Quantum Integrable Systems and Special Functions

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Abstract

The class of quantum integrable systems associated with root systems was introduced in [OP 1977] as a generalization of the Calogero–Sutherland systems [Ca 1971], [Su 1972]. For the potential $v(q) = \kappa(\kappa - 1)\sin^{-2} q$, the wave functions of such systems are related to polynomials in $l$ variables ($l$ is a rank of root system) and they are a generalization of Gegenbauer polynomials and Jack polynomials [Ja 1970]. In [Pe 1998a], it was proved that the series for the product of two such polynomials is a $\kappa$-deformation of the Clebsch–Gordan series. This yields recurrence relations for these polynomials, in particular, for generalized zonal polynomials on symmetric spaces.

The present paper follows papers [Pe 1998a], [PRZ 1998]. In last of them, the recurrence relations were used to compute the explicit expressions for $A_2$ type polynomials, i.e., for the wave functions of the three-body Calogero–Sutherland system.

As it was shown by Ragoucy, Zaugg and the author of this paper (see [Pe 1999] and Appendix B), the similar results are also valid in $A_2$ case for the more general two-parameter deformation ($(q, t)$-deformation) introduced by Macdonald [Ma 1988].

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1 Introduction

The class of quantum integrable systems associated with root systems was introduced in [OP 1977] (see also [OP 1978] and [OP 1983]) as a generalization of the Calogero–Sutherland systems [Ca 1971], [Su 1972]. Such systems depend on one real parameter $\kappa$ (for root systems of the type $A_n$, $D_n$ and $E_6$, $E_7$, $E_8$), on two parameters (for the type $B_n$, $C_n$, $F_4$ and $G_2$) and on three parameters for the type $BC_n$. These parameters are related to the coupling constants of the quantum system.

For the potential $v(q) = \kappa(\kappa - 1) \sin^{-2} q$ and special values of parameter $\kappa$, the wave functions correspond to the characters of the compact simple Lie groups ($\kappa = 1$) [We 1925/26] or to zonal spherical functions on symmetric spaces ($\kappa = 1/2, 2, 4$) [Ha 1958], [He 1978]. At arbitrary values of $\kappa$, they provide an interpolation between these objects.

This class has many remarkable properties. Here we mention only one: the wave functions of such systems are a natural generalization of special functions (hypergeometric functions) to the case of several variables. The history of this problem and some results may be found in [OP 1983]. In [Pe 1998a], it was shown that the product of two wave functions is a finite linear combination of analogous functions, namely, of functions that appear in the corresponding Clebsch–Gordan series. In other words, this deformation ($\kappa$-deformation) does not change the Clebsch–Gordan series. For rank 1, we obtain the well-known cases of the Legendre, Gegenbauer and Jacobi polynomials, and the limiting cases of the Laguerre and Hermite polynomials (see for example [Vi 1968]). Some other cases were also considered in [He 1955], [Ja 1970], [Ko 1974], [Ja 1975], [Vr 1976], [Se 1977], [KS 1978], [Ma 1982], [Pr 1984], [Vr 1984], [HO 1987], [He 1987], [Op 1988a], [Op 1988b], [Ma 1988], [La 1989], [Op 1989], [St 1989], [KS 1995] and [CP 1997]. In [PRZ 1998], $\kappa$-deformed Clebsch–Gordan series was used in order to obtain the explicit expressions for the generalized Gegenbauer polynomials$^3$ of type $A_2$ what gives the explicit solution of the three-body Calogero–Sutherland model. For special values of $\kappa = 1/2, 2, 4$, these formulae give the explicit expressions for zonal polynomials of type $A_2$.

In [Pe 1999] and Appendix B, are presented analogous results obtained by E. Ragoucy, Ph. Zaugg and the author for two-parameter family of polynomials of type $A_2$ introduced by Ruijsenaars [Ru 1987] and Macdonald [Ma 1988].

$^3$ In some papers, the name Jack polynomials [Ja 1970] is used. However, the Jack polynomials are a very special case of polynomials under consideration. So, we prefer to use the name generalized Gegenbauer polynomials for the general case and Jack polynomials for the special case.
2 General description

The systems under consideration are described by the Hamiltonian (for more details, see [OP 1983])

\[ H = \frac{1}{2} p^2 + U(q), \quad p^2 = (p, p) = \sum_{j=1}^{t} p_j^2, \]  

(2.1)

where \( p = (p_1, \ldots, p_t) \), \( p_j = -i \partial / \partial q_j \), is a momentum operator, and \( q = (q_1, \ldots, q_t) \) is a coordinate vector in the \( l \)-dimensional vector space \( V \sim \mathbb{R}^l \) with standard scalar product \((\cdot, \cdot)\). The potential \( U(q) \) is constructed by means of a certain system of vectors \( R^+ = \{\alpha\} \) in \( V \) (the so-called root systems):

\[ U = \sum_{\alpha \in R^+} g_{\alpha}^2 v(q_{\alpha}), \quad q_{\alpha} = (\alpha, q), \quad g_{\alpha}^2 = \kappa_{\alpha}(\kappa_{\alpha} - 1), \quad g_{\alpha} = g_{\beta}, \quad \text{if} \ (\alpha, \alpha) = (\beta, \beta). \]  

(2.2)

Such systems are completely integrable for potentials of five types (see [OP 1983] for \( A_l \); [HO 1987], [He 1987], [Op 1988a], [Op 1988b] and [Op 1989] for a general case). They are a generalization of the Calogero–Sutherland systems [Ca 1971], [Su 1972] for which \( \{\alpha\} = \{e_i - e_j\}, \{e_j\} \) being a standard basis in \( V \).

