Weak $(1,1)$ Boundedness of Riesz Transforms on Vector Bundles

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Abstract

We prove weak $(1,1)$ boundedness of (local) Riesz transforms corresponding to a large class of Schrödinger operators on vector bundles mainly under the generalized volume doubling condition, either Gaussian or sub-Gaussian upper bounds for the heat kernel only in short time, and derivative estimates for semigroups on vector bundles. Consequently, neither Gaussian nor sub-Gaussian upper estimates for the heat kernel are necessary for weak $(1,1)$ boundedness of the Riesz transform on vector bundles.

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1 Introduction

Let $M$ be be a complete, noncompact Riemannian manifold, $\text{vol}$ be the Riemannian volume measure, and $\Delta$ be the Laplace–Beltrami operator. Let $(q_t)_{t>0}$ be the heat kernel corresponding to $\Delta$ and $B(x,r)$ be the open ball in $M$ with center $x$ and radius $r > 0$. Denote $V(x,r) = \text{vol}(B(x,r))$. The main theme of the Riesz transform on Riemannian manifolds, denoted by $\nabla(-\Delta)^{-1/2}$, is on the weak $(1,1)$ boundedness, i.e.,

$$\text{vol}\{x \in M: |\nabla(-\Delta)^{-1/2}f(x)| \geq \sigma\} \lesssim \frac{1}{\sigma} \int_M |f| \text{dvol}, \quad \forall f \in C_c^\infty(M),$$

and the $L^p$ boundedness, i.e., for which $p \in (1,\infty)$,

$$\|\nabla(-\Delta)^{-1/2}f\|_{L^p(M,\text{vol})} \lesssim \|f\|_{L^p(M,\text{vol})}, \quad \forall f \in C_c^\infty(M).$$

For instance, in [8], Strichartz asked, on what non-compact Riemannian manifolds and for which $p \in (1,\infty)$, the Riesz transform $\nabla(-\Delta)^{-1/2}$ is $L^p$ bounded.

In [3, Theorem 1.1], under the volume doubling condition, i.e., there exists a constant $C > 0$ such that

$$V(x,2r) \leq CV(x,r), \quad \forall x \in M, \ r > 0, \tag{1.1}$$

and the Gaussian upper bound for the heat kernel, i.e., for any $x,y \in M$,

$$q_t(x,y) \leq \frac{C_1}{V(x,\sqrt{t})} \exp\left\{-C_2 \frac{d^2(x,y)}{t}\right\}, \quad \forall t > 0,$$

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for some constants $C_1, C_2 > 0$, the Riesz transform was proved to be weak $(1, 1)$ bounded (and hence $L^p$ bounded for all $p \in (1, 2]$ by interpolation since the $L^2$ boundedness trivially holds). Let $m > 1$. Recently, under the volume doubling condition (1.1) and the sub-Gaussian upper bound for the heat kernel, estimates on the gradient of the heat kernel play a crucial role. However, similar gradient estimates seem not easy to get for heat kernels on vector bundles since they are just linear operators between fibers. Due to this gap, the derivative formula for semigroups on vector bundles was established, and then applied to prove the weak $(1, 1)$ boundedness of the (local) Riesz transform corresponding to a large class of Schrödinger operators on vector bundles under a generalized volume doubling condition and the Gaussian upper bound for the heat kernel. In proofs of the aforementioned results, estimates on the gradient of the heat kernel $q_t$ play a crucial role. However, similar gradient estimates seem not easy to get for heat kernels on vector bundles since they are just linear operators between fibers. Due to this gap, the derivative formula for semigroups on vector bundles was established, and then applied to prove the weak $(1, 1)$ boundedness of the (local) Riesz transform corresponding to a large class of Schrödinger operators on vector bundles under a generalized volume doubling condition and the Gaussian upper bound for the heat kernel $q_t$; see [7, THEOREMS 2.1 and 4.1].

In this work, we consider the weak $(1, 1)$ boundedness of the (local) Riesz transform corresponding to a large class of Schrödinger operators on vector bundles under the volume doubling condition (1.1) and the Gaussian or sub-Gaussian upper bound for small times. In Section 2, we introduce the framework and recall the derivative estimate of semigroups on vector bundles. In Section 3, we present the main result (Theorem 3.1) and its full proof.

2 Preliminaries

Let $E \to M$ and $F \to M$ be Riemannian bundles over the same (not necessarily complete) Riemannian manifold $M$, equipped with metric connections $\nabla^E$ and $\nabla^F$ respectively. Denote $TM$ and $T^*M$ the tangent and the cotangent bundle of $M$, respectively. We use $\Gamma_{C^\infty}(\bullet)$ (resp. $\Gamma_{C^\infty,c}(\bullet)$ and $\Gamma_{C^\infty,c,\text{loc}}(\bullet)$) to denote the class of smooth (resp. the bounded smooth and the compactly supported smooth) sections of a vector bundle “$\bullet$”.

Let $\omega \in \Gamma_{C^\infty}(\text{Hom}(T^*M \otimes E, F))$ be a multiplication map, where $\text{Hom}(\bullet, \bullet)$ is the Hom-bundle. Introduce the Dirac type operator from $\Gamma_{C^\infty}(E)$ to $\Gamma_{C^\infty}(F)$ defined by

$$D_{\omega} := \omega \nabla^E,$$

which is a first order differential operator and can be regarded as the composition:

$$\Gamma_{C^\infty}(E) \xrightarrow{\nabla^E} \Gamma_{C^\infty}(TM \otimes E) \xrightarrow{\omega} \Gamma_{C^\infty}(F).$$

The Bochner Laplacian $(\nabla^E)^* \nabla^E : \Gamma_{C^\infty}(E) \to \Gamma_{C^\infty}(E)$ is the second order elliptic differential operator given by the composition:

$$\Gamma_{C^\infty}(E) \xrightarrow{\nabla^E} \Gamma_{C^\infty}(TM \otimes E) \xrightarrow{\nabla^TM \otimes 1 + 1 \otimes \nabla^E} \Gamma_{C^\infty}(TM \otimes TM \otimes E) \xrightarrow{\text{tr}} \Gamma_{C^\infty}(E),$$

where tr is the trace operator with respect to the Riemannian metric of $M$ and $\nabla^TM$ is the Riemannian connection on $TM$, and so is the Bochner Laplacian $(\nabla^F)^* \nabla^F : \Gamma_{C^\infty}(F) \to \Gamma_{C^\infty}(F).$
Consider Schrödinger type operators
\[ L = -\nabla^2 + V - \mathcal{U}_E \]
on \( \Gamma_{C^\infty}(E) \), and
\[ T = -\nabla^2 + V - \mathcal{U}_F \]
on \( \Gamma_{C^\infty}(F) \). Note in passing that if \( E \) and \( F \) are the trivial bundle \( M \times \mathbb{R} \), then \( L \) and \( T \) are just Schrödinger operators on \( M \) of the type \( \Delta + \nabla V + U \), where \( U : M \to \mathbb{R} \) is a real potential.

