Mean-Field Backward Stochastic Differential Equations Driven by Fractional Brownian Motion

Yu Feng SHI
Institute for Financial Studies and School of Mathematics, Shandong University,
Jinan 250100, P. R. China
E-mail: yfshi@sdu.edu.cn

Jia Qiang WEN
Department of Mathematics, Southern University of Science and Technology,
Shenzhen 518055, P. R. China
E-mail: wenjq@sustech.edu.cn

Jie XIONG
Department of Mathematics and SUSTech International Center for Mathematics,
Southern University of Science and Technology, Shenzhen 518055, P. R. China
E-mail: xiongj@sustech.edu.cn

Abstract In this paper, we study a new class of equations called mean-field backward stochastic differential equations (BSDEs, for short) driven by fractional Brownian motion with Hurst parameter $H > 1/2$. First, the existence and uniqueness of this class of BSDEs are obtained. Second, a comparison theorem of the solutions is established. Third, as an application, we connect this class of BSDEs with a nonlocal partial differential equation (PDE, for short), and derive a relationship between the fractional mean-field BSDEs and PDEs.

Keywords Mean-field backward stochastic differential equation, fractional Brownian motion, partial differential equation

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1 Introduction

A centered Gaussian process $B^H = \{B^H_t, t \geq 0\}$ is called a fractional Brownian motion (fBm, for short) with Hurst parameter $H \in (0, 1)$ if its covariance is

$$E(B^H_t B^H_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$
When $H = 1/2$, this process becomes a classical Brownian motion. For $H > 1/2$, $B^H_t$ exhibits the property of long-range dependence, which makes fBm an important driving noise in many fields such as finance, telecommunication networks, and physics.

In 1990, Pardoux and Peng [22] introduced the nonlinear backward stochastic differential equations (BSDEs, for short). In the next two decades, it had been widely used in different fields of mathematical finance (see El Karoui et al. [10]), stochastic control (see Yong and Zhou [29]), and partial differential equations (see Bardoux and Peng [23]). At the same time, for better applications, BSDE itself has been developed into many different branches. For example, recently, BSDEs driven by fractional Brownian motion, also known as fractional BSDEs, were studied by Hu and Peng [16], where they proved the existence and uniqueness of solutions when the Hurst parameter $H > 1/2$. Then Maticiuc and Nie [20] obtained some general results of fractional BSDEs through a rigorous approach. Buckdahn and Jing [7] studied the fractional mean-field stochastic differential equations (SDEs, for short) with $H > 1/2$ and a stochastic control problem. Some other recent developments of fractional BSDEs can be found in Bender [1], Borkowska [2], Han et al. [12], Hu et al. [13], Hu et al. [14], Hu, Nualart and Song [15], Jing [17], Wen et al. [25], Wen and Shi [26–28], etc., among theory and applications. Furthermore, as a natural extension of BSDEs, Buckdahn et al. [3] and Buckdahn et al. [5] introduced the so-called mean-field BSDEs, owing to that mathematical mean-field approaches play an important role in many fields, such as Economics, Physics and Game Theory (see Lasry and Lions [19] and the papers therein). For a recent development of this theory, we refer the reader to Buckdahn et al. [6], Li [18], and Shi et al. [24], etc.

As another important development of BSDEs, the fractional mean-field BSDEs have important applications in stochastic optimal control problems and partial differential equations (PDEs, for short). However, to our best knowledge, few works about the fractional mean-field BSDEs are available up to now. Therefore, in this paper, we introduce and study the fractional mean-field BSDEs. Specifically, we consider the following equations:

$$Y_t = \xi + \int_t^T E^f(f(s, \eta_s, Y'_s, Z'_s, Y_s, Z_s))ds - \int_t^T Z_s dB_s^H, \quad 0 \leq t \leq T. \quad (1.1)$$

We point out that the stochastic integral in (1.1) is the divergence type integral (see Decreusefond and Ústünel [8], and Nualart [21]). In particular, we consider the case of the Hurst parameter $H > 1/2$. First, two different methods are proposed to prove the existence and uniqueness of Eq. (1.1). Interestingly, the conditions required by the first method, which is introduced by Maticicu and Nie [20], are weaker than the second one, which is introduced in this paper; and inverse, however, the second method is more convenient than the first one. Second, for its wide applications to BSDEs, a comparison theorem of such fractional BSDEs is obtained. Finally, as an application, we connect this mean-field BSDE with a nonlocal PDE. Furthermore, it should be pointed out that, similar to fractional mean-field SDEs studied in [7], our fractional mean-field BSDEs can also be applied to stochastic optimal controls problem (see Douissi et al. [9]).

We organize this article as follows. In Section 2, some preliminaries about fBm are presented. The existence and uniqueness of the fractional mean-field BSDEs are proved by two different methods in Section 3. We derive a comparison theorem for such fractional BSDE in Section 4,
and connect this class of BSDEs with a nonlocal PDE in Section 5.

2 Preliminaries

We recall, in this section, some basic results of fractional Brownian motion. For a deeper discussion, the readers may refer to the articles such as Decreusefond and Üstünel [8], Hu [11] and Nualart [21].

Assume $B^H = \{B^H_t, t \geq 0\}$ is an fBm defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and the filtration $\mathcal{F}$ is generated by $B^H$. Let $H > 1/2$ throughout this paper. Moreover, we denote $\phi(x) = H(2H - 1)|x|^{2H - 2}$, where $x \in \mathbb{R}$, and suppose $\xi$ and $\eta$ be two continuous functions defined in $[0, T]$. Define

$$\langle \xi, \eta \rangle_T = \int_0^T \int_0^T \phi(u - v)\xi_u\eta_v\,dudv,$$

and $\|\xi\|^2_T = \langle \xi, \xi \rangle_T$. (2.1)

Then, $\langle \xi, \eta \rangle_T$ is a Hilbert scalar product. Under this scalar product, we denote by $H$ the completion of the continuous functions. Besides, denote by $\mathcal{P}_T$ the set of all polynomials of fBm in $[0, T]$, i.e., every element of $\mathcal{P}_T$ is of the form

$$\Phi(\omega) = h\left(\int_0^T \xi_1(t)dB^H_t, \ldots, \int_0^T \xi_n(t)dB^H_t\right),$$

where $h$ is a polynomial function and $\xi_i \in \mathcal{H}, i = 1, 2, \ldots, n$. In addition, Malliavin derivative operator $D^H_s$ of $\Phi \in \mathcal{P}_T$ is defined by

$$D^H_s\Phi = \sum_{i=1}^n \frac{\partial h}{\partial x_i}\left(\int_0^T \xi_1(t)dB^H_t, \ldots, \int_0^T \xi_n(t)dB^H_t\right)\xi_i(s), \quad s \in [0, T].$$

