INTRODUCTION

To extract localized information via selective excitation and/or spatial encoding, MRI sequences generate spatiotemporally modulated transverse magnetization and reconstruct the recorded signal. In principle, any MRI experiment can be fully described by solving the local dynamic equations. However, the vast majority of MRI sequences comprises unbalanced gradients, effecting significant intravoxel modulations of the magnetization density, which are not related to spatial encoding. For the whole voxel, one therefore has to integrate numerically over many independent local solutions. This is possible, but provides little to no intuition to which extent different magnetization pathways contribute to the reconstructed signal and how the latter precisely depends on tissue properties and sequence design. This deficiency motivated the search for alternative descriptions, which are better adapted to MR imaging.

Already prior to the uprise of MRI, Kaiser et al.1 analyzed field gradient effects in NMR Fourier transform...
s spectroscopy by two independent techniques, namely Fourier expansion with respect to local off-resonance and the partition method. Hennig refined the latter by introducing extended phase graphs (EPG) to MRI.\textsuperscript{2–4} Partition method and EPG are closely related and split the transverse magnetization into two complex conjugate components with opposed phase evolution, as first proposed by Woessner.\textsuperscript{5} Instantaneous RF pulses effect transitions between these two and the longitudinal component. While the partition method keeps track of each magnetization pathway separately, EPG collects equivalent phase evolutions in \textit{configurations} $F_n^V$ (transverse components) and $Z_n$ (longitudinal component).

In its original formulation, the EPG framework constitutes a genuine imaging theory of wide scope, applicable to periodic and non-periodic sequences alike. It not only allows to predict the appearance of \textit{echoes}, but also to calculate their amplitudes. Echoes are formally associated with (partial) rephasing of transverse magnetization in a superposition of \textit{isochromats} (= spins with equivalent phase accumulation), while remaining flexible about the physical origin (bulk off-resonance, gradients) of the latter. In the general non-periodic case, echoes can be classified based upon the contributing magnetization pathways (cf. fig. 11 in Ref. \textsuperscript{[6]}), but since the number of generated echoes increases rapidly with the number of RF pulses,\textsuperscript{3} this approach is only practical for a moderate number of RF pulses.

Fortunately, the most prevalent families of MRI sequences comprise free precession intervals of constant duration $\tau$ and zero-order gradient moment $p$, such that spins accumulate a constant phase $\theta = -\omega \tau - px$ in each interval. Although the bulk off-resonance frequency $\omega$ can usually be considered as already determined by the position $x$, we will remain flexible and (mostly) treat it as an independent parameter to simplify the discussion of inhomogeneous broadening. In that interpretation, isochromats are defined by the pair $(x, \omega)$. Due to the assumed periodicity, we may write the magnetization density immediately before or after any RF pulse as a Fourier series with phase factors $e^{i n \theta}$ ($n \in \mathbb{Z}$) and the EPG configurations as coefficients. Again, the configuration, which survives the voxel average ($\int d\theta \cdots$), defines the EPG echo.

For the remainder of this article, we will restrict our notion of EPG to this periodic, Fourier-based specialization, which has been successfully applied to GRE and (multi-)SE sequences, including effects due to bulk motion,\textsuperscript{7} isotropic,\textsuperscript{7} and anisotropic\textsuperscript{3} diffusion, magnetization exchange,\textsuperscript{9} and susceptibility variations.\textsuperscript{10,11} An in-depth review of EPG variants has been given by Weigel.\textsuperscript{6}

The Fourier series representation is particularly valuable, since it isolates spatial modulations due to bulk off-resonance and gradients in the phase factors $e^{i n \theta}$ and therefore allows for a concise and rigorous treatment of signal localization. In this article, we derive a generalization, which is applicable to arbitrary MRI sequences. In a sloppy notation (the rigorous treatment follows below), we will switch to an \textit{integral} representation of the isochromat $m(x, \omega, t)$

$$\sum_{n \in \mathbb{Z}} e^{-i \omega \tau \cdot px} m^{(n)}(x, t) \rightarrow \int dp \, d\tau \, e^{-i \omega \tau \cdot px} \hat{m}(p, \tau, x, t)$$

which we call the \textit{continuous configuration model} (CCM).

