INVERSE PROBLEM FOR SINGULAR DIFFUSION OPERATOR

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Abstract. In this study, singular diffusion operator with jump conditions is considered. Integral representations have been derived for solutions that satisfy boundary conditions and jump conditions. Some properties of eigenvalues and eigenfunctions are investigated. Asymptotic representation of eigenvalues and eigenfunctions have been obtained. Reconstruction of the singular diffusion operator have been shown by the Weyl function.

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1. INTRODUCTION

Let’s define the following boundary value problem which will be denoted by $L$ in the sequel all the paper

$$l(y) := -y'' + [2\lambda p(x) + q(x)]y = \lambda^2 \delta(x)y, \ x \in [0,\pi] \backslash \{p_1, p_2\}$$

(1.1)

with the boundary conditions

$$y'(0) = 0, \quad y(\pi) = 0$$

(1.2)

and the jump conditions

$$y(p_1 + 0) = \alpha_1 y(p_1 - 0),$$

(1.3)

$$y'(p_1 + 0) = \beta_1 y'(p_1 - 0) + i\lambda \gamma_1 y(p_1 - 0),$$

(1.4)

$$y(p_2 + 0) = \alpha_2 y(p_2 - 0),$$

(1.5)

$$y'(p_2 + 0) = \beta_2 y'(p_2 - 0) + i\lambda \gamma_2 y(p_2 - 0),$$

(1.6)

where $\lambda$ is a spectral parameter, $q(x) \in L_2[0,\pi],$ $p(x) \in W_2'[0,\pi],$ $p_1, p_2 \in (0,\pi),$ $p_1 < p_2,$ $|\alpha_1 - 1|^2 + \gamma_1^2 \neq 0,$ $|\alpha_2 - 1|^2 + \gamma_2^2 \neq 0.$

$$\delta(x) = \begin{cases} 
1 & x \in (0, p_1); \\
\alpha^2 & x \in (p_1, p_2); \\
\beta^2 & x \in (p_2, \pi);
\end{cases}$$

to be $\alpha > 0, \alpha \neq 1, \beta > 0, \beta \neq 1$ real numbers.

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Direct and inverse problems are important in mathematics, physics and engineering. The inverse problem is called the reconstruction of the operator whose spectral characteristics are given in sequences. For example; to learn the distribution of density in the nonhomogeneous arc according to the wavelengths in mechanics and finding the field potentials according to scattering data in the quantum physics are examples of inverse problems. The first study on inverse problems for differential equations was made by Ambartsumyan in [25]. A significant study in the spectral theory of the singular differential operators was carried out by Levitan in [4]. An important method in the solution of inverse problems is the transformation operators. In [14], Guseinov studied the regular differential equation and the direct spectral problem of the operator under certain initial conditions. In recent years, Weyl function has frequently been used to solve inverse problems. The Weyl function was introduced by H. Weyl in 1910 in the literature. Many studies have been made on direct or inverse problems [1–28]. The solution of discontinuous boundary value problem can be given as an example of concrete problem of mathematical physics. Boundary value problems with discontinuous coefficients are important for applied mathematics and applied sciences.

In [17], Koyunbakan and Panakhov proved that the potential function can be determined on \([\pi, \pi]\) while it is known on \([0, \pi]\) by single spectrum in [12]. In [26], Yang showed that can be determined uniquely diffusion operator from nodal data.

2. Preliminaries

Let \(\phi(x, \lambda), \psi(x, \lambda)\) be solutions of (1.1) respectively under the boundary conditions

\[
\phi(0, \lambda) = 1, \quad \phi'(0, \lambda) = 0 \\
\psi(\pi, \lambda) = 0, \quad \psi'(\pi, \lambda) = 1
\]

and discontinuity conditions (1.3) − (1.6), where \(Q(t) = 2\lambda p(t) + q(t)\).

It is obvious that the function \(\phi(x, \lambda)\) is similar to [8] satisfies the following integral equations if \(0 \leq x < p_1\):

\[
\phi(x, \lambda) = e^{\lambda x} + \frac{1}{\lambda} \int_0^x \sin \lambda (x - t) Q(t) y(t, \lambda) dt, \tag{2.1}
\]

if \(p_1 < x < p_2\):

\[
\phi(x, \lambda) = \beta_1^+ e^{\lambda \zeta^+(x)} + \beta_1^- e^{\lambda \zeta^-(x)} + \frac{\gamma_1}{2\alpha} e^{\lambda \zeta^+(x)} - \frac{\gamma_1}{2\alpha} e^{\lambda \zeta^-(x)} \\
+ \frac{\beta_1^+}{\lambda} \int_0^{p_1} \sin \lambda \zeta^+(x) - t J(t) y(t, \lambda) dt \\
+ \frac{\beta_1^-}{\lambda} \int_0^{p_1} \sin \lambda \zeta^-(x) - t J(t) y(t, \lambda) dt \tag{2.2}
\]
\[-i \frac{\gamma_I}{2\alpha} \int_{0}^{p_1} \frac{\cos \lambda (\xi^+ (x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ i \frac{\gamma_I}{2\alpha} \int_{0}^{p_1} \frac{\cos \lambda (\xi^- (x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ \int_{p_1}^{p_2} \frac{\sin \lambda (x - t)}{\lambda} J(t) y(t, \lambda) \, dt , \]

if \( p_2 < x \leq \pi :\)

\[
\phi(x, \lambda) = e^{i \alpha b^+ (x)} \xi^+ + e^{i \beta b^- (x)} \xi^- + e^{i \lambda x^+ (x)} \theta^+ + e^{i \lambda x^- (x)} \theta^- \]
\[+ \left( \beta_1^+ \beta_2^+ + \frac{\gamma_I \gamma_2}{4\alpha \beta} \right) \int_{0}^{p_1} \frac{\sin \lambda (b^+ (x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ \left( \beta_1^+ \beta_2^- - \frac{\gamma_I \gamma_2}{4\alpha \beta} \right) \int_{0}^{p_1} \frac{\sin \lambda (s^+ (x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ \left( \beta_1^- \beta_2^- - \frac{\gamma_I \gamma_2}{4\alpha \beta} \right) \int_{0}^{p_1} \frac{\sin \lambda (s^- (x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ i \left( \frac{\gamma_I \beta_2^+}{2\alpha} - \frac{\gamma_I \beta_1^+}{2\beta} \right) \int_{0}^{p_1} \frac{\cos \lambda (b^+ (x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ i \left( \frac{\gamma_I \beta_2^-}{2\alpha} - \frac{\gamma_I \beta_1^+}{2\beta} \right) \int_{0}^{p_1} \frac{\cos \lambda (s^+ (x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ i \left( \frac{\gamma_I \beta_2^+}{2\alpha} + \frac{\gamma_I \beta_1^-}{2\beta} \right) \int_{p_1}^{p_2} \frac{\cos \lambda (b^- (x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ \beta_2^+ \int_{p_1}^{p_2} \frac{\sin \lambda (\beta x - \beta p_2 + \alpha p_2 - \alpha x)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[- \beta_2^- \int_{p_1}^{p_2} \frac{\sin \lambda (\beta x - \beta p_2 - \alpha p_2 + \alpha x)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[- i \frac{\gamma_2}{2\beta} \int_{p_1}^{p_2} \frac{\cos \lambda (\beta x - \beta p_2 + \alpha p_2 - \alpha x)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ i \frac{\gamma_2}{2\beta} \int_{p_1}^{p_2} \frac{\cos \lambda (\beta x - \beta p_2 - \alpha p_2 + \alpha x)}{\lambda} J(t) y(t, \lambda) \, dt \]
\[+ \int_{p_2}^{\pi} \frac{\sin \lambda (x - t)}{\lambda} J(t) y(t, \lambda) \, dt , \]
\[ \phi(x, \lambda) = e^{\lambda b^+(x)} + e^{\lambda b^-(x)} + \vartheta e^{\lambda s^+(x)} + \vartheta e^{\lambda s^-(x)} \]

