Stability criteria for open downslope flows under oblique perturbations

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Abstract. We consider the stability problem for wide uniform stationary open flows down a slope with constant inclination under gravity. The depth-averaged equations are used with arbitrary dependence of the bottom friction on the flow depth and depth-averaged velocity. The stability conditions relatively to perturbations propagating along the flow are widely known. In this paper we focus on the effect of oblique perturbations that propagate at an arbitrary angle to the velocity of the undisturbed flow. We have found that under certain conditions the oblique perturbations can grow even when the perturbations propagating along the flow are damped. This means that the stability conditions found in the investigation of the one-dimensional problem are insufficient for the stability of the flow if oblique perturbations are possible.

1. Introduction

We consider the stability problem of wide uniform stationary open flows down a plane with constant inclination under the gravity. Equations in hydraulic approximation (i.e., depth-averaged) are used to describe the flow:

\[ \frac{\partial h}{\partial t} + \text{div} h \vec{q} = 0, \quad (1) \]
\[ \frac{d\vec{q}}{dt} = g \sin \alpha \vec{e} - \frac{c^2}{h} \text{grad} h - F\vec{q}, \quad (2) \]

where \( h \) is the flow depth in the direction normal to the bottom, \( \vec{q} \) is the vector of the depth-averaged velocity, \( \alpha \) is an inclination angle of the slope, unit vector \( \vec{e} \) is directed along the line of the maximum incline of the bottom (Figure 1), \( g \) is the gravity force acceleration, \( c = \sqrt{gh \cos \alpha} \), \( F\vec{q} = \tau/\rho h \), \( \tau \) is the friction force at the bottom per unit area, \( \tau = \tau(\rho, h, \vec{q}) \), \( \rho \) is the flow density. In the undisturbed flow \( h = h_0 \), \( \vec{q} = \vec{q}_0 \), \( \vec{q}_0 \) is parallel to \( \vec{e} \).

Equation (2) can be simplified, first, in the case of flows with large time and length scales (large-scale approximation or kinematic waves theory). Then differential terms in (2) can be neglected. Second, (2) can be simplified in the case of small-scale approximation, when the gravity force and the friction force can be neglected. It is known that in the uniform flow with \( h = h_0 \), \( \vec{q} = \vec{q}_0 \) small perturbations both in large- and small-scale approximations propagate...
Figure 1. Coordinate system and the scheme of the flow.

without change of the shape. Their velocities relatively to the moving medium are $N_0$ and $c_0$, respectively, where

$$N_0 = -\frac{h_0 q_0 \frac{\partial F}{\partial h_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}}, \quad c_0 = \sqrt{gh_0 \cos \alpha}.$$ 

In general case we had to use full momentum equation (2) to describe behavior of small perturbations. 1D problem where disturbances propagate along the basic flow velocity have been widely studied (see, for example, [1, 2, 3, 4, 5]). The stability condition for the 1D problem is formulated in [2] in the following form:

$$-c_0 \leq N_0 \leq c_0. \quad (3)$$

In the following, we will investigate 2D problem on oblique perturbations, for which the propagation direction makes an angle $\theta$ with the undisturbed flow velocity $\vec{q}_0$.

2. Equations for small perturbations and dispersion equation

Let us set

$$h = h_0 + h', \quad u = u_0 + u', \quad v = v_0 + v',$$

where $h', u', v'$ are small disturbances. The system (1),(2), linearized near the undisturbed flow with $h = h_0$, $\vec{q} = \vec{q}_0$, is:

$$\frac{\partial h'}{\partial t} + u_0 \frac{\partial h'}{\partial x} + v_0 \frac{\partial h'}{\partial y} + h_0 \frac{\partial v'}{\partial y} + h_0 \frac{\partial u'}{\partial x} = 0,$$

$$\frac{\partial u'}{\partial t} + u_0 \frac{\partial u'}{\partial x} + v_0 \frac{\partial u'}{\partial y} + \frac{c_0^2}{h_0^2} \frac{\partial h'}{\partial x} = -\frac{\partial F}{\partial q_0} u_0 \cos \theta \frac{\partial u'}{\partial x} - \frac{\partial F}{\partial h_0} u_0 h_0 \frac{\partial u'}{\partial x} + u_0 h_0 \frac{\partial \frac{\partial F}{\partial q_0}}{\partial h_0} - F_0 u',$$

