Algebras of distributions
for binary semi-isolating formulas
of a complete theory*

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Abstract

We define a class of algebras describing links of binary semi-isolating formulas on a set of realizations for a family of 1-types of a complete theory. These algebras include algebras of isolating formulas considered before. We prove that a set of labels for binary semi-isolating formulas on a set of realizations for a 1-type $p$ forms a monoid of a special form with a partial order inducing ranks for labels, with set-theoretic operations, and with a composition. We describe the class of these structures. A description of the class of structures relative to families of 1-types is given.

Key words: type, complete theory, algebra of binary semi-isolating formulas, join of monoids, deterministic structure.

In [1], a series of constructions is introduced admitting to realize key properties of countable theories and to obtain a classification of countable models of small (in particular, of Ehrenfeucht) theories with respect to two basic characteristics: Rudin–Keisler preorders and distribution functions for numbers of limit models. The construction of these theories is essentially based on the definition of special directed graphs with colored vertices and arcs as well as on the definition of $(n + 1)$-ary predicates that turn prime models

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over realizations of \( n \)-types to prime models over realizations of 1-types and reducing links between prime models over finite sets to links between prime models over elements such that these links are given by definable sets of arcs and edges.

In the paper, we develop a general approach to the description of binary links between realizations of 1-types in terms of labels of pairwise non-equivalent isolating formulas \([2]\) to sets of labels of semi-isolating formulas.

We use the standard relation algebraic, model-theoretical, semigroup, and graph-theoretic terminology \([3]–[14]\) as well as some notions, notations, and constructions in \([1, 2]\).

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1. Preliminary notions, notations, and properties

**Definition** \([1, 2, 15, 16]\). Let \( T \) be a complete theory, \( \mathcal{M} \models T \). Consider types \( p(x), q(y) \in S(\emptyset) \), realized in \( \mathcal{M} \), and all \((p, q)\)-preserving \((p, q)\)-semi-isolating, \((p \rightarrow q)\)-, or \((q \leftarrow p)\)-formulas \( \varphi(x, y) \) of \( T \), i.e., formulas for which there is \( a \in M \) such that \( \models p(a) \) and \( \varphi(a, y) \vdash q(y) \). Now, for each such a formula \( \varphi(x, y) \), we define a binary relation \( R_{p,\varphi,q} \equiv \{ (a, b) \in \mathcal{M} : \models p(a) \land \varphi(a, b) \} \). If \( (a, b) \in R_{p,\varphi,q} \), \((a, b)\) is called a \((p, \varphi, q)\)-arc. If \( \varphi(a, y) \) is principal (over \( a \)), the \((p, \varphi, q)\)-arc \((a, b)\) is also principal.

If, in addition, \( \varphi(x, y) \) is a \((p \leftrightarrow q)\)-formula, i.e., it is both a \((p \rightarrow q)\)- and a \((q \rightarrow p)\)-formula then the set \( [a, b] \equiv \{ (a, b), (b, a) \} \) is said to be a \((p, \varphi, q)\)-edge. If the \((p, \varphi, q)\)-edge \([a, b]\) consists of principal \((p, \varphi, q)\)- and \((q, \varphi(y, x), p)\)-arcs then \([a, b]\) is a principal \((p, \varphi, q)\)-edge.

\((p, \varphi, q)\)-arcs and \((p, \varphi, q)\)-edges are called arcs and edges respectively if we say about fixed or some \((p \rightarrow q)\)-formula \( \varphi(x, y) \). If \((a, b)\) is a principal \((p, \varphi, q)\)-arc such that the pair \((b, a)\) is not a principal arc (on any formula), that is \((b, a) \notin SI(p,q)\), then \((a, b)\) is called irreversible. If \((a, b)\) is a \((p, \varphi, q)\)-arc and \((b, a)\) is not a \((q, \varphi, p)\)-arc then \((a, b)\) is also an irreversible arc.

For types \( p(x), q(y) \in S(\emptyset) \), we denote by SICF\((p, q)\) the set of \((p \rightarrow q)\)-formulas \( \varphi(x, y) \) such that \( \{ \varphi(a, y) \} \) is consistent for \( \models p(a) \). Let SICE\((p, q)\) be the set of pairs of formulas \((\varphi(x, y), \psi(x, y)) \in SICF(p, q)\) such that for any (some) realization \( a \) of \( p \) the sets of solutions for \( \varphi(a, y) \) and \( \psi(a, y) \) coincide. Clearly, SICE\((p, q)\) is an equivalence relation on the set SICF\((p, q)\). Notice that each SICE\((p, q)\)-class \( E \) corresponds to either a set of \((p, \varphi, q)\)-edges, or a set of irreversible \((p, \varphi, q)\)-arcs, or simultaneously a set of \((p, \varphi, q)\)-edges and of irreversible \((p, \varphi, q)\)-arcs linking realizations of \( p \) and \( q \) by any (some)
formula $\varphi$ in $E$. Thus the quotient $\text{SICF}(p,q)/\text{SICE}(p,q)$ is represented as a disjoint union of sets $\text{SICFE}(p,q)$, $\text{SICFA}(p,q)$, and $\text{SICFM}(p,q)$, where $\text{SICFE}(p,q)$ consists of $\text{SICE}(p,q)$-classes correspondent to sets of edges, $\text{SICFA}(p,q)$ consists of $\text{SICE}(p,q)$-classes correspondent to sets of irreversible arcs, and $\text{SICFM}(p,q)$ consists of $\text{SICE}(p,q)$-classes correspondent to sets containing edges and irreversible arcs.

The sets $\text{SICF}(p,p)$, $\text{SICE}(p,p)$, $\text{SICFE}(p,p)$, $\text{SICFA}(p,p)$, and $\text{SICFM}(p,p)$ are denoted by $\text{SICF}(p)$, $\text{SICE}(p)$, $\text{SICFE}(p)$, $\text{SICFA}(p)$, and $\text{SICFM}(p)$ respectively.

Let $T$ be a complete theory without finite models, $U = U^- \cup \{0\} \cup U^+ \cup U'$ be an alphabet of cardinality $\geq |S(T)|$ and consisting of negative elements $u^- \in U^-$, positive elements $u^+ \in U^+$, neutral elements $u' \in U'$, and zero 0. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$. The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$. Elements of $U$ are called labels.

Let $\nu(p,q) \colon \text{SICF}(p,q)/\text{SICE}(p,q) \to U$ be injective labelling functions, $p(x), q(y) \in S(\varnothing)$, for which negative elements correspond to the classes in $\text{SICFA}(p,q)/\text{SICE}(p,q)$, positive elements and 0 correspond to the classes in $\text{SICFE}(p,q)/\text{SICE}(p,q)$ such that 0 is defined only for $p = q$ and is represented by the formula $(x \approx y)$, and neutral elements code the classes in $\text{SICFM}(p,q)/\text{SICE}(p,q)$, $\nu(p) := \nu(p,p)$. We additionally assume that $\rho_\nu(p) \cap \rho_\nu(q) = \{0\}$ for $p \neq q$ where, as usual, we denote by $\rho_f$ the image of the function $f$ and $\rho_\nu(p,q) \cap \rho_\nu(p',q') = \emptyset$ if $p \neq q$ and $(p,q) \neq (p',q')$. Labelling functions with the properties above as well families of these functions are said to be regular. Further we shall consider only regular labelling functions and their regular families.

The labels, correspondent to isolating formulas, are said to be isolating whereas each label in $\bigcup_{p,q \in S(\varnothing)} \rho_\nu(p,q)$ is semi-isolating. By the definition, each isolating label belongs to $U^- \cup \{0\} \cup U^+$, i.e., it is not neutral.

We denote by $\theta_{p,u,q}(x,y)$ formulas in $\text{SICF}(p,q)$ with a label $u \in \rho_\nu(p,q)$. If the type $p$ is fixed and $p = q$ then the formula $\theta_{p,u,q}(x,y)$ is denoted by $\theta_u(x,y)$.

Note that if $\theta_{p,u,q}(x,y)$ and $\theta_{q,v,p}(x,y)$ are formulas witnessing that for realizations $a$ and $b$ of $p$ and $q$ respectively the pairs $(a,b)$ and $(b,a)$ belong to $\text{SI}(p,q)$, then the formula $\theta_{p,u,q}(x,y) \land \theta_{q,v,p}(y,x)$ witnesses that $[a,b]$ is a $(p,\varphi,q)$-edge. If the edge $[a,b]$ is principal and $\theta_{p,u,q}(a,y)$ is an isolating formula such that $\models \theta_{p,u,q}(a,b), \models p(a)$, then the label $u$ is invertible and a
label \( v \in U^{\geq 0} \) corresponds uniquely to \( u \) such that \( \theta_{q,v,p}(b,y) \) is an isolating formula with \( \models \theta_{q,v,p}(b,a) \), and vice versa. The labels \( u \) and \( v \) are reciprocally inverse and are denoted by \( v^{-1} \) and \( u^{-1} \) respectively. In general case, each label \( u \in U^{\geq 0} \) has a (nonempty) set of inverse labels in \( U^{\geq 0} \), denoted also by \( u^{-1} \). Note that independently on a label \( u \in U^{\geq 0} \), for which a formula \( \theta_{p,u,q}(x,y) \) witnesses that \([a,b]\) is a \((p,\varphi,q)\)-edge, the set \( u^{-1} \) includes all labels \( v \in U^{\geq 0} \) such that \([b,a]\) is a \((q,\theta_{q,v,p},p)\)-edge.

Neutral labels correspond, for instance, the formulas \( \theta_{p,u,q}(x,y) \sqcup \theta_{p,v,q}(x,y) \), where \( u < 0 \) and \( v \geq 0 \).

For types \( p_1, p_2, \ldots, p_{k+1} \in S^1(\varnothing) \) and sets \( X_1, X_2, \ldots, X_k \subseteq U \) of labels we denote by

\[
\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})
\]

the set of all labels \( u \in U \) correspondent to formulas \( \theta_{p_1,u,p_{k+1}}(x,y) \) satisfying, for realizations \( a \) of \( p_1 \) and some \( u_1 \in X_1, \ldots, u_k \in X_k \), the following condition:

\[
\theta_{p_1,u,p_{k+1}}(a,y) = \theta_{p_1,u_1,p_2,u_2,\ldots,p_{k+1}}(a,y),
\]

where

\[
\theta_{p_1,u_1,p_2,u_2,\ldots,p_{k+1}}(x,y) = \exists x_2, x_3, \ldots, x_{k-1}(\theta_{p_1,u_1,p_2}(x,x_2) \land \theta_{p_2,u_2,p_3}(x_2,x_3) \land \ldots \land \theta_{p_{k-1},u_{k-1},p_k}(x_{k-1},x_k) \land \theta_{p_k,u_k,p_{k+1}}(x_k,y)).
\]

In view of transitivity of semi-isolation, each formula \( \theta_{p_1,u_1,p_2,u_2,\ldots,p_{k+1}}(x,y) \) has a label in \( \rho_{\nu(p_1,p_{k+1})} \).

Thus the Boolean \( \mathcal{P}(U) \) of \( U \) is the universe of an algebra \( \mathfrak{A} \) of distributions of binary semi-isolating formulas with \( k \)-ary operations

\[
\text{SI}(p_1, \cdot, p_2, \cdot, \ldots, p_k, \cdot, p_{k+1}),
\]

where \( p_1, \ldots, p_{k+1} \in S^1(\varnothing) \). This algebra has a natural restriction to any family \( R \subseteq S^1(\varnothing) \) as well as to the algebras of distributions of binary isolating formulas \[2\]. Besides, if \( U_0 \) is a subalphabet of \( U \) then the restriction of the universe of \( \mathfrak{A} \) to the set \( \mathcal{P}(U_0) \) and the restrictions for values of operations to the set \( U_0 \) forms, possibly partial, algebra \( \mathfrak{A} \upharpoonright U_0 \).

Note that if some set \( X_i \) is disjoint with \( \rho_{\nu(p_1,p_{k+1})} \), in particular, if it is empty then

\[
\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1}) = \varnothing,
\]

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and if each $X_i$ has common elements with $\rho_{\nu(p_i,p_{i+1})}$ then

$$\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1}) \neq \emptyset.$$  

Note also that if $X_i \not\subseteq \rho_{\nu(p_i,p_{i+1})}$ for some $i$ then

$$\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1}) =$$

$$= \text{SI}(p_1, X_1 \cap \rho_{\nu(p_1,p_2)}, p_2, X_2 \cap \rho_{\nu(p_2,p_3)}, \ldots, p_k, X_k \cap \rho_{\nu(p_k,p_{k+1})}, p_{k+1}).$$

In view of the previous equation, further, considering values

$$\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1}),$$

we shall assume that $X_i \subseteq \rho_{\nu(p_i,p_{i+1})}$, $i = 1, \ldots, k$.

If each set $X_i$ is a singleton consisting of an element $u_i$ then we use $u_i$ instead of $X_i$ in $\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$ and write

$$\text{SI}(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}).$$

By the definition the following equality holds:

$$\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1}) =$$

$$= \bigcup \{ \text{SI}(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \mid u_1 \in X_1, \ldots, u_k \in X_k \}.$$  

Hence the specification of $\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$ is reduced to the specifications of $\text{SI}(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$. Note also that $\text{SI}(p, X, q) = X$ for any $X \subseteq \rho_{\nu(p,q)}$.

