Using the Finite Product Method for solving
eigenvalue problems formulated in cylindrical
coordinates

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Abstract. Analysis of free/forced wave propagation in multi-layered structures and structures
under heavy fluid loading constitute classical problems of fluid-structure interaction. These
waveguides and many similar ones support infinitely many waves and in some cases e.g. at
high-frequency excitations or in near-field analysis it is necessary to account for a large number
of them. Finding the dispersion curves from these transcendental dispersion relations is not a
trivial task due to their ill-conditioned/unstable nature as well as numerical algorithms ability to
solve transcendental equations. These issues can, however, be circumvented by using the Finite
Product Method (FPM). The FPM is generic and has been used to solve dispersion equations of
homogeneous waveguides derived in Cartesian coordinates with sine/cosine functions. However,
it is yet to be formulated in cylindrical coordinates when Bessel functions are involved in
dispersion equations. We focus in this paper on extending the method to the cylindrical
problems and compound waveguides illustrated here by a fluid-filled cylindrical shell. The
great advantage of the FPM is that it reduces the transcendental dispersion equation to a
polynomial one easily solved by numerical algorithms but more importantly it delivers only
authentic roots of the dispersion equation i.e. no spurious roots as often encountered when
using Taylor approximations etc.

1. Introduction
The eigenvalue problems formulated in cylindrical coordinates condense to solving transcen-
dental characteristic equations containing Bessel functions. These characteristic equations may be
written on the general form;

\[ h(\Omega, k, m) - g(\Omega, k, m) \frac{G_m[\kappa(\Omega, k)]}{G'_m[\kappa(\Omega, k)]} = 0 \]  (1)

where \( h \) and \( g \) are some functions containing the physical properties of the problem, \( G_m \) any
Bessel function and \( G'_m \) its derivative. In general, \( \Omega, k \) and \( m \) are unknown parameters. This
equation, presented here in a rather general format, encloses a wide variety of challenging
physical and engineering problems, conveniently formulated in cylindrical coordinates, see e.g.
[1–4]. A broad range of these problems is concerned with waveguides in various realms of physics
such as; acoustics, optics, electrics and structural dynamics, so that the characteristic equation,
Eq.(1), encounters the meaning of a dispersion equation for a waveguide supporting an infinite
number of waves.
To analyse performance of a waveguide the \((\Omega, k)\) dispersion diagram is needed. Thus, we need to find the solutions, \(k\), to the dispersion equation as a function of \(\Omega\). However, since such an equation system often evolve from interacting fields of different nature (compound waveguides) e.g. vibro-acoustics or electro-magnetics (essentially rigid/compliant), finding (or approximating) these roots is not a trivial task as discussed in [5]. Various methods to accommodate this exist, such as Wave Based (WB), Semi-Analytical Finite Element (SAFE), Partial Wave Root Finding (PWRF), Pseudo-spectral Collocation (PSC) methods etc. In engineering applications, the preferred choice is the methods based on geometrical discretisation into elements such as the Wave Finite Element (WFE) or Spectral Element (SE) method, see e.g. [6; 7]. In particular, these methods are suitable for retrieving dispersion diagrams for complicated waveguides but require CAD models for element discretisation. However, when the dispersion equation is on analytical form as in Eq.(1), approximations are nevertheless necessary, either as a substitute for the exact equation or as input to find the exact solutions. Some of the most used approximations are polynomials such as Chebyshev or Taylor series, where the former is good but tends to mask the physical nature and analytical structure of the problem, [5], while the latter offers a poor range of applicability and introduce spurious roots, [5]. In light of these drawbacks the Finite Product Method (FPM) developed in [5] was proposed as a powerful alternative. Several benefits of this method can be listed from [5], for instance, it offers arbitrarily high accuracy at negligible expense, introduce no spurious roots and preserves the physical nature of the problem at hand — but most important; it is surprisingly simple. Further, the FPM also exhibits strength similar to the Chebyshev polynomials by escaping Runge’s Phenomenon.

