CHERN-SIMONS GAUGE THEORY AS A STRING THEORY

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ABSTRACT

Certain two dimensional topological field theories can be interpreted as string theory backgrounds in which the usual decoupling of ghosts and matter does not hold. Like ordinary string models, these can sometimes be given space-time interpretations. For instance, three-dimensional Chern-Simons gauge theory can arise as a string theory. The world-sheet model in this case involves a topological sigma model. Instanton contributions to the sigma model give rise to Wilson line insertions in the space-time Chern-Simons theory. A certain holomorphic analog of Chern-Simons theory can also arise as a string theory.

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1. Introduction

In this paper, I will describe how Chern-Simons gauge theory in three dimensions can be viewed as a string theory. The string theory in question will be constructed using a topological sigma model [1] (related to Floer/Gromov theory) in which the target space is $T^*M$, $M$ being a three-manifold. The perturbation theory of this string theory will coincide with Chern-Simons perturbation theory, in the form that this has been presented by Axelrod and Singer [4] and further studied by Kontsevich [5]. Mathematically, the idea is roughly that there are no instantons with target space $T^*M$ and boundary values in $M$, so in an appropriate topological field theory, the usual counting of instantons is vacuous. However, there are virtual instantons at infinity; their proper counting leads to Chern-Simons perturbation theory. Chern-Simons theory enters in this particular string theory in much the same way that ordinary space-time physics (with general relativity as the long wavelength limit) arises in conventional string theory.

Physically, one might take the following as the starting point. String theorists usually construct two dimensional field theories describing particular classical solutions of string theory by constructing a “matter” system, of total central charge 26, and coupling it to the ghosts, of central charge $-26$. The vanishing total central charge ensures the existence of a BRST operator $Q$, obeying $Q^2 = 0$, and playing a crucial role in world-sheet and space-time gauge invariance. One knows, however [7], that $Q$ can be interpreted as a generator of linearized gauge transformations, mixing ghosts and matter, so the assumption that the matter and ghosts are decoupled cannot be valid as a fundamental principle; it is merely a (partial) gauge condition and very likely cannot be imposed at all in some situations, perhaps time dependent ones.

So in §2, we will ask what is left of the standard structure if one does not assume decoupling of matter from ghosts. From the discussion, it will become obvious that exotic realizations of the same basic structure can be constructed using topological sigma models; we therefore do this in §3. The resulting models
can be considered for either open or closed strings, but the open string case is in some ways easier to understand. In §4, we use open-string field theory as a shortcut to determining the space-time interpretation of the open-string version of our models, with the result - alluded to above - that one type of model is equivalent in perturbation theory to three dimensional Chern-Simons gauge theory. This is possibly the first time that the background independent, gauge invariant space-time interpretation of a string theory has been completely determined. (But see [8] for a previous investigation of the space-time interpretation of some topological string theories of a rather different flavor.) As one application, our result explains certain observations by Kontsevich about Chern-Simons gauge theory, and we will make a small digression on that account. Another version of the theory has for its space-time interpretation a sort of holomorphic version of Chern-Simons theory. In §5, we attempt to discuss the closed string sector (which among other things should be more closely related to usual manifestations of mirror symmetry). This concluding section is brief since I do not understand it.

To keep this paper within reasonable length, it has not been possible to give a self-contained explanation of all of the relevant background. The relevant material on topological sigma models can be found in [6]; for their coupling to topological gravity see [1]. The relation of Riemann surfaces to gauge theories will be briefly recalled presently. Apart from this, it is helpful to have some familiarity with construction of string models, the relation between world-sheet and space-time physics, string field theory, and Chern-Simons perturbation theory. The main point of the paper is the unexpected relation of those latter topics to each other and to topological sigma models.

Riemann Surfaces And Gauge Theories

Riemann surfaces enter our story in two ways. On the one hand, when we study two dimensional quantum field theory, Riemann surfaces are present at the beginning. Then we wish to show how gauge theory in the target space emerges.
Fig. 1. A two loop Feynman diagram (a) and its thickening (b), in which the boundary components are labeled by gauge indices.

Fig. 2. The cell decomposition of open string moduli space depends on building Riemann surfaces by gluing together flat strips $S_i$ of width $\pi$ and variable length $T_i$, $0 \leq T_i \leq \infty$. The strips are glued together in groups of three along the dotted lines; their midpoints meet in conical singularities of deficit angle $-\pi$ that are marked as solid dots. The resemblance of this figure to figure (1) is central in this paper.

On the other hand, oriented two dimensional manifolds arise in gauge theories with $U(N)$ gauge group in the following fashion, due originally to 't Hooft [9]; see also [10,§4] for a recent explanation. ($SO(N)$ and $Sp(N)$ can be considered similarly and lead to not necessarily orientable Riemann surfaces.) Though the following general comments also go through for ordinary Yang-Mills theory in any dimension, let us to be definite consider the Chern-Simons action for a $U(N)$-valued connection $A$ on a three manifold $M$:

$$I = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

(1.1)

Consider expanding this theory in perturbation theory in $1/k$, say around the trivial connection. The successive terms are all topological invariants. The coefficient
of $1/k^r$ comes from Feynman diagrams with $r$ loops, and is a function of $N$. For given $r$, by considering the dependence on $N$, how many invariants can be extracted? This question can be conveniently answered as follows. The gauge field $A$ is essentially a one-form valued in hermitian $N \times N$ matrices. Write $A$ as $A^i_j$, $i, j = 1 \ldots N$, making the matrix indices explicit while leaving implicit the fact that $A$ is a one-form. The interaction

$$\text{Tr} A \wedge A \wedge A = A^i_j \wedge A^j_k \wedge A^k_i \quad (1.2)$$

involves sewing the “lower” index of one $A$ field to the “upper” index of the next. To exhibit this in drawing Feynman diagrams, represent the $A$ propagator not as a line but as a slightly thickened band in which one edge represents the “upper” index and the other line represents the “lower” index. The index flow at the cubic vertex (1.2) is then neatly incorporated by smoothly joining the bands, as in figure (1(b)). In the process, the Feynman diagram of figure (1(a)) has been replaced by the Riemann surface $\Sigma$ of figure (1(b)). Each boundary component is labeled by a gauge index $i$ running from 1 to $N$. If there are $h$ boundary components, the sum over gauge indices gives a factor of $N^h$. The coefficients of $N^h/k^r$ are the three-manifold invariants that can be extracted from Chern-Simons theory with gauge group $U(N)$.

The genus of $\Sigma$ is $g = (r - h + 1)/2$. As $r$ and $h$ vary, $g$ and $h$ vary independently, and $\Sigma$ varies over all topological types of oriented two dimensional surface with boundary. Thus, $U(N)$ gauge theory gives us a three manifold invariant $\Gamma_{g,h}(M)$ for every topological type of oriented two dimensional surface with boundary. On the other hand, in §3-4 we will consider a two dimensional topological field theory (closely related to the counting of almost holomorphic curves) in which one can conveniently take the target space to be $T^*M$, the cotangent bundle of a three-manifold $M$. The partition function $Z_{g,h}(M)$ of this theory formulated on a Riemann surface of genus $g$ with $h$ holes is, again, a three-manifold invariant depending on $g$ and $h$. Our main conclusion is that $\Gamma_{g,h}(M) = Z_{g,h}(M)$. In reaching this conclusion, the link between the Riemann surfaces of the two dimensional
field theory and those of $U(N)$ gauge theory is provided by the fatgraph [11–13] or open string field theory [14,15] description of the moduli space of Riemann surfaces with boundary; in this description, complex Riemann surfaces are built up from pictures similar to that of figure (1(b)), as we will recall in more detail in §4.

2. Axioms

In this section, we will discuss what remains of the usual structures in string theory if one does not assume that ghosts and matter are decoupled. $\Sigma$ will be a Riemann surface of genus $g$ with local coordinates $x^\alpha, \alpha = 1, 2$. The symbols $J$ and $h$ will be used to denote a complex structure and a metric on $\Sigma$; of course a metric determines a complex structure. The space of all complex structures will be called $J$; the group of diffeomorphisms of $\Sigma$ will be called $G$. The quotient $J/G$ is the moduli space of complex structures on $\Sigma$. The space of metrics on $\Sigma$ will be called $K$.

Usually one considers conformally invariant world-sheer theories consisting of “matter,” of central charge $c = 26$, and “ghosts,” of $c = -26$. As explained in the introduction, we want to drop the assumption of this decoupling; so if conformal invariance is retained, we preserve the fact that the total central charge is $c = 0$, but no longer build this from a cancellation between different contributions. We will, however, not necessarily retain the assumption of conformal invariance.

One part of the structure that must be kept for all that follows is the existence of a BRST operator $Q$ with $Q^2 = 0$, and with the further property that the stress-tensor $T_{\alpha\beta}$ can be written as

$$T_{\alpha\beta} = \{Q, b_{\alpha\beta}\}, \quad (2.1)$$

for some field $b_{\alpha\beta}$ – which in the usual case is called the antighost field. This ensures general covariance – or topological invariance – of the two dimensional
theory. In the conformally invariant case, $b$ and $T$ are traceless. The stress tensor is conserved, $D_\alpha T^\alpha_\beta = 0$, and so we will assume that the $b$ field obeys

$$D_\alpha b^\alpha_\beta = 0$$

which when $b$ is traceless is its usual equation of motion. In addition, in the usual constructions, there are no short-distance singularities in operator products of $b$ fields, and there are only the standard short-distance singularities in products of $b$ and $T$. Concerning the generalization, topological sigma models are an example showing that these assumptions are too strong. In those models, there is a delta function contact term in the $b \cdot T$ product. (This results in the $\rho \psi \psi$ term in the coupling to topological gravity, eqn. (4.18) of [1].) I do not know what sort of singularities should be allowed in $b \cdot b$ and $b \cdot T$ operator products, in general, except to say that they must be such that a coupling of the model to topological gravity must be possible.

Another feature of the usual case that we wish to preserve is that there is a “ghost number” operator $G$ (mathematically it would usually be called the dimension), with $Q$ and $b$ having $G = 1$ and $G = -1$, respectively. Moreover, we wish to preserve the usual fact that the “ghost number of the vacuum” is $-3\chi(\Sigma)$ where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$; this means that only an operator product of total ghost number $3\chi(\Sigma)$ can have a non-zero vacuum expectation value. For instance, for an orientable Riemann surface of genus $g$, the case that we will consider in this section for definiteness, $3\chi(\Sigma) = -(6g - 6).

A particularly important case (in genus $\geq 2$ for simplicity, to avoid ghost zero modes) is the expectation value of a product of $6g - 6$ $b$’s. Let $\delta^{(k)}h$, $k = 1, \ldots, 6g - 6$ be $6g - 6$ variations of $h$ (that is, $6g - 6$ tangent vectors to the space $\mathcal{K}$ of metrics). Let

$$b^{(k)} = \int_\Sigma d^2x \sqrt{h} h^{\alpha\alpha'} h^{\beta\beta'} \delta^{(k)} h_{\alpha\beta} \cdot b_{\alpha'\beta'} \quad (2.3)$$
be the corresponding modes of $b$, and let

$$\Theta(\delta^{(1)}h,\ldots,\delta^{(6g-6)}h) = \langle b^{(1)}\ldots b^{(6g-6)} \rangle,$$  \hspace{1cm} (2.4)

with $\langle \quad \rangle$ the expectation value in the “measure” determined by the genus $g$ world-sheet path integral. By fermi statistics, $\Theta$ is skew-symmetric in the $\delta^{(k)}h$’s and so can be interpreted as a $6g-6$ form on $\mathcal{K}$. In the usual case, one proves that $\Theta$ is closed starting with

$$0 = \langle \{Q, b^{(1)} \ldots b^{(6g-5)} \} \rangle = \sum_{j=1}^{6g-5} (-1)^{j-1} \langle b^{(1)} \ldots \{Q, b^{(j)} \} \ldots b^{(6g-5)} \rangle.$$  \hspace{1cm} (2.5)

Then using (2.1) to write $\{Q, b^{(j)} \}$ as a moment of $T$, and interpreting the insertion of $T$ as a derivative on $\mathcal{K}$, one can interpret (2.5) as a first order differential equation for $\Theta$ which amounts to

$$d\Theta = 0.$$  \hspace{1cm} (2.6)

This proof goes through in general if the $b\cdot b$ and $b\cdot T$ operator products are standard; however, topological sigma models are an example in which the definition of $\Theta$ must be modified by addition of contact terms to ensure $d\Theta = 0$. A successful coupling of the model to topological gravity will always ensure the existence of a suitable modification (by addition of contact terms) of the definition of $\Theta$.

$\Theta$ is obviously diffeomorphism invariant. Moreover, in the usual case it is “basic”; that is, it vanishes if any $\delta^{(k)}h$ is of the form $\delta^{(k)}h_{\alpha\beta} = D_{\alpha}v_{\beta} + D_{\beta}v_{\alpha}$ induced by an infinitesimal diffeomorphism $x^{\alpha} \rightarrow x^{\alpha} + \epsilon v^{\alpha}$. This is proved by integrating by parts, using (2.2) and the absence of short distance singularities of the $b$’s. In general (because of the relation of the topological gravity multiplet to the equivariant cohomology of the diffeomorphism group), a successful coupling to topological gravity will always ensure that $\Theta$ is basic.
The fact that $\Theta$ is basic and diffeomorphism invariant means that it can be interpreted as the pullback of a closed $6g-6$ form $\nu$ on the space $K/G$ of metrics modulo diffeomorphisms. In the conformally invariant case, $\Theta$ is actually a pullback from a form $\mu$ on the moduli space $M$ of complex structures; as $M$ is $6g-6$ dimensional, $\mu$ is a top form or measure on $M$. If conformal invariance does not hold, there is no natural way to construct from $\mu$ a measure on $M$, but the fact that $\nu$ is closed, $d\nu = 0$, is almost as good. It means that if $s$ is a section of the bundle $K/G \to M$, and $\mu = s^*(\nu)$, then $\mu$ is independent of the choice of $s$, modulo an exact form. $\mu$ is the desired $6g-6$-form on $M$.