In this paper, we consider in details only the case of \( A_2 \) with potential \( v(q) = \sin^{-2} q \). For the description of other cases see [Pe 1999].

3 The Clebsch–Gordan series

Let us recall the main results of [Pe 1998a] and specialize them to the \( A_2 \) case with potential \( v(q) = \sin^{-2} q \).

The Schrödinger equation for this quantum system has the form

\[ H \Psi^\kappa = E(\kappa) \Psi^\kappa; \quad H = -\Delta_2 + U(q_1, q_2, q_3), \quad \Delta_2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial q_j^2} \]  

(3.1)

with potential

\[ U(q_1, q_2, q_3) = \kappa(\kappa - 1) \left( \sin^{-2}(q_1 - q_2) + \sin^{-2}(q_2 - q_3) + \sin^{-2}(q_3 - q_1) \right). \]  

(3.2)

The ground state wave function and its energy are

\[ \Psi_0^\kappa(q) = \left( \prod_{j<k}^{3} \sin(q_j - q_k) \right)^\kappa, \quad E_0(\kappa) = 8 \kappa^2. \]  

(3.3)

Substituting \( \Psi^\kappa = \Phi^\kappa \Psi_0^\kappa \) in (3.1), we obtain

\[ -\Delta^\kappa \Phi^\kappa = \varepsilon(\kappa) \Phi^\kappa, \quad \Delta^\kappa = \Delta_2 + \Delta_1^\kappa, \quad \varepsilon(\kappa) = E(\kappa) - E_0(\kappa). \]  

(3.4)
Here the operator $\Delta^\kappa_1$ takes the form
\[
\Delta^\kappa_1 = \kappa \sum_{j<k}^3 \cot(q_j - q_k) \left( \frac{\partial}{\partial q_j} - \frac{\partial}{\partial q_k} \right).
\] (3.5)

It is easy to see that the set of symmetric polynomials in variables $\exp(2iq_j)$ is invariant under the action of $\Delta^\kappa$. Such polynomial $m_\lambda$ is labelled by the $SU(3)$ highest weight $\lambda = m\lambda_1 + n\lambda_2$, with $m, n$ being non-negative integers, and $\lambda_{1,2}$ being two fundamental weights. In general,
\[
\Phi^\kappa_\lambda = \sum_{\mu \leq \lambda} C^\mu_\mu(\kappa) m_\mu, \quad \mu, \lambda \in P^+, \quad m_\mu = \sum_{\nu \in W \cdot \mu} e^{2i(q,\nu)},
\] (3.6)

where $P^+$ denotes the cone of dominant weights, $W$ is the Weyl group, and $C^\mu_\mu(\kappa)$ are some constants.

As it was shown in [Pe 1998a], the product of two wave functions is a finite sum of wave functions (a sort of the $\kappa$-deformed Clebsch–Gordan series):
\[
\Phi^\kappa_\mu \Phi^\kappa_\lambda = \sum_{\nu \in D_\mu(\lambda)} C^\nu_{\mu\lambda}(\kappa) \Phi^\kappa_{\nu}.
\] (3.7)

In this equation, $D_\mu(\lambda) = (D_\mu + \lambda) \cap P^+$, where $D_\mu$ is a weight diagram of the representation with the highest weight $\mu$.

Since $\Phi^\kappa_\mu$ are symmetric functions of $\exp(2iq_j)$, it is convenient to use a new set of variables:
\[
\begin{align*}
    z_1 &= e^{2i q_1} + e^{2i q_2} + e^{2i q_3}, \\
    z_2 &= e^{2i(q_1+q_2)} + e^{2i(q_2+q_3)} + e^{2i(q_3+q_1)}, \\
    z_3 &= e^{2i(q_1+q_2+q_3)}.
\end{align*}
\] (3.8)

In the centre-of-mass frame ($\sum_i q_i = 0$), the wave functions depend only on two variables chosen as $z_1$ and $z_2$ (in this case, $z_3 = 1$). In these variables, up to a normalization factor, we have
\[
\Delta^\kappa = (z_1^2 - 3z_2) \partial^2_1 + (z_2^2 - 3z_2) \partial^2_2 + (z_1 z_2 - 9) \partial_1 \partial_2 + (3\kappa + 1) (z_1 \partial_1 + z_2 \partial_2),
\] (3.9)

where $\partial_i = \partial/\partial z_i$. Corresponding eigenvalues are
\[
\varepsilon_{m,n}(\kappa) = m^2 + n^2 + mn + 3\kappa(m+n).
\] (3.10)

We shall use the normalization for polynomials $\Phi^\kappa_\lambda$ such that the coefficient at the highest monomial is equal to one. Denoting them by $P^\kappa_{m,n}$, we have
\[
P^\kappa_{m,n}(z_1, z_2) = \sum_{p,q} C^p_{m,n}(\kappa) z_1^p z_2^q = z_1^m z_2^n + \text{lower terms},
\] (3.11)
with \( p + q \geq m + n \) and \( p - q \equiv m - n \mod 3 \). As it is easy to see, the first polynomials are

\[
P_{0,0}^\kappa = 1, \quad P_{1,0}^\kappa = z_1, \quad P_{0,1}^\kappa = z_2. \tag{3.12}
\]