As in [7], we make the standing assumption that
\[ \vartheta = TD_\omega - D_\omega L \]
is of zeroth order, i.e., \( \vartheta \in \Gamma_{C^\infty}(\text{Hom}(E,F)) \), and \( \omega \) is compatible with the Riemannian connection, which means that for any \( Z \in \Gamma_{C^\infty}(TM) \), \( \alpha \in \Gamma_{C^\infty}(E) \) and \( v \in TM \), it holds that
\[ \nabla^E_v (\omega(Z^b \otimes \alpha)) = \omega((\nabla^TMZ)^b \otimes \alpha) + \omega(Z^b \otimes \nabla^E_v \alpha), \]
where \( b : TM \to T^*M \) is the music isomorphism.

Let \( (\bullet, \bullet)_E \) (resp. \( (\bullet, \bullet)_F \)) denote the scalar product and \( |\bullet|_E \) (resp. \( |\bullet|_F \)) the induced norm on fibers of \( E \) (resp. \( F \)). Denote \( d\mu = e^V \text{dvol} \), where \( \text{dvol} \) is the Riemannian volume measure on \( M \). Let \( p \in [1, \infty) \). Denote \( \Gamma_{L^p}(E) \) the real Banach space of measurable sections \( \alpha : M \to E \) such that \( \|\alpha\|_p < \infty \), where
\[ \|\alpha\|_p := \begin{cases} \left( \int_M |\alpha(x)|_E^p \, d\mu(x) \right)^{1/p}, & p \in [1, \infty), \\ \inf \{ c \geq 0 : |\alpha|_E \leq c \mu\text{-a.e.} \}, & p = \infty. \end{cases} \]

It turns out that for every \( p \in [1, \infty) \), \( \Gamma_{L^p}(E) \) is the closure of \( \Gamma_{C^\infty}(E) \) with respect to the norm \( \| \cdot \|_p \).

We further assume that \( \mathcal{U}_E \) is symmetric, i.e., for every \( x \in M \), \( \mathcal{U}_E(x) \) is a symmetric linear operator from fiber \( E_x \) to itself. It is well known that if \( \mathcal{U}_E \) is lower bounded, then \( (L, \Gamma_{C^\infty}(E)) \) is upper bounded in \( \Gamma_{L^p}(E) \) and hence has a canonical self-adjoint extension, namely, the Friedrich extension in \( \Gamma_{L^p}(E) \), still denoted by \( L \). Let \( (P_t)_{t \geq 0} \) be the semigroup corresponding to \( L/2 \).

Denote \( (P^0_t)_{t \geq 0} \) the semigroup corresponding to the Friedrich extension of \( (\Delta + \nabla V)/2, C^\infty(M) \) in \( L^2_{\mu}(M) \).

Now we recall a derivative estimate of \( P_t \), which was established under some further assumptions (see [7, THEOREM 2.1]). Let \( \| \cdot \| \) denote the operator norm.

**Hypothesis (I).** There exist some constants \( a_1, a_2, a_3 \in \mathbb{R} \) and \( C(\vartheta), C(\omega) \geq 0 \) such that
\begin{align*}
& (I.1) \quad a_1 |\alpha|^2_E \leq (\mathcal{U}_E \alpha, \alpha)_E \leq a_2 |\alpha|^2_E, \text{ for every } \alpha \in E, \\
& (I.2) \quad (\mathcal{U}_F \beta, \beta)_F \geq a_3 |\beta|^2_F, \text{ for every } \beta \in F, \\
& (I.3) \quad \|\vartheta\| \leq C(\vartheta), \|\omega\| \leq C(\omega).
\end{align*}

Let \( d \) be the Riemannian distance on \( M \). For \( x \in M \), let \( d_x := d(x, \bullet) \) and denote \( \text{cut}(x) \) the cut locus of \( x \) in \( M \).
Theorem 2.1. Let $M$ be a complete Riemannian manifold and assume that for every $x \in M$, there exist constants $c > 0$ and $\delta \in (0, 1)$ such that

$$ (\Delta + \nabla V) d_x \lesssim c(d_x^{-1} + d_x^\delta) $$

outside $\{x\} \cup \text{cut}(x)$. Suppose that Hypothesis (I) holds. Then, for every $\beta \in \Gamma_{C^\infty}(E)$ and $x \in M$,

$$ |D_\omega P_t \beta|^2(x) \leq e^{-a_1 t} \|\beta\|_\infty \frac{a(C(\omega) + C(\vartheta) \sqrt{t}/2)^2}{1 - e^{-at}} P_t^0 |\beta|(x), $$

for all $t > 0$, where $a := \max\{a_2 - a_3, 0\}$ and $a/(1 - e^{-at}) := 1/t$ if $a = 0$.

For some concrete examples included in the above context, we refer the reader to [7, Section 2].

3 Riesz transforms on vector bundles

From now on, we assume that $M$ is a complete and noncompact Riemannian manifold, and Hypothesis (I) holds as well as the other assumptions in Section 2. For some suitable nonnegative constant $\lambda$, let us define the (local) Riesz transform $R_\lambda$ associated with the operator $L$ by

$$ R_\lambda \alpha = D_\omega (\lambda - L)^{1/2} \alpha, \quad \alpha \in \Gamma_{C^\infty}(E). $$

We shall consider the weak $(1, 1)$ boundedness of $R_\lambda$, i.e.,

$$ \mu\{|R_\lambda \alpha| > \sigma\} \lesssim \frac{\mu(|\alpha|)}{\sigma}, \quad \text{for any } \sigma > 0 \text{ and any } \alpha \in \Gamma_{C^\infty}(E). $$

Here and in the sequel, we use the notation $f \lesssim g$ if there exists some universal constant $C > 0$ such that $f \leq Cg$.

