Fractional diffusion equation with distributed-order material derivative. Stochastic foundations

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Abstract
In this paper, we present the stochastic foundations of fractional dynamics driven by the fractional material derivative of distributed-order type. Before stating our main result, we present the stochastic scenario which underlies the dynamics given by the fractional material derivative. Then we introduce the Lévy walk process of distributed-order type to establish our main result, which is the scaling limit of the considered process. It appears that the probability density function of the scaling limit process fulfills, in a weak sense, the fractional diffusion equation with the material derivative of distributed-order type.

Keywords: fractional material derivative, Lévy walk, distributed-order derivative, weak convergence

1. Introduction
The scaling limit of the standard random walk with jumps having finite variance leads to the following classical diffusion equation given in dimensionless\(^2\) form:

\[
\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}.
\]

By adding the heavy-tailed waiting times, which are independent of the jumps, we obtain the continuous-time random walk (CTRW), whose long-time limit is governed by the fractional diffusion equation [1]:

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\(^2\)In this paper, all fractional diffusion equations are given in dimensionless form. To get the physical meaning of the equations, one should include the appropriate diffusion constants carrying the dimension, see [1].
\[
\frac{\partial p}{\partial t} = 0D_1^{-\alpha} \frac{\partial^2 p}{\partial x^2},
\]

(1)

where the operator \( 0D_1^{-\alpha}, 0 < \alpha < 1, f \in C^1([0,\infty)) \), is a fractional derivative of the Riemann–Liouville type [2]. In addition, if we assume that both jumps and waiting times are independent and chosen from the power-law distribution, the density of the corresponding scaling limit solves the space-time fractional diffusion equation [1]:

\[
\frac{\partial p}{\partial t} = \sigma D_1^{-\alpha} \nabla^\mu p.
\]

where \( \nabla^\mu \) denotes a one-dimensional Riesz operator with \( 0 < \mu < 2 \).

The main difficulty underlying such models with heavy-tailed jumps is the infinite second moment. To overcome this problem, the so-called Lévy walks were introduced [3, 4] (see also [5] for a recent review). The wait-first Lévy walk (denoted for short as wait-first LW) is a special type of CTRW, for which the length of each jump is equal to the length of the preceding waiting time, meaning therefore that this strong spatiotemporal dependence assures finite moments of all orders. Thus, even if the jumps are heavy-tailed, the ensemble mean square displacement is well-defined. We use here the notion wait-first to emphasise the fact that in the trajectories of the process, every waiting time is preceded by a respective jump. The dynamics of the wait-first LW densities is governed by the so-called fractional material derivatives \( \left( \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} \right)^\alpha \), see [6]. In the Fourier–Laplace space they are given by

\[
\mathcal{F}_t \mathcal{L}_x \left\{ \left( \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} \right)^\alpha p(x,t) \right\} = (s \pm ik)^\alpha p(k,s).
\]

Apart from the wait-first LW model, one can also investigate its modification, called the jump-first Lévy walk (denoted for short as jump-first LW), for which every random waiting time is preceded by a random jump. Recently, it was shown [7] that based on the wait-first LW and the jump-first LW, one can introduce another type of Lévy walk, namely the continuous Lévy walk, introduced in [3, 4]. The trajectories of the last model are continuous and can be obtained using some linear interpolation between the trajectories of the wait-first and jump-first LWs.

In this paper, we introduce a generalization of the wait-first and jump-first LW models, which leads to the scaling limits with their density dynamics governed by a fractional material derivative of distributed-order type. The concept of the fractional derivative of the distributed-order type was proposed by Caputo [8, 9], and it can be presented by means of equation (1). Namely, we assume that the dynamics of the density involves different fractional orders, and therefore the fractional diffusion equation (1) can be generalized to the following one:

\[
\frac{\partial p}{\partial t} = \int_0^1 w(\alpha)0D_1^{-\alpha} \frac{\partial^2 p}{\partial x^2} \, d\alpha,
\]

(2)

where \( w(\alpha) \) is the probability density function on \( (0,1) \) of the fractional order \( \alpha \). As one can notice, the considered distributed-order generalization allows different fractional orders to be involved in the dynamics of the diffusion equation. In general, the concept of a distributed-order fractional derivative can be applied in the time and/or space domain [10]. By adding some specific assumptions to the probability function \( w(\alpha) \), we can arrive at the dynamics that is characterized by the logarithmic decay of the ensemble mean square displacement [11, 12]. It appears that the idea of distributed-order fractional operators governing the dynamics is supported by several applications in polymer physics, such as particles in a quenched force field, accelerating/decelerating anomalous diffusion and random walks in random media [13–16]. The stochastic scenario which underlies the distributed-order fractional derivative in
equation (2) assumes that the waiting times are heavy-tailed with the index $\alpha$ being randomized according to some probability density $w(\alpha)$, $0 < \alpha < 1$. A stochastic interpretation can be given by means of the CTRW scenario in the potential well, in which the shallowness of the potential is given by a random $\alpha$ related to the power-law index of the waiting time distribution, see [11].

