Tricritical point induced by discontinuous and Berezinskii-Kosterlitz-Thouless phase transitions

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We examine classical two-dimensional 6-state Potts and clock spin models combined into a single PC model. Within the PC model, we define a parameter $0 \leq \alpha \leq 1$ which linearly interpolates between the 6-state Potts model ($\alpha = 0$) and the 6-state clock model ($\alpha = 1$). The corner transfer matrix renormalization group method is used to analyze the phase transitions on the square lattice in the thermodynamic limit. Three different phases and phase transitions are identified. The phase diagram is constructed, and we determine a tricritical point at $\alpha_c = 0.214$ and $T_c = 0.834$. Analyzing the latent heat and the entanglement entropy in the vicinity of the $\alpha_c$, we observe a single discontinuous phase transition and two Berezinskii-Kosterlitz-Thouless (BKT) phase transitions meeting in the tricritical point. The tricritical point exhibits the continuous phase transition of the second order with the critical exponents $\beta \approx 0.1$ and $\delta \approx 14$. We conjecture that an infinitesimal surrounding of the tricritical point consists of the three fundamental phase transitions, in which the first and BKT orders gradually weaken into the tricritical point of the second order.

I. INTRODUCTION

The multi-state spin models provide sufficient information about various types of phase transitions. For classical statistical systems, phase transitions of the first, second, and infinite order reflect the microscopic properties of spin models. Number of spin states as well as spin interactions play an important role in our study of phase transition orders. Among the typical spin models considered, we choose the two-dimensional $q$-state Potts and $q$-state clock models.

The $q$-state Potts models exhibit second-order (continuous) phase transitions if $q = 2, 3, 4$ whereas first-order (discontinuous) phase transitions occur if $q \geq 5$.\textsuperscript{1,3} The $q$-state clock model also reveals the second-order (continuous) phase transition if $q = 2, 3, 4$, however, for $q \geq 5$ there are two infinite-order Berezinskii-Kosterlitz-Thouless (BKT) phase transitions in the $q$-state clock model\textsuperscript{4,9}.

If sorted into universality classes (only for $q = 2, 3, 4$), the 2-state Ising universality and the 3-state Potts universality are common for both models. However, the 4-state Potts universality with logarithmic corrections differs from the Ising universality of the 4-state clock model which decouples into the two Ising models yielding again the Ising universality. The correlation length diverges ($\xi \rightarrow \infty$) for the second-order transitions, whereas the first-order transitions result in the finite $\xi < \infty$.

There is a dramatic change of the phase-transition nature between $q = 4$ and $q = 5$. For instance, long-range correlations of the 5-state Potts model weaken the typical first-order phase transitions due to the proximity of the second-order transition ($q = 4$). Hence, the discontinuous transition at $q = 5$ is similar to the continuous transition. As a consequence, the correlation length, $\xi_q$, becomes extraordinary large ($\xi_q \approx 2512$) at the phase transition, although it is finite.\textsuperscript{2} If compared to the Potts models with $q \geq 6$, we find a rapid decrease of the correlation length, i.e., $\xi_6 \approx 159$, $\xi_7 \approx 48$, etc.\textsuperscript{10}. Therefore, numerical studies of the discontinuous phase transitions are more feasible for larger $q$, as they require lower accuracy around the phase transition. On the other hand, the clock model with $q \geq 5$ is known for the existence of an intermediate critical phase, which separates the low-temperature ferromagnetic phase and the high-temperature paramagnetic one. The intermediate phase is the BKT phase. Again, the proximity of second-order phase transition at $q = 4$ makes the 5-state clock model rather difficult to be treated with sufficient accuracy since the BKT phase is narrow, as we show in the phase diagram later.

