STRONG NUMERICAL METHODS OF ORDERS 2.0, 2.5, AND 3.0 FOR ITO STOCHASTIC DIFFERENTIAL EQUATIONS BASED ON THE UNIFIED STOCHASTIC TAYLOR EXPANSIONS AND MULTIPLE FOURIER–LEGENDRE SERIES

DMITRIY F. KUZNETSOV

Abstract. The article is devoted to the construction of explicit one-step strong numerical methods with the orders of convergence 2.0, 2.5, and 3.0 for Itô stochastic differential equations with multidimensional non-commutative noise. We consider the numerical methods based on the unified Taylor–Itô and Taylor–Stratonovich expansions. For numerical modeling of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 we apply the method of multiple Fourier–Legendre series converging in the sense of norm in Hilbert space $L^2([t,T]^k)$, $k = 1, \ldots, 6$. The article is addressed to engineers who use numerical modeling in stochastic control and for solving the non-linear filtering problem. The article can be interesting for mathematicians who working in the field of high-order strong numerical methods for Itô stochastic differential equations.

CONTENTS

1. Introduction 2
2. Explicit One-Step Strong Numerical Schemes of Orders 2.0, 2.5, and 3.0 Based on the Unified Taylor–Itô expansion 3
3. Approximation of iterated Itô Stochastic Integrals Based on Multiple Fourier–Legendre Series 8
4. Explicit One-Step Strong Numerical Schemes of Orders 2.0, 2.5, and 3.0 Based on the Unified Taylor–Stratonovich expansion 28
5. Fourier–Legendre Expansions of Iterated Stratonovich Stochastic Integrals of multiplicities 1 to 6 31
References 40

Mathematics Subject Classification: 60H05, 60H10, 42B05, 42C10.
Keywords: Explicit one-step strong numerical method, Unified Taylor–Itô expansion, Unified Taylor–Stratonovich expansion, Iterated Itô stochastic integral, Iterated Stratonovich stochastic integral, Method of generalized multiple Fourier series, Multiple Fourier–Legendre series, Strong convergence, Mean-square convergence, Approximation, Expansion.
1. Introduction

Let \((\Omega, F, P)\) be a complete probability space, let \(\{F_t, t \in [0, T]\}\) be a nondecreasing right-continuous family of \(\sigma\)-algebras of \(F_t\) and let \(f_t\) be a standard \(m\)-dimensional Wiener stochastic process, which is \(F_t\)-measurable for any \(t \in [0, T]\). We assume that the components \(f^{(i)}_t\) \((i = 1, \ldots, m)\) of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

\[
\begin{align*}
x_t &= x_0 + \int_0^t a(x_\tau, \tau) d\tau + \int_0^t \Sigma(x_\tau, \tau) d\mathbf{f}_\tau, \\
x_0 &= \mathbf{x}(0, \omega).
\end{align*}
\]

Here \(x_t\) is some \(n\)-dimensional stochastic process satisfying the equation (1). The nonrandom functions \(a : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n, \Sigma : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^{n \times m}\) guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) \((1)\). The second integral on the right-hand side of (1) is interpreted as the Ito stochastic integral. Let \(x_0\) be an \(n\)-dimensional random variable, which is \(F_0\)-measurable and \(\mathbb{M}\{x_0^2\} < \infty\) \((\mathbb{M} \text{ denotes a mathematical expectation})\). We assume that \(x_0\) and \(f_t - f_0\) are independent when \(t > 0\).

It is well known \([2, 5]\) that Ito SDEs are adequate mathematical models of dynamic systems of different physical nature under the influence of random disturbances. One of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions \([2]-[10]\). The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs and have the following form

\[
\begin{align*}
J[\psi^{(k)}]_{T,t} &= \int_t^T \psi_k(t_k) \int_{t_1}^{t_k} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k}, \\
J^*[\psi^{(k)}]_{T,t} &= \int_t^T \psi_k(t_k) \int_{t_1}^{t_k} \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k},
\end{align*}
\]

where every \(\psi_l(\tau) \ (l = 1, \ldots, k)\) is a non-random function on \([t, T]\), \(\mathbf{w}^{(i)}_\tau = f^{(i)}_\tau\) for \(i = 1, \ldots, m\) and \(\mathbf{w}^{(0)}_\tau = \tau, \ i_1, \ldots, i_k = 0, 1, \ldots, m,\)

\[
\int_t^T \int_{t_1}^{t_k} d\mathbf{w}^{(i)}_{t_1} \ldots d\mathbf{w}^{(i_k)}_{t_k}
\]

denote Ito and Stratonovich stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from \([2]\)).

Note that \(\psi_l(\tau) \equiv 1 \ (l = 1, \ldots, k)\) and \(i_1, \ldots, i_k = 0, 1, \ldots, m\) in \([2]-[4], [6], [7]\). At the same time \(\psi_l(\tau) \equiv (t - \tau)^{q_l} \ (l = 1, \ldots, k; q_1, \ldots, q_k = 0, 1, 2, \ldots)\) and \(i_1, \ldots, i_k = 1, \ldots, m\) in \([8]-[10]\).

Effective solution of the problem of combined mean-square approximation of the iterated Ito and Stratonovich stochastic integrals \([2]\) and \([3]\) of multiplicities 1 to 6 composes one of the subjects of this article.

We want to mention in short that there are two main criteria of numerical methods convergence for Ito SDEs \([2]-[4]\): a strong or mean-square criterion and a weak criterion where the subject of approximation is not the solution of Ito SDE, simply stated, but the distribution of Ito SDE solution.

Using strong numerical methods, we may build sample paths of Ito SDEs numerically. These methods require combined mean-square approximation of the iterated Ito and Stratonovich stochastic integrals \([2]\) and \([3]\).
Also numerical integration of Ito SDEs based on the strong convergence criterion of approximation is widely used for the numerical solution of different mathematical problems connected with Ito SDEs. Among these problems, we note the following: signal filtering under influence of random noises in various statements, optimal stochastic control, testing estimation procedures of parameters of stochastic systems, stochastic stability and bifurcations analysis [2]-[4].

The problem of effective jointly numerical modeling (in accordance to the mean-square convergence criterion) of the iterated Ito and Stratonovich stochastic integrals (2) and (3) is difficult from theoretical and computing point of view [2]-[66].

The only exception is connected with a narrow particular case, when \( i_1 = \ldots = i_k \neq 0 \) and \( \psi_1(s), \ldots, \psi_k(s) \equiv \psi(s) \). This case allows the investigation with using of the Ito formula [2]-[4].

Note that even for the mentioned coincidence \( (i_1 = \ldots = i_k \neq 0) \), but for different functions \( \psi_1(s), \ldots, \psi_k(s) \) the mentioned difficulties persist, and relatively simple families of iterated Ito and Stratonovich stochastic integrals, which can be often met in the applications, cannot be represented effectively in a finite form (for the mean-square approximation) using the system of standard Gaussian random variables.

Note that for a number of special types of Ito SDEs the problem of approximation of iterated stochastic integrals can be simplified but cannot be solved. The equations with additive scalar noise, with additive vector noise, with non-additive scalar noise, with a small parameter are related to such types of equations [2]-[4]. For the mentioned types of equations, simplifications are connected with the fact that either some coefficient functions from stochastic analogues of the Taylor formula identically equal to zero, or scalar noise has a strong effect, or due to the presence of a small parameter we may neglect some members from the stochastic analogues of the Taylor formula, which include difficult for approximation iterated stochastic integrals [2]-[4], [13]. In this article, we consider Ito SDEs with multidimensional, non-additive and non-commutative noise.

Seems that iterated stochastic integrals may be approximated by multiple integral sums of different types [3], [4], [14]. However, this approach implies partitioning of the interval of integration \([t,T]\) of iterated stochastic integrals (the length \( T - t \) of this interval is a small value, because it is a step of integration of numerical methods for Ito SDEs) and according to numerical experiments this additional partitioning leads to significant calculating costs [5].

In [3] (also see [2], [4], [12], [13]), Milstein proposed to expand (2) or (3) into iterated series in terms of products of standard Gaussian random variables by representing the Wiener process as a trigonometric Fourier series with random coefficients (the version of the so-called Karhunen–Loeve expansion of the Brownian bridge process). To obtain the Milstein expansion of (3), the truncated Fourier expansions of components of the Wiener process \( f_s \) must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of (3) valid for an arbitrary multiplicity \( k \).

For this reason, only expansions of single, double, and triple stochastic integrals of the form (3) were presented in [2], [11]-[13] \((k = 1, 2, 3)\) and in [3], [4] \((k = 1, 2)\) for the simplest case \( \psi_1(s), \psi_2(s), \psi_3(s) \equiv 1; i_1, i_2, i_3 = 0, 1, \ldots, m \).

Moreover, in [2] (Sect. 5.8, pp. 202-204), [11] (pp. 82-84), [12] (pp. 438-439), [13] (pp. 263-264) the authors use (without rigorous proof) the Wong–Zakai approximation [67]-[69] within the frames of the Milstein approach [3] based on the Karhunen–Loeve expansion of the Brownian bridge process (see discussions in [54] (Sect. 2.18, 6.2), [55], [56] (Sect. 2.6.2, 6.2), [14]).

Note that in [60] the method of expansion of the iterated Ito stochastic integrals (2) \((k = 2; \psi_1(s), \psi_2(s) \equiv 1; i_1, i_2 = 1, \ldots, m)\) based on expansion of the Wiener process using Haar functions and trigonometric functions has been considered.

It is necessary to note that the Milstein approach [3] excelled in several times or even in several orders the methods of multiple integral sums [3], [4], [14] considering computational costs in the sense of their diminishing.

An alternative strong approximation method was proposed for (3) in [15]-[17] (also see [24]-[30]), where \( J^* [\psi^{(k)}]_{T,t} \) was represented as a multiple stochastic integral from the certain discontinuous
non-random function of \( k \) variables, and the function was then expressed as an iterated generalized Fourier series in a complete systems of continuous functions that are orthonormal in \( L_2([t, T]) \). In [15-17] (also see [24-30]) the cases of Legendre polynomials and trigonometric functions are considered in details. As a result, a general iterated series expansion of [3] in terms of products of standard Gaussian random variables was obtained in [15-17] (also see [24-30]) for an arbitrary multiplicity \( k \). Hereinafter, this method is referred to as the method of generalized iterated Fourier series.

It was shown in [14], [10] (also see [18-28]) that the method of generalized iterated Fourier series leads to the Milstein expansion [3] of (3) in the case of trigonometric functions (at least for \( k = 2 \); \( \psi_1(s), \psi_2(s) \equiv 1 \); \( i_1, i_2 = 1, \ldots, m \)) and to a substantially simpler expansion of (3) in the case of Legendre polynomials.

Note that the method of generalized iterated Fourier series as well as the Milstein approach [3] lead to iterated application of the operation of limit transition. This problem appears for triple stochastic integrals \( (i_1, i_2, i_3 = 1, \ldots, m) \) or even for some double stochastic integrals in the case, when \( \psi_1(\tau), \psi_2(\tau) \neq 1 (i_1, i_2 = 1, \ldots, m) \) [15], [10] (also see [18-28]).

The mentioned problem (iterated application of the operation of limit transition) not appears in the method, which is considered for (2) in Theorems 1, 2 (see below) [3], [18-28], [31-56]. The idea of this method is as follows: the iterated Ito stochastic integral (2) of multiplicity \( k \) is represented as the multiple stochastic integral from the certain discontinuous non-random function of \( k \) variables defined on the hypercube \([t, T]^k\) where \([t, T]\) is the interval of integration of the iterated Ito stochastic integral (2). Then, the indicated non-random function of \( k \) variables is expanded in the hypercube into the generalized multiple Fourier series converging in the mean-square sense in the space \( L_2([t, T]^k) \).

After a number of nontrivial transformations we come (see Theorems 1, 2 below) to the mean-square converging expansion of the iterated Ito stochastic integral (2) into the multiple series in terms of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned non-random function of \( k \) variables, which can be calculated using the explicit formula regardless of the multiplicity \( k \) of the iterated Ito stochastic integral (2). Hereinafter, this method is referred to as the method of generalized multiple Fourier series.

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see (8)) for calculation of expansion coefficients of the iterated Ito stochastic integral (2) with any fixed multiplicity \( k \).

2. We have new possibilities for exact calculation of the mean-square error of approximation of the iterated Ito stochastic integral (2) (see (19), (20), [31-34], [36], [37], [39], [49-52], [54-56]).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space \( L_2([t, T]) \), then we have new possibilities for approximation — we may use not only the trigonometric functions as in (2-4) but the Legendre polynomials.

4. As it turned out (see [3], [17], [10], [18-37], [40], [42], [43], [46], [50-52], [54-56]), it is more convenient to work with Legendre polynomials for constructing of approximations of the iterated Ito stochastic integrals (2). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions. Another advantages of Legendre polynomials in the framework of the mentioned problem are considered in [34], [32], [53-56].

5. The approach based on the Karhunen–Loeve expansion of the Brownian bridge process (also see [66]) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorems 1, 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case \( \psi_1(s), \psi_2(s), \psi_3(s) \equiv 1; i_1, i_2, i_3 = 1, \ldots, m \)) of iterated Ito stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as \( p_1, \ldots, p_k \)).
example, when \( p_1 = \ldots = p_k = p \to \infty \). For iterated series, the condition \( p_1 = \ldots = p_k = p \to \infty \) obviously does not guarantee the convergence of this series.

However, the authors of the works [2] (Sect. 5.8, pp. 202–204), [11] (pp. 82–84), [12] (pp. 438–439), [13] (pp. 263-264) unreasonably use the condition \( p = p_2 = p_3 = p \to \infty \) within the frames of the Milstein approach [3] based on the series expansion of the Brownian bridge process together with the Wong–Zakai approximation [67]-[69].

2. EXPLICIT ONE-STEP STRONG NUMERICAL SCHEMES OF ORDERS 2.0, 2.5, AND 3.0 BASED ON THE UNIFIED TAYLOR–ITO EXPANSION

Consider the partition \( \{\tau_j\}_{j=0}^N \) of the interval \([0, T]\) such that

\[
0 = \tau_0 < \ldots < \tau_N \leq T, \quad \Delta N = \max_{0 \leq j \leq N-1} \Delta \tau_j, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\]

Let \( y_{\tau_j}^{\text{def}} = y_j; \quad j = 0, 1, \ldots, N \) be a time discrete approximation of the process \( x_s, s \in [0, T] \), which is a solution of the Ito SDE (1).