Our idea here is to include the fractional derivatives of a distributed-order type into the LW framework. The stochastic scenario which underlies the distributed-order concept within LWs assumes that the waiting time sequences are heavy-tailed and distributed with a randomized index $\beta$, and the jumps are equal to the waiting times up to some constant. Again, due to the strong coupling between the waiting times and jumps, the introduced generalized wait-first LW has finite moments of all orders and jumps that can be extremely long. Moreover, it explains the stochastic origins of the fractional dynamics driven by the fractional material derivative of distributed-order type.

The paper is organized as follows: in the next section, we recall the notion of the fractional material derivative and its relation with the wait-first and jump-first LWs. Next, we define the generalization of the Lévy walk models, which assumes that the scaling limit of the jumps and the scaling limit of the waiting times have logarithmic decay of the ensemble MSD. We derive the corresponding scaling limit in the case of the newly introduced wait-first and jump-first scenarios. Finally, we introduce a pseudo-differential operator called the fractional material derivative of distributed-order type, and show that the density of the obtained diffusion limit solves the corresponding fractional diffusion equation.

2. Stochastic foundations of fractional dynamics with fractional material derivative

In the following section, we recall the formal definition of the wait-first and jump-first LWs. The scaling limits of these processes lead to fractional dynamics that involves fractional material derivatives.

2.1. Definitions of wait-first LW and jump-first LW

The wait-first LW process arises as a special case of CTRW. CTRW is a stochastic process, which generalizes the classical random walk [17, 18] by assuming that consecutive random jumps are separated by random waiting times [1, 18–21].

**Definition 2.1.** Let $\{(T_i, J_i)\}_{i \geq 1}$ be a sequence of independent and identically distributed (IID) random vectors, such that the following conditions hold:

(a) $P(T_i > 0) = 1$ and $T_i$ is heavy-tailed with the index $\alpha \in (0; 1)$, i.e.

$$P(T_i > t) \approx t^{-\alpha} \quad \text{as} \quad t \to \infty,$$

(b) $P(J_i \in \mathbb{R}^d) = 1$ and for each $J_i$ we have

$$J_i = V_i T_i,$$

where $\{V_i\}_{i \geq 1}$ is an IID sequence of nondegenerate unit vectors from $\mathbb{R}^d$, governing the direction of the jumps. Sequences $\{V_i\}_{i \geq 1}$ and $\{T_i\}_{i \geq 1}$ are assumed mutually independent.

Next, let $N_t = \max\{n : T_1 + T_2 + \ldots + T_n \leq t\}$ be the counting process corresponding to $\{T_i\}_{i \geq 1}$. Then, the processes
\[ R(t) = \sum_{i=0}^{N_i} J_i, \quad \hat{R}(t) = \sum_{i=0}^{N_i+1} J_i \]  

are called the wait-first LW and the jump-first LW, respectively.

Physically, the processes defined in (5) have very intuitive behavior, and can be used to model random phenomena at the microscopic level. That is to say, a particle whose position is described by the wait-first LW, starts its motion at the origin and stays there for the random time \( T_1 \), then it makes the first jump \( J_1 \). Next, it stays at the new location for the random time \( T_2 \) before making the next jump \( J_2 \); then the scheme repeats. Due to relation (4), the length of each jump \( J_i \) is equal to the length of the preceding waiting time \( T_i \), while the direction of each jump \( J_i \) is governed by the random vector \( \nu_a \). Therefore, the position of the particle \( R(t) \) at any moment \( t \) satisfies \( \| R(t) \| \leq t \), where \( \| \cdot \| \) is the d-dimensional Euclidean norm.

As the reader can notice for \( \hat{R}(t) \), the counting process is \( N_t + 1 \), and is always larger by 1 than the counting process of the corresponding \( R(t) \). This difference has an important influence on the trajectories and properties of the \( \hat{R}(t) \) process. In contrast to the wait-first LW process, the particle running jump-first LW performs the first jump \( J_1 \) immediately after it starts the motion, waits at the new location for the time \( T_1 \), then performs the second jump \( J_2 \) to relocate to the next position, and waits there for the time \( T_2 \); then the scheme repeats. In fact, the \( \hat{R}(t) \) process consists of ‘jump-wait’ events, while the \( R(t) \) process consists of ‘wait-jump’ events. One of the main differences between wait-first and jump-first LWs is that the second moment of the wait-first LW process is finite, while for the jump-first LW process it is infinite [22–25].

2.2. Asymptotic properties of wait-first and jump-first LWs

In this section, we present the asymptotic behavior of wait-first and jump-first LW processes, and we also give a detailed description of the corresponding scaling limit processes.

