Existence of solutions for a quasilinear elliptic system with local nonlinearity on $\mathbb{R}^N$

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1 | INTRODUCTION

In this paper, we investigate the existence of solutions for a class of quasilinear elliptic system. By developing the Moser iteration technique, we obtain that the system has a nontrivial solution $(u_\lambda, v_\lambda)$ with $\| (u_\lambda, v_\lambda) \|_\infty \leq 2$ for every $\lambda$ large enough when the nonlinear term $F$ satisfies some growth conditions only in a circle with center 0 and radius 4, and the families of solutions $\{ (u_\lambda, v_\lambda) \}$ satisfy that $\| (u_\lambda, v_\lambda) \| \to 0$ as $\lambda \to \infty$. Moreover, because the interaction of $u$ and $v$ in the elliptic system causes that the estimate of $\| u \|_\infty$ cannot vary with $\| u \|$, the conclusion for the elliptic system is weaker than the corresponding result for the quasilinear elliptic equation, which is given in the end as a comparison.

KEYWORDS  
cutoff technique, local nonlinearity near origin, Moser iteration technique, mountain pass theorem, quasilinear elliptic system

In this paper, we investigate the existence of solutions for a class of quasilinear elliptic problem with the following form

$$\left\{ \begin{array}{l} -\text{div} (\phi(|\nabla u|) \nabla u) + V(x)\phi(|u|)u = f(x, u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial \Omega \end{array} \right. $$

(1)

have been investigated extensively (e.g., see previous works$^{1-9}$ and references therein), where $\Omega \subset \mathbb{R}^N$ is an open set, $N \geq 2, V, f$ are continuous functions, and $\phi : [0, \infty) \to [0, \infty)$ satisfies some suitable monotonicity and growth conditions. Equation (1) arises from some fields of physics, for example,

1. nonlinear elasticity: $\Phi(t) = (1 + t^\gamma)^\gamma - 1, \gamma > 1/2$;
2. plasticity: $\Phi(t) = t^\alpha (\log(1 + t))^\beta, \alpha \geq 1, \beta > 0$;
3. generalized Newtonian fluids: $\Phi(t) = \int_0^t s^{1-\alpha} (\sinh^{-1}s)^\beta ds, 0 \leq \alpha \leq 1, \beta > 0$,

where $\Phi(t) = \int_0^t \phi(s) ds$ (see previous studies$^{1,5,6,10,11}$).
Specifically, Alves et al.\(^1\) considered Equation (1) with \(\Omega = \mathbb{R}^N\). They assumed that \(V \in C(\mathbb{R}^N, \mathbb{R}), \inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0\), \(V\) is a radial function or is a \(\mathbb{Z}^N\) periodic function, and \(f \in C(\mathbb{R}^N, \mathbb{R})\) and satisfies

\[
\lim_{|t| \to 0} \frac{f(t)}{\phi(|t|) |t|} = 0, \quad \lim_{|t| \to +\infty} \frac{f(t)}{\phi_\ast(|t|) |t|} = 0, \tag{2}
\]

and there exists \(\nu > m\) such that

\[
0 < \nu F(t) \leq tf(t) \quad \text{for all } t \in \mathbb{R}/\{0\}, \tag{3}
\]

which is usually called as Ambrosetti–Rabinowitz condition (\(\text{AR}\) for short), where \(F(t) = \int_0^t f(s)ds\). After developing a Strauss-type result and a Lions-type result, the obtained Equation (1) has a nontrivial solution. Recently, Liu\(^8\) considered the case that \(V\) has an infinite potential well, that is, \((V1)\) for all \(M > 0\), \(\mu(V^{-1}(\infty, M)) < \infty\), where \(\mu\) is the Lebesgue measure or \(V\) has a finite potential well, that is, \((V1)'\) for all \(x \in \mathbb{R}^N\), \(V(x) < \lim_{|x| \to \infty} V(x) < \infty\).

He also considered the case that \(V\) is a steep potential well; that is, \(V(x) = \lambda a(x) + 1\), where \(\lambda\) is a parameter and \(a \in C(\mathbb{R}^N, \mathbb{R})\). For all these cases, he assumed that (2) and (AR) hold. Then he obtained some existence and multiplicity results of solutions for system (1).

It is easy to see that (2) and (AR) imply that \(f\) satisfies some conditions near both 0 and \(\infty\). So it is natural to ask if it is possible to restrict those conditions for \(f\) to either of them. To this end, Costa and Wang\(^12\) used a cutoff technique together with energy estimates to study the multiplicity of both signed and sign-changing solutions for one-parameter family of elliptic problem (1) with \(\phi = 1\), \(V = 0\), \(f(x, u) = \lambda f(u)\), where \(\lambda > 0\) is a parameter, \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N (N \geq 3)\), and \(f \in C^1(\mathbb{R}, \mathbb{R})\). By such ingenious method, the nonlinearity \(f(u)\) was assumed to satisfy the superlinear growth only in a neighborhood of \(u = 0\). Afterwards, Medeiros and Severo\(^13\) applied the idea in Costa and Wang\(^12\) to the problem (1) with \(\phi(t) = |t|^{p-2}\) and \(f(x, u) = \lambda f(u)\) on the whole space \(\mathbb{R}^N\), that is, the following \(p\)-Laplacian equation:

\[
-\Delta_p u + V(x)|u|^{p-2}u = \lambda f(u) \quad \text{in } \mathbb{R}^N, \tag{4}
\]

where \(1 < p < N\) and \(\lambda > 0\). They assumed that \(V \in C(\mathbb{R}^N, \mathbb{R}), \inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0\), and \((V1)\) holds. Moreover, \(f\) satisfies the following conditions: (f1) there exists \(r \in (p, p^\ast)\) such that

\[
\limsup_{|s| \to 0} \frac{f(s)s}{|s|^r} < +\infty;
\]

(f2) there exists \(q \in (p, p^\ast)\) such that

\[
\limsup_{|s| \to 0} \frac{F(s)}{|s|^q} > 0;
\]

(f3) there exists \(\nu \in (p, p^\ast)\) such that

\[
0 < \nu F(s) \leq sf(s) \quad \text{for } |s| \neq 0\] small,

where \(p^\ast = \frac{Np}{N-p}\). With developing Moser iteration technique, they proved that equation (4) has one positive solution and one negative solution for all \(\lambda\) large enough. (f1)–(f3) show that \(f(s)\) satisfies the superlinear growth only in a neighborhood of \(s = 0\). Here, it needs to be emphasized that (f1)–(f3) with \(p = 2\) were given first in Costa and Wang.\(^12\) The idea in Costa and Wang\(^12\) has been applied to various differential equations and we cite these previous studies,\(^14,15,16\) as some examples.

Inspired by Costa and Wang\(^12\) and Medeiros and Severo,\(^13\) in this paper, we investigate the existence of solutions for the following quasilinear elliptic problem with a parameter

\[
\begin{align*}
-\text{div}(\phi_1(|\nabla u|)\nabla u) + V_i(x)\phi_1(|u|)u &= \lambda F_i(x, u, v), \ x \in \mathbb{R}^N, \\
-\text{div}(\phi_2(|\nabla v|)\nabla v) + V_i(x)\phi_2(|v|)v &= \lambda F_i(x, u, v), \ x \in \mathbb{R}^N, \\
u \in W^{1,\Phi_i}(\mathbb{R}^N)), \ \nu \in W^{1,\Phi_i}(\mathbb{R}^N),
\end{align*}
\tag{5}
\]

where \(N \geq 2\), \(\lambda\) is a parameter with \(\lambda > 0\), \(F \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \phi_i\) and \(V_i \in C(\mathbb{R}^N, \mathbb{R}^+), i = 1, 2\) satisfy the following assumptions:

(\(\phi_1\)) \(\phi_i \in C^1(0, \infty), t \to \phi_i(t)t\) are strictly increasing, \(i = 1, 2\);

(\(\phi_2\)) \(1 < l := \inf_{t > 0} \frac{\phi_i(t)}{\phi_i(t)} \leq \sup_{t > 0} \frac{\phi_i(t)}{\phi_i(t)} =: m_i < \min\{N, l_i^\ast\}\), where \(\Phi_i(t) := \int_0^t \phi_i(s)ds, t \in \mathbb{R}\) and \(l_i^\ast := \frac{IN}{N-l_i}, i = 1, 2\).
Recently, Wang et al.\textsuperscript{17,18} investigated the quasilinear elliptic system (5) with $\lambda = 1$. In Wang et al.\textsuperscript{17} when $F$ has a sublinear growth, by using the least action principle, they obtained that system (5) has at least one nontrivial solution and when $F$ satisfies an additional symmetric condition, by using the genus theory, they obtained that system (5) has infinitely many solutions. In Wang et al.\textsuperscript{18} by using the mountain pass theorem, when $F$ satisfies some superlinear conditions, they obtained that system (5) has a ground state solution. Especially, they obtained the following theorem.

**Theorem A.** (\textsuperscript{18}) Assume that $(\phi_1), (\phi_2)$, and the following conditions hold:

\begin{enumerate}
\item[(V1)] $V_i$ are one-periodic functions in $x_1, \ldots, x_N$ for all $x \in \mathbb{R}^N$, $i = 1, 2$ (called 1-periodic for short);
\item[(V2)] there exist two positive constants $\alpha_1$ and $\alpha_2$ such that
\[
\alpha_1 \leq \min\{V_1(x), V_2(x)\} \leq \max\{V_1(x), V_2(x)\} \leq \alpha_2
\]
for all $x \in \mathbb{R}^N$;
\item[(H0)] $F \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $F$ is 1-periodic in $x \in \mathbb{R}^N$ and $F(x, 0, 0) = 0$ for all $x \in \mathbb{R}^N$;
\item[(H1)]
\[
\lim_{\|t\| \to 0} \frac{F_i(x, t, s)}{\|t\|^\alpha} = 0, \quad \lim_{\|t\| \to 0} \frac{F_i(x, t, s)}{\|t\|^\beta} = 0
\]
\[
\lim_{\|t\| \to 0} \frac{F_i(x, t, s)}{\|t\|^\gamma} = 0, \quad \lim_{\|t\| \to 0} \frac{F_i(x, t, s)}{\|t\|^\delta} = 0
\]
uniformly in $x \in \mathbb{R}^N$, where $\Phi_i$ and $\Phi_{1i}$, $i = 1, 2$ are defined in Section 2 below;
\item[(H2)] there exist $\mu_i > m_i(i = 1, 2)$ such that
\[
0 < F(x, t, s) \leq \frac{1}{\mu_1}tF_i(x, t, s) + \frac{1}{\mu_2}sF_i(x, t, s), \quad \text{for all } (t, s) \neq 0.
\]
\end{enumerate}

Then system (5) with $\lambda = 1$ has a ground state solution.

It is easy to see (H0)–(H2) imply $F$ satisfies some growth conditions near both 0 and $\infty$. In this paper, by applying the method in Costa and Wang\textsuperscript{12} and developing the Moser iteration technique, we only need to make some assumptions on the nonlinearity $F(x, t, s)$ in a circle with center 0 and radius 4. If we assume that $V_i$, $i = 1, 2$ satisfies (V1) instead of (V1) in Theorem 1.1 below, then Theorem 1.1 can be seen as an extension of the result in Medeiros and Severo\textsuperscript{13} to system (5) in some sense. Moreover, we shall find that the elliptic system (5) is more complex and more general than the scalar equation (1) with $x \in \mathbb{R}^N$, which directly leads to some stronger restrictions for the nonlinearity $F$ in Theorem 1.1 because of a different Moser iteration result; see Section 5 for more details. To be precise, we obtain the following theorem for system (5):

**Theorem 1.1.** Assume that $(\phi_1) – (\phi_2)$, (V1) and the following conditions hold:

\begin{enumerate}
\item[(\phi_3)] there exist positive constants $q_i$ such that $t^2\phi_i(|t|) \geq q_i|t|^k_i$ for all $t \in \mathbb{R}$, $i = 1, 2$;
\item[(V0)] $V_i \in C(\mathbb{R}^N, \mathbb{R}^+) \quad \text{and} \quad V_i^\infty := \inf_{\mathbb{R}^N} V_i(x) > 0$, $i = 1, 2$;
\item[(F0)] $F \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $F$ is 1-periodic in $x \in \mathbb{R}^N$;
\item[(F1)] there exist $k_i \in (m_i, l_i^1)$ and $M_i > 0$, $i = 1, 2$ such that
\[
F(x, t, s) \geq M_1|t|^{k_1} + M_2|s|^{k_2}
\]
for all $|t, s| < 4$ and $x \in \mathbb{R}^N$;
\item[(F2)] there exist $M_3 > 0$, $M_4 > 0$, $r_1 \in \left(\max\left\{m_1, m_2, 1 + \frac{(1+m_2)(\Theta_1-1)}{\Theta_1 \Theta_2} \right\}, \min\left\{l_1^1, 1 + \frac{r_1^{(\Theta_1-1)} - (\Theta_1-1)(1+m_2)}{\Theta_1 l_1^1} \right\}\right)$ and $r_2 \in \left(\max\left\{m_1, m_2, 1 + \frac{(1+m_2)(\Theta_1-1)}{\Theta_1 \Theta_2} \right\}, \min\left\{l_2^1, 1 + \frac{r_2^{(\Theta_2-1)} - (\Theta_2-1)(1+m_2)}{\Theta_2 l_2^1} \right\}\right)$ for some $\Theta_1 > 1$ and $\Theta_2 > 1$ such that
\[
|F_i(x, t, s)| \leq M_3 |t|^{r_1-1} + M_4 |s|^{r_2-1}, \quad |F_i(x, t, s)| \leq M_3 |t|^{r_1-1} + M_4 |s|^{r_2-1}
\]
for all $|t, s| < 4$ and $x \in \mathbb{R}^N$;
\end{enumerate}
(F3) there exist \( \mu_i > \max\{m_1, m_2\} \), \( i = 1, 2 \) such that

\[
0 < F(x, t, s) \leq \frac{1}{\mu_1} t F_t(x, t, s) + \frac{1}{\mu_2} s F_s(x, t, s)
\]

for all \( x \in \mathbb{R}^N \) and \( |(t, s)| < 4 \) with \( (t, s) \neq (0, 0) \).

