Gauge theory of a massive relativistic spinning point particle

J. H. Lorentsen† and N. K. Nielsen*
Fysisk Institute, Odense University, DK-5230 Odense M, Denmark

A massive relativistic spinning point particle in any number of dimensions has in a previous article been shown to be described by first class constraints, which define a gauge theory. In the present paper we find the corresponding finite gauge transformations. By comparing the integrated gauge transformations to transformation equations found by Pryce, we conclude that the selection of gauge corresponds to selection of the relativistic center of mass frame in the model of Pryce, where a spinning particle is considered a composite object. The Lorentz group is identified as the gauge group, and as gauge field we identify the relativistic angular velocity. We also show that an analogous physical interpretation is possible for the relativistic spherical top of Hanson and Regge.

PACS numbers: 11.30.Ly, 03.65.Fd., 11.30.Cp

I. INTRODUCTION

The problem of defining a relativistic center of mass coordinate frame was considered by Pryce many years ago. Demanding that the definition of the center of mass in the non-relativistic limit should reduce to the non-relativistic result, he found three different ways in which the center of mass could be defined. The view adapted by Pryce is that a massive spinning particle is composed of several objects. Spin is generated by the relative motion of the objects, and their center of mass defines the position of the spinning particle. Thus the orbital angular momentum, and hence also the spin, depends on the choice of center of mass, since the total angular momentum has to be independent of this choice. However, the center of mass is not a unique concept in special relativity.

To describe a particle with spin formally, one can attach a rotating frame to the particle. This corresponds to using the Lorentz transformation matrix $\Lambda_{\mu\nu}$ as a dynamical variable on the same footing as the particle position $x^\mu$. In it was shown to be convenient to use canonical coordinates on the Poincaré group manifold as dynamical variables instead of $\Lambda_{\mu\nu}$ and $x^\mu$.

Pryce determined three ways in which the choice of center of mass reference frame can be expressed through a set of constraints on the spinning particle. The so-called Pryce constraint is

$$\hat{S}^{\mu\nu} P_\nu = 0,$$

where $\hat{S}^{\mu\nu}$ is the spin matrix, and $P^\mu$ are the momenta. This corresponds to a center of mass computed in the rest frame of the composite body and next Lorentz transformed to an arbitrary frame.

Hanson and Regge Dirac quantized the point particle using the Pryce constraint and the gauge condition (in the following referred to as the Pryce gauge condition)

$$\Lambda_0^\mu + \frac{P_\mu}{m} = 0$$

where $m$ is the mass. The resulting Dirac brackets between coordinates $\hat{x}^\mu$, momenta $P^\mu$, and spin $\hat{S}^{\mu\nu}$ are the same as the Poisson brackets Pryce found for the corresponding quantities in his spinning particle model. The constraints are second class and hence do not define a gauge theory.

The second set of spin constraints found by Pryce are

$$\hat{S}^{0\mu} = 0,$$

where $S^{\mu\nu}$ is the corresponding spin matrix. These constraints correspond to a center of energy computed in the reference frame of an arbitrary observer. We will in the following refer to these constraints as the $S^{0\mu}$-constraints.

The third set of spin the constraints found by Pryce are

$$(P^0 + m)S^{0\mu} + S^{\mu i} P_i = 0,$$

(referred to in the following as the Wigner constraints) where $S^{\mu\nu}$ is the corresponding spin matrix. These constraints arise when the center of mass is a weighted average of the previous two centers of mass; they were also found by Wigner when determining the representations of the Poincaré group. If one uses Wigner’s construction of the representations of the Poincaré group by means of little groups, and takes the one corresponding to the rest frame, an identification of the spin matrix reveals the constraints.

Modified constraints that include all three cases were found in . The constraints are

$$\psi^\mu = S^{\mu\nu} (P_\nu - m \Lambda_0^\nu).$$

These constraints were found to be first class constraints, and thus define a gauge theory. Gauges should be selected by fixing $\Lambda_0^\mu$. The constraints are thus gauge dependent. If we select the gauge, eq. reduces to
We show below that the finite gauge transformation equations one gets from the general constraints are the transformation equations that Pryce found for the transformation between the different sets of variables. Thus the selection of gauges in the gauge theory defined by the constraints corresponds to a definition of the relativistic center of mass frame.

