Littlewood’s Algorithm and Quaternion Matrices

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Abstract

A strengthened form of Schur’s triangularization theorem is given for quaternion matrices with real spectrum (for complex matrices it was given by Littlewood). It is used to classify projectors ($A^2 = A$) and self-annihilating operators ($A^2 = 0$) on a quaternion unitary space and examples of unitarily wild systems of operators on such a space are presented. Littlewood’s algorithm for reducing a complex matrix to a canonical form under unitary similarity is extended to quaternion matrices whose eigenvalues have geometric multiplicity 1.

1
1 Introduction and Definitions

We denote the set of $m$-by-$n$ matrices by $M_{m,n}(\mathbb{F})$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$, the skew field of real quaternions with involution

$$a + bi + cj + dk = a - bi - cj - dk, \quad a, b, c, d \in \mathbb{R},$$

and write $M_n \equiv M_{n,n}$; $A^*$ denotes the conjugate transpose; the $n$-by-$n$ upper triangular Jordan block with eigenvalue $\lambda$ is denoted by $J_n(\lambda)$.

A matrix $U \in M_n(\mathbb{F})$ is called unitary if $U^*U = I$. Two matrices $A$ and $B \in M_n(\mathbb{F})$ are unitarily similar (over $\mathbb{F}$) if there exists a unitary $U$ such that $A = U^*BU$; they are called unitarily equivalent if there exist unitary $U$ and $V$ such that $A = UV^*V$.

Let $A$ be a quaternion $n$-by-$n$ matrix; $\lambda \in \mathbb{H}$ is a (right) eigenvalue of $A$ if there exists a nonzero $v \in \mathbb{H}^n$ such that $Av = v\lambda$. The eigenvalues are defined only up to similarity: $Avh = vh \cdot h^{-1}\lambda h$ for each nonzero $h \in \mathbb{H}$, so $h^{-1}\lambda h$ is an eigenvalue of $A$ whenever $\lambda$ is. Every eigenvalue $\lambda = a + bi + cj + dk$ is similar to exactly one complex number with nonnegative imaginary part, namely $a + \sqrt{b^2 + c^2 + d^2}i$ [15, Lemma 2.1]; this complex number is called a standard eigenvalue of $A$. There exists a nonsingular $S \in M_n(\mathbb{H})$ such that $S^{-1}AS$ is a Jordan matrix

$$J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k), \quad \lambda_j = a_j + b_ji \in \mathbb{C}, \quad b_j \geq 0,$$

with standard eigenvalues, determined up to permutation of Jordan blocks [6, Chapter 3]. We will assume that $\lambda_1 \geq \cdots \geq \lambda_k$ with respect to the following ordering in $\mathbb{C}$:

$$a + bi \succeq c + di \quad \text{if either} \quad a \geq c \quad \text{and} \quad b = d, \quad \text{or} \quad b > d. \quad (2)$$

Performing the Gram-Schmidt orthogonalization on the columns of $S$ gives a unitary matrix $U = ST$, where $T$ is an upper triangular matrix with positive diagonal elements; this is the QR decomposition of $S$. Therefore, $A$ is unitarily similar to an upper triangular matrix $U^*AU = T^{-1}JT$ having the same diagonal as $J$ (Schur’s theorem for quaternion matrices). For a survey of quaternions and matrices of quaternions, see [15]. Canonical matrices of sesquilinear forms and pairs of hermitian forms on a quaternion vector space, and selfadjoint and isometric operators on a quaternion vector space with indefinite scalar product, are given in [13].
This article is a result of attempts of the authors to extend quaternion matrices Littlewood’s algorithm [9] for reducing a complex matrix to a canonical form under unitary similarity. This algorithm was discussed in [2] and [12]; see also [14] and the survey [11]. Littlewood’s algorithm is based on two statements:

(A) Strengthened Schur Theorem. Each square complex matrix $A$ is unitarily similar to an upper triangular matrix of the form

$$
F = \begin{bmatrix}
\lambda_1 I_{n_1} & F_{12} & F_{13} & \cdots & F_{1s} \\
0 & \lambda_2 I_{n_2} & F_{23} & \cdots & F_{2s} \\
0 & 0 & \lambda_3 I_{n_3} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & F_{s-1,s} \\
0 & 0 & 0 & \cdots & \lambda_s I_{n_s}
\end{bmatrix}, \tag{3}
$$

where $\lambda_1 \geq \cdots \geq \lambda_s$ and if $\lambda_i = \lambda_{i+1}$ then the columns of $F_{i,i+1}$ are linearly independent; subject to the foregoing conditions, the diagonal blocks $\lambda_i I_{n_i}$ are uniquely determined by $A$. If $F'$ is any other upper triangular matrix that is unitarily similar to $A$ and satisfies the foregoing conditions, then $F' = V^* F V$, where $V$ is complex unitary and $V = V_1 \oplus \cdots \oplus V_s$, where each $V_i \in M_{n_i}(\mathbb{C})$.

(B) Singular value decomposition. Each nonzero complex matrix $A$ is unitarily equivalent to a nonnegative diagonal matrix of the form

$$
D = a_1 I_{n_1} \oplus \cdots \oplus a_{t-1} I_{n_{t-1}} \oplus 0, \quad a_i \in \mathbb{R}, \quad a_1 > \cdots > a_{t-1} > 0. \tag{4}
$$

If $U^* DV = D$, where $U$ and $V$ are complex unitary matrices, then $U = U_1 \oplus \cdots \oplus U_{t-1} \oplus U'$, $V = U_1 \oplus \cdots \oplus U_{t-1} \oplus V'$, and each $U_i \in M_{n_i}$.

