A RECURSION FOR A SYMMETRIC FUNCTION GENERALIZATION OF THE \( q \)-DYSON CONSTANT TERM IDENTITY

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Abstract. In 2000, Kadell gave an orthogonality conjecture for a symmetric function generalization of the \( q \)-Dyson constant term identity or the Zeilberger–Bressoud \( q \)-Dyson theorem. The non-zero part of Kadell’s orthogonality conjecture is a constant term identity indexed by a weak composition \( v = (v_1, \ldots, v_n) \) in the case when only one \( v_i \neq 0 \). This conjecture was first proved by Károlyi, Lascoux and Warnaar in 2015. They further formulated a closed-form expression for the above mentioned constant term in the case when all the parts of \( v \) are distinct. Recently we obtain a recursion for this constant term provided that the largest part of \( v \) occurs with multiplicity one in \( v \). In this paper, we generalize our previous result to all compositions \( v \).

Keywords: \( q \)-Dyson constant term identity, Kadell’s orthogonality conjecture, symmetric function, recursion

1. Introduction

In 1975, Andrews [1] conjectured that for non-negative integers \( a_1, a_2, \ldots, a_n \),

\[
\text{CT}_x \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_{a_i}(qx_j/x_i; q)_{a_j} = \frac{(q; q)_{a_1+\cdots+a_n}}{(q; a_1)(q; a_2)\cdots(q; q)_{a_n}},
\]

where for \( k \) a non-negative integer \((z; q)_k := (1-z)(1-zq)\cdots(1-zq^{k-1})\) is a \( q \)-shifted factorial, and \( \text{CT}_x \) denotes taking the constant term with respect to \( x := (x_1, x_2, \ldots, x_n) \). When \( q \to 1^- \), the constant term identity (1.1) reduces to the Dyson constant term identity [6].

Andrews’ \( q \)-Dyson conjecture was first proved combinatorially in 1985 by Zeilberger and Bressoud [15]. Twenty years later Gessel and Xin [9] gave a second proof using formal Laurent series, and in 2014, Károlyi and Nagy [12] discovered a very short and elegant proof using multivariable Lagrange interpolation. Finally, Cai [4] found an inductive proof. These days, Andrews’ ex-conjecture is usually referred as the Zeilberger–Bressoud \( q \)-Dyson theorem or the \( q \)-Dyson constant term identity.

There are many generalizations of the equal parameter case of the \( q \)-Dyson constant term identity, i.e., \( a_1 = a_2 = \cdots = a_n \). For example, the \( q \)-Morris identity [15] or the \( q \)-Selberg integral [8, 17], and the Macdonald constant term ex-conjecture [5, 13, 14, 16]. In the theory of Macdonald polynomials [14], the equal parameter case of the \( q \)-Dyson identity is equivalent to a scalar product identity. This provides a satisfactory explanation for the equal parameter
case of the \(q\)-Dyson identity in terms of orthogonal polynomials. Giving a similar such explanation for the full \(q\)-Dyson identity is an important open problem.

The first step towards a resolution of this problem was made by Kadell [10]. He formulated a symmetric function generalization of the \(q\)-Dyson identity and gave an orthogonal conjecture we will describe next. Let \(X = (x_1, x_2, \ldots)\) be an alphabet of countably many variables. Then the \(r\)th complete symmetric function \(h_r(X)\) may be defined in terms of its generating function as

\[
\sum_{r \geq 0} z^r h_r(X) = \prod_{i \geq 1} \frac{1}{1 - z x_i}.
\]

More generally, for the complete symmetric function indexed by a composition (or partition) \(v = (v_1, v_2, \ldots, v_k)\)

\[
h_v := h_{v_1} \cdots h_{v_k}.
\]

For \(a := (a_1, a_2, \ldots, a_n)\) a sequence of non-negative integers, let \(x^{(a)}\) denote the alphabet

\[
x^{(a)} := (x_1, x_1q, \ldots, x_1q^{a_1-1}, \ldots, x_n, x_nq, \ldots, x_nq^{a_n-1})
\]

of cardinality \(|a| := a_1 + \cdots + a_n\). Define a generalized \(q\)-Dyson constant term

\[
D_{v,\lambda}(a) := \text{CT}_x x^{-v} h_{\lambda}(x^{(a)}) \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_{a_i}(qx_j/x_i; q)_{a_j}.
\]

Here \(v = (v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n\), \(x^v\) denotes the monomial \(x_1^{v_1} \cdots x_n^{v_n}\), and \(\lambda\) is a partition such that \(|v| = |\lambda|\). (Note that if \(|v| \neq |\lambda|\) then \(D_{v,\lambda}(a) = 0\).) For the constant term \(1.3\), Kadell [10, Conjecture 4] conjectured that for \(r\) a positive integer and \(v\) a composition such that \(|v| = r\),

\[
D_{v,(r)}(a) = \begin{cases} 
\sum_{i=k+1}^n a_i \left[ a_i + r - 1 \right] \prod_{1 \leq i \leq j \leq n} \left[ a_i + \cdots + a_n \right] & \text{if } v = (0^{k-1}, r, 0^{n-k}), \\
0 & \text{otherwise},
\end{cases}
\]

