DIFFERENTIAL INEQUALITIES AND A MARTY-TYPE CRITERION FOR QUASI-NORMALITY

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Abstract

We show that the family of all holomorphic functions \( f \) in a domain \( D \) satisfying

\[
\frac{|f^{(k)}|}{1 + |f|^3}(z) \leq C \quad \text{for all } z \in D
\]

(where \( k \) is a natural number and \( C > 0 \)) is quasi-normal. Furthermore, we give a general counterexample to show that for \( \alpha > 1 \) and \( k \geq 2 \) the condition

\[
\frac{|f^{(k)}|}{1 + |f|^\alpha}(z) \leq C \quad \text{for all } z \in D
\]

does not imply quasi-normality.

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1. Introduction and statement of results

One of the key results in the theory of normal families of meromorphic functions is Marty’s theorem [11], which says that a family \( \mathcal{F} \) of meromorphic functions in a domain \( D \) in the complex plane \( \mathbb{C} \) is normal (in the sense of Montel) if and only if the family \( \{f^\#: f \in \mathcal{F}\} \) of the corresponding spherical derivatives \( f^\# := |f'|/(1 + |f|^2) \) is locally uniformly bounded in \( D \).

A substantial (and best possible) improvement of the direction ‘\( \Longleftrightarrow \)’ in Marty’s theorem is due to Hinkkanen [7]: a family of meromorphic (respectively holomorphic) functions is already normal if the corresponding spherical derivatives are bounded on the preimages of a set consisting of five (respectively three) elements. (An analogous result for normal functions was earlier proved by Lappan [8].)
In several previous papers [1–6, 10], we studied the question of how normality (or quasi-normality) can be characterized in terms of the more general quantity
\[
\frac{|f^{(k)}|}{1 + |f|^\alpha}, \quad \text{where } k \in \mathbb{N}, \alpha > 0
\]
rather than the spherical derivative \(f^\#\).

Before summarizing the main results from these studies we would like to recall the definition of quasi-normality and also to introduce some notations.

A family \(F\) of meromorphic functions in a domain \(D \subseteq \mathbb{C}\) is said to be quasi-normal if from each sequence \(\{f_n\}_n\) in \(F\) one can extract a subsequence which converges locally uniformly (with respect to the spherical metric) on \(D \setminus E\), where the set \(E\) (which may depend on \(\{f_n\}_n\)) has no accumulation point in \(D\). If the exceptional set \(E\) can always be chosen to have at most \(q\) points, yet for some sequence there actually occur \(q\) such points, then we say that \(F\) is quasi-normal of order \(q\).

We set \(\Delta(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}\) for the open disk with center \(z_0 \in \mathbb{C}\) and radius \(r > 0\). By \(\mathcal{H}(D)\) we denote the space of all holomorphic functions and by \(M(D)\) the space of all meromorphic functions in a domain \(D\). We write \(P_f\) and \(Z_f\) for the set of poles, respectively for the set of zeros, of a meromorphic function \(f\), and we use the notation \(f_n \xrightarrow{\chi} f\) (in \(D\)) to indicate that the sequence \(\{f_n\}_n\) converges to \(f\) locally uniformly in \(D\) (with respect to the spherical metric).

The Marty-type results known so far can be summarized as follows.

**Theorem A.** Let \(k\) be a natural number, \(\alpha > 0\) be a real number and \(F\) be a family of functions meromorphic in a domain \(D\). Consider the family
\[
F_{k,\alpha}^* := \left\{ \frac{|f^{(k)}|}{1 + |f|^\alpha} : f \in F \right\}.
\]
Then the following holds.

(a) [6, 10] If each \(f \in F\) has zeros only of multiplicity \(\geq k\) and if \(F_{k,\alpha}^*\) is locally uniformly bounded in \(D\), then \(F\) is normal.