**Vertex Operators And Ghosts**

The most traditional string theory vertex operators can be written $W = c\bar{c}V$ where $c$ and $\bar{c}$ are the ghosts and $V$ is a spin $(1,1)$ conformal field constructed from matter fields only. It is well known in the operator formalism [16] that these conditions can be relaxed. The only really essential properties of $W$ in conventional string theory are that it have ghost number 2, spin zero, and be annihilated by $Q$, and that the Fock space state corresponding to $W$ should be annihilated by an operator usually written as $b_0^- = b_0 - \bar{b}_0$ (here $b_0$ and $\bar{b}_0$ are the zero modes of left- and right-moving ghosts). The requirement that $W$ have spin zero and be annihilated by $b_0^-$ ensures that the analogs of $\Theta$ defined with an insertion of $W$ have the essential properties that entered above (diffeomorphism invariant and basic). The ghost number two condition shifts the ghost number of the vacuum, leading in a natural way to measures on the moduli space of Riemann surfaces with marked points.

These conditions make sense in our abstract setting, and can be borrowed bodily, except that if we do not assume conformal invariance, the definition of $b_0^-$ must be written $b_0^- = \oint dx^\alpha b_{\alpha 0}$. (The integration is over a parametrized circle and “0” is the normal direction to the circle.) Note, therefore, that in including vertex operators, we need make no explicit mention of the ghosts; only the product $W = c\bar{c}V$, and not the separate factors in that product, needs to be generalized.
Where, therefore, do the ghosts appear in the discussion? Of course, the antighosts entered in the definition of the measure for $g \geq 2$; but what about the ghosts?

We perhaps could “find” the ghosts by trying to generalize the usual definition of the genus one measure – as $c$ and $\overline{c}$ each has a zero mode on a surface of genus one. To get a different perspective, I will instead discuss the issue from the standpoint of closed string field theory.

In closed string field theory as we know it today (see [17] for a review), the string field $\Psi$ is a vector in a string Hilbert space obtained by quantization on a circle $S$. Picking a parametrization of the circle, let $L_0^- = L_0 - \overline{L}_0$ be the generator of a rotation of the circle, and let $c_0^- = (c_0 - \overline{c}_0)/2$. $\Psi$ is required to obey

$$L_0^- \Psi = b_0^- \Psi = 0. \quad (2.7)$$

The quadratic part of the Lagrangian is

$$(\Psi, c_0^- Q \Psi), \quad (2.8)$$

and the linearized gauge invariance is

$$\delta \Psi = Q \epsilon, \quad (2.9)$$

where $\epsilon$ has ghost number 1 and

$$b_0^- \epsilon = 0. \quad (2.10)$$

This is proved to be a gauge invariance as follows. In the usual theory, (2.10) implies that

$$\epsilon = b_0^- \alpha \quad (2.11)$$

for some $\alpha$. Inserting this in (2.8), and using

$$\{b_0^-, c_0^-\} = 1, \quad (2.12)$$
together with
\[ \{ Q, b_0^- \} = L_0^- \tag{2.13} \]
(which is a consequence of (2.1)) along with (2.7), one verifies the invariance of (2.8).

What are the essential points that should be retained if the ghosts and matter fields are not decoupled? In conventional string theory, one has
\[ (b_0^-)^2 = 0, \tag{2.14} \]
a consequence of the absence of singularities in the $b \cdot b$ operator products. This property also holds in topological sigma models, and it seems reasonable to insist on it in general even if one permits some kind of $b \cdot b$ singularities. The ability to write $\epsilon$ as in (2.11) is then the assertion that the cohomology of $b_0^-$ vanishes in ghost number 1. If the cohomology of $b_0^-$ is altogether zero, then there exists an operator $c_0^-$ obeying (2.12). Moreover, $c_0^-$ is uniquely determined modulo $c_0^- \rightarrow c_0^- + \{ b_0^-, f \}$ for some $f$; using (2.7) and (2.13), one sees that the Lagrangian (2.8) is invariant under such a shift of $c_0^-$. Thus, (2.14) and the trivial cohomology of $b_0^-$ are sufficient requirements for free closed string field theory (and I believe also for the interacting case).

What if the cohomology of $b_0^-$ is not zero? Then using (2.13) and the fact that the cohomology can be represented by states invariant under the compact group generated by $L_0^-$ (and in any case since the physical field $\Psi$ is required to have this invariance), we see that $Q$ generates a linear transformation of the cohomology of $b_0^-$. If this linear transformation is zero, there will be an operator $c_0^-$ obeying not (2.12) but
\[ \{ b_0^-, c_0^- \} Q = Q \{ b_0^-, c_0^- \} = Q. \tag{2.15} \]
This actually is enough to ensure the gauge invariance of the free Lagrangian (2.8).
but the fact that $Q$ annihilates the cohomology of $b_0^-$ means that for $\epsilon$ annihilated by $b_0^-$ and $L_0^-$, one can write $Q\epsilon = b_0^-\beta$ for some $\beta$; this is good enough.) Topological sigma models give an example in which the cohomology of $b_0^-$ is non-zero, but annihilated by $Q$. We therefore can write down closed string field theory in this case, but as we will see in §5, it seems difficult to understand it.

A Comundrum

The conundrum that I want to state is obvious. The properties of conventional string backgrounds that I have cited are the only ones I know of that make sense in the general case in which ghosts and matter are not decoupled. Yet they are so general as to permit bizarre realizations, like the topological sigma models that we will consider presently. Are they adequate and if not how should they be supplemented?

The following example will perhaps serve to sharpen the puzzle. In Type II superstring theory, it is usually supposed that a classical solution is described by a superconformal world-sheet theory. One therefore might expect and hope to supplement the above discussion with general properties of superconformal symmetry. Yet it has been pointed out [18] that Ramond-Ramond vertex operators do not commute with the left- or right-moving superconformal currents, and consequently that when such vertex operators are added to the world-sheet Lagrangian (as one would expect in a generic time-dependent situation), the world-sheet theory is not superconformal. Therefore, the structure of Type II backgrounds, at the general level of our above discussion, looks hard to distinguish from that of bosonic string theory. If we do not wish to claim that bosonic strings and Type II superstrings are the same theory (and I would be skeptical of that interpretation), we apparently must find general properties of the world-sheet theory that go well beyond the ones I have cited.
3. Realizations Via Topological Field Theories

We will now use topological sigma models to make some realizations of the structure just explained. There are two classes of such models [6,19,20], which I will call the A and B models following some of the papers just cited.

These models govern maps from a Riemann surface $\Sigma$ to a target space $X$, which must be presented with an almost complex structure, in the case of the A model, or an actual complex structure, for the B model. The A model is concerned with almost holomorphic maps from $\Sigma$ to $X$, while the B model is related to periods of differential forms on $X$. The B model is only well-defined for $X$ obeying the Calabi-Yau condition $c_1(X) = 0$. The A model is defined without that condition, but obeys the axioms of §2 (definite ghost number of the vacuum) only if $c_1(X) = 0$. Therefore, we will limit ourselves to Calabi-Yau target spaces. This is, of course, also the case in which the sigma model is conformally invariant.

For either the A or B models, the ghost number of the vacuum is $-d \cdot \chi(\Sigma)$, with $d = \dim \mathbb{C}(X)$. This coincides with the desired valued $-3 \chi(\Sigma)$ that played such an important role in §2 precisely if $d = 3$, and therefore we will limit ourselves to this case. It is curious to note that this is the same value of $d$ that arises in the usual “physical” applications of Calabi-Yau manifolds in superstring compactification.

General tangent space indices to $X$ will be written as $I, J, K$, while indices of type $(1,0)$ or $(0,1)$ will be written as $i, j, k$ or $\bar{i}, \bar{j}, \bar{k}$, respectively. We consider $X$ as a Kahler manifold endowed with a Ricci-flat Kahler metric $g_{i\bar{j}}$. The bosonic fields of the A or B models are simply a map $\Phi : \Sigma \rightarrow X$; if we pick local coordinates $\phi^I$ on $X$, this map can be described by giving functions $\phi^I(x^\alpha)$. It is also convenient to pick a local complex coordinate $z$ on $\Sigma$.

The A Model

The detailed construction of the A and B models is explained in [6]. In the A model, the fermi fields are a section $\chi^I$ of $\Phi^*(TX)$, and a one-form $\psi$ with values in $\Phi^*(TX)$. $\psi$ obeys a self-duality condition which says that its $(1,0)$ part $\psi^I_z$ has
values in $\Phi^*(T^0,1X)$, and its $(0,1)$ form $\psi_z^i$ has values in $\Phi^*(T^{1,0}X)$. The BRST transformation laws are

$$
\begin{align*}
\delta \phi^I &= i\alpha \chi^I \\
\delta \chi^I &= 0 \\
\delta \psi_z^i &= -\alpha \partial_z \phi^i - i\alpha \chi^j \Gamma_{jm}^i \psi_z^m \\
\delta \bar{\psi}_z^i &= -\alpha \partial_\bar{\sigma} \phi^i - i\alpha \chi^j \Gamma_{jm}^i \bar{\psi}_z^m 
\end{align*}
\tag{3.1}
$$

with $\alpha$ an anticommuting parameter and $\Gamma$ the affine connection of $X$. The BRST operator $Q$ is defined by writing $\delta \Lambda = -i\alpha \{Q, \Lambda\}$ for any field $\Lambda$. The Lagrangian can be written in the form $L = i\{Q, V\}$ with any $V$ such that $L$ is nondegenerate. A suitable choice of $V$ is

$$V = t \int_{\Sigma} d^2z \ g_{ij} \left( \psi_z^i \partial_z \phi^j + \partial_z \phi^i \bar{\psi}_z^j \right) \tag{3.2}$$

(with $d^2z = |dz \wedge d\sigma|$) and gives

$$L = 2t \int_{\Sigma} d^2z \left( \frac{1}{2} g_{I\bar{J}} \partial_z \phi^I \partial_{\bar{\sigma}} \phi^J + i \psi_z^i D_z \chi^i g_{\alpha}^I + i \bar{\psi}_z^i D_{\bar{\sigma}} \bar{\chi}^i \bar{g}_{\alpha}^I - R_{\alpha\beta} \psi_z^i \bar{\psi}_z^j \chi^j \bar{\chi}^I \right) \tag{3.3}$$

with $t$ a constant.

The fact that $L = i\{Q, V\}$ means that the $t$ dependence (and dependence on the target space metric) of (3.3) is of the form $\{Q, \ldots\}$ and so does not affect the BRST invariant physics. As weak coupling means large $t$, the fact that there is no $t$ dependence means that weak coupling is exact.

The dependence on the metric of $\Sigma$ is likewise of the form $\{Q, \ldots\}$, so we can write $T_{\alpha\beta} = \{Q, b_{\alpha\beta}\}$ where at the classical level

$$b_{\alpha\beta} = itg_{IJ} \left( \psi_{\alpha}^I \partial_\beta \phi^J + \psi_{\beta}^I \partial_\alpha \phi^J - h_{\alpha\beta} h^{\sigma\tau} \psi_{\alpha}^I \partial_\tau \phi^J \right). \tag{3.4}$$

Quantum mechanically, the formula for $b$ requires modification except in the Calabi-Yau case ($T$ and $b$ will no longer be traceless), but an appropriate $b$ still ex-
ists, as the theory admits a higher derivative regularization preserving the fermionic symmetry.

**The B Model**

In the B model, the fermi fields include first of all sections $\vec{\eta}^i, \theta^i$ of $\Phi^*(T^{0,1}X)$; actually, it is convenient to write the formulas in terms of not $\theta^i$ but $\theta_j = g_{ij}\theta^i$. The other fermi fields are a one-form $\rho^i$ with values in $\Phi^*(T^{1,0}X)$. The transformation laws are

$$
\begin{align*}
\delta \phi^i &= 0 \\
\delta \phi^i &= i\alpha \eta^i \\
\delta \eta^i &= \delta \theta_i = 0 \\
\delta \rho^i &= -\alpha d\phi^i
\end{align*}
$$

and the Lagrangian is again of the form $L = i\{Q, V\}$; for a suitable $V$, one gets

$$
L = t \int \Sigma d^2z \left( g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i\eta^i (D_z \rho^i_z + D_{\bar{z}} \rho^i_{\bar{z}}) g_{i\bar{i}} \\
+ i\theta_i (D_z \rho^i - D_{\bar{z}} \rho^i_{\bar{z}}) + R_{i\bar{i}j} \rho^i_z \rho^j_{\bar{z}} \eta^\bar{k} g^{k\bar{j}} \right).
$$

The physics is independent of the coupling parameter $t$ for the same reason as in the A model.

In this case, we can write $T_{\alpha\beta} = \{Q, b_{\alpha\beta}\}$ with

$$
b_{\alpha\beta} = it g_{IJ} \left( \rho^I_{\alpha} \partial_{\beta} \phi^J + \rho^I_{\beta} \partial_{\alpha} \phi^J - h_{\alpha\beta\gamma} h^{\sigma\tau} \rho^I_{\sigma} \partial_{\tau} \phi^J \right).$$

(3.7)
3.1. Boundary Conditions

In our applications, we will want to formulate these theories on Riemann surfaces $\Sigma$ that may have non-empty boundary. In doing so, we require a boundary condition. We want local, elliptic boundary conditions that preserve the fermionic symmetry. I will explain natural boundary conditions for both the A and B model, but there may very well be other natural boundary conditions – perhaps required for mirror symmetry – that should be considered.

