Proto-jet configurations in RADs orbiting a Kerr SMBH: symmetries and limiting surfaces

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Abstract
Ringed accretion disks (RADs) are agglomerations of perfect-fluid tori orbiting around a single central attractor that could arise during complex matter inflows in active galactic nuclei. We focus our analysis to axi-symmetric accretion tori orbiting in the equatorial plane of a supermassive Kerr black hole; equilibrium configurations, possible instabilities, and evolutionary sequences of RADs were discussed in our previous works. In the present work we discuss special instabilities related to open equipotential surfaces governing the material funnels emerging at various regions of the RADs, being located between two or more individual toroidal configurations of the agglomerate. These open structures could be associated to proto-jets. Boundary limiting surfaces are highlighted, connecting the emergency of the jet-like instabilities with the black hole dimensionless spin. These instabilities are observationally significant for active galactic nuclei, being related to outflows of matter in jets emerging from more than one torus of RADs orbiting around supermassive black holes.

Keywords: black hole physics, jets, gravitation, hydrodynamics, accretion, accretion disks, galaxies: active, galaxies: jets

(Some figures may appear in colour only in the online journal)

1. Introduction
Spacetime symmetries play a dominant role in Astrophysics. The case of accretion disks, and especially the constrained axi-symmetric (coplanar) tori orbiting a central Kerr supermassive black hole (SMBH) represents an example where both stable and unstable configurations can be predominantly determined by the geometric properties of the background.
More generally the physics of accretion disks is regulated by the balance of many factors influencing their evolution and morphology; magnetic fields, thermal or viscous processes for example play a combined and different role, from the phase of disk formation to accretion, eventually characterizing different accretion disk models. Nevertheless it is possible to trace a correlation between accretion disk models, determined by this special balance and the disk attractor itself. In the Polish-doughnut (P-D) thick accretion disk model [1], the role played by the curvature effects and spacetime symmetries is essential in setting constraints on axi-symmetric accretion tori and in these situations, the geometric properties of the spacetime become relevant with respect to other ingredients in the determination of the disk forces balance. In here we consider ringed accretion disks (RADs), introduced in [2] and detailed in [3–6], featuring the formation and interaction of several, both corotating and counterrotating, axi-symmetric (coplanar) tori orbiting a central Kerr SMBH.

These structures may form around SMBH in active galactic nuclei (AGNs), where the attractor, interacting with its environment during its life-time can give rise to different accretion periods. The case of a single orbiting torus stands as limiting case of the RAD.

The spacetime symmetries determined by the central attractor strongly constrain the configurations existence and stability. Properties of each torus are determined by an effective potential function enclosing the background Kerr geometry and the centrifugal effects. Equipotential surfaces, being also equipressure surfaces, are associated with critical points identifying the toroidal surfaces of the disk. The cusped surfaces are the critical topologies associated to the torus unstable phases. The outflow of matter through the cusp occurs by the Paczynski mechanism of violation of mechanical equilibrium of the tori [7], i.e. an instability in the balance of the gravitational and inertial forces and the pressure gradients in the fluid [1]. The instabilities inside the ringed accretion disk may give rise to accretion into the attractor, whereas the open equipotential surfaces have been related to ‘proto jet-shell’ structures, a general discussion of these special configurations can be found for example in [8–19]. The interaction between several launching points of jets, or other topologies of the decompositions, leads to collision in a couple, inducing a second instability phase with accretion or a further proto-jet formation, finally a phase, or a cycle, of a ‘drying-feeding’ process between two RADs sub-configurations. In this work we concentrate our attention on these open configurations and their constraints.

From phenomenological viewpoint, the dynamics of the unstable phases of this system is significant for the high energy phenomena related to accretion onto super-massive black holes in AGNs, and the extremely energetic phenomena in quasars which could be observable in their x-ray emission, as the x-ray obscuration and absorption by one of the RAD ring. The radially oscillating tori of the ringed disk could be related to the high-frequency quasi periodic oscillations observed in non-thermal x-ray emission from compact objects (QPOs), a still obscure feature of the x-ray astronomy related to the inner parts of the disk. BH accretion rings models may be revealed by future x-ray spectroscopy, from the study of relatively indistinct excesses on top of the relativistically broadened spectral line profile [20–22]. The predicted relatively indistinct excesses of the relativistically broadened emission-line components, shall arise in a well-confined radial distance in the accretion disk, envisaging therefore a sort of rings model which may be adapted as a special case of the model discussed in [3, 4].

In section 2 we introduce the RADs model discussing the tori morphology, sections 2.1 and 2.2, while instabilities are considered in section 3. Geometrical correlations of unstable configurations are discussed in section 3.1. Section 3.2 is devoted to the parameter setting and discussion on the system rotational symmetries, in this section we provide also exact form of fluid specific angular momentum ĝ introduced in [4] and also considered in [6]. Analysis of the open configurations is in section 4, while in section 4.1 restrictions on the existence of the
open configurations are investigated introducing some limiting surfaces. Concluding remarks follow in section 5.

2. Tori in the Kerr spacetime

We start by presenting the Kerr metric tensor in the Boyer–Lindquist (BL) coordinates, \( \{t, r, \theta, \phi\} \)

\[
d^s = -dt^2 + \frac{L^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (\rho^2 + a^2) \sin^2 \theta d\phi^2 + \frac{2M}{\rho^2} r (dt - a \sin^2 \theta d\phi)^2,
\]

where \( \rho^2 \equiv r^2 + a^2 \cos^2 \theta \) and \( \Delta \equiv r^2 - 2Mr + a^2 \).

\( a = J/M \in [0, 1] \) is the specific angular momentum, \( M \) a mass parameter and \( J \) is the total angular momentum of the gravitational source. The non-rotating limiting case \( a = 0 \) is the Schwarzschild metric while the extreme Kerr black hole has dimensionless spin \( a/M = 1 \).

Radii

\[
r_h \equiv M + \sqrt{M^2 - a^2}; \quad r_c \equiv M - \sqrt{M^2 - a^2}
\]

are the horizons \( r_c < r_h \) and the outer static limit \( r_c^+ \) respectively, being \( r_h < r_c^+ \) on \( \theta \neq 0 \) and \( r_c^+ = 2M \) in the equatorial plane \( \theta = \pi/2 \).

Ringed accretion disks are toroidal configurations of perfect fluids orbiting a central Kerr black hole (BH) attractor. Quantities

\[
E \equiv -g_{\alpha\beta} r^\alpha p^\beta, \quad L \equiv g_{\alpha\beta} \xi_\alpha p^\beta,
\]

are constants of motion, where the covariant components \( p_\alpha \) and \( p_\beta \) of a particle four-momentum are conserved along the geodesics. The constant \( L \) in equation (4) may be interpreted as the axial component of the angular momentum of a particle for timelike geodesics and \( E \) as representing the total energy of the test particle coming from radial infinity, as measured by a static observer at infinity. This is a property derived from the presence of the two Killing vector fields \( \xi_\phi = \partial_\phi \), rotational Killing field, and \( \xi_\theta = \partial_\theta \), which is the Killing field representing the stationarity of the spacetime. In the test particle circular motion one can limit the analysis to the case of positive values of \( a \) for corotating \( (L > 0) \) and counterrotating \( (L < 0) \) orbits with respect to the black hole. In fact the metric tensor (1) is invariant under the application of any two different transformations: \( x^\alpha \rightarrow -x^\alpha \) where \( x^\alpha = (t, \phi) \), thus the metric is invariant for exchange of couple \( (t, \phi) \rightarrow (-t, -\phi) \), or \( (a, t) \rightarrow (-a, -t) \), or after a change \( (a, \phi) \rightarrow (-a, -\phi) \), consequently, the test particle dynamics is invariant under the mutual transformation of the parameters \( (a, L) \rightarrow (-a, -L) \).

It will be important to consider in the analysis of the ringed disks the notable radii \( r_{\text{K}}^\pm \equiv \{r_{\gamma}^\pm, r_{\text{sho}}^\pm, r_{\text{mao}}^\pm\} \), defining the geodesic structure of the Kerr spacetime with respect to

\footnote{We adopt the geometrical units \( c = 1 = 1 \) and the \((-++,++)\) signature, Greek indices run in \( \{0, 1, 2, 3\} \). The four-velocity satisfy \( u^a u_a = -1 \). The radius \( r \) has unit of mass \( [M] \), and the angular momentum units of \( [M^2] \), the velocities \( [u] = [u] = 1 \) and \( [r^\alpha] = [a^\alpha] = [M]^{-1} \) with \( [r^\alpha u^\alpha] = [M]^{-1} \) and \( [a^\alpha u^\alpha] = [M] \). For the sake of convenience, we always consider the dimensionless energy and effective potential \( |V_{\text{eff}}| = 1 \) and an angular momentum per unit of mass \( |L|[M] = |M| \).}
the matter distribution. Specifically, for timelike particle orbits, \( r^+_{\gamma} \) is the marginally circular orbit or the photon circular orbit, timelike circular orbits can fill the spacetime region \( r > r^+_{\gamma} \). The marginally stable circular orbit \( r^+_{\min} \): stable orbits are in \( r > r^+_{\min} \) for counterrotating and corotating particles respectively. The marginally bounded circular orbit is \( r^\pm_{\mbo} \), where \( E^\pm (r^\pm_{\mbo}) = 1 \) [23–29]. The effective potential, regulating the motion of test particle circular geodesics can admit, as function of \( rM \), minimum points correspondent to particle stable circular orbits only in the region \( r > r^\pm_{\mbo} \), respectively for counterrotating and corotating motion with respect to the central Kerr black hole; the effective potential admits the maximum points, correspondent to particle unstable circular orbits, only in the region \( r \in [r^\pm_{\mbo}, r^\pm_{\mbo}] \). The energy \( E \) of the particle, as measured by an observer at infinity, will be greater that the value at infinity \( E = \mu \), for the particle of mass \( \mu \) on orbits in \( r \in [r^\pm_{\gamma}, r^\pm_{\mbo}] \), and the energy will be lower then limiting \( E = \mu \), in the region \( r \in [r^\pm_{\mbo}, r^\pm_{\mbo}] \).

