DIFFERENTIAL EQUATION APPROXIMATION AND ENHANCING CONTROL METHOD FOR FINDING THE PID GAIN OF A QUARTER-CAR SUSPENSION MODEL WITH STATE-DEPENDENT ODE

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(Communicated by Kok Lay Teo)

Abstract. Active suspension control strategy design in vehicle suspension systems has been a popular issue in road vehicle applications. In this paper, we consider a quarter-car suspension problem. A nonlinear objective function together with a system of state-dependent ODEs is involved in the model. A differential equation approximation method, together with the control parametrization enhancing transform (CPET), is used to find the optimal proportional-integral-derivative (PID) feedback gains of the above model. Hence, an approximated optimal control problem is obtained. Proofs of convergences of the state and the optimal control of the approximated problem to those of the original optimal control problem are provided. A numerical example is solved to illustrate the efficiency of our method.

1. Introduction. Vehicle suspension systems have been a popular issue in road vehicle applications. Many researchers have dedicated effort in the modelling and the design of the computer-controlled suspension system: active or semi-active, quarter-car or full-vehicle models, enhancing the ride comfort and road-handling facilities. In the last three decades, most of the vehicle suspension problems are tackled by linear optimal control theory ([4]-[6])). However, the vehicle suspension systems usually consist of a lot of nonlinearities. More recently, some researchers ([7], [22] and [28]) began to approximate the non-linear suspension system by a linear

2020 Mathematics Subject Classification. Primary: 49M15, 65M60; Secondary: 35Q92.
Key words and phrases. Car suspension, control parametrization enhancing technique, differential equation approximations, optimal control, PID controllers, time-varying feedback gains.
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system by ignoring the dynamics in the actuator. In this way, the Proportional-Integral-Derivative (PID) controller can be obtained. The adoption of these PID controllers is due to its ability to minimize the suspension travel, together with its derivative and its accumulated value at all time. Moreover, due to the simple structure and flexibility, the PID controllers can be easily combined with linear, nonlinear or intelligent controllers to achieve improved performance.

Albeit the usefulness of the PID controller, the linear approximations in the above papers cannot provide an adequate model. Thus, [24] presented a quarter car suspension model without ignoring the dynamics of the actuator. In this way, the Proportional-Integral-Derivative (PID) controller can be obtained. The adoption of these PID gains in our controllers is time-dependent. Numerical results show that the oscillation of the four measures (the suspension travel, the sprung mass acceleration, the actuator force, and the control voltage) obtained by the time-varying controllers are less severe than those obtained by the static controllers. Thus, the use of the time-varying controllers can enhance the stability of the suspension system.

In this paper, we focus on a quarter-car suspension system similar to those of [1], [2], [3], [24] and [27]. Similar to the model of [24], our model also consists of all the nonlinearities arising from the state-dependent ODE of the actuator dynamic; moreover, our objective is also to enhance ride comfort as well as to save cost. However, unlike the controllers used in [24], each of the proportional, integral and derivative gains in our controllers is time-dependent. Numerical results show that the oscillation of the four measures (the suspension travel, the sprung mass acceleration, the actuator force, and the control voltage) obtained by the time-varying controllers are less severe than those obtained by the static controllers. Thus, the use of the time-varying controllers can enhance the stability of the suspension system.

The state-dependent ODE in the quarter-car suspension model in this paper has the complex form:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), \text{sign}(x_4(t) - x_3(t)), \text{sign}(x_5(t)), \\
&\quad \text{sign}(P_s - x_5(t) \text{sign}(x_5(t))), u(t), t), \\
x(0) &= \text{given},
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state variable, \( u(t) \in \mathbb{R}^m \) is the control variable, \( f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n \), \( n=6 \) and \( m=1 \) for the open-loop control problem and \( n=8 \) and \( m=3 \) for the feedback PID controller problem, \( P_s \) is a given constant, \( \text{sign} : \mathbb{R} \rightarrow \mathbb{R} \) is defined by

\[
\text{sign}(z) = \begin{cases} 
1, & \text{if } z > 0, \\
0, & \text{if } z = 0, \\
-1, & \text{if } z < 0.
\end{cases}
\]

Hence, the above state-dependent ODE consists of 8 regions of analyticity, namely:

\[
\begin{align*}
\Omega_1 &= \{ x \in \mathbb{R}^n : x_4 - x_3 \geq 0, x_6 \geq 0, x_5 \geq P_s \}; \\
\Omega_2 &= \{ x \in \mathbb{R}^n : x_4 - x_3 \geq 0, x_6 \geq 0, x_5 \leq P_s \}; \\
\Omega_3 &= \{ x \in \mathbb{R}^n : x_4 - x_3 \geq 0, x_6 \leq 0, x_5 \geq -P_s \}; \\
\Omega_4 &= \{ x \in \mathbb{R}^n : x_4 - x_3 \geq 0, x_6 \leq 0, x_5 \leq -P_s \}; \\
\Omega_5 &= \{ x \in \mathbb{R}^n : x_4 - x_3 \leq 0, x_6 \geq 0, x_5 \geq P_s \}; \\
\Omega_6 &= \{ x \in \mathbb{R}^n : x_4 - x_3 \leq 0, x_6 \geq 0, x_5 \leq P_s \}; \\
\Omega_7 &= \{ x \in \mathbb{R}^n : x_4 - x_3 \leq 0, x_6 \leq 0, x_5 \geq -P_s \}; \\
\Omega_8 &= \{ x \in \mathbb{R}^n : x_4 - x_3 \leq 0, x_6 \leq 0, x_5 \leq -P_s \}.
\end{align*}
\]
In view of the above characteristic, we develop a state equation approximation method similar to that used in [21] for solving the state-dependent ODE system. More precisely, we extend the application of the differential equation approximation method used in [21] from solving a simple state-dependent ODE with one non-smooth component to a more complex state-dependent ODE with many non-smooth components. By approximating the non-smooth function \( \text{sign}(y) \) by a smooth function \( \text{sign}_\delta(y) \), we obtain a more smooth transition of \( \dot{x}(t) \) whenever \( x(t) \) is very close to the boundary of any region \( \Omega_i (i = 1, \ldots, 8) \). (The transition of the state \( x(t) \) directly from \( \Omega_i \) to \( \Omega_j \), where \( i \neq j \), is possible if and only if \( \partial \Omega_i \cap \partial \Omega_j \) is non-empty. For instance, it is not possible for the state to transit directly from \( \Omega_1 \) to \( \Omega_4 \).) Using the above approximation method, the problem of finding the optimal time-varying PID feedback gains of the AVSS can be cast into an approximated optimal control problem whose state-space equations are governed by smooth functions only. This optimal control problem can be solved by the classical control parametrization enhancing transform (CEPT) ([9], [14], [15], [16], [18], [26] and [29]). Furthermore, proofs of convergences of the state and the optimal control of the approximated problem to those of the original optimal control problem are obtained.

On the other hand, our optimal control problem involving state-dependent ODEs can also be formulated as an optimal control problem of switched system involving state-dependent switching conditions, such as those appearing in [17]-[20]. However, the switched systems of the above papers have only one non-smooth component, such as \( x_1(t) \). Moreover, the method for finding the optimal control of the switched system in the above papers usually involves solving a two-phase optimization problem, such that the inner optimization concerns with finding the optimal control with the number of switchings \( N \) being fixed, and the outer optimization concerns with finding the optimal \( N \). Furthermore, to solve the inner optimization problem, one also needs to introduce boundary conditions at all the switching points. As the problem in this paper involves solving state-dependent ODE with many non-smooth components, the differential equation approximation method developed in this paper, which does not require imposing boundary conditions at the switching points or using global optimization technique, appears to be more appropriate for handling the non-smoothness in the differential equations.

In the other direction of study, apart from the car suspension model, the PID controller has also been successfully used in a lot of real-life situation. For instance, it has been used in [10] for controlling the direction of ship steering and in [13] for stabilizing the spacecraft attitude. In the above papers, the classical control parametrization method was used to obtain the PID controllers with fixed switching times. In this paper, we use the control parametrization enhancing technique (i.e. CEPT, a time scaling transformation technique) to obtain the PID controller with varying switching times. In this way, the performance of our optimal PID controller is better than those obtained in the above papers.