(b) (Xu [16]) Assume that there are a value \(w^* \in \mathbb{C}\) and a constant \(M < \infty\) such that for each \(f \in F\) we have \(|f'(z)| + \cdots + |f^{(k-1)}(z)| \leq M\) whenever \(f(z) = w^*\) and that there exists a set \(E \subset \overline{\mathbb{C}}\) consisting of \(k + 4\) elements such that for all \(f \in F\) and all \(z \in D\),
\[
f(z) \in E \implies \frac{|f^{(k)}|}{1 + |f|^{k+1}}(z) \leq M.
\]
Then \(F\) is normal.
If all functions in \(F\) are holomorphic, then this also holds if one merely assumes that \(E\) has at least three elements.

(c) [4] If \(\alpha > 1\) and if each \(f \in F\) has poles only of multiplicity \(\geq k/(\alpha - 1)\), then the normality of \(F\) implies that \(F_{k,\alpha}^*\) is locally uniformly bounded.
This does not hold in general for \(0 < \alpha \leq 1\).
Remarks.

(1) In (a) and (b) the assumption on the multiplicities of the zeros, respectively the (slightly weaker) condition on the existence of the value $w^*$, is essential. The condition $|f^{(k)}(z)|/1 + |f(z)|^\alpha \leq C$ itself does not imply normality. Indeed, each polynomial of degree at most $k-1$ satisfies this condition, but those polynomials only form a quasi-normal, but not a normal family.

(2) It is worthwhile to mention two special cases of Theorem A(c).

- If $\alpha \geq k+1$ and if $F$ is normal, then the conclusion that $F^*_{k,\alpha}$ is locally uniformly bounded holds without any further assumptions on the multiplicities of the poles. This had been proved already by Li and Xie [9].

- If all functions in $F$ are holomorphic, then for any $\alpha > 1$ the normality of $F$ implies that $F^*_{k,\alpha}$ is locally uniformly bounded [6, Theorem 1(c)].

In this paper, we further study the differential inequality $|f^{(k)}(z)|/(1 + |f(z)|^\alpha) \leq C$, but this time without any additional assumptions on the multiplicities of the zeros of the functions under consideration. It turns out that for $\alpha = 1$ (and hence trivially for $\alpha < 1$), this differential inequality implies quasi-normality, but that this does not hold for $\alpha > 1$.

**Theorem 1.1.** Let $k \geq 2$ be a natural number, $C > 0$ and $D \subseteq \mathbb{C}$ a domain. Then the family

$$F_k := \{ f \in \mathcal{H}(D) : \frac{|f^{(k)}(z)|}{1 + |f(z)|^\alpha} \leq C \}$$

is quasi-normal.

Remarks.

(1) In Theorem 1.1, we restrict to holomorphic rather than meromorphic functions, since if a meromorphic function $f$ has a pole at $z_0$, then $|f^{(k)}(z)|/(1 + |f(z)|^\alpha) \leq C$ is clearly violated in a certain neighborhood of $z_0$.

(2) The result also holds for $k = 1$, and in this case we can even conclude that $F$ is normal. However, this is just a trivial consequence of Hinkkanen’s extension of Marty’s theorem since the condition $|f'(z)|/(1 + |f(z)|) \leq C$ clearly implies that the derivatives $f'$ (and hence the spherical derivatives $f^\#$) are uniformly bounded on the preimages of five finite values.

(3) In Theorem 1.1, for $k \geq 2$, the order of quasi-normality can be arbitrarily large. This is demonstrated by the sequence of the functions

$$f_n(z) := n(e^{z} - e^{\zeta z})$$

(where $\zeta := e^{2\pi i/k}$) on the strip $D := \{ z \in \mathbb{C} : -1 < \text{Re}((1 - \zeta)z) < 1 \}$. Indeed, $f_n^{(k)} = f_n$, so the differential inequality from Theorem 1.1 trivially holds, but every subsequence of $\{ f_n \}$ is not normal exactly at the infinitely many common zeros $z_j = (2\pi i j)/(1 - \zeta) \in D$ ($j \in \mathbb{Z}$) of the $f_n$, so $\{ f_n \}$ is quasi-normal of infinite order.
In the spirit of Bloch’s heuristic principle, one might ask for a corresponding result for entire functions. However, since the exponential function (and, more generally, entire solutions of the linear differential equation \( f^{(k)} = C \cdot f \)) satisfy the condition \( |f^{(k)}(z)|/(1 + |f(z)|) \leq C \), there does not seem to be a natural analogue for entire functions.