However, the FPM is yet to be developed for problems formulated in cylindrical coordinates as well as for compound waveguides and, given its advantages, it is the obvious objective of this paper. As an example to illustrate the formulation of the FPM for solving Eq.(1) we consider wave propagation in a conservative elastic fluid-filled cylindrical shell (using \(\text{exp}[kx - im\theta - i\omega t]\)) in which case; Eq.(1) encounters the interpretation of a dispersion relation, the Bessel-functions become of first kind and order \(m \in \mathbb{Z}\) which incurs the meaning of circumferential modes, \(k \in \mathbb{C}\) are axial wavenumbers, \(\Omega \in |\mathbb{R}|\) (conservative) the frequency parameter and \(J'_m(\kappa) = \frac{1}{\kappa} \frac{dJ_m(\kappa r)}{dr} \bigg|_{r=1} = \frac{dJ_m(\kappa)}{d\kappa}\) such that

\[
g(\Omega, k, m) = \frac{\alpha(\Omega)}{\kappa(\Omega, k)} f_1(\Omega, k, m) \quad h(\Omega, k, m) = f_2(\Omega, k, m)
\]

where \(\kappa(\Omega, k) = \sqrt{k^2 + \Omega^2\gamma^2}\) \(\wedge\) \(\alpha(\Omega) = \frac{\rho}{\mu} \Omega^2\)

all formulated in their non-dimensional form (unless otherwise stated). Here \(\kappa\) is the radial wavenumber, \(f_2 = 0\) corresponds to the dispersion equation for the in vacuo shell, \(G_m(\kappa) \rightarrow J_m(\kappa) = 0\) to the soft baffle dispersion equation (pressure release) and \(G'_m(\kappa) \rightarrow J'_m(\kappa) = 0\) to the rigid baffle dispersion equation. Since \(f_1\) and \(f_2\) are rather cumbersome for this problem they are not presented here but can be deduced from e.g. [3; 4]. For the example used in this paper we use the non-dimensional parameters; \(\mu = 0.0175\), \(\rho = 0.128\), \(\gamma = 3.7773\), corresponding to e.g. a water-filled steel shell. Here \(\mu\) is the thickness-to-radius ratio, \(\rho\) the fluid-to-structure density ratio and \(\gamma\) the structure-to-fluid sound speed ratio, see e.g. [3; 4] for details.

2. The Finite Product Method for problems involving Bessel functions

To formulate the shell problem in the framework of FPM we follow [5] and introduce, first, an equivalent infinite product formulation of the transcendental terms (Bessel functions). This
form is standard and can be found in e.g. [1; 8; 9] as;

\[ J_m(\kappa) = \frac{(\frac{\kappa}{2})^m}{\Gamma(m+1)} \prod_{n=1}^{\infty} \left[ 1 - \frac{\kappa^2}{j_{m,n}^2} \right] = \frac{(\frac{\kappa}{2})^m}{\Gamma(m+1)} \prod_{n=1}^{N_1} \left[ 1 - \frac{\kappa^2}{j_{m,n}^2} \right] \prod_{n=N_1+1}^{\infty} \left[ 1 - \frac{\kappa^2}{j_{m,n}^2} \right] \] (3)

\[ J'_m(\kappa) = \frac{(\frac{\kappa}{2})^{m-1}}{2\Gamma(m)} \prod_{n=1}^{\infty} \left[ 1 - \frac{\kappa^2}{j_{m,n}^2} \right] = \frac{(\frac{\kappa}{2})^{m-1}}{2\Gamma(m)} \prod_{n=1}^{N_2} \left[ 1 - \frac{\kappa^2}{j_{m,n}^2} \right] \prod_{n=N_2+1}^{\infty} \left[ 1 - \frac{\kappa^2}{j_{m,n}^2} \right] \] \quad \text{for } m \neq 0 \] (4)

where \( j_{m,n} \) and \( j'_{m,n} \) are the zeros of the transcendental functions, also found in e.g. [8; 9] and is otherwise standard in most mathematical software. Further, we split the product into a finite and an infinite one characterised by the approximation order \((N_1, N_2) \in \mathbb{N}\). Note that Eq.(4) holds only for \( m \neq 0 \) as indicated, however, at \( m = 0 \) we use instead the identity; \( J'_0(\kappa) = -J_1(\kappa) \) so that Eq.(3) apply.

