Swap portfolios and reverse-weighted portfolios, with an application to commodity futures

Ricardo T. Fernholz \(^1\) \hspace{1cm} \text{Robert Fernholz} \(^2\)

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Abstract

A market portfolio is a portfolio in which each asset is held at a weight proportional to its market value. A swap portfolio is a portfolio in which each one of a pair of assets is held at a weight proportional to the market value of the other. A reverse-weighted index portfolio is a portfolio in which the weights of the market portfolio are swapped pairwise by rank. Swap portfolios are functionally generated, and in a coherent market they have higher asymptotic growth rates than the market portfolio. Although reverse-weighted portfolios with two or more pairs of assets are not functionally generated, in a market represented by a first-order model with symmetric variances, they will grow faster than the market portfolio. This result is applied to a market of commodity futures.

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\(^1\)Claremont McKenna College, 500 E. Ninth St., Claremont, CA 91711, rfernholz@cmc.edu.
\(^2\)Intech Investment Management, LLC, One Palmer Square, Suite 441, Princeton, NJ 08542, bob@bobfernholz.com.
1 Introduction

Stochastic portfolio theory considers the behavior of markets comprised of assets represented by continuous semimartingales and the behavior of portfolios within these markets (Fernholz, 2002). The question of relative arbitrage, that is, a situation where a particular portfolio will almost surely outperform the market portfolio over a fixed period of time, has been considered rather thoroughly (Fernholz, 2002; Fernholz et al., 2005, 2018), but the question of asymptotic long-term portfolio behavior has perhaps received less attention. Here we consider long-term portfolio behavior and show that quite simple portfolios can be constructed to asymptotically outperform the market.

Functionally generated portfolios are portfolios with weights derived from a positive $C^2$ function of the market weights. These portfolios were introduced by Fernholz (1999), and can be constructed to outperform a capitalization-weighted stock market portfolio under realistic conditions (Fernholz, 2002; Fernholz et al., 2005, 2018). Here we consider a simple example of a functionally generated portfolio, a swap portfolio. A swap portfolio holds only two assets, and the weight of each is proportional to the market weight of the other. Nevertheless, under the weak condition of market coherence, i.e., that all the assets in the market have the same asymptotic growth rate, we show that a swap portfolio will almost surely outperform the market portfolio over the long term.

The weights of a functionally generated portfolio are functions of the market weights, but the converse is not true. Indeed, a reverse-weighted portfolio, in which the weights of the market portfolio are reversed according to rank, is not functionally generated (at least if the market contains more than a single pair of assets). We consider a reverse-weighted portfolio in a market represented by a first-order model, a system of continuous semimartingales with growth and variance parameters that are determined by rank alone (Fernholz, 2002; Banner et al., 2005). We show that in a market represented by a first-order model with rank-symmetric variance parameters, the reverse-weighted portfolio will almost surely grow faster than the market. We apply this result to a market of commodity futures, a market that can be approximated by a first-order model with rank-symmetric variance parameters.

For this application to commodities, we construct implied two-month futures prices and then normalize by setting these prices to be the same on the starting date for the data in a manner similar to Asness et al. (2013). We show that the first-order model for implied two-month commodity futures prices from 1995-2018 has rank-symmetric variance parameters and growth rate parameters that are substantially lower at top ranks than at bottom ranks. These estimated parameters are similar to the first-order parameters estimated for spot commodity prices by Fernholz (2017). Consistent with our theoretical results, we show that the reverse-weighted portfolio of commodity futures outperforms the price-weighted market portfolio of commodity futures from 1977-2018. We also show that over this same time period the reverse-weighted portfolio outperforms the diversity-weighted portfolio with a negative parameter (Vervuurt and Karatzas, 2015) as well as the equal-weighted portfolio of commodity futures.

2 Markets and Market Portfolios

In this section we introduce some of the basic ideas of stochastic portfolio theory; for further details we refer the reader to Fernholz (2002) and Fernholz and Karatzas (2009). For $n > 1$, consider a market represented by a system $\{X_1, \ldots, X_n\}$ of positive continuous semimartingales such that $X_i(t)$ represents the market value of the $i$th asset at time $t \in [0, \infty)$. Let $\pi$ be a portfolio with weight processes $\pi_1, \ldots, \pi_n$, which are bounded measurable processes adapted to the underlying filtration, and which add up to one. For a portfolio $\pi$, the portfolio value process $Z_\pi$ will satisfy

$$dZ_\pi(t) \triangleq Z_\pi(t) \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)},$$
or, in logarithmic terms,
\[ d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) \, d \log X_i(t) + \gamma^*_\pi(t) \, dt, \quad \text{a.s.,} \quad (2.1) \]
with the excess growth rate process
\[
\begin{align*}
\gamma^*_\pi(t) &\triangleq \frac{1}{2} \left( \sum_{i=1}^{n} \pi_i(t) \sigma_{ii}(t) - \sigma^2_\pi(t) \right) \\
&= \frac{1}{2} \left( \sum_{i=1}^{n} \pi_i(t) \sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t) \pi_j(t) \sigma_{ij}(t) \right), \quad \text{a.s.,} \quad (2.2)
\end{align*}
\]
where
\[
\begin{align*}
\sigma_{ij}(t) \, dt &\triangleq d \langle \log X_i, \log X_j \rangle_t, \\
\sigma^2_\pi(t) \, dt &\triangleq d \langle \log Z_\pi \rangle_t,
\end{align*}
\]
\(\langle \cdot, \cdot \rangle_t\) represents the quadratic variation process, and \(\langle \cdot, \cdot \rangle_t\) represents the cross variation process. It can be shown that
\[
\gamma^*_\pi(t) \geq 0, \quad \text{a.s.,} \quad (2.4)
\]
if the \(\pi_i(t) \geq 0, \text{ for } i = 1, \ldots, n\), and this provides a measure of the efficacy of diversification in the portfolio.

