Non-existence of non-topological solitons in some types of gauge field theories in Minkowski space

Mikhail N. Smolyakov

Skobeltsyn Institute of Nuclear Physics, Moscow State University, 119991, Moscow, Russia

Abstract

In this paper the conditions, under which non-topological solitons are absent in the Yang-Mills theory coupled to a non-linear scalar field in Minkowski space, are obtained in a very simple way. It is also shown that non-topological solitons are absent in the theory describing the massive complex vector field coupled to the electromagnetic field in Minkowski space.

1 Introduction

In this paper we present some non-existence results for gauge field theories, which can be obtained with the help of the so-called "scaling arguments". These scaling arguments are based on the use of an appropriate form of the variations of fields. For the first time this technique was used in [1, 2] to show the absence (or to find necessary conditions for existence, see [3]) of solitons in a non-linear scalar field theory (see also discussions in [4, 5]). Later such methods were used in models with more complicated static configurations of fields, for example, in the case of Yang-Mills field coupled to a scalar field [5] and for skyrmions, monopoles and instantons [6, 7, 8], as well as in models admitting time-independent effective Lagrangians [9].

In this paper we will use the scaling arguments to show the absence of non-topological solitons (when all the fields vanish at spatial infinity) in some types of gauge field theories in Minkowski space even if configurations of fields are time-dependent. We restrict ourselves to the case when solutions to equations of motion are periodic in time with a period $T < \infty$, but not necessarily of the simplest form $\sim e^{i\omega t}$ (it should be noted, that the scaling arguments were used long ago in [3] for obtaining restrictions on the existence of soliton solutions periodic in time in a non-linear scalar field theory). This restriction is necessary to make the proof mathematically rigorous, below we will discuss it in more detail. We will obtain the conditions for the scalar field potential which ensure the absence of non-topological solitons in a theory describing Yang-Mills field coupled to a non-linear scalar field (analogous scaling arguments were applied to the case of Klein-Gordon-Maxwell system with some simplifications, see [10]). For non-negative scalar field potentials we get a restriction, which is in agreement with that obtained in [11] using a different technique. Another consequence of our analysis is the absence of non-topological solitons in pure Yang-Mills theory, which is of course a well-known result and was obtained in [12]-[18] also using different techniques. Finally we will apply our methods to the theory of U(1)-charged massive vector field to show the absence of non-topological solitons in this case.

2 Yang-Mills field coupled to a non-linear scalar field

Let us consider the following form of the four-dimensional action:
\[ S = \int d^4x \left[ \eta^{\mu\nu} (D_\mu \phi)^\dagger D_\nu \phi - V(\phi^\dagger \phi) - \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \right], \tag{1} \]

where \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the flat Minkowski metric,

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g C^{abc} A^b_\mu A^c_\nu, \tag{2} \]

\[ D_\mu \phi = \partial_\mu \phi - ig T^a A^a_\mu \phi, \tag{3} \]

where \( C^{abc} \) are the structure constants of a compact gauge group and \( T^a \) are generators of the group in the representation space of field \( \phi \). We also suppose that

\[ V(\phi^\dagger \phi)|_{\phi^\dagger \phi = 0} = 0, \quad \left. \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \right|_{\phi^\dagger \phi = 0} = C, \quad |C| < \infty. \tag{4} \]

The latter condition ensures that the trivial solution is \( \phi \equiv 0, A_\mu = g T^a A^a_\mu \equiv 0 \). The condition \( V(\phi^\dagger \phi)|_{\phi^\dagger \phi = 0} = 0 \) is imposed for simplicity, in order not to deal with additive constant s. We also suppose that:

1. there are no sources which are external to the system described by action (1);
2. solutions to equations of motion are periodic in time with a period \( \tilde{T} < \infty \) up to a coordinate shift and a spatial rotation, i.e. for all fields on the solution the relation \( \Psi(t + \tilde{T}, \vec{x}) \equiv \Lambda(\Omega) \Psi(t, \Omega^{-1} \vec{x} - \vec{l}) \) must hold for any \( t \), where \( \Psi(t, \vec{x}) \) schematically represents the field under consideration, \( \tilde{T} \) is a constant vector, \( \Omega \in SO(3) \) and \( \Lambda(\Omega) \) denotes the representation of the rotation group carried by the field \( \Psi(t, \vec{x}) \) (this means that we also consider solitons which may not be at rest and spinning solitons).