Simple consequences of (3.7) for \( P_{\lambda}^\kappa = P_{1,0}^\kappa \) or \( P_{0,1}^\kappa \) are \cite{Pe 1998}

\[
z_1 P_{m,n}^\kappa = P_{m+1,n}^\kappa + a_{m,n}(\kappa) P_{m,n-1}^\kappa + c_m(\kappa) P_{m-1,n+1}^\kappa,
\]

\[
z_2 P_{m,n}^\kappa = P_{m,n+1}^\kappa + \tilde{a}_{m,n}(\kappa) P_{m-1,n}^\kappa + c_n(\kappa) P_{m+1,n-1}^\kappa,
\]

where

\[
a_{m,n}(\kappa) = \tilde{a}_{n,m}(\kappa) = c_n(\kappa) c_{m+n+\kappa}(\kappa),
\]

\[
c_m(\kappa) = \frac{e(m)}{e(\kappa + m)}, \quad e(m) = \frac{m}{m - 1 + \kappa}.
\]

Below we shall construct such polynomials using these recurrence relations.

## 4 \( A_2 \) case

Now we proceed to the case of \( A_2 \sim su(3) \). In this case, the representation \( d \) of \( A_2 \) is characterized by two non-negative numbers \( d = d_{mn} \). We have two fundamental representations

\[
d_{10}, \quad d_{01}; \quad (\dim d_{10} = \dim d_{01} = 3),
\]

and also representations

\[
d_{n0}, \quad d_{0n}; \quad \left(\dim d_{n0} = \dim d_{0n} = \frac{1}{2} (n + 1)(n + 2)\right),
\]

\[
d_{n1} \left(\dim d_{n1} = (n + 1)(n + 3)\right), \quad d_{nn} \left(\dim d_{nn} = (n + 1)^2\right).
\]

We start with the Clebsch–Gordan series

\[
d_{10} \otimes d_{n+1,0} = d_{n+2,0} \oplus d_{n,1},
\]

\[
d_{01} \otimes d_{n0} = d_{n,1} \oplus d_{n-1,0}. \tag{4.1}
\]

Excluding \( d_{n,1} \), we obtain

\[
d_{n-1,0} \oplus (d_{01} \otimes d_{n0}) \oplus (d_{10} \otimes d_{n+1,0}) \oplus d_{n+2,0} = 0,
\]

or

\[
\chi_{n-1,0} - z_2 \chi_{n,0} + z_1 \chi_{n+1,0} - \chi_{n+2,0} = 0, \tag{4.2}
\]
where are introduced the notations:
\[ z_1 = \chi_{10} = e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}, \]
\[ z_2 = \chi_{01} = e^{-i\theta_1} + e^{-i\theta_2} + e^{-i\theta_3}. \]  

From this, we obtain the expression for the generating function
\[ F_0^1(z_1, z_2; u) = \sum_{n=0}^{\infty} \chi_{n0}(z_1, z_2) u^n, \]
\[ F_0^1(z_1, z_2; u) = (1 - z_1 u + z_2 u^2 - u^3)^{-1}. \]  

Let us define now the \( \kappa \)-deformed functions \( \tilde{P}_{n,0}(z_1, z_2) \) by the formula
\[ F^\kappa(z_1, z_2; u) = (1 - z_1 u + z_2 u^2 - u^3)^{-\kappa} = \sum_{n=0}^{\infty} \tilde{P}_{n,0}(z_1, z_2) u^n. \]  

Differentiating \( F^\kappa \) on \( u, z_1 \) and \( z_2 \), we get
\[ F^\kappa_u = \kappa \left( z_1 - 2z_2 u + 3u^2 \right) F^{\kappa+1}, \]
\[ F^\kappa_{z_1,z_1} = \kappa \left( \kappa + 1 \right) u^2 F^{\kappa+2}, \]
\[ F^\kappa_{z_2,z_2} = \kappa \left( \kappa + 1 \right) u^4 F^{\kappa+2}, \]
\[ F^\kappa_{z_1,z_2} = -\kappa \left( \kappa + 1 \right) u^3 F^{\kappa+2}, \]
\[ u F^\kappa_u = \kappa \left( z_1 u - 2 z_2 u^2 + 3 u^3 \right) F^{\kappa+1}, \]  
and
\[ (1 - z_1 u + z_2 u^2 - u^3) F^\kappa_u = \kappa \left( z_1 - 2z_2 u + 3u^2 \right) F^\kappa. \]  

From this it follows the important recurrence formula
\[ (n + 3) \tilde{P}_{n+3,0}^\kappa = (n + 2 + \kappa) z_1 \tilde{P}_{n+2,0}^\kappa + (n + 1 + 2\kappa) z_2 \tilde{P}_{n+1,0}^\kappa + (n + 3\kappa) \tilde{P}_{n,0}^\kappa. \]  

We have also
\[ F^\kappa_{z_1} = \kappa u F^{\kappa+1}, \]
\[ F^\kappa_{z_2} = -\kappa u^2 F^{\kappa+1}. \]  

Hence
\[ \partial_{z_1} P_{n,0}^\kappa = n P_{n+1,0}^{\kappa+1}, \quad \partial_{z_2} P_{n,0}^\kappa = -\frac{n(n-1)}{\kappa + n - 1} P_{n-2,0}^{\kappa+1}. \]  

