The number of degrees of freedom of three-dimensional
Navier–Stokes turbulence

Chuong V. Tran
School of Mathematics and Statistics, University of St Andrews
St Andrews KY16 9SS, United Kingdom

Abstract

In Kolmogorov’s phenomenological theory of turbulence, the energy spectrum in the
inertial range scales with the wave number $k$ as $k^{-5/3}$ and extends up to a dissipation
wave number $k_{\nu}$, which is given in terms of the energy dissipation rate $\epsilon$ and viscosity $\nu$ by
$k_{\nu} \propto (\epsilon/\nu^3)^{1/4}$. This result leads to Landau’s heuristic estimate for the number of degrees of
freedom that scales as $Re^{9/4}$, where $Re$ is the Reynolds number. Here we consider the possibility of establishing a quantitative basis for these results from first principles. In particular,
we examine the extent to which they can be derived from the three-dimensional Navier–
Stokes system, making use of Kolmogorov’s hypothesis of finite and viscosity-independent
energy dissipation only. It is found that the Taylor microscale wave number $k_T$ (a close
cousin of $k_{\nu}$) can be expressed in the form $k_T \leq CU/\nu = (CU/\|u\|)^{1/2}(\epsilon/\nu^3)^{1/4}$. Here $U$
and $\|u\|$ are, respectively, a “microscale” velocity and the root mean square velocity, and
$C \leq 1$ is a dynamical parameter. This result can be seen to be in line with Kolmogorov’s
prediction for $k_{\nu}$. Furthermore, it is shown that the minimum number of greatest Lyapunov
exponents whose sum becomes negative does not exceed $Re^{9/4}$, where $Re$ is defined in terms
of an average energy dissipation rate, the system length scale, and $\nu$. This result is in a
remarkable agreement with the Landau estimate, up to a presumably slight discrepancy
between the conventional and the present energy dissipation rates used in the definition of
$Re$. 

chuong@mcs.st-and.ac.uk
1 Introduction

For the past several decades, Kolmogorov’s phenomenological theory of turbulence\(^1\) has been a starting point for a majority of theoretical, numerical, and experimental studies of fluid motion. A cornerstone of the theory is the notion of energy cascade and dissipation anomaly. This means that no matter how small the viscosity, the large-scale kinetic energy that generates and sustains the turbulence can be transferred to the correspondingly small scales for removal by viscous effects. More precisely, the energy dissipation rate has been conjectured to remain finite and nonzero and to become independent of viscosity in the limit of small viscosity. This conjecture was first extended by Obukhov\(^2\) and Corrsin\(^3\) to the case of passive scalar transport and mixing in turbulent flows, for which the dissipation of the scalar variance by molecular diffusion has been thought to be diffusivity independent in the limit of vanishingly small diffusivity. In a similar manner, Batchelor\(^4\) adapted Kolmogorov’s theory for two-dimensional turbulence, by hypothesizing that the dissipation of enstrophy (half mean square vorticity) becomes independent of viscosity (and remains nonzero) in the inviscid limit. Like the energy dissipation rate in the Kolmogorov theory, the supposedly finite dissipation rates of the scalar variance and enstrophy are key parameters in the Obukhov–Corrsin and Batchelor theories. Since then, the concept of cascade and dissipation anomaly has been thought to apply to a variety of fluid systems in other contexts. Thus, the Kolmogorov theory of turbulence evidently has become one of the most influential theories in science.

An important prediction of Kolmogorov’s theory is that the energy cascading range (known as energy inertial range) scales with the wave number \(k\) as \(k^{-5/3}\) and extends up to a dissipation wave number \(k_\nu\), which is given in terms of the viscosity-independent energy dissipation rate \(\epsilon\) and the viscosity \(\nu\) by \(k_\nu \propto (\epsilon/\nu^3)^{1/4}\). This wave number marks the end of the energy inertial range, beyond which viscous forces become significant and the dissipa-
tion of energy mainly occurs. Heuristically, if the turbulence is homogeneous and isotropic, consisting predominantly of vortices of volume \( k_\nu^{-3} \), then there are \( (Lk_\nu)^3 \propto (L^{4/3} \epsilon^{1/3}/\nu)^{9/4} \) such vortices within a domain region of size \( L \). This is the basis for the Landau\(^5\) estimate \( N \propto \text{Re}^{9/4} \) for the number of degrees of freedom \( N \), where \( \text{Re} = L^{4/3} \epsilon^{1/3}/\nu \) is the Reynolds number. A number of authors\(^6\)–\(^11\) have attempted to address these results from first principles, pending a verdict on solution regularity. The reported upper bounds for \( N \) in the literature have a wide-ranging dependence on \( \text{Re} \), depending on how \( \text{Re} \) is defined (and to some extent on how \( N \) is defined). Constantin \textit{et al.}\(^6\) (see also Foias \textit{et al.}\(^7\)) showed that the number of determining modes is proportional to \( \text{Re}^{9/4} \), where \( \epsilon \) in \( \text{Re} \) is replaced by the asymptotic average of the supremum of the energy dissipation rate. They also found that the attractor dimension scales as \( \text{Re}^3 \), where \( \text{Re} \) is defined in terms of the maximum fluid velocity. This dimension was found by Gibbon and Titi\(^9\) to scale with the presently defined \( \text{Re} \) as \( \text{Re}^{18/5} \). Gibbon\(^11\) showed that the local number of degrees of freedom scales as \( \text{Re}^3 \), where \( \text{Re} \) was defined in terms of the local fluid velocity. Among these results, the first one appears to be closest to the Landau estimate, differing from it by the definition of \( \epsilon \), not by the power of \( \text{Re} \).