**Definition 1** [2]. We will say that a time discrete approximation \( y_j; \quad j = 0, 1, \ldots, N \), corresponding to the maximal step of discretization \( \Delta N \), converges strongly with order \( \gamma > 0 \) at time moment \( T \) to the process \( x_s, s \in [0, T] \) if there exists a constant \( C > 0 \), which does not depend on \( \Delta N \), and a \( \delta > 0 \) such that

\[
M\{|x_T - y_T|\} \leq C(\Delta N)^\gamma
\]

for each \( \Delta N \in (0, \delta) \).

Consider the following explicit one-step strong numerical scheme of order 3.0 based on the so-called unified Taylor–Ito expansion [5], [10], [19], [24], [54]-[56]

\[
y_{p+1} = y_p + \sum_{i_1=1}^{m} \Sigma_{s, s, s, s} i_1 \tilde{y}_{i_1}^{(s)} 0_{p+1, p} \Delta a + \sum_{i_1, i_2=1}^{m} G_0^{(i_2)} \Sigma_{s, s, s, s} \tilde{I}_{0000, 0000}^{(i_1 i_2)} \Delta \tau + \sum_{i_1=1}^{m} \left[ G_0^{(i_1)} a \left( \Delta \tilde{y}_{1000}^{(i_1)} + \tilde{I}_{0000, 0000}^{(i_1 i_2)} \right) - L \Sigma_{s, s, s, s} \tilde{I}_{1000}^{(i_1)} \right] + \sum_{i_1, i_2=1}^{m} \left[ G_0^{(i_2)} \Sigma_{s, s, s, s} \tilde{I}_{1000, 0000}^{(i_2 i_2)} \Delta a + \sum_{i_1, i_2, i_3=1}^{m} \left[ G_0^{(i_2)} \Sigma_{s, s, s, s} \tilde{I}_{1000, 0000}^{(i_2 i_2)} \right] + \sum_{i_1, i_2=1}^{m} \left[ G_0^{(i_2)} L \Sigma_{s, s, s, s} \tilde{I}_{1000, 0000}^{(i_2)} \right] \right] + \sum_{i_1, i_2, i_3=1}^{m} \left[ G_0^{(i_1 i_2)} \Sigma_{s, s, s, s} \tilde{I}_{1000, 0000}^{(i_1 i_2 i_2)} \right] + \sum_{i_1, i_2, i_3, i_4=1}^{m} \left[ G_0^{(i_4)} G_0^{(i_3)} G_0^{(i_2)} G_0^{(i_1)} \Sigma_{s, s, s, s} \tilde{I}_{1000, 0000}^{(i_1 i_2 i_3 i_4)} \right]
\]
\[ u_{p+1} = \sum_{i_1=1}^{m} \left[ G_0^{(i_1)} L a \left( \frac{1}{2} \hat{j}_{p+1, r_p}^{(i_1)} + \Delta \hat{j}_{p+1, r_p}^{(i_1)} + \frac{\Delta^2}{2} \hat{I}_{p+1, r_p}^{(i_1)} \right) + \frac{1}{2} L L \sum_{i_1} \hat{j}_{p+1, r_p}^{(i_1)} - L G_0^{(i_1)} a \left( \hat{j}_{p+1, r_p}^{(i_1)} + \Delta \hat{j}_{p+1, r_p}^{(i_1)} \right) \right] + \sum_{i_1, i_2, i_3=1}^{m} \left[ G_0^{(i_3)} L G_0^{(i_2)} \sum_{i_1} \left( \hat{i}_{p+1, r_p}^{(i_3 i_2 i_1)} - \hat{i}_{p+1, r_p}^{(i_3 i_2 i_1)} \right) + \frac{1}{2} L L G_0^{(i_2)} \sum_{i_1} \hat{i}_{p+1, r_p}^{(i_2 i_1)} + L G_0^{(i_2)} a \left( \hat{i}_{p+1, r_p}^{(i_2 i_1)} + \Delta \hat{i}_{p+1, r_p}^{(i_2 i_1)} \right) - L G_0^{(i_2)} \sum_{i_1} \hat{i}_{p+1, r_p}^{(i_2 i_1)} \right] + \sum_{i_1, i_2, i_3, i_4, i_5=1}^{m} \left[ G_0^{(i_5)} G_0^{(i_4)} G_0^{(i_3)} G_0^{(i_2)} \sum_{i_1} \hat{i}_{p+1, r_p}^{(i_5 i_4 i_3 i_2 i_1)} + \frac{\Delta^3}{6} L L a, \right] \]

\[ v_{p+1} = \sum_{i_1, i_2=1}^{m} \left[ G_0^{(i_2)} G_0^{(i_1)} L a \left( \frac{1}{2} \hat{j}_{p+1, r_p}^{(i_2 i_1)} + \Delta \hat{j}_{p+1, r_p}^{(i_2 i_1)} + \frac{\Delta^2}{2} \hat{I}_{p+1, r_p}^{(i_2 i_1)} \right) + \frac{1}{2} L L G_0^{(i_2)} \sum_{i_1} \hat{j}_{p+1, r_p}^{(i_2 i_1)} + G_0^{(i_2)} L G_0^{(i_1)} a \left( \hat{j}_{p+1, r_p}^{(i_2 i_1)} - \hat{i}_{p+1, r_p}^{(i_2 i_1)} + \Delta \left( \hat{i}_{p+1, r_p}^{(i_2 i_1)} - \hat{i}_{p+1, r_p}^{(i_2 i_1)} \right) \right) + L G_0^{(i_2)} L \sum_{i_1} \left( \hat{i}_{p+1, r_p}^{(i_2 i_1)} - \hat{i}_{p+1, r_p}^{(i_2 i_1)} + \frac{1}{2} \hat{i}_{p+1, r_p}^{(i_2 i_1)} - \frac{1}{2} \hat{i}_{p+1, r_p}^{(i_2 i_1)} \right) - L G_0^{(i_2)} a \left( \hat{i}_{p+1, r_p}^{(i_2 i_1)} + \hat{i}_{p+1, r_p}^{(i_2 i_1)} \right) \right] + \sum_{i_1, i_2, i_3, i_4=1}^{m} \left[ G_0^{(i_4)} G_0^{(i_3)} G_0^{(i_2)} \sum_{i_1} \left( \hat{i}_{p+1, r_p}^{(i_4 i_3 i_2 i_1)} - \hat{i}_{p+1, r_p}^{(i_4 i_3 i_2 i_1)} \right) + G_0^{(i_4)} L G_0^{(i_3)} \sum_{i_1} \left( \hat{i}_{p+1, r_p}^{(i_4 i_3 i_2 i_1)} - \hat{i}_{p+1, r_p}^{(i_4 i_3 i_2 i_1)} \right) - \Delta G_0^{(i_4)} a \left( \hat{i}_{p+1, r_p}^{(i_4 i_3 i_2 i_1)} + \hat{i}_{p+1, r_p}^{(i_4 i_3 i_2 i_1)} \right) \right] \]
where $\Delta = T/N$ ($N > 1$) is a constant (for simplicity) step of integration, $\tau_p = p\Delta$ ($p = 0, 1, \ldots, N$), $\hat{I}^{(i_1 \ldots i_k)}_{l_1 \ldots l_k}$ is an approximation of the iterated Ito stochastic integral

$$I^{(i_1 \ldots i_k)}_{l_1 \ldots l_k} = \sum_{i=1}^{n} a_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{j=1}^{m} \sum_{l_i=1}^{n} \Sigma_{ij}(x, t) \Sigma_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_j},$$

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} a_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{j=1}^{m} \sum_{l_i=1}^{n} \Sigma_{ij}(x, t) \Sigma_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_j}.$$

$l_1, \ldots, l_k = 0, 1, 2, \ldots, i_1, \ldots, i_k = 1, \ldots, m, k = 1, 2, \ldots$, are the columns of the matrix function $\Sigma$, and $\Sigma_{ij}$ is the $(i,j)$th component of the matrix function $\Sigma$, $a_i$ is the $i$th component of the vector function $a$, and $x_i$ is the $i$th component of the column vector $x$.

It is well known [2] that under the standard conditions the numerical scheme [4] has strong order of convergence 3.0. Among these conditions we consider only the condition for approximations of iterated Ito stochastic integrals from the numerical scheme [4] [2, 5]

$$M \left\{ \left( I^{(i_1 \ldots i_k)}_{l_1 \ldots l_k} - \hat{I}^{(i_1 \ldots i_k)}_{l_1 \ldots l_k} \right)^2 \right\} \leq C \Delta^7,$$

where $M$ is a constant.
where \( \hat{j}^{(i_1, \ldots, i_k)}_{t_{i_1} \ldots t_{i_k} \tau_{p+1} \tau_p} \) is an approximation of \( j^{(i_1, \ldots, i_k)}_{t_{i_1} \ldots t_{i_k} \tau_{p+1} \tau_p} \), constant \( C \) does not depend on \( \Delta \).

Note that if we exclude \( u_{p+1,p} + v_{p+1,p} \) from the right-hand side of (4), then we will have the explicit one-step strong numerical scheme of order 2.0. The right-hand side of (4) but without the value \( v_{p+1,p} \) define the explicit one-step strong numerical scheme of order 2.5.

Note that the truncated unified Taylor–Ito expansion \([5],[8],[10],[18]-[28],[54]-[56]\) contains the less number of various types of iterated Ito stochastic integrals (moreover, their major part will have less multiplicities) in comparison with the classical Taylor–Ito expansion \([2],[7]\).

Furthermore, some iterated Ito stochastic integrals from the Taylor–Ito expansion \([2],[7]\) are connected by linear relations. However, the iterated stochastic integrals from the unified Taylor–Ito expansion \([5],[8],[10],[18]-[28],[54]-[56]\) cannot be connected by linear relations. Therefore, we call these families of stochastic integrals from the unified Taylor–Ito expansion as the stochastic Fourier–Legendre Series \([5],[8],[10],[16],[18]-[28],[54]-[56]\) cannot be connected by linear relations. However, the iterated stochastic integrals from the unified Taylor–Ito expansion as the stochastic Fourier–Legendre Series \([5],[8],[10],[16],[18]-[28],[54]-[56]\) contains 29 different types of iterated Ito stochastic integrals. At the same time, the analogue of (4) based on the classical Taylor–Ito expansion \([2],[7]\) contains 29 different types of iterated Ito stochastic integrals.

3. APPROXIMATION OF ITERATED ITO STOCHASTIC INTEGRALS BASED ON MULTIPLE FOURIER–LEGENDRE SERIES

Suppose that every \( \psi_l(\tau) \) (\( l = 1, \ldots, k \)) is a non-random function from the space \( L_2([t,T]) \). Define the following function on the hypercube \([t,T]^k\)

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \ldots \psi_k(t_k) & \text{for } t_1 < \ldots < t_k \\
0 & \text{otherwise}
\end{cases}, \quad t_1, \ldots, t_k \in [t,T], \quad k \geq 2,
\]

and \( K(t_1) = \psi_1(t_1) \) for \( t_1 \in [t,T] \).

Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of functions in the space \( L_2([t,T]) \). The function \( K(t_1, \ldots, t_k) \) belongs to the space \( L_2([t,T]^k) \). At this situation it is well known that the generalized multiple Fourier series of \( K(t_1, \ldots, t_k) \in L_2([t,T]^k) \) is converging to \( K(t_1, \ldots, t_k) \) in the hypercube \([t,T]^k\) in the mean-square sense, i.e.

\[
\lim_{p_1, \ldots, p_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_1, \ldots, j_k} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right\|_{L_2([t,T]^k)} = 0,
\]

where

\[
C_{j_1, \ldots, j_k} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k,
\]

\[
\|f\| = \left( \int_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.
\]

Consider the partition \( \{\tau_j\}_{j=0}^{N} \) of \([t,T]\) such that

\[
t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \to 0 \text{ if } N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\]
Theorem 1 [5] (2006), [18]-[28], [31]-[56]. Suppose that every \( \psi_i(t) \) \((i = 1, \ldots, k)\) is a continuous non-random function on \([t, T]\) and \(\{\phi_j(x)\}_{j=0}^{\infty}\) is a complete orthonormal system of continuous functions in the space \(L_2([t, T])\). Then

\[
J[\psi^{(k)}_{i,j}]_{T,t} = \lim_{p_1,\ldots,p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k\ldots,j_1} \left( \prod_{l=1}^k \phi_{j_l}(\tau_{l_1}) \right)
\]

(10)

where \(J[\psi^{(k)}]_{T,t}\) is defined by (2),

\[
G_k = H_k \setminus L_k, \quad H_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1\},
\]

\[
L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1; l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},
\]

\(\lim\) is a limit in the mean-square sense, \(i_1, \ldots, i_k = 0, 1, \ldots, m\),

\[
\zeta_j^{(i)} = \int_t^T \phi_j(s) dw_s^{(i)}
\]

are independent standard Gaussian random variables for various \(i\) or \(j\) \((i \neq 0)\), \(C_{j_k\ldots,j_1}\) is the Fourier coefficient \(\mathfrak{s}_k\), \(\Delta w_{\tau_j}^{(i)} = w_{\tau_j}^{(i)} - w_{\tau_{j+1}}^{(i)}\) \((i = 0, 1, \ldots, m\), \(\{\tau_j\}_{j=0}^{N}\) is a partition of the interval \([t, T]\), which satisfies the condition (4).

The convergence in the mean of degree 2\(n\) \((n \in \mathbb{N})\) \([18]-[20], [22]-[28], [54]-[56]\) as well as the convergence with probability 1 \([38]-[40], [54]-[56]\) of approximations from Theorem 1 are proved. Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in \(L_2([t, T])\) can also be applied in Theorem 1 \([5], [18]-[28], [54]-[56]\). The modification of Theorem 1 for complete orthonormal with weight \(r(x) \geq 0\) systems of functions in the space \(L_2([t, T])\) can be found in \([20], [31]-[50]\). Application of Theorem 1 and Theorem 2 (see below) for the mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional \(Q\)-Wiener process can be found in the monographs \([54]-[56]\) (Chapter 7) and in \([30], [37], [38], [62]-[64]\).

