COHERENT COCHAIN COMPLEXES AND BEILINSON t-STRUCTURES, WITH AN APPENDIX BY ACHIM KRAUSE

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Abstract. We define and study coherent cochain complexes in arbitrary stable ∞-categories, following Joyal. Our main result is that the ∞-category of coherent cochain complexes in a stable ∞-category C is equivalent to the ∞-category of complete filtered objects in C.

We then show how the Beilinson t-structure can be interpreted in light of such equivalence, and analyze its behavior in the presence of symmetric monoidal structures. We also examine the relationship between the notion of (higher) Toda brackets and coherent cochain complexes. Finally, we prove how every coherent cochain complex gives rise to a spectral sequence and illustrate some examples.

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1. INTRODUCTION

Recall that a filtered chain complex (resp. a filtered spectrum) consists of a \( \mathbb{Z} \)-indexed sequence

\[ \cdots \to F^n \to F^{n-1} \to F^{n-2} \to \cdots \]

where each \( F^i \) is a chain complex (resp. a spectrum) and the morphisms between them are chain maps\(^1\) (resp. morphisms of spectra).

Two of the motivating reasons for the study of filtered derived categories and filtered spectra are the construction of spectral sequences, and the existence of the Beilinson t-structure (firstly introduced in [Bei87]; see Definition 5.5). The objective of this paper is to present a new perspective on such objects (and more generally, filtered objects in stable ∞-categories) that allows to generalize at once both constructions in a homotopy coherent fashion, and to gain some insight on other related constructions, like the obstruction theory for the realizability of spectra with prescribed homotopy groups and k-invariants (see Section 8). This perspective will be realized by using coherent cochain complexes (originally introduced in their bounded flavor in [Joy08a, 35.1], in relation to the ∞-categorical Dold-Kan correspondence).

\(^1\)in the case of chain complexes it is common to allow only monomorphisms, but as one can always replace a chain map by a monomorphic one up to quasi-isomorphism, this extra condition will not be relevant to us
A coherent cochain complex in a stable ∞-category \( C \) is a homotopy coherent version of an ordinary cochain complex, and consists of a \( \mathbb{Z} \)-indexed sequence of objects \( C^i \in C \) and differentials
\[
\partial : \cdots \to C^n \to C^{n+1} \to C^{n+2} \to \cdots
\]
together with nullhomotopies \( \partial^2 \simeq 0 \), and further coherence data making all the nullhomotopies mutually compatible. Concretely, a coherent cochain complex will be defined as a pointed functor from (the nerve of) a 1-category having as objects the integers together with an extra base point, and being generated by morphisms \( \partial : n \to n+1 \) such that \( \partial^2 = 0 \) (see Definition 2.1 for a rigorous formulation). For the sake of simplicity, in this introduction we will restrict to the case of spectra, and the corresponding ∞-category of coherent cochain complexes \( \text{Ch}^*(\text{Sp}) \), although the results in the rest of the paper are discussed in the wider generality of stable ∞-categories equipped with some t-structure.

At their core, both the Beilinson t-structure and the spectral sequence associated to a filtered spectrum are in a sense “blind” to the information stored at the limit of the relevant filtered object. As it is customary, we say that a filtered object \( F \) is complete if \( \varprojlim F \simeq 0 \), and we call the cofiber of the canonical map \( \varprojlim F \to \varprojlim F \) the completion of \( \varprojlim F \). If \( F \) is a filtered spectrum, the spectral sequence it generates only abuts to the completion of its colimit. Something similar is true for the Beilinson t-structure, although saying precisely what this means requires more work; the failure of it being left complete is a good starting point: all and only the constant objects are the ones that are ∞-connected with respect to the Beilinson t-structure, and constant objects are in a precise sense orthogonal to complete ones (see Proposition 2.21). In Theorem 5.11, we prove that the Beilinson t-structure can be glued out of its restriction to the full subcategory of complete objects and the trivial t-structure on the full subcategory of constant objects.

As it turns out, the ∞-category of filtered spectra is equivalent to that of coherent cochain complexes of spectra:

**Theorem** (see Theorems 3.19, 4.7 and §6) There exists a symmetric monoidal adjunction
\[
\begin{array}{ccc}
\text{Fun}(\mathbb{Z}^{op}, \text{Sp}) & \xleftarrow{\hat{A}} & \text{Ch}^*(\text{Sp}) \\
\downarrow \approx & & \downarrow \\
\end{array}
\]
that restricts to an equivalence on complete filtered spectra. Moreover, this adjunction sends the Day convolution symmetric monoidal structure on filtered spectra, to a symmetric monoidal structure given componentwise by
\[
(C \otimes D)^n \simeq \bigoplus_{s+t=n} C^s \otimes D^t.
\]

As already noted in [Lur17, 1.2], given a filtered spectrum \( F \) one can construct a sequence of morphisms on suitable shifts of the graded pieces
\[
gr^n F[-n] \to gr^{n+1} F[-n-1]
\]
such that all pairwise compositions are nullhomotopic, and the homotopies are suitably compatible (this is in fact the first step for the construction of the spectral sequence of \( F \)). The left adjoint defined in the theorem sends \( F \) to the coherent cochain complex
\[
\cdots \to \text{gr}^{-2} F[-2] \to \text{gr}^{-1} F[-1] \to \text{gr}^0 F \to \text{gr}^1 F[1] \to \text{gr}^2 F[2] \to \cdots
\]
where the differentials are precisely the maps of (1). Although above we only represented the components and the differentials of the object \( \hat{A}F \), it also encodes much more data: \( \text{Ch}^*(\text{Sp}) \) is defined using a pointed diagram 1-category and pointed functors, hence its objects keep track also of the nullhomotopies for the two-fold compositions of their differentials, and of all the recursive coherences between them. The compatibilities between all the nullhomotopies can be made explicit by means of
higher Toda brackets (see Section 7); in order to do so, we use an explicit $E_1$ presentation of exterior algebras over $\mathbb{Z}$ due to Achim Krause, presented in Appendix B. The left adjoint above is really a homotopy coherent version of the construction of homotopy objects in Beilinson’s $t$-structure; such $t$-structure corresponds to the pointwise one along the left adjoint functor discussed above; that is, for a filtered spectrum $F$,

$$\pi^B_n F \cong \pi^l_0 A F,$$

where $\pi^l_0 C$ denotes the functor applying $\pi_0$ to all the components of $C$.

The right adjoint can be thought of as the functor iteratively solving all the extension problems needed to reconstruct the filtered object from its graded pieces and the differentials. Notice that, in order for such a reconstruction to be possible in general, one really needs to use all the information stored in the higher morphisms. In fact, the differentials appearing in a coherent cochain complex encode precisely the differentials of the $E_1$ page of a spectral sequence abutting to the total homology of the complex (that is, the object underlying its associated filtered object). The higher pages together with their differentials can be recovered from the complex itself, by means of an incarnation of Deligne’s décalage construction, as explained in Section 9.

1.1. Outline of the paper. In Section 2, we put the basis for the rest of the paper, by recalling the main definitions and some structural results about the $\infty$-categories at hand. In particular, we show that for a stable $\infty$-category $\mathcal{C}$, both $\text{Fun}(\mathbb{Z}^{\text{op}}, \mathcal{C})$ and $\text{Ch}^* \mathcal{C}$ sit in suitable recollements of stable $\infty$-categories.

Section 3 is the technical core of the paper, in which we construct the adjunction between filtered spectra and cochain complexes of spectra, proving the equivalence of the latter with the $\infty$-category of complete filtered spectra. Then, in Section 4, we generalize the result to all stable $\infty$-categories with sequential (inverse) limits.

In Section 5, we analyze the interplay between the Beilinson $t$-structure and the levelwise $t$-structure on the relevant categories, when the base $\infty$-category $\mathcal{C}$ comes equipped with a $t$-structure. We continue the analysis of $\infty$-categories with extra structure in Section 6, where we study the interplay of the equivalence of Theorem 4.7 with symmetric monoidal structures, and the compatibility of the relevant $t$-structures in the case where the base $\mathcal{C}$ is also equipped with a $t$-structure.

In Section 7, we use an explicit $E_1$ presentation of exterior algebras over $\mathbb{Z}$, proven by Achim Krause in Appendix B, to show that $\text{Ch}^* \mathcal{C}$ is the universal $\infty$-category of $\mathbb{Z}^{\text{op}}$-indexed sequences of morphisms in $\mathcal{C}$ such that any pairwise composition is trivial, as are all the possible Toda brackets.

In Section 8, we show how coherent cochain complexes encode a form of obstruction theory, and we use the formalism developed there to recover a spectral sequence from a coherent cochain complex in Section 9.

Finally, in Section 10, we have a look at a few examples from the recent literature that we believe benefit from the perspective of coherent cochain complexes.

Some of the main tools for constructing the adjunction and proving the equivalence of Theorem 4.7 is the stable nerve-realization paradigm. We recollect and prove many results about the $\infty$-categorical incarnation of this topic in Appendix A.

1.2. Notational conventions. Throughout, we will use the following notations and terminology:

1. We will denote by $\mathcal{P}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \delta)$

the $\infty$-category of presheaves on $\mathcal{C}$.

2. We will denote by $\mathcal{P}_{st}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \text{Sp})$

the $\infty$-category of spectra-valued presheaves on $\mathcal{C}$, and sometimes refer to it as the $\infty$-category of stable presheaves on $\mathcal{C}$.
(3) We will denote by \( \jmath : \mathcal{C} \to \mathcal{P}(\mathcal{C}) \)
the Yoneda embedding, and by \( \jmath_C \) the functor represented by an object \( C \in \mathcal{C} \).
(4) Given any functor \( \infty \)-category \( \text{Fun}(\mathcal{C}, \mathcal{D}) \), we will denote by
\( \text{cns} : \mathcal{D} \to \text{Fun}(\mathcal{C}, \mathcal{D}) \)
the (fully faithful) functor obtained by precomposition with the terminal functor \( \mathcal{C} \to \Delta^0 \), and by \( \text{cns}_C \) its value at \( C \in \mathcal{C} \).
(5) When considering \( \mathbb{Z} \) or \( \mathbb{N} \) as categories, we will always implicitly assume that
they are given the structure of a poset category, with their usual ordering \( \leq \). We will use the notations \( \mathbb{Z}^\delta \) and \( \mathbb{N}^\delta \) to denote their underlying discrete categories.
(6) Unless otherwise stated, whenever we refer to \( \mathbb{Z} \), \( \mathbb{Z}^{\text{op}} \) or \( \mathbb{Z}^\delta \) as symmetric monoidal categories, we consider them endowed with the symmetric monoidal structure induced by addition.
(7) We refer to a presentable \( \infty \)-category endowed with a symmetric monoidal structure whose tensor product preserves colimits in each variable as a presentably symmetric monoidal \( \infty \)-category.
(8) We refer to a stable \( \infty \)-category endowed with a symmetric monoidal structure whose tensor product is exact in each variable as a stably symmetric monoidal \( \infty \)-category.
(9) We refer to a presentable stable \( \infty \)-category endowed with a symmetric monoidal structure whose tensor product preserves colimits in each variable as a stable presentably symmetric monoidal \( \infty \)-category.
(10) We will most of the times omit the Eilenberg–MacLane functor from the notations, when considering an Abelian group as a spectrum.

Remark 1.1 Throughout, when dealing with t-structures, we will use the homological grading convention. When dealing with spectral sequences, we will use the cohomological Serre grading convention. For the sake of clarity, we decided to stick with the choice of working exclusively with decreasing filtered objects and coherent cochain complexes. The results of this paper translate immediately to the case of increasing filtrations and coherent chain complexes, with the caveat that the equivalence of Theorem 4.7 always inverts the direction of the \( \mathbb{Z}^\delta \)-indexed arrows, hence complete increasing filtered objects are equivalent to coherent chain complexes.

Remark 1.2 Throughout, we (mostly implicitly) work in \( \text{ZFC+U} \), where “U” is Tarski’s axiom: “For each set \( x \), there exists a Grothendieck universe \( U \) such that \( x \in U \)”. Equivalently we assume the existence of infinitely many strongly inaccessible cardinals, and we fix one as the cardinality of our universe of small sets. As we are agnostic about the size of the universe fixed, all the arguments that do not rely on dealing with a bigger universe hold regardless of the actual size of the objects referred to as “small sets”. In those few cases where we need to deal with more than one universe at a time, we are going to be explicit about relative sizes.

1.3. Related works. Some of the structural results about filtered objects in stable \( \infty \)-categories contained in Section 3 have been presented for the case of filtered spectra in [Lur15, Sections 2-3]. The \( \infty \)-category of filtered objects of a stable \( \infty \)-category has also been studied in [GP18, Section 2], and some results in Section 3 and Section 6 overlap with loc. cit. In [Rak20], Raksit considers an alternative formulation for coherent chain complexes; although we don’t delve into a detailed comparison, the results of Section 7, should give a clear idea about the relation between the formulation in op. cit. and the one in this paper. In [Wal19], Walde proves the equivalence between \( \text{Fun}(\mathbb{N}, \mathcal{C}) \) and \( \text{Ch}_{\geq 0}(\mathcal{C}) \) through their equivalence with \( \text{Fun}(\Delta^{\text{op}}, \mathcal{C}) \). A

\[ \text{the reader should note that in [GP18] the authors refer to what we call filtered objects as “sequences”, and reserve the term “filtered objects” for what we call complete filtered objects.} \]
similar approach to the construction of the spectral sequence of a filtered spectrum through the décalage functor that does not use the language of coherent cochain complexes appeared recently in Hedenlund’s Ph.D. thesis [Hed21]. The décalage functor introduced in Section 9 is further analyzed and discussed in forthcoming work by Hedenlund–Krause–Nikolaus.

1.4. Prerequisites. We assume the reader is familiar with the theory of ∞-categories as developed in [Lur09] and with the contents of [Lur17]. In particular, we assume the reader is thoroughly acquainted with the theory of t-structures as developed in [Lur17, 1.2.1] (see also [AN20, Appendix A]). Some of the terminology and notations in this paper differ from the ones used in [Lur09] and [Lur17], but we use the same notations and terminology of op. cit. for the concepts we do not explicitly recall or introduce here.

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2. Definitions and preliminaries

In this section we introduce and discuss a few key facts about coherent cochain complexes and filtered objects in ∞-categories. In particular, in Proposition 2.20 and Proposition 2.21 we prove that both constructions figure in suitable recollements (see Definition 2.13) of stable ∞-categories.

Definition 2.1 [Joy08a, 35.1] Let Ch be the pointed (ordinary) category given by

\[ \text{ob } Ch = \mathbb{Z} \cup \{ \text{pt} \} \]

\[ Ch(n, m) = \begin{cases} \{ \text{id}, 0 \} & \text{if } m = n; \\ \{ \partial_n, 0 \} & \text{if } m = n - 1; \\ \{ 0 \} & \text{otherwise}. \end{cases} \]

where pt ∈ Ch is a zero object, and 0 is the zero map.

Given any pointed ∞-category C

(1) the ∞-category of coherent chain complexes in C is the full subcategory

\[ Ch_\ast(C) := \text{Fun}^0(Ch, C) \subset \text{Fun}(Ch, C) \]

spanned by pointed functors.

(2) The ∞-category of coherent cochain complexes is the full subcategory

\[ Ch^\ast(C) := \text{Fun}^0(Ch^{op}, C) \subset \text{Fun}(Ch^{op}, C) \]

spanned by pointed functors.

(3) An object C ∈ Ch^\ast(C) is bounded above if there exists an n ∈ Z such that

\[ C_k \simeq 0 \text{ for all } k > n. \]

(4) An object C ∈ Ch^\ast(C) is bounded below if there exists an n ∈ Z such that

\[ C_k \simeq 0 \text{ for all } k < n. \]

(5) An object C ∈ Ch^\ast(C) is bounded if it is bounded above and bounded below.

Remark Notice that 0 ∈ Z is an object of Ch, but it is not its zero object.

Example 2.2 Let A be an Abelian category, then Ch^\ast(A) is the usual category of cochain complexes of A.
In what follows, we will exclusively focus our attention on coherent cochain complexes.

**Notation 2.3** Given a coherent cochain complex $C \in \text{Ch}^* \mathcal{C}$, we will denote $C(\partial_n)$ by $\partial^n$, or just by $\partial^n$ if there is no risk of confusion.

**Definition 2.4** Let $\mathcal{C}$ be any $\infty$-category.

1. The category $\text{Fil}^{\downarrow} \mathcal{C} := \text{Fun}(\mathbb{Z}^{\text{op}}, \mathcal{C})$ is called the $\infty$-category of (decreasing) filtered objects of $\mathcal{C}$.
2. The category $\text{Fil}^{\uparrow} \mathcal{C} := \text{Fun}(\mathbb{Z}, \mathcal{C})$ is called the $\infty$-category of (increasing) filtered objects of $\mathcal{C}$.
3. Given $F \in \text{Fil}^{\downarrow} \mathcal{C}$, we call $F^{-\infty} := \text{colim}_i F_i$ the underlying object of $F$.
4. Given $F \in \text{Fil}^{\downarrow} \mathcal{C}$, we say that $F$ is complete if $F^{+\infty} := \text{lim}_i F_i \simeq 0$. We will denote by $\text{Fil}^{\downarrow} \hat{\mathcal{C}}$ the full subcategory of $\text{Fil}^{\downarrow} \mathcal{C}$ spanned by complete objects.

Throughout, we will concentrate on the case of descending filtrations.

**Definition 2.5** Let $\mathcal{C}$ be an $\infty$-category with cofibers and countable coproducts. Precomposition with the inclusion map $\iota_n : \Delta^\{n+1, n\} \to \mathbb{Z}^{\text{op}}$ induces functors $(\iota_n)^* : \text{Fil}^\downarrow \mathcal{C} \to \text{Fun}(\Delta_1^\text{op}, \mathcal{C})$.

1. We will define the $n$-th graded functor $\text{gr}^n$ as the composite functor $\text{Fil}^\downarrow \mathcal{C} \xrightarrow{(\iota_n)^*} \text{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{\text{cof}} \mathcal{C}$.
2. We define the associated graded functor $\text{gr}$ as the composite functor $\text{Fil}^\downarrow \mathcal{C} \xrightarrow{\text{gr}^n} \prod_{n \in \mathbb{Z}} \mathcal{C} \xrightarrow{\bigoplus} \mathcal{C}$.

**Definition 2.6** We say that a map $\alpha : F \to G$ in $\text{Fil}^\downarrow \mathcal{C}$ is a graded equivalence if $\alpha : F \to G$ is such that $\text{gr}(\alpha)$ is an equivalence. Equivalently, if for all $n \in \mathbb{Z}$, the dotted map

$$
\begin{array}{ccc}
F^{n+1} & \longrightarrow & F^n \\
\downarrow & & \downarrow \\
G^{n+1} & \longrightarrow & G^n
\end{array}
$$

induced by universality of cofibers is an equivalence.

**Notation 2.7** Let $\mathcal{C}$ be an $\infty$-category. For $n, m \in \mathbb{Z} \cup \{+\infty, -\infty\}$, with $n \leq m$, we use the notation $F^n/F^m := \text{cof}(F^m \to F^n)$ to denote the cofiber of the evident map.

One of the nice features of ordinary cochain complexes is the possibility to write them as limits of bounded above (or below) ones. This feature is still present in the coherent setting, as we now show.

**Construction 2.8** Given any integer $n$, let $\text{Ch}^{\{n\}} \mathcal{C}$ denote the full subcategory of $\text{Ch}$ spanned by $\{\text{pt}, n, n-1, n-2, \cdots\}$. Let now $\mathcal{C}$ denote a pointed complete $\infty$-category. Then, each inclusion $\iota^n : \text{Ch}^{\{n\}} \to \text{Ch}$ induces, by right Kan extension along it, an adjunction

$$
\text{Ch}^*(\mathcal{C}) \xleftarrow{\iota^n} \text{Fun}^0(\text{Ch}^{\text{op}}^{\{n\}}, \mathcal{C}) \xrightarrow{(\iota^n)^*}
$$

\(^3\text{the position of the arrows in the notation is meant to remind the common convention of using upper indices for decreasing filtrations, and lower indices for increasing filtrations.}

\(^4\text{or, more generally, such that all the relevant Kan extensions exist and are pointwise.} \)
For any given $C \in \text{Ch}^*(\mathcal{C})$, we can compute explicitly the values of $\iota^n_* C$. As $\iota^n$ is fully faithful, the only values we need to determine are the ones for $M > n$:

$$\iota^n_* C(M) \overset{(A.12)}{=} \int_{s \in \text{Ch}^*_s} [\text{Map}_{\text{Ch}^s}(M, s), C^s]$$

$$\overset{(A.9)}{=} \lim_{\to} C$$

where the last limit is 0 as the indexing category is pointed, and $C$ preserves the zero object. If we now denote by $(-)_{\leq m} := \iota^m_*(\iota^m)^*$, we have that the unit of the adjunction $\iota^n \dashv \iota^n_*$ induces an endofunctor of $\text{Ch}^*(\mathcal{C})$, denoted $(-)_{\leq n}$, sending a complex $C$ to the complex $C_{\leq n}$, which in degree $m$ is given by

$$\begin{cases}
C^m & \text{if } m \leq n \\
0 & \text{else}.
\end{cases}$$

Similarly, the inclusions $\iota^{n,n+1}_*: \text{Ch}_{(-n,n]} \to \text{Ch}_{(-n,n+1]}$ induce adjunctions

$$\text{Fun}^0 \left( \text{Ch}^*_{(-n,n+1]}, \mathcal{C} \right) \leftrightarrow \text{Fun}^0 \left( \text{Ch}^*_{(-n,n+1]}, \mathcal{C} \right).$$

Notice that we have a natural equivalence

$$\left( \iota^{n+1} \circ \iota^{n,n+1} \right)^* \simeq \left( \iota^{n,n+1} \right)^* \left( \iota^{n+1} \right)^* \simeq \left( \iota^n \right)^*;$$

by passing to adjoints, we get a natural transformation

$$\left( \iota^{n+1} \right)^* \Rightarrow \iota^{n,n+1}_* (\iota^n)^*;$$

if we now precompose (2) with $\iota^{n+1}_*$ (using that adjunctions compose), we get a natural transformation

$$\iota^{n+1}_*: (-)_{\leq n+1} \Rightarrow (-)_{\leq n}.$$ 

By inspection, the above is given pointwise by

$$(\iota^{n+1}_*)^m \overset{\text{id}}{\simeq} \begin{cases}
\text{id}: C^m \to C^m & \text{if } m \leq n \\
0: C^{n+1} \to 0 & \text{if } m = n + 1 \\
0: 0 \to 0 & \text{if } m \geq n + 2.
\end{cases}$$

**Remark 2.9** Similarly to what we did in Construction 2.8, one can truncate below a certain integer. It is also possible to consider left Kan extensions along the $\iota^n$’s, and the induced counits to obtain different truncations with $\text{cof} \partial^m$ in degree $n + 1$, for truncations above $n$, or $\text{fib} \partial^m$ in degree $n - 1$, for truncations below $n$. In what follows, we won’t need any of such variants.

**Lemma 2.10** Let $\mathcal{C}$ be a pointed complete $\infty$-category. Then, for any $C \in \text{Ch}^* \mathcal{C}$

$$C \simeq \lim \left( \cdots \to C_{\leq n+1} \xrightarrow{\iota^n_*} C_{\leq n} \to \cdots \right)$$

(where the $\iota^n$’s are the natural transformations defined in Construction 2.8).

**Proof.** It follows from Proposition 2.20 that limits in $\text{Ch}^* \mathcal{C}$ can be computed object-wise. As for any $m \in \mathbb{Z}$ the sequence

$$\cdots \to \left( C_{\leq n+1} \right)^m \xrightarrow{\iota^n_*} \left( C_{\leq n} \right)^m \to \cdots$$

is eventually constant on the left, the result follows.

**Lemma 2.11** Given any pointed $\infty$-category $\mathcal{C}$, the functor $u: \text{Ch}^*(\mathcal{C}) \to \prod_{\mathbb{Z}} \mathcal{C}$ induced by precomposition with $\mathbb{Z}^\partial \to \text{Ch}^\text{op}$ is conservative.
Proof. As equivalences in ∞-categories are detected at the level of homotopy categories, this is clear. □

Proposition 2.12 Let C be a complete and cocomplete semiadditive ∞-category, and let n* : Fun(Ch^op, C) → Fun(Δ^0, C) denote precomposition with n : Δ^0 → Ch^op. The “evaluation at n” functor ev_n : Ch^*(C) ⊂ Fun(Ch^op, C) n* −→ C, admits both a left and a right adjoint. Such adjoints are given objectwise by

\[(ev_n)^* X \simeq \begin{cases} X & \text{if } m = n - 1, n \\ 0 & \text{else.} \end{cases} \]

with the identity as the only nontrivial differential, and

\[(ev_n)! X \simeq \begin{cases} X & \text{if } m = n, n + 1 \\ 0 & \text{else.} \end{cases} \]

with the identity as the only nontrivial differential.

Proof. The functor n* admits both adjoints n! and n*, given respectively by left and right Kan extension. As adjoint functors compose, it follows from Proposition 2.20 that the right Kan extension (ev_n)^* is given by the composite pt ⋙ n* , and the left Kan extension (ev_n)! is given by pt ⋙ n. By Remark A.12, n* is given by

\[n_* X^m = [\text{Map}_{\text{Ch}^\text{op}}(m, n), X] = \begin{cases} X \oplus X & \text{if } m = n - 1, n \\ X & \text{else.} \end{cases} \]

with the differentials n_* X^m → n_* X^{m+1} determined by

\[\text{Map}_{\text{Ch}^\text{op}}(m + 1, n) \xrightarrow{(\partial^m)^*} \text{Map}_{\text{Ch}^\text{op}}(m, n)\]

and thus given by

\[\cdots \xrightarrow{id} X \xrightarrow{id} X \oplus X \xrightarrow{(id)} X \oplus X \xrightarrow{(id)} X \xrightarrow{id} \cdots \] (3)

in degrees n - 2 to n + 1, and by identities elsewhere.

By Lemma A.11, n! is given by

\[n_! X^m = \text{Map}_{\text{Ch}^\text{op}}(n, m) \otimes X = \begin{cases} X \oplus X & \text{if } m = n + 1 \\ X & \text{else.} \end{cases} \]

with the differentials n_! X^m → n_! X^{m+1} determined by

\[\text{Map}_{\text{Ch}^\text{op}}(n, m + 1) \xrightarrow{(\partial^m)!} \text{Map}_{\text{Ch}^\text{op}}(n, m)\]

and thus given by

\[\cdots \xrightarrow{id} X \xrightarrow{id} X \oplus X \xrightarrow{(id)} X \oplus X \xrightarrow{\text{id}} X \xrightarrow{id} \cdots \] (4)

in degrees n - 1 to n + 2, and by identities elsewhere.

By the explicit description of pt given in Proposition 2.20, we get the formulas for (ev_n)^* and (ev_n)!.

We now recall some definitions and facts about recollements, which we will use extensively in the following sections; we consider only recollements in the case of stable ∞-categories, but the theory holds in greater generality; see also [FL15], [BG16] and [Lur17, A.8].

Definition 2.13 Let C be a stable ∞-category, and let i : C_0 ↪ C and j : C_1 ↪ C be full subcategories. We say that C is a recollement of the essential image of i and the essential image of j if:

\[\cdots \xrightarrow{id} X \xrightarrow{id} X \oplus X \xrightarrow{(id)} X \oplus X \xrightarrow{id} X \xrightarrow{id} \cdots \]
(1) Both $i$ and $j$ admit left adjoints:

$$
\begin{array}{ccc}
\mathcal{C}_0 & \overset{i_L}{\leftarrow} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C}_1 & \overset{j_L}{\rightarrow} & \mathcal{C}
\end{array}
$$

(2) The functor $j_L$, left adjoint to $j$, carries every object of $\mathcal{C}_0$ to zero;

(3) If $\alpha$ is a morphism of $\mathcal{C}$ such that $i_L(\alpha)$ and $j_L(\alpha)$ are equivalences, then $\alpha$ is an equivalence.

**Remark 2.14** It follows from [Lur17, A.8.5, A.8.19] that if $\mathcal{C}$ is a recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$, then we actually have the following adjunctions

$$
\begin{array}{ccc}
\mathcal{C}_0 & \overset{i_L}{\leftarrow} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C}_1 & \overset{j_L}{\rightarrow} & \mathcal{C}
\end{array}
$$

where $(j_L)_!$ is fully faithful, and $i_R$ is such that

$$ii_R \rightarrow \text{id}_{\mathcal{C}} \rightarrow jj_L$$

is a co/fiber sequence.

**Proposition 2.15** [Lur17, Proposition A.8.20] Let $\mathcal{C}$ be a stable $\infty$-category, and let $i: \mathcal{C}_0 \rightarrow \mathcal{C}$ be a fully faithful functor. The following are equivalent:

1. The functor $i$ admits a left adjoint and a right adjoint;
2. There exists a full subcategory $j: \mathcal{C}_1 \hookrightarrow \mathcal{C}$, closed under equivalences, such that $\mathcal{C}$ is the recollement of the essential images of $i$ and $j$.

Moreover, if the conditions above hold, we can identify $\mathcal{C}_1$ with the full subcategory $\mathcal{C}_1 \subseteq \mathcal{C}$ spanned by those objects $X \in \mathcal{C}$ such that for all $C \in \mathcal{C}_0$, $\text{Map}_{\mathcal{C}}(C, X) \simeq \text{pt}$.

Recollements are strictly related to semiorthogonal decompositions.

**Definition 2.16** Let $\mathcal{C}$ be a stable $\infty$-category. A semiorthogonal decomposition of $\mathcal{C}$ is the datum of two full subcategories $\mathcal{C}_0$ and $\mathcal{C}_1$ of $\mathcal{C}$ such that

1. $\mathcal{C}_1 \simeq \mathcal{C}_0^\perp$;
2. Every object $C \in \mathcal{C}$ sits in a cofiber sequence

$$
C_0 \rightarrow C \rightarrow C_1
$$

where $C_0 \in \mathcal{C}_0$ and $C_1 \in \mathcal{C}_1$.

**Remark 2.17** It follows from Remark 2.14 that every recollement determines a semiorthogonal decomposition.

We learned the following argument from [BG16, Proof of Lemma 3], and we present it here almost verbatim for the reader’s convenience.

**Proposition 2.18** The commutative square

$$
\begin{array}{ccc}
\text{id}_{\mathcal{C}} & \overset{\eta}{\rightarrow} & ii_L \\
\downarrow & & \downarrow \\
jj_L & \overset{\eta jj_L}{\rightarrow} & ii_L jj_L
\end{array}
$$

is Cartesian in $\text{Fun}(\mathcal{C}, \mathcal{C})$.

**Proof.** It follows from Remark 2.14 that the fiber of the vertical maps is given by $\eta ii_R: ii_R \rightarrow ii_L ii_R$, which, as $ii_L \simeq \text{id}$, is an equivalence. $\square$
Remark 2.19 In particular, every recollement determines a “fracture square”

\[
\begin{array}{ccc}
C & \longrightarrow & i_L C \\
\downarrow & & \downarrow \\
j_L C & \longrightarrow & i_L j_L C
\end{array}
\]

for any object \( C \in \mathcal{C} \).

We are now ready to prove the main results of this section.

Proposition 2.20 Let \( \mathcal{C} \) be a stable \( \infty \)-category that is both complete and cocomplete. The \( \infty \)-category \( \text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C}) \) is the recollement of the essential images of \( \text{cns} : \mathcal{C} \to \text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C}) \) and \( \text{Ch}^*(\mathcal{C}) \):

\[
\begin{array}{ccc}
\mathcal{C} & \overset{\text{colim}}{\longrightarrow} & \text{cns} \longrightarrow \text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C}) \overset{\text{lim}}{\longleftarrow} \text{Ch}^*(\mathcal{C})
\end{array}
\]

Moreover, the inclusion \( i : \text{Ch}^*(\mathcal{C}) \to \text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C}) \) is both left and right adjoint to a functor \( \text{pt} \), given by

\[
\text{pt} F(n) \simeq F(n)/F(\text{pt})
\]
on objects. Finally, the \( \infty \)-category \( \text{Ch}^*(\mathcal{C}) \) is stable, complete and cocomplete; if \( \mathcal{C} \) is presentable, \( \text{Ch}^*(\mathcal{C}) \) is presentable as well.

