MAJORISATION WITH APPLICATIONS TO THE CALCULUS OF VARIATIONS

MARIUS BULIGA

Abstract. This paper explores some connections between rank one convexity, multiplicative quasiconvexity and Schur convexity. Theorem 5.1 gives simple necessary and sufficient conditions for an isotropic objective function to be rank one convex on the set of matrices with positive determinant. Theorem 6.2 describes a class of possible non-polyconvex but multiplicative quasiconvex isotropic functions. This class is not contained in a well known theorem of Ball (6.3 in this paper) which gives sufficient conditions for an isotropic and objective function to be polyconvex. We show here that there is a new way to prove directly the quasiconvexity (in the multiplicative form). Relevance of Schur convexity for the description of rank one convex hulls is explained.

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1. Introduction

There is a strong resemblance between the following two theorems. The first theorem is (Horn, (1954), Thompson (1971), theorem 1.4):

**Theorem 1.1.** Let $X, Y$ be any two positive definite $n \times n$ matrices and let $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$ denote the respective sets of eigenvalues. Then there is an unitary matrix $U$ such that $XU$ and $Y$ have the same spectrum if and only if:

$$\prod_{i=1}^{k} x_i \geq \prod_{i=1}^{k} y_i , \quad k = 1,..., n-1$$

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i$$

The second theorem is (Dacorogna, Tanteri (2001), theorem 20, see also Dacorogna, Marcellini (1999)):

**Theorem 1.2.** Let $0 \leq \lambda_1(A) \leq ... \leq \lambda_n(A)$ denote the singular values of a matrix $A \in R^{n \times n}$ and

$$E(a) = \left\{ A \in R^{n \times n} : \lambda_i(A) = a_i , \quad i = 1,..., n , \quad \det A = \prod_{i=1}^{n} a_i \right\}$$

The following then holds

$$P_{co} E = R_{co} E(a) = \left\{ A \in R^{n \times n} : \prod_{i=\nu}^{n} \lambda_i(A) \leq \prod_{i=\nu}^{n} a_i , \quad \nu = 2,..., n , \quad \det A = \prod_{i=1}^{n} a_i \right\}$$

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where $PCo$, $Rco$ stand for polyconvex, rank one convex envelope.

Both theorems can be understood as describing the set \( \{ y : y \preccurlyeq x \} \) where \( \preccurlyeq \) is some order relation connected to the products which rise in each theorem.

It turns out that a common frame of these apparently scattered results is the notion of majorisation. Majorisation comes in pair with Schur convexity. The purpose of this note is to study the monotonicity (Schur convexity in particular) properties of rank one convex functions with respect to majorisation relation.

The content of the paper is described further. After the setting of notations in section 2, section 3 gives a brief passage trough basic properties of the majorisation relations. Section 4 lists some properties of singular values and eigenvalues of matrices connected to majorisation. Section 5 concerns simple necessary and sufficient conditions for an objective, isotropic function to be rank one convex on the set of matrices with positive determinant. In section 6 is described a class of objective isotropic functions which are multiplicative quasiconvex. This class is interesting because most of its elements seem to be non-polyconvex. This class is not contained in a well known theorem of Ball (6.3 in this paper) which gives sufficient conditions for an isotropic and objective function to be polyconvex. It turns out that Schur convexity can be used to prove quasiconvexity. In section 7 the resemblance between theorems 1.1 and 1.2 is explained.

2. Notations

\begin{align*}
A,B, \ldots & \quad \text{real or complex matrices} \nonumber \\
x,y,u,v, \ldots & \quad \text{real or complex vectors} \nonumber \\
\lambda(A) & \quad \text{the vector of eigenvalues of } A 
\nonumber \\
\sigma(A) & \quad \text{the vector of singular values of } A 
\nonumber \\
A^* & \quad \text{the conjugate transpose of } A 
\nonumber \\
A^T & \quad \text{the transpose of } A 
\nonumber \\
diag(A) & \quad \text{the diagonal of } A, \text{ seen as a vector} 
\nonumber \\
Diag(v) & \quad \text{the diagonal matrix constructed from the vector } v 
\nonumber \\
S_n & \quad \text{the set of permutation (of coordinates) matrices} 
\nonumber \\
\text{conv}(A) & \quad \text{the convex hull of the set } A 
\nonumber \\
\circ & \quad \text{function composition} 
\end{align*}

For any matrix $A \in gl(n, C)$, the matrix $A^*A$ is Hermitian. The eigenvalues of the square root of $A^*A$ are, by definition, the singular values of $A$.

Matrices are identified with linear transformations.

For a vector $x \in R^n$ we denote by $x^\downarrow$, $x^\uparrow$, the vectors obtained by rearranging the coordinates of $x$ in decreasing, respectively increasing orders.

3. Basics about majorisation

We have used Bhatia [4], Chapter 2, and Marshall, Olkin [13], Chapters 1-3. The results are given in the logical order.

