ACTIVITY PHASE TRANSITION FOR CONSTRAINED DYNAMICS

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Abstract. We consider two cases of kinetically constrained models, namely East and FA-1f models. The object of interest of our work is the activity $A(t)$ defined as the total number of configuration changes in the interval $[0, t]$ for the dynamics on a finite domain. It has been shown in [GJLPDW1, GJLPDW2] that the large deviations of the activity exhibit a non-equilibrium phase transition in the thermodynamic limit and that reducing the activity is more likely than increasing it due to a blocking mechanism induced by the constraints. In this paper, we study the finite size effects around this first order phase transition and analyze the phase coexistence between the active and inactive dynamical phases in dimension 1. In higher dimensions, we show that the finite size effects are also determined by the dimension and the choice of boundary conditions.

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1. Introduction

Kinetically constrained spin models (KCSM) are interacting particle systems which have been introduced and very much studied in the physics literature to model liquid/glass transition and more generally glassy dynamics (see [RS, GST] and references therein). A configuration is given by assigning to each vertex $x$ of a (finite or infinite) connected graph $G$ its occupation variable $\eta_x \in \{0, 1\}$ which corresponds to an empty or filled site, respectively. The evolution is given by Markovian stochastic dynamics of Glauber type. Each site with rate one refreshes its occupation variable to a filled or to an empty state with probability $\rho$ or $1 - \rho$ respectively provided that the current configuration satisfies an a priori specified local constraint. Here we focus on two of the most studied KCSM, the East [JE] and FA-1f models [FA1, FA2] on hypercubic lattices ($G \subset \mathbb{Z}^d$): the constraint at $x$ requires for East model its right nearest neighbour to be empty, for FA-1f model at least one of its nearest neighbours to be empty. Note that in both cases (and this is a general feature of KCSM) the constraint which should be satisfied to allow creation/annihilation of a particle at $x$ does not involve $\eta_x$, thus detailed balance w.r.t. the Bernoulli product measure at density $\rho$ is an invariant reversible measure for the process. Both models are ergodic on $G = \mathbb{Z}^d$ for any $\rho \in (0, 1)$ with a positive spectral gap which shrinks to zero as $\rho \to 1$ corresponding to the occurrence of diverging mixing times [CMRT].

Several numerical works and approximated analytical treatments have shown that relaxation for both models occurs in a more and more spatially heterogenous way as density is increased (see Section 1.5 of [GST] and references therein). For example when measuring the persistence field $p_x(t)$ which equals to one if site $x$ has never changed its state up to
time \( t \) and zero otherwise a clear spatial segregation is observed among sites with 0/1 values of \( p \) at time scales corresponding to the typical relaxation time of the persistence function which corresponds to the spatial average of the persistence field. More quantitatively, if one measures the spatial correlation function of this persistence field, a dynamical correlation length corresponding to the extent of these heterogeneities can be extracted. This length increases as the density is increased. The occurrence of these dynamical heterogeneities has lead to the idea that the dynamics of KCM takes place on a first-order coexistence line between active and inactive dynamical phases [MGC, JGC]. In order to exploit this idea in [GJLPDW1, GJLPDW2] the fluctuation of the dynamical activity \( A(t) \) defined as the number of microscopic configuration changes on a volume of linear size \( N \) in the time interval \([0, t]\) has been investigated. The mean activity scales as

\[
\lim_{N \to \infty} \lim_{t \to \infty} \frac{\langle A(t) \rangle}{Nt} = \bar{A},
\]

where \( \bar{A} \) depends on the density and on the choice of the constraints, as we will detail in Section 2. Thus one could expect that the probability of observing a deviation from the mean value scales as

\[
\lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{Nt} \log \left( \frac{\langle A(t) \rangle}{Nt} \right) \simeq -f(a),
\]

with \( 0 < f(a) < \infty \) for \( a \neq \bar{A} \) as it occurs for the models without constraints. However, as it has been observed in [GJLPDW1, GJLPDW2], due to the presence of the constraint it is possible to realize at a low cost a trajectory with zero activity by starting from a completely filled configuration and imposing that a single site does not change its state (see Section 2.2 for a detailed explanation of the mechanism behind this phenomenon). Analogously one can obtain a smaller activity than the mean one by blocking for a fraction of time a single site. As a consequence of this sub-extensive cost for lowering the activity \( f(a) = 0 \) for \( a < \bar{A} \). For the same reason, the moment generating function

\[
\psi(\lambda) = \lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{Nt} \log \langle \exp(\lambda A(t)) \rangle
\]

is non analytic at \( \lambda = 0 \) with a discontinuous first order derivative [GJLPDW1, GJLPDW2].

In this paper, we investigate the finite size scaling of the first order transition (1.2). Our main results are estimates of the cost of phase coexistence between the active and inactive dynamical phases. From these estimates, the relevant scaling asymptotic in (1.2) can be determined. For East and FA1f in one dimension, we prove (Theorem 2.1) that

\[
\varphi(\alpha) := \lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \left( \exp \left( \frac{\alpha A(t)}{N} \right) \right)
\]

satisfies \( \varphi(\alpha) = \alpha \bar{A} \) if \( \alpha > \alpha_0 \) and \( \varphi(\alpha) = -\Sigma \) if \( \alpha < \alpha_1 \). This shows that a transition in (1.2) occurs for a value \( \lambda = \frac{\alpha_0}{N} \) with \( \alpha_1 < \alpha_c < \alpha_0 \). As a consequence, the scaling of the large deviations differs for increasing or decreasing the activity (see Theorem 2.2).

We also analyze the measure on the space-time configurations

\[
\mu_{\alpha, T}^N := \frac{\langle \cdot \exp(\frac{\alpha}{N} A(T)) \rangle}{\langle \exp(\frac{\alpha}{N} A(T)) \rangle}
\]
which corresponds to the conditional measure with a fixed activity \( \frac{A(t)}{t} = \bar{A} \) where \( \alpha \) is the parameter conjugated to \( \bar{A} \) in Legendre transform and prove (Theorem 2.1) that depending on the value of \( \alpha \) this measure has very different typical configurations which can be interpreted as active and inactive dynamical phases: for \( \alpha > \alpha_0 \), \( \mu_{\alpha,T}^N \) concentrates on trajectories with the mean activity and for \( \alpha < \alpha_1 \), it concentrates on trajectories with zero activity.

Finally, we investigate the higher dimensional cases and show that the finite size scaling depends not only on the dimension but also on the boundary conditions (Theorem 2.5). This leads to a variety of scalings for the large deviations when the activity is reduced (Theorem 2.6).

2. Models and results

2.1. East and FA-1f in \( d = 1 \): the phase transition. The East and FA-1f models in one dimension are Glauber type Markov processes on the configuration space \( \Omega = \{0, 1\}^\Lambda \) where \( \Lambda \subset \mathbb{Z} \). Both models depend on a parameter \( \rho \), with \( \rho \in (0, 1) \), which we will call the density. Here we will consider the models in finite volumes \( \Lambda = \Lambda_N := [1, N] \) and we will be interested in the thermodynamic limit \( N \to \infty \). We call \( \Omega_N \) the configuration space correspondent to \( \Lambda_N \) and denote by greek letters \( \eta, \omega \) the elements of \( \Omega_N \). Then for any site \( i \in \Lambda_N \) we let \( \eta_i \in (0, 1) \) be the value of configuration \( \eta \) at site \( i \) and we say that \( i \) is empty (filled) if \( \eta_i = 0 \) (\( \eta_i = 1 \), respectively).

The Markov process corresponding to both models can be informally described as follows. Each site \( i \in \Lambda_N \) waits an independent mean one exponential time and then, provided the current configuration satisfies a proper local constraint, we refresh the value of the configuration at \( i \) by setting it to 1 with probability \( \rho \) and to zero with probability \( 1 - \rho \). Instead if the constraint is not satisfied nothing occurs. Then the procedure starts again. The specific choice of the constraint identifies the model: for East one requires that the right nearest neighbour of \( i \) is empty; for FA-1f model one requires that at least one among the right and left nearest neighbours of \( i \) is empty. In formulas the constraint at \( i \) is satisfied for East and FA-1f in the configuration \( \eta \) iff \( \eta_{i+1} = 0 \) and \( \eta_{i+1} \eta_{i-1} = 0 \), respectively. Note that in both models the constraint is local (it depends on the configuration on a finite neighborhood of the to-be-updated site) and does not depend on the value of the configuration on the to-be-updated site. Both models belongs to the larger class of Kinetically Constrained Models (KCM in short), which have been introduced and widely studied in physics literature (see for reviews [RS, GST]).

In order for the above description to be complete, we need to specify what happens at sites near the boundary. A standard choice in statistical mechanics is to defined the dynamics in finite volume by imposing a fixed boundary condition. Here the choice of this boundary condition is very delicate, indeed due to the presence of the constraints both models are very sensitive to the specific choice of these conditions even on large volumes. For example for the East model it is easy to verify that if we fix a boundary condition equal to one at \( N + 1 \), we start the evolution from \( \eta \in \Omega_N \) and we let \( x \) be the position of the rightmost zero of \( \eta \), then at any subsequent time site \( x \) stays empty and sites \( [x + 1, N] \) stay filled. In this case we say that the configuration is frozen on \( [x, N] \) meaning that under the evolution, the configuration on these sites remains unchanged. On the other hand, for a boundary
condition equal to zero at $N + 1$, it can be easily verified that there is no site on which the configuration is frozen no matter which is the choice of the initial configuration. From the above observation it follows that in the case of filled boundary condition the configuration space (even on finite volume) is not irreducible. Indeed there exists configurations $\sigma, \eta \in \Omega_N$ such that it is not possible to devise a path of elementary moves with strictly positive rates which connects $\sigma$ to $\eta$. Instead if we take an empty boundary condition the configuration space is irreducible. This can be easily verified by constructing a path which completely empties any configuration starting from the right boundary. Analogously for FA-1f model if one imposes filled boundary conditions both at 0 and $N + 1$ the configuration space is reducible. On the other hand any of the choices which has at least one empty site in the couple $(0, N + 1)$ is sufficient to guarantee irreducibility. Here for both models we will only be interested on choices of the boundary conditions which guarantee irreducibility (and therefore ergodicity as we consider finite systems).

Note also that for both models, no matter which choice we perform for the boundary condition, the dynamics satisfy detailed balance with respect to Bernoulli product measure $\nu$ at density $\rho$, namely $\nu(\eta) := \prod_{i \in \Lambda_N} \nu_i(\eta_i)$ with $\nu_i(1) = \rho$ (this is a direct consequence of the above observed fact that the constraint at $i$ does not depend on the value of $\eta_i$). Therefore $\nu$ is an invariant measure for the process and in the irreducible case this is the unique invariant measure.

Let us now give a formal definition of these processes via the action of their generator $L_N$ on local functions $f : \Omega_N \to \mathbb{R}$. We introduce

$$L_N f(\eta) = \sum_{i \in \Lambda_N} c_i(\eta)(f(\eta^i) - f(\eta)) \quad (2.1)$$

where $\eta^i$ stands for the configuration $\eta$ changed at $i$, namely

$$\eta^i_j = \begin{cases} 
\eta_j & \text{if } j \neq i \\
1 - \eta_j & \text{if } j = i
\end{cases} \quad (2.2)$$

and we let

$$c_i(\eta) := r_i(\eta)[\eta_i(1 - \rho) + (1 - \eta_i)\rho] \quad (2.3)$$

with $r_i$ the function that encodes the constraint at site $i$, namely $r_i(\eta) = 1$ ($r_i(\eta) = 0$) iff the constraint at $i$ is (is not) satisfied. Thus $r_i$ is model dependent and for the East model with frozen empty boundary condition at the right boundary we set

$$r_i(\eta) := (1 - \eta_{i+1}) \quad \text{if } i \in [1, N - 1]; \quad r_N = 1 \quad (2.4)$$

for FA-1f with empty boundary condition at the right and left boundary we set

$$r_i(\eta) := (1 - \eta_{i+1}\eta_{i-1}) \quad \text{if } i \in [2, N - 1]; \quad r_1 = 1, \; r_N = 1 \quad (2.5)$$

for FA-1f with empty boundary condition at the right boundary and occupied boundary condition at the left boundary we set

$$r_i(\eta) := (1 - \eta_{i+1}\eta_{i-1}) \quad \text{if } i \in [2, N - 1]; \quad r_1 = (1 - \eta_2), \; r_N = 1 \quad (2.6)$$

and finally for FA-1f with empty boundary condition at the left boundary and occupied boundary condition at the right boundary we set

$$r_i(\eta) := (1 - \eta_{i+1}\eta_{i-1}) \quad \text{if } i \in [2, N - 1]; \quad r_1 = 1, \; r_N = (1 - \eta_{N-1}). \quad (2.7)$$
As already mentioned our analysis will focus on the above choices which are the only ones which guarantee ergodicity. Therefore in the following when we refer to the East model, to FA-1f with two empty boundaries and to FA-1f with one empty boundary we mean respectively the choice (2.4), (2.5) and (2.6) (by symmetry reasons the choices (2.6) and (2.7) for FA-1f are equivalent therefore we never consider the case (2.7)). Also, when we state results referred to FA-1f model without further specifying the boundary conditions it means that these results hold for both the choices (2.5) and (2.6).

Note that the generator (2.1) can be equivalently rewritten as

$$
L_N f(\eta) = \sum_{i \in \Lambda_N} r_i(\eta) \left( \nu_i(f) - f(\eta) \right)
$$

(2.8)

with $\nu_i(f) = \int d\nu_i(\eta_i) f(\eta)$ the local mean at site $i$.

The object of interest of our work is the total activity

$$
A(t) := \sum_{i \in \Lambda_N} A_i(t)
$$

(2.9)

where $A_i(t)$ is the random variable which corresponds to the number of configuration changes on site $i$ during the time interval $[0, t]$. It is easy to verify that $A_i(t) - \int_0^t c_i(\eta(s))ds$ is a martingale and therefore $A(t)$ satisfies a law of large numbers with

$$
\lim_{N \to \infty} \lim_{t \to \infty} \frac{A(t)}{Nt} = A
$$

where $A$, which will be referred to in the following as the mean instantaneous activity, is defined as

$$
A := \nu(c_j(\eta)), \quad j \in \Lambda_N \setminus (1, N)
$$

(2.10)

note that the definition is well posed since by translation invariance $\nu(c_j(\eta)) = \nu(c_{j'}(\eta))$ for $j, j' \in [2, N-1]$. For East, we get $A = 2\rho(1-\rho)^2$, for FA-1f instead $A = 2\rho(1-\rho)(1-\rho^2)$.

Here we will be interested in the study of the generating function which controls the fluctuations of $A(t)$ for a given $N$

$$
\varphi^{(N)}(\lambda) = \lim_{t \to \infty} \frac{1}{t} \log \langle \exp(\lambda A(t)) \rangle
$$

(2.11)

where here and in the following $\langle \rangle$ denotes the mean over the evolution of the process and over the initial configuration which is distributed with the equilibrium Bernoulli measure $\nu$ at density $\rho$ (where the density $\rho$ is fixed by the rates (2.3)). With a slight abuse of notation for any event $E$ we will also denote by $\langle E \rangle$ the probability of $E$, namely we set $\langle E \rangle := \langle 1_E \rangle$. The main result of this paper is that in the scaling $\lambda = \alpha/N$ a phase transition occurs for this generating function. More precisely if we define

$$
\varphi(\alpha) := \limsup_{N \to \infty} \varphi^{(N)} \left( \frac{\alpha}{N} \right)
$$

(2.12)

then the following holds:

**Theorem 2.1.** Consider East or FA-1f model in $d = 1$ at any $\rho \in (0, 1)$. There exists $\alpha_1 < \alpha_0 < 0$ and a constant $\Sigma > 0$ such that

(i) for $\alpha > \alpha_0$ it holds $\varphi(\alpha) = A\alpha$;
(ii) for $\alpha < \alpha_1$ it holds $\varphi(\alpha) = -\Sigma$.

As a consequence of this theorem, estimates on the large deviations for a reduced activity can be obtained.

**Theorem 2.2.** Consider East or FA-1f model in $d = 1$ at any $\rho \in (0, 1)$. For any $u \in [0, 1)$ it holds

$$-\Sigma(1-u) \leq \liminf_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{A(t)}{Nt} \simeq uA \right) \leq \limsup_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{A(t)}{Nt} \simeq uA \right) \leq \alpha_0 A(1-u).$$

**Remark 2.3.** We conjecture that in (2.12) the $\limsup_{N \to \infty}$ can be replaced by $\lim_{N \to \infty}$. In the regime $\alpha > \alpha_0$ and $\alpha < \alpha_1$, this follows from the proof of Theorem 2.1.