The paper is organized as follows: In Section 2, we first describe the formulation of the state-dependent ODE quarter-car suspension problem, using the PID controller. Then we show how to convert the optimal feedback PID controller problem into an equivalent non-smooth open-loop optimal control problem. In Section 3, we use a differential equation approximation method to convert the non-smooth open-loop optimal control problem into a standard optimal control problem whose state-space description is governed by differential equations with smooth functions.
only. In Section 4, we investigate the convergence results of the differential equation approximation method. In Section 5, we describe the CEPT for solving the quarter-car suspension problem formulated in Section 4. In Section 6, we provide a detailed discussion of a real-life quarter-car suspension problem. In Section 7, we solve a numerical example to illustrate the efficiency of our method. In Section 8, we extend our method to handle the situation that both the time horizon and the road disturbance cannot be accurately determined. Finally, the conclusion is presented in Section 9.

2. Formulation of the state dependent ODE quarter-car suspension problem, using PID controller. In this section, we first describe the form of the state-dependent ODE optimal feedback PID control for the quarter car suspension model that appears in [24]. Then, we show how to convert the optimal feedback PID control problem into an equivalent non-smooth open-loop optimal control problem. Consider the following form of the state-dependent ODE that appeared in [24]:

\[
\dot{x}(t) = f(x(t), \text{sign}(x_4(t) - x_3(t)), \text{sign}(x_6(t)), \\
\text{sign}(P_s - x_5(t) \text{sign}(x_6(t))), u(t), t),
\]

\[
x(0) = x_0,
\]

where \(x(t) \in R^6\) is the state variable, \(u(t) \in R\) is the control variable, \(f : R^6 \times R \times R \times R \times R \times R \to R^6, \text{sign} : R \to R\) is as defined in (3). Then the output function is an error function defined by

\[
y(t) = e(x(t)),
\]

where \(e : R^6 \to R\) is a given continuous function.

Let \(T\) be the final time. Let us impose the continuous state inequality constraint to the problem as follows:

\[
h_i(x(t), t) \geq 0 \quad \text{for } i = 1, \ldots, q, \quad t \in [0, T],
\]

where \(h_i : R^6 \times R \to R, i = 1, \ldots, q\), are given functions. The objective function to be minimized is

\[
J(u) = \int_0^T L(x(t), u(t), t)dt,
\]

where \(L : R^6 \times R \times R \to R\) is a given function.

Now, we implement a control system with a time-varying PID controller such that the “gains” of the controller are functions of time. That is, we implement a controller \(u(t)\) such that

\[
u(t) = K_p(t)y(t) + K_i(t) \int_0^t y(s) ds + K_d(t)\dot{y}(t),
\]

where \(K_p(t)\), \(K_i(t)\), and \(K_d(t)\) are, respectively, the proportional gain function, the integral gain function, and the derivative gain function, acting on the error function

\[
y(t) = e(x(t)),
\]

and \(K_p(t), K_i(t), K_d(t)\) are piecewise continuous functions of time \(t\).

We can now state our optimal feedback control problem, denoted by \((P)\), as follows:
(Problem ($P$)) Subject to the dynamic system (12)-(13) and the constraint (15), find an optimal feedback control of the form (17) such that the objective function (16) is minimized over the set of all feedback controls of the form (17).

Now, we convert the above optimal feedback control problem into an equivalent open-loop control problem. Let

$$
\mathbf{u}(t) = (\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t))^T = (K_P(t), K_I(t), K_D(t))^T.
$$

(19)

For the integral term of the controller, we define

$$
z(t) = \int_0^t y(s)ds.
$$

(20)

Then $z(t)$ satisfies the differential equation

$$\begin{align*}
\dot{z}(t) &= y(t), \\
z(0) &= 0.
\end{align*}
$$

(21)

(22)

Let $\mathbf{x}(t) = (x_1(t), \ldots, x_8(t))^T$, where

$$
x_i(t) = \begin{cases} x_e(t), & \text{for } i = 1, \ldots, 6, \\ y(t), & i = 7, \\ z(t), & i = 8. \end{cases}
$$

(23)

Then from (17), (23) and (18), we have

$$
u(t) = g(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{u}_1(t)\mathbf{x}_7(t) + \mathbf{u}_2(t)\mathbf{x}_8(t) + \mathbf{u}_3(t)\dot{\mathbf{x}}(t),
$$

(24)

where $g : R^8 \times R^3 \to R$, and $\mathbf{v} : R^8 \to R$ is the trivial extension of $e : R^6 \to R$ such that

$$\mathbf{v}(\mathbf{x}(t)) = e(x_1(t), \ldots, x_6(t)) + 0 \times \mathbf{x}_7(t) + 0 \times \mathbf{x}_8(t).
$$

(25)

Then, from (12), (13), (23), (24) and (18), the state equation with the PID controller $u(t)$ given by (17) becomes

$$\begin{align*}
\dot{x}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)), \\
\dot{x}_7(t) &= \mathbf{f}_7(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)), \\
\dot{x}_8(t) &= \mathbf{f}_8(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)),
\end{align*}
$$

(26)

(27)

where

$$
\mathbf{f}_i(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)) = f_i(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)) + g(\mathbf{x}(t), \mathbf{u}(t), t), \\
\mathbf{f}_7(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)) = f_7(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)) + g(\mathbf{x}(t), \mathbf{u}(t), t), \\
\mathbf{f}_8(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)) = f_8(\mathbf{x}(t), \mathbf{u}(t), \mathbf{x}_7(t), \mathbf{x}_8(t)) + g(\mathbf{x}(t), \mathbf{u}(t), t),
$$

(28)

and

$$
x_{0,i} = \begin{cases} x_{0,i}, & i = 1, \ldots, 6, \\ e(x_0), & i = 7, \\ 0, & i = 8. \end{cases}
$$

(29)

Hence, the objective of the PID controller has the form

$$
\mathcal{J}(\mathbf{u}) = \int_0^T \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)dt,
$$

(30)

where $L : R^8 \times R^3 \times R \to R$ is given by

$$
\mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t) = L(x_1(t), \ldots, x_6(t), g(\mathbf{x}(t), \mathbf{u}(t), t)).
$$

(31)
Let $U$ be a compact subset of $R^4$. A piecewise constant function $\overline{u}$ from $[0,T]$ into $R^4$ is said to be an admissible control if $\overline{u}(t) \in U$ for almost all $t \in [0,T]$. Let $U$ be the set of all admissible controls.

Let $\pi(t|u)$ be the solution of the system (26)-(27) corresponding to each admissible control $\pi \in U$. Let $F \subset U$ be the set of all feasible controls of Problem $(P1)$ to be defined later. Then

$$F = \{ \pi \in U : \overline{h}_i(\pi(t|u), t) \geq 0, \quad i = 1, \ldots, q, \quad t \in [0,T] \}, \quad (32)$$

where $\overline{h}_i : R^6 \times R \to R, i = 1, \ldots, q$, is the trivial extension of $h_i : R^6 \to R$ such that

$$\overline{h}_i(\pi(t), t) = h_i(\pi_1(t), \ldots, \pi_6(t), t) + 0 \times \pi_7(t) + 0 \times \pi_8(t). \quad (33)$$

We can now state our optimal feedback control problem, denoted by $(P1)$, as follows: (Problem $(P1)$) Subject to the dynamic system (26)-(27), find an optimal piecewise constant control $u^* = (K_p^*(t), K_d^*(t), \quad (35))$ which minimizes the objective function $\overline{J}(\overline{u})$ defined by (30) over the set of all piecewise constant controls in $F$.

3. The differential equation approximation method for solving the quarter-car suspension problem. In order to solve Problem $(P1)$, we first develop an approximate method for solving the above state dependent ODE system (26)-(27). To achieve the above purpose, we replace the non-smooth function $\text{sign}(y)$ by a sufficiently smooth approximation $\text{sign}_\delta(y)$, where

$$\text{sign}_\delta(y) = \begin{cases} 1, & \text{if } y > \delta, \\ I_\delta(y), & \text{if } -\delta \leq y \leq \delta, \\ -1, & \text{if } y < -\delta, \end{cases} \quad (34)$$

and

$$I_\delta(y) = -0.5\left(\frac{y}{\delta}\right)^3 + 1.5\left(\frac{y}{\delta}\right), \quad (35)$$

where $I_\delta(y) \in [-1,1]$.

