Uniform Convergence for Local Linear Regression Estimation of the Conditional Distribution

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Abstract

This paper studies the local linear regression (LLR) estimation of the conditional distribution function $F(y|x)$. We derive three uniform convergence results: the uniform bias expansion, the uniform convergence rate, and the uniform asymptotic linear representation. The uniformity of the above results is not only with respect to $x$ but also $y$, and therefore are not covered by the current developments in the literature of local polynomial regressions. Such uniform convergence results are especially useful when the conditional distribution estimator is the first stage of a semiparametric estimator. We demonstrate the usefulness of these uniform results with an example.

Keywords: Conditional Distribution, Empirical Process Theory, Local Linear Regression, Uniform Convergence Rate.

1 Introduction

This paper studies the nonparametric estimation of the conditional distribution function. The analysis concerns a random variable $Y \in \mathbb{R}$ and a random vector of covariates $X \in \mathbb{R}^d$. The conditional distribution function of $Y$ given $X = x$ is denoted by $F(\cdot|x)$, that is,

$$F(y|x) = P(Y \leq y | X = x), y \in \mathbb{R}.$$
When the conditional distribution function $F(\cdot|\cdot)$ is assumed to be smooth, it is natural to consider using the local linear regression (LLR) method to estimate $F$.

The main subject of the paper is the uniform convergence of the LLR estimator, where the uniformity is with respect to both $y$ and $x$. In particular, we derive the uniform bias expansion, characterize the uniform convergence rate, and give the uniform asymptotic linear representation for the estimator. As explained in, for example, Hansen (2008) and Kong et al. (2010), these uniform results are often useful for semiparametric estimation based on nonparametrically estimated components.

The estimation of the conditional distribution is an important research question. Hansen (2004) studies the asymptotic properties of both the Nadaraya-Watson (local constant) estimator and the LLR estimator and obtains pointwise convergence results. It is well-known that the LLR estimator has better boundary properties, but unlike the Nadaraya-Watson estimator, the LLR estimator is not guaranteed to be a proper distribution function. To solve this problem Hall et al. (1999) propose a weighted Nadaraya-Watson estimator that has the same asymptotic distribution as the LLR estimator, but the weights are computation-demanding. More recently, Das and Politis (2020) propose a way to correct the LLR estimator.

The local polynomial estimators have been studied extensively, but the uniform convergence results in the estimation of $F$ are new to the literature. The reason is that in a general setup, there is only one regressand $Y$ while in the setting of conditional distribution estimation, there is a class of regressand $1\{Y \leq y\}, y \in \mathbb{R}$. For example, Masry (1996) establishes the uniform convergence rate for general local polynomial estimators, but the uniformity is with respect to the values of the regressors. Therefore, the results there can only be applied to an estimate of $F(y|\cdot)$ for a fixed $y \in \mathbb{R}$. In a similar way, the results in Kong et al. (2010) cannot be used to provide a uniform asymptotic linear representation for $y \in \mathbb{R}$. Our paper aims at solving these issues and proving that under suitable conditions, the desired results are uniform with respect to both $y$ and $x$. We make use of the recent discovery by Fan and Guerre (2016) on the support of the covariates so that the uniform results are valid over the entire support.

The second contribution of the paper is that we provide a novel way to prove the uniform convergence rate by using empirical process theory. The methodology is based on the theory developed by Giné and Guillou (2001) and Giné and Guillou (2002) to show the uniform almost
sure convergence of the kernel density estimator. In this paper, we simplify their method and make it more accessible to users that are only concerned with the notion of uniform convergence in probability.

The remaining part of the paper is organized as follows. Section 2 introduces the statistical model and the assumptions. Section 3 establishes the uniform bias expansion result. Section 4 introduces the empirical process theory to prove the uniform convergence rate. Section 5 presents the uniform asymptotic linear representation and provides a simple example to illustrate the result. The proofs are contained in the Supplementary Material.

2 Model and Assumptions

Let \{ (Y_i, X_i), 1 \leq i \leq n \} be a random sample of \((Y, X)\). The estimation procedure is described as follows. Let \( w \) and \( k \) be two kernel functions and \( K(v) = \int_{-\infty}^{v} k(u)du \). Let \( h_1 = h_1n = o(1) \) and \( h_2 = h_2n = o(1) \) be two scalar sequences of bandwidths. Let \( r(u) = (1, u')', u \in \mathbb{R}^d \) and \( e_0 = (1, 0, \cdots, 0) \) be the first \((d + 1)\)-dimensional unit vector. The proposed estimator is \( \hat{F}(y|x) = e_0' \hat{\beta}(y, x, h_1, h_2) \), where

\[
\hat{\beta}(y, x, h_1, h_2) = \left( \hat{\beta}_0(y, x, h_1, h_2), \hat{\beta}_1(y, x, h_1, h_2), \cdots, \hat{\beta}_d(y, x, h_1, h_2) \right)'
\]

\[
= \arg \min_{\beta \in \mathbb{R}^{d+1}} \sum_{i=1}^{n} \left( K \left( \frac{y - Y_i}{h_2} \right) - r(X_i - x)' \beta \right)^2 w \left( \frac{X_i - x}{h_1} \right).
\]

Let \( H_1 \) be the \((d + 1) \times (d + 1)\) diagonal matrix with diagonal elements: \((1, h_1, \cdots, h_1)\). The first-order condition of the above minimization problem gives

\[
H_1 \hat{\beta}(y, x, h_1, h_2) = \hat{\Xi}(x, h_1)^{-1} \hat{\upsilon}(y, x, h_1, h_2),
\]

where

\[
\hat{\Xi}(x, h_1) = \frac{1}{n h_1^d} \sum_{i=1}^{n} r \left( \frac{X_i - x}{h_1} \right) r \left( \frac{X_i - x}{h_1} \right)' w \left( \frac{X_i - x}{h_1} \right),
\]

\[
\hat{\upsilon}(y, x, h_1, h_2) = \frac{1}{n h_1^d} \sum_{i=1}^{n} r \left( \frac{X_i - x}{h_1} \right) K \left( \frac{y - Y_i}{h_2} \right) w \left( \frac{X_i - x}{h_1} \right).
\]
The following assumptions are maintained throughout the paper.

**Assumption X** (Distribution of $X$). The support of $X$, denoted by $\mathcal{X}$, is compact. The marginal density $f_X$ is twice continuously differentiable and bounded away from zero on $\mathcal{X}$. There exist some $\lambda_0, \lambda_1 \in (0, 1]$ such that for any $x \in \mathcal{X}$ and all $\epsilon \in (0, \lambda_0]$, there is $x' \in \mathcal{X}$ satisfying $B(x', \lambda_1 \epsilon) \subset B(x, \epsilon) \cap \mathcal{X}$, where $B(x, \epsilon)$ denotes the ball centered at $x$ with radius $\epsilon$.

**Assumption Y** (Conditional distribution of $Y|X$). The support of $Y$, $\mathcal{Y}$, is bounded. The conditional distribution function $F(y|x)$ is twice continuously differentiable over $(y, x) \in \text{supp}(Y, X)$.