Theorem 2.1 (i) and (ii) will be proved in Section 5.1 and 6.1 respectively. Theorem 2.2 will be proven in Section 7.

We also analyze the measure on the space-time configurations defined as

$$\mu_{\alpha,T}^N = \frac{\langle \cdot \exp(\frac{\alpha}{N}A(T)) \rangle}{\langle \exp(\frac{\alpha}{N}A(T)) \rangle}$$

and Theorem 2.4 states that depending on the value of $\alpha$ this measure has very different typical configurations. For any configuration $\eta \in \Omega_N$, we call $\{\eta(s)\}_{s \geq 0}$ the trajectory of the Markov process generated by $L_N$ starting at time zero from $\eta$. Then the following holds

**Theorem 2.4.** Consider East model and FA-1f model in $d = 1$ with $\rho \in (0, 1)$. Then there exists $\alpha_1 < \alpha_0 < 0$ and a sequence $\gamma_N$ with $\lim_{N \to \infty} \gamma_N = 0$ such that

(i) for $\alpha > \alpha_0$

$$\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^N \left( \int_0^T dt \sum_{i \in \Lambda_N} \eta_i(t) - N\rho \right) \leq \gamma_N NT = 1; \quad (2.15)$$

$$\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^N \left( \sum_{i \in \Lambda_N} c_i(\eta(t)) - NA \leq \gamma_N NT \right) = 1. \quad (2.16)$$

(ii) if $\alpha < \alpha_1$

$$\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^N \left( \int_0^T dt \sum_{i \in \Lambda_N} \eta_i(t) \geq (1-\gamma_N) NT \right) = 1; \quad (2.17)$$

$$\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^N \left( \sum_{i \in \Lambda_N} c_i(\eta(t)) \leq \gamma_N NT \right) = 1. \quad (2.18)$$

Theorem 2.4(i) and (ii) will be proven in Section 5.2 and 5.3 respectively where stronger results (Lemma 5.1 and 6.8) concerning the concentration of the number of particles and the activity on mesoscopic boxes (and not on the whole volume) will also be established.
2.2. Heuristics of the phase transition and open problems. As we already mentioned in the introduction, the occurrence of a phase transition for the activity large deviations was first discovered in [GJLPDW2], where it was shown that

\[ \lambda \in \mathbb{R}, \quad \psi(\lambda) = \lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{Nt} \log \langle \exp(\lambda A(t)) \rangle \quad (2.19) \]

has a critical value at \( \lambda = 0 \) (see the left part of figure 1). We recall below the mechanism of this phase transition in the case of the one-dimensional East model (the case of FA-1f model is analogous). When \( \lambda > 0 \), the activity is increased and the large deviation functional is expected to be smooth. Negative values of \( \lambda \) lead to a decay of the activity and the constraint will play a crucial role. To have no activity, a possible strategy is to start at time \( t = 0 \) from a configuration totally filled (\( \eta_i = 1 \) for all \( i \)) and to remain in this configuration at any time. This can be achieved by preventing \( \eta_N \) from flipping to 0, indeed the site \( N \) is the only site allowed to flip due to the constraints and if it is maintained equal to 1 then the rest of the configuration is blocked. This leads to the lower bound

\[ \langle A(t) = 0 \rangle \geq \rho^N \exp(-\rho t), \quad (2.20) \]

where \( \rho^N \) stands for the cost of the initial configuration and the last term is the probability that a Poisson process of intensity \( 1 - \rho \) has no jump up to time \( t \). After rescaling (2.19), this shows that \( \psi(\lambda) \geq 0 \) for \( \lambda < 0 \) and since \( \psi(0) = 0 \) and \( \psi \) is increasing in \( \lambda \) it follows that \( \psi(\lambda) = 0 \) for \( \lambda \leq 0 \). On the other hand by convexity \( \psi(\lambda) \geq \lambda A \) and therefore at \( \lambda = 0 \) the first order derivative of \( \psi \) has a jump. As noted in [GJLPDW1, GJLPDW2], it is remarkable that the phase transition occurs at \( \lambda = 0 \) which corresponds to the unperturbed dynamics. Thus one may wonder if the singularity of the large deviation functional would lead to specific properties of the constrained systems.

In this paper, we investigate the finite size scaling of this first order phase transition through the function \( \varphi(\alpha) \) introduced in (2.12). This refined thermodynamic scaling corresponds to a blow up of the region \( \lambda = 0 \). The results of Theorem 2.1 are depicted in the right part of figure 1. In a range of \( \lambda \) of order \( 1/N \), we have shown that the transition is shifted from 0. To understand this, we first suppose that no transition takes place at 0 and

![Figure 1](image_url)

**Figure 1.** The function \( \psi(\lambda) \) is depicted on the left. The right figure represents the graph of the function \( \varphi(\alpha) \): the results of Theorem 2.1 are in thick line and the conjectured behavior in dashed lines.

that \( \psi \) could be expanded (analytically). If this was the case then we would expect that for
large $N$

$$\varphi(\alpha) \simeq N\psi(\frac{\alpha}{N}) = N \left[ \psi(0) + \psi'(0) \frac{\alpha}{N} + O \left( \frac{1}{N^2} \right) \right] = A\alpha + O \left( \frac{1}{N} \right),$$

where we used that $\psi(0) = 0$ and $\psi'(0)$ is equal to the mean activity $A$ (in fact $\psi$ is not differentiable at 0, but its right-derivative is equal to $A$). Part (i) of Theorem 2.1 shows that this behavior persists for negative values of $\alpha$ provided $\alpha > \alpha_0$. In this regime, we expect that the total activity is shifted from its mean value $A N t$ by an order $\alpha t \psi''(0)$ which does not scale with $N$. For such small shifts of the activity, the system remains very close to its typical state (when $N$ is large). In particular, Theorem 2.4 asserts that the mean density is very concentrated close to its equilibrium value $\rho$.

A phase transition occurs for smaller values of $\alpha$ and $\varphi$ becomes equal to the constant $-\Sigma > -(1 - \rho)$. The estimate (2.20) leads to the lower bound $-(1 - \rho)$ and thus it is too crude to justify the claimed behavior. Indeed for $\lambda = \frac{\alpha}{N}$, an activity of order $o(N)$ will not contribute to the scaling limit (2.12), thus it is more favorable to leave a small portion of the system active as depicted in figure 2 instead of forcing the whole configuration to remain totally filled. By analogy with equilibrium, one can interpret $\Sigma$ as a surface tension between the inactive and the active region (per unit of time)

$$\langle \frac{1}{Nt} \mathcal{A}(t) \rangle \approx 0 \simeq \exp(-\Sigma t),$$

where $\frac{1}{Nt} \mathcal{A}(t) \approx 0$ means that the rescaled activity is close to 0 in the thermodynamic limit.

Contrary to the strategy in (2.20), the interface between the inactive and the active region is now allowed to fluctuate and the probabilistic cost is lowered. The surface tension $\Sigma$ can be obtained from a variational problem which is specified for FA-1f with two empty boundaries in Section 6.1 and for East and FA-1f with one empty boundary in Section 6.2. However, our results do not provide a complete description of the system and it remains to prove that the typical configurations look like figure 2. Nevertheless, Theorem 2.4 ensures that in the inactive regime almost all the sites are equal to 1. This confirms the conjectured picture.

Our results (Theorems 2.1 and 2.4) do not provide the entire phase diagram for the generating function $\varphi$ (only for $\alpha \not\in [\alpha_1, \alpha_0]$). However, this is enough to deduce (see Theorem 2.4)
the correct order of the scaling for the large deviations of the activity below the mean value

\[ \forall u \in [0, 1], \quad -\Sigma(1 - u) \leq \lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{A(t)}{Nt} \simeq uA \right) < \alpha_0 A(1 - u). \]

This scaling is anomalous compared to the extensive scaling in \( N \) of the unconstrained models \([1,1]\).

We conjecture that there is a unique critical value \( \alpha_c \) and that the two regimes remain valid up to \( \alpha_c \) as depicted in figure 1, namely \( \varphi = -\Sigma \) for \( \alpha \leq \alpha_c \) and \( \varphi = \alpha A \) for \( \alpha \geq \alpha_c \). This would imply

\[ \alpha_c = -\frac{\Sigma}{A}. \]

This conjecture is supported by numerical simulations and we refer to \([BLT]\) for an account on these numerical results. If this conjecture is verified, then Theorem 2.2 can be improved and the large deviations for reducing the activity would be given by

\[ \forall u \in [0, 1], \quad \lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{A(t)}{Nt} \simeq uA \right) = -\Sigma(1 - u). \]

![Figure 3. On the left, the graph of the magnetization \( m(h) \) for the infinite volume Ising model. On a finite domain of size \( N \) (large) and with + boundary conditions, the finite size scaling shows that the discontinuity occurs close to \( \frac{\alpha}{N} \) as depicted on the right.](image)

In \([GJLPDW1, GJLPDW2]\), it was suggested that the large deviation approach could provide a natural way to define a dynamical free energy characterizing glassiness. It is currently an open question to understand if (and how) the dynamical phase transition in \( \psi(\lambda) \) (2.19) can lead to quantitative predictions on the model at \( \lambda = 0 \). Equilibrium statistical mechanics could serve as a guide to clarify this issue. Indeed a similar phenomenon to the one depicted above for East and FA-1f occurs in the finite size scaling of the ferromagnetic Ising model. For an Ising model in the phase transition regime (\( T < T_c \)), a first order phase transition occurs in the magnetic field \( h \) and the magnetization \( m(h) \) is discontinuous at \( h = 0 \) (see figure 3). On a finite domain, say a square of size \( N \), with external boundary conditions +, then the magnetization \( m_N(h) \) is continuous and approaches the graph of \( m(h) \). A finite size scaling \([SS]\) shows that up to rescaling \( h = \alpha/N \), the magnetization \( m_N(\frac{\alpha}{N}) \) converges to a step function with a jump at a critical value \( \alpha_c \neq 0 \) (see figure 3). The shift of the transition is reminiscent of the shift for the constrained models and it can
be understood as follows. For $\alpha \in [\alpha_c, 0]$, the magnetization is slightly lowered but remains close to the magnetization $m^* = \lim_{h \to 0^+} m(h)$ imposed by the + boundary conditions, then for $\alpha < \alpha_c$ the negative magnetic field forces a droplet of the $-$ phase which fills the system. The creation of this droplet has a cost proportional to a surface order $\tau N$ (where $\tau$ is the surface tension term), but leads to an energy gain $-2m^*hN^2$. Thus, the critical value is obtained for $-2m^*hN^2 = \tau N \Rightarrow \alpha_c = -\frac{\tau}{2m^*}$.

In this analogy, the magnetization plays a role similar to the activity and $h, \lambda$ are the conjugate parameters. Even so a first order phase transition occurs for the Ising model at $h$ equal 0, it is known that the + pure phase, i.e. the Gibbs measure obtained from the + boundary conditions after the thermodynamic limit, is well behaved and that the cumulants of the magnetization can be obtained by taking the successive derivative of the pressure for $h \to 0^+$. As $\lambda$ has no physical meaning (contrary to $h$), it is not clear from the mere knowledge of the first order phase transition how to deduce precise informations on the constrained dynamics at $\lambda = 0$.

2.3. East and FA-1f in $d \geq 2$. In the previous sections we have considered one dimensional East and FA-1f models. Both models can be extended to higher dimensions in a very natural way. Let us set some notation. Let $d > 1$, then $\Lambda_N^d := [1, N]^d$ and $\Omega_N^d := \{0, 1\}^{\Lambda_N^d}$ and let $\vec{e}_j$ with $j \in [1, d]$ be the Euclidean basis vectors. The East and FA-1f models in dimension $d$ at density $\rho \in (0, 1)$ are Glauber type Markov processes with generator $L_N^d$ which acts on $f : \Omega_N^d \to \mathbb{R}$ exactly as in (2.1) with the sum over $i$ running now on $i \in \Lambda_N^d$, with $c_i$ defined again as in (2.3) and $r_i$ defined for the East model as

$$r_i(\eta) = 1 - \eta_i + \eta_{i+\vec{e}_1} \quad \text{if} \quad i \cdot \vec{e}_1 \in [1, N-1]; \quad r_i(\eta) = 1 \quad \text{otherwise} \quad (2.21)$$

and for FA-1f model with completely empty boundary conditions as

$$r_i(\eta) = 1 - \prod_{j=1}^d \eta_{i+\vec{e}_j} \eta_{i-\vec{e}_j} \quad \text{if} \quad i \cdot \vec{e}_j \in [2, N-1] \quad \forall j \in [1, d]; \quad r_i(\eta) = 1 \quad \text{otherwise} . \quad (2.22)$$

As in the one dimensional case, the above choice of the boundary condition is the only ergodic choice for the East constraints, while in the FA-1f case any choice with at least one empty boundary site is ergodic. We will be interested here in all the choices of the boundary conditions which correspond to requiring a completely empty hyperplane of linear size $N$ and dimension $c$ with $c \in [0, d-1]$ ($c = 0$ corresponds to the choice of a single empty boundary site). In short we will say that we consider a boundary condition of dimension $c$ in this case (note that the only ergodic choice for East corresponds to a particular boundary condition of dimension $d-1$). Note that as in the one dimensional case both dynamics satisfy detailed balance with respect to Bernoulli product measure $\nu$ at density $\rho$, namely $\nu(\eta) := \prod_{i \in \Lambda_N^d} \nu_i(\eta_i)$ with $\nu_i(1) = \rho$.

Let $A(t)$ be defined as in (2.9) where now the sum runs over all sites inside $\Lambda_N^d$. As for the one dimensional case it is immediate to verify that $A(t)$ satisfies the law of large number

$$\lim_{t \to \infty} \lim_{N \to \infty} \frac{A(t)}{N^d} = A$$
with $A$ defined as in (2.10) for $j \in \Lambda_N^d$ with $i \cdot \vec{e}_j \in [2, N - 1]$ for all $j \in [1, d]$. Note that $A$ coincides with the one for the corresponding one dimensional model at the same density. Let $\varphi^{(N)}$ be defined as in (2.11) and let the rescaled generating function $\varphi_d(\alpha)$ and the measure $\mu_{\alpha,T}^{N,d}$ be defined as

$$\varphi_d(\alpha) := \limsup_{N \to \infty} \frac{1}{N^c} \varphi^{(N)} \left( \frac{\alpha}{N^{d-c}} \right)$$

$$\mu_{\alpha,T}^{N,d} := \frac{\langle \exp \left( \frac{\alpha}{N^{d-c}} A(T) \right) \rangle}{\langle \exp \left( \frac{\alpha}{N^{d-c}} A(T) \right) \rangle}.$$  

(2.23)  

(2.24)

Note that the above definitions if we set $d = 1$ and $c = 0$ are compatible with the definitions used in Section 2 for the one dimensional case, namely $\varphi_1(\alpha) = \varphi(\alpha)$ and $\mu_{\alpha,T}^{N,1} = \mu_{\alpha,T}^N$ with $\varphi(\alpha)$ and $\mu_{\alpha,T}^N$ defined by equations (2.12) and (2.14) respectively. We stress that the finite size scaling depends on the choice of the boundary conditions. We expect this to be the choice which leads to a phase transition for the generating function, as in the one dimensional case. This conjecture, as will be further clarified by the proof of Theorem 2.7, is related to the fact that in order to have no activity for a $d$-dimensional model a possible strategy is to start at time zero from a completely filled configuration and prevent all the $O(N^c)$ sites which are in contact with the boundary empty set from flipping.

The $d$-dimensional East model corresponds to $N^{d-1}$ independent one dimensional East models so that $\varphi_d(\alpha) = \varphi_1(\alpha)$ for any $d$. Thus the phase transition results of Theorem 2.1 hold for $\varphi_d$ and Theorem 2.4 applies as well in this case.