Figure 1 depicts three graphs of $\text{sign}_\delta(x)$ versus $x$ with three different values of $\delta$, namely, $\delta = 0.0001, \delta = 0.00005$ and $\delta = 0.00001$, to illustrate the relative shapes of this non-smooth function.

By replacing $\text{sign}$ by $\text{sign}_\delta$ and $\pi$ by $\pi_\delta$ in (26)-(27), we obtain the approximated state equations as follows:

$$\pi_\delta(t) = \overline{J}(\pi_\delta(t), \text{sign}_\delta(\pi_{\delta,4}(t) - \pi_{\delta,3}(t)), \text{sign}_\delta(\pi_{\delta,6}(t))),$$

$$\text{sign}_\delta(P_s - \pi_{\delta,5}(t) \text{sign}_\delta(\pi_{\delta,6}(t)), \pi(t), t), \quad (36)$$

$$\pi_\delta(0) = \pi_0. \quad (37)$$

Remark 1. From (36) and (37), it is clear that for any $\delta > 0$, $\text{sign}_\delta(y) \in C_\infty(-\infty, \infty)$. Moreover, when the value of $\delta$ decreases, the approximation of $\text{sign}(y)$ by $\text{sign}_\delta(y)$ becomes more accurate; however, the ODE associated with the approximation scheme becomes stiffer.

Although $\text{sign}_\delta(y)$ does not converge uniformly to $\text{sign}(y)$ as $\delta \to 0$, we can prove in the next section that the values of the state variables of (36)-(37) converge to that of (26)-(27) as $\delta \to 0$. 
Let $\delta(t|\bar{u})$ be the solution of the system (36)-(37) corresponding to each admissible control $\bar{u} \in \mathcal{U}$. Let $\mathcal{F}_\delta \subset \mathcal{U}$ be the set of all feasible controls of the approximated problem $(P1^\delta)$ to be defined later. Then

$$
\mathcal{F}_\delta = \{ \bar{u} \in \mathcal{U} : \overline{h}_i (x_\delta(t|\bar{u}), t) \geq 0, \quad i = 1, \ldots, q, \quad t \in [0, T] \}.
$$

Replacing the system (26)-(27) by the approximated system (36)-(37), we obtain the new objective function to be minimized as follows:

$$
J_\delta(u) = \int_0^T L(x_\delta(t|\bar{u}), u(t), t) dt.
$$

Then we can define the approximated problem $(P1^\delta)$ as follows:

(Problem $(P1^\delta)$) Subject to the dynamic system (36)-(37), find an optimal piecewise constant PID control $\pi_\delta^*(t) \in \mathcal{F}_\delta$ which minimizes the objective function $J_\delta(\pi)$ defined by (39) over the set of all piecewise constant controls in $\mathcal{F}_\delta$.

4. **Convergence result of the state approximation equation method.** Same as in Section 3, we let $\pi(t|\bar{u})$ and $x_\delta(t|\bar{u})$ be the solutions of system (26)-(27) and system (36)-(37) corresponding to each admissible control $\bar{u} \in \mathcal{U}$, respectively. We wish to prove that $x_\delta(t|\bar{u})$ converges to $\pi(t|\bar{u})$ a.e. on $[0, T]$ as $\delta \to 0$. Moreover, we
also wish to prove a result concerning the convergence of the optimal control of the approximated problem \((P1^\delta)\) to that of the original problem \((P1)\).

Let \(\|\cdot\|\) be the usual norm in any finite-dimensional Euclidean space. Let \(L_\infty([0,T],\mathbb{R}^8)\) be the set of all measurable function \(H : [0,T] \to \mathbb{R}^8\) such that \(\text{ess sup}_{t \in [0,T]} |H(t)| < \infty\).

In order to prove the convergence result, we first need to convert \((26)-(27)\) into a form which consists of 8 regions of analyticity. The new form of the state equations is as follows:

\[
\begin{aligned}
\hat{\varvec{\pi}}(t) &= \begin{cases}
\hat{f}_1(\varvec{\pi}(t), \varpi(t), t) & \text{when } \varvec{\pi}(t) \in \Omega_1, \\
\hat{f}_2(\varvec{\pi}(t), \varpi(t), t) & \text{when } \varvec{\pi}(t) \in \Omega_2, \\
\hat{f}_3(\varvec{\pi}(t), \varpi(t), t) & \text{when } \varvec{\pi}(t) \in \Omega_3, \\
\hat{f}_4(\varvec{\pi}(t), \varpi(t), t) & \text{when } \varvec{\pi}(t) \in \Omega_4, \\
\hat{f}_5(\varvec{\pi}(t), \varpi(t), t) & \text{when } \varvec{\pi}(t) \in \Omega_5, \\
\hat{f}_6(\varvec{\pi}(t), \varpi(t), t) & \text{when } \varvec{\pi}(t) \in \Omega_6, \\
\hat{f}_7(\varvec{\pi}(t), \varpi(t), t) & \text{when } \varvec{\pi}(t) \in \Omega_7, \\
\hat{f}_8(\varvec{\pi}(t), \varpi(t), t) & \text{when } \varvec{\pi}(t) \in \Omega_8,
\end{cases}
\end{aligned}
\]

\(\varvec{\pi}(0) = \varvec{\pi}_0,\)

where \(\Omega_1, \ldots, \Omega_8\) are as defined in equations \((4)-(11)\) and for each \(i = 1, \ldots, 8\), the function \(\hat{f}\) satisfies

\[
\hat{f}_i(\varvec{\pi}(t), \varpi(t), t) = \hat{f}_i(\varvec{\pi}(t), \text{sign}(\varvec{\pi}_4(t) - \varvec{\pi}_3(t)), \text{sign}(\varvec{\pi}_6(t)), \\
\text{sign}(P_8 - \varvec{\pi}_5(t) \text{sign}(\varvec{\pi}_6(t))), \varpi(t), t)
\]

in its corresponding region.

Furthermore, we also need to impose the following assumptions:

**Assumption A**

\((A1)\) \(\hat{f} : \mathbb{R}^8 \times R \times R \times R^3 \times R \to \mathbb{R}^8\) in system \((36)-(37)\) is piecewise continuously differentiable with respect to each of its argument and for each \(i = 1, \ldots, 8\), \(\hat{f}_i : \mathbb{R}^8 \times R^3 \times R \to \mathbb{R}^8\) in system \((40)-(42)\) is continuously differentiable with respect to each of its argument. (In other words, \(\hat{f}\) is continuously differentiable whenever \(\varvec{\pi} \notin \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \ldots \cup \partial\Omega_8\).) Moreover, for each \(t\) and for each control \(\varvec{\pi} \in U\), there is no ambiguity as to which region \(\varvec{\pi}(t)\) belongs to and when \(\varvec{\pi}(t)\) enters any region \(\Omega_i\), \(i = 1, \ldots, 8\), it will remain in it for a finite time.

\((A2)\) For any compact subset \(V \subset \mathbb{R}^3\), there exists a positive constant \(K\) such that

\[
|\hat{f}(\varvec{\pi}, \varpi, t)| \leq K(1 + |\varvec{\pi}|), \quad i = 1, \ldots, 8
\]

for any \((\varvec{\pi}, \varpi, t) \in \mathbb{R}^8 \times V \times [0,T]\).

\((A3)\) \(\bar{f}_i(\varvec{\pi}, t), i = 1, \ldots, q\), is continuous with respect to \(\varvec{\pi}\) and piecewise continuous with respect to \(t\).

\((A4)\) \(\hat{f}(\varvec{\pi}, \varpi, t)\) is continuous with respect to each of its argument.

**Lemma 4.1.** Let \(\delta\) be a very small number. Then both \(\{\varvec{\pi}(t; \varpi) : u \in U\}\) of the solutions of system \((26)-(27)\) (or equivalently, system \((40)-(42)\)) and \(\{\varpi(t; \varpi) : \varpi \in U, 0 \leq \delta \leq \delta\}\) of the solutions of system \((36)-(37)\) are bounded in \(L_\infty([0,T], \mathbb{R}^8)\).

In other words, there exists a \(K_1 > 0\) such that

\[(i) \ |\varvec{\pi}(t; \varpi)| \leq K_1\]

\[(44)\]
for all $\overline{\eta} \in \mathcal{U}$ and for all $t \in [0, T]$, and

(ii) $|\overline{\eta}_s(t|\overline{\eta})| \leq K_1$ \hspace{1cm} (45)

for all $\overline{\eta} \in \mathcal{U}$, for all $\delta \in [0, \overline{\delta}]$ and for all $t \in [0, T]$.