Let us denote the total value of the market by \(X(t) = X_1(t) + \cdots + X_n(t)\). The market portfolio \(\mu\) is the portfolio with weights \(\mu_1, \ldots, \mu_n\) such that each asset is weighted proportionally to its market value:
\[
\mu_i(t) = X_i(t)/X(t). \quad (2.5)
\]
It can be shown that, with appropriate initial conditions, the value process of the market portfolio satisfies
\[
Z_\mu(t) = X(t). \quad (2.6)
\]
In the case that \(\pi = \mu\), the left-hand side vanishes, and we have
\[
\sum_{i=1}^{n} \mu_i(t) \, d \log \mu_i(t) = -\gamma^*_\mu(t) \, dt \leq 0, \quad \text{a.s.} \quad (2.7)
\]
From this we see that whatever benefit the market has from diversification is lost in the weighted average of the \(\log \mu_i\) terms. This suggests that market weights might not always be “optimal”, and some kind of improvement may be possible.

In order to understand the long-term behavior of portfolios, we shall need to impose some asymptotic stability conditions. The market is coherent if for \(i = 1, \ldots, n\),
\[
\lim_{t \to \infty} \frac{1}{t} \log \mu_i(t) = 0, \quad \text{a.s.} \quad (2.8)
\]
The market has positive asymptotic diversification if

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \gamma^n_k(s) \, ds > 0, \quad \text{a.s.}$$  \hspace{1cm} (2.9)$$

It will also be convenient to consider portfolio behavior in terms of rank.

For \( t \in [0, \infty) \), let \( r_i \), a permutation acting on \( \{1, \ldots, n\} \), be the rank function for \( X_1(t), \ldots, X_n(t) \), with \( r_i(i) < r_i(j) \) if \( X_i(t) > X_j(t) \) or if \( X_i(t) = X_j(t) \) and \( i < j \). The corresponding rank processes \( X^{(1)} \geq \cdots \geq X^{(n)} \) are defined by \( X_{(r_i)(i)}(t) = X_i(t) \). We have assumed that the semimartingales \( X_i(t) \) are positive, so we can consider the logarithmic processes \( \log X_1, \ldots, \log X_n \). For \( 1 \leq k < \ell \leq n \), let \( \Lambda^X_{k,\ell} \) denote the local time at the origin for \( \log X_k - \log X_\ell \), with \( \Lambda^X_{0,1} = \Lambda^X_{n,n+1} = 0 \) (see Karatzas and Shreve [1991], Section 3.7). The processes \( \log X_1, \ldots, \log X_n \) have a triple point at time \( t > 0 \) if there exist \( j < k < \ell \) such that \( \log X_j(t) = \log X_k(t) = \log X_\ell(t) \). Multidimensional Brownian motion almost surely has no triple points (see Karatzas and Shreve [1991], Proposition 3.22, page 161), but some of the systems we consider satisfy only the weaker condition that the processes \( \log X_1, \ldots, \log X_n \) spend no local time at triple points, by which we mean that for all \( \ell \geq k + 2 \), we have \( \Lambda^X_{k,\ell} = 0 \). If the \( \log X_i \) spend no local time at triple points, then Theorem 2.5 of Banner and Ghomrasni [2008] shows that the rank processes \( \log X_{(k)} \) satisfy

$$d \log X_{(k)}(t) = \sum_{i=1}^n \mathbb{1}_{(r_i(i) = k)} \left( d \log X_i(t) + \frac{1}{2} d\Lambda^X_{k,k+1}(t) - \frac{1}{2} d\Lambda^X_{k-1,k}(t) \right), \quad \text{a.s.,}$$  \hspace{1cm} (2.10)

for \( k = 1, \ldots, n \). In a coherent market, the rank processes share the same asymptotic growth rate as each of the assets and the market portfolio (see Fernholz [2002], Proposition 2.1.2).

## 3 Swap portfolios

A positive \( C^2 \) function \( S \) defined on the unit simplex \( \Delta^n \subset \mathbb{R}^n \) generates a portfolio \( \pi \) if

$$\log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = \log S(\mu(t)) + \Theta(t), \quad \text{a.s.,}$$  \hspace{1cm} (3.1)

where the drift process \( \Theta \) is of locally bounded variation. It was shown in Fernholz [2002], Theorem 3.1.5, that the portfolio \( \pi \) will have weights

$$\pi_i(t) = \left( \int_0^t \log S(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) D_j \log S(\mu(t)) \right) \mu_i(t), \quad \text{a.s.,}$$  \hspace{1cm} (3.2)

for \( i = 1, \ldots, n \), with

$$d\Theta = \frac{-1}{2S(\mu(t))} \sum_{i,j=1}^n D_{ij} S(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) \, dt, \quad \text{a.s.}$$  \hspace{1cm} (3.3)

For \( n \geq 2 \), we see that (3.2) and (3.3) indicate that for \( 1 \leq i < j \leq n \), the function

$$S(x) = \frac{x_i x_j}{x_i + x_j},$$  \hspace{1cm} (3.4)

generates the swap portfolio \( \pi \) with weight processes

$$\pi_i(t) = \frac{\mu_j(t)}{\mu_i(t) + \mu_j(t)} \quad \text{and} \quad \pi_j(t) = \frac{\mu_i(t)}{\mu_i(t) + \mu_j(t)}, \quad \text{a.s.,}$$  \hspace{1cm} (3.5)

and drift process

$$\Theta(t) = \int_0^t \frac{\mu_i(s) \mu_j(s)}{(\mu_i(s) + \mu_j(s))^2} (\tau_{ii}(s) - 2 \tau_{ij}(s) + \tau_{jj}(s)) \, ds, \quad \text{a.s.}$$  \hspace{1cm} (3.6)

We see that the weights \( \pi_i, \pi_j \) are proportional to “swapped” market weights \( \mu_i, \mu_j \). We would like to see if this might improve on the negative effect seen in (2.7).
Proposition 3.1. Suppose that for \( n \geq 2 \) the market \( \{X_1, \ldots, X_n\} \) is coherent and for \( 1 \leq i < j \leq n \) the submarket \( \{X_i, X_j\} \) has positive asymptotic diversification. Then the swap portfolio \( \pi \) with weight processes \( \pi_i \) and \( \pi_j \) as in (3.5) will have a higher asymptotic growth rate than the market portfolio.

