The line graph of the crown graph is distance integral

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Abstract

The distance eigenvalues of a connected graph $G$ are the eigenvalues of its distance matrix $D(G)$. A graph is called distance integral if all of its distance eigenvalues are integers. Let $n \geq 3$ be an integer. A crown graph $Cr(n)$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. Let $L(Cr(n))$ denote the line graph of the crown graph $Cr(n)$. In this paper, by using the orbit partition method in algebraic graph theory, we determine the set of all distance eigenvalues of $L(Cr(n))$ and show that this graph is distance integral.

1 Introduction and Preliminaries

In this paper, a graph $G = (V, E)$ is considered as an undirected simple graph where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For all the terminology and notation not defined here, we follow [3,4,5,6,7].

Let $G = (V, E)$ be a graph and $A = A(G)$ be an adjacency matrix of $G$. The characteristic polynomial of $G$ is defined as $P(G; x) = P(x) = |xI - A|$. A zero of $p(x)$ is called an eigenvalue of the graph $G$. A graph is called integral, if all the eigenvalues are integers. The study of integral graphs was initiated by

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The characteristic polynomial \( P \) where \( d \) characteristic polynomial of \( G \) been published in the last twenty years. Let \( n \) be the number of vertices of the graph \( G \). The distance matrix \( D = D(G) \) is an \( n \times n \) matrix indexed by \( V \), such that \( D_{u,v} = d_G(u,v) = d(u,v) \), where \( d_G(u,v) \) is the distance between the vertices \( u \) and \( v \) in the graph \( G \). The characteristic polynomial \( P(D;x) = |xI - D| = D_G(x) \) is the distance characteristic polynomial of \( G \). Since \( D \) is a real symmetric matrix, the distance characteristic polynomial \( D_G(x) \) has real zeros. Every zero of the polynomial \( D_G(x) \) is called a distance eigenvalue of the graph \( G \). A survey on the distance spectra of graphs has been appeared in [1]. A graph \( G \) is distance integral (briefly, \( D \)-integral) if all the distance eigenvalues of \( G \) are integers. Although there are many papers that study distance spectrum of graphs and their applications, the \( D \)-integral graphs are studied only in a few number of papers (see [14]).

Let \( n \geq 3 \) be an integer. A crown graph \( Cr(n) \) is a graph obtained from the complete bipartite graph \( K_{n,n} \) by removing a perfect matching. The bipartite Kneser graph \( H(n,k), 1 \leq k \leq n - 1 \), is a bipartite graph with the vertex-set consisting of all \( k \)-subsets and \( (n - k) \)-subsets of the set \( [n] = \{1, 2, 3, \ldots, n\} \), in which two vertices \( v \) and \( w \) are adjacent if and only if \( v \subset w \) or \( w \subset v \). Recently, this class of graphs has been studied form several aspects [9,11,12,13]. It is easy to see that the crown graph \( Cr(n) \) is isomorphic with the bipartite graph \( H(n,1) \). The crown graph \( Cr(n) \) is a vertex and edge-transitive graph of order \( 2n \) and regularity \( n - 1 \) with diameter 3.

In [14] it has been shown that the graph \( Cr(n) \) is a distance integral graph. Let \( L(Cr(n)) \) denote the line graph of the crown graph \( Cr(n) \). It is not hard to see that the graph \( L(Cr(n)) \) is a vertex-transitive graph of order \( n(n-1) \) and regularity \( 2(n - 1) - 2 = 2n - 4 \) with diameter 3.

From the various interesting properties of the graph \( L(Cr(n)) \), we interested in its distance eigenvalues. In this paper, we wish to show that the graph \( L(Cr(n)) \) is distance integral. We explicitly determine all the distinct distance eigenvalues of the graph \( L(Cr(n)) \). The method which we use in this paper is completely different from what has been used in [14]. The main tool which we use in our work is the orbit partition method in algebraic graph theory which we have already employed it in determining the adjacency eigenvalues of a particular family of graph [10]. In this paper, We show how we can find, by using this method, the set of all distinct distance eigenvalues of the graph \( L(Cr(n)) \).

The set of all permutations of a set \( V \) is denoted by \( Sym(V) \). A permutation group on \( V \) is a subgroup of \( Sym(V) \). If \( G = (V,E) \) is a graph, then we can view each automorphism as a permutation of \( V \), and so \( Aut(G) \) is a permutation group. A permutation representation of a group \( \Gamma \) is a homomorphism from \( \Gamma \) into \( Sym(V) \) for some set \( V \). A permutation representation is also referred to as an action of \( \Gamma \) on the set \( V \), in which case we say that \( \Gamma \) acts on \( V \). A permutation group \( \Gamma \) on \( V \) is transitive if given any two elements \( x \) and \( y \) from \( V \) there is an element \( g \in \Gamma \) such that \( x^g = y \). For each \( v \in V \), the set \( v^\Gamma = \{v^g \mid g \in \Gamma\} \) is called an orbit of \( \Gamma \). It is easy to see that if \( \Gamma \) acts on \( V \), then \( \Gamma \) is transitive on \( V \) (or \( \Gamma \) acts transitively on \( V \)), when there is just one orbit. It is easy to see
that the set of orbits of \( \Gamma \) on \( V \) is a partition of the set \( V \).