Littlewood’s algorithm. Let $A \in M_n(\mathbb{C})$. Littlewood’s algorithm has the following steps. The first step is to reduce $A$ to the form (3); notice that the diagonal blocks and all sub-diagonal blocks of $F$ have been completely reduced. We restrict the set of unitary similarities to those that preserve the

\[\begin{aligned}
\text{This formulation is not go with the algorithm: we reduce a matrix to the form (3), restrict the set of admissible transformations to those that preserve all diagonal and sub-diagonal blocks, then the preserving them matrices have the block-diagonal form. I propose the following version: “where $\lambda_1 \geq \cdots \geq \lambda_s$ and if $\lambda_i = \lambda_{i+1}$ then the columns of $F_{i,i+1}$ are linearly independent. The diagonal blocks $\lambda_i I_{n_i}$ are uniquely determined by $A$. If $V^{-1} F V = F'$, where $V$ is complex unitary and $F'$ differs from $F$ only in over-diagonal blocks, then $V = V_1 \oplus \cdots \oplus V_s$, where each $V_i$ is $n_i \times n_i$.”}
\end{aligned}\]
block structure of (3), \( s \) that is, to the transformations

\[
F \mapsto V^* F V, \quad V = V_1 \oplus \cdots \oplus V_s.
\]  

(5)

The second step is to take the first nonzero superdiagonal block \( F_{ij} \) with respect to lexicographically ordered indices, and reduce it to the form (4) by unitary equivalence \( F_{ij} \mapsto V_i^* F_{ij} V_j \). We make an additional partition of \( F \) into blocks conformal with the partition of the obtained block \( F'_{ij} = D \), and restrict the set of admissible transformations (5) to those that preserve \( D \) (i.e., \( V_i = U_1 \oplus \cdots \oplus U_{t-1} \oplus U' \) and \( V_j = U_1 \oplus \cdots \oplus U_{t-1} \oplus V' \)). The \( i \)th step of the algorithm is to take the first block that changes under admissible transformations and reduce it by unitary similarity or equivalence to the form (3) or (4). We restrict the set of admissible transformations to those that preserve the reduced part and make additional block partitions conformal to the block that has just been reduced. Since we have finitely many blocks, the process ends on a certain matrix \( A^\infty \) with the property that \( A \) is unitarily similar to \( B^\infty \) if and only if \( A^\infty = B^\infty \). The matrix \( A^\infty \) is called the canonical form of \( A \) with respect to unitary similarity.

Statement (B) holds for all quaternion matrices, that is, if \( A \) is an \( m \)-by-\( n \) quaternion matrix, then there exist an \( m \)-by-\( m \) unitary matrix \( U \), an \( n \)-by-\( n \) unitary matrix \( V \), and a diagonal matrix \( \Sigma \) with nonnegative entries such that \( A = U \Sigma V \) [15, Theorem 7.2].

In Section 2 we prove statement (A) for quaternion matrices with real spectrum; it need not hold for quaternion matrices with nonreal eigenvalues. The proof is based on modified Jordan matrices [1], which we associate with the Weyr characteristic [10, p. 73] of a matrix.

In Section 3 we show that Littlewood’s algorithm can be applied to quaternion matrices with real spectrum that reduce to the form (3) with \( s = 2 \); Littlewood’s process then consists of at most two steps. This two-step Littlewood’s process can be used to obtain the canonical forms of projectors \( (\mathcal{A}^2 = \mathcal{A}) \) and self-annihilating operators \( (\mathcal{A}^2 = 0) \) on a quaternion unitary space. A canonical form of a complex projector was given by Dokovic [3] and Ikramov [5]; Ikramov’s proof is based on Littlewood’s algorithm.

Unfortunately, Littlewood’s algorithm cannot always be applied to quaternion matrices with real spectrum that reduce to the form (3) with \( s \geq 3 \). The reason is that in the process of reduction, one can meet a block

\[ \text{‘Block structure’ is not clear, the transformation must preserve all diagonal and sub-diagonal blocks.} ]
with nonreal eigenvalues. The problem of classifying such matrices has the same complexity as the problem of classifying all quaternion matrices up to unitary similarity since two quaternion matrices

\[
M_A = \begin{bmatrix}
3I_n & I_n & A \\
0 & 2I_n & I_n \\
0 & 0 & I_n
\end{bmatrix}
\quad \text{and} \quad
M_B = \begin{bmatrix}
3I_n & I_n & B \\
0 & 2I_n & I_n \\
0 & 0 & I_n
\end{bmatrix}
\]

are unitarily similar if and only if \(A\) and \(B\) are unitarily similar. That is, if \(V\) is unitary and \(V^*M_AV = M_B\), then \(V = V_1 \oplus V_2 \oplus V_3\) by statement (A) for quaternion matrices with real spectrum; furthermore, \(V_1 = V_2 = V_3\) and \(V_1^*AV_1 = B\).

Moreover, the problem of classifying quaternion matrices up to unitary similarity (and even the problem of classifying, up to unitary similarity, quaternion matrices with Jordan Canonical Form \(I \oplus 2I \oplus \cdots \oplus tI\)) has the same complexity as the problem of classifying an arbitrary system of linear mappings on quaternion unitary spaces. For example, the problem of classifying systems of four linear mappings

\[
(U, V, W)
\]

\((U, V, W)\) are arbitrary quaternion unitary spaces) is the canonical form problem for quaternion matrices of the form

\[
M(A, B, C, D) = \begin{bmatrix}
5I & I & A & C & B \\
0 & 4I & I & 0 & 0 \\
0 & 0 & 3I & 0 & 0 \\
0 & 0 & 0 & 2I & D \\
0 & 0 & 0 & 0 & I
\end{bmatrix}
\]

under unitary similarity. Indeed, by statement (A) for quaternion matrices with real spectrum, if \(V\) is unitary then \(V^*M(A, B, C, D)V = M(A', B', C', D')\) implies \(V = V_1 \oplus \cdots \oplus V_5\). It also follows that \(V_1 = V_2 = V_3\), hence \((A, B, C, D)\) and \((A', B', C', D')\) are the matrices of the same system of linear mappings \((A, B, C, D)\) in different orthogonal bases of \(U, V, W\); compare with [14 Sect. 2.3].
In particular, the problem of classifying quaternion matrices up to unitary similarity is equivalent to the problem of classifying \( m \)-tuples of quaternion matrices up to simultaneous unitary similarity

\[
(A_1, \ldots, A_m) \mapsto (V^{-1}A_1V, \ldots, V^{-1}A_mV).
\]

The case for \( m \)-tuples of complex matrices was proved in [7]. Other examples of classification problems that have the same complexity as classifying arbitrary systems of linear operators on unitary spaces are given in Section 3.

In Section 4 we prove statement (A) for nonderogatory quaternion matrices—those matrices all of whose eigenvalues have geometric multiplicity 1 [4, Section 1.4.4]. We then extend Littlewood’s algorithm to such matrices. We also study the structure of their canonical matrices.

## 2 A Strengthened Schur Theorem for Quaternion Matrices with Real Spectrum

In this section we prove the following theorem.

**Theorem 2.1.** Let \( A \) be a given square quaternion matrix and suppose that \( A \) has only real eigenvalues.

(a) Then there exists a quaternion unitary matrix \( U \) such that \( F \equiv U^*AU \) has the form (3), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \) are real numbers; when \( \lambda_i = \lambda_{i+1} \), then \( n_i \geq n_{i+1} \), and \( F_{i,i+1} \) is an upper triangular matrix whose diagonal entries are positive real numbers.