Proof. The functor \( \text{cns} \) admits a left and a right adjoint, given respectively by left and right Kan extension along the terminal morphism (i.e. the colim and lim functors). Hence, by Proposition 2.15, \( \text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C}) \) is the recollement of (the essential image of) \( \text{cns} \) and the full subcategory \( \text{cns}(\mathcal{C}) \) orthogonal to it. We claim that

\[
\text{cns}(\mathcal{C})^\perp \simeq \text{Ch}^*(\mathcal{C}).
\]

(5)
To see this, recall that \( \text{cns}(\mathcal{C})^\perp \) is spanned by those objects \( F \in \text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C}) \) such that for all \( X \in \mathcal{C} \) the mapping space \( \text{Map}_{\text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C})}(\text{cns}(\mathcal{C}), F) \) is contractible. But, as

\[
\text{Map}_{\text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C})}(\text{cns}(\mathcal{C}), F) \simeq \text{Map}_{\mathcal{C}}(X, F(\text{pt})),
\]

we see that (5) holds. We denote by \( \text{pt} \) the left adjoint to the fully faithful inclusion

\[
i : \text{Ch}^*(\mathcal{C}) \rightarrow \text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C}).
\]

Now, \( \text{Ch}^*(\mathcal{C}) \) is stable by [Lur17, A.8.17] (or just because of the fact that (co)limits in functor categories are computed pointwise) and, if \( \mathcal{C} \) is presentable (as \( \text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C}) \) is presentable by [Lur09, 5.5.3.6]) \( \text{Ch}^*(\mathcal{C}) \) is presentable by [Lur17, 1.4.4.9]. By [Lur17, A.8.5], we see that

\[
\text{pt} F \simeq \text{cof} \left( \text{cns}(\mathcal{C})/F(\text{pt}) \right)
\]

(6)
and in particular \( \text{pt} F(n) \simeq F(n)/F(\text{pt}) \). The existence of a left adjoint for \( \text{pt} \) follows from Remark 2.14. To give a description, let’s first note that, by (6) and the fully faithfulness of the inclusion \( i \) we can write

\[
\text{Map}_{\text{Ch}^*(\mathcal{C})}(C, \text{pt} X) \simeq \text{Map}_{\text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C})}(C, \text{cof}(\text{cns}(\mathcal{C}))/X)
\]

\[
\simeq \text{Map}_{\text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C})}(C[-1], \text{fib}(\text{cns}(\mathcal{C}))/X)
\]

and, as corepresentable functors commute with limits, the latter is equivalent to

\[
\text{fib} \left( \text{Map}_{\text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C})}(C[-1], \text{cns}(\mathcal{C})) \rightarrow \text{Map}_{\text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C})}(C[-1], X) \right)
\]

which in turn, as \( \text{cns} \) is right adjoint to colim, can be computed as

\[
\text{fib} \left( \text{Map}_{\mathcal{C}}(C[-1], \text{pt}) \rightarrow \text{Map}_{\text{Fun}(\text{Ch}^{\text{op}}, \mathcal{C})}(C[-1], X) \right).
\]

(7)
But, as the colimit of a coherent cochain complex is always 0, we have that

\[
\text{Map}_{\mathcal{C}}(C[-1], \text{pt}) \simeq \text{pt}
\]
and thus (7) is equivalent to
\[
\text{fib} \left( \text{pt} \rightarrow \text{Map}_{\text{Fun}(\text{Ch}^{op}, \mathcal{E})} (C[-1], X) \right) \\
\simeq \Omega \text{Map}_{\text{Fun}(\text{Ch}^{op}, \mathcal{E})} (C[-1], X) \\
\simeq \text{Map}_{\text{Fun}(\text{Ch}^{op}, \mathcal{E})} (C, X).
\]

In particular, as \( \text{pt} \) is both left and right adjoint to the inclusion, we have that \( \text{Ch}^{+}(\mathcal{E}) \) is closed under all limits and colimits in \( \text{Fun}(\text{Ch}^{op}, \mathcal{E}) \), which is by hypothesis complete and cocomplete. \( \square \)

**Proposition 2.21** Let \( \mathcal{C} \) be a stable \( \infty \)-category that is both complete and cocomplete. The \( \infty \)-category \( \text{Fil}^{l} \mathcal{C} \) is the recollement of the essential images of \( \text{cns}: \mathcal{C} \rightarrow \text{Fil}^{l} \mathcal{C} \) and \( i: \text{Fil}^{l} \mathcal{C} \rightarrow \text{Fil}^{\hat{l}} \mathcal{C} \):

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{i} & \text{Fil}^{l} \mathcal{C} \\
\colim & \searrow & \text{Fil}^{l} \mathcal{C} \\
\lim & \nearrow & \text{Fil}^{\hat{l}} \mathcal{C}
\end{array}
\]

The left adjoint to the inclusion \( i \) is given by Bousfield localization at the class of graded equivalences, and computed as
\[
LF^{n} \simeq F^{n}/F^{+\infty}.
\]
Moreover, the \( \infty \)-category \( \text{Fil}^{\hat{l}} \mathcal{C} \) is stable, complete and cocomplete; if \( \mathcal{C} \) is presentable, \( \text{Fil}^{\hat{l}} \mathcal{C} \) is presentable as well.

**Proof.** The proof is completely analogous to the proof of Proposition 2.20. The only new element is the identification of the local maps for the Bousfield localization determined by \( L \). In order to prove it, let us notice that \( L\alpha \) is an equivalence if and only if \( \text{cof} L\alpha \simeq L(\text{cof} \alpha) \simeq 0 \), which in turn, by [Lur17, A.8.5], is the case if and only if \( \text{cof} \alpha \) is essentially constant; by inspection of the following diagram

\[
\begin{array}{ccc}
F^{n+1} & \rightarrow & F^{n} \\
\downarrow & & \downarrow \\
G^{n+1} & \rightarrow & G^{n} \\
\downarrow & & \downarrow \\
(\text{cof} \alpha)^{n+1} & \rightarrow & (\text{cof} \alpha)^{n}
\end{array}
\]

where all the rows and columns are co/fiber sequences, we see that this is the case if and only if \( \alpha \) is a graded equivalence. \( \square \)

We conclude this section with the following fact about fully faithful adjoint functors, which we will use later.

**Proposition 2.22** Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \infty \)-categories, and let

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{F} & \mathcal{D} \\
\downarrow & & \downarrow \eta \\
\mathcal{C} & \xrightarrow{G} & \mathcal{D}
\end{array}
\]

be an adjunction. If there exists a natural isomorphism \( \alpha: \text{id}_{\mathcal{C}} \Rightarrow GF \) then the unit of the adjunction is an equivalence (equivalently, \( F \) is fully faithful).

**Proof.** By [RV16], it is possible to associate to any adjunction a homotopy coherent monad on \( GF \), whose 1-skeletal part looks as follows

\[
\begin{array}{ccc}
\text{id}_{\mathcal{C}} & \xrightarrow{\eta} & GF \\
\text{GF} & \xleftarrow{\eta} & \text{GF}GF \\
\text{GF} & \xrightarrow{\eta} & \text{GF}GF \\
\text{GF} & \xrightarrow{\eta} & \text{GF}GF \\
& & \cdots
\end{array}
\]

as a diagram in \( \text{Fun}(\mathcal{C}, \mathcal{D}) \). As, by hypothesis, \( \text{id}_{\mathcal{C}} \simeq GF \) via some \( \alpha \), the homotopy coherent monad structure on \( GF \) transfers to a homotopy coherent monad on \( \text{id}_{\mathcal{C}} \),
whose 1-skeletal part looks as follows
\[
\begin{array}{c}
\text{id}_\mathbb{C} \xrightarrow{\tilde{\eta}} \text{id}_\mathbb{C} \xrightarrow{\mu} \text{id}_\mathbb{C} \xrightarrow{\tilde{\eta}} \text{id}_\mathbb{C} \xrightarrow{\mu} \text{id}_\mathbb{C} \\
\end{array}
\]
Now, by unitality, \(\mu \tilde{\eta} \simeq \text{id}_{\text{id}_\mathbb{C}}\), and as the following diagram commutes by naturality of \(\mu\) (or of \(\tilde{\eta}\)),
\[
\begin{array}{c}
\text{id}_\mathbb{C} \xrightarrow{\tilde{\eta}} \text{id}_\mathbb{C} \\
\mu \downarrow \quad \downarrow \mu \\
\text{id}_\mathbb{C} \xrightarrow{\tilde{\eta}} \text{id}_\mathbb{C}
\end{array}
\]
we see that \(\tilde{\eta} \mu \simeq \text{id}_{\text{id}_\mathbb{C}}\). Thus, \(\tilde{\eta}\) is an equivalence, and so is \(\eta\). \(\square\)

3. Filtered spectra and cochain complexes of spectra

Our goal in this section is to construct an equivalence between the \(\infty\)-categories \(\text{Fil}^\dagger_{\mathbb{S}p}\) and \(\text{Ch}^\ast(\mathbb{S}p)\). In order to do so, we will first construct a pair of adjoint functors between \(\text{Fil}^\dagger_{\mathbb{S}p}\) and \(\text{Ch}^\ast(\mathbb{S}p)\) using the machinery of Appendix A, and then prove that the right adjoint is a fully faithful functor having \(\text{Fil}^\dagger_{\mathbb{S}p}\) as its essential image. Along the way, we compute explicitly the values of the pair of adjoints constructed abstractly.

In order to construct the left adjoint, let us start with the following observation.

**Lemma 3.1** Restriction along \(\Sigma_{\infty}^+ : \mathbb{Z} \rightarrow \mathcal{P}_{st}(\mathbb{Z})\) induces an equivalence of \(\infty\)-categories
\[
\text{Fun}^L(\text{Fil}^\dagger(\mathbb{S}p), \text{Ch}^\ast(\mathbb{S}p)) \simeq \text{Fun}(\mathbb{Z}, \text{Ch}^\ast(\mathbb{S}p))
\]
whose inverse is given by associating to every functor \(F : \mathbb{Z} \rightarrow \text{Ch}^\ast(\mathbb{S}p)\) its stable realization functor (see Definition A.17).

**Proof.** As, by Proposition 2.20, \(\text{Ch}^\ast(\mathbb{S}p)\) is a stable \(\infty\)-category, this is just an application of Lemma A.18, together with the observation that \(\text{Fil}^\dagger(\mathbb{S}p) \simeq \mathcal{P}_{st}(\mathbb{Z})\). \(\square\)

According to the previous lemma, we will have to provide a functor \(\mathbb{Z} \rightarrow \text{Ch}^\ast(\mathbb{S}p)\) in order to get the wanted left adjoint. A priori, constructing such a functor would require to keep track of an infinite amount of coherences, but it turns out that such coherences are essentially trivial. This idea is made precise by the following results. The content of Proposition 3.3 is a corollary of [Lur19, Tag 00J6]; we report a direct proof here for the reader’s convenience.

**Notation 3.2** For any \(m, n \in \mathbb{Z}\), let \(I_{[m,n]}\) denote the simplicial set
\[
\Delta^{\{m,m-1\}} \coprod_{\Delta^{\{n-1\}}} \cdots \coprod_{\Delta^{\{n-2\}}} \Delta^{\{n-2,n-1\}} \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}
\]
which can be informally represented as
\[
\begin{array}{c}
m \quad \cdots \quad n-1 \\
\end{array}
\]
Similarly, we denote by \(I_{[-\infty,n]}\) and \(I_{[n,\infty]}\) respectively, the simplicial sets
\[
\cdots \coprod_{\Delta^{\{n-2\}}} \Delta^{\{n-2,n-1\}} \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \coprod_{\Delta^{\{n\}}} \Delta^{\{n,n+1\}} \coprod_{\Delta^{\{n+1\}}} \Delta^{\{n+1,n+2\}} \cdots
\]
and finally by
\[
I_{[-\infty,\infty]} := \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \coprod_{\Delta^{\{n\}}} \Delta^{\{n,n+1\}} \coprod_{\Delta^{\{n+1\}}} \Delta^{\{n+1,n+2\}} \cdots
\]
which can be depicted as
\[
\begin{array}{c}
n-1 \quad \cdots \quad n \quad n+1 \quad \cdots
\end{array}
\]
Proposition 3.3 The inclusion
\[ I_{[−∞, +∞]} \rightarrow \mathbb{Z} \]
is inner anodyne.

Proof. Let us begin observing that, as shown in the proof of [Joy08b, Proposition 2.13], for any \( s > 0 \), both
\[ \Delta^{(0,1)} \amalg \Delta^{(1,\ldots,s)} \rightarrow \Delta^s \]
and
\[ \Delta^{s-1} \amalg \Delta^{(s-1,\ldots,s)} \rightarrow \Delta^s \]
are inner anodyne. By virtue of the following commutative diagram of simplicial sets
\[ \begin{array}{ccc} \Delta\{0,1\} \amalg \Delta\{1,\ldots,s\} & \rightarrow & \Delta^s \\
\downarrow & & \downarrow \\
\Delta\{0,1\} \amalg \Delta\{1,\ldots,s\} & \rightarrow & \Delta^s \amalg \Delta^{(s+1)} \end{array} \]
where the square is a pushout, since the left vertical map is inner anodyne, the right vertical map is inner anodyne, and, as the right horizontal map is inner anodyne, the composite \( \alpha \) is inner anodyne as well.

Now, for any \( m \in \mathbb{Z} \), let \( P_m \) denote the following pushout of simplicial sets
\[ \begin{array}{ccc} I_{[−m,m]} & \rightarrow & \Delta\{−m,\ldots,m\} \\
\downarrow & & \downarrow \\
I_{[−∞, +∞]} & \rightarrow & P_m. \end{array} \]
As, by what we have seen above, the left vertical map in the following pushout square of simplicial sets
\[ \begin{array}{ccc} \Delta^{(0,1)} \amalg \Delta\{1,\ldots,s\} \amalg \Delta\{s,s+1\} & \rightarrow & P_m \\
\downarrow & & \downarrow \\
\Delta\{−m,\ldots,m\} & \rightarrow & P_{m+1} \end{array} \]
is inner anodyne, we have that \( \beta_m: P_m \rightarrow P_{m+1} \) is inner anodyne for any \( m > 0 \).

Finally, as the colimit projection \( \gamma \) below
\[ \begin{array}{ccc} I_{[−∞, +∞]} \simeq P_0 & \xrightarrow{\beta_0} & P_1 \xrightarrow{\beta_1} P_2 \rightarrow \cdots \\
\downarrow & & \downarrow \\
& & \colim_m P_m \simeq \mathbb{Z} \end{array} \]
is a transfinite composition of inner anodyne maps, it is inner anodyne. \( \square \)

Corollary 3.4 Let \( \mathcal{C} \) be an \( \infty \)-category. To specify a functor \( \mathbb{Z} \rightarrow \mathcal{C} \) it is sufficient to specify its value on objects and on morphisms of the form \( n \rightarrow n + 1 \) for all \( n \in \mathbb{Z} \).

Construction 3.5 Let \( \tilde{\mathcal{A}}: \mathbb{Z} \rightarrow Ch^*(Sp) \) be the functor defined on objects as \( \tilde{\mathcal{A}}(n) = S^n_{[n]} \), where
\[ S^n_{[n]} : Ch^{op} \rightarrow Sp \]
\[ m \mapsto \begin{cases} S[n] & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \]
We define \( \tilde{\mathcal{A}} \) on morphisms as follows. First of all, let us note that, as
\[ Fun(\Delta^1, Ch^*(Sp)) \simeq Fun(\Delta^1, Fun^{op}(Ch^{op}, Sp)) \]
\[ \subset Fun(\Delta^1, Fun(Ch^{op}, Sp)) \]
\[ \simeq Fun(\Delta^1 \times Ch^{op}, Sp) \]
it is equivalent to determine a map in $\text{Ch}^*(\text{Sp})$ or a map from $\Delta^1 \times \text{Ch}^{\text{op}}$ landing in spectra, such that both restrictions to $\{0\} \times \text{Ch}^{\text{op}}$ and $\{1\} \times \text{Ch}^{\text{op}}$ preserve zero objects (recall that $\text{Fun}^{0}(\text{Ch}^{\text{op}}, \text{Sp})$ is a full subcategory of $\text{Fun}(\text{Ch}^{\text{op}}, \text{Sp})$). Let $\iota_m: \text{Ch}^{\text{op}}_{[m,m+1]} \to \text{Ch}^{\text{op}}$ be the inclusion functor. Then, we define $\tilde{\mathcal{A}}(\iota_m)$ as the morphism corresponding to the following left Kan extension

$$
\begin{array}{ccc}
\Delta^1 \times \text{Ch}^{\text{op}}_{[m,m+1]} & \xrightarrow{\alpha} & \text{Sp} \\
\text{id} \times \iota_m & \downarrow & \\
\Delta^1 \times \text{Ch}^{\text{op}} & \xrightarrow{\beta} & \\
\end{array}
$$

where $\alpha$ is obtained from the defining square for the suspension $S[0] \simeq \text{S}[m+1]$, and extending this map $\Delta^1 \times \Delta^1 \to \text{Sp}$ to $\Delta^1 \times \text{Ch}^{\text{op}}$ in the only possible way that sends $(0, \text{pt})$ and $(1, \text{pt})$ to the zero spectrum. As $\text{id} \times d_m$ is fully faithful, the values of $\beta(0, \text{pt})$ and $\beta(1, \text{pt})$ are determined by $\alpha$, and are zero by construction. By Corollary 3.4, this completes the construction of $\tilde{\mathcal{A}}$.

**Definition 3.6** Let $\hat{\mathcal{A}} := [-]_{\hat{\mathcal{A}}}^*: \text{Fil}^i(\text{Sp}) \to \text{Ch}^*(\text{Sp})$ denote the stable $\mathcal{A}$-realization functor associated, by the equivalence of Lemma 3.1, to the functor $\tilde{\mathcal{A}}$ given in Construction 3.5. Let us denote by $I := N^i_{\hat{\mathcal{A}}}$ its right adjoint

$$
\begin{array}{cccc}
\text{Fil}^i(\text{Sp}) & \xleftarrow{\hat{\mathcal{A}}} & \text{Ch}^*(\text{Sp}) & \xrightarrow{I}
\end{array}
$$

(the reason for the hat in the notation will be clarified in Proposition 3.9). We will refer to $\hat{\mathcal{A}}$ as the shelling functor and to the $I$ as the piling-up functor; we will use the terms associated shelled complex and piled-up filtered object for objects of the form $\hat{\mathcal{A}}F$ and $IC$, respectively.

**Remark 3.7** It follows from Proposition A.23 that, for any coherent cochain complex $C$, the filtered object $IC^*$ is given by

$$
IC^* \simeq \text{map}_{\text{Ch}^*}((S[m]^*)_n, C).
$$

Our next goal is to show that $\hat{\mathcal{A}}$ factors through the localization $\text{Fil}^i(\text{Sp}) \to \text{Fil}^i(\text{Sp})$. In order to do so, it will be useful to identify its graded pieces $\hat{\mathcal{A}}(-)^n$.

**Lemma 3.8** There is a natural equivalence

$$
\hat{\mathcal{A}}(-)^n \simeq \text{gr}^n(-)[n]
$$

of functors $\text{Fil}^i(\text{Sp}) \to \text{Sp}$.

**Proof.** By virtue of Lemma A.18, it suffices to prove that both functors preserve colimits and agree on elements of the form $\Sigma^\infty \wedge x$.

As both $\hat{\mathcal{A}}$ and $\text{ev}_n$ admit right adjoints (see Proposition 2.12), $\hat{\mathcal{A}}(-)^n \simeq \text{ev}_n \circ \hat{\mathcal{A}}$ preserves colimits. By Definition 2.5, $\text{gr}^n$ is given by $\text{cof} \circ (\iota_n)^*$; by [Lur17, 1.1.1.8], $\text{cof}$ admits a right adjoint, and since $\text{Sp}$ is complete, $(\iota_n)^*$ admits a right adjoint, given by right Kan extension along $\iota_n$.

It follows from Remark A.16 that for all $m \in \mathbb{Z}$

$$
\hat{\mathcal{A}}(\Sigma^\infty \wedge x_m) \simeq x_m \wedge_{\hat{\mathcal{A}}} \simeq \hat{\mathcal{A}}(m) \simeq S[m]^n
$$

where $\hat{\mathcal{A}}$ stands for “Aushülen”, German for hull shelling

where $I$ stands for “Impilare”, Italian for piling up.
(see Appendix A for a detailed discussion) and thus that
\[ \widehat{\mathcal{A}} \left( \Sigma^\infty_{+} \hat{m} \right)^n \simeq \begin{cases} S[n] & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \]
whereas a direct check shows that
\[ \text{gr}^n \left( \Sigma^\infty_{+} \hat{m} \right) \simeq \begin{cases} S & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \]
concluding the proof.

**Proposition 3.9** The adjunction $\widehat{\mathcal{A}} \dashv I$ given in Definition 3.6 factors as
\[
\begin{array}{ccc}
\text{Fil}^i \text{Sp} & \xleftarrow{L} & \widehat{\text{Fil}}^i \text{Sp} & \xrightarrow{\mathcal{A}} & \text{Ch}^*(\text{Sp}) \\
& \downarrow & \downarrow & \downarrow & \\
& \xrightarrow{I} & \text{Ch}^*(\text{Sp}) & \end{array}
\]
through the localization of Proposition 2.21.

**Proof.** Let us first show that $\widehat{\mathcal{A}}$ factors through the localization. By virtue of Proposition 2.21, it is enough to show that it sends maps inducing equivalences on associated gradeds to equivalences. But, as per Lemma 2.11, the functor $u: \text{Ch}^*(\text{Sp}) \to \prod_{\mathbb{Z}} \text{Sp}$ is conservative. It is thus enough to know that on each component, $\widehat{\mathcal{A}}(-)^n$ sends local maps to equivalences, which is true by virtue of Lemma 3.8. As $\widehat{\mathcal{A}}$ preserves colimits, and $\widehat{\text{Fil}}^i \text{Sp}$ is a full subcategory of $\text{Fil}^i \text{Sp}$ closed under colimits (see Proposition 2.21), we get that $\mathcal{A}$ preserves colimits, and hence admits a right adjoint. Since adjoints compose, this just means that $I$ takes values in the full subcategory of complete objects.

Our next goal is to prove that the induced adjunction $\mathcal{A} \dashv I$ is an equivalence of $\infty$-categories. To this end, we need to prove a few key lemmata before, and to get a better understanding of the functor $I$.

**Lemma 3.10** The functor $\mathcal{A}$ is conservative.

**Proof.** Let $\alpha: F \to G$ be a map in $\widehat{\text{Fil}}^i \text{Sp}$ such that $\mathcal{A}(\alpha): \mathcal{A}F \to \mathcal{A}G$ is an equivalence. As $\text{Ch}^*(\text{Sp})$ is a full subcategory of a functor category, equivalences are given pointwise, hence we have that, for all $n \in \mathbb{Z}$
\[ \mathcal{A}F^n \simeq \mathcal{A}G^n. \]
But, by Lemma 3.8 this is just means that $\alpha$ induces an equivalence on associated gradeds, hence, by Proposition 2.21, it is an equivalence in $\widehat{\text{Fil}}^i \text{Sp}$.

**Lemma 3.11** There is a natural equivalence $(ev_n)_! \simeq (ev_{n-1})_*$ (with notation as in Proposition 2.12) of functors $\text{Sp} \to \text{Ch}^*(\text{Sp})$.

**Proof.** It follows from the pointwise description of Proposition 2.12 that $(ev_n)_*$ preserves all colimits. As $S \simeq P_{\text{et}}(\Delta^0) \simeq \text{St}(P(\Delta^0))$, Lemma A.18 implies that it is enough to check that they take the same value on $\Sigma^\infty_{+} \hat{pt} \simeq \Sigma^\infty \hat{pt} \simeq S$. But this follows immediately again from description given in Proposition 2.12.

**Lemma 3.12** We have equivalences
\[ \text{pt} \circ \Sigma^\infty_{+} \hat{m} \simeq (ev_n)_! S \simeq (ev_{n+1})_* S \]
of objects in $\text{Ch}^*(\text{Sp})$.

**Proof.** The second equivalence is an instance of Lemma 3.11. Regarding the first one, we will show that they represent the same functor. In fact:
\[ \text{Map}_{\text{Ch}^*(\text{Sp})}((ev_n)_! S, C) \simeq \text{Map}_S(S, C^n) \simeq \Omega^\infty C^n \]
and, using Proposition A.22:

\[
\text{Map}_{\text{Ch}^*(\text{Sp})}(\text{pt} \circ \Sigma^\infty_+ \to \text{Ch}^*, C) \simeq \text{Map}_{\text{P}_{\text{st}}(\text{Ch})}(\Sigma^\infty_+ \to \text{Ch}^*, C) \\
\simeq \Omega^\infty \text{map}_{\text{P}_{\text{st}}(\text{Ch})}(\Sigma^\infty_+ \to \text{Ch}^*, C) \\
\simeq \Omega^\infty C^n
\]

concluding the proof. □

**Notation 3.13** Motivated by Lemma 3.12, we introduce the notation

\[ y_n \cdot S := \text{pt} \circ \Sigma^\infty_+ \cdot \text{Ch}^* \to \text{Ch}^*(\text{Sp}) \]

to emphasize the connection with the pointed stabilization of the Yoneda embedding.

We have the following “density” result for elements of the form \( y_n S \) in \( \text{Ch}^*(\text{Sp}) \).

**Lemma 3.14** Given any \( C \in \text{Ch}^*(\text{Sp}) \), there is a natural equivalence

\[ C \simeq \int_{n \in \text{Ch}} \text{map}_{\text{Ch}^*(\text{Sp})}(y_n S, C) \otimes y_n S \]

where \( \otimes \) denotes the canonical tensoring of stable \( \infty \)-categories over spectra (see [Lur17, 4.8.2.20]).

**Proof.** Using Proposition A.22

\[
\text{Map}_{\text{Ch}^*(\text{Sp})}(\int_{n \in \text{Ch}} \text{map}_{\text{Ch}^*(\text{Sp})}(y_n S, C) \otimes y_n S, D) \\
\simeq \int_{n \in \text{Ch}} \text{Map}_{\text{Ch}^*(\text{Sp})}(\text{map}_{\text{Ch}^*(\text{Sp})}(y_n S, C) \otimes y_n S, D) \\
\simeq \int_{n \in \text{Ch}} \text{Map}_{\text{Ch}^*(\text{Sp})}(\text{map}_{\text{Ch}^*(\text{Sp})}(y_n S, C), \text{map}_{\text{Ch}^*(\text{Sp})}(y_n S, D)) \\
\simeq \int_{n \in \text{Ch}} \text{Map}_{\text{Ch}^*(\text{Sp})}(\text{map}_{\text{Ch}^*(\text{Sp})}((\Sigma^\infty_+ \to \text{Ch}^*, C), \text{map}_{\text{Ch}^*(\text{Sp})}(\Sigma^\infty_+ \to \text{Ch}^*, D)) \\
\simeq \int_{n \in \text{Ch}} \text{Map}_{\text{Ch}^*(\text{Sp})}(C^n, D^n) \\
\simeq \int_{n \in \text{Ch}} \text{Map}_{\text{Ch}^*(\text{Sp})}(C^n, D^n).
\]

Now, using the end formula for the space of natural transformations (see [Gla16, 2.3], [GHN17, 5.1]), we get that

\[
\int_{n \in \text{Ch}} \text{Map}_{\text{Ch}^*(\text{Sp})}(C^n, D^n) \\
\simeq \text{Map}_{\text{Ch}^*(\text{Sp})}(C, D) \\
\simeq \text{Map}_{\text{Ch}^*(\text{Sp})}(C, D)
\]

hence both objects corepresent the same functor. □

**Lemma 3.15** There is a co/fiber sequence in \( \text{Ch}^*(\text{Sp}) \) given by

\[ S_{[n-1]} \to S_{[n]} \to y_{n-1} S_{[n]} \]

where the first map is the structure map defined in Construction 3.5.

**Proof.** By Proposition 2.20, the inclusion of \( \text{Ch}^*(\text{Sp}) \) into \( \mathcal{P}_{\text{st}}(\text{Ch}) \) preserves colimits, therefore we can understand the colimit in the latter \( \infty \)-category, where it can be computed pointwise (see [Lur09, 5.1.2.3]). The result then follows by inspection of
the following diagram
\[
\begin{array}{cccccccc}
S_{[n-1]}^{n-1} & = & \cdots & \rightarrow & 0 & \rightarrow & S_{[n-1]} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \cdots \\
\downarrow & & & & & & \downarrow & & & & & & \downarrow \\
S_{[n]}^{n} & = & \cdots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & S_{[n]} & \rightarrow & 0 & \rightarrow & \cdots \\
\downarrow & & & & & & \downarrow & & & & & & \downarrow \\
y_{n-1}S[n] & = & \cdots & \rightarrow & 0 & \rightarrow & S[n] & \rightarrow & S[n] & \rightarrow & 0 & \rightarrow & \cdots.
\end{array}
\]

\[\square\]

The previous result lets us get an explicit description of the graded pieces of \(I\).

**Proposition 3.16** Let \(C \in \text{Ch}^*(\Sp)\); then
\[
gr^n(I C) \simeq C^n[-n].
\]
**Proof.** By Lemma 3.15, we have a co/fiber sequence
\[
S_{[n]}^n \rightarrow S_{[n+1]}^{n+1} \rightarrow y_n S[n+1].
\]
By applying map_{Ch^*(\Sp)}(-, C) to it, we get
\[
\text{map}_{Ch^*(\Sp)}(y_n S[n+1], C) \rightarrow IC(n+1) \rightarrow IC(n),
\]
(see Remark 3.7) and, by Lemma 3.12,
\[
\text{map}_{Ch^*(\Sp)}(y_n S[n+1], C) \simeq \text{map}_{Ch^*(\Sp)}(\Sigma_+^\infty \yon^* \yon, C[-n-1])
\]
\[
\simeq \text{map}_{Ch}(\Sigma_+^\infty \yon, C[-n-1])
\]
\[
\simeq C^n[-n-1]
\]
(where the last equivalence follows from Proposition A.22). \[\square\]

**Corollary 3.17** Let \(C \in \text{Ch}^*(\Sp)\). Then
\[ theorem 3.10 \]
(1) If \(C^m \simeq 0\), then \(IC^m \simeq IC^{m+1}\).
(2) If there exists an \(N\) such that \(C^m \simeq 0\) for all \(m < N\), then \(IC^m \simeq IC^N\) for all \(m \leq N\).
(3) If there exists an \(N\) such that \(C^m \simeq 0\) for all \(m \geq N\), then \(IC^m \simeq IC^N\) for all \(m \geq N\).
(4) If there exists an \(N\) such that \(C^m \simeq 0\) for all \(m \geq N\), then \(IC^m \simeq 0\) for all \(m \geq N\) and \(IC^{N-1} \simeq C^{N-1}[-N+1]\).
(5) For all \(n \in \mathbb{Z}\), we have that \(IS_{[n]}^n \simeq \Sigma_+^\infty \yon [n] - n\).
**Proof.** The first three points are straightforward consequences of Proposition 3.16, whereas (4) follows immediately from (3), together with the completeness of \(IC\) (see Proposition 3.9). Point (5) follows from the definition of \(S_{[n]}^n\) together with (1-4). \[\square\]

**Proposition 3.18** The composite \(\widehat{AI}\) is equivalent to id_{Ch^*(\Sp)}.

**Proof.** It follows from Remark 3.7 that \(I\) commutes with arbitrary coproducts, and is thus cocontinuous. Hence, \(\widehat{AI}\) is cocontinuous as well. By Lemma 3.14, we have that every coherent cochain complex is canonically the colimit of elements in the essential image of \(y_n S\). In particular, it is enough to check that the two functors agree on elements of the form \(y_n S\). Using Lemma 3.15 and Corollary 3.17.5, we have that
\[
\widehat{AI}y_n S \simeq \text{cof} \left(\widehat{AI}S_{[n]}^n \rightarrow \widehat{AI}S_{[n+1]}^{n+1}\right)
\]
\[
\simeq \text{cof} \left(\Sigma_+^\infty \yon \rightarrow \Sigma_+^\infty \yon [n+1]\right)
\]
\[
\simeq \text{cof} \left(S_{[n]}^n \rightarrow S_{[n+1]}^{n+1}\right)
\]
\[
\simeq y_n S
\]
We can now prove the following theorem.