For FA-1f it is not immediate to generalize the one dimensional results since the model cannot be decoupled into independent one dimensional FA-1f models. In this case we prove

**Theorem 2.5.** Consider FA-1f model in dimension $d \geq 2$ with boundary condition of dimension $c$ with $c \in [0, d - 1]$. There exists $\alpha_0 < 0$ and a sequence $\gamma_N$ with $\lim_{N \to \infty} \gamma_N = 0$ such that for $\alpha > \alpha_0$

$$\varphi_d(\alpha) = A\alpha,$$

$$\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^{N,d} \left( \int_0^T dt \left| \sum_{i \in \Lambda_N^d} \eta_i(t) - N^d \rho \right| \leq \gamma_N N^d T \right) = 1,$$

$$\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^{N,d} \left( \int_0^T dt \left| \sum_{i \in \Lambda_N^d} c_i(\eta(t)) - N^d A \right| \leq \gamma_N N^d T \right) = 1.$$



**Theorem 2.6.** Consider East or FA-1f model in $d \geq 2$ at any $\rho \in (0, 1)$. For any $u \in [0, 1]$ it holds

$$-(1 - \rho)(1 - u) \leq \liminf_{N \to \infty} \lim_{t \to \infty} \frac{1}{t N^c} \log \langle \frac{A(t)}{N^{d_t}} \simeq u A \rangle \leq \limsup_{N \to \infty} \lim_{t \to \infty} \frac{1}{t N^c} \log \langle \frac{A(t)}{N^{d_t}} \simeq u A \rangle \leq \alpha_0 A (1 - u),$$

(2.25)

where $\alpha_0 < 0$ was introduced in Theorem 2.5.
These results correspond to those of Theorems 2.1(i) and 2.4(i) and 2.2 in the one-dimensional case. For the large negative $\alpha$ regime, the result for dimension larger than one is much less precise.

**Theorem 2.7.** Consider FA-1f model in dimension $d \geq 2$ with boundary condition of dimension $c \in [0, d - 1]$. For any $\delta > 0$ there exists $\alpha_1(\delta) < 0$ such that

$$
\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha, T}^{N,d} \left( \int_0^T dt \sum_{i \in \Lambda_N} \eta_i(t) \geq (1 - \delta)N^d T \right) = 1. \tag{2.26}
$$

Theorem 2.5 and 2.7 will be proven in Section 5.3 and 6.4 respectively.

3. Preliminary results

In Section 3.1 we recall some basic tools from the Donsker-Varadhan large deviation theory and we establish a variational formula for $\varphi(N)(\lambda)$ (3.10). We underline that this formula is valid in any dimension. Finally, in Section 3.2 we recall a result on the spectral gap of the generator for KCM which will be used in some of our proofs.

3.1. Donsker-Varadhan theory. The Donsker-Varadhan theory for large deviations [DZ] will be a basic tool to derive our results. Fix the dimension $d$ and let $\mathcal{D}_N$ be the Dirichlet form corresponding to the generator $\mathcal{L}_N^d$ (2.1) which is defined on any function $g$ as

$$
\mathcal{D}_N(g) := -\nu(g \mathcal{L}_N^d g). \tag{3.1}
$$

For future use we note that the Dirichlet form can be rewritten by using the definition (2.1) as

$$
\mathcal{D}_N(g) = \sum_{i \in \Lambda_N} \nu \left( c_i(\eta) (g(\eta^i) - g(\eta))^2 \right) = \sum_{i \in \Lambda_N} \nu (r_i(\eta) \text{Var}_i(g)) \tag{3.2}
$$

with $\text{Var}_i(g) = \nu_i(f - \nu_i(g))^2$.

For any smooth function $V : \Omega_N^d \to \mathbb{R}$, we define the time average of $V$ over the process as

$$
\pi_t(V) := \frac{1}{t} \int_0^t V(\eta(s)) ds
$$

where $\eta(s)$ is a trajectory of the Markov process starting at time zero from $\eta$. The dynamics is reversible with respect to the measure $\nu$, thus the Donsker-Varadhan theory asserts that for any $\gamma \in \mathbb{R}$

$$
\lim_{t \to \infty} \frac{1}{t} \log \left\{ \exp \left( \gamma t \pi_t(V) \right) \right\} = \sup_f \left\{ \gamma \nu(fV) - \mathcal{D}_N(\sqrt{f}) \right\}, \tag{3.3}
$$

where we recall that $\langle \cdot \rangle$ is the mean over the evolution of the process and over the initial configuration which is distributed with the equilibrium Bernoulli measure $\nu$ and the supremum is over the positive functions $f$ which satisfy $\nu(f) = 1$. Note that the r.h.s of (3.3) corresponds to the largest eigenvalue of the modified operator $\mathcal{L} + \gamma V$. Furthermore, if one defines the empirical measure in $\Omega_N^d$ by setting for any $A \subset \Omega_N^d$ and any $t \geq 0$

$$
\pi_t(A) = \frac{1}{t} \int_0^t ds \mathbbm{1}_A(\eta(s)), \tag{3.4}
$$
Donsker-Varadhan theory establishes that the large deviation functional of the empirical measure is the Dirichlet form $D_N$. Thus if we let $\psi$ be any function from $\Omega^d_N \rightarrow \mathbb{R}$, for any $[a,b] \subset \mathbb{R}$ it holds
\[
\lim_{t \rightarrow \infty} \frac{1}{t} \log \left( \frac{1}{t} \int_0^t ds \psi(\eta(s)) \in [a,b] \right) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \langle \pi_t(\psi) \in [a,b] \rangle = - \inf_{g \cdot \nu(g) = 1, g \geq 0} D_N(\sqrt{g}).
\] (3.5)

For any $\lambda \in \mathbb{R}$, we consider the modified dynamics obtained by rescaling the time by $\exp(\lambda)$. The generator reads
\[
\mathcal{L}_{N,\lambda} f(\eta) = \sum_{i \in \Lambda^d_N} \exp(\lambda) c_i(\eta) (f(\eta^i) - f(\eta)) = \exp(\lambda) \mathcal{L}_N f(\eta),
\] (3.6)
which is again reversible with respect to $\nu$. Then by evaluating the Radon-Nykodim derivative $dP_t / dP_{\lambda t}$ where $P_t$ and $P_{\lambda t}$ denote respectively the probability of the trajectory up to time $t$ for the process evolving under $\mathcal{L}_N$ and $\mathcal{L}_{N,\lambda}$, we obtain
\[
\frac{dP_t}{dP_{\lambda t}} = \exp \left( -\lambda A(t) + (\exp(\lambda) - 1) \int_0^t ds \mathcal{H}(\eta(s)) \right),
\] (3.7)
with
\[
\mathcal{H}(\eta) = \sum_{i \in \Lambda^d_N} c_i(\eta).
\] (3.8)
where we recall that $A(t)$ is the total activity up to time $t$ and $c_i$ has been defined in (2.3).

This implies in particular for any function $\psi(t) : (\eta_s)_{s \leq t} \rightarrow \mathbb{R}$
\[
\langle \psi(t) \exp(\lambda A(t)) \rangle = \left\langle \psi(t) \exp \left( (\exp(\lambda) - 1) \int_0^t ds \mathcal{H}(\eta(s)) \right) \right\rangle_{\lambda}
\] (3.9)
where here and in the following $\langle \cdot \rangle_{\lambda}$ is the expectation over the modified dynamics, i.e. the mean over the initial configuration distributed with $\nu$ and over the evolution of the process with generator $\mathcal{L}_{N,\lambda}$. Note that (3.9) together with definition (2.11) and formula (3.3) ($\mathcal{L}_{N,\lambda}$ is reversible with respect to $\nu$ thus Donsker-Varadhan theory applies), leads to the following variational formula
\[
\varphi^{(N)}(\lambda) = \sup_f \left\{ (\exp(\lambda) - 1) \nu(f \mathcal{H}) - \exp(\lambda) D_N(\sqrt{f}) \right\}
\] (3.10)
where the supremum is taken over the positive functions $f : \Omega_N \rightarrow \mathbb{R}$ such that $\nu(f) = 1$.

### 3.2. Positivity of the spectral gap.
Finally, another tool which we will use is the knowledge of a positive lower bound uniform on $N$ for the spectral gap of $\mathcal{L}_N$ which is defined as
\[
\text{gap}(\mathcal{L}_N) := \inf_{f \neq \text{const}} \frac{D_N(f)}{\text{Var}(f)},
\] (3.11)
where $\text{Var}(f)$ is the variance w.r.t. the invariant Bernoulli measure $\nu$ on $\Lambda_N$ and the minimization is over the functions $f : \Omega_N \rightarrow \mathbb{R}$ which are not constant. The following holds
Proposition 3.1 ([AD], [CMRT1], [CMRT2]). Consider East or FA-1f model in any dimension and with any choice of the boundary condition which guarantees ergodicity. For any \( \rho \in (0, 1) \) there exists \( S_\rho > 0 \) such that
\[
\inf_N \text{gap}(\mathcal{L}_N) > S_\rho.
\]
From this result and Definition (3.11) it follows immediately that
\[
D_N(f) \geq S_\rho \text{Var}(f). \tag{3.12}
\]

Let \( \mathcal{G} \) be a generic connected subset of \( \mathbb{Z}^d \) and consider FA-1f with a single empty site at the boundary on \( \mathcal{G} \). We call \( \mathcal{L}_G \) the generator and \( \text{gap}(\mathcal{L}_G) \) the corresponding spectral gap defined as in (3.11) with \( D_N \) substituted by \( D_G(f) := -\nu(f, \mathcal{L}_G f) \). Then
\[
\text{gap}(\mathcal{L}_G) > S_\rho.
\]

The result for the East model has been derived in [AD] and subsequently proven in [CMRT1] for a larger class of constraints including FA-1f in any dimension with completely empty boundary conditions. Actually, the latter result could be easily derived by the result in [AD] without using the technique of [CMRT1]. Instead, the result for FA-1f with generic boundary condition and on a generic graph has been derived in [CMRT2].

4. Local equilibrium

4.1. East and FA-1f in \( d = 1 \). Throughout this section, we consider East and FA-1f models in one dimension with mean density \( \rho \in (0, 1) \). Let us start by defining the coarse grained activity. Let \( K \) be such that \( N/K \) is integer and partition \( \Lambda_N \) into boxes \( B_i \) with \( i \in [1, N/K] \) of size \( K \), namely \( B_i = [(i-1)K+1, iK] \). We define the activity in the interior of \( B_i \) as
\[
\mathcal{H}_i(\eta) = \sum_{i \in \tilde{B}_i} c_i(\eta), \tag{4.1}
\]
where \( \tilde{B}_i \subset B_i \) are the sites such that the corresponding constraints depend only on the configuration inside \( B_i \), namely \( \tilde{B}_i := B_i \setminus iK \) for East and \( \tilde{B}_i := B_i \setminus \{(i-1)K+1, iK\} \) for FA-1f. We also define the coarse grained density as
\[
\mathcal{R}_i(\eta) = \sum_{i \in \tilde{B}_i} \eta_i K. \tag{4.2}
\]
Fix \( \epsilon > 0 \), we define the activity-density associated to a configuration \( \eta \) as
\[
u_{K, \epsilon}^i(\eta) = \begin{cases} 1 & \text{ if } \eta_j = 1 \quad \forall j \in B_i, \\ -1 & \text{ if } |\mathcal{H}_i(\eta) - \mathcal{A} | \tilde{B}_i | \leq \epsilon K \text{ and } |\mathcal{R}_i(\eta) - \rho| \leq \epsilon, \\ 0 & \text{ otherwise} \end{cases} \tag{4.3}
\]
where the mean instantaneous activity \( \mathcal{A} \) has been defined in formula (2.10) and in order for the above definition to be well posed we restrict \( \epsilon \) to the values such that \( \rho + \epsilon < 1 \) and \( \mathcal{A} | \tilde{B}_i | - \epsilon K > 0 \). In the rest of the paper in any result that uses these labels we imply that \( N \) and \( K \) are integers chosen in order that \( N/K \) is also integer and that the above restriction on \( \epsilon \) is satisfied.
The main result of this section is Lemma 4.2 which states that the probability of finding under the empirical measure $\pi_T$ a density of boxes with activity-density label equal to zero, namely with activity different from the mean activity and from zero and/or particle density different from $\rho$ and one is suppressed exponentially in $T$ and $N/K$. In other words locally we are equilibrated either in the completely filled state or in the mean state identified by $\nu$. Instead Lemma 4.3 guarantees that the probability of finding a density of boxes with activity-density label equal to one, namely completely filled boxes, is suppressed exponentially in $T$ (but not wrt $N/K$).

Before stating and proving these results we give some inequalities which immediately follow from the above definition of the labels and which will be used in the subsequent sections. Recall the definition of $H(\eta)$ (3.8), then

\begin{align*}
H(\eta) & \geq \sum_{i=1}^{N/K} H_i(\eta) \geq (KA_2 - 2A_2 - \epsilon K) \sum_{i=1}^{N/K} 1_{u^i_{K,\epsilon} = -1} \\
H(\eta) & \leq \sum_{i=1}^{N/K} H_i(\eta) + \frac{2N}{K} \leq \sum_{i=1}^{N/K} 1_{u^i_{K,\epsilon} = -1} (KA_2 - 2A_2 + \epsilon K) + K \sum_{i=1}^{N/K} 1_{u^i_{K,\epsilon} = 0} + \frac{2N}{K}.
\end{align*}

(4.4) (4.5)

Recall the definition of the empirical measure $\pi_T$, see equation (3.4), we define the following events:

Definition 4.1. Fix $T > 0$, $N, K$ integers, $\delta \in [0, 1]$ and $\epsilon > 0$. Then for $j \in \{-1, 0, 1\}$ we define the event $W_{j,\delta}$ by requiring a density at least $\delta$ of $j$ activity-density labels. In formulas $W_{j,\delta}$ is verified iff

\[ \pi_T \left( \sum_{i=1}^{N/K} 1 \{ u^i_{K,\epsilon} = j \} \right) \geq \frac{\delta N}{K}. \]

For any integer $\ell$ we also define $V_{j,\ell}$ which is verified iff

\[ \pi_T \left( \sum_{i=1}^{N/K} 1 \{ u^i_{K,\epsilon} = j \} \right) \in [\ell, \ell + 1). \]

We stress that, even if it is not explicated for simplicity of notation, the event $W_{j,\delta}$ and $V_{j,\ell}$ depend on the choice of $T, N, K, \epsilon$.

Lemma 4.2. There exists $C(\rho), C'(\rho) > 0$ such that for any $\delta, \epsilon > 0$, any $\lambda \in \mathbb{R}$ and any integer $N, K$ provided $K \geq K(\delta, \epsilon) = C' \frac{\log(\delta)}{\epsilon^2}$ it holds

\[ \lim_{T \to \infty} \frac{1}{T} \log \langle W_{0,\delta} \rangle_{\lambda} \leq - C \delta^2 \frac{N}{K} \exp(\lambda). \]

Lemma 4.3. For any $\delta > 0$, there is $K(\delta)$ such that for any $K \geq K(\delta)$, for any $\lambda \in \mathbb{R}$, any $N \geq K$ and any $\epsilon > 0$ it holds

\[ \lim_{T \to \infty} \frac{1}{T} \log \langle W_{1,\delta} \rangle_{\lambda} \leq - \frac{S_\rho \delta}{4} \exp(\lambda), \]

where $S_\rho > 0$ has been defined in Proposition 3.7.
Remark 4.1. The blocking mechanism described in Section 2.2 implies that
\[
\lim_{T \to \infty} \frac{1}{T} \log \langle W_{1,\delta} \rangle_\lambda \geq - (1 - \rho) \exp(\lambda),
\] (4.8)
thus the scaling in (4.7) cannot be improved.

As a consequence of Lemma 4.2, we will see that

\[\text{Lemma 4.4. There exists } C(\rho) > 0 \text{ s.t. for any } \epsilon, \delta > 0, \text{ any } \alpha \in \mathbb{R} \text{ and any } K \geq \bar{K} \text{ with } K \text{ specified in Lemma 4.2 and any } N \geq K \text{ it holds}
\]
\[
\lim_{T \to \infty} \frac{1}{T} \log \mu_{\alpha,T}^{N,N/\delta} (W_{0,\delta}) \leq - C \delta^2 \frac{N}{K} \exp\left(\frac{\alpha}{N}\right) + |\alpha|(1 + \frac{C}{N}).
\] (4.9)

We start by proving a preliminary result. For \(i \in [1, N/K]\), let \(Z_i\) be the event that there exists at least one empty site inside the box \(B_i\) and \(E_i\) be the event that is verified iff \(u_{K,\epsilon}^i \in \{1, 0\}\) and define the function \(V : \Omega_N \to \mathbb{R}\) as
\[
V(\eta) := \sum_{i=1}^{N/K-1} \mathbb{1}_{E_i}(\eta)\mathbb{1}_{Z_{i+1}}(\eta) + \mathbb{1}_{E_N/K}(\eta).
\] (4.10)

\[\text{Lemma 4.5. Set } m := \nu(E_i). \text{ There exists a constant } C = C(\rho) > 0 \text{ such that uniformly in } N \text{ for any } \lambda \in \mathbb{R} \text{ it holds}
\]
\[
\lim_{T \to \infty} \frac{1}{T} \log \left(\pi_T(V) \geq x + m \frac{N}{K} \right)_\lambda \leq - \frac{C}{N} \frac{K}{x^2} \exp(\lambda).
\] (4.11)

\textbf{Proof.} For any } \gamma > 0
\[
\left(\pi_T(V) \geq x + m \frac{N}{K} \right)_\lambda \leq \exp\left(-T \gamma \left( x + m \frac{N}{K} \right) \right) \left( \exp\left( \gamma \int_0^T dt V(\eta(t)) \right) \right)_\lambda.
\] (4.12)

Then using (3.3) we get
\[
\lim_{T \to \infty} \frac{1}{T} \log \left(\pi_T(V) \geq x + \frac{N}{K} \right)_\lambda \leq - \gamma \left( x + m \frac{N}{K} \right) + \sup_f \left\{ \gamma \nu(fV) - \exp(\lambda) \mathbb{D}_N(\sqrt{f}) \right\},
\] (4.13)

where the supremum is over the \(f : \Omega_N \to \mathbb{R}\) such that \(\nu(f) = 1\) and \(f \geq 0\).