For \( \alpha > 1 \) and \( k \geq 2 \), the condition \( |f^{(k)}(z)|/(1 + |f(z)|^\alpha) \leq C \) does not imply quasi-normality. In Section 3 we will construct a general counterexample for arbitrary \( k \geq 2 \), \( \alpha > 1 \) and \( C > 0 \). (For \( k = 2 \) and \( \alpha = 3 \), we had given such a counterexample already in [6].) In fact, it turns out that this condition does not even imply \( Q_\beta \)-normality for any ordinal number \( \beta \). (For the exact definition of \( Q_\beta \)-normality, we refer to [12].) So, there is no chance to extend Theorem 1.1 to the case \( \alpha > 1 \) even if one replaces the concept of quasi-normality by a weaker concept.

The same counterexample also shows that Theorem 1.1 cannot be extended in the spirit of the afore-mentioned results of Hinkkanen and Xu (Theorem A(b)). More precisely, a condition like

\[
f(z) \in E \implies \frac{|f^{(k)}|}{1 + |f|^\alpha}(z) \leq C,
\]

where \( E \) is any finite subset of \( \mathbb{C} \), does not imply quasi-normality (and not even \( Q_\beta \)-normality). This is due to the fact that this condition is even weaker than \( |f^{(k)}(z)|/(1 + |f(z)|^\alpha) \leq C' \) for suitable \( C' > 0 \).

One crucial step in our proof of Theorem 1.1 consists in using the fact that also the reverse inequality \( |f^{(k)}(z)|/(1 + |f(z)|) \geq C \) implies quasi-normality [5]. This is one of the main results from our studies [1, 2, 5, 10] on meromorphic functions satisfying differential inequalities of the form \( |f^{(k)}(z)|/(1 + |f(z)|^\alpha) \geq C \). These investigations were inspired by the observation that there is a counterpart to Marty’s theorem in the following sense: a family of meromorphic functions whose spherical derivatives are bounded away from zero has to be normal [3, 14]. For the sake of completeness, we summarize the main results from those studies.

**Theorem B.** Let \( k \geq 1 \) and \( j \geq 0 \) be integers and \( C > 0 \) and \( \alpha > 1 \) be real numbers. Let \( \mathcal{F} \) be a family of meromorphic functions in some domain \( D \).

(a) [2] If

\[
\frac{|f^{(k)}|}{1 + |f|^\alpha}(z) \geq C \quad \text{for all } z \in D \text{ and all } f \in \mathcal{F},
\]

then \( \mathcal{F} \) is normal.

(b) [5, 10] If

\[
\frac{|f^{(k)}|}{1 + |f|}(z) \geq C \quad \text{for all } z \in D \text{ and all } f \in \mathcal{F},
\]

then \( \mathcal{F} \) is quasi-normal, but in general not normal.
If \( k > j \) and
\[
\frac{|f^{(k)}(z)|}{1 + |f^{(j)}(z)|^\alpha} \geq C \quad \text{for all } z \in D \text{ and all } f \in \mathcal{F},
\]
then \( \mathcal{F} \) is quasi-normal in \( D \). If all functions in \( \mathcal{F} \) are holomorphic, \( \mathcal{F} \) is quasi-normal of order at most \( j - 1 \). (For \( j = 0 \) and \( j = 1 \), this means that it is normal.) This does not hold for \( \alpha = 1 \) if \( j \geq 1 \).

## 2. Proof of Theorem 1.1

We apply induction. As mentioned above, the quasi-normality (in fact, even normality) of \( \mathcal{F}_1 \) follows from Hinkkanen’s generalization of Marty’s theorem.

Let some \( k \geq 2 \) be given and assume that it is already known that (on arbitrary domains) each of the conditions
\[
|f^{(j)}(z)| \leq C, \quad \text{where } j \in \{1, \ldots, k-1\}
\]
implies quasi-normality.