**Proof.** The proofs of both parts (i) and (ii) of this lemma are similar to that of Lemma 6.4.2 of [25]. \qed

For each $\overline{\eta} \in \mathcal{U}$, let

$$F(t|\overline{\eta}) = f(\overline{\eta}_4(t|\overline{\eta}), \text{sign}(\overline{\eta}_3(t|\overline{\eta}) - \overline{\eta}_3(t|\overline{\eta})), \text{sign}(\overline{\eta}_5(t|\overline{\eta})), \text{sign}(P_s - \overline{\eta}_5(t|\overline{\eta}) \text{sign}(\overline{\eta}_6(t|\overline{\eta})), \overline{\eta}(t), t)$$ \hspace{1cm} (46)

and

$$F_3(t|\overline{\eta}) = f(\overline{\eta}_3(t|\overline{\eta}), \text{sign}(\overline{\eta}_5(t|\overline{\eta}) - \overline{\eta}_5(t|\overline{\eta})), \text{sign}(\overline{\eta}_6(t|\overline{\eta})), \text{sign}(P_s - \overline{\eta}_5(t|\overline{\eta}) \text{sign}(\overline{\eta}_6(t|\overline{\eta})), \overline{\eta}(t), t).$$ \hspace{1cm} (47)

**Remark 2.** In view of Lemma 4.1 and Remark 1, it is clear that all the arguments of $F_3(t|\overline{\eta})$, including the complex argument $\text{sign}(P_s - \overline{\eta}_5(t|\overline{\eta}) \text{sign}(\overline{\eta}_6(t|\overline{\eta})))$ are continuous on $[0, T]$ for all $0 \leq \delta \leq \overline{\delta}$, where $\overline{\delta}$ is as defined in Lemma 4.1. Thus, by (A1), $F_3(t|\overline{\eta})$ is continuous on $[0, T]$ for all $0 \leq \delta \leq \overline{\delta}$. However, $F(t|\overline{\eta})$ is only piecewise continuous on $[0, T]$.

**Remark 3.** In view of Lemma 4.1 and (A2), it is clear that both $\{F(t|\overline{\eta}) : \overline{\eta} \in \mathcal{U}, t \in [0, T]\}$ and $\{F_3(t|\overline{\eta}) : \overline{\eta} \in \mathcal{U}, \delta \in [0, \overline{\delta}], t \in [0, T]\}$ are bounded. In other words, there exists $K_2 > 0$ such that

$$|F(t|\overline{\eta})| \leq K_2$$ \hspace{1cm} (48)

for all $\overline{\eta} \in \mathcal{U}$ and for all $t \in [0, T]$ and

$$|F_3(t|\overline{\eta})| \leq K_2$$ \hspace{1cm} (49)

for all $\overline{\eta} \in \mathcal{U}$, for all $\delta \in [0, \overline{\delta}]$, and for all $t \in [0, T]$.

**Remark 4.** In view of Remark 2 and Remark 3, the existences of the complex arguments $\text{sign}(P_s - \overline{\eta}_5(t|\overline{\eta}) \text{sign}(\overline{\eta}_6(t|\overline{\eta})))$ in (26) and $\text{sign}(P_s - \overline{\eta}_5(t|\overline{\eta}) \text{sign}(\overline{\eta}_6(t|\overline{\eta})))$ in (36) do not complicate the proof of Theorem 4.1 given below.

**Theorem 4.1.** For each $\overline{\eta} \in \mathcal{U}$, the trajectory $\overline{\eta}_3(t|\overline{\eta})$ converges to $\overline{\eta}(t|\overline{\eta})$ everywhere on $[0, T]$ as $\delta \to 0$.

**Proof.** Let $\overline{\eta} \in \mathcal{U}$ and let $\delta < \overline{\delta}$ be another very small number. We assume that both $\overline{\eta}(t|\overline{\eta})$ and $\overline{\eta}_3(t|\overline{\eta})$ oscillate between two regions only once throughout $[0, T]$. (We can extend our discussion to include all the complicated situations easily.) However, the oscillation of the state between regions $\Omega_i$ and $\Omega_j$, where $i \neq j$, is possible if and only if $\partial \Omega_i \cap \partial \Omega_j$ is non-empty. For instance, it is not possible for the state to oscillate between regions $\Omega_1$ and $\Omega_4$. In view of Remark 2, without loss of generality, we assume that both $\overline{\eta}(t|\overline{\eta})$ and $\overline{\eta}_3(t|\overline{\eta})$ oscillate between the regions $\Omega_1$ and $\Omega_3$ once throughout $[0, T]$, where

$$\overline{\eta}(t) \in \Omega_1 = \{x \in R^8 : x_4 - x_3 \geq 0, x_6 \geq 0, x_5 \geq P_s\}$$ \hspace{1cm} (50)

and

$$\overline{\eta}(t) \in \Omega_3 = \{x \in R^8 : x_4 - x_3 \geq 0, x_6 \leq 0, x_5 \geq P_s\}.$$ \hspace{1cm} (51)

(All the possible oscillations between regions $\Omega_i$ and $\Omega_j$, where $i \neq j$, can be considered in a similar manner.)
Then, both \( \tilde{\pi}(t|\overline{\pi}) \) and \( \tilde{\pi}_\delta(t|\overline{\pi}) \) will first transverse from the interior of \( \Omega_1 \) to the interior of \( \Omega_3 \), i.e., from \( \Omega_1 \setminus \Omega^1_3 \) to \( \Omega^1_3 \) to \( \Omega^3_3 \). Let \( t_1 \) be the time that both \( \tilde{\pi}(t|\overline{\pi}) \) and \( \tilde{\pi}_\delta(t|\overline{\pi}) \) first arrive the boundary of \( \Omega^1_3 \) for the first time. Without loss of generality, we assume that \( \tilde{\pi}(t|\overline{\pi}) \) arrives the boundary of \( \Omega^1_3 \) faster than \( \tilde{\pi}_\delta(t|\overline{\pi}) \). Let \( t_2 \) be the time that \( \tilde{\pi}_\delta(t|\overline{\pi}) \) arrives the boundary of \( \Omega^1_3 \) for the first time. Then \( \tilde{\pi}_\delta(t_1|\overline{\pi}) = \tilde{\pi}_\delta(t_1|\overline{\pi}) = \delta \) and \( \tilde{\pi}_\delta(t_2|\overline{\pi}) = -\delta \). Both \( \tilde{\pi}(t|\overline{\pi}) \) and \( \tilde{\pi}_\delta(t|\overline{\pi}) \) will remain in \( \Omega^3_3 \setminus \Omega^3_3 \) for a finite time and then transverse from \( \Omega^3_3 \) to \( \Omega^3_3 \). Let \( t_3 \) and \( t_4 \) be, respectively, the times that \( \tilde{\pi}_\delta(t|\overline{\pi}) \) arrive the boundary of \( \Omega^3_3 \) the second time and the boundary of \( \Omega^3_3 \) the second time. We shall investigate the convergence of \( \tilde{\pi}_\delta(t|\overline{\pi}) \) to \( \tilde{\pi}(t|\overline{\pi}) \) in each of the following periods as \( \delta \to 0 \).

**Period 1** For \( t \in [0, t_1] \)

It is clear that for all \( t \in [0, t_1] \), \( \tilde{\pi}(t|\overline{\pi}) = \tilde{\pi}_\delta(t|\overline{\pi}) \) because the systems of O.D.E. for both \( \tilde{\pi}(t|\overline{\pi}) \) and \( \tilde{\pi}_\delta(t|\overline{\pi}) \) are the same.

