Bilinear Strichartz estimates for the Schrödinger map problem

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Abstract: In this paper we prove bilinear Strichartz estimates for a solution to the Schrödinger map problem whose size is small in the critical Strichartz space \( \| | \nabla |^{d/2 - 2} \psi_x \|_{L_t^2 L_x^{d+2}} \). These estimates will be useful in an upcoming paper in proving a local well-posedness result. Bilinear estimates make use of an argument similar to the argument found in [15]. We use the same gauges as in [1], [2], and [17].

1 Introduction

The Schrödinger map problem

\[
\partial_t \phi = \phi \times \Delta_x \phi, \\
\phi(0) = \phi_0, \\
\phi : I \times \mathbb{R}^d \to S^2 \hookrightarrow \mathbb{R}^3
\]

is a problem which has been a subject of a great deal of recent attention. This is a problem with a rich geometric structure that arises naturally in a number of different ways. See [12] or [14] for more details.

This system (1.1) enjoys conservation of energy,

\[
E(\phi(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\partial_x \phi(t, x)|^2 dx
\]

and mass

\[
M(\phi(t)) = \int_{\mathbb{R}^d} |\phi(t, x) - Q|^2 dx,
\]

where \( Q \in S^2 \) is some fixed base point. When \( d = 2 \) both (1.1) and (1.2) are invariant with respect to the scaling
\[ \phi(t, x) \mapsto \phi(\lambda^2 t, \lambda x), \quad \lambda > 0. \] (1.4)

When \( d = 2 \) is called energy critical. \([1], [2], [3], [17]\), studied the partial differential equation satisfied by the derivatives of a solution to (1.1). The derivatives of \( \psi(t, x) \), \( \psi_l = \partial_x l \psi(t, x) \) satisfy an equation that is a perturbation of the free Schrödinger equation

\[ (i\partial_t + \Delta)\psi_l = -2iA_m\partial_m\psi_l - i(\partial_m A_m)\psi_l + (A_t + A_m A_m)\psi_l - i\psi_m \text{Im}(\bar{\psi}_m \psi_l). \] (1.5)

**Remark:** In this paper we adopt the usual convention that Latin letters \( l, m = 1, \ldots, d \) and we sum over repeated indices. \( A_m, A_t \) are the connection coefficients.

Using the Coulomb gauge in dimensions \( d \geq 4 \) \([1]\) proved global well-posedness of (1.1) for initial data sufficiently small in \( \dot{H}^{d/2} \). \([2]\) proved global well-posedness for small data in \( d \geq 2 \) using the caloric gauge. This result was subsequently extended by \([17]\) to data with energy below the energy of the ground state and \( d = 2 \), provided the data satisfies certain other smallness assumptions.

The chief difficulty in the study of the derivative Schrödinger maps equation arises from the magnetic term \( A_m \partial_m \psi_l \) when \( \psi_l \) is at a high frequency and \( A_m \) is at a low frequency. This term cannot be treated perturbatively using only the Strichartz estimates. Instead \([1], [2], [17]\) utilized bilinear Strichartz estimates to move half of the derivative from the high frequency term to the low frequency term. This combined with local smoothing results is enough to close the bootstrap under the smallness conditions of \([1], [2], \) and \([17]\).

In this paper we prove some bilinear Strichartz estimates for a solution to (1.1). We start by recalling a bilinear Strichartz estimate for the linear Schrödinger equation.

**Theorem 1.1** If \( u \) solves the free Schrödinger equation

\[
\begin{align*}
iu_t + \Delta u &= 0, \\
u(0) &= u_0,
\end{align*}
\] (1.6)

then for \( M \ll N \), when \( P_N \) is a Littlewood-Paley operator,

\[
\|(P_M u)(P_N \bar{u})\|_{L^2_{t,u}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \frac{M^{(d-1)/2}}{N^{1/2}} \|P_M u_0\|_{L^2(\mathbb{R}^d)} \|P_N u_0\|_{L^2(\mathbb{R}^d)}. \] (1.7)

This can be proved using Fourier analytic techniques. \([4]\) used the Fourier transform to prove this theorem when \( d = 2 \). The result was subsequently extended to all dimensions (see for example \([?]\)). One can also prove a similar result on \( I \) if \( u \) solves

\[
\begin{align*}
iu_t + \Delta u &= \pm|u|^2 u,
\end{align*}
\] (1.8)
\[ \| \nabla \|^{|d-2|/2} u \|_{L^2(I \times \mathbb{R}^d)} < \infty. \]  

(1.9)

[15] proved theorem [13] via an interaction Morawetz estimate. This method is useful to this paper because it is very robust under perturbations of (1.6). In particular, if \( \psi_l \) solves (1.5) then (1.7) holds under a slight strengthening of (1.9).

First define a Sobolev space for \( \phi : I \times \mathbb{R}^d \to S^2 \).

**Definition 1.1** Let \( \mathcal{F}(d) \) denote the Fourier transform on \( L^2(\mathbb{R}^d) \). For \( \sigma \geq 0 \) define the inhomogeneous Sobolev spaces on \( \mathbb{R}^d \) for vector valued functions.

\[ H^\sigma(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C}^n : \| f \|_{H^\sigma(\mathbb{R}^d)} = \left( \sum_{l=1}^{n} \| \mathcal{F}(f_l)(\xi)(1 + |\xi|^2)^\sigma/2 \|^2_{L^2(\mathbb{R}^d)} \right)^{1/2} < \infty \}, \]  

(1.10)

as well as the homogeneous Sobolev spaces

\[ \dot{H}^\sigma(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C}^n : \| f \|_{\dot{H}^\sigma(\mathbb{R}^d)} = \left( \sum_{l=1}^{n} \| \mathcal{F}(f_l)(\xi) \cdot |\xi|^\sigma \|^2_{L^2(\mathbb{R}^d)} \right)^{1/2} < \infty \}. \]  

(1.11)

For \( \sigma \geq 0, Q = (Q_1, Q_2, Q_3) \in S^2 \) define the complete metric space

\[ H^\sigma_Q(\mathbb{R}^d; S^2) = \{ f : \mathbb{R}^d \to \mathbb{R}^3 : |f(x)| \equiv 1, f - Q \in H^\sigma \}. \]  

(1.12)

This metric has the induced distance

\[ d^\sigma_Q(f, g) = \| f - g \|_{H^\sigma(\mathbb{R}^d)}. \]  

(1.13)

Let \( \| f \|_{H^\sigma_Q} = d^\sigma_Q(f, Q) \) for \( f \in H^\sigma_Q \). Define the complete metric spaces

\[ H^\infty = H^\infty(\mathbb{R}^d; \mathbb{C}^n) = \bigcap_{\sigma \in \mathbb{Z}_+} H^\sigma(\mathbb{R}^d) \quad \text{and} \quad H^\infty_Q(\mathbb{R}^d; S^2) = \bigcap_{\sigma \in \mathbb{Z}_+} H^\sigma_Q(\mathbb{R}^d; S^2) \]  

(1.14)

with the induced distances.

Choose a small constant \( \delta > 0 \), say \( \delta = \frac{1}{10} \). Let \( \psi_x \) be the vector \( \psi_x = (\psi_1, \ldots, \psi_d) \). Let \( \beta(k) \) be a frequency envelope that majorizes \( 2^{(d-2)/2} \| P_k \psi_x(0) \|_{L^2(\mathbb{R}^d)} \), satisfying

\[ 2^{(d-2)/2} \| P_k \psi_x(0) \|_{L^2(\mathbb{R}^d)} \leq \beta(k), \quad \beta(k) \leq 2^\delta |k-l| \beta(l), \quad \sum_k \beta(k)^2 \lesssim \| \psi \|^2_{H^{d/2}_Q}. \]  

(1.15)
These results will be used in a subsequent paper to prove well-posedness of (1).

Remark: The time variable is usually assigned to \( t \), where \( t \geq 0 \).

The time variable under the harmonic map heat flow in the caloric gauge. Moreover suppose \( \psi \) solves (1.5), satisfies (1.15) and (1.17), and \( A \) satisfies the Coulomb gauge. Then

\[
\| (P_k \psi_x) (P_l \psi_t) \|_{L^2_t(x)} \lesssim 2^{-|l-k|/2} (\alpha(k) + \beta(k)) (\alpha(l) + \beta(l)).
\]

Theorem 1.2 Suppose \( d \geq 4 \) and \( k - l \geq 10 \), \( \psi \) solves (1.5), satisfies (1.15) and (1.17), and \( A \) satisfies the Coulomb gauge. Then

\[
\| (P_k \psi_x) (P_l \psi_t) \|_{L^2_t(x)} \lesssim 2^{-|l-k|/2} (\alpha(k) + \beta(k)) (\alpha(l) + \beta(l)).
\]

Theorem 1.3 Suppose \( d = 2 \), \( k - l \geq 10 \), \( \psi \) solves (1.5), satisfies (1.15) and (1.17), and \( A \) satisfies the Coulomb gauge. Moreover suppose \( \psi(s, t, x) \) is the solution of the harmonic map heat flow with initial data \( \psi(0, t, x) \). Then

\[
\| (P_k \psi_x(s)) (P_l \psi_t(\bar{s})) \|_{L^2_t(x)} \lesssim 2^{-|l-k|/2} (\alpha(k) + \beta(k)) (\alpha(l) + \beta(l))(1 + s 2^{k-4})^{-4}(1 + \bar{s} 2^{l-4} ).
\]

These results will be used in a subsequent paper to prove well-posedness of (1.1).

2 Gauge Field Equations

Let \( \phi \) be any function such that \( \phi : \mathbb{R}^2 \times (-T, T) \to S^2 \). Denote space and time derivatives of \( \phi \) as \( \partial_\alpha \phi \), where \( \alpha = 1, ..., d + 1 \) and \( \partial_{d+1} \phi = \partial_t \phi \).

Remark: The time variable is usually assigned to \( t = 0 \). However this index will be reserved for time variable under the harmonic map heat flow in the caloric gauge.
As in [1], [2], and [17] select an orthonormal frame \((v(t, x), w(t, x)) \in T_{\phi(t,x)} S^2\), i.e. smooth functions \(v, w : \mathbb{R}^2 \times (-T, T) \to S^2\) such that at each point \((x, t)\) the vectors \(v(t, x), w(t, x)\) form an orthonormal basis \(T_{\phi(t,x)} S^2\). As a matter of convention assume \(v \) and \(w \) are chosen so that \(v \times w = \phi\).

