A new low-cost meshfree method for two and three dimensional problems in elasticity

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Abstract
In this paper, we develop a new meshless method based on the generalized moving least squares approximation for elasto-static problems. It is called Direct Meshless Local Petrov-Galerkin (DMLPG) method. The most significant advantage of the new method in comparison with the original MLPG lies on the complexity side, where DMLPG reduces the computational costs, significantly. Although, the “Petrov-Galerkin” strategy is used to build up the primary local weak forms, but finally the role of trial space is ignored and direct approximations for local weak forms and boundary conditions are performed to construct the final stiffness matrix. This modification shifts the numerical integrations over polynomials rather than over complicated MLS shape functions. In this paper, DMLPG is applied for two and three dimensional problems in elasticity. Some variations of new method are developed and their efficiencies are reported. Finally, we will conclude that DMLPG can replace the original MLPG in many situations.

Key words: DMLPG methods, MLPG methods, MLS approximation, GMLS approximation, Direct approximation, Elasticity problems

1. Introduction

The Meshless Local Petrov-Galerkin (MLPG) methods were widely applied to find the numerical solutions of elasto-static and elasto-dynamic problems. MLPG was first introduced in [1], and was first applied for elasticity in [2]. Afterward, many papers were appeared for different types of mechanical problems. For example see [3] [4] [5] [6]. MLPG is based on local weak forms and it is known as a truly meshless method, because it uses no global background mesh to evaluate integrals and everything breaks down to some regular, well-shaped and independent sub-domains. This is in contrast to methods which are based on global weak forms, such as Element-free Galerkin (EFG) method [7], where triangulation is again required for numerical integration. But MLPG still suffers
from the cost of numerical integration. This is due to the complexity of the integrands. In MLPG and all MLS based methods, integrations are down over complicated MLS shape functions, and this leads to high computational costs in comparison with the finite elements method (FEM), where integrands are simple and close form polynomials. There are some researches concerning the numerical integration and trying to give accurate and fast quadratures for MLPG. For example see [8, 9]. This is the reason why this method, and of course the other meshless methods, have found very limited application to three-dimensional problems, which are routine applications of FEM.

A tricky modification has been applied to MLPG in [10], which shifts the numerical integrations over low-degree polynomial basis functions rather than complicated MLS shape functions. This reduces the computational costs of MLPG, significantly. In the new method, local weak forms are considered as functionals and directly approximated from nodal data using a generalized moving least squares (GMLS) approximation. Thus this method is called Direct MLPG (DMLPG). Although DMLPG uses the same local forms, but it is theoretically different form MLPG, because it eliminates the role of trial space. DMLPG can be considered as a generalized finite difference method (GFDM), not only in its usual strong form, but also in a weak formulation. It is worthy to note that, by this modification we do not lose the order of convergence. This has been analytically proven in [11, 12] for different definitions of functionals, specially for the local weak forms of DMLPG.

DMLPG has been applied to the heat conduction problem in [13] and has been numerically investigated for 2D and 3D potential problems in [14].

In this paper, the application of DMLPG is provided for elasto-static problems for the first time. We consider both two and three dimensional problems to show the efficiency of the new method. The paper is written in an engineering-oriented style, and can be easily extended to the other problems in elasticity.

2. Generalized moving least squares

Generalized moving least squares (GMLS) approximation was presented in [11] in details. Here we briefly discuss this concept. Let $\Omega$ be a bounded subset in $\mathbb{R}^d$, $d \in \mathbb{Z}_+$, and $X = \{x_1, x_2, \ldots, x_N\} \subset \Omega$ be a set of meshless points scattered (with certain quality) over $\Omega$. The MLS method approximates the function $u \in U$ (with certain smoothness) by its values at points $x_j$, $j = 1, \ldots, N$, by

$$u(x) \approx \hat{u}(x) = \sum_{j=1}^{N} a_j(x)u(x_j), \quad x \in \Omega,$$

where $a_j(x)$ are MLS shape functions obtained in such way that $\hat{u}$ be the best approximation of $u$ in polynomial subspace $P_m(\mathbb{R}^d) = \text{span}\{p_1, \ldots, p_Q\}$, $Q = (m+d)^{\frac{d}{2}}$, with respect
to a weighted, discrete and moving $\ell^2$ norm. The weight function governs the influence of the data points and assumed to be a function $w: \Omega \times \Omega \to \mathbb{R}$ which becomes smaller the further away its arguments are from each other. Ideally, $w$ vanishes for arguments $x,y \in \Omega$ with $\|x - y\|_2$ greater than a certain threshold, say $\delta$. Such a behavior can be modeled by using a translation-invariant weight function. This means that $w$ is of the form $w(x,y) = \varphi(\|x - y\|_2/\delta)$ where $\varphi$ is a compactly supported function supported in $[0,1]$. If we define