\[ + \left( \beta_1^+ \beta_2^- + \frac{\gamma_1 \gamma_2}{4 \alpha \beta} \right) \int_0^{p_1} \frac{\sin \lambda (b^+(x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ + \left( \beta_1^- \beta_2^+ - \frac{\gamma_1 \gamma_2}{4 \alpha \beta} \right) \int_0^{p_1} \frac{\sin \lambda (s^+(x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ + \left( \beta_1^- \beta_2^- - \frac{\gamma_1 \gamma_2}{4 \alpha \beta} \right) \int_0^{p_1} \frac{\sin \lambda (b^-(x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ + \left( \beta_1^+ \beta_2^+ + \frac{\gamma_1 \gamma_2}{4 \alpha \beta} \right) \int_0^{p_1} \frac{\sin \lambda (s^-(x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ - i \left( \frac{\gamma_1 \beta_2^+}{2 \alpha} + \frac{\gamma_2 \beta_1^+}{2 \beta} \right) \int_0^{p_1} \frac{\cos \lambda (b^+(x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ - i \left( \frac{\gamma_1 \beta_2^-}{2 \alpha} - \frac{\gamma_2 \beta_1^-}{2 \beta} \right) \int_0^{p_1} \frac{\cos \lambda (s^+(x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ + i \left( \frac{\gamma_1 \beta_2^-}{2 \alpha} - \frac{\gamma_2 \beta_1^-}{2 \beta} \right) \int_0^{p_1} \frac{\cos \lambda (b^-(x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ + i \left( \frac{\gamma_1 \beta_2^+}{2 \alpha} + \frac{\gamma_2 \beta_1^+}{2 \beta} \right) \int_0^{p_1} \frac{\cos \lambda (s^-(x) - t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ + \beta_2^+ \int_{p_1}^{p_2} \frac{\sin \lambda (\beta x - \beta p_2 + \alpha p_2 + \alpha t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ - \beta_2^- \int_{p_1}^{p_2} \frac{\sin \lambda (\beta x - \beta p_2 - \alpha p_2 + \alpha t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ - \gamma_2 \int_{p_1}^{p_2} \frac{\cos \lambda (\beta x - \beta p_2 + \alpha p_2 - \alpha t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ + i \gamma_2 \int_{p_1}^{p_2} \frac{\cos \lambda (\beta x - \beta p_2 - \alpha p_2 + \alpha t)}{\lambda} J(t) y(t, \lambda) \, dt \]

\[ + \int_{p_2}^{\pi} \frac{\sin \lambda (x - t)}{\lambda} J(t) y(t, \lambda) \, dt, \]

and it is obvious that the function \( \psi(x, \lambda) \) satisfies the following integral equations if \( p_2 < x \leq \pi \):

\[ \psi(x, \lambda) = \frac{\sin \lambda \beta (x - \pi)}{\lambda \beta} + \int_x^{\pi} \frac{\sin \lambda \beta (x - t)}{\lambda \beta} Q(t) y(t, \lambda) \, dt, \]  \hspace{1cm} (2.5)

if \( p_1 < x < p_2 \):

\[ \psi(x, \lambda) = \left( \frac{\alpha \beta \gamma_2 - \gamma_2}{2 \alpha \beta \lambda \alpha \gamma_2 \beta} - \frac{1}{2 \alpha \beta} \right) e^{-\lambda (\beta (p_2 - \pi) + \alpha (p_2 - x))} \]
\[ \psi(x, \lambda) = \left( \xi^+ + \frac{\alpha}{2\beta_1} \right) \eta^+ e^{-\lambda (b^- (\pi) + x)} + \left( \xi^- - \frac{\alpha}{2\beta_1} \right) \eta^- e^{-\lambda (b^+ (\pi) + x)} \]

\[ + \left( \frac{\alpha\beta_2 + \gamma_2}{2\alpha\beta_2\lambda \alpha \gamma \beta} + \frac{1}{2\alpha\beta_2 \lambda} \right) e^{-\lambda (b^- (\pi) - \alpha (p_2 - x))} \]

\[ - \left( \frac{\alpha\beta_2 - \gamma_2}{2\alpha\beta_2} \right) \frac{1}{2} \int_{p_1}^{p_2} \frac{\sin \lambda (x - p_2 + \alpha - \alpha p_2)}{\lambda \alpha} Q(t, \lambda) y(t, \lambda) \, dt \]

\[ + \left( \frac{\alpha\beta_2 - \gamma_2}{2\alpha\beta_2} + \frac{1}{2} \right) \int_{p_1}^{p_2} \frac{\sin \lambda (x - p_2 - \alpha + \alpha p_2)}{\lambda \alpha} Q(t, \lambda) y(t, \lambda) \, dt \]  

(2.6)

\[ + \frac{1}{2} \left( \frac{\alpha\beta_2 - \gamma_2}{\alpha\beta_2 \alpha \beta} - \frac{1}{\alpha \beta_2} \right) \int_{p_2}^{\pi} \frac{\sin \lambda (x - p_2 + \beta (t - p_2))}{\lambda \beta} Q(t, \lambda) y(t, \lambda) \, dt \]

\[ - \frac{1}{2} \left( \frac{\alpha\beta_2 - \gamma_2}{\alpha\beta_2 \alpha \beta} - \frac{1}{\alpha \beta_2} \right) \int_{p_2}^{\pi} \frac{\sin \lambda (x - p_2 - \beta (t - p_2))}{\lambda \beta} Q(t, \lambda) y(t, \lambda) \, dt \]

\[ + \frac{\gamma_2}{2\alpha\beta_2 \lambda} \int_{p_1}^{p_2} \frac{\cos \lambda (x - p_2 + \alpha - \alpha p_2)}{\lambda \alpha} Q(t, \lambda) y(t, \lambda) \, dt \]

\[ - \frac{\gamma_2}{2\alpha\beta_2 \lambda} \int_{p_1}^{p_2} \frac{\cos \lambda (x - p_2 - \alpha + \alpha p_2)}{\lambda \alpha} Q(t, \lambda) y(t, \lambda) \, dt \]

\[ + \int_{p_1}^{\pi} \frac{\sin \lambda \alpha (x - t)}{\lambda \alpha} Q(t, \lambda) y(t, \lambda) \, dt, \]

if \( 0 \leq x < p_1 \):
\[ + \frac{i\gamma_1}{2\alpha_i\beta_1} \int_{p_1}^{p_2} \frac{\pi \cos \lambda (x - 2p_1 + p_2 + \beta t - \beta p_2)}{\lambda} Q(t) y(t, \lambda) dt \\
+ A \int_{p_1}^{p_2} \frac{\sin \lambda (x - p_2 + \alpha - \alpha p_2)}{\lambda \alpha} Q(t) y(t, \lambda) dt \\
+ A \int_{p_1}^{p_2} \frac{\sin \lambda (x - 2p_1 + p_2 + \alpha - \alpha p_2)}{\lambda \alpha} Q(t) y(t, \lambda) dt \\
+ C \int_{p_1}^{p_2} \frac{\cos \lambda (x - p_2 - \alpha + \alpha p_2)}{\lambda \alpha} Q(t) y(t, \lambda) dt \\
+ C \int_{p_1}^{p_2} \frac{\cos \lambda (x - 2p_1 + p_2 - \alpha + \alpha p_2)}{\lambda \alpha} Q(t) y(t, \lambda) dt \\
+ D \int_{p_1}^{p_2} \frac{\cos \lambda (x - p_2 - \alpha + \alpha p_2)}{\lambda \alpha} Q(t) y(t, \lambda) dt \\
+ D \int_{p_1}^{p_2} \frac{\cos \lambda (x - 2p_1 + p_2 + \alpha - \alpha p_2)}{\lambda \alpha} Q(t) y(t, \lambda) dt \\
+ \int_0^x \frac{\sin \lambda (x - t)}{\lambda} Q(t) y(t, \lambda) dt, \]