Then introduce the derivative fields. Set
\[
\psi_\alpha = v \cdot \partial_\alpha \phi + iw \cdot \partial_\alpha \phi.
\] (2.1)
Then \(\partial_\alpha \phi\) admits the representation
\[
\partial_\alpha \phi = v \text{Re}(\psi_\alpha) + w \text{Im}(\psi_\alpha).
\] (2.2)
Rewrite the vector \(\partial_\alpha \phi\) with respect to the orthonormal basis \((v, w)\), then identify \(\mathbb{R}^2\) with the complex numbers \(\mathbb{C}\) according to \(v \leftrightarrow 1, w \leftrightarrow i\). This identification respects the complex structure of the target manifold. The Riemannian connection on \(S^2\) pulls back to a covariant derivative on \(\mathbb{C}\), which we denote by
\[
D_\alpha = \partial_\alpha + iA_\alpha.
\] (2.3)
The connection coefficients \(A_\alpha\) are defined via
\[
A_\alpha = w \cdot \partial_\alpha v.
\] (2.4)
Because the Riemannian connection on \(S^2\) is torsion free the derivative fields satisfy the equations
\[
D_\beta \psi_\alpha = D_\alpha \psi_\beta.
\] (2.5)
Equivalently,
\[
\partial_\beta A_\alpha - \partial_\alpha A_\beta = \text{Im}(\psi_\beta \bar{\psi}_\alpha) = q_{\beta\alpha}.
\] (2.6)
If \(\phi\) is a smooth solution to the Schödinger map problem [1,1] then the derivatives satisfy the equation
\[
\psi_t = iD_t \psi_t.
\] (2.7)
This is because
\[
\phi \times \Delta \phi = J(\phi) (\phi^* \nabla)_j \partial_j \phi,
\] (2.8)
where \(J(\phi)\) denotes the complex structure \(\phi \times\) and \((\phi^* \nabla)_j\) the pullback of the Levi - Cevita connection \(\nabla\) on the sphere. This implies
\[(i\partial_t + \Delta)\psi_l = -2iA_m\partial_m\psi_l - i(\partial_mA_m)\psi_l + (A_t + A_mA_m)\psi_l - i\psi_lIm(\bar{\psi}_m\psi_l),\]
\[D_\alpha\psi_\beta = D_\beta\psi_\alpha,\]
\[Im(\psi_\alpha\bar{\psi}_\beta) = \partial_\alpha A_\beta - \partial_\beta A_\alpha.\]

A solution \(\psi_m\) to (2.7) cannot be determined uniquely without choosing an orthonormal frame \((v, w)\). Changing a given choice of orthonormal frame induces a gauge transformation and may be represented as
\[\psi_m \mapsto e^{i\theta}\psi_m \quad A_m \mapsto A_m + \partial_m\theta.\]

The system (2.7) is invariant with respect to such gauge transformations.

In this paper we will discuss bilinear Strichartz estimates for two choices of gauge, the Coulomb gauge and the caloric gauge. The Coulomb gauge is a gauge which is quite useful in high dimensions (see [1]) and in low dimensions when some additional symmetry is imposed on the problem (see [10] and [3]). In this paper we will discuss the Coulomb gauge for dimensions \(d \geq 4\).

However, the Coulomb gauge becomes very difficult to use in low dimensions for a general Schrödinger map problem. Therefore for dimension \(d = 2\) we will consider the caloric gauge. This gauge was introduced in [22] to study wave maps in hyperbolic space. The series of papers [23], [24], [25], [26], [27] then used this gauge to establish global regularity of wave maps in hyperbolic space. [21] suggested that the caloric gauge would be a suitable gauge in which to study Schrödinger maps. [2] utilized this gauge to establish global well-posedness in the setting of initial data with small critical norm. This result was further expanded by [17].

### 2.1 Coulomb Gauge:

Under the Coulomb gauge
\[\sum_{m=1}^{d} \partial_mA_m = 0.\]

In view of (2.6) this leads to
\[A_m = \Delta^{-1} \sum_{l=1}^{d} \partial_lIm(\bar{\psi}_l\psi_m).\]

Also by (2.6)
\[
\Delta A_0 = \sum_{l=1}^{d} \partial_l (\partial_0 A_l + \text{Im}(\psi_{l} \bar{\psi}_{l+1})) = \sum_{l=1}^{d} \partial_l \text{Im}(\psi_{l} \bar{\psi}_{l+1}).
\] (2.13)

Using (2.5), (2.7),
\[
= -\sum_{m=1}^{d} \partial_m \text{Re}(\bar{\psi}_{m} D_{m} \psi_{m}) = -\sum_{m,l=1}^{d} \partial_l \partial_m \text{Re}(\bar{\psi}_{l} \psi_{m}) + \frac{1}{2} \Delta (\sum_{m=1}^{d} \psi_{m} \bar{\psi}_{m}).
\] (2.14)

The caloric gauge will be discussed in an upcoming section.

3 Proof of theorem 1.1

Everything in this section can be found in [15]. Theorem 1.1 will be proved here for the reader’s convenience, since the proof will be modified to deal with the case when \( \psi \) solves (1.5).

Suppose \( u \) solves
\[
(i\partial_t + \Delta)u = 0.
\] (3.1)

The argument of [15] is more useful for this paper than the argument of [4] because it is very robust under perturbations of the Laplacian \( \Delta \) or perturbations of (3.1). Define the Morawetz potential
\[
M(t) = \int |u_{M}(t,y)|^2 \frac{(x-y)j}{|x-y|} \text{Im}[\bar{u}_{N}(t,x)\partial_j u_{N}(t,x)]dxdy
+ \int |u_{N}(t,y)|^2 \frac{(x-y)j}{|x-y|} \text{Im}[\bar{u}_{M}(t,x)\partial_j u_{M}(t,x)]dxdy.
\] (3.2)

Because \( e^{it\Delta} \) is a Fourier multiplier and \( |e^{-it\xi^2}| = 1, \)
\[
\|u_{M}(t,x)\|_{L^2(\mathbb{R}^d)} = \|u_{M}(0,x)\|_{L^2(\mathbb{R}^d)},
\] (3.3)

and therefore since \( \frac{(x-y)}{|x-y|} \leq 1, \)
\[
|M(t)| \lesssim (M + N)\|u_{M}(0,x)\|^2_{L^2(\mathbb{R}^d)}\|u_{N}(0,x)\|^2_{L^2(\mathbb{R}^d)}. \] (3.4)

Lemma 3.1 For \( \omega \in S^{d-1} \) let \( x_{\omega} = x \cdot \omega, \partial_{\omega} = (\omega \cdot \nabla), \)
\[
\int_{S^{d-1}} \frac{x_{\omega}}{|x_{\omega}|} f(x) \partial_{\omega} g(x) d\omega = \frac{1}{|x|} f(x)(x \cdot \nabla) g(x).
\] (3.5)
Proof: Without loss of generality suppose $x = (x_1, 0, ..., 0)$.

\[
\frac{x_\omega}{|x_\omega|} = \frac{\omega_1}{|\omega_1|} \frac{x_1}{|x_1|}.
\] (3.6)

\[
\int_{S^{d-1}} \frac{x_1}{|x_1|} \frac{\omega_1}{|\omega_1|} f(x) \omega_j \partial_j f(x) = C(d) \frac{x_1}{|x_1|} f(x) \partial_1 g(x) = C(d) \frac{x_j}{|x|} f(x) \partial_j g(x).
\] (3.7)

Therefore, $M(t) = \frac{1}{x_0} \int_{S^{d-1}} M_\omega(t) d\omega$, where

\[
M_\omega(t) = \int |u_M(t, y)|^2 \frac{(x - y)_\omega}{|(x - y)_\omega|} Im[\bar{u}_N(t, x) \partial_\omega u_N(t, x)] dx dy
+ \int |u_N(t, y)|^2 \frac{(x - y)_\omega}{|(x - y)_\omega|} Im[\bar{u}_M(t, x) \partial_\omega u_M(t, x)] dx dy.
\] (3.8)

By the fundamental theorem of calculus

\[
M_\omega(T) - M_\omega(0) = \int_0^T \frac{d}{dt} M_\omega(t) dt.
\] (3.9)

Without loss of generality take $\omega = (1, 0, ..., 0)$.

\[
\frac{d}{dt} M_\omega(t) = -2 \int \partial_k Im(\bar{u}_M \partial_k u_M)(t, y) \frac{(x - y)_1}{|(x - y)_1|} Im[\bar{u}_N \partial_1 u_N](t, x) dx dy
\] (3.10)

\[
-2 \int \partial_k Im(\bar{u}_N \partial_k u_N)(t, y) \frac{(x - y)_1}{|(x - y)_1|} Im[\bar{u}_M \partial_1 u_M](t, x) dx dy
\] (3.11)

\[
+ \frac{1}{2} \int |u_M(t, y)|^2 \frac{(x - y)_1}{|(x - y)_1|} \partial_1 \partial_{k_1}^2 (|u_N(t, x)|^2) dx dy
\] (3.12)

\[
+ \frac{1}{2} \int |u_N(t, y)|^2 \frac{(x - y)_1}{|(x - y)_1|} \partial_1 \partial_{k_1}^2 (|u_M(t, x)|^2) dx dy
\] (3.13)

\[
-2 \int |u_M(t, y)|^2 \frac{(x - y)_1}{|(x - y)_1|} \partial_1 Re(\partial_1 \bar{u}_N \partial_k u_N)(t, x) dx dy
\] (3.14)

\[
-2 \int |u_N(t, y)|^2 \frac{(x - y)_1}{|(x - y)_1|} \partial_1 Re(\partial_1 \bar{u}_M \partial_k u_M)(t, x) dx dy.
\] (3.15)

Integrating by parts

\[
= -2 \int Im[\bar{u}_M \partial_1 u_M](t, x_1, y_2, ..., y_d) Im[\bar{u}_N \partial_1 u_N](t, x_1, x_2, ..., x_d) dx dy
\] (3.16)
\[-2 \int Im[\pi_N \partial_1 u_N](t, x_1, y_2, ..., y_d) Im[\bar{u}_M \partial_1 u_M](t, x_1, x_2, ..., x_d) dxdy \] (3.17)

\[+ \frac{1}{2} \int \partial_1(|u_M(t, x_1, y_2, ..., y_d)|^2) \partial_1(|u_N(t, x_1, x_2, ..., x_d)|^2) dxdy \] (3.18)

\[+ \frac{1}{2} \int \partial_1(|u_N(t, x_1, y_2, ..., y_d)|^2) \partial_1(|u_M(t, x_1, x_2, ..., x_d)|^2) dxdy \] (3.19)

\[+ 2 \int |u_M(t, x_1, y_2, ..., y_d)|^2 |\partial_1 u_N(t, x_1, x_2, ..., x_d)|^2 dxdy \] (3.20)

\[+ 2 \int |u_N(t, x_1, y_2, ..., y_d)|^2 |\partial_1 u_N(t, x_1, x_2, ..., x_d)|^2 dxdy. \] (3.21)

\[= \int \int |\partial_1(\bar{u}_N(t, x_1, x_2, ..., x_d) u_M(t, x_1, y_2, ..., y_d))|^2 dx_1 dx_2 \cdots dx_d dy_2 \cdots dy_d. \] (3.22)

In one dimension this implies

\[\int \int |\partial_x(\bar{u}_N u_M)(t, x)|^2 dx dt \lesssim (M + N)\|u_M(0)\|_{L^2(R)}^2 \|u_N(0)\|_{L^2(R)}^2. \] (3.23)

Therefore Bernstein’s inequality implies that when \( M < N \),

\[\|\bar{u}_M u_N\|_{L^2_{0,d}(R \times R)} \lesssim \frac{1}{N^{1/2}} \|u_M(0)\|_{L^2(R)} \|u_N(0)\|_{L^2(R)}, \] (3.24)

which concludes the proof of Theorem 1 when \( d = 1 \). In higher dimensions let \( \tilde{P}_M \) be the Littlewood - Paley projection onto frequencies \( |\xi_2 + \cdots + \xi_d| \leq 100M \). This implies that for some \( \tilde{\phi}(x) \), \(|\tilde{\phi}(x)| \lesssim 1\), \(|\phi(x)| \lesssim (1 + |x|)^{-N} \) for any \( N \),

\[u_M = \tilde{P}_M u_M = \int_{R^{d-1}} u_M(x_1, x_2 - \bar{y}) \phi(M \bar{y}) M^{d-1} d\bar{y}. \] (3.25)

\[\partial_1(\bar{u}_N(t, x_1, x') u_M(t, x_1, x' + y_0)) = \int_{R^{d-1}} \partial_1(\bar{u}_N(t, x_1, x') u_M(t, x_1, x' + y_0 + \bar{y})) \phi(M \bar{y}) M^{d-1} d\bar{y}. \] (3.26)