The geodesic structure represents a geometric property of the spacetime consisting of the union of the orbital regions with boundaries in \( rN \) [4, 5, 30]. It can be decomposed, for \( a \neq 0 \), into the geodesic structures for corotating \( (r_N^+ \) ) and counterrotating \( (r_N^- \) ) matter according to the convention adopted here and in [3]. Given \( r_i \in r_N^\pm \), we adopt the notation for any function \( \mathbf{Q}(r) : \mathbf{Q}_i \equiv \mathbf{Q}(r_i) \), therefore for example \( \ell^+_\mbo \equiv \ell_+(r^+_\mbo) \), and more generally given the radius \( r_i \), and the function \( \mathbf{Q}(r) \), there is \( \mathbf{Q}_i \equiv \mathbf{Q}(r_i) \).

For the symmetries of the problem, we assume \( \partial_r \mathbf{Q} = 0 \) and \( \partial_r \mathbf{Q} = 0, \) with \( \mathbf{Q} \) being a generic spacetime tensor [2, 31], and a one-species-particle perfect fluid system described by the energy momentum tensor

\[
T_{\alpha\beta} = (\varrho + p)u_\alpha u_\beta + pg_{\alpha\beta},
\]

where \( \varrho \) and \( p \) are the total energy density and pressure, respectively, as measured by an observer comoving with the fluid whose four-velocity \( u^\alpha \) is a timelike flow vector field. Continuity equation and the Euler equation are respectively:

\[
\begin{align*}
\varrho u^\alpha \nabla_\alpha \varrho + (p + \varrho)\nabla^\alpha u_\alpha &= 0, \\
(p + \varrho)u^\alpha \nabla_\alpha u^\gamma + h^{\beta\gamma} \nabla_\beta p &= 0, \\
h_{\alpha\beta} &= g_{\alpha\beta} + u_\alpha u_\beta, \\
\nabla^\alpha g_{\beta\gamma} &= 0.
\end{align*}
\]

We consider the fluid toroidal configurations (with \( u^\theta = 0 \)) centered on the plane \( \theta = \pi/2 \), and defined by the constraint \( u^r = 0 \). Assuming a barotropic equation of state \( p = p(\varrho) \), the Euler equation (6) provides the following equation

\[
\frac{\partial_\ell p}{\varrho + p} = \partial_r W + \frac{Q \partial_\mu \ell}{1 - \Omega \ell}, \quad \ell \equiv \frac{L}{E} \quad W \equiv \ln V_{\ell}(\ell)
\]

\[
V_{\ell}(\ell) = u_t = \pm \sqrt{\frac{2_\phi^2 - 8a_\phi}{2g^2_{\phi\phi} + 2fg_{\phi\phi} + k^2 g_{\phi}}}, \quad \text{where on} \quad \theta = \pi/2
\]

\[
V_{\ell}(\ell) = \frac{\Delta \mu^2}{a^2 + \Delta \ell^2 + 2r^2 + r(r^2 - 4aM) - \Delta (a^2 + \ell^2)}
\]

(note that the effective potential \( V_{\ell}(\ell) \) is dimensionless). This potential is regulated by the radial profile of the geodesic specific angular momentum.
\[ \ell = \frac{a^3 M + a M r (3r - 4M) \pm \sqrt{Mr^3 \left[a^2 + (r - 2M)r\right]^2}}{[M^2 a^2 - (r - 2M)^2 r]M} \]  

(9)

while the continuity equation in equation (6) is identically satisfied as consequence of the applied conditions and symmetries. We note that equation (7) is a rearrangement of the Euler equation in equation (6). We singled out the variation of the specific angular momentum (\(\ell\)) from the first term in r.h.s of equation (7), defining the effective potential and enclosing the information on the gravitational component of the torus forces balance. A relevant aspect of equation (7) is the shape of the toroidal configurations that can be found as the associated exact integrals. This is possible due to symmetries of the system and the assumption of a barotropic equation of state \(p = p(\rho)\). In this model, the fluid flow is not iso-entropic, the entropy \(S\) is not constant in space and time but it is constant along the fluid flow, i.e. \(u^a \nabla_a S = 0\). In the \(V_{\text{eff}}\) definition, the normalization condition for the fluid four-velocity, \(u^a u_a = -1\), has been taken into account, together with the definition of the parameterized specific angular momentum \(\ell\). Assuming \(\ell = \text{constant}\), the second term of r.h.s of equation (7) vanishes. This special re-writing of the forces balance allows also identification of the critical points of the pressure. Both these aspects make the model extremely effective and versatile. On the other hand, there are several possible generalizations of this set-up: 1. the first we mention here concerns the inclusion of other components in the balance of forces. The perfect fluid energy momentum tensor (5) can include for example a magnetic field component [32, 33]. 2. Secondly, we might consider the influence of the last term of equation (7) by adopting a varying fluid angular momentum \(\ell(r)\) as, for example, in [34, 35].

Notice that we singled out definition of specific angular momentum of the fluid \(\ell \equiv E/L\) in terms of the functions \(E\) and \(L\), introduced in equation (4); these quantities are constants of motion for test particle circular orbits, according to the presence of the two Killing fields \(\xi_t\) and \(\xi_\phi\), and in the test particle scenario in the BL coordinate frame they have an immediate interpretation as quantities measured by a static observer at infinity. Nevertheless, for orbiting extended matter configurations, \((E, L)\) are not (generally) constants of motion and an interpretation of these quantities has to be properly given. Concerning the choice of \(\ell = \text{constant}\), it is clear that in equation (5) we considered a simple fluid, i.e. made up by one species particles, which is subjected to pressure forces according to a barotropic equation of state. Other forces may have to be included as the magnetic or electric components in the energy momentum tensor. For the detailed discussion on relation between \(L, \ell\) and \(E\) in the tori construction see for example [2].

We have therefore introduced the effective potential function \(V_{\text{eff}}(\ell)\) for the fluid which reflects the contribution of the background Kerr geometry and the centrifugal effects. \(\Omega\) is the relativistic angular frequency of the orbiting fluid relative to the distant observer. The specific angular momentum \(\ell\) is considered here constant and conserved (see also [34, 36]).

As for the case of the test particle dynamics, due to the problem symmetries we can limit the analysis to positive values of \(a > 0\), for corotating \((\ell > 0)\) and counterrotating \((\ell < 0)\) fluids and we adopt the notation \((\pm)\) for counterrotating or corotating matter respectively (as \(V_{\text{eff}}(\ell)\) in equation (7) is invariant under the mutual transformation of the parameters \((a, \ell) \rightarrow (-a, -\ell)\).

2.1. Ringed accretion disks

We consider a fully general relativistic model of ringed accretion disk (RADs) made by several corotating and counterrotating toroidal rings orbiting a supermassive Kerr attractor [3].
General relativity hydrodynamic Boyer condition of equilibrium configurations of rotating perfect fluids governs the single torus. As a consequence of this many properties of the orbiting tori are determined by the effective potential. The toroidal surfaces are the equipotential surfaces of the effective potential (and equipressure surfaces) $V_{\text{eff}}(\ell)$, considered as function of $r$, solutions of $V_{\text{eff}} = K = \text{constant}$ or $\ln(V_{\text{eff}}) = c = \text{constant}$ [37]. These correspond also to the surfaces of constant density, specific angular momentum $\ell$, and constant relativistic angular frequency $\Omega$, where $\Omega = \Omega(\ell)$ as a consequence of the von Zeipel theorem\(^2\) [32, 38].

Each Boyer surface turns to be identified by the couple of parameters $(\ell, K)$.

### 2.2. RADs morphology

A torus in the RAD agglomerate can be corotating, $\ell a > 0$, or counterrotating, $\ell a < 0$, with respect to the central black hole rotation $a > 0$. Consequently, given a couple $(C_a, C_b)$ with specific angular momentum $(\ell_a, \ell_b)$, orbiting in the equatorial plane of a central Kerr SMBH, we can introduce the concept of corotating tori, defined by the condition $\ell_a \ell_b > 0$, and the counterrotating tori defined by the relations $\ell_a \ell_b < 0$. In other words, a couple of corotating tori can be both corotating $\ell a > 0$ or counterrotating $\ell a < 0$ with respect to the central attractor\(^1\) On the other hand, corotating couples are made of a corotating torus and a counterrotating torus, consequently the following two cases can occur: a couple can be composed by an inner corotating torus, with the respect to the BH and an outer counterrotating torus, or viceversa, the inner torus can be counterrotating and the outer one corotating.