where

\[ \xi^\pm(x) = \pm \alpha x \mp \alpha p_1 + p_1, \quad \beta_1^\pm = \frac{1}{2} \left( \alpha_i \pm \beta_1 \right), \]

\[ b^\pm(x) = \beta x - \beta p_2 + \mu^\pm(p_2), \quad s^\pm(x) = -\beta x + \beta p_2 + \mu^\pm(p_2), \]

\[ \beta_2^\pm = \frac{1}{2} \left( \alpha_2 \pm \frac{\alpha \beta_2}{\beta} \right), \quad \xi^\pm = \frac{1}{2} \left( \beta^\pm + \frac{\gamma_1}{2\alpha} \right) \left( \alpha_2 \pm \frac{\alpha \beta_2}{\beta} - \gamma_2 \right), \quad \mu^\pm = \left( \alpha \beta^2 + \gamma \frac{\gamma_2}{\beta} + \frac{1}{\beta} \right), \]

\[ A = \left[ \frac{\gamma_1 \gamma_2}{4\lambda \alpha \alpha_1 \beta_1 \beta_2} + \left( \frac{-1}{2\alpha_1} - \frac{1}{4\beta_1} \right) \left( \alpha \beta_2 - \gamma_2 \right) - \frac{1}{2} \right], \]

\[ B = \left[ \frac{-\gamma_1}{2\alpha \beta_1} \left( \frac{\alpha \beta_2 - \gamma_2}{2\alpha \beta_2} - \frac{1}{2} \right) + \frac{1}{2\alpha} \frac{\gamma_2}{\beta} \right], \]

\[ C = \left[ \frac{1}{2\alpha} + \frac{1}{2\beta} \right] \left( \frac{\alpha \beta_2 - \gamma_2}{2\alpha \beta_2} + \frac{1}{2} \right) + \frac{i\gamma_1 \gamma_2}{4\lambda \alpha \alpha_1 \beta_1 \beta_2} \right] \]
Theorem 1. If \( p(x) \in W^1_2(0, \pi) \) and \( q(x) \in L_2^2(0, \pi) \); \( y_0(x, \lambda) \) be solutions of (1.1), that satisfies conditions (1.2) – (1.6), has the form

\[
y_0(x, \lambda) = y_0(x, \lambda) + \int_{-x}^{x} K_0(x, t) e^{\alpha \beta} dt \quad (v = 1, 3)
\]

where

\[
y_0(x, \lambda) = \begin{cases} 
R_0(x) e^{\beta x} & 0 \leq x < p_1; \\
R_1(x) e^{\beta x} & p_1 < x < p_2; \\
R_3(x) e^{\beta x} & p_2 < x \leq \pi;
\end{cases}
\]

\[
R_0(x) = e^{-\int_0^x p(x) dx}, \quad R_1(x) = \left( \beta^+_1 + \frac{\gamma_1}{2\alpha} \right) R_0(p_1) e^{-\frac{\gamma_1}{2\pi} \int_0^{p_1} p(t) dt}, \\
R_2(x) = \left( \beta^-_1 - \frac{\gamma_1}{2\alpha} \right) R_0(p_1) e^{\frac{\gamma_1}{2\pi} \int_0^{p_1} p(t) dt}, \quad R_3(x) = \left( \beta^+_2 + \frac{\gamma_2}{2\beta} \right) R_1(p_2) e^{-\frac{\gamma_1}{2\pi} \int_0^{p_2} p(t) dt}, \\
R_4(x) = \left( \beta^-_2 - \frac{\gamma_2}{2\beta} \right) R_2(p_2) e^{\frac{\gamma_1}{2\pi} \int_0^{p_2} p(t) dt}, \quad R_5(x) = \left( \beta^-_2 - \frac{\gamma_2}{2\beta} \right) R_3(p_2) e^{\frac{\gamma_1}{2\pi} \int_0^{p_2} p(t) dt}, \\
R_6(x) = \left( \beta^+_2 - \frac{\gamma_2}{2\beta} \right) R_6(p_2) e^{\frac{\gamma_1}{2\pi} \int_0^{p_2} p(t) dt}.
\]

and \( \Phi(x) = \int_0^x \left( 2 |p(t)| + (x-t)|q(t)| \right) dt \) and the functions \( K_0(x, t) \) satisfies the inequality

\[
\int_{-x}^{x} |K_0(x, \lambda)| dt \leq e^{c_\Phi(x)} - 1
\]

with

\[
c_1 = 1, \quad c_2 = \left( \beta^+_1 + |\beta^-_1| + \frac{\gamma_1}{\alpha} + \frac{2}{\alpha} \right), \\
c_3 = \left( \alpha_2 (\beta^+_1 + |\beta^-_1|) + \frac{1}{\alpha} (\beta^+_2 + |\beta^-_2|) + \frac{\beta^+_2}{\beta} + \frac{\gamma_2}{\beta} \right).
\]

where

\[
\zeta^+ (x) = \pm \alpha x \mp \alpha p_1 + p_1, \quad \beta^+_1 = \frac{1}{2} \left( \alpha_1 \pm \beta_1 \right), \\
\beta^- (x) = \beta x - \beta p_2 + \zeta^+ (p_2), \quad s^+ (x) = -\beta x + \beta p_2 + \zeta^+ (p_2), \\
\beta^+_2 = \frac{1}{2} \left( \alpha_2 \pm \alpha \beta_2 \right), \quad \xi^+ = \frac{1}{2} \left( \beta^+_1 + \frac{\gamma_1}{2\alpha} \right) \left( \alpha_2 \pm \alpha \beta_2 \beta + \frac{\gamma_2}{\beta} \right), \\
\beta^2 = \frac{1}{2} \left( \beta^+_1 + \frac{\gamma_1}{2\alpha} \right) \left( \alpha_2 \pm \alpha \beta_2 \beta + \frac{\gamma_2}{\beta} \right), \quad \beta^\pm = \frac{1}{2} (1 \pm 1 \beta).
\]
The proof is done as in [8].

**Theorem 2.** Let \( p(x) \in W^1_1(0, \pi) \) and \( q(x) \in L^2(0, \pi) \). The functions \( A(x,t) \), \( B(x,t) \), whose first order partial derivatives, are summable on \([0, \pi]\), for each \( x \in [0, \pi] \) such that representation

\[
\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x A(x,t) \cos \lambda t dt + \int_0^x B(x,t) \sin \lambda t dt
\]

is satisfied.

If \( p_1 < x < p_2 \):

\[
\begin{align*}
\varphi(x, \lambda) &= \left( \beta_1^+ + \frac{\gamma_1}{2\alpha} \right) R_0(p_1) \cos \left[ \lambda \xi^+(x) - \frac{1}{\alpha} \int_{p_1}^x p(t) dt \right] \\
&\quad + \left( \beta_1^- - \frac{\gamma_1}{2\alpha} \right) R_0(p_1) \cos \left[ \lambda \xi^-(x) + \frac{1}{\alpha} \int_{p_1}^x p(t) dt \right] \\
&\quad + \int_0^{\xi^+(x)} A(x,t) \cos \lambda t dt + \int_0^{\xi^+(x)} B(x,t) \sin \lambda t dt,
\end{align*}
\]

where \( \beta_1^+ = \frac{1}{2} \left( \alpha_1 + \frac{\beta_1}{\alpha} \right) \). If \( p_2 < x \leq \pi \):

\[
\begin{align*}
\varphi(x, \lambda) &= \left( \beta_2^+ + \frac{\gamma_2}{2\beta} \right) R_1(p_2) \cos \left[ \lambda b^+(x) - \frac{1}{\beta} \int_{p_2}^x p(t) dt \right] \\
&\quad + \left( \beta_2^- - \frac{\gamma_2}{2\beta} \right) R_1(p_2) \cos \left[ \lambda b^-(x) + \frac{1}{\beta} \int_{p_2}^x p(t) dt \right] \\
&\quad + \int_{p_2}^{\xi^+(x)} A(x,t) \cos \lambda t dt + \int_{p_2}^{\xi^+(x)} B(x,t) \sin \lambda t dt,
\end{align*}
\]