Then, following [5] we may write the infinite product in terms of a so-called gamma-conversion factor (converting Bessel functions to polynomials), which constitute the amplitude-modulation of the Bessel-approximation. However, since there are no explicit form of the zeros of the Bessel-function, one needs to apply approximate asymptotic zeros so that the conversion factor is accurate only up to some power, \( p; \mathcal{O}\left(\frac{1}{N^p}\right) \). This was shown in [1] for \( p = 1 \) with the asymptotic zeros;

\[ j_{m,n} = \tilde{j}_{m,n} + \mathcal{O}\left(\frac{1}{N}\right) = \left(n + \frac{1}{2}m - \frac{1}{4}\right)\pi + \mathcal{O}\left(\frac{1}{N}\right) \]

\[ j'_{m,n} = \tilde{j}'_{m,n} + \mathcal{O}\left(\frac{1}{N}\right) = \left(n + \frac{1}{2}m - \frac{3}{4}\right)\pi + \mathcal{O}\left(\frac{1}{N}\right) = \tilde{j}_{m-1,n} - \frac{1}{2}\pi + \mathcal{O}\left(\frac{1}{N}\right) \] (5)

where we see that the asymptotic zeros corresponds to a shifted sine/cosine dispersion and so they follow: \( \tilde{j}'_{m,n} = \tilde{j}_{m,n} - \frac{1}{2}\pi = \tilde{j}_{m-1,n} \) and incidentally \( \tilde{j}'_{m,n+1} = \tilde{j}_{m+1,n} \). Also, as given in e.g. [8; 9], the actual zeros distribute as;

\[ j_{m,n} < j_{m+1,n} < j_{m,n+1} \quad \& \quad j'_{m,n} < j'_{m+1,n} < j'_{m,n+1} \]

\[ j_{m,n} < j'_{m,n} < j_{m,n+1} < j'_{m,n+1} \] (6)

Obviously, we may always improve this approximation taking increasing asymptotic orders of \( p \). However, as the essential part of the FPM is that the conversion factors of \( J_m(\kappa) \) and \( J'_m(\kappa) \) cancel almost exactly for proper choice of \( N_1 \) and \( N_2 \), increasing the accuracy is indeed redundant. As in [5] this cancellation and corresponding choice of \( N_1/N_2 \) can be shown using e.g. Stirling’s approximation, but as this is more or less obvious from Eq.(3) and (4) using (5) or (6) we take it for granted until returning to it in Sec. 3 and use the approximation order \((N_1, N_2) = (N_1 + 1, N_1)\). Thus, to get the finite products, \( \tilde{J}_m(\kappa) \) and \( \tilde{J}'_m(\kappa) \), we simply discard the transcendental terms and substitute the Bessel-functions with their FP-approximations in Eq.(7), assuming \( N_1 \) large enough that the approximation remains good in some region of the real/complex \((\Omega, k)\)-space.

\[ \tilde{J}_m(\kappa) = \frac{(\frac{\kappa}{2})^m}{\Gamma(m+1)} \prod_{n=1}^{N_1} \left[ 1 - \frac{\kappa^2}{j_{m,n}^2} \right] \quad \tilde{J}'_m(\kappa) = \frac{(\frac{\kappa}{2})^{m-1}}{2\Gamma(m)} \prod_{n=1}^{N_2} \left[ 1 - \frac{\kappa^2}{j_{m,n}^2} \right] \] (7)

with \( \tilde{J}_m(\kappa) \equiv 1 \) and \( \tilde{J}'_m(\kappa) \equiv 1 \) for \( N_1 = N_2 = 0 \). In contrast, to Taylor approximations the FPM requires no derivatives nor does it introduce spurious zeros. Further, Runge’s phenomenon cancels (by way of the fraction, see e.g. Fig. 3) as in Chebyshev polynomials, yet it preserves.
the physical nature of the equation system. In essence, the FPM is extremely simple in that we use only an equivalent infinite product representation of the transcendental terms (available in literature), introduce the approximation order \((N_1, N_2)\), discard the transcendental part of the product formulation and find the relation between the approximation orders to ensure correct limit behaviour. As we shall see in Sec. 3 finding the approximation orders are equally simple. Though it is illustrated here for the Bessel functions of first kind the FPM is generic and thus applies equally well to any other Bessel function.

In Fig. 1 dispersion diagrams for different approximation orders \((N_1, N_1 + 1)\) are shown for a water-filled steel shell in bending \((m = 1)\). From the figures we find that even for low order approximations the dispersion curves are surprisingly accurate in a fairly large region of the real/complex \((\Omega, k)\)-space bounded approximately by; \(\Omega = \frac{j'_m N_2}{\pi} \approx \frac{(N_1 + \frac{3}{4})\pi}{2}\) and \(k = j'_m N_2 \approx (N_1 + \frac{3}{4})\pi\) for the cases shown here \((m = 1)\). This is discussed further in Sec. 3. Fortunately, as seen from the dimensional frequency scale on figure (e,f) this constitutes a large applicability range for waveguide problems. Note also from figure (f) that a real-branch of the dispersion diagram is not captured at all; explained by a too low approximation order since the branch originates from \(j'_m N_2 + 1\).

