Hamilton-Jacobi Equations for Two Classes of State-Constrained Zero-Sum Games

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Abstract—This paper presents Hamilton-Jacobi (HJ) formulations for two classes of two-player zero-sum games: one with a maximum cost value over time, and one with a minimum cost value over time. In the zero-sum game setting, player A minimizes the given cost while satisfying state constraints, and player B wants to prevent player A’s success. For each class of problems, this paper presents two HJ equations: one for time-varying dynamics, cost, and state constraint; the other for time-invariant dynamics, cost, and state constraint. Utilizing the HJ equations, the optimal control for each player is analyzed, and a numerical algorithm is presented to compute the solution to the HJ equations. A two-dimensional water system is introduced as an example to demonstrate the proposed HJ framework.

I. INTRODUCTION

In two-player zero-sum games, one player’s control signal minimizes a cost while satisfying a state constraint, while the second player’s control signal tries either to maximize the cost or to violate the state constraint. An optimal control problem may be considered a special case of the zero-sum game: a control signal that minimizes the given cost while satisfying the constraint is to be determined.

The Hamilton-Jacobi (HJ) partial differential equation (PDE) can be used to represent zero-sum games for dynamical systems. The HJ formulation includes a cost function in the form of the integration of a stage cost and a terminal cost, nonlinear dynamics, control constraints, and state constraints.

The zero-sum game can be classified according to 1) whether the terminal time is a given constant or a variable to be determined, and 2) whether or not state constraints exist. If the problem is state-unconstrained and the terminal time is a given constant, the Hamilton-Jacobi-Isaacs (HJI) PDE [1] applies. For the state-constrained problem where the terminal time is given, [2] presents the corresponding HJ equation. For problems where the terminal time is a variable to be determined, [3] deals with the state-unconstrained problem where the stage cost is zero, and [4], [5] deal with the zero-stage-cost and state-constrained problems. This paper generalizes the previous work to deal with the case of non-zero stage-cost and state constraint.

This paper proposes HJ equations for two classes of state-constrained problems where the terminal time is a variable to be determined and the stage cost is non-zero. In the first class of problems, player A wants to minimize the maximum cost over time while satisfying the state constraint, and player B wants to prevent player A’s success. This class of problems can be interpreted as a robust control problem on the time and disturbances that optimizes the maximum cost over time with respect to the worst disturbances. In the second class of problems, player A wants to minimize the minimum cost over time while satisfying the state constraint, and player B again wants to prevent player A’s success. This class of problems can be interpreted as another robust control problem on the disturbances that optimizes the minimum cost over time with respect to the worst disturbances.

The proposed HJ equations can generally deal with both time-varying and time-invariant dynamics, cost, and state constraint. Furthermore, this paper presents additional HJ equations equivalent to the proposed HJ equations for the time-invariant case.

A. Contribution

This paper presents four HJ equations for zero-sum games: two classes, and time-varying and time-invariant. Among the four HJ equations, three equations are proposed by this paper, and the other one is presented in [6] and reviewed here for completeness. Also, this paper provides and presents analysis for the optimal control signal for each player, numerical algorithm to compute the proposed HJ equation, as well as a practical example.

B. Organization

The organization of this paper is as follows. Section II presents a mathematical formulation for two classes of state-constrained zero-sum games. Section III presents the HJ equations for the first class of problems both time-varying and time-invariant. Section IV presents the HJ equations for the second class of problems for both the time-varying and time-invariant cases. Section V presents analysis for an optimal control signal based on the solution to the HJ equations. Section VI presents a numerical algorithm to compute the solution to the HJ equations for each class of problem. Section VII provides a practical example where our HJ formulation can be utilized, and Section VIII concludes this paper. Proofs are detailed in the Appendices.
II. Two Classes of State-Constrained Zero-Sum Games

We first present the two classes of two-player zero-sum games, called Problems 1 and 2. Consider a dynamical system:
\[\dot{x}(s) = f(s, x(s), \alpha(s), \beta(s)), s \in [t, T], \text{ and } x(t) = x, \]
where \((t, x)\) are the initial time and state, \(x : [t, T] \rightarrow \mathbb{R}^n\) is the state trajectory, \(f : [0, T] \times \mathbb{R}^n \times A \times B \rightarrow \mathbb{R}^n\) is the dynamics, \(A \subseteq \mathbb{R}^m_a\), \(B \subseteq \mathbb{R}^m_b\) are the control constraints, \(\alpha \in \mathcal{A}(t)\), \(\beta \in \mathcal{B}(t)\) are the control signals, in each, player A controls \(\alpha\) and player B controls \(\beta\), and the sets of measurable control signals are
\[
\mathcal{A}(t) := \{\alpha : [t, T] \rightarrow A | \|\alpha\|_{L^\infty(t,T)} < \infty\},
\]
\[
\mathcal{B}(t) := \{\beta : [t, T] \rightarrow B | \|\beta\|_{L^\infty(t,T)} < \infty\}.
\]

In each zero-sum game, we specify each player’s control signal or strategy: player A wants to minimize the cost under the state constraint, and player B wants to prevent player A’s success, although the cost is defined in different ways for Problems 1 and 2. In each problem, we introduce two value functions depending on which players play first or second.