$$P = P(x) = (p_k(x_j)) \in \mathbb{R}^{N \times Q},$$

$$W = W(x) = \text{diag}\{w(x_j, x)\} \in \mathbb{R}^{N \times N},$$

then a simple calculation gives the shape functions

$$a(x) := [a_1(x), \ldots, a_N(x)] = p(x)(P^TWP)^{-1}P^TW. \quad (2.3)$$

where $p = [p_1, \ldots, p_Q]$. If $X_x = \{x_j : \|x - x_j\| \leq \delta\}$ is $P_m(\mathbb{R}^d)$-unisolvent then $A(x) = P^TWP$ is positive definite [15] and the MLS approximation is well-defined at sample point $x$. Of course if $\|x - x_j\| \geq \delta$ then $a_j(x) = 0$. Thus, in programming we can only form $P$ and $W$ for active points $X_x$ instead of $X$. Derivatives of $u$ are usually approximated by derivatives of $\hat{u}$,

$$D^\alpha u(x) \approx D^\alpha \hat{u}(x) = \sum_{j=1}^N D^\alpha a_j(x)u(x_j), \quad x \in \Omega, \quad \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d_0. \quad (2.4)$$

These derivatives are sometimes called standard or full derivatives. Details are in [16, 17] and any other text containing the application of MLS approximation.

The GMLS approximation can be introduced as bellow. Suppose that $\lambda$ is a linear functional from the dual space $U^*$. The problem is the recovery of $\lambda(u)$ from nodal values $u(x_1), \ldots, u(x_N)$. The functional $\lambda$ can, for instance, describe point evaluations of $u$, its derivatives up to order $m$, and the weak formulations which involve $u$ or a derivative against some test function. The approximation $\hat{\lambda}(u)$ of $\lambda(u)$ should be a linear function of the data $u(x_j)$, i.e., it should have the form

$$\lambda(u) \approx \hat{\lambda}(u) = \sum_{j=1}^N a_j(\lambda)u(x_j), \quad (2.5)$$

where $a_j(\lambda)$ are shape functions associated to the functional $\lambda$. If $\lambda$ is chosen to be the point evaluation functional $\delta_x$, where $\delta_x(u) := u(x)$, then the classical MLS approximation [2.1] is resulted. If we assume $\lambda$ is finally evaluated at sample point $x$, then the same weight function $w(x,y)$ as in the classical MLS can be used independent of the choice of $\lambda$. Using this assumption, analogous to (2.3), [11] proves,

$$a(\lambda) := [a_1(\lambda), \ldots, a_N(\lambda)] = \lambda(P)(P^TWP)^{-1}P^TW, \quad (2.6)$$
where \( \lambda(P) = [\lambda(p_1), \ldots, \lambda(p_Q)] \). In fact, we have a direct approximation for \( \lambda(u) \) from nodal values \( u(x_1), \ldots, u(x_N) \), without any detour via classical MLS shape functions. One can see, \( \lambda \) acts only on polynomial basis functions. This is the central idea in this GMLS approximation which finally speeds up our numerical algorithms. If \( \lambda \) contains derivatives of \( u \), (2.6) shows that derivatives of weight functions are not required. This paves the way for generalizing the forthcoming schemes for discontinuous problems.

In particular, if \( \lambda(u) = D^\alpha(u) \) then derivatives of \( u \) are recovered. They are different from the standard derivatives (2.4), and in meshless literatures are called diffuse or uncertain derivatives. But [11] and [12] prove the optimal rate of convergence for them toward the exact derivatives, and thus there is no diffuse or uncertain about them. As suggested in [11], they can be called GMLS derivative approximations.

In next sections, we deliberately choose \( \lambda \) in such way that MLPG methods speed up, significantly.

The GMLS approximation of this section is different from one presented in [18]. In that paper a Hermite-type MLS approximation has been used to solve the forth order problems of thin beams. Here we approximate the general functional \( \lambda(u) \) from values \( u(x_1), \ldots, u(x_N) \), where information of \( D^\alpha u \) is not required. In more general situation, the GMLS approximation of [18] can be written as

\[
\hat{u}(x) = \hat{\lambda}(x) = \sum_{k=1}^{K} \sum_{j=1}^{N} a_{k,j}(x) \mu_{k,j}(u),
\]

where \( \mu_{k,j} \) are linear functionals from \( U^* \) and should be chosen properly to ensure the solvability of the problem.

In a more and more general situation, both these generalizations can be used simultaneously

\[
\lambda(u) \approx \tilde{\lambda}(u) = \sum_{k=1}^{K} \sum_{j=1}^{N} a_{k,j}(\lambda) \mu_{k,j}(u).
\]

Up to the now, there is no rigorous error analysis for such generalized approximation, even when \( \lambda \) and \( \mu_{k,j} \) are some special functionals. Throughout, we leave the above recent formulations and focus on GMLS approximation (2.5) together with (2.6).

