Transformation of a generalized Harry Dym equation into the Hirota–Satsuma system

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Abstract

The new generalized Harry Dym equation, recently introduced by Z. Popowicz in Phys. Lett. A 317, 260–264 (2003), is transformed into the Hirota–Satsuma system of coupled KdV equations.

1 Introduction

Recently, Popowicz \[1\] introduced the following new generalization of the Harry Dym (HD) equation:

\[
\begin{align*}
  u_t &= u^3 \left( u^{-1/2} v^{3/2} \right)_{xxx}, \\
  v_t &= v^3 \left( v^{-1/2} u^{3/2} \right)_{xxx}.
\end{align*}
\]

This remarkably simple and symmetric system of coupled equations possesses a Lax pair and a Hamiltonian structure \[1\].

In the present paper, we construct a chain of transformations which relates the new generalized HD equation of Popowicz \[1\] with the well-studied Hirota–Satsuma system of coupled KdV equations \[2\]. There are no general methods of transforming a given nonlinear system into another one, less complicated or better studied, and the usual way of finding necessary transformations is based on experience, guess and good luck. For this reason, we give no comments on how these transformations were found in the present case.

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2 Transforming the generalized HD equation

First, we transform the dependent variables $u$ and $v$ of the generalized HD equation (1) in the following way:

$$u = a \exp(b), \quad v = a \exp(-b).$$ \hspace{1cm} (2)

In the new dependent variables, $a(x,t)$ and $b(x,t)$, the system (1) takes the form

$$a_t = a^3 a_{xxx} + 12 a^3 a_x b_x^2 + 12 a^4 b_x b_{xx},$$
$$b_t = -2a^3 b_{xxx} - 6a^2 a_x b_{xx} - 6a^2 a_{xx} b_x - 8a^3 b_x^2,$$ \hspace{1cm} (3)

where the (original) HD equation is clearly seen at $b = \text{constant}$. Note that the separatants of the coupled evolution equations (3), i.e. the coefficients at the third-order $x$-derivatives in (3), have the form constant $\times a^3$, and thus the role played by the invertible transformation (2) is to prepare the system (1) for the next step of transforming, which brings (3) into a constant-separant form.

Second, we try to transform $x$, $a$ and $b$ in (3) as follows:

$$x = p(y,t), \quad a(x,t) = p_y(y,t), \quad b(x,t) = q(y,t).$$ \hspace{1cm} (4)

This is an extension of the transformation used by Ibragimov [3] to relate the (original) HD equation with the Schwarzian-modified KdV equation. The transformation (4) really works and relates the system (3) with the constant-separant system

$$p_t = p_{yyy} - \frac{3p_{yy}^2}{2p_y} + 6p_y q_y^2,$$ \hspace{1cm} (5a)
$$q_t = -2q_{yyy} - 2q_y^3 - 3 \left( \frac{p_{yy}}{p_y} - \frac{3p_{yy}^2}{2p_y^2} \right) q_y.$$ \hspace{1cm} (5b)

To verify this, one may use the identities

$$a \partial_x = \partial_y, \quad a_t = p_{yt} - \frac{p_{yy} p_t}{p_y}, \quad b_t = q_t - \frac{q_y p_t}{p_y},$$

which follow from (4) straightforwardly. Note that (4) is not an invertible transformation: it maps the system (3) into the system (5), whereas its
application in the opposite direction, from (8) to (5), requires one integration by $x$. We have omitted the terms $\alpha(t)p_y$ and $\alpha(t)q_y$ in the right-hand sides of (5a) and (5b), respectively, where this arbitrary function $\alpha(t)$ appeared as a ‘constant’ of that integration. The Schwarzian-modified KdV equation is clearly seen in the system (5) at $q = \text{constant}$. Nevertheless, further steps of transforming the generalized HD equation (1) do not involve the Schwarzian derivative.

Third, we make the transformation

$$f(y, t) = \frac{p_{yy}}{p_y}, \quad g(y, t) = q_y,$$

admitted by the equations (5) owing to their form, and obtain the system

$$f_t = (f_{yy} + 12gg_y - \frac{1}{2}f^3 + 6fg^2)_y,$$
$$g_t = (-2g_{yy} - 3gf_y + \frac{3}{2}f^2g - 2g^3)_y$$

which belongs to the well-studied class of coupled mKdV equations.

Fourth, we apply the Miura-type transformation

$$r(y, t) = f_y + \frac{1}{2}f^2 + 2g^2, \quad s(y, t) = g_y + fg$$

to the coupled mKdV equations (7) and obtain the Hirota–Satsuma system of coupled KdV equations

$$r_t = r_{yyy} - 3rr_y + 24ss_y, \quad s_t = -2s_{yyy} + 3rs_y.$$  \hspace{1cm} (9)

Note that the system (7) and the transformation (8) (up to a linear change of variables) were introduced in [4].

Consequently, we have constructed the chain of transformations (2), (4), (6) and (8) which relates the generalized HD equation (1) with the Hirota–Satsuma system (9).

3 Conclusion

In this paper, we transformed the new generalized HD equation of Popowicz (1) into the well-known Hirota–Satsuma system of coupled KdV equations (9). Possible applications of this, which were completely out of the scope of our work, may include relating Lax pairs and Hamiltonian structures of
deriving a recursion operator for (11) from the known one of (9), explaining analytic properties of (11) from the standpoint of the Painlevé property of (9), etc. We can add that chains of transformations, similar to the one obtained in this paper, have been successfully used in [5] for relating two new generalized HD equations of Brunelli, Das and Popowicz [6] with known integrable systems as well.

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