**Period 2** For \( t \in [t_1, t_2] \)

Due to the fact that \( F_{\delta}(t|\overline{\pi}) \) defined by (47) is continuous on \( [t_1, t_2] \) and \( \hat{\delta} \) is a very small number, it is clear that \( \tilde{\pi}_\delta(t|\overline{\pi}) = F_{\delta}(t|\overline{\pi}) < 0 \) a.e. on \( [t_1, t_2] \). Thus, the time required by \( \tilde{\pi}_\delta(t|\overline{\pi}) \) to travel from the boundary \( \Omega^1_3 \) to the boundary of \( \Omega^3_3 \) (i.e., \( t_2 - t_1 \)) is less than \( 2\hat{\delta}/C_1 \), where

\[
C_1 = \inf_{t \in [t_1, t_2]} |F_{\delta}(t|\overline{\pi})| > 0. \tag{52}
\]

Thus, from (26)-(27), (36)-(37), (46)-(49) and (52), we have

\[
|\tilde{\pi}(t|\overline{\pi}) - \tilde{\pi}_\delta(t|\overline{\pi})| \leq \int_{t_1}^{t_2} |F(t|\overline{\pi}) - F_{\delta}(t|\overline{\pi})| dt \leq C_1 \hat{\delta} \tag{53}
\]

for all \( t \in [t_1, t_2] \), where

\[
C_1 = \frac{4K_2}{C_1}. \tag{54}
\]

**Period 3** For \( t \in [t_2, t_3] \)

When \( t \in [t_2, t_3] \), both \( \tilde{\pi}(t|\overline{\pi}) \) and \( \tilde{\pi}_\delta(t|\overline{\pi}) \) belong to the interior of \( \Omega_3 \), i.e. \( \Omega_3 \setminus \Omega^3_3 \). Thus, from (40), we have

\[
|\tilde{\pi}(t|\overline{\pi}) - \tilde{\pi}_\delta(t|\overline{\pi})| \leq |\tilde{\pi}(t_2|\overline{\pi}) - \tilde{\pi}_\delta(t_2|\overline{\pi})|

+ \int_{t_2}^{t_3} |f_3(\tilde{\pi}(\tau|\overline{\pi}), \overline{\pi}(\tau), \tau) - f_3(\tilde{\pi}_\delta(\tau|\overline{\pi}), \overline{\pi}(\tau), \tau)| d\tau \tag{55}
\]

for all \( t \in [t_2, t_3] \).

From (A1), \( \frac{\partial f_3(\tilde{\pi}, \overline{\pi}, t)}{\partial x} \) is continuous on \(\mathcal{U} \times [t_2, t_3] \) for each \( x \in B \) and continuous on \( B \) for each \( u \in \mathcal{U} \) and each \( t \in [t_2, t_3] \), where \( B = \{ y \in \mathbb{R}^n : |y| \leq K_1 \} \) and \( K_1 \) is as defined in Lemma 4.1. Thus, in view of (55) and (53), there exists a constant \( C_2 > 0 \) such that

\[
|\tilde{\pi}(t|\overline{\pi}) - \tilde{\pi}_\delta(t|\overline{\pi})| \leq C_1 \hat{\delta} + C_2 \int_{t_2}^{t_3} |\tilde{\pi}(\tau|\overline{\pi}) - \tilde{\pi}_\delta(\tau|\overline{\pi})| d\tau \tag{56}
\]

for all \( t \in [t_2, t_3] \).
From Gronwall-Bellman’s Lemma (Theorem 2.8.6 of [25]), we have
\[ |\pi(t|\bar{u}) - \pi_\delta(t|\bar{u})| \leq C_2 \delta \]
for all \( t \in [t_2, t_3] \), where
\[ C_2 = C_1 C_2 \exp(C_2 (t_3 - t_2)). \tag{58} \]

**Period 4** For \( t \in [t_3, t_4] \)
Using a similar argument as that used in obtaining (52) in Period 2, we have
\[ t_4 - t_3 \leq \frac{2\delta}{C_3}, \tag{59} \]
where
\[ C_3 = \inf_{t \in [t_3, t_4]} |F_{\delta\delta}(t | \bar{u})| > 0. \tag{60} \]
Thus, from (26)-(27), (36)-(37), (46)-(49) and (57), we have
\[ |x(t|u) - x_\delta(t|u)| \leq |x(t_3|u) - x_\delta(t_3|u)| + \int_{t_3}^{t_4} |F(t|u) - F_\delta(t|u)| dt \leq C_3 \delta \tag{61} \]
for all \( t \in [t_3, t_4] \), where
\[ C_3 = C_2 + \frac{4K_2}{C_3}. \tag{62} \]

**Period 5** For \( t \in [t_4, T] \)
Using an argument as that used in obtaining (57) in Period 3, we have
\[ |\pi(t|\bar{u}) - \pi_\delta(t|\bar{u})| \leq C_4 \delta \tag{63} \]
for all \( t \in [t_4, T] \), where
\[ C_4 = C_3 C_2 \exp(C_2 (T - t_4)). \tag{64} \]

Since \( \delta \) can be arbitrarily small, we conclude that the \( \pi_\delta(t|\bar{u}) \) converges to \( \pi(t|\bar{u}) \) everywhere on \([0, T]\) as \( \delta \to 0 \).

\[ \lim_{p \to \infty} \pi(t|\pi_p) = \pi(t|\bar{u}). \tag{65} \]

**Lemma 4.2** Let \( \{\pi_p\}_{p=1}^\infty \) be a bounded sequence of functions in \( L_\infty([0, T], R^3) \) that converges to a function \( \bar{u} \) a.e. on \([0, T]\). Then
\[ \lim_{p \to \infty} \pi(t|\pi_p) = \pi(t|\bar{u}). \tag{66} \]

**Proof.** The proof is similar to that of Lemma 6.4.3 of [25].

**Lemma 4.3** Let \( \pi_\delta \) be a feasible control of Problem (P1). Suppose that \( \pi_\delta \) converges to \( \bar{u} \) a.e. on \([0, T]\) as \( \delta \to 0 \). Then \( \bar{u} \) is also a feasible control of Problem (P1).

**Proof.** Let \( \{\delta_p\}_{p=1}^\infty \) be a sequence of \( \delta \) which decreases monotonically to 0 with \( \delta_1 = \bar{\delta} \), where \( \bar{\delta} \) is as defined in Lemma 4.1. Then we obtain from Theorem 4.1 that for any integer \( p_1 > 0 \), \( \pi_{\delta_{p_1}}(t|\pi_{\bar{u}}) \) converges to \( \pi(t|\pi_{\bar{u}}) \) as \( p \to \infty \). On the other hand, we obtain from Lemma 4.2 that for any integer \( p_2 > 0 \), \( \pi_{\delta_{p_2}}(t|\pi_{\bar{u}}) \) converges to \( \pi_{\delta_{p_2}}(t|\pi_{\bar{u}}) \) as \( p \to \infty \). Thus, the diagonal sequence \( \pi_{\delta_{p_2}}(t|\pi_{\bar{u}}) \) converges to \( x(t|\bar{u}) \) as \( p \to \infty \).

Since the above statement is true for any sequence \( \{\delta_p\}_{p=1}^\infty \), we conclude that
\[ \lim_{\delta \to \infty} \pi_\delta(t|\bar{u}) = \pi(t|\bar{u}) \]
for all $t \in [0, T]$. Thus, from (A3), we have
\[ h_i(x(t|u), t) = \lim_{\delta \to 0} h_i(x_\delta(t|u_\delta), t) \geq 0 \] (67)
for all $i = 1, \ldots, q$ and for all $t \in [0, T]$.

Thus, from (67), we conclude that $\pi$ is also a feasible control of Problem (P1). \hfill \Box

Let $\mathcal{F}$ be the interior of the set $\mathcal{F}$ defined by
\[ \mathcal{F} = \{ u \in U : h_i(x(t|u), t) > 0, \quad i = 1, \ldots, q, \quad t \in [0, T] \} . \] (68)

In order to prove that the sequence of optimal control of Problem ($P_{1\delta}$) converges to that of Problem (P1) as $\delta \to 0$, we need the following condition:

(A5) For any $u \in \mathcal{F}$, there exists $\bar{u} \in \mathcal{F}$ such that $\alpha \bar{u} + (1 - \alpha) u \in \mathcal{F}$ for all $\alpha \in (0, 1]$.

Theorem 4.2. Let $\bar{u}^*$ be an optimal control of Problem (P1) and let $u_{1\delta}^*$ be an optimal control of Problem ($P_{1\delta}$). Then

(i) $\lim_{\delta \to 0} J(\bar{u}_{1\delta}^*) = J(\bar{u}^*)$. (69)

(ii) Suppose that $u_{1\delta}^*$ converges to $\bar{u}^*$ a.e. on $[0, T]$ as $\delta \to 0$. Then $\bar{u}^*$ is also an optimal control of Problem (P1).