In this study, we revisit the Kolmogorov prediction \( k_\nu \propto (\epsilon/\nu^3)^{1/4} \) and Landau estimate \( N \propto \text{Re}^{9/4} \), by further examining the extent to which these can be deduced from the three-dimensional Navier–Stokes system, making use of Kolmogorov’s hypothesis of finite and viscosity-independent energy dissipation only. It is found that the Taylor microscale wave number \( k_T \), which can be thought of as a dissipation wave number, satisfies \( k_T \leq CU/\nu \). Here \( C \) is a nondimensional dynamical parameter satisfying \( C \leq 1 \) and \( U \) denotes a “microscale” velocity—a time-dependent velocity scale associated with the enstrophy production term. This result is derived without an \textit{a priori} knowledge of \( \epsilon \). Upon invoking Kolmogorov’s hypothesis, we obtain \( k_T \leq (CU/\|u\|)^{1/2}(\epsilon/\nu^3)^{1/4} \), where \( \|u\| \) is the root mean square velocity, which can be seen to be in line with the prediction for \( k_\nu \).
Furthermore, it is shown that along any given bounded trajectory, the minimum number of greatest local Lyapunov exponents whose sum becomes negative is bounded from above by $\text{Re}^{9/4}$, where $\text{Re}$ is defined in terms of a new domain-average energy dissipation rate. This result means that finite-dimensional volume elements in the infinite-dimensional phase space (solution space) contract exponentially if their dimensions exceed $\text{Re}^{9/4}$. This is also an upper bound for generalized dimensions, such as the box-counting or Hausdorff dimensions, of a nontrivial attractor (in forced turbulence) if one exists. This result is consistent with the Landau estimate, up to a difference between the usual and present average energy dissipation rates used in the definition of $\text{Re}$. This difference, which is arguably slight, is the “extent” to which the present analysis needs to reach for a complete agreement with the Landau estimate. Thus, the present result is a step closer to that predicted by the classical theory.

2 Preliminaries

The motion of an incompressible fluid is governed by the Navier–Stokes equations,

$$\dot{u} + (u \cdot \nabla) u + \nabla p = \nu \Delta u, \tag{1}$$

$$\nabla \cdot u = 0,$$

where $u(x, t)$ is the fluid velocity, $p$ is the pressure, and $\nu$ is the viscosity. We consider unforced dynamics for convenience. The results can be seen to carry over to the forced case without change, except for a minor modification to section 3, where the forcing term would be ignored if the injected enstrophy is negligible compared with that produced by the vortex stretching mechanism. This is the case for forcing that injects energy at large scales. Equation (1) is considered in a periodic domain $D = [0, 2\pi L]^3$, and all fields are assumed to have zero spatial average. This allows each component of $u$ to be expressible as a Fourier series in terms of $\exp\{ik \cdot x/L\}$. Here $k = (k_1, k_2, k_3)$, where $k_1, k_2, k_3$ are integers.
not simultaneously zero. The incompressible velocity field $u(x, t)$ can then be represented by

$$u(x, t) = \sum_k \hat{u}(k, t) \exp \left\{ \frac{i k \cdot x}{L} \right\},$$

(2)

where $k \cdot \hat{u}(k, t) = 0$ for incompressibility and $\hat{u}(k, t) = \hat{u}^*(-k, t)$ for reality. In other words, the solution vector-valued function space (phase space) can be spanned by the infinite basis $\{\exp[i k \cdot x/L]\}_k$. Throughout this study, we assume strong solutions up to the time under consideration. The possibility of subsequent development of singularities is not an issue. This allows us to make use of usual assumptions within the realm of the classical theory, such as bounded velocity and vorticity.

