}), \tag{2.42}$$

with any real constant $l$ and real positive constant $k$. It gives the transformation

$$F \rightarrow k F, \tag{2.43}$$

$$d^2 x^\nu + \Gamma^\nu_{\alpha \beta} dx^\alpha dx^\beta \rightarrow k^2 (d^2 x^\nu + \Gamma^\nu_{\alpha \beta} dx^\alpha dx^\beta) + l dx^\nu, \tag{2.44}$$

$$\lambda dx^\nu \rightarrow k^2 (\lambda dx^\nu) + l dx^\nu. \tag{2.45}$$

Thus, equation (2.37) is invariant under this transformation (2.42). This invariance shows the Euler–Lagrange equation (2.40) is a covariant (reparametrization invariant) equation. That is, it is independent of parameter choices.

The Finsler connection (2.25) is calculated as

$$2G^\mu = \left( \frac{dx^\nu}{dT} \frac{\partial F}{\partial x^\nu} \right) F^{-1} + F^{ab} \left( \frac{\partial G^a}{\partial x^\nu} + \frac{\partial G^b}{\partial x^\mu} \frac{\partial F}{\partial x^\nu} \right) d^2 x^\mu d^2 x^\nu,$$

$$= \frac{1}{2F} \left( \frac{\partial g_{\nu \beta}}{\partial x^\sigma} dx^\nu dx^\beta dx^\sigma \right) F^{-1} = \frac{1}{2F} \left( g^{\mu \beta} - g^a_{\mu \beta} \frac{dx^a}{dx^\sigma} \frac{dx^a}{dx^\beta} \right) d^2 x^\mu d^2 x^\nu,$$

where we have used

$$F_{ab} = \frac{1}{F} \left( g_{ab} - \frac{dx_a dx_b}{F^2} \right), \quad F^{ab} = F \left( g^{ab} - g^{a \gamma} \frac{dx^a}{dx^\gamma} \frac{dx^b}{dx^\gamma} + g^{a \gamma} \frac{dx^a}{dx^\gamma} \frac{dx^b}{dx^\gamma} \right). \tag{2.47}$$

$$\ell^\nu_a = \delta^\nu_a - \frac{P_a dx^\nu}{F}, \quad \ell^\nu_b = \frac{g^{ab} dx^\nu}{F}, \quad F^{ab} \ell^\nu_a = F \left( g^{ab} - g^a_{\mu \nu} \frac{dx^a}{dx^\mu} \right). \tag{2.48}$$

$2G^\mu$ is related to the ordinary Christoffel symbol as $2G^\mu = \Gamma^\mu_{\alpha \beta} dx^\alpha dx^\beta$. Equation (2.28) is equivalent to
\[ d^2x^\mu + \Gamma_{\alpha\beta}^\mu dx^\alpha dx^\beta - \lambda dx^\mu = 0, \tag{2.49} \]

where \( \lambda = \lambda(x, dx, d^2x) \) is supposed to be an arbitrary homogeneity function of first degree, satisfying \( \lambda(x, kdx, k^2d^2x) = k\lambda(x, dx, d^2x), \ k > 0 \). The pull-back of the above equation by a parametrization \( \epsilon \) gives

\[ 0 = \epsilon^*\left(d^2x^\mu + \Gamma_{\alpha\beta}^\mu dx^\alpha dx^\beta - \lambda dx^\mu \right) = \left\{ \frac{d^2x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} - \lambda \left( x, \frac{dx}{d\tau}, \frac{d^2x}{d\tau^2} \right) \frac{dx^\mu}{d\tau} \right\}d\tau^2, \tag{2.50} \]

which determines the solution curve \( \epsilon \) for the relevant point particle. By using the covariant differential operator \( d \) defined in (2.13), we can give a direct mathematical meaning for equation (2.49) itself, not through the process of the pullback. The homogeneity transformation (2.16) transforms equation (2.49) into

\[ k^2\{d^2x^\mu + \Gamma_{\alpha\beta}^\mu dx^\alpha dx^\beta - \lambda dx^\mu \} = 0, \ \ \ \ \ \ \ \ \ \ \lambda' := \lambda \left( x, dx, d^2x + \frac{l}{k^2} dx \right) - \frac{l}{k^2}. \tag{2.51} \]

However, the arbitrariness of \( \lambda \) function means both equations (2.49) and (2.51) derive geometrically the same oriented curve \( c \) as a solution without its parametrization. Actually, the reparametrization \( \tau \to \tau' = f(\tau) \Leftrightarrow \tau = g(\tau') \) of \( \epsilon \) derives

\[ x^\mu(c(\tau)) = x^\mu(c(g(\tau')))) = x^\mu(c'(\tau')), \ \ \ \ \ \ c(g(\tau')) \equiv c'(\tau'), \tag{2.52} \]

\[ \frac{dx^\mu(c(\tau))}{d\tau} = \frac{dx^\mu(c'(\tau'))}{d\tau'} \frac{df(\tau)}{d\tau}, \tag{2.53} \]

\[ \frac{d^2x^\mu(c(\tau))}{d\tau^2} = \frac{d^2x^\mu(c'(\tau'))}{d\tau'^2} \left( \frac{df(\tau)}{d\tau} \right)^2 + \frac{dx^\mu(c'(\tau'))}{d\tau'} \frac{d^2f(\tau)}{d\tau^2}, \tag{2.54} \]

which is represented by the transformation (2.16) with \( k = \frac{df}{d\tau} \) and \( l = \frac{d^2f}{d\tau^2} \).