Accurate calculation of the free energy is needed when finding the correct phase-transition temperature $T_1$ of the first order. To avoid sticking to a false local free-energy minimum, the global free-energy minimum can be reached by imposing two different boundary conditions. The first-order phase transition leads to a discontinuous jump in the internal energy, which is associated with the nonzero latent heat for $q \geq 5$.\textsuperscript{2} Alternatively, to locate the two BKT phase-transition temperatures $T_1$ and $T_2$ in the $q \geq 5$ clock model accurately, an additional finite-size scaling is required. It turns out that the entanglement-entropy analysis has been found effective and accurate.\textsuperscript{8,9}

To accomplish the task successfully, we apply the corner transfer matrix renormalization group (CTMRG) method, as a reliable and the stable numerical algorithm.\textsuperscript{11,12} Being motivated by complexity of the phase transitions, we propose a model, where two types of phase transitions meet and can form a tricritical point. For this purpose, we have combined the Potts and clock models into one model, the PC model, which we simulate.
numerically with CTMRG algorithm. The complexity of the PC model calls for an additional intensive analysis. We, therefore, leave some questions unanswered in this work for future analysis.

Having observed the three fundamental types of the phases (ferromagnetic, paramagnetic, and BKT), the PC model enables gradual modification of the latent-heat for the given $q$-state Potts model down to zero at the tricritical point. On the other hand, we can observe a gradual formation of the BKT phase starting from the tricritical point toward the $q$-state clock model. The BKT (infinite-order) phase transition weakens if approaching the tricritical point from the opposite direction. The type of the phase transition at the tricritical point is explored within this article.

The paper is organized as follows. We specify the $q$-state spin models and construct the phase diagram of the Potts and clock models in Sec. II. A brief introduction to the CTMRG method is given in Sec. III, where the reader can find summarized thermodynamic quantities used in the analysis section. Section IV is devoted to numerical results, where we mainly focus on the analysis of latent heat and entanglement entropy, respectively, when studying the discontinuous and the BKT transitions. We locate the tricritical point $T_c(\alpha_c)$ in the $T$-$\alpha$ phase diagram followed by the classification of its critical exponents $\beta$ and $\delta$ analyzing the spontaneous and induced magnetization. The results are concluded in Sec. V.

II. SPIN MODEL

Consider $q$-state spins $\sigma_{i,j} = 0, 1, \ldots, q - 1$ and each spin $\sigma_{i,j}$ lies on the two-dimensional square lattice of the infinite size at position $i$ and $j$. The nearest-neighbor spin pairs interact via $J$. The spins can enter the $q$-state Potts-model Hamiltonian

$$\mathcal{H}_{\text{total}}^q = -J \sum_{i,j=-\infty}^{\infty} \delta(\sigma_{i,j}; \sigma_{i+1,j}) + \delta(\sigma_{i,j}; \sigma_{i,j+1})$$

and the $q$-state clock-model Hamiltonian

$$\mathcal{H}_{\text{total}}^{q-C} = -J \sum_{i,j=-\infty}^{\infty} \cos(\theta_{i,j} - \theta_{i+1,j}) + \cos(\theta_{i,j} - \theta_{i,j+1})$$

where $\theta_{i,j} = \frac{2\pi}{q}\sigma_{i,j}$. The entire square lattice is built up of horizontally and vertically interacting spin pairs. When $q = 2$, both models belong to the Ising universality class (including the 4-state clock model) \cite{12}. Within the models, the phase transitions of the first, second, and infinite (BKT) orders can be accessed by varying $q$.

Figure 1 sketches the phase diagram of the $q$-state Potts and the $q$-state clock models in the ferromagnetic regime $J > 0$. The phase diagrams of both models are plotted in the same graph using the CTMRG method for $2 \leq q \leq 20$. For $q > 20$ we used analytic formula for the Potts model only, where the phase-transition temperature $T_c = 1/\ln(1 + \sqrt{q})$ are known exactly \cite{1}. The squares in the phase diagram separate the ferromagnetic phase and the high-temperature paramagnetic phase. This is true for finite $q \geq 5$. Taking the limit $q \to \infty$ leads to the suppression of the ferromagnetic phase in both cases. The clock model with $q \to \infty$ describes the classical XY model in Eq. \cite{2} with continuous spin $0 \leq \theta < 2\pi$ \cite{17}.