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for \(k = 1, \ldots, 6\) \([5], [18]-[28], [31]-[56]\)

\[
J[\psi^{(1)}]_{T,t} = \lim_{p_1 \to \infty} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_j^{(1)}
\]

(12)

\[
J[\psi^{(2)}]_{T,t} = \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \left( \zeta_j^{(1)} \zeta_j^{(2)} - 1_{\{j_1 = j_2\neq 0\}} 1_{\{j_1 = j_2\}} \right)
\]

(13)

\[
J[\psi^{(3)}]_{T,t} = \lim_{p_1, p_2, p_3 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3j_2j_1} \left( \zeta_j^{(1)} \zeta_j^{(2)} \zeta_j^{(3)} - \zeta_j^{(3)} \zeta_j^{(1)} \zeta_j^{(2)} \right)
\]
\begin{align}
(14) \quad & -1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} \zeta_j^{(i_3)} - 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \zeta_j^{(i_1)} - 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} \zeta_j^{(i_2)} \\
& \quad \quad J_{[\psi(4)]}_{t, t} = \text{l.i.m.}_{p_1, \ldots, p_4 \to \infty} \sum_{j_0 = 0}^{p_1} \cdots \sum_{j_4 = 0}^{p_4} C_{j_4 \ldots j_1} \left( \prod_{l=1}^{4} \zeta_{j_l}^{(i_l)} - \\
& \quad -1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} \zeta_j^{(i_3)} \zeta_j^{(i_4)} - 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} \zeta_j^{(i_2)} \zeta_j^{(i_4)} - \right. \\
& \quad \left. -1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \zeta_j^{(i_1)} \zeta_j^{(i_4)} - 1_{\{i_2 = i_4 \neq 0\}} 1_{\{j_2 = j_4\}} \zeta_j^{(i_3)} \zeta_j^{(i_1)} + \\
& \quad +1_{\{i_4 = i_3 \neq 0\}} 1_{\{j_4 = j_3\}} 1_{\{i_4 = i_3\neq 0\}} 1_{\{j_4 = j_3\}} \right)
\end{align}

\begin{align}
(15) \quad & \quad + 1_{\{i_1 = i_4 \neq 0\}} 1_{\{j_1 = j_4\}} 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \\
& \quad J_{[\psi(5)]}_{t, t} = \text{l.i.m.}_{p_1, \ldots, p_5 \to \infty} \sum_{j_0 = 0}^{p_1} \cdots \sum_{j_5 = 0}^{p_5} C_{j_5 \ldots j_1} \left( \prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} - \\
& \quad -1_{\{i_1 = i_2 \neq 0\}} 1_{\{j_1 = j_2\}} \zeta_j^{(i_3)} \zeta_j^{(i_4)} \zeta_j^{(i_5)} - 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} \zeta_j^{(i_2)} \zeta_j^{(i_4)} \zeta_j^{(i_5)} - \right. \\
& \quad \left. -1_{\{i_1 = i_4 \neq 0\}} 1_{\{j_1 = j_4\}} \zeta_j^{(i_3)} \zeta_j^{(i_5)} - 1_{\{i_1 = i_5 \neq 0\}} 1_{\{j_1 = j_5\}} \zeta_j^{(i_2)} \zeta_j^{(i_4)} \zeta_j^{(i_5)} - \right. \\
& \quad \left. -1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} \zeta_j^{(i_1)} \zeta_j^{(i_5)} - 1_{\{i_2 = i_4 \neq 0\}} 1_{\{j_2 = j_4\}} \zeta_j^{(i_3)} \zeta_j^{(i_5)} - \right. \\
& \quad \left. -1_{\{i_2 = i_5 \neq 0\}} 1_{\{j_2 = j_5\}} \zeta_j^{(i_3)} \zeta_j^{(i_4)} - 1_{\{i_3 = i_4 \neq 0\}} 1_{\{j_3 = j_4\}} \zeta_j^{(i_1)} \zeta_j^{(i_5)} - \right. \\
& \quad \left. +1_{\{i_3 = i_5 \neq 0\}} 1_{\{j_3 = j_5\}} \zeta_j^{(i_1)} \zeta_j^{(i_4)} \zeta_j^{(i_5)} \\
& \quad \quad \quad \quad \quad \quad +1_{\{i_4 = i_5 \neq 0\}} 1_{\{j_4 = j_5\}} \zeta_j^{(i_1)} \zeta_j^{(i_3)} \zeta_j^{(i_5)} \\
& \quad \end{align}

\begin{align}
(16) \quad & \quad + 1_{\{i_2 = i_5 \neq 0\}} 1_{\{j_2 = j_5\}} 1_{\{i_1 = i_3 \neq 0\}} 1_{\{j_1 = j_3\}} \zeta_j^{(i_1)} \\
& \quad J_{[\psi(6)]}_{t, t} = \text{l.i.m.}_{p_1, \ldots, p_6 \to \infty} \sum_{j_0 = 0}^{p_1} \cdots \sum_{j_6 = 0}^{p_6} C_{j_6 \ldots j_1} \left( \prod_{l=1}^{6} \zeta_{j_l}^{(i_l)} - \\
& \quad -1_{\{i_1 = i_6 \neq 0\}} 1_{\{j_1 = j_6\}} \zeta_j^{(i_2)} \zeta_j^{(i_3)} \zeta_j^{(i_4)} \zeta_j^{(i_5)} \zeta_j^{(i_6)} - 1_{\{i_2 = i_6 \neq 0\}} 1_{\{j_2 = j_6\}} \zeta_j^{(i_1)} \zeta_j^{(i_3)} \zeta_j^{(i_4)} \zeta_j^{(i_5)} \zeta_j^{(i_6)} - \right. \\
& \quad \left. -1_{\{i_3 = i_6 \neq 0\}} 1_{\{j_3 = j_6\}} \zeta_j^{(i_1)} \zeta_j^{(i_2)} \zeta_j^{(i_4)} \zeta_j^{(i_5)} \zeta_j^{(i_6)} - 1_{\{i_4 = i_6 \neq 0\}} 1_{\{j_4 = j_6\}} \zeta_j^{(i_1)} \zeta_j^{(i_2)} \zeta_j^{(i_3)} \zeta_j^{(i_5)} \zeta_j^{(i_6)} - \right. \\
& \quad \left. +1_{\{i_5 = i_6 \neq 0\}} 1_{\{j_5 = j_6\}} \right)
\end{align}
\[ -1 \{i_6 = i_3 \neq 0\} \{j_6 = j_3\} \{i_4 = i_3 \neq 0\} \{j_3 = j_4\} \{i_2 = i_3 \neq 0\} \{j_2 = j_3\} - \\
-1 \{i_3 = i_6 \neq 0\} \{j_3 = j_6\} \{i_4 = i_3 \neq 0\} \{j_4 = j_2\} \{i_2 = i_3 \neq 0\} \{j_2 = j_3\} - \\
-1 \{i_6 = i_4 \neq 0\} \{j_6 = j_4\} \{i_2 = i_3 \neq 0\} \{j_1 = j_3\} \{i_1 = i_3 \neq 0\} \{j_2 = j_3\} - \\
-1 \{i_6 = i_3 \neq 0\} \{j_6 = j_3\} \{i_1 = i_3 \neq 0\} \{j_1 = j_2\} \{i_2 = i_3 \neq 0\} \{j_2 = j_3\} - \\
-1 \{i_6 = i_4 \neq 0\} \{j_6 = j_4\} \{i_1 = i_2 \neq 0\} \{j_1 = j_3\} \{i_3 = i_5 \neq 0\} \{j_3 = j_5\} - \\
-1 \{i_6 = i_5 \neq 0\} \{j_6 = j_5\} \{i_1 = i_4 \neq 0\} \{j_1 = j_2\} \{i_2 = i_3 \neq 0\} \{j_2 = j_3\} - \\
-1 \{i_6 = i_5 \neq 0\} \{j_6 = j_5\} \{i_1 = i_2 \neq 0\} \{j_1 = j_2\} \{i_3 = i_4 \neq 0\} \{j_3 = j_4\} - \\
-1 \{i_6 = i_5 \neq 0\} \{j_6 = j_5\} \{i_1 = i_3 \neq 0\} \{j_1 = j_3\} \{i_2 = i_4 \neq 0\} \{j_2 = j_4\} \]

where \(1_A\) is the indicator of the set \(A\).

Note that we will consider the case \(i_1, \ldots, i_6 = 1, \ldots, m\). This case corresponds to the numerical method \([4]\).

For further consideration, let us consider the generalization of formulas \([12]–[17]\) for the case of an arbitrary multiplicity \(k\) \((k \in \mathbb{N})\) of the iterated Ito stochastic integral \(J_{[t,T]}^{(k)}\) defined by \([2]\). In order to do this, let us introduce some notations. Consider the unordered set \(\{1, 2, \ldots, k\}\) and separate it into two parts: the first part consists of \(r\) unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining \(k - 2r\) numbers. So, we have

\[ (\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}, \{q_1, \ldots, q_{k-2r}\}) \]

where \(\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}\), braces mean an unordered set, and parentheses mean an ordered set.

We will say that \([18]\) is a partition and consider the sum with respect to all possible partitions

\[ \sum_{\{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}, \{q_1, \ldots, q_{k-2r}\}\}} a_{g_1 g_2 \ldots g_{2r-1} g_{2r} q_1 \ldots q_{k-2r}} \]

Below there are several examples of sums in the form \([19]\)

\[ \sum_{\{\{g_1, g_2\}\}} a_{g_1 g_2} = a_{12}, \]

\[ \sum_{\{\{g_1, g_2\}\}, \{g_1, q_4\}} a_{g_1 g_2 g_3 g_4} = a_{1234} + a_{1324} + a_{2314}, \]

\[ \sum_{\{\{g_1, q_2\}\}, \{g_1, q_4\}} a_{g_1 g_2 g_3 g_4} = \\
= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12}, \]
\[ \sum_{((s_1, g_2), (q_1, g_2, q_3)) \in \{1, 2, 3, 4, 5\}} a_{g_1, g_2, g_1, q_3} = \]
\[ = a_{12, 124} + a_{12, 124} + a_{14, 235} + a_{15, 254} + a_{23, 145} + a_{24, 135} + a_{25, 134} + a_{34, 125} + a_{35, 124} + a_{45, 123}, \]
\[ \sum_{((s_1, g_2), (q_1, g_4, q_1)) \in \{1, 2, 3, 4, 5\}} a_{g_1, g_2, g_4, q_1} = \]
\[ = a_{12, 34} + a_{12, 34} + a_{14, 23} + a_{15, 25} + a_{23, 14} + a_{24, 13} + a_{25, 13} + a_{34, 23} + a_{34, 23} + a_{45, 23} + a_{54, 23}. \]

Now we can write (10) as

\[ J[\psi^{(k)}]_{T, t} = \lim \sum_{p_1, \ldots, p_k \to \infty} \prod_{j_1=0}^{p_1} \cdots \prod_{j_k=0}^{p_k} C_{j_k \cdots j_1} \left( \prod_{l=1}^{k/2} \zeta_{j_l}^{(n_l)} + \sum_{r=1}^{[k/2]} (-1)^r \right) \times \]
\[ \sum_{(\{(s_1, g_2), (q_1, g_4, q_1)\}, (q_1)) \in \{1, 2, 3, 4, 5\}} \prod_{s=1}^{r} 1_{i_{q_2-s-1} = i_{q_2}} 1_{j_{q_2-s-1} = j_{q_2}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(n_{q_l})}, \]

where \([x]\) is an integer part of a real number; another notations are the same as in Theorem 1.

In particular, from (20) for \(k = 5\) we obtain

\[ J[\psi^{(5)}]_{T, t} = \lim \sum_{p_1, \ldots, p_5 \to \infty} \prod_{j_1=0}^{p_1} \cdots \prod_{j_5=0}^{p_5} C_{j_5 \cdots j_1} \left( \prod_{l=1}^{5} \zeta_{j_l}^{(n_l)} - \right) \]
\[ - \sum_{(\{(s_1, g_2), (q_1, g_4, q_1)\}, (q_1)) \in \{1, 2, 3, 4, 5\}} 1_{i_{q_1} = i_{q_2} \neq 0} 1_{j_{q_1} = j_{q_2}} \prod_{l=1}^{3} \zeta_{j_{q_l}}^{(n_{q_l})} + \]
\[ + \sum_{(\{(s_1, g_2), (q_1, g_4, q_1)\}, (q_1)) \in \{1, 2, 3, 4, 5\}} 1_{i_{q_1} = i_{q_2} \neq 0} 1_{j_{q_1} = j_{q_2}} 1_{i_{q_3} = i_{q_4} \neq 0} 1_{j_{q_3} = j_{q_4}} \zeta_{j_{q_1}}^{(n_{q_1})}, \]

The last equality obviously agrees with (16).

Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space \(L_2([t, T])\) and \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])\).

**Theorem 2** [54] (Sect. 1.11), [38] (Sect. 1.5). Suppose that \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])\) and \(\{\phi_j(x)\}_{j=0}^{\infty}\) is an arbitrary complete orthonormal system of functions in the space \(L_2([t, T])\). Then the following expansion
\[ J[\psi^{(k)}]_{T,t} = \lim_{p_1,\ldots,p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \cdots j_1} \left( \prod_{i=1}^{k} \zeta^{(i)}_{j_i} \right) \sum_{r=1}^{[k/2]} (-1)^r \times \]

converging in the mean-square sense is valid, where \([x]\) is an integer part of a real number \(x\); another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in [70]. Note that we use another notations [54] (Sect. 1.11), [38] (Sect. 15) in comparison with [70]. Moreover, the proof of an analogue of Theorem 2 from [70] is somewhat different from the proof given in [54] (Sect. 1.11), [38] (Sect. 15).

Let us consider the exact calculation and effective estimation of the mean-square error of approximation \( J[\psi^{(k)}]_{T,t}^q \). Here \( J[\psi^{(k)}]_{T,t}^q \) is the expression on the right-hand side of (21) before passing to the limit \( \lim_{p_1,\ldots,p_k \to \infty} \) for the case \( p_1 = \ldots = p_k = q \)

\[ J[\psi^{(k)}]_{T,t}^q = \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k \cdots j_1} \left( \prod_{i=1}^{k} \zeta^{(i)}_{j_i} \right) \sum_{r=1}^{[k/2]} (-1)^r \times \]

\[ \times \sum_{\substack{(q_1, q_2, \ldots, q_{2r-1}, q_{2r-1}, q_1, q_2, \ldots, q_{2r}) = (1,2,\ldots,k) \}} \prod_{s=1}^{r} \mathbf{1}_{i_{q_2s-1} = i_{q_2s} \neq 0} \mathbf{1}_{j_{q_2s-1} = j_{q_2s}} \prod_{t=1}^{k-2r} \zeta^{(i_{q_t})}_{j_{q_t}} \]

where \([x]\) is an integer part of a real number \(x\); another notations are the same as in Theorems 1, 2.

