PSEUDOSUPERSYMMETRIC QUANTUM MECHANICS: GENERAL CASE, ORTHOSUPERSYMMETRIES, REDUCIBILITY, AND BOSONIZATION

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Abstract

Pseudosupersymmetric quantum mechanics (PsSSQM), based upon the use of pseudofermions, was introduced in the context of a new Kemmer equation describing charged vector mesons interacting with an external constant magnetic field. Here we construct the complete explicit solution for its realization in terms of two superpotentials, both equal or unequal. We prove that any orthosupersymmetric quantum mechanical system has a pseudosupersymmetry and give conditions under which a pseudosupersymmetric one may be described by orthosupersymmetries of order two. We propose two new matrix realizations of PsSSQM in terms of the generators of a generalized deformed oscillator algebra (GDOA) and relate them to the cases of equal or unequal superpotentials, respectively. We demonstrate that these matrix realizations are fully reducible and that their irreducible components provide two distinct sets of bosonized operators realizing PsSSQM and corresponding to nonlinear spectra. We relate such results to some previous ones obtained for a GDOA connected with a $C_3$-extended oscillator algebra (where $C_3 = \mathbb{Z}_3$) in the case of linear spectra.

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1 Introduction

During the past two decades, supersymmetry, based upon a symmetry between bosons and fermions [1], has found a lot of applications in quantum mechanics [2]. The success of this new field, supersymmetric quantum mechanics (SSQM), has triggered the search for generalizations by extending the symmetry to some exotic statistics. Replacing fermions by parafermions [3], pseudofermions [4], or orthofermions [5], for instance, has led to parasupersymmetric (PSSQM) [6, 7], pseudosupersymmetric (PsSSQM) [4, 8], or orthosupersymmetric quantum mechanics (OSSQM) [9], respectively. Substituting a $\mathbb{Z}_k$ grading to the $\mathbb{Z}_2$ one characteristic of SSQM has also given rise to fractional supersymmetric quantum mechanics (FSSQM) [10]. More recently, from a somewhat different viewpoint, extending Witten’s index [11] to more general topological invariants has resulted in the concept of topological symmetries (TS) [12].

In the present paper, we will come back to one of the generalizations of SSQM, namely PsSSQM, which has been introduced in the context of a new relativistic Kemmer equation describing charged vector mesons interacting with an external constant magnetic field [4]. This equation, which has solved for the first time the longstanding problems of reality of energy eigenvalues and causality of propagation, leads in the nonrelativistic limit to a pseudosupersymmetric oscillator Hamiltonian, which can be realized in terms of boson-like operators and pseudofermionic ones, where the latter are intermediate between fermionic and order-two parafermionic operators.

Later on, PsSSQM has been reformulated in terms of two superpotentials $W_1$ and $W_2$, but only a special case corresponding to the choice $W_1 = W_2 = W$ has actually been studied in detail [8]. One of the purposes of the present paper is to derive explicit forms of the pseudosupersymmetric Hamiltonian in the general case, including both the choices $W_1 = W_2$ and $W_1 \neq W_2$.

Another aim is to reconsider the connections between PsSSQM and OSSQM, partly analyzed in Ref. [8]. Here they will be thoroughly discussed by using a slightly
different approach, thereby emphasizing some similarities with the links between PSSQM or FSSQM and OSSQM, which have been recently established [13].

Still another purpose is to examine the realizations of PsSSQM in terms of superpotentials at the light of a recent work, wherein we have provided a bosonization (i.e., a realization in terms of only boson-like operators without fermion-like ones) of several variants of SSQM [14], generalizing a well-known result for SSQM [13] in terms of the Calogero-Vasiliev algebra [18]. For the SSQM variants, the algebras used are some generalized deformed oscillator algebras (GDOAs) (see Ref. [17] and references quoted therein) related to $C_\lambda$-extended oscillator ones [14, 18], where $C_\lambda = \mathbb{Z}_\lambda$ denotes the cyclic group of order $\lambda$ ($\lambda \in \{3, 4, 5, \ldots\}$). Such GDOAs reduce to the Calogero-Vasiliev algebra for $\lambda = 2$. In the case of PsSSQM, we plan to establish here a correspondence between the two matrix realizations obtained for $W_1 = W_2$ and $W_1 \neq W_2$, respectively, and the two different types of bosonization obtained in Ref. [14].

A fourth purpose of the present work is to go further than the oscillator spectra considered in Ref. [14] by providing two new matrix realizations of PsSSQM in terms of GDOA generators in the case of general nonlinear spectra. Such matrix realizations will prove fully reducible and will lead to two different types of PsSSQM bosonization valid for nonlinear spectra.

This paper is organized as follows. In Sec. 2, we review the physical motivation for the introduction of PsSSQM. In Sec. 3, we obtain two different matrix realizations in terms of superpotentials. In Sec. 4, we provide an entirely new analysis of the connections between PsSSQM and OSSQM. We introduce two matrix realizations of PsSSQM in terms of GDOA generators in Sec. 5 and use them in Sec. 6 to establish the reducibility and bosonization of PsSSQM. In Sec. 7, the results of Secs. 5 and 6 are specialized to GDOAs associated with $C_3$-extended oscillator algebras. Finally, Sec. 8 contains a summary of the main results.
2 Physical Motivation for the Introduction of Pseudosupersymmetric Quantum Mechanics

Until recently, the problem of the interaction of relativistic vector mesons with an external electromagnetic field has been plagued with two main difficulties: the existence of complex energy eigenvalues and the violation of the causality principle (see [19] and references quoted therein). To eliminate such drawbacks, it has been proposed to add a new term characterized by some real parameter $\lambda$ to the Kemmer equation describing the phenomenon [4].

In the simplest context of a constant magnetic field, the modified Kemmer equation, when reduced to its Sakata-Taketani form, gives rise to a six-component Klein-Gordon type equation

$$P_0^2 \chi(x) = \left( \Pi^2 + 1 - 2eB \Sigma_3 + \lambda eB \right) \chi(x), \quad (2.1)$$

in units wherein $\hbar = m = c = 1$. Here $e$ is the charge of the vector meson, $\Pi = (\Pi_1, \Pi_2, \Pi_3)$ comes from the minimum coupling substitution $\vec{P} \rightarrow \vec{\Pi} = \vec{P} - e\vec{A}$, with the gauge symmetric potential $A_1 = -\frac{1}{2}By$, $A_2 = \frac{1}{2}Bx$, $A_3 = 0$, and $\vec{B} = (0, 0, B)$, while

$$\Sigma_3 = \begin{pmatrix} S_3 & 0 & 0 \\ 0 & S_3 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (2.2)$$

By using Johnson-Lippmann arguments, one can distinguish the so-called perpendicular and parallel parts of Eq. (2.1) and, through the connection of the former with a one-dimensional oscillator, obtain the squared relativistic energies as

$$E^2 = 1 + 2\omega \left[ n + \frac{1}{2}(\lambda + 1) - s \right], \quad (2.3)$$

where $\omega \equiv eB$, $n$ is the Landau-level quantum number, and $s$ refers to the eigenvalues of $S_3$. In the standard formulation of the Kemmer equation corresponding to $\lambda = 0$, some energy eigenvalues become complex for $\omega > 1$ [19]. On the contrary, for $\lambda \geq 1$, all the energies remain real for any $\omega$. For such $\lambda$ values, it can also be checked that the causality principle is fulfilled [4].
In the nonrelativistic limit, the perpendicular part of the Hamiltonian corresponding to (2.3) is given by

\[ H_{NR}^\perp = \frac{1}{2} \left( \Pi_1^2 + \Pi_2^2 \right) + \frac{1}{2} \omega (\vec{r} - 2S_3), \]  

(2.4)

where \( \Pi \) denotes the 3 \times 3 unit matrix and the simple choice \( \lambda = 1 \) has been made.