3. Accuracy analysis and discussion

Proper relation between \(N_1\) and \(N_2\) may be found from a Stirling approximation and/or a limit study to ensure the asymptotic behaviour. However, using just the distribution of zeros from Eq.(6) the relation appear to be obvious from the fraction the finite products;

\[
\frac{\tilde{J}_m(\kappa)}{J'_m(\kappa)} = \frac{\left(\frac{\pi}{2}\right)^m}{m} \prod_{n=1}^{N_1} \left[1 - \frac{\kappa^2}{j'_m n}\right] = \frac{\kappa}{m} \prod_{n=1}^{N_1} \left[1 - \frac{\kappa^2}{j'_m n}\right] \approx \frac{\kappa}{m} \prod_{n=1}^{N_1} \left[1 - \kappa^2 \left(\frac{n}{j'_m n}\right)^2\right] \tag{8}
\]

Then using Eq.(6) we see from the equation that the fraction is lead by its zero \(j_m,n\) (or \(\tilde{j}_m,n\)) for fixed \(n\) (seen explicitly using the first order asymptotic zeros, Eq.(5)). This leaves only two obvious choices; either \(N_1 = N_2\), (fraction lead by zero, \(j_m,n\)) or \(N_2 = N_1 + 1\) (fraction lead by pole, \(\tilde{j}_m,n\)). From this we may deduce a conservative validity range for \(k \in \mathbb{R}\) based on the last zeros included i.e. \(|\kappa| \leq \max\{|j_m,N_1|, |\widetilde{j}_m,N_1|\}\) corresponding to;

\[
|\kappa| \leq j_m,N_1 = \left(N_1 + \frac{1}{2} - \frac{1}{4}\right) \pi + \mathcal{O}\left(\frac{1}{N}\right) \quad \text{for} \quad N_2 = N_1
\]

\[
|\kappa| \leq j'_m,N_1 + 1 = \left(N_1 + \frac{1}{2} + \frac{1}{4}\right) \pi + \mathcal{O}\left(\frac{1}{N}\right) \quad \text{for} \quad N_2 = N_1 + 1
\tag{9}

For \(|\kappa| > \max\{|j_m,N_1|, |\widetilde{j}_m,N_2|\}\) the FP-fraction in Eq.(8) diverges rapidly from the exact, see Fig. 3b. Hence, the threshold of Eq.(9) constitute the arc; \(k(\Omega) = \sqrt{\max\{|j_m,N_1|, |\widetilde{j}_m,N_2|\}^2 - \gamma^2 \Omega^2}\) in the \((\Omega, k)\)-space, seen in Fig. 1 as \(\kappa_{\text{Thres}}\). Though the validity range for \(k \in i\mathbb{R}\) (purely imaginary) extends beyond this as seen from Fig. 1 we may use Eq.(9) also as a conservative range for the complex domain.

For any other choice of approximation order, zeros (and poles) are ’left out’ resulting in the approximation being no better than \(\min\{j_m,N_1, \widetilde{j}_m,N_2\}\) since Eq.(8) will diverge significantly already in this region as a consequence of the amplitude modulation. Then by the same argument...
Figure 1. A comparison between the exact and FPM-dispersion diagrams for a fluid-filled shell in bending ($m = 1$). Left figures show imaginary parts (propagating waves), right shows real parts with approximation orders (0,1) for (a,b), (1,2) for (c,d) and (3,4) for (e,f). Grid points according to Sec. 3 are shown and $i\mathbb{R}$ indicate purely imaginary wavenumbers.

all zeros up to the chosen approximation order should obviously be included. In general the
approximation order should be chosen as the zeros, \( r_n \) and \( q_n \), up to any two consecutive zeros of the sorted set:

\[
\{ \ldots, r_{N_1}, q_{N_2}, r_{N_1+1}, \ldots \} \quad \text{where} \quad \ldots < r_{N_1} < q_{N_2} < r_{N_1+1} < \ldots
\]

(10)
corresponding to a lead of either \( q_{N_2} \) or \( r_{N_1+1} \). Indeed this corresponds to \( N_2 = N_1 \) and \( N_2 = N_1 + 1 \) for the shell and using this generalisation it is easy from Eq.(5) to show for some hypothetical fraction \( \frac{J_{m+2}(\kappa)}{J_m(\kappa)} \) that the proper relation between \( N_1/N_2 \) is either; \( N_2 = N_1 + 1 \) or \( N_2 = N_1 + 2 \).