Proof. The proofs of parts (i) and (ii) of this theorem are similar to those of Theorem 8.5.1 and Theorem 8.5.2 of [25], respectively. \hfill \Box

5. Control parametrization enhancing technique. The Control Parametrization Enhancing Technique (CPET) for optimal control problems was introduced in [9]. CPET provides a computationally simple and numerically accurate solution without the assumption that the optimal control is of pure bang-bang type. If the optimal control is bang-bang, CPET computes the exact switching times using the control parametrization technique.

We consider the optimal control Problem ($P_{1\delta}$) as described in Section 2. Let
\[ U = \{ \pi(t) \in \mathbb{R}^3 : \text{is a step function} : u_l \leq \pi(t) \leq u_u \} . \]
We further introduce an independent variable which varies from 0 to $M$ for some positive integer $M$. The transformation CPET is then defined by the following differential equations:
\[ \frac{dt(s)}{ds} = \pi(s), \] (70)
\[ t(0) = 0, \] (71)
\[ t(M) = T, \] (72)
\[ \pi(s) \geq 0 \text{ for } s \in [0, M], \] (73)
where the scalar function $\pi(s)$ is called the enhancing control. Under this transformation, the time control problem ($P_{1\delta}$) becomes
\[ \min_{\pi(t)} J_3(\pi, \pi) = \int_0^M \pi(s)L(x_\delta(s), \pi(s), s) ds + \varepsilon \int_0^M (\pi), \] (74)
subject to the following differential equations

\[
\frac{d\pi(s)}{ds} = \nabla(s) \tilde{f}(\pi(s), \text{sign} \delta (\pi_4(s) - \pi_3(s)), \text{sign} \delta (\pi_6(s))),
\]
\[
\text{sign} \delta (P_s - \pi_5(s) \text{sign} \delta (\pi_6(s))), \nabla(s), s),
\]
(75)

\[
\frac{dt(s)}{ds} = \nabla(s),
\]
(76)

\[
\pi_3(0) = \pi_0, \quad t(0) = 0,
\]
(77)
(78)

together with the following constraints

\[
\int_0^M \nabla(s) \varphi_{i, \epsilon} (\pi_3(s)) ds = 0 \quad \text{for } i = 1, \ldots, q,
\]
(79)

\[
t(M) = T,
\]
(80)

\[
w_l \leq \nabla(s) \leq w_u \quad \text{for } s \in [0, M],
\]
(81)

\[
\nabla(s) \geq 0 \quad \text{for } s \in [0, M],
\]
(82)

where

\[
\varphi_{i, \epsilon} (\pi_3(s)) = \begin{cases} 
\pi_i (\pi_3(s), s), & \text{if } \pi_i (\pi_3(s), s) < -\epsilon, \\
(\pi_i (\pi_3(s), s) - \epsilon)^2 / 4\epsilon, & \text{if } -\epsilon \leq \pi_i (\pi_3(s), s) \leq \epsilon, \\
0, & \text{otherwise},
\end{cases}
\]
(83)

where \(\epsilon\) denotes some small positive number and \(\sqrt[3]{\nabla(s)} = \sum_{i=1}^{M} \sqrt[3]{\nabla_i(s)}\) denotes the bounded variation of the control \(\nabla(s)\), which is the sum of the total variation of each component of \(\nabla(s)\) on \([0, M]\). The details of this formulation are given in [25].

In the above formulation, the continuous state constraint

\[
\pi_i (\pi_3(s), s) \geq 0, \quad i = 1, \ldots, q, \quad t \in [0, T]
\]
(84)

is handled by the traditional constraint transcription method together with a local smoothing technique. However, a more efficient method, called the exact penalty method, has been developed in [11] and [12] for handling the above continuous state constraint. The main idea of this method is to incorporate the exact penalty function constructed from the equality constraints and continuous inequality constraints into the objective function, forming a new objective function. This gives rise to a sequence of unconstrained optimization problems. For sufficiently large penalty parameter, any local minimizer of the unconstrained optimization problem is a local minimizer of the original optimization problem with equality constraints and continuous inequality constraints.
6. A detailed discussion of a real-life quarter-car suspension problem. In this Section, we provide a detailed discussion of the real-life quarter-car suspension problem that appeared in [24]. The following system of differential equations from [24] is obtained via Newton’s law of motion:

\[
\dot{x}_1(t) = x_3, \\
\dot{x}_2(t) = x_4(t), \\
\dot{x}_3(t) = \frac{1}{m_s} \left[ k_s^l (x_2(t) - x_1(t)) + k_{nl}^l (x_2(t) - x_1(t))^3 + b_s^l (x_4(t) - x_3(t)) \\
- b_s^{sym} |x_4(t) - x_3(t)| + b_s^{nl} \sqrt{|x_4(t) - x_3(t)|} \text{sign}(x_4(t) - x_3(t)) - A x_5(t) \right], \\
\dot{x}_4(t) = -m_s \dot{x}_3(t) + w(t) k_l, \\
\dot{x}_5(t) = \dot{\Phi} x_6(t) - \beta x_5(t) + \alpha A (x_3(t) - x_4(t)), \\
\dot{x}_6(t) = \frac{-x_6(t) + K_s u(t)}{\tau},
\]

where \( \Phi = \text{sign}(P_s - x_5(t) \text{sign}(x_6(t))) \sqrt{|P_s - x_5(t) \text{sign}(x_6(t))|} \)

and the initial conditions of \( x_1(t), \ldots, x_6(t) \) are given. Thus, the general state-dependent differential equation described in Section 2 covers the above system as a special case. In (85)-(91), \( x_1(t) \) and \( x_2(t) \) denote the heave displacement of the chassis and the wheel, respectively; \( x_3(t) \) and \( x_4(t) \) denote the heave velocity of the chassis and the wheel, respectively; \( x_5(t) \) denotes the applied pressure developed by the motion of the piston; \( x_6(t) \) denotes the displacement of the spool valve. By using a time-varying input voltage \( u(t) \), we test the robustness of our suspension system against the road disturbance \( w(t) \). All the parameters used in the model are given in Appendix A and a schematic diagram is depicted in Figure 2.

All the terms involving the sign function can be explained as follows:

(i) \( b_s^{nl} \sqrt{|x_4(t) - x_3(t)|} \text{sign}(x_4(t) - x_3(t)) \) in (87) indicates that the direction of the suspension force depends on whether the relative velocity between the chassis travel and the wheel travel is positive or negative.

(ii) \(-x_5(t) \text{sign}(x_6(t)) \) in (91) indicates that the direction of the applied pressure generated from the piston depends on whether the displacement of the spool valve is positive or negative.

(iii) \( \text{sign}(P_s - x_5(t) \text{sign}(x_6(t))) \sqrt{|P_s - x_5(t) \text{sign}(x_6(t))|} \) in (91) indicates that the direction of the hydraulic load flow depends on whether the difference between the hydraulic pressure and the applied pressure is positive or negative.

The objective is to minimize

\[
J = \frac{1}{T} \int_0^T \left[ \left( \frac{\ddot{x}_1(t)}{\ddot{x}_{1,\text{max}}(t)} \right)^2 + \left( \frac{x_2(t) - w(t)}{(x_2(t) - w(t))_{\text{max}}} \right)^2 + \left( \frac{y(t)}{y_{\text{max}}} \right)^2 + \left( \frac{u(t)}{u_{\text{max}}} \right)^2 + \left( \frac{F_a(t)}{F_{a,\text{max}}} \right)^2 \right] dt,
\]

where \( y(t) = x_3(t) - x_1(t) \) is the difference between the displacement of the chassis and the wheel, \( F_a(t) = |x_5(t) \times A| \) is the actuation force, \( T \) is the final time and the values of \( \ddot{x}_{1,\text{max}}(t), \ (x_2(t) - w(t))_{\text{max}}, \ y_{\text{max}}, \ u_{\text{max}} \) and \( F_{a,\text{max}} \) are given.

The first, second, third, fourth and fifth term in the objective function denote the cost associated with the ride comfort, the vehicle road holding properties, the
vehicle suspension travel, the consumption power due to the control input voltage and that due to the actuation force, respectively. The terminal time $T$ is chosen in the following manner:

Suppose that the vehicle is travelling at a speed of $V$ meter per second starting from time $t = 0$ at sea level, where the road is level. After a short moment, it passes over the sinusoidal profile such that the amplitude of the bump is equal to $h$ meter and the wavelength of the bump is equal to $\lambda$ meter. Then we should choose a terminal time $T$ so that at time $t = T$, the car has passed through the sinusoidal profile completely and near $t = T$, the oscillations of the four measures (i.e., the suspension travel, the sprung mass acceleration, the actuator force, and the control voltage) produced by some PID controllers after the road disturbance are almost negligible. In this way, we can test how the above road disturbance $w(t)$ can affect the stability of the four measures produced by each of the PID controllers.