**Assumption K** (Kernel functions).

(i) The kernel function $w$ is a product kernel: $w(u) = w_1(u_1)w_2(u_2)\cdots w_d(u_d)$. Each $w_\ell$ (1) is a symmetric density function with a compact support $[-1, 1]$, (2) has its second moment normalized to one, that is, $\int u_\ell^2 w_\ell(u_\ell) du_\ell = 1$, (3) is positive in the interior of the support $(-1, 1)$, and (4) is of bounded variation.

(ii) The kernel function $k$ (1) is a symmetric density function with a compact support and (2) has its second-moment normalized to one, that is, $\int v^2 k(v) dv = 1$.

A brief discussion of the above assumptions is in order. **Assumption X** is introduced by Fan and Guerre (2016) as a regularity condition on the support $\mathcal{X}$. It ensures enough observations around every estimation location, including the boundary points. **Assumption Y** imposes smoothness conditions on the conditional distribution function $F$. Under this assumption, the Hessian matrix of $F$ is uniformly continuous on the compact support $\text{supp}(Y, X)$. **Assumption K** contains standard conditions on the kernel functions $k$ and $w$. The bounded variation condition is imposed because it is needed for applying the empirical process theory.

### 3 Uniform Bias Expansion

We denote the true value of the conditional distribution function and its derivative with respect to $x$ as

$$\beta^*(y, x) = (\beta_0^*(y, x), \beta_1^*(y, x), \cdots, \beta_d^*(y, x))' = (F(y|x), \nabla_x F(y|x))'',$$


where $\nabla_x F(y|x)' = \left( \frac{\partial}{\partial x_1} F(y|x), \ldots, \frac{\partial}{\partial x_d} F(y|x) \right)$ is the gradient of $F(y|x)$ with respect to $x$. A convenient way to analyze the estimator $\tilde{\beta}(y, x, h_1, h_2)$ is to consider it as an estimator of the pseudo-true value defined as

$$\bar{\beta}(y, x, h_1, h_2) = (\tilde{\beta}_0(y, x, h_1, h_2), \tilde{\beta}_1(y, x, h_1, h_2), \ldots, \tilde{\beta}_d(y, x, h_1, h_2))'$$

$$= \arg \min_{\beta \in \mathbb{R}^{d+1}} \mathbb{E} \left[ (K \left( \frac{y - Y}{h_2} \right) - r(X - x)' \beta)^2 w \left( \frac{X - x}{h_1} \right) \right]. \quad (2)$$

We can break the asymptotic analysis of $\tilde{\beta}(y, x, h_1, h_2) - \beta^*(y, x)$ into two parts:

$$\tilde{\beta}(y, x, h_1, h_2) - \beta^*(y, x) = \underbrace{\tilde{\beta}(y, x, h_1, h_2) - \bar{\beta}(y, x, h_1, h_2)}_{\text{stochastic term}} + \underbrace{\bar{\beta}(y, x, h_1, h_2) - \beta^*(y, x)}_{\text{bias term}}.$$

In this section, we study the bias term, which the difference between the pseudo-true value and the true value. The first-order condition of (2) gives an explicit expression of the pseudo-true value: $H_1 \bar{\beta}(y, x, h_1, h_2) = \Xi(x, h_1)^{-1}v(y, x, h_1, h_2)$, where

$$\Xi(x, h_1) = \frac{1}{h_1^d} \mathbb{E} \left[ r(X - x) r(X - x)' w \left( \frac{X - x}{h_1} \right) \right],$$

$$v(y, x, h_1, h_2) = \frac{1}{h_1^d} \mathbb{E} \left[ r(X - x) K \left( \frac{y - Y}{h_2} \right) w \left( \frac{X - x}{h_1} \right) \right].$$

Define $\Omega(x, h_1) = \int r(u)r(u)'w(u)1\{x + h_1 u \in X\}du$. The following lemma shows that the matrices $\Xi(x, h_1)$ and $\Omega(x, h_1)$ are always bounded and invertible.

**Lemma 1.** There exists $C > 0$ such that the eigenvalues of $\Xi(x, h_1)$ and $\Omega(x, h_1)$ are in $[1/C, C]$ for all $x \in X$ and $h_1 \geq 0$ small enough.

**Theorem 1.** Let Assumptions $X$, $Y$, and $K$ hold. Then

$$H_1 (\bar{\beta}(y, x) - \beta^*(y, x))$$

$$= \frac{h_1^2}{2} \Xi(x, h_1)^{-1} \sum_{\ell, \ell' = 1}^d \frac{\partial^2}{\partial x_\ell \partial x_{\ell'}} F(y|x) \int r(u) u_\ell u_{\ell'} w(u) 1\{x + h_1 u \in X\} du$$

$$+ \frac{h_1^2}{2} \Omega(x, h_1)^{-1} \frac{\partial^2}{\partial y^2} F(y|x) \int r(u) w(u) 1\{x + h_1 u \in X\} du + o(h_1^2 + h_2^2), \quad (3)$$
uniformly over $y \in \mathbb{R}$ and $x \in \mathcal{X}$. In particular, we have

$$
\hat{\beta}_0(y, x) - \beta_0^*(y, x) = \frac{h_1^2}{2} \sum_{\ell=1}^{d} \frac{\partial^2}{\partial x_\ell^2} F(y|x) + \frac{h_2^2}{2} \frac{\partial^2}{\partial y^2} F(y|x) + o(h_1^2 + h_2^2),
$$

(4)

uniformly over $y \in \mathbb{R}$ and $x \in \hat{\mathcal{X}}_{h_1}$, where $\hat{\mathcal{X}}_{h_1} = \{ x \in \mathcal{X} : x \pm h_1 = (x_1 \pm h_1, \ldots, x_d \pm h_1) \in \mathcal{X} \}$ denotes the set of interior points with respect to the bandwidth $h_1$.

The novelty of Theorem 1 is that it provides a uniform bias expansion for the LLR estimator over the entire region $(y, x) \in \mathbb{R} \times \mathcal{X}$. For the boundary points $x \notin \hat{\mathcal{X}}_{h_1}$, the bias is $O(h_1^2 + h_2^2)$. For the interior points $x \in \hat{\mathcal{X}}_{h_1}$, the bias expression (4) is the same as in Hansen (2004) and Chapter 6 of Li and Racine (2007), which contains the curvature of $F(y|x)$.

4 Uniform Convergence Rate

In this section, we derive the uniform convergence rate of the stochastic term $\hat{\beta}(y, x, h_1, h_2) - \hat{\beta}(y, x, h_1, h_2)$. We make use of the empirical process theory, which is a powerful tool for studying the uniform convergence of random sequences. Some auxiliary concepts and results are introduced below.