We have a freedom to choose time parameter \( \tau \) which may be any function on \( M \) or first-order homogeneity function such as \( \tau = F(x, dx) \). Dividing equation (2.50) formally by \( d\tau^2 \), a classical representation of auto-parallel equation is reproduced:

\[ 0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} - \lambda \left( x, \frac{dx}{d\tau}, \frac{d^2x}{d\tau^2} \right) \frac{dx^\mu}{d\tau}, \tag{2.55} \]

which is the equation of geodesic (with \( \lambda \) taken as 0) appearing in all textbooks in physics. Our \( d^2x^\mu \) notation, which may look strange for some mathematicians, is familiar to physicists.

**2.3.2. Potential system on Euclidean space.** The Finsler–Lagrangian of the potential system in three dimensional Euclidean space is given by

\[ F(x, dx) = F(x^0, x^i, dx^0, dx^i) = L \left( x^i, \frac{dx^i}{dx^0} \right) dx^0 \]

\[ = \frac{m}{2} \left( \frac{dx^1}{dx^0} \right)^2 + \left( \frac{dx^2}{dx^0} \right)^2 + \left( \frac{dx^3}{dx^0} \right)^2 - V(x) dx^0, \tag{2.56} \]
which gives
\[
0 = \frac{\partial F}{\partial x^0} - d\left( \frac{\partial F}{\partial x^i} \right) = \begin{cases} 
-d^i \left\{ \frac{m}{2} (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right\} + V 
\end{cases} \quad (\mu = 0) 
\]
\[
- \frac{\partial V}{\partial x^i} dx^0 - d\left( m \frac{dx^i}{dx^0} \right) \quad (i = 1, 2, 3) 
\]  
(2.57)

The covariant Euler–Lagrange equation is given by the pull-back of the above equation through a parametrization of \( c \). Our covariant description of the system derives one additional relation, energy conservation equation, as an equation of motion for \( \mu = 0 \), compared with the usual approach. This additional equation must not be independent of the other three equations with \( i = 1,2,3 \). Actually, the energy conservation is derived from the reparametrization invariance schematically in our treatment \[15\].

Our notation of \( F \) using \( dx^\mu \) in equation (2.56) may seem odd at its first look. However, the result of equation (2.57)
\[
0 = \frac{\partial V}{\partial x^0} dx^0 + d\left( m \frac{dx^i}{dx^0} \right) = \frac{\partial V}{\partial x^i} dx^0 + m \frac{dx^i dx^0 - dx^0 dx^i}{(dx^0)^2}, 
\]  
(2.58)

has clear mathematical meanings in pullback by parametrization of \( c \). Furthermore, in its division by time parameter function \( d\tau \),
\[
0 = \frac{\partial V}{\partial x^0} dx^0 + m \frac{dx^i dx^0 - dx^0 dx^i}{(dx^0)^2},
\]  
(2.59)

the quantities
\[
\frac{dx^i}{dx^0} = \frac{dx^i}{d\tau}, \quad d\left( \frac{dx^i}{d\tau} \right) = d\left( \frac{dx^0}{d\tau} \right) = \frac{dx^i}{d\tau} \frac{dx^0}{d\tau} = \frac{dx^i dx^0}{(dx^0)^2}.
\]  
(2.60)

are well-known as covariant (reparametrization invariant) forms of the derivative in the historical reference by Cauchy \[29\]. As was already discussed, equation (2.12) is covariant under the homogeneity transformation (2.16), and its solution curve \( c \) is uniquely determined independently of its parametrizations. The Finsler non-linear connection is given by
\[
2G^i = \left\{ \frac{dx^\nu}{d\tau} \right\} \frac{\partial F}{\partial x^\nu} dx^u + F^a_{\ b} \left( \frac{\partial F}{\partial x^b} + dx^\nu_\partial + \frac{\partial^2 F}{\partial x^a \partial x^\nu} \right) 
\]
\[
= \left\{ \frac{\partial V}{\partial x^a} dx^b \right\} dx^u + \left( \frac{dx^\nu dx^b}{m} + \frac{dx^0 dx^\nu}{F} \right) \left( \frac{\partial V}{\partial x^b} dx^0 \right) 
\]
\[
= \frac{\partial V}{\partial x^b} \left\{ (dx^0)^2 \delta_{ab} - 2 \frac{dx^0 dx^b dx^a}{F} \right\}, 
\]  
(2.61)

where we have used
\[
F^a_{\ b} = \frac{m}{dx^0} \delta_{ab}, \quad F^a_{\ b} = \frac{dx^0}{m} \delta_{ab}, \quad p_b = \frac{\partial F}{\partial x^b} = m \frac{dx^b}{dx^0},
\]  
(2.62)
\[ \ell^\mu_a = \delta^\mu_a - \frac{m \delta_b \delta^b dx^\mu}{F dx^0}, \quad F^ab \ell^a_a = \frac{\delta^b dx^0}{m} - \frac{dx^b dx^\mu}{F}. \]  

(2.63)

Similarly to the free particle motion in Lorentzian spacetime, the covariant Euler Lagrange equation is given by the pull-back of the equation

\[ d^2 x^\mu + 2G^\mu - \lambda dx^\mu = 0, \]  

(2.64)

by a parametrization of \( c \), where \( \lambda = \lambda(x, dx, d^2x) \) is a first-order homogeneity function satisfying \( \lambda(x, kdx, k^2d^2x^0) = k\lambda(x, dx, d^2x) \).