In addition to the above, some restrictions are imposed for the suspension system:

\[
|x_2(t) - x_1(t)| \leq 0.1, \quad (93)
\]
\[
|u(t)| \leq 10, \quad (94)
\]
\[
|F_a(t)| \leq (m_s + m_u)g, \quad (95)
\]
Definition 7.1 A 10-control-switchings (optimal switching times) scenario refers to the following situation:

(i) Each of the PI controller, PD controller, and PID controller can have at most 10 switching times throughout [0, T].
(ii) Both the magnitudes and the switching times of each of three feedback gains (i.e., the proportional feedback gain $K_P(t)$, the integral feedback gain $K_I(t)$ and the derivative feedback gain $K_D(t)$) are decision variables; however, $K_P(t)$, $K_I(t)$ and $K_D(t)$ are restricted to have the same switching times.

Similar definitions can be given to 2 control-switchings (optimal switching times) scenario and 1 control-switching (optimal switching time) scenario.

**Definition 7.2** A 2-control-switchings (fixed switching times) scenario refers to the following situation:

(i) Each of the PI controller, PD controller and PID controller can have at most 2 switching times throughout $[0, T]$.
(ii) Only the magnitude of each of three feedback gains are decision variables; the switching times of $K_P(t)$, $K_I(t)$, and $K_D(t)$ are fixed at $t = 1/3$ and $t = 2/3$.

Applying the combined state approximation and CPET method with $\delta = 0.001$, we first solve Problem ($Q$) using 3 different types of time-varying controllers, namely, the PI controller, the PD controller, and the PID controller under the 10-control-switchings (optimal switching times) scenario. Figure 3(a) – Figure 3(d) shows, respectively, the graphs of the suspension travel vs. time, the sprung mass acceleration vs. time, the actuator force vs. time, and the control voltage vs. time, obtained by using the above three controllers. It can be seen that the four measures obtained by each of the three controllers in this 10-control-switchings (optimal switching times) scenario are nearly the same. Due to the fact that time-varying feedback gains are used for each of the $K_P(t)$, $K_I(t)$, and $K_D(t)$ gains, (All the feedback gains are unbounded.) the control (i.e. the input voltage) has a full flexibility in searching the optimal gain of each of the three controllers. Albeit with the similar results, the PID controller produces the smallest amount of oscillation in each of the four measures. For comparison purpose, the scenario obtained by using constant control gains (the results plotted with dash-dotted-line) is also included in Figure 3(a) – Figure 3(d). From Figure 3(a) – Figure 3(d), it is clear that the stabilities of the four measures obtained by the static controller are much worse than those obtained by the time-varying controllers with 10 control-switchings.

Figure 4(a) – Figure 4(d) shows the impact of the different switching-time scenarios of the PID controller on the above four measures (the suspension travel, the sprung mass acceleration, the actuator force, and the control voltage), whereas Table 1 depicts the optimal switching times and the optimal values of the three feedback gains of the PID controller under different switching-time scenarios.

From Figure 4(a) – Figure 4(d), it can be seen that for the 1-control-switching scenario, the four measures keep on oscillating throughout the time interval $[0, T]$. (From the first row of Table 1, the optimal switching time of the three feedback gains of this scenario occurs at 1.516 second.) Thus, it is desirable to use a 2 control-switchings for each of the feedback gains so that more stable measures can be obtained. (From the second row of Table 1, the optimal switching times of the three feedback gains of this scenario occur at 0.781 second and 1.608 seconds, respectively.) The stabilities of the four measures are greatly improved. From the solid lines in Figure 3(a) – Figure 3(d) and the solid lines in Figure 4(a) – Figure 4(d), it can be seen that the degrees of oscillation of the four measures in the 2-control-switchings (optimal switching times) scenario are only slightly worse than those in the 10-control-switchings (optimal switching times) scenario. However, the computational time of the 2-control-switchings (optimal switching times) scenario is less than that of the 10-control-switchings (optimal switching times) scenario. Thus,
FIGURE 3(a) and FIGURE 3(b)  Comparison of the impact of the various controllers on the first two measures (i.e., the suspension travel and the sprung mass acceleration):  (i) PI controller under the 10-control-switchings (optimal switching time) scenario (represented by dotted lines);  (ii) PD controller under the 10-control-switchings (optimal switching time) scenarios (represented by dash lines);  (iii) PID controller under the 10-control-switchings (optimal switching times) scenarios (represented by solid lines);  (iv) the static controller (represented by dash-dotted lines)
FIGURE 3(c) and FIGURE 3(d) Comparison of the impact of the various controllers on the last two measures (i.e., the actuator force and the control voltage): (i) PI controller under the 10-control-switchings (optimal switching time) scenario (represented by dotted lines); (ii) PD controller under the 10-control-switchings (optimal switching time) scenarios (represented by dash lines); (iii) PID controller under the 10-control-switchings (optimal switching times) scenarios (represented by solid lines); (iv) the static controller (represented by dash-dotted lines)
FIGURE 4(a) and FIGURE 4(b)  Comparison of the impact of the different switching-time scenarios of the PID controllers on the first two measures (i.e., the suspension travel and the sprung mass acceleration) : (i) PID controller under the 1-control-switching (optimal switching time) scenario (represented by dotted lines); (ii) PID controller under the 2-control-switchings (optimal switching time) scenario (represented by solid lines); (iii) PID controller under the 2-control-switchings (fixed switching times) scenario (represented by dash lines)
FIGURE 4(c) and FIGURE 4(d) Comparison of the impact of the different switching-time scenarios of the PID controller on the last two measures (i.e., the actuator force and the control voltage): (i) PID controller under the 1-control-switching (optimal switching time) scenario (represented by dotted lines); (ii) PID controller under the 2-control-switchings (optimal switching time) scenario (represented by solid lines); (iii) PID controller under the 2-control-switchings (fixed switching times) scenario (represented by dash lines)
by allowing the feedback gains to have two-control-switchings, it is sufficient to solve this quarter-car suspension problem very efficiently. For comparison purpose, we also include a new 2-control-switchings scenario, which is called 2-control-switchings (fixed switching times) scenario, in Figure 4(a) – Figure 4(d). From Figure 4(a) – Figure 4(d), the oscillating behaviors of the four measures persist, although the degree of oscillating obtained under this new 2-control-switchings scenario is less severe than the l-control-switching (optimal switching time) scenario.

Finally, we give a description of the performance specifications, (i.e., rise time, setting time, overshoot percentage, and the undershoot percentage) of the suspension travel obtained by the PID controller. Thus, we need the definition of these performance specifications, together with their illustrations, which have been plotted in Figure 3(a).

**Definition 7.3** Let \( y(t) \) denote the response of the suspension travel produced by the controller. Let the amplitude of this response at time \( t \) be \( x_2(t) - x_1(t) + 0.1 \). In other words, at the lower bound of the suspension travel, (i.e., when \( x_2(t) - x_1(t) = -0.1 \)), the amplitude is zero. Let this response at the steady-state be \( y_{\text{Steady}} \). (Since the steady-state suspension occurs when \( x_2(t) - x_1(t) = 0 \), the amplitude at the steady-state suspension \( y_{\text{Steady}} \) is equal to 0.1.) Let \( \text{SettlingMax} \) (respectively, \( \text{SettlingMin} \)) be the maximum amplitude (respectively, the minimum amplitude) once the response has risen after the road disturbance. Then we can define the performance specifications as follow:

- **Rise time** is the time taken for the response to rise from the minimum of the response to \( y_{\text{Steady}} \).
- **Settling time** is the time taken for the error between amplitude \( y(t) \) and amplitude(\( y_{\text{Steady}} \)) (i.e., \( |\text{amplitude}(y(t)) - \text{amplitude}(y_{\text{Steady}})| \)) to fall to within 2% of amplitude(\( y_{\text{Steady}} \)). (As mentioned earlier, the response \( y_{\text{Steady}} \) occurs when \( x_2(t) - x_1(t) = 0 \).)
Overshoot percentage (respectively, undershoot percentage) is the maximum percentage of the overshoot (respectively, the undershoot), relative to \( y_{\text{Steady}} \) after the response has risen above \( y_{\text{Steady}} \).

