Spanning trees with at most 5 leaves and branch vertices in total of $K_{1,5}$-free graphs

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Abstract
In this paper, we prove that every $n$-vertex connected $K_{1,5}$-free graph $G$ with $\sigma_4(G) \geq n-1$ contains a spanning tree with at most 5 leaves and branch vertices in total. Moreover, the degree sum condition $\sigma_4(G) \geq n-1$ is best possible.

Keywords: spanning tree; $K_{1,5}$-free; degree sum
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1 Introduction

In this paper, we only consider finite simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, we use $N_G(v)$ and $d_G(v)$ (or $N(v)$ and $d(v)$ if there is no ambiguity) to denote the set of neighbors of $v$ and the degree of $v$ in $G$, respectively. For any $X \subseteq V(G)$, we denote by $|X|$ the cardinality of $X$. We define $N(X) = \bigcup_{x \in X} N(x)$ and $d(X) = \sum_{x \in X} d(x)$. For an integer $k \geq 1$, we let $N_k(X) = \{x \in V(G) \mid |N(x) \cap X| = k\}$. We use $G - X$ to denote the graph obtained from $G$ by deleting the vertices in $X$ together with their incident edges. The subgraph of $G$ induced by $X$ is denoted by $G[X]$. We define $G - uv$ to be the graph obtained from $G$ by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph obtained from $G$ by adding an edge $uv$ between two non-adjacent vertices $u$ and $v$ of $G$. We write $A := B$ to rename $B$ as $A$.

A subset $X \subseteq V(G)$ is called an independent set of $G$ if no two vertices of $X$ are adjacent in $G$. The maximum size of an independent set in $G$ is denoted by $\alpha(G)$. For $k \geq 1$, we define $\sigma_k(G) = \min\{\sum_{i=1}^{k} d(v_i) \mid \{v_1, \ldots, v_k\} \text{ is an independent set in } G\}$. For $r \geq 1$, a graph is said to be $K_{1,r}$-free if it does not contain $K_{1,r}$ as an induced subgraph. A $K_{1,3}$-free graph is also called a claw-free graph.

Let $T$ be a tree. A vertex of degree one is a leaf of $T$ and a vertex of degree at least three is a branch vertex of $T$. The set of leaves of $T$ is denoted by $L(T)$ and the set of branch vertices of $T$ is denoted by $B(T)$. For two distinct vertices $u, v$ of $T$, we denote by $P_T[u, v]$ the unique path in $T$ connecting $u$ and $v$ and denote by $d_T[u, v]$ the distance between $u$ and $v$ in $T$. We define the orientation of $P_T[u, v]$ is from $u$ to $v$.

There are many known results on the independence number conditions and the degree sum conditions to ensure that a connected graph $G$ contains a spanning tree with a bounded number of leaves or branch vertices. Win [20] obtained a sufficient condition related to the independence number for $k$-connected graphs having a few leaves, which confirms a conjecture of Las Vergnas [14]. On the other hand, Broersma and Tuinstra [1] gave a degree sum condition for a connected graph to contain a spanning tree with a bounded number of leaves. Beside that, recently, the first named author [7] stated an improvement of Win’s result by giving an independence number condition for a graph having a spanning tree which covers a certain subset of $V(G)$ and has at most $l$ leaves.

In 2012, Kano et al. [11] presented a degree sum condition for a connected claw-free graph to have a spanning tree with at most $l$ leaves, which generalizes a result of Matthews and Sumner [17] and a result of Gargano et al. [5]. Later, Chen et al. [2], Matsuda et al. [16] and Gould and Shull [6] also considered the sufficient conditions for a connected claw-free graph to have a spanning tree with few leaves or few branch vertices, respectively.

On the other hand, Kyaw [12, 13] obtained the sharp sufficient conditions for connected $K_{1,4}$-free graphs to have a spanning tree with few leaves. After that, many researchers also studied sufficient conditions for existence of spanning trees with few leaves or few branch vertices in connected $K_{1,4}$-free graphs (see Chen et al. [8] and Ha [8] for examples).

For the $K_{1,5}$-free graphs, some results were obtained as follows.

**Theorem 1.1 (Chen et al.)** Let $G$ be a connected $K_{1,5}$-free graph with $n$ vertices. If $\sigma_5(G) \geq n - 1$, then $G$ contains a spanning tree with at most 4 leaves.
Theorem 1.2 ([10, Hu and Sun]) Let $G$ be a connected $K_{1,5}$-free graph with $n$ vertices. If $\sigma_6(G) \geq n - 1$, then $G$ contains a spanning tree with at most 5 leaves.

Moreover, many researchers have also studied the degree sum conditions for graphs to have spanning trees with a bounded number of branch vertices and leaves.

Theorem 1.3 ([18, Nikoghosyan], [19, Saito and Sano]) Let $k \geq 2$ be an integer. If a connected graph $G$ satisfies $\deg_G(x) + \deg_G(y) \geq |G| - k + 1$ for every two non-adjacent vertices $x, y \in V(G)$, then $G$ has a spanning tree $T$ with $|L(T)| + |B(T)| \leq k + 1$.

In 2019, Maezawa et al. improved the previous result by proving the following theorem.

Theorem 1.4 ([15, Maezawa et al.,]) Let $k \geq 2$ be an integer. Suppose that a connected graph $G$ satisfies $\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G| - k + 1}{2}$ for every two non-adjacent vertices $x, y \in V(G)$, then $G$ has a spanning tree $T$ with $|L(T)| + |B(T)| \leq k + 1$.

Recently, Hanh and the first named author also gave sharp results for the case of claw-free graphs and $K_{1,4}$-free graphs, respectively.

Theorem 1.5 ([9, Hanh]) Suppose that a connected claw-free graph $G$ of order $n$ satisfies $\sigma_5(G) \geq n - 2$. Then $G$ has a spanning tree $T$ with $|B(T)| + |L(T)| \leq 5$.

