GROUND STATES FOR KIRCHHOFF-TYPE EQUATIONS WITH CRITICAL GROWTH

QUANQING LI
Department of Mathematics, Honghe University
Mengzi, Yunnan 661100, China
Department of Mathematics, Tsinghua University
Beijing, Beijing 100084, China

KAIMIN TENG
Department of Mathematics, Taiyuan University of Technology
Taiyuan, Shanxi 030024, China

XIAN WU
Department of Mathematics, Yunnan Normal University
Kunming, Yunnan 650092, China

(Communicated by Rafael Ortega)

Abstract. In this paper, we study the following Kirchhoff-type equation with critical growth

\[-(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = \lambda f(x,u) + |u|^4u, \quad x \in \mathbb{R}^3,\]

where constants \(a > 0, b > 0, \lambda > 0\) and \(f\) is a continuous superlinear but subcritical nonlinearity. When \(V\) and \(f\) are asymptotically periodic in \(x\), we prove that the equation has a ground state solution for large \(\lambda\) by Nehari method. Moreover, we regard \(b\) as a parameter and obtain a convergence property of the ground state solution as \(b \downarrow 0\).

1. Introduction and preliminaries. Consider the following Kirchhoff-type equation with critical growth

\[-(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = \lambda f(x,u) + |u|^4u, \quad x \in \mathbb{R}^3,\]  \hspace{1cm} (1.1)

where constants \(a > 0, b > 0, \lambda > 0\) and \(f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})\). If we set \(V(x) = 0\) and replace \(\mathbb{R}^3\) and \(\lambda f(x,u) + |u|^4u\) by a bounded domain \(\Omega \subset \mathbb{R}^3\) and a new nonlinearity \(g(x,u)\) in (1.1), respectively, then we obtain the following Kirchhoff Dirichlet problem

\[-(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = g(x,u), \quad \text{in} \quad \Omega,\]
\[u = 0, \quad \text{on} \quad \partial \Omega,\]  \hspace{1cm} (1.2)

2000 Mathematics Subject Classification. 35J20, 35J70, 35P05, 35P30,34B15, 58E05, 47H04.

Key words and phrases. Kirchhoff-type equation, critical growth, ground state solutions.

This work is supported in part by the National Natural Science Foundation of China (11501403; 11461023; 11701322; 11561072) and the Shanxi Province Science Foundation for Youths under grant 2013021001-3 and the Honghe University Doctoral Research Programs (XJ17B11, XJ17B12) and the Yunnan Province Local University (Part) Basic Research Joint Project (2017FH001-013).
which is related to the stationary analogue of the Kirchhoff equation

\[ u_{tt} - (a + b \int_{\Omega} |\nabla u|^2 \, dx) \Delta u = g(x, u), \]

which was proposed by Kirchhoff [15] as an extension of the classical D’Alembert wave equation for free vibrations of elastic strings. In (1.2), \( u \) denotes the displacement, \( g(x, u) \) the external force and \( b \) the initial tension while \( a \) is related to the intrinsic properties of the string, such as Young’s modulus. We have to point out that such nonlocal problems also appear in other fields as biological systems, where \( u \) describes a process which depends on the average of itself, for example, population density. After the pioneer work of Lions [16], in which a functional analysis approach was introduced, (1.2) has been paid much attention to by many researchers. Some early studies of the Kirchhoff equation may refer [2, 3, 5, 7, 9, 16, 20]. Recently, some scholars have studied the existence of nontrivial solutions for the Kirchhoff-type problem

\[ -(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx) \Delta u + V(x) u = g(x, u), \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \quad (1.3) \]

where \( a > 0 \), \( b \geq 0 \), \( N = 1, 2, 3 \), \( V : \mathbb{R}^N \to \mathbb{R} \) and \( g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \). For example, Wu in [23] studied the sequence of high energy solutions for Eq. (1.3) and four main results were given. By some special techniques, the authors in [18] improved and united the above four results. In [6], problem (1.3) with concave and convex nonlinearities was studied by using Nehari manifold and fiber map methods, and multiple positive solutions were obtained. Li and Ye in [19] proved that Eq. (1.3) with \( g(x, u) = |u|^{p-1}u \) had a ground state solution, where \( 2 < p \leq 3 \). For more results, we refer the readers to the papers [8, 10, 11, 12, 21, 24] and the references therein.

In this paper, we shall study the existence of ground states for critical problem (1.1).

Let \( \mathcal{H} \) be the class of functions \( h \in L^\infty(\mathbb{R}^3) \) such that, for every \( \varepsilon > 0 \) the set \( \{ x \in \mathbb{R}^3 : |h(x)| \geq \varepsilon \} \) has finite Lebesgue measure. In order to reduce the statements for main results, we list the assumptions as follows:

1. (\( V_1 \)) \( V \in L^\infty(\mathbb{R}^3) \) and \( V_0 := \inf_{x \in \mathbb{R}^3} V(x) > 0 \).
2. (\( V_2 \)) There exists a function \( V_p \in L^\infty(\mathbb{R}^3) \), which is 1-periodic in \( x_i \) (\( i = 1, 2, 3 \)), such that \( V - V_p \in \mathcal{H} \) and \( V(x) \leq V_p(x) \) for all \( x \in \mathbb{R}^3 \).
3. (\( f_1 \)) \( f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}) \) and there exists \( 4 < q < 6 \) such that

\[ |f(x, t)| \leq C(1 + |t|^{q-1}) \]

for all \( (x, t) \in \mathbb{R}^3 \times \mathbb{R} \), where \( C \) is a positive constant.
4. (\( f_2 \)) \( f(x, t) = o(|t|) \) uniformly in \( x \in \mathbb{R}^3 \) as \( |t| \to 0 \).
5. (\( f_3 \)) \( tf(x, t) - 4F(x, t) \geq sf(x, st) - 4F(x, st) \) for all \( (x, t) \in \mathbb{R}^3 \times \mathbb{R} \) and \( s \in [0, 1] \), where \( F(x, t) := \int_0^t f(x, \tau) \, d\tau \).
6. (\( f_4 \)) \( f(x, t)t > 0 \) for all \( (x, t) \in \mathbb{R}^3 \times \mathbb{R} \setminus \{0\} \).
7. (\( f_5 \)) There exists a function \( f_p \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}) \), which is 1-periodic in \( x_i \) (\( i = 1, 2, 3 \)), such that

\[ \begin{align*}
&(i) \quad |f_p(x, t)| \leq |f(x, t)|, \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}.
&(ii) \quad |f_p(x, t) - f(x, t)| \leq |h(x)|(1 + |t|^{q-1}), \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}, \quad \text{where } h \in \mathcal{H} \quad \text{and} \quad q \text{ is given by } (f_1).
\end{align*} \]
(iii) \( tf_p(x, t) - 4F_p(x, t) \geq stf_p(x, st) - 4F_p(x, st) \) for all \((x, t) \in \mathbb{R}^3 \times \mathbb{R}\) and \(s \in [0, 1]\), where \( F_p(x, t) := \int_0^t f_p(x, \tau) d\tau \).

(iv) \( f_p(x, t) t \geq 0 \) for all \((x, t) \in \mathbb{R}^3 \times \mathbb{R}\).

Since \( \{ h \in L^\infty(\mathbb{R}^3) : \lim_{|x| \to \infty} h(x) = 0\} \subset \mathcal{H} \), the case \(+\infty > V_\infty := \liminf_{|x| \to \infty} V(x) \geq V(x) \in C(\mathbb{R}^3)\) verifies the assumption \((V_2)\). A simple example of \( f \) satisfying the hypotheses \((f_1) - (f_3)\) is the function \( f(x, t) = |t|^{q-2}t \) and we may choose \( f_p = f = |t|^{q-2}t \) in \((f_3)\). We point out that this kind of hypothesis \((f_3)\) was first introduced by Jeanjean in \([14]\). Latter, it was used by Liu and Li \([17]\) for the general case. Furthermore, Remark 1.1 in \([25]\) implies that \((f_3)\) is weaker than the following assumption:

\( \tilde{f}_3 \) the function \( t \mapsto \frac{f(-t)}{t} \) is non-increasing on \((-\infty, 0)\) and non-decreasing on \((0, +\infty)\).