Let us denote

\[ \mathbb{M}\left( J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^q \right)^2 \stackrel{\text{def}}{=} E_k^q, \quad \int_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k \stackrel{\text{def}}{=} I_k. \]

In [19], [20], [31], [39], [54]–[56] it was shown that

\[ E_k^q \leq k! \left( I_k - \sum_{j_1,\ldots,j_k=0}^{q} C_{j_k \cdots j_1}^2 \right), \]

where \(i_1,\ldots,i_k = 1,\ldots,m\) \((T-t \in (0, +\infty))\) or \(i_1,\ldots,i_k = 0,1,\ldots,m\) \((T-t \in (0,1))\).

The value \(E_k^q\) can be calculated exactly.

**Theorem 3** [54] (Sect. 1.12), [39] (Sect. 6). Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \(L_2([t,T])\) and \(\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T]), i_1,\ldots,i_k = 1,\ldots,m\). Then
where \( i_1, \ldots, i_k = 1, \ldots, m; \) the expression

\[
\sum_{(j_1, \ldots, j_k)}
\]

means the sum with respect to all possible permutations \((j_1, \ldots, j_k)\). At the same time if \( j_r \) swapped with \( j_q \) in the permutation \((j_1, \ldots, j_k)\), then \( i_r \) swapped with \( i_q \) in the permutation \((i_1, \ldots, i_k)\); another notations are the same as in Theorems 1, 2.

Note that

\[
M \left\{ J[\psi^{(k)}]_{T,t} \sum_{(j_1, \ldots, j_k)} \int_t^T \phi_{j_k}(t_k) \cdots \int_t^{t_2} \phi_{j_1}(t_1) df^{(i_1)}_{t_1} \cdots df^{(i_k)}_{t_k} \right\} = C_{j_k \ldots j_1}.
\]

Then from Theorem 3 for pairwise different \( i_1, \ldots, i_k \) and for \( i_1 = \ldots = i_k \) we obtain [20, 31, 39, 54–56]

\[
E^q_k = I_k - \sum_{j_1, \ldots, j_k=0}^{q} C_{j_k \ldots j_1}^2
\]

\[
E^q_k = I_k - \sum_{j_1, \ldots, j_k=0}^{q} C_{j_k \ldots j_1} \left( \sum_{(j_1, \ldots, j_k)} C_{j_k \ldots j_1} \right),
\]

where

\[
\sum_{(j_1, \ldots, j_k)}
\]

is a sum with respect to all possible permutations \((j_1, \ldots, j_k)\).

Consider some examples [20, 31, 39, 54–56] of application of Theorem 3 \((i_1, \ldots, i_5 = 1, \ldots, m)\)

\[
E^q_2 = I_2 - \sum_{j_1, j_2=0}^{q} C_{j_2 j_1}^2 - \sum_{j_1, j_2=0}^{q} C_{j_2 j_1} C_{j_1 j_2} \quad (i_1 = i_2),
\]

\[
E^q_3 = I_3 - \sum_{j_3, j_2, j_1=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^{q} C_{j_3 j_2 j_1} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3),
\]

\[
E^q_3 = I_3 - \sum_{j_3, j_2, j_1=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^{q} C_{j_2 j_3 j_1} C_{j_3 j_2 j_1} \quad (i_1 \neq i_2 = i_3),
\]

\[
E^q_3 = I_3 - \sum_{j_3, j_2, j_1=0}^{q} C_{j_3 j_2 j_1}^2 - \sum_{j_3, j_2, j_1=0}^{q} C_{j_3 j_2 j_1} C_{j_1 j_3 j_2} \quad (i_1 = i_3 \neq i_2),
\]
\[ E_4^q = I_4 - \sum_{j_1, \ldots, j_4 = 0}^q C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_2 \neq i_3, i_4; \ i_3 \neq i_4), \]

\[ E_4^q = I_4 - \sum_{j_1, \ldots, j_4 = 0}^q C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_3 \neq i_2, i_4), \]

\[ E_4^q = I_4 - \sum_{j_1, \ldots, j_4 = 0}^q C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_2 = i_3 \neq i_1), \]

\[ E_4^q = I_4 - \sum_{j_1, \ldots, j_4 = 0}^q C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_3 \neq i_2, i_4), \]

\[ E_4^q = I_4 - \sum_{j_1, \ldots, j_4 = 0}^q C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_2 \neq i_3 = i_4), \]

\[ E_4^q = I_4 - \sum_{j_1, \ldots, j_4 = 0}^q C_{j_4 \ldots j_1} \left( \sum_{(j_1, j_2, j_3)} C_{j_4 \ldots j_1} \right) \quad (i_1 = i_2 \neq i_3 = i_4), \]

The values \( E_4^q \) and \( E_5^q \) were calculated exactly for all possible combinations of \( i_1, \ldots, i_5 = 1, \ldots, m \) in \([19], [20], [39], [54]-[56]\).

Let us consider the approximations of iterated Ito stochastic integrals from \([4]\) using \([12]-[17]\) and complete orthonormal system of Legendre polynomials in the space \( L_2(\tau_p, \tau_{p+1}) \) \( (\tau_p = p\Delta, N\Delta = T, \ p = 0, 1, \ldots, N) \) \([20]\) (also see \([15]-[19], [21]-[56]\))

\[
I_{0_p, \tau_{p+1}, \tau_p}^{(i_1)} = \sqrt{\Delta} \zeta_0^{(i_1)},
\]

\[
I_{0_p, \tau_{p+1}, \tau_p}^{(i_1, i_2)} = \frac{\Delta}{2} \left( \zeta_0^{(i_1)} \zeta_0^{(i_2)} + \sum_{i=1}^q \frac{1}{\sqrt{i^2 - 1}} \left( \zeta_{i-1}^{(i_1)} \zeta_{i-1}^{(i_2)} - \zeta_i^{(i_1)} \zeta_i^{(i_2)} \right) - \mathbf{1}_{\{i_1 = i_2\}} \right),
\]
\[ I_{1,p+1..p}^{(i_1)} = -\frac{\Delta^{3/2}}{2} \left( \psi_{0}^{(i_1)} + \frac{1}{\sqrt{3}} \psi_{1}^{(i_1)} \right) , \]
\[ I_{00}^{(i_1+i_2)} = \sum_{j_1,j_2,j_3 = 0}^{q} C_{j_1,j_2,j_3} \left( \psi_{1}^{(i_1+i_2)} \psi_{2}^{(i_2)} - 1_{\{i_1+i_2\}} 1_{\{j_1+j_2\}} \psi_{3}^{(i_3)} - 1_{\{i_2+i_3\}} 1_{\{j_2+j_3\}} \psi_{3}^{(i_3)} \right) , \]
\[ I_{000}^{(i_1+i_2)} = \sum_{j_1,j_2,j_3,j_4 = 0}^{q} C_{j_1,j_2,j_3,j_4} \left( \prod_{l=1}^{4} \psi^{(i_l)} - 1_{\{i_1+i_2\}} \psi_{3}^{(i_3)} - 1_{\{i_1+i_2\}} \psi_{3}^{(i_3)} + 1_{\{i_1+i_2\}} \psi_{3}^{(i_3)} \psi_{3}^{(i_3)} + \psi_{3}^{(i_3)} \right) , \]
\[ I_{01}^{(i_1+i_2)} = -\frac{\Delta}{2} I_{00}^{(i_1+i_2)} - \frac{\Delta^2}{4} \left( \frac{1}{\sqrt{3}} \psi_{0}^{(i_1+i_2)} + \frac{(i+2)\psi_{1}^{(i_1+i_2)} - (i+1)\psi_{1}^{(i_1)} \psi_{1}^{(i_2)} - \frac{i \psi_{1}^{(i_1+i_2)} \psi_{1}^{(i_2)}}{(2i+1)(2i+5)(2i+3)} \right) , \]
\[ I_{10}^{(i_1+i_2)} = -\frac{\Delta}{2} I_{00}^{(i_1+i_2)} - \frac{\Delta^2}{4} \left( \frac{1}{\sqrt{3}} \psi_{0}^{(i_1+i_2)} + \frac{(i+1)\psi_{1}^{(i_1)} \psi_{1}^{(i_2)}}{(2i+1)(2i+5)(2i+3)} + \frac{i \psi_{1}^{(i_1+i_2)} \psi_{1}^{(i_2)}}{(2i-1)(2i+3)} \right) , \]
\[ I_{2}^{(i_1+i_2)} = \frac{\Delta^5/2}{3} \left( \psi_{0}^{(i_1+i_2)} + \frac{\sqrt{3}}{2} \psi_{1}^{(i_1+i_2)} + \frac{1}{2\sqrt{2}} \psi_{2}^{(i_1+i_2)} \right) , \]
\[ I_{001}^{(i_1+i_2)} = \sum_{j_1,j_2,j_3 = 0}^{q} C_{j_1,j_2,j_3} \left( \psi_{1}^{(i_1+i_2)} \psi_{2}^{(i_2)} - 1_{\{i_1+i_2\}} 1_{\{j_1+j_2\}} \psi_{3}^{(i_3)} - 1_{\{i_2+i_3\}} 1_{\{j_2+j_3\}} \psi_{3}^{(i_3)} - 1_{\{i_1+i_3\}} 1_{\{j_1+j_3\}} \psi_{3}^{(i_3)} \right) , \]
\[
I_{010}^{(i_1,i_2,i_3)} q = \sum_{j_1,j_2,j_3=0}^{q} C_{010}^{j_1,j_2,j_3} \left( \phi^{(i_1)} j_{j_1} \phi^{(i_2)} j_{j_2} \phi^{(i_3)} j_{j_3} - 1_{\{i_1=i_2\}} 1_{\{j_1=j_2\}} \phi^{(i_3)} j_{j_3} - 1_{\{i_2=i_3\}} 1_{\{j_2=j_3\}} \phi^{(i_1)} j_{j_1} \right),
\]

\[
I_{100}^{(i_1,i_2,i_3)} q = \sum_{j_1,j_2,j_3=0}^{q} C_{100}^{j_1,j_2,j_3} \left( \phi^{(i_1)} j_{j_1} \phi^{(i_2)} j_{j_2} \phi^{(i_3)} j_{j_3} - 1_{\{i_1=i_2\}} 1_{\{j_1=j_2\}} \phi^{(i_3)} j_{j_3} - 1_{\{i_1=i_3\}} 1_{\{j_1=j_3\}} \phi^{(i_2)} j_{j_2} \right),
\]

\[
I_{000}^{(i_1,i_2,i_3)} q = \sum_{j_1,j_2,j_3=0}^{q} C_{j_1,j_2,j_3}^{j_1,j_2,j_3} \left( \prod_{i=1}^{5} \phi^{(i)} j_{j_i} - 1_{\{j_1=j_2\}} 1_{\{i_1=i_2\}} 1_{\{j_1=j_3\}} 1_{\{i_3=i_4\}} 1_{\{j_3=j_4\}} 1_{\{i_4=i_5\}} \phi^{(i_5)} j_{j_5} - 1_{\{j_1=j_3\}} 1_{\{i_1=i_3\}} 1_{\{j_1=j_4\}} 1_{\{i_2=i_5\}} 1_{\{j_3=j_5\}} 1_{\{i_4=i_5\}} \phi^{(i_4)} j_{j_4} - 1_{\{j_2=j_3\}} 1_{\{i_2=i_3\}} 1_{\{j_2=j_4\}} 1_{\{i_3=i_5\}} 1_{\{j_4=j_5\}} 1_{\{i_5=i_5\}} \phi^{(i_2)} j_{j_2} - 1_{\{j_2=j_4\}} 1_{\{i_2=i_3\}} 1_{\{j_2=j_3\}} 1_{\{i_3=i_5\}} 1_{\{j_4=j_5\}} 1_{\{i_5=i_5\}} \phi^{(i_1)} j_{j_1} + \sum_{i=0}^{q} \left( (i+2)(i+3) \frac{\phi^{(i)} j_{j_1} \phi^{(i)} j_{j_2} \phi^{(i)} j_{j_3} \phi^{(i)} j_{j_4} \phi^{(i)} j_{j_5}}{\sqrt{(2i+1)(2i+7)(2i+3)(2i+5)}} \right) \right),
\]

\[
I_{021}^{(i_1,i_2)} q = -\frac{\Delta^2}{4} I_{001}^{(i_1,i_2)} q - \Delta I_{011}^{(i_1,i_2)} q + \Delta^3 \left[ \frac{2}{3\sqrt{5} \phi^{(i_2)} j_{j_0}} \phi^{(i_1)} j_{j_1} \phi^{(i_2)} j_{j_2} \phi^{(i_3)} j_{j_3} \phi^{(i_4)} j_{j_4} \phi^{(i_5)} j_{j_5} \phi^{(i_6)} j_{j_6} \right],
\]

\[
1_{\{i_2=i_3\}} 1_{\{j_2=j_3\}} 1_{\{i_3=i_4\}} \phi^{(i_1)} j_{j_1} \phi^{(i_2)} j_{j_2} \phi^{(i_3)} j_{j_3} \phi^{(i_4)} j_{j_4} \phi^{(i_5)} j_{j_5} \phi^{(i_6)} j_{j_6},
\]

\[
t^2 + 3 \phi^{(i_2)} j_{j_0} \phi^{(i_1)} j_{j_1} \phi^{(i_2)} j_{j_2} \phi^{(i_3)} j_{j_3} \phi^{(i_4)} j_{j_4} \phi^{(i_5)} j_{j_5} \phi^{(i_6)} j_{j_6} - \frac{(i+2)(i+3)}{\sqrt{(2i+1)(2i+7)(2i+3)(2i+5)}} \phi^{(i_2)} j_{j_0} \phi^{(i_2)} j_{j_2} \phi^{(i_3)} j_{j_3} \phi^{(i_4)} j_{j_4} \phi^{(i_5)} j_{j_5} \phi^{(i_6)} j_{j_6} + \frac{(i^2 + 3) \phi^{(i_2)} j_{j_0} \phi^{(i_1)} j_{j_1} \phi^{(i_2)} j_{j_2} \phi^{(i_3)} j_{j_3} \phi^{(i_4)} j_{j_4} \phi^{(i_5)} j_{j_5} \phi^{(i_6)} j_{j_6}}{\sqrt{(2i+1)(2i+7)(2i+3)(2i+5)}} \right) - \frac{1}{24} 1_{\{i_1=i_2\}} \Delta^4,
\]