Let the total Hamiltonian depend on number of the spin states $q$ and be parameterized by a real number $\alpha$ such that $0 \leq \alpha \leq 1$. The parameter $\alpha$ linearly interpolates between the Potts model if $\alpha = 0$ and the clock model if $\alpha = 1$. The thermodynamic properties are governed by the partition function

$$Z_q(\alpha) = \sum_{\text{spin configs}} \exp \left( -\frac{\mathcal{H}_{\text{total}}^{q-PC}(\alpha)}{k_B T} \right)$$

$$= \sum_{\text{spin configs}} \exp \left( -\frac{1}{k_B T} \sum_{i,j=-\infty}^{\infty} H_{i,j,\theta}^{q-PC}(\alpha) \right)$$

$$= \sum_{\text{spin configs}} \prod_{i,j=-\infty}^{\infty} \exp \left( -\frac{H_{i,j,\theta}^{q-PC}(\alpha)}{k_B T} \right),$$

where the summation runs over all spin configurations of the total Hamiltonian $\mathcal{H}_{\text{total}}^{q-PC}(\alpha)$ on the infinite square.
At the end, we divide the local Hamiltonian into the Potts part $\mathcal{H}_{i,j,\vartheta}^{q-\text{Potts}}$ and the clock part $\mathcal{H}_{i,j,\vartheta}^{q-\text{clock}}$. The three-spin local Hamiltonians $H_{i,j,\vartheta}^{q-\text{PC}}(\alpha)$ cover the entire square lattice, see Fig. 2 (if the lattice is finite, periodic boundary conditions are useful).

In addition, we introduced index $k$ to split the three-spin local Hamiltonian between one spin pair oriented vertically ($k = 0$) and another oriented horizontally ($k = 1$). We have chosen this form to keep the vertical and horizontal Hamiltonians symmetric and identical, i.e., $\mathcal{H}_{i,j,0,\vartheta}^{q-\text{Potts}} \equiv \mathcal{H}_{i,j,1,\vartheta}^{q-\text{Potts}}$. For the Potts model we get

$$\mathcal{H}_{i,j,k,\vartheta}^{q-\text{Potts}} = -J \delta(\sigma_{i,j} + 1, \sigma_{i+k,j+1-k}) - \frac{h}{4} \left[ \delta(\sigma_{i,j}, \vartheta) + \delta(\sigma_{i+k,j+1-k}, \vartheta) \right].$$

(5)

The clock model also satisfies $\mathcal{H}_{i,j,0,\vartheta}^{q-\text{clock}} = \mathcal{H}_{i,j,1,\vartheta}^{q-\text{clock}}$ and

$$\mathcal{H}_{i,j,k,\vartheta}^{q-\text{clock}} = -J \cos \frac{2\pi}{q} (\sigma_{i,j} - \sigma_{i+k,j+1-k}) - \frac{h}{4} \cos \frac{2\pi}{q} (\sigma_{i,j} - \sigma_{i+k,j+1-k} - \vartheta).$$

(6)

Selecting $q = 6$, the 6-state Potts model can be expressed in form of the $6 \times 6$ diagonal matrix

$$\mathcal{H}_{i,j,0}^{6-\text{Potts}} = \begin{pmatrix}
\tilde{j} & 0 & 0 & 0 & 0 & 0 \\
0 & J & 0 & 0 & 0 & 0 \\
0 & 0 & J & 0 & 0 & 0 \\
0 & 0 & 0 & J & 0 & 0 \\
0 & 0 & 0 & 0 & J & 0 \\
0 & 0 & 0 & 0 & 0 & J
\end{pmatrix}.$$  

(7)

for any $i, j, k$. The magnetic field $h$, acting in the direction $\vartheta = 0$, is included in $\tilde{j} = J + \frac{h}{4}$. Analogously, the 6-state clock model has the form

$$\mathcal{H}_{i,j,0}^{6-\text{clock}} = -\frac{1}{2} \begin{pmatrix}
2\tilde{j} & \tilde{j} & -\tilde{j} & -2\tilde{j} & -\tilde{j} & \tilde{j} \\
\tilde{j} & 2J & J & -J & -2J & -J \\
-\tilde{j} & J & J & J & -2J & -J \\
-2\tilde{j} & -J & J & J & J & -J \\
-\tilde{j} & -2J & -J & J & J & J \\
\tilde{j} & -J & -2J & -J & J & J
\end{pmatrix}.$$  