For \( H_{NR}^\perp \), one can construct two charge operators \( Q_1, Q_2 \), defined by \[ Q_1 = \mathcal{A} \Pi_1 + \mathcal{B} \Pi_2, \quad Q_2 = -\mathcal{B} \Pi_1 + \mathcal{A} \Pi_2, \]  

(2.5)

where \( \mathcal{A} \) and \( \mathcal{B} \) are 3 \times 3 odd matrices given by

\[
\mathcal{A} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 1+i \\ 0 & 0 & -1+i \\ 1-i & -1-i & 0 \end{pmatrix}, \quad \mathcal{B} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 1-i \\ 0 & 0 & 1+i \\ 1+i & 1-i & 0 \end{pmatrix}.
\]

(2.6)

They satisfy the relations

\[
Q_i^3 = Q_i H_{NR}^\perp, \quad [H_{NR}^\perp, Q_i] = 0, \quad i = 1, 2, \]

(2.7)

\[
Q_i Q_j = -Q_j Q_i = Q_j H_{NR}^\perp, \quad i, j = 1, 2, \quad i \neq j,
\]

(2.8)

which differ from those characterizing either SSQM \[1\] or PSSQM \[6, 7\]. The charges \( Q_1, Q_2 \) correspond to the superposition of usual bosons (associated with the even operators \( \Pi_1 \) and \( \Pi_2 \)) and pseudofermions (associated with the odd matrices \( \mathcal{A} \) and \( \mathcal{B} \)).

In terms of linear combinations of the type

\[
Q = c(Q_1 - iQ_2), \quad Q^\dagger = c(Q_1 + iQ_2),
\]

(2.9)

where \( c \) is some real constant (not to be confused with the velocity of light), the algebra defined in (2.7) and (2.8) takes the form

\[
Q^2 = 0, \quad [H_{NR}^\perp, Q] = 0, \quad QQ^\dagger Q = 4c^2 QH_{NR}^\perp,
\]

(2.10)

together with the Hermitian conjugate relations. The first two equations in (2.10) are the same as those occurring in SSQM \[1\], whereas the third one is rather similar to
the multilinear relation valid in PSSQM of order two. Actually, for \( c = 1 \) or \( c = 1/2 \), it is compatible with the multilinear relation appearing in Rubakov-Spiridonov-Khare \( [6] \) or Beckers-Debergh \( [7] \) version of PSSQM, respectively.

With the choice \( c = 1/2 \), one can rewrite \( Q, Q^\dagger \), and \( H_{NR}^\perp \) as

\[
Q = \sqrt{\omega} b a^\dagger, \quad Q^\dagger = \sqrt{\omega} b^\dagger a, \quad H_{NR}^\perp = \omega \left[ \frac{1}{2} \{ a, a^\dagger \} \mathbb{I} + \frac{1}{2} \text{diag}(-1, 1, 3) \right],
\]

where

\[
a = \frac{1}{\sqrt{2\omega}}(\Pi_1 + \text{i}\Pi_2), \quad a^\dagger = \frac{1}{\sqrt{2\omega}}(\Pi_1 - \text{i}\Pi_2)
\]

are bosonic creation and annihilation operators, while

\[
b = \frac{1}{\sqrt{2}} (\mathcal{A} + \text{i}\mathcal{B}) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 + \text{i} \\ 0 & 0 & -1 + \text{i} \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
b^\dagger = \frac{1}{\sqrt{2}} (\mathcal{A} - \text{i}\mathcal{B}) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 - \text{i} & -1 - \text{i} & 0 \end{pmatrix}
\]

are pseudofermionic ones, satisfying the relations

\[
b^2 = (b^\dagger)^2 = 0, \quad bb^\dagger b = b, \quad b^\dagger bb^\dagger = b^\dagger.
\]

Such operators describe small violations of the Pauli principle.

The new symmetry described by Eq. (2.10) is termed pseudosupersymmetry, while the Hamiltonian in Eq. (2.11) is considered as the pseudosupersymmetric oscillator \( [4, 8] \).

### 3 Pseudosupersymmetric Quantum Mechanics in Terms of Two Superpotentials

As reviewed in Sec. 2, PsSSQM is characterized by a pseudosupersymmetric Hamiltonian \( \mathcal{H} \) and pseudosupercharge operators \( Q, Q^\dagger \) satisfying the relations

\[
Q^2 = 0, \quad [\mathcal{H}, Q] = 0, \quad QQ^\dagger Q = 4c^2 Q\mathcal{H},
\]
and their Hermitian conjugates, where \( c \) is some real constant \( \mathbb{R} \).

One may now look for a realization of the two pseudosupercharges \( Q, Q^\dagger \) as

\[
Q = \frac{c}{\sqrt{2}} \begin{pmatrix}
0 & 0 & (1 - i)[P + iW_1(x)] \\
0 & 0 & (1 + i)[P + iW_2(x)] \\
0 & 0 & 0
\end{pmatrix},
\]

\[
Q^\dagger = \frac{c}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & (1 + i)[P - iW_1(x)] \\
(1 + i)[P - iW_1(x)] & (1 - i)[P - iW_2(x)] & 0
\end{pmatrix},
\]

where \( P \) is the momentum operator \( (P = -i\frac{d}{dx}) \) and \( W_1(x), W_2(x) \) are two superpotentials. It is obvious that with this choice the nilpotency (3.1) is verified.

Equation (3.2) is fulfilled if the pseudosupersymmetric Hamiltonian \( \mathcal{H} \) is realized by a \( 3 \times 3 \) Hermitian matrix of the form

\[
\mathcal{H} = \begin{pmatrix}
H_1 & H_4 & 0 \\
H_4^\dagger & H_2 & 0 \\
0 & 0 & H_3
\end{pmatrix}, \quad H_i = H_i^\dagger, \quad i = 1, 2, 3,
\]

where \( H_1, H_2, H_3, \) and \( H_4 \) are constrained by the conditions

\[
H_1(P + iW_1) + iH_4(P + iW_2) = (P + iW_1)H_3,
\]

\[
-iH_4^\dagger(P + iW_1) + H_2(P + iW_2) = (P + iW_2)H_3.
\]

Finally, Eq. (3.3) determines \( H_3 \) in terms of the two superpotentials \( W_1, W_2 \) as

\[
H_3 = \frac{1}{4}[P - iW_1)(P + iW_1) + (P - iW_2)(P + iW_2)]
\]

\[
= \frac{1}{2} \left[ P^2 + \frac{1}{2} \left( W_1' + W_2' + W_1' + W_2' \right) \right].
\]

By combining (3.3) with (3.7) and (3.8), the latter are transformed into

\[
(P + iW_1)[(P - iW_1)(P + iW_1) + (P - iW_2)(P + iW_2)]
\]

\[
= 4H_1(P + iW_1) + 4iH_4(P + iW_2),
\]

\[
(P + iW_2)[(P - iW_1)(P + iW_1) + (P - iW_2)(P + iW_2)]
\]

\[
= -4iH_4^\dagger(P + iW_1) + 4H_2(P + iW_2).
\]

The problem left amounts to solving this system of equations in order to express \( H_1, H_2, H_4 \) in terms of \( W_1 \) and \( W_2 \). For such a purpose, we shall distinguish between the two cases \( W_1 = W_2 \) and \( W_1 \neq W_2 \).
3.1 The case of equal superpotentials

Whenever the two superpotentials are equal,

\[ W_1(x) = W_2(x) = W(x), \quad (3.12) \]

Eqs. (3.10) and (3.11) assume a very simple form

\[ (P + iW)(P - iW) = 2(H_1 + iH_4) = 2(H_2 - iH_4^\dagger). \quad (3.13) \]

Since the left-hand side of Eq. (3.13) is Hermitian, we immediately get

\[ H_4^\dagger = -H_4, \quad (3.14) \]
\[ H_1 = H_2 = \frac{1}{2}(P^2 + W^2 - W') - iH_4. \quad (3.15) \]

In the special case (3.12), Eq. (3.9) becomes

\[ H_3 = \frac{1}{2}(P^2 + W^2 + W'), \quad (3.16) \]

so that Eq. (3.13) may also be written as

\[ H_1 = H_2 = H_3 - i(H_4 - iW'). \quad (3.17) \]

We conclude that in the case of equal superpotentials, the general solution of Eqs. (3.10), (3.11) is given by (3.14), (3.15) (or (3.17)), and (3.16). Hence \( H_4 \) remains undetermined except for its antihermitian character. Beckers and Debergh [8] restricted themselves to the choice \( H_4 = iW' \), in which case the three Hamiltonians \( H_1, H_2, \) and \( H_3 \) become identical.