Let $p_t^0$ be the heat kernel of $P_t^0$ with respect to $\mu$. Denote $cB(x, r) = B(x, cr)$ for every ball $B(x, r)$.

Hypothesis (II). $m \geq 2$ and $M$ is complete and noncompact satisfying that

(I.1) for any $x \in M$, there exists a constant $\delta \in (0, 1)$ such that

$$ (\Delta + \nabla V) d_x \lesssim d_x^{-1} + d_x^\delta $$

outside $\{x\} \cup \text{cut}(x)$;

(I.2) there exists constants $D \geq 1$ and $\kappa \in [0, \frac{m}{m-1})$ such that

$$ V(x, \tau r) \leq D \tau^D V(x, r) \exp(\tau^\kappa + r^\kappa), \quad \forall x \in M, \tau \geq 1, r > 0; $$

(I.3) there exists a constant $c > 0$ such that

$$ p_t^0(x, y) \lesssim \frac{1}{V(x, t^{1/m})} \exp\left\{-c\left(\frac{d^{\text{mid}}(x, y)}{t}\right)\frac{m}{m-1}\right\}, \quad \forall t \in (0, 1], x, y \in M. $$

(I.1) is just condition B1 appeared on page 115 in [7]. It was pointed out that (see lines 1-2 on page 114 of [7]), if there exists some point $o \in M$ such that $\text{Ric} \gtrsim -(1 + d_o^{2\delta})$ and $|\nabla V| \lesssim 1 + d_o^\delta$, then (I.1) holds, where Ric is the Ricci curvature of $M$. We should mention that, (II.2) is not comparable with the local condition assumed in [3, Theorem...
since \( \kappa \) is allowed to be bigger than 1, and in particular, when \( m = 2 \), (II.2) is just condition B2 on page 115 in [7]. It is well known that, under the completeness assumption, (II.2) can be derived from \( \text{Ric} - \text{Hess}_V \geq 0 \) with \( V \) bounded, where \( \text{Hess}_V \) is the Hessian of \( V \).

For \( m > 2 \), the sub-Gaussian upper bound (II.3) appears naturally as the upper bound of the transition density of a canonical diffusion process on fractal sets with respect to a proper Hausdorff measure. For instance, on the Sierpiński gasket in \( \mathbb{R}^2 \), the upper and the lower bounds for the transition density of the natural Brownian motion are comparable with

\[
\frac{1}{t^{v/m}} \exp \left\{ -c \left( \frac{d^m(x,y)}{t} \right)^{m-1} \right\}, \quad \forall \ t > 0,
\]

where \( d(x,y) = |x - y| \), \( v = \log 3/\log 2 \) and \( m = \log 5/\log 2 > 2 \); see e.g. [1]. Compared with assumptions on the sub-Gaussian upper bound both in [2] and [6] (see also (1.2) above), which are indeed Gaussian for \( t \in (0,1) \), the assumption (II.3) is more natural.

Let \( \Upsilon \in (0, \infty) \]. By [4, Theorem 1.1], under the volume doubling condition (1.1), the on-diagonal upper bound

\[
p_0^0(x,x) \lesssim \frac{1}{V(x, \sqrt{t})}, \quad \forall \ x \in M, \ t \in (0, \Upsilon),
\]

self-improves to the Gaussian upper bound, i.e., (II.3) with \( m = 2 \) for all \( t \in (0, \Upsilon) \). However, this self-improving property for \( m > 2 \), more precisely, from (3.1) to (II.3) for all \( t \in (0, \Upsilon) \) under the assumption (1.1), may not be true. It seems that to find an example to illustrate that the self-improving property does not hold in the sub-Gaussian situation is quite an interesting open problem.

Now we present the main result of this paper.

**Theorem 3.1.** Suppose that Hypotheses (I) and (II) hold and \( R_\lambda \) is bounded in \( \Gamma_{L^2_p}(E) \). Then,

1. for \( \lambda > -a_1 \) (which is specified in (I.1)), \( R_\lambda \) is weak \((1,1)\) and bounded in \( \Gamma_{L^p_p}(E) \) for all \( 1 < p \leq 2 \);
2. if \( \vartheta = 0 \), and either \( a_1 > 0 \) or \( a_1 = \kappa = 0 \), and (II.3) holds for all \( t > 0 \), then \( R_0 \) is weak \((1,1)\) and bounded in \( \Gamma_{L^p_p}(E) \) for all \( 1 < p \leq 2 \).

We should mention that \( R_\lambda \) being bounded in \( \Gamma_{L^2_p}(E) \) is not such a restrictive condition, since in many geometric applications, \( D_\omega \) is just the Dirac operator and \( -L \) is just the square of the Dirac operator, for instance, the Hodge Laplacian acting on differential forms, and in that case, \( R_\lambda \) is trivially bounded in \( \Gamma_{L^2_p}(E) \). See also [7, REMARK 4.5].

We have pointed out in Section 1 that the case when \( m = 2 \) and \( \kappa \in [0,2) \) is covered by [7, THEOREM 4.1]. However, the method below effectively deals with this particular situation and the general one when \( m > 2 \) and \( \kappa \in [0, \frac{m}{m-1}) \) simultaneously.

In order to prove Theorem 3.1, we need some lemmata. First we present the following one, which shows that a basic estimate similar to [3, Lemma 2.1] also holds under the generalized volume doubling property (II.2).

**Lemma 3.2.** Suppose that (II.2) holds. Then, for any \( r \geq 0 \) and any \( \eta > 0 \),

\[
\int_{M \setminus B(y,r)} \exp \left\{ -\eta \left( \frac{d^m(x,y)}{t} \right)^{m-1} \right\} \, d\mu(x) \\
\lesssim V(y, t^{1/m}) \exp \left\{ -\eta \left( \frac{t^{m}}{2} \right)^{m-1} \right\}, \quad \forall \ y \in M.
\]
Lemma 3.3.

\[ \text{Lemma.} \]

there exist constants \( c \) and \( \kappa < \frac{m}{m-1} \) in (II.2).