Recall that \( \{ X(t) \}_{t \geq 0} \) is a d-dimensional Lévy process if its Fourier transform is given by the Lévy–Khinchin representation, see [26]:

\[ \mathbb{E} \exp \{ i \langle k, X(t) \rangle \} = \exp \{ t \psi(k) \}, \quad k \in \mathbb{R}^d \]

with the characteristic exponent \( \psi(k) \) given by:

\[ \psi(k) = i \langle k, a \rangle + \frac{1}{2} \langle k, Q k \rangle + \int_{x \neq 0} \left( \exp i \langle k, x \rangle - 1 - \frac{i \langle k, x \rangle}{1 + \| x \|^2} \right) \nu(dx). \]

Here, \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathbb{R}^d \), \( a \in \mathbb{R}^d \) is the drift parameter, \( Q \) is the symmetric and positively defined \( d \)-dimensional square matrix determining the Gaussian part of \( X(t) \), and \( \nu \) is the so-called Lévy measure. This is the \( \sigma \)-finite Borel measure on \( \mathbb{R}^d \setminus \{ 0 \} \), such that \( \int_{x \neq 0} \min(\| x \|^2, 1) \nu(dx) < \infty \). The triplet \( [a, Q, \nu] \), which is called the Lévy triplet, uniquely determines the Lévy process \( X(t) \).

Let us introduce the following notation: denote by \( X^-(t) = \lim_{s \nearrow t} X(s) \) the left-continuous version of the right-continuous process \( X(t) \), and by \( Y^+(t) = \lim_{s \searrow t} Y(s) \) the right-continuous version of any left-continuous process \( Y(t) \).
**Theorem 2.1.** Let $\mathbf{R}(t)$ and $\tilde{\mathbf{R}}(t)$ be the wait-first and jump-first LWs, defined in definition 2.1, respectively. The following convergences hold in $\mathbb{Y}_1$ topology as $n \to \infty$:

\[
\begin{align*}
n^{-1/\alpha} \mathbf{R} \left( n^{1/\alpha} t \right) & \Rightarrow \left( \mathbf{L}_{\alpha}^{-1}(S_{\alpha}^{-1}(t)) \right)^+. \\
n^{-1/\alpha} \tilde{\mathbf{R}} \left( n^{1/\alpha} t \right) & \Rightarrow \mathbf{L}_{\alpha}(S_{\alpha}^{-1}(t)).
\end{align*}
\]

Here, the process $S_{\alpha}^{-1}(t)$ is defined as follows:

\[
S_{\alpha}^{-1}(t) = \inf \{ \tau \geq 0 : S_{\alpha}(\tau) > t \}.
\]

The dependence structure between the Lévy processes $\mathbf{L}_{\alpha}(t)$ and $S_{\alpha}(t)$ is given by their joint Lévy triplet,

\[
\int_{x \neq 0} \frac{x}{1 + \|x\|^2} \nu(\mathbf{L}_{\alpha}, S_{\alpha})(dx), 0, \nu(\mathbf{L}_{\alpha}, S_{\alpha})
\]

with the following Lévy measure

\[
\nu(\mathbf{L}_{\alpha}, S_{\alpha})(dx) = \int_{S^{d-1}} \delta_{\mathbf{x}}(dx_1) \nu(dt) \Lambda(du).
\]

Here, $\mathbf{x} = (x_1, t) \in \mathbb{R}^d \setminus \{0\} \times (0, \infty), \mathbf{u} = x_1/\|x_1\| \in S^{d-1}, S^{d-1}$ is the $(d-1)$-dimensional sphere, $\delta_{\mathbf{x}}(\cdot)$ is a Dirac delta function at point $s$, $\nu$ is the Lévy measure of an $\alpha$-stable subordinator $\nu(dt) = \alpha^{-\alpha-1}/\Gamma(1-\alpha)$ and $\Lambda(du) = P(V_1 \in du)$, where the sequence $\{V_i\}_{i \geq 1}$ is defined in point (b) of definition 2.1.

**Proof.** For a proof of theorem 2.1 we refer to the first part of appendices B and C in [22].

The results follow immediately when we put $\gamma = 1$ (using the notation therein). \hfill $\Box$

The above-presented result needs an extended comment. First of all, the scaling limit processes for $\mathbf{R}(t)$ and $\tilde{\mathbf{R}}(t)$, although they both have the form of subordination, differ significantly. In the case of the wait-first LW, the limit process is a right-continuous version of the left-continuous $\alpha$-stable process $\mathbf{L}_{\alpha}^{-1}(s)$, subordinated to the inverse $\alpha$-subordinator $S_{\alpha}^{-1}(t)$.

For the jump-first LW scaling limit we obtain the right-continuous $\alpha$-stable process $\mathbf{L}_{\alpha}(s)$ subordinated to the inverse $\alpha$-subordinator $S_{\alpha}^{-1}(t)$. In both scenarios, the inverse $\alpha$-stable subordinator $S_{\alpha}^{-1}(t)$ plays the role of the internal operational time of the system (see [27–30] for similar results). We emphasize that the difference between the scaling limits of the considered processes is due to the fact that the $\mathbf{R}(t)$ process is composed of ‘wait-jump’ events, while $\tilde{\mathbf{R}}(t)$ is from ‘jump-wait’ events. For an extended discussion on this matter see [22–25, 31].

