The primitive derivation and freeness of multi-Coxeter arrangements

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Abstract

We will prove the freeness of multi-Coxeter arrangements by constructing a basis of the module of vector fields which contact to each reflecting hyperplanes with some multiplicities using K. Saito’s theory of primitive derivation.

Key words: Hodge filtration; Finite reflection group; Coxeter arrangement; Adjoint quotient.

1 Introduction

Let $V$ be a Euclidean space over $\mathbb{R}$ with finite dimension $\ell$ and inner product $I$. Let $W \subset O(V, I)$ be a finite irreducible reflection group and $\mathcal{A}$ the corresponding Coxeter arrangement i.e. the collection of all reflecting hyperplanes of $W$. For each $H \in \mathcal{A}$, we fix a defining equation $\alpha_H \in V^*$ of $H$.

In [5], H. Terao constructed a free basis of $\mathbb{R}[V]$-module

$$D^m(\mathcal{A}) := \{ \delta \in \text{Der}_V \mid \delta \alpha_H \in (\alpha_H^m), \forall H \in \mathcal{A} \}, \quad (1)$$

($m \in \mathbb{Z}_{\geq 0}$). The purpose of this paper is to construct a basis by a simpler way using Saito’s result and give a generalization.

For given multiplicity $\bar{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, we say that the multi-Coxeter arrangement $\mathcal{A}^{(\bar{m})}$ is free if the module

$$D(\mathcal{A}^{(\bar{m})}) := \{ \delta \in \text{Der}_V \mid \delta \alpha_H \in (\alpha_H^{\bar{m}(H)}), \forall H \in \mathcal{A} \} \quad (2)$$

is a free $\mathbb{R}[V]$-module [8]. Then our main result is

**Theorem 1** Let $\bar{m}$ be a multiplicity satisfying $\bar{m}(H) \in \{0, 1\}$ for all $H \in \mathcal{A}$. Suppose the multi-Coxeter arrangement $\mathcal{A}^{(\bar{m})}$ is free, then $\mathcal{A}^{(\bar{m}+2k)}$ ($k \in \mathbb{Z}_{\geq 0}$) is also free, where the new multiplicity $\bar{m} + 2k$ take value $\bar{m}(H) + 2k$ at $H \in \mathcal{A}$. □
We construct a basis in Theorem 7. We note that $\mathcal{A}(\tilde{m})$ is not necessarily free for $\tilde{m} : \mathcal{A} \to \{0, 1\}$. If we apply Theorem 1 for $\tilde{m}(H) \equiv 0$ or $\tilde{m}(H) \equiv 1$, we obtain the freeness of $D^{2k}(\mathcal{A})$ or $D^{2k+1}(\mathcal{A})$. Terao’s basis is expected to coincide with that of ours.

The original motivation to study the module $D(\mathcal{A}(\tilde{m}))$ came from the study of structures of the relative de Rham cohomology $H^*(\Omega^\bullet_{g/S})$ of the adjoint quotient map $\chi : g \to S := g/\text{ad}(G)$ of a simple Lie algebra $g$. In the case of ADE type Lie algebras, an isomorphism as $C[S](= C[g]^G = C[h]^W)$-modules (where $h$ is a Cartan subalgebra)

$$H^2(\Omega^\bullet_{g/S}) \cong D^5(\mathcal{A})^W$$

is obtained [7]. But for BCFG type Lie algebras, because the $W$ action on $\mathcal{A}$ is not transitive, $H^2(\Omega^\bullet_{g/S})$ is expected to be isomorphic to the module $D(\mathcal{A}(\tilde{m}))^W$ with a suitable multiplicity $\tilde{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ which is not constant.

## 2 K. Saito’s results on primitive derivation

In this section, we fix notations and recall some results.

Let $x_1, \ldots, x_\ell \in V^*$ be a basis of $V^*$ and $P_1, P_2, \ldots, P_\ell \in \mathbb{R}[V]^W$ be the homogeneous generators of $W$-invariant polynomials on $V$ such that $\mathbb{R}[V]^W = \mathbb{R}[P_1, P_2, \ldots, P_\ell]$ with

$$\text{deg} \, P_1 \leq \text{deg} \, P_2 \leq \cdots \leq \text{deg} \, P_\ell =: h.$$ 