Theorem 1.6 ([8, Ha]) Let $k, m$ be two non-negative integers ($m \leq k + 1$) and let $G$ be a connected $K_{1,4}$-free graph of order $n$. If $\sigma_{m+2}(G) \geq n - k$, then $G$ has a spanning tree with at most $m + k + 2$ leaves and branch vertices.

In this paper, we further consider connected $K_{1,5}$-free graphs. We give a sufficient condition for a connected $K_{1,5}$-free graph to have a spanning tree with few leaves and branch vertices in total. More precisely, we prove the following.

Theorem 1.7 Let $G$ be a connected $K_{1,5}$-free graph with $n$ vertices. If $\sigma_4(G) \geq n - 1$, then $G$ contains a spanning tree with at most 5 leaves and branch vertices in total.

It is easy to see that if a tree has at least 2 branch vertices then it has at least 4 leaves. Therefore, we immediately obtain the following corollary from Theorem 1.7.

Corollary 1.8 Let $G$ be a connected $K_{1,5}$-free graph with $n$ vertices. If $\sigma_4(G) \geq n - 1$, then $G$ contains a spanning tree with at most 1 branch vertices.

We end this section by constructing an example to show that the degree sum condition “$\sigma_4(G) \geq n - 1$” in Theorems 1.7 is sharp. For an integer $m \geq 1$, let $D_1, D_2, D_3, D_4$ be vertex-disjoint copies of the complete graph $K_m$ with $m$ vertices. Let $xy$ be an edge such that neither $x$ nor $y$ is contained in $\bigcup_{i=1}^{4} V(D_i)$. Join $x$ to all the vertices in $V(D_1) \cup V(D_2)$ and join $y$ to all the vertices in $V(D_3) \cup V(D_4)$. The resulting graph is denoted by $G$. Then it is easy to check that $G$ is a connected $K_{1,5}$-free graph with $n = 4m + 2$ vertices and $\sigma_4(G) = 4m = n - 2$. However, every spanning tree of $G$ contains at least 6 leaves and branch vertices in total.
2 Proof of the main result

In this section, we extend the idea of Chen-Ha-Hanh in [4] to prove Theorem 1.7. For this purpose, we need the following lemma.

Lemma 2.1 Let $G$ be a connected graph such that $G$ does not have a spanning tree with at most 5 leaves and branch vertices in total, and let $T$ be a maximal tree of $G$ with $|L(T)| + |B(T)| \in \{6,7\}$. Then there does not exist a tree $T'$ in $G$ such that $|L(T')| + |B(T')| \leq 5$ and $V(T') = V(T)$.

Proof. Suppose for a contradiction that there exists a tree $T'$ in $G$ with at most 5 leaves and branch vertices in total and $V(T') = V(T).$ Since $G$ has no spanning tree with at most 5 leaves and branch vertices in total, we see that $V(G) - V(T') \neq \emptyset$. Hence there must exist two vertices $v$ and $w$ in $G$ such that $v \in V(T')$ and $w \in N(v) \cap (V(G) - V(T'))).$ Let $T_1$ be the tree obtained from $T'$ by adding the vertex $w$ and the edge $vw$. Then $|L(T_1)| + |B(T_1)| - |L(T')| - |B(T')| \in \{0,1,2\}.$

If $|L(T_1)| + |B(T_1)| \in \{6,7\}$, then $T_1$ contradicts the maximality of $T$ (since $|V(T_1)| = |V(T)| + 1 > |V(T)|$). So we may assume that $|L(T_1)| + |B(T_1)| \leq 5$. By repeating this process, we can recursively construct a set of trees $\{T_i \mid i \geq 1\}$ in $G$ such that $|L(T_i)| + |B(T_i)| \leq 5$ and $|V(T_i+1)| = |V(T_i)| + 1$ for each $i \geq 1$. Since $G$ has no spanning tree with at most 5 leaves and branch vertices in total and $|V(G)|$ is finite, the process must terminate after a finite number of steps, i.e., there exists some $k \geq 1$ such that $T_{k+1}$ is a tree in $G$ such that $|L(T_{k+1})| + |B(T_{k+1})| \in \{6,7\}$. But this contradicts the maximality of $T$. So the lemma holds.

Proof of Theorem 1.7. We prove the theorem by contradiction. Suppose to the contrary that $G$ contains no spanning tree with at most 5 leaves and branch vertices in total. Then every spanning tree of $G$ contains at least 6 leaves and branch vertices in total. We choose a maximal tree $T$ of $G$ with $|L(T)| + |B(T)| \in \{6,7\}$.

We consider four cases according to the number of branch vertices in $T$. (Note that $T$ contains at most two branch vertices.)

Case 1. $T$ contains two branch vertices and four leaves.

Let $s$ and $t$ be the two branch vertices in $T$ and let $U = \{u_1; u_2; u_3; u_4\}$ be the set of leaves of $T$. Then $d_T(s) = d_T(t) = 3$. Moreover, by the maximality of $T$, we have $N(U) \subset V(T)$. For simplifying notation, let $[k]$ be the set of $\{1, 2, \ldots, k\}$ for some positive integer $k$.

For each $i \in [4]$, let $B_i$ be the vertex set of the connected component of $T - \{s,t\}$ containing $u_i$ and let $v_i$ be the unique vertex $B_i \cap N_T(\{s,t\})$. Without loss of generality, we may assume that $\{v_1,v_2\} \subset N_T(s)$ and $\{v_3,v_4\} \subset N_T(t)$. For each $1 \leq i \leq 4$ and $x \in B_i$, we use $x^-$ and $x^+$ to denote the predecessor and the successor of $x$ on $P_T[s,u_i]$ or $P_T[t,u_i]$, respectively (if such a vertex exists). Let $s^+$ be the successor of $s$ on $P_T[s,t]$. Define $P := V(P_T[s,t]) - \{s,t\}$.

For this case, we further choose $T$ such that

(C1) $d_T[s,t]$ is as small as possible.