Set

\[
H^1_1(\mathbb{R}^3) = \{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]dx < +\infty \}
\]

with the inner product

\[
\langle u, v \rangle = \int_{\mathbb{R}^3} [a\nabla u \cdot \nabla v + V(x)uv]dx
\]

and the norm

\[
\| u \| = \| u \|_{H^1(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]dx \right)^{1/2}
\]

In view of \((V_1) - (V_2)\), the norms \( \| u \| \) and \( \| u \|_p := \left( \int_{\mathbb{R}^3} [a|\nabla u|^2 + V_p(x)u^2]dx \right)^{1/2} \) and \( \| u \|_{H^1(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} [a|\nabla u|^2 + u^2]dx \right)^{1/2} \) are equivalent. By the condition \((V_1)\) we know that the embedding \( H^1_1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3) \) is continuous for each \( 2 \leq s \leq 6 \) and locally compact for each \( 2 \leq s < 6 \).

Our main results are the following:

**Theorem 1.1.** Suppose that \((V_1) - (V_2)\) and \((f_1) - (f_5)\) are satisfied. Then there exists \( \lambda^* > 0 \) such that for each \( \lambda > \lambda^* \), problem \((1.1)\) has a ground state solution \( u_\lambda \).

**Remark 1.** When \( f \) is a function of \( C^1 \) class, He and Zou in \([13]\) studied the existence of positive solutions for problem \((1.1)\) with autonomous case. When \( f \) is merely continuous, Wang et al. in \([26]\) obtained a ground state solution for the same problem. But, they assume that \( f \) satisfies \( \tilde{f}_3 \). Hence our result Theorem 1.1 improves and extends their results.

**Remark 2.** For the following class of nonlocal problem

\[
M \int_{\mathbb{R}^3} |\nabla u|^2dx + \int_{\mathbb{R}^3} V(x)u^2dx|\triangle u + V(x)u| = \lambda f(u) + u^5,
\]

the authors in \([1]\) proved that there exists \( \lambda^* > 0 \) such that the above equation has a positive solution for all \( \lambda \geq \lambda^* \), where \( V \) is asymptotically periodic and \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfies the following assumptions

\( (M_1) \) There exists \( a > 0 \) such that \( M(t) \geq a, \forall t > 0 \).

\( (M_2) \) There exists \( b \geq 0 \) such that \( \lim_{t \to +\infty} \frac{M(t)}{t} = b \).

\( (M_3) \) The function \( t \mapsto \frac{1}{2} \tilde{M}(t^2) - \frac{1}{2+2H(b)} M(t^2) t^2 \) is increasing in \([0, +\infty)\), where \( \tilde{M}(t) := \int_0^t M(s)ds \) with \( H(t) = 1 \) if \( t > 0 \) and \( H(t) = 0 \) if \( t \leq 0 \).
(M₄) There exists θ ∈ (2 + 2H(b), 6) such that \( \lim_{t \to +\infty} \frac{1}{t} \left[ \frac{1}{2} M(t^2) - \frac{1}{θ} M(t^2) t^2 \right] = +\infty. \)

(M₅) The function \( M \) is increasing in \((0, +\infty)\).

(M₆) The function \( t \mapsto \frac{M(t)}{t^2} \) is nonincreasing in \((0, +\infty)\).

Moreover, \( f \) satisfies

\( (f₁)* \) \( f(t) = 0, \forall t \leq 0. \)

\( (f₂)* \) \( \lim_{|t| \to 0} \frac{|f(t)|}{|t|} = 0. \)

\( (f₃)* \) There exists \( q \in (2 + 2H(b), 6) \) such that \( \lim_{|t| \to +\infty} \frac{|f(t)|}{|t|^q} = 0. \)

\( (f₄)* \) \( 0 < θF(t) \leq tf(t), \forall t > 0 \), where \( θ \) is the constant given in \((M₄)\).

\( (f₅)* \) The function \( t \mapsto \frac{f(t)}{t^{2+2H(b)}} \) is increasing in \((0, +\infty)\).

Especially, take \( M(t) = a + bt, \forall t > 0 \), where \( a > 0, b > 0 \). Then \( H(b) = 1 \) and the above problem reduces to the problem

\[ [a + b(\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} V(x) u^2 dx)] (-\triangle u + V(x) u) = \lambda f(u) + u^5, \]

but at this stage, the condition \((f₃)\) in the present article is weaker than \((f₅)\).

Fixed \( λ > λ^* \). Obviously, \( u_λ \) obtained in Theorem 1.1 depends on \( b \), we next denote \( u_λ \) by \( u_λ^b \) to emphasize this dependence. As like [8, 21], we give a convergence property of \( u_λ^b \) as \( b \downarrow 0 \), which reflects some relationship between \( b > 0 \) and \( b = 0 \) in problem (1.1). Our main result in this direction can be stated as the following theorem.

**Theorem 1.2.** If the assumptions of Theorem 1.1 are verified, then, for any sequence \( \{b_n\} \) with \( b_n \downarrow 0 \) as \( n \to \infty \), there exists \( λ^* > 0 \), independent of \( b_n \), such that for each \( λ > λ^* \), problem (1.1) has a ground state solution \( u_λ^{b_n} \) with \( u_λ^{b_n} \rightharpoonup u_λ^0 \) in \( H^1_v(\mathbb{R}^3) \) as \( n \to \infty \), where \( u_λ^0 \) is a weak solution of the problem

\[ -a \triangle u + V(x) u = \lambda f(x, u) + |u|^4 u, \quad x \in \mathbb{R}^3. \]

It is well known to us that a weak solution of problem (1.1) is a critical point of the following functional

\[ I_λ(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx - \lambda \int_{\mathbb{R}^3} F(x, u) dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx. \]

Let us define

\[ N := \{ u \in H^1_v(\mathbb{R}^3) \setminus \{0\} : \langle I_λ'(u), u \rangle = 0 \} \]

and

\[ N_\rho := \{ u \in H^1_v(\mathbb{R}^3) \setminus \{0\} : \langle I_{λ,ρ}'(u), u \rangle = 0 \}, \]

where

\[ I_{λ,ρ}(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_ρ(x) u^2 dx - \lambda \int_{\mathbb{R}^3} F_ρ(x, u) dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx. \]

Clearly, by \((V₂)\), \((f₄)\), \((f₅)-(i)\) and \((iv)\) we have \( I_λ(u) \leq I_{λ,ρ}(u) \) for all \( u \in H^1_v(\mathbb{R}^3) \).
2. Proof of Theorem 1.1. To begin with, we give some lemmas.

Lemma 2.1. For \( \lambda > 0 \) we have

(i) for each \( u \in H^1_0(\mathbb{R}^3) \setminus \{0\} \), there exists a unique \( t_u > 0 \) such that \( \alpha(t_u) = \max_{t \geq 0} \alpha(t) \), \( \alpha'(t) > 0 \) for \( 0 < t < t_u \) and \( \alpha'(t) < 0 \) for \( t_u < t \). Moreover, \( tu \in \mathcal{N} \) if and only if \( t = t_u \). Here \( \alpha(t) := I_{\lambda}(tu) \).

(ii) for each \( v \in H^1_0(\mathbb{R}^3) \setminus \{0\} \), there exists a unique \( t_v > 0 \) such that \( \beta(t_v) = \max_{t \geq 0} \beta(t) \), \( \beta'(t) > 0 \) for \( 0 < t < t_v \) and \( \beta'(t) < 0 \) for \( t_v < t \). Moreover, \( tv \in \mathcal{N}_p \) if and only if \( t = t_v \). Here \( \beta(t) := I_{\lambda,p}(tv) \).