Following this setup we focus on the solution of equation (7), $W = \text{constant}$, associated to the critical points, i.e. the extrema of the effective potential as functions of $rM$, thus the minimum and maximum points of the effective potential, with angular momentum and parameter $K$ in the ranges, $(K_0, L_i)$ with $i = \{1, 2, 3\}$, and $(K_j, L_2)$, $j = \{0, 1\}$, represented in figure 2. In fact, only for $K$ and $\ell$ parameters in these ranges, the effective potential function has extreme points. Specifically there are minima for $K \in K_0$ and $\ell \in L_i$, or $K \in K_1$, and $\ell \in L.2$. In section 4 we briefly discuss the solutions of $W = \text{constant}$ which are not associated to the extreme points of the effective potential, therefore we shall consider the other parameter regions of figure 2—see for details [2].

Thus, more specifically we explore the orbital region $\Delta r_{\text{crit}} \equiv [r_{\text{Max}}, r_{\text{Min}}]$, whose boundaries correspond to the maximum and minimum points of the effective potential respectively. The inner edge of the Boyer surface, the torus, must be at $r_{\text{Min}} \in \Delta r_{\text{crit}}$, while the outer edge of the torus is at $r_{\text{Max}} > r_{\text{Min}}$. Then, there is a further matter configuration, which is closest to the central black hole and it is located at $r_{\text{Max}} < r_{\text{Max}}$. The limiting case of $K_{\ell} = K_{\text{Max}}$ corresponds to a one-dimensional ring of matter located in $r_{\text{Min}}$.

The centers, $r_{\text{cent}}$, of the closed configurations $C_{\pm}$, where the hydrostatic pressure is maximum, are located at the minimum points $r_{\text{Min}} > r_{\text{Max}}$ of the effective potential. The toroidal surfaces are characterized by parameters $K_{\pm} \in [K_{\text{Min}}, K_{\text{Max}}] \subset [K_{\text{Min}}, K_{\text{Max}}]$, $i \equiv K_0$ and specific angular momentum $\ell_{\pm} \equiv \ell_{\text{Min}} \leq \ell_{\text{Max}} < 0$ for counterrotating and corotating fluids respectively.

Therefore, the configurations have critical points, corresponding to the minimum and maximum of the hydrostatic pressure, more specifically the maximum points of the effective potential $r_{\text{Max}}$ correspond to minimum points of the hydrostatic pressure and the

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\(^1\) More generally $\Sigma_q$ is the surface $Q = \text{constant}$ for any quantity or set of quantities $Q$. Therefore in this case $\Sigma_{\ell} = \text{constant}$ for $i \in \{p, \rho, \ell, \Omega\}$, where the angular frequency is indeed $\Omega = \Omega(\ell)$ and $\Sigma_i = \Sigma_{\ell}$ for $i \in \{p, \rho, \ell, \Omega\}$.

\(^2\) In the following we will adopt often, when not required to do otherwise, the notation which does not explicit the fluid sign rotation. Then the discussion is intended to be independent from this and each ordering relation must be understood for each corotating sequence.
points of gravitational and hydrostatic instability. An accretion overflow of matter from the closed, cusped configurations in $C^\pm$ (see figures 3 and 4) can occur from the instability point $r^+_I \equiv r_{\text{Max}} \in [r^+_\text{mbo}, r^+_\text{mas}]$ towards the attractor, if $K^\pm_{\text{Max}} \in K^0_\pm$ with proper angular momentum $\ell \in [\ell^+_\text{mas}, \ell^+_\text{mas}] \cup [\ell^-\text{mas}, \ell^-\text{mas}]$ (\textbf{L1} $^+ \cup \textbf{L1}$ $^-$), respectively, for counterrotating or corotating matter. Otherwise, there can be funnels of material, associated to matter jets, along an open configuration $O^\pm$ having $K^\pm_{\text{Max}} \geq 1$ (\textbf{K1} $^\pm$), launched from the point $r^+_J \equiv r_{\text{Max}} \in [r^+_\gamma, r^+_\text{mbo}]$ with proper angular momentum $\ell \in [\ell^+_\gamma, \ell^+_\text{mbo}] \cup [\ell^-\gamma, \ell^-\text{mbo}]$ (\textbf{L2} $^+ \cup \textbf{L2}$ $^-$). We could refer to the surfaces $O^\pm$ as proto-jets, or for brevity jets$^4$.

A second class of solutions are the equilibrium, not-accreting, closed configurations, with topology $C$, defining for $\pm \ell^- > \pm \ell^+_\text{mas}$ and centered in $r > r^+_\text{mas}$ respectively. However there are no maximum points of the effective potential for specific angular momentum $\pm \ell^- > \ell^+_\gamma$

$^4$The role of ‘proto-jet’ configurations, corresponding to limiting topologies for the closed or closed cusped solutions associated with equilibrium or accretion, is still under investigation. More generally, in this model the open surfaces with $\ell \in \textbf{L1}$ have been always associated with the jet emission along the attractor symmetry axis—see for a general discussion [8–11, 14, 15, 18]. Although in section 4 we will briefly discuss the more general open configurations, in this work we mainly analyze the cusped open configurations considered as limiting surfaces associated with the two critical points of the effective potential, the inner one where the hydrostatic pressure is minimum, which is the instability point, and the outer one where the hydrostatic pressure is maximum, which is defined as ‘center’ of the open configuration. The Boyer surfaces with sufficiently high specific angular momentum and elongation, i.e. $(\ell, K) \in \textbf{L2} \times \textbf{K1}$ (corresponding to centers which are far enough from the attractor, sufficiently high centrifugal component of the potential and sufficiently high density) are not capable to close forming an outer disk edge, but there is still a point of instability, $r_J$, at the inner edge. Since the nature of these configurations is a limiting evolutionary stage with respect to the other open topologies as well as to the equilibrium disk or closed disk in accretion, we referred to them shortly as proto-jet. It is clear that the problem to interpret these configurations invests the more general problem of the accretion-jet correlation which we also address in this work. In terms of proto-jet configurations ($\ell \in \textbf{L2}$) it is necessary to investigate the relation between jets emission and accretion, particularly with respect to the location of the inner edge of the accreting disk (involving transition from the instability regions, for the parameter ranges $(\textbf{L1}, \textbf{L2})$), the magnitude of the disk specific angular momentum, and disk extension ($K \in \textbf{K0}$ here also it linked to the size of the disk and its density and enthalpy). In this work, we attack the problem for the first time in framework of the ringed accretion disk, while a more accurate analysis of the interpretation of these open solutions, as well as of the emergence of other open configurations is left for further investigations.
and therefore, only equilibrium configurations are possible for fluids having specific angular momenta in these ranges—see figures 3 and 4.

To simplify our discussion in the following we use label \((i)\) with \(i \in \{1, 2, 3\}\) respectively, for any quantity \(Q\) relative to the range of specific angular momentum \(L_i\).

### Figure 2

Scheme of the variation ranges \(K \in K_j\) where \(j \in \{\ast, 0, 1\}\) for the corotating \((-\) and counterrotating \((+\) cases respectively. \(K_{\text{mio}}^\pm\) is the value of the parameters \(K\) on the marginally stable orbit \(r_{\text{mio}}^\pm\) while \(\ell_{\text{mio}}^\pm \equiv \ell_\pm (r_{\text{mio}}^\pm)\), \(\ell_{\gamma}^\pm \equiv \ell_\pm (r_{\gamma}^\pm)\), \(\ell_{\text{mbo}}^\pm \equiv \ell_\pm (r_{\text{mbo}}^\pm)\) respectively. Radius \(r_{\gamma}^\pm\) is the photon orbit, and \(r_{\text{mbo}}^\pm\) is the marginally bounded orbit. White regions indicate the ranges \(L_i - K_j\) where critical points of the hydrostatic pressure are allowed. The topology of the Boyer surface is also indicated: \(C^\pm\) are the closed regular toroidal surface in equilibrium. \(C_{\ast}^\pm\) are the closed cusped surfaces with accretion point. \(O^\pm\) are open surfaces with a cusp (proto-jets) associated to the jet launch. The surfaces \(B^\pm\) and \(O^\pm\) are not associated to the critical points of the pressure. Sequences of \(B^\pm\) and \(O^\pm\) configurations for increasing \(K\) with \(\ell^\pm \neq \pm \ell_{\text{mbo}}^\pm\) constant are more articulated. In \(L_3 - Ki\) no \(B\) surfaces appear \([-B\]) and in \(L_2 - Ki\) the \(B\) surfaces are associated also with values \(V_{\text{eff}} > 1\), thus this configuration could also be as \(B_{\text{in}}\) or \(B_{\text{ext}}\) kinds. Surfaces not associated to the critical points of the effective potential are discussed of section 4—see figures 3 and 4, a general discussion can is in section 4 and also [2]. The superscript, or equivalently subscript \(\{Q, Q\}_i\) with \(i \in \{1, 2, 3\}\) respectively, is for any quantity \(Q\) relative to the range of specific angular momentum \(L_i\).
Figure 3. Upper: spacetime spin $a = 0.77M$, \( \ell \) counterrotating sequences, \( \ell_i \ell_j < 0 \). The outer horizon is at \( r_h = 1.638 04M \), the region \( r < r_h \) is black-colored, the region \( r < r^*_m \) is gray colored, radius \( r^*_m \) marks the static limit. Surfaces in the \( x - y \) plane, for different values of specific angular momentum \( \ell \) for corotating (\( \ell > 0 \)-continuum lines) and counterrotating (\( \ell < 0 \)-dashed lines) fluids. \( r^*_\gamma \) are the photon circular orbits were \( \ell_{\text{isco}} = \ell(r_{\text{isco}}) \) and \( \ell_{\gamma} = \ell(r_{\gamma}) \). Radius \( r_{\text{isco}} \) is the marginally stable orbit. The superscript (\( \pm \)) is for corotating (\( - \)) and counterrotating (\( + \)) matter respectively. The numbers close to the curve are the values of the specific angular momentum [4].