Finally we have the basic differential equation for \( F^\kappa(z_1, z_2; u) \):
\[ \left( \left( D_{z_1}^2 + D_{z_2}^2 + D_{z_1} D_{z_2} \right) - 3 z_2 \partial_{z_1}^2 - 3 z_1 \partial_{z_2}^2 - 9 \partial_{z_1} \partial_{z_2} + 3 \kappa (D_{z_1} + D_{z_2}) \right) F^\kappa = \left( D_u^2 + 3 \kappa D_u \right) F^\kappa(z_1, z_2; u); \]
\[ D_{z_1} = z_1 \partial_{z_1}, \quad D_{z_2} = z_2 \partial_{z_2}, \quad D_u = u \partial_u. \]

Let us note that the normalization of polynomials \( \tilde{P}_{n,0}^\kappa(z_1, z_2) \) follows from the expression (4.5) for the generating function. Namely,
\[ \tilde{P}_{n,0}^\kappa(z_1, z_2) = \frac{(\kappa)_n}{n!} z_1^n + \cdots = \frac{(\kappa)_n}{n!} P_{n,0}^\kappa(z_1, z_2), \]
where

\[(\kappa)_n = (\kappa)(\kappa + 1) \cdots (\kappa + n - 1).\]

The main property of this normalization is that \(\tilde{P}_{n,0}^\kappa(z_1, z_2)\) has a polynomial dependence on the parameter \(\kappa\).

Now we shall consider other Clebsch–Gordan series for \(\kappa = 1\):

\[d_{1,0} \otimes d_{n+1,0} = d_{n+2,0} \oplus d_{n,1}.\]

According to [Pe 1998], the analogous formula is valid for an arbitrary value of \(\kappa\), i.e.,

\[a_n z_1 \tilde{P}_{n+1,0}^\kappa = b_n \tilde{P}_{n+2,0}^\kappa + c_n \tilde{P}_{n,1}^\kappa.\]  \hspace{1cm} (4.11)

Here

\[a_n = \kappa + n + 1, \quad b_n = n + 2,\]

and we have

\[c_n(\kappa) \tilde{P}_{n,1}^\kappa = (\kappa + n + 1)z_1 \tilde{P}_{n+1,0}^\kappa - (n + 2) \tilde{P}_{n+2,0}^\kappa.\]  \hspace{1cm} (4.12)

Let us calculate now the generating function for both left-hand and right-hand sides of this equation,

\[G^\kappa = \sum_{n=0}^\infty c_n(\kappa) \tilde{P}_{n,1}^\kappa(z_1, z_2),\]

\[G^\kappa = \kappa z_1 \left( F_0^\kappa - 1 \right) + \frac{z_1}{u} F_1^\kappa - \frac{1}{u^2} (F_1^\kappa - \kappa z_1 u),\]

where

\[F_0^\kappa = \sum_{n=0}^\infty \tilde{P}_{n,0}^\kappa u^n, \quad F_1^\kappa = \sum_{n=0}^\infty n \tilde{P}_{n,0}^\kappa u^n = D_u F_0^\kappa,\]

and

\[G^\kappa = \frac{1}{u^2} (\kappa z_1 u - (1 - z_1 u)D_u) F_0^\kappa.\]

Finally,

\[G^\kappa = \kappa \left( 2z_2 - (z_1 z_2 + 3) u + 2 z_1 u^2 \right) F_0^{\kappa+1}.\]  \hspace{1cm} (4.14)

From this, it follows three-term recurrence relation

\[\tilde{P}_{n,1}^\kappa = \kappa \left( 2z_2 \tilde{P}_{n,0}^{\kappa+1} - (z_1 z_2 + 3) \tilde{P}_{n-1,0}^{\kappa+1} + 2 z_1 \tilde{P}_{n-2,0}^{\kappa+1} \right).\]

Now let us follow [PRZ 1998] and construct the general polynomials in terms of the simplest polynomials (Jack polynomials) \(P_{m,0}^\kappa\) and \(P_{0,n}^\kappa\). We get

\[P_{m,0}^\kappa P_{0,n}^\kappa = \sum_{i=0}^{\min(m,n)} \gamma_{m,n}^i P_{m-i,n-i}^\kappa.\]  \hspace{1cm} (4.15)
This is a consequence of equation (3.7), with the notable difference that the sum on the right-hand side is over a restricted domain (actually, it parallels exactly the \( SU(3) \) Clebsch–Gordan decomposition).

To prove this, let us assume that (4.15) is valid up to \((m, n)\). Then using (3.13) and (3.14), we get

\[
P_{\kappa,0}^\kappa P_{0,n+1}^\kappa = \sum_{i=0}^{\min(m,n+1)} \gamma_{m,n+1}^i P_{m-i,n+1-i}^\kappa + c_n \delta_{m,n+1}^i P_{m+1-i,n-1-i}^\kappa,
\]

where we defined

\[
\begin{align*}
\gamma_{m,n+1}^i &= \gamma_{m,n}^i + \tilde{a}_{m-i+1,n-i+1} \gamma_{m,n}^{i-1} - c_n c_{m-i+1} \gamma_{m,n-1}^{i-1}, \\
\delta_{m,n+1}^i &= c_{n-i} c_{m,n} - \gamma_{m,n-1}^i - a_{m-i+1,n-i} \gamma_{m,n-1}^{i-1}
+ c_{n-i} c_{m,n-1} \gamma_{m,n-2}^{i-1}.
\end{align*}
\]

From the polynomial normalization, we already know that \( \gamma_{m,n}^0 = 1 \). After a straightforward computation, the solution to (4.17) proved to be equal

\[
\gamma_{m,n}^i = e(2\kappa + m + n + 1 - i)_{-i} / e(1)_i e((m+1)_{-i} e((n+1)_{-i} = e(1) e((m+1)_{-i} e((n+1)_{-i},
\]

where

\[
e(m) = \frac{m}{m-1+\kappa}.
\]

It implies that \( \delta_{m,n+1}^i = 0 \) in (4.18). Let us give also the more explicit expression

\[
\gamma_{m,n}^i = \frac{(\kappa)_i (m)_i (n)_i (3\kappa + m + n - 1 - i)_i}{i! (\kappa + m - 1)_i (\kappa + n - 1)_i (2\kappa + m + n - i)_i},
\]

where

\[
\begin{align*}
(x)_i &= x(x+1) \cdots (x+i-1), \\
(x)_i &= x(x-1) \cdots (x-i+1).
\end{align*}
\]

The constructive aspect of this formula is in its inverted form.