The A model is closely related to Floer theory, and the boundary condition that I want to consider for the A model is the one that was studied by Floer. For each component $C_i$ of $\partial \Sigma$, we pick a Lagrangian submanifold $M_i$ of the Kahler manifold $X$. For instance, in the nicest case, there may be a real involution $\tau_i$ of $X$, and $M_i$ may be a component of the fixed point set of $\tau_i$. Let $TM_i$ and $NM_i$ be the tangent and normal bundles to $M_i$ in $X$; we regard them as the real and imaginary subbundles of $TX|_{M_i}$. Then regarding $C_i$ as a real locus in the complex Riemann surface $\Sigma$, we require that the boundary values should be real in the following sense: $\Phi|_{C_i}$ is real, that is it maps $C_i$ to $M_i$; the normal derivative to $\Phi$ at $C_i$ is imaginary (it takes values in $\Phi^*(NM_i)$); $\chi$ and the pullback of $\psi$ to $M_i$ are real (they take values in $\Phi^*(TM_i)$). These boundary conditions make sense even if $M_i$ does not arise as the fixed point set of a real involution.

For the B model, we pick instead “free” boundary conditions that do not require anything analogous to the choice of the $M_i$. We require that the normal derivative of $\Phi$ vanishes on $\partial \Sigma$, that $\theta$ vanishes on $\partial \Sigma$, and that the pullback to $\partial \Sigma$ of $\star \rho$ vanishes ($\star$ being here the Hodge star operator).

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* For some generalities about topological field theories on Riemann surfaces with boundary, and some specifics about the A model, see [21].
† Readers to whom these boundary conditions may seem strange at first sight are invited to try to find other local, elliptic, $Q$-invariant boundary conditions for the A model.
3.2. Boundary Terms

Now, we further wish to couple this system to target space gauge fields in the sense of string theory, that is to gauge fields on \( X \). This will be done by introducing Chan-Paton factors, that is by coupling the gauge fields to “charges” that propagate on \( \partial \Sigma \). These charges are string theory analogs of the labels on the boundaries in figure (1(b)), and will reduce to the latter when we make contact with target space Chern-Simons theory. Just as in the construction leading to figure (1(b)), we must take the gauge group to be \( U(N) \) if we want to consider oriented Riemann surfaces only; \( SO(N) \) and \( Sp(N) \) are possible if one wishes to permit unoriented surfaces. The restriction on the gauge group is not needed in the preliminary discussion of the present section but is essential in §4.

The quantum version of either of the models we have introduced is described by a Feynman path integral

\[
\int D\Psi_i \exp(-L(\Psi_i)),
\]

where \( \Psi_i \) are the various fields and \( L \) is the Lagrangian. Let now \( A = A_I d\phi^I \) be a connection with structure group \( U(N) \) on a rank \( N \) complex vector bundle \( E \) over \( X \). Let \( \Sigma \) be an oriented Riemann surface whose boundary is a disjoint union of circles \( \Sigma_i \). The orientation of \( \Sigma \) induces an orientation of each \( \Sigma_i \). Given \( \Phi : \Sigma \to X \), for each \( \Sigma_i \) we can take the trace of the holonomy of \( \Phi^*(A) \) around \( \Sigma_i \); we write this as

\[
\text{Tr} \, P \exp \oint_{\Sigma_i} \Phi^*(A).
\]

(The trace is taken in the defining \( N \) dimensional representation of \( U(N) \).) Then, tentatively, the coupling to gauge fields in the target space is accomplished by replacing (3.8) by

\[
\int D\Psi_i \exp(-L(\Psi_i)) \cdot \prod_i \text{Tr} \, P \exp \oint_{\Sigma_i} \Phi^*(A).
\]
(For the B model, this will require some modification.) If $A$ is trivial, this is simply a factor of $N$ for each boundary component, just as in the evaluation of the Feynman amplitude of figure (1(b)).

We must determine whether this modification of the path integral preserves the fermionic symmetry. In general the variation of the trace of the holonomy about a circle $C$ is

$$\delta \text{Tr} P \exp \oint_C \Phi^*(A) = \text{Tr} \oint_C \delta \phi^I d\phi^J F_{IJ}(\tau) d\tau \cdot P \exp \oint_{C,\tau} \Phi^*(A).$$

(3.11)

Here $\tau$ is a coordinate on $C$, and $\exp \oint_{C,\tau} \Phi^*(A)$ is the holonomy of $\Phi^*(A)$ around $C$, starting and ending at $\tau$; and $F_{IJ}$ (or more fastidiously $\Phi^*(F_{IJ})$) are the components of the pullback by $\Phi$ of the space-time curvature $F = dA + A \wedge A$.

The A Model

In the case of the A model, since $\Phi^*(C_i) \subset M_i$, the bundle $E$ and connection $A$ actually need only be defined on the union of the $M_i$, not on all of $X$.

Inserting the A model transformation law $\delta \phi^I = i\alpha \chi^I$ in (3.11), we see that the holonomy factors in (3.10) are invariant under the fermionic symmetry if and only if the space-time curvature $F$ vanishes. Thus, in the A model, it is possible to couple only to flat connections on the $M_i$. We will give this a more intuitive explanation in §4, where we will see that the target space physics of the A model is Chern-Simons gauge theory on the $M_i$. The classical solutions of Chern-Simons gauge theory are precisely the flat connections. Thus, the requirement of the A model that the target space connection must be flat is a special case of the fact that in string theory, the background fields that can be incorporated in the world-sheet theory are always classical solutions of the space-time theory.

Since $A$ is flat, its role is from some points of view almost trivial. The homotopy factors in (3.10) depend only on the topology of $\Sigma$ and the choice of a homotopy class of maps $\Phi : \Sigma \to X$, in which one is evaluating the functional integral.
But deeper aspects of the theory involve summing over the topology of $\Sigma$ and the homotopy type of $\Phi$, and then the factors coming from holonomies of a flat connection on $X$ fit together coherently (to give, as we will see, Chern-Simons theory expanded around the given flat connection $A$).

**The B Model**

The situation in the $B$ model is similar but more subtle. Plugging the transformation laws $\delta \phi^i = 0$, $\delta \phi^{\tilde{i}} = i \alpha \eta^{\tilde{i}}$ into (3.11), we find that the holonomy is invariant under the fermionic symmetry if and only if the $(1, 1)$ and $(0, 2)$ parts of the curvature vanish. The vanishing $(0, 2)$ curvature asserts that the $(0, 1)$ part of $A$ defines a holomorphic structure on the bundle $E$; the additional vanishing of the $(1, 1)$ curvature says that $A$ is a holomorphic connection (locally it can be represented by a holomorphic one-form of type $(1, 0)$). The latter condition is rather restrictive and we wish to eliminate it. This can be done as follows. Replace $\Phi^*(A)$ by the following “improved” connection on the bundle $\Phi^*(E)$ over $\Sigma$:

$$\tilde{A} = \Phi^*(A) - i \eta^{\tilde{i}} F_{ij} \rho^j. \quad (3.12)$$

Then using the transformation laws of the $B$ model, one readily sees that for any circle $C \subset \Sigma$, the trace of the holonomy

$$\text{Tr} \ P \exp \oint_C \tilde{A} \quad (3.13)$$

of $\tilde{A}$ is invariant under the fermionic symmetry, provided that the $(0, 2)$ part of the curvature of $A$ vanishes, that is provided $A$ determines a holomorphic structure on $E$. Thus, the formula (3.10) for coupling external gauge fields to the $A$ model should be modified, in the case of the $B$ model, to

$$\int D\Psi_i \exp(-L(\Psi_i)) \cdot \prod_i \text{Tr} \ P \exp \oint_{C_i} \tilde{A}. \quad (3.14)$$

I will leave it to the interested reader to verify that up to terms of the form $\{Q, \ldots\}$, the coupling to gauge fields in (3.14) depends only on the holomorphic structure.
of $E$ (i.e., the $(0, 1)$ part of $A$), not on the $(1, 0)$ part of $A$. This is in accord with the fact that, as we will see in §4, the $B$ model has for its classical solutions in space-time the holomorphic vector bundles. As always, the classical solutions are the objects to which the world-sheet theories can be coupled.

3.3. LARGE $t$ LIMIT

To set up the Hamiltonian version of these theories, for open strings, we take $\Sigma$ to be an infinite strip $0 \leq \sigma \leq \pi, -\infty \leq \tau \leq \infty$ with metric $ds^2 = d\sigma^2 + d\tau^2$. We consider the $A$ and $B$ models with the boundary conditions just described. In the case of the $A$ model, we use the same Lagrangian submanifold $M$ at the two ends of the strip. A quantum Hilbert space $\mathcal{H}$ is introduced in the usual way by quantizing on the initial value surface $\tau = 0$. We want to compute the cohomology of $Q$ and certain aspects of the large $t$ behavior that will be essential in the next section.

The $A$ Model

If we write $\psi = \psi_\sigma d\sigma + \psi_\tau d\tau$, then, as the self-duality condition determines $\psi_\sigma$ in terms of $\psi_\tau$, we can regard $\chi$ and $\psi_\tau$ as the independent fermi variables. The canonical commutation relations are

$$\left[\frac{d\phi^I}{d\tau}(\sigma), \phi^J(\sigma')\right] = -\frac{i}{t} g^{IJ} \delta(\sigma - \sigma')$$

$$\{\psi_\tau(\sigma), \chi(\sigma')\} = \frac{1}{t} \delta(\sigma - \sigma').$$

(3.15)

The Hilbert space $\mathcal{H}$ consists of functionals $\mathcal{A}(\Phi, \ldots)$ where now $\Phi$ is a map of the interval $I = [0, \pi]$ to $X$ mapping $\partial I$ to $M$, and “…” is a subset of half the fermi variables, depending on a choice of representation of the canonical anticommutators.
The Hamiltonian is
\[ L_0 = \int_0^\pi d\sigma\ T_{00}. \]  
(3.16)

Using the canonical commutation relations to write \( d\phi/d\tau \) in terms of \( \delta/\delta\phi \), this can be written
\[ L_0 = \frac{1}{2} \int_0^\pi \left( -\frac{1}{t} g^{IJ} \frac{\delta^2}{\delta\phi^I \delta\phi^J(\sigma)} + t g_{IJ} \frac{d\phi^I}{d\sigma} \frac{d\phi^J}{d\sigma} \right) + \text{terms with fermions}. \]  
(3.17)

The fundamental relation \( T_{\alpha\beta} = \{ Q, b_{\alpha\beta} \} \) implies that if we introduce the zero mode of the \( b \) field,
\[ b_0 = \int_0^\pi d\sigma\ b_{00} \]  
(3.18)

then
\[ L_0 = \{ Q, b_0 \}. \]  
(3.19)

(3.19) implies that the \( Q \) cohomology can be computed in the subspace of \( \mathcal{H} \) annihilated by \( L_0 \). Since the \( Q \) cohomology (and everything else of essential interest) is independent of \( t \), it is enough to study the kernel of \( L_0 \) for large \( t \). Actually, with other applications in mind, we want to understand not just the kernel of \( L_0 \) but all eigenvalues that are of order \( 1/t \) for large \( t \).

Looking at (3.17), we see that such eigenfunctions must be localized near the region with \( d\phi^I/d\sigma = 0 \), that is, the space of constant maps \( \Phi : I \rightarrow X \). Because of the boundary condition that \( \partial\Sigma \) is mapped to \( M \), the constant map in question must in fact map \( I \) to \( M \). The non-zero modes of the fermions make contributions of order 1 to \( L_0 \). Combining these observations, the low-lying eigenfunctions of \( L_0 \) can be described by a functional \( \mathcal{A} \) of the bose and fermi zero modes, with other modes in their Fock vacuum. (One must check that the energy of this Fock vacuum is zero, but this follows from \( Q \)-invariance or better from the supersymmetry of the
untwisted theory.) The bose and fermi zero modes are all tangent to $M$; denote them as $q^a, \chi^a, \psi^a, a = 1 \ldots 3$, with $q^a$ being coordinates on $M$. (We reserve the letter $\phi$ for coordinates on $X$.) The canonical commutation relations (3.15) show that $\psi^a$ can be represented as $\partial/\partial \chi^a$; then $\mathcal{A}$ reduces to a function $\mathcal{A}(q^a, \chi^a)$. This has an expansion in powers of $\chi^a$

$$\mathcal{A} = c(q) + \chi^a A_a(q) + \chi^a \chi^b B_{ab}(q) + \ldots$$

(3.20)

in which the successive terms can be interpreted as $p$-forms on $M$ of degrees $0 \leq p \leq 3$. If Chan-Paton factors are included in the discussion, these differential forms take values in $\text{End}(E)$, the endomorphisms of some flat vector bundle $E$ over $M$.

The commutation relations $[Q, \phi^I] = -\chi^I$, $\{Q, \chi^I\} = 0$ show that, if we interpret $\chi^I$ as $-d\phi^I$, we can identify $Q$ with the exterior derivative $d$ on $M$ (with values in $\text{End}(E)$). The $Q$ cohomology is thus $H^*(M, \text{End}(E))$.

Moreover, (3.17) shows that acting on differential forms on $M$, $L_0$ reduces to

$$L_0 = \frac{\pi}{2t} \Delta,$$

(3.21)

with $\Delta = dd^* + d^*d$ the usual Laplacian on forms. The underlying relation $L_0 = \{Q, b_0\}$ leads one to guess that in the same approximation

$$b_0 = \frac{\pi}{2t} d^*,$$

(3.22)

and this can be verified using the commutation relations.

