A quasilinear problem with fast growing gradient*

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Abstract

In this paper we consider the following Dirichlet problem for the $p$-Laplacian in the positive parameters $\lambda$ and $\beta$:

$$
\begin{cases}
-\Delta_p u &= \lambda h(x,u) + \beta f(x,u,\nabla u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
$$

where $h,f$ are continuous nonlinearities satisfying $0 \leq \omega_1(x)u^{q-1} \leq h(x,u) \leq \omega_2(x)u^{q-1}$ with $1 < q < p$ and $0 \leq f(x,u,v) \leq \omega_3(x)u^{a}|v|^b$, with $a, b > 0$, and $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 3$. The functions $\omega_i$, $1 \leq i \leq 3$, are nonnegative, continuous weights in $\Omega$. We prove that there exists a region $\mathcal{D}$ in the $\lambda \beta$-plane where the Dirichlet problem has at least one positive solution. The novelty in this paper is that our result is valid for nonlinearities with growth higher than $p$ in the gradient variable.

keywords: $p$-Laplacian, positive solution, nonlinearity depending on the gradient, sub- and super-solution method.

1 Introduction

Dependence on the gradient in problems involving quasilinear operators as the $p$-Laplacian have been challenging researchers of elliptic PDE’s in questions of existence and uniqueness. The approach used to handle this problems varies, ranging from change of variables in order to eliminate the dependence on the gradient to a combination of topological and blow-up arguments.

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In a nutshell, no general method to deal with this kind of problem has been established.

In this paper we intend to show how simple techniques (sub- and super-solution method combined with a global estimate on the gradient) are able to solve some quasilinear problems involving nonlinearities with fast growing gradient, that is, nonlinearities where the exponent of $|\nabla u|$ is greater than $p$. This type of problem is rare in the literature. The method we choose allows us to make simple hypotheses, also in contrast with papers in the area.

We consider the following Dirichlet problem in the positive parameters $\lambda$ and $\beta$:

$$\begin{cases}
-\Delta_p u &= \lambda h(x, u) + \beta f(x, u, \nabla u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{cases}$$

Our hypotheses on $f$ (see below) include nonlinearities that depend on the gradient with an exponent higher than $p$ and thus the application of variational methods (even in combination with topological techniques, see [7]) can not handle directly this kind of problem. Known versions of the sub- and super-solution method developed for equations depending on the gradient (see [3, 6]) require nonlinearities with the gradient term growing at most as $|\nabla u|^p$.

Here, inspired by the classical paper of Ambrosetti, Brezis and Cerami [2] we define a fixed point operator for each $(\lambda, \beta)$ in a region $D$ of the $\lambda\beta$-plane and use global $C^1,\alpha$ estimates on the solution of the Poisson equation $-\Delta_p u = g$ with Dirichlet boundary conditions on $\Omega$ to obtain an invariant subset by this operator. Hence, by applying Schauder’s fixed point theorem we prove the existence of at least one positive solution for the Dirichlet problem above if $(\lambda, \beta) \in D$.

## 2 Existence of a positive solution

In this section we consider the existence of positive solutions for the following problem in two positive parameters in the bounded, smooth domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases}
-\Delta_p u &= \lambda h(x, u) + \beta f(x, u, \nabla u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{cases}$$

where $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator, $p > 1$, and $h, f$ are continuous nonlinearities satisfying

(H1) $0 \leq \omega_1(x)u^{q-1} \leq h(x, u) \leq \omega_2(x)u^{q-1}, \ 1 < q < p$;

(H2) $0 \leq f(x, u, v) \leq \omega_3(x)u^{a}|v|^b, \ a, b > 0$,

and $\omega_i : \overline{\Omega} \to [0, \infty), \ 1 \leq i \leq 3$, are nonnegative continuous functions (with $\omega_i \neq 0$) that we call weights.