By Holder’s inequality

\[|\partial_1(\bar{u}_N(t, x_1, x') u_M(t, x_1, x' + y_0))| \lesssim M^{\frac{d-1}{2}} \left( \int_{R^{d-1}} |\partial_1(\bar{u}_N(t, x_1, x') u_M(t, x_1, x' + y_0 + \bar{y}))|^2 d\bar{y} \right)^{1/2} \left( \int |\phi(M \bar{y})|^2 M^{d-1} d\bar{y} \right)^{1/2}. \] (3.27)
Therefore by (3.4), (3.22),
\[ \| \partial_1(\bar{u}_N(t, x_1, x')u_M(t, x_1, x' + y_0)) \|_{L^2_t L^2_x}^2 \lesssim M^{d-1}N\|u_M(0)\|_{L^2(\mathbb{R}^d)}^2\|u_N(0)\|_{L^2(\mathbb{R}^d)}^2. \]
(3.28)

Integrating over \( \omega \in S^{d-1} \) implies
\[ \| \nabla(\bar{u}_N(t, x_1, x')u_M(t, x_1, x' + y_0)) \|_{L^2_t L^2_x}^2 \lesssim M^{d-1}N^{1/2}\|u_M(0)\|_{L^2(\mathbb{R}^d)}\|u_N(0)\|_{L^2(\mathbb{R}^d)}. \]
(3.29)

Applying Bernstein’s inequality proves theorem 1. □

An identical computation would produce the same result with \( u_M \) replaced by \( u_M(x + x_0) \) for some \( x_0 \in \mathbb{R}^d \). Therefore,

**Corollary 3.2** If \( u \) solves the free Schrödinger equation then for \( M << N \),
\[ \| (P_M u(t, x))(P_N \bar{u}(t, x + x_0)) \|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \frac{M^{(d-1)/2}N^{1/2}}{N^{1/2}}\|P_M u_0\|_{L^2(\mathbb{R}^d)}\|P_N u_0\|_{L^2(\mathbb{R}^d)}. \]
(3.30)

### 4 Almost Conserved Quantities

Conservation of energy implies
\[ \| \psi_x(t) \|_{L^2(\mathbb{R}^d)} = \| \psi_x(0) \|_{L^2(\mathbb{R}^d)}. \]
(4.1)

Therefore consider \( d \geq 4 \).

**Theorem 4.1** For \( d \geq 4 \), \( \epsilon(\|\psi\|_{\dot{H}^{d/2}_Q}) > 0 \) sufficiently small,
\[ \| \nabla |\frac{d-2}{2} \psi_x \|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)} \lesssim \| \nabla |\frac{d-2}{2} \psi_x(0) \|_{L^2(\mathbb{R}^d)}. \]
(4.2)

**Proof:** Suppose \( \psi_x \) solves (1.5). Take the inner product
\[ \langle u, v \rangle = Re \int u(x)\bar{v}(x)dx. \]
(4.3)

\[ \frac{1}{2} \frac{d}{dt} \langle \nabla \left| \frac{d-2}{2} \psi_x \right|, \nabla \left| \frac{d-2}{2} \psi_x \right| \rangle = \langle i\Delta \left| \frac{d-2}{2} \psi_x \right|, \nabla \left| \frac{d-2}{2} \psi_x \right| \rangle \]
\[ + \langle \nabla \left| \frac{d-2}{2} \right. \left( -2A_m \partial_m \psi_t - (\partial_m A_m) \psi_t - i(A_t + A_m A_m) \psi_t - \psi_m \text{Im} \left( \bar{\psi}_m \psi_t \right) \right|, \nabla \left| \frac{d-2}{2} \psi_t \right| \rangle. \]
(4.4)
The first term on the right hand side of (4.4) is \( \equiv 0 \).

\[
\| \nabla \|^{\frac{d-2}{2}} (-(\partial_m A_m)\psi_t - i(A_t + A_m A_m)\psi_t - \psi_m Im(\bar{\psi}_m \psi_l)) \|_{L^{\frac{2(d+2)}{d+4}}(I \times \mathbb{R}^d)} 
\]

\[
\lesssim \| \nabla \cdot A \| + |A_t| + |A_x \cdot A_x| + |\psi_x|^2 \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+4}}(I \times \mathbb{R}^d)}
\]

\[
+ \| \psi_x \|_{L^{d+2}(I \times \mathbb{R}^d)} (\| \nabla \|^{\frac{d-2}{2}} (\nabla \cdot A) \|_{L^2(I \times \mathbb{R}^d)} + \| \nabla \|^{\frac{d-2}{2}} A_t \|_{L^2(I \times \mathbb{R}^d)} + \| \nabla \|^{\frac{d-2}{2}} (A_x) \| A_x \|_{L^2(I \times \mathbb{R}^d)})
\]

\[
\lesssim \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} (\| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^2(I \times \mathbb{R}^d)}^{2-4/d} + \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^2(I \times \mathbb{R}^d)}^{4-4/d}).
\]

Therefore by (1.17),

\[
\int_I \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} (\| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^2(I \times \mathbb{R}^d)}^{2-4/d} + \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^2(I \times \mathbb{R}^d)}^{4-4/d}) dt
\]

Finally evaluate

\[
\int_I \| \nabla \|^{\frac{d-2}{2}} A_m \partial_m \psi_t, \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} dt = \int_I \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} dt
\]

Integrate the right hand side of (4.11) by parts.

\[
\int_I \| \nabla \|^{\frac{d-2}{2}} A_m (t, x) \partial_m \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} dt = \int_I \nabla \cdot (A) \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} dt
\]

Therefore,

\[
(4.12) \lesssim \| \nabla \|^{\frac{d-2}{2}} A_m \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} + \| \nabla \|^{\frac{d-2}{2}} A_m \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)}
\]

\[
\lesssim \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)} + \| \nabla \|^{\frac{d-2}{2}} \psi_x \|_{L^{\frac{2(d+2)}{d+2}}(I \times \mathbb{R}^d)}^{2-4/d}.
\]

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Proof: Begin with the easiest term and move to the most difficult. By (1.10),

\[ \epsilon^{4/d} \| \nabla \left| \frac{d-2}{2} \psi_x(0) \right| \|_{L^2} \leq 1, \quad (4.16) \]

the theorem is proved. \( \Box \)

In order to make use of the interaction Morawetz estimates of the previous section we need to estimate \( \| P_N \psi_x(t) \|_{L_x^2} \) when \( t \in I \).

\[ \frac{1}{2} \frac{d}{dt} \langle P_M \psi_x, P_M \psi_x \rangle = -2 P_M (A_m \partial_m \psi_x, P_M \psi_x) \quad (4.17) \]

\[ -\langle P_M((\partial_m A_m) \psi_x), P_M \psi_x \rangle - \langle i P_M((A_t + A_x \cdot A_x) \psi_x), P_M \psi_x \rangle - \langle P_M(\psi_m \Im(\bar{\psi}_m \psi)), P_M \psi \rangle. \quad (4.18) \]

**Lemma 4.2** When \( d \geq 4 \), \( \psi \) solves (1.15) and satisfies (1.15) and (1.17), \( M = 2^k \) for some \( k \in \mathbb{Z} \),

\[ \| P_M((P_{\geq 100} A_m) \partial_m \psi_x + (\partial_m A_m) \psi_x + i(A_t + A_x \cdot A_x) \psi_x + \psi_m \Im(\bar{\psi}_m \psi)) \|_{L_x^{2(d+2)}} \lesssim 2^{-k(d-2)/2} \alpha(k). \quad (4.19) \]

**Remark:** If \( M = 2^k \), \( N = 2^j \), let \( \alpha(M) = \alpha(k) \) and \( \alpha(N) = \alpha(j) \).

**Proof:** Begin with the easiest term and move to the most difficult. By (1.17)

\[ \| P_M(\psi_m \Im(\bar{\psi}_m \psi)) \|_{L_x^{2(d+2)}} \lesssim \| \psi_x \|_{L_x^{2(d+2)}} \| P_{\geq 100} \psi_x \|_{L_x^{2(d+2)}} \lesssim \epsilon^{4/d} \| \nabla \left| \frac{d-2}{2} \psi_x \right| \|_{L_x^2} \| P_{\geq 100} \psi_x \|_{L_x^{2(d+2)}}. \quad (4.20) \]

By (4.16)

\[ \lesssim \sum_{l \geq k-10} 2^{-l(d-2)/2} \alpha(l) \lesssim 2^{-k(d-2)/2} \alpha(k). \quad (4.21) \]

Likewise,

\[ \| P_M((\partial_m A_m) \psi_x + i(A_t + A_x \cdot A_x) \psi_x) \|_{L_x^{2(d+2)}} \lesssim 2^{-k(d-2)/2} \alpha(k) \quad (4.22) \]

\[ + \| P_{\geq 100}((\partial_m A_m) + i(A_t + A_x \cdot A_x)) \|_{L_x^{2(d+2)}} \| P_{\geq 100} \psi_x \|_{L_x^{2(d+2)}}. \quad (4.23) \]

By (2.12), (2.14), and (4.21),
\[ \| P_{\geq M}^4 ( (\partial_m A_m) + i (A_t + A_x \cdot A_x) ) \|_{L^2_{t,x}} \lesssim \| P_{\geq M} \psi_x \|_{L^2_{t,x}} \| \psi_x \|_{L^4_{t,x}}^{4/d} \left( \| \nabla |^{d-2}_{2} \psi_x \|_{L^\infty_{t} L^2_{x}}^{2} + \| \nabla |^{d-2}_{2} \psi_x \|_{L^\infty_{t} L^2_{x}}^{4/d} \right) \lesssim 2^{-(d-2)/2} \alpha(k). \] (4.24)

□

Now consider
\[
\int_I \langle 2 P_M ((P_{\leq M}^4 A_m) \partial_m \psi_x), P_M \psi_x \rangle dt = 2 \int_I \langle (P_{\leq M}^4 A_m) \partial_m (P_M \psi_x), P_M \psi_x \rangle dt \tag{4.25}
\]
\[
+ 2 \int_I \langle P_M ((P_{\leq M}^4 A_m) \partial_m \psi_x), P_M \psi_x \rangle dt - 2 \int_I \langle (P_{\leq M}^4 A_m) \partial_m (P_M \psi_x), P_M \psi_x \rangle dt. \tag{4.26}
\]
Integrating the right hand side of (4.25) by parts
\[
\int_I \int (P_{\leq M}^4 A_m) \partial_m |P_M \psi_x|^2 dxdt \lesssim \| \nabla \cdot A \|_{L^4_{t,x}}^{4/d} \| P_M \psi_x \|_{L^{2(d+2)}_{t,x}}^{4/(d+2)} \lesssim \epsilon^{4/d} \| \nabla |^{d-2}_{2} \psi_x \|_{L^\infty_{t} L^2_{x}}^{2} \| \nabla |^{d-2}_{2} \psi_x \|_{L^\infty_{t} L^2_{x}}^{2(d-2)} \lesssim \alpha(k)^2 2^{-k(d-2)}. \tag{4.27}
\]
By the fundamental theorem of calculus we have the estimate on the Littlewood - Paley multiplier for |\eta| << M, |\xi| \sim M.

\[ |\phi\left( \frac{\xi}{M} \right) - \phi\left( \frac{\xi + \eta}{M} \right)| \lesssim \frac{1}{M} |\eta|. \tag{4.28} \]
Therefore,
\[
\| P_M ((P_{\leq M}^4 A_m) \partial_m \psi_x) - (P_{\leq M}^4 A_m) \partial_m P_M \psi_x \|_{L^2_{t,x}} \lesssim \| P_M \psi_x \|_{L^{2(d+2)}_{t,x}} \| \partial_x A \|_{L^{4/d}_{t,x}}^{4/d} \lesssim \alpha(k)^2 2^{-k(d-2)/2}. \tag{4.29}
\]
Combining lemma 4.2, 4.27, and 4.29

