Abstract

In this note, we extend some recent results on the local and global existence of solutions for 3D magneto-hydrodynamics equations to the more general setting of the intermittent initial data, which is characterized through a local Morrey space. This large initial data space was also exhibited in a contemporary work [3] in the context of 3D Navier-Stokes equations.

Keywords : MHD equations; Local Morrey spaces; Global weak solutions; Suitable solutions.

AMS classification : 35Q30, 76D05.

1 Introduction

In a recent work [9], P. Fernandez-Dalgo & P.G. Lemarié-Rieusset obtained new energy controls for the homogeneous and incompressible Navier-Stokes (NS) equations, which allowed them to develop a theory to construct weak solutions for initial data $u_0$ belonging to the weighted space $L^2_{w,\gamma} = L^2(w, dx)$, where, for $0 < \gamma < 2$ we define $w_\gamma(x) = (1 + |x|)^{-\gamma}$. Moreover, this method also gives a new proof of the existence of discretely self-similar solutions.

This new approach has attracted the interest in the research community and more recently, in the paper [3] written by Bradshaw, Tsai & Kukavika, the main theorem on global existence given in [9] is improved with respect to the
initial data \( u_0 \) which belongs to a larger space than the weighted Lebesgue space above. More precisely, the authors prove that if \( u_0 \) verifies
\[
\lim_{R \to +\infty} R^{-2} \int_{|x| \leq R} |u_0(x)|^2 \, dx = 0,
\]
then the (NS) system, with a zero forcing tensor, has a global solution.

Due to the structural similarity between the (NS) equations and the magneto-hydrodynamics equations (see equations (MHD) below) it is quite natural to extend those recent results obtained for the (NS) equations to the more general setting of the coupled magneto-hydrodynamics system which writes down as follows:

\[
\begin{align*}
\partial_t u &= \Delta u - (u \cdot \nabla) u + (b \cdot \nabla) b - \nabla p + \nabla \cdot F, \\
\partial_t b &= \Delta b - (u \cdot \nabla) b + (b \cdot \nabla) u - \nabla q + \nabla \cdot G, \\
\nabla \cdot u &= 0, \quad \nabla \cdot b = 0, \\
u(0, \cdot) &= u_0, \quad b(0, \cdot) = b_0.
\end{align*}
\]

Here the fluid velocity \( u : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3 \), the fluid magnetic field \( b : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3 \), the fluid pressure \( p : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R} \) and the term \( q : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R} \) (which appears in physical models considering Maxwell’s displacement currents [1], [18]) are the unknowns. On the other hand, the data of the problem are given by the fluid velocity at \( t = 0 \): \( u_0 : \mathbb{R}^3 \to \mathbb{R}^3 \); the magnetic field at \( t = 0 \), \( b_0 : \mathbb{R}^3 \to \mathbb{R}^3 \); and the tensors \( F = (F_{ij})_{1 \leq i,j \leq 3} \), \( G = (G_{ij})_{1 \leq i,j \leq 3} \) (where \( F_{ij}, G_{ij} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R} \)) whose divergences: \( \nabla \cdot F, \nabla \cdot G \), represent volume forces applied to the fluids.

In the setting of this coupled system, in a previous work [7], we adapted the energy controls given in [9] for the (NS) equations to the (MHD) equations and this approach allowed us to establish the existence of discretely self-similar solutions for discretely self-similar initial data belonging to \( L^2_{\text{loc}} \), and moreover, the existence of global suitable weak solutions when the initial data \( u_0, b_0 \) belong to the weighted spaces \( L^2_w(\mathbb{R}^3) \), for \( 0 < \gamma \leq 2 \), and the tensor forces \( F, G \) belong to the space \( L^2((0, +\infty), L^2_w(\mathbb{R}^3)) \). For all the details see Theorem 1 and Theorem 2 in [7].

In this paper, we continue with the research program started in [7] for the (MHD) equations; and we relax the method developed in [9] to enlarge the initial data space. Indeed, following some ideas of [2] (for the (NS) equations)
we define $B_2(\mathbb{R}^3) \subset L^2_{\text{loc}}(\mathbb{R}^3)$ as the Banach space of all functions $u \in L^2_{\text{loc}}$ such that:

$$
\|u\|_{B_2}^2 = \sup_{R \geq 1} R^{-2} \int_{|x| \leq R} |u|^2 \, dx < +\infty.
$$

Moreover, we denote $B_2L^2(0, T)$ the Banach space defined as the space of all functions $u \in L^2_{\text{loc}}((0, T) \times \mathbb{R}^3)$ such that

$$
\|u\|_{B_2L^2(0, T)}^2 = \sup_{R \geq 1} R^{-2} \int_{|x| \leq R} \int_0^T |u|^2 \, dt \, dx < +\infty.
$$

In this framework, our main theorem reads as follows:

**Theorem 1** Let $0 < T < +\infty$. Let $u_0, b_0 \in B_2(\mathbb{R}^3)$ be divergence-free vector fields. Let $F$ and $G$ be tensors belonging to $B_2L^2(0, T)$. Then, there exists a time $0 < T_0 < T$ such that the system (MHD) has a solution $(u, b, p, q)$ which satisfies:

- **u, b** belong to $L^\infty((0, T_0), B_2)$ and $\nabla u, \nabla b$ belong to $B_2L^2(0, T_0)$.
- The pressure $p$ and the term $q$ are related to $u, b, F$ and $G$ by:

$$
p = \sum_{1 \leq i, j \leq 3} \mathcal{R}_i \mathcal{R}_j (u_i u_j - b_i b_j - F_{i,j}) \quad \text{and} \quad q = -\sum_{1 \leq i, j \leq 3} \mathcal{R}_i \mathcal{R}_j (G_{i,j}),
$$

where $\mathcal{R}_i = \frac{\partial}{\sqrt{-\Delta}}$ denotes the Riesz transform.
- The map $t \in [0, T) \mapsto (u(t, \cdot), u(t, \cdot))$ is $\ast$-weakly continuous from $[0, T)$ to $B_2(\mathbb{R}^3)$, and for all compact set $K \subset \mathbb{R}^3$ we have:

$$
\lim_{t \to 0} \|(u(t, \cdot) - u_0, b(t, \cdot) - b_0)\|_{L^2(K)} = 0.
$$
- The solution $(u, b, p, q)$ is suitable: there exists a non-negative locally finite measure $\mu$ on $(0, T) \times \mathbb{R}^3$ such that:

$$
\partial_t \left( \frac{|u|^2 + |b|^2}{2} \right) = \Delta \left( \frac{|u|^2 + |b|^2}{2} \right) - |\nabla u|^2 - |\nabla b|^2 - \nabla \cdot \left( \frac{|u|^2}{2} + \frac{|b|^2}{2} + p \right) u \\
+ \nabla \cdot ((u \cdot b) + q |b|) + u \cdot (\nabla \cdot F) + b \cdot (\nabla \cdot G) - \mu.
$$

In particular we have the global control on the solution: for all $0 \leq t \leq T_0$,

$$
\max \{\|(u, b)(t)\|_{B_2}^2, \|\nabla (u, b)\|_{B_2L^2(0, T_0)}^2\} \leq C\|(u_0, b_0)\|_{B_2}^2 \\
+ C\|(F, G)\|_{B_2L^2(0, T)}^2 + C \int_0^t \|(u, b)(s)\|_{B_2}^2 + \|(u, b)(s)\|_{B_2}^6 \, ds.
$$

(1)
Finally, if the data verify:

\[
\lim_{R \to +\infty} R^{-2} \int_{|x| \leq R} |u_0(x)|^2 + |b_0(x)|^2 \, dx = 0,
\]

and

\[
\lim_{R \to +\infty} R^{-2} \int_0^{+\infty} \int_{|x| \leq R} |F(t, x)|^2 + |G(t, x)|^2 \, dx \, ds = 0,
\]

then \((u, b, p, q)\) is a global weak solution.

**Remark 1.1** A vector field \(u\) denotes the vector \((u_1, u_2, u_3)\) and for a tensor \(\mathbb{F} = (F_{i,j})\) we use \(\nabla \cdot \mathbb{F}\) to denote the vector \((\sum_i \partial_i F_{i,1}, \sum_i \partial_i F_{i,2}, \sum_i \partial_i F_{i,3})\).

Thus, if \(\nabla \cdot u = 0\) then we can write \((b \cdot \nabla) u = \nabla \cdot (b \otimes u)\).

It is worth to make the following comments on this result. Remark first that we prove a global control on the solutions (1) which is not exhibited in [3]. This new control is also valid for the (NS) equations (taking \(b = 0, b_0 = 0\) and \(G = 0\) in the (MHD) system). On the other hand, it is interesting to note that the main difference between this result and our previous work [7] is that, in the more general setting of the space \(B^2_2(\mathbb{R}^3)\), the control on the pressure \(p\) and the term \(q\) is a little more technical, and so the method seems not to be adaptable to study the existence of self-similar solutions of equations (MHD) as done in Theorem 2 in [7].

Getting back to the (NS) equations, the global existence and uniqueness of solutions for the 2D case with initial data \(u_0 \in B_2(\mathbb{R}^2)\) is an open problem proposed by A. Basson in [2]. In further research, we thing that it would be interesting to study this problem in the simplest and closest cases with an initial data in \(u_0 \in B_{2,0}(\mathbb{R}^2)\) (see Section 2 for a definition) or \(u_0 \in L^2_{w,\infty}(\mathbb{R}^2)\) with \(0 < \gamma \leq 2\).