Below: spacetime spin \( a = 0.77M \), corotating sequences, \( \ell_i \ell_j > 0 \), of counterrotating open configurations \( J^+ - J^+ \). Decomposition including open-crossed sub-configurations (\( \gamma \)-surface) \( O^*_\gamma \), open cusped with angular momentum \( \ell_{\gamma} \). The outer horizon is at \( r_h = 1.638 04M \), black region is \( r < r_h \), gray region is \( r < r^*_m \), where \( r^*_m \) is the static limit, and \( r^*_\gamma \) is the photon orbit on \( \Sigma_{\pi/2} \). For \( O^+_\gamma \) there is \( \ell_{\gamma} = -5.625 51 \) and \( K_\gamma = 1.287 75 \), for \( O^+_{i} \) there is \( \ell_i = -4.664 87 \) and \( K_i = 1.002 72 \). The \( h\gamma \)-surfaces, \( O^+_i \) and \( O^+_j \) are also plotted—see also section 4.
points of accretion, \( r_s = r_{\text{Max}} \) for \( C_i \) topologies, or of launching of jets, \( r_{\text{Max}} = r_f \) for the open cusped topologies (proto-jets). Therefore it is always \( \ell'_{\text{min}} < \ell''_{\text{min}} \). We note that only for an \( \ell \) corotating sequence this definition implies also \( r_{\text{Max}} > \ell''_{\text{Max}} \). In other words, for the \( \ell \) corotating sequences there is always \( (\ell_i < (\ell_o = (\ell_o) , \text{ whereas for an } \ell \text{ counterrotating couple this is not always verified} [4–6]. \text{ Here } (\ell \) means any closed or open configuration.

As for the sequences of open configurations, \( O_o \), we shall mainly deal with the relations between the critical points of the configurations, then it will be convenient to introduce the criticality indices \( i \), which are associated univocally to the couple \( (r_{\text{Max}}^i, \ell_i) \), where \( i \) is as usual the configuration index univocally associated to \( (r_{\text{Max}}^i, \ell_i) \). We can note that \( \ell i(\ell) \) is a decreasing function of the configuration index \( i \) (ordering the maximum of pressures) or \( \ell \) is a univocally determined \( \ell \).

Thus, for example, \( O_{\ell} > O_{\ell'} \) and \( O_{\ell'} > O_{\ell''} \) where we have \( \ell_i = |\ell_o| < |\ell_o| = |\ell_o| \).

Finally, it should be noted that the simple case of barotropic tori with constant distribution of specific angular momentum provides all the relevant details of the RAD structure, as made up by thick toroidal disks, and at the present level of our knowledge it is a proper approximation to more realistic models. The effects on the adoption of more general laws for the fluid specific angular momentum in the tori model has been extensively discussed in the literature-see for example \[34, 35\]. Then, concerning the piecewise structure of the distribution of specific angular momenta (the differential rotation of the ringed disk [3]), the rings are supposed to be formed in different accretion regimes and therefore it is natural (and necessary for the tori separation) that each torus is characterized by a different specific angular momentum.

3. RADs instabilities

The existence of a minimum of the hydrostatic pressure \( (r_{\text{Max}}) \) implies the existence of a critical topology for the fluid configuration-figures 2–4. To sum up the situation for a RAD we may say there are two classes of points associated to the instability of the macro-structures: 1. The Paczynski instability points (corresponding to violation of mechanical equilibrium in orbiting fluids see also \[1, 39\]), maxima of the effective potential \( r_{\text{Max}} \in \{r_s, r_f\} \) for a \( O_o \) or a \( C_i \) topology and 2. The contact points \( r_{\ell} \) \( \ell \) between two surfaces of the decomposition featuring tori collision and associated to the second mode of instability for a \( C_o \) ringed disk.

Each ring may admit a maximum of two contact points \( r_{\ell} \).

Consequently, we can identify two successive instability phases of the macro-configuration: the first (I) with the formation of one or more points of instability or involve two sub-configurations in the case of formation of a macro-configuration \( C_{\ell o} \). (II) The second phase consists in the global instability of the ringed disk which follows the first phase. For both instability modes, in the second phase a penetration of matter from one unit to another is expected, and this can result in possible destabilization of the entire macro-structure by collisions between fluids. Then, although the second phase in the two modes may be similar, the essential difference is in the first phase of instability which is, in the first mode induced by an instability point, while in the second mode, the presence of contact points is necessary. The Paczynski mechanism in the first mode may be involved as the cause of the second phase, while in the second mode, starting with a \( C_o \) topology, of Paczynski instability could emerge.

5 The cusps \( (r_s, r_f) \) of the corotating surfaces can also occur in the ergoregion in the spacetimes with \( a > a_1 \), while for spin \( a = a_1 \equiv 1/\sqrt{2M} \approx 0.707 107 M \) the unstable point is on the static limit-see [30] and also [40] and Figure 3. These surfaces are detailed in section 4.
in the second phase as a possible effect of the first phase of instability. In this work, we analyze the situation only in the first phase.

### 3.1. Geometrical correlations of unstable configurations

Concerning the possibility of collision and emergence of instability we introduce the concept of geometrical and causal correlation for unstable configurations of the RAD. Two sub-configurations of a ringed disk are said to be geometrically correlated if they are in contact, or they may be in contact according to some constraints settled on their morphological or topological evolution and the geodesic structure of the spacetime, i.e. according to the effective potentials they are subjected to [3]. We note that, since the intersection of the geodesic structures defined by $r_{\pm}$, and introduced in section 2, is not empty, the analysis of the geodesic structure will be particularly important for the characterization of the geometrical correlation for the corotating sequences and particularly for the mixed subsequences. Thus part of this analysis is devoted to figure out whether and when two sub-configurations of a decomposition can be in contact, or, in fact, geometrically correlated according to a number of features set in advance, or which we seek to establish. A contact in this model causes collision and penetration of matter, eventually with the feeding of one sub-configuration with material and supply of specific angular momentum of another consecutive ring of the decomposition. This mechanism could possibly end in a change of the ring disk morphology and topology. In fact, several instability points can be geometrically correlated and present in a first phase of macro-structure instability, and being causally correlated, arising in a second phase of the

![Figure 4. Spacetime spin $a = 0.25M$, $\ell_{\alpha} \alpha > 0$, of corotating disks $\ell_{\alpha} \alpha > 0 \forall \alpha$. Decompositions including open-crossed sub-configurations $O_{\alpha}$. The outer horizon is at $r_{h} = 1.968 25M$, the region $r < r_{h}$ is black-colored. In L3 – Ki no B surfaces appear ([!B]). Surfaces not associated to the critical points of the effective potential are discussed of section 4.](image)
macro-structure instability. The existence of a contact point in this model is governed by the geodesic structure of the Kerr geometry. It is clear that a geometrical correlation in a ringed structure induces a causal correlation in a couple, when the morphology or topology of an element can be regarded as a result of that correlation, which arises by the geometry of the common attractor\(^6\).

### 3.2. Parameter setting, rotational symmetry and \(\ell\)

In this section we investigate some symmetric properties of the angular momenta \(\ell\) : \(V_{\text{eff}}(\ell, r) = 1\) introduced throughout the discussion of \([4, 6]\), for a general radius \(r\) on the equatorial plane. This special specific angular momentum allows us to discuss some properties of the effective potential inherited from the background symmetries and the toroidal motion of the orbiting system. An important role of the momentum \(\ell\) has been widely studied as \(\ell_{\text{nhbo}} \colon V_{\text{eff}}(\ell_{\text{nhbo}}, r_{\text{nhbo}}) \equiv K_{\text{nhbo}} = 1\) and \(\partial_r V_{\text{eff}} = 0\) in \(r = r_{\text{nhbo}}\). Then clearly this is the asymptotic value of the effective potential for large \(rM\) (or large \(R = r/a\)).