where \( \beta_2^+ = \frac{1}{2} \left( \alpha_2 + \frac{\beta_2}{\beta} \right) \). Moreover, the equations

\[
\begin{align*}
A(x, \xi^+(x)) \cos \frac{\beta(x)}{\alpha} + B(x, \xi^+(x)) \sin \frac{\beta(x)}{\alpha} &= \left( \beta_1^+ + \frac{\gamma_1}{2\alpha} \right) R_0(p_1) \int_0^x q(t) dt + \frac{p^2(t)}{2\alpha^2} dt \tag{2.10}
\end{align*}
\]

\[
\begin{align*}
A(x, \xi^+(x)) \sin \frac{\beta(x)}{\alpha} - B(x, \xi^+(x)) \cos \frac{\beta(x)}{\alpha} &= \left( \beta_1^+ + \frac{\gamma_1}{2\alpha} \right) R_0(p_1) (p(x) - p(0)) \tag{2.11}
\end{align*}
\]
\begin{align*}
A (x, \zeta^- (x) + 0) - A (x, \zeta^- (x) - 0) &= \left( \beta_1 - \frac{\gamma_1}{2\alpha} \right) \frac{R_0 (p_1)}{2\alpha^2} \sin \frac{\beta (x)}{\alpha} (p (x) - p (0)) \\
&+ \left( \beta_1 - \frac{\gamma_1}{2\alpha} \right) \frac{R_0 (p_1)}{2\alpha^2} \cos \frac{\beta (x)}{\alpha} \int_0^x \left( q (t) + \frac{p^2 (t)}{\alpha^2} \right) dt \tag{2.12} \\
B (x, \zeta^- (x) + 0) - B (x, \zeta^- (x) - 0) &= \left( \beta_1 - \frac{\gamma_1}{2\alpha} \right) \frac{R_0 (p_1)}{2\alpha^2} \cos \frac{\beta (x)}{\alpha} (p (x) - p (0)) \\
&- \left( \beta_1 - \frac{\gamma_1}{2\alpha} \right) \frac{R_0 (p_1)}{2\alpha^2} \sin \frac{\beta (x)}{\alpha} \int_0^x \left( q (t) + \frac{p^2 (t)}{\alpha^2} \right) dt \tag{2.13} \\
B (x, 0) &= \frac{\partial A (x, t)}{\partial t} \bigg|_{t=0} = 0 \tag{2.14} \\
A (x, s^- (x) + 0) - A (x, s^- (x) - 0) &= - \left( \beta_1 - \frac{\gamma_2}{2\beta} \right) \frac{R_2 (p_2)}{2\beta^2} (p (x) - p (0)) \sin \frac{\omega (x)}{\beta} \\
&- \left( \beta_1 - \frac{\gamma_2}{2\beta} \right) \frac{R_2 (p_2)}{2\beta^2} \int_0^x \left( q (t) + \frac{p^2 (t)}{\beta^2} \right) dt \cos \frac{\omega (x)}{\beta} \tag{2.15} \\
B (x, s^- (x) + 0) - B (x, s^- (x) - 0) &= - \left( \beta_1 - \frac{\gamma_2}{2\beta} \right) \frac{R_2 (p_2)}{2\beta^2} (p (x) - p (0)) \cos \frac{\omega (x)}{\beta} \\
&+ \left( \beta_1 - \frac{\gamma_2}{2\beta} \right) \frac{R_2 (p_2)}{2\beta^2} \int_0^x \left( q (t) + \frac{p^2 (t)}{\beta^2} \right) dt \sin \frac{\omega (x)}{\beta} \tag{2.16} \\
A (x, s^+ (x) + 0) - A (x, s^+ (x) - 0) &= - \left( \beta_1 - \frac{\gamma_2}{2\beta} \right) \frac{R_1 (p_2)}{2\beta^2} (p (x) - p (0)) \sin \frac{\omega (x)}{\beta} \\
&- \left( \beta_1 - \frac{\gamma_2}{2\beta} \right) \frac{R_1 (p_2)}{2\beta^2} \int_0^x \left( q (t) + \frac{p^2 (t)}{\beta^2} \right) dt \cos \frac{\omega (x)}{\beta} \tag{2.17} \\
B (x, s^+ (x) + 0) - B (x, s^+ (x) - 0) &= - \left( \beta_1 - \frac{\gamma_2}{2\beta} \right) \frac{R_1 (p_2)}{2\beta^2} (p (x) - p (0)) \cos \frac{\omega (x)}{\beta} \\
&+ \left( \beta_1 - \frac{\gamma_2}{2\beta} \right) \frac{R_1 (p_2)}{2\beta^2} \int_0^x \left( q (t) + \frac{p^2 (t)}{\beta^2} \right) dt \sin \frac{\omega (x)}{\beta} \tag{2.18} \\
A (x, b^- (x) + 0) - A (x, b^- (x) - 0) &= - \left( \beta_1 + \frac{\gamma_2}{2\beta} \right) \frac{R_2 (p_2)}{2\beta^2} (p (x) - p (0)) \sin \frac{\omega (x)}{\beta} \\
&- \left( \beta_1 + \frac{\gamma_2}{2\beta} \right) \frac{R_2 (p_2)}{2\beta^2} \int_0^x \left( q (t) + \frac{p^2 (t)}{\beta^2} \right) dt \cos \frac{\omega (x)}{\beta} \tag{2.19}
\end{align*}
\[ B(x, b^-(x) + 0) - B(x, b^-(x) - 0) = \left( \beta_2^+ + \frac{\gamma_2}{2\beta} \right) \frac{R_2(p_2)}{2\beta^2} (p(x) - p(0)) \cos \frac{\omega(x)}{\beta} \]
\[ - \left( \beta_2^- - \frac{\gamma_2}{2\beta} \right) \frac{R_2(p_2)}{2\beta^2} \int_0^\pi \left( q(t) + \frac{p^2(t)}{\beta^2} \right) dt \sin \frac{\omega(x)}{\beta} \quad (2.20) \]

\[ A(x, b^+(x) + 0) - A(x, b^+(x) - 0) = \frac{\beta_2^+ + \frac{\gamma_2}{2\beta} \cos \omega(x)}{2\beta^2} (p(x) - p(0)) \sin \frac{\omega(x)}{\beta} \]
\[ - \left( \beta_2^- + \frac{\gamma_2}{2\beta} \right) \frac{R_1(p_2)}{2\beta} \int_0^\pi \left( q(t) + \frac{p^2(t)}{\beta^2} \right) dt \cos \frac{\omega(x)}{\beta} \quad (2.21) \]

\[ B(x, b^+(x) + 0) - B(x, b^+(x) - 0) = \left( \beta_2^+ + \frac{\gamma_2}{2\beta} \right) \frac{R_1(p_2)}{2\beta^2} (p(x) - p(0)) \cos \frac{\omega(x)}{\beta} \]
\[ - \left( \beta_2^- + \frac{\gamma_2}{2\beta} \right) \frac{R_1(p_2)}{2\beta^2} \int_0^\pi \left( q(t) + \frac{p^2(t)}{\beta^2} \right) dt \sin \frac{\omega(x)}{\beta} \quad (2.22) \]

are held. If in addition we suppose that \( p(x) \in W_2^2(0, \pi), q(x) \in W_2^1(0, \pi) \), the functions \( A(x, t), B(x, t) \) the following system are provided.

\[
\begin{cases}
\frac{\partial^2 A(x, t)}{\partial x^2} - q(x) A(x, t) - 2p(x) \frac{\partial^2 B(x, t)}{\partial x^2} = \eta \frac{\partial^2 A(x, t)}{\partial t^2} \\
\frac{\partial^2 B(x, t)}{\partial x^2} - q(x) B(x, t) + 2p(x) \frac{\partial^2 A(x, t)}{\partial t^2} = \eta \frac{\partial^2 B(x, t)}{\partial t^2}
\end{cases} \quad (2.23)
\]

where

\[ \eta = \begin{cases} 
\alpha^2 & p_1 < x < p_2; \\
\beta^2 & p_2 < x < \pi.
\end{cases} \]

The proof is done as in [7].

