Orbital stabilization of underactuated mechanical systems without Euler-Lagrange structure after of a collocated pre-feedback

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Abstract—In this note we study the generation of attractive oscillations of a class of mechanical systems with underactuation. The proposed design consists of two terms, i.e., a partial linearizing state feedback, and an immersion and invariance orbital stabilization controller. The first step is adopted to simplify analysis and design, however, bringing additional difficulty that the model losses Euler-Lagrange structures after the collocated pre-feedback. To address this, we propose a constructive solution to the orbital stabilization problem via a smooth controller in an analytic form, and the model class identified in the paper is characterized via some easily *apriori* verifiable assumptions on the inertia matrix and potential energy.

Index Terms—Nonlinear systems, Immersion and Invariance, mechanical systems, orbital stabilization.

I. INTRODUCTION

Oscillating behaviour of dynamical systems is ubiquitous in biology, physics and engineering [3], [21], [27]. For the latter, we are concerned with the constructive perspective to generate stable oscillators for closed loop—known as the orbital stabilization problem [10]—which widely appears in many engineering areas, including bipedal robots [19], exoskeletons [11], electrical motors [23], AC power converters [13], path following of closed orbits [24], [29], combustion oscillations, etc. Despite the fact that these control problems may be formulated as trajectory tracking, it brings the following merits to consider them as orbital stabilization:

1) There is no need to ensure phase synchronization, i.e. asymptotic convergence of tangential coordinates (also known as angular variable or isochrons), making control design more flexible;
2) The closed-loop dynamics are autonomous, unlike time-varying in trajectory tracking, as a result, there is no need to design motion planning algorithms to generate reference trajectories [15];
3) The error dynamics in tracking control, unfortunately, fails to preserve geometric properties of the original control system. For example, the tracking error dynamics of mechanical systems in general are not in Euler-Lagrange (EL) or Hamiltonian forms, unless imposing additional assumptions [14].

Unlike regulation of equilibria, there are only a few tools for orbital stabilization, which may be generally classified into the following two categories. The first class is based on the concept of energy, see for examples [4], [7], [19], [20], [28], which rely on either shaping the total energy with minima on the desired orbit, or regulating the energy to some value. In [28], these two ideas are systematically studied for port-Hamiltonian systems, and their equivalence has been revealed. Another widely studied technical route is from a geometric perspective. In [6], the authors consider the case that in the neighborhood of a given orbit a change of coordinate may be found locally to decompose the systems state into transverse coordinate and phase variable, and then the transverse feedback linearization technique is applicable to deal with orbital stabilization. Later, such an idea was elaborated for underactuated EL mechanical systems in [15], [16], known as the virtual holonomic constraint (VHC) approach, which may conceptually date back to [2] published in 1911. The key step is to use a partial feedback linearization to impose a geometric constraint of configuration variables with respect to a new independent variable, with the obtained zero dynamics behaving as oscillations. Recently, the immersion and invariance (I&I) technique, which was originally proposed for nonlinear and adaptive control and the design of observers [5], is adopted for orbital stabilization of nonlinear systems [12]. This method can be viewed as an extension of VHC to general nonlinear systems, but with a key difference of the design procedure—in the former a lower dimensional-oscillator needs to be selected in advance, then solving the Francis-Byrnes-Isidori (FBI) equations.

In this paper, the I&I orbital stabilization technique is tailored for a class of underactuated mechanical systems. To be precise, the main contributions are summarized as follows.

C1 We provide a constructive solution to a class of underactuated EL systems by integrating collocated feedback linearization (w.r.t. the actuated configuration) [18] with a well elaborated I&I controller. Note that the system is not in the EL structure after pre-feedback.

C2 The proposed controller is given in a compact analytic formulation with assumptions which can be verified as a priori, as a consequence, neither requiring to verify “transverse controllability” on-line [8], [9], [16] nor calculating functions numerically [17] as done in the VHC method.

C3 Under the proposed controller, the subsystem of partial states can be regarded as an undamped (or conservative) EL system perturbed by a decaying term, the boundedness of which is analyzed comprehensively.

The remainder of the paper is organized as follows. In Section II we recall the main results of I&I orbital stabilization in [12]. It is followed by the EL model which we are interested in and the precise problem formulation in Section III. We present the main result of the paper in Section IV, and then we show the performance of the proposed controller using two benchmarks in Section V. The paper is wrapped up with some concluding remarks in Section VI.

Notation. Throughout the paper, we assume all mappings are sufficiently smooth. $I_n$ is the $n \times n$ identity matrix and $0_{n \times n}$ is an $n \times s$ matrix of zeros. For $x \in \mathbb{R}^n$, $S \in \mathbb{R}^{n \times n}$, $S = S^\top > 0$, we denote the Euclidean norm $|x|^2 := x^\top x$, and the weighted-
norm $\|x\|^2 := x^T S x$. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ we define the differential operators $\nabla f := \left( \frac{\partial f}{\partial x} \right)^T$, $\nabla x_i f := \left( \frac{\partial f}{\partial x_i} \right)^T$, in which $x_i$ is an element of the vector $x$. We use $S$ to denote the unit circle. For a multi-variable smooth function $V(x, \xi)$, when clear we use $\frac{\partial V}{\partial \xi}(a, b)$ to denote $\frac{\partial V(x, \xi)}{\partial \xi}|_{x=a, \xi=b}$.