Proof. For any \( r \geq 0 \) and any \( \eta > 0 \),

\[
\int_{M \setminus B(y,r)} \exp \left\{ -\eta \left( \frac{d^m(x,y)}{t} \right)^{\frac{1}{m-1}} \right\} \, d\mu(x) \\
\lesssim e^{-\frac{\eta}{2} \left( \frac{m}{m-1} \right)^{\frac{1}{m-1}}} \int_M \exp \left\{ -\frac{\eta}{2} \left( \frac{d^m(x,y)}{t} \right)^{\frac{1}{m-1}} \right\} \, d\mu(x). \tag{3.2}
\]

For \( t \in (0,1] \), applying (II.2), we have

\[
\frac{\int_M \exp \left\{ -\frac{\eta}{2} \left( \frac{d^m(x,y)}{t} \right)^{\frac{1}{m-1}} \right\} \, d\mu(x)}{\int_M \mu} \leq \mu(B(0)) + \sum_{i=1}^{\infty} \mu(B_i) \exp \left\{ -\frac{\eta}{2} \left( \frac{1}{m-1} \right) \right\} \\
\leq \mu(B_0) + \sum_{i=1}^{\infty} \mu(B_i) \exp \left\{ -\frac{\eta}{2} \left( i - 1 \right)^{\frac{1}{m}} \right\} \\
\leq \mu(B_0) \left[ 1 + \sum_{i=1}^{\infty} (i-1)^{D} \exp \left\{ (i-1)^{\frac{m}{m-1}} - \frac{\eta}{2} \left( i - 1 \right)^{\frac{1}{m-1}} \right\} \right] \\
\lesssim V(y,1) \exp \left\{ (2\kappa + 1)t^{\kappa/m} - \frac{\eta}{2} \left( i - 1 \right)^{\frac{1}{m-1}} \right\} \\
\lesssim V(y,t^1/m), \tag{3.3}
\]

where the last line is due to the assumption that \( \kappa < \frac{m}{m-1} \) in (II.2).

For \( t > 1 \), letting \( B_i = B(y,(i+1)t^{1/m}) \setminus B(y,it^{1/m}) \), \( i = 0, 1, 2, \ldots \), we have

\[
\frac{\int_M \exp \left\{ -\frac{\eta}{2} \left( \frac{d^m(x,y)}{t} \right)^{\frac{1}{m-1}} \right\} \, d\mu(x)}{\int_M \mu} \leq \mu(B_0) + \sum_{i=1}^{\infty} \mu(B_i) \exp \left\{ -\frac{\eta}{2} \left( i - 1 \right)^{\frac{1}{m}} \right\} \\
\leq \mu(B_0) + \sum_{i=1}^{\infty} \mu(B_i) \exp \left\{ -\frac{\eta}{2} \left( i - 1 \right)^{\frac{1}{m}} \right\} \\
\leq \mu(B_0) \left[ 1 + \sum_{i=1}^{\infty} (i^{1/m} + 1)^D \exp \left\{ (i^{1/m} + 1)^{\kappa} + \frac{\kappa}{m} - \frac{\eta}{2} \left( i - 1 \right)^{\frac{1}{m}} \right\} \right] \\
\lesssim V(y,1) \exp \left\{ (2\kappa + 1)t^{\kappa/m} - \frac{\eta}{2} \left( i - 1 \right)^{\frac{1}{m-1}} \right\} \\
\lesssim V(y,t^1/m), \tag{3.4}
\]

where the last line is again due to that \( \kappa < \frac{m}{m-1} \).

Combing the estimates (3.2), (3.3) and (3.4), we complete the proof.

Next we establish the upper bound for the heat kernel \( p_t^0 \) for all \( t > 0 \) in the following lemma.

Lemma 3.3. Let \( m \geq 2 \). Suppose that (II.2) and (II.3) hold with \( \kappa \in [0,\frac{m}{m-1}) \). Then, there exist constants \( c_1, c_2 > 0 \) such that

\[
p_t^0(x,y) \lesssim \frac{1}{V(y,t^{1/m})} \exp \left\{ -c_1 \left( \frac{d^m(x,y)}{t} \right)^{\frac{1}{m-1}} + c_2 t^{\frac{\kappa}{m}} \right\}, \quad \forall x, y \in M, t > 0.
\]

Proof. The main idea of proof is based on the method of induction.

(1) Let \( 0 < t \leq 1 \). By the symmetry and semigroup property of the heat kernel \( p_t^0 \), we have

\[
p_{2t}^0(x,y) = \int_M p_t^0(x,z)p_t^0(y,z) \, d\mu(z).
\]

6
Applying the inequality

\[ a^q + b^q \geq 2^{1-q}(a + b)^q, \quad \forall a, b \geq 0, q \geq 1, \]

and the triangle inequality \( d(x, z) + d(y, z) \geq d(x, y) \), we obtain that, for any \( 0 < \gamma < 2c \) with the same \( c \) in (II.3),

\[
p^0_{2t}(x, y) \leq \int_M p^0_t(x, z) \exp \left\{ \frac{\gamma}{2} \left( \frac{d^m(x, z)}{t} \right)^{\frac{1}{m-1}} \right\} p^0_t(y, z) \exp \left\{ \frac{\gamma}{2} \left( \frac{d^m(y, z)}{t} \right)^{\frac{1}{m-1}} \right\} \\
\times \exp \left\{ -c_{\gamma, m} \left( \frac{d^m(x, y)}{t} \right)^{\frac{1}{m-1}} \right\} d\mu(z),
\]

where \( c_{\gamma, m} = \gamma 2^{-m/(m-1)} \). Set

\[
N_\gamma(x, t) = \int_M p^0_t(x, z)^2 \exp \left\{ \gamma \left( \frac{d^m(x, z)}{t} \right)^{\frac{1}{m-1}} \right\} d\mu(z).
\]

By the Cauchy–Schwarz inequality, we have

\[
p^0_{2t}(x, y) \leq (N_\gamma(x, z, t)N_\gamma(y, z, t))^{1/2} \exp \left\{ -c_{\gamma, m} \left( \frac{d^m(x, y)}{t} \right)^{\frac{1}{m-1}} \right\}.
\]