(iii) there exists \( t_0 > 0 \) such that \( t_u \geq t_0 \) for each \( u \in S_1 := \{ u \in H^1_0(\mathbb{R}^3) : \|u\| = 1 \} \) and for each compact subset \( W \subset S_1 \), there exists \( C_W > 0 \) such that \( t_u \leq C_W \) for all \( u \in W \).

(iv) there exists \( \rho > 0 \) such that \( c_\lambda := \inf_{\mathcal{N}} I_{\lambda} \geq \inf_{S_\rho} I_{\lambda} > 0 \), where \( S_\rho := \{ u \in H^1_0(\mathbb{R}^3) : \|u\| = \rho \} \).

Proof. (i) By \((f_1) - (f_2)\), for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[
|f(x,u)| \leq \varepsilon |u| + C_\varepsilon |u|^{\eta-1}
\]

and

\[
|F(x,u)| \leq \varepsilon |u|^2 + C_\varepsilon |u|^\eta
\]

for all \((x,u) \in \mathbb{R}^3 \times \mathbb{R}\). Consequently, for \( \varepsilon > 0 \) sufficiently small, (2.2) implies that

\[
\alpha(t) = I_{\lambda}(tu) \geq \frac{1}{2} t^2 \|u\|^2 - \lambda \int_{\mathbb{R}^3} [\varepsilon t^2 u^2 + C_\varepsilon t^\eta |u|^\eta] dx - C_1 t^6 \|u\|^6
\]

\[
\geq \frac{1}{2} t^2 \|u\|^2 - \lambda \varepsilon C_2 t^2 \|u\|^2 - \lambda C_3 C_\varepsilon t^\eta \|u\|^\eta - C_1 t^6 \|u\|^6
\]

\[
\geq \frac{1}{4} t^2 \|u\|^2 - \lambda C_3 C_\varepsilon t^\eta \|u\|^\eta - C_1 t^6 \|u\|^6
\]

\[
> 0
\]

for small \( t > 0 \), and (2.1) implies that

\[
\alpha'(t) = \langle I'_{\lambda}(tu), u \rangle \geq t \|u\|^2 - \lambda \int_{\mathbb{R}^3} [\varepsilon tu^2 + C_\varepsilon t^{\eta-1} |u|^\eta] dx - C_1 t^5 \|u\|^6
\]

\[
\geq t \|u\|^2 - \lambda \varepsilon C_2 t \|u\|^2 - \lambda C_3 C_\varepsilon t^{\eta-1} \|u\|^\eta - C_1 t^5 \|u\|^6
\]

\[
\geq \frac{1}{2} t \|u\|^2 - \lambda C_3 C_\varepsilon t^{\eta-1} \|u\|^\eta - C_1 t^5 \|u\|^6
\]

\[
> 0
\]

for small \( t > 0 \). Moreover, by virtue of \((f_4)\) we have

\[
\alpha(t) = I_{\lambda}(tu) \leq \frac{1}{2} t^2 \|u\|^2 + C_4 t^4 \|u\|^4 - \frac{1}{6} t^6 \int_{\mathbb{R}^3} |u|^6 dx \rightarrow -\infty
\]

as \( t \rightarrow +\infty \). Hence \( \alpha \) has a positive maximum and there exists \( t_u > 0 \) such that \( \alpha'(t_u) = 0 \) and \( \alpha'(t) > 0 \) for \( 0 < t < t_u \).

We claim that \( \alpha'(t) \neq 0 \) for all \( t > t_u \). Indeed, if the conclusion is false, then, from the above arguments, there exists \( t_u < t_2 < +\infty \) such that \( \alpha'(t_2) = 0 \) and...
\( \alpha(t_u) \geq \alpha(t_2) \). But, (f3) implies that

\[
\alpha(t_2) = \alpha(t_2) - \frac{t_2}{4} \alpha'(t_2)
\]

\[
= \frac{t_2^2}{4} \int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]dx + \lambda \int_{\mathbb{R}^3} \left[ \frac{1}{4} t_2 uf(x, t_2 u) - F(x, t_2 u) \right]dx
\]

\[
+ \frac{1}{12} t_2^6 \int_{\mathbb{R}^3} |u|^6 dx
\]

\[
> \frac{t_2^2}{4} \int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]dx + \lambda \int_{\mathbb{R}^3} \left[ \frac{1}{4} t_2 uf(x, t_2 u) - F(x, t_2 u) \right]dx
\]

\[
+ \frac{1}{12} t_2^6 \int_{\mathbb{R}^3} |u|^6 dx
\]

\[
= \alpha(t_u) - \frac{t_n}{4} \alpha'(t_u)
\]

\[
= \alpha(t_u)
\]

a contradiction.

Combining the claim with prior arguments, we obtain the first conclusion of (i). The second conclusion is an immediate consequence of the fact that \( \alpha'(t) = t^{-1}(I'_\lambda(tu), tu) \). This completes the proof of (i). Similarly, we can prove that (ii) holds.

(iii) For \( u \in S_1 \), by (i) there exists \( t_u > 0 \) such that \( t_u u \in \mathcal{N} \). Hence for \( \varepsilon > 0 \) small, (2.1) implies that

\[
0 = \langle I'_\lambda(t_u u), t_u u \rangle \geq t_u^2 \|u\|^2 - \lambda \int_{\mathbb{R}^3} [\varepsilon t_u^2 u^2 + C_2 t_u^6 |u|^q]dx - C_1 t_u^6 \|u\|^6
\]

\[
\geq t_u^2 \|u\|^2 - \lambda \varepsilon t_u^2 \|u\|^2 - \lambda C_3 C_2 t_u^6 \|u\|^q - C_1 t_u^6 \|u\|^6
\]

\[
\geq \frac{1}{2} t_u^2 - \lambda C_3 C_2 t_u^6 - C_1 t_u^6.
\]

As a consequence, there exists \( t_0 > 0 \) such that \( t_u \geq t_0 \) for all \( u \in S_1 \). To prove that \( t_u \leq C_W \) for all \( u \in W \subset S_1 \). We argue by contradiction. Suppose there exists \( \{u_n\} \subset W \subset S_1 \) such that \( t_n := t_{u_n} \to +\infty \) as \( n \to \infty \). Since \( W \) is compact, there exists \( u \in W \) such that \( u_n \to u \) in \( H_1^1(\mathbb{R}^3) \). Consequently, by (f4) we deduce that

\[
I_\lambda(t_n u_n) = \frac{1}{2} t_n^2 \|u_n\|^2 + b \frac{1}{4} t_n^4 \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 - \lambda \int_{\mathbb{R}^3} F(x, t_n u_n) dx - \frac{1}{6} t_n^6 \int_{\mathbb{R}^3} |u_n|^6 dx
\]

\[
\leq t_n^2 \left( \frac{1}{2} \right) + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 - \frac{1}{6} t_n^6 \int_{\mathbb{R}^3} |u_n|^6 dx
\]

\[
\to -\infty
\]

as \( n \to \infty \). But, (f3) implies that

\[
I_\lambda(t_n u_n) = I_\lambda(t_n u_n) - \frac{1}{4} \langle I'_\lambda(t_n u_n), t_n u_n \rangle
\]

\[
= \frac{1}{4} t_n^2 \|u_n\|^2 + \lambda \int_{\mathbb{R}^3} \left[ \frac{1}{4} t_n u_n f(x, t_n u_n) - F(x, t_n u_n) \right]dx
\]

\[
+ \frac{1}{12} t_n^6 \int_{\mathbb{R}^3} |u_n|^6 dx \geq 0,
\]

a contradiction. So the conclusion (iii) follows.
(iv) For $\lambda > 0$ and $u \in S_\rho$, by the proof of (i) we know that

$$I_\lambda(u) \geq \frac{1}{4} \|u\|^2 - \lambda C \|u\|^q - C \|u\|^6$$

$$= \frac{1}{4} \rho^2 - \lambda C \rho^q - C \rho^6$$

$$\geq \frac{1}{8} \rho^2 > 0$$

for small $\rho > 0$. Furthermore, for every $u \in \mathcal{N}$, there exists $t_1 > 0$ such that $t_1 u \in S_\rho$. Hence by (i) we have

$$0 < \frac{1}{8} \rho^2 \leq \inf_{S_\rho} I_\lambda \leq I_\lambda(t_1 u) \leq \max_{t > 0} I_\lambda(tu) = I_\lambda(u),$$

which implies that $c_\lambda = \inf_{\mathcal{N}} I_\lambda \geq \inf_{S_\rho} I_\lambda > 0$. This completes the proof.