**Definition 3.1.** The following majorisation notions are partial order relations in $R^n$. Let $x, y \in R^n$ be arbitrary vectors. Then:

- $x \preceq y$ if $x_i \leq y_i$ for any $i \in \{1, \ldots, n\}$.
- $x \prec_w y$ if

\[
\sum_{j=1}^{k} x_j^\downarrow \leq \sum_{j=1}^{k} y_j^\downarrow
\]
for any \( k \in \{1, ..., n\} \). We say that \( x \) is submajorised by \( y \).

- \( x \prec y \) if \( x \prec_w y \) and

\[
\sum_{j=1}^{n} x_j^+ = \sum_{j=1}^{n} y_j^+
\]

We say that \( x \) is majorised by \( y \).

The notion of majorisation, the last in definition 3.1, is the most interesting. See Marshall, Olkin [13], Chapter 1, for the various places where one can encounter it. The majorisation is in closed relationship with the notions of a T-transform and a doubly-stochastic matrix.

**Definition 3.2.** A linear map \( A \) on \( \mathbb{R}^n \) is a T-transform if there are \( t \in [0, 1] \), \( i, j \in \{1, ..., n\} \) such that

\[
(Ax)_k = x_k
\]

for any \( k \) different from \( i, j \),

\[
(Ax)_i = tx_i + (1-t)x_j, \quad (Ax)_j = tx_j + (1-t)x_i
\]

A matrix \( A \in gl(n, \mathbb{R}) \) is called doubly-stochastic if

\[
A_{ij} \geq 0
\]

\[
\sum_{k=1}^{n} A_{kj} = 1, \quad \sum_{k=1}^{n} A_{ik} = 1
\]

for all \( i, j \). Any T-transform is doubly-stochastic. Matrices which correspond to permutation of coordinates are also doubly-stochastic.

The property of a matrix of being doubly stochastic can be formulated in terms of majorisation.

**Theorem 3.1.** A matrix \( A \) is doubly stochastic if and only if \( Ax \prec x \) for any \( x \in \mathbb{R}^n \).

Conversely, we have:

**Theorem 3.2.** The following statements are equivalent:

(i) \( x \prec y \)

(ii) \( x \) is obtained from \( y \) by a finite number of T-transforms

(iii) \( x = Ay \) for some doubly stochastic matrix \( A \)

There are strong connections between majorisation, doubly stochastic matrices and convexity. These will make the subject relevant for the Calculus of Variations further.

**Theorem 3.3.** (Birkhoff) The set of doubly stochastic matrices is the convex hull of the set of permutation matrices.

**Theorem 3.4.** (Hardy, Littlewood, Polya) The following statements are equivalent:

(i) \( x \prec y \)

(ii) \( x \) is in the convex hull of \( S_n x \)
(iii) for any convex function $\phi$ from $\mathbb{R}$ to $\mathbb{R}$ we have
$$\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i)$$

With any order relation comes an associated monotonicity notion.

**Definition 3.3.** Consider a map $\Phi$ defined from an $S_n$ invariant set in $\mathbb{R}^n$, with range in $\mathbb{R}^m$. We say that $\Phi$ is:
- increasing if $x \leq y \implies \Phi(x) \leq \Phi(y)$
- convex if for all $t \in [0,1]$
  $$\Phi(tx + (1-t)y) \leq t\Phi(x) + (1-t)\Phi(y)$$
- isotone if $x \prec y \implies \Phi(x) \prec_w \Phi(y)$
- strongly isotone if $x \prec_w y \implies \Phi(x) \prec_w \Phi(y)$
- strictly isotone if $x < y \implies \Phi(x) < \Phi(y)$

Any isotone $\Phi$ with range in $\mathbb{R}$ is called Schur-convex. Note that convexity in the sense of this definition matches with the classical notion for functions $\Phi$ with range in $\mathbb{R}$.

The next theorem shows that symmetric convex maps are isotone.

**Theorem 3.5.** Let $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ be convex. If for any $P \in S_n$ there is $P' \in S_m$ such that $\Phi \circ P = P' \circ \Phi$ then $\Phi$ is isotone. If in addition $\Phi$ is monotone increasing then $\Phi$ is strictly isotone.

In particular any $L^p$ norm on $\mathbb{R}^n$ is Schur-convex. Not any isotone function is convex, though. Important examples are the elementary symmetric polynomials, which are not convex but they are Schur-concave.

One can give three characterizations of isotone (or Schur convex) functions $f : \mathbb{R}^n \to \mathbb{R}$. Before that we need some notations.

Let us begin by noticing that the permutation group $S_n$ acts on $GL(n, \mathbb{R})^+$ in this obvious way: for any $P \in S_n$ and any $F \in GL(n, \mathbb{R})^+$ the matrix $P.F \in GL(n, \mathbb{R})^+$ has components $(P.F)_{ij} = F_{P(i)P(j)}$.