3. Local weak forms of the elasticity problem

Let \( \Omega \subset \mathbb{R}^d \) (usually \( d = 2, 3 \)) is a bounded domain with boundary \( \Gamma \). From here on, integers \( i \) and \( j \) are assumed to vary from 1 to \( d \). Consider the following \( d \)-dimensional elasto-static problem

\[
\sigma_{ij,j} + b_i = 0, \quad \text{in } \Omega \tag{3.1}
\]
where \( \sigma_{ij} \) is the stress tensor, which corresponds to the displacement field \( u_i \), and \( b_i \) is the body force. The corresponding boundary conditions are given by

\[
\begin{align*}
  u_i = \pi_i, & \quad \text{on } \Gamma_u, \\
  t_i = \sigma_{ij}n_j = \overline{t}_i, & \quad \text{on } \Gamma_t,
\end{align*}
\]

where \( \pi_i \) and \( \overline{t}_i \) are the prescribed displacement and traction on the boundaries \( \Gamma_u \) and \( \Gamma_t \), respectively. \( n \) is the unit outward normal to the boundary \( \Gamma \).

Many numerical methods such as FEM, FVM, BEM, EFG and etc. are based on a global weak form of (3.1) over entire \( \Omega \), which can be derived using the integration by parts. However, the MLPG method starts from weak forms over sub-domains \( \Omega_k \) inside the global domain \( \Omega \). Sub-domains usually cover the entire domain \( \Omega \) and have simple geometries in order of doing the numerical integrations as much as possible.

Let \( X = \{x_1, x_2, \ldots, x_N\} \subset \Omega \) are scattered meshless points, where some points are located on the boundary \( \Gamma \) to enforce the boundary conditions. In this work, spherical (circular in 2D) subdomains \( \Omega_k = B(x_k, r_k) \cap \Omega \) with radius \( r_k \) centered at \( x_k \), and cubical (rectangular in 2D) subdomains \( \Omega_k = C(x_k, s_k) \cap \Omega \) with side-length \( s_k \) centered at \( x_k \) are employed. Of course, for boundary points, \( \partial \Omega_k \) intersects with the global boundary \( \Gamma \).

A local weak form of the equilibrium equation over \( \Omega_k \) is written as

\[
\int_{\Omega_k} (\sigma_{ij,j} + b_i)v_i \, d\Omega = 0,
\]

where \( v_i \) are appropriate test functions. We are not introduced Lagrange multiplier or penalty parameter in the weak form, because in our numerical method the essential boundary conditions are imposed in a suitable collocation form. Thus we assume \( x_k \) is located either inside \( \Omega \) or on \( \Gamma_t \) where the tractions are prescribed. Using \( \sigma_{ij,j}v_i = (\sigma_{ij}v_i)_j - \sigma_{ij}v_{i,j} \) and the Divergence Theorem, \( (3.4) \) yields

\[
\int_{\partial \Omega_k \setminus \Gamma_t} \sigma_{ij}n_jv_i \, d\Gamma - \int_{\Omega_k} \sigma_{ij}v_{i,j} \, d\Omega = \int_{\Omega_k} b_i v_i \, d\Omega,
\]

where \( n \) is the outward unit normal to the boundary \( \partial \Omega_k \). Imposing the natural boundary conditions \( \sigma_{ij}n_j = \overline{t}_i \) on \( \partial \Omega_k \cap \Gamma_t \), we have

\[
\int_{\partial \Omega_k \setminus \Gamma_t} \sigma_{ij}n_jv_i \, d\Gamma - \int_{\Omega_k} \sigma_{ij}v_{i,j} \, d\Omega = \int_{\Omega_k} b_i v_i \, d\Omega - \int_{\partial \Omega_k \cap \Gamma_t} \overline{t}_i v_i \, d\Gamma.
\]

In Petrov-Galerkin methods, the trial functions and the test functions come from different spaces. Thus there will be many choices for test functions \( v_i \), and this leads to a list of MLPG methods labeled from 1 to 6. But this may cause some difficulties in mathematical analysis. Up to the here, the new procedure is identical to the classical MLPG method. In the next section we pave the way of going from MLPG to DMLPG using the concept of GMLS approximation.
4. DMLPG formulation

Although, DMLPG uses the same local weak forms obtained from a Petrov-Galerkin formulation, but it is mathematically different from MLPG because direct approximations for local weak forms are provided to rule out the action of trial space.

Using the same labels as in MLPG, here we discuss DMLPG1 and 5 and leave the others for a new research. Note that there are some difficulties to develop DMLPG3 and 6 because they are based on a Galerkin formulation [10, 13].

We use the same scheme to impose the essential boundary conditions in all types of DMLPG. The MLS collocation method is applied on points located on \( \Gamma_u \),

\[
\sum_{\ell=1}^{N} a_{\ell}(x_k) u_i(x_k) = \overline{u}_i(x_k), \quad x_k \in \Gamma_u. \tag{4.1}
\]

In fact, the functional \( \lambda \) in GMLS is taken to be \( \delta_{x_k} \), the point evaluation functionals at \( x_k \). In following subsections, we consider the local weak forms around the points located either inside \( \Omega \) or over Neumann parts of the boundary \( \Gamma \).

