Topological Strings and $D$-Branes

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Abstract

In this talk we give a brief review of the algebraic structure behind the open and closed topological strings and $D$-branes and emphasize the role of tensor category and the Frobenius algebra. Also, we speculate on the possibility of generalizing the topological strings and the $D$-branes through the subfactor theory.

Keywords: topological strings; D-branes; Frobenius algebra

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1 Introduction

The $D$-branes, discovered by J. Polchinski in 1995 [1], are extended objects in string theory characterized by physical properties such as tension, charge and supersymmetry and by certain geometric properties which depend on the chosen string theory. In some spacetime backgrounds, as the flat spacetime, the geometry of $D$-branes can be determined postulating that the $D$-branes are those submanifolds on which the open strings can end. From that, one infers that in the low energy limit of string theory the $D$-branes are solutions of supergravity consistent with $T$-duality and supporting supersymmetry in various degrees. The description of $D$-branes as submanifolds of spacetime is valid for other string theories, e. g. described by certain $\sigma$-models. Also, a microscopic description of $D$-branes as coherent states of infinite norm from the Fock space of closed strings can be given in the perturbative limit of string theory [2, 3]. However, the vacuum of string theory is highly degenerate. Consequently, an infinite number of theories on spacetime manifolds of various dimensions from eleven to four can be produced. Therefore, it is desirable to have a general definition of $D$-branes which could be used in any string theory and any background. Also, since different (dual) string theories are valid in various regions of the moduli space which should be entirely covered by string field theory, one would like to have this general definition lifted to string field theory.

Formulating a general definition of $D$-branes represents an open problem and an active line of research. Progress has been made in this direction mainly in the context of Calabi-Yau backgrounds which represent a viable alternative for compactification down to a spacetime with four dimensions and extensions of Standard Model living in it. In this endeavor, new mathematical tools have been employed and results from topological field theory have been used (see, e. g. [4, 5, 6].)

The aim of this talk is to give a brief review of some basic concepts which have been useful so far in studying the topological strings and $D$-branes. Also, we are going to speculate on the possible generalization of these objects by using the subfactor theory applied to the Frobenius algebras of the topological strings. The organization of this work is as follows. In Section 2 we are going to review the relationship between the string theory, the topological quantum field theory (TQFT) and $D$-branes and emphasize the algebraic structure behind it. In Section 3 we speculate on the extension of the notions topological strings and $D$-branes through the subfactor theory and suggest that this extension, if possible, could be interpreted as a (topological) string field theory of open and closed strings. We justify the introduction of the mathematical structures in an intuitive manner and refer the reader for technical details and more information on the topics to [4, 5, 6, 7, 8, 9]. Also, we assume that the reader is conversant with basic category theory and abstract algebra as well as with some well established results from string theory and $D$-branes.

2 TQFT, Topological Strings and D-Branes

In this section, we present the mathematical structure of TQFT and topological strings and we give the relationship between TQFT and the Frobenius algebra. The basic references used for this section are [8, 9, 10].
2.1 Categories and TQFT

The scattering processes in string theory can be derived, in principle, by borrowing the intuitive graphical concepts from quantum field theory. This amounts to calculate the partition function $Z$ as the Polyakov sum over all "loops" in the perturbation theory

$$Z \sim \sum_{\text{topologies}} \int D\phi_i e^{S(\phi_i)},$$

where $\phi_i$ stand for all string degrees of freedom, bosonic as well as fermionic. The "loops" that should be considered in string theory are $2d$ Euclidean surfaces classified by their genus. In the case of closed strings, all $2d$ surfaces can be obtained by gluing together three fundamental surfaces: the Hartle-Hawking tadpole (the disk $B^2$), the propagator (the cylinder) and the three-vertex (pants diagram) [8]. The topology changing element is the three vertex since by gluing $2n$ vertices on two branches one can construct $n$-tori. In flat spacetime background, the modular invariance between the open string theory at one-loop and the closed string theory at tree level (open-closed string interpretation of the cylinder diagram) and the $T$-duality allow to pass from the open string sector to the closed string sector. In particular, the $D$-branes which lies on the boundary of the cylinder in the open string theory continue to lie on the boundary in the closed string theory. However, the role of the Dirichlet and Neumann boundary conditions in the two sectors is interchanged [1]. This construction can be used as a guide to define the $D$-branes in a different class of string models, the topological strings. The basic requirements are: i) the string partition function $Z$ be computed as the Polyakov sum over all $2d$ homeomorphic invariant Euclidean world-sheet surfaces of the closed string; ii) the $D$-branes be associated to the boundaries of these surfaces; iii) to the $D$-branes are associated spaces of fields from the open string spectra. In order to make prediction in the topological theory, one has to give the computation rules. A simple inspection of the way in which the fundamental surfaces are joined to give other surfaces suggests that these rules be algebraic. Also, they should act on the spaces of fields on the boundaries. Since the world-sheets connect various spaces of fields, they can be viewed as mappings among these spaces. From these remarks, one recognizes that a suitable mathematical framework to describe $D$-branes in topological string theory is that of categories. Another way to see that is from the observation that the TQFT’s can be formulated in terms of modular tensor categories [8].