The representation becomes unambiguous and meaningful, by demanding that the position dependence of the configurations $\hat{m}(p, \tau, x, t)$ only reflects intrinsic tissue properties, bulk motion and/or $B_1^\uparrow$ variations. Just by inserting the CCM into the dynamic equations for the local magnetization density $m(x, \omega, t)$, we then obtain an equivalent set of partial differential equations for $\hat{m}(p, \tau, x, t)$.

To become more familiar with the CCM, we discuss several examples and topics:

- relation between CCM and EPG
- signal localization in arbitrary MRI sequences
- signal formation in frequency swept NMR (SWIFT)\textsuperscript{12}

with additional exemplary material provided in the supporting information (SI), which is available online.

The article finishes with a general discussion how the CCM might benefit the development and optimization of novel imaging sequences.

2 | THEORY

2.1 | Configuration space representation

We assume that the state of the local spin density is specified by some vector $s(x, t)$, such that the magnetization density $m(x, t)$

$$m := \begin{bmatrix} m_x + im_y \\ m_x - im_y \\ m_z \\ m_z \end{bmatrix} =: \begin{bmatrix} m_+ \\ m_- \\ m_x \\ m_z \end{bmatrix}$$

can be extracted from the state vector $s$ by a suitable linear operation $m = Ls$. We further demand that the temporal evolution of the state vector is governed by a differential equation

$$\partial_t s = Xs - i (\gamma Gx + \omega) Ps - R (s - s_{eq})$$

$G(t)$ denotes the applied gradient and $\omega(x)$ refers to bulk off-resonance.
As discussed in the introduction, the local state \( s(x, t) \) provides no information about its immediate spatial vicinity. We therefore propose a different representation,

\[
s(x, t) = \int dp \, d\tau \, e^{-i[\omega(p)x + p\tau]} \tilde{s}(p, \tau, x, t)
\]

(4)

in which spatial modulations due to gradients and bulk off-resonance are expressed by the phase factors \( e^{-i[\omega(p)x + p\tau]} \), while the residual position dependence of \( \tilde{s}(\cdot, \cdot, x, \cdot) \) is solely due to intrinsic tissue properties, bulk motion effects or RF pulses. The configurations \( \tilde{s} \) represent the time-dependent state of the local spin ensembles by a vector field in configuration space \( (p, \tau, x) \) and we will refer to the representation (4) as **continuous configuration model** (CCM) in the following.

We insert the CCM (4) into the differential equation (3), use partial integration to remove the explicit dependence on \( x \), and thereby transform Equation (3) into

\[
\partial_t \tilde{s} = \hat{X} \tilde{s} - (\gamma G \nabla_p + \partial_r) P \tilde{s} - R (\tilde{s} - \tilde{s}_eq)
\]

(5)

The thermal equilibrium configuration \( \tilde{s}_eq \) contains no history of accumulated phases:

\[
\tilde{s}_eq(p, \tau, x) := \delta(p) \delta(\tau) s_{eq}(x)
\]

(6)

Technically, we are done with theory.

In the remainder of this article, we will try to justify the more complicated representation (4) of the spin state density.

### 3 | METHODS

#### 3.1 | Tissue model

In this article, we consider a pure, static tissue, such that the state vector \( s \) can be identified with the magnetization density \( m \) and Equation (3) with the Bloch equations:

- \( P = \text{diag}[1, -1, 0] \) (precession)
- \( R(x) = \text{diag} \{ R_x, R_y, R_z \} \) (relaxation, repolarization)
- \( s_{eq}(x) = m_{eq}(x) \) (thermal equilibrium magnetization density)
- \( X(x, t) = \hat{X}(x, t) = -i \Omega_1(x, t) \) (matrix, describing spin rotation by RF pulses)

The interested reader is referred to the (SI) for possible extensions (diffusion, magnetization exchange/transfer, quantum mechanical models), which can be also be described by Equation (3) but require a different interpretation of (at least some of) its components \( (s, s_{eq}, X, P, R) \).

Including effects due to arbitrary bulk motion is also possible by adding a term on the RHS of Equations (3) and (5). Again, the details can be found in the (SI).

### 3.2 | Numerical simulations

The numerical implementation of the CCM made use of the hard pulse approximation and allows to include effects due to magnetization exchange/transfer, diffusion and/or bulk motion, as derived in the (SI).

For various examples, we confirm in the supporting information (SI) that the CCM is fully consistent with established theory.

