Vector-valued Privacy-Preserving Average Consensus

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Abstract

Achieving average consensus without disclosing sensitive information can be a critical concern for multi-agent coordination. This paper examines privacy-preserving average consensus (PPAC) for vector-valued multi-agent networks. In particular, a set of agents with vector-valued states aim to collaboratively reach an exact average consensus of their initial states, while each agent’s initial state cannot be disclosed to other agents. We show that the vector-valued PPAC problem can be solved via associated matrix-weighted networks with the higher-dimensional agent state. Specifically, a novel distributed vector-valued PPAC algorithm is proposed by lifting the agent-state to higher-dimensional space and designing the associated matrix-weighted network with dynamic, low-rank, positive semi-definite coupling matrices to both conceal the vector-valued agent state and guarantee that the multi-agent network asymptotically converges to the average consensus. Essentially, the convergence analysis can be transformed into the average consensus problem on switching matrix-weighted networks. We show that the exact average consensus can be guaranteed and the initial agents’ states can be kept private if each agent has at least one “legitimate” neighbor. The algorithm, involving only basic matrix operations, is computationally more efficient than cryptography-based approaches and can be implemented in a fully distributed manner without relying on a third party. Numerical simulation is provided to illustrate the effectiveness of the proposed algorithm.

Keywords: Privacy-preserving average consensus, matrix-weighted networks, vector-valued state, positive semi-definite coupling, dynamic edge weights, agent-state lifting.

1. Introduction

Distributed average consensus plays a crucial role in distributed estimation, control, and optimization of networked multi-agent systems Mesbahi and Egerstedt [18], Kia et al. [11]. Specifically, a set of agents iteratively exchange information with their neighbors over a communication graph such that, under certain conditions on the inter-agent couplings, all the agents’ states converge to the average of their initial states. Traditional average consensus algorithms admit the exchange of explicit states between neighboring agents, which may disclose initial states of the agents to malicious ones. In certain average consensus applications, however, an agent’s initial state contains sensitive information and must be kept confidential. One example is opinion consensus in social networks, where a set of agents holding respective opinions leverage consensus algorithms to reach a common opinion but are unwilling to reveal their own opinions as these opinions may reflect sensitive personal preferences Mo and Murray [19].

Another example is dynamic formation control, where a set of mobile agents employ consensus algorithms to agree on, say, the time-varying geometric center of their formation Porfiri et al. [26]. However, these agents may not want to expose their own locations during this process, as that could reveal their habits, interests, activities, and relationships Krontiris et al. [13]. This necessitates new class of algorithms that can simultaneously achieve average consensus while protecting the privacy of individual agents’ initial states.

Literature Review

The problem of privacy-preserving average consensus (PPAC) has been extensively examined in recent literature. Perturbing the original data prior to sharing it on the network is a typical option for privacy-enhancing coordination Mo and Murray [19], He et al. [17], Manitara and Hadjicostis [12]. Here, differential privacy is a formal framework for the design and analysis of privacy-preserving algorithms by persistently adding random noises into agents’ states such that, with high probability, the initial state of an agent cannot be inferred by adversarial agents Zhao et al. [20], Nozari et al. [17], He et al. [8]. However, there is a fundamental trade-off between privacy and accuracy.
of the consensus value as persistent random noises are incorporated He et al. [8], Cortés et al. [3], Lu and Zhu [15]. Notably, some of the works have proposed utilizing decaying or correlated noises to obfuscate exchanged states such that individual agents’ initial states cannot be uniquely determined while average consensus can be maintained Manitara and Hadjicostis [17], Gade and Vaidya [5], He et al. [7]. However, there is a risk of revealing an interval of an agent’s initial state, corresponding to the magnitude of the added noises He et al. [7]. Cryptography-based approaches are also employed for privacy-preserving algorithm design. In fact, the PPAC problem is closely related to secure multi-party computation, where all participants compute a joint function while preserving their respective inputs private Lu and Zhu [15], Pailler [21], Lu and Zhu [16], Legendijk et al. [14], Kogiso and Fujita [12]. In this line of work, additive properties of homomorphic encryption are employed to encrypt exchanged states such that the desired computations can be carried out on encrypted states, thus generating an encrypted result that, when decrypted, matches the result of computations performed on original states Ruan et al. [28]. Nevertheless, cryptography-based approaches may suffer from communication and computation overhead caused by the encryption process; moreover a centralized authority is necessary to carry out aggregation over encrypted data Lu and Zhu [15], Chong et al. [2]. In Wang [33], a state-decomposition-based protocol has been proposed where the state of a node is randomly decomposed into two subsates such that the mean of these states remains the same and only one of the substates is revealed to neighboring nodes. Privacy-preserving problems for continuous-time dynamical systems have also been examined recently by Altafini [1], Xiong and Li [34].

To the best of our knowledge, most existing privacy-preserving average consensus algorithms are specifically designed for scalar-valued agent states; however, vector-valued information exchange is ubiquitous in multi-agent networks. For instance, vector-valued local estimates of the optimal solution are exchanged amongst neighboring agents for multi-agent optimization Yang et al. [35]; point vectors of neighboring agents in a global reference frame are exchanged to construct relative bearing vector in the distributed control law for bearing-based multi-agent formation Zhao and Zelazo [37], Trinh et al. [30].

Inspired by these recent developments in matrix-weighted networks, where the couplings amongst vector-valued agent states are characterized by square matrices Trinh et al. [30], Tuna [31], Pan et al. [23], Wang et al. [32], Pan et al. [25], [22], [24], this paper proposes a novel distributed vector-valued PPAC algorithm by employing matrix-valued state coupling amongst the agents. In this view, the state coupling of neighboring agents in traditional scalar-weighted networks can be regarded as a $d$-order identity matrix where $d$ refers to the dimension of the agents’ states. We show that vector-valued PPAC can in fact be transformed into a properly constructed matrix-weighted networks.

Contributions. In this paper, a novel distributed vector-valued PPAC algorithm, based on matrix-valued state coupling, is proposed. The idea of the algorithm is to utilize properly constructed, low-rank, semi-definite coupling matrices, together with agent-state lifting (introducing virtual states), to conceal the vector-valued agent states, and in the meantime, employ a periodic edge weight switching mechanism to guarantee that the multi-agent network asymptotically converges to the exact average consensus. We show that the exact average consensus can be guaranteed and the initial agents’ states can be preserved by the proposed algorithm if each agent has at least one “legitimate” neighbor. Since our algorithm only involves basic matrix multiplication, it is more computationally efficient than cryptography-based algorithms. Moreover, the algorithm can be implemented in a distributed manner without the involvement of a third party.

The results in this paper have the following immediate implications. Firstly, the vector-valued PPAC framework can be immediately applied in distributed control and optimization algorithms with privacy-preservation concerns Dibaji et al. [4], Kia et al. [11]. Secondly, we present general results on the average asymptotic consensus problems on time-varying discrete-time matrix-weighted networks, which is of independent interest in distributed control and optimization Trinh et al. [24], Wang et al. [32], Tuna [31]. Moreover, the proposed algorithm can be applied to scalar-valued PPAC problem, where the virtual states can serve as an effective means of privacy-preservation.

The remainder of the paper is organized as follows. Preliminaries and problem formulation are presented in §2 where notation and a brief introduction to the matrix-weighted network are introduced. Vector-valued PPAC algorithm, based on matrix-valued inter-agent state coupling, is proposed in §3 followed by the analysis of average consensus and privacy-preserving performance in §4 and §5 respectively. Simulation results are presented in §6 we provide concluding remarks in §7.

2. Notation and Problem Formulation

We first introduce the notation. Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}_+$ be the set of real numbers, natural numbers and positive integers, respectively. For $n \in \mathbb{Z}_+$, denote $n = \{1, 2, \ldots, n\}$. We use $M \succ 0$ (respectively, $M \succeq 0$) to denote that a symmetric matrix $M$ is positive definite (respectively, positive semi-definite). The null space and range space of a matrix $M$ is denoted by null$(M)$ and range$(M)$, respectively. Let $I_n, 0_{n \times n}$ and $J_n$ designate the $n$-dimensional column vector whose components are all 1’s, the $n \times n$ matrix whose components are all 0’s, and the $n \times n$ identity matrix, respectively. The eigenspace of $M \in \mathbb{R}^{n \times n}$ corresponding to eigenvalue $\lambda \in \mathbb{R}$ is denoted by $\mathbb{E}_\lambda = \{v \in \mathbb{R}^n | Mv = \lambda v\}$. We use $a \mod b$ to refer to the remainder of the Euclidean division where $a \in \mathbb{Z}_+$ is the dividend and $b \in \mathbb{Z}_+$ is the divisor. Let $\delta(x) : \mathbb{R} \to \{0, 1\}$ denote the sign function such that $\delta(x) = 1$ for $x > 0$ and $\delta(x) = 0$ for $x \leq 0$. For a vector $x \in \mathbb{R}^n$ and $i_1 < i_2 \in \mathbb{Z}_+$,
we write $x^{[i_1:i_2]}$ to denote a $\mathbb{R}^{(i_2-i_1+1)}$ vector formed by entries of $x$ in sequence from $i_1$ to $i_2$.