(b) The diagonal blocks \( F_{ii} = \lambda_i I_{n_i} \) are uniquely determined. The off-diagonal blocks \( F_{ij} \) are determined up to the following equivalence. If \( V \) is a quaternion unitary matrix, then \( F' \equiv V^*FV \) has the form (3) with \( F'_{ii} = F_{ii} \) (and without conditions on \( F'_{i,i+1} \)) if and only if \( V \) has the form

\[
V = V_1 \oplus V_2 \oplus \cdots \oplus V_s,
\]

where each \( V_i \) has size \( n_i \)-by-\( n_i \).

The matrix (3) is a unitary variant of a modified Jordan matrix, which was proposed by Belitskiǐ [1] and is obtained from the Jordan matrix by a simultaneous permutation of rows and columns. We define it through the Weyr characteristic of a matrix.
A list of positive integers \((m_1, m_2, \ldots, m_k)\) is said to be \textit{decreasingly ordered} if \(m_1 \geq m_2 \geq \cdots \geq m_k\). Given a decreasing list \((m_1, m_2, \ldots, m_k)\), its \textit{conjugate} is the decreasingly ordered list \((r_1, r_2, \ldots, r_s)\) in which \(s = m_1\) and \(r_i\) is the number of \(m_j\)'s larger than or equal to \(i\).

The Jordan Canonical Form \(J_{m_1}(0) \oplus J_{m_2}(0) \oplus \cdots \oplus J_{m_k}(0)\) of a nilpotent matrix \(A\) can be arranged so that the sizes of its Jordan blocks form a decreasingly ordered list \(m \equiv (m_1, m_2, \ldots, m_k)\), which is called the \textit{Segre characteristic} of \(A\); its conjugate \(r \equiv (r_1, r_2, \ldots, r_s)\) is called the \textit{Weyl characteristic} of \(A\) \[10, p. 73\]. Notice that \(\text{rank}(A^l) = r_{l+1} + \cdots + r_s\) for \(1 \leq l < s\).

\section*{Lemma 2.1}

Let \(A \equiv J_{m_1}(0) \oplus J_{m_2}(0) \oplus \cdots \oplus J_{m_k}(0)\) be given, and suppose that \(m_1 \geq m_2 \geq \cdots \geq m_k\). Let \((r_1, r_2, \ldots, r_s)\) be the conjugate of \((m_1, m_2, \ldots, m_k)\). Then \(A\) is similar to

\[
B \equiv \begin{bmatrix}
0_{r_1} & G_{12} & 0 & \cdots & 0 \\
0 & 0_{r_2} & G_{23} & \cdots & 0 \\
0 & 0 & 0_{r_3} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & G_{s-1,s} \\
0 & 0 & 0 & \cdots & 0_{r_s}
\end{bmatrix}
\]

where \(G_{i,i+1} \equiv \begin{bmatrix} I_{r_{i+1}} \\ 0 \end{bmatrix}\) is \(r_i\)-by-\(r_{i+1}\).

\section*{Proof}

Notice that

\[
\text{rank}(G_{i,i+1}G_{i+1,i+2} \cdots G_{i+t,i+t+1}) = \text{rank}(G_{i+t,i+t+1}) = r_{i+t+1}.
\]

One checks that \(\text{rank}(A^l) = \text{rank}(B^l)\) for all \(l\). It follows that \(A\) is similar to \(B\). \qed

\section*{Remark 2.1}

The two matrices \(A\) and \(B\) in Lemma 2.1 are permutation similar. To get \(B\) from \(A\), permute the first columns of \(J_{m_1}(0)\), \(J_{m_2}(0)\), \ldots, and \(J_{m_k}(0)\) into the first \(k\) columns, then permute the corresponding rows. Next permute the second columns into the next columns and permute the corresponding rows; continue the process until \(B\) is achieved.

Let \(A \in M_n(\mathbb{H})\) be given, and let \(J(A)\) be its Jordan Canonical Form \[11\]. A repeated application of Lemma 2.1 to the nilpotent part of \(J(A) - \lambda_jI\) for each of the distinct eigenvalues \(\lambda_j\) gives the following.
Lemma 2.2. Let $A \in M_n(\mathbb{H})$ be given. Then $A$ is similar to a unique matrix of the form

$$B \equiv \begin{bmatrix}
\lambda_1 I_{n_1} & G_{12} & 0 & \cdots & 0 \\
0 & \lambda_2 I_{n_2} & G_{23} & \cdots & 0 \\
0 & 0 & \lambda_3 I_{n_3} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & G_{s-1,s} \\
0 & 0 & 0 & \cdots & \lambda_s I_{n_s}
\end{bmatrix}$$

with $\lambda_1 \geq \cdots \geq \lambda_s$. If $\lambda_i \neq \lambda_{i+1}$, then $G_{i,i+1} = 0$; otherwise, $n_i \geq n_{i+1}$ and $G_{i,i+1} \equiv \begin{bmatrix} I_{n_{i+1}} & 0 \end{bmatrix}$ is $n_i$-by-$n_{i+1}$.

Belitskii [1] called the matrix (6) a modified Jordan matrix and proved that all matrices commuting with $B$ have an upper block-triangular form; this fact plays a central role in his algorithm for reducing $m$-tuples of complex matrices to a canonical form by simultaneous similarity.

Proof of Theorem 2.1. (a) Let $A \in M_n(\mathbb{H})$ be given and suppose that $A$ has only real eigenvalues, say $\lambda_1 > \cdots > \lambda_s$. Lemma 2.2 guarantees that $S^{-1}AS = B$ for some nonsingular matrix $S$, and $B$ has the form (6). Perform a Gram-Schmidt orthogonalization on the columns of $S$ so that $U = ST$ is unitary and $T$ is an upper triangular matrix with positive diagonal elements. $T^{-1}$ is necessarily upper triangular, and its diagonal elements are also positive.