(8)

The full PC model parameterized by $\alpha$ in Eq. (1) gives

$$\mathcal{H}_{i,j,0}^{6-\text{PC}}(\alpha) = -\frac{J}{2} \begin{pmatrix}
2 & \alpha & -\alpha & -2\alpha & -\alpha & \alpha \\
\alpha & 2 & \alpha & -\alpha & -2\alpha & -\alpha \\
-\alpha & \alpha & 2 & -\alpha & -2\alpha & -\alpha \\
-2\alpha & -\alpha & \alpha & 2 & -\alpha & -\alpha \\
-\alpha & -2\alpha & \alpha & -\alpha & 2 & \alpha \\
\alpha & -\alpha & -2\alpha & -\alpha & \alpha & 2
\end{pmatrix}.$$  

(9)
We can generalize the local Hamiltonian for any q-state PC model by $q \times q$ matrix. If we index the q-state matrix subsystems with $\sigma, \sigma' = 0, 1, 2, \ldots, q - 1$, each matrix element has the expression

$$[H_{i,j,k,q}^q(\alpha)]_{\sigma,\sigma'} = - \alpha J \cos \left[ \frac{2\pi}{q} (\sigma - \sigma') \right] - \frac{\alpha h}{4} \cos \left[ \frac{2\pi}{q} (\sigma - \vartheta) \right] \delta_{(\sigma',\vartheta)} - \frac{\alpha h}{4} \cos \left[ \frac{2\pi}{q} (\sigma' - \vartheta) \right] \delta_{(\sigma,\vartheta)} + (\alpha - 1) \left( J + \frac{h}{2} \delta_{(\sigma,\vartheta)} \right) \delta_{(\sigma,\sigma')} + \frac{h}{4} \delta_{(\sigma,\vartheta)} \delta_{(\sigma,\sigma')}.$$ (10)

For brevity, we have omitted the lattice positions $i, j$ for both spins $\sigma, \sigma'$ such that we simplify them into $\sigma \equiv \sigma_{i,j}$ and $\sigma' \equiv \sigma_{i+k,j+1-k}$ for either $k$.

While varying $0 \leq \alpha \leq 1$ with a given $q \geq 5$, the first-order phase transition, characteristic for the Potts model ($\alpha = 0$), gradually changes and, at some point, splits into two infinite-order (BKT) phase transitions, which are typical for the clock model ($\alpha = 1$).

If there exists a point $\alpha_c$, possibly a tricritical point, a continuous (second-order) phase transition is very likely at $\alpha_c$. It is our task to locate the point $\alpha_c$ and determine its critical exponents. There could be another scenario, according to which there is no continuous transition at $\alpha_c$, and some discontinuities in thermodynamic functions may possibly occur.

In the following we claim that with increasing $0 \leq \alpha \leq \alpha_c$, the first-order transition gradually weakens when approaching the tricritical point $\alpha_c$. At $\alpha_c$, the first order discontinuity vanishes and changes into the second-order phase transition. Right above $\alpha_c$ the two infinite-order (BKT) transitions coexist and start moving away with increasing $\alpha > \alpha_c$.

### III. CTMRG METHOD

The CTMRG method is designed to built-up the square lattice iteratively. As the iterations increase, the lattice gradually expands the size. If the integer variable $k = 1, 2, 3, \ldots$ enumerates the CTMRG iteration number, then the lattice size is identical to the total number of spins $N_k = (2k + 1)^2$. Numerical accuracy of CTMRG is controlled by maximal dimension of the spin-configuration space. The dimension denote number of the states kept, and we assign it to the integer variable $m$ [11] [12]. The higher the $m$, the higher the numerical accuracy is reached. Ideally, all thermodynamic quantities are equivalent to the exact solution only if keeping $m = q^k$ which is not computationally feasible. However, when evaluating the thermodynamic limit ($k \to \infty$ i.e., $N_k \to \infty$), in addition, we need to extrapolate thermodynamic quantities with respect to $m \to \infty$ in order to obtain accurate results.