Proof. Let

\[
\eta_i = \frac{\mu_i}{\mu_i + \mu_j} \quad \text{and} \quad \eta_j = \frac{\mu_j}{\mu_i + \mu_j}
\]

be the market weight processes for the submarket \( \{X_i, X_j\} \). We see from (3.6) the drift process satisfies

\[
d\Theta(t) = \frac{\mu_i(t)\mu_j(t)}{(\mu_i(t) + \mu_j(t))^2} (\tau_{ii}(t) - 2\tau_{ij}(t) + \tau_{jj}(t)) dt
\]

\[
= \eta_i(t)\eta_j(t) (\tau_{ii}(t) - 2\tau_{ij}(t) + \tau_{jj}(t)) dt
\]

\[
= \eta_i(t)\eta_j(t) (\sigma_{ii}(t) - 2\sigma_{ij}(t) + \sigma_{jj}(t)) dt
\]

\[
= (\eta_i(t)\sigma_{ii}(t) + \eta_j(t)\sigma_{jj}(t) - \sigma_{ij}^2(t)) dt
\]

\[
= 2\gamma^*_i(t) dt, \quad \text{a.s.}
\]

Hence,

\[
\log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = \log \mu_i(t) + \log \mu_j(t) - \log (\mu_i(t) + \mu_j(t)) + 2 \int_0^t \gamma^*_i(s) ds, \quad \text{a.s.} \quad (3.7)
\]

Since the market \( \{X_1, \ldots, X_n\} \) is coherent and the submarket \( \{X_i, X_j\} \) has positive asymptotic diversification,

\[
\lim_{t \to \infty} \frac{1}{t} \log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = 2 \lim_{t \to \infty} \frac{1}{t} \int_0^t \gamma^*_i(s) ds
\]

\[
> 0, \quad \text{a.s.},
\]

and the proposition follows. \( \square \)

It would seem reasonable that if we swapped the weights of more than one pair of assets, the resulting portfolio would also grow faster than the market. This may be true, but the proof will have to rely on other methods, since portfolios in which the weights of more than a single pair of assets are swapped is not functionally generated.

Let \( \{X_1, \ldots, X_n\} \) be a market with \( n \geq 2 \) and let \( U \subset \mathbb{R}^n \) be a neighborhood of \( \Delta^n \subset \mathbb{R}^n \). Suppose that \( f : U \to \mathbb{R} \) is a \( C^1 \) function such that \( f_1(x) + \cdots + f_n(x) = 1 \), for \( x \in U \), and that \( \pi \) is a portfolio with \( \pi_i(t) = f_i(\mu(t)) \), for \( i = 1, \ldots, n \). In this case Fernholz (2002), Proposition 3.1.11, implies that if \( \pi \) is functionally generated then there exists a \( C^1 \) function \( h \) defined on \( U \) such that

\[
D_j(f_i(x)/x_i + h(x)) = D_i(f_j(x)/x_j + h(x)), \quad (3.8)
\]

for \( i, j = 1, \ldots, n \) and \( x \in U \).

Example 3.2. Suppose that \( n = 4 \), and consider the “double-swap” portfolio \( \pi \) with weights

\[
\pi_1(t) = \mu_2(t), \quad \pi_2(t) = \mu_1(t), \quad \pi_3(t) = \mu_4(t), \quad \pi_4(t) = \mu_3(t).
\]

For \( \pi \) to be functionally generated, we see from (3.8) that we must have

\[
D_1(x_1/x_2 + h(x)) = D_2(x_2/x_1 + h(x))
\]

\[
D_1(x_4/x_3 + h(x)) = D_3(x_2/x_1 + h(x))
\]

\[
D_2(x_4/x_3 + h(x)) = D_3(x_1/x_2 + h(x)).
\]

The first of these equations implies that

\[
D_1 h(x) - D_2 h(x) = 1/x_1 - 1/x_2. \quad (3.9)
\]
In the two-asset case, where we have a generating function, we could solve this with \( h(x) = \log(x_1 x_2) \), however here we have four assets. In this case, the second and third equations imply that
\[
D_1 h(x) = D_2 h(x) = D_3 h(x),
\]
which is incompatible with (3.9). Hence, \( \pi \) is not functionally generated.

Even though Example 3.2 shows that multiple-swap portfolios are not functionally generated, similar portfolios may be functionally generated. For example, following the example towards the bottom of page 50 of Fernholz (2002) we see that for \( m \geq 1 \) and \( n = 2m \), if \( p_1, \ldots, p_m \) are constants such that \( p_1 + \cdots + p_m = 1 \), then the function
\[
S(x) = \prod_{k=1}^{m} \left( \frac{x_{2k-1} x_{2k}}{x_{2k-1} + x_{2k}} \right)^{p_k},
\]
generates the portfolio \( \pi \) with weights
\[
\pi_{2k-1}(t) = p_k \left( \frac{\mu_{2k}(t)}{\mu_{2k-1}(t) + \mu_{2k}(t)} \right), \quad \pi_{2k}(t) = p_k \left( \frac{\mu_{2k-1}(t)}{\mu_{2k-1}(t) + \mu_{2k}(t)} \right),
\]
for \( k = 1, \ldots, m \). While this may be similar to a swap portfolio, it is not a true swap portfolio.