The solution space is equipped with the scalar product $\langle u \cdot v \rangle$ and energy norm $\|u\|$ given, respectively, by

$$\langle u \cdot v \rangle = (2\pi L)^{-3} \left( \int_D u \cdot v \, dx \right)$$

(3)

and

$$\|u\| = \langle |u|^2 \rangle^{1/2}.$$

(4)

The advection term $(u \cdot \nabla)u$ conserves the kinetic energy $\|u\|^2/2$. More generally, one has by integration by parts the identity

$$\langle \nu \cdot (u \cdot \nabla)w \rangle = -\langle w \cdot (u \cdot \nabla)v \rangle,$$

(5)

for all admissible $v$ and $w$. Also by integration by parts followed by the Cauchy–Schwarz inequality, we have

$$\|\nabla u\|^2 = -\langle u \cdot \Delta u \rangle \leq \|u\| \|\Delta u\|.$$

(6)

Consider a set of mutually orthonormal functions $\{v_1, v_2, \ldots, v_n\}$, i.e., $\langle v_i \cdot v_j \rangle = \delta_{ij}$ with $\delta_{ij}$ being the Kronecker delta symbol. For large $n$, the number of Fourier modes within
the wave number radius \( n^{1/3}/L \) is approximately \( n \). Their (repeated) eigenvalues under \(-\Delta\) are \( k^2/L^2 = (k_1^2 + k_2^2 + k_3^2)/L^2 \leq n^{2/3}/L^2 \) and sum up to approximately \( n^{5/3}/L^2 \). Since these constitute the smallest eigenvalues of \(-\Delta\), the Rayleigh–Ritz principle implies that

\[
\sum_{j=1}^{n} \|\nabla v_j\|^2 \geq c \frac{n^{5/3}}{L^2},
\]

where \( c \) is a nondimensional constant independent of the orthonormal set \( \{v_1, v_2, \ldots, v_n\} \).

3 Taylor microscale and number of active modes

In this section we consider the equation governing the evolution of enstrophy \( \|\nabla u\|^2/2 \), from which the enstrophy production and dissipation terms are compared. From this comparison during the growing phase of the enstrophy, up to a maximum, we derive two expressions for the Taylor microscale wave number and compare them with \( k_\nu \). We then rephrase the classical argument leading to the Landau estimate for the number of degrees of freedom.

By taking the scalar product of the momentum equation of Eq. (1) with \( \Delta u \) and integrating by parts the resulting time derivative term, we obtain

\[
\|\nabla u\| \frac{d}{dt} \|\nabla u\| = \langle \Delta u \cdot (u \cdot \nabla) u \rangle - \nu \|\Delta u\|^2.
\]

The triple-product term in Eq. (8) is responsible for enstrophy production, which holds the key to our understanding of turbulence. As it stands, this term is strongly dominated by the small scales because of the factors \( \nabla u \) and \( \Delta u \). Indeed, one can readily appreciate this claim by rewriting the enstrophy production term in the more familiar form \( \langle \omega \cdot (\omega \cdot \nabla) u \rangle \),

where \( \omega = \nabla \times u \) is the vorticity, which clearly involves small-scale quantities only. This means that the “macroscale” velocity field plays a largely insignificant role in the enstrophy dynamics. In other words, the interactions among the small scales are responsible for intense enstrophy production rather than the advection of small-scale eddies and filaments by the large-scale flow. This can be true even if the turbulence has energetic large-scale structures.
superimposed on a sea of small-scale vortices. Motivated by this observation, we define a “microscale” velocity $U$ by

$$
U = \frac{|\langle (\Delta u \cdot (u \cdot \nabla) u) \rangle|}{\|\nabla u\|}{\|\Delta u\|} = \frac{|\langle \omega \cdot (\omega \cdot \nabla) u \rangle|}{\|\nabla u\|}{\|\Delta u\|}.
$$

(9)

As it stands, $U$ can be considered as the typical velocity at the dissipation scale. Since $U$ is essentially the domain average of the velocity weighted by two functions of unity norm, it satisfies $U \leq \|u\|_{\infty}$, where $\|u\|_{\infty}$ is the maximum fluid velocity. Given the steep energy spectrum of the classical theory, it is likely that $U \ll \|u\|_{\infty}$. It may probably be the case that $U \ll \|u\|$. Anyhow, with definition (9), Eq. (8) can be rewritten as

$$
\|\nabla u\| \frac{d}{dt} \|\nabla u\| \leq U \|\Delta u\| \|\nabla u\| - \nu \|\Delta u\|^2
= \frac{\|\Delta u\|^2}{\|\nabla u\|^2} \left( U \|\nabla u\|^3 - \nu \|\nabla u\|^2 \right).
$$

(10)

During the period of enstrophy increase, up to a local maximum or global maximum, i.e., maximum energy dissipation, Eq. (10) implies

$$
k_T = \frac{\|\nabla u\|}{\|u\|} \leq \frac{U \|\nabla u\|^2}{\nu \|u\| \|\Delta u\|} = \frac{CU}{\nu},
$$

(11)

where $k_T$ is the Taylor microscale wave number mentioned earlier and $C = \|\nabla u\|^2 / (\|u\| \|\Delta u\|)$ is a nondimensional parameter. By Eq. (6), $C$ satisfies $C \leq 1$, but this bound can be highly excessive. For example, for the classical spectrum, $C$ tends to zero quite rapidly (as $k_\nu^{-1/3}$) in the limit $k_\nu \to \infty$. Note that without a prior knowledge of $k_\nu$, one could define $k_T$ as the energy dissipation wave number. Within the context of Kolmogorov’s theory, we have $k_T \leq k_\nu$; and the equality would occur for a $k^{-1}$ energy spectrum.