Returning to the shell example, the two choices are in general equally good, in that; taking \( N_2 = N_1 + 1 \) the validity range of \( \kappa \) extends by \( \frac{1}{2}\pi + O\left(\frac{1}{N}\right) \), however, at the expense of an increase in the polynomial order of power two. Thus, we cannot immediately argue one choice over the other as this depends on the problem at hand. Though both choices are good and equally valid we wish to find the best choice. To do so we study the nature of the problem. From this we find that \( f_2 \) is quartic in \( k^2 \) and \( f_1 \) quadratic, so that from the dispersion equation;

\[
\frac{f_1(\Omega, k, m)}{f_2(\Omega, k, m)} = \frac{\mu}{\rho k^2} \kappa(\Omega, k) \frac{J'_m(\kappa(\Omega,k))}{J_m(\kappa(\Omega,k))}
\]

(11)
we get that \( \frac{f_1}{f_2} = \mathcal{O}\left(\frac{1}{k^4}\right) \rightarrow 0 \) as \( k \rightarrow \infty \). Hence, \( J'_m\left(\kappa(\Omega,k)\right) \) must tend to zero for \( k \rightarrow \infty \) and we may conclude that \( \kappa \rightarrow j'_{m,n} \) so that the asymptotic solution for \( k \) becomes;

\[
k_0 \approx \sqrt{\frac{J^2_{m,n} - \Omega^2}{4}}.
\]

For this reason it is indeed favourable to have \( j'_{m,n} \) in the lead, since we, besides increasing the approximation order, capture also another branch of the dispersion equation. This, along with the asymptotic behaviour, is easily seen from Fig. 2 at \( \Omega = 0 \) where new branches depart only from \( j'_{m,n} \). For the waveguide considered, it means that higher order waves converge towards the rigid acoustic duct, as discussed in [3]. Essentially, this means that the higher order waves do not notice the compliance of the shell. For more details on this problem attention should be drawn to e.g. [3; 4].

### 3.1. Grid points

Because the FP-approximation of the dispersion equation is concerned only with approximation of the transcendental terms we indeed have a number of solutions (denoted grid points) for which the solution of the approximate dispersion equation belongs to the solution set of the exact one. The grid points are readily available from Eq.(1) (with the transcendental functions substituted by their FP-approximation) when each term is simultaneously zero. In general, this gives the four cases;

1) \( h(\Omega, k, m) = g(\Omega, k, m) = 0 \) 
2) \( h(\Omega, k, m) = G_m[\kappa(\Omega, k)] = 0 \)
3) \( g(\Omega, k, m) = G'_m[\kappa(\Omega, k)] = 0 \) 
4) \( G_m[\kappa(\Omega, k)] = G'_m[\kappa(\Omega, k)] = 0 \)
in which not all cases necessarily hold in the real/complex \( (\Omega, k) \)-space. In terms of the shell this translates, by way of the definitions in Eq.(2), to nine cases;

(i) \( (\times) \) \( \Omega = \kappa = 0 \Rightarrow (\Omega, k) = (0, 0) \)
(ii) \( (\circ) \) \( \Omega = f_2(\Omega, k, m) = 0 \Rightarrow (\Omega, k) = (0, f_2(m, k, 0) = 0) \) — Structure originated stationary wavenumbers (in vacuo shell)
(iii) \( (\diamond) \) \( \Omega = j'_{m}(\kappa) = 0 \Rightarrow (\Omega, k) = (0, \pm j'_{m,n}) \) — Fluid originated stationary wavenumbers (rigid duct)

\[
\text{where } \ldots < r_{N_1} < q_{N_2} < r_{N_1+1} < \ldots
\]
(iv) $f_2(\Omega, k, m) = \tilde{J}_m(\kappa) = 0 \Rightarrow (\Omega, k) = \left( f_2(\Omega, \sqrt{j_{m,n}^2 - \gamma^2 \Omega^2}, m) = 0, \sqrt{j_{m,n}^2 - \gamma^2 \Omega^2} \right)$