Let $\mathcal{G}$ be a class of uniformly bounded measurable functions defined on some subset of $\mathbb{R}^d$, that is, there exists $M > 0$ such that $|g| \leq M$ for all $g \in \mathcal{G}$. We say $\mathcal{G}$ is Euclidean with coefficients $(A, v)$, where $A, v > 0$, if for every probability measure $P$ and every $\epsilon \in (0, 1]$, $N(\mathcal{G}, P, \epsilon) \leq A/\epsilon^v$, where $N(\mathcal{G}, P, \epsilon)$ is the the $\epsilon$-covering of the metric space $(\mathcal{G}, L_2(P))$, that is, $N(\mathcal{G}, P, \epsilon)$ is defined as the minimal number of open $\| \cdot \|_{L_2(P)}$-balls of radius $\epsilon$ and centers in $\mathcal{G}$ required to cover $\mathcal{G}$. By definition, if $\mathcal{G}$ is Euclidean with coefficients $(A, v)$, then any subset of $\mathcal{G}$ is also Euclidean with coefficients $(A, v)$.

The above definition of Euclidean classes is introduced by Nolan and Pollard (1987). The same concept is also studied by Giné and Guillou (1999) but they refer to what we call “Euclidean” as “VC.” There is a slight difference that Nolan and Pollard (1987) use the $L_1$-norm while Giné and Guillou (1999) use the $L_2$-norm. We ignored the envelope in their definition since we only work with uniformly bounded $\mathcal{G}$. The following lemma is useful for deriving uniform convergence results.
Lemma 2. Let \( X_1, \ldots, X_n \) be an iid sample of a random vector \( X \) in \( \mathbb{R}^d \). Let \( G_n \) be a sequence of classes of measurable real-valued functions defined on \( \mathbb{R}^d \). Assume that there is a uniformly bounded Euclidean class \( G \) with coefficients \( A \) and \( v \) such that \( G_n \subset G \) for all \( n \). Let \( \sigma_n^2 \) be a positive sequence such that \( \sigma_n^2 \geq \sup_{g \in G_n} \mathbb{E}[g(X)^2] \). Then

\[
\Delta_n = \sup_{g \in G_n} \left| \sum_{i=1}^n (g(X_i) - \mathbb{E} g(X_i)) \right| = O_p \left( \sqrt{n \sigma_n^2 | \log \sigma_n |} + | \log \sigma_n | \right).
\]

In particular, if \( n \sigma_n^2 / | \log \sigma_n | \to \infty \), then

\[
\Delta_n = O_p \left( \sqrt{n \sigma_n^2 | \log \sigma_n |} \right).
\]

The above lemma is based on the results developed by Giné and Guillou (2001) and Giné and Guillou (2002). Those two papers focus on proving the almost sure convergence of kernel density estimators based on the empirical process theory. We simplify their method and make it available to users who are only interested in convergence in probability. Based on 2, deriving the uniform convergence rate of kernel-based nonparametric estimators boils down to two parts: prove the relevant function classes are Euclidean and compute a uniform bound of the variance.

The following theorem establishes the uniform convergence rate for the stochastic term of the LLR estimator. Then combining Theorem 1, we obtain the uniform convergence rate of the LLR estimator as a corollary.

Theorem 2. Let Assumptions \( X, Y, \) and \( K \) hold. If the bandwidth satisfies \( nh_1^d / | \log h_1 | \to \infty \), then

\[
\sup_{y \in \mathbb{R}, x \in \mathcal{X}} \left| H_1 \left( \hat{\beta}(y, x, h_1, h_2) - \bar{\beta}(y, x, h_1, h_2) \right) \right| = O_p \left( \sqrt{\frac{| \log h_1 |}{nh_1^d}} \right). \tag{5}
\]

Corollary 1. Under the assumptions of Theorem 2, we have

\[
\sup_{y \in \mathbb{R}, x \in \mathcal{X}} \left| \hat{F}(y|x) - F(y|x) \right| = O_p \left( \frac{h_1^2 + h_2^2}{nh_1^d} + \sqrt{\frac{| \log h_1 |}{nh_1^d}} \right).
\]

We want to compare the above uniform convergence result with the one in Masry (1996). In Masry (1996), the covariates \( X \) are supported on the entire space \( \mathbb{R}^d \) while the uniform result is valid only over a compact subset of \( \mathbb{R}^d \) and a single \( y \) value. In our case, the support \( \mathcal{X} \) is compact, and the uniform result is valid over the entire support \( \mathcal{X} \) and \( y \in \mathcal{Y} \).

Corollary 1 shows that the uniformity over \( y \in \mathbb{R} \) does not have an impact on the convergence
rate. This is similar to the fact that we can uniformly estimate the unconditional distribution function under the \( n^{-1/2} \)-rate. The conditional distribution estimation is a nonparametric problem only concerning the covariates.

5 Uniform Asymptotic Linear Representation

This section derives the uniform asymptotic linear representation of the LLR estimator. These results are particularly useful in deriving the asymptotic normality of complicated estimators.

**Theorem 3.** Let Assumptions \( X, Y, \) and \( K \) hold. If the bandwidth satisfies that \( nh_1^d / |\log h_1| \to \infty, nh_1^{d+4} / |\log h_1| \text{ bounded}, \) and \( h_2 = O(h_1) \), then

\[
H_1 \left( \hat{\beta}(y, x, h_1, h_2) - \beta(y, x, h_1, h_2) \right) = \Xi(x, h_1)^{-1} \frac{1}{nh_1^d} \sum_{i=1}^n s(Y_i, X_i; y, x, h_1, h_2) + O_p \left( \frac{|\log h_1|}{nh_1^d} \right),
\]

uniformly over \( y \in \mathbb{R} \) and \( x \in X \), where

\[
s(Y_i, X_i; y, x, h_1, h_2) = r \left( \frac{X_i - x}{h_1} \right) \left( K \left( \frac{y - Y_i}{h_2} \right) - \tilde{F}(y|X_i) \right) w \left( \frac{X_i - x}{h_1} \right).
\]

The uniform asymptotic order of the remainder term, \( O_p \left( |\log h_1| / nh_1^d \right) \), is the same as in Equation (13) of Kong et al. (2010). This again shows that the uniformity over \( y \in \mathbb{R} \) does not have an impact on the convergence rate. Combining the results in Theorem 1 and 3 and applying the central limit theorem for triangular arrays, we can show that the LLR estimator is asymptotic normal with some asymptotic bias.

Let’s consider a simple example to demonstrate the usefulness of the uniform asymptotic linear representation. Suppose that \( d = 1, X = [x, \bar{x}] \), and \( Y = [y, \bar{y}] \). We want to estimate the integrated conditional distribution \( \theta = \int_y^\bar{y} \int_x^\bar{x} F(y|x) dxdy \) with the estimator \( \hat{\theta} = \int_y^\bar{y} \int_x^\bar{x} \hat{F}(y|x) dxdy \). The theorem below gives the asymptotic distribution of the estimator \( \hat{\theta} \).