We begin establishing a version of a result on the regularity of solutions of the $p$-Laplacian, which was proved by Tolksdorf [13] and Liebermann [12]. The proof is given, since the result is not explicitly stated in those papers.
Lemma 1 Let $\Omega$ be a bounded, smooth domain of $\mathbb{R}^N$ and $g \in L^\infty(\Omega)$. Assume that $u \in W_0^{1,p}(\Omega)$ is a weak solution of
\[
\begin{cases}
-\Delta_p u = g & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]
(2)

Then there exists a positive constant $K$, depending only on $p$, $N$ and $\Omega$, such that
\[
\|\nabla u\|_\infty \leq K(\|g\|_\infty)^{\frac{1}{p-1}}.
\]
(3)

Proof. Let us firstly assume that $\|g\|_\infty = 1$. By applying a simple comparison principle, one can easily verify that $|u| \leq \phi$ where $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is the $p$-torsion function of $\Omega$, that is, $-\Delta_p \phi = 1$ in $\Omega$. Therefore,
\[
\|u\|_\infty \leq L := \|\phi\|_\infty.
\]
(4)

It follows from global regularity results by Lieberman (see [12]) that there exist constants $\alpha \in (0,1)$ and $K > 0$ such that $u \in C^{1,\alpha}(\Omega)$ and $\|u\|_{1,\alpha} \leq K$ and, moreover, $\alpha$ and $K$ depend only on $p$, $N$ and $\Omega$. (In principle, these constants could also depend on the bound $L$ for $\|u\|_\infty$, but as we easily see, the bound in (4) is uniform with respect to $u$ whenever $\|g\|_\infty = 1$).

Since $\|\nabla u\|_\infty \leq \|u\|_{1,\alpha}$ we obtain (3) in the case $\|g\|_\infty = 1$. If $0 < \|g\|_\infty \neq 1$ we apply the previous argument to the function $v := \frac{u}{\|g\|_\infty^{\frac{1}{p-1}}}$ since this function satisfies
\[
\begin{cases}
-\Delta_p v = g/\|g\|_\infty & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega.
\end{cases}
\]
(5)

Thus, we obtain
\[
\frac{\|\nabla u\|_\infty}{\|g\|_\infty^{\frac{1}{p-1}}} \leq K.
\]

Therefore, we have proved (3) for any $0 \neq g \in L^\infty(\Omega)$ where the positive constant $K$ depends only on $p$, $N$ and $\Omega$. Obviously, (3) remains valid for the same constant $K$ if $g \equiv 0$. \qed

To solve problem (1) we define
\[
r := a + b + 1, \quad \omega(x) := \max_{i \in \{1,2,3\}} \omega_i(x)
\]
and denote by $\lambda_1$ and $u_1$ the first eigenpair of the $p$-Laplacian with weight $\omega_1$, that is,
\[
\begin{cases}
-\Delta_p u_1 = \lambda_1 \omega_1 u_1^{p-1} & \text{in } \Omega, \\
u_1 = 0 & \text{on } \partial\Omega,
\end{cases}
\]
with $u_1$ positive satisfying $\|u_1\|_\infty = 1$.

Let also $\phi \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ be the solution of the problem
\[
\begin{cases}
-\Delta_p \phi = \omega & \text{in } \Omega \\
\phi = 0 & \text{on } \partial\Omega
\end{cases}
\]
and define

\[ \gamma := \frac{\mathcal{K} \| \omega \|_{\infty}^{\frac{1}{p-q}}}{\| \phi \|_{\infty}}, \]

where \( \mathcal{K} \) satisfies (3). We stress that \( \gamma \) depends only on \( \omega, p, N \) and \( \Omega \).

**Lemma 2** There exists a region \( \mathcal{D} \) in the \( \lambda \beta \)-plane such that, if \((\lambda, \beta) \in \mathcal{D}\) then

\[ \lambda M^{q-1} + \beta \gamma b M^{r-p} \leq (M/\| \phi \|_{\infty})^{p-1}, \]  

(6)

for some positive constant \( M \).