Notice that
\[
\nu(f \mathbb{1}_{E_i} \mathbb{1}_{Z_{i+1}}) = \nu(\mathbb{1}_{E_i} \mathbb{1}_{Z_{i+1}}) \nu_{B_i}(\sqrt{f})^2 + \nu \left( \mathbb{1}_{E_i} \mathbb{1}_{Z_{i+1}} \left[ f - \nu_{B_i}(\sqrt{f})^2 \right] \right).
\] (4.14)

The first term in (4.14) can be bounded from above by
\[
\nu(\mathbb{1}_{E_i} \mathbb{1}_{Z_{i+1}}) \nu_{B_i}(\sqrt{f})^2 \leq \nu(\mathbb{1}_{E_i}) \nu_{B_i}(\sqrt{f})^2 \leq \nu(f)m = m,
\] (4.15)

where we use the fact that \(\nu_{B_i}(\sqrt{f})^2\) does not depend on the variables inside \(B_i\), \(E_i\) does not depend on the variables outside \(B_i\) and the fact that \(\nu(\nu_{B_i}(\sqrt{f})^2) \leq \nu(\nu_{B_i}(f)) = \nu(f) = 1\).
On the other hand for the second term in (4.14) we have
\[
\nu \left( \mathbf{1}_{\xi_i} \mathbf{1}_{Z_{i+1}^c} \left[ f - \nu B_i(\sqrt{f})^2 \right] \right) \leq \nu \left( \mathbf{1}_{Z_{i+1}}(\eta) \left[ \sqrt{f} - \nu B_i(\sqrt{f}) \right]^2 \right)^{1/2} \nu \left( \left[ \sqrt{f} + \nu B_i(\sqrt{f}) \right]^2 \right)^{1/2} \\
\leq 2 \nu \left( \mathbf{1}_{Z_{i+1}} \operatorname{Var} B_i(\sqrt{f}) \right)^{1/2},
\]
where to obtain the first inequality we upper bound \( \mathbf{1}_{\xi_i} \) by one and we use Cauchy-Schwartz while for the second inequality we use the fact that the event \( Z_{i+1} \) depends only on the variables inside \( B_{i+1} \), thus it is independent on the variables in the block \( B_i \). Then we notice that \( \mathbf{1}_{Z_{i+1}} = 1 \) guarantees the existence of (at least) one zero inside \( B_{i+1} \) and we let \( \xi \) be the position of the first zero starting from the right border of this box. Thus
\[
\nu \left( \mathbf{1}_{Z_{i+1}} \operatorname{Var} B_i(\sqrt{f}) \right) = \sum_{j=1}^K \nu \left( \mathbf{1}_{\xi = j+iK} \operatorname{Var} B_i(\sqrt{f}) \right),
\]
and by letting \( L_{i,j} \) and \( R_{i,j} \) be the subset of \( B_i \cup B_{i+1} \) to the left (respectively right) of \( j+iK \), namely \( L_{i,j} := B_i \cup [iK, \ldots, j+iK - 1] \) and \( R_{i,j} := B_{i+1} \setminus L_{i,j} \) we have
\[
\nu \left( \mathbf{1}_{\xi = j+iK} \operatorname{Var} B_i(\sqrt{f}) \right) \leq \sum_{\eta^i} \nu(\eta^i) \mathbf{1}_{\xi = j+iK} \sum_{\eta^j} \nu(\eta^j) \operatorname{Var} B_i(\sqrt{f}) \\
\leq \sum_{\eta^i} \nu(\eta^i) \mathbf{1}_{\xi = j+iK} \operatorname{Var} L_{i,j}(\sqrt{f}),
\]
where \( \eta^i \) (\( \eta^j \)) is the configuration restricted to \( B^i \) (\( B^j \)) and we use the product form of \( \nu \) and, in the last passage, the convexity of the variance and the fact that \( B_i \subset B^j \). Then by using the spectral gap inequality (3.12) for the model on \( L_{i,j} \) with a frozen zero at the right boundary together with the expression of the Dirichlet form (3.2) we get (recall that \( S_\rho > 0 \) is the lower bound on the infimum over \( N \) of the spectral gap of \( \Lambda_N \) at density \( \rho \))
\[
\operatorname{Var} L_{i,j}(\sqrt{f}) \leq S_\rho^{-1} \nu(\eta^i) \left( \sum_{x \in L_{i,j}} r_x(\eta^j) \operatorname{Var}_x(\sqrt{f}) \right),
\]
where we denote by \( r_x \) the constraints for the model on \( L_{i,j} \) with empty boundary condition on the right boundary, namely \( r_x(\eta) = 1 \) if \( x = j+iK+1 \) and otherwise \( r_x(\eta^j) = 1 - \eta^j_{x+1} \) if we are considering East or \( r_x(\eta^j) = 1 - \eta^j_{x-1} \eta^i_{x+1} \) if we are considering FA-1f. Then we note that for any \( \eta \) such that \( \xi(\eta) = j+iK \) and which equals \( \eta^i \) on \( B^i \), it holds \( c_x(\eta^j) = c_x(\eta) \) for any \( x \in L_{i,j} \). Thus we can insert (4.19) into (4.17) and use this observation to get
\[
\nu \left( \mathbf{1}_{Z_{i+1}} \operatorname{Var} B_i(\sqrt{f}) \right) \leq \frac{1}{S_\rho} \nu \left( \sum_{x \in B_i \cup B_{i+1}} r_x \operatorname{Var}_x(\sqrt{f}) \right).
\]
Then (4.14), (4.15), (4.16) and (4.20) yield
\[
\nu(f \mathbf{1}_{\xi_i} \mathbf{1}_{Z_{i+1}}) \leq m + 2 \sqrt{\frac{\nu(\sum_{x \in B_i \cup B_{i+1}} r_x \operatorname{Var}_x(\sqrt{f}))}{S_\rho}}.
\]
and for $\nu(f1_{K/N}(\eta))$ the same upper bound can be obtained along the same lines (actually easily because the boundary condition guarantees a zero at the right border of $B_{N/K}$). Thus for any function $f$ s.t. $\nu(f) = 1$ and $f > 0$ it holds

$$\gamma\nu(fV) - \exp(\lambda)D(\sqrt{T}) \leq \gamma m\frac{N}{K} + \sum_{i=1}^{N/K} \left[ \frac{2\sqrt{\gamma}}{\sqrt{S_\rho}} \sqrt{D_{K,i}(\sqrt{T})} - \exp(\lambda)D_{K,i}(\sqrt{T}) \right]$$

$$= \frac{N}{K} \left[ \gamma m + \frac{2\gamma^2}{S_\rho \exp(\lambda)} - \frac{K}{N} \sum_{i=1}^{N/K} \left( \exp(\lambda/2)\sqrt{D_{K,i}(\sqrt{T})} - \frac{\sqrt{2\gamma}}{\sqrt{\lambda} \sqrt{S_\rho \exp(\lambda/2)}} \right)^2 \right] \leq \frac{N}{K} \left[ \gamma m + \frac{2\gamma^2}{S_\rho \exp(\lambda)} \right]$$

with $D_{K,i}(\sqrt{T})$ the contribution to the Dirichlet form coming from the sites in the box $B_i$, namely

$$D_{K,i}(\sqrt{f}) = \nu \left( \sum_{j \in B_i} c_j(\eta) \left( \sqrt{f(\eta')} - \sqrt{f(\eta)} \right)^2 \right).$$

Then by using (4.13) and optimizing over $\gamma$ we get

$$\lim_{T \to \infty} \frac{1}{T} \log \left\langle \frac{1}{T} \int_0^T dt \, V(\eta(t)) \right\rangle \geq x + m\frac{N}{K} \lambda \leq - \frac{K S_\rho \exp(\lambda)}{N} \frac{\delta}{8} x^2. \quad (4.23)$$

This completes the proof. \hfill \Box

We are now ready to prove the main results of this section.

**Proof of Lemma 4.2.** We recall the definition (4.11) for the function $V$, where $\mathcal{E}_i$ is the event which is verified iff $u_{K,i}^\epsilon \in \{0,1\}$. Thus from the definition of the activity-density labels it follows immediately that the probability of $\mathcal{E}_i$ goes to zero as the size of the box, $K$, goes to infinity and it is bounded from above by $\exp(-K\epsilon^2C)$. Thus provided $K \geq \tilde{K}$ it holds $\nu(\mathcal{E}_i) < \delta/6$. Thanks to these facts we can apply Lemma 4.5 with the choice $x = \delta N/(6\tilde{K})$ to obtain that

$$\lim_{T \to \infty} \frac{1}{T} \log \left\langle \pi_T(V) \right\rangle \geq \frac{\delta N}{3\tilde{K}} \lambda \leq - C \frac{N}{K} \delta^2 \exp(\lambda), \quad (4.24)$$

where $V$ is defined in (4.10). We will now prove that the following inequality holds for any $\eta \in \Omega_N$

$$\sum_{i=1}^{N/K} 1_{u_{K,i}^\epsilon = 0}(\eta) \leq V(\eta). \quad (4.25)$$

Then collecting (4.24) and (4.25) the proof of (4.6) is completed.

We are therefore left with the proof of (4.25) which immediately follows from the following observation. Let $i < j$ be such that $u_{K,i}^\epsilon = u_{K,j}^\epsilon = 0$ and $u_{K,\epsilon}^k \neq 0$ for all $k \in [i+1, j-1]$. Then there exists $k \in [i, j-1]$ such that $1_{\mathcal{E}_k} 1_{Z_{k+1} = 1} = 1$. In order to prove this statement we consider separately the case (a) $j = i + 1$ and (b) $j > i + 1$. In case (a) the result holds since $u_{K,i}^\epsilon = 0$ implies $1_{\mathcal{E}_i} = 1$ and $u_{K,j}^{i+1} = 0$ implies $1_{Z_{i+1} = 1} = 1$, thus $1_{\mathcal{E}_i} 1_{Z_{i+1} = 1} = 1$. In case (b) we distinguish subcases (b1) $u_{K,\epsilon}^k = 1$ for all $k \in [i+1, j-1]$ and (b2) there exists at least one site $k \in [i+1, j-1]$ such that $u_{K,\epsilon}^k = -1$. If (b1) holds then $1_{\mathcal{E}_{j-1}} 1_{Z_j = 1}$ (since $u_{K,\epsilon}^0 = 1$ implies $1_{\mathcal{E}_{j-1}} = 1$ and $u_{K,\epsilon}^j = 0$ implies $1_{Z_j = 1}$) and the desired result is proven. In case
(b2) if we let $\ell$ be the smallest index in $[i + 1, j - 1]$ such that $u_{K,\ell}^i = -1$ then $1_{\mathcal{F}_{\ell-1}} 1_{Z_{\ell}} = 1$ (indeed $u_{K,\ell}^{\ell-1} \in \{0, 1\}$ and therefore $1_{\mathcal{F}_{\ell-1}} = 1$ and $u_{K,\ell}^i = -1$ implies $1_{Z_{\ell}} = 1$) and again the desired result is proven.

\[
\square
\]

**Proof of Lemma 4.3.** By Donsker-Varadhan large deviation principle (3.5) and the spectral gap inequality (3.12) we get

\[
\lim_{T \to \infty} \frac{1}{T} \log \langle W_1, \delta \rangle \lambda \leq - \exp(\lambda) \inf_f \left\{ \mathcal{D}_N(\sqrt{f}) \right\} \leq - \exp(\lambda) S\inf_f \left\{ \text{Var}(\sqrt{f}) \right\},
\]

where the infimum is over the positive functions $f$ such that $\nu(f) = 1$, $\nu \left( \sum_{i=1}^{N/K} 1_{\{u_{K,i+1}^i = 1\}} \right) \geq \delta N/K$.

We will now show that under the latter constraint

\[
\text{Var}(\sqrt{f}) \geq \delta - \rho^K - \rho^K/2.
\]

Then by choosing $K$ sufficiently large so that $\rho^K \leq \delta^2/4$ by collecting (4.26) and (4.28) the desired result follows (note that $\rho^K \leq \delta^2/4$ implies $\rho^K \leq \delta/4$ since we only have to deal with the case $\delta \leq 1$). We are therefore left with proving that under conditions (4.27) the inequality (4.28) holds.

For each box $B_i$ with $i \in [1, N/K]$ we define the coarse grained variable $\omega_i$ by

\[
\omega_i = \mathbf{1}_{\{u_{K,i+1}^i = 1\}},
\]

and let $p := \nu(u_{K,i+1}^i = 1) = \rho^K$ and $m_p$ be the Bernoulli product measure with density $p$ on $\Omega_{N/K}$. The marginal of $\nu(\eta) f(\eta)$ on the coarse grained variables is given by $m_p(\omega) g(\omega)$ with

\[
g(\omega) := \frac{1}{m_p(\omega)} \sum_{\eta \sim \omega} f(\eta) \nu(\eta),
\]

where the sum is over the $\eta$’s compatible with $\omega$. Then

\[
\sqrt{g(\omega)} = \sqrt{\sum_{\eta \sim \omega} \frac{\nu(\eta)}{m_p(\omega)} f(\eta)} \geq \sum_{\eta \sim \omega} \frac{\nu(\eta)}{m_p(\omega)} \sqrt{f(\eta)} ,
\]

where we used the fact that for each fixed $\omega$ it holds $\sum_{\eta \sim \omega} \frac{\nu(\eta)}{m_p(\omega)} = 1$ and the concavity of the square root. Then from (4.30) we get

\[
\text{Var}(\sqrt{f}) = 1 - \nu(\sqrt{f})^2 \geq 1 - m_p(\sqrt{g})^2.
\]

Thus in order to prove (4.28) it is sufficient to show that

\[
m_p(\sqrt{g})^2 \leq 1 - \delta + p + \sqrt{p},
\]

when $g$ is defined as in (4.29) and $f$ satisfies conditions (4.27) which imply

\[
m_p(\sum_{i=1}^{N/K} \omega_i) \geq \delta N/K.
\]
The latter inequality implies that there exists (at least) a box $B_j$ with $j \in [1, N/K]$ such that $\vartheta_j := m_p(g(\omega)\omega_j) \geq \delta$. Let $j$ be the rightmost box which verifies this constraint and rewrite each configuration $\omega$ via the couple $(\omega_j, \sigma)$ with $\sigma = \{\omega_i\}_{i \neq j}$. Thus

$$m_p(\sqrt{g}) = pZ_1 + (1-p)Z_2$$

(4.33)

where

$$Z_1 := \sum_{\sigma} m_p^j(\sigma)\sqrt{g(1, \sigma)} \quad \text{and} \quad Z_2 := \sum_{\sigma} m_p^j(\sigma)\sqrt{g(0, \sigma)}.$$

where now $m_p^j$ denotes the product measure with density $p$ on $\{1, \ldots, N/K\} \setminus j$. By Jensen inequality

$$Z_1^2 \leq \sum_{\sigma} m_p^j(\sigma)g(1, \sigma) = \frac{\vartheta_j}{p}, \quad Z_2^2 \leq \sum_{\sigma} m_p^j(\sigma)g(0, \sigma) = \frac{1 - \vartheta_j}{1 - p}.$$  

(4.34)

Thus (4.33) yields

$$m_p(\sqrt{g})^2 = p^2Z_1^2 + (1-p)^2Z_2^2 + 2p(1-p)Z_1Z_2 \leq p\vartheta_j + (1-p)(1-\vartheta_j) + 2\sqrt{p(1-p)}\sqrt{\vartheta_j(1-\vartheta_j)} \leq 1 - \vartheta_j + p + \sqrt{p} \leq 1 - \delta + p + \sqrt{p},$$

where we used (4.34) for the first inequality, the fact that $\vartheta_j \leq 1$ thus $\vartheta_j(1-\vartheta_j) \leq 1/4$ and that $p < 1$ for the second inequality and finally the bound $\vartheta_j \geq \delta$ for the last inequality. Thus (4.32) is proven and the proof of the Lemma is concluded.