Applying Lemma 3.2 with \( r = 0 \), we deduce that

\[
N_\gamma(x, t) \lesssim \frac{1}{[V(x, t^{1/m})]^2} \int_M \exp \left\{ -(2\gamma - \gamma) \left( \frac{d^m(x, z)}{t} \right)^{\frac{1}{m-1}} \right\} d\mu(z)
\lesssim \frac{1}{V(x, t^{1/m})},
\]

and

\[
N_\gamma(y, t) \lesssim \frac{1}{V(y, t^{1/m})}.
\]

Hence,

\[
p^0_{2t}(x, y) \lesssim \left( \frac{1}{V(x, t^{1/m})} \frac{1}{V(y, t^{1/m})} \right)^{1/2} \exp \left\{ -c_{\gamma, m} \left( \frac{d^m(x, y)}{t} \right)^{\frac{1}{m-1}} \right\}.
\]

By (II.2), \( V(y, (2t)^{1/m}) \lesssim V(y, t^{1/m}) e^{x/m} \), and

\[
V(y, (2t)^{1/m}) \leq V(x, d(x, y) + (2t)^{1/m}) \lesssim V(x, t^{1/m}) \left( \frac{d(x, y) + (2t)^{1/m}}{t^{1/m}} \right)^D
\times \exp \left\{ t^{\kappa/m} + \left( \frac{d(x, y) + (2t)^{1/m}}{t^{1/m}} \right)^\kappa \right\},
\]

which imply that

\[
p^0_{2t}(x, y) \lesssim \left( e^{x/m} \left( 1 + \frac{d(x, y)^D}{t^{1/m}} \right) \exp \left[ t^{\kappa/m} + \left( \frac{d(x, y)^\kappa}{t^{1/m}} \right) \right] \right)^{1/2}
\]

\[
\left( \frac{1}{[V(y, (2t)^{1/m})]^2} \right)^{1/2}.
\]
By the method of induction, similar as the calculation in steps (1) and (2), we complete
and

\[
\frac{1}{V(y, (2t)^{1/m})} \exp \left\{ -c_{\kappa,\gamma, m} \left( \frac{d^m(x, y)}{2t} \right)^{\frac{1}{m-1}} + (2t)^{\kappa/m} \right\},
\]

for some constant \( c_{\kappa, \gamma, m} > 0 \), where the last inequality of (3.5) holds by the assumption
that \( \kappa \in (0, \frac{m}{m-1}) \), and the inequality \((1 + \xi)^{D/2} e^{-C\xi} \leq C_1 e^{-C_2 \xi}\), for some \( C_1, C_2 > 0 \) and
any \( \xi \geq 0 \), where \( C \) is a positive constant.

(2) Similar as step (1) above, for any \( t \in (0, 1] \) and any \( \epsilon \in (0, 2c_{\kappa, \gamma, m}) \), we have

\[
p^0_{d_4}(x, y) \leq \left( N_\epsilon(x, 2t) N_\epsilon(y, 2t) \right)^{1/2} \exp \left\{ -c_{\epsilon, m} \left( \frac{d^m(x, y)}{2t} \right)^{\frac{1}{m-1}} \right\},
\]

for some constants \( c_{\epsilon, m} > 0 \), where, by applying (3.5) and Lemma 3.2,

\[
N_\epsilon(x, 2t) \leq \int_M \exp \left\{ -(2c_{\kappa, \gamma, m} - \epsilon) \left( \frac{d^m(x, z)}{2t} \right)^{\frac{1}{m-1}} + (2t)^{\kappa/m} \right\} d\mu(z)
\]

\[
\leq \frac{e^{(2t)^{\kappa/m}}}{V(x, (2t)^{1/m})},
\]

and

\[
N_\epsilon(y, 2t) \leq \frac{e^{(2t)^{\kappa/m}}}{V(y, (2t)^{1/m})}.
\]

Hence, by (II.2),

\[
p^0_{d_4}(x, y) \leq \left( \frac{e^{(2t)^{\kappa/m}}}{V(x, (2t)^{1/m})} \frac{e^{(2t)^{\kappa/m}}}{V(y, (2t)^{1/m})} \right)^{1/2} \exp \left\{ -c_{\epsilon, m} \left( \frac{d^m(x, y)}{2t} \right)^{\frac{1}{m-1}} \right\}
\]

\[
\leq \left( \frac{2^{1/m} + \frac{d(x, y)}{(2t)^{1/m}}}{V(y, (2t)^{1/m})} \right)^{D/2} \exp \left\{ (2t)^{\kappa/m} + \left( 2^{1/m} + \frac{d(x, y)}{(2t)^{1/m}} \right)^{\kappa} \right\}
\]

\[
\times \exp \left\{ -c_{\epsilon, m} \left( \frac{d^m(x, y)}{2t} \right)^{\frac{1}{m-1}} \right\}
\]

\[
\leq \frac{1}{V(y, (4t)^{1/m})} \exp \left\{ -c_{\kappa, \epsilon, m} \left( \frac{d^m(x, y)}{4t} \right)^{\frac{1}{m-1}} + (4t)^{\frac{\kappa}{m}} \right\},
\]

for some constant \( c_{\kappa, \epsilon, m} > 0 \).

(3) Finally, for any \( t \in (0, \infty) \), there exists a positive integer \( N \) such that \( t/N \in (0, 1] \). By the method of induction, similar as the calculation in steps (1) and (2), we complete the proof. \( \square \)

Now we begin to prove Theorem 3.1. In fact, the main idea of proof is from [3] and [7]. However, we need some key modifications; see e.g. (3.6) and the proof of (3.9) below. Let \( 1_A \) denote the indicator function of a set \( A \).

**Proof of Theorem 3.1.** Let \( \alpha \in \Gamma_{C^\infty_0}(E) \) and \( \sigma > 0 \). There exists a partition of the support of \( \alpha \), denoted by \( (E_n)_{n=1}^N \), such that each \( E_n \) is a bounded domain of diameter no bigger than 1. For each \( n \in \{1, 2, \cdots, N\} \), we take use of the Calderón–Zygmund decomposition (see [7, LEMMA 4.3]) for \( |\alpha|1_{E_n} \), and then patch them together to obtain that

\[
|\alpha| = g + b := g + \sum_i b_i,
\]
where $g$ and $b_i$ are functions on $M$, and find a sequence of balls $B_i = B(x_i, r_i)$ with $x_i \in M$ and $r_i \in (0, 1]$ such that

(a) $0 \leq g(x) \lesssim \sigma$ for $\mu$-a.e. $x \in M$;
(b) each $b_i$ is supported in $B_i$ and $\|b_i\|_1 \lesssim \sigma \mu(B_i)$;
(c) $\sum_i \mu(B_i) \lesssim \|\alpha\|_1/\sigma$;
(d) there exists a positive integer $n$ so that every point of $M$ is contained in at most $n$ balls $B_i$;

see [7, LEMMA 4.3] again. It follows immediately from (b) and (c) that $\|b\|_1 \leq \sum_i \|b_i\|_1 \lesssim \|\alpha\|_1$, and hence $\|g\|_1 \lesssim \|\alpha\|_1$.