Such a restriction is however not needed. For an arbitrary solution (3.14) – (3.16), we indeed note that \( \mathcal{H} \), as given by (3.6), can be diagonalized through a unitary transformation

\[ U_1 = \begin{pmatrix} 
\frac{1-i}{\sqrt{2}} & -\frac{1+i}{\sqrt{2}} & 0 \\
0 & 0 & 1 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 
\end{pmatrix}. \quad (3.18) \]
The equivalent pseudosupersymmetric Hamiltonian and charges are given by

\[ H' \equiv U_1 H U_1^\dagger = \frac{1}{2} (P^2 + W^2) \mathbb{1} + \frac{1}{2} W' \text{diag}(-1, 1, -1) - 2i H_4 \text{diag}(0, 0, 1), \] (3.19)

\[ Q' \equiv U_1 Q U_1^\dagger = -ic\sqrt{2} \begin{pmatrix} 0 & P + iW & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \] (3.20)

\[ Q'^\dagger \equiv U_1 Q'^\dagger U_1^\dagger = ic\sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ P - iW & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \] (3.21)

where \( \mathbb{1} \) again denotes the \( 3 \times 3 \) unit matrix.

The Hamiltonian \( H' \) can be alternatively written as

\[ H' = \begin{pmatrix} H_{SS} & 0 \\ 0 & 0 \end{pmatrix}, \] (3.22)

where

\[ H_{SS} = \begin{pmatrix} \frac{1}{2} (P^2 + W^2 - W') & 0 \\ 0 & \frac{1}{2} (P^2 + W^2 + W') \end{pmatrix} \] (3.23)

is a standard supersymmetric Hamiltonian corresponding to the superpotential \( W(x) \) and

\[ H_0 = H_1 - iH_4 = H_3 - W' - 2iH_4 \] (3.24)

may be any Hermitian operator due to the arbitrariness of \( H_4 \).

From (3.20) and (3.21), it is clear that the pseudosupercharges \( Q', Q'^\dagger \) do not connect the eigenstates of \( H_0 \) with those of \( H_{SS} \). We also note that they may be related to the orthosupercharges \( Q_1, Q_1^\dagger \) of Khare et al. \[9\] in OSSQM of order two. We shall come back to the connections between PsSSQM and OSSQM in Sec. 4.

In the special case \( H_4 = iW' \) considered by Beckers and Debergh \[8\], Eq. (3.19) becomes

\[ H' = \frac{1}{2} (P^2 + W^2) \mathbb{1} + \frac{1}{2} W' \text{diag}(-1, 1, 3). \] (3.25)

For the oscillator-like superpotential \( W(x) = x \) (in units wherein \( \omega = 1 \)), one then gets the pseudosupersymmetric oscillator Hamiltonian

\[ H^{(1)}_{osc} = \frac{1}{2} (P^2 + x^2) \mathbb{1} + \frac{1}{2} \text{diag}(-1, 1, 3), \] (3.26)
already encountered in Eq. (2.11) and whose spectrum contains the eigenvalues $n$, $n + 1$, and $n + 2$ for $n = 0, 1, 2, \ldots$. The corresponding pseudosupercharges are

$$Q^{(1)\prime}_{\text{osc}} = -ic\sqrt{2} \begin{pmatrix} 0 & P + ix & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (Q^{(1)\prime}_{\text{osc}})^\dagger = ic\sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ P - ix & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.27)$$

The choice of (3.26) in Ref. [8] to describe the pseudosupersymmetric oscillator was dictated by the characteristics of the physical problem at hand, namely that of relativistic vector mesons in a constant magnetic field, as reviewed in Sec. 2.

If we disregard the application to such a problem, a natural choice for the pseudosupersymmetric oscillator Hamiltonian and corresponding pseudosupercharges would be

$$\mathcal{H}^{(2)}_{\text{osc}} = \frac{1}{2}\{a^\dagger, a\} \mathbb{I} + \frac{1}{2}\{b^\dagger, b\}, \quad Q^{(2)}_{\text{osc}} = 2cba^\dagger, \quad (Q^{(2)}_{\text{osc}})^\dagger = 2cb^\dagger a, \quad (3.28)$$

where $a^\dagger = (x - iP)/\sqrt{2}$, $a = (x + iP)/\sqrt{2}$ are bosonic creation and annihilation operators, $b^\dagger$, $b$ are the pseudofermionic operators of Eqs. (2.13), (2.14), and both types of operators are assumed to commute with one another. This choice corresponds to $W(x) = x$, $H_4 = i/4$ in Eqs. (3.14) – (3.16) and leads to the diagonalized pseudosupersymmetric Hamiltonian

$$\mathcal{H}^{(2)\prime}_{\text{osc}} = \frac{1}{2}(P^2 + x^2) \mathbb{I} + \frac{1}{2} \text{diag}(-1, 1, 0), \quad (3.29)$$

while the corresponding pseudosupercharges $Q^{(2)\prime}_{\text{osc}}$, $(Q^{(2)\prime}_{\text{osc}})^\dagger$ coincide with $Q^{(1)\prime}_{\text{osc}}$, $(Q^{(1)\prime}_{\text{osc}})^\dagger$, given in (3.27). The spectrum of $\mathcal{H}^{(2)\prime}_{\text{osc}}$ contains the eigenvalues $n$, $n + 1$, and $n + \frac{1}{2}$ for $n = 0, 1, 2, \ldots$. Contrary to what happens for $\mathcal{H}^{(1)\prime}_{\text{osc}}$ whose levels starting from the second excited state are threefold degenerate, those of $\mathcal{H}^{(2)\prime}_{\text{osc}}$ are either nondegenerate or twofold degenerate. This type of spectrum is characteristic of the generic case for the Hamiltonian (3.19) or (3.22) in view of the arbitrariness of $H_4$ or $H_0$.

Still another interesting special case corresponds to $H_4 = 0$ and $W(x) = -x$, for which we get

$$\mathcal{H}^{(3)}_{\text{osc}} = \frac{1}{2}(P^2 + x^2) \mathbb{I} + \frac{1}{2} \text{diag}(1, 1, -1), \quad (3.30)$$

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The Hamiltonian $H^{(3)_r}_{\text{osc}}$ coincides with that of the Beckers-Debergh parasupersymmetric oscillator [7] and its spectrum contains the eigenvalues $n + 1$, $n$, and $n + 1$ for $n = 0, 1, 2, \ldots$. The corresponding charges are of course different from the Beckers-Debergh parasupercharges.

### 3.2 The case of unequal superpotentials

Whenever the superpotentials are unequal, we are left with the general equations (3.10), (3.11). To solve them, let us write the first two diagonal matrix elements of $H$ as the sum of a kinetic and a potential energy terms,

$$H_i = \frac{1}{2} P_i^2 + V_i(x), \quad i = 1, 2.$$  \hfill (3.34)

On inserting these expressions into Eqs. (3.10) and (3.11), writing the latter in normal order form with $P$ on the right of $x$, and equating the coefficients of equal powers of $P$, we obtain that $H_4$ does not depend on $P$ and that $V_1$, $V_2$, $H_4$ are related to $W_1$ and $W_2$ through the consistency conditions

$$V_1 + iH_4 = \frac{1}{4}(W_1^2 + W_2^2 - 3W_1' + W_2'),$$  \hfill (3.35)

$$V_1 W_1 + iH_4 W_2 = \frac{1}{4}(W_1^3 + W_1 W_2^2 - W_1 W_1' + W_1 W_2' - 2W_2 W_2' + W_1' - W_2''),$$  \hfill (3.36)

$$V_2 - iH_4^\dag = \frac{1}{4}(W_1^2 + W_2^2 + W_1' - 3W_2'),$$  \hfill (3.37)

$$V_2 W_2 - iH_4^\dag W_1 = \frac{1}{4}(W_2^2 W_2 + W_3^2 - 2W_1 W_1' + 2W_2 W_2' - W_1'' + W_2'').$$  \hfill (3.38)

Since the right-hand sides of these equations are real functions of $x$, the same must be true for their left-hand sides. This implies that $H_4$ must be an imaginary
function of $x$, hence it satisfies Eq. (3.14) again. When taking the latter into account, the set of Eqs. (3.35) – (3.38) becomes a nonhomogeneous system of four linear equations in three unknowns $V_1$, $V_2$, and $H_4$. Such a system turns out to be compatible and for $W_1 \neq W_2$ its solution is given by

\[ V_1 = \frac{1}{4} \left[ W_1^2 + W_2^2 - \frac{W_1 - 3W_2}{W_1 - W_2} (W_1' - W_2') + \frac{W_1'' - W_2''}{W_1 - W_2} \right], \quad (3.39) \]

\[ V_2 = \frac{1}{4} \left[ W_1^2 + W_2^2 + \frac{3W_1 - W_2}{W_1 - W_2} (W_1' - W_2') + \frac{W_1'' - W_2''}{W_1 - W_2} \right], \quad (3.40) \]

\[ H_4 = \frac{i}{4} \frac{2W_1W_1' - 2W_2W_2' + W_1'' - W_2''}{W_1 - W_2}. \quad (3.41) \]