Consider a multi-agent system consisting of $n > 1$ ($n \in \mathbb{Z}_+$) agents whose interaction network is characterized by a communication graph $G = (V,E,A)$. The node and edge sets of $G$ are denoted by $V = \{1,2,\ldots,n\}$ and $E \subseteq V \times V$, respectively. A path in $G$ is a sequence of edges of the form $(i_1,i_2),(i_2,i_3),\ldots,(i_{p-1},i_p)$, where nodes $i_1,i_2,\ldots,i_p \in V$ are distinct; in this case we say that node $i_p$ is reachable from $i_1$. A graph $G$ is connected if any two distinct nodes in $G$ are reachable from each other. A tree is a connected graph with $n \geq 2$ nodes and $n-1$ edges where $n \in \mathbb{Z}_+$. For matrix-weighted switching networks, we adopt the following terminology. An edge $(i,j) \in E$ is positive definite (semi-definite) if $A_{ij}$ is positive definite (semi-definite). A positive tree of $G$ is a tree such that every edge in this tree is positive definite. A positive spanning tree of $G$ is a positive tree containing all nodes in $G$.

### 2.1. Matrix-weighted Average Consensus

In a matrix-weighted network $G = (V,E,A)$, each edge $(i,j) \in E$ is assigned a matrix-valued weight encoded by a matrix $A_{ij} \in \mathbb{R}^{d \times d}$ such that $A_{ij} \neq 0_{d \times d}$ if $(i,j) \in E$, and $A_{ij} = 0_{d \times d}$ otherwise. We shall assume that all non-zero matrix-valued weights are either positive definite or positive semi-definite unless otherwise stated. Thereby, the matrix-valued adjacency matrix $A = (A_{ij}) \in \mathbb{R}^{d \times d}$ is a block matrix such that the block located in its $i$-th row and the $j$-th column is $A_{ij}$. Each agent $i$ has a vector-valued initial state $x_i(0) = (x_{i1}(0),\ldots,x_{id}(0))^\top \in \mathbb{R}^d$. The agents aim to agree on the average of their initial states. To this end, each agent $i$ updates its state by the protocol,

$$x_i(k+1) = x_i(k) + \sigma \sum_{j \in \mathcal{N}_i} A_{ij}(x_j(k) - x_i(k)), \quad (1)$$

where $\sigma > 0$, $k \in \mathbb{N}$ is the step index, and 

$$\mathcal{N}_i = \{ j \in V \mid (i,j) \in E \}$$

denotes the neighbor set of agent $i$. The matrix-valued Laplacian matrix of $G$ is defined as $L = D - A$, where $D = \text{diag}\{D_1,\ldots,D_n\} \in \mathbb{R}^{d \times d}$ and $D_i = \sum_{j \in \mathcal{N}_i} A_{ij} \in \mathbb{R}^{d \times d}$. Subsequently, the overall dynamics of (1) can be compactly written as,

$$x(k+1) = (I_d - \sigma L)x(k), \quad (2)$$

where $x(k) = (x_1(k)^\top,\ldots,x_n(k)^\top)^\top \in \mathbb{R}^{dn}$.

The definition of asymptotic average consensus is formalized next.

**Definition 1.** The matrix-weighted network (2) achieves asymptotic average consensus if

$$\lim_{k \to \infty} x(k) = \mathbb{I}_n \otimes \text{Avg}(x(0)), \quad (3)$$

where $\text{Avg}(x(0)) = \frac{1}{n} \sum_{i=1}^n x_i(0)$ denotes the average value of all the $n$ agents’ initial states.

The following lemmas characterizes the null space of matrix-weighted Laplacian which plays a central role in achieving asymptotic average consensus under (2).

**Lemma 1.** Pan et al. [22], Trinh et al. [23] Let $G = (V,E,A)$ be a matrix-weighted network. Then the associated matrix-valued Laplacian matrix $L$ of $G$ is positive semi-definite and the structure of its null space can be characterized by $\text{null}(L) = \text{span}\{R,H\}$, where

$$R = \text{range}\{I_n \otimes I_d\}, \quad (4)$$

and

$$H = \{v = (v_1^\top,v_2^\top,\ldots,v_n^\top)^\top \in \mathbb{R}^{dn} \mid (v_i - v_j) \in \text{null}(A_{ij}), (i,j) \in E\}. \quad (5)$$

**Lemma 2.** Trinh et al. [22], Pan et al. [23] Let $G = (V,E,A)$ be a matrix-weighted network. If $G$ has a positive spanning tree, then the matrix-valued Laplacian $L$ satisfies $\text{null}(L) = R$.

A notable feature of continuous-time multi-agent networks on scalar-weighted networks is it that network connectivity can translate into achieving consensus, which is not valid for matrix-weighted networks. In this case, it is therefore intricate to obtain a purely graph-theoretic condition for achieving consensus without any assumptions on the matrix-valued edge weights, the null space of which play a paramount role in achieving consensus on continuous-time matrix-weighted networks Trinh et al. [29], Pan et al. [23].

Specifically, according to Lemma 1 the null space of a matrix-valued Laplacian is not only determined by the network connectivity, but also by the null space of each matrix-valued edge weight. In this case, the null space of the matrix-valued Laplacian associated with a connected matrix-weighted network may not be $R$, implying that the discrete-time multi-agent system (2) on connected matrix-weighted networks may also achieve cluster consensus due to the complexity of $\text{null}(L)$, regardless of the selection of $\sigma$. We have the similar observations; see Example 2.

### 2.2. Consensus with Privacy

For preserving privacy, it is not preferable to transmit agents’ states in (1). Instead, by a rearrangement of (1),

$$x_i(k+1) = x_i(k) + \sigma \sum_{j \in \mathcal{N}_i} (A_{ij}x_j(k) - A_{ij}x_i(k)), \quad (6)$$

one can immediately see that each $j \in \mathcal{N}_i$ only needs to send

$$y_{j\rightarrow i}(k) = A_{ij}x_j(k) \quad (7)$$

2A positive spanning tree in a matrix-weighted network $G = (V,E,A)$ is a spanning tree $T$ of $G$ such that all edge weights in $T$ are positive definite matrices.
to $i$ for its state update. Therefore, the state $x_j(k)$ can be concealed by a properly designed weight matrix $A_{ij}$, yielding the following protocol,

$$x_i(k + 1) = x_i(k) + \sigma \sum_{j \in N_i} (y_{ji}(k) - A_{ij}x_i(k)). \tag{8}$$

Note that the weight matrix $A_{ij}$ cannot be positive definite, otherwise, agent $i$ can immediately infer the state of agent $j$ by $A_{ij}^{-1}(A_{ij}x_i(k)) = x_j(k)$. On the other hand, according to Lemma 2, the design of each $A_{ij}$ also has to guarantee the existence of positive spanning tree in $G$ for average consensus. The aforementioned two criteria for edge weight matrices can conflict with each other. This motivates us to consider dynamic matrix-valued edge weight mechanism, examined in §3.

2.3. Attacker Model and Privacy Definition

This paper is concerned with the honest-but-curious (or semi-honest) attacker model, i.e., an adversarial agent follows the designed algorithm but attempts to use its received data to infer other agents’ private data (Hazay and Lindell [6]). We shall refer to this type of adversarial agent as an honest-but-curious agent. This attacker model has been widely used in PP AC Ruan et al. [28], He et al. [7], Huang et al. [10]. Moreover, a legitimate agent is an agent who follows the consensus protocol faithfully by

$$\sum_{j \in N_i} \sigma = 1,$$

and

$$\sum_{j \in N_i} \sigma y_{ji}(k) = 0.$$

Definition 2 also implies that even a bound of $\sigma$ is indistinguishable from $d$-dimensional virtual state for each agent. Wang [33], Rezaazadeh and Kia [27].

Definition 2. Let $M$ be a privacy-preserving algorithm $\theta$. We say the privacy of the initial value of agent $b \in V_B$, denoted by $x_b(0)$, is preserved if for any $x_b(0) \in \mathbb{R}^d$, there exist all the observations of agents $S \subseteq V \setminus \{b\}$ can gain through the execution of $M$ over the input $\{x_i(0), I_i\}_{i \in V}$. We are now ready to formalize the notion of privacy Wang [33], Rezaazadeh and Kia [27].