Claim 2.2 For all $1 \leq i,j \leq 4$ and $i \neq j$, if $x \in N(u_j) \cap B_i$, then $x \neq u_i$, $x \neq v_i$ and $x^- \notin N(U - \{u_j\})$. 


Lemma 2.1. By applying Claim 2.2, we have that this contradicts the condition (C1). So the claim holds.

By Claim 2.2, we know that $U$ is an independent set in $G$.

Claim 2.3 $N(u_i) \cap P = \emptyset$ for each $i \in \{2\}$.

Proof. Suppose the assertion of the claim is false. Then there exists some vertex $x \in P$ such that $xu_i \in E(G)$ for some $i \in \{2\}$. Let $T' := T - v_i v_i^- + xu_i$, then $T'$ is a tree in $G$ such that $V(T') = V(T)$, $T'$ has 4 leaves and 1 branch vertex such that $d_T[s', t'] < d_T[s, t]$. But this contradicts the condition (C1). So the claim holds.

Claim 2.4 $N(u_i) \cap \{t\} = \emptyset$ for each $i \in \{2\}$.

Proof. Suppose $su_i \in E(G)$ for some $i \in \{2\}$. Consider the tree $T' := T - v_i v_i^- + tu_i$ is a tree in $G$ with 4 leaves and 1 branch vertex such that $V(T') = V(T)$, contradicting Lemma 2.1. This proves Claim 2.4

Similarly, we also have

Claim 2.5 $N(u_i) \cap \{s\} = \emptyset$ for each $3 \leq i \leq 4$.

Claim 2.6 $N_2(U - u_i) \cap B_i = \emptyset$ for each $i \in \{4\}$. In particular, $N_3(U) = (N_2(U) - N(u_i)) \cap B_i = \emptyset$ for each $i \in \{4\}$.

Proof. For the sake of convenience, we may assume by symmetry that $i \in \{2\}$.

Suppose this is false. Then there exists some vertex $x \in (N_2(U - u_i)) \cap B_i$ for some $i \in \{2\}$. By applying Claim 2.2, we have $x \neq u_i$ and $x \neq v_i$.

Since $x \in N_2(U - u_i) \cap B_i$ there must exist two distinct indices $j, k \in \{4\} - \{i\}, j < k$, such that $xu_j, xu_k \in E(G)$. Set

$$T' := \begin{cases} T - \{v_j v_j^-, v_k v_k^-\} + \{xu_j, xu_k\}, & \text{if } j = 3 - i, \\ T - \{ss^+, v_k v_k^-\} + \{xu_j, xu_k\}, & \text{if } 3 \leq j < k \leq 4, \end{cases}$$

Then $T'$ is a tree in $G$ with 1 branch vertex and 4 leaves such that $V(T') = V(T)$, contradicting Lemma 2.1.

By Claims 2.2 and 2.6, $\{u_i\}, N(u_i) \cap B_i$, and $(N(U - \{u_i\}) \cap B_i)^-$ are pairwise disjoint subsets in $B_i$ for each $i \in \{4\}$ (where $(N(U - \{u_i\}) \cap B_i)^- = \{x^- | x \in N(U - \{u_i\}) \cap B_i\}$) and $N_3(U) = (N_2(U) - N(u_i)) \cap B_i = \emptyset$ for each $i \in \{4\}$. Then for each $i \in \{4\}$, we conclude that

$$|B_i| \geq 1 + |N(u_i) \cap B_i| + |(N(U - \{u_i\}) \cap B_i)^-|$$

$$= 1 + |N(u_i) \cap B_i| + |N(U - \{u_i\}) \cap B_i|$$

$$= 1 + \sum_{j=1}^{4} |N(u_j) \cap B_i|.$$
By applying Claim 2.3, we obtain

$$
\sum_{i=1}^{4} |N(u_i) \cap P| = 0.
$$

On the other hand, by Claims 2.4, 2.5 we obtain that

$$
\sum_{i=1}^{4} |N(u_i) \cap \{s\}| \leq 2, \sum_{i=1}^{4} |N(u_i) \cap \{t\}| \leq 2.
$$

Note that $N(U) \subseteq V(T)$. Now, we conclude that

$$
|V(T)| = \sum_{i=1}^{4} |B_i| + |V(P_T[s, t])| \geq \sum_{i=1}^{4} \left( \sum_{j=1}^{4} |N(u_j) \cap B_i| + 1 \right) + \left( \sum_{i=1}^{4} |N(u_i) \cap \{s\}| + \sum_{i=1}^{4} |N(u_i) \cap \{t\}| - 2 + \sum_{i=1}^{4} |N(u_i) \cap P| \right) = 2 + \sum_{i=1}^{4} \sum_{j=1}^{4} |N(u_j) \cap B_i| + \sum_{i=1}^{4} |N(u_i) \cap \{s\}| + \sum_{i=1}^{4} |N(u_i) \cap \{t\}| + \sum_{i=1}^{4} |N(u_i) \cap P| = \sum_{j=1}^{4} |N(u_j) \cap V(T)| + 2 = \sum_{j=1}^{4} d(u_j) + 2 = d(U) + 2.
$$

Since $U$ is an independent set in $G$, we have

$$
n - 1 \leq \sigma_4(G) \leq d(U) \leq |V(T)| - 2 \leq n - 2,
$$

a contradiction.

Case 2. $T$ contains two branch vertices and five leaves.