Lemma 2.2. $I_\lambda$ is coercive on $\mathcal{N}$, i.e., $I_\lambda(u) \to +\infty$ as $u \in \mathcal{N}$ and $\|u\| \to \infty$.

Proof. If the conclusion is false, then there exist a sequence $\{u_n\} \subset \mathcal{N}$ and a positive number $d$ such that $\|u_n\| \to \infty$ and $I_\lambda(u_n) \leq d$. Set $w_n = \frac{u_n}{\|u_n\|}$. Then, up to a subsequence, we have

$$w_n \to w \text{ in } H^1_0(\mathbb{R}^3),$$

$$w_n \to w \text{ in } L^s_{\text{loc}}(\mathbb{R}^3) \text{ for } 2 \leq s < 6,$$

$$w_n(x) \to w(x) \text{ a.e. on } \mathbb{R}^3.$$

Since $w_n \neq 0$, there exists a point $y \in \mathbb{R}^3$ such that $\int_{B_1(y)} |w_n|^6 dx := \delta > 0$. Set $\beta(z) := \int_{B_1(z)} |w_n|^6 dx$. Then the integral absolute continuity implies that $\beta(z)$ is continuous on $\mathbb{R}^3$. Set a large $R > 0$ with $\int_{\mathbb{R}^3 \setminus B_R(0)} |w_n|^6 dx < \delta$. Then, for any $z \in \mathbb{R}^3 \setminus B_{R+1}(0)$,

$$\beta(z) = \int_{B_1(z)} |w_n|^6 dx < \delta.$$

Hence

$$\sup_{z \in \mathbb{R}^3} \beta(z) = \sup_{z \in B_{R+1}(0)} \beta(z).$$

By the continuity of $\beta$ and the compactness of $\overline{B}_{R+1}(0)$, there exists $y_n \in \overline{B}_{R+1}(0)$ such that $\beta(y_n) = \sup_{z \in \overline{B}_{R+1}(0)} \beta(z)$. Therefore,

$$\int_{B_1(y_n)} |w_n|^6 dx = \sup_{z \in \mathbb{R}^3} \int_{B_1(z)} |w_n|^6 dx.$$

We assert that $\limsup_{n \to \infty} \int_{B_1(y_n)} |w_n|^6 dx > 0$. Otherwise, we have

$$\lim_{n \to \infty} \int_{B_1(y_n)} |w_n|^6 dx = 0.$$

In view of Lemma 3.8 in [4] we know that $w_n \to 0$ in $L^6(\mathbb{R}^3)$. By the interpolation inequality, one has

$$\|w_n\|_{L^6(\mathbb{R}^3)} \leq \|w_n\|_{L^2(\mathbb{R}^3)}^{1-\theta} \|w_n\|_{L^6(\mathbb{R}^3)}^\theta,$$
where $s \in (2,6)$ and $\theta = \frac{3(s-2)}{2s}$. Then $w_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Hence by (2.2), for large $t$ and small $\varepsilon > 0$, we have

$$
\frac{1}{2} t^2 \|w_n\|^2 - \lambda \int_{\mathbb{R}^3} F(x, t_n w_n) dx - \frac{1}{6} t^6 \int_{\mathbb{R}^3} |w_n|^6 dx
$$

as $n \to \infty$, a contradiction. Consequently, by $(f_4)$ we obtain

$$
0 \leq \frac{I_\lambda(u_n)}{\|u_n\|^6} \leq \frac{1}{6} \int_{\mathbb{R}^3} |w_n|^6 dx + o(1) \leq \frac{1}{6} \int_{B_1(y_n)} |w_n|^6 dx + o(1) < 0
$$

for large $n$, a contradiction. This completes the proof. \hfill \Box

**Lemma 2.3.** There exists $\lambda^* > 0$ such that $0 < c_\lambda < \frac{1}{4} (aS)^2$ for all $\lambda > \lambda^*$, where $S$ is the best constant of the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, i.e.,

$$
S = \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|_6^6}{\|\nabla u\|_2^2}.
$$

**Proof.** Take $u \in H^1_0(\mathbb{R}^3) \setminus \{0\}$. Then by $(f_1) - (f_3)$, there exists a unique $t_\lambda > 0$ such that $\max_{t \geq 0} I_\lambda(tu) = I_\lambda(t_\lambda u)$. Hence $(f_4)$ implies that

$$
t_\lambda^2 \|u\|^2 + t_\lambda^4 b \int_{\mathbb{R}^3} |\nabla u|^2 dx = \lambda \int_{\mathbb{R}^3} f(x, t_\lambda u) t_\lambda u dx + t_\lambda^6 \int_{\mathbb{R}^3} |u|^6 dx + \frac{1}{6} t_\lambda^6 \int_{\mathbb{R}^3} |u|^6 dx,
$$

which implies that $\{t_\lambda\}$ is bounded. Hence, for the sequence $\lambda_n \to +\infty$ as $n \to \infty$, up to a subsequence, there exists $t_0 \geq 0$ such that $\lambda_n \to t_0$ as $n \to \infty$. If $t_0 > 0$, then by $(f_4)$ and Fatou Lemma we have

$$
\lim_{n \to \infty} \lambda_n \int_{\mathbb{R}^3} f(x, t_{\lambda_n} u) t_{\lambda_n} u dx + t_{\lambda_n}^6 \int_{\mathbb{R}^3} |u|^6 dx = +\infty.
$$

But, on the other hand,

$$
t_\lambda^2 \|u\|^2 + t_\lambda^4 b \int_{\mathbb{R}^3} |\nabla u|^2 dx \to t_0^2 \|u\|^2 + t_0^4 b \int_{\mathbb{R}^3} |\nabla u|^2 dx,
$$

a contradiction. Then $t_0 = 0$. Consequently, again by $(f_4)$ one has

$$
c_\lambda \leq \inf_{v \in D^{1,2}_0(\mathbb{R}^3) \setminus \{0\}} \sup_{t > 0} I_\lambda(t v) = \sup_{t > 0} I_\lambda(t \lambda u)
$$

$$
\leq \frac{1}{2} t_\lambda^2 \|u\|^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx \to 0
$$

as $\lambda \to +\infty$. Thus there exists $\lambda^* > 0$ such that $0 < c_\lambda < \frac{1}{4} (aS)^2$ for all $\lambda > \lambda^*$. This completes the proof. \hfill \Box

**Remark 3.** If we regard $b > 0$ in problem (1.1) as a parameter and denote $c_\lambda$ by $c_\lambda^b$, then from the above proof we see that $c_\lambda^b \to 0$ as $\lambda \to +\infty$ uniformly in $b \in (0,1]$. Hence $\lambda^*$ in Lemma 2.3 is independent on $b \in (0,1]$. This will be used to prove Theorem 1.2 in section 3.
Define the mapping \( m : S_1 \rightarrow \mathcal{N} \) by setting \( m(w) := t_w w \), where \( t_w \) is as in Lemma 2.1.

**Lemma 2.4 (22).** The mapping \( m \) is a homeomorphism between \( S_1 \) and \( \mathcal{N} \), and the inverse of \( m \) is given by \( m^{-1}(u) = \frac{u}{\|u\|} \).

Considering the functional \( \psi_\lambda : S_1 \rightarrow \mathbb{R} \) given by
\[
\psi_\lambda(w) := I_\lambda(m(w)),
\]
then we have the following lemma.