We can then reduce this problem to find out the solutions of the following equations

\[
2\Delta^2 - (\alpha^2 + \Delta_- \Delta_+) r + 2r^2 = 0 \quad \text{where}
\]

\[
V_{\text{eff}}(\Delta_\pm, r; \alpha)^2 \equiv \frac{r \Delta}{2\Delta^2 - \Delta_- \Delta_+ r + r^2} > 0,
\]

\[
\Delta_\pm \equiv \ell \pm a, \quad \Delta_- \Delta_+ > 0, \quad \Delta_Q \not\equiv 0 \quad \text{if} \quad \ell \not\equiv 0,
\]

\[
\Delta_Q = \{\Delta_+, \Delta_-, \} , \quad a > 0, \quad |\ell/a| > 1. \quad (10)
\]

For sake of simplicity in this section we mainly use dimensionless quantities where \(r \to r/M\) and \(a \to a/M\). Equation (10) is obtained by the condition \(V_{\text{eff}}^2 - 1 = 0\) on the equatorial plane \((\theta = \pi/2)\), while \(V_{\text{eff}}(\Delta_\pm, r; \alpha)^2\) is a rearrangement and a new parametrization of \(V_{\text{eff}}(r; \ell, a)^2\) of equation (7) using new \(\Delta_\pm\). Note that \(\Delta_\pm (-\ell) \not\equiv 0\) for \(\ell \not\equiv 0\), and the quantity \(\Delta_\pm\) in equation (10) should not be confused with \(\Delta\) in the metric given by equation (1). No critical points exist, therefore, no Boyer surface exists, for \(|\ell/a| < 1\)\(^2\). We will use equation (11), exploiting the symmetries in the couple \(\Delta_\pm\) for sign reversal in the specific angular momentum. We have:

\[
\ell = \frac{\Delta_- + \Delta_+}{2}, \quad a = \frac{(\Delta_+ - \Delta_-)}{2},
\]

\[
V_{\text{eff}}(\Delta_\pm, r) \equiv \frac{1}{2} \sqrt{\frac{\ell}{[\Delta_-(\Delta_+ - \ell)^2 + 4(\ell - 2)r]}}, \quad (12)
\]

Then we can introduce, a single rotation parameter for the system accretor-disk rather than two, defined by:

\[
A_\pm \equiv \frac{\Delta_\pm + \Delta_\mp}{2} : \quad A_+ = a > 0, \quad A_- = \ell, \quad (13)
\]

where the two rotational parameters \(\ell\) and \(a/M\) are now replaced by the two \(\Delta_\pm\).

\(^6\)We note that the problem of inferring and even to define a causal correlation between events or objects can be indeed subtle and it is certainly relevant in a variety of scientific disciplines-for a very general discussion we refer to [41]. In the case of jet-proto-jet correlation and the jet-accretion correlation proposed here we should consider that generally correlation does not necessary imply causation, and this latter aspect of the correlation should be faced more deeply together with a more general discussion of the correlation definition used here.
The difference in the two forms of the potential in equations (10) and (12), respectively. This definition could be important in the analysis of accretion disks, because this re-parametrization may result in the identification of significant traceable quantities, when the fluid specific angular momentum neither the attractor spin are not made explicit. Therefore, we expect these symmetries will be deepened further in a future work. Solving explicitly the problem (10) in terms of $\ell$, one gets the two solutions

$$\ell_+ \equiv \mp \frac{2a + \sqrt{2r_\Delta}}{r-2}, \quad \ell_- < 0 \quad \forall r, \quad \ell_+ < 0 \quad \text{in} \quad r < r_\ell^+,$$

where $\ell_{\pm} = \ell_{\pm}(r_{\text{min}})$ and $\ell_{\pm} = \ell_{\pm}(r_{\text{max}})$.

Therefore, the specific angular momentum $\ell$ is provided by equation (14) and, considering the form of the radii $r_{\pm}$, the symmetries between of the counterrotating fluid configurations at the marginally stable orbits are clear. However, we can investigate more deeply this aspect, using the variables $\Delta_{\pm}$. Then the solutions (10) are the couples:

$$(\Delta_{-}^{[1]}, \Delta_{+}^{[1]}) \quad \text{and} \quad (\Delta_{-}^{[+]}, \Delta_{+}^{[-]})$$

where

$$\Delta_{\pm} \equiv \frac{a(\ell \pm \sqrt{r_\Delta})}{r-2}, \quad \Delta_{\pm} \equiv \frac{a(\ell - 4) \pm \sqrt{r_\Delta}}{r-2}.$$ (15)

Considering the following symmetries:

$$\Delta_{+, \ell} = -\Delta_{-, \ell}, \quad \Delta_{+, -\ell} = -\Delta_{-, \ell} \quad \text{(16)}$$

$$\Delta_{-}(-\ell)^2 = \Delta_{+}(\ell)^2, \quad \Delta_{-}(\ell)^2 = \Delta_{+}(-\ell)^2,$$

$$\Delta_{+}(\ell)\Delta_{-}(-\ell) = \Delta_{-}(-\ell)\Delta_{+}(\ell) \quad \text{(17)}$$

and having in mind also equation (11), we can say that a change in sign $\Delta Q$ acts in exchanging the subscript sign $\Delta_{\pm}$ and $\ell$. Then we can solve the Boyer problem to find out the ring surfaces in terms of the resolving couples $\Delta_{\pm}$. By considering effective potential $V_{\text{eff}}(\Delta_{\pm}, r)^2$ in equation (10), we get the following solutions for the specific angular momenta of the critical points for the pressures where solutions of the Boyer problem exist:

$$\Delta_{-} = \frac{a(\ell - 1)^2 - \sqrt{r^3_\Delta}}{\Delta + X}, \quad \text{and} \quad \Delta_{-} = \frac{a(\ell - 4) \pm \sqrt{r_\Delta}}{r-2},$$ (18)

$$\Delta_{+} = \frac{a(\ell + 1)^2 + r^3_\Delta}{\Delta + X}, \quad \text{and} \quad \Delta_{+} = \frac{a(\ell + 4) \pm \sqrt{r_\Delta}}{r-2},$$

where $\Delta_{+, \ell} = 2a$. Similar considerations are also possible considering the situation on different planes, for example by considering the quantity $\ell/a \sin \theta$, see for example [2], and possibly generalizing the specific angular momentum definition [34].

However, it is convenient to take advantage of the symmetries of the configuration in the treatment of extended systems of matter in the axissymmetric fields, and particularly the
symmetry of reflection on the equatorial plane. In the re-parametrization \((\ell, a/M) \mapsto \Delta_{\pm}\) we can take advantage of the symmetries in equation (16), managing only one sign, then reducing to one only, always positive variable \(\Delta_+\) or \(\Delta_-\), without considering the corotation or counter-rotation nature of the fluid (that could indeed be difficult to be assessed) but considering equation (17). The square \(\Delta^2\) is the only term of the potential in equation (10) that changes subscript after a change of the specific angular momentum. Suppose for simplicity \(\ell > 0\), then, being \(\Delta_{\pm}\), the variable, we can solve for one of the two \(\Delta_{\pm}\) and then use any of the symmetries in equation (16) to infer information on the other one.

4. Open equipotential surfaces and fluid configurations

The existence of the fluid configurations is schematically summarized in figure 2 in terms on the ranges of the specific angular momentum and the \(K\) parameter.

In [4] we discussed the configurations associated with the pressure critical points, occurring for the parameter ranges \(L_i \cup K_j\), with \(K_i \in \{K_0, K_1\}\), and \(L_i \in \{L_1, L_2\}\), or \(L_3 \cup K_0\). Here, we add some comments on the surfaces not associated to the critical points of the effective potential function but the solutions of \(V_{\text{eff}}(\ell, r) = K\). In the following, we address the discussion in terms of the specific angular momentum magnitude unless the fluid rotation with respect to the black hole is not explicitly specified.

We start our considerations by noting that the hydrostatic pressure has a monotonic behavior as a function of \(r/M\) for specific angular momentum in \(L_0\) : \(\ell < \ell_{\text{max}}\), where \(V_{\text{eff}} < 1\) on the equatorial plane. For these values of specific angular momentum \(\ell\), there are no critical points of the fluid pressure. As proved in [2], the case \(\ell \equiv |\ell/a| < 1\) is a restriction of \(L_0\) on the equatorial plane \(^7\). At \(K < 1\), range \(K_0\), and \(\ell < \ell_{\text{max}}\), range \(L_0\), no critical points of the effective potential occur on the equatorial plane: the pressure generally decreases with the radius (on the plane \(\Sigma^{+}\)). However, at \(L_0 \cup K_0\) there are the \(B_{\text{in}}\) surfaces, fat closed tori as shown in figure 4. Funnels are not associated to these solutions. The unique solution of the problem \(V_{\text{eff}} = K < 1\) corresponds to the outer edge of this configurations. The surface becomes smaller for decreasing \(K \in K_0\). The surface area increases by decreasing the specific angular momentum magnitude \(\ell \in L_0\). Qualitatively we could conclude that the rotation relative to the attractor plays an irrelevant role in the determination of the morphology of these surfaces, which confirms a symmetry between the \(\ell\) counterrotating sequences as pointed out in [2] and [3]. However, these features distinguish morphologically the \(B_{\text{in}}\) from the lobe \(B_{\text{ext}}\) which appears at higher specific angular momenta. Decreasing the specific angular momentum from starting value \(\ell \in L_3\), when there is a maximum of the hydrostatic pressure, a first surface \(B_{\text{ext}}\) occurs. At \(K \geq K_{\text{min}}\), a toroidal ring is formed, this grows and approaches the attractor as the specific angular momentum magnitude decreases. If \(K_{\text{Max}} < 1\), i.e. \(\ell \in L_1\), then with decreasing of the specific angular momentum in \(L_1\) towards values in \(L_0\), a \(B_{\text{in}}\) inner surface appears (i.e. one has the sequence of configurations \(\{B_{\text{ext}}, C, C_x, B_{\text{in}}\}\)), with no open funnel-figure 4.