□

Before proving the last result of this section, Lemma 4.4, we state separately a result which will be also useful in other proofs.

**Lemma 4.6.** Consider East and FA-1f model in $d = 1$. There exists $C > 0$ s.t. for any $N$ if $\alpha < 0$

$$\langle \exp \left( \frac{\alpha}{N} A(T) \right) \rangle \geq \exp \left( \alpha A(T) \left( 1 + \frac{C}{N} \right) \right)$$

(4.35)

if $\alpha > 0$

$$\langle \exp \left( \frac{\alpha}{N} A(T) \right) \rangle \geq \exp \left( \alpha A(T) \left( 1 - \frac{C}{N} \right) \right)$$

(4.36)

**Proof.** By Jensen inequality it holds

$$\langle \exp \left( \frac{\alpha}{N} A(T) \right) \rangle \geq \exp \left( \frac{\alpha}{N} \langle A(T) \rangle \right)$$

(4.37)

Then from the definition of the process and recalling (2.10) it follows immediately that

$$\langle A(t) \rangle = \frac{t}{N} \sum_{i=1}^{N} \nu(c_i(\eta)) = t(N-2)A + tv(c_1(\eta)) + tv(c_N(\eta))$$

(4.38)

where $A = \nu(c_1(\eta)) = \rho(1-\rho)^2$ and $\nu(c_N(\eta)) = \rho(1-\rho)$ for East; $A = \rho(1-\rho)(1-\rho^2)$, $\nu(c_1(\eta)) = \nu(c_N(\eta)) = \rho(1-\rho)$ for FA-1f with two zeros at the boundary; $A = \rho(1-\rho)(1-\rho^2)$,
\( \nu(c_1(\eta)) = \rho(1-\rho)^2 \) and \( \nu(c_N(\eta)) = \rho(1-\rho) \) for FA-1f with one zero at the (right) boundary. Thus
\[
\mathcal{A}_N \left( 1 - \frac{C}{N} \right) \leq <\mathcal{A}(t)> \leq \mathcal{A}_N \left( 1 + \frac{C}{N} \right)
\]
(4.39)
and (4.36) immediately follows from these bounds and inequality (4.37).

\[\square\]

Proof of Lemma 4.4. From (2.14) and Lemma 4.6 for any \( \alpha \leq 0 \)
\[\mu^N_{\alpha,T}(W_{0,\delta}) \leq \exp(-\alpha \lambda T) \langle 1 \rangle \] 
if instead \( \alpha > 0 \) by using the fact that \( \mathcal{H}(\eta(s)) \leq N \) we get
\[\mu^N_{\alpha,T}(W_{0,\delta}) \leq \exp(-\alpha \lambda T + \alpha T) \langle 1 \rangle .\]

The result immediately follows from the above inequalities and applying Lemma 4.2

4.2. FA-1f in \( d \geq 2 \). We start by extending to the higher dimensional case the notion of coarse grained activity. We let again \( N/K \) be integer and partition \( \Lambda^K_0 \) into \( (N/K)^d \) boxes of linear size \( K \). We let \( B_i \) be the boxes (with \( i \in [1,(N/K)^d] \)) numbered in such a way that for any \( i \in [1,(N/K)^d-1] \) there is \( j \in [1,\ldots,d] \) such that \( B_{i+1} \) is obtained by shifting \( B_i \) of \( K \vec{c}_j \).

Then we define the activity-density labels as in the one dimensional case and the event \( W_{i,\delta} \) as in Definition 4.1 with \( (N/K)^d \) substituted by \( (N/K)^d \). The following holds

Lemma 4.7. Consider FA-1f model in dimension \( d \geq 2 \) with any boundary condition which guarantees ergodicity and \( \rho \in (0,1) \). The results in Lemma 4.2 hold also for FA-1f in dimension \( d \) if we substitute \( N/K \) with \( (N/K)^d \).

Proof. The proof follows the same lines as for Lemma 4.2 with some new ingredients that we detail here. Since there is at least one empty boundary condition, there exists at least one box which has an empty site on its boundary. We let \( j \) be the index of the smallest such box. Then we let \( V(\eta) := \sum_{i=1}^{j-1} 1_{\xi_1(\eta)} 1_{\Lambda_{i+1}}(\eta) + \sum_{i=j+1}^{(N/K)^d} 1_{\xi_1}(\eta) 1_{\Lambda_{i+1}}(\eta) + 1_{\xi_j} \) and notice (along the same lines as for the unidimensional case) that \( \sum_{i=1}^{(N/K)^d} 1_{\Lambda_{i+1}}(\eta) \leq V(\eta) \) for any \( \eta \). Thus provided we can establish for \( V \) the validity of Lemma 4.2 with \( N/K \) substituted by \( (N/K)^d \) we can conclude along the same lines as for the one dimensional case. The validity of this modified Lemma 4.2 also follows along the same lines as the one dimensional case with a different point that we detail here. Recall that in one dimension under the event \( Z_{i+1} \) which guarantees that there exists at least one empty site in \( B_{i+1} \) we identified the rightmost such site, which we denoted by \( \xi \). Then under the event that \( \xi = j + iK \) we somehow extended the variance up to \( L_{i,j} := B_i \cup [iK,\ldots,j+iK-1] \) and used the positivity of the spectral gap on \( L_{i,j} \) (which is a volume of the type \( \Lambda_{K+i} \)) with fixed empty boundary condition at \( j+iK \).

Now instead under the event \( Z_{i+1} \) we number the sites of \( B_{i+1} \) as \( x_1,\ldots,x_{K^d} \) in a way that \( x_{i+1} \) is nearest neighbour of \( x_i \) and then let \( \xi \) be the empty site with the biggest label. Then under the event \( \xi = x_j \) we let \( L_{i,j} := B_i \cup B_{i+1} \setminus [x_j,\ldots,x_{K^d}] \) and use the positivity of the spectral gap on \( L_{i,j} \) with fixed empty boundary condition at \( x_j \).

Note that now \( L_{i,j} \) is not an hypercube and the boundary condition is a single empty boundary condition even if we are considering the dynamics on \( \Lambda_N \) with, for example, completely empty boundary conditions. Nevertheless we can use the positivity of the spectral gap on a generic connected graph with
empty boundary condition (see Proposition 3.1) to bound again the variance on \( L_{i,j} \) with the Dirichlet form and then proceed as in the one dimensional case.

\[ \lim_{T \to \infty} \frac{1}{T} \log \mu_{\alpha,T}^{N,d}(W_0,\delta) \leq - C\delta^2 \left( \frac{N}{K} \right)^d \exp(\alpha/N^{d-c}) + |\alpha|N^c(1 + CN^{c-d}) \quad (4.40) \]

**Proof.** The proof uses Lemma 4.7 and follows exactly the same lines as the proof for Lemma 4.4. \( \square \)

**Lemma 4.9.** Consider FA-1f model in dimension \( d \geq 2 \) with \( \rho \in (0, 1) \) and boundary condition of dimension \( c \in [0, d - 1] \). Then

\[ \lim_{T \to \infty} \frac{1}{T} \log \langle W_{1,\delta} \rangle_\lambda \leq - \frac{N^c S_\rho \delta}{4}, \quad (4.41) \]

with \( S_\rho \) defined in Proposition 3.1.

**Proof.** We detail the proof in the case \( d = 2, c = 1 \) with the specific choice that all the sites in the boundary set which have first coordinate equal to \( N + 1 \) are empty. The other cases can be proven analogously. As in the proof of Lemma 4.3 we start by applying Donsker-Varadhan large deviation principle (3.5) which leads to

\[ \lim_{T \to \infty} \frac{1}{T} \log \langle W_{1,\delta} \rangle_\lambda \leq - \exp(\lambda) \inf_{f \in \mathcal{C}} \left\{ D_N^{(2)}(\sqrt{f}) \right\}, \quad (4.42) \]

where \( \mathcal{C} \) is the set of positive functions \( f \) s.t.

\[ \nu(f) = 1, \quad \nu \left( \int f \sum_{i=1}^{(N/K)^2} 1_{u_{K,i}=1} \right) \geq \delta \left( \frac{N}{K} \right)^2, \quad (4.43) \]

and we added the index (2) to explicitate the fact that we refer here to the Dirichlet form of the two dimensional model. By the monotonicity of the rates for any function \( f \), one has

\[ D_N^{(2)}(\sqrt{f}) = \sum_{i=1}^{N} D_N^{(2,i)}(\sqrt{f}) \geq \sum_{i=1}^{N} \mu \left( \mathcal{D}_N^{(1,i)}(\sqrt{f}) \right) \geq S_\rho \sum_{i=1}^{N} \mu \left( \text{Var}_i(\sqrt{f}) \right), \]

where \( D_N^{(2,i)} \) is the contribution of the \( i - th \) line to the Dirichlet form and \( D_N^{(1,i)} \) is instead the Dirichlet form of the one dimensional FA-1f model on the \( i - th \) line with empty boundary condition on the right border (note that \( D_N^{(1,i)}(\sqrt{f}) \) is a function of the configuration on all the sites that do not belong to the \( i - th \) line) and \( \text{Var}_i \) denotes the variance w.r.t. the Bernoulli measure restricted to the \( i - th \) line with the other variables held fixed. The first inequality follows from the fact that for any site belonging to the \( i - th \) line if the constraint is satisfied for the one dimensional model so it is for the two dimensional model (but the converse is not true), indeed for any \( x \) it holds

\[ 1 - \eta_{x+e_1} \eta_{x-e_1} \leq 1 - \eta_{x+e_1} \eta_{x-e_1} \eta_{x+e_2} \eta_{x-e_2}. \]
second inequality follows by using the spectral gap inequality \((3.12)\) for the one dimensional model. Then we notice that the condition \((4.43)\) implies that

\[
\nu \left( \sum_{i=1}^{N} \sum_{j=1}^{(N/K)} 1_{\tilde{u}_{K,\epsilon}^{i,j}=1} \right) \geq \delta N \frac{N}{K},
\]

where \(\tilde{u}_{K,\epsilon}^{i,j}\) stands for the activity label on the \(j\)-th one-dimensional box (of size \(K\)) of the line \(i\). Thus we get

\[
\inf_{f \in C} D_{N}^{(2)}(\sqrt{f}) \geq S_{\rho} \inf_{\sum_{i=1}^{N} \delta_{i} \geq \delta N} \sum_{i=1}^{N} \mu(\inf_{f \in C(\delta_{i})} \text{Var}_{i}(\sqrt{f})) ,
\]

where we denote by \(C(\delta)\) the set of positive functions which satisfy the conditions \(\nu(f) = 1\) and

\[
\nu \left( \sum_{j=1}^{N/K} 1_{\tilde{u}_{K,\epsilon}^{i,j}=1} \right) \geq \delta N \frac{N}{K}
\]

Following the lines of Lemma 4.3 we get for any \(i \in [1, N]\)

\[
\inf_{f \in C(\delta_{i})} \text{Var}_{i}(\sqrt{f}) \geq \delta_{i} - \rho K - \rho^{K/2},
\]

and inserting this result in \((4.45)\) and \((4.42)\) yields the desired result provided \(K\) is chosen sufficiently large so that \(\rho K + \rho^{K/2} \leq 3/4\).

## 5. Active regime

In this section we prove Theorem 2.1 (i) which establishes the linearity of the moment generating function \(\varphi\) for FA-1f and East model in the small \(\alpha\) regime. Then, we prove Theorem 2.4 (i) and the stronger result Lemma 5.1 on the conditional measure. In section 5.3 we generalize these results to higher dimensions. The key ingredients which will be used in these (quite technical) proofs are the main results obtained in the previous section, namely Lemma 4.2 and Lemma 4.3.

### 5.1. Linearity of the generating function: proof of Theorem 2.1 (i).

In this section we will prove the following proposition

**Proposition 5.1.** There exists \(C < \infty\) s.t. for any \(\delta > 0\) and \(\alpha > \alpha_{0} = -\frac{S_{\rho}}{4K}\) there is \(N(\delta, \alpha) < \infty\) such that for all \(N \geq N(\delta, \alpha)\) it holds

\[
\lim_{T \to \infty} \frac{1}{T} \log \left\langle \exp \left( \frac{\alpha}{N} A(T) \right) \right\rangle \leq A\alpha + C\delta + C\frac{\alpha}{\sqrt{N}} + C\frac{\alpha^{2}}{\sqrt{N}}. \tag{5.1}
\]

From this proposition, we deduce

**Proof of Theorem 2.1(i).** The result follows from the definition \((2.12)\) using the lower bound of Lemma 4.6 and the upper bound of Proposition 5.1.

We are therefore left with proving Proposition 5.1.
Proof of Proposition 5.1. We distinguish two cases.

Case $\alpha \leq 0$.

Choose $\epsilon = A_2^\delta$ and $K$ such that $K \geq \bar{K}(\frac{\delta}{2}, A_2^\delta)$ with $\bar{K}$ defined in Lemma 4.2, then

$$1_{W_{0,\delta/2}} + 1_{W_{-1,1-\delta}} + \sum_{\ell=[\delta N/(2K)]}^{N/K} 1_{V_{1,\ell} \cap W_{0,\delta/2}} \geq 1.$$  (5.2)

where the sets $V$ and $W$ were introduced in Definition 4.1. From (3.9) it follows that

$$\langle \exp \left( \frac{\alpha}{N} A(T) \right) \rangle \leq F_1 + F_2 + \sum_{\ell=[\delta N/(2K)]}^{N/K} F_{3,\ell}.$$  (5.3)

where

$$F_1 := \langle 1_{W_{0,\delta/2}} \exp \left( (\exp \left( \frac{\alpha}{N} \right) - 1) \int_0^T ds \mathcal{H}(\eta(s)) \right) \rangle_{\alpha/N}$$  (5.4)

$$F_2 := \langle 1_{W_{-1,1-\delta}} \exp \left( (\exp \left( \frac{\alpha}{N} \right) - 1) \int_0^T ds \mathcal{H}(\eta(s)) \right) \rangle_{\alpha/N}$$  (5.5)

and

$$F_{3,\ell} := \langle 1_{V_{1,\ell} \cap W_{0,\delta/2}} \exp \left( (\exp \left( \frac{\alpha}{N} \right) - 1) \int_0^T ds \mathcal{H}(\eta(s)) \right) \rangle_{\alpha/N}.$$  (5.6)

By using the fact that $\alpha \leq 0$, we get from (5.4), (5.5) and (5.6)

$$F_1 \leq \langle 1_{W_{0,\delta/2}} \rangle_{\alpha/N}.$$  (5.7)

$$F_2 \leq \langle 1_{W_{-1,1-\delta}} \exp \left( \frac{\alpha}{N} \int_0^T ds \mathcal{H}(\eta(s)) \right) \rangle_{\alpha/N} \exp \left( \frac{T \alpha^2}{N} \right).$$  (5.8)

and

$$F_{3,\ell} \leq \langle 1_{V_{1,\ell} \cap W_{0,\delta/2}} \exp \left( \frac{\alpha}{N} \int_0^T ds \mathcal{H}(\eta(s)) \right) \rangle_{\alpha/N} \exp \left( \frac{T \alpha^2}{N} \right).$$  (5.9)

Recall Lemma 4.2 then since we have chosen $K \geq \bar{K}(\frac{\delta}{2}, A_2^\delta)$ there is $C > 0$ such that

$$\lim_{T \to \infty} \frac{1}{T} \log(F_1) \leq \lim_{T \to \infty} \frac{1}{T} \log \langle 1_{W_{0,\delta/2}} \rangle_{\alpha/N} \leq -\delta^2 C \frac{N}{K} \exp\left( \frac{\alpha}{N} \right).$$  (5.10)

Then recalling inequality (4.4), Definition 4.1 for the event $W_{-1,1-\delta}$ and our choice $\epsilon = A_2^\delta$, we get from (5.8)

$$F_2 \leq \exp \left( T \alpha \left( A_1 - \frac{\delta}{2} \right) (1 - \delta) - \frac{2A_1}{K} + \frac{T \alpha^2}{N} \right).$$  (5.11)

