LAPLACE TRANSFORM, DYNAMICS AND SPECTRAL GEOMETRY

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Abstract. We consider vector fields $X$ on a closed manifold $M$ with rest points of Morse type. For such vector fields we define the property of exponential growth. A cohomology class $\xi \in H^1(M;\mathbb{R})$ which is Lyapunov for $X$ defines counting functions for isolated instantons and closed trajectories. If $X$ has exponential growth property we show, under a mild hypothesis generically satisfied, that these counting functions can be recovered from the spectral geometry associated to $(M,g,\omega)$ where $g$ is a Riemannian metric and $\omega$ is a closed one form representing $\xi$, cf Theorems 3 and 4 in section 1.6. This is done with the help of Dirichlet series and their Laplace transform.

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1. INTRODUCTION

1.1. Vector fields with zeros of Morse type and Lyapunov cohomology class. Let $X$ be a smooth vector field on a smooth manifold $M$. A point $x \in M$ is called a rest point or a zero of $X$ if $X(x) = 0$. Denote by $X := \{x \in M|X(x) = 0\}$ the set of rest points of $X$.

Recall that:

(i) A parameterized trajectory is a map $\theta : \mathbb{R} \to M$ so that $\theta'(t) = X(\theta(t))$. A trajectory is an equivalence class of parameterized trajectories with $\theta_1 \equiv \theta_2$ iff $\theta_1(t + a) = \theta_2(t)$ for some real number $a$. Any representative $\theta$ of a trajectory is called a parametrization.

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(ii) An instanton from the rest point $x$ to the rest point $y$ is an isolated trajectory with the property that for one and then any parameterization $\theta$, $\lim_{t \to -\infty} \theta(t) = x$, $\lim_{t \to +\infty} \theta(t) = y$.

(iii) A parameterized closed trajectory is a pair $\tilde{\theta} = (\theta, T)$, with $\theta$ a parameterized trajectory and $T$ a positive real number so that $\theta(t+T) = \theta(t)$. A parameterized closed trajectory gives rise to a smooth map $\theta: S^1 := \mathbb{R}/TZ \to M$.

A closed trajectory is an equivalence class $[\tilde{\theta}]$ of parameterized closed trajectories with $(\theta_1, T_1) \equiv (\theta_2, T_2)$ iff $\theta_1 \equiv \theta_2$ and $T_1 = T_2$.

Recall that a rest point $x \in X$ is said to be of Morse type if there exist coordinates $(t_1, \ldots, t_n)$ around $x$ so that $X = 2 \sum_{i=1}^{q} t_i \partial_{x_i} - 2 \sum_{i=q+1}^{n} t_i \partial_{x_i}$. The integer $q$ is called the Morse index of $x$ and denoted by $\text{ind}(x)$. A rest point of Morse type is non-degenerate and its Hopf index is $(-1)^{n-q}$. It is independent of the chosen coordinates $(t_1, \ldots, t_n)$. Then $X = \bigsqcup_q X_q$ where $X_q$ denotes the set of rest points of index $q$.

For any rest point of Morse type $x$, the stable resp. unstable set is defined by:

$$W_x^\pm := \{ y | \lim_{t \to \pm \infty} \Psi_t(y) = x \}$$

where $\Psi_t : M \to M$ denotes the flow of $X$. The stable and unstable sets are images of injective smooth immersions $i_\pm^x: W_x^\pm \to M$. By abuse of notation we denote the source manifold also by $W_x^\pm$. The manifold $W_x^-$ resp. $W_x^+$ is diffeomorphic to $\mathbb{R}^{\text{ind}(x)}$ resp. $\mathbb{R}^{n-\text{ind}(x)}$.

**Convention.** Unless explicitly mentioned all the vector fields in this paper are assumed to have all rest points of Morse type, hence isolated.

**Definition 1.** A vector field $X$ is said to have the exponential growth property at a rest point $x$ if for some (and then any) Riemannian metric $g$ there exists a positive constant $C$ so that $\text{Vol}(D_r(x)) \leq e^{Cr^r}$ for all $r \geq 0$. Here $D_r(x) \subseteq W_x^-$ denotes the disk of radius $r$ with respect to the induced Riemannian metric $(i_x^-)^* g$ on $W_x^-$ centered at $x \in W_x^-$. A vector field $X$ is said to have the exponential growth property if it has the exponential growth property at all rest points.

**Definition 2.** A cohomology class $\xi \in H^1(M; \mathbb{R})$ is called Lyapunov class for a vector field $X$ if there exits a Riemannian metric $g$ and a closed one form $\omega$ representing $\xi$ so that $X = -\text{grad}_g \omega$.

**Remark 1.** (to Definition 2) 1. An equivalent definition is the following: There exists a closed one form $\omega$ representing $\xi$ so that $\omega(X) < 0$ on $M \setminus X$ and such that in a neighborhood of any rest point the vector field $X$ is equal to $-\text{grad}_g \omega$ for some Riemannian metric $g$. It is proved in section 3 that the two definitions are actually equivalent.

2. The closed form $\omega$ is a Morse form, i.e. locally it is the differential of a smooth function whose critical points are non-degenerate.

3. Not all vector fields admit Lyapunov cohomology classes.

**Definition 3.** The vector field $X$ is said to satisfy the Morse–Smale property, MS for short, if for any $x, y \in X$ the maps $i_x^-$ and $i_y^+$ are transversal.

We expect that every vector field which has a Lyapunov cohomology class, and satisfies the Morse–Smale property, has the exponential growth property, cf. the conjecture in section 3.2. For the sake of Theorem 4 we introduce in section 6.1, cf.
Definition 4. the strong exponential growth property. If the conjecture is true both concepts are superfluous for the results of this paper.

In this paper we will show that a vector field $X$ and a Lyapunov class $ξ$ for $X$ provide counting functions for the instantons from $x$ to $y$ when $\text{ind}(x) − \text{ind}(y) = 1$ and counting functions for closed trajectories. Moreover these counting functions can be interpreted as Dirichlet series. If the vector field has exponential growth property these series have a finite abscissa of convergence, hence have a Laplace transform, cf section 1.2. Their Laplace transform can be read off from the spectral geometry of a pair $(g, ω)$ where $g$ is a Riemannian metric and $ω$ is a closed one form representing $ξ$.

We will describe these counting functions and prove our results under the hypotheses that properties MS and NCT defined below are satisfied. Generically these properties are always satisfied, cf. Proposition 2 below.

Also in this paper, for any vector field $X$ and cohomology class $ξ ∈ H^1(M; R)$ we define an invariant $ρ(ξ, X) ∈ R \cup \{±∞\}$ and show that if $ξ$ is Lyapunov for $X$ then exponential growth property is equivalent to $ρ(ξ, X) < ∞$.

If the vector field $X$ satisfies MS then the set $M(x, y) = W_x^- \cap W_y^+$, $x, y ∈ X$ is the image by an injective immersion of a smooth manifold of dimension $\text{ind}(x) − \text{ind}(y)$ on which $R$ acts freely. The quotient is a smooth manifold $T(x, y)$ of dimension $\text{ind}(x) − \text{ind}(y) − 1$ called the manifold of trajectories from $x$ to $y$. If $\text{ind}(x) − \text{ind}(y) = 1$ then $T(x, y)$ is zero dimensional and its elements are isolated trajectories called instantons.

Choose $O = \{O_x\}_{x ∈ X}$ a collection of orientations of the unstable manifolds of the critical points, with $O_x$ an orientation of $W_x^-$. Any instanton $[θ]$ from $x ∈ X_q$ to $y ∈ X_{q−1}$ has a sign $\epsilon([θ]) = ϵ^O([θ]) = ±1$ defined as follows: The orientations $O_x$ and $O_q$ induce an orientation on $[θ]$. Take $\epsilon([θ]) = +1$ if this orientation is compatible with the orientation from $x$ to $y$ and $\epsilon([θ]) = −1$ otherwise.

Let $Ψ_t$ denote the flow of $X$. The closed trajectory $[θ]$ is called non-degenerate if for some (and then any) $t_0 ∈ R$ and representative $\hat{θ} = (θ, T)$ the differential $D_{θ(t_0)}Ψ_T : T_{θ(t_0)}M → T_{θ(t_0)}M$ is invertible with the eigenvalue 1 of multiplicity one.

Definition 4. The vector field $X$ is said to satisfies the non-degenerate closed trajectories property, NCT for short, if all closed trajectories of $X$ are non-degenerate.

Any non-degenerate closed trajectory $[\hat{θ}]$ has a period $p([\hat{θ}]) ∈ N$ and a sign $\epsilon([\hat{θ}]) := ±1$ defined as follows:

(i) $p([\hat{θ}])$ is the largest positive integer $p$ such that $θ : S^1 → M$ factors through a self map of $S^1$ of degree $p$.

(ii) $\epsilon([\hat{θ}]) := \text{sign det } D_{θ(t_0)}Ψ_T$ for some (and hence any) $t_0 ∈ R$ and parameterization $θ$.

A cohomology class $ξ ∈ H^1(M; R)$ induces the homomorphism $ξ : H_1(M; Z) → R$ and then the injective group homomorphism

$$ξ : \Gamma_ξ → R, \text{ with } \Gamma_ξ := H_1(M; Z)/ \ker ξ.$$

For any two points $x, y ∈ M$ denote by $P_{x,y}$ the space of continuous paths from $x$ to $y$. We say that $α ∈ P_{x,y}$ is equivalent to $β ∈ P_{x,y}$ iff the closed path $β^{-1} ∗ α$ represents an element in $\ker ξ$. (Here $∗$ denotes the juxtaposition of paths. Precisely
if \( \alpha, \beta : [0, 1] \to M \) and \( \beta(0) = \alpha(1) \), then \( \beta \circ \alpha : [0, 1] \to M \) is given by \( \alpha(2t) \) for \( 0 \leq t < 1/2 \) and \( \beta(1 - 2t) \) for \( 1/2 \leq t < 1 \).

We denote by \( \mathcal{P}_{x,y} = \mathcal{P}_{x,y}^\xi \) the set of equivalence classes of elements in \( \mathcal{P}_{x,y} \). Note that \( \Gamma_\xi \) acts freely and transitively, both from the left and from the right, on \( \mathcal{P}_{x,y}^\xi \).

The action \( \ast \) is defined by juxtaposing at \( x \) resp. \( y \) a closed curve representing an element \( \gamma \in \Gamma_\xi \) to a path representing the element \( \tilde{\alpha} \in \mathcal{P}_{x,y}^\xi \).

Any closed one form \( \omega \) representing \( \xi \) defines a map, \( \overline{\omega} : \mathcal{P}_{x,y} \to \mathbb{R} \), by

\[
\overline{\omega}(\alpha) := \int_{[0,1]} \alpha^* \omega
\]

which in turn induces the map \( \overline{\omega} : \mathcal{P}_{x,y}^\xi \to \mathbb{R} \). We have:

\[
\overline{\omega}(\gamma \ast \tilde{\alpha}) = \xi(\gamma) + \overline{\omega}(\tilde{\alpha})
\]

\[
\overline{\omega}(\tilde{\alpha} \ast \gamma) = \overline{\omega}(\tilde{\alpha}) + \xi(\gamma)
\]

Note that for \( \omega' = \omega + dh \) we have \( \overline{\omega'} = \overline{\omega} + h(y) - h(x) \).

**Proposition 1.** Suppose \( \xi \in H^1(M; \mathbb{R}) \) is a Lyapunov class for the vector field \( X \).

(i) If \( X \) satisfies MS, \( x \in \mathcal{X}_q \) and \( y \in \mathcal{X}_{q-1} \) then the set of instantons from \( x \) to \( y \) in each class \( \tilde{\alpha} \in \mathcal{P}_{x,y}^\xi \) is finite.

(ii) If \( X \) satisfies both MS and NCT then for any \( \gamma \in \Gamma_\xi \) the set of closed trajectories representing the class \( \gamma \) is finite.

The proof is a straightforward consequence of the compacity of space of trajectories of bounded energy, cf. [3] and [9].

Suppose \( X \) is a vector field which satisfies MS and NCT and suppose \( \xi \) is a Lyapunov class for \( X \). In view of Proposition 1 we can define the counting function of closed trajectories by

\[
\mathbb{Z}_X^\xi : \Gamma_\xi \to \mathbb{Q}, \quad \mathbb{Z}_X^\xi(\gamma) := \sum_{[\theta] \in \gamma} \frac{(-1)^{r([\theta])}}{p([\theta])} \in \mathbb{Q}.
\]

If a collection of orientations \( \mathcal{O} = \{ \mathcal{O}_x \}_{x \in \mathcal{X}} \) is given one defines the counting function of the instantons from \( x \) to \( y \) by

\[
\ell_{x,y}^{X,\mathcal{O},\xi} : \mathcal{P}_{x,y}^\xi \to \mathbb{Z}, \quad \ell_{x,y}^{X,\mathcal{O},\xi}(\tilde{\alpha}) := \sum_{[\theta] \in \tilde{\alpha}} \varepsilon([\theta]). \tag{1}
\]

Note that the change of the orientations \( \mathcal{O} \) might change the function \( \ell_{x,y}^{X,\mathcal{O},\xi} \) but only up to multiplication by \( \pm 1 \). A key observation in this work is the fact that the counting functions \( \ell_{x,y}^{X,\mathcal{O},\xi} \) and \( \mathbb{Z}_X^\xi \) can be interpreted as Dirichlet series.

As long as Hypotheses MS and NCT are concerned we have the following generality result. For a proof consult [8] and the references in [3] page 211.

**Proposition 2.** Suppose \( X \) has \( \xi \in H^1(M; \mathbb{R}) \) as a Lyapunov cohomology class.

(i) One can find a vector fields \( X' \) arbitrarily close to \( X \) in the \( C^1 \)-topology which satisfy MS and have \( \xi \) as Lyapunov cohomology class. Moreover one can choose \( X' \) equal to \( X \) in some neighborhood of \( \mathcal{X} \) and away from any given neighborhood of \( \mathcal{X} \).

(ii) If in addition \( X \) above satisfies MS one can find vector fields \( X' \) arbitrary closed to \( X \) in the \( C^1 \)-topology which satisfy MS and NCT, and have \( \xi \) as
Lyapunov cohomology class. Moreover one can choose \( X' \) equal to \( X \) in some neighborhood of \( X \).

(iii) Consider the space of vector fields which have the same set of rest points as \( X \), and agree with \( X \) in some neighborhood of \( X \). Equip this set with the \( C^1 \)-topology. The subset of vector fields which satisfy MS and NCT is Baire residual set.

1.2. Dirichlet series and their Laplace transform. Recall that a Dirichlet series \( f \) is given by a pair of finite or infinite sequences:

\[
\left( \lambda_1, a_1 \quad \lambda_2, a_2 \quad \cdots \quad \lambda_k, a_k \quad \lambda_{k+1}, a_{k+1} \quad \cdots \right)
\]

The first sequence is a sequence of real numbers with the property that \( \lambda_k \to \infty \) if the sequences are infinite. The second sequence is a sequence of non-zero complex numbers. The associated series

\[
L(f)(z) := \sum_i e^{-z\lambda_i}a_i
\]

has an abscissa of convergence \( \rho(f) \leq \infty \), characterized by the following properties, cf. [18] and [19]:

(i) If \( \Re z > \rho(f) \) then \( f(z) \) is convergent and defines a holomorphic function.

(ii) If \( \Re z < \rho(f) \) then \( f(z) \) is divergent.

A Dirichlet series can be regarded as a complex valued measure with support on the discrete set \( \{ \lambda_1, \lambda_2, \ldots \} \subseteq \mathbb{R} \) where the measure of \( \lambda_i \) is equal to \( a_i \). Then the above series is the Laplace transform of this measure, cf. [19]. The following proposition is a reformulation of results which lead to the Novikov theory and to the work of Hutchings–Lee and Pajitnov etc, cf. [6] and [9] for more precise references.

Proposition 3.

(i) (Novikov) Suppose \( X \) is a vector field on a closed manifold \( M \) which satisfies MS and has \( \xi \) as a Lyapunov cohomology class. Suppose \( \omega \) is a closed one form representing \( \xi \). Then for any \( x \in X_q \) and \( y \in X_{q-1} \) the collection of pairs of numbers

\[
\mathbb{I}_{x,y}^{X,\omega} := \left\{ (-\omega(\hat{\alpha}), \mathbb{I}_{x,y}^{X,\xi}(\hat{\alpha})) \mid \mathbb{I}_{x,y}^{X,\xi}(\hat{\alpha}) \neq 0, \hat{\alpha} \in \hat{\mathcal{P}}_{x,y}^{\xi} \right\}
\]

defines a Dirichlet series. The sequence of \( \lambda \)'s consists of the numbers \( -\omega(\hat{\alpha}) \) when \( \mathbb{I}_{x,y}^{X,\xi}(\hat{\alpha}) \) is non-zero, and the sequence \( a \)'s consists of the numbers \( \mathbb{I}_{x,y}^{X,\xi}(\hat{\alpha}) \in \mathbb{Z} \).

(ii) (D. Fried, M. Hutchings) If in addition \( X \) satisfies NCT then the collection of pairs of numbers

\[
\mathbb{Z}_{\xi}^{X} := \left\{ (-\xi(\gamma), \mathbb{Z}_{\xi}^{X}(\gamma)) \mid \mathbb{Z}_{\xi}^{X}(\gamma) \neq 0, \gamma \in \Gamma_{\xi} \right\}
\]

defines a Dirichlet series. The sequence of \( \lambda \)'s consists of the real numbers \( -\xi(\gamma) \) when \( \mathbb{Z}_{\xi}^{X}(\gamma) \) is non-zero and the sequence of \( a \)'s consists of the numbers \( \mathbb{Z}_{\xi}^{X}(\gamma) \in \mathbb{Q} \).

We will show that if \( X \) has exponential growth property then the Dirichlet series \( \mathbb{I}_{x,y}^{X,\omega} \) and \( \mathbb{Z}_{\xi}^{X} \) have the abscissa of convergence finite and therefore Laplace transform. The main results of this paper, Theorems 3 and 4 below, will provide
explicit formulae for these Laplace transforms in terms of the spectral geometry of \((M, g, \omega)\). To explain such formulae we need additional considerations and results.

1.3. The Witten–Laplacian. Let \(M\) be a closed manifold and \((g, \omega)\) a pair consisting of a Riemannian metric \(g\) and a closed one form \(\omega\). We suppose that \(\omega\) is a Morse form. This means that locally \(\omega = dh, h\) smooth function with all critical points non-degenerate. A critical point or a zero of \(\omega\) is a critical point of \(h\) and since non-degenerate, has an index, the index of the Hessian \(d^2_h\), denoted by \(\text{ind}(x)\). Denote by \(\mathcal{X}\) the set of critical points of \(\omega\) and by \(\mathcal{X}_q\) be the subset of critical points of index \(q\).

For \(t \in \mathbb{R}\) consider the complex \((\Omega^\ast(M), d_q^\ast(t))\) with differential \(d_q^\ast(t) : \Omega^q(M) \rightarrow \Omega^{q+1}(M)\) given by

\[
d_q^\ast(t) (\alpha) := d\alpha + t\omega \wedge \alpha.
\]

Using the Riemannian metric \(g\) one constructs the formal adjoint of \(d_q^\ast(t)\), \(d_q^{\ast}(t)^\ast : \Omega^{q+1}(M) \rightarrow \Omega^q(M)\), and one defines the Witten–Laplacian \(\Delta_q^\ast(t) : \Omega^q(M) \rightarrow \Omega^q(M)\) associated to the closed 1–form \(\omega\) by:

\[
\Delta_q^\ast(t) := d_q^\ast(t)^2 \circ d_q^\ast + d_q^{\ast-1}(t) \circ d_q^{\ast-1}(t)^2.
\]

Thus, \(\Delta_q^\ast(t)\) is a second order differential operator, with \(\Delta_q^\ast(0) = \Delta^q\), the Laplace–Beltrami operator. The operators \(\Delta_q^\ast(t)\) are elliptic, selfadjoint and nonnegative, hence their spectra \(\text{Spect} \, \Delta_q^\ast(t)\) lie in the interval \([0, \infty)\). It is not hard to see that

\[
\Delta_q^\ast(t) = \Delta^q + t(L + L^\ast) + t^2||\omega||^2 \text{Id},
\]

where \(L\) denotes the Lie derivative along the vector field \(-\text{grad}_g \omega\), \(L^\ast\) the formal adjoint of \(L\) and \(||\omega||^2\) is the fiber wise norm of \(\omega\).

The following result extends a result due to E. Witten (cf. [20]) in the case that \(\omega\) is exact and its proof was sketched in [3].