**Theorem 1 [PRZ 1998].** The generalized Gegenbauer polynomials \( P_{m,n}^\kappa \) of type \( A_2 \) are given by the formula

\[
P_{m,n}^\kappa = \sum_{i=0}^{\min(m,n)} \beta_{m,n}^i P_{m-i,0}^\kappa P_{0,n-i}^\kappa,
\]

\footnote{Note that this expression for \( \gamma_{m,n}^i \) may be obtained from the general Macdonald formula [Ma 1995], however, the way of proof given in [PRZ 1998] is more convenient here.}
where the constants $\beta_{m,n}^i$ are

$$\beta_{m,n}^i = \frac{(-1)^i 3\kappa + m + n - 2i}{i!} \frac{(m)_i (n)_i (3\kappa + m + n - 1)_i}{(k + m - 1)_i (k + n - 1)_i (2\kappa + m + n - 1)_i}. \tag{4.23}$$

Note that $\beta_{m,n}^i$ are obtained by using of the relation

$$\beta_{m,n}^i = -\sum_{j=0}^{i-1} \beta_{m,n}^j \gamma_{m-j,n-j}^{i-j}. \tag{4.24}$$

From this theorem, we see that the construction of a general polynomial $P^\kappa_{m,n}$ is similar to the construction of $SU(3)$ representations from tensor products of two fundamental representations.

Likewise, we can consider other types of decompositions, such as

$$P^\kappa_{m,0} P^\kappa_{n,0} = \sum_{i=0}^{\min(m,n)} \beta_{m,n}^i P^\kappa_{m+2i,n-2i}. \tag{4.25}$$

The proof is analogous to (4.15) (see footnote 3). The coefficients $\tilde{\gamma}_{m,n}^i$ are given by the formula

$$\tilde{\gamma}_{m,n}^i = \frac{(m)_i (n)_i (2\kappa + m + n - 1 - i)_i}{i! (k + m - 1)_i (k + n - 1)_i (2\kappa + m + n - i)_i}. \tag{4.26}$$

**Theorem 2** [PRZ 1998]. There is another formula for polynomials $P^\kappa_{m,n}$ at $m \geq n$:

$$\tilde{\gamma}_{m,n}^{n} P^\kappa_{m,n} = \sum_{i=0}^{n} \beta_{m,n}^i P^\kappa_{m+n+i,0} P^\kappa_{n-i,0}. \tag{4.27}$$

where

$$\tilde{\beta}_{m,n}^i = \frac{(-1)^i (m+2i) (k+m+n)^i}{i! (m+n+1)^i} \frac{(m)_i}{(k+m+1)^i (k+n-1)_i}. \tag{4.28}$$

This theorem follows directly from equation (4.25). The coefficients $\tilde{\beta}_{m,n}^i$ are found by using of the relation

$$\tilde{\beta}_{m,n}^i = -\left(\tilde{\gamma}_{m,n+i,n-i}^{i-n-i}\right)^{-1} \sum_{j=0}^{i-1} \beta_{m,n}^j \tilde{\gamma}_{m+n+j,n-j}^{n-i-j}. \tag{4.29}$$

As a by-product, let us specialize equation (4.22) to the case $\kappa = 1$, where $P^\kappa_{m,n}$ are nothing but the $SU(3)$ characters. We get

$$P^1_{m,n} = P^1_{m,0} P^1_{0,n} - P^1_{m-1,0} P^1_{0,n-1}. \tag{4.30}$$
From this, we easily deduce the generating function for $SU(3)$ characters (see e.g. [PS 1978])

$$G^1(u, v) = \sum_{m,n=0}^{\infty} u^m v^n P^1_{m,n} = \frac{1 - uv}{(1 - z_1 u + z_2 u^2 - u^3)(1 - z_2 v + z_1 v^2 - v^3)}. \quad (4.31)$$

Closing this section, let us note that for $\kappa=1/2, 1, 2$ and 4, the obtained formulae yield the explicit expression of zonal polynomials for certain symmetric spaces listed below.

Let us mention also the papers [CP 1997], [Pe 1998b] where the integral representations for the case $N=3$ were obtained.