$$\frac{\partial v'}{\partial t} + u_0 \frac{\partial v'}{\partial x} + v_0 \frac{\partial v'}{\partial y} + \frac{c_0^2}{h_0^2} \frac{\partial h'}{\partial y} = -\frac{\partial F}{\partial q_0} u_0 \sin \theta \frac{\partial v'}{\partial y} - \frac{\partial F}{\partial h_0} v_0 \frac{\partial v'}{\partial y} + v_0 h_0 \frac{\partial \frac{\partial F}{\partial q_0}}{\partial h_0} - F_0 v'.$$

Here and further we use the following notations:

$$F_0 = F(h_0, q_0),$$

$$\frac{\partial F}{\partial q_0} = \frac{\partial F}{\partial q} \bigg|_{h=h_0, q=q_0}, \quad \frac{\partial F}{\partial h_0} = \frac{\partial F}{\partial h} \bigg|_{h=h_0, q=q_0}, \quad \beta = q_0 \frac{\partial F}{\partial q_0}, \quad A = F_0 + q_0 \frac{\partial F}{\partial q_0}.$$
Consider small perturbations in the form:

\[ k' = H e^{ikx - \omega t}, \]
\[ u' = U e^{ikx - \omega t}, \]
\[ v' = V e^{ikx - \omega t}, \]  

(5)

\( x \) axis is directed along the wave vector \( \vec{k} \), so \( \vec{k} = \{ k_x, k_y \} = \{ k, 0 \} \).

Substitution of (2) into (4) leads to the system of linear algebraic equations for \( U, V, H \).

For each real \( k \) and each angle \( \theta \) we can find \( C \), i.e., \( \omega \), by the equation (6). The disturbances do not grow if \( ImC \leq 0 \) for all \( k > 0 \) and \( ImC \geq 0 \) for all \( k < 0 \).

Let us note that if for a certain \( k = k_p \) the dispersion equation (6) has the roots \( \omega_{p1} = p_1 + iq_1 \), \( \omega_{p2} = p_2 + iq_2 \), \( \omega_{p3} = p_3 + iq_3 \), the roots of the equation (6) for \( k = k_n = -k_p \), \( \omega_{n1} = -p_1 + iq_1 \), \( \omega_{n2} = -p_2 + iq_2 \), \( \omega_{n3} = -p_3 + iq_3 \). So it is enough to investigate the roots of the dispersion equation (10) for all real \( k > 0 \). The flow is stable, if for all real \( k > 0 \) imaginary parts of the dispersion equation roots \( ImC \leq 0 \).

3. Stability criterion with account of oblique perturbations

Existence of roots of (6) with positive imaginary part is investigated using the argument principle. This principle states that the number of roots of analytical function inside the contour in the plane of its argument is equal to the argument change of the function under running along this contour, divided by \( 2\pi \). In our case we have to find conditions, under which there are no roots \( C \) of the equation (6) in the upper semiplane \( ImC > 0 \). The contour, consisting of a semicircle of infinitely large radius and the real axis, is chosen for this reason (Figure 2,a).

We start to move along the semicircle \( 1 \) from the point \( A \) to the point \( B \). The behavior of the function \( D \), while running this part of the contour, is defined by the major term in the equation (6) that is the polynomial of the third order. So the argument of \( D \) increases by \( 3\pi \) (Figure 2,b). Then we move along the real axis from the point \( B \) to the point \( A \), and it is useful to analyze locations of points \( D(a_-), D(u_0), D(a_0x), D(a_+) \) at the complex plane \( D \). Here \( u_0 = q_0 \cos \theta \), \( a_- = u_0 - q_0 \), \( a_+ = u_0 + q_0 \). There is only one possibility to connect points \( D(B) \) and \( D(A) \) so that the argument of the function \( D \) would decrease by \( 3\pi \), and the total change of the function \( D \) argument will be equal to zero. This possibility is shown in the Figure 2,b. The stability condition can be formulated as follows: the flow is stable if and only if (1) for all real \( k > 0 \), for real \( C \) the equation \( ImD(C, k) = 0 \) has two roots: \( C_1 < C_2 \), (2) \( ReD(C_1) > 0 \) and \( ReD(C_2) < 0 \).