3. Killing symmetry on Finsler manifold

From the variational calculation explained in equations (2.8)–(2.10) in the last chapter, it is convenient to define the Lie derivative of \( F \).

**Definition 3.1.** Lie derivative of Finsler metric \( F \) along a vector field \( \nu \) on \( M \) is defined by

\[ \mathcal{L}_\nu F := \nu^\mu \frac{\partial F}{\partial x^\mu} + dx^\mu \frac{\partial F}{\partial dx^\mu}, \]  

(3.1)

where \( \nu \) is a generalized vector field \( \nu = \nu^\mu(x, dx) \frac{\partial}{\partial x^\mu} \).

**Definition 3.2.** When the Lie derivative of Finsler metric \( F \) by a generalized vector field \( K = K^\mu(x, dx) \frac{\partial}{\partial x^\mu} \) satisfies the relation

\[ \mathcal{L}_K F = K^\mu \frac{\partial F}{\partial x^\mu} + dK^\mu \frac{\partial F}{\partial dx^\mu} = dB(x, dx), \]  

(3.2)

\( K \) is called quasi-Killing vector field. When \( B = 0 \), \( K \) is just called Killing vector field on Finsler manifold. Here \( K^\mu \) and \( B \) are homogeneity of degree 0 with respect to \( dx \).

**Theorem 3.1.** When \( K = K^\mu(x, dx) \frac{\partial}{\partial x^\mu} \) is the quasi-Killing vector field satisfying \( \mathcal{L}_K F = dB \), we have the conserved quantity \( J \) on the solution curve \( c \) determined by covariant Euler–Lagrange equation (2.12).

\[ J \equiv K^\mu \frac{\partial F}{\partial dx^\mu} - B, \quad c^\ast dJ = 0. \]  

(3.3)

**Proof.** Denoting \( c \) as an arbitrary parametrization of the solution curve \( c \),

\[ c^\ast dJ = c^\ast \left( dK^\mu \frac{\partial F}{\partial dx^\mu} + K^\mu \frac{\partial F}{\partial dx^\mu} d\left( \frac{\partial F}{\partial dx^\mu} \right) - dB \right) \]

\[ = c^\ast \left( dK^\mu \frac{\partial F}{\partial dx^\mu} + K^\mu \frac{\partial F}{\partial dx^\mu} - dB \right) \]

\[ = c^\ast (\mathcal{L}_K F - dB) = 0, \]  

(3.4)

where we have used equation (2.12) in the second equality. \( \square \)

In this proof we should note that \( dJ \) is defined independently of parametrizations \( c \) since \( J \) is a homogeneity function with respect to \( dx \) of degree 0.
Killing symmetry (or vector fields) and the corresponding conservation laws, called
the first Noether’s theorem, are given in Olver [26] in the standard jet bundle formulation. If we take \( x^0 \) as a time parameter, we have

\[
\mathcal{L}_KF = K^0 \frac{\partial F}{\partial x^0} + K^i \frac{\partial F}{\partial x^i} + dK^0 \frac{\partial F}{\partial dx^0} + dv^j \frac{\partial F}{\partial dx^j} = 0,
\]

where \( K = (K^0, K^i) \) and \( v = (v^0, v^i) \) and we used the homogeneity condition of \( F \),

\[
\frac{dx^0}{d\xi^0} = \frac{dx^0}{d\xi^0} + \frac{dx^i}{d\xi^0} = \frac{1}{F} \frac{\partial}{\partial x^0}.
\]

This is the very condition that \( K \) is a variational symmetry as seen in [26, 30]. Therefore, we give their covariant (reparametrization invariant) formulation in a simpler way by using Finsler geometry.

**Definition 3.3.** By using the non-linear connection \( N^\alpha_{\mu\nu} \), spray operator \( S \) to a general function \( F(x, d^2) \) is defined by

\[
S = dx^0 \frac{\partial H}{\partial x^0} - 2G^\mu \frac{\partial H}{\partial dx^\mu} - 2G^\mu = N^\mu_{\alpha} dx^\alpha.
\]

From the definition of Finsler non-linear connection [12] given in the introduction,

\[
0 = \frac{\partial F}{\partial x^\mu} - N^\mu_{\alpha}(x, d^2) \frac{\partial F}{\partial dx^\alpha} = \frac{\partial G^\alpha}{\partial dx^\mu} = N^\mu_{\alpha},
\]

we have the identity

\[
SF = dx^\mu \frac{\partial F}{\partial x^\mu} - 2G^\mu \frac{\partial F}{\partial dx^\mu} = dx^\mu \left( \frac{\partial F}{\partial x^\mu} - N^\mu_{\alpha} \frac{\partial F}{\partial dx^\alpha} \right) = 0.
\]

Its derivative by \( dx^\alpha \) leads

\[
0 = \frac{\partial F}{\partial x^\mu} + dx^\mu \frac{\partial^2 F}{\partial dx^\mu \partial x^\mu} - 2 \frac{\partial G^\mu}{\partial dx^\mu} \frac{\partial F}{\partial dx^\mu} - 2G^\mu \frac{\partial^2 F}{\partial dx^\mu \partial dx^\mu} = \frac{\partial F}{\partial x^\mu} + dx^\mu \frac{\partial^2 F}{\partial dx^\mu \partial x^\mu} - 2G^\mu \frac{\partial^2 F}{\partial dx^\mu \partial dx^\mu}.
\]
where we used equation (3.10) in the last equality.