Equation (11) has been derived independently of the Kolmogorov hypothesis of finite energy dissipation $\epsilon$. This hypothesis immediately implies that $k_T = (\epsilon/\nu)^{1/2} / \|u\|$. Here, we are interested in its dynamical consequences rather than this immediate implication. Together with the predicted $k^{-5/3}$ spectrum, the hypothesis has a profound implication on
$U$, i.e., on the enstrophy production term. One can see from Eq. (10) that a finite and viscosity-independent energy dissipation rate requires that $U \|\nabla u\|^3 / \|\Delta u\|$ remain finite. This implies the scaling $U \propto k_\nu^{-1/3}$ since $\|\nabla u\|^3 / \|\Delta u\| \propto k_\nu^{1/3}$. Thus, our suggestion that $U \ll \|u\|$ is fully justified within the framework of the classical theory. In any case, Eq. (11) can be rewritten in terms of $\epsilon$ as

$$k_T \leq \left( \frac{CU}{\|u\|} \right)^{1/2} \left( \frac{\epsilon}{\nu^3} \right)^{1/4}. \quad (12)$$

This brings $k_T$ closer to $k_\nu$ in form. Note that if we invoke the Cauchy-Schwarz inequality, i.e., replacing $C$ by its upper bound of unity and use $U \leq \|u\|_\infty$, we would have $k_T \leq (\|u\|_\infty / \|u\|)^{1/2}(\epsilon/\nu^3)^{1/4}$, which is apparently an excessive estimate.

The dissipation wave number $k_\nu$ naturally defines the number of degrees of freedom, being the number of dynamically active Fourier modes having wave numbers not exceeding $k_\nu$. This number, denoted by $N_c$, is approximately given by

$$N_c = \left( \frac{k_\nu}{k_0} \right)^3 = \left( \frac{L^{4/3} \epsilon^{1/3}}{\nu} \right)^{9/4} = \text{Re}^{9/4}, \quad (13)$$

where $k_0 = 1/L$ is the lowest wave number. Equation (13) essentially rephrases Landau’s estimate mentioned earlier based on the number of vortices having volume $k_\nu^{-3}$ within the domain $D$. In what follows, we show that this result largely agrees with an estimate for the number of degrees of freedom defined within the context of dynamical systems theory.

4 Number of degrees of freedom

This section revisits the notion of number of degrees of freedom from the perspective of dynamical systems theory. This number is then estimated for three-dimensional Navier–Stokes turbulence, using what is essentially equivalent to the “trace formula,” which was derived in the 1980s.\textsuperscript{13,14} Since then, this formula has become the tool for virtually every study on attractor dimension of turbulence in both two and three dimensions. The present
formulation is a simple version (with other advantages in addition to simplicity, see below) of the highly technical formulation that leads to the trace formula. It is an extension of a recent study by Tran and Blackbourn\textsuperscript{15} on two-dimensional turbulence and is summarized in what follows.

Chaotic dynamics are characterized by the sensitive dependence of solutions on initial conditions, resulting in a rapid separation of nearby trajectories—solution “curves”—in phase space. In a neighborhood of a given point on a given trajectory (a given solution at a given time), this separation is greatest in the orthogonal “directions” corresponding to the greatest local Lyapunov exponents. These constitute the most unstable directions of the dynamics linearized about the solution under consideration. In general, these directions and the associated exponents can change continuously along the trajectory, an underpinning feature of dynamical complexity. In an infinite-dimensional dissipative system, the number of positive local Lyapunov exponents along a given bounded trajectory is presumably finite, followed by a spectrum of negative exponents corresponding to stable orthogonal directions. The smallest number of greatest exponents whose sum becomes negative (hereafter denoted by $N$) is significant as phase space $n$-dimensional volume elements along the trajectory contract exponentially for $n \geq N$. When $N$ is common for all points on an arbitrary bounded trajectory, volume contraction becomes universal on bounded sets of phase space. This number is an upper bound for the so-called Lyapunov or Kaplan and Yorke dimension\textsuperscript{16,17} of an attractor if one exists. It is also an upper bound for other generalized dimensions, such as the box-counting and Hausdorff dimensions, of the attractor. Such an $N$ represents the number of degrees of freedom of the dynamical system in question, in the sense that its chaotic dynamics can be adequately described by an $N$-dimensional model. This makes sense even for cases in which no nontrivial attractors are known to exist. Furthermore, $N$ is well defined regardless of whether or not conventional Lyapunov exponents exist. Another advantage of the present formulation is that the
problem of global regularity of solutions of the Navier–Stokes system is not an issue. The reason is that $N$ is determined pointwise in time, therefore remaining valid up to the time of solution blowup should this turn out to be the case.