This paper is organized as follows. In Section 2 we state some useful tools on the local Morrey spaces. Section 3 is devoted to some *a priori* estimates and stability results on the (MHD) equations, which will allow us to prove our main result in the last Section 4.

## 2 The local Morrey space \(B^p_\gamma\)

In order to understand how Theorem 1 generalizes the results obtained by [9], we recall some useful results obtained in [8]. We consider the space \(\mathbb{R}^d\) only in this section.
Definition 2.1 Let $\gamma \geq 0$ and $1 < p < +\infty$. We denote $B^p_\gamma(\mathbb{R}^d)$ the Banach space of all functions $u \in L^p_{\text{loc}}(\mathbb{R}^d)$ such that:

$$
\|u\|_{B^p_\gamma} = \sup_{R \geq 1} \left( \frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p \, dx \right)^{1/p} < +\infty.
$$

Moreover, for $0 < T \leq +\infty$, $B^p_{\gamma,L}((0,T))$ is the Banach space of all functions $u \subset (L^p_{\gamma,L})_{\text{loc}}([0,T] \times \mathbb{R}^d)$ such that

$$
\|u\|_{B^p_{\gamma,L}((0,T))} = \sup_{R \geq 1} \left( \frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u(t,x)|^p \, dx \, dt \right)^{1/p} < +\infty.
$$

In what follows, we will denote $B^p_{\mathbb{R}}(\mathbb{R}^d) = B^p_0$ and $B^2_2 = B_2$.

Also, the space $B^p_{\gamma,0}$ is defined as the subspace of all functions $u \in B^p_\gamma$ such that $\lim_{R \to +\infty} \frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p \, dx = 0$; and similar, $B^p_{\gamma,0,L}((0,T))$ is the subspace of all functions $u \in B^p_{\gamma,L}((0,T))$ such that $\lim_{R \to +\infty} \frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u(t,x)|^p \, dx \, dt = 0$.

The following result shows how $B^p_\gamma$ is strongly lied with the weighted spaces $L^p_{w_{\gamma}} = L^p(w_{\gamma} \, dx)$ (where $w_{\gamma} = (1 + |x|)\gamma$) considered in [7] and [9].

Lemma 2.1 Consider $\gamma \geq 0$ and let $\gamma < \delta < +\infty$. We have the continuous embedding

$$
L^p_{w_{\gamma}} \subset B^p_{\gamma,0} \subset B^p_{\gamma} \subset L^p_{w_{\delta}}.
$$

Moreover, for all $0 < T \leq +\infty$ we have:

$$
L^p((0,T)), L^p_{w_{\gamma}} \subset B^p_{\gamma,0,L}((0,T)) \subset B^p_{\gamma,L}((0,T)) \subset L^p((0,T)), L^p_{w_{\delta}}.
$$

Proof. Only the embedding $L^p((0,T)), L^p_{w_{\gamma}} \subset B^p_{\gamma,0,L}((0,T))$ is not proved in [8] and we prove it. Let $\lambda > 1$ and $n \in \mathbb{N}$, let $u_n(t,x) = u(t,\lambda^nx)$. We have:

$$
\sup_{R \geq 1} \left( \frac{1}{\lambda^nR^\gamma} \int_0^T \int_{|x| \leq \lambda^nR} |u(t,x)|^p \, dx \, dt \right) = \lambda^{(d-\gamma)n} \sup_{R \geq 1} \left( \int_0^T \int_{|x| \leq R} |u(t,\lambda^n x)|^p \, dx \, dt \right) \leq C \lambda^{(d-\gamma)n} \|u_n\|_{B^p_{\gamma,L}((0,T))} \leq C \lambda^{(d-\gamma)n} \|u_n\|_{L^p_{w_{\gamma}}} \leq C \int_0^T \int_0^T \int_{|x| \leq |s|} |u(s,x)|^p \frac{1}{(\lambda^n + |x|)^\gamma} \, dx \, ds \, dt,
$$

and we conclude by dominated convergence.

Thereafter, we have the following result involving the interpolation theory of Banach spaces:
Theorem 2 ([8]) The space $B^p_\gamma$ can be obtained by interpolation: for all $0 < \gamma < \delta < \infty$ we have $B^p_\gamma = [L^p, L^p_{w_\delta}]^\gamma_\delta \infty$; and the norms $\| \cdot \|_{B^p_\gamma}$ and $\| \cdot \|_{[L^p, L^p_{w_\delta}]^\gamma_\delta \infty}$ are equivalents.

This theorem has a useful corollary and in order to state it we need first the following result on the Muckenhoupt weights (see [10] for a definition).

Lemma 2.2 (Muckenhoupt weights, [9]) If $0 < \delta < d$ and $1 < p < +\infty$. Then, $w_\delta(x) = (1 + |x|)^{-\delta}$ belongs to the Muckenhoupt class $A_p(\mathbb{R}^3)$.

Moreover we have:

• The Riesz transforms $R_j$ are bounded on $L^p_{w_\delta}$: $\| R_j f \|_{L^p_{w_\delta}} \leq C_{p,\delta} \| f \|_{L^p_{w_\delta}}$
• The Hardy–Littlewood maximal function operator is bounded on $L^p_{w_\delta}$:

$$\| \mathcal{M} f \|_{L^p_{w_\delta}} \leq C_{p,\delta} \| f \|_{L^p_{w_\delta}}.$$  

With this lemma at hand, the next important corollary of Theorem 2 follows:

Corollary 2.1 If $0 < \delta < d$ and $1 < p < +\infty$, then we have:

• The Riesz transforms $R_j$ are bounded on $B^p_\delta$: $\| R_j f \|_{B^p_\delta} \leq C_{p,\delta} \| f \|_{B^p_\delta}$
• The Hardy–Littlewood maximal function operator is bounded on $B^p_\delta$:

$$\| \mathcal{M} f \|_{B^p_\delta} \leq C_{p,\delta} \| f \|_{B^p_\delta}.$$  

Proof. Remark that Theorem 2 implies $B^p_{\delta_0} = [L^p, L^p_{w_{\delta_0}}]^{\delta_0_\delta_0 \infty}$, for some $\delta < \delta_0 < d$. So, we conclude directly by Lemma 2.2.

3 Some results for the $(MHD^*)$ system

Our main theorem bases on the two following results for the equations:

$$(MHD^*)\begin{cases}
\partial_t u = \Delta u - (v \cdot \nabla) u + (c \cdot \nabla) b - \nabla p + \nabla \cdot F, \\
\partial_t b = \Delta b - (v \cdot \nabla) b + (c \cdot \nabla) u - \nabla q + \nabla \cdot G, \\
\nabla \cdot u = 0, \ \nabla \cdot b = 0, \\
u(0, \cdot) = u_0, \ b(0, \cdot) = b_0.
\end{cases}$$

In this system, the functions $(v, c)$ are defined as follows:
when we will consider the (MHD) equations we have \((v, c) = (u, b)\).

- when we will consider the regularized (MHD) equations we have \((v, c) = (u + \theta_\epsilon, b + \theta_\epsilon)\), where, for \(0 < \epsilon < 1\) and for a fixed, non-negative and radially non increasing test function \(\theta \in \mathcal{D}(\mathbb{R}^3)\) which is equals to 0 for \(|x| \geq 1\) and \(\int \theta \, dx = 1\); we define \(\theta_\epsilon(x) = \frac{1}{\epsilon^3} \theta(x/\epsilon)\).

### 3.1 A priori estimates

**Theorem 3** Let \(0 < T < +\infty\). Let \(u_0, b_0 \in B_2\) be a divergence-free vector fields and let \(F, G\) be tensors such that \(F, G \in B_2 L^2(0, T)\). Moreover, let \((u, b, p, q)\) be a solution of the problem (MHD*).

We suppose that:

- \(u, b\) belongs to \(L^\infty((0, T), B_2)\) and \(\nabla u, \nabla b\) belongs to \(B_2 L^2(0, T)\).
- The pressure \(p\) and the term \(q\) are related to \(u, b, F\) and \(G\) by
  \[
  p = \sum_{1 \leq i,j \leq 3} \mathcal{R}_i \mathcal{R}_j (v_i u_j - c_i b_j - F_{i,j}) \quad \text{and} \quad q = \sum_{1 \leq i,j \leq 3} \mathcal{R}_i \mathcal{R}_j (v_i b_j - c_j u_i - G_{ij}).
  \]
- The map \(t \in [0, T) \mapsto u(t, \cdot)\) is \(*\)-weakly continuous from \([0, T)\) to \(B_2\), and for all compact set \(K \subset \mathbb{R}^3\) we have:
  \[
  \lim_{t \to 0} \|(u(t, \cdot) - u_0, b(t, \cdot) - b_0)\|_{L^2(K)} = 0.
  \]
- The solution \((u, b, p, q)\) is suitable: there exists a non-negative locally finite measure \(\mu\) on \((0, T) \times \mathbb{R}^3\) such that
  \[
  \partial_t \left(\frac{|u|^2 + |b|^2}{2}\right) = \Delta \left(\frac{|u|^2 + |b|^2}{2}\right) - |\nabla u|^2 - |\nabla b|^2 - \nabla \cdot \left(\frac{|u|^2 + |b|^2}{2}\right) v + p u
  \]
  \[
  + \nabla \cdot ((u \cdot b) c + qb) + u \cdot (\nabla \cdot F) + b \cdot (\nabla \cdot G) - \mu.
  \]

(2)

Then, exists a constant \(C \geq 1\), which does not depend on \(T\), and not on \(u_0, b_0, u, b, F, G\) nor \(\epsilon\), such that:

- We have the following control on \([0, T)\):
  \[
  \max \{\|(u, b)(t)\|_{B_2}, \|\nabla (u, b)\|_{B_2 L^2(0, T)}\} \leq C \|(u_0, b_0)\|_{B_2}^2 + C \int_0^T \|(u, b)(s)\|_{B_2}^2 + \|(u, b)(s)\|_{B_2}^2 \, ds.
  \]

(3)
• Moreover, if \( T_0 < T \) is small enough:

\[
C \left( 1 + \| (u_0, b_0) \|_{B_2}^2 + \| (F, G) \|_{B_2 L^2(0, T_0)}^2 \right)^2 T_0 \leq 1,
\]

then the following control respect to the data holds:

\[
\sup_{0 \leq t \leq T_0} \max \{\| (u, b)(t, \cdot) \|_{B_2}^2, \| \nabla (u, b) \|_{B_2 L^2(0, t)}^2 \} \\
\leq C \left( 1 + \| (u_0, b_0) \|_{B_2}^2 + \| (F, G) \|_{B_2 L^2(0, T_0)}^2 \right).
\]

**Proof.** In this proof, we will focus only in the case \((v, c) = (u \ast \theta_\varepsilon, b \ast \theta_\varepsilon)\) (the case \((v, c) = (u, b)\) can be treated in a similar way). The proof of this theorem follows similar ideas of the proof of Theorem 3 in [7] and we will only detail the main computations.