Let \( \{f_n\}_n \) be a sequence in \( \mathcal{F}_k \) and \( z^* \) an arbitrary point in \( D \). Suppose to the contrary that \( \{f_n\}_n \) is not quasi-normal at \( z^* \).

**Case 1:** There are an \( m \in \{1, \ldots, k-1\} \) and a subsequence \( \{f_{n_\ell}\}_\ell \) such that both \( \{f^{(m)}_{n_\ell}\}_\ell \) and \( \{f^{(m)}_{n_\ell}/f_{n_\ell}\}_\ell \) are normal at \( z^* \).

Then (after turning to an appropriate subsequence, which we again denote by \( \{f_n\}_n \) rather than \( \{f_{n_\ell}\}_\ell \)), without loss of generality we may assume that in a certain disk \( \Delta(z^*, r) =: U \) both sequences \( \{f^{(m)}_n\}_n \) and \( \{f^{(m)}_n/f_n\}_n \) converge uniformly (with respect to the spherical metric) to limit functions \( H \in \mathcal{H}(U) \cup \{\infty\} \) and \( L \in \mathcal{M}(U) \cup \{\infty\} \), respectively.

**Case 1.1:** \( H \) is holomorphic.

For each \( n \) we choose \( p_n \) to be the \((m-1)\)th Taylor polynomial of \( f_n \) at \( z^* \), that is, \( p_n \) has degree at most \( m-1 \) and satisfies \( p^{(j)}_n(z^*) = f^{(j)}_n(z^*) \) for \( j = 0, \ldots, m-1 \). Then \( f_n \) has the representation
\[
f_n(z) = p_n(z) + \int_{z^*}^z \int_{z^*}^{\xi_1} \cdots \int_{z^*}^{\xi_{m-1}} f^{(m)}_n(\xi_m) d\xi_m \cdots d\xi_1.
\]

Here for \( n \to \infty \)
\[
\int_{z^*}^{\xi} \int_{z^*}^{\xi_1} \cdots \int_{z^*}^{\xi_{m-1}} f^{(m)}_n(\xi_m) d\xi_m \cdots d\xi_1 \Rightarrow \int_{z^*}^{\xi} \int_{z^*}^{\xi_1} \cdots \int_{z^*}^{\xi_{m-1}} H(\xi_m) d\xi_m \cdots d\xi_1 =: F(z),
\]
where \( F \) is holomorphic in \( U \). Since the family of polynomials of degree at most \( m-1 \) is quasi-normal (cf. [13, Theorem A.5]), we obtain the quasi-normality of \( \{f_n\}_n \) at \( z^* \).
Case 1.2: $L(z^*) \neq \infty$.

We choose $r_0 \in (0; r)$ such that $|L(z)| \leq |L(z^*)| + 1$ for all $z \in \Delta(z^*, r_0) =: U_0$.

Then for all $z \in U_0$ and all $n$ large enough,

$$\frac{|f_n^{(m)}|}{1 + |f_n|}(z) \leq \frac{|f_n^{(m)}|}{|f_n|}(z) \leq |L(z)| + 1 \leq |L(z^*)| + 2,$$

so by the induction hypothesis we obtain the quasi-normality of $\{f_n\}_n$ at $z^*$.

Case 1.3: $H \equiv \infty$ and $L(z^*) = \infty$. (This comprises the cases that $L \equiv \infty$ and that $L$ is meromorphic with a pole at $z^*$.)