We evaluate the partition function $Z_q(\alpha)$ by means of the corner-transfer-matrix formalism [11] [13]. The free energy $F$ normalized per number of spins has the expression in the thermodynamic limit

$$F_q(\alpha) = \lim_{N \to \infty} - \frac{k_B T}{N} \ln Z_q(\alpha) = \lim_{N \to \infty} - \frac{k_B T}{N} \ln \left[ \text{Tr} C_q^4(\alpha) \right].$$ (11)

where $N$ is the number of spins on the square lattice and $C$ is a corner transfer matrix. The structure of CTMRG enables us to construct a reduced density matrix $\rho$ by taking partial trace out of the four corner transfer matrices, $\rho = \text{Tr} C_q^4(\alpha)$ [11].

We use CTMRG for the analysis of the phase transitions, especially, we detect the tricritical point. Therefore, we evaluate the order parameter (magnetization $M$) in arbitrary spin direction $\vartheta$

$$M = \langle \sigma \rangle = \text{Tr} \left[ \rho \delta_{(\sigma,\vartheta)} \right].$$ (12)

and the internal energy $U = -T^2 \partial_T (F/T)$, being also proportional to the nearest-neighbor spin-spin correlation,

$$U = -2J \langle \sigma \sigma' \rangle = -2J \text{Tr} \left\{ \rho \cos \left[ \frac{2\pi}{q} (\sigma - \sigma') \right] \right\}.$$ (13)

The latent heat $\lambda$ is directly obtained from the internal energy by taking left and right limits at the phase-transition temperature $T_i$

$$\lambda = \lim_{T \to T_i^+} U(T) - \lim_{T \to T_i^-} U(T).$$ (14)

If the first-order phase transition is present, then $\lambda > 0$. The specific heat (capacity)

$$c = \frac{\partial U}{\partial T} = -T \frac{\partial^2 F}{\partial T^2}$$ (15)

exhibits two peaks in the proximity of both BKT transitions. Finally, we calculate the von-Neumann entanglement entropy

$$S = - \text{Tr} (\rho \ln \rho) = - \sum_{j=1}^m \omega_j \ln \omega_j$$ (16)

by the eigenvalues $\omega_j$ of the reduced density matrix $\rho$.

The phase transitions are characterized by singular behavior of the above thermodynamic quantities. Typically, they (or their derivatives) exhibit singular behavior with sharp maxima. The maxima typically diverge in the thermodynamic limit $N \to \infty$ such as $S$ and $c$. Or, we can observe discontinuities in phase transitions of the first order, in which those quantities are accompanied by either non-diverging maxima ($S$ and $c$) or non-analytic behavior, such as sudden jumps in $M$ and $U$. 


We can uniquely identify the first-order phase transitions by calculating the nonzero latent heat $\lambda > 0$ in Eq. (14). Let us recall that the precise location of the discontinuous jump in the internal energy $U$ can be determined only by minimizing the free-energy from a crossover caused by imposing open and fixed boundary conditions.

IV. RESULTS

We use dimensionless units and set $k_B = 1$. To simplify the problem, we consider the ferromagnetic ordering at low temperatures and we set the interaction $J = 1$. As we have mentioned earlier, we introduce the parameter $0 \leq \alpha \leq 1$ in order to explore how a discontinuous phase transition gradually transforms onto the BKT phase transition, i.e., we define the PC model linearly interpolating between the Potts ($\alpha = 0$) and the clock ($\alpha = 1$) models. We can observe such nontrivial features only when $q \geq 5$, see Fig. 3. The higher the $q$, the more computational cost is required to reach desired accuracy.

On the other hand, when $q = 5$, the correct determination of the two BKT phase transitions encounters extensive numerical effort with high numerical accuracy to detect the two closely placed BKT phase transitions. We, therefore, select $q = 6$ in the following, as emphasized by the three asterisks in Fig. 3.