Clearly, if $u_i = 0$ then $p_i = p_{i+1}$ for nonempty sets

$$\text{SI}(p_1, u_1, p_2, u_2, \ldots, p_i, 0, p_{i+1}, \ldots, p_k, u_k, p_{k+1})$$

and the following conditions hold:

$$\text{SI}(p_1, 0, p_1) = \{0\},$$

$$\text{SI}(p_1, u_1, p_2, u_2, \ldots, p_i, 0, p_{i+1}, \ldots, p_k, u_k, p_{k+1}) =$$

$$= \text{SI}(p_1, u_1, p_2, u_2, \ldots, p_i, u_{i+1}, p_{i+2}, \ldots, p_k, u_k, p_{k+1}).$$

If all types $p_i$ equal to a type $p$ then we write $\text{SI}_p(X_1, X_2, \ldots, X_k)$ and $\text{SI}_p(u_1, u_2, \ldots, u_k)$ as well as $[X_1, X_2, \ldots, X_k]_p$ and $[u_1, u_2, \ldots, u_k]_p$ instead of

$$\text{SI}(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$$
and
\[ \text{SI}(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \]
respectively. We omit the index \( \cdot_p \) if the type \( p \) is fixed. In this case, we write \( \theta_{u_1, u_2, \ldots, u_k}(x, y) \) instead of \( \theta_{p, u_1, p, u_2, \ldots, p, u_k, p}(x, y) \).

**Proposition 1.1.** (1) If \( p, q \in S^1(T) \) are principal types then \( \rho_{\nu(p,q)} \cup \rho_{\nu(q,p)} \subseteq U^\geq 0 \).

(2) If \( p, q \in S^1(T) \), \( p \) is a principal type and \( q \) is a non-principal type then \( \rho_{\nu(p,q)} = \emptyset \) and \( \rho_{\nu(q,p)} \subseteq U^- \).

**Proof.** (1) If \( \rho_{\nu(p,q)} \) contains a label \( u \notin U^\geq 0 \) then there are realizations \( a \) and \( b \) of \( p \) and \( q \) respectively such that \( (a, b) \in \text{SI}(p,q) \) and \( (b, a) \notin \text{SI}(p,q) \). But since \( p(x) \) contains a principal formula \( \varphi(x) \), this formula witnesses that \( (b, a) \in \text{SI}(p,q) \). The contradiction implies that \( \rho_{\nu(p,q)} \subseteq U^\geq 0 \). Similarly we obtain \( \rho_{\nu(q,p)} \subseteq U^- \).

(2) Let \( \varphi(x) \) be a principal formula of \( p(x) \). If \( \models p(a), \models q(b) \), and \( (a, b) \in \text{SI}(p,q) \) that witnessed by a formula \( \theta_u(x, y) \), the formula \( \exists x(\varphi(x) \land \theta_u(x, y)) \) isolates \( q(y) \). Since \( q \) is not isolated we obtain \( \rho_{\nu(p,q)} = \emptyset \). By the same reason, \( \rho_{\nu(q,p)} \subseteq U^- \). □

**Corollary 1.2.** If \( p(x) \) is a principal type then \( \rho_{\nu(p)} \subseteq U^\geq 0 \).

**Proposition 1.3.** Let \( p_1, p_2, \ldots, p_{k+1} \) be types in \( S^1(\emptyset) \). The following assertions hold.

(1) If \( u_i \in \rho_{\nu(p_i, p_{i+1})}, i = 1, \ldots, k, \) and some \( u_i \) is negative then
\[ \text{SI}(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \subseteq U^- . \]

(2) If \( u_i \in \rho_{\nu(p_i, p_{i+1})} \cap U^\geq 0, i = 1, \ldots, k, \) then
\[ \text{SI}(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \subseteq U^\geq 0 . \]

(3) If \( u_i \in \rho_{\nu(p_i, p_{i+1})} \cap (U^\geq 0 \cup U'), i = 1, \ldots, k, \) and some \( u_i \) belongs to \( U' \) then
\[ \text{SI}(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \subseteq U' . \]

(4) If \( u_i \in \rho_{\nu(p_i, p_{i+1})} \cap U^\geq 0, i = 1, \ldots, k, \) then all elements of the set \( X = \text{SI}(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \) are invertible and the set \( X^{-1} = \cup \{v^{-1} | v \in X\} \) coincides with the set \( \text{SI}(p_{k+1}, u_k^{-1}, p_k, u_k^{-1}, \ldots, p_2, u_2^{-1}, p_1) \).

**Proof.** (1)–(3) follow by the transitivity of semi-isolation.
(4) All elements in $X$ are invertible by (2). Let $v'$ be an element in $v^{-1} \subseteq X^{-1}$. Then for any $(p_1, \theta_{p_1,v,p_{k+1}}, p_{k+1})$-edge $[a, b]$ the following conditions hold:

(a) there are realizations $a_i$ of $p_i$, $i = 1, \ldots, k + 1$, such that $a_0 = a$, $a_{k+1} = b$, $\models \theta_{p_i,u_i,p_{i+1}}(a_i, a_{i+1})$, $i = 1, \ldots, k$;

(b) $[b, a]$ is a $(p_{k+1}, \theta_{p_{k+1},v,p_1}, p_1)$-edge.

Since $[a_{i+1}, a_i]$ is an $u'_i$-edge for any $u'_i \in u_i^{-1}$, $i = 1, \ldots, k$, then

$$\theta_{p_{k+1},v',p_1}(b, x) \vdash \theta_{p_{k+1},u'_{k},p_k,u'_k,\ldots,p_2,u'_1,p_1}(b, x),$$

whence, $v' \in SI(p_{k+1}, u_k^{-1}, p_k, u_{k-1}^{-1}, \ldots, p_2, u_1^{-1}, p_1)$.

If $v' \in SI(p_{k+1}, u'_k, p_k, u'_{k-1}, \ldots, p_2, u'_1, p_1)$, $u'_i \in u_i^{-1}$, $i = 1, \ldots, k$, then $v' \geq 0$ and for any $(p_{k+1}, \theta_{p_{k+1},v',p_1}, p_1)$-edge $[b, a]$ there are realizations $b_i$ of $p_i$, $i = 1, \ldots, k + 1$, such that $b_{k+1} = b$, $b_1 = a$, $\models \theta_{p_i,u_i,p_{i+1}}(b_i, b_{i+1})$, $i = 1, \ldots, k$. We have $\models \theta_{p_i,u_i,p_{i+1}}(b_i, b_{i+1})$, $i = 1, \ldots, k$, and so the elements $b_1, \ldots, b_{k+1}$ witness that $[a, b]$ is a $(p_1, \theta_{p_1,u_1,p_2,u_2,\ldots,U,p_k,p_{k+1}}, p_{k+1})$-edge. Then $v'$ belongs to $v^{-1}$, where $v \in X$ is a label for the formula $\theta_{p_1,u_1,p_2,u_2,\ldots,U,p_k,p_{k+1}}$. Thus, $v' \in X^{-1}$. □

**Corollary 1.4.** Restrictions of $U$ to the sets $U^{\leq 0}$, $U^{\geq 0}$, and $U^{\geq 0} \cup U'$ form subalgebras of the algebra of distributions of binary semi-isolating formulas. The operation of inversion is coordinated with the operations of the algebra.

2. Preordered algebras of distributions of binary semi-isolating formulas

For the set $U$ of labels in the algebra $\mathfrak{A}$ of binary semi-isolating formulas of theory $T$, we define the following relation $\preceq$: if $u, v \in U$ then $u \preceq v$ if and only if $u = v$, or $u, v \in \rho_\nu(p,q)$ for some types $p, q \in S^1(\emptyset)$ and $\theta_{p,u,q}(a, y) \vdash \theta_{p,v,q}(a, y)$ for some (any) realization $a$ of $p$. If $u \preceq v$ and $u \neq v$ we write $u \prec v$.

By the definition the relation $\preceq$ is reflexive and transitive. It is antisymmetric since distinct labels correspond to non-equivalent formulas.

Below we consider some properties for the substructures $\langle \rho_\nu(p,q); \preceq \rangle$ of the partially ordered set $\langle U; \preceq \rangle$.

**Proposition 2.1.** (1) For any types $p, q \in S^1(\emptyset)$ the partially ordered set $\langle \rho_\nu(p,q); \preceq \rangle$ forms an upper semilattice.

(2) An element $u \in \rho_\nu(p,q)$ is $\preceq$-minimal if and only if for a realization $a$ of $p$, the formula $\theta_{p,u,q}(a, y)$ is isolating.
(3) (monotony). If \( u, v \in \rho_{p,q} \) and \( u \leq v \) then \( v \in U^\delta, \delta \in \{-, +\}, \)
implies \( u \in U^\delta, \) and if \( u \in U' \) then \( v \in U' \).

Proof. (1) If \( u_1, u_2 \in \rho_{p,q} \) then for the formulas \( \theta_{p,u_1,q}(x, y) \) and \( \theta_{p,u_2,q}(x, y) \)
the label \( v \) for the formula \( \theta_{p,u_1,q}(x, y) \lor \theta_{p,u_2,q}(x, y) \) is a supremum for the
labels \( u_1 \) and \( u_2 \).

(2) If \( \theta_{p,u,q}(a, y) \) is an isolating formula then the label \( u \) is \( \preceq \)-minimal
by the definition. If the formula \( \theta_{p,u,q}(a, y) \) is not isolating then there is
a formula \( \varphi(a, y) \) such that the semi-isolating formulas \( \theta_{p,u,q}(a, y) \land \varphi(a, y) \)
and \( \theta_{p,u,q}(a, y) \land \neg \varphi(a, y) \) are consistent. For the labels \( v_1 \) and \( v_2 \) of these
formulas, we have \( v_1 \neq v_2, v_1 \triangleleft u, \) and \( v_2 \preceq u \).

(3) If \( v \in \rho_{p,q} \cap U^- \) then for any solution \( b \) of the formula \( \theta_{p,v,q}(a, y) \),
where \( \models p(a) \), the pair \( (a, b) \) is an irreversible arc. Hence, for any solution
\( b \) of \( \theta_{p,u,q}(a, y) \), where \( u \leq v \), the pair \( (a, b) \) is also an irreversible arc and so
\( u \) belongs to \( U^- \). Replacing arcs by edges, the same arguments show that
\( u \preceq v \) and \( v \in U^+ \) imply \( u \in U^+ \). If \( u \in U' \) then the set of pairs \( (a, b) \) for
the formula \( \theta_{p,u,q}(a, y) \) contains both irreversible and reversible arcs. This
property is preserved for any label \( v \) with \( u \leq v \), whence \( v \in U' \). \( \square \)

The partial order \( \preceq \) has a natural extension to a preorder on the set
\( \mathcal{P}(U) \): for any sets \( X, Y \in \mathcal{P}(U) \) we put \( X \preceq Y \) if \( X = \emptyset \), or for any \( x \in X \)
there is \( y \in Y \) with \( x \preceq y \) and for any \( y \in Y \) there is \( x \in X \) with \( x \preceq y \).
Thus, the algebra \( \mathfrak{A} \) is transformed to the preordered algebra \( \langle \mathfrak{A}; \preceq \rangle \)
with the monotonic property with respect its restrictions to the sets \( U^{\leq 0}, U^{\geq 0}, \) and
\( U' \).

Another natural expansion of the now preordered algebra \( \langle \mathfrak{A}; \preceq \rangle \) is based
on the the properties mentioned that if \( u_1, u_2 \in \rho_{p,q} \) and \( v \in \rho_{q,r} \)
then the formulas \( \theta_{p,u_1,q,v,r}(x, y) \) and \( \theta_{p,u_1,q}(x, y) \lor \theta_{p,u_2,q}(x, y) \) as well as \( \theta_{p,u,q}(x, y) \land
\theta_{p,u,q}(x, y) \land \neg \theta_{p,u_2,q}(x, y) \) (if the formulas \( \theta_{p,u,q}(a, y) \land \theta_{p,u_2,q}(a, y) \)
and \( \theta_{p,u_1,q}(a, y) \land \neg \theta_{p,u_2,q}(a, y) \) are consistent for \( \models p(a) \)) have labels in \( U \).
We denote these labels by \( u_1 \circ v, u_1 \lor u_2, u_1 \land u_2, \) and \( u_1 \land \neg u_2 \) respectively.
The last label is also denoted by \( \neg u_2 \land u_1 \). The label \( u_1 \circ v \) is the composition
of labels \( u_1 \) and \( v \); \( u_1 \lor u_2 \) is the union or the disjunction of labels \( u_1 \) and \( u_2 \);
\( u_1 \land u_2 \) is their intersection or conjunction; \( u_1 \land \neg u_2 \) is the relative complement
of \( u_2 \) in \( u_1 \).

Clearly, \( u_1 \preceq u_1 \lor u_2, u_2 \preceq u_1 \lor u_2, u_1 \land u_2 \preceq u_1, u_1 \land u_2 \preceq u_2, u_1 \land \neg u_2 \preceq u_1 \).

We set

\[
(p, (u_1 \circ v), r) \ iff \ \begin{cases} 
\{u_1 \circ v\}, & \text{if } u_1 \in \rho_{p,q} \text{ and } v \in \rho_{q,r}; \\
\emptyset, & \text{if } u_1 \notin \rho_{p,q} \text{ or } v \notin \rho_{q,r};
\end{cases}
\]
\[(p, (u_1 \lor u_2), q) \begin{cases} \{u_1 \lor u_2\}, & \text{if } u_1 \in \rho_{v(p,q)} \text{ and } u_2 \in \rho_{v(p,q)}, \\ \{u_1\}, & \text{if } u_1 \in \rho_{v(p,q)} \text{ and } u_2 \notin \rho_{v(p,q)}, \\ \{u_2\}, & \text{if } u_1 \notin \rho_{v(p,q)} \text{ and } u_2 \in \rho_{v(p,q)}, \\ \emptyset, & \text{if } u_1 \notin \rho_{v(p,q)} \text{ and } u_2 \notin \rho_{v(p,q)}, \end{cases}\]

\[(p, (u_1 \land u_2), q) \begin{cases} \{u_1 \land u_2\}, & \text{if } u_1 \in \rho_{v(p,q)} \text{ and } u_2 \in \rho_{v(p,q)}, \\ \emptyset, & \text{otherwise}, \end{cases}\]

\[(p, (u_1 \land \neg u_2), q) \begin{cases} \{u_1 \land \neg u_2\}, & \text{if } u_1 \in \rho_{v(p,q)} \text{ and } u_2 \in \rho_{v(p,q)}, \\ \emptyset, & \text{otherwise}, \end{cases}\]

\[(p, (X_1 \tau X_2), q) \begin{cases} \{p, (u_1 \tau u_2), q) \mid u_1 \in X_1, u_2 \in X_2\}, & \tau \in \{\land, \lor, \land\}, \\ \{p, (X_1 \neg X_2), q\} \begin{cases} \emptyset, & \text{if } X_1 = X_2 \land X_1, \end{cases} \end{cases}\]

\[(p, (X_1 \neg X_2), q) = \{p, (\neg X_2 \land X_1), q\} = \emptyset \cup \{p, (u_1 \land \neg u_2), q) \mid u_1 \in X_1, u_2 \in X_2\}, \ X_1, X_2 \in \mathcal{P}(U).\]

Labels \(u_1\) and \(u_2\) are consistent if \(u_1 \land u_2 \in U\). If \(u_1 \land u_2 = \emptyset\) the labels \(u_1\) and \(u_2\) are called inconsistent.