\(^7\) On the planes different from the equatorial one, we can generalize the limit by assuming \(\bar{\ell} \equiv \ell/|a\sigma| < 1\) or, as discussed below, \(\bar{\ell} \equiv \ell/|a\sigma| < -1, 1|\), with \(\sigma = \sin \theta\), where no critical points are possible—see also [2].
For $K \geq 1$, open surfaces of funnels of matter appear, close to the rotation axis as the specific angular momentum magnitude decreases in $\ell \in \mathbf{LO}$. These configurations will be referred as $O_{in}$. This surfaces (where $\partial_y y > 0$) do not cross the axes $y = 0$—see figure 4.

4.1. Restrictions on the existence of the open configurations: limiting surfaces

The set of $O$ and $B_{ext}$ configurations which are not related to the critical points of the effective potential correspond to the solutions $I = 0$ of:

$$\Pi(\ell) = g_{\phi\phi} + 2g_{\phi t} + \ell^2 g_{tt}, \quad (20)$$

—see figures 3 and 4. The effective potential, related to the four-velocity component $u_t = g_{\phi t}u^\phi + g_{tt}u^t$, is not well defined on the zeros of $\Pi(\ell)$.

Decomposing explicitly the function $\ell$ in terms of the quantities $\Sigma = u'$ and $\Phi = u^\phi$ then, the quantity $\Pi$ can be written as $\Pi(\Sigma, \Phi) = g_{tt}(\Sigma)^2 + 2g_{\phi t}\Sigma\Phi + g_{\phi\phi}\Phi^2$. Thus, $\Pi$, is related to the normalization factor $\gamma$ for the stationary observers, establishing thereby the light-surfaces (for example [42, 43]). Alternatively, by expressing the effective potential in terms of $L(\ell)$, and by using definition equation (4) and definition of $\ell$ in equation (7), one obtains $\Pi(L, E) = E^2g_{\phi\phi} + 2Eg_{\phi t}L(\ell) + g_{tt}L(\ell)^2$.

The attractor-ringed-disk system shows various symmetry properties with respect to the rationalization $\ell = \ell/|a\sigma|$ and $R = r/a$, where $\sigma = \sin \theta$ [2]. These quantities are dimensionless and, assuming $a > 0$ with $\ell$ positive or negative according to the fluids rotation, the parameter $\ell$, takes care of the symmetry for reflection on the equatorial plane through $\sigma$. Noticeably, many properties of the RAD depend mainly on the rationalized specific angular momentum $\ell$.

We discuss this kind of symmetry in section 3.2. To characterize the dependence on $a$, and the RAD symmetry properties with respect to the equatorial plane, it is convenient to rewrite the quantity $\Pi$ in terms of $\ell$ and $\ell \equiv \ell/\sigma$ as follows

$$\Pi(\ell, R) = -a^2\sigma^2 \left[2(\ell - 1)^2R\sigma^2 + a \left(1 + R^2 - \ell^2\sigma^2\right)\right], \quad (21)$$

8 Increasing $K \in \mathbf{K1}$, or decreasing $|\ell| \in \mathbf{LO}$ a collimation occurs, i.e. there would be $\partial_y |y| < 0$ but $\partial_y |y| > 0$.

As in [6], we can define collimation of the funnels along the rotation axis if there is at least one $x_c : \partial_y |y| < 0$ for $x > x_c$ where the rotation axis is located at $y = 0$. The open surface will be said collimated, the coordinate $x_c$, and the corresponding $y_c$, give the collimation points. If $\partial_y |y| = 0$, in a given finite region of $x$, then the funnel structure is tubular, formed by matter rotating around the $x$ axis at each $\Sigma_o$ (in this model $i = 0$). In the tubular structure the radius of the cylinder remains constant for each $y$. In the proto-jets there should be some mechanism pushing the matter to induce a $\partial_y |y| < 0$ by changing also accordingly the specific angular momentum magnitude or $K$ parameter. For example, one might ask if a change of spin of the attractor could constitute such a factor, or also one can equally consider the case of the open critical configurations $O_i$. For a general discussion on the role of the open surfaces and their connection with jet emission we refer to [8–11, 14, 18]. Then, given an $O_{cor}$ of $O_{cor}'$, it is always $O_{cor}' \supset O_{cor}$ at $r < r_j'$ and $O_{cor}' \subset O_{cor}$ for $r > r_j'$, and there is a couple of cross points ($O_{cor}' \cap O_{cor} \neq 0$) in $|y|',|r|'$—see figure 3. This means that, given $x < r_j'$ on the equatorial plane, there is $|O_{cor}'| \equiv |y|' > |O_{cor}| \equiv |y|$. In other words, the curve $O_{cor}'$ is contained in the region of the $y \times x$ plane with boundary $O_{cor}$ for $r < r_j'$. Viceversa, at $r > r_j'$ the (open) surface $O_{cor}'$ is contained in the region of the plane cut by $O_{cor}'$, or for $x > r_j'$ on the equatorial plane, there is $|O_{cor}'| \equiv |y|' < |O_{cor}| \equiv |y|$. This means, in particular, that the funnels of $O_{cor}$ couple of open surfaces do not cross each other in the region $r > r_j'$ and generally the separation $|y|' - |y|$ increases with $r > r_j$. But they approx to the axis in the region close to the instability launch point $r_{\infty}$ and for higher specific angular momentum $|\ell|$. Though, in this model of axissymmetric flows with constant specific angular momentum, there is no ‘collimation’ of the funnels or, if $x = 0$ is the rotation axis, it is $\partial_y |y| > 0$ on the equatorial plane, where $|y| > r_j$. Nevertheless any matter in open funnels, with specific angular momentum in $|\ell|', |r|'$, will be bounded by the couple of configurations $O_{cor}' \subset O_{cor}$. Then $\partial_y (y_c - y_j') > 0$ (on the section $x > 0$ and $y > 0$), the minimum value of the distance $(y_c - y_j')$ is reached in the equatorial plane, where $x = 0$. 15
in dimensionless quantities. At fixed specific angular momentum $\ell$, the zeros of the $\Pi$ function define \textit{limiting surfaces} of the fluid configurations—figures 3 and 4. For fluids with specific angular momentum $\ell \in \{L3\}$, the limiting surfaces are the cylinder-like surfaces $O_{\text{at}}$, crossing the equatorial plane on a point which is increasingly far from the attractor with increasing specific angular momentum magnitude. A second $B$-like surface, embracing the BH, appears, matching the outer surface at the cusp $r_\gamma$. Decreasing the specific angular momentum magnitude $\ell$ towards the limiting value $\ell_*$, the surface $\gamma(x)$, symmetric with respect to the equatorial plane, has a minimum at $x = 0$—equation (21). Then, for $\ell = \ell_*$ a cusp appears together with a inner surface closed on the BH.

The morphology of these surfaces is analogue to the matter funnels of the open topologies having $\ell \in \{L2\}$ and $K \in \{K1\}$. In this sense, the light-surfaces, for $\ell = \ell_*$, can be interpreted as ‘limiting surfaces’ of the open Boyer surfaces. Decreasing the specific angular momentum $\ell$, from a starting $\ell \in \{L3\}$, the $O_{\text{at}}$ surfaces open up at the radius $r_\gamma$ in $r < r_\gamma$, and approaching the horizon as a $O_{\text{in}}$ topology towards the rotation axes as shown in figures 3 and 4. Comparing with the corotating surfaces, because of their rotation with respect to the black hole, the light-surfaces for the counterrotating fluids form in more distant orbital regions. This fact implies the emergence of a broad diversification of the unstable rotating structures for these fluids—see figure 3.

More generally, we can introduce the limiting geometric surfaces, or $\gamma$-\textit{surfaces}, related to the geometric properties of the Kerr spacetimes, formed by the surfaces given by the solution of equation (7) and associated to the specific angular momentum $\{\ell_{\text{mao}}, \ell_{\text{mbo}}, \ell_{\gamma}^\pm\}$.

On the other hand, the solutions of $\Pi(\ell) = 0$, for fixed parameters $\ell$ and $a/M$, define the limiting hydrostatic surfaces, $h\gamma$-\textit{surfaces}, (with the notation $\{\ell\}$), which are associated with each hypersurface $\Sigma_{\ell}$, whose topology and morphology changes with the variation of one of the two rotational parameters $(\ell, a/M)$. More specifically, the $h\gamma$-surfaces are the limiting surfaces to which the fluid configuration belonging to the same topological class, and determined by the Euler problem on $\Sigma_{\ell} \cup \Sigma_{a}$, approaches, at the variation of the free $K$-parameter. The $\gamma$-surfaces, on the other hand, constrain the fluid at variation of $\ell$. At $\Sigma_{a/M}$, the $\gamma$-surfaces are the limits of the $h\gamma$-surfaces, approached by varying $\ell$; the $h\gamma$-surfaces in turn limit the matter fluid surfaces as described below. In figure 3 an example of an $\ell$-rotating sequence of counterrotating $h\gamma$-surfaces, the inner one $O_{\ell}^\gamma +$ and outer $O_{\ell}^\gamma -$ one, is shown.