Then it is classically known [1] that

$$|\mathcal{A}| = \frac{h\ell}{2} \quad (3)$$

and

$$\text{deg} \, P_{\ell-1} < h. \quad (4)$$

It follows from (3) that the rational vector field (with pole along $\bigcup_{H \in \mathcal{A}} H$) $D := \frac{\partial}{\partial P_\ell}$ on $V$ is uniquely determined up to non-zero constant factor independently on the generators $P_1, \ldots, P_\ell$. We call $D$ the primitive vector field. If we fix generators $P_1, \ldots, P_\ell$, then $\frac{\partial}{\partial P_1}, \ldots, \frac{\partial}{\partial P_{\ell-1}}$ are able to be considered as rational vector fields on $V$. Since the Jacobian is

$$Q := \prod_{H \in \mathcal{A}} \alpha_H \cdot \partial(P_1, \ldots, P_\ell) \partial(x_1, \ldots, x_\ell),$$

$D$ is symbolically expressed as

$$D \hat{=} \frac{1}{Q} \det \left( \begin{array}{cccc}
\frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_1}{\partial x_\ell} & \frac{\partial}{\partial x_1} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial P_\ell}{\partial x_1} & \cdots & \frac{\partial P_\ell}{\partial x_\ell} & \frac{\partial}{\partial x_\ell}
\end{array} \right).$$

Next we define an affine connection $\nabla : \text{Der}_V \times \text{Der}_V \to \text{Der}_V.$
Definition 2 For given \( \delta_1, \delta_2 \in \text{Der}_V \) with \( \delta_2 = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i} \),

\[ \nabla_{\delta_1} \delta_2 := \sum_{i=1}^{\ell} (\delta_1 f_i) \frac{\partial}{\partial x_i}. \]

\( \Box \)

The connection \( \nabla \) can be also characterized by the formula:

\[ (\nabla_{\delta_1} \delta_2)\alpha = \delta_1 (\delta_2 \alpha), \quad \forall \text{ linear function } \alpha \in V^*. \] (5)

This formula plays an important role in our computations.

The derivation \( \nabla_D \) by the primitive vector field is particularly important. Define \( \mathbb{R}[V]^W,\tau := \{ f \in \mathbb{R}[V]^W \mid Df = 0 \} = \mathbb{R}[P_1, \ldots, P_{\ell-1}] \). Then \( \nabla_D \) is an \( \mathbb{R}[V]^W,\tau \)-homomorphism. The following decomposition of \( \text{Der}_V^W = \text{D}^1(\mathcal{A})^W \) has been obtained in [2,3].

Theorem 3 Let \( n \geq 1 \), define

\[ \mathcal{G}_n := \left\{ \delta \in \text{Der}_V^W \mid (\nabla_D)^n \delta \in \sum_{i=1}^{\ell} \mathbb{R}[V]^W,\tau \frac{\partial}{\partial P_i} \right\}, \]

then for every \( n \geq 0 \), \( \nabla_D \) induces an \( \mathbb{R}[V]^W,\tau \)-isomorphism \( \mathcal{G}_{n+1} \to \mathcal{G}_n \) and

\[ \text{D}^1(\mathcal{A})^W = \bigoplus_{n \geq 1} \mathcal{G}_n. \]

If we define \( \mathcal{H}^k := \bigoplus_{n \geq k} \mathcal{G}_n \), then it becomes a rank \( \ell \) free \( \mathbb{R}[V]^W \)-submodule of \( \text{Der}_V^W \), which is called the Hodge filtration. \( \Box \)

In particular, \( \nabla_D : \mathcal{H}^2 \to \mathcal{H}^1 = \text{D}^1(\mathcal{A})^W \) is an \( \mathbb{R}[V]^W,\tau \)-isomorphism. All we need in the sequel is the existence of an injection \( \nabla_D^{-1} : \text{Der}_V^W \to \text{Der}_V^W \).

3 Construction of a basis

We construct a basis of \( \text{D}(\mathcal{A}^{(2k+\tilde{m})}) \). The following is a key lemma which connects two filtrations, the Hodge filtration and the contact-order filtration.