**The B Model**

For the $B$ model we can be brief, as the arguments are so similar. In the large $t$ limit, the wave-functional $\mathcal{A}$ reduces to a function of the zero modes of $\phi^I$ and $\eta^I$ (the zero mode of $\rho$ being represented as $\partial/\partial \eta$). Expanding

$$\mathcal{A}(\phi^I, \eta^I) = c(\phi^I) + \eta^I A_I(\phi^I) + \eta^I \eta^J B_{IJ}(\phi^I) + \ldots,$$

(3.23)

we see that for the low-lying states, $\mathcal{A}$ reduces to a sum of $(0, q)$ forms on $X$ (valued in $\text{End}(E)$, $E$ being a holomorphic vector bundle on $X$), for $0 \leq q \leq 3$. 

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The commutation relations \([Q, \phi^i] = 0, [Q, \phi^i] = -\eta^i, \{Q, \eta^i\} = 0\) show that, if we interpret \(\eta^i\) as \(-d\phi^i\), we can identify \(Q\) with the \(\bar{\partial}\) operator (with values in \(\text{End}(E)\)). So the cohomology of \(Q\) is \(H^{0,*}(X, \text{End}(E))\). Likewise, by arguments similar to those for the A model, \(L_0\) and \(b_0\) reduce to

\[
\begin{align*}
L_0 &= \frac{\pi}{2t} \left( \bar{\partial} \partial^* + \partial^* \bar{\partial} \right) \\
b_0 &= \frac{\pi}{2t} \bar{\partial}^*.
\end{align*}
\]

(3.24)

4. Space-Time Interpretation Via String Field Theory

In this section we will determine the space-time interpretation of the models just constructed in the case of open strings, that is in the case of orientable world-sheets with boundary. The reason for focussing on this case is that a simple and beautiful answer will arise, as we will see. The more perplexing closed string case (Riemann surfaces without boundary) is the subject of §5.

Usually, closed strings are inescapable even when one tries to do open string physics, for elementary topological reasons. The reader may therefore be surprised at the lengthy discussion of open strings in this section ignoring closed strings. In fact, as we discuss in §4.2, our main points can be formulated at the classical level, where closed strings cannot appear; also, we argue in §5 that in these topological theories, the open and closed strings are decoupled.

The simplicity of the open string topological theories is closely related to the comparative simplicity of open string field theory [14], which in turn is intimately tied up [15] with the existence of a simple cell decomposition of the moduli space of Riemann surfaces with boundary [11–13]. This cell decomposition involves the construction of Riemann surfaces by gluing together flat strips \(S_i\) of width \(\pi\) and length \(T_i\); they are glued, as sketched in figure (2), at vertices that generically (unless some \(T_i = 0\)) are trivalent. The resulting pictures have been called “fatgraphs”
Fig. 3. The multiplication law (a) and integration law (b) of open string field theory are defined by pictures involving gluing of strings.

by Penner. In the limit when the $T_i$ are all large, or equivalently when the widths of the strips can be neglected, the fatgraphs of the open-string field theory reduce to ordinary graphs corresponding to Feynman diagrams of ordinary field theory.

Construction Of Open String Field Theory

We recall now the highlights of the construction of open string field theory. A key part of the structure is a $\mathbb{Z}$-graded associative algebra $\mathcal{B}$, with a multiplication law that we denote as $\star$, and a derivation $Q$ of degree 1 with $Q^2 = 0$. The degree is usually called ghost number.* There is a linear functional $\int : \mathcal{B} \to \mathbb{C}$, of degree $-3$ (that is, if $b$ is an element of $\mathcal{B}$ of definite ghost number, then $\int b$ vanishes unless $b$ has ghost number three), obeying $\int a \star b = (-1)^{\text{deg } a \text{ deg } b} \int b \star a$, $\int Qb = 0$, for all $a, b \in \mathcal{B}$. The string field is a ghost number 1 element $A \in \mathcal{B}$. The Lagrangian is

$$L = \frac{1}{2} \int \left( A \star QA + \frac{2}{3} A \star A \star A \right).$$

(4.1)

This is invariant under gauge transformations generated by

$$\delta A = Q \epsilon - \epsilon \star A + A \star \epsilon.$$  

(4.2)

Chan-Paton factors are introduced as follows (in the case of a trivial rank $N$ bundle in space-time with flat connection; the generalization is discussed presently).

* The ghost number as I will count it here is the ghost number as defined in [14] plus 3/2; thus, the $SL(2, \mathbb{R})$ invariant vacuum has ghost number zero.
Let $M_N(\mathbb{C})$ be the associative algebra of $N \times N$ complex matrices. One simply replaces $\mathcal{B}$ with $\mathcal{B} \otimes M_N(\mathbb{C})$ (and $\int$ by $\int \otimes \text{Tr}$, $\text{Tr}$ being the usual trace on $M_N(\mathbb{C})$). This preserves the basic structures. In this process, $\mathcal{A}$ acquires matrix indices. If a suitable reality condition is imposed, $\mathcal{A}$ takes values in $N \times N$ hermitian matrices – the Lie algebra of $U(N)$.

Conventionally, $\mathcal{B}$ is taken to be the space of open string states in some critical string theory; the multiplication and integration operations $\star$ and $\int$ are defined by operations of gluing strings that we recall in figure (3). $Q$ is the BRST operator of the critical string theory. The integration law is of ghost number $-3$ since the Euler characteristic of a disc is $-1$, and in critical string theory the ghost number of the vacuum is $-3\chi(\Sigma)$.

Now we want to consider open string field theory with the conventional string models replaced by the more exotic ones discussed in §2-3. In particular, for the world-sheet theory we take a topological sigma model with a Calabi-Yau target space $X$ of complex dimension three; this ensures the correct ghost number of the vacuum – or of the integration law. Multiplication and integration are defined by the standard gluing operations; $Q$ will now be the BRST operator of whatever topological field theory we consider. This framework enables us to construct an open-string field theory from any world-sheet theory (obeying the usual axioms). For instance, instead of tensoring with $M_N(\mathbb{C})$ to introduce trivial Chan-Paton factors, we can use the world-sheet theories with boundary interactions constructed in §3.2 to couple to a flat bundle on $M$ in the case of the $A$ model, or a holomorphic bundle on $X$ in the case of the $B$ model.

### 4.1. The $A$ Model

We want to understand the physical content of these models. First we consider the $A$ model. It is necessary to pick boundary conditions, and as in §3 we follow Floer and pick boundary conditions associated with Lagrangian submanifolds of $X$. In §3, in studying a particular surface $\Sigma$, we introduced a separate Lagrangian
submanifold $M_i$ for each boundary component $C_i$ of $\Sigma$. In string field theory, one generates all possible $\Sigma$'s via a Feynman diagram expansion, and the $M_i$'s must be built in universally at the outset. We will do this in the simplest way by picking a single $M$ once and for all. Thus the free boundaries of all our strings and surfaces will be mapped to $M \subset X$. (Generalizations exist; by correlating the choice of $M$ with the Chan-Paton factors one could make a gauge invariant string field theory with more than one $M$.)

Since $X$ is of complex dimension three, $M$ is a three-manifold. A neighborhood of $M$ in $X$ is equivalent topologically and even symplectically to a neighborhood of $M$ in its cotangent bundle $T^*M$. The topological string theory with target space $X$ involves, roughly, two ingredients. One is the instantons with target $X$ (and boundary values in $M$); these are the usual subjects of study in Floer theory. The other side of the story, as we will see, involves Chern-Simons theory with target space $M$. I want first to isolate this “new” ingredient – new in the sense that it is not usually coupled with Floer theory. The instantons can be suppressed and the new ingredient isolated by replacing $X$ by $T^*M$, since a simple vanishing theorem (discussed presently) shows that there are no non-constant instantons mapping $\Sigma$ to $T^*M$. Later we will generalize to arbitrary $X$ and determine the instanton corrections to the space-time Chern-Simons theory.

So until further notice, in discussing the $A$ model, our target space will be the cotangent bundle $T^*M$ of an oriented three-manifold $M$. Like any symplectic manifold, $T^*M$ can be given an almost complex structure such that the symplectic structure is positive and of type $(1,1)$; indeed, this is essential in Floer/Gromov theory of symplectic manifolds. This is good enough for formulating the $A$ model with target space $T^*M$; though the transformation laws and Lagrangian of the $A$ model were written in §3 in a way that assumed the integrability of the almost complex structure, this assumption can be relaxed, as explained in detail in [1].

\* The orientation is needed to consistently define the sign of the fermion determinant in the world-sheet theory of §3.
The Vanishing Theorem

Now let us briefly explain the vanishing theorem, which asserts that instantons mapping \( \Sigma \) to \( T^*M \) and mapping \( \partial \Sigma \) to \( M \) are necessarily constant.

Consider temporarily a general symplectic manifold \( X \) with symplectic form \( \omega \) and an almost complex structure \( J \) such that \( \omega \) is of type \((1,1)\) and positive. Positivity means that the metric \( g_{IK} = J^S_I \omega_{SK} \) is positive definite; the \((1,1)\) condition means that if \( i, j, k \) and \( \overline{i}, \overline{j}, \overline{k} \) are indices of types \((1,0)\) and \((0,1)\), respectively, then \( g_{ij} = \overline{g_{ij}} = 0 \), and \( g_{ij} = \overline{g_{ij}} = -i \omega_{ij} \). If \( \Sigma \) is a Riemann surface, an instanton or almost holomorphic map is a map \( \Phi : \Sigma \to X \) with \( \overline{\partial} \phi^i = 0 \). Consider the bosonic sigma model action

\[
I = i \int_{\Sigma} dz \wedge d\overline{z} \ g_{IJ} \partial_z \phi^I \overline{\partial_{\overline{z}} \phi^J}. \tag{4.3}
\]

Instantons minimize this action for a given homotopy class since

\[
I = 2i \int_{\Sigma} dz \wedge d\overline{z} \ g_{IJ} \partial_z \phi^I \overline{\partial_{\overline{z}} \phi^J} - i \int_{\Sigma} dz \wedge d\overline{z} \ g_{IJ} \left( \partial_z \phi^i \overline{\partial_{\overline{z}} \phi^j} - \partial_z \phi^j \overline{\partial_{\overline{z}} \phi^i} \right) = 2i \int_{\Sigma} dz \wedge d\overline{z} \ g_{IJ} \partial_z \phi^I \overline{\partial_{\overline{z}} \phi^J} + \int_{\Sigma} \Phi^*(\omega). \tag{4.4}
\]

The first term on the right hand side of (4.4) is positive semi-definite and vanishes precisely for instantons; so for instantons, the action reduces to \( I = \int_{\Sigma} \Phi^*(\omega) \). The vanishing theorem comes by showing that if \( X = T^*M \), and \( \Phi(\partial \Sigma) \subset M \), then

\[
\int_{\Sigma} \Phi^*(\omega) = 0. \tag{4.5}
\]

If this is known, then \( I \) vanishes for instantons that map \( \partial \Sigma \) to \( M \); but from the definition (4.3) it is clear that \( I \) vanishes only for constant maps.
To justify (4.4), pick on $M$ local coordinates $q^a$, $a = 1 \ldots 3$. The symplectic structure of $T^*M$ can be written as $\omega = \sum_{a=1}^3 dp_a \wedge dq^a$, with $p_a$ linear coordinates in the fibers that vanish on $M$. This is $\omega = d\rho$, where $\rho = \sum p_a dq^a$ vanishes on $M$. So $\int_\Sigma \Phi^*(\omega) = \int_{\partial \Sigma} \Phi^*(\rho) = 0$, if $\Phi(\partial \Sigma) \subset M$.

**Low Energy Expansion**

As explained following equation (3.3), the key simplification of the $A$ model is that the essential physics is independent of the coupling parameter $t$. As $t$ and the target space metric $g_{IJ}$ appear only in the combination $tg_{IJ}$, large $t$ is simply the limit in which the target space metric is scaled up; it is the limit of large distances or long wavelengths. This is the limit in which ordinary string theory reduces approximately to field theory. Since ordinary string theory is $t$-dependent, the large $t$ behavior is only an approximation. The topological string theories that we are studying are $t$-independent, so one can hope for an exact description by looking at the large $t$ behavior.

We have analyzed the large $t$ behavior of the string states in §3.3. In particular, as we saw there, the low-lying modes can be described as functions $A(q^a, \chi^b)$ of the center of mass variables. In §3.3, we computed the cohomology for all ghost numbers, but in the string field theory context, we want to insist that $A$ have ghost number 1. This means that it must be linear in $\chi$, so we write

$$A = \chi^a A_a(q), \quad (4.6)$$

with $A_a dq^a$ a one-form on $M$, valued possibly in $\text{End}(E)$, the endomorphisms of a flat vector bundle $E$.

In §3.3, we identified $Q$, in the same large $t$ limit, with the exterior derivative $d$. Then the quadratic part of the string field Lagrangian $\frac{1}{2} \int A \star Q A$ reduces (for the low-lying modes in the large $t$ limit) to

$$\frac{1}{2} \int_M \text{Tr} A \wedge dA, \quad (4.7)$$
with \( \text{Tr} \) a trace on the Chan-Paton factors. This is the quadratic part of the ordinary Chern-Simons action for a gauge field \( A \).

What about the cubic part of the string field action? If \( A^{(j)} \) is a mode of the gauge field \( A \), the corresponding vertex operator is \( V^{(j)} = \chi^a A^{(j)}_a(q) \). Let us evaluate by world-sheet path integrals the coupling of three such modes on the disc. This is done by inserting three vertex operators \( V^{(i)} \), \( i = 1 \ldots 3 \) at boundary points of the disc, as in figure (4). Because of \( SL(2, \mathbb{R}) \) symmetry, all that matters in the choice of the three points is the cyclic order. So we can take \( \langle V^{(1)}(0)V^{(2)}(1)V^{(3)}(\infty) \rangle \). In the large \( t \) limit, the path integral reduces to an integral over the constant zero modes; this integral gives

\[
\int dq^1 \ldots dq^3 d\chi^1 \ldots d\chi^3 \text{Tr} \chi^a A^{(1)}_a(q) \chi^b A^{(2)}_b(q) \chi^c A^{(3)}_c(q) = \int_M \text{Tr} A^{(1)} \wedge A^{(2)} \wedge A^{(3)}
\]

(4.8)

This is simply a matrix element of the cubic part of the usual Chern-Simons action. So putting the pieces together, the string field action (4.1) reduces to the ordinary Chern-Simons Lagrangian

\[
L_T = \frac{1}{2} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).
\]

(4.9)

The string field gauge invariances can similarly be seen in the same approximation to reduce to conventional gauge invariance. So, under the conditions that we have formulated, the abstract Chern-Simons Lagrangian of string field theory reduces to the ordinary Chern-Simons Lagrangian.
Now usually, string theories reduce to field theories at low energy only approximately. For instance, ordinary string theories reduce approximately at low energies to general relativity plus massless matter fields, but there are $O(\alpha')$ corrections.