**Appendix A. List of polynomials $P^{\kappa}_{m,n}$ with $m+n \leq 4$**

Following [PRZ 1998], we list here the polynomials $P^{\kappa}_{m,n}$ with $m+n \leq 4$:

$$P^{\kappa}_{2,0} = z_1^2 - \frac{2}{\kappa + 1} z_2,$$
$$P^{\kappa}_{1,1} = z_1 z_2 - \frac{3}{2\kappa + 1},$$
$$P^{\kappa}_{3,0} = z_1^3 - \frac{6}{\kappa + 2} z_1 z_2 + \frac{6}{(\kappa + 1)(\kappa + 2)} z_1,$$
$$P^{\kappa}_{2,1} = z_1^2 z_2 - \frac{2}{\kappa + 1} z_2^2 - \frac{3\kappa + 1}{(\kappa + 1)^2} z_1,$$
$$P^{\kappa}_{4,0} = z_1^4 - \frac{12}{\kappa + 3} z_1^2 z_2 + \frac{12}{(\kappa + 2)(\kappa + 3)} z_2^2 + \frac{24}{(\kappa + 2)(\kappa + 3)} z_1,$$
$$P^{\kappa}_{3,1} = z_1^3 z_2 - \frac{6}{\kappa + 2} z_1^2 z_2 - \frac{3(3\kappa + 2)}{(\kappa + 2)(2\kappa + 3)} z_1^2 + \frac{30}{(\kappa + 2)(2\kappa + 3)} z_2,$$
$$P^{\kappa}_{2,2} = z_1^2 z_2^2 - \frac{2}{\kappa + 1} (z_1^3 + z_2^3) - \frac{12(\kappa - 1)}{(\kappa + 1)(2\kappa + 3)} z_1 z_2 + \frac{9(\kappa - 1)}{(\kappa + 1)^2(2\kappa + 3)}.$$

**Appendix B. Some formulae for Macdonald polynomials for $A_2$ case (by A.M. Perelomov, E. Ragoucy and Ph. Zaugg)**

Here we give only some necessary for us information. For other results and details, see [Ma 1995].

The Macdonald polynomials of type $A_2$ may be defined as polynomial eigenfunctions of the Macdonald difference equation

$$M^1 P^{(q,t)}_{m,n}(x_1, x_2, x_3) = \lambda P^{(q,t)}_{m,n}(x_1, x_2, x_3), \quad P_{m,n} = z_1^m z_2^n + \text{lower terms}, \quad (B.1)$$
\[ z_1 = x_1 + x_2 + x_3, \quad z_2 = x_1x_2 + x_2x_3 + x_3x_1, \quad z_3 = x_1x_2x_3, \]

where

\[ M^1 = \prod_{j \neq k} \left( \frac{tx_j - x_k}{x_j - x_k} \right) T_j, \quad T_1 f(x_1, x_2, x_3) = f(qx_1, x_2, x_3), \ldots \quad (B.2) \]

The Macdonald polynomials satisfy the following recurrence relation

\[ z_1 P_{m,n}^{(q,t)}(z_1, z_2, z_3) = P_{m+1,n}^{(q,t)} + a_{m,n}(q,t) P_{m,n-1}^{(q,t)} + b_{m,n}(q,t) P_{m-1,n+1}^{(q,t)}, \quad (B.3) \]

where

\[ a_{m,n} = c_n \tilde{c}_{m+n}, \quad b_{m,n} = c_m, \]

\[ c_m(q,t) = \frac{(1 - q^m)(1 - t^2q^{m-1})}{(1 - tq^m)(1 - t^2q^{m-1})}, \quad (B.4) \]

\[ \tilde{c}_{m+n}(q,t) = \left( \frac{1 - tq^{m+n}}{1 - t^2q^{m+n}} \right) \left( \frac{1 - t^4q^{m+n-1}}{1 - t^2q^{m+n-1}} \right). \]

Note that

\[ c_m(q,1) = c_m(q, q) = 1, \quad \tilde{c}_{m+n}(q,1) = \tilde{c}_{m+n}(q, q) = 1. \quad (B.5) \]

Now let us consider the case \( t = q^k \), \( k \) being an integer. In this case, the explicit expression for the generating function of polynomials \( P_{n,0}^{(k)} = P_{n,0}^{(q,t)} \) has the form

\[ G^{(k)}(u) = \prod_{j=0}^{k-1} F^1(q^j u), \]

\[ G^{(k)}(u) = \sum C_n^{(k)} P_{n,0}^{(k)} u^n, \quad (B.6) \]

\[ C_n^{(k)} = \frac{[k]_n}{[1]_n}, \quad [x]_n = [x][x+1] \cdots [x+n-1], \quad [x] = \frac{1 - q^x}{1 - x}. \]

From this we get a three-term recurrence relation being analogous to (4.7)

\[ [n + 1] \tilde{P}_{n+1}^k = [k + n] z_1 \tilde{P}_n^k - [2k + n - 1] z_2 \tilde{P}_{n-1}^k + [3k + n - 2] z_3 \tilde{P}_{n-2}^k, \quad (B.7) \]

which follows from the recurrence relation

\[ (1 - z_1 u + z_2 u^2 - z_3 u^3) G^{(k)}(u) = (1 - z_1 u q^k + z_2 u^2 q^{2k} - z_3 u^3 q^{3k}) G(qu). \quad (B.8) \]

From here, we obtain

\[ \tilde{P}_0^k = 1, \quad \tilde{P}_1^k = [k] z_1, \quad \tilde{P}_2^k = \left[ \frac{[k + 1][k]}{[1][2]} \right] z_1^2 - \left[ \frac{[2k]}{[2]} \right] z_2, \ldots \quad \tilde{P}_{m,0}^k = \left[ \frac{[1]^m}{[k]^m} \right] P_{m,0}, \]
Let us give also the formula for the generating function for the case of arbitrary \( t \):

\[
G^{(q,t)}(u) = \prod_{j=0}^{\infty} \frac{(1 - q^j t u)}{(1 - q^j u)} = \sum_{j=0}^{\infty} c_j(q, t) u^j .
\]