Before studying the general solution (3.3), (3.34), (3.39) – (3.41) in more detail, it is worth considering the special case of equal and opposite superpotentials, $W_1(x) = -W_2(x) = W(x)$. The solution then reduces to

\[ H_1 = \frac{1}{2} \left( P^2 + W^2 - 2W' + \frac{W''}{2W} \right), \quad (3.42) \]

\[ H_2 = \frac{1}{2} \left( P^2 + W^2 + 2W' + \frac{W''}{2W} \right), \quad (3.43) \]

\[ H_3 = \frac{1}{2} (P^2 + W^2), \quad H_4 = \frac{i}{4} W''. \quad (3.44) \]

The corresponding pseudosupersymmetric Hamiltonian $H$ cannot be diagonalized through the unitary transformation $U_1$ anymore, but we note that for the oscillator-like superpotential $W(x) = x$, $H$ is diagonal so that we still get another pseudosupersymmetric oscillator Hamiltonian

\[ H_{\text{osc}}^{(4)} = \frac{1}{2} (P^2 + x^2)I + \text{diag}(-1, 1, 0). \quad (3.45) \]

By a permutation of rows and columns corresponding to the unitary matrix

\[ U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3.46) \]

the latter may be transformed into the Rubakov-Spiridonov-Khare parasupersymmetric oscillator Hamiltonian $[11]$,

\[ H_{\text{osc}}^{(4)''} \equiv U_2 H_{\text{osc}}^{(4)} U_2^\dagger = \frac{1}{2} (P^2 + x^2)I + \text{diag}(-1, 0, 1), \quad (3.47) \]
whose spectrum contains the eigenvalues \( n - \frac{1}{2}, n + \frac{1}{2}, \) and \( n + \frac{3}{2} \) for \( n = 0, 1, 2, \ldots \).

The same transformation leads to the pseudosupercharge
\[
Q^{(d)\text{osc}}_\text{osc} = U_2 Q^{(d)\text{osc}}_\text{osc} U_2^\dagger = \frac{c}{\sqrt{2}} \begin{pmatrix}
0 & (1 - i)(P + ix) & 0 \\
0 & 0 & 0 \\
0 & (1 + i)(P - ix) & 0
\end{pmatrix}
\] (3.48)
and its Hermitian conjugate, which of course differ from Rubakov-Spiridonov-Khare parasupercharges. Contrary to the pseudosupercharges (3.20), (3.21), those given in (3.48) and its Hermitian conjugate connect the eigenstates of all the components of the pseudosupersymmetric Hamiltonian. Hence the model described by (3.43) or (3.44) cannot be split into two isospectral models, one scalar and one 2 \(\times\) 2, as it was the case for that corresponding to (3.20).

Going back now to the general solution for \( W_1 \neq W_2 \), we may ask which choice of superpotentials makes \( \mathcal{H} \) diagonal. From Eq. (3.41), we immediately see that the condition \( H_4 = 0 \) leads to the differential equation
\[
W_1'' + 2W_1W_1' = W_2'' + 2W_2W_2',
\] (3.49)
which is equivalent to
\[
W_1' + W_1^2 = W_2' + W_2^2 + C,
\] (3.50)
where \( C \) is some real integration constant. By replacing \( W_1 \) and \( W_2 \) by the linear combinations \( W_\pm = W_1 \pm W_2 \), Eq. (3.50) is transformed into
\[
W_-' + W_+W_- = C,
\] (3.51)
which can be easily solved to yield \( W_- \) in terms of \( W_+ \). The results for \( W_1 \) and \( W_2 \) read
\[
W_1(x) = \frac{1}{2} \left( W_+(x) + \frac{C}{\exp(\int^x W_+(u)du) dt + D} \left( \frac{d}{dt} W_+(x) \right) \right), \quad (3.52)
\]
\[
W_2(x) = \frac{1}{2} \left( W_+(x) - \frac{C}{\exp(\int^x W_+(u)du) dt + D} \left( \frac{d}{dt} W_+(x) \right) \right), \quad (3.53)
\]
where \( D \) is another real integration constant. We conclude that for any real function \( W_+(x) \) and any real constants \( C, D \), the choice (3.52), (3.53) for \( W_1 \) and \( W_2 \) ensures that \( \mathcal{H} \) is diagonal.
The simplest choice for $W_+$ is $W_+ = 0$. We then obtain from (3.52) and (3.53) that $W_1(x) = -W_2(x) = \frac{1}{2}(Cx + D)$, which for $C = 2$ and $D = 0$ leads to the pseudosupersymmetric Hamiltonian $\mathcal{H}_{\text{osc}}^{(4)}$ given in Eq. (3.45).

4 Connection with Orthosupersymmetric Quantum Mechanics

In Sec. 3, we have already established some connections between PsSSQM and PSSQM for oscillator-like superpotentials. In the present section, we turn ourselves to OSSQM and study its relationship with PsSSQM.

To start with, we note that pseudofermion operators can be constructed in terms of orthofermion creation and annihilation operators of order two $c_\alpha^\dagger, c_\alpha$, $\alpha = 1, 2$, defined by the relations

\[ c_\alpha c_\beta = 0, \quad c_\alpha c_\beta^\dagger + \delta_{\alpha,\beta} \sum_{\gamma=1}^{2} c_\gamma c_\gamma^\dagger = \delta_{\alpha,\beta}, \quad \alpha, \beta = 1, 2, \quad (4.1) \]

and their Hermitian conjugates [3]. It is indeed clear that the linear combinations with complex coefficients

\[ \tilde{b} = \xi c_1^\dagger + \eta c_2^\dagger, \quad \tilde{b}^\dagger = \xi^* c_1 + \eta^* c_2, \quad |\xi|^2 + |\eta|^2 = 1, \quad (4.2) \]

satisfy Eq. (2.14).

In the standard three-dimensional matrix representation of the orthofermion algebra, wherein

\[ c_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_1^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_2^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (4.3) \]

$\tilde{b}$ and $\tilde{b}^\dagger$ are represented by

\[ \tilde{b} = \begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ \eta & 0 & 0 \end{pmatrix}, \quad \tilde{b}^\dagger = \begin{pmatrix} 0 & \xi^* & \eta^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.4) \]
Such matrices are unitarily equivalent to the standard matrix realization (2.13) of the pseudofermionic operators $b$, $b^\dagger$ since the matrices

$$b \equiv U_3 \tilde{b} U_3^\dagger = \begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{pmatrix}, \quad b^\dagger \equiv U_3 \tilde{b}^\dagger U_3^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \xi^* & \eta^* & 0 \end{pmatrix},$$

(4.5)

where

$$U_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

(4.6)

coincide with (2.13) provided we make the choice

$$\xi = \frac{1}{2}(1 + i), \quad \eta = \frac{1}{2}(-1 + i)$$

(4.7)
in (1.2).

Let us now consider an orthosupersymmetric Hamiltonian $H^K$ of order two, satisfying the relations

$$Q^K_\alpha Q^K_\beta = 0, \quad \{H^K, Q^K_\alpha \} = 0, \quad Q^K_\alpha \left(Q^K_\beta \right)^\dagger + \delta_{\alpha,\beta} \sum_{\gamma=1}^{2} \left(Q^K_\gamma \right)^\dagger Q^K_\gamma = 2\delta_{\alpha,\beta} H^K,$$

(4.8) \hspace{1cm} (4.9) \hspace{1cm} (4.10)

where $Q^K_\alpha$, $(Q^K_\alpha)^\dagger$, $\alpha = 1, 2$, are the orthosupercase operators $\mathbb{F}$. The above relationship between orthofermions of order two and pseudofermions suggests the construction of the operators

$$\tilde{Q} = \zeta \left(Q^K_1\right)^\dagger + \rho \left(Q^K_2\right)^\dagger, \quad \tilde{Q}^\dagger = \zeta^* Q^K_1 + \rho^* Q^K_2, \quad \tilde{H} = H^K, \quad |\zeta|^2 + |\rho|^2 = 2c^2,$$

(4.11)

which can be checked to satisfy the defining relations (3.1) – (3.3) of PsSSQM. We conclude that any order-two orthosupersymmetric quantum mechanical system has a pseudosupersymmetry generated by $\tilde{Q}$ and $\tilde{Q}^\dagger$, given in (4.11).

