Extremal limit of the regular charged black holes in nonlinear electrodynamics

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The near horizon limit of the extreme nonlinear black hole is investigated. It is shown that resulting geometry belongs to the $\text{AdS}_2 \times S^2$ class with different modules of curvatures of subspaces and could be described in terms of the Lambert functions. It is demonstrated that the considered class of Lagrangians does not admit solutions of the Bertotti-Robinson type.

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Still growing interest in the extremal black holes that is seen recently is motivated by their unusual and not fully understood nature. The problems of entropy, semiclassical configurations, interactions with matter, or information paradox – to name a few – have not been resolved yet. Apart from their global structure and behaviour, the near-horizon region is also of interest. Indeed, applying appropriate limiting procedure in the geometry of the extreme and near extreme black holes one can generate new exact solutions [1, 2, 3, 4, 5, 6].

On the other hand, of the equal importance are the questions of the nature of singularities that reside, hidden to external observer, in the centers of most black holes. According to the widespread opinion, singularities that plague such models are symptoms of illness of the theory rather than its health and the role of the regular solutions in our understanding of the black hole physics could not be overestimated. One of the methods which can be used in constructions of non-singular models is replacing black hole interior by a regular core. This idea appeared almost forty years ago, in mid sixties [7, 8, 9] and is actively investigated today (see Ref. [10] for a discussion and a comprehensive list of references). The exact analytical solution for a specific sources has been constructed in Refs. [11, 12, 13].

Among various regular models known to date, especially intriguing are the solutions to the coupled equations of nonlinear electrodynamics and general relativity found by Ayón–Beato and García [14] and by Bronnikov [15]. The latter describes magnetically charged configuration. It should be noted however that the Ayón–Beato and García solution has been constructed with the use of different Lagrangians valid in different regions [15]. Attempts to circumvent the conditions of the no go theorem have been undertaken by Burinskii and Hildebrandt [16]. They demonstrated that modifications of the Ayón–Beato and García model, in which electric field does not extend to the central region are in concord with the no go theorem.

The coupled system of equations obtained from the total action

$$S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} [R - L(F)]$$

(1)

for the static and spherically symmetric configuration described by the line element of the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2,$$

(2)

$$f(r) = 1 - \frac{2m(r)}{r},$$

(3)

with nonlinear magnetic field reduce to simple quadrature

$$m(r) = \frac{1}{4} \int L(F) r^2 dr + C,$$

(4)

where $C$ is an integration constant. Integrating Eq. (4),

$$m(r) = \frac{1}{4} \int L(F) r^2 dr + C,$$

(4)

Let us consider general relativistic formulation of nonlinear electrodynamics described by a class of a gauge invariant Lagrangians $L(F)$, where $F = F_{ab} F^{ab}$ and $F_{ab} = \partial_a A_b - \partial_b A_a$. According to the well-known theorem [17, 18], if $L(F)$ has a Maxwell asymptotic for weak fields, i.e., $L(F) \sim F$ and $\mathcal{L}_F = d\mathcal{L}/dF \rightarrow \text{const}$ as $F \rightarrow 0$, then any static and spherically symmetric solutions to the coupled equations of the nonlinear electrodynamics and general relativity with nonzero electric charge cannot have a regular center. It does not mean that this no go theorem forbids existence of regular black holes in general. Simple example of a regular magnetic solution has been presented by Bronnikov in Ref. [15]. It bears a formal resemblance to the solution found earlier by Ayón–Beato and García [14] describing electrically charged configuration. It should be noted however that the Ayón–Beato and García solution has been constructed with the use of different Lagrangians valid in different regions [15]. Attempts to circumvent the conditions of the no go theorem have been undertaken by Burinskii and Hildebrandt [16]. They demonstrated that modifications of the Ayón–Beato and García model, in which electric field does not extend to the central region are in concord with the no go theorem.

In this note we shall investigate the extremal magnetically charged black hole with the special emphasis put on the near horizon geometry and its relation with the exact solutions of the Einstein field equations with topology $\text{AdS}_2 \times S^2$. To begin with, however, we remind a few basic facts.