Then there exists \( \Lambda_s > 0 \) such that system (5) has a nontrivial solution \( (u_s, v_s) \) with \( \|(u_s, v_s)\|_\infty \leq 2 \) for each \( \lambda > \Lambda_s \) and \( \|(u_s, v_s)\| \to 0 \) as \( \lambda \to \infty \).

**Remark 1.1.** We have to emphasize that we do not eliminate the possibility of semi-nontrivial solutions for the nontrivial solution \( (u_s, v_s) \) in Theorem 1.1; that is, it is possible that \( (u_s, v_s) = (0, v_s) \) or \( (u_s, v_s) = (u_s, 0) \).

**Remark 1.2.** There exist examples satisfying Theorem 1.1. For example, let

\[
F(x, t, s) = \sigma(t, s)b(x)G_1(t, s) + (1 - \sigma(t, s))b(x)G_2(t, s)
\]

for all \( (x, t, s) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \), where \( b(x) \) satisfies (V0), \( G_1 \in C^1(\mathbb{R}^2, \mathbb{R}) \) satisfies (F1)-(F3), \( G_2 \in C^1(\mathbb{R}^2, \mathbb{R}) \) and

\[
\sigma(t, s) = \begin{cases} 
1, & \text{if } |(t, s)| < 4, \\
\sin \frac{\pi(t^2 + s^2 - 64)^2}{4608}, & \text{if } 4 \leq |(t, s)| \leq 8, \\
0, & \text{if } |(t, s)| > 8. 
\end{cases}
\]

In particular, \( F \) satisfies Theorem 1.1 but not satisfying Theorem A if we let \( N = 6, \phi_1(t) = 4|t|^2 + 5|t|^3 \) and \( \phi_2(t) = 4|t|^2 \log(2 + |t|) + \frac{|t|^4}{1 + |t|} \), \( b(x) = \left(1 + \sum_{i=1}^{6} \cos^2 \pi x_i \right) \) (or \( b(x) \equiv 1 \)), \( G_1(t, x) = |t|^{\frac{12}{7}} + |s|^{\frac{6}{7}} + |t|^3 |s|^7 \) and \( G_2(t, s) = |t|^3 + |s|^3 \). The readers can see the detail computation in Section 4.

**Remark 1.3.** In Theorem 1.1, we only assume that \( V_i(i = 1, 2) \) are periodic functions. In fact, it is also possible that similar results can be established if \( V_i(i = 1, 2) \) are radial functions or satisfy \( (V1) \) or \( (V1)' \) by combinig those arguments in Alves et al and Liu.

We organize the paper as follows. In Section 2, we recall some knowledge for Orlicz and Orlicz–Sobolev spaces. In Section 3, we complete the proof of Theorem 1.1. In Section 4, we present some detail arguments for the example mentioned in Remark 1.1. In Section 5, corresponding to Theorem 1.1, we present a result for a quasilinear elliptic equation with a parameter \( \lambda \), which shows some differences between Equation (1) with \( \Omega = \mathbb{R}^N \) and system (5).

## 2 | PRELIMINARIES

In this section, we recall some notions and properties about Orlicz and Orlicz–Sobolev spaces and some useful lemmas. The readers can see these details in previous studies.\(^{1,6,19-24}\)

We will start with some properties about \( N \) function. Assume that \( a : [0, \infty) \to [0, \infty) \) is a right continuous, monotone increasing function with

(i) \( a(0) = 0 \);
(ii) \( \lim_{t \to \infty} a(t) = \infty \);
(iii) \( a(t) > 0 \) whenever \( t > 0 \).

Then the integral \( A(t) = \int_0^t a(s)ds \) is called an \( N \) function, which is defined on \([0, \infty)\).

Define the complement of \( A \) by

\[
\tilde{A}(t) = \max_{s \geq 0} \{ts - A(s)\}, \quad \text{for } t \geq 0.
\]

Then \( \tilde{A} \) is also an \( N \) function and \( \tilde{A} = A \), and Young’s inequality holds, that is,

\[
st \leq A(s) + \tilde{A}(t), \quad \text{for all } s, t \geq 0.
\] (2.1)
If
\[ \sup_{t > 0} \frac{A(2t)}{A(t)} < \infty, \]
then we call \( A \) satisfies a \( \Delta_2 \) condition globally. When \( A \) satisfies \( \Delta_2 \) condition globally, \( A(t) > A^\delta(t) \) \hspace{1cm} (2.2)
for any \( \beta > 1 \) (see previous study\(^{21}\)) and the Orlicz space \( L^A(\Omega) \) is defined by the vectorial space consisting of the measurable functions \( u : \Omega \to \mathbb{R} \) satisfying
\[ \int_{\Omega} A(|u|) \, dx < \infty, \]
where \( \Omega \subset \mathbb{R}^N \) is an open set. Define
\[ ||u||_A := \inf \left\{ \alpha > 0 : \int_{\Omega} A \left( \frac{|u|}{\alpha} \right) \, dx < 1 \right\}, \quad \text{for } u \in L^A(\Omega), \]
which is called Luxemburg norm. Then \( (L^A(\Omega), ||\cdot||_A) \) is a Banach space. If \( A(t) = |t|^p (1 < p < +\infty) \), \( (L^A(\Omega), ||\cdot||_A) \) corresponds to the classical Lebesgue space \( L^p(\Omega) \) with the norm
\[ ||u||_p := \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}}. \]
Define
\[ W^{1,A}(\Omega) = \left\{ u \in L^A(\Omega) : \frac{\partial u}{\partial x_i} \in L^A(\Omega), i = 1, \ldots, N \right\} \]
with the norm
\[ ||u||_{1,A} = ||u||_A + ||\nabla u||_A. \]
Then \( W^{1,A}(\Omega) \) is a Banach space, which is called Orlicz–Sobolev space. Denote the closure of \( C_0^\infty(\Omega) \) in \( W^{1,A}(\Omega) \) by \( W_0^{1,A}(\Omega) \). \( W_0^{1,A}(\mathbb{R}^N) = W^{1,A}(\mathbb{R}^N) \) if \( \Omega = \mathbb{R}^N \).

**Lemma 2.1** \(^6\). If \( A \) is an \( N \) function, then the following conditions are equivalent:
(i) \[ l = \inf_{t > 0} \frac{ta(t)}{A(t)} \leq \sup_{t > 0} \frac{ta(t)}{A(t)} = m; \] \hspace{1cm} (2.3)
(ii) let \( \zeta_0(t) = \min\{t^l, t^m\}, \zeta_1(t) = \max\{t^l, t^m\} \), for \( t \geq 0 \). \( A \) satisfies
\[ \zeta_0(t) A(\rho) \leq A(\rho t) \leq \zeta_1(t) A(\rho), \text{ for all } \rho, t \geq 0; \]
(iii) $A$ satisfies a $\Delta_2$-condition globally.

**Lemma 2.2** ($\varphi$). If $A$ is an $N$ function and (2.3) holds, then $A$ satisfies

$$
\zeta_0(\|u\|_A) \leq \int_{\mathbb{R}^N} A(|u|) \, dx \leq \zeta_1(\|u\|_A), \text{ for all } u \in L^A(\mathbb{R}^N).
$$

**Lemma 2.3** ($\varphi$). If $A$ is an $N$ function, $l, m \in (1, \infty)$ and (2.3) holds. Let $\tilde{A}$ be the complement of $A$ and $\zeta_2(t) = \min\{t^l, t^m\}$, $\zeta_3(t) = \max\{t^l, t^m\}$, for $t \geq 0$, where $\tilde{l} := \frac{l}{l-1}$ and $\tilde{m} := \frac{m}{m-1}$. Then $\tilde{A}$ satisfies (i)

$$
\tilde{m} = \inf_{t > 0} \frac{\tilde{t}\tilde{A}'(t)}{\tilde{A}(t)} \leq \sup_{t > 0} \frac{\tilde{t}\tilde{A}'(t)}{\tilde{A}(t)} = \tilde{l};
$$

(ii)

$$
\zeta_2(t)\tilde{A}(\rho) \leq \tilde{A}(\rho t) \leq \zeta_3(t)\tilde{A}(\rho), \text{ for all } \rho, t \geq 0;
$$

(iii)

$$
\zeta_1(\|u\|_{\tilde{A}}) \leq \int_{\mathbb{R}^N} \tilde{A}(|u|) \, dx \leq \zeta_3(\|u\|_{\tilde{A}}), \text{ for all } u \in L^A(\mathbb{R}^N).
$$

**Lemma 2.4** ($\varphi$). If $A$ is an $N$ function, $l, m \in (1, N)$ and (2.3) holds. Let $\zeta_4(t) = \min\{t^N, t^m\}$, $\zeta_5(t) = \max\{t^N, t^m\}$, for $t \geq 0$, where $l^* := \frac{ln}{N-l}$, $m^* := \frac{mn}{N-m}$. Then $A_*$ satisfies

(i)

$$
l^* = \inf_{t > 0} \frac{tA_*'(t)}{A_*(t)} \leq \sup_{t > 0} \frac{tA_*'(t)}{A_*(t)} = m^*;
$$

(ii)

$$
\zeta_4(t)A_*(\rho) \leq A_*(\rho t) \leq \zeta_5(t)A_*(\rho), \text{ for all } \rho, t \geq 0;
$$

(iii)

$$
\zeta_4(\|u\|_{A_*}) \leq \int_{\mathbb{R}^N} A_*(|u|) \, dx \leq \zeta_5(\|u\|_{A_*}), \text{ for all } u \in L^{A_*}(\mathbb{R}^N),
$$

where $A_*$ is the Sobolev conjugate function of $A$, which is defined by

$$
A_*^{-1}(t) = \int_0^t \frac{A^{-1}(s)}{s^{\frac{N}{N-1}}} \, ds, \text{ for } t \geq 0.
$$

**Proposition 2.5** ($\varphi$). Under the assumptions of Lemma 2.4, the embedding

$$
W^{1,A}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)
$$

is continuous for any $N$ function $B$ satisfying

$$
\limsup_{r \to 0} \frac{B(r)}{A(r)} < \infty \text{ and } \limsup_{r \to 0} \frac{B(r)}{A_*(r)} < \infty.
$$

Therefore, there exists a constant $C$ such that

$$
\|u\|_B \leq C\|u\|_{1,A}, \text{ for all } u \in W^{1,A}(\mathbb{R}^N).
$$

**Remark 2.1.** By Lemma 2.1 and Lemma 2.3, $(\varphi_1) - (\varphi_2)$ imply that $\Phi_i$ and $\tilde{\Phi}_i$, $i = 1, 2$, are $N$ functions that satisfy $\Delta_2$ condition globally. Thus, $L^{\Phi_i}(\mathbb{R}^N)$ and $W^{1,\Phi_i}(\mathbb{R}^N)$ are separable and reflexive Banach spaces (see previous studies$^{19,23}$).

By (ii) in Lemma 2.1, (ii) in Lemma 2.4, and Proposition 2.5, it is easy to obtain the embedding

$$
W^{1,\Phi_i}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)
$$
is continuous with \( p_i \in [m_i, l_i^*], i = 1, 2 \), if we let \( A(r) := \Phi_i(r) \) and \( B(r) := r^\beta \). Hence, there exist positive constants \( C_{0,i}, i = 1, 2 \), such that

\[
\|u\|_{p_i} \leq C_{0,i} \|u\|_{1, \Phi_i},
\] (2.4)

whenever \( p_i \in [m_i, l_i^*], i = 1, 2 \).