In order to make transparent the mathematical structure behind the above physical ideas, let us neglect for the moment the $D$-branes and look at the topological strings which are described by $1 + 1$ TQFT’s. Let us recall some basic notions from algebra. Let $k$ be a commutative ring and $L$ a Lie algebra over $k$ endowed with a surgenerete bilinear form $\beta$, i. e. a mapping $\beta : L \times L \to k$ which defines an isomorphism of $k$-modules

$$\beta(x, y) = \langle \psi(x), y \rangle,$$

where $L^*$ is the dual to $L$, i. e. $L^* = \text{Hom}_k(L, k)$.

**Definition 1.** $\langle L, \beta \rangle$ is said to be quasi-Frobenius algebra if $L$ is finitely generated projective $k$-module and

$$\beta([x, y], z) + \beta([y, z], x) + \beta([z, x], y) = 0, \ \forall x, y, z, \in L.$$
Definition 2. The quasi-Frobenius algebra \((L, \beta)\) is said to be Frobenius algebra if there is an element \(\phi \in L^*\) such that the bilinear form
\[
\beta_\phi(x, y) = \phi([x, y]), \quad \forall x, y \in L
\] (4)
is surgenerate.

The mapping \(\phi\) is called the Frobenius homomorphism of \(L\). The definition of Frobenius algebra does not apply only to Lie algebras. In fact, the following more general definition can be given.

Definition 3. An associative \(k\)-algebra \(A\) is said to be Frobenius if \(A\) is finitely generated \(k\)-module and there exists \(\phi \in A^*\) such that the bilinear form \(\beta_\phi(x, y) = \phi(xy), \quad \forall x, y \in A\), where \(\phi\) is a Frobenius homomorphism.

The importance of these notions for TQFT is revealed by the following theorem [9, 10]

Theorem 1.: There is a one-to-one correspondence between the set of all 1+1 TQFT’s and the finite-dimensional Frobenius algebras.

The content of the above theorem can be better understood if we recall the definition of a TQFT.

Definition 4. A \(d + 1\) dimensional TQFT is defined by the following collection of data [10]:
1. To any \(d\) dimensional manifold \(N\) without boundary is assigned a finite-dimensional vector space \(\tau(N)\).
2. To any \((d + 1)\) manifold \(M\) (possibly with boundary) is assigned a vector \(\tau(M)\) in the vector space \(\tau(\partial M)\).
3. To any homeomorphism of \(d\) dimensional manifolds \(f : N \to N'\) is assigned an isomorphism of vector spaces \(f_* : \tau(N) \to \tau(N')\).
4. The following functorial isomorphisms hold
\[
\tau(\bar{N}) \rightarrow \tau(N)^*, \quad \tau(\emptyset) \rightarrow k, \quad \tau(N_1 \sqcup N_2) \rightarrow \tau(N_1) \otimes \tau(N_2),
\] (5) (6) (7)
where \(\bar{N}\) is the manifold \(N\) with the opposite orientation. The isomorphisms given by the relations (5), (6) and (7) are compatible with each other and with the commutativity, associativity and unit morphism. The above set of data define a \(d + 1\) TQFT if they satisfy the axioms of functoriality, gluing, normalization and border (see the Appendix.)

The abstract definition of a TQFT formalizes the more intuitive concepts that the spacetime manifold interpolates among its boundaries and that the physical fields belong to some spaces defined on the respective manifolds. The physical content of the theory is contained in the partition function which should satisfy some tensor properties derived from the additive property of the action and should associate a number to the spacetime manifold.
and to any of its boundaries. This number depends only on the topological properties of
the manifold. Mathematically, this definition asserts that TQFT is a category (a bicategory)
and Theorem 1 allows us to compute the their properties in terms of the Frobenius algebra.