**Theorem 3.19** The adjunction

$$
\xymatrix{ \mathcal{F}_I^{\dagger} \mathbf{Sp} \ar[r]_\mathcal{A} & \mathbf{Ch}(\mathbf{Sp}) \ar[l]^I }
$$

is an equivalence.

**Proof.** Putting together Proposition 3.18, and Proposition 3.9, we get that $\mathcal{A} I \simeq \text{id}_{\mathbf{Ch}(\mathbf{Sp})}$. This, together with (the dual of) Proposition 2.22 implies that the counit $\varepsilon : \mathcal{A} I \Rightarrow \text{id}_{\mathbf{Ch}(\mathbf{Sp})}$ of the adjunction $\mathcal{A} + I$ is an equivalence. If we now consider the triangular identity

$$
\mathcal{A} \xrightarrow{\eta_0} \mathcal{A} I \xrightarrow{\varepsilon} \mathcal{A}
$$

we see that, as both $\text{id}_{\mathcal{A}}$ and $\varepsilon_{\mathcal{A}}$ are equivalences, $\mathcal{A} \eta$ is an equivalence as well; but, as by Lemma 3.10 $\mathcal{A}$ is conservative, then $\eta$ has to be an equivalence, concluding the proof. □

We conclude the section giving more information about the functor $I$; we can leverage from Lemma 3.15 our understanding of it, by getting a recursive description of its components, which in turn gives a complete description of its values in the case of bounded above cochain complexes.

In the rest of the section, we will make free use of some results about cubic diagrams in stable $\infty$-categories, as presented in [DJW19], and refer to op. cit. for all the related concepts, notations and terminology we use and do not introduce here. The following fact is somewhat implicit in [DJW19], but as we will use it crucially, we report it here for convenience.

**Proposition 3.20** Let $\mathcal{C}$ be a stable $\infty$-category, let $a \in \mathbb{N}$ and let $C : (\Delta^a)^{op} \to \mathcal{C}$ be a functor. Moreover, if $(\Delta^1)^a$ denotes the $a$-fold product of $\Delta^1$ with itself, and we denote by $\vec{v} = (\vec{v}_a, \vec{v}_{a-1}, \ldots, \vec{v}_1)$ the objects of $(\Delta^1)^a$ (where each $\vec{v}_i$ can either be 0 or 1), let $F$ be the $a$-cube having for vertices:

$$
F(\vec{v}) = \begin{cases} 
C(a) & \text{if } \vec{v} = (0, \ldots, 0) \\
C(a-b) & \text{if } 0 < b \leq a, \vec{v}_i = 0 \text{ for } i > b \text{ and } \vec{v}_i = 1 \text{ for } i \leq b \\
0 & \text{else}
\end{cases}
$$

where all the nonzero maps are determined by $C$ (that is, $F$ is an $a$-cube having the $C(i)$’s on its “spine” and zero objects elsewhere). Then,

$$
tot\text{-cof} F \simeq \text{cof}^a(C) \simeq \text{cof} \left( \underbrace{\text{cof} \left( C(a) \to C(a-1) \right) \to C(1) \cdots \to C(0) }_{a \text{ times}} \right).
$$

**Proof.** By [DJW19, A.24], one can extend $F$ to a coCartesian $(a+1)$-cube $\tilde{F}$ such that $\tilde{F}|_{(0) \times \Delta^a} \simeq F$, and with $\tilde{F}|_{(1) \times \Delta^a}$ having tot-cof $F$ as its terminal vertex, and 0 elsewhere (see op. cit. for a precise statement). As $\tilde{F}$ is coCartesian, by iteratively applying [DJW19, A.11], $\text{cof}^a(\tilde{F})$ is a coCartesian 1-cube, i.e. an equivalence. We conclude by observing that its source is given by $\text{cof}^a(F)$, and its target is just tot-cof $F$. □
Remark 3.21  
By virtue of Lemma 3.7 and Lemma 3.15, we see that
\[ IC^n \simeq \text{map}(S_{[n]}^n, C) \]
\[ \simeq \text{map}(\text{fib}(y_{n-1}S_{[n]} \to S_{[n]}^{n-1}), C) \]
\[ \simeq \text{cof}(\text{map}(S_{[n]}^{n-1}, C) \to \text{map}(y_{n-1}S_{[n]}, C)) \]
\[ \simeq \text{cof}(\text{map}(S_{[n]-1}^{n-1}, C)[-1] \to C^{n-1}[-n]) \]
\[ \simeq \text{cof}(\text{map}(S_{[n]-1}^{n-1}, C) \to C^{n-1}[-n+1][-1]) \]
\[ \simeq \text{fib}(IC^{n-1} \to C^{n-1}[-n+1]) \]

hence, by iteratively applying the above, we get that, for all \( a \in \mathbb{N} \), \( IC^n \) is naturally equivalent to
\[ \underbrace{\text{fib}(\cdots \text{fib}(IC^{n-a} \to C^{n-a}[-n+a]))}_{a \text{ times}} \to C^{n-a+1}[-n+a-1] \cdots \to C^{n-1}[-n+1] \].

By stability, the above is equivalent to
\[ \underbrace{\text{cof}(\cdots \text{cof}(\text{cof}(IC^{n-a} \to C^{n-a}[-n+a])[-1])}_{a \text{ times}} \to C^{n-a+1}[-n+a-1] \cdots \to C^{n-1}[-n+1][-1] \]

which, in turn, is equivalent to
\[ \underbrace{\text{cof}(\cdots \text{cof}(\text{cof}(IC^{n-a}[-a] \to C^{n-a}[-n]))}_{a \text{ times}} \to C^{n-a+1}[-n] \cdots \to C^{n-1}[-n] \].

By 3.20, the latter is equivalent to the total cofiber of a suitable cube \( F_a \), that is
\[ IC^n \simeq \text{tot-cof} F_a \]

and hence, by [DJW19, A.31],
\[ IC^n \simeq (\text{tot-fib} F_a)[a]. \]

We can describe \( F_a \) explicitly as follows; if \((\Delta^1)^a\) denotes the \( a \)-fold product of \( \Delta^1 \) with itself, and we denote by \( \bar{v} = (v_{a-1}, v_{a-2}, \ldots, v_0) \) the objects of \((\Delta^1)^a\) (where each \( v_i \) can either be 0 or 1) \( F_a \) is the \( a \)-cube having for vertices:

\[ F_a(\bar{v}) = \begin{cases} 
  IC^{n-a}[-a] & \text{if } \bar{v} = (0, \ldots, 0) \\
  C^{n-a+b}[-n] & \text{if } 0 \leq b < a, \ v_i = 0 \text{ for } i > b \text{ and } v_i = 1 \text{ for } i \leq b \\
  0 & \text{else.}
\end{cases} \]
As an example, in the case $a = 3$, the formula above looks as follows

$$IC^3 \simeq \text{tot-cof} \begin{pmatrix} IC^3[-3] & \rightarrow & C^3[-n] \\ 0 & \rightarrow & C^2[-n] \\ 0 & \rightarrow & C^1[-n] \end{pmatrix}.$$ 

The previous remark takes a particularly pleasant form when the complex is bounded.

**Corollary 3.22** Let $C \in \text{Ch}^*(Sp)$, and assume there exists an $N$ such that $C^m \simeq 0$ for all $m > N$; then for all $a > 0$

$$IC^{N-a} \simeq \text{cof} \left( \cdot \text{cof} \left( \text{cof} \left( \ldots \text{cof} \left( C^{N-a}[-N] \rightarrow C^{N-a+2}[-N] \rightarrow C^N[-N] \right) \ldots \rightarrow C^N[-N] \right) \right) \right).$$

**Proof.** It follows from Remark 3.21, Corollary 3.17.4 and [DJW19, Proposition A.24] that the $(a+1)$-cube defined by

$$F(\bar{v}) = \begin{cases} IC^{N-a}[-a] & \text{if } \bar{v} = (0, \ldots, 0) \\ C^{N-a+b}[-N] & \text{if } 0 \leq b < a, \bar{v}_i = 0 \text{ for } i > b \text{ and } \bar{v}_i = 1 \text{ for } i \leq b \\ IC^N \simeq C^N[-N] & \text{if } \bar{v} = (1, \ldots, 1) \\ 0 & \text{else} \end{cases}$$

(where $\bar{v} = (\bar{v}_0, \bar{v}_a, \ldots, \bar{v}_b)$) is coCartesian. In particular, again by [DJW19, Proposition A.24], $IC^{N-a}[-a]$ is the total fiber of $F|_{\Delta^a \times \{1\}}$; the thesis follows from [DJW19, Proposition A.31].

**Example 3.23** In the cases $a = 2$ and $a = 3$, the previous Corollary specializes to

$$IC^{N-2} \simeq \text{tot-cof} \begin{pmatrix} C^{N-2} & \rightarrow & C^{N-1} \\ \downarrow & & \downarrow \\ 0 & \rightarrow & C^N \end{pmatrix} [-N].$$
and

\[
IC^{N-3} \simeq \text{tot-cof} \begin{pmatrix}
C^{N-3} & \rightarrow & C^{N-2} \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & C^N
\end{pmatrix} \quad \left[ -N \right]
\]

respectively.

The following remark, inspired by the “Gap objects” considered in \[Lur17, 1.2.2\], will be useful later, and sheds some light on the relation between coherent cochain complexes and the objects called \(Z\)-complexes in \textit{op. cit.}

\textbf{Remark 3.24} Proposition 3.16 and the calculus of total cofibers of cubic diagrams allow us to understand also all the intermediate subquotients of \(IC\). We can consider the diagram consisting of co/Cartesian squares

\[
IC^n \longrightarrow IC^{n-1} \longrightarrow IC^{n-2} \longrightarrow IC^{n-3} \longrightarrow IC^{n-4} \longrightarrow \cdots
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow C^{n-1}[-n+1] \rightarrow IC^{n-2}/IC^n \rightarrow IC^{n-3}/IC^n \rightarrow IC^{n-4}/IC^n \rightarrow \cdots
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow C^{n-2}[-n+2] \rightarrow IC^{n-3}/IC^{n-1} \rightarrow IC^{n-4}/IC^{n-1} \rightarrow \cdots
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow C^{n-3}[-n+3] \rightarrow IC^{n-4}/IC^{n-2} \rightarrow \cdots
\]

\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\]

from which we can deduce that, for any \(n \in \mathbb{Z}\)

\[
IC^{n-2}/IC^n \simeq \text{cof}(\partial^n)[-n+1];
\]

moreover, we can proceed inductively and apply \[DJW19, A.26\] to the cofiber sequences

\[
IC^{n-k+1}/IC^n \rightarrow IC^{n-k}/IC^n \rightarrow C^{n-k}[-n+k]
\]

to identify \(IC^{n-k}/IC^n\) with the total cofiber of a cube having the truncation

\[
\left( C^{n-k} \rightarrow C^{n-k+1} \rightarrow \cdots \rightarrow C^{n-1} \right) [-n+1]
\]

of \(C[-n+1]\) on its “spine” and zeroes elsewhere; thus, by Proposition 3.20 we have

\[
IC^{n-k}/IC^n \simeq \text{cof}^{k-1}(\ldots \text{cof}(\text{cof}(C^{n-k} \rightarrow C^{n-k+1}) \rightarrow C^{n-k+2} \rightarrow \cdots \rightarrow C^{n-1})[-n+1]
\]

for any \(k \geq 2\).
4. The general equivalence

In this section, we extend the result of Theorem 3.19 to all stable ∞-categories with sequential limits. Along the way, we prove that the explicit formulas given for \( \mathcal{A} \) and \( I \) in the previous section hold also in the general case.

**Lemma 4.1** Let \( \mathcal{C} \) be a small stable ∞-category. Then, the following equivalences hold:

\[
\text{Fun}(\mathcal{C}, \text{Ch}^\ast(\mathcal{Sp})) \simeq \text{Ch}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp}))
\]

and

\[
\text{Fun}(\mathcal{C}, \text{Fil}^\ast(\mathcal{Sp})) \simeq \text{Fil}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})).
\]

**Proof.** We prove this for \( \text{Ch}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})) \), the case of \( \text{Fil}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})) \) being entirely analogous. We have that, by definition of \( \text{Ch}^\ast(\mathcal{Sp}) \),

\[
\text{Fun}(\mathcal{C}, \text{Ch}^\ast(\mathcal{Sp})) \simeq \text{Fun}(\mathcal{C}, \text{Fun}^0(\text{Ch}^{\mathcal{op}}, \mathcal{Sp})).
\] (8)

Now, as

\[
\text{Fun}(\mathcal{C}, \text{Fun}(\text{Ch}^{\mathcal{op}}, \mathcal{Sp})) \simeq \text{Fun}(\mathcal{C} \times \text{Ch}^{\mathcal{op}}, \mathcal{Sp})
\]

we have that the ∞-categories of (8) are equivalent to the full subcategory of \( \text{Fun}(\mathcal{C} \times \text{Ch}^{\mathcal{op}}, \mathcal{Sp}) \) spanned by functors that are pointed in the second variable (that is, sending any pair \((C,0) \in \mathcal{C} \times \text{Ch}^{\mathcal{op}} \) to 0). Thus, as

\[
\text{Fun}(\mathcal{C} \times \text{Ch}^{\mathcal{op}}, \mathcal{Sp}) \simeq \text{Fun}(\text{Ch}^{\mathcal{op}}, \text{Fun}(\mathcal{C}, \mathcal{Sp}))
\]

the ∞-categories of (8) are in turn equivalent to the full subcategory

\[
\text{Fun}^0(\text{Ch}^{\mathcal{op}}, \text{Fun}(\mathcal{C}, \mathcal{Sp})) \subset \text{Fun}(\text{Ch}^{\mathcal{op}}, \text{Fun}(\mathcal{C}, \mathcal{Sp}))
\]

spanned by pointed functors, which, by definition, is \( \text{Ch}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})) \). \( \square \)

**Corollary 4.2** Let \( \mathcal{C} \) be a small stable ∞-category. Then, the equivalence of Theorem 3.19 extends to an equivalence between \( \text{Ch}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})) \) and \( \text{Fil}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})) \) which we will again denote by

\[
\text{Fil}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})) \xrightarrow{\mathcal{A}} \text{Ch}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})).
\]

**Remark 4.3** One way to re-phrase Lemma 4.1 and Corollary 4.2 is to say that objects in \( \text{Ch}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})) \) can be described as bifunctors \((C,n) \mapsto G_C(n)\) such that \( G_C(pt) \simeq 0 \) for any \( C \in \mathcal{C} \), and similarly objects in \( \text{Fil}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})) \) can be described as bifunctors \((C,n) \mapsto F_C(n)\) such that \( \lim_n F_C(n) \simeq 0 \) for any \( C \in \mathcal{C} \). The equivalence of Corollary 4.2 is then given on objects by pointwise (in \( \mathcal{C} \)) postcomposition with \( \mathcal{A} \) and \( I \); that is, we have

\[
\mathcal{A}((C,n) \mapsto F_C^n) \simeq ((C,n) \mapsto \mathcal{A}(F_C^n))
\]

and

\[
I((C,n) \mapsto G_C^n) \simeq ((C,n) \mapsto I(G_C^n)).
\]

**Remark 4.4** Let \( \mathcal{C} \) be a small stable ∞-category, and let \( \mathcal{A} \subseteq \text{Fun}(\mathcal{C}, \mathcal{Sp}) \) be a full stable subcategory closed under sequential limits. Postcomposition with the inclusion \( \iota : \mathcal{A} \to \text{Fun}(\mathcal{C}, \mathcal{Sp}) \) induces fully faithful functors

\[
\iota_{\text{Fil}} : \text{Fil}^\ast(\mathcal{A}) \to \text{Fil}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp}))
\]

and

\[
\iota_{\text{Ch}} : \text{Ch}^\ast(\mathcal{A}) \to \text{Ch}^\ast(\text{Fun}(\mathcal{C}, \mathcal{Sp})).
\]

By inspection, an object \( F \) lies in the essential image of \( \iota_{\text{Fil}} \) if and only if \( F^n \) lies in \( \mathcal{A} \) for all integers \( n \), and an object \( C \) lies in the essential image of \( \iota_{\text{Ch}} \) if and only if \( C^n \) lies in \( \mathcal{A} \) for all integers \( n \).

**Lemma 4.5** Let \( \mathcal{A} \subseteq \text{Fun}(\mathcal{C}, \mathcal{Sp}) \) be a full, stable subcategory closed under sequential limits.
(1) If $F$ is a coherent cochain complex in $\hat{\text{Fil}}^i(\text{Fun}(\mathcal{C}, \text{Sp}))$ that lies in the essential image of $\hat{\text{Fil}}^i(\mathcal{A})$, then $\mathcal{A}F$ lies in the essential image of $\text{Ch}^\ast(\mathcal{A})$.

(2) If $C$ is a coherent cochain complex in $\text{Ch}^\ast(\text{Fun}(\mathcal{C}, \text{Sp}))$ that lies in the essential image of $\text{Ch}^\ast(\mathcal{A})$, then $FC$ lies in the essential image of $\hat{\text{Fil}}^i(\mathcal{A})$. 

\textbf{Proof.} (1): Let $F: (C, n) \mapsto F^n_C$ lie in (the essential image of) $\hat{\text{Fil}}^i(\mathcal{A})$. By Lemma 3.8 together with Remark 4.3, we have that, for any integer $m$, $(\mathcal{A}F)^m$ is given by $\text{cof}(F^{m+1}_C \to F^m_C)$. By Remark 4.4, each $F^n_C$ lies in $\mathcal{A}$, hence, as $\mathcal{A}$ is a stable subcategory each $(\mathcal{A}F)^m$ lies in $\mathcal{A}$, and thus $\mathcal{A}F$ lies in the essential image of $\text{Ch}^\ast(\mathcal{A})$.

(2): Let $G: (C, n) \mapsto G^n_C$ lie in (the essential image of) $\text{Ch}^\ast(\mathcal{A})$. By Lemma 2.10, $G$ can be expressed as the limit of bounded above complexes:

$$G \simeq \lim_{j \in \mathbb{Z}^{op}} G^\leq j;$$

notice that, by Remark 4.4 and by the definition of the $G^\leq j$’s given in Construction 2.8, all the objects of the form $(G^\leq j)_k$ belong to $\mathcal{A}$. By Remark 3.21 together with Remark 4.3, we have that, for any choice of integers $j$ and $m$, $(IG^\leq j)^m$ is either zero or can be expressed as a suitable total fiber for a finite diagram $K^{j,m}$ whose entries are either zeroes or of the form $(G^\leq j)^k[i,j]$ for different values of $k$; in particular, $G^\leq j$ is a finite limit of a diagram with values in $\mathcal{A}$. Putting everything together (and keeping in mind that by Proposition 2.12 evaluation commutes with limits), we have that

$$(IG)^n \simeq \left( I \left( \lim_j G^\leq j \right) \right)^n \simeq \left( \lim_j (IG^\leq j) \right)^n \simeq \lim_j (IG^\leq j)^n \simeq \lim_j (\text{tot-cof} K^{j,n})$$

is given by a sequential limit of finite colimits of elements of $\mathcal{A}$, and thus lies in $\mathcal{A}$ for any given $n$. This in turn proves that $IG$ lies in the essential image of $\hat{\text{Fil}}^i(\mathcal{A})$. $\square$

\textbf{Lemma 4.6} Let $\mathcal{A} \subseteq \text{Fun}(\mathcal{C}, \text{Sp})$ be a full, stable subcategory closed under sequential limits. Then the equivalence between $\hat{\text{Fil}}^i(\text{Fun}(\mathcal{C}, \text{Sp}))$ and $\text{Ch}^\ast(\text{Fun}(\mathcal{C}, \text{Sp}))$ restricts to an equivalence

$$\hat{\text{Fil}}^i(\mathcal{A}) \simeq \text{Ch}^\ast(\mathcal{A}).$$

\textbf{Proof.} It follows from Lemma 4.5 that (in the notation of Remark 4.4) $\mathcal{A} \circ \iota_{\text{Fil}}$ factors through $\iota_{\text{Ch}}$ and $I \circ \iota_{\text{Ch}}$ factors through $\iota_{\text{Fil}}$, i.e. there exist functors (which we’ll temporarily denote $A$ and $I$) such that $\mathcal{A} \circ \iota_{\text{Fil}} \simeq \iota_{\text{Ch}} \circ A$ and $I \circ \iota_{\text{Ch}} \simeq \iota_{\text{Fil}} \circ I$. In particular, as

$$\iota_{\text{Ch}} \simeq \mathcal{A} \circ I \circ \iota_{\text{Ch}} \quad \iota_{\text{Fil}} \simeq I \circ \mathcal{A} \circ \iota_{\text{Fil}}$$

$$\simeq \mathcal{A} \circ \iota_{\text{Fil}} \circ I \quad \quad \quad \simeq I \circ \iota_{\text{Ch}} \circ A$$

$$\simeq \iota_{\text{Ch}} \circ A \circ I \quad \quad \quad \simeq \iota_{\text{Fil}} \circ I \circ A$$

$A$ and $I$ are mutually inverse. $\square$

\textbf{Theorem 4.7} Let $\mathcal{C}$ be a stable $\infty$-category having sequential limits. Then there exists an equivalence of stable $\infty$-categories

$$\hat{\text{Fil}}^i(\mathcal{C}) \xrightarrow{\sim} \text{Ch}^\ast(\mathcal{C}).$$

\textbf{Proof.} Let $U_0$ be our universe of small sets, and let $U_1$ denote a universe containing $U_0$ as an element. Let us denote by $\text{Sp}_{U_1}$ the stabilization of the $\infty$-category of $U_1$-small spaces. The stable Yoneda embedding (see Definition A.19) provides a fully faithful (see [Nik16, Section 6]) exact functor

$$\mathbb{J}^{\text{st}}: \mathcal{C} \to \text{Fun}(\mathcal{C}^{op}, \text{Sp}_{U_1}).$$
The results now follow from Lemma 4.6 applied to $\mathcal{A}$.

Remark 4.8 In particular, by composing the equivalence of Theorem 4.7 with the adjunction $L \dashv i$ of Proposition 2.21, we get an induced adjunction

$$\text{Fil}^i \mathcal{C} \overset{\phi}{\longrightarrow} \text{Ch}^* \mathcal{C}$$

for any stable $\infty$-category $\mathcal{C}$ with sequential limits.

Remark 4.9 In the proof of Lemma 4.5 we also showed that the pointwise descriptions already given for $\mathcal{A}$ and $\mathcal{I}$ in the case $\mathcal{C} = \text{Sp}$ hold also in the general case of Theorem 4.7; that is, for $F \in \text{Fil}^i \mathcal{C}$

$$\mathcal{A} F \simeq \text{gr}^n F[n]$$

and, for $C \in \text{Ch}^* \mathcal{C}$,

$$\mathcal{I} C \simeq \text{tot-fib} F_a[a]$$

for a suitable cube $F_a$, given explicitly by

$$F_a(\vec{v}) =
\begin{cases}
  \mathcal{I} C(n-a)[-a] & \text{if } \vec{v} = (0, \ldots, 0) \\
  C^{n-a+b}[-n] & \text{if } 0 \leq b < a, \ \vec{v}_i = 0 \text{ for } i > b \text{ and } \vec{v}_i = 1 \text{ for } i \leq b \\
  0 & \text{else.}
\end{cases}$$

(see 3.7 for details about the notation). In particular, Remark 3.24 generalizes as well, giving

$$\mathcal{I} C^{n-k}/\mathcal{I} C^n \simeq \text{cof}(\cdot \cdot \cdot \text{cof}(\text{cof}(C^{n-k} \rightarrow C^{n-k+1})\rightarrow C^{n-k+2})\cdots \rightarrow C^{n-1})[-n+1]$$

for any $n \in \mathbb{Z}$ and $k \geq 2$.

5. COHERENT COCHAIN COMPLEXES AND BEILINSON $t$-STRUCTURES

In this section, we study the connection between the pointwise $t$-structure on coherent cochain complexes and the Beilinson $t$-structure on filtered objects and show how the former can in some sense be interpreted as an easier to understand version of the latter.

In fact, if $\text{Fil}^i \mathcal{C}$ is equipped with the Beilinson $t$-structure, then the full subcategory of complete filtered objects inherits a $t$-structure from it. It turns out that such inherited $t$-structure is equivalent to the one obtained by carrying the pointwise $t$-structure on $\text{Ch}^* \mathcal{C}$ along the equivalence of Theorem 4.7; moreover, the Beilinson $t$-structure on $\text{Fil}^i \mathcal{C}$ is in some sense characterized by this property and by carrying “trivial information” on essentially constant objects (see Theorem 5.11 for a precise statement). In particular, $\text{Fil}^i \mathcal{C}$ and $\text{Fil}^{\hat{i}} \mathcal{C}$ have the same heart (Remark 5.14).

Definition 5.1 Let $\mathcal{C}$ be a stable $\infty$-category equipped with a $t$-structure. We define the pointwise $t$-structure on $\text{Ch}^* \mathcal{C}$ to be the one defined by

$$(\text{Ch}^* \mathcal{C})_{\geq 0} = \{ C \in \text{Ch}^* \mathcal{C} \mid \forall n \ C^n \in \mathcal{C}_{\geq 0} \}$$

$$(\text{Ch}^* \mathcal{C})_{\leq 0} = \{ C \in \text{Ch}^* \mathcal{C} \mid \forall n \ C^n \in \mathcal{C}_{\leq 0} \}.$$

We will denote the truncation functors for this $t$-structure by $\tau_{\geq n}^\text{lvl}$ and $\tau_{\leq n}^\text{lvl}$, and the homotopy objects by $\pi_{n}^\text{lvl}$ for all $n \in \mathbb{Z}$.

Remark 5.2 It follows immediately from the definitions that $\text{Ch}^* \mathcal{C}$ has precisely the same separatedness and completeness properties that $\mathcal{C}$ has.

\footnote{This fact can easily be proved directly, but will be an immediate consequence of Theorem 4.7}
Definition 5.3 Let $\mathcal{C}$ be a stable $\infty$-category with sequential limits. The transferred t-structure on $\text{Fil}^i\mathcal{C}$ is the t-structure $(\mathcal{I}(\text{Ch}^+\mathcal{C})_{\geq 0}, \mathcal{I}(\text{Ch}^+\mathcal{C})_{\leq 0})$ transferred along the equivalence of Theorem 4.7.

Remark 5.4 As a direct consequence of the definitions, we have that
\[
\left(\text{Fil}^i\mathcal{C}\right)^\vee \simeq (\text{Ch}^+\mathcal{C})^\vee \simeq \text{Ch}^+(\mathcal{C}^\vee).
\]

The following definition is a straightforward generalization of the t-structure first introduced by Beilinson in [Bei87].

Definition 5.5 Let $\mathcal{C}$ be a stable $\infty$-category having all sequential limits\(^8\), equipped with a right separated t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.

The Beilinson t-structure $(\text{Fil}^i_{\geq 0}\mathcal{C}, \text{Fil}^i_{\leq 0}\mathcal{C})$ on $\text{Fil}^i\mathcal{C}$ is defined as follows:

- $\text{Fil}^i_{\geq 0}\mathcal{C}$ is the full subcategory spanned by the objects $F \in \text{Fil}^i\mathcal{C}$ such that
  \[\text{gr}^i(F) \in \mathcal{C}_{\geq -i}\text{ for all } i.\]
- $\text{Fil}^i_{\leq 0}\mathcal{C}$ is the full subcategory spanned by the objects $F \in \text{Fil}^i\mathcal{C}$ such that
  \[F^i \in \mathcal{C}_{\leq -i}\text{ for all } i.\]

(Note the asymmetry in the definition). This t-structure appeared first, in a slightly different setting, in [Bei87]. Its existence in this generality will be a consequence of [FL15, Theorem 2.19] together with Theorem 5.11. We will denote the truncation functors for this t-structure by $\tau^B_{\geq n}$ and $\tau^B_{\leq n}$, and the homotopy objects by $\pi^B_n$ for all $n \in \mathbb{Z}$.

Remark 5.6 The hypothesis of $\mathcal{C}$ having all sequential limits in Definition 5.5 is there just because we will use Theorem 4.7 to prove its existence. We believe that the t-structure can exist even without this assumption on $\mathcal{C}$, but as all the examples that arise in practice satisfy this extra hypothesis, we didn’t bother finding a proof that does not use it.

Remark 5.7 We can infer a few properties of the Beilinson t-structure from its definition:

1. Even if we are assuming $\mathcal{C}$ to be right separated, $\text{Fil}^i\mathcal{C}$ need not be so; in fact, the full subcategory of $\infty$-coconnective objects $\cap_n (\text{Fil}^i\mathcal{C})_{\leq n}$ consists of all filtered objects whose associated graded is trivial, hence of all the essentially constant objects.

2. Since the full subcategory of $\infty$-connective objects consists of the levelwise $\infty$-connective ones, $\text{Fil}^i\mathcal{C}$ is left separated if and only if $\mathcal{C}$ is so.

We learned the following fact from [BMS19, 5.4]; although the result in loc. cit. is stated in less generality, the proof carries verbatim to the general case. We report the argument here for the reader’s convenience.

Proposition 5.8 Let $\mathcal{C}$ be a stable $\infty$-category having all sequential limits, equipped with a right separated t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, and let $\tau^B_{\geq n}$ denote its Whitehead truncation functors. Let $\text{Fil}^i(\mathcal{C})$ be equipped with the Beilinson t-structure, and let $\tau^B_{\geq n}$ denote its Whitehead truncation functors. Then, there is a natural equivalence of functors $\text{Fil}^i\mathcal{C} \to \mathcal{C}$
\[\text{gr}^i \circ \tau^B_{\geq n} \simeq \tau^B_{\geq -i} \circ \text{gr}^i\]
for all $i \in \mathbb{Z}$.

Proof. Notice that for any $i$, the exact functor $\text{gr}^i : \text{Fil}^i\mathcal{C} \to \mathcal{C}$ carries $(\text{Fil}^i\mathcal{C})_{\geq 0}$ to $\mathcal{C}_{\geq -i}$. Moreover, as by [Lur17, 1.2.1.16] each $\mathcal{C}_{\geq -i}$ is closed under extensions, the fiber sequence
\[F^i \to \text{gr}^i F \to F^{i+1}[1]\]
\[\text{this assumption is likely superfluous; see Remark 5.6}\]

\[\text{unlike the previous one, this assumption is crucial; see Remark 5.16}\]
proves that $gr^i F \in \mathcal{C}_{\leq -i}$, and thus $gr^i$ carries also $(\text{Fil}^i \mathcal{C})_{\leq 0}$ to $\mathcal{C}_{\leq -i}$. That is, each $gr^i$ is $t$-exact with respect to the Beilinson t-structure on $\text{Fil}^i \mathcal{C}$ and the shifted t-structure $(\mathcal{C}_{\geq -i}, \mathcal{C}_{\leq -i})$ on $\mathcal{C}$. As any exact and $t$-exact functor between stable $\infty$-categories equipped with $t$-structures commutes with the truncation functors associated to the $t$-structures, the result follows. \hfill $\square$

**Corollary 5.9** In particular, in the hypotheses of Proposition 5.8, if we denote by $\pi_n$ the $t$-structure homotopy object functors of $\mathcal{C}$ and by $\pi_n^B$ the $t$-structure homotopy object functors of $\text{Fil}^i \mathcal{C}$, we have the equivalence (natural in $F$)

$$gr^i \pi_n^B F \simeq (\pi_n (gr^i F))[-i]$$

for all $i, n \in \mathbb{Z}$.

We recall the following theorem from [FL15] (which is an $\infty$-categorical generalization of [BBD83, 1.4.10]).