Next, we recall a variant of mountain pass theorem. Let \( X \) be a Banach space. \( \varphi \in C^1(X, \mathbb{R}) \) and \( c \in \mathbb{R} \). A sequence \( \{u_n\} \subset X \) is called (PS)\(_c\) sequence if the sequence \( \{u_n\} \) satisfies

\[
\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0.
\]

**Lemma 2.5** (Mountain pass theorem\(^{20,22,24}\)). Let \( X \) be a Banach space, \( \varphi \in C^1(X, \mathbb{R}) \), \( w \in X \) and \( r > 0 \) such that \( \|w\| > r \) and

\[
b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \geq \varphi(w).
\]

Then there exists a (PS)\(_c\) sequence with

\[
c := \inf \max_{\gamma \in \Gamma} \varphi(\gamma(t))
\] (2.5)

and

\[
\Gamma := \{ \gamma \in ([0, 1], X) : \gamma(0) = 0, \gamma(1) = w \}.
\]

### 3 PROOFS

Define \( W := W^{1, \Phi_1}(\mathbb{R}^N) \times W^{1, \Phi_2}(\mathbb{R}^N) \) with the norm

\[
\|(u, v)\| = \|u\|_{1, \Phi_1} + \|v\|_{1, \Phi_2} = \|\nabla u\|_{\Phi_1} + \|u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2} + \|v\|_{\Phi_2}.
\]

Then \((W, \|\cdot\|)\) is a separable and reflexive Banach space.

Next, we use the idea in Costa and Wang\(^{12}\) to prove our theorem, on the whole, which is the cutoff technique together with energy estimates. In order to adapt system \((5)\), we make an extension to \( \mathbb{R}^2 \) for the cutoff function in Costa and Wang.\(^{12}\) For some \( \delta > 0 \), let \( \rho_\delta \in C^1(\mathbb{R} \times [0, 1]) \) be a cutoff function defined by

\[
\rho_\delta(t, s) = \begin{cases} 
1, & \text{if } |(t, s)| \leq \delta/2, \\
0, & \text{if } |(t, s)| \geq \delta
\end{cases}
\] (3.1)

and \( tp_\delta'(t, s) + sp_\delta'(t, s) \leq 0 \) for all \((t, s) \in \mathbb{R}^2\). We give some examples and their figures (Figures 1–4) for the cutoff functions as follows:

1. \( \rho_\delta(t, s) = \begin{cases} 
1, & \text{if } |(t, s)| \leq \delta/2, \\
\sin \frac{8\pi(t^2 + s^2 - \delta^2)}{9\delta^2}, & \text{if } \frac{\delta}{2} < |(t, s)| < \delta,
\end{cases} \) (3.2)

2. \( \rho_\delta(t, s) = \begin{cases} 
1, & \text{if } |(t, s)| \leq \delta/2, \\
\sin \frac{2\pi(t^2 + s^2 - \delta^2)}{3\delta^2}, & \text{if } \frac{\delta}{2} < |(t, s)| < \delta.
\end{cases} \) (3.3)

3. \( \rho_\delta(t, s) = \begin{cases} 
1, & \text{if } |(t, s)| \leq \delta/2, \\
\cos \frac{8\pi(t^2 + s^2 - \delta^2)}{9\delta^2}, & \text{if } \frac{\delta}{2} < |(t, s)| < \delta,
\end{cases} \) (3.4)

4. \( \rho_\delta(t, s) = \begin{cases} 
1, & \text{if } |(t, s)| \leq \delta/2, \\
\cos \frac{2\pi(t^2 + s^2 - \delta^2)}{3\delta^2}, & \text{if } \frac{\delta}{2} < |(t, s)| < \delta.
\end{cases} \) (3.5)
Remark 3.1. For the examples of $\rho_\delta$, it seems to be natural to use exponential functions as a link between 1 and 0 because of their infinite differentiability. However, we cannot find such examples with exponential functions because it seems to be difficult to ensure that $\rho$ is differentiable at both $|t| = \frac{\delta}{2}$ and $|t, s| = \delta$. From the characteristic or shape of $\rho_\delta$, sine functions and cosine functions seem to be better choices.
By (F0)–(F3) and similar to the argument of Remark 2.8 in Wang et al., there exist positive constants \( M_5 \) and \( M_6 \) such that
\[
|F(x, t, s)| \leq M_5 |t|^{r_1} + M_6 |s|^{r_2}
\]
for all \(|t, s| < 4 \) and \( x \in \mathbb{R}^N \). In fact, it follows from (F0) and (F3) that \( F(x, 0, 0) = 0 \) for all \( x \in \mathbb{R}^N \). Then if \( r_2 \geq r_1 \), \( t \in (-4, 0) \) and \( s \in \left( -\sqrt{16 - t^2}, 0 \right) \), by (F2), we have
\[
|F(x, t, s)| \leq \int_t^0 |F_r(x, \tau, s)| d\tau + \int_s^0 |F_\zeta(x, 0, \zeta)| d\zeta
\]
\[
\leq \int_t^0 (M_3 |\tau|^{r_1 - 1} + M_4 |s|^{r_2 - 1}) d\tau + \int_s^0 M_4 |\zeta|^{r_2 - 1} d\zeta
\]
\[
\leq \frac{M_3}{r_1} |t|^{r_1} + M_4 |s|^{r_2 - 1} |t| + M_4 |s|^{r_2}
\]
\[
\leq \frac{M_3}{r_1} |t|^{r_1} + \frac{M_4 (r_2 - 1)}{r_2} |s|^{r_2} + \frac{M_4}{r_2} |t|^{r_2}
\]
\[
= \left( \frac{M_3}{r_1} + \frac{M_4}{r_2} |t|^{r_2 - r_1} \right) |t|^{r_1} + M_4 |s|^{r_2}
\]
\[
\leq \left( \frac{M_3}{r_1} + \frac{2^{r_2-r_1} M_4}{r_2} \right) |t|^{r_1} + M_4 |s|^{r_2}
\]
for all \( x \in \mathbb{R}^N \). If \( r_1 > r_2 \), \( t \in (-4, 0) \) and \( s \in \left( -\sqrt{16 - t^2}, 0 \right) \), we have
\[
|F(x, t, s)| \leq \int_s^0 |F_\zeta(x, t, \zeta)| d\zeta + \int_t^0 |F_r(x, \tau, 0)| d\tau
\]
\[
\leq \int_s^0 (M_3 |\tau|^{r_1 - 1} + M_4 |s|^{r_2 - 1}) d\zeta + \int_t^0 M_3 |\tau|^{r_1 - 1} d\tau
\]
\[
\leq M_3 |t|^{r_1 - 1} |s| + \frac{M_4}{r_2} |s|^{r_2} + \frac{M_4}{r_1} |t|^{r_1}
\]
\[
\leq \frac{M_3 (r_1 - 1)}{r_1} |t|^{r_1} + \frac{M_3}{r_2} |s|^{r_2} + \frac{M_4}{r_1} |t|^{r_1}
\]
\[
= \frac{M_3}{r_1} |t|^{r_1} + \left( \frac{M_4}{r_2} + \frac{M_3}{r_1} |s|^{r_2 - r_1} \right) |s|^{r_2}
\]
\[
\leq \frac{M_3}{r_1} |t|^{r_1} + \left( \frac{M_4}{r_2} + \frac{2^{r_2-r_1} M_3}{r_1} \right) |s|^{r_2}
\]
for all \( x \in \mathbb{R}^N \). Combing (3.7) and (3.8), let \( M_3 = M_3 + 2^{r_2-r_1} M_4 \) and \( M_6 = M_4 + 2^{r_2-r_1} M_3 \). Then (3.6) holds. Similar arguments can be done for other cases that \(|(t, s)| < 4\).

Let \( \delta = 4 \) in (3.1). Define \( F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by
\[
F(x, t, s) = \rho_4(t, s) F(x, t, s) + (1 - \rho_4(t, s)) M_5 |t|^{r_1} + (1 - \rho_4(t, s)) M_6 |s|^{r_2}.
\]
Then by (F0)–(F3), the definition of \( \rho_4 \) and a direct computation, it is easy to obtain the following lemma:

**Lemma 3.1.** Assume that (F0)–(F3) hold. Then

\( \text{(F0)} \quad F \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \), \( F \) is 1–periodic in \( x \in \mathbb{R}^N \) and \( F(x, 0, 0) = 0 \) for all \( x \in \mathbb{R}^N \);

\( \text{(F1)} \quad 0 \leq F(x, t, s) \leq M_5 |t|^{r_1} + M_6 |s|^{r_2} \) for all \( (t, s) \in \mathbb{R}^2 \) and \( x \in \mathbb{R}^N \);

\( \text{(F2)} \quad \) there exist \( M_7 > 0 \) and \( M_8 > 0 \) such that
\[
|\tilde{F}(x, t, s)| \leq M_7 |t|^{r_1 - 1} + M_8 |s|^{r_2 - 1}, \quad |\hat{F}(x, t, s)| \leq M_7 |t|^{r_1 - 1} + M_8 |s|^{r_2 - 1}
\]
for all \( (t, s) \in \mathbb{R}^2 \) and \( x \in \mathbb{R}^N \);
Consider the modified problem

\[
\begin{align*}
-\text{div}(\phi_1(|\nabla u|)|\nabla u|) + V_1(x)\phi_1(|u|)u &= \lambda \tilde{F}_u(x, u, v), \quad x \in \mathbb{R}^N, \\
-\text{div}(\phi_2(|\nabla v|)|\nabla v|) + V_2(x)\phi_2(|v|)v &= \lambda \tilde{F}_v(x, u, v), \quad x \in \mathbb{R}^N, \\
u &\in W^{1,\Phi}_1(\mathbb{R}^N), \quad v \in W^{1,\Phi}_2(\mathbb{R}^N).
\end{align*}
\]

Define the functional \( \tilde{J}_\lambda : W \to \mathbb{R} \) by

\[
\tilde{J}_\lambda(u, v) = \int_{\mathbb{R}^N} \Phi_1(|\nabla u|)dx + \int_{\mathbb{R}^N} \Phi_2(|\nabla v|)dx + \int_{\mathbb{R}^N} V_1(x)\Phi_1(|u|)dx + \int_{\mathbb{R}^N} V_2(x)\Phi_2(|v|)dx - \lambda \int_{\mathbb{R}^N} F(x, u, v)dx.
\]

By (F1)' and a standard procedure, we can obtain that \( \tilde{J}_\lambda \) is well defined and \( \tilde{J}_\lambda \in C^1(W, \mathbb{R}) \) and

\[
\langle \tilde{J}_\lambda'(u, v), (\tilde{u}, \tilde{v}) \rangle = \int_{\mathbb{R}^N} (\phi_1(|\nabla u|)|\nabla u|, \nabla \tilde{u})dx + \int_{\mathbb{R}^N} V_1(x)\phi_1(|u|)u\tilde{u}dx + \int_{\mathbb{R}^N} \phi_2(|\nabla v|)|\nabla v|, \tilde{v})dx + \int_{\mathbb{R}^N} V_2(x)\phi_2(|v|)v\tilde{v}dx - \lambda \int_{\mathbb{R}^N} \tilde{F}_u(x, u, v)\tilde{u}dx - \lambda \int_{\mathbb{R}^N} \tilde{F}_v(x, u, v)\tilde{v}dx
\]

for all \((\tilde{u}, \tilde{v}) \in W\).

**Lemma 3.2.** \( \tilde{J}_\lambda \) satisfies the mountain pass geometry, that is,

(i) there exist two positive constants \( \gamma, \eta \) such that \( \tilde{J}_\lambda(u, v) \geq \eta \) for all \( ||(u, v)|| = \gamma \);

(ii) there exists \((u_0, v_0) \in C^0(\mathbb{R}^N) \cup \{0\} \times C^0(\mathbb{R}^N) \cup \{0\} \) with \( u_0 > 0, v_0 > 0 \) and \( ||(u_0, v_0)||_\infty := \max_{x \in \mathbb{R}^N} (u_0^2 + v_0^2)^{\frac{1}{2}} < 1 \) such that \( \tilde{J}_\lambda(u_0, v_0) < 0 \).