Since the derivative operator $D^H : L^2(\Omega, \mathcal{F}, P) \rightarrow (\Omega, \mathcal{F}, \mathcal{H})$ is closable, one can denote by $\mathbb{D}^{1,2}$ the completion of $\mathcal{P}_T$ under the following norm

$$\|\Phi\|^2_{1,2} = E|\Phi|^2 + E\|D^H_s\Phi\|^2_T.$$

Furthermore, we introduce the following derivative

$$\mathbb{D}^H_t \Phi = \int_0^T \phi(t - s)D^H_s\Phi\,ds, \quad t \in [0, T].$$

Now, let us consider the adjoint operator of Malliavin derivative operator $D^H$. We call this operator the divergence operator, which represents the divergence type integral and is denoted by $\delta(\cdot)$.

**Definition 2.1** A process $u \in L^2(\Omega \times [0, T]; \mathcal{H})$ is said to belong to the domain $\text{Dom}(\delta)$, if there exists $\delta(u) \in L^2(\Omega, \mathcal{F}, P)$ satisfying the following duality relationship:

$$E(\Phi\delta(u)) = E(\langle D^H_s\Phi, u \rangle_T), \quad \text{for every } \Phi \in \mathcal{P}_T.$$

Moreover, if $u \in \text{Dom}(\delta)$, the divergence type integral of $u$ with respect to $B^H$ is defined by putting $\int_0^T u_s\,dB^H_s =: \delta(u)$.

It should be pointed out that, in this paper, unless otherwise specified, the $dB^H$-integral represents the divergence type integral.
Proposition 2.2 (Hu [11, Proposition 6.25])  Let $L_H^{1,2}$ be the space of all processes $F : \Omega \times [0, T] \to \mathcal{H}$ satisfying $E(\|F\|^2_T + \int_0^T \|D_s^H F_t\|^2 dsdt) < \infty$. Then, if $F \in L_H^{1,2}$, the divergence type integral $\int_0^T F_s dB^H_s$ exists in $L^2(\Omega, \mathcal{F}, P)$, and

$$E\left(\int_0^T F_s dB^H_s\right) = 0; \quad E\left(\int_0^T F_s dB^H_s\right)^2 = E\left(\|F\|_T^2 + \int_0^T \int_0^T D_s^H F_t D_t^H F_s dsdt\right).$$

Proposition 2.3 (Hu [11, Theorem 10.3])  Suppose $g$ and $f$ are two deterministic continuous functions. Let

$$X_t = X_0 + \int_0^t g_s ds + \int_0^t f_s dB^H_s, \quad t \in [0, T],$$

where $X_0$ is a constant. Then, if $F \in C^{1,2}([0, T] \times \mathbb{R})$, one has

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) g_s ds$$

$$+ \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) \left[\frac{d}{ds}\|f_s\|_2^2\right] ds, \quad t \in [0, T].$$

Proposition 2.4 (Hu [11, Theorem 11.1])  For $i = 1, 2$, let $g_i$ and $f_i$ be two real-valued processes satisfying $E\int_0^T (|g_i(s)|^2 + |f_i(s)|^2) ds < \infty$. Moreover, assume that $D^H_t f_i(s)$ is continuously differentiable in its arguments $(s, t) \in [0, T]^2$ for almost every $\omega \in \Omega$, and $E\int_0^T \int_0^T \|D^H_t f_i(s)\|^2 dsdt < \infty$. Denote

$$X_i(t) = \int_0^t g_i(s) ds + \int_0^t f_i(s) dB^H_s, \quad t \in [0, T].$$

Then

$$X_1(t)X_2(t) = \int_0^t X_1(s) g_2(s) ds + \int_0^t X_1(s) f_2(s) dB^H_s + \int_0^t X_2(s) g_1(s) ds$$

$$+ \int_0^t X_2(s) f_1(s) dB^H_s + \int_0^t \int_0^t \|D^H_{s} X_1(s) g_2(s) + \int_0^t \int_0^t \|D^H_{s} X_2(s) g_1(s) ds.$$

3 Existence and Uniqueness

In this section, we study the existence and uniqueness of the fractional mean-field BSDEs. For simplify of presentation, the case of one dimension is investigated. Let

$$\eta_t = \eta_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB^H_s, \quad (3.1)$$

where $\eta_0$ is a constant, and $b$ and $\sigma$ are two deterministic differentiable functions, such that $\sigma_t \neq 0$ (then either $\sigma_t < 0$ or $\sigma_t > 0$, $t \in [0, T]$). We recall that (see (2.1))

$$\|\sigma\|_2^2 = H(2H - 1) \int_0^t \int_0^t |u - v|^{2H-2} \sigma_u \sigma_v du dv.$$ 

So $\delta_t(\|\sigma\|_2^2) = 2\delta_t \sigma_t > 0$ for $t \in (0, T]$, where $\delta_t = \int_0^t \phi(t - v) \sigma_v dv$.

Now, we denote the (non-completed) product space of $(\Omega, \mathcal{F}, P)$ by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P)$, and denote the filtration of this product space by $\mathcal{F}_t = \mathcal{F} \otimes \mathcal{F}_t, 0 \leq t \leq T$. A random variable, originally defined on $\Omega$, $\xi \in L^0(\Omega, \mathcal{F}, P; \mathbb{R})$ is canonically extended to $\tilde{\Omega}$: $\xi'(\omega', \omega) = \xi(\omega'), (\omega', \omega) \in \tilde{\Omega} = \Omega \times \Omega$. On the other hand, for every $\theta \in L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, the
random variable $\theta(\cdot, \omega) : \Omega \rightarrow \mathbb{R}$ is in $L^1(\Omega, \mathcal{F}, P)$, $P(d\omega)$, a.s., and its expectation is denoted by

$$E'[\theta(\cdot, \omega)] = \int_{\Omega} \theta(\omega', \omega)P(d\omega').$$

Then we have $E'[\theta] = E'[\theta(\cdot, \omega)] \in L^1(\Omega, \mathcal{F}, P)$. In addition,

$$\tilde{E}[\theta] = \int_{\Omega} \theta d\tilde{P} = \int_{\Omega} E'[\theta(\cdot, \omega)]P(d\omega) = E[E'[\theta]].$$