4 Reverse-weighted portfolios and first-order models

For a market \( \{X_1, \ldots, X_n\} \), the reverse-weighted portfolio is the portfolio \( \pi \) with weight processes
\[
\pi_i(t) = \mu_{n+1-r_i(i)}(t), \quad (4.1)
\]
for \( i = 1, \ldots, n \). For \( n = 2 \), the swap portfolio of Proposition 3.1 is the reverse-weighted portfolio, but as we have seen in Example 3.2 generating functions cannot be applied if more than a single pair of assets is swapped. Hence, in order to study reverse-weighted portfolios in markets with \( n \geq 4 \) it is convenient to introduce some simplifying assumptions.

**Definition 4.1.** (Fernholz 2002; Banner et al. 2005) For \( n > 1 \), a first-order model is a system of positive continuous semimartingales \( \{X_1, \ldots, X_n\} \) defined by
\[
d \log X_i(t) = g_{r_i(i)}(t) \, dt + \sigma_{r_i(i)} \, dW_i(t), \quad (4.2)
\]
where \( \sigma_1^2, \ldots, \sigma_n^2 \) are positive constants, \( g_1, \ldots, g_n \) are constants satisfying
\[
g_1 + \cdots + g_n = 0 \quad \text{and} \quad g_1 + \cdots + g_k < 0 \quad \text{for} \quad k < n, \quad (4.3)
\]
and \( (W_1, \ldots, W_n) \) is a Brownian motion.

First-order models are simplified versions of markets that retain certain important characteristics of actual markets. By Proposition 2.3 of Banner et al. (2005), each of the processes \( X_i \) in a first-order-model asymptotically spends equal time in each rank and hence has zero asymptotic log-drift. These processes spend zero local time at triple points, so (2.10) will hold for them (see Lemma 1 of Ichiba et al. 2011). For a portfolio \( \pi \) in a market \( \{X_1, \ldots, X_n\} \) represented by a first-order model, the **portfolio growth rate** \( \gamma_\pi \) will satisfy
\[
\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t) g_{r_i(i)} + \gamma_\pi^* \quad (4.4)
\]
\[
= \sum_{k=1}^{n} \pi_{p_k(i)}(t) g_k + \gamma_\pi^*(t), \quad \text{a.s.},
\]
where \( p_t \in \Sigma_n \) is the inverse permutation to the rank function \( r_t \). By Proposition 1.3.1 of Fernholz (2002), the value process \( Z_\pi \) of \( \pi \) satisfies
\[
\lim_{t \to \infty} \frac{1}{t} \left( Z_\pi(t) - \int_0^t \gamma_\pi(s) \, ds \right) = 0, \quad \text{a.s.}
\]

**Proposition 4.2.** Suppose that \( n > 1 \) and \( \{X_1, \ldots, X_n\} \) is a market represented by a first-order model \( \mathcal{M}_n \) for which the excess growth rate of the reverse-weighted portfolio \( \mathcal{S}_\pi \) satisfies \( \gamma_\pi^*(t) \geq \gamma_\mu^*(t) \), a.s., for \( t \in (0, \infty) \). Then the growth rate of the reverse-weighted portfolio will be greater than that of the market,
\[
\gamma_\pi(t) > \gamma_\mu(t), \quad \text{a.s.,}
\]
for \( t \in [0, \infty) \), except on a set of Lebesgue measure zero.

**Proof.** For the first-order model \( \{X_1, \ldots, X_n\} \), it follows from \( (4.4) \) that the market growth rate is
\[
\gamma_\mu(t) = \sum_{i=1}^n \mu_i(t) g_{r_i(t)} + \gamma_\mu^*(t)
\]
\[
= \sum_{k=1}^n \mu_{(k)}(t) g_k + \gamma_\mu^*(t), \quad \text{a.s.}
\]
Similarly, the growth rate for the reverse-weighted portfolio \( \pi \) is
\[
\gamma_\pi(t) = \sum_{i=1}^n \pi_i(t) g_{r_i(t)} + \gamma_\pi^*(t)
\]
\[
= \sum_{k=1}^n \mu_{(n+1-k)}(t) g_k + \gamma_\pi^*(t), \quad \text{a.s.}
\]
Hence,
\[
\gamma_\pi(t) - \gamma_\mu(t) = \sum_{k=1}^n \left( \mu_{(n+1-k)}(t) - \mu_{(k)}(t) \right) g_k + \gamma_\pi^*(t) - \gamma_\mu^*(t)
\]
\[
\geq \sum_{k=1}^n \left( \mu_{(n+1-k)}(t) - \mu_{(k)}(t) \right) g_k
\]
\[
= \sum_{k=1}^n \varphi_k(t) g_k, \quad \text{a.s.,}
\]
where \( \varphi_k(t) = \mu_{(n+1-k)}(t) - \mu_{(k)}(t) \). From the definition of rank it follows that \( \varphi_{k+1}(t) > \varphi_k(t) \), a.s., for \( k = 1, \ldots, n-1 \) and \( t \in [0, \infty) \) except on a set of Lebesgue measure zero. With this inequality and \( (4.3) \), we can apply summation by parts and obtain
\[
\sum_{k=1}^n \varphi_k(t) g_k = \varphi_1(t) \sum_{k=1}^n g_k + \sum_{k=1}^{n-1} \left( \varphi_{k+1}(t) - \varphi_k(t) \right) \sum_{\ell=k+1}^n g_\ell
\]
\[
> 0, \quad \text{a.s.,}
\]
for \( t \in [0, \infty) \) except on a set of Lebesgue measure zero, and \( (4.5) \) follows.

We can apply this to a first-order model with rank-symmetric variance parameters, \( \sigma_k^2 = \sigma_{n+1-k}^2 \).

