COMPACT QUANTUM STABILIZER SUBGROUPS

HUICHI HUANG

Abstract. We generalize the concept of stabilizer subgroups to compact quantum groups.

1. Introduction

We introduce the concept of compact quantum stabilizer subgroups, which are generalizations of compact stabilizer subgroups.

This short note was finished during my PhD study. I put it on my personal webpage for several years and finally decide to put it on arxiv with no intent of submission in the near future.

2. Formulation of definition

Consider a compact quantum group $G$ acting on a compact Hausdorff space $X$. Let $H$ be a subgroup of $G$ and $I_H$ be the corresponding Woronowicz $C^*$-ideal of $H$. Use $\pi_H : A \to A/I_H$ to denote morphism from $H$ to $G$, and $\Delta_H$ to denote the coproduct of $H$.

Use $\alpha_H : B \to B \otimes (A/I_H)$ to denote the induced action of $H$ on $X$, i.e.,

$$\alpha_H(f) = (id \otimes \pi_H)\alpha(f)$$

for $f \in B$.

Let $G/H$ be the quotient space, which is the dual space of the $C^*$-algebra

$$C(G/H) = \{ a \in A | (\pi_H \otimes id)\Delta(a) = 1 \otimes a \}.$$

Then $G \curvearrowright G/H$ by $\Delta|_{C(G/H)}$.

Definition 2.1. Let $Y \subseteq X$. We say that $H$ fixes $Y$ (under the action $\alpha$) if

$$(ev_y \otimes id)\alpha_H(f) = f(y)1$$

for all $f \in B$ and $y \in Y$, i.e., $(ev_y \otimes id)\alpha(f) - f(y)1$ is in $\ker \pi_H$. If a subgroup $H_Y$ of $G$ fixes $Y$ and every subgroup $H$ of $G$ fixing $Y$ is a

Date: May 25, 2018.
2010 Mathematics Subject Classification. Primary: 46L65; Secondary: 16W22.
Key words and phrases. Compact quantum group, stabilizer subgroup.
subgroup of $\mathcal{H}_Y$, i.e., there is an epimorphism $\pi_{\mathcal{H}_Y} : A/I_Y \to A/I_\mathcal{H}$ such that

$$(\pi_{\mathcal{H}_Y} \otimes \pi_{\mathcal{H}_Y})\Delta_Y = \Delta_\mathcal{H}\pi_{\mathcal{H}_Y},$$

then $\mathcal{H}_Y$ is called the stabilizer of $Y$.

**Remark 2.2.** If the counit of $\mathcal{G}$ is bounded, then $\mathbb{C}$ is a subgroup fixing any $Y \subseteq X$.

Suppose $Y$ is an $\alpha$-invariant subspace of $X$ containing $x$.

**Proposition 2.3.** $\mathcal{H}$ fixes $x$ under $\alpha$ if and only if $\mathcal{H}$ fixes $x$ under $\alpha_Y$.

**Proof.** Let $\tilde{ev}_x$ to be evaluation functional of $C(Y)$ at $x$. Note that $C(Y) \cong C(X)/J_Y$. Hence $(\tilde{ev}_x \otimes id)\alpha_Y(f + J_Y) = (ev_x \otimes id)\alpha(f)$ for all $f \in B$. It follows that $(\tilde{ev}_x \otimes id)\alpha_Y(f + J_Y) = \tilde{ev}_x(f + J_Y)1$ if and only if $(ev_x \otimes id)\alpha(f) = f(x)1$ for all $f \in B$, which completes the proof. \hfill $\square$

**Proposition 2.4.** If $\mathcal{H}$ fixes $Y$, then $(ev_x \otimes id)\alpha(B) \subseteq C(\mathcal{G}/\mathcal{H})$ for all $x \in Y$.

**Proof.** Take any $f \in B$. We have

$$\Delta((ev_x \otimes id)\alpha(f)) = (ev_x \otimes id \otimes id)(\alpha \otimes id)\alpha(f).$$

Applying $\pi_\mathcal{H} \otimes id$ on both sides of the above identity, we have

$$(\pi_\mathcal{H} \otimes id)\Delta((ev_x \otimes id)\alpha(f)) = (((ev_x \otimes \pi_\mathcal{H})\alpha) \otimes id)\alpha(f).$$

Since $\mathcal{H}$ fixes $x$, we have $(ev_x \otimes \pi_\mathcal{H})\alpha(\cdot) = ev_x(\cdot)1$. This implies

$$(\pi_\mathcal{H} \otimes id)\Delta((ev_x \otimes id)\alpha(f)) = ((ev_x \otimes \pi_\mathcal{H})\alpha \otimes id)\alpha(f) = 1 \otimes ((ev_x \otimes id)\alpha(f)).$$

Therefore $(ev_x \otimes id)\alpha(f) \in C(\mathcal{G}/\mathcal{H})$ for all $f \in B$. \hfill $\square$

**Corollary 2.5.** If $\mathcal{H}$ fixes $x$, then $(ev_x \otimes id)\alpha : B \to C(\mathcal{G}/\mathcal{H})$ is equivariant.