The preordered algebra \((\mathfrak{A}; \leq)\) equipped with binary operations \((p, (\cdot \tau \cdot), q), \tau \in \{\lor, \land, \land\}, \) and \((p, (\cdot \land \neg \cdot), q), p, q \in S^1(\emptyset),\) is called a preordered algebra with relative set-theoretic operations and the composition or briefly a POSTC-algebra.

For any types \(p, q \in S^1(\emptyset)\) the structure \(\langle \rho_{v(p,q)} \cup \{\emptyset\}; \lor, \land, \emptyset\rangle\) with operations \(\lor\) and \(\land\) on labels, being extended by equalities \(u \lor \emptyset = u, u \land \emptyset = \emptyset,\) where \(u \in \rho_{v(p,q)} \cup \{\emptyset\},\) is an Ershov algebra, i.e., a distributive lattice with zero \(\emptyset\) and relative complements \(\neg\) such that for any \(u, v \in \rho_{v(p,q)}\) if \(u \leq v\) and \(u' = \neg u \land v\) is a label then \(u \land u' = \emptyset\) and \(u \lor u' = v,\) and if the label \(u'\) does not exist then \(u = v.\)

A label \(u \in U\) is an atom or an atomic label if \(u\) is a \(\leq\)-minimal element in \(U,\) i.e., for any label \(v \in U\) if \(v \leq u\) then \(v = u.\)

By Proposition 2.1, the set of atoms equals the set of isolating labels and, thus, each atom \(u \in \rho_{v(p,q)}\) is represented by an isolated formula \(\theta_{p,u,q}(a, y),\) where \(\models p(a).\)

Let \(R\) be a nonempty family of types in \(S^1(\emptyset), \mathfrak{A}_R\) be a restriction of POSTC-algebra \(\mathfrak{A}\) to the family \(R.\) The structure \(\mathfrak{A}_R\) is atomic if for any types \(p, q \in R\) and for any label \(u \in \rho_{v(p,q)}\) there is an atom \(v \in \rho_{v(p,q)}\) such that \(v \leq u.\) The POSTC-algebra \(\mathfrak{A}\) is called \(R\)-atomic if \(\mathfrak{A}_R\) is atomic. If \(R = S^1(\emptyset)\) then the \(R\)-atomic POSTC-algebra is called atomic.
Using the definition of atomic structure, of \( R \)-atomic POSTC-algebra, and of small theory we obtain the following assertions.

**Proposition 2.2.** If \( R \) is a nonempty family of types in \( S^1(\emptyset) \) and for any type \( p \in R \), there is an atomic model \( M_p \) over a realization of \( p \), then the POSTC-algebra \( \mathfrak{A} \) is \( R \)-atomic.

**Corollary 2.3.** If \( T \) is a small theory then the POSTC-algebra \( \mathfrak{A} \) is atomic.

### 3. Ranks and degrees of semi-isolation

The following definition is a local variation of Morley rank [18].

**Definition.** For triples \((p,u,q)\), where \( p,q \in S^1(\emptyset) \), \( u \in U \cup \{\emptyset\} \), we define inductively the rank \( si(p,u,q) \) of semi-isolation:

1. \( si(p,u,q) = 0 \) if \( u \not\in \rho_{\nu}(p,q) \);
2. \( si(p,u,q) \geq 1 \) if \( u \in \rho_{\nu}(p,q) \);
3. for a positive ordinal \( \alpha \), \( si(p,u,q) \geq \alpha + 1 \) if there is a set \( \{v_i \mid i \in \omega\} \) of pairwise inconsistent labels such that \( v_i < u \) and \( si(p,v_i,q) \geq \alpha \), \( i \in \omega \);
4. for a limit ordinal \( \alpha \), \( si(p,u,q) \geq \alpha \) if \( si(p,u,q) \geq \beta \) for any \( \beta \in \alpha \).

As usual, we write \( si(p,u,q) = \alpha \) if \( si(p,u,q) \geq \alpha \) and \( si(p,u,q) \not\geq \beta \) for any \( \beta \in \alpha \).

If types \( p \) and \( q \) are fixed, we write \( si(u) \) instead of \( si(p,u,q) \) and this value is said to be the rank of semi-isolation or the si-rank of the label \( u \) or of the element \( u = \emptyset \) (with respect to the pair \((p,q)\)). For a formula \( \theta_{p,u,q}(x,y) \) we set \( si(\theta_{p,u,q}(x,y)) \equiv si(u) \).

Clearly, if the theory is small then the si-rank of any label is an ordinal (having a label \( u \) with \( si(p,u,q) = \infty \), we get continuum many complete types \( r(x,y) \supseteq p(x) \cup q(y) \)).

By the definition we have the following inequality for any formula \( \theta_{p,u,q}(x,y) \) and any realization \( a \) of \( p \) giving a low bound for Morley rank of the formula \( \theta_{p,a,q}(a,y) \) by the si-rank:

\[
si(\theta_{p,u,q}(x,y)) \leq MR(\theta_{p,u,q}(a,y)) + 1. \tag{1}
\]

The inequality (1) implies

**Remark 3.1.** If a theory \( T \) has a finite Morley rank then si-ranks of labels \( \bigcup_{p,q \in S^1(T)} \rho_{\nu}(p,q) \) are bounded by the value \( MR(x \approx x) + 1 \).
We set \( \text{si}(p, q) \equiv \sup \{ \text{si}(p, u, q) \mid u \in U \cup \{\emptyset\} \} \), \( \text{si}(p) \equiv \text{si}(p, p) \). For a nonempty family \( R \) of 1-types, we put \( \text{si}(R) \equiv \sup \{ \text{si}(p, q) \mid p, q \in R \} \). A family \( R \) is called \text{si-minimal} if \( \text{si}(R) = 1 \). The value \( \text{si}(p, q) \) is said to be the \text{rank of semi-isolation} or the \text{si-rank} of pair \((p, q)\), and \( \text{si}(R) \) is the \text{rank of semi-isolation} or the \text{si-rank} of the family \( R \).

Since there are \( |T| \) formulas of a theory \( T \) and the inequality (1) holds we obtain

**Proposition 3.2.** Each \( \text{si-rank} \) in a theory \( T \) is either equal to \( \infty \) or less than \( \min\{ |T|^+, (\text{MR}(x \approx x) + 1)^+ \} \). If Morley rank \( \text{MR}(x \approx x) \) is equal to an ordinal \( \alpha \) then any \( \text{si-rank} \) in \( T \) is not more than \( \alpha + 1 \).

The estimation for \( \text{si-ranks} \) in Proposition 3.2 can be far from exact. For instance, \( \text{si-ranks} \) in \( \omega \)-categorical theories are finite while there are non-\( \omega \)-stable \( \omega \)-categorical theories.

**Proposition 3.3.** For any types \( p, q \in S^1(\emptyset) \) the following assertions are satisfied.

1. If \( u, v \in \rho_{\nu(p, q)} \cup \{\emptyset\} \) and \( u \preceq v \) then \( \text{si}(u) \leq \text{si}(v) \).
2. If \( u, v \in \rho_{\nu(p, q)} \cup \{\emptyset\} \) then \( \text{si}(u \lor v) = \max\{\text{si}(u), \text{si}(v)\} \) and \( \text{si}(u \land v) \leq \min\{\text{si}(u), \text{si}(v)\} \). The last inequality is transformed to the equality if and only if there is a label \( v' \) such that \( v' \preceq u \), \( v' \preceq v \), and \( \text{si}(v') = \text{si}(u) \) or \( \text{si}(v') = \text{si}(v) \).
3. The equality \( \text{si}(p, q) = 0 \) holds if and only if there are no realizations of \( p \) semi-isolating realizations of \( q \).
4. The equality \( \text{si}(p, q) = 1 \) holds if and only if there is a \((p \rightarrow q)\)-formula and each \((p \rightarrow q)\)-formula \( \varphi(x, y) \) is equivalent to a disjunction of formulas \( \varphi_i(x, y) \) such that each formula \( \varphi_i(a, y) \) is isolating, where \( \models p(a) \).

Proof is obvious. \( \square \)

**Proposition 3.4.** For any nonempty family \( R \subseteq S^1(\emptyset) \) the following assertions are satisfied.

1. \( \text{si}(R) \geq 1 \).
2. The family \( R \) is \text{si-minimal} if and only if for any types \( p, q \in R \) each \((p \rightarrow q)\)-formula \( \varphi(x, y) \) is equivalent to a disjunction of formulas \( \varphi_i(x, y) \) such that each formula \( \varphi_i(a, y) \) is isolating, where \( \models p(a) \).

Proof. (1) is implied by the inequality \( \text{si}(p) \geq 1 \) for any type \( p \in S^1(\emptyset) \) since the formula \( (a \approx y) \) witnesses that \( a \) semi-isolates itself, where \( \models p(a) \). (2) is an obvious corollary of (1) and Proposition 3.3, (4). \( \square \)
Remark 3.5. Since for a strongly minimal theory $T$ the set of solutions for any formula $\varphi(a, y)$ is finite or cofinite, any semi-isolating formula $\psi(a, y)$ is represented as a finite disjunction of some isolating formulas $\psi_i(a, y)$ or as a negation of a finite disjunction of isolating formulas $\psi_i(a, y)$. If $\psi(a, y) \models p(y)$ and $p(y)$ is a non-principal type then the representation of $\psi(a, y)$ is possible only as a finite disjunction of isolating formulas. It means that $\text{si}(p) = 1$. If $\text{si}(p)$ is a principal type and there are finitely many pairwise non-equivalent isolating formulas $\psi(a, y)$ with $\models p(a)$ and $\psi(a, y) \models p(y)$ then $\text{si}(p) = 1$ too. If there are infinitely many these pairwise non-equivalent isolating formulas $\psi(a, y)$ then $\text{si}(p) = 2$. \(\square\)

Definition. Let $\alpha$ be a positive ordinal, $u_1$ and $u_2$ be labels in $\rho_{\nu(p, q)}$ such that $\text{si}(u_1) = \text{si}(u_2) = \alpha$. The labels $u_1$ and $u_2$ are $\alpha$-almost identically or $\sim_\alpha$-equivalent (denoted by $u_1 \sim_\alpha u_2$) if $\text{si}(u_1 \bowtie u_2) < \alpha$, where $u_1 \bowtie u_2 := (u_1 \land \neg u_2) \lor (-u_1 \lor u_2)$.

Proposition 3.6. The relation $\sim_\alpha$ is an equivalence relation for any set of labels in $\rho_{\nu(p, q)}$ having the si-rank $\alpha$.

Proof. Clearly the relation $\sim_\alpha$ is reflexive and symmetric. For the checking of transitivity we assume that $u_1 \sim_\alpha u_2$ and $u_2 \sim_\alpha u_3$. Since $(u_1 \land \neg u_2 \land u_3) \leq (u_1 \land \neg u_2) \leq (u_1 \bowtie u_2)$ we have

$$\text{si}(u_1 \land \neg u_2 \land u_3) \leq \text{si}(u_1 \bowtie u_2) < \alpha.$$  

As $u_1 \land u_3 = (u_1 \land u_2 \land u_3) \lor (u_1 \land \neg u_2 \land u_3) \lor (u_1 \land \neg u_2 \land u_3) < \alpha$, for $u_1 \sim_\alpha u_3$, it is enough to prove that $\text{si}(u_1 \land u_2 \land u_3) = \alpha$. Suppose on contrary that $\text{si}(u_1 \land u_2 \land u_3) < \alpha$. Then $\text{si}(u_1 \land u_2) = \alpha$ and

$$u_1 \land u_2 = (u_1 \land u_2 \land u_3) \lor (u_1 \land u_2 \land \neg u_3)$$

imply $\text{si}(u_1 \land u_2 \land \neg u_3) = \alpha$. But $(u_1 \land u_2 \land \neg u_3) \leq (u_2 \land \neg u_3) \leq (u_2 \bowtie u_3)$, and $\text{si}(u_2 \bowtie u_3) < \alpha$ gives $\text{si}(u_1 \land u_2 \land \neg u_3) < \alpha$. The obtained contradiction means that $u_1 \sim_\alpha u_3$. \(\square\)

By the definition, for any label $u \in \rho_{\nu(p, q)}$ having the si-rank $\alpha$, there is a greatest number $n \in \omega \setminus \{0\}$ of pairwise inconsistent (or, that equivalent, of pairwise non-$\sim_\alpha$-equivalent) labels $u_1, \ldots, u_n$ such that $u_i \leq u$ and $\text{si}(u_i) = \alpha$, $i = 1, \ldots, n$. This number $n$ is called the degree of semi-isolation or the si-degree of label $u$ and it is denoted by $\deg(p, u, q)$ or by $\deg(u)$. We have $\text{si}(\emptyset) = 0$ and put $\deg(\emptyset) = 1$. 

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Proposition 3.7. (1) If \( u \in \rho_{\nu(p,q)} \) and \( \text{si}(u) = \alpha \) then \( \text{deg}(u) \) is equal to the number of pairwise inconsistent labels \( u_1, \ldots, u_n \in \rho_{\nu(p,q)} \) having the si-rank \( \alpha \), the si-degree 1, and such that \( u = u_1 \lor \ldots \lor u_n \).