The layout of the paper is the following: In sec. II we review the Poisson brackets found in [3] and [4], with emphasis on the first class constraint algebra found in [3]. In sec. III we determine the infinitesimal gauge transformations generated by the constraints. In sec. IV the infinitesimal gaugetransformations are integrated and the connection to the transformation formulas of Pryce established. Finally the analysis is extended to the relativistic spherical top in sec. V. Concluding remarks are contained in sec. VI.

II. POISSON BRACKETS

The determination of whether a set of constraints is first or second class begins by the determination of the Poisson brackets involving the variables in the theory. In [4] the relevant Poisson brackets were found from the Poincaré group structure equations and the canonical commutation relations for canonical coordinates on the group manifold and their momenta:

\[ \{ S_{\mu\nu}, S_{\lambda\rho} \} = \frac{1}{2} C^\delta_{\mu
u,\lambda\rho} S_{\delta \delta} \]

\[ \{ S_{\mu\nu}, A^\lambda_{\mu} \} = C^\kappa_{\mu
u,\rho} A^\lambda_{\kappa} \]  \hspace{1cm} (12)

where \( S_{\mu\nu} \) denotes the spin matrix in generality. The rest of the brackets vanish

\[ \{ \Lambda^\mu_{\nu}, \Lambda^\rho_{\lambda} \} = \{ \Lambda^\mu_{\nu}, P^\rho \} = \{ \Lambda^\mu_{\nu}, x^\lambda \} = 0, \]

\[ \{ S_{\mu\nu}, P^\lambda \} = \{ S_{\mu\nu}, x^\lambda \} = 0, \]

\[ \{ \phi, \psi \} = 0 \]  \hspace{1cm} (13)

Here the Poincaré group structure constants are

\[ C^\mu_{\lambda\rho,\kappa} = \delta^\mu_{\kappa} C^\nu_{\lambda\rho,\delta} - \delta^\mu_{\delta} C^\nu_{\lambda\rho,\kappa} + \delta^\nu_{\kappa} C^\mu_{\lambda\rho,\delta} - \delta^\nu_{\delta} C^\mu_{\lambda\rho,\kappa} \]

\[ C^\lambda_{\mu\nu,\rho} = \delta^\lambda_{\nu} \eta_{\mu\rho} - \delta^\lambda_{\mu} \eta_{\nu\rho}. \]  \hspace{1cm} (14)

The constraints are

\[ \phi = P^2 + m^2, \quad \psi^\mu = S^{\mu\lambda}(P_\lambda - m\Lambda^0_{\lambda}). \]  \hspace{1cm} (15)

\( \phi \) is the mass-shell condition. This is a system of first class constraints since \( \{ \phi, \psi \} = 0 \) and

\[ \{ \psi^\mu, \psi^\nu \} = P^\mu \psi^\nu - P^\nu \psi^\mu + S^{\mu\nu} \phi. \]  \hspace{1cm} (16)

The constraint algebra is closed (the constraints are first class) and therefore they define a gauge theory.
The relativistic spherical top of Hanson and Regge \cite{3} is defined by a system of constraints where the mass-shell constraint is replaced by the condition

\[ \phi = P^2 + f(\frac{1}{2}S_{\mu\nu}S^{\mu\nu}) = 0 \]  

(17)

which incorporates Regge trajectories. Here the function \( f \) is real but otherwise unspecified. In \cite{4} the mass-shell condition was shown to have the general form

\[ \phi = P^2 + f(m^2), \quad m^2 = \frac{1}{2}S_{\mu\nu}S^{\mu\nu} + S_{\mu\lambda}A^0 \Lambda_{\lambda}^{\rho}S^{\rho}\nu, \]

\[ \psi^\mu = S^{\mu\nu}(P_\nu - mA_\nu). \]