**Corollary 4.3.** Suppose that \( \{X_1, \ldots, X_n\} \) is a market represented by a first-order model \( \mathcal{M}_n \) for which \( \sigma_k^2 = \sigma_{n+1-k}^2 > 0 \), for \( k = 1, \ldots, n \). Then the growth rate of the reverse-weighted portfolio \( \pi \) will be greater than that of the market,
\[
\gamma_\pi(t) > \gamma_\mu(t), \quad \text{a.s.,}
\]
for \( t \in [0, \infty) \), except on a set of Lebesgue measure zero.
Proof. Since both the weights (4.1) and the variances $\sigma_k^2 = \sigma_{n+1-k}^2$ are reversed by rank, we see from (2.3) that the excess growth rates $\gamma_k^e(t) = \gamma_k^e(t)$, a.s., for $t \in [0, \infty)$. Hence, Proposition 4.2 can be applied.

5 First-order approximation of asymptotically stable markets

In general, we consider markets with some level of asymptotic stability, and markets with stable asymptotic characteristics can often be approximated by first-order models. Here we shall introduce asymptotic stability conditions on a market that will allow us to approximate the market by a first-order model. We shall represent the market by an asymptotically stable system of positive continuous semimartingales $\{X_1, \ldots, X_n\}$, for $n > 1$, and we shall let $\{X_1, \ldots, X_n\}$ denote the first-order model that approximates the market. We shall let $\{X_1, \ldots, X_n\}$ represent the rank processes corresponding to $\{X_1, \ldots, X_n\}$.

**Definition 5.1.** [Fernholz 2002] The system $\{X_1, \ldots, X_n\}$ of positive continuous square-integrable semimartingales is **asymptotically stable** if

1. $\lim_{t \to \infty} \frac{1}{t} \left( \log X_{(1)}(t) - \log X_{(n)}(t) \right) = 0$, a.s. (coherence);
2. $\lambda_{k,k+1} = \lim_{t \to \infty} \frac{1}{t} \Lambda_{k,k+1}(t) > 0$, a.s.;
3. $\sigma_{k,k+1}^2 = \lim_{t \to \infty} \frac{1}{t} \left< \log X_{(k)} - \log X_{(k+1)} \right>_t > 0$, a.s.;

for $k = 1, \ldots, n-1$, where $\Lambda_{k,k+1}$ is the local time at the origin for $\log (X_{(k)}/X_{(k+1)})$. By convention, let $\lambda_{0,1} = \lambda_{n,n+1} = 0, \sigma_0^2 = \sigma_{1,2}^2$, and $\sigma_n^2 = \sigma_{n-1,n}^2$.

The first order model $\{X_1, \ldots, X_n\}$ defined by (4.2) is asymptotically stable with

$$\lambda_{k,k+1} = \lim_{t \to \infty} \frac{1}{t} \Lambda_{k,k+1}(t) = -2(g_1 + \cdots + g_k), \quad a.s., \quad (5.1)$$

for $k = 1, \ldots, n-1$, and

$$\sigma_{k,k+1}^2 = \lim_{t \to \infty} \frac{1}{t} \left< \log X_{(k)} - \log X_{(k+1)} \right>_t = \sigma_k^2 + \sigma_{k+1}^2, \quad a.s., \quad (5.2)$$

for $k = 1, \ldots, n-1$ (see Section 3 of Banner et al. [2005]). The **gap processes** $(\log X_{(k)} - \log X_{(k+1)})$, for $k = 1, \ldots, n-1$, have stable distributions that are exponentially distributed and satisfy

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T (\log X_{(k)}(t) - \log X_{(k+1)}(t)) dt = \frac{\sigma_{k,k+1}^2}{2\lambda_{k,k+1}}, \quad a.s., \quad (5.3)$$

(see Section 4 of Banner et al. [2005] or Theorem 1 of Ichiba et al. [2011]).

**Definition 5.2.** [Fernholz 2002] Let $\{X_1, \ldots, X_n\}$ be an asymptotically stable system of positive continuous semimartingales with parameters $\lambda_{k,k+1}$ and $\sigma_{k,k+1}^2$, for $k = 1, \ldots, n$, defined as in Definition 5.1. Then the **first-order approximation** for $\{X_1, \ldots, X_n\}$ is the first-order model $\{X_1, \ldots, X_n\}$ with

$$d \log X_i(t) = g_{r_t(i)} dt + \sigma_{r_t(i)} dW_i(t), \quad (5.4)$$

for $i = 1, \ldots, n$, where $r_t \in \Sigma_n$ is the rank function for the $X_i$, the parameters $g_k$ and $\sigma_k$ are defined by

$$g_k = \frac{1}{2} \lambda_{k-1,k} - \frac{1}{2} \lambda_{k,k+1}, \quad \text{for } k = 1, \ldots, n,$$

$$\sigma_k^2 = \frac{1}{4} \left( \sigma_{k-1,k}^2 + \sigma_{k,k+1}^2 \right), \quad \text{for } k = 1, \ldots, n, \quad (5.5)$$

$\sigma_k$ is the positive square root of $\sigma_k^2$, and $(W_1, \ldots, W_n)$ is a Brownian motion.
For the first-order model (5.4) with parameters (5.5), equations (5.1) and (5.2) imply that
\[ \lambda_{k,k+1} = -2(g_1 + \cdots + g_k) = \lambda_{k,k+1}, \quad \text{a.s.,} \] (5.6)
for \( k = 1, \ldots, n - 1 \), and
\[ \sigma^2_{k,k+1} = \sigma^2_k + \sigma^2_{k+1} = \frac{1}{4}(\sigma^2_{k-1,k} + 2\sigma^2_{k,k+1} + \sigma^2_{k+1,k+2}), \quad \text{a.s.,} \] (5.7)
for \( k = 1, \ldots, n - 1 \). Hence, (5.3) becomes
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T (\log X_{(k)}(t) - \log X_{(k+1)}(t)) dt = \frac{\sigma^2_{k,k+1}}{2\lambda_{k,k+1}} = \frac{\sigma^2_{k-1,k} + 2\sigma^2_{k,k+1} + \sigma^2_{k+1,k+2}}{8\lambda_{k,k+1}}, \quad \text{a.s.,} \] (5.8)
for \( k = 1, \ldots, n - 1 \). If the \( X_i \) satisfy
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T (\log X_{(k)}(t) - \log X_{(k+1)}(t)) dt \equiv \frac{\sigma^2_{k,k+1}}{2\lambda_{k,k+1}}, \] (5.9)
for \( k = 1, \ldots, n - 1 \), then the stable distribution (5.8) for the first-order approximation will be a smoothed version of the stable distribution (5.2) for the \( X_i \).