Let us discuss the second assumption. It is not difficult to show that the periodicity condition presented above corresponds to a rotation in the plane orthogonal to some axis of rotation and a rectilinear motion along this axis. For this reason, one can always choose a suitable coordinate system in which the axis of rotation passes through the point \( \vec{x} = 0 \) and \( \tilde{l} \) belongs to this axis, i.e. \( \Omega \tilde{l} = \tilde{l} \). In this coordinate system the velocity of the rectilinear motion is \( \tilde{l}/\tilde{T} \) and we suppose that \( |\tilde{l}|/\tilde{T} < 1 \) (i.e. smaller than the speed of light). Next, with the help of a Lorentz transformation we can always pass to the coordinate system where \( \tilde{l} = 0 \). In this new coordinate system \( \Psi(t + T, \vec{x}) \equiv \Lambda(\Omega) \Psi(t, \Omega^{-1} \vec{x}) \) with a new period \( T \) and we can rewrite the initial action as

\[ S = \int_{-\infty}^\infty dt \int d^3x L[\Psi(t, \vec{x})] = \sum_{n = -\infty}^{\infty} \int_{nT}^{(n+1)T} dt \int d^3x L[\Psi(t, \vec{x})] = \tag{5} \]

\[ = \sum_{n = -\infty}^{\infty} \int_0^T dt \int d^3x L[\Psi(t + nT, \vec{x})] = \sum_{n = -\infty}^{\infty} \int_0^T dt \int d^3x L[\Lambda^n(\Omega) \Psi(t, \Omega^{-n} \vec{x})] = \]

\[ = \sum_{n = -\infty}^{\infty} \int_0^T dt \int d^3x L[\Psi(t, \vec{x})], \]

where we have made change of variables \( \vec{x} \rightarrow \Omega^n \vec{x} \) in each integral. Thus, we can use the effective action

\[ S_{\text{eff}} = \int_0^T dt \int d^3x L[\Psi(t, \vec{x})] \tag{6} \]
instead of the initial one. We will discuss the necessity of our restriction to consider such types of solutions at the end of this section.

It is interesting to note that if we suppose that for a solution there is no \( N \neq 0 \) such that \( \Omega^N = 1 \), where \( N \) is integer, then this solution will not be periodic in the usual sense, but nevertheless we can use the effective action of form (6) for examining such types of solitons.

Now let us proceed to the system described by action (1). The corresponding effective action takes the form (here and below we omit the subscript "eff" for effective actions)

\[
S = \int_0^T dt \int d^3x \left[ (D_0\phi)^\dagger D_0\phi - (D_i\phi)^\dagger D_i\phi - V(\phi^\dagger \phi) + \frac{1}{2} F_{0i}^a F_{0i}^a - \frac{1}{4} F_{ij}^a F_{ij}^a \right],
\]

where \( i, j = 1, 2, 3 \). Let us denote

\[
\begin{align*}
\int_0^T dt \int d^3x (D_0\phi)^\dagger D_0\phi &= \Pi_0 \geq 0, \\
\int_0^T dt \int d^3x (D_i\phi)^\dagger D_i\phi &= \Pi_1 \geq 0, \\
\int_0^T dt \int d^3x \frac{1}{2} F_{0i}^a F_{0i}^a &= \Pi_{A0} \geq 0, \\
\int_0^T dt \int d^3x \frac{1}{4} F_{ij}^a F_{ij}^a &= \Pi_{A1} \geq 0.
\end{align*}
\]

We suppose that all these integrals are finite.