**Theorem 5.10** [FL15, Theorem 2.19] Given a recollement $\mathcal{C}_0 \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{C}_1$, suppose both $\mathcal{C}_0$ and $\mathcal{C}_1$ are equipped with $t$-structures, then there exists a $t$-structure on $\mathcal{C}$, called the *glued $t$-structure*, given by

- $\mathcal{C}_{\geq 0} = \{ X \in \mathcal{C} \mid j_L X \in (\mathcal{C}_1)_{\geq 0} \text{ and } i_L X \in (\mathcal{C}_0)_{\geq 0} \}$
- $\mathcal{C}_{\leq 0} = \{ X \in \mathcal{C} \mid j_L X \in (\mathcal{C}_1)_{\leq 0} \text{ and } i_R X \in (\mathcal{C}_0)_{\leq 0} \}$

(with notations as in Remark 2.14).

**Theorem 5.11** Let $\mathcal{C}$ be a stable $\infty$-category with all sequential limits equipped with a right separated $t$-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Then the glued $t$-structure on $\text{Fil}^i \mathcal{C}$ (via the recollement of Remark 2.14) obtained by considering

1. the trivial $t$-structure$^{10}$ on $\mathcal{C}$,
2. the transferred $t$-structure on $\text{Fil}^i \mathcal{C}$

is the Beilinson $t$-structure (with respect to $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$) on $\text{Fil}^i \mathcal{C}$, as per Definition 5.5.

Before proving Theorem 5.11, we state one immediate consequence of it (and the definition of glued $t$-structure given in Theorem 5.10).

**Corollary 5.12** The Beilinson $t$-structure on $\text{Fil}^i \mathcal{C}$ induces a $t$-structure on the full subcategory of complete filtered objects $\text{Fil}^i \mathcal{C}$ that is equivalent to the transferred $t$-structure of Definition 5.3.

**Proof of Theorem 5.11.** Let us start by identifying the subcategory of connective objects for the transferred $t$-structure on $\text{Fil}^i \mathcal{C}$. We have that $F \in (\text{Fil}^i \mathcal{C})_{\geq 0}$ if and only if $AF \in (\text{Ch}^* \mathcal{C})_{\geq 0}$. By Definition 5.1, this is the case if and only if

$$(AF)^n \simeq gr^n F[n] \in \mathcal{C}_{\geq 0} \text{ for all } n$$

hence

$$(\text{Fil}^i \mathcal{C})_{\geq 0} = \left\{ F \in \text{Fil}^i \mathcal{C} \mid \forall n \text{ } gr^n F \in \mathcal{C}_{\geq -n} \right\}.$$  

Now, according to Theorem 5.10, the connective objects in the glued $t$-structure on $\text{Fil}^i \mathcal{C}$ are given by all the $G \in \text{Fil}^i \mathcal{C}$ such that

$$(\text{Fil}^i \mathcal{C})_{\geq 0} = \left\{ F \in \text{Fil}^i \mathcal{C} \mid \forall n \text{ } gr^n F \in \mathcal{C}_{\geq -n} \right\}.$$  

(the condition on $G^{-\infty}$ being empty, as we are considering the trivial $t$-structure on $\mathcal{C}$); but, as for all $n$ we have $gr^n LG \simeq gr^n G$, the above is equivalent to the condition

$$gr^n G \in \mathcal{C}_{\geq -n} \text{ for all } n$$

which in turn determines exactly the class of connective objects for the Beilinson $t$-structure of Definition 5.5; as, by [Lur17, 1.2.1.3], the class of connective objects completely determines the $t$-structure, provided its existence, the only thing left is to

---

$^{10}$That is, the one given by $(\mathcal{C}, \{0\})$, where all objects are connective, and only the zero object is coconnective.
chek is that the description of coconnective objects given in Theorem 5.10 coincides with the one given in Definition 5.5; that is, we have to prove that

\[(\forall n \ gr^n F \in \mathcal{C}_{\leq -n} \text{ and } F^{+\infty} \simeq 0) \iff (\forall n \ F^n \in \mathcal{C}_{\leq -n}).\]

For the “only if” direction: [Lur17, 1.2.1.16] implies that \(\mathcal{C}_{\leq -n}\) is closed under extensions, hence if \(F^{n+1} \in \mathcal{C}_{\leq -n-1}\) and \(F^n \in \mathcal{C}_{\leq -n}\) the fiber sequence

\[F^n \rightarrow gr^n F \rightarrow F^{n+1}[1]\]

proves that \(gr^n F \in \mathcal{C}_{\leq -n}\). To check that \(F^{+\infty} \simeq 0\), observe that any subset of the form \(\mathbb{Z}^\oplus_n\) is an initial subcategory of \(\mathbb{Z}^\oplus\), hence for any \(n \in \mathbb{Z}\) we have

\[F^{+\infty} := \lim (\cdots \rightarrow F^{n+1} \rightarrow F^n \rightarrow \cdots) \simeq \lim (\cdots \rightarrow F^{n+1} \rightarrow F^n)\]

as (by [Lur17, 1.2.1.6]), each \(\mathcal{C}_{\leq n}\) is closed under all limits existing in \(\mathcal{C}\),

\[F^{+\infty} \in \mathcal{C}_{\leq -n} \forall n \in \mathbb{Z}\]

(recall that \(\mathcal{C}_{\leq m} \subseteq \mathcal{C}_{\leq n}\) for all pairs of integers \(m \leq n\)) and thus \(F^{+\infty} \simeq 0\) by the right separatedness hypothesis.

For the “if” direction, let us start by noticing that as \(gr^{n+1} F[1] \in \mathcal{C}_{\leq -n}\), the fiber sequence

\[F^n / F^{n+2} \rightarrow gr^n F \rightarrow gr^{n+1} F\]

proves \(F^n / F^{n+2} \in \mathcal{C}_{\leq -n}\) (again, as the latter \(\infty\)-category is closed under limits in \(\mathcal{C}\)). We can now proceed inductively for \(m \geq 2\) to show that (as \(gr^{m+n} F[1] \in \mathcal{C}_{\leq -n-m+1} \subseteq \mathcal{C}_{\leq -n}\)) the fiber sequence

\[F^n / F^{n+m+1} \rightarrow F^n / F^{n+m} \rightarrow gr^{n+m} F[1]\]

implies all objects \(F^n / F^k\) for \(k > n\) lie in \(\mathcal{C}_{\leq -n}\). Since (again, by [Lur17, 1.2.1.6]) we know we can compute colimits in \(\mathcal{C}_{\leq -n}\) just by computing them in \(\mathcal{C}\) and then reflecting along the left adjoint to the inclusion (in particular, the colimit is the same in both categories if the object already happened to land in \(\mathcal{C}_{\leq -n}\) when computed in \(\mathcal{C}\)), we have that

\[\lim_k F^n / F^k \simeq \lim_k \cof (F^k \rightarrow F^n)\]

\[\simeq \cof (\lim_k F^k \rightarrow F^n)\]

\[\simeq F^n / F^{+\infty}\]

lies in \(\mathcal{C}_{\leq -n}\). But, as by hypothesis \(F^{+\infty} \simeq 0\), we have that \(F^n \in \mathcal{C}_{\leq -n}\) as desired. \(\square\)

**Notation 5.13** Motivated by the previous results, we will refer to the transferred \(t\)-structure on \(\text{Fil}^+ \mathcal{C}\) also as the **Beilinson \(t\)-structure**.

**Remark 5.14** In the situation of Theorem 5.10, passing to hearts one gets a “recollement” of Abelian categories:

\[
\begin{array}{ccc}
\mathcal{C}_0 & \xrightarrow{i_L} & \mathcal{C}^\triangledown \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{j_L} & \mathcal{C}_1
\end{array}
\]

where \(i, j\) and \((j_L)_!\) are fully faithful. As shown already in [BBD83, 1.4.18], in this situation \(j_L\) characterizes \(\mathcal{C}_1\) as the quotient category \(\mathcal{C}^\triangledown / \mathcal{C}_0^\triangledown\).

In particular, in the hypothesis of Theorem 5.11, we have that (as \(\mathcal{C}\) is endowed with the trivial \(t\)-structure) \(\mathcal{C}^\triangledown \simeq 0\) and thus (using Remark 5.4)

\[(\text{Fil}^+ \mathcal{C})^\triangledown \simeq \left(\text{Fil}^+ \mathcal{C}\right)^\triangledown \simeq \text{Ch}^* (\mathcal{C}^\triangledown).\]
Remark 5.15 It follows from Definition 5.3 together with Corollary 5.12 that for any $F \in \widehat{\text{Fil}}\mathcal{C}$ and $C \in \text{Ch}^*\mathcal{C}$ we have

$$\mathcal{A}\left(\tau_{\leq n}^B F\right) \simeq \tau_{\leq n}^\text{lv} (\mathcal{A}F)$$

and

$$I \left(\tau_{\leq n}^\text{lv} C\right) \simeq \tau_{\leq n}^B (IC)$$

for all $n \in \mathbb{Z}$, and similar formulae for the truncations above $n$. Moreover, by Remark 5.4, we also have

$$\pi_n^B F \simeq \pi_n^\text{lv} (\mathcal{A}F)$$

and

$$\pi_n^\text{lv} C \simeq \pi_n^B IC$$

for all $n \in \mathbb{Z}$.

Remark 5.16 One can construct on $\text{Fil}^i\mathcal{C}$ the glued t-structure as in Theorem 5.10 regardless of the right separatedness hypothesis; if $\mathcal{C}$ is not right separated, the class of coconnective objects will be the full subcategory

$$\{ F \in \text{Fil}^i_{\leq 0} \mid \tilde{A}F \in (\text{Ch}^*\mathcal{C})_{\leq 0} \text{ and } F^{-\infty} \simeq 0 \}$$

but this class will in general not coincide with the one described in Definition 5.5 (and, as by [Lur17, 1.2.1.3] the class of connective objects completely determines the t-structure, there cannot exist a t-structure exactly as in Definition 5.5 if the two classes of “candidate coconnective objects” do not coincide). We are not aware of any application for this t-structure.

6. Symmetric monoidal structures

The ∞-categorical Day convolution (introduced in [Gla13] and further developed in [Lur17, 2.2.6]) provides a way to equip with a symmetric monoidal structure any functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$, provided that both $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal, and $\mathcal{D}$ is presentably so. In particular (see Remark 6.1), we can endow $\text{Fil}^i\mathcal{C}$ with a symmetric monoidal structure whenever $\mathcal{C}$ is presentably symmetric monoidal. As it turns out, such monoidal structure induces one on $\widehat{\text{Fil}}^i\mathcal{C}$, and thus on $\text{Ch}^*\mathcal{C}$, whenever Theorem 4.7 applies. In this section, we analyze these induced symmetric monoidal structures, and their interaction with the t-structures introduced in the previous section. In particular, we prove that the Day convolution structure on both $\text{Fil}^i\mathcal{C}$ and $\widehat{\text{Fil}}^i\mathcal{C}$ is compatible with Beilinson t-structures, and that the symmetric monoidal structure on $\text{Ch}^*\mathcal{C}$ provides a homotopy coherent refinement of the usual tensor product of cochain complexes.

Remark 6.1 Let $\mathcal{C}$ be a presentably symmetric monoidal $\infty$-category. By [Gla13, 2.11] (see also [Lur17, 2.2.6.17], with $\kappa$ chosen to be the strongly inaccessible cardinal determining the size of our universe of small sets) we can endow $\text{Fil}^i(\mathcal{C})$ with the structure of a symmetric monoidal $\infty$-category, given by Day convolution (where the symmetric monoidal structure on $\mathbb{Z}^\mathbb{op}$ is given by addition). Again, by [Lur17, 2.2.6.17], if $F$ and $G$ are filtered objects in $\mathcal{C}$, their Day convolution product is given by $\otimes \circ (F,G)$ along +, hence by [Lur09, 4.3.3.2] and [Lur09, 4.3.2.2] it is pointwise given by

$$\left(\frac{F \otimes G}{\text{Day}}\right)^p \simeq \text{colim}_{(s,t) \in \mathbb{Z}^\mathbb{op} \times \mathbb{Z}^\mathbb{op}} F^s \otimes G^t$$

where the shape of the colimit follows from inspection of the comma category $(+ \downarrow n)$ (see Definition A.1). In particular, $\text{Fil}^i(\mathcal{C})$ is presentably symmetric monoidal. Similarly, we can endow $\prod_{\mathbb{Z}} \mathcal{C} \simeq \text{Fun}(\mathbb{Z}^\mathbb{op}, \mathcal{C})$ with a presentably symmetric monoidal structure given by Day convolution, whose product is pointwise given by

$$\left(\left(\prod_{\mathbb{Z}} X_u \otimes_{\text{Day}} Y_v\right)_{u \in \mathbb{Z}}\right)_{n} \simeq \bigoplus_{s+t=n} X_s \otimes Y_t$$

where once again the shape of the colimit follows from inspection of the comma category $(+ \downarrow n)$. 
**Proposition 6.2** Let $\mathcal{C}$ be a presentably symmetric monoidal $\infty$-category, whose unit we’ll denote by $1$. Then, the unit object for $(\text{Fil}^i \mathcal{C}, \otimes_{\text{Day}})$ is given by the filtered object

$$1_{\langle \leq 0 \rangle} := \cdots \to 0 \to 0 \to 1 \xrightarrow{id} 1 \xrightarrow{id} \cdots$$

consisting of copies of $1$ and identity morphisms for $n \leq 0$, and of copies of $0$ for $n > 0$.

**Proof.** First, let us note that it is enough to prove the following claim:

(♠) For any $F \in \text{Fil}^i \mathcal{C}$ and any $n \in \mathbb{Z}$ we have

$$(F \otimes_{\text{Day}} 1_{\langle \leq 0 \rangle})^n \simeq F^n.$$  

In fact, the existence of the unitor equivalence $1_{\text{Day}} \otimes 1_{\langle \leq 0 \rangle} \sim 1_{\langle \leq 0 \rangle}$ together with the claim implies the existence of equivalences

$$1_{\text{Day}}^n \simeq (1_{\text{Day}} \otimes 1_{\langle \leq 0 \rangle})^n \simeq 1^n_{\langle \leq 0 \rangle}$$

for all $n \in \mathbb{Z}$. As equivalences in $\text{Fil}^i \mathcal{C}$ can be checked pointwise, this is enough to conclude $1_{\text{Day}} \simeq 1_{\langle \leq 0 \rangle}$.

We now turn to the proof of claim (♠). By (9), this boils down to proving that the colimit of the following diagram

$$\cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots$$

$$\cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots$$

$$\cdots \to F_{n+3} \to F_{n+2} \to F_{n+1} \to F^n \to \cdots$$

$$\cdots \to F_{n+3} \to F_{n+2} \to F_{n+1} \to \cdots$$

$$\cdots \to F_{n+3} \to F_{n+2} \cdots$$
is equivalent to $F^n$. By finality, it is enough to check that the colimit of the following diagram

\[
\begin{array}{ccccccc}
& & & & & & F^n+1 \\
& & & & & \downarrow & \downarrow \\
& & & & F^n & \downarrow & \\
& & & & \downarrow & \downarrow \\
& & 0 & \rightarrow & 0 & \rightarrow & 0 \\
& 0 & \rightarrow & F^n & \rightarrow & F^n+1 \\
F^n+1 & \rightarrow & F^n & \rightarrow & F^n+1 & \rightarrow & F^n+2 \\
\ldots & \rightarrow & \ldots & \rightarrow & \ldots & \rightarrow & \ldots
\end{array}
\]  

(10)

is equivalent to $F^n$. If we denote by $A$ the colimit of the diagram

\[
\begin{array}{ccccccc}
& & & & & & F^n+1 \\
& & & & & \downarrow & \\
& & & & F^n & \downarrow & \\
& & & & \downarrow & \\
& & 0 & \rightarrow & 0 & \rightarrow & 0 \\
& 0 & \rightarrow & F^n & \rightarrow & F^n+1 \\
F^n+1 & \rightarrow & F^n & \rightarrow & F^n+1 & \rightarrow & F^n+2 \\
\ldots & \rightarrow & \ldots & \rightarrow & \ldots & \rightarrow & \ldots
\end{array}
\]  

(11)

and by $B$ the colimit of the diagram

\[
\begin{array}{ccccccc}
& & & & & & F^n+1 \\
& & & & & \downarrow & \downarrow \\
& & & & F^n & \downarrow & \downarrow \\
& & & & \downarrow & \downarrow \\
& & 0 & \rightarrow & F^n & \rightarrow & F^n+1 \\
& 0 & \rightarrow & F^n+2 & \rightarrow & F^n+1 & \rightarrow & F^n+2 \\
F^n+1 & \rightarrow & F^n & \rightarrow & F^n+1 & \rightarrow & F^n+2 \\
\ldots & \rightarrow & \ldots & \rightarrow & \ldots & \rightarrow & \ldots
\end{array}
\]  

(12)

we have that, by [Lur09, 4.2.3.10], we can decompose the colimit of (10) as the coproduct $A \oplus B$. As (11) consists only of zero objects, its colimit is zero. Thus, the colimit of (10) is equivalent to the colimit of (12). By finality, we can omit the top right arrow $0 \rightarrow F^n$ from the diagram in order to compute its colimit. By applying inductively [Lur09, 4.2.3.10], we see that $B$ can be computed as the iterated pushout

\[
F^n \coprod_{F^{n+1}} F^{n+1} \coprod_{F^{n+2}} F^{n+2} \ldots
\]

As each of the pushouts is taken along an equivalence, we have $B \simeq F^n$, as desired. □

Recall the following definition.
Definition 6.3 [Lur17, 2.2.1.7] Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category, and let \( \mathcal{L} : \mathcal{C} \to \mathcal{C} \) be a localization functor. The functor \( \mathcal{L} \) is compatible with the symmetric monoidal structure if for every \( \mathcal{L} \)-equivalence \( X \to Y \), and every \( Z \in \mathcal{C} \), the induced \( X \otimes Z \to Y \otimes Z \) is an \( \mathcal{L} \)-equivalence.

The following proposition already appeared as [GP18, 2.25], we present an alternative proof.

Proposition 6.4 Let \( \mathcal{C} \) be a presentably symmetric monoidal \( \infty \)-category. Then, the localization functor \( \mathcal{L} : \text{Fil}^\mathbb{C} \downarrow \mathcal{C} \to \text{Fil}^\mathbb{C} \) is compatible with Day convolution.

Proof. Note that, by [Nik16, 3.11], \( \text{Fil}^\mathbb{C} \downarrow \mathcal{C} \) admits an internal mapping object given by

\[
\lim_{n \in \mathbb{Z}^{op}} \text{Map}(F, G)(n) \simeq \int_{m \in \mathbb{Z}^{op}} \text{map}_\mathbb{C}(F(m), G(m + n))
\]

(\( \text{map}_\mathbb{C} \) denotes the internal mapping object of \( \mathbb{C} \)). By [Nik16, 2.12 (3)], it suffices to prove that, given any \( F \in \text{Fil}^\mathbb{C} \) and any \( G \in \text{Fil}^\mathbb{C} \), the internal mapping object is complete; by the following computation

\[
\lim_{n \in \mathbb{Z}^{op}} \text{Map}(F, G)(n) \simeq \lim_{n \in \mathbb{Z}^{op}} \int_{m \in \mathbb{Z}^{op}} \text{map}_\mathbb{C}(F(m), G(m + n)) \\
\simeq \int_{m \in \mathbb{Z}^{op}} \lim_{n \in \mathbb{Z}^{op}} \text{map}_\mathbb{C}(F(m), G(m + n)) \\
\simeq \int_{m \in \mathbb{Z}^{op}} \text{map}_\mathbb{C}(F(m), 0) \\
\simeq 0.
\]

this is in fact the case. \( \Box \)

Remark 6.5 [GP18, 2.25] It follows from Proposition 6.4 and [Lur17, 2.2.1.9] that if \( \mathcal{C} \) is presentably symmetric monoidal, we have an induced presentably symmetric monoidal structure on \( \text{Fil}^\mathbb{C} \), which we’ll refer to as the completed Day convolution monoidal structure and will denote by \( \hat{\otimes} \). In particular, we have that

\[
F \hat{\otimes} G \simeq \mathcal{L}(F \otimes \text{Day} G).
\]

Remark 6.6 From Proposition 6.4 and Remark 6.5, we have that the unit for \( \hat{\otimes} \) is given by

\[
\mathcal{L}1_{(\leq 0)} \simeq 1_{(\leq 0)}.
\]

Definition 6.7 Let \( \mathcal{C} \) be a stable presentably symmetric monoidal \( \infty \)-category. We refer to the symmetric monoidal structure induced on \( \text{Ch}^*(\mathcal{C}) \) by the equivalence of Theorem 4.7 as the coherent cochains tensor product, and we will denote it just by \( \otimes \).

Remark 6.8 It follows from Theorem 4.7 together with Remark 4.9 that the functor \( u \) defined in Lemma 2.11 fits into the following commutative diagram

\[
\begin{array}{ccc}
\text{Fil}^\mathbb{C} & \xrightarrow{\hat{\mathcal{A}}} & \text{Ch}^*(\mathcal{C}) \\
\xrightarrow{\text{gr}^n} & & \xrightarrow{\text{gr}^n} \\
\prod_{\mathbb{Z}} \mathcal{C} & \xrightarrow{(\Sigma^n)_{n \in \mathbb{Z}}} & \prod_{\mathbb{Z}} \mathcal{C}.
\end{array}
\]

that is, \( u \circ \hat{\mathcal{A}}F \simeq (\text{gr}^n F[n])_{n \in \mathbb{Z}} \) naturally in \( F \in \text{Fil}^\mathbb{C} \).

The symmetric monoidal structure given in Definition 6.7 is really a homotopy coherent generalization of the usual tensor product of cochain complexes, in a sense made precise by the following results.
**Proposition 6.9** The functor \( u \circ \hat{A} \) is symmetric monoidal.

*Proof.* This is a direct consequence of [GP18, 2.26] together with Remark 6.8 and \((\Sigma^n)_{n \in \mathbb{Z}}\) being an equivalence. \( \square \)

**Corollary 6.10** Let \( \mathcal{C} \) be a stable presentably symmetric monoidal \( \infty \)-category. Let \( C \) and \( D \) be elements of \( \text{Ch}^\ast (\mathcal{C}) \). Then, we have that

\[
(C \otimes D)^n \simeq \bigoplus_{s + t = n} C^s \otimes D^t.
\]

*Proof.* By definition of \( u \) (see Lemma 2.11), \((C \otimes D)^n \simeq u(C \otimes D)^n\). The result follows from the following computation

\[
\begin{align*}
(C \otimes D)^n & \overset{(3.19)}{=} u\left( AIC \otimes AID \right)^n \overset{(6.7)}{=} u\left( IC \circledast ID \right)^n \\
& \overset{(6.5)}{=} u\left( IC \otimes ID \right)^n \\
& \overset{(3.9)}{=} uA\left( IC \otimes ID \right)^n \\
& \overset{(3.9)}{=} \left( u\hat{A}IC \otimes u\hat{A}ID \right)^n \\
& \overset{(3.18)}{=} \left( uC \otimes uD \right)^n \\
& \overset{(6.1)}{=} \bigoplus_{s + t = n} C^s \otimes D^t.
\end{align*}
\]

\( \square \)

We now analyze the interaction between the symmetric monoidal structures introduced in this section, and the t-structures introduced in Section 5. We start by recalling the following definition (see [Lur17, 2.2.1] and [AN20, A.2] for more details about the general theory of the interaction between t-structures and symmetric monoidal structures).

**Definition 6.11** Let \( \mathcal{C} \) be a stably symmetric monoidal \( \infty \)-category equipped with a t-structure \((\mathcal{C} \leq 0, \mathcal{C} \geq 0)\). The t-structure is said to be **compatible** with the symmetric monoidal structure if the following conditions hold:

1. The unit object for \( \otimes \) lies in \( \mathcal{C} \geq 0 \);
2. Given any pair of connective objects \( X, Y \in \mathcal{C} \geq 0 \), their product \( X \otimes Y \) lies in \( \mathcal{C} \geq 0 \) as well.

**Remark 6.12** Conditions (1) and (2) guarantee that \( \mathcal{C} \geq 0 \) inherits a symmetric monoidal structure from \( \mathcal{C} \) such that the fully faithful inclusion \( \mathcal{C} \geq 0 \hookrightarrow \mathcal{C} \) is a (strong) symmetric monoidal functor.

**Proposition 6.13** Let \( \mathcal{C} \) be a presentably symmetric monoidal \( \infty \)-category equipped with a t-structure that is compatible with the monoidal structure. Then:

1. The Beilinson t-structure on \( \text{Fil}^i \mathcal{C} \) is compatible with the Day convolution product \( \circledast_{\text{Day}} \);
2. The Beilinson t-structure on \( \hat{\text{Fil}}^i \mathcal{C} \) is compatible with the completed Day convolution product \( \hat{\circledast} \);
3. If \( \mathcal{C} \) is moreover stable, the pointwise t-structure on \( \text{Ch}^\ast \mathcal{C} \) is compatible with the coherent cochains tensor product \( \otimes \)
Remark such that all pairwise composable morphisms are nullhomotopic (i.e. for all complexes in suitable pointed \( \infty \)-coherent cochain complexes. Our main result will be characterizing coherent cochain is the usual one.

Proof. By Proposition 6.2, we have that
\[
gr^i \mathbb{I}_{\leq 0} \simeq \begin{cases} 0 & \text{for } i \neq 1; \\ 1 & \text{for } i = 1. \end{cases}
\]
and thus the unit is Beilinson connective. Let us now consider \( F, G \in \text{Fil}^i \mathcal{C}_{\geq 0} \).

By [GP18, 2.26] we have that
\[
gr^i (F \otimes_{\text{Day}} G) \simeq \bigoplus_{s + t = i} \gr^{s} F \otimes \gr^{t} G
\]
from which we immediately have that \( F \otimes_{\text{Day}} G \) lies in \( \text{Fil}^i \mathcal{C}_{\geq 0} \). This completes the proof of (1). By Proposition 2.21, applying the localization functor \( L \) has no effect on associated gradeds, hence (1) immediately implies (2). Finally, (3) is a trivial consequence of (2) and the definition of \( \otimes \) on \( \text{Ch}^* \mathcal{C} \).

\[ \square \]

Remark 6.14 It follows from [Lur17, 2.2.1.10] that Day convolution, and hence also the tensor product of coherent chain complexes, induce a symmetric monoidal structure on the hearts of the respective Beilinson t-structures. From Corollary 6.10, we have that the induced symmetric monoidal structure on
\[
\text{Fil}^i (\text{Sp})^\otimes \simeq \text{Ch}^* (\text{Sp})^\otimes \simeq \text{Ch}^* (\mathbb{Z})
\]
is the usual one.

7. COHERENT COCHAIN COMPLEXES AND TODA BRACKETS

In this section, we will have a closer look at the relation between Toda brackets and coherent cochain complexes. Our main result will be characterizing coherent cochain complexes in suitable pointed \( \infty \)-categories as being precisely the sequences of objects
\[
\ldots \to X^n \xrightarrow{f^n} X^{n+1} \xrightarrow{f^{n+1}} X^{n+2} \ldots
\]
such that all pairwise composable morphisms are nullhomotopic (i.e. for all \( n \in \mathbb{Z} \) we have \( f^{n+1} \circ f^n \simeq 0 \)) and all possible Toda brackets are compatibly trivial (see Remark 7.20).

Let us first start with a short recollection about Toda brackets. Given a pointed \( \infty \)-category \( \mathcal{C} \), let \( f^0: X^0 \to X^1, f^1: X^1 \to X^2 \) and \( f^2: X^2 \to X^3 \) be morphisms in \( \mathcal{C} \), such that \( f^1 f^0 \simeq f^2 f^1 \simeq 0 \), and let \( \alpha: 0 \Rightarrow f^1 f^0 \) and \( \beta: 0 \Rightarrow f^1 f^2 \) be two choices of nullhomotopies:
\[
\begin{array}{c}
X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} X^3.
\end{array}
\]
In such a situation, the two whiskerings \( f^2 \alpha \) and \( \beta f^0 \) determine two paths \( 0 \Rightarrow f^2 f^1 f^0 \) in \( \text{Map}_*(X^0, X^3) \). By gluing them along their endpoints, we obtain a map (pointed at zero) \( T: S^1 \to \text{Map}_*(X^0, X^3) \), which in turns determines a class \( \langle f^2, f^1, f^0 \rangle_{(\alpha, \beta)} \in \pi_1 (\text{Map}(X^0, X^3)) \). The homotopy class \( \langle f^2, f^1, f^0 \rangle_{(\alpha, \beta)} \) is known as the Toda bracket determined by \( \alpha \) and \( \beta \).

The customary approach to Toda brackets would be not fix \( \alpha \) and \( \beta \) as part of the datum, and to instead define the Toda bracket to be a subset of \( \pi_1 (\text{Map}(X^0, X^3)) \), whose elements are given by all the possible choices of homotopy classes of paths \(([\alpha], [\beta])\). According to this definition, the object referred above as \( \langle h, g, f \rangle_{(\alpha, \beta)} \) is a specified element of this subset.

The notion can be generalized to longer sequences of maps, provided that the Toda brackets of the shorter sub-sequences are nullhomotopic; to illustrate how the generalization works, let us consider the case of a sequence of length 4
\[
X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} X^3 \xrightarrow{f^3} X^4.
\]
and choices of nullhomotopies $\alpha$, $\beta$ and $\gamma$ for all consecutive pairs of maps, for which $(f^3, f^1, f^0)(\alpha, \beta) = 0$ and $(f^3, f^2, f^1)(\beta, \gamma) = 0$; in such a situation, we obtain a pair of nullhomotopies for a pair of maps $X^0 \to \Sigma X^4$, one for the composite

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{(f^3, f^2, f^1)(\beta, \gamma)} \Sigma X^4,$$

and the other one for composite

$$X^0 \xrightarrow{(f^3, f^2, f^1)(\alpha, \beta)} \Sigma X^3 \xrightarrow{\Sigma f^3} \Sigma X^4,$$

putting together these two as before, we obtain a pointed map $S^1 \to \text{Map}_*(X^0, \Sigma X^4)$, and thus a class $(f^3, f^2, f^1, f^0) \in \pi_2(\text{Map}_*(X^0, X^3))$, the $4$-fold Toda bracket of the sequence\footnote{Again, we are working with a coherent notion, depending also on the choices of all the five nullhomotopies involved}. This idea clearly generalizes to longer sequences of maps, provided that the shorter subsequences have nullhomotopic brackets.

Although it is possible to work with these notions using the approach sketched above, in the rest of the section we will use a slightly different perspective to define and generalize Toda brackets. Our approach will be to define these classes by means of the actions of certain algebra objects in graded pointed spaces; the main advantage for us will be the case of working at once with all the possible $n$-fold Toda brackets in a $\mathbb{Z}$-indexed sequence (provided that all the possible $(n-1)$-fold Toda brackets are trivial); of course, one can recover the case of finite sequences by considering a $\mathbb{Z}$-indexed sequence where only finitely many objects are non-zero (see Definition 7.16 for the details). One other pleasant feature of our approach will be to obtain an inherently coherent notion of Toda brackets (that is, encoding also choices for all the involved homotopies, rather than defining the notion up to a coset of “indeterminacies”).