**Theorem 4.** Let Assumptions \( X, Y, \) and \( K \) hold. If the bandwidth satisfies that \( \sqrt{n}h_1 / |\log h_1| \to \infty, \sqrt{nh_1^2} \to 0 \text{ bounded}, \) and \( h_2 = O(h_1) \), then \( \sqrt{n} (\hat{\theta} - \theta) \xrightarrow{d} N(0, V) \), where

\[
V = \int \left( \int_y^\bar{y} \int_{-1}^1 (1 \{s \leq y\} - \hat{F}(y|t)) w(u) du \right)^2 f(s, t) dt ds,
\]
and \( f(y, x) \) denotes the joint density of \((Y, X)\).

## A Proofs

**Proof of Lemma 1.** This Lemma is almost the same as Lemma 11 in Fan and Guerre (2016). The only difference is that in this paper we allow the kernel to diminish at the boundary of the support but the proof of Fan and Guerre (2016) nonetheless goes through. In fact, following their steps, we can show that the eigenvalues of \( \Xi(x, h_1) \) and \( \Omega(x, h_1) \) are larger than

\[
\inf_{x \in B(0, 1)} \min b' \left( \int r(u)r(u)'w(u)1\{u \in B(x, \lambda_1)\}du \right) b,
\]

which is strictly positive since \( w > 0 \) on \([-1, 1]^d\).

**Proof of Theorem 1.** By the standard change of variable and the law of iterated expectations, we can write

\[
\Xi(x, h_1) = \int r(u)r(u)'w(u)f_X(x + h_1u)du,
\]

\[
v(y, x, h_1, h_2) = \int r(u)w(u)\bar{F}(y| x + h_1u)f_X(x + h_1u)du,
\]

where \( \bar{F}(y|x) = \mathbb{E}[K((y - Y)/h_2) | X = x] \). Because \( f_X \) is continuously differentiable on \( \mathcal{X} \), we have \( \Xi(x, h_1) = f_X(x)\Omega(x, h_1) + o(1) \), uniformly over \( x \in \mathcal{X} \). Applying change of variable and integration by parts to \( \bar{F}(y|x + h_1u) \), we have

\[
\bar{F}(y|x + h_1u) = \int K((y - y')/h_2)F(y'| x + h_1u)dy'
\]

\[
= \int K(v)F(y - h_2v|x + h_1u)h_1dv
\]

\[
= \int k(v)F(y - h_2v|x + h_1u)dv.
\]
For any \( y \in \mathcal{Y} = \text{supp}(Y) \), the following expansion holds:

\[
F(y - h_1 v | x) = F(y|x + h_1 u) - \frac{\partial}{\partial y} F(y|x + h_1 u) h_2 v + \frac{1}{2} \frac{\partial^2}{\partial y^2} F(y|x + h_1 u) h_2 v^2 \\
+ \frac{1}{2} h_2^2 \left( \frac{\partial^2}{\partial y^2} F(\tilde{y}|x + h_1 u) v^2 - \frac{\partial^2}{\partial y^2} F(y|x + h_1 u) v^2 \right),
\]

where \( \tilde{y} \) is between \( y \) and \( y - h_2 v \). Notice that we do not need to assume that \( F(\cdot|x + h_1 u) \) is differentiable on \( \mathbb{R} \) for the above expansion. This is because if \( y - h_2 v \notin \mathcal{Y} \), then there always exists \( \tilde{y} \in \mathcal{Y} \) such that \( \tilde{y} \) is between \( y \) and \( y - h_2 v \) and \( F(\tilde{y}|x + h_1 u) = F(y - h_2 v|x + h_1 u) \). This is true since \( F(\cdot|x + h_1 u) \) is a constant function outside of \( \mathcal{Y} \). Therefore,

\[
F(y|x + h_1 u) = F(y|x + h_1 u) + \frac{h_2^2}{2} \frac{\partial^2}{\partial y^2} F(y|x + h_1 u) \int v^2 k(v) dv + o(h_2^2),
\]

uniformly over \( y \in \mathbb{R} \) and \( x + h_1 u \in \mathcal{X} \). The remainder term is uniformly \( o(h_2^2) \) because \( \frac{\partial^2}{\partial y^2} F \) is a continuous function on the compact set \( \text{supp}(Y, X) \). Next, by the smoothness of \( F(y|x) \) with respect to \( x \), we have

\[
F(y|x + h_1 u) = F(y|x) + h_1 u' \nabla_x F(y|x) + \frac{h_1^2}{2} u' \nabla^2_x F(y|x) u
\]

\[
= r(u)' H_1 \beta^* + \frac{h_1^2}{2} u' \nabla^2_x F(y|x) u + \frac{h_1^2}{2} u' \left( \nabla^2_x F(y|x) u - \nabla^2_x F(y|x) u \right)
\]

\[
= r(u)' H_1 \beta^* + \frac{h_1^2}{2} u' \nabla^2_x F(y|x) u + o(h_1^2),
\]

uniformly over \( y \in \mathbb{R} \) and \( x + h_1 u \in \mathcal{X} \). The remainder term is uniformly \( o(h_1^2) \) because \( \nabla_x F \) is a continuous function on the compact set \( \text{supp}(Y, X) \). Similarly,

\[
\frac{\partial^2}{\partial y^2} F(y|x + h_1 u) = \frac{\partial^2}{\partial y^2} F(y|x) + o(1), \text{ and } f_X(x + h_1 u) = f_X(x) + o(1),
\]

uniformly over \( y \in \mathbb{R} \) and \( x + h_1 u \in \mathcal{X} \). Therefore,

\[
v(y, x, h_1, h_2) = \Xi(x, h_1) H_1 \beta^*(y, x) + \frac{h_1^2}{2} f_X(x) \int r(u) w(u) u' \nabla^2_x F(y|x) u 1\{x + h_1 u \in \mathcal{X}\} du
\]

\[
+ \frac{h_2^2}{2} \frac{\partial^2}{\partial y^2} F(y|x) f_X(x) \int v^2 k(v) dv \int r(u) w(u) 1\{x + h_1 u \in \mathcal{X}\} du + o(h_1^2 + h_2^2),
\]

uniformly over \( y \in \mathbb{R} \) and \( x + h_1 u \in \mathcal{X} \). Therefore,
uniformly over $y \in \mathbb{R}$ and $x \in \mathcal{X}$. Therefore,

$$
H_1(\hat{\beta}(y, x, h_1, h_2) - \beta^*(y, x)) = \frac{h_1^2}{2} \Omega(x, h_1)^{-1} \sum_{\ell, \ell'} \frac{\partial^2}{\partial x_\ell \partial x_{\ell'}} F(y|x) \int r(u) u_\ell u_{\ell'} w(u) 1\{x + h_1 u \in \mathcal{X}\} du + \frac{h_2^2}{2} \Omega(x, h_1)^{-1} \frac{\partial^2}{\partial y^2} F(y|x) \int v^2 k(v) dv \int r(u) w(u) 1\{x + h_1 u \in \mathcal{X}\} du + o(h_1^2 + h_2^2).
$$

Then the first claim of the theorem follows.