**Proof.** It follows from $(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$ that

$$\Delta|_{C(\mathcal{G}/\mathcal{H})}(ev_x \otimes id)\alpha = [(ev_x \otimes id)\alpha \otimes id]\alpha.$$ 

\hfill $\square$

**Lemma 2.6.** Let $x \in X$. If $\pi$ is a $*$-homomorphism from $A$ to a unital $C^*$-algebra $A'$ such that $\pi((ev_x \otimes id)\alpha(f)) = f(x)1$ for all $f \in B$, then

$$(1) \quad (\pi \otimes \pi)\Delta((ev_x \otimes id)\alpha(f)) = f(x)1 \otimes 1$$

for all $f \in B$. 

Proof. By Proposition 2.4, we have
\[(\pi \otimes id)\Delta((ev_x \otimes id)\alpha(f)) = 1 \otimes (ev_x \otimes id)\alpha(f).\]
Apply \(id \otimes \pi\) on both sides of the above equation, we get
\[(\pi \otimes \pi)\Delta((ev_x \otimes id)\alpha(f)) = 1 \otimes (ev_x \otimes \pi)\alpha(f) = f(x)1 \otimes 1.\]
□

Suppose \(G\) has the bounded counit \(\varepsilon\). Let \(I_Y\) be the ideal of \(A\) generated by elements of the form \((ev_x \otimes id)\alpha(f) - f(x)1\) for all \(x \in Y\) and \(\pi_Y : A \rightarrow A/I_Y\) be the quotient map. Note that \((ev_x \otimes id)\alpha(f) - f(x)1\) is in ker \(\varepsilon\). Hence \(I_Y \subseteq \ker \varepsilon\) is a proper ideal of \(A\).

Under the assumption of boundedness of counit, we show the existence of stabilizer subgroups.

Lemma 2.7. \(I_0\) is a Woronowicz C*-ideal.

Proof. \(I_Y\) is a Woronowicz C*-ideal if and only if \(I_Y \subseteq \ker (\pi_Y \otimes \pi_Y)\Delta\). So it suffices to show
\[(\pi_Y \otimes \pi_Y)\Delta((ev_x \otimes id)\alpha(f) - f(x)1) = 0,
\]
for all \(f \in B\), which follows directly from Lemma 2.6. □

Hence \(A/I_Y\) is a subgroup of \(G\), denote it by \(H_0\). Use \(\Delta_0\) to denote the coproduct of \(H_0\). Then

Lemma 2.8. \(H_0\) fixes \(x\).

Proof. From the definition of \(H_0\), \(\pi_Y((ev_x \otimes id)\alpha(f)) = f(x)1\) for all \(f \in B\). So \(H_0\) fixes \(x\). □

Theorem 2.9. \(H_0\) is the stabilizer of \(x\).

Proof. Suppose a subgroup \(H\) of \(G\) fixes \(x\). Then \((ev_x \otimes id)\alpha(f) - f(x)1\) is in ker \(\pi_H\). Hence \(I_Y \subseteq I_H\), which proves the existence of \(\pi_{H,0} : A/I_Y \rightarrow A/I_H\) and that \(\pi_{H,0}\pi_Y = \pi_H\).

For \(a \in A\), we have
\[
(\pi_{H,0} \otimes \pi_{H,0})\Delta_0(a + I_Y)
= (\pi_{H,0} \otimes \pi_{H,0})(\pi_Y \otimes \pi_Y)\Delta(a)
= (\pi_H \otimes \pi_H)\Delta(a)
= \Delta_H(a + I_H)
= \Delta_H\pi_{H,0}(a + I_Y)
\]
□

Lemma 2.10. Suppose a subgroup \(H\) of \(G\) acts on \(B\) by \(\alpha_H\) and fixes \(x\). Under the action of \(H\), \(\text{Orb}_x = \{x\}\).
Proof. Suppose there exists a $y \in \text{Orb}_x$ other than $x$. Since $\mathcal{B}$ is a dense $*$-subalgebra of $B$, there exists $0 \leq f \in \mathcal{B}$ such that $f(y) = 0$ and $f(x) > 0$. So $(ev_x \otimes h)\alpha_H(f) = h(f(x)1) = f(x) = 0$. Moreover it follows from the faithfulness of $h$ in $B$ and $(ev_y \otimes id)\alpha_H(f) \in \mathcal{B}$ that $(ev_y \otimes h)\alpha_H(f) = h((ev_y \otimes id)\alpha_H(f)) > 0$. This leads to a contradiction to that $(ev_y \otimes h)\alpha_H = (ev_x \otimes h)\alpha_H$. □

3. A CONCRETE EXAMPLE

Next, we give a concrete example of stabilizer subgroups.

Theorem 3.1. Let the quantum permutation group $A_s(n)$ act on $X_n = \{x_1, x_2, ..., x_n\}$ by $\alpha$ such that

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes a_{ji}$$

for $1 \leq i \leq n$. For any $x \in X_n$, the stabilizer of $x$, denoted by $A_x$, is isomorphic to the quantum permutation group $A_s(n-1)$.