Both properties (1.2) and (4.11) can actually be extended to order-$p$ orthofermionic operators and order-$p$ orthosupersymmetric quantum mechanical systems, respectively. The corresponding equations read

$$\tilde{b} = \sum_{\alpha=1}^{p} \xi_\alpha c^\dagger_\alpha, \quad \tilde{b}^\dagger = \sum_{\alpha=1}^{p} \xi^*_\alpha c_\alpha, \quad \sum_{\alpha=1}^{p} |\xi_\alpha|^2 = 1,$$

(4.12)

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\[ \tilde{Q} = \sum_{\alpha=1}^{p} \zeta_{\alpha} (Q_{\alpha}^{K})^{\dagger}, \quad \tilde{Q}^{\dagger} = \sum_{\alpha=1}^{p} \zeta_{\alpha} Q_{\alpha}^{K}, \quad \tilde{\mathcal{H}} = \mathcal{H}^{K}, \quad \sum_{\alpha=1}^{p} |\zeta_{\alpha}|^2 = 2c^2, \quad (4.13) \]

where \( c_{\alpha}, c_{\alpha}^{\dagger} \) and \( Q_{\alpha}^{K}, (Q_{\alpha}^{K})^{\dagger}, \mathcal{H}^{K} \) satisfy equations similar to (4.1) and (4.8) – (4.10) with all indices \( \alpha, \beta, \gamma \) running from 1 to \( p \).

To study in more detail the connections between order-two OSSQM and PsSSQM, let us consider the OSSQM realization of Khare et al. in terms of two superpotentials \( W_{1}^{K}(x), W_{2}^{K}(x) \) and compare it with the corresponding realization of PsSSQM, given in Sec. 3. On using the former, we get the following realization for the operators of Eq. (4.11),

\[ \tilde{Q} = \begin{pmatrix} 0 & 0 & 0 \\ \zeta(P + iW_{1}^{K}) & 0 & 0 \\ \rho(P + iW_{2}^{K}) & 0 & 0 \end{pmatrix}, \quad \tilde{Q}^{\dagger} = \begin{pmatrix} 0 & \zeta^{*}(P - iW_{1}^{K}) & \rho^{*}(P - iW_{2}^{K}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.14) \]

\[ \tilde{\mathcal{H}} = \begin{pmatrix} h_{1} & 0 & 0 \\ 0 & h_{2} & 0 \\ 0 & 0 & h_{3} \end{pmatrix}, \quad (4.15) \]

where

\[ h_{1} = \frac{1}{2} \left[ p^2 + (W_{1}^{K})^2 + (W_{1}^{K})' \right], \quad (4.16) \]
\[ h_{2} = \frac{1}{2} \left[ p^2 + (W_{1}^{K})^2 - (W_{1}^{K})' \right], \quad (4.17) \]
\[ h_{3} = \frac{1}{2} \left[ p^2 + (W_{2}^{K})^2 - (W_{2}^{K})' \right], \quad (4.18) \]

and \( W_{1}^{K}, W_{2}^{K} \) are constrained by the relation

\[ (W_{1}^{K})^2 + (W_{1}^{K})' = (W_{2}^{K})^2 + (W_{2}^{K})'. \quad (4.19) \]

Applying now the unitary transformation \( U_{3}^{L} \), we obtain the matrices

\[ Q \equiv U_{3}^{L}\tilde{Q}U_{3}^{\dagger} = \begin{pmatrix} 0 & 0 & \zeta(P + iW_{1}^{K}) \\ 0 & \rho(P + iW_{2}^{K}) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.20) \]

\[ Q^{\dagger} \equiv U_{3}^{L}\tilde{Q}^{\dagger}U_{3}^{\dagger} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \zeta^{*}(P - iW_{1}^{K}) & \rho^{*}(P - iW_{2}^{K}) & 0 \end{pmatrix}, \quad (4.21) \]

\[ \mathcal{H} \equiv U_{3}^{L}\tilde{\mathcal{H}}U_{3}^{\dagger} = \begin{pmatrix} h_{2} & 0 & 0 \\ 0 & h_{3} & 0 \\ 0 & 0 & h_{1} \end{pmatrix}, \quad (4.22) \]
which have to be compared with the standard realization of PsSSQM in terms of two superpotentials $W_1, W_2$, given in (3.4) – (3.6).

More precisely, we would like to determine under which conditions a PsSSQM system described by Eqs. (3.4) – (3.6) may be characterized by Eqs. (4.20) – (4.22) constructed from a matrix realization of OSSQM. Comparison between Eqs. (3.4), (3.5) and (4.20), (4.21) directly leads to the identifications

$$
\zeta = -ic\sqrt{2}\xi = \frac{c(1 - i)}{\sqrt{2}}, \quad \rho = -ic\sqrt{2}\eta = \frac{c(1 + i)}{\sqrt{2}},
$$

and

$$
W^K_1(x) = W_1(x), \quad W^K_2(x) = W_2(x),
$$

where we have taken Eq. (4.7) into account. Furthermore, Eqs. (3.6) and (4.22) give rise to the constraints

$$
h_1 = H_3, \quad h_2 = H_1, \quad h_3 = H_2, \quad H_4 = 0,
$$

which have to be compatible with the expressions of $h_1, h_2, h_3$ and $H_1, H_2, H_3, H_4$ in terms of $W^K_1, W^K_2$ and $W_1, W_2$, respectively.

In the case of equal superpotentials $W_1 = W_2 = W$ considered in Subsec. 3.1, we note that Eq. (4.24) implies that Eq. (4.19) is automatically satisfied. It then remains to impose the sole condition $H_4 = 0$, because if the latter is satisfied, it results from Eqs. (3.15), (3.16), and (4.16) – (4.18) that the same is true for the remaining constraints in Eq. (4.25).

In the case of unequal superpotentials $W_1 \neq W_2$ discussed in Subsec. 3.2, we know that the condition $H_4 = 0$ leads to the constraint (3.50), where $C$ may be any real constant. On taking Eq. (4.24) into account, it is clear that the two constraints (3.50) and (4.19) are compatible only if $C = 0$. It is then straightforward to check that if we impose such a restriction, the remaining conditions in Eq. (4.25) are fulfilled by the operators (3.9), (3.34), (3.39), (3.40) and (4.16) – (4.18).

We conclude that any PsSSQM system for which $H_4 = 0$ may be considered as an order-two OSSQM one with the same superpotentials provided either $W_1 = W_2$ or
$W_1 \neq W_2$ and the integration constant $C$ in Eq. (3.50) vanishes. Order-two OSSQM systems are therefore a subclass of PsSSQM ones. Examples of PsSSQM systems that cannot be considered as OSSQM ones are provided by the pseudosupersymmetric oscillator Hamiltonians $\mathcal{H}^{(1)}_{\text{osc}}, \mathcal{H}^{(2)}_{\text{osc}},$ and $\mathcal{H}^{(4)}_{\text{osc}},$ defined in Sec. 3. The first two correspond to $W_1 = W_2$ and $H_4 \neq 0$, while the third one is obtained for $W_1 \neq W_2$ and $C = 2$. Such a Hamiltonian being unitarily equivalent to the Rubakov-Spiridonov-Khare parasupersymmetric oscillator has a negative-energy ground state, whereas such a phenomenon cannot occur in OSSQM.

5 Pseudosupersymmetric Quantum Mechanics in Terms of Generalized Deformed Oscillator Algebra Generators

The purpose of the present section is to propose two new realizations of PsSSQM in terms of the generators of a GDOA. The latter may be defined as a nonlinear associative algebra $\mathcal{A}(G(N))$ generated by the operators $N = N^\dagger$, $a^\dagger$, and $a = (a^\dagger)^\dagger$, satisfying the commutation relations

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = G(N), \quad (5.1)$$

where $G(N) = [G(N)]^\dagger$ is some Hermitian function of $N$ [7].