Since the event $V_{1,\ell} \cap W_{0,\delta/2}$ is a subset of the event $W_{-1,1-\delta'}$ with $\delta' = \frac{\delta}{2} + (\ell + 1) \frac{K}{N}$, by using again (4.4) we get from (5.9)

$$F_{3,\ell} \leq \exp \left( T \alpha \left( A_1 - \frac{\delta}{2} - \frac{2A_1}{K} \right) \left( 1 - \delta - (\ell + 1) \frac{K}{N} \right) + \frac{T \alpha^2}{N} \right) \langle 1_{W_{1,\ell}} \rangle_{\alpha/N}.$$  (5.12)
where we used the fact that the event $\mathcal{V}_{1,\ell}$ is a subset of the event $\mathcal{W}_{1,\ell K_N}$. Then by using Lemma 4.3 it holds for any $\ell \geq \lceil \delta N/(2K) \rceil$ that
\[
\lim_{T \to \infty} \frac{1}{T} \log \left( \mathbb{1}_{\mathcal{W}_{1,\ell K_N}} \right)_{\alpha/N} \leq - \frac{S \rho \ell K_N}{4N} \exp\left( \frac{\alpha}{N} \right).
\] (5.13)
Thus collecting (5.12) and (5.13) we get
\[
\lim_{T \to \infty} \frac{1}{T} \log \left( \mathcal{F}_3,\ell \right) \leq \alpha A \left( 1 - \frac{\delta}{2} \right)^2 - \left( \alpha A \left( 1 - \frac{\delta}{2} \right) + \frac{S \rho}{4} \exp\left( \frac{\alpha}{N} \right) \right) \frac{\ell K}{N} + C \frac{K}{N} + C \frac{K}{N}.
\] (5.14)
Since $\alpha > \alpha_0 = -\frac{S \rho A}{4}$ and $\frac{K}{N} \geq \frac{\delta}{2}$, one gets (for $\delta$ small enough)
\[
\lim_{T \to \infty} \frac{1}{T} \log \left( \mathcal{F}_3,\ell \right) \leq \alpha A - \alpha A \delta - \left( \alpha A + \frac{S \rho}{4} \exp\left( \frac{\alpha}{N} \right) \right) \frac{\delta}{2} + C \frac{K}{N} + C \frac{K}{N}.
\] (5.15)
Thus under this hypothesis by using (5.3) and collecting (5.10), (5.11) and (5.14) if we now set $N \geq \bar{K} \left( \frac{\delta}{2}, A \frac{\delta}{2} \right)^2$ and $K = \sqrt{N}$ we get the desired result.

**Case $\alpha > 0$.**

Recall Definition 4.1 and the definition (5.4) for $\mathcal{F}_1$. Then it holds
\[
\left\langle \exp \left( \frac{\alpha}{N} A(T) \right) \right\rangle \leq F_1 + F_4,
\] (5.16)
where we let
\[
F_4 := \left\langle \mathbb{1}_{\mathcal{W}_{1,\delta/2}} \exp \left( \left( \exp\left( \frac{\alpha}{N} \right) - 1 \right) \int_0^T ds \, \mathcal{H}(\eta(s)) \right) \right\rangle_{\alpha/N}.
\] (5.17)
For $N > \alpha/\log 2$, then $\exp(\alpha/N) - 1 < \alpha/N + (\alpha/N)^2$. Thus via definition (5.4) since $\mathcal{H} \leq N$, using Lemma 4.2 we get
\[
\lim_{T \to \infty} \frac{1}{T} \log(F_1) \leq - \delta^2 C \frac{N}{K} + \alpha + \frac{\alpha^2}{N}.
\] (5.18)
Then using definition (5.10) and recalling inequality (4.3) we get
\[
\lim_{T \to \infty} \frac{1}{T} \log(F_4) \leq \alpha A + \alpha \delta + \frac{3\alpha}{K} + \frac{\alpha^2}{N}.
\] (5.19)
Thus by using (5.15) and collecting (5.17) and (5.18) if we now set $N \geq \max(\bar{K} \left( \frac{\delta}{2}, A \frac{\delta}{2} \right)^2, \frac{\alpha}{\log 2})$ and $K = \sqrt{N}$ we get the desired result.

5.2. **Conditional measure: proof of Theorem 2.4 (i) and a stronger result.** The main result of this section is the following Lemma 5.1 which, as we will explain, is a stronger version of Theorem 2.4(i).

**Lemma 5.1.** Let $K_N$ and $\delta_N$ be two sequences such that
\[
\lim_{N \to \infty} \delta_N = \lim_{N \to \infty} 1/K_N = \lim_{N \to \infty} K_N/(N\delta_N^2) = \frac{\log(\delta_N)}{K_N\delta_N^2} = 0.
\]
Note that there are subsequences which verify the above conditions, e.g. the choice $K_N = \sqrt{N}$ and $\delta_N = N^{-1/8}$. For each $N$ let the activity-density labels be defined with $K = K_N$ and $\epsilon_N = A \frac{\delta_N}{2}$. Then there exists $\alpha_0 < 0$ such that if $\alpha > \alpha_0$ it holds

$$
\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^N(W_{-1,1,-\delta_N}) = 1. \quad (5.19)
$$

**Proof.** Fix $N$ sufficiently large in order that it holds $K_N \geq \bar{K}(\frac{\delta_N}{2}, A \frac{\delta_N}{2})$ with $\bar{K}$ defined in Lemma 4.2 (this is possible thanks to the hypothesis $\lim_{N \to \infty} \frac{\delta_N}{\log(\delta_N)} = \infty$). We distinguish two cases.

**Case** $\alpha < 0$.

Since $\mu_{\alpha,T}^N$ is a measure from inequality (5.2), it holds

$$
1 - \mu_{\alpha,T}^N(W_{0,\delta_N/2}) \leq \sum_{\ell = \lceil \frac{K_N N}{2} \rceil}^{N/K} \mu_{\alpha,T}^N(V_{1,\ell} \cap W_{0,\delta_N/2}) \leq \mu_{\alpha,T}^N(W_{-1,1,-\delta_N}) \leq 1 \quad (5.20)
$$

Recall equation (2.14) which defines the conditional measure $\mu_{\alpha,T}^N$. If we apply Proposition 4.6 to bound the denominator and (3.9) to rewrite the numerator we get

$$
\mu_{\alpha,T}^N(W_{0,\delta_N/2}) \leq \exp \left( -\alpha A T(1 + \frac{C}{N}) \right) F_1, \quad (5.21)
$$

$$
\mu_{\alpha,T}^N(V_{1,\ell} \cap W_{0,\delta_N/2}) \leq \exp \left( -\alpha A T(1 + \frac{C}{N}) \right) F_{3,\ell}, \quad (5.22)
$$

where $F_1$ and $F_{3,\ell}$ are the functions that have been defined respectively in (5.4) and (5.6) and here we set $\delta = \delta_N$. Then we use (5.10) and the assumption $\lim_{N \to \infty} \frac{\delta_N^2}{K_N} = \infty$ to conclude that

$$
\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^N(W_{0,\delta_N/2}) = 0. \quad (5.23)
$$

Then by using (5.14), we get for $\frac{K_N}{N} \geq \frac{\delta}{2}$

$$
\lim_{T \to \infty} \frac{1}{T} \log \mu_{\alpha,T}^N(V_{1,\ell} \cap W_{0,\delta_N/2}) \leq -3 \alpha A \frac{\delta_N}{2} - \frac{S \delta_N}{4} + \frac{C}{N} - 2\alpha A \frac{\delta}{K_N}. \quad (5.24)
$$

By construction $\delta_N \gg \frac{1}{K_N} \gg \frac{1}{N}$. Thus for $\alpha > \alpha_0 = -\frac{S}{12A}$ and $N$ large enough

$$
\lim_{T \to \infty} \mu_{\alpha,T}^N(V_{1,\ell} \cap W_{0,\delta_N/2}) = 0. \quad (5.24)
$$

Note that the threshold $\alpha_0$ obtained here is not as sharp as in Proposition 5.1. The proof is then completed via (5.20), (5.23) and (5.24).

**Case** $\alpha \geq 0$.

Recall Definition 4.1 then it holds

$$
\mathbb{1}_{W_{-1,1,-\delta_N}} + \mathbb{1}_{W_{0,\delta_N/4}} + \mathbb{1}_{W_{0,\delta_N/4} \cap W_{-1,1,-\delta_N}} \geq 1. \quad (5.25)
$$

Since $\mu_{\alpha,T}^N$ is a measure from inequality (5.25) it holds

$$
1 - \mu_{\alpha,T}^N(W_{0,\delta_N/4}) - \mu_{\alpha,T}^N(W_{0,\delta_N/4} \cap W_{-1,1,-\delta_N}) \leq \mu_{\alpha,T}^N(W_{-1,1,-\delta_N}) \leq 1. \quad (5.26)
$$
Recall equation (2.14) which defines the measure $\mu_{\alpha,T}^N$. Applying Proposition 4.6 to bound the denominator and (3.9) to rewrite the numerator, there is $C > 0$ such that

$$\mu_{\alpha,T}^N(W_{0,\delta_N/4}) \leq \exp\left(-\alpha A_T(1 - \frac{C}{N})\right) F_1,$$  \hspace{1cm} (5.27)

where $F_1$ was defined in (5.4) and here we set $\delta = \frac{\delta_N A}{4}$. Thus (5.10) together with the hypothesis on $\delta_N$ and $K_N$ imply that

$$\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^N(W_{0,\delta_N/4}) = 0.$$  \hspace{1cm} (5.28)

On the other hand, by using inequality (4.5) and again Proposition 4.6 to bound the denominator and (3.9) to rewrite the numerator of the conditional measure, we get with $\varepsilon = A\frac{\delta_N}{2}$

$$\lim_{T \to \infty} \frac{1}{T} \log \left< \exp \left( \frac{\alpha}{N} A(t) \right) \right> \leq \alpha \delta_N A + \alpha (1 - \delta_N) A (1 + \frac{\delta_N}{2}) + \frac{2\alpha}{K_N} \leq \alpha A \left( 1 - \delta_N - \frac{\delta_N^2}{2} \right) + \frac{2\alpha}{K_N}.$$

Thus

$$\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha,T}^N(W_{0,\delta_N/4} \cap W_{-1,1-\delta_N}) = 0.$$  \hspace{1cm} (5.29)

The result is then proved thanks to (5.26), (5.28) and (5.29). \hfill $\blacksquare$

**Proof of Theorem (2.4)** (i). It is immediate to verify that the event $W_{-1,1-\delta}$ implies that $\pi_T(\sum_{i=1}^N \eta_i - N \rho) \leq (\epsilon + \delta) N$. Therefore (2.13) is proven by using Lemma 5.1 and taking $\gamma_N = \frac{3}{4} A \delta_N$. A similar argument leads to result (2.16). \hfill $\blacksquare$

### 5.3. FA-1f in dimension $d > 1$: proof of Theorem 2.5

The proof follows by using the results of Section 4.2 along exactly the same lines as the proof of Theorem 2.1(i) and 2.4(i).

### 6. Inactive regime

In this section we prove Theorem 2.1 (ii). We analyze in detail the case of FA-1f with two empty boundaries in Section 6.1 and then we sketch how the proof is extended to FA-1f with one empty boundary and East model in Section 6.2. Our proof will provide a variational characterization of the constant $\Sigma$ which appears in the theorem and which, as explained in section 2.2, plays the role of an interface energy. We underline that this variational problem, and therefore the value of $\Sigma$, depends not only on the choice of the constraints (the interface energy for East and FA-1f at the same density are different) but also on the choice of the boundary conditions (the interface energy for FA-1f with one and two empty boundaries are also different, see Remark 6.7). Finally, we prove Theorem 2.7 for FA-1f in higher dimensions.
6.1. FA-1f with two empty boundaries: proof of Theorem 2.1 (ii). Fix integer 
$L, L' > 0$ and let $C_{L,L'}$ be the set of probability densities on $\Omega_{L+L'+2}$ such that $\eta_{L+1} = \eta_{L+2} = 1$ with probability one, namely 

$$C_{L,L'} = \left\{ f : \nu(f) = 1, \nu(f \eta_{L+1} \eta_{L+2}) = 1 \right\}. \quad (6.1)$$

Let $\Sigma_{L,L'} = \inf_{f \in C_{L,L'}} D_{L+L'+2}(\sqrt{f})$ with $D$ the Dirichlet form of FA-1f model with two empty boundaries and define the interface energy $\Sigma$ as 

$$\Sigma := \lim_{L \to \infty} \lim_{L' \to \infty} \Sigma_{L,L'}. \quad (6.2)$$

The definition is well posed thanks to the following Lemma 6.1.

**Lemma 6.1.** $\Sigma_{L,L'}$ is non-increasing in $L$ and in $L'$. For any $L, L'$ it holds $\Sigma_{L,L'} \geq 0$. Therefore the limit $\lim_{L \to \infty} \lim_{L' \to \infty} \Sigma_{L,L'}$ exists.

**Proof.** The positivity of $\Sigma_{L,L'}$ immediately follows from its definition. Let $f(\eta_1, \ldots, \eta_{L+L'+2})$ be the function s.t. $\Sigma_{L,L'} = D_{L+L'+2}(\sqrt{f})$. Set $g(\eta_1, \ldots, \eta_{L+L'+3}) := f(\eta_2, \ldots, \eta_{L+L'+3})$. Then $g \in C_{L+1,L'}$ and

$$\Sigma_{L+1,L'} \leq D_{L+L'+3}(\sqrt{g}) = \sum_{i=3}^{L+L'+3} \sum_{\omega \in \Omega_{L+L'+3}} \nu(\omega) c_i(\omega) \left( \sqrt{g(\omega)} - \sqrt{g(\omega')} \right)^2 + \sum_{\omega \in \Omega_{L+L'+3}} \nu(\omega)(1 - \omega_1 \omega_3)(\rho(1 - \omega_2) + (1 - \rho)\omega_2) \left( \sqrt{g(\omega)} - \sqrt{g(\omega')} \right)^2 \leq \sum_{i=2}^{L+L'+2} \sum_{\omega \in \Omega_{L+L'+2}} \nu(\omega) c_i(\omega) \left( \sqrt{f(\omega)} - \sqrt{f(\omega')} \right)^2 + \sum_{\omega \in \Omega_{L+L'+2}} \nu(\omega)(\rho(1 - \omega_1) + (1 - \rho)\omega_1) \left( \sqrt{f(\omega)} - \sqrt{f(\omega')} \right)^2 = D_{L+L'+2}(\sqrt{f}) = \Sigma_{L,L'}. \quad (6.3)$$

Note that we have used the fact that the occupation variable at position 1 is unconstrained. Analogously if we set $h(\eta_1, \ldots, \eta_{L+L'+3}) := f(\eta_1, \ldots, \eta_{L+L'+2})$. Then $h \in C_{L,L'+1}$ and $D_{L+L'+3}(\sqrt{h}) \leq D_{L+L'+2}(\sqrt{f})$. Thus $\Sigma_{L,L'+1} \leq \Sigma_{L,L'}$ follows.

We split the proof of Theorem 2.1 (ii) into upper and lower bounds which are stated in the two following lemmas.

**Lemma 6.2.** Let $\Sigma$ be defined as in (6.2). Then for FA-1f model with two empty boundaries and any $\alpha < 0$ it holds

$$\lim_{N \to \infty} \lim_{T \to \infty} \frac{1}{T} \log \left\langle \exp \left( \frac{\alpha}{N} A(T) \right) \right\rangle \geq - \Sigma. \quad (6.4)$$
Lemma 6.3. Let \( \Sigma \) be defined as in (6.2). Then for FA-1f model with two empty boundaries and any \( \alpha < -\frac{\Sigma + 8\sqrt{\rho(1-\rho)}}{T} \) it holds

\[
\lim_{N \to \infty} \lim_{T \to \infty} \frac{1}{T} \log \left\langle \exp \left( \frac{\alpha}{N} A(T) \right) \right\rangle \leq -\Sigma.
\]

(6.5)

Then

Theorem 2.1 (ii) for FA-1f model with two empty boundaries. The result follows immediately from Lemma 6.2 and Lemma 6.3. \( \square \)

We are now left with the proof of the two above Lemmas.