where \(\binom{n}{k}\) is a \(q\)-binomial coefficient. In fact Kadell only considered \(v = (r, 0^{n-1})\) in his conjecture, but the more general statement given above is what was proved by Károlyi, Lascoux and Warnaar in [11, Theorem 1.3] using multivariable Lagrange interpolation and key polynomials. If for a sequence \(u = (u_1, \ldots, u_n)\) of integers we denote by \(u^+\) the sequence obtained from \(u\) by ordering the \(u_i\) in weakly decreasing order (so that \(u^+\) is a partition if \(u\) is a composition), then Károlyi et al. also proved a closed-form expression for \(D_{v,v^+}(a)\) in the case when \(v\) is a composition all of whose parts are distinct, i.e., \(v_i \neq v_j\) for all \(1 \leq i < j \leq n\). Subsequently, Cai [4] gave an inductive proof of Kadell’s conjecture. Recently, we [19] obtained a recursion for \(D_{v,v^+}(a)\) if \(v\) is a composition such that its largest part has multiplicity one, see Corollary [1.2] below.

In this paper, we obtain a recursion for \(D_{v,v^+}(a)\) for \(v\) an arbitrary non-zero composition. Given a sequence \(s = (s_1, \ldots, s_n)\) and an integer \(k \in \{1, 2, \ldots, n\}\), define \(s^{(k)} := (s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_n)\). Furthermore, for a subset \(I = \{k_1, k_2, \ldots, k_l\}\) of \(\{1, 2, \ldots, n\}\), define \(s^{(I)} := (s_1, s_2, \ldots, \hat{s}_{k_1}, \ldots, \hat{s}_{k_1}, \ldots, \hat{s}_{k_l}, \ldots, s_n)\), where \(\hat{t}\) denotes the omission of \(t\).
Theorem 1.1. Let $D_{0,0}(0) := 1$. Then for $v = (v_1, \ldots, v_n)$ a non-zero composition such that its largest part equals $r$ and $I := \{ i \in \{1, 2, \ldots, n \} : v_i = r \}$, the following recursion holds:

\[ D_{v,v^+}(a) = D_{v(i),v(v)}(a^{(I)}) \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|} q^{L_{I,J}(a)} \frac{1 - q^{a_J}}{1 - q^{a_i - a_J + r}}, \]

where $|S|$ denotes the number of elements in the set $S$, $a_S := \sum_{j \in S} a_j$, $\binom{n}{c}$ is a $q$-multinomial coefficient for the composition $c$, and

\[ L_{I,J}(a) = \sum_{\substack{1 \leq i \leq j \leq n \atop i \in I, j \notin J}} a_j. \]

For example, if $v = (0, 2, 3, 2, 3, 1)$, then $v^+ = (3, 3, 2, 2, 1, 0)$, $I = \{3, 5\}$ and $v^{(I)} = (0, 2, 2, 1)$. For $v$ a composition the constant term $D_{v,\lambda}(a) = 0$ if $a_i = 0$ and $v_i \neq 0$ for some $i$. On the other hand, if $a_i = v_i = 0$ for some $i$, then the constant term $D_{v,\lambda}(a)$ reduces to the $n-1$ variable case $D_{v(i),\lambda}(a^{(I)})$. Hence, since we only concern the constant term $D_{v,v^+}(a)$ for $v$ a composition in this paper, we can assume that all the $a_i$ are positive integers from now on. We note that for $v = (0, \ldots, 0)$ the constant term $D_{v,v^+}(a)$ corresponds to the $q$-Dyson constant term (the left-hand side of (1.1)). Using the recursion (1.5) and the $q$-Dyson identity (1.1), we can obtain a closed-form formula for $D_{v,v^+}(a)$ for arbitrary compositions $v$. If the largest part of $v$ has multiplicity one in $v$, then Theorem 1.1 reduces to [19, Theorem 1.3].

Corollary 1.2. Let $v = (v_1, \ldots, v_n)$ be a composition such that its largest part has multiplicity one in $v$. Fix a positive integer $k$ by $v_k = \max\{v\}$. Then

\[ D_{v,v^+}(a) = q^{\sum_{i=k+1}^n a_i \left[ \frac{v_i + |a| - 1}{a_k - 1} \right]} D_{v(k),v(k)}(a^{(k)}). \]

The method employed to prove Theorem 1.1 is based on Cai’s splitting formula (see (3.2) below) for the following rational function.

\[ F(a, w) := \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_{a_i} (qx_j/x_i; q)_{a_j} \prod_{i=1}^n \prod_{j=1}^s (x_i/w_j; q)_{a_i}^{-1}, \]

where $w := (w_1, \ldots, w_s)$ is a sequence of parameters. Throughout this paper, we assume that all terms of the form $c x_i/w_j$ in (1.8) satisfy $|c x_i/w_j| < 1$, where $c \in \mathbb{C}(q) \setminus \{0\}$. Hence,

\[ \frac{1}{1 - c x_i/w_j} = \sum_{k \geq 0} (c x_i/w_j)^k. \]

By the generating function of complete symmetric functions (1.2), the constant term $D_{v,\lambda}(a)$ equals a certain coefficient of $F(a, w)$. That is

\[ D_{v,\lambda}(a) = CT_{x,w} x^{-v} w^\lambda F(a, w), \]

where we assume $s$ in (1.8) is fixed by $\ell(\lambda)$ the length of $\lambda$. In [4], Cai gave a splitting formula for $F(a, w)$. Using his formula, we obtain an inductive formula for $D_{v,v^+}(a)$ (see Lemma 1.2). Then, by the inductive formula we prove the recursion (1.5) for $D_{v,v^+}(a)$. 