‘Periodic’ grid points governed by structure (interaction between structure and soft baffle)

(v) $f_1(\Omega, k, m) = \tilde{J}'_m(\kappa) = 0 \Rightarrow (\Omega, k) = \left( f_1(\Omega, \sqrt{j_{m,n}^2 - \gamma^2 \Omega^2}, m) = 0, \sqrt{j_{m,n}^2 - \gamma^2 \Omega^2} \right)$

‘Periodic’ grid points governed by rigid baffle (points where structural waves act rigid)

(vi) $f_2(\Omega, k, m) = f_1(\Omega, k, m) = 0$ - Cut-on of structure originated wave

(vii) $\kappa = f_1(\Omega, k, m) = 0 \Rightarrow (\Omega, k) = (f_1(\Omega, i\gamma \Omega, m) = 0, i\gamma \Omega)$ - Solutions belong to the $\Omega \in \mathbb{C}$-space

(viii) $\tilde{J}_m(\kappa) = 0 \Rightarrow \tilde{J}_m(0) = 0 \Rightarrow (\Omega, k) = (\Omega, i\gamma \Omega)$ for $m \neq 0$

(ix) $\tilde{J}_m(\kappa) = \tilde{J}_m(0) = 0 \Rightarrow (\Omega, k) = (\Omega, i\gamma \Omega)$ for $m \neq \{0, 1\}$

From these cases we effortlessly find the grid points shown in Fig. 1 and 2 plotted with symbols corresponding to those above. Here the colours indicate whether the cases are feasible (green) or infeasible (red). In particular, case(viii, ix) both originate from $J'_m(\kappa)$ (Eq.(2)) and constitute in fact spurious roots as a consequence of rearranging the dispersion equation (true both for exact and FP-dispersion equation), while case (vii) belongs to the complex $\Omega$-space and is thus invalid for conservative systems where $(\Omega, k) \in (\mathbb{R}, \mathbb{C})$. From case (ii, iii) we see that only at the stationary frequency $(\Omega = 0)$ is all fluid and structural waves fully uncoupled simultaneously and behave respectively as a rigid duct and an empty shell. Case (vi) gives only the cut-on for the second structural wave.

In addition, note that the grid points of case(iii – v) depends on the approximation order meaning that all fluid governed wave branches attain grid points lying on; $k_{iii} = \pm j'_m$, $k_{iv}(\Omega) = \sqrt{j_{m,n}^2 - \gamma^2 \Omega^2}$ and $k_{v}(\Omega) = \sqrt{j_{m,n}^2 - \gamma^2 \Omega^2}$.

Figure 2. Comparison of dispersion diagrams to locate grid points (case(viii) discarded). Figure shows terms from Eq.(2) corresponding to exact, FP-approximation $(N_1, N_2) = (7, 8)$, in vacuo shell, rigid duct, soft baffle and $f_1$. For higher order waves exact and FPM are inseparable by eye. (a) Imaginary part (propagating waves) (b) real part.

Note from the figure that the ‘periodic’ grid points appear exactly when the soft baffle and
'in vacuo' shell intersect (move in phase) and when \( f_1 \) intersect with the rigid duct, which eventually also becomes in-phase 'movement' of structure and duct. In addition, we note that when a dispersion curve veers away (introduced in [10]) in the presence of veering (or cut-on) of others it 'looses' the ability to attain future grid points, yet it remains surprisingly accurate. For compound waveguides this veering appears in the transition zone for a wave being structure to becoming fluid governed – established already in Eq.(11) by the limit behaviour of the dispersion equation. Effectively, a veering wave branch becomes fluid governed and so the wave of the compound waveguide solution is bounded always by its counter parts, respectively, the rigid and soft baffle acoustic problems as seen from the figure. This ensures on its own a high accuracy range for these wave branches and does in fact improve as \( \Omega \to \infty \) since the asymptotes of these waves converge (compound, soft and rigid baffle).