**Theorem 4.3** If \( \psi_x \) satisfies (1.15), (1.17), and (4.16) also holds, for \( t \in I, \)
\[
\| P_M \psi_x (t) \|_{L^2_x} \lesssim M^{-d/2} \alpha(M) + \beta(M). \tag{4.30} \]
5 Proof of theorem \ref{thm:1.2}

Now suppose $\psi_x$ solves (1.5) and $d \geq 4$. Theorem \ref{thm:1.3} (1.15), and (1.17) imply

$$\sup_{t \in I} |M(t)| \lesssim N(\alpha(M) + \beta(M))^2(\alpha(N) + \beta(N))^2. \quad (5.1)$$

By corollary 3.2 this would automatically imply theorem 1.2 if

$$ (\partial_t - i\Delta)\psi_x = 0. \quad (5.2)$$

Therefore it suffices to bound the errors arising from the right hand side of (1.5). These errors are quite similar to the errors in the proof of theorem 4.3. Without loss of generality it suffices to consider the error terms in

$$ \int \int |P_M\psi_x(t,y)|^2 \frac{(x-y)_j}{|x-y|} Im[P_N\bar{\psi}_x(t,x)\partial_j\psi_x(t,x)] dxdy. \quad (5.3)$$

The error is given by

$$ E = 2 \int \int \left| P_M(\psi_x(t,y))\right|^2 \frac{(x-y)_j}{|x-y|} Im[P_N\bar{\psi}_x(t,x)\partial_j\psi_x(t,x)] dxdydt \quad (5.4)$$

$$ + \int \int \int |P_M\psi_x(t,y)|^2 \frac{(x-y)_j}{|x-y|} Im[(P_N\bar{\psi}_x(t,x)\partial_j\psi(t,x)] dxdydt \quad (5.5)$$

$$ + \int \int \int |P_M\psi_x(t,y)|^2 \frac{(x-y)_j}{|x-y|} Im[(P_N(\partial_t - i\Delta)\psi_x(t,x)] dxdydt. \quad (5.6)$$

By lemma 4.2 since

$$ \|P_N(-2(\sum_{\leq N} A_m)\partial_m\psi_x - (\partial_m A_m)\psi_x - i(A_t + A_x \cdot A_x)\psi_x)\|_{L_{t,x}^{2d+2}} \lesssim N^{-\frac{d+1}{2}} \alpha(N), \quad (5.7)$$

$$ E = -4 \int \int \left| P_M(\psi_x(t,y))P_M((\sum_{\leq N} A_m)\partial_m\psi_x(t,y))] \right| \frac{(x-y)_j}{|x-y|} Im[(P_N\bar{\psi}_x(t,x)\partial_j\psi_x(t,x)] dxdydt \quad (5.8)$$

$$ - 2 \int \int |P_M\psi_x(t,y)|^2 \frac{(x-y)_j}{|x-y|} Im[(P_N\bar{\psi}_x(t,x)\partial_jP_N(A_m)\partial_m\psi_x(t,x))] dxdydt \quad (5.9)$$

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- 2 \int \int |P_M \psi_x(t, y)|^2 \frac{(x - y)^j}{|x - y|} \text{Im}[|P_N| \frac{P}{\Delta_{m \Lambda_m}} \psi_x(t, x) \partial_j P_N \psi_x(t, x)|dxdydt \tag{5.10}

\text{+ } O(NN^{-(d-2)}M^{-(d-2)}(\alpha(M) + \beta(M))^2(\alpha(N) + \beta(N))^2. \tag{5.11}

\text{Integrate (5.8) and (5.9) + (5.10) by parts.}

\int \int (P_{\leq \frac{M}{100}} A_m) \partial_m |P_M \psi_x|^{\frac{2}{|x - y|}} \text{Im}[P_N \psi_x \partial_j P_N \psi_x]|dxdydt \tag{5.12}

= - \int \int \partial_m (P_{\leq \frac{M}{100}} A_m)|P_M \psi_x|^{\frac{2}{|x - y|}} \text{Im}[P_N \psi_x \partial_j P_N \psi_x]|dxdydt \tag{5.13}

- \int \int (P_{\leq \frac{M}{100}} A_m)|P_M \psi_x|^{\frac{2}{|x - y|}} \partial_m (\frac{(x - y)^j}{|x - y|}) \text{Im}[P_N \psi_x \partial_j P_N \psi_x]|dxdydt. \tag{5.14}

\tag{5.13} \lesssim N \|P_N \psi_x\|_{L^\infty_{t,x}}^2 \|\partial_x A\|_{L^2_{t,x}} \|P_M \psi_x\|_{L^2_{t,x}} \|P_N \psi_x\|_{L^2_{t,x}} \tag{5.15}

\lesssim NN^{-(d-2)}M^{-(d-2)}(\alpha(M) + \beta(M))^2(\alpha(N) + \beta(N))^2.

\text{The Hardy - Littlewood - Sobolev inequality implies}

\tag{5.14} \lesssim \|P_M \psi_x\|_{L^2_{t,x}}^{\frac{2}{(d+2)}} \|P_{\leq \frac{M}{100}} A_m\|_{L^\infty_{t,x}}^{\frac{2}{d}} \|P_N \psi_x\|_{L^\infty_{t,x}}^{\frac{4}{d}} \|P_N \psi_x\|_{L^2_{t,x}}^{\frac{2-4}{d}}. \tag{5.16}

\text{Likewise}

\int \int |P_M \psi_x(t, y)|^2 \frac{(x - y)^j}{|x - y|} \text{Im}[(P_N \psi_x)(P_{\leq \frac{M}{100}} A_m) \partial_j \partial_m P_N \psi_x](t, x)|dxdy \tag{5.17}

+ \int \int |P_M \psi_x(t, y)|^2 \frac{(x - y)^j}{|x - y|} \text{Im}[(P_{\leq \frac{M}{100}} A_m) \partial_m (P_N \psi_x) \partial_j P_N \psi_x](t, x)|dxdy \tag{5.18}

= - \int \int |P_M \psi_x(t, y)|^2 \frac{(x - y)^j}{|x - y|} \partial_m (P_{\leq \frac{M}{100}} A_m) \text{Im}[(P_N \psi_x) \partial_j P_N \psi_x](t, x)|dxdydt \tag{5.19}

= - \int \int |P_M \psi_x(t, y)|^2 \partial_m (\frac{(x - y)^j}{|x - y|}) (P_{\leq \frac{M}{100}} A_m) \text{Im}[(P_N \psi_x) \partial_j P_N \psi_x](t, x)|dxdydt. \tag{5.20}

\text{Once again use the Hardy - Littlewood - Sobolev theorem for (5.20). As in the proof of theorem 4.3}
\[ \| P_M (P \leq M A_m ) \partial_m \psi_x (P_M \bar{\psi}_x) - (P \leq M A_m ) (P_M \partial_m \bar{\psi}_x) (P_M \bar{\psi}_x) \|_{L^1_{t,x}} \lesssim M^{-(d-2)} \alpha (M)^2. \] (5.21)

\[ \| (P_N \bar{\psi}_x) \partial_j (P \leq N A_m ) \partial_m (P_N \psi_x) - (P_N \bar{\psi}_x) (P \leq N A_m ) \partial_j \partial_m (P_N \psi_x) \|_{L^1_{t,x}} \lesssim N \cdot N^{-(d-2)} \alpha (N)^2. \] (5.22)

\[ \| (P_N \bar{\psi}_x) \partial_j (P \leq N A_m ) \partial_m (P_N \psi_x) - P_N (P \leq N A_m ) \partial_m \psi_x) \|_{L^1_{t,x}} \lesssim N \cdot N^{-(d-2)} \alpha (N)^2. \] (5.23)

\[ \| ((P \leq N A_m ) \partial_m (P_N \psi_x) - P_N (P \leq N A_m ) \partial_m \psi_x) \|_{L^1_{t,x}} \lesssim N \cdot N^{-(d-2)} \alpha (N)^2. \] (5.24)

This proves theorem 1.2.2 □

6 Caloric Gauge

The caloric gauge was proposed in [22] in the context of wave maps and then in [21] in the context of Schrödinger maps. Precisely, at each time \( t \) we solve the covariant heat equation with \( \phi (t) \) as the initial data on \([0, \infty) \times \mathbb{R}^d\),

\[ \partial_s \tilde{\phi} = \Delta_x \tilde{\phi} + \tilde{\phi} \cdot \sum_{m=1}^{d} |\partial_m \tilde{\phi}|^2 \] (6.1)

\[ \tilde{\phi}(0, t, x) = \phi(t, x). \]

[16] proved that (6.1) is well-posed on \( \mathbb{R}^2 \) for \( s > 0 \) when the energy of \( \phi \) is less than the energy of the ground state. Moreover, \( \phi (s) \) approaches the equilibrium state \( Q \) as \( s \to \infty \). Therefore we can choose \((v_\infty, w_\infty)\) at \( s = \infty \) as an arbitrary orthonormal base in \( T_Q S^2 \). Pulling back \((v_\infty, w_\infty)\) along the backward heat flow by parallel transport gives an orthonormal frame \((v, w)\) for all \( s \geq 0 \). Moreover,

\[ w \cdot \partial_s v = A_s = 0. \] (6.2)

In the gauge the harmonic map heat flow is given by

\[ (\partial_s - \Delta_x) \psi_m = 2i A_l \partial_l \psi_m - (A_l \cdot A_l - i \partial_l A_l) \psi_m + i \text{Im}(\psi_m \bar{\psi}_l) \psi_l, \] (6.3)

\[ (\partial_s - \Delta_x) \psi_l = 2i A_l \partial_l \psi_l - (A_l \cdot A_l - i \partial_l A_l) \psi_l + i \text{Im}(\psi_l \bar{\psi}_l) \psi_l. \] (6.4)
(2.6) implies
\[ \partial_0 A_s = Im(\psi_0 \bar{\psi}_m). \]  
(6.5)

Integrating backward from \( s = \infty \), for any \( m = 1, \ldots, d + 1 \),
\[ A_m(s) = -\int_s^{\infty} Im(\bar{\psi}_m (\partial_l \psi_l + i A_l \psi_l))(r) dr. \]  
(6.6)

**Theorem 6.1** Let \( \phi \) be a heat flow with classical initial data whose energy \( E_0 \) is less than \( E_{\text{crit}} \). Let \( e \) be a caloric gauge for \( \phi \), and let \( A_x \) denote the connection fields. Then we have the pointwise bounds

\[ \sup_{s>0} s^{(k+1)/2} \| \partial_x^k A_x(s) \|_{L^\infty_x(R^2)} \lesssim E_{0,k} 1, \]  
(6.7)

\[ \sup_{s>0} s^{k/2} \| \partial_x^k A_x(s) \|_{L^2_x(R^2)} \lesssim E_{0,k} 1, \]  
(6.8)

for all \( k \geq 0 \) and \( s > 0 \), as well as

\[ \int_0^{\infty} s^{(k-1)/2} \| \partial_x^k A_x(s) \|_{L^\infty_x(R^2)} ds \lesssim E_{0,k} 1, \]  
(6.9)

\[ \int_0^{\infty} s^{k-1} \| \partial_x^k A_x(s) \|_{L^2_x(R^2)} ds \lesssim E_{0,k} 1. \]  
(6.10)

For all \( k \geq 0 \).

