In search of a variational formulation of the relativistic Navier-Stokes equations

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Drawing an analogy with Maxwell theory a new Lagrangian is proposed for a variational formulation of the relativistic Navier-Stokes equations which to-date has remained elusive. A key feature is the use of tensor potentials, whose degrees of gauge freedom allow for the reformulation of the energy-momentum equations in a self-adjoint form. An already existing potential-based representation of the relativistic field equations is a suitable starting point for the present considerations, which in turn are guided by the already successfully solved case of non-relativistic, stationary and incompressible flow.

1 Introduction

In [1, 2] a potential-based formulation of the relativistic Navier-Stokes (NS) equations has been developed, an essential underpinning being analogies drawn with the methodical reduction of Maxwell’s equations. We assume a flat space-time and the d’Alembertian (\(\Box\)) with the d’Alembertian (\(\Box\)) follows that the tensor (\(\nu_{\alpha\beta}\)) is taking irreversible effects such as viscosity and heat conduction into account. Both (e) and (\(R^{\alpha\beta}\)) depend on the constitutive relationships chosen to underpin the fluid model. The original form of the energy-momentum balance results from the 4-divergence of (3), \(\partial_{\alpha}(tu^{\alpha}) = 0\), the set of field equations is completed. Among the decisive advantages of the use of potentials in classical Maxwell theory is the achievement of self-adjointness and the finding of an associated Lagrangian. The question of whether this can also be achieved analogously for the potential-based equations (3) is pursued in the context of this paper. Important clues are provided around an index pair denote symmetrisation, with square brackets indicating skew-symmetrisation. From its symmetries, it follows that the tensor (\(\nu_{\alpha\beta}\)) may contain up to 21 independent entries, but by intelligent gauging this number can be reduced significantly. As secondary fields a symmetric second rank tensor (\(\tilde{a}^{\alpha\mu}\)) and a vector field (\(A^\mu\)) are defined as:

\[
\tilde{a}^{\alpha\beta} := 4\eta^{\beta\alpha}a_{\alpha\lambda}a_{\lambda\nu} - \eta^{\alpha\beta}a_{\alpha\lambda}a_{\lambda\nu},
\]

allowing to reformulate the relativistic energy-momentum balance in the potential-based first integral form as [1, 2]:

\[
\left(nmc^2 + \nu + p\right)u^\alpha u^\nu - p\eta^{\alpha\beta} + R^{\alpha\beta} = \Box \tilde{a}^{\alpha\beta} - 2\tilde{a}^{\beta\alpha}A^\beta + \eta^{\beta\alpha}\partial_{\alpha}A^\mu + 4\eta^{\beta\lambda}\partial_{\beta}\partial_{\alpha}a_{\alpha\lambda\nu},
\]

with the d’Alembertian (\(\Box := \partial_{\mu}\partial^{\mu} = \frac{1}{c^2}\partial_t^2 - \nabla^2\)) the particle density \(n\), the specific internal energy \(e = e(n, s)\), the pressure \(p = n^2\partial_t e/\partial n\) and the specific entropy \(s\). By \((u^\alpha) = (1, u_1/c, u_2/c, u_3/c)\), with \(\gamma^{-1} = \sqrt{1 - (u_1^2 + u_2^2 + u_3^2)/c^2}\) the 4-velocity is denoted and the friction tensor (\(R^{\alpha\beta}\)) is taking irreversible effects such as viscosity and heat conduction into account. Both \(e\) and (\(R^{\alpha\beta}\)) depend on the constitutive relationships chosen to underpin the fluid model. The original form of the energy-momentum balance results from the 4-divergence of (3), \(\partial_{\alpha}(u^\alpha u^\beta) = 0\). By the continuity equation:

\[
\partial_{\alpha}(nu^\alpha) = 0,
\]

the set of field equations is completed. Among the decisive advantages of the use of potentials in classical Maxwell theory is the achievement of self-adjointness and the finding of an associated Lagrangian. The question of whether this can also be achieved analogously for the potential-based equations (3) is pursued in the context of this paper. Important clues are provided by the earlier work [4], where a Lagrangian density was constructed for steady viscous flow in the non-relativistic case.