A graph \( G = (V, E) \) is called vertex-transitive if \( \text{Aut}(G) \) acts transitively on \( V \). We say that \( G \) is edge-transitive if the group \( \text{Aut}(G) \) acts transitively on the edge set \( E \), namely, for any \( \{x, y\}, \{v, w\} \in E(G) \), there is some \( a \in \text{Aut}(G) \) such that \( a(\{x, y\}) = \{v, w\} \). We say that \( G \) is distance-transitive if for all vertices \( u, v, x, y \) of \( G \) such that \( d(u, v) = d(x, y) \), where \( d(u, v) \) denotes the distance between the vertices \( u \) and \( v \) in \( G \), there is an automorphism \( a \) in \( \text{Aut}(G) \) such that \( a(u) = x \) and \( a(v) = y \).

## 2 Main Results

Let \( G = (V, E) \) be a graph with the vertex-set \( V = \{v_1, \ldots, v_n\} \) and distance matrix \( D(G) = D = (d_{ij})_{n \times n} \), where \( d_{ij} = d(v_i, v_j) \). Let \( H \leq \text{Aut}(G) \) and \( \pi = \{w_1^H = C_1, \ldots, w_m^H = C_m\} \) be the orbit partition of \( H \), where \( \{w_1, \ldots, w_m\} \subset V \). Let \( Q = Q_\pi = (q_{ij})_{m \times m} \) be the matrix which its rows and columns is indexed by \( \pi \) such that,

\[
q_{ij} = \sum_{w \in C_j} d(v, w),
\]

where \( v \) is a fixed element in the cell \( C_i \). It is easy to check that this sum is independent of \( v \), that is, if \( u \in C_i \), then \( q_{ij} = \sum_{w \in C_j} d(v, w) = \sum_{w \in C_j} d(u, w) \).

Hence, the matrix \( Q \) is well defined. We call the matrix \( Q \) the quotient matrix of \( D \) over \( \pi \). We claim that every eigenvalue of \( Q \) is an eigenvalue of the distance matrix \( D \). In fact we have the following fact.

**Theorem 2.1.** Let \( G = (V, E) \) be a graph with the distance matrix \( D \). Let \( \pi \) be an orbit partition of \( V \) with \( m \) cells and \( Q \) be a quotient matrix of \( D \) over \( \pi \). Then, every eigenvalue of \( Q \) is an eigenvalue of the distance matrix \( D \).

**Proof.** Let \( \lambda \) be an eigenvalue of the quotient matrix \( Q \) with a none zero eigenvector \( f \). Let \( f(\pi_j) = x_j \). Thus for every \( i, 1 \leq i \leq m \), we have,

\[
\sum_{j=1}^m q_{ij} \lambda = \sum_{j=1}^m q_{ij} x_j = \lambda x_i = \lambda f(\pi_i).
\]

We define the function \( \hat{f} : V(G) \to \mathbb{R} \) by the rule \( \hat{f}(v) = f(\pi_i) \) if and only if \( v \in \pi_i \). The function \( \hat{f} \) is well defined since \( \pi \) is a partition of \( V(G) \). Also, \( \hat{f} \) is a none zero function since the function \( f \) is non zero. If \( v \in V(G) = V \), then there is a unique \( i \) such that \( v \in \pi_i \). Now we have,

\[
\sum_{w \in V} D_{vw} \hat{f}(w) = \sum_{w \in \pi_1} D_{vw} \hat{f}(w) + \sum_{w \in \pi_2} D_{vw} \hat{f}(w) + \cdots + \sum_{w \in \pi_m} D_{vw} \hat{f}(w) = \\
\sum_{w \in \pi_1} D_{vw} f(\pi_1) + \sum_{w \in \pi_2} D_{vw} f(\pi_2) + \cdots + \sum_{w \in \pi_m} D_{vw} f(\pi_m) = \\
f(\pi_1)(\sum_{w \in \pi_1} D_{vw}) + f(\pi_2)(\sum_{w \in \pi_2} D_{vw}) + \cdots + f(\pi_m)(\sum_{w \in \pi_m} D_{vw}) = \\
f(\pi_1)q_{i1} + f(\pi_2)q_{i2} + \cdots + f(\pi_m)q_{im} = \lambda f(\pi_i)
\]
\[ \sum_{j=1}^{m} q_{ij} f(\pi_j) = \lambda f(\pi_i) = \lambda \hat{f}(v), \text{ since } v \in \pi_i. \]

Thus \( \lambda \) is an eigenvalue of the matrix \( D \).