**Theorem 1.** Let \(M\) be a closed manifold and \((g, \omega)\) be a pair as above. Then there exist constants \(C_1, C_2, C_3, T > 0\) so that for \(t > T\) we have:

\[
\begin{align*}
(\text{i}) & \quad \text{Spect} \, \Delta_q^\ast(t) \cap [C_1 e^{-C_2 t}, C_3 t] = \emptyset. \\
(\text{ii}) & \quad \sharp(\text{Spect} \, \Delta_q^\ast(t) \cap [0, C_1 e^{-C_2 t}]) = \sharp\mathcal{X}_q. \\
(\text{iii}) & \quad 1 \in (C_1 e^{-C_2 t}, C_3 t).
\end{align*}
\]

Here \(\sharp A\) denotes cardinality of the set \(A\).

Theorem 1 can be complemented with the following proposition, see Lemma 1.3 in [2].

**Proposition 4.** For all but finitely many \(t\) the dimension of \(\ker \Delta_q^\ast(t)\) is constant in \(t\).

Denote by \(\Omega^\ast_{sm}(M)(t)\) the \(\mathbb{R}\)–linear span of the eigen forms which correspond to eigenvalues smaller than 1 referred bellow as the small eigenvalues. Denote by \(\Omega^\ast_{la}(M)(t)\) the orthogonal complement of \(\Omega^\ast_{sm}(M)(t)\) which, by elliptic theory, is a closed subspace of \(\Omega^\ast(M)\) with respect to \(C^\infty\)–topology, in fact with respect to any Sobolev topology. The space \(\Omega^\ast_{la}(M)(t)\) is the closure of the span of the eigen forms which correspond to eigenvalues larger than one. As an immediate consequence of Theorem 1 we have for \(t > T\):

\[
(\Omega^\ast(M), d_\omega(t)) = (\Omega^\ast_{sm}(M)(t), d_\omega(t)) \oplus (\Omega^\ast_{la}(M)(t), d_\omega(t)) \quad (2)
\]
With respect to this decomposition the Witten–Laplacian is diagonalized
\[ \Delta^q_{\omega}(t) = \Delta^q_{\omega,sm}(t) \oplus \Delta^q_{\omega,la}(t). \]  
(3)

and by Theorem [13], we have for \( t > T \)
\[ \dim \Omega^q_{sm}(M)(t) = 2\chi_q. \]

The cochain complex \( (\Omega^q_{la}(M)(t), d_\omega(t)) \) is acyclic and in view of Theorem [13] of finite codimension in the elliptic complex \( (\Omega^q(M), d_\omega(t)) \). Therefore we can define the function
\[ \log T_{an,la}(t) = \log T_{an,la}^{\omega,g}(t) := \frac{1}{2} \sum_q (-1)^{q+1} \log \det \Delta^q_{\omega,la}(t) \]
(4)
where \( \det \Delta^q_{\omega,la}(t) \) is the zeta-regularized product of all eigenvalues of \( \Delta^q_{\omega,la}(t) \) larger than one. This quantity will be referred to as the large analytic torsion.

1.4. Canonical base of the small complex. Let \( M \) be a closed manifold and \((g, g', \omega)\) be a triple consisting of two Riemannian metrics \( g \) and \( g' \) and a Morse form \( \omega \). The vector field \( X = -\text{grad}_g \omega \) has \( \omega \) as a Lyapunov cohomology class.

Suppose that \( X \) satisfies MS and has exponential growth. Choose \( O = \{ O_x \}_{x \in X} \) a collection of orientations of the unstable manifolds with \( O_x \) orientation of \( W^-_x \). Let \( h_x : W^-_x \to \mathbb{R} \) be the unique smooth map defined by \( dh_x = (i_x^-)^*\omega \) and \( h_x(x) = 0 \). Clearly \( h_x \leq 0 \).

In view of the exponential growth property, cf. section 1, there exists \( T \) so that for \( t > T \) the integral
\[ \text{Int}^q_{X,\omega,O}(t)(a)(x) := \int_{W^-_x} e^{th_x(i_x^-)^*a}, \quad a \in \Omega^q(M), \]
(5)
is absolutely convergent, cf. section 1 and defines a linear map:
\[ \text{Int}^q_{X,\omega,O}(t) : \Omega^q(M) \to \text{Maps}(\mathcal{X}_q, \mathbb{R}). \]

**Theorem 2.** Suppose \((g, g', \omega)\) is a triple as above with \( X \) of exponential growth and satisfying MS. Equip \( \Omega^q(M) \) with the scalar product induced by \( g \) and \( \text{Maps}(\mathcal{X}_q, \mathbb{R}) \) with the unique scalar product which makes \( E_x \in \text{Maps}(\mathcal{X}_q, \mathbb{R}) \), the characteristic functions of \( x \in \mathcal{X}_q \), an orthonormal base.

Then there exists \( T \) so that for any \( q \) and \( t \geq T \) the linear map \( \text{Int}^q_{X,\omega,O}(t) \) defined by (5), when restricted to \( \Omega^q_{sm}(M)(t) \), is an isomorphism and an \( O(1/t) \) isometry. In particular \( \Omega^q_{sm}(M)(t) \) has a canonical base \( \{ E^q_x(t)|x \in \mathcal{X}_q \} \) with \( E^q_x(t) = (\text{Int}^q_{X,\omega,O}(t))^{-1}(E_x) \).

As a consequence we have
\[ d^{q-1}_{\omega} E^q_x(t) := \sum_{x \in \mathcal{X}_q} I^{X,\omega,O}_{x,y}(t) \cdot E^q_x(t), \]
(6)
where \( I^{X,\omega,O}_{x,y} : [T, \infty) \to \mathbb{R} \) are smooth, actually analytic functions, cf. Theorem [4] below.

In addition to the functions \( I^{X,\omega,O}_{x,y}(t) \) defined for \( t \geq T \), cf. [4], we consider also the function
\[ \log V(t) = \log V_{\omega,g,X}(t) := \sum_q (-1)^q \log \text{Vol}\{ E_x(t)|x \in \mathcal{X}_q \}. \]
(7)
Observe that the change in the orientations $\mathcal{O}$ does not change the right side of (6), so $\mathcal{O}$ does not appear in the notation $\mathbb{V}(t)$.

1.5. **A geometric invariant associated to $X$, $\omega, g$** and a smooth function associated with the triple $(g, g', \omega)$. Recall that Mathai–Quillen [12] (cf. also [1]) have introduced a differential form $\Psi_g \in \Omega^{n-1}(TM \setminus M; \mathcal{O}_M)$ for any Riemannian manifold $(M, g)$ of dimension $n$. Here $\mathcal{O}_M$ denotes the orientation bundle of $M$ pulled back to $TM$. For any closed one form $\omega$ on $M$ we consider the form $\omega \wedge X^*\Psi_g \in \Omega^n(M \setminus X; \mathcal{O}_M)$. Here $X = -\text{grad}_g \omega$ is regarded as a map $X : M \setminus X \to TM \setminus M$ and $M$ is identified with the image of the zero section of the tangent bundle.

The integral

$$\int_{M \setminus X} \omega \wedge X^*\Psi_g$$

is in general divergent. However it does have a regularization defined by the formula

$$R(X, \omega, g) := \int_M \omega_0 \wedge X^*\Psi_g - \int_M fE_g + \sum_{x \in X}(\omega) \text{ind}(x) f(x)$$

where

(i) $f$ is a smooth function whose differential $df$ is equal to $\omega$ in a small neighborhood of $X$ and therefore $\omega_0 := \omega - df$ vanishes in a small neighborhood of $X$ and

(ii) $E_g \in \Omega^n(M; \mathcal{O}_M)$ is the Euler form associated with $g$.

It will be shown in section 1.6 below that the definition is independent of the choice of $f$; see also [2]. Finally we introduce the function

$$\log \hat{T}_{an}^{X, \omega, g}(t) := \log T_{an, la}(t) - \log \mathbb{V}_{\omega, g, X}(t) + tR(X, \omega, g)$$

where $X = -\text{grad}_g \omega$.

1.6. **The main results.** The main results of this paper are Theorems 3 and 4 below.

**Theorem 3.** Suppose $X$ is a vector field which is MS and has exponential growth and suppose $X$ is a Lyapunov cohomology class for $X$. Let $(g, g', \omega)$ be a system as in Theorem 1 so that $X = -\text{grad}_g \omega$ and $\omega$ a Morse form representing $X$. Let $I_{X, g, \omega}^{\mathcal{O}, \omega, g} : [T, \infty) \to \mathbb{R}$ be the functions defined by (6). Then the Dirichlet series $I_{X, g, \omega}^{\mathcal{O}, \omega, g}$ have finite abscissa of convergence and their Laplace transform are exactly the functions $I_{X, g, \omega}^{\mathcal{O}, \omega, g}(t)$. In particular $I_{X, g, \omega}^{\mathcal{O}, \omega, g}(t)$ is the restriction of a holomorphic function on $\{ z \in \mathbb{C} | \Re z > T \}$.

**Theorem 4.** Suppose $X$ is a vector field with $X$ a Lyapunov cohomology class which satisfies MS and NCT. Let $(g, g', \omega)$ be a system as in Theorem 1 so that $X = -\text{grad}_g \omega$ and $\omega$ a Morse form representing $X$. Let $\log \hat{T}_{an}^{X, \omega, g}(t)$ be the function defined by (9).

If in addition $X$ has exponential growth and $H^*(M, t[\omega]) = 0$ for $t$ sufficiently large or $X$ has strong exponential growth then the Dirichlet series $I_X$ has finite abscissa of convergence and its Laplace transform is exactly the function $\log \hat{T}_{an}^{X, \omega, g}(t)$. In particular $\log \hat{T}_{an}^{X, \omega, g}(t)$ is the restriction of a holomorphic function on $\{ z \in \mathbb{C} | \Re z > T \}$. 


(One can replace the acyclicity hypothesis by "\(H^*_{\text{sing}}(M; \Lambda_{\xi, \rho})\) is a free \(\Lambda_{\xi, \rho}\)-module for \(\rho\) large enough").

If the conjecture in section 3.2 is true, then the additional hypothesis (exponential growth resp. strong exponential growth) are superfluous.

Remark 2. The Dirichlet series \(Z_X\) depends only on \(X\) and \(\xi = [\omega]\), while \(r^{X, O, \xi}_{\tau, y}\) depends only on \(X\) and \(\xi\) up to multiplication with a constant (with a real number \(r\) for the sequence of \(\lambda\)'s and with \(\epsilon = \pm 1\) for the sequence of \(a\)'s).

Corollary 1 (J. Marcsik cf. [11] or [7]). Suppose \(X\) is a vector field with no rest points, \(\xi \in H^1(M; \mathbb{R})\) a Lyapunov class for \(X\), \(\omega\) a closed one form representing \(\xi\) and let \(g\) a Riemannian metric on \(M\). Suppose all closed trajectories of \(X\) are non-degenerate and denote by

\[
\log T_{\text{an}}(t) := 1/2 \sum (-1)^{q+1} q \log \det(\Delta^q(t)).
\]

Then

\[
\log T_{\text{an}}(t) + t \int_M \omega \wedge X^*\Psi_g
\]

is the Laplace transform of the Dirichlet series \(Z_X\) which counts the set of closed trajectories of \(X\) with the help of \(\xi\).

Remark 3. In case that \(M\) is the mapping torus of a diffeomorphism \(\phi : N \to N\), \(M = N_\phi\) whose periodic points are all non-degenerate, the Laplace transform of the Dirichlet series \(Z_X\) is the Lefschetz zeta function \(\text{Lef}(\phi)\) of \(\phi\), with the variable \(Z\) replaced by \(e^{-Z}\).

Theorems 3, 4 and Corollary 1 can be routinely extended to the case of a compact manifolds with boundary.

In section 2 we discuss one of the main topological tools in this paper, the completion of the unstable sets and of the space of unparameterized trajectories, cf. Theorem 5. This theorem was also proved in [18]. In this paper we provide a significant short cut in the proof and a slightly more general formulation.

In section 3 we define the invariant \(\rho\) and discuss the relationship with the exponential growth property. Additional results of independent interest pointing toward the truth of the conjecture in section 3.2 are also proved. The results of this section are not needed for the proofs of Theorems 2–4.

The proof of Theorem 1 as stated is contained in [6] and so is the proof of Theorem 2 but in a slightly different formulation and (apparently) less generality. For this reason and for the sake of completeness we will review and complete the arguments (with proper references to [18] when necessary) in section 4. Section 4 contains the proof of Theorem 2 and 3. Section 5 treats the numerical invariant \(R(X, \omega, g)\). The proof of Theorem 4 is presented in section 6 and relies on some previous work of Hutchings–Lee, Pajitnov [10], [9], [16] and the work of Bismut–Zhang and Burghelea–Friedlander–Kappeler [1].

2. Topology of the space of trajectories and unstable sets

In this section we discuss the completion of the unstable manifolds and of the manifolds of trajectories to manifolds with corners, which is a key topological tool in this work. The main result, Theorem 5 is of independent interest.
Definition 5. Suppose $\xi \in H^1(M; \mathbb{R})$. We say a covering $\pi : \tilde{M} \to M$ satisfies property $P$ with respect to $\xi$ if $\tilde{M}$ is connected and $\pi^*\xi = 0$.

Let $X$ be vector field on a closed manifold $M$ which has $\xi \in H^1(M; \mathbb{R})$ as a Lyapunov cohomology class, see Definition 2. Suppose that $\pi : \tilde{M} \to M$ be a covering satisfying property P with respect to $\xi$. Since $\xi$ is Lyapunov there exists a closed one form $\omega$ representing $\xi$ and a Riemannian metric $g$ so that $X = -\operatorname{grad}_g \omega$. Since the covering has property P we find $h : \tilde{M} \to \mathbb{R}$ with $\pi^*\omega = dh$.

Denote by $\tilde{X}$ the vector field $\tilde{X} := \pi^*X$. We write $\tilde{X} = \pi^{-1}(X)$ and $\tilde{X}_q = \pi^{-1}(X_q)$. Clearly $\operatorname{Cr}(h) = \pi^{-1}(\operatorname{Cr}(\omega))$ are the zeros of $\tilde{X}$.

Given $\tilde{x} \in \tilde{X}$ let $i_+^x : W_x^+ \to \tilde{M}$ and $i_-^x : W_x^- \to \tilde{M}$, denote the one to one immersions whose images define the stable and unstable sets of $\tilde{x}$ with respect to the vector field $\tilde{X}$. The maps $i_+^x$ and $i_-^x$ are actually smooth embeddings because $\tilde{X}$ is gradient like for the function $h$, and the manifold topology on $W_x^\pm$ coincides with the topology induced from $\tilde{M}$. For any $\tilde{x}$ with $\pi(\tilde{x}) = x$ one can canonically identify $W_x^+ \to W_x^\pm$ and then we have $\pi \circ i_+^x = i_+^x$.

As the maps $i_+^x$ and $i_-^x$ are transversal, $\mathcal{M}(\tilde{x}, \tilde{y}) := W_x^- \cap W_{\tilde{y}}^+$ is a submanifold of $\tilde{M}$ of dimension $\operatorname{ind}(\tilde{x}) - \operatorname{ind}(\tilde{y})$. The manifold $\mathcal{M}(\tilde{x}, \tilde{y})$ is equipped with the action $\mu : \mathbb{R} \times \mathcal{M}(\tilde{x}, \tilde{y}) \to \mathcal{M}(\tilde{x}, \tilde{y})$, defined by the flow generated by $\tilde{X}$. If $\tilde{x} \neq \tilde{y}$ the action $\mu$ is free and we denote the quotient $\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$ by $\mathcal{T}(\tilde{x}, \tilde{y})$. The quotient $\mathcal{T}(\tilde{x}, \tilde{y})$ is a smooth manifold of dimension $\operatorname{ind}(\tilde{x}) - \operatorname{ind}(\tilde{y}) - 1$, possibly empty, which, in view of the fact that $\tilde{X}(h) = \omega(X) < 0$ is diffeomorphic to the submanifold $h^{-1}(c) \cap \mathcal{M}(\tilde{x}, \tilde{y})$, where $c$ is any regular value of $h$ with $h(\tilde{x}) > c > h(\tilde{y})$.

Note that if $\operatorname{ind}(\tilde{x}) \leq \operatorname{ind}(\tilde{y})$, and $\tilde{x} \neq \tilde{y}$, in view the transversality required by the Hypothesis MS, the manifolds $\mathcal{M}(\tilde{x}, \tilde{y})$ and $\mathcal{T}(\tilde{x}, \tilde{y})$ are empty. We make the following convention: $\mathcal{T}(\tilde{x}, \tilde{y}) := \emptyset$. This is very convenient for now $\mathcal{T}(\tilde{x}, \tilde{y}) \neq \emptyset$ implies $\operatorname{ind}(\tilde{x}) > \operatorname{ind}(\tilde{y})$ and in particular $\tilde{x} \neq \tilde{y}$.

An unparameterized broken trajectory from $\tilde{x} \in \tilde{X}$ to $\tilde{y} \in \tilde{X}$, is an element of the set $\mathcal{B}(\tilde{x}, \tilde{y}) := \bigcup_{k \geq 0} \mathcal{B}(\tilde{x}, \tilde{y})_k$, where

$$\mathcal{B}(\tilde{x}, \tilde{y})_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1})$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{X}$ with $\tilde{y}_0 = \tilde{x}$ and $\tilde{y}_{k+1} = \tilde{y}$.

For $\tilde{x} \in \tilde{X}$ introduce the completed unstable set $\tilde{W}_x^- := \bigcup_{k \geq 0} (\tilde{W}_x^-)_k$, where

$$(\tilde{W}_x^-)_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times \tilde{W}_{\tilde{y}_k}$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{X}$ with $\tilde{y}_0 = \tilde{x}$.

To study $\tilde{W}_x^-$ we introduce the set $\mathcal{B}(\tilde{x}; \lambda)$ of unparameterized broken trajectories from $\tilde{x} \in \tilde{X}$ to the level $\lambda \in \mathbb{R}$ as $\mathcal{B}(\tilde{x}; \lambda) := \bigcup_{k \geq 0} \mathcal{B}(\tilde{x}; \lambda)_k$ where

$$\mathcal{B}(\tilde{x}; \lambda)_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times (\tilde{W}_{\tilde{y}_k} \cap h^{-1}(\lambda))$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{X}$ with $\tilde{y}_0 = \tilde{x}$. Clearly, if $\lambda > h(\tilde{x})$ then $\mathcal{B}(\tilde{x}; \lambda) = \emptyset$.

Since any broken trajectory of $\tilde{X}$ intersects each level of $h$ in at most one point one can view the set $\mathcal{B}(\tilde{x}, \tilde{y})$ resp. $\mathcal{B}(\tilde{x}; \lambda)$ as a subset of $C^0([h(\tilde{y}), h(\tilde{x})], \tilde{M})$ resp. $C^0([\lambda, h(\tilde{x})], \tilde{M})$. One parameterizes the points of a broken trajectory by the value
of the function \( h \) on these points. This leads to the following characterization (and implicitly to a canonical parameterization) of an unparameterized broken trajectory.

**Remark 4.** Let \( \tilde{x}, \tilde{y} \in \tilde{X} \) and set \( a := h(\tilde{y}), b := h(\tilde{x}) \). The parameterization above defines a one to one correspondence between \( \mathcal{B}(\tilde{x}, \tilde{y}) \) and the set of continuous mappings \( \gamma : [a, b] \to \tilde{M} \), which satisfy the following two properties:

(i) \( h(\gamma(s)) = a + b - s, \gamma(a) = \tilde{x} \) and \( \gamma(b) = \tilde{y} \).

(ii) There exists a finite collection of real numbers \( a = s_0 < s_1 < \cdots < s_{r-1} < s_r = b \), so that \( \gamma(s_i) \in \tilde{X} \) and \( \gamma \) restricted to \( (s_i, s_{i+1}) \) has derivative at any point in the interval \((s_i, s_{i+1})\), and the derivative satisfies

\[
\gamma'(s) = \frac{\tilde{X}}{-X \cdot h}(\gamma(s)).
\]

Similarly the elements of \( \mathcal{B}(\tilde{x}; \lambda) \) correspond to continuous mappings \( \gamma : [\lambda, b] \to \tilde{M} \), which satisfies \((i)\) and \((ii)\) with \( a \) replaced by \( \lambda \) and the condition \( \gamma(b) = \tilde{y} \) ignored.

We have the following proposition, which can be found in [6].

**Proposition 5.** For any \( \tilde{x}, \tilde{y} \in \tilde{X} \) and \( \lambda \in \mathbb{R} \), the spaces

(i) \( \mathcal{B}(\tilde{x}, \tilde{y}) \) with the topology induced from \( C^0(\gamma(\tilde{y}), h(\tilde{x})), \tilde{M}) \), and

(ii) \( \mathcal{B}(\tilde{x}; \lambda) \) with the topology induced from \( C^0(\lambda, h(\tilde{x})), \tilde{M}) \)

are compact.