The $\alpha$-stable process $\mathbf{L}_{\alpha}(s)$ occurs as a scaling limit of the sum of jumps $\{J_i\}_{i \geq 1}$ (see definition 2.1). Since the jumps of the underlying wait-first and jump-first LWs are heavy-tailed with parameter $\alpha$, see (3) and (4), $\mathbf{L}_{\alpha}(s)$ belongs to the domain of attraction of $\alpha$-stable law. The spatial properties of $\mathbf{L}_{\alpha}(s)$ are also inherited from the sequence $\{V_i\}_{i \geq 1}$ introduced in equation (4). It appears that the distribution $\Lambda$ of unit vector $V_1$, which governs the possible directions of the LW jumps, also controls the directions of the $d$-dimensional jumps in the limiting process $\mathbf{L}_{\alpha}(s)$.

The scaling limits of wait-first and jump-first LWs are composed of two processes: $\mathbf{L}_{\alpha}(t)$ and $S_{\alpha}(t)$, which according to the joint Lévy measure (6), are strongly dependent. The Lévy measure can be considered as the intensity of jumps of the joint Lévy process $(\mathbf{L}_{\alpha}(t), S_{\alpha}(t))$. 
Therefore, we can conclude from (6) that the processes $L_\alpha(t)$ and $S_\alpha(t)$ have jumps of the same length occurring in the same epochs of time.

2.3. The fractional dynamics of the wait-first and jump-first LW scaling limits, including the fractional material derivative

In this part, we use the results of the previous sections to establish a link between the scaling limits of the wait-first and jump-first LWs, and the dynamics driven by the fractional material derivative.

It follows from the results presented in [22, 31] that the joint Lévy process $(L_\alpha(t), S_\alpha(t))$ has the following Fourier–Laplace exponent

$$
\psi(k, s) = \int_{u \in \mathbb{S}^{d-1}} (s - i\langle k, u \rangle)^\alpha \Lambda(du).
$$

The Fourier–Laplace transform presented in equation (7) uniquely determines the distribution of the scaling limits obtained in theorem 2.1. Moreover, it appears that it is closely related to the $d$-dimensional fractional material derivative introduced in [6]. The $d$-dimensional fractional material derivative is defined as

$$
\mathbb{D}^\alpha_{x,t}p(x,t) = \int_{u \in \mathbb{S}^{d-1}} \left( \frac{\partial}{\partial t} - \langle \nabla, u \rangle \right)^\alpha p(x, t)\Lambda(du)
$$

where its Fourier–Laplace transform is equal to

$$
\mathcal{F}_t L_x \{ \mathbb{D}^\alpha_{x,t}p(x,t) \} = \int_{u \in \mathbb{S}^{d-1}} (s - i\langle k, u \rangle)^\alpha \Lambda(du) p(k, s).
$$

In the above formula, $p(k, s)$ is the Fourier–Laplace transform of $p(x, t)$ and $\mathbb{S}^{d-1}$ denotes the $(d - 1)$-dimensional sphere. Let us notice that the $d$-dimensional fractional material derivative introduced in equation (8) depends strongly on $\Lambda(\cdot)$. Before we give details of the $d$-dimensional case, let us establish the link to the one-dimensional case described in the introduction of this paper. Namely, in this case $\mathbb{S}^0$ in fact consists of two points $\{-1, 1\}$, such that $\Lambda(\{1\}) = p$ and $\Lambda(\{-1\}) = 1 - p$ for some $p \in [0; 1]$. Therefore, in this case, operator (8) is equal to

$$
\mathbb{D}^\alpha_{x,t}p(x,t) = \left( p \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^\alpha + (1 - p) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^\alpha \right) p(x, t),
$$

which agrees with the results obtained for the one-dimensional case [23]. Now let us present examples of the fractional material derivative in the multidimensional case. Namely, let us assume that $\Lambda(e_j) = 1/d$ for $j = 1, \ldots, d$, where $e_1, e_2, \ldots, e_d$ are standard coordinate vectors ($e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0)$ and so on). This leads to the following form of the fractional material derivative:

$$
\mathbb{D}^\alpha_{x,t}p(x,t) = \frac{1}{d} \left( \sum_{j=1}^{d} \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x_j} \right)^\alpha \right) p(x, t).
$$

Another interesting example includes the case of $\Lambda(\cdot)$, which is symmetric around the origin; that is, for simplicity $\Lambda(e_j) = \Lambda(-e_j) = 1/(2d)$ for $j = 1, 2, \ldots, d$, leading to another possible formulation of the fractional material derivative:
\[ \mathbb{D}^{\alpha, \Lambda}_{x,t} p(x,t) = \frac{1}{2d} \left( \sum_{j=1}^{d} \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x_j} \right)^\alpha + \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial (-x_j)} \right)^\alpha \right) p(x,t). \]