We start by proving the global control (3). The idea is to apply the energy balance (2) to a suitable test function. Let \( 0 < t_0 < t_1 < T \) and such that \( \partial_t \alpha_{t_0, t_1} \) is the difference between two identity approximations, the first one in \( t_0 \) and the second one in \( t_1 \). For this, we take a non-decreasing function \( \alpha \in C^\infty(\mathbb{R}) \) which is equals to 0 on \((-\infty, \frac{t_0}{2})\) and is equals to 1 on \((1, +\infty)\). Then, for \( 0 < \eta < \min(\frac{t_0}{2}, T - t_1) \) we set the function \( \alpha_{t_0, t_1}(t) = \alpha( \frac{t - t_0}{\eta} ) - \alpha( \frac{t - t_1}{\eta} ) \). On the other hand, we consider a non-negative function \( \phi \in \mathcal{D}(\mathbb{R}^3) \) which is equals to 1 for \(|x| \leq 1/2\) and is equals to 0 for \(|x| \geq 1\); and for \( R \geq 1 \) we set \( \phi_R(x) = \phi(\frac{x}{R}) \).

Thus, by the energy balance (2) we can write

\[
- \int \int \frac{|u|^2}{2} + \frac{|b|^2}{2} \partial_t \alpha_{t_0, t_1} \phi_R \, dx \, ds + \int \int |\nabla u|^2 + |\nabla b|^2 \alpha_{t_0, t_1} \phi_R \, dx \, ds \\
\leq \int \int \frac{|u|^2}{2} + \frac{|b|^2}{2} \alpha_{t_0, t_1} \Delta \phi_R \, dx \, ds \\
+ \sum_{i=1}^3 \int \int (\frac{|u|^2}{2} + \frac{|b|^2}{2})v_i + pu_i \alpha_{t_0, t_1} \partial_i \phi_R \, dx \, ds \\
+ \sum_{i=1}^3 \int \int (u \cdot b) c_i + qb_i \alpha_{t_0, t_1} \partial_i \phi_R \, dx \, ds \\
- \sum_{1 \leq i,j \leq 3} ( \int \int F_{i,j} u_j \alpha_{t_0, t_1} \partial_i \phi_R \, dx \, ds + \int \int F_{i,j} \partial_i u_j \alpha_{t_0, t_1} \phi_R \, dx \, ds ) \\
- \sum_{1 \leq i,j \leq 3} ( \int \int G_{i,j} b_j \alpha_{t_0, t_1} \partial_i \phi_R \, dx \, ds + \int \int G_{i,j} \partial_i b_j \alpha_{t_0, t_1} \phi_R \, dx \, ds ),
\]
and taking the limit when \( \eta \) goes to 0, by the dominated convergence theorem we obtain (when the limit in the left side is well-defined):

\[
- \lim_{\eta \to 0} \int \left( \frac{|u|^2}{2} + \frac{|b|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R \right) dx \, ds + \int_{t_0}^{t_1} \int |\nabla u|^2 + |\nabla b|^2 \phi_R dx \, ds \\
\leq \int_{t_0}^{t_1} \int \frac{|u|^2 + |b|^2}{2} \Delta \phi_R dx \, ds \\
+ \sum_{i=1}^{3} \int_{t_0}^{t_1} \int [(\frac{|u|^2}{2} + \frac{|b|^2}{2})v_i + pu_i] \partial_i \phi_R dx \, ds \\
+ \sum_{i=1}^{3} \int_{t_0}^{t_1} \int [(u \cdot b)c_i + qb_i] \partial_i \phi_R dx \, ds \\
- \sum_{1 \leq i,j \leq 3} (\int_{t_0}^{t_1} \int F_{i,j} u_j \partial_i \phi_R dx \, ds + \int_{t_0}^{t_1} \int F_{i,j} \partial_i u_j \phi_R dx \, ds) \\
- \sum_{1 \leq i,j \leq 3} (\int_{t_0}^{t_1} \int G_{i,j} b_j \partial_i \phi_R dx \, ds + \int_{t_0}^{t_1} \int G_{i,j} \partial_i b_j \phi_R dx \, ds).
\]

We define now the quantity

\[
A_R(t) = \int (|u(t,x)|^2 + |b(t,x)|^2)\phi_R(x) \, dx,
\]

hence, if \( t_0 \) and \( t_1 \) are Lebesgue points of \( A_R(t) \) and moreover, due to the fact that

\[
- \int \left( \frac{|u|^2}{2} + \frac{|b|^2}{2} \right) \partial_t \alpha_{\eta,t_0,t_1} \phi_R dx \, ds = -\frac{1}{2} \int \partial_t \alpha_{\eta,t_0,t_1} A_R(s) \, ds,
\]

we have

\[
\lim_{\eta \to 0} - \int \left( \frac{|u|^2}{2} + \frac{|b|^2}{2} \right) \partial_t \alpha_{\eta,t_0,t_1} \phi_R dx \, ds = \frac{1}{2}(A_R(t_1) - A_R(t_0)).
\]

Then, since \( \phi_R \) is a support compact function we can let \( t_0 \) go to 0 and thus we can replace \( t_0 \) by 0 in this inequality. Moreover, if we let \( t_1 \) go to \( t \), then by the \(*\)-weak continuity we have \( A_R(t) \leq \lim_{t_1 \to t} A_R(t_1) \), and thus we
may replace $t_1$ by $t \in (0, T)$. In this way, for every $t \in (0, T)$ we can write:

$$
\int \left[ \frac{|u(t, x)|^2}{2} + \frac{|b(t, x)|^2}{2} \right] \phi_R \, dx + \int_0^t \int \left( |\nabla u|^2 + |\nabla b|^2 \right) \phi_R \, ds \, dx
$$

$$
\leq \int \left[ \frac{|u_0(x)|^2}{2} + \frac{|b_0(x)|^2}{2} \right] \phi_R \, dx + \int_0^t \int \left( |u|^2 + |b|^2 \right) \Delta \phi_R \, ds \, dx
$$

$$
+ \sum_{i=1}^3 \int_0^t \int \left[ \frac{|u|^2}{2} + \frac{|b|^2}{2} \right] v_i + pu_i \partial_i \phi_R \, ds \, dx
$$

$$
+ \sum_{i=1}^3 \int_0^t \int \left[ (u \cdot b) v_i + q b_i \partial_i \phi_R \right] \, ds \, dx
$$

$$
- \sum_{1 \leq i, j \leq 3} \left( \int_0^t \int F_{i,j} u_j \partial_i \phi_R \, ds \, dx + \int_0^t \int F_{i,j} \partial_i u_j \phi_R \, ds \, dx \right)
$$

$$
- \sum_{1 \leq i, j \leq 3} \left( \int_0^t \int G_{i,j} b_j \partial_i \phi_R \, ds \, dx + \int_0^t \int G_{i,j} \partial_i b_j \phi_R \, ds \, dx \right).$$

In this inequality, we still need to estimate the terms in the right-hand side. For the second term, as $R \geq 1$ we write

$$
\frac{1}{R^2} \int \left( |u|^2 + |b|^2 \right) \Delta \phi_R \, dx \leq \frac{C}{R^2} \int_{B(0, R)} \left( |u|^2 + |b|^2 \right) \, dx \leq C(\|u\|_{B^2} + \|b\|_{B_2}^2).
$$