Conversely, if the second order derivatives of functions \( A(x, t), B(x, t) \) are summable on \([0, \pi]\) and \( A(x, t), B(x, t) \) provides (2.23) system and equations (2.10) – (2.22), then the function \( \varphi(x, \lambda) \) which is defined by (1.3) – (1.6) is a solution of (1.1) satisfying boundary conditions (1.2).

**Definition 1.** If there is a nontrivial solution \( y_0(x) \) that provides the (1.2) conditions for the (1.1) problem, then \( \lambda_0 \) is called eigenvalue. Additionally, \( y_0(x) \) is called the eigenfunction of the problem corresponding to the eigenvalue \( \lambda_0 \).

Let us assume that \( q(x) \) satisfies the following conditions.

\[ \int_0^\pi \left\{ |y'(x)|^2 + q(x) |y(x)|^2 \right\} dx > 0. \quad (2.24) \]

For all \( y(x) \in W_2^2[0, \pi] \) such that \( y(x) \neq 0 \) and \( y'(0) \cdot \overline{y(0)} - y'(\pi) \cdot \overline{y(\pi)} = 0. \)

**Lemma 1.** The eigenvalues of the boundary value problem \( L \) are real.
Proof. We set \( l(y) := [-y'' + q(x)y] \). Integration by part yields
\[
(l(y), y) = \int_0^\pi l(y) \cdot \overline{y(x)} \, dx = \int_0^\pi \left\{ |y'(x)|^2 + q(x) |y(x)|^2 \right\} \, dx. \tag{2.25}
\]
Since condition (2.24) holds, it follows that \((l(y), y) > 0\). \( \square \)

**Lemma 2.** Eigenfunction corresponding to different eigenvalues of problem \( L \) are orthogonal in the sense of the equality
\[
(\lambda_n + \lambda_k) \int_0^\pi \delta(x) y(x, \lambda_n)y(x, \lambda_k) \, dx - 2 \int_0^\pi p(x)y(x, \lambda_n)y(x, \lambda_k) \, dx = 0. \tag{2.26}
\]
The proof of Lemma 2 carried out as claim [14].

3. Properties of the spectrum

Let \( \psi(x, \lambda) \) and \( \varphi(x, \lambda) \) be any two solutions of equation (1.1),
\[
W[\psi(x, \lambda), \varphi(x, \lambda)] = \psi(x, \lambda) \varphi'(x, \lambda) - \psi'(x, \lambda) \varphi(x, \lambda),
\]
Wronskian doesn’t depend on \( x \). In this case, it depends only on the \( \lambda \) parameter. Although it is shown as \( W[\psi, \varphi] = \Delta(\lambda) \). \( \Delta(\lambda) \) is called the characteristic function of \( L \). Clearly, the function \( \Delta(\lambda) \) is entire in \( \lambda \). It follows that, \( \Delta(\lambda) \) has at most a countable set of zeros \( \{\lambda_n\} \).

**Lemma 3.** The zeros \( \{\lambda_n\} \) of the characteristic function \( \Delta(\lambda) \) coincide with the eigenvalues of the boundary value problem \( L \). The functions \( \psi(x, \lambda_0) \) and \( \varphi(x, \lambda_0) \) are eigenfunctions corresponding to the eigenvalue \( \lambda_n \), and there exist a sequence \( \{\beta_n\} \) such that
\[
\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n), \quad \beta_n \neq 0. \tag{3.1}
\]
The proof of the Lemma 3 is done as in [27].

Let use denote
\[
\alpha_n = \int_0^\pi \delta(x) \varphi^2(x, \lambda_n) \, dx - \frac{1}{\lambda_n} \int_0^\pi p(x) \varphi^2(x, \lambda_n) \, dx, \quad n = 1, 2, 3, \ldots. \tag{3.2}
\]
The numbers \( \{\alpha_n\} \) are called normalized numbers of the problem \( L \).

**Lemma 4.** The equality \( \dot{\Delta}(\lambda_n) = 2\lambda_n \beta_n \alpha_n \) is obtained. Here \( \dot{\Delta} = \frac{d\Delta}{d\lambda} \).

Proof. Since \( \varphi(x, \lambda) \) and \( \psi(x, \lambda) \) are the solutions of (1.1),
\[
-\varphi''(x, \lambda) + [2\lambda p(x) + q(x)] \varphi(x, \lambda) = \lambda^2 \delta(x) \varphi(x, \lambda),
-\psi''(x, \lambda) + [2\lambda p(x) + q(x)] \psi(x, \lambda) = \lambda^2 \delta(x) \psi(x, \lambda)
\]
equations are provided. Hence, we differentiate the equalities with respect to
\[
-\dot{\varphi''}(x, \lambda) + [2\lambda p(x) + q(x)] \dot{\varphi}(x, \lambda) = \lambda^2 \delta(x) \dot{\varphi}(x, \lambda),
-\dot{\psi''}(x, \lambda) + [2\lambda p(x) + q(x)] \dot{\psi}(x, \lambda) = \lambda^2 \delta(x) \dot{\psi}(x, \lambda),
\]
\(-\Psi''(x, \lambda) + [2\lambda p(x) + q(x)] \Psi(x, \lambda) = \lambda^2 \delta(x) \Psi(x, \lambda) + [2\lambda \delta(x) - 2p(x)] \psi(x, \lambda)\).

Thanks to these equations
\[
\frac{d}{dx} \left\{ \phi(x, \lambda) \cdot \Psi'(x, \lambda) - \phi'(x, \lambda) \cdot \Psi(x, \lambda) \right\} = -[2\lambda \delta(x) - 2p(x)] \phi(x, \lambda) \psi(x, \lambda),
\]
\[
\frac{d}{dx} \left\{ \phi(x, \lambda) \cdot \Psi'(x, \lambda) - \phi'(x, \lambda) \cdot \Psi(x, \lambda) \right\} = [2\lambda \delta(x) - 2p(x)] \phi(x, \lambda) \psi(x, \lambda).
\]
If the last equations are integrated from \(x\) to \(\pi\) and from 0 to \(x\), respectively, by the discontinuity conditions, we obtain
\[
- \left\{ \phi'(\xi, \lambda) \cdot \Psi'(\xi, \lambda) - \phi'(\xi, \lambda) \cdot \Psi(\xi, \lambda) \right\} \bigg|_{x}^{\pi} = \int_{x}^{\pi} [2\lambda \delta(\xi) - 2p(\xi)] \phi(\xi, \lambda) \psi(\xi, \lambda) \, d\xi
\]
and
\[
\int_{0}^{x} [2\lambda \delta(\xi) - 2p(\xi)] \phi(\xi, \lambda) \psi(\xi, \lambda) \, d\xi.
\]
If we add the last equalities side by side, we get
\[
W \left[ \phi(\xi, \lambda), \dot{\Psi}(\xi, \lambda) \right] + W \left[ \phi(\xi, \lambda), \Psi(\xi, \lambda) \right] = -\Delta(\lambda)
\]
\[
= \int_{0}^{\pi} [2\lambda \delta(\xi) - 2p(\xi)] \phi(\xi, \lambda) \psi(\xi, \lambda) \, d\xi
\]
for \(\lambda \rightarrow \lambda_n\), this yields
\[
\Delta(\lambda_n) = -\int_{0}^{\pi} [2\lambda_n \delta(\xi) - 2p(\xi)] \beta_n \phi^2(\xi, \lambda_n) \, d\xi
\]
\[
= 2\lambda_n \beta_n \left\{ \int_{0}^{\pi} \delta(\xi) \phi^2(\xi, \lambda_n) \, d\xi - \frac{1}{\lambda_n} \int_{0}^{\pi} p(\xi) \phi^2(\xi, \lambda_n) \, d\xi \right\} = 2\lambda_n \beta_n \alpha_n.
\]