Let \( \hat{i}_{\tilde{x}} : \hat{W}^-_{\tilde{x}} \to \hat{M} \) denote the map whose restriction to \( \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times \hat{W}^-_{\tilde{y}_k} \) is the composition of the projection on \( \hat{W}^-_{\tilde{y}_k} \) with \( i_{\tilde{y}_k} \). Moreover let \( \tilde{h} := h_{\tilde{x}} \circ \hat{i}_{\tilde{x}} : \hat{W}^-_{\tilde{x}} \to \mathbb{R} \), where \( h_{\tilde{x}} = h - h(\tilde{x}) \).

Recall that an \( n \)-dimensional manifold with corners \( P \), is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts \( \varphi : U \to \varphi(U) \subseteq \mathbb{R}^n_+ \), where \( \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \} \). The collection of points of \( P \) which correspond by (some and hence every) chart to points in \( \mathbb{R}^n \) with exactly \( k \) coordinates equal to zero is a well defined subset of \( P \) called the \( k \)-corner of \( P \) and it will be denoted by \( P_k \). It has a structure of a smooth \((n-k)\)-dimensional manifold. The union \( \partial P = P_1 \cup P_2 \cup \cdots \cup P_n \) is a closed subset which is a topological manifold and \((P, \partial P)\) is a topological manifold with boundary \( \partial P \).

The following theorem was proven in [11] for the case that \( \tilde{M} \) is the minimal covering which has property \( P \).

**Theorem 5.** Let \( M \) be a closed manifold, \( X \) a vector field which is MS and suppose \( \xi \) is a Lyapunov class for \( X \). Let \( \pi : \tilde{M} \to M \) be a covering which satisfies property \( P \) with respect to \( \xi \) and let \( h : \tilde{M} \to \mathbb{R} \) be a smooth map as above. Then:

(i) For any two rest points \( \tilde{x}, \tilde{y} \in \tilde{X} \) the smooth manifold \( \mathcal{T}(\tilde{x}, \tilde{y}) \) has \( \mathcal{B}(\tilde{x}, \tilde{y}) \) as a canonical compactification. Moreover there is a canonical way to equip \( \mathcal{B}(\tilde{x}, \tilde{y}) \) with the structure of a compact smooth manifold with corners, whose \( k \)-corner is \( \mathcal{B}(\tilde{x}, \tilde{y})_k \) from (11).

(ii) For any rest point \( \tilde{x} \in \tilde{X} \), the smooth manifold \( \hat{W}^-_{\tilde{x}} \) has \( \hat{W}^-_{\tilde{x}} \) as a canonical completion. Moreover there is a canonical way to equip \( \hat{W}^-_{\tilde{x}} \) with the structure of a smooth manifold with corners, whose \( k \)-corner coincides with \( (W^-_{\tilde{x}})_k \) from (11).

(iii) \( \hat{i}_{\tilde{x}} : \hat{W}^-_{\tilde{x}} \to \tilde{M} \) is smooth and proper, for all \( \tilde{x} \in \tilde{X} \).
(iv) \( \hat{h}_x : \hat{W}_x^- \rightarrow \mathbb{R} \) is smooth and proper, for all \( \hat{x} \in \hat{X} \).

Proof. In view of Lemma 4 in section 3, the set of Lyapunov classes for \( X \) is open in \( H^1(M; \mathbb{R}) \). So we can find a closed one form \( \omega \) and a Riemannian metric \( g \) such that \( X = -\text{grad}_g \omega \) and such that \( \omega \) has degree of rationality one. Consider the minimal covering on which \( \xi = [\omega] \) becomes exact. Since \( \xi \) has degree of rationality one the critical values of \( h \) form a discrete set. Recall that the closed one form \( \omega \) has degree of rationality \( k \) if the image of \([\omega](\Gamma) \subset \mathbb{R}\) is a free abelian group of rank \( k \). In [5, paragraphs 4.1–4.3] one can find all details of the proof of Theorem 5 for this special \( \xi \) and this special covering.

Note that as long as properties (i) through (iii) are concerned they clearly remain true when we pass to the universal covering of \( M \) which obviously has property \( P \). One easily concludes that they also remain true for every covering which has property \( P \). So we have checked (i) through (iii) in the general situation.

Next observe that \( \hat{h}_x = h^x \circ \hat{i}_x^- \) is certainly smooth as a composition of two smooth mappings. The properness of \( h_x \) follows from Proposition 5(ii). \( \square \)

It will be convenient to formulate Theorem 5 without any reference to the covering \( \pi : \hat{M} \rightarrow M \) or to lifts \( \hat{x} \) of rest points \( x \).

Let \( \xi \in H^1(M; \mathbb{R}) \) be a one dimensional cohomology class so that \( \pi^\ast \xi = 0 \). As in section 4 denote by \( P_{x,y} \) the set of continuous paths from \( x \) to \( y \) and by \( P^\hat{M}_{\hat{x},\hat{y}} \) the equivalence classes of paths in \( P_{\hat{x},\hat{y}} \) with respect to the following equivalence relation.

**Definition 6.** Two paths \( \alpha, \beta \in P_{x,y} \) are equivalent if for some (and then for any) lift \( \hat{x} \) of \( x \) the lifts \( \hat{\alpha} \) and \( \hat{\beta} \) of \( \alpha \) and \( \beta \) originating from \( \hat{x} \) end up in the same point \( \hat{y} \).

The reader might note that the present situation is slightly more general than the one considered in introduction which correspond to the case the covering \( \pi \) is the \( \Gamma \)-principal covering with \( \Gamma \) induced from \( \xi \) as described in section 1. For this covering we have \( \hat{P}^\hat{M}_{\hat{x},\hat{y}} = \hat{P}_\xi^{\hat{M}_{\hat{x},\hat{y}}} \).

Note that any two lifts \( \hat{x}, \hat{y} \in \hat{M} \) determine an element \( \hat{\alpha} \in \hat{P}^\hat{M}_{\hat{x},\hat{y}} \) and the set of trajectories from \( \hat{x} \) to \( \hat{y} \) identifies to the set \( T(x,y,\hat{\alpha}) \) of trajectories of \( X \) from \( x \) to \( y \) in the class \( \hat{\alpha} \).

Theorem 5 can be reformulated in the following way:

**Theorem 6** (Reformulation of Theorem 5). Let \( M \) be a smooth manifold, \( X \) a smooth vector field which is MS and suppose \( \xi \) is a Lyapunov class for \( X \). Let \( M \) be a covering of \( \hat{M} \) which has property \( P \) with respect to \( \xi \). Then:

(i) For any two rest points \( x, y \in \mathcal{X} \) and every \( \hat{\alpha} \in \hat{P}^\hat{M}_{\hat{x},\hat{y}} \) the set \( T(x,y,\hat{\alpha}) \) has the structure of a smooth manifold of dimension \( \text{ind}(x) - \text{ind}(y) - 1 \) which admits a canonical compactification to a compact smooth manifold with corners \( B(x,y,\hat{\alpha}) \). Its \( k \)-corner is

\[
B(x,y,\hat{\alpha})_k = \bigcup T(y_0,y_1,\hat{\alpha}_0) \times \cdots \times T(y_k,y_{k+1},\hat{\alpha}_k)
\]

where the union is over all (tuples of) critical points \( y_i \in \mathcal{X} \) and \( \hat{\alpha}_i \in \hat{P}^\hat{M}_{\hat{y}_i,\hat{y}_{i+1}} \) with \( y_0 = x \), \( y_{k+1} = y \) and \( \hat{\alpha}_0 \ast \cdots \ast \hat{\alpha}_k = \hat{\alpha} \).
(ii) For any rest point \( x \in \mathcal{X} \) the smooth manifold \( W^x_- \) has a canonical completion to a smooth manifold with corners \( \tilde{W}^-_x \). Its \( k \)-corner is
\[
(W^x_x)_k = \bigcup \mathcal{T}(y_0, y_1, \hat{\alpha}_0) \times \cdots \times \mathcal{T}(y_{k-1}, y_k, \hat{\alpha}_{k-1}) \times W^-_{y_k}
\]
where the union is over all (tuples of) critical points \( y_i \in \mathcal{X} \) and \( \hat{\alpha}_i \in \mathcal{P}^M_{y_{i-1}, y_i} \) with \( y_0 = x \).

(iii) The mapping \( i^-_x : \tilde{W}^-_x \to M \) which on \( (W^-_x)_k \) is given by the composition of the projection onto \( W^-_{y_k} \) with \( i^-_{y_k} : W^-_{y_k} \to M \) is smooth, for all \( x \in \mathcal{X} \).

(iv) Let \( \omega \) be a closed one form representing \( \xi \). Then the mappings \( \hat{h}_x : W^-_x \to \mathbb{R} \) which on \( (W^-_x)_k \) is given by the composition of the projection onto \( W^-_{y_k} \) with \( h^\prime_{y_k} : W^-_{y_k} \to \mathbb{R} \) plus \( \alpha \) is smooth and proper, for all \( x \in \mathcal{X} \).

The above results can be easily extended to the case of compact manifolds with boundary.

2.1. Appendix to section 2. Given a compact smooth manifold \( M \) with boundary \( \partial M \) we will consider only admissible metrics, i.e. Riemannian metrics \( g \) which are product like near the boundary. In this case denote by \( g_0 \) the induced metric on the boundary. This means that there exists a collar neighborhood \( \varphi : \partial M \times [0, \epsilon) \to M \) with \( \varphi \) equal to the identity when restricted to \( \partial M \times \{0\} \) and \( \varphi^* g = g_0 + ds^2 \).

Convention. Unless explicitly mentioned in this paper all the vector fields on a compact manifold with boundary are assumed to be tangent to the boundary and have rest points of Morse type.

Definition 7. The vector field \( X \) has \( \xi \in H^1(M; \mathbb{R}) \) as Lyapunov cohomology class if the following conditions hold:

(i) There exists a closed one form representing \( \xi \) and an admissible metric so that \( X = -\text{grad}_g \omega \). In particular \( X_{\partial M} = -\text{grad}_{g_0} \omega_{\partial M} \), where \( \omega_{\partial M} \) denotes the pullback of \( \omega \) to \( \partial M \).

(ii) If we set
\[
\mathcal{X}'' := \{ x \in \mathcal{X} \cap \partial M | \text{ind}_{\partial M}(x) = \text{ind}(x) \}
\]
\[
\mathcal{X}''' := \{ x \in \mathcal{X} \cap \partial M | \text{ind}_{\partial M}(x) = \text{ind}(x) - 1 \}
\]
then \( \mathcal{X}'' \) resp. \( \mathcal{X}''' \) lie in different components \( \partial M_\pm \) resp. \( \partial M_0 \) of \( M \).

This definition implies that \( \mathcal{X} = \mathcal{X}' \cup \mathcal{X}'' \), where \( \mathcal{X}' \) is the set of rest points inside \( M \) and \( \mathcal{X}'' \) of the rest points on \( \partial M \) which is the same as the set of rest points of \( X_{\partial M} \). For \( x \in \mathcal{X}'' \) denote by \( i^-_x : \tilde{W}^-_{x} \to M \) the unstable manifold with respect to \( X \) and by \( j^-_x : W^\prime_{\partial M,x} \to \partial M \) the unstable manifold with respect to \( X_{\partial M} \).

Remark 5.

(i) If \( x \in \mathcal{X}'' \) then the unstable manifold of \( x \) with respect to \( X \) and \( X_{\partial M} \) are the same. More precisely \( i^-_x : \tilde{W}^-_{x} \to M \) identifies to \( j^-_x : W^\prime_{\partial M,x} \to \partial M \) followed by the inclusion of \( \partial M \subset M \).

(ii) If \( x \in \mathcal{X}''' \) then
\[
\text{(a)} \quad (W^-_x, W^\prime_{\partial M,x}) \text{ is a smooth manifold with boundary diffeomorphic to } (\mathbb{R}_+, \mathbb{R}^{k-1}) \text{ with } k = \text{ind}(x); \text{ and}
\]
(b) \( i^-_M : W^-_M \to M \) is transversal to the boundary of \( M \) and \( i^-_M : (i^-_M)^{-1}(\partial M) \to \partial M \) can be identified to \( j^-_M : W^-_{\partial M,x} \to \partial M \).

Theorems 5 and 6 remain true as stated with the following specifications. Set \( P_y^- := W^-_y \setminus W^-_{\partial M,y} \) for \( y \in X'_n \), and \( P_y^- := W^-_y \) for \( y \in X' \cup X'' \). For \( x \in X'' \) the \( k \)-corner of \( W^-_x \) then is

\[
(W^-_x)_k = (\hat{W}_{\partial M,x})_{k-1} \cup \bigcup T(y_0, y_1, \hat{\alpha}_0) \times \cdots \times T(y_k-1, y_k, \hat{\alpha}_{k-1}) \times P^-_{y_k}
\]

where the big union is over all (tuples of) \( y_i \in X \) and \( \hat{\alpha}_i \in \hat{\mathcal{P}}_{y_i,y_{i+1}} \) with \( y_0 = x \).

3. Exponential growth property and the invariant \( \rho \)

In this section we introduce for a pair \((X, \xi)\) consisting of a vector field \( X \) and a cohomology class \( \xi \in H^1(M; \mathbb{R}) \) an invariant \( \rho(\xi, X) \in \mathbb{R} \cup \{\pm \infty\} \). For the purpose of this paper we are interested in the case this invariant is smaller than \( \infty \). One expects that this is always the case if \( \xi \) is Lyapunov for \( X \) at least in the case \( X \) satisfies MS. If \( X \) has \( \xi \) as a Lyapunov cohomology class we prove that the exponential growth and \( \rho < \infty \) are equivalent. The discussion of this section is not needed for the proofs of Theorem 2, 3 and 4.

Throughout this section \( M \) will be a closed manifold and \( X \) a vector field with Morse zeros.

3.1. The invariant \( \rho \). For a critical point \( x \in X \), i.e. a zero of \( X \), we let \( i^-_x : W^-_x \to M \) denote the smooth immersion of the unstable manifold into \( M \). If \( M \) is equipped with a Riemannian metric we get an induced Riemannian metric \( g_x := (i^-_x)^*g \) on \( W^-_x \) thus a volume density \( \mu(g_x) \) on \( W^-_x \) and hence the spaces \( L^p(W^-_x) \) for \( p \geq 1 \). Though the \( L^p \)-norm depends on the metric \( g \) the space \( L^p(W^-_x) \) and its topology does not. Indeed for another Riemannian metric \( g' \) on \( M \) we find a constant \( C > 0 \) so that \( 1/C \leq \frac{\mu(g_x)}{\mu(g'_x)} \leq C \) for all tangent vectors \( X \) and \( Y \) which implies \( 1/C' \leq \frac{\mu(g_x)}{\mu(g'_x)} \leq C' \) for some constant \( C' > 0 \).

Given a closed 1–form \( \omega \) on \( M \) we let \( h^\omega_x \) denote the unique smooth function on \( W^-_x \) which satisfies \( dh^\omega_x = (i^-_x)^*\omega \) and \( h^\omega_x(x) = 0 \). We are interested in the space of 1–forms for which \( e^{b\omega_x} \in L^1(W^-_x) \). This condition actually only depends on the cohomology class of \( \omega \). Indeed we have \( h^{\omega+x+df} = h^\omega_x + (i^-_x)^*f - f(x) \) and so \( |h^{\omega+x+df} - h^\omega_x| \leq C'' \) and \( e^{-C''} \leq e^{b\omega_x} / e^{b\omega_x} \leq e^{C''} \) for some constant \( C'' > 0 \). So we define

\[
R_x(X) := \{ [\omega] \in H^1(M) \mid e^{b\omega_x} \in L^1(W^-_x) \}
\]

and set \( R(X) := \bigcap x \in \text{Crit}(X) R_x(X) \). Let us also define

\[
\rho_x(\xi, X) := \inf \{ t \in \mathbb{R} \mid t\xi \in R_x(X) \} \in \mathbb{R} \cup \{\pm \infty\}
\]

as well as:

\[
\rho(\xi, X) := \inf \{ t \in \mathbb{R} \mid t\xi \in R(X) \} \in \mathbb{R} \cup \{\pm \infty\}.
\]

Observe that \( \rho(\xi, X) = \max_{x \in X} \rho_x(\xi, X) \).

**Lemma 1.** The sets \( R_x(X) \) and \( R(X) \) are convex. Particularly

\[
\rho(\lambda \xi_1 + (1 - \lambda) \xi_2, X) \leq \max \{ \rho(\xi_1, X), \rho(\xi_2, X) \}
\]

for all \( 0 \leq \lambda \leq 1 \).
Proof. Indeed let \([\omega_0], [\omega_1] \in H^1(M), \lambda \in [0, 1]\) and set \(\omega_\lambda := \lambda \omega_0 + (1 - \lambda) \omega_1\). Then \(h^\omega_\lambda = \lambda h^\omega_0 + (1 - \lambda) h^\omega_1\). For \(\lambda \in (0, 1)\) we set \(p := 1/\lambda > 1\) and \(q := 1/(1 - \lambda)\). Then \(1/p + 1/q = 1\) and by Hölder’s inequality
\[
\|e^{h^\omega_\lambda}\|_1 = \|e^{\lambda h^\omega_0} e^{(1 - \lambda) h^\omega_1}\|_1 \\
\leq \|e^{\lambda h^\omega_0}\|_p \|e^{(1 - \lambda) h^\omega_1}\|_q \\
= \|e^{h^\omega_0}\|_1^p \|e^{h^\omega_1}\|_1^{1 - \lambda}
\]
So, if \([\omega_0] \text{ and } [\omega_1] \in R_x(X)\) then \([\omega_\lambda] \in R_x(X)\), and thus \(R_x(X)\) is convex. As an intersection of convex sets \(R(X)\) is convex too.

Next we introduce:
\[
B_x(X) := \{ [\omega] \in \Omega^1(M) \mid e^{h^\omega} \in L^\infty(W^-) \}
\]
and set \(B(X) := \bigcap_{x \in \text{Crit}(X)} B_x(X)\). Note if \(\xi\) is a Lyapunov cohomology class for \(X\) then \(\xi \in B(X)\), cf. Lemma 4 below.

Most obviously we have:

**Lemma 2.** The sets \(B_x(X)\) and \(B(X)\) are convex cones. Moreover we have \(R_x(X) + B_x(X) \subseteq R_x(X)\) and \(R(X) + B(X) \subseteq R(X)\).

Next define
\[
L(X) := \{ \xi \in \Omega^1(M) \mid \xi \text{ is Lyapunov class for } X \}
\]
Recall from Definition 2 that \(\xi \in L(X)\) if there exists a closed one form \(\omega\) representing \(\xi\) and a Riemannian metric \(g\) such that \(X = -\text{grad}_g \omega\).

**Lemma 3.** Let \(M\) be a smooth manifold, \(X\) a vector field, \(\omega\) a closed one form and \(g\) a Riemannian metric. Suppose \(U \subset M\) is an open set and

(i) the vector fields \(X\) and \(\text{grad}_g \omega\) agree on \(U\) and

(ii) \(\omega(X) < 0\) on a neighborhood of \(M \setminus U\).

Then there exists a Riemannian metric \(g'\) so that:

(i) \(X = -\text{grad}_{g'} \omega\)

(ii) \(g\) and \(g'\) agree on \(U\).

**Proof.** Let \(N\) be an open neighborhood of \(M \setminus U\) so that \(\omega(X) < 0\) and therefore \(X_x \neq 0, x \in N\). For \(x \in N\) the tangent space \(T_x\) decomposes as the direct sum \(T_x = V_x \oplus [X_x]\) where \([X_x]\) denotes the one dimensional vector space generated by \(X_x\) and \(V_x = \ker(\omega(x) : T_xM \to \mathbb{R})\). Clearly on \(U\) the function \(-\omega(X)\) is the square of the length of \(X_x\) with respect to the metric \(g\) and \(X_x\) is orthogonal to \(V_x\) and on \(N\) it is strictly negative. Define a new Riemannian metric \(g'\) on \(M\) as follows: For \(x \in U\) the scalar product in \(T_xM\) is the same as the one defined by \(g\). For \(x \in N\) the scalar product on \(T_xM\) agrees to the one defined by \(g\) but make \(V_x\) and \([X_x]\) perpendicular and the length of \(X_x\) equal to \(\sqrt{-\omega(X)(x)}\). It is clear that the new metric is well defined and smooth.

**Corollary 2.** Let \(X\) be a vector field on \(M\) and let \(\xi \in \Omega^1(M)\). Then \(\xi\) is Lyapunov for \(X\) if and only if there is a closed one form \(\omega\) representing \(\xi\) and a Riemannian metric \(g\) such that the following hold:

(i) \(\omega(X) < 0\) on \(M \setminus X\).