(2) If \( u, v \in \rho_{\nu(p,q)} \), \( \text{si}(u) = \text{si}(v) \), and \( u \leq v \) then \( \text{deg}(u) \leq \text{deg}(v) \).

(3) If \( u, v \in \rho_{\nu(p,q)} \) and \( \text{si}(u) = \text{si}(v) \) then

\[
\text{deg}(u \lor v) \leq \text{deg}(u) + \text{deg}(v).
\]

The equality in this inequality holds if and only if \( \text{si}(u \land v) < \text{si}(u) \). If \( \text{si}(u \land v) = \text{si}(u) \) then

\[
\text{deg}(u \lor v) = \text{deg}(u) + \text{deg}(v) - \text{deg}(u \land v).
\]

(4) If \( u \in \rho_{\nu(p,q)} \) is a label for an isolating formula, i.e., \( u \) is an atom, then \( \text{si}(u) = 1 \) and \( \text{deg}(u) = 1 \).

(5) If for a label \( u \in \rho_{\nu(p,q)} \), \( \text{si}(u) = 1 \) and \( \text{deg}(u) = 1 \), then \( u \) is not neutral.

(6) If \( u \in \rho_{\nu(p,q)} \) and \( \text{si}(u) = 1 \) then \( \text{deg}(u) \) is equal to the number of pairwise inconsistent labels \( u_1, \ldots, u_n \in \rho_{\nu(p,q)} \) for isolating formulas such that \( u = u_1 \lor \ldots \lor u_n \).

Proof is obvious. \( \square \)

If there is a label \( u \in \rho_{\nu(p,q)} \) with \( \text{si}(p,q) = \text{si}(u) \) then the degree of semi-isolation or the si-degree \( \text{deg}(p,q) \) of pair \( (p,q) \) is

\[
\text{deg}(p) \equiv \text{deg}(p,p).
\]

If for a nonempty family \( R \) of 1-types there is a label \( u \in \rho_{\nu(p,q)}, p,q \in R \), with \( \text{si}(R) = \text{si}(u) \) then the degree of semi-isolation or the si-degree \( \text{deg}(R) \) of \( R \) is

\[
\text{sup}\{\text{deg}(u) \mid u \in \rho_{\nu(R)}, \text{si}(R) = \text{si}(u)\}.
\]

Clearly, if \( \text{deg}(p,q) \) or \( \text{deg}(R) \) exist then these values are positive natural numbers or equal \( \omega \).

For an ordinal \( \alpha \), a natural number \( n \geq 1 \), and a set \( X \in \{U, U \cup \{\emptyset\}\} \) we put

\[
X \upharpoonright (\alpha, n) \equiv \{u \in X \mid \text{si}(u) \leq \alpha \text{ and if } \text{si}(u) = \alpha \text{ then } \text{deg}(u) < n\},
\]
Clearly, if $\alpha = \beta + 1$ then $X \upharpoonright (\alpha, 1) = X \upharpoonright \beta$, and if $\alpha$ is a limit ordinal then $X \upharpoonright (\alpha, 1) = \bigcup_{\beta < \alpha} X \upharpoonright \beta$.

For ordinals $\alpha, \beta, \beta \in (\omega + 1) \setminus \{0\}$, and for the algebra $\mathfrak{A}$ of distributions of binary semi-isolating formulas of theory $T$ as well as for expansions and restrictions $\mathfrak{A}'$ of $\mathfrak{A}$, defined in previous sections, we denote by $\mathfrak{A} \upharpoonright (\alpha, \beta)$ and $\mathfrak{A}' \upharpoonright (\alpha, \beta)$ as well as by $\mathfrak{A}_{\alpha, \beta}$ and $\mathfrak{A}'_{\alpha, \beta}$ the restrictions of these algebras to the set $(U \cup \{\emptyset\}) \upharpoonright (\alpha, \beta)$. If $\beta = \omega$, these restrictions are denoted by $\mathfrak{A} \upharpoonright \alpha$, $\mathfrak{A}' \upharpoonright \alpha$, $\mathfrak{A}_\alpha$, and $\mathfrak{A}'_\alpha$. The restrictions are called the $(\alpha, \beta)$-restrictions and the $\alpha$-restrictions respectively.

Since the si-rank of each label is positive, non-trivial restrictions (i. e., with nonempty sets of used labels) are only the restrictions of algebras with $\alpha > 0$. If $\text{si}(S^1(\emptyset)) = \alpha_0$ then, taking into consideration the inequality $\alpha > 0$, all essential (i. e., reflecting links of sets of labels of semi-isolating formulas with respect to their si-ranks) restrictions of these algebras are formed only for $0 < \alpha < \alpha_0$.

In view of Proposition 3.7 we obtain

**Proposition 3.8.** The algebra of distributions of binary isolating formulas of theory $T$ coincides with the algebra $\mathfrak{A} \upharpoonright (1, 2)$. The algebra $\mathfrak{A} \upharpoonright 1$ consists of labels being disjunctions of labels of isolating formulas.

4. **Monoid of distributions of binary semi-isolating formulas on a set of realizations of a type**

Consider a complete theory $T$, a type $p(x) \in S(T)$, a regular labelling function $\nu(p): \text{PF}(p)/\text{PE}(p) \to U$, and a family of sets $\text{SI}_p(u_1, \ldots, u_k)$ of labels of binary semi-isolating formulas, $u_1, \ldots, u_k \in \rho_{\nu(p)}$, $k \in \omega$.

Below we show some basic properties for sets

$$[u_1, \ldots, u_k] \vdash \text{SI}_p(u_1, \ldots, u_k).$$

**Proposition 4.1** (Associativity). For any $u_1, u_2, u_3 \in \rho_{\nu(p)}$, the following equalities hold:

$$[[u_1, u_2], u_3] = [u_1, u_2, u_3] = [u_1, [u_2, u_3]].$$
Proof of inclusions \([\{u_1, u_2\}, u_3] \subseteq [u_1, u_2, u_3]\) and \([u_1, [u_2, u_3]] \subseteq [u_1, u_2, u_3]\) repeats [2] Proposition 3.1, 4.

The reverse inclusions are satisfied since, taking labels \(v_1\) and \(v_2\) for the formulas \(\theta_{u_1,u_2}(x, y)\) and \(\theta_{u_2,u_3}(x, y)\), we obtain, for \(\models p(a)\), that the formulas \(\theta_{v_1,u_3}(a, y)\), \(\theta_{u_1,u_2,u_3}(a, y)\), and \(\theta_{u_1,v_2}(a, y)\) are pairwise equivalent, i. e.,
\[
[v_1, u_3] = [u_1, u_2, u_3] = [u_1, v_2]. \Box
\]

In view of associativity, using the induction on number of parenthesis, we prove that all operations \([\cdot, \ldots, \cdot]\) acting on sets in \(\mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}\) are generated by the binary operation \([\cdot, \cdot]\) on the set \(\mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}\) and the values \([X_1, X_2, \ldots, X_k], X_1, X_2, \ldots, X_k \subseteq \rho_{\nu(p)}\), do not depend on the sequence of adding of brackets for
\[
X_{i,i+1,\ldots,i+m+n} = [X_{i,i+1,\ldots,i+m}, X_{i+m+1,\ldots,i+m+2,\ldots,i+m+n}],
\]
where \(X_{1,2,\ldots,k} = [X_1, X_2, \ldots, X_k]\).

Thus the structure \(\mathfrak{S}J_{\nu(p)} := \langle \mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}; [\cdot, \cdot]\rangle\) is a semigroup admitting the representation of all operations \([\cdot, \ldots, \cdot]\) by terms of the language \([\cdot, \cdot]\). Further the operation \([\cdot, \cdot]\) will be denoted also by \(\cdot\) and we shall use the record \(uv\) instead of \(u \cdot v\).

Since by the choice of the label 0 for the formula \((x \approx y)\) the equalities \(X \cdot \{0\} = X\) and \(\{0\} \cdot X = X\) are true for any \(X \subseteq \rho_{\nu(p)}\), the semigroup \(\mathfrak{S}J_{\nu(p)}\) has the unit \(\{0\}\), and it is a monoid. We have
\[
Y \cdot Z = \bigcup\{yz \mid y \in Y, z \in Z\}
\]
for any sets \(Y, Z \in \mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}\) in this structure.

Thus the following proposition holds.

**Proposition 4.2.** For any complete theory \(T\), any type \(p \in S(T)\), and the regular labelling function \(\nu(p)\), any operation \(\text{SI}_{p}(\cdot, \cdot, \ldots, \cdot)\) on the set \(\mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}\) interpretable by a term of the monoid \(\mathfrak{S}J_{\nu(p)}\).

The monoid \(\mathfrak{S}J_{\nu(p)}\) is called the monoid of binary semi-isolating formulas over the labelling function \(\nu(p)\) or the \(\text{SI}_{\nu(p)}\)-monoid.

In view of Propositions 1.3 and 4.1 we obtain

**Proposition 4.3.** For any complete theory \(T\), any type \(p \in S(T)\), and the regular labelling function \(\nu(p)\), the restriction \(\mathfrak{S}J_{\nu(p)}^{\leq 0}\) (respectively \(\mathfrak{S}J_{\nu(p)}^{\geq 0}\),
\( \mathcal{S} \mathcal{I}^{\geq 0, \text{neu}}_{\nu(p)} \) of the monoid \( \mathcal{S} \mathcal{I}_{\nu(p)} \) to the set \( U^{\leq 0} \cup U^{\geq 0} \cup U' \) is a submonoid of \( \mathcal{S} \mathcal{I}_{\nu(p)} \).

By Proposition 3.8, the (1, 2)-restriction of the monoid \( \mathcal{S} \mathcal{I}_{\nu(p)} \) coincides with the \( I_{\nu(p)} \)-groupoid \( \mathcal{P}_{\nu(p)} \). Besides, the (1, 2)-restrictions of monoids \( \mathcal{S} \mathcal{I}^{\leq 0}_{\nu(p)} \) and \( \mathcal{S} \mathcal{I}^{\geq 0}_{\nu(p)} \) equal respectively to the groupoid \( \mathcal{P}^{\leq 0}_{\nu(p)} \) and the monoid \( \mathcal{P}^{\geq 0}_{\nu(p)} \).

5. \( \alpha \)-deterministic and almost \( \alpha \)-deterministic \( \mathcal{S} \mathcal{I}_{\nu(p)} \)-monoids

In the following definition, we generalize the notions of deterministic and almost deterministic structure \( \mathcal{P}_{\nu(p)} \) proposed in [2].

**Definition.** Let \( U_0 \) be a subalphabet of the alphabet \( U \), \( \alpha \) be a positive ordinal, and \( n \geq 1 \) be a natural number. We put

\[
\rho_{\nu(p), \alpha, n} = \{ u \in \rho_{\nu(p)} \mid \text{si}(u) \leq \alpha, \deg(u) < n \text{ for } \text{si}(u) = \alpha \},
\]

\[
\rho_{\nu(p), \alpha} = \bigcup_{n \in \omega} \rho_{\nu(p), \alpha, n}.
\]

The partial subalgebra \( \mathcal{S} \mathcal{I}_{\nu(p)} \upharpoonright U_0 \) of the monoid \( \mathcal{S} \mathcal{I}_{\nu(p)} \) is called \( (\alpha, n) \)-deterministic if for any labels \( u_1, u_2 \in \rho_{\nu(p), \alpha, n} \cap U_0 \), the set \( [u_1, u_2] \cap U_0 \) consists of labels having the si-ranks \( \leq \alpha \) and contains less than \( n \) pairwise non\( \sim_{\alpha} \)-equivalent labels of si-rank \( \alpha \).

The partial subalgebra \( \mathcal{S} \mathcal{I}_{\nu(p)} \upharpoonright U_0 \) of the monoid \( \mathcal{S} \mathcal{I}_{\nu(p)} \) is called \( \alpha \)-deterministic if \( \mathcal{S} \mathcal{I}_{\nu(p)} \upharpoonright U_0 \) is \( (\alpha, 2) \)-deterministic.

The partial subalgebra \( \mathcal{S} \mathcal{I}_{\nu(p)} \upharpoonright U_0 \) of the monoid \( \mathcal{S} \mathcal{I}_{\nu(p)} \) is called almost \( \alpha \)-deterministic or \( (\alpha, \omega) \)-deterministic if for any labels \( u_1, u_2 \in \rho_{\nu(p), \alpha} \cap U_0 \), the set \( [u_1, u_2] \cap U_0 \) consists of labels having the si-ranks \( \leq \alpha \) and contains finitely many pairwise non\( \sim_{\alpha} \)-equivalent labels of si-rank \( \alpha \).

By the definition, each \( (\alpha, \omega) \)-deterministic structure \( \mathcal{S} \mathcal{I}_{\nu(p)} \upharpoonright U_0 \) is a union of its \( (\alpha, n) \)-deterministic substructures, \( n \geq 1 \). So each \( \alpha \)-deterministic structure \( \mathcal{S} \mathcal{I}_{\nu(p)} \upharpoonright U_0 \) is almost \( \alpha \)-deterministic.

If \( U_0 = U \) we shall not point out restrictions to the set \( U_0 \) for considered structures.

Below we show some basic properties of (almost) \( \alpha \)-deterministic partial algebras \( \mathcal{S} \mathcal{I}_{\nu(p)} \upharpoonright U_0 \).
Proposition 5.1. (Monotony) If a structure $\mathcal{S}(\nu \upharpoonright p) \upharpoonright U_0$ is (almost) $\alpha$-deterministic and $\beta$ is a positive ordinal then the structure $(\mathcal{S}(\nu \upharpoonright p) \upharpoonright U_0) \upharpoonright \beta$ is also (almost) $\alpha$-deterministic.