2.4. Objectives

This paper aims to develop a vector-valued PPAC algorithm such that the following properties are guaranteed simultaneously:

- Exact average consensus: For each agent $i \in V$, its state asymptotically converges to $\text{Avg}(x(0))$.
- Privacy: The privacy of the initial states of legitimate agents is protected in the sense of Definition 2.

3. Algorithm Design

This work intends to design a novel vector-valued PPAC algorithm by employing the framework of matrix-weighted networks (MWNs).

We make the following assumption on the communication networks throughout this paper.

Assumption 1: The communication graph $G$ is undirected and connected.

In the following, we shall refer to our algorithm as the MWN-PPAC algorithm; let us first motivate its structure.

We denote by $A = \{A_{ij} \mid (i, j) \in E\}$ the profile of matrix-valued edge weights of $G$. Then profile $A$ needs to ensure the objectives in §2.4. Essentially, our MWN-PPAC algorithm is to achieve a fundamental tradeoff between privacy preservation and average consensus in designing profile $A$. In fact, the positive semi-definite edge weight matrices are plausible for privacy-preservation purposes but not for the average consensus, since according to Lemma 2 positive definite edge weight matrices are more preferable to guarantee the existence of a positive spanning tree in matrix-weight networks.

We shall now proceed to achieve the aforementioned design objectives, that of privacy and consensus, by agent state lifting and dynamic matrix-valued weights.

3.1. Agent State Lifting

First, in designing profile $A$, the rank of each $A_{ij}$ needs to be as low as possible for privacy preservation. Then, less information with respect to the correlation of entries in $x_i(k)$ can be inferred from $y_{j \rightarrow i}(k)$. Note that designing low-rank matrix-weighted edge weights is the most preferable for this purpose. However, although one cannot infer each element in $x_i(0)$, a linear correlation amongst the entries in $x_i(0)$ can be deduced. To this end, the algorithm is designed to first lift the individual state space by introducing a $d'$-dimensional virtual state for each agent. The idea of this design is essential to sacrifice the possible disclosure of linear correlation of entries in virtual states instead of that in real agents’ states. We shall technically demonstrate this point in the proof of Theorem 4. For a glimpse, here we consider the following example.
Example 1. Consider a connected matrix-weighted network \( \mathcal{G} \) with \( n = 5 \) agents in Figure 1. Choose the following semi-definite matrix as edge weight at time \( k = 0 \), namely,

\[
A_{ij} = \begin{pmatrix}
\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\
-\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
\frac{1}{2} & -\frac{1}{2} & 1
\end{pmatrix}, \quad (i,j) \in \mathcal{E}.
\]

Then the following linear correlation amongst the entries in \( x_j(0) \) can be inferred by agent \( i \),

\[
A_{ij} x_j(0) = \frac{1}{4} x_{j1}(0) - \frac{1}{4} x_{j2}(0) + \frac{1}{2} x_{j3}(0)
= y_{j \rightarrow i}(0).
\]

Therefore, the agent state lifting procedure is employed in our design to avoid disclosing the linear correlations amongst the entries of real agents’ states; we summarize this below.

Step 1: Agent State Lifting.
Each agent \( i \in \mathcal{V} \) lifts the state space by introducing a \( d' \)-dimensional virtual state,

\[
x'_i = (x'_{i1}(k), \ldots, x'_{id}(k))^\top \in \mathbb{R}^{d'}.
\]

That is, the state of each agent \( i \in \mathcal{V} \) is lifted from

\[
x_i(k) = (x_{i1}(k), \ldots, x_{id}(k))^\top \in \mathbb{R}^d
\]

into

\[
\tilde{x}_i(k) = (\tilde{x}_{i1}(k), \tilde{x}_{i2}(k), \ldots, \tilde{x}_{i,d+d'}(k))^\top \in \mathbb{R}^{d+d'},
\]

where

\[
\tilde{x}_{i,l}(k) = x'_{il}(k) \quad \text{for} \quad l \in \mathbb{N} \quad \text{and} \quad \tilde{x}_{i,d+i_l}(k) = x_{il}(k) \quad \text{for} \quad l_2 \in \mathbb{N}.
\]

Remark 3. The virtual state mechanism not only renders the vector-valued PPAC algorithm proposed in this paper applicable to scalar-weighted networks, but provides freedom to protect the correlation of entries in the initial state of each agent from being disclosed.

For simplicity, in the following discussion, we shall use the symbol \( \tilde{x}_i(k) \) to denote the lifted agent \( \tilde{x}_i(k) \). Therefore, the virtual state and real state of an agent \( i \) in the lifted agent state vector \( \tilde{x}_i(k) \) can be represented by \( x'_{i1}(k) \) and \( x'_{i,d'+1}(k) \), respectively.

3.2. Dynamic Matrix-valued Edge Weight

In order to achieve a tradeoff between privacy-preservation and average consensus, a periodic switching mechanism for matrix-valued edge weights is employed, where the period satisfies \( T = d + d' - 1 \). Here, the edge weight matrix will become time-dependent and denoted as \( A_{ij}(k) \). The purpose of this design is to guarantee average consensus of the multi-agent system [2] and the matrix-valued edge weights will be given subsequently. The necessity of introducing switching mechanism for matrix-valued edge weight is that the average consensus cannot be guaranteed on a connected time-invariant matrix-weighted network with positive semi-definite matrix-valued edge weights, preferable for privacy-preservation purposes. We provide the following example to illustrate this point.

Example 2. Consider the matrix-weighted network \( \mathcal{G} \) shown in Figure 1. Choose the following positive semi-definite matrix

\[
A_{ij} = \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

as edge weight for all \( (i,j) \in \mathcal{E} \) and \( \sigma = 0.1 \) that satisfies Lemma 3. Although the network in Figure 1 is connected, the multi-agent system on the connected network in Figure 1 with the matrix-valued edge weight as in (14).
which case average consensus of multi-agent [2] cannot be achieved Pan et al. [23].

We shall proceed to present the second ingredient regarding the dynamic matrix-valued weight design.

Step 2: Dynamic Matrix-valued Edge Weight Design.

To design dynamic matrix-valued edge weights, we shall employ the following auxiliary vector set, which is distributed to each agent at the initialization of the algorithm.

Choose $d' \geq 3$ and design orthogonal vector set

$$\mathcal{V} = \left\{ v_i \in \mathbb{R}^{d+d'} \right\}_{i=1}^{d+d'}$$

satisfying,

1) all vectors in $\mathcal{V}$ are mutually perpendicular,
2) the number of non-zero elements in each $v_i \in \mathbb{R}^{d+d'}$ is not less than 2,
3) the number of non-zero entries in $v_1$ is less than $d'$,
4) all entries in $v_{d+d'}$ are non-zero.

Based on the vector set $\mathcal{V}$, we now design the following mechanism for the periodic switching of matrix-valued edge weights.

For $k = 0$, if $(i,j) \in \mathcal{E}$ and $i \in V_B$ and $j \in \mathcal{N}_i \cap \mathcal{N}_j$, choose

$$A_{ij}(0) = A_{ji}(0) = \frac{1}{\alpha_{ij}}v_i v_j^\top + \frac{1}{\beta_{ij}}v_{d+d'} v_{d+d'}^\top,$$

where $\alpha_{ij} > 0$ and $\beta_{ij} > 0$; else choose $A_{ij}(0) \in \mathbb{R}^{d \times d}$ arbitrarily.

For $k \in \mathbb{Z}_+$, if $(i,j) \in \mathcal{E}$, choose

$$A_{ij}(k) = A_{ji}(k) = \gamma_{ij}(k) v_i v_j^\top + \zeta_{ij}(k) v_{d+d'} v_{d+d'}^\top,$$

where

$$\gamma(k) = \begin{cases} k \mod d', & k \mod d' \neq 0 \\ d', & k \mod d' = 0 \end{cases},$$

$$\frac{1}{4(n-1)\sigma} > \gamma_{ij}(k) > 0,$$

$$\frac{1}{4(n-1)\sigma} > \zeta_{ij}(k) > 0,$$

and

$$d' = d + d' - 1.$$

The above procedures can be summarized in the Algorithm 1.