2.2 Topological Strings

One can think of the topological strings as 1 + 1 TQFT. However, a full string theory
contains products of two topologically distinct sectors: the open string and the closed
string sector, respectively. In these sectors the fields are described by vector spaces on the
2d manifold and different vector spaces on the boundary and the orientation of the surfaces
play an important role. To take into account this new data, one should modify the category
used in the previous section and which is appropriate to describe only the closed string
sector. To this end, one has to define a vector space $V(\Sigma)$ where $\Sigma$ is the two-dimensional
world-sheet with $\partial\Sigma_1$ and $\partial\Sigma_2$ compact, oriented, one-dimensional boundaries of $\Sigma$ and

$$V(\partial\Sigma_1 \sqcup \partial\Sigma_2) = V(\partial\Sigma_1) \otimes V(\partial\Sigma_2). \quad (8)$$

In the open string sector the boundaries are closed intervals and the vector space is $V(I)$
while in the closed string sector the boundaries are circles and the vector space is $V(S^1)$. However, open and closed strings interact and products of intervals and circles represent
possible boundaries. The physical interpretation of the vector space $V(\partial\Sigma)$ is that of the
space of states on the boundary. Eventually, we would like that some of these states live
on the $D$-branes.

As we have seen in the previous subsection, there is a Frobenius algebra that dictates the
way in which the 2d manifold be composed such that the 2d TQFT be compatible with the
geometric gluing of 2d surfaces. Thus, the algebraic structure obtained in this way results
from the very definition of the TQFT. Therefore, one should impose some consistency
conditions on the state spaces such that they be compatible with the Frobenius algebra
of the world-sheet composition. Let us analyze what are the elements of this construction.
Firstly, we have the pure open string sector and the closed string sector. In this sectors the
world-sheets are two-dimensional manifolds which interpolate among boundaries of just
one type, i. e. mappings like

$$\Sigma_{\text{open}} : I \to I, \quad \Sigma_{\text{closed}} : S^1 \to S^1. \quad (9)$$

Beside these sectors the TQFT contains open-closed and closed-open sectors and the world-
sheets in these sectors are cobordisms among boundaries which are disjoint unions of open
and closed string boundaries, respectively,

$$\partial\Sigma = (\sqcup_m(I)^m) \sqcup (\sqcup_n(S^1)^n). \quad (10)$$

The basic scattering processes in the open-closed and closed-open sectors are given by
mappings like

$$\Sigma_{\text{open-closed}} : I \to S^1, \quad \Sigma_{\text{closed-open}} : S^1 \to I. \quad (11)$$

The following relation between the above mappings hold

$$\bar{\Sigma}_{\text{open-closed}} = \Sigma_{\text{closed-open}}, \quad \bar{\Sigma} \circ \Sigma = 1. \quad (12)$$
The relations (9), (11) and (12) show that the state spaces satisfy a certain algebra. In order to find what this algebra is, note that $V(I)$ and $V(S^1)$ are commutative and that there is a neutral element with respect to multiplication (unit). If we take now $V(S^1)$, there are two remarkable maps from and to the field of complex numbers, respectively,

$$E_c : C \to V(S^1), \quad \beta_c : V(S^1) \to C.$$ (13)

The interpretation of $E_c$ is that it defines the unit element for the algebra of state spaces. The map $\beta_c$ is a non-degenerate bilinear trace. The properties of the two maps can be verified if they are represented as the cobordisms generated by the disk

$$\beta_c :: B^2 : S^1 \to \emptyset, \quad E_c :: B^2 : \emptyset \to S^1.$$ (14)

From these properties we recognize the structure of the Frobenius algebra. The bilinear trace $\beta_o$ for the state space $V(I)$ is given by the following relation [8]

$$\beta_o = \beta_c \circ i^*, \quad i : V(S^1) \to V(I), \quad i^* : V(I) \to V(S^1).$$ (15)

Here, $i^*$ denotes the homomorphism of modules adjoint to the map $i$. If one considers the open strings, one can prove the following [8]

**Theorem 2.** A two-dimensional open topological string theory is given by the following set of data:

1. A commutative Frobenius algebra $(V(S^1), \beta_c)$.
2. A commutative Frobenius algebra $(V(I), \beta_o)$.
3. A map $i : V(S^1) \to V(I)$ such that $i(1) = 1$ and $\text{Im}(i) \subset C(V(I))$ where $C(V(I))$ is the center of $V(I)$, i.e. all commuting elements of $V(I)$.

The result of computations in a topological open string theory is usually a number associated to all surfaces topologically equivalent with any of the surfaces from the loop-expansion of the partition function defined in (1) for open string. From the physical point of view, these numbers correspond to the elements of the $S$-matrix that describes scattering processes among arbitrary number of strings on a determined world-sheet and in a fixed spacetime background. The open strings that enter in the process are described by boundaries of $I$ type. There are also closed string described by boundaries of the $S^1$ type. Thus, the elements of the $S$-matrix depend on the number of circles and the genus $g$ of the world-sheet.