4.1. DMLPG1

If test functions \( v_i \) are chosen such that they all vanish over \( \partial \Omega_k \setminus \Gamma_t \), then the first integral in (3.6) vanishes and if we define

\[
\lambda_k^{(i)}(u) := -\int_{\Omega_k} \sigma_{ij} v_i \, d\Omega, \quad \beta_k^{(i)} := \int_{\Omega_k} b_i v_i \, d\Omega - \int_{\partial \Omega_k \cap \Gamma_t} t_i v_i \, d\Gamma, \quad x_k \in \text{int}(\Omega) \cup \Gamma_t, \tag{4.2}
\]

then (3.5) becomes

\[
\lambda_k^{(i)}(u) = \beta_k^{(i)}, \quad x_k \in \text{int}(\Omega) \cup \Gamma_t.
\]

Now, the GMLS can be applied to approximate the above functionals. To simplify the notation, let

\[
\lambda_k = \begin{bmatrix} \lambda_k^{(1)} \\ \vdots \\ \lambda_k^{(d)} \end{bmatrix}, \quad \beta_k = \begin{bmatrix} \beta_k^{(1)} \\ \vdots \\ \beta_k^{(d)} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix}, \quad A_{k\ell} = \begin{bmatrix} a_{k\ell}^{(11)} & \cdots & a_{k\ell}^{(1d)} \\ \vdots & \ddots & \vdots \\ a_{k\ell}^{(d1)} & \cdots & a_{k\ell}^{(dd)} \end{bmatrix},
\]

where \( A = (A_{k\ell}) \) is introduced as a block matrix for reserving the acts of GMLS functions. Blocks of \( A \) are not diagonal, because \( \lambda_k^{(i)}(u) \) depends not only on \( u_i \) (for specified \( i \)) but also on all \( u_i \) for \( i = 1, \ldots, d \). The GMLS approximation can be used to write

\[
\lambda_k(u) \approx \tilde{\lambda}_k(u) = \sum_{\ell=1}^{N} A_{k\ell} u(x_\ell). \tag{4.3}
\]
According to (2.6), if $A_k,$ represents the $k$-th block row of $A$, then

$$A_k = \lambda_k(P) \Phi \in \mathbb{R}^{d \times dN}, \quad (4.4)$$

where $\Phi \in \mathbb{R}^{dQ \times dN}$ is a block matrix obtained from $\phi := (P^T WP)^{-1} WP^T \in \mathbb{R}^{Q \times N}$ by

$$\Phi_{ij} = \begin{bmatrix} \phi_{ij} & 0 \\ \vdots & \ddots \\ 0 & \phi_{ij} \end{bmatrix} \in \mathbb{R}^{d \times d}.$$

Matrices $P$ and $W$ are defined in (2.2), and

$$\lambda_k(P) = - \left[ \int_{\Omega_k} \varepsilon_v D P_1(x) d\Omega, \int_{\Omega_k} \varepsilon_v D P_2(x) d\Omega, \ldots, \int_{\Omega_k} \varepsilon_v D P_Q(x) d\Omega \right] \in \mathbb{R}^{d \times dQ}, \quad (4.5)$$

where for a two dimensional problem ($d = 2$) of isotropic material, the stress-strain matrix $D$ is defined by

$$D = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix},$$

where

$$E = \begin{cases} E & \text{for plane stress} \\ \frac{E}{1 - \nu^2} & \text{for plane strain} \end{cases}, \quad \nu = \begin{cases} \nu & \text{for plane stress} \\ \frac{\nu}{1 - \nu} & \text{for plane strain} \end{cases},$$

in which $E$ and $\nu$ are Young’s modulus and Poisson’s ratio, respectively. The strain matrix for test functions $v_i$ is

$$\varepsilon_v = \begin{bmatrix} v_{1,1} & 0 & v_{1,2} \\ 0 & v_{2,2} & v_{2,1} \end{bmatrix},$$

and

$$P_n(x) = \begin{bmatrix} p_{n,1}(x) & 0 \\ 0 & p_{n,2}(x) \\ p_{n,2}(x) & p_{n,1}(x) \end{bmatrix}, \quad n = 1, 2, \ldots, Q.$$