II. IMMERSION AND INVARIANCE METHOD

The Immersion and Invariance (I&I) technique is a constructive methodology used to design nonlinear and adaptive control, and state observers for dynamical systems [5]. In the following proposition we summarize the I&I methodology for orbital stabilization.

**Proposition 1:** [12] Consider the system

$$\dot{x} = f(x) + g(x)u,$$  (1)

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ ($m < n$), and $g(x)$ full rank. Assume we can find mappings

$$\alpha : \mathbb{R}^p \to \mathbb{R}^p, \quad \pi : \mathbb{R}^p \to \mathbb{R}^n, \quad \phi : \mathbb{R}^n \to \mathbb{R}^{n-p},$$

with $p < n$, such that the following assumptions hold.

**A1** (Target systems) The dynamical system

$$\dot{\xi} = \alpha(\xi)$$

with state $\xi \in \mathbb{R}^p$, has non-trivial, periodic solutions $\xi_*(t) = \xi_*(t + T)$, $\forall t \geq 0$ for some $T > 0$, which are parameterized by the initial conditions $\xi(0)$.

**A2** (Immersion condition) For all $\xi \in \mathbb{R}^p$,

$$g^+ (\pi(\xi)) \varpi = 0$$

with

$$\varpi(\xi) := f(\pi(\xi)) - \nabla \pi^T(\xi) \alpha(\xi),$$

where $g^+ : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is a full-rank left-annihilator of $g(x)$.

**A3** (Implicit manifold) The following set identity holds

$$M := \{x \in \mathbb{R}^n \mid \phi(x) = 0\} = \{x \in \mathbb{R}^n \mid x = \pi(\xi), \xi \in \mathbb{R}^p\}.$$  (6)

**A4** (Attractivity and boundedness) All the trajectories of the system

$$\dot{z} = \nabla \phi^T(x)[f(x) + g(x)v(x, z)]$$

$$\dot{x} = f(x) + g(x)v(x, z)$$

with the initial condition $z(0) = \phi(x(0))$ and the constraint

$$v(\pi(\xi), 0) = c(\pi(\xi)), $$

where

$$c(\pi(\xi)) := -[g^T(\pi(\xi))g(\pi(\xi))]^{-1} g^T(\pi(\xi))\varpi(\xi),$$

are bounded and satisfy

$$\lim_{t \to \infty} z(t) = 0.$$  (10)

Then the system

$$\dot{x} = f(x) + g(x)v(x, \phi(x))$$  (11)

ensures that the periodic solution $x_*(t) = \pi(\xi_*(t))$ is orbitally attractive.

III. MECHANICAL SYSTEMS AND PROBLEM FORMULATION

In this note we consider underactuated mechanical systems with 2-DOF, the dynamics of which is described by the EL equations of motion

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla V(q) = G\tau,$$  (12)

where $q \in \mathbb{R}^2$ are the configuration variables, $\tau \in \mathbb{R}^1$ is the control signal, $M(q) > 0$ is the generalized inertia matrix, $C(q, \dot{q})\dot{q}$ represents the Coriolis and centrifugal forces, $V(q)$ is the systems potential energy and, without loss of generality, we assume the input matrix is of the form

$$G = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$  (13)

To simplify the notation, we define $q = \text{col}(q_u, q_\theta)$, with $q_u$ as the actuated coordinate and $q_\theta$ the unactuated one. In addition, we assume that the inertia matrix depends only on the unactuated variable $q_\theta$ and has the form

$$M(q) = \begin{bmatrix} m_{uu}(q_u) & m_{u\theta}(q_u) \\ m_{u\theta}(q_u) & m_{\theta\theta}(q_u) \end{bmatrix},$$  (14)

in which $m_{uu}$, $m_{u\theta}$, $m_{\theta\theta}$ are positive functions. Thus, there exists a partial linearizing control $u^\text{pl} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, such that the system (12) in closed loop with the static state–feedback control law $\tau = u^\text{pl}(q, \dot{q}) + u$ takes the form

$$\ddot{q}_u = u - m_{uu}(q_u)\dot{q}_u + c_u(q_u, \dot{q}_u, \dot{q}_\theta)\dot{q}_\theta + \nabla V_u(q_u, \dot{q}_u) = -m_{uu}(q_u)u.$$  (15)

At this point, it is important to highlight the properties of the system (15):

i) There is not a Lagrangian function, such that the system satisfies the EL equations.

ii) For 2 DOF’s, the term $m_{uu}$ does not verify the following property

$$m_{uu}(q_u) = 2c_u(q_u, \dot{q}_u),$$

which is equivalent to the skew-symmetric matrix for mechanical systems with n-DOF’s.

iii) There is not an energy function verifying the passivity of the dynamical model (15).

iv) The potential energy is not separable, i.e., there are not functions $V_u \in \mathbb{R}$ and $V_\theta \in \mathbb{R}$, so that $V_u(q_u, \dot{q}_u) = V_x(q_x) + V_\theta(q_\theta)$. See the model of the Pendubot system in Section [7].