Our strategy for proving that coherent chain complexes are precisely those $\mathbb{Z}$-indexed sequences of morphisms where consecutive maps compose to zero and all Toda brackets are coherently nullhomotopic requires a few steps: in §7.1, we will prove that for a pointed $\infty$-category $\mathcal{C}$ (which we will for simplicity assume to be also presentable) both $\text{Fun}(\mathbb{Z}_{\text{op}}, \mathcal{C})$\footnote{In the present section we will use the notation $\text{Fun}(\mathbb{Z}_{\text{op}}, \mathcal{C})$ instead of the previously introduced $\text{Fil}^*(\mathcal{C})$ to stress that here we want to think not of structured objects of independent interest, but of sequences of morphisms upon which we intend to put relations; in particular, the reader should keep in mind that the appearance of this $\infty$-category here has nothing to do with the equivalence of Theorem 4.7} and $\text{Ch}^*(\mathcal{C})$ are $\infty$-categories of modules for suitable $\mathbb{E}_1$-algebra objects in $\text{Gr} \mathcal{S}_{\bullet}$. Then, in §7.2 we will construct a sequence of $\mathbb{E}_1$-algebras in $\text{Gr} \mathcal{S}_{\bullet}$

$$R_1 \to R_2 \to R_3 \to \cdots$$

(13)

such that the $\infty$-category of modules over $R_1$ will be equivalent to $\text{Fun}(\mathbb{Z}_{\text{op}}, \mathcal{C})$, the $\infty$-category of modules over colim$_n R_n$ will be equivalent to $\text{Ch}^*(\mathcal{C})$, and such that the sequence induced by (13) on homology will be equivalent to the resolution of $\Lambda(\mathcal{C})$ described in Theorem B.1, with $e$ of degree 1. Modules over $R_n$ for a fixed $n \geqslant 2$ will be sequences where all pairwise composable morphisms will be nullhomotopic, all possible $m$-fold Toda brackets for $m \leqslant n$ will be defined and trivial, and all the relevant nullhomotopies will be encoded in the module structure (see Remark 7.15 for the details). In order to prove the existence of the sequence (13), we will use the results of Appendix B. As it turns out, the construction of the above sequence up to $R_3$ is slightly more subtle than its extension to all the other $R_n$’s, hence we will first construct the sequence inductively for $n \geqslant 4$, and defer the construction of the beginning of the sequence to §7.3.

7.1. Algebras in graded pointed spaces. The main goal of this section is to prove that, given a pointed $\infty$-category $\mathcal{C}$, both $\text{Fun}(\mathbb{Z}_{\text{op}}, \mathcal{C})$ and $\text{Ch}^*(\mathcal{C})$ can be expressed as $\infty$-categories of modules over suitably defined $\mathbb{E}_1$-algebras in graded pointed spaces. For simplicity, we will assume that $\mathcal{C}$ is also presentable, but with some extra effort...
the results can be generalized further, using methods along the lines of those employed in Section 4.

The stable analogue of the following proposition appeared as [Lur15, 2.4.4].

Proposition 7.1 Let $\mathcal{C}$ be a pointed $\infty$-category. The following data are equivalent:

1. self-equivalences $\mathcal{C} \to \mathcal{C}$;
2. (left or right) actions of $\mathbb{Z}^\delta$ on $\mathcal{C}$;
3. monoidal functors $\mathbb{Z}^\delta \to \text{Fun}(\mathcal{C}, \mathcal{C})$;
4. monoidal functors $\mathbb{Z}^\delta \to \text{Fun}^{\text{Rex}}(\mathcal{C}, \mathcal{C})$;
5. right exact monoidal functors $\mathcal{P}^{\text{fin}}(\mathbb{Z}^\delta) \to \text{Fun}^{\text{Rex}}(\mathcal{C}, \mathcal{C})$;
6. (left or right) actions of $\mathcal{P}^{\text{fin}}_0(\mathbb{Z}^\delta)$ on $\mathcal{C}$ such that the action map commutes with finite colimits in each variable.

If $\mathcal{C}$ is pointed presentable, then the above are equivalent to:

7. left adjoint monoidal functors $\mathcal{P}_*(\mathbb{Z}^\delta) \to \text{Fun}^{\text{Rex}}(\mathcal{C}, \mathcal{C})$;
8. actions of $\mathcal{P}_*(\mathbb{Z}^\delta)$ on $\mathcal{C}$ such that the action map commutes with small colimits in each variable.

Here $\mathcal{P}_*(\mathbb{Z}^\delta)$ and $\mathcal{P}^{\text{fin}}_*(\mathbb{Z}^\delta)$ are defined as in §A.3.

Proof. The equivalence (1) $\Leftrightarrow$ (3) follows from [Lur15, 2.4.3]: loc. cit. gives an equivalence

$$\text{Fun}^\circ(\mathbb{Z}^\delta, \text{Fun}(\mathcal{C}, \mathcal{C})) \simeq \text{Aut}(\mathcal{C})$$

where the monoidal structure on $\text{Fun}(\mathcal{C}, \mathcal{C})$ is given by composition, and $\text{Aut}(\mathcal{C}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{C})$ denotes the full subcategory spanned by self-equivalences. (3)$\Leftrightarrow$(2) and (5)$\Leftrightarrow$(6) are just reformulations of the definition of action of a symmetric monoidal $\infty$-category. (3)$\Leftrightarrow$(4) holds trivially, as $\mathbb{Z}^\delta$ is a discrete category. (4)$\Leftrightarrow$(5) follows from Proposition A.28 and the monoidality of the pointed Yoneda embedding (see [Nik16, §6]). (7)$\Leftrightarrow$(5) and (8)$\Leftrightarrow$(6) follow at once from Proposition A.28.

Remark 7.2 In what follows, unless otherwise specified, we will implicitly work with left actions whenever we apply Proposition 7.1. Recall that, in the terminology of [Lur17], a left action of a monoidal $\infty$-category $\mathcal{C}$ on some $\infty$-category $\mathcal{M}$ is equivalent to exhibiting $\mathcal{M}$ as left-tensored over $\mathcal{C}$, and to exhibiting $\mathcal{M}$ as a left $\mathcal{C}$-module in (a suitably sized) $\infty$-category of $\infty$-categories (see [Lur15, p. 8]).

Example 7.3 Let $\mathcal{C}$ be a pointed $\infty$-category. Then, precomposition with $-1: \mathbb{Z} \to \mathbb{Z}$ induces an automorphism

$$(-)\{1\}: \text{Ch}^*\mathcal{C} \to \text{Ch}^*\mathcal{C} : C^* \mapsto C\{1\}^* \simeq C^{*-1}$$

by virtue of Proposition 7.1, this endows $\text{Ch}^*\mathcal{C}$ with the structure of an $\infty$-category left-tensored over $\mathcal{P}^{\text{fin}}_0(\mathbb{Z}^\delta) \simeq \text{Gr}(\delta^{\text{fin}}_*)$ (and when $\mathcal{C}$ is also presentable, the left-tensoring extends over $\text{Gr}(\delta_*)$); we will use the notation:

$$(-)\{n\}: \text{Ch}^*\mathcal{C} \to \text{Ch}^*\mathcal{C} \quad \forall n \in \mathbb{Z}$$

to denote the (right exact) endofunctor induced by $n$.

Example 7.4 Let $\mathcal{C}$ be a pointed $\infty$-category. Similarly to the previous example, precomposition with $-1: \mathbb{Z}^{\text{op}} \to \mathbb{Z}^{\text{op}}$ induces an automorphism

$$(-)\{1\}: \text{Fil}^*\mathcal{C} \to \text{Fil}^*\mathcal{C}$$

hence, $\text{Fil}^*\mathcal{C}$ is also left-tensored over $\text{Gr}(\delta_*^{\text{fin}})$ (and when $\mathcal{C}$ is also presentable, the left-tensoring extends over $\text{Gr}(\delta_*)$); also in this case, we will use the notation:

$$(-)\{n\}: \text{Fil}^*\mathcal{C} \to \text{Fil}^*\mathcal{C} \quad \forall n \in \mathbb{Z}$$

to denote the automorphism induced by $n$.

Notation 7.5 Given any $n \in \mathbb{N}$ and any $t \in \mathbb{Z}$, we will denote by $S^{n,t} \in \text{Ch}^*\delta_*$ the graded pointed space consisting of a copy of $S^n$ in degree $t$, and copies of pt in all other degrees.
Proposition 7.6 Let $C \in \text{Ch}^* \mathcal{C}$ be a coherent cochain complex in a complete\footnote{or, more generally, in a pointed $\infty$-category where all the relevant right Kan extensions exist and are pointwise}, pointed $\infty$-category. The functor $- \otimes C : \text{Gr} \delta_\ast \to \text{Ch}^* \mathcal{C}$ induced by the left-tensoring of Example 7.3 admits a right adjoint, denoted $\text{Map}_\ast (C, -)$, whose value on any $D \in \text{Ch}^* \mathcal{C}$ is pointwise given by

$$\text{Map}_\ast (C, D)^n \simeq \text{Map}_\ast (C[n], D).$$

Proof. By unraveling the definitions, the $\text{Gr} \delta_\ast$-action on $\text{Fun}^{\text{Res}}(\text{Ch}^* \mathcal{C}, \text{Ch}^* \mathcal{C})$ is obtained by Kan extension of the functor $\{\} : \mathbb{Z}^\delta \to \text{Fun}^{\text{Res}}(\text{Ch}^* \mathcal{C}, \text{Ch}^* \mathcal{C})$ along the functor $y : \mathbb{Z}^\delta \to \text{Gr} \delta_\ast : n \mapsto S^0[n]$. As the Kan extension is pointwise,

$$- \otimes C \simeq \text{Lan}_y (ev_C \circ \{\})$$

where $ev_C : \text{Fun}^{\text{Res}}(\text{Ch}^* \mathcal{C}, \text{Ch}^* \mathcal{C}) \to \text{Ch}^* \mathcal{C}$ denotes the evaluation at $C$ functor. In particular, if we put $F := ev_C \circ \{\}$, we have that $- \otimes C$ is, in the language of Appendix A, the pointed $F$-realization functor. Thus, it admits as a right adjoint given by the pointed $F$-nerve functor $N^\text{pt}_F$, which by Proposition A.29 is given by

$$N^\text{pt}_F (D) \simeq \text{Map}_\ast (F(-), D) \simeq \text{Map}_\ast (C[-], D)$$

as desired. \hfill $\square$

Definition 7.7 We will refer to the right adjoint functor defined in Proposition 7.6 as the graded pointed mapping space functor.

Theorem 7.8 Let $\mathcal{C}$ be a pointed presentable $\infty$-category. Then, there exists an $E_1$-algebra in graded pointed spaces $A \in \text{Alg}_{\text{Gr} \delta_\ast}$ such that the $\infty$-category of coherent cochain complexes in $\mathcal{C}$ is equivalent to the $\infty$-category of right $A$-modules in graded objects of $\mathcal{C}$:

$$\text{Ch}^* \mathcal{C} \simeq \text{RMod}_A (\text{Gr} \mathcal{C}).$$

Moreover, the underlying graded pointed space of $A$ is given by $S^{0,1}$. \hfill $\square$

Proof. Let us first notice that the claim for a generic pointed presentable $\infty$-category $\mathcal{C}$ can be deduced from the case $\mathcal{C} \simeq \delta_\ast$, by means of Lurie’s tensor product (see [Lur17, §4.8]): by [Lur17, 4.8.1.17],

$$\text{Fun} (\text{Ch}^{\text{op}}, \mathcal{C}) \simeq \text{Fun} (\text{Ch}^{\text{op}}, \delta_\ast) \otimes \mathcal{C};$$

since preserving the zero object is a property, the equivalence above induces an equivalence $\text{Ch}^* (\mathcal{C}) \simeq \text{Ch}^* (\delta_\ast) \otimes \mathcal{C}$. Now, by [Lur17, 4.8.4.6],

$$\text{RMod}_A (\text{Gr} \delta_\ast) \otimes \mathcal{C} \simeq \text{RMod}_A (\text{Gr} \mathcal{C});$$

it thus suffices to prove our claim for the case of pointed spaces.

By Example 7.3, together with Proposition 7.1, we have that $\text{Ch}^* \delta_\ast$ is left-tensored over $\text{Gr} \delta_\ast$ (see Remark 7.2). By [Lur17, 4.8.5.8], it is thus enough to prove that:

1. $\text{Ch}^* (\delta_\ast)$ admits geometric realizations of simplicial objects.
2. The action map $\text{Gr} \delta_\ast \times \text{Ch}^* (\delta_\ast) \to \text{Ch}^* (\delta_\ast)$ preserves geometric realizations of simplicial objects.
3. There exists an $M \in \text{Ch}^* (\delta_\ast)$ such that the functor $- \otimes M : \text{Gr} \delta_\ast \to \text{Ch}^* (\delta_\ast)$ admits a right adjoint, denoted $\text{Map}_\ast (M, -)$.
4. The functor $\text{Map}_\ast (M, -)$ preserves geometric realizations of simplicial objects.
5. The functor $\text{Map}_\ast (M, -)$ is conservative.
6. For every coherent chain complex $C$ and every graded pointed space $X$, the map

$$X \otimes \text{Map}_\ast (M, C) \otimes M \xrightarrow{X \otimes cC} X \otimes C$$

is adjoint to an equivalence

$$X \otimes \text{Map}_\ast (M, C) \to \text{Map}_\ast (M, X \otimes C).$$
We take $M$ to be the coherent cochain complex consisting of a copy of $S^0$ sitting in degree 0, a copy of $S^0$ sitting in degree 1, copies of pt elsewhere, and the identity as its only possibly nontrivial differential. (1) follows from the cocompleteness of $S_*$, (2) follows from presentability of $S_*$ and Proposition 7.1, (3) and (4) hold for any choice of $M$, and follow from Proposition 7.6. Using the explicit description for the right adjoint given in Proposition 7.6, we get that for our choice of $M$, $\Map_*(M,C)^n \simeq \Map_*(M\{n\},C)$ is equivalent to the space of choices of $f$ and $g$’s making the square

$$
\begin{array}{ccc}
S^0 & \xrightarrow{id} & S^0 \\
\downarrow & & \downarrow \\
C^n & \xrightarrow{\partial^n} & C^{n+1}
\end{array}
$$

commute; this space is in turn equivalent to $\Map_*(S^0, C^n) \simeq C^n$, and thus the functor $\Map_*(M,-)$ is equivalent to the forgetful functor $u_\ast \colon Ch^* S_* \to Gr S_*$ introduced in Lemma 2.11, implying (5).

For (6), by definition of adjoint map, the adjoint map of interest can be factored as

$$
X \otimes \Map_*(M,C) \xrightarrow{\eta_X \otimes \Map_*(M,C)} \Map_*(M,X \otimes \Map_*(M,C) \otimes M) \xrightarrow{\Map_*(M,X \otimes \varepsilon_c)} \Map_*(M, X \otimes C).
$$

By a pointwise check, we have that $\eta_X \otimes \Map_*(M,C) \simeq \Map_*(M, \eta_X \otimes C)$, and $X \otimes \varepsilon_c \simeq \varepsilon_X \otimes C$; by the conservativity of $\Map_*(M,-)$, it is sufficient to show that the composite

$$
X \otimes C \xrightarrow{\eta_X \otimes C} \Map_*(M, X \otimes C) \otimes M \xrightarrow{\varepsilon_X \otimes C} X \otimes C
$$

is an equivalence; but, as the latter is one of the triangle identities, the desired condition holds.

Finally, in the proof of [Lur17, 4.8.5.8] is shown that $A$ can be identified with the internal hom object $\Map_*(M,M)$, thence we get the desired characterization for the graded pointed space underlying $A$.

\begin{remark}
As, by [GGN15, 5.5], the stabilization functor is symmetric monoidal on presentable \(\infty\)-categories, we can stabilize Theorem 7.8, to obtain an equivalence

$$
Ch^*(\mathcal{C}) \simeq RMod_{30,0,30,1}(\text{Gr } \mathcal{C})
$$

for any stable presentable \(\infty\)-category $\mathcal{C}$.
\end{remark}

\begin{theorem}
Let $\mathcal{C}$ be a pointed presentable \(\infty\)-category. Then, the \(\infty\)-category of filtered objects of $\mathcal{C}$ is equivalent to the \(\infty\)-category of right $\text{Free}_{\mathbb{E}_1}(S^{0,1})$-modules in graded objects:

$$
\text{Fun}(\mathbb{Z}^{op}, \mathcal{C}) \simeq RMod_{\text{Free}_{\mathbb{E}_1}(S^{0,1})}(\text{Gr } \mathcal{C}).
$$

\end{theorem}

\begin{proof}
Let us first prove the claim for the case $\mathcal{C} \simeq S_*$. This is basically an adaptation of [Lur15, 3.1.6] to the case of pointed spaces.

First of all, notice that, as the functor $u_\ast \colon \text{Fun}(\mathbb{Z}^{op}, S_*) \to \text{Gr } S_*$ given by precomposition with $\mathbb{Z}^1 \to \mathbb{Z}^{op}$: $n \mapsto -n$ is symmetric monoidal, it induces a functor

$$
\theta \colon \text{Fun}(\mathbb{Z}^{op}, S_*) \simeq \text{Mod}_{\Delta_{	ext{Day}}}(\text{Fun}(\mathbb{Z}^{op}, S_*)) \to \text{Mod}_{\Delta_{	ext{Day}}}(\text{Gr } S_*);
$$

by Proposition 6.2 together with [Lur17, 4.1.1.18], we have that $u\mathbb{1}_{\leq 0}$ is an $\mathbb{E}_\infty$-algebra whose underlying $\mathbb{E}_1$-algebra is equivalent to $\text{Free}_{\mathbb{E}_1}(S^{0,1})$.

It is thus sufficient to show that the functor $\theta$ is an equivalence. Let us start with fully faithfulness. That is, we want to prove that the maps

$$
\varphi_{X,Y} \colon \text{Map}_*(X,Y) \to \text{Map}_*(\theta X, \theta Y)
$$

are all equivalences. As the functor $\varphi_{-,Y} \colon \text{Fun}(\mathbb{Z}^{op}, S_*) \to \text{Fun}(\Delta^1, S_*)$ sending a graded pointed space $X$ to the map $\varphi_{X,Y}$ sends colimits to limits, the full subcategory spanned by those $X$ such that $\varphi_{X,Y}$ is an equivalence is a pointed subcategory of
Fun(\mathbb{Z}^{\text{op}}, \Sigma_\ast) \) closed under colimits. Thus, as \( \text{Fun}(\mathbb{Z}^{\text{op}}, \Sigma_\ast) \) is generated under colimits by objects of the form \( k_n^\ast \) for \( n \in \mathbb{Z} \) (see Proposition A.28), it suffices to show that \( \varphi_{-Y} \) is an equivalence for such generators; this is indeed the case, as for \( X \simeq k_n^\ast \) we have \( \varphi_{X,Y} \simeq \text{id}_{Y \ast} \). Essential surjectivity now follows from the observation that the essential image of \( \theta \) is closed under colimits, as \( \theta \) is fully faithful and commutes with colimits.

Similarly to the proof of Theorem 7.8, the claim for general pointed presentable \( \infty \)-categories follows from the properties of Lurie’s tensor product. By [Lur17, 4.8.1.17],

\[
\text{Fun}(\mathbb{Z}^{\text{op}}, \mathcal{C}) \simeq \text{Fun}(\mathbb{Z}^{\text{op}}, \Sigma_\ast) \otimes \mathcal{C}
\]

and by [Lur17, 4.8.4.6],

\[
\text{RMod}_{\text{Free}_E}(s_{\ast}^n)_{\text{pt}}(\text{Gr} \Sigma_\ast) \otimes_{\text{Gr} \Sigma_\ast} \mathcal{C} \simeq \text{RMod}_{\text{Free}_E}(s_{\ast}^0)_{\text{pt}}(\text{Gr} \mathcal{C})
\]

we thence have the desired claim.

**Remark 7.11** We can describe more explicitly the equivalence of Theorem 7.10. Let us first observe that the action of the free generator of \( R_1 := \text{Free}_E(S_{\ast}^0) \) determines the action of all other spaces of \( R_1 \). By unraveling the definitions, specifying a map \( t^\ast : S_{\ast}^{0,n} \otimes X \to X \) on a graded object \( X \) is equivalent to specifying maps \( X^n \to X^{n+1} \) for all \( n \in \mathbb{Z} \). Hence, given an \( R_1 \)-module, the associated filtered object is the one having as maps between consecutive objects the maps determined by \( t^\ast \), and the actions of the \( s \)-th powers of the generator \( t^\ast : S_{\ast}^{0,s} \otimes X \to X \) correspond to choices for all the possible \( s \)-fold composites for the structure maps of the resulting filtered object (i.e., maps \( X^n \to X^{n+s} \) for all \( n > 1 \)). This description makes also clear the behavior of the functor going in the opposite direction.

**7.2. The resolution sequence.** We will now move to the construction of the sequence (13). By Theorem 7.10, we can take \( R_1 \simeq \text{Free}_E(S_{\ast}^0) \). We will define the first map of the sequence to be the one obtained by killing the square of the free generator of \( R_1 \).

**Construction 7.12** By [Lur17, 4.1.1.18], we have \( (\text{Free}_E(S_{\ast}^0))^n \simeq S^n \) for all \( n \geq 0 \). In particular, the inclusion of pointed spaces \( S_{\ast}^{0,2} \to \text{Free}(S_{\ast}^0) \) induces a map of \( E_1 \)-algebras \( t^2 : \text{Free}_E(S_{\ast}^{0,2}) \to \text{Free}_E(S_{\ast}^0) \). We define \( R_2 \) to be the \( E_1 \)-algebra in graded pointed spaces obtained as the pushout of \( t^2 \) along the augmentation map for the free algebra \( \text{Free}_E(S_{\ast}^{0,2}) \):

\[
\begin{array}{ccc}
\text{Free}_E(S_{\ast}^{0,2}) & \longrightarrow & S_{\ast}^{0,0} \\
\downarrow^{t^2} & & \downarrow \\
\text{Free}_E(S_{\ast}^{0,1}) & \longrightarrow & R_2
\end{array}
\]

We pick the bottom horizontal map \( \text{Free}_E(S_{\ast}^{0,1}) \simeq R_1 \to R_2 \) to be the first map in the sequence (13).

**Remark 7.13** Notice that, as passing to homology preserves colimits, the pushout square (14) induces on homology the square of Theorem B.2 for the case \( n = 2 \).

In Lemma 7.25, we will show that there exists a map \( S_{\ast}^{1,3} \to R_2 \), inducing on homology the map \( \text{Free}_E(r_3) \to A^{(2)} \) of Theorem B.2. Given such a map, we can consider the pushout diagram of \( E_1 \)-algebras

\[
\begin{array}{ccc}
\text{Free}_E(S_{\ast}^{1,3}) & \longrightarrow & S_{\ast}^{0,0} \\
\downarrow & & \downarrow \\
R_2 & \longrightarrow & R_3
\end{array}
\]

and pick the bottom horizontal map to be the second map in the sequence (13).

Similarly, in Lemma 7.27, we will show that there exists a map \( S_{\ast}^{2,4} \to R_2 \), inducing on homology the map \( \text{Free}_E(r_4) \to A^{(3)} \) of Theorem B.2.

Given such maps, we can construct the rest of the sequence by induction.
Construction 7.14 Let us assume we are given a sequence \( R_1 \to R_2 \to \cdots \to R_n \) with \( n \geq 3 \), such that every \( R_j \) comes equipped with a map \( S^{j-1,j+1} \to R_j \) inducing on homology the map \( \text{Free} \_E_1(r_{j+1}) \to \Lambda^{(j)} \) of Theorem B.2, and that each map \( R_{j-1} \to R_j \) for \( 2 \leq j \leq n \) fits in a commutative square of \( \mathbb{E}_1 \)-algebras

\[
\begin{array}{ccc}
\text{Free} \_E_1(S^{j-2,j}) & \to & S^{0,0} \\
\downarrow & & \downarrow \\
R_{j-1} & \to & R_j,
\end{array}
\]

Our goal is to produce inductively an \( \mathbb{E}_1 \)-algebra \( R_{n+1} \) and a map \( S^{n,n+2} \to R_{n+1} \) inducing on homology the map \( \text{Free} \_E_1(r_{n+2}) \to \Lambda^{(n+1)} \) of Theorem B.2.

Let us denote by \( R_{n+1} \) the object obtained pushing out the map \( \text{Free} \_E_1(S^{n-1,n+1}) \to R_n \) we have by inductive hypothesis along the augmentation map of the free algebra:

\[
\begin{array}{ccc}
\text{Free} \_E_1(S^{n-1,n+1}) & \to & S^{0,0} \\
\downarrow & & \downarrow \\
R_n & \to & R_{n+1}.
\end{array}
\]

We pick the bottom map of the above square to be the one for the sequence (13). By passing to homology, we get precisely the pushout square of Theorem B.2. By Lemma B.14, we have an element \( r_{n+1} \) of bidegree \( (n+1, n-1) \) in \( \Lambda^{(n)} \simeq \Lambda(c) \otimes \mathbb{Z}[r_{n+1}] \). By Hurewicz’s Theorem, the cohomology class \( r_{n+1} \) determines in an essentially unique way a map \( S^{n,n+2} \to R_{n+1} \), which we will denote \( \langle t \rangle^{n+2} \) and call the \( (n+2) \)-fold Toda power of \( t \). This completes the induction step needed to extend the sequence for \( m \geq 4 \).

Remark 7.15 We can now understand the relations between \( R_m \)-modules for varying \( n \):

1. It follows from Construction 7.12 together with Remark 7.11 that the \( R_1 \)-module associated to a filtered space \( X \) can be given the structure of an \( R_2 \)-module if and only if all the composites of pairs of consecutive maps of \( X \) are nullhomotopic, and specifying the structure of an \( R_2 \)-module is equivalent to choosing a nullhomotopy for each such pair.

2. It follows from Construction 7.14 together with Remark 7.11 that given any \( R_{n-1} \)-module \( C \), it can be given the structure of an \( R_n \)-module if and only if the map \( \langle t \rangle^m : S^{2,m-n} \to \text{Map}_*(C,C) \) of Construction 7.14 is trivial. By unraveling the definitions, we see that if we denote by \( X \) the filtered space associated to \( C \) (obtained by restriction of scalars along \( R_1 \to R_{n-1} \)), the components of \( \langle t \rangle^m \) are maps

\[
S^{2,m-n} \to \text{Map}_*(X^m, X^{m+n})
\]

for all \( m \in \mathbb{Z} \); specifying the structure of an \( R_n \)-module on \( C \) is equivalent to choosing nullhomotopy for each such map.

Definition 7.16 Let \( X^0 \to X^1 \to \cdots \to X^n \) be a sequence of pointed spaces and maps between them, and assume that the filtered pointed space \( X \) obtained by extending the given sequence by zeroes can be given the structure of an \( R_m \)-module.

Let us moreover fix one such structure and denote the resulting object by \( C_X \). We define the \( n \)-fold Toda bracket of \( C_X \) to be the class

\[
S^{n-2} \to \text{Map}_*(X^0, X^n)
\]

given by the only nontrivial component of the map \( \langle t \rangle^n : S^{n-2,n} \to \text{Map}_*(C_X, C_X) \) of Construction 7.14, or, in the case of \( n = 3 \) (resp. \( n = 4 \)), the map \( \langle t \rangle^3 \) defined in Construction 7.24 (resp. the map \( \langle t \rangle^4 \) defined in Construction 7.26).
Definition 7.17 Given any filtered pointed space \( X \in \text{Fun}(\mathbb{Z}^{op}, \Delta_+) \), we say it is a naive cochain complex if its corresponding \( R_1 \)-module can be given a structure of \( R_2 \)-module. Given a naive cochain complex \( X \), we say that it has uniformly trivial \( n \)-fold Toda brackets if it can be recursively given the structure of an \( R_m \)-module after choosing \( R_m \)-module structures for all \( m < n \).

Remark 7.18 As all pointed \( \infty \)-categories are canonically enriched over pointed spaces, the definition of Toda brackets generalizes in an obvious way to sequences of maps in an arbitrary pointed \( \infty \)-category. In particular, it extends to sequences of maps in any stable \( \infty \)-category.

Proposition 7.19 The \( \mathbb{E}_1 \)-algebra \( A \) of Theorem 7.8 is equivalent to the colimit \( \text{colim}_n R_n \) of the sequence (13); i.e. for \( \mathcal{C} \) pointed presentable with a special extremal separator,

\[
\text{Ch}^*(\mathcal{C}) \simeq \text{RMod}_{\text{colim}_{n \to \infty}} R_n(\text{Gr} \mathcal{C}).
\]

Proof. It follows from the resolution sequence (13), together with the associated sequence in homotopy, that the underlying graded space of \( \text{colim}_n R_n \) is equivalent to \( S^{0,0} \sqcup S^{0,1} \), as is also the underlying space of the \( \mathbb{E}_1 \)-algebra \( A \) of Theorem 7.8. As \( S^{0,0} \sqcup S^{0,1} \) is 0-truncated, the \( \mathbb{E}_1 \)-algebra structures on it are determined by the associative ring structures on the discrete object \( S^{0,0} \sqcup S^{0,1} \) (living in the 1-category \( \text{Ch}^*(\text{Top}) \)). But, by a direct check, there exists only one nontrivial associative ring structure on \( S^{0,0} \sqcup S^{0,1} \). As \( \text{colim}_n R_n \) is augmented over \( S^{0,0} \), it cannot have trivial multiplication; likewise, \( A \) cannot be trivial by Theorem 7.8. Therefore \( A \) and \( \text{colim}_n R_n \) must be equivalent as \( \mathbb{E}_1 \) algebras.

Remark 7.20 The above proposition can be informally rephrased by saying that the datum of a coherent cochain complex is equivalent to the datum of a naive chain complex together with choices for all the nullhomotopies of pairs of consecutive maps, and recursively defined choices of nullhomotopies for all possible \( n \)-fold Toda brackets, for all \( n \geq 3 \).

7.3. The resolution sequence for \( n \leq 3 \).

Notation 7.21 Let \( \mathcal{C} \) be a pointed \( \infty \)-category. Let \( X = (X^n)_{n \in \mathbb{Z}} \in \text{Gr} \mathcal{C} \). Given any \( s \in \mathbb{Z} \) and any \( t \in \mathbb{N} \) will denote by \( \Omega^f s \) the composite functor \( \Omega^f_\mathcal{C} \circ \{ -s \} \simeq \{ -s \} \circ \mathcal{C}_\mathcal{C} \). That is,

\[
(\Omega^f s X)^n \simeq \Omega^f (X^{n+s}).
\]

Notice that, for \( S^{0,0} \in \text{Gr} \Delta_* \), we have \( \Omega^f s (S^{0,0}) \simeq S^{0,-s} \).

Remark 7.22 Let \( X \in \text{Ch}^* \Delta_* \); we have that:

(1) For all \( t \in \mathbb{N}, s \in \mathbb{Z} \),
\[
\text{Map}_*(S^{f,s}, X) \simeq (\Omega^f X)\{ -s \} \simeq \Omega^{f+s} X;
\]

(2) For all \( t \in \mathbb{N}, s \in \mathbb{Z} \),
\[
S^{f,s} \underset{\text{Day}}{\boxtimes} X \simeq (\Sigma^t X)\{ s \}.
\]

Remark 7.23 The structure map \( \text{Free}_{\mathbb{E}_1}(S^{0,1}) \to R_2 \) defined in Construction 7.12 determines a degree 1 square-zero element \( t \in (\pi_0(R_2))^1 \), whose action induces the following diagram of right \( R_2 \)-modules:
**Construction 7.24** In the situation of Remark 7.23, let us denote by $F_2$ the fiber (in $R_2$-modules) of multiplication by $t$, considered as a map $R_2 \to \Omega^{0,1} R_2$. Choosing a nullhomotopy $\alpha: 0 \Rightarrow t^2$ is equivalent to choosing a factorization of $t$ through $\Omega^{0,1} F_2$. By the following diagram (where all squares are Cartesian)

we see that there exists a map $R_2 \to \Omega^{1,3} R_2$ induced by such a factorization. We call this map the 3-fold Toda power of $t$, and denote it by $\langle t \rangle^3$. With a little abuse of terminology, we will reserve the same name for the map $S_1,3 \to \text{Hom}(R_2, R_2)$ obtained by adjoining over twice:

As $R_2$ is a free $R_2$-module of rank 1, we have that $\text{Hom}(R_2, R_2) \simeq R_2$, hence the triple Toda power can be equivalently expressed as a map $S_1,3 \to R_2$.

**Lemma 7.25** The map $S^{1,3} \to R_2$ constructed above induces on homology the map $\text{Free}_{2,1}(r_3) \to \Lambda^{(2)}$ defined in Theorem B.2.