We will be looking for smooth solutions to equations of motion, following from (1), with the asymptotic form

\[
\begin{align*}
&\lim_{x^i \to \pm \infty} \phi(t, \vec{x}) = 0, \\
&\lim_{x^i \to \pm \infty} A_\mu(t, \vec{x}) = 0.
\end{align*}
\]

**Theorem 1** For potentials satisfying (4) non-topological solitons of form (12), (13), periodic in time up to a spatial rotation and a coordinate shift such that \( \Omega^\dagger \vec{l} = \vec{l} \), \( \vec{l}/\vec{T} < 1 \), with integrals (8)-(11) and integrals \( \int_0^T dt \int d^3x V(\phi^\dagger \phi) \) finite, are absent in the theory with action (7) if there exists \( \gamma : \frac{1}{2} < \gamma \leq \frac{3}{2} \) for which the inequality

\[
2\gamma \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi - 3V(\phi^\dagger \phi) \geq 0
\]

is fulfilled for any \( \phi \) (or at least for the range of values of the field \( \phi \), which is supposed to occur in the solution).

**Proof:**

Let us suppose that there exists a solution \( \phi(t, \vec{x}), A_\mu(t, \vec{x}) \). With the help of (8)-(11) we get from (7)

\[
S = \Pi_0 - \Pi_1 - \int_0^T dt \int d^3x V(\phi^\dagger \phi) + \Pi_{A0} - \Pi_{A1}.
\]
Now let us consider the following transformation of our solution
\[
\phi(t, \vec{x}) \rightarrow \lambda^\gamma \phi(t, \lambda \vec{x}),
\]
(16)
\[
A_\mu^0(t, \vec{x}) \rightarrow A_\mu^0(t, \lambda \vec{x}),
\]
(17)
\[
A_\mu^i(t, \vec{x}) \rightarrow \lambda A_\mu^i(t, \lambda \vec{x})
\]
(18)
with a real parameter \(\lambda\). The action on this transformed solution takes the form
\[
S = \lambda^{2\gamma-3} \Pi_0 - \lambda^{2\gamma-1} \Pi_1 - \lambda^{-3} \int_0^T dt \int d^3x V \left( \lambda^{2\gamma} \phi^\dagger(t, \lambda \vec{x}) \phi(t, \lambda \vec{x}) \right) + \lambda^{-1} \Pi_{A0} - \lambda \Pi_{A1}.
\]
(19)
Since we suppose that \(\phi(t, \vec{x}), A_\mu(t, \vec{x})\) is a solution to the equations of motion, the variation of the action on this solution should vanish for any variations of the fields. For the case of the transformations described by (16)-(18) it means that
\[
\frac{dS}{d\lambda}|_{\lambda=1} = (2\gamma - 3) \Pi_0 - (2\gamma - 1) \Pi_1 - \int_0^T dt \int d^3x \left( 2\gamma \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi - 3V(\phi^\dagger \phi) \right) - \Pi_{A0} - \Pi_{A1} = 0.
\]
(20)
Indeed, \(\lambda = 1\) in (16)-(18) corresponds to the solution.

Now we are ready to consider the consequences following from equation (20).

1. \(\frac{1}{2} < \gamma < \frac{3}{2}\). If
\[
2\gamma \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi - 3V(\phi^\dagger \phi) \geq 0
\]
(21)
for any \(\phi\), then \(\Pi_0 = \Pi_1 = \Pi_{A0} = \Pi_{A1} \equiv 0\) (2\gamma \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} - 3V(\phi^\dagger \phi) = 0\) also, in this case \(F_{\mu \nu}^a \equiv 0\) (this equality means that \(A_\mu\) is a pure gauge and we can set \(A_\mu \equiv 0\)). From \(\Pi_0 = \Pi_1 \equiv 0\) with \(A_\mu \equiv 0\) and according to (12) we get \(\phi \equiv 0\). Thus, solitons of form (12), (13) are absent in the theory.