**Problem 1** For given initial time and state \((t, x)\), solve
\[
\partial^+_1(t, x) := \sup_{\delta \in \Delta(t)} \inf_{\alpha \in \mathcal{A}(t)} \max_{\tau \in [t,T]} \int_t^\tau L(s, x(s), \alpha(s), \delta(s))ds + g(\tau, x(\tau)),
\]
subject to \(c(s, x(s)) \leq 0, \ s \in [t, \tau]\),
where \(x\) solves (1) for \((\alpha, \delta)\); and solve
\[
\partial^-_1(t, x) := \inf_{\gamma \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \max_{\tau \in [t,T]} \int_t^\tau L(s, x(s), \gamma(s), \beta(s))ds + g(\tau, x(\tau)),
\]
subject to \(c(s, x(s)) \leq 0, \ s \in [t, \tau]\),
where \(x\) solves (1) for \((\gamma, \beta)\).

\(\Delta(t)\) is a set of non-anticipative strategies for player B, and \(\Gamma(t)\) is a set of non-anticipative strategies for player A. The non-anticipative strategy outputs a control signal for the second player as a reaction to the first player’s control signal without using the future information. The non-anticipative strategy has been introduced by Elliott and Kalton [7]:
\[
\Delta(t) := \{\delta : \mathcal{A}(t) \rightarrow \mathcal{B}(t) | \forall s \in [t, \tau] \text{ and } \alpha, \alpha \in \mathcal{A}(t), \text{ if } \alpha(\tau) = \alpha(\tau) \text{ a.e. } \tau \in [t, s], \text{ then } \delta(\alpha)(\tau) = \delta(\alpha)(\tau) \text{ a.e. } \tau \in [t, s]\},
\]
\[
\Gamma(t) := \{\gamma : \mathcal{B}(t) \rightarrow \mathcal{A}(t) | \forall s \in [t, \tau], \beta, \beta \in \mathcal{B}(t), \text{ if } \beta(\tau) = \beta(\tau) \text{ a.e. } \tau \in [t, s], \text{ then } \gamma(\beta)(\tau) = \gamma(\beta)(\tau) \text{ a.e. } \tau \in [t, s]\}.
\]

**Problem 2** For given initial time and state \((t, x)\), solve
\[
\partial^+_2(t, x) := \sup_{\delta \in \Delta(t)} \inf_{\alpha \in \mathcal{A}(t)} \min_{\tau \in [t,T]} \int_t^\tau L(s, x(s), \alpha(s), \delta(s))ds + g(\tau, x(\tau)),
\]
subject to \(c(s, x(s)) \leq 0, \ s \in [t, \tau]\),
where \(x\) solves (1) for \((\alpha, \delta)\); and solve
\[
\partial^-_2(t, x) := \inf_{\gamma \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \min_{\tau \in [t,T]} \int_t^\tau L(s, x(s), \gamma(s), \beta(s))ds + g(\tau, x(\tau)),
\]
subject to \(c(s, x(s)) \leq 0, \ s \in [t, \tau]\),
where \(x\) solves (1) for \((\gamma, \beta)\).

For both problems, \(L : [t, T] \times \mathbb{R}^n \times A \times B \rightarrow \mathbb{R}\) is the stage cost, \(g : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}\) is the terminal cost, \(f : [t, T] \times \mathbb{R}^n \times A \times B \rightarrow \mathbb{R}^n\) is the system dynamics, and \(c : [t, T] \times \mathbb{R}^n \rightarrow \mathbb{R}\) is the state constraint function.

The difference between \(\partial^+_i\) and \(\partial^-_i\) \((i = 1, 2)\) is play order. In \(\partial^+_1(t, x)\), at each time \(s \in [t, \tau]\), player A plays first with its strategy \(\alpha(s)\), and then player B reacts by following its own strategy \(\delta(\alpha)\). Despite this play order at each time, the choice of player B’s strategy comes first since it should be chosen without information about player A’s control signal. In other words, player B first chooses its strategy, and then player A chooses its control signal. In \(\partial^-_1(t, x)\), at each time \(s\), player B first plays \(\beta(s)\), and then player A reacts with its strategy \(\gamma(\beta)(s)\). Similarly to \(\partial^+_1(t, x)\), in \(\partial^-_1(t, x)\), player A first chooses its strategy, and then player B chooses its control signal.

Problems 1 and 2 are representative of many practical problems. For Problem 1, consider two water systems where player A controls the water level of pond 1 that is connected to pond 2. Suppose player B is precipitation. Player A needs to minimize the highest water level of pond 1 over time while satisfying constraints for water level of pond 1 and 2 under the worst precipitation assumption. For Problem 2, consider a car that tries to change its lane while avoiding collision with other cars. Here, the cost is the distance to the goal lane, and the car wants to successfully change lanes at some time in the given time interval, while other cars might bother to lane change.