The remainder of this paper is organised as follows. In the next section we introduce some basic notation used throughout this paper. In Section 3 we present Cai’s splitting formula for $F(a, w)$. In Section 4 we obtain an inductive formula for $D_{v,v^+}(a)$ using Cai’s splitting formula. In Section 5 we prepare some results used in the proof of Theorem 1.1. In Section 6, we complete the proof of the recursion for $D_{v,v^+}(a)$.

2. Basic notation

In this section we introduce some basic notation used throughout this paper. For $v = (v_1, \ldots, v_n)$ a sequence, we write $|v|$ for the sum of its entries, i.e., $|v| = v_1 + \cdots + v_n$. Moreover, if $v \in \mathbb{R}^n$ then we write $v^+$ for the sequence obtained from $v$ by ordering its elements in weakly decreasing order. If all the entries of $v$ are non-negative integers, we refer to $v$ as a (weak) composition. A partition is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and only finitely-many $\lambda_i$ are positive. The length of a partition $\lambda$, denoted $\ell(\lambda)$ is defined to be the number of non-zero $\lambda_i$ (such $\lambda_i$ are known as the parts of $\lambda$). We adopt the convention of not displaying the tails of zeros of a partition.

For $k$ a non-negative integer, the $q$-shifted factorial is defined as

$$(z)_k = (z; q)_k := \prod_{i=0}^{k-1} (1 - zq^i),$$

where, typically, we suppress the base $q$. Using the above we can define the $q$-binomial coefficient as

$$\left[\begin{array}{c} n \\ k \end{array}\right] = \frac{(q^{n-k+1})_k}{(q)_k}$$

for $n$ and $k$ non-negative integers. Furthermore, for $n$ a non-negative integer and $s = (s_1, s_2, \ldots, s_k)$ a composition such that $|s| = n$, we define the $q$-multinomial coefficient as

$$\left[\begin{array}{c} n \\ s \end{array}\right] = \frac{(q)_n}{(q)^{s_1}(q)_s^2 \cdots (q)^{s_k}_k}.$$ 

In particular,

$$\left[\begin{array}{c} n \\ (k, n-k) \end{array}\right] = \left[\begin{array}{c} n \\ k \end{array}\right].$$

3. A splitting formula for $F(a, w)$

In this section, we present Cai’s splitting formula [4, Proposition 4.2] for $F(a, w)$. To make the paper self-contained, we include a proof of this splitting formula using partial fraction decomposition.

We need the following simple result in the proof of the splitting formula mentioned above.

Lemma 3.1. Let $i$ and $j$ be positive integers. Then, for $k$ an integer such that $0 \leq k \leq j-1$,

$$(3.1a) \quad \frac{(1/z)_i(qz)_j}{(q^{-k}/z)_i} = q^i(q^{1-i}z)_k(q^{k+1}z)_{j-k},$$
and
\[(3.1b) \quad \frac{(z)_j(q/z)_i}{(q^{-k}/z)_i} = q^{(k+1)i}(q^{-i}z)_{k+1}(q^{k+1}z)_{j-k-1}.\]

Note that the \(j = k\) case of (3.1a) (taking \(z \mapsto z/q\)) is the standard fact in [8, Equation (I.13)].

Proof. For \(0 \leq k \leq j\),
\[
\frac{(1/z)_i(qz)_j}{(q^{-k}/z)_i} = \frac{(q^{i-k}/z)_k(qz)_j}{(q^{-k}/z)_k} = \frac{(-1/z)^k q^{j-k+1}(q^{-1}z)_k(qz)_j}{(-1/z)^k q^{j+1}(q^{-1}z)_k} = q^k(q^{-1}z)_k(q^{k+1}z)_{j-k}.
\]

Taking \(z \mapsto z/q\) and \(k \mapsto k+1\) in (3.1a) yields (3.1b) for \(-1 \leq k \leq j-1\). Hence, both (3.1a) and (3.1b) hold simultaneously for \(0 \leq k \leq j-1\). \(\square\)

In [4], Cai showed that \(F(a, w)\) admits the following partial fraction expansion.