For the elasticity problem of isotropic material in 3D (i.e. $d = 3$), we have $D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \in \mathbb{R}^{6 \times 6}$ where

$$D_1 = \frac{E}{(1 - 2\nu)(1 + \nu)} \begin{bmatrix} 1 - \nu & \nu & \nu \\ \nu & 1 - \nu & \nu \\ \nu & \nu & 1 - \nu \end{bmatrix}, \quad D_2 = \frac{E}{2(1 + \nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
In addition, the strain matrix of test functions is
\[
\varepsilon_v = \begin{bmatrix}
v_{1,1} & 0 & 0 & 0 & v_{1,3} & v_{1,2} \\
0 & v_{2,2} & 0 & v_{2,3} & 0 & v_{2,1} \\
0 & 0 & v_{3,3} & v_{3,2} & v_{3,1} & 0
\end{bmatrix},
\]
and finally
\[
P_n(x) = \begin{bmatrix}
p_{n,1}(x) & 0 & 0 \\
0 & p_{n,2}(x) & 0 \\
0 & 0 & p_{n,3}(x) \\
p_{n,3}(x) & p_{n,2}(x) & 0 \\
p_{n,3}(x) & 0 & p_{n,1}(x) \\
p_{n,2}(x) & p_{n,1}(x) & 0
\end{bmatrix}, \quad n = 1, 2, \ldots, Q.
\]

For simplicity we choose \( v_1 = \cdots = v_d =: v \) in following numerical algorithms. To set up the final linear system, we first assume
\[
\mathbf{u} = [u_1(x_1), \ldots, u_d(x_1), u_1(x_2), \ldots, u_d(x_2), \ldots, u_1(x_N), \ldots, u_d(x_N)]^T \in \mathbb{R}^{dN \times 1}.
\]

Without loss of generality, let the first \( N_b \) meshless points are located on \( \Gamma_u \). The boundary matrix \( B \in \mathbb{R}^{dN_b \times dN} \) corresponds to the essential boundary conditions is a block matrix in which
\[
B_{k\ell} = \begin{bmatrix}
a_{\ell}(x_k) & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & a_{\ell}(x_k)
\end{bmatrix}_{d \times d},
\]
where \( a_{\ell}(x_k) \) are the values of GMLS shape functions defined in (4.1). Finally if we set
\[
K = \begin{bmatrix}
B \\
A
\end{bmatrix}_{dN \times dN}, \quad R = \begin{bmatrix}
\mathbf{u}(x_1) & \cdots & \mathbf{u}(x_{N_b}) & \beta_{N_b+1} & \cdots & \beta_N
\end{bmatrix}_{dN \times 1}^T,
\]
then we have the final system of linear equations
\[
K \mathbf{u} = R. \quad (4.6)
\]

Sometimes, in a boundary point \( x_k \) both tractions and displacements are prescribed, i.e. for some \( i \), tractions \( t_i \) and for the others, displacements \( u_i \) are known. Let for such point \( x_k \), displacements \( u_{i_1}, \ldots, u_{i_s} \) for indices \( \{i_1, \ldots, i_s\} \subset \{1, 2, \ldots, d\} \) are prescribed. Since the essential boundary conditions are applied using the collocation method, in the \( k \)-th block row of \( A \), rows \( i_1, \ldots, i_s \) should be replaced by corresponding MLS shape function vectors, say \( a^{(i_m)}_k, 1 \leq m \leq s \), of size \( dN \). These vectors are introduces as bellow: first we define \( a^{(i_m)}_k \) as zero \( dN \)-vectors. Then vector \( [a_1(x_k), a_2(x_k), \ldots, a_N(x_k)] \) of MLS shape functions is settled down in places \( i_m, i_m + d, \ldots, i_m + (N-1)d \) of \( a^{(i_m)}_k \). Of course the
corresponding right-hand sides should form by known boundary values $u_{im}$ instead of 
$\beta_{im}^{(1)}$.

In DMLPG process, integrations are only appeared in (4.5), where they are done over polynomials rather that MLS shape functions. This is the main idea behind the DMLPG approach. In fact, DMLPG shifts the numerical integration into the MLS itself, rather than into an outside loop over calls to MLS routines. Thus DMLPG is extremely faster than original MLPG.

Moreover, in some situations, we can get the exact numerical integrations with limited number of Gaussian points. For example, if cubical subdomains with polynomial test function $v$ are used in DMLPG1, the integrands are $d$-variate polynomials of degree $(m-1)\times(n-1)$, where $n$ is the degree of the polynomial test function. Thus a $\left\lceil \frac{(m-1)(n-1)+1}{2} \right\rceil$-point Gauss quadrature in each axis is enough for doing the exact numerical integration.

As a polynomial test function on the square or cube for DMLPG1 with $n = 2$, we can use
$$v = v(x; x_k) = \begin{cases} \prod_{i=1}^{d} \left(1 - \frac{1}{4k_s(x_i - x_k)^2}\right), & x \in C(x_k, s_k), \\ 0, & \text{otherwise} \end{cases} \quad (4.7)$$
where $x = (x_1, \ldots, x_d)$ and $x_k = (x_{k1}, \ldots, x_{kd})$. Note that, we should be careful for points located on the curved parts of the boundary.