In the case of topological string theories (such as the $A$ model considered here) the situation is quite different. The analog of the usual $O(\alpha')$ corrections would be perturbative corrections in $1/t$. These are absent, because the model is $t$-independent. Instanton corrections are also absent, because for target space $T^*M$, there are no non-constant instantons, as we have seen. So the reduction of the $A$ model to (4.9) is exact. If we replace $T^*M$ by some Calabi-Yau threefold $X$ (having $M$ as a Lagrangian subspace), there are instanton corrections to (4.9) that we will discuss later.

### 4.2. Reconsideration From The Hamiltonian Point Of View

Now I want to reconcile what we have said above with previous discussions of the $A$ model such as [6] (or [1, §4.1] where the coupling to topological gravity is considered).

Formally, the $A$ model is a quantum field theory that counts holomorphic curves obeying various conditions. (This is familiar in applications of the $A$ model to mirror symmetry, for instance.) However, in the vanishing theorem that we have discussed above, we have seen that for target space $T^*M$, there are no non-constant instantons. How therefore can the model possibly be non-trivial, and in particular equivalent as we have claimed to perturbative Chern-Simons gauge theory?

To answer this question, it is necessary to recall some background. Formally, in topological quantum field theories of this general nature ("cohomological field theories"), one has an infinite dimensional function space $S$, and over it an infinite dimensional vector bundle $W$; one wishes to make sense of the Euler class of this bundle. To tame the situation, one has a section $w$ of $W$. In finite dimensions, if $S$ is compact and without boundary, the Euler class of $W$ would be Poincaré dual to a homology class $[w]$ supported on the zeros of $w$. The Euler class of $W$, integrated
over $S$, can be represented by various integral formulas; one can construct an integral formula which is localized near the zeros of $w$ [22]. The Feynman path integral of a cohomological field theory is a function space version of such a formula. This point of view is developed (in an analogous four dimensional gauge theory) in [23].

To identify $S$, $W$, and $w$ in the case at hand, let $\Sigma$ be a compact oriented two dimensional surface, perhaps with boundary, not endowed with any \textit{a priori} complex structure. Let $X$ be a symplectic manifold, with an almost complex structure obeying the usual conditions, and with a Lagrangian submanifold $M$. Fix a homotopy class $H$ of maps of $\Phi : \Sigma \to X$ with $\Phi(\partial \Sigma) \subset M$. Such a map can be described locally by complex functions $\phi^i$ on $\Sigma$ (corresponding to local complex coordinates on $X$). Then we set $S$ to be the space of pairs $(J, \Phi)$, where $J$ is a complex structure on $\Sigma$, and $\Phi$ is a map $\Sigma \to X$ in the class $H$, with two such pairs considered equivalent if they differ by a diffeomorphism of $\Sigma$. For $W$ we take the vector bundle over $S$ whose fiber is the space of sections of $\Omega^{0,1}(\Sigma) \otimes \Phi^*(T^{1,0}X)$, where $\Omega^{0,1}(\Sigma)$ is the space of $(0, 1)$-forms on $\Sigma$ in the complex structure $J$, and $T^{1,0}X$ is the holomorphic tangent bundle of $X$. For $w$, we pick a natural section of $W$ given by the equation for an almost holomorphic curve; it can be written $w = \overline{\partial}_J \phi^i$, with $\overline{\partial}_J$ the $\overline{\partial}$ operator on $\Sigma$ determined by $J$.

Now, let us focus on the case $X = T^*M$, endowed with an almost complex structure of the usual type and so in particular with a metric. The vanishing theorem discussed earlier implies that for $H$ a non-trivial homotopy class, there are no instantons; $w$ has no zeros. If $S$ were compact, we would conclude that the Euler class of $W$ is zero.

However, $S$ is non-compact, and its non-compactness is essential. In fact, in a certain sense, there are instantons or zeros of $w$ at infinity. These arise in a limit in which the Riemann surface degenerates to a graph $\Gamma$. This possibility is sketched in figure (5); from the point of view of open-string field theory, the degeneration arises in the limit in which all $T_i$ go to infinity in figure (2).
Fig. 5. Degeneration of a three-holed sphere to a trivalent graph can occur in two ways.

In a given homotopy class of maps of the graph $\Gamma$ to $X$, there will be a map $\Phi$ for which $\Phi(\Gamma)$ is a “geodesic graph” of least total length. $\Phi$ can be regarded as a limit of maps from almost degenerate Riemann surfaces (obtained by slight thickening of $\Gamma$) that almost obey the instanton equation. It is an “instanton at infinity.” The sigma model action (4.3) (which should vanish for instantons obeying our boundary conditions, as we saw in the proof of the vanishing theorem) vanishes in the limit of such a geodesic graph.

The existence of zeros at infinity means that there is no straightforward evaluation of the world-sheet path integral by counting zeros of $w$. There are no real zeros, but there are virtual zeros at infinity. While the contributions of real zeros are integers that can be counted topologically, there is no apparent reason for this to be true for the virtual zeros at infinity. In any event, the contributions of the virtual zeros, summed over all homotopy classes, can be evaluated by computing the measure on the moduli space (of complex structures on $\Sigma$) that comes from the world-sheet path integral.$^*$

$^*$ When $\pi_1(M)$ has infinitely many conjugacy classes, so that the number of homotopy classes is infinite, it is scarcely plausible that the full answer can be written as a sum over homotopy classes with each class contributing an integer. For convergence, the contributions would have to be almost all zero. The possible noncommutativity of the sum over homotopy classes and the integration over moduli space may also mean that the final answer (which should be obtained by summing over homotopy classes first) cannot be written as a sum of contributions of homotopy classes, though the measure on moduli space can be so written. In this case, the question of whether the individual contributions are integers does not even make sense. This seems to be the case for the one loop contributions, as one can see by comparing to D. Fried’s formulas expressing the logarithm of analytic torsion as a sum over geodesics [24].
The essential simplification of the A model always arises because for large \( t \), this measure is concentrated near zeros of \( w \). In the case that the only zeros are virtual zeros at infinity, the measure is concentrated for large \( t \) on the region near infinity; but this is the limit in which string theory reduces to field theory, and world-sheet path integrals reduce to Feynman graphs. So the non-vanishing contributions to the world-sheet path integral come entirely from the regions in the moduli space (of complex structures on \( \Sigma \)) in which the string theory reduces to the evaluation of Feynman graphs. This is the equivalence of the string theory to a field theory that was claimed earlier. Of course, one must remember that (as in the figure, where there are two possible degenerations) one Riemann surface will in general be capable of degenerating to several possible Feynman graphs.

**Perturbation Theory**

To make this a little more explicit, we will now discuss how to extract the field theoretic Feynman rules from the string theory.

We want to describe the measure on the moduli space of Riemann surfaces that follows from the description of moduli space via fatgraphs. This can be worked out systematically by quantizing open string field theory. The necessary techniques were given by Thorn and Boccicchio [25,26] and involve an elegant application of the Batalin-Vilkovisky approach to quantization. Instead of following that road, I will merely use the fatgraph description of moduli space and the definition of the measure in equation (2.4).

The basic fact that we will use is therefore the fact that every Riemann surface
Σ with boundary has a canonical flat metric (with some conical singularities) built as in figure (2). This metric is obtained by gluing together some standard strips. The strips are flat rectangles of width π in the σ − τ plane, say 0 ≤ σ ≤ π, 0 ≤ τ ≤ T, with metric $ds^2 = d\sigma^2 + d\tau^2$. Such a strip, say $S$ (figure (6)), has one real modulus, namely $T$. A deformation of this modulus from $T$ to $T + \delta T$ can be made by changing the metric to

$$ds^2 = d\sigma^2 + d\tau^2 (1 + 2f(\tau)),$$

(4.10)

with $f(\tau)$ any function with $\int_0^T d\tau f(\tau) = \delta T$. It is convenient to take $f(\tau) = \delta T \cdot \delta(\tau - \tau_0)$, for some $\tau_0$. If one of the $\delta^{(k)} h$ in equation (2.4) is the deformation just described of the length of the strip $S$, then the corresponding mode $b^{(k)}$ of equation (2.3) is simply

$$b^{(k)} = \int_0^\pi d\sigma \ b_{00}(\sigma, \tau_0).$$

(4.11)

This is in other words an insertion of the zero mode $b_0$ at $\tau = \tau_0$.

We also have to worry about the propagation of the string through the proper time $T$; this gives a factor of $\exp(-TL_0)$, with $L_0$ the string Hamiltonian discussed in §3.3. Allowing for the integral over $T$ and the insertion of $b_0$, we get for each strip a factor

$$\int_0^\infty dT \ b_0 \exp(-TL_0).$$

(4.12)

Let $\mathcal{V}$ be the kernel of $L_0$, and let $1/L_0$ be the inverse of $L_0$ in the subspace orthogonal to $\mathcal{V}$. Since $b_0$ annihilates the kernel of $L_0$, the integral in (4.12) can

* The topological sigma model in question was constructed by twisting an ordinary supersymmetric model (as explained in detail in [6]) by a power of the canonical bundle. On the flat strip under discussion here, the canonical bundle is naturally trivialized, and the twisting does nothing. In the underlying supersymmetric model, $b_0$ is the adjoint of $Q$, and as in Hodge theory, the relation $L_0 = \{Q, b_0\}$ implies that $Q$ and $b_0$ annihilate the kernel of $L_0$. 

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be evaluated to give
\[ \int_0^\infty dT \ b_0 \exp(-T L_0) = b_0 \frac{1}{L_0}. \] (4.13)

In equations (3.21) and (3.22), we determined the large \( t \) limits of \( L_0 \) and \( b_0 \) to be \( \pi \Delta/2t \) and \( \pi d^*/2t \). (Thus, \( b_0 \) vanishes for large \( t \), but \( b_0/L_0 \) has a non-vanishing limit because of the small eigenvalues of \( L_0 \).) So the large \( t \) limit of (4.13) is
\[ d^* \frac{1}{\Delta}. \] (4.14)

Now, (4.13) is the open string propagator, while (4.14) is the propagator of Chern-Simons field theory.

In the fatgraph approach to construction of moduli space, the strips we have just analyzed are glued together at cubic vertices to make Riemann surfaces. The structure of the interaction vertices is the same as in the open string multiplication law, figure (3(a)). We have already seen in connection with figure (4) how these vertices reduce at large \( t \) to wedge products of differential forms, as in Chern-Simons field theory.

While the underlying string field \( \mathcal{A} \) had ghost number 1, and the underlying Chern-Simons gauge field \( A \) is a one-form, in (4.13) and (4.14) states of all ghost number – or differential forms of all degrees – are propagating. In either case, the vertices of the gauge fixed theory have in a natural sense the same structure (gluing of strings or wedge products) as that in the classical theory, though the constraint on the ghost number or degree is absent. In the case of string theory, this structure was explained by Thorn and Boccicchio [25,26] by solving the master equation and gauge fixing of string field theory. In Chern-Simons theory, the analogous formulation of the gauge fixed theory, combining the fields of different ghost number, is due to Axelrod and Singer [4].
Fig. 7. Coupling of four states $\alpha_{(1)}, \ldots, \alpha_{(4)}$ on the boundary of a disc (a) with an antighost insertion around $\alpha_{(2)}$; the Feynman diagrams corresponding to the two possible “channels” (b).

The Tree Level $S$-Matrix

I would now like to make this more explicit in a particular example. I have picked the example to anticipate the following question that may perplex some readers. We have extracted Chern-Simons theory from the open string degenerations of moduli space, but are there additional closed string contributions? It will be clear in §5 that the role of the closed strings is not fully understood, though I will argue there that the closed and open strings are decoupled. I therefore want to demonstrate that the statement “the topological string theory of the $\textbf{A}$ model is equivalent to Chern-Simons field theory” can be tested in a situation in which closed strings are definitely not relevant. This is the case for scattering amplitudes (or the analog of what in ordinary string theory would be scattering amplitudes) for the case that $\Sigma$ is a disc.