Motivated by Buckdahn et al. [3, 5], we investigate the mean-field BSDEs driven by fBm as follows:

$$Y_t = \xi + \int_t^T E'[f(s, \eta_s, Y'_s, Z'_s, Y_s, Z_s)]ds - \int_t^T Z_s dB^H_s, \quad 0 \leq t \leq T. \quad (3.2)$$

**Remark 3.1** Owing to our notation, we mark that the coefficient of BSDE (3.2) is explained by:

$$E'[f(s, \eta_s, Y'_s, Z'_s, Y_s, Z_s)](\omega) = E'[f(s, \eta_s(\omega), Y'_s(\omega), Z'_s(\omega), Y_s(\omega), Z_s(\omega))] = \int_{\Omega} f(s, \eta_s(\omega), Y_s(\omega'), Z_s(\omega'), Y_s(\omega), Z_s(\omega))P(d\omega').$$

From the above remark, combining the definition of expectation, we have the following two special cases:

$$E'[f(s, Y'_s, Z'_s)] = E[f(s, Y_s, Z_s)], \quad E'[f(s, \eta_s, Y_s, Z_s)] = f(s, \eta_s, Y_s, Z_s). \quad (3.3)$$

Now before giving the definition of the solutions of BSDE (3.2), we introduce the following sets: for $p, q \in \mathbb{N}$,

- $C_{pol}^{p,q}([0, T] \times \mathbb{R}) = \{ \varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \mid \varphi(t, x) \text{ is } p \text{ times derivative in } t \text{ and } q \text{ times derivative in } x, \text{ and all the derivatives of } \varphi \text{ are polynomial growth}\}$;
- $\mathcal{V}_{[0, T]} = \{ Y = \varphi(\cdot, \eta(\cdot)) \mid \varphi \in C_{pol}^{1,3}([0, T] \times \mathbb{R}) \text{ with } \varphi \in C_{pol}^{0,1}([0, T] \times \mathbb{R}), \ t \in [0, T] \}$.

Moreover, by $\overline{\mathcal{V}}_{[0, T]}$ and $\overline{\mathcal{V}}_{[0, T]}^H$ denote the completion of $\mathcal{V}_{[0, T]}$ under the following norm respectively,

$$\|Y\| \triangleq \left( E \int_0^T e^{\beta t}|Y(t)|^2 \, dt \right)^{\frac{1}{2}}, \quad \|Z\| \triangleq \left( E \int_0^T t^{2H-1}e^{\beta t}|Z(t)|^2 \, dt \right)^{\frac{1}{2}},$$

where $\beta \geq 0$ is a constant. It is easy to know $\overline{\mathcal{V}}_{[0, T]}^H \subseteq \overline{\mathcal{V}}_{[0, T]} \subseteq L^2_{x}([0, T]; \mathbb{R})$.

**Definition 3.2** We call $(Y, Z)$ a solution of BSDE (3.2), if they satisfy the following conditions:

(i) $(Y, Z) \in \overline{\mathcal{V}}_{[0, T]} \times \overline{\mathcal{V}}_{[0, T]}^H$;

(ii) $Y_t = \xi + \int_t^T E'[f(s, \eta_s, Y'_s, Z'_s, Y_s, Z_s)]ds - \int_t^T Z_s dB^H_s, \ a.s. \ 0 \leq t \leq T$.

Next, we shall propose two different methods to prove the existence and uniqueness of BSDE (3.2).

3.1 The First Method

In this subsection, the first method, initially introduced by Maticiuc and Nie [20], is used to establish the existence and uniqueness of Eq. (3.2). In order to find the solution of BSDE (3.2), the following assumptions are needed.

(H1) Suppose that $\xi = g(\eta_T)$, where $g \in C^3_{pol}(\mathbb{R})$;
(H2) Assume that for the coefficient \( f = f(t, x, y', z', y, z) : [0, T] \times \mathbb{R}^5 \to \mathbb{R} \) with \( f \in C_{\text{pol}}^0([0, T] \times \mathbb{R}^5) \), there is a constant \( C \geq 0 \) such that, for every \( t \in [0, T] \), \( x, y_1, y_2, z_1, z_2, y'_1, y'_2, z'_1, z'_2 \in \mathbb{R} \), we have

\[
|f(t, x, y'_1, z'_1, y_1, z_1) - f(t, x, y'_2, z'_2, y_2, z_2)| \\
\leq C(|y'_1 - y'_2| + |z'_1 - z'_2| + |y_1 - y_2| + |z_1 - z_2|).
\]

For notational simplicity, we denote \( f_0(t, x) = f_0(t, x, 0, 0, 0, 0) \). Now we present the main result of this part.

**Theorem 3.3** Under (H1) and (H2), BSDE (3.2) admits a unique solution. Moreover,

\[
E\left(e^{\beta T}|Y_t|^2 + \int_t^T e^{\beta s} s^{2H-1}|Z_s|^2 ds\right) \leq R\Theta(t, T, K), \quad t \in [0, T],
\]

where \( R \) is a positive constant which may change line to line, and

\[
\Theta(t, T, K) = E\left(e^{\beta T}|g(\eta_T)|^2 + \int_t^T e^{\beta s}|f_0(s, \eta_s)|^2 ds\right).
\]

**Proof** For every \( (y, z) \in \hat{V}_{[0, T]} \times \hat{V}^H_{[0, T]} \), consider the following simple BSDE:

\[
Y_t = g(\eta_T) + \int_t^T E'[f(s, \eta_s, y'_s, z'_s, y_s, z_s)] ds - \int_t^T Z_s dB^H_s, \quad 0 \leq t \leq T.
\]

From Proposition 17 of Maticiuc and Nie [20], we know BSDE (3.5) has a unique solution \( (Y, Z) \in \hat{V}_{[0, T]} \times \hat{V}^H_{[0, T]} \). Now define a mapping \( I : \hat{V}_{[0, T]} \times \hat{V}^H_{[0, T]} \to \hat{V}_{[0, T]} \times \hat{V}^H_{[0, T]} \) such that

\[
I((y, z)) = (Y, Z).
\]

Let \( n \in \mathbb{N} \) and \( t_i = \frac{i-1}{n}T, \quad i = 1, \ldots, n + 1 \). First we shall solve (3.2) in \([t_n, T]\). In order to do this, we show \( I \) is a contraction on \( \hat{V}_{[t_n, T]} \times \hat{V}^H_{[t_n, T]} \).