Consider the case, when

\[ F_0 > 0, \quad \partial F / \partial q_0 > 0, \quad \partial F / \partial u_0 < 0, \quad \text{so that} \quad F_0 + q_0 \frac{\partial F}{\partial q_0} > 0, \quad 0 < \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} < 1. \]  

(7)

For example, turbulent flows with the bottom friction defined by the Chézy or Manning formulae and laminar flow of dilatant power-law fluid satisfy the conditions (7). In the case
Figure 2. Illustration of the argument method in application to the dispersion equation.

Figure 3. Stability regions at the plane of parameters $\xi$ and $\beta/A$. Instability regions 2 and 3 are shadowed, non-shadowed region 1 correspond to the stability of downslope flow.

(7), stability and instability areas are shown at the plane of parameters $\xi$, $\gamma$ (Figure 3), where $\xi = N_0/c_0$. For $|\xi| > 1$ the flow is unstable. However, perturbations with propagation angles

$$\arccos\left(\frac{\xi - \sqrt{\xi^2 - 4 \left(\frac{q_0 \frac{\partial F_0}{\partial q_0}}{F_0 + q_0 \frac{\partial F_0}{\partial q_0}} - \left(\frac{q_0 \frac{\partial F_0}{\partial q_0}}{F_0 + q_0 \frac{\partial F_0}{\partial q_0}}\right)^2\right)} + \pi n \leq \theta \leq \pi - \arccos\left(\frac{\xi - \sqrt{\xi^2 - 4 \left(\frac{q_0 \frac{\partial F_0}{\partial q_0}}{F_0 + q_0 \frac{\partial F_0}{\partial q_0}} - \left(\frac{q_0 \frac{\partial F_0}{\partial q_0}}{F_0 + q_0 \frac{\partial F_0}{\partial q_0}}\right)^2\right)} + \pi n, \ n = 0, 1 \right)$$

(8)

(regions 3 in the Figure 3 and white sectors in the Figure 4, a) do not grow. For the parameters $\xi$ and $\beta/A$ from the region 1 (white region and the boundary of this region, shown by thick
lines, in the Figure 3) perturbations do not grow for all angles of propagation.

\[
\begin{align*}
\arccos \left( \frac{\xi^2 - 4 \left( \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} - \left( \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} \right)^2 \right)}{2 \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}}} + \pi n < \theta < 
\end{align*}
\]

\[
\begin{align*}
< \arccos \left( \frac{\xi^2 - 4 \left( \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} - \left( \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} \right)^2 \right)}{2 \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}}} + \pi n, \quad n = 0, 1 \quad (9)
\end{align*}
\]

\[
\begin{align*}
\arccos \left( \frac{\xi^2 - 4 \left( \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} - \left( \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} \right)^2 \right)}{2 \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}}} + \pi n < \theta < 
\end{align*}
\]

\[
\begin{align*}
< - \arccos \left( \frac{\xi^2 - 4 \left( \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} - \left( \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}} \right)^2 \right)}{2 \frac{q_0 \frac{\partial F}{\partial q_0}}{F_0 + q_0 \frac{\partial F}{\partial q_0}}} + \pi n, \quad n = 0, 1 \quad (10)
\end{align*}
\]

In the regions 2 longitudinal perturbations do not grow, but oblique perturbations with the angles of propagation described by formulae (9) and (10) grow (grey sectors in the Figure 4, b).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Sectors of propagation angles for growing (grey sectors) and not growing (white sectors) perturbations.}
\end{figure}

4. Conclusions

Behavior of the downslope flows under small oblique perturbations is studied. The flow is described by the continuity and momentum equations in the hydraulic approximation. The dispersion equation is derived and analyzed using the argument method for complex analytical functions. It is shown that the absence of amplitudes growth of longitudinal perturbations does
not guarantee the stability of the flow. The range of the flow parameters is found, for which oblique perturbations grow, while longitudinal perturbations do not grow. Ranges of angles $\theta$ for the growing perturbations are indicated.

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