Denote,
\[
\Gamma_n = \{ \lambda : |\lambda| = |\lambda_n^0| + \delta, \delta > 0, n = 0, 1, 2, \ldots \},
\]
\[
G_n = \{ \lambda : |\lambda - \lambda_n^0| \geq \delta, \delta > 0, n = 0, 1, 2, \ldots \},
\]
where \(\delta\) is sufficiently small positive number. For sufficiently large values of \(n\), one has
\[
|\Delta(\lambda) - \Delta_0(\lambda)| < \frac{C_8}{2} e^{\pi(|\beta_0 - \beta_2 + \alpha_2 - \alpha_1| + p_1)}, \quad \lambda \in \Gamma_n.
\]
As it is shown in [19], \(|\Delta_0(\lambda)| \geq C_5 e^{i m \lambda |\pi|} \) for all \(\lambda \in G_\delta\), where \(C_5 > 0\)
\[
\lim_{|\lambda| \rightarrow \infty} e^{-i m \lambda |\pi|} (\Delta(\lambda) - \Delta_0(\lambda))
\]
\[
\lim_{|\lambda| \to \infty} e^{-im\lambda \pi} \left( \int_0^\pi \tilde{A}(\pi, t) \cos \lambda t dt + \int_0^\pi \tilde{B}(\pi, t) \sin \lambda t dt \right) = 0
\]

is constant. On the other hand, since for sufficiently large values of \( n \) (see[23]) we get (3.3). The Lemma 4 is proved.

**Lemma 5.** The problem \( L(\alpha, p_1, p_2) \) has countable set of eigenvalues. If one denotes by \( \lambda_1, \lambda_2, \ldots \) the positive eigenvalues arranged in increasing order and by \( \lambda_{-1}, \lambda_{-2}, \ldots \) the negative eigenvalues arranged in decreasing order, then eigenvalues of the problem \( L(\alpha, p_1, p_2) \) have the asymptotic behavior

\[
\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{\lambda_n^0} n \to \infty,
\]

where \( k_n \in l_2, d_n \) is a bounded sequence and

\[
\lambda_n^0 = \frac{n\pi}{\beta \pi - \beta p_2 + \alpha p_2 - \alpha p_1 + p_1} + \psi_1(n); \quad \sup_n |\psi_1(n)| = c < +\infty.
\]

**Proof.** According to previous lemma, if \( n \) is a sufficiently large natural number and \( \lambda \in \Gamma_n \), we have \( |\Delta_0(\lambda)| \geq C_\delta e^{im\lambda \pi} > C_\delta e^{im\lambda |\pi|} > |\Delta(\lambda) - \Delta_0(\lambda)|. \) Applying Rouche’s theorem, we conclude that for sufficiently large \( n \) inside the contour \( \Gamma_n \) the functions \( \Delta_0(\lambda) \) and \( \Delta_0(\lambda) + \{\Delta(\lambda) - \Delta_0(\lambda)\} = \Delta(\lambda) \) have the same number of zeros. That is, there are exactly \((n + 1)\) zeros \( \lambda_1, \lambda_2, \ldots, \lambda_{n}. \) Analogously, it is shown by Rouche’s theorem that, for sufficiently large values of \( n \), the function \( \Delta(\lambda) \) has a unique zero inside each circle \( \lambda = \lambda_n^0 + \varepsilon \), where \( \lim_{n \to \infty} \varepsilon_n = 0. \) If \( \Delta(\lambda_n) = 0 \), we have

\[
\Delta_0(\lambda_n^0 + \varepsilon_n) + \int_0^\pi A(\pi, t) \cos (\lambda_n^0 + \varepsilon_n) t dt + \int_0^\pi B(\pi, t) \sin (\lambda_n^0 + \varepsilon_n) t dt = 0, \tag{3.4}
\]

\[
\Delta_0(\lambda_n^0 + \varepsilon_n) = \left( \beta_2 + \frac{\gamma_2}{2\beta} \right) R_1(\varepsilon_n) \cos \left( \frac{\lambda_n^0 + \varepsilon_n}{\beta} \right) - \frac{1}{\beta} \int_0^\pi p(t) dt \tag{3.5}
\]

\[
+ \left( \beta_2 + \frac{\gamma_2}{2\beta} \right) R_2(\varepsilon_n) \cos \left( \frac{\lambda_n^0 + \varepsilon_n}{\beta} \right) - \frac{1}{\beta} \int_0^\pi p(t) dt
\]

\[
+ \left( \beta_2 - \frac{\gamma_2}{2\beta} \right) R_1(\varepsilon_n) \cos \left( \frac{\lambda_n^0 + \varepsilon_n}{\beta} \right) + \frac{1}{\beta} \int_0^\pi p(t) dt
\]

\[
+ \left( \beta_2 - \frac{\gamma_2}{2\beta} \right) R_2(\varepsilon_n) \cos \left( \frac{\lambda_n^0 + \varepsilon_n}{\beta} \right) + \frac{1}{\beta} \int_0^\pi p(t) dt.
\]

Since \( \Delta_0(\lambda) \) is an analytical function,

\[
\Delta_0(\lambda_n^0 + \varepsilon_n) = \Delta_0(\lambda_n^0) \varepsilon_n + \Delta(\lambda_n^0) \varepsilon_n + \frac{\Delta(\lambda_n^0)}{2!} \varepsilon_n^2 + \ldots,
\]

\[
\lim_{n \to \infty} \varepsilon_n = 0.
\]
\( \lambda_n^0 \) is the roots of the \( \Delta_0 (\lambda) = 0 \) equation \( \Delta_0 (\lambda_n^0 + \varepsilon_n) = \left[ \frac{\Delta_0 (\lambda_n^0)}{\Delta_0 (\lambda_n^0)} + o (1) \right] \varepsilon_n, n \rightarrow \infty \) is provided.

\[
\begin{align*}
\left[ \frac{\Delta_0 (\lambda_n^0)}{\Delta_0 (\lambda_n^0)} + o (1) \right] \varepsilon_n + & \int_{p_2}^{s_+ (x) - 0} A (\pi, t) \cos (\lambda_n^0 + \varepsilon_n) t \, dt \\
+ & \int_{s_- (x) + 0}^{s_+ (x) - 0} A (\pi, t) \cos (\lambda_n^0 + \varepsilon_n) t \, dt + \int_{s_+ (x) + 0}^{b^- (x) - 0} A (\pi, t) \cos (\lambda_n^0 + \varepsilon_n) t \, dt \\
+ & \int_{b^- (x) + 0}^{b^- (x) - 0} A (\pi, t) \cos (\lambda_n^0 + \varepsilon_n) t \, dt + \int_{b^- (x) + 0}^{s_+ (x) - 0} A (\pi, t) \cos (\lambda_n^0 + \varepsilon_n) t \, dt \\
+ & \int_{s_- (x) + 0}^{s_+ (x) - 0} B (\pi, t) \sin (\lambda_n^0 + \varepsilon_n) t \, dt + \int_{s_+ (x) + 0}^{b^- (x) - 0} B (\pi, t) \sin (\lambda_n^0 + \varepsilon_n) t \, dt \\
+ & \int_{b^- (x) + 0}^{b^- (x) - 0} B (\pi, t) \sin (\lambda_n^0 + \varepsilon_n) t \, dt + \int_{b^- (x) + 0}^{s_+ (x) - 0} B (\pi, t) \sin (\lambda_n^0 + \varepsilon_n) t \, dt \\
+ & \int_{b^- (x) + 0}^{s_+ (x) - 0} B (\pi, t) \sin (\lambda_n^0 + \varepsilon_n) t \, dt = 0
\end{align*}
\]

It is easy to see that the function \( \Delta_0 (\lambda) = 0 \) is type of [16], so there is a \( \eta_\delta > 0 \) such that \( \left| \frac{\Delta_0 (\lambda_n^0)}{\lambda_n^0} \right| \geq \eta_\delta > 0 \) is satisfied for all \( \eta \). We also have

\[
\lambda_n^0 = \frac{n\pi}{\beta - \beta p_2 + \alpha p_2 - \alpha p_1 + p_1 + \psi_1 (n)},
\]

where \( \sup |\psi_1 (n)| < M \) is for some constant \( M > 0 \) [18]. Further, substituting (3.6) into (3.5) after certain transformations, we reach \( \varepsilon_n \in l_2 \).