For two arbitrary given elements \((y, z)\) and \((\overline{y}, \overline{z})\) in \( \hat{V}_{[t_n, T]} \times \hat{V}^H_{[t_n, T]} \), let \((Y, Z) = I((y, z))\) and \((\overline{Y}, \overline{Z}) = I((\overline{y}, \overline{z}))\). We denote their differences by

\[
(Y - \overline{Y}, Z - \overline{Z}) = (y - \overline{y}, z - \overline{z}).
\]

By applying Itô formula (Proposition 2.4), one has

\[
e^{\beta t}Y_t^2 + \beta \int_t^T e^{\beta s} Y_s^2 ds + 2 \int_t^T e^{\beta s} \mathbb{D}^H_s Y_s \hat{Z}_s ds + 2 \int_t^T e^{\beta s} \hat{Y}_s \hat{Z}_s dB^H_s
\]

\[
= 2 \int_t^T e^{\beta s} \hat{Y}_s E'[f(s, \eta_s, y'_s, z'_s, y_s, z_s)] ds - f(s, \eta_s, y'_s, z'_s, y_s, z_s) ds.
\]

We know (see Hu and Peng [16], Maticiuc and Nie [20]) that

\[
\mathbb{D}^H_s Y_s = \frac{\sigma_s}{\sigma_s} \hat{Z}_s.
\]

Moreover, by Remark 6 in Maticiuc and Nie [20], there is a constant \( M > 0 \) such that for every \( t \in [0, T] \),

\[
\frac{s^{2H-1}}{M} \leq \frac{\sigma^2_s}{\sigma^2_s} \leq M t^{2H-1}.
\]

From (3.6), noting Proposition 2.2, we have

\[
E\left(e^{\beta T}|Y_t|^2 + \beta \int_t^T e^{\beta s} Y_s^2 ds + \frac{2}{M} \int_t^T e^{\beta s} s^{2H-1} Z_s^2 ds\right)
\]

\[
\leq 2E \int_t^T e^{\beta s} \hat{Y}_s E'[f(s, \eta_s, y'_s, z'_s, y_s, z_s)] ds.
\]

From assumption (H2), noting (3.3), we obtain

\[
E\left(e^{\beta T}|Y_t|^2 + \beta \int_t^T e^{\beta s} Y_s^2 ds + \frac{2}{M} \int_t^T e^{\beta s} s^{2H-1} Z_s^2 ds\right)
\]

\[
= 2E \int_t^T e^{\beta s} \hat{Y}_s E'[f(s, \eta_s, y'_s, z'_s, y_s, z_s)] ds.
\]
\[ \leq 2CE \int_t^T e^{\beta s} |\hat{Y}_s| E[(\hat{y}_s^2 + |\hat{z}_s|)] ds \]
\[ = 2CE \int_t^T e^{\beta s} |\hat{Y}_s| [E(|\hat{y}_s| + |\hat{z}_s|)] ds. \tag{3.8} \]

Therefore by choosing \( \beta \geq 1 \), and using H"older inequality and Jensen inequality, we get
\[ E\left(e^{\beta t}\hat{Y}_t^2 + \int_t^T e^{\beta s}\hat{Y}_s^2 ds + \frac{2}{M} \int_t^T e^{\beta s} s^{2H-1} \hat{Z}_s^2 ds\right) \leq 4C \int_t^T (e^{\beta s} E[|\hat{y}_s|^2]) \hat{z}_s ds. \tag{3.9} \]

Denote \( x(t) = (e^{\beta t}E[\hat{Y}_t^2])^{1/2} \). Then from (3.9),
\[ x(t)^2 \leq 4C \int_t^T x(s)(e^{\beta s}E(|\hat{y}_s|^2 + |\hat{z}_s|^2))^{1/2} ds. \]

Applying Lemma 20 in Maticiuc and Nie [20] to the above inequality, one has
\[ x(t) \leq 2C \int_t^T [e^{\beta s}E(|\hat{y}_s| + |\hat{z}_s|)]^{1/2} ds. \]

Therefore for \( t \in [t_n, T] \),
\[ x(t)^2 \leq 4C^2 \left( \int_{t_n}^T [e^{\beta s}E(|\hat{y}_s|^2 + |\hat{z}_s|^2)]^{1/2} ds \right)^2. \tag{3.10} \]

Now we compute
\[ \int_{t_n}^T x(s)^2 ds \leq 4C^2 (T - t_n) \left( \int_{t_n}^T [e^{\beta s}E(|\hat{y}_s|^2 + |\hat{z}_s|^2)]^{1/2} ds \right)^2. \]

From H"older inequality,
\[ \left( \int_{t_n}^T [e^{\beta s}E(|\hat{y}_s|^2 + |\hat{z}_s|^2)]^{1/2} ds \right)^2 \leq \left( \int_{t_n}^T [e^{\beta s}E(|\hat{y}_s|^2)]^{1/2} ds + \int_{t_n}^T [e^{\beta s}E(|\hat{z}_s|^2)]^{1/2} ds \right)^2 \leq 2 \left( \int_{t_n}^T [e^{\beta s}E(|\hat{y}_s|^2)]^{1/2} ds \right)^2 + 2 \left( \int_{t_n}^T \left[ \frac{1}{s^{2H-1}} e^{\beta s} s^{2H-1} E|\hat{z}_s|^2 \right]^{1/2} ds \right)^2 \leq 2(T - t_n) \int_{t_n}^T e^{\beta s} E|\hat{y}_s|^2 ds + \frac{2(T^{2-2H} - t_n^{2-2H})}{2 - 2H} \int_{t_n}^T e^{\beta s} s^{2H-1} E|\hat{z}_s|^2 ds \leq \left[ 2(T - t_n) + \frac{T^{2-2H} - t_n^{2-2H}}{1 - H} \right] E \int_{t_n}^T e^{\beta s} (|\hat{y}_s|^2 + s^{2H-1} |\hat{z}_s|^2) ds. \tag{3.11} \]