**Proof:** This was proved in theorem 7.4 of [16]. \( \square \)

**Corollary 6.2** Let \( \phi \) be a heat flow with classical initial data with energy \( E_0 \) less than \( E_{\text{crit}} \). Let \( e \) be a caloric gauge for \( \phi \). Then for all \( k \geq 1 \),

\[ \int_0^{\infty} s^{k-1} \| \partial_x^k \psi_x \|_{L^2_x(R^2)}^2 \lesssim E_{0,k} 1, \]  
(6.11)

\[ \sup_{s>0} s^{(k-1)/2} \| \partial_x^{k-1} \psi_x \|_{L^2_x(R^2)} \lesssim E_{0,k} 1, \]  
(6.12)

\[ \int_0^{\infty} s^{k-1} \| \partial_x^{k-1} \psi_x \|_{L^\infty_x(R^2)}^2 ds \lesssim E_{0,k} 1, \]  
(6.13)

\[ \sup_{s>0} s^{k/2} \| \partial_x^{k-1} \psi_x \|_{L^\infty_x(R^2)} \lesssim E_{0,k} 1. \]  
(6.14)

Analogous estimates hold if one replaces \( \partial_x \psi_x \) with \( \psi_s \), \( \partial_x^2 \) with \( \partial_s \), and/or \( \partial_x \) with \( D_x \).
Proof: This is corollary 7.5 in [15]. □

**Theorem 6.3** For \( t \in I \), \( \alpha \) and \( \beta \) satisfy (1.15) and (1.17), \( d = 2 \),

\[
\|P_M \psi_x(t)\|_{L^2(R^2)}^2 \leq \alpha(M)^2 + C(E_0)(\beta(M)^2 + \|P_M \bar{\psi}_x\|_{L^2_{t,x}}^2).
\]

(6.15)

Proof:

\[
\frac{d}{dt}\|P_M \psi_x\|_{L^2}^2 = -\langle 2P_M(A_m \partial_m \psi_x), P_M \psi_x \rangle
\]

(6.16)

\[
-\langle P_M((\partial_m A_m) \psi_x), P_M \psi_x \rangle - \langle iP_M((A_t + A_x \cdot A_x) \psi_x), P_M \psi_x \rangle - \langle P_M(\psi_m \Im(\bar{\psi}_m \psi_x)), P_M \psi_x \rangle.
\]

(6.17)

This implies that for \( t \in I \),

\[
\|P_M \psi_x(t)\|_{L^2(R^2)}^2 \leq \beta(M)^2 - \int \langle 2P_M(A_m \partial_m \psi_x), P_M \psi_x \rangle dt
\]

(6.18)

\[
+ \|P_M \bar{\psi}_x\|_{L^2_{t,x}} (\|\partial_x A\|_{L^2_{t,x}} + \|A_t\|_{L^2_{t,x}} + \|A_x\|_{L^4_{t,x}} + \|\psi_x\|_{L^4_{t,x}}^2).
\]

(6.19)

As in the Coulomb gauge

\[
-2 \int (P_M((P_{\lesssim M} A_m) \partial_m \psi_x), P_M \psi_x) dt + 2 \int (P_{\lesssim M} A_m) \partial_m (P_M \psi_x), P_M \psi_x) dt
\]

(6.20)

\[
\lesssim \|\partial_x A_x\|_{L^2_{t,x}}^2 ||P_{\lesssim M} \psi_x|_{L^4_{t,x}} ||P_M \psi_x|_{L^2_{t,x}} \lesssim \|\partial_x A_x\|_{L^2_{t,x}} \alpha(M)^2.
\]

(6.21)

Integrating by parts

\[
- \int \int (P_{\lesssim M} A_m) \partial_m |P_M \psi_x|^2 dx dt \lesssim \|\partial_x A_x\|_{L^2_{t,x}} \alpha(M)^2.
\]

(6.22)

Therefore,

\[
\|P_M \psi_x(t)\|_{L^2(R^2)}^2 \leq \beta(M)^2 + C(E_0)\alpha(M)^2 \|\nabla A_x\|_{L^2_{t,x}}
\]

(6.23)

\[
- \langle (P_M \bar{\psi}_x) \psi_x L^2_{t,x} (\|\partial_x A_x\|_{L^2_{t,x}} + \|A_t\|_{L^2_{t,x}} + \|A_x\|_{L^4_{t,x}} + \|\psi_x\|_{L^4_{t,x}}^2).
\]

This proves the theorem assuming

\[
\|\partial_x A_x(0)\|_{L^2_{t,x}} + \|A_t(0)\|_{L^2_{t,x}} + \|A_x(0)\|_{L^4_{t,x}} + \|\psi_x(0)\|_{L^4_{t,x}}^2 \lesssim \varepsilon^2.
\]

(6.24)
\[
\|\psi_x\|^2_{L^4_{t,x}} \lesssim \sum_k \alpha(k)^2 \lesssim \varepsilon^2. \tag{6.25}
\]

Combining (6.7) and (6.9),
\[
\|A_x(s)\|_{L^2_s L^\infty_x([0,\infty) \times \mathbb{R}^2)} \lesssim E_0. \tag{6.26}
\]

**Remark:** For the rest of this section \(A \lesssim B\) means \(A \lesssim E_0 B\).

**Lemma 6.4** For any \(k \geq 0\),
\[
\|\partial_x^k \psi_x(s)\|_{L^4_x(\mathbb{R}^2)} \lesssim k s^{-k/2} \|\psi_x(0)\|_{L^4_x(\mathbb{R}^2)}. \tag{6.27}
\]
\[
\|\partial_x \psi_x(s)\|^2_{L^2_x H^4_x} \lesssim \sum_k \alpha(k)^2 \lesssim \|\psi_x(0)\|^2_{L^4_{t,x}}. \tag{6.28}
\]

**Proof:** This is proved by Duhamel’s principle.
\[
\psi_x(s) = e^{s \Delta} \psi_x(0) + \int_0^s e^{(s-s') \Delta} \left[ 2i \partial_t (A_t \psi_x) - (A_x \cdot A_x + i \partial_t A_t) \psi_x + i \text{Im}(\psi_x \bar{\psi}_1) \right] (s') ds'. \tag{6.29}
\]
\[
\|\psi_x(s)\|_{L^\infty_x L^4_x} \lesssim \|\psi_x(0)\|_{L^4_x} + \|\psi_x\|^2_{L^2_s L^\infty_x} \|\psi_x\|_{L^\infty_s L^4_x} + \|A_x\|^2_{L^2_s L^\infty_x} \|\psi_x\|_{L^\infty_s L^4_x}
\]
\[
+ \|\partial_x A\|_{L^1_s L^\infty_x} \|\psi_x\|_{L^\infty_s L^4_x} + C(\delta_0) \|A_x\|_{L^1_s L^4_x} \|\psi_x\|_{L^\infty_s L^4_x} + \delta \|\psi_x\|_{L^\infty_s L^4_x}. \tag{6.30}
\]

The last inequality follows from corollary \([6.2]\) and splitting
\[
\int_0^s e^{(s-s') \Delta} \partial_t (A_t \psi_x)(s') ds' = \int_0^{(1-\delta)s} e^{(s-s') \Delta} \partial_t (A_t \psi_x)(s') ds' + \int_{(1-\delta)s}^s e^{(s-s') \Delta} \partial_t (A_t \psi_x)(s') ds'. \tag{6.32}
\]
\[
\int_{(1-\delta)s}^s \frac{1}{(s-s')^{1/2}} \frac{1}{(s')^{1/2}} ds' \lesssim \delta^{1/2}. \tag{6.33}
\]
\[
\int_0^{(1-\delta)s} \frac{1}{(s-s')^{1/2}} f(s') ds' \lesssim C(\delta) \|f\|_{L^2_s}. \tag{6.34}
\]

By theorem \([6.1]\) and corollary \([6.2]\) after partitioning \([0, \infty]\) into finitely many pieces and iterating, \(6.30\) and \(6.31\) imply
\[ \| \psi_x(s) \|_{L^4_x(R^2)} \lesssim \| \psi_x(0) \|_{L^4_x(R^2)}, \quad (6.35) \]

Likewise since the kernel of \( \partial^k_x e^{(s-s')D} \) has \( L^1 \) norm bounded by \( \frac{1}{(s-s')^{\delta}} \),

\[ \| \int_0^{(1-\delta)s} e^{(s-s')D} \partial^k_x [2i\partial_t(A_t \psi_x) - (A_x \cdot A_x + i \partial_t A_t) \psi_x + i \text{Im}(\psi_x \bar{\psi}) \psi_t](s') ds' \|_{L^4_x(R^2)} \lesssim_k s^{-k/2} \| \psi_x(0) \|_{L^4_x}. \quad (6.36) \]

Next, theorem 6.1, corollary 6.2, and an induction imply

\[ \| \int_0^s \partial^k_x e^{(s-s')D}[-(A_x \cdot A_x + i \partial_t A_t) \psi_x + i \text{Im}(\psi_x \bar{\psi}) \psi_t](s') ds' \|_{L^4_x} \lesssim \| \psi_x(0) \|_{L^4_x}. \quad (6.37) \]

\[ \lesssim \int_j^s \frac{1}{(s-s')^{1/2}} \| \psi_x(s) \|_{L^4_x} \lesssim_k s^{-k/2} \| \psi_x(0) \|_{L^4_x}. \quad (6.38) \]

Finally,

\[ \| \int_0^s \partial^k_x e^{(s-s')D} [2i\partial_t(A_t \psi_x)](s') ds' \|_{L^4_x} \lesssim_{E_0, k} s^{-k/2} \delta + s^{-1/2} s^{-k/2} \| \psi_x(0) \|_{L^4_x}. \quad (6.39) \]

Combining (6.36), (6.38), (6.39), and \( \| \partial^k_x e^{sD} \psi_x(0) \|_{L^4_x} \lesssim s^{-k/2} \| \psi_x(0) \|_{L^4_x} \) proves (6.27).

Now to prove (6.28). Estimate

\[ \| \int_{s/2}^{s} e^{(s-s')D} \partial_x [2i\partial_t(A_t \psi_x) - (A_x \cdot A_x + i \partial_t A_t) \psi_x + i \text{Im}(\psi_x \bar{\psi}) \psi_t](s') ds' \|_{L^4_x L^4_{t, \delta}([2^j, 2^{j+1}] \times R^2)} \quad (6.40) \]

By theorem 6.1, corollary 6.2, and (6.27),

\[ \lesssim \left( \| \nabla A \|_{L^4_x L^4_{t, \delta}([2^j, 2^{j+1}] \times R^2)} + \| \psi_x \|_{L^4_x L^4_{t, \delta}([2^j, 2^{j+1}] \times R^2)} + \| A_x \|_{L^4_x L^4_{t, \delta}([2^j, 2^{j+1}] \times R^2)} \right) \| \psi_x \|_{L^4_x}. \quad (6.41) \]

Next, by Sobolev embedding, theorem 6.1, corollary 6.2, and Holder’s inequality,

\[ \| \int_0^{s/2} e^{(s-s')D} \partial_x [2i\partial_t(A_t \psi_x) - (A_x \cdot A_x + i \partial_t A_t) \psi_x + i \text{Im}(\psi_x \bar{\psi}) \psi_t](s') ds' \|_{L^4_x L^4_{t, \delta}([2^j, 2^{j+1}] \times R^2)} \quad (6.42) \]
\[
\lesssim \|\psi_x\|_{L^\infty_t L^2_x} 2^{-j/2} \sum_{k \leq j} 2^{k/2} (\|A_x\|_{L^2_{t,x}}^2 |[2^k,2^{k+1}] \times \mathbb{R}^2| + \|\partial_x A\|_{L^2_{t,x}}^2 |[2^k,2^{k+1}] \times \mathbb{R}^2| + \|\psi_x\|_{L^2_{t,x}}^2 |[2^k,2^{k+1}] \times \mathbb{R}^2|).
\]