**Lemma 2.5 (22).** (i) If \( \{w_n\} \) is a Palais-Smale sequence for \( \psi_\lambda \), then \( \{m(w_n)\} \) is a Palais-Smale sequence for \( I_\lambda \). If \( \{u_n\} \subset \mathcal{N} \) is a bounded Palais-Smale sequence for \( I_\lambda \), then \( \{m^{-1}(u_n)\} \) is a Palais-Smale sequence for \( \psi_\lambda \).

(ii) \( w \in S_1 \) is a critical point of \( \psi_\lambda \) if and only if \( m(w) \) is a nontrivial critical point of \( I_\lambda \). Moreover, the corresponding values of \( \psi_\lambda \) and \( I_\lambda \) coincide and \( \inf_{S_1} \psi_\lambda = \inf_{\mathcal{N}} I_\lambda \).

(iii) A minimizer of \( I_\lambda \) on \( \mathcal{N} \) is a ground state of Eq. (1.1).

**Lemma 2.6.** Let \((V)\) and \((f_5)-(ii)\) hold. If \( \{u_n\} \subset H^1(\mathbb{R}^3) \) satisfies \( u_n \rightarrow 0 \) in \( H^1(\mathbb{R}^3) \) and \( \varphi_n \in H^1(\mathbb{R}^3) \) is bounded in \( H^1(\mathbb{R}^3) \). Then
\[
\int_{\mathbb{R}^3} [V(x) - V_p(x)] u_n \varphi_n dx \rightarrow 0 \tag{2.3}
\]
and
\[
\int_{\mathbb{R}^3} [f(x, u_n) - f_p(x, u_n)] \varphi_n dx \rightarrow 0 \tag{2.4}
\]
and
\[
\int_{\mathbb{R}^3} [F(x, u_n) - F_p(x, u_n)] dx \rightarrow 0. \tag{2.5}
\]

**Proof.** For any \( \varepsilon > 0 \) and \( R > 0 \), set
\[
D_\varepsilon(R) := \{x \in \mathbb{R}^3 : |h(x)| \geq \varepsilon, |x| \geq R\},
\]
where \( h \) appears in \((f_5)-(ii)\). By \( h \in \mathcal{H} \) we can deduce that there exists \( R_1 > 0 \) such that \( |D_\varepsilon(R_1)| < \varepsilon \). Indeed, assuming for a moment the conclusion is false, we deduce that there exists a sequence \( \{R_n\} := \{n\} \) such that \( |D_\varepsilon(R_n)| \geq \varepsilon > 0 \). Set
\[
\Omega := \{x \in \mathbb{R}^3 : |h(x)| \geq \varepsilon\}
\]
and
\[
\Omega_n := \Omega \setminus D_\varepsilon(R_n) = \{x \in \mathbb{R}^3 : |h(x)| \geq \varepsilon, |x| < R_n\}.
\]
Then by \( h \in \mathcal{H} \) we obtain \( |\Omega| : = \alpha < +\infty \). Furthermore, \( \Omega_n \subset \Omega_{n+1} \) and \( \Omega = \bigcup_{n=1}^\infty \Omega_n \). Thereby,
\[
\alpha = |\Omega| = \lim_{n \rightarrow \infty} |\Omega_n| = \lim_{n \rightarrow \infty} |\Omega \setminus D_\varepsilon(R_n)| = \lim_{n \rightarrow \infty} |\Omega| - |D_\varepsilon(R_n)| \leq \alpha - \varepsilon < \alpha,
\]
a contradiction. By \( u_n \rightarrow 0 \) in \( H^1(\mathbb{R}^3) \) we have
\[
|u_n|_{L^2(B_{R_1}(0))} < \varepsilon \quad \text{and} \quad |u_n|_{L^q(B_{R_1}(0))}^{q-1} < \varepsilon \quad \text{(2.6)}
\]
for large $n$. Note that

$$I := \int_{\mathbb{R}^3} [f(x, u_n) - f_p(x, u_n)] \varphi_n dx \leq \int_{\mathbb{R}^3} |f(x, u_n) - f_p(x, u_n)| \varphi_n dx$$

$$\leq \int_{\mathbb{R}^3} |h(x)(|u_n| + |u_n|^{q-1})| \varphi_n dx$$

$$= \left[ \int_{D_n(R_1)} + \int_{B_1(0)} + \int_{\mathbb{R}^3 \setminus (D_n(R_1) \cup B_1(0))} |h(x)(|u_n| + |u_n|^{q-1})| \varphi_n dx \right]$$

$$:= I_1 + I_2 + I_3.$$  

By Hölder inequality, (2.6) and the boundedness of $\{||u_n||\}$ and $\{||\varphi_n||\}$ we can derive that

$$I_1 \leq |h|_\infty |D_\varepsilon(R_1)|^{\frac{1}{q}} |u_n|_2 |\varphi_n|_6 + |h|_\infty |D_\varepsilon(R_1)|^{\frac{a_0}{q'}} |u_n|_q^{-1} |\varphi_n|_6$$

$$\leq C(\varepsilon^{\frac{1}{q}} + \varepsilon^{-\frac{a_0-q}{q'}})$$

and

$$I_2 \leq |h|_\infty |u_n|_{L^2(B_1(0))} |\varphi_n|_2 + |h|_\infty |u_n|_{L^2(B_1(0))} |\varphi_n|_q$$

$$\leq C \varepsilon$$

for large $n$ and

$$I_3 \leq \varepsilon |u_n|_2 |\varphi_n|_2 + \varepsilon |u_n|_q^{-1} |\varphi_n|_q$$

$$\leq C \varepsilon.$$

Hence $I \leq (\varepsilon^{\frac{1}{q}} + \varepsilon^{-\frac{a_0-q}{q'}})$ for large $n$, which implies that

$$\int_{\mathbb{R}^3} [f(x, u_n) - f_p(x, u_n)] \varphi_n dx \to 0,$$

i.e., (2.4) holds. Similarly, by $(V_2)$ we can conclude that (2.3) holds and

$$\int_{\mathbb{R}^3} [f(x, tu_n) - f_p(x, tu_n)] u_n dx \to 0$$

for all $t \in [0, 1]$. Consequently,

$$\int_{\mathbb{R}^3} [F(x, u_n) - F_p(x, u_n)] dx = \int_{\mathbb{R}^3} \int_0^1 (f(x, tu_n) - f_p(x, tu_n)) u_n dt dx$$

$$= \int_0^1 \left[ \int_{\mathbb{R}^3} (f(x, tu_n) - f_p(x, tu_n)) u_n dx \right] dt \to 0,$$

i.e., (2.5) holds. This completes the proof.

**Proof of Theorem 1.1.** By Lemma 2.5 $(iii)$, it suffices to prove that the infimum $c_\lambda$ is attained for fixed $\lambda > \lambda^*$. For fixed $\lambda > \lambda^*$, let $\{\psi_{\lambda}(w_n)\} \subset S_1$ be a minimizing sequence satisfying $\psi_{\lambda}(w_n) \to c_\lambda$ by Lemma 2.5 $(ii)$. By the Ekeland variational principle, we suppose $\psi_{\lambda}(u_n) \to 0$ in $H_V^{-1}(\mathbb{R}^3)$. Set $u_n = m(w_n) \in \mathcal{N}$. Then Lemma 2.5 $(i)$ implies that $I_1(u_n) = \psi_{\lambda}(w_n) \to c_\lambda$ and $I'_1(u_n) \to 0$ in $H_V^{-1}(\mathbb{R}^3)$. By Lemma 2.2 we see that $\{u_n\}$ is bounded in $H_V^1(\mathbb{R}^3)$. Therefore, up to a subsequence, there exists $u \in H_V^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H_V^1(\mathbb{R}^3)$, $u_n \to u$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$. Hence $I'_1(u) = 0$. In the following, we distinguish two cases.
Case 1: $u \neq 0$. Then $u \in \mathcal{N}$ and $c_\lambda \leq I_\lambda(u)$. Consequently, by weakly lower semi-continuity of the norm, Fatou Lemma and $(f_3)$ one has

$$c_\lambda = \lim_{n \to \infty} [I_\lambda(u_n) - \frac{1}{4} \langle I'_\lambda(u_n), u_n \rangle]$$

$$= \liminf_{n \to \infty} \left\{ \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 + V(x)u_n^2 \right\} dx + \lambda \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(x, u_n)u_n - F(x, u_n) \right] dx$$

$$+ \frac{1}{12} \int_{\mathbb{R}^3} |u_n|^6 dx$$

$$\geq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 \right\} dx + \lambda \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(x, u)u - F(x, u) \right] dx + \frac{1}{12} \int_{\mathbb{R}^3} |u|^6 dx$$

$$= I_\lambda(u) - \frac{1}{4} \langle I'_\lambda(u), u \rangle$$

$$= I_\lambda(u) \geq c_\lambda,$$

which implies that $I_\lambda(u) = c_\lambda$.