**Proposition 3.2.** Let \(F(a, w)\) be defined as in (1.8). Then
\[(3.2) \quad F(a, w) = \sum_{i=1}^{n} \sum_{j=0}^{a_i-1} \frac{A_{ij}}{1-q^j x_i/w_1},\]

where
\[(3.3) \quad A_{ij} = \frac{1}{(q^{-j})_j(q)_{a_i-j-1}} \prod_{1 \leq v < u \leq n \atop v, u \neq i} (x_v/x_u)_{a_v} (qx_u/x_v)_{a_u} \prod_{u=1}^{n} \prod_{v=2}^{s} (x_u/w_v)_{a_u}^{-1} \times \prod_{l=1}^{i-1} q^{j u} (q^{-a_u} x_i/x_l)_j (q^{j+1} x_i/x_l)_{a_i-j} \prod_{l=i+1}^{n} q^{j+1} (q^{-a_u} x_i/x_l)_{a_i-j+1}.\]

Note that \(A_{ij}\) is a power series in \(x_i\).

**Proof.** By partial fraction decomposition of \(F(a, w)\) with respect to \(w_1\), we can rewrite \(F(a, w)\) as (3.2) and
\[(3.4) \quad A_{ij} = F(a, w)(1-q^j x_i/w_1)|_{w_1=q^j x_i}.
\]

Carrying out the substitution \(w_1 = q^j x_i\) in \(F(a, w)(1-q^j x_i/w_1)\) yields
\[(3.5) \quad A_{ij} = \frac{1}{(q^{-j})_j(q)_{a_i-j-1}} \prod_{l=1}^{i-1} (x_l/x_i)_{a_l} (qx_i/x_l)_{a_l} \prod_{l=i+1}^{n} (x_l/x_i)_{a_l} (qx_l/x_i)_{a_l} \times \prod_{1 \leq v < u \leq n \atop v, u \neq i} (x_v/x_u)_{a_v} (qx_u/x_v)_{a_u} \prod_{u=1}^{n} \prod_{v=2}^{s} (x_u/w_v)_{a_u}^{-1}.\]
Using (3.1) with \((i, j, k, z) \mapsto (a_i, a_i, j, x_i/x_i)\), we have
\[
(3.6a) \quad \frac{(x_i/x_i)_{a_i}(qx_i/x_i)_{a_i}}{(q^{-j}x_i/x_i)_{a_i}} = q^{a_i} (q^{1-a_i}x_i/x_i)_{j} (q^{j+1}x_i/x_i)_{a_i-j},
\]
and
\[
(3.6b) \quad \frac{(x_i/x_i)_{a_i}(qx_i/x_i)_{a_i}}{(q^{-j}x_i/x_i)_{a_i}} = q^{(j+1)a_i} (q^{-a_i}x_i/x_i)_{j+1} (q^{j+1}x_i/x_i)_{a_i-j-1}
\]
respectively. Substituting (3.6) into (3.5) we obtain (3.3).

4. An inductive formula for \(D_{v,v^+}(a)\)

In this section, we utilize Cai’s splitting formula (3.2) for \(F(a, w)\) to obtain an inductive formula for \(D_{v,v^+}(a)\).

We begin by giving an easy result deduced from the \(q\)-binomial theorem.

**Proposition 4.1.** Let \(n\) and \(t\) be non-negative integers. Then
\[
(4.1) \quad \sum_{k=0}^{t} \frac{q^{k(n-t)}}{(q^{-k})_k (q)_t} = \left[ n \right]_t.
\]

**Proof.** We can rewrite the left-hand side of (4.1) as
\[
(4.2) \quad \sum_{k=0}^{t} \frac{(-1)^k q^{k(n-t)+(k+1)}}{(q)_k (q)_t} = \frac{1}{(q)_t} \sum_{k=0}^{t} q^{\left( \frac{k}{2} \right)} \left[ \frac{t}{k} \right] (-q^{n-t+1})^k.
\]

Using the well-known \(q\)-binomial theorem [2, Theorem 3.3]
\[
(z)_t = \sum_{k=0}^{t} q^{\left( \frac{k}{2} \right)} \left[ \frac{t}{k} \right] (-z)^k
\]
with \(z \mapsto q^{n-t+1}\), we have
\[
\frac{1}{(q)_t} \sum_{k=0}^{t} q^{\left( \frac{k}{2} \right)} \left[ \frac{t}{k} \right] (-q^{n-t+1})^k = (q^{n-t+1})_t/(q)_t = \left[ n \right]_t.
\]

By the splitting formula for \(F(a, w)\), we obtain the following inductive formula for \(D_{v,v^+}(a)\).

**Lemma 4.2.** Let \(v = (v_1, \ldots, v_n)\) be a non-zero composition such that its largest part equals \(r\), and set \(I := \{ i \in \{1, 2, \ldots, n\} : v_i = r \}\). Then
\[
(4.3) \quad D_{v,v^+}(a) = \sum_{i \in I} q^{\sum_{j=i+1}^{n} a_j} \left[ \frac{|a| + r - 1}{a_i - 1} \right] D_{v^{(i)},(v^{(i)})^+}(a^{(i))}).
\]