### 4 | RESULTS

In the following, we first discuss how the CCM generalizes EPG theory, investigate general aspects of signal localization in MRI and then take a closer look at signal formation in frequency swept NMR (SWIFT).

#### 4.1 | Relation to EPG

Within the **hard pulse approximation**, any RF pulse or complete MRI sequence is decomposed into an alternating series of instantaneous small-angle rotations and free precession periods without RF transmission \( (\Omega_1 \equiv 0) \).

Instantaneous rotations are expressed by linear transformations

\[
\tilde{s}(p, \tau, x, t^+) = U(x, t) \tilde{s}(p, \tau, x, t^-)
\]

with the rotation matrix \( U \) defined precisely as for the state vector \( s \).

For any free precession interval \( I := [t_a, t_b] \), the differential Equation (5) can be integrated in closed form, but the details depend on the tissue model. Specifically for the Bloch equations (see the (SI) for more examples), we obtain for any \( t =: \delta t + t_e \in I \) the following recurrence relations for the transverse and longitudinal components:

\[
\hat{m}_z(p, \tau, x, t) = e^{-R_z(x)\delta t} \hat{m}_z(p \mp \delta p, \tau \mp \delta \tau, x, t_a)
\]

(8)

\[
\begin{align*}
\hat{m}_z(p, \tau, x, t) &= e^{-R_z(x)\delta t} \hat{m}_z(p, \tau, x, t_a) \\
&+ (1-e^{-R_z(x)\delta t}) \hat{m}_{eqz}(p, \tau, x)
\end{align*}
\]

(9)

The accumulated gradient moment \( \delta p(t) \) follows the common trajectory:

\[
\delta p(t) := \gamma \int_{t_a}^{t_b} dt' G(t')
\]

(10)
In the hard pulse approximation, any MRI sequence has a sparse support in configuration space, described by a finite set of $\delta$-functions

$$\hat{s}(p, \tau, x, t) = \sum_n \delta(p - p_n(t)) \delta(\tau - \tau_n(t)) s^{(n)}(x, t)$$

(11)

This is so because

- the origin of each magnetization pathway—freshly re-polarized longitudinal magnetization in each precession interval—is proportional to $\delta(p)\delta(\tau)$, cf. Equation (6).
- the configurations change their location in configuration space during free precession intervals, cf. Equations (8) and (9). Step size and direction depend on the diagonal elements of $P$, which ultimately increases the number of occupied configurations.
- instantaneous RF pulses (7) mix the configuration vector components, but do not affect their location in configuration space.

The representation (11) turns the integral (4) into a (possibly multi-periodic) Fourier series.

For periodic sequences, the recurrence relations (8) and (9) reproduce the known EPG expressions$^{14}$

$$s^{(n)} = \begin{bmatrix} m_{x_+}^{(n)} \\ m_{x_-}^{(n)} \\ m_{z}^{(n)} \end{bmatrix} = \begin{bmatrix} F_n \\ F_s \\ -n \end{bmatrix} Z_n$$

(12)

For non-periodic sequences, the multi-periodic Fourier representation (4) and (11), in combination with explicit recurrence relations (such as (8) and (9)) constitutes a (discrete) extension of the EPG framework.

4.2 Signal localization in MRI

Besides selective excitation, MRI relies on postprocessing of data to generate localized information. The signal $d_x(t)$, acquired in coil element $y$ with sensitivity $c_y$, depends on the transverse magnetization density $m_x$ and noise $\eta_y$

$$d_y(t) = \int dx \ c_y(x) m_x(x, t) + \eta_y(t)$$

(13)

Because of $m_x = \hat{L}_y s$, we may apply the CCM (4) and obtain with $\hat{m}_x := \hat{L}_y \hat{s}$

$$d_y(t) = \int dx \ dp \ d\tau \ e^{-i[\omega(x)\tau + px]} \ c_y(x) \ \hat{m}_x(p, \tau, x, t) + \eta_y(t)$$

(14)

(This is the point, where the linearity of $L$ matters, since we pulled $L_y$ inside the integral on the RHS of Equation (4).)