Combining (6.41) and (6.43) implies

\[
\left\| \int_0^s e^{(s-s') \Delta} \partial_t [2i \partial_t (A_t \psi_x)] - (A_x \cdot \partial_t A_t) \psi_x + i \text{Im}(\psi_x \bar{\psi}_l) \psi_l (s') ds' \right\|_{L^2_t L^2_x} \lesssim E_0\|\psi_x(0)\|_{L^4_x}. \tag{6.44}
\]

\[
\left\| \nabla e^{s \Delta} \psi_x(0) \right\|_{L^4_{t,x}}^2 \lesssim \sum_{2^k \leq \frac{s}{j}} 2^k \|P_k \psi_x(0)\|_{L^4_{t,x}}^2 + \sum_{2^k > \frac{s}{j}} \frac{2^k}{(s2^{2k})^3} \|P_k \psi_x(0)\|_{L^4_{t,x}}^2. \tag{6.45}
\]

\[
\left\| \nabla e^{s \Delta} \psi_x(0) \right\|_{L^2_t L^4_{t,x}(|[2^{-2j},2^{-2j+2}] \times \mathbb{R}^2)} \lesssim \sum_{k \geq j} 2^{k-j} \|P_k \psi_x(0)\|_{L^4_{t,x}}^2 + \sum_{k > j} 2^{5(j-k)} \|P_k \psi_x(0)\|_{L^4_{t,x}}^2. \tag{6.46}
\]

Therefore,

\[
\sum_j \|\nabla e^{s \Delta} \psi_x(0)\|_{L^4_{t,x}(|[2^{-2j},2^{-2j+2}] \times \mathbb{R}^2)}^2 \lesssim \sum_k \beta(k)^2 \lesssim \|\psi_x(0)\|_{L^4_{t,x}}^2. \tag{6.47}
\]

This gives (6.28). □

Corollary 6.5

\[
\|A_x(s)\|_{L^4_{t,x}} \lesssim \epsilon. \tag{6.48}
\]

\textbf{Proof:} This follows from lemma 6.4, theorem 6.1, corollary 6.2, and the formula

\[
A_x(s) = - \int_s^\infty \text{Im}(\bar{\psi}_x (\partial_l \psi_l + i A_l \psi_l))(r) dr. \tag{6.49}
\]

\[
\|A_x\|_{L^4_{t,x}} \lesssim \|\psi_x\|_{L^2_t L^\infty_x} (\|\partial_x \psi_x\|_{L^2_t L^4_x} + \|A_x\|_{L^2_t L^\infty_x} \|\psi_x\|_{L^\infty_t L^4_x}) \lesssim \|\psi_x(0)\|_{L^4_{t,x}}. \tag{6.50}
\]

□

Theorem 6.6

\[
\|P_k A_x(s)\|_{L^4_1(\mathbb{R}^2)} \lesssim 2^{-k}. \tag{6.51}
\]
Proof: By Bernstein’s inequality

\[ 2^{2k} \| P_k A_x(s) \|_{L^1_2} \lesssim \int_s^\infty \| \nabla^2 \psi_x(r) \|_{L^2_2} \| \nabla \psi_x(r) \|_{L^2_2} + \| \psi_x(r) \|_{L^2_2} \| \nabla^3 \psi_x(r) \|_{L^2_2} \, dr \]  

(6.52)

\[ + \int_s^\infty \| \nabla^2 \psi_x \|_{L^2_2} \| A_x \|_{L^\infty_2} \| \psi_x \|_{L^2_2} + \| \psi_x \|_{L^2_2} \| \nabla^2 A_x \|_{L^2_2} \| \psi_x \|_{L^\infty_2} + \| \psi_x \|_{L^2_2} \| A_x \|_{L^2_2} \| \nabla^2 \psi_x \|_{L^\infty_2} \, dr \]  

(6.53)

\[ \lesssim \int_s^\infty r^{-3/2} \, dr \lesssim s^{-1/2}. \]  

(6.54)

The first inequality in (6.54) follows from theorem 6.1 and corollary 6.2. So for \( s > 2^{-2k} \),

\[ \| P_k A_x(s) \|_{L^1_4(\mathbb{R}^2)} \lesssim 2^{-k}. \]  

(6.55)

For \( s < 2^{-2k} \) Holder’s inequality and (6.55) imply

\[ \| P_k A_x(s) \|_{L^1_4(\mathbb{R}^2)} \lesssim \int_s^{2^{-2k}} \| \psi_x \| \| \nabla \psi_x \|_{L^2_2} \, dr \]  

(6.56)

\[ \lesssim 2^{-k} \| \psi_x \|_{L^\infty_2} \| \nabla \psi_x \|_{L^2_2} + 2^{-k} \| \psi_x \|_{L^\infty_2} \| A_x \|_{L^2_2} \| \nabla \psi_x \|_{L^\infty_2} \lesssim 2^{-k}. \]  

(6.57)

□

Lemma 6.7

\[ \| \psi_t \|_{L^2_4 L^4_{t,x}}^2 \lesssim \sum_k \beta(k)^2 \lesssim \epsilon^2. \]  

(6.58)

Proof:

\[ \psi_t(0) = i \partial_t \psi_t(0) - A_t(0) \psi_t(0). \]  

(6.59)

As in lemma 6.4

\[ \| e^{s \Delta} (\partial_t \psi_t(0)) \|_{L^2_4 L^4_{t,x}} \lesssim \epsilon. \]  

(6.60)

By Sobolev embedding and theorem 6.6

\[ \| P_N (A_t(0) \psi_t(0)) \|_{L^4_{t,x}} \lesssim N \| P_{N \leq 20} \|_{L^4_{t,x}} \| A_t(0) \|_{L^2_2} \]  

(6.61)

\[ + N^{1/2} \| P_{N \leq 4} \psi_t(0) \|_{L^4_{t} L^\infty_x} \| P_{N \leq 4} \|_{L^2_2} \lesssim 20 \]  

(6.62)
\[ N^{3/2} \sum_{k > 0} \| P^k N \psi_x(0) \|_{L^4_{t,x}} \| P^k N A(x)(0) \|_{L^{4/3}} \]
\[ \lesssim N \alpha(N) + N \sum_{k \leq 0} 2^{k/2} \alpha(2^k N) + N \sum_{k > 0} 2^{-k/2} \alpha(2^k N) \lesssim N \alpha(N). \] (6.63)

This implies
\[ \| e^{s \Delta} \psi_t(0) \|_{L^2 L^4_{t,x}} + \| s^{1/2} e^{s \Delta} \psi_t(0) \|_{L^4_{t,x}} \lesssim \epsilon. \] (6.65)

\[ \psi_t(s) = e^{s \Delta} \psi_t(0) + \int_0^s e^{(s-s') \Delta} [2i \partial \psi_t] - (A_x \cdot A_x + i \partial \psi_t + i \text{Im}(\psi_t \psi_t^\dagger)](s')ds', \] (6.66)

Making an argument identical to the proof of lemma 6.4 proves the lemma. \( \Box \)

**Corollary 6.8**
\[ \| A_t(s) \|_{L^2_{t,x}} \lesssim \epsilon^2. \] (6.67)

**Proof:**
\[ \| A_t \|_{L^2_{t,x}} \lesssim \| \psi_t \|_{L^2_{t,x}} \| \partial_x \psi \|_{L^4_{t,x}} + \| A_x \|_{L^2 L^\infty} \| \psi_x \|_{L^2_L^\infty L^4_{t,x}} \lesssim \epsilon^2. \] (6.68)

Recall the choice of frequency envelope
\[ \alpha(k) = \sum_j 2^{-j} \| P^j \psi_x(0) \|_{L^4_{t,x}}. \] (6.69)

Let
\[ \alpha(t, k) = \sum_j 2^{-j} \| P^j \psi_x(t, 0) \|_{L^4_{t,x}}. \] (6.70)

\[ \left( \int \alpha(t, k)^4 dt \right)^{1/4} \lesssim \sum_j 2^{-j} \| P^j \psi_x(0) \|_{L^4_{t,x}} = \alpha(k). \] (6.71)

\[ \left( \sum_k \| P \psi_x(t, 0) \|_{L^4_L}^2 dt \right)^{1/2} \lesssim \sum_k \| P \psi_x(t, 0) \|_{L^4_{L^2}}^{1/2} \lesssim \sum \alpha(k)^2 \lesssim \epsilon^2. \] (6.72)

**Theorem 6.9**
\[ \| P_k \psi_x(s) \|_{L^4} \lesssim (1 + s 2^{2k})^{-4} \alpha(t, k). \] (6.73)
Proof: We start by proving \( \| P_k \psi_x(s) \|_{L^1_x} \lesssim \alpha(k) \).

\[
\| e^{\Delta} P_k \psi_x(0) \|_{L^1_x} \lesssim (1 + s 2^{2k})^{-4} \alpha(t, k).
\]

(6.74)

Make the bootstrap assumption

\[
\| P_k \psi_x(s) \|_{L^1_x} \leq C \alpha(t, k).
\]

(6.75)

\[
\| \int_0^{(1-\delta)s} e^{(s-s') \Delta} P_k [2i \partial_t (A_1 \psi_x) - (A_1 A_1 + i \partial_t A_1) \psi_x + i \text{Im} (\psi_x \bar{\psi_t} \psi)] ds' \|_{L^1_x} \lesssim e^{-\delta s 2^{2k}} \|
\]

(6.76)

\[
\lesssim e^{-\delta s 2^{2k}} \| P_{k-5} \leq s \|_{L^\infty_x \| L^1_x \| s^{1/2} 2^k \| A_x \|_{L^2_x L^\infty} + \| \partial_x A_x \|_{L^1_x L^\infty} + \| \psi_x \|_{L^2_x L^\infty} \|
\]

(6.77)

\[
+ e^{-\delta s 2^{2k}} ( \sum_{j \leq k-5} 2^{j/2} \| P_j \psi_x \|_{L^\infty_x \| L^1_x \| s^{1/2} 2^k \| P_{s-k-5} A_x \|_{L^2_x L^1_x})
\]

(6.78)

\[
+ \| P_{s-k-5} A_x \|_{L^2_x L^1_x} \| A_x \|_{L^2_x L^\infty} + \| \partial_x P_{s-k-5} A_x \|_{L^1_x L^\infty} + \| P_{s-k-5} \psi_x \|_{L^2_x L^1_x} \| \psi_x \|_{L^2_x L^\infty} \|
\]

(6.79)

Next, estimate

\[
\| \int_0^s e^{(s-s') \Delta} P_k [2i \partial_t (A_1 \psi_x) - (A_1 A_1 + i \partial_t A_1) \psi_x + i \text{Im} (\psi_x \bar{\psi_t} \psi)] ds' \|_{L^1_x} \leq \alpha(k).
\]

(6.80)

By Sobolev embedding and integration,

\[
\lesssim \| P_{k-5} \leq s \|_{L^\infty_x \| L^1_x \| \left( \int_0^{\delta s} e^{-s'^2 2^{2k}} ds' \right)^{1/2} \| s^{1/2} A_x \|_{L^\infty_x} + \delta \| s \|_{L^\infty_x} + \| \partial_x A_x \|_{L^\infty_x} + \| \psi_x \|_{L^\infty_x}^2
\]

(6.81)

\[
+ \left( \sum_{j \leq k-5} 2^{j/2} \| P_j \psi_x \|_{L^\infty_x \| L^1_x \| s^{1/2} P_{s-k-5} A_x \|_{L^\infty_x \| L^1_x} \left( \int_0^{\delta s} e^{-s'^2 2^{2k}} ds' \right)^{1/2}
\]

(6.82)
\[ +2^k \sum_{j \geq k} \|P_j \psi_x\|_{L^\infty \times L^2} [\delta^{1/2}\|s^{1/2}P_j A_x\|_{L^\infty \times L^2} (\int_0^{\delta s} e^{-s^{2}2^k} 2^k ds')^{1/2} + \delta]\|s^{1/2}P_{\geq j} A_x\|_{L^\infty \times L^2} + \delta\|s\partial_x P_{\geq j} A_x\|_{L^\infty \times L^2} + \delta\|s^{1/2}P_{\geq j} \psi_x\|_{L^\infty \times L^2}\|s^{1/2} \psi_x\|_{L^\infty \times L^2} \]

\[ \lesssim C\alpha(t, k)\delta^{1/2}. \]  