Since we are addressing underactuated systems with 2 DOF’s, the dynamical model (15) can be written in the form of (1) with $x = \text{col}(q_u, q_\theta, q_\theta)$ $\in \mathbb{R}^4$ and

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ \frac{1}{m_{uu}(x_1)}(c_u(x_1)x_3 + \nabla x_1 V_u(x_1, x_2)) \\ 0 \end{bmatrix},$$  (16)

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{m_{uu}(x_1)}{m_{uu}(x_1)} \end{bmatrix},$$  (17)

We adopt this assumption to simplify the presentation. Indeed, the proposed approach can be extended straightforwardly to underactuated mechanical systems with arbitrary DOF’s and codimension one.

2 See [18] for the explicit expression of $u^\text{pl}(q, \dot{q})$. 

where to write \[16\] we define the Coriolis forces as \(c_u(x_1, x_3)x_4 = \bar{c}_u(x_1)x_2^2\) with \(\bar{c}_u \in \mathbb{R}\), which always exists for a given \(M(q)\) as the form \[14\].

In this paper, we are interested in the following orbital stabilization problem.

**Problem Formulation.** Find a mapping \(u : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\) such that, in closed-loop with \[15\] and \(f(x), g(x)\) given by \[16\] and \[17\] respectively, the system has a periodic solution \(X : \mathbb{R}_+ \to \mathbb{R}^2\) that is orbitally attractive. That is,

\[
\begin{align*}
\dot{X}(t) &= F(X(t)) \\
X(t) &= X(t + T), \quad \forall t \geq 0,
\end{align*}
\]

and the set defined by its associated closed orbit

\[
\{x \in \mathbb{R}^4 \mid x = X(t), \quad 0 \leq t \leq T\},
\]

is attractive \[10, Definition 8.2\].

### IV. MAIN RESULT

Before presenting the main result, the following assumptions are imposed.

**A5.** The coordinate \(x_1\) lives on the unit circle such that \(m_{uu}(x_1), m_{uu}(x_1)\) and \(c_u(x_1)\) are bounded signals.

**A6.** The mapping \(s(x_1) \neq 0\) is chosen such that the following conditions hold.

(a) For an interval of \(x_1\) the mapping \(K(x_1)\) is smooth and bounded. In addition, it verifies

\[
K'(x_1) = \frac{s(x_1) - m_{uu}(x_1)}{m_{uu}(x_1)}
\]

or equivalently

\[
m_{uu}(x_1) + m_{uu}(x_1)K'(x_1) = s(x_1).
\]

(b) The function \(U(x_1)\) globally\[4\] has an isolated minimum point, around which is of interest to generate periodic oscillations, and it is given by

\[
U(x_1) = -\int_0^{x_1} \rho(s)m(s)ds
\]

with

\[
\rho(x_1) = -\frac{\nabla U(x_1, K(x_1))}{s(x_1)}.
\]

(c) The function

\[
m(x_1) = \exp(-2\int_0^{x_1} \beta(s)ds)
\]

is positive on an interval of \(x_1\) with function \(\beta(x_1)\) given by

\[
\beta(x_1) = -\frac{\bar{c}_u(x_1)K'(x_1)x_2^2 + m_{uu}(x_1)K''(x_1)}{s(x_1)}.
\]

Now, we are in position to describe the main result of the note.

**Proposition 2:** Consider the system \[11\] with \(f(x)\) and \(g(x)\) given by \[16\] and \[17\] respectively, with the controller

\[
u(x) = -\frac{K'(x_1)x_2^2\bar{c}_u(x_1) - m_{uu}(x_1)K''(x_1)x_3^2}{s(x_1)} \\
m_{uu}(x_1)\phi_1(x_1) + \phi_2(x_1) + \nabla U(x_1, x_2)K'(x_1)
\]

\[
\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} := \begin{pmatrix} x_2 - K(x_1) \\ x_4 - K'(x_1)x_3 \end{pmatrix}
\]

and the mapping \(s(x_1), K(x_1)\) verifying \(A6(a)\). Then, the controller solves the problem of orbital stabilization with non-trivial orbits.

**Proof:** We verify the assumptions \(A1-A4\) making use of the target system

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \rho(\xi_1) + \beta(\xi_1)\xi_2^2,
\end{align*}
\]

with \(\rho(\xi_1)\) and \(\beta(\xi_1)\) given by \[21\] and \[23\], respectively, and the mapping

\[
\pi(\xi) = \begin{pmatrix} \xi_1 \\ K(\xi_1) \\ \beta(\xi_1)\xi_2 \\ \xi_2 \end{pmatrix}.
\]

First, we need to verify \(A1\), i.e., the target dynamics \[26\] has periodic solutions. To the end, we construct a function

\[
H(\xi) := \frac{1}{2}m(\xi_1)\xi_2^2 + U(\xi_1)
\]

with \(U\) defined in \[20\] and \(m\) defined in \[22\], then yielding

\[
\frac{\partial H}{\partial \xi_1} = \frac{1}{2}m(\xi_1)\beta(\xi_1)\xi_2^2 + \frac{\partial U}{\partial \xi_1} = -m_1(\xi_1)(\beta(\xi_1)\xi_2^2 + \rho(\xi_1))
\]

\[
\frac{\partial H}{\partial \xi_2} = m(\xi_1)\xi_2.
\]