We propose another formulation of Killing symmetry on Finsler manifold.

**Definition 3.4.** $K^\flat$ is a function of $(x, dx)$, and is the first homogeneity function with respect to $dx$, satisfying

$$SK^\flat = 0. \quad (3.13)$$

Then, we call $K^\flat$ a Killing 1-form.

**Theorem 3.2.** When $K^\flat$ is Killing 1-form, $SK^\flat = 0$, we have a conserved quantity:

$$J := \frac{K^\flat}{F}, \quad c^*dJ = 0. \quad (3.14)$$

**Proof.** Both $K^\flat$ and $F$ are first-order homogeneity function and satisfy $SK^\flat = 0$, $SF = 0$. Thus, $J = K^\flat F$ satisfies

$$SJ = dx^\mu \frac{\partial J}{\partial x^\mu} = 2G^{\mu\nu} \frac{\partial J}{\partial dx^\mu} = 0,$$

and it has homogeneity of degree 0, $dx^\mu \frac{\partial J}{\partial dx^\mu} = 0$. Then,

$$c^*dJ = c^* \left\{ dx^\mu \frac{\partial J}{\partial x^\mu} + d^2 x^\mu \frac{\partial J}{\partial dx^\mu} \right\} = c^* \left\{ (d^2 x^\mu + 2G^{\mu\nu}) \frac{\partial J}{\partial dx^\mu} \right\} = c^* \left\{ \lambda dx^\mu \frac{\partial J}{\partial dx^\mu} \right\} = 0. \quad (3.16)$$

The conserved quantity $J$ is independent of parameter choices. Resorting to this theorem, by using special ansatz of the form of $K^\flat$, we can find out hidden conserved quantities, which will be discussed in the next chapter. The Killing 1-form $K^\flat$ as well as the Killing vector field $K$ describes the conserved quantity. Both are related with each other:

**Proposition 3.1.** When Killing 1-form $K^\flat$ is represented in the form

$$K^\flat = F \left\{ K^\flat(x, dx) \frac{\partial F}{\partial dx^\mu} - B(x, dx) \right\}, \quad (3.17)$$

with some functions $K^\flat(x, dx)$ and $B(x, dx)$ which are zeroth homogeneity functions with respect to $dx$, correspondingly we obtain the quasi-Killing vector field as $K = K^\flat(x, dx) \frac{\partial}{\partial x^\mu}$.

That is,

$$SK^\flat = 0 \iff \mathcal{L}_K F = dB \text{ on shell}, \quad (3.18)$$

where ‘on shell’ means that $x^\mu$ and $dx^\mu$ are related with each other by the Euler–Lagrange equation (2.18), or they are considered to be on a solution curve $c$ determined by equation (2.12).
Proof. Since $SF = 0$,

\[ SK^K = F \left( dx^\mu \frac{\partial K^K}{\partial x^m} \frac{\partial F}{\partial x^m} - 2G^\nu \frac{\partial K^K}{\partial x^\nu} \frac{\partial F}{\partial x^\nu} + K^K dx^\nu \frac{\partial F}{\partial x^\nu} \frac{\partial^2 F}{\partial x^\nu \partial x^\rho} - 2K^K dx^\nu \frac{\partial^2 F}{\partial x^\nu \partial x^\rho} \right) \]

\[ - dx^\nu \frac{\partial B}{\partial x^\nu} + 2G^\mu \frac{\partial B}{\partial x^\mu} \right) \]

\[ = F \left( K^K \frac{\partial F}{\partial x^\mu} + dx^\nu \frac{\partial K^K}{\partial x^\nu} \frac{\partial F}{\partial x^\nu} - 2G^\nu \frac{\partial K^K}{\partial x^\nu} \frac{\partial F}{\partial x^\nu} - dx^\nu \frac{\partial B}{\partial x^\nu} + 2G^\mu \frac{\partial B}{\partial x^\mu} \right) \]

\[ = F \left( K^K \frac{\partial F}{\partial x^\mu} + dx^\nu \frac{\partial K^K}{\partial x^\nu} \frac{\partial F}{\partial x^\nu} - dB - (dx^\nu + 2G^\nu) \left( \frac{\partial K^K}{\partial x^\nu} \frac{\partial F}{\partial x^\nu} - \frac{\partial B}{\partial x^\nu} \right) \right). \]  (3.19)

where by using equation (3.12) the third and fourth terms in the RHS of the first equation are rewritten to the first term of the next. When on-shell condition is $0 = d^2 x^\nu + 2G^\nu - \lambda dx^\nu$, and the $K^K$ and $B$ are homogeneity 0 functions, we obtain

\[ SK^K = F \left( \mathcal{L}_K F - dB \right) \text{ on shell.} \]  (3.20)

When $SK^K = 0$ the conserved quantity $J = \frac{K^K}{F}$ with equation (3.13) coincides with equation (3.3) from the first Noether’s theorem. □

4. Generalization of killing tensors on Finsler manifold

4.1. Killing tensors

The Killing tensors defined in Riemannian manifold can be extended on Finsler manifold $M$. First, we extend the covariant derivative by using the Finsler non-linear connection.