(6.84)

The last inequality follows from Bernstein’s inequality, the bootstrap assumption, theorem 6.1, and corollary 6.2. Partitioning \([0, \infty)\) into finitely many intervals \(I_j\) for each \(t\) such that

\[ \|A_x\|_{L^2 \times L^\infty(I_j \times \mathbb{R}^2)} + \|\partial_x A_x\|_{L^1 \times L^\infty(I_j \times \mathbb{R}^2)} + \|\psi_x\|_{L^2 \times L^\infty(I_j \times \mathbb{R}^2)} + \|\partial_x A_x\|_{L^2 \times L^\infty(I_j \times \mathbb{R}^2)} \]

is small on each \(I_j\) and iterating,

\[ \|P_k \psi_x(s, t)\|_{L^4} \lesssim \alpha(t, k). \]  

(6.86)

This in turn implies

\[ \|P_k \psi_x(s)\|_{L^4} \lesssim \alpha(k). \]  

(6.87)

To prove (6.73) it only remains to consider \(s > 2^{-2k} \) \( e^{-s^{2}2^k} s^{1/2}2^k \lesssim (1 + s2^k)^{-4}\), which takes care of (6.77), (6.78), and (6.79). Now make the bootstrap assumption

\[ \|P_k \psi_x(s)\|_{L^4} \leq C(1 + s2^k)^{-4}\alpha(k). \]  

(6.88)

Plugging this in to (6.81) and (6.83)

\[ (6.81) + (6.83) \lesssim C\delta^{1/2}(1 + s2^k)^{-4}\alpha(k). \]  

(6.89)

By theorem 6.1, corollary 6.2, and Bernstein’s inequality,

\[ \delta^{1/2}\|s^{1/2}P_{\geq k-5} A_x\|_{L^\infty \times L^2} + \delta\|s^{1/2}P_{\geq k-5} A_x\|_{L^\infty \times L^2}\|s^{1/2} A_x\|_{L^\infty \times L^2} \]

\[ \lesssim s^{-4}2^{-17k/2}\delta^{1/2}\|s^{9/2}\partial_x^9 A_x\|_{L^\infty \times L^2}\|s^{9/2}\partial_x^8 A_x\|_{L^\infty \times L^2} \]

\[ + s^{-4}2^{-17k/2}\delta^{1/2}\|s^{9/2}\partial_x^9 A_x\|_{L^\infty \times L^2}\|s^{9/2}\partial_x^8 A_x\|_{L^\infty \times L^2}\|s^{1/2} A_x\|_{L^\infty \times L^2} \]

(6.90)

(6.91)

(6.92)

(6.93)
Proof:

$$
\| P_k \psi_x(s) \|_{L^4_{t,x}} \lesssim \alpha(k), \quad \text{when } s > 2^{-2k}
$$

(6.82) \lesssim (1 + s 2^{2k})^{-4}. \tag{6.96}

This completes the proof of the theorem. \(\Box\)

**Corollary 6.10**

$$
2^k \| P_k A_x(s) \|_{L^2_{t,x}} \lesssim \epsilon \alpha(k). \tag{6.97}
$$

**Proof:**

$$
A_x(s) = - \int_s^\infty \text{Im}(\bar{\psi}_x (\partial_t \psi_l + i A_l \psi_l)) dr.
$$

(6.98)

$$
\| P_k A_x(s) \|_{L^2_{t,x}} \lesssim \int_s^\infty \| P_{k-5 \leq r \leq k+5} \psi_x(r) \|_{L^4_{t,x}} (\| \partial_x \psi_x(r) \|_{L^4_{t,x}} + \| A_x(r) \|_{L^\infty} \| \psi_x(r) \|_{L^4_{t,x}}) dr
$$

+ \int_s^\infty \| P_{k-5 \leq r \leq k+5} \psi_x(r) \|_{L^4_{t,x}} \| \partial_x P_{k-5 \leq r \leq k+5} \psi_x(r) \|_{L^4_{t,x}} dr \tag{6.100}

(6.99)

+ \int_s^\infty \| P_{k-5 \leq r \leq k+5} \psi_x(r) \|_{L^4_{t,x}} \| P_{k-5 \leq r \leq k+5} \psi_x(r) \|_{L^4_{t,x}} \| A_x(r) \|_{L^\infty} + \| P_{k-5 \leq r \leq k+5} \psi_x(r) \|_{L^4_{t,x}}^2 \| P_{k-5 \leq r \leq k+5} A_x(r) \|_{L^2_{t,x}} \tag{6.101}

+ \sum_{j \geq k+5} \int_s^\infty 2^j \| P_j \psi_x(r) \|_{L^4_{t,x}}^2 + \| P_j \psi_x(r) \|_{L^4_{t,x}} \| P_{j \leq r \leq k+5} \psi_x(r) \|_{L^4_{t,x}} \| A_x(r) \|_{L^\infty} \tag{6.102}

+ \| P_j \psi_x(r) \|_{L^4_{t,x}} \| \psi_x(r) \|_{L^4_{t,x}} \| P_{j \leq r \leq k+5} A_x(r) \|_{L^\infty} dr.
$$

(6.99) \lesssim \left( \int_0^\infty (1 + s 2^{2k}) \alpha(k) ds \right)^{1/2} (\| \partial_x \psi_x \|_{L^2_{t,x} L^4_{x}} + \| A_x \|_{L^2_{t,x} L^\infty_{x}} \| \psi_x \|_{L^\infty_{t,x} L^4_{x}}) \lesssim 2^{-k} \alpha(k) \epsilon. \tag{6.103}

(6.100) \lesssim \sum_{j \geq k-5} \epsilon \int_0^\infty (1 + s 2^{2j}) \alpha(j) ds \lesssim 2^{-k} \epsilon \alpha(k). \tag{6.104}

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By Bernstein’s inequality, theorem 6.1, corollary 6.2,

\begin{equation}
(6.101) \lesssim \epsilon \left( \sum_j \int_0^\infty (1 + s 2^j)^{-\theta} ds \right)^{1/2} \| A_x \|_{L^2_t L^\infty_x} + \alpha(k)^2 2^{-k} \| \partial^2_x A_x \|_{L^1_t L^2_x} \lesssim 2^{-k} \epsilon \alpha(k).
\end{equation}

\begin{equation}
(6.102) \lesssim \sum_{j \geq k+5} 2^j \epsilon \left( \int_0^\infty (1 + s 2^j)^{-4} ds \right) + \epsilon \left( \int_0^\infty (1 + s 2^j)^{-8} ds \right)^{1/2} \| A_x \|_{L^2_t L^\infty_x} \lesssim 2^{-k} \alpha(k) \epsilon.
\end{equation}

\[\square\]

In conclusion this proves

\[\| A_x \|_{L^4_{t,x}}^2 + \| \psi_x \|_{L^4_{t,x}}^2 + \| \partial_x A_x \|_{L^2_{t,x}} + \| A_t \|_{L^2_{t,x}} \lesssim \epsilon^2. \]  

(6.107)

This completes the proof of theorem 6.3. \[\square\]

Performing an identical calculation to the one performed in the case of the Coulomb gauge, the error involving terms of the form

\[P_M((P_{\leq \frac{M}{100}} A_t) \partial_t \psi_x)\]

is bounded by \(C(E_0)\epsilon(\alpha(M) + \beta(M))^2(\alpha(N) + \beta(N))^2\).

\[M\| P_{\geq \frac{M}{100}} A_x \|_{L^2_{t,x}} + \| \partial_x A_x \|_{L^2_{t,x}} + \| A_t \|_{L^2_{t,x}} + \| A_x \|_{L^4_{t,x}}^2 + \| \psi_x \|_{L^4_{t,x}}^2 \lesssim \epsilon^2. \]  

(6.109)

Therefore in the caloric gauge the error is bounded by

\[\| (P_M \psi_x)(P_N \bar{\psi}_x) \|_{L^2_{t,x}} \lesssim \left( \frac{M}{N} \right)^{1/2} (\alpha(M) + \beta(M))(\alpha(N) + \beta(N))\]

\[+ \left( \frac{M}{N} \right)^{1/2} C(E_0) \epsilon \sum_k \| (P_M \psi_x)(P_{2^k M} \bar{\psi}_x) \|_{L^2_{t,x}} (\alpha(N) + \beta(N)) \]

\[+ \left( \frac{M}{N} \right)^{1/2} C(E_0) \epsilon \sum_k \| (P_N \psi_x)(P_{2^k N} \bar{\psi}_x) \|_{L^2_{t,x}} (\alpha(M) + \beta(M)). \]  

(6.110)  

(6.111)  

(6.112)

**Theorem 6.11** For \(M \ll N\),

\[\| (P_M \psi_x)(P_N \bar{\psi}_x) \|_{L^2_{t,x}} \lesssim \left( \frac{M}{N} \right)^{1/2} (\alpha(M) + \beta(M))(\alpha(N) + \beta(N)). \]  

(6.113)
Proof: From the Morawetz estimates if \( M << N \),

\[
\| (P_M \psi_x)(P_N \bar{\psi}_x) \|_{L_{t,x}^2} \lesssim \left( \frac{M}{N} \right)^{1/2} (\alpha(M) + \beta(M)) (\alpha(N) + \beta(N))
\] (6.114)

\[
+ \left( \frac{M}{N} \right)^{1/2} C(E_0) \epsilon \sum_k \| (P_M \psi_x)(P_{2^k M} \bar{\psi}_x) \|_{L_{t,x}^2} (\alpha(N) + \beta(N))
\] (6.115)

\[
+ \left( \frac{M}{N} \right)^{1/2} C(E_0) \epsilon \sum_k \| (P_N \psi_x)(P_{2^k N} \bar{\psi}_x) \|_{L_{t,x}^2} (\alpha(M) + \beta(M)).
\] (6.116)

Therefore,

\[
\sum_{M,N,M << N} \| (P_M \psi_x)(P_N \bar{\psi}_x) \|_{L_{t,x}^2} \lesssim \sum_M (\alpha(M) + \beta(M))^2
\] (6.117)

\[
+ C(E_0) \epsilon \sum_{M \leq N} \left( \frac{M}{N} \right)^{1/2} \sum_k \| (P_M \psi_x)(P_{2^k M} \bar{\psi}_x) \|_{L_{t,x}^2} (\alpha(N) + \beta(N))
\] (6.118)

\[
+ C(E_0) \epsilon \sum_{M \leq N} \left( \frac{M}{N} \right)^{1/2} \sum_k \| (P_N \psi_x)(P_{2^k N} \bar{\psi}_x) \|_{L_{t,x}^2} (\alpha(M) + \beta(M))
\] (6.119)

\[
\lesssim \sum_M (\alpha(M) + \beta(M))^2 + C(E_0) \epsilon \sum_M \sum_k \| (P_M \psi_x)(P_{2^k M} \bar{\psi}_x) \|_{L_{t,x}^2} (\alpha(M) + \beta(M)) \lesssim \epsilon^2.
\] (6.120)

The last inequality follows from taking \( \epsilon > 0 \) sufficiently small and absorbing the second term into the right hand side. This in turn implies

\[
\sum_N \| (P_M \psi_x)(P_N \bar{\psi}_x) \|_{L_{t,x}^2} \lesssim (\alpha(M) + \beta(M))^2
\] (6.121)

\[
+ (\alpha(M) + \beta(M)) C(E_0) \epsilon \sum_N \| (P_M \psi_x)(P_N \bar{\psi}_x) \|_{L_{t,x}^2} + C(E_0) \epsilon^2 (\alpha(M) + \beta(M)).
\] (6.122)