**Proof.** The inequality (6) can be written as

\[ \Phi(M) := \lambda A M^{q-p} + \beta B M^{r-p} \leq 1, \]

(7)

where the coefficients

\[ A = \| \phi \|_{\infty}^{p-1} \quad \text{and} \quad B := \mathcal{K} b \| \phi \|_{\infty}^{p-1-b} \| \omega \|_{\infty}^{\frac{b}{p-q}} \]

(8)

clearly depend only on \( \omega, p \) and \( \Omega \).

In order to determine an adequate value for \( M \), we consider the possibilities for the sign of \( r - p \).

**Case 1:** \( r - p > 0 \). In this case we have

\[ \lim_{t \to 0^+} \Phi(t) = \lim_{t \to +\infty} \Phi(t) = +\infty \]

implying that \( \Phi \) has a minimum value. Since the only critical point \( M \) of \( \Phi \) is given by

\[ M := \left[ \frac{\lambda A(p - q)}{\beta B(r - p)} \right]^{\frac{1}{r-q}}, \]

(9)

we obtain

\[ \Phi(M) = \frac{\beta B(r - p) M^{r-p}}{p - q} + \beta B M^{r-p} = \beta B M^{r-p} \left( \frac{r - q}{p - q} \right) \leq \Phi(t) \quad \text{for all} \quad t \geq 0. \]

Now we need to find sufficient conditions on \( \lambda \) and \( \beta \) in order to obtain \( \Phi(M) \leq 1 \) or, equivalently,

\[ \beta B \left[ \frac{\lambda A(p - q)}{\beta B(r - p)} \right]^{\frac{r-q}{r-q}} \left( \frac{r - q}{p - q} \right) \leq 1. \]

After rewriting this last inequality we arrive at

\[ \lambda^{r-p} \beta^{p-q} \frac{1}{A} \left( \frac{r - p}{A} \right)^{r-p} \left( \frac{p - q}{B} \right)^{p-q} \left( \frac{1}{(r-q)^{r-q}} \right) =: \mathcal{K}. \]

(10)
Thus, if the positive parameters $\lambda$ and $\beta$ satisfy (11), we conclude that $\pi := (M/ \|\phi\|_{\infty})\phi$ is a super-solution for (16), where $M$ is given by (9).

**Case 2:** $r - p = 0$. In this case $\Phi(t) := \lambda A t^{r-p} + \beta B$ is positive, strictly decreasing and satisfies
\[
\lim_{t \to 0^+} \Phi(t) = +\infty \quad \text{and} \quad \lim_{t \to +\infty} \Phi(t) = \beta B.
\]
So, in order to have $\Phi(M) \leq 1$ for some $M > 0$ it is necessary that $\beta B < 1$. Thus, if $\lambda > 0$ and $\beta < B^{-1}$ we can take $M > 0$ such that $\Phi(M) = 1$, that is
\[
M = \left( \frac{\lambda A}{1 - \beta B} \right)^{\frac{1}{p-q}}.
\]
Thus, if $\lambda$ and $\beta$ satisfy (11) then $u := \frac{M}{\|\phi\|_{\infty}}\phi$ is a super-solution for (16), where $M$ is given by (12).

**Case 3:** $r - p < 0$. It follows from (7) that $\Phi$ is strictly decreasing and
\[
\lim_{t \to 0^+} \Phi(t) = +\infty \quad \text{and} \quad \lim_{t \to +\infty} \Phi(t) = 0.
\]
Hence, for any positive parameters $\lambda$ and $\beta$, there always exists $M > 0$ such that $\Phi(M) = \lambda A M^{r-p} + \beta B M^{r-p} = 1$ and for such a $M$ the function $\pi = (M/ \|\phi\|_{\infty})\phi$ is a super-solution of (16).