We choose $r_0 \in (0; r)$ such that $|L(z)| \geq 3$ for all $z \in \Delta(z^*, r_0) =: U_0$. Then for sufficiently large $n$, say for $n \geq n_0$, and all $z \in U_0$,

$$\left|\frac{f_n^{(m)}}{f_n}(z)\right| \geq |L(z)| - 1 \geq 2 \quad \text{and} \quad |f_n^{(m)}(z)| \geq 2.$$

Now fix an $n \geq n_0$ and a $z \in U_0$. If $|f_n(z)| \leq 1$,

$$\frac{|f_n^{(m)}|}{1 + |f_n|}(z) \geq \frac{|f_n^{(m)}|}{2}(z) \geq 1.$$

If $|f_n(z)| \geq 1$,

$$\frac{|f_n^{(m)}|}{1 + |f_n|}(z) \geq \frac{|f_n^{(m)}|}{2|f_n|}(z) \geq 1.$$

Combining both cases, we conclude that

$$\left|\frac{f_n^{(m)}}{f_n}(z)\right| \geq 1 \quad \text{for all} \quad z \in U_0 \quad \text{and} \quad n \geq n_0,$$

so by Theorem B(b) we obtain the quasi-normality of $\{f_n\}_n$ at $z^*$.

Case 2: For each $j = 1, \ldots, k - 1$ and each subsequence $\{f_{n_\ell}\}_\ell$, at least one of the sequences $\{f_n^{(j)}\}_\ell$ and $(f_n^{(j)})/(f_n)$ is not normal at $z^*$.

Then, after turning to an appropriate subsequence, which we again denote by $\{f_n\}_n$, by Montel’s theorem for all $j = 1, \ldots, k - 1$ we find sequences $\{w_{j,n}\}_n$ such that $\lim_{n \to \infty} w_{j,n} = z^*$ and such that for each $n$ we have $|f_n^{(j)}(w_{j,n})| \leq 1$ or $(f_n^{(j)})/(f_n)(w_{j,n}) \leq 1$. Both cases can be unified by writing

$$|f_n^{(j)}(w_{j,n})| \leq 1 + |f_n(w_{j,n})| \quad \text{for all} \quad j = 1, \ldots, k - 1 \quad \text{and} \quad n. \quad (2.1)$$

Furthermore, since $\{f_n\}_n$ is not quasi-normal and hence not normal at $z^*$, we may also assume that there is a sequence $\{w_{0,n}\}_n$ such that $\lim_{n \to \infty} w_{0,n} = z^*$ and $|f_n(w_{0,n})| \leq 1$ for all $n$.

We choose $r > 0$ sufficiently small such that $\overline{\Delta(z^*, r)} \subseteq D$, $2r < 1$ and $(4r(1 + C))/(1 - 2r) \leq 1$. Then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $j = 0, \ldots, k - 1$ we have $w_{j,n} \in \Delta(z^*, r)$. 

\[ \text{Differential inequalities and a Marty-type criterion for quasi-normality} \]
We use the notation
\[ M(r, f) := \max_{|z - z^*| \leq r} |f(z)| \quad \text{for} \quad f \in \mathcal{H}(\Delta(z^*, r)) \]
and obtain for all \( n \geq n_0 \), all \( j = 1, \ldots, k - 1 \) and all \( z \in \Delta(z^*, r) \),
\[
|f^{(j)}_n(z)| = |f^{(j)}_n(w_{jn}) + \int_{[w_{jn}; z]} f^{(j+1)}_n(\zeta) \, d\zeta|
\leq |f^{(j)}_n(w_{jn})| + |z - w_{jn}| \cdot \max_{\zeta \in [w_{jn}; z]} |f^{(j+1)}_n(\zeta)|
\leq 1 + |f_n(w_{jn})| + 2r \cdot M(r, f^{(j+1)}_n),
\]
where for the last estimate we have applied (2.1).

Since this holds for any \( z \in \Delta(z^*, r) \), we conclude that for all \( n \geq n_0 \) and all \( j = 1, \ldots, k - 1 \),
\[ M(r, f_n^{(j)}) \leq 1 + M(r, f_n) + 2r \cdot M(r, f^{(j+1)}_n). \]
Similarly, in view of \( |f_n(w_{0,n})| \leq 1 \), we also have
\[ M(r, f_n) \leq 1 + 2r \cdot M(r, f'_n). \]
Induction yields
\[
M(r, f_n) \leq 1 + \sum_{j=1}^{k-1} (2r)^j \cdot (1 + M(r, f_n)) + (2r)^k \cdot M(r, f^{(k)}_n)
\leq \sum_{j=0}^{k-1} (2r)^j + \sum_{j=1}^{k-1} (2r)^j \cdot M(r, f_n) + (2r)^k \cdot C \cdot (1 + M(r, f_n))
\leq C + \frac{1}{1 - 2r} + \frac{2r \cdot (1 + C)}{1 - 2r} \cdot M(r, f_n)
\leq C + \frac{1}{1 - 2r} + \frac{1}{2} \cdot M(r, f_n).
\]
Hence,
\[ M(r, f_n) \leq 2C + \frac{2}{1 - 2r} \]
for all \( n \geq n_0 \). Thus, \( \{f_n\}_{n \geq n_0} \) is uniformly bounded in \( \Delta(z^*, r) \) and hence normal at \( z^* \) by Montel’s theorem.