**Algorithm 1 MWN-PPAC algorithm**

**Initialization:**
1. Set $k = 0$ and choose $d' \geq 3$.
2. Each agent $i \in \mathcal{V}$ constructs $\mathcal{G}$ in (15) and lifts its state.
3. for $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$ do
4. if $i \in V_B$ and $j \in \mathcal{N}_i \cap \mathcal{N}_A$ then
5. Construct $A_{ji}(0)$ by (16).
6. else
7. Construct arbitrary $A_{ji}(0) \in \mathbb{R}^{d \times d}$.
8. end if
9. Agent $i$ sends $A_{ji}(0)v_i(0)$ to $j$.
10. end for
11. Set $k = k + 1$

**Loop:**
12. while $0 < k < \max\text{\_\_iteration\_number}$ do
13. for $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$ do
14. Construct $A_{ij}(k)$ by (17).
15. Agent $i$ sends $A_{ij}(k)v_i(k)$ to $j$.
16. end for
17. Agent $i$ updates $x_i(k)$ using protocol [1].
18. Set $k = k + 1$.
19. end while

Choose $d' = 3$ and one can construct the orthogonal vector set by choosing

$$v_i = \left( \frac{1}{i}, \frac{1}{i}, \ldots, \frac{1}{i}, \frac{1}{i}, \frac{1}{i}, \ldots, 0, \ldots, 0 \right)^\top,$$

for all $i \in d + 3$ where $g_i = \max\{d + 2 - i, 0\}$ and $f_i = \delta(d + 3 - i)$. Specifically, consider a matrix-weighted network $G$ consisting $n = 5$ agents where the state dimension of each agent is $d = 3$. Then, by using (21), one can obtain the elements in $\mathcal{V} = \{ v_i \in \mathbb{R}^6 \}_{i=1}^{d+d'}$ as follows,

$$v_1 = (1, 1, 0, 0, 0, 0)^\top,$$

$$v_2 = \left( \frac{1}{2}, -\frac{1}{2}, 1, 0, 0, 0 \right)^\top,$$

$$v_3 = \left( \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1, 0, 0 \right)^\top,$$

$$v_4 = \left( \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 1, 0 \right)^\top,$$

$$v_5 = \left( \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, 1 \right)^\top,$$

and

$$v_6 = \left( \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6} \right)^\top.$$
mechanism is plausible since it renders Algorithm 1 efficient in terms of both communication and computation.

**Remark 6.** According to Algorithm 1, the observations of an agent $i \in V$ satisfies
\[
\mathcal{O}_i^M(\{x_i(0), T_i\}_{i \in V}) = \{\text{form of Algorithm 1}\} \cup \mathcal{P} \cup \mathcal{S}',
\]
where
\[
\mathcal{P} = \{\sigma, \mathcal{U}, d, d'\}
\]
and
\[
\mathcal{S}' = \{x_i(k), N_i, A_{ij}(k), A_{ji}(k), y_{j \rightarrow i}(k)\}_{j \in N_i, k \in \mathbb{N}}.
\]

In the following, we shall first analyze the asymptotic convergence of the Algorithm 1 towards average consensus.

### 4. Average Consensus Analysis

In this section, we shall show the asymptotic average consensus of matrix-weighted network (2) can be achieved under Algorithm 1. Since the matrix-valued edge weights are dynamically switching, the analysis is equivalent to the average consensus problem of multi-agent system (2) on matrix-weighted switching networks; the dynamic edge weights constructed in Algorithm 1 are just a special case of our convergence analysis for average consensus. To keep the following presentation concise, we shall occasionally write $A_k$ and $L_k$ for matrix-valued weight matrix $A(k)$ and matrix-valued Laplacian $L(k)$, respectively.

The main result in this section is stated in the following theorem.

**Theorem 1.** The matrix-weighted network (2) using Algorithm 1 achieves asymptotic average consensus, namely,
\[
\lim_{k \to \infty} x(k) = I_n \otimes \text{Avg}(x(0)).
\]

In order to prove Theorem 1, we need to establish the following auxiliary results.

First, we note that the structure of the null space of a matrix-valued Laplacian matrix for the time-invariant network essentially determines the steady-state of the multi-agent system (2); refer to equations (4) and (5). According to Lemma 1 and Lemma 2, it is reasonable to examine the null space of the matrix-weighted Laplacian corresponding to a series of matrix-weighted networks. For $k', k'' \in \mathbb{Z}_+$ and $k' < k''$, the union of graphs $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}, A_k)$ over a time interval $[k', k''] \subset [0, \infty)$ is,
\[
\bigcup_{k=k'}^{k''-1} \mathcal{G}(k) = \left(\mathcal{V}, \mathcal{E}, \sum_{k=k'}^{k''-1} A_k\right).
\]

The following result reveals the connection between the null space of matrix-valued Laplacian corresponding to the union of a series of matrix-weighted networks and the intersection of the null space of matrix-valued Laplacians corresponding to each separate matrix-weighted network.

**Theorem 2.** Let $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}, A_k)$ be a matrix-weighted switching network with matrix-weighted Laplacian $L_k$. Then
\[
\text{null} \left( \sum_{i=1}^{k' - k'} L_{k'+i-1} \right) = \mathcal{R}
\]
if and only if
\[
\bigcap_{i \in k'' - k'} \text{null} (L_{k'+i-1}) = \mathcal{R},
\]
where $k' < k'' \in \mathbb{Z}_+$.

**Proof.** (Necessity) From the definition of matrix-valued Laplacian matrix, one has
\[
\mathcal{R} \subseteq \bigcap_{i \in k'' - k'} \text{null} (L_{k'+i-1}).
\]
Assume that $\bigcap_{i \in k'' - k'} \text{null} (L_{k'+i-1}) \neq \mathcal{R}$, then there exists an $\eta \notin \mathcal{R}$ such that $L_{k'+i-1} \eta = 0$ for all $i \in k'' - k'$, which would imply,
\[
\left( \sum_{i=1}^{k' - k'} L_{k'+i-1} \right) \eta = \sum_{i=1}^{k' - k'} (L_{k'+i-1} \eta) = 0,
\]
contradicting the fact that
\[
\text{null} \left( \sum_{i=1}^{k' - k'} L_{k'+i-1} \right) = \mathcal{R}.
\]
Therefore,
\[
\bigcap_{i \in k'' - k'} \text{null} (L_{k'+i-1}) = \mathcal{R}.
\]

(Sufficiency) Assume that
\[
\text{null} \left( \sum_{i=1}^{k' - k'} L_{k'+i-1} \right) \neq \mathcal{R}.
\]
Then there exists $\eta \notin \mathcal{R}$ such that
\[
\left( \sum_{i=1}^{k' - k'} L_{k'+i-1} \right) \eta = 0.
\]
Hence,
\[
\eta^\top \left( \sum_{i=1}^{k' - k'} L_{k'+i-1} \right) \eta = 0
\]
implying that
\[
\eta^\top \left( \sum_{i=1}^{k' - k'} L_{k'+i-1} \right) \eta = \sum_{i=1}^{k' - k'} \eta^\top L_{k'+i-1} \eta = 0.
\]
We note that $L_{k' + 1}$ is positive semi-definite for all $i \in k'' - k'$, and $\eta \top L_{k' + 1} \eta = 0$, one has $L_{k' + 1} \eta = 0$; this, on the other hand, contradicts the premise that

$$\bigcap_{i \in k'' - k'} \text{null} (L_{k' + 1}) = \mathcal{R}.$$  

Thus

$$\text{null} \left( \sum_{i=1}^{k'' - k'} L_{k' + 1} \right) = \mathcal{R}.$$

\[ \square \]

Remark 7. Remarkably, Theorem 2 establishes a quantitative connection between the null space of matrix-valued Laplacian of a set of matrix-weighted networks and that of their union. This is eventually a result that is valid for general switching matrix-weighted networks whose edge weight matrices can be either positive definite or positive semi-definite. Essentially, the null space of matrix-valued Laplacian of a set of matrix-weighted networks is equal to that of their union, i.e., $\text{null} \left( \sum_{i=1}^{k'' - k'} L_{k' + 1} \right) = \bigcap_{i \in k'' - k'} \text{null} (L_{k' + 1})$.

We now proceed to prove the asymptotic average consensus of the matrix-weighted network 2 under Algorithm 1. The idea is to construct the error vector

$$\omega(k) = x(k) - 1_n \otimes (\text{Avg}(x(0))), \tag{31}$$

and then show that $\omega(k)$ converges to the origin as $k$ goes to infinity.

To this end, we first need to employ the state transition matrix from time $k'$ to time $k''$, denoted by

$$\Phi(k'', k') = \prod_{i=1}^{k'' - k'} (I - \sigma L_{k' + 1}).$$

Then $x(k'') = \Phi(k'', k') x(k')$, where $k' < k'' \in \mathbb{Z}_+$. 