2. \(\gamma = \frac{3}{2}\). If
\[
\frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi - V(\phi^\dagger \phi) \geq 0
\]
(22)
for any \(\phi\), then \(\Pi_1 = \Pi_{A0} = \Pi_{A1} \equiv 0\), in this case \(A_\mu \equiv 0\), \(\phi = \phi(t) \equiv 0\) (see (12)) and solitons of form (12), (13) are also absent in the theory.

3. \(\gamma = \frac{1}{2}\). If
\[
\frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi - 3V(\phi^\dagger \phi) \geq 0
\]
(23)
for any \(\phi\), then \(\Pi_0 = \Pi_{A0} = \Pi_{A1} \equiv 0\), in this case \(A_\mu \equiv 0\), \(\phi = \phi(\vec{x})\). To obtain restrictions for this case, we can use the results of the well-known Derrick theorem [2], which states that in the scalar field theories described by the standard action static solitons are absent if \(V(\phi^\dagger \phi) \geq 0\). Thus, solitons of form (12), (13) are absent in the theory if
\[
\frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi \geq 3V(\phi^\dagger \phi) \geq 0
\]
(24)
for any \(\phi \neq 0\). But the latter inequality is more stringent than that following from (22) for \(V(\phi^\dagger \phi) \geq 0\) (i.e. (24) covers a narrower range of potentials).
End of the proof.

**Remark:** When considering static configuration of fields we can also take the following transformations of the fields:

\[
\phi(t, \vec{x}) = \phi(\vec{x}) \rightarrow \lambda^\gamma \phi(\vec{x}),
\]

\[
A_0^a(t, \vec{x}) = A_0^a(\vec{x}) \rightarrow \lambda^\beta A_0^a(\vec{x}),
\]

\[
A_i^a(t, \vec{x}) = A_i^a(\vec{x}) \rightarrow A_i^a(\vec{x})
\]

with \(\gamma > 0, \beta < -\gamma\). Then we get

\[
\frac{dS^\phi}{d\lambda} \bigg|_{\lambda=1} = 2(\gamma + \beta)\Pi_0 - 2\gamma \left[ \Pi_1 + \int_0^T dt \int d^3x \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi \right] + 2\beta \Pi_{A0} = 0.
\]

Thus if

\[
\frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi \geq 0,
\]

then \(\Pi_0 = \Pi_1 = \Pi_{A0} \equiv 0\). Relation \(\Pi_{A0} \equiv 0\) implies \(A_0 \equiv 0\), from \(\Pi_1 \equiv 0\) it follows that \(D_i \phi \equiv 0\) and thus \(\partial_i (\phi^\dagger \phi) \equiv 0\), which implies \(\phi \equiv 0\) (see (12)). Then it is very easy to show that \(A_i \equiv 0\), for example, applying (18) to the action containing \(A_i\) only. Thus, we get the absence of static solitons if condition (29) is fulfilled. This restriction was previously obtained in [19] (see also [20]).

**Corollary 1**

1. For \(V(\phi^\dagger \phi) \geq 0\), solitons of form (12), (13) are absent if

\[
\frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi - V(\phi^\dagger \phi) \geq 0.
\]

2. For \(V(\phi^\dagger \phi) \leq 0\), solitons (even unstable) of form (12), (13) are absent if

\[
\frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi - 3V(\phi^\dagger \phi) > 0.
\]

3. The restrictions presented above are valid for the models with the scalar field only, i.e. if we drop the gauge field from the theory.

4. Non-topological solitons satisfying the conditions presented above are absent in the pure Yang-Mills theory.

The proof of this corollary follows directly from Theorem 1.