(ii) \(X = -\text{grad}_g \omega\) on a neighborhood of \(X\).
denote the distance to $x$. Exponential growth versus manifold, let Corollary 3. Suppose $\xi$. On the other hand, starting at the origin which is contained in $L$. Pick $r$.$\xi$.$X$. Proof. Suppose Lemma 5. This does not depend on the Riemannian metric $g$. We have $\xi$. Lemma 6. Suppose we have $\xi$. Next note that both conditions (i) and (ii) in Corollary 2 are convex and homogeneous conditions on $\omega$. Thus $L(X)$ is a convex cone. $\Box$

**Corollary 3.** Suppose $\xi_0$ and $\xi$ are Lyapunov for $X$. Then $\rho(\xi_0, X) < \infty$ implies $\rho(\xi, X) < \infty$.

### 3.2. Exponential growth versus $\rho$.

Let $x \in X$ be a zero of $X$, $W_x^-$ the unstable manifold, let $g$ be a Riemannian metric on $M$ and let $r := \text{dist}(x, \cdot) : W_x^- \to [0, \infty)$ denote the distance to $x$ with respect to the induced metric $g_x = (i_x^-)^*g$ on $W_x^-$. Clearly $r(x) = 0$. Moreover let $B_s(x) := \{y \in W_x^- | r(y) \leq s\}$ denote the ball of radius $s$ centered at $x$.

Recall from Definition 11 that $X$ has the exponential growth property at a zero $x$ if there exists a constant $C \geq 0$ such that $\text{Vol}(B_s(x)) \leq e^{Cs}$ for all $s \geq 0$. Clearly this does not depend on the Riemannian metric $g$ on $M$ even though the constant $C$ will depend on $g$.

**Lemma 6.** Suppose we have $C \geq 0$ such that $\text{Vol}(B_s(x)) \leq e^{Cs}$ for all $s \geq 0$. Then $e^{-(C+\epsilon)r} \in L^1(W_x^-)$ for every $\epsilon > 0$.

**Proof.** We have

$$\int_{W_x^-} e^{-(C+\epsilon)r} = \sum_{n=0}^{\infty} \int_{B_{n+1}(x) \setminus B_n(x)} e^{-(C+\epsilon)r}$$

(13)

On $B_{n+1}(x) \setminus B_n(x)$ we have $e^{-(C+\epsilon)r} \leq e^{-(C+\epsilon)n}$ and thus

$$\int_{B_{n+1}(x) \setminus B_n(x)} e^{-(C+\epsilon)r} \leq \text{Vol}(B_{n+1}(x)) e^{-(C+\epsilon)n} \leq e^{C(n+1)} e^{-(C+\epsilon)n} = e^C e^{-\epsilon n}$$

So (13) implies

$$\int_{W_x^-} e^{-(C+\epsilon)r} \leq e^C \sum_{n=0}^{\infty} e^{-\epsilon n} = e^C (1 - e^{-\epsilon})^{-1} < \infty$$

and thus $e^{-(C+\epsilon)r} \in L^1(W_x^-)$. $\Box$
Lemma 7. Suppose we have $C \geq 0$ such that $e^{-Cr} \in L^1(W^-)$. Then there exists a constant $C_0$ such that $\text{Vol}(B_s(x)) \leq C_0e^{Cs}$ for all $s \geq 0$.

Proof. We start with the following estimate for $N \in \mathbb{N}$:

$$\text{Vol}(B_{N+1}(x))e^{-C(N+1)} = \sum_{n=0}^{N} \text{Vol}(B_{n+1}(x))e^{-C(n+1)} - \text{Vol}(B_n(x))e^{-Cn}$$

$$\leq \sum_{n=0}^{\infty} (\text{Vol}(B_{n+1}(x)) - \text{Vol}(B_n(x)))e^{-C(n+1)}$$

$$= \sum_{n=0}^{\infty} \text{Vol}(B_{n+1}(x) \setminus B_n(x))e^{-C(n+1)}$$

$$\leq \sum_{n=0}^{\infty} \int_{B_{n+1}(x) \setminus B_n(x)} e^{-Cr} = \int_{W^-} e^{-Cr}$$

Given $s \geq 0$ we choose an integer $N$ with $N \leq s \leq N+1$. Then $\text{Vol}(B_s(x))e^{-Cs} \leq \text{Vol}(B_{N+1}(x))e^{-CN} = e^{C} \text{Vol}(B_{N+1}(x))e^{-C(N+1)}$. So the computation above shows

$$\text{Vol}(B_s(x))e^{-Cs} \leq e^{C} \int_{W^-} e^{-Cr} =: C_0 < \infty$$

and thus $\text{Vol}(B_s(x)) \leq C_0e^{Cs}$ for all $s \geq 0$. \hfill \square

As immediate consequence of the two preceding lemmas we have

Proposition 6. Let $x$ be a zero of $X$. Then the following are equivalent:

(i) $X$ has the exponential growth property at $x$ with respect to one (and hence every) Riemannian metric on $M$.

(ii) For one (and hence every) Riemannian metric on $M$ there exists a constant $C \geq 0$ such that $e^{-Cr} \in L^1(W^-)$.

Let $g$ be a Riemannian metric on $M$, $\omega$ a closed one form and consider $X = -\text{grad}_g \omega$. Assume $X$ has all zeros of Morse type and let $x$ be one of them. Recall that we have a smooth function $h^\omega: W^- \to (-\infty, 0]$ defined by $(i_x^\omega)^*\omega = dh^\omega_x$ and $h^\omega_x(x) = 0$. The next two lemmas tell, that $-h^\omega: W^- \to [0, \infty)$ is comparable with $r : W^- \to [0, \infty)$.

Lemma 8. In this situation there exists a constant $C_{\omega, g} \geq 0$ such that $r \leq 1 - C_{\omega, g}h^\omega_x$ on $W^-$.\hfill \square

Proof. The proof is exactly the same as the one in [3, Lemma 3(2)]. Note that the MS property is not used there.

Lemma 9. In this situation there exists a constant $C'_{\omega, g} \geq 0$ such that $-h^\omega \leq C'_{\omega, g}r$.\hfill \square

Proof. Let $\gamma : [0, 1] \to W^-$ be any path starting at $\gamma(0) = x$. For simplicity set $h := h^\omega$. Since $h(x) = 0$ we find

$$|h(\gamma(1))| = \left| \int_{0}^{1} (dh)(\gamma'(t))dt \right| \leq ||\omega||_\infty \int_{0}^{1} |\gamma'(t)|dt = ||\omega||_\infty \text{length}(\gamma)$$
with $||\omega||_{\infty}$ the supremum norm of $\omega$. We conclude

$$||\omega||_{\infty} r(\gamma(1)) = ||\omega||_{\infty} \text{dist}(x, \gamma(1)) \geq |h(\gamma(1))| \geq -h(\gamma(1))$$

and thus $-h \leq C_{\omega, g}^r$ with $C_{\omega, g}^r := ||\omega||_{\infty}$. □

Let us collect what we have found so far.

**Proposition 7.** Suppose $\xi$ is Lyapunov for $X$ and let $x$ be a zero of $X$. Then the following are equivalent.

(i) $X$ has the exponential growth property at $x$ with respect to one (and hence every) Riemannian metric on $M$.

(ii) For one (and hence every) Riemannian metric on $M$ there exists a constant $C \geq 0$ such that $e^{-Cr} \in L^1(W_x^-)$.

(iii) $\rho_x(\xi, X) < \infty$.

**Proof.** The equivalence of (i) and (iii) was established in Proposition 6 without the assumption that $\xi$ is Lyapunov for $X$. The implication (iii) $\Rightarrow$ (ii) follows from Lemma 8; the implication (ii) $\Rightarrow$ (iii) from Lemma 9. □

Note that this again implies Corollary 3. We expect that these equivalent statements hold true, at least in the generic situation. More precisely:

**Conjecture** (Exponential growth). Let $g$ be a Riemannian metric on a closed manifold $M$, $\omega$ a closed one form (and assume $X = -\text{grad}_g \omega$ is Morse–Smale.) Let $x$ be a zero and let $i_x^- : W_x^- \to M$ denote its unstable manifold. Let $\text{Vol}(B_s(x))$ denote the volume of the ball $B_s(x) \subseteq W_x^-$ of radius $s$ centered at $x \in W_x^-$ with respect to the induced Riemannian metric $(i_x^-)^* g$ on $W_x^-$. Then there exists a constant $C \geq 0$ such that $\text{Vol}(B_s(x)) \leq e^{Cs}$ for all $s \geq 0$.

**3.3. A criterion for exponential growth.** The rest of the section is dedicated to a criterion which guarantees that the exponential growth property, and hence $\rho < \infty$, holds in simple situations.

Suppose $x \in X$. Let $B \subseteq W_x^-$ denote a small ball centered at $x$. The submanifold $i_x^- ((W_x^- \setminus B) \subseteq M$ gives rise to a submanifold $\text{Gr}(W_x^- \setminus B) \subseteq \text{Gr}_q(TM)$ in the Grassmannian tangent bundle, the space of $q$-dimensional subspaces in $TM$. For a critical point $y \in X$ we define

$$K_x(y) := \text{Gr}_q(T_y W_y^-) \cap \overline{\text{Gr}(W_x^- \setminus B)}$$

where $\text{Gr}_q(T_y W_y^-) \subseteq \text{Gr}_q(T_y M) \subseteq \text{Gr}_q(TM)$. Note that $K_x(y)$ does not depend on the choice of $B$.

**Remark 6.**

(i) Even though we removed a neighborhood of $x$ from the unstable manifold $W_x^-$ the set $K_x(x)$ need not be empty. However if we did not remove $B$ the set $K_x(x)$ would never be vacuous for trivial reasons.

(ii) If $q = \text{ind}(x) > \text{ind}(y)$ we have $K_x(y) = \emptyset$, for $\text{Gr}_q(T_y W_y^-) = \emptyset$ since $q > \dim(T_y W_y^-) = \text{ind}(y)$.

(iii) If $\dim(M) = n = q = \text{ind}(x)$ we always have $K_x(y) = \emptyset$ for all $y \in X$.

**Proposition 8.** Let $\xi$ be Lyapunov for $X$ and suppose $K_x(y) = \emptyset$ for all $y \in X$. Then $\rho_x(\xi, X) < \infty$.

We start with a little
Lemma 10. Let \( (V, g) \) be an Euclidean vector space and \( V = V^+ \oplus V^- \) an orthogonal decomposition. For \( \kappa \geq 0 \) consider the endomorphism \( A_\kappa := \kappa \text{id} \oplus - \text{id} \in \text{End}(V) \) and the function
\[
\delta^{A_\kappa}: \text{Gr}_q(V) \to \mathbb{R}, \quad \delta^{A_\kappa}(W) := \text{tr}_{g|_W} (p_W^\perp \circ A_\kappa \circ i_W),
\]
where \( i_W: W \to V \) denotes the inclusion and \( p_W^\perp: V \to W \) the orthogonal projection. Suppose we have a compact subset \( K \subseteq \text{Gr}_q(V) \) for which \( \text{Gr}_q(V^+) \cap K = \emptyset \). Then there exists \( \kappa > 0 \) and \( \epsilon > 0 \) with \( \delta^{A_\kappa} \leq -\epsilon \) on \( K \).

Proof. Consider the case \( \kappa = 0 \). Let \( W \in \text{Gr}_q(V) \) and choose a \( g|_W \)-orthonormal base \( e_i = (e_i^+, e_i^-) \in V^+ \oplus V^-, 1 \leq i \leq q \), of \( W \). Then
\[
\delta^{A_0}(W) = \sum_{i=1}^q g(e_i, A_0 e_i) = -\sum_{i=1}^q g(e_i^-, e_i^-).
\]
So we see that \( \delta^{A_0} \leq 0 \) and \( \delta^{A_\kappa}(W) = 0 \) iff \( W \in \text{Gr}_q(V^+) \). Thus \( \delta^{A_\kappa}|_K < 0 \). Since \( \delta^{A_\kappa} \) depends continuously on \( \kappa \) and since \( K \) is compact we certainly find \( \kappa > 0 \) and \( \epsilon > 0 \) so that \( \delta^{A_\kappa}|_K \leq -\epsilon \). \( \square \)

Proof of Proposition \( \S \) Let \( S \subseteq W^- \) denote a small sphere centered at \( x \). Let \( \tilde{X} := (i^-)^* X \) denote the restriction of \( X \) to \( W^- \) and let \( \Phi_t \) denote the flow of \( \tilde{X} \) at time \( t \). Then
\[
\varphi: S \times [0, \infty) \to W^- \quad \varphi(x, t) = \varphi(t) = \Phi_t(x)
\]
parameterizes \( W^- \) with a small neighborhood of \( x \) removed.

Let \( \kappa > 0 \). For every \( y \in \mathcal{X} \) choose a chart \( u_y: U_y \to \mathbb{R}^n \) centered at \( y \) so that
\[
X|_{U_y} = \kappa \sum_{i \leq \text{ind}(y)} u_y^i \frac{\partial}{\partial u_y^i} - \sum_{i > \text{ind}(y)} u_y^i \frac{\partial}{\partial u_y^i}.
\]
Let \( g \) be a Riemannian metric on \( M \) which restricts to \( \sum_i du_y^i \otimes du_y^i \) on \( U_y \) and set \( g_x := (i^-)^* g \). Then
\[
\nabla X|_{U_y} = \kappa \sum_{i \leq \text{ind}(y)} du_y^i \otimes \frac{\partial}{\partial u_y^i} - \sum_{i > \text{ind}(y)} du_y^i \otimes \frac{\partial}{\partial u_y^i}.
\]
In view of our assumption \( K_x(y) = \emptyset \) for all \( y \in \mathcal{X} \) Lemma 15 permits us to choose \( \kappa > 0 \) and \( \epsilon > 0 \) so that after possibly shrinking \( U_y \) we have
\[
\text{div}_{g_x}(\tilde{X}) = \text{tr}_{g_x}(\nabla \tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0, \infty)) \cap (i^-)^{-1}(\bigcup_{y \in \mathcal{X}} U_y).
\]
(14)

Next choose a closed 1–form \( \omega \) so that \( |\omega| = \xi \) and \( \omega(X) < 0 \) on \( M \setminus \mathcal{X} \). Choose \( \tau > 0 \) so that
\[
\tau \omega(X) + \text{ind}(x)||\nabla X||_g \leq -\epsilon < 0 \quad \text{on} \quad M \setminus \bigcup_{y \in \mathcal{X}} U_y.
\]
(15)

Using \( \tau \tilde{X} \cdot h_x^\omega \leq 0 \) and
\[
\text{div}_{g_x}(\tilde{X}) = \text{tr}_{g_x}(\nabla \tilde{X}) \leq \text{ind}(x)||\nabla \tilde{X}||_{g_x} \leq \text{ind}(x)||\nabla X||_g
\]
Lemma 15 and (16) yield
\[
\tau \tilde{X} \cdot h_x^\omega + \text{div}_{g_x}(\tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0, \infty)).
\]
(16)
Choose an orientation of \( W_x^- \) and let \( \mu \) denote the volume form on \( W_x^- \) induced by \( g_x \). Consider the function 
\[
\psi : [0, \infty) \to \mathbb{R}, \quad \psi(t) := \int_{\varphi(S \times [0, t])} e^{\tau h^x_x} \mu \geq 0.
\]
For its first derivative we find 
\[
\psi'(t) = \int_{\varphi(S(S))} e^{\tau h^x_x} i_X \mu > 0
\]
and for the second derivative, using (16), 
\[
\psi''(t) = \int_{\varphi(S(S))} \left( \tau X \cdot h^x_x + \text{div}_{g_x(\tilde{X})} \right) e^{\tau h^x_x} i_X \mu \leq -\epsilon \int_{\varphi(S(S))} e^{\tau h^x_x} i_X \mu = -\epsilon \psi'(t).
\]
So \( \ln \circ \psi' \)'(t) \leq -\epsilon \) hence \( \psi'(t) \leq \psi'(0)e^{-\epsilon t} \) and integrating again we find 
\[
\psi(t) \leq \psi(0) + \psi'(0)(1 - e^{-\epsilon t})/\epsilon \leq \psi'(0)/\epsilon.
\]
So we have \( e^{\tau h^x_x} \in L^1(\varphi(S \times [0, \infty)) \) and hence \( e^{\tau h^x_x} \in L^1(W_x^-) \) too. We conclude \( \rho_x(\xi, X) \leq \tau < \infty \). 

**Remark 7.**

(i) In view of Remark 6(ii) Proposition 8 implies \( \rho_x(\xi, X) < \infty \) whenever \( \xi \) is Lyapunov for \( X \) and \( \text{ind}(x) = \dim(M) \). However there is a much easier argument for this special case. Indeed, in this case \( W_x^- \) is an open subset of \( M \) and therefore its volume has to be finite. Since \( \xi \) is Lyapunov for \( X \) we immediately even get \( \rho_x(\xi, X) \leq 0 \).

(ii) In the case \( \text{ind}(x) = 1 \) we certainly have \( \rho_x(\xi, X) \leq 0 \).

(iii) Throughout the whole section we did not make use of a Morse–Smale condition.

Using Proposition 7, Proposition 8, Remark 6(ii) and Remark 7(ii) we get

**Corollary 4.** Suppose \( \xi \) is Lyapunov for \( X \) and \( x \) a zero of \( X \). If \( 1 < \text{ind}(x) < \dim(M) \) assume moreover that \( K_x(y) = \emptyset \) for all zeros \( y \) of \( X \) with \( \dim(M) > \text{ind}(y) \geq \text{ind}(x) \). Then \( X \) has the exponential growth property at \( x \) and \( \rho_x(\xi, X) < \infty \).

In particular if \( \dim(M) = 2 \) the exponential growth conjecture is true.

**4. Proof of Theorems 2 and 3**

Let \( X \) be a vector field with \( \xi \in H^1(M; \mathbb{R}) \) a Lyapunov cohomology class. Recall that in Section 1 we have defined the instanton counting function (or the Novikov incidence) \( i^x_y : \mathcal{P}^x_y \to \mathbb{Z} \), cf. (1).

The following proposition is a reformulation of a basic observation made by S.P. Novikov [14] in order to define his celebrated complex.

**Proposition 9.**

(i) For any \( x \in \mathcal{X}_q, y \in \mathcal{X}_{q-1} \) and every real number \( R \) the set 
\[
\{ \tilde{a} \in \mathcal{P}^x_y \mid i^x_y(\tilde{a}) \neq 0, \omega(\tilde{a}) \geq R \}
\]
is finite. Here \( \omega \) is any closed one form representing \( \xi \).
(ii) For any \( x \in \mathcal{X}_q, z \in \mathcal{X}_{q-2} \) and \( \hat{\gamma} \in \hat{P}^\xi_{z,x} \) one has
\[
\sum I^X_{x,y} \xi(\hat{\alpha}) \cdot \xi_{y,z} = 0.
\]
where the sum is over all \( y \in \mathcal{X}_{q-1}, \hat{\alpha} \in \hat{P}^\xi_{z,y} \) and all \( \hat{\beta} \in \hat{P}^\xi_{y,z} \) with \( \hat{\alpha} \cdot \hat{\beta} = \hat{\gamma} \).

Formula (17) implicitly states that the left side of the equality contains only finitely many non-zero terms.

Proposition 9 above is equivalent to Theorem 2 parts 1 and 2 in [6]. The proof, originally due to Novikov can be also found in [6].

The following proposition will be the main tool in the proof of Theorem 2.

**Proposition 10.** Suppose \( t \in \mathbb{R}, \omega \) a closed one form representing \( \xi \) and \( t > \rho(\xi, X) \). Then:

(i) For every \( a \in \Omega^q(M) \) and every \( x \in \mathcal{X}_q \), the integral
\[
\int_{X,\omega}^q(t)(a)(x) := \int_{W_x^{-}} e^{t\xi}(i_x)^* a
\]
converges absolutely. In particular it defines a linear map \( \int_{X,\omega}^q(t) : \Omega^q(M) \to \text{Maps}(\mathcal{X}_q, \mathbb{R}) \).

(ii) The map \( \int_{X,\omega}^q(t) : \Omega^q(M) \to \text{Maps}(\mathcal{X}_q, \mathbb{R}) \) is surjective and
\[
\int_{X,\omega}^{q+1}(t)(d_\omega(t))a(x) = \sum_{y \in \mathcal{X}_q} e^{\xi(a)} \xi_{x,y}(\hat{\alpha}) \int_{X,\omega}^q(t)(a)(y)
\]
where the right side of (18) is a potentially infinite sum which is convergent.

Recall that for an oriented \( n \)-dimensional manifold \( N \) and \( a \in \Omega^n(N) \) one has \(|a| := |a'| \text{Vol} \in \Omega^n(M)\), where \( \text{Vol} \in \omega^n(N) \) is any volume form and \( a' \in C^\infty(N, \mathbb{R}) \) is the unique function satisfying \( a = a' \cdot \text{Vol} \). The integral \( \int_N a \) is called absolutely convergent, if \( \int_N |a| \) converges.