(18)

which by the Pryce gauge condition reduces to \( \{ \psi, \psi \} = 0 \). If we use the Poisson brackets \( \{ \phi, \phi \} = 0 \), \( \{ m, S^{\mu\nu} \} = 0 \), \( \{ m, A^{\mu\nu} \} = \frac{1}{2}(\Lambda^{\mu\lambda}S^\nu_\lambda + \eta^{\mu\nu}(S^0_\rho + \Lambda_\mu^\rho S^\rho_\nu - \Lambda^\mu_\nu S_\alpha^\rho A^\rho_\alpha)). \)

(19)

From the above equations we have \( \{ m, A^{0\mu} \} = 0 \). This means that the Poisson brackets \( \{ \phi, \phi \}, \{ \phi, \psi^\mu \}, \) and \( \{ \psi^\mu, \psi^\nu \} \) are the same as the corresponding ones in the point particle case. The constraints are thus still first class and still define a gauge theory. The main difference to the point particle case is that the mass is now a function of the spin.

**III. INFINITESIMAL GAUGE TRANSFORMATIONS**

In this section we determine the infinitesimal gauge transformations generated by the system of constraints \( \{ \phi, \phi \} = 0 \), in order to find the gauge transformation equations. We first consider the case of a spinless particle where we only have the mass shell constraint. This simple and well-known example serves to display the general strategy that will be used for the spinning particle.

**A. The spinless point particle**

The spinless point particle is described by the coordinates \( x^\mu \) and momenta \( P^\mu \). The only constraint is the mass shell condition \( \phi \) in eq. \( \{ \phi, \phi \} = 0 \). Since \( \{ \phi, \phi \} = 0 \) this is a first class system. Time development is found from the Hamiltonian

\[ H = \frac{1}{2}(P^2 + m^2) \]  

(20)

where \( e \) is a Lagrange multiplier, such that for an arbitrary dynamical variable \( X \) the time derivative is

\[ \dot{X} = \{ X, H \}. \]  

(21)

In a similar way we find the gauge transformations \( \delta X \) is found from the generator \( Q \):

\[ Q = \frac{1}{2}(P^2 + m^2) \]  

(22)

(with \( e \) infinitesimal and real) by the formula

\[ \delta X = \{ X, Q \}. \]  

(23)

Using these equations we find that the equations of motion are

\[ \ddot{x}^\mu = eP^\mu, \quad \dot{P}^\mu = 0, \quad \dot{e} = 0. \]  

(24)

and the gauge transformations are

\[ \delta x^\mu = \alpha P^\mu, \quad \delta P^\mu = 0, \quad \delta e = 0. \]  

(25)

From \( \delta (\dot{x}^\mu) = (\delta \dot{x}^\mu) \) follows immediately

\[ \delta e = \dot{\alpha}, \]  

(26)

which is the gauge transformation of an Abelian gauge field in one dimension. Thus we have an Abelian gauge theory, where \( e \) is the gauge field. It is a constant, and \( \alpha \) is consequently linear in \( \tau \).

We can immediately integrate the infinitesimal gauge transformations to obtain finite gauge transformations. As a specific example we can take \( \alpha = 0 \) in the covariant gauge where \( x = x_{\text{cov}} \) with \( x_{\text{cov}}^2 + 1 = 0 \), and thus \( e = \frac{1}{m} \) from \( \delta P^\mu = \alpha P^\mu \) and the mass-shell condition. Equation \( \{ \phi, \phi \} = 0 \) can be integrated to

\[ x^\mu(\alpha) = x_{\text{cov}}^\mu + \alpha P^\mu. \]  

(27)

The selection of the gauge determines the Lagrange multiplier \( e \). Taking the time derivative of \( x_{\text{cov}}^\mu \), we get

\[ eP^\mu = \ddot{x}_{\text{cov}}^\mu + \dot{\alpha}P^\mu. \]  

(28)

The equation of motion of \( x_{\text{cov}}^\mu \) then gives us that

\[ e = \frac{1}{m} + \dot{\alpha}. \]  

(29)

As an example we could choose the proper time gauge \( x^0 = \tau \). The Lagrange multiplier and gauge parameter are then by \( \{ \phi, \phi \} = 0 \)

\[ e = \frac{1}{P^\mu}, \quad \dot{\alpha} = \frac{1}{P^\mu} - \frac{1}{m}. \]  

(30)

We found above that the constraint gives an Abelian gauge theory. We then determined the general transformation formula \( \{ \phi, \phi \} = 0 \) allowing a transformation from the covariant gauge to any other gauge.