The third and fourth terms are estimates as follows. We consider first the expressions where the pressure terms $p$ and $q$ do not appear. Using the H"older inequalities and the Sobolev embeddings we have:

$$
\sum_{i=1}^3 \int \frac{(u \cdot b)}{2} \partial_i \phi_R \, dx \leq \|u\|_{L^{12}(B(0, R))} \|b\|_{L^{12}(B(0, R))} \|b \ast \partial_i \phi_R\|_{L^\infty} \leq \frac{C}{R} \|u\|_{L^2(B(0, R))}^{3/4} \|u\|_{L^6(B(0, R))}^{1/4} \|b\|_{L^2(B(0, R))}^{3/4} \|b\|_{L^6(B(0, R))}^{1/4} \leq \frac{C}{R} \|u\|_{L^2(B(0, R))}^{3/4} \|b\|_{L^2(B(0, R))}^{3/4} U^{1/4} B^{5/4},
$$

where we have denoted the quantities

$$
U = \left( \int \phi_{2R} |\nabla u|^2 \, dx \right)^{1/2} + \left( \int_{|x| \leq 2R} |u|^2 \, dx \right)^{1/2}
$$

and

$$
B = \left( \int \phi_{2(R+1)} |\nabla b|^2 \, dx \right)^{1/2} + \left( \int_{|x| \leq 2(R+1)} |b|^2 \, dx \right)^{1/2}.$$
Thus, we can write (by the Young’s inequalities for products with $1 = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{5}{8}$):

$$\frac{1}{R^2} \sum_{i=1}^{3} \int \frac{(\mathbf{u} \cdot \mathbf{b})}{2} (b_i * \theta_{\epsilon}) \partial_i \phi_R \, dx$$

$$\leq C \left( \frac{\|\mathbf{u}\|_{L^2(B(0, R))}}{R} \right)^{3/4} \left( \frac{\|\mathbf{b}\|_{L^2(B(0, R))}}{R} \right)^{3/4} \left( \frac{U}{R} \right)^{1/4} \left( \frac{B}{R} \right)^{5/4}$$

$$\leq C \| (\mathbf{u}, \mathbf{b}) \|_{B_2} + C \| (\mathbf{u}, \mathbf{b}) \|_{B_2}^2 + \frac{C_0}{R^2} \int (\phi_{2R} \nabla \mathbf{u})^2 + (\phi_{2(R+1)} \nabla \mathbf{b})^2 \, dx$$

where $C_0 > 0$ is an arbitrarily small constant.

Now, in order to estimate the expressions where the pressure terms $p$ and $q$ appear, we need the following technical lemma which will be proved at the end of this section.

**Lemma 3.1** Within the hypothesis of Theorem 3, the terms $p$ and $q$ belong $L^{3/2}_{\text{loc}}$. Moreover, there exist an arbitrarily small constant $C_0 > 0$ and a constant $C > 0$, which do not depend on $T$, $\mathbf{u}$, $\mathbf{b}$, $\mathbf{u}_0$, $\mathbf{b}_0$, $\mathbf{F}$, $\mathbf{G}$ nor $\epsilon$; such that for all $R \geq 1$ and for all $0 \leq t \leq T$ we have:

$$\frac{1}{R^2} \sum_{i=1}^{3} \int_0^t \int \left( pu_i + qb_i \right) \partial_i \phi_R \, dx \, ds$$

$$\leq C \| (\mathbf{F}, \mathbf{G}) \|_{B_2L^2(0, t)}^2 + C \int_0^t \| (\mathbf{u}, \mathbf{b})(s) \|_{B_2}^2 + \| (\mathbf{u}, \mathbf{b})(s) \|_{B_2}^6$$

$$+ \frac{C_0}{R^2} \int_0^t \int |\varphi_{2(5R+1)} \nabla \mathbf{u}|^2 + |\varphi_{2(5R+1)} \nabla \mathbf{b}|^2 \, dx \, ds.$$
where $C_0 > 0$ always denote a small enough constant.

Once we dispose of all these estimates, we are able to write

$$
\int \left( \frac{|\mathbf{u}(t, x)|^2}{2} + \frac{|\mathbf{b}(t, x)|^2}{2} \right) \phi_R \, dx + \int_0^t \int \left( |\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2 \right) \phi_R \, ds \, dx
\leq \int \left( \frac{|\mathbf{u}(0, x)|^2}{2} + \frac{|\mathbf{b}(0, x)|^2}{2} \right) \phi_R \, dx + C \|(\mathbf{F}, \mathbf{G})\|_{B_2 L^2(0, t)}^2 ds
+ C \int_0^t \| (\mathbf{u}, \mathbf{b})(s, \cdot) \|^2_{B_2} + \| (\mathbf{u}, \mathbf{b})(s, \cdot) \|_{B_2}^6 \, ds
+ \frac{C_0}{R^2} \int \int \varphi_{2(5R+1)} |\nabla \mathbf{u}|^2 + |\nabla \varphi_{2(5R+1)}| \, dx,
$$

where the desired energy control (3) follows. To finish this proof, the estimate (4) follows directly from (3) and the Lemma 3.1 in [7] (see the proof of Corollary 3.3, page 17, for all the details).

**Proof of Lemma 3.1.** As in the proof of the theorem above, we will consider only the case $(v, c) = (u \ast \theta_\epsilon, b \ast \theta_\epsilon)$. Moreover, we will focus only on the expression which involves the pressure $p$, since the computations for the other expression, where the term $q$ appears, are completely similar.

We write

$$
\frac{1}{R^2} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |p_{u_k}| \, \partial_k \phi_R \, dx \, ds \leq \frac{c}{R^3} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |p_{u_k}| \, dx \, ds,
$$

and recalling that $p = \sum_{1 \leq i, j \leq 3} \mathcal{R}_i \mathcal{R}_j ((u_i \ast \theta_\epsilon)u_j - (b_i \ast \theta_\epsilon)b_j - F_{i,j})$, the last expression allow us to write

$$
\frac{1}{R^2} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |p_{u_k}| \, \partial_k \phi_R \, dx \, ds
\leq \frac{c}{R^3} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |u_{k \sum_{i, j=1}^3 \mathcal{R}_i \mathcal{R}_j ((u_i \ast \theta_\epsilon)u_j)\, dx \, ds
+ \frac{c}{R^3} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |u_{k \sum_{i, j=1}^3 \mathcal{R}_i \mathcal{R}_j ((b_i \ast \theta_\epsilon)b_j - F_{i,j})\, dx \, ds,
$$

and since we have the same information on $u$ and $b$ it is enough to study the last term above. For $R \geq 1$ we define the following expressions:

$$
p_1 = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbb{1}_{|y| \leq 5R}(\theta_\epsilon \ast b_j)) \, p_2 = - \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbb{1}_{|y| \geq 5R}(\theta_\epsilon \ast b_i))b_j,
$$

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and 
\[ p_3 = - \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbf{1}_{|y| < 50R} F_{i,j}), \quad p_4 = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbf{1}_{|y| \geq 50R} F_{i,j}), \]

and then, by the Young’s inequalities (for products), we have

\[
\frac{c}{R^3} \sum_{k=1}^{3} \int_{0}^{t} \int_{|x| \leq R} |u_k|^3 |\sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j ((b_i \ast \theta_{|x|} b_j - F_{i,j})| \leq C \int_{|x| \leq R} (|p_1|^{3/2} + |p_2|^{3/2} + |u|^3 + |p_3|^2 + |p_4|^{2} + |u|^2) \, dx \, ds,
\]

where we will study each term separately.

To study \( p_1 \), by the continuity of \( \mathcal{R}_i \) on \( L^\frac{7}{2}(\mathbb{R}^3) \), since the test function \( \theta_{|x|} \) verifies \( \int \theta_{|x|}(x) \, dx = 1 \) and \( \text{supp}(\theta_{|x|}) \subset B(0, 1) \) and moreover, by the Fubini’s theorem we can write

\[
\int_{|x| \leq R} |p_1|^{3/2} \, dx \leq C \int |p_1|^{3/2} \, dx \leq C \int |(\mathbf{1}_{|x| < 50R} (\theta_{|x|} \ast \mathbf{b}) \otimes \mathbf{b})|^{3/2} \, dx
\]

\[
\leq C \left( \int |\mathbf{1}_{|x| < 50R} (\theta_{|x|} \ast \mathbf{b})|^3 \, dx \right)^{1/2} \left( \int |\mathbf{1}_{|y| < 50R} \mathbf{b}|^3 \, dx \right)^{1/2}
\]

\[
\leq C \left( \int_{|x| \leq 50R} \int \theta_{|x|}(x-z) |\mathbf{b}(z)|^3 \, dz \, dx \right)^{1/2} \left( \int |\mathbf{1}_{|y| < 50R} \mathbf{b}|^3 \, dx \right)^{1/2}
\]

\[
\leq C \int_{|x| \leq 50R+1} |\mathbf{b}|^3 \, dz.
\]

With this estimate at hand, we see that

\[
\int_{|x| \leq R} |\mathbf{u}|^3 + |p_1|^{3/2} \, dx \leq C \int_{|x| \leq 50R+1} |\mathbf{u}|^3 + |\mathbf{b}|^3 \, dx,
\]

and using the Sobolev embedding we write

\[
\frac{C}{R^3} \int_{|x| \leq 50R+1} |\mathbf{u}|^3 \, dx \leq \frac{C}{R^3} ||\mathbf{u}||^{3/2}_{L^2(B(0, 5R+1))} ||\mathbf{u}||^{3/2}_{L^6(B(0, 5R+1))}
\]

\[
\leq \frac{C}{R^{3/2}} ||\mathbf{u}||^{3/2}_{L^2(B(0, 5R+1))} \left( \left( \frac{1}{R^2} \int |\phi_{2(5R+1)} \nabla \mathbf{u}|^2 \, dx \right)^{1/2} + \left( \frac{1}{R^2} \int_{|x| \leq 2(5R+1)} |\mathbf{u}|^2 \, dx \right)^{1/2} \right)^{3/2}
\]

\[
\leq C ||\mathbf{u}||^6_{B_2} + C_0 ||\mathbf{u}||^{2}_{B_2} + C_0 \frac{R}{R^2} \int |\phi_{2(5R+1)} \nabla \mathbf{u}|^2 \, dx,
\]
where \( C_0 > 0 \) is a arbitrarily small constant. Similar bounds works for \( b \).