Lemma 4 Let \( \delta', \delta \in \text{Der}_V \) be vector fields on \( V \) and assume \( \nabla_D \delta' = \delta \). Then for any \( H \in \mathcal{A} \), \( \delta \alpha_H \) is divisible by \( \alpha_H^m \) if and only if \( \delta' \alpha_H \) is divisible by \( \alpha_H^{m+2} \).
Proof.
Suppose \( \delta' \alpha = \alpha^{m'} f \) (where \( \alpha = \alpha_H \)). Then from (3),
\[
(\nabla_D^2 \delta') \alpha = D(\delta' \alpha) = \frac{1}{Q} \det \left( \begin{array}{ccc}
\frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial P_1}{\partial x_{\ell-1}} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_{\ell-1}}
\end{array} \right) \left( \frac{\partial}{\partial \alpha_m}(\alpha^{m'} f) \right). 
\] (6)
Thus \( \delta \alpha \) is divisible by \( \alpha^{m' - 2} \). Further, assume \( f \) is not divisible by \( \alpha \), let us show that \( \delta \alpha \) is not divisible by \( \alpha^{m' - 1} \). Take a coordinate system \( x_1, \cdots, x_{\ell-1}, x_{\ell} \) such that \( x_{\ell} = \alpha \), then it suffices to show that
\[
\det \left( \begin{array}{ccc}
\frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial P_1}{\partial x_{\ell-1}} & \cdots & \frac{\partial P_{\ell-1}}{\partial x_{\ell-1}}
\end{array} \right)
\] is not divisible by \( \alpha \).

After taking \( \otimes \mathbb{C} \) and restricting to \( H_{\mathbb{C}} := H \otimes \mathbb{C} \), determinant above can be interpreted as the Jacobian of the composed mapping
\[
\phi : H_{\mathbb{C}} \to \text{Spec} \mathbb{C}[V]^W, \tau \\
(x_1, \cdots, x_{\ell-1}, 0) \mapsto (P_1, \cdots, P_{\ell-1}).
\]
On the other hand, since the set
\[
\{ x \in V \otimes \mathbb{C} \mid P_1(x) = \cdots = P_{\ell-1}(x) = 0 \}
\]
is a union of some eigenspaces of Coxeter transformations in \( W \), which are regular, that is, they intersect with \( H_{\mathbb{C}} \) only at \( 0 \in H_{\mathbb{C}} \). Hence \( \phi^{-1}(0) = \{0\} \subset H_{\mathbb{C}} \), and the Jacobian of \( \phi \) cannot be identically zero. \( \Box \)

Remark 5 The precise expression of the Jacobian of \( \phi \) is obtained in [4]. It is equal to the reduced defining equation of the union of hyperplanes \( \bigcup_{H' \in A \setminus \{H\}} (H \cap H') \), on \( H \).

Because of Theorem [3] the operator \( \nabla_D^{-1} \) is well defined on \( \text{Der}_V^W = D^1(A)^W \), we have

Lemma 6 Let \( \delta \in \text{Der}_V^W \) be a \( W \)-invariant vector field on \( V \). Then for any \( H \in A \), \( \delta \alpha_H \) is divisible by \( \alpha_H^{m+2} \) if and only if \( (\nabla_D^{-1} \delta) \alpha_H \) is divisible by \( \alpha_H^{m+2} \). \( \Box \)

By induction with \( H^1 = D^1(A)^W \), Lemma [3] indicates
\[
H^k = \nabla_D^{-k+1} D^1(A)^W \subset D^{2k+1}(A)^W. \tag{7}
\]
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The converse is also true, which will be proved in §4. We denote by 
\[ E := -\sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i} \]
the Euler vector field. Note that \( E \) is contained in \( D(\mathcal{A})^{W} \), \( \nabla_{\delta} E = \delta \) and \( \nabla_{\alpha} \delta = (\deg \delta) \delta \) for any homogeneous vector field \( \delta \in \text{Der}_{\mathcal{V}} \). By Theorem 3, we have a “universal” vector field \( \nabla_{\delta} E \).

As in §1, let \( \tilde{m} : \mathcal{A} \to \{0, 1\} \) be a multiplicity and assume that \( \delta_1, \delta_2, \ldots, \delta_\ell \in D(\mathcal{A}(\tilde{m})) \) be a free basis of the multiarrangement \( \mathcal{A}(\tilde{m}) \).

**Theorem 7** Under the above hypothesis, \( \nabla_{\delta_1} \nabla_{\delta}^{-k} E, \ldots, \nabla_{\delta_\ell} \nabla_{\delta}^{-k} E \) form a free basis of \( D(\mathcal{A}(\tilde{m}+2k)) \).