**B. A point particle with spin**

For a particle with spin, we generate infinitesimal gauge transformations from \( \delta X = \{ X, Q \} \) where \( X \) is an arbitrary variable, and \( Q \) is a linear combination of the constraints in eq. \( \{ \phi, \phi \} = 0 \)
We determine the time derivative of any parameter $X$ means of $\dot{X}$ where

$$\dot{\Lambda}$$

formation on $\dot{\Lambda}$ angular momentum. By means of the Jacobi identity and infinitesimal gauge transformations:

$$\delta x^\mu = a^\mu (P^\nu - m\Lambda_{0}^\nu) - a^\nu (P^\mu - m\Lambda_{0}^\mu).$$

We notice that the generators of the Poincaré group are invariant. We also notice that all the equations, except eq. (44), are linear differential equations. (34) is nonlinear because the constraints are imposed.

Since the constraints are first class, they are invariant under gauge transformations, after the constraints are imposed.

The Hamiltonian is a linear combination of the constraints

$$H = \frac{1}{2}(P^2 + m^2) + v^\mu S_{\mu\nu}(P^\nu - m\Lambda_{0}^\nu).$$

We determine the time derivative of any parameter $X$ by means of $X = \{X, H\}$. The outcome is (cf. (32)-(35)):

$$\dot{x}^\mu = e P^\mu + v^\lambda S^\lambda_{\mu}$$

$$\dot{S}^{\mu\nu} = v^\mu \dot{v}^\nu - \dot{v}^\nu v^\mu + v^\lambda S^\lambda_{\mu} P^\nu - v^\lambda S^\lambda_{\nu} P^\mu$$

$$\dot{\Lambda}^{\mu\nu} = \Lambda^\mu_{\rho \nu} (\alpha)^{\rho\nu} - \alpha^\mu (P^\nu - m\Lambda_{0}^\nu) - \alpha^\nu (P^\mu - m\Lambda_{0}^\mu).$$

$$\delta P^\mu = 0, \delta M^{\mu\nu} = 0.$$ where we defined

$$\Lambda(\alpha)^{\mu\nu} = \alpha^\mu (P^\nu - m\Lambda_{0}^\nu) - \alpha^\nu (P^\mu - m\Lambda_{0}^\mu).$$

They can be linear in $\tau$ and proportional to $v^\mu$. Gauge transformations in this class have the same effect on the variables as the time development described by (38), (39), (40) and (42), and can thus also be used to eliminate $v^\mu$ in these equations.

They can be constant (independent of $\tau$). These global gauge gauge transformations are integrated in the following section to obtain generalizations of the transformation formulas mentioned in the introduction (8), (9), (10) and (11).

IV. FINITE GAUGE TRANSFORMATIONS

In this section we consider integration of the nontrivial infinitesimal gauge transformation formulas found in sec. IIB to obtain the corresponding finite gauge transformations.

We assume $\dot{\alpha}^\mu = 0$ (global gauge transformations).

A. Finite gauge transformations of spin matrix, coordinates, and Lagrange multipliers

In order to find the finite transformations for the spin matrix, we use eq. (33) with constraints taken into account ($\psi^\mu = 0$)

$$\delta S^{\mu\nu} = \alpha^\lambda S^\lambda_{\mu} P^\nu - \alpha^\lambda S^\lambda_{\nu} P^\mu.$$ We write the infinitesimal variables $a^\mu$ as follows

$$a^\mu = \delta s a^\mu,$$

where $s$ is a real “gauge development” parameter, and $a^\mu$ is independent of $s$. This gives

$$\frac{dS^{\mu\nu}(s)}{ds} = a^\lambda S^\lambda_{\nu} P^\mu - a^\lambda S^\lambda_{\mu} P^\nu.$$ Whence

$$\frac{d(S^{\mu\nu}(s)a^\rho)}{ds} = a^\mu P^\rho S^\nu_{\lambda} (s)a^\lambda.$$ that is integrated