Summarizing, we have proved that there exists a positive constant $M$ satisfying (10) whenever the pair $(\lambda, \beta)$ belongs to the set $D$ defined by:
\[
D = \begin{cases} 
\{ \lambda, \beta > 0 : \lambda \beta^{r-p} \beta^{p-q} \leq K \} & \text{if } r - p > 0, \\
\{ \lambda, \beta > 0 : \beta < B^{-1} \} & \text{if } r - p = 0, \\
\{ \lambda, \beta > 0 \} & \text{if } r - p < 0,
\end{cases}
\] (13)
where $K$ and $B$ were determined by (11) and (8), respectively.

For each $u \in C^1(\Omega)$ we define the continuous nonlinearity $F^u : \Omega \times \mathbb{R} \to \mathbb{R}$ by
\[
F^u(x, \xi) := \lambda \omega_1 \xi^{q-1} + \lambda \left( h(x, u(x)) - \omega_1 u(x)^{q-1} \right) + \beta f(x, u(x), \nabla u(x)).
\] (14)
and observe that $F^u(x, u) = \lambda h(x, u) + \beta f(x, u, \nabla u)$.

**Theorem 3** Assume that $h$ and $f$ are continuous and satisfy (H1) and (H2). There exists a region $D$ in the $\lambda \beta$-plane such that if $(\lambda, \beta) \in D$ the Dirichlet problem (1) has at least one positive solution $u$ satisfying, for some positive constants $\epsilon$ and $M$:
\[
\epsilon u_1 \leq u \leq (M/ \|\phi\|_{\infty})\phi \quad \text{and} \quad \|\nabla u\|_{\infty} \leq \gamma M.
\]
Proof. Let $(\lambda, \beta) \in D$ where the region $D$ is defined by (13) and take $M > 0$ satisfying (9) from Lemma 2. Let us define the subset
\[ F := \{ u \in C^1(\Omega) : \epsilon u_1 \leq u \leq (M/\|\phi\|_\infty)\phi \text{ and } \|\nabla u\|_\infty \leq \gamma M \} \subset C^1(\Omega) \] (15)
where
\[ 0 < \epsilon \leq \min \left\{ \left( \lambda/\lambda_1 \right)^{\frac{1}{p-q}}, (M\lambda_1^{-1})^{\frac{1}{p-q}} \right\} / \|\phi\|_\infty \} . \]

We divide this proof into five steps.

**Step 1.** We prove that for each $u \in F$ there exists a positive solution $U$ of the problem
\[
\begin{cases}
-\Delta_p U &= F^u(x, U) \quad \text{in } \Omega \\
U &= 0 \quad \text{on } \partial \Omega
\end{cases}
\]
satisfying
\[ \epsilon u_1 \leq u \leq (M/\|\phi\|_\infty)\phi. \]
In order to do this we firstly verify that the functions
\[ u := \epsilon u_1 \quad \text{and} \quad \overline{u} := (M/\|\phi\|_\infty)\phi \]
constitute an ordered pair of sub- and super-solutions of (16). This fact implies, by applying a standard iteration process, that there exists a weak solution $U$ of (16) satisfying $u \leq U \leq \overline{u}$.

Since $\overline{u}$ satisfies
\[
\begin{cases}
-\Delta_p \overline{u} &= \omega(M/\|\phi\|_\infty)^{p-1}, \quad \text{in } \Omega \\
\overline{u} &= 0 \quad \text{on } \partial \Omega
\end{cases}
\]
we obtain from the weak comparison principle again that $u$ is a sub-solution for (16). This principle still produces the ordering $u \leq \overline{u}$ in $\Omega$.