This completes the proof of Theorem 1.1. \( \square \)

3. A general counterexample

In this section, we will show that for \( \alpha > 1 \) and \( k \geq 2 \) the differential inequality
\[ |f^{(k)}(z)|/(1 + |f(z)|^\alpha) \leq C \]
does not imply quasi-normality. In [6], we had already given
a counterexample for the case \( k = 2 \) and \( \alpha = 3 \). We generalize this example to arbitrary \( k \geq 2, \alpha > 1 \) and \( C > 0 \).

For given \( k_0 \geq 2, C > 0 \) and \( \alpha > 1 \), we construct a sequence \( \{f_n\}_n \) of holomorphic functions in \( D := \Delta(0; 2) \) such that \( |f_n^{(k_0)}(z)|/(1 + |f_n(z)|^\alpha) \leq C \) for all \( z \in D \) and all \( n \), but \( \{f_n\}_n \) is not quasi-normal in \( D \).

First, take \( p, q \in \mathbb{N} \) such that \( 1 < (p/q) < \min\{\alpha; 2\} \). The real function \( h(x) := (1 + x^{p/q})/(1 + x^\alpha) \) is continuous in \([0, \infty)\) with \( \lim_{x \to \infty} h(x) = 0 \) and hence there exists an \( M > 0 \) such that

\[
\frac{1 + x^{p/q}}{1 + x^\alpha} \leq M \quad \text{for all } x \geq 0. \tag{3.1}
\]

Let \( g_n(z) := z^n - 1 \) for \( n \geq 1 \). The zeros of \( g_n \) are the \( n \)th roots of unity \( z_{\ell}^{(n)} = e^{2\pi i \ell/n} \) \( (\ell = 0, 1, \ldots, n - 1) \), and they are all simple, \( g_n'(z_{\ell}^{(n)}) \neq 0 \). We consider the functions

\[
h_n := g_n \cdot e^{p_n},
\]

where the \( p_n \) are polynomials yet to be determined. Then

\[
h_n' = e^{p_n}(g_n' + g_n p_n')
\]

and

\[
h_n'' = e^{p_n}(2g_n'p_n' + g_n p_n'^2 + g_n'' + g_n p_n''). \tag{3.2}
\]

Our aim is to choose the \( p_n \) in such a way that for \( \ell = 0, \ldots, n - 1 \),

\[
h_n''(z_{\ell}^{(n)}) = h_n^{(3)}(z_{\ell}^{(n)}) = \cdots = h_n^{(k_0+1)}(z_{\ell}^{(n)}) = 0. \tag{3.3}
\]

We first deduce several constraints on the \( p_n \) that are sufficient for (3.3), and then—by an elementary result on Hermite interpolation—we will see that it is possible to satisfy these constraints with polynomials \( p_n \) of sufficiently large degree.

First, in order to get \( h_n''(z_{\ell}^{(n)}) = 0 \), in view of (3.2), we will require that

\[
p_n'(z_{\ell}^{(n)}) = -\frac{g_n''(z_{\ell}^{(n)})}{2g_n'(z_{\ell}^{(n)})} \quad (\ell = 0, 1, \ldots, n - 1). \tag{3.4}
\]

In order to proceed we need the following lemma.