4.1 Local Lyapunov exponents

Consider the linear evolution of a disturbance $v$ to the solution $u(x, t)$ of Eq. (1) commencing from a smooth initial field $u_0 = u(x, 0)$. The governing equations for $v$ are

$$v_t + (u \cdot \nabla)v + (v \cdot \nabla)u + \nabla p' = \nu \Delta v, \quad (14)$$

$$\nabla \cdot u = 0 = \nabla \cdot v,$$

where $p'$ is the perturbed pressure. By taking the scalar product of $v$ with Eq. (14) and noting that both $\langle v \cdot (u \cdot \nabla)v \rangle$ and $\langle v \cdot \nabla p' \rangle$ vanish, we obtain the equation governing the evolution of $\|v\|$,

$$\|v\| \frac{d}{dt} \|v\| = -\langle v \cdot (v \cdot \nabla)u \rangle - \nu \|\nabla v\|^2. \quad (15)$$

It follows that

$$\lambda = \frac{d}{dt} \ln \|v\| = -\frac{1}{\|v\|^2} \left( \langle v \cdot (v \cdot \nabla)u \rangle + \nu \|\nabla v\|^2 \right). \quad (16)$$

Here $\lambda$ is the exponential growth or decay rate of $\|v\|$.

The greatest local Lyapunov exponent and the corresponding most unstable direction can be found by maximizing $\lambda$ with respect to all admissible disturbances $v$. We denote by $(\lambda_1, v_1)$ the solution of this problem, where for convenience (and without loss of generality) $v_1$ has been normalized, i.e., $\|v_1\| = 1$. The second greatest exponent $\lambda_2$ and the corresponding second most unstable direction $v_2$ orthogonal to $v_1$ can be obtained by maximizing $\lambda$ with respect to all disturbances $v$ subject to the orthogonality constraint $\langle v \cdot v_1 \rangle = 0$. Similarly, the pair of third greatest exponent $\lambda_3$ and third most unstable direction $v_3$ can be obtained by solving the same maximization problem, where the admissible disturbances
\( \mathbf{v} \) satisfy the constraint \( \langle \mathbf{v} \cdot \mathbf{v}_1 \rangle = \langle \mathbf{v} \cdot \mathbf{v}_2 \rangle = 0 \). By repeating this procedure \( n \) times, we obtain the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) of mutually orthonormal functions and the corresponding ordered set of exponents \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). These can be described more formally by
\[
\lambda_j = \max_{\| \mathbf{v} \| = 1} \left\{ -\langle \mathbf{v} \cdot (\mathbf{v} \cdot \nabla) \mathbf{u} \rangle - \nu \| \nabla \mathbf{v} \|^2 \right\}
= -\langle \mathbf{v}_j \cdot (\mathbf{v}_j \cdot \nabla) \mathbf{u} \rangle - \nu \| \nabla \mathbf{v}_j \|^2
\]
for \( 1 \leq j \leq n \), where the maximization is subject to the constraint \( \langle \mathbf{v} \cdot \mathbf{v}_i \rangle = 0 \) for \( i < j \). These exponents provide a complete picture of solution stability with respect to disturbances. In passing, it is worth mentioning that in the maximization problem, the solutions \( \mathbf{v}_j \) arise as compromises between the triple-product and viscous dissipation terms. That means that they do not necessarily maximize the former. It would be interesting to have a knowledge of the maximizers \( (\lambda'_j, \mathbf{v}'_j) \) of the triple-product term alone. A comparison between the two Lyapunov spectra \( \lambda_j \) and \( \lambda'_j \) and between \( \| \nabla \mathbf{v}_j \| \) and \( \| \nabla \mathbf{v}'_j \| \), conceivably by numerical methods, could provide some invaluable dynamical insights.