Let us turn our attention to the full open and closed string theory. As we have seen in the previous subsection, the closed strings can be described by a TQFT which, at its turn, can be interpreted either as a category or as a commutative Frobenius algebra. Therefore, the open string theory as stated in the Theorem 2 should be supplemented with the Frobenius algebra from the closed string sector and with the compatibility conditions for all sectors. Summarizing these observations, let us state the theorem that defines a open and closed topological string theory [7]

**Theorem 3** A two-dimensional (oriented) open and closed topological string theory is given by the following data:
1. A commutative Frobenius algebra $A$ (for the TQFT.)
2. A commutative Frobenius algebra $V(\partial \Sigma)$ for each boundary condition.
3. A homomorphism $i_{V(\partial \Sigma)} : \mathcal{C} \to C(V(\partial \Sigma))$ where $C(V(\partial \Sigma))$ is the center of $V(\partial \Sigma)$ and $i_{V(\partial \Sigma)} = 1$.
4. For any two algebras $V(\partial \Sigma_1)$ and $V(\partial \Sigma_2)$ the corresponding morphism is given by the composition
   \[
   \pi_2^1 : V(\partial \Sigma_1) \to V(\partial \Sigma_2), \quad \pi_2^1 = i_{V(\partial \Sigma_2)} \circ i_{V(\partial \Sigma_1)}^{-1}.
   \]

The known examples of open and closed string theories have a semi-simple Frobenius algebra $A$. Important examples are represented by simplified versions of topological strings on Calaby-Yau manifolds, useful for understanding the compactification down to four dimensions (see [4, 5, 6, 8] and references therein.)

Theorem 3 and the known examples show that there is an algebraic elementary structure behind topological string theory. Physical properties of such theories are encoded in certain algebraic objects as K-theory groups.

### 3 Topological Field String Theories and $D$-branes

In the perturbative limit of string theory, the $D$-branes are represented by a set of Neumann and Dirichlet boundary conditions which break the Lorentz invariance of spacetime but preserve the conformal invariance of the world-sheet theory. Since the flat spacetime theory is the guiding principle in generalizing the string theory, it is postulated that the $D$-branes in any CFT in two dimensions are defined by the same property. Namely, they are arbitrary local boundary conditions of the two-dimensional CFT that preserve the conformal symmetry on the boundary. A natural technique to extract information from such concept is the renormalization group (RG) when applied to the space of all two-dimensional theories which are conformally invariant in the bulk but may have the conformal invariance broken on the boundary. This point of view is extensively presented in [7] and we refer the reader to this reference for details. Instead, let me speculate on a different line of thought on which we would like to generalize the definitions of topological strings and $D$-branes.

Let us systematize the ideas from the previous section. A two-dimensional topological string theory is a choice of two Frobenius algebras $A_1$ and $A_2$ for the TQFT and the other one for the set of state spaces associated to boundary conditions, respectively, and a map $\mathcal{I} = \{i_{V(\partial \Sigma)}\}$ (the set of all maps $i$ for all boundaries) with adjoint. These algebras are related to the modular category of vector spaces over one-dimensional compact boundaries (freely constructed from unions of elements of the basis $\{I, S^1\}$) with 2d surfaces (morphisms) among them. The $D$-branes are boundary conditions that preserve the conformal invariance and, therefore, induce a subset

\[
\mathcal{D}(\partial \Sigma) \leftrightarrow \mathcal{V}(\partial \Sigma)
\]  

of the set of all state spaces for a given boundary $\partial \Sigma$. This implies that the $D$-branes select a subcategory from the original modular category in which the objects are consistent with the conformal invariance. Next, note that since the composition of surfaces can hold at left and right and the same hold for states in the state spaces, the category of open and closed
strings $\mathcal{S}$ is actually a bicategory. Furthermore, defining a tensor product in a natural way makes $\mathcal{S}$ a tensor category, too, and since the tensor product is associative the category is also strict (see the Appendix.) Now let us recall the following result [12]

**Theorem 4** Let $\mathcal{A}$ be a (strict) tensor category and $A$ be a Frobenius algebra in $\mathcal{A}$. Then there exists and almost bicategory $\Omega$ such that:

1. $\text{Obj}(\Omega) = \{\Gamma, \Lambda\}$ where $\Gamma, \Lambda$ are 2- or 1-morphism, respectively, in the 2-category objects from $\mathcal{A}$.
2. There is an isomorphism $I: \mathcal{A} \to \text{Hom}_{\Omega}(\Gamma, \Gamma)$ of tensor categories.
3. There is an 1-morphism $J: \Lambda \to \Gamma$ and $\bar{J}: \Gamma \to \Lambda$ such the $J\bar{J} = I(A)$.