**Lemma 3.1.** For every \( k \geq 2 \),

\[
h_n^{(k)} = e^{p_n}[k g_n p_n^{(k-1)} + g_n \varphi_k(p_n', \ldots, p_n^{(k-1)}) + \psi_k(g_n', \ldots, g_n^{(k)}), p_n', \ldots, p_n^{(k-2)}) + g_n p_n^{(k)}],
\]

where \( \varphi_k \in \mathbb{C}[x_1, \ldots, x_{k-1}] \) and \( \psi_k \in \mathbb{C}[y_1, \ldots, y_k, x_1, \ldots, x_{k-2}] \) are polynomials.

**Proof.** We prove the lemma by induction on \( k \). The base case \( k = 2 \) follows from (3.2) with \( \varphi_2(x_1) = x_1^2 \) and \( \psi_2(y_1, y_2) = y_2 \). Assume that the lemma holds for some \( k \geq 2 \).
Then differentiating gives

\[
h^{(k+1)}_n = e^{p_n} \left[ k g''_n p_n^{(k+1)} + k g'_n p_n^{(k)} + g'_n \varphi_k(p'_n, \ldots, p^{(k-1)}_n) \right. \\
+ g_n \sum_{m=1}^{k-1} \frac{\partial \varphi_k}{\partial x_m}(p'_n, \ldots, p^{(k-1)}_n) \cdot p^{(m+1)}_n \\
+ \left. \sum_{m=1}^{k} \frac{\partial \psi_k}{\partial y_m}(g'_n, \ldots, g^{(k)}_n, p'_n, \ldots, p^{(k-2)}_n) \cdot p^{(m+1)}_n \right]
\]

\[
+ g_n p^{(k+1)}_n + k g'_n p'_n + g'_n p'_n \varphi_k(p'_n, \ldots, p^{(k-1)}_n) \\
+ p'_n \psi_k(g'_n, \ldots, g^{(k)}_n, p'_n, \ldots, p^{(k-2)}_n) + g_n p^{(k+1)}_n \right].
\]

= \left. e^{p_n} \cdot \right. \left[ (k + 1)g'_n p_n^{(k)} + g_n \varphi_{k+1}(p'_n, \ldots, p^{(k)}_n) \\
+ \psi_{k+1}(g'_n, \ldots, g^{(k+1)}_n, p'_n, \ldots, p^{(k-1)}_n) + g_n p^{(k+1)}_n \right].
\]

where

\[
\varphi_{k+1}(x_1, \ldots, x_k) := \sum_{m=1}^{k-1} \frac{\partial \varphi_k}{\partial x_m}(x_1, \ldots, x_{k-1}) \cdot x_{m+1} + x_1 \varphi_k(x_1, \ldots, x_{k-1}) + x_1 x_k
\]

and

\[
\psi_{k+1}(y_1, \ldots, y_{k+1}, x_1, \ldots, x_{k-1}) := ky_{k+1} + y_1 \varphi_k(x_1, \ldots, x_{k-1}) \\
+ \sum_{m=1}^{k} \frac{\partial \psi_k}{\partial y_m}(y_1, \ldots, y_k, x_1, \ldots, x_{k-2}) \cdot y_{m+1} \\
+ \sum_{m=1}^{k-2} \frac{\partial \psi_k}{\partial x_m}(y_1, \ldots, y_k, x_1, \ldots, x_{k-2}) \cdot x_{m+1} \\
+ ky_1 x_1 x_{k-1} + x_1 \psi_k(y_1, \ldots, y_k, x_1, \ldots, x_{k-2})
\]

are indeed polynomials of the requested form.

Hence, the lemma holds for \( k + 1 \) as well. \( \square \)
Now we inductively determine the required values of $p_n^{(k)}(z^{(n)}_\ell)$ for $k = 2, \ldots, k_0$ and $\ell = 0, \ldots, n - 1$. For given $k \in \{2, \ldots, k_0\}$, let us assume that we already know the values of $p_n(z^{(n)}_\ell), \ldots, p_n^{(k-1)}(z^{(n)}_\ell)$ that ensure that $h^{(k)}_n(z^{(n)}_\ell) = \cdots = h^{(k-1)}_n(z^{(n)}_\ell) = 0$ for all admissible $\ell$. (Note that the required values of $p_n^{(k)}(z^{(n)}_\ell)$ have been found in (3.4).)