If the behavior of the first-order approximation \{ \( X_1, \ldots, X_n \) \} is close enough to that of the original market \{ \( X_1, \ldots, X_n \) \}, then the stable distributions (5.8) and (5.9) will be close to each other. If the \( \sigma^2_{k,k+1} \) are fairly constant in \( k \), then Proposition 4.2 will probably hold, and the reverse-weighted portfolio should have a higher growth rate than the market portfolio. In the next section we shall test this methodology on a set of empirical data.

6 An application to commodity futures markets

We examine the applicability of our theoretical results using monthly historical futures prices data for 26 commodities from 1977-2018. Table 1 lists the start month and trading market for the 26 commodity futures contracts in our data set. We construct equal-weighted, price-weighted, and reverse price-weighted portfolios of commodity futures and examine their relative performance over this time period. Following Vervuurt and Karatzas (2015), we also construct and examine the performance of the diversity-weighted portfolio of commodity futures with a negative diversity parameter of -0.5. For this application to commodity futures, the price-weighted portfolio corresponds to the market portfolio \( \mu \) defined in Section 2, and the reverse price-weighted portfolio in this application corresponds to the reverse-weighted portfolio (4.1) defined in Section 3.

Implied commodity futures prices and returns

The most liquid commodity futures contracts are usually those with expiration dates approximately one or two months in the future. Accordingly, we use two-month commodity futures contracts to generate our data set whenever such prices exist. We define implied two-month futures prices to fill the gaps when the available data do not include actual two-month futures prices. From these implied prices we can define time series that can be approximated with a first-order model following the methodology of the previous section.

Suppose that commodity \( i \) has a futures contract with expiration date \( \tau \in \mathbb{N} \). For \( t \in \mathbb{N} \), with \( t \leq \tau \), let
\[ F_i(t, \tau) = \text{futures price for commodity } i \text{ at time } t \text{ for the contract with expiration at } \tau. \]
In this case, \( F_i(\tau, \tau) \) is the spot price for commodity \( i \) at time \( \tau \). For \( t \in \mathbb{N} \), let \( \nu_2 > \nu_1 \geq 0 \), \( \nu_1, \nu_2 \neq 2 \), be the smallest integers closest to two such that \( t + \nu_1 \) and \( t + \nu_2 \) are expiration dates of futures contracts for the \( i \)th commodity. We define the carry factor for commodity \( i \) at time \( t \) to be
\[ \Delta_i(t) \triangleq \frac{\log F_i(t, t + \nu_2) - \log F_i(t, t + \nu_1)}{\nu_2 - \nu_1}. \] (6.1)
Note that the carry factor $\Delta_i$ is commodity-specific and can vary over time, and it is calculated using futures contracts with expirations as close as possible, but not equal to, two months in the future.

**Definition 6.1.** For $t \in \mathbb{N}$, let $\nu \geq 0$, be the smallest of the closest integers to two such that $t + \nu$ is an expiration date of a futures contract for the $i$th commodity. Then the implied two-month futures price at time $t$ for commodity $i$ is

$$\tilde{F}_i(t, t + 2) \triangleq e^{(2-\nu)\Delta_i(t)} F_i(t, t + \nu).$$

(6.2)

We generate monthly time series $X_1(t), \ldots, X_n(t)$, for $t \in \mathbb{N}$, from the implied two-month futures prices of each of the $n = 26$ commodities in our data set. We use these data to compare and rank the commodities over the 1977-2018 time period. In terms of Definition 6.1, we let

$$X_i(t) \triangleq \tilde{F}_i(t, t + 2),$$

(6.3)

for $t = 1, 2, \ldots, 492$, and $i = 1, \ldots, 26$. It follows from Definition 6.1 and 6.3 that if $F_i(t, t + 2)$ exists, then

$$X_i(t) = \tilde{F}_i(t, t + 2) = F_i(t, t + 2),$$

so that in this case the two-month futures price is equal to the two-month implied futures price.

When forming equal-weighted, price-weighted, diversity-weighted, and reverse-weighted portfolios, we hold two-month futures contracts in all months in which such contracts exist. For those months in which there are no two-month futures contracts, we hold those contracts with the next expiration horizon greater than two months in the future. In both cases, the change in the two-month implied futures price of commodity $i$, $d \log X_i(t)$, is not necessarily equal to the return from holding the underlying commodity futures contract, $d \log F_i(t, \tau)$, where $\tau \geq t + 2$ is a futures contract expiration date. We refer to the difference between the log change in the two-month implied futures price and the return from holding the underlying futures contract as the carry. This carry satisfies

$$C_i(t) \, dt = d \log F_i(t, \tau) - d \log X_i(t),$$

(6.4)

for $t \in [0, \infty)$.

In order to meaningfully compare and rank the two-month implied futures prices of the 26 different commodities in our data set, it is necessary to normalize these prices. We set the initial implied futures prices of all commodities with available futures contracts on the November 1968 data start month — soybean meal, soybean oil, and soybeans — equal to each other. All subsequent monthly price changes occur without modification, meaning that implied futures price dynamics are unaffected by our normalization. This method of normalizing prices is similar to Asness et al. (2013), who rank commodity futures based on each commodity’s current spot price relative to its average spot price 4.5 to 5.5 years in the past.