4.2 Upper bounds for the number of degrees of freedom

We now calculate the number of degrees of freedom \( N \) described earlier by minimizing \( n \) such that the sum \( \sum_{j=1}^n \lambda_j \) is negative. By Eq. (17), we have
\[
\sum_{j=1}^n \lambda_j = -\sum_{j=1}^n \left( \langle \mathbf{v}_j \cdot (\mathbf{v}_j \cdot \nabla) \mathbf{u} \rangle + \nu \| \nabla \mathbf{v}_j \|^2 \right)
= \sum_{j=1}^n \left( \langle \mathbf{u} \cdot (\mathbf{v}_j \cdot \nabla) \mathbf{v}_j \rangle - \nu \| \nabla \mathbf{v}_j \|^2 \right),
\]
where Eq. (5) has been used. Similar to the definition of \( U \) in the preceding section, we define two average quantities \( U' \) and \( \Omega \), respectively, by
\[
U' = \frac{1}{\left( n \sum_{j=1}^n \| \nabla \mathbf{v}_j \|^2 \right)^{1/2}} \left| \sum_{j=1}^n \langle \mathbf{u} \cdot (\mathbf{v}_j \cdot \nabla) \mathbf{v}_j \rangle \right|
\]
and
\[
\Omega = \frac{1}{n} \left| \sum_{j=1}^n \langle \mathbf{v}_j \cdot (\mathbf{v}_j \cdot \nabla) \mathbf{u} \rangle \right|. \]
Like $U$ in Eq. (9), $U' \leq \|u\|_\infty$ represents a small-scale velocity by virtue of its very definition. The reason is that the orthonormal set $\{v_1, v_2, \ldots, v_n\}$ consists of progressively smaller-scale functions $v_j$, i.e., increasingly greater $\|\nabla v_j\|$ as the index $j$ increases. Note that a suitable rearrangement of the set may be necessary if it is not already in that order. Furthermore, similar to the $j$-th eigenvalue of $-\Delta$, we have $\|\nabla v_j\|_2^2 \propto j^{2/3}$. This implies that $U'$ is more strongly weighted by the smaller-scale $v_j$'s. It is, however, not known with precision how $U'$ and $U$ compare. Now for definition (20), $\Omega$ can be thought of as the domain average of $|\nabla u|$ weighted by $\sum_{j=1}^n |v_j|^2/n$. By definition, the inequality $\Omega \leq \|\nabla u\|_\infty$ holds. On physical grounds, this bound can become excessive for large Re because intense velocity gradients are known to be highly concentrated in space, effectively getting “moderated” under the spatial average in the definition of $\Omega$. Moreover, unless the spatial distribution of a majority of $v_j$’s is strongly correlated to that of $\nabla u$ (i.e., locally peaking in the same small regions as $\nabla u$), this moderation can be more effective than that in $\|\nabla u\| = \langle |\nabla u|^2 \rangle^{1/2}$. The reason is that $|\nabla u|^2$ is more resistant to such moderation than $|\nabla u|$ as is reflected in the fact that $\langle |\nabla u| \rangle \leq \|\nabla u\|$. Hence, even though $\Omega$ is undetermined, it is expected to be closer to $\|\nabla u\|$ rather than to $\|\nabla u\|_\infty$. This “conjecture” could be readily tested numerically, given the linear and kinematic nature of the maximization problem. On an optimistic note, it is worth mentioning that one cannot rule out the possibility $\Omega \leq \|\nabla u\|$, even though that might seem unlikely.

Upon substituting Eqs. (19) and (7) into the second equation of Eq. (18) we obtain

$$
\sum_{j=1}^n \lambda_j \leq \left( \sum_{j=1}^n \|\nabla v_j\|^2 \right)^{1/2} \left( U' n^{1/2} - \nu \left( \sum_{j=1}^n \|\nabla v_j\|^2 \right)^{1/2} \right) \leq \left( n \sum_{j=1}^n \|\nabla v_j\|^2 \right)^{1/2} \left( U' - \nu \frac{c^{1/2} n^{1/3}}{L} \right).
$$

The condition $\sum_{j=1}^n \lambda_j \leq 0$ requires a straightforward lower bound for $n$, from which we
deduce the bound

\[ N \leq c^{-3/2} \left( \frac{U'L}{\nu} \right)^3 = Re^3, \]  

(22)

where \( c^{-3/2} \) has been incorporated into the newly defined Reynolds number. Similar results have been reported by Constantin et al.\(^6\) and Gibbon,\(^11\) where their Reynolds numbers were defined in terms of \( \|u\|_\infty \) and of the local velocity \( |u(x,t)| \), respectively. The Gibbon estimate (local number of degrees of freedom) becomes the Constantin estimate where \( |u(x,t)| \) peaks.

The above estimate has made no use of the assumption of finite energy dissipation. Now if we identify \( \nu \Omega^2 \) with \( \epsilon \), then upon substituting Eqs. (20) and (7) into the first equation of Eq. (18), we obtain

\[ \sum_{j=1}^{n} \lambda_j \leq n\Omega - \nu \sum_{j=1}^{n} \|\nabla v_j\|^2 \leq n \left( \frac{\epsilon^{1/2}}{\nu^{1/2}} - \nu \frac{cn^{2/3}}{L^2} \right). \]  

(23)

The condition \( \sum_{j=1}^{n} \lambda_j \leq 0 \) requires a straightforward lower bound for \( n \), from which we deduce the bound

\[ N \leq c^{-3/2} \left( \frac{L^{4/3} \epsilon^{1/3}}{\nu} \right)^{9/4} = Re^{9/4}, \]  

(24)

where again \( c^{-3/2} \) has been incorporated into the Reynolds number Re. This result differs from the Landau estimate by the use of \( \Omega \) instead of \( \|\nabla u\| \) in the definition of Re. This difference can be slight as argued above. In the classical picture of homogeneous turbulence, there would hardly be any distinction between \( \|\nabla u\| \) and \( \Omega \).