**Definition 4.1.** $K$ is a symmetric $p$-rank tensor field with components $K_{\mu_1 \cdots \mu_p}(x)$. The covariant derivative $\nabla$ of $K$ is extended to Finsler manifold $(M, F)$ which is defined by using Finsler non-linear connection. Using components it is written by

\[ \nabla_{\mu_1} K_{\mu_1 \cdots \mu_p} := \partial_{\mu_1} K_{\mu_1 \cdots \mu_p} - N^\rho_{\mu_1} K_{\rho \mu_2 \cdots \mu_p} - \cdots - N^\rho_{\mu_1} K_{\mu_1 \cdots \rho}, \]  (4.1)

where $N^\rho_{\mu_1} := \frac{\partial G^\rho}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^{\mu_1}}$.

**Definition 4.2.** The $p$-rank tensors $K$, when its components satisfy the relation $\nabla_{\mu_1} K_{\mu_1 \cdots \mu_p} = 0$, is called Killing tensors. Here $(\mu_1 \cdots \mu_p)$ means symmetrization.

**Proposition 4.1.** Killing 1-form $K^K$ is given by using Killing tensors $K$ with components $K_{\mu_1 \cdots \mu_p}$:

\[ K^K = K_{\mu_1 \cdots \mu_p}(x) dx^{\mu_1} \cdots dx^{\mu_p} \frac{F^{p-1}}{F}, \]  (4.2)

which satisfies $SK^K = 0$, then

\[ J := \frac{K^K}{F} = K_{\mu_1 \cdots \mu_p}(x) dx^{\mu_1} \cdots dx^{\mu_p} \frac{F}{F}, \]  (4.3)

is conserved.
Proof.

\[ SK^p = \frac{dx^\rho}{dx^p} \frac{\partial K^\rho}{\partial x^p} - 2G^\rho \frac{\partial K^\rho}{\partial x^p} \]

\[ = \left( \frac{dx^\mu dx^{\mu_1} \cdots dx^{\mu_P}}{F^{p-1}} \right) \left\{ \frac{\partial K_{\mu_{\mu_1} \cdots \mu_P}(x)}{\partial x^\mu} - N^\rho_{\mu \rho} K_{\mu_{\rho_{\mu_1} \cdots \rho P}}(x) - \cdots - N^\rho_{\mu \rho} K_{\mu_{\mu_1} \cdots \mu_{P-\rho}}(x) \right\} \]

\[ = \frac{dx^\mu dx^{\mu_1} \cdots dx^{\mu_P}}{F^{p-1}} \nabla_\mu K_{\mu_{\mu_1} \cdots \mu_P}(x), \]  

(4.4)

where we used \(2G^\rho = N^\rho_{\mu \rho} dx^\mu dx^\rho\). The conservation has already been proved in equation (3.14).

\[ \square \]

4.2. Riemannian case and Carter constant

In Riemannian case \(F = \sqrt{g_{\mu \nu}(x)} dx^\mu dx^\nu\), \(N^\rho_{\mu \rho} = \Gamma^\rho_{\mu \rho}\) and \(\nabla\) becomes the usual Riemannian covariant derivative. Thus, \(K_{\mu_1 \cdots \mu_P}(x)\) in equation (4.2) which satisfies \(\nabla_\mu K_{\mu_{\mu_1} \cdots \mu_P} = 0\) becomes the ordinary Killing tensors in this case. By taking the arc length parameter \(\tau = dx^\nu / F\), \(J = K_{\mu_1 \cdots \mu_P}(x) u^{\mu_1} \cdots u^{\mu_P}\) where \(u^\mu = dx^\mu / \tau\). This is the ordinary conserved quantity made from Killing tensors along geodesics.

Proposition 4.2. When \(K_{\mu_1 \cdots \mu_P}(x)\) is Killing tensor on Riemannian manifold, \(\nabla_\mu K_{\mu_{\mu_1} \cdots \mu_P} = 0\),

\[ K := K^\rho_{\mu_1 \cdots \mu_P}(x) \frac{dx^{\mu_1} \cdots dx^{\mu_{P-1}}}{F^{p-1}} \frac{\partial}{\partial x^\rho}, \]  

is the quasi-Killing vector field, satisfying \(\mathcal{L}_K F = dB\) with \(B = \frac{p+1}{p} J\), where \(K^\rho_{\mu_1 \cdots \mu_P}(x) = g^{\rho\sigma}(x) K_{\mu_1 \cdots \mu_P}(x)\).