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with function $\mathcal{L}(F)$ given by
\[ \mathcal{L}(F) = F \cosh^{-2} \left[ a \left( \frac{F}{2} \right)^{1/4} \right], \tag{5} \]
where $F = 2Q^2/r^4$ and $a$ is a free parameter, one obtains
\[ m(r) = C - \frac{|Q|^{3/2}}{2a} \tanh \left( \frac{a|Q|^{1/2}}{r} \right). \tag{6} \]

Employing boundary condition $m(\infty) = M$ and setting $a = |Q|^{3/2}/2M$ yields
\[ f(r) = 1 - \frac{2M}{r} \left( 1 - \tanh \frac{Q^2}{2Mr} \right). \tag{7} \]

One of the most attractive features of this solution is possibility to express the location of the horizons in terms of the Lambert functions, $W_i(\xi)$. The Lambert functions $W_i$ defined by means of the general formula
\[ \exp(W(\xi))W(\xi) = \xi, \tag{8} \]
have two real branches, $W_0(\xi)$ and $W_{-1}(\xi)$ with a branch point $-1/e$. The run of the real branches of the Lambert functions is displayed in Fig. 1. Since the value of the principal branch of the Lambert function at $1/e$ plays an important role in our considerations we put
\[ W_0(1/e) = w_0. \tag{9} \]

To solve equation $f(r) = 0$ let us introduce a new ‘radial’ coordinate $x$ and a new parameter $q$ by means of $r = Mx$ and $Q^2 = q^2M^2$, respectively. Subsequently introducing a new unknown function $\tilde{W}$
\[ x = -\frac{4q^2}{4W - q^2}, \tag{10} \]
one arrives at
\[ \exp(\tilde{W})\tilde{W} = -\frac{q^2}{4} \exp(q^2/4). \tag{11} \]

As the result of simple manipulations one can relate the exact location of the event horizon $r_+ (= Mx_+)$ and the inner horizon $r_- (= Mx_-)$ with the the real branches of the Lambert functions
\[ x_+ = -\frac{4q^2}{4W_0(-\frac{q^2}{4} \exp(q^2/4)) - q^2}, \tag{12} \]
and
\[ x_- = -\frac{4q^2}{4W_{-1}(-\frac{q^2}{4} \exp(q^2/4)) - q^2}. \tag{13} \]
The horizons $r_+$ and $r_-$ for
\[ q_{\text{extr}} = 2w_0^{1/2}, \tag{14} \]
merge at
\[ x_{\text{extr}} = \frac{4w_0}{1 + w_0} \tag{15} \]
into a degenerate event horizon. Numerically one has
\[ x_{\text{extr}} = 0.871, \quad \text{and} \quad q_{\text{extr}} = 1.056. \tag{16} \]

The three types of solutions therefore are: the regular black hole with the inner and event horizons for $q < q_{\text{extr}}$, the extremal black hole for $q = q_{\text{extr}}$, and the regular configuration for $q > q_{\text{extr}}$. At large distances as well as for small charges the geometry resembles that of the Reissner-Nordström with one notable distinction: for $q > 1$ the Reissner-Nordström solution describes naked singularity whereas the regular geometry could be interpreted as a particle like solution. Qualitative behaviour of $f(r)$ is displayed in Fig. 2.

**FIG. 1:** Location of the horizons $x_+$ (upper curve) and $x_-$ (bottom curve) plotted against $q$. The run of the real branches of the Lambert function $W_0(\xi)$ (upper curve) and $W_{-1}(\xi)$ (lower curve) is displayed for comparison. The degenerate event horizon at $x_+ = 2w_0^{1/2}$ corresponds to the branch point of the Lambert function at $\xi = -1/e$. 