Let $s$ and $t$ be the two branch vertices in $T$ such that $d_T(s) = 4$ and $d_T(t) = 3$. Let $U = \{u_1; u_2; u_3; u_4; u_5\}$ be the set of leaves of $T$. For each $i \in [5]$, let $B_i$ be the vertex set of the connected component of $T - \{s, t\}$ containing $u_i$ and let $v_i$ be the unique vertex in $B_i \cap N_T(\{s, t\})$. Without loss of generality, we may assume that $\{v_1, v_2, v_3\} \subseteq N_T(s)$ and $\{v_4, v_5\} \subseteq N_T(t)$. For each $i \in [5]$ and $x \in B_i$, we use $x^-$ and $x^+$ to denote the predecessor and the successor of $x$ on $P_T[s, u_i]$ or $P_T[t, u_i]$, respectively (if such a vertex exists). Let $s^+$ and $t^+$ be the successor of $s$ and the predecessor of $t$ on $P_T[s, t]$, respectively. Define $P := V(P_T[s, t]) - \{s, t\}$.

For this case, we choose $T$ such that

(D1) $d_T[s, t]$ is as small as possible, and
(D2) $\sum_{i=1}^{3} |B_i|$ is as large as possible, subject to (D1).

**Claim 2.7** For all $1 \leq i, j \leq 5$ and $i \neq j$, if $x \in N(u_j) \cap B_i$, then $x \neq u_i, x \neq v_i$ and $x^- \notin N(U - \{u_j\})$.

**Proof.** Suppose $x = u_i$ or $x = v_i$. Then $T' := T - v_i v_i^- + x u_j$ is a tree in $G$ with 4 leaves and at most 2 branch vertices such that $V(T') = V(T)$. Then this contradicts either Lemma 2.1 or the proof of Case 1. So we have $x \neq u_i, x \neq v_i$.

Next, assume $x^- \in N(U - \{u_j\})$. Then there exists some $k \in [5] - \{j\}$ such that $x^- u_k \in E(G)$. Now, $T' := T - \{v_i v_i^-, x x^-\} + \{x u_j, x^- u_k\}$ is a tree in $G$ with 4 leaves and at most 2 branch vertices such that $V(T') = V(T)$, also contradicting either Lemma 2.1 or the proof of Case 1. This proves Claim 2.7.

By Claim 2.7 we know that $U$ is an independent set in $G$. Since $G$ is $K_{1,5}$-free, we have $N_5(U) = \emptyset$.

**Claim 2.8** $N(u_i) \cap P = \emptyset$ for each $4 \leq i \leq 5$.

**Proof.** Suppose the assertion of the claim is false. Then there exists some vertex $x \in P$ such that $x u_j \in E(G)$ for some $i \in \{4, 5\}$. Let $T' := T - tv_i + xu_j$, then $T'$ is a tree in $G$ with 5 leaves such that $V(T') = V(T)$, $T'$ has two branch vertices $s$ and $x$, $d_T(s) = 4, d_T(x) = 3$ and $d_T[x, s] < d_T[s, t]$. But this contradicts the condition (D1). So the claim holds.

**Claim 2.9** If $P \neq \emptyset$, then $\sum_{i=1}^{3} |N(u_i) \cap \{x\}| \leq 1$ for each $x \in P$.

**Proof.** Suppose to the contrary that there exists some vertex $x \in P$ such that $\sum_{i=1}^{3} |N(u_i) \cap \{x\}| \geq 2$. Then there exist two distinct indices $j, k \in [3]$ such that $x u_j, x u_k \in E(G)$. Let $T' := T - \{x u_j, x u_k\} + \{x u_j, x u_k\}$, then $T'$ is a tree in $G$ with 5 leaves such that $V(T') = V(T)$, $T'$ has two branch vertices $x$ and $t$, $d_T(s) = 4, d_T(t) = 3$ and $d_T[x, t] < d_T[s, t]$, contradicting the condition (D1). This completes the proof of Claim 2.9.

**Claim 2.10** $N(u_i) \cap \{s\} = \emptyset$ for each $4 \leq i \leq 5$.

**Proof.** Suppose $s u_i \in E(G)$ for some $i \in \{4, 5\}$. If $P = \emptyset$, then we have $s t \in E(T)$ and $T' := T - st + s u_i$ is a tree in $G$ with $|L(T')| + |B(T')| = 5$ and $V(T') = V(T)$, contradicting Lemma 2.1. So we may assume that $P \neq \emptyset$ and hence $s^+ \neq t$. By applying Claims 2.7 and 2.8 we deduce that $N(u_i) \cap \{s^+, v_1, v_2, v_3\} = \emptyset$.

Suppose that $s^+ v_j \in E(G)$ for some $j \in [3]$. Then $T' := T - \{ss^+, s u_j\} + \{s u_i, s^+ v_j\}$ is a tree in $G$ with 4 leaves and 2 branch vertices such that $V(T') = V(T)$. By the same argument as in the proof of Case 1, we can derive a contradiction. So we conclude that $N(s^+) \cap \{v_1, v_2, v_3\} = \emptyset$.

Now, assume there exits two distinct $j, k \in [3]$ such that $v_j v_k \in E(G)$. Then by Claim 2.7 we see that $u_k \neq v_k$. Let $T'' := T - \{s v_j, t v_i\} + \{s u_i, v_j v_k\}$, then $T''$ is a tree in $G$ with 2 branch vertices and 5 leaves such that $V(T'') = V(T)$, $T''$ has two branch vertices $s$ and $v_k, d_T(s) = 4,$
Proof. Suppose for a contradiction that \( \sum_{i=1}^{5} |N(u_i) \cap \{t\}| \geq 4 \).

If \( P = \emptyset \) then we have \( st \in E(G) \). Since \( \sum_{i=1}^{5} |N(u_i) \cap \{t\}| \geq 4 \), there exists some \( j \in [3] \) such that \( tu_j \in E(G) \). Let \( T' := T - st + tu_j \), then \( T' \) is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \). Repeating the same argument as in the proof of Case 1, we can deduce a contradiction.