The $\gamma$-\textit{surfaces} define, for the counterrotating sequences on a $\Sigma_{a/M}$, three geometric regions depending on the attractor spin: 1. An external region, or $r > r_\gamma^+$, confined by the $\gamma$-surface $O_{\ell}^{\gamma^+}$ associated to the counterrotating photon-orbit with a cusp at $r_\gamma^+$. 2. The region in $]r_\gamma^+, r_\gamma^-[$, and finally 3. An internal region at $]r_\gamma^-, r_\gamma^+[$. The open $\gamma$-surface $O_{\ell}^{\gamma^-}$ is cusped in $r_\gamma^-$, and it corresponds to the corotating photon orbit, see figure 3. The two critical surfaces $O_{\ell}^{\gamma^\pm}$, have one lobe closed on the black hole and the location on the corotating closed lobe of $O_{\ell}^{\gamma^-}$ is \textit{inside} the $O_{\ell}^{\gamma^+}$ configuration and, for $a < a_1 = 0.707 107M$ this is inside the ergoregion of the Kerr spacetime, see [30].

Since there is $\ell_{\text{mbo}} \in [\ell_{\text{mao}}, \ell_{\gamma}]$, the $\gamma$-surfaces associated to the specific angular momenta $\ell_{\text{mao}}$ and $\ell_{\text{mbo}}$ belong to the internal region, and thus they are configurations of the $O_{\ell}^\gamma$ type—see figures 1–4. There are no $\gamma$-surfaces in the external region. For the specific angular momentum $\ell_1$ on each hyperplane $\Sigma_{a/M}$, these surfaces are in turn determined \textit{a priori} on each $\Sigma_{a/M}$, whatever the specific angular momentum $\ell$ of the fluid is on that plane. The $h\gamma$-surfaces have a cusped topology on the equatorial plane at $r_\gamma^\mp$ for counterrotating and corotating fluids respectively. We note that the investigation of these regions is useful in particular for the analysis of the counterrotating sequences at each $\Sigma_{a}$. In fact, the limiting hydrostatic surfaces are never cusped but at $\ell = \ell_\gamma$, where we have the topological class $O_{\ell}^{\gamma}$, which coincide with the
geometric light surfaces, or \( O^x_\gamma \equiv O^x_\ell \), closed on the black hole and opened outwards. For the specific angular momentum \( \ell = \ell_\gamma^x \), the matter configurations, limited by \( O^x_\ell \), can be closed, centered in \( r_c > r_{\text{mso}} \) for \( K \in K_0 \), or can be open in \( O^{\ast}_\ell > O^x_\ell \) for \( K \in K_1 \). The surface \( O^x_\gamma \) is a boundary surface reached by lowering the specific angular momentum towards \( \ell_\gamma \), or

\[
\lim_{|\ell| \to |\ell_\gamma|} O_{\text{ext}} \approx O^\gamma_\ell, \quad r_{\text{ext}} > r_\gamma, \quad O_{\text{ext}} > O^\gamma_\ell, \quad |\ell_{\text{ext}}| > |\ell_\gamma|, \tag{22}
\]

where \( r_{\text{ext}} \) is the crossing point of \( O \) on the equatorial plane. This is not a cusp but it is a regular, minimum point of the curve \( y(x) \). Therefore, the matter surfaces \( O_{\text{ext}} \) have a critical point of the hydrostatic pressure, a maximum in \( r_s \) but, like the correspondent \( h\gamma \)-surface \( O^{\ell}_{\text{ext}} \), it is open and regular.

Concluding, there are three classes of open matter configurations, \( \mathcal{O}_i \in \{ O_a, O_b, O_{\text{ext}} \} \), bounded by the limiting hydrostatic surfaces \( O^\ell_\ell \in \{ O^\ell_a, O^\ell_b, O^{\ast}_{\text{ext}} \} \) respectively, where \( O^\ell_a \equiv O^\ell_b \). Each class of limiting surface \( O^\ell_\ell \) is bounded by the correspondent open \( h\gamma \)-surface, \( O^\ell_\gamma \), and it approaches \( O^\ell_\gamma \) varying \( K \). Therefore the two classes \( O^\ell_{\text{ext}} \) and \( O^\ell_b \) of open surfaces are separated by \( O^\ell_a = O^\gamma_\ell \), so that there is

\[
O^\ell_a \prec O_{\text{in}}(\ell) \prec O^\ell_b = O^\gamma_\ell \prec O^{\ast}_{\text{ext}} \prec O_{\text{ext}}(\ell_{\text{ext}}),
\]

with \( \ell_{\text{in}} < \ell_\gamma < \ell_{\text{ext}} \) and

\[
O^\ell_a = O^\gamma_\ell \prec O_{\text{in}}(\tilde{\ell}) \prec O^{\ast}_{\text{ext}} \prec O_{\text{ext}}(\ell_{\text{ext}}),
\]

with \( \tilde{\ell} < \ell_{\text{in}} < \ell_\gamma < \ell_{\text{ext}} \). \tag{24}

On the other hand, there are the limiting specific angular momenta at \( \ell_{\text{mso}} > \ell_{\text{mso}} \), leading both to \( O^x_\gamma \); or \( O^\ell_{\text{in}}(\ell_{\text{mso}}) = O^\ell_a(\ell_{\text{mso}}) = O^\gamma_\ell(\ell_{\text{mso}}) = O^\ell_{\text{ext}}(\ell_{\text{mso}}) \).

The matter surfaces \( O_i \) are bounded by the limiting \( O^\gamma_\ell \), solution of \( \Pi = 0 \). The matter surfaces \( O_{\text{ext}} \succ O^\gamma_\ell \) and \( O_{\text{in}} \prec O^\gamma_\ell \) are bounded by \( O^\gamma_\ell \), with specific angular momentum fixed in \( \ell_\gamma \). They are also bounded by the solutions of \( \Pi(\ell) = 0 \), for the same specific angular momentum, and having equal topology \( O_{\text{ext}} \succ O^\ell_{\text{ext}} \succ O^\gamma_\ell \). The \( \gamma \)-surfaces are approached by changing the specific angular momentum (see equation (23)). While the \( h\gamma \)-surfaces are generally approached by an asymptotic limit of \( K \), as it is clear from figure 2.

We note that the second and third inequality in equation (23) (and the first and second of equation (24)) are ensured by the relations among the specific angular momentum of the \( \ell \) corotating sequences, as the following general relations hold

\[
\partial Q r^{[\pm]}_{\text{crit}} \leq 0, \quad r^{[\pm]}_{\text{crit}} = r_{\text{min}} = r_{\text{cent}}, \tag{25}
\]

\[
r^{[-]}_{\text{crit}} \in \{ r_{\text{max}} = r_s, \quad r_{\text{max}} = r_J, \quad r_{\text{in}}, \quad r_{\text{ext}} \}, \quad Q \in \{ |\ell|, |K| \}, \tag{26}
\]
in particular for \( Q = |\ell| \), where equation (25), for \( Q = |\ell| \), does not apply to the first and last inclusion relations of equation (23) because the two couples of open surfaces \( (O_{\text{ext}}, O^{\ast}_{\text{ext}}) \) and \( (O_{\text{in}}, O^\ell_{\text{in}}) \), respectively, have the same topology and the same specific angular momentum, the limit indeed is approached changing the \( K \) parameter or for \( Q = K \). Thus, the non-cusped limiting surfaces can be the regular couples \( B^\gamma_{\text{ext}} < O^\gamma_{\text{ext}} \), when \( \ell \in L_3 \), that is \( \ell > \ell_\gamma \), in the external region, according to figures 2–4.

9In these cases the sequentiality will be intended according to the ordered sequences of the equatorial crosses \( r_{\text{crit}} \), for the open, not-cusped, \( O_{\text{ext}} \) surfaces.
Associated with these configurations, there is the couple $B_{\text{ext}} < O_{\text{ext}}$ in $L_3 \cup K_1$. Otherwise there can be, in the internal region, also a $O_{\text{in}}^0$ surface for specific angular momentum $\ell < \ell_\gamma$ embracing the horizon. The $O_{\text{in}}$ surfaces are due to the cusp opening occurring when the magnitude of the specific angular momentum decreases (where $\partial^2_\ell x > 0$). There are therefore the $h\gamma$-surfaces $O_{\text{in}}^\ell$, approaching the proper limits on the specific angular momentum (starting by initial data in $L_0$, $L_1$ or $L_2$) the $\gamma$-surfaces $O_{\text{in}}^\ell$, zeros of equation (20) for $\ell = \{\ell_{\text{mho}}, \ell_{\text{mbo}}\}$. Indeed, decreasing $\ell_0 \in L_3$, we have the $h\gamma$-surfaces sequences: \[
\{ (B_{\text{ext}}^\ell < O_{\text{ext}}^\ell)|_{L_3}, O_{\text{in}}^\ell|_{\ell < \ell_\gamma}, O_{\text{in}}^\ell|_{\ell = \ell_\gamma}\}.\]
For the open crossed surfaces $O_{\text{in}}$ (for $\ell \in L_2$) there are \[
\lim_{\ell \to + \ell_\gamma} O_{\text{in}} = O_{\text{in}}^\ell, \quad O_{\text{in}} > O_{\text{in}}^\ell \quad r_x = r_j > r_j^\ell, \quad O_{\text{in}} < O_{\text{in}}^\ell, \tag{27}
\]
see also equation (22) and figure 3. The open surfaces $O_{\text{ext}}$ are limited by the configurations $O_{\text{in}}^\ell$, in other words $r_{\text{ext}} \equiv y_3 > r_{\text{ext}}^\ell$, where $O_{\text{ext}} > O_{\text{ext}}^\ell$-figure 3. There is also $B_{\text{ext}}^\ell < O_{\text{ext}}^\ell$, and \[
\partial_{\ell,\ell} B_{\text{ext}}^\ell > 0, \quad \partial_{\ell,\ell} B_{\text{ext}}^\ell > 0. \] For specific angular momentum $\ell < \ell_\gamma$, there are the open surfaces ($O_{\text{in}}^\ell, O_{\text{in}}^\ell, O_{\text{in}}^\ell$). Whereas, for $\ell \in L_2$, there are the surfaces $\{ O_{\text{in}}, O_{\text{in}}, B_{\text{in}} \}$. The $B_{\text{in}}$ configuration occurs occurs for $K < K_{\text{min}}$ (then also $K \in K_{\text{K}}, K_{\text{K}} \in K_{\text{K}}$, $K_{\text{K}} \in K_{\text{K}}$). At fixed $\ell \in L_1$, there can be the inner $B_{\text{in}}$ in $K_{\text{K}}$: i.e. increasing $K > 0$, there is the sequence $\{ B_{\text{in}}, (B_{\text{in}}, C_{\text{in}}, C_{\text{in}}) \}$. If the starting point is $\ell \in L_0$, then, increasing $K > 0$, there is the sequence $B_{K} = \{ B_{\text{in}}, O_{\text{in}} \}$.