Let
$$\mathcal{D} = \{ x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n \}$$

We shall call a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ symmetric if for any permutation matrix $P \in S_n$ $P(A) \subset A$ and $f \circ P = f$. The partial derivative of $f$ with respect to $x_i$ will be denoted by $f_i$.

**Theorem 3.6.** Let $I$ be an open interval in $\mathbb{R}$ and let $f : I^n \to \mathbb{R}$ be continuously differentiable. Then $f$ is Schur convex if and only if one of the following conditions is true:

(a) (Schur) $f$ is symmetric and $f_i$ is decreasing in $x_i$ for all $x \in \mathcal{D} \cap I^n$.

(b) (Schur) $f$ is symmetric and for all $i \neq j$
$$ (x_i - x_j)(f_i(x) - f_j(x)) \geq 0 $$
Eliminate from the hypothesis the differentiability of \( f \) and consider
\[
f : A \subset \mathbb{R}^n \to \mathbb{R}
\]
with \( A \) symmetric. Then \( f \) is Schur convex if and only if:
\( f \) is symmetric and
\[
x_1 \mapsto f(x_1, s - x_1, x_2, \ldots, x_n)
\]
is increasing in \( x_1 \geq s/2 \), for any fixed \( s, x_3, \ldots, x_n \).

For weak majorisation and strongly isotone functions we have the following theorem:

**Theorem 3.7.** Let \( I \) be an open interval in \( \mathbb{R} \) and let \( f : I^n \to \mathbb{R} \).

(a) (Ostrowski) Let \( f \) be continuously differentiable. Then \( f \) is strongly isotone if and only if \( f \) is symmetric and for all \( x \in D \cap I^n \) we have \( Df(x) \in D \cap R^n_+ \), that is:
\[
f,1(x) \geq f,2(x) \geq \ldots \geq f,n(x) \geq 0
\]
(b) Without differentiability assumptions, \( f \) is strongly isotone if and only if \( f \) increasing and Schur convex.

4. ORDER RELATIONS FOR MATRICES

The results from this section have deep connections with Lie group theory, symplectic geometry and sub-Riemannian geometry. I shall give here only a minimal presentation, for matrix groups.

Main references are again Bhatia [4], Chapter 2, and Marshall, Olkin [13], Chapter 3; also Thompson [16]. The paper Kostant [11] gives an image of what’s really happening from the Lie group point of view.

**Definition 4.1.** We denote by \( \mathcal{P}(n) \) the cone of Hermitian, positive definite matrices. In the class of Hermitian matrices we have the order relation \( A \geq B \) if \( A - B \in \mathcal{P}(n) \).

The order relation \( \leq \) between Hermitian matrices reflects into the order relation between the eigenvalues. The next theorem, belonging to Weyl, is theorem F1, chapter 16, Marshall, Olkin [13].

**Theorem 4.1.** (Weyl) If \( A, B \) are Hermitian matrices such that \( A \leq B \) then
\[
\lambda^\downarrow(A) \leq \lambda^\downarrow(B)
\]

For \( A, B \) matrices, their Schur product is the matrix \( A \odot B \) given by
\[
(A \odot B)_{ij} = A_{ij}B_{ij} \quad \text{(no summation)}
\]

**Theorem 4.2.** (Schur) If \( A \geq B \) and \( C \geq 0 \) then \( A \odot C \geq B \odot C \).

Next theorem shows a first connection between majorisation and symmetric matrices.

**Theorem 4.3.** (Schur) For any symmetric matrix \( A \) we have \( \text{diag}(A) \prec \lambda(A) \).

(Horn) Conversely, given vectors \( a, b \in \mathbb{R}^n \) such that \( a \prec b \), there is a symmetric matrix \( A \) such that \( \text{diag}(A) = a \) and \( \lambda(A) = b \).
In \( GL(n, R)^+ \) we have the order relation (introduced by Thompson [16])

\[ X \prec Y \text{ if } \log \sigma(X) \prec \log \sigma(y) \]

Horn-Thompson theorem [1.1] can be reformulated as:

**Theorem 4.4.** (Horn, Thompson, theorem [1.1 reformulated]) Let \( X, Y \) be any two positive definite \( n \times n \) matrices and let \( x_1 \geq x_2 \geq \ldots \geq x_n \) and \( y_1 \geq y_2 \geq \ldots \geq y_n \) denote the respective sets of eigenvalues. Then there is an unitary matrix \( U \) such that \( XU \) and \( Y \) have the same spectrum if and only if \( Y \prec X \).

Another interesting majorisation occurs between the absolute value of eigenvalues and singular values respectively.

**Theorem 4.5.** (Weyl) For any matrix \( F \in GL(n, C) \) we have the inequality:

\[ \log |\lambda(F)| \prec \log \sigma(F) \]

Finally, denote by \( |A| \) the spectral radius of \( A \), i.e. the maximum over the modulus of singular values of \( A \). Next theorem shows the algebraic deep of Thompson’s order relation.