Once again absorbing the second term into the right hand side

\[
\sum_N \| (P_M \psi_x)(P_N \bar{\psi}_x) \|_{L_{t,x}^2} \lesssim \epsilon (\alpha(M) + \beta(M)).
\] (6.123)

Plugging in this inequality gives the theorem. □
7 Bilinear Estimates for $s > 0$

Next we seek to estimate

$$\| (P_k \psi_x(t)) (P_j \tilde{\psi}_x(t)) \|_{L^2_t L^2_x}$$

for $s > 0$. Define the double envelope at $s = 0$

$$\gamma(t, k, l) = \sum_{j_1,j_2} 2^{-2\delta |j_1-k|} 2^{-2\delta |j_2-l|} 2^{j_1-j_2} 2^{s/2} \| (P_{j_1} \psi_x(t)) (P_{j_2} \tilde{\psi}_x(t)) \|_{L^2_t}.$$  

(7.2)

This implies

$$2^{l-1/2} \| (P_k \psi_x)(P_j \tilde{\psi}_x) \|_{L^2_t L^2_x} \lesssim (\int \gamma(t, k, l)^2 dt)^{1/2}.$$  

(7.4)

Also,

$$\int \gamma(t, k, l)^2 dt)^{1/2} \lesssim \sum_{j_1,j_2} 2^{-2\delta |j_1-k|} 2^{-2\delta |j_2-l|} 2^{j_1-j_2} \| (P_{j_1} \psi_x(t)) (P_{j_2} \tilde{\psi}_x(t)) \|_{L^2_t t,x}.$$  

(7.3)

We also have the estimates

$$\gamma(t, k + 1, l), \gamma(t, k - 1, l) \leq 2^{2\delta} \gamma(t, k, l) \quad \gamma(t, k + 1, l), \gamma(t, k, l - 1) \leq 2^{2\delta} \gamma(t, k, l).$$  

(7.6)

Now recall Duhamel's principle. If $\psi_x(s)$ solves the harmonic map heat flow

$$\| (P_k \psi_x(s,t)) (P_j \tilde{\psi}_x(0,t)) \|_{L^2_t L^2_x} \lesssim \| P_k (e^{s\Delta} \psi_x(0,t)) (P_j \tilde{\psi}_x(0,t)) \|_{L^2_t}$$

(7.7)

$$+ \| P_k \partial_t (\int_0^s e^{(s-s') \Delta} A_t \psi_x(s', t) ds') (P_j \tilde{\psi}_x(0,t)) \|_{L^2_t}.$$  

(7.8)

$$+ \| P_k (\int_0^s e^{(s-s') \Delta} (\partial_t A_t) \psi_x(s', t) ds') (P_j \tilde{\psi}_x(0,t)) \|_{L^2_t}.$$  

(7.9)

$$+ \| P_k (\int_0^s e^{(s-s') \Delta} A_t A_t \psi_x(s', t) ds') (P_j \tilde{\psi}_x(0,t)) \|_{L^2_t}.$$  

(7.10)

$$+ \| P_k (\int_0^s e^{(s-s') \Delta} Im (\psi_x \bar{\psi}_l) \psi_l(s', t) ds') (P_j \tilde{\psi}_x(0,t)) \|_{L^2_t}.$$  

(7.11)
Theorem 7.1

\[ \|(P_k \psi_x(s))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \lesssim 2^{-|k-l|/2}(1 + s2^{2k})^{-4}(\alpha(k) + \beta(k))(\alpha(l) + \beta(l)). \]  

(7.12)

Proof: We have already proved this theorem when \(|k - l| \leq 10\) and for any \(k, l\) when \(s = 0\). As usual we start by proving

\[ \|(P_k \psi_x(s))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \lesssim 2^{-|k-l|/2} \gamma(t, k, l). \]  

(7.13)

Next, by Bernstein’s inequality, Sobolev embedding, theorem 6 and corollary 6.1.

\[ \|(P_k \psi_x(s))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \leq C \gamma(t, k, l). \]  

(7.15)

\[ (\int_0^s e^{-s'2^{2k}}ds')^{1/2} \left( \sup_{0 < s' < s} \|(P_{-5 \leq \leq k+5} \psi_x(s'))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \right) |A_x| \|A_x\|_{L^2_{t} L^\infty} \]  

(7.16)

\[ + \left( \sup_{0 < s' < s} \|(P_{-5 \leq \leq k+5} \psi_x(s'))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \right) |\partial_x A_x| \|A_x\|_{L^1_{t} L^\infty} \]  

(7.17)

\[ + \left( \sup_{0 < s' < s} \|(P_{-5 \leq \leq k+5} \psi_x(s'))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \right) |\psi_x| \|\psi_x\|_{L^2_{t} L^\infty} \]  

(7.18)

\[ \lesssim C \gamma(t, k, l) 2^{-|k-l|/2} (\|A_x\|_{L^2_{t} L^\infty} + |\partial_x A_x| \|L^1_{t} L^\infty| + \|A_x\|_{L^2_{t} L^\infty} + \|\psi_x\|_{L^2_{t} L^\infty}). \]  

(7.20)

Next, by Bernstein’s inequality, Sobolev embedding, theorem 6.1 and corollary 6.2

\[ \sum_{j \geq k+5} 2^j \left( \sup_{0 < s' < s} \|(P_j \psi_x(s'))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \right) |P_j A_x| \|A_x\|_{L^1_{t} L^\infty} \]  

(7.21)

\[ + \sum_{j \geq k+5} 2^j \left( \sup_{0 < s' < s} \|(P_j \psi_x(s'))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \right) |\partial_x P_j A_x| \|A_x\|_{L^1_{t} L^\infty} \]  

(7.22)

\[ + \sum_{j \geq k+5} 2^j \left( \sup_{0 < s' < s} \|(P_j \psi_x(s'))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \right) |P_{\geq j} A_x| \|A_x\|_{L^2_{t} L^\infty} \]  

(7.23)

\[ + \sum_{j \geq k+5} 2^j \left( \sup_{0 < s' < s} \|(P_j \psi_x(s'))(P_l \tilde{\psi}_x(0))\|_{L^2_x} \right) |P_{\geq j} \psi_x| \|\psi_x\|_{L^2_{t} L^\infty} \]  

(7.24)
Also by Bernstein’s inequality, Sobolev embedding, theorem 6.1 and corollary 6.2

\[ \sum_{j<k-5} \inf(2^j + 2^l, 2^k)(\sup_{0<s'<s}(P_j \psi_x(s'))(P_l \bar{\psi}_x(0))\|P_{>k-5} A_x\|_{L^2_x}) \]  
\[ + \sum_{j<k-5} \inf(2^j + 2^l, 2^k)(\sup_{0<s'<s}(P_j \psi_x(s'))(P_l \bar{\psi}_x(0))\|P_{>k-5} \partial_x A_x\|_{L^2_x}) \]  
\[ + \sum_{j<k-5} \inf(2^k, 2^j + 2^l)(\sup_{0<s'<s}(P_j \psi_x(s'))(P_l \bar{\psi}_x(0))\|A_x\|_{L^2_x L^\infty}) \]  
\[ + \sum_{j<k-5} \inf(2^k, 2^j + 2^l)(\sup_{0<s'<s}(P_j \psi_x(s'))(P_l \bar{\psi}_x(0))\|\psi_x\|_{L^2_x L^\infty}) \]  
\[ \leq C \gamma(t, k, l)2^{-|k-l|/2}(\|\partial_x A_x\|_{L^2_x} + \|A_x\|_{L^2_x L^\infty} + \|\partial_x A_x\|_{L^2_x} + \|\psi_x\|_{L^2_x L^\infty}). \]  

(7.26)

(7.27)

(7.28)

(7.29)

Partitioning \([0, \infty)\) and iterating over each piece proves (7.13). To prove (7.12) it remains to prove some decay in \(s\) when \(s > 2^{-2k}\).

\[ \|P_k(\int_0^{(1-\delta)s} e^{(s-s')\Delta}[2iA_t \partial_t \psi_x(s') - (A_t A_t + i \partial_t A_t) + Im(\psi_x \bar{\psi}_x)\psi_x]ds')(P_l \psi_x(0))\|_{L^2_x} \]  
\[ \leq e^{-\delta s2^{2k}}2^{-|k-l|/2}\gamma(t, k, l) \leq (1 + s2^{2k})^{-4}2^{-|k-l|/2}\gamma(t, k, l). \]  

(7.31)

Make the bootstrap assumption

\[ \|P_k(\psi_x(s))P_l(\psi_x(0))\|_{L^2_x} \leq C(1 + s2^{2k})^{-4}\gamma(t, k, l). \]  

(7.32)

When \(k \leq l\),

\[ \|P_l(\psi_x(0))(\int_0^{s} e^{(s-s')\Delta}P_k[2i\partial_t(A_t \psi_x) - (A_t A_t + i \partial_t A_t)\psi_x + Im(\psi_x \bar{\psi}_x)\psi_x](s')ds')\|_{L^2_x} \]  
\[ \leq C(1 + s2^{2k})^{-4}\gamma(t, k, l)[\delta^{1/2}\|s^{1/2} A_x\|_{L^\infty} + \delta\|s \partial_x A_x\|_{L^\infty} + \delta\|s^{1/2} A_x\|_{L^\infty}^2 + \delta\|s^{1/2} \psi_x\|_{L^\infty}^2]. \]  

(7.33)

(7.34)
When \( k > l \),

\[
\sum_{j \leq k-5} 2^{-(j-l)/2} C \gamma(t, j, l) 2^{-sk} s^{-4} [s^{9/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} A_x]_{L^\infty_{s,x}} \leq C(1 + s2^{2k})^{-4} \gamma(t, k, l) [s^{1/2} A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \psi_x]_{L^\infty_{s,x}} \leq \sum_{j \leq k-5} 2^{-(j-l)/2} C \gamma(t, j, l) 2^{-sk} s^{-4} [s^{9/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} A_x]_{L^\infty_{s,x}} \leq C \gamma(t, k, l) [s^{1/2} A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \psi_x]_{L^\infty_{s,x}}.
\] (7.35)

\[

\sum_{j \leq k-5} 2^{-(j-l)/2} C \gamma(t, j, l) 2^{-sk} s^{-4} [s^{9/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} A_x]_{L^\infty_{s,x}} \leq C(1 + s2^{2k})^{-4} \gamma(t, k, l) [s^{1/2} A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \psi_x]_{L^\infty_{s,x}} \leq \sum_{j \leq k-5} 2^{-(j-l)/2} C \gamma(t, j, l) 2^{-sk} s^{-4} [s^{9/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} A_x]_{L^\infty_{s,x}} \leq C \gamma(t, k, l) [s^{1/2} A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \psi_x]_{L^\infty_{s,x}}.
\] (7.36)

\[

\sum_{j \leq k-5} 2^{-(j-l)/2} C \gamma(t, j, l) 2^{-sk} s^{-4} [s^{9/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} A_x]_{L^\infty_{s,x}} \leq C(1 + s2^{2k})^{-4} \gamma(t, k, l) [s^{1/2} A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \psi_x]_{L^\infty_{s,x}} \leq \sum_{j \leq k-5} 2^{-(j-l)/2} C \gamma(t, j, l) 2^{-sk} s^{-4} [s^{9/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} A_x]_{L^\infty_{s,x}} \leq C \gamma(t, k, l) [s^{1/2} A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \partial_x^8 A_x]_{L^\infty_{s,x}} + \delta [s^{1/2} \psi_x]_{L^\infty_{s,x}}.
\] (7.37)

\ \ □

We can integrate from 0 to \( s' \) with a fixed \( s > 0 \) in exactly the same manner. This completes the proof of theorem (7.3). □

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