**Proof.** Upon passing to homology degree-wise, using that $H(R_2) \simeq \Lambda^{(2)}$ in the notation of Theorem B.2, and that, by Remark B.15, we can use $\tilde{P}_2 := \mathbb{Z}(e_1, e_2)/(\partial e_2 = e_1^2) \simeq \Lambda^{(2)}$ as a representative for the homology in the 1-category of dg-modules, the defining diagram can be represented by the strictly commuting diagram of dg-modules

(\text{where the negative signs are due to the direction of the homotopy, going from 0 to } e_2). Thus, the diagonal map is given by the action of $e_1 e_2 - e_2 e_1$, which by Remark B.15 represents exactly the homology class pointed by $r_3$ in the defining diagram for $\Lambda^{(3)}$. $\square$
Construction 7.26 Let us now consider the pushout square

\[
\begin{array}{ccc}
\text{Free}_{\mathbb{Z}_1}(S^{1,3}) & \rightarrow & S^{0,0} \\
\downarrow & & \downarrow \\
R_2 & \rightarrow & R_3
\end{array}
\]

where the vertical map is the one induced by the 3-fold Toda power constructed above. We pick the bottom horizontal map to be the second map in the sequence (13).

By the commutativity of the following diagram of $R_3$-modules (where we know that $\langle t \rangle^3 \simeq 0$ by the defining pushout for $R_3$)

\[
\begin{array}{ccc}
R_3 & \rightarrow & \Omega^{0,1}F_3 \\
\downarrow & & \downarrow \\
\Omega^{0,1}R_3 & \rightarrow & \Omega^{0,2}F_3
\end{array}
\]

we see that $t$ factors through $L_3 := \text{fib}(\Omega^{0,1}R_3 \rightarrow \Omega^{0,2}F_3)$. If we denote by $G_3 := \text{fib}(\Omega^{0,1}F_3 \rightarrow \Omega^{1,3}R_3)$, we have the following diagram (where all squares are Cartesian)

\[
\begin{array}{ccc}
R_3 & \rightarrow & L_3 \\
\downarrow & & \downarrow \\
\Omega^{2,4}R_3 & \rightarrow & \Omega^{0,1}G_3 \\
\downarrow & & \downarrow \\
\text{pt} & \rightarrow & \Omega^{0,2}F_3 \\
\downarrow & & \downarrow \\
\text{pt} & \rightarrow & \Omega^{1,4}R_3
\end{array}
\]

and we define the composite

\[R_3 \rightarrow L_3 \rightarrow \Omega^{2,4}R_3\]

to be the 4-fold Toda power of $t$, denoted $\langle t \rangle^4$. As for the 3-fold case, the datum of a 4-fold Toda bracket uniquely determines a map $S^{2,4} \rightarrow R_3$ that we will refer to by the same name.

Lemma 7.27 The map $S^{2,4} \rightarrow R_3$ constructed above induces on homology the map $\text{Free}_{\mathbb{Z}_1}(r_4) \rightarrow \Lambda^{(3)}$ defined in Theorem B.2.
Proof. As in Lemma 7.25, we can consider the diagram of dg-modules obtained from the defining diagram of the 4-fold Toda power by passing to homology:

\[
\begin{array}{c}
\Sigma^0 \bigoplus \Sigma^{-1} \tilde{P}_3 & \xrightarrow{\begin{pmatrix} e_1 \\ -e_2 \\ e_3 \end{pmatrix}} & \Sigma^0 \bigoplus \Sigma^{-1} \tilde{P}_3 \\
\Sigma^0 \bigoplus \Sigma^{-1} \tilde{P}_3 & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}} & \Sigma^0 \bigoplus \Sigma^{-1} \tilde{P}_3 \\
\Sigma^0 \bigoplus \Sigma^{-1} \tilde{P}_3 & \xrightarrow{\begin{pmatrix} e_2 \\ e_1 \end{pmatrix}} & 0 \\
\end{array}
\]

(again, the signs are consistent with the choice of having the nullhomotopies starting at 0) and we see that the 4-fold Toda power induces on homology the map given by the action of

\[e_1 e_3 - e_2^2 + e_3 e_1\]

which by Remark B.15 represents exactly the homology class pointed by \(\tau_4\) in the defining diagram for \(\Lambda^{(4)}\).

□

8. Total homology and k-decompositions

The Postnikov tower construction makes clear how a spectrum is uniquely determined by its homotopy groups together with its k-invariants. In particular, any spectrum \(X\) uniquely determines co/fiber sequences of the form

\[\tau_{[n,n+1]} X \rightarrow \pi_n X[n] \xrightarrow{k_n} \pi_{n+1} X[n+2]\]

for each \(n \in \mathbb{Z}\), and it’s easy to verify that the \(k_n\)’s are such that

\[k_n \circ k_{n-1} [-1] \simeq 0\]

for all \(n \in \mathbb{Z}\). It is widely known among the experts that the above observation admits a converse precisely when the nullhomotopies for such pairwise compositions of “truncated k-invariants” are suitably compatible (see e.g. [Sag08, Section 4], where an instance of this fact is discussed for objects with finite Postnikov filtrations in the setting of triangulated categories); to be precise, a spectrum is uniquely determined by the datum of its homotopy groups together with a collection of maps \(\{k_n: \pi_n X[n] \rightarrow \pi_{n+1} X[n+2]\}_{n \in \mathbb{Z}}\) such that all pairs of composable maps compose to a nullhomotopic one, and all the possible higher Toda brackets are trivial.

In this section we provide a rigorous formulation of the idea discussed above using the language of coherent cochain complexes developed in the previous sections, generalizing it to the context of stable \(\infty\)-categories satisfying some mild hypothesis\(^{15}\) and general homotopy objects in (ordinary) Abelian categories. The process of reconstructing an object from its decomposition in homotopy objects and maps between their shifts will be given by the total homology construction (see 8.8), a concept further

\(^{15}\)that is, admitting sequential limits, sequential colimits and equipped with a right complete t-structure
investigated in the next section, in relation to the spectral sequence generated by a coherent cochain complex.

Let us begin by constructing an \( \infty \)-category that encapsulates the data needed to reconstruct the objects from their homotopy groups and their \( k \)-invariants.

**Definition 8.1** Let \( \mathcal{C} \) be a stable \( \infty \)-category equipped with a t-structure. We will denote by \( \text{Ch}_+^+ (\mathcal{C}^\vee) \) the full subcategory of \( \text{Ch}^+ (\mathcal{C}) \) spanned by those objects \( C \) such that \( C_n \in \mathcal{C}^\vee [2n] \) for all \( n \in \mathbb{Z} \), and refer to it as the \( \infty \)-category of degenerate coherent cochain complexes of \( \mathcal{C} \).

**Remark 8.2** The notation \( \text{Ch}_+^+ (\mathcal{C}^\vee) \) is slightly abusive, as the object depends also on the t-structure \( \mathcal{C} \) is equipped with.

We can now construct a functor that incarnates the idea discussed in the introduction of this section.

**Remark 8.3** Let \( \mathcal{C} \) be a stable \( \infty \)-category equipped with a right complete t-structure. By the definition of right completeness, there exists an equivalence
\[
\mathcal{C} \simeq \lim \left( \cdots \to (\mathcal{C})_n \overset{\tau_{\geq n+1}}{\longrightarrow} (\mathcal{C})_{n+1} \to \cdots \right).
\]

By the right completeness of \( \text{Fil}^i \mathcal{C} \) (see Remark 5.2, together with Theorem 4.7), the fully faithful inclusion \( \mathcal{C} \hookrightarrow \text{Fil}^i (\mathcal{C}) \) factors through the subcategory of complete objects:
\[
\tau_{\geq i} : \mathcal{C} \hookrightarrow \text{Fil}^i (\mathcal{C}).
\]

**Definition 8.4** Let \( \mathcal{C} \) be a stable presentable \( \infty \)-category equipped with a right complete t-structure. Notice that, by Remark 8.3, the composite \( \mathcal{A} \circ \tau_{\geq i} : \mathcal{C} \to \text{Ch}^+ (\mathcal{C}) \) factors through the inclusion \( \text{Ch}_+^+ (\mathcal{C}^\vee) \subset \text{Ch}^+ (\mathcal{C}) \). We define the \( k \)-decomposition functor
\[
\mathcal{K} : \mathcal{C} \to \text{Ch}_+^+ (\mathcal{C}^\vee)
\]
\[
C \mapsto (\cdots \to \pi_{n-1} X [2n-2] \to \pi_n X [2n] \to \pi_{n+1} [2n+2] \to \cdots)
\]
to be such factorization of \( \mathcal{A} \circ \tau_{\geq i} \) through \( \text{Ch}_+^+ (\mathcal{C}^\vee) \). We refer to the differentials of the complex \( \mathcal{K} C \) as the \( k \)-invariants of \( C \).

**Example 8.5** Let \( R \) be an ordinary commutative ring, and let \( D(R) \) denote its derived category (considered as a stable \( \infty \)-category equipped with the usual t-structure). Given any object \( X \in D(R) \), its \( k \)-decomposition \( \mathcal{K} X \) is given by the coherent cochain complex
\[
\cdots \to (H^{n+1} X) [2n-2] \to (H^{n} X) [2n] \to (H^{n-1} X) [2n+2] \to \cdots
\]
with \( H^{n-1} X \in \text{Ab} \).

**Example 8.6** Given any \( X \in \text{Sp} \), its \( k \)-decomposition \( \mathcal{K} X \) is given by the coherent cochain complex
\[
\cdots \to (\pi_{n-1} X) [2n-2] \to (\pi_n X) [2n] \to (\pi_{n+1} X) [2n+2] \to \cdots
\]
with \( \pi_n X \in \text{Ab} \). The equivalence \( \text{Ch}_+^+ (\text{Sp}) \simeq \text{Sp} \) can be interpreted as saying that any spectrum can be expressed as a coherent chain complex of (suitably shifted) Abelian groups, and, viceversa, one can construct a spectrum from the datum of objects in \( \bigoplus_n \text{Ab} [2n] \) and maps between consecutive objects \( A_n [2n] \to A_{n+1} [2n+2] \) such that all possible Toda brackets between them are trivial.

**Remark 8.7** The previous two examples make clear how, from the point of view of coherent chain complexes, the only difference between an object in \( D(\mathbb{Z}) \) and a spectrum lies in the \( \mathbb{Z} \)-linearity of the \( k \)-invariants:
\[
D(\mathbb{Z}) \simeq \text{Ch}_+^+ (\mathbb{Z}) \subset \text{Ch}_+^+ (\text{Sp}) \simeq \text{Sp}.
\]

Of course, this is just a consequence of a theorem of Shipley’s (see [Shi07]) showing that the derived \( \infty \)-category \( D(R) \) of a discrete commutative ring \( R \) is equivalent to the \( \infty \)-category of \( HR \)-modules \( \text{Mod}_{HR} \).
We now turn our attention to the functor inverse to $\mathcal{K}$. It turns out that the process reconstructing an object of $\mathcal{C}$ from its $k$-decomposition is a particular case of the more general construction assigning to every coherent cochain complex $C$ the object underlying its piled-up filtration $IC$.

**Definition 8.8** Let $\mathcal{C}$ be a stable $\infty$-category with sequential limits and sequential colimits. We define the total homology functor to be the composite functor $\mathcal{H} := \text{colim} \circ I$

$$\mathcal{H} : \text{Ch}^+ \mathcal{C} \to \mathcal{C}$$

$$C \mapsto (IC)^{-\infty}.$$  

**Theorem 8.9** Let $\mathcal{C}$ be a stable $\infty$-category with sequential limits and sequential colimits, equipped with a right complete t-structure. The restriction of $\mathcal{H}$ to $\text{Ch}_k^+ (\mathcal{C}^\circ)$ induces an equivalence

$$\text{Ch}_k^+ (\mathcal{C}^\circ) \xrightarrow{\sim} \mathcal{C}$$

whose inverse is given by the $k$-decomposition functor of Definition 8.4.

**Proof.** By the definition of right completeness, there exists an equivalence

$$\mathcal{C} \simeq \lim \left( \cdots \to (\mathcal{C})_{\geq n} \xrightarrow{\tau_{\geq n+1}} (\mathcal{C})_{\geq n+1} \to \cdots \right).$$

By the discussion in [Lur17] right before Proposition 1.2.1.17, we can give an alternative description of the right hand side as the full subcategory of $\text{Fil}^k (\mathcal{C})$ spanned by those filtered objects for which

1. For each $n \in \mathbb{Z}$, $F^n \in \mathcal{C}_{\geq n}$;
2. For each $m \geq n$, the associated map $F^m \to F^n$ induces an equivalence $F^n \xrightarrow{\sim} \tau_{\geq m} F^n$.

We first notice that, as long as (1) holds, we can replace (2) with the weaker assumption that

(♠) For each $n \in \mathbb{Z}$, the associated map $F^{n+1} \to F^n$ induces an equivalence $F^{n+1} \xrightarrow{\sim} \tau_{\geq n+1} F^n$.

By induction, assume that for a fixed $n$ and some $m \geq n F^m \to F^n$ induces an equivalence $F^m \to \tau_{\geq m} F^n$. We want to prove that $F^{m+1} \to F^n$ induces an equivalence $F^{m+1} \to \tau_{\geq m+1} F^n$. By (♠), $F^{m+1} \to F^n$ induces an equivalence $F^{m+1} \to \tau_{\geq m+1} F^n$, and by our inductive hypothesis

$$\tau_{\geq m+1} F^n \to \tau_{\geq m+1} \circ \tau_{\geq m} F^n \simeq \tau_{\geq m+1} F^n$$

is an equivalence, hence the composite map $F^{m+1} \to \tau_{\geq m+1} F^n \to \tau_{\geq m+1} F^n$ (which, by (1), is precisely the image under $\tau_{\geq m+1}$ of the composite $F^{m+1} \to F^n \to F^n$) is an equivalence as desired.

As $\tau_{\geq k}$ is fully faithful, it induces an equivalence with its essential image in $\text{Fil}^k (\mathcal{C})$ (see Remark 8.3). As $\mathcal{A}$ is an equivalence, it is sufficient to check that $I \left( \text{Ch}_k^+ (\mathcal{C}^\circ) \right) \simeq \tau_{\geq k} \mathcal{C}$. By the definition of $\text{Ch}_k^+ (\mathcal{C}^\circ)$ and Lemma 3.8, all objects in $I \left( \text{Ch}_k^+ (\mathcal{C}^\circ) \right)$ satisfy (1) and (2). On the other hand, again by Lemma 3.8, any filtered object satisfying conditions (1) and (2) is such that its shelled complex lies in $\text{Ch}_k^+ (\mathcal{C}^\circ)$. Hence, $I \left( \text{Ch}_k^+ (\mathcal{C}^\circ) \right)$ and $\tau_{\geq k} \mathcal{C}$ are two full subcategories of $\text{Fil}^k (\mathcal{C})$ spanned by the same objects. To identify its inverse, it now suffices to notice that

$$\mathcal{H} \circ \mathcal{K} \simeq \text{colim} \circ I \circ \mathcal{A} \circ \tau_{\geq k}$$

$$\simeq \text{colim} \circ \text{id} \circ \tau_{\geq k}$$

$$\simeq \text{id}$$

as desired. □

We now turn our attention to the behavior of $\mathcal{H}$ on objects coming from the (ordinary) category of chain complexes in the heart.
Proposition 8.10  Let $\mathcal{C}$ be as in Theorem 8.9. Then the t-structure homotopy objects of the total homology of an ordinary chain complex in $C_\circ \in \text{Ch}^*(\mathcal{C}^\vee)$ (considered as an element of $(\text{Ch}^* \mathcal{C})^\vee$) are given by the cohomology of $C_\circ$:

$$\pi_n H C_\circ \simeq H^n C_\circ.$$ 

Proof. By Proposition 3.16, together with the hypothesis that $C_\circ$ lies in $\text{Ch}^*(\mathcal{C}^\vee)$, we have that $\text{gr}^n I C_\circ \simeq C_\circ[-n]$; in particular, (as $C_\circ$ is complete) we see that for any integer $n$, $I C_\circ$ is $(-n)$-coconnective, and that

$$\pi_m (I C_\circ^n) \cong \pi_m (I C_\circ^{n-1}) \quad \text{for all } m \leq n. \quad (15)$$

By Remark 3.24, we have a co/fiber sequence

$$C_\circ^{n+1}[-n-1] \to I C_\circ^n/I C_\circ^{n+2} \to C_\circ^n[-n]$$

whose associated long exact sequence on homotopy objects lets us identify

$$\pi_m\left(I C_\circ^n/I C_\circ^{n+2}\right) \cong \begin{cases} 
\ker(d^n_{C_\circ}) & \text{for } m = -n; \\
\coker(d^n_{C_\circ}) & \text{for } m = -n - 1; \\
0 & \text{else.}
\end{cases}$$

Again by Remark 3.24, we have a co/fiber sequence

$$I C_\circ^n/I C_\circ^{n+2} \to I C_\circ^{n-1}/I C_\circ^{n+2} \to C_\circ^{n-1}[-n+1]$$

whose associated long exact sequence starting at $-n + 1$ looks as follows

$$\cdots 0 \to \pi_{-n+1}(I C_\circ^{n-1}/I C_\circ^{n+2}) \to C_\circ^{n-1} \to \ker(d^n) \to \pi_{-n}(I C_\circ^{n-1}/I C_\circ^{n+2}) \to 0 \cdots$$

letting us identify $\pi_{-n}(I C_\circ^{n-1}/I C_\circ^{n+2})$ as

$$\coker\left(C_\circ^{n-1} \xrightarrow{d_{C_\circ}^{n-1}} \ker(d^n)\right) \cong H^n C_\circ.$$ 

As $I C_\circ^{n+2}$ is $(-n-2)$-coconnective, the long exact sequence associated to the co/fiber sequence

$$I C_\circ^{n+2} \to I C_\circ^{n-1} \to I C_\circ^{n-1}/I C_\circ^{n+2}$$

shows that

$$\pi_m(I C_\circ^{n+2}) \cong (\pi_m I C_\circ^{n-1}) \quad \text{for } m \geq -n - 1$$

and thus in particular that $\pi_{-n}(I C_\circ^{n-1}) \cong H^n C_\circ$. Finally, (15) implies the desired result. \hfill \square

Remark 8.11  In the special case of $\mathcal{C} = \text{Sp}$, the proof of Proposition 8.10 shows that $I C_\circ$ for an ordinary chain complex of Abelian groups $C_\circ$ gives precisely the tower obtained through the “brutal truncations”\footnote{sometimes referred to also as the “stupid truncation” in the literature} of the complex $C_\circ$. That is,

$$I C_\circ^n \simeq H(\tau^{\geq n} C_\circ),$$

where $\tau^{\geq n}$ here denotes the brutal truncation $(1\text{-})$functor and $H$ denotes the Eilenberg–MacLane $\infty$-(co)functor $\text{Ch}^* (\text{Ab}) \to \text{Sp}$.

In particular, we have that in the case of spectra the total homology funtor $H$ restricted to $\text{Ch}^* (\text{Ab}) \subset \text{Ch}^* (\text{Sp})$ coincides with the Eilenberg–MacLane functor.

We conclude the section with the following variant of Definition 8.4.

Variant 8.12  Let $\mathcal{C}$ be a stable presentable $\infty$-category equipped with a right separated t-structure, and let $q \in \mathbb{N}$ be fixed. Let $\text{Ch}^{\leq q}(\mathcal{C}^\vee)$ denote the full subcategory of $\text{Ch}^* \mathcal{C}$ spanned by those objects $C$ such that $C^n \in \mathcal{C}^\vee[n(q+1)]$ for all $n \in \mathbb{Z}$. Consider the full subcategory $\mathcal{C}_{q\text{-periodic}} \subset \mathcal{C}$ spanned by the objects $X$ such that the t-structure homotopy objects $\pi_n X$ are isomorphic to 0 for $n$ not a multiple of $q$:

$$\mathcal{C}_{q\text{-periodic}} := \{ X \mid \pi_n X \cong 0 \text{ for } n \not\equiv 0 \mod q \}.$$ 

We define $\mathcal{K}_q$ to be the factorization of $\mathcal{A} \circ \tau\tau_{q\bullet}$ (where $\tau\tau_{q\bullet}$ denotes the sub-filtration of $\tau_{q\bullet}$ obtained by skipping all the stages of the Whitehead tower that are not multiples
of $q$) through the inclusion $\text{Ch}^*_q(\mathbb{C}^\vee) \subset \text{Ch}^*(\mathbb{C})$, and refer to it as the $q$-periodic $k$-decomposition functor. Similarly to what happens for $\mathcal{K}$, the functor $\mathcal{K}_q$ induces an equivalence $\mathcal{E}_{q,\text{periodic}} \simeq \text{Ch}^*_q(\mathbb{C}^\vee)$.

We learned about the following example from Achim Krause.

**Example 8.13** As an instance of Variant 8.12, we can consider the $2(p^n - 1)$-periodic $k$-decomposition of the $n$-th Morava K-theory ($n \geq 1$) spectrum, for some fixed prime $p$. As

$$\mathbb{F}_p[2m(p^n - 1) + m] \simeq \mathbb{F}_p[m(p^n - 1)]$$

the coherent cochain complex $\mathcal{K}_{2(p^n - 1)}K(n)$ looks as follows

$$\cdots \to \mathbb{F}_p[(m - 1)(p^n - 1)] \to \mathbb{F}_p[m(p^n - 1)] \to \mathbb{F}_p[(m + 1)(p^n - 1)] \to \cdots$$

(where $\mathbb{F}_p[m(p^n - 1)]$ sits in degree $m$), and the differentials are given by suitable shifts of the $n$-th Milnor primitive for the mod $p$ Steenrod algebra:

$$\partial^m \simeq Q_n[m] \in \mathcal{A}_p.$$

9. **The spectral sequence associated to a coherent cochain complex**

In this section, we discuss how coherent cochain complexes give rise to spectral sequences. We have the following result.

**Theorem 9.1** Let $\mathbb{C}$ be a stable $\infty$-category with sequential limits and sequential colimits, equipped with a right complete $t$-structure. Then, every coherent cochain complex $C \in \text{Ch}^* \mathbb{C}$ generates a spectral sequence

$$E_1^{i,j} \cong \pi_{-j}C^i$$

whose $E_1$ page is given by the homotopy groups of the components of $C$ and whose $E_1$ differentials

$$d_1^{i,j} = \pi_{-j}\partial_C^i$$

are obtained from the coherent differentials of $C$ by passing to homotopy. When the spectral sequence collapses at a finite stage, it converges strongly to the homotopy groups of the total homology of $C$

$$E_\infty^{i,j} \cong \pi_{-j}C^i \Rightarrow \pi_{-i-j}HC.$$

We defer the proof to later in this section. Of course, the above theorem follows at once from Theorem 4.7 together with Theorem 5.11 and the existence of the spectral sequence associated to a filtered object in a suitable stable $\infty$-category (whose incarnation in the case of spectra has been known by the experts for a long time and an account of which can be found in [Lur17, 1.2.2], in the generality of this paper). The goal of this section is to give a self-contained construction of the above spectral sequence.

**Remark 9.2** The convergence statement in Theorem 9.1 is far from being optimal; it is somewhat of a folk result that the corresponding spectral sequence for a filtered object is conditionally convergent. In forthcoming work by Hedenlund–Krause–Nikolaus, the authors give a new proof of this fact, using the filtered counterpart of the décalage construction we illustrate in this section using the coherent cochain complexes perspective.

We learned about the relation between the Beilinson $t$-structure and Deligne’s décalage functor from Benjamin Antieau. Most of the ideas discussed in this section are already present in some form in [Ant19] and [BMS19, 5.5]; we believe that the language of coherent cochain complexes gives a particularly pleasant perspective on the topic.

The key ingredient in the construction of the spectral sequence of a coherent cochain complex is given by the following construction.
Definition 9.3 Let $\mathcal{C}$ be a stable $\infty$-category with sequential limits and sequential colimits, equipped with a right separated $t$-structure, and let $C \in \text{Ch}^* \mathcal{C}$ be a coherent cochain complex. We can apply the functor $K$ levelwise, to obtain a bicomplex (denoted $K_{\text{lvl}} C$)

\[
\cdots \xrightarrow{} K C^0 \xrightarrow{} K C^1 \xrightarrow{} K C^2 \xrightarrow{} \cdots
\]

\[
\cdots \xrightarrow{} \pi_0 C^0[0] \xrightarrow{} \pi_0 C^1[0] \xrightarrow{} \pi_0 C^2[0] \xrightarrow{} \cdots
\]

\[
\cdots \xrightarrow{} \pi_1 C^0[2] \xrightarrow{} \pi_1 C^1[2] \xrightarrow{} \pi_1 C^2[2] \xrightarrow{} \cdots
\]

\[
\cdots \xrightarrow{} \pi_2 C^0[4] \xrightarrow{} \pi_2 C^1[4] \xrightarrow{} \pi_2 C^2[4] \xrightarrow{} \cdots
\]

i.e. such that $(K_{\text{lvl}} C)^{i,j} \simeq \pi_j C^i[2j]$, where the horizontal differentials are the $k$-invariants for the relevant objects, and the vertical ones are induced from the differentials of $C$. We define the décalé complex\footnote{Our choice of terminology is justified by the construction of the spectral sequence of Theorem 9.1 (see the proof at the end of the section): the functor Déc provides an incarnation in coherent chain complexes of Deligne’s décalé filtration (see [Del71, 1.3])} denoted $\text{Déc} C$ to be the levelwise total homology of the bicomplex $K_{\text{lvl}} C$ with respect to the horizontal maps of the above diagram. That is,

\[
\text{Déc} C^0 = \mathcal{H}\left( \cdots \xrightarrow{} \pi_0 C^0[0] \xrightarrow{} \pi_0 C^1[0] \xrightarrow{} \pi_0 C^2[0] \xrightarrow{} \cdots \right)
\]

\[
\text{Déc} C^1 = \mathcal{H}\left( \cdots \xrightarrow{} \pi_1 C^0[2] \xrightarrow{} \pi_1 C^1[2] \xrightarrow{} \pi_1 C^2[2] \xrightarrow{} \cdots \right)
\]

\[
\text{Déc} C^2 = \mathcal{H}\left( \cdots \xrightarrow{} \pi_2 C^0[4] \xrightarrow{} \pi_2 C^1[4] \xrightarrow{} \pi_2 C^2[4] \xrightarrow{} \cdots \right)
\]

or, more precisely:

$$\text{Déc} := \text{Ch}^* \mathcal{H} \circ K_{\text{lvl}}.$$ 

As $\text{Déc}$ gives an endofunctor for $\text{Ch}^* \mathcal{C}$, we will denote by

$$\text{Déc}^n : \text{Ch}^* \mathcal{C} \to \text{Ch}^* \mathcal{C}, \text{ with } n \geq 1$$

its iterations. We will also use the convention $\text{Déc}^0 := \text{id}.$
Remark 9.4 In the situation of Definition 9.3, using Remark 5.15 together with Remark 8.11 we see that (omitting as usual the Eilenberg–Maclane functor from the notations)

\[(\text{Dec} C)^n \simeq \mathcal{H} \left( (\pi^\text{ lvl} n C) [2n] \right) \]

\[\simeq \colim I \left( (\pi^\text{ lvl} n C) [2n] \right) \]

\[\simeq (\mathcal{H} \left( (\pi^\text{ lvl} n C) \right)) [2n] \]

\[\simeq (\pi^\text{ lvl} n C) [2n] \]

\[\simeq (\pi^B n C) [2n] \]

that is, Dec \(C^n\) is equivalent (up to a shift) to the \(n\)-th Beilinson homotopy chain complex of the piled up filtered object of \(C\).

Remark 9.5 In the situation of Definition 9.3, we have that \(\mathcal{H} C \simeq \mathcal{H} \text{Dec} C\) for any \(C \in \text{Ch}^* \mathcal{G}\), and thus

\(\mathcal{H} C \simeq \mathcal{H} \text{Dec} n C\)

for any \(n \in \mathbb{N}\). To see this, let us consider the bifiltered object

\[I^2(\mathcal{K}^\text{ lvl} C) := \text{Fil}^i(\mathcal{I}) \circ I \left( (\mathcal{K}^\text{ lvl} C) \right) \simeq I \circ \text{Ch}^* (\mathcal{I}) \left( (\mathcal{K}^\text{ lvl} C) \right)\]

associated to the levelwise \(k\)-decomposition of \(C\) (where we are using the fact that the diagram

\[
\begin{array}{ccc}
\text{Ch}^* (\text{Ch}^* \mathcal{G}) & \xrightarrow{\text{Ch}^* (\mathcal{I})} & \text{Ch}^* (\text{Fil}^i \mathcal{G}) \\
\downarrow & & \downarrow I \\
\text{Fil}^i (\text{Ch}^* \mathcal{G}) & \xrightarrow{\text{Fil}^i (\mathcal{I})} & \text{Fil}^i (\text{Fil}^i \mathcal{G})
\end{array}
\]

commutes). It follows from Theorem 4.7 that given any \(F \in \text{Fil}^i (\mathcal{G})\), we have an equivalence of bifiltered objects

\[\tau^B_{\geq m} F \simeq I \mathcal{A} \tau^B_{\leq m} F \simeq I \left( \pi^B n F [2*] \right) \in \text{Fil}^i \left( \text{Fil}^i \mathcal{G} \right).\]

In particular, for \(F \simeq IC\), we have

\[\tau^B_{\geq m} IC \simeq I \left( \pi^B n IC [2*] \right) \simeq I \left( I \pi^\text{ lvl} n C [2*] \right) \simeq I^2 \left( (\mathcal{K}^\text{ lvl} C) \right).\]

Now, we can use the above observation to deduce our initial claim:

\[\mathcal{H} C \simeq \colim IC^n \]

\[\simeq \colim_m (\tau^B_{\geq m} IC)^n \]

\[\simeq \colim_m I^2 (\mathcal{K}^\text{ lvl} C)^{m,n} \]

\[\simeq \colim_m I \left( \colim_n (\pi^\text{ lvl} n C [2*])^n \right)^m \]

\[\simeq \colim_m I \left( \text{Dec} C \right)^m \]

\[\simeq \mathcal{H} \text{Dec} C.\]

Proof of Theorem 9.1. Given any \(n \in \mathbb{Z}\), we have that \(\pi_n C^* \in \text{Ch}^* (\mathcal{G}^\mathcal{O})\) is an ordinary chain complex. Let

\[\pi_{-j} (\text{Dec}^n C) := E_{n+1}^{(n+1) i + nj, -n+i \pm(n+1)j}\]

we want to prove that the above objects define the pages of a spectral sequence. We start by defining the differentials of the \(E_1\) page to be

\[d^1_{i,j} := \pi_{-j} (\partial^0_C)\]
Using Proposition 8.10, we have
\[
\pi_{-j}(\mathrm{D}\acute{e}c C)^i \cong \pi_{-j} \mathcal{H} \left( \pi_1 \mathcal{C} \right)[2i] \\
\cong \pi_{-j-2i} \mathcal{H} \left( \pi_1 \mathcal{C} \right) \\
\cong H^{j+2i} \left( \pi_1 \mathcal{C} \right);
\]
the above isomorphism allows us to define
\[
\partial_{2i}^{2i+j,-i} := \pi_{-j}(\partial_{\mathrm{D}\acute{e}c C})
\]
and it’s easy to check that \(d_2\) has the correct bidegree. As
\[
\pi_{-j}(\mathrm{D}\acute{e}c^n \mathcal{C})^i \cong H^{j+2i} \left( \pi_1 \mathrm{D}\acute{e}c^{n-1} \mathcal{C} \right)
\]
we can now proceed by induction to show that the \(E_{n+1}\) page is given by the cohomology of the \(E_n\) page: using that
\[
\pi_i(\mathrm{D}\acute{e}c^{n-1} \mathcal{C})^{j+2i} := E_n^{(n+1)i+nj,-ni+(-n+1)j} \\
\cong E_n^{(n+1)i+nj,-ni+(-n+1)j}
\]
we see that
\[
E_{n+1}^{(n+1)i+nj,-ni+(-n+1)j} := \pi_{-j}(\mathrm{D}\acute{e}c^n \mathcal{C})^i \\
\cong H^{j+2i} \left( \pi_1 \mathrm{D}\acute{e}c^{n-1} \mathcal{C} \right) \\
\cong \ker \left( \pi_i(\mathrm{D}\acute{e}c^{n-1} \mathcal{C})^{j+2i} \to \pi_i(\mathrm{D}\acute{e}c^{n-1} \mathcal{C})^{j+2i+1} \right) \\
\cong \ker \left( E_n^{(n+1)i+nj,-ni+(-n+1)j} \to E_n^{(n+1)i+nj+1,-ni+(-n+1)(j+1)} \right) \\
\cong H^{(n+1)i+nj,-ni+(-n+1)j} E_n
\]
concluding the existence proof.