The almost bicategory $\Omega$ can be turned into a bicategory if a further requirement is imposed on the Frobenius algebra to produce the lacking unit morphism $\Lambda \to \Lambda$.

Now let us apply the Theorem 4 to the topological strings. According to it, one can construct two almost bicategories $\Omega_1$ and $\Omega_2$. The objects in this category are morphisms inside the objects in the tensor categories. At their turn, these objects are $2d$ TQFT and spaces of states, respectively. Therefore, it is natural to think of morphisms between TQFT and between spaces of states which are associated to open and closed topological strings as belonging to the category of topological string field theories. In particular, from the Theorem 4 we have the following identifications for each Frobenius algebra

\[
\text{Hom}_{\Omega}(\Gamma, \Gamma) = \text{Obj}(\mathcal{A}),
\]

\[
\text{Hom}_{\Omega}(\Lambda, \Gamma) = \{XJ, \ X \in \text{Obj}(\mathcal{A})\},
\]

\[
\text{Hom}_{\Omega}(\Lambda, \Lambda) = \{JXJ, \ X \in \text{Obj}(\mathcal{A})\}.
\]

From the first relation (18) we see that if the objects of the new category are maps between $2d$ topological open and closed strings (topological string fields) then the corresponding morphisms are TQFT, too. It would be interesting to investigate what is the exact interpretation of the lift of the $\mathcal{I}$ map in the category $\Omega$ and the implication of the structure $V(\partial\Sigma_1 \sqcup \partial\Sigma_2) \sim \text{Mor}(\partial\Sigma_1, \partial\Sigma_2)$ which is an additive category [8], since it is known that the monoidal Morita equivalence implies equivalent quantum doubles and the same state sum of closed oriented three-manifolds [12].

Let us briefly discuss the implications of the above speculations on the $D$-branes. From the relation (17) we see that introducing $D$-branes in a given topological string theory actually restricts the Frobenius algebra associated with the state spaces to certain closed set of spaces compatible with the conformal symmetry. The exact content of this set is not know, but by using the renormalization group methods one can identify, in principle, the points in the moduli space where the conformal symmetry is unbroken. By the construction given in the Theorem 4, the $D$-branes are lifted to a subset of the almost bicategory $\Omega_2$ which has as objects the morphisms among the spaces of states consistent with the conformal symmetry. In particular, one could interpret these morphisms as $D$-branes in the topological string field theory. The maps among the $D$-branes are, according with the relation (18), objects from the tensor category of state spaces. It is of interest to see if there is any Morita equivalence inherited by $D$-branes from the monoidal equivalence mentioned above. We hope to report in more detail on the above ideas elsewhere.
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4 Appendix

In this appendix we are going to give basic mathematical definitions which have not been included in the text. See for more details [11].

Functoriality Axiom. If $f : M \to M'$ is a homeomorphism of $d + 1$ dimensional manifolds then $(f|_{\partial M})_*(\tau(M)) = \tau(M')$.

Gluing Axiom. If $M$ is a $d + 1$ dimensional manifold, $\partial M = N_1 \sqcup N_2 \sqcup N_3$ and $f : N_1 \to N_2$ is a homeomorphism, then $\tau(M') = \psi(\tau(M))$, where $\psi$ is the map $\psi : \tau(N_1) \otimes \tau(N_2) \otimes \tau(N_3) \to \tau(N_2)^* \otimes \tau(N_2) \otimes \tau(N_3) \to \tau(N_3)$ and $M' = M/f$ is a $d + 1$ dimensional manifold obtained from $M$ by identifying $N_1$ with $\bar{N}_2$ using $f$, i.e. by gluing $N_1$ to $N_2$.

Normalization Axiom. Let $I$ be an interval and $N$ be a $d$ dimensional manifold. Then $\partial(I \times N) = N \sqcup \bar{N}$ and we require that $\tau(I \times N)$ equals the image of $id_{\tau(N)}$ in $\tau(\bar{N}) \otimes \tau(N)$.

Border Axiom. If $B^{d+1}$ is the unit ball in $R^{d+1}$ and $S^d = \partial B^{d+1}$ then $\tau(S^d) = k$ and $\tau(B^{d+1}) = 1 \in k$.

Strict Tensor Category A tensor category is said to be strict if the tensor product $\otimes$ is associative, i.e. $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$.

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