We study now the term \( p_2 \). Remark first that there exist a constant \( C > 0 \) (which does not depend on \( R > 1 \)) such that for all \(|x| \leq R \) and all \(|y| \geq 5R\), the kernel \( K_{i,j} \) of the operator \( R_i R_j \) verifies \(|K_{i,j}(x-y)| \leq \frac{C}{|y|} \) (see [10] for a proof) and then we write:

\[
\left( \int_{|x| \leq R} |p_2|^{3/2} \, dx \right)^{2/3} \\
\leq C \sum_{i,j} \left( \int_{|x| \leq R} \left( \int |K_{i,j}(x-y)| \left| (\theta \ast b_i)(y) b_j(y) 1_{|y| \geq 5R} \right| \, dy \right)^{3/2} \, dx \right)^{2/3} \\
\leq C \left( \int_{|x| \leq R} \left( \int_{|y| \geq 5R} \frac{1}{|y|^3} |(\theta \ast b) \otimes b| \, dy \right)^{3/2} \, dx \right)^{2/3} \\
\leq CR^2 \int_{|y| \geq 5R} \frac{1}{|y|^3} |(\theta \ast b) \otimes b| \, dy \\
\leq CR^2 \left( \int_{|y| \geq 5R} \frac{1}{|y|^3} |\theta \ast b|^2 \, dy \right)^{1/2} \left( \int_{|y| \geq 5R} \frac{1}{|y|^3} |b|^2 \, dy \right)^{1/2} \\
\leq CR^2 \left( \int_{|y| \geq 5R} \frac{1}{|y|^3} \int_{|y-z| < 1} \theta_\epsilon(y-z) |b(z)|^2 \, dz \, dy \right)^{1/2} \left( \int_{|y| \geq 5R} \frac{1}{|y|^3} |b|^2 \, dy \right)^{1/2} \\
\leq CR^2 \left( \int_{|y| \geq 5R} \frac{1}{|y|^3} \int_{|z| < 1} \theta_\epsilon(y-z) |b(z)|^2 \, dz \, dy \right)^{1/2} \left( \int_{|y| \geq 5R} \frac{1}{|y|^3} |b|^2 \, dy \right)^{1/2} \\
\leq CR^2 \int_{|z| \geq 5R-1} \frac{1}{|z|^3} |b|^2 \, dz.
\]

With this estimate, and the fact that \( B_2(\mathbb{R}^3) \subset L^2_{w_0}(\mathbb{R}^3) \), we finally obtain

\[
\frac{C}{R^3} \int_{|y| \leq R} |p_2|^{3/2} \, dx \leq C \left( \int \frac{1}{(1+|z|)^3} |b|^2 \right)^{3/2} \leq C \|b\|_{B_2}^3.
\]

It remains to estimate the terms \( p_3 \) and \( p_4 \) which involve the tensor \( F \). For \( p_3 \), using the continuity of the Riesz transform \( R_i \) on \( L^2 \), we obtain directly:

\[
\frac{c}{R^3} \int_0^t \int_{|x| \leq R} |p_3|^2 \, dx \, ds \leq \frac{C}{R^3} \sum_{i,j} \int_0^t \int_{|x| < 5R} |F_{i,j}|^2 \, dx \, ds \leq C \|F\|_{B_2 L^2(0,t)}^2.
\]

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For the term $p_4$, remark first that we have
\[
\left( \int_{|x| \leq R} |p_4|^2 \, dx \right)^{1/2} \leq \sum_{i,j} \left( \int_{|x| \leq R} \left( \int_{|y| \geq 5R} |K_{i,j}(x-y)F_{i,j}| \, dy \right)^2 \, dx \right)^{1/2}
\leq \sum_{i,j} R^{3/2} \int_{|y| \geq 5R} \frac{1}{|y|^3} |F_{i,j}| \, dy,
\]
and then, for $0 < \delta < 1$, and by the Hölder inequalities we can write:
\[
\frac{C}{R^{3\delta}} \int_0^t \int_{|x| \leq R} |p_4|^2 \, dx \, ds \leq \sum_{i,j} \int_0^t \left( \int \frac{1}{(1+|x|)^3} |F_{i,j}| \, dx \right)^2 \, ds \leq \sum_{i,j} \int_0^t \left( \int \frac{1}{(1+|x|)^{2+\delta}} |F_{i,j}|^2 \, dx \right)^2 \, ds \leq \sum_{i,j} \int (1+|x|)^{2+\delta} \int_0^t |F_{i,j}|^2 \, ds \, dx \leq C \|F\|^2_{B_2L^2(0,T)}.
\]

The lemma is proven. $\diamond$

### 3.2 A stability result

**Theorem 4** Let $0 < T < +\infty$. Let $u_{0,n}, b_{0,n}$ be divergence-free vector fields such that $(u_{0,n}, b_{0,n}) \in B_2$. Let $F_n$ and $G_n$ be tensors such that $(F_n, G_n) \in B_2L^2(0,T)$. Let $(u_n, b_n, p_n, q_n)$ be a solution of the $(MHD^*)$ problem:

\[
\left\{
\begin{array}{l}
\partial_t u_n = \Delta u_n - (v_n \cdot \nabla) u_n + (c_n \cdot \nabla) b_n - \nabla p_n + \nabla \cdot F_n, \\
\partial_t b_n = \Delta b_n - (v_n \cdot \nabla) b_n + (c_n \cdot \nabla) u_n - \nabla q_n + \nabla \cdot G_n, \\
\nabla \cdot u_n = 0, \quad \nabla \cdot b_n = 0, \\
u_n(0, \cdot) = u_{0,n}, \quad b_n(0, \cdot) = b_{0,n},
\end{array}
\right.
\]

which verifies the same hypothesis of Theorem 3.

If $(u_{1,n}, b_{1,n})$ is strongly convergent to $(u_{0,\infty}, b_{0,\infty})$ in $B_2$, and if the sequence $(F_n, G_n)$ is strongly convergent to $(F_\infty, G_\infty)$ in $B_2L^2(0,T)$; then there exists $(u_\infty, b_\infty, p_\infty, q_\infty)$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ with values in $\mathbb{N}$ such that:
• \((u_{nk}, b_{nk})\) converges \(^\ast\)-weakly to \((u_\infty, b_\infty)\) in \(L^\infty((0, T), B_2)\), \((\nabla u_{nk}, \nabla b_{nk})\) converges weakly to \((\nabla u_\infty, \nabla b_\infty)\) in \(B_2L^2(0, T)\).

• \((u_{nk}, b_{nk})\) converges strongly to \((u_\infty, b_\infty)\) in \(L^2_{\text{loc}}([0, T) \times \mathbb{R}^3)\).

• For \(2 < \gamma < 5/2\), the sequence \((p_{nk}, q_{nk})\) converges weakly to \((p_\infty, q_\infty)\) in \(L^3((0, T), L^{6/5}_{w_{\alpha}}) + L^2((0, T), L^2_{w_{\alpha}})\).

Moreover, \((u_\infty, b_\infty, p_\infty, q_\infty)\) is a solution of the problem \((MHD^*)\):

\[
\begin{align*}
\partial_t u_\infty &= \Delta u_\infty - (u_\infty \cdot \nabla) u_\infty + (b_\infty \cdot \nabla) b_\infty - \nabla p_\infty + \nabla \cdot F_\infty, \\
\partial_t b_\infty &= \Delta b_\infty - (u_\infty \cdot \nabla) b_\infty + (b_\infty \cdot \nabla) u_\infty - \nabla q_\infty + \nabla \cdot G_\infty, \\
\nabla \cdot u_\infty &= 0, \quad \nabla \cdot b_\infty = 0, \\
(u_\infty(0, \cdot) &= u_{0,\infty}, \quad b_\infty(0, \cdot) = b_{0,\infty},
\end{align*}
\]

and verifies all the hypothesis of Theorem \(\Box\).

**Proof.** We will verify that the sequence \((u_n, b_n)\) satisfy the hypothesis of the Rellich lemma (see Lemma 6 in [9]). Remark first that: since for \(2 < \gamma \leq 5/2\) we have that \(u_n, b_n\) is bounded in \(L^\infty((0, T), B_2) \subset L^\infty((0, T), L^2_{w_{\alpha}})\) and moreover, since we have that \(\nabla u_n, \nabla b_n\) is bounded in \(B_2L^2(0, T) \subset L^2((0, T), L^2_{w_{\alpha}})\), then for all \(\phi \in D(\mathbb{R}^3)\) we have that \((\phi u_n, \phi b_n)\) are bounded in \(L^2((0, T), H^1)\). On the other hand, for the pressure \(p_n\) and the term \(q_n\) we write \(p_n = p_{n,1} + p_{n,2}\) with

\[
p_{n,1} = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (v_{n,i} u_{n,j} - c_{n,j} b_{n,j}), \quad p_{n,2} = -\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (F_{n,i,j}),
\]

and we write \(q_n = q_{n,1} + q_{n,2}\) with

\[
q_{n,1} = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (v_{n,i} b_{n,j} - c_{n,i} u_{n,j}), \quad q_{n,2} = -\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (G_{n,i,j}).
\]