Using the results from the book [Ma 1995], it is not difficult to get the formulae

\[
P^{(q,t)}_{m,0} P^{(q,t)}_{0,n} = \sum_{i=0}^{\min(m,n)} \gamma^i_{m,n} P^{(q,t)}_{m-i,n-i} ,
\]

where

\[
\gamma^i_{m,n} = \frac{(t; q)_i (q^m; q^{-1})_i (q^n; q^{-1})_i (q^{m+n-1-i}t^3; q^{-1})_i}{(q; q)_i (q^{m-1}t; q^{-1})_i (q^{n-1}t; q^{-1})_i (q^{m+n-i}t^2; q^{-1})_i} ,
\]

and

\[
P^{(q,t)}_{m,0} P^{(q,t)}_{n,0} = \sum_{i=0}^{\min(m,n)} \tilde{\gamma}^i_{m,n} P^{(q,t)}_{m+n-2i,n} ,
\]

where

\[
\tilde{\gamma}^i_{m,n} = \frac{(t; q)_i (q^m; q^{-1})_i (q^n; q^{-1})_i (q^{m+n-i}t^2; q^{-1})_i}{(q; q)_i (q^{m-1}t; q^{-1})_i (q^{n-1}t; q^{-1})_i (q^{m+n-i}t; q^{-1})_i} .
\]

Inverting formulae (B.10) and (B.11) as in the [PRZ 1998], we come to

**Theorem 1a.** The Macdonald polynomials \( P^{(q,t)}_{m,n} \) of type \( A_2 \) are given by the formula

\[
P^{(q,t)}_{m,n} = \sum_{i=0}^{\min(m,n)} \beta^i_{m,n} P^{(q,t)}_{m-i,0} P^{(q,t)}_{0,n-i} ,
\]

where constants have the form

\[
\beta^i_{m,n} = (-1)^i q^{i(i-1)/2} \frac{(t; q^{-1})_i (1 - q^{m+n-2i}t^3; q^{-1})_i}{(q; q)_i (1 - q^{m+n-i}t^3; q^{-1})_i} \times \frac{(q^m; q^{-1})_i (q^n; q^{-1})_i (q^{m+n-i}t^3; q^{-1})_i}{(q^{m-1}t; q^{-1})_i (q^{n-1}t; q^{-1})_i (q^{m+n-i}t; q^{-1})_i} .
\]

In similar way, we get

**Theorem 2a.** The Macdonald polynomials \( P^{(q,t)}_{m,n} \) of type \( A_2 \) are given by the formula

\[
\tilde{\gamma}^i_{m+n,n} P^{(q,t)}_{m,n} = \sum_{i=0}^{n} \tilde{\beta}^i_{m,n} P^{(q,t)}_{m+n+i,0} P^{(q,t)}_{n-i,0} , \quad m \geq n
\]

where constants \( \tilde{\gamma}^i_{m,n} \) are given by formula (B.13), and

\[
\tilde{\beta}^i_{m,n} = (-1)^i q^{i(i-1)/2} \frac{(t; q^{-1})_i (1 - q^{m+2i}) (q^m; q)_i (q^n; q^{-1})_i (q^{m+n+i}t; q)_i}{(q; q)_i (1 - q^m) (q^{m+n+i}t; q)_i (q^{m+n+i}t; q)_i} .
\]
From Theorems 1a and 2a, many interesting identities may be obtained. Here we give one of them:

\[
S_{n,l}^k = \sum_{i=0}^{l} (-1)^i q^{(i-1)/2} \frac{[3k+n-2i]}{[3k+n-i]} \left( \prod_{j=1}^{i} \frac{[k-j+1]}{[j]} \frac{[3k+n-j]}{[2k+n-j]} \right) \\
\times \left( \prod_{j=0}^{l-i-1} \frac{[k+j]}{[j+1]} \frac{[3k+n-l-i-j-1]}{[2k+n-l-i-j]} \right) = 0, \tag{B.17}
\]

where \([n] = (1 - q^n)/(1 - q)\).

The simplest version of this formula is

\[
\sum_{i=0}^{l} (-1)^i q^{(i-1)/2} \frac{[k+l-i-1]}{[i]! \ [l-i]! \ [k-i]!} = 0. \tag{B.18}
\]

We give below the list of polynomials \(P_{m,n}^{(q,t)}\) at \(m + n \leq 4\)

\[
P_{0,0}^{(q,t)} = 1, \quad P_{1,0}^{(q,t)} = z_1, \quad P_{0,1}^{(q,t)} = z_2,
\]

\[
P_{2,0}^{(q,t)} = z_1^2 - \frac{(1-q)(1+t)}{(1/qt)} z_2, \\
P_{1,1}^{(q,t)} = z_1 z_2 - \frac{(1-q) (1+t+t^2)}{(1-q t^2)} z_3, \\
P_{3,0}^{(q,t)} = z_1^3 - \frac{(1-q) (2+q+t+2qt)}{1-q^2 t} z_1 z_2 + \frac{(1-q)^2 (1+q)(1+t+t^2)}{(1-q t)(1-q^2 t)} z_3,
\]

\[
P_{2,1}^{(q,t)} = z_1^2 z_2 - \frac{(1-q)(1+t)}{1-q t} z_2^2 + \frac{(1-q^2)(1-q t^3)}{(1-q t^2)(1+q t)} z_1 z_3, \\
P_{4,0}^{(q,t)} = z_1^4 - \frac{(1-q)(3+2q+q^2+t+2qt+3q^2 t)}{1-q^3 t} z_2^2 z_3 \\
- \frac{(1-q)^2 (1+q)(1+t)(1+q t)}{(1-q^2 t)(1-q^3 t)} z_2 \\
+ \frac{(1-q)^2 (1+q)(2+q+q^2+t+2qt+q^2 t+t^2+q t^2+q^2 t^2)}{(1-q^2 t)(1-q^3 t)} z_1 z_3,
\]