Write

$$T^{-1} = \begin{bmatrix}
C_1 & C_{12} & C_{13} & \cdots & C_{1s} \\
0 & C_2 & C_{23} & \cdots & C_{2s} \\
0 & 0 & C_3 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & C_{s-1,s} \\
0 & 0 & 0 & \cdots & C_s
\end{bmatrix}$$

and

$$T = \begin{bmatrix}
D_1 & D_{12} & D_{13} & \cdots & D_{1s} \\
0 & D_2 & D_{23} & \cdots & D_{2s} \\
0 & 0 & D_3 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & D_{s-1,s} \\
0 & 0 & 0 & \cdots & D_s
\end{bmatrix}$$
conformal to $B$. Since each $\lambda_i$ is real, direct computation of the product $U^*AU = T^{-1}BT$ shows that

$$U^*AU = T^{-1}BT = \begin{bmatrix}
\lambda_1 I_{n_1} & F_{12} & F_{13} & \cdots & F_{1s} \\
0 & \lambda_2 I_{n_2} & F_{23} & \cdots & F_{2s} \\
0 & 0 & \lambda_3 I_{n_3} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & F_{s-1,s} \\
0 & 0 & 0 & \cdots & \lambda_s I_{n_s}
\end{bmatrix}. $$

Since all the eigenvalues are real, the off-diagonal blocks $F_{i,i+1}$ satisfy

$$F_{i,i+1} = \lambda_i C_i D_{i,i+1} + C_i G_{i,i+1} D_{i+1} + \lambda_{i+1} C_{i+1} D_{i+1}. $$

Since $T^{-1}T = I$, we have

$$C_i D_{i,i+1} + C_{i+1} D_{i+1} = 0. $$

Hence, when $\lambda_i = \lambda_{i+1}$,

$$F_{i,i+1} = C_i G_{i,i+1} D_{i+1}. $$

If $\lambda_i = \lambda_{i+1}$, Lemma 2.2 guarantees that $n_i \geq n_{i+1}$. Moreover, the form of $G_{i,i+1}$ shows that $F_{i,i+1}$ is an upper triangular matrix whose diagonal entries are positive real numbers.

(b) We now prove the uniqueness part. That the eigenvalues of $F$ and their multiplicity are determined is clear. The sizes $n_i$ are also determined by looking at powers of $F - \lambda_i I$. We can also look at a decreasingly ordered list of the sizes of Jordan blocks corresponding to $\lambda_i$ and notice that the conjugate of this list gives us the sizes needed.

Let $V$ be unitary. Suppose that $F' \equiv V^* F V$ has the form (3) and suppose further that $F'_{ii} = F_{ii}$. We claim that $V$ is block diagonal conformal to $F$.

Write $V = [V_{ij}]$ conformal to $F$ (and $F'$). Form the products $F V = V F'$. Suppose that $\lambda_s \neq \lambda_1$. The $(s,1)$ block satisfies the equation

$$\lambda_s V_{s1} = \lambda_1 V_{s1}. $$

Hence, $V_{s1} = 0$. If $\lambda_{s-1} \neq \lambda_1$, we look at the $(s-1,1)$ block and conclude that $V_{s-1,1} = 0$. We proceed until $\lambda_j = \lambda_1$.

Now, we check if $\lambda_s \neq \lambda_2$. If so, then we look at the $(s,2)$ block, and proceed as before.
We conclude that $V$ is block upper triangular, but since $V$ is unitary, $V$ is block diagonal. Hence, it suffices to prove the claim when all the eigenvalues are the same, say $\lambda$.

As before, we write $V = [V_{ij}]$ conformal to $F$, and look at the equation $FV = VF'$.

The $(s,1)$ block satisfies $\lambda V_{s1} = \lambda V_{s1}$. However, the $(s-1,1)$ block satisfies the equation

$$\lambda V_{s-1,1} + F_{s-1,s} V_{s1} = \lambda V_{s-1,1}.$$

Hence, $F_{s-1,s} V_{s1} = 0$. Since $F_{s-1,s}$ is upper triangular with positive diagonal entries, $V_{s1} = 0$.

Next, we look at the $(s-2,1)$ block to get

$$\lambda V_{s-2,1} + F_{s-2,s-1} V_{s-1,1} = \lambda V_{s-2,1},$$

and similarly, we conclude that $V_{s-1,1} = 0$. Notice that the same argument can be used to reach the conclusion that $V_{i,1} = 0$ for all $i = 2, \ldots, s$.

We then look at the $(s,2)$ block, $(s-1,2)$ block, and so on. The conclusion is that $V$ is block upper triangular. Since $V$ is also unitary, $V$ is in fact block diagonal and the sizes of the blocks in $V$ match those of $F$.

### 3 Applications of the Strengthened Schur Theorem

A square matrix $A$ is called a projection or idempotent if $A^2 = A$; it is called self-annihilating if $A^2 = 0$. A canonical form of a complex idempotent matrix under unitary similarity was obtained in [3, 5], see also Section 2.3 of [14].

**Theorem 3.1.** (a) Let $A$ be a quaternion idempotent matrix ($A^2 = A$). Then $A$ is unitarily similar to a direct sum that is uniquely determined up to permutation of summands of matrices of the form

$$\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} \text{ (b positive), } [1], [0].$$

(b) Let $A$ be a self-annihilating quaternion matrix ($A^2 = 0$). Then $A$ is unitarily similar to a direct sum that is uniquely determined up to permutation of summands of matrices of the form

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \text{ (b positive), } [0].$$
Proof. (a) Let $A$ be a quaternion idempotent matrix. Theorem 2.1(a) ensures that $A$ is similar to a matrix $F$ that has the form (3). Since $A^2 = A$, we also have $F^2 = F$. Hence, $F$ must have the form

$$F = UAU^* = \begin{bmatrix} I & F_{12} \\ 0 & 0 \end{bmatrix}.$$  

By Theorem 2.1(b), $F_{12}$ is determined up to unitary equivalence. Now let $F_{12} = V_1 \Sigma V_2^*$ be the singular value decomposition of $F_{12}$, where $V_1$ and $V_2$ are (quaternion) unitary matrices, $\Sigma = \text{diag}(b_1, \ldots, b_l) \oplus 0$, and $b_1 \geq \cdots \geq b_l > 0$. Take $V = V_1 \oplus V_2$ and notice that

$$V^* F V = \begin{bmatrix} I & \Sigma \\ 0 & 0 \end{bmatrix}.$$  

(7)

The conclusion follows by noting that the block matrix (7) is permutation similar to a matrix that is a sum of the desired matrices.