As we know from §3.3, the “physical states” of the open string field theory are elements of $H^1(\Sigma, \text{End}(E))$, $E$ being a flat vector bundle that we will keep fixed in
the discussion. The exterior derivative with values in that bundle will be denoted as $d$. Let $\alpha(1), \ldots, \alpha(4)$ be four such states. We consider their coupling on the disc in that cyclic order, as in figure (7). To calculate the four point function in field theory, all that we really need to know is that there are two possible Feynman diagrams, indicated in figure (7), and that the propagator is $d^*/\Delta$. The amplitude is therefore

$$I(\alpha(1), \ldots, \alpha(4)) = \int_M \text{Tr} \left( \alpha(1) \wedge \alpha(2) \frac{d^*}{\Delta} \alpha(3) \wedge \alpha(4) + \alpha(2) \wedge \alpha(3) \frac{d^*}{\Delta} \alpha(4) \wedge \alpha(1) \right).$$

(4.15)

To do the same calculation in string theory, we introduce vertex operators $V(i) = \alpha(i) a \chi^a$ for the external states, and one antighost insertion corresponding to the one real modulus of the disc. We can map $V(1), V(3)$, and $V(4)$ to 0, 1, and $\infty$, and take the modulus to be the position of $V(2)$. In that case, the antighost insertion can be taken to be a small contour integral around the position of $V(2)$, as in the figure. The contour integral can be evaluated by thinking of $V(2)$ as an operator inserted on the end of an open string in a Hamiltonian formulation; the antighost contour computes the commutator of this operator with the antighost zero mode $b_0$, and so replaces the zero form operator $V(2)$ by its one-form descendant

$$W(2) = [b_0, V(2)] = A_a(q) \frac{dq^a}{d\tau}.$$

(4.16)

The string theory amplitude is therefore

$$\int_0^1 d\sigma \langle V(1)(0) \cdot A_a \frac{dq^a}{d\tau}(\sigma) \cdot V(3)(1)V(4)(\infty) \rangle.$$

(4.17)

Now we wish to evaluate this, as usual, in the limit of $t \to \infty$. If the integral were limited to a range of $\sigma$ bounded away from 0 and 1, it would vanish as $1/t$ for $t \to \infty$. This is so because the $\phi$ and $\psi - \chi$ propagators are both proportional to $1/t$. A non-zero result arises only because if the result is blindly expanded in
powers of $1/t$, the endpoint contributions are infinite. This is because the term of lowest order in $1/t$ in the operator product $V^{(1)}(0) \cdot A_\sigma \frac{d\sigma}{d\tau} (\sigma)$ is proportional to $1/\sigma$, coming from $\phi - \partial \phi$ contractions. A simple-minded $1/t$ expansion of (4.17), evaluating those contractions (and discarding a term proportional to $\{Q, \ldots\}$), leads to a divergent contribution near $\sigma = 0$ that looks like

$$\frac{\pi}{2t} \int_0^\epsilon d\sigma \frac{1}{\sigma} \langle \chi^a (d^\ast (\alpha^{(1)} \land \alpha^{(2)}))_a (0) V^{(3)}(1) V^{(4)}(\infty) \rangle,$$

with $\epsilon$ an arbitrary fixed positive number.

To cure the infinity, we must remember the anomalous dimensions of the various operators, which are given by the eigenvalues of $L_0$. To order $1/t$, we had $L_0 = \pi \Delta/2t$ in §3.3. The physical states have harmonic representatives, and if we pick such representatives (discarding terms of the form $\{Q, \ldots\}$), the $V^{(i)}$ have $L_0 = 0$ in order $1/t$. However, the vertex operator $\chi^a (d^\ast (\alpha^{(1)} \land \alpha^{(2)}))_a$ in (4.18) corresponds to a state not annihilated by $L_0$. Including powers of $\sigma$ coming from the anomalous dimensions, and replacing the expectation values $\langle \ldots \rangle$ by zero mode integrals, (4.18) is replaced by

$$\frac{\pi}{2t} \int_0^\epsilon d\sigma \frac{1}{\sigma} \int_M \text{Tr} \left( d^\ast (\alpha^{(1)} \land \alpha^{(2)}) \sigma^{\pi \Delta/2t} \land \alpha^{(3)} \land \alpha^{(4)}(\infty) \right) \tag{4.19}$$

The endpoint contribution of the $\sigma$ integral is now finite as $t \to \infty$, and reproduces one of the two Feynman diagram contributions in equation (4.15). The other arises from the endpoint at $\sigma = 1$.

Now let us discuss briefly the physical significance of these couplings. Given a flat bundle $E$ corresponding to a solution $A^{(0)}$ about which one wishes to expand, the space of linearized deformations is $H^1(M, \text{End}(E))$. Writing $A = A^{(0)} + B$, impose the gauge condition $d^\ast B = 0$ (with $d^\ast$ being defined with respect to $A^{(0)}$). In turn, write $B = \sum_i u_i \alpha^{(i)} + \sum_\sigma v_\sigma \beta^{(\sigma)}$, where $\alpha^{(i)}$ are a basis of zero modes of

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the Laplacian $\Delta$ and $\beta_{(\sigma)}$ are the non-zero modes; and $u_i, v_\sigma$ are real coefficients.

The underlying Lagrangian

$$L = \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

(4.20)

can be regarded as a function $L(u_i, v_\sigma)$. The tree diagrams that we have discussed above are a calculational technique for “integrating out” the $v_\sigma$ to get an “effective potential” $V(u_i)$. It can be written in terms of the $n$-point functions on the disc discussed above for $n = 4$ as

$$V(u_i) = \sum_i \frac{u_i u_j u_k}{3!} \langle \alpha(i) \alpha(j) \alpha(k) \rangle + \sum_{i,j,k,l} \frac{u_i u_j u_k u_l}{4!} \langle \alpha(i) \alpha(j) \alpha(k) \alpha(l) \rangle + \ldots$$

(4.21)

Since $V$ is obtained from the underlying Lagrangian $L$ by “integrating out the massive modes,” its stationary points, \textit{i.e.}, solutions of $\partial V/\partial u_i = 0$, are in one-to-one correspondence with the stationary points of the underlying Lagrangian.

The process of “integrating out the massive modes” can be described as follows. Let $L$ be a function with a critical point at the origin, and let $n$ be the dimension of the kernel of the Hessian or matrix of second derivatives of $L$. Then by a generalization of the Morse lemma, it is possible to pick coordinates $u_i, i = 1 \ldots n$ and $v_\sigma$ in a neighborhood of the origin so that

$$L(u_i, v_\sigma) = V(u_i) + \sum_\sigma a_\sigma v_\sigma^2$$

(4.22)

with non-zero constants $a_\sigma$; $V(u_i)$ is then the “effective potential with massive modes integrated out.” The $u_i$ are of course only uniquely determined up to diffeomorphism. Describing the possible canonical forms of $V(u_i)$ up to a reparametrization of the variables is more or less the problem of singularity theory.

Since the $u_i$ can be described in first order as coordinates near the origin on the naturally defined vector space $H^1(M, \text{End}(E))$, it is natural to think of the
effective potential $V$ as a function defined on that space, and as such it is uniquely defined up to a coordinate transformation

$$u_i \rightarrow \tilde{u}_i = u_i + c_{ijk} u_j u_k + \ldots$$  \hspace{1cm} (4.23)

that is trivial in first order. The quartic coupling $\langle \alpha_{(i)} \alpha_{(j)} \alpha_{(k)} \alpha_{(l)} \rangle$, for particular values of $i, j, k, l$, is invariant under such transformations if and only if the cubic coupling $\langle \alpha_{(m)} \alpha_{(n)} \alpha_{(p)} \rangle$ vanishes for $m, n$ equal to two consecutive values of $i, j, k, l$; by Poincaré duality, this is so precisely if $\alpha_{(m)} \wedge \alpha_{(n)}$ vanishes in $H^2(M, \text{End}(E))$ whenever $m$ and $n$ equal two consecutive values of $i, j, k$, and $l$.

Under such conditions, the four point coupling can be given the following interpretation. Since $\alpha_{(i)} \wedge \alpha_{(j)}$ and $\alpha_{(j)} \wedge \alpha_{(k)}$ vanish in cohomology, there are $\text{End}(E)$-valued one-forms $y, z$ with

$$dy = \alpha_{(i)} \wedge \alpha_{(j)}, \quad dz = \alpha_{(j)} \wedge \alpha_{(k)}. \hspace{1cm} (4.24)$$

In fact, we can set

$$y = d^* \frac{1}{\Delta} (\alpha_{(i)} \wedge \alpha_{(j)}) \hspace{1cm} (4.25)$$

$$z = d^* \frac{1}{\Delta} (\alpha_{(j)} \wedge \alpha_{(k)}).$$

(Because $1/\Delta$ is the inverse of $\Delta$ only in the orthocomplement of the kernel of $\Delta$, in proving that (4.25) obeys (4.24) one needs the fact that $\alpha_{(i)} \wedge \alpha_{(j)}$ and $\alpha_{(j)} \wedge \alpha_{(k)}$ vanish in cohomology and so are orthogonal to the kernel.) Let

$$w(\alpha_{(i)}, \alpha_{(j)}, \alpha_{(k)}) = y \wedge \alpha_{(k)} + \alpha_{(i)} \wedge z. \hspace{1cm} (4.26)$$

Then $dw = 0$, so $w$ determines an element of $H^2(M, \text{End}(E))$. Considering how $w$ transforms under a shift in $y$ and $z$, one sees that it is not well-defined as an

---

* One considers $i, j, k, l$ arranged on a circle in that order, so $l$ and $i$ are consecutive.
element of $H^2(M, \text{End}(E))$ but only as an element of the quotient group

$$H^2(M, \text{End}(E))/(H^1(M, \text{End}(E)) \wedge \alpha(\ell) \oplus \alpha(i) \wedge H^1(M, \text{End}(E))).$$  \hspace{1cm} (4.27)

The element $w$ of that quotient space is known as the Massey triple product of $\alpha(i), \alpha(j),$ and $\alpha(k)$. Under the further hypothesis that $\alpha(k) \wedge \alpha(\ell)$ and $\alpha(\ell) \wedge \alpha(i)$ vanish in cohomology,

$$\int_M \text{Tr} w(\alpha(i), \alpha(j), \alpha(k)) \wedge \alpha(\ell)$$  \hspace{1cm} (4.28)

is well-defined and in fact equals coefficient of the quartic term in the effective potential as computed above. The contribution of the quartic coupling to $\partial V/\partial u_\ell$ is therefore

$$\frac{1}{3!} \sum_{i,j,k} u_i u_j u_k \int_M \text{Tr} w(\alpha(i), \alpha(j), \alpha(k)) \cdot \alpha(\ell).$$  \hspace{1cm} (4.29)

As a flat connection is a critical point of the Chern-Simons action, this is in keeping with the observation [27] that the Massey triple product and its higher order analogs are the obstructions to deformation of a flat connection.

4.3. Comparison To Remarks Of Kontsevich

What do we learn by interpreting Chern-Simons gauge theory as a string theory? Here I will only point out one obvious consequence. Other applications may involve, for instance, a deeper relation of Chern-Simons theory to mirror manifolds.

The fact that Chern-Simons theory can be interpreted as a topological sigma model coupled to topological gravity means that one can introduce the usual observables of topological gravity – the stable cohomology classes on the moduli space of Riemann surfaces introduced by Mumford, Morita, and Miller. In the present context, this would amount to the following. As was sketched in §2, the topological sigma model determines a differential form $\Theta$ on moduli space; for open strings, the relevant moduli space is the moduli space $\mathcal{M}_{g,s,h}$ of Riemann surfaces of genus

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with \( s \) punctures (where vertex operators are inserted) and \( h \) boundary components. Instead of integrating the closed form \( \Theta \) over \( \mathcal{M}_{g,s,h} \), we can integrate it over a homology cycle \( C \) in \( \mathcal{M}_{g,s,h} \) of codimension, say, \( n \).

The trouble is that we have fixed \( X \) to have complex dimension three precisely to ensure that \( \Theta \) is a top form. It therefore cannot be integrated over a cycle of positive codimension unless the definition is modified somehow. The following way of doing this will enable us to make contact with observations by Kontsevich.

Instead of considering a single target space \( X \) with a fixed metric, we will consider a family of \( X \)'s with a family of metrics. The partition function of the theory will be not a number but a closed differential form on the space of metrics on \( X \). To implement this idea, introduce along with the metric \( g_{IJ}(\phi) \) of \( X \) a fermionic variable \( \zeta_{IJ}(\phi) \) (of ghost number one) which one can think of as (up to a constant) the exterior derivative of \( g_{IJ}(\phi) \). Take the new transformation laws

\[
\delta g_{IJ} = i\alpha\zeta_{IJ}, \quad \delta \zeta_{IJ} = 0. \tag{4.30}
\]

Recalculating the Lagrangian starting from (3.2), we find that (3.3) is replaced by

\[
\tilde{L} = L + L', \quad L' = -t \int_{\Sigma} d^2z \zeta_{ij} \left( \psi^*_i \partial_z \phi^j + \partial_z \phi^i \psi^*_j \right). \tag{4.31}
\]

Since \( L' \) is of ghost number \(-1\) in the “matter fields” \( \phi, \chi, \psi \), every insertion of \( L' \) shifts the ghost number by one unit. If we want to integrate not over moduli space but over a cycle \( C \) of codimension \( n \), the non-vanishing contributions will be precisely \( n^{th} \) order in \( L' \). As \( L' \) is linear in \( \zeta_{IJ} \), these contributions will be \( n^{th} \) order in \( \zeta_{IJ} \) and so will define an \( n \)-form \( \Omega \) on the space \( \mathcal{R} \) of metrics on \( X \). \( Q \)-invariance means that \( \Omega \) is closed. Let \( \mathcal{F} \) be the group of diffeomorphisms of \( X \). \( \Omega \) is invariant under the natural action of \( \mathcal{F} \) on \( \mathcal{R} \) and moreover is basic (to show the later one notes that if \( \zeta_{IJ} = D_I v_J + D_J v_I \) for some \( v^I \), then \( L' \) is of the form

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\( \{Q, \ldots\} \) up to terms that vanish by the equations of motion). So if one has a fiber bundle

\[
\begin{align*}
X & \rightarrow Y \\
\downarrow & \\
B,
\end{align*}
\]

with an arbitrary base \( B \), then by picking a metric on the total space \( Y \) one gets a family of metrics on \( X \), parametrized by \( B \), and \( \Omega \) determines an \( n \)-dimensional cohomology class of \( B \).

We have found, therefore, a map from the codimension \( n \) homology of \( M_{g,s,h} \) to the \( n \)-dimensional cohomology of \( B \). This map was described by Kontsevich [5] by examining Chern-Simons perturbation theory. The considerations just explained give a more conceptual explanation for its existence.