4.2. DMLPG5

If $v = v_i \equiv 1$ are chosen over $\Omega_k$, then the second integral in (3.6) vanishes, and by defining
$$\lambda_k^{(i)}(u) := \int_{\partial \Omega_k \setminus \Gamma_t} \sigma_{ij} n_j d\Gamma, \quad \beta_k^{(i)} = \int_{\Omega_k} b_i d\Omega - \int_{\partial \Omega_k \cap \Gamma_t} t_i d\Gamma, \quad x_k \in \text{int}(\Omega) \cup \Gamma_t, \quad (4.8)$$
we have
$$\lambda_k^{(i)}(u) = \beta_k^{(i)}.$$  

As before, we apply the GMLS to find direct approximations for functionals $\lambda_k^{(i)}$. Equations are the same as those where obtained for DMLPG1, unless (4.5) which should be replaced by
$$\lambda_k(P) = \left[ \int_{\partial \Omega_k \setminus \Gamma_t} N D P_1(x) d\Gamma, \int_{\partial \Omega_k \setminus \Gamma_t} N D P_2(x) d\Gamma, \ldots, \int_{\partial \Omega_k \setminus \Gamma_t} N D P_Q(x) d\Gamma \right] \in \mathbb{R}^{d \times dQ}, \quad (4.9)$$
where $N$ is reserved for matrix of components of normal vector, which is defined for two dimensional problem by
$$N = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}.$$
and for three dimensional problem by

\[
\mathcal{N} = \begin{bmatrix}
n_1 & 0 & 0 & n_3 & n_2 \\
0 & n_2 & 0 & n_3 & 0 \\
0 & 0 & n_3 & n_2 & n_1 \\
\end{bmatrix}.
\]

One can see, the integrals in (4.9) are all boundary integrals. Thus DMLPG5 is even faster. Again if cubes are used as subdomains, a \( \left\lceil \frac{m}{2} \right\rceil \)-point Gauss quadrature in each axis gives the exact solution for local boundary integrals.

In the following section, some numerical experiments in two and three dimensional elasticity are presented to show the efficiencies of new methods.

5. Numerical results

The following compactly supported Gaussian weight function is used

\[
w(x, y) = \varphi(r) = \frac{\exp(-(-\epsilon r)^2) - \exp(-\epsilon^2)}{1 - \exp(-\epsilon^2)}, \quad 0 \leq r = \frac{\|x - y\|_2}{\delta} \leq 1,
\]

where the shape parameter \( \epsilon \) is taken to be 4 in this paper. Here \( \delta = \delta(x) \) is the radius of circular (in 2D) or spherical (in 3D) support of weight function \( w \) at point \( x \) in question. \( \delta \) should be large enough to ensure the regularity of moment matrix \( PTP \) in MLS/GMLS approximation. Thus \( \delta \) is proportional to \( h \) (mesh-size) and \( m \), say \( \delta = cmh \). If we have a varying-density data point, the support size \( \delta \) can vary from point to point in \( \Omega \). The polynomial degree \( m = 2 \) and both spherical and cubical subdomains are used. For spheres, the above Gaussian weight function with \( \delta \) being replaced by the radius \( r_k \) of the local domain \( \Omega_k \), is used as a test function, while for cubes, (4.7) is applied.

Displacement and strain energy relative errors will be presented in the following numerical examples. They are defined as

\[
r_u = \frac{\|u_{\text{exact}} - u_{\text{numerical}}\|}{\|u_{\text{exact}}\|}, \quad r_\epsilon = \frac{\|\epsilon_{\text{exact}} - \epsilon_{\text{numerical}}\|}{\|\epsilon_{\text{exact}}\|},
\]

where \( \|\cdot\| \) is a discrete 2-norm on a very fine mesh point in domain \( \Omega \).

All routines are written using MATLAB® and run on a Pentium 4 PC with 2.00 GB of Memory and a twin-core 2.80 GHz CPU.

5.1. Cantilever beam

As a benchmark problem in 2D elasticity, a cantilever beam loaded by a tangential traction on the free end, as shown in Fig. 1, is now considered. The exact solution of
this problem is given in Timoshenko and Goodier [19] as follow:

\[ u_1 = -\frac{P}{6EI} \left( x_2 - \frac{D}{2} \right) \left( 3x_1(2L - x_1) + (2 + \nu)x_2(x_2 - D) \right), \]

\[ u_2 = \frac{P}{6EI} \left[ x_1^2(3L - x_1) + 3\nu(L - x_1) \left( x_2 - \frac{D}{2} \right)^2 + \frac{4 + 5\nu}{4} D^2 x_1 \right], \]

where \( I = \frac{D^3}{12} \) and \( x = (x_1, x_2) \in \mathbb{R}^2 \). The corresponding exact stresses are

\[ \sigma_{11} = -\frac{P}{I} \left( L - x_1 \right) \left( x_2 - \frac{D}{2} \right), \]

\[ \sigma_{22} = 0, \]

\[ \sigma_{12} = -\frac{Px_2}{2I} (x_2 - D). \]