$$S^\mu_{\lambda}(s)a^\lambda = S^\mu_{\lambda}(0)a^\lambda e^{a^\mu P^\nu}. $$ Insertion back into eq. (47), which next is integrated, gives

$$S^{\mu\nu}(s) - S^{\mu\nu}(0) = \frac{a^\lambda}{a^\cdot P} (S^\mu_{\lambda}(0) P^\nu - S^\nu_{\lambda}(0) P^\mu)(e^{a^\cdot P^\nu} - 1).$$ This equation has already the same form as (10) and (11). It takes a simpler form in the limit $s \to \infty$, if we select the sign of $a^\mu$ such that $a^\cdot P < 0$:
\[ S^{\mu \nu}(\infty) = S^{\mu \nu}(0) - \frac{a_\lambda}{a \cdot P} (S^{\alpha \lambda}(0) P^\nu - S^{\nu \lambda}(0) P^\mu) \]  

(51)

From this equation we see that \( S^{\mu \nu}(\infty) \) fulfill the constraints \( S^{\alpha \lambda}(\infty) a_\lambda = 0 \). If we compare these constraints with the constraints \( 3 \) we conclude

\[ P^\mu - m a^{0 \mu}(\infty) = C a^\mu \]  

(52)

where \( C \) is a real proportionality factor that is determined in terms of \( a^\mu \) and \( P^\mu \) in sec. IV.C.

For the coordinates we proceed as we did for the spin matrix. If we use \( a^\mu = \delta s a^\mu \) and \( \alpha = 0 \) in \( 52 \) and insert the result we found for the spin matrix \( S^{\mu \nu}(s) \), we get

\[ \frac{d x^\mu(s)}{d s} = a^\rho S^{\mu \rho}(0) e^{a \cdot P s}. \]  

(53)

that is integrated

\[ x^\mu(s) = \frac{a^\rho}{a \cdot P} S^{\mu \rho}(0)(e^{a \cdot P s} - 1) + x^\mu(0). \]  

(54)

If we again use that \( a \cdot P < 0 \), and take the \( s \to \infty \) limit, we get the transformation equations

\[ x^\mu(\infty) = x^\mu(0) + \frac{a^\rho}{a \cdot P} S^{\mu \rho}(0). \]  

(55)

We finally find the finite gauge transformations of the Lagrange multipliers. From eq. \( 44 \) follows that the quantity

\[ \zeta = v \cdot P \]  

(56)

is invariant under global gauge transformations Consequently \( 44 \) implies

\[ \frac{d v^\mu(s)}{d s} = v^\mu(s)(a \cdot P) - \zeta a^\mu. \]  

(57)

that is integrated

\[ v^\mu(s) = \frac{\zeta a^\mu}{a \cdot P} + (v^\mu(0) - \frac{\zeta a^\mu}{a \cdot P}) e^{a \cdot P s} \]  

(58)

that for \( a \cdot P < 0 \) in the limit \( s \to \infty \) reduces to

\[ v^\mu(\infty) = \frac{\zeta a^\mu}{a \cdot P}. \]  

(59)

The proportionality factor \( \zeta \) is undetermined.

**B. Connection to the Pryce transformation formulas**

In the previous subsection a general formalism for finite gauge transformations was constructed. We next demonstrate that the transformation formulas of Pryce \( 1 \) quoted in section \( 1 \) come out as special cases.

We begin by considering the gauge transformations, where the Pryce gauge \( 3 \) corresponds to \( s = \infty \). This means \( a^\mu = \alpha P^\mu \) , with \( \alpha \) positive to ensure \( a \cdot P < 0 \), but otherwise arbitrary. From \( 1 \) and \( 55 \) we get

\[ \hat{S}^{\mu \nu} = S^{\mu \nu}(0) + \frac{P_\lambda}{m^2} (S^{\alpha \lambda}(0) P^\nu - S^{\nu \lambda}(0) P^\mu) \]  

(60)

\[ \hat{x}^\mu = x^\mu(0) - \frac{P_\lambda}{m^2} S^{\mu \lambda}(0). \]  

(61)