(b) The proof is similar to that of (a) except that $A^2 = 0$ means that

$$F = UAU^* = \begin{bmatrix} 0 & F_{12} \\ 0 & 0 \end{bmatrix},$$

and $\Sigma = V_1^* F_{12} V_2$ has no zero columns. \qed

A self-annihilating quaternion matrix has the Jordan Canonical Form $J_2(0) \oplus \cdots \oplus J_2(0) \oplus 0$ and a simple canonical form under unitary similarity, as was shown in Theorem 3.1(b). What about quaternion matrices $A$ with the Jordan Canonical Form $J_2(\lambda) \oplus \cdots \oplus J_2(\lambda) \oplus \lambda I_k$? If $\lambda$ is real, then $(A - \lambda I)^2 = 0$, and hence $A$ is unitarily similar to a direct sum of matrices of the form

$$\begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix} \text{ (} b \text{ positive), and } [\lambda].$$

However, when $\lambda \notin \mathbb{R}$, notice that $(A - \lambda I)^2$ need not equal 0. Part (a) of the next theorem shows that the class of such matrices is unitarily wild, that is, it contains the problem of classifying square complex matrices up to (complex) unitary similarity and hence (see Section 1) it has the same complexity as the problem of classifying arbitrary systems of linear mappings on (complex) unitary spaces. Parts (b)–(d) for complex matrices were given in [8, 14].
Theorem 3.2. The problem of classifying each of the following classes of matrices and pairs of matrices under unitary similarity is unitarily wild:

(a) square quaternion matrices whose Jordan Canonical Form consists only of Jordan blocks $J_2(\lambda)$ and $[\lambda]$, where $\lambda \notin \mathbb{R}$ is the same for all the matrices in the class;

(b) square quaternion matrices $A$ satisfying $A^3 = 0$;

(c) pairs of quaternion idempotent matrices $(A, B)$, even if $A$ is self-adjoint, that is $A^2 = A^* = A$ and $B^2 = B$;

(d) pairs of mutually- and self-annihilating quaternion matrices $(A, B)$, that is $AB = BA = A^2 = B^2 = 0$.

Proof. (a) Let $\lambda \notin \mathbb{R}$ be a given eigenvalue, which we may assume is standard, so $\lambda = x + yi$ with $y > 0$. To prove (a), we exhibit a mapping $M \mapsto A_M \in M_{8n}(\mathbb{H})$ such that $M, N \in M_n(\mathbb{C})$ are (complex) unitarily similar if and only if $A_M$ and $A_N$ are (quaternion) unitarily similar.

For such a given $\lambda$ and $M \in M_n(\mathbb{C})$, we define

$$X_M \equiv \begin{bmatrix} 4I_n & 0 & I_nj & Mj \\ 0 & 3I_n & I_nj & I_nj \\ 0 & 0 & 2I_n & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix} \in M_{4n}(\mathbb{H})$$

and

$$A_M \equiv \begin{bmatrix} \lambda I_{4n} & X_M \\ 0 & \lambda I_{4n} \end{bmatrix} \in M_{8n}(\mathbb{H}).$$

Notice that $A_M$ is similar to a direct sum of Jordan blocks $J_2(\lambda)$.

Now, suppose that $M$ is unitarily similar to $N$, say $U^* M U = N$ for some unitary $U \in M_n(\mathbb{C})$. Let $V \equiv U \oplus U \oplus U \oplus U$ and notice that

$$(V \oplus V)^* A_M (V \oplus V) = A_N$$

since $jU = Uj$. Hence, $A_M$ is unitarily similar to $A_N$.

Conversely, suppose that $A_M$ is (quaternion) unitarily similar to $A_N$, that is $V^* A_M V = A_N$ for some (quaternion) unitary matrix $V$. We claim that $M$ and $N$ are (complex) unitarily similar. Partition the unitary matrix $V$ conformal to $A_M$, and rewrite the given condition to get

$$\begin{bmatrix} \lambda I_{4n} & X_M \\ 0 & \lambda I_{4n} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \lambda I_{4n} & X_N \\ 0 & \lambda I_{4n} \end{bmatrix},$$

which yields the following equalities:
(i) $\lambda V_{11} + X_M V_{21} = V_{11}\lambda$, 

(ii) $\lambda V_{12} + X_M V_{22} = V_{11}X_N + V_{12}\lambda$,

(iii) $\lambda V_{21} = V_{21}\lambda$,

(iv) $\lambda V_{22} = V_{21}X_N + V_{22}\lambda$.

Writing $\lambda = x + yi$, and using (iii) gives $iV_{21} = V_{21}i$. It follows that $V_{21}$ has complex entries.

From (i), we get $y(iV_{11} - V_{11}i) = -X_M V_{21}$. Write $V_{11} = P + Qj$, where $P$ and $Q$ have complex entries, so that the equality becomes $2yQk = -X_M V_{21}$.

Write $Q = [Q_{ij}]$ and $V_{21} = [A_{ij}]$ conformal to $X_M$ to get

$$2y \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix} k = \begin{bmatrix} 4I_n & 0 & I_{n,j} & Mj \\ 0 & 3I_n & I_{n,j} & I_{n,j} \\ 0 & 0 & 2I_n & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

Since $A_{ij}$ and $Q_{ij}$ have complex entries, we must have $A_{4j} = 0 = A_{3j}$ for $j = 1, 2, 3, 4$. Equating the first two rows gives $V_{21} = 0$. Moreover, $Q = 0$ as well, which means that $V_{11}$ has only complex entries.

Since $V$ is unitary and since $V_{21} = 0$, we must also have $V_{12} = 0$. Moreover, (iv) reduces to $\lambda V_{22} = V_{22}\lambda$, so $V_{22}$ has only complex entries.

Now, (ii) reduces to $X_M V_{22} = V_{11}X_N$. Write $X_M = P + Q_mj$ and $X_N = P + Q_Nj$, where $P = \text{diag}(4I_n, 3I_n, 2I_n, I_n)$, and $Q_M$ and $Q_N$ have complex entries. Since $V_{11}$ and $V_{22}$ have complex entries, we have $PV_{22} = V_{11}P$.

Multiplying this equality by $V_{22}^*P = PV_{11}^*$ gives $P^2 \equiv (PV_{22})(V_{22}^*P) = V_{11}P^2V_{11}^*$. It follows that $V_{11}$ is block diagonal; that is, it has the form $V_{11} = C_1 \oplus C_2 \oplus C_3 \oplus C_4$. Similarly, $V_{22}$ is block diagonal, and since $PV_{22} = V_{11}P$, we must have $V_{22} = V_{11} = C_1 \oplus C_2 \oplus C_3 \oplus C_4$.

Equating the noncomplex part of (ii) gives the equality $Q_mjV_{22} = V_{11}Q_Nj$. Hence, we have the following equalities:

$$\overline{C}_3 = C_1, \quad \overline{C}_3 = C_2, \quad \overline{C}_4 = C_2, \quad \text{and} \quad M\overline{C}_4 = C_1N.$$
Therefore, \( MC_1 = C_1N \) and \( M \) is unitarily similar to \( N \).