**Proof.** If \(|t| \leq 1\), by (F2)', \((\phi_2)\) and Lemma 2.1, we have

\[
\frac{F_t(x, t, s)}{\phi_1(|t|)|t| + \Phi_1^{-1}(\Phi_2(|s|))} \leq \frac{F_t(x, t, s)|t|}{\phi_1(|t|)|t|^2 + \Phi_1^{-1}(\Phi_2(|s|))} \leq \frac{M_2|t|^\gamma + M_3|t||s|^{\gamma-1}}{\phi_1(|t|)|t|^2} \leq \frac{M_7|t|^\gamma + M_8|t||s|^{\gamma-1}}{l_1 \Phi_1(|t|)} \leq \frac{M_7|t|^\gamma + M_8|t||s|^{\gamma-1}}{l_1 \min\{|t|^l, |l|^{m_1}\} \Phi(1)} \leq \frac{M_7|t|^\gamma + M_8|t||s|^{\gamma-1}}{\Phi(1)|t|^{m_1}} = \frac{M_7|t|^{\gamma-1} + M_8|s|^{\gamma-1}}{l_1 \Phi(1)|t|^{m_1-1}}.
\]

Because \( r_i > \max\{m_1, m_2\}, i = 1, 2, \) (3.10) implies that

\[
\lim_{|t|, |s| \to 0} \frac{F_t(x, t, s)}{\phi_1(|t|)|t| + \Phi_1^{-1}(\Phi_2(|s|))} = 0.
\]
Similarly, we also get that
\[
\lim_{|t(s)| \to 0} \frac{\tilde{F}_1(x, t, s)}{\Phi_{1s}^{-1}(\Phi_k(x)) + \Phi_{2s}(s)} = 0.
\]

Next, we prove that
\[
\lim_{|t(s)| \to -\infty} \frac{\tilde{F}_1(x, t, s)}{\Phi_{1s}^{-1}(\Phi_k(x)) + \Phi_{2s}(s)} = 0.
\] (3.11)

We divide the proof into three cases. For the case that \(|t| \to \infty\) and \(|s|\) is bounded, we assume that \(|t| \geq 1\). Then by (F2)', Lemma 2.4 and \(l_1^* \leq m_1^*\), we have
\[
\frac{\tilde{F}_1(x, t, s)}{\Phi_{1s}^{-1}(\Phi_k(x)) + \Phi_{2s}(s)} \leq \frac{\tilde{F}_1(x, t, s)|t|}{\Phi_{1s}^{-1}(\Phi_k(x)) + |t|} = \frac{\Phi_{1s}^{-1}(\Phi_k(x)) + M_6|s|^{r_1-1}}{l_1^* |t|^{r_1-1} + M_6|s|^{r_1-1}}. 
\] (3.12)

Note that \(l_1^* > r_1\). Then (3.11) holds. For the case that \(|t|\) is bounded and \(|s| \to \infty\), we assume that \(|s| \geq 1\). Because \(\Phi_{1s}^{-1}(s)\) is nondecreasing on \((0, \infty)\), we have \(\Phi_{1s}^{-1}(s) \geq \Phi_{1s}^{-1}(1)\) for all \(s \geq 1\). Hence, by Lemma 2.4, we have
\[
\Phi_{1s}^{-1}(\Phi_k(x)) + \Phi_{2s}(s) \geq \Phi_{1s}^{-1}(\Phi_k(x)) + \Phi_{2s}(s) \geq \Phi_{1s}^{-1}(1) \Phi_{2s}(1)|s|^{\frac{r_2}{2}}. 
\] (3.13)

Then by (F2)', Lemma 2.4, (3.13), and \(l_1^* \leq m_1^*\), similar to the argument of (3.12), we have
\[
\frac{\tilde{F}_1(x, t, s)}{\Phi_{1s}^{-1}(\Phi_k(x)) + \Phi_{2s}(s)} \leq \frac{M_7|t|^{r_1-1} + M_6|s|^{r_2-1}}{\Phi_{1s}^{-1}(1) \Phi_{2s}(1)|s|^{\frac{r_2}{2}}}. 
\]

Note that \(l_2^* > r_2\). Then (3.11) holds. For the case that \(|t| \to \infty\) and \(|s| \to \infty\), we assume that \(|t| \geq 1\) and \(|s| \geq 1\). Then
\[
\frac{\tilde{F}_1(x, t, s)}{\Phi_{1s}^{-1}(\Phi_k(x)) + \Phi_{2s}(s)} \leq \frac{\tilde{F}_1(x, t, s)|t|}{\Phi_{1s}^{-1}(\Phi_k(x)) + |t|} = \frac{\Phi_{1s}^{-1}(\Phi_k(x)) + M_6|s|^{r_1-1}}{l_1^* |t|^{r_1-1} + M_6|s|^{r_1-1}}. 
\]

Note that \(l_1^* > r_1\) and \(l_2^* > r_2\). Then (3.11) holds. Hence, (F2)' implies (F2) of Lemma 3.14 in Wang et al.\(^{18}\) So the conclusion holds by Lemma 3.14 and Lemma 3.15 in Wang et al.\(^{18}\) 

\[\square\]

Remark 3.1. There is a similar result of Lemma 3.2 in Wang et al\(^{18}\) (see Wang et al\(^{18}\), Corollary 3.3) where the authors did not show the detail proof. Here, we present the proof for readers' convenience.
Because (F0)‘–(F3)’ imply those conditions of Theorem 3.1 in Wang et al., it follows from Lemma 3.14–Lemma 3.16 in Wang et al. that system (3.9) has a nontrivial solution \((u_t, v_t)\) such that \(\tilde{J}(u, v) = c_\tau\) with
\[
c_\tau := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \tilde{J}(\gamma(t))
\]
and
\[
\Gamma := \{ \gamma \in (0, 1, W) : \gamma(0) = (0, 0), \gamma(1) = (u_0, v_0) \}.
\]

Next, we make an estimate for \(\|u_t\|_{1, \phi_t}\) and \(\|v_t\|_{1, \phi_t}\). We introduce the functional \(J_t : W \to \mathbb{R}\) as follows:
\[
J_t(u) = \int_{\mathbb{R}^N} \Phi_1(|\nabla u|)dx + \int_{\mathbb{R}^N} V_t(x)\Phi_1(|u|)dx + \int_{\mathbb{R}^N} \Phi_2(|\nabla v|)dx + \int_{\mathbb{R}^N} V_t(x)\Phi_2(|v|)dx - \lambda M_1 \int_{\mathbb{R}^N} |u|^{k_1}dx - \lambda M_2 \int_{\mathbb{R}^N} |v|^{k_2}dx.
\]

**Lemma 3.3.** There exist \(\Lambda_0 > 0\) such that for each \(\lambda > \Lambda_0\), there exists \(C_* > 0\) such that
\[
\|u_t\|_{1, \phi_t} \leq C_* \max \left\{ \lambda^{-\frac{1}{k_1-1}} + \lambda^{-\frac{1}{k_2-1}}, \lambda^{-\frac{1}{m_1+1}}, \lambda^{-\frac{1}{m_2+1}} \right\},
\]
\[
\|v_t\|_{1, \phi_t} \leq C_* \max \left\{ \lambda^{-\frac{1}{k_1-1}} + \lambda^{-\frac{1}{k_2-1}}, \lambda^{-\frac{1}{m_1+1}}, \lambda^{-\frac{1}{m_2+1}} \right\}.
\]

**Proof.** By (V0) and (V1), we know that \(V_i^\infty := \max_{t \in [0, T]} V_t(x), i = 1, 2\) exist. Then by Lemma 2.2, we have
\[
\tilde{J}(u_0, v_0) = \int_{\mathbb{R}^N} \Phi_1(|\nabla u_0|)dx + \int_{\mathbb{R}^N} V_t(x)\Phi_1(|u_0|)dx + \int_{\mathbb{R}^N} \Phi_2(|\nabla v_0|)dx
\]
\[
+ \int_{\mathbb{R}^N} V_t(x)\Phi_2(|v_0|)dx - \lambda M_1 \int_{\mathbb{R}^N} |u_0|^{k_1}dx - \lambda M_2 \int_{\mathbb{R}^N} |v_0|^{k_2}dx
\]
\[
\leq \max \left\{ s_1^1 \|\nabla u_0\|_{\phi_t}^{\frac{1}{k_1}}, \|\nabla v_0\|_{\phi_t}^{\frac{1}{k_2}}, \|\nabla v_0\|_{\phi_t}^{m_1}, \|\nabla v_0\|_{\phi_t}^{m_2}, \|V_t\|_{1, \phi_t} \right\} + \lambda \max \left\{ s_1^1 \|u_0\|_{\phi_t}^{\frac{1}{k_1}}, \|u_0\|_{\phi_t}^{m_1}, \|u_0\|_{\phi_t}^{m_2} \right\}
\]
\[
+ \max \left\{ s_2^1 \|\nabla v_0\|_{\phi_t}^{\frac{1}{k_1}}, \|\nabla v_0\|_{\phi_t}^{m_1}, \|\nabla v_0\|_{\phi_t}^{m_2}, \|V_t\|_{1, \phi_t} \right\} + \lambda \max \left\{ s_2^1 \|v_0\|_{\phi_t}^{\frac{1}{k_2}}, \|v_0\|_{\phi_t}^{m_1}, \|v_0\|_{\phi_t}^{m_2} \right\}
\]
\[
- \lambda M_1 \int_{\mathbb{R}^N} |u_0|^{k_1}dx - \lambda M_2 \int_{\mathbb{R}^N} |v_0|^{k_2}dx
\]
\[
\leq s_1^1 \left( \|\nabla u_0\|_{\phi_t}^{\frac{1}{k_1}}, \|\nabla v_0\|_{\phi_t}^{\frac{1}{k_1}}, \|\nabla v_0\|_{\phi_t}^{m_1}, \|\nabla v_0\|_{\phi_t}^{m_2}, \|V_t\|_{1, \phi_t} \right)
\]
\[
+ s_2^1 \left( \|\nabla v_0\|_{\phi_t}^{\frac{1}{k_2}}, \|\nabla v_0\|_{\phi_t}^{\frac{1}{k_2}}, \|\nabla v_0\|_{\phi_t}^{m_1}, \|\nabla v_0\|_{\phi_t}^{m_2}, \|V_t\|_{1, \phi_t} \right)
\]
\[
- \lambda M_1 \int_{\mathbb{R}^N} |u_0|^{k_1}dx - \lambda M_2 \int_{\mathbb{R}^N} |v_0|^{k_2}dx.
\]

for \(s \in [0, 1]\) and \((u_0, v_0)\) obtained in Lemma 3.2. Let
\[
g_1(s) = s^1 \left( \|\nabla u_0\|_{\phi_t}^{\frac{1}{k_1}}, \|\nabla v_0\|_{\phi_t}^{\frac{1}{k_1}}, \|\nabla v_0\|_{\phi_t}^{m_1}, \|\nabla v_0\|_{\phi_t}^{m_2}, \|V_t\|_{1, \phi_t} \right) - \lambda M_1 s^1 \int_{\mathbb{R}^N} |u_0|^{k_1}dx.
\]
\[
g_2(s) = s^1 \left( \|\nabla v_0\|_{\phi_t}^{\frac{1}{k_2}}, \|\nabla v_0\|_{\phi_t}^{\frac{1}{k_2}}, \|\nabla v_0\|_{\phi_t}^{m_1}, \|\nabla v_0\|_{\phi_t}^{m_2}, \|V_t\|_{1, \phi_t} \right) - \lambda M_2 s^1 \int_{\mathbb{R}^N} |v_0|^{k_2}dx.
\]

Obviously, there exist \(\Lambda_1 > 0\) and \(\Lambda_2 > 0\) such that
\[
s_{\lambda_1} := \left( \frac{1}{\lambda M_1 \int_{\mathbb{R}^N} |u_0|^{k_1}dx} \lambda M_1 \int_{\mathbb{R}^N} |u_0|^{k_1}dx \right)^{\frac{1}{k_1-1}} \in (0, 1)
\]
\[
s_{\lambda_2} := \left( \frac{1}{\lambda M_2 \int_{\mathbb{R}^N} |v_0|^{k_2}dx} \lambda M_2 \int_{\mathbb{R}^N} |v_0|^{k_2}dx \right)^{\frac{1}{k_2-1}} \in (0, 1)
\]
when \(s = s_{\lambda_1}, g_1(s)\) attains the
maximum on \([0, 1]\). Then there exists \(C_{1,*} > 0\) such that

\[
\max_{s \in (0, 1)} g_1(s) = \lambda^{-\frac{l_1}{2s-1}} k_1^{-\frac{l_1}{2s-1}} l_1^{-\frac{l_1}{2s-1}} (k_1 - l_1) A^{\frac{l_1}{2s-1}} \left( M_1 \int_{\mathbb{R}^N} |u_0|^{k_1} dx \right)^{-\frac{l_1}{2s-1}} \leq C_{1,*} \lambda^{-\frac{l_1}{2s-1}},
\]

where \(A = \|\nabla u_0\|_{\Phi_1}^l + V_1^\infty \|u_0\|_{\Phi_1}^m + \|\nabla u_0\|_{\Phi_1}^m + V_1^\infty \|u_0\|_{\Phi_1}^m\). When \(s = s_{\alpha, *}, g_2(s)\) attains the maximum on \([0, 1]\) and there exists \(C_{2,*} > 0\) such that

\[
\max_{s \in (0, 1)} g_2(s) = \lambda^{-\frac{l_2}{2s-1}} k_2^{-\frac{l_2}{2s-1}} l_2^{-\frac{l_2}{2s-1}} (k_2 - l_2) B^{\frac{l_2}{2s-1}} \left( M_2 \int_{\mathbb{R}^N} |v_0|^{k_2} dx \right)^{-\frac{l_2}{2s-1}} \leq C_{2,*} \lambda^{-\frac{l_2}{2s-1}},
\]

where \(B = \|\nabla v_0\|_{\Phi_2}^l + V_2^\infty \|v_0\|_{\Phi_2}^m + \|\nabla v_0\|_{\Phi_2}^m + V_2^\infty \|v_0\|_{\Phi_2}^m\). Notice that \(s(u_0(x), v_0(x)) \leq \|u_0, v_0\|_{\infty} < 1\) for all \(x \in \mathbb{R}^N\) and \(s \in [0, 1]\). Then by the definition of \(\bar{F}\) and (F1), we have \(\bar{F}(s_{u_0}, s_{v_0}) = F(s_{u_0}, s_{v_0}) \geq M_1 |s_{u_0}(x)|^{k_1} + M_2 |s_{v_0}(x)|^{k_2}\). Thus,

\[
c_{\alpha} := \inf_{r \in \Gamma} \max_{s \in [0, 1]} \bar{J}_r(s) \leq \max_{s \in [0, 1]} \bar{J}_s(s_{u_0}, s_{v_0}) \leq \max_{s \in [0, 1]} \bar{J}_s(s_{u_0}, s_{v_0}) \leq C_{1,*} \lambda^{-\frac{1}{2s_1-1}} + C_{2,*} \lambda^{-\frac{1}{2s_2-1}}.
\]