Hence, the target dynamics \[26\] can be written in a Hamiltonian form

\[
\dot{\xi} = J(\xi)\nabla H(\xi), \quad J := \begin{pmatrix} 0 & -\frac{1}{m(\xi_1)} \\ -1/m(\xi_1) & 0 \end{pmatrix}
\]

in which \(H\) is the total energy, and \(U(\xi_1)\) may be referred as the potential energy. Since the time derivative of \(H\) along the trajectories of \[29\] verifies \(\dot{H} = 0\), thus making it a undamped (conservative) system. For a given initial condition \(\xi(0)\), its trajectory is invariant in the set

\[
\Omega_{\xi(0)} := \{\xi \in \mathbb{R}^2 \mid H(\xi) = H(\xi(0))\}
\]

Now, let us characterize the topology of level sets of the function \(H(\xi)\). By solving \(\nabla H = 0\) and invoking the assumption \(A6(b)\) and the fact \(m(\xi_1) > 0\), the Hamiltonian \(H(\xi)\) has a global (isolated) minimum point at \((\xi_1^*, 0)\) for some \(\xi_1^*\). We consider the following auxiliary system

\[
\dot{\xi} = J(\xi) - R\nabla H(\xi), \quad R := rI_2
\]

for some \(r > 0\). The time derivative of \(H\) along the auxiliary system \[30\] satisfying

\[
\dot{H} = -||\nabla H(\xi)||_2^2 \leq 0.
\]

From some basic Lyapunov analysis, we may obtain that \((\xi_1^*, 0)\) is a globally asymptotically stable equilibrium for the system \[30\] and \(H(\xi)\) is qualified as the Lyapunov function. According to \[26, Theorem 1.2\], for any \(c\) satisfying

\[
H(\xi_1^*, 0) < c < \sup H(\xi),
\]

the level set \(\{\xi \in \mathbb{R}^2 \mid H(\xi) = c\}\) is diffeomorphic to \(S^1\). Now, let us come back to the target dynamics \[29\]. Since there is no equilibrium for the level set \(\{\xi \in \mathbb{R}^2 \mid H(\xi) = c\}\) with \(c > H(\xi_1^*, 0)\), all the trajectories of \[29\], excluding the isolated equilibrium, are non-trivial periodic solutions. Thus, we have verified the assumption \(A1\).
Regarding the assumption A2, a feasible (full-rank) left annihilator of (37) is
\[ G^\perp = \begin{bmatrix} m_{aa}(\xi_1) \\ m_{uu}(\xi_1) \end{bmatrix}, \] (31)
then, with mapping \( \pi \) given by (27) Assumption A2 of proposition 1 takes the form
\[ 0 = -\frac{\bar{c}_a(\xi_1)K''(\xi_1)\xi_2^2 + \nabla V_u(\xi_1,K(\xi_1)) - \rho(\xi_1) \beta(\xi_1)\xi_2}{m_{uu}(\xi_1)} \]
\[ - \frac{m_{aa}(\xi_1)K'(\xi_1)\beta(\xi_1)\xi_2}{m_{uu}(\xi_1)} + \frac{m_{aa}(\xi_1)K'(\xi_1)\beta(\xi_1)\xi_2}{m_{uu}(\xi_1)}, \] (32)
which can be written as
\[ 0 = \frac{\bar{c}_a(\xi_1)K''(\xi_1)\xi_2^2 + \beta(\xi_1) + \frac{m_{aa}(\xi_1)K'(\xi_1)\beta(\xi_1) + K''}{m_{uu}(\xi_1)}}{m_{uu}(\xi_1)} \]
\[ 0 = -\frac{\nabla V_u(\xi_1,K(\xi_1))}{m_{uu}(\xi_1)} - \frac{\bar{c}_a(\xi_1)K'(\xi_1)\beta(\xi_1)}{m_{uu}(\xi_1)}. \] (33)

We can notice that these two equations may be written as
\[ \beta(\xi_1) = -\frac{m_{aa}(\xi_1)K''(\xi_1)\xi_2^2 + \bar{c}_a(\xi_1)K'(\xi_1)\beta(\xi_1) + K''}{m_{aa}(\xi_1) + m_{aa}(\xi_1)K'(\xi_1)}, \] (34)
\[ \rho(\xi_1) = -\frac{m_{aa}(\xi_1)K''(\xi_1)\xi_2^2 + \bar{c}_a(\xi_1)K'(\xi_1)\beta(\xi_1)}{m_{aa}(\xi_1) + m_{aa}(\xi_1)K'(\xi_1)}. \] (35)

Moreover, both equations are non-singular if the denominator is different from zero, which can be satisfied if
\[ m_{aa}(\xi_1) + m_{aa}(\xi_1)K'(\xi_1) = \bar{s}(\xi_1) \] (36)
with free mapping \( \bar{s}(\xi_1) \neq 0 \), or equivalently expressed as
\[ K'(\xi_1) = \frac{\bar{s}(\xi_1) - m_{aa}(\xi_1)}{m_{aa}(\xi_1)}. \] (37)

These equations correspond to A6(a).