Otherwise, \( P \neq \emptyset \), then \( t^- \neq s \). It follows from Claim 2.8 that \( N(u_i) \cap \{t^-\} = \emptyset \) for each \( 4 \leq i \leq 5 \). Suppose that \( t^- u_j \in E(G) \) for some \( j \in [3] \). Since \( t \in N_4(U) \), there exists some \( k \in [3] - \{j\} \) such that \( tu_k \in E(G) \). Let \( T' := T - \{sv_j, tt^-\} + \{tu_k, t^- u_j\} \), then \( T' \) is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \), contradicting the proof of Case 1. Therefore, we deduce that \( N(U) \cap \{t^-\} = \emptyset \). But then, \( (N(t) \cap U) \cup \{t^-\} \) is an independent set and \( G[(N(t) \cap U) \cup \{t, t^-\}] \) is an induced \( K_{1,5} \) of \( G \), again a contradiction. So the claim holds.

Claim 2.12 We have \( N_3(U - \{u_i\}) \cap B_i = \emptyset \) for every \( i \in [5] \). In particular, we obtain \( N_4(U) = \emptyset \).

Proof. Suppose to the contrary that there exists some vertex \( x \in N_3(U - \{u_i\}) \) for some \( i \in [5] \). By Claim 2.7, we know that \( x^-, x^+ \notin N_3(U - \{u_i\}) \).

Suppose that \( x^- x^+ \in E(G) \). Since \( x \in N_3(U - \{u_i\}) \), there must exist two distinct \( j, k \in [5] - \{i\} \) such that \( xu_j, xu_k \in E(G) \). Then \( T' := T - \{v_j v_j^-, xx^-, xx^+\} + \{xu_j, xu_k, x^- x^+\} \) is a tree in \( G \) with 4 leaves and at least 2 branch vertices such that \( V(T') = V(T) \), contradicting either Lemma 2.1 or the proof of Case 1. Hence \( x^- x^+ \notin E(G) \).

Then \( (N(x) \cap (U - \{u_i\})) \cup \{x^-, x^+\} \) is an independent set and \( G[(N(x) \cap (U - \{u_i\})) \cup \{x, x^-, x^+\}] \) is an induced \( K_{1,5} \) of \( G \), contradicting the assumption that \( G \) is \( K_{1,5} \)-free. This completes the proof of Claim 2.12.

Claim 2.13 We have \( (N_3(U) - N(u_i)) \cap B_i = \emptyset \) for each \( 1 \leq i \leq 5 \).

Proof. Suppose this is false. Then there exists some vertex \( x \in (N_3(U) - N(u_i)) \cap B_i \) for some \( 1 \leq i \leq 5 \). By applying Claim 2.7, we have \( x \neq u_i, x \neq v_i \) and \( x^-, x^+ \notin N(U - \{u_i\}) \).

Suppose that \( x^- x^+ \in E(G) \). Since \( x \in N_3(U) - N(u_i) \), there must exist two distinct indices \( j, k \in [5] - \{i\} \) such that \( xu_j, xu_k \in E(G) \). Then \( T' := T - \{v_j v_j^-, xx^-, xx^+\} + \{xu_j, xu_k, x^- x^+\} \) is a tree in \( G \) with 4 leaves and at least 2 branch vertices such that \( V(T') = V(T) \), contradicting either Lemma 2.1 or the proof of Case 1. Hence \( x^- x^+ \notin E(G) \).

Now, \( (N(x) \cap U) \cup \{x^-, x^+\} \) is an independent set and \( G[(N(x) \cap U) \cup \{x, x^-, x^+\}] \) is an induced \( K_{1,5} \) of \( G \), giving a contradiction. So the assertion of the claim holds.
Claim 2.14 \( N(u_j) \cap B_i = \emptyset \) for all \( 4 \leq i \leq 5 \) and \( 1 \leq j \leq 3 \). In particular, \( N_3(U) \cap N(u_i) \cap B_i = \emptyset \) for each \( 4 \leq i \leq 5 \).

Proof. Suppose the assertion of the claim is false. Then there exists some vertex \( x \in B_i \) such that \( xu_j \in E(G) \) for some \( i \in \{4, 5\} \) and \( j \in [3] \). By Claim 2.7, we have \( x \neq u_i \) and \( x \neq v_i \). Let \( T' := T - xx^- + xu_j \), and let \( B_k \) be the vertex set of the connected component of \( T' - \{s, t\} \) containing \( u_k \) for each \( 1 \leq k \leq 3 \). It is easy to check that \( T' \) is a tree in \( G \) with 5 leaves such that \( V(T') = V(T) \), \( T' \) has two branch vertices \( s \) and \( t \), \( d_T'(s) = 4 \), \( d_T'(t) = 3 \), \( d_T'[s, t] = d_T[s, t] \) and \( \sum_{k=1}^{3} |B_k'| = \sum_{k=1}^{3} |B_k| + |V(P_T[x, u_i])| > \sum_{k=1}^{3} |B_k| \). But this contradicts the condition (D2). This proves Claim 2.14.

Claim 2.15 \( |N_3(U) \cap N(u_i) \cap B_i| \leq 1 \) for each \( 1 \leq i \leq 3 \).

Proof. Suppose for a contradiction that there exist two distinct vertices \( x, y \in N_3(U) \cap N(u_i) \cap B_i \) for some \( i \in [3] \). Without loss of generality, we may assume that \( x \in V(P_T[s, y]) \). By Claim 2.7, we have \( x \neq u_i, x \neq v_i, y \neq u_i, y \neq v_i, x^- \notin N(U) \) and \( x^+ \notin N(U - \{u_i\}) \). In particular, \( x^+ \neq y \). Since \( x, y \in N_3(U) \cap N(u_i) \), there exist two distinct \( j, k \in [5] - \{i\} \) such that \( xu_j, yu_k \in E(G) \). We may assume that \( x^-x^+, x^-u_i \notin E(G) \); for otherwise,

\[
T' := \begin{cases} 
T - \{sv_i, xx^-, xx^+, yy^+\} + \{xu_i, xu_j, x^-x^+, yu_k\}, & \text{if } x^-x^+ \in E(G), \\
T - \{sv_i, xx^+, yy^-\} + \{xu_j, x^-u_i, yu_k\}, & \text{if } x^-u_i \in E(G), 
\end{cases}
\]

is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \). By the same argument as in the proof of Case 1, we can deduce a contradiction. But then, \( (N(x) \cap U) \cup \{x^-, x^+\} \) is an independent set and \( G[(N(x) \cap U) \cup \{x^-, x^+\}] \) is an induced \( K_1, 5 \) of \( G \), again a contradiction. So the claim holds.