Proof of Lemma 6.2. We fix \( K \) and take \( N \geq K \). Let \( \mathcal{O} \) be the event which is verified iff \( \pi_T(\eta_{K+1}, \ldots, \eta_{N-K}) = 1 \). Then from (3.9) we get

\[
\left\langle \exp \left( \frac{\alpha}{N} A(T) \right) \right\rangle \geq \left\langle \exp \left( \frac{\alpha}{N} \int_0^T ds \mathcal{H}(\eta(s)) \right) \right\rangle = \left\langle \exp \left( \frac{T\alpha^2}{N} \right) \right\rangle
\]

(6.6)

By using the fact that on the event \( \mathcal{O} \), it holds \( \int_0^T \eta_{i-1}(s)\eta_{i+1}(s) ds = T \) for any \( i \in [K + 2, N - K - 1] \) we get

\[
\int_0^T ds \mathcal{H}(\eta(s)) \leq 2(K + 1)T + \sum_{i=K+2}^{N-K-1} \int_0^T ds \ c_i(\eta(s)) = 2(K + 1)T + \sum_{i=K+2}^{N-K-1} \int_0^T ds(1 - \eta_{i-1}(s)\eta_{i+1}(s)) = 2(K + 1)T
\]

which together with (6.6) yields

\[
\left\langle \exp \left( \frac{\alpha}{N} A(T) \right) \right\rangle \geq \exp \left( \frac{\alpha}{N} 2(K + 1)T + \frac{\alpha^2}{N} T \right) \left\langle 1_{\mathcal{O}} \right\rangle_{\alpha/N}.
\]

(6.7)

From the Donsker-Varadhan large deviation principle (3.5) it holds

\[
\lim_{T \to \infty} \frac{1}{T} \log \left\langle 1_{\mathcal{O}} \right\rangle_{\alpha/N} = -\exp \left( \frac{\alpha}{N} \right) \inf_{f: \nu(f) = 1, \nu(f\eta_{K+1} \ldots \eta_{N-K}) = 1} \mathcal{D}_N(\sqrt{f}) \geq -\Sigma_{K,K}, \quad (6.8)
\]

where in order to obtain the last inequality we proceed as follows. Denote by \( g \) the function that belongs to \( \mathcal{C}_{K,K} \) and s.t. \( \Sigma_{K,K} = \mathcal{D}_{K+K+2}(\sqrt{g}) \). Set

\[
h(\eta_1, \ldots, \eta_N) := \frac{1}{\rho^{N-2K-2}} g(\eta_1, \ldots, \eta_K, \eta_{K+1}, \eta_{N-K}, \eta_{N-K+1}, \ldots, \eta_N) \prod_{j=K+2}^{N-K-1} \eta_j.
\]

Then it can be verified that \( \nu(h) = 1, \nu(h\eta_{K+1} \ldots \eta_{N-K}) = 1 \) and \( \mathcal{D}_N(\sqrt{h}) = \mathcal{D}_{K+K+2}(\sqrt{g}) \). Thus (6.8) follows. From (6.7) and (6.8) we therefore obtain

\[
\lim_{T \to \infty} \frac{1}{T} \log \left\langle \exp \left( \frac{\alpha}{N} A(T) \right) \right\rangle \geq -\Sigma_{K,K} + \frac{\alpha}{N} 2(K + 1) + \frac{\alpha^2}{N}.
\]

(6.9)

The result follows by taking \( N \) to infinity and then \( K \) to infinity. \( \square \)
Before proving Lemma 6.3, we state and prove an auxiliary result. Fix integers \( L, L' > 0 \) and \( u \in (0, 1) \) and consider \( C_{L,L'}^u \) the set of probability densities on \( \{0, 1\}^{L+L'+4} \) such that 
\[
\eta_{L+1} = \eta_{L+2} = \eta_{L+3} = \eta_{L+4} = 1 \text{ with probability at least } 1 - u, \text{ namely }
\]
\[
C_{L,L'}^u = \left\{ f : \nu(f(\eta)) = 1, \nu(f(\eta)\eta_{L+1}\eta_{L+2}\eta_{L+3}\eta_{L+4}) \geq 1 - u \right\}. \quad (6.10)
\]
We will now prove that provided \( u \) is sufficiently small the interface energy is well approximated by the infimum of \( D_{L+L'+2}(\sqrt{f}) \) restricted to \( C_{L,L'}^u \). More precisely,

**Lemma 6.4.** For any \( u > 0 \)
\[
\inf_{f \in C_{L+1,L'+1}^u} D_{L+L'+2}(\sqrt{f}) \geq \Sigma_{L,L'} - \left( 8\sqrt{\rho(1 - \rho)} + \Sigma_{L,L'} \right) u. \quad (6.11)
\]

**Proof.** Fix \( f \in C_{L+1,L'+1}^u \). The Dirichlet form (3.2) can be bounded from below by
\[
D_{L+L'+4}(\sqrt{f}) \geq \sum_{i=1}^{L+1} \sum_{\eta_i\eta_{i+2}=\eta_{i+3}=1} \nu(\eta)c_i(\eta)(\sqrt{f(\eta)} - \sqrt{f(\eta)})^2 + \sum_{i=L+4}^{L+L'+4} \sum_{\eta_i\eta_{i+2}=\eta_{i+3}=1} \nu(\eta)c_i(\eta)(\sqrt{f(\eta)} - \sqrt{f(\eta)})^2 + \sum_{\eta_i\eta_{i+2}=\eta_{i+3}=1} \nu(\eta)c_{L+2}(\eta)(\sqrt{f(\eta^{L+2})} - \sqrt{f(\eta)})^2 + \sum_{\eta_i\eta_{i+2}=\eta_{i+3}=1} \nu(\eta)c_{L+3}(\eta)(\sqrt{f(\eta^{L+3})} - \sqrt{f(\eta)})^2. \quad (6.12)
\]
We define a new probability density \( g(\eta) = \frac{f(\eta)\eta_{L+2}\eta_{L+3}}{c(u)} \),

with \( c(u) := \nu(f(\eta)\eta_{L+2}\eta_{L+3}) \). Note that \( \nu(g\eta_{L+2}\eta_{L+3}) = \nu(g) = 1 \), thus \( g \) belongs to \( C_{L+1,L'+1}^u \). Furthermore, since \( f \) belongs to \( C_{L+1,L'+1}^u \) one has \( c(u) \geq 1 - u \). Note that the Dirichlet form of \( g \) satisfies
\[
c(u)D_{L+L'+4}(\sqrt{g}) = \sum_{i=1}^{L+1} \sum_{\eta_i\eta_{i+2}=\eta_{i+3}=1} \nu(\eta)c_i(\eta)(\sqrt{f(\eta)} - \sqrt{f(\eta)})^2 + \sum_{i=L+4}^{L+L'+4} \sum_{\eta_i\eta_{i+2}=\eta_{i+3}=1} \nu(\eta)c_i(\eta)(\sqrt{f(\eta)} - \sqrt{f(\eta)})^2 + 2(1 - \rho) \sum_{\eta_i\eta_{i+2}=\eta_{i+3}=1} \nu(\eta)(1 - \eta_{L+1})f(\eta) + 2(1 - \rho) \sum_{\eta_i\eta_{i+2}=\eta_{i+3}=1} \nu(\eta)(1 - \eta_{L+4})f(\eta). \quad (6.13)
\]
Decompose $\eta$ as $\eta = (\omega_l, \eta_{L+2}, \eta_{L+3}, \omega_r)$ where $\omega_l = \eta_1, \ldots, \eta_{L+1}$ and $\omega_r = \omega_{L+4}, \ldots, \omega_{L+L'+4}$, then
\[
\sum_{\eta L+3=1}^\nu \nu(\eta) c_{L+2}(\eta) \left( \sqrt{f(\eta^{L+2})} - \sqrt{f(\eta)} \right)^2 \geq 2(1 - \rho) \sum_{\eta L+3=1}^\nu \nu(\eta)(1 - \eta_{L+1}) f(\eta) - 4\rho(1 - \rho) \sum_{\omega_l, \omega_r} \nu(\omega_l) \nu(\omega_r) \rho(1 - \eta_{L+1}) \sqrt{f(\omega_l, 1, \omega_r)} \sqrt{f(\omega_l, 0, 1, \omega_r)}.
\] (6.14)
Then by using Cauchy-Schwartz
\[
\sum_{\omega_l, \omega_r} \nu(\omega_l) \nu(\omega_r) \rho(1 - \eta_{L+1}) \sqrt{f(\omega_l, 1, \omega_r)} \sqrt{f(\omega_l, 0, 1, \omega_r)} \leq \frac{1}{\sqrt{\rho(1 - \rho)}} \nu(f(1 - \eta_{L+1})) \nu(f(1 - \eta_{L+1})) \leq \frac{u}{\sqrt{\rho(1 - \rho)}}
\] (6.15)
where to obtain the last inequality we used the fact that $f$ belongs to $C^n_{L, L'}$. Thus
\[
\sum_{\eta L+3=1}^\nu \nu(\eta) c_{L+2}(\eta) \left( \sqrt{f(\eta^{L+2})} - \sqrt{f(\eta)} \right)^2 \geq 2(1 - \rho) \sum_{\eta L+3=1}^\nu \nu(\eta)(1 - \eta_{L+1}) f(\eta) - 4\rho(1 - \rho) u.
\] (6.16)
Similarly it can be verified that
\[
\sum_{\eta L+3=1}^\nu \nu(\eta) c_{L+3}(\eta) \left( \sqrt{f(\eta^{L+3})} - \sqrt{f(\eta)} \right)^2 \geq 2(1 - \rho) \sum_{\eta L+3=1}^\nu \nu(\eta)(1 - \eta_{L+4}) f(\eta) - 4\rho(1 - \rho) u.
\] (6.17)
Thus by using (6.12), (6.13), (6.16) and (6.17) we get
\[
\mathcal{D}_{L+L'+4}(\sqrt{f}) \geq c(u) \mathcal{D}_{L+L'+4}(\sqrt{g}) - 8\sqrt{\rho(1 - \rho)} u \geq \Sigma_{L+1, L'+1} - (8\sqrt{\rho(1 - \rho)} + \Sigma_{L+1, L'+1}) u,
\]
where for the last inequality we used that, as noted above, $g$ belongs to $C_{L+1, L'+1}$ and $c(u) \geq (1 - u)$. \hfill \Box

Proof of Lemma 6.3: Recall Definition 4.1 and set $\epsilon = K \delta$ and choose $K \geq \bar{K}(\delta, \epsilon)$ with $\bar{K}$ defined in Lemma 4.2 and let $a_n = N^{-1/16}$. Then the following inequality holds
\[
\mathbf{1}_{W_0, \delta a_n} + \mathbf{1}_{W_1, 1 - \delta} + \sum_{\ell = [\delta(1 - a_n) N/K]}^{N/K} \mathbf{1}_{W_{1, 1 - \delta} \cap W_{0, \delta a_n}} \geq 1
\] (6.18)
which implies
\[
\left\langle \exp \left( \frac{\alpha}{N} \mathcal{A}(T) \right) \right\rangle \leq G_1 + G_2 + \sum_{\ell = [\delta(1 - a_n) N/K]}^{N/K} G_{3, \ell}
\] (6.19)
where
\[
G_1 := \left\langle \mathbf{1}_{W_0, \delta a_n} \exp \left( \left( \exp \left( \frac{\alpha}{N} \right) - 1 \right) \int_0^T ds \mathcal{H}(\eta(s)) \right) \right\rangle_{\alpha/N}
\] (6.20)
\[ G_2 := \left\langle \mathbf{1}_{W_{1,-\delta}} \exp \left( \left( \exp \left( \frac{\alpha}{N} \right) - 1 \right) \int_0^T ds \, \mathcal{H}(\eta(s)) \right) \right\rangle_{\alpha/N}. \tag{6.21} \]

\[ G_{3,\ell} := \left\langle \mathbf{1}_{\mathcal{V}_{-1,\ell} \cap W_{0,\delta \alpha_N}} \exp \left( \left( \exp \left( \frac{\alpha}{N} \right) - 1 \right) \int_0^T ds \, \mathcal{H}(\eta(s)) \right) \right\rangle_{\alpha/N}. \tag{6.22} \]

As in (5.10), \( G_1 \) is bounded by
\[
\lim_{T \to \infty} \frac{1}{T} \log(G_1) \leq -a_N^2 \delta^2 C N K. \tag{6.23}
\]

On the other hand since \( \alpha < 0 \), we have
\[
G_2 \leq \left\langle \mathbf{1}_{W_{1,-\delta}} \right\rangle_{\alpha/N}. \tag{6.24}
\]

We notice that the event \( W_{1,-\delta} \) implies that there exists at least one box \( i \in [1, N/K] \) such that \( \pi_T(\mathbf{1}_{w_{K_i}^{(i)} = 1}) \geq 1 - \delta \). Thus in this box, there are at least 4 consecutive sites occupied with high probability
\[
\pi_T(\eta((i-1)K+[K/2]+1) \cap [K/2]+2 \eta((i-1)K+[K/2]+3) \eta((i-1)K+[K/2]+4)) \geq 1 - \delta. \tag{6.25}
\]

Let \( R_i \) be the event that is verified if (6.25) holds. We get
\[
\left\langle \mathbf{1}_{W_{1,-\delta}} \right\rangle_{\alpha/N} \leq \sum_{i=1}^{N/K} \left\langle \mathbf{1}_{R_i} \right\rangle_{\alpha/N}. \tag{6.26}
\]

Donsker-Varadhan large deviation principle (3.5) implies
\[
\lim_{T \to \infty} \frac{1}{T} \log \left\langle \mathbf{1}_{R_i} \right\rangle_{\alpha/N} = -\exp(\alpha/N) \inf_{f \in C^2_{L,L'}} \left\{ D_{L+L'+2}(\sqrt{f}) \right\},
\]

where we have set \( L = (i-1)K + [K/2] \) and \( L' = N - 2 - (i-1)K - [K/2] \). By using Lemma 6.4, noticing that \( L, L' \geq [K/2] - 2 \) and recalling the monotonicity property stated by Lemma 6.1, we obtain
\[
\lim_{T \to \infty} \frac{1}{T} \log \left\langle \mathbf{1}_{R_i} \right\rangle_{\alpha/N} \leq -\Sigma_{L,L'} + \left[ 8 \sqrt{\rho(1 - \rho)} + \Sigma_{L,L'} \right] \delta - \frac{\alpha}{N} \Sigma_{L,L'} \leq -\Sigma + \left[ 8 \sqrt{\rho(1 - \rho)} + \Sigma_{\frac{K}{2}, \frac{K}{2}} \right] \delta + \frac{\left| \alpha \right|}{N} \Sigma_{1,1}. \tag{6.27}
\]

Thus
\[
\lim_{T \to \infty} \frac{1}{T} \log G_2 \leq -\Sigma + C \delta + \frac{\left| \alpha \right| \Sigma_{1,1}}{N}, \tag{6.28}
\]

where \( C = 8 \sqrt{\rho(1 - \rho)} + \Sigma_{\frac{K}{2}, \frac{K}{2}} \).

By using definition (6.22) and inequality (4.4), we get
\[
G_{3,\ell} \leq \left\langle \mathbf{1}_{\mathcal{V}_{-1,\ell} \cap W_{0,\delta \alpha_N}} \right\rangle_{\alpha/N} \exp \left( \frac{T \alpha^2}{N} \right) \exp \left( \frac{\alpha}{N} T \ell (KA - 2A - K\epsilon) \right). \tag{6.29}
\]
Note that \( \mathcal{V}_{-1, \ell} \cap \mathcal{W}_{0, \delta a N}^c \) is a subset of the event \( \mathcal{W}_{1, 1- (\delta a N + (\ell + 1) K N)} \). Thus, we can use an estimate similar to (6.28) in order to bound from above \[ \frac{1}{T} \log \left\langle 1_{\mathcal{V}_{-1, \ell} \cap \mathcal{W}_{0, \delta a N}^c} \right\rangle_{\alpha/N} \leq -\Sigma + C \left( \delta a N + (\ell + 1) K N \right) + \frac{C''}{N}, \] (6.30)
where \( C'' > 0 \) is a constant. Combining (6.29) and (6.30), we obtain
\[ \lim_{T \to \infty} \frac{1}{T} \log G_{3, \ell} \leq -\Sigma + C \left( \delta a N + (\ell + 1) K N \right) + \frac{C''}{N}, \] (6.31)
where we used that \( \epsilon = A \delta \). Recall \( \ell \geq \lceil \delta (1 - a N) N K \rceil \) and \( C = 8 \sqrt{\rho (1 - \rho)} + \Sigma K^2 \). Thus for \( \alpha < -\frac{8 \sqrt{\rho (1 - \rho)} + \Sigma}{A} \), we can choose \( \delta \) small enough and \( K \) large enough such that
\[ \lim_{T \to \infty} \frac{1}{T} \log G_{3, \ell} \leq -\Sigma + C' \frac{K}{N}. \] (6.32)
Sending \( N \to \infty \) and then \( K \to \infty \), we get the desired result by collecting (6.19), (6.23), (6.28) and (6.32).