**Remark:** for the first time the no-go condition (30) for \(V(\phi^\dagger \phi) \geq 0\) and the potentials \(V(\phi^\dagger \phi)\) including a positive mass term was obtained in a different way in [11] (for the case of a single complex scalar field coupled to the electromagnetic field this condition was obtained in [21]). The absence of solitons ("classical lumps") in the pure Yang-Mills theory was shown long time ago for the static [12], periodic [13] and general cases [14, 15, 16, 17, 18] (the corresponding conjecture was discussed earlier, see, for example [22] and references therein).
It should be noted that the results presented above can be obtained by taking the equations of motion following from action (11), multiplying them by variations of fields following from (16)-(18) (or from (25)-(27), integrating over the four-volume and combining the results coming from different equations of motion (as an example see [3], where such an alternative way was presented). Indeed, the variational principle gives us the equations of motion and thus all the results obtained above can be also obtained with the help of the equations of motion. In this sense the proof presented in formulas (25)-(29) does not differ from that of [19, 20]. Moreover, as it was noted in [8], in principle one can get an infinite number of integral identities using the equations of motion (or local identities such as the conservation of the energy-momentum tensor which follow from the equations of motion). But it seems that considering the direct transformations of the action under rescalings of the fields is technically simpler at least for obtaining no-go results.

Now let us consider several examples. First, let us take the simplest polynomial potential of the form \( V(\phi^\dagger \phi) = q(\phi^\dagger \phi)^n \) with \( q > 0 \) and \( n \geq 1 \). From (30) (which is nothing but (14) with \( \gamma = \frac{3}{2} \)) we get \((n - 1)(\phi^\dagger \phi)^n \geq 0 \) and thus such a potential does not provide the existence of a non-topological soliton solution satisfying (12) and (13). Second, let us take a more complicated potential of the form \( V(\phi^\dagger \phi) = q_1(\phi^\dagger \phi)^2 - q_2 \phi^\dagger \phi \) with \( q_1 > 0, q_2 > 0 \) (this is the Higgs potential up to a constant). Substituting this potential into (31) (of course, we can use (31) not only for non-negative potentials) we get \( q_1(\phi^\dagger \phi)^2 \geq 0 \) and thus this potential does not provide the existence of a non-topological soliton solution satisfying (12) and (13). (We can also take (14) with, for example, \( \gamma = 1 \) and get \( q_1(\phi^\dagger \phi)^2 + q_2 \phi^\dagger \phi \geq 0 \), which leads to the same conclusion).

Meanwhile, for the "opposite" potential \( V(\phi^\dagger \phi) = q_1 \phi^\dagger \phi - q_2(\phi^\dagger \phi)^2 \) with \( q_1 > 0, q_2 > 0 \) there is no \( \gamma : \frac{1}{2} < \gamma \leq \frac{3}{2} \), for which (14) holds for any \( \phi \) and thus there is no restriction for the existence of solitons. Moreover, at least in the absence of the gauge field a soliton is shown to exist in this case [23]. The same is valid for the potential \( V(\phi^\dagger \phi) = -q_1 \phi^\dagger \phi \ln(q_2 \phi^\dagger \phi) \) with \( q_1 > 0, q_2 > 0 \) there is no \( \gamma : \frac{1}{2} < \gamma \leq \frac{3}{2} \), for which (14) holds for any \( \phi \), and again at least in the absence of the gauge field a soliton is shown to exist in this case [24]. And finally, let us take a potential of the form \( V(\phi^\dagger \phi) = q_1(\phi^\dagger \phi)^4 - q_2(\phi^\dagger \phi)^2 \) with \( q_1 > 0, q_2 > 0 \). Substituting this potential into (14) and taking \( \gamma = \frac{3}{4} \) we get \( q_1(\phi^\dagger \phi)^4 \geq 0 \) and thus this potential does not provide the existence of a non-topological soliton solution satisfying (12) and (13). It should be noted that the cases \( V(\phi^\dagger \phi) = q_1(\phi^\dagger \phi)^2 - q_2 \phi^\dagger \phi \) and \( V(\phi^\dagger \phi) = q_1(\phi^\dagger \phi)^4 - q_2(\phi^\dagger \phi)^2 \) with \( q_1 > 0, q_2 > 0 \) can be considered only as illustrative examples. Indeed, although \( \phi \equiv 0, A_\mu \equiv 0 \) is a solution to the corresponding equations of motion and in principle it can be considered as the "vacuum" from the mathematical point of view, \( \phi = 0 \) is the local maximum of the scalar field potential and thus such a configuration is unstable and can not be considered as the vacuum from the physical point of view. Even if there were non-topological solitons in these cases, obviously they would be unstable.