From a signal processing perspective, spatial discretization can be viewed as projection to an approximation space,$^{15}$ spanned by a set of basis functions $\phi_p(x)$. For smooth coil sensitivities $c_y$, we then get

$$c_y(x) \ \hat{m}_x(p, \tau, x, t) \approx \sum_p c_{r\rho} \ \hat{m}_{\rho\rho}(p, \tau, t) \ \phi_p(x)$$

(15)

Scaled sinc functions $\phi_p(x) := \prod_s \text{sinc}((x_j - x_{p_j}) / \Delta x_j)$, located at a regular grid $x_p$ of spacing $\Delta x$, constitute the most natural and popular choice for the basis functions $\phi_p$ in MRI. Since their Fourier transform $\phi_p(p) = e^{-i p x} \prod_s \theta(\pi / \Delta x_j - |p_j|)$ is well localized ($\theta$ denotes the Heaviside step function), only configurations $\hat{m}_{\rho\rho}$ with $|p_j| \leq \pi / \Delta x_j$ contribute to the signal. This is not a stringent criterion though, since the magnetization is usually not perfectly represented by the basis $\phi_p$ (e.g., due to partial volume effects) and configurations with $|p_j| > \pi / \Delta x_j$ may leak into the signal.

In summary, the finite set of acquired data $d_{\rho}(t_x) := d_{\rho}(t_x)$ is approximately linked to the set of discretized configurations $\hat{m}_{\rho\rho}$ via

$$d_{\rho}(t_x) \approx \sum_p c_{\rho\rho} \ \int d\tau \ e^{-i\omega_p \tau} \ \int dp \ e^{-i p x} \ \hat{m}_{\rho\rho}(p, \tau, t_x) + \eta_{\rho}(t_x)$$

(16)

The terms in the first integral of Equation (16) relate to bulk off-resonance and inhomogeneous broadening. $\omega_p = \langle \omega \rangle_p$ denotes the average bulk resonance frequency over the (voxel) volume, covered by the function $\phi_p(x)$. With respect to susceptibility effects, intra-voxel frequency variations $\omega - \omega_p$ are usually treated as independent from gradient-induced modulations and described by a frequency distribution $r_p(\omega)$. For a (truncated) Lorentzian,$^{16}$ we obtain the familiar attenuation factor $r_p(\tau) \approx e^{-|\tau| / r_p}$ for not too small $|\tau|$. Note that the absolute value is not superfluous, since $\tau$ can become negative.)

In Equation (16), we only cared about the spatial encoding aspect—what about selective excitation? This aspect of localization is also encoded in the $p$-dependence of $\hat{m}_{\rho\rho}$. For 3D imaging, Equation (16) still applies and takes care of the shape of the excited block. For 2D imaging, we have to integrate over the slice profile, which can be formally achieved by regarding $\phi_p$ as constant along the slice normal (indicated by "\*"). We then have $\phi_p(p_1 + p_\perp \propto \delta(p_\perp)$ and it suffices to add a factor $\delta(p_\perp)$ to the integrand in Equation (16).

Conventional, pulsed FT NMR relies on a sparse occupation of k-space, based upon a clear functional separation of (balanced) spatial encoding gradients and unbalanced crushers/spoilers. With respect to the hard
pulse approximation (11), this means that all $p_n(t_c)$, which satisfy the condition $|p_n(t_c)| < \pi/\Delta x$ must have the same value $k_c$. Any other occupied configurations are shifted outside of k-space with crusher gradients or do not exist at all (bSSFP).

For pulsed FT NMR, Equation (16) therefore simplifies to the more familiar expression

$$d_{x} \approx \sum_{n} e^{-ik \cdot x} c_{p,n} e^{-i\omega_{r} t_{p,n}} \rho(t_{n}) \widetilde{m}_{c}(t_{c}) + \eta_{fx}(17)$$

where the sum over $n$ is restricted to those configurations, which satisfy $p_{n}(t_{c}) = k_{c}$.

Let us finish this section with a few comments:

- The sum over $n$ is mainly relevant for bSSFP, since usually only a single $n$ is present in sequences with unbalanced gradients.
- The connection between acquired data $d_{x}$ and localized configurations in Eqs. (16) and (17) is not limited to steady-state conditions, but holds generally and can be used for spatiotemporal reconstructions.
- The functional separation of spatial encoding and crusher gradients is not a mandatory requirement for MRI sequences, as we will see below in our brief discussion of frequency swept NMR (SWIFT). In such cases, we have to work with the more general Equation (16).