From now on we fix \(\gamma \in (2, \frac{5}{2})\), and using the interpolation inequalities and the continuity of the Riesz transforms in the Lebesgue weighted spaces we get that the sequence \((p_{n,1}, q_{n,1})\) is bounded in \(L^3((0, T), L^{6/5}_{w_{\alpha}})\). Indeed, for the term \(p_{n,1}\) recall that by Lemma [22] we have that for \(0 < \gamma < 5/2\) the weight \(w_{6/5}\) belongs to the Muckenhoupt class \(A_p(\mathbb{R}^3)\) (with \(1 < p < +\infty\) and then we can write:

\[
\| \sum_{i,j} R_i R_j (u_{n,i} u_{n,j}) w_{\gamma} \|_{L^{6/5}} \leq \|(u_n \otimes u_n) w_{\gamma} \|_{L^{6/5}} \leq \| \sqrt{w_{\gamma} u_n} \|_{L^2}^2 \| \sqrt{w_{\gamma} u_n} \|_{L^2}^{3/2} \\
\leq \| \sqrt{w_{\gamma} u_n} \|_{L^2}^{3/2} (\| \sqrt{w_{\gamma} u_n} \|_{L^2} + \| \sqrt{w_{\gamma} \nabla u_n} \|_{L^2})^{1/2}.
\]
The term \( q_{n,1} \) is estimated in a similar way. Moreover we have that the sequence and \((p_{n,2}, q_{n,2})\) is bounded in \(L^2((0,T), L^2_{w_{1/2}})\). With these information, by equation (6) we obtain that \((\varphi \partial_t u_n, \varphi \partial_t b_n)\) are bounded in the space \(L^2 L^2 W^{-1.6/5} + L^2 H^{-1} \subset L^2((0,T), H^{-2})\). Thus, we can apply the Rellich lemma and there exists an increasing sequence \((n_k)_{k \in \mathbb{N}}\) in \(\mathbb{N}\), and there exist a couple of functions \((u_\infty, b_\infty)\) such that \((u_{n_k}, b_{n_k})\) converges strongly to \((u_\infty, b_\infty)\) in \(L^2_{\text{loc}}([0,T)\times \mathbb{R}^3)\). We also have that \((v_{n_k}, c_{n_k}) = (v_{n_k} \ast \theta_{\epsilon_{n_k}}, c_{n_k} \ast \theta_{\epsilon_{n_k}})\) converges strongly to \((u_\infty, b_\infty)\) in \(L^2_{\text{loc}}([0,T)\times \mathbb{R}^3)\).

As \((u_n, b_n)\) are bounded in \(L^\infty((0,T), L^2_{w_{1/2}})\) and \((\nabla u_n, \nabla b_n)\) are bounded in \(L^2((0,T), L^2_{w_{1/2}})\), we have that \((u_{n_k}, b_{n_k})\) converges \(*\)-weakly to \((u_\infty, b_\infty)\) in \(L^\infty((0,T), L^2_{w_{1/2}})\) and \((\nabla u_{n_k}, \nabla b_{n_k})\) converges weakly to \((\nabla u_\infty, \nabla b_\infty)\) in \(L^2((0,T), L^2_{w_{1/2}})\). Moreover, by the Sobolev embeddings and the interpolation inequalities we have that \((u_{n_k}, c_{n_k}) = (v_{n_k} \ast \theta_{\epsilon_{n_k}}, c_{n_k} \ast \theta_{\epsilon_{n_k}})\) converges weakly to \((u_\infty, b_\infty)\) in \(L^3((0,T), L^3_{w_{3/2}})\), since it is bounded in \(L^3((0,T), L^3_{w_{3/2}})\). In particular, we may observe that the terms \(v_{n_k,i} u_{n_k,j}, c_{n_k,i} b_{n_k,j}, v_{n_k,i} b_{n_k,j}\) and \(c_{n_k,i} u_{n_k,j}\) are weakly convergent in \((L^{6/5} L^{6/5})_{\text{loc}}\) and thus in \(\mathcal{D}'((0,T)\times \mathbb{R}^3)\).

As those terms are bounded in \(L^3((0,T), L^6_{w_{3/2}})\), they are weakly convergent in \(L^3((0,T), L^6_{w_{3/2}})\); and defining \(p_\infty = p_{\infty,1} + p_{\infty,2}\) with

\[
\begin{align*}
p_{\infty,1} &= \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (v_{\infty,i} u_{\infty,j} - c_{\infty,i} b_{\infty,j}), \\
p_{\infty,2} &= \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (F_{\infty,i,j}),
\end{align*}
\]

and \(q_\infty = q_{\infty,1} + q_{\infty,2}\) with

\[
\begin{align*}
q_{\infty,1} &= \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (v_{\infty,i} b_{\infty,j} - c_{\infty,i} u_{\infty,j}), \\
q_{\infty,2} &= \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (G_{\infty,i,j}),
\end{align*}
\]

we obtain that \((p_{n_k,1}, q_{n_k,1})\) are weakly convergent in \(L^3((0,T), L^6_{w_{3/2}})\) to \((p_\infty,1, q_\infty,1)\), and moreover, we get that \((p_{n_k,2}, q_{n_k,2})\) is strongly convergent in \(L^2((0,T), L^2_{w_{1/2}})\) to \((p_\infty,2, q_\infty,2)\). So, we have that \((u_\infty, b_\infty, p_\infty, q_\infty)\) verify the three first equations in the system \((MHD^*)\) in \(\mathcal{D}'((0,T)\times \mathbb{R}^3)\).

It remains to verify the conditions at the time \(t = 0\). Remark that \((\partial_t u_\infty, \partial_t b_\infty)\) are locally in \(L^2 H^{-2}\), and then \((u_\infty, b_\infty)\) have representatives such that \(t \mapsto (u_\infty(t,\cdot), b_\infty(t,\cdot))\) is continuous from \([0,T)\) to \(\mathcal{D}'(\mathbb{R}^3)\) (hence \(*\)-weakly continuous from \([0,T)\) to \(B_2\)) and moreover, they coincide with
\( u_\infty(0, .) + \int_0^t \partial_t u_\infty \, ds \) and \( b_\infty(0, .) + \int_0^t \partial_t b_\infty \, ds \). Thus, in \( \mathcal{D}'((0, T) \times \mathbb{R}^3) \), we have that
\[
u_{\infty}(0, .) + \int_0^t \partial_t \nu_{\infty} \, ds = \nu_{\infty} = \lim_{n_k \to +\infty} \nu_{n_k} = \lim_{n_k \to +\infty} \nu_{n_k, 0} + \int_0^t \partial_t \nu_{n_k} \, ds
= \nu_{\infty, 0} + \int_0^t \partial_t \nu_{\infty} \, ds,
\]
which implies that \( \nu_{\infty}(0, .) = \nu_{\infty, 0} \). Similar we have the identity \( b_{\infty}(0, .) = b_{\infty, 0} \). We conclude that \( (\nu_{\infty}, b_{\infty}, p_{\infty}, q_{\infty}) \) is a solution of the \((MHD^*)\) equations.