Note that \((u_1, v_1)\) is a critical point of \(\bar{J}_s\) with critical value \(c_{\alpha}\). Because \((\bar{J}'_s(u_1, v_1), (u_1, v_1)) = 0\) and \(\theta \geq \max\{m_1, m_2\}\), by \((\phi_1), (F3)'\) and Lemma 2.2, we have

\[
\theta c_{\alpha} = \theta \bar{J}_s(u_1, v_1) - \bar{J}_s'(u_1, v_1) - (u_1, v_1)) = \theta \int_{\mathbb{R}^N} \Phi_1(|\nabla u_1|) dx + \theta \int_{\mathbb{R}^N} V_1(x) \Phi_1(|u_1|) dx + \theta \int_{\mathbb{R}^N} \Phi_2(|\nabla v_1|) dx + \theta \int_{\mathbb{R}^N} V_2(x) \Phi_2(|v_1|) dx
\]

\[
- \theta \lambda \int_{\mathbb{R}^N} F(u, v_1, v_1) dx - \theta \int_{\mathbb{R}^N} V_1(x) \Phi_1(|u_1|) u_1^2 dx - \theta \int_{\mathbb{R}^N} V_2(x) \Phi_2(|v_1|) v_1^2 dx + \lambda \int_{\mathbb{R}^N} F(u, v_1, v_1) u_1 dx + \lambda \int_{\mathbb{R}^N} F(v, u_1, u_1) v_1 dx
\]

\[
\geq (\theta - m_1) \int_{\mathbb{R}^N} \Phi_1(|\nabla u_1|) dx + (\theta - m_1) \int_{\mathbb{R}^N} V_1(x) \Phi_1(|u_1|) dx + (\theta - m_2) \int_{\mathbb{R}^N} \Phi_2(|\nabla v_1|) dx + (\theta - m_2) \int_{\mathbb{R}^N} V_2(x) \Phi_2(|v_1|) dx
\]

\[
\geq (\theta - m_1) \min \{\|\nabla u_1\|_{\Phi_1}^l, \|\nabla u_1\|_{\Phi_1}^{m_1}\} + (\theta - m_1) V_{1, \infty} \min \{\|u_1\|_{\Phi_1}^l, \|u_1\|_{\Phi_1}^{m_1}\}
\]

\[
+ (\theta - m_2) \min \{\|\nabla v_1\|_{\Phi_2}^l, \|\nabla v_1\|_{\Phi_2}^{m_1}\} + (\theta - m_2) V_{2, \infty} \min \{\|v_1\|_{\Phi_2}^l, \|v_1\|_{\Phi_2}^{m_1}\},
\]

which, together with (3.14), implies that

\[
\min \{\|\nabla u_1\|_{\Phi_1}^l, \|\nabla u_1\|_{\Phi_1}^{m_1}\} + V_{1, \infty} \min \{\|u_1\|_{\Phi_1}^l, \|u_1\|_{\Phi_1}^{m_1}\}
\]

\[
\leq \frac{\theta}{\theta - m_1} c_{\alpha} \leq \frac{\theta}{\theta - m_1} C_{1,*} \lambda^{-\frac{l_1}{2s_1-1}} + \frac{\theta}{\theta - m_1} C_{2,*} \lambda^{-\frac{l_2}{2s_2-1}}
\]

and

\[
\min \{\|\nabla v_1\|_{\Phi_2}^l, \|\nabla v_1\|_{\Phi_2}^{m_1}\} + V_{2, \infty} \min \{\|v_1\|_{\Phi_2}^l, \|v_1\|_{\Phi_2}^{m_1}\}
\]

\[
\leq \frac{\theta}{\theta - m_2} c_{\alpha} \leq \frac{\theta}{\theta - m_2} C_{1,*} \lambda^{-\frac{l_1}{2s_1-1}} + \frac{\theta}{\theta - m_2} C_{2,*} \lambda^{-\frac{l_2}{2s_2-1}}.
\]
Hence, there exists $C_*>0$ such that

$$
\|u_1\|_{1,\Phi_1} = \|\nabla u_1\|_{\phi_1} + \|u_1\|_{\phi_1} \\
\leq \left(\frac{1}{V_{1,\infty}} + 1\right) \max \left\{ \frac{1}{\theta - m_1} \left( C_{1,*} \right)^{1/\theta - 1/\theta} m_1 \lambda^{-1/\theta} \leq \frac{1}{\theta - m_1} \left( C_{2,*} \right)^{1/\theta - 1/\theta} m_2 \lambda^{-1/\theta} \right\},
$$

and

$$
\|v_1\|_{2,\Phi_2} = \|\nabla v_1\|_{\phi_2} + \|v_1\|_{\phi_2} \\
\leq \left(\frac{1}{V_{2,\infty}} + 1\right) \max \left\{ \frac{1}{\theta - m_2} \left( C_{1,*} \right)^{1/\theta - 1/\theta} m_2 \lambda^{-1/\theta} \leq \frac{1}{\theta - m_2} \left( C_{2,*} \right)^{1/\theta - 1/\theta} m_2 \lambda^{-1/\theta} \right\}.
$$

Let $\Lambda_0 = \max\{\Lambda_1, \Lambda_2\}$. Then we complete the proof. \(\square\)

**Lemma 3.4.** There exists $\Lambda_* > \Lambda_0$ such that for all $\lambda > \Lambda_*$,

$$
\|u_\lambda\|_{\infty} \leq 1, \|v_\lambda\|_{\infty} \leq 1.
$$

**Proof.** The proof is motivated by Lemma 2.6 in Medeiros and Severo, which originates from Moser and is often called Moser iteration technique (see also Chabrowski and Yang and Figueiredo and Furtado for some related arguments). However, because of the interaction of $u$ and $v$, our proof is more difficult than those in previous studies, where some scalar equations were investigated.

Next, we start the proof. Because $(u_\lambda, v_\lambda)$ is a critical point of $I_\lambda$, we have

$$
\int_{\mathbb{R}^N} (\phi_1(|\nabla u_\lambda|)\nabla u_\lambda, \nabla u_\lambda) dx + \int_{\mathbb{R}^N} V_1(x)|u_\lambda|u_\lambda \hat{\mu} dx = \lambda \int_{\mathbb{R}^N} \hat{F}_{u_\lambda}(x, u_\lambda, v_\lambda) \hat{\mu} dx,
$$

$$
\int_{\mathbb{R}^N} (\phi_2(|\nabla v_\lambda|)\nabla v_\lambda, \nabla v_\lambda) dx + \int_{\mathbb{R}^N} V_2(x)|v_\lambda|v_\lambda \hat{\nu} dx = \lambda \int_{\mathbb{R}^N} \hat{F}_{v_\lambda}(x, u_\lambda, v_\lambda) \hat{\nu} dx
$$

for all $(\hat{\mu}, \hat{\nu}) \in W$. Next, we prove $\|u_\lambda\|_{\infty} \leq 1$. Without loss of generality, for each $k > 0$, we define

$$
u_k = \begin{cases} u_\lambda, & \text{if } u_\lambda \leq k, \\ k, & \text{if } u_\lambda > k, \end{cases}
$$

$$
\phi_k = |u_k|^{1/\beta_1 - 1}u_k \quad \text{and} \quad w_k = u_k|u_k|^{\beta_1 - 1} \quad \text{with } \beta_1 > 1.
$$

By $(\phi_k)$, we have $\phi_k(|\nabla u_k|)|\nabla u_k|^{2} \geq q_1|\nabla u_k|^{1}$. Then it follows from (2.1), (2.2), Hölder inequality, Lemma 2.2, and (2.4) that
\[ q_1 \int_{\mathbb{R}^N} |\nabla u_4|^{\alpha_1} |u_4|^{\gamma_1} dx \\
\leq \int_{\mathbb{R}^N} \phi_1(|\nabla u_4|)(\nabla u_4, \nabla \varphi) dx - l_1(\beta_1 - 1) \int_{\mathbb{R}^N} |u_4|^{\gamma_1 \beta_1} u_4 \phi_1(|\nabla u_4|)(\nabla u_4, \nabla u_4) dx \\
\leq - \int_{\mathbb{R}^N} V_1(x) \phi(|u_4|) u_4 \varphi_0 dx + \lambda \int_{\mathbb{R}^N} F(u_4, x, u_4, \varphi_0) dx \\
\leq \lambda M \int_{\mathbb{R}^N} |u_4|^{\gamma_1 \beta_1} |u_4|^{l_1(\beta_1 - 1)} dx + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |u_4| |u_4|^{l_1(\beta_1 - 1)} dx \\
= \lambda M \int_{\mathbb{R}^N} |u_4|^{\gamma_1 \beta_1} |w_1|^{l_1} dx + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |w_1| |u_4|^{l_1(\beta_1 - 1)} dx \\
\leq \lambda M \int_{\mathbb{R}^N} |u_4|^{\gamma_1 \beta_1} |w_1|^{l_1} dx + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |w_1| |u_4|^{l_1(\beta_1 - 1)} dx \\
= \lambda M \int_{\mathbb{R}^N} |u_4|^{\gamma_1 \beta_1} |w_1|^{l_1} dx + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |u_4|^{l_1(\beta_1 - 1)} dx \\
\leq \lambda M \int_{\mathbb{R}^N} |u_4|^{\gamma_1 \beta_1} |w_1|^{l_1} dx + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |u_4|^{l_1(\beta_1 - 1)} dx \\
\leq \lambda M \left( \int_{\mathbb{R}^N} |v_1|^{\gamma_1 \beta_1} |w_1|^{l_1} dx \right)^{\frac{1}{l_1}} + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |u_4|^{l_1(\beta_1 - 1)} dx \\
= \lambda M \int_{\mathbb{R}^N} |u_4|^{\gamma_1 \beta_1} |w_1|^{l_1} dx + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |u_4|^{l_1(\beta_1 - 1)} dx \\
\leq \lambda M \left( \int_{\mathbb{R}^N} |v_1|^{\gamma_1 \beta_1} |w_1|^{l_1} dx \right)^{\frac{1}{l_1}} + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |u_4|^{l_1(\beta_1 - 1)} dx \\
\leq \lambda M \int_{\mathbb{R}^N} |v_1|^{\gamma_1 \beta_1} |w_1|^{l_1} dx + \lambda M \int_{\mathbb{R}^N} |v_2|^{l_1 - 1} |u_4|^{l_1(\beta_1 - 1)} dx.
\[
\begin{align*}
&= \lambda M_2 r_2^{-1} \left\{ \begin{array}{c}
2 \int_{\Gamma_{\delta_1-1}} |u_\omega| |u_\omega|^{n-1} dx \\
2 \int_{\Gamma_{\delta_2-1}} |u_\omega| |u_\omega|^{m_1-1} dx
\end{array} \right\}^{\frac{1}{n}} \\
&\times \left( \int_{\mathbb{R}^N} \Phi_2(|v_\omega|) |v_\omega|^{m_2} dx + \int_{\mathbb{R}^N} (\Phi_2^{-1}(1))_{\frac{m_2}{\theta_2}} |v_\omega|^{m_2} dx \right) \\
&\times \left( \int_{\mathbb{R}^N} \Phi_1(|u_\omega|) |u_\omega|^{m_1} dx + \int_{\mathbb{R}^N} (\Phi_1^{-1}(1))_{\frac{m_1}{\theta_1}} |u_\omega|^{m_1} dx \right) \\
&+ \lambda M_7 \int_{\mathbb{R}^N} |u_\omega| |u_\omega|^{n-1} |w_k| dx \\
&\leq \lambda M_2 r_2^{-1} \left\{ \begin{array}{c}
2 \int_{\Gamma_{\delta_1-1}} |u_\omega| |u_\omega|^{n-1} dx \\
2 \int_{\Gamma_{\delta_2-1}} |u_\omega| |u_\omega|^{m_1-1} dx
\end{array} \right\}^{\frac{1}{n}} \\
&\times \left( \int_{\mathbb{R}^N} \Phi_2^{-1}(1)_{\frac{m_1}{\theta_1}} (|v_\omega|^{m_2} dx) \right) \\
&\times \left( \int_{\mathbb{R}^N} \Phi_1^{-1}(1)_{\frac{m_1}{\theta_1}} (|u_\omega|^{m_1} dx) \right) \\
&+ \lambda M_7 \int_{\mathbb{R}^N} |u_\omega| |u_\omega|^{n-1} |w_k| dx \\
&\leq \lambda M_2 r_2^{-1} \left\{ \begin{array}{c}
2 \int_{\Gamma_{\delta_1-1}} |u_\omega| |u_\omega|^{n-1} dx \\
2 \int_{\Gamma_{\delta_2-1}} |u_\omega| |u_\omega|^{m_1-1} dx
\end{array} \right\}^{\frac{1}{n}} \\
&\times \left( \int_{\mathbb{R}^N} \Phi_2^{-1}(1)_{\frac{\theta_2}{m_2}} (|v_\omega|^{m_2} dx) \right) \\
&\times \left( \int_{\mathbb{R}^N} \Phi_1^{-1}(1)_{\frac{\theta_1}{m_1}} (|u_\omega|^{m_1} dx) \right) \\
&+ \lambda M_7 \int_{\mathbb{R}^N} |u_\omega| |u_\omega|^{n-1} |w_k| dx \\
&\leq \lambda M_2 r_2^{-1} \left\{ \begin{array}{c}
2 \int_{\Gamma_{\delta_1-1}} |u_\omega| |u_\omega|^{n-1} dx \\
2 \int_{\Gamma_{\delta_2-1}} |u_\omega| |u_\omega|^{m_1-1} dx
\end{array} \right\}^{\frac{1}{n}} \\
&\times \left\{ \begin{array}{c}
\|v_\omega\|_{\Phi_2}^{\frac{m_2}{\theta_2}} \\
\|u_\omega\|_{\Phi_2}^{\frac{m_1}{\theta_1}}
\end{array} \right\} \\
&\times \max \left\{ \left\{ \|v_\omega\|_{\Phi_2}^{\frac{m_2}{\theta_2}} , \|v_\omega\|_{\Phi_2}^{\frac{m_1}{\theta_1}} \right\} \right\} \\
&\times \left( \int_{\mathbb{R}^N} \Phi_2^{-1}(1)_{\frac{m_1}{\theta_1}} (|u_\omega|^{m_1} dx) \right) \\
&\times \left( \int_{\mathbb{R}^N} \Phi_1^{-1}(1)_{\frac{m_1}{\theta_1}} (|u_\omega|^{m_1} dx) \right) \\
&\times \left( \int_{\mathbb{R}^N} \Phi_1^{-1}(1)_{\frac{m_1}{\theta_1}} (|u_\omega|^{m_1} dx) \right)
\end{align*}
\]
\[ + \max \left\{ \| u \|_{\Phi_1}, \| u \|_{\Phi_1} \right\} \]