Claim 2.16 For each \( 1 \leq i \leq 3 \), if \( u_iv_i \in E(G) \), then \( N_3(U) \cap N(u_i) \cap B_i = \emptyset \).

Proof. Suppose to the contrary that \( u_iv_i \in E(G) \) and there exists some vertex \( x \in N_3(U) \cap N(u_i) \cap B_i \) for some \( i \in [3] \). By Claim 2.7, we have \( x \neq v_i \). Since \( x \in N_3(U) \cap N(u_i) \), there exists some \( j \in [5] - \{i\} \) such that \( xu_j \in E(G) \). Let \( T' := T - \{sv_i, xx^-\} + \{u_iv_i, xu_j\} \), then \( T' \) is a tree in \( G \) with 4 leaves and two branch vertices such that \( V(T') = V(T) \). By the same argument as in the proof of Case 1, we give a contradiction. This completes the proof of Claim 2.16.

Claim 2.17 For each \( 1 \leq i \leq 3 \), if \( sv_i \in E(G) \), then \( N_3(U) \cap N(u_i) \cap B_i = \emptyset \).

Proof. For the sake of convenience, we may assume by symmetry that \( i = 1 \). Suppose the assertion of the claim is false. Then there exists some vertex \( x \in N_3(U) \cap N(u_1) \cap B_1 \). By applying Claims 2.7 and 2.16, we know that \( x \notin \{u_1, v_1\} \) and \( N(u_1) \cap \{v_1, v_2, v_3\} = \emptyset \).

Suppose \( v_1v_j \in E(G) \) for some \( j \in \{2, 3\} \). Then \( T' := T - \{sv_1, sv_j\} + \{sv_4, v_1v_j\} \) is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \). By the same argument as in the proof of Case 1, we can deduce a contradiction. So we have \( v_1v_2, v_1v_3 \notin E(G) \).

Next, assume that \( v_2v_3 \in E(G) \). Then \( u_2 \neq v_2 \) and \( u_3 \neq v_3 \) by Claim 2.7. If there exists some \( j \in \{2, 3\} \) such that \( xu_j \in E(G) \), then \( T' := T - \{sv_2, sv_3\} + \{v_2v_3, xu_j\} \) is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \), contradicting Case 1. Hence
Therefore, we have\( xu_2, xu_3 \notin E(G) \). Then, since \( x \in N_3(U) \cap N(u_1) \), we conclude that \( xu_4, xu_5 \in E(G) \). Let \( T' := T - \{ sv_2, tv_-, xx^- \} + \{ su_1, tv_2, xu_4 \} \). If \( P = \emptyset \), then \( t^- = s \), and \( T' \) is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \), giving a contradiction with the proof of Case 1. So we deduce that \( P \neq \emptyset \). But then, \( T' \) is a tree in \( G \) with 5 leaves such that \( V(T') = V(T) \), \( T' \) has two branch vertices \( s \) and \( v_3 \), \( d_{T'}(s) = 4 \), \( d_{T'}(v_3) = 3 \) and \( d_{T'}[s, v_3] = 1 < d_{T}[s, t] \), contradicting the condition (D1). Therefore, \( v_1, v_2 \) and \( v_3 \) are pairwise non-adjacent in \( G \).

We now consider the vertex \( s^+ \). We will show that \( N(s^+) \cap \{ u_1, v_1, v_2, v_3 \} = \emptyset \).

We first prove that \( s^+ u_1 \notin E(G) \). Suppose this is false. Let \( T' := T - \{ ss^+, \{ s^+ u_1 \} \} \), then \( T' \) is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \). By the same argument as in the proof of Case 1, we can deduce a contradiction.

Finally, we show that \( s^+ v_2, s^+ v_3 \notin E(G) \). Suppose not, and let \( s^+ v_j \in E(G) \) for some \( j \in \{ 2, 3 \} \). If there exists some \( k \in \{ 4, 5 \} \) such that \( xu_k \in E(G) \), then \( T' := T - \{ ss^+, sv_j \} + \{ s^+ v_j, xu_k \} \) is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \). Repeating the same argument as in the proof of Case 1, we can deduce a contradiction. Therefore, we have \( xu_4, xu_5 \notin E(G) \). Since \( x \in N_3(U) \cap N(u_1) \), we deduce that \( xu_2, xu_3 \in E(G) \). Let \( T' := T - \{ ss^+, sv_j, xx^-, xx^+ \} + \{ su_1, s^+ v_j, xu_2, xu_3 \} \), then \( T' \) is a tree in \( G \) with 4 leaves and 2 branch vertices such that \( V(T') = V(T) \), again a contradiction. Hence \( N(s^+) \cap \{ u_1, v_1, v_2, v_3 \} = \emptyset \).

Now, \( \{ s^+, u_1, v_1, v_2, v_3 \} \) is an independent set and \( G[\{ s, s^+, \{ u_1, v_1, v_2, v_3 \} \} \) is an induced \( K_{1, 5} \) of \( G \), giving a contradiction. So the assertion of the claim holds.

By Claim 2.7, \( \{ u_i \}, N(u_i) \cap B_i, (N(U - \{ u_i \}) \cap B_i) \) and \( (N_3(U) - N(u_i)) \cap B_i \) are pairwise disjoint subsets in \( B_i \) for each \( i \in \{ 5 \} \), where \( (N(U - \{ u_i \}) \cap B_i) = \{ x^- \mid x \in N(U - \{ u_i \}) \cap B_i \} \). Recall that \( N_5(U) = N_4(U) = N_3(U) - N(u_i) \cap B_i = \emptyset \) (for each \( 1 \leq i \leq 5 \)) by Claims 2.12 and 2.13.