Our next task is to verify the local energy equality. We define the quantity
\[
A_{n_k} = -\partial_t \left( \frac{|u_{n_k}|^2 + |b_{n_k}|^2}{2} \right) + \Delta \left( \frac{|u_{n_k}|^2 + |b_{n_k}|^2}{2} \right) - \nabla \cdot \left( \left( \frac{|u_{n_k}|^2 + |b_{n_k}|^2}{2} \right) v_{n_k} \right)
- \nabla \cdot (p_{n_k} u_{n_k}) - \nabla \cdot (q_{n_k} b_{n_k}) + \nabla \cdot ((u_{n_k} \cdot b_{n_k}) c_{n_k})
+ u_{n_k} \cdot (\nabla \cdot F_{n_k}) + b_{n_k} \cdot (\nabla \cdot G_{n_k}).
\]
Remark that by the information on \((u_n, b_n)\) and by interpolation we have \((u_n, b_n)\) are bounded in \(L^{10/3}((0, T), L^{10/3}_w)\) and then \((u_{n_k}, b_{n_k})\) are locally bounded in \(L^{10/3}_t L^{10/3}_x\) and locally strongly convergent in \(L^2_t L^2_x\). So, \((u_{n_k}, b_{n_k})\) converges strongly in \((L^{3}_t L^{3}_x)_{loc}\). Moreover, by Lemma 3.1 we have that \((p_{n_k}, q_{n_k})\) are locally bounded in \(L^{3/2}_t L^{3/2}_x\). Thus the quantity \(A_{n_k}\) converges in the distributional sense to
\[
A_{\infty} = -\partial_t \left( \frac{|u_{\infty}|^2 + |b_{\infty}|^2}{2} \right) + \Delta \left( \frac{|u_{\infty}|^2 + |b_{\infty}|^2}{2} \right) - \nabla \cdot \left( \left( \frac{|u_{\infty}|^2 + |b_{\infty}|^2}{2} \right) v_{\infty} \right)
- \nabla \cdot (p_{\infty} u_{\infty}) - \nabla \cdot (q_{\infty} b_{\infty}) + \nabla \cdot ((u_{\infty} \cdot b_{\infty}) c_{\infty})
+ u_{\infty} \cdot (\nabla \cdot F_{\infty}) + b_{\infty} \cdot (\nabla \cdot G_{\infty}).
\]
Moreover, recall that by hypothesis of this theorem there exist \(\mu_{n_k}\) a non-negative locally finite measure on \((0, T) \times \mathbb{R}^3\) such that
\[
\partial_t \left( \frac{|u_{n_k}|^2 + |b_{n_k}|^2}{2} \right) = \Delta \left( \frac{|u_{n_k}|^2 + |b_{n_k}|^2}{2} \right) - |\nabla u_{n_k}|^2 - |\nabla b_{n_k}|^2
- \nabla \cdot \left( \left( \frac{|u_{n_k}|^2 + |b_{n_k}|^2}{2} \right) v_{n_k} \right) - \nabla \cdot (p_{n_k} u_{n_k}) - \nabla \cdot (q_{n_k} b_{n_k})
+ \nabla \cdot ((u_{n_k} \cdot b_{n_k}) c_{n_k}) + u_{n_k} \cdot (\nabla \cdot F_{n_k}) + b_{n_k} \cdot (\nabla \cdot G_{n_k}) - \mu_{n_k}.
\]
Then, by definition of \(A_{n_k}\) we can write \(A_{n_k} = |\nabla u_{n_k}|^2 + |\nabla b_{n_k}|^2 + \mu_{n_k}\), and thus we have \(A_{\infty} = \lim_{n_k \to +\infty} |\nabla u_{n_k}|^2 + |\nabla b_{n_k}|^2 + \mu_{n_k}\).
Now, let $\Phi \in D((0, T) \times \mathbb{R}^3)$ be a non-negative function. As $\sqrt{\Phi}(\nabla u_n + \nabla b_n)$ is weakly convergent to $\sqrt{\Phi}(\nabla u_\infty + \nabla b_\infty)$ in $L^2_tL^2_x$, we have
\[
\int A_\infty \Phi \, dx \, ds = \lim_{n_k \to +\infty} \int A_{n_k} \Phi \, dx \, ds \geq \limsup_{n_k \to +\infty} \int (|\nabla u_{n_k}|^2 + |\nabla b_{n_k}|^2) \Phi \, dx \, ds
\]
\[
\geq \int (|\nabla u_\infty|^2 + |\nabla b_\infty|^2) \Phi \, dx \, ds.
\]
Thus, there exists a non-negative locally finite measure $\mu_\infty$ on $(0, T) \times \mathbb{R}^3$ such that $A_\infty = (|\nabla u_\infty|^2 + |\nabla b_\infty|^2) + \mu_\infty$, and then we obtain the desired local energy equality:
\[
\partial_t (\frac{|u_\infty|^2 + |b_\infty|^2}{2}) = \Delta (\frac{|u_\infty|^2 + |b_\infty|^2}{2}) - |\nabla u_\infty|^2 - |\nabla b_\infty|^2
\]
\[
- \nabla \cdot (\frac{|u_\infty|^2}{2} + \frac{|b_\infty|^2}{2}v_\infty) - \nabla \cdot (p_\infty u_\infty) - \nabla \cdot (q_\infty b_\infty)
\]
\[
+ \nabla \cdot ((u_\infty \cdot b_\infty)c_\infty) + u_\infty \cdot (\nabla \cdot F_\infty) + b_\infty \cdot (\nabla \cdot G_\infty) - \mu_\infty.
\]
In order to finish this proof, it remains to prove the convergence to the initial data $(u_{0,\infty}, b_{0,\infty})$. Once we dispose of this local energy equality, as in (5) we can write:
\[
\int \frac{|u_n(t,x)|^2 + |b_n(t,x)|^2}{2} \phi_R \, dx + \int_0^t \int (|\nabla u|^2 + |\nabla b|^2) \phi_R \, dx \, ds
\]
\[
\leq \int \frac{|u_{0,n}(x)|^2 + |b_{0,n}(x)|^2}{2} \phi_R \, dx + \int_0^t \int \frac{|u_n|^2 + |b_n|^2}{2} \Delta \phi_R \, dx \, ds
\]
\[
+ \sum_{i=1}^3 \int_0^t \int [(\frac{|u_n|^2}{2} + \frac{|b_n|^2}{2})v_{n,i} + p_n u_{n,i}] \partial_i \phi_R \, dx \, ds
\]
\[
+ \sum_{i=1}^3 \int_0^t \int [(u_n \cdot b_n)c_{n,i} + q_n b_{n,i}] \partial_i \phi_R \, dx \, ds
\]
\[
- \sum_{1 \leq i, j \leq 3} \int_0^t \int F_{n,i,j} u_{n,j} \partial_i \phi_R \, dx \, ds + \int_0^t \int F_{n,i,j} \partial_i u_{n,j} \phi_R \, dx \, ds
\]
\[
- \sum_{1 \leq i, j \leq 3} \int_0^t \int G_{n,i,j} b_{n,j} \partial_i \phi_R \, dx \, ds + \int_0^t \int G_{n,i,j} \partial_i b_{j} \phi_R \, dx \, ds.
\]
Then we have:

\[
\limsup_{n_k \to +\infty} \int \frac{|u_{n_k}(t, x)|^2 + |b_{n_k}(t, x)|^2}{2} \phi_R \, dx + \int_0^t \int (|\nabla u_{n_k}|^2 + |\nabla b_{n_k}|^2) \phi_R \, dx \, ds \\
\leq \int \frac{|u_0(t)|^2 + |b_0(t)|^2}{2} \phi_R \, dx + \int_0^t \int \frac{|u_{\infty}|^2 + |b_{\infty}|^2}{2} \Delta \phi_R \, dx \, ds \\
+ \sum_{i=1}^3 \int_0^t \int \left( \frac{|u_{\infty}|^2}{2} + \frac{|b_{\infty}|^2}{2} \right) v_{\infty,i} + p_{\infty} u_{\infty,i} \partial_i \phi_R \, dx \, ds \\
+ \sum_{i=1}^3 \int_0^t \int \left( \frac{|u_{\infty} \cdot b_{\infty}|}{2} \right) c_{\infty,i} + q_{\infty} b_{\infty,i} \partial_i \phi_R \, dx \, ds \\
- \sum_{1 \leq i, j \leq 3} \left( \int_0^t \int F_{\infty,i,j} u_{\infty,j} \partial_i \phi_R \, dx \, ds + \int_0^t \int F_{\infty,i,j} \partial_i u_{\infty,j} \phi_R \, dx \, ds \right) \\
- \sum_{1 \leq i, j \leq 3} \left( \int_0^t \int G_{\infty,i,j} b_{\infty,j} \partial_i \phi_R \, dx \, ds + \int_0^t \int G_{\infty,i,j} \partial_i b_j \phi_R \, dx \, ds \right).
\]

Recalling that \( u_{n_k} = u_{0,n_k} + \int_0^t \partial_t u_{n_k} \, ds \), we may observe that \( u_{n_k}(t, .) \) converges to \( u_{\infty}(t, .) \) in \( \mathcal{D}'(\mathbb{R}^3) \), hence, it converges weakly in \( L^2_{\text{loc}}(\mathbb{R}^3) \) and we can write:

\[
\int \frac{|u_{\infty}(t, x)|^2}{2} \phi_R \, dx \leq \limsup_{n_k \to +\infty} \int \frac{|u_{n_k}(t, x)|^2}{2} \phi_R \, dx.
\]

Moreover, this weakly convergence gives

\[
\int_0^t \int \frac{|\nabla u_{\infty}(s, x)|^2}{2} \phi_R \, dx \, ds \leq \limsup_{n_k \to +\infty} \int_0^t \int \frac{|\nabla u_{n_k}(s, x)|^2}{2} \phi_R \, dx \, ds,
\]

and we have the same estimates for \( b_{\infty} \). In this way we get

\[
\int \frac{|u_{\infty}(t,x)|^2 + |b_{\infty}(t,x)|^2}{2} \phi_R \, dx + \int_0^t \int (|\nabla u_{\infty}|^2 + |\nabla b_{\infty}|^2) \phi_R \, dx \, ds \\
\leq \int \frac{|u_0(t)|^2 + |b_0(t)|^2}{2} \phi_R \, dx + \int_0^t \int \frac{|u_{\infty}|^2 + |b_{\infty}|^2}{2} \Delta \phi_R \, dx \, ds \\
+ \sum_{i=1}^3 \int_0^t \int \left( \frac{|u_{\infty}|^2}{2} + \frac{|b_{\infty}|^2}{2} \right) v_{\infty,i} + p_{\infty} u_{\infty,i} \partial_i \phi_R \, dx \, ds
\]
Thus we have the strong convergence to initial data in the Hilbert space $L^p$. Following the ideas of [7], for the given function $\phi_R(x) = \phi(\frac{x}{R})$ and the Leray’s projector $P$, we define $u_{0,R} = P(\phi_R u_0)$, $b_{0,R} = P(\phi_R b_0)$, $F_R = \phi_R F$, $G_R = \phi_R G$; and we consider the approximated problem $(MHD_{R,\epsilon})$:

$$
\begin{align*}
\partial_t u_{R,\epsilon} &= \Delta u_{R,\epsilon} - ((u_{R,\epsilon} \ast \theta_\epsilon) \cdot \nabla) u_{R,\epsilon} + ((b_{R,\epsilon} \ast \theta_\epsilon) \cdot \nabla) b_{R,\epsilon} + \nabla p_{R,\epsilon} + \nabla \cdot F_R, \\
\partial_t b_{R,\epsilon} &= \Delta b_{R,\epsilon} - ((u_{R,\epsilon} \ast \theta_\epsilon) \cdot \nabla) b_{R,\epsilon} + ((b_{R,\epsilon} \ast \theta_\epsilon) \cdot \nabla) u_{R,\epsilon} - \nabla q_{R,\epsilon} + \nabla \cdot G_R, \\
\nabla \cdot u_{R,\epsilon} &= 0, \quad \nabla \cdot b_{R,\epsilon} = 0, \\
u_{R,\epsilon}(0, \cdot) &= u_{0,R}, \quad b_{R,\epsilon}(0, \cdot) = b_{0,R}.
\end{align*}
$$

Finally, letting $t$ go to 0, we have:

$$\limsup_{t \to 0} \|(u_{\infty}, b_{\infty})(t, \cdot)\|_{L^2(\phi_R(x)dx)}^2 \leq \|(u_{0,\infty}, b_{0,\infty})\|_{L^2(\phi_R(x)dx)}^2.$$

On the other hand, by weakly convergence we also have

$$\|(u_{0,\infty}, b_{0,\infty})\|_{L^2(\phi_R(x)dx)}^2 \leq \liminf_{t \to 0} \|(u_{\infty}, b_{\infty})(t, \cdot)\|_{L^2(\phi_R(x)dx)}^2.$$

Thus we have the strong convergence to initial data in the Hilbert space $L^2(\phi_R(x)dx)$.