\[ + \lambda M_7 \int_{\mathbb{R}^n} |u|^{r-1} |w_k|^i \, dx \]

\[ = \lambda M_8 C_0 C_{0.2} \frac{\mu_2 (l_1 - 1) \lambda_1}{\theta_1 l_1} \| v \|_{\Phi_2} \frac{\mu_2 (l_1 - 1) \lambda_1}{\theta_1 l_1} \| u \|_{\Phi_1} \]

\[ \times \left( \max \left\{ \| v \|_{\Phi_2}, \| v \|_{\Phi_2} \right\} \right) + (\Phi_2^{-1}(1))^{r-1} \frac{\mu_2 (l_1 - 1) \lambda_1}{\theta_1 l_1} \]

\[ \leq \lambda M_7 \int_{\mathbb{R}^n} |u|^{r-1} |w_k|^i \, dx + \lambda M_6 C_1 \left( \int_{\mathbb{R}^n} |w_k|^i |u|^{r-1} \, dx \right) \]

for some \( \sigma_1 \in \left(1, \frac{r}{m_1}\right), \sigma_2 \in \left(1, \frac{r}{m_1}\right) \) and any given \( \beta_1 \in I \beta := \left(1 + \frac{r_1}{l_1 l_1}, +\infty\right) \), where

\[ \Phi_1^{(l_1 (\beta_1 - 1) - \Theta (l_1 - 1)) - m_1} (|u|) < \Phi_1 (|u|) \]

which holds by (2.2). Moreover, for the validity of the interval of \( \Theta_1 \), we need the following restriction:

\[ r_2 > 1 + \frac{(1 + m_2)(l_1 - 1)(\Theta_1 - 1)}{l_1 \Theta_1} \]

to make

\[ \Phi_2^{(\Theta_1 \lambda_1 (r_2 - 1)) - m_1} (|v|) < \Phi_2 (|v|) \]

hold by (2.2). In addition, the range of \( \sigma_1 \) and \( \sigma_2 \) is determined by (2.4), and we assume that

\[ C_1 = C_0 C_{0.1} C_{0.2} \frac{\mu_2 (l_1 - 1) \lambda_1}{\theta_1 l_1} \frac{\mu_2 (l_1 - 1) \lambda_1}{\theta_1 l_1} \| u \|_{\Phi_1} \]

\[ \times \left( \max \left\{ \| v \|_{\Phi_2}, \| v \|_{\Phi_2} \right\} \right) + (\Phi_2^{-1}(1))^{r-1} \frac{\mu_2 (l_1 - 1) \lambda_1}{\theta_1 l_1} \]

\[ \times \left( \max \left\{ \| u \|_{\Phi_1}, \| u \|_{\Phi_1} \right\} \right) + (\Phi_1^{-1}(1))^{(\beta_1 - 1)(l_1 - 1) - \frac{m_1}{\Theta_1 \lambda_1}} \]

where \( C_0 = 2^{r_2 - 1 + (l_1 - 1)(\beta_1 - 1) - \frac{(m_2 + 1)l_1 (\Theta_1 - 1) + (m_2 + 1)l_1 (\Theta_1 - 1)}{\theta_1 l_1}} - \frac{r_2 - 1}{l_1} \).
By Gagliard–Nirenberg-Sobolev inequality, (3.16) and Hölder inequality, we have

\[
\left( \int_{\mathbb{R}^n} |w_k|^{\frac{r}{l_i}} dx \right)^{\frac{l_i}{r}} \leq D_1 \int_{\mathbb{R}^n} |\nabla w_k|^{l_i} dx
\]

\[
\leq \lambda D_2 \beta_1 \left( \int_{\mathbb{R}^n} |u_\lambda|^{r_{l_i}} dx \right)^{\frac{r_{l_i}}{l_i}} \left( \int_{\mathbb{R}^n} |w_k|^{\frac{r_{l_i}}{l_i} \frac{\alpha_{l_i}}{l_i} dx} \right)^{\frac{\alpha_{l_i}}{l_i}} + \lambda D_3 \beta_1 \left( \int_{\mathbb{R}^n} |u_\lambda|^{r_{l_i}} dx \right)^{\frac{r_{l_i}}{l_i}} \left( \int_{\mathbb{R}^n} |w_k|^{\frac{r_{l_i}}{l_i} \frac{\alpha_{l_i}}{l_i} dx} \right)^{\frac{\alpha_{l_i}}{l_i}}
\]

Note that \( r_1 \in \left( \max \left\{ m_1, m_2, 1 + \frac{(1+m_1)(l_1-1)(\Theta(1, -1))}{l_1 \Theta_1}, \min \left\{ l_1^*, 1 + \frac{\rho_{l_1} l_1}{l_1 - r_{l_i}} - \frac{\rho_{l_1} l_1 + m_1 l_1}{\Theta l_1} \right\} \right\} \). Let \( \beta_1 = 1 + \frac{r_{l_i}}{l_i} \). Then \( \beta_1 \in I_\beta \). Thus \( \frac{\rho_{l_1} l_1}{l_1 - r_{l_i}} = l_1^* \). Thus, we have

\[
\left( \int_{\mathbb{R}^n} |w_k|^{\frac{r}{l_i}} dx \right)^{\frac{l_i}{r}} \leq \lambda D_2 \beta_1 \left\| u_\lambda \right\|_{1, \Phi_1}^{\frac{r_{l_i}}{l_i} \frac{\alpha_{l_i}}{l_i}} \left\| u_j \right\|_{1, \Phi_1}^{\frac{\alpha_{l_i}}{l_i}} + \lambda D_3 \beta_1 \left\| u_\lambda \right\|_{1, \Phi_1}^{\frac{r_{l_i}}{l_i} \frac{\alpha_{l_i}}{l_i}} \left\| u_j \right\|_{1, \Phi_1}^{\frac{\alpha_{l_i}}{l_i}} \right.
\]

where \( \alpha_{l_i}^* = \frac{l_i}{l_i - r_{l_i}} \). Lemma 3.3 implies that there exists sufficient large \( \Lambda_3 > \Lambda_0 \) such that \( \left\| u_\lambda \right\|_{1, \Phi_1} < \min \left\{ 1, \frac{1}{C_{l_1}} \right\} \) for all \( \lambda > \Lambda_3 \) and then (2.4) implies that \( \left\| u_j \right\|_{\beta_1 \alpha_{l_i}^*} = \left\| u_j \right\|_{l_i} < \min \left\{ C_{l_1, 1}, 1 \right\} \). Because \( r_1 - l_i > \frac{r_{l_i} l_1}{l_i} \) and \( l_i \beta_1 > \beta_1 \). Hence, by (3.17), we have

\[
\left( \int_{\mathbb{R}^n} |w_k|^{\frac{r}{l_i}} dx \right)^{\frac{l_i}{r}} \leq \lambda (D_2 + D_3) \beta_1 \left\| u_\lambda \right\|_{1, \Phi_1}^{\frac{r_{l_i} l_1}{l_i} \frac{\alpha_{l_i}}{l_i}} \left\| u_j \right\|_{1, \Phi_1}^{\frac{\alpha_{l_i}}{l_i}} \right.
\]

Then it follows from the definition of \( w_k \), Fatou’s lemma, and (3.18) that

\[
\left\| u_j \right\|_{\beta_1 \alpha_{l_i}^*} \leq \left( \lambda (D_2 + D_3) \beta_1 \left\| u_\lambda \right\|_{1, \Phi_1}^{\frac{r_{l_i} l_1}{l_i} \frac{\alpha_{l_i}}{l_i}} \right)^{\frac{\alpha_{l_i}}{l_i}} \left\| u_j \right\|_{\beta_1 \alpha_{l_i}^*}.
\]
Next, we start the iteration process. For each $n = 0, 1, 2, \ldots$, we define $\beta_1^{(n+1)} = \frac{\xi}{a_1^n} \beta_1^{(n)}$, where $\beta_1^{(0)} = \beta_1$.

$$
\|u_3\|_{\frac{\gamma}{\beta_1^2}} \leq \left( \lambda(D_2 + D_3) \beta_1^{(\frac{1}{\gamma})} \|u_4\|_1 \phi_1 \right)^{\frac{1}{\beta_1^2}} \|u_4\|_1 \phi_1 \left( 1 - \frac{\xi}{a_1^2} \beta_1^{(\frac{1}{\gamma})} \|u_4\|_1 \phi_1 \right)^{\frac{1}{\beta_1^2}} \|u_4\|_1 \phi_1
$$

and then

$$
\|u_3\|_{\frac{\gamma}{\beta_1^2}} \leq \left( \lambda(D_2 + D_3) \beta_1^{(\frac{1}{\gamma})} \|u_4\|_1 \phi_1 \right)^{\frac{1}{\beta_1^2}} \|u_4\|_1 \phi_1 \left( 1 - \frac{\xi}{a_1^2} \beta_1^{(\frac{1}{\gamma})} \|u_4\|_1 \phi_1 \right)^{\frac{1}{\beta_1^2}} \|u_4\|_1 \phi_1
$$

Repeating such process, we obtain that

$$
\|u_3\|_{\frac{\gamma}{\beta_1^2}} \leq \left( \lambda(D_2 + D_3) \|u_4\|_1 \phi_1 \right)^{\frac{1}{\beta_1^2}} \|u_4\|_1 \phi_1 \left( 1 - \frac{\xi}{a_1^2} \beta_1^{(\frac{1}{\gamma})} \|u_4\|_1 \phi_1 \right)^{\frac{1}{\beta_1^2}} \|u_4\|_1 \phi_1
$$

where $D_4 = \beta_1^{(\frac{1}{\gamma})} \sum_{i=0}^{n-1} \left( \frac{1}{a_1^2} \right)^i \left( \frac{\gamma}{\beta_1^2} \right)^i \left( \frac{\xi}{a_1^2} \right)^i$.