This paper assumes the following.

**Assumption 1** (Lipschitz continuity and compactness)

1) \(A \text{ and } B \text{ are compact;}
2) \(f : [0, T] \times \mathbb{R}^n \times A \times B \rightarrow \mathbb{R}^n, f = f(t, x, a, b) \text{ is Lipschitz continuous in } (t, x) \text{ for each } (a, b) \in A \times B; \)
3) the stage cost \(L : [0, T] \times \mathbb{R}^n \times A \times B \rightarrow \mathbb{R}, L = L(t, x, a, b) \text{ is Lipschitz continuous in } (t, x) \text{ for each } (a, b) \in A \times B; \)
4) for all \((t, x) \in [0, T] \times \mathbb{R}^n, \{f(t, x, a, b) | a \in A, b \in B\} \text{ and } \{L(t, x, a, b) | a \in A, b \in B\} \text{ are compact and convex;}
5) the terminal cost \(g : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}, g = g(t, x) \text{ is Lipschitz continuous in } (t, x); \)
6) the state constraint $c : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $c = c(t, x)$ is Lipschitz continuous in $(t, x)$;  
7) the stage cost $(L)$ and the terminal cost $(g)$ are bounded below.

### III. HAMILTON-JACOBI EQUATIONS FOR PROBLEM 1

#### A. HJ equation for Problem 1 (time-varying case)

In this subsection, we derive an HJ equation for Problem 1 $(\vartheta^\pm_1)$. Unfortunately, for some initial time and state $(t, x)$, there is no control $\alpha$ (or strategy $\gamma$) of player A that satisfies the state constraint for all strategies $\delta$ of player B (or control signal $\beta$). In this case, $\vartheta^+_1(t, x)$ is infinity. Thus, $\vartheta^+_1$ is neither continuous nor differentiable in $(0, T) \times \mathbb{R}^n$. To overcome this issue, we utilize an additional variable $z \in \mathbb{R}$ to define continuous value functions $V^\pm_1$ in (13) and (14) that combine the cost $\vartheta$ in (3) or (5), and the constraint in (4) or (6). We call this method the augmented-$z$ method. This method has been utilized to handle state constraints to solve other HJ problems [2], [6]. $V^\pm_1$ is well-defined in $[0, T] \times \mathbb{R}^n \times \mathbb{R}$.

$$V^+_1(t, x, z) := \sup_{\vartheta \in \Delta(t) \alpha \in \Lambda(t)} \inf_{\gamma \in \Gamma(t)} J^1(t, x, z, \alpha, \delta[\alpha]), \quad (13)$$

$$V^-_1(t, x, z) := \inf_{\gamma \in \Gamma(t)} \sup_{\vartheta \in \Delta(t) \alpha \in \Lambda(t)} J^1(t, x, z, \alpha, \delta[\alpha]), \quad (14)$$

where cost $J^1(t, x, z, \alpha, \delta[\alpha]) \to \mathbb{R}$ is defined as follows:

$$J^1(t, x, z, \alpha, \delta[\alpha]) := \max_{\tau \in [t, T]} \max_{s \in [t, \tau]} c(s, x(s)) - L(s, x(s), \alpha(s), \delta[\alpha]), \quad (15)$$

where $x$ solves (1). Define the auxiliary state trajectory $z$

$$\dot{z}(t) = -L(s, x(s), \alpha(s), \delta[\alpha]), \quad s \in [t, T], \quad z(t) = z. \quad (16)$$

Then, (1) and (16) are the joint ODEs whose solution is the augmented state trajectories: $(x, z) : [t, T] \to \mathbb{R}^{n+1}$

$$\begin{bmatrix} \dot{x}(s) \\ \dot{z}(s) \end{bmatrix} = \begin{bmatrix} f(s, x(s), \alpha(s), \delta[\alpha]) \\ -L(s, x(s), \alpha(s), \delta[\alpha]) \end{bmatrix}, \quad s \in [t, T], \quad \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix}. \quad (17)$$

Then, $J^1(t)$ in (15) becomes

$$J^1(t) = \max_{\tau \in [t, T]} \max_{s \in [t, \tau]} c(s, x(s)) - L(s, x(s), \alpha(s), \delta[\alpha]), \quad (18)$$

The last equality is derived by the distributivity property of the maximum operations.