Proof:

\[ \mathcal{L}_K F = \frac{dx^{\mu_1} \cdots dx^{\mu_{P-1}}}{F^{p-1}} \frac{\partial F}{\partial x^\rho} \]

\[ + \left\{ \frac{dx^{\mu_1} \cdots dx^{\mu_{P-1}}}{F^{p-1}} \frac{\partial F}{\partial x^\mu} + (p-1) K^\rho_{\mu_1 \cdots \mu_P} \frac{dx^{\mu_1} \cdots dx^{\mu_{P-1}}}{F^{p-1}} \frac{\partial F}{\partial x^\rho} \right\} \]

\[ = \frac{dx^{\mu_1} \cdots dx^{\mu_{P-1}}}{F^{p-1}} \frac{\partial F}{\partial x^\mu} + d(g^{\rho\sigma} K_{\mu_1 \cdots \mu_P}) \frac{dx^{\mu_1} \cdots dx^{\mu_{P-1}} \frac{\partial F}{\partial x^\rho}}{F^{p-1}} \]

\[ + (p-1) K^\rho_{\mu_1 \cdots \mu_P} \frac{dx^{\mu_1} \cdots dx^{\mu_{P-1}}}{F^{p-1}} \frac{\partial F}{\partial x^\rho} - (p-1) K^\rho_{\mu_1 \cdots \mu_P} \frac{dx^{\mu_1} \cdots dx^{\mu_{P-1}}}{F^{p+1}} \frac{\partial F}{\partial x^\rho}, \]  

(4.6)

Here
Using the above relation, the third and fourth terms of the last equation of (4.6) are rewritten. Then, we get

\[
L_k F = K^\alpha_{\nu\rho\sigma\cdots} \frac{\partial F}{\partial x^\alpha} + d(g^{\mu\nu}K^\rho_{\nu\nu\cdots}) \frac{\partial F}{\partial x^\rho} + \frac{p - 1}{p} \frac{\partial K^\rho_{\nu\nu\cdots}}{\partial x^\rho} + \frac{1}{p} \frac{\partial \nu_{\nu\cdots}}{\partial x^\nu} = \frac{p - 1}{p} \frac{\partial K^\rho_{\nu\nu\cdots}}{\partial x^\rho} + \frac{1}{p} \frac{\partial \nu_{\nu\cdots}}{\partial x^\nu}
\]

(4.8)

Since \( K_{\nu\cdots} \) satisfies \( \nabla_{\nu}(\xi_{\nu\cdots}) = 0 \), the generalized vector field (4.5) is the quasi-Killing, satisfying \( L_k F = dB \), even without using on-shell condition.

Carter constant is the conserved quantity along the geodesic motion of point particle under Kerr spacetime. It is derived from the second-rank Killing tensor in Riemannian geometry. The Kerr metric is simply given by using vierbein as

\[
g_{\mu\nu}(x)dx^\mu dx^\nu = -(\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 + (\theta^0)^2,
\]

\[
\theta^1 = \frac{\rho}{\sqrt{\Delta}}, \quad \theta^2 = \rho d\theta, \quad \theta^3 = \frac{S}{\rho}((r^2 + a^2)d\phi - adt), \quad \theta^0 = \frac{\sqrt{\Delta}}{\rho}(dt - aS^2 d\phi),
\]

(4.10)

where \( \rho^2 = r^2 + a^2C^2, \Delta = r^2 + a^2 - 2Mr, \) and \( C, S = \cos \theta, \sin \theta. \) The Killing tensor, denoted as \( K^\mu_{\nu\nu\cdots} \), is read off from the Carter constant \([31, 32]\) as

\[
K^\nu = K^\nu_{\mu\rho\nu\nu\cdots} \frac{dx^\nu}{F} = a^2C^2 \frac{1}{F} ((\theta^0)^2 - (\theta^1)^2) + r^2 \frac{1}{F} ((\theta^2)^2 + (\theta^3)^2)
\]

(4.11)

The quasi-Killing vector field corresponding to the Carter constant is given by

\[
K^\nu_{\text{Carter}} = g^{\nu\rho}K^\rho_{\mu\nu\nu\cdots} \frac{\partial}{\partial x^\nu}.
\]

(4.12)
which satisfies, from the last proposition, the relation (4.8) with \( p = 2 \) as

\[
\mathcal{L}_{K^{\text{Carter}}} F = d \left( \frac{1}{2} K_{\text{Carter}} \frac{dx^\mu}{F} \frac{dx^\nu}{F} \right). \tag{4.13}
\]

We checked the above equation is actually derived from Lie derivative of the vector field equation (4.12) without using on-shell condition. This shows \( \nabla_x K_{\text{Carter}} = 0 \). The generalized vector field (4.12) represents a hidden symmetry \([26]\) on Kerr spacetime.