The variables \( S^{\mu \nu}(0) \) and \( x^\mu(0) \) have to be specified in order to get Pryce’s transformation formulas. If they are \( S^{\mu \nu}_{0 \nu} \)-gauge quantities, we get by taking \( S^{\mu \nu}_{0}(0) = 0 \):

\[ \hat{S}^{\mu \nu} = S^{\mu \nu} + \frac{P_\lambda}{m(P^0 + m)} (S^{\alpha \lambda}(0) P^\nu - S^{\nu \lambda}(0) P^\mu) \]  

(62)

and

\[ \hat{x}^\mu = x^\mu - \frac{P_\lambda}{m(P^0 + m)} S^{\mu \lambda}. \]  

(63)

while by taking the initial variables to be Wigner-gauge quantities we get by the Wigner constraint:

\[ \hat{S}^{\mu \nu} = S^{\mu \nu} + \frac{P_\lambda}{m(P^0 + m)} (S^{\lambda \mu}(0) P^\nu - S^{\nu \lambda}(0) P^\mu) \]  

(64)

\[ \hat{x}^\mu = x^\mu - \frac{P_\lambda}{m(P^0 + m)} S^{\mu \lambda}. \]  

(65)

where \( i \) only runs over spatial indices. These equations are generalizations to arbitrary dimensionality of those Pryce \( 1 \) found for the corresponding center of mass reference frames (cf. \( 8 \) and \( 10 \)).

Transformation equations, where the Wigner gauge corresponds to \( s = \infty \), can also be found. From the Wigner gauge conditions and \( 52 \) one finds \( a^\mu = \alpha(P^\mu - m\delta^{0 \mu}) \), where again \( \alpha > 0 \) to ensure \( a \cdot P < 0 \). The transformation equations \( 1 \) and \( 55 \) in this case imply:

\[ S^{\mu \nu} = S^{\mu \nu}(0) + \frac{(P_\lambda - m\delta^{0 \lambda})}{m(P^0 + m)} (S^{\alpha \lambda}(0) P^\nu - S^{\nu \lambda}(0) P^\mu) \]  

(66)

\[ x^\mu = x^\mu(0) - \frac{(P_\lambda - m\delta^{0 \lambda})}{m(P^0 + m)} S^{\mu \lambda}(0). \]  

(67)

Since we already have found the finite gauge transformation leading from the Wigner gauge to the Pryce gauge, we only consider the case where we start in the \( S^{\mu \nu}_{0 \nu} \)-gauge. Here we get:

\[ S^{\mu \nu} = S^{\mu \nu} + \frac{P_\lambda}{m(P^0 + m)} (S^{\mu \lambda}(0) P^\nu - S^{\nu \lambda}(0) P^\mu) \]  

(68)

\[ x^\mu = x^\mu - \frac{P_\lambda}{m(P^0 + m)} S^{\mu \lambda}. \]  

(69)

which again are some of Pryce’s transformation formulas.
C. Finite gauge transformation of the Lorentz transformations

Up to this point we have considered all variables with nontrivial gauge transformation laws, except the Lorentz transformations $\Lambda^{\mu \nu}$. As noted earlier we can not expect the transformation equation for them to be as easy to integrate as the other ones, because they involve nonlinear differential equations. On the other hand, it is important to demonstrate that the finite gauge transformations can be determined, because gauge fixing is done by fixing $\Lambda^{0 \mu}$. From eq. (34) with $\alpha^\mu = \delta s a^\mu$ follows

$$\frac{d \Lambda^{0 \mu}}{ds} = \Lambda^0_\lambda (a^\lambda P^\mu - a^\mu P^\lambda) - m (a^\mu + \Lambda^{0 \mu} \Lambda^0_\lambda a^\lambda).$$

(70)

If we consider a boost in any direction we can write $\Lambda^{0 \mu}$ as

$$\Lambda^{0 \mu}(s) = - \cosh \psi(s) \quad \Lambda^{0 \nu}(s) = c^i(s) \sinh \psi(s)$$

$$\bar{c}(s)^2 = 1$$

(71)

where $\psi$ and $\bar{c}$ are functions of $s$. When this is inserted into eq. (74), we get

$$d \psi = c^i (a^0 P^i - a^i P^0) - ma^0 \sinh \psi - m \cosh \psi \sinh \psi$$

(72)

and

$$d \bar{c}^i \sinh \psi = \delta^i - c^i \sinh \psi \left( a^0 P^i - a^i P^0 \right)$$

$$+ (a^j P^i - a^i P^j) c^j \sinh \psi.$$  

(73)

Eqs. (72) and (73) are the final general equations.