On the other hand, after some straightforward calculations one finds that the control \( c(\pi(\xi)) \) defined by (9) is given by
\[ c(\xi) = -\frac{K'(\xi_1)\bar{c}_a(\xi_1)(\bar{K}(\xi_1)\xi_2^2 - m_{aa}(\xi_1)\xi_2)}{\bar{s}(\xi_1)} \]
\[ - \frac{\nabla V_u(\xi_1,K(\xi_1))}{\bar{s}(\xi_1)} \] (38).

The implicit manifold description of Assumption A3 is satisfied selecting the mapping \( \phi(x) \) as (25). To complete the design we need to verify Assumption A4, using (25) we define the off-the-manifold coordinate
\[ z_1 = x_2 - K(x_1), \]
\[ z_2 = x_4 - K'(x_1)x_3, \] (39)

taking its time derivative yields
\[ \dot{z}_1 = \dot{x}_2 \]
\[ \dot{z}_2 = u \left[ m_{uu} + m_{uu}K'' \right] + \frac{K'}{m_{uu}} \left[ \bar{c}_a(x_1)^2 + \nabla V_u(x_1,x_2) \right] \]
\[ - K''x_3^2 \] (40)
in which we have used (19) to get the second identity. We choose the control \( u = v(x,z) \) as
\[ v(x,z) = \frac{-K''x_3^2\bar{c}_a(x_1) - K''\nabla V_u + m_{uu}K''x_3^2}{\bar{s}(x_1)} \]
\[ - \frac{m_{uu}(\gamma_1z_1 + \gamma_2z_2)}{\bar{s}(x_1)} \] (41)
which, considering (38), verifies the constraint (8). Furthermore, from the fact that \( z = \phi(x) \), the controller (41) is equivalent to (24).

Now, the closed loop dynamics is
\[ \dot{z}_1 = \dot{x}_2 \]
\[ \dot{z}_2 = -\gamma_1z_1 - \gamma_2z_2 \]
\[ \dot{z}_3 = x_3 \]
\[ \dot{z}_4 = x_4 \]
\[ \dot{z}_3 = \beta(x_1)x_2^2 - \nabla x_1 V_u(x_1,K(x_1) + z_1) \]
\[ - \frac{\bar{c}_a(x_1)}{\bar{s}(x_1)} [z_2 + 2K'(x_1)x_3] z_2 \]
\[ + \frac{m_{uu}(x_1)}{\bar{s}(x_1)} (\gamma_1z_1 + \gamma_2z_2) \]
\[ \dot{x}_4 = v(x,z). \] (42)

The first two equations ensures that \( z(t) \to 0 \) exponentially fast. Moreover, from A5 and using the first equality of (39) the variable \( x_2 \) remains bounded.

In the above, we have verified the exponential convergence of \( |z(t)| \) to zero, as well as the boundedness of \( (x_1,x_2) \). The remainder is to show the boundedness of the other system states.

To prove the boundedness of \( x_3 \) we make use of A6 and the subsystem \( x_1 \) and \( x_3 \) of (42). Hence, from A6(a) and A6(c) there exist constants \( s_{\min}, s_{\max}, m_{\min} \) and \( m_{\max} \) verifying the following inequalities
\[ 0 < s_{\min} \leq s(\xi) \leq s_{\max} \]
\[ 0 < m_{\min} \leq m(\xi) \leq m_{\max} \]
\[ \forall \xi \in (-x_1^+, x_1^+) \] (43)
with \( x_1^+ \) a positive number.

Now, computing the time derivative of the energy function \( \mathcal{H}_x(x_1,x_3) := \mathcal{H}(\xi_1,\xi_2)|_{\xi_1=x_1,\xi_2=x_3} \) given by (29), along the dynamics \( x_1 \) and \( x_3 \) defined in (42) we obtain
\[ \dot{\mathcal{H}}_x = m(x_1)\dot{x}_3 - \frac{\beta}{x_1}(x_1) m(x_1) + \frac{\nabla x_1 V_u(x_1,K(x_1))}{\bar{s}(x_1)} m(x_1) \]
\[ = \nabla x_1 V_u(x_1,K(x_1)) - \nabla x_1 V_u(x_1,K(x_1) + z_1) m(x_1)x_3 \]
\[ - \frac{\bar{c}_a(x_1)}{\bar{s}(x_1)} [z_2 + 2K'(x_1)x_3] z_2 m(x_1)x_3 \]
\[ + \frac{m(x_1)}{\bar{s}(x_1)} x_3 m_{uu}(x_1)(\gamma_1z_1 + \gamma_2z_2). \] (44)