Both MLPG1 and DMLPG1 are applied with \( L = 8, D = 1, P = 1, E = 1, \nu = 0.25 \) for the plane stress case. The uniform mesh sizes \((33 \times 5), (65 \times 9)\) and \((129 \times 17)\) are used to detect the rates of convergence and computational costs of both techniques. Circular domains with radius \( r_k = 0.7h \), and rectangular domains with height-length \( h \times h \) are employed as sub-domains \( \Omega_k \) for all \( k \). As pointed before, for \( m = 2 \) a 2-point Gaussian quadrature in each axis is enough to get the exact numerical integrations over rectangles in DMLPG. But 10-point quadrature in each axis is used for circles \((r \text{ and } \theta \text{ directions})\) in both methods and rectangles in MLPG. The sufficiently large number of Gaussian points should be used to get the high accuracy for integration against MLS shape functions in MLPG. However, DMLPG works properly with fewer integration points, because there is no shape function incorporated in integrands. Here, to perform the comparisons in complexity side, we use the same number of Gaussian points for both methods with circular subdomains. Results are presented in Figs. 2 and 3 to compare the accuracy of numerical displacements, numerical strains in MLPG1 and DMLPG1 for rectangular and circular sub-domains. The rates are more or less the same, but the results of DMLPG with rectangles are surprising, because we have exact numerical integration.

As discussed, DMLPG is superior to MLPG in complexity side. To confirm this numerically, the CUP times used are compared in Fig. 4 for rectangular and circular subdomains.

Finally, the DMLPG solutions of normal stress \( \sigma_{11} \) and shear stress \( \sigma_{12} \) at \( x_1 = L/2 = 4 \) are plotted in Fig. 5 and compared with the exact solutions.

5.2. Infinite plate with circular hole

Consider an infinite plate with a central hole \( x_1^2 + x_2^2 \leq a^2 \) of radius \( a \), subjected to a unidirectional tensile load of \( \sigma = 1 \) in the \( x_1 \)-direction at infinity. There is an analytical
solution for stress in the polar coordinate \((r, \theta)\)

\[
\sigma_{11} = \sigma \left[ 1 - \frac{a^2}{r^2} \left( \frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3a^4}{2r^4} \cos 4\theta \right],
\]

\[
\sigma_{12} = \sigma \left[ -\frac{a^2}{r^2} \left( \frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3a^4}{2r^4} \sin 4\theta \right],
\]

\[
\sigma_{22} = \sigma \left[ -\frac{a^2}{r^2} \left( \frac{1}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3a^4}{2r^4} \cos 4\theta \right],
\]

with the corresponding displacements

\[
u_1 = \frac{1 + \nu}{E} \sigma \left[ \frac{1}{1 + \nu} r \cos \theta + \frac{2 \alpha^2}{1 + \nu} \cos \theta + \frac{1}{2} \frac{a^2}{r^2} \cos 3\theta - \frac{1}{2} \frac{a^4}{r^3} \cos 3\theta \right],
\]

\[
u_2 = \frac{1 + \nu}{E} \sigma \left[ -\nu \frac{1}{1 + \nu} r \sin \theta - \frac{1 - \nu}{1 + \nu} \frac{a^2}{r} \sin \theta + \frac{1}{2} \frac{a^2}{r^2} \sin 3\theta - \frac{1}{2} \frac{a^4}{r^3} \sin 3\theta \right].
\]

In computations, we consider a finite plate of length \(b = 4\) with a circular hole of radius \(a = 1\) (see Fig. [6]), where the solution is very close to that of the infinite plate [20]. Due to symmetry, only the upper right quadrant of the plate is modelled. The traction boundary conditions given by the exact solution are imposed on the right and top edges (see Fig. [6]). Symmetry conditions are imposed on the left and bottom edges, i.e., \(u_1 = 0, t_2 = 0\) are prescribed on the left edge and \(u_2 = 0, t_1 = 0\) on the bottom edge, and the inner boundary at \(a = 1\) is traction free, i.e. \(t_1 = t_2 = 0\). Numerical results are presented for a plane stress case with \(E = 1.0\) and \(\nu = 0.25\). The initial set point is depicted in Fig. [6] where we use more points near the hole. Thus the support size \(\delta\) varies according to the density of neighboring points. Here \(\delta = 2mh\) and \(\delta = 2.5mh\) are used for points near the hole and points far away from the hole, respectively. Mesh-size \(h\) is defined to be \(\min\{hr, h\theta\}\) for near points. In DMLPG, we use circular subdomains for points located on the arc boundary \(r = a\), and square subdomains for other points. Computations are repeated by halving \(hr\) and \(h\theta\), twice. Results are presented in Figs. [7] and [8] which compare the displacement errors, the strain energy errors, and the CPU times used. Moreover, the exact normal stress \(\sigma_{11}\) at \(\chi = 0\) is plotted in Fig. [9] and compared with the DMLPG solution.