When $x \in \mathcal{X}_{h_1}$, $x + h_1 u \in \mathcal{X}$ for all $u \in [-1, 1]^d$. In that case, $\Omega(x, h_1)$ becomes the identity matrix because $w_\ell$ is symmetric and has variance one. Then the second claim of the theorem follows.

Proof of Lemma 2. Let $M > 0$ be the uniform bound of $\mathcal{G}$. Notice that each $\mathcal{G}_n$ is a uniformly bounded (by $M$) Euclidean class with the same coefficients $(A, v)$. Denote

$$
\Delta_n^o = \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n Rad_i f(X_i) \right|
$$

where $Rad_i, 1 \leq i \leq n$, is a sequence of iid Rademacher variables. By Proposition 2.1 in Giné and Guillou (2001), we have for all $n \geq 1$,

$$
\mathbb{E}\Delta_n^o \leq C \left( v M \log(AM/\sigma_n) + \sqrt{v} \sqrt{n \sigma_n^2 \log(AM/\sigma_n)} \right) = O \left( \sqrt{n \sigma_n^2 \log \sigma_n} + |\log \sigma_n| \right).
$$

By the symmetrization result in, for example, Lemma 2.3.1 of van der Vaart and Wellner (1996), we know that $\mathbb{E}\Delta_n \leq 2\mathbb{E}\Delta_n^o = O \left( \sqrt{n \sigma_n^2 \log \sigma_n} + |\log \sigma_n| \right)$. Then the claimed result follows from the Chebyshev inequality.

Proof of Theorem 2. We proceed with two steps. Recall the expression of $H_1 \hat{\beta}$ in Equation (1). In Step 1, we derive the uniform convergence rate of the numerator $\hat{\psi}(y, x, h_1, h_2)$. In Step 2, we derive the uniform convergence rate of the denominator $\hat{\mathcal{E}}(x, h_1)$.
Step 1. To avoid repetition in the proof, we consider a generic element of the vector $\vec{\nu}(y, x, h_1, h_2)$:

$$\hat{\nu}(y, x, h_1, h_2) = \frac{1}{nh_1^d} \sum_{i=1}^{n} (X_i - x)^\pi K \left( \frac{y - Y_i}{h_2} \right) w \left( \frac{X_i - x}{h_1} \right),$$

$$= \frac{1}{nh_1^d} \sum_{i=1}^{n} K \left( \frac{y - Y_i}{h_2} \right) \prod_{\ell=1}^{d} (X_{i\ell} - x_\ell)^\pi \omega_\ell \left( \frac{X_{i\ell} - x_\ell}{h_1} \right), \quad (A.1)$$

where $\pi = (\pi_1, \ldots, \pi_d), \pi_\ell \in \{0, 1\}, \sum \pi_\ell \leq 1$. We want to derive the following uniform convergence rate of $\hat{\nu}(y, x, h_1, h_2)$:

$$\sup_{y \in \mathbb{R}, x \in \mathcal{X}} |\hat{\nu}(y, x, h_1, h_2) - \mathbb{E}\hat{\nu}(y, x, h_1, h_2)| = O_p \left( \sqrt{\log h_1 / (nh_1^d)} \right). \quad (A.2)$$

By defining

$$\psi_n(Y, X, y, x) = K \left( \frac{y - Y_i}{h_2} \right) \prod_{\ell=1}^{d} (X_{i\ell} - x_\ell)^\pi \omega_\ell \left( \frac{X_{i\ell} - x_\ell}{h_1} \right)$$

and $\Psi_n = \{\psi_n(\cdot, \cdot, y, x) : y \in \mathbb{R}, x \in \mathcal{X}\}$, we can write the LHS of (A.2) as

$$\sup_{\psi_n \in \Psi_n} \left| \frac{1}{nh_1^d} \sum_{i=1}^{n} (\psi_n(Y_i, T_i, R_i; y, t) - \mathbb{E}\psi_n(Y_i, T_i, R_i; y, t)) \right|,$$

which can be studied with the empirical process theory introduced previously. Notice that $\psi_n$ and $\Psi_n$ depend on $n$ through the bandwidth $h_1$ and $h_2$.

Consider a larger class $\Psi$ that does not depend on $n$ defined by the following product:

$$\Psi = \Psi_Y \Psi_{X_1} \Psi_{X_2} \cdots \Psi_{X_d},$$

where

$$\Psi_Y = \{ (Y, X) \mapsto K ((y - Y) / h) : y \in \mathbb{R}, h > 0 \},$$

$$\Psi_{X_\ell} = \{ (Y, X) \mapsto ((X_\ell - x_\ell) / h)^\pi \omega_\ell ((X_\ell - x_\ell) / h) : x \in \mathcal{X}, h > 0 \}, \ell = 1, \ldots, d.$$

For all $n \geq 1$, $\Psi_n$ is a subset of the product class $\Psi$. Then we want to show that $\Psi$ is uniformly bounded and Euclidean. If that is true, then we can appeal to Lemma 2.
In view of Lemma \( B.5 \), we only need to show that \( \Psi_Y \) and \( \Psi_{X_\ell} \) are uniformly bounded and Euclidean. The class \( \Psi_Y \) is uniformly bounded by 1. The function \( K \) is of bounded variation on \( \mathbb{R} \) since it is the integral of the integrable function \( k \) (Corollary 3.33 in Folland (1999)). Then by Lemma \( B.1 \), we know that \( \Psi_Y \) is Euclidean. The class \( \Psi_{X_\ell} \) is uniformly bounded by \( \|w_\ell\|_\infty \). This is because \( w_\ell \) is support on \([-1, 1]\) and hence the term in front of \( w_\ell, (X_\ell - x_\ell)/h_1 \), cannot exceed one in magnitude. To show that \( \Psi_{X_\ell} \) is Euclidean, notice that the function \( u_\ell \mapsto u_\ell^{\pi_\ell} w_\ell(u_\ell) \) is of bounded variation. This is because on the support of \( w_\ell \), \([-1, 1]\), both \( u_\ell \mapsto u_\ell^{\pi_\ell} \) and \( w_\ell \) are of bounded variation. Then their product is also of bounded variation (Theorem 6.9, Apostol (1974)). Then we know \( \Psi_{X_\ell} \) is Euclidean by appealing to Lemma \( B.1 \).

Next, we want to derive a uniform variance bound for each \( \Psi_n \). By the standard change of variable, we know that \( \sup_{\psi_n \in \Psi_n} \mathbb{E}[\psi(Y, X; y, x)^2] \) is bounded by

\[
\sup_{x \in X} \mathbb{E} \left[ w \left( \frac{X - x}{h_1} \right)^2 \right] \leq \sup_{x \in X} h_1^d \int w(u)^2 f_X(x + h_1 u) du \leq 2 h_1^d \|f_X\|_\infty \prod_{\ell=1}^d \|w_\ell\|_\infty^2,
\]

where we have used the fact that \( K \in [0, 1] \) and \( w_\ell \) is supported on \([-1, 1]\). Therefore, we can define \( \sigma^2_{\Psi_n} = 2 h_1^d \|f_X\|_\infty \prod_{\ell=1}^d \|w_\ell\|_\infty^2 \) as a uniform variance bound for \( \Psi_n \). Under the assumption that \( nh_1^d / |\log h_1| \to \infty \), we can apply Lemma 2 to the sequence \( \Psi_n \) and obtain that

\[
\sup_{\psi_n \in \Psi_n} \left| \sum_{i=1}^n (\psi_n(Y_i, X_i; y, x) - \mathbb{E}\psi_n(Y_i, X_i; y, x)) \right| = O_p \left( \sqrt{n \sigma^2_{\Psi_n} / \log \sigma_{\Psi_n}} (nh_1^d) \right) = O_p \left( \sqrt{|\log h_1| / (nh_1^d)} \right),
\]

which is the desired result specified in Equation (A.2).