In passing, it is worth mentioning that the sum of the triple-product terms in the first equation of Eq. (18) is quite susceptible to sophisticated (and potentially excessive) estimates, which we have thus far deliberately avoided. Consider, for example, the Lieb–
Thirring inequality\textsuperscript{20,21} concerning the orthonormal set \( \{v_1, v_2, \ldots, v_n\} \),
\[
\| \sum_{j=1}^{n} |v_j|^2 \| \leq c' L^{3/2} \left( \sum_{j=1}^{n} \| \nabla v_j \|^2 \right)^{3/4},
\]
where \( c' \) is a nondimensional constant independent of the set \( \{v_1, v_2, \ldots, v_n\} \). By applying Eq. (25) to the first equation of Eq. (18), via the intermediate step
\[
\sum_{j=1}^{n} \langle v_j \cdot (v_j \cdot \nabla) u \rangle \leq \left| \sum_{j=1}^{n} |v_j|^2 \right| \| \nabla u \|,
\]
we would arrive at
\[
\sum_{j=1}^{n} \lambda_j \leq c' L^{3/2} \left( \sum_{j=1}^{n} \| \nabla v_j \|^2 \right)^{3/4} \| \nabla u \| - \nu \sum_{j=1}^{n} \| \nabla v_j \|^2 \\
\leq \left( \sum_{j=1}^{n} \| \nabla v_j \|^2 \right)^{3/4} \left( c' L^{3/2} \frac{\epsilon^{1/2}}{\nu^{1/2}} - \nu \frac{c_1^{1/4} \epsilon^{5/12}}{L^{1/2}} \right),
\]
where Eq. (7) has been used in the second step. It follows that
\[
N \leq \left( \frac{c' \epsilon^{1/4}}{c} \right)^{3/5} \left( \frac{L^{4/3} \epsilon^{1/3}}{\nu} \right)^{18/5} = \text{Re}^{18/5},
\]
where the constant prefactor has been absorbed into \( \text{Re} \), which is now defined in terms of \( \| \nabla u \| \) instead of \( \Omega \). The price for this is the scaling \( \text{Re}^{18/5} \) instead of \( \text{Re}^{9/4} \) for \( N \). This result (given in a quite different form) was derived by Gibbon and Titi\textsuperscript{9} as an upper bound for the attractor dimension.

### 4.3 Discussion

In two-dimensional turbulence, \( N \) has been found to satisfy\textsuperscript{15}
\[
N \leq C' \text{Re}(1 + \ln \text{Re})^{1/3},
\]
where \( C' \) is an absolute constant and the Reynolds number \( \text{Re} \) is defined in terms of the materially conserved vorticity, the domain size, and \( \nu \). Apart from the difference in the level of rigor in the definition of \( \text{Re} \), there is a sharp contrast between the nearly linear
scaling of \( N \) with \( \text{Re} \) in two-dimensional turbulence and the highly superlinear scaling of \( N \) with \( \text{Re} \) in the present case. This is due to fundamental differences between the two cases. We discuss two most apparent discrepancies in what follows.

One of these is due to the dimension of the physical space and is easy to recognize. Given \( n \) Fourier modes of lowest wave numbers, the sum of their eigenvalues under \(-\Delta-\) a collective measure of viscous dissipation strength—are \( \propto n^{5/3} \) and \( \propto n^2 \), in three and two dimensions, respectively. This means that for the same Reynolds number, three-dimensional turbulence is expected to have more dynamically active modes than its two-dimensional counterpart. This makes an intuitively obvious contribution to the difference between Eqs. (24) and (29).

The other contributing factor can be attributed to the discrepancy in the “effective degree” of nonlinearity of the small-scale dynamics of the Navier–Stokes equations in these cases. In three dimensions, the dynamics are highly nonlinear, effectively quadratic. In principle, the vortex stretching term \((\omega \cdot \nabla)u\) can give rise to an explosive vorticity growth\(^{22,23}\). On the contrary, the two-dimensional Navier–Stokes system is effectively nearly linear, rendering far less intense dynamics of the small scales—a widely recognized fact. One can readily appreciate this claim by a quick inspection of the equation governing the vorticity gradient \( \nabla \omega \),

\[
\nabla \omega_t + (u \cdot \nabla) \nabla \omega = \omega \times \nabla \omega - (\nabla \omega \cdot \nabla)u + \nu \Delta \nabla \omega, \tag{30}
\]

\[
\nabla \cdot u = 0,
\]

where \( n \) is the normal to the fluid domain. In Eq. (30), the sole effect of the first term on the right-hand side is to rotate \( \nabla \omega \) without changing its magnitude, and the second term alone is responsible for vorticity gradient amplification. By ignoring the viscous term for convenience, we can deduce from Eq. (30) the equation

\[
|\nabla \omega|_t + u \cdot \nabla |\nabla \omega| = -\frac{\nabla \omega \cdot (\nabla \omega \cdot \nabla)u}{|\nabla \omega|^2} \leq |\nabla u||\nabla \omega|. \tag{31}
\]
This means that following the fluid motion, $|\nabla \omega|$ can grow no more rapidly than exponentially in time at the instantaneous rate $|\nabla u|$. Now, since vorticity is conserved in the inviscid dynamics, $|\nabla u|$ is relatively well behaved because $\|\nabla u\| = \|\omega\|$ is conserved. Indeed, numerical evidence shows that for an initial vorticity reservoir at large scales, $|\nabla u|$ remains largely unchanged up to and beyond the instance of peak enstrophy dissipation.\(^{24}\)