Since $R_\alpha = D_m(-L + \lambda)^{-1/2}$, we need to prove that

$$
\mu\{|R_\alpha \alpha| > \sigma\} \lesssim \frac{\|\alpha\|_1}{\sigma}, \text{ for any } \sigma > 0 \text{ and any } \alpha \in \Gamma_{C^\infty}(E).
$$

For any function $h$ defined on $M$, we let $\tilde{h} := h\frac{\sigma}{|\alpha|}1_{\{|\alpha| > 0\}}$. Applying the Calderón–Zygmund decomposition of $|\alpha|$ at the level $\sigma$, we have that

$$
\mu\{|R_\alpha \alpha| > \sigma\} \leq \mu\{|R_\alpha \tilde{g}| > \sigma/2\} + \mu\{|R_\alpha \tilde{b}| > \sigma/2\}.
$$

Since $R_\alpha$ is assumed to be bounded in $\Gamma_{L_2}(E)$ and $0 \leq g \lesssim \sigma$ $\mu$-a.e., we derive that

$$
\mu\{|R_\alpha \tilde{g}| > \sigma/2\} \lesssim \|R_\alpha \tilde{g}\|_2/\sigma^2 \lesssim \|g\|_2^2/\sigma^2 \lesssim \|g\|_1/\sigma \lesssim \|\alpha\|_1/\sigma.
$$

Hence, it remains to prove that

$$
\mu\{|R_\alpha\left(\sum_i \tilde{b}_i\right)| > \sigma/2\} \lesssim \frac{\|\alpha\|_1}{\sigma}.
$$

Let $t_i = r_i^m$. We can write

$$
R_\alpha \tilde{b}_i = R_\alpha e^{-\lambda t_i} P_{2t_i} \tilde{b}_i + R_\alpha (I - e^{-\lambda t_i} P_{2t_i}) \tilde{b}_i.
$$

(3.6)

Note that we introduced an extra term $e^{-\lambda t_i}$ in equation (3.6), which is important for achieving our aim but unfortunately missing in the equation in line -3 on page 117 of [7]. Then, we have

$$
\mu\{|R_\alpha\left(\sum_i \tilde{b}_i\right)| > \sigma/2\} \leq \mu\{|R_\alpha\left(\sum_i e^{-\lambda t_i} P_{2t_i} \tilde{b}_i\right)| > \sigma/4\}
$$

$$
+ \mu\{|R_\alpha\left(\sum_i (I - e^{-\lambda t_i} P_{2t_i}) \tilde{b}_i\right)| > \sigma/4\}.
$$

(3.7)

We start to estimate the first term on the right hand of (3.7). Since $R_\alpha$ is bounded in $\Gamma_{L_2}(E)$, we get

$$
\mu\{|R_\alpha\left(\sum_i e^{-\lambda t_i} P_{2t_i} \tilde{b}_i\right)| > \sigma/4\} \lesssim \frac{1}{\sigma^2} \sum_i e^{-\lambda t_i} P_{2t_i} \tilde{b}_i
$$

$$
\lesssim \frac{1}{\sigma^2} \sum_i e^{-\{a_1 + \lambda\}t} P_{2t_i} \tilde{b}_i
$$

$$
\lesssim \frac{1}{\sigma^2} \sum_i e^{-\{a_1 + \lambda\}t} P_{2t_i} \tilde{b}_i
$$

$$
$$

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where, in the last but one line, we used the semigroup domination property (see e.g. [5])

\[ |P_s \tilde{b}_i| \leq e^{-a_1 s/2} P_{s}^0 |b_i|, \quad \text{for any } s > 0, \]

and in the last line, we used the assumption \( a_1 + \lambda > 0 \). By duality,

\[ \| \sum_i P_{2t_i}^0 |b_i| \|_2 = \sup_{\|f\|_2 = 1} \left| \int_M \sum_i (P_{2t_i}^0 |b_i|) f \, d\mu \right| \]
\[ = \sup_{\|f\|_2 = 1} \left| \sum_i \int_M |b_i| P_{2t_i}^0 f \, d\mu \right| \]
\[ \leq \sup_{\|f\|_2 = 1} \sum_i \|b_i\|_1 \left( \sup_{B_i} P_{2t_i}^0 f \right). \]

We claim that

\[ \sup_{B_i} P_{2t_i}^0 f \lesssim \inf_{B_i} \mathcal{M}(f), \quad (3.8) \]

where \( \mathcal{M} \) is the Hardy–Littlewood maximal operator defined as

\[ \mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{V(x,r)} \int_{B(x,r)} |f(y)| \, d\mu(y). \]

Let \( y \in B_i \). Set \( G_1 = 2^{2/m} B_i \) and \( G_j = 2^{(j+1)/m} B_i \setminus 2^{j/m} B_i \) when \( j = 2, 3, \cdots \). Applying Lemma 3.3, we derive that, for any \( z \in G_j \),

\[ P_{2t_i}^0(y,z) \lesssim \frac{1}{V(y, (2t_i)^{1/m})} e^{-c_1 2^{j-1} + c_2 (2t_i)^{m/m}} \lesssim \frac{1}{V(y, (2t_i)^{1/m})} e^{-c_1 2^{j-1}}, \]

since \( t_i \in (0,1] \), where \( c_1, c_2 > 0 \) are constants from Lemma 3.3. Hence, by \( (II.2) \),

\[ P_{2t_i}^0 f(y) = \int_M P_{2t_i}^0(y,z) f(z) \, d\mu(z) = \sum_{j=1}^{\infty} \int_{G_j} P_{2t_i}^0(y,z) f(z) \, d\mu(z) \]
\[ \lesssim \sum_{j=1}^{\infty} \frac{\mu(2^{j+1} B_i)}{V(y, (2t_i)^{1/m})} \frac{e^{-c_1 2^{j-1}}}{\mu(2^{j+1} B_i)} \int_{2^{j+1} B_i} f(z) \, d\mu(z) \]
\[ \lesssim \sum_{j=1}^{\infty} 2^{(j/m+1)D} e^{2(j/m+1)n} e^{-c_1 2^{j-1}} \left( \inf_{B_i} \mathcal{M}(f) \right) \]
\[ \lesssim \inf_{B_i} \mathcal{M}(f). \]