The proof of Proposition 10 is given in details in [6] section 5, (cf. Proposition 4) and uses in an essential way Theorem 5 and Stokes’ theorem. Particular care is necessary in view of the fact that \( W_x^{-} \) are not compact. The integration has to be performed on a non compact manifold and Stokes’ theorem applied to non-compact manifolds with corners.

The proof of Theorem 2 boils down to the verification of the following claims:

**Claim 1.** For any \( t > \text{sup}\{\rho(\xi, X), T\} \), \( x \in \mathcal{X}_{q+1} \) and \( y \in \mathcal{X}_q \) the possibly infinite sum
\[
\sum_{\hat{\alpha} \in \hat{P}^\xi_{x,y}} \xi_{x,y}^\xi(\hat{\alpha}) e^{\xi(\hat{\alpha})}
\]
is convergent and the formula
\[
\delta_{X,\omega,\xi}^q(t)(E_y) = \sum_{x \in \mathcal{X}_{q+1}, \hat{\alpha} \in \hat{P}^\xi_{x,y}} (I_{x,y}^X \xi(\hat{\alpha}) e^{\xi(\hat{\alpha})}) E_x
\]
defines a linear map \( \delta_{X,\omega,\xi}^q(t) : C^q(X) = \text{Maps}(\mathcal{X}_q, \mathbb{R}) \to C^{q+1}(X) = \text{Maps}(\mathcal{X}_{q+1}, \mathbb{R}) \) which makes \( \{C^*(X), \delta_{X,\omega,\xi}^q(t)\} \) a smooth (actually analytic) family of cochain complexes of finite dimensional Euclidean spaces. Recall that \( \{E_x\}_{x \in \mathcal{X}} \) denotes the
characteristic functions of \( x \in \mathcal{X} \) and \( \{ E_x \}_{x \in \mathcal{X}} \) provide the canonical base of \( C^*(X) \) which, implicitly equips each component \( C^q(X) \) with a scalar product, the unique scalar product which makes this base orthonormal. Recall also that \( \overline{\gamma} : \mathcal{P} \rightarrow \mathbb{R} \) was defined in section \( \text{I} \) before Proposition \( \text{I} \) and makes sense even when \( \omega \) is not a representative of \( \xi \) but still, its pull back on \( M \) is exact.

**Claim 2.** The linear maps \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) \) are surjective and define a morphism of cochain complexes.

**Claim 3.** There exists \( T \) (larger than \( \rho(\omega, X) \)) so that for \( t > T \) the linear map \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) \) when restricted to \( \Omega^q_{\text{sm}}(M)(t) \) is an isomorphism and actually an \( O(1/t) \)-isometry.

Everything but the \( O(1/t) \)-isometry statement in Claim 3 is a straightforward consequence of Theorem 2 and Proposition 1 above. To check this part of Claim 3 we have to go back to the proof of Theorems 3 and 4 in [6], section 6. We observe that if \( t \) is large enough the restriction of \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) \) to the subspace \( H_1(t) \subset \Omega^q(M) \) defined in [6], section 4 page 172 (cf. Proof of Theorem 3) is surjective and then by Lemma 7 in [6] so is the restriction of \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) \) to \( \Omega^q_{\text{sm}}(M)(t) \). This because \( H_1(t) \) and \( \Omega^q_{\text{sm}}(M)(t) \) are, by Lemma 7 in [6] section 6, as close as we want for \( t \) large enough. Since the spaces \( \Omega^q_{\text{sm}}(M)(t) \) and \( C^q(X) \) have the same finite dimension, by the surjectivity in Claim 2 \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) \) is an isomorphism and, as shown in [6], section 4 page 172 an \( O(1/t) \)-isometry. We take as the base \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) \) the differential forms \( E_x^q(t) = \text{Int}^q_{X,\omega,\mathcal{O}}(t)^{-1}(E_x) \). This finishes the proof of Theorem 2 and of Claim 3.

We conclude this section with the following remarks. Let \( X \) be a vector field which has \( \xi \) as Lyapunov cohomology class. Suppose \( X \) satisfies MS and \( \rho(\xi, X) < \infty \). Let \( \omega \) be a closed one form representing \( \xi \).

For \( t > \rho(\xi, X) \) the finite dimensional vector spaces

\[
C^q(X) := \text{Maps}(\mathcal{X}_q, \mathbb{R})
\]

and the linear maps

\[
\delta^q_{X,\omega,\mathcal{O}}(t) : \text{Maps}(\mathcal{X}_q, \mathbb{R}) \rightarrow \text{Maps}(\mathcal{X}_{q+1}, \mathbb{R})
\]

defined by

\[
\delta^q_{X,\omega,\mathcal{O}}(t)(E_x) := \sum_{y \in \mathcal{X}_{q+1}, \hat{a} \in \mathcal{P}_{\mathcal{X}_{q+1}}^y} I_{y,x}^X,\omega,\mathcal{O}^q(\hat{a}) e^{\nabla(\hat{a})} E_y
\]

give rise to a cochain complex of finite dimensional vector spaces

\[
\mathcal{C}^*(X, \omega, \mathcal{O})(t) := \{ C^q(X), \delta^q_{X,\omega,\mathcal{O}}(t) \},
\]

and to a morphism of such complexes:

\[
\text{Int}^q_{X,\omega,\mathcal{O}}(t) : (\Omega^*(M), d_\omega(t)) \rightarrow \mathcal{C}^*(X, \omega, \mathcal{O})(t)
\]

One can show (implicit in Theorem 3) that \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) \) induces an isomorphism in cohomology. This fact will be used in this paper only when \( X \) is the gradient of a smooth function in which case it is a straightforward consequence of deRham’s theorem with local coefficients.

Let \( \omega_1 \) and \( \omega_2 \) be two closed one forms representing the same cohomology class \( \xi \) and let \( f : M \rightarrow \mathbb{R} \) be a smooth function so that \( \omega_1 - \omega_2 = df \). The collections...
of linear maps

\[ m^q_f(t) : \Omega^q(M) \to \Omega^q(M), \quad m^q_f(t)(a) := e^{tf}a, \]

where \( a \in \Omega^q(M) \), and

\[ s^q_f(t) : C^q(X, \omega_1, \mathcal{O}) \to C^q(X, \omega_2, \mathcal{O}), \quad s^q_f(t)(E_x) := e^{tf(x)}E_x, \]

where \( E_x \in \text{Maps}(X, \mathbb{R}) \) denotes the characteristic function of \( x \in X \), define morphisms of cochain complexes making the diagram commutative for any \( t > \rho \).

The Mathai–Quillen form has the following properties:

In this section we discuss the numerical invariant \( R(X, \omega, g) \) associated to a vector field \( X \), a closed one form \( \omega \) and a Riemannian metric \( g \). The invariant is defined by a possibly divergent integral but regularizable and is implicit in the work of [11]. More on this invariant is contained in [17].

In section 1.5, we have considered the Mathai–Quillen form \( \Psi_g \in \Omega^{n-1}(TM \setminus M; \mathcal{O}_M) \) of an \( n \)-dimensional Riemannian manifold \( (M, g) \). The Mathai–Quillen form (see [12]) is actually associated to a pair \( \nabla = (\nabla, \mu) \) consisting of a connection and a parallel Euclidean structure on a vector bundle \( E \to M \). If \( E \) is of rank \( k \) it is a \( k-1 \) form \( \Psi_\nabla \in \Omega^{k-1}(E \setminus M; \mathcal{O}_E) \) with values in the pull back of the orientation bundle \( \mathcal{O}_E \) of \( E \) to the total space of \( E \). Here \( M \) is identified with the zero section in the bundle \( E \). If \( g \) is a Riemannian metric let \( \nabla^g = (\nabla, \mu) \) denote the Levi–Civita pair associated to \( g \) and write \( \Psi_g := \Psi_{\nabla^g} \).

The Mathai–Quillen form has the following properties:

(i) For the Euler form \( E_\nabla \in \Omega^k(M; \mathcal{O}_E) \) associated to \( \nabla \) we have \( d\Psi_\nabla = \pi^*E_\nabla \).

(ii) For two \( \nabla^1 \) and \( \nabla^2 \) we have \( \Psi_{\nabla^2} - \Psi_{\nabla^1} = \pi^*cs(\nabla^1, \nabla^2) \) modulo exact forms. Here \( cs(\nabla^1, \nabla^2) \in \Omega^{k-1}(M; \mathcal{O}_E)/d\Omega^{k-2}(M; \mathcal{O}_E) \) is the Chern–Simon invariant.

(iii) For every \( x \in M \) the form \( -\Psi_\nabla \) restricts to the standard generator of \( H^{k-1}(E_x \setminus 0; \mathcal{O}_E) \), where \( E_x \) denotes the fiber over \( x \in M \). Note that the restriction of \( -\Psi_\nabla \) is closed by [8].

(iv) Suppose \( E = TM \), \( \nabla^g \) is the Levi–Civita pair, and suppose that on the open set \( U \) we have coordinates \( x^1, \ldots, x^n \) in which the Riemannian metric \( g_{ij} \) is given by \( g_{ij} = \delta_{ij} \). Then, with respect to the induced coordinates \( x^1, \ldots, x^n, \xi^1, \ldots, \xi^n \) on \( TU \), the form \( \Psi_g \) is given by

\[ \Psi_g = \frac{\Gamma(n/2)}{2\pi^{n/2}} \sum_i (-1)^i \frac{\xi^i}{(\sum_j (\xi^j)^2)^{n/2}} d\xi^1 \wedge \cdots \wedge d\xi^i \wedge \cdots \wedge d\xi^n, \]

cf. [12].
Let $X$ be a vector field on $M$, i.e. a section of the tangent bundle $TM$. We suppose that it has only isolated zeros, that is its zero set $\mathcal{X}$ is a discrete subset of $M$. The vector field defines an integer valued map $\text{IND} : \mathcal{X} \to \mathbb{Z}$, where $\text{IND}(x)$ denotes the Hopf index of the vector field $X$ at the zero $x \in \mathcal{X}$. This integer $\text{IND}(x)$ is the degree of the map $(U, U \setminus x) \to (T_xM, T_xM \setminus 0)$ obtained by composing $X : U \to TU$ with the projection $p : TU \to T_xM$ induced by a local trivialization of the tangent bundle on a small disk $U \subseteq M$ centered at $x$.

Choose coordinates around $x$ so that we can speak of the disk $U_\epsilon$ with radius $\epsilon > 0$ centered at $x$. It is well known that we have:

$$\text{IND}_x = -\lim_{\epsilon \to 0} \int_{\partial U_\epsilon} X^*\Psi_g$$

Indeed, by (iii) we may assume that $g$ is flat on $U_\epsilon$. Thus $E\Psi_g = 0$ and $\Psi_g$ is closed on $U_\epsilon$ by (ii). Using (iii) we see that $-\Psi_g$ gives the standard generator of $H^{n-1}(TU_\epsilon \setminus U_\epsilon; \mathcal{O}_{U_\epsilon})$ and thus certainly $\text{IND}(x) = -\int_{\partial U_\epsilon} X^*\Psi_g$.

The vector field $X$ has its rest points (zeros) non-degenerate and in particular isolated, if the map $X$ is transversal to the zero section in $TM$. In this case $\mathcal{X}$ is an oriented zero dimensional manifold, whose orientation is specified by $\text{IND}(x)$. Moreover we have

$$\text{IND}(x) = \text{sign} \det H \in \{\pm 1\},$$

where $H : T_xM \to T_xM$ denotes the Hessian. Particularly, if there exist coordinates $x^1, \ldots, x^n$ centered at $x$ so that

$$X = -\sum_{1 \leq i \leq k} x^i \frac{\partial}{\partial x^i} + \sum_{i > k} x^i \frac{\partial}{\partial x^i},$$

we get $\text{IND}(x) = (-1)^k$.

Let $X^1$ and $X^2$ be two vector fields and $\mathcal{X} := \{X_s\}_{s \in [-1, 1]}$ a smooth homotopy from $X^1$ to $X^2$, i.e. $X_s = X^1$ for $s \leq -1 + \epsilon$ and $X_s = X^2$ for $s \geq 1 - \epsilon$. The homotopy is called non-degenerate if the map $\mathcal{X} : [-1, 1] \times M \to TM$ defined by $\mathcal{X}(s, x) := X_s(x)$ is transversal to the zero section of $TM$. In this case necessarily $X^1$ and $X^2$ are vector fields with non-degenerate zeros and so are all but finitely many $X_s$. Moreover all $X_s$ have isolated zeros with indexes in $\{0, 1, -1\}$ and the zero set $\tilde{\mathcal{X}}$ of $\mathcal{X}$ is an oriented one dimensional smooth submanifold of $[-1, 1] \times M$. Note that we have

$$\partial \tilde{\mathcal{X}} = \sum_{y \in X^2} \text{IND}(y)y - \sum_{x \in X^1} \text{IND}(x)x.$$

If $\mathcal{X}'$ is a second homotopy joining $X^1$ with $X^2$ then $\tilde{\mathcal{X}}' - \tilde{\mathcal{X}}$ is the boundary of a smooth $2$–cycle. Indeed, if we choose a homotopy of homotopies joining $\mathcal{X}$ with $\mathcal{X}'$ which is transversal to the zero section, then its zero set will do the job.

Given a closed one form $\omega$ on $M$ denote by

$$I_{\mathcal{X}, \omega} := \int_{\tilde{\mathcal{X}}} p_2^*\omega,$$

where $p_2 : \tilde{\mathcal{X}} \to M$ denotes the restriction of the projection $[-1, 1] \times M \to M$. It follows from the previous paragraph that $I_{\mathcal{X}, \omega}$ does not depend on the homotopy $\mathcal{X}$ — only on $X^1, X^2$ and $\omega$, and therefore will be denoted from now on by $I(X^1, X^2, \omega)$. 
Remark 8. If there exists a simply connected open set $V \subset M$ so that $\mathcal{X}_s \subset V$ for all $s \in [-1,1]$ then one can calculate $I_{X,\omega}$ as follows: Choose a smooth function $f : V \to \mathbb{R}$ so that $\omega|_V = df$. Then

$$I_{X,\omega} = \sum_{y \in \mathcal{X}_2} \text{IND}(y)f(y) - \sum_{x \in \mathcal{X}_1} \text{IND}(x)f(x).$$

The proof of this equality is a straightforward application of Stokes’ theorem.

With these considerations we will describe now the regularization referred to in Section 1.5, cf. (8). First note that for a non-vanishing vector field $X$, a closed one form $\omega$ and a Riemannian metric $g$ the quantity

$$R(X,\omega,g) := \int_M \omega \wedge X^* \Psi_g$$

has the following two properties.

$$R(X,\omega + df,g) - R(X,\omega,g) = - \int_M f E_g$$

for every smooth function $f$. If $g^1$ and $g^2$ are two Riemannian metrics then

$$R(X,\omega,g^2) - R(X,\omega,g^1) = \int_M \omega \wedge \text{cs}(g^1,g^2)$$

where $\text{cs}(g^1,g^2) = \text{cs}(\overline{\nabla}g^1,\overline{\nabla}g^2)$. This follows from properties \[ and \[ of the Mathai–Quillen form.

If $X$ has zeros, then the form $\omega \wedge X^* \Psi_g$ is well defined on $M \setminus \mathcal{X}$ but the integral $\int_{M \setminus \mathcal{X}} \omega \wedge X^* \Psi_g$ might be divergent unless $\omega$ is zero on a neighborhood of $\mathcal{X}$.

We will define below a regularization of the integral $\int_{M \setminus \mathcal{X}} \omega \wedge X^* \Psi_g$ which in case $\mathcal{X} = \emptyset$ is equal to the integral \[. For this purpose we choose a smooth function $f : M \to \mathbb{R}$ so that the closed 1-form $\omega' := \omega - df$ vanishes on a neighborhood of $\mathcal{X}$, and put

$$R(X,\omega,g;f) := \int_{M \setminus \mathcal{X}} \omega' \wedge X^* \Psi_g - \int_M f E_g + \sum_{x \in \mathcal{X}} \text{IND}(x)f(x)$$

Proposition 11. The quantity $R(X,\omega,g;f)$ is independent of $f$.

Therefore $R(X,\omega,g;f)$ can be denoted by $R(X,\omega,g)$ and will be called the regularization of $\int_{M \setminus \mathcal{X}} \omega \wedge X^* \Psi_g$.

Proof. Suppose $f^1$ and $f^2$ are two functions such that $\omega^i := \omega - df^i$ vanishes in a neighborhood $U$ of $\mathcal{X}$, $i = 1,2$. For every $x \in \mathcal{X}$ we choose a chart and let $D_\epsilon(x)$ denote the $\epsilon$–disk around $x$. Put $D_\epsilon := \bigcup_{x \in \mathcal{X}} D_\epsilon(x)$. 

For $\varepsilon$ sufficiently small $D_\varepsilon \subseteq U$ and $f^2 - f^1$ is constant on each $D_\varepsilon(x)$. From (28), Stokes’ theorem and we conclude that

$$R(X, \omega; g; f^2) - R(X, \omega; g; f^1) - \sum_{x \in X} \text{IND}(x) (f^2(x) - f^1(x)) =$$

$$= - \int_{M \setminus X} d((f^2 - f^1) \wedge X^* \Psi_g)$$

$$= - \lim_{\varepsilon \to 0} \int_{M \setminus D_\varepsilon} d((f^2 - f^1) \wedge X^* \Psi_g)$$

$$= \sum_{x \in X} (f^2(x) - f^1(x)) \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon(x)} X^* \Psi_g$$

$$= - \sum_{x \in X} \text{IND}(x) (f^2(x) - f^1(x))$$

and thus $R(X, \omega; g; f^1) = R(X, \omega; g; f^2)$. □

**Proposition 12.** Suppose that $X$ is a non-degenerate homotopy from the vector field $X^1$ to $X^2$ and $\omega$ is a closed one form. Then

$$R(X^2, \omega, g) - R(X^1, \omega, g) = I(X^1, X^2, \omega). \quad (24)$$

**Proof.** We may assume that there exists a simply connected $V \subseteq M$ with $X_s \subseteq V$ for all $s \in [-1, 1]$. Indeed, since both sides of (24) do not depend on the homotopy $X$ we may first slightly change the homotopy and assume that no component of $X$ lies in a single $\{s\} \times M$. Then we find $-1 = t_0, \ldots, t_k = 1$ so that for every $0 \leq i < k$ we find a simply connected $V_i \subseteq M$ such that $X_s \subseteq V_i$ for all $s \in [t_i, t_{i+1}]$.

Assuming $V$ as above we choose a function $f$ so that $\omega' := \omega - df$ vanishes on a neighborhood of every $X_s$, i.e. $p_2^* \omega'$ vanishes on a neighborhood of $\tilde{X}$. Here $p_2 : [-1, 1] \times M \to M$ denotes the canonical projection. Moreover let $\tilde{p}_2 : [-1, 1] \times TM \to TM$ denote the canonic projection and note that $p_2^* \omega' \wedge X^* \tilde{p}_2^* \Psi_g$ is a globally defined form on $[-1, 1] \times TM$. Using Stokes’ theorem and Remark 8 we then get:

$$R(X^2, \omega, g) - R(X^1, \omega, g) - I_{X, \omega} = \int_{[-1,1] \times M} d(p_2^* \omega' \wedge X^* \tilde{p}_2^* \Psi_g)$$

$$= \int_{[-1,1] \times M} p_2^* (\omega' \wedge E_g)$$

$$= 0$$

For the second equality we used $dX^* \tilde{p}_2^* \Psi_g = p_2^* E_g$. The integrand of the last integral vanishes because of dimensional reasons. □

With little effort, using Stokes’ theorem and the properties of the Mathai–Quillen form, one can prove

$$R(X, \omega + df, g) - R(X, \omega, g) = - \int_M f E_g + \sum_{x \in X} \text{IND}(x) f(x)$$

for every smooth function $f$, and

$$R(X, \omega, g^2) - R(X, \omega, g^1) = \int_M \omega \wedge \text{cs}(g^1, g^2)$$
for any two Riemannian metrics $g^1$ and $g^2$. It is also not difficult to generalize the regularization to vector fields with isolated singularities, cf. [2].

6. Proof of Theorem [1]

The proof of Theorem [1] presented here combines results of Hutchings, Pajitnov and others (cf. [2], [13]) with results of Bismut–Zhang, cf. [1], [7] and [4]. A recollection of these results, additional notations and preliminaries are necessary. They will be collected in four preliminary subsections. These subsections will be followed by the fifth where Theorem [1] is proven.