**Proof.** Substituting the splitting formula (3.2) for \(F(a, w)\) into
\[
(4.4) \quad D_{v,v^+}(a) = CT_{x,w} \frac{w^{v^+}}{x^v} F(a, w),
\]
we have
\[ D_{v,v^+}(a) = \sum_{i=1}^{n} \sum_{j=0}^{a_i-1} \frac{C_T}{x^v} \frac{w^v}{1 - q^j x_i/w_1} A_{ij}, \]
where we take \( w = (w_1, \ldots, w_n) \). Taking the constant term with respect to \( w_1 \), we obtain
\[ (4.5) \quad D_{v,v^+}(a) = \sum_{i=1}^{n} \sum_{j=0}^{a_i-1} \frac{C_T}{x^v} \frac{q^{j} x_i^1 w_2^{\mu_2} \cdots w_n^{\mu_n}}{1 - q^j x_i/w_1} A_{ij}, \]
where \( \mu := v^+ = (\mu_1, \ldots, \mu_n) \) and \( \mu_1 = \max\{v\} = r \). Since \( A_{ij} \) is a power series in \( x_i \) for all \( i \),
\[ \frac{C_T}{x^v} \frac{q^{j} x_i^1 w_2^{\mu_2} \cdots w_n^{\mu_n}}{1 - q^j x_i/w_1} A_{ij} = 0 \quad \text{for} \; i \notin I, \]
and
\[ \frac{C_T}{x^v} \frac{q^{j} x_i^1 w_2^{\mu_2} \cdots w_n^{\mu_n}}{1 - q^j x_i/w_1} A_{ij} = 0 \quad \text{for} \; i \in I. \]
Hence, (4.5) reduces to
\[ D_{v,v^+}(a) = \sum_{i=1}^{n} \sum_{j=0}^{a_i-1} \frac{q^{j} (a_1 + \cdots + a_{i-1} + r + (j+1)(a_{i+1} + \cdots + a_n))}{(q^{-j})_j (q)_{a_i - j - 1}} \]
\[ \times \frac{w_2^{\mu_2} \cdots w_n^{\mu_n}}{x_1^{v_1} \cdots x_{i-1}^{v_{i-1}} x_i^{v_i+1} \cdots x_n^{v_n}} \prod_{1 \leq v < u \leq n, v,u \neq i} (x_v/x_u)_{a_v} (q x_u/x_v)_{a_u} \prod_{u=1}^{n} \prod_{v=2, v \neq i}^{n} (x_u/w_v)_{a_u}^{-1}. \]

Using (4.1) with \( k \mapsto j, t \mapsto a_i - 1 \) and \( n \mapsto |a| + r - 1 \), we find
\[ D_{v,v^+}(a) = \sum_{i \in I} q^{a_i + \cdots + a_n} \left[ \frac{|a| + r - 1}{a_i - 1} \right] \frac{w_2^{\mu_2} \cdots w_n^{\mu_n}}{x_1^{v_1} \cdots x_{i-1}^{v_{i-1}} x_i^{v_i+1} \cdots x_n^{v_n}} \]
\[ \times \prod_{1 \leq v < u \leq n, v,u \neq i} (x_v/x_u)_{a_v} (q x_u/x_v)_{a_u} \prod_{u=1}^{n} \prod_{v=2, v \neq i}^{n} (x_u/w_v)_{a_u}^{-1}. \]

By (4.4) again with \( a \mapsto a^{(i)}, v \mapsto v^{(i)}, x \mapsto x^{(i)} \) and \( w \mapsto w^{(1)} \), the constant term in the above sum equals \( D_{v^{(i)},v^{(i)^+}}(a^{(i)}), \) completing the proof. \( \square \)
5. Preliminaries for the proof of Theorem 1.1

In this section, we prepare some results used in the proof of Theorem 1.1.

Proposition 5.1. Let $I$ be a non-empty subset of $\{1, 2, \ldots, n\}$ and $J$ be a non-empty subset of $I$. Let $L_{I,J}(a)$ be defined as in (1.6). Then, for an element $i \in J$,

$$(5.1) \quad L_{I \setminus \{i\}, J \setminus \{i\}}(a^{(i)}) = L_{I,J}(a) - \sum_{j=i}^{n} a_j + \sum_{\substack{j \in J \setminus \{i\} \geq i}} a_j.$$ 

Note that if $I$ is a one-element subset of $\{1, 2, \ldots, n\}$, then both sides of (5.1) vanish.

Proof. By the definition of $L_{I,J}(a)$ in (1.6) with $I \mapsto I \setminus \{i\}$, $J \mapsto J \setminus \{i\}$ and $a \mapsto a^{(i)}$, we have

$$L_{I \setminus \{i\}, J \setminus \{i\}}(a^{(i)}) = \sum_{1 \leq u \leq v \leq n} a_v.$$ 

Hence, for $i \in J$

$$L_{I \setminus \{i\}, J \setminus \{i\}}(a^{(i)}) = \sum_{1 \leq u \leq v \leq n} a_v = L_{I \setminus \{i\}, J}(a).$$

We can rewrite $L_{I \setminus \{i\}, J}(a)$ as

$$L_{I,J}(a) - \sum_{\substack{j=i} \in J \setminus \{i\}}^{n} a_j.$$ 

It is not hard to see that this equals the right-hand side of (5.1). \qed

The next two lemmas concern sums related to $L_{I,J}(a)$. In these sums, large cancellations occur and the sums reduce to one or two terms.