By Theorem 2.1, we can find some of the distance eigenvalues of the graph \( G \), but we cannot determine all the eigenvalues, since it is possible that \( G \) has a distance eigenvalue \( \theta \) such that \( \theta \) is not an eigenvalue of the matrix \( Q \). By the next theorem, we give a condition, holding that guarantees that the eigenvalue \( \theta \) to be an eigenvalue of the matrix \( Q \).

**Proposition 2.2.** Let \( G = (V, E) \) be a graph with the distance matrix \( D \). Let \( \pi \) be an orbit partition of \( V \) with \( m \) cells and \( Q \) be the quotient matrix of \( D \) over \( \pi \). Let \( \theta \) be an eigenvalue of the distance matrix \( D \), with the non zero eigenvector \( f \) such that \( f \) is constant on every cell of \( \pi \). Then \( \theta \) is an eigenvalue of the matrix \( Q \).

**Proof.** We define the function \( \tilde{f} : \pi \to \mathbb{R} \) by the rule \( \tilde{f}(\pi_j) = f(v) \), where \( v \) is an element in the cell \( \pi_j \). Note that since \( f \) is constant on the set \( \pi_j \), then \( \tilde{f}(\pi_j) \) is independent of \( v \in \pi_j \), and hence \( \tilde{f} \) is a well defined function. Also, since \( f \) is non zero, then \( \tilde{f} \) is non zero. If \( u \in V \), since \( f \) is an eigenvector with the eigenvalue \( \theta \), then we have,

\[ \sum_{w \in V} D_{uw} f(w) = \sum_{w \in V} d(u, w) f(w) = \theta f(u). \]

There is a unique \( i \) such that \( u \in \pi_i \). Now we have,

\[
\sum_{w \in V} d(u, w) f(w) = \sum_{w \in \pi_1} d(u, w) f(w) + \sum_{w \in \pi_2} d(u, w) f(w) + \cdots + \sum_{w \in \pi_m} d(u, w) f(w)
\]

\[
= \sum_{w \in \pi_1} d(u, w) \tilde{f}(\pi_1) + \sum_{w \in \pi_2} d(u, w) \tilde{f}(\pi_2) + \cdots + \sum_{w \in \pi_m} d(u, w) \tilde{f}(\pi_m)
\]

\[
= \tilde{f}(\pi_1) \left( \sum_{w \in \pi_1} d(u, w) \right) + \tilde{f}(\pi_2) \left( \sum_{w \in \pi_2} d(u, w) \right) + \cdots + \tilde{f}(\pi_m) \left( \sum_{w \in \pi_m} d(u, w) \right)
\]

\[
= \tilde{f}(\pi_1) q_{i1} + \tilde{f}(\pi_2) q_{i2} + \cdots + \tilde{f}(\pi_m) q_{im} = \theta f(u) = \theta \tilde{f}(\pi_i). \quad (\ast)
\]

From (\ast) it follows that \( \theta \) is an eigenvalue of the matrix \( Q \) with the eigenvector \( \tilde{f} \).

Let \( G = (V, E) \) be a graph with an adjacency matrix \( A = (a_{vw}) \) and automorphism group \( \Gamma = Aut(G) \). We recall that every eigenvector \( f \) of \( G \) with the eigenvalue \( \theta \) is a real function on \( V \) such that \( \sum_{w \in V} a_{vw} f(w) = \theta f(v) \), for every
The argument shows that $D$ is an eigenvector of $G$ with eigenvalue $\lambda$ [7, chap 9].

Let $D = (d_{vw})_{n \times n}$, $d_{vw} = d(v, w)$, be a distance matrix for $G$. Let $h$ be an eigenvector of $D$ with the eigenvalue $\lambda$. Thus for every $v \in V$, we have,

$$\sum_{w \in V} d_{vw} h(w) = \sum_{w \in V} d(v, w) h(w) = \lambda h(v).$$

We claim that if $g \in \text{Aut}(\Gamma)$, then $h^g$ is an eigenvector of $D$ with the eigenvalue $\lambda$. Note that if $x, y \in V$, then $d(x, y) = d(x^g, y^g)$. We now have,

$$\sum_{w \in V} d_{vw} h^g(w) = \sum_{w \in V} d(v, w) h^g(w) = \sum_{w \in V} d(v^g, w^g) h(w^g) = \lambda h(v^g) = \lambda h^g(v).$$

The argument shows that $h^g$ is really an eigenvector of $D$ with the eigenvalue $\lambda$. We now formally state the obtained result.

**Proposition 2.3.** Let $G = (V, E)$ be a graph and $D$ be a distance matrix for $G$. Let $f$ be an eigenvector with the eigenvalue $\lambda$ for $D$. If $g$ is an automorphism of the graph $G$, then the function $f^g$ defined by the rule $f^g(v) = f(g^v)$, $v \in V$, is an eigenvector for $D$ with the eigenvalue $\lambda$.