(b) Notice that using (a), the problem of classifying square quaternion matrices up to unitary similarity is unitarily wild. Hence it suffices to prove that two \( n \times n \) quaternion matrices \( M \) and \( N \) are unitarily similar if and only if the two \( 3n \times 3n \) matrices

\[
A_M = \begin{bmatrix} 0 & I_n & M \\ 0 & 0 & I_n \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_N = \begin{bmatrix} 0 & I_n & N \\ 0 & 0 & I_n \\ 0 & 0 & 0 \end{bmatrix}
\]

are unitarily similar. One checks that \( A_M^3 = A_N^3 = 0 \).

Suppose \( V^* A_M V = A_N \), where \( V \) is unitary. By Theorem 2.1(b), \( V \) has the form \( V_1 \oplus V_2 \oplus V_3 \). The equality \( A_M V = V A_N \) now gives \( V_1 = V_2 = V_3 \), and thus \( V_1^* M V_1 = N \).

(c) We look at the pairs of quaternion idempotent matrices

\[
\left( \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M & I_n - M \\ M & I_n - M \end{bmatrix} \right) \quad \text{and} \quad \left( \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} N & I_n - N \\ N & I_n - N \end{bmatrix} \right),
\]

which are unitarily similar if and only if \( M \) and \( N \) are unitarily similar.

(d) The pairs of quaternion matrices

\[
\left( \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \right) \quad \text{and} \quad \left( \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix} \right)
\]

are unitarily similar if and only if \( M \) and \( N \) are unitarily similar.

\[\blacksquare\]

4 Littlewood’s Algorithm for Nonderogatory Matrices

A square matrix is called \textit{nonderogatory} if every (standardized) eigenvalue has geometric multiplicity 1, that is, its Jordan Canonical Form does not contain two Jordan blocks having the same standard eigenvalue [4, Section 1.4.4]. In this section, we give an algorithm for reducing a nonderogatory quaternion matrix \( A \) by unitary similarity to a certain matrix \( A^\infty \), which has the property that \( A \) and \( B \) are unitarily similar if and only if \( A^\infty = B^\infty \). We call such a matrix \( A^\infty \) the \textit{canonical form of \( A \) with respect to unitary similarity}.

We denote by \( U(\mathbb{F}) \equiv \{ f \in \mathbb{F} \mid \bar{f} = f^{-1} \} \) the set of unitary elements of \( \mathbb{F} \), where \( \mathbb{F} \) is \( \mathbb{H} \), \( \mathbb{C} \), or \( \mathbb{R} \).
Theorem 4.1. (a) Each nonderogatory quaternion matrix $M$ is unitarily similar to an upper triangular matrix of the form

$$A = \begin{bmatrix}
\lambda_1 & a_{12} & \cdots & a_{1n} \\
\lambda_2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \lambda_{n-1} & \lambda_n
\end{bmatrix}, \quad \lambda_l = x_l + y_l i \in \mathbb{C}, \; y_l \geq 0,
\lambda_1 \geq \cdots \geq \lambda_n,
\lambda_{l,t+1} \notin \mathbb{C} j \text{ if } \lambda_l = \lambda_{l+1}.
\tag{8}
$$

(b) The diagonal elements of $A$ are uniquely determined. Moreover, for every quaternion unitary matrix $S$, the matrices $A$ and $A' = S^*AS$ have the form $\text{(8)}$ if and only if $\lambda_1 = \lambda_1, \ldots, \lambda_n = \lambda_n$ and

$$S = \text{diag}(s_1, \ldots, s_n), \quad s_1, \ldots, s_p \in \mathcal{U}(\mathbb{C}), \; s_{p+1}, \ldots, s_n \in \mathcal{U}(\mathbb{H}),
\tag{9}
$$

where $p$ is such that $\lambda_1, \ldots, \lambda_p \notin \mathbb{R}$ and $\lambda_{p+1}, \ldots, \lambda_n \in \mathbb{R}$.

Proof. (a) The proof follows that of Theorem 2.1(a), that is, we write $S^{-1}MS = J$, where $J$ is the Jordan canonical form of $M$; and all the eigenvalues lie in the upper half-plane. We then apply the $QR$ factorization to $S$ to obtain a unitary matrix $U = ST$ with $T$ an upper triangular matrix whose diagonal elements are positive real numbers. The matrix $A \equiv U^*MU = T^{-1}JT$ has the desired form $\text{(8)}$. Notice that when $\lambda_l = \lambda_{l+1}$, $a_{l,l+1}$ has the form $-\lambda_l x + x \lambda_l + a$, with $a > 0$. Now, if $\lambda \in \mathbb{C}$ and $x \in \mathbb{H}$, then $-\lambda x + x \lambda \in \mathbb{C} j$. Hence, when $\lambda_l = \lambda_{l+1}$, $a_{l,l+1} \notin \mathbb{C} j$.

(b) The proof also follows that of Theorem 2.1(b), and makes use of the techniques used in Theorem 3.2(a). First, notice that by the uniqueness of the Jordan form and the fact that the eigenvalues are ordered we must have $\lambda_1 = \lambda_1, \ldots, \lambda_n = \lambda_n$.

Now, we show that $S$ is block diagonal. If $\lambda_1 \neq \lambda_n$, then the $(n, 1)$ entries of $AS = SA'$ give $\lambda_n s_{n1} = s_{n1} \lambda_1$. We express $s_{n1} = p + q j$, with $p, q \in \mathbb{C}$ and conclude that $s_{n1} = 0$ since $\lambda_1$ and $\lambda_n$ are complex numbers with nonnegative imaginary components. Another way to look at it is that otherwise we would have $s_{n1}^{-1} \lambda_n s_{n1} = \lambda_1$, contradicting the fact that $\lambda_1 \neq \lambda_n$ and $\lambda_1$ and $\lambda_n$ are complex numbers with nonnegative imaginary components.

Next, we check if $\lambda_1 \neq \lambda_{n-1}$. If so, then we look at the $(n-1, 1)$ entries of $AS = SA'$ to obtain $\lambda_{n-1} s_{n-1,1} = s_{n-1,1} \lambda_1$, so that $s_{n-1,1} = 0$. We proceed in this manner and conclude that $s_{ij} = 0$ whenever $i > j$ and $\lambda_i \neq \lambda_j$.

It follows that $S$ is upper block triangular, and since $S$ is unitary, it must be block diagonal. Hence, it suffices to prove that the claim holds when all the eigenvalues coincide, that is, $\lambda \equiv \lambda_1 = \lambda_2 = \cdots = \lambda_n$. 