**FIG. 2:** Qualitative behaviour of the function $f(x)$ for various values of the parameter $q$. Top to bottom the curves are for $q > q_{\text{extr}}$, $q = q_{\text{extr}}$, and $q < q_{\text{extr}}$. The double root corresponds to the extremal event horizon.
To investigate the near-horizon geometry of the regular magnetic black hole further let us postpone analyses of the solution (2) for a while and consider the line element of the form
\[ ds^2 = \frac{1}{hy^2} (-dt^2 + dy^2) + r_0^2 dt^2. \] (17)

Other useful representations obtained through a redefinition of coordinates according to formulas
\[ h^{1/2} t = e^r \coth \chi, \]
\[ h^{1/2} y = e^r \sinh^{-1} \chi \] (18)
and
\[ \sinh^2 \chi = R h - 1, \]
\[ \tau h = \tau \] (19)
are
\[ ds^2 = \frac{1}{h} (-\sinh^2 \chi dt^2 + d\chi^2) + r_0^2 d\Omega^2 \] (20)
and
\[ ds^2 = -(R^2 h - 1) d\tau^2 + \frac{dR^2}{R^2 h - 1} + r_0^2 d\Omega^2, \] (21)
respectively. Topologically it is AdS$_2 \times$ S$^2$, i.e., a direct product of the two-dimensional anti-de Sitter geometry and two-sphere of constant curvature. The curvature scalar for the line element (17) is simply a sum of the curvatures of the subspaces AdS$_2$ and S$^2$ :
\[ R = K_{\text{AdS}_2} + K_{\text{S}^2}, \] (22)
where $K_{\text{AdS}_2} = -2h$ and $K_{\text{S}^2} = 2/r_0^2$.

We intend to solve the coupled system of equations of the nonlinear electrodynamics and general relativity. For the line element (17) the Einstein tensor is given by
\[ G^a_b = \text{diag} \left[ -\frac{1}{r_0^2}, -\frac{1}{r_0^2}, h, h \right], \] (23)
whereas the electromagnetic tensor compatible with assumed symmetries is simply
\[ F = Q \sin \theta d\theta \wedge d\phi, \] (24)
where $Q$ is, as before, the magnetic charge. It could be easily verified that
\[ F = F_{ab} F^{ab} = \frac{2Q^2}{r_0^2} > 0. \] (25)
As the stress-energy tensor of the nonlinear electrodynamics is
\[ T^b_a = \frac{1}{4\pi} \left( \frac{d\mathcal{L}(F)}{dF} F_{ac} F^{bc} - \frac{1}{4} \delta^b_a \mathcal{L}(F) \right), \] (26)
the Einstein fields equations reduce to two independent equations:
\[ \frac{1}{r_0^2} = \frac{1}{2} \mathcal{L}(F) \] (27)
and
\[ h = 2 \left( \frac{d\mathcal{L}(F)}{dF} F_{23} F^{23} - \frac{1}{4} \mathcal{L}(F) \right). \] (28)

Now we shall select the particular form of the Lagrangian. Let us choose $\mathcal{L}(F)$ as given by (5) with
\[ a = \frac{|Q|^{3/2}}{2\alpha}, \] (29)
where $\alpha$ is some positive parameter of dimension of length. Introducing a new coordinate $x$ and a new parameter $q$ by means of the formulas $r_0 = \alpha x$ and $Q = \alpha q$, Eqs. (27) and (28), after some manipulations, could be rewritten as
\[ \frac{q^2}{x^2} \cosh^{-2} \left( \frac{q^2}{2x} \right) = 1 \] (30)
and
\[ h = \frac{q^2}{x^2 \alpha^2} \cosh^{-2} \left( \frac{q^2}{2x} \right) - \frac{q^4}{2x^5 \alpha^2} \sinh^2 \left( \frac{q^2}{2x} \right). \] (31)

One can relegate hyperbolic functions combining Eqs. (30) and (31)
\[ \frac{1}{x^2} - \frac{q^2}{2x} \left( 1 - \frac{x^2}{q^2} \right)^{1/2} = h \alpha^2. \] (32)