Notice that

$$
\alpha_1^i < l_1^i, \quad \lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{1}{\beta_1 l_1^i l_1^{-i} \left( \frac{\xi}{a_1^2} \right)^i} = 0, \quad \lim_{n \to \infty} \sum_{i=0}^{n-1} \left( \frac{1}{l_1} \right)^{n-1} \left( \frac{\alpha_1^i}{l_1^i} \right)^i = 0, \quad \lim_{n \to \infty} \frac{1}{l_1^i} = 0.
$$

So (3.19) implies that $\|u_4\|_1 \phi_1 \leq 1$ for all $\lambda > \Lambda_3$.

Similarly, there exists $\Lambda_4 > \Lambda_0$ such that $\|u_4\|_1 \phi_1 \leq 1$ for all $\lambda > \Lambda_4$. Let $\Lambda_* = \max\{\Lambda_3, \Lambda_4\}$. Then we complete the proof.

Proof of Theorem 1.1. By Lemma 3.4, for each $\lambda > \Lambda_*$, we have

$$
\|(u_3, v_3)\|_1 \phi_1 \leq \|u_4\|_1 \phi_1 + \|v_4\|_1 \phi_1 \leq 2
$$

which implies that $F(x, u_3, v_3) = F(x, u_4, v_4)$ for all $x \in \mathbb{R}^N$. Hence, $(u_3, v_3)$ is a nontrivial weak solution of system (5) and Lemma 3.3 implies that $\|u_4\|_1 \phi_1 \to 0$ and $\|v_4\|_1 \phi_1 \to 0$ as $\lambda \to \infty$. \qed
\begin{equation}
\begin{aligned}
-\text{div}[(4|\nabla u|^2 + 5|\nabla u|^3)\nabla u] + \left(1 + \sum_{i=1}^{6}\cos^2\pi x_i\right)(4|u|^2 + 5|u|^3)u \\
= \lambda F_\nu(x, u, v), \ x \in \mathbb{R}^6, \\
-\text{div}\left[(4|\nabla v|^2 \log(2 + |\nabla v|) + \frac{|\nabla v|^2}{1+|\nabla v|})\nabla v\right] + \left(1 + \sum_{i=1}^{6}\sin^2\pi x_i\right)(4|v|^2 \log(1 + |v|) + \frac{|v|^2}{1+|v|})v
\end{aligned}
\end{equation}

where

\[F(x, t, s) = \sigma(t, s)b(x) \left(|t|^{\frac{17}{2}} + |s|^{\frac{12}{7}} + |t|^3|s|^7\right) + (1 - \sigma(t, s))b(x) \left(|t|^3 + |s|^3\right)\]

for all \((x, t, s) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}\) with \(\sigma(t, s)\) defined by (6) and \(b(x) = \left(1 + \sum_{i=1}^{6}\cos^2\pi x_i\right)\) (or \(b(x) \equiv 1\)). So,

\[F(x, t, s) = \begin{cases} 
    b(x) \left(|t|^{\frac{17}{2}} + |s|^{\frac{12}{7}} + |t|^3|s|^7\right), & \text{if } |(t, s)| \leq 4, \\
    \sin \frac{\pi(t^2+s^2-64)^2}{4608}b(x) \left(|t|^{\frac{17}{2}} + |s|^{\frac{12}{7}} + |t|^3|s|^7\right) + \left(1 - \sin \frac{\pi(t^2+s^2-64)^2}{4608}\right)b(x)(|t|^3 + |s|^3), & \text{if } 4 < |(t, s)| \leq 8, \\
    b(x)(|t|^3 + |s|^3), & \text{if } |(t, s)| > 8,
\end{cases}
\]

and we also draw the figure of \(F\) (see Figures 5–8 below).

Let \(N = 6, \phi_1(t) = 4|t|^2 + 5|t|^3\) and \(\phi_2(t) = 4|t|^2 \log(2 + |t|) + \frac{|t|^3}{1+|t|}\). Then \(\phi_i(t) = 1, 2\) satisfy \((\phi_1) - (\phi_3)\) and \(l_1 = l_2 = 4, m_1 = m_2 = 5\) (see Wang et al\(^{17}\)). So, \(l_1^2 = l_2^2 = 12\).

Let \(V_1(x) = 1 + \sum_{i=1}^{6}\cos^2\pi x_i\) and \(V_2(x) = 1 + \sum_{i=1}^{6}\sin^2\pi x_i\) for all \((x, t, s) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}\). Then it is obvious that \(V_i\) satisfies (V0) and (V1).

Let

\[A(t, s) := \frac{t\pi(t^2+s^2-64)}{1152}\cos\frac{\pi(t^2+s^2-64)^2}{4608}b(x) \left(|t|^{\frac{17}{2}} + |s|^{\frac{12}{7}} + |t|^3|s|^7\right) + \frac{\pi(t^2+s^2-64)^2}{4608}b(x) \left(\frac{17}{2} |t|^{\frac{13}{2}} |t| + 7|t|^2|s|^7 t\right) - \frac{t\pi(t^2+s^2-64)}{1152}\cos\frac{\pi(t^2+s^2-64)^2}{4608}b(x)(|t|^3 + |s|^3) + 3 \left(1 - \sin \frac{\pi(t^2+s^2-64)^2}{4608}\right)\sigma(x)|t|.
\]

**FIGURE 5** \((t, s) \in ((-0.5, 0.5), (-0.5, 0.5))\) [Colour figure can be viewed at wileyonlinelibrary.com]
and

\[
B(t, s) := \frac{s\pi(t^2 + s^2 - 64)}{1152} \cos \frac{\pi(t^2 + s^2 - 64)^2}{4608} b(x) \left( |t|^{17 \over 2} + |s|^{17 \over 2} + |t|^7 |s|^7 \right) \\
+ \sin \frac{\pi(t^2 + s^2 - 64)^2}{4608} b(x) \left( \frac{17}{2} |s|^{17 \over 2} s + 7 |s|^5 |t|^7 s \right) \\
- \frac{s\pi(t^2 + s^2 - 64)}{1152} \cos \frac{\pi(t^2 + s^2 - 64)^2}{4608} b(x) \left( |t|^3 + |s|^3 \right) + 3 \left( 1 - \sin \frac{\pi(t^2 + s^2 - 64)^2}{4608} \right) b(x) |s| s.
\]
It is easy to see that \( F \) satisfies (F0) and
\[
F_t(x, t, s) = \begin{cases} \frac{b(x)}{2} |t|^{\frac{17}{2}} t + 7|t|^7|s|^7 t, & \text{if } |(t, s)| \leq 4, \\ A(t, s), & \text{if } 4 < |(t, s)| \leq 8 \\ 3b(x)|t|t, & \text{if } |(t, s)| > 8, \end{cases}
\]
\[
F_s(x, t, s) = \begin{cases} \frac{b(x)}{2} |s|^{\frac{17}{2}} s + 7|s|^7|t|^7 s, & \text{if } |(t, s)| \leq 4, \\ B(t, s), & \text{if } 4 < |(t, s)| \leq 8 \\ 3b(x)|s|s, & \text{if } |(t, s)| > 8. \end{cases}
\]

Thus, we get
\[
F(x, t, s) = b(x) \left( |t|^{\frac{17}{2}} + |s|^{\frac{17}{2}} + |t|^7|s|^7 \right)
\]
\[
\geq \begin{cases} |t|^9 + |s|^9, & \text{if } |(t, s)| < 1, \\ \frac{1}{16} |t|^2 + |s|^2 \left( |t|^7 + |s|^7 \right), & \text{if } 1 \leq |(t, s)| \leq 4, \\ \frac{1}{16} |t|^9 + \frac{1}{16} |s|^9, & \text{for all } x \in \mathbb{R}^N \text{ and } |(t, s)| \leq 4. \end{cases}
\]

Then (F1) holds with \( k_1 = k_2 = 9 \in (m_1, l_1) = (m_2, l_2) = (5, 12), M_1 = M_2 = \frac{1}{16}. \) We also have
\[
|F_t(x, t, s)| = b(x) \left( \frac{17}{2} |t|^{\frac{17}{2}} + 7|t|^6|s|^7 \right)
\]
\[
\leq \begin{cases} \left( \frac{17}{2} + 7 \right) \times 7|t|^6 + |s|^6, & \text{if } |(t, s)| < 1 \\ \left( \frac{17}{2} + 7 \right) \times 7(t^2 + s^2)^4(|t|^6 + |s|^6), & \text{if } 1 \leq |(t, s)| \leq 4 \\ \leq 2^{41}(|t|^6 + |s|^6), & \text{for all } x \in \mathbb{R}^N \text{ and } |(t, s)| \leq 4. \end{cases}
\]

Similarly,
\[
|F_s(x, t, s)| \leq 2^{41}(|t|^6 + |s|^6), \text{ for all } x \in \mathbb{R}^N \text{ and } |(t, s)| \leq 4.
\]

Then (F2) holds with \( M_3 = M_4 = 2^{41} \) and \( r_1 = r_2 = 7 \in \left( \max \left\{ m_1, m_2, 1 + \frac{(1+m_2)(l_1-1)(\Theta_2-1)}{l_1 l_2} \right\}, 1 + \frac{(l_1-1)(1+m_2)}{\Theta_1 l_1} \right) = \left( \max \left\{ 5, \frac{9(\Theta_2-1)}{20}, 10 \right\}, \frac{9}{20} \right) \) and \( r_2 = 7 \in \left( \max \left\{ m_1, m_2, 1 + \frac{(1+m_2)(l_1-1)(\Theta_1-1)}{l_1 l_2} \right\}, 1 + \frac{(l_1-1)(1+m_2)}{\Theta_2 l_2} \right) = \left( \max \left\{ 5, \frac{9(\Theta_2-1)}{20}, 10 \right\}, \frac{9}{20} \right) \) for some \( \Theta_1, \Theta_2 > 1. \) In particular, taking \( \Theta_1 = \Theta_2 = 6, \) it is easy to see that \( r_1 = r_2 = 7 \in \left( 5, \frac{37}{4} \right). \) Note that
\[
b(x) \left( |t|^{\frac{17}{2}} + |s|^{\frac{17}{2}} + |t|^7|s|^7 \right) \leq b(x) \left( |t|^{\frac{17}{2}} + |s|^{\frac{17}{2}} + \frac{28}{17} |t|^7|s|^7 \right).
\]

So
\[
F(x, t, s) \leq \frac{2}{17} tF_t(x, t, s) + \frac{2}{17} sF_s(x, t, s) = \frac{1}{\mu_1} tF_t(x, t, s) + \frac{1}{\mu_2} sF_s(x, t, s)
\]

for all \( x \in \mathbb{R}^N \) and \( |(t, s)| \leq 4, \) where \( \mu_1 = \mu_2 = \frac{17}{2} > 5 = m_1 = m_2. \) Thus, we have verified that system (4.1) satisfies all the conditions of Theorem 1.1. Hence, there exists \( \Lambda > 0 \) such that system (5) has a nontrivial solution \((u_1, v_1)\) with \( \|(u_1, v_1)\|_\infty \leq 2 \) for each \( \lambda > \Lambda, \) and \( \|(u_1, v_1)\| \to 0 \) as \( \lambda \to \infty. \)

Next, we show that (4.2) does not satisfy Theorem A. In fact, for (4.2), we have
\[
\frac{1}{\mu_1} tF_t(x, t, s) + \frac{1}{\mu_2} sF_s(x, t, s) \leq \frac{1}{5} \left( tF_t(x, t, s) + sF_s(x, t, s) \right)
\]
\[
\leq \frac{3}{5} b(x)|t|^3 + \frac{3}{5} b(x)|s|^3
\]
\[
< b(x)|t|^3 + b(x)|s|^3
\]
for all \( x \in \mathbb{R}^N, |(t, s)| > 8 \) and any \( \mu_i \in (5, +\infty), i = 1, 2 \), which shows that \( F \) does not satisfy (H2) in Theorem A.