Hence, from (3.10) and (3.11),
\[ \int_{t_n}^T x(s)^2 ds \leq (T - t_n) G \cdot E \int_{t_n}^T e^{\beta s} (|\hat{y}_s|^2 + s^{2H-1} |\hat{z}_s|^2) ds, \tag{3.12} \]
where \( G = 4C^2[2(T - t_n) + \frac{T^{2-2H} - t_n^{2-2H}}{1 - H}] \). Similarly,
\[ \int_{t_n}^T \frac{1}{s^{2H-1}} x(s)^2 ds \leq \frac{T^{2-2H} - t_n^{2-2H}}{1 - H} G \cdot E \int_{t_n}^T e^{\beta s} (|\hat{y}_s|^2 + s^{2H-1} |\hat{z}_s|^2) ds. \tag{3.13} \]
From (3.8) and the inequality $2ab \leq \frac{1}{\delta}a^2 + \delta b^2$, where $\delta > 0$ is a constant, combining Jensen inequality, we deduce

$$E\left(\int_t^T e^{\beta s} \hat{Y}_s^2 \, ds + \frac{2}{M} \int_t^T e^{\beta s} s^{2H-1} \hat{Z}_s^2 \, ds\right)$$

$$\leq 2CE \int_t^T e^{\beta s} \left(1 + \frac{1}{s^{2H-1}}\right) |\hat{Y}_s|^2 + \delta |\hat{y}_s|^2 + \delta s^{2H-1}|\hat{z}_s|^2 \, ds$$

$$\leq 2C\delta E \int_t^T e^{\beta s} \left(1 + \frac{1}{s^{2H-1}}\right) |\hat{Y}_s|^2 \, ds + 2C\delta E \int_t^T e^{\beta s} (|\hat{y}_s|^2 + s^{2H-1}|\hat{z}_s|^2) \, ds.$$ 

Applying the inequalities (3.12) and (3.13), and choosing $M \geq 2$, one has

$$E \int_t^T e^{\beta s} (|\hat{Y}_s|^2 + s^{2H-1}|\hat{Z}_s|^2) \, ds \leq \tilde{G} E \int_t^T e^{\beta s} (|\hat{y}_s|^2 + s^{2H-1}|\hat{z}_s|^2) \, ds,$$

where

$$\tilde{G} = \frac{CGM}{\delta} (T - t_n) + \frac{CGM}{\delta(1 - H)} (T^{2-2H} - t_n^{2-2H}) + CM\delta.$$ 

Now, by choosing $\delta$ such that $CM\delta < \frac{1}{4}$, and taking $n$ large enough such that

$$\frac{CGM}{\delta} (T - t_n) < \frac{1}{4}, \quad \frac{CGM}{\delta(1 - H)} (T^{2-2H} - t_n^{2-2H}) < \frac{1}{4}.$$ 

Then we obtain

$$E \int_t^T e^{\beta s} (|\hat{Y}_s|^2 + s^{2H-1}|\hat{Z}_s|^2) \, ds \leq \frac{3}{4} E \int_t^T e^{\beta s} (|\hat{y}_s|^2 + s^{2H-1}|\hat{z}_s|^2) \, ds.$$ 

Consequently, $I$ is a contraction on $\tilde{V}_{[t_n, T]} \times \tilde{V}^H_{[t_n, T]}$. Discussing as in the proof of Theorem 22 of Maticiuc and Nie [20], we know BSDE (3.2) admits a unique solution on $\tilde{V}_{[t_n, T]} \times \tilde{V}^H_{[t_n, T]}$. Next, the procedure is to solve (3.2) in $[t_{n-1}, t_n]$. Repeating the above technique and discussion, we obtain that (3.2) has a unique solution in $\tilde{V}_{[0, T]} \times \tilde{V}^H_{[0, T]}$.

Now we prove the estimate (3.4). Suppose $(Y, Z)$ is the solution of Eq. (3.2). Similarly to (3.7), and from (H2), we obtain

$$E\left(e^{\beta T} Y_T^2 + \beta \int_t^T e^{\beta s} Y_s^2 \, ds + \frac{2}{M} \int_t^T e^{\beta s} s^{2H-1} Z_s^2 \, ds\right)$$

$$\leq E\left(e^{\beta T} |g(\eta_T)|^2 + 2 \int_t^T e^{\beta s} Y_s E[f(s, \eta_s, Y'_s, Z'_s, Y_s, Z_s)] \, ds\right)$$

$$\leq E e^{\beta T} |g(\eta_T)|^2 + E \int_t^T e^{\beta s} |f(s, \eta_s)|^2 \, ds$$

$$+ E \int_t^T \left(1 + 4C + \frac{4C^2 M}{s^{2H-1}}\right) e^{\beta s} |Y_s|^2 \, ds + \frac{1}{M} E \int_t^T e^{\beta s} s^{2H-1} |Z_s|^2 \, ds.$$ 

Therefore

$$E\left(e^{\beta T} Y_T^2 + \frac{1}{M} \int_t^T e^{\beta s} s^{2H-1} Z_s^2 \, ds\right)$$

$$\leq R\Theta(t, T, K) + E \int_t^T \left(1 + 4C + \frac{4C^2 M}{s^{2H-1}}\right) e^{\beta s} |Y_s|^2 \, ds. \quad (3.14)$$
Then, by Gronwall inequality,
\[ Ee^{\beta t}Y_t^2 \leq R\Theta(t, T, K) \exp \left\{ (1 + 4C)(T - t) + 4C^2M \frac{T^{2-2H} - t^{2-2H}}{2 - 2H} \right\}. \]
Again from (3.14), we have
\[ E \int_t^T e^{\beta s}s^{2H-1}|Z_s|^2 \, ds \leq R\Theta(t, T, K). \]
Therefore the estimate (3.4) is obtained. This completes the proof.

3.2 The Second Method
Here, we introduce another method to prove the existence and uniqueness of BSDE (3.2). It should be pointed out that this method is more convenient than the above one. However, the price of doing so is that we should strengthen the condition of the coefficient \( f \) with respect to \( z \).

(H3) For the coefficient \( f = f(t, x, y', z', y, z) : [0, T] \times \mathbb{R}^5 \to \mathbb{R} \) with \( f \in C_{\text{pol}}^{0,1}([0, T] \times \mathbb{R}^5) \), there is a constant \( C \geq 0 \) such that, for every \( t \in [0, T] \), \( x, y_1, y_2, z_1, z_2, y'_1, y'_2, z'_1, z'_2 \in \mathbb{R} \), we have
\[
|f(t, x, y'_1, z'_1, y_1, z_1) - f(t, x, y'_2, z'_2, y_2, z_2)|
\leq C(|y'_1 - y'_2| + t^{H-\frac{1}{2}}|z'_1 - z'_2| + |y_1 - y_2| + t^{H-\frac{1}{2}}|z_1 - z_2|).
\]