Here we understand \( \frac{\partial f}{\partial (-x)} \) as the inverse of the Fourier transform \( F^{-1}(-ikf(k)) \). For details and examples of the possible multidimensional fractional derivatives, see section 6.5 in [32]. Now let us formulate the fractional diffusion equations that are fulfilled by the densities of the scaling limits obtained in theorem 2.1. The densities of the wait-first LW and jump-first LW satisfy the following fractional diffusion equations \[22\], respectively:

\[ \mathbb{D}^{\alpha, \Lambda}_{x,t} p_1(x,t) = \delta_0(x) \frac{I^{-\alpha}}{\Gamma(1-\alpha)}, \quad (9) \]

\[ \mathbb{D}^{\alpha, \Lambda}_{x,t} p_2(x,t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty \int_{u \in \mathbb{R}^{d-1}} \delta_{u}(dx)s^{-\alpha}\Lambda(du)ds. \quad (10) \]

The solutions of these equations are fulfilled in a weak sense, meaning that they are satisfied in the Fourier–Laplace space [31].

In the next section, we will introduce the stochastic scheme, which leads to the fractional dynamics given by equations similar to (9) and (10), but with the operator \( \mathbb{D}^{\alpha, \Lambda}_{x,t} \) replaced by the fractional material derivative of distributed-order type.

3. The stochastic foundations of fractional dynamics with a fractional material derivative of distributed-order type

3.1. The definitions of the generalized wait-first LW and generalized jump-first LW

Before we present our main results of this section, we briefly recall the results concerning the stochastic scheme underlying the distributed-order derivatives.

Let us consider an IID sequence \( \{B_i\}_{i \geq 1} \) of random variables, such that \( B_1 \) is distributed on interval \([0,1]\) according to the probability density function \( p(\beta) \), which regularly varies at 0 with exponent \( \gamma - 1 \) (here \( \gamma > 0 \)), i.e.

\[ \lim_{t \to 0} \frac{p(\lambda t)}{p(t)} = \lambda^{\gamma - 1}. \]

We also assume that the following property holds

\[ \int_{0}^{1} \frac{p(\beta)}{1 - \beta} d\beta < \infty. \]

Then, for any \( n \geq 1 \) let \( \{T_i^{(n)}\}_{i \geq 1} \) be an IID sequence of nonnegative random variables, which has the following distribution conditionally on \( B_i = \beta \):

\[ P(T_i^{(n)} > t | B_i = \beta) = \begin{cases} 1 & \text{for } 0 \leq t < n^{-1/\beta}, \\ n^{-1} t^{-\beta} & \text{for } t \geq n^{-1/\beta}. \end{cases} \quad (11) \]

By simple arguments we get that for \( t > n^{-1/\beta} \)

\[ P(T_i^{(n)} > t) = \int_{0}^{1} n^{-1} t^{-\beta} p(\beta) d\beta. \]
Therefore, in view of remark 3.2 [11], the above-defined sequence of waiting times \( T^{(n)}_1 \) has slowly varying tails. This property is manifested by the logarithmic decay in the tails as \((\log(t))^{-\alpha}\), and is also described as ultraslow decay.

The asymptotic properties of the random variables \( \{T^{(n)}_i\}_{i \geq 1} \) have been investigated in [11]; we summarize them in the following lemma:

**Lemma 3.1.** For any \( n \) let \( \{T^{(n)}_i\}_{i \geq 1} \) be an IID sequence, such that the condition (11) holds. Then, the following convergence holds in the \( J_1 \) topology as \( n \to \infty \):

\[
\sum_{i=1}^{[nt]} T^{(n)}_i \Rightarrow S^*_\beta(t),
\]

where the process is a subordinator such that its Lévy triplet is given by

\[
\begin{bmatrix}
\int_{x>0} \frac{x}{1+x^2} \nu^S_\beta(dx), 0, \nu^S_\beta
\end{bmatrix},
\]

(12)

with the Lévy measure \( \nu^S_\beta \) given by

\[
\nu^S_\beta(dr) = \int_0^1 \beta t^{-\beta-1} p(\beta) d\beta dr \quad \text{for} \ t > 0.
\]

(13)

**Proof.** The proofs follows immediately from theorem 3.4 and corollary 3.5 of [11].

The specific construction of sequence \( \{T^{(n)}_i\}_{i \geq 1} \) leads to the fractional dynamics given by the distributed-order derivative [11]. Therefore, we will use it to construct the appropriate modifications of the wait-first and jump-first LWs in order to obtain the dynamics governed by the fractional material derivative of distributed-order type.

**Definition 3.1.** For any \( n \geq 1 \) let \( \{(J^{(n)}_i, T^{(n)}_i)\}_{i \geq 1} \) be a sequence of independent and identically distributed (IID) random vectors, such that all the following conditions hold:

(a) \( T^{(n)}_i \) fulfill the condition (11),
(b) \( P(J^{(n)}_i \in \mathbb{R}^d) = 1 \) and for each \( J^{(n)}_i \) we put

\[
J^{(n)}_i = V_i T^{(n)}_i,
\]

where \( \{V_i\}_{i \geq 1} \) is an IID sequence of nondegenerate unit vectors from \( \mathbb{R}^d \), and sequences \( \{V_i\}_{i \geq 1} \) and \( \{T^{(n)}_i\}_{i \geq 1} \) are mutually independent.