A. Phase diagram

Figure 3 shows the entanglement entropy $S$ and the magnetization $M$ (in the inset) with respect to temperature $T$ within the selected spectrum $\alpha = 0.0, 0.2, 0.4, 0.6, 0.8, \text{and } 1.0$. The first-order phase transition is accompanied with a discontinuous jump in $S$ and $M$ at temperature $T_1(\alpha = 0) = 1/\ln(1 + \sqrt{6})$ (dotted lines) and $T_1(\alpha = 0.2) \approx 0.833$ (dashed line). When $\alpha > 0.2$ (full lines), the typical shapes of $S$ and $M$ are signatures of the BKT phase transitions.

We remark here that the entanglement entropy and magnetization for $\alpha = 0.4$ seem to exhibit such behavior which is typical for the continuous (second-order) phase transition. However, the detailed specification of $S$ and $M$ at $\alpha = 0.4$ results in two BKT phase transitions, as we discuss in the following.

In Fig. 4, the entanglement entropy is shown in thermodynamic limit (at finite precision $m = 100$) to examine the BKT phase transitions in detail. The entanglement entropy $S$ logarithmically diverges in the BKT phase.

We have thus sketched the divergence of $S$ by the two hatched areas on top of Fig. 4. Each area consists of the BKT phase-transition temperatures $T_1(\alpha)$ and $T_2(\alpha)$, where $S$ is infinite. The entire BKT phase is critical because the correlation length is also infinite in the entire interval $T_1(\alpha) \leq T \leq T_2(\alpha)$. The dashed and dotted lines, respectively, estimate the extrapolated critical temperatures $T_1(\alpha)$ and $T_2(\alpha)$ in the limit $N, m \to \infty$.

In order to locate the tricritical point, we return back to Fig. 3 which indicates the existence of $\alpha_c$ within the interval $0.2 < \alpha_c < 0.4$. The first-order phase transition at $\alpha = 0.2$ dramatically changes into a subtly curved entanglement entropy describing two BKT transitions at $\alpha = 0.4$ (better visible in Fig. 4). Specifically, we expect that the tricritical point $T_c(\alpha_c)$ has to be within the interval $0.2 < \alpha_c < 0.4$ such that it satisfies the condition

$$
\lim_{\alpha \to \alpha_c} T_1'(\alpha) = \lim_{\alpha \to \alpha_c} T_1(\alpha) = \lim_{\alpha \to \alpha_c} T_2(\alpha) \equiv T_c(\alpha_c).
$$
The free energy is a smooth monotonic function of $\alpha$ in the dynamic limit. The free energy is calculated using the nonlinear least-square method to minimize the free energy (not shown) \[ \text{(3)}. \] Consequently, we determine the remaining quantities, such as $U$, $M$, and $S$ at the correct phase transition temperature $T_i(\alpha)$.

Once $T_i(\alpha)$ has been specified out of $\mathcal{F}(\alpha)$, we focus on a narrow region around the tricritical point. The internal energy $U$ (cf. Eq. (14)) shown in Fig. 6 is discontinuous if $0.210 \leq \alpha \leq 0.214$ and continuous if $0.216 \leq \alpha \leq 0.222$. The discontinuity exhibits phase transition of the first order. Since the latent heat $\lambda > 0$ when obtained from Eq. (14). If approaching the limit $\alpha \rightarrow \alpha_\text{c}^-$ (from left), we found out gradually closing continuous jump in $\mathcal{F}$, i.e., $\lambda(\alpha) \rightarrow \lambda(\alpha_\text{c}) = 0$, as shown in Fig. 6. It happens at the tricritical point at $T_i \approx 0.834$ and $\alpha_\text{c} \approx 0.214$, as marked in Fig. 5.