Lemma 1 (Equivalence of two value functions) Suppose Assumption 1 holds. For all $(t, x) \in [0, T] \times \mathbb{R}^n$, $V^+_1$ subject to (3), $V^-_1$ subject to (4), $V^+_1$ in (13), and $V^-_1$ in (14) have the following relationship.

$$\vartheta^+_1(t, x) = \min z \text{ subject to } V^+_1(t, x, z) \leq 0. \quad (19)$$

This implies that

$$\vartheta^+_1(t, x) = \sup_{\vartheta \in \Delta(t) \alpha \in \Lambda(t)} \inf_{\gamma \in \Gamma(t)} \max_{s \in [t, T]} c(s, x(s)) - L(s, x(s), \alpha(s), \delta[\alpha]), \quad (20)$$

$$\vartheta^-_1(t, x) = \inf_{\gamma \in \Gamma(t)} \sup_{\vartheta \in \Delta(t) \alpha \in \Lambda(t)} \max_{s \in [t, T]} c(s, x(s)) - L(s, x(s), \alpha(s), \delta[\alpha]), \quad (21)$$

where $x$ solves (1) for $(\alpha, \delta[\alpha])$, and

$$\vartheta^-_1(t, x) = \inf_{\gamma \in \Gamma(t)} \sup_{\vartheta \in \Delta(t) \alpha \in \Lambda(t)} \max_{s \in [t, T]} c(s, x(s)) - L(s, x(s), \alpha(s), \delta[\alpha]), \quad (22)$$

$$\vartheta^+_1(t, x) = \sup_{\vartheta \in \Delta(t) \alpha \in \Lambda(t)} \inf_{\gamma \in \Gamma(t)} \max_{s \in [t, T]} c(s, x(s)) - L(s, x(s), \alpha(s), \delta[\alpha]), \quad (23)$$

where $x$ solves (1) for $(\gamma[\beta], \beta)$.

**Proof.** See Appendix A

The rest of this subsection focuses on the derivation of the corresponding HJ equation for $V^\pm_1$. The HJ equation is based on the principle of dynamic programming in Lemma 2.

**Lemma 2 (Optimality condition)** Fix $(t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}$. Consider a small step $h > 0$ such that $t + h \leq T$, $V^+_1$ in (13) has the following property:

$$V^+_1(t, x, z) = \sup_{\vartheta \in \Delta(t) \alpha \in \Lambda(t)} \inf_{\gamma \in \Gamma(t)} \max_{s \in [t, t + h]} c(s, x(s)) - L(s, x(s), \alpha(s), \delta[\alpha]), \quad (24)$$

$$\text{where } (x, z) \text{ solves (17) for } (\alpha, \delta[\alpha]). \text{ Similarly, for } V^-_1 \text{ (14).} \quad (25)$$

**Proof.** See Appendix B

Theorem 1 presents the corresponding HJ equations for $V^\pm_1$ in (13) and (14) using viscosity theory. Intuitively, the HJ equation in Theorem 1 is derived as $h$ in Lemma 2 converges to zero.

**Theorem 1 (HJ equation for Problem 1)** For all $(t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}$, $V^\pm_1$ in (13) and (14) is the unique viscosity solution to the HJ equation:

$$\max \left\{ c(t, x) - V^\pm_1(t, x, z), g(t, x) - z - V^\pm_1(t, x, z), \right\} = 0 \quad (26)$$
in $(0, T) \times \mathbb{R}^n \times \mathbb{R}$, where $\tilde{H}^\pm: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$

$$
\tilde{H}^+(t, x, z, p, q) := \max_{a \in A} \min_{b \in B} -p \cdot f(t, x, a, b) + qL(t, x, a, b),
$$
(27)

$$
\tilde{H}^-(t, x, z, p, q) := \min_{b \in B} \max_{a \in A} -p \cdot f(t, x, a, b) + qL(t, x, a, b),
$$
(28)

and

$$
V_1^\pm(T, x, z) = \max\{c(T, x), g(T, x) - z\}
$$
(29)
on $\{t = T\} \times \mathbb{R}^n \times \mathbb{R}$. Denote $V_1^\pm = \frac{\partial V_1^\pm}{\partial t}$, $D_x V_1^\pm = \frac{\partial V_1^\pm}{\partial x}$, and $D_z V_1^\pm = \frac{\partial V_1^\pm}{\partial z}$.

**Proof.** See Appendix [C].

### B. HJ equation for Problem 1 (time-invariant case)

We define the problem as time-invariant if the stage cost, terminal cost, dynamics, and state constraints are all independent of time.

In this section, we convert $\tilde{\vartheta}_1^\pm$ (4) subject to (4) and (5) subject to (6) to a fixed-terminal-time problem for the time-invariant case of Problem 1, which allows us to utilize methods for the fixed-terminal-time problems [2]. In the fixed-terminal-time problem, optimal control signals of players have to be determined, but the terminal time does not need to be specified but is given.

The conversion of Problem 1 to a fixed-terminal-time problem by introducing a freezing control signal $\mu: [t, T] \to [0, 1]$ to the dynamics and a set of control signals controls signals:

$$
\dot{x}(s) = f(x(s), \alpha(s), \beta(s))\mu(s), s \in [t, T], (30)
\mathcal{M}(t) = \{\mu: [t, T] \to [0, 1] \mid \|\mu\|_{L^\infty([t, T])} < \infty\}. \quad (31)
$$

This freezing control signal controls the contribution of the two players to the system. For example, $\mu(s) = 0$ implies that the state stops at $s$, and the two players do not contribute to the system. On the other hand, $\mu(s) = 1$ allows the state evoludes by the control signals of the players. The maximum over $\tau$ operation in Problem 1 can be replaced by the maximum over the freezing control signal if it eliminates contribution of the two players after the maximal terminal time.