(30)
\[ I_{20r_{p+1}}^{(i_{1}i_{2})q} = -\frac{\Delta^2}{4} I_{00r_{p+1}}^{(i_{1}i_{2})q} - \Delta I_{10r_{p+1}}^{(i_{1}i_{2})q} + \frac{\Delta^3}{8} \left[ \frac{2}{\sqrt{3}} \sum_{i=0}^{q} \left( (i + 1)(i + 2) - (i + 2)(i + 3) \right) \frac{\zeta_{i+3}^{(i_{1})}}{\zeta_{i+1}^{(i_{2})}} + \right. \\
\left. + \frac{1}{3} \sum_{i=0}^{q} \left( (i + 1)(i + 2) - (i + 2)(i + 3) \right) \frac{\zeta_{i+3}^{(i_{1})}}{\zeta_{i+1}^{(i_{2})}} \right] - \frac{1}{24} I_{(i_{1}=i_{2})}^{(i_{1}i_{2})} \Delta^3, \]

\[ I_{11r_{p+1}}^{(i_{1}i_{2})q} = -\frac{\Delta^2}{4} I_{00r_{p+1}}^{(i_{2}i_{1})q} - \Delta \left( I_{01r_{p+1}}^{(i_{1}i_{2})q} + I_{10r_{p+1}}^{(i_{1}i_{2})q} \right) + \frac{\Delta^3}{8} \left[ \frac{1}{3} \sum_{i=1}^{q} \left( (i + 1)(i + 2) - (i + 2)(i + 3) \right) \frac{\zeta_{i+3}^{(i_{1})}}{\zeta_{i+1}^{(i_{2})}} + \right. \\
\left. + \frac{1}{24} I_{(i_{1}=i_{2})}^{(i_{1}i_{2})} \Delta^3, \right. \]
\[ I_{01000,p+1}^{(i_1, i_2, i_3, i_4)} = \sum_{j_1, j_2, j_3, j_4=0}^q C_{01000} \left( \prod_{l=1}^{4} \gamma_{j_l} \right) - 1_{\{i_1 = i_2\}} 1_{\{j_1 = j_2\}} \gamma_{j_3} \gamma_{j_4} - 1_{\{i_1 = i_3\}} 1_{\{j_1 = j_3\}} \gamma_{j_2} \gamma_{j_4} - 1_{\{i_1 = i_4\}} 1_{\{j_1 = j_4\}} \gamma_{j_2} \gamma_{j_3} - 1_{\{i_2 = i_4\}} 1_{\{j_2 = j_4\}} \gamma_{j_1} \gamma_{j_3} + 1_{\{i_1 = i_2\}} 1_{\{j_1 = j_2\}} 1_{\{i_3 = i_4\}} 1_{\{j_3 = j_4\}} + 1_{\{i_1 = i_3\}} 1_{\{j_1 = j_3\}} 1_{\{i_2 = i_4\}} 1_{\{j_2 = j_4\}} + 1_{\{i_1 = i_4\}} 1_{\{j_1 = j_4\}} 1_{\{i_2 = i_3\}} 1_{\{j_2 = j_3\}} \right), \]

\[ I_{1000,p+1}^{(i_1, i_2, i_3, i_4)} = \sum_{j_1, j_2, j_3, j_4=0}^q C_{1000} \left( \prod_{l=1}^{4} \gamma_{j_l} \right) - 1_{\{i_1 = i_2\}} 1_{\{j_1 = j_2\}} \gamma_{j_3} \gamma_{j_4} - 1_{\{i_1 = i_3\}} 1_{\{j_1 = j_3\}} \gamma_{j_2} \gamma_{j_4} - 1_{\{i_1 = i_4\}} 1_{\{j_1 = j_4\}} \gamma_{j_2} \gamma_{j_3} - 1_{\{i_2 = i_4\}} 1_{\{j_2 = j_4\}} \gamma_{j_1} \gamma_{j_3} + 1_{\{i_1 = i_2\}} 1_{\{j_1 = j_2\}} 1_{\{i_3 = i_4\}} 1_{\{j_3 = j_4\}} + 1_{\{i_1 = i_3\}} 1_{\{j_1 = j_3\}} 1_{\{i_2 = i_4\}} 1_{\{j_2 = j_4\}} + 1_{\{i_1 = i_4\}} 1_{\{j_1 = j_4\}} 1_{\{i_2 = i_3\}} 1_{\{j_2 = j_3\}} \right), \]

\[ I_{1000000,p+1}^{(i_1, i_2, i_3, i_4, i_5)} = \sum_{j_1, j_2, j_3, j_4, j_5, j_6=0}^q C_{1000000} \left( \prod_{l=1}^{6} \gamma_{j_l} \right) - 1_{\{j_1 = j_6\}} 1_{\{i_1 = i_6\}} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \gamma_{j_5} - 1_{\{j_2 = j_6\}} 1_{\{i_2 = i_6\}} \gamma_{j_1} \gamma_{j_3} \gamma_{j_4} \gamma_{j_5} - 1_{\{j_3 = j_6\}} 1_{\{i_3 = i_6\}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_4} \gamma_{j_5} - 1_{\{j_4 = j_6\}} 1_{\{i_4 = i_6\}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_5} - 1_{\{j_5 = j_6\}} 1_{\{i_5 = i_6\}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} - 1_{\{j_6 = j_6\}} 1_{\{i_6 = i_6\}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} + 1_{\{i_1 = i_2\}} 1_{\{j_1 = j_2\}} 1_{\{i_3 = i_4\}} 1_{\{j_3 = j_4\}} + 1_{\{i_1 = i_3\}} 1_{\{j_1 = j_3\}} 1_{\{i_2 = i_4\}} 1_{\{j_2 = j_4\}} + 1_{\{i_1 = i_4\}} 1_{\{j_1 = j_4\}} 1_{\{i_2 = i_3\}} 1_{\{j_2 = j_3\}} + 1_{\{i_1 = i_6\}} 1_{\{j_1 = j_6\}} 1_{\{i_2 = i_5\}} 1_{\{j_2 = j_5\}} + 1_{\{i_1 = i_5\}} 1_{\{j_1 = j_5\}} 1_{\{i_2 = i_6\}} 1_{\{j_2 = j_6\}} \right). \]
\[ C_{j_3 j_2 j_1} = \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{z} \int_{\tau_p}^{y} \phi_{j_3} (x) \phi_{j_2} (y) \phi_{j_1} (x) dx dy dz = \]

where
\[
C_{j_4 j_3 j_2 j_1} = \frac{\tau_{p+1}}{\tau_{p}} \int_{\tau_{p}}^{u} \int_{\tau_{p}}^{z} \int_{\tau_{p}}^{y} \phi_{j_4}(u) \phi_{j_3}(z) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz du = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)}}{8} \Delta^{3/2} C_{j_4 j_3 j_2 j_1},
\]

\[
C^{001}_{j_3 j_2 j_1} = \frac{\tau_{p+1}}{\tau_{p}} \int_{\tau_{p}}^{u} \int_{\tau_{p}}^{z} \phi_{j_3}(z) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}{16} \Delta^{5/2} C^{001}_{j_3 j_2 j_1},
\]

\[
C^{010}_{j_3 j_2 j_1} = \frac{\tau_{p+1}}{\tau_{p}} \int_{\tau_{p}}^{u} \int_{\tau_{p}}^{z} (\tau_p - y) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}{16} \Delta^{5/2} C^{010}_{j_3 j_2 j_1},
\]

\[
C^{100}_{j_3 j_2 j_1} = \frac{\tau_{p+1}}{\tau_{p}} \int_{\tau_{p}}^{u} \int_{\tau_{p}}^{z} \phi_{j_3}(z) \phi_{j_2}((\tau_p - y) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}{16} \Delta^{5/2} C^{100}_{j_3 j_2 j_1},
\]

\[
C_{j_4 j_3 j_2 j_1} = \frac{\tau_{p+1}}{\tau_{p}} \int_{\tau_{p}}^{u} \int_{\tau_{p}}^{z} \int_{\tau_{p}}^{y} \phi_{j_4}(u) \phi_{j_3}(z) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz du = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1)}}{32} \Delta^{5/2} C_{j_4 j_3 j_2 j_1},
\]

\[
C^{0001}_{j_4 j_3 j_2 j_1} = \frac{\tau_{p+1}}{\tau_{p}} \int_{\tau_{p}}^{u} \int_{\tau_{p}}^{z} \phi_{j_4}(u) \phi_{j_3}(z) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz du = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)}}{32} \Delta^{3} C^{0001}_{j_4 j_3 j_2 j_1},
\]

\[
C^{0010}_{j_3 j_2 j_1} = \frac{\tau_{p+1}}{\tau_{p}} \int_{\tau_{p}}^{u} \int_{\tau_{p}}^{z} \phi_{j_4}(u) \phi_{j_3}(z) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz du = \]

where

\[
C_{j_4j_3j_2j_1}^{0100} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \phi_{j_4}(u) \phi_{j_3}(z) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz du = \
\frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)}}{32} \Delta^3 \tilde{C}_{j_4j_3j_2j_1}^{0100},
\]

\[
C_{j_4j_3j_2j_1}^{1000} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \phi_{j_4}(u) \phi_{j_3}(z) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz du = \
\frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)}}{32} \Delta^3 \tilde{C}_{j_4j_3j_2j_1}^{1000},
\]

\[
C_{j_6j_5j_4j_3j_2j_1} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \phi_{j_6}(w) \phi_{j_5}(v) \phi_{j_4}(u) \phi_{j_3}(z) \phi_{j_2}(y) \phi_{j_1}(x) dx dy dz du dv dw = \
\frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(2j_5 + 1)(2j_6 + 1)}}{64} \Delta^3 \tilde{C}_{j_6j_5j_4j_3j_2j_1},
\]
where $P_i(x)$ $(i = 0, 1, 2, \ldots)$ is the Legendre polynomial and

$$\phi_i(x) = \sqrt{\frac{2i + 1}{\Delta}} P_i \left( \left( x - \tau_p - \frac{\Delta}{2} \right) \frac{2}{\Delta} \right), \quad i = 0, 1, 2, \ldots$$

Let us consider the exact relations and some estimates for the mean-square errors of approximations of iterated Ito stochastic integrals.

Using Theorem 3, we get \[13\]–\[28\], \[39\] (also see \[5\], \[15\], \[16\], \[29\]–\[37\], \[54\]–\[56\])

\[
M \left\{ \left( \Phi_{10}^{(i_1, i_2)} - \Phi_{10}^{(i_1, i_2)q} \right)^2 \right\} = \frac{\Delta^2}{2} \left( \frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2 - 1} \right) \quad (i_1 \neq i_2),
\]

\[
M \left\{ \left( \Phi_{10}^{(i_1, i_2)} - \Phi_{10}^{(i_1, i_2)q} \right)^2 \right\} = M \left\{ \left( \Phi_{01}^{(i_1, i_2)} - \Phi_{01}^{(i_1, i_2)q} \right)^2 \right\} = \frac{\Delta^4}{16} \left( \frac{5}{9} - 2 \sum_{i=2}^{q} \frac{1}{4i^2 - 1} - \sum_{i=1}^{q} \frac{1}{(2i - 1)^2(2i + 3)^2} - \sum_{i=0}^{q} \frac{(i + 2)^2 + (i + 1)^2}{(2i + 1)(2i + 5)(2i + 3)^2} \right) \quad (i_1 \neq i_2),
\]

\[
M \left\{ \left( \Phi_{10}^{(i_1, i_1)} - \Phi_{10}^{(i_1, i_1)q} \right)^2 \right\} = M \left\{ \left( \Phi_{01}^{(i_1, i_1)} - \Phi_{01}^{(i_1, i_1)q} \right)^2 \right\} = \frac{\Delta^4}{16} \left( \frac{1}{9} - \sum_{i=0}^{q} \frac{1}{(2i + 1)(2i + 5)(2i + 3)^2} - 2 \sum_{i=1}^{q} \frac{1}{(2i - 1)^2(2i + 3)^2} \right).
\]

Applying \[24\], \[25\]–\[28\], we obtain
where

\[ C_{j_2,j_1}^{20} = \int_{\tau_p}^{\tau_{p+1}} \phi_{j_2}(y) \int_{\tau_p}^{y} \phi_{j_1}(x)(\tau_p - x)^2 dx dy = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)}}{16} \Delta^3 C_{j_2,j_1}^{20}, \]

\[ C_{j_2,j_1}^{02} = \int_{\tau_p}^{\tau_{p+1}} \phi_{j_2}(y)(\tau_p - y)^2 \int_{\tau_p}^{y} \phi_{j_1}(x) dx dy = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)}}{16} \Delta^3 C_{j_2,j_1}^{02}. \]
\[ C_{j_2j_1}^{11} = \int_{\tau_p}^{\tau_{p+1}} \phi_{j_2}(y)(\tau_p - y) \int_{\tau_p}^{y} \phi_{j_1}(x)(\tau_p - x) dx dy = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)}}{16} \Delta_3 C_{j_2j_1}^{11}, \]

\[ \tilde{C}_{j_2j_1}^{20} = \int_{-1}^{1} P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x)(x + 1)^2 dx dy, \]

\[ \tilde{C}_{j_2j_1}^{02} = \int_{-1}^{1} P_{j_2}(y)(y + 1)^2 \int_{-1}^{y} P_{j_1}(x) dx dy, \]

\[ \tilde{C}_{j_2j_1}^{11} = \int_{-1}^{1} P_{j_2}(y)(y + 1) \int_{-1}^{y} P_{j_1}(x)(x + 1) dx dy, \]

where \( P_i(x) \ (i = 0, 1, 2, \ldots) \) is the Legendre polynomial and

\[ \phi_i(x) = \sqrt{\frac{2i + 1}{\Delta}} P_i \left( \left( x - \frac{\alpha}{2} \right) \frac{2}{\Delta} \right), \quad i = 0, 1, 2, \ldots \]

At the same time using the estimate (22) for \( i_1, \ldots, i_6 = 1, \ldots, m \), we get

\[ M \left\{ \left( \int \left( I_{10}^{01} - I_{01}^{10} \right) \right)^2 \right\} \leq \frac{\Delta^4}{4} - \sum_{j_1,j_2=0}^{q} \left( C_{j_2j_1}^{01} \right)^2, \]

\[ M \left\{ \left( \int \left( I_{10}^{01} - I_{01}^{10} \right) \right)^2 \right\} \leq \frac{\Delta^4}{12} - \sum_{j_1,j_2=0}^{q} \left( C_{j_2j_1}^{10} \right)^2, \]

\[ M \left\{ \left( \int \left( I_{10}^{01} - I_{01}^{10} \right) \right)^2 \right\} \leq \frac{\Delta^3}{6} - \sum_{j_1,j_2=0}^{q} \left( C_{j_2j_1}^{01} \right)^2, \]

\[ M \left\{ \left( \int \left( I_{10}^{01} - I_{01}^{10} \right) \right)^2 \right\} \leq \frac{\Delta^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_3j_2j_1}^{12}, \]

\[ M \left\{ \left( \int \left( I_{10}^{01} - I_{01}^{10} \right) \right)^2 \right\} \leq \frac{\Delta^5}{60} - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}^{100}, \]

\[ M \left\{ \left( \int \left( I_{10}^{01} - I_{01}^{10} \right) \right)^2 \right\} \leq \frac{\Delta^5}{20} - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}^{100}, \]

\[ M \left\{ \left( \int \left( I_{10}^{01} - I_{01}^{10} \right) \right)^2 \right\} \leq \frac{\Delta^5}{10} - \sum_{j_1,j_2,j_3=0}^{q} C_{j_3j_2j_1}^{100}, \]
The Fourier–Legendre coefficients

\[ C_{j_3,j_2,j_1}, \tilde{C}_{j_3,j_2,j_1}, \tilde{C}^{001}_{j_3,j_2,j_1}, \tilde{C}^{010}_{j_3,j_2,j_1}, \tilde{C}^{010}_{j_3,j_2,j_1}, \tilde{C}^{100}_{j_3,j_2,j_1}, \tilde{C}^{001}_{j_3,j_2,j_1}, \tilde{C}^{001}_{j_3,j_2,j_1}, \tilde{C}^{100}_{j_3,j_2,j_1}, \tilde{C}^{100}_{j_3,j_2,j_1}, \tilde{C}^{100}_{j_3,j_2,j_1} \]

can be calculated exactly before start of the numerical method \([41]\) using DERIVE or MAPLE (computer algebra systems). In \([5, 18, 28, 40, 54-56]\) several tables with these coefficients can be found. Note that the mentioned Fourier–Legendre coefficients are independent of the integration step \(\tau_{p+1} - \tau_p\) of the numerical scheme, which can be not a constant in a general case.