Note that the matrix-valued Laplacian matrix $L_k$ has at least $d$ zero eigenvalues. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{dn}$ be the eigenvalues of $L_k$. Then we have

$$0 = \lambda_1 = \cdots = \lambda_d \leq \lambda_{d+1} \leq \ldots \leq \lambda_{dn}.$$

Let $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_{dn}$ denote the eigenvalues of $I - \sigma L_k$; then $\beta_i (I - \sigma L_k) = 1 - \sigma \lambda_i (L_k)$, namely,

$$1 = \beta_1 = \cdots = \beta_d \geq \beta_{d+1} \geq \ldots \geq \beta_{dn}.$$ 

In the meantime, the eigenvector corresponding to the eigenvalue $\beta_i (I - \sigma L_k)$ is equal to that associated with $\lambda_i (L_k)$.

Note that $\Phi(k'', k') \top \Phi(k'', k')$ has at least $d$ eigenvalues at 1. Let $\mu_j$ be the eigenvalues of $\Phi(t_{k''}, t_{k'}) \top \Phi(t_{k''}, t_{k'})$, where $j \in \mathbb{Z}_+$ such that $\mu_1 = \cdots = \mu_d = 1$ and $\mu_{d+1} \geq \mu_{d+2} \geq \cdots \geq \mu_{dn}$.

We now present the following lemma to establish the connection between the null space of the matrix

$$\sum_{i=1}^{k'' - k'} L_{k' + 1},$$

and the specific eigenvalue $\mu_{d+1}$ of $\Phi(k'', k') \top \Phi(k'', k')$ which turns out to be crucial for determining the convergence of error vector $\omega(k)$ in (31).

Theorem 3. Let $G_k = (\mathcal{V}, \mathcal{E}, A_k)$ be a matrix-weighted switching network with matrix-weighted Laplacian $L_k$. Then

$$\text{null} \left( \sum_{i=1}^{k'' - k'} L_{k' + 1} \right) = \mathcal{R}$$

if and only if

$$\mu_{d+1} (\Phi(k'', k') \top \Phi(k'', k')) < 1,$$

where $k' < k'' \in \mathbb{Z}_+$.

Proof. (Sufficiency) Assume that $\text{null} \left( \sum_{i=1}^{k'' - k'} L_{k' + 1} \right) \neq \mathcal{R}$; then according to Rayleigh theorem (Horn and Johnson), there exists an $\eta \notin \mathcal{R}$ such that $L_{k' + 1} \eta = 0$ for all $i \in k'' - k'$. Thus one can obtain $(I - \sigma L_{k' + 1}) \eta = \eta$ for all $i \in k'' - k'$ and $\Phi(k'', k') \eta = \eta$. Using Rayleigh theorem again, one has

$$\mu_{d+1} (\Phi(k'', k') \top \Phi(k'', k')) \geq \frac{\eta \top \Phi(k'', k') \top \Phi(k'', k') \eta}{\eta \top \eta} = 1,$$

contradicting,

$$\mu_{d+1} (\Phi(k'', k') \top \Phi(k'', k')) < 1.$$ 

Therefore $\text{null} \left( \sum_{i=1}^{k'' - k'} L_{k' + 1} \right) = \mathcal{R}$ holds.

(Necessity) Assume that $\mu_{d+1} (\Phi(k'', k') \top \Phi(k'', k')) \geq 1$. Again, by Rayleigh theorem, there exists a $\eta \notin \mathcal{R}$ and $\eta \neq 0$ such that

$$\mu_{d+1} (\Phi(k'', k') \top \Phi(k'', k')) = \frac{\eta \top \Phi(k'', k') \top \Phi(k'', k') \eta}{\eta \top \eta} \geq 1.$$ 

Thus

$$\| \eta \| \leq \| \Phi(k'', k') \eta \|.$$ 

Let $\eta_k = \eta$ and $\eta_k = (I - \sigma L_{k' + 1}) \eta_{k' + 1}$ for $i \in k'' - k'$. Due to the fact $\beta_j (I - \sigma L_{k' + 1}) \leq 1$ and $\lambda_j (L_{k' + 1}) \leq 1$ (according to Lemma 4 in [1]) for $j \in \mathbb{Z}_+$ and $i \in k'' - k'$, then

$$\| (I - \sigma L_{k' + 1}) \eta_{k' + 1} \| \leq \| \eta_k \|,$$

which implies that,

$$\| \eta \| \leq \| \Phi(k'', k') \eta \|$$

$$\| \eta_k \| \leq \| \eta_{k' + 1} \| \leq \ldots \leq \| \eta_k \|$$

$$\| \eta \|.$$
Hence, 
\[ \| (I - \sigma L_{k''-k'}) \eta_{k''-k'} \| = \| \eta_{k''-k'} \|, \]
for any \( i \in k'' - k' \). Then 
\[ \eta_{k''-k'}^i (I - \sigma L_{k''-k'}) (I - \sigma L_{k''-k'}) \eta_{k''-k'} = \eta_{k''-k'} \]
By Rayleigh theorem, 
\[ (I - \sigma L_{k''-k'}) \eta_{k''-k'} = 0 \]
and thereby \( L_{k''-k'} \eta_{k''-k'} = 0 \). Thus 
\[ \eta_{k''-k'} \in \text{null}(L_{k''-k'}). \]
Since, 
\[ \| \eta_{k''-k'} \| = \| (I - \sigma L_{k''-k'}) \eta_{k''-k'} - \eta_{k''-k'} \| = \| \sigma L_{k''-k'} \eta_{k''-k'} \| \]
one can further obtain \( \eta_{k''-k'} = \eta_{k''-k'} \) for any \( i \in k'' - k' \), which implies that 
\[ \eta \in \bigcap_{i \in k'' - k'} \text{null}(L_{k''-k'}) \]
and 
\[ \text{null} \left( \sum_{i=1}^{k''-k'} L_{k''-k'} \right) \neq \mathcal{R}, \]
which is a contradiction. The proof is thus concluded. \( \square \)

**Remark 8.** Similarly, Theorem 3 is also a rather general result that is valid for switching matrix-weighted networks whose edge weight matrices can be either positive definite or positive semi-definite.

We are now ready to examine the connection between achieving asymptotic average consensus and the dynamic matrix-valued edge weights constructed by Algorithm 1. In Algorithm 1, \( \rho(k) \) is employed to denote the periodic switching signal for matrix-valued edge weights. Then, the matrix-valued weight matrix at time \( k > 0 \) satisfies \( A(k) = A(\rho(k)) \) and the matrix-valued Laplacian also has \( L(k) = L(\rho(k)) \). The set of candidate matrix-valued edge weights is hence, 
\[ A_c = \{ A(1), A(2), \ldots, A(d + d' - 1) \}. \] (32)
We proceed to present the following lemma related to the null space of \( L(\rho(k)) \) and the existence of a positive spanning tree in a periodic switching matrix-weighted network constructed by Algorithm 1.

**Lemma 3.** Let \( L(\rho(k)) \) be the matrix-valued Laplacian corresponding to matrix-valued edge weights constructed in Algorithm 1. Then 
\[ \text{null} \left( \sum_{k=1}^{d+d'-1} L_{\rho(k)} \right) = \mathcal{R}. \] (33)

**Proof.** According to the construction \( \{ A_{ij}(k) | k > 0 \} \), one can see that: for \( k > 0 \), \( \psi_\rho(k) \in \mathcal{G} \) is an eigenvector corresponding to the eigenvalue \( \gamma_{\rho(k)}^1 \) of matrix \( A_{ij}(k) \) and \( \psi_{d+d'} \in \mathcal{G} \) is an eigenvector corresponding to the eigenvalue \( \gamma_{\rho(k)}^{d+d'} \) of matrix \( A_{ij}(k) \) and elements in \( \mathcal{G} \setminus \{ \psi_\rho(k), \psi_{d+d'} \} \) are eigenvectors corresponding to zero eigenvalue of matrix \( A_{ij}(k) \), i.e., \( \text{rank}(A_{ij}(k)) = 2 \) for \( k > 0 \). Consequently, the null spaces of \( A_{ij}(k) \)'s lead to having, 
\[ \bigcap_{k=1}^{d+d'-1} \text{null}(A_{ij}(k)) = \{0\}, \] (34)
for all \( (i, j) \in \mathcal{E} \). Therefore, \( \sum_{k=1}^{d+d'-1} A_{ij}(k) \) is positive definite for all \( (i, j) \in \mathcal{E} \), i.e., the union of \( \mathcal{G}(k) \) over the time interval \( [1, d + d' - 1] \) has a positive spanning tree, according to Lemma 2 \( \text{null} \left( \sum_{k=1}^{d+d'-1} L(\rho(k)) \right) = \mathcal{R}, \) thus completing the proof. \( \square \)

**Lemma 4.** Let \( L(\rho(k)) \) be the matrix-valued Laplacian corresponding to matrix-valued edge weights constructed in Algorithm 1, where \( k \in \mathbb{Z}_+ \). If 
\[ \frac{1}{4(n-1)\sigma} > \gamma_{\rho(k)}^1 > 0, \] (35)
and 
\[ \frac{1}{4(n-1)\sigma} > \gamma_{\rho(k)}^{d+d'} > 0, \] (36)
for any \( (i, j) \in \mathcal{E} \), then \( \lambda_{\max}(L(\rho(k))) < \frac{1}{2} \).