**Theorem 4.6.** (Kostant) \( X \prec Y \) if and only if for any linear representation \( \pi \) of the group \( GL(n, R)^+ \) we have \( |\pi(X)| \leq |\pi(Y)| \).

5. **Objective isotropic elastic potentials**

We are interested in functions \( w : GL(n, R)^+ \to R \) which are objective

\[ \forall Q \in SO(n) \ w(QF) = W(F) \]

and isotropic

\[ \forall Q \in SO(n) \ w(FQ) = W(F) \]

If \( w \) is \( C^2 \), then we call it rank one convex if it satisfies the ellipticity condition:

\[ \sum_{i,j,k,l=1}^{n} \frac{\partial^2 w}{\partial F_{ij} \partial F_{kl}}(F) a_i b_j a_k b_l \geq 0 \tag{1} \]

for any \( F \in GL(n, R)^+ \), \( a, b \in R^n \).

For \( A, B \) matrices, we denote by \( [[A, B]] \) the segment

\[ [[A, B]] = \{(1-t)A + tB : t \in [0,1]\} \]

We have the more general definition of rank one convexity:

**Definition 5.1.** The function \( w : GL(n, R)^+ \to R \) is rank one convex if for any \( A, B \in GL(n, R)^+ \) such that rank \( (A - B) = 1 \) and \( [[A, B]] \subset GL(n, R)^+ \) the function \( t \in [0,1] \to w((1-t)A + tB) \in R \) is convex.

It is straightforward that the ellipticity condition is equivalent to rank one convexity for smooth functions.

If \( w \) is objective and isotropic then there is a symmetric function \( g : R^n_+ \to R \) such that \( w(F) = g(\sigma(F)) \). If \( w \) is \( C^2 \) then \( g \) is too.

We shall introduce two auxiliary functions. They are the following:

\[ h : R^n \to R , \quad h(x) = g(\exp x) \]

\[ l : R^n_+ \to R , \quad l(x) = g(\sqrt{x}) \]
The auxiliary function \( h \) will be called "the diagonal of \( w \)."

Now, let \( x \in \mathbb{R}^n \) or \( \mathbb{R}^n_+ \) such that \( x_i \neq x_j \) for \( i \neq j \). The following quantities will help.

\[
\Gamma_{ij}(x) = \frac{h_i(x) - h_j(x)}{x_i - x_j}
\]

Any function \( \Gamma_{ij} \) can be prolonged by continuity in \( x_i = x_j \). In order to shorten the notation we shall put in the functions arguments only the terms that count. For example \( f(x_i, x_j) \) means \( f(x) \) and \( f(x_i, x_i) \) is \( f(x) \) for an \( x \) such that \( x_i = x_j \). With this notation one can define by continuity:

\[
\Gamma_{ij}(x_i, x_i) = h_{ij}(x_i, x_i) - h_{jj}(x_i, x_i)
\]

A straightforward computation shows that

\[
\Gamma_{ij}(x_i, x_j) = G_{ij}(\exp x_i, \exp x_j)(\exp x_i + \exp x_j)
\]

where

\[
G_{ij}(x_i, x_j) = \frac{x_ig_i(x_i, x_j) - x_jg_j(x_i, x_j)}{x_i - x_j}
\]

Therefore the coefficients \( G_{ij} \) can be prolonged by continuity to \( x_i = x_j \). We shall put \( \Gamma_{ii} = G_{ii} = 0 \).

We shall introduce also the symmetric matrix \( \Xi \). We shall define it first for vectors \( x \in \mathbb{R}^n_+ \) with all components different and then extend it by continuity to all vectors. For \( i \neq j \) we define:

\[
\Xi_{ij}(x_i, x_j) = \frac{l_i(x_i, x_j) - l_j(x_i, x_j)}{x_i - x_j}
\]

If \( x_i = x_j \) then the prolongation by continuity of \( \Xi_{i,j} \) is

\[
\Xi_{ij}(x_i, x_i) = l_{ii}(x_i, x_i) - l_{ij}(x_i, x_i)
\]

For \( i = j \) and all \( x \), we define:

\[
\Xi_{ii}(x) = 2l_{ii}(x) + l_i(x)/x_i
\]

Again by straightforward computation we find that, for any \( x \) with all components different:

\[
\Xi(x) \odot (\sqrt{x} \odot \sqrt{x}) = \tilde{H}(\sqrt{x})
\]

where the matrix \( \mathcal{P} \) is defined by:

\[
\mathcal{P}_{ij}(x_i, x_j) = \frac{x_ig_i(x_i, x_j) - x_jg_j(x_i, x_j)}{x_i - x_j}
\]

for \( i \neq j \) and

\[
\mathcal{P}_{ii}(x) = 0
\]

As previously, the function \( \mathcal{P}_{ij} \) can be prolonged by continuity to \( x_i = x_j \).