5.3. 3D Boussinesq problem

The Boussinesq problem can be describe as a concentrated load acting on a semi-infinite elastic medium with no body force. The exact displacement field within the semi-infinite medium is given by Timoshenko and Goodier [19]

\[
u_x = \frac{(1 + \nu)P}{2E\pi\rho} \left( \frac{2\pi}{\rho^2} - \frac{(1 - 2\nu)r}{\rho + z} \right),
\]

\[
w = \frac{(1 + \nu)P}{2E\pi\rho} \left( \frac{\rho^2}{\rho^2 + 2(1 - \nu)} \right).
\]
where \( u_r \) is the radial displacement, \( w \) (or \( u_3 \)) is the vertical displacement, \( \rho = \sqrt{x_1^2 + x_2^2 + x_3^2} \) is the distance to the loading point and \( r = \sqrt{x_1^2 + x_2^2} \) is the projection of \( \rho \) on the loading surface. The exact stresses field is

\[
\sigma_r = \frac{P}{2\pi\rho^2} \left[ -\frac{3zr^2}{\rho^3} + \frac{(1 - 2\nu)\rho}{\rho + z} \right],
\]

\[
\sigma_\theta = \frac{(1 - 2\nu)P}{2\pi\rho^2} \left[ \frac{z}{\rho} - \frac{\rho}{\rho + z} \right],
\]

\[
\sigma_{zz} = -\frac{3\pi z^3}{2\pi\rho^3},
\]

\[
\tau_{zr} = \tau_{rz} = -\frac{3\pi rz^2}{2\pi\rho^3}.
\]

It is clear that the displacements and stresses are strongly singular and approach to infinity; with the displacement being \( O(1/\rho) \) and the stresses being \( O(1/\rho^2) \). MLPG has been applied to this problem in [3].

In numerical simulation, a finite sphere with large radius \( b = 10 \) is used. Due to the symmetry, a first one-eighth of the sphere is considered and symmetry boundary conditions are applied on planes \( xz \) and \( yz \) (see Fig. 10). In fact we impose \( t_1 = u_2 = t_3 = 0 \) on plane \( xz \), and \( u_1 = t_2 = t_3 = 0 \) on plane \( yz \). In order to avoid direct encounter with the singular loading point, the theoretical displacement is applied on a small spherical surface with radius \( b/40 = 0.25 \). An isotropic material of \( E = 1000, \nu = 0.25 \) and \( P = 1 \) is used. The number of meshless points is 1386, which are scattered inside the domain and on the boundary. The density of nodes depends on the distance from the loading points, where we have many points near the small sphere and few points far from it (see Fig. 10). Thus the support size \( \delta \) varies and depends on \( \rho \), correspondingly. Analytical and DMLPG solutions of the radial displacement \( u_r \) and vertical displacement \( w \) on the surface \( xy \) are plotted in Fig. 11. The Von Mises stress on the surface \( xy \) is also shown in Fig. 12. These are the results of DMLPG1 with cubes as sub-domains where the CPU time used is around 5 seconds. Again we note that a 2-point Gaussian quadrature in each axis gives the exact numerical integration. The same results will be obtained by DMLPG5.

Finally, for comparison we apply both MLPG1 and MLPG5 to this problem with the same meshless points and MLS parameters. The accuracy of results are far less than DMLPG solutions and the CPU run times are about 7400 sec. for MLPG1 and 450 sec. for MLPG5. In computations, a 10-point Gauss formula is employed in each axis. In fact, for MLPG1, the MLS shape function subroutines should be called 1000 times to integrate over a sub-domain \( \Omega_k \). In MLPG5 this number reduces to 100, because the integrals are all boundary integrals in this example. Compare with DMLPG where the MLS subroutines are not called for integrations at all, leading to 5 sec. running time in this example.
6. Conclusion

In this paper we developed a new meshfree method for elasticity problems, which is a weak form method in the cost-level of collocation (integration-free) methods. Integrations have been shifted into the MLS itself, rather than into an outside loop over calls to MLS routines. In fact, we need to integrate against low-degree polynomials basis functions instead of complicated MLS shape functions. Besides, in some situations we can perform exact numerical integrations. We applied DMLPG1 and 5 for problems in two and three dimensional elasticity in this paper. The new methods can be easily applied to other problems in solid engineering. On a downside, DMLPG1 and 5 do not work for linear basis functions ($m = 1$). In addition, because of symmetry properties of polynomials in local sub-domains, [10] shows that the convergence rates do not increase when going from $m = 2k$ to $m = 2k + 1$. But the results show that this observation affects MLPG and DMLPG in the same way. DMLPG4 can be formulated using the strategy presented in [21] to make the second unsymmetric local weak forms and then applying the GMLS approximation of this paper. Finally, we believe that DMLPG methods have great potential to replace the original MLPG methods in many situations, specially for three dimensional problems.

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Figure 1: A cantilever beam
Figure 2: Relative displacement and strain errors for beam, rectangular subdomains
Figure 3: Relative displacement and strain errors for beam, circular subdomains
Figure 4: Computational costs for beam, rectangular (left) and circular (right) subdomains
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