**Step 2.** Following the same procedure as in Step 1, we can show that the uniform convergence rate for each element of the matrix \( \hat{\Sigma}(x, h_1) \) is also \( \sqrt{|\log h_1| / (nh_1^d)} \). We omit the details for brevity. Then by Lemma 1, we know that with probability approaching one, the eigenvalues of \( \hat{\Sigma}(x, h_1) \) is in \([1/C, C]\). In particular, with probability approaching one, the inverse matrix \( \hat{\Sigma}(x, h_1)^{-1} \) is well-defined, and its induced 2-norm \( \|\hat{\Sigma}(x, h_1)^{-1}\|_2 \) is bounded. Then applying Lemma 1 once again,
we have

\[
\sup_{x \in X} \left\| \hat{\Sigma}(x, h_1)^{-1} - \Sigma(x, h_1)^{-1} \right\|_2 = \sup_{x \in X} \left\| \hat{\Sigma}(x, h_1)^{-1}(\Sigma(x, h_1) - \hat{\Sigma}(x, h_1))\Sigma(x, h_1)^{-1} \right\|_2 \\
\leq \sup_{x \in X} \left\| \hat{\Sigma}(x, h_1)^{-1} \right\|_2 \left\| \Sigma(x, h_1) - \hat{\Sigma}(x, h_1) \right\| \left\| \Sigma(x, h_1)^{-1} \right\|_2 \\
= O_p \left( \sqrt{\log h_1 / (nh_1^d)} \right),
\]

where the second line follows from the submultiplicativity of the induced 2-norm. Combining the above result with Step 1, we obtain

\[
\sup_{y \in Y, x \in X} \left\| \hat{\Sigma}(t)^{-1} \hat{\nu}(y, t) - \Sigma(t)^{-1} \nu(y, t) \right\|_2 \\
\leq \sup_{y \in Y, x \in X} \left\| \hat{\Sigma}(t)^{-1} - \Sigma(t)^{-1} \right\|_2 \left\| \hat{\nu}(y, t) \right\|_2 + \sup_{y \in Y, x \in X} \left\| \hat{\nu}(y, t) - \nu(y, t) \right\|_2 \left\| \Sigma(t)^{-1} \right\|_2 \\
= O_p \left( \sqrt{\log h_1 / (nh_1^d)} \right),
\]

where the last line uses the fact that \( \hat{\nu} \) is uniformly bounded. This proves Equation (5).

\( \square \)

**Proof of Theorem 3.** Notice that we can write \( H_1 \left( \hat{\beta}(y, x, h_1, h_2) - \bar{\beta}(y, x, h_1, h_2) \right) \) as

\[
\hat{\Sigma}(x, h_1)^{-1} \left( \hat{\nu}(y, x, h_1, h_2) - \hat{\Sigma}(x, h_1) H_1 \bar{\beta}(y, x, h_1, h_2) \right) \\
= \hat{\Sigma}(x, h_1)^{-1} \frac{1}{nh_1^d} \sum_{i=1}^n s(Y_i, X_i; y, x, h_1, h_2) \\
+ \hat{\Sigma}(x, h_1)^{-1} \frac{1}{nh_1^d} \sum_{i=1}^n r \left( \frac{X_i - x}{h_1} \right) \left( \hat{F}(y|X_i) - r \left( \frac{X_i - x}{h_1} \right) ' \right) H_1 \bar{\beta}(y, x, h_1, h_2) \left( \frac{X_i - x}{h_1} \right) w \left( \frac{X_i - x}{h_1} \right) \\
= \Sigma(x, h_1)^{-1} \frac{1}{nh_1^d} \sum_{i=1}^n s(Y_i, X_i; y, x, h_1, h_2) + err_1(y, x) + err_2(y, x)
\]

where

\[err_1(y, x) = \hat{\Sigma}(x, h_1)^{-1} \frac{1}{nh_1^d} \sum_{i=1}^n r \left( \frac{X_i - x}{h_1} \right) \left( \hat{F}(y|X_i) - r \left( \frac{X_i - x}{h_1} \right) ' \right) H_1 \bar{\beta}(y, x, h_1, h_2) \left( \frac{X_i - x}{h_1} \right) w \left( \frac{X_i - x}{h_1} \right),\]

\[err_2(y, x) = \left( \hat{\Sigma}(x, h_1)^{-1} - \Sigma(x, h_1)^{-1} \right) \frac{1}{nh_1^d} \sum_{i=1}^n s(Y_i, X_i; y, x, h_1, h_2).\]

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We use the empirical process theory to derive the uniform convergence rates of $\text{err}_1$ and $\text{err}_2$ respectively in the following Step 1 and Step 2.

**Step 1.** Define a sequence of function classes $\Phi_n = \{\phi_n(\cdot, y; x) : y \in \mathbb{R}, x \in X\}$, where

$$
\phi_n(y, X; y, x) = \left( \hat{F}(y|X) - r\left(\frac{X - x}{h_1}\right)' H_1 \bar{\beta}(y, x, h_1, h_2) \right) \prod_{\ell=1}^{d} \left( \frac{X_\ell - x_\ell}{h_1} \right)^{\pi_\ell} w_\ell \left( \frac{X_\ell - x_\ell}{h_1} \right)
$$

with $\sum \pi_\ell \leq 1$ as before. We want to derive the convergence rate of

$$
\sup_{\phi_n \in \Phi_n} \left| \frac{1}{nh_1^d} \sum_{i=1}^{n} \phi_n(Y_i, X_i; y, x) \right|
$$

Notice that $\phi_n$ is already centered, that is, $E\phi_n(Y, X; y, x) = 0$, by the first-order condition of (2).

Define a larger product class $\Phi$ that does not vary with $n$ by $\Phi = \Phi_Y \Psi_X \Psi_X \cdots \Psi_X$, where $\Psi_X$ is defined in the proof of Theorem 2 and

$$
\Phi_Y = \{(Y, X) \mapsto (E[K((y - Y)/h)|X] - r(X - x)' \beta \times 1\{|X_\ell - x_\ell| \leq 1, 1 \leq \ell \leq d\} : y \in \mathbb{R}, x \in X, h > 0, \|\beta\|_2 \leq C\}.
$$

To understand the expression of $\Phi_Y$, recall that by definition $\hat{F}(y|X) = E[K((y - Y)/h^2) | X]$. The term $\bar{\beta}(y, x, h_1, h_2)$ is replaced by a general $\beta \in \mathbb{R}^{d+1}$ with a bounded norm. This can be done as both the numerator and denominator of $\bar{\beta}(y, x, h_1, h_2)$ is bounded. The indicator term $1\{|X_\ell - x_\ell| \leq 1, 1 \leq \ell \leq d\}$ comes from the support of $w$. This indicator term is needed for deriving the uniform boundedness.