We present fixed-terminal-time problems as below:

$$
\tilde{\vartheta}_1^\pm(t, x) := \sup_{\delta \in \Delta(t)} \inf_{\nu, \nu_A \in \mathcal{N}_A(t)} \int_t^T L(x(s), \alpha(s), \delta(s))\nu_A[\alpha](s)ds + g(x(T)), \quad (32)
$$
subject to $c(x(s)) \leq 0, s \in [t, T], (33)$

where $x$ solves (30) for $(\alpha, \delta[\alpha], \nu_A[\alpha]);$

$$
\tilde{\vartheta}_1^+(t, x) := \inf_{\tilde{\gamma} \in \mathcal{F}(t)} \sup_{\tilde{\beta} \in \mathcal{B}(t)} \int_t^T L(x(s), \tilde{\gamma}[\beta, \tilde{\beta}], \beta(s))\mu(s)ds + g(x(T)), \quad (34)
$$
subject to $c(x(s)) \leq 0, s \in [t, T], (35)$

where $x$ solves (30) for $(\tilde{\gamma}[\beta, \tilde{\beta}], \mu, \beta).$, $N_A$ is a set of non-anticipative strategies for the freezing control to player A, and

$\tilde{\vartheta}_1^+$ is a set of non-anticipative strategies for player A to player B and the freezing control:

$$
\nu_A(t) := \nu_A : (\mathcal{A}(t) \to \mathcal{M}(t) \mid \forall s \in [t, \tau], \alpha, \tilde{\alpha} \in \mathcal{A}(t), (36)
$$

if $\alpha(t) = \tilde{\alpha}(t)$ a.e. $\tau \in [t, s], (37)$

then $\nu_A[\alpha](\tau) = \nu_A[\tilde{\alpha}](\tau)$ a.e. $\tau \in [t, s], (38)$

This freezing control signal controls the contribution of the two players after the maximal terminal time.

We present fixed-terminal-time problems as below:

$$
\tilde{\vartheta}_1^+(t, x) := \inf_{\nu[\alpha]} \sup_{\delta \in \Delta(t)} \int_t^T L(x(s), \alpha(s), \delta[\alpha](s))\nu_A[\alpha](s)ds + g(x(T)), \quad (39)
$$
subject to $c(x(s)) \leq 0, s \in [t, T], (40)$

where $x$ solves (30) for $\tilde{\gamma}(s)$ solves, for $s \in [t, T], (41)

$$
\tilde{\vartheta}_1^+(t, x) := \inf_{\tilde{\gamma} \in \mathcal{F}(t)} \sup_{\tilde{\beta} \in \mathcal{B}(t), \mu \in \mathcal{M}(t)} \int_t^T L(x(s), \tilde{\gamma}[\beta, \tilde{\beta}], \beta(s))\mu(s)ds + g(x(T)), \quad (42)
$$
subject to $c(x(s)) \leq 0, s \in [t, T], (43)$

where $\tilde{\vartheta}_1^+$ is (3) subject to (4), $\tilde{\vartheta}_1^+$ is (5) subject to (6), $\tilde{\vartheta}_1^+$ is (5) subject to (5), and $\tilde{\vartheta}_1^+$ is (5) subject to (5).

**Proof.** See Appendix [D].

### Corollary 1 (Equivalent fixed-terminal-time game to the time-invariant Problem 1)

$$
\tilde{\vartheta}_1^+ = \tilde{\vartheta}_1^+, \quad (42)
$$

where $\tilde{\vartheta}_1^+$ is (3) subject to (4), $\tilde{\vartheta}_1^+$ is (5) subject to (6), $\tilde{\vartheta}_1^+$ is (5) subject to (5), and $\tilde{\vartheta}_1^+$ is (5) subject to (5).

**Proof.** Let the right hand terms in (40) and (41) be denoted as $W_1^\pm$ By Corollary 5.3 in [2], $\tilde{\vartheta}_1^+(t, x, z) = \min_{\tilde{\vartheta}_1^+} \leq 0$. This fact and Lemma 1 allow us to conclude (42).

This corollary remarks that the free-terminal-time games $(\tilde{\vartheta}_1^+)$ can be converted to fixed-terminal-time games $(\tilde{\vartheta}_1^+)$, in which only control signals and strategies have to be specified, since the terminal time is fixed.

In Lemma 3 $V_1^\pm$ is converted to a fixed-terminal-time game, whose corresponding HJ equation has been investigated in [2]. This allows us to derive an HJ equation for the time-invariant Problem 1 in Theorem 2.