3.2. Accuracy

Though we have already discussed the choice of \( N_1/N_2 \) as well as a simple accuracy limit, Eq.(9), we shall discuss here rather brief, following [5], other convenient properties of the FPM and FP-fraction, causing the accuracy range to extend beyond the conservative one based on the fraction itself, Eq.(9). First, consider the discarded fraction, Eq.(12).

\[
\frac{\prod_{n=N_1+1}^{\infty} \left[ 1 - \left( \frac{\kappa}{j_{m,n}} \right)^2 \right]}{\prod_{n=N_2+1}^{\infty} \left[ 1 - \left( \frac{\kappa}{j'_{m,n}} \right)^2 \right]} \approx \frac{\prod_{n=N_1+1}^{\infty} \left[ 1 - \left( \frac{\kappa}{\tilde{j}_{m,n}} \right)^2 \right]}{\prod_{n=N_2+1}^{\infty} \left[ 1 - \left( \frac{\kappa}{\tilde{j}_{m,n}-\frac{1}{2}\pi} \right)^2 \right]} \approx 1 \tag{12}
\]

Obviously, the FP-approximations require this fraction to be close to one (\( \approx 1 \)) for the finite product to be a good approximation. From the equation and the chosen approximation order it is obvious that the product tends monotonically to 1 for any fixed \( \kappa \) as \( N_1,N_2 \to \infty \). Similar for \( \kappa \) from \( \max \{|j_{m,N_1}|,|j'_{m,N_2}|\} \) to 0 the fraction is uniform meaning that the product tends monotonically to 1 as \( \kappa \to 0 \) for any approximation order. This is seen from Fig. 3(a) showing Eq.(12) approximated using Gamma functions and asymptotic zeros from Eq.(5) following [1]. However, as discussed in [5] (and supported by the figure) this error measure does not fully comprehend the excellent accuracy obtained by the FP-dispersion equation. For one, it does not capture the grid points which indeed belongs to the exact solution set. On the other hand, it was argued in [5] that a proper error measure is to consider the difference in tangents of the dispersion curves at the grid points. This fact appear fairly obvious from the approximation methodology, find details in [5, Sec. 4]. This is a generic property and does thus immediately apply to cylindrical problems as well. Do, however, note that for compound waveguides this measure does likely not explain properly the accuracy of the wave branches after veering (loss of grid points), which is, nonetheless discussed with Fig. 2.

Now, instead of reproducing these results we consider the FP-fraction plotted in Fig. 2(b) from which we see that Runge’s phenomenon cancels almost exactly in the presence of the Bessel fraction so that the amplitude modulation in the FP-approximations vanish. As a consequence this ensures that the validity range extends beyond the simplified on in Eq.(9) and in effect makes the FPM particularly powerful in the presence of fractions. By this simple consideration we may safely extend the accuracy range from the arc in Eq.(9) to the entire square

\[
|\Omega,k| \leq j_{m,N_1} = \left( N_1 + \frac{1}{2} m - \frac{1}{4} \right) \pi + \mathcal{O}\left( \frac{1}{N} \right) \quad \text{for} \quad N_2 = N_1
\]
\[
|\Omega,k| \leq j'_{m,N_1+1} = \left( N_1 + \frac{1}{2} m + \frac{1}{4} \right) \pi + \mathcal{O}\left( \frac{1}{N} \right) \quad \text{for} \quad N_2 = N_1 + 1 \tag{13}
\]
As seen from Fig. 2 this proves well for propagating and decaying waves (real and imaginary), while the attenuating waves (complex) seem to require higher approximation orders.

Finally, as suggested by the figures, the validity range may in some cases, as for the fluid-filled shell, extend significantly beyond the simplified measures deduced here. Indeed better estimates may be derived for the specific problems following the methodology in [5], however, those derived here are generic and based alone on the zeros included in the finite products; related directly to the approximation order.

4. Conclusion
In this paper we have developed and illustrated the Finite Product Method for problems formulated in cylindrical coordinates and exemplified it using a conservative time-harmonic elastic cylindrical fluid-filled shell. The strength of the FPM is that it introduces no spurious roots but most important; it is extremely simple. Essentially, in the FPM, we simply replace the transcendental terms with their equivalent infinite products (available in literature), truncate the products to finite ones and determine proper approximation orders based on the sorted set of roots of the original transcendental terms. In addition, a region of high accuracy for all wave branches is defined directly from the chosen approximation order.

Given the simplicity and excellent accuracy of the FPM it is a particularly powerful tool in the realm of waveguide theory and by the generality it is easily extend to consider more complicated dispersion equations for multi-layered compound waveguides either in the format of circular plates or layered shells. Also the example presented here is indeed not restricted to integer $m$, hence pipes/profiles of open cross-sections can be studied straightforwardly using the FPM.

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