Let $G = (V, E)$ be a graph and $D$ be a distance matrix for $G$. Let $f$ be an eigenvector with the eigenvalue $\lambda$ for $D$. Let $H$ be a subgroup of $\text{Aut}(G)$. We can construct from $f$ an eigenvector $p$ with the eigenvalue $\lambda$ for $D$ such that $p$ is constant on every orbit of $H$ on $V$. In fact if we define the function $p$, by the rule,

$$p = \sum_{h \in H} f^h, \quad (**)$$

then from Proposition 2.3, it follows that if $p \neq 0$, then $p$ is an eigenvector of $D$ with the eigenvalue $\lambda$. If $O = v^H$, $v \in V$, is an orbit of $H$ on $V$, then $p$ is constant on $O$. Note that if $w \in O$, then $w = v^{h_1}$ for some $h_1 \in H$. Hence we have,

$$p(w) = \sum_{h \in H} f^h(w) = \sum_{h \in H} f(w^h) = \sum_{h \in H} f(v^{h_1}) = \sum_{h \in H} f(v^h) = \sum_{h \in H} f^h(v) = p(v).$$

In other words, $p$ is an eigenvector of $D$ with the eigenvalue $\lambda$ such that it is constant on every cell of $\pi$. Hence if $p \neq 0$, as we saw in proposition 2.2, we can construct the function $\tilde{p}$ from $p$ such that $\tilde{p}$ is an eigenvector for the matrix $Q$ with the eigenvalue $\lambda$, where $Q$ is a quotient matrix of $D$ over $\pi$.

We now can deduce that if $\lambda$ is an eigenvalue of the distance matrix $D$ with the eigenvector $f \neq 0$ such that $\lambda$ is not an eigenvalue of the matrix $Q$, then the function $p$ is the zero function. Thus we have the following result.
Theorem 2.4. Let $G = (V, E)$ be a vertex-transitive graph and $D$ be a distance matrix for $G$. Let $f \neq 0$ be an eigenvector with the eigenvalue $\lambda$ for $D$. Let $H$ be a subgroup of $\text{Aut}(G)$ and $\pi$ be its orbit partition on $V$ and $Q$ is a quotient matrix of $D$ over $\pi$. If $\lambda$ is not an eigenvalue of $Q$, then the sum of the values of $f$ on each cell of $\pi$ is zero.

Now consider the function $p$ defined in (**) . As we saw, it is favorite that $p \neq 0$. In the next theorem we state some conditions, holding those guarantee that $p \neq 0$.

Proposition 2.5. Let $G = (V, E)$ be a vertex-transitive graph and $D$ be a distance matrix for $G$. Let $H$ be a subgroup of $\text{Aut}(G)$ with the orbit partition $\pi$ on $V$ such that $\pi$ has a singleton cell $\{x\}$. Let $\lambda$ be an eigenvalue of $D$. Then $\lambda$ is an eigenvalue for the matrix $Q$, where $Q$ is a quotient matrix of $D$ over $\pi$.

Proof. Let $0 \neq f$ be an eigenvector with the eigenvalue $\lambda$ for $D$. Since $0 \neq f$, hence there is an element $w \in V$ such that $f(w) \neq 0$. Since $G$ is a vertex-transitive graph, then $x^g = w$, for some $g \in \text{Aut}(G)$. Let $t = f^g$. Thus we have,

$$t(x) = f^g(x) = f(x^g) = f(w) \neq 0.$$ 

Let $p = \sum_{h \in H} t^h$. Then by Proposition 2.3, $p$ is an eigenvector with the eigenvalue $\lambda$ for $D$ such that it is constant on each cell of the partition $\pi$. On the other hand we have,

$$p(x) = \sum_{h \in H} t^h(x) = \sum_{h \in H} t(x^h) = |H|t(x) \neq 0.$$ 

Hence by proposition 2.2, $\lambda$ is an eigenvalue for the matrix $Q$. \hfill \square

From Theorem 2.1, Proposition 2.2, and Proposition 2.5, we obtain the following important result.

Theorem 2.6. Let $G = (V, E)$ be a vertex-transitive graph with the distance matrix $D$. Let $H$ be a subgroup of $\text{Aut}(G)$ with the orbit partition $\pi$ on $V$ such that $\pi$ has a singleton cell $\{x\}$. Let $Q = Q_{\pi}$ be a quotient matrix of $D$ over $\pi$. Then the set of distinct eigenvalues of $D$ is equal to the set of distinct eigenvalues of $Q$.

In the sequel, we will see how Theorem 2.6, help us in finding the set of distance eigenvalues of the line graph of the crown graph.

Let $n \geq 3$ be an integer and $[n] = \{1, 2, \ldots, n\}$. Let $X = \{x_1, x_2, \ldots, x_n\}$ be an $n$-set disjoint from $[n]$. The crown graph $Cr(n)$ is a graph with the vertex-set $[n] \cup X$ and the edge-set $E_0 = \{e_{ij} = \{i, x_j\} \mid i, j \in [n], i \neq j\}$. Thus, $L(Cr(n))$, the line graph of $Cr(n)$, is a graph with the vertex-set $E_0$ in which two vertices $e_{ij}$ and $e_{rs}$ are adjacent if and only if $i = r$ or $j = s$. 