## 5 | A RESULT FOR ELLIPTIC EQUATION

Notice that (F1)–(F3) hold for all \(|(t, s)| \leq 4\), which are stronger than (f1)–(f3) where \(|t| \leq \delta\) for some positive constant \( \delta \). This is because we cannot obtain an estimate for \( |u|_\infty \) and \( |v|_\infty \) like Lemma 2.6 in Medeiros and Severo\(^{13} \) where \( |u|_\infty \) grows with \( |u| \). Instead, we obtain \( |u|_\infty \leq 1 \) and \( |v|_\infty \leq 1 \) (see Lemma 3.4), which are caused by the relation of \( u \) and \( v \) from the proof of Lemma 3.4. Hence, in this section, for the following quasilinear scalar equation

\[
\begin{cases}
-\text{div}(\phi(|Vu|)Vu) + V(x)\phi(|u|)u = \lambda f(x, u), & x \in \mathbb{R}^N, \\
u \in W^{1,\Phi}(\mathbb{R}^N),
\end{cases}
\]  

(5.1)

where \( N \geq 2 \) and \( \lambda > 0 \), we shall obtain a result similar to Theorem 1 in Costa and Wang\(^{12} \) and Theorem 1.2 in Medeiros and Severo\(^{13} \). To be precise, we have the following theorem:

**Theorem 5.1.** Assume that \( \phi \) satisfies (\( \phi_1 \)), (\( \phi_2 \)) and the following conditions hold:

(\( \phi_3 \)) there exists a positive constant \( q \) such that \( t^2 \phi(|t|) \geq q|t|^4 \) for all \( t \in \mathbb{R} \);

(V0) \( V \in C(\mathbb{R}^N, \mathbb{R}^+), V_\infty := \inf_{\mathbb{R}^N} V(x) > 0 \) and \( V \) is a 1-periodic function;

(C1) \( f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \) and \( f \) is 1-periodic in \( x \in \mathbb{R}^N \);

(C2) there exist \( \delta > 0 \), \( k \in (m, K) \) and \( D_1 > 0 \) such that

\[
F(x, t) \geq D_1 |t|^k
\]

for all \( t \in (-\delta, \delta) \) and \( x \in \mathbb{R}^N \), where \( K = \min \left\{ \frac{\nu t^* m^* t^* - m}{m}, \frac{1}{\delta, l} \right\} \) and \( F(x, t) = \int_0^t f(x, \xi) d\xi \);

(C3) there exist \( r \in (m, l^*) \) and \( D_2 > 0 \) such that

\[
|f(x, t)| \leq D_2 |t|^{r - 1}
\]

for all \( t \in (-\delta, \delta) \) and \( x \in \mathbb{R}^N \);

(C4) there exists \( \mu > m \) such that

\[
0 < F(x, t) \leq \frac{1}{\mu} tf(x, t)
\]

for all \( x \in \mathbb{R}^N \) and \( t \in (-\delta, \delta) \) with \( t \neq 0 \).

Then there exists \( \Lambda_0 > 0 \) such that Equation (5.1) has a nontrivial solution \( u_\lambda \) for each \( \lambda > \Lambda_0, \|u_\lambda\| \to 0 \) and \( \|u_\lambda\|_\infty \to 0 \) as \( \lambda \to \infty \).

The proof is similar to Theorem 1.1. The main difference is the result of Moser iteration. Next, we outline the proof. We work on the space \( W^{1,\Phi}(\mathbb{R}^N) \) with the norm \( \| \cdot \|_{1,\Phi} \). Define the cutoff function \( \rho \in C^1(\mathbb{R}, [0, 1]) \)

\[
\rho(t) = \begin{cases} 
1, & \text{if } |t| \leq \delta/2, \\
0, & \text{if } |t| \geq \delta
\end{cases}
\]

for all \( t \in \mathbb{R} \) and \( t\rho'(t) \leq 0 \). Similar to (3.2)–(3.5), some examples of \( \rho(t) \) can also be given, for example,

\[
\rho(t) = \begin{cases} 
1, & \text{if } |t| < \delta/2, \\
\sin \frac{8\pi|t|^{\delta/2}2^\nu}{9\delta}, & \text{if } \frac{\delta}{2} \leq |t| \leq \delta, \\
0, & \text{if } |t| > \delta.
\end{cases}
\]

Let

\[
\tilde{F}(x, t) = \rho(t)F(x, t) + (1 - \rho(t))D_3|t|^r.
\]

**Lemma 5.1.** Assume that (C1)–(C4) hold. Then
(C1) $\tilde{F} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\tilde{F}$ is $1$–periodic in $x \in \mathbb{R}^N$ and $\tilde{F}(x, 0) = 0$ for all $x \in \mathbb{R}^N$;

(C2) $0 \leq \tilde{F}(x, t) \leq D_3|t|$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$;

(C3) there exists $D_4 > 0$ such that

$$|\hat{f}(x, t)| \leq D_4|t|^{r-1}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$;

(C3) $\theta \tilde{F}(x, t) \leq \hat{f}(x, t)t$ for all $t \in \mathbb{R}/\{0\}$ and $x \in \mathbb{R}^N$,

where $\theta = \min \{r, \mu\}$.

Consider the modified problem

$$\begin{cases}
-\text{div}(\phi(|\nabla u|)\nabla u) + V(x)\phi(|u|)u = \lambda \hat{f}(x, u), & x \in \mathbb{R}^N, \\
u \in W^{1, \Phi}(\mathbb{R}^N).
\end{cases} \tag{5.2}$$

Define the functional $\tilde{J}_\lambda : W^{1, \Phi}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\tilde{J}_\lambda(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx - \lambda \int_{\mathbb{R}^N} \tilde{F}(x, u)dx.$$ 

It is easy to see that $\tilde{J}_\lambda$ is well defined and $\tilde{J}_\lambda \in C^1(W^{1, \Phi}(\mathbb{R}^N), \mathbb{R})$ and

$$\langle \tilde{J}'_\lambda(u), \tilde{u} \rangle = \int_{\mathbb{R}^N} (\phi(|\nabla u|)\nabla u, \nabla \tilde{u})dx + \int_{\mathbb{R}^N} V(x)\phi(|u|)u\tilde{u}dx - \lambda \int_{\mathbb{R}^N} \tilde{F}_u(x, u)\tilde{u}dx.$$

Lemma 5.2. $\tilde{J}_\lambda$ satisfies the mountain pass geometry, that is,

(i) there exist two positive constants $\gamma, \eta$ such that $\tilde{J}_\lambda(u) \geq \eta$ for all $\|u\| = \gamma$;

(ii) there exists $u_0 \in C^\infty_0(\mathbb{R}^N)/\{0\}$ with $u_0 > 0$ and $0 < \|u_0\|_\infty < \frac{\mu}{2}$ such that $\tilde{J}_\lambda(u_0) < 0$.

Proof. Notice that (C1)’(C2)’ imply those conditions of Theorem 1.5 in Alves et al. Then the proof is completed easily.

By Lemma 4.1–Lemma 4.3 in Alves et al.

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \tilde{J}_\lambda(\gamma(t))$$

and

$$\Gamma := \{\gamma \in ([0, 1], W^{1, \Phi}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = u_0\}.$$ 

Lemma 5.3. For each $\lambda > 0$, there exists $C_{ss} > 0$ such that

$$\|u_\lambda\|_{1, \Phi} \leq C_{ss} \max \left\{ \lambda^{\frac{1}{q-1}}, \lambda^{\frac{r-1}{r-m-N}} \right\}.$$ 

Proof. It is easy to obtain the conclusion from the proof of Lemma 3.3.

Lemma 5.4. For each $\lambda > 0$, there exists a positive constant $C$, which only depends on $q, r, l, N$, such that

$$\|u_\lambda\|_\infty \leq C(\lambda\|u_\lambda\|_{1, \Phi}^{r-1})^{\frac{1}{r-2}} \|u_\lambda\|_{1, \Phi}.$$ 

Proof. It is easy to obtain the conclusion from the proof of Lemma 3.3.
Proof. The proof is a direct generalization of Lemma 2.6 in Medeiros and Severo. Because $u_\lambda$ is a critical point of $\tilde{J}_\lambda$, we have

$$
\int_{\mathbb{R}^N} (\phi(|\nabla u_\lambda|)\nabla u_\lambda, \nabla \tilde{u})dx + \int_{\mathbb{R}^N} V(x)\phi(|u_\lambda|)u_\lambda \tilde{u}dx = \lambda \int_{\mathbb{R}^N} \tilde{f}(x, u_\lambda)\tilde{u}dx
$$

(5.3)

for all $\tilde{u} \in W^{1,\Phi}(\mathbb{R}^N)$. Without loss of generality, for each $k > 0$, define

$$
u_k = \begin{cases} u_\lambda, & \text{if } u_\lambda \leq k, \\ k, & \text{if } u_\lambda > k, \end{cases}
$$

$\varphi_k = |u_k|^{(p-1)}u_k$ and $w_k = u_\lambda |u_k|^{(p-1)}$ with $p > 1$. By $(\Phi_4)$, we have $\Phi(|\nabla u_\lambda|)|\nabla u_\lambda|^2 \geq q|\nabla u_\lambda|^l$. Then taking $\tilde{u} = \varphi_k$ in (5.3), by the definition of $u_k$ and $(C3)$, we have

$$
q \int_{\mathbb{R}^N} |\nabla u_\lambda|^l |u_k|^{(p-1)}dx \\
\leq \int_{\mathbb{R}^N} \phi(|\nabla u_\lambda|)\nabla u_\lambda, \nabla \varphi_k)dx - l(p-1) \int_{\mathbb{R}^N} |u_k|^{(p-1)-2}u_ku_\lambda \phi(|\nabla u_\lambda|)(\nabla u_\lambda, \nabla u_k)dx \\
\leq - \int_{\mathbb{R}^N} V(x)\phi(|u_\lambda|)u_\lambda \varphi_kdx + \lambda \int_{\mathbb{R}^N} \tilde{f}(x, u_\lambda)\varphi_kdx \\
\leq \lambda D_4 \int_{\mathbb{R}^N} |u_\lambda|^l |u_k|^{(p-1)}dx \\
= \lambda D_4 \int_{\mathbb{R}^N} |u_\lambda|^l |w_k|^{l^*} dx.
$$

The rest proof is the same as Lemma 2.6 in Medeiros and Severo with replacing $p$ with $l$ and $p^*$ with $l^*$. \qed

Proof of Theorem 5.1. By Lemma 5.3 and Lemma 5.4, we have

$$
\|u_\lambda\|_\infty \leq C\lambda^{\frac{1}{l^*}} C_{a_+}^{l^*} \max \left\{ \lambda^{\frac{1}{l^*}} \frac{p^*}{m}, \lambda^{\frac{1}{l^*}} \frac{p^*}{m-l^*} \right\}.
$$

Notice that $k > l, l^* > l, l^* > r$ and

$$
k < K = \min \left\{ l^*, \frac{ml - l + l^*}{m} \right\}.
$$

Then there exists a large $\Lambda_0 > 0$ such that $\|u_\lambda\|_\infty < \frac{\delta}{2}$ for all $\lambda > \Lambda_0$, which implies that $\tilde{F}(x, u_\lambda) = F(x, u_\lambda)$ for all $x \in \mathbb{R}^N$. Hence, $u_\lambda$ is a nontrivial weak solution of system (5.1) and Lemma 5.3 and Lemma 5.4 imply that $\|u_\lambda\|_{1, \Phi} \to 0$ and $\|u_\lambda\|_\infty \to 0$ as $\lambda \to \infty$, respectively. \qed

Remark 5.1. Comparing Theorem 1.1 with Theorem 5.1, it is easy to see Theorem 5.1 for the scalar equation (5.1) is better because $\tilde{f}$ satisfies growth conditions for $|t| \leq \delta$ with some positive constant $\delta$ rather than just for $|t| \leq 4$. The proof of Theorem 1.1 for the elliptic system (5) present more complex derivation. Especially, Moser iteration for system (5) is more difficulties than that for the scalar equation (5.1). Moreover, in Theorem 1.1, we assume that $F$ satisfies $(F1)$–$(F3)$ for all $|\langle t, s \rangle| \leq 4$, which is a circular domain. However, if we assume that there exist two positive constants $\delta_1$ and $\delta_2$ such that $F$ satisfies $(F1)$–$(F3)$ for all $|t| \leq \delta_1$ and $|s| \leq \delta_2$, which is a rectangular domain, the arguments will become more complex and it is unknown if Theorem 1.1 holds in such rectangular domain. One can consider the proof of Lemma 3.1 and the examples of cutoff functions to see such complexity. Finally, we would like to mention that if we let

$$
\tilde{F}^+(u) = \begin{cases} \tilde{F}(u) & \text{if } u \geq 0, \\ 0 & \text{if } u < 0 \end{cases}, \quad \tilde{F}^-(u) = \begin{cases} \tilde{F}(u) & \text{if } u \leq 0, \\ 0 & \text{if } u > 0 \end{cases}
$$

(5.4)

and consider the functional

$$
\tilde{J}_\lambda(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx - \lambda \int_{\mathbb{R}^N} \tilde{F}^+(x, u)dx
$$
and
\[ J_\lambda^*(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx - \lambda \int_{\mathbb{R}^N} \tilde{F}^-(x,u)dx, \]

respectively, then we can obtain Equation (5.2) has a positive solution and a negative solution. Thus, Theorem 5.1 can be seen as the generalization of Theorem 1.2 in Medeiros and Severo if we assume that (V1) holds instead of the periodicity of V.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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