4.3 Frequency swept NMR (SWIFT)

While transmit and receive periods are well separated in conventional pulsed FT NMR, a different concept is pursued in SWIFT. Here, short signal readouts are interspersed into frequency swept RF pulses in the presence of slowly modulated gradients, the latter of which simultaneously encode spatial information and (as we will see) suppress magnetization pathways. As the echo time is essentially zero, SWIFT sequences are predominantly used to visualize tissues with very short $T_2$ and acquired with rather small (effective) flip angles. Under these conditions, measurable transverse magnetization approximately originates from direct excitation of an essentially time-invariant reservoir of longitudinal magnetization $m_z(x)$ and $\delta \gamma G_{\text{v}} + \Delta_{z} + R_{2}$.

We specified the frequency swept RF pulse by

$$\Omega_{i}(t) := \omega_{i}(t) e^{i\varphi(t)}(19)$$

in the phase-modulated frame, such that $\omega_{i}$ and $\varphi$ denote the time-dependent RF amplitude and phase, respectively.

The integral of (18) reads

$$\hat{m}_{c}(p, r, x, t) \approx -i \Omega_{i}(t) \delta(p - rG) \theta(\tau) e^{-R_{2}(x)} m_{z}(x)(20)$$

(Interpreted as a distribution, the Heaviside step function $\theta$ has a derivative: $\theta' = \delta$)

After inserting this solution into the CCM (4), the acquired signal (13) can be written as a convolution

$$d(t) \approx h \ast \Omega_{i}(t)(21)$$

of the impulse response function ($= \text{FID}$)

$$h(t) := -i \theta(t) \int dx e^{-i[\omega(x) + \gamma Gx] t} e^{-R_{2}(x) t} m_{z}(x)(22)$$

and the RF pulse $\Omega_{i}(t)$. In the original SWIFT article, essentially the same convolution formula (Equation 4 therein) was motivated with ideas from stochastic NMR.

Since the signal purely relies on the recovery of longitudinal magnetization, it depends on $T_1$ only and may be expressed by an Ernst formula. On the other hand, the bright CSF in fig. 7 of Ref. [21] indicates that transverse coherences should be taken into account for $T_2 > TR$ and moderate flip angles.

In order to take a closer look at signal formation, let us consider a sequence of frequency swept RF pulses of duration $T_{p} = TR$ in presence of a constant gradient $G$, as depicted in Figure 1. The magnetization density at the local resonance frequency $\tilde{\omega} := \omega + \gamma Gx$ approaches a $T_{2}$-periodic steady-state

$$m_{c}(\tilde{\omega}, x, t) \approx \sum_{n} e^{-i2\pi t/\text{TR}} f_{j}(\tilde{\omega}, x)(23)$$

In the rapid passage linear region, excitations occur approximately instantaneously at those time points $t(\tilde{\omega})$, when the swept RF frequency $\omega_{SF}$ matches the local resonance frequency $\tilde{\omega}$. In this simplified view, the local steady state allows for a conventional EPG decomposition in terms of bSSFP configurations $m_{c}^{(n)}(x)$, leading to a closed form estimate of the Fourier coefficients $f_{j}$.
The window function \( w(\tilde{\omega}) \approx \theta(\Omega - |\tilde{\omega}|) \) reflects the finite bandwidth of the frequency sweep.

For a constant gradient \( G \), a configuration space representation \((4)\), which satisfies the differential equation \((5)\) in the free precession periods between the instantaneous RF pulses, is given by

\[
\hat{m}_+(\mathbf{p}, \tau, \mathbf{x}, t) \approx \delta(\mathbf{p} - \tau \mathbf{G}) \cdot \int \frac{d\tilde{\omega}}{2\pi} e^{i\tilde{\omega} \cdot \mathbf{z}_j(\tilde{\omega})} m_+(\tilde{\omega}, \mathbf{x}, t) \quad (25)
\]

In the bSSFP approximation \((24)\), the integral in Equation \((25)\) turns into a contour integral along the imaginary axis after variable substitution \( \mathbf{z}_j := i \tilde{\omega} \). To estimate its value, we make use of the residue theorem, noting that the analytic continuation \( \hat{f}(\mathbf{z}, \mathbf{x}) := f(\tilde{\omega}, \mathbf{x}) \) exhibits singularities at \( \mathbf{z}_j := ij \frac{2\pi}{\Delta \mathbf{G}} - R_j(\mathbf{x}) \). The behavior at large \( |\mathbf{z}| \) is dominated by exponentials of the form \( e^{x_j} \), and, as usual, we close the integration contour to the side where they become small, as shown in Figure 2. Since \( \text{Re}(\mathbf{z}_j) < 0 \), this implies that the poles are only enclosed for \( x_j > 0 \) as expressed by the \( \theta \)-functions. We further assumed that the chosen window function \( w \) permits an analytical continuation \( \hat{w} \) with \( \hat{w}(\mathbf{z}_j) \approx w(j \frac{2\pi}{\Delta \mathbf{G}}) \), such that the summation is approximately restricted to \( |\text{Im}(\mathbf{z}_j)| < \Omega \).