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Let $V = \{(i, j) \mid i, j \in [n], i \neq j\}$. Let $G$ be a graph with the vertex-set $V$ in which two vertices $(i, j)$ and $(r, s)$ are adjacent if and only if $i = r$ or $j = s$. It is easy to check that the graph $G$ is isomorphic with the graph $L(Cr(n))$. Hence in the sequel we work on the graph $G$ and call it the line graph of the crown graph and denote it by $L(Cr(n))$.

It is easy to see that $L(Cr(3))$ is the cycle graph $C_6$, which its structure is known and its line graph is again $C_6$. Hence in the rest of the paper we assume that $n \geq 4$. It is easy to check that two non adjacent vertices $(i, j)$ and $(r, s)$ are at distance 2 from each other whenever $i = s$ or $j = r$ or \{i, j\} \cap \{r, s\} = \emptyset. Moreover, vertices $(i, j)$ and $(j, i)$ are at distance 3 from each other $(P : (i, j), (x, j), (x, i), (j, i)$ is a shortest path between the vertices $(i, j)$ and $(j, i)$. Thus the diameter of the graph $L(Cr(n))$ is 3.

Note that since the crown graph $Cr(n)$ is a regular graph and its adjacency spectrum is known [4], hence the adjacency spectrum of its line graph, that is, $L(Cr(n))$ is known [3,5,7].

**Remark 2.7.** Although the crown graph $Cr(n)$ is a distance-transitive graph (and consequently it is distance-regular [4,9]), it is easy to check that the graph $L(Cr(n))$ is not distance-regular. Hence we can not use the theory of distance-regular graphs for determining the set of distance eigenvalues of this graph.

Figure 1, shows the graph $L(Cr(4))$. Note that in this figure the vertex $(i, j)$ is denoted by $ij$.

![Figure 1. The graph $L(Cr(4))$](image)

Since the graph $Cr(n)$ is distance-transitive, hence it is edge-transitive. Thus the graph $L(Cr(n))$ is a vertex-transitive graph. For each $\alpha \in Sym([n])$, let $f_\alpha$ be the function on the vertex-set of the graph $L(Cr(n))$ defined by the rule, $f_\alpha(i, j) = (\alpha(i), \alpha(j))$. Let $\beta$ be the function on the vertex-set of the graph $L(Cr(n))$ defined by the rule, $\beta(i, j) = (j, i)$. Now we can check that $Aut(L(Cr(n))) \cong Sym([n]) \times \langle \beta \rangle$ [9,11,13], where $\langle \beta \rangle$ is the subgroup generated by the automorphism $\beta$ in the automorphism group of the graph $L(Cr(n))$. 
Theorem 2.8. Let \( n > 3 \) be an integer. Then the line graph of the crown graph, that is, the graph \( L(Cr(n)) \) is a distance integral graph with distinct distance eigenvalues, \(-n - 1, -n + 3, -1, 1, 2n^2 - 4n + 3\).

Proof. Let \( D \) be a distance matrix of the graph \( L(Cr(n)) \). In the first step, we proceed to construct an orbit partition \( \pi \) for the vertex-set of \( L(Cr(n)) \), such that this partition has a singleton cell. If we construct such a partition, then since the graph \( L(Cr(n)) \) is a vertex-transitive graph, then by Theorem 2.6, every distance eigenvalue of the graph \( L(Cr(n)) \) is an eigenvalue of the matrix \( Q \) and vice versa, where \( Q \) is the quotient matrix of \( D \) over \( \pi \). Let \( H = \{ f_\alpha \mid \alpha \in Sym([n]), \; \alpha(1) = 1, \; \alpha(2) = 2 \} \). Then \( H \) is a subgroup of \( Aut(L(Cr(n))) \), the automorphism group of \( L(Cr(n)) \). Let \( H_1 = \{ \alpha \mid f_\alpha \in H \} \). Note that \( H_1 \) is a subgroup of \( Sym([n]) \) isomorphic with \( Sym([n - 2]) \). In the sequel, we want to determine the orbit partition of the subgroup \( H \). In fact, \( H \) generates the following orbits;

\[
O_1 = H((1, 2)) = \{ h((1, 2)) \mid h \in H \} = \{ (\alpha(1), \alpha(2)) \mid \alpha \in H_1 \} = \{(1, 2)\}.
\]

\[
O_2 = H((1, 3)) = \{ h((1, 3)) \mid h \in H \} = \{ (\alpha(1), \alpha(3)) \mid \alpha \in H_1 \} = \{(1, i) \mid 3 \leq i \leq n\}.
\]

\[
O_3 = H((3, 1)) = \{ h((3, 1)) \mid h \in H \} = \{ (\alpha(3), \alpha(1)) \mid \alpha \in H_1 \} = \{(i, 1) \mid 3 \leq i \leq n\}.
\]

\[
O_4 = H((2, 1)) = \{ h((2, 1)) \mid h \in H \} = \{ (\alpha(2), \alpha(1)) \mid \alpha \in H_1 \} = \{(2, 1)\}.
\]