The solution of (73) is

$$c^i = \frac{P^i}{|P|}, \epsilon = \pm 1$$

(74)

for

$$a^\mu = \alpha P^\mu + \beta \eta^{0 \mu}$$

(75)

with $\alpha$ and $\beta$ arbitrary. These gauge choices include the three gauges mentioned in the introduction.

The general solution of (72) in these gauges is found by quadrature

$$e^{\psi(s)} = \frac{Y(s)}{a^0 + \epsilon \frac{\bar{c}}{|\bar{c}|}} Y(s) + \left( a^0 - \epsilon \frac{\bar{c}}{|\bar{c}|} \right) \frac{\epsilon \frac{\bar{c}}{|\bar{c}|}}{a^0 + \epsilon \frac{\bar{c}}{|\bar{c}|} - Y(s)}$$

(76)

where

$$Y(s) = \frac{\left( a^0 - \epsilon \frac{\bar{c}}{|\bar{c}|} \right) e^{\psi(0)} - \left( a^0 - \epsilon \frac{\bar{c}}{|\bar{c}|} \right) \epsilon \frac{\bar{c}}{|\bar{c}|} e^{\psi(0)} + \frac{\epsilon \frac{\bar{c}}{|\bar{c}|}}{a^0 + \epsilon \frac{\bar{c}}{|\bar{c}|}}}{\epsilon \frac{\bar{c}}{|\bar{c}|} e^{\psi(0)} - \epsilon \frac{\bar{c}}{|\bar{c}|} e^{\psi(0)} + \frac{\epsilon \frac{\bar{c}}{|\bar{c}|}}{a^0 + \epsilon \frac{\bar{c}}{|\bar{c}|}}} e^{\psi(0)}.$$

(77)

We shall only give explicit expressions for $\Lambda^{0 \mu}(s)$ in the limit $s \to \infty$. In this limit we have for $a \cdot P < 0$ that $Y(s) \to 0$ and consequently

$$e^{\psi(\infty)} = \frac{(a^0 - \epsilon \frac{\bar{c}}{|\bar{c}|} P_{\mu}^0 + \epsilon |\bar{c}| P_{\mu})}{a^0 + \epsilon \frac{\bar{c}}{|\bar{c}|} P_{\mu}}.$$  

(78)

Inserting this result into (71) and using (74)-(75) we obtain

$$P_{\mu} - m \Lambda^{0 \mu}(\infty) = - 2a^\mu \frac{a \cdot P}{(a^0)^2 - (\epsilon \frac{\bar{c}}{|\bar{c}|} P_{\mu})^2}$$

(79)

in conformity with (82), with

$$C = - \frac{2a \cdot P}{(a^0)^2 - (\epsilon \frac{\bar{c}}{|\bar{c}|} P_{\mu})^2}.$$  

(80)

The remaining components of $\Lambda^{\mu \nu}$ can be determined from (74) once the components of $\Lambda^{0 \mu}$ are known. The resulting differential equations are linear.

V. THE RELATIVISTIC SPHERICAL TOP

We noticed in section II that the relativistic spherical top is defined from a set of first class constraints (51). This means that the set of constraints define a gauge theory. We generate infinitesimal gauge transformations through

$$\delta X = \{ X, Q \} \quad Q = \alpha \frac{\phi}{2} + \alpha^\mu \psi^\mu$$

(81)

where $\phi$ and $\psi^\mu$ are defined in equations (18). Using (19) one sees that the infinitesimal gauge transformations and equations of motion of $x^\mu$, $P^\mu$, $M^{\mu \nu}$, and $S^{\mu \nu}$ are the same as the ones (29) for the point particle. The mass $m$ defined in (18) is gauge invariant.