We restrict ourselves here to GDOAs possessing a bosonic Fock space representation. In the latter, we may write

$$a^\dagger a = F(N), \quad aa^\dagger = F(N + 1), \quad (5.2)$$

where the structure function $F(N) = [F(N)]^\dagger$ is such that

$$G(N) = F(N + 1) - F(N) \quad (5.3)$$

and is assumed to satisfy the conditions $F(0) = 0$ and $F(n) > 0$ if $n = 1, 2, 3, \ldots$. The carrier space $\mathcal{F}$ of such a representation can be constructed from a vacuum state
\(|0\rangle\) (such that \(a|0\rangle = N|0\rangle = 0\)) by successive applications of the creation operator \(a^\dagger\). Its basis states

\[
|n\rangle = \left(\prod_{i=1}^{n} F(i)\right)^{-1/2} (a^\dagger)^n|0\rangle, \quad n = 0, 1, 2, \ldots,
\]

(5.4)
satisfy the relations

\[
N|n\rangle = n|n\rangle, \quad a^\dagger|n\rangle = \sqrt{F(n+1)}|n+1\rangle, \quad a|n\rangle = \sqrt{F(n)}|n-1\rangle.
\]

(5.5)

Note that for \(G(N) = I, F(N) = N\), the algebra \(A(G(N))\) reduces to the standard (bosonic) oscillator algebra \(A(I)\), for which the creation and annihilation operators may be written as \(a^\dagger = (x - iP)/\sqrt{2}, a = (x + iP)/\sqrt{2}\) or, alternatively, as in Eq. (2.12).

Let us first consider the pseudosupercharges (3.20), (3.21), and the pseudosymmetric Hamiltonian (3.22), obtained in the case of equal superpotentials. For \(W(x) = x\), they become

\[
Q' = 2c \begin{pmatrix} 0 & a^\dagger & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q'^\dagger = 2c \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H' = \begin{pmatrix} a^\dagger a & 0 & 0 \\ 0 & a a^\dagger & 0 \\ 0 & 0 & H_0 \end{pmatrix},
\]

(5.6)

where \(a^\dagger, a\) are standard bosonic operators belonging to \(A(I)\). Inspired by this remark and some results for SSQM [21], let us introduce the matrices

\[
\bar{Q} = 2c \begin{pmatrix} 0 & f(N)a^\dagger & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{Q}^\dagger = 2c \begin{pmatrix} 0 & 0 & 0 \\ f(N+1)a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

(5.7)

\[
\bar{H} = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix},
\]

(5.8)

where \(N, a^\dagger, a\) are the generators of some GDOA \(A(G(N))\), \(f(N)\) is some real function of \(N\), and \(\bar{H}_i, i = 1, 2, 3\), are some \(N\)-dependent Hermitian operators. It is straightforward to show that such operator-valued matrices satisfy the defining relations (3.1) – (3.3) of PsSSQM provided we choose

\[
\bar{H}_1 = f^2(N)F(N), \quad \bar{H}_2 = f^2(N+1)F(N+1),
\]

(5.9)
while $H_3$ remains arbitrary. For $f(N) = 1$ and $F(N) = N$, we get back $H'$ in Eq. (7.6) with $H_3 = H_0$. For arbitrary $f(N)$ and $F(N)$, the spectrum of $H$ contains the eigenvalues $f^2(n)F(n)$ and $f^2(n + 1)F(n + 1)$, $n = 0, 1, 2, \ldots$, as well as those of $H_3$, and is clearly nonlinear. For generic $H_3$, the levels are either nondegenerate or twofold degenerate.

Let us next consider the pseudosupersymmetric Hamiltonian $H^{(4)\nu}_{\text{osc}}$ and the pseudosupercharges $Q^{(4)\nu}_{\text{osc}}$, $(Q^{(4)\nu}_{\text{osc}})^\dagger$, corresponding to the choice $W_1(x) = -W_2(x) = x$ and given in (3.47) and (3.48), respectively. By a procedure similar to that used above, we are led to propose another realization of PsSSQM,

$$\tilde{Q} = c\sqrt{2} \begin{pmatrix} 0 & f_1(N)a\dagger & 0 \\ 0 & 0 & 0 \\ 0 & if_2(N+1)a & 0 \end{pmatrix}, \quad (5.10)$$

$$\tilde{Q}^\dagger = c\sqrt{2} \begin{pmatrix} f_1(N+1)a & 0 & -if_2(N)a\dagger \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (5.11)$$

$$\tilde{H} = \begin{pmatrix} \tilde{H}_1 & 0 & 0 \\ 0 & \tilde{H}_2 & 0 \\ 0 & 0 & \tilde{H}_3 \end{pmatrix}, \quad (5.12)$$

where $N$, $a\dagger$, $a$ are the generators of $\mathcal{A}(G(N))$ again, $f_1(N)$, $f_2(N)$ some real functions of $N$, and $\tilde{H}_i$, $i = 1, 2, 3$, are some $N$-dependent Hermitian operators. This time we find that $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$ are constrained by the relations

$$\tilde{H}_1 = \frac{1}{2}[f_1^2(N)F(N) + f_2^2(N-1)F(N-1)], \quad (5.13)$$

$$\tilde{H}_2 = \frac{1}{2}[f_2^2(N+1)F(N+1) + f_2^2(N)F(N)], \quad (5.14)$$

$$\tilde{H}_3 = \frac{1}{2}[f_1^2(N+2)F(N+2) + f_2^2(N+1)F(N+1)]. \quad (5.15)$$

For $f_1(N) = f_2(N) = 1$ and $F(N) = N$, $\tilde{H}$ reduces to $H^{(4)\nu}_{\text{osc}} = \text{diag}(N - \frac{1}{2}, N + \frac{1}{2}, N + \frac{3}{2})$, while $\tilde{Q}$ and $\tilde{Q}^\dagger$ only differ from $Q^{(4)\nu}_{\text{osc}}$ and $(Q^{(4)\nu}_{\text{osc}})^\dagger$ by an irrelevant phase factor. For arbitrary $f_1(N)$, $f_2(N)$, and $F(N)$, the spectrum of $\tilde{H}$ is nonlinear and contains the eigenvalues $\frac{1}{2}[f_1^2(n)F(n) + f_2^2(n-1)F(n-1)]$, $\frac{1}{2}[f_2^2(n+1)F(n+1) + f_2^2(n)F(n)]$, and $\frac{1}{2}[f_1^2(n+2)F(n+2) + f_2^2(n+1)F(n+1)]$ for $n = 0, 1, 2, \ldots$. Except for a single nondegenerate level and a single twofold-degenerate one, all the levels are threefold degenerate.
It is worth mentioning that the Hamiltonian $\mathcal{H}$ of Eqs. (5.12) – (5.15) is also obtained with a different set of charges in a realization of Rubakov-Spiridonov-Khare PSSQM in terms of $\mathcal{A}(G(N))$ generators [22].

6 Reducibility and Bosonization of Pseudosupersymmetric Quantum Mechanics

We now plan to show that the two matrix realizations of PsSSQM in terms of GDOA generators, defined in Eqs. (5.7) – (5.9) and (5.10) – (5.15), respectively, are fully reducible.

For such a purpose, let us introduce the operators

$$P_\mu = \frac{1}{3} \sum_{\nu=0}^{2} e^{-\frac{2\pi i \mu \nu}{3}} T^\nu, \quad \mu = 0, 1, 2,$$  \hspace{1cm} (6.1)

where

$$T = e^{2\pi i N/3}. \hspace{1cm} (6.2)$$

It is straightforward to show that in the Fock representation $T^3 = I$ and the operators $P_\mu$ project on the subspaces $\mathcal{F}_\mu \equiv \{|k\lambda + \mu\rangle \mid k = 0, 1, 2, \ldots\}$ of the Fock space $\mathcal{F}$. Here $|n\rangle = |k\lambda + \mu\rangle$ are the basis states (5.4). We actually obtain a decomposition of $\mathcal{F}$ into three mutually orthogonal subspaces, $\mathcal{F} = \sum_{\mu=0}^{2} \mathcal{F}_\mu$. In other words, the operators $P_\mu$ satisfy the relations

$$P_\mu^\dagger = P_\mu, \quad P_\mu P_\nu = \delta_{\mu,\nu} P_\mu, \quad \sum_{\mu=0}^{2} P_\mu = I \hspace{1cm} (6.3)$$

in $\mathcal{F}$. From Eqs. (5.1), (6.1), and (6.2), we also derive the additional relations

$$[N, T] = 0, \quad a^\dagger T = e^{-2\pi i/3} T a^\dagger, \hspace{1cm} (6.4)$$

and

$$[N, P_\mu] = 0, \quad a^\dagger P_\mu = P_{\mu+1} a^\dagger, \hspace{1cm} (6.5)$$

where we use the convention $P_{\mu'} = P_\mu$ if $\mu' - \mu = 0 \mod 3$. 