Regarding the first term, it yields
\[ \left| \nabla x_1 V_u(x_1,K(x_1)) - \nabla x_1 V_u(x_1,K(x_1) + z_1) \right| \]
\[ = \int_{x_1}^{x_1 + z_1} \left| \partial^2 V_u(x_1,K(x_1) + z_1) / \partial Kx_1 \right| \]
\[ \leq \int_{x_1}^{x_1 + z_1} \left| \partial^2 V_u(x_1,K(x_1) + z_1) / \partial Kx_1 \right| ds \cdot |z_1|. \]
Due to \( x_1 \in S, K(x_1) + s z_1 \in L_\infty \) for \( s \in [0,1] \) and the continuity of \( \partial^2 V_u / \partial Kx_1 \), we have that
\[ \left| \partial^2 V_u(x_1,K(x_1) + z_1) / \partial Kx_1 \right| \]
is uniformly bounded. Since \( z \) is exponentially convergent to zero, the term
\[ \nabla x_1 V_u(x_1,K(x_1)) - \nabla x_1 V_u(x_1,K(x_1) + z_1) \]
in \([45]\) also converges to zero exponentially fast. On the other hand, the terms \(s(x_1), \, \dot{c}_u(x_1), \, K'(x_1) \) and \(m(x_1) \) are all bounded due to the continuity of these functions and \(x_1 \in S\). Hence, the time derivative of \(H_x\) given by \([44]\) takes the form
\[
H_x = \epsilon_1(t)x_3^2 + \epsilon_2(t)x_3^2
\]
with two exponentially convergent terms \(\epsilon_1\) and \(\epsilon_2\) defined as
\[
\epsilon_1 := -\frac{\dot{c}_u(x_1)K'(x_1)}{s(x_1)^2} + 2m(x_1)\frac{\dot{x}_3}{s(x_1)} + \frac{m(x_1)}{s(x_1)}m_{uu}(x_1)(\gamma_1 z_1 + \gamma_2 z_2).
\]

Thus, there exist \(k_1, k_2, a_1\) and \(a_2 > 0\) such that
\[
|\epsilon_1(t)| \leq a_1 e^{-k_1 t}, \quad |\epsilon_2(t)| \leq a_2 e^{-k_2 t}.
\]

Invoking the definition of \(H_x\) and \(H_x \geq [\mathbb{R}\rangle\) as well as the equality \([45]\), we have
\[
H_x \leq |\epsilon_1|H_x + |\epsilon_2|\sqrt{H_x} \leq a_1 e^{-k_1 t}H_x + a_2 e^{-k_2 t}\sqrt{H_x}.
\]

Now, we construct the auxiliary system
\[
r' = a_1 e^{-k_1 t}r + a_2 e^{-k_2 t}\sqrt{r},
\]
with initial condition \(r(0) > 0\). From the comparison lemma \([10]\), if the system state \(r\) in \([46]\) is essentially bounded, then the Hamiltonian function \(H_x(x)\) would be bounded as \(t \to \infty\). Solving the ordinary differential equation \([46]\), we obtain the solution
\[
r(t) = \left(\int_0^t a_2 e^{\frac{1}{2k_1} e^{-k_1 s}} \frac{1}{a_1} \frac{1}{k_1} e^{-k_1 s} ds + C_1\right)^2
\]
with some constant \(C_1\), for which the following hold
\[
\lim_{t \to \infty} \exp\left(\frac{1}{2k_1} e^{-k_1 t}\right) = 1
\]
\[
\limsup_{t \to \infty} \int_0^t a_2 e^{\frac{1}{2k_1} e^{-k_1 s}} e^{-k_2 s} ds < \infty.
\]

Indeed, the second limit follows by
\[
\exp\left(\frac{1}{2k_1} e^{-k_1 s}\right) \leq \exp\left(\frac{1}{2k_1} e^{-k_1 t}\right) = a_3, \quad \forall s \geq 0
\]
due to \(k_1 > 0\) and
\[
\left|\int_0^t a_2 e^{\frac{1}{2k_1} e^{-k_1 s}} e^{-k_2 s} ds\right| \leq \int_0^\infty a_2 a_3 e^{-k_2 s} ds
\]
\[
\leq a_2 a_3 \int_0^\infty e^{-k_2 s} ds < \infty,
\]
due to \(k_2 > 0\). Therefore, we have that \(r \in L_\infty\). As a consequence, \(H_x\) and \(x_3\) are also bounded. On the other hand, since the variable \(z_2 = x_4 - K'(x_1) x_3\) is bounded, the state \(x_4\) is also bounded. We complete the proof. ■

Remark 1: In the assumption \(A6(b)\), we assume that there is a single isolated minimum point of the function \(U\), in order to simplify the analysis. However, it can be extended, but for the case with the multiple isolated minima straightforwardly, because the topological property of Lyapunov function also holds true for local asymptotic stability in the domain of attraction, see [26, Section 2] for examples. For this case with a given \(c\), the level set \(H(x)\) contains several disconnected closed orbits, and we still achieve orbital stabilization with the proposed controller. The orbit, which the trajectory ultimately converges to, depends on the initial condition\(x(0)\).

Remark 2: Proposition \(A6(c)\) does not claim that \(x(t)\) converges to a particular periodic orbit \(\pi(x)\), but only to one of all possible periodic orbits of the target dynamics, whose existence has been assured in the proof of \(A1\).

V. ILLUSTRATIVE EXAMPLES

In this section we validate our approach through simulations of two well-known models: The Furuta pendulum and the Pendubot system.