$m$ has non-vanishing Poisson brackets with $\Lambda^{\mu \nu}$, and we must therefore consider the gauge transformation $\delta \Lambda^{\mu \nu}$, which will lead to the gauge field, in more detail. Using (12), (3), (19) and (8), we find

$$\delta \Lambda^{\mu \nu} = \Lambda^{\mu \rho} \sigma(\alpha)^{\rho \nu}$$

(82)

where we introduced

$$\sigma(\alpha)^{\mu \nu} = \lambda(\alpha)^{\mu \nu} + \tilde{\alpha} \left( S^{\mu \nu} - \Lambda^{0 \rho} (\Lambda^{0 \mu} S^{\nu \rho} - \Lambda^{0 \nu} S^{\rho \mu}) \right)$$

(83)

with

$$\tilde{\alpha} = (\alpha f^\prime (m^2) - \frac{1}{m} \alpha^\sigma S^{\sigma \alpha} \Lambda^{0 \alpha}).$$

(84)

and where $\lambda(\alpha)^{\mu \nu}$ is given in (38). This transformation formula is more complicated than (34). However, for $\Lambda^{0 \nu}$ it reduces to eq. (34). Consequently, the determination
of finite gauge transformations of $\Lambda^{0\nu}$ reported in sec. IV also applies here.

The equation of motion

$$\dot{\Lambda}^{\mu\nu} = \{\Lambda^{\mu\nu}, H\}, \quad H = \frac{e}{2} \phi + v_{\mu} \psi^{\mu}$$  \hspace{1cm} (85)

gives, in analogy with the above calculation

$$\dot{\Lambda}^{\mu\nu} = \Lambda^{\mu\nu} \sigma(v)^{\mu\nu}$$  \hspace{1cm} (86)

where $\sigma(v)^{\mu\nu}$ is a relativistic angular velocity defined by

$$\sigma(v)^{\mu\nu} = \lambda(v)^{\mu\nu} + \tilde{v}(S^{\mu\nu} - \Lambda^{0\rho}(\Lambda^{0\nu} S^{\mu\rho} - \Lambda^{0\rho} S^{\mu\nu}))$$  \hspace{1cm} (87)

with $\lambda(v)^{\mu\nu}$ given by (43), and where

$$\tilde{v} = \left(\frac{e}{m} f'(m^2) - \frac{1}{m} v_{\rho} S^{\sigma\lambda} \Lambda^{0\lambda}\right).$$  \hspace{1cm} (88)

Eq. (45) is found to be valid also in this case, and the Lagrange multipliers are still constants. We can thus take over the analysis of finite gauge transformations of sec. IV. This means that if we follow the same way of integrating the transformation equations as in the point particle case, we can give the same physical interpretation to the result: The selection of gauge corresponds to a selection of center of mass. We identify $\sigma^{\mu\nu}(v)$ as the gauge field and the Lorentz group $SO(D - 1, 1)$ as the gauge group.

VI. CONCLUSION

We have shown that the gauge theory defined by the constraints (5) for the relativistic spinning particle is connected to the problem of selecting a relativistic center of mass. The gauge theory has the Lorentz group as gauge group, and the gauge field is the angular velocity $\lambda^{\mu\nu}(v)$. The gauge theory of the relativistic spherical top defined by the constraints (15) can be given a similar physical interpretation; it has the same gauge group and a similar gauge field as the relativistic spinning particle model.

Acknowledgement: We are grateful to U. J. Quaade for pointing out the possible relevance of Pryce’s papers to the choice of gauge.

[1] H. M. L. Pryce, Proc. Roy. Soc. London A, 195, 62 (1948)
[2] H. M. L. Pryce, Proc. Roy. Soc. London, 150, 166 (1935)
[3] C. Itzykson and A. Voros, Phys. Rev. D 5, 2939 (1972)
[4] A. J. Hanson and T. Regge, Ann. Phys. 87, 498 (1974)
[5] N. K. Nielsen and U. J. Quaade, Phys. Rev. D 52, 1204 (1995)
[6] E. P. Wigner, Ann. Math. 40, 204 (1939)

IU. M. Shirokov, Sov. Phys. JETP 6 919 (1958)