For each $n$, we have $\Phi_n \in \Phi$. We want to show that $\Phi$ is a uniformly bounded Euclidean class. Since $\Psi_X$ is proven to be uniformly bounded and Euclidean in Theorem 2, we only need to focus on the class $\Psi_Y$. First notice that the class

$$
\{(Y, x) \mapsto r(X - x)' \beta 1\{|X_\ell - x_\ell| \leq 1, 1 \leq \ell \leq d\} : y \in \mathbb{R}, x \in X, h > 0, \|\beta\|_2 \leq C\}
$$

is uniformly bounded and Euclidean in view of Lemma B.2. By Lemma B.3, we know that the
following class is uniformly bounded and Euclidean:

\[
\{ (Y, X) \mapsto \mathbb{E}[K((y - Y)/h) \mid X] | X \mathbf{1}\{|X_\ell - x_\ell| \leq 1, 1 \leq \ell \leq d \} : y \in \mathbb{R}, h > 0 \}
\]

Then by Lemma B.4, we know that \( \Phi_\gamma \) is uniformly bounded and Euclidean. Hence, \( \Phi \) is uniformly bounded and Euclidean.

Then we want to derive a variance bound for each \( \Phi_n \). By the usual change of variable, we have for any \( y \in \mathbb{R} \) and \( x \in \mathcal{X} \),

\[
\mathbb{E}[\phi_n(Y, x; y, x)^2] \leq h_1^d \int \left( \tilde{f}^{-\gamma}_{1|X}(y | x + h_1 u) - r(u) H_1 \beta(y, x, h_1, h_2) \right)^2 w(u)^2 f_X(x + h_1 u) du.
\]

from the uniform bias expansion results in Theorem 1, we have

\[
\sup_{y \in \mathbb{R}, x \in \mathcal{X}} \left| \tilde{f}^{-\gamma}_{1|X}(y | x + h_1 u) - r(u) H_1 \beta(y, x, h_1, h_2) \right| = O(h_1^2 + h_2) = O(h_1^2).
\]

Therefore, we can construct a uniform variance bound \( \sigma^2_{\phi_n} = O(h_1^{d+4}) \) for the class \( \Phi_n \). Then by Lemma 2, we can show that

\[
\sup_{\phi_n \in \Phi_n} \left| \frac{1}{nh_1^d} \sum_{i=1}^n \phi_n(Y_i, X_i; y, x) \right| = O_p \left( \frac{\sqrt{h_1^d |\log h_1|/n + |\log h_1|/(nh_1^d)}}{nh_1^d} \right).
\]

where the second line follows from the assumption that \( nh_1^{d+4} / |\log h_1| \leq C \). Therefore,

\[
\sup_{y \in \mathbb{R}, x \in \mathcal{X}} \| \text{err}_1(y, x) \|_2 = \sup_{x \in \mathcal{X}} \left\| \hat{\xi}(x, h_1)^{-1} \right\|_2 O_p \left( |\log h_1|/nh_1^d \right) = O_p \left( |\log h_1|/nh_1^d \right).
\]

**Step 2.** Similar as before, we can show that

\[
\sup_{y \in \mathbb{R}, x \in \mathcal{X}} \left\| \frac{1}{nh_1^d} \sum_{i=1}^n s(Y_i, X_i; y, x, h_1, h_2) \right\|_2 = O_p \left( \sqrt{|\log h_1|/(nh_1^d)} \right).
\]

It is straightforward to see that the summand is centered and the relevant function classes are uniformly bounded and Euclidean. For the variance bound, we can simply bound the term \( (K((y - Y_i)/h_2) - \tilde{f}(y|X_i))^2 \) by 1. We omit the details of the derivation. Then by the uniform
convergence rate of $\hat{\Xi}(t)^{-1}$ derived in the proof of Theorem 2, we have

$$\sup_{y \in \mathbb{R}, x \in X} \| err_2(y, t) \| = O_p \left( \frac{|\log h_1|}{(nh_1)} \right).$$

Therefore, we have shown that both the terms $err_1(y, x)$ and $err_2(y, x)$ are $O_p \left( \frac{|\log h_1|}{(nh_1)} \right)$ uniformly. Then the desired result follows.

Proof of Theorem 4. Consider the following bias-variance decomposition of $\hat{\theta} - \theta$:

$$\int_{\bar{y}} \int_{\bar{x}} \left( \hat{\beta}_0(y, x, h_1, h_2) - \beta_0^*(y, x, h_1, h_2) \right) dy dx + \int_{\bar{y}} \int_{\bar{x}} \left( \tilde{\beta}_0(y, x, h_1, h_2) - \beta_0^*(y, x, h_1, h_2) \right) dy dx$$

bias term

stochastic term

By Theorem 1 and the assumption that $\sqrt{n}h_1^2 = o(1)$, we know that the bias term is $o(n^{1/2})$. For the stochastic term, we first want to take care of the matrix $\Xi(x, h_1)$. Recall that when $x \in \bar{X}_{\bar{h}} = [\bar{x} + h, \bar{x} - h]$, $\Xi(x, h_1)$ is equal to the identity matrix $I$. In the proof of Theorem 3, we have shown that

$$\sup_{y \in \mathbb{R}, x \in X} \left\| \frac{1}{n h_1} \sum_{i=1}^{n} s(Y_i, X_i; y, x, h_1, h_2) \right\|_2 = O_p \left( \frac{1}{\sqrt{n}} \right).$$

Therefore, we have

$$\left\| \int_{\bar{y}} \int_{\bar{x}} \left( \Xi(x, h_1) - I \right) \frac{1}{nh_1} \sum_{i=1}^{n} s(Y_i, X_i; y, x, h_1, h_2) \right\|_2 = o_p(1/\sqrt{n}).$$

By Theorem 3 and the assumption that $\sqrt{n}h_1/|\log h_1| \to \infty$, we can write the stochastic term as

$$\int_{\bar{y}} \int_{\bar{x}} \left( \tilde{\beta}_0(y, x, h_1, h_2) - \beta_0^*(y, x, h_1, h_2) \right) dy dx = \frac{1}{n} \sum_{i=1}^{n} Z_i + o_p(1/\sqrt{n}).$$
where

\[
Z_i = \frac{1}{h_1} \int_y^g \int_x^{h_2} e_0 I s(Y_i, X_i; y, x, h_1, h_2) dx dy
\]

\[
= \frac{1}{h_1} \int_y^g \int_x^{h_2} \left( K \left( \frac{y - Y_i}{h_2} \right) - \tilde{F}(y|X_i) \right) w \left( \frac{X_i - x}{h_1} \right) dx dy.
\]