For \( T_2 \gg 1 \)K, moderate flip angles and sufficient distance from \( t = \cdot \tau(\tilde{\omega}) \), the bSSFP picture approximates the local magnetization density \( m_+ \) and the configuration \( \hat{m}_+ \) reasonably well, cf. Figure 3.

Since only configurations with \( |p_j| < \frac{\pi}{\Delta x_j} \) (in all directions \( j \)) contribute to the reconstructed localized signal, only two configurations from Equation \((26)\) are observable in each acquired spoke, if the gradient amplitude is adapted to the desired resolution \( G = \frac{2\pi}{\gamma \Delta x} \):

- **Prior** to the effective excitation time \( \tau(\tilde{\omega}) \), we observe an ECHO \((n = -1)\) moving toward the center at \( \mathbf{p} = \mathbf{0} \).
- **After** the effective excitation time \( \tau(\tilde{\omega}) \), we observe a FID \((n = 0)\) moving away from the center.

\[
\hat{m}_+(\mathbf{p}, \tau, \mathbf{x}, t) \approx \delta(\mathbf{p} - \tau \mathbf{G}) \cdot \frac{1}{\tau} \sum_{n}^{\infty} \left| \frac{\text{Im}(\mathbf{z}_j)}{\Omega} \right| < \Omega \quad e^{i \frac{2\pi}{\Delta \mathbf{G}} (\tau - n \Delta \mathbf{G}) / \tau R_j(\mathbf{x})} e^{-R_j(\mathbf{x})(\tau - n \Delta \mathbf{G})} \quad (26)
\]

\[
\approx \left[ \theta(\tau - n \Delta \mathbf{G}) - \theta(\tau - (n+1) \Delta \mathbf{G}) \right] m_+^{(n)}(\mathbf{x})
\]
Figure 4 shows how the configurations spread in configuration space as a function of time. To compare with the sparse k-space occupation in pulsed FT NMR, we also include a corresponding plot for the DESS24 sequence, depicted in Figure 1. The script `swift.m`, which was used to generate Figures 3 and 4, also illustrates the sudden transition from the ECHO to the FID signal around the k-space center.

In a real SWIFT acquisition, the gradient $G$ must be reoriented from pulse to pulse in order to allow for a 3D reconstruction. This can also be simulated with `swift.m` (consistent with our choice $T_p = 1\text{K}$, under the assumption of a constant rotation speed) and we observe a signal decay, increasing with speed of rotation and distance from the isocenter. It is therefore important to ensure a sufficiently adiabatic reorientation of $G$.

In summary, with respect to signal formation, SWIFT and SSFP are strongly connected, in accordance with the observed similar contrast. The main difference lies in the spatial encoding, as visualized in Figure 4B (SWIFT) versus Figure 4C (DESS). In DESS, reconstruction of FID and ECHO poses no problems, since both signals are well separated from each other and from the RF pulses. In contrast, the transition ECHO $\rightarrow$ FID in SWIFT coincides with effective excitation near the center of k-space.

Together with the superposition of different k-space locations in the recorded data, spatial reconstruction becomes more challenging (cf. Discussion).

5 DISCUSSION

In this article, we proposed the continuous configuration model (CCM) (4) as a representation of the magnetization density, devised to simplify the handling of signal localization in arbitrary MRI sequences. Just by inserting the CCM into any appropriate set of dynamic equations (3), governing the state vector $s$ of the spin density, we obtained a related set of differential equations (5) for the configurations $\hat{s}$.

It was shown that the data $d$, acquired in any MRI sequence, are directly linked to the discretized configurations $\hat{m}_{sp} = L_s \hat{s}_p$ via Equation (16).

We convinced ourselves that the CCM generalizes the extended phase graph (EPG) formalism, in particular, its most commonly encountered Fourier-based variant for periodic sequences. While the Fourier integral definition of EPG states 6 can be considered as applicable to arbitrary MRI sequences, the CCM shows how to update them without having to resort to local solutions.