**Proof.** Let 
\[ \Gamma^{\rho(k)} = \begin{bmatrix} \gamma_{\rho(k)}^{1,1} & -\gamma_{\rho(k)}^{1,2} & \cdots & -\gamma_{\rho(k)}^{1,n} \\ -\gamma_{\rho(k)}^{2,1} & \gamma_{\rho(k)}^{2,2} & \cdots & -\gamma_{\rho(k)}^{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{\rho(k)}^{n,1} & -\gamma_{\rho(k)}^{n,2} & \cdots & \gamma_{\rho(k)}^{n,n} \end{bmatrix}, \]
and 
\[ \Pi^{\rho(k)} = \begin{bmatrix} -\gamma_{\rho(k)}^{1,1} & -\gamma_{\rho(k)}^{1,2} & \cdots & -\gamma_{\rho(k)}^{1,n} \\ -\gamma_{\rho(k)}^{2,1} & \gamma_{\rho(k)}^{2,2} & \cdots & -\gamma_{\rho(k)}^{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{\rho(k)}^{n,1} & -\gamma_{\rho(k)}^{n,2} & \cdots & \gamma_{\rho(k)}^{n,n} \end{bmatrix}. \]
Note that matrix-valued Laplacian corresponding to matrix-valued edge weights constructed in Algorithm 1 satisfy
\[
L_{\rho(k)} = \Gamma_{\rho(k)} \otimes \frac{v_{\rho(k)} v_{\rho(k)}^\top}{v_{\rho(k)} v_{\rho(k)}^\top} + \Pi_{\rho(k)} \otimes \frac{v_{d+d'} v_{d+d'}^\top}{v_{d+d'} v_{d+d'}^\top},
\]
therefore,
\[
\lambda_{\text{max}}(L_{\rho(k)}) = \|L_{\rho(k)}\|_2 \\
\leq \left\| \Gamma_{\rho(k)} \otimes \frac{v_{\rho(k)} v_{\rho(k)}^\top}{v_{\rho(k)} v_{\rho(k)}^\top} \right\|_2 + \left\| \Pi_{\rho(k)} \otimes \frac{v_{d+d'} v_{d+d'}^\top}{v_{d+d'} v_{d+d'}^\top} \right\|_2 \\
= \lambda_{\text{max}}(\Gamma_{\rho(k)}) \lambda_{\text{max}} \left( \frac{v_{\rho(k)} v_{\rho(k)}^\top}{v_{\rho(k)} v_{\rho(k)}^\top} \right) + \lambda_{\text{max}}(\Pi_{\rho(k)}) \lambda_{\text{max}} \left( \frac{v_{d+d'} v_{d+d'}^\top}{v_{d+d'} v_{d+d'}^\top} \right).
\]
Note that,
\[
\lambda_{\text{max}} \left( \frac{v_{\rho(k)} v_{\rho(k)}^\top}{v_{\rho(k)} v_{\rho(k)}^\top} \right) = \lambda_{\text{max}} \left( \frac{v_{d+d'} v_{d+d'}^\top}{v_{d+d'} v_{d+d'}^\top} \right) = 1.
\]

Then one has,
\[
\lambda_{\text{max}}(L_{\rho(k)}) \leq \lambda_{\text{max}}(\Gamma_{\rho(k)}) + \lambda_{\text{max}}(\Pi_{\rho(k)}).
\]
According to (35) and (36), one has,
\[
\left\| \Gamma_{\rho(k)} \right\|_1 = \left\| \Gamma_{\rho(k)} \right\|_\infty = \max_{1 \leq i < n} \left( 2 \sum_{i=1}^{n} |\gamma_{i,j}^{(k)}| \right) < \frac{1}{2\sigma},
\]
and
\[
\left\| \Pi_{\rho(k)} \right\|_1 = \left\| \Pi_{\rho(k)} \right\|_\infty = \max_{1 \leq i < n} \left( 2 \sum_{i=1}^{n} |\pi_{i,j}^{(k)}| \right) < \frac{1}{2\sigma}.
\]
Using the facts that
\[
\lambda_{\text{max}}(\Gamma_{\rho(k)}) = \left\| \Gamma_{\rho(k)} \right\|_2 \\
\leq \sqrt{\left\| \Gamma_{\rho(k)} \right\|_1 \left\| \Gamma_{\rho(k)} \right\|_\infty},
\]
and
\[
\lambda_{\text{max}}(\Pi_{\rho(k)}) = \left\| \Pi_{\rho(k)} \right\|_2 \\
\leq \sqrt{\left\| \Pi_{\rho(k)} \right\|_1 \left\| \Pi_{\rho(k)} \right\|_\infty}
\]
yields \( \lambda_{\text{max}}(\Gamma_{\rho(k)}) < \frac{1}{2\sigma} \) and \( \lambda_{\text{max}}(\Pi_{\rho(k)}) < \frac{1}{2\sigma} \). Therefore,
\[
\lambda_{\text{max}}(L_{\rho(k)}) < \frac{1}{\sigma}.
\]

Note that \( \rho(k) \) in Lemma 4 is eventually a periodic function of time index \( k \). Therefore, \( L_{\rho(k)} \) in Lemma 4 is related to the matrix-value Laplacian at time \( k \) and can also be referred to as \( L(k) \).

We are now ready to prove Theorem 1

**Proof.** Let \( \omega(k) = x(k) - x_f \), where \( x_f = I_n \otimes (\text{Avg}(\{0\})) \). Then we have \( \omega(k+1) = (I - \sigma L(k)) \omega(k) \).

Denote
\[
\Phi(d+d',1) = (I - \sigma L(d+d'-1)) \ldots (I - \sigma L(1)),
\]
If \( \omega(1) \in \mathcal{R} \), then \( x(k) = x_f \) for any \( k \geq 1 \); else if \( \omega(1) \notin \mathcal{R} \), observe that,
\[
\mu_{d+d'+1}(\Phi(d+d',1)^\top \Phi(d+d',1)) \\
\geq (\omega(1)^\top \Phi(d+d',1)^\top \Phi(d+d',1) \omega(1)) \\
= \frac{\omega(d+d')^\top \omega(d+d')}{\omega(1)^\top \omega(1)},
\]
which implies,
\[
\omega(d+d')^\top \omega(d+d') \\
\leq \mu_{d+d'+1}(\Phi(d+d',1)^\top \Phi(d+d',1)) \omega(1)^\top \omega(1),
\]
Therefore,
\[
\| \omega(d+d') \| \\
\leq \mu_{d+d'+1}(\Phi(d+d',1)^\top \Phi(d+d',1))^{\frac{1}{2(p+1)}} \omega(1)^\top \omega(1),
\]
if there exists \( k_0 \in \mathbb{N} \) such that \( \omega((k_0+1)(d+d')) \in \mathcal{R} \), then \( x(k) = x_f \) for any \( k \geq (k_0+1)(d+d') \). Otherwise, for any \( p \in \mathbb{Z}_+ \), one has,
\[
\| \omega(p(d+d')) \| \\
\leq \mu_{d+d'+1}(\Phi(d+d',1)^\top \Phi(d+d',1))^{\frac{1}{2(p+1)}} \omega(1)^\top \omega(1).
\]
Moreover,
\[
\| \omega(k+1) \| - \| \omega(k) \| = \omega(k) ^ \top (I - \sigma L(k)) (I - \sigma L(k)) \omega(k) - \omega(k) ^ \top \omega(k) \\
= - \omega(k) ^ \top (2 \sigma L(k) - \sigma^2 L^2(k)) \omega(k),
\]
and
\[
\frac{\omega(k) ^ \top L^2(k) \omega(k)}{\omega(k) ^ \top L(k) \omega(k)} \leq \lambda_{\text{max}}(L(k)).
\]
From Lemma 3 \( \lambda_{\text{max}}(L(k)) \leq \frac{1}{\sigma} \), thus,
\[
\sigma^2 \omega(k) ^ \top L^2(k) \omega(k) < \sigma \omega(k) ^ \top L(k) \omega(k),
\]
and
\[
\| \omega(k+1) \| - \| \omega(k) \| \leq 0.
\]
Hence,
\[
\| \omega(k) \| \leq \| \omega(p(d+d')) \| \leq \mu_{d,d'+1}(\Phi(d+d',1) ^ \top \Phi(d+d',1)) \frac{\| \omega(1) \|}{p}
\]
for \( k \in [p(d+d'), (p+1)(d+d')] \) and any \( p \in \mathbb{Z}_+ \).