For those commodities that enter into our data set after November 1968, we set the initial implied futures log-price equal to the average log-price of those commodities already in our data set on that month. After a commodity enters into the data set with a normalized price, all subsequent monthly price changes occur without modification. The resulting normalized two-month implied futures prices for the 26 commodities in our data set are plotted in Figure 1 with the log-prices reported relative to the average for all commodities in each month.

**Portfolios of commodity futures**

We analyze and compare the performance of price-weighted, equal-weighted, diversity-weighted, and reverse price-weighted portfolios of commodity futures. The price-weighted portfolio is simply the market portfolio $\mu$ defined in Section 2 with weight processes

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}.$$
\[ i = 1, \ldots, n, \text{ where } \mathbf{X}_1, \ldots, \mathbf{X}_n \text{ are the two-month implied futures prices defined above.} \]

The equal-weighted portfolio is the portfolio with weights constant and equal to \(1/n\) for all \(t\). The diversity-weighted portfolio is the portfolio with weights

\[ \pi_i(t) = \frac{\mathbf{X}_i^p(t)}{\mathbf{X}_1^p(t) + \cdots + \mathbf{X}_n^p(t)}, \]

for \(i = 1, \ldots, n\), with the diversity parameter \(p\) set equal to \(-0.5\) following Vervuurt and Karatzas (2015).

Finally, the reverse price-weighted portfolio is the reverse-weighted portfolio defined in Section 4, with

\[ \pi_i(t) = \mu_{(n+1-r_t(i))}, \]

for \(i = 1, \ldots, n\). Although our commodity futures data cover 1968-2018, the fact that we normalize implied futures prices by setting them equal to each other on the November 1968 start date, as discussed above, implies that these prices cannot be meaningfully compared and ranked until they have time to disperse. Thus, for each commodity in our data set, we wait five years after the start of its price data before including that commodity futures contract in our equal-weighted, price-weighted, diversity-weighted, and reverse-weighted portfolios. Furthermore, we do not start forming portfolios until we have at least ten commodities with at least five years of price data, which occurs in November 1977. As a consequence, all results we report for the different portfolios run from November 1977 to January 2018.

In Table 2, we report the average and standard deviation of the annual log-returns for equal-weighted, price-weighted, diversity-weighted, and reverse-weighted portfolios of commodity futures from 1977-2018. The cumulative returns for these three portfolios are shown in Figure 2. This figure clearly shows that the reverse-weighted portfolio grows faster than the price-weighted market portfolio over the 1977-2018 time period, consistent with the result in Proposition 4.2. The reverse-weighted portfolio also grows faster than the equal-weighted and diversity-weighted portfolios.

Table 2 also reports the Sharpe ratios of the equal-weighted, diversity-weighted, and reverse-weighted portfolios, defined as the average log-returns of each of the three portfolios minus the log-returns of the price-weighted portfolio divided by the standard deviation of these relative returns. The cumulative relative returns for the equal-weighted, diversity-weighted, and reverse-weighted portfolios are shown in Figure 3. The results in the table and figure confirm the faster growth of the reverse-weighted portfolio relative to the equal-weighted and diversity-weighted portfolios, and also reveal a higher Sharpe ratio for the reverse-weighted portfolio.

According to (2.1), the log-returns of the three commodity futures portfolios shown in Figure 2 can be decomposed into the weighted growth rate of the individual futures contracts and the excess growth rate process, \(\gamma^*\), which is given by (2.2). Figure 4 plots the cumulative values of the excess growth rate process for the price-weighted, reverse-weighted, equal-weighted, and diversity-weighted portfolios of commodity futures from 1977-2018. These excess growth rates are calculated using the decomposition (2.1) together with the log-returns of each portfolio and the weighted log-returns of the individual futures contracts held in each portfolio. According to the figure, the processes \(\gamma^*\) for the reverse-weighted and price-weighted portfolios are approximately equal to each other, which is an important condition needed to apply Proposition 4.2 regarding the growth rates of the two portfolios. Therefore, Figure 4 together with Proposition 4.2 explains the outperformance of the reverse-weighted portfolio relative to the price-weighted portfolio shown in Figure 2.

As discussed earlier, returns from holding commodity futures contracts are not equal to changes in implied futures prices, with the difference being the carry as defined by (6.4). Because the price-weighted, reverse-weighted, equal-weighted, and diversity-weighted portfolios weight each futures contract differently, it follows that the carry for each portfolio will also be different. Figure 5 plots the cumulative carry for these four portfolios of commodity futures. According to the figure, the carry for all four portfolios is consistently negative, with the most negative carry for the reverse-weighted portfolio and the least negative carry for the price-weighted portfolio. Furthermore, the magnitude of the cumulative effects of this carry on returns is meaningfully large — the reverse-weighted portfolio outperforms the price-weighted, equal-weighted, and diversity-weighted portfolios despite a substantially more negative carry.
First-order approximation of implied futures market

In order to understand the outperformance of the reverse-weighted portfolio relative to the price-weighted portfolio as shown in Table 2 and Figure 2, we estimate the first-order approximation of the two-month normalized implied futures market plotted in Figure 1. According to Definition 4.1, a first-order model is defined only for a fixed number of ranked assets. Therefore, we estimate the first-order approximation of the implied futures market using commodity futures price data starting on April 1995, since this date is five years after the last commodity futures contract price data begin (Table 1) and hence it is also the last date on which a new commodity enters our data set. The total number of commodity implied futures prices is thus fixed at 26 from April 1995 to January 2018, and we are able to estimate the first-order approximation of this market over that time period.