**Step 2.** Now we complete the verification that $U \in F$ by proving that $|\nabla U| \leq \gamma M$. Indeed, it follows from (3) of Lemma 1 that
\[ \|\nabla U\|_\infty^{p-1} \leq K^{p-1} \|F^u(x, U)\|_\infty \]
and from (H1), (H2) and (6) that
\[ 0 \leq F^u(x, U) = \lambda_1 U^{q-1} + \lambda (h(x, u) - \omega_1 u^{q-1}) + \beta f(x, u, \nabla u) \]
\[ \leq \lambda_1 U^{q-1} + \lambda (\omega_2 - \omega_1) u^{q-1} + \beta \omega_3 u^a |\nabla u|^b \]
\[ \leq \lambda_2 (M/\|\phi\|_\infty)^{a-1} + \beta \omega_3 (M/\|\phi\|_\infty)^a (\gamma M)^b \]
\[ \leq \|\omega\|_\infty (M/\|\phi\|_\infty)^{p-1} = (\gamma M/K)^{(p-1)/p}. \]

**Step 3.** We prove the uniqueness of \( U \). It is a consequence of a result proved in [9], but it also follows from Picone’s inequality (see [1])
\[ |\nabla u|^p \geq |\nabla v|^{p-2} \nabla v \cdot \nabla \left( \frac{u}{v} \right), \]
which is valid for all differentiable \( u \geq 0 \) and \( v > 0 \). In fact, if \( U \) and \( V \) are both positive solutions of problem (16), we have
\[ \int_\Omega F^u(x, U) U^{p-1} dx = \int_\Omega |\nabla U|^p dx \geq \int_\Omega |\nabla V|^{p-2} \nabla V \cdot \nabla \left( \frac{U^p}{V^{p-1}} \right) dx = \int_\Omega F^u(x, V) V^{p-1} dx, \]
from what follows
\[ \int_\Omega \left( \frac{F^u(x, U)}{U^{p-1}} - \frac{F^u(x, V)}{V^{p-1}} \right) U^p dx \geq 0. \]

An analogous inequality is also true for \( V \):
\[ - \int_\Omega \left( \frac{F^u(x, U)}{U^{p-1}} - \frac{F^u(x, V)}{V^{p-1}} \right) V^p dx \geq 0, \]
and so
\[ \int_\Omega \left( \frac{F^u(x, U)}{U^{p-1}} - \frac{F^u(x, V)}{V^{p-1}} \right) (U^p - V^p) dx \geq 0. \]
(18)
Since \( q < p \), it follows from (14) that \( F^u(x, \xi)/\xi^{p-1} \) is decreasing with respect to \( \xi \). Therefore, the last integrand is non-positive and so (18) yields
\[ \left( \frac{F^u(x, U)}{U^{p-1}} - \frac{F^u(x, V)}{V^{p-1}} \right) (U^p - V^p) = 0 \text{ in } \Omega \]
from what we obtain \( U = V \).

**Step 4.** The regularity \( U \in C^{1,\alpha}(\Omega) \) for some \( 0 < \alpha < 1 \) uniform with respect to \( u \in \mathcal{F} \) follows from the uniform boundedness of both \( U \) and \( |\nabla U| \) together with classical results (see [10, 12, 13]). We emphasize that the bounds for \( U \) and \( |\nabla U| \) are determined by the positive constant \( M \) which, in its turn, is fixed according with the pair \((\lambda, \beta) \in D\).

**Step 5.** In this last step we complete the proof. As consequence of the previous steps the operator
\[ T: \mathcal{F} \subset C^{1,\alpha}(\Omega) \quad \mapsto \quad C^{1,\alpha}(\Omega) \cap W^{1,p}_0(\Omega) \subset C^{1,\alpha}(\Omega) \]
\[ u \mapsto U, \]
is well-defined, $U$ being the unique positive solution of (16). Moreover, it follows clearly from the compactness of the immersion $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ that $T$ is continuous and compact. Thus, since $T$ leaves invariant the set $\mathcal{F}$ defined by (15) and this set is bounded and convex we can apply Schauder’s Fixed Point Theorem to obtain a fixed point $u$ for $T$. Of course, such a fixed point $u$ satisfies
\[
\begin{cases}
-\Delta_p u = F'(x, u) = \lambda h(x, u) + \beta f(x, u, \nabla u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

\[\square\]

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