In other words,

\[
\begin{align*}
\text{Overshoot percentage} &= \frac{\text{SettlingMax} - \text{amplitute}(y_{\text{Steady}})}{\text{amplitute}(y_{\text{Steady}})} \times 100, \\
\text{Undershoot percentage} &= \frac{\text{amplitute}(y_{\text{Steady}}) - \text{SettlingMin}}{\text{amplitute}(y_{\text{Steady}})} \times 100.
\end{align*}
\]

From the above definitions together with Figure 3(a), it can be seen that the rise time of the suspension travel obtained by the PID controller with 10 control-switchings (i.e., the time required by the suspension to travel from its minimum point where \( y(t) \) is about \(-0.005 \) to \( y(t) = 0 \)) is 0.20 second; the settling time is 1.5 second, which is 0.6 second after the road disturbance has occurred; the overshoot percentage is 1.54 and the undershoot percentage is 0.6.

Also from Figure 3(a), it can be seen that both the rise time and the settling time of the suspension travel obtained by the PID controller with 10-control switchings (shown by the solid line) are much shorter than that obtained by the static controller (shown by the dash-dotted line); the overshoot percentage and undershoot percentage obtained by the PID controller with 10-control switchings are also much smaller than that obtained by the static controller. Thus, we can conclude that the performance of the PID controller with 10-control switchings is much better than that of the static controller.

Pedro J. O. and Dahuni O. A. in [23] has mentioned that during the process of tracking a generated desired suspension travel in the presence of deterministic road disturbance, the design of the controller should satisfy the condition that ‘the maximum overshoot and the maximum undershoot should not be greater than 5%’. Since both the maximum overshoot and the maximum undershoot obtained by our PID controller with 10-control switchings are less than 2%, we conclude that the design of our PID controller is very satisfactory.

8. Extension of our method to handle the situation that both the time horizon and the road disturbance cannot be accurately determined. Suppose that we have a very long road of distance, say, more than 10 km. Then both the time horizon and the road disturbance of our problem cannot be accurately determined. In this situation, we first roughly divide the whole road into several sections, (i.e., Section \( k, k = 1, \ldots, m \)) so that the spectral density of the road irregularities within each section is the same. Then we can assume that the speed of the car within each section of the road is uniform. From the distance and the speed of the car, we can estimate the time taken by the car to travel from one section of the road to the next section. In this way, the time horizon of our problem can be estimated. Next, we also need to estimate the road disturbance in each section of the road. Suppose the spectral density of the road irregularities in Section \( k, k = 1, \ldots, m \), is given by \( S_{w_k}(w) = (\sigma^2/\pi)a_k v_k/w_k^2 + \sigma_k^2 v_k^2 \), where \( w_k \) is the angular power spectral density of the profile in Section \( k \), \( a_k \) is the coefficient depending on the type of the road surface of Section \( k \), and \( v_k \) is the speed of the car in Section \( k \). Then, according to [6], the process of the road disturbance in Section \( k \), denoted by \( w_k(t) \), satisfies the following differential equation

\[
\dot{w}_k(t) + a_k v_k w_k(t) = \xi_k, \tag{101}
\]

where \( \xi_k(t) \) is a white noise process with mean equal to zero and covariance function
\[ E(\xi_k(t)\xi_k(\tau)) = 2\sigma_k^2 a_k v_k \delta(t - \tau), \quad (102) \]

and \(\sigma_k^2\) denotes the variance of the road irregularities. Now, by assuming that the variance of the road irregularities in each section of the road is very small, we can assume that \(\sigma_k^2, k = 1, \ldots, m\), is zero. Thus, we obtain \(\xi_k(t) = 0\) in (101). Hence, the stochastic differential equation (101) is converted into the following deterministic differential equation:

\[ \dot{w}_k(t) + a_k v_k w_k(t) = 0 \quad \text{in Section } k \text{ of the long road.} \quad (103) \]

(Note that if Section \(k\) is flat, then \(a_k = 0\) and hence \(w_k(t) = c_k\), where \(c_k\) is a constant.) Hence, in view of (101) and the fact that the car is travelling at the speed \(v_k\) in Section \(k\) of the long road, we have

\[
\dot{w}(t) = \begin{cases} 
  a_1 v_1 w_1(t), & \text{if } t \in \left[0, \frac{d_1}{v_1}\right), \\
  a_k v_k w_k(t), & \text{if } t \in \left[\sum_{i=1}^{k-1} \frac{d_i}{v_i}, \sum_{i=1}^{k} \frac{d_i}{v_i}\right), i = 2, \ldots, m, 
\end{cases}
\quad (104)
\]

where \(w(0)\) is given, \(d_i\) is the length of Section \(i\) of the long road such that

\[
\sum_{i=1}^{m} \frac{d_i}{v_i} = \hat{T}, \quad (105)
\]

where \(\hat{T}\) is the terminal time of our problem. Thus, by treating \(w(t)\) as an additional state variable, we obtain a new optimal control problem, whereby we find the optimal control which simultaneously minimizes the ride comfort, the vehicle holding properties, the suspension travel, and the consumption of power, under the situation that both the time horizon and the road disturbance cannot be accurately determined. This new optimal control problem, denoted by Problem \((\hat{Q})\), can be stated as follows:

(Problem \((\hat{Q})\)) Subject to the dynamic system (85) – (91) and (104), the constraints (93) – (97), find an optimal feedback control of the form (98) such that the objective function \(\hat{J}\) is minimized over the set of all feedback controls of the form (98), where \(\hat{J}\) is the same as \(J\) in (92), except that the time horizon \([0, T]\) in (92) is now being replaced by \([0, \hat{T}]\).

Similar to Problem \((Q)\) in Section 6, Problem \((\hat{Q})\) can also be easily converted into an equivalent open-loop optimal control problem \((\hat{Q}1)\) which can be solved by the combined state equation approximation and CPET method. After we have solved Problem \((\hat{Q}1)\), we can easily get back the optimal feedback controller in exactly the same way as that described for solving Problem \((Q)\).

9. Conclusion. In this paper, we consider a nonlinear model of a quarter-car suspension problem with state-dependent, non-smooth ordinary differential equations. To solve the non-smooth ODEs, a differential equation approximation method is proposed, and the proof of convergence is provided. A computational method involving enhanced switching controls is used to solve an optimal vehicle suspension problem involving non-smooth ODE. A numerical example is presented to show the effectiveness of the method used. From the numerical results, it is clear that finding the exact switching times for the PID controllers is essential for the efficient management of the road-adaptive suspension system. For further investigation, the
model can be extended to a full vehicle suspension problem with stochastic road signal.

Acknowledgments. The authors would like to thank three referees for their valuable comments and suggestions. This research was supported by the General Research Fund (Project no. G. YBCM) of the Research Grants Council, Hong Kong.

Appendix A. Parameters used in the model

| Parameter                      | Value                      |
|--------------------------------|----------------------------|
| Sprung Mass                    | $m_s = 290kg$              |
| Unsprung Mass                  | $m_u = 40kg$               |
| Linear Suspension Stiffness    | $k_l = 23500\text{N/m}$    |
| Nonlinear Suspension Stiffness | $k_{nl} = 2350000\text{N/m}$ |
| Tyre Stiffness                 | $k_t = 190000\text{N/m}$   |
| Linear Suspension Damping      | $b_l = 700\text{Ns/m}$     |
| Nonlinear Suspension Damping   | $b_{nl} = 400\text{Ns/m}$  |
| Asymmetric Suspension Damping  | $b_{sym} = 400\text{Ns/m}$ |
| Actuator Parameter             | $\alpha = 4.51 \times 10^{13}$, $\beta = 1$, $\gamma = 1.545 \times 10^9$ |
| Piston Area                    | $A = 0.000335\text{m}^2$   |
| Supply Pressure                | $P_s = 10342500\text{Pa}$  |
| Time Constant                  | $\tau = 0.0333\text{s}$   |
| Servo-valve Gain               | $K_v = 0.001\text{m/V}$    |
| Gravity                        | $g = 9.81\text{m/s}^2$    |

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Received June 2018; 1st revision December 2018; final revision December 2018.