15
We consider two cases: $\lambda \in \mathbb{R}$ and $\lambda \notin \mathbb{R}$.

Suppose $\lambda \in \mathbb{R}$. Notice that $\lambda s_{n1} = s_{n1}\lambda$ always holds. Now, look at the $(n-1,1)$ entries of $AS = SA'$ to obtain $\lambda s_{n-1,1} + a_{n-1,n}s_{n1} = s_{n-1,1}\lambda$. Hence, $a_{n-1,n}s_{n1} = 0$, and $s_{n1} = 0$ since $a_{n-1,n} \neq 0$ by (8). Now, look at the $(n-2,1)$ entries, then the $(n-3,1)$ entries, and so on and conclude that $s_{ni} = 0$ whenever $i > 1$.

Similarly, we look at the $(n,2)$ entries, the $(n-1,2)$ entries, and so on to conclude that in fact, $S$ is upper triangular. Since $S$ is also unitary, $S$ is also diagonal.

Now, suppose $\lambda \notin \mathbb{R}$. Then $\lambda = x + yi$, $y > 0$; the equality $\lambda s_{n1} = s_{n1}\lambda$ implies that $is_{n1} = s_{n1}i$ and $s_{n1} \in \mathbb{C}$. Furthermore, $\lambda s_{n-1,1} + a_{n-1,n}s_{n1} = s_{n-1,1}\lambda$ implies $y(is_{n-1,1} - s_{n-1,1}i) + a_{n-1,n}s_{n1} = 0$. Write $s_{n-1,1} = p + qj$ and $a_{n-1,n} = u + vj$, where $p, q, u, v \in \mathbb{C}$ to get $2yqk + us_{n1} + v\bar{s}_{n1}j = us_{n1} + (v\bar{s}_{n1} + 2yqi)j = 0$. Since $us_{n1}$ and $v\bar{s}_{n1} + 2yqi$ are complex numbers, and $u \neq 0$ (since $a_{n-1,n} \notin \mathbb{R}$ by (8)), we must have $s_{n1} = 0$ and $q = 0$ (i.e., $s_{n-1,n} \in \mathbb{C}$).

Now, $\lambda s_{n-2,1} + a_{n-2,n-1}s_{n-1,1} = s_{n-2,1}\lambda$ implies $s_{n-1,1} = 0$ and $s_{n-2,1} \in \mathbb{C}$. We repeat this process until we obtain $s_{ij} = 0$ for all $i > j$ and $s_{ii} \in \mathbb{C}$. Since $S$ is unitary, $S$ is diagonal.

An algorithm for reducing a matrix $A$ of the form (8) to canonical form with respect to unitary similarity

By Theorem 4.1(b), the diagonal entries of $A$ are uniquely determined. Furthermore, all unitary similarity transformations that preserve the triangular form of $A$ and its diagonal entries have the form:

$$A \mapsto S^*AS, \quad S \in \mathcal{G}_0 \equiv \bigtimes_{p} \mathcal{U}(\mathbb{C}) \times \cdots \times \mathcal{U}(\mathbb{C}) \times \bigtimes_{n-p} \mathcal{U}(\mathbb{C}^n) \times \cdots \times \mathcal{U}(\mathbb{C}^n).$$

(10)

We successively reduce the off-diagonal entries $a_{ij}$ ($i < j$) to a canonical form in the following order:

$$a_{12}, a_{23}, \ldots, a_{n-1,n}; \ a_{13}, a_{24}, \ldots, a_{n-2,n}; \ldots; \ a_{1n}.$$  

(11)

On each step, we use only those transformations (10) that preserve the already reduced entries.
Suppose that all entries that precede $a_{tr}$ in the sequence (11) have been reduced, and let all the transformations (10) that preserve the entries preceding $a_{tr}$ have the form

$$A \mapsto S^*AS, \quad S \in \mathcal{G} \equiv \{S = \text{diag}(s_1, \ldots, s_n) \in \mathcal{G}_0 \mid \mathcal{R}\},$$

(12)

where $\mathcal{R}$ is a set of relations of the form

$$s_i \in \mathbb{C}, \quad s_i \in \mathbb{R}, \quad s_i = s_j, \text{ or } s_i = s_j^{-1} \in \mathbb{C}.$$  

(13)

We reduce $a_{tr}$ to canonical form $a'_{tr}$ by transformations (12) and show that all transformations (12) that preserve $a'_{tr}$ have the form $A \mapsto S^*AS, \quad S \in \mathcal{G}' = \{S \in \mathcal{G} \mid \Delta \mathcal{R}\}$, where $\Delta \mathcal{R}$ consists of relations of the form (13); this is required for the correctness of the induction step.

As follows from the form of relations (13), for every $i \in \{1, \ldots, n\}$ there exists $\mathbb{F}_i \in \{\mathbb{H}, \mathbb{C}, \mathbb{R}\}$ such that

$$\{s_i \mid S \in \mathcal{G}\} = \mathbb{U}(\mathbb{F}_i).$$

If $a_{tr}$ is not changed by transformations (12), we set $a'_{tr} = a_{tr}$ and $\Delta \mathcal{R} = \emptyset$.

Denote by $\mathbb{P}$ the set of positive real numbers and suppose that

$$a_{tr} = z_1 + z_2j, \quad z_1, z_2 \in \mathbb{C},$$

(14)

was changed by transformations (12). We have the following cases.

1) $\mathbb{F}_l = \mathbb{H}$ or $\mathbb{F}_r = \mathbb{H}$. If $\mathcal{R}$ does not imply $s_l = s_r$, then we reduce $a_{tr}$ to the form $a'_{tr} = s_i^{-1}a_{tr}s_i \in \mathbb{P}$ and obtain $\Delta \mathcal{R} = \{s_l = s_r\}$. If $s_l = s_r$, follow from $\mathcal{R}$, then $\mathbb{F}_l = \mathbb{F}_r = \mathbb{H}$, take $a'_{tr} = s_i^{-1}a_{tr}s_i \in \mathbb{C}$ with a nonnegative imaginary component (note that $a'_{tr} \notin \mathbb{R}$, otherwise $a_{tr}$ is not changed by admissible transformations) and obtain $\Delta \mathcal{R} = \{s_l \in \mathbb{C}\}$.