6.2. East and FA-1f with one empty boundaries: proof of Theorem 2.1 (ii). Here we explain how to extend the results of the previous section to the case of East and FA-1f model with one empty boundary, thus completing the proof of Theorem 2.1 (ii). We start by giving the definition of the interface energy \( \Sigma \) for these models. Fix integer \( L > 0 \) and consider \( \mathcal{C}_L \) the set of probability densities on \( \Omega_L \) such that \( \eta_1 = 1 \) with probability one, namely
\[ \mathcal{C}_L = \left\{ f; \quad \nu(f) = 1, \quad \nu(f \eta_1) = 1 \right\}. \] (6.33)
Let \( \Sigma_L = \inf_{f \in \mathcal{C}_L} \mathcal{D}_L(\sqrt{f}) \), then we define the interface energy \( \Sigma \) as
\[ \Sigma := \lim_{L \to \infty} \Sigma_L \] (6.34)
The definition is well posed thanks to the following Lemma 6.1.

**Lemma 6.5.** Let \( \mathcal{D}_L \) be either the Dirichlet form of FA-1f with one empty boundary or the Dirichlet form of East. Then \( \Sigma_L \) is non-increasing in \( L \) and it holds \( \Sigma_L \geq 0 \). Therefore the limit \( \lim_{L \to \infty} \Sigma_L \) exists.

**Proof.** Let \( f(\eta_1, \ldots, \eta_L) \) be the function in \( \mathcal{C}_L \) s.t. \( \mathcal{D}(\sqrt{f}) = \Sigma_L \). Then set \( g(\eta_1, \ldots, \eta_{L+1}) := f(\eta_1, \ldots, \eta_L) \). Then \( g \in \mathcal{C}_{L+1} \) and, as for inequality (6.3), one can verify that \( \mathcal{D}_{L+1}(\sqrt{g}) \leq \mathcal{D}_L(\sqrt{f}) \) which implies \( \Sigma_{L+1} \leq \Sigma_L \) (since \( \Sigma_{L+1} \leq \mathcal{D}_{L+1}(\sqrt{g}) \) and \( \Sigma_L = \mathcal{D}_L(\sqrt{f}) \)).

We will now state a result which allows to approximate the interface energy in the spirit of Lemma 6.4. Let for any integer \( L > 0 \) and \( u \in (0, 1) \)
\[ \mathcal{C}_L^u := \left\{ f; \quad \nu(f(\eta)) = 1, \quad \nu(f(\eta) \eta_1 \eta_2) \geq 1 - u \right\} \]
then
Lemma 6.6. For any \( u > 0 \)
\[
\inf_{f \in \mathcal{C}_{L}^{u}} \mathcal{D}_{L}(\sqrt{f}) \geq \Sigma_{L} - \left(4 \sqrt{\rho(1 - \rho)} + \Sigma_{L}\right) u.
\] (6.35)

Proof. The proof is analogous to the proof of Lemma 6.4, therefore we sketch only the main points. Let \( f \in \mathcal{C}_{L}^{u} \) and set \( g(\eta) = \eta_{1} f(\eta)/c(u) \) with \( c(u) = \nu(f \eta_{1}) \geq 1 - u \). Then \( g \) belongs to \( \mathcal{C}_{L} \) and it holds
\[
\mathcal{D}_{L}(\sqrt{f}) \geq c(u) \mathcal{D}_{L}(\sqrt{g}) - 4 \sqrt{\rho(1 - \rho)} u \geq (1 - u) \Sigma_{L} - 4 \sqrt{\rho(1 - \rho)} u
\] (6.36)
where the first inequality is obtained following the same lines as in Lemma 6.4.

By using the above definitions and results we are now ready to prove Theorem 2.1 (ii).

Theorem 2.1 (ii) for East and for FA-If model with one empty boundary. It is enough to prove that if \( \Sigma \) is defined as in (6.24) then for FA-If with one empty boundary and for East the same inequalities as in Lemma 6.2 and Lemma 6.3 with \( \alpha_{0} = -4 \sqrt{\rho(1 - \rho) + \Sigma} \).

In order to prove the lower bound one introduces the event \( \mathcal{O} \) which is verified iff \( \pi_{T}(\eta_{1} \ldots \eta_{N - K}) = 1 \). Along the same lines used to obtain (6.9) on can verify that
\[
\lim_{N \to \infty} \lim_{T \to \infty} \frac{1}{T} \log \langle \exp \left( \frac{\alpha}{N} A(T) \right) \rangle = - \Sigma_{K} + \frac{\alpha}{N} (K + 1).
\]
The lower bound follows again by taking \( N \) to infinity and then \( K \) to infinity.

In order to prove the lower bound we follow exactly the same lines as in Lemma 6.3, the only difference being that in inequality (6.26) now the event \( \mathcal{R}_{i} \) is substituted by an event \( \tilde{\mathcal{R}}_{i} \) which verified iff
\[
\pi_{T}(\eta_{(i - 1)K + 1} \ldots \eta_{(i - 1)K + 2}) \geq 1 - \delta.
\]
Then Donsker-Varhadan principle (3.5) yields
\[
\lim_{T \to \infty} \frac{1}{T} \log \langle \mathbf{1}_{\tilde{\mathcal{R}}_{i}} \rangle_{\alpha/N} = - \exp(\alpha/N) \inf_{f \nu(f) = 1, \nu(f \eta_{(i - 1)K + 1} \eta_{(i - 1)K + 2}) \geq 1 - \delta} \mathcal{D}_{N}(\sqrt{f})
\] (6.37)
Given \( f \) on \( \Omega_{N} \) we define a new function function \( g \) on \( \Omega_{N - (i - 1)K} \) as follows
\[
g(\eta_{1}, \ldots, \eta_{N - (i - 1)K}) := \sum_{\eta'_{1} \ldots \eta'_{(i - 1)K}} \prod_{j=1}^{(i - 1)K} \rho \nu(1 - \rho)^{1 - \nu_{j}'} f(\eta_{1}' \ldots \eta'_{(i - 1)K}, \eta_{1}, \ldots, \eta_{N - (i - 1)K}).
\]
Then one can verify that it holds \( \mathcal{D}_{N}(\sqrt{f}) \geq \mathcal{D}_{K}(\sqrt{g}) \) and if \( f \) satisfies \( \nu(f) = 1 \) and \( \nu(\tilde{f}_{(i - 1)K + 1} \eta_{(i - 1)K + 2}) \geq 1 - \delta \) then \( g \) satisfies \( \nu(g) = 1 \) and \( \nu(\tilde{g}_{\eta_{1} \eta_{2}}) \geq 1 - \delta \), therefore \( g \) belongs to \( \mathcal{C}_{K}^{u} \). Therefore from (6.37) we get
\[
\lim_{T \to \infty} \frac{1}{T} \log \langle \mathbf{1}_{\tilde{\mathcal{R}}_{i}} \rangle_{\alpha/N} \leq - \Sigma_{N - (i - 1)K} + (4 \sqrt{\rho(1 - \rho)} + \Sigma_{N - (i - 1)K}) \delta + \frac{|\alpha|}{N} \Sigma_{1}
\]
\[
\leq - \Sigma + (4 \sqrt{\rho(1 - \rho)} + \Sigma_{K}) \delta + \frac{|\alpha|}{N} \Sigma_{1}
\] (6.38)

□
Remark 6.7. Fix \( \rho \in (0,1) \) and let \( \Sigma_1 \) and \( \Sigma_2 \) be the interface energies defined in formulas (6.2) and (6.3) by using the Dirichlet form of FA-1f with two empty boundaries and one empty boundary respectively. Then it can be easily verified that \( \Sigma_1 = 2 \Sigma_2 \).

6.3. Proof of Theorem 2.4 (ii) and a stronger result. We start by establishing a result which is stronger than Theorem 2.4 (ii).

Lemma 6.8. Consider East or FA-1f model in one dimension. Let \( K_N = \sqrt{N} \) and \( \delta_N = N^{-1/8} \) and for each \( N \) let the activity-density labels be defined with \( \epsilon_N = A \delta_N / 2 \). Then if \( \alpha < \alpha_0 \) with \( \alpha_0 \) defined in Lemma 6.3 it holds

\[
\lim_{N \to \infty} \lim_{T \to \infty} \mu_{\alpha, T}^N(\mathcal{W}_{1, 1-\delta_N}) = 1. \tag{6.39}
\]

Proof. Fix \( N \) sufficiently large such that \( K_N \geq \bar{K} (\frac{\delta_N}{2}, A \delta_N / 2) \) with \( \bar{K} \) defined in Lemma 1.2. Since \( \mu_{\alpha, T}^N \) is a measure from inequality (6.18) it holds

\[
1 - \mu_{\alpha, T}^N(\mathcal{W}_{0, \delta_N a_N}) - \sum_{\ell = [\delta_N (1-\alpha_N) N / K]}^{N/K} \mu_{\alpha, T}^N(\mathcal{W}_{1-\ell} \cap \mathcal{W}_{0, \delta_N a_N}) \leq \mu_{\alpha, T}^N(\mathcal{W}_{1, 1-\delta_N}) \leq 1, \tag{6.40}
\]

where we set \( a_N = N^{-1/16} \). We get from Lemma 6.2

\[
\lim_{N \to \infty} \lim_{T \to \infty} \frac{1}{T} \log \mu_{\alpha, T}^N(\mathcal{W}_{0, a_N \delta_N}) \leq \Sigma + \lim_{N \to \infty} \lim_{T \to \infty} \frac{1}{T} \log(G_1) \leq \Sigma - C \lim_{N \to \infty} a_N^2 \delta_N^2 \frac{N}{K_N}, \tag{6.41}
\]

where \( G_1 \) has been defined in (6.20) and we used inequality (6.23).

Choose \( \alpha \) such that \( \alpha < -2 \sqrt{\rho (1-\rho) + \Sigma / A} - \delta_N \). For \( G_{3, \ell} \) defined in (6.22) with \( \ell \geq [\delta_N (1-\alpha_N) N / K] \), one has by using (6.31) for any \( N \) large enough

\[
\lim_{T \to \infty} \frac{1}{T} \log \mu_{\alpha, T}^N(\mathcal{W}_{1-\ell} \cap \mathcal{W}_{0, a_N \delta_N}) \leq \Sigma + \lim_{T \to \infty} \frac{1}{T} \log(G_{3, \ell}) \leq - \frac{A}{2} \delta_N + C'' \frac{K_N}{N}. \tag{6.42}
\]

Thanks to the hypothesis on \( K_N, a_N, \delta_N \), it holds \( \lim_{N \to \infty} \delta_N \frac{N}{K_N} = \infty \) and \( \lim_{N \to \infty} a_N^2 \delta_N^2 \frac{N}{K_N} = \infty \) thus by using (6.40), (6.41) and (6.42) the proof is concluded.

Proof of Theorem 2.4 (ii). The event \( \mathcal{W}_{1, 1-\delta} \) implies

\[
\pi_T (\sum_i \eta_i) \geq N(1-\delta), \quad \text{and} \quad \pi_T (\sum_i c_i(\eta)) \leq \delta N.
\]

Thus the result follows by taking \( \gamma_N = \delta_N \) with \( \delta_N \) defined in Lemma 6.8.

6.4. FA-1f in \( d > 1 \): proof of Theorem 2.7.

Proof of Theorem 2.7. Recall that we have extended definition 4.1 to the higher dimensional case by substituting \( N / K \) with \( (N / K)^d \). We start from inequality

\[
1 - \mu_{\alpha, T}^{N_d}(\mathcal{W}_{0, \delta / 2}) - \mu_{\alpha, T}^{N_d}(\mathcal{W}_{1, 1-\delta}) + \mu_{\alpha, T}^{N_d}(\mathcal{W}_{1-\delta / 2}) \geq 1
\]

which leads to

\[
1 - \mu_{\alpha, T}^{N_d}(\mathcal{W}_{0, \delta / 2}) - \mu_{\alpha, T}^{N_d}(\mathcal{W}_{1, 1-\delta}) \leq \mu_{\alpha, T}^{N_d}(\mathcal{W}_{1-\delta / 2}) \leq 1. \tag{6.44}
\]
Lemma 4.8 guarantees that

$$\lim_{T \to \infty} \lim_{N \to \infty} \mu_{N,d}^{\alpha,T} 1_{W_{0,5/2}} = 0 \quad (6.45)$$

Then we notice that if at time zero the configuration is completely filled and the clocks on all the $N^c$ sites which are unconstrained do not ring up to time $T$ then $A(T) = 0$. Therefore

$$\langle \exp \left( \frac{\alpha}{N^{d-c}} A(T) \right) \rangle \geq \exp(-N^c T) \rho N^d$$

and by inserting this bound in the denominator of the definition (2.24) of the measure $\mu_{N,d}^{\alpha,T}$ and using inequality (4.4) (extended to dimension $d$), we get

$$\lim_{T \to \infty} \lim_{N \to \infty} \mu_{N,d}^{\alpha,T} 1_{W_{1,5/2}} = 0$$

provided $\alpha < -2/(\delta A)$ and the result is proven by inserting (6.45) and (6.46) into (6.44). \(\Box\)

### 7. Large deviations for a reduced activity

As we already recalled in the introduction, in absence of constraints (i.e. for the model defined by (2.1) and (2.3) with $r_i(\eta) \equiv 1$), the probability of observing a large deviation from the mean value scales as

$$\lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{Nt} \log \langle A(t) \rangle \simeq -f(a),$$

with $0 < f(a) < \infty$ for $a \neq 2\rho(1-\rho)$. In this section we will prove Theorems 2.2 and 2.6 which establish that a different scaling occurs for the large deviations of the activity below the mean value in the presence of constraints.

**Proof of Theorem 2.2.** Let us start by the upper bound. Let $\alpha_0$ be defined as in Theorem 2.1, then

$$\langle \frac{A(t)}{Nt} \simeq a \rangle \leq \exp(-\alpha_0 u A t) \langle \exp \left( \frac{\alpha_0}{N} A(t) \right) \rangle .$$

Therefore by taking the lim sup $N \to \infty$ on the right and left hand side and using Theorem 2.1 (i), we obtain the desired upper bound.

For the lower bound, we consider FA-1f with two empty boundaries (the proof in the other cases is analogous). The contribution to $A(t)/(Nt)$ can be decomposed into the contributions coming respectively from the configuration changes during the time interval $[0, ut]$ and $[ut, t]$. With probability (w.r.t. the mean over the initial distribution $\nu$ and the evolution of the process) which goes to one as $t$ goes to infinity the first contribution goes to $u A$ and, thanks to the reversibility of $\nu$, the distribution at time $ut$ is still $\nu$. Then we can impose that during the second time interval $[ut, t]$ the contribution to $A(t)/(Nt)$ is at most $2/\sqrt{N}$ by requiring that $\eta_{\sqrt{N}+1}(s) \ldots \eta_{N-\sqrt{N}}(s) = 1$ for any time $s$ in $[ut, t]$. Thus

$$\liminf_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \langle A(t) \rangle \simeq u A \geq \liminf_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log (\mathbb{1}_{B}),$$

where $B$ is the event which is verified iff starting from $\nu$ it holds $\pi_{(1-\rho)t}(\eta_{\sqrt{N}+1} \ldots \eta_{N-\sqrt{N}}) = 1$. As we did for the event $\mathcal{O}$ in (6.8) we get here

$$\liminf_{N \to \infty} \lim_{t \to \infty} \frac{1}{(1-\rho)t} \log (1_{B}) \geq -\Sigma,$$
and the proof is completed. □

**Proof of Theorem 2.6.** The proof of the upper bound follows along the same line as for Theorem 2.2 starting now from the inequality

\[
\left\langle \frac{A(t)}{N^d t} \sim u A \right\rangle \leq \exp(-\alpha_0 u A t N^c) \left\langle \exp\left(\frac{\alpha_0}{N^d - c} A(t)\right)\right\rangle.
\]  

(7.2)

The lower bound is derived in the same way by freezing the configuration during the time interval \([ut, t]\). The probability of realizing this event can be bounded from below by the probability that the \(N^c\) unconstrained sites (those which are in contact with the empty boundary) remain frozen equal to 1. □

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