The CCM isolates spatial modulations due to gradients and bulk off-resonance in the phase factors $e^{-i\omega(x)p \cdot x}$. Precisely as in EPG, the Fourier variables $(p, \tau)$ characterize the magnetization pathways, which contribute to a given configuration:

- $p = \text{accumulated net gradient moment}$
- $\tau = \text{accumulated net precession time}$

The configuration $\hat{s}(p, \cdot, \cdot, \cdot)$ therefore determines the

- shape of the selectively excited slice/volume
- spatial encoding for $|p_j| < \pi/\Delta x_j$ (k-space)
- suppression of magnetization pathways for $|p_j| > \pi/\Delta x_j$ (crushers)

On the other hand, $\hat{s}(\cdot, \tau, \cdot, \cdot)$ controls

- the local signal phase (voxel average)
- signal loss due to inhomogeneous broadening (susceptibility effects)

Finally, the spatial dependence $\hat{s}(\cdot, \cdot, x, \cdot)$ is relevant for discretization and relates to

- local tissue properties
- bulk motion
- $B_1^+$ inhomogeneity
With respect to gradients, pulsed FT NMR sequences share a clear functional separation between localization and contrast generation (cf. Equation (17)) and this is the reason, why spatial encoding gradients can usually be ignored in simulations. The EPG framework has been successfully applied to virtually any relevant pulsed FT NMR sequence, up to randomized variants like unbalanced magnetic resonance fingerprinting (MRF).25 Additional value, generated...
by the flexibility of the CCM, should therefore mainly be expected in the assessment of experimental imperfections, for example, due to

- variable eddy currents (e.g., caused by rotating spirals in MRF)
- the leakage of unwanted magnetization pathways into the reconstructed signal.

To demonstrate that the CCM is not limited to pulsed FT NMR sequences, we took a closer look at frequency swept NMR. SWIFT\(^{12}\) is predominantly recognized as a silent zero echo time sequence, but the analysis of signal formation revealed a close connection to unbalanced SSFP and a related contrast, which can be expected to persist under more general conditions such as variable flip angles and/or phase cycling. Beyond its current applications, SWIFT could therefore become of interest for silent conventional or quantitative MRI.

Image reconstruction is more involved though, since the acquired data \(d_{\alpha}\) no longer correspond to well-defined locations in \(k\)-space (compare Figure 4B and C). Since each gradient serves more than one purpose in frequency swept NMR, we further need to disentangle tissue properties from spatial encoding in the function \(\hat{m}_{+\alpha}(p, \cdot, \cdot)\) in Equation (16). We actually did so in the bSSFP approximation (26), when we attributed the observed contrast of SWIFT to the SSFP configurations \(\hat{m}^{(0)}_{+\alpha}(FID)\) and \(\hat{m}^{(-1)}_{+\alpha}(ECHO)\). But to construct a proper encoding matrix, Equation (26) does not suffice as an input to Equation (16), because of the oversimplistic assumption of instantaneous excitation at \(t = t_0\) (cf. Figure 3C). Since the latter coincides with the transition \(ECHO \rightarrow FID\) at \(p = 0\), the most pivotal region for spatial encoding, the detailed properties of the RF pulse need to be taken into account, as done in the convolution method\(^{12}\) or the more \(k\)-space oriented algebraic approach.\(^{17}\) A detailed analysis of the options to improve existing reconstructions along these lines is beyond the scope of this article though.

In summary, the CCM constitutes a generalization of the EPG framework, applicable to arbitrary MRI sequences and tissues, in which the differential equations (5) describe the transport, modulation, relaxation and recreation of signal “energy.” Signal localization is treated as an inverse problem, with data consistency defined by Equation (16). This signal processing perspective could be advantageous to explore the available options for encoding and reconstructing diagnostic information with MRI.

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DATA AVAILABILITY STATEMENT

To support reproducible research, the related code (MATLAB R2020b, Natick, Massachusetts: The MathWorks Inc.), together with all scripts, used to produce the figures of this manuscript and the (SI), is freely available.\(^{26}\) The corresponding script names are given in the figure captions. Actual versions of the software can be obtained at https://github.com/cganten/CoMoTk.