A consequence of theorem 6.4 Ball is:

**Proposition 5.1.** For \( x \) with all components different, the ellipticity condition for the objective isotropic function \( w \) can be expressed in terms of the associated function \( g \) as

\[
\sum_{i,j=1}^{n} g_{ij}a_i a_j b_i b_j + \sum_{i \neq j} G_{ij} a_i^2 b_j^2 + \sum_{i \neq j} \mathcal{P}_{ij} a_i a_j b_i b_j \geq 0
\]
By continuity arguments it follows that one can write the ellipticity condition for all \( x \in \mathbb{R}^n \) like this:

\[
\sum_{i,j=1}^{n} H_{ij} a_i a_j b_i b_j + \sum_{i,j=1}^{n} G_{ij} a_i^2 b_j^2 \geq 0
\]

where \( H \) is the matrix \( H = \mathcal{H} + D^2 g \).

**Theorem 5.1.** Necessary and sufficient conditions for \( w \in C^2 \) to be rank one convex are:

(a) \( h \) is Schur convex and

(b) for any \( x \in \mathbb{R}^n \) we have

\[
H_{ij} x_i x_j + G_{ij} |x_i| |x_j| \geq 0
\]

**Remark 5.1.** The condition (a) is equivalent to the Baker-Ericksen \([1]\) set of inequalities

\[
\frac{x_i g_i(x_i, x_j) - x_j g_j(x_i, x_j)}{x_i^2 - x_j^2} \geq 0
\]

for all \( i \neq j \) and \( x_i \neq x_j \). Indeed, by theorem \([3]\) (b), the function \( h \) is Schur convex if and only if

\[
(h_i(x_i, x_j) - h_j(x_i, x_j))(x_i - x_j) \geq 0
\]

for all \( i \neq j \) and \( x_i \neq x_j \). But the definition of the "diagonal" \( h \) and obvious computation show the equivalence between the two sets of inequalities. Silhavy \([4]\) expresses Baker-Ericksen inequalities using multiplication instead division, but apparently he does not make this obvious connection with Schur convexity.

**Proof.** We prove first the sufficiency. The hypothesis is that for all \( i, j \) \( G_{ij} \geq 0 \) and for all \( x \in \mathbb{R}^n \) the relation (3) holds. We claim that for any \( a, b \in \mathbb{R}^n \) the inequality

\[
G_{ij} a_i a_j b_i b_j \leq G_{ij} a_i^2 b_j^2
\]

is true. The ellipticity condition follows then from (3) by the choice \( x_i = a_i b_i \), for each \( i = 1, \ldots, n \). Indeed, we have the chain of inequalities

\[
0 \leq H_{ij} a_i b_i a_i b_j + G_{ij} a_i b_i |a_j b_j| \leq H_{ij} a_i b_i a_i b_j + G_{ij} a_i^2 b_j^2
\]

In order to prove the claim note that \( G_{ij} \geq 0 \) implies

\[
-G_{ij} (a_j b_i - a_i b_j)^2 \leq 0
\]

A straightforward computation which uses the relations \( G_{ij} = G_{ji} \) gives

\[
0 \geq -G_{ij} (a_j b_i - a_i b_j)^2 = 2G_{ij} (a_j b_i - a_i b_j) a_i b_j
\]

The sufficiency part is therefore proven.

For the necessity part choose first in the ellipticity condition \( a_i = \delta_{iI}, b_i = \delta_{iJ} \). For \( I \neq J \) we obtain \( G_{ij} \geq 0 \), which means the Schur convexity of \( h \). (For \( I = J \) we obtain \( g_{ii} \geq 0 \), interesting but with no use in this proof.)

Next, suppose that \( x, a \in (\mathbb{R}^*)^n \) and choose \( b_i = x_i / a_i \) for each \( i = 1, \ldots, n \). The ellipticity condition gives:

\[
\sum_{i,j} H_{ij} x_i x_j + \sum_{i,j=1}^{n} G_{ij} \left( \frac{a_i}{a_j} \right)^2 x_j^2 \geq 0
\]
Take $a_i^2 = |x_i|$ and get (3), but only for $x \in (R^*)^n$. The expression from the left of (3) makes sense for any $x$. By continuity with respect to $x$ we prove the thesis.

There is a certain interest in giving necessary and sufficient conditions for an objective isotropic $w$ to be rank one convex, especially in the cases $n = 2$ and $n = 3$. These conditions have been expressed in copositivity terms as in Simpson and Spector \[15\] for $n = 3$, Silhavy \[14\] and Dacorogna \[6\] for arbitrary $n$ (for an account on the history of results related to this problem see the Silhavy or Dacorogna op. cit.). The conditions given in theorem 5.1 have some advantages. The relation between rank one convexity and Schur convexity, which is rather obvious, can be used to obtain quasiconvexity results. As for the condition (b), it contains one inequality instead of a $2^n$ family of (equally complex) inequalities expressing copositivity. Moreover, for $n = 2$ or $n = 3$, it can be used to obtain explicit conditions, as in Dacorogna \[6\]. These explicit conditions (for $n = 3$), contained in theorem 5, Dacorogna, op. cit., are clearly not independent and have a rather involved form. I think that for practical purposes it is much easier to think in other terms. The next proposition, with a straightforward proof, is relevant.