A. Furuta pendulum

The Furuta pendulum is a 2-DOF underactuated system and it is described by an arm shaft (corresponding to the angle \(q_a\) referred to the \(z\)-axis) that is subject to an input control and a pendulum shaft (angle \(q_u\) referred to the \(z\)-axis) which is not actuated (see Fig. 1). The inertia matrix of the system is given by

\[
M(q_a) = \begin{bmatrix}
1 & a_1 \cos(q_u) & a_2 + \sin^2(q_u)
\end{bmatrix},
\]

with Coriolis matrix

\[
C(q_u, q_a) = \begin{bmatrix}
0 & -\dot{q}_u \sin(q_u) \cos(q_u) \\
\dot{q}_u \sin(q_u) \cos(q_u) & \dot{q}_u \sin(q_u) \cos(q_u)
\end{bmatrix},
\]

potential energy \(V(q_u) = a_3 \cos(q_u)\) and constants \(a_1 = \frac{ml^2}{J_a + m^2}, \quad a_2 = \frac{J_a + m^2}{J_a}, \quad a_3 = mgl\), where \(m\) is the mass of the pendulum, \(r\) is the length of the pendulum, \(2l\) is the length of the pendulum, \(J\) is the moment of inertia of the pendulum and \(J_a\) is the moment of inertia if the arm and the motor. For further details of the model and the parameters, see [1], in particular, derivations of Section 2.1 therein.

![Fig. 1: Furuta’s pendulum](image-url)
Consequently, the functions $\rho(x_1)$ and $\beta(x_1)$ given by (21) and (23) respectively, take the form

$$\rho(x_1) = -\frac{a_3}{J} \sin(x_1)$$

(47)

$$\beta(x_1) = -\kappa_1 \tan(x_1)$$

(48)

with $\kappa_1 = (1+k_1)(1+k_1+a_1^2)/a_1^2$. Hence, using these functions and after some straightforward calculations, the functions (20) and (22) have the form

$$U(x_1) = \frac{a_3}{Jk_2} \left( \frac{1}{\cos(x_1)c_2} - 1 \right)$$

(49)

$$m(x_1) = \cos(x_1)^{-\kappa_1}$$

(50)

with $k_2 = 2^{4}k_1 + 2a_1^2 + 2k_1^2 + a_1^2$.

Since the desired objective is to oscillate the pendulum in the upper half plane we impose, $k_1 > 0$, so that (49) is positive for $x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and has a minimum at zero with $x_1 = 0$, that corresponds to the upright equilibrium of the pendulum.

Now, we present several simulations to validate the performance of the proposed controller. We use the parameters of (1) which are $m = 0.0679Kg$, $l = 0.14m$, $r = 0.235$ and $J = 0.0012Kgm^2$.

The simulations were carried out to assess the impact on the transient performance of the proposed controller of the free gain $k_1$ and the tuning gains $\gamma_1, \gamma_2$ of the $z$-dynamics. We use $x(0) = \begin{bmatrix} \frac{\pi}{3} \text{ rad}, 0.6 \text{ rad}, 0 \end{bmatrix}$ as initial condition for these scenarios. In addition, we show the effects produced by different initial conditions.

In Figs. 2a and 2b we appreciate that the oscillation amplitude in both links can be modified with respect to $\gamma_1$ and $\gamma_2$ (and $k_1 = 5$). In particular, we have that their amplitudes and velocities decreasing when $\gamma_1$ increases (and $\gamma_2$ decreases). In all cases the oscillation of the unactuated link is around of the upright position. On the other hand, from Fig. 2c we notice that preserving $\gamma_1, \gamma_2 = 5$ and increasing $k_1$, the unactuated link oscillates with the same amplitude—whose value is given by the initial condition—while its velocity decreases. Regarding the actuated link, both—oscillation amplitude and velocities—are modified. These behaviors are shown in Fig. 2d.

Finally, for different initial conditions, $\gamma_1, \gamma_2$ as above and $k_1 = 9$, the oscillation amplitude of the unactuated link is also determined by the initial condition, while for the actuated one is not. It has sense that the velocities increase as initial positions increase because a greater torque must be applied to generate the oscillations of the unactuated link on the upright position, see Figs. 2e and 2f.

B. Pendubot system

The Pendubot is an underactuated mechanical system consisting of two links which can move freely on a vertical plane through a pair of revolute joints, where the first one has an actuator that applies a torque on it, while the second joint is passive. A schematic picture of the Pendubot is shown in Fig. 3, where $q_1$ and $q_2$ are the actuated and unactuated coordinates, respectively.

The inertia and Coriolis matrices and potential energy are describing as follows

$$M(q_a) = \begin{bmatrix} \frac{c_2}{c_2 + c_3 \cos(q_a)} & c_2 + c_3 \cos(q_a) \\ c_2 + c_3 \cos(q_a) & \frac{c_1 + c_2 + 2c_3 \cos(q_a)}{c_2 + c_3 \cos(q_a)} \end{bmatrix}$$

$$C(q, \dot{q}) = c_3 \sin(q_a) \begin{bmatrix} \dot{q}_a & -\dot{q}_a - \dot{q}_a \end{bmatrix}$$

and

$$V(q) = -c_4g \cos(q_a) - c_5g \cos(q_a + q_b).$$

An animation of the system behaviors may be found at youtu.be/DPHtccgYQ.