Since \( \left( \int_0^\pi A_1 (\pi, t) \sin (\lambda_n^0 + \varepsilon_n) t \, dt \right) \in l_2 \) and \( \left( \int_0^\pi B_1 (\pi, t) \cos (\lambda_n^0 + \varepsilon_n) t \, dt \right) \in l_2 \), we have

\[
\begin{align*}
\varepsilon_n = & \frac{1}{2\lambda_n^0 \Delta_0 (\lambda_n^0)} \left[ \left( \beta_2 - \frac{\gamma_2}{2\beta} \right) \frac{R_2 (p_2)}{2\beta} \sin \left[ \lambda_n^0 s^- (\pi) + \omega (x) \right] \right] + \\
& \left[ \left( \beta_2 - \frac{\gamma_2}{2\beta} \right) \frac{R_1 (p_2)}{2\beta} \sin \left[ \lambda_n^0 s^+ (\pi) + \omega (x) \right] \right] \\
& \left[ \left( \beta_2 - \frac{\gamma_2}{2\beta} \right) \frac{R_1 (p_2)}{2\beta} \sin \left[ \lambda_n^0 b^- (\pi) - \omega (x) \right] \right] \\
& \left[ \left( \beta_2 - \frac{\gamma_2}{2\beta} \right) \frac{R_1 (p_2)}{2\beta} \sin \left[ \lambda_n^0 b^+ (\pi) - \omega (x) \right] \right] \int_0^\pi \left( q (t) + p^2 (t) \right) \, dt \\
& \left[ \left( \beta_2 - \frac{\gamma_2}{2\beta} \right) \frac{R_2 (p_2)}{2\beta^2} \cos \left[ \lambda_n^0 s^- (\pi) + \omega (x) \right] \right].
\end{align*}
\]
is bounded sequence. The proof is completed. □

where

\[
\phi(x, \lambda) = \frac{1}{2} \left( \beta^+_2 + \gamma_2 \right) \exp \left( -i (\lambda b^- (s) - w(x)) \right) \left( 1 + O \left( \frac{1}{\lambda} \right) \right) \quad |\lambda| \to \infty
\]

it has an asymptotic representation where \( w(x) = \int_{p_2}^x p(t) \, dt \) and \( \beta^+_2 = \frac{1}{2} (\alpha_2 t + \frac{\phi^+_2}{\beta}) \).
4. **Inverse Problem**

Let us consider the boundary value problem \( \tilde{L} : \)

\[
\tilde{L} := \begin{cases}
    I(y) := -y'' + [2\lambda \tilde{p}(x) + \tilde{q}(x)]y = \lambda^2 \delta(x)y, & x \in (0, \pi) \\
    y'(0) = 0, y(\pi) = 0 \\
    y(\tilde{p}_1 + 0) = \alpha_{1y}(\tilde{p}_1 - 0) \\
    y'(\tilde{p}_1 + 0) = \beta_{1y}'(\tilde{p}_1 - 0) + i\lambda \gamma_{1y}(\tilde{p}_1 - 0) \\
    y(\tilde{p}_2 + 0) = \alpha_{2y}(\tilde{p}_2 - 0) \\
    y'(\tilde{p}_2 + 0) = \beta_{2y}'(\tilde{p}_2 - 0) + i\lambda \gamma_{2y}(\tilde{p}_2 - 0)
\end{cases}
\]

Let the function \( \Phi(x, \lambda) \) denote solution of (1.1) that satisfy the conditions \( \Phi'(0) = 1, \Phi(\pi) = 0 \) respectively and jump conditions (1.3) – (1.6). Let us define it as \( M(\lambda) := \Phi(0, \lambda) \). The \( \Phi(x, \lambda) \) and \( M(\lambda) \) functions are called the Weyl solution and the Weyl function, respectively.

\[
\Phi(x, \lambda) = M(\lambda) \varphi(x, \lambda) + S(x, \lambda) \quad \lambda \neq \lambda_n, \quad n = 1, 2, 3, \ldots
\]

is true. Because of \( W[\varphi, S]|_{x=0} = \varphi(0, \lambda)S'(0, \lambda) - \varphi'(0, \lambda)S(0, \lambda) = 1 \neq 0, \varphi(x, \lambda) \) and \( S(x, \lambda) \) solutions are linear independent. When \( \psi(x, \lambda) \) is solution (1.1),

\[
\psi(x, \lambda) = A(\lambda) \varphi(x, \lambda) + B(\lambda) S(x, \lambda), \quad \psi'(x, \lambda) = A(\lambda) \varphi'(x, \lambda) + B(\lambda) S'(x, \lambda).
\]

Due to boundary conditions, \( A(\lambda) = \psi(0, \lambda), B(\lambda) = \psi'(0, \lambda) = -\Delta(\lambda) \). Then \( \psi(x, \lambda) = \psi(0, \lambda) \varphi(x, \lambda) - \Delta(\lambda) S(x, \lambda) \) is obtained. Hence,

\[
\Phi(x, \lambda) := -\frac{\psi(x, \lambda)}{\Delta(\lambda)} = S(x, \lambda) + M(\lambda) \varphi(x, \lambda), \quad M(\lambda) = -\frac{\psi(0, \lambda)}{\Delta(\lambda)}.
\]

The \( M(\lambda) \) function is a meromorphic function.

**Theorem 3.** If \( M(\lambda) = \tilde{M}(\lambda) \), then \( L = \tilde{L} \).

**Proof.** Let us define the matrix \( P(x, \lambda) = [P_{j,k}(x, \lambda)] \), \((j, k = 1, 2)\) by the formula

\[
P(x, \lambda) \cdot \begin{pmatrix}
    \tilde{\varphi}(x, \lambda) \\
    \tilde{\varphi}'(x, \lambda)
\end{pmatrix} = \begin{pmatrix}
    \Phi(x, \lambda) \\
    \Phi'(x, \lambda)
\end{pmatrix}.
\]

In this case

\[
P_{11}(x, \lambda) = -\varphi(x, \lambda) \frac{\psi'(x, \lambda)}{\Delta(\lambda)} + \Phi(x, \lambda) \frac{\psi(x, \lambda)}{\Delta(\lambda)}.
\]

\[
P_{12}(x, \lambda) = -\tilde{\varphi}(x, \lambda) \frac{\psi(x, \lambda)}{\Delta(\lambda)} + \varphi(x, \lambda) \frac{\tilde{\psi}(x, \lambda)}{\Delta(\lambda)}.
\]
The Liouville theorem provides that the \( \lambda \) are obtained. When \( M(\lambda) = M(\lambda) \equiv \tilde{M}(\lambda) \), it is clear that the \( P_{j,k}(x,\lambda), (j,k = 1,2) \) functions are full functions according to \( \lambda \). From (3.3); for \( \forall x \in [0,\pi] \), \( c_6 \), \( C_6 \) constants that provide \( |P_{11}(x,\lambda)| \leq C_6 \) and \( |P_{12}(x,\lambda)| \leq C_6 \) inequalities can be shown. From the Liouville theorem \( P_{11}(x,\lambda) \equiv A(x) \) and \( P_{12}(x,\lambda) \equiv 0 \). From

\[
\begin{align*}
P_{21}(x,\lambda) &= -\varphi'(x,\lambda) \frac{\psi'(x,\lambda)}{\Delta(\lambda)} - \varphi'(x,\lambda) \frac{\psi'(x,\lambda)}{\Delta(\lambda)}, \\
P_{22}(x,\lambda) &= -\tilde{\varphi}(x,\lambda) \frac{\psi'(x,\lambda)}{\Delta(\lambda)} + \varphi'(x,\lambda) \frac{\psi'(x,\lambda)}{\Delta(\lambda)}.
\end{align*}
\]