Note that in \([57, 60]\) the database with 270,000 exactly calculated Fourier–Legendre coefficients was described. This database was used in the software package \([57, 60]\), which is written in the Python programming language for the implementation of explicit one-step numerical schemes with strong orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence for Ito SDEs. The optimization of the mean-square approximation procedures for iterated Ito stochastic integrals from these numerical schemes can be found in \([59]\).
On the basis of the presented approximations of iterated Ito stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to \( \tau_{p+1} - \tau_{p} \) (\( \tau_{p+1} - \tau_{p} \ll 1 \)) in the mean-square sense for iterated Ito stochastic integrals. This leads to a sharp decrease of member quantities in the approximations of iterated Ito stochastic integrals (see the number \( q \) in Theorem 3), which are required for achieving the acceptable accuracy of approximation.

4. **Explicit One-Step Strong Numerical Schemes of Orders 2.0, 2.5, and 3.0 Based on the Unified Taylor–Stratonovich Expansion**

Consider the following explicit one-step strong numerical scheme of order 3.0 based on the so-called unified Taylor–Stratonovich expansion [9] (also see [5], [18]-[28], [54]-[56])

\[
y_{p+1} = y_{p} + \sum_{i_{1}=1}^{m} \Sigma_{i_{1}} \hat{i}_{0}^{(i_{1})} \Sigma_{0} + \Delta \hat{a} + \sum_{i_{1}, i_{2}=1}^{m} G_{0}^{(i_{2})} \Sigma_{i_{1}} \hat{i}_{0}^{(i_{2}i_{1})} + \\
+ \sum_{i_{1}=1}^{m} \left[ G_{0}^{(i_{1})} \hat{a} \left( \hat{i}_{0}^{(i_{1})} \Sigma_{0} + \hat{i}_{1}^{(i_{1})} \right) - L \Sigma_{i_{1}} \hat{i}_{1}^{(i_{1})} \right] + \\
+ \sum_{i_{1}, i_{2}, i_{3}=1}^{m} \left[ G_{0}^{(i_{2})} \Sigma_{i_{1}} \hat{i}_{0}^{(i_{2}i_{1})} - \hat{i}_{01}^{(i_{2}i_{1})} \right] - \hat{L} \Sigma_{i_{1}} \hat{i}_{1}^{(i_{2}i_{1})} + \\
+ G_{0}^{(i_{2})} \Sigma_{0} \left( \hat{i}_{01}^{(i_{2}i_{1})} + \Delta \hat{i}_{001}^{(i_{2}i_{1})} \right) \\
+ \sum_{i_{1}, i_{2}, i_{3}, i_{4}=1}^{m} G_{0}^{(i_{4})} G_{0}^{(i_{2})} G_{0}^{(i_{2})} \Sigma_{i_{1}} \hat{i}_{000}^{(i_{4}i_{3}i_{2}i_{1})} + q_{p+1} + r_{p+1}, 
\]

(33)
\[-\tilde{L}G_0^{(t_3)} G_0^{(t_2)} G_0^{(t_1)} \hat{I}_{100}^{*} \left( r_{p+1} \right) + \]

\[+ \sum_{i_1, i_2, i_3, i_4, i_5 = 1}^m G_0^{(t_5)} G_0^{(t_4)} G_0^{(t_3)} G_0^{(t_2)} G_0^{(t_1)} \sum_{i_1} \hat{I}_{100}^{*} \left( r_{p+1} \right) + \]

\[+ \frac{\Delta^3}{6} \tilde{L}L a, \]

\[\tilde{r}_{p+1} = \sum_{i_1, i_2 = 1}^m \left[ G_0^{(t_2)} G_0^{(t_1)} \tilde{L} a \left( \frac{1}{2} \hat{I}_{02}^{*} + \Delta \hat{I}_{11}^{*} + \frac{\Delta^2}{2} \hat{I}_{11}^{*} \right) + \right. \]

\[+ \frac{1}{2} \tilde{L}L G_0^{(t_2)} \sum_{i_1} \tilde{I}_{11}^{*} + \]

\[+ G_0^{(t_2)} \tilde{L} G_0^{(t_1)} \tilde{a} \left( \tilde{I}_{11}^{*} + \tilde{I}_{11}^{*} + \tilde{I}_{11}^{*} \right) + \]

\[+ G_0^{(t_2)} \tilde{L} G_0^{(t_1)} \tilde{a} \left( \Delta \tilde{I}_{11}^{*} + \Delta \tilde{I}_{11}^{*} \right) + \]

\[+ \sum_{i_1, i_2, i_3, i_4, i_5 = 1}^m \left[ G_0^{(t_5)} G_0^{(t_4)} G_0^{(t_3)} G_0^{(t_2)} G_0^{(t_1)} \tilde{a} \left( \Delta \tilde{I}_{11}^{*} + \Delta \tilde{I}_{11}^{*} \right) + \right. \]

\[+ G_0^{(t_4)} \tilde{L} G_0^{(t_3)} G_0^{(t_2)} \sum_{i_1} \tilde{I}_{11}^{*} + \]

\[+ \tilde{L} G_0^{(t_2)} \tilde{a} \left( \Delta \tilde{I}_{11}^{*} + \Delta \tilde{I}_{11}^{*} \right) \]
where $\Delta = \bar{T}/N$ ($N > 1$) is a constant (for simplicity) step of integration, $\tau_p = p\Delta$ ($p = 0, 1, \ldots, N$), $\hat{I}^{*(i_1, \ldots, i_k)}_{t_1, \ldots, t_k, s, t}$ is an approximation of the iterated Stratonovich stochastic integral

\[
I^{*(i_1, \ldots, i_k)}_{t_1, \ldots, t_k, s, t} = \int_t^{t_1} (t - \tau_1)^{l_1} \ldots \int_t^{t_k} (t - \tau_k)^{l_k} \, d\mathbf{f}^{(i_1)}_{\tau_1} \ldots d\mathbf{f}^{(i_k)}_{\tau_k},
\]

$$
\hat{a}(x, t) = a(x, t) - \frac{1}{2} \sum_{j=1}^{m} G_j^{(j)} \Sigma_j(x, t),
$$

$\hat{L} = L - \frac{1}{2} \sum_{j=1}^{m} G_j^{(j)} G_0^{(j)} = \frac{\partial}{\partial t} + \sum_{j=1}^{n} \hat{a}^{(j)}(x, t) \frac{\partial}{\partial x_j},$

$L = \frac{\partial}{\partial t} + \sum_{i=1}^{m} a_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{m} \Sigma_{ij}(x, t) \Sigma_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_i},$

$G_0^{(i)} = \sum_{j=1}^{m} \Sigma_{ji}(x, t) \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, m,$

$l_1, \ldots, l_k = 0, 1, \ldots, i_1, \ldots, i_k = 1, \ldots, m, \quad k = 1, 2, \ldots, \Sigma_i$ is the $i$th component of the matrix function $\Sigma$ and $\Sigma_{ij}$ is the $ij$th component of the matrix function $\Sigma$, $a_i$ is the $i$th component of the vector function $a$ and $x_i$ is the $i$th component of the column $x$, the columns

$$
\Sigma_{i1}, \hat{a}, G_0^{(i2)} \Sigma_{i1}, G_0^{(i1)} \hat{a}, \tilde{L} \Sigma_{i1}, G_0^{(i3)} c_0^{(i2)} \Sigma_{i1}, \tilde{L} \tilde{a}, G_0^{(i2)} \tilde{L} \Sigma_{i1}, \tilde{L} a, G_0^{(i3)} \tilde{L} \Sigma_{i1}, G_0^{(i1)} \tilde{L} \Sigma_{i1}, G_0^{(i2)} G_0^{(i1)} \hat{a},
$$

$$
G_0^{(i4)} G_0^{(i3)} G_0^{(i2)} \Sigma_{i1}, G_0^{(i1)} \tilde{L} \Sigma_{i1}, \tilde{L} \Sigma_{i1}, G_0^{(i2)} G_0^{(i1)} G_0^{(i3)} \hat{a}, G_0^{(i3)} G_0^{(i2)} G_0^{(i1)} \hat{a}, G_0^{(i4)} G_0^{(i3)} G_0^{(i2)} L \Sigma_{i1}, G_0^{(i1)} G_0^{(i2)} G_0^{(i3)} G_0^{(i4)} \Sigma_{i1},
$$

$$
G_0^{(i4)} \tilde{L} G_0^{(i3)} G_0^{(i2)} G_0^{(i1)} \Sigma_{i1}, G_0^{(i4)} G_0^{(i3)} G_0^{(i2)} L \Sigma_{i1}, G_0^{(i1)} G_0^{(i2)} G_0^{(i3)} G_0^{(i4)} G_0^{(i5)} G_0^{(i4)} G_0^{(i3)} G_0^{(i2)} \Sigma_{i1},
$$

are calculated at the point $(y_p, p)$.

It is well known [2] that under the standard conditions the numerical scheme [33] has strong order of convergence 3.0. Among these conditions we consider only the condition for approximations of iterated Stratonovich stochastic integrals from the numerical scheme [33] [2], [9]

$$
M \left\{ \left( I^{*(i_1, \ldots, i_k)}_{t_1, \ldots, t_k, \tau_p+1, \tau_p} - \hat{I}^{*(i_1, \ldots, i_k)}_{t_1, \ldots, t_k, \tau_p+1, \tau_p} \right)^2 \right\} \leq C \Delta^7,
$$

where $\hat{I}^{*(i_1, \ldots, i_k)}_{t_1, \ldots, t_k, \tau_p+1, \tau_p}$ is an approximation of $I^{*(i_1, \ldots, i_k)}_{t_1, \ldots, t_k, \tau_p+1, \tau_p}$, constant $C$ does not depend on $\Delta$.

Note that if we exclude $q_{p+1, p} + r_{p+1, p}$ from the right-hand side of [33], then we will have the explicit one-step strong numerical scheme of order 2.0. The right-hand side of [33] but without the value $r_{p+1, p}$ and with replacing the value $\Delta^3 LL\hat{a}/6$ by the value $\Delta^3 LLa/6$ define the explicit one-step strong numerical scheme of order 2.5.

Note that the truncated unified Taylor–Stratonovich expansion [9] (also see [5], [18]-[28], [51]-[56]) contains the less number of various types of iterated Stratonovich stochastic integrals (moreover,
their major part will have less multiplicities) in comparison with the classical Taylor–Stratonovich expansion \([2, 7]\).

Furthermore, some iterated stochastic integrals from the Taylor–Stratonovich expansion \([2, 7]\) are connected by linear relations. However, the iterated stochastic integrals from the unified Taylor–Stratonovich expansion \([9]\) (also see \([5, 18–28, 54–56]\)) cannot be connected by linear relations. Therefore, we call these families of stochastic integrals as the stochastic bases \([5, 18–28, 54–56]\).

Note that \((33)\) contains 20 different types of iterated Stratonovich stochastic integrals. At the same time, the analogue of \((33)\) based on the classical Taylor–Stratonovich expansion \([2, 7]\) contains 29 different types of iterated stochastic integrals.

5. **Fourier–Legendre Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 6**

As noted above, in a number of works of the author Theorems 1, 2 have been adapted for the iterated Stratonovich stochastic integrals \((3)\) of multiplicities 1 to 6 (the case of multiplicity 1 is given by \((12)\)). Let us first present some old results.

**Theorem 4** \([18–20, 25–28, 41, 47, 54–56]\). Assume that the following conditions are fulfilled:

1. The function \(\psi_2(\tau)\) is continuously differentiable at the interval \([t, T]\) and the function \(\psi_1(\tau)\) is two times continuously differentiable at the interval \([t, T]\).

2. \(\{\phi_j(x)\}_{j=0}^\infty\) is a complete orthonormal system of Legendre polynomials or system of trigonometric functions in the space \(L^2([t, T])\).

Then, the iterated Stratonovich stochastic integral of multiplicity 2

\[
J^*[\psi^{(2)}]_{T,t} = \int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) df_t^{(i_1)} df_{t_2}^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m)
\]

is expanded into the converging in the mean-square sense multiple series

\[
J^*[\psi^{(2)}]_{T,t} = \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1}^{i_1 i_2} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},
\]

where the meaning of notations introduced in the formulation of Theorem 1 is remained.