**Proposition 5.2.** Let $H, G$ be two symmetric $n \times n$ matrices, such that $G$ has positive entries. The following statements are equivalent:

(a) for any $x \in R^n$
\[ H_{ij}x_i x_j + G_{ij} \| x_i \| x_j \| \geq 0 \]

(b) we have the set inclusion $A(G) \subset A(-H)$, where
\[ A(G) = \{ x \in R^n : G_{ij} \| x_i \| x_j \| \leq 1 \} \]
\[ A(-H) = \{ x \in R^n : -H_{ij}x_i x_j \leq 1 \} \]

The matter of finding conditions upon $H, G$ such that inclusion (b) happens is one of comparing asymptotic and extremal properties of the sets $A(G)$ and $A(-H)$. These sets have simple descriptions and the algebraic conditions upon $H, G$ reflects nothing but the geometrical effort to put $A(G)$ inside $A(-H)$. If $H$ is positive definite then $A(-H) = R^n$. Otherwise the inclusion $A(G) \subset A(-H)$ can be expressed in terms of eigenvalues and eigenvectors of $H, G$. I don’t pursue this path here, because it is separate from the purpose of this note.

6. **Majorisation and quasiconvexity**

The goal of this section is to give a class of multiplicative quasiconvex isotropic functions which seem to be complementary to the polyconvex isotropic ones. We quote the following result of Thompson and Freede \[17\], Ball \[3\] (for a proof coherent with this paper see Le Dret \[12\]).

**Theorem 6.1.** Let $g : [0, \infty)^n \to R$ be convex, symmetric and nondecreasing in each variable. Define the function $w$ by
\[ w : gl(n, R) \to R, \quad w(F) = g(\sigma(F)). \]
Then $w$ is convex.

The main result of this section is:
Theorem 6.2. Let \( g : (0, \infty)^n \to R \) be a continuous symmetric function and \( h : R^n \to R, h = g \circ \exp \). Consider also the function \( p : R^n \to R \)
\[
p\left( \sum_{i=1}^{k} x_i \right) = h(x_k)
\]
Suppose that:
(a) \( h \) is convex,
(b) \( p \) is nonincreasing in each argument.
Let \( \Omega \subset R^n \) be bounded, with piecewise smooth boundary and \( \phi : \Omega \to R \) be any Lipschitz function such that \( D\phi(x) \in GL(n, R)^+ \) a.e. and \( \phi(x) = x \) on \( \partial \Omega \). Define the function
\[
w : GL(n, R)^+ \to R , \ w(F) = g(\sigma(F))
\]
Then for any \( F \in GL(n, R)^+ \) we have:
\[
\int_{\Omega} w(F D\phi(x)) \geq |\Omega| w(F)
\]

Remark 6.1. In the above definition \( \Omega \) can be replaced by any bounded open set with piecewise smooth boundary.

The notion of multiplicative quasiconvexity is given further.

Definition 6.1. Let \( w : GL(n, R)^+ \to R \) be a function and \( \Omega = (0,1)^n \). \( w \) is multiplicative quasiconvex if for any \( F \in GL(n, R)^+ \) and for any Lipschitz function \( u : \Omega \to R \), such that for almost any \( x \in \Omega \) \( \det Du(x) > 0 \) and \( u(x) = x \) on \( \partial \Omega \), we have the inequality:
\[
\int_{\Omega} w(F Du(x)) \geq \int_{\Omega} w(F)
\]

Lemma 6.1. Let \( h : R^n \to R \) be continuous, Schur convex and \( g = h \circ \log \). Define
\[
w : GL(n, R)^+ \to R, \ w(F) = g(\sigma(F))
\]
\[
\tilde{w} : GL(n, C) \to R, \ \tilde{w}(F) = g(|\lambda(F)|)
\]
Then for any \( F \)
\[
w(F) \geq \tilde{w}(F)
\]
Proof. This is a straightforward consequence of the Weyl inequality (theorem 4.3)
\[ \log | \lambda(F) | \prec \log \sigma(F) \]
and of the Schur convexity of \( h \).

**Lemma 6.2.** With the notations from the lemma 6.1, for any two symmetric matrices \( A, B \), we have
\[ \tilde{w}(\exp A \exp B) \geq \tilde{w}(\exp(A + B)) \]

Proof. We have to check the conditions from Thompson [16], Lemma 6, which gives sufficient conditions on the function \( \tilde{w} \) in order to satisfy the inequality we are trying to prove. These conditions are:

1. for any \( X \) and any symmetric positive definite \( Y \)
   \[ \tilde{w}(XY) = \tilde{w}(YX) \]
   This is satisfied by definition of \( \tilde{w} \).
2. for any \( X \) and any \( m = 1, 2, ... \)
   \[ \tilde{w} ([XX^*]^m) \geq \tilde{w} (X^{2m}) \]
   From the definition of \( \tilde{w} \) and Lemma 6.1 we find that \( \tilde{w} \) satisfies this condition too.