**A Digression**

Let us now make a small digression to examine some related observations by Kontsevich in the light of standard quantum field theory ideas. We want to consider the standard Fadde’ev-Popov-BRST quantization of three dimensional Chern-Simons gauge theory, with gauge group \( G \), on a three-manifold \( M \). In doing so, in addition to the gauge field \( A_a \), one introduces a ghost field \( c \) (anticommuting, of ghost number one, transforming in the adjoint representation). The usual BRST transformation laws are

\[
\begin{align*}
\delta A_a &= -D_a c \\
\delta c &= \frac{1}{2}[c, c].
\end{align*}
\]

We temporarily postpone introducing the antighosts and auxiliary fields that enter the gauge fixing.

Just from the \( A - c \) system, it is possible to construct new observables (discussed, for instance, in [28], where more information can be found). Let \( T \) be an invariant, antisymmetric polynomial on the Lie algebra \( \mathcal{G} \) of \( G \). If \( P \in M \) is any
point, let
\[ O_T^{(0)}(P) = T(c(P)). \] (4.34)

It is evident that \( O^{(0)} \) is BRST invariant and cannot be written in the form \( \delta(\ldots) \).

By solving the "descent equations"
\[
\begin{align*}
    dO^{(0)} &= \delta O^{(1)} \\
    dO_T^{(1)} &= \delta O^{(2)} \\
    dO_T^{(2)} &= \delta O^{(3)} \\
    dO_T^{(3)} &= 0,
\end{align*}
\] (4.35)

one finds for each \( i, 0 \leq i \leq 3 \), an operator-valued \( i \)-form \( O_T^{(i)} \) that is BRST invariant up to \( \delta(\ldots) \). Hence, if \( Y \subset M \) is a \( i \)-dimensional cycle, then
\[
\int_Y O_T^{(i)} \] (4.36)

is BRST invariant (and is easily seen, by virtue of (4.35), to depend only on the homology class of \( Y \)). In particular, setting \( i = 3 \) and \( Y = M \), we get new terms that can be added to the Chern-Simons Lagrangian. To be precise, if \( L \) is the usual Chern-Simons Lagrangian and \( T_\alpha \) are the antisymmetric invariants on the Lie algebra, we can take
\[
L \rightarrow L + \sum_\alpha t_\alpha \int_M d^3x \ O_T^{(3)\alpha}. \] (4.37)

This modification of the standard Chern-Simons theory is implicit in the work of Kontsevich, who describes the situation in terms of a certain class of homotopy Lie algebras.
One could also, as in Donaldson theory, pick closed submanifolds $M_\alpha$ of $M$, of dimension $d_\alpha$, and generalize (4.37) to

$$L \to L + \sum_\alpha t_\alpha \int_{M_\alpha} d^3 x \; O^{(d_\alpha)}_{T_\alpha}.$$  \hspace{1cm} (4.38)

As it stands, (4.37) is not very useful. Since the usual Chern-Simons theory conserves ghost number, and the $O^{(3)}$'s all have ghost number +3, the results will be independent of the $t_\alpha$ unless we also introduce some interaction of negative ghost number. To get something interesting, we will now modify the usual gauge fixing.

Gauge fixing requires the introduction of antighost and auxiliary fields. The standard procedure is to introduce the antighost $\bar{c}$ (anticommuting, of ghost number $-1$, in the adjoint representation) and the auxiliary field $w$ (commuting, of ghost number 0, in the adjoint representation), with

$$\delta \bar{c} = iw, \quad \delta w = 0.$$  \hspace{1cm} (4.39)

Gauge fixing is then carried out by

$$L \to L + \delta \Gamma,$$  \hspace{1cm} (4.40)

with any convenient $\Gamma$. A standard choice involves picking a metric $g_{ab}$ on $M$. Writing also $A = A_{(0)} + B$, where $A_{(0)}$ is a solution of the classical equations about which one wishes to expand, and denoting the covariant derivative with respect to $A_{(0)}$ as $D_{(0)}$, we take

$$\Gamma = - \int_M d^3 x \sqrt{g} g^{ab} \text{Tr} D_{(0)}^a B_b.$$  \hspace{1cm} (4.41)

Computing $\delta \Gamma$, one gets the usual gauge fixing Lagrangian

$$L_{GF} = \int_M d^3 x \sqrt{g} \text{Tr} \left( iw D_{(0)}^a B^b - \bar{c} D_{(0)}^a D^b c \right).$$  \hspace{1cm} (4.42)

Just as in our field theoretic discussion, to modify this, we will consider not a
fixed target $M$ but a family of $M$’s with a variable metric. Instead of considering the metric $g_{ab}$ of $M$ to be “inert” under the BRST transformations, we introduce corresponding fermi variables $\zeta_{ab}$ (of ghost number 1), with

$$\delta g_{ab} = \zeta_{ab}, \quad \delta \zeta_{ab} = 0.$$  \hspace{1cm} (4.43)

Then (4.42) is replaced by $\tilde{L}_{GF} = L_{GF} + \Delta L_{GF}$ with

$$\Delta L_{GF} = -\int_M d^3 x \sqrt{g} \left( \zeta^{ab} - \frac{1}{2} g^{ab} \zeta^c_c \right) \text{Tr} \tau D_a^{(0)} B_b.$$ \hspace{1cm} (4.44)

Since $\Delta L_{GF}$ is of ghost number $-1$ in the matter fields, insertions of $O^{(3)}$’s can be balanced by insertions of $\Delta L_{GF}$. As $\Delta L_{GF}$ is linear in $\zeta$, the resulting amplitudes, just as in our string theoretic discussion, will be naturally not numbers but differential forms on the base space of a fibration.

### 4.4. General Target Spaces

So far we have only considered the A model with target space $X = T^*M$. Now we want to generalize the discussion to consider an arbitrary symplectic target manifold $X$ (of $c_1 = 0$), with $M$ as a Lagrangian submanifold.

The first consequence of replacing $T^*M$ with a more general $X$ is that there may be nonconstant instantons. The same argument that we used in proving the vanishing theorem for $T^*M$ shows that a nonconstant instanton would necessarily have a positive value of the instanton number

$$q = \int_\Sigma \Phi^*(\omega).$$ \hspace{1cm} (4.45)

To improve the convergence of our formulas, we pick a positive number $\theta$, and weight instantons of instanton number $q$ with a factor of $\exp(-\theta q)$. This can be
naturally built into the formulas by adding to the Lagrangian a suitable multiple of $q$:

$$L \rightarrow L + \theta \int_{\Sigma} \Phi^* (\omega).$$

(4.46)

In the absence of non-constant instantons, the space-time theory of the $A$ model was ordinary Chern-Simons theory. We want to determine the corrections to this coming from the non-constant instantons. In doing so, our goal is to find the classical Lagrangian underlying the space-time physics of the $A$ model. To this end, we concentrate on the case that the world-sheet is a disc $\Sigma$. (However, corrections due to higher topologies can be described similarly.) The target space Lagrangian $L_T$ is equal to a world-sheet path integral on the disc:

$$L_T = \int DX \ldots D\psi \exp(-L)$$

(4.47)

We already know that the contribution to (4.47) of instanton number $q = 0$ is the ordinary Chern-Simons action. We want to determine the contribution for some non-zero value of $q$.

Consider the moduli space of holomorphic maps $\Phi : \Sigma \rightarrow X$ with $\Phi(\partial \Sigma) \subset M$ and with two such maps identified if they differ by an $SL(2, \mathbb{R})$ transformation. The fact that $c_1(X) = 0$ and $\dim \mathcal{C}^e(X) = 3$ means that in the moduli problem, the dimensions of the appropriate $H^0$ and $H^1$ are zero. "Generically" this means that there are only finitely many such instantons for each value of $q$. For simplicity, we will consider only this case. If $\Phi$ is such an instanton, let $C = \Phi(\partial \Sigma)$. Generically, $C$ is a knot in the three-manifold $M$. Let us work out the contribution of $\Phi$ to the path integral (4.47). If we are expanding around a background connection $A$ in $M$, the contribution of the Chan-Paton factors is

$$\text{Tr} \ P \exp \int_C A.$$  

(4.48)

Note that though $C$ bounds a disc in $X$, it may not do so in $M$, so (4.48) can be non-trivial even if $A$ is flat. We also get a factor of $\exp(-\theta q)$ from the instanton-counting
term in (4.46). The remaining contributions are nearly trivial since (i) they are
independent of $t$; (ii) they reduce in the large $t$ limit to a ratio of determinants;
(iii) except for a possible sign, the boson and fermion determinants cancel because
of the $Q$ symmetry. The contribution of an instanton is hence

$$\eta \exp(-\theta q) \text{Tr} P \exp \int C A$$

(4.49)

where $\eta = \pm 1$ is the ratio of determinants.

The total action is therefore easy to evaluate. If $\Phi_i, \ i = 1, 2, 3, \ldots$ are the
instantons of non-zero instanton number, with instanton numbers $q_i$, boundaries
$C_i$, and determinant factors $\eta_i$, then the action is

$$L_T = \frac{1}{2} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) + \sum_{i=1}^{\infty} \eta_i \exp(-\theta q_i) \text{Tr} P \exp \int C A.$$

(4.50)

For instance, for $\theta \gg 0$, the factors $\exp(-\theta q_i)$ are small, and the instanton
corrections to amplitudes can be evaluated perturbatively. Their evaluation would
involve calculating expectation values of products of Wilson lines on the three-
manifold $M$.

4.5. The B Model

Now, we would like, in a similar spirit, to identify the space-time field theory
that is equivalent to the B model, with the “free” boundary conditions of §3.1. To
be more precise, we consider the open string sector of the B model, and we use the
same general framework of open string field theory as in §4.1. We can be brief, as
the arguments are so similar.

As in the case of the A model, the main simplification comes from the invariance under rescaling the metric of the target space $X$ by an arbitrary factor $t$. In
§3.3, we saw that the low-lying modes of the string are functions $A(\phi^I, \eta^5)$ of the
zero modes. As we now wish $\mathcal{A}$ to have ghost number 1, we take it to be linear in $\eta$, so in fact

$$A = \eta^i A_i(\phi^I).$$  \hspace{1cm} (4.51)

So the physical field $A$ is a one-form of type $(0, 1)$. If gauge fields are included via Chan-Paton factors, then $A$ takes values in $N \times N$ matrices or more generally in the endomorphisms of some holomorphic vector bundle $E$.

The linearized gauge transformation law $\delta A = Q \epsilon$ reduces for large $t$ to $\delta A = \overline{\partial} \epsilon$, so $A$ must be interpreted as the $(0, 1)$ part of a connection on $E$. What should be the field equation for $A$? This can be anticipated from the discussion at the end of §3.2, where we showed that the background connections to which the B model can be coupled are precisely those for which the $(0, 2)$ part of the curvature vanishes, in other words those that define holomorphic structures on $E$. To write a Lagrangian from which this equation can be derived, let $\lambda$ be an everywhere non-zero holomorphic three-form on $X$. Then up to an undetermined constant, the Lagrangian whose solutions are connections of vanishing $(0, 2)$ curvature is

$$L_T = \frac{1}{2} \int_X \lambda \wedge \text{Tr} \left( A \wedge \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A \right).$$  \hspace{1cm} (4.52)

Arguments similar to those that we have given for the A model show that in the large $t$ limit, the open string field theory of the B model reduces to (4.52).

The quantum field theory with Lagrangian (4.52) is unrenormalizable by power counting. However, it has the following all-but-unique property: there are no possible counterterms that respect the classical symmetries. The symmetries of (4.52) include complex gauge transformations, $\overline{\partial} A \rightarrow g \overline{\partial} A g^{-1}$, with $g$ an arbitrary gauge transformation of $E$ not respecting any reality or unitarity condition; and local complex changes of coordinates that preserve $\lambda$. There is no local density constructed from $A$ that is invariant under these symmetries. (Even $L_T$ itself, though possessing these invariances at least if one considers only gauge transformations
Infinities in quantum field theory are ordinarily integrals of invariant local densities, so if (4.52) could be quantized preserving the symmetries, one would expect this theory to be finite, though superficially unrenormalizable.

Relying only on usual field theory arguments, it is not at all clear that (4.52) can be quantized preserving its symmetries. However, the equivalence of (4.52) to a string theory strongly suggests that it in fact is finite. One might worry about whether closed string poles can ruin the finiteness; at the end of §5 we will argue that this does not occur. One might also wonder whether the finite theory given by the string theory is really (4.52) or some more elaborate theory with (4.52) coupled to closed strings. In §5 we argue that the closed strings are decoupled.

5. The Closed String Sector

We have discovered that – with some reasonable boundary conditions – the open string sector of the topological A and B string theories has an elegant interpretation in terms of a space-time field theory. The extension to closed strings does not work so nicely; the first point of this section is to sketch what the problem is. After doing this, I will conclude by trying to show that open and closed strings are decoupled in these models; this is intended as partial justification for studying the open strings separately in §4.
For open strings, the propagator is

\[
\frac{b_0}{L_0},
\]

with \(L_0\) the Hamiltonian of the string and \(b_0\) the antighost zero mode. For closed strings, one has separate zero modes \(b_0\) and \(\bar{b}_0\) for right- and left-moving antighosts. It is convenient to set \(b_0^\pm = b_0 \pm \bar{b}_0\). The formula analogous to (5.1) is that the closed string propagator is

\[
\frac{b_0 \bar{b}_0^+}{2L_0^+} \Pi = \frac{b_0 \bar{b}_0}{L_0^+} \Pi.
\]

(5.2)

Here \(L_0^+ = L_0 - \overline{\mathcal{L}}_0\) is the total string Hamiltonian, and \(\Pi\) is the projection operator on states invariant under rotation of the circle. This formula is fairly well known, and in any case can be derived similarly to (5.1), replacing the strip in figure (6) with the cylinder of figure (8); the extra ghost field in the numerator and the projection on rotation-invariant states come from the twist symmetry of the cylinder, indicated in the figure.