Recall from [4] that a generalized triangulation $\tau = (f, g)$ on a closed manifold $M$ is a pair consisting of a Morse function $f$ and a Riemannian metric $g$ so that $X = -\text{grad}_g f$ satisfies MS.

6.1. Homotopy between vector fields. Let $\xi \in H^1(M; \mathbb{R})$, and $\pi: \tilde{M} \to M$ be a covering so that $\pi^*\xi = 0$.

Recall that a smooth family of sections $X := \{X_s\}_{s \in [-1, 1]}$, of the tangent bundle will be called a homotopy from the vector field $X^1$ to the vector field $X^2$ if there exists $\epsilon > 0$ so that $X_s = X^1$ for $s < -1 + \epsilon$ and $X_s = X^2$ for $s > 1 - \epsilon$.

To a homotopy $X := \{X_s\}_{s \in [-1, 1]}$ one associates the vector field $Y$ on the compact manifold with boundary (cf. appendix to section [2] for definition) $N := M \times [-1, 1]$ defined by

$$Y(x, s) := X(x, s) + 1/2(s^2 - 1) \frac{\partial}{\partial s}.$$  

(25)

With this notation we have the following.

Proposition 13. If $X$ is a homotopy between two vector fields $X^1$ and $X^2$ which both have $\xi$ as a Lyapunov cohomology class. Then the vector field $Y$ has $p^*\xi$ as a Lyapunov cohomology class, cf. Definition [7] where $p: N = M \times [-1, 1] \to M$ is the first factor projection.

Proof. Since $X^1$ and $X^2$ are both vector fields with $\xi$ as Lyapunov cohomology class we can choose closed 1–forms $\omega_i$ representing $\xi$ and Riemannian metrics $g_i$ on $M$ such that $X^i = -\text{grad}_{g_i} \omega_i$, $i = 1, 2$. Choose an admissible Riemannian metric $g$ on $N$ inducing $g_i$ on the boundaries; for example take

$$g := (1 - \lambda)p^*g_1 + \lambda p^*g_2 + ds^2,$$

where $\lambda: [-1, 1] \to \mathbb{R}$ is a non-negative smooth function satisfying

$$\lambda(s) = \begin{cases} 0 & \text{for } s \leq -1 + \epsilon \text{ and} \\ 1 & \text{for } s \geq 1 - \epsilon. \end{cases}$$

Next choose a closed 1–form $\omega$ on $N$ which restricts to $p^*\omega_1$ on $M \times [-1, -1 + \epsilon]$ and which restricts to $p^*\omega_2$ on $M \times [1 - \epsilon, 1]$. This is possible since $\omega_1$ and $\omega_2$ define the same cohomology class $\xi$ and can be achieved in the following way. Choose a function $h$ on $M$ with $\omega_2 - \omega_1 = dh$ and set $\omega := p^*\omega_1 + d(\lambda p^*h)$. Choose a function $u: [-1, 1] \to \mathbb{R}$, such that:

(i) $u(s) = -\frac{1}{2}(s^2 - 1)$ for all $s \leq -1 + \epsilon$ and all $s \geq 1 - \epsilon$.

(ii) $u(s) \geq \frac{-\omega(Y)(x, s)}{\frac{1}{2}(s^2 - 1)}$ for all $s \in [-1 + \epsilon, 1 - \epsilon]$ and all $x \in M$. 

This is possible since \( \frac{-\omega(Y)(x,s)}{\frac{2}{2(s^2-1)}} \leq 0 \) for \( s = -1 + \epsilon \) and \( s = 1 - \epsilon \).

Then \( \tilde{\omega} := \omega + u(s)ds \) represents the cohomology class \( p^*\xi \) and one can verify that \( Y' \) is a vector field which coincides with \( -\text{grad}_Y \tilde{\omega} \) in a neighborhood of \( \partial N \) and for which \( \tilde{\omega}(Y) < 0 \) on \( N \setminus Y \).

Let \( X \) be a homotopy between two vector fields \( X^1 \) and \( X^2 \) which satisfies MS. Let \( Y \) be the vector field defined in (25). With the notations from the appendix to section 2 (the case of a compact manifold with boundary) we have

\[
Y = Y'' = Y''_1 \cup Y''_2
\]

with

\[
Y''_1 = X^1 \times \{-1\} \quad \text{and} \quad Y''_2 = X^2 \times \{1\}.
\]

**Definition 8.** The homotopy \( X \) is called MS if the vector field \( Y \) is MS, i.e. \( X^1 \) and \( X^2 \) are MS and for any \( y \in Y''_1 \) and \( z \in Y''_2 \), the maps \( i^+_y \) and \( i^-_z \) are transversal. The homotopy \( X \) has exponential growth if \( Y \) has exponential growth.

**Proposition 14.** Let \( X^1 \) and \( X^2 \) be two vector fields which satisfy MS and \( X \) a homotopy from \( X^1 \) to \( X^2 \). Then there exists a MS homotopy \( X' \) from \( X^1 \) to \( X^2 \), arbitrarily close to \( X \) in the \( C^1 \)-topology.

**Proof.** First we modify the vector field \( Y \) into \( Y' \) by a small change in the \( C^1 \)-topology, and only in the neighborhood of \( M \times \{0\} \), in order to have the Morse–Smale condition satisfied for any \( y \in Y''_1 \) and \( z \in Y''_2 \). This can be done using Proposition 2. Unfortunately \( Y' \) might not have the \( I \)-component equal to \( 1/2(s^2-1)\partial/\partial s \), it is nevertheless \( C^1 \)-close, so by multiplication with a function which is \( C^1 \)-close to 1 and equal to 1 on the complement of a small compact neighborhood of the locus where \( Y \) and \( Y' \) are not the same, one obtains a vector field \( Y'' \) whose \( I \)-component is exactly \( 1/2(s^2-1)\partial/\partial s \). The \( M \)-component of \( Y'' \) defines the desired homotopy. By multiplying a vector field with a smooth positive function the stable and unstable sets do not change, and their transversality continues to hold. \( \square \)

In view of Theorem 6 for compact manifolds with boundary we have the following

**Remark 9.** For any \( y = (x,1) \in Y''_2 \) the 1–corner of \( \tilde{W}^-_y \) is given by

\[
\partial_1(\tilde{W}^-_y) = V_0 \cup V_1 \cup V_2
\]

where

\[
V_0 \simeq W^-_x \quad V_1 \simeq \bigcup_{v \in Y''_1 \text{ and } \alpha \in \mathbb{P}(v, \hat{\alpha}) \text{ and } \text{ind}(v) = \text{ind}(y) - 1} T(y, v, \hat{\alpha}) \times (W^-_v \setminus \partial N) \quad V_2 \simeq \bigcup_{w \in Y''_2 \text{ and } \alpha \in \mathbb{P}(w, \hat{\alpha}) \text{ and } \text{ind}(w) = \text{ind}(y) - 1} T(y, w, \hat{\alpha}) \times W^-_u
\]

It is understood that \( W^-_x \) represents the unstable manifold in \( M = M \times \{1\} \) if \( x \in X^2 \).

In view of (26) we introduce the invariant \( \rho(\xi, X) \in \mathbb{R} \cup \{\pm \infty\} \) for any homotopy \( X \) by defining

\[
\rho(\xi, X) := \rho(p^*\xi, Y)
\]
Clearly $\rho(\xi, X) \geq \rho(\xi, X^1)$ for $i = 1, 2$.

Suppose $X$ is a MS homotopy from the MS vector field $X^1$ to the MS vector field $X^2$. For each $X^i$ choose the orientations $O^i$, $i = 1, 2$. Observe that the set $\mathcal{P}_{x', x}$ identifies to $\mathcal{P}_{(x', 1), (x', -1)}$. The orientations $O^1$ and $O^2$ define the orientations $O$ for the unstable manifolds of the rest points of $Y$. For $x^1 \in \mathcal{X}^1$, $x^2 \in \mathcal{X}^2$ and $\hat{\alpha} \in \mathcal{P}_{x^2, x^1}$ define the incidences

$$\Gamma^2 \mathcal{X}^1, O^2, O^1 (\hat{\alpha}) := \mathcal{Y}^\omega((x^2, 1), (x^1, -1))(\hat{\alpha}).$$

(26)

Suppose in addition that $\rho(\xi, X) < \infty$. For any $t > \rho(\xi, X)$ and $\omega$ a closed one form representing $\xi$, define the linear maps

$$u^\omega(t) := u^\omega(t) : \text{Maps}(\mathcal{X}^1, \mathbb{R}) \to \text{Maps} \left( \mathcal{X}^2, \mathbb{R} \right)$$

and the linear maps

$$h^\omega(t) := h^\omega(t) : \Omega^\omega(M) \to \text{Maps}(\mathcal{X}^2, \mathbb{R})$$

by

$$u^\omega(t)(E_{x^1}) := \sum_{z^2 \in x^2} \Gamma^2 \mathcal{X}^1, O^1 (\hat{\alpha}) e^\omega(\hat{\alpha}) E_{x^2}, \quad x^1 \in \mathcal{X}^1$$

(27)

and

$$(h^\omega(t)(a))(x^2) = \int_{W_y^\omega} e^{t \mathcal{F}_y(i_y^\omega)^* p^\ast a}, \quad x^2 \in \mathcal{X}^2 and y = (x^2, 1).$$

The right side of (27) is a convergent infinite sum since it is a sub sum of the right hand side of (18) when applied to the vector field $Y$.

**Proposition 15.** Suppose $X^1, X^2$ are two MS vector fields having $\xi$ as a Lyapunov cohomology class and suppose $X$ is a MS homotopy. Suppose that $\rho := \rho(\xi, X) = \rho(p^\ast \xi, Y) < \infty$ and $\omega$ is a closed one form with $p^\ast \omega$ exact; here $p : M \to M$ is the $\Gamma$–principal covering associated with $\Gamma$. Then for $t > \rho$ we have:

(i) The collection of linear maps $\{u^\omega(t)\}$ defines a morphism of cochain complexes:

$$u^\omega(t) := u^\omega(t) : C^\ast (X^1, O^1, \omega)(t) \to C^\ast (X^2, O^2, \omega)(t)$$

(ii) The collection of linear maps $h^\omega(t)$ defines an algebraic homotopy between

$$\text{Int}^\ast \mathcal{X}^2, O^2, \omega(t) and u^\omega(t) \circ \text{Int}^\ast \mathcal{X}^1, O^1, \omega(t).$$

Precisely, we have:

$$h^\omega(t) \circ d^\omega(t) + d^\omega(t-1) \circ h^\omega(t) = u^\omega(t) \circ \text{Int}^\ast \mathcal{X}^1, O^1, \omega(t) - \text{Int}^\ast \mathcal{X}^2, O^2, \omega(t)$$

**Proof.** Statement (i) follows from the equality

$$\sum_{z^2 \in \mathcal{X}^1} \Gamma^2 \mathcal{X}^1, O^1 (\hat{\alpha}) \Gamma^1 \mathcal{X}^1, O^1 (\hat{\beta}) = \sum_{z^2 \in \mathcal{X}^2} \Gamma^2 \mathcal{X}^2, O^2 (\hat{\alpha}) \Gamma^2 \mathcal{X}^2, O^2 (\hat{\beta}) = 0$$

for any $x^1 \in \mathcal{X}^1, z^2 \in \mathcal{X}^2$ and $\hat{\alpha} \in \mathcal{P}_{x^1, x^2}$, $\hat{\beta} \in \mathcal{P}_{x^1, x^2}$.

When applied to the vector field $Y$, the rest points $(x, -1)$ and $(z, 1)$ and $\hat{\gamma} \in \mathcal{P}_{x, z} = \mathcal{P}_{(x, 1), (z, -1)}$. The sign stems from the fact that the sign associated to a trajectory from $z$ to $z'$ changes when it is considered as trajectory in $M \times [-1, 1]$ instead of $M \times \{1\}$.

To verify statement (ii) we first observe that:
Proposition 16. □

(a) If \( y, u \in \mathcal{Y} \) the restriction of \( F_y \) to \( \mathcal{T}(y, u)(\check{\alpha}) \times W^u_x, \check{\alpha} \in \check{\mathcal{P}}_{y,u} \), when this set is viewed as a subset of \( W^u_x \), is given by

\[
F_u \circ \text{pr}_{W^u_x} + \mathbf{w}(\check{\alpha}).
\]

(b) If \( y = (x, -1), x \in \mathcal{X} \), via the identification of \( W^u_x \) to \( W^u_y \), we have \( F_y = h_x \).

In view of the uniform convergence of all integrals which appear in the formulae below, guaranteed by the hypothesis \( t > \rho \), the Stokes theorem for manifolds with corners gives for any \( a \in \Omega^q(M) \) and \( y \in (\mathcal{Y}^\alpha)_q \)

\[
\int_{W^u_y} d(e^{tF_y}c) = \int_{V_0} e^{tF_y}c + \int_{V_2} e^{tF_y}c + \int_{V_2} e^{tF_y}c,
\]

where \( c := (i^\ast_y)^p a \in \Omega^q(W^u_y) \).

In view of the Remark 9 we have

\[
\int_{V_0} e^{tF_y}c = \mathrm{Int}^q_{\mathcal{X}^1, \mathcal{O}^1, \omega}(t)(a),
\]

and

\[
\int_{V_2} e^{tF_y}c = \sum_{u \in \mathcal{Y}^\alpha, \check{\alpha} \in \check{\mathcal{P}}_{y,u} \text{ind}(u) = \text{ind}(y) - 1} \| Y_{y,u}^{\mathcal{O}}(\check{\alpha}) \| e^{\omega(\check{\alpha})} \int_{W^u_x} e^{tF_y} (i^\ast_y)^p a
\]

\[
= - (\delta^\omega_y)^{-1}(t) \circ h^\omega_x(t)(a) \quad (30)
\]

and

\[
\int_{V_1} e^{tF_y}c = - \sum_{v \in \mathcal{Y}^\alpha, \check{\alpha} \in \check{\mathcal{P}}_{y,v} \text{ind}(v) = \text{ind}(y) - 1} \| Y_{y,v}^{\mathcal{O}}(\check{\alpha}) \| e^{\omega(\check{\alpha})} \int_{W^u_v} e^{tF_y} (i^\ast_v)^p a
\]

\[
= - (u^\omega_y(t) \circ \mathrm{Int}^q_{\mathcal{X}^1, \mathcal{O}^1, \omega}(t))(a). \quad (31)
\]

Moreover

\[
(h^\omega_y(t) \circ d_\omega(t))(b) = \int_{W^u_y} e^{tF_y} (i^\ast_y)^p (db + t\omega \wedge b)
\]

\[
= \int_{W^u_y} d(e^{tF_y} (i^\ast_y)^p b)
\]

and the statement follows combining the equalities (28)–(32) □

The following proposition will be important in the proof of Theorem 4.

Proposition 16.

(i) Let \( (f, g) \) be a pair consisting of a Morse function and a Riemannian metric. Then the vector field \(-\text{grad}_g f\) has any \( \xi \in H^1(M; \mathbb{R}) \) as a Lyapunov cohomology class.

(ii) Let \( X \) be a vector field which has MS property and has \( \xi \) as Lyapunov cohomology class. Let \( \tau = (f, g) \) be a generalized triangulation. Then there exists a homotopy \( \mathcal{X} \) from \( X^1 := X \) to \( X^2 \) which is MS and is \( \mathcal{C}^0 \)–close to the family \( X_s = \frac{1}{2} X - \frac{1}{2} \text{grad}_g f \). One can choose \( \mathcal{X} \) to be \( \mathcal{C}^1 \)–close to a family \( l(s)X - (1 - l(s)) \text{grad}_g f \) where \( l : [-1, 1] \to [0, 1] \) is a smooth function with \( l'(s) \leq 0 \) and \( l'(s) = 0 \) in a neighborhood of \( \{ \pm 1 \} \).
Proof of (iii). Let $\omega$ be a closed one form representing $\xi$ with support disjoint from a neighborhood of $\text{Cr}(f)$. Clearly for $C$ a large constant the form $\omega' := \omega + Cdf$ represents $\xi$ and satisfies $\omega'(-\text{grad}_g f) < 0$. 

Proof of (ii). First consider the family $X_s := (\frac{1-s}{2})X - (\frac{1+s}{2})\text{grad}_g f$. Change the parametrization to make this family locally constant near $\{\pm 1\}$, hence get a homotopy and apply Proposition 14 to change this homotopy into one which satisfies MS. 

Definition 9. A vector field $X$ which satisfies MS and has $\xi$ as a Lyapunov cohomology class is said to have strong exponential growth if for one (and then any) generalized triangulation $\tau = (f, g)$ there exists a homotopy $X$ from $X$ to $-\text{grad}_g f$ which has exponential growth, cf Definition 8, equivalently, with $\rho(\xi, X), \infty$.

To summarize the discussion in this subsection consider:

(i) a vector field $X^1 = -\text{grad}_g \omega$ with $\omega$ a Morse form representing $\xi$ and $g'$ a Riemannian metric so that $X^1$ satisfies MS,

(ii) A vector field $X^2 = -\text{grad}_g f$, $\tau = (f, g')$ a generalized triangulation,

(iii) A homotopy $X$ from $X^1$ to $X^2$ which satisfies MS.

Since $\rho(\xi, X^2) = -\infty$, for any $t \in \mathbb{R}$ we have a well defined morphism of cochain complexes

$$\text{Int}^*_X(O^2, \omega^2)(t) : (\Omega^*(M), d_\omega(t)) \rightarrow C^*(X^2, O^2, \omega(t)).$$

If $X^1$ has $\rho(\xi, X^1) < \infty$, equivalently $X^1$ has exponential growth, then for $t$ large enough we have a well defined morphism of cochain complexes

$$\text{Int}^*_X(O^1, \omega^1)(t) : (\Omega^*(M), d_\omega(t)) \rightarrow C^*(X^1, O^1, \omega(t)).$$

If $X$ has $\rho(\xi, X) < \infty$ then for $t$ large enough we have the morphism of cochain complexes

$$u^*_X(O^1, \omega^1)(t) : C^*(X^1, O^1, \omega)(t) \rightarrow C^*(X^2, O^2, \omega)(t)$$

and the algebraic homotopy $h^*_X(O^1, \omega^1)(t)$ making the diagram below homotopy commutative

$$\begin{array}{ccc}
\text{Int}^*_X(O^1, \omega^1)(t) & \xrightarrow{\text{Id}} & (\Omega^*(M), d_\omega(t)) \\
\downarrow & & \downarrow \\
C^*(X^1, O^1, \omega)(t) & \xrightarrow{u^*_X(O^1, \omega^1)(t)} & C^*(X^2, O^2, \omega)(t)
\end{array}$$

with all arrows inducing isomorphisms in cohomology.

6.2. A few observations about torsion. Consider cochain complexes $(C^*, d^*)$ of free $A$–modules of finite rank. The cohomology $H^* := H^*(C^*, d^*)$ is also a graded $A$–free module of finite rank. Here $A$ is a commutative ring with unit.

Recall that the bases $m^I \equiv \{m^I_1, \ldots, m^I_k\}$ and $m^J \equiv \{m^J_1, \ldots, m^J_k\}$ of the free $A$–module $M$ are equivalent iff the isomorphism $T : M \rightarrow M$ defined by $T(m^I_i) = m^J_i$ has determinant $\pm 1$.

For two equivalence classes of bases, $[\mathcal{I}]$ of $C^*$ and $[\mathcal{H}]$ of $H^*$, Milnor, cf. [13], has defined the torsion $\tau((C^*, d^*), [\mathcal{I}], [\mathcal{H}]) \in A^+ / \{\pm 1\}$ where $A^+$ denotes the multiplicative group of invertible elements of $A$.

Recall that the bases $m^I \equiv \{m^I_1, \ldots, m^I_k\}$ and $m^J \equiv \{m^J_1, \ldots, m^J_k\}$ of the free $A$–module $M$ are equivalent iff the isomorphism
If the complex \((C^*, d^*)\) is acyclic there is no need of the base \(h\) and one has \(\tau((C^*, d^*), [\mathcal{C}], [h^0]) \in A^+ / \{ \pm 1 \}\). If \(\alpha : A \to B\) is a unit preserving ring homomorphism, by tensoring \((C^*, d^*), [\mathcal{C}], [h^0]\) with \(B\), regarded as an \(A\)–module via \(\alpha\), one obtains \((\mathcal{C}^*, (d^*)^*), [\mathcal{C}'], [h']\) a cochain complex of free \(B\)–modules whose cohomology is a free \(B\)–module and the bases \([\mathcal{C}'], [h']\). Clearly

\[
\tau(((\mathcal{C}^*, (d^*)^*), [\mathcal{C}'], [h'])) = \alpha(\tau((C^*, d^*), [\mathcal{C}], [h^0])).
\]

If \(A\) is the field \(\mathbb{R}\) or \(\mathbb{C}\), hence \((C^*, d^*)\) is a cochain complex of finite dimensional vector spaces, and \(\langle \cdot, \cdot \rangle\) are scalar products in \(C^*\) one can define the \(T\)–torsion,

\[
T((C^*, d^*), \langle \cdot, \cdot \rangle) \in \mathbb{R}_+, \text{ by the formula}
\]

\[
\log T((C^*, d^*), \langle \cdot, \cdot \rangle) = 1/2 \sum (-1)^{i+1} i \log \det '\Delta_i
\]

where \(\det '\Delta_i\) is the product of the non-zero eigen values of \(\Delta_i := (d^{i+1})^* \cdot d^i + d^{i-1} \cdot (d^i)^*\). Here \((d^i)^*\) denotes the adjoint of \(d^i\) with respect of the scalar product \(\langle \cdot, \cdot \rangle\).