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Let us next consider the $3 \times 3$ matrix

$$U_4 = \begin{pmatrix} P_0 & P_1 & P_2 \\ P_1 & P_0 & P_2 \\ P_2 & P_1 & P_0 \end{pmatrix},$$

(6.6)

whose elements are $P_\mu$ operators. From (6.3), it results that

$$U_4^\dagger = \begin{pmatrix} P_0 & P_1 & P_2 \\ P_2 & P_0 & P_1 \\ P_1 & P_2 & P_0 \end{pmatrix},$$

(6.7)

and

$$U_4 U_4^\dagger = U_4^\dagger U_4 = \mathbb{I},$$

(6.8)

showing that $U_4$ is a unitary matrix.

Through the unitary transformation represented by $U_4$, we then easily get diagonal realizations of PsSSQM equivalent to (5.7) – (5.9) and (5.10) – (5.15),

$$Q' \equiv U_4 \tilde{Q} U_4^\dagger = \begin{pmatrix} \tilde{Q}_0 & 0 & 0 \\ 0 & \tilde{Q}_1 & 0 \\ 0 & 0 & \tilde{Q}_2 \end{pmatrix},$$

(6.9)

$$\tilde{Q}^\dagger' \equiv U_4 \tilde{Q}^\dagger U_4^\dagger = \begin{pmatrix} \tilde{Q}_0^\dagger & 0 & 0 \\ 0 & \tilde{Q}_1^\dagger & 0 \\ 0 & 0 & \tilde{Q}_2^\dagger \end{pmatrix},$$

(6.10)

$$\tilde{H}' \equiv U_4 \tilde{H} U_4^\dagger = \begin{pmatrix} \tilde{H}_0 & 0 & 0 \\ 0 & \tilde{H}_1 & 0 \\ 0 & 0 & \tilde{H}_2 \end{pmatrix},$$

(6.11)

where $\tilde{Q}_\mu$, $\tilde{Q}^\dagger_\mu$, $\tilde{H}_\mu$, $\mu = 0, 1, 2$, are given by

$$\tilde{Q}_\mu = 2c f(N) a^\dagger P_{\mu+2}, \quad \tilde{Q}^\dagger_\mu = 2c f(N+1)a P_\mu,$$

(6.12)

$$\tilde{H}_\mu = \sum_{\nu=0}^{2} g_{\nu}(N) P_{\mu+3-\nu},$$

(6.13)

$$g_{\nu}(N) = \tilde{H}_{\nu+1} = f^2(N+\nu)F(N+\nu), \quad \nu = 0, 1, \quad g_2(N) = \tilde{H}_3,$$

(6.14)

in the former case and

$$\tilde{Q}_\mu = c \sqrt{2} \left[ f_1(N) a^\dagger + if_2(N+1)a \right] P_{\mu+2},$$

(6.15)

$$\tilde{Q}^\dagger_\mu = c \sqrt{2} \left[ f_1(N+1) a P_\mu - if_2(N) a^\dagger P_{\mu+1} \right],$$

(6.16)
\[ H_\mu = \sum_{\nu=0}^{2} g_\nu(N) P_{\mu+3-\nu}, \quad (6.17) \]
\[ g_\nu(N) = \bar{H}_{\nu+1} = \frac{1}{2} \left[ f_1^2(N+\nu)F(N+\nu) + f_2^2(N+\nu-1)F(N+\nu-1) \right], \quad (6.18) \]
in the latter. Note that in (6.14), \( g_2(N) \) remains arbitrary.

It is straightforward to check that the operators \( \bar{Q}_{\mu}, \bar{Q}_{\mu}^\dagger, \bar{H}_\mu \) satisfy the defining relations (3.1) – (3.3) of PsSSQM for any \( \mu \in \{0, 1, 2\} \), any GDOA \( \mathcal{A}(G(N)) \) (i.e., any structure function \( F(N) \)), and any choice of functions \( f(N) \) in (5.7) or \( f_1(N), f_2(N) \) in (5.10) and (5.11). Both sets of equations (6.12) – (6.14) and (6.15) – (6.18) provide us with a bosonization of PsSSQM similar to that known for SSQM \[15\] in terms of the Calogero-Vasiliev algebra \[16\]. It should be stressed that such results remain valid in the case of the standard oscillator algebra \( \mathcal{A}(I) \).

### 7 Pseudosupersymmetric Quantum Mechanics in Terms of \( C_3 \)-Extended Oscillator Algebra Generators

Let us specialize the results of Secs. 5 and 6 by selecting the GDOA \( \mathcal{A}^{(3)}(G(N)) \) associated with a \( C_3 \)-extended oscillator algebra \( \mathcal{A}^{(3)}_{\alpha_0,\alpha_1} \), where \( C_3 = \mathbb{Z}_3 \) is the cyclic group of order three \[14, 18\]. As generator of the latter we take the operator \( T \) defined in (6.2), hence \( C_3 = \{T, T^2, T^3 = I\} \).\(^{23}\) The GDOA \( \mathcal{A}^{(3)}(G(N)) \) corresponds to the choice

\[ G(N) = I + \kappa_1 T + \kappa_2 T^2 \quad (7.1) \]
in Eq. (5.1). Here \( \kappa_1 \) is some complex constant and \( \kappa_2 = \kappa_1^2 \).

The function \( G(N) \) of Eq. (7.1) can be alternatively written in terms of the projection operators (6.1) as

\[ G(N) = I + \sum_{\mu=0}^{2} \alpha_\mu P_\mu, \quad (7.2) \]

\(^{23}\)In \( \mathcal{A}^{(3)}_{\alpha_0,\alpha_1} \), \( T \) is considered as an operator independent of the remaining ones, so that the property \( T^3 = I \) and Eq. (5.4) (or alternatively Eqs. (6.3) and (6.5)) have to be postulated in addition to (5.1) and (7.1) (or (7.2)). The GDOA \( \mathcal{A}^{(3)}(G(N)) \) corresponds to the realization (6.2) of \( T \).
where
\[ \alpha_\mu = \sum_{\nu=1}^{2} e^{2\pi i \mu \nu / 3} \kappa_\nu, \quad \mu = 0, 1, 2, \tag{7.3} \]
are some real parameters subject to the condition
\[ \sum_{\mu=0}^{2} \alpha_\mu = 0. \tag{7.4} \]
The \( P_\mu \)'s can now be interpreted as the projection operators on the carrier spaces of the three inequivalent unitary irreducible representations of the cyclic group \( C_3 \).

From (7.2) and (7.4), it follows that the algebra \( A^{(3)}(G(N)) \) depends upon two independent, real parameters \( \alpha_0, \alpha_1 \), and goes over to the standard oscillator algebra \( A(I) \) for \( \alpha_0, \alpha_1 \to 0 \). Its structure function \( F(N) \), which is a solution of Eq. (5.3) with \( G(N) \) given by (7.2), can be expressed as
\[ F(N) = N + \sum_{\mu=0}^{2} \beta_\mu P_\mu, \quad \beta_0 \equiv 0, \quad \beta_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu, \quad \mu = 1, 2. \tag{7.5} \]
The existence of a bosonic Fock space representation is guaranteed by the constraints
\[ \alpha_0 > -1, \quad \alpha_0 + \alpha_1 > -2 \tag{7.6} \]
on the parameters, ensuring that \( F(n) > 0 \) for \( n = 1, 2, 3, \ldots \).