\[
O_5 = H((2, 3)) = \{ h((2, 3)) \mid h \in H \} = \{ (\alpha(2), \alpha(3)) \mid \alpha \in H_1 \} = \{(2, i) \mid 3 \leq i \leq n\}.
\]

\[
O_6 = H((3, 2)) = \{ h((3, 2)) \mid h \in H \} = \{ (\alpha(3), \alpha(2)) \mid \alpha \in H_1 \} = \{(i, 2) \mid 3 \leq i \leq n\}.
\]

\[
O_7 = H((3, 4)) = \{ h((3, 4)) \mid h \in H \} = \{ (\alpha(3), \alpha(4)) \mid \alpha \in H_1 \} = \{(i, j) \mid 3 \leq i, j \leq n, \; i \neq j\}.
\]

If we let \( \pi = \{ O_1, O_2, O_3, \ldots, O_7 \} \), then \( O_1 \cup O_2 \cup \cdots \cup O_7 = V = V(L(Cr(n))) \). Let \( Q = (q_{ij})_{7 \times 7} \), be the quotient matrix of \( D \) over \( \pi \), that is, \( q_{ij} \) is the sum of the distances of a vertex in the cell \( O_i \) from all the vertices in the cell \( O_j \). Then the following hold.

\[
q_{11} = 0.
\]

\[
q_{12} = n - 2. \quad \text{Because the vertex } (1, 2) \in O_1 \text{ is adjacent to all the } n - 2 \text{ vertices in } O_2.
\]

\[
q_{13} = 2(n - 2) = 2n - 4. \quad \text{Because the distance of the vertex } (1, 2) \in O_1 \text{ is 2 from every vertex in } O_3. \quad \text{Note that } |O_3| = n - 2.
\]
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$q_{14} = 3$. Because the distance of the vertex $(1, 2) \in O_1$ is 3 from the vertex $(2, 1) \in O_4$.

$q_{15} = 2(n - 2) = 2n - 4$. Because the distance of the vertex $(1, 2) \in O_1$ is 2 from every vertex in $O_5$. Note that $|O_5| = n - 2$.

$q_{16} = n - 2$. Because the distance of the vertex $(1, 2) \in O_1$ is 1 from every vertex in $O_6$.

$q_{17} = 2(n - 2)(n - 3)$. Because the distance of the vertex $(1, 2) \in O_1$ is 2 from every vertex in $O_7$. Note that $|O_7| = (n - 2)(n - 3)$.

$q_{21} = 1$. Because the distance of the vertex $(1, 3) \in O_2$ is 1 from the vertex $(1, 2) \in O_4$.

$q_{22} = n - 3$. Because the distance of the vertex $(1, 3) \in O_2$ is 1 from each of the other vertices in $O_2$. Note that $|O_2| = n - 2$.

$q_{23} = 2n - 3$. Because the distance of the vertex $(1, 3) \in O_2$ is 3 from the vertex $(3, 1)$, and is 2 from each of the other vertices in $O_3$. Hence we have $q_{23} = 3 + 2(n - 3) = 2n - 3$. Note that $|O_3| = n - 2$.

$q_{24} = 2$. Because the distance of the vertex $(1, 3) \in O_2$ is 2 from the vertex $(2, 1) \in O_4$.

$q_{25} = 2n - 5$. Because the distance of the vertex $(1, 3) \in O_2$ is 1 from the vertex $(2, 3)$, and is 2 from each of the other vertices in $O_5$. Hence we have $q_{25} = 1 + 2(n - 3) = 2n - 5$. Note that $|O_5| = n - 2$.

$q_{26} = 2n - 4$. Because the distance of the vertex $(1, 3) \in O_2$ is 2 from every vertex in $O_6$. Note that $|O_6| = n - 2$.

$q_{27} = (n - 3)(2n - 5)$. Because the distance of the vertex $(1, 3) \in O_2$ from every vertex of the form $(j, 3)$, $4 \leq j \leq n$, in $O_7$ is 1 and from each of the other vertices in $O_7$ is 2. Hence we have $q_{27} = (n - 3) + 2((n - 3) + (n - 4)(n - 3)) = (n - 3) + 2(n - 3)(n - 3) = (n - 3)(2n - 5)$.

$q_{31} = 2$. Because the distance of the vertex $(3, 1) \in O_3$ is 2 from the vertex $(1, 2) \in O_1$.

$q_{32} = 2n - 3$. Because the distance of the vertex $(3, 1) \in O_3$ is 3 from the vertex $(1, 3) \in O_2$ and from each of the other vertices is 2. Hence we have $q_{32} = 3 + 2(n - 3)$.

$q_{33} = n - 3$. Because $|O_3| = n - 2$ and the vertex $(3, 1) \in O_3$ is adjacent to each of the other vertices in $O_3$. 
$q_{34} = 1.$

$q_{35} = 2n - 4$. Because the distance of the vertex $(3, 1) \in O_3$ is 2 from every vertex in $O_5$.