Lemma 5.2. Let $I$ be a subset of $\{1, 2, \ldots, n\}$ of cardinality at least two. Then, for an element $i \in I$,

$$(5.2) \quad \sum_{\emptyset \neq J \subseteq I \setminus \{i\}} (-1)^{|J|} q^{L_{I \setminus \{i\}, J}(a^{(i)}) + a_J} = -q^{L_{I \setminus \{i\}, J}(a) - \sum_{j=i+1}^{n} a_j},$$ 

where $a_S := \sum_{j \in S} a_j$ and $L_{I,J}(a)$ is defined as in (1.6).

Proof. Denote by $k$ the least element in $I \setminus \{i\}$. We can then rewrite the left-hand side of (5.2) as

$$(5.3) \quad \sum_{\substack{J \subseteq I \setminus \{i\} \setminus \emptyset \text{ and } k \notin J}} (-1)^{|J|} q^{L_{I \setminus \{i\}, J}(a^{(i)}) + a_J} + (-1)^{|J \cup \{k\}|} q^{L_{I \setminus \{i\}, J \cup \{k\}}(a^{(i)}) + a_{J \cup \{k\}}} - q^{L_{I \setminus \{i\}, \{k\}}(a^{(i)}) + a_k}.$$ 

Note that if there is no \( J \subseteq I \setminus \{i\} \) such that \( J \neq \emptyset \) and \( k \notin J \), then we set the above sum to be zero. We complete the proof by showing that the sum in (5.3) equals zero and

\[
L_{I \setminus \{i\}, \{k\}}(a^{(i)}) + a_k = L_{I, \{i\}}(a) - \sum_{j=i+1}^{n} a_j. \tag{5.4}
\]

To show that the sum in (5.3) equals zero, it suffices to show that

\[
\sum_{i \in J} (-1)^{|J \setminus \{i\}| + 1} q^{\sum_{j=i+1}^{n} a_j + L_{I \setminus \{i\}, J \setminus \{i\}}(a^{(i)})}(1 - q^{a_i}) = (-1)^{|J|} q^{L_{I, J}(a)}(1 - q^{a_j}), \tag{5.6}
\]

where \( a_S := \sum_{j \in S} a_j \) and \( L_{I, J}(a) \) is defined as in (1.6).

**Lemma 5.3.** Let \( I \) be a non-empty subset of \( \{1, 2, \ldots, n\} \) and \( J \) be a non-empty subset of \( I \). Then

\[
\sum_{i \in J} (-1)^{|J \setminus \{i\}| + 1} q^{\sum_{j=i+1}^{n} a_j + L_{I \setminus \{i\}, J \setminus \{i\}}(a^{(i)})}(1 - q^{a_i}) = (-1)^{|J|} q^{L_{I, J}(a)}(1 - q^{a_j}), \tag{5.6}
\]

Proof. We prove (5.6) by telescoping. Let \( J = \{k_1, k_2, \ldots, k_s\} \) such that \( 1 \leq k_1 < k_2 < \cdots < k_s \leq n \). Write the left-hand side of (5.6) as

\[
\sum_{i=1}^{s} t_{k_i}(1 - q^{a_{k_i}}), \tag{5.7}
\]

where

\[
t_{k_i} := (-1)^{|J \setminus \{k_i\}| + 1} q^{\sum_{j=k_i+1}^{n} a_j + L_{I \setminus \{k_i\}, J \setminus \{k_i\}}(a^{(k_i)})}. \tag{5.8}
\]

We will show that

\[
t_{k_{i-1}} = q^{a_{k_i}} t_{k_i} \quad \text{for } i = 2, \ldots, s,
\]

and the sum (5.7) reduces to

\[
t_{k_s} - q^{a_{k_1}} t_{k_1}.
\]

Then we will show that this equals the right-hand side of (5.6).
Since $|J \setminus \{k_{i-1}\}| = |J \setminus \{k_i\}| = |J| - 1$, to show $t_{k_{i-1}} = q^{a_{k_i}} t_{k_i}$ for $i = 2, \ldots, s$, it suffices to show that for a fixed integer $i$ such that $2 \leq i \leq s$,

$$\sum_{j=k_{i-1}+1}^{n} a_j + L_{I \setminus \{k_{i-1}\}, J \setminus \{k_{i-1}\}}(a^{(k_{i-1})}) = \sum_{j=k_{i-1}+1}^{n} a_j + L_{I \setminus \{k_i\}, J \setminus \{k_i\}}(a^{(k_i)}) + a_{k_i}.$$  

(5.10)

By (5.1)

$$\sum_{j=k_{i-1}+1}^{n} a_j + L_{I \setminus \{k_{i-1}\}, J \setminus \{k_{i-1}\}}(a^{(k_{i-1})}) = \sum_{j=k_{i-1}+1}^{n} a_j + L_{I, J}(a) - \sum_{j=k_{i-1}+1}^{n} a_j + \sum_{j \geq k_i} a_j$$

$$= L_{I, J}(a) - a_{k_{i-1}} + \sum_{j \geq k_i} a_j = L_{I, J}(a) + \sum_{j \geq k_i} a_j.$$

Using (5.1) again, we find

$$L_{I, J}(a) + \sum_{j \geq k_i} a_j = L_{I \setminus \{k_i\}, J \setminus \{k_i\}}(a^{(k_i)}) + \sum_{j = k_i}^{n} a_j - \sum_{j \in J \setminus \{k_i\}} a_j + \sum_{j \geq k_i} a_j$$

$$= L_{I \setminus \{k_i\}, J \setminus \{k_i\}}(a^{(k_i)}) + \sum_{j = k_i}^{n} a_j = L_{I \setminus \{k_i\}, J \setminus \{k_i\}}(a^{(k_i)}) + \sum_{j = k_{i+1}}^{n} a_j + a_{k_i}.$$  

Hence, we obtain (5.10) and (5.9) follows.