We give now the proof of the theorem 6.2.

Proof. To any \( F \in GL(n, R) \) we associate its polar decomposition \( F = R_F U_F = V_F R_F \). For any function \( \phi \) such that \( D\phi(x) \in GL(n, R) \) we shall use the (similar) notation
\[ D\phi(x) = R\phi(x)U\phi(x) = V\phi(x)R\phi(x) \]
With the notations from the theorem, we have from the isotropy of \( w \), hypothesis (a) and theorem 3.5 that \( h \) is Schur convex. From lemma 6.1 and lemma 6.2 we obtain the chain of inequalities
\[ \int_{\Omega} w(FD\phi(x)) \geq \int_{\Omega} \tilde{w}(U_FV\phi(x)) \geq \int_{\Omega} \tilde{w}(\exp (\log U_F + \log V\phi(x))) \]
The chain of inequalities continues by using the convexity hypothesis (a) (suppose that \( |\Omega| = 1 \)):
\[ \int_{\Omega} \tilde{w} (\exp (\log U_F + \log V\phi(x))) \geq \tilde{w} \left( \exp \left( \log U_F + \int_{\Omega} \log V\phi(x) \right) \right) \]
Now, I claim that the matrix
\[ \int_{\Omega} \log V\phi(x) \]
is negative definite. Then, from theorem 4.3, we find that
\[ \lambda^+(\log U_F + \int_{\Omega} \log V\phi(x)) \leq \lambda^+(\log U) \]
We use now the nonincreasing condition (b) to finish the chain of inequalities
\[ \tilde{w} \left( \exp \left( \log U_F + \int_{\Omega} \log V\phi(x) \right) \right) \geq \tilde{w}(U_F) = w(U_F) = w(F) \]
Let us see, finally, why the matrix \( \int_{\Omega} \log V \phi(x) \) is negative definite. The function \( \log \lambda^+_i(F) \) is well known polyconcave, hence all the functions \( \log \lambda^+_i(F) \) satisfy the inequality:

\[
\int_{\Omega} \log \lambda^+_i(D\phi(x)) \leq |\Omega| \log \lambda^+_i(I_n) = 0
\]

Take now any vector \( v \in \mathbb{R}^n, v \neq 0 \). Remember that \( V \phi(x) \) is a symmetric matrix which admits the decomposition

\[
V \phi(x) = Q \phi(x) \text{Diag}(\lambda(D\phi(x)))Q^T \phi(x)
\]

hence

\[
\log V \phi(x) = Q \phi(x) \text{Diag}(\log \lambda(D\phi(x)))Q^T \phi(x)
\]

Therefore

\[
\sum_{i,j=1}^{n} [\log V \phi(x)]_{ij} v_i v_j \leq \log \lambda^+_1 |v|^2
\]

Use the inequality given by polyconcavity to deduce the claim. \( \square \)

A consequence of the theorem 6.2 is:

**Proposition 6.1.** *In the hypothesis of the theorem 6.2, the function \( w \) is rank one convex.*

The family of functions satisfying the hypothesis of theorem 6.2 is non void. Two examples are given further.

For the first example take the polar decomposition \( F = RFU_F \) and define the function: \( w(F) = \log \text{trace } U_F^{-1} \). It satisfies the hypothesis, by straightforward computation. Indeed, using the notations of theorem 6.2, the associated function \( g : (0, +\infty)^n \rightarrow \mathbb{R} \) is

\[
g(y_1, ..., y_n) = \log \left( \sum_{i=1}^{n} \frac{1}{y_i} \right)
\]

hence the function \( h(x) = g(\exp x) \) has the expression:

\[
h(x_1, ..., x_n) = \log \left( \sum_{i=1}^{n} \exp(-x_i) \right)
\]

which is easy to check that is convex and the associated function \( p \) is decreasing in each argument.

For the second example consider a modified Ogden potential. Set

\[
\|F\|_k = \left( \prod_{i=1}^{k} (\sigma_i)^{1/k} \right)
\]

and define:

\[
w(F) = \sum_{i=1}^{n} \frac{1}{\|F\|_i^\alpha}
\]

for some \( \alpha \geq 2 \). The associated function \( h \) is then

\[
h(x_1, ..., x_n) = \sum_{k=1}^{n} \exp \left( (-\alpha/k) \sum_{i=1}^{k} x_i^\alpha \right)
\]

which again satisfies the hypothesis of the theorem.
Both functions are not known to be polyconvex. In fact they do not satisfy the following sufficient condition for polyconvexity, due to Ball \cite{3}, given here for simplicity for $n = 3$ (see also Le Dret \cite{12}).