Proof is obvious. □

Proposition 5.2. For any monoid $\mathcal{S}(\nu \upharpoonright p)$ and ordinals $\alpha, \beta$, where $\alpha, \beta > 0$, $\beta \in \omega + 1$, the following conditions are equivalent:

(1) the monoid $\mathcal{S}(\nu \upharpoonright p)$ is $(\alpha, \beta)$-deterministic;

(2) $\text{si}(u_1 \circ u_2) \leq \alpha$ for any labels $u_1, u_2 \in \rho_{\nu(p), \alpha, \beta}$ and if $\text{si}(u_1 \circ u_2) = \alpha$ then $\text{deg}(u_1 \circ u_2) < \beta$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Consider arbitrary labels $u_1, u_2 \in \rho_{\nu(p), \alpha, \beta}$. Since, by hypothesis, $\text{si}(u_1 \circ u_2) \leq \alpha$ and $v \leq (u_1 \circ u_2)$ for any label $v \in [u_1, u_2]$, $[u_1, u_2]$ consists of labels of $\text{si}$-ranks $\leq \alpha$, and if $\text{si}(v) = \text{si}(u_1 \circ u_2) = \alpha$ then $\text{deg}(v) \leq \text{deg}(u_1 \circ u_2) < \beta$. Thus, the monoid $\mathcal{S}(\nu \upharpoonright p)$ is $(\alpha, \beta)$-deterministic. □

Proposition 5.2 immediately implies

Corollary 5.3. For any monoid $\mathcal{S}(\nu \upharpoonright p)$ and a positive ordinal $\alpha$ the following conditions are equivalent:

(1) the monoid $\mathcal{S}(\nu \upharpoonright p)$ is almost $\alpha$-deterministic;

(2) $\text{si}(u_1 \circ u_2) \leq \alpha$ for any labels $u_1, u_2 \in \rho_{\nu(p), \alpha}$.

Corollary 5.4. If $\text{si}(p)$ is an ordinal then the monoid $\mathcal{S}(\nu \upharpoonright p)$ is almost $\text{si}(p)$-deterministic.

Proposition 5.5. If a monoid $\mathcal{S}(\nu \upharpoonright p)$ is $(\alpha, \beta)$-deterministic then the structure $\mathcal{S}(\nu \upharpoonright p) \upharpoonright \alpha = \mathcal{S}(\nu \upharpoonright p) \upharpoonright \alpha$ is also an $(\alpha, \beta)$-deterministic monoid.

Proof. Since for any $\alpha$-restriction the associativity, the presence of unit $\{0\}$, and the $(\alpha, \beta)$-determinacy is preserved, it is enough to note that for any labels $u_1$ and $u_2$ in $\mathcal{S}(\nu \upharpoonright p, \alpha, \beta)$ there is a label $v$ in $\mathcal{S}(\nu \upharpoonright p, \alpha, \beta)$ belonging $[u_1, u_2]$. We can take $u_1 \circ u_2$ for $v$ since, by hypothesis, $\text{si}(u_1 \circ u_2) \leq \alpha$ and if $\text{si}(u_1 \circ u_2) = \alpha$ then $\text{deg}(u_1 \circ u_2) < \beta$. □

Proposition 5.6. If $\text{si}(p)$ is an ordinal then the monoid $\mathcal{S}(\nu \upharpoonright p)$ is $\text{si}(p)$-deterministic if and only if the value $\text{deg}(p)$ is not defined or equals 1.

Proof. If $\text{deg}(p)$ is not defined the ordinal $\alpha = \text{si}(p)$ is limit and can not be achieved by labels in $\rho_{\nu(p)}$. In particular, for any $u_1, u_2 \in \rho_{\nu(p)}$ the set $[u_1, u_2]$ does not contain labels of $\text{si}$-rank $\alpha$. If $\text{deg}(p) \geq 2$ then there are
non-$\sim_\alpha$-equivalent labels $u_1, u_2 \in \rho_\nu(p)$ of si-rank $\alpha$. Then $[u_1 \lor u_2, 0]$ contains the labels $u_1$ and $u_2$, whence the monoid $\mathcal{G}_\nu(p)$ is not $\alpha$-deterministic. If $\deg(p) = 1$ then there is unique, up to $\sim_\alpha$-equivalence, label in $\rho_\nu(p)$ having the si-rank $\alpha$. Since such a label is unique, the monoid $\mathcal{G}_\nu(p)$ is $\alpha$-deterministic. □

**Proposition 5.7.** The structure $\mathcal{P}_\nu(p)$ is (almost) deterministic if and only if the structure $\mathcal{G}_\nu(p),1,2$ is (almost) 1-deterministic.

*Proof.* follows by the equality $\mathcal{G}_\nu(p),1,2 = \mathcal{P}_\nu(p)$. □

**Proposition 5.8.** Let $p(x)$ be a complete type of a theory $T$, $\nu(p)$ be a regular labelling function, and $\text{si}(p) < \omega$. The following conditions are equivalent:

1. the monoid $\mathcal{G}_\nu(p)$ is $(1,n)$-deterministic for some $n \in \omega$;
2. the set $\rho_\nu(p)$ is finite;
3. the set $\rho_\nu(p),1$ finite;
4. the set $\rho_\nu(p),1,2$ (consisting of all atoms $u \in \rho_\nu(p)$) is finite.

*Proof.* If $\text{si}(p) > 1$ then, by $\text{si}(p) < \omega$, the set $\rho_\nu(p),1$ is infinite and so the set $\rho_\nu(p)$ is also infinite. Since each label in $\rho_\nu(p),1$ is a disjunction of labels in $\rho_\nu(p),1,2$ and for any labels $u_1, \ldots, u_n \in \rho_\nu(p),1$ the label $u_1 \lor \ldots \lor u_n$ belongs also to $\rho_\nu(p),1$, the set $\rho_\nu(p),1,2$ is infinite and the monoid $\mathcal{G}_\nu(p)$ is not $(1,n)$-deterministic for $n \in \omega$. Thus, none of the conditions (1)–(4) is not satisfied.

If $\text{si}(p) = 1$ then each label in $\rho_\nu(p)$ has the si-rank 1 and is represented as a disjunction of labels in $\rho_\nu(p),1,2$. Thus, the conditions (2)–(4) are equivalent. If the set $\rho_\nu(p),1,2$ contains $m \in \omega$ labels then there are $2^m - 1$ labels forming the set $\rho_\nu(p)$. Hence, the monoid $\mathcal{G}_\nu(p)$ is $(1,2^m - 1)$-deterministic. If the set $\rho_\nu(p),1,2$ is infinite then, for pairwise distinct labels $u_1, \ldots, u_m \in \rho_\nu(p),1,2$, the set $[u_1 \lor \ldots u_m, 0]$ contains $2^m - 1$ labels and, since $m$ is not bounded, the monoid $\mathcal{G}_\nu(p)$ is not $(1,n)$-deterministic for any $n$. Thus, the condition (1) is equivalent to each of the conditions (2)–(4). □

Proposition 5.8 and [2 Corollary 7.4] imply

**Corollary 5.9.** Let $p(x)$ be a complete type of a theory $T$, $\nu(p)$ be a regular labelling function, $\text{si}(p) < \omega$, and $\mathcal{G}_\nu(p)$ is a $(1,n)$-deterministic monoid, for some $n \in \omega$, having a negative label. Then the groupoid $\mathcal{G}_\nu(p),1,2$ generates the strict order property.
Definition [19]. Let $p(x)$ be a type in $S(T)$. A type $q(x_1, \ldots, x_n) \in S(T)$ is called a $(n, p)$-type if $q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^{n} p(x_i)$. The set of all $(n, p)$-types of $T$ is denoted by $S_{n,p}(T)$ and elements of the set $S_p(T) \equiv \bigcup_{n \in \omega \setminus \{0\}} S_{n,p}(T)$ are called $p$-types.

A type $q(\bar{y})$ in $S_p(T)$ is called $p$-principal if there is a formula $\varphi(\bar{y}) \in q(\bar{y})$ such that $\bigcup \{p(y_i) \mid y_i \in \bar{y}\} \cup \{\varphi(\bar{y})\} \vdash q(\bar{y})$.

Lemma 5.10 [19]. For any type $p$ and a natural number $n \geq 1$ the following conditions are equivalent:

1. the set of $(n, p)$-types with a tuple $(x_1, \ldots, x_n)$ of free variables is infinite;
2. there is a non-$p$-principal $(n, p)$-type.

Proposition 5.8 and Lemma 5.10 imply Corollary 5.11. If $p(x)$ is a complete type of a theory $T$, $\nu(p)$ is a regular labelling function, and all $(2, p)$-types are $p$-principal, then the monoid $\mathfrak{S}J_{\nu(p)}$ is $(1, n)$-deterministic for some $n \in \omega$.

By Corollaries 5.9 and 5.11, we obtain

Corollary 5.12. Let $p(x)$ be a complete type of a theory $T$, $\nu(p)$ be a regular labelling function, $\rho_{\nu(p)} \cap U^- \neq \emptyset$, and all $(2, p)$-types are $p$-principal. Then the groupoid $\mathfrak{G}J_{\nu(p),1,2}$ generates the strict order property.

For a type $p(x)$ and a positive ordinal $\alpha$, we denote by $SI_{p,\alpha}$ (in a model $\mathcal{M}$ of $T$) the relation of semi-isolation (over $\emptyset$) on a set of realizations of $p$ restricted to the set of formulas of si-rank $\leq \alpha$:

$$SI_{p,\alpha} = \{(a, b) \mid \mathcal{M} \models p(a) \land p(b) \text{ and } a \text{ semi-isolates } b \}.$$  

by a formula $\theta_{\alpha}(x, y)$ with asi-rank $\leq \alpha$.

Clearly, $I_p = SI_{p,1}$ for any type $p \in S^1(\emptyset)$. Seeing this equality and $\mathfrak{G}J_{\nu(p),1,2} = \mathfrak{I}_{\nu(p)}$ the following proposition generalizes Proposition 4.3 in [2].

Proposition 5.13. Let $p(x)$ be a complete type of a theory $T$, $\nu(p)$ be a regular labelling function, and $\alpha$ be a positive ordinal. The following conditions are equivalent:

1. the relation $SI_{p,\alpha}$ (on a set of realizations of $p$ in any model $\mathcal{M} \models T$) is transitive;
2. the structure $\mathfrak{G}J_{\nu(p),\alpha}$ is an almost $\alpha$-deterministic monoid.
Proof. Let $a$, $b$, and $c$ be realizations of $p$ such that $(a, b) \in \mathrm{SI}_{p, \alpha}$ and $(b, c) \in \mathrm{SI}_{p, \alpha}$ by semi-isolating formulas $\theta_{u_1}(a, y)$ and $\theta_{u_2}(b, y)$. If the structure $\mathcal{GJ}_{\nu(p), \alpha}$ is an almost $\alpha$-deterministic monoid then $\mathrm{si}(u_1 \circ u_2) \leq \alpha$ and the pair $(a, c)$ belongs to $\mathrm{SI}_{p, \alpha}$ by the semi-isolating formula $\theta_{u_1, u_2}(x, y)$. Since elements $a$, $b$, and $c$ are arbitrary we have $(2) \Rightarrow (1)$.

Assume now that for some $u_1, u_2 \in \rho_{\nu(p), \alpha}$ the set $\mathrm{SI}_p(u_1, u_2)$ contains a label $u$ such that $\mathrm{si}(u) > \alpha$. Then by compactness the set $q(a, y) \models \{ \theta_{u_1, u_2}(a, y) \} \cup \{ \neg \theta_v(a, y) \mid v \in \mathrm{SI}_p(u_1, u_2), \mathrm{si}(v) \leq \alpha \}$
is consistent, where $\models p(a)$. Consider realizations $b$ and $c$ of $p$ such that $\models \theta_{u_1}(a, b) \land \theta_{u_2}(b, c)$ and $\models q(a, c)$. We have $(a, b) \in \mathrm{SI}_{p, \alpha}$ and $(b, c) \in \mathrm{SI}_{p, \alpha}$ but $(a, c) \notin \mathrm{SI}_{p, \alpha}$ by the construction of $q$. Thus, the relation $\mathrm{SI}_{p, \alpha}$ is not transitive and the implication $(1) \Rightarrow (2)$ holds. □

Note that for any ordinal $\alpha > 0$ there are no $(p, \theta_u, p)$-edges, linking distinct realizations of $p$ and satisfying the conditions $u > 0$, $\mathrm{si}(u) \leq \alpha$, and $\mathrm{si}(u^{-1}) \leq \alpha$, if and only if the relation $\mathrm{SI}_{p, \alpha}$ is antisymmetric. Since $\mathrm{SI}_{p, \alpha}$ is reflexive, the definition of $\nu(p)$ and Propositions 1.3, 5.13 imply

**Corollary 5.14.** Let $p(x)$ be a complete type of a theory $T$, $\nu(p)$ be a regular labelling function, and $\alpha$ be a positive ordinal. The following conditions are equivalent:

1. The relation $\mathrm{SI}_{p, \alpha}$ is a partial order on a set of realizations of $p$ in any model $\mathcal{M} \models T$;
2. The structure $\mathcal{GJ}_{\nu(p), \alpha}$ is an almost $\alpha$-deterministic monoid and $\rho_{\nu(p), \alpha} \subseteq U^{\leq 0}$.

This partial order $\mathcal{GJ}_{\nu(p), \alpha}$ is identical if and only if $\rho_{\nu(p), \alpha} = \{0\}$. If $\mathrm{SI}_{p, \alpha}$ is not identical, it has infinite chains.

Propositions 1.3 and 5.13 imply also

**Corollary 5.15.** Let $p(x)$ be a complete type of a theory $T$, $\nu(p)$ be a regular labelling function, and $\alpha$ be a positive ordinal. The following conditions are equivalent:

1. The relation $\mathrm{SI}_{p, \alpha}$ is an equivalence relation on the set of realizations of $p$ in any model $\mathcal{M} \models T$;
2. The structure $\mathcal{GJ}_{\nu(p), \alpha}$ is an almost $\alpha$-deterministic monoid and consists of labels in $U^{\geq 0}$. 