As mentioned above, by (5.9)

$$\sum_{i=1}^{s} t_{k_i} (1 - q^{a_{k_i}}) = t_{k_s} - q^{a_{k_1}} t_{k_1}.$$  

We complete the proof by showing that

$$t_{k_s} - q^{a_{k_1}} t_{k_1} = (-1)^{|J|} q^{L_{I, J}(a)} (1 - q^{a_J}).$$

By the expression (5.8) for $t_{k_1}$, we have

$$t_{k_s} - q^{a_{k_1}} t_{k_1} = (-1)^{|J \setminus \{k_s\}|+1} q^{\sum_{j = k_{s+1}}^{n} a_j + L_{I \setminus \{k_s\}, J \setminus \{k_s\}}(a^{(k_s)})}$$

$$- (-1)^{|J \setminus \{k_1\}|+1} q^{\sum_{j = k_1}^{n} a_j + L_{I \setminus \{k_1\}, J \setminus \{k_1\}}(a^{(k_1)}) + a_{k_1}}.$$  

By (5.1)

$$t_{k_s} - q^{a_{k_1}} t_{k_1} = (-1)^{|J \setminus \{k_s\}|+1} q^{\sum_{j = k_{s+1}}^{n} a_j + L_{I, J}(a) - \sum_{j = k_s}^{n} a_j + \sum_{j \in J \setminus \{k_s\}} a_j}

- (-1)^{|J \setminus \{k_1\}|+1} q^{\sum_{j = k_1}^{n} a_j + L_{I, J}(a) - \sum_{j = k_1}^{n} a_j + \sum_{j \in J \setminus \{k_1\}} a_j + a_{k_1}}.$$  

(5.11)
Using
\[ |J \setminus \{k_s\}| + 1 = |J \setminus \{k_1\}| + 1 = |J|, \quad \sum_{j \in J, j \geq k_s} a_j = a_{k_s} \quad \text{and} \quad \sum_{j \in J, j \geq k_1} a_j = a_J, \]
and extracting out the same factor from the right-hand side of (5.11), we obtain
\[ t_{k_s} - q^{a_{k_1}} t_{k_1} = (-1)^{|J|} q^{L_{I,J}(a)} (1 - q^{a_J}). \]

By Lemma 5.2 and Lemma 5.3, we can rewrite the next double sum.

**Proposition 5.4.** Let \( I \) be a non-empty subset of \( \{1, 2, \ldots, n\} \) of cardinality at least two. Then, for \( r \) an integer

\[
(5.12) \quad \sum_{i \in I} \sum_{\emptyset \neq J \subseteq I \setminus \{i\}} (-1)^{|J|+1} q^{\sum_{j=i+1}^n a_j + L_{I \setminus \{i\}, J(a^{(i))}} \frac{(1 - q^{a_i}) (1 - q^{a_J})}{(1 - q^{a_i} - a_i + r)(1 - q^{a_i} - a_J + r)}
\]

\[ = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|} q^{L_{I,J}(a)} \frac{1 - q^{a_J}}{1 - q^{a_i} - a_i - a_J + r}, \]

where \( a_S := \sum_{j \in S} a_j \) for the set \( S \) and \( L_{I,J}(a) \) is defined as in (1.6).

**Proof.** By a partial fraction decomposition,
\[
\frac{1}{(1 - q^{a_i} - a_i + r)(1 - q^{a_i} - a_J + r)} = \frac{1}{(1 - q^{a_J})(1 - q^{a_i} - a_i + r)} + \frac{1}{(1 - q^{a_J})(1 - q^{a_i} - a_J + r)}.
\]
Denote by \( L \) the left-hand side of (5.12). Using the above equation, we find
\[
L = \sum_{i \in I} \sum_{\emptyset \neq J \subseteq I \setminus \{i\}} (-1)^{|J|+1} q^{\sum_{j=i+1}^n a_j + L_{I \setminus \{i\}, J(a^{(i))}} \left( -\frac{q^{a_J}(1 - q^{a_i})}{1 - q^{a_i} - a_i + r} + \frac{1 - q^{a_i}}{1 - q^{a_i} - a_i - a_J + r} \right)
\]
\[ = \sum_{i \in I} \left( \frac{q^{\sum_{j=i+1}^n a_j} (1 - q^{a_i})}{1 - q^{a_i} - a_i + r} \right) \sum_{\emptyset \neq J \subseteq I \setminus \{i\}} (-1)^{|J|} q^{L_{I \setminus \{i\}, J(a^{(i))}}+a_J}
\]
\[ + \sum_{i \in I} \sum_{\emptyset \neq J \subseteq I \setminus \{i\}} (-1)^{|J|+1} q^{\sum_{j=i+1}^n a_j + L_{I \setminus \{i\}, J(a^{(i))}} \frac{1 - q^{a_i}}{1 - q^{a_i} - a_i - a_J + r}. \]