2) $\mathbb{F}_l = \mathbb{F}_r = \mathbb{C}$ and $\mathcal{R}$ does not imply $s_l = s_r$ or $s_l = s_r^{-1}$. Then by (12) $a'_{tr} = s_i^{-1}a_{tr}s_i = s_i^{-1}z_1s_r + (s_i^{-1}z_2s_r)j = z_1s_i^{-1}s_r + z_2s_i^{-1}s_r^{-1}j$. If $z_1z_2 \neq 0$, we make $z_1' \in \mathbb{P}$, then $s_l = s_r$ (to preserve $z_1'$), next make $z_2' \in \mathbb{P}$, then $s_l = s_r = \pm 1$; we obtain $a'_{tr} = \mathbb{P}^1 + \mathbb{P}^1$ and $\Delta \mathcal{R} = \{s_l = s_r \in \mathbb{R}\}$.

If $z_1 \neq 0 = z_2$, we make $a'_{tr} \in \mathbb{P}$ and obtain $\Delta \mathcal{R} = \{s_l = s_r\}$. If $z_1 = 0$, then $z_2 \neq 0$ (otherwise $a_{tr}$ is not changed by admissible transformations) we make $a'_{tr} \in \mathbb{P}^1$ and obtain $\Delta \mathcal{R} = \{s_l = s_r^{-1}\}$.

3) $\mathbb{F}_l = \mathbb{F}_r = \mathbb{C}, \ s_l = s_r$ or $s_l = s_r^{-1}$. If $s_l = s_r$, then $a'_{tr} = z_1 + z_2s_i^{-2}j$, make $a'_{tr} \in \mathbb{C} + \mathbb{P}j$ and obtain $\Delta \mathcal{R} = \{s_l \in \mathbb{R}\}$.

17
If \( s_l = s_r^{-1} \), then \( a'_{lr} = z_1 s_l^{-2} + z_2 j \), make \( a'_{lr} \in \mathbb{P} + \mathbb{C}j \) and obtain \( \Delta \mathcal{R} = \{ s_l \in \mathbb{R} \} \).

4) Either \( \mathbb{F}_l = \mathbb{C} \) and \( \mathbb{F}_r = \mathbb{R} \), or \( \mathbb{F}_l = \mathbb{R} \) and \( \mathbb{F}_r = \mathbb{C} \). Make \( a'_{lr} = 1 + z'_2 j \) or \( a'_{lr} = j \) and obtain \( \Delta \mathcal{R} = \{ s_l = s_r \in \mathbb{R} \} \).

5) \( \mathbb{F}_l = \mathbb{F}_r = \mathbb{R} \). Make \( a'_{lr} = z'_1 + z'_2 j \) with \( z'_1 > 0 \) (see (2)), or \( a'_{lr} = z'_2 j \) with \( z'_2 > 0 \), and obtain \( \Delta \mathcal{R} = \{ s_l = s_r \} \).

The process ends with the reduction of \( a_{1n} \). We denote the matrix obtained by \( A^\infty \); it is the canonical form of \( A \) with respect to unitary similarity. At each step we reduced an entry to a form that is uniquely determined by the already reduced entries and the class of (quaternion) unitarily similar to \( A \), and so we obtain the following theorem:

**Theorem 4.2.** Two nonderogatory quaternion matrices \( A \) and \( B \) are unitarily similar if and only if \( A^\infty = B^\infty \).

For a canonical \( n \times n \) matrix \( A = A^\infty \), its graph \( \Gamma(A) \) has vertices 1, \ldots, \( n \), and \( l \) and \( r \) are jointed by an edge if and only if the relations \( s_l = s_r \) and \( s_l = s_r^{-1} \) do not follow from the condition of preserving the entries of \( A \) that precede \( a_{lr} \), but one of them follows from the condition of preserving \( a_{lr} \) (i.e., \( s_l = s_r \) or \( s_l = s_r^{-1} \) is contained in \( \Delta \mathcal{R} \); see the cases 1–5). Notice that there is an edge \( (i, i+1) \) if \( \lambda_i = \lambda_{i+1} \) in (8) since then \( a_{i,i+1} \notin \mathbb{C}j \).

A square matrix \( A \) is called **unitarily indecomposable** if it is not unitarily similar to a direct sum of square matrices.

**Theorem 4.3.** (a) The graph of each canonical matrix is a union of trees. Any union of trees with numbered vertices can be the graph of a canonical matrix.

(b) A canonical matrix is unitarily indecomposable if and only if its graph is a tree. Moreover, let the graph \( \Gamma \) of a canonical matrix \( A \) be the union of \( m \) trees \( \Gamma_i \) (1 \( \leq i \leq m \)) with the vertices \( v_{i1} < v_{i2} < \cdots < v_{ir_i} \). Rearrange the columns of \( A \) such that their old numbers form the sequence

\[ v_{11}, \ldots, v_{1r_1}; \; v_{21}, \ldots, v_{2r_2}; \; \ldots; \; v_{m1}, \ldots, v_{mr_m}, \]

then rearrange its rows in the same manner. The matrix obtained has the form \( A_1 \oplus \cdots \oplus A_m \), where each \( A_i \) is a unitarily indecomposable canonical \( r_i \times r_i \) matrix.

**Proof.** (a) Let the graph \( \Gamma(A) \) of a canonical matrix \( A \) have a cycle \( v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_p \rightarrow v_1 \) (\( p \geq 2 \)), and let, say, \( a_{\{v_{1}v_{2}\}}, \ldots, a_{\{v_{p-1}v_{p}\}} \) precede \( a_{\{v_{p}v_{1}\}} \) in the
sequence $a_{ij}$, where $a_{ij}$ denotes $a_{ij}$ if $i < j$ and $a_{ji}$ if $i > j$. Then the equality $s_{vp} = s_{v_1}^{\pm 1}$ follows from the condition of preserving the entries of $A$ that precede $a_{v_pv_1}$, a contradiction to the existence of the edge $v_pv_1$.

Let a graph $\Gamma$ with vertices $1, \ldots, n$ be a joint of trees. Take $A$ of the form 

\[
(\begin{array}{cccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
1 & (n-1) & \cdots & i
\end{array})
\]

in which $\lambda_1 = ni$, $\lambda_2 = (n-1)i$, \ldots, $\lambda_n = i$, and, for every $l < r$, $a_{lr} = 1$ if there is the edge $l \rightarrow r$ and $a_{lr} = 0$ otherwise. Clearly, $A$ is a canonical matrix and $\Gamma(A) = \Gamma$.

(b) Let $A$ be a canonical matrix. Since $a_{ij} = 0$ whenever $i$ and $j$ are not connected in $\Gamma(A)$, the graph of a unitarily indecomposable canonical matrix is a tree. It follows from the algorithm of reduction to canonical form that if $\Gamma(A)$ is not a connected graph, then $A$ can be reduced to a direct sum of unitarily indecomposable canonical matrices by simultaneous permutation of its rows and columns.

\begin{proof}

\end{proof}

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