Figure 6 plots the parameters $g_k$, defined by (5.5), for the first-order approximation of the two-month implied futures market. This figure plots the values of these parameters after applying a reflected Gaussian filter with a bandwidth of six ranks together with the unfiltered values, which are represented by the red circles in the figure. Figure 6 shows that the first-order approximation of this market features mostly higher growth rates $g_k$ for lower-ranked commodity futures than for higher-ranked futures, and that the sum of these growth rates is negative for top-ranked subsets. This pattern is consistent with the stability condition (4.3) for the $g_k$ parameters of a first-order model.

Figure 7 plots both filtered and unfiltered values of the parameters $\sigma_k$, defined by (5.5), for this same first-order approximation. This figure shows that the filtered values of the volatility parameters $\sigma_k$ are roughly constant across ranks, which implies that the conditions of Corollary 4.3 are approximately satisfied by the commodity implied futures market. According to Corollary 4.3, then, the growth rate of the reverse-weighted portfolio of commodity futures should be greater than that of the price-weighted portfolio, before adjusting for the carry of each portfolio. In fact, we find that the growth rate of reverse-weighted implied commodity futures is enough larger than that of price-weighted implied commodity futures that the reverse-weighted portfolio substantially outperforms the price-weighted portfolio, despite its significantly more negative carry as shown in Figure 5.

A closer inspection of Figure 7 reveals that the unfiltered estimates of the parameters $\sigma_k$ are highest and approximately equal to each other at the top two and bottom two ranks. Furthermore, these volatility parameters form a roughly symmetric U-shape when plotted versus rank. The results of Corollary 4.3 still apply to such a first-order model, since $\sigma^2_k = \sigma^2_{n+1-k} > 0$ for all $k = 1, \ldots, n$ in the case of a symmetric U-shape for the volatility parameters. Thus, both the filtered and unfiltered estimates of the volatility parameters $\sigma_k$ are approximately consistent with the outperformance of the reverse-weighted portfolio shown in Figure 2 at least in the absence of a different carry for the two portfolios.

Finally, Figure 8 presents a log-log plot of average relative implied futures prices for different ranked prices versus rank for 1995-2018 together with the predicted relative prices according to (5.8) using the estimated parameters $g_k$ and $\sigma_k$ from Figures 6 and 7. This figure shows that the predicted relative prices according to the first-order approximation using the estimated parameters $g_k$ and $\sigma_k$ provides a reasonably accurate match to the actual average relative commodity implied futures prices observed over this time period.

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| Commodity         | Exchange where Traded | Start Date |
|-------------------|-----------------------|------------|
| Soybean Meal      | CBOT                  | 11/1968    |
| Soybean Oil       | CBOT                  | 11/1968    |
| Soybeans          | CBOT                  | 11/1968    |
| Wheat             | CBOT                  | 1/1969     |
| Corn              | CBOT                  | 1/1969     |
| Live Hogs         | CME                   | 12/1969    |
| Cotton            | NYBOT                 | 10/1972    |
| Silver            | COMEX                 | 10/1972    |
| Orange Juice      | CEC                   | 11/1972    |
| Platinum          | NYMEX                 | 11/1972    |
| Sugar             | CSC                   | 1/1973     |
| Lumber            | CME                   | 7/1973     |
| Coffee            | CSC                   | 10/1973    |
| Oats              | CBOT                  | 10/1974    |
| Gold              | COMEX                 | 1/1975     |
| Live Cattle       | CME                   | 4/1976     |
| Wheat, K.C.       | KCBT                  | 5/1976     |
| Feeder Cattle     | CME                   | 11/1977    |
| Heating Oil       | NYMEX                 | 10/1979    |
| Cocoa             | CSC                   | 1/1981     |
| Wheat, Minn.      | MGE                   | 1/1981     |
| Palladium         | NYMEX                 | 1/1983     |
| Crude Oil         | NYMEX                 | 4/1983     |
| Rough Rice        | CBOT                  | 9/1986     |
| Copper            | COMEX                 | 11/1988    |
| Natural Gas       | NYMEX                 | 4/1990     |

Table 1: List of commodity futures contracts along with the exchange where each commodity is traded and the date each commodity started trading.

|                | Price-Weighted Portfolio | Equal-Weighted Portfolio | Diversity-Weighted Portfolio | Reverse Price-Weighted Portfolio |
|----------------|--------------------------|--------------------------|-------------------------------|---------------------------------|
| Average        | -1.43%                   | 0.43%                    | 1.09%                         | 1.83%                           |
| Standard Deviation | 15.38%                   | 13.79%                   | 13.68%                        | 13.85%                          |
| Sharpe Ratio   | 0.40                     | 0.41                     | 0.41                          | 0.47                            |

Table 2: Annual average and standard deviation of log-returns for price-weighted, equal-weighted, diversity-weighted, and reverse price-weighted portfolios, and Sharpe ratio of equal-weighted, diversity-weighted, and reverse price-weighted portfolios relative to the price-weighted portfolio, 1977-2018.
Figure 1: Two-month implied futures log-prices relative to the average, 1968-2018.

Figure 2: Cumulative log-returns for the price-weighted, reverse price-weighted, equal-weighted, and diversity-weighted portfolios, 1977-2018.
Figure 3: Cumulative log-returns for the reverse price-weighted, equal-weighted, and diversity-weighted portfolios relative to the price-weighted portfolio, 1977-2018.

Figure 4: Cumulative values of the excess growth rate process $\gamma^*$ for the price-weighted, reverse price-weighted, equal-weighted, and diversity-weighted portfolios, 1977-2018.
Figure 5: Cumulative carry for the price-weighted, reverse price-weighted, equal-weighted, and diversity-weighted portfolios, 1977-2018.

Figure 6: Estimated parameters $g_k$ for the first-order approximation of the two-month implied-futures market, 1995-2018.
Figure 7: Estimated parameters $\sigma_k$ for the first-order approximation of the two-month implied-futures market, 1995-2018.

Figure 8: Average relative two-month implied futures prices for different ranks and predicted relative prices for different ranks according to (5.8), 1995-2018.