At the end of this section let us discuss why we restrict ourselves to considering solutions periodic in time up to translations and spatial rotations. Of course, it is reasonable to examine such a type of solutions when considering solitons. But there is another reason, which concerns the technical side of the method used above. When we obtain equations of motions we suppose that the variations of fields vanish at spatial and time infinity. But the variations of the fields coming from (16)-(18) clearly do not tend to zero at time infinity in the general case and we have a contradiction between the form of the variations used to obtain equations of motion and the form of the variations used to show the absence of some solutions to these equations of motion (it should be noted that there is another problem, which can occur when considering the full action with \( t \in (-\infty, \infty) \) – the method described above can not be used for infinite
integrals; for periodic solutions we can take the effective action of form (6), which is finite by construction. In principle, in the case of non-vanishing variations there can arise additional terms in the action coming from the surface terms at time infinity (an analogous problem, but in the case of the Derrick theorem [2] applied to a finite space-time domain, was discussed in [25]). Considering solutions periodic in time up to a spatial rotation and a coordinate shift makes the surface terms at $t=0$ and $t=T$, which arise when obtaining equations of motion from effective action (7) with the help of variations of fields following from (16)-(18), be modulo equal and cancel each other (note, that if, from the very beginning, one wants to consider solitons which are not at rest, it is necessary to use more complicated transformations, which can be obtained from (16)-(18) with the help of the corresponding Lorentz transformations, and, of course, the effective action with $\tilde{T}$ instead of $T$, to get the canceling surface terms and the same no-go result). The latter ensures that the equations of motion obtained from the initial action and those obtained from the effective action coincide, as well as possible solutions. In this case the procedure of obtaining the no-go results described in this section appears to be consistent with the equations of motion coming from the original action.

3 Charged massive vector field

The scaling arguments, used above for the theory describing Yang-Mills field coupled to a non-linear scalar field, can be used to show the absence of solitons in other gauge field theories. As an illustrative example, let us consider the massive complex vector field coupled to the electromagnetic field. The action of this theory has the form

$$S = \int d^4x \left[ -\frac{1}{2} \eta^{\mu\rho} \eta^{\nu\sigma} W_\mu^- W_\nu^+ + m^2 \eta^{\mu\nu} W_\mu^- W_\nu^+ - \frac{1}{4} F_{\mu\nu}^+ F_{\mu\nu}^- \right]$$  \hspace{1cm} (32)

with $m \neq 0$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \hspace{1cm} (33)$$

$$D_\mu W_\nu^\pm = \partial_\mu W_\nu^\pm \mp ieA_\nu W_\nu^\pm, \hspace{1cm} (34)$$

$$W_\mu^\pm = D_\mu W_\nu^\pm - D_\nu W_\mu^\pm. \hspace{1cm} (35)$$

Again we suppose that:

1. there are no sources which are external to the system described by action (32);
2. all fields are smooth and vanish at spatial infinity;
3. solutions to equations of motion are periodic in time up to a spatial rotation and a coordinate shift (see previous section for details).