By the standard change of variable, we can write \(Z_i\) as

\[
Z_i = \int_y^g \int_x^{h_2} \left( K \left( \frac{y - Y_i}{h_2} \right) - \tilde{F}(y|X_i) \right) w \left( \frac{u - x}{h_1} \right) dx dy \]

The random variables \(\{Z_i : 1 \leq i \leq n\}\) forms an iid triangular array. Each \(Z_i\) is centered, that is, \(\mathbb{E}[Z_i] = 0\). The variance of \(Z_i\) can be calculated based on change of variables:

\[
\mathbb{E}[Z_i^2] = \int \left( \int_{y}^{g} \int_{(t-x)/h_1}^{(t-x)/h_1} \left( K \left( \frac{y - s}{h_2} \right) - \tilde{F}(y|t) \right) w \left( u \right) dudt \right)^2 f(s, t)dt ds
\]

\[
\sim \int \left( \int_{y}^{g} \int_{-1}^{1} \left( 1 \{s \leq y \} - \tilde{F}(y|t) \right) w \left( u \right) dudt \right)^2 f(s, t)dt ds = V.
\]

It is straightforward to see that \(Z_i\) is bounded, and hence any moment of \(|Z_i|\) is finite. Then we can apply the Lyapnov central limit theorem (for example, Theorem 5.11 in White (2001)) to obtain that \(\sum Z_i / \sqrt{n}\) converges in distribution to \(N(0, V)\). The desired result is thus proved.

B Preliminary Results in Empirical Process Theory

Lemma B.1. Let \(K : \mathbb{R} \to \mathbb{R}\) be a function of bounded variation. Then the following class is Euclidean:

\[
\{ K \left( (\cdot - x)/h \right) : x \in \mathbb{R}, h > 0 \}.
\]

Proof of Lemma B.1. This is a direct application of Lemma 22(i) in Nolan and Pollard (1987).

Lemma B.2. Any uniformly bounded and finite-dimensional vector space of functions is Euclidean.

Proof of Lemma B.2. This follows from Lemma 2.6.15 and Theorem 2.6.7 in van der Vaart and Wellner.
Andrews (1996).

**Lemma B.3.** Let \( \mathcal{G} \) be a uniformly bounded Euclidean class with coefficients \((A, v)\). Then the class \( \{ \mathbb{E}[g(\cdot) | X] : g \in \mathcal{G} \} \) is also uniformly bounded and Euclidean with coefficients \((A, v)\).

*Proof of Lemma B.3.* This follows from the fact that the conditional expectation is a projection in the Hilbert space \( L_2(P) \) and hence reduces the norm.

**Lemma B.4.** Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be two classes of functions that are uniformly bounded and Euclidean with coefficients \((A_1, v_1)\) and \((A_2, v_2)\) respectively. Then the class \( \mathcal{G}_1 \oplus \mathcal{G}_2 = \{ g_1 + g_2 : g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2 \} \) is also uniformly bounded and Euclidean with coefficients \((A_1A_2A_2^{v_1+v_2}, v_1 + v_2)\).

*Proof.* By Inequalities (A.4) in Andrews (1994), we have

\[
N(\mathcal{G}_1 \oplus \mathcal{G}_2, L_2(P), \epsilon) \leq N(\mathcal{G}_1, L_2(P), \epsilon/2)N(\mathcal{G}_2, L_2(P), \epsilon/2) \\
\leq A_1(2/\epsilon)^{v_1}A_2(2/\epsilon)^{v_2} = A_1A_2^{v_1+v_2}/\epsilon^{v_1+v_2}.
\]

**Lemma B.5.** Let \( \mathcal{G}_1 \) be a class of functions that is uniformly bounded by \( M_1 \) and Euclidean with coefficients \((A_1, v_1)\) and \( \mathcal{G}_2 \) a class of functions that is uniformly bounded by \( M_2 \) and Euclidean with coefficients \((A_2, v_2)\). Then the class \( \mathcal{G}_1 \mathcal{G}_2 = \{ g_1 \cdot g_2 : g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2 \} \) is uniformly bounded by \( M_1M_2 \) and Euclidean with coefficients \((A_1A_2(M_1 + M_2)^{v_1+v_2}, v_1 + v_2)\).

*Proof of Lemma B.5.* The proof is similar to that of Theorem 3 in Andrews (1994). By definition, for every measure \( P \) and every \( \epsilon \in (0, 1] \), \( N(\mathcal{G}_1, P, \epsilon) \leq A_1/\epsilon^{v_1} \) and \( N(\mathcal{G}_2, P, \epsilon) \leq A_2/\epsilon^{v_2} \). We can construct \( \{ \tilde{g}_{1,j_1} : 1 \leq j_1 \leq J_1 \} \) and \( \{ \tilde{g}_{2,j_2} : 1 \leq j_2 \leq J_2 \} \) to be the \( \epsilon \)-covering of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), respectively, where \( J_1 = N(\mathcal{G}_1, P, \epsilon) \) and \( N(\mathcal{G}_1, P, \epsilon) \). For any \( g_1 \in \mathcal{G}_1 \) and \( g_2 \in \mathcal{G}_2 \), suppose \( g_1 \) is in the \( \epsilon \)-neighborhood of \( \tilde{g}_{1,j_1} \), and \( g_2 \) is in the \( \epsilon \)-neighborhood of \( \tilde{g}_{2,j_2} \). Then the \( L_2(P) \) distance between \( g_1g_2 \) and \( \tilde{g}_{1,j_1} \tilde{g}_{2,j_2} \) is

\[
\| g_1g_2 - \tilde{g}_{1,j_1} \tilde{g}_{2,j_2} \|_{L_2(P)} \leq \| g_1g_2 - g_1\tilde{g}_{2,j_2} \|_{L_2(P)} + \| g_1\tilde{g}_{2,j_2} - \tilde{g}_{1,j_1} \tilde{g}_{2,j_2} \|_{L_2(P)} \\
\leq M_1 \| g_2 - \tilde{g}_{2,j_2} \|_{L_2(P)} + M_2 \| g_1 - \tilde{g}_{1,j_1} \|_{L_2(P)} \leq (M_1 + M_2)\epsilon.
\]
This means that \( \{ g_{1,j_1} g_{2,j_2} : 1 \leq j_1 \leq J_1, 1 \leq j_2 \leq J_2 \} \) forms a \((M_1 + M_2)\varepsilon\)-cover of \( G_1 G_2 \). Therefore,

\[
N(G_1 G_2, L_2(P), \varepsilon) \leq N(G_1, L_2(P), \varepsilon/(M_1 + M_2)) N(G_2, L_2(P), \varepsilon/(M_1 + M_2)) \\
\leq A_1 A_2 (M_1 + M_2)^{\alpha_1 + \alpha_2} / \varepsilon^{\alpha_1 + \alpha_2}.
\]

This proves the result. \qed

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