### 4.3. Runge–Lentz vectors in classical mechanics

We can invent a way of finding the Runge–Lentz vectors in the Newtonian system, by using the Killing 1-form \( \flat K \). The classical Lagrangian system is treated in Finsler geometry with the metric

\[
F(x, dx) = \frac{m}{2} \left( \frac{(dx^1)^2}{dx^0} + \frac{(dx^2)^2}{dx^0} + \frac{(dx^3)^2}{dx^0} \right) + \frac{G M m}{r} dx^0,
\]

which satisfies the first-order homogeneity relation \( F(x, kdx) = kF(x, dx) \). Following the formulas in the last chapter, the non-linear connection is given by

\[
2G^0 = -\frac{2 G M m}{r^2} \frac{(dx^0)^2}{F}, \quad 2G^i = -\frac{2 G M m}{r^2} \frac{dx^i dx^a dx^0}{F} + \frac{G M m}{r^3} (dx^0)^2,
\]

with space component \( a = 1, 2, 3 \). We adopt a different ansatz of Killing vector field using second-rank tensor \( K_{\mu \nu} \) from equation (4.2),

\[
K^a = F K_{\mu \nu} (x) \frac{dx^\mu}{dx^0} \frac{dx^\nu}{dx^0},
\]

by considering the non-relativistic nature of the system. Then,

\[
SK^a = F \left[ \frac{dx^a}{dx^0} \frac{\partial K_{\mu \nu} (x)}{\partial x^a} \frac{dx^\mu}{dx^0} \frac{dx^\nu}{dx^0} - 2G^0 \frac{\partial}{\partial x^a} \left\{ \frac{dx^\mu}{dx^0} \frac{dx^\nu}{dx^0} \right\} K_{\mu \nu} (x) \right]
\]

\[
= F \left[ \frac{\partial K_{\mu \nu}}{\partial x^a} \frac{dx^\mu}{dx^0} \frac{dx^\nu}{dx^0} \frac{(dx^0)^2}{(dx^0)^2} - \frac{G M m}{r^3} (dx^0)^2 \frac{\partial}{\partial x^a} \left\{ \frac{dx^\mu}{dx^0} \frac{dx^\nu}{dx^0} \right\} K_{\mu \nu} (x) \right]
\]

\[
= F \left[ \frac{\partial K_{\mu \nu}}{\partial x^a} \frac{dx^\mu}{dx^0} \frac{dx^\nu}{dx^0} \frac{GMm}{r^3} K_{\mu \nu} \frac{dx^a}{dx^0} \right], \tag{4.17}
\]

where in the second equality we used the fact that \( 2G^0 \) and the first term of \( 2G^a \) in equation (4.15) cancel since \( J \) in \( K^a = FJ \) is homogeneity zero. The symmetry condition \( \delta K^a = 0 \) requires the relations

\[
\frac{\partial K_{00}}{\partial x^0} - 2K_{0a} \frac{G M m}{r^3} = 0, \quad \frac{\partial K_{ab}}{\partial x^0} + \frac{\partial K_{ba}}{\partial x^a} + \frac{\partial K_{0b}}{\partial x^a} = 0,
\]

\[
\frac{\partial K_{00}}{\partial x^a} + \frac{2}{r} \frac{\partial K_{0a}}{\partial x^0} - 2K_{ab} \frac{G M m}{r^3} = 0, \quad \frac{\partial K_{ab}}{\partial x^a} + \frac{\partial K_{bc}}{\partial x^b} + \frac{\partial K_{ca}}{\partial x^c} = 3 \partial_0 K_{abc} = 0. \tag{4.18}
\]

By solving these equations we can find a hidden conserved quantity. Requiring the static condition, \( \frac{\partial K_{\mu \nu}}{\partial x^0} = 0, K_{0a} = 0 \) from the first equations. The second equations are automatically satisfied.
The point is the fourth equations $\partial_i K_{ab}^{(4)} = 0$. There are $3H_3 = 10$ equations. By taking the tensorial form $K_{ab} = \delta_{ab} f(x) + x^a x^b g(x)$, we obtain the energy and total angular momentum. Thus, we must seek the non-tensorial form. $K_{11} = K_{11}(x^2, x^3)$ since $\partial_1 K_{11} = 0; \partial_2 K_{11} + 2\partial_3 K_{11} = 0$ and $\partial_2 K_{22} + 2\partial_3 K_{22} = 0$ leads $(K_{11}, K_{12}, K_{22}) = (-f(x^2), \frac{x^1}{2}f'(x^2), \frac{x^1}{2}f''(x^2))$ with arbitrary function $f(x^2)$. The simplest, non-trivial solution is $(K_{11}, K_{12}, K_{22}) = c_1(-x^2, x^1, 0)$ with constant $c_1$. There are 6 independent solutions of this type. Then, the solutions with 6 constants $c_1 (i = 1, \ldots, 6)$ are $(K_{11}, K_{12}, K_{33}, K_{12}, K_{33}, K_{23}) = (-c_1 x^2 - c_2 x^3 - c_3 x^3 - c_4 x^4 - c_5 x^5 - c_6 x^6; c_1 x^2 + c_2 x^2 + c_3 x^2 + c_4 x^2 + c_5 x^2 + c_6 x^2)$. The 3rd equation $\partial_0 K_{ab} = 2K_{ab}x^h \frac{\partial GM}{x^a} x^0$ is regarded as an integrability condition requiring outer derivative of $\partial_0 K_\alpha$ should vanish. It leads three solutions:

$$(4.19)$$

\begin{align*}
(c_4 = c_5 = 1) & \quad K_{00} = \frac{GMx^1}{r} \quad K_{22} = K_{33} = -x^1 \quad K_{12} = \frac{x^2}{2} \quad K_{13} = \frac{x^3}{2} \\
(c_6 = c_1 = 1) & \quad K_{00} = \frac{GMx^2}{r} \quad K_{33} = K_{11} = -x^2 \quad K_{12} = \frac{x^3}{2} \quad K_{13} = \frac{x^1}{2} \quad K_{23} = \frac{x^2}{2} \\
(c_2 = c_3 = 1) & \quad K_{00} = \frac{GMx^3}{r} \quad K_{11} = K_{22} = -x^3 \quad K_{12} = \frac{x^4}{2} \quad K_{13} = \frac{x^5}{2} \quad K_{23} = \frac{x^6}{2} 
\end{align*}$$

where the other $c_i$’s and $K_{\mu\nu}$’s are 0 in each line. These are called Runge–Lentz tensors which lead Runge–Lentz vectors through the standard procedure $J = \frac{\partial x^i}{\partial \tau} = K_{ab} x^b \frac{\partial}{\partial x^a}$. They are denoted as $J_i$ ($i = 1, 2, 3$), which are given by

$$
J_i = \frac{GMmx^i}{r} - mx^i \left\{ \frac{dx^2}{dx^0} \right\}^2 + \frac{dx^3}{dx^0} \left\{ \frac{dx^2}{dx^0} \right\}^2 + m \frac{dx^i}{dx^0} \left\{ \frac{x^1 dx^2}{dx^0} + \frac{x^3 dx^3}{dx^0} \right\} \left\{ \frac{dx^2}{dx^0} \right\}^2 + \frac{dx^i}{dx^0} \left\{ \frac{x^1 dx^2}{dx^0} + \frac{x^3 dx^3}{dx^0} \right\} \left\{ \frac{dx^3}{dx^0} \right\}^2,
$$

where we took $m$ as the overall proportional constant. Runge–Lentz vectors are also derived, similarly to our method, using Hamiltonian formalism [33]. Finally, we give the generalized Killing vector fields corresponding to equation (4.19). We take

$$K = K^a x^a - \frac{\partial}{\partial x^a} = 2K_{ab}(x^a) \frac{dx^b}{dx^0} \frac{\partial}{\partial x^a}. \quad (4.21)$$

which gives three quasi-Killing vector fields corresponding to three lines of (4.19):

$$
K_i = \left( x^2 \frac{dx^2}{dx^0} + x^3 \frac{dx^3}{dx^0} \right) \frac{\partial}{\partial x^2} + \left( x^2 \frac{dx^1}{dx^0} - 2x^1 \frac{dx^2}{dx^0} \right) \frac{\partial}{\partial x^3} + \left( x^3 \frac{dx^1}{dx^0} - 2x^1 \frac{dx^3}{dx^0} \right) \frac{\partial}{\partial x^3},
K_2 = \left( x^1 \frac{dx^1}{dx^0} - 2x^2 \frac{dx^1}{dx^0} \right) \frac{\partial}{\partial x^1} + \left( x^3 \frac{dx^3}{dx^0} + x^1 \frac{dx^1}{dx^0} \right) \frac{\partial}{\partial x^2} + \left( x^3 \frac{dx^2}{dx^0} - 2x^2 \frac{dx^3}{dx^0} \right) \frac{\partial}{\partial x^3},
K_3 = \left( x^2 \frac{dx^3}{dx^0} - 2x^3 \frac{dx^2}{dx^0} \right) \frac{\partial}{\partial x^1} + \left( x^1 \frac{dx^1}{dx^0} - 2x^2 \frac{dx^1}{dx^0} \right) \frac{\partial}{\partial x^2} + \left( x^1 \frac{dx^2}{dx^0} + x^2 \frac{dx^2}{dx^0} \right) \frac{\partial}{\partial x^3}. \quad (4.22)
$$
They satisfy the relation
\[
\mathcal{L}_K F = d B, \quad B = -mK_{00} + K_{ab} \frac{dx^a}{dx^0} \frac{dx^b}{dx^0},
\]
and represent hidden symmetries of the system. The corresponding conserved currents
\[
J = K^a \frac{\partial F}{\partial x^a} - B = m\left(K_{00} + K_{ab} \frac{dx^a}{dx^0} \frac{dx^b}{dx^0}\right)
\]
lead equation (4.20).

5. Concluding remarks

We have formulated the Killing vector field \( K \) and the conserved quantity on Finsler manifold in a covariant form. The Killing symmetry on the Finsler manifold is represented in another form by using the Killing 1-form \( K^* \), which is a generalization of the concept of Killing tensors. In Riemannian case we have shown that Killing tensors are represented as Killing symmetries on Riemannian manifold, as a generalized vector field. Using the spray operator \( S \) and the non-linear connection, the condition \( SK^* = 0 \) is proved to be equivalent to the conservation law \( c^* d\left(\frac{K^*}{c^0}\right) = 0 \) on the solution curve \( c \). They give higher derivative conserved quantities on Finsler manifold, which may be called ‘hidden’ conserved quantities. The Runge–Lentz vectors are of this kind. We have also proposed a technique of finding these quantities using our ansatz of the Killing 1-form. The equation \( SK^* = 0 \) leads partial differential equations. By solving them we obtain hidden conserved quantities straightforwardly. This method gives a new analytical way of finding the conserved quantity in Lagrangian systems which cannot be found easily in the standard Noether procedure.

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