\[
P_{3,1}^{(q,t)} = z_1 z_2 - \frac{(1-q)(2+q+t+2qt)}{1-q^2 t} z_1 z_2^2 - \frac{(1-q^3)(1-q^2 t^3)}{(1-q^2 t)(1-q^3 t^2)} z_1 z_2 z_3 \\
+ \frac{(1-q)^2}{(1-q^2 t)(1-q^3 t^2)} \times (2+2q+q^2+2t+4qt+3q^2 t+q^3 t+t^2+3q t^2+4q^2 t^2+q^3 t^2+2q^2 t^3+q^3 t^3+2q^3 t^3) z_2 z_3;
\]
\[ P_{2,2}^{(q,t)} = \frac{z_1^2 z_2^2 - (1-q)(1+t)}{1-qt} \left( z_2^3 + z_1^3 z_3 \right) - \frac{(1-q)(3q + q^2 - 3t + 2q^2 t + q^3 t - t^2 - 2qt^2 + 3q^3 t^2 - qt^3 - 3q^2 t^3)}{(1-qt)(1 - q^3 t^2)} z_1 z_2 z_3 + \frac{(1-q)^2 (1+q)(1+t + t^2)(q - t + q^2 t - qt^2 + q^3 t^2 - q^2 t^3)}{(1-qt)(1 - q^2 t^2)(1 - q^3 t^2)} z_3^2. \]

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**References**

[CP 1997] Cariñena J.M. and Perelomov A.M., J. Phys. A30, L495–L501
[Ca 1971] Calogero F., J. Math. Phys. 12, 419–436
[HO 1987] Heckman G. and Opdam E., Compositio Math. 64, 329–352
[Ha 1958] Harish–Chandra, Amer. J. Math. 80, 241–310 and 553–613
[He 1955] Herz C., Ann. of Math. 61, 474–523
[He 1978] Helgason S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Acad. Press: N.Y.
[He 1987] Heckman G., Composito Math. 64, 353–373
[Hu 1972] Humphreys J.E., *Introduction to Lie Algebras and Representation Theory*, Springer
[Ja 1970] Jack H., Proc. Roy. Soc. Edinb. A69, 1–18
[Ja 1975] James A., pp 497–520 in: *Theory and Application of Special Functions*, Acad. Press: N.Y.
[KS 1978] Koornwinder T. and Sprinkhuizen–Kuyper I., SIAM J. Math. Anal. 9, 457–483
[KS 1995] Kuznetsov V.B. and Sklyanin E.K., RIMS Kokyuroku 919, 27–34
[Ko 1974] Koornwinder T., Proc. Ned. Akad. Wet. A77, 48–66, 357–381
[La 1989] Lassalle M., C.R. Acad. Sci., ser. I, 309, 941–944
[Ma 1982] Macdonald I.G., SIAM J.Math.Anal. 13, 988–1007

[Ma 1988] Macdonald I.G., *Orthogonal polynomials associated with root systems*, preprint

[Ma 1995] Macdonald I.G., *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press

[OP 1977] Olshanetsky M.A. and Perelomov A.M., Lett. Math. Phys. 2, 7–13

[OP 1978] Olshanetsky M.A. and Perelomov A.M., Funct. Anal. Appl. 12, 121–128

[OP 1983] Olshanetsky M.A. and Perelomov A.M., Phys. Rep. 94, 313–404

[OV 1990] Onishchik A.L. and Vinberg E.B., *Lie Groups and Algebraic Groups*, Springer–Verlag

[Op 1988a] Opdam E., Composito Math. 67, 21–49

[Op 1988b] Opdam E., Composito Math. 67, 191–209

[Op 1989] Opdam E., Inv. Math. 98, 1–18

[PRZ 1998] Perelomov A.M., Ragoucy E. and Zaugg Ph., J. Phys. A31, L559–L565

[PS 1978] Patera J. and Sharp R.T., Proceedings of the VII International Colloquium on Group Theoretical Methods in Physics, Austin, Texas.

[Pe 1998a] Perelomov A.M., J. Phys. A31, L31–L37

[Pe 1998b] Perelomov A.M., in: Proc. of the II International Workshop *Lie Theory and its Applications in Physics*. II, pp.171–183, Singapore: World Scientific

[Pe 1999] Perelomov A.M., J. Phys. A32, 8563–8576

[Pr 1984] Prati M.–C., Lett. Nuovo Cim. 41, 275

[Ru 1987] Ruijsenaars S., Comm. Math. Phys. 110, 191–213

[Se 1977] Sekiguchi J., Publ. RIMS Kyoto Univ. Suppl. 12, 455–459

[St 1989] Stanley R.P., Adv. Math. 77, 76–115

[Su 1972] Sutherland B., Phys. Rev. A4, 2019–2021

[Vi 1968] Vilenkin N.Ja., *Special Functions and the Theory of Group Representations*, Amer. Math. Soc. Transl. Monographs 22, Providence, RI
[Vr 1976] Vretare L., Math. Scand. 39, 343–358

[Vr 1984] Vretare L., SIAM J. Math. Anal. 15, 805–833

[We 1925] Weyl H., Math. Zs. 23, 271–309

[We 1926] Weyl H., Math. Zs. 24, 328–376 and 377–395