**Theorem 6.3.** Let $\phi : \mathbb{R}^3_+ \times \mathbb{R}^3_+ \to \mathbb{R}$ which is nondecreasing in the first six variables and such that for any pair of permutations $\sigma, \tau \in S_3$

$$\phi(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, v_{\tau(1)+3}, v_{\tau(2)+3}, v_{\tau(3)+3}, v_7) = \phi(v_1, ..., v_7)$$

Then the function:

$$w(F) = \phi(\sigma_1(F), ..., \sigma_1(F)\sigma_2(F), ..., \det F)$$

is polyconvex.

This is the reason for thinking that the functions described in theorem 6.2 are complementary to objective, isotropic, polyconvex ones. Remark nevertheless that the function

$$w(F) = -\log \det F$$

satisfies the hypothesis of theorem 6.2 and it is also polyconvex.

Let us consider only the Schur convexity and componentwise convexity hypothesis related to $w$.

**Proposition 6.2.** Let $h : \mathbb{R}^n \to \mathbb{R}$ be Schur convex and the function $x \in \mathbb{R} \mapsto h(\log(x), ..., \log(x))$ be convex, continuous. Let $\phi : \Omega \to \mathbb{R}$ be such that almost everywhere we have $D\phi(x) \in GL(n, \mathbb{R})^+$,

$$\int_{\Omega} D\phi(x) = I_n$$

and the map $x \mapsto w(D\phi(x))$ is integrable. Then

$$\int_{\Omega} w(D\phi(x)) \geq |\Omega| w(I_n)$$

**Proof.** Because $h$ is Schur convex and for almost any $x \in \Omega$

$$\frac{1}{n} \log \det D\phi(x)(1, ..., 1) < \log \sigma(D\phi(x))$$

we have the inequality

$$w(D\phi(x)) \geq w((\det D\phi(x))^{1/n} I_n)$$

Use the convexity hypothesis to obtain the desired inequality.

7. RANK ONE CONVEX HULLS AND MAJORISATION

In this section it is explained how majorisation appears in the representation of some rank one convex hulls. What would be really nice to understand are the implications of Lie group aspects of majorisation onto calculus of variations. The fact that such implications should exist is straightforward, but far from being self evident.

Further is given a proof of theorem 1.2 using majorisation. In this proof we use the fact that majorisation relation

$$x \prec\prec y \text{ if } \log x \prec \log y$$
is defined using polyconvex maps. The isotropy of the set $E(a)$ from theorem 1.2 implies that the description of its rank one convex hull reduces to the description of the set of matrices $B \prec \text{Diag}(a)$, where $\prec$ is Thompson’s order relation. These facts (partially) explain the resemblance between theorems 1.1 and 1.2.

Let $a \in (0, \infty)^n$. Denote by $E(a)$ the set of matrices $F$ with positive determinant such that $\sigma(F) = Pa$ for some $P \in S_n$. We have to prove that

$$Pco E(a) = Rco E(a) = K(a)$$

where

$$K(a) = \{ B \in \text{GL}(n, R)^+ : B \prec \text{Diag}(a) \}$$

The set $K(a)$ is polyconvex, being an intersection of preimages of $(-\infty, 0]$ by polyconvex functions. Therefore

$$Rco E(a) \subset Pco E(a) \subset K(a)$$

It is left to prove that $K(a) \subset Rco E(a)$. For this remark that $E(a)$ can be written as:

$$E(a) = \{ R P \text{Diag}(a) Q : R, Q \in \text{SO}(n) , P \in S_n \}$$

Consider the convex cone of functions ($Rco$ denotes the class of rank one convex functions)

$$Rco(a) = \{ \phi \in Rco : \forall A \in E(a) \phi(A) = 0 \}$$

This cone is closed to $\text{sup}$ operation. Moreover, it has the same symmetries as $E(a)$, that is for any $R, Q \in \text{SO}(n)$ and any $P \in S_n$ we have

$$\phi \in Rco(a) \implies [F \in \text{GL}(n, R)^+ \mapsto (R, Q, P).\phi(F) = \phi(R P F Q)] \in Rco(a)$$

Hence if $\phi \in Rco(a)$ then $\tilde{\phi} \in Rco(a)$, where $\tilde{\phi}$ is the objective isotropic function

$$\tilde{\phi}(F) = \sup \{(R, Q, P).\phi(F) : R, Q \in \text{SO}(n) , P \in S_n \}$$

Objective isotropic rank one convex functions have Schur convex diagonal, as a consequence of theorem 5.1 (a) (if the rank one convex $w$ is not $C^2$ use a convolution argument). Therefore $F \in K(a)$ and $\phi \in Rco(a)$ imply

$$\phi(F) \leq \tilde{\phi}(F) \leq \tilde{\phi}(\text{Diag}(a)) = 0$$

This proves the inclusion $K(a) \subset Rco(a)$.

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