If in addition a scalar product \(\langle \cdot, \cdot \rangle\) in cohomology \(H^* := H^*(C^*, d^*)\) is given one defines the positive real numbers \(\text{Vol}(H^i, \langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle)\) to be the volume of the isomorphism

\[
\ker d^i / d^{i-1}(C^{i-1}) \to H^i.
\]

Here the first vector space is equipped with the scalar product induced from \(\langle \cdot, \cdot \rangle\) and the second with the scalar product \(\langle \cdot, \cdot \rangle\). (Recall that the volume of an isomorphism \(\theta : (V_1, \langle \cdot, \cdot \rangle_1) \to (V_2, \langle \cdot, \cdot \rangle_2)\) between two Euclidean vector spaces is the positive real number defined by:\(\log \text{Vol}(\theta) := 1/2 \log \det \theta^t \cdot \theta\).)

If \(A\) is \(\mathbb{R}\) or \(\mathbb{C}\) then any base \(\mathcal{C}\) resp. \(h\) induce a scalar product \(\langle \cdot, \cdot \rangle\) \(\langle \cdot, \cdot \rangle\) resp. \(\langle \cdot, \cdot \rangle\), the unique scalar product which makes the base orthonormal. Although equivalent bases do not necessary provide the same scalar products they do however lead to the same \(T\)–torsions. This follows by inspection from Milnor’s definition. Moreover one has

\[
|\tau((C^*, d^*), [\mathcal{C}], [h^0])| = T((C^*, d^*), \langle \cdot, \cdot \rangle) + \sum (-1)^i \log \text{Vol}(H^i, \langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle)
\]

Let \(u^* : (C_1^*, d_1^*) \to (C_2^*, d_2^*)\) be a morphism of cochain complexes of free \(A\)–modules of finite rank which induce isomorphism in cohomology. Then the mapping cone \(Cu^*\) is an acyclic cochain complex of free \(A\)–modules of finite rank.

Two equivalence classes of bases \([\mathcal{C}_1]\) of \(C_1^*\) and \([\mathcal{C}_2]\) of \(C_2^*\) provide an equivalence class of bases \([\mathcal{C}]\) of \(Cu^*\), and permit to define

\[
\tau(u^*, [\mathcal{C}_1], [\mathcal{C}_2]) := \tau(Cu^*, [\mathcal{C}]).
\]

If \(A\) is \(\mathbb{R}\) or \(\mathbb{C}\) the scalar products \(\langle \cdot, \cdot \rangle_1\) and \(\langle \cdot, \cdot \rangle_2\) in \(C_1^*\) and \(C_2^*\) provide the scalar product \(\langle \cdot, \cdot \rangle\) in \(Cu^*\) and permit to define

\[
T(u^*, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2) := T(Cu^*, \langle \cdot, \cdot \rangle).
\]

If the scalar products \(\langle \cdot, \cdot \rangle_i := \langle \cdot, \cdot \rangle_{\mathcal{C}_i}, i = 1, 2\) are induced from the bases \(c_i, i = 1, 2\) we also have

\[
|\tau(u^*, [\mathcal{C}_1], [\mathcal{C}_2])| = T(u^*, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2)
\]

(33)

It is a simple exercise of linear algebra (cf. [3]) to check that:
Proposition 17. 

(i) Let \( u^* : (C_1^*, d_1^*) \to (C_2^*, d_2^*) \) be a morphism of cochain complexes of free \( A \)-modules whose cohomology modules \( H_i^* = H^*(C_i^*, d_i^*) \), \( i = 1,2 \) are also free. Suppose that \( u^* \) induces isomorphisms in cohomology and denote these isomorphisms by \( H^*_u \). Suppose \( C_1 \) and \( C_2 \) are bases of \( C_1^* \) and \( C_2^* \) and \( h_1 \) and \( h_2 \) are bases in \( H_1^* \) and \( H_2^* \). If
\[
\prod (\det H^*_u)^{(-1)^i} = \pm 1,
\]
with \( (\det H^*_u) \) calculated with respect to the bases \( h_1 \) and \( h_2 \), then
\[
\tau(u^*, [C_1], [C_2]) = \tau(C_2^*, [C_2], [h_2]) \cdot \tau(C_1^*, [C_1], [h_1])^{-1}.
\]

(ii) Suppose \( A \) is \( \mathbb{R} \) or \( \mathbb{C} \) and \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) are scalar products on \( C_1^* \) and \( C_2^* \). Then
\[
\log T(u^*, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2) = \log T((C_1^*, d_1^*), \langle \cdot, \cdot \rangle_1)
- \log \tau((C_2^*, d_2^*), \langle \cdot, \cdot \rangle_2) + \log \text{Vol} H^*_u \tag{34}
\]
where \( \log \text{Vol} H^*_u = \sum_i (-1)^i \log \text{Vol} H^*_u \) and \( \text{Vol} H^*_u \) is calculated with respect to the scalar product induced from \( \langle \cdot, \cdot \rangle_i \), \( i = 1,2 \) in cohomology. Moreover if \( u^* \) is an isomorphism then
\[
\log T(u^*, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2) = \sum_i (-1)^i \log \text{Vol} u^i.
\]

We conclude this subsection by recalling the following result of Bismut–Zhang, see [1] and [3].

Suppose that \((M, g)\) is a closed Riemannian manifold \( X = -\text{grad}_g f \) with \( \tau = (f, g') \) a generalized triangulation, \( \rho \) a representation of \( \pi_1(M) \) and \( \mu \) a Hermitian structure in the flat vector bundle associated with \( \rho \). Consider \( \text{Int}_\tau^* : (\Omega^* (M, \rho), d_\rho) \to (C^* (\tau, \rho), d_\tau, \rho) \) and equip each of these complexes with a scalar product, the first complex with the scalar product induced from the Riemannian metric \( g \) and the Hermitian structure \( \mu \) and the second with the scalar product \( \langle \cdot, \cdot \rangle_\mu, \tau \) induced from the generalized triangulation \( \tau \) and the Hermitian structure \( \mu \), cf. [3]. In this notation \( C^q(\tau, \rho) \) can be viewed as the vector space of sections above \( \mathcal{X}_q \) of the vector bundle \( E_\rho \to M \) equipped with the hermitian structure \( \mu \). This is a finite dimensional vector space with a scalar product.

Equip the cohomology of these cochain complexes with the induced scalar product. Denote by \( H^* \text{Int} \) the isomorphism induced in cohomology and write
\[
\log V H(\rho, \mu, g, \tau) = \sum (-1)^q \log \text{Vol}(H^* \text{Int}) \tag{35}
\]

Let \( \omega(\mu) \) be the closed one form induced by \( \mu \) as described in [1] and [3]. To recall its definition denote by \( \text{det} E_\rho \to M \) the determinant line bundle of \( E_\rho \to M \) and equip it with the induced flat connection and hermitian structure. The form \( \omega(\mu) \) is the logarithmic differential of the norm (with respect to the induced hermitian structure) of a parallel section in \( \text{det} E_\rho \to M \). Such sections exist only locally but their logarithmic differential is defined on the entire manifold \( M \) and independent of the chosen section.

We have the following result due to Bismut–Zhang, cf. [3].
Theorem 7. With the hypothesis above we have
\[ \log T_{an}(M, g, \rho, \mu) = \log T(C^*(\tau, \rho), \delta_{O, \rho}, \langle \cdot, \cdot \rangle_{\mu, \rho}) + \log VH(\rho, \mu, g, \tau) + R(X, \omega(\mu), g) \]

6.3. A summary of Hutchings–Pajitnov results. We begin by recalling the results of Hutchings cf. [1].

Let \( M \) be a compact smooth manifold, \( m \in M \) a base point and \( \xi \in H^1(M; \mathbb{R}) \). Recall from section 1.1 that \( \xi \) defines the free abelian group \( \Gamma \) and induces the injective homomorphism \( \xi : \Gamma \to \mathbb{R} \). Denote by \( \pi : M \to M \) the principal \( \Gamma \)-covering canonically associated with \( x \) and \( \pi_1(M, m) \to \Gamma \). To \( \xi \) we associate

(i) the Novikov ring \( \Lambda_\xi \) with coefficients in \( \mathbb{R} \) which is actually a field,
(ii) the subring \( \Lambda_{\xi, \rho} \subset \Lambda_\xi \), for any \( \rho \in [0, \infty) \), cf. below,
(iii) the multiplicative groups of invertible elements \( \Lambda_{\xi, \rho}^* \subset \Lambda_\xi \) and \( \Lambda_{\xi, \rho}^+ \subset \Lambda_{\xi, \rho}^* \).

The Novikov ring \( \Lambda_\xi \) consists of functions \( f : \Gamma \to \mathbb{R} \) which satisfy the property that for any real number \( R \in \mathbb{R} \) the cardinality of the set \( \{ \gamma \in \Gamma | f(\gamma) \neq 0, \xi(\gamma) \leq R \} \) is finite. The multiplication in \( \Lambda_\xi \) is given by convolution, cf. [1].

Note \( \mathbb{Z}[\Gamma] \subset \Lambda_{\xi, \rho} \subset \Lambda_\xi \) and \( H^*_{\text{sing}}(M; \Lambda_\xi) := H^*_{\text{sing}}(\hat{M}; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} \Lambda_\xi \) is a finite dimensional vector space over the field \( \Lambda_\xi \). Let \( \det H^*_{\text{sing}}(M; \Lambda_\xi) \) denote the one dimensional vector space over \( \Lambda_\xi \) defined by
\[
\det H^*_{\text{sing}}(M; \Lambda_\xi) = \bigotimes_i (\Lambda^i(H^i_{\text{sing}}(M; \Lambda_\xi)))^{\epsilon(i)}
\]

where \( V^{\epsilon(i)} = V \) if \( i \) is even and \( V^{\epsilon(i)} \) is the dual of \( V \) if \( i \) is odd. Let \( X \) be a vector field which satisfies MS and has \( \xi \) as a Lyapunov cohomology class and let \( \hat{X} \) be the pullback of \( X \) on \( \hat{M} \). Choose \( O \) a collection of orientations for the unstable manifolds of the rest points of \( X \) and therefore of the rest points of \( \hat{X} \). Denote by \( (NC^\partial_{X, \xi, \rho}, \partial^\partial) \) the Novikov cochain complex of free \( \Lambda_\xi \) modules (vector spaces since \( \Lambda_\xi \) is a field) as defined in [1] and by \( H^*_{\partial}(M; \Lambda_\xi) \) its cohomology. There exists a canonical isomorphism
\[
HV^*_{X, \xi} : H^*_{\partial}(M; \Lambda_\xi) \to H^*_{\text{sing}}(M; \Lambda_\xi)
\]
described below.

The isomorphism \( HV^*_{X, \xi} \) : Note that in view of Proposition 14 for any two vector fields \( X^1 \) and \( X^2 \) which satisfy MS and have \( \xi \) as Lyapunov cohomology class, there exists homotopies \( \hat{X} \) from \( X^1 \) to \( X^2 \) which satisfy MS. The incidences, \( \overline{\partial^1_{x_2, x_1}(\hat{\alpha})} \) defined in subsection 6.1 provide a morphism
\[
u_u^* : (NC^\partial_{X^1, \xi, \rho^1}, \partial^\partial_{x_1}) \to (NC^\partial_{X^2, \xi, \rho^2}, \partial^\partial_{x_2})
\]
of cochain complexes which induces isomorphism between their cohomology. This cohomology isomorphism is independent of the homotopy \( \hat{X} \) and will be denoted by
\[
H^* u(X^1, X^2) : H^*_{\partial}(X^1, M; \Lambda_\xi) \to H^*_{\partial}(X^2, M; \Lambda_\xi).
\]

To show this one proceeds in the following way. One introduces the concept of homotopy between two homotopies \( X^1 \) and \( X^2 \) (both homotopies from the vector field
Let \( \tau = (f, g) \) be generalized triangulation, and \( \xi \in H^1(M; \mathbb{R}) \). Let \( X' := -\grad_g f \). By Proposition \[ \text{a} \] \( X \) has \( \xi \) as Lyapunov cohomology class. The Novikov complex \((NC_{X,\xi}^q, \partial_C^q)\) identifies to the geometric cochain complex associated to \((\tilde{f}, g)\) to \( \tilde{M} \), tensored (over \( \mathbb{Z}[\Gamma] \)) by \( \Lambda_{\xi} \). Recall that the geometric (or Morse) cochain complex associated to \((\tilde{f}, g)\) is a cochain complex of free \( \mathbb{Z}[\Gamma] \) modules which can be regarded as a quotient of \( C^*_\text{sing}(\tilde{M}) \). When tensored with \( \Lambda_{\xi} \) over \( \mathbb{Z}[\Gamma] \) it calculates the same cohomology as \( C^*_\text{sing}(\tilde{M}) \otimes \mathbb{Z}[\Gamma] \Lambda_{\xi} \). We have therefore a well defined isomorphism from \( H^*_{\tilde{X}}(M; \Lambda_{\xi}) \) to \( H^*_{\text{sing}}(M; \Lambda_{\xi}) \).

The composition of the this isomorphism with \( H^* u(X, X') \) provides the canonical isomorphism \( HV^*_{\tilde{X}} : H^*_{\tilde{X}}(M; \Lambda_{\xi}) \rightarrow H^*_{\text{sing}}(M; \Lambda_{\xi}) \).

Denote by \( E(M, m) \) the set of Euler structures based at \( m \in M \) cf. \[ \text{b} \] or \[ \text{c} \] for a definition, and let \( e \in E(M, m) \). Recall that in the presence of \( X \) an Euler structure \( e \) is represented by an Euler chain (cf. \[ \text{c} \]) which consists of a collection of paths \( \alpha_q \) from \( m \) to \( x \in X \). Each such path provides a lift \( \tilde{x} \) of \( x \) (i.e. \( \pi(\tilde{x}) = x \)) and therefore a base \( \{ E_x | x \in X \} \) with \( E_x \) the characteristic function of \( \tilde{x} \) regarded as an element of \( NC_{X,\xi,\rho}^q, q = \text{ind}(x) \). Conversely, any lift \( s : X \rightarrow \tilde{X}, s(x) = \tilde{x} \) defines an Euler chain and therefore together with \( X \) an Euler structure \( e \). The path \( \alpha_x \) is the image by \( \pi \) of a smooth path from \( m \) to \( \tilde{x} \). Different Euler chains representing the same Euler structure might provide nonequivalent bases. All theses bases will be named \( e \)-compatible and denoted by \( \underline{e} \). Any lift \( s \) which defines with \( X \) the Euler structure \( e \) will be also called \( e \)-compatible.

Choose an element \( o_H \in \det H^*_{\text{sing}}(M; \Lambda_{\xi}) \setminus \{0\} \), and consider bases \( h^* \in H^*(NC_{X,\xi}^q, \partial_C^q) \) which represent via the isomorphism \( HV^*_X \) the element \( o_H \). They all will be called \( o_H \)-compatible. Again the \( o_H \)-compatible bases might not be equivalent, however an inspection of Milnor definition of torsion \[ \text{d} \] implies that the element

\[
\tau((NC_{X,\xi}^q, \partial_C^q), [\underline{e}], [\underline{h}]) \in \Lambda_{\xi}^+ / \{\pm 1\}
\]
as defined in section \[ \text{e} \] for \( \underline{e} \) resp. \( \underline{h} \) \( e \)-compatible resp. \( o_H \)-compatible bases depends only on \( e \) and \( o_H \); therefore denoted by \( \tau_e(X, e, o_H) \).

If \( H^*_{\text{sing}}(M; \Lambda_{\xi}) = 0 \) there is no need of \( o_H \) and we have \( \tau_e(X, e) \in \Lambda_{\xi}^+ / \{\pm 1\} \).

Note that if \( X \) has exponential growth, in view of Theorem \[ \text{f} \] the complex \((NC_{X,\xi,\rho}^q, \partial_C^q)\) contains, for \( \rho \) large enough, a subcomplex of free \( \Lambda_{\xi,\rho} \) modules \((NC_{X,\xi,\rho}^q, \partial_C^q)\) cf. \[ \text{g} \] so that

\[
(NC_{X,\xi,\rho}^q, \partial_C^q) \otimes_{\Lambda_{\xi,\rho}} \Lambda_{\xi} = (NC_{X,\xi}^q, \partial_C^q).
\]

Moreover an \( e \)-compatible base will provide a base of \( \Lambda_{\xi,\rho} \)-modules in this subcomplex. If \( H^*_{\text{sing}}(M; \Lambda_{\xi}) = 0 \) then \( \tau_e(X, e) \in \Lambda_{\xi,\rho}^+ / \{\pm 1\} \) for \( \rho \) large enough. More general, if \( H^*_{\text{sing}}(M; \Lambda_{\xi,\rho}) \) is a free \( \Lambda_{\xi,\rho} \)-module and \( o_H \in \det H^*_{\text{sing}}(M; \Lambda_{\xi,\rho}) := \bigotimes_i (\Lambda^i(H^*_X(M; \Lambda_{\xi,\rho})))^{(i)} \) we will have \( \tau_e(X, e, o_H) \in \Lambda_{\xi,\rho}^+ / \{\pm 1\} \).
If the homotopy $X$ has exponential growth then, for $\rho$ big enough, we have $u^q_{X,\mathcal{O}_1,\mathcal{O}_2}(NC^q_{X_1,\xi,\rho}, \partial_{\mathcal{O}_1}) \subset (NC^q_{X_2,\xi,\rho}, \partial_{\mathcal{O}_2})$ and $\tau(u^q_{X,\mathcal{O}_1,\mathcal{O}_2}, [\xi_1], [\xi_2])$ which depends only on $X^1, X^2$ and $e$, lies in $\Lambda^+_\xi,\rho/\{\pm 1\}$.

Note that for $t > \rho$ we denote by $e_{\mathcal{O}} : \Lambda^+_\xi,\rho \to \mathbb{R}$ the ring homomorphism which associates to each $f \in \Lambda^+_\xi,\rho$ interpreted as a Dirichlet series $f$, the value of the Laplace transform $L(f)$ at $t$, cf. section 1.2. When applied to torsion it calculates the torsion of the corresponding complex tensored by $\mathbb{R}$.

Suppose now that $X$ is MS and satisfies also NCT. As noticed in [9], $Z_X \in \Lambda^+\xi$ and then $e^{Z_X} \in \Lambda^+\xi$. The main result of Hutchings can be formulated as follows

**Theorem 8.** If $X^1$ and $X^2$ are two vector fields which are MS and NCT and have $\xi$ as a Lyapunov cohomology class then

$$e^{Z_{X^1}} \cdot \tau_{\xi}(X^1, e, o_H) = e^{Z_{X^2}} \cdot \tau_{\xi}(X^2, e, o_H).$$

The proof of this theorem is given in [9]. The author considers only the acyclic case (in which case $o_H$ is not needed). The acyclicity hypothesis is used only to insure that the Milnor torsion (cf. [13]) $\tau_{\xi}(X, e)$ can be defined. This can be also defined in the non-acyclic case at the expense of the orientation $o_H$. The orientation $o_H$ induces via $\nu_{\xi}$ an orientation in the cohomology of the Novikov complex associated to $X$ and together with the Euler structure $e$ a class of bases in the Novikov complex. From this moment on the arguments in [9] can be repeated word by word.

Let $X$ be a homotopy from the vector field $X^1$ to the vector field $X^2$ which is MS and suppose that both vector fields have $\xi$ as a Lyapunov cohomology class. The incidences $X^{\mathcal{O}_1,\mathcal{O}_2}$, cf. (26), induced from $X$ and the orientation $\mathcal{O} = \mathcal{O}_1 \sqcup \mathcal{O}_2$ provide a morphism $u^*_{X,\mathcal{O}_1,\mathcal{O}_2} : (NC^*_X, \partial_{\mathcal{O}_1}) \to (NC^*_X, \partial_{\mathcal{O}_2})$ which induces an isomorphism in cohomology as already indicated.