Proving the theorem 4 \([18–20, 25–28, 41, 47, 54–56]\) we used Theorem 1 and double integration by parts. This procedure leads to the condition of double continuous differentiability of the function \(\psi_1(\tau)\) at the interval \([t, T]\). The mentioned condition can be weakened \([17, 35, 42, 49, 54–56]\) and Theorem 4 will be valid for continuously differentiable functions \(\psi_l(\tau)\) \((l = 1, 2)\) at the interval \([t, T]\).

**Theorem 5** \([18–20, 25–28, 41, 47, 54–56]\). Assume, that \(\{\phi_j(x)\}_{j=0}^\infty\) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L^2([t, T])\). Furthermore, the function \(\psi_2(\tau)\) is continuously differentiable at the interval \([t, T]\) and the functions \(\psi_1(\tau), \psi_3(\tau)\) are twice continuously differentiable at the interval \([t, T]\). Then, for the iterated Stratonovich stochastic integral of multiplicity 3

\[
J^*[\psi^{(3)}]_{T,t} = \int_t^T \psi_3(t_3) \int_t^{t_3} \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) df_t^{(i_1)} df_{t_2}^{(i_2)} df_{t_3}^{(i_3)} \quad (i_1, i_2, i_3 = 1, \ldots, m)
\]
the following expansion

\[ J^* [\psi^{(3)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1} \psi^{(i_3)}_{j_3} \psi^{(i_2)}_{j_2} \psi^{(i_1)}_{j_1} \]

converging in the mean-square sense is valid, where

\[ C_{j_3j_2j_1} = \int_{t}^{T} \psi_3(t_3) \phi_{j_3}(t_3) \int_{t}^{t_2} \psi_2(t_2) \phi_{j_2}(t_2) \int_{t}^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3; \]

another notations are the same as in Theorems 1, 2.

**Theorem 6** [18–20, 23–25, 41, 43, 51–53]. Assume that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t,T]) \). Then, for the iterated Stratonovich stochastic integral of multiplicity 4

\[ I_{T,t}^{(i_1i_2i_3i_4)} = \lim_{p \to \infty} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1} \psi^{(i_1)}_{j_1} \psi^{(i_2)}_{j_2} \psi^{(i_3)}_{j_3} \psi^{(i_4)}_{j_4} \]

converging in the mean-square sense is valid, where

\[ C_{j_4j_3j_2j_1} = \int_{t}^{T} \phi_{j_4}(t_4) \int_{t}^{t_4} \phi_{j_3}(t_3) \int_{t}^{t_3} \phi_{j_2}(t_2) \int_{t}^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4; \]

\( w^{(i)}_\tau = \mathbf{f}^{(i)}_\tau \) \( (i = 1, \ldots, m) \) are independent standard Wiener processes and \( w^{(0)}_\tau = \tau \); another notations are the same as in Theorems 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [54] (Sect. 2.10–2.16), [41] (Sect. 13–19), [43] (Sect. 5–11), [44] (Sect. 7–13), [71] (Sections 4–9). Let us formulate four theorems that were obtained using this approach.

**Theorem 7** [18, 20, 23, 41, 43, 51–53]. Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t,T]) \). Furthermore, let \( \psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \) are continuously differentiable nonrandom functions on \([t,T]\). Then, for the iterated Stratonovich stochastic integral of third multiplicity

\[ J^* [\psi^{(3)}]_{T,t} = \int_{t}^{T} \psi_3(t_3) \int_{t}^{t_3} \psi_2(t_2) \int_{t}^{t_2} \psi_1(t_1) dw^{(i_1)}_{t_1} dw^{(i_2)}_{t_2} dw^{(i_3)}_{t_3} \]

the following relations
(36) \[ J^* [\psi^{(3)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3,j_2,j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \]

(37) \[ M \left\{ \left( J^* [\psi^{(3)}]_{T,t} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3,j_2,j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \right)^2 \right\} \leq \frac{C}{p} \]

are fulfilled, where \( i_1, i_2, i_3 = 0, 1, \ldots, m \) in (36) and \( i_1, i_2, i_3 = 1, \ldots, m \) in (37), constant \( C \) is independent of \( p \),

\[ C_{j_3,j_2,j_1} = \int_t^T \psi_3(t_3) \phi_{j_3}(t_3) \int_t^{t_2} \psi_2(t_2) \phi_{j_2}(t_2) \int_t^{t_1} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 \]

and

\[ \zeta_j^{(i)} = \int_t^T \phi_j(\tau) d\tau \]

are independent standard Gaussian random variables for various \( i \) or \( j \) (in the case when \( i \neq 0 \)); another notations are the same as in Theorems 1, 2.

**Theorem 8** Let \( \{ \phi_j(x) \}_{j=0}^{\infty} \) be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L^2([t,T]) \). Furthermore, let \( \psi_1(\tau), \ldots, \psi_4(\tau) \) be continuously differentiable nonrandom functions on \([t,T]\). Then, for the iterated Stratonovich stochastic integral of fourth multiplicity

(38) \[ J^* [\psi^{(4)}]_{T,t} = \int_t^T \psi_4(t_4) \int_t^{t_3} \psi_3(t_3) \int_t^{t_2} \psi_2(t_2) \int_t^{t_1} \psi_1(t_1) dw_{i_1}(t_1) dw_{i_2}(t_2) dw_{i_3}(t_3) dw_{i_4}(t_4) \]

the following relations

(39) \[ J^* [\psi^{(4)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4,j_3,j_2,j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \]

(40) \[ M \left\{ \left( J^* [\psi^{(4)}]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4,j_3,j_2,j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \right)^2 \right\} \leq \frac{C}{p^{1-\varepsilon}} \]

are fulfilled, where \( i_1, \ldots, i_4 = 0, 1, \ldots, m \) in (38) \( (39) \) and \( i_1, \ldots, i_4 = 1, \ldots, m \) in (40), constant \( C \) does not depend on \( p, \varepsilon \) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \( L^2([t,T]) \) and \( \varepsilon = 0 \) for the case of complete orthonormal system of trigonometric functions in the space \( L^2([t,T]) \),
\( C_{j_1 j_2 j_3 j_4} = \)
\[
= \int_t^T \psi_4(t_4) \phi_{j_4}(t_4) \int_t^{t_4} \psi_3(t_3) \phi_{j_3}(t_3) \int_t^{t_2} \psi_2(t_2) \phi_{j_2}(t_2) \int_t^{t_1} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;
\]

another notations are the same as in Theorem 7.

**Theorem 9** [41], [43], [44], [54], [71]. Assume that \( \{ \phi_j(x) \}_{j=0}^\infty \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_5(\tau) \) are continuously differentiable nonrandom functions on \([t, T]\). Then, for the iterated Stratonovich stochastic integral of fifth multiplicity
\[ J^*[\psi^{(5)}]_{T,t} = \int_t^T \psi_5(t_5) \ldots \int_t^{t_5} \psi_1(t_1) d\mathbf{w}_{i_1}^{(i_1)} \ldots d\mathbf{w}_{i_5}^{(i_5)} \]
the following relations
\[ J^*[\psi^{(5)}]_{T,t} = \lim_{p\to\infty} \sum_{j_1,\ldots,j_5=0}^p C_{j_5\ldots j_1}^{(i_1)} \ldots \phi_{j_5}^{(i_5)}, \]
\[ M \left\{ \left( J^*[\psi^{(5)}]_{T,t} - \sum_{j_1,\ldots,j_5=0}^p C_{j_5\ldots j_1}^{(i_1)} \ldots \phi_{j_5}^{(i_5)} \right)^2 \right\} \leq \frac{C}{p^{1-\varepsilon}} \]
are fulfilled, where \( i_1,\ldots,i_5 = 0, 1, \ldots, m \) in [41], [42] and \( i_1,\ldots,i_5 = 1, \ldots, m \) in [43], constant \( C \) is independent of \( p, \varepsilon \) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \( L_2([t, T]) \) and \( \varepsilon = 0 \) for the case of complete orthonormal system of trigonometric functions in the space \( L_2([t, T]) \),
\[ C_{j_5\ldots j_1} = \int_t^T \psi_5(t_5) \phi_{j_5}(t_5) \ldots \int_t^{t_5} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_5; \]
another notations are the same as in Theorems 7, 8.

**Theorem 10** [41], [43], [44], [54]. Suppose that \( \{ \phi_j(x) \}_{j=0}^\infty \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \). Then, for the iterated Stratonovich stochastic integral of sixth multiplicity
\[ I_{T,t}^{(i_1\ldots i_6)} = \int_t^T \ldots \int_t^{t_5} d\mathbf{w}_{i_1}^{(i_1)} \ldots d\mathbf{w}_{i_6}^{(i_6)} \]
the following expansion
\[ I_{T,t}^{(i_1...i_6)} = \lim_{p \to \infty} \sum_{j_1,...,j_6=0}^{p} C_{j_6...j_1} \zeta_{j_1}^{(i_1)} \cdots \zeta_{j_6}^{(i_6)} \]

that converges in the mean-square sense is valid, where \( i_1, \ldots, i_6 = 0, 1, \ldots, m \),

\[ C_{j_6...j_1} = \int_{t_1}^{t_2} \phi_{j_6}(t_6) \cdots \int_{t}^{t_2} \phi_{j_1}(t_1) dt_1 \cdots dt_6; \]

another notations are the same as in Theorems 1, 2.

On the base of Theorems 4–10 the following hypothesis was formulated in [18–20, 25–28, 44, 54–56].

Hypothesis 1 [18–20, 25–28, 44, 54–56]. Assume that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t,T]) \). Moreover, every \( \psi_l(\tau) \) (\( l = 1, \ldots, k \)) is an enough smooth nonrandom function on \([t,T]\). Then, for the iterated Stratonovich stochastic integral \( J^*_{\psi(k)}_{T,t} \) defined by (3) the following expansion

\[ J^*_{\psi(k)}_{T,t} = \lim_{p \to \infty} \sum_{j_1,...,j_k=0}^{p} C_{j_k...j_1} \zeta_{j_1}^{(i_1)} \cdots \zeta_{j_k}^{(i_k)} \]

converging in the mean-square sense is valid, where the notations are the same as in Theorems 1, 2.

Hypothesis 1 allows to approximate the iterated Stratonovich stochastic integral \( J^*_{\psi(k)}_{T,t} \) by the sum

\[ J^*_{\psi(k)}_{T,t} = \lim_{p \to \infty} \sum_{j_1,...,j_k=0}^{p} C_{j_k...j_1} \zeta_{j_1}^{(i_1)} \cdots \zeta_{j_k}^{(i_k)} \]

where

\[ \lim_{p \to \infty} M \left\{ \left( J^*_{\psi(k)}_{T,t} - J^*_{\psi(k)}_{T,t}^{(p)} \right)^2 \right\} = 0. \]

Note that Hypothesis 1 is proved in [54] (Sect. 2.10) under the condition of convergence of trace series (also see [41, 43, 44]). In [44, 54–56] a more general hypothesis is formulated.

Applying Theorems 4–10, we obtain the following approximations of iterated Stratonovich stochastic integrals from (33)

\[ I_{p+1,\tau_p}^{(i_1)} \]

\[ I_{0,\tau_p}^{(i_1,i_2)} = \Delta^3/2 \left( \zeta_{i_2}^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_{i_1}^{(i_1)} \right); \]

\[ I_{0,\tau_p}^{(i_1,i_2)} = \Delta^2/2 \left( \zeta_{i_1}^{(i_1)} \right); \]

\[ I_{1,\tau_p}^{(i_1)} = \Delta^3/2 \left( \zeta_{i_1}^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_{i_1}^{(i_1)} \right). \]
\[ I^{*(i_1i_2i_3)}_{001, p+1, r_p} = \sum_{j_1, j_2, j_3=0}^q C_{j_3j_2j_1} \xi_{j_3}^{(i_1)} \xi_{j_2}^{(i_2)} \xi_{j_1}^{(i_3)}, \]

\[ I^{*(i_1i_2)}_{01, p+1, r_p} = -\frac{\Delta}{2} I^{*(i_1i_2)}_{001, p+1, r_p} - \frac{\Delta^2}{4} \left[ \frac{1}{\sqrt{3}} \zeta_{s_1}^{(i_1)} + \sum_{i=0}^{q} \left( \frac{(i+2)\xi_{s_i}^{(i_1)}\xi_{s_{i+2}}^{(i_2)}}{\sqrt{(2i+1)(2i+5)(2i+3)}} - \frac{\xi_{s_i}^{(i_1)}\xi_{s_{i+2}}^{(i_2)}}{(2i-1)(2i+3)} \right) \right], \]

\[ I^{*(i_1i_2)}_{10, p+1, r_p} = -\frac{\Delta}{2} I^{*(i_1i_2)}_{001, p+1, r_p} - \frac{\Delta^2}{4} \left[ \frac{1}{\sqrt{3}} \zeta_{s_1}^{(i_1)} + \sum_{i=0}^{q} \left( \frac{(i+1)\xi_{s_i}^{(i_1)}\xi_{s_{i+2}}^{(i_2)}}{\sqrt{(2i+1)(2i+5)(2i+3)}} + \frac{\xi_{s_i}^{(i_1)}\xi_{s_{i+2}}^{(i_2)}}{(2i-1)(2i+3)} \right) \right], \]

\[ I^{*(i_1i_2i_3i_4)}_{0000, p+1, r_p} = \sum_{j_1, j_2, j_3, j_4=0}^q C_{j_4j_3j_2j_1} \xi_{j_4}^{(i_1)} \xi_{j_3}^{(i_2)} \xi_{j_2}^{(i_3)} \xi_{j_1}^{(i_4)}, \]

\[ I^{*(i_1)}_{2, p+1, r_p} = \frac{\Delta^{5/2}}{3} \left( \frac{\zeta_{s_0}^{(i_1)}}{s_0} + \frac{\sqrt{5}}{2} \zeta_{s_1}^{(i_1)} + \frac{1}{2\sqrt{5}} \zeta_{s_2}^{(i_1)} \right), \]

\[ I^{*(i_1i_2i_3)}_{100, p+1, r_p} = \sum_{j_1, j_2, j_3=0}^q C_{j_3j_2j_1} \xi_{j_3}^{(i_1)} \xi_{j_2}^{(i_2)} \xi_{j_1}^{(i_3)}, \]

\[ I^{*(i_1i_2i_3)}_{010, p+1, r_p} = \sum_{j_1, j_2, j_3=0}^q C_{j_3j_2j_1} \xi_{j_3}^{(i_1)} \xi_{j_2}^{(i_2)} \xi_{j_1}^{(i_3)}, \]