2 Methodical approach and result

2.1 Modified gauging of the tensor potential

According to [4, 5], a decisive condition for self-adjointness is the appropriate gauging of the potentials. Different from the gauging used in [1, 2] in order to eliminate the last term in the field equations (3), we assume that by gauging the following form of the fourth rank potential:

\[
a^{\nu\lambda\kappa\rho} = \frac{1}{4} \left[ \tilde{a}^{\nu\lambda\rho} - a^{\nu\rho}\eta^{\lambda\kappa} + a^{\nu\kappa}\eta^{\lambda\rho} \right] + \frac{1}{12} \left[ \eta^{\rho\kappa}\eta^{\lambda\rho} - \eta^{\rho\rho}\eta^{\lambda\kappa} \right] \Phi
\]

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can be reached with a traceless symmetric tensor potential $\bar{a}^{\alpha\kappa}$ and a scalar potential $\Phi$. Thus, (3) results in the tensor equation:

$$\begin{align*}
(nmc^2 + ne + p) \, u^\alpha u^\beta - p\eta^{\alpha\beta} + R^{\alpha\beta} &= \square \bar{a}^{\alpha\beta} - 2\partial(\alpha \, A^\beta) + \eta^{\alpha\beta} \partial_\mu A^\mu + \frac{1}{3} \left[ \partial_\alpha \partial^\beta \Phi - \eta^{\alpha\beta} \square \Phi \right] \\
A^\mu &:= \partial_\mu \bar{a}^{\alpha\mu}
\end{align*}$$

which is conveniently decomposed into its trace (via contraction with $\eta_{\alpha\beta}$):

$$nmc^2 + ne - 3p + R^{\alpha\alpha} = 2\partial_\mu A^\mu - \square \Phi$$

(8)

and the remaining traceless tensor equation:

$$\begin{align*}
(nmc^2 + ne + p) \left[ u^\alpha u^\beta - \frac{1}{4} \eta^{\alpha\beta} \right] + R^{\alpha\beta} &= \square \bar{a}^{\alpha\beta} - 2\partial(\alpha \, A^\beta) + \frac{1}{2} \eta^{\alpha\beta} \partial_\mu A^\mu + \frac{1}{3} \left[ \partial_\alpha \partial^\beta - \frac{1}{4} \eta^{\alpha\beta} \square \right] \Phi \\
(9)
\end{align*}$$

with $R^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{4} R^\lambda \eta_{\alpha\lambda}$. The decomposition of the tensor equation in its trace and a remaining traceless equation was a crucial step in [4] for finding a Lagrangian for steady incompressible viscous flow.

### 2.2 Suggestion of a Lagrangian

Lagrangian densities are not uniquely determined for a given set of field equations. Additional requirements are invariant to the Poincare group and occurrence of derivatives of at most first order. These are fulfilled by the following proposal:

$$\begin{align*}
\ell &= \left[ (nmc^2 + ne + p) \, u^\alpha u^\beta + R^{\alpha\beta} \right] \bar{a}^{\alpha\beta} + \frac{1}{2} \partial_\mu \bar{a}^{\alpha\beta} \partial_\alpha \bar{a}_{\beta\lambda} - A^\beta A_\beta - \frac{1}{3} A^\mu \partial_\mu \Phi \\
&+ \frac{1}{12} \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{6} \left( nmc^2 + ne - 3p + R^{\alpha\alpha} \right) \Phi - nu^\alpha \partial_\alpha \chi .
\end{align*}$$

(10)

Variation of the related action integral $\int \ell \, dx^0 \cdots dx^3$ with respect to the traceless symmetric tensor potential $\bar{a}^{\alpha\beta}$ delivers the traceless symmetric tensor equation (9) as associated Euler-Lagrange equations, while variation w.r.t. the scalar potential $\Phi$ reproduces the trace (8). Recombination of both leads to the full symmetric tensor equation (6), the 4-divergence of which recovers the original energy-momentum equations $\partial_\alpha T^{\alpha\beta} = 0$. In contrast to the paper [4], where due to incompressibility the continuity equation is automatically fulfilled by introducing a stream function vector, in the present case the continuity equation (4) is taken into account via a Lagrange multiplier $\chi$; it results as the Euler-Lagrange equation with respect to the latter. Finally, variation w.r.t. $n, s$ and the three independent components $u_1, u_2, u_3$ of the velocity deliver gauging conditions for the potentials $\bar{a}^{\alpha\beta}$, $\Phi$ and the evolution equation for the Lagrange multiplier $\chi$. These are not discussed here.

### 3 Conclusions and remaining issues

It has been demonstrated that the use of potentials in the relativistic theory of viscous flows enables a variational formulation — just as in Maxwell’s theory. This is already remarkable because comparable investigations for the classical Galilean invariant NS equation led to a Lagrangian density with discontinuities that only approximately reproduces the NS equation [6].

In contrast to Maxwell’s theory, however, a special gauging of the potentials is required. A detailed analysis of the latter and clarification of the meaning of the Lagrange multiplier $\chi$ is the primary goal of further investigations, also with a view to reducing the number of potentials. In the course of the same, the question also arises whether the variational formulation implies restrictions on the form of the friction tensor $R^{\alpha\beta}$ with regard to its dependence on temperature and velocity gradient. The detailed form of this tensor is still the subject of scientific dispute, especially with regard to the preservation of the principle of causality [7].

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