Case 2: $u = 0$. We shall apply the concentration-compactness principle due to P. L. Lions to the sequence of $L^1$ functions $u_n^2$ and we know that two cases may happen.

Case (i): Vanishing, i.e.,

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} u_n^2(x) dx = 0;$$

Case (ii): Nonvanishing, i.e., there exist a sequence $\{y_n\} \subset \mathbb{R}^3$ and a constant $\delta > 0$ such that

$$\liminf_{n \to \infty} \int_{B_1(y_n)} u_n^2(x) dx \geq \delta. \quad (2.7)$$

If the former occurs, by virtue of Lemma 3.8 in [4] we have $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Consequently, by $(f_1) - (f_2)$ we know that

$$\int_{\mathbb{R}^3} F(x, u_n) dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} f(x, u_n) dx \to 0.$$ 

As a consequence,

$$o(1) = \|u_n\|^2 + b(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2 - \int_{\mathbb{R}^3} |u_n|^6 dx.$$

Assume that $\|u_n\|^2 \to l_1 \geq 0$ and $b(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2 \to l_2 \geq 0$. Then $\int_{\mathbb{R}^3} |u_n|^6 dx \to l_1 + l_2$. We claim that $l_1 > 0$ if and only if $l_2 > 0$. Indeed, if $l_2 > 0$, then

$$b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 = b \left( \int_{\mathbb{R}^3} a|\nabla u_n|^2 dx \right)^2$$

$$\leq \frac{b}{a^2} \int_{\mathbb{R}^3} a|\nabla u_n|^2 dx + \int_{\mathbb{R}^3} V(x)u_n^2 dx = \frac{b}{a^2} \|u_n\|^4,$$

which implies that $0 < l_2 \leq \frac{b}{a^2} l_1^2$. Thus $l_1 > 0$. Conversely, for $l_1 > 0$, if $l_2 = 0$, then by the Sobolev embedding theorem we deduce that

$$bS^2 \left( \int_{\mathbb{R}^3} |u_n|^6 dx \right)^\frac{2}{3} \leq b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2,$$

which implies that $0 \leq bS^2 (l_1 + l_2)^\frac{3}{2} \leq l_2 = 0$. Hence $l_1 = 0$, a contradiction. Thus the assertion holds. In the sequel, let $l_1 > 0$. By $I_\lambda(u_n) \to c_\lambda$ we derive that

$$c_\lambda + o(1) = \frac{1}{2} \|u_n\|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 - \frac{1}{6} \int_{\mathbb{R}^3} |u_n|^6 dx,$$
which implies that $c_\lambda = \frac{1}{3} l_1 + \frac{1}{12} l_2$. Noting that the Sobolev embedding theorem implies that $l_1 \geq aS(l_1 + l_2)^2$ and $l_2 \geq bS^2(l_1 + l_2)^2$. Hence $l_1 \geq (aS)^2$ and $l_2 \geq bS^6$. So $c_\lambda \geq \frac{1}{3} (aS)^2 + \frac{1}{12} bS^6$, which contradicts with Lemma 2.3. Therefore, $l_1 = l_2 = 0$. Thus $c_\lambda = 0$, a contradiction. As a consequence, the latter occurs. Without loss of generality, we may assume that $y_n \in \mathbb{Z}^N$. Set $\bar{u}_n(\cdot) = u_n(\cdot + y_n)$.

Up to a subsequence, there exists $\bar{u} \in H^1_{**}(\mathbb{R}^3)$ such that $\bar{u}_n \rightharpoonup \bar{u}$ in $H^1_{**}(\mathbb{R}^3)$, $\bar{u}_n \rightarrow \bar{u}$ in $L^s_{loc}(\mathbb{R}^3)$ for $2 \leq s < 6$ and $\bar{u}_n(x) \rightarrow \bar{u}(x)$ a.e. on $\mathbb{R}^3$. By (2.7) we have $\bar{u} \neq 0$. In the sequel, we divide the proof into three steps.

**Step 1:** We prove that $I_{\lambda,p}'(\bar{u}) = 0$. Indeed, for all $\varphi \in H^1_{**}(\mathbb{R}^3)$, set $\varphi_n(\cdot) := \varphi(\cdot - y_n)$. In view of Lemma 2.6 we conclude that

$$\int_{\mathbb{R}^3} [V(x) - V_p(x)]u_n \varphi_n dx \rightarrow 0$$

and

$$\int_{\mathbb{R}^3} [f(x, u_n) - f_p(x, u_n)]\varphi_n dx \rightarrow 0.$$ 

Consequently,

$$\langle I_{\lambda}'(u_n), \varphi_n \rangle - \langle I_{\lambda,p}'(u_n), \varphi_n \rangle$$

$$= \int_{\mathbb{R}^3} [V(x) - V_p(x)]u_n \varphi_n dx - \lambda \int_{\mathbb{R}^3} [f(x, u_n) - f_p(x, u_n)]\varphi_n dx$$

$$\rightarrow 0.$$ 

Hence $\langle I_{\lambda,p}'(u_n), \varphi_n \rangle \rightarrow 0$. Moreover, by the periodicity of $V_p$ and $f_p$ in the variable $x$ and $y_n \in \mathbb{Z}^N$ we derive that $\langle I_{\lambda,p}'(\bar{u}_n), \varphi \rangle = \langle I_{\lambda,p}'(u_n), \varphi_n \rangle$, which implies that $\langle I_{\lambda,p}'(\bar{u}_n), \varphi \rangle \rightarrow 0$. As a consequence, $I_{\lambda,p}'(\bar{u}) = 0$.

**Step 2:** We claim that $I_{\lambda,p}'(\bar{u}) \leq c_\lambda$. Indeed, by the boundedness of $\{\|u_n\|\}$ and Lemma 2.6 we have

$$\int_{\mathbb{R}^3} [V(x) - V_p(x)]u_n^2 dx \rightarrow 0$$

and

$$\int_{\mathbb{R}^3} [f(x, u_n) - f_p(x, u_n)]u_n dx \rightarrow 0$$

and

$$\int_{\mathbb{R}^3} [F(x, u_n) - F_p(x, u_n)] dx \rightarrow 0.$$ 

Consequently,

$$\int_{\mathbb{R}^3} \frac{1}{4} f(x, u_n)u_n - F(x, u_n) dx = \int_{\mathbb{R}^3} \frac{1}{4} f_p(x, u_n)u_n - F_p(x, u_n) dx + o(1).$$