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REFERENCES

1. Kaiser R, Bartholdi E, Ernst RR. Diffusion and field-gradient effects in NMR Fourier spectroscopy. J Chem Phys. 1974;60:2966-2979.
2. Hennig J. Multiecho imaging sequences with low refocusing flip angles. J Magn Reson. 1969;1988(78):397-407.
3. Hennig J. Echoes—how to generate, recognize, use or avoid them in MR-imaging sequences. Part I: fundamental and not so fundamental properties of spin echoes. Concepts Magn Reson. 1991;3:125-143.
4. Hennig J. Echoes—how to generate, recognize, use or avoid them in MR-imaging sequences. Part II: echoes in imaging sequences. Concepts Magn Reson. 1991;3:179-192.
5. Woessner DE. Effects of diffusion in nuclear magnetic resonance spin-echo experiments. J Chem Phys. 1961;4:2057-2061.
6. Weigel M. Extended phase graphs: dephasing, RF pulses, and echoes - pure and simple. J Magn Reson Imaging. 2014;1:266-295.
7. Sodickson A. A generalized k-space formalism for treating the spatial aspects of a variety of NMR experiments. Prog Nucl Magn Reson Spectrosc. 1998;3:77-108.
8. Weigel M, Schwenko S, Kiselev V, Scheffler K, Hennig J. Extended phase graphs with anisotropic diffusion. J Magn Reson. 2010;205:276-285.
9. Malik SJ, Teixeira RPA, Hajnal JV. Extended phase graph formalism for systems with magnetization transfer and exchange. Magn Reson Med. 2017;767-779.
10. Ganten C. Static susceptibility effects in balanced SSFP sequences. Magn Reson Med. 2006;6:687-691.
11. Leupold J. Steady-state free precession signals of arbitrary dephasing order and their sensitivity to \(T_2^*\). Con Magn Reson Part A. 2017: c21435.
12. Idiyatullin D, Corum C, Park JY, Garwood M. Fast and quiet MRI using a swept radiofrequency. J Magn Reson. 2006;342-349.
13. Subramanian VH, Eleff SM, Rehn S, Leigh JS. An exact synthesis procedure for frequency selective pulses. Proc Int Soc Magn Reson Med. 1986;5:1452.
14. Scheffler K. A pictorial description of steady-states in rapid magnetic resonance imaging. Con Magn Reson. 1999;11:291-304.
15. Mallat S, Peyré G. A Wavelet Tour of Signal Processing. Elsevier; 1999. ISBN 9780124666061.
16. Yablonskiy DA, Haacke EM. Theory of NMR signal behavior in magnetically inhomogeneous tissues: the static dephasing regime. Magn Reson Med. 1994;749-763.
17. Weiger M, Hennel F, Pruessmann KP. Sweep MRI with algebraic reconstruction. Magn Reson Med. 2010;1685-1695.
18. Lighthill MJ. *Introduction to Fourier Analysis and Generalised Functions*. Cambridge University Press; 1958. ISBN 9781139171427.
19. Ernst RR. Magnetic resonance with stochastic excitation. *J Magn Reson*. 1969;1970:10-27.
20. Kaiser R. Coherent spectrometry with noise signals. *J Magn Reson* (1969). 1970:28-43.
21. Idiyatullin D, Corum C, Moeller S, Garwood M. Gapped pulses for frequency-swept MRI. *J Magn Reson*. 2008;267-273.
22. Pipe JG. Spatial encoding and reconstruction in MRI with quadratic phase profiles. *Magn Reson Med*. 1995;24-33.
23. Park JY, Delabarre L, Garwood M. Improved gradient-echo 3D magnetic resonance imaging using pseudo-echoes created by frequency-swept pulses. *Magn Reson Med*. 2006; 848-857.
24. Bruder H, Fischer H, Graumann R, Deimling M. A new steady-state imaging sequence for simultaneous acquisition of two MR images with clearly different contrasts. *Magn Reson Med*. 1988;35-42.
25. Jiang Y, Ma D, Seiberlich N, Gulani V, Griswold MA. MR fingerprinting using fast imaging with steady state precession (FISP) with spiral readout. *Magn Reson Med*. 2015;1621-1631.
26. Ganter C. Configuration Model Toolkit (CoMoTk). 2021.

**SUPPORTING INFORMATION**

Additional supporting information may be found in the online version of the article at the publisher’s website.

**Text S1** Configuration space representation of MRI sequences supporting information.

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