**Theorem 2 (HJ equation for Problem 1 (time-invariant version))** Consider Problem 1 for the time-invariant case. For
all \((t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}\), \(V_1^\pm\) in (13) and (14) is the unique viscosity solution to the HJ equation:
\[
\max \left\{ c(x) - V_1^\pm(t, x, z),
V_1^\pm - \min \left\{ 0, \bar{H}^\pm(x, z, D_x V_1^\pm, D_z V_1^\pm) \right\} \right\} = 0
\]
(43)
in \((0, T) \times \mathbb{R}^n \times \mathbb{R}\), where \(\bar{H}^+\) and \(\bar{H}^-\) are defined in (27) and (28), respectively, without the time dependency, and
\[
V_1^\pm(T, x, z) = \max \{ c(x), g(x) - z \}
\]
(44)
on \(\{ t = T \} \times \mathbb{R}^n \times \mathbb{R}\).

**Proof.** See Appendix \[2\]

Note that the Hamiltonian \(\bar{H}^\pm\) in (43) is time-invariant.

We observe that the right two terms in the HJ equation (43) become \(V_1^\pm - \min \{ 0, \bar{H}^\pm \}\) in (43). Note that these two terms are not algebraically equal.

### C. HJ equation for Problem 1 (optimal control setting)

In this subsection, we solve Problem 1 in the optimal control problem setting: for given initial time and state \((t, x)\),
\[
\vartheta_1(t, x) := \inf_{\alpha \in \mathcal{A}(t)} \max_{\tau \in [t, T]} \int_t^\tau L(s, x(s), \alpha(s))ds + g(\tau, x(\tau)),
\]
subject to \(c(s, x(s)) \leq 0, \quad s \in [t, \tau]\),
(45)
where \(x\) solves
\[
\dot{x}(s) = f(s, x(s), \alpha(s)), \quad s \in [t, T], \quad \text{and} \quad x(t) = x.
\]
(47)
Section III-A and III-B present the HJ equations for Problem 1 in the zero-sum game setting. By removing player B in the zero-sum game, we can get HJ equations for Problem 1 in the optimal control setting. Thus, Theorem 1 and 2 imply the following remark.

**Remark 1 (HJ equation for Problem 1 (optimal control setting))** Let \(V_1\) be the unique viscosity solution to the HJ equation:
\[
\max \left\{ c(x) - V_1(t, x, z),
V_1 - H(t, x, z, D_x V_1, D_z V_1) \right\} = 0
\]
(48)
in \((0, T) \times \mathbb{R}^n \times \mathbb{R}\), where \(H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}\)
\[
\bar{H}(t, x, z, p, q) := \max_{a \in \mathcal{A}} -p \cdot f(t, x, a) + q L(t, x, a),
\]
(49)
and
\[
V_1(T, x, z) = \max \{ c(T, x), g(T, x) - z \}
\]
(50)
on \(\{ t = T \} \times \mathbb{R}^n \times \mathbb{R}\). Then,
\[
\vartheta_1(t, x) = \min z \text{ subject to } V_1(t, x, z) \leq 0,
\]
(51)
where \(\vartheta_1\) is defined in (45) subject to (40).

If Problem 1 is time-invariant, \(V_1\) is the unique viscosity solution to the HJ equation:
\[
\max \left\{ c(x) - V_1(t, x, z),
V_1 - \min \left\{ 0, \bar{H}^\pm(x, z, D_x V_1, D_z V_1) \right\} \right\} = 0
\]
(52)
in \((0, T) \times \mathbb{R}^n \times \mathbb{R}\), where \(\bar{H}^\pm\) is defined in (49) with ignoring the time dependency, and
\[
V_1(T, x, z) = \max \{ c(x), g(x) - z \}
\]
(53)
on \(\{ t = T \} \times \mathbb{R}^n \times \mathbb{R}\).

### IV. HAMILTON-JACOBI EQUATIONS FOR PROBLEM 2

Problem 2 in the zero-sum game setting is defined in Section II and Section IV-A, and IV-B present HJ equations for the time-varying and time-invariant Problem 2, respectively. Section IV-C presents HJ equations for Problem 2 and the time-invariant Problem 2 in the optimal control setting.

For Problem 2 in the zero-sum game and optimal control settings, the corresponding HJ equations have been presented in the authors’ previous work [6]. This section first presents this previous work and then proposes HJ equations for the time-invariant version.