Thus, if \( \|f\|_2 = 1 \), then by (b) above,

\[ \sum_i \|b_i\|_1 \left( \sup_{B_i} P_{2t_i}^0 f \right) \lesssim \sum_i \sigma \mu(B_i) \inf_{B_i} \mathcal{M}(f) \]
\[ \leq \sigma \sum_i \int_{B_i} \mathcal{M}(f) \, d\mu \leq \sigma \left[ \mu \left( \cup B_i \right) \right]^{1/2} \|\mathcal{M}(f)\|_2 \]
\[ \lesssim (\sigma \|\alpha\|_1)^{1/2}, \]
where we used (c) above and the fact that \( \mathcal{M} \) is bounded in \( L^2(\mathcal{M}, \mu) \). Thus,

\[
\mu \{ |R_\lambda(\sum_i e^{-\lambda t_i} P_{2t_i} \tilde{b}_i)| > \sigma/4 \} \lesssim \frac{\|a\|_1}{\sigma}.
\]

It remains to estimate the second term on the right hand side of (3.7). Obviously,

\[
\mu \{ |R_\lambda(\sum_i (I - e^{-\lambda t_i} P_{2t_i}) \tilde{b}_i)| > \sigma/4 \}
\]

\[
\leq \sum_i \mu(2B_i) + \mu(1_{M \setminus \cup_i 2B_i} \{ |R_\lambda(\sum_i (I - e^{-\lambda t_i} P_{2t_i}) \tilde{b}_i)| > \sigma/4 \})
\]

\[
\leq \sum_i \mu(2B_i) + \frac{4}{\sigma} \sum_i \int_{M \setminus 2B_i} |R_\lambda(I - e^{-\lambda t_i} P_{2t_i}) \tilde{b}_i| \, d\mu,
\]

where, by (II.2) and (c) above,

\[
\sum_i \mu(2B_i) \lesssim \sum_i \mu(B_i) e^{\epsilon r_i} \lesssim \sigma^{-1}\|a\|_1,
\]

since \( r_i \in (0, 1] \). Hence, from (b) and (c), it is sufficient to prove that

\[
\int_{M \setminus 2B_i} |R_\lambda(I - e^{-\lambda t_i} P_{2t_i}) \tilde{b}_i| \, d\mu \lesssim \|b_i\|_1. \quad (3.9)
\]

Since

\[
(-L + \lambda)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\lambda s} P_{2s} \frac{ds}{\sqrt{s}},
\]

we obtain that

\[
(-L + \lambda)^{-1/2}(I - e^{-\lambda t_i} P_{2t_i})
\]

\[
= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\lambda s} P_{2s} \frac{ds}{\sqrt{s}} - \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\lambda(s+t_i)} P_{2(s+t_i)} \frac{ds}{\sqrt{s}}
\]

\[
= \frac{1}{\sqrt{\pi}} \int_0^\infty \left( \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s-t_i}} \right) e^{-\lambda s} P_{2s} \, ds.
\]

Hence, applying Theorem 2.1, we immediately deduce that

\[
\frac{|R_\lambda(I - e^{-\lambda t_i} P_{2t_i}) \tilde{b}_i|}{\sigma} \lesssim \int_0^\infty \left( \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s-t_i}} \right) e^{-\lambda s} \|P_s \tilde{b}_i\|_\infty \|P_0^0 \tilde{b}_i\|_\infty \, ds,
\]

where

\[
C(s) := \frac{a[C(\omega) + C(\vartheta) \sqrt{s}/2]^2}{1 - e^{-as}}
\]

with \( a = \max\{a_2 - a_3, 0\} \). By the semigroup domination property again,

\[
P_s^0 \tilde{b}_i \leq e^{-a_1 s/2} P_{2s} \tilde{b}_i,
\]

and by Lemma 3.3, we have that

\[
|P_s \tilde{b}_i(x)| \leq e^{-a_1 s/2} \int_{B_i} p_0^0(x, y) |\tilde{b}_i|(y) \, d\mu(y)
\]
Lemma 3.2 implies that

\[ |R_{\lambda}(I - e^{-\lambda t} P_{2t}) b_i| \leq \int_0^\infty \left| \frac{1}{\sqrt{s}} - \frac{1_{(s>t_1)}}{\sqrt{s-t_1}} \right| e^{-(\lambda + a_1)s} \left( C(s) \|b_i\|_1 P_{2s}^0 \|b_i\|_1 \sup_{y \in B_i} \frac{e^{c_2 s^{\kappa/m}}}{V(y, s^{1/m})} \right)^{1/2} ds. \] (3.10)

Let \( \eta = c_1/2 \). By (b) above, the Cauchy–Schwarz inequality, Lemma 3.3 and Lemma 3.2, we have

\[ \int_{M \setminus 2B_i} (P_{2s}^0 b_i(x))^{1/2} d\mu(x) = \int_{M \setminus 2B_i} \left( \int_{B_i} P_{2s}^0(x, y) b_i(y) d\mu(y) \right)^{1/2} d\mu(x) \]
\[ \leq \int_{M \setminus 2B_i} \left( \sup_{y \in B_i} e^{-\eta \left( \frac{d(y, x)}{2s} \right)^{m-1}} \right)^{1/2} \left( \int_{B_i} e^{\eta \left( \frac{d(y, x)}{2s} \right)^{m-1}} P_{2s}^0(x, y) b_i(y) d\mu(y) \right)^{1/2} d\mu(x) \]
\[ \leq \left( \int_{M \setminus 2B_i} \int_{B_i} e^{\eta \left( \frac{d(y, x)}{2s} \right)^{m-1}} P_{2s}^0(x, y) b_i(y) d\mu(y) d\mu(x) \right)^{1/2} \]
\[ \times \left( \sup_{y \in B_i} \int_{M \setminus 2B_i} e^{-\eta \left( \frac{d(y, x)}{2s} \right)^{m-1}} d\mu(x) \right)^{1/2} \]
\[ =: J_1 \times J_2. \]