\[ I^{*(i_1i_2i_3)}_{001, p+1, r_p} = \sum_{j_1, j_2, j_3=0}^q C_{j_3j_2j_1} \xi_{j_3}^{(i_1)} \xi_{j_2}^{(i_2)} \xi_{j_1}^{(i_3)}, \]

\[ I^{*(i_1i_2i_3i_4i_5)}_{00000, p+1, r_p} = \sum_{j_1, j_2, j_3, j_4, j_5=0}^q C_{j_5j_4j_3j_2j_1} \xi_{j_5}^{(i_1)} \xi_{j_4}^{(i_2)} \xi_{j_3}^{(i_3)} \xi_{j_2}^{(i_4)} \xi_{j_1}^{(i_5)}, \]

\[ I^{*(i_1i_2)}_{02, p+1, r_p} = -\frac{\Delta^2}{4} I^{*(i_1i_2)}_{001, p+1, r_p} - \frac{\Delta^2}{8} I^{*(i_1i_2)}_{01, p+1, r_p} + \frac{\Delta^3}{8} \left[ \frac{2}{3\sqrt{5}} \zeta_{s_2}^{(i_1)} \zeta_{s_0}^{(i_2)} \right]. \]
\[ I^{(i_1 i_2)}_{0001} = \sum_{j_1, j_2, j_3, j_4 = 0}^{q} C^{0001}_{j_1 j_2 j_3 j_4} \gamma_{j_1}^{(i_1)} \gamma_{j_2}^{(i_2)} \gamma_{j_3}^{(i_1)} \gamma_{j_4}^{(i_4)} \]  
\[ I^{(i_1 i_2)}_{0010} = \sum_{j_1, j_2, j_3, j_4 = 0}^{q} C^{0010}_{j_1 j_2 j_3 j_4} \gamma_{j_1}^{(i_1)} \gamma_{j_2}^{(i_2)} \gamma_{j_3}^{(i_3)} \gamma_{j_4}^{(i_4)} \]  
\[ I^{(i_1 i_2)}_{0100} = \sum_{j_1, j_2, j_3, j_4 = 0}^{q} C^{0100}_{j_1 j_2 j_3 j_4} \gamma_{j_1}^{(i_1)} \gamma_{j_2}^{(i_2)} \gamma_{j_3}^{(i_3)} \gamma_{j_4}^{(i_4)} \]  
\[ I^{(i_1 i_2)}_{0100} = \sum_{j_1, j_2, j_3, j_4 = 0}^{q} C^{0100}_{j_1 j_2 j_3 j_4} \gamma_{j_1}^{(i_1)} \gamma_{j_2}^{(i_2)} \gamma_{j_3}^{(i_3)} \gamma_{j_4}^{(i_4)} \]  
\[ I^{(i_1 i_2 i_3 i_4 i_5 i_6)}_{000000} = \sum_{j_1, j_2, j_3, j_4, j_5, j_6 = 0}^{q} C^{000000}_{j_1 j_2 j_3 j_4 j_5 j_6} \gamma_{j_1}^{(i_1)} \gamma_{j_2}^{(i_2)} \gamma_{j_3}^{(i_3)} \gamma_{j_4}^{(i_4)} \gamma_{j_5}^{(i_5)} \gamma_{j_6}^{(i_6)} \]
where formulas for the Fourier–Legendre coefficients

\[ C_{j_1 j_2 1}, C_{j_4 j_3 j_2 1}, C_{001}^{j_1 j_2 1}, C_{100}^{j_3 j_2 1}, C_{001}^{j_5 j_4 j_3 j_2 1}, C_{0010}^{j_4 j_3 j_2 1}, C_{0010}^{j_6 j_5 j_4 j_3 j_2 1} \]

can be found in Sect. 3.

On the basis of the presented approximations of iterated Stratonovich stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to \( \tau_{p+1} / \tau_p \) \( (\tau_{p+1} - \tau_p \ll 1) \) in the mean-square sense for iterated Stratonovich stochastic integrals. This leads to a sharp decrease of member quantities in the approximations of iterated Stratonovich stochastic integrals (see the numbers \( q \) in the approximations of iterated Stratonovich stochastic integrals from this section), which are required for achieving the acceptable accuracy of approximation.

From (24) \( (i_1 \neq i_2) \) we have

\[
M \left\{ \left( I_{00}^{(i_1 i_2)} - I_{00}^{(i_1 i_2)} \right)^2 \right\} = \frac{\Delta^2}{2} \sum_{i=q+1}^{\infty} \frac{1}{4i^2 - 1} \leq \\
\frac{\Delta^2}{2} \int_{q}^{\infty} \frac{1}{4x^2 - 1} dx = - \frac{\Delta^2}{8} \ln \left| 1 - \frac{2}{2q + 1} \right| \leq \frac{C \Delta^2}{q},
\]

where \( C_1 \) is a constant.

Since the value \( \Delta = \tau_{p+1} - \tau_p \) plays the role of integration step in the numerical scheme (43), then this value is a sufficiently small.

Keeping in mind this circumstance, it is easy to notice that there exists such a constant \( C_2 \) that

\[
M \left\{ \left( I_{l_1 \ldots l_k}^{(i_1 \ldots i_k)} - I_{l_1 \ldots l_k}^{(i_1 \ldots i_k)} \right)^2 \right\} \leq C_2 M \left\{ \left( I_{00}^{(i_1 i_2)} - I_{00}^{(i_1 i_2)} \right)^2 \right\},
\]

where \( I_{l_1 \ldots l_k}^{(i_1 \ldots i_k)} \) is an approximation of the iterated Stratonovich stochastic integral \( I_{l_1 \ldots l_k}^{(i_1 \ldots i_k)} \).

From (44) and (46) we finally obtain

\[
M \left\{ \left( I_{l_1 \ldots l_k}^{(i_1 \ldots i_k)} - I_{l_1 \ldots l_k}^{(i_1 \ldots i_k)} \right)^2 \right\} \leq \frac{C \Delta^2}{q},
\]

where constant \( C \) does not depend on \( \Delta \).

The same idea can be found in [24] for the case of trigonometric functions. Note that, in contrast to the estimate (48), the constant \( C \) in Theorems 7–9 does not depend on \( q \).

Since

\[ J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} \text{ w. p. 1} \]

for pairwise different \( i_1, \ldots, i_k = 1, \ldots, m \), then we can write for pairwise different \( i_1, \ldots, i_6 = 1, \ldots, m \) (see (24))

\[
M \left\{ \left( I_{01}^{(i_1 i_2)} - I_{01}^{(i_1 i_2)} \right)^2 \right\} = \frac{\Delta^4}{4} - \sum_{j_1, j_2 = 0}^{q} (C_{j_1 j_2}^{010})^2,
\]
\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^4}{12} - \sum_{j_1, j_2 = 0}^{q} (C_{10}^{10,j_2,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^3}{6} - \sum_{j_1, j_2, j_3 = 0}^{q} C_{j_3}^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^4}{24} - \sum_{j_1, j_2, j_3 = 0}^{q} C_{j_3}^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^4}{60} - \sum_{j_1, j_2, j_3 = 0}^{q} (C_{100}^{100,j_3,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^5}{20} - \sum_{j_1, j_2, j_3 = 0}^{q} (C_{010}^{010,j_3,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^5}{10} - \sum_{j_1, j_2, j_3 = 0}^{q} (C_{001}^{001,j_3,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^5}{120} - \sum_{j_1, j_2, j_3, j_4 = 0}^{q} C_{j_5}^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^6}{30} - \sum_{j_2, j_1 = 0}^{q} (C_{20}^{20,j_2,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^6}{18} - \sum_{j_2, j_1 = 0}^{q} (C_{11}^{11,j_2,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^6}{6} - \sum_{j_2, j_1 = 0}^{q} (C_{02}^{02,j_2,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^6}{360} - \sum_{j_1, j_2, j_3, j_4 = 0}^{q} (C_{1000}^{1000,j_4,j_3,j_2,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^6}{120} - \sum_{j_1, j_2, j_3, j_4 = 0}^{q} (C_{0100}^{0100,j_4,j_3,j_2,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^6}{60} - \sum_{j_1, j_2, j_3, j_4 = 0}^{q} (C_{0010}^{0010,j_4,j_3,j_2,j_1})^2, \]

\[ M \left\{ \left( I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} - I^{\ast(1,1,2)}_{0000,\tau_{p+1,\tau_{p}}} q \right)^2 \right\} = \frac{\Delta^6}{36} - \sum_{j_1, j_2, j_3, j_4 = 0}^{q} (C_{0001}^{0001,j_4,j_3,j_2,j_1})^2, \]
For example [5] (see also [18]-[28], [53]-[64])

\[
M \left\{ \left( I^{(i_1 i_2 i_3 i_4)}_{0000} - I^{(i_1 i_2 i_3 i_4)}_{1000} \right)^2 \right\} = \frac{\Delta^6}{720} - \sum_{j_1, j_2, j_3, j_4, j_5, j_6 = 0}^q C_{j_5 j_3 j_4 j_3 j_2 j_1}^2.
\]

The theory presented in this article was realized in the form of a software package in the Python programming language. The mentioned software package implements the strong numerical methods with convergence orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs with multidimensional non-commutative noise based on the unified Taylor–Itô and Taylor–Stratonovich expansions. At that for the numerical simulation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 we applied the formulas based on multiple Fourier–Legendre series [57], [60]. Moreover, we used [57], [60] the database with 270,000 exactly calculated Fourier–Legendre coefficients.

**References**

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Kiev, Naukova Dumka, 1982, 612 pp.
[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Berlin, Springer, 1992, 632 pp.
[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk, Ural University Press, 1988, 225 pp.
[4] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Berlin, Springer, 2004, 616 pp.
[5] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBP/U2/z17-227
Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[6] Platen E., Wagner W. On a Taylor formula for a class of Itô processes. Probab. Math. Statist. 3 (1982), 37-51.

[7] Kloeden P.E., Platen E. The Stratonovich and Itô-Taylor expansions. Math. Nachr. 151 (1991), 33-50.

[8] Kulchitskiy O.Yu., Kuznetsov D.F. The unified Taylor-Itô expansion. Journal of Mathematical Sciences (N. Y.). 99, 2 (2000), 1130-1140. DOI: http://doi.org/10.1007/BF02673635

[9] Kuznetsov D.F. New representations of the Taylor-Stratonovich expansions. Journal of Mathematical Sciences (N. Y.). 118, 6 (2003), 5586-5596. DOI: http://doi.org/10.1023/A:102638522239

[10] Kuznetsov D.F. Four new forms of the Taylor-Itô and Taylor-Stratonovich expansions and its application to the high-order strong numerical methods for Itô stochastic differential equations. arXiv:2001.10192 [math.PR], 2020, 90 pp. [In English].

[11] Kloeden, P.E., Platen, E., Schurz, H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994. 292 pp.

[12] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl. 10, 4 (1992), 431-441.

[13] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Berlin, Heidelberg, Springer-Verlag, 2010. 868 pp.

[14] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.

[15] Kuznetsov D. F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal ”Differential Equations and Control Processes” ISSN 1817-2172 (online), 1 (1997), 18-77.
Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[16] Kuznetsov D.F. Problems of the numerical analysis of Itô stochastic differential equations. [In Russian]. Electronic Journal ”Differential Equations and Control Processes” ISSN 1817-2172 (online), 1 (1998), 60-367. Available at: http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html
Hard Cover Edition: SPbGTU Publishing House, 1998, 204 pp. (ISBN 5-7422-0045-5)

[17] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. arXiv:1801.00784 [math.PR]. 2018, 77 pp. [In English].

[18] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal ”Differential Equations and Control Processes” ISSN 1817-2172 (online), 1 (2017), 385 pp. (A.1-A.385).
DOI: http://doi.org/10.18720/SPBP/U2/z17-3
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[19] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal ”Differential Equations and Control Processes” ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBP/U2/z17-4
Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[20] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal ”Differential Equations and Control Processes” ISSN 1817-2172 (online), 4 (2018), A.1-A.1073.
Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[21] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBP/U2/s17-228
Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[22] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Matlab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBP/U2/s17-229
Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[23] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Matlab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBP/U2/s17-230
Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[24] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Matlab programs, 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+768 pp. DOI: http://doi.org/10.18720/SPBP/U2/s17-231
Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

[25] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier series. [In Russian]. Electronic Journal ”Differential Equations and Control Processes” ISSN 1817-2172 (online), 3 (2010), A.1-A.257.
Kuznetsov D.F. Numerical simulation of 2.5-set of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 from the Taylor–Stratonovich expansion. arXiv: 1806.10705 [math.PR], 2018, 28 pp. [In English].

Kuznetsov D.F. Explicit one-step strong numerical methods of orders 2.0 and 2.5 for Ito stochastic differential equations based on the unified Taylor-Ito and Taylor-Stratonovich expansions. arXiv: 1802.04841 [math.PR]. 2018, 36 pp. [In English].

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. arXiv:1801.01564 [math.PR]. 2018, 65 pp. [In English].

Kuznetsov D.F. Expansions of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 4. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.05654 [math.PR]. 2018, 46 pp. [In English].

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2. Combined approach based on generalized multiple and iterated Fourier series. arXiv:1801.07248 [math.PR], 2018, 20 pp. [In English].

Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Ito and Stratonovich stochastic integrals. arXiv:1712.08991 [math.PR], 2017, 56 pp. [In English].

Kuznetsov D.F. To numerical modeling with strong orders 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. arXiv:1802.00888 [math.PR], 2018, 28 pp. [In English].

Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. arXiv:1901.02345 [math.GM], 2019, 40 pp. [In English].

Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR], 2018, 40 pp. [In English].

Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Itô SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2022, 912 pp. [In English].

Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

Kuznetsov, D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor–Ito and Taylor–Stratonovich expansions and multiple Fourier–Legendre series. arXiv:2009.14011 [math.PR], 2020, 342 pp. [In English].

Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020), MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR]. 2020, 63 pp. [In English].

Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Samoilyov, R.E., Koryev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html
[64] Kuznetsov D.F. Application of multiple Fourier–Legendre series to implementation of strong exponential Mil-stein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02612 [math.PR], 2019, 32 pp. [In English].

[65] Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Itô stochastic integrals based on generalized multiple Fourier series. arXiv:2006.16040 [math.PR], 2020, 33 pp. [In English].

[66] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].

[67] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 36, 5 (1965), 1560-1564.

[68] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[69] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989, 555 pp.

[70] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal “Differential Equations and Control Processes” ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[71] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal “Differential Equations and Control Processes” ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, SAINT-PETERSBURG, RUSSIA

Email address: sde.kuznetsov@inbox.ru