Therefore, from Lemma 3 and Theorem 3 one has
\[
\mu_{d,d'+1}(\Phi(d+d',1) ^ \top \Phi(d+d',1)) < 1. \tag{37}
\]
Then \( \lim_{k \to \infty} \| \omega(k) \| = 0 \), implying that the multi-agent system 2 using Algorithm 1 admits average consensus. \( \square \)

Then, we have shown that the asymptotic average consensus of multi-agent system 2 can be guaranteed using the profile of matrix-valued edge weight generated from Algorithm 1. We shall proceed to examine the privacy-preservation performance of the Algorithm 1 in the subsequent discussions.

5. Privacy-preserving Analysis

In this section, we shall discuss the privacy-preservation performance of Algorithm 1. We first provide the main result.

Theorem 4. By implementing Algorithm 1 for an agent \( b \in V_B \), the privacy of \( x_b(0) \) is preserved if \( b \) has at least one legitimate neighbor \( m \in V_B \).

Proof. We shall prove the result via the indistinguishability of a private value’s arbitrary variation to the honest-but-curious agents. Without loss of generality, assume that agent \( b \) has only one legitimate neighbor \( m \in V_B \). Here, there are two cases, namely, \((a, m) \in E \) (Case 1) and \((a, m) \not\in E \) (Case 2) as shown in Figure 3.

Figure 3: The local communication structures amongst a legitimate agent \( b \) and its honest-but-curious neighbor \( a \) and legitimate neighbor \( m \).

Case 1: We need to show that the privacy of the initial value \( x_b^{[d+1:d+d']}(0) \) can be preserved against any honest-but-curious agent \( a \in V_A \), namely, agent \( a \) cannot infer the exact value of \( x_b^{[d+1:d+d']}(0) \) via \( \mathcal{O}_H(\{x_i(0), z_i\} \in V) \). According to Definition 2, it suffices to show that for arbitrary initial value of real state \( x_b^{[d+1:d+d']}(0) \) \( \neq x_b^{[d+1:d+d']}(0) \), there exist initial values of agent \( m \) and associated matrix-valued edge weights such that the equality (37) holds, and the agents’ states still converge to the original average value \( \text{Avg}(\bar{x}(0)) \) even if \( x_b(0) \) is changed into \( \bar{x}_b(0) \). We shall present the selection of initial value of \( x_m(0) \) and associated matrix-valued coupling weights, respectively.

We first examine the selection of initial value of \( x_m(0) \). First, we choose \( \bar{x}_m(0) \) as,
\[
\bar{x}_m(0) = x_m(0) + \bar{x}_b(0) - \bar{x}_b(0). \tag{38}
\]
Second, the virtual state of \( \bar{x}_b(0) \) satisfies
\[
\begin{bmatrix}
\mathbf{v}_1^{[1:d']}
\end{bmatrix} ^ \top \begin{bmatrix}
x_b^{[1:d']}(0) - x_b^{[1:d']}(0)
\end{bmatrix} = 0, \tag{39}
\]
and
\[
\begin{bmatrix}
\mathbf{v}_1^{[1:d']}
\end{bmatrix} ^ \top \begin{bmatrix}
x_b^{[d+1:d+d']}(0)
\end{bmatrix} = - \begin{bmatrix}
\mathbf{v}_1^{[1:d']}
\end{bmatrix} ^ \top \begin{bmatrix}
x_b^{[d+1:d+d']}(0)
\end{bmatrix} - \begin{bmatrix}
x_b^{[d+1:d+d']}(0)
\end{bmatrix}, \tag{40}
\]
and \( x_b^{[1:d']}(0) - x_b^{[1:d']}(0) \) and \( x_b^{[1:d']}(0) - x_b^{[1:d']}(0) \) are linearly independent. According to 2) and 3) of Step 2 in 3.2, we notice that the initial value subject to the aforementioned constraints (38), (39) and (40) are feasible.

Finally, we choose \( \bar{x}_p(0) = x_p(0) \), \( \forall p \in V \setminus \{a, b, m\} \).

Subsequently, we proceed to examine the selection of associated matrix-valued coupling weights as follows. For \( k = 0 \), we first choose \( \bar{A}_{bm}(0) \) such that
\[
\bar{A}_{bm}(0) (x_m(0) - x_b(0)) = A_{bm}(0) (x_m(0) - x_b(0)), \tag{42}
\]
and

\[ \tilde{A}_{bn}(0) (\bar{x}_b(0) - \bar{x}_b(0)) = \frac{1}{2\sigma} (x_b(0) - \bar{x}_b(0)). \tag{43} \]

One can see that there are \((d + d')^2\) free variables in matrix \(\tilde{A}_{bn}(0)\) with \(2(d + d')\) equations in the equalities \((42)\) and \((43)\), also we note that \(x_b^{[d]}(0) - x_b^{[d]}(0)\) and \(x_b^{[d]}(0) - x_b^{[d]}(0)\) are linearly independent. Therefore, there exists a matrix \(\tilde{A}_{bn}(0) \in \mathbb{R}^{n \times n}\) satisfying \((42)\) and \((43)\) simultaneously.

Second, let \(E^*\) denote edges between agents \(m\) and \(p\) where \(p \in V_B \setminus \{b, m\}\). We choose \(A_{pm}(0)\) such that,

\[ \tilde{A}_{pm}(0) (\bar{x}_m(0) - x_p(0)) = A_{pm}(0) (\bar{x}_m(0) - x_p(0)), \tag{44} \]

for all edges in \(E^*\). One can see that there are \((d + d')^2\) free variables in matrix \(\tilde{A}_{pm}(0)\) with \(d + d'\) equations in the equalities \((43)\), therefore, there exist a matrix \(\tilde{A}_{pm}(0)\) satisfying \((43)\).

Third, we choose

\[ \tilde{A}_{pq}(0) = A_{pq}(0), \tag{45} \]

for all \((p, q) \in E \setminus \{(m, b) \cup E^*\}\).

Finally, for \(k \geq 1\), we choose \(\tilde{A}_{pq}(k) = A_{pq}(k)\) for all \((p, q) \in E\).

From \((38)\) to \((40)\), one can conclude that,

\[ \tilde{A}_{ab}(0) \bar{x}_b(0) = A_{ab}(0) x_b(0), \tag{46} \]

and

\[ \tilde{A}_{am}(0) \bar{x}_m(0) = A_{am}(0) x_m(0), \tag{47} \]

for all \(a \in V_A\).

To sum up the above analysis, one can verify that \(\bar{x}_a(1) = \bar{x}_a(1)\) for all \(q \in V\). Specifically, according to \((31)\), \((41)\), \((45)\) and \((47)\), one has,

\[
\begin{align*}
\bar{x}_a(1) &= \bar{x}_a(0) + \sigma A_{ab}(0) \bar{x}_b(0) - \sigma A_{ab}(0) \bar{x}_a(0) \\
&+ \sigma A_{am}(0) \bar{x}_m(0) - \sigma A_{am}(0) \bar{x}_a(0) \\
&+ \sigma \sum_{p \in V_A \setminus \{b, m\}} \left( A_{ap}(0) \bar{x}_p(0) - A_{ap}(0) \bar{x}_a(0) \right) \\
&= \bar{x}_a(0) + \sigma A_{ab}(0) \bar{x}_b(0) - \sigma A_{ab}(0) \bar{x}_a(0) \\
&+ \sigma A_{am}(0) \bar{x}_m(0) - \sigma A_{am}(0) \bar{x}_a(0) \\
&+ \sigma \sum_{p \in V_A \setminus \{b, m\}} \left( A_{ap}(0) \bar{x}_p(0) - A_{ap}(0) \bar{x}_a(0) \right) \\
&= \bar{x}_a(1),
\end{align*}
\]

for all \(a \in V_A\).

According to \((31)\), \((42)\), \((45)\) and \((47)\), one has,

\[
\begin{align*}
\bar{x}_m(1) &= \bar{x}_m(0) + \sigma A_{mb}(0) \bar{x}_b(0) - \sigma A_{mb}(0) \bar{x}_m(0) \\
&+ \sigma \sum_{p \in V_B \setminus \{b, m\}} \left( A_{mp}(0) \bar{x}_p(0) - A_{mp}(0) \bar{x}_m(0) \right) \\
&= \bar{x}_m(0) + \sigma A_{mb}(0) \bar{x}_b(0) - \sigma A_{mb}(0) \bar{x}_m(0) \\
&+ \sigma \sum_{p \in V_B \setminus \{b, m\}} \left( A_{mp}(0) \bar{x}_p(0) - A_{mp}(0) \bar{x}_m(0) \right) \\
&= \bar{x}_m(1),
\end{align*}
\]

for all \(m \in V_B\).