Again by the periodicity of $V_p$ and $f_p$ in the variable $x$, weakly lower semi-continuity of the norm, $(f_5)$ and Fatou Lemma we deduce that
\[ c_\lambda = \lim_{n \to \infty} [I_\lambda(u_n) - \frac{1}{4} \langle I'_\lambda(u_n), u_n \rangle] \]

\[ \geq \frac{1}{4} \liminf_{n \to \infty} \int_{\mathbb{R}^3} [a |\nabla u_n|^2 + V(x)u_n^2] \, dx + \lambda \liminf_{n \to \infty} \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(x, u_n)u_n - F(x, u_n) \right] \, dx \]

\[ + \frac{1}{12} \liminf_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^6 \, dx \]

\[ = \frac{1}{4} \liminf_{n \to \infty} \int_{\mathbb{R}^3} [a |\nabla u_n|^2 + V(x)u_n^2] \, dx + \lambda \liminf_{n \to \infty} \int_{\mathbb{R}^3} \left[ \frac{1}{4} f_p(x, u_n)u_n - F_p(x, u_n) \right] \, dx \]

\[ + \frac{1}{12} \liminf_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^6 \, dx \]

\[ \geq \frac{1}{4} \| \bar{u} \|_p^2 + \lambda \int_{\mathbb{R}^3} \left[ \frac{1}{4} f_p(x, \bar{u}) \bar{u} - F_p(x, \bar{u}) \right] \, dx + \frac{1}{12} \int_{\mathbb{R}^3} |\bar{u}|^6 \, dx \]

\[ = I_{\lambda,p}(\bar{u}) - \frac{1}{4} \langle I'_{\lambda,p}(\bar{u}), \bar{u} \rangle \]

\[ = I_{\lambda,p}(\bar{u}). \]

Step 3: We prove that \( \max_{t > 0} I_{\lambda,p}(t \bar{u}) = I_{\lambda,p}(\bar{u}) \). Indeed, by \( \bar{u} \neq 0 \) and \( I'_{\lambda,p}(\bar{u}) = 0 \) we have \( \bar{u} \in \mathcal{N}_p \). Consequently, \( \text{Lemma 2.1 (ii)} \) implies that the conclusion holds.

Since \( \bar{u} \neq 0 \), by \( \text{Lemma 2.1 (i)} \) there exists \( t_0 > 0 \) such that \( t_0 \bar{u} \in \mathcal{N} \). Consequently,

\[ c_\lambda = \inf_{\mathcal{N}} I_\lambda \leq I_\lambda(t_0 \bar{u}) \leq I_{\lambda,p}(t_0 \bar{u}) \leq \max_{t > 0} I_{\lambda,p}(t \bar{u}) = I_{\lambda,p}(\bar{u}) \leq c_\lambda, \]

which implies that \( I_\lambda(t_0 \bar{u}) = c_\lambda \).

All in all, \( c_\lambda \) is attained. And then the corresponding minimizer is a ground state of Eq. (1.1) by \( \text{Lemma 2.5 (iii)} \). This completes the proof.

3. Proof of Theorem 1.2. In the sequel, we regard \( b \in (0, 1] \) in problem (1.1) as a parameter, then by Remark 2.1 and Theorem 1.1 we know that there exists \( \lambda^* > 0 \), independent of \( b \), such that for each \( \lambda > \lambda^* \), problem (1.1) has a ground state solution \( u_\lambda^b \). Now, we denote \( u_\lambda, I_\lambda, c_\lambda \) by \( u_\lambda^b, I_\lambda^b, c_\lambda^b \), respectively. We shall analyze the convergence property of \( u_\lambda^b \) as \( b \searrow 0 \).

Proof of Theorem 1.2. Since \( b_n \to 0 \), we can assume \( \{b_n\} \subset (0, 1) \), then by Remark 2.1 and Theorem 1.1, there exists \( \lambda^* > 0 \), independent of \( b_n \), such that for each \( \lambda > \lambda^* \), problem (1.1) has a ground state solution \( u_\lambda^{b_n} \). We claim that \( \{u_\lambda^{b_n}\} \) is bounded in \( H_1^1(\mathbb{R}^3) \). Indeed, by the above arguments we know that \( I_{\lambda}^{b_n}(u_\lambda^{b_n}) = c_\lambda^{b_n} \leq c_\lambda \) and \( (I_{\lambda}^{b_n})'(u_\lambda^{b_n}) = 0 \). Hence

\[ c_\lambda \geq I_{\lambda}^{b_n}(u_\lambda^{b_n}) = I_{\lambda}^{b_n}(u_\lambda^{b_n}) - \frac{1}{4} \langle (I_{\lambda}^{b_n})'(u_\lambda^{b_n}), u_\lambda^{b_n} \rangle \]

\[ = \frac{1}{4} \| u_\lambda^{b_n} \|^2 + \lambda \int_{\mathbb{R}^3} \left[ \frac{1}{4} f(x, u_\lambda^{b_n})u_\lambda^{b_n} - F(x, u_\lambda^{b_n}) \right] \, dx + \frac{1}{12} \int_{\mathbb{R}^3} |u_\lambda^{b_n}|^6 \, dx, \]
which combining with \((f_3)\) implies that \(\{u_{\lambda}^{b_n}\}\) is bounded in \(H^1_\sigma(\mathbb{R}^3)\). Consequently, up to a subsequence, there exists \(u_{\lambda}^0 \in H^1_\sigma(\mathbb{R}^3)\) such that \(u_{\lambda}^{b_n} \rightharpoonup u_{\lambda}^0\) in \(H^1_\sigma(\mathbb{R}^3)\). Thus \((I^\alpha_\lambda)'(u_{\lambda}^0) = 0\), i.e., \(u_{\lambda}^0\) is a weak solution of the problem

\[
-a \Delta u + V(x) u = \lambda f(x, u) + |u|^4 u, \quad x \in \mathbb{R}^3,
\]

where \(I^\alpha_\lambda\) is the energy functional of (3.1), i.e.,

\[
I^\alpha_\lambda(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 dx - \lambda \int_{\mathbb{R}^3} F(x, u) dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx.
\]

Indeed, for any \(\varphi \in H^1_\sigma(\mathbb{R}^3)\), we have

\[
0 = \langle (I^\alpha_\lambda)'(u_{\lambda}^0), \varphi \rangle
\]

\[
= a \int_{\mathbb{R}^3} \nabla u_{\lambda}^0 \cdot \nabla \varphi dx + b_n \int_{\mathbb{R}^3} |\nabla u_{\lambda}^{b_n}|^2 dx \int_{\mathbb{R}^3} \nabla u_{\lambda}^{b_n} \cdot \nabla \varphi dx + \int_{\mathbb{R}^3} V(x) u_{\lambda}^{b_n} \varphi dx
\]

\[
- \lambda \int_{\mathbb{R}^3} f(x, u_{\lambda}^{b_n}) \varphi dx - \int_{\mathbb{R}^3} |u_{\lambda}^{b_n}|^4 u_{\lambda}^0 \varphi dx.
\]

Since \(u_{\lambda}^{b_n} \rightharpoonup u_{\lambda}^0\) in \(H^1_\sigma(\mathbb{R}^3)\), by the definition of weak convergence we know that

\[
\int_{\mathbb{R}^3} [a \nabla u_{\lambda}^{b_n} \cdot \nabla \varphi + V(x) u_{\lambda}^{b_n} \varphi] dx \rightarrow \int_{\mathbb{R}^3} [a \nabla u_{\lambda}^0 \cdot \nabla \varphi + V(x) u_{\lambda}^0 \varphi] dx.
\]

By \(b_n \rightarrow 0\) and the boundedness of \(\{|u_{\lambda}^{b_n}|\}\), one has

\[
b_n \int_{\mathbb{R}^3} |\nabla u_{\lambda}^{b_n}|^2 dx \int_{\mathbb{R}^3} \nabla u_{\lambda}^{b_n} \cdot \nabla \varphi dx \rightarrow 0.
\]

In the following, we shall prove that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} f(x, u_{\lambda}^{b_n}) \varphi dx = \int_{\mathbb{R}^3} f(x, u_{\lambda}^0) \varphi dx
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda}^{b_n}|^4 u_{\lambda}^0 \varphi dx = \int_{\mathbb{R}^3} |u_{\lambda}^0|^4 u_{\lambda}^0 \varphi dx.
\]