### A. HJ equation for the time-varying Problem 2

This subsection provides an HJ formulation for Problem 2: solve \(\vartheta_2^+\) in (9) subject to (10) and \(\vartheta_2^-\) in (11) subject to (12). For \((t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}\), define the augmented value functions corresponding to the upper and lower value functions (\(\vartheta_2^\pm\)):
\[
V_2^+(t, x, z) := \sup_{\delta \in \Delta(t)} \inf_{\alpha \in \mathcal{A}(t)} J_2(t, x, a, \alpha, \delta|\alpha),
\]
(54)
\[
V_2^-(t, x, z) := \inf_{\gamma \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} J_2(t, x, \alpha, \gamma|\beta),
\]
(55)
where cost \(J_2 : (t, x, z, \alpha, \beta) \rightarrow \mathbb{R}\) is defined as follows:
\[
J_2(t, x, z, \alpha, \beta) := \min_{\tau \in [t, T]} \max_{\beta \in \mathcal{B}(t)} \left\{ \min_{\alpha \in \mathcal{A}(t)} \int_t^\tau L(s, x(s), \alpha(s), \beta(s))ds + g(\tau, x(\tau)) - z \right\},
\]
(56)
where \(x\) solves (1) for \((\alpha, \beta)\). [6] proved that, for all \((t, x) \in [t, T] \times \mathbb{R}^n\),
\[
\vartheta_2^\pm(t, x) = \min z \text{ subject to } V_2^\pm(t, x, z) \leq 0,
\]
(57)
and \(V_2^\pm\) are the unique viscosity solutions to the HJ equations in Theorem 3.

**Theorem 3 (HJ equation for Problem 2)** [6] For all \((t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}\), \(V_2^\pm\) in (54) and (55) are the unique viscosity solutions to the HJ equations:
\[
\max \left\{ c(t, x) - V_2^\pm(t, x, z), \min \left\{ g(t, x) - z - V_2^\pm(t, x, z),
\bar{H}^\pm(t, x, z, D_x V_2^\pm, D_z V_2^\pm) \right\} \right\} = 0
\]
(58)
in \((0, T) \times \mathbb{R}^n \times \mathbb{R}\), where \(\bar{H}^\pm\) are defined in (27) and (28), and
\[
V_2^\pm(T, x, z) = \max \{ c(T, x), g(T, x) - z \}
\]
(59)
We observe that the difference between the two types of HJ equations for $V_1^+$ and $V_2^+$ is that the minimum operation in (58) for $V_1^+$ is replaced by the maximum operation in (26). This is from the difference between $\sigma_1^+$ and $\sigma_2^+$; $\sigma_1^+$ in (3) and (5) have max$_\tau$ operation, and $\sigma_2^+$ in (9) and (11) have min$_\tau$ operation.

B. HJ equation for Problem 2 (time-invariant case)

Through similar analysis to that in Section III-B, this subsection derives the HJ equations for the time-invariant Problem 2. For Problem 1, the freezing control signal $\mu : [t, T] \to [0, 1]$ allows to convert to the fixed-terminal-time problems by replacing the maximum over $\tau$ operation in Problem 1 to the supremum over the freezing control signal or strategy. Instead, Problem 2 is specified in terms of the minimum over $\tau$ operation, which will be replaced by the infimum over the freezing control signal or strategy.

Consider two fixed-terminal-time problems:

$$\hat{\sigma}_2^+(t, x) := \sup_{\delta \in \Delta(t)} \inf_{\alpha \in A(t), \mu \in M(t)} \int_t^T L(x(s), \alpha(s), \delta(\alpha, \mu)(s)) \mu(s) ds + g(x(T)),$$

subject to $c(x(s)) \leq 0, s \in [t, T]$,

Consider two fixed-terminal-time problems:

$$\hat{\sigma}_2^-(t, x) := \inf_{\gamma \in \Gamma(t), \nu_B \in N_B(t), B \in B(t)} \sup_{\beta \in B(t)} \int_t^T L(x(s), \gamma(\beta)(s), \beta(s)) \nu_B(\beta)(s) ds + g(x(T)),$$

subject to $c(x(s)) \leq 0, s \in [t, T]$,

where $x$ solves (30) for $(\alpha, \delta(\alpha, \mu), \mu, \beta)$, $M$ is defined in (31), and $\delta$ is the non-anticipative strategy for player B to both player A and the freezing control:

$$\hat{\Delta}(t) := \{ \hat{\delta} : A(t) \times M(t) \to B(t) \mid \forall s \in [t, T], \alpha, \hat{\alpha} \in A(t), \mu, \mu \in M(t) \text{ if } \alpha(\tau) = \hat{\alpha}(\tau), \mu(\tau) = \hat{\mu}(\tau) \text{ a.e. } \tau \in [t, s],$$

then $\delta(\alpha, \mu)(\tau) = \delta(\hat{\alpha}, \hat{\mu})(\tau) \text{ a.e. } \tau \in [t, s]$;

By combining the HJ formulation for the fixed-terminal-time problems [2] and Lemma 4, the HJ equation for the time-invariant Problem 2 is derived in Theorem 4. The proof for Theorem 4 is analogous to the proof for Theorem 2.

Theorem 4 (HJ equation for Problem 2 (time-invariant version)) Consider Problem 2 in the time-invariant case. For all \((t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}, V_2^+ in (54) and (55) are the unique viscosity solutions to the HJ equations:

$$\max \{ c(x) - V_2^+(t, x, z), V_2^+ - \max \{ 0, H^+(x, z, D_x V_2^+, D_z V_2^+) \} \} = 0$$

in \((0, T) \times \mathbb{R}^n \times \mathbb{R}, \text{ where } H^+ \text{ and } H^- \text{ are defined in (27) and (28), respectively, without the time dependency, and}$$

$$V_2^+(T, x, z) = \max \{ c(x), g(x) - z \}$$

on \( \{ t = T \} \times \mathbb{R}^n \times \mathbb{R} \).