More precisely, the ratio of the irrotational strain to $\|\omega\|$, initially at $\approx 2$, has been found to remain within the range $[2, 3.5]$ throughout the said period, during which $\|\nabla \omega\|$ grows approximately exponentially by several orders of magnitude. Hence, the vorticity gradient stretching term can be said to be marginally nonlinear, if it is to be considered \textit{nonlinear} at all. The same remark can be made about a broad family of fluid systems in the geophysical context. For this case, the gradient $\nabla q$ of the materially conserved potential vorticity $q$ is governed by Eqs. (30) and (31), with $q$ replacing $\omega$. For this family, the velocity gradient $\nabla u$ is also well behaved. In fact, it is presumably better behaved than its counterpart in two-dimensional Navier–Stokes turbulence because $\|\nabla u\| < \|q\|$. Thus, the small-scale dynamics of this family are effectively marginally nonlinear.

The enstrophy production in three-dimensional turbulence is a fundamental problem in fluid mechanics and has always been a centre of attraction for the turbulence community.\(^{25-34}\) This is a formidable problem, being virtually intractable as we have come to realize. In the limit of large Reynolds number, analytic and dynamically independent upper bounds for $\langle \Delta u \cdot (u \cdot \nabla)u \rangle$ tend to become so excessive that they render no practical value. The reason behind these excessive estimates is that in order to bound the norm of a quantity, say $\nabla u$, one usually resorts to norms of its derivatives, such as $\Delta u$. As a result, when $\|\nabla u\| \to \infty$, its upper bounds usually diverge far more rapidly. Known inequalities applicable to the enstrophy production term invariably reduce to the form “$1 \leq \infty$” (or equivalently “$0 \leq 1$”) as $\|\nabla u\| \to \infty$. An example is the Cauchy–Schwarz inequality (6), which was given in section 2 and briefly discussed in section 3. There, we bypassed this
inequality by introducing the dynamical parameter $C$. Another example is the Agmond\textsuperscript{35,36} inequality $\|u\|_\infty \leq c'' L^{3/2} \|\nabla u\|^{1/2} \|\Delta u\|^{1/2}$, where $c''$ is a nondimensional constant. This could be one of the most generous estimates in the present context. Given the excessive nature of the available inequalities (when applied to high-Reynolds number turbulence), it is desirable, if not crucial, to develop new techniques that could derive dynamically binding estimates from the governing equations.

5 Conclusion

This study has examined a possible route toward a quantitative basis in support of the Kolmogorov prediction for the viscous dissipation wave number $k_\nu$ in three-dimensional Navier–Stokes turbulence and the associated Landau estimate for its number of degrees of freedom $N$. For $k_\nu$, we have taken an indirect approach, by estimating the Taylor microscale wave number $k_T$, which is a close cousin of $k_\nu$. It has been found that $k_T \leq CU/\nu$, where $U$ is a “microscale” velocity and $C \leq 1$ is a dynamical parameter. When expressed in terms of the energy dissipation rate $\epsilon$, this result becomes $k_T \leq (CU/\|u\|)^{1/2}(\epsilon/\nu^3)^{1/4}$, where $\|u\|$ is the root mean square velocity. The latter can be seen to be in line with Kolmogorov’s prediction for $k_\nu$. For $N$, we have taken a direct approach, by deriving an upper bound for the minimum number of greatest local Lyapunov exponents whose sum becomes negative. The calculations of this bound have been carried out in a general manner, therefore the obtained result is universal for bounded trajectories. It is an upper bound for generalized dimensions, such as the box-counting and Haussdorff dimensions, of a nontrivial attractor (for the forced case) if one exists. It has been found that $N$ satisfies $N \leq \text{Re}^{9/4}$, where $\text{Re}$ is defined in terms of an average energy dissipation rate, the system length scale, and $\nu$. This result is in a remarkable agreement with the Landau estimate if one identifies the conventional energy dissipation rate $\epsilon = \nu \|\nabla u\|^2$ with the newly defined rate $\nu \Omega^2$, where $\Omega$ is effectively the spatial average of the velocity gradient $|\nabla u|$ weighted by the average
of the squares of mutually orthonormal functions. Although $\Omega$ is essentially undetermined, we have argued that it can be close to $\|\nabla u\|$. This is the “extent” to which the present analysis needs to reach for a complete agreement with the Landau estimate of the number of degrees of freedom on the basis of the Kolmogorov theory. In the classical picture of homogeneous and isotropic turbulence, there would be virtually no distinction between $\|\nabla u\|$ and $\Omega$.

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