4 Proof of Theorem

4.1 Local in time existence

Following the ideas of [7], for the given function $\phi_R(x) = \phi(\frac{x}{R})$ and the Leray’s projector $P$, we define $u_{0,R} = P(\phi_R u_0)$, $b_{0,R} = P(\phi_R b_0)$, $F_R = \phi_R F$, $G_R = \phi_R G$; and we consider the approximated problem $(MHD_{R,\epsilon})$:

By the Appendix in [7] (see the page 35) we know that $(MHD_{R,\epsilon})$ has a unique solution $(u_{R,\epsilon}, b_{R,\epsilon})$ in $L^\infty((0, +\infty), L^2) \cap L^2((0, +\infty), \dot{H}^1)$, and moreover, this solution belongs to $C([0, +\infty), L^2)$ and it fulfills the hypothesis of the Theorem. Applying this result (for the case $(\nu, c) = (u \ast \theta_\epsilon, b \ast \theta_\epsilon)$) there exists a constant $C > 0$ such that for every time $T_0$ small enough:

$$C \left(1 + \|(u_{0,R}, b_{0,R})\|_{\dot{B}_2^1}^2 + \|(F_{R,\epsilon}, G_{R,\epsilon})\|_{\dot{B}_2^1 L^2(0,T_0)}^2\right)^2 T_0 \leq 1.$$
we have the controls:
\[
\sup_{0 \leq t \leq T_0} \| (u_{R,t}, b_{R,t})(t) \|_{B_2}^2 \leq C \left( 1 + \| (u_{0,R}, b_{0,R}) \|_{B_2}^2 + \| (F_{R,t}, G_{R,t}) \|_{B_2 L^2(0,T_0)}^2 \right),
\]
and
\[
\| \nabla (u_{R,t}, b_{R,t}) \|_{B_2 L^2(0,T_0)}^2 \leq C \left( 1 + \| (u_{0,R}, b_{0,R}) \|_{B_2}^2 + \| (F_{R,t}, G_{R,t}) \|_{B_2 L^2(0,T_0)}^2 \right).
\]

Then, in the setting of Theorem 4.2, we set \((u_{0,n}, b_{0,n}) = (u_{0,R_n}, b_{0,R_n}), F_n = F_{R_n}, G_n = G_{R_n}\) and \((u_n, b_n) = (u_{R_n,t,n}, b_{R_n,t,n})\); and letting \(R_n \to +\infty\) and \(\epsilon_n \to 0\) we find a local solution of the (MHD) equations which verifies the desired properties stated in Theorem 4.2.

### 4.2 Global in time existence

Let \(\lambda > 1\). For \(n \in \mathbb{N}\) we consider the (MHD) equations with initial value
\[
(u_{0,n}, b_{0,n}) = (\lambda^n u_0(\lambda^n \cdot), \lambda^n b_0(\lambda^n \cdot)),
\]
and the forcing tensors
\[
(F_n, G_n) = (\lambda^{2n} F(\lambda^{2n} \cdot, \lambda^n \cdot), \lambda^{2n} G(\lambda^{2n} \cdot, \lambda^n \cdot)).
\]
Then, by the local in time existence proved above, there exists a solution \((v_n, c_n)\) on \((0, T_n)\), with
\[
C \left( 1 + \| (v_{0,n}, c_{0,n}) \|_{B_2}^2 + \| (F_n, G_n) \|_{B_2 L^2(0,T_n)}^2 \right)^2 T_n = 1.
\]
Remark also that by the well-known scaling properties of the (MHD) equations we have
\[
(v_n(t, x), c_n(t, x)) = (\lambda^n u_n(\lambda^{2n} t, \lambda^n x), \lambda^n b_n(\lambda^{2n} t, \lambda^n x)),
\]
where \((u_n, b_n)\) is a solution of the (MHD) on \((0, 2\lambda^n T_n)\) associated with the initial data \((u_0, b_0)\) and then forcing tensors \(F\) and \(G\).

At this point, we need the following simple remark which will be proved at the end of this section.

**Remark 4.1** If \(u_0, b_0 \in B_{2,0}\) and \(F, G \in B_{2,0} L^2(0, +\infty)\), then for all \(\lambda > 1\) we have:
\[
\lim_{n \to +\infty} \frac{\lambda^n}{1 + \| (v_{0,n}, c_{0,n}) \|_{B_2}^2 + \| (F_n, G_n) \|_{B_2 L^2}} = +\infty.
\]
Thus, for $\lambda > 1$ fix we have $\lim_{n \to +\infty} \lambda^{2n} T_n = +\infty$. Then, for $T > 0$, let $n_T$ such that $\lambda^{2n_T} T_n > T$ for $n \geq n_T$, then $(u_n, b_n)$ is a solution of the (MHD) equations on $(0, T)$.

We set now $(w_n(t, x), d_n(t, x)) = (\lambda^{n_T} u_n(\lambda^{2n_T} t, \lambda^{n_T} x), \lambda^{n_T} b_n(\lambda^{2n_T} t, \lambda^{n_T} x))$, where we observe that for $n \geq n_T$ the couple $(w_n, d_n)$ is a solution of (MHD) equations on $(0, \lambda^{-2n_T} T)$ with initial value $(v_{0,n_T}, c_{0,n_T})$ and forcing tensor $(F_{n_T}, G_{n_T})$. But, since we have $\lambda^{-2n_T} T \leq T_{n_T}$, then we obtain

$$C \left( 1 + \| (v_{0,n_T}, c_{0,n_T}) \|_{B_2}^2 + \| (F_{n_T}, G_{n_T}) \|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2 \right)^2 \lambda^{-2n_T} T \leq 1,$$

and thus, by Theorem 3 we are able to write:

$$\sup_{0 \leq t \leq \lambda^{-2n_T} T} \| (w_n, d_n)(t, \cdot) \|_{L^2_w}^2 \leq C (1 + \| (v_{0,n_T}, c_{0,n_T}) \|_{B_2}^2 + \| (F_{n_T}, G_{n_T}) \|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2),$$

and

$$\| \nabla (w_n, d_n) \|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2 \leq C (1 + \| (v_{0,n_T}, c_{0,n_T}) \|_{B_2}^2 + \| (F_{n_T}, G_{n_T}) \|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2).$$

From these estimates we get the following uniforms controls for $u_n$ and $b_n$:

$$\| (w_n, d_n)(t) \|_{B_2}^2 \geq \lambda^{n_T} \| (u_n, b_n)(\lambda^{2n_T} t, \cdot) \|_{B_2}^2,$$

and

$$\| \nabla (w_n, d_n) \|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2 \geq \lambda^{n_T} \| \nabla (u_n, b_n) \|_{B_2 L^2(0, T)}^2.$$

In order to finish this proof, observe that we have controlled uniformly $u_n, b_n$ and $\nabla u_n, \nabla b_n$ on $(0, T)$ for $n \geq n_T$. Then, we may apply Theorem 4 to obtain a solution on $(0, T)$. As $T > 0$ is an arbitrary time, we can use a diagonal argument to obtain a solution $u, b$ on $(0, +\infty)$. Finally, the control for the solution $(u, b, p, q)$ on $(0, +\infty)$ is given by Theorem 3.

Proof of Remark 4.1. It is enough to detail the computations for the functions $u_{0,n}$ and $F_n$ since the computations for $b_{0,n}$ and $G_n$ follows the same lines.

It is straightforward to see that we have

$$\| v_{0,n} \|_{B_2^2}^2 = \sup_{R \geq 1} \frac{1}{\lambda^n R^2} \int_{|x| \leq R} |\lambda^n u_0(\lambda^n x)|^2 \, dx = \sup_{R \geq 1} \frac{1}{(\lambda^n R)^2} \int_{|x| \leq \lambda^n R} |u_0(x)|^2 \, dx,$$
\[ \lim_{R \to +\infty} \sup_{R \geq P} \frac{1}{(\lambda^n R)^2} \int_{|x| \leq \lambda^n R} |u_0(x)|^2 \, dx = \lim_{R \to +\infty} \frac{1}{R^2} \int_{|x| \leq R} |u_0(x)|^2 \, dx = 0. \]

Moreover, remark that we have:

\[ \frac{\|F_n\|_{B_2 L^2(0, +\infty)}}{\lambda^n} = \sup_{R \geq 1} \frac{1}{\lambda^n R^2} \int_0^{+\infty} \int_{|x| \leq R} |\lambda^{2n} F(\lambda^2 t, \lambda^n x)|^2 \, dx \, ds \]

\[ = \sup_{R \geq 1} \frac{1}{(\lambda^n R)^2} \int_0^{+\infty} \int_{|x| \leq \lambda^n R} |F(t, x)|^2 \, dx, \]

and

\[ \lim_{P \to +\infty} \sup_{R \geq P} \frac{1}{R^2} \int_0^{+\infty} \int_{|x| \leq R} |F(t, x)|^2 \, dx \, ds = \lim_{R \to +\infty} \frac{1}{R^2} \int_0^{+\infty} \int_{|x| \leq R} |F(t, x)|^2 \, dx \, ds = 0. \]

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