For the renewal process \( N^{(n)}_t = \max\{i \in \mathbb{N}_0 : T^{(n)}_1 + T^{(n)}_2 + \ldots + T^{(n)}_i \leq t\} \), we define the following processes:

\[
\begin{align*}
R^{(n)}(t) &= \sum_{i=1}^{N^{(n)}_t} J^{(n)}_i, \\
\tilde{R}^{(n)}(t) &= \sum_{i=1}^{N^{(n)}_t+1} J^{(n)}_i,
\end{align*}
\]

which are called here the generalized wait-first LW and the generalized jump-first LW, respectively.
Let us observe that according to the fact that the waiting times $T_i^{(n)}$ have slowly varying tails, the jumps due to the specific construction (definition 3.1) almost straightforwardly inherit the property of slowly varying tails. It is important that in the generalized wait-first scenario, in a similar way to in the classical case of the wait-first scenario, the jumps with slowly varying tails are preceded by waiting times characterized by the same property, which makes all the moments of the generalized wait-first LW finite. Again, as in the classical case, the moments of the generalized jump-first LW are infinite due to the slowly varying tails of the first jump in their trajectories.

3.2. The asymptotic properties of the generalized wait-first LW and generalized jump-first LW

In the next part, we present the asymptotic behavior of the generalized wait-first LW and the generalized jump-first LW processes introduced in section 3.1. We also give a detailed description of the resulting scaling limit processes.

Let us start with the technical lemma concerning the asymptotic properties of the joint process of the partial sums for $(J_i^{(n)}, T_i^{(n)})_{i \geq 1}$.

**Lemma 3.2.** Let us consider an IID sequence $(J_i^{(n)}, T_i^{(n)})_{i \geq 1}$ defined in definition 3.1. Then, the following convergence holds in the $J_1$ topology as $n \to \infty$:

$$\left( \sum_{i=1}^{\lfloor nt \rfloor} J_i^{(n)}, \sum_{i=1}^{\lfloor nt \rfloor} T_i^{(n)} \right) \Rightarrow \left( L^*_\beta(t), S^*_\beta(t) \right),$$

(14)

where the process $(L^*_\beta(t), S^*_\beta(t))$ is a $(d+1)$-dimensional Lévy process defined by the following Lévy triplet:

$$\left[ \int_{x \neq 0} \frac{x}{1+\|x\|^2} \nu'(L^*_\beta, S^*_\beta)(dx), 0, \nu(L^*_\beta, S^*_\beta) \right].$$

(15)

with the following Lévy measure:

$$\nu'(L^*_\beta, S^*_\beta)(dx) = \int_0^1 \left( \int_{u \in S^{d-1}} \delta_{u}(dx_1) \Lambda(du) \beta r^{-\beta-1} dt \right) p(\beta) d\beta.$$  

(16)

Here, $x = (x_1, t) \in \mathbb{R}^d \setminus \{0\} \times (0, \infty)$, $u = x_1/\|x_1\| \in S^{d-1}$, $S^{d-1}$ is a $(d-1)$-dimensional sphere, $\delta_s(\cdot)$ is a Dirac delta function at point $s$ and $\Lambda(du) = P(V_1 \in du)$, where the sequence $\{V_i\}_{i \geq 1}$ is defined in point b) of definition 3.1.

**Proof.** First, we recall a result concerning the finite-dimensional convergence of the waiting times defined in (11). Namely, from lemma 3.1 it follows that

$$\sum_{i=1}^n T_i^{(n)} \overset{d}{\longrightarrow} S^*_\beta(1),$$

holds as $n \to \infty$. Here $S^*_\beta(1)$ is the distribution of the process $S^*_\beta(t)$ at $t = 1$. Process $S^*_\beta(t)$ is fully described by its Lévy triplet given by (12), with the Lévy measure $\nu_{S^*_\beta}$ given by (13). Based on theorem 3.2.2 in [33], the above-mentioned result implies that as $n \to \infty$
\[ nP(T_1^{(n)} \in dt) \to \nu_{S_\beta^*}(dr) = \int_0^1 \beta t^{-\beta-1} p(\beta) d\beta dr \quad \text{for any } t > 0 \] (17)

and the Gaussian part in the Lévy–Khinchin representation of the process \( S_\beta^*(t) \) is equal to 0.