Remark 3.4 Suppose \( \alpha \) and \( \gamma \) are two square integrable, jointly measurable stochastic processes. Then we can define for all \( t \in [0, T] \), \( x, y, z \in \mathbb{R} \),
\[
f^{\alpha, \gamma}(t, x, y, z) := E'[f(t, x, \alpha'_t, \gamma'_t, y, z)] = \int_{\Omega} f(t, x, \alpha_t(\omega'), \gamma_t(\omega'), y, z) P(d\omega').
\]
Indeed, due to the assumptions on the coefficient \( f \in C_{\text{pol}}^{0,1}([0, T] \times \mathbb{R}^5) \), we have that \( f^{\alpha, \gamma} \in C_{\text{pol}}^{0,1}([0, T] \times \mathbb{R}^3) \). In addition, with the same constant \( C \) of assumption (H3), for every \( t \in [0, T] \), \( x, y_1, y_2, z_1, z_2 \in \mathbb{R} \), we have
\[
|f^{\alpha, \gamma}(t, x, y_1, z_1) - f^{\alpha, \gamma}(t, x, y_2, z_2)| \leq C(|y_1 - y_2| + t^{H-\frac{1}{2}}|z_1 - z_2|).
\]

Theorem 3.5 Under (H1) and (H3), BSDE (3.2) admits a unique solution.

We point out that in the proof of Theorem 3.5, the following result is used.

Lemma 3.6 (Wen and Shi [26, Lemma 3.1]) Suppose \( g \) is a given differentiable function with polynomial growth, and \( f(t, x) \) is a \( C_{\text{pol}}^{0,1} \)-continuous function. Then the following BSDE:
\[
Y_t = g(\eta_T) + \int_t^T f(s, \eta_s) \, ds - \int_t^T Z_s dB_s^H, \quad t \in [0, T],
\]
admits a unique solution \((Y, Z) \in \tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H \). Moreover,
\[
E\left(e^{\beta t}|Y_t|^2 + \frac{\beta}{2} \int_t^T e^{\beta s}|Y_s|^2 \, ds + \frac{2}{M} \int_t^T s^{2H-1}e^{\beta s}|Z_s|^2 \, ds\right)
\leq E\left(e^{\beta T}|g(\eta_T)|^2 + \frac{2}{\beta} \int_t^T e^{\beta s}|f(s, \eta_s)|^2 \, ds\right),
\]
where \( M > 0 \) is a suitable constant and \( \beta > 0 \).
Proof of Theorem 3.5  For every $(y, z) \in \tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H$, we consider BSDE as follows:

$$Y_t = g(\eta_T) + \int_t^T E'[f(s, \eta_s, y_s', z_s', y_s, z_s)] ds - \int_t^T Z_s dB^H_s, \quad 0 \leq t \leq T. \tag{3.16}$$

From Lemma 3.6, Eq. (3.16) admits the unique solution $(Y, Z) \in \tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H$. Once again, define $I : \tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H \to \tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H$ such that $I[(y_z, z)] = ((Y), (Z))$. Now we directly show that $I$ is a contraction mapping on $\tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H$. For two arbitrary elements $(y_1(\cdot), z_1(\cdot))$ and $(y_2(\cdot), z_2(\cdot)) \in \tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H$, set $(Y_1(\cdot), Z_1(\cdot)) = I[(y_1(\cdot), z_1(\cdot))]$ and $(Y_2(\cdot), Z_2(\cdot)) = I[(y_2(\cdot), z_2(\cdot))]$. Denote $(\hat{y}(\cdot), \hat{z}(\cdot)) = (y_1(\cdot) - y_2(\cdot), z_1(\cdot) - z_2(\cdot))$, $(\hat{Y}(\cdot), \hat{Z}(\cdot)) = (Y_1(\cdot) - Y_2(\cdot), Z_1(\cdot) - Z_2(\cdot))$. Then, from the estimate (3.15), we have

$$E \int_0^T e^{\beta s} \left( \frac{\beta}{2} |\hat{Y}(s)|^2 + \frac{2}{M} s^{2H-1} |\hat{Z}(s)|^2 \right) ds \leq \frac{2}{\beta} E \int_0^T e^{\beta s} (E'[f(s, \eta_s, y_s', z_s', y_s, z_s) - f(s, \eta_s, y_2(s), z_2(s), y_2(s), z_2(s))])^2 ds.$$

From assumption (H3) we obtain

$$E'[f(s, \eta_s, y_s', z_s', y_s, z_1(s)) - f(s, \eta_s, y_2(s), z_2(s), y_2(s), z_2(s))]$$

$$\leq CE'[|\hat{y}(s)| + s^{H-\frac{1}{2}} |\hat{z}(s)| + |\hat{y}(s)| + s^{H-\frac{1}{2}} |\hat{z}(s)|]$$

$$= C(E|\hat{y}(s)| + s^{H-\frac{1}{2}} E|\hat{z}(s)| + |\hat{y}(s)| + s^{H-\frac{1}{2}} |\hat{z}(s)|).$$

Therefore, by Jensen inequality and Fubini theorem, one has

$$E \int_0^T e^{\beta s} \left( \frac{\beta}{2} |\hat{Y}(s)|^2 + \frac{2}{M} s^{2H-1} |\hat{Z}(s)|^2 \right) ds$$

$$\leq \frac{2C^2}{\beta} E \int_0^T e^{\beta s} (E|\hat{y}(s)| + s^{H-\frac{1}{2}} E|\hat{z}(s)| + |\hat{y}(s)| + s^{H-\frac{1}{2}} |\hat{z}(s)|)^2 ds$$

$$\leq \frac{16C^2}{\beta} E \int_0^T e^{\beta s} (|\hat{y}(s)|^2 + s^{2H-1} |\hat{z}(s)|^2) ds.$$

In other words,

$$E \int_0^T e^{\beta s} \left( \frac{M \beta}{4} |\hat{Y}(s)|^2 + s^{2H-1} |\hat{Z}(s)|^2 \right) ds \leq \frac{8MC^2}{\beta} E \int_0^T e^{\beta s} (|\hat{y}(s)|^2 + s^{2H-1} |\hat{z}(s)|^2) ds.$$

Thus, by taking $\beta = 16MC^2 + \frac{1}{M}$, we get

$$E \int_0^T e^{\beta s} (|\hat{Y}(s)|^2 + s^{2H-1} |\hat{Z}(s)|^2) ds \leq \frac{1}{2} E \int_0^T e^{\beta s} (|\hat{y}(s)|^2 + s^{2H-1} |\hat{z}(s)|^2) ds.$$

Therefore, $I$ is a contraction mapping on $\tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H$. Consequently, Eq. (3.2) admits a unique solution $(Y, Z) \in \tilde{V}_{[0, T]} \times \tilde{V}_{[0, T]}^H$. \hfill \Box

Remark 3.7 Now, we make a comparison for the above two methods. It is easy to see that (H2) is weaker than (H3). So from the point of view of the condition, the first method is better than the second one. On the other hand, thanks to the concise proof, the second method is convenient than the first method. So from this point of view, the second method is better.
4 Comparison Theorem

In this section, we study a comparison theorem of the fractional mean-field BSDEs of the following form:

$$Y_t = g(\eta_T) + \int_t^T E'[f(s, \eta_s, Y'_s, Y_s, Z_s)] ds - \int_t^T Z_s dB^H_s, \quad 0 \leq t \leq T. \quad (4.1)$$

Under (H1) and (H3), it is easy to know that the above equation admits a unique solution. Here we use (H3), not (H2), because it is more convenient for the proof of the following comparison theorem.