$q_{36} = 2n - 5$. Because the distance of the vertex $(3, 1) \in O_3$ is 1 from the vertex $(3, 2)$ in $O_6$ and from each of the other vertices is 2. Hence we have $q_{36} = 1 + 2(n - 3) = 2n - 5$.

$q_{37} = (n - 3)(2n - 5)$. Because the distance of the vertex $(3, 1) \in O_3$ from every vertex of the form $(3, j), 4 \leq j \leq n$, in $O_7$ is 1 and from each of the other vertices in $O_7$ is 2. Hence we have $q_{37} = (n - 3) + 2((n - 3)(n - 3)) = (n - 3)(2n - 5)$.

$q_{41} = 3$.

$q_{42} = 2n - 4$. Because the distance of the vertex $(2, 1) \in O_4$ is 2 from every vertex in $O_2$, hence we have $q_{42} = 2(n - 2)$.

$q_{43} = n - 2$. Because the vertex $(2, 1) \in O_4$ is adjacent to every vertex in $O_5$.

$q_{44} = 0$.

$q_{45} = n - 2$. Because the vertex $(2, 1) \in O_4$ is adjacent to every vertex in $O_5$.

$q_{46} = 2n - 4$. Because the distance of the vertex $(1, 2) \in O_4$ is 2 from every vertex in $O_6$. Hence we have $q_{46} = 2(n - 2)$.

$q_{47} = 2(n - 2)(n - 3)$. Because the distance of the vertex $(2, 1) \in O_4$ is 2 from every vertex in $O_7$. Since $|O_7| = (n - 2)(n - 3)$, thus we have $q_{47} = 2(n - 2)(n - 3)$.

$q_{51} = 2$. Because the distance of the vertex $(2, 3) \in O_5$ is 2 from the vertex $(1, 2)$ in $O_1$.

$q_{52} = 2n - 5$. Because the distance of the vertex $(2, 3) \in O_5$ is 1 from the vertex $(1, 3) \in O_2$, and is 2 from each of the other vertices in $O_2$. Hence we have $q_{52} = 1 + 2(n - 3) = 2n - 5$.

$q_{53} = 2n - 4$. Because the distance of the vertex $(2, 3) \in O_5$ is 2 from every vertex in $O_3$.

$q_{54} = 1$. Because the vertex $(2, 3) \in O_5$ is adjacent to vertex $(2, 1)$ in $O_4$.

$q_{55} = n - 3$. Because the vertex $(2, 3) \in O_5$ is adjacent to every other vertex in $O_5$. 
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$q_{56} = 2n - 3$. Because the distance of the vertex $(2, 3) \in O_5$ is 3 from the vertex $(3, 2)$ in $O_6$ and is 2 from every other vertex in $O_6$. Thus we have $q_{56} = 3 + 2(n - 3)$.

$q_{57} = (n - 3)(2n - 5)$. Because the distance of the vertex $(2, 3) \in O_5$ from every vertex of the form $(j, 3)$, $4 \leq j \leq n$, in $O_7$ is 1 and from each of the other vertices in $O_7$ is 2. Hence we have $q_{57} = (n - 3) + 2((n - 3) + (n - 4)(n - 3)) = (n - 3) + 2(n - 3)(n - 3) = (n - 3)(2n - 5)$.

$q_{61} = 1$. Because the vertex $(3, 2) \in O_6$ is adjacent to the vertex $(1, 2)$ in $O_1$.

$q_{62} = 2n - 4$. Because the distance of the vertex $(3, 2) \in O_6$ is 2 from every vertex in $O_2$. Hence we have $q_{62} = 2(n - 2)$.

$q_{63} = 2n - 5$. Because the distance of the vertex $(3, 2) \in O_6$ is 1 from the vertex $(3, 1)$ in $O_3$ and is 2 from every other vertex in $O_3$. Thus we have $q_{63} = 1 + 2(n - 3)$.

$q_{64} = 2$. Because the distance of the vertex $(3, 2) \in O_6$ is 2 from the vertex $(2, 1)$ in $O_4$.

$q_{65} = 2n - 3$. Because the distance of the vertex $(3, 2) \in O_6$ is 3 from the vertex $(2, 3)$ in $O_5$, and is 2 from every other vertex in $O_5$. Thus we have, $q_{65} = 3 + 2(n - 3)$.

$q_{66} = n - 3$. Because the vertex $(3, 2) \in O_6$ is adjacent to every other vertex in $O_6$. Note that $|O_6| = n - 2$.

$q_{67} = (n - 3)(2n - 5)$. Because the distance of the vertex $(3, 2) \in O_6$ from every vertex of the form $(3, j)$, $4 \leq j \leq n$, in $O_7$ is 1 and from each of the other vertices in $O_7$ is 2. Hence we have $q_{67} = (n - 3) + 2((n - 3) + (n - 4)(n - 3)) = (n - 3) + 2(n - 3)(n - 3) = (n - 3)(2n - 5)$.