For the \(A\) model, for instance, the long wavelength limits of \(b_0\) and \(\bar{b}_0\) are the \(\partial^*\) and \(\overline{\partial}^*\) operators of the target space \(X\). With \(L_0\) reducing at long wavelengths to the Laplacian \(\Delta\), the string propagator looks like

\[
\frac{\partial^* \overline{\partial}^*}{\Delta}.
\]

(5.3)

This propagator, however, does not seem to arise by gauge fixing of any local Lagrangian. It is a pseudodifferential operator of degree zero, so for it to arise as the inverse of a differential operator, that operator would have to be of degree zero, that is a constant.

In fact, by repeating for the closed string the analysis of §3.3, one finds that the low energy modes of the closed string sector of the \(A\) model are naturally represented by a two-form \(h\) in space-time.* (For large \(t\), winding sectors of the

* For the \(B\) model, one gets instead a sum of \((0, i)\) forms with values in \(\bigwedge^j T^{(1,0)}\), with \(i + j = 2\).
closed string cannot have small eigenvalues of $L_0$.) The free Lagrangian for $h$ that one might guess by analogy with our open string results would be

$$L = \int_X h \wedge \partial \bar{\partial} h.$$

(5.4)

This has

$$\frac{\partial^* \bar{\partial}^*}{\Delta^2}$$

(5.5)

for a gauge fixed propagator. The extra factor of $\Delta$ in the denominator, which is in sharp variance with (5.3), of course, makes (5.5) a pseudodifferential operator of degree $-2$, in keeping with the fact that the kinetic operator in (5.4) is second order.

What sort of Lagrangian do we get by taking the low energy limit of closed string field theory? As we discussed in §2 in a related context, the free part of the closed string Lagrangian is

$$(\Psi, c_0^- Q \Psi),$$

(5.6)

where ideally

$$\{b_0^-, c_0^-\} = 1.$$  

(5.7)

Such a $c_0^-$ does not exist, since $b_0^-$, whose field theory limit is $\partial^* - \bar{\partial}^*$, has a non-trivial cohomology. However, one can pick a $c_0^-$ such that $\{b_0^-, c_0^-\} = 1 - T$, where $T$ is the projection operator onto a subspace (say the kernel of $L_0$) annihilated by $Q$; this is good enough to ensure gauge invariance of (5.7). Such a $c_0^-$ is, in the field theory limit,

$$\frac{\partial - \bar{\partial}}{2\Delta}.$$  

(5.8)

With this choice of $c_0^-$, and recalling that the field theory limit of $Q$ is $\partial + \bar{\partial}$, the
field theory limit of the Lagrangian is not (5.4) but

$$\int_X h \wedge \frac{1}{\Delta} \partial \bar{\partial} h.$$  (5.9)

When this is gauge-fixed and inverted to get a propagator, the factor of $\Delta^{-1}$ migrates to the numerator, canceling a factor of $\Delta^{-1}$ in (5.5) and reproducing the field theory limit of the closed string propagator (5.3).

**A General Puzzle**

The conclusion seems to be that closed string field theory of the A model – or similarly of the B model – would be non-local in space-time. Sometimes such apparent non-localities can be eliminated by introducing additional fields (such that the apparent non-locality arises in integrating them out). I have no evidence that that can be done here. In any event, certain puzzling arguments seem to show that the closed string A and B theories do not behave as one would expect of space-time field theories.

In either the A or the B model, there are non-trivial cubic and higher order couplings of the physical modes. In studying the open string sector of the A model, there was at the classical level (the world-sheet being a disc) a cubic coupling of three physical fields given by the classical formula $\int_M \text{Tr} A \wedge A \wedge A$, and various higher interactions involving Massey products, as we saw in §4.2. As always in field theory, these couplings are Taylor series coefficients of a natural potential $V(t_i)$ for sources $t_i$ coupling to the physical modes. The classical solutions of the space-time theory – or equivalently the possible world-sheet theories – are in one-to-one correspondence with the critical points of $V$.

What about the closed string sectors? There are analogous cubic and higher couplings of physical modes. For instance, for the closed string sector of the A model, the cubic coupling in the large volume limit is $\int_X h \wedge h \wedge h$. Experience with both field theory and with the open string sectors of the A and B models lead us to
form a generating function $V(t_i)$ from these couplings, with the expectation that the allowed world-sheet theories will correspond to the critical points of $V$. This latter expectation proves to be false. For the A model, for instance, the part of the $h$ field that is a $(1, 1)$ form in space-time represents a displacement in the Kahler class of the metric of $X$. The A model makes sense for any choice of this Kahler class, so in contrast to what one would have anticipated, having $V'(t_i) \neq 0$ is not an obstruction to being able to define the world-sheet theory. Likewise, for the B model, the low energy modes include a displacement in the complex structure of $X$. The B model makes sense for any complex structure on $X$, even though there are non-zero cubic and higher order couplings for the fields representing a displacement in the complex structure.

5.1. Closed String Contributions To Amplitudes

Leaving this puzzle as food for thought, I will conclude by making some simple comments about the closed string contributions to open string amplitudes of the A and B models.

Let us look briefly at the couplings of open and closed strings in these models. To do so we will use the fatgraph or open string field theory description of external open and closed strings. An external open string (a marked point on the boundary of a Riemann surface) should have one real modulus, while an external closed string (a marked point in the interior) should have two. In the fatgraph description, as one might expect, external open strings are represented by open string propagators going off to infinity, as for $O_1$ and $O_2$ in figure (9(b)). The one real modulus of the open string is, roughly, the location at which its propagator is attached to the rest of the figure. The proper coupling of external closed strings (which can be deduced by seeing how closed string poles arise in open string diagrams, as in [29]) is as follows. One attaches to the open string diagram an external open string propagator of finite length $T$, closes it up by folding together its free end, and inserts the closed string at the resulting conical singularity. The two real moduli
Fig. 9. In the fatgraph description of a Riemann surface coupling open and closed strings, the closed strings are incorporated as in (a). A standard open string strip of width $\pi$ and length $T$, open at one end (where it attaches to the rest of the diagram) is closed at the other end by folding it over on itself; the closed string is attached at the resulting conical singularity, as shown. A typical fatgraph coupling open and closed strings is shown in (b). It consists of five flat strips glued on the dotted lines, with conical singularities shown as solid dots. The open strings $O_1$ and $O_2$ are attached at the ends of infinite flat strips, while the closed strings appear as conical singularities on strips of finite lengths $T_1$ and $T_2$. This particular world-sheet has three real moduli: $T_1$, $T_2$, and the length $T_3$ of the one internal propagator.

of the closed string are, roughly, $T$ and the position at which the propagator is attached to the rest of the diagram. This is sketched in figures (9(a,b)).

To compute the classical couplings of $n$ open strings and $m$ closed strings, we consider the moduli space $\mathcal{D}_{n,m}$ of a disc with $n$ marked points on the boundary and $m$ in the interior; its real dimension is $n+m-3$. (The analysis goes through the same way in higher genus.) Comparing the dimension of $\mathcal{D}_{n,m}$ to that of $\mathcal{D}_{n+m,0}$, there is one extra modulus for each marked point in the interior. As explained in the last paragraph, the fatgraph description of $\mathcal{D}_{n,m}$ (figure (9(b))) is similar to the fatgraph description of $\mathcal{D}_{n+m,0}$ except that, while the external open strings are attached to outgoing propagators of infinite length, the external closed strings are attached to propagators of variable length; the one extra real modulus for each external closed string is precisely the length of the propagator to which it is attached.
In the A model, for instance, an external open string is represented by an (End(E)-valued) one-form \( \alpha \), while an external closed string is represented by a two-form \( h \). Integration over the extra modulus of the propagator by which the closed string is attached to the rest of the diagram multiplies \( h \) by the open string propagator \( b_0/L_0 \). In the large \( t \) limit, this simply turns \( h \) into a one-form

\[
\frac{d^*}{L_0} \cdot h.
\]

(5.10)

This one-form then couples as just one more external open string state (which happens to be valued in the center of End(E)). But the one-form in (5.10) is exact (since, for instance, the open string propagator annihilates harmonic forms), so the corresponding open string state decouples. Consequently the on-shell couplings of open and closed strings are all zero – in either the A or B model.

If the closed strings are, then, decoupled from the open strings, what is their role? The following conjecture seems natural to me. The field theories we have extracted from the A and B models all have \( c \)-number anomalies, analogous to the central charge in two-dimensional conformal field theory. For instance, the anomalies of Chern-Simons theory are connected with framings of three-manifolds and of knots [30]. One-loop anomalies of the field theory related to the B model were calculated long ago by Ray and Singer [31]. These anomalies are possible only because of the ultraviolet difficulties of quantum field theory, which of course are greatly ameliorated in string theory. It therefore seems reasonable to suspect that the closed string contributions in the A and B theories cancel the anomalies of these theories, without, in view of the decoupling argued above, having much effect on the open string “physics.”

In any event, whatever the closed string contributions, they are finite. In usual string theory, possible infinities come from physical closed string poles, but in these topological models, there are no such poles since the closed string propagator annihilates the physical states or harmonic forms; this is because (in contrast to conventional string backgrounds) the \( b_0 \) operator annihilates the kernel of \( L_0 \).
Thus, for instance, in the case of the B model, whose finiteness perhaps comes as a surprise (since the field theory related to the low energy limit of the open strings is superficially unrenormalizable), the closed strings will not ruin this finiteness.

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REFERENCES

1. E. Witten, “Topological Sigma Models,” Comm. Math. Phys. 118 (1988) 411.

2. M. Gromov, “Pseudo-Holomorphic Curves In Symplectic Manifolds,” Invent. Math. 82 (1985) 307.

3. A. Floer, “Symplectic Fixed Points And Holomorphic Spheres,” Comm. Math. Phys. 120 (1989) 575.

4. S. Axelrod and I. M. Singer, “Chern-Simons Perturbation Theory,” MIT preprint (1991).

5. M. Kontsevich, lecture at Institute for Advanced Study (January, 1992).

6. E. Witten, “Mirror Manifolds and Topological Field Theory,” in Essays On Mirror Manifolds, ed. S.-T. Yau (International Press, 1992).

7. W. Siegel, “Covariantly Second-Quantized Strings, II,III,” Phys. Lett. 151B (1985) 391,396.

8. S. Elitzur, A. Forge, and E. Rabinovici, “On Effective Theories Of Topological Strings,” preprint CERN-TH.6326 (1991).

9. G. ’t Hooft, Nucl. Phys. B72 (1974) 461.

10. E. Witten, “Two Dimensional Gravity And Intersection Theory On Moduli Space,” Surv. Diff. Geom. 1 (1991) 243.

11. R. Penner, “The Teichmuller Space Of A Punctured Surface,” Comm. Math. Phys. (1987), “Perturbative Series And The Moduli Space Of Riemann Surfaces,” J. Diff. Geom. 27 (1988) 35.

12. J. Harer, “The Virtual Cohomological Dimension Of The Mapping Class Group Of Orientable Surfaces,” Inv. Math. 84 (1986) 157.

13. B. H. Bowditch and D. B. A. Epstein, “Natural Triangulations Associated To A Surface,” Topology 27 (1988) 91.
14. E. Witten, “Non-Commutative Geometry And String Field Theory,” Nucl. Phys. B268 (1986) 253.

15. S. Giddings, E. Martinec, and E. Witten, “Modular Invariance In String Field Theory,” Phys. Lett. 176B (1986) 362.

16. J. Distler and P. Nelson, “Topological Couplings And Contact Terms In 2d Field Theory,” Comm. Math. Phys. 138 (1991) 273.

17. B. Zwiebach, “Closed String Field Theory: Quantum Action And The BV Master Equation,” preprint IASSNS-HEP-92/41 (June, 1992).

18. L. Dixon, lecture at Princeton University, ca. 1987.

19. C. Vafa, “Topological Mirrors And Quantum Rings,” in Essays on Mirror Manifolds, ed S.-T. Yau (International Press, 1992).

20. J. M. F. Labastida and P. M. Llatas, “Topological Matter in Two Dimensions” (preprint, 1991).

21. P. Horava, “Equivariant Topological Sigma Models,” preprint (1991).

22. V. Mathai and D. Quillen, “Superconnections, Thom Classes, and Equivariant Differential Forms,” Topology 25 (1986) 85.

23. M. F. Atiyah and L. Jeffrey, “Topological Lagrangians And Cohomology,” J. Geom. Phys. 7 (1990) 119.

24. D. Fried, Inv. Math. 84 (1986) 523.

25. C. B. Thorn, “Perturbation Theory For Quantized String Fields,” Nucl. Phys. B287 (1987) 61.

26. M. Bochicchio, “Gauge Fixing For The Field Theory Of The Bosonic String,” Phys. Lett. 193B (1987) 31.

27. D. Johnson and J. Millson, “Deformation Spaces Associated To Compact Hyperbolic Manifolds,” in Discrete Groups In Geometry And Analysis: Papers In Honor Of G. D. Mostow, ed R. Howe (Birkhauser, Boston, 1987).
28. O. Piguet and S. P. Sorella, “On The Finiteness Of BRS Modulo-$d$ Cocycles,” Univ. of Geneva preprint UGVA-DPT 1992/3-759.

29. D. Z. Freedman, S. B. Giddings, J. A. Shapiro, and C. B. Thorn, Nucl. Phys. B298 (1988) 253.

30. E. Witten, “Quantum Field Theory And The Jones Polynomial,” Comm. Math. Phys. 121 (1989) 351.

31. D. Ray and I. M. Singer, “Analytic Torsion And The Laplacian On Complex Manifolds” Ann. of Math. 98 (1973) 154.