Let us denote (again in the coordinate system where $\vec{l} = 0$)

$$\int_0^T dt \int d^3x W_{0i}^- W_{0i}^+ = \Pi W_0 \geq 0, \hspace{1cm} (36)$$

$$\int_0^T dt \int d^3x W_{ij}^- W_{ij}^+ = \Pi W_1 \geq 0. \hspace{1cm} (37)$$

\footnote{For example, in the simple case $\vec{l} = (l, 0, 0)$ for the scalar field one should take transformation of the form

$$\phi(t, x, y, z) \rightarrow \lambda^{v} \phi \left( \frac{1}{1-v^2} (l-v)^2 \right) \left( \lambda-1 \right) \phi, \hspace{1cm} \left( \lambda-1 \right) \phi, \hspace{1cm} (36)$$

with $v = l/T < 1$ instead of (16). The variation of the field, following from such a transformation, possesses the same periodicity properties as the field itself.}
\[
m^2 \int_0^T dt \int d^3 x W_0 W_0^+ = V_0 \geq 0, \tag{38}
\]

\[
m^2 \int_0^T dt \int d^3 x W_i W_i^+ = V_1 \geq 0, \tag{39}
\]

\[
\int_0^T dt \int d^3 x \frac{1}{2} F_{0i} F_{0i} = \Pi_{A0} \geq 0, \tag{40}
\]

\[
\int_0^T dt \int d^3 x \frac{1}{4} F_{ij} F_{ij} = \Pi_{A1} \geq 0. \tag{41}
\]

We also suppose that all these integrals are finite.

Analogously to what was made above, we also suppose that there exists a solution to the corresponding equations of motion and consider the following transformation of this solution:

\[
W_0^\pm(t, \vec{x}) \rightarrow \lambda^{\beta-1} W_0^\pm(t, \lambda \vec{x}), \tag{42}
\]

\[
W_i^\pm(t, \vec{x}) \rightarrow \lambda^{\beta} W_i^\pm(t, \lambda \vec{x}), \tag{43}
\]

\[
A_0^\mu(t, \vec{x}) \rightarrow A_0^\mu(t, \lambda \vec{x}), \tag{44}
\]

\[
A_i^\mu(t, \vec{x}) \rightarrow \lambda A_i^\mu(t, \lambda \vec{x}), \tag{45}
\]

with a real parameter \( \lambda \). For the action on the transformed fields we get

\[
S = \lambda^{2\beta-3} \Pi_{W0} - \lambda^{2\beta-1} \Pi_{W1} + \lambda^{2\beta-5} V_0 - \lambda^{2\beta-3} V_1 + \lambda^{-1} \Pi_{A0} - \lambda \Pi_{A1}. \tag{46}
\]

Now let us take

\[
\beta = \frac{3}{2}. \tag{47}
\]

We get

\[
\frac{dS}{d\lambda}\bigg|_{\lambda=1} = -2 \Pi_{W1} - 2 V_0 - \Pi_{A0} - \Pi_{A1} = 0. \tag{48}
\]

From this equation it follows that

\[
\Pi_{W1} = V_0 = \Pi_{A0} = \Pi_{A1} \equiv 0, \tag{49}
\]

which means that \( F_{\mu\nu} \equiv 0 \) and we can set \( A_\mu \equiv 0 \), we also get \( W_0^\pm \equiv 0 \) and \( W_{ij}^\pm \equiv 0 \). With \( A_\mu \equiv 0 \) we can rewrite \( W_{ij}^\pm \equiv 0 \) as

\[
\partial_i W_j^\pm - \partial_j W_i^\pm \equiv 0. \tag{50}
\]

Now let us take the equations of motion for the field \( W_{i}^\pm \) with \( A_\mu \equiv 0 \). It follows from these equations that

\[
\partial^\mu W_{i}^\pm = 0. \tag{51}
\]

Using the fact that \( W_0^\pm \equiv 0 \) we get

\[
\partial^i W_i^\pm = 0. \tag{52}
\]