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Recall \[2\] that an element \( u \in \rho_\nu(p) \) is called (almost) deterministic if for any/some realization \( a \) of \( p \) the formula \( \theta_u(a, y) \) has unique solution (has finitely many solutions).

Since each semi-isolating formula \( \theta_u(a, y) \) with finitely many solutions is equivalent to a disjunction of isolating formulas \( \theta_{u_i}(a, y) \), each almost deterministic element has the si-rank 1 and so belongs to the set of labels in the structure \( \mathfrak{SI}_{\nu(p),1,n+1} \), where \( n \) is the number of solutions for \( \theta_u(a, y) \). In particular, each deterministic element belongs to the set of labels in the structure \( \mathfrak{SI}_{\nu(p),1,2} \).

It is shown in \[2\], Proposition 4.7] that if elements \( u \) and \( v \) are (almost) deterministic then each element \( v' \) in \( u \cdot v \) is also (almost) deterministic. Hence, the si-rank 1 is preserved for compositions \( u \circ v \) of (almost) deterministic elements \( u \) and \( v \). Moreover, the si-degree 1 is preserved for compositions of deterministic elements.

In Figure 1, the fragments of Hasse diagram are presented illustrating the links of the structure \( \mathfrak{SI} = \mathfrak{SI}_{\nu(p)} \) with structures above, being restrictions of \( \mathfrak{SI} \) to subalphabets of \( U \). Here the superscripts \( \cdot \leq 0 \) and \( \cdot \geq 0 \) point out on restrictions of \( \mathfrak{SI} \) to the sets \( U^{\leq 0} \) and \( U^{\geq 0} \) respectively, and the subscripts to the upper estimates for si-ranks and si-degrees of labels. In Figure 1, a, a hierarchy of structures \( \mathfrak{SI}_\alpha, \alpha \leq \text{si}(p) \), is depicted starting with the trivial substructure; in Figure 1, b, links between substructures of \( \mathfrak{SI}_{\nu(p),1} \) are presented; in Figure 1, c, links between substructures of \( \mathfrak{SI}_{\alpha+1} \) for \( 1 \leq \alpha < \text{si}(p) \) are shown. For a limit ordinal \( \beta \leq \text{si}(p) \), the Hasse diagram for substructures of \( \mathfrak{SI}_\beta \) is obtained by union of presented diagrams for \( \alpha < \beta \). If an ordinal \( \beta \leq \text{si}(p) \) is not limit, the Hasse diagram corresponds to the union of presented diagrams for \( \alpha < \beta \) with the removal of structures \( \mathfrak{SI}^{\leq 0}_{\beta+1,2} \) and \( \mathfrak{SI}^{\geq 0}_{\beta+1,2} \).

### 6. POSTC-monoids

In this Section, we shall consider both the monoids \( \mathfrak{SI}_{\nu(p)} \) and their expansions (with the addition of empty set to the universe such that \( X \cdot \varnothing = \varnothing \cdot X = \varnothing \) for \( X \in \mathcal{P}(\rho_\nu(p)) \)) by operations and relations of POSTC-algebras containing these monoids. These expansions

\[
\mathfrak{M}_{\nu(p)} = \langle \mathcal{P}(\rho_\nu(p)); \cdot, \leq, \lor, (\cdot \land \neg \cdot), o \rangle
\]

1This extension forms also a monoid with the unit \( \varnothing \) instead of \( \{0\} \).
are called \textit{preordered monoids with relative set-theoretic operations and compositions} over regular labelling functions $\nu(p)$, or briefly $\text{POSTC}_{\nu(p)}$-monoids.

We collect basic structural properties of $\text{POSTC}_{\nu(p)}$-monoids and show that any expanded monoid $\mathfrak{S}\mathfrak{I}$, satisfying the following list of properties, coincides with some $\text{POSTC}_{\nu(p)}$-monoid $\mathfrak{M}_{\nu(p)}$.

Let $U = U^- \cup \{0\} \cup U^+ \cup U'$ be an alphabet consisting of a set $U^-$ of negative elements, a set $U^+$ of positive elements, a set $U'$ of neutral elements, and zero 0. As above, we write $u < 0$ for any element $u \in U^-$, $u > 0$ for any element $u \in U^+$, and $u \cdot v$ instead of $\{u\} \cdot \{v\}$ considering an operation $\cdot$ on the set $\mathcal{P}(U)$; $U^0 \doteq U^- \cup \{0\}$, $U^1 \doteq U^+ \cup \{0\}$.

A structure $\mathfrak{M} = \langle \mathcal{P}(U); \cdot, \leq, \lor, \land, (\cdot \land \cdot), \circ \rangle$ is called a $\text{POSTC}$-monoid if it satisfies the following conditions:
the operation \(\cdot\) of the monoid \(\langle P(U) \setminus \{\emptyset\}; \cdot \rangle\) with the unit \(\{0\}\) is generated by the function \(\cdot\) on elements in \(U\) such that each elements \(u, v \in U\) define a nonempty set \((u \cdot v) \subseteq U\): for any sets \(X, Y \in P(U) \setminus \{\emptyset\}\) the following equality holds:

\[X \cdot Y = \bigcup \{u \cdot v \mid u \in X, v \in Y\};\]

if \(X \in P(U)\) then \(X \cdot \emptyset = \emptyset \cdot X = \emptyset;\)

the relation \(\leq\) on the set \(P(U)\) is a preorder with the least element \(\emptyset\); this preorder is induced by the partial order \(\leq'\) on the set \(U\) of labels (forming a upper semilattice) by the following rule: if \(X, Y \in P(U)\) then \(X \leq Y\) if and only if \(X = \emptyset\), or for any label \(u \in X\) there is a label \(v \in Y\) with \(u \leq' v\) and for any label \(v \in Y\) there is a label \(u \in X\) with \(u \leq' v\);

- a label \(u \in U\) is called an atom if \(v \leq u\) implies \(v = u\) for any label \(v \in U\); only labels in \(U^- \cup \{0\} \cup U^+\) may be atoms; the label \(0\) is an atom; some labels in \(U^{\geq 0}\) lay under each label in \(U'\), moreover, if only labels \(v \in U^{\geq 0}\) lay under a label \(u \in U'\) then there is no greatest labels among labels \(v\); only labels in \(U'\) lay over each label in \(U'\);

- the operations \(\lor, \land, (\land \neg \cdot)\) on the set \(U \cup \{\emptyset\}\) form a distributive lattice with relative complements, moreover, for any elements \(u, v \in U \cup \{\emptyset\}\),

\[u \leq' v \iff u \land v = u \iff u \lor v = v,\]

\[(u \land \neg v) = \emptyset \iff u \leq v;\]

- the operation \(\circ\) is defined on the set \(U\) such that for any labels \(u, v \in U\) the label \(u \circ v\) is the greatest element of the set \(u \cdot v\);

- the operations \(\lor, \land, \circ\) on the set \(P(U)\) are induced by the correspondent operations on the set \(U \cup \{\emptyset\}\): if \(X, Y \in P(U)\) and \(\tau \in \{\lor, \land, \circ\}\) then \(X \tau Y = \{u \tau v \mid u \in X, v \in Y\}\); the operation \((\land \neg \cdot)\) on the set \(P(U)\) is also induced by the correspondent operation on the set \(U \cup \{\emptyset\}\): if \(X, Y \in P(U)\) then \(X \land \neg Y = \{u \land \neg v \mid u \in X, v \in Y\}\);

- the sets \(U^- \cup \{\emptyset\}\) and \(U^{\geq 0} \cup \{\emptyset\}\) are closed with respect to the operations \(\lor, \land, (\land \neg \cdot)\); the set \(U'\) is closed under the operation \(\lor\); if \(u \in U^-\) and \(v \in U^{\geq 0}\) then \((u \lor v) \in U'\);

- repeating the definition in Section 2, for each label \(u \in U\), the \textit{rank of semi-isolation} \(\text{si}(u) \geq 1\) and the \textit{degree of semi-isolation} \(\text{deg}(u)\) of label \(u\) is
defined inductively, $\text{si}(\emptyset) = 0, \deg(\emptyset) = 1$, as well as equivalence relations $\sim_\alpha$, restrictions $X_\alpha, X_{\alpha, \beta}$ of sets $X \in \{U, U \cup \{\emptyset\}\}$ and restrictions $M'_\alpha, M'_{\alpha, \beta}$ for restrictions $M'$ of $M$ to sets of labels of si-ranks $\leq \alpha$, and for labels of si-rank $\alpha$ to sets of labels of si-degree $< \beta$;

- the restriction $\langle \mathcal{P}(U) \setminus \{\emptyset\}; \cdot \rangle_{1,2}$ of the monoid $\langle \mathcal{P}(U) \setminus \{\emptyset\}; \cdot \rangle$ is a $\mathcal{I}$-groupoid;

- if $u < 0$ then sets $u \cdot v$ and $v \cdot u$ consist of negative elements for any $v \in U$;
  - if $u > 0$ and $v > 0$ then $(u \cdot v) \subseteq U^\geq$;
  - if $u, v \in U^\geq \cup U'$, and $u \in U'$ or $v \in U'$, then $(u \cdot v) \subseteq U'$;

- for any element $u > 0$ there is a nonempty set $u^{-1}$ of inverse elements $u' > 0$ such that $0 \in (u \cdot u') \cap (u' \cdot u)$; in this case if $u \leq' v$ and $v \in U^+$ then $u^{-1} \subseteq v^{-1}$;

- if a positive element $u$ belongs to a set $v_1 \cdot v_2$, where $v_1 \circ v_2 \in U^+$, then $u^{-1} \subseteq v_2^{-1} \cdot v_1^{-1}$.

By the definition each POSTC-monoid $M$ contains POSTC-submonoids $M^\leq$ and $M^\geq$ with the universes $\mathcal{P}(U^- \cup \{0\})$ and $\mathcal{P}(U^+ \cup \{0\})$ respectively, being also POSTC-monoids (with $U^+ \cup U' = \emptyset$ and $U^- \cup U' = \emptyset$ respectively).

A POSTC-monoid $M$ is called atomic if for any label $u \in U$ there is an atom $v \in U$ such that $v \leq u$.

**Theorem 6.1.** For any (at most countable and having an ordinal $\sup\{\text{si}(u) \mid u \in U\}$) POSTC-monoid $M$ there is a (small) theory $T$ with a type $p(x) \in S(T)$ and a regular labelling function $\nu(p)$ such that $M_{\nu(p)} = M$.

**Proof** follows the same scheme as the proof of [2, Theorem 6.1] and, for the structure $\langle \mathcal{P}(U) \setminus \{\emptyset\}; \cdot \rangle_{1,2}$, it repeats this proof word for word. Since the proof of [2, Theorem 6.1] is voluminous we only point out the distinctive features leading to the proof of this theorem.

1. A binary predicate $Q_u$ is defined for each element $u \in U \cup \{\emptyset\}$. This predicate links only elements of the same colors if $u \geq 0$, and defines a $Q_u$-ordered coloring Col if $u \in U^- \cup U'$; $Q_\emptyset = \emptyset$.

2. For any elements $u, v \in U \cup \{\emptyset\}$ the following condition is satisfied: $u \leq' v \iff Q_u \subseteq Q_v$. 

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3. For any elements \( u, v \in U \cup \{ \emptyset \} \) the following conditions hold:

\[
\begin{align*}
  u_1 \lor u_2 &= v \iff Q_{u_1} \cup Q_{u_2} = Q_v, \\
  u_1 \land u_2 &= v \iff Q_{u_1} \cap Q_{u_2} = Q_v, \\
  u_1 \land \neg u_2 &= v \iff Q_{u_1} \setminus Q_{u_2} = Q_v, \\
  u_1 \circ u_2 &= v \iff Q_{u_1} \circ Q_{u_2} = Q_v.
\end{align*}
\]

In particular, the predicates \( Q_{u_1} \) and \( Q_{u_2} \) are disjoint if and only if \( u_1 \land u_2 = \emptyset \). \( \square \)

**Remark 6.2.** Since labels \( u \in \rho_{\nu(p)} \) for semi-isolating formulas admit complements in \( \rho_{\nu(p)} \) only for principal types \( p \) (and these complements are defined relative to the isolating formula of \( p \)), unlike \( I \)-groupoids, if a POSTC-monoid \( \mathfrak{M} \) is constructed by a set \( U \geq 0 \), it admits a representation in a transitive theory \( T \) with a (unique) type \( p(x) \in S(T) \) and a regular labelling function \( \nu(p) \) such that \( \mathfrak{M}_{\nu(p)} = \mathfrak{M} \) if and only if the set-theoretic operations in \( \mathfrak{M} \) form a Boolean algebra.

7. Partial POSTC-monoid

on a set of realizations for a family of 1-types
of a complete theory

In this section, the results above for a structure of a type, as well as results in [2] for isolating formulas, are generalized for a structure on a set of realizations for a family of types.