The convergence statement follows at once from Remark 9.5. \(\square\)

10. Examples

In what follows, we present some results which appeared recently in the literature using the language developed in the rest of the paper. The aim of this section is to give a new perspective on known results, hence we make no claims of originality regarding the results presented; still, we believe that looking at these results through the lenses of coherent cochain complexes buys some insight about these results and adds to the clarity of their statement. In fact, some of the examples discussed here motivated the investigation of the formalism of coherent chain complexes in the first place.

Example 10.1 [Ant19] Let \(k\) be a commutative ring, and let \(\mathbf{Sch}_k^\mathrm{op}\) denote the category of qcqs \(k\)-schemes whose cotangent complex \(L_{X/k}\) has Tor-amplitude concentrated in \([0, 1]\). Then

1. there exists a functor \(\mathbb{H}C^{-}\)\((-/k)\): \(\mathbf{Sch}_k^\mathrm{op} \to \mathrm{Ch}^*(\mathrm{D}(k))\) sending each scheme \(X\) to a coherent cochain complex \(\mathbb{H}C^{-}(X/k)\) such that
   (a) the total homology of \(\mathbb{H}C^{-}(X/k)\) is given by negative cyclic homology:
   \[
   \mathcal{H}(\mathbb{H}C^{-}(X/k)) \simeq HC^{-}(X/k)
   \]
   (b) the components of \(HC^{-}(X)\) are given up to a shift by truncations of the Hodge-completed derived de Rham complex:
   \[
   HC^{-}(X/k)^n \simeq \widehat{L}^nOmega^n_{X/k}[3n]
   \]
(c) the coherent cochain complex $\mathbb{H}C^{-}(X/k)$ generates a conditionally convergent spectral sequence

$$E_{1}^{i,j} \cong H^{i+j}_c\left(\hat{\Omega}_{X/k}^{\geq i}\right) \implies \pi_{-i-j}\mathbb{H}C^{-}(X/k)$$

(2) there exists a functor $\mathbb{H}P(-/k): Sch^\text{op}_k \rightarrow Ch^\ast(D(k))$ sending each scheme $X$ to a coherent cochain complex $\mathbb{H}P(X/k)$ such that

(a) the total homology of $\mathbb{H}P(X/k)$ is given by periodic homology:

$$\mathcal{H}(\mathbb{H}P(X/k)) \simeq \mathbb{H}P(X/k)$$

(b) the components of $\mathbb{H}P(X/k)$ are given by shifts of the Hodge-completed derived de Rham complex:

$$\mathbb{H}P(X/k)_n \simeq \hat{\Omega}_{X/k}[3n]$$

(c) the coherent cochain complex $\mathbb{H}P(X/k)$ generates a conditionally convergent spectral sequence

$$E_{1}^{i,j} \cong H^{i+j}_dR(X/k) \implies \pi_{-i-j}\mathbb{H}P(X/k).$$

**Remark 10.2** Let $k$ be an animated ring$^{18}$; then, the hull-shelled complex of the Hodge-completed derived de Rham complex $\hat{\Omega}_{X/k}$ with the Hodge filtration

$$L\Omega_{X/k} := \mathcal{A}\left(\hat{\Omega}_{X/k}^{\geq \ast}\right)$$

is such that its components are successive exterior powers of the cotangent complex:

$$\left(L\Omega_{X/k}\right)_n \simeq \Lambda^n L_{R/k};$$

thus, we can represent $L\Omega_{X/k}$ as

$$\cdots 0 \rightarrow R \rightarrow L_{R/k} \rightarrow \Lambda^2 L_{R/k} \rightarrow \Lambda^3 L_{R/k} \rightarrow \cdots$$

showing how it can be seen as a homotopy coherent generalization of the usual algebraic de Rham complex.

**Example 10.3** $^{[\text{Rak20}]}$ With notation as in $^{\text{10.2}}$, $L\Omega_{X/k}$ can be given the structure of a derived commutative algebra in coherent cochain complexes (i.e. a module over a suitably defined $\infty$-operad $L\text{Sym}$; see $^{[\text{Rak20}, \S 4]}$ for details); the resulting object is initial in the full subcategory of $LMod_{L\text{Sym}} Ch^\ast$ consisting of objects $C$ that are concentrated in non-negative degrees and are equipped with a map of $k$-algebras $R \rightarrow C^n$. Moreover, the first differential of the complex $\partial^0: R \rightarrow L_{R/k}$ is given by the universal derivation.

**Example 10.4** $^{[\text{BMS19}], [\text{AN20}]}$ Let $k$ be a perfect field of characteristic $p$, and let $\text{Alg}^\text{ind-sm}_k$ denote the category of ind-smooth $k$-algebras. Then, there exists a functor $\mathbb{T}P: \text{Alg}^\text{ind-sm}_k \rightarrow Ch^\ast(Sp)$ sending a $k$-algebra $R$ to a coherent cochain complex $\mathbb{T}P(R)$ such that:

1. the total homology of $\mathbb{T}P(R)$ is given by topological periodic homology:

$$\mathcal{H}(\mathbb{T}P(R)) \simeq \mathbb{T}P(R).$$

2. the components of $\mathbb{T}P(R)$ are given by the de Rham–Witt complex

$$\mathbb{T}P(R)_n \simeq \mathbb{W}Ω^\ast_{R}[3n];$$

in particular, when $R$ is smooth, the components are given by crystalline cohomology

$$\mathbb{T}P(R)_n \simeq R\Gamma_{\text{cris}}(R/W(k))[3n];$$

$^{18}$Following $^{[\text{CS19}]}$, an element of the $\infty$-category obtained by Dwyer–Kan localization of the 1-category of simplicial commutative rings at the subcategory of weak equivalences
(3) the coherent cochain complex $\mathbb{T}^p(R)$ generates a conditionally (strongly, if $\text{Spec } R$ has finite dimension over $\text{Spec } k$) convergent spectral sequence

$$E_{1}^{i,j} \simeq H_{\text{MW}}^{j+3i}(R) \Rightarrow \pi_{-i-j} \mathbb{T}^p(R);$$

in particular, for $R$ smooth, we get

$$E_{1}^{i,j} \simeq H_{\text{crys}}^{j+3i}(R/W(k)) \Rightarrow \pi_{-i-j} \mathbb{T}^p(R).$$

**Appendix A. $\infty$-categorical and stable nerve-realization paradigm**

The content of this appendix is well-known to experts, but to the best of our knowledge there is no systematic presentation of these results in the existing literature. Thus we thought it appropriate to include them here for the convenience of the reader.

We begin by showing a “density formula” for $\infty$-categorical presheaves, then proceed to describe the nerve-realization paradigm in the unstable and in the stable case. For the unstable case, we closely follow the presentation of the analogue results in ordinary category theory given in [Lor21, Chapter 3], adapting the few things that need to be adapted in order to translate the arguments to the $\infty$-categorical setting.

For the stable case, we leverage on results of [BGT13] and [Nik16] to give a description of stable nerves.

**A.1. A “density formula” for presheaves.** One of the features making the $\infty$-category of presheaves of a small $\infty$-category $\mathcal{C}$ so important, is the fact that the Yoneda embedding is in some sense the “free cocompletion functor” (see [Lur09, 5.1.5.6] for a precise statement and a proof). In particular, every presheaf is the colimit of representable ones (see [Lur09, 5.1.5.8]). The proof contained in [Lur09] of this last fact is somewhat indirect, as it does not explicitly give a way to construct for any presheaf $P$, a diagram having $P$ as a colimit. As later we will need such an explicit diagram, we will now show that, similarly to what happens in the ordinary case, any presheaf is the colimit of the inclusion of a suitable comma category in the category of presheaves.

The following is just a special case of the most general definition of a comma $\infty$-category it is possible to give, but it is general enough for our purposes.

**Definition A.1** Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories, and $d \in \mathcal{D}$ an object. We will denote by $(F \downarrow d)$ the comma category obtained as the following pullback

$$
\begin{array}{ccc}
(F \downarrow d) & \rightarrow & \mathcal{D}/d \\
\downarrow & & \downarrow \\
\mathcal{C} & \rightarrow & \mathcal{D}.
\end{array}
$$

**Lemma A.2** The map $(F \downarrow d) \to \mathcal{C}$ is a right fibration classifying $\text{Map}_{\mathcal{D}}(F(-), d)$.

*Proof.* As the projection $\mathcal{D}/d \to \mathcal{D}$ classifies $\mathcal{J}_d$, [Lur09, 3.2.1.4] implies that the pullback (16) classifies the composite $\mathcal{J}_d \circ F$. □

The above proposition, together with the Yoneda lemma, specialize to presheaves as follows.

**Remark A.3** By Lemma A.2, the right fibration $(\mathcal{J} \downarrow P) \to \mathcal{C}$ classifies the functor $\text{Map}_{\mathcal{P}(\mathcal{C})}(\mathcal{J} \downarrow P)$, but, by [Lur09, 5.5.2.1], this is precisely $P$, hence $(\mathcal{J} \downarrow P) \to \mathcal{C}$ is a right fibration classifying $P$.

**Proposition A.4** Let $\mathcal{C}$ be a small $\infty$-category, and $P$ any presheaf on $\mathcal{C}$. Then

$$P \simeq \text{colim} \left( (\mathcal{J} \downarrow P) \xrightarrow{\mathcal{J}} \mathcal{C} \xrightarrow{\eta} \mathcal{P}(\mathcal{C}) \right).$$
Proof. We will prove that $P$ and $X$ corepresent the same functor in $\mathcal{P}(\mathcal{E})$. Given any presheaf $Y$, have that

$$\text{Map}_{\mathcal{P}(\mathcal{E})}(\text{colim}_{\alpha \in (\downarrow)} \cdot \circ \pi(\alpha), Y) \simeq \lim_{\alpha \in (\downarrow)} \text{Map}_{\mathcal{P}(\mathcal{E})}(\cdot \circ \pi(\alpha), Y) \quad (17)$$

Let us now notice that, by Lemma A.2, the functor

$$\text{Map}_{\mathcal{P}(\mathcal{E})}(\cdot \circ \pi(\cdot), Y): (\downarrow) \to S$$

is classified by the right fibration

$$(\cdot \circ \pi \downarrow Y) \to (\cdot \downarrow P)$$

hence, by [Lur09, 3.3.3.4], (17) is equivalent to the space of sections

$$\text{Map}_{/(\cdot \downarrow P)}((\cdot \downarrow P), (\cdot \circ \pi \downarrow Y)). \quad (18)$$

Let us note that, as $(\cdot \circ \pi \downarrow Y)$ fits in the following pasting of pullback squares

$$\begin{array}{ccc}
(\cdot \circ \pi \downarrow Y) & \to & (\cdot \downarrow Y) \\
\downarrow & & \downarrow \\
(\cdot \downarrow P) & \to & \mathcal{E}
\end{array} \quad \begin{array}{ccc}
\mathcal{E} & \to & \mathcal{P}(\mathcal{E})
\end{array}$$

we have that (18) is equivalent to

$$\text{Map}_{/\mathcal{E}}((\cdot \downarrow P), (\cdot \downarrow Y)),$$

and, by Remark A.3, and by virtue of the straightening/unstraightening equivalence, we have that

$$\text{Map}_{/\mathcal{E}}((\cdot \downarrow P), (\cdot \downarrow Y)) \simeq \text{Map}_{/\mathcal{E}}(\text{Un}(P), \text{Un}(Y))$$

$$\simeq \text{Map}_{\mathcal{P}(\mathcal{E})}(\text{St}(\text{Un}(P)), Y)$$

$$\simeq \text{Map}_{\mathcal{P}(\mathcal{E})}(P, Y),$$

completing the proof. □

A.2. The unstable nerve-realization paradigm. With the density formula at our disposal, we can now easily adapt the results on nerves and realizations from ordinary category theory to the coherent setting. In order to do so, it will be useful to express Kan extensions using co/ends, hence we will start by recollecting some definitions and proving a few useful lemmata.

**Definition A.5** [Lur17, 5.2.1.1], [GHN17, 2.2] Let $\mathcal{C}$ be an $\infty$-category, and let $\varepsilon: \Delta \to \Delta$ be the functor given by $[n] \mapsto [n] \star [n]^{\text{op}}$. The twisted arrow category $\text{tw}\mathcal{C}$ is the simplicial set given by $\varepsilon^*\mathcal{C}$.

**Remark A.6** In particular, we have

$$(\text{tw}\mathcal{C})_n \cong \text{Hom}(\Delta^n \star (\Delta^n)^{\text{op}}, \mathcal{C})$$

and two canonical projections $\text{tw}\mathcal{C} \to \mathcal{C}$ and $\text{tw}\mathcal{C} \to \mathcal{C}^{\text{op}}$ induced by the natural transformations $\Delta^* \to \Delta^* \star (\Delta^*)^{\text{op}}$ and $(\Delta^*)^{\text{op}} \to \Delta^* \star (\Delta^*)^{\text{op}}$. By [Lur17, 5.2.1.3], the induced map $\text{tw}\mathcal{C} \to \mathcal{C} \times \mathcal{C}^{\text{op}}$ is a right fibration, and by [Lur17, 5.2.1.11] it classifies the bifunctor $\text{Map}_{\mathcal{E}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$. Equivalently, the opposite map $(\text{tw}\mathcal{C})^{\text{op}} \to \mathcal{C}^{\text{op}} \times \mathcal{C}$ is the left fibration classifying the bifunctor $\text{Map}_{\mathcal{E}}$.

**Definition A.7** [GHN17, 2.5] If $F: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ is a functor of $\infty$-categories, the end (resp. coend) of $F$ is defined to be the limit (resp. colimit) of the composite functor

$$\text{tw}\mathcal{C} \to \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$$
and is denoted by

$$\int_{c \in \mathcal{C}} F \quad \text{(resp. } \int_{c \in \mathcal{C}} F).$$

**Lemma A.8** Both projection maps $\text{tw} \mathcal{C} \to \mathcal{C}^{\text{op}}$ and $\text{tw} \mathcal{C} \to \mathcal{C}$ are final and initial functors.  \(^{19}\)

**Proof.** We will prove that the projection $\text{tw} \mathcal{C} \to \mathcal{C}^{\text{op}}$ is both final and initial, the other case being entirely analogous. Let us start by proving finality. By [Lur09, 4.1.3.1], it is enough to check the fibers of $\text{tw} \mathcal{C} \to \mathcal{C}^{\text{op}}$ are weakly contractible (notice that, for any $C \in \mathcal{C}$, as $\mathcal{C}/C$ has an initial object, it is weakly contractible, and thus $\text{tw} \mathcal{C} \times_{\mathcal{C}} \mathcal{C}/C$ has the same homotopy type of $\text{tw} \mathcal{C} \times_{\mathcal{C}} \{C\}$). Given any $C \in \mathcal{C}$, let us denote the fiber $\text{tw} \mathcal{C} \times_{\mathcal{C}} \mathcal{C}/C \{C\}$ by $(\text{tw} \mathcal{C})_C$. We claim that each $(\text{tw} \mathcal{C})_C$ is equivalent to $\mathcal{C}/C$ as an $\infty$-category, and thus weakly contractible.

By pasting of pullbacks, we know that $(\text{tw} \mathcal{C})_C$ fits into a diagram

$$\begin{array}{ccc}
\Delta^0 \times \mathcal{C} & \overset{C \times \text{id}}{\longrightarrow} & \mathcal{C}^{\text{op}} \times \mathcal{C} \\
\downarrow & & \downarrow \\
\Delta^0 \times \mathcal{C} & \overset{\text{id}}{\longrightarrow} & \mathcal{C}^{\text{op}}
\end{array}$$

and, again by pasting of pullbacks, we know we have the following diagram

$$\begin{array}{ccc}
\text{Map}_\mathcal{C}(C, D) & \longrightarrow & (\text{tw} \mathcal{C})_C \longrightarrow \text{tw} \mathcal{C} \\
\downarrow & & \downarrow \\
\Delta^0 & \overset{D}{\longrightarrow} & \Delta^0 \times \mathcal{C} \overset{C \times \text{id}}{\longrightarrow} \mathcal{C}^{\text{op}} \times \mathcal{C}
\end{array}$$

where both squares are Cartesian. As $\text{tw} \mathcal{C} \to \mathcal{C}^{\text{op}} \times \mathcal{C}$ is a right fibration, hence $(\text{tw} \mathcal{C})_C \to \mathcal{C}$ is a right fibration as well. But, as we identified the fibers over any $D \in \mathcal{C}$ as $\text{Map}(C, D)$, this has to be the classifying fibration for $\text{Map}(C, -)$, and thus $(\text{tw} \mathcal{C})_C$ has to be equivalent to $\mathcal{C}/C$ as desired. Recall that, by definition, a functor $F: \mathcal{C} \to \mathcal{D}$ is initial if and only if its opposite functor $F^{\text{op}}: \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ is final. The initiality statement follows from an analogous argument applied to the opposite projection $(\text{tw} \mathcal{C})^{\text{op}} \to \mathcal{C}$, using that the map $(\text{tw} \mathcal{C})^{\text{op}} \to \mathcal{C}^{\text{op}} \times \mathcal{C}$ is a left fibration.

The above lemma has as an immediate consequence that a co/end with a dummy variable is just a co/limit.

**Corollary A.9** Let $F: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor that factors through the projection $\mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$ (resp. $\mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}^{\text{op}}$), and let $\tilde{F}: \mathcal{C} \to \mathcal{D}$ (resp. $\tilde{F}: \mathcal{C}^{\text{op}} \to \mathcal{D}$) denote such factorization. Then

$$\int_{c \in \mathcal{C}} F \simeq \text{colim}_{\mathcal{C}} \tilde{F} \quad \text{and} \quad \int_{c \in \mathcal{C}} F \simeq \text{lim}_{\mathcal{C}} \tilde{F},$$

$$\left(\text{resp.} \int_{c \in \mathcal{C}} F \simeq \text{colim}_{\mathcal{C}^{\text{op}}} \tilde{F} \quad \text{and} \quad \int_{c \in \mathcal{C}^{\text{op}}} F \simeq \text{lim}_{\mathcal{C}^{\text{op}}} \tilde{F}\right).$$

**Lemma A.10** Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories, with $\mathcal{D}$ cocomplete. Then

$$F \simeq \int_{c \in \mathcal{C}} \text{Map}_\mathcal{C}(c, -) \otimes F(c)$$

where $S \otimes X := \text{colims} X$ denotes the canonical tensoring in spaces for $\infty$-categories.

\(^{19}\)Sometimes in the literature the terms “cofinal functor” and “final functor” are used in place of what we call respectively “final functor” and “initial functor”.


Proof. The following computation shows that the two objects corepresent the same functor
\[
\text{Map}_{\text{Fun}(\mathfrak{E}, \mathfrak{D})} \left( \int_{c \in \mathfrak{E}} \text{Map}_{\mathfrak{E}}(c, -) \otimes F(c), G \right) \simeq \\
\simeq \int_{c \in \mathfrak{E}} \text{Map}_{\mathfrak{D}} \left( \int_{c' \in \mathfrak{E}} \text{Map}_{\mathfrak{E}}(c, c') \otimes F(c'), G(c') \right) \\
\simeq \int_{c \in \mathfrak{E}} \int_{c' \in \mathfrak{E}} \text{Map}_{\mathfrak{D}} \left( \text{Map}_{\mathfrak{E}}(c, c') \otimes F(c), G(c') \right) \\
\simeq \int_{c \in \mathfrak{E}} \text{lim}_{c' \in \mathfrak{E}} \text{Map}_{\mathfrak{D}} \left( F(c), G(c') \right) \\
\simeq \int_{c \in \mathfrak{E}} \text{Map}_{\mathfrak{D}} \left( F(c'), G(c') \right) \\
\simeq \text{Map}_{\text{Fun}(\mathfrak{E}, \mathfrak{D})} \left( F, G \right).
\]

We now turn to the discussion of Kan extensions in terms of co/ends.

Lemma A.11 Let \( \mathfrak{C}, \mathfrak{D}, \) and \( \mathfrak{E} \) be \( \infty \)-categories, with \( \mathfrak{C} \) cocomplete, and let \( G : \mathfrak{C} \to \mathfrak{D} \) and \( F : \mathfrak{C} \to \mathfrak{E} \) be functors. Then
\[
\text{Lan}_G F(d) \simeq \int_{c \in \mathfrak{E}} \text{Map}_{\mathfrak{D}}(Gc, d) \otimes F(c).
\]

Proof. Unraveling the definitions (see [Lur09, 4.3.3.2] and [Lur09, 4.3.2.2]), we know that
\[
\text{Lan}_G F(d) \simeq \text{colim} \left( (G \downarrow d) \xrightarrow{\pi} \mathfrak{C} \xrightarrow{\xi} \mathfrak{E} \right).
\]
Now, the result holds from the following computation
\[
\int_{c \in \mathfrak{E}} \text{Map}_{\mathfrak{D}}(Gc, d) \otimes F(c) \overset{(A.4)}{=} \int_{c \in \mathfrak{E}} \left( \text{colim}_{c \in (G \downarrow d)} \xi \circ \pi(a) \right) (c) \otimes F(c) \\
\overset{(A.10)}{=} \text{colim}_{c \in (G \downarrow d)} \int_{c' \in \mathfrak{E}} \left( \xi \circ \pi(a) \right) (c) \otimes F(c) \\
\overset{(19)}{=} \text{colim}_{c \in (G \downarrow d)} F \circ \pi(a) \overset{(19)}{=} \text{Lan}_G F(d).
\]

Remark A.12 A suitable dualization of the above arguments shows that
\[
\text{Ran}_G F(d) \simeq \int_{c \in \mathfrak{E}} \left[ \text{Map}_{\mathfrak{D}}(d, Gc), F(c) \right]
\]
where \([S, X] := \text{lim}_S X\) denotes the canonical cotensoring in spaces for \( \infty \)-categories.

Definition A.13 Let \( F : \mathfrak{C} \to \mathfrak{D} \) be a functor from a small \( \infty \)-category \( \mathfrak{C} \) to a cocomplete \( \infty \)-category \( \mathfrak{D} \). The left Kan extension of \( F \) along the Yoneda embedding \( \xi : \mathfrak{C} \to \mathcal{P}(\mathfrak{C}) \) is called the \( F \)-realization, denoted
\[
|-F : \mathcal{P}(\mathfrak{C}) \to \mathfrak{D}.
\]
Proposition A.14 Let \( F: \mathcal{C} \to \mathcal{D} \) be a functor from a small \( \infty \)-category \( \mathcal{C} \) to a cocomplete \( \infty \)-category \( \mathcal{D} \). Then, the \( F \)-realization admits a right adjoint, denoted \( N_F: \mathcal{D} \to \mathcal{P}(\mathcal{C}) \). Moreover, the value of \( N_F \) on an object \( d \in \mathcal{D} \) is given by the presheaf \( \chi_d \circ F \), or, more informally
\[
N_F(d): c \mapsto \text{Map}_C(F(c), d).
\]

Proof. By [Lur09, 5.1.5.5, 5.1.5.6], precomposition with \( \chi_d \) and left Kan extension along it give mutually inverse functors
\[
\text{Fun}^L(P(C), D) \overset{\sim}{\longrightarrow} \text{Fun}(C, D)
\]
which hence \( N_F \) has a right adjoint. The objectwise description follows formally from the following computation
\[
\text{Map}_\mathcal{D}(|P|_F, d) \overset{\text{(A.11)}}{=} \text{Map}_\mathcal{D}\left(\int_{c \in \mathcal{C}} \text{Map}_{\mathcal{P}(\mathcal{C})}(\chi_c, P) \otimes F(c), d\right)
\]
\[
\simeq \int_{c \in \mathcal{C}} \text{Map}_\mathcal{D}\left(P(c) \otimes F(c), d\right)
\]
\[
\simeq \int_{c \in \mathcal{C}} \text{Map}_\mathcal{D}\left(P(c), \text{Map}_\mathcal{D}(F(c), d)\right)
\]
\[
\simeq \text{Map}_{\mathcal{P}(\mathcal{C})}(P, \text{Map}_\mathcal{D}(F(-), d))
\]
from which we see that \( \chi_d \circ F \) is right adjoint to \( F \)-realization. \( \square \)

Definition A.15 In the situation of Proposition A.14, we refer to the right adjoint
\[
N_F: \mathcal{D} \to \mathcal{P}(\mathcal{C})
\]
as the \( F \)-nerve.

A.3. The stable nerve-realization paradigm. When, in the situation of Definition A.13 the \( \infty \)-category \( \mathcal{D} \) is stable and presentable, we can take the nerve-realization paradigm one step further.

Remark A.16 Let \( F: \mathcal{C} \to \mathcal{D} \) be a functor from a small \( \infty \)-category to a stable presentable \( \infty \)-category \( \mathcal{D} \). By [Lur17, 1.4.4.5], we have an equivalence
\[
\text{Fun}^L(\mathcal{P}_{\text{st}}(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D})
\]
given by precomposition with \( \Sigma_\infty^+: \mathcal{P}(\mathcal{C}) \to \text{St}(\mathcal{P}(\mathcal{C})) \), and, equivalently, an equivalence
\[
\text{Fun}^R(\mathcal{D}, \mathcal{P}_{\text{st}}(\mathcal{C})) \simeq \text{Fun}^R(\mathcal{D}, \mathcal{P}(\mathcal{C}))
\]
given by postcomposition with \( \Omega_\infty \). In particular, we get an extension of the nerve-realization adjunction
\[
\mathcal{P}(\mathcal{C}) \leftrightarrow \mathcal{D}
\]
to an adjunction
\[
\mathcal{P}_{\text{st}}(\mathcal{C}) \leftrightarrow \mathcal{D}
\]
(20)
Notice that, again by [Lur17, 1.4.4.5], we have that
\[
|-|_F \simeq |\Sigma_\infty^+(-)|_F^{\text{st}} \quad \text{and} \quad N_F \simeq \Omega_\infty(N_F^{\text{st}}(-)).
\]

Definition A.17 We refer to the adjunction (20) as the stable nerve-realization paradigm.

Putting together what we got this far, we get the following equivalence.
Lemma A.18  Let $\mathcal{C}$ be a small $\infty$-category and $\mathcal{D}$ be a stable, presentable $\infty$-category. Then, the stable nerve-realization paradigm gives an equivalence of $\infty$-categories
\[ \text{Fun}(\mathcal{C}, \mathcal{D}) \to \text{Fun}^L(\mathcal{P}_\text{st}(\mathcal{C}), \mathcal{D}) \]
\[ F \mapsto |-|_{F}^{st} \]

Proof. As $\mathcal{C}$ is small, and $\mathcal{D}$ is cocomplete, by [Lur09, 5.1.5.7], we have
\[ \text{Fun}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \]
and the equivalence is given by $F \mapsto |-|^{|}_{F}$. Moreover, as both $\text{Fun}(\mathcal{C}^{\text{op}}, \Omega)$ and $\mathcal{D}$ are presentable and $\mathcal{D}$ is stable, by [Lur17, 1.4.4.5], we have a further equivalence
\[ \text{Fun}^L(\mathcal{P}_\text{st}(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \]
as reviewed in Remark A.16, completing the proof. □

It is also possible to give an explicit description of stable nerves, similar to the one we gave in the unstable case, using mapping spectra instead of mapping spaces. To this end, let us first recall the stable Yoneda embedding.

Definition A.19 (See [BGT13, Definition 2.15]20) Let $\mathcal{D}$ be a stable $\infty$-category. Recall that, by [Lur17, 1.4.2.23], postcomposition with $\Omega^\infty$ induces an equivalence
\[ \text{Fun}^{\text{lex}}(\mathcal{D}, \mathcal{P}_\text{st}(\mathcal{D})) \simeq \text{Fun}^{\text{lex}}(\mathcal{D}, \mathcal{P}(\mathcal{D})) \]
We call the stable Yoneda embedding the map corresponding to $\mathcal{Y}$ under the above equivalence and denote it as $\mathcal{Y}^{st}$. We denote the adjoint functor $\mathcal{D}^{\text{op}} \times \mathcal{D} \to \text{Sp}$ as $\text{map}_{\mathcal{D}}(\_ , -)$ and call it the mapping spectrum functor.

Remark A.20  In particular, the functor $\text{map}_{\mathcal{D}}(\_ , -)$ is such that
\[ \Omega^\infty \text{map}_{\mathcal{D}}(\_ , -) \simeq \text{Map}_{\mathcal{D}}(\_ , -) \]
and is the unique such functor sending finite colimits in the first variable to limits and finite limits in the second variable to limits. As shown in [Nik16, Remark 6.2], the functor actually sends all (small) colimits in the first variable to limits, and all (small) limits in the second variable to limits.

Remark A.21  The functor $\mathcal{Y}^{st}$ is extensively studied in [Nik16]. In particular, in op. cit. Section 6 it is proven that it is fully faithful.

We have the following variant of Yoneda’s Lemma for spectral presheaves of general $\infty$-categories.

Proposition A.22  Let $\mathcal{C}$ be a small $\infty$-category; then
\[ \text{map}_{\mathcal{P}_\text{st}(\mathcal{C})}(\Sigma^\infty_{+} \mathcal{Y}, -) \simeq \text{id}_{\mathcal{P}_\text{st}(\mathcal{C})} \]

Proof. By [Lur17, 1.4.2.23], we know that
\[ \text{Fun}^{\text{lex}}(\mathcal{P}_\text{st}(\mathcal{C}), \mathcal{P}_\text{st}(\mathcal{C})) \simeq \text{Fun}^{\text{lex}}(\mathcal{P}_\text{st}(\mathcal{C}), \mathcal{P}(\mathcal{C})) \]  (21)
and that $\text{id}_{\mathcal{P}_\text{st}(\mathcal{C})}$ on the left hand side, corresponds to $\Omega^\infty$ on the right hand side. Now, by the usual $\infty$-categorical Yoneda lemma (see [Lur09, 5.5.2.1]) we have
\[ \Omega^\infty \simeq \text{Map}_{\mathcal{P}(\mathcal{C})}(\_ , \Omega^\infty) \simeq \text{Map}_{\mathcal{P}_\text{st}(\mathcal{C})}(\Sigma^\infty_{+} \mathcal{Y}, -) \]
But, by Remark A.20, we know that the latter corresponds to
\[ \text{map}_{\mathcal{P}_\text{st}(\mathcal{C})}(\Sigma^\infty_{+} \mathcal{Y}, -) \]
under the equivalence (21). □

We conclude the section with an explicit description of stable nerves.

20notice that we use the opposite capitalization convention of [BGT13] to distinguish between mapping spaces and the mapping spectra.
Proposition A.23 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a small $\infty$-category $\mathcal{C}$ to a stable presentable $\infty$-category $\mathcal{D}$. Then, the value of the stable $F$-nerve on an object $d \in \mathcal{D}$ is given by the stable presheaf map$_\mathcal{D}(F(-), d)$.

Proof. The functor $N^\ast_F$ is adjoint to a functor $\chi : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \text{Sp}$, which in turn is adjoint to a functor $\xi : \mathcal{C}^{\text{op}} \to \mathcal{P}_d(\mathcal{D}^{\text{op}})$. By Remark A.16, together with Proposition A.14, we have that

$$\Omega^\infty \xi(c) \simeq \text{Map}_\mathcal{D}(Fc, -) : \mathcal{D} \to \mathcal{S}$$

and that each $\xi(c)$ is left exact (as $N^\ast_F$ is so), thus, by Remark A.20, $\xi(c) \simeq \text{map}_\mathcal{D}(F(-), -)$, from which it follows that $\chi \simeq \text{map}_\mathcal{D}(F(=), -)$, and thus that

$$N^\ast_F(d) \simeq \text{map}_\mathcal{D}(F(-), d)$$

as requested. \qed

A.4. The case of pointed $\infty$-categories. We recollect some variations of the results of §A.3 that apply to pointed $\infty$-categories. We will not give full proofs for them, as the arguments are entirely analogue to the ones used in the stable case.

We will denote by $\mathcal{P}_*(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S}_*) \simeq \mathcal{P}(\mathcal{C}) \otimes \mathcal{S}_*$ the $\infty$-category of presheaves with values in pointed spaces.