Next, we estimate $\alpha_\text{c}$ from the viewpoint of the latent heat $\lambda$. Figure 7 shows the latent heat $\lambda$ as a function of $\alpha$ in the log-linear scale. It is obvious from the graph that the latent heat drops to zero fast. Having applied the nonlinear least-square method to $\lambda(\alpha) = \eta(\alpha_\text{c} - \alpha)^2$, we found the fitting parameters $\eta = 0.54945$, $\alpha_\text{c} = 0.21405$, and $\varepsilon = 0.52631$. The inset shows the $\alpha$-dependence of $\lambda^{1/2}$ to stress that the latent heat linearly decreases when approaching the tricritical point. Thus, we observe an approximative square-root dependence $\lambda(\alpha) \propto \sqrt{\alpha_\text{c} - \alpha}$. The first-order phase transition in the PC model terminates at $\alpha_\text{c} = 0.21405$, where we expect the second-order phase transition which smoothly changes into the two BKT phase transitions, as $\alpha$ further increases above $\alpha_\text{c}$.

Right region: In the BKT regime, $\lambda(\alpha \geq \alpha_\text{c}) = 0$. Notice tiny ripples in $U$ in Fig. 6 which are more visible right below the phase-transition temperature for $\alpha \geq 0.218$. They indicate proximity of the lower-temperature BKT phase transitions at $T_1(\alpha)$ and $T_2(\alpha)$. Since the ripples are sensitive to temperature derivative, they affect the specific heat $c = \partial U/\partial T$ which yields two peaks corresponding to the two BKT transition temperatures.
There is no image in the provided text.
\[ T_m(0) \sim m^{-1} \] if plotted with respect to \( m^{-2} \). Here, \( \delta_m, d_m, \) and \( \rho_m \) are the unknown parameters found by the least-square fitting. The extrapolation critical exponents yields \( \delta_\infty \equiv \delta = 14.0362 \).

V. CONCLUSIONS

We have proposed the PC model in which \( \alpha \) parameterizes the linearly interpolated 6-state Potts model (\( \alpha = 0 \)) with the 6-state clock (\( \alpha = 1 \)) model. Using the CTMRG method, we have constructed the phase diagram of the PC model, as depicted in Fig. 9. We determined the tricritical point at \( \alpha_c = 0.21405 \) and \( T_c = 0.845017 \), where the second-order phase transition takes place. The associated critical exponents are \( \beta = 0.09699 \) and \( \delta = 14.0362 \). Table I summarizes our results and compares them with the exact results known for the Potts and clock models for \( q = 2, 3, 4 \).

The phase diagrams of any \( q \)-state PC models with finite \( q > 6 \) are expected to be qualitatively similar to the current study (\( q = 6 \)), due to the similarities of the models at \( \alpha = 0, 1 \). This is evident from the phase diagram in Fig. 10. However, if \( q = 5 \), we observe a weak first-order transition and a narrow BKT phase, which make the task nontrivial to be numerically reliable. Moreover, the phase-diagram structure would differ significantly since \( T_1(0) < T_1(1) < T_2(1) \), and the task needs to be clarified.

The question whether the infinite-order phase transition persists for \( \alpha_c < \alpha \leq 1 \) or continuously varies from the second-order at \( \alpha = \alpha_c \) towards the infinite order at \( \alpha = 1 \) (via third, fourth, etc.), is left unanswered yet. Comparing the shapes of entanglement entropy in Figs. 3 and 4 for \( \alpha \geq 0.4 \), we recognize that the BKT transition is of the infinite order. However, specific heat \( c \) versus temperature \( T \) in Fig. 5 suggests a tiny discontinuity at \( T_2(\alpha_c < \alpha \lesssim 0.6) \) which may refer to persisting traces of the first-order transition even inside the infinite-order (BKT) regime (or, alternatively, a sequence of higher-order phase transitions). We cannot resolve the problem yet due to numerical limitations of the CTMRG method.

On the top of the analysis, the PC model allows additional interaction corrections (most likely artificial) so as to freely modify the interaction matrices, as in Eq. 15. By adjusting the interaction matrices, novel physical properties could be brought into the light at the macroscopic level.

Our original motivation for the study originated in the proposal of a simple spin model capable of producing phase transitions of the first, second, and infinite orders...
(including those of third, fourth, fifth, and the higher orders). However, the complexity of the task does not enable us to proceed and concisely clarify if the model can describe the full set of all higher-order phase transitions. We keep this task open for further investigations.

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