Hence,

\[
\begin{align*}
P_{11}(x,\lambda) &= \varphi(x,\lambda) \left[ \tilde{S}'(x,\lambda) + \tilde{M}(\lambda) \cdot \tilde{\varphi}'(x,\lambda) \right] - \tilde{\varphi}'(x,\lambda) \left[ \tilde{S}(x,\lambda) + M(\lambda) \cdot \varphi(x,\lambda) \right] \\
&= \varphi(x,\lambda) \tilde{S}'(x,\lambda) - \tilde{\varphi}'(x,\lambda) S(x,\lambda) + \left[ \tilde{M}(\lambda) - M(\lambda) \right] \varphi(x,\lambda) \tilde{\varphi}'(x,\lambda), \\
P_{12}(x,\lambda) &= \varphi(x,\lambda) \left[ \tilde{S}'(x,\lambda) + \tilde{M}(\lambda) \cdot \tilde{\varphi}'(x,\lambda) \right] - \tilde{\varphi}'(x,\lambda) \left[ \tilde{S}(x,\lambda) + M(\lambda) \cdot \varphi(x,\lambda) \right] \\
&= \varphi(x,\lambda) S(x,\lambda) - \varphi(x,\lambda) S(x,\lambda) + \left[ M(\lambda) - \tilde{M}(\lambda) \right] \varphi(x,\lambda) \tilde{\varphi}'(x,\lambda), \\
P_{21}(x,\lambda) &= \varphi'(x,\lambda) \left[ \tilde{S}'(x,\lambda) + \tilde{M}(\lambda) \cdot \tilde{\varphi}'(x,\lambda) \right] - \varphi'(x,\lambda) \left[ \tilde{S}'(x,\lambda) + \tilde{M}(\lambda) \cdot \tilde{\varphi}'(x,\lambda) \right] \\
&= \varphi'(x,\lambda) \tilde{S}'(x,\lambda) - \varphi'(x,\lambda) S'(x,\lambda) + \left[ \tilde{M}(\lambda) - M(\lambda) \right] \varphi'(x,\lambda) \tilde{\varphi}'(x,\lambda), \\
P_{22}(x,\lambda) &= \varphi'(x,\lambda) \left[ S'(x,\lambda) + M(\lambda) \cdot \varphi'(x,\lambda) \right] + \varphi'(x,\lambda) \left[ \tilde{S}'(x,\lambda) + \tilde{M}(\lambda) \cdot \tilde{\varphi}'(x,\lambda) \right] \\
&= \varphi'(x,\lambda) S'(x,\lambda) - \varphi'(x,\lambda) \tilde{S}'(x,\lambda) + \left[ M(\lambda) - \tilde{M}(\lambda) \right] \varphi'(x,\lambda) \tilde{\varphi}'(x,\lambda).
\end{align*}
\]

are obtained. When \( M(\lambda) \equiv \tilde{M}(\lambda) \), it is clear that the \( P_{j,k}(x,\lambda), (j,k = 1,2) \) functions are full functions according to \( \lambda \). From (3.3); for \( \forall x \in [0,\pi] \), \( c_6 \), \( C_6 \) constants that provide \( |P_{11}(x,\lambda)| \leq c_6 \) and \( |P_{12}(x,\lambda)| \leq C_6 \) inequalities can be shown. From the Liouville theorem \( P_{11}(x,\lambda) \equiv A(x) \) and \( P_{12}(x,\lambda) \equiv 0 \). From

\[
\begin{align*}
\varphi(x,\lambda) \cdot \varphi'(x,\lambda) - \tilde{\varphi}'(x,\lambda) \cdot \Phi(x,\lambda) &= A(x), \\
\tilde{\varphi}(x,\lambda) \cdot \Phi(x,\lambda) - \varphi(x,\lambda) \cdot \tilde{\varphi}(x,\lambda) &= A(x)
\end{align*}
\]

are obtained and

\[
W [\varphi, \Phi] = W \left[ \varphi(x,\lambda), -\frac{\psi(x,\lambda)}{\Delta(\lambda)} \right]
\]
\[ \frac{1}{\Delta(\lambda)} W [\varphi(x, \lambda), -\psi(0, \lambda) \varphi(x, \lambda) + \Delta(\lambda) S(x, \lambda)] \]

\[ = -\frac{\psi(0, \lambda)}{\Delta(\lambda)} W [\varphi(x, \lambda), \varphi(x, \lambda)] + W [\varphi(x, \lambda), S(x, \lambda)] = 1. \]

And similarly \( W [\tilde{\varphi}, \tilde{\Phi}] = 1 \) is obtained. If this equation is written in place of (4.1),

\[ 1 = W [\varphi(x, \lambda), \Phi(x, \lambda)] = W \left[ A(x) \tilde{\varphi}(x, \lambda), A(x) \tilde{\Phi}(x, \lambda) \right] \]

\[ = A^2(x) W \left[ \tilde{\varphi}(x, \lambda), \tilde{\Phi}(x, \lambda) \right] = A^2(x) \]

is obtained.

Therefore, \( \left( \beta_1^* + \frac{\gamma_1}{2p} \right) \neq 1; \ p_1 = \tilde{p}_1, \ p_2 = \tilde{p}_2 \). We have \( A(x) = 1 \) from (4.1) \( \varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda) \) and \( \Phi(x, \lambda) \equiv \tilde{\Phi}(x, \lambda) \).

When \( \varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda) \),

\[ -\varphi'' + [2\lambda p(x) + q(x)] \varphi = \lambda^2 \delta(x) \varphi, \]

\[ -\varphi'' + [2\lambda p(x) + q(x)] \varphi = \lambda^2 \delta(x) \varphi \]

are obtained.

\[ \left\{ \lambda^2 \left( \delta(x) - \tilde{\delta}(x) \right) + 2\lambda \left( p(x) - \tilde{p}(x) \right) + (q(x) - \tilde{q}(x)) \right\} \varphi \equiv 0 \quad (\text{for } \forall \lambda) \]

\( \delta(x) = \tilde{\delta}(x), \ p(x) = \tilde{p}(x) \) and \( q(x) = \tilde{q}(x) \) a.e. For every \( \lambda \) in discontinuity conditions,

\[ (\alpha_1 - \bar{\alpha}) \varphi(p_1 - 0, \lambda) = 0 \]

\[ \left( \beta_1 - \bar{\beta}_1 \right) \varphi'(p_1 - 0, \lambda) + (\gamma_1 - \bar{\gamma}_1) \varphi(p_1 - 0, \lambda) = 0 \]

\[ (\alpha_2 - \bar{\alpha}_2) \varphi(p_2 - 0, \lambda) = 0 \]

\[ \left( \beta_2 - \bar{\beta}_2 \right) \varphi'(p_2 - 0, \lambda) + (\gamma_2 - \bar{\gamma}_2) \varphi(p_2 - 0, \lambda) = 0 \]

\( \alpha_1 = \alpha_1, \beta_1 = \bar{\beta}_1, \gamma_1 = \bar{\gamma}_1 \) and \( \alpha_2 = \bar{\alpha}_2, \beta_2 = \bar{\beta}_2, \gamma_2 = \bar{\gamma}_2 \).

Consequently \( L = \tilde{L} \). The proof is completed. \( \square \)

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