Before proceeding to prove the result, we first recall some facts from [4, Proposition 2.5] about morphisms between compact quantum groups.

Proposition 3.2. Let $\Psi : A \rightarrow B$ be a morphism of compact quantum groups. That is, $(\Psi \otimes \Psi)\Delta_A = \Delta_B \Psi$. Then we have that $\Psi$ preserves the Hopf $*$-algebra structures. Namely, $\Psi(\mathcal{A}) \subseteq \mathcal{B}$, $\kappa_B \Psi = \Psi \kappa_A$, and $\varepsilon_B \Psi = \varepsilon_A$, where for instance, $\kappa_A$ is the antipode and $\varepsilon_A$ is the counit on $\mathcal{A}$.

For convenience, let $x = x_n$ and $I_n$ be the ideal of $A_s(n)$ generated by $\{(ev_n \otimes id)\alpha(f) - f(x_n)1|f \in B\}$. If we choose $f$ to be $e_i$ for $1 \leq i \leq n$, then we have $I_n$ is the ideal generated by $a_{ni}$’s and $a_{nn} - 1$ for all $1 \leq i \leq n - 1$. Furthermore, we have

Lemma 3.3. For all $1 \leq i \leq n - 1$, $a_{in}$’s are in $I_n$.

Proof. Let $\pi$ be the quotient map from $A_s(n)$ onto $A_s(n)/I_n$, which is a morphism between compact quantum groups. Let $\kappa$ and $\kappa_n$ be the corresponding antipodes of $A_s(n)$ and $A_s(n)/I_n$. By Proposition 3.2, we have that for any $1 \leq i \leq n - 1$,

$$\kappa_n \pi(a_{in}) = \pi \kappa(a_{in}) = \pi(a_{ni}) = 0.$$ 

Note that $a_{in}$ is in the Hopf $*$-subalgebra of $A_s(n)$. Hence Proposition 3.2 tells us that $\pi(a_{in})$ is in the Hopf $*$-subalgebra of $A_s(n)/I_n$ on which $\kappa_n$ is injective. So $\pi(a_{in}) = 0$, which proves that $a_{in} \in I_n$ for all $1 \leq i \leq n - 1$. □
Now we are ready to prove Theorem 3.1.

**Proof.** Let \( \{b_{ij}\}_{1 \leq i, j \leq n-1} \) be the set of generators of \( A_s(n-1) \). By the universality of \( A_s(n) \), there exists a \(*\)-homomorphism \( S : A_s(n) \to A_s(n-1) \) such that \( S(a_{ij}) = b_{ij} \) for \( 1 \leq i, j \leq n-1 \), otherwise \( S(a_{ij}) = \delta_{ij} \). Since \( a_{ni} \)'s and \( a_{nn-1} \) are in \( \ker S \) for all \( 1 \leq i \leq n-1 \), it follows that \( I_n \subseteq \ker S \). Therefore, naturally \( S \) induces a \(*\)-homomorphism \( \Phi : A_s(n)/I_n \to A_s(n-1) \).

On the other hand, by the universality of \( A_s(n-1) \) and Lemma 3.3, there exists a \(*\)-homomorphism \( \Psi : A_s(n-1) \to A_s(n)/I_n \) such that \( \Psi(b_{ij}) = a_{ij} + I_n \). It is easy to check that \( \Psi = \Phi^{-1} \), which proves that \( \Phi \) is an isomorphism and ends the proof. \( \square \)

**Remark 3.4.** (1) In fact, the quantum subgroup stabilizing a character \( \chi \) is already introduced by C. Pinzari [3, Theorem 7.3] without assuming the boundedness of the counit. However, this assumption could not be omitted since we need this to guarantee that the ideal generated by \( (\chi \otimes id)\alpha(b) - \chi(b)1_A \) is not \( A \).

(2) Theorem 3.1 is first stated (without proof) by S. Wang in [5, Concluding remarks (2) preceding Appendix].

**References**

[1] E. Bédos, G. J. Murphy and L. Tuset. Co-amenability of compact quantum groups. *J. Geom. Phys.* **40** (2001), no. 2, 130–153.

[2] E. C. Lance. *Hilbert C*-modules. A toolkit for operator algebraists.* London Mathematical Society Lecture Note Series, **210**. Cambridge University Press, Cambridge, 1995.

[3] C. Pinzari. Embedding ergodic actions of compact quantum groups on C-algebras into quotient spaces. *Internat. J. Math.* **18** (2007), no. 2, 137-164.

[4] S. Wang. Free products of compact quantum groups. *Comm. Math. Phys.* **167** (1995), no. 3, 671–692.

[5] S. Wang. Quantum symmetry groups of finite spaces. *Comm. Math. Phys.* **915** (1998), no. 1, 195–211.

College of Mathematics and Statistics, Chongqing University, Chongqing, 401332, PR. China

E-mail address: huanghuichi@cqu.edu.cn