In other words, the specific angular momentum is low enough not to lead to the formation of an outer lobe, but it eventually opens in $O_{\text{in}}$. The $B_{\text{ext}}$, associated to $O_{\text{ext}}$ for high values of the specific angular momenta, has morphology similar to the $B_{\text{in}}$ surface. The $B_{\text{configurations}}$ for $\ell \in L_2$, where the cusped surfaces $O_{\text{in}}$ can appear, are classified as $B_{\text{ext}}$, in fact the $B_{\text{ext}}$ ones are separated by $O_{\text{ext}}$ for each value of $\ell \in L_3$ and any $K$ value-see figures 2 and 4.

For parameter $K \in K_0 > K_{\text{max}}$, there are $B_{\text{in}}$ surfaces opening for $K \geq 1$ as $O_{\text{in}} \in [O_{\text{in}}^\ell, O_{\text{in}}^\ell]$, and, similarly to $O_{\text{in}}$ with $\ell \in L_1$, and $O_{\text{in}}$ with $\ell \in L_0$. The presence of a minimum point of the hydrostatic pressure always implies, in a Kerr black hole geometry, the presence also of a maximum pressure point. The inverse is not true, for example for $\ell \in L_3$, when there is only one family of non-critical open surfaces $O_{\text{ext}}$ at $r > r_\gamma$.

We summarizing saying that for $K \in K_1$ there are open surfaces for any specific fluid angular momentum $\ell$. The cusped open configurations $O_{\text{in}}$, closed on the BH, are associated to parameters $\ell \in L_2 \cup K \in K_1$, where the limiting surface is \[
K \in K_1 \quad O_{\text{in}} > O_{\text{in}}^\ell \quad \ell \in L_2 \quad r_j \in [r_j^\ell = r_\gamma, r_{\text{mbo}}] \tag{28}
\]
\[
\partial_{\ell,\ell} r_j < 0 \quad \partial_{\ell,\ell} r_j = 0, \quad \lim_{\ell \to + \ell_\gamma} O_{\text{in}} \approx O_{\text{in}}^\ell. \tag{29}
\]
However, we have \[
\lim_{k \to \infty} O_{\text{ext}} \approx O_{\text{ext}}^k \quad \text{while} \quad \lim_{\ell \to + \ell_\gamma} O_{\text{ext}}^\ell \approx O_{\text{ext}}^\ell. \tag{30}
\]

The open surfaces, of $O_{\text{in}}$ class with $\ell \in L_2$ are limited by $O_{\text{in}}^\ell$ at equal $\ell$. These are limited by the boundary surfaces with specific angular momentum $\ell_{\text{mho}}$ and $\ell_{\text{mbo}}$. However, as clear from the figure 3, for $x$ large enough, the funnels of $h\gamma$-surfaces go far from the source and, independently by the magnitude $\ell_{\text{ho}}/\ell_{\text{bo}} = -1$, the two $l$-counterrotating funnels are getting closer.

On the other hand, for sufficiently high magnitude of the specific angular momenta, i.e. $\ell \in L_3$ and $K \in K_0$, there are maximum pressure points but not the minimum of the hydrostatic pressure. Consequently there are closed stable $C$ or open $O_{\text{ext}}$, topologies. The Paczyński-Wiita instability can occur only after a reduction of the specific angular momentums, when the disk center approaches the attractor. If the specific angular momentum decreases from the
\( \ell \in L_3 \cup K \subseteq K^0 \) to \( \ell \in L_1 \cup K \subseteq K^0 \), then the unstable phase will necessarily correspond to an accretion. If the decrease of specific angular momentum occurs with an increase of \( K \), then it can give rise to the open cusped \( O_x \) surface. The high values of the specific angular momentum, \( \ell \in L_3 \) or also \( \ell \in L_2 \), are associated to closed equilibrium configurations or open surfaces. If the disk stretches sufficiently, the pressure at the inner edge can reach the proper minimum value to be open. On the other hand if there is \( \ell \in L_3 \), then the elongation of the configurations and fluid angular momentum magnitude are too high causing the opening of a \( O_{\text{ext}} \) surface.

5. Concluding remarks

We focused on the open, unstable solutions of Euler equations for RADs, aggregations of several tori of one-species particle perfect fluid centered on the equatorial plane of a Kerr SMBH. These structures, have been variously associated with jet of proto-jet emission. This investigation then fits into the more broad discussion on the role and significance of open surfaces in relation to (matter) jets emission and collimation, as well as jet-accretion correlation—see also [8–11, 14, 15, 18].

The issue of the location of the inner edge of a single torus is a relevant aspect of the possible jet-accretion correlation, relating jet-emission to the inner part of the accretion disk—see for a discussion of the inner edge problem [39, 44–47]. Attempts to characterized narrow, relatively long matter funnels of jets, here considered as proto-jets configurations, are for example in [45, 48–55]. For updated investigations on jet emission, detection and jet-accretion disk correlation see for example [56–64] and also [65–71].

After introducing the model for the single torus orbiting of the RAD, in section 2 we discussed the main features of the ringed accretion disks, and the emergence of the different instabilities for the systems. The problem to find the solutions of the Euler equations for the RADs has been reduced to the study of the critical points for an effective potential describing the orbiting fluids with proper boundary conditions; in this way we could estimate very precisely the diverse contributions of the background and the centrifugal forces on dynamics of the single torus as well as on the set of configurations in the RAD and the entire agglomeration. In section 2.2, we have deepened this aspect by taking advantage of system symmetries, considering a different parametrization for the effective potential function highlighting the symmetry of motion and background represented by Kerr axi-symmetric solution. Eventually we introduced different rotational parameters. This has allowed us to highlight the role of the dimensionless radius \( R \equiv r/a \) and of the fluid specific angular momentum with respect to the black hole spin, the ratio \( \ell/a \sin \theta \), pointed out also in [2], and the quantities \( A^\pm = \ell \pm a \).

The analysis has ultimately singled out the role of the limiting surfaces, the \( \gamma \)-surfaces and \( h_\gamma \)-surfaces, for the open configurations. We proved there is a strict correlation between different \( \gamma \)-surfaces (geometric surfaces of the spacetime structure) and the \( h_\gamma \)-surfaces having roles in the matter models. The \( h_\gamma \)-surfaces limit the fluid configuration, while the \( \gamma \)-surfaces constrain the fluid, at variation of \( \ell \). The limiting geometric surfaces, \( \gamma \)-surfaces, are related to the geometric properties of the Kerr spacetimes inherited by the solution of equation (7), associated to the specific angular momentum \( \{ \ell^\pm_{\text{mso}}, \ell^\pm_{\text{mbo}}, \ell^\pm_{\gamma} \}_\pi/2 \). Given the significance from the phenomenological point of view of different instabilities which characterize the toroidal population in the RAD, it is clear that a study of these constraints (especially in relation to the central attractor) is important. The limiting hydrostatic surfaces, \( h_\gamma \)-surfaces, are associated with each hypersurface \( \Sigma_\ell \) of constant specific fluid angular momentum, whose topology and morphology changes with the variation of one of the two system rotational parameters.
At $\Sigma_{a/M}$, the $\gamma$-surfaces are the limits of the $h\gamma$-surfaces, approached by varying $\ell$; the $h\gamma$-surfaces in turn limit the matter fluid surfaces.

The role of these surfaces in the RADs, their origins and the destabilizing effects on the system were briefly addressed, possible magnetic effects in a magnetized RAD contests are discussed in [33], while a more thorough discussion of the boundary conditions for the open solutions are postponed to future analysis. We expect these considerations can play an important part in the phenomenological aspects connected with x-ray emission in SMBH-AGNs and in the study of the possibility for a jet-accretion correlation.

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