Indeed, by \((f_1) - (f_2)\), for any \(\delta > 0\), there exists \(C_\delta > 0\) such that

\[
|f(x, t)| \leq \delta |t| + C_\delta |t|^{q-1}
\]

for all \((x, t) \in \mathbb{R}^3 \times \mathbb{R}\). Consequently, by Young inequality we derive that

\[
||f(x, u_{\lambda}^{b_n}) - f(x, u_{\lambda}^0)|| \varphi|
\]

\[
\leq |u_{\lambda}^{b_n}| ||\varphi| + C_1 |u_{\lambda}^{b_n}|^{q-1} ||\varphi| + |u_{\lambda}^0||\varphi| + C_1 |u_{\lambda}^0|^{q-1} ||\varphi|
\]

\[
\leq |u_{\lambda}^0| - u_{\lambda}^{b_n}||\varphi| + 2 |u_{\lambda}^0||\varphi| + C_1 |u_{\lambda}^{b_n} - u_{\lambda}^0|^{q-1} ||\varphi| + C_2 |u_{\lambda}^0|^{q-1} ||\varphi|
\]

\[
\leq \delta |u_{\lambda}^{b_n} - u_{\lambda}^0|^2 + C_\delta |\varphi|^2 + 2 |u_{\lambda}^0||\varphi| + \delta |u_{\lambda}^{b_n} - u_{\lambda}^0|^q + C_1 C_\delta |\varphi|^q + C_2 |u_{\lambda}^0|^{q-1} ||\varphi|.
\]

Set

\[
G_{\delta, n}(x) = \max\{||f(x, u_{\lambda}^{b_n}) - f(x, u_{\lambda}^0)|| \varphi| - \delta |u_{\lambda}^{b_n} - u_{\lambda}^0|^2 - \delta |u_{\lambda}^{b_n} - u_{\lambda}^0|^q, 0\}.
\]

Then \(0 \leq G_{\delta, n}(x) \leq C_3 |\varphi|^2 + 2 |u_{\lambda}^0||\varphi| + C_1 C_\delta |\varphi|^q + C_2 |u_{\lambda}^0|^{q-1} ||\varphi| \in L^1(\mathbb{R}^3)\) and \(G_{\delta, n}(x) \rightarrow 0\) a.e. on \(\mathbb{R}^3\). By Lebesgue dominated convergence theorem we have
\[ \int_{\mathbb{R}^3} G_{\delta,n}(x) \, dx \to 0 \quad \text{as} \quad n \to \infty. \]  
Hence
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^3} [f(x, u_n^{b(\lambda)}) - f(x, u_\lambda^0)] \varphi \, dx \]
\[ \leq \limsup_{n \to \infty} \int_{\mathbb{R}^3} G_{\delta,n}(x) \, dx + \delta \limsup_{n \to \infty} \int_{\mathbb{R}^3} |u_n^{b(\lambda)} - u_\lambda^0|^2 \, dx + \delta \limsup_{n \to \infty} \int_{\mathbb{R}^3} |u_n^{b(\lambda)} - u_\lambda^0|^4 \, dx \]
\[ \leq C_3 \delta. \]

By the arbitrariness of \( \delta \) we know that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^3} f(x, u_n^{b(\lambda)}) \varphi \, dx = \int_{\mathbb{R}^3} f(x, u_\lambda^0) \varphi \, dx. \]

Similarly, we can deduce that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^{b(\lambda)}|^4 u_n^{b(\lambda)} \varphi \, dx = \int_{\mathbb{R}^3} |u_\lambda^0|^4 u_\lambda^0 \varphi \, dx. \]

Thus
\[ a \int_{\mathbb{R}^3} \nabla u_\lambda^0 \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^3} V(x) u_\lambda^0 \varphi \, dx - \lambda \int_{\mathbb{R}^3} f(x, u_\lambda^0) \varphi \, dx - \int_{\mathbb{R}^3} |u_\lambda^0|^4 u_\lambda^0 \varphi \, dx = 0, \]

i.e., \( \langle (I_\lambda^0)'(u_\lambda^0), \varphi \rangle = 0 \), which implies that \( (I_\lambda^0)'(u_\lambda^0) = 0 \). This completes the proof. \( \square \)

REFERENCES

[1] C. O. Alves and G. M. Figueiredo, Nonlinear perturbations of a periodic Kirchhoff equation in \( \mathbb{R}^N \), Nonlinear Anal., 75 (2012), 2750–2759.
[2] A. Arosio and S. Panizzi, On the well-posedness of the Kirchhoff string, Trans. Amer. Math. Soc., 348 (1996), 305–330.
[3] S. Bernstein, Sur une classe d’équations fonctionnelles aux dérivées partielles, Bull. Acad. Sci. URSS, Sér. Math. [Izv. Akad. Nauk SSSR], 4 (1940), 17–26.
[4] T. Bartsch, Z. Q. Wang and M. Willem, The Dirichlet problem for superlinear elliptic equations, In Stationary Partial Differential Equations. Handb. Differ. Equ., vol. II, pp. 1–55. Elsevier/North-Holland, Amsterdam (2005).
[5] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Global existence and uniform decay for the Kirchhoff-Carrier equation with nonlinear dissipation, Adv. Differential Equations, 6 (2001), 701–730.
[6] C. Chen, Y. Kuo and T. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differential Equations, 250 (2011), 1876–1908.
[7] M. Chipot and B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, Nonlinear Anal., 30 (1997), 4619–4627.
[8] Y. Deng, S. Peng and W. Shuai, Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in \( \mathbb{R}^3 \), J. Funct. Anal., 269 (2015), 3500–3527.
[9] P. D’Ancona and S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math., 108 (1992), 247–262.
[10] G. M. Figueiredo and H. R. Quoirin, Ground states of elliptic problems involving non-homogeneous operators, Indiana University Mathematics Journal, 65 (2016), 779–795.
[11] Z. Guo, Ground states for Kirchhoff equations without compact condition, J. Differential Equations, 259 (2015), 2884–2902.
[12] X. He and W. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in \( \mathbb{R}^3 \), J. Differential Equations, 252 (2012), 1813–1834.
[13] X. He and W. Zou, Ground states for nonlinear Kirchhoff equations with critical growth, Ann. Mat. Pura Appl., 193 (2014), 473–500.
[14] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman-Lazer type problem set on \( \mathbb{R}^N \), Proc. Roy. Soc. Edinburgh, 129 (1999), 787–809.
[15] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.
J. L. Lions, On some questions in boundary value problems of mathematical physics, in Contemporary Developments in Continuum Mechanics and Partial Differential Equations, Proc. Internat. Sympos. Inst. Mat. Univ. Fed. Rio de Janeiro, 1997, in: North-Holland Math. Stud., 30 (1978), 284–346.

S. Liu and S. Li, Infinitely many solutions for a superlinear elliptic equation, Acta Math. Sinica (Chin. Ser.), 46 (2003), 625–630 (in Chinese).

Q. Li and X. Wu, A new result on high energy solutions for Schrödinger-Kirchhoff type equations in $\mathbb{R}^N$, Appl. Math. Lett., 30 (2014), 24–27.

G. Li and H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, J. Differential Equations, 257 (2014), 566–600.

S. I. Pohožaev, A certain class of quasilinear hyperbolic equations, Mat. Sb. (N.S.), 96 (1975), 152–168.

W. Shuai, Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains, J. Differential Equations, 259 (2015), 1256–1274.

A. Szulkin, T. Weth, The method of Nehari manifold, in D. Y. Gao, D. Motreanu (Eds), Handbook of Nonconvex Analysis and Applications, International Press, Boston, (2010), 597–632.

X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in $\mathbb{R}^N$, Nonlinear Anal., 12 (2011), 1278–1287.

X. Wu, High energy solutions of systems of Kirchhoff-type equations in $\mathbb{R}^N$, J. Math. Phy., 53 (2012), 063508.

X. Wu and K. Wu, Geometrically distinct solutions for quasilinear elliptic equations, Nonlinearity, 27 (2014), 987–1001.

J. Wang, L. Tian, J. Xu and F. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, J. Differential Equations, 253 (2012), 2314–2351.

Received January 2017; Revised July 2017.

E-mail address: wuxian2042@163.com