**Theorem 4.1** For $i = 1, 2$, suppose $g_i$ satisfies (H1), and $f_i(t, x, y', y, z)$ and $\partial_y f_i(t, x, y', y, z)$ satisfy (H3), for each $(t, x, y', y, z) \in [0, T] \times \mathbb{R}^4$. Moreover, assume $f_i$ is increasing in $y'$. Then, if $g_1(x) \leq g_2(x)$ and $f_1(t, x, y', y, z) \leq f_2(t, x, y', y, z)$ for each $(t, x, y', y, z) \in [0, T] \times \mathbb{R}^4$, it holds that $Y_1(t) \leq Y_2(t)$ almost surely.

**Proof** For $i = 1, 2$, we define $f_i(x, y, z) = E'[f_i(s, x, y', y, z)]$. By virtue of Remark 3.4, $f_i$ and $\partial_y f_i$ satisfy (H3), moreover, $f_i \leq f_2$, and $f_i$ is increasing in $v$.

Let $\tilde{Y}_0(\cdot) = Y_2(\cdot)$. We consider BSDE:

$$\tilde{Y}_1(t) = g_1(\eta_T) + \int_t^T E'[f_1(s, \eta_s, Y'_1(s), \tilde{Y}_1(s), \tilde{Z}_1(s))] ds - \int_t^T \tilde{Z}_1(s) dB^H_s, \quad t \in [0, T].$$

By Theorem 3.5, the above equation admits a unique solution $(\tilde{Y}_1(\cdot), \tilde{Z}_1(\cdot)) \in \tilde{V}_{[0, T]} \times \tilde{V}^H_{[0, T]}$. Now since

$$\begin{align*}
&f_1^0(t, x, y, z) \leq f_2^0(t, x, y, z), \quad \forall (t, x, y, z) \in [0, T] \times \mathbb{R}^3; \\
g_1(x) \leq g_2(x), \quad \forall x \in \mathbb{R},
\end{align*}$$

from Theorem 12.3 of Hu et al. [15], we deduce

$$\tilde{Y}_1(t) \leq \tilde{Y}_0(t) = Y_2(t), \quad \text{a.s.}$$

Next, we consider the following BSDE:

$$\tilde{Y}_2(t) = g_1(\eta_T) + \int_t^T E'[f_1(s, \eta_s, \tilde{Y}'_1(s), \tilde{Y}_2(s), \tilde{Z}_2(s))] ds - \int_t^T \tilde{Z}_2(s) dB^H_s.$$

Let $(\tilde{Y}_2(\cdot), \tilde{Z}_2(\cdot)) \in \tilde{V}_{[0, T]} \times \tilde{V}^H_{[0, T]}$ be the unique solution of the above equation. Thanks to that $f_1^0$ is increasing in $v$, one has

$$f_1^0(t, x, y, z) \leq \tilde{f}_1^0(t, x, y, z), \quad \forall (t, x, y, z) \in [0, T] \times \mathbb{R}^3.$$

Therefore, similar to the above, we deduce

$$\tilde{Y}_2(t) \leq \tilde{Y}_1(t), \quad \text{a.s.}$$

By induction, one can construct a sequence $\{ (\tilde{Y}_n(\cdot), \tilde{Z}_n(\cdot)) \}_{n \geq 1} \subset \tilde{V}_{[0, T]} \times \tilde{V}^H_{[0, T]}$ such that for every $t \in [0, T]$,

$$\tilde{Y}_n(t) = g_1(\eta_T) + \int_t^T E'[f_1(s, \eta_s, \tilde{Y}'_{n-1}(s), \tilde{Y}_n(s), \tilde{Z}_n(s))] ds - \int_t^T \tilde{Z}_n(s) dB^H_s.$$

Similarly, we obtain

$$Y_2(t) = \tilde{Y}_0(t) \geq \tilde{Y}_1(t) \geq \tilde{Y}_2(t) \geq \cdots \geq \tilde{Y}_n(t) \geq \cdots, \quad \text{a.s.}$$
In the following, we show \( \{ (\tilde{Y}_n(\cdot), \tilde{Z}_n(\cdot)) \}_{n \geq 1} \) is a Cauchy sequence. Denote \( \hat{Y}_n(t) = \tilde{Y}_n(t) - \tilde{Y}_{n-1}(t) \), and \( \hat{Z}_n(t) = \tilde{Z}_n(t) - \tilde{Z}_{n-1}(t), n \geq 4 \). Then, from the estimate (3.15), we have
\[
E \int_0^T e^{\beta s} \left( \frac{\beta}{2} |\hat{Y}_n(s)|^2 + \frac{2}{M} s^{2H-1} |\hat{Z}_n(s)|^2 \right) ds \\
\leq \frac{2}{\beta} E \int_0^T e^{\beta s} |\hat{Y}_{n-1}(s, \eta_s, \tilde{Y}_n(s), \tilde{Z}_n(s)) - f_1 \hat{Y}_{n-2}(s, \eta_s, \tilde{Y}_{n-1}(s), \tilde{Z}_{n-1}(s))|^2 ds.
\]
From assumption (H3), one has
\[
E \int_0^T e^{\beta s} \left( \frac{\beta}{2} |\hat{Y}_n(s)|^2 + \frac{2}{M} s^{2H-1} |\hat{Z}_n(s)|^2 \right) ds \\
\leq \frac{6C^2}{\beta} E \int_0^T e^{\beta s} (|\hat{Y}_n(s)|^2 + s^{2H-1} |\hat{Z}_n(s)|^2 + |\hat{Y}_{n-1}(s)|^2) ds.
\]
Let \( \beta = 12MC^2 + \frac{4}{M} \). Then we obtain
\[
E \int_0^T e^{\beta s} (|\hat{Y}_n(s)|^2 + s^{2H-1} |\hat{Z}_n(s)|^2) ds \\
\leq \frac{1}{4} E \int_0^T e^{\beta s} (|\hat{Y}_n(s)|^2 + s^{2H-1} |\hat{Z}_n(s)|^2 + |\hat{Y}_{n-1}(s)|^2) ds.
\]
Hence
\[
E \int_0^T e^{\beta s} (|\hat{Y}_n(s)|^2 + s^{2H-1} |\hat{Z}_n(s)|^2) ds \\
\leq \frac{1}{3} E \int_0^T e^{\beta s} |\hat{Y}_{n-1}(s)|^2 ds \\
\leq \frac{1}{3} E \int_0^T e^{\beta s} (|\hat{Y}_{n-1}(s)|^2 + s^{2H-1} |\hat{Z}_{n-1}(s)|^2) ds.
\]
Therefore
\[
E \int_0^T e^{\beta s} (|\hat{Y}_n(s)|^2 + s^{2H-1} |\hat{Z}_n(s)|^2) ds \\
\leq \left( \frac{1}{3} \right)^{n-4} E \int_0^T e^{\beta s} (|\hat{Y}_4(s)|^2 + s^{2H-1} |\hat{Z}_4(s)|^2) ds.
\]
So \( (\hat{Y}_n(\cdot))_{n \geq 4} \) and \( (\hat{Z}_n(\cdot))_{n \geq 4} \) are Cauchy sequences in \( \tilde{\mathcal{V}}_{[0,T]} \) and \( \tilde{\mathcal{V}}_{H, [0,T]} \), respectively. Denote their limits by \( \hat{Y} \) and \( \hat{Z} \), respectively. From Theorem 3.5, we have \( \hat{Y}(t) = Y_1(t) \), a.s., which deduce that
\[
Y_1(t) \leq Y_2(t), \quad \text{a.s.}
\]
Therefore, our conclusion follows. \( \square \)