$q_{71} = 2$. Because the distance of the vertex $(3, 4) \in O_7$ is 2 from the vertex $(1, 2)$ in $O_1$.

$q_{72} = 2n - 5$. Because the distance of the vertex $(3, 4) \in O_7$ is 1 from the vertex $(2, 4)$ and is 2 from every other vertex in $O_2$. Thus we have $q_{72} = 1 + 2(n - 3)$.

$q_{73} = 2n - 5$. Because the distance of the vertex $(3, 4) \in O_7$ is 1 from the vertex $(3, 1)$, and is 2 from every other vertex in $O_3$. Thus we have $q_{73} = 1 + 2(n - 3)$.

$q_{74} = 2$. Because the distance of the vertex $(3, 4) \in O_7$ is 2 from the vertex $(2, 1)$ in $O_4$.

$q_{75} = 2n - 5$. Because the distance of the vertex $(3, 4) \in O_7$ is 1 from the vertex $(2, 4)$ in $O_5$ and is 2 from every other vertex in $O_5$. Thus we have $q_{75} = 1 + 2(n - 3)$.

$q_{76} = 2n - 5$. Because the distance of the vertex $(3, 4) \in O_7$ is 1 from the
vertex (3, 2) in \( O_6 \), and is 2 from every other vertex in \( O_6 \). Thus we have \( q_{76} = 1 + 2(n - 3) \).

\[ q_{77} = 2(n - 4)(n - 2) + 3. \]

Because the distance of the vertex (3, 4) ∈ \( O_7 \) from every vertex in \( O_7 \) of the form (3, j), 5 ≤ j ≤ n, is 1, and from every vertex in \( O_7 \) of the form (i, 4), 5 ≤ i ≤ n, is 1, and from the vertex (4, 3) is 3, and from every vertex in \( O_7 \) of the form (i, j), 4 ≤ i ≤ n, j ≠ 4, is 2. Hence we have \( q_{77} = (n - 4) + (n - 4) + 3 + 2(n - 4)(n - 3) = 2(n - 4)(1 + n - 3) + 3 = 2(n - 4)(n - 2) + 3 \).

Therefore, we obtain the following quotient matrix \( Q \) of \( D \) over \( \pi \).

\[
Q = \begin{pmatrix}
0 & n - 2 & 2n - 4 & 3 & 2n - 4 & n - 2 & 2(n - 2)(n - 3) \\
1 & n - 3 & 2n - 3 & 2 & 2n - 5 & 2n - 4 & (n - 3)(2n - 5) \\
2 & 2n - 3 & n - 3 & 1 & 2n - 4 & 2n - 5 & (n - 3)(2n - 5) \\
3 & 2n - 4 & n - 2 & 0 & n - 2 & 2n - 4 & 2(n - 2)(n - 3) \\
2 & 2n - 5 & 2n - 4 & 1 & n - 3 & 2n - 3 & (n - 3)(2n - 5) \\
1 & 2n - 4 & 2n - 5 & 2 & 2n - 3 & n - 3 & (n - 3)(2n - 5) \\
2 & 2n - 5 & 2n - 5 & 2 & 2n - 5 & 2n - 5 & 2(n - 4)(n - 2) + 3
\end{pmatrix}
\]

We can use the Wolfram Mathematica [15] for finding the eigenvalues of the matrix \( Q \). Using the Wolfram Mathematica we have,

\[
Q = \{\{0, n - 2, 2n - 4, 3, 2n - 4, n - 2, 2(n - 2)(n - 3)\}, \\
\{1, n - 3, 2n - 3, 2, 2n - 5, 2n - 4, (n - 3)(2n - 5)\}, \\
\{2, 2n - 3, n - 3, 1, 2n - 4, 2n - 5, (n - 3)(2n - 5)\}, \\
\{3, 2n - 4, n - 2, 0, n - 2, 2n - 4, 2(n - 2)(n - 3)\}, \\
\{2, 2n - 5, 2n - 4, 1, n - 3, 2n - 3, (n - 3)(2n - 5)\}, \\
\{1, 2n - 4, 2n - 5, 2, 2n - 3, n - 3, (n - 3)(2n - 5)\}, \\
\{2, 2n - 5, 2n - 5, 2, 2n - 5, 2n - 5, 2(n - 4)(n - 2) + 3\}\}
\]

**Eigenvalues [Q]** = \{-1, 1, -1 - n, -1 - n, 3 - n, 3 - n, 3 - 4n + 2n^2\}.

We now conclude that the set \{-n - 1, -n + 3, -1, 1, 2n^2 - 4n + 3\}, is the set of all distinct eigenvalues of the graph \( L(\mathcal{C}r(n)) \).

**Remark 2.9.** It is not true that if \( G = (V, E) \) is an integral distance-transitive graph, then its line graph \( L(G) \) is distance integral. Using Wolfram Mathematica [15], one can see that the Johnson graph \( J(6, 2) \) which is an integral distance-transitive graph, is distance integral. But its line graph, that is, \( L(J(6, 2)) \) is not distance integral.

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