In general the problem should be treated numerically: for a particular choice of $q$ the equation (30) gives concrete values of $x$ and hence $h$. First, observe that depending on $q$, Eq. (30) has two, one or has no solutions at all. Numerical calculations indicate that for $q_c = 1.325$ there is only one solution at $x = 0.735$, whereas for $q < q_c$ Eq. (30) has two solutions. Moreover, it should be noted that there is a particular combination of $q$ and $x$ expressible in terms of the Lambert functions satisfying Eq. (30). Indeed, it could be easily checked that $q = q_{\text{extr}}$ and $x = x_{\text{extr}}$ (given respectively by Eqs. (14) and (15)) comprises such a solution with $h$ given by
\[ h = \frac{1}{32 \alpha^3} \left( 1 + w_0^3 \right)^{3}. \] (33)

Results of numerical calculations are displayed in Fig. 3.

Although Bertotti-Robinson line element [21, 22] is a special case of (17) with $h = 1/r_0^2$ it does not belong to a family of solutions of Eq. (20). It could be easily shown that vanishing of the trace of the stress-energy tensor (20) is, by (25), possible only for Lagrangians $\mathcal{L}(F) \propto F,$
Figure 3: Curvature radii of $S^2$ (solutions of Eq. (30)) plotted against parameter $q$.

and, consequently, $L(F)$ in the form given by [1] does not admit solutions of the Bertotti-Robinson type.

Now, let us return to the black hole solution. The extremal black hole is described by a line element [2] with

$$f(r) = 1 - \frac{2M}{r} \left[1 - \tanh \left(\frac{2Mw_0}{r}\right)\right].$$

(34)

It could be easily shown that $x_+$ given by [10] is the degenerate event horizon. In order to investigate the geometry of the vicinity of the event horizon and to obtain uniform approximation we introduce new coordinates $\tilde{t}$ and $y$:

$$\tilde{t} = \frac{t}{\varepsilon}, \quad r = r_+ + \frac{\varepsilon}{A y},$$

(35)

where

$$A = \frac{1}{32M^2} \left(1 + \frac{w_0}{w_0^3}\right).$$

(36)

Expanding the function $f(r)$ in terms of $\varepsilon$, retaining quadratic terms and subsequently taking the limit $\varepsilon = 0$ we obtain

$$ds^2 = \frac{1}{A y^2} (-dt^2 + dy^2) + r_+^2 d\Omega^2.$$  

(37)

Since $A^{-1} > r_+^2$, the line element does not belong to the Bertotti-Robinson class, as expected. This result could be easily anticipated as the stress-energy tensor of the nonlinear electromagnetic field has nonvanishing trace at the event horizon. It should be noted that putting $\alpha = M$ in [17] with [30] one obtains [30].

It is evident that the procedure of constructing the near horizon geometry is insensitive to the nature of the central region of the black hole. One expects, therefore, that similar consideration could be carried out for the electrically charged Ayón–Bento and García solution as modified by Burinskii and Hildebrandt.

Finally let us return to the problem of boundary conditions and restrict discussion to the solutions describing black holes. Assuming that the exact location of the event horizon is known, the boundary condition $m(r_+) = r_+/2$ gives

$$m(r) = \frac{r_+}{2} + \frac{|Q|^{3/2}}{2a} \tanh \left(\frac{a|Q|^{1/2}}{r_+}\right)$$

$$- \frac{|Q|^{3/2}}{2a} \tanh \left(\frac{a|Q|^{1/2}}{r}\right).$$

(38)

Denoting first two (constant) terms in the right hand side of the above equation by $M_H$ and demanding $\alpha = |Q|^{3/2}/2M_H$ results in the equation which relates $r_+, Q$, and $M_H$

$$1 - \frac{2M_H}{r_+} \left(1 - \tanh \frac{Q^2}{2M_Hr_+}\right) = 0,$$

(39)

and the line element [2] with [10]. The location of the inner horizon is given by Eq. [10] and the near horizon geometry is, of course, identical to the one considered previously.
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