In order to find the values of $p_n^{(k)}(z^{(n)}_\ell)$ (which ensure that $h^{(k+1)}_n(z^{(n)}_\ell) = 0$), we apply Lemma 3.1 with $k + 1$ in place of $k$ and obtain the condition

$$p_n^{(k)}(z^{(n)}_\ell) = -\frac{\psi_{k+1}(g_{n+1}, \ldots, g_n, \ldots, g_{n+1},(k+1)g_n'(z^{(n)}_\ell)).}{(3.5)}$$

(Observe that evaluating the right-hand side requires only the knowledge of values of $p_n', \ldots, p_n^{(k-1)}$ that have been previously determined.)

It is well known (see, for example, [15, page 52]) that for every $n \geq 1$ the conditions (3.4) and (3.5) (for $k = 2, \ldots, k_0$) can be achieved with a polynomial $p_n$ of degree at most $n k_0$.

In this way,

$$h^{(n)}_n(z^{(n)}_\ell) = \cdots = h^{(n+1)}_n(z^{(n)}_\ell) = 0.$$

In particular, each $z^{(n)}_\ell$ is a zero of $h^{(n)}_n$ of multiplicity $\geq 2$.

Now the functions $(h^{(n)}_n)^q/h^p_n$ are entire: $h^p_n$ is entire and its zeros $z^{(n)}_\ell$ ($\ell = 0, 1, \ldots, n - 1$) have multiplicity $p$, while $(h^{(n)}_n)^q$ has zeros at $z^{(n)}_\ell$ of multiplicity at least $2q > p$. Thus, $c_n := \max_{z \in D} \left| \left( (h^{(n)}_n)^q / (h^p_n)(z) \right) \right| < \infty$. Define now for every $n \geq 1$,

$$f_n := a_n \cdot h_n,$$

where $a_n > 0$ is a large enough constant such that both

$$a_n \geq \left( \frac{c_n \cdot M^q}{C^q} \right)^{1/(p-q)}$$

that is, $\frac{c_n}{a_n^{p-q}} \leq \left( \frac{C}{M} \right)^q$ (3.6)

and $f_n \xrightarrow{\text{X}} \infty$ on $\mathbb{C} \backslash \partial \Delta(0;1)$; the latter can be achieved by choosing

$$a_n \geq \min \left\{ |h_n(z)| : |z| \leq 1 - \frac{1}{n} \text{ or } 1 + \frac{1}{n} \leq |z| \leq n \right\}.$$

Then $f_n$ is not quasi-normal in $D$ (as it is not normal at any point of $\partial \Delta(0;1)$), yet satisfies

$$\frac{|f_n^{(n)}(z)|}{1 + |f_n(z)|^p} \leq C \quad \text{for all } z \in D.$$
where the last inequality is just (3.6). Therefore,

$$\frac{|f_{n}^{(k_{0})}|}{1 + |f_{n}|^{p/q}(z)} \leq \frac{C}{M} \quad \text{for all } z \in D,$$

and together with (3.1) we conclude that

$$\frac{|f_{n}^{(k_{0})}|}{1 + |f_{n}|^{p/q}(z)} = \frac{|f_{n}^{(k_{0})}|}{1 + |f_{n}|^{p/q}(z)} \cdot \frac{1 + |f_{n}|^{p/q}(z)}{1 + |f_{n}|^{p/q}(z)} \leq \frac{C}{M} \cdot M = C \quad \text{for all } z \in D,$$

as desired.

**Remark.** Actually, we have shown something stronger: the condition $((f^{(k)}(z))) / (1 + |f(z)|^q) \leq C$ does not even imply $Q_{\beta}$-normality for any ordinal number $\beta$ since the constructed sequence $\{f_{n}\}_{n}$ and all of its subsequences are not normal at any point of the continuum $\partial\Delta(0; 1)$.

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