Equations (50) and (52) can be rewritten as

\[
\text{div} \, W^\pm = 0, \quad \text{rot} \, W^\pm = 0, \tag{53}
\]

\[
\text{div} \, \vec{F} = 0, \quad \text{rot} \, \vec{F} = 0,
\]

\[
\text{div} \, \vec{A} = 0, \quad \text{rot} \, \vec{A} = 0.
\]
where $\vec{W}^\pm = (W_1^\pm, W_2^\pm, W_3^\pm)$. Equations \((53)\) imply that
\[
\vec{W}^\pm = \nabla \varphi^\pm,
\]
\[
\Delta \varphi^\pm = 0,
\]
where $\varphi^\pm = \varphi^\pm(t, \vec{x})$, $(\varphi^\pm)^* = \varphi^-$. The condition $\int d^3x W_i^+ - W_i^- = \int d^3x \partial_i \varphi^- \partial_i \varphi^+ < \infty$ clearly leads to $\varphi^\pm = \varphi^\pm(t)$ (one can simply use the results of \([1, 2]\) to show it) and therefore
\[
\vec{W}^\pm \equiv 0.
\]
Thus
\[
W_\mu^\pm \equiv 0.
\]
Finally, we can make the following statement:

Proposition 1 Non-topological solitons, periodic in time up to a spatial rotation and a coordinate shift such that $\Omega \vec{l} = \vec{l}$, $|\vec{l}|/\tilde{T} < 1$ and with finite integrals \((36)-(41)\), are absent in the theory described by action \((32)\).

Acknowledgements

The author is grateful to I.P. Volobuev for valuable discussions and to an unknown referee of Journal of Mathematical Physics for suggesting to consider rotations in addition to translations. The work was supported by FASI state contract 02.740.11.0244 and by grant of Russian Ministry of Education and Science NS-4142.2010.2.

References

[1] R.H. Hobart, Proc. Phys. Soc. 82, 201 (1963).
[2] G.H. Derrick, J. Math. Phys. 5, 1252 (1964).
[3] G. Rosen, J. Math. Phys. 7, 2071 (1966).
[4] R. Rajaraman, Solitons and instantons: an introduction to solitons and instantons in quantum field theory (North-Holland), (1982).
[5] V. Rubakov, Classical theory of gauge fields (Princeton: Princeton University press), (2002).
[6] S. Dimopoulos, T. Eguchi, Phys. Lett. B 66, 480 (1977).
[7] N.V. Krasnikov, Phys. Lett. B 72, 455 (1978).
[8] N.S. Manton, J. Math. Phys. 50, 032901 (2009).
[9] V.G. Makhankov, Phys. Lett. A 61, 431 (1977).
[10] M.N. Smolyakov, J. Phys. A: Math. Gen. 43, 455202 (2010).
[11] R.T. Glassey, W.A. Strauss, Commun. Math. Phys. 67, 51 (1979).
[12] S. Deser, *Phys. Lett. B* 64, 463 (1976).

[13] H. Pagels, *Phys. Lett. B* 68, 466 (1977).

[14] S.R. Coleman, *Commun. Math. Phys.* 55, 113 (1977).

[15] S.R. Coleman, L. Smarr, *Commun. Math. Phys.* 56, 1 (1977).

[16] R. Weder, *Commun. Math. Phys.* 57, 161 (1977).

[17] M. Magg, *J. Math. Phys.* 19, 991 (1978).

[18] R.T. Glassey, W.A. Strauss, *Commun. Math. Phys.* 65, 1 (1979).

[19] E. Malec, *Acta Phys. Polon. B* 18, 1017 (1987).

[20] E. Malec, *Acta Phys. Polon. B* 14, 581 (1983).

[21] G. Rosen, *J. Math. Phys.* 9, 999 (1968).

[22] G. Rosen, *J. Math. Phys.* 13, 595 (1972).

[23] D.L.T. Anderson, G.H. Derrick, *J. Math. Phys.* 11, 1336 (1970).

[24] G. Rosen, *Phys. Rev.* 183, 1186 (1969).

[25] A.B. Adib, *Nonexistence of static scalar field configurations in finite systems*, arXiv:hep-th/0208168.