Then for each \( i \in \{ 3 \} \), we conclude that

\[
|B_i| \geq 1 + |N(u_i) \cap B_i| + |(N(U - \{ u_i \}) \cap B_i) - |(N_3(U) - N(u_i)) \cap B_i|
\]

\[
= 1 + |N(u_i) \cap B_i| + |N(U - \{ u_i \}) \cap B_i| + |(N_3(U) - N(u_i)) \cap B_i|
\]

\[
= 1 + \sum_{j=1}^{5} |N(u_j) \cap B_i| - |N_3(U) \cap N(u_i) \cap B_i|
\]

\[
\geq \sum_{j=1}^{5} |N(u_j) \cap B_i| + |N(u_i) \cap \{ s \}|,
\]

(1)

where the last inequality follows from Claims 2.17 and 2.19. Similarly, for each \( 4 \leq i \leq 5 \), we have

\[
|B_i| \geq 1 + |N(u_i) \cap B_i| + |(N(U - \{ u_i \}) \cap B_i) - |(N_3(U) - N(u_i)) \cap B_i|
\]

\[
= 1 + |N(u_i) \cap B_i| + |N(U - \{ u_i \}) \cap B_i| + |(N_3(U) - N(u_i)) \cap B_i|
\]

\[
= 1 + \sum_{j=1}^{5} |N(u_j) \cap B_i| - |N_3(U) \cap N(u_i) \cap B_i|
\]

\[
= 1 + \sum_{j=1}^{5} |N(u_j) \cap B_i| + |N(u_i) \cap \{ s \}|,
\]

(2)
where the last equality follows from Claims 2.10 and 2.14.

For each $1 \leq i \leq 5$, we define $d_i = |N(u_i) \cap P|$. Then $d_4 = d_5 = 0$ by Claim 2.3. By applying Claim 2.9 we know that $N(u_1) \cap P, N(u_2) \cap P$ and $N(u_3) \cap P$ are pairwise disjoint. Therefore,

$$|P| \geq \sum_{i=1}^{5} d_i = \sum_{i=1}^{5} |N(u_i) \cap P|.$$  

By combining Claim 2.11 we have

$$|V(P_T[s, t])| = 2 + |P| \geq \sum_{i=1}^{5} |N(u_i) \cap \{t\}| + \sum_{i=1}^{5} |N(u_i) \cap P| - 1. \quad (3)$$

Note that $N(U) \subseteq V(T)$. By (1), (2) and (3), we conclude that

$$|V(T)| = \sum_{i=1}^{3} |B_i| + \sum_{i=4}^{5} |B_i| + |V(P_T[s, t])|$$

$$\geq \sum_{i=1}^{3} \left( \sum_{j=1}^{5} |N(u_j) \cap B_i| + |N(u_i) \cap \{s\}| \right) + \sum_{i=4}^{5} \left( 1 + \sum_{j=1}^{5} |N(u_j) \cap B_i| + |N(u_i) \cap \{s\}| \right)$$

$$+ \left( \sum_{i=1}^{5} |N(u_i) \cap \{t\}| + \sum_{i=1}^{5} |N(u_i) \cap P| - 1 \right)$$

$$= 2 + \sum_{i=1}^{5} \sum_{j=1}^{5} |N(u_j) \cap B_i| + \sum_{i=1}^{5} |N(u_i) \cap \{s, t\}| + \sum_{i=1}^{5} |N(u_i) \cap P|$$

$$= \sum_{j=1}^{5} |N(u_j) \cap V(T)| + 1$$

$$= \sum_{j=1}^{5} d(u_j) + 1$$

$$= d(U) + 1.$$  

Since $U$ is an independent set in $G$, we have

$$n - 1 \leq \sigma_4(G) \leq \sigma_5(G) - 1 \leq d(U) - 1 \leq |V(T)| - 2 \leq n - 2,$$

a contradiction.

**Case 3.** $T$ contains one branch vertex and five leaves.

Let $r$ be the unique branch vertex in $T$ with $d_T(r) = 5$ and let $N_T(r) = \{v_1, v_2, v_3, v_4, v_5\}$. For each $i \in [5]$, let $B_i$ be the vertex set of the connected component of $T - \{r\}$ containing $u_i$ and let $v_i$ be the unique vertex in $B_i \cap N_T(\{r\})$. For each $i \in [5]$ and $x \in B_i$, we use $x^−$ and $x^+$ to denote the predecessor and the successor of $x$ on $P_T[r, u_i]$, respectively (if such a vertex exists).
Since $G$ is $K_{1,5}$-free, there exist two distinct indices $i, j \in [5]$ such that $u_iv_j \in E(G)$. Let $T' := T - rv_i + v_iv_j$. If $v_j$ is a leaf of $T$, then $T'$ is a tree in $G$ with 4 leaves and 1 branch vertex such that $V(T') = V(T)$, which contradicts Lemma 2.1. So we may assume that $v_j$ has degree two in $T$. Then $T'$ is a tree in $G$ with 5 leaves such that $V(T') = V(T), T'$ has two branch vertices $r$ and $v_j$, $d_{T'}(r) = 4$ and $d_{T'}(v_j) = 3$. By the same argument as in the proof of Case 2, we can also derive a contradiction.

Case 4. $T$ contains one branch vertex and six leaves.

Let $r$ be the unique branch vertex in $T$ with $d_T(r) = 6$ and let $U = \{u_i\}_{i=1}^6$ be the set of leaves of $T$. For each $i \in [6]$, let $B_i$ be the vertex set of the connected component of $T - \{r\}$ containing $u_i$ and let $v_i$ be the unique vertex in $B_i \cap N_T(\{r\})$. For each $i \in [6]$ and $x \in B_i$, we use $x^-$ and $x^+$ to denote the predecessor and the successor of $x$ on $P_T[r, u_i]$, respectively (if such a vertex exists).