Definition A.24 Let $\mathcal{C}$ be a pointed $\infty$-category. By [Lur17, 4.8.2.12], postcomposition with the functor forgetting the basepoint induces an equivalence

$$\text{Fun}^0(\mathcal{C}, \mathcal{P}_*(\mathcal{C})) \simeq \text{Fun}'(\mathcal{C}, \mathcal{P}(\mathcal{C}))$$

where $\text{Fun}^0$ denotes the $\infty$-category of pointed functors, and $\text{Fun}'$ denotes the $\infty$-category of functors carrying the zero objects of $\mathcal{C}$ to terminal objects of $\mathcal{P}(\mathcal{C})$. We call the pointed Yoneda embedding the map corresponding to $\mathcal{C}$ under the above equivalence, and denote it as $\mathcal{P}_e$. We denote the adjoint functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{S}_*$ as $\text{Map}_{\mathcal{C}_e}(=, -)$ (or simply as $\text{Map}_*(=, -)$, when there is no risk of confusion) and call it the pointed mapping space functor.

Definition A.25 Let $\mathcal{C}$ be a pointed essentially small $\infty$-category. Then $\mathcal{P}_*^{\text{fin}}(\mathcal{C})$ denotes the smallest full subcategory of $\mathcal{P}_*(\mathcal{C})$ containing the essential image of $\mathcal{P}_e : \mathcal{C} \to \mathcal{P}_*(\mathcal{C})$ that is closed under finite colimits.

Remark A.26 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a small $\infty$-category to a pointed presentable $\infty$-category $\mathcal{D}$. Then, we have an extension of the nerve-realization adjunction

$$\mathcal{P}_*(\mathcal{C}) \xrightarrow{|-|^\ast_F} \mathcal{D}$$

(22)

such that

$$|-|^\ast_F \simeq |(-)_+|^\ast_F$$

and $N_F \simeq U(N^\ast_F(=))$ where $U$ denotes the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ (usually omitted from notations), and $(-)_+$ denotes its left adjoint.

Definition A.27 We refer to the adjunction (22) as the pointed nerve-realization paradigm.

Proposition A.28 Let $\mathcal{C}$ be an essentially small $\infty$-category. For any pointed and finitely cocomplete $\infty$-category $\mathcal{D}$, composition with $\mathcal{P}_*^{\text{fin}} : \mathcal{C} \to \mathcal{P}_*^{\text{fin}}(\mathcal{C})$ induces an equivalence

$$\text{Fun}^{\text{Rex}}(\mathcal{P}_*^{\text{fin}}(\mathcal{C}), \mathcal{D}) \to \text{Fun}(\mathcal{C}, \mathcal{D})$$

If $\mathcal{D}$ is also presentable, we have

$$\text{Fun}^I(\mathcal{P}_*(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}^{\text{Rex}}(\mathcal{P}_*^{\text{fin}}(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}(\mathcal{C}, \mathcal{D})$$

and the right-to-left composite is given by

$$F \mapsto |-|^\ast_F.$$
Proposition A.29 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a small $\infty$-category $\mathcal{C}$ to a pointed presentable $\infty$-category $\mathcal{D}$. Then, the value of the pointed $F$-nerve on an object $d \in \mathcal{D}$ is given by the pointed presheaf $\text{Map}_*(F(-), d)$.

Appendix B. An $E_1$ presentation of exterior algebras (by Achim Krause)

In this appendix, we produce an explicit $E_1$ presentation of exterior algebras over $\mathbb{Z}$. The main result is the following:

**Theorem B.1** Let $\Lambda(e_k)$ denote the $E_1$-algebra in $D(\mathbb{Z})$ represented by the dga which is exterior on a generator in degree $k$, with zero differential. Then $\Lambda(e_k)$ admits a description as a colimit

$$\colim_{\mathbb{Z} \geq 1} \Lambda^{(n)} \simeq \Lambda(e_k)$$

where $\Lambda^{(1)}$ is a free $E_1$-algebra on a generator of degree $n$, and each of the maps $\Lambda^{(n-1)} \to \Lambda^{(n)}$ is an $E_1$-pushout

$$\text{Free}_{E_1}(r_n) \to \Lambda^{(n-1)} \to \Lambda^{(n)}$$

Here $r_n$ is an element of degree $nk + n - 2$.

The significance of Theorem B.1 is that it allows us to describe an obstruction theory for $E_1$-maps out of exterior algebras. Indeed, to find a map $\Lambda(e_k) \to R$, we need to provide a cycle $e$ of degree $k$ in $R$, a nullhomotopy of $e \cdot e$, a nullhomotopy of a certain element of degree $3k + 1$ (depending on the chosen nullhomotopy of $e \cdot e$), and so on. Having provided the first $i$ pieces of this data, i.e. a map $\Lambda^{(i)} \to R$, the next obstruction, i.e. the image of $r_{i+1}$, is a version of an $i+1$-fold Massey power of $e$.

Theorem B.1 will follow from the following bigraded version, where the degree of the generator is abstracted away into a second, formal grading. We work in $\text{gr } D(\mathbb{Z}) = \text{Fun}(\mathbb{Z}^\delta, D(\mathbb{Z}))$ (where $\mathbb{Z}^\delta$ denotes the “discrete” category $\mathbb{Z}$, i.e. without nontrivial morphisms). An object is thus simply a list of objects of $D(\mathbb{Z})$. The monoidal structure is Day convolution. Homology groups now have two degrees: If our object is given by $(X_n)_{n \in \mathbb{Z}}$, we write $H_{k,n}(X) = H_k(X_n)$, and similarly we can shift in both those degrees, writing $\Sigma_{k,n}$. Note that $\Sigma_{1,0}$ is ordinary suspension. Whenever confusion is imminent, we will refer to the second grading (i.e. the $n$ in $(k,n)$ above) as “formal” degree. Similarly, we will call an object of $D(\mathbb{Z})$ “formally $n$-connective” if it vanishes in formal degrees $< n$, regardless of “topological” connectivity of the individual components $X_n$. Note that the notion of formal connectivity is somewhat simpler than usual topological connectivity: The $n$-connective objects are closed under any limits and colimits, since those are formed pointwise. (In particular, the notion of formal connectivity is unrelated to t-structures.)

**Theorem B.2** Let $\Lambda(e)$ denote the $E_1$-algebra in $\text{gr } D(\mathbb{Z})$ represented by the graded exterior algebra on one generator $e$ in bidegree $(0,1)$. Then $\Lambda(e)$ admits a description as colimit

$$\colim_{\mathbb{Z} \geq 1} \Lambda^{(n)} \simeq \Lambda(e)$$

where $\Lambda^{(1)}$ is a free $E_1$ algebra on a generator of bidegree $(0,1)$, and each of the maps $\Lambda^{(n-1)} \to \Lambda^{(n)}$ is an $E_1$-pushout

$$\text{Free}_{E_1}(r_n) \to \Lambda^{(n-1)} \to \Lambda^{(n)}$$

Here $r_n$ is an element of bidegree $(n-2, n)$. 
The exterior algebra discussed in the theorem has underlying object of the form \((X_n)_{n \in \mathbb{Z}}\) with \(X_0\) and \(X_1\) both given by the complexes \(\mathbb{Z}[0]\). Another way to express this is to say that there is a functor

\[ \text{gr Ab} \to \text{gr D(\mathbb{Z})}, \]

simply given by applying the “discrete complex” functor pointwise, and it is lax symmetric monoidal. The \(E_1\) algebra \(\Lambda(e)\) is precisely the image of the ordinary graded exterior algebra on a degree 1 generator in \(\text{gr Ab}\).

Theorem B.2 should be regarded as a universal version of B.1. Indeed, the following lemma shows how B.1 is a special case of Theorem B.2:

**Lemma B.3** Let \(C\) be a closed symmetric-monoidal \(\infty\)-category and colimit-preserving symmetric-monoidal functor \(D(\mathbb{Z}) \to C\), and let \(L \in C\) be an invertible object. Then there is a colimit-preserving monoidal functor

\[ \text{gr D(\mathbb{Z})} \to C \]

taking the graded object \(X\) to \(\bigoplus_n L \otimes^n X_n\).

**Proof.** An invertible object determines a unique monoidal functor \(\mathbb{Z} \to C\), since \(\mathbb{Z}\) is the free \(E_1\) monoid on a single invertible object. Together with the symmetric-monoidal functor \(D(\mathbb{Z}) \to C\), we obtain a monoidal functor \(\mathbb{Z} \times D(\mathbb{Z}) \to C\), as the composite

\[ F : \mathbb{Z} \times D(\mathbb{Z}) \to C \times C \overset{\otimes}{\to} C. \]

We also have a symmetric-monoidal functor \(\mathbb{Z} \times D(\mathbb{Z}) \to \text{Fun}(\mathbb{Z}, D(\mathbb{Z}))\), which simply takes \((n, C)\) to the graded object which is \(C\) in degree \(n\). The left Kan extension of \(F\) along this embedding is the desired functor, as one directly computes from the pointwise formula for Kan extensions. \(\square\)

For example, we have a monoidal functor from \(\text{gr D(\mathbb{Z})}\) to \(D(\mathbb{Z})\) sending \(\mathbb{Z}\) in bidegree \((0, 1)\) to \(\mathbb{Z}\) in arbitrary degree \(n\). This sends the exterior algebra \(\Lambda(z)\) to the exterior algebra in \(D(\mathbb{Z})\) on a degree \(n\) generator, and our \(E_1\)-presentation to an \(E_1\)-presentation in \(D(\mathbb{Z})\).

The proof of Theorem B.2 is based on bar-cobar duality. Specifically, we say that an algebra \(A\) in \(\text{gr D(\mathbb{Z})}\) is formally connected if the unit map \(\mathbb{Z} \to \text{gr D(\mathbb{Z})}\) induces an equivalence of the “formal degree” 0 part, and the algebra \(A\) vanishes in negative formal degrees. Analogously, we call a coalgebra \(C\) formally connected if the counit map \(C \to \mathbb{Z}\) induces an equivalence in nonpositive formal degrees. More generally, we call an algebra \(A\) formally \(n\)-connective if the cofiber \(\tilde{A}\) of \(\mathbb{Z} \to A\) is formally \(n\)-connective as an object of \(\text{gr D(\mathbb{Z})}\), and analogously for coalgebras. With this convention, an algebra or coalgebra is formally 1-connective. We write \(\text{Alg}^{cn}(\text{gr D(\mathbb{Z})})\) for the full subcategory of \(\text{Alg}(\text{gr D(\mathbb{Z})})\) on all formally connected algebras, and analogously for coalgebras.

**Proposition B.4** There is an equivalence

\[ \text{Alg}^{cn}(\text{gr D(\mathbb{Z})}) \to \text{CoAlg}^{cn}(\text{gr D(\mathbb{Z})}) \]

given by the bar-cobar adjunction.

This is an instance of more general results on Koszul duality, and can for example be extracted out of [FG12, Proposition 4.1.2], where the requisite pronilpotency condition is guaranteed by our graded connected setup. Since this in particular involves an identification of the usual 2-sided bar construction with the operadic one for \(E_1\), we also give a slightly more elementary argument, relying on connectivity behaviour of the bar and cobar constructions which we will require anyways. The overall structure of this argument is similar to the structure of the argument in Section 4 of loc. cit.

**Lemma B.5** Let \(X \to Y\) be a map of formally 1-connective objects of \(\text{gr D(\mathbb{Z})}\) whose fiber is formally \((k + 1)\)-connective. Then the fiber of

\[ X^\otimes n \to Y^\otimes n \]
is formally \((n + k)\)-connective.

**Proof.** The map \(X^i \otimes Y^{n-i} \to X^{i-1} \otimes Y^{n-i+1}\) is obtained by tensoring the map \(X \to Y\) with the formally \((n-1)\)-connective object \(X^{i-1} \otimes Y^{n-i}\). Thus its fiber is formally \((n+k)\)-connective, i.e. it induces isomorphisms in formal degrees \(\leq (n + k)\). Since we can write the map \(X \otimes^n Y \to Y \otimes^n\) as a composite of \(n\) such maps, the claim follows. \(\square\)

For a coalgebra \(C\), \(\text{Cobar}(C)\) is the limit of a cosimplicial diagram with \(C \otimes^n\) in level \(n\). The \(\text{Tot}^n\)-tower for that diagram takes the form

\[
\text{Cobar}(C) \simeq \lim(\ldots \to \text{Tot}^n(\text{Cobar}^\bullet(C)) \to \text{Tot}^{n-1}(\text{Cobar}^\bullet(C)) \to \ldots)
\]

with fiber of \(\text{Tot}^n \to \text{Tot}^{n-1}\) given by \(\Omega^n \tilde{C} \otimes^n\). Here \(\tilde{C}\) is the fiber of the counit map \(C \to \mathbb{Z}\). If \(C\) is connected, note that \(\Omega^n C \otimes^n\) is formally \(n\)-connective (since \(\Omega\) does not change formal degree!), so it follows that \(\text{Cobar}(C) \to \text{Tot}^n(\text{Cobar}^\bullet(C))\) is an equivalence in formal degrees \(\leq n\).

The connectivity of the \(\text{Tot}^n\)-tower has the following two immediate consequences:

**Lemma B.6** \(\text{Cobar} : \text{CoAlg}_{gr\ D(\mathbb{Z})} \to \text{gr\ D(\mathbb{Z})}\) commutes with sifted colimits.

**Proof.** \(\text{Tot}^n(\text{Cobar}^\bullet)\) commutes with sifted colimits, since it is a finite limit of terms of the form \(C \otimes^k\). Since the \(\text{Tot}^n\)-tower stabilizes degreewise, the result follows. \(\square\)

**Lemma B.7** For a map of connected coalgebras \(C \to D\), the formal connectivity of \(C \to D\) (i.e. the lowest degree in which \(C \to D\) is not an equivalence) agrees with the formal connectivity of \(\text{Cobar}(C) \to \text{Cobar}(D)\). Dually the formal connectivity of a map of connected algebras agrees with the formal connectivity after Bar. In particular both \(\text{Cobar}\) and \(\text{Bar}\) are conservative.

**Proof.** Assume \(C \to D\) is (at least) \(k\)-connective, with \(k \geq 1\). Then so is \(\tilde{C} \to \tilde{D}\), and by Lemma B.5, the map \(\tilde{C} \otimes^n \to \tilde{D} \otimes^n\) is \(n + k - 1\)-connective. By looking at the total homotopy fiber of the square

\[
\begin{array}{ccc}
\text{Tot}^n(\text{Cobar}^\bullet(C)) & \longrightarrow & \text{Tot}^n(\text{Cobar}^\bullet(D)) \\
\downarrow & & \downarrow \\
\text{Tot}^{n-1}(\text{Cobar}^\bullet(C)) & \longrightarrow & \text{Tot}^{n-1}(\text{Cobar}^\bullet(D)),
\end{array}
\]

we now see that the fiber of \(\text{Tot}^n(\text{Cobar}^\bullet(C)) \to \text{Tot}^n(\text{Cobar}^\bullet(D))\) agrees with the fiber on \(\text{Tot}^1\) in degree \(k\). This fiber is simply \(\Omega \text{fib}(\tilde{C} \to \tilde{D})\). In the limit, we see that

\[
\Omega \text{fib}(\tilde{C} \to \tilde{D})_k \simeq \text{fib}(\text{Cobar}(C) \to \text{Cobar}(D))_k.
\]

This shows inductively that \(\text{Cobar}(C) \to \text{Cobar}(D)\) has exactly the same formal connectivity as \(C \to D\). The argument for Bar proceeds analogously, using the skeletal filtration. \(\square\)

**Proof of Proposition B.4.** Since formally connected algebras are canonically augmented (the map \(A \to \mathbb{Z}\) is determined as the inverse equivalence to the unit in degree 0, and as 0 in all other degrees, both of which are contractible choices), and formally connected coalgebras are canonically coaugmented, we can equivalently state that \(\text{Bar}\) and \(\text{Cobar}\) constitute inverse equivalences between

\[
\text{Alg}(\text{gr\ D(\mathbb{Z})})_{/\mathbb{Z}} \to \text{CoAlg}(\text{gr\ D(\mathbb{Z})})_{/\mathbb{Z}},
\]

where the subscript denotes the “double slice” category of pointed augmented objects, and the formal connectivity assumption is now stated on the pointing or augmentation morphism. Recall that formally connected objects are actually closed under limits and colimits.

In this setting, [Lur17, 5.2.2.19] guarantees that we at least have an adjunction, with the left adjoint taking an algebra to a coalgebra with underlying object given by the bar construction, and right adjoint taking a coalgebra to an algebra having underlying object given by the cobar construction.
To check that this adjunction is an equivalence, it suffices to check that the unit 
\[ A \to \text{Cobar}(\text{Bar}(A)) \] is an equivalence, and that Cobar is conservative. The latter follows from Lemma B.7. For the former, since we can write every algebra \( A \) as sifted colimit of free ones, and Lemma B.6 tells us that Cobar commutes with sifted colimits (while Bar is even a left adjoint), it suffices to check that

\[ A \to \text{Cobar}(\text{Bar}(A)) \]

is an equivalence for \( A \) free. If \( A \) is free on a single element of bidegree \((k, n)\), then \( \text{Bar}(A) \) has homology concentrated in degree \((k + 1, n)\) (and \((0, 0)\)). For degree reasons, it is thus a square-zero coalgebra. Since both Free and Bar commute with colimits (the latter because it is left adjoint), it follows that \( \text{Bar}(\text{Free}(X)) = Z \oplus \Sigma^{1,0}X \) with square-zero coalgebra structure for any \( X \in \text{gr}_+ D(Z) \). Finally, for such a square-zero coalgebra, we have that \( \text{Cobar}(Z \oplus \Sigma^{1,0}X) = \bigoplus X^\otimes n \simeq \text{Free}(X) \). \qed

In the proof, we have also shown

**Lemma B.8** Bar and Cobar take the free algebra \( \text{Free}_{E_1}(\Sigma^{0,n}X) \) to the square-zero coalgebra \( Z \oplus \Sigma^{1,n}X \) and vice-versa. \qed

Under this bar-cobar equivalence, the statement of Theorem B.2 amounts to giving a similar decomposition of \( \text{Bar}(\Lambda(e)) \). Morally, since Bar takes a free \( E_1 \)-algebra to a square zero coalgebra, and pushouts in coalgebras can be computed underlying, where Bar turns an \( E_1 \)-presentation of an algebra into a sort of square-zero cell structure on the resulting coalgebra. In particular, just the underlying homology of \( \text{Bar}(A) \) should already tell us how many cells we need.

**Lemma B.9** Let \( C \to D \) be a map of connected coalgebras which is an equivalence in formal degrees \( \leq n \). Assume

\[
\begin{array}{ccc}
F & \longrightarrow & C \\
\downarrow & & \downarrow \\
Z & \longrightarrow & D
\end{array}
\]

is a pullback diagram of coalgebras. Then \( F \) is a formally \((n+1)\)-connective coalgebra, and in formal degree \( n + 1 \) the diagram is a pullback (and pushout) square in \( D(Z) \).

**Proof.** By an induction on \( n \), we may automatically assume the first part of the claim, namely that \( F \) is zero in formal degrees \( \leq n \). From bar-cobar duality, we see that the diagram

\[
\begin{array}{ccc}
\text{Cobar}(F) & \longrightarrow & \text{Cobar}(C) \\
\downarrow & & \downarrow \\
Z & \longrightarrow & \text{Cobar}(D)
\end{array}
\]

(23)

is a pullback diagram of algebras, in particular an underlying pullback diagram.

Since the fiber of the map \( \tilde{C} \to \tilde{D} \) is formally \((n+1)\)-connective by assumption, the fiber of the map \( \tilde{C}^{\otimes k} \to \tilde{D}^{\otimes k} \) is formally \( n + k \)-connective by Lemma B.5. As \( \tilde{F}^{\otimes k} \) is even formally \((n+1)k = nk + k\)-connective, it follows that the total homotopy fiber of the diagram

\[
\begin{array}{ccc}
\tilde{F}^{\otimes k} & \longrightarrow & \tilde{C}^{\otimes k} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{D}^{\otimes k}
\end{array}
\]

is formally \( n + k \)-connective. It follows that the total homotopy fiber of the diagram (23) agrees in formal degree \((n+1)\) with the total homotopy fiber of the corresponding diagram for \( \text{Tot}^3(\text{Cobar}(\cdot)) \). Since the total fiber of the corresponding diagram for \( \text{Tot}^3 \) is simply 0, we get that the total fiber of (23) agrees in formal degree \((n+1)\)
with $\Sigma^{0,-1}$ of the total fiber of

\[
\begin{array}{ccc}
\tilde{F} & \longrightarrow & \tilde{C} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{D}.
\end{array}
\]

But since (23) is a pullback diagram in $\text{gr} \, D(\mathbb{Z})$, this proves the claim. \qed

**Definition B.10** By a **generalized cell structure** on a connected coalgebra $C$ we mean a sequence of $C^{(n)} \to C$, equivalences in formal degrees $\leq n$, with $C^{(0)} = \mathbb{Z}$, and compatible pushout diagrams

\[
\begin{array}{ccc}
\mathbb{Z} \oplus \Sigma^{-1,n}X_n & \longrightarrow & C^{(n-1)} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & C^{(n)}.
\end{array}
\]

We will call $X_n$ the **complex of n-cells**.

Analogously, we define on the algebra side:

**Definition B.11** By a **generalized $E_1$-cell structure** on a connected algebra $A$ we mean a sequence of $A^{(n)} \to A$, equivalences in formal degrees $\leq n$, with $A^{(0)} = \mathbb{Z}$, and compatible pushout diagrams

\[
\begin{array}{ccc}
\text{Free}_{E_1} (\Sigma^{-1,n}X_n) & \longrightarrow & A^{(n-1)} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & A^{(n)}.
\end{array}
\]

We will call $X_n$ the **complex of n-cells**.

**Theorem B.12** Every connected coalgebra $C$ admits a generalized cell structure whose complex of n-cells is given by $C_n$. Every connected algebra $A$ admits a generalized $E_1$-cell structure whose complex of n-cells is given by $\Sigma^{-1} \text{Bar}(A)_n$.

**Proof.** We first discuss the coalgebra case. We let $C^{(n)}$ be the formal truncation which is 0 in formal degrees $> n$. Just like t-structure truncation, the corresponding functor $\tau^{f}_{\leq n} : \text{gr}_{\geq 0} D(\mathbb{Z}) \to \text{gr}_{\geq 0} D(\mathbb{Z})$ arises from an adjunction between $\text{gr}_{\geq 0} D(\mathbb{Z})$ and $\text{gr}_{[0,n]} D(\mathbb{Z})$, where the functors are the obvious ones (forgetting degrees $> n$, and extending by 0). Here we write $\text{gr}_{[0,n]}(D(\mathbb{Z}))$ for objects concentrated in formal degrees between and including 0 and $n$. Unlike the case of t-structures, these functors are adjoint in both possible ways! Since the functor $\text{gr}_{\geq 0} D(\mathbb{Z}) \to \text{gr}_{[0,n]} D(\mathbb{Z})$ is strict monoidal, we obtain a lax and an oplax structure on the composite $\tau^{f}_{\leq n}$. This gives us a canonical coalgebra structure on $C^{(n)}$. Informally, it is given by the composite

\[
C^{(n)} \to C \to C \otimes C \to C^{(n)} \otimes C^{(n)}.
\]

If we define the coalgebra $F$ by the pullback diagram

\[
\begin{array}{ccc}
F & \longrightarrow & C^{(n-1)} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & C,
\end{array}
\]

Lemma B.9 implies that $F$ is concentrated in formal degrees $\geq n$, and that in degree $n$ the diagram is a pullback in $D(\mathbb{Z})$, given by

\[
\begin{array}{ccc}
F_n & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C_n.
\end{array}
\]
In particular, \( F_n = \Sigma^{-1} C_n \). If we compose with the canonical map of coalgebras \( \tau_{\leq n}^F F \to F \) (again using the discussion of adjoints and truncation at the beginning of this proof), we thus obtain a diagram of coalgebras

\[
\begin{array}{ccc}
\tau_{\leq n}^F F & \to & C^{(n-1)} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \to & C,
\end{array}
\]

which is a pullback (and pushout) of objects of \( D(\mathbb{Z}) \) in degree \( n \). Since pushouts of coalgebras are formed underlying, the pushout vanishes in degrees \( > n \), and maps equivalently to \( C \) in degrees \( \leq n \), i.e., agrees with \( C^{(n)} \). For degree reasons, \( \tau_{\leq n}^F F \) is \( \mathbb{Z} \oplus \Sigma^{-1,n} C_n \), and so the claim about coalgebras follows.

For the algebra claim, we simply apply the coalgebra statement to \( \text{Bar}(\Lambda(e)) \) and apply Lemma B.8 to the resulting cell structure.

**Lemma B.13** \( \text{Bar}(\Lambda(e)) \) can be described by the graded dg coalgebra \( \mathbb{Z}\{x_1, x_2, \ldots\} \), with zero differential, \( x_i \) in bidegree \((i, i)\) and comultiplication given by

\[
\Delta(x_n) = \sum_{i+j=n} x_i \otimes x_j.
\]

**Proof.** The bigraded homology of \( \text{Bar}(\Lambda(e)) \) is given by the bigraded Tor groups \( \text{Tor}^{\Lambda(e)}(\mathbb{Z}, \mathbb{Z}) \). The standard free resolution over an exterior algebra allows us to compute these to be given by \( \mathbb{Z} \) in degree \((n, n)\) for each \( n \). The coalgebra structure can also be computed either directly from the resolution, or by dualizing a nd thinking about the algebra structure given by composition on \( \text{Ext}^{\Lambda(e)}(\mathbb{Z}, \mathbb{Z}) \) (which is polynomial). Since the dual algebra has polynomial homology, it is equivalent to the algebra given by the dga \( \mathbb{Z}[x] \) with zero differential, from which we get our description by dualizing back. \( \square \)

**Proof of Theorem B.2.** We simply apply Theorem B.12 to \( \Lambda(e) \), using Lemma B.13 to see that the complex of \( n \)-cells in the resulting \( \mathbb{E}_1 \)-cell structure is \( \mathbb{Z}[n-1] \). \( \square \)

In the resulting cell structure on the coalgebra side, the \( n \)-skeleton \( C^{(n)} \) explicitly represents the sub-coalgebra of \( \mathbb{Z}\{x_1, \ldots, x_n\} \subseteq \mathbb{Z}\{x_1, \ldots\} \).

We can also study the \( \mathbb{E}_1 \)-algebras \( \Lambda(n) \cong \text{Cobar}(C^{(n)}) \) arising as skeleta on the algebra side more closely:

**Lemma B.14** For \( n = 1 \), the homology \( H_*(\Lambda^{(1)}) \) is of the form \( \mathbb{Z}[e] \), with \( e \) in bidegree \((0, 1)\). For \( n > 1 \), the homology \( H_*(\Lambda^{(n)}) \) is of the form \( \Lambda(e) \otimes \mathbb{Z}[r_{n+1}] \), with \( r_{n+1} \) of bidegree \((n-1, n+1)\).

**Proof.** Since \( \Lambda(n) = \text{Cobar}(C^{(n)}) \), we can compute its homology groups as derived cotensor product \( \text{Cotor}_*^{C^{(n)}}(\mathbb{Z}, \mathbb{Z}) \), or equivalently (since the cotensor product with \( \mathbb{Z} \) and \( \text{Hom}_{C^{(n)}}(\mathbb{Z}, -) \) agree, they both simply take a comodule to its primitives), \( \text{Ext}_*^{C^{(n)}}(\mathbb{Z}, \mathbb{Z}) \). Since everything is finite type and thus dualizable, this agrees with \( \text{Ext}(\mathbb{Z}, \mathbb{Z}) \) over the graded algebra dual to the coalgebra \( C^{(n)} \). Since \( C^{(n)} \) is represented by the graded coalgebra \( \mathbb{Z}\{x_1, x_2, \ldots, x_n\} \), its dual agrees with the truncated polynomial algebra \( \mathbb{Z}[x]/x^{n+1} \), with \( x \) now in bidegree \((-1, -1)\). This truncated polynomial algebra admits a 2-periodic minimal resolution, and from the Yoneda description of the product structure on \( \text{Ext}_*(\mathbb{Z}, \mathbb{Z}) \) we recover the claimed multiplicative structure. \( \square \)

Since \( \Lambda^{(n)} \) arises from \( \Lambda^{(n-1)} \) by a pushout

\[
\begin{array}{ccc}
\text{Free}_{\mathbb{E}_1}(\mathbb{Z}[n, n-2]) & \to & \Lambda^{(n-1)} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \to & \Lambda^{(n)}.
\end{array}
\]
and the degree \( n \)-part of \( \Lambda^{(n)} \) vanishes by Lemma B.14, the upper horizontal attaching map needs to be an equivalence in formal degree \( n \). Thus, it is given by a choice of generator for the polynomial part of \( H_*(\Lambda^{(n-1)}) \) as in B.14.

**Remark B.15** From the explicit description of \( \text{Bar}(\Lambda(e)) \) in Lemma B.13 one can actually explicitly describe the skeleton of the \( E_1 \)-cell structure on \( \Lambda(e) \): The coalgebra skeleta of \( \text{Bar}(\Lambda(e)) \) are represented by the sub-coalgebras \( \mathbb{Z}\{x_1, x_2, \ldots, x_n\} \), i.e. the skeleta of the \( E_1 \)-cell structure of \( \Lambda(e) \) are obtained by applying the cobar construction to these coalgebras. Given an explicit dg coalgebra which is flat over \( \mathbb{Z} \) with differential \( \partial e \), corresponds precisely to a choice of element of \( H_*(\Lambda(e)) \).

For example, we know abstractly from B.15 that \( \Lambda(e) \) is generated by \( \partial e \) in the presentation of \( \Lambda(e) \). Using the explicit description from Remark B.14, we can also fix the sign in the choice of generator in Lemma B.14, and equivalently in the choice of attaching maps in the presentation of \( \Lambda(e) \).

The description of \( \Lambda(e) \) as sequence of cell-attachments along these elements \( r_n \) gives us an obstruction theory for \( E_1 \)-maps out of \( \Lambda(e) \). Indeed, an \( E_1 \)-map \( \Lambda^{(1)} \to R \) corresponds precisely to a choice of element of \( H_*(\Lambda^{(1)}) \). Having fixed an \( E_1 \)-map \( \Lambda^{(n-1)} \to R \), we obtain a well-defined element of \( H_*(\Lambda^{(n)}) \) as the image of \( r_n \), and we get a further extension to \( \Lambda^{(n)} \) if and only of this element vanishes (with choices of extensions being in bijection to “nullhomotopies” of a representing cycle, i.e. a torsor over \( H_{(n-1,n)}(R) \)).

**Definition B.16** Given a map \( \Lambda^{(n-1)} \to R \) sending \( e \in H_{0,1}(\Lambda^{(n-1)}) \) to \( a \in R \), we call the image of \( r_n \) in \( H_{(n-2,n)}(\Lambda^{(n-1)}) \) an \( n \)-fold Massey power of \( a \), and write \( \langle a \rangle^n \).

Looking at the explicit dga description in Remark B.15, we see that \( \langle a \rangle^n \) indeed is a representative of the \( n \)-fold Massey product \( \langle a, \ldots, a \rangle \). However, it is somewhat more tightly defined: An arbitrary Massey product depends of choices of nullhomotopies for every adjacent product, nullhomotopies for the resulting 3-fold Massey products, and so on. In the Massey power above, we choose the same homotopy \( \langle e_2 \rangle \) for all the adjacent products \( a \cdot a \), a single homotopy for all the resulting 3-fold Massey products, and so on.

The obstruction theory discussed above can thus be summarized as follows: Having fixed nullhomotopies for all Massey powers \( \langle a \rangle^i \) for \( 2 \leq i < n \), we get a well-defined \( n \)-fold Massey power \( \langle a \rangle^n \). A map \( \Lambda(e) \to R \) taking \( e \) to an element \( a \) in bidegree \((0,1)\) corresponds precisely to a coherent choice of nullhomotopies for all Massey powers of \( a \).
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