*Proof.*

Let \( \delta \in D(\mathcal{A}(\tilde{m})) \), we first prove \( \nabla_{\delta} \nabla_{\delta}^{-k} E \in D(\mathcal{A}(\tilde{m}+2k)) \). From (7), \( \nabla_{\delta} \nabla_{\delta}^{-k} E \in D(2k+1) \), we may assume

\[
(\nabla_{\delta}^{-k} E) = \alpha^{2k+1} f
\]

for \( \alpha = \alpha_\mathcal{H}, \ (H \in \mathcal{A}) \). Applying \( \delta \) to the both sides of (8), we have

\[
(\nabla_{\delta} \nabla_{\delta}^{-k} E) = \alpha^{2k} ((2k+1)(\delta \alpha) f + \alpha(\delta f)).
\]

Since \( \delta \alpha \) is divisible by \( \alpha \) with multiplicity \( \tilde{m}(H) \leq 1 \), hence \( (\nabla_{\delta} \nabla_{\delta}^{-k} E) \alpha \) is divisible by \( \alpha \tilde{m}(H)+2k \).

Here we recall G. Ziegler’s criterion on freeness of multiarrangements.

**Theorem 8** \[8 \]

Let \( \tilde{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0} \) be a multiplicity and \( \delta_1, \ldots, \delta_\ell \in D(\mathcal{A}(\tilde{m})) \) be homogeneous and linearly independent over \( \mathbb{C}[\mathcal{V}] \). Then \( \mathcal{A}(\tilde{m}) \) is free with basis \( \delta_1, \ldots, \delta_\ell \) if and only if

\[
\sum_{i=1}^{\ell} \deg \delta_i = \sum_{H \in \mathcal{A}} \tilde{m}(H).
\]

We compute the degrees of \( \nabla_{\delta_1} \nabla_{\delta}^{-k} E, \ldots, \nabla_{\delta_\ell} \nabla_{\delta}^{-k} E, \)

\[
\sum_{i=1}^{\ell} \deg(\nabla_{\delta_i} \nabla_{\delta}^{-k} E) = \sum_{i=1}^{\ell} (kh + \deg \delta_i)
\]

\[
= kh \ell + \sum_{i=1}^{\ell} \deg \delta_i,
\]

where \( h = \deg P_\ell \) is the Coxeter number. On the other hand, the sum of multiplicities is

\[
\sum_{H \in \mathcal{A}} (\tilde{m}(H) + 2k) = 2k|\mathcal{A}| + \sum_{H \in \mathcal{A}} \tilde{m}(H)
\]

The assumption implies \( \sum_{H \in \mathcal{A}} \tilde{m}(H) = \sum_{i=1}^{\ell} \deg \delta_i \) and because of (3), we conclude that (10) coincides with (11).
4 Some conclusions

Lemma 9 \( \nabla_{\partial_i} \mathbb{D}^{2k+1}(A)^W \subset \mathbb{D}^{2k-1}(A)^W \) \((k > 0)\).

Proof.
We only prove for \( i = \ell \), remaining cases can be proved similarly. It is sufficient to show that \( (\nabla_D \delta) \alpha_{H_0} \) has no poles for any \( \delta \in \mathbb{D}^{2k+1}(A)^W \) and \( H_0 \in A \). By (3), \( QD\delta \alpha_{H_0} \) can be divided by \( \alpha_{H_0} \), so all we have to show is that \( QD\delta \alpha_{H_0} \) is divisible by \( \beta := \alpha_{H'} \) for all \( H' \in A \setminus \{H_0\} \). We denote by \( s_\beta \in W \) the reflection with respect to the hyperplane \( H' \subset V \), then \( s_\beta(\alpha) \) is expressed in the form \( s_\beta(\alpha) = \alpha + 2c\beta \) for some \( c \in \mathbb{R} \). Apply \( s_\beta \) to the function \( QD\delta \alpha \), since \( D \) and \( \delta \) are \( W \)-invariant, and \( s_\beta(Q) = -Q \), \( s_\beta(QD\delta \alpha) = -QD\delta \alpha - 2cQD\delta \beta \).

by using the equation \( s_\beta(QD\delta \beta) = QD\delta \beta \), we have \( s_\beta(QD\delta \alpha + cQD\delta \beta) = -(QD\delta \alpha + cQD\delta \beta) \).

So \( QD\delta \alpha + cQD\delta \beta \) is divisible by \( \beta \), but from the first half of this proof, \( cQD\delta \beta \) is divisible by \( \beta \), and the other term \( QD\delta \alpha \) is also divisible by \( \beta \).

As a consequence of induction, we have

Corollary 10 \[ \mathcal{H}^k = \mathbb{D}^{2k+1}(A)^W \].

Finally, we apply Theorem 4 to \( \tilde{m} \equiv 0 \) or \( \tilde{m} \equiv 1 \), since both \( D^0(A) = \text{Der}_V \) and \( D^1(A) \) are free, we obtain

Corollary 11 \[ \mathbb{D}^m(A) \) is free for all \( m \geq 0 \).\]

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