Claim 2.18 For all $1 \leq i, j \leq 6$ and $i \neq j$, if $x \in N(u_j) \cap B_i$, then $x \neq u_i, x \neq v_i$ and $x^- \notin N(U - \{u_j\})$.

Proof. Suppose $x = u_i$ or $x = v_i$. Then $T' := T - v_i v_j^- + xu_j$ is a tree in $G$ with 5 leaves and 1 branch vertex such that $V(T') = V(T)$. By the same argument as in the proof of Case 3, we can derive a contradiction. So we have $x \neq u_i, x \neq v_i$.

Next, assume $x^- \in N(U - \{u_j\})$. Then there exists some $k \in [6] - \{j\}$ such that $x^- u_k \in E(G)$. Now, $T' := T - \{v_i v_j^-, x^-\} + \{xu_j, x^- u_k\}$ is a tree in $G$ with 5 leaves and 1 branch vertex such that $V(T') = V(T)$. By the same argument as in the proof of Case 3, we can deduce a contradiction. This proves Claim 2.18.

By applying Claims 2.18 we deduce that $U$ is an independent set in $G$.

Claim 2.19 For every $1 \leq i, j \leq 6$ and $i \neq j$, if $v_i v_j \in E(G)$ then $N(u_i) \cap B_k = \emptyset$ and $N(u_k) \cap B_i = \emptyset$ for each $k \in [6] - \{i, j\}$.

Proof. Suppose to the contrary that there exists some vertex $x \in B_k$ such that $xu_i \in E(G)$. By Claim 2.18 we have $x \neq u_k$. Let $T' := T - \{v_i v_j^-, x^-\} + \{xu_j, v_i v_j\}$. Then $T'$ is a tree in $G$ with 5 leaves such that $V(T') = V(T), T'$ has two branch vertices $x, r$. This implies a contradiction by using the proof of Case 2.

Now, suppose that there exists some vertex $x \in B_i$ such that $xu_k \in E(G)$. By Claim 2.18 we have $x \neq u_i$. Let $T' := T - \{v_i v_j^-, v_j v_j^-\} + \{xu_k, v_i v_j\}$. Then $T'$ is a tree in $G$ with 5 leaves such that $V(T') = V(T), T'$ has two branch vertices $x, r$. This also gives a contradiction by using the same arguments as in the proof of Case 2.

The proof of Claim 2.19 is completed.

Since $G$ is $K_{1,5}$-free, there exist two distinct indices $i, j \in [5]$ such that $v_i v_j \in E(G)$. Without loss of generality, we may assume that $v_1 v_2 \in E(G)$.

Set $U_1 = \{u_1, u_2, u_3, u_4\}$. By Claim 2.19 we obtain that

$$N(U_1) \cap B_j = N(\{u_i\}_{i=1}^2) \cap B_j \text{ for all } j \in \{1, 2\},$$

and

$$N(U_1) \cap B_j = N(\{u_i\}_{i=3}^4) \cap B_j \text{ for all } j \in \{3, 4, 5, 6\}.$$

By Claim 2.18, $\{u_i\}, N(u_i) \cap B_i$ and $(N(U_1 - \{u_i\}) \cap B_i)^-$ are pairwise disjoint subsets in $B_i$ for each $i \in [4]$, where $(N(U_1 - \{u_i\}) \cap B_i)^- = \{x^- \mid x \in N(U_1 - \{u_i\}) \cap B_i\}$. Recall that
$N_4(U_1) = (N_3(U_1) - N(u_i)) \cap B_i = (N_2(U_1) - N(u_i)) \cap B_i = \emptyset$ (for each $1 \leq i \leq 4$). Then for each $i \in \{4\}$, we conclude that

$$|B_i| \geq 1 + |N(u_i) \cap B_i| + |(N(U_1 - \{u_i\}) \cap B_i)|$$

$$= 1 + |N(u_i) \cap B_i| + |N(U_1 - \{u_i\}) \cap B_i| + |(N_2(U_1) - N(u_i)) \cap B_i|$$

$$= 1 + \sum_{j=1}^{4} |N(u_j) \cap B_i|.$$  \hspace{1cm} (6)

On the other hand, by Claim 2.18, $\{u_i\}$, $N(u_3) \cap B_i$, and $(N(u_4) \cap B_i)^-$ are pairwise disjoint subsets in $B_i$ for each $i \in \{5, 6\}$. Then for each $i \in \{5, 6\}$, we conclude that

$$|B_i| \geq 1 + |N(u_3) \cap B_i| + |(N(u_4) \cap B_i)^-|$$

$$= 1 + |N(u_3) \cap B_i| + |N(u_4) \cap B_i|$$

$$= 1 + \sum_{j=1}^{4} |N(u_j) \cap B_i|.$$  \hspace{1cm} (7)

By (6) and (7), we conclude that

$$|V(T)| = 1 + \sum_{i=1}^{6} |B_i|$$

$$\geq 1 + \sum_{i=1}^{6} \left( 1 + \sum_{j=1}^{6} |N(u_j) \cap B_i| \right)$$

$$= 7 + \sum_{i=1}^{6} \sum_{j=1}^{4} |N(u_j) \cap B_i|$$

$$\geq 3 + \sum_{i=1}^{6} \sum_{j=1}^{4} |N(u_j) \cap B_i| + \sum_{j=1}^{4} |N(u_j) \cap \{r\}|$$

$$= \sum_{j=1}^{4} |N(u_j) \cap V(T)| + 3$$

$$= \sum_{j=1}^{4} d(u_j) + 3$$

$$= d(U_1) + 3.$$

Since $U_1$ is an independent set in $G$, we have

$$n - 1 \leq \sigma_4(G) \leq d(U_1) \leq |V(T)| - 3 \leq n - 3.$$

This also gives a contradiction.

This completes the proof of Theorem 1.7.
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