We can further rewrite \( L \) as

\[
(5.13) \quad L = \sum_{i \in I} \left( \frac{q^{\sum_{j=i+1}^n a_j} (1 - q^{a_i})}{1 - q^{a_i} - a_i + r} \right) \sum_{\emptyset \neq J \subseteq I \setminus \{i\}} (-1)^{|J|} q^{L_{I \setminus \{i\}, J(a^{(i))}}+a_J}
\]
\[ + \sum_{J \subseteq I, |J| > 1} \left( \frac{1}{1 - q^{a_J} - a_J + r} \right) \sum_{i \in J} (-1)^{|J|} q^{L_{I \setminus \{i\}, J(a^{(i))}}+a_J} q^{\sum_{j=i+1}^n a_j + L_{I \setminus \{i\}, J(a^{(i))}} (1 - q^{a_i})}. \]

By Lemma 5.2

\[
(5.14a) \quad \sum_{\emptyset \neq J \subseteq I \setminus \{i\}} (-1)^{|J|} q^{L_{I \setminus \{i\}, J(a^{(i))}}+a_J} = -q^{L_{I,J}(a)} - \sum_{j=i+1}^n a_j.
\]
By Lemma 5.3 for $J \subseteq I$ and $|J| > 1$ we have
\[(5.14b) \quad \sum_{i \in J} (-1)^{|J \setminus \{i\}|+1} q^{\sum_{j=i+1}^{n} a_j + L_{I \setminus \{i\}, J \setminus \{i\}}(a^{(i)})} (1 - q^{a_i}) = (-1)^{|J|} q^{L_{I, J}(a)} (1 - q^{a_J}).\]
Substituting (5.14) into (5.13) yields (5.12). \qed

6. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 using the inductive formula (4.3) for $D_{v,v^+}(a)$ and Proposition 5.4.

Proof of Theorem 1.1 Let $s = |I|$ be the cardinality of $I$. We then proceed by induction on $s$.

If $s = 1$, then by (4.3) we have
\[(6.1) \quad D_{v,v^+}(a) = q^{\sum_{j=1}^{n} a_j} \left[ \frac{|a| + r - 1}{a_k - 1} \right] D_{v^{(k)}, v^{(k)^+}}(a^{(k)}),\]
where $k$ is the index of the unique largest element in $v$. It is easy to check that
\[\left[ \frac{|a| + r - 1}{a_k - 1} \right] = \frac{1 - q^{a_k}}{1 - q^{|a| - a_k + r}}, \quad \left[ \frac{|a| - a_i + r - 1}{a_i - 1} \right].\]

Substituting the above equation into (6.1) we obtain the $s = 1$ case of (1.3).

Suppose $2 \leq s \leq n$. By (4.3) and the induction hypothesis, we find
\[(6.2) \quad D_{v,v^+}(a) = \sum_{i \in I} q^{\sum_{j=i+1}^{n} a_j} \left[ \frac{|a| + r - 1}{a_i - 1} \right] D_{v^{(i)}, v^{(i)^+}}(a^{(i)}) \frac{1 - q^{a_j}}{1 - q^{|a| - a_i + r + J_J \setminus \{i\}} q^{L_{I \setminus \{i\}, I}(a^{(i)})}}.\]

Since
\[\left[ \frac{|a| + r - 1}{a_i - 1} \right] \left[ \frac{|a| - a_i + r - 1}{a_i - 1} \right] = \frac{1 - q^{a_i}}{1 - q^{|a| - a_i + r}}, \quad \left[ \frac{|a| + r - 1}{a_i - 1} \right],\]
we can rewrite (6.2) as
\[(6.3) \quad D_{v,v^+}(a) = (-1)^{|I|} D_{v^{(i)}, v^{(i)^+}}(a^{(i)}) \frac{1 - q^{a_j}}{1 - q^{|a| - a_i + r + J_J \setminus \{i\}} q^{L_{I \setminus \{i\}, J}(a^{(i)})}} \sum_{i \notin J \subseteq I} \sum_{J \subseteq I \setminus \{i\}} (-1)^{|J|+1} q^{\sum_{j=i+1}^{n} a_j + L_{I \setminus \{i\}, J}(a^{(i)})} \frac{1 - q^{a_i}}{1 - q^{|a| - a_i + r + J_J \setminus \{i\}} q^{L_{I \setminus \{i\}, J}(a^{(i)})}} \frac{1 - q^{a_j}}{1 - q^{|a| - a_i + r + J_J \setminus \{i\}} q^{L_{I \setminus \{i\}, J}(a^{(i)})}}.\]

Then the theorem follows by Proposition 5.4. \qed
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