To prove lemma 3.2, let us observe that for any Borel sets \( A_1 \in \mathcal{B}(\mathbb{R}^d) \) and \( A_2 \in \mathcal{B}(\mathbb{R}_+) \) such that \( A_1 = A_1(R, D) = \{ ru \in \mathbb{R}^d : r \in R, u \in D \} \), where \( R \in \mathcal{B}(\mathbb{R}_+) \) and \( D \in \mathcal{B}(\mathbb{S}^{d-1}) \), due to the independence between \( T_1^{(n)} \) and \( V_1 \) we have that

\[ nP(J_1^{(n)} \in A_1, T_1^{(n)} \in A_2) = n \int_{\mathbb{S}^{d-1}} P(T_1^{(n)} u \in A_1, T_1^{(n)} \in A_2) P(V_1 \in du) \]

\[ = n \int_{A_1} \int_{\mathbb{S}^{d-1}} 1(ru \in A_1) P(V_1 \in du) P(T_1^{(n)} \in dt) \] (18)

where \( 1(\cdot) \) is equal to 1 if the condition in the brackets is fulfilled and 0 if it is not. Let us observe that for \((x, t) \in \mathbb{R}^d \setminus \{0\} \times \mathbb{R}_+ \) and \( u \in \mathbb{S}^{d-1} \), by taking the advantage of Fubini’s theorem and the asymptotic property (17), it follows from formula (18) that

\[ nP(J_1^{(n)} \in A_1, T_1^{(n)} \in A_2) = n \int_{A_1} \int_{\mathbb{S}^{d-1}} 1(ru \in A_1) P(V_1 \in du) P(T_1^{(n)} \in dt) \]

\[ \overset{\text{as } n \to \infty}{\rightarrow} \int_0^1 \int_{A_1} \int_{\mathbb{S}^{d-1}} 1(ru \in A_1) \beta t^{-\beta-1} \Lambda(du) p(\beta) d\beta d\beta, \]

as \( n \to \infty \), where \( \Lambda(du) = P(V_1 \in du) \). Thus, the Lévy measure of the joint distribution \((L_\beta^*(1), S_\beta^*(1))\) is equal to

\[ \nu_{(L_\beta^*(1), S_\beta^*(1))}(dx) = \int_0^1 \left( \int_{u \in \mathbb{S}^{d-1}} \delta_{ru}(dx_1) \Lambda(du) \beta t^{-\beta-1} dt \right) p(\beta) d\beta. \]

Here, \( x = (x_1, t) \in \mathbb{R}^d \setminus \{0\} \times (0, \infty), u = x_1/\|x_1\| \in \mathbb{S}^{d-1} \).

Since the distribution \( S_\beta^*(1) \) has a Gaussian component equal to 0, one can conclude that due to the tight relation between \( J_1^{(n)} \) and \( T_1^{(n)} \), the joint limit distribution \((L_\beta^*(1), S_\beta^*(1))\) has no Gaussian component either. Theorem 3.2.2 [33] assures that the convergence in (14) holds in the sense of distributions. Based on theorem 4.1 in [34], we conclude that the convergence is also valid in the functional sense (here convergence in the Skorokhod \( J_1 \) topology, see [35, 36]), which finishes the proof.

In the next theorem, we present the main results of this section, which are the scaling limits of the generalized wait-first and generalized jump-first LWs.

**Theorem 3.1.** Let \( \mathbf{R}^{(n)}(t) \) and \( \mathbf{R}^{(n)}(t) \) be the generalized wait-first LW and generalized jump-first LW defined in definition 3.1. The following convergences hold in the \( J_1 \) topology as \( n \to \infty \):

\[ \mathbf{R}^{(n)}(t) \Rightarrow \left( \mathbf{L}_{\beta}^* \left( S_{\beta}^{-1}(t) \right) \right)^+, \]

\[ \mathbf{R}^{(n)}(t) \Rightarrow \mathbf{L}_{\beta}^* \left( S_{\beta}^{-1}(t) \right). \]
where the process \( S_{\beta}^{-1}(t) \) is defined as follows:
\[
S_{\beta}^{-1}(t) = \inf\{\tau \geq 0 : S_{\beta}(\tau) > t\}.
\]

The dependence structure between the \( L_{\beta}(t) \) and \( S_{\beta}(t) \) processes is given by their joint \( \text{Lévy} \) triplet in (15).

**Proof.** In lemma 3.2, we showed the functional convergence of the joint process of the cumulated jumps and waiting times to \((L_{\beta}(t), S_{\beta}(t))\), which is fully described by the \( \text{Lévy} \) triplet given by (15). Moreover, from the \( \text{Lévy} \) triplet of \( S_{\beta}(t) \), we can conclude that it is a process with unbounded, non-negative and strictly increasing trajectories. Therefore, the convergence in the Skorokhod \( J_1 \) topology in theorem 3.1 can now be easily concluded from theorem 3.6 in [27].

The general structure of the scaling limit processes in theorem 3.1 is similar to the one in theorem 2.1. For the scaling limit of \( \tilde{R}^{(0)}(t) \) a \( \text{Lévy} \) process \( L_{\beta}^{*-}(s) \) is subordinated to the inverse subordinator \( S_{\beta}^{-1}(t) \), while for the scaling limit of \( \tilde{R}^{(0)}(t) \) we obtain the right-continuous \( \text{Lévy} \) process \( L_{\beta}(s) \) subordinated to the inverse subordinator \( S_{\beta}^{-1}(t) \). Similarly, as for \( S_{\beta}(t) \) in theorem 2.1, here the inverse subordinator \( S_{\beta}^{-1}(t) \) plays the role of the internal operational time of the resulting scaling limit processes.