Choose bases $\xi_i$ in each of the Novikov complexes $(NC^*_X, \partial_{\mathcal{O}_i})$, $i = 1, 2$, which are $e$–compatible. By the same inspection of the Milnor definition of torsion one concludes that $\tau(u^*_{X}, [\xi_1], [\xi_2]) \in \Lambda^+_H/\{\pm 1\}$ defined in section 6.2 depends only on $X^1, X^2$ and $e$. In view of Proposition 17 and of Theorem 8 one obtains

**Proposition 18.** Suppose $X^2$ and $X^1$ are two vector fields which satisfy MS and NCT and have $\xi$ as a Lyapunov cohomology class. Let $e$ be an Euler structure as above. Then

$$\tau(u^*_{X,\mathcal{O}_1,\mathcal{O}_2}, [\xi_1], [\xi_2]) = e^{Z_{X^2}} \cdot e^{-Z_{X^1}}.$$

As a consequence $\tau(u^*_{X}, [\xi_1], [\xi_2])$ depends only on $X^1, X^2$ and then can be denoted by $\tau(X^1, X^2)$.

Suppose $X^2 = -\operatorname{grad}_{g''} f$, $\tau = (f, g'')$ a generalized triangulation.

**Corollary 5.** $\tau(X^1, X^2) = e^{Z_{X^1}}.$

It is not hard to see that Hutchings theorem is equivalent to this corollary. In this form the result was also established by Pajitnov [10].

Suppose $X$ is a vector field with $\xi$ a Lyapunov cohomology class which satisfies MS and in addition has exponential growth. The exponential growth implies that any $e$–compatible base of $NC^q_{X,\xi}$ is actually a base of $NC^q_{X,\xi,\rho}$ for $\rho$ large enough.
For \( t > \rho \) the \( \mathbb{R} \)-linear maps \( \text{Ev}_t^\omega : NCq_{X^1,\xi,\rho} \to \text{Maps}(X_q, \mathbb{R}) \) defined by
\[
\text{Ev}_t^\omega(f) := \sum_{\tilde{x} \in \pi^{-1}(x)} f(\tilde{x}) e^{-t \hat{h}(\tilde{x})}
\]
provide a morphism of cochain complexes \( \text{Ev}_t^\omega : (NC^*_{X^1,\xi,\rho}, \partial^*_C) \to \mathbb{C}^*(X, \mathcal{O}, \omega)(t) \) with \( \text{Ev}_t^\omega(\mathcal{E}_x) = e^{-t \hat{h}(\tilde{x})} \mathcal{E}_x \). We regard \( \mathbb{C}^*(X, \mathcal{O}, \omega)(t) \) equipped with the canonical base \( \{ \mathcal{E}_x \} \) with \( E_x \) the characteristic function of \( x \in \mathcal{X} \). The isomorphism \( \text{Ev}_t^\omega \) factors through an isomorphism from \( (NC^*_{X^1,\xi,\rho}, \partial^*_C) \otimes_{\text{ev}_t} \mathbb{R} \) to \( \mathbb{C}^*(X, \mathcal{O}, \omega)(t) \). If the Novikov complex \((NC^*_{X,\xi,\rho}, \partial^*_C)\) is acyclic so is \( \mathbb{C}^*(X, \mathcal{O}, \omega)(t) \).

Let \( s : \mathcal{X} \to \tilde{\mathcal{X}} \) be a compatible lift and \( \xi \) the associated \( e \)-compatible base. A simple inspection of Milnor definition of torsion leads to
\[
\text{ev}_t(\tau(X, \xi, e)) = \text{ev}_t(\tau((NC^*_{X,\xi,\rho}, \partial^*_C), [\xi]) = \tau(\mathbb{C}^*(X, \mathcal{O}, \omega)(t), [E_x]) \cdot e^{-t \sum_{x \in \mathcal{X}} \text{IND}(x) \hat{h}(\tilde{x})}
\]
Suppose now that \( X^i, i = 1, 2 \), are two vector fields which have \( \xi \) as a Lyapunov cohomology class and \( \mathcal{X} \) is a homotopy from \( X^1 \) to \( X^2 \). Suppose in addition that \( X^1 \) and \( \mathcal{X} \) satisfy MS and have exponential growth. Then we obtain the morphism of Novikov cochain complexes \( u^*_{\xi,\rho} : (NC^*_{X^1,\xi,\rho}, \partial^*_C) \to (NC^*_{X^1,\xi,\rho}, \partial^*_C) \) which induces an isomorphism in cohomology. When tensored by \( \mathbb{R} \) via \( \text{ev}_t : \Lambda_{\xi,\rho} \to \mathbb{R} \) this morphism identifies to \( u^*_{\xi,\rho} : (NC^*_{X^1,\xi,\rho}, \partial^*_C) \to (NC^*_{X^1,\xi,\rho}, \partial^*_C) \).

The Euler structure \( e \in E(M, \rho) \) permits to choose \( e \)-compatible lifts of the rest points and then \( e \)-compatible bases \( e_1 \) and \( e_2 \). The inspection of Milnor definition of torsion leads to
\[
\text{ev}_t(\tau(u^*_{\xi,\rho}, \mathcal{O}^1, \mathcal{O}^2), [\mathcal{E}_1], [\mathcal{E}_2]) = \tau(u^*_{\xi,\rho}, \mathcal{O}^1, \mathcal{O}^2, \omega)(t), [E_{x_1}], [E_{x_2}]) \cdot e^{-t \sum_{x \in \mathcal{X}^1} \text{IND}(x) \hat{h}(\tilde{x}) - \sum_{x \in \mathcal{X}^2} \text{IND}(x) \hat{h}(\tilde{x})}
\]
where \( E_{x_1} \), resp. \( E_{x_2} \) are the canonical base provided by the rest points of \( X^1 \) resp. \( X^2 \).

Note that in view of Proposition \( \text{(Additional property)} \) for any \( e \)-compatible lifts of \( \mathcal{X}^1 \) and \( \mathcal{X}^2 \) we have:
\[
\sum_{x \in \mathcal{X}^1} \text{IND}(x) \hat{h}(\tilde{x}) - \sum_{x \in \mathcal{X}^2} \text{IND}(x) \hat{h}(\tilde{x}) = I(X^1, X^2, \omega).
\]
Different lifts which define the same the Euler structure keep the left side of \( I \) unchanged.

### 6.4. The geometry of closed one form
Suppose \( M \) is a connected smooth manifold and \( p \in M \) is a base point. The homomorphism \( [\omega] : H_1(M; \mathbb{Z}) \to \mathbb{R} \) induces the one dimensional representation \( \rho = \rho_\omega : \pi_1(M, p) \to \text{GL}_1(\mathbb{R}) \) defined by the composition \( \pi_1(M, p) \to H_1(M; \mathbb{Z}) \to [\omega] : \mathbb{R} \to \mathbb{R} \to \text{GL}_1(\mathbb{R}) \). The representation \( \rho \) provides a flat rank one vector bundle \( \xi_\rho : E_\rho \to M \) with the fiber above \( p \) identified with \( \mathbb{R} \). This bundle is the quotient of trivial flat bundle \( \tilde{M} \times \mathbb{R} \to \tilde{M} \) by the group \( \Gamma \) which acts diagonally on \( \tilde{M} \times \mathbb{R} \). Here \( \tilde{M} \) denotes the principal \( \Gamma \)-covering associated with \( [\omega] \) and constructed canonically with respect to \( p \) (from the set of continuous paths originating from \( p \)). Note that \( \tilde{M} \) is equipped with a base point \( \tilde{p} \) corresponding to the constant path in \( p \). The group \( \Gamma \) acts freely on
\[ \tilde{M} \] with quotient space \( M \). The action of \( \Gamma \) on \( \mathbb{R} \) is given by the representation \( \Gamma \to \mathbb{R} \xrightarrow{\exp} \mathbb{R}_+ = GL_1(\mathbb{R}) \).

There is a bijective correspondence between the closed one forms \( \omega \) in the cohomology class represented by \( \rho \) and the Hermitian structures \( \mu \) in the vector bundle \( \xi_p \) which agree with a given Hermitian structure on the fiber above \( p \) (identified to \( \mathbb{R} \)).

Given \( \omega \) in the cohomology class \( [\omega] \in H^1(M; \mathbb{R}) \), one constructs a Hermitian structure \( \tilde{\mu}_\omega \) on the trivial bundle \( \tilde{M} \times \mathbb{R} \to \tilde{M} \) which is \( \Gamma \)-invariant. Therefore, by passing to quotients one obtains a Hermitian structure \( \mu_\omega \) in \( \xi_p \). The Hermitian structure \( \tilde{\mu}_\omega \) is defined as follows:

(i) Observe that the pull back \( \tilde{\omega} \) of \( \omega \) on \( \tilde{M} \) is exact and therefore equal \( d\tilde{h} \)
where \( \tilde{h} : \tilde{M} \to \mathbb{R} \) is the unique function with \( \tilde{h}(\tilde{p}) = 0 \) and \( d\tilde{h} = \tilde{\omega} \).

(ii) Define \( \tilde{\mu}(\tilde{x}) \) by specifying the length of the vector \( 1 \in \mathbb{R} \). We put \( ||1_{\tilde{x}}||_{\tilde{\mu}(\tilde{x})} := e^{h(\tilde{x})} \).

Given a Hermitian structure \( \mu \) one construct a closed one form \( \omega_\mu \) as follows: Denote by \((\tilde{E}_\rho \to \tilde{M}, \tilde{\mu})\) the pair consisting of the flat line bundle \( \tilde{E}_\rho \to \tilde{M} \) and the Hermitian structure \( \tilde{\mu} \) the pullback of the pair \((E_\rho \to M, \mu)\) to \( \tilde{M} \) by the map \( \tilde{M} \to M \). Let \( \overline{\pi} \) be the Hermitian structure obtained by parallel transporting the scalar product \( \tilde{\mu}_\rho \). Denote by \( \alpha : M \to \mathbb{R} \) the function \( \alpha(\tilde{x}) := ||v||_{\tilde{\mu}(\tilde{x})}/||v||_{\overline{\pi}(\tilde{x})} \) for \( v \) a nonzero vector in \( \tilde{E}_\tilde{x} \).

Define \( \tilde{\omega}_\mu := d\log(\alpha) \) and observe that this is a \( \Gamma \) invariant closed one form, hence descends to a closed one form \( \omega \) on \( M \).

To simplify the writing below we denote (by a slight abuse of notation):

(i) \( \rho(t) := \rho_{t\omega} \)
(ii) \( \mu(t) := \mu_{t\omega} \)

Remark 10. The cochain complex \((\Omega^*(M), d_{\omega}(t))\) equipped with the scalar product induced from \( g \) is isometric to the cochain complex \((\Omega^*(M, \rho(t)))\) equipped with the scalar product induced from \( g \) and \( \mu(t) = \mu \) as defined in [4].

In particular we have
\[
\log T_{an, \omega}^\omega g(\omega, t) = \log T_{an}(M, \rho(t), g, \mu(t))
\]
where \( \log T_{an}(M, \rho, g, \mu) \) is the analytic torsion considered in [4] and associated with the Riemannian manifold \((M, g)\) and the representation \( \rho \) and the Hermitian structure \( \mu \) in the flat bundle induced from \( \rho \).

Remark 11. Let \( \xi \in H^1(M; \mathbb{R}) \) and \( \omega \) a closed one form representing \( \xi \). Suppose \( X = -\text{grad}_g \tau \) where \( \tau = (f, g^n) \) is a generalized triangulation. Choose orientations \( \mathcal{O} \) for the unstable manifolds of \( X \). The morphism
\[
\text{Int}_{X, \mathcal{O}, \omega}(t) : (\Omega^*(M), d_{\omega}(t)) \to C^*(X, \mathcal{O}, \omega)(t)
\]
defined in [5] where \((\Omega^*(M))\) is equipped with the scalar product induced from \( g \) and \( \text{Maps}(\mathcal{X}_q, \mathbb{R}) \) is equipped with the obvious scalar product, i.e. associated with the base \( \{E_x\} \), is isometrically conjugate to
\[
\text{Int}^*(\omega) : (\Omega^*(M, \rho(t)), d_{\rho(t)}(t)) \to (C^*(\tau, \rho(t)), \delta_{\mathcal{O}, \rho(t)}(t))
\]
defined in [4] where \((\Omega^*(M, \rho(t)))\) is equipped with the scalar product induced by \((g, \mu(t))\) and \( C^*(\tau, \rho(t)) \) is equipped with the scalar product induced from \( \tau \) and \( \rho(t) \).
6.5. **Proof of Theorem 4** We begin with a triple \((g, g', \omega)\) with \(X^1 = X = \nabla' g\omega\) as in the hypothesis of Theorem 4. We choose orientations \(O^1\) for the unstable manifolds of \(X^1\). We also choose \(X^2 = \nabla' g\) so that \(\tau = (f, g'')\) is a generalized triangulation and choose orientations \(O^2\) for the unstable manifolds of \(X^2\).

For simplicity of the writing, we will use the following abbreviations: \(I_1(t) := \text{Int}_{X^1, O^1, \omega}(t)|_{\Omega^\omega_{\text{sm}}(M)}\) and \(I_2(t) := \text{Int}_{X^2, O^2, \omega}(t)|_{\Omega^\omega_{\text{sm}}(M)}\).

In view of Proposition 17 and Remark 11 in section 6.4, one has

\[
\log \text{Vol}(H^*(I_1)) = \log \text{Vol}(H^*(I_2)) + t R(\omega, g, X^2)
\]

where \(\langle \cdot, \cdot \rangle_1\) is the scalar product induced from the canonical base \(\{E_x\}, x \in X^1\).

In view of Theorem 4 and Remark 11 in section 6.4, one has

\[
\log T_{an, g}^\omega(t) = \log T(C(X^2, O^2, \omega)(t), \langle \cdot, \cdot \rangle_2) + t R(\omega, g, X^2)
\]

where \(\langle \cdot, \cdot \rangle_2\) is the scalar product induced from the canonical base \(\{E_x\}, x \in X^2\).

Combining (39) and (40) one obtains

\[
\log(\text{Vol}(t)) - \log T_{an, \omega}(t) = \log \text{Vol}(H^*(I_1(t))) - \log \text{Vol}(H^*(I_2(t))) + t \log \text{Vol}(H^*(I_2(t))) + t R(\omega, g, X^2)
\]

First consider the case that \(X = X^1\) has exponential growth and \(H^*(M, t\omega) = 0\) for \(t\) large enough. Note that \(X^2\) has exponential growth too by Proposition 16. Clearly then \(\log \text{Vol}(H^*(I_1(t))) = \log \text{Vol}(H^*(I_2(t))) = 0\). By (39), (40) and (43) we have

\[
\log T(C(X^1, O^1, \omega)(t), \langle \cdot, \cdot \rangle_1) - \log T(C(X^2, O^2, \omega)(t), \langle \cdot, \cdot \rangle_2) =
\]

\[
= \log (\text{ev}_t(\tau(X^1, \xi, e^*_1))) - \log (\text{ev}_t(\tau(X^2, \xi, e^*_2))) - t I(X^1, X^2, \omega)
\]

By Theorem 4 in section 6.3, we have

\[
\log (\text{ev}_t(\tau(X^1, \xi, e^*_1))) - \log (\text{ev}_t(\tau(X^2, \xi, e^*_2))) =
\]

\[
= \log (\text{ev}_t(e^{\mathbb{Z}X^1} \cdot e^{-ZX^2})) = \log (\text{ev}_t e^{-ZX^1}) \]

\[
= -L(\mathbb{Z}_X^1)(t)
\]

Combining (41), (42) and (43) one obtains the result.

Second consider the case \(X\) has (strong) exponential growth property. Then choose a homotopy \(X\) which satisfy: \(\rho(\xi, X) < \infty\). Then for \(t\) big enough, we have the following (algebraically) homotopy commutative diagram of finite dimensional cochain complexes whose arrows induce isomorphisms in cohomology.

\[
\begin{array}{ccc}
\Omega^\omega_{\text{sm}}(M)(t), d_\omega(t) & \xrightarrow{\text{Id}} & \Omega^\omega_{\text{sm}}(M)(t), d_\omega(t) \\
\text{Int}_{X^1, O^1, \omega}(t) & \downarrow & \text{Int}_{X^2, O^1, \omega}(t) \\
C^*(X^1, O^1, \omega)(t) & \xrightarrow{u^*_{X^1, O^1, \omega}} & C^*(X^2, O^1, \omega)(t)
\end{array}
\]

For simplicity of the writing, we will use the following abbreviations: \(I_1(t) := \text{Int}_{X^1, O^1, \omega}(t)|_{\Omega^\omega_{\text{sm}}(M)}\) and \(I_2(t) := \text{Int}_{X^2, O^1, \omega}(t)|_{\Omega^\omega_{\text{sm}}(M)}\).

In view of Proposition 17 and Remark 11 in section 6.4, one has

\[
\log(\text{Vol}(t)) - \log T_{an, g}^\omega(t) = \log \text{Vol}(H^*(I_1(t))) - \log \text{Vol}(H^*(I_2(t))) + t \log \text{Vol}(H^*(I_2(t))) + t R(\omega, g, X^2)
\]

First consider the case that \(X = X^1\) has exponential growth and \(H^*(M, t\omega) = 0\) for \(t\) large enough. Note that \(X^2\) has exponential growth too by Proposition 16. Clearly then \(\log \text{Vol}(H^*(I_1(t))) = \log \text{Vol}(H^*(I_2(t))) = 0\). By (39), (40) and (43) we have

\[
\log T(C(X^1, O^1, \omega)(t), \langle \cdot, \cdot \rangle_1) - \log T(C(X^2, O^2, \omega)(t), \langle \cdot, \cdot \rangle_2) =
\]

\[
= \log (\text{ev}_t(\tau(X^1, \xi, e^*_1))) - \log (\text{ev}_t(\tau(X^2, \xi, e^*_2))) - t I(X^1, X^2, \omega)
\]

By Theorem 4 in section 6.3, we have

\[
\log (\text{ev}_t(\tau(X^1, \xi, e^*_1))) - \log (\text{ev}_t(\tau(X^2, \xi, e^*_2))) =
\]

\[
= \log (\text{ev}_t(e^{\mathbb{Z}X^1} \cdot e^{-ZX^2})) = \log (\text{ev}_t e^{-ZX^1}) \]

\[
= -L(\mathbb{Z}_X^1)(t)
\]

Combining (41), (42) and (43) one obtains the result.

Second consider the case \(X\) has (strong) exponential growth property. Then choose a homotopy \(X\) which satisfy: \(\rho(\xi, X) < \infty\). Then for \(t\) big enough, we have the following (algebraically) homotopy commutative diagram of finite dimensional cochain complexes whose arrows induce isomorphisms in cohomology.
For simplicity we write \( u^*(t) := u^*_{X, \mathcal{O}^1, \mathcal{O}^2}(t) \) and observe that in view of the homotopy commutativity of the above diagram and of Proposition \ref{thm:strong_growth} we have

\[
\log T(u^*(t), \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2) = \log \mathrm{Vol}(H^*(I_1(t))) - \log \mathrm{Vol}(H^*(I_2(t))) + \log T(\mathbb{C}(X^2, \mathcal{O}^1, \omega)(t), \langle \cdot, \cdot \rangle_1)
\]

\[
- \log T(\mathbb{C}(X^2, \mathcal{O}^1, \omega)(t), \langle \cdot, \cdot \rangle)
\]

(44)

As noticed \((\mathbb{C}^*(X^2, \mathcal{O}^2, \omega)(t), \langle \cdot, \cdot \rangle)\) is isometric to \((\mathbb{C}(M, \tau, \rho(t), \mu(t))\).

By \ref{thm:strong_growth} and Proposition \ref{thm:singularity_free} combined with the observations that \(X^2\) has no closed trajectories we have

\[
\log T(u^*(t), \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2) = \log \tau(u^*(t), e_1^*, e_2^*)
\]

\[
= \ev_t(\mathbb{Z}_{X^1}) + I(X^1, X^2, t\omega)
\]

(45)

Combining \ref{eq:strong_growth} and \ref{eq:weak_growth} we obtain

\[
\log T + \ev_t(\mathbb{Z}_{X^1}) + t I(X^1, X^2, \omega) = \log T^\om_{\an, \la}(t) + t R(X^2, \omega, g)
\]

which in view of Proposition \ref{thm:strong_growth} implies the result.

When \( H^*(M; t\Lambda_\xi) \) is acyclic we do not need the morphism \( u^* \) and a simple consequence of Corollary \ref{cor:acyclic} implies the result. It turns out that the strong exponential growth can be weaken to the (apparently) weaker hypothesis \( H^*_{\sing}(M; \Lambda_{\xi, \rho}) \) is a free module over \( \Lambda_{\xi, \rho} \) for some \( \rho \). As in acyclic case one can circumvent the morphism \( u^* \).

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