Finally, for all \(p \in V_B \setminus \{b, m\}\), according to \((31)\), \((41)\) and \((47)\), one has,

\[
\begin{align*}
\bar{x}_b(1) &= \bar{x}_b(0) + \sigma \sum_{p \in V_B \setminus \{b, m\}} \left( A_{bp}(0) \bar{x}_p(0) - A_{bp}(0) \bar{x}_b(0) \right) \\
&+ \sigma \sum_{p \in V_B \setminus \{b, m\}} \left( A_{bp}(0) \bar{x}_p(0) - A_{bp}(0) \bar{x}_b(0) \right) \\
&= \bar{x}_b(1).
\end{align*}
\]
\[
\begin{align*}
\bar{x}_p(1) &= \bar{x}_p(0) + \sigma \sum_{k=0}^{t-1} (A_{pq}(0)x_q(0) - A_{pq}(0)x_p(0)) \\
&= \bar{x}_p(0) + \sigma \sum_{k=0}^{t-1} (A_{pq}(0)x_q(0) - A_{pq}(0)x_p(0)) \\
&= x_p(1).
\end{align*}
\]

We notice that \(A_{pq}(k) = A_{pq}(k)\) for all \((p,q) \in E\) and \(k \geq 1\); As such in Definition 2 can be straightforward verified.

Case 2: The proof of this case is similar in spirit with the Case 1, hence omitted for brevity. □

Remark 9. Consider the multi-agent system adopting Algorithm 1. If an agent \(b\) is connected to the remaining agents in the network only through an (or a group of colluding) honest-but-curious agent \(a\), then the initial state of agent \(b\) can be uniquely inferred by agent \(a\) in an asymptotic sense. To see this, we notice that
\[
x_b(0) = x_b(l)
\]

\[\begin{align*}
&\quad - \sigma \sum_{k=0}^{t-1} A_ab(k)x_a(k) + \sigma \sum_{k=0}^{t-1} A_ab(k)x_b(k).
\end{align*}\]

The average consensus is achieved as \(l\) goes to infinity, namely, \(\lim_{l \to \infty} x_a(l) = \lim_{l \to \infty} x_b(l)\). Therefore, agent \(a\) can uniquely infer the initial state of agent \(b\) through (48).

Remark 10. For \(k \geq 1\), recall that the information of agent \(i\) derived from agent \(j\) is \(y_{j\to i}(k) = A_{ij}(k)x_j(k)\); then one has
\[
V_{\rho(k)} \text{diag} \left( \gamma_{ij}^\rho, \chi_{ij}^\rho, 0, \cdots, 0 \right) V_{\rho(k)}^\top x_j(k) = y_{j\to i}(k),
\]

where
\[
V_{\rho(k)} = \left( \frac{v_{\rho(k)}}{\| v_{\rho(k)} \|} \left\| \frac{v_{d+d'}}{\| v_{d+d'} \|} \right\| P_0 \right)_{1}.
\]

Thus,
\[
\begin{align*}
\bar{v}_{\rho(k)}^\top x_j(k) &= \frac{1}{\gamma_{ij}^\rho} \left( \left( \frac{v_{\rho(k)}}{\| v_{\rho(k)} \|} \left\| \frac{v_{d+d'}}{\| v_{d+d'} \|} \right\| P_0 \right)_{1} y_{j\to i}(k) \right), \\
\bar{v}_{d+d'}^\top x_j(k) &= \frac{1}{\chi_{ij}^\rho} \left( \left( \frac{v_{\rho(k)}}{\| v_{\rho(k)} \|} \left\| \frac{v_{d+d'}}{\| v_{d+d'} \|} \right\| P_0 \right)_{2} y_{j\to i}(k) \right).
\end{align*}
\]

We notice that there exists \(d+d'\) free variables in at most two equations, therefore agent \(i\) can not infer \(x_j(k)\) associated with agent \(j\) for \(k \geq 1\).

\[
A_{ij}(0) \quad A_{ij}(1) \quad A_{ij}(2) \quad A_{ij}(3) \quad A_{ij}(4) \quad A_{ij}(5) \quad A_{ij}(1)
\]

\[\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots
\end{array}\]

Figure 4: An illustration for the switching pattern of the matrix-valued weights on edge \((i,j) \in E\) with period \(T = 5\). The boundaries between neighboring period are highlighted by vertical red bars.

6. Simulation

We now provide a simulation example to illustrate the effectiveness of the proposed MWN-PPAC algorithm. Consider the 5-node network in Figure 1 where each agent holds a 3-dimensional state. Choose the dimension of virtual state \(d' = 3\) and the matrix-valued weight on each edge \((i,j) \in E\) adopting a periodic switching pattern with period \(T = 5\), as illustrated in Figure 4. We choose the construction of the orthogonal vector set \(\Psi\) as the same as that discussed in Remark 4 namely, by (22) to (27). We choose the initial states
\[
x(0) = (x_1^T(0), x_2^T(0), x_3^T(0), x_4^T(0), x_5^T(0))^\top,
\]

as follows,
\[
\begin{align*}
x_1(0) &= (0.20, 0.30, 0.25, 0.60, 0.32, 0.65)^\top, \\
x_2(0) &= (0.60, 0.72, 0.57, 0.24, 0.91, 0.95)^\top, \\
x_3(0) &= (0.52, 0.71, 0.80, 0.20, 0.12, 0.62)^\top, \\
x_4(0) &= (0.02, 0.04, 0.12, 0.82, 0.38, 0.23)^\top, \\
x_5(0) &= (0.37, 0.17, 0.77, 0.33, 0.32, 0.72)^\top.
\end{align*}
\]

Note that the virtual state can be randomly chosen. Choose \(\sigma = 2\) and randomly choose \(\gamma_{ij}^\rho, \chi_{ij}^\rho \in (0, \frac{1}{4(n-1)n})\), \(\alpha_{ij} > 0, \beta_{ij} > 0\) for \((i,j) \in E\) and \(k > 0\). The average value of \(x(0)\) is
\[
\text{Avg}(x(0)) = \frac{1}{5} \sum_{j=1}^{5} x_j(0)
\]

\[\begin{align*}
&= (0.34, 0.39, 0.50, 0.44, 0.41, 0.63)^\top.
\end{align*}\]

As one can see, average consensus is asymptotically achieved; see Figure 5.

Moreover, consider privacy preservation on the information channel from agent 3 to its neighboring agent 2. The transmitted information versus the state of the agent are shown in Figure 6. One can see that the information exchanged amongst neighbor agents is not related to the true state of the corresponding agents.

We further examine another arbitrary initial value, namely,
\[
x(0) = (x_1^T(0), x_2^T(0), x_3^T(0), x_4^T(0), x_5^T(0))^\top,
\]

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots
\end{array}\]
such that $\hat{x}_i(0) = x_i(0)$ for $i \in \{2, 4, 5\}$ and
\[
\hat{x}_1(0) = (0.30 \ 0.20 \ 0.65 \ 0.50 \ 0.12 \ 0.75)^\top,
\hat{x}_3(0) = (0.42 \ 0.81 \ 0.40 \ 0.30 \ 0.32 \ 0.52)^\top.
\]
A comparison of the state evolution of Algorithm 1 initiated from $\bar{x}(0)$ and $\hat{x}(0)$, respectively, is shown in Figures 7 and 8 where only $k = 0, 1, 2, 3, 4, 5$ are shown. One can see that although Algorithm 1 is initiated from different initial values, the real states in each dimension coincide at $k = 1$ and therefore at all $k > 1$.

Figure 7: Trajectories of real agent states initiated from $\bar{x}(0)$ ((a)-(c)) and that by $\hat{x}(0)$ ((d)-(f)), respectively.

Figure 8: Trajectories of virtual agent states initiated from $\bar{x}(0)$ ((a)-(c)) and that by $\hat{x}(0)$ ((d)-(f)), respectively.

7. Conclusion Remarks

In this paper, we proposed an algorithmic framework for vector-valued privacy-preserving average consensus. The algorithm is essentially built on two principles, namely, agent state lifting and dynamic matrix-valued weight design. A self-contained analysis in terms of convergence and privacy preservation of the algorithm was then provided. The proposed algorithm is simple and efficient, and can be implemented in a distributed manner. The proposed algorithm provides a new dimension for PPAC algorithm design by utilizing the proper lifting of agents’ state space.

Future works in this direction include examining privacy-preserving distributed algorithms on matrix-weighted networks in the context of estimation, control, and optimization.

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