It is worth noticing that the processes \( L_{\beta}(s) \) and \( S_{\beta}(t) \) are no longer stable. Due to the specific construction of the waiting times—see (11) and (3)—where the heavy-tailed index \( \beta \) is defined conditionally on the sequence \( \{B_t\}_{t \geq 1} \) and \( L_{\beta}(s) \) and \( S_{\beta}(t) \) belong to a more general class of \( \text{Lévy} \) processes. Processes \( L_{\beta}(s) \) and \( S_{\beta}(t) \) are strongly dependent, which can be deduced from their joint \( \text{Lévy} \) triplet. It appears that their joint \( \text{Lévy} \) measure (16) is an integral over the distribution of the parameter \( \beta \) of the joint \( \text{Lévy} \) measures (6) in the wait-first and jump-first LW case.

### 3.3. The fractional dynamics of the generalized wait-first and generalized jump-first LW scaling limits, including the fractional material derivative of the distributed-order type

It follows from the results presented in [22, 31] that the scaling limit of the process \( \left( L_{\beta}(t), S_{\beta}(t) \right) \) has the following Fourier–Laplace exponent:
\[
\psi(k, s) = \int_0^1 \left( \int_{u \in \mathbb{R}^{d-1}} (s - i\langle k, u \rangle)^{\beta} \Gamma(1 - \beta) \Lambda(du) \right) p(\beta) d\beta. \tag{19}
\]

The Fourier–Laplace transform presented in equation (19) uniquely determines the distributions of the corresponding scaling limits obtained in theorem 3.1. Therefore, one can introduce the pseudodifferential operator of a fractional material derivative of distributed-order type:
\[
\mathcal{D}_{x,t}^{p(\beta), \Lambda} p(x, t) = \int_0^1 \left( \int_{u \in \mathbb{R}^{d-1}} \left( \frac{\partial}{\partial t} - \langle \nabla, u \rangle \right)^{\beta} \Gamma(1 - \beta) p(x, t) \Lambda(du) \right) p(\beta) d\beta \tag{20}
\]

which according to (19) is defined for some density function \( p(x, t) \) in the Fourier–Laplace space as
\[
\mathcal{F}_{x,t} \{ \mathcal{D}_{x,t}^{p(\beta), \Lambda} p(x, t) \} = \int_0^1 \left( \int_{u \in \mathbb{R}^{d-1}} (s - i\langle k, u \rangle)^{\beta} \Gamma(1 - \beta) \Lambda(du) \right) p(k, s) p(\beta) d\beta.
\]
In formula (20), the first integral is the one with respect to the distribution \( p(\beta) \) over the integrand, which is the usual fractional material derivative operator (8).

In analogy to the result of section 2, the densities of the scaling limits of the generalized wait-first and generalized jump-first LWs in theorem 3.1 satisfy the following fractional diffusion equations, respectively:

\[
D^{p(\beta),\Lambda}_{x,t} p_1(x,t) = \delta_0(x) \int_0^1 t^{-\beta} p(\beta) d\beta, \\
D^{p(\beta),\Lambda}_{x,t} p_2(x,t) = \int_0^1 \int_0^\infty \int_{\mathbb{R}^{d-1}} \delta_{\eta u}(x)s^{-\beta} \Lambda(du) ds p(\beta) d\beta,
\]

where the fractional material derivative operator of distributed-order is on the left-hand side of the above equations.

It is worth noticing that the results obtained for generalized wait-first and generalized jump-first LWs can be reduced after appropriate scaling to the results for wait-first and jump-first LWs, when we assume the distribution of the \( \beta \) index to be \( p(\beta) = \delta(\beta - \beta_0) \) for some fixed \( \beta_0 \in (0, 1) \). In this case, one can compare the formulas (19) and (20) with those obtained in (7), (8) and in [7].

4. Conclusion

In this paper, we introduced the wait-first and jump-first Lévy walk models, which underlie the fractional dynamics involving fractional material derivatives of distributed-order type. Our approach was based on a strongly coupled CTRW, with the distribution of waiting times displaying ultraslow (logarithmic) decay of the tails. This type of distribution is closely related to the so-called ultraslow diffusion. We have derived the diffusion limits of the considered generalization of the Lévy walk models. It appears that the limit has the form of subordination (time change) of certain Lévy processes, with the inverse subordinator corresponding to the waiting times with ultra-heavy tails. This type of superdiffusive dynamics is characterized by trajectories that have very long jumps (with logarithmic decay of their tails) and finite mean square displacement (in the case of the wait-first scenario).

The introduced model explains the stochastic origins of the fractional dynamics driven by the fractional material derivative of distributed-order-type. This is manifested by the fact that the probability density function of the obtained diffusion limit process solves the fractional diffusion equation with such a material derivative in Fourier–Laplace space.

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