The constant parameters $c_i$ with $i = 1 : 5$ are defined as $c_1 = m_1l_1^2 + m_2l_2^2 + l_1$, $c_2 = m_2l_2^2 + l_2$, $c_3 = m_2l_1l_2$, $c_4 = m_1l_1 + m_3l_2$, and $c_5 = m_2l_1$, in which $l_j$ is the length of the links, $l_2$ is the length at the center of mass and $m_i$ are the mass of the links with $j = 1 : 2$. For additional details of the model, see [25, 26].

The control objective is to generate stable oscillations of the unactuated link starting above the horizontal. Hence, Following Proposition 2, we choose

$$s(x_1) = -k_2[c_2 + c_3 \cos(x_1)] + c_2$$

with $k_2$ a free parameter to be chosen. As consequence, we get $K = -k_2x_1, K' = -k_2$ and $K'' = 0$. Using these mappings, the functions $\rho(x_1)$ and $\beta(x_1)$ are given by

$$\rho(x_1) = -\frac{c_5g \sin((1 - k_2)x_1)}{k_2[c_2 + c_3 \cos(x_1)]} - c_2$$

(51)

$$\beta(x_1) = \frac{c_3k_2^2 \sin(x_1)}{k_2[c_2 + c_3 \cos(x_1)]} - c_2.$$  (52)

We note that the motion of the unactuated link is relative to the actuated one, thus, to ensure that the oscillations are generated on the upper plane we select $k_2$ so that the new potential energy has a minimum around of

$$q_u = \pm n\pi, \quad n > 1, \quad n \in \mathbb{N},$$

which forces to the actuated link to be in an upright position. Hence, using (51) and (52), $k_2 = -1$ and making simple calculations we have that the new potential energy (20) and function (22) take the form

$$U(x_1) = \frac{2c_5g(c_2 + c_3 \cos(x_1))}{c_2^2(4c_2^2 + 4c_2c_3 \cos(x_1) + c_3^2 \cos(x_1)^2)}$$

(54)

$$m(x_1) = \frac{1}{2c_2 + c_3 \cos(x_1))^2}.$$  (55)

To corroborate the effectiveness of our approach, simulation results are presented using the parameters given in [25], which are $l_1 = 0.2m$, $l_2 = 0.28m$, $m_1 = 0.2Kg$, $m_2 = 0.052Kg$, $l_1 = 0.13m$, $l_2 = 0.15m$, $I_1 = 3.38 \times 10^{-4}Kgm^2$ and $I_2 = 1.17 \times 10^{-3}Kgm^2$. As above mentioned, the second link is unactuated and its motion depends of the first one. Consequently, the oscillations of both links are "synchronized". Since our orbital controller with (54) and (55) ensures oscillations on the upper plane, Figs. 4a and 4b show the performance of both links under different $\gamma_1, \gamma_2$ values and using as initial conditions $x(0) = \begin{bmatrix} \frac{\pi}{3}, \frac{\pi}{3}, 0 \end{bmatrix}$. We notice that the oscillating frequency is increasing when $\gamma_1$ increases and $\gamma_2$ decreases. Now, we set $\gamma_1, \gamma_2 = (10, 5)$ and propose different initial conditions verifying $x_1(0) + x_2(0) \geq \pi$. This condition allows us to have the unactuated pendulum close to the upright position independently that the actuated link is placed in the lower or upper plane. Thus, the transitory behavior of both links are appreciated in Figs. 4c and 4d and as expected the oscillations are generated on the upper plane around of $x_1 = \pi$. This contrasts radically with respect to [9], where the oscillations for low positions for both links or in the top position for link actuated and low position for the unactuated link are distinguished by satisfying some (numerical) quantities of each case. An animation of the system behaviors is also available at the same link as the one in the first example.

$^6$Compared with the classical model representation of the Pendubot system—where the actuated link is the first one—we have rewritten the Inertia and Coriolis matrices to match them with the notation of the paper (see Eqs. (13) and (14)).
VI. CONCLUSION

We tailored in this paper the I&I orbital stabilization approach, which was recently proposed in [12], to a class of underactuated mechanical models. First, we use the widely popular partial feedback linearization [18] to simplify the sequential design; then, it is combined with an elaborated I&I controller. The full design is simple and in a compact form, and all the assumptions can be easily verified as a priori rather than being done numerically on-line. In the proof of the main result, we had an encounter with a conservative EL system perturbed by a decaying term, the boundedness of which has been studied carefully and comprehensively. At the end, we apply the approach to two benchmarks, i.e., the models of Furuta pendulum and Pendubot, showing good performance, as well as verifying all the theoretical results. Though the approach is given to the 2-DOF systems, it can be seamlessly extended to the arbitrary-DOF case with underactuation one. On the other hand, the extensions to the systems with more than one passive configurations, and to the hybrid models, are interesting topics to be explored in future, which should be practically useful in application to robotics.

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Fig. 4: Simulation results for the model of Pendubot

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