**Example 4.2** Suppose we are facing with the following two mean-field BSDEs,
\[
Y_1(t) = g_1(\eta_T) + \int_t^T [Y_1(s) + EY_1(s) + Z_1(s) - 1] ds - \int_t^T Z_1(s) dB_s^H,
\]
\[
Y_2(t) = g_2(\eta_T) + \int_t^T [Y_2(s) + EY_2(s) + Z_2(s) + 1] ds - \int_t^T Z_2(s) dB_s^H,
\]
where \( t \in [0, T] \), \( g_1 \) and \( g_1 \) satisfy (H1) with \( g_1(x) \leq g_2(x), \forall x \in \mathbb{R} \). Then, according to Theorem 4.1, one has

\[
Y_1(t) \leq Y_2(t), \quad \text{a.s.}
\]

5 Connection with PDEs

As an important application of the fractional mean-field BSDEs, we connect the mean-field BSDEs driven by fBm with the following PDE:

\[
\begin{aligned}
&v_t(t, x) + v_x(t, x)b_t + \frac{1}{2}v_{xx}(t, x)\tilde{\sigma}_t \\
&+ E[f(t, x, v(t, \eta), v(t, x), v_x(t, x)\sigma_t)] = 0, \quad (t, x) \in [0, T) \times \mathbb{R}; \quad (5.1)
\end{aligned}
\]

where \( \tilde{\sigma}_t := \frac{d}{dt}(\|\sigma_t\|^2) \). From (3.3), one has that

\[
E[f(t, x, v(t, \eta), v(t, x), v_x(t, x)\sigma_t)] = E'[f(t, x, v(t, \eta'), v(t, x), v_x(t, x)\sigma_t)]. \quad (5.2)
\]

It is easy to see PDE (5.1) is a special case of Eq. (6.1) of Buckdahn et al. [5]. As presented in [5], PDE (5.1) is nonlocal and under sufficient conditions, it has a unique viscosity solution. In other words, PDE (5.1) is well-defined.

**Theorem 5.1** If PDE (5.1) has a solution \( v(t, x) \) with \( v \in C^{1,2}([0, T] \times \mathbb{R}) \), then \((Y_t, Z_t) := (v_x(t, \eta), v_x(t, \eta)\sigma_t)\) satisfy the fractional mean-field BSDE:

\[
Y_t = g(\eta_T) + \int_t^T E'[f(s, \eta_s, Y_s', Y_s, Z_s)]ds - \int_t^T Z_sdB^H_s, \quad 0 \leq t \leq T. \quad (5.3)
\]

**Proof** By applying Itô formula (Proposition 2.3), one has

\[
dv(t, \eta_t) = v_t(t, \eta_t)dt + v_x(t, \eta_t)b_tdt + v_x(t, \eta_t)\sigma_tdB^H_t + \frac{1}{2}v_{xx}(t, \eta_t)\tilde{\sigma}_tdt
\]

\[
= \left( v_t(t, \eta_t) + v_x(t, \eta_t)b_t + \frac{1}{2}v_{xx}(t, \eta_t)\tilde{\sigma}_t \right)dt + v_x(t, \eta_t)\sigma_t dB^H_t.
\]

Since \( v \) satisfies PDE (5.1), and noting (5.2), we have

\[
dv(t, \eta_t) = E'[f(t, \eta_t, v_x(t, \eta_t'), v(t, \eta_t), v_x(t, \eta_t)\sigma_t)]dt + v_x(t, \eta_t)\sigma_t dB^H_t.
\]

Therefore, \((Y_t, Z_t) := (v_x(t, \eta), v_x(t, \eta)\sigma_t)\) satisfy the fractional mean-field BSDE (5.3). \qed

**Remark 5.2** Theorem 5.1 establishes a relationship between the fractional mean-field BSDEs and PDEs, and extends Theorem 4.1 of Hu and Peng [16] to mean-field circumstance. For the general result of the connection between the fractional mean-field BSDEs and PDEs, we will give some further studied in the future.

6 Conclusions

The innovation of this paper is that we introduced a new class of BSDEs, called the fractional mean-field BSDEs, and used a new method to obtain the existence and uniqueness of this class of equations. As an application, a relationship between such BSDE and a nonlocal PDE is established. Moreover, a comparison theorem is also obtained. We pointed out that the application of this class of equations in the stochastic optimal control problem is presented in our forthcoming paper in Douissi, Wen and Shi [9]. The deficiency of this paper is that the
Hurst parameter $H$ is required to be greater than $1/2$. In the coming future researches, we hope to publish some results about this topic when the Hurst parameter $H < 1/2$.

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