Let \( R \) be a nonempty family of types in \( S^1(T) \). We denote by \( \nu(R) \) a regular family of labelling functions

\[
\nu(p, q): PF(p, q)/PE(p, q) \to U, \; p, q \in R,
\]

\[
\rho_{\nu(R)} = \bigcup_{p, q \in R} \rho_{\nu(p, q)}.
\]

As in Proposition 4.1, the partial (for \( |R| > 1 \)) function SI on the set \( R \times \mathcal{P}(U) \times R \), which maps each tuple of triples \((p_1, u_1, p_2), \ldots, (p_k, u_k, p_{k+1})\), where \( u_1 \in \rho_{\nu(p_1, p_2)} \cup \{ \emptyset \} \), \ldots, \( u_k \in \rho_{\nu(p_k, p_{k+1})} \cup \{ \emptyset \} \), to the set of triples \((p_1, v, p_{k+1})\), where \( v \in SI(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \), is associative:

\[
\begin{align*}
SI(SI(p_1, u_1, p_2, u_2, p_3, u_3, p_4)) &= SI(p_1, u_1, p_2, u_2, p_3, u_3, p_4) = \\
&= SI(p_1, u_1, SI(p_2, u_2, p_3, u_3, p_4))
\end{align*}
\]
for \( u_1 \in \rho_{\nu(p_1,p_2)} \cup \{ \emptyset \}, u_2 \in \rho_{\nu(p_2,p_3)} \cup \{ \emptyset \}, u_3 \in \rho_{\nu(p_3,p_4)} \cup \{ \emptyset \}. \)

Consider the structure

\[
\mathfrak{M}_{\nu(R)} = \langle R \times \mathcal{P}(U) \times R; \cdot, \leq, \lor, \land, (\cdot \land \neg \cdot), \circ \rangle
\]

with the partial operations \( \cdot \) and \( \circ \) such that

\[
(p_1, X_1, p_2) \cdot (p_2, X_2, p_3) = \bigcup \{(p_1, u_1, p_2) \cdot (p_2, u_2, p_3) \mid u_1 \in X_1, u_2 \in X_2 \},
\]

\[
(p_1, u_1, p_2) \cdot (p_2, u_2, p_3) = \{ (p_1, v, p_3) \mid v \in \text{SI}(p_1, u_1, p_2, u_2, p_3) \},
\]

\[
(p_1, X_1, p_2) \circ (p_2, X_2, p_3) = \bigcup \{(p_1, u_1, p_2) \circ (p_2, u_2, p_3) \mid u_1 \in X_1, u_2 \in X_2 \},
\]

\[
(p_1, u_1, p_2) \circ (p_2, u_2, p_3) = \{ (p_1, u \circ v, p_3) \},
\]

\[
u \in \rho_{\nu(p_1,p_2)} \cup \{ \emptyset \}, u_2 \in \rho_{\nu(p_2,p_3)} \cup \{ \emptyset \},
\]

as well as the relation \( \leq \) of preorder, being induced by the partial order, of the same name, on the set of labels and the partial operations \( \lor, \land, (\cdot \land \neg \cdot) \) such that

\[
(p, X, q) \lor (p, Y, q) = \bigcup \{(p, u, q) \lor (p, v, q) \mid u \in X, v \in Y \},
\]

\[
(p, u, q) \lor (p, v, q) = \{ (p, u \lor v, q) \},
\]

\[
(p, X, q) \land (p, Y, q) = \bigcup \{(p, u, q) \land (p, v, q) \mid u \in X, v \in Y \},
\]

\[
(p, u, q) \land (p, v, q) = \{ (p, u \land v, q) \},
\]

\[
(p, X, q) \land \neg (p, Y, q) = \bigcup \{(p, u, q) \land \neg (p, v, q) \mid u \in X, v \in Y \},
\]

\[
(p, u, q) \land \neg (p, v, q) = \{ (p, u \land \neg v, q) \},
\]

\[
\quad u, v \in \rho_{\nu(p,q)} \cup \{ \emptyset \}.
\]

The POSTC-monoids \( \mathfrak{M}_{\nu(p)} \), \( p \in R \), are naturally embeddable in this structure. The structure \( \mathfrak{M}_{\nu(R)} \) is called a join of POSTC-monoids \( \mathfrak{M}_{\nu(p)} \), \( p \in R \), relative to the family \( \nu(R) \) of labelling functions and it is denoted by \( \bigoplus_{\nu(R)} \mathfrak{M}_{\nu(p)} \). If \( \rho_{\nu(p,q)} = \emptyset \) for all \( p \neq q \) the join \( \bigoplus_{\nu(R)} \mathfrak{M}_{\nu(p)} \) is free, it is represented as the disjoint union of POSTC-monoids \( \mathfrak{M}_{\nu(p)} \) and denoted by \( \bigcup_{p \in R} \mathfrak{M}_{\nu(p)} \).

By (2) we have

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Proposition 7.1. For any complete theory $T$, for any nonempty family $R \subseteq S(T)$ of 1-types, and for any regular family $\nu(R)$ of labelling functions, each $n$-ary partial operation $SI(p_1, \cdots, p_n+1)$ on the set $\mathcal{P}(U)$ is interpretable by a term of the structure $\bigoplus_{p \in \nu(R)} M_{\nu(p)}$ with fixed types $p_1, \ldots, p_{n+1} \in R$.

Denote by $SI_{\nu(R)}$ the restriction of $M_{\nu(R)} \upharpoonright R \times (\mathcal{P} \setminus \emptyset) \times R$ to the partial operation $\cdot$.

Using Proposition 1.3 we obtain the following analogue of Proposition 4.3.

Proposition 7.2. For any complete theory $T$, for any nonempty family $R \subseteq S(T)$ of 1-types, and for any regular family $\nu(R)$ of labelling functions, the restriction of the structure $SI_{\nu(R)}$ to the set $U \leq 0$ (respectively $U \geq 0$, $U \geq 0 \cup U'$) is closed under the partial operation $\cdot$.

By Proposition 7.2 the structure $SI_{\nu(R)}$ has substructures $SI_{\leq 0,\nu(R)}$, $SI_{\geq 0,\nu(R)}$ and $SI_{\geq 0,\nu(R)}^\text{neu}$, generated by triples $(p, u, q)$ with $u \leq 0$, $u \geq 0$, and $u \in U \geq 0 \cup U'$ respectively, $p, q \in R$. Here, for any triple $(p, u, q)$ in $SI_{\geq 0,\nu(R)}$ the triple $(q, u-1, p)$ is also attributed to $SI_{\geq 0,\nu(R)}$.

Replacing for the definition in Section 5 the function $\nu(p)$ to the family $\nu(R)$ of functions we obtain the notions of $(\alpha, n)$-deterministic, $\alpha$-deterministic, almost $\alpha$-deterministic, and $(\alpha, \omega)$-deterministic structures $SI_{\nu(p)} \upharpoonright U_0$.

Below we formulate a series of assertions that immediately transform from the class of structures $SI_{\nu(p)}$ to the class of structures $SI_{\nu(R)}$.

Proposition 7.3. (Monotony) If a structure $SI_{\nu(R)} \upharpoonright U_0$ is (almost) $\alpha$-deterministic and $\beta$ is a positive ordinal then the structure $(SI_{\nu(R)} \upharpoonright U_0) \upharpoonright \beta$ is also (almost) $\alpha$-deterministic.

Proposition 7.4. For a structure $SI_{\nu(R)}$ and ordinals $\alpha, \beta$, where $\alpha, \beta > 0$, $\beta \in \omega + 1$, the following conditions are equivalent:

1. the structure $SI_{\nu(R)}$ is $(\alpha, \beta)$-deterministic;
2. for any types $p, q, r \in R$ and labels $u_1 \in \rho_{\nu(p,q)\alpha,\beta}$, $u_2 \in \rho_{\nu(q,r)\alpha,\beta}$ the inequality $si(u_1 \circ u_2) \leq \alpha$ holds and is $si(u_1 \circ u_2) = \alpha$ then $\deg(u_1 \circ u_2) < \beta$.

Corollary 7.5. For a structure $SI_{\nu(R)}$ and a positive ordinal $\alpha$, the following conditions are equivalent:

1. the structure $SI_{\nu(R)}$ is $\alpha$-deterministic;
(2) \( \operatorname{si}(u_1 \circ u_2) \leq \alpha \) for any types \( p, q, r \in R \) and labels \( u_1 \in \rho_{\nu(p,q),\alpha}, u_2 \in \rho_{\nu(q,r),\alpha} \).

**Corollary 7.6.** If \( \operatorname{si}(R) \) is an ordinal then the structure \( \mathcal{G}_\nu(R) \) is almost \( \operatorname{si}(R) \)-deterministic.

**Proposition 7.7.** If a structure \( \mathcal{G}_\nu(R) \) is \( (\alpha, \beta) \)-deterministic then \( \mathcal{G}_\nu(R), \alpha \models \mathcal{G}_\nu(R) \upharpoonright \alpha \) is also \( (\alpha, \beta) \)-deterministic.

**Proposition 7.8.** If \( \operatorname{si}(R) \) is an ordinal then the structure \( \mathcal{G}_\nu(R) \) is \( \operatorname{si}(R) \)-deterministic if and only if the value \( \deg(R) \) is not defined or equals 1.

**Proposition 7.9.** A structure \( \mathcal{P}_\nu(R) \) is \( (\alpha, n) \)-deterministic if and only if \( \mathcal{S}_\nu(R, 1, 2) \) is \( (\alpha, n) \)-deterministic.

Let \( R \) be a nonempty family of complete 1-types of a theory \( T \), \( \nu(R) \) be a regular family of labelling functions, and \( \alpha \) be an ordinal, \( \alpha > 0 \). The structure \( \mathcal{G}_\nu(R) \) is called **locally \( \alpha \)-deterministic** if for any nonempty finite set \( R_0 \subseteq R \) there is a natural number \( n \geq 2 \) such that the structure \( \mathcal{G}_\nu(R_0) \) is \( (\alpha, n) \)-deterministic.

Repeating the proof of Proposition 5.8 we obtain

**Proposition 7.10.** Let \( R \) be a nonempty family of complete 1-types of a theory \( T \), \( \nu(R) \) be a regular family of labelling functions, \( \operatorname{si}(R) < \omega \). The following conditions are equivalent:

1. the structure \( \mathcal{G}_\nu(R) \) is locally \( 1 \)-deterministic;
2. the set \( \rho_{\nu(p,q)} \) is finite for any \( p, q \in R \);
3. the set \( \rho_{\nu(p,q),1} \) is finite for any \( p, q \in R \);
4. the set \( \rho_{\nu(p,q),1,2} \) (consisting of all atoms \( u \in \rho_{\nu(p,q)} \)) is finite for any \( p, q \in R \).

The notion of \( (n, p) \)-type is generalized in the following definition.

**Definition** (K. Ikeda, A. Pillay, A. Tsuboi [20]). Let \( p_1(x_1), \ldots, p_n(x_n) \) be types in \( S(T) \) with disjoint free variables. A type \( q(x_1, \ldots, x_n) \in S(T) \) is said to be a \( (p_1, \ldots, p_n) \)-type if \( q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^{n} p_i(x_i) \). The set of all \( (p_1, \ldots, p_n) \)-types of \( T \) is denoted by \( S_{p_1, \ldots, p_n}(T) \). A theory \( T \) is **almost \( \omega \)-categorical** if for any types \( p_1(x_1), \ldots, p_n(x_n) \in S(T) \) there are only finitely many types \( q(x_1, \ldots, x_n) \in S_{p_1, \ldots, p_n}(T) \).

**Definition** (B. S. Baizhanov, S. V. Sudoplatov, V. V. Verbovskiy [16]). A type \( q(\bar{x}) \) in \( S_{p_1, \ldots, p_n}(T) \) is said to be \( (p_1, \ldots, p_n) \)-principal if there is a
formula $\varphi(\bar{y}) \in q(\bar{x})$ such that

$$\cup\{p_i(x_i) \mid i = 1, \ldots, n\} \cup \{\varphi(\bar{x})\} \vdash q(\bar{x}).$$

The following lemma obviously generalizes Lemma 5.10.

**Lemma 7.11** [16]. For any types $p_1(x_1), \ldots, p_n(x_n) \in S(\emptyset)$ the following conditions are equivalent:

1. the set of $(p_1, \ldots, p_n)$-types with free variables in $(x_1, \ldots, x_n)$ is finite;
2. any $(p_1, \ldots, p_n)$-type is $(p_1, \ldots, p_n)$-principal.

By Lemma 7.11, a theory $T$ is almost $\omega$-categorical if and only if for any types $p_1(x_1), \ldots, p_n(x_n) \in S^1(T)$, each $(p_1, \ldots, p_n)$-type is $(p_1, \ldots, p_n)$-principal.

Proposition 7.10 and Lemma 7.11 imply

**Corollary 7.12.** If $R$ is a nonempty family of complete 1-types of a theory $T$, $\nu(R)$ is a regular family of labelling functions, and all $(p_1, p_2)$-types, where $p_1, p_2 \in R$, are $(p_1, p_2)$-principal then the structure $S_{\nu(R)}$ is locally 1-deterministic.

**Corollary 7.13.** If $T$ is an almost $\omega$-categorical theory and $\nu(S^1(\emptyset))$ is a regular family of labelling functions then the structure $S_{\nu(R)}$ is locally 1-deterministic.

For a nonempty family $R$ of 1-types in $S(T)$ and a positive ordinal $\alpha$, we denote by $S_{R,\alpha}$ (in a model $M$ of $T$) the restriction of $S_R$ to the set of formulas of si-ranks $\leq \alpha$:

$$S_{R,\alpha} = \{(a, b) \mid \text{tp}(a), \text{tp}(b) \in R \text{ and } a \text{ semi-isolates } b \}$$

by a formula $\theta_{\text{tp}(a), \text{tp}(b)}(x, y)$, with a si-rank $\leq \alpha$.

Clearly, $I_R = S_{R,1}$ for any nonempty family $R$ of 1-types. Considering this equality and the equality $S_{\nu(R),1,2} = \mathcal{P}_{\nu(R)}$, the following proposition generalizes Proposition 5.13 as well as Propositions 4.3 and 8.3 in [2].

**Proposition 7.14.** Let $R$ be a nonempty family of complete 1-types of a theory $T$, $\nu(R)$ be a regular family of labelling functions, and $\alpha$ be a positive ordinal. The following conditions are equivalent:

1. the relation $S_{R,\alpha}$ (on a set of realizations of types $p \in R$ in any model $M \models T$) is transitive;
(2) the structure $\mathcal{S}_{\nu(R),\alpha}$ is almost $\alpha$-deterministic.

Proof repeats the proof of Proposition 5.13 almost word for word. □

Propositions 1.3 and 7.14 imply the following assertions.