Lemma 3.2 implies that

\[ J_2 \lesssim e^{-\frac{1}{8} \left( \frac{2s}{m} \right)^m} \sup_{y \in B_i} \left[ V\left(y, \left(2s^{1/m}\right)\right) \right]^{1/2}. \]

Lemma 3.2 and Lemma 3.3 imply that

\[ J_1 \lesssim \left( \int_{M \setminus 2B_i} e^{c_2 (2s)^{\kappa/m}} |b_i(y)| d\mu(x) \int_{B_i} e^{-c_1 - \eta \left( \frac{d(y, x)}{2s} \right)^{m-1}} d\mu(y) \right)^{1/2} \]
\[ \lesssim (e^{c_2 (2s)^{\kappa/m}} \|b_i\|_1)^{1/2}. \]

Hence,

\[ \int_{M \setminus 2B_i} (P_{2s}^0 b_i(x))^{1/2} d\mu(x) \]
\[ \lesssim \exp \left[ - \frac{c_1}{8} \left( \frac{2s^{1/m}}{2s} \right)^m + \frac{c_2}{2} (2s)^{\frac{\kappa}{m}} \right] \left[ \|b_i\|_1 \sup_{y \in B_i} V\left(y, \left(2s^{1/m}\right)\right) \right]^{1/2}. \] (3.11)

Combining (3.9), (3.10) and (3.11), we arrive at

\[ \int_{M \setminus 2B_i} |R_{\lambda}(I - e^{-\lambda t} P_{2t}) b_i| d\mu \]
\[ \lesssim \|b_i\|_1 \int_0^\infty \left| \frac{1}{\sqrt{s}} - \frac{1_{(s>t_1)}}{\sqrt{s-t_1}} \right| h(s) \left( \sup_{x, y \in B_i} \frac{V(x, \left(2s^{1/m}\right))}{V(y, s^{1/m})} \right)^{1/2}, \]

where

\[ h(s) = \sqrt{C(s)} \exp \left[ 2c_2 s^{\frac{\kappa}{m}} - \frac{c_1}{4} \left( \frac{t_1}{s} \right)^{\frac{1}{m}} - (a_1 + \lambda)s \right]. \]

By (II.2), for any \( x, y \in B_i = B(x, t_i^{1/m}) \),

\[ V\left(x, \left(2s^{1/m}\right)\right) \leq V\left(y, 2t_i^{1/m} + \left(2s^{1/m}\right)\right) \]

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\[
\begin{align*}
&\lesssim \left( \frac{2^{1/m} + (2s)^{1/m}}{s^{1/m}} \right)^{D} V(y, s^{1/m}) \exp \left[ \left( \frac{2^{1/m} + (2s)^{1/m}}{s^{1/m}} \right)^{\kappa} + s^{\kappa/m} \right] \\
&\leq C_{\varepsilon, D} V(y, s^{1/m}) \exp \left[ \varepsilon \left( \frac{t_{1}}{s} \right)^{\frac{1}{m-1}} + s^{\kappa/m} \right],
\end{align*}
\]
for any \( \varepsilon > 0 \), where \( C_{\varepsilon, D} \) is a positive constant. Hence, for any \( \varepsilon > 0 \),
\[
\int_{M \setminus 2B_i} |R_{\lambda} (I - e^{-\lambda t_{i}} P_{2t_{i}}) \tilde{b}_i| \, d\mu \\
\lesssim \|b_i\|_1 \left( \int_{0}^{\infty} \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s - t_{i}}} \left| k(s) \right| \, ds \right) \\
= \|b_i\|_1 \left( \int_{t_{i}}^{\infty} \frac{k(s)}{\sqrt{s}} \, ds + \int_{t_{i}}^{\infty} \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s - t_{i}}} \left| k(s) \right| \, ds \right) \\
=: \|b_i\|_1 (K_1 + K_2),
\]
where
\[
k(s) = \sqrt{C(s)} \exp \left[ - \frac{(c_1 - 2\varepsilon)}{4} \left( \frac{t_{1}}{s} \right)^{\frac{1}{m-1}} - (a_1 + \lambda)s + (1 + 2c_2)s^{\frac{\kappa}{m}} \right].
\]
It is straightforward to check that,
\[K_1 \lesssim \int_{0}^{t_{i}} \frac{1}{s} \exp \left[ - \frac{(c_1 - 2\varepsilon)}{4} \left( \frac{t_{1}}{s} \right)^{\frac{1}{m-1}} \right] \, ds,
\]
which is finite for any \( \varepsilon \in (0, c_1/2) \), and since \( a_1 + \lambda > 0 \) and \( \kappa < \frac{m}{m-1} \),
\[K_2 \lesssim \int_{t_{i}}^{\infty} \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s - t_{i}}} \left[ 1 + \sqrt{s} \right] \exp \left[ - (a_1 + \lambda)s + (1 + 2c_2)s^{\frac{\kappa}{m}} \right] \, ds < \infty.
\]
Thus, (3.9) is proved, and hence (1) is proved.

Now let \( \vartheta = 0 \). We need to prove \( R_0 = D_u (-L)^{-1/2} \) is weak \((1,1)\) bounded. From the above study, the only difference is the estimation of \( K_2 \). If either \( a_1 > 0 \) or \( a_1 = \kappa = 0 \), then
\[K_2 \lesssim \int_{t_{i}}^{\infty} \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s - t_{i}}} \left[ 1 + \sqrt{s} \right] \, ds \]
\[= \int_{0}^{\infty} \frac{1}{\sqrt{u}} - \frac{1}{\sqrt{u + 1}} \left( \frac{1}{\sqrt{u + 1}} + \sqrt{t_{i}} \right) \, ds < \infty.
\]
Thus, (2) is proved.

Therefore, we complete the proof of Theorem 3.1.

Finally, we should mention that some extensions as in [6] of our main results are possible. However, we leave these for interested readers.

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