By using the explicit form of the structure function of \( A^{(3)}(G(N)) \), given in (7.5), the eigenvalues \( \bar{\mathcal{E}}^{(\mu)}_n \) of the bosonized pseudosupersymmetric Hamiltonian \( \bar{H}_\mu \), defined in (6.13) and (6.14), can be written in the form
\[ \bar{\mathcal{E}}^{(0)}_{3k} = 3k f_2(3k), \quad \bar{\mathcal{E}}^{(0)}_{3k+2} = \bar{\mathcal{E}}^{(0)}_{3k+3}, \quad \bar{\mathcal{E}}^{(0)}_{3k+1} \text{ arbitrary}, \tag{7.7} \]
\[ \bar{\mathcal{E}}^{(1)}_{3k} = \bar{\mathcal{E}}^{(1)}_{3k+1} = (3k + 1 + 2\gamma_0) f_2^2(3k + 1), \quad \bar{\mathcal{E}}^{(1)}_{3k+2} \text{ arbitrary}, \tag{7.8} \]
\[ \bar{\mathcal{E}}^{(2)}_{3k+1} = \bar{\mathcal{E}}^{(2)}_{3k+2} = (3k + 2 + 2\gamma_2) f_2^2(3k + 2), \quad \bar{\mathcal{E}}^{(2)}_{3k} \text{ arbitrary}, \tag{7.9} \]
where
\[ \gamma_\mu \equiv \frac{1}{2} (\beta_\mu + \beta_{\mu+1}) = \begin{cases} \frac{1}{2} \alpha_0 & \text{if } \mu = 0 \\ \alpha_0 + \frac{1}{2} \alpha_1 & \text{if } \mu = 1 \\ \frac{1}{2} (\alpha_0 + \alpha_1) & \text{if } \mu = 2 \end{cases}. \tag{7.10} \]
Similarly, for $\mathcal{H}_\mu$, defined in (6.17) and (6.18), we obtain

$$
\bar{\mathcal{E}}^{(0)}_{3k} = \frac{1}{7}(3k f_1^2(3k) + (3k - 1 + 2\gamma_2) f_2^2(3k - 1)),
$$

$$
\bar{\mathcal{E}}^{(0)}_{3k+1} = \bar{\mathcal{E}}^{(0)}_{3k+2} = \bar{\mathcal{E}}^{(0)}_{3k+3},
$$

$$
\bar{\mathcal{E}}^{(1)}_{3k} = \bar{\mathcal{E}}^{(1)}_{3k+1} = \frac{1}{2}[(3k + 1 + 2\gamma_0) f_1^2(3k + 1) + 3k f_2^2(3k)],
$$

$$
\bar{\mathcal{E}}^{(1)}_{3k+2} = \bar{\mathcal{E}}^{(1)}_{3k+3},
$$

$$
\bar{\mathcal{E}}^{(2)}_{3k} = \bar{\mathcal{E}}^{(2)}_{3k+1} = \bar{\mathcal{E}}^{(2)}_{3k+2} = \frac{1}{2}[(3k + 2 + 2\gamma_2) f_1^2(3k + 2) + (3k + 1 + 2\gamma_0) f_2^2(3k + 1)].
$$

Such relations show that the existence of the two different types of bosonization (6.12) – (6.14) and (6.15) – (6.18), as well as the arbitrariness of $f(N)$ and $f_1(N)$, $f_2(N)$ make it virtually possible to reproduce any nonlinear pseudosupersymmetric spectra.

In the particular cases where $f(N) = 1$ or $f_1(N) = f_2(N) = 1$, we get back the results previously obtained for linear spectra [14]. Equation (6.15) indeed corresponds to Eq. (5.20) of Ref. [14] (for the choice $\varphi = \pi/2$ in Eq. (5.18) of the same reference), while Eq. (6.12) can be deduced from Eq. (5.23) of Ref. [14] by interchanging the roles of $\bar{Q}_\mu$ and $\bar{Q}_\mu^\dagger$ (which leaves the PsSSQM algebra invariant) and changing the label $\mu$. The corresponding pseudosupersymmetric Hamiltonians and their eigenvalues can be similarly related.

More generally, it results from Eqs. (7.6) and (7.10) that if $f_2^2(N)$ is chosen to be an increasing function of $N$ and $\bar{H}_3$ is a positive-definite operator, the spectrum (7.7) has a nondegenerate ground state at vanishing energy, whereas for the spectra (7.8) and (7.9), the ground state at a positive energy may be nondegenerate or twofold degenerate. Furthermore, if $f_1^2(N)$ and $f_2^2(N)$ are chosen to be increasing functions of $N$, the ground state of the spectrum (7.11) is nondegenerate and its energy may have any sign, while that of the spectra (7.12) and (7.13) is twofold or threefold degenerate and has a positive energy.
8 Summary

In the present paper, we have constructed the complete explicit solution for the realization of PsSSQM in terms of two superpotentials and we have established that it can be separated into two branches corresponding to equal or unequal superpotentials, respectively.

We have then proved that any order-\(p\) orthosupersymmetric system has a pseudosupersymmetry, but that the reverse is not true. We have actually given conditions under which a pseudosupersymmetric system may be described by orthosupersymmetries of order two. In this way, we have extended to PsSSQM some recent results valid for PSSQM and FSSQM \cite{13}, thereby establishing that OSSQM is contained in the other variants of SSQM mentioned in Sec. 1.

Next, we have proposed two new matrix realizations of PsSSQM in terms of GDOA generators and we have related them to the two distinct realizations of the same in terms of superpotentials.

Finally we have demonstrated that such matrix realizations are fully reducible and that their irreducible components provide two distinct sets of bosonized operators realizing PsSSQM and corresponding to nonlinear spectra. These two sets reduce to those found in Ref. \cite{14} when we choose a GDOA associated with a \(C_3\)-extended oscillator algebra and restrict ourselves to linear spectra. Such results are part of a more general study, wherein we plan to prove the full reducibility of SSQM variants and their resultant bosonization when they are realized in terms of GDOA generators (see also Ref. \cite{22}).
References

[1] E. Witten, *Nucl. Phys.* B188, 513 (1981).

[2] F. Cooper, A. Khare and U. Sukhatme, *Phys. Rep.* 251, 267 (1995); B. Bagchi, *Supersymmetry in Quantum and Classical Mechanics* (Chapman and Hall / CRC, Florida, 2000).

[3] H. S. Green, *Phys. Rev.* 90, 270 (1953); Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and Parastatistics* (Springer, Berlin, 1982).

[4] J. Beckers, N. Debergh and A. G. Nikitin, *Fortschr. Phys.* 43, 67, 81 (1995).

[5] A. K. Mishra and G. Rajasekaran, *Pramana (J. Phys.)* 36, 537 (1991); *ibid.* 37, 455(E) (1991).

[6] V. A. Rubakov and V. P. Spiridonov, *Mod. Phys. Lett.* A3, 1337 (1988); A. Khare, *J. Math. Phys.* 34, 1277 (1993).

[7] J. Beckers and N. Debergh, *Nucl. Phys.* B340, 767 (1990).

[8] J. Beckers and N. Debergh, *Int. J. Mod. Phys.* A10, 2783 (1995).

[9] A. Khare, A. K. Mishra and G. Rajasekaran, *Int. J. Mod. Phys.* A8, 1245 (1993).

[10] S. Durand, *Phys. Lett.* B312, 115 (1993); *Mod. Phys. Lett.* A8, 1795 (1993).

[11] E. Witten, *Nucl. Phys.* B202, 253 (1982).

[12] A. Mostafazadeh and K. Aghababaei Samani, *Mod. Phys. Lett.* A15, 175 (2000); K. Aghababaei Samani and A. Mostafazadeh, *Nucl. Phys.* B595, 467 (2001).

[13] A. Mostafazadeh, *J. Phys.* A34, 8601 (2001); K. Aghababaei Samani and A. Mostafazadeh, *Mod. Phys. Lett.* A17, 131 (2002).
[14] C. Quesne and N. Vansteenkiste, *Int. J. Theor. Phys.* **39**, 1175 (2000).

[15] M. S. Plyushchay, *Ann. Phys. (N.Y.)* **245**, 339 (1996); J. Beckers, N. Debergh and A. G. Nikitin, *Int. J. Theor. Phys.* **36**, 1991 (1997).

[16] M. A. Vasiliev, *Int. J. Mod. Phys.* **A6**, 1115 (1991).

[17] J. Katriel and C. Quesne, *J. Math. Phys.* **37**, 1650 (1996); C. Quesne and N. Vansteenkiste, *J. Phys. A* **28**, 7019 (1995); *Helv. Phys. Acta* **69**, 141 (1996).

[18] C. Quesne and N. Vansteenkiste, *Phys. Lett.* **A240**, 21 (1998); *Helv. Phys. Acta* **72**, 71 (1999).

[19] B. Vijayalakshmi, M. Seetharaman and P. M. Mathews, *J. Phys. A* **12**, 665 (1979); J. Daicic and N. E. Frankel, *ibid.* **A26**, 1397 (1993).

[20] N. Vansteenkiste, Thèse, Université Libre de Bruxelles (2001), unpublished.

[21] D. Bonatsos and C. Daskaloyannis, *Phys. Lett.* **B307**, 100 (1993).

[22] C. Quesne and N. Vansteenkiste, *Mod. Phys. Lett.* **A17**, 839 (2002).