Corollary 7.15. Let $R$ be a nonempty family of complete 1-types of a theory $T$, $\nu(R)$ be a regular family of labelling functions, and $\alpha$ be a positive ordinal. The following conditions are equivalent:

1. the relation $S_{I,R,\alpha}$ in any model $\mathcal{M} \models T$ is a partial order;
2. the structure $\mathcal{S}_{\nu(R),\alpha}$ is almost $\alpha$-deterministic and $\rho_{\nu(R),\alpha} \subseteq U_{\leq 0}$.

The partial order $S_{I,R,\alpha}$ is identical if and only if $\rho_{\nu(R),\alpha} = \{ 0 \}$. The non-identical partial order $S_{I,R,\alpha}$ has infinite chains if and only if $|\rho_{\nu(p),\alpha}| > 1$ for some type $p \in R$ or there is a sequence $p_n$, $n \in \omega$, of pairwise distinct types in $R$ such that $|\rho_{\nu(p_n,p_{n+1}),\alpha}| \geq 1$, $n \in \omega$.

Corollary 7.16. Let $R$ be a nonempty family of complete 1-types of a theory $T$, $\nu(R)$ be a regular family of labelling functions, and $\alpha$ be a positive ordinal. The following conditions are equivalent:

1. the relation $S_{I,R,\alpha}$ on a set of realizations of types $p \in R$ in any model $\mathcal{M} \models T$ is an equivalence relation;
2. the structure $\mathcal{S}_{\nu(R),\alpha}$ is almost $\alpha$-deterministic and $\rho_{\nu(R),\alpha} \subseteq U_{\geq 0}$.

The results above substantiate that the diagram in Figure 1 admits the transformation replacing the type $p$ by a nonempty family $R \subseteq S^1(\emptyset)$.

8. POSTC$_R$-structures

Definition. Let $\mathcal{R}$ be a nonempty set,

$$U = U^- \cup \{ 0 \} \cup U^+ \cup U'$$

be an alphabet consisting of a set $U^-$ of negative elements, a set $U^+$ of positive elements, a set $U'$ of neutral elements, and zero 0. If $p$ and $q$ are elements in $\mathcal{R}$, we write $u < 0$ and $(p,u,q) < 0$ for any element $u \in U^-$, $u > 0$ and $(p,u,q) > 0$ for any element $u \in U^+$; $U_{\leq 0} = U^- \cup \{ 0 \}$, $U_{\geq 0} = U^+ \cup \{ 0 \}$. For the set $\mathcal{R}^2$ of all pairs $(p,q)$, $p,q \in \mathcal{R}$, we consider a regular family $\mu(\mathcal{R})$ of sets $\mu(p,q) \subseteq U$ such that

- $0 \in \mu(p,q)$ if and only if $p = q$;
- $\mu(p,p) \cap \mu(q,q) = \{ 0 \}$ for $p \neq q$;
- $\mu(p,q) \cap \mu(p',q') = \emptyset$ if $p \neq q$ and $(p,q) \neq (p',q')$;
\[ \bigcup_{p,q \in \mathcal{R}} \mu(p, q) = U. \]

Further we write \( \mu(p) \) instead of \( \mu(p, p) \), and considering a partial operation \( \cdot \) on the set \( \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R} \) we shall write, as above, \( (p, u, q) \cdot (q, v, r) \) instead of \( (p, \{u\}, q) \cdot (q, \{v\}, r) \). A structure

\[ \mathcal{M} = \langle \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}; \cdot, \subseteq, \lor, (\cdot \land \cdot), \circ \rangle \]

with a regular family \( \mu(\mathcal{R}) \) of sets is said to be a POSTC\(_\mathcal{R}\)-structure if the following conditions hold:

- the partial operation \( \cdot \) of the structure \( \langle \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}; \cdot \rangle \) has values \( (p, X, q) \cdot (p', Y, q') \) only for \( p' = q \), \( X \subseteq \mu(p, q) \), \( Y \subseteq \mu(p', q') \), and it is generated by the function \( \cdot \) for elements in \( U \): for any sets \( X, Y \in \mathcal{P}(U) \), \( \varnothing \neq X \subseteq \mu(p, q) \), \( \varnothing \neq Y \subseteq \mu(q, r) \), the following equality is satisfied:

\[ (p, X, q) \cdot (q, Y, r) = \bigcup \{(p, x, q) \cdot (q, y, r) \mid x \in X, y \in Y\}, \]

and if some of \( X, Y \) is empty then \( (p, X, q) \cdot (q, Y, r) = \varnothing \);

- each restriction \( \mathcal{M}_{\mu(p)} \) of \( \mathcal{M} \) to \( \{p\} \times \mathcal{P}(\mu(p)) \times \{p\} \) is isomorphic to a POSTC-monoid with the universe \( \mathcal{P}(\mu(p)) \), \( p \in \mathcal{R} \); atoms \( u \in \mu(p) \) in \( \mathcal{M}_{\mu(p)} \) are called \( p \)-atoms;

- each restriction \( \mathcal{M}_{\mu(p, q)} \), \( p \neq q \), of \( \mathcal{M} \) to \( \{p\} \times \mathcal{P}(\mu(p, q)) \times \{q\} \) has empty partial operations \( \cdot \) and \( \circ \); the restriction of \( \mathcal{M}_{\mu(p, q)} \) to the relation \( \subseteq \) is a preordered set \( \{\{p\} \times \mathcal{P}(\mu(p, q)) \times \{q\}\} \subseteq_{p, q} \) with the least element \( (p, \varnothing, q) \), the preorder \( \subseteq_{p, q} \) of this structure is induced by the partial order \( \subseteq_{p, q} \) on the set \( \mu(p, q) \) of labels (forming a upper semilattice if \( \mu(p, q) \neq \varnothing \)) by the following rule: if \( X, Y \in \mathcal{P}(\mu(p, q)) \) then \( X \subseteq_{p, q} Y \) if and only if \( X = \varnothing \), or for any label \( u \in X \) there is a label \( v \in Y \) with \( u \subseteq_{p, q} v \) and for any label \( v \in Y \) there is a label \( u \in X \) with \( u \subseteq_{p, q} v \);

- a label \( u \in \mu(p, q) \), where \( p \neq q \), is said to be a \( (p, q) \)-atom if \( v \subseteq_{p, q} u \) implies \( v = u \) for any label \( v \in \mu(p, q) \); only labels in \( \mu(p, q) \cap (U^- \cup U^+) \) may be \( (p, q) \)-atoms; some labels in \( \mu(p, q) \cap U^{\geq 0} \) lay under each label in \( \mu(p, q) \cap U' \), moreover, if only labels \( v \in \mu(p, q) \cap U^{\geq 0} \) lay under a label \( u \in \mu(p, q) \cap U' \) then there are no greatest labels among labels \( v \); only labels in \( \mu(p, q) \cap U' \) lay over each label in \( \mu(p, q) \cap U' \);
'the operations $\lor, \land, (\cdot \land \neg \cdot)$ are defined on each set $\mu(p, q) \cup \{\emptyset\}$ in the structure $M_{\mu(p,q)}$ and form a distributive lattice with relative complements on $\mu(p, q) \cup \{\emptyset\}$, moreover, for any elements $u, v \in \mu(p, q) \cup \{\emptyset\}$,

\[
u \leq_{p,q} v \Leftrightarrow u \land v = u \Leftrightarrow u \lor v = v \Leftrightarrow u \land \neg v = \emptyset;
\]

- the relation $\leq$ on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ is a preorder with minimal elements $(p, \emptyset, q), p, q \in \mathcal{R}$; this preorder is induced by the union $\leq_U$ of preorders $\leq_p$ in the structures $M_{\mu(p)}$, $p \in \mathcal{R}$, and of preorders $\leq_{p,q}$ in the structures $M_{\mu(p,q)}$, $p, q \in \mathcal{R}$, $p \neq q$, on sets of labels in these structures: if $X, Y \in \mathcal{P}(U)$ then $(p, X, q) \leq_U (p', Y, q')$ if and only if $p = p'$, $q = q'$, and $X = \emptyset$ or for any label $u \in X$ there is a label $v \in Y$ with $u \leq_U v$ and for any label $v \in Y$ there is a label $u \in X$ with $u \leq_U v$;

- the partial operations $\lor, \land, (\cdot \land \neg \cdot)$ are defined on the set $\mathcal{R} \times (U \cup \{\emptyset\}) \times \mathcal{R}$ in the structure $M$ being unions of correspondent operations on the sets $\mu(p) \cup \{\emptyset\}$ in $M_{\mu(p)}$ and on the sets $\mu(p, q) \cup \{\emptyset\}$ in $M_{\mu(p,q)}$, $p \neq q$;

- the partial operation $\circ$ is defined on the set $\mathcal{R} \times (U \cup \{\emptyset\}) \times \mathcal{R}$ in the structure $M$ being obtained from the union of correspondent operations in the structures $M_{\mu(p)}$, $p \in \mathcal{R}$, by the following extension: if $u_1 \in \mu(p, q)$ and $u_2 \in \mu(q, r)$ then there is unique element $v \in \mu(p, r)$, such that $(p, u_1, q) \circ (q, u_2, r) = (p, v, r)$; this element $v$ is the $\leq_{p,r}$-greatest label in the set $(p, u_1, q) \cdot (q, u_2, r)$, it is called a composition of elements $u_1$ and $u_2$ and it is denoted by $u_1 \circ u_2$:

\[(p, u_1, q) \circ (q, \emptyset, r) = (p, \emptyset, q) \cdot (q, u_2, r) = (p, \emptyset, q) \cdot (q, \emptyset, r) = (p, \emptyset, r);\]

- the partial operations $\lor, \land, \circ$ on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ are induced by the correspondent partial operations on the set $\mathcal{R} \times (U \cup \{\emptyset\}) \times \mathcal{R}$: if $(p, X, q), (p', Y, q') \in \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ and $\tau \in \{\lor, \land, \circ\}$ then the value $(p, X, q) \tau (p', Y, q')$ is not defined or it is defined and coincides with the value $(p, X, q) \tau (p', Y, q')$ in the set $\{ (p, u, q) \tau (p', v, q') \mid u \in X, v \in Y \}$, in which all values are defined; the partial operation $(\cdot \land \neg \cdot)$ on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ is also induced by the correspondent partial operation on the set $\mathcal{R} \times (U \cup \{\emptyset\}) \times \mathcal{R}$: if $(p, X, q), (p', Y, q') \in \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ then the value $(p, X, q) \land \neg (p', Y, q')$ is defined only for $p = p'$, $q = q'$, $X, Y \subseteq \mu(p, q)$ and it is equal to $\{(p, u, q) \land \neg (p, v, q) \mid u \in X, v \in Y \}$;
such that $M$ consists of negative elements for any $sup$ elements inverse $I \in \mathbb{R}_{p,q}$ consists of elements in $\beta$.

The set $v$ of the set of labels of si-ranks $\nu$ where $fined$: the equivalence relations $\sim$.

By the definition, each POSTC $R$-structure $M$ to the set of labels of si-ranks $\leq \alpha$, and for labels of si-rank $\alpha$ to the set of labels of si-degree $< \beta$;

- the restriction $(\mathcal{R} \times (\mathcal{P}(U) \setminus \{\emptyset\}) \times \mathcal{R}; \cdot)_1,2$ of the structure $M$ is an $I_{\mathcal{R}}$-structure;
  - if $u \in \mu(p,q)$ and $u < 0$ then the set $(p, u, q, (q, v, r))$ and $(r, v', p, (p, u, q))$ consists of negative elements for any $v \in \mu(q, r)$ and $v' \in (r, p)$;
  - if $u \in \mu(p,q), v \in \mu(q, r), u > 0$, and $v > 0$, then the set $(p, u, q, (q, v, r))$ consists of elements in $U_{\geq 0}$;
  - if $u \in \mu(p,q) \cap (U_{\geq 0} \cup U')$, $v \in \mu(q, r) \cap (U_{\geq 0} \cup U')$, and $u \in U'$ or $v \in U'$, then $(p, u, q, (q, v, r)) \subseteq U'$;
  - for any element $u \in \mu(p,q)$ with $u > 0$ there is a nonempty set $u^{-1}$ of inverse elements $u' > 0$ such that $(p, 0, p) \in (p, u, q, (q, u', p))$ and $(q, 0, q) \in (q, u', p, (p, u, q)), moreover, if $u \leq_{p,q} v$ and $v \in U^+$ then $u^{-1} \subseteq v^{-1}$;
  - if an element $(p, u, r)$, where $u > 0$, belongs to a set $(p, v_1, q, (q, v_2, r), v_1 \circ v_2 \in U^+, then (r, u^{-1}, p) \subseteq (r, v_2^{-1}, q, (q, v_1^{-1}, p)$.

By the definition, each POSTC$_{\mathcal{R}}$-structure $M$ contains POSTC$_{\mathcal{R}}$-substructures $M_{\leq 0}$ and $M_{\geq 0}$ being restrictions of $M$ to the sets $U_{\leq 0}$ and $U_{\geq 0}$ respectively.

A POSTC$_{\mathcal{R}}$-structure $M$ is called atomic if for any label $u \in \mu(p)$, $p \in \mathcal{R}$, there is a $p$-atom $v \in U$ such that $v \leq_p u$, and for any label $u \in \mu(p,q)$, $p, q \in \mathcal{R}, p \neq q$, there is a $(p, q)$-atom $v \in U$ such that $v \leq_{p,q} u$.

Combining the proof of Theorems 6.1 and 9.1 in [2] as well as the proof of Theorem 6.1, we obtain the following theorem.

**Theorem 8.1.** For any (at most countable and having an ordinal sup$\{si(u) \mid u \in U\}$) POSTC$_{\mathcal{R}}$-structure $M$ there is a (small) theory $T$ with a family of 1-types $R \subset S(T)$ and a regular family $v(R)$ of labelling functions such that $M_{v(R)} = M$.  

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In conclusion, we note that, using the operation \( \cdot^{\text{eq}} \), the constructions above can be transformed for an arbitrary family of types in \( S(T) \).

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