In comparison between the HJ equations for Problem 2 and its time-invariant version, $\min \{ g - z - V_2^+, V_2^- - H^+ \}$ in (58) becomes $V_2^+ - \max \{ 0, H^+ \}$ in (58). Note that the difference between Problem 1 and 2 leads to the difference in HJ equations for the time-invariant problems: (57) has the term $V_1^+ - \min \{ 0, H^+ \}$, but (56) has the term $V_2^+ - \max \{ 0, H^+ \}$.

C. HJ equation for Problem 2 (optimal control setting)

In this subsection, we solve Problem 2 in the optimal control setting: for given initial time and state \((t, x)\),

$$\hat{\sigma}_2(t, x) := \inf_{\alpha \in A(t)} \sup_{\nu_B \in N_B(t)} \min_{B \in B(t)} \int_t^T L(s, x(s), \alpha(s)) ds + g(\tau, x(\tau)),$$

subject to $c(x(s)) \leq 0, s \in [t, T]$,

where $x$ solves (30). By removing the contribution of player B in Theorem 3 and 4, we solve $\hat{\sigma}_2$ using the HJ equations in the following remark.

Remark 2 (HJ equation for Problem 2 (optimal control setting)) Let $V_2$ be the unique viscosity solution to the HJ equation (6):

$$\max \{ c(T, x) - V_2(T, x, z), \min \{ g(T, x) - z - V_2(T, x, z) \} \} = 0$$

in \((0, T) \times \mathbb{R}^n \times \mathbb{R}, \text{ where } H \text{ is defined in (49), and}$$

$$V_2(T, x, z) = \max \{ c(T, x), g(T, x) - z \}$$

on \( \{ t = T \} \times \mathbb{R}^n \times \mathbb{R} \). Then,

$$\hat{\sigma}_2(t, x) = \min \{ z \} \text{ subject to } V_2(t, x, z) \leq 0,$$

where $\hat{\sigma}_2$ is (70) subject to (71).
If Problem 2 is time-invariant, $V_2$ is the unique viscosity solution to the HJ equation:

$$\max\left\{c(x) - V_2(t,x,z), V_{2,i} - \max\left\{0, \mathcal{H}(x,z), D_x V_2, D_z V_2\right\}\right\} = 0$$

in $(0,T) \times \mathbb{R}^n \times \mathbb{R}$, where $\mathcal{H}$ is defined in (49) without the time dependency, and

$$V_2(T, x, z) = \max\{c(x), g(x) - z\}$$

on $\{t = T\} \times \mathbb{R}^n \times \mathbb{R}$.

V. OPTIMAL CONTROL SIGNAL AND STRATEGY

The optimal control signal or strategy for Problems 1 and 2 are specified by the HJ equations in Sections III and IV. This section utilizes the HJ equations in Theorems 3 and 5 and the method in this section can be simply extended for the other HJ equations in Theorems 2 and 3 and Remarks 1 and 2.

Recall $V^\pm_i$ $(i = 1, 2)$ defined in (15), (14), (54), (55), and suppose $V^\pm_i$ is computed from the HJ equations in Theorems 1 and 3.

Lemmas 3 and 4 imply the following remark.

Remark 3 (Find $\tilde{\vartheta}^\pm_i$ from $V^\pm_i$) For initial time $t = 0$ and state $x \in \mathbb{R}^n$,

$$(x_*, z_*) = (x, \tilde{\vartheta}^\pm_i(0, x)),$$

where $(x_*, z_*)$ is an optimal trajectory for $V^\pm_i$.

With the initial augmented state $(x_*(0), z_*(0))$, the optimal control and strategy can be found at $(t, x_*(t), z_*(t))$, and the optimal state trajectory is also updated by solving the ODE (17).

Define $\mathcal{H}^\pm_i : A \times B \to \mathbb{R}$ for a fixed $(t, x, z) \in (0, T) \times \mathbb{R}^n \times \mathbb{R}$

$$\mathcal{H}^\pm_i(a,b) := -D_x V^\pm_i(t, x, z) \cdot f(t, x, a, b) + D_z V^\pm_i(t, x, z) L(t, x, a, b)$$

for $i = 1, 2$, thus

$$\mathcal{H}^+ (t, x, z, D_x V^+, D_z V^+) = \min_{a \in A} \max_{b \in B} \mathcal{H}^+_i(a,b),$$

$$\mathcal{H}^- (t, x, z, D_x V^-, D_z V^-) = \min_{b \in B} \max_{a \in A} \mathcal{H}^-_i(a,b),$$

where $\mathcal{H}^+$ and $\mathcal{H}^-$ are defined in (27) and (28), respectively. In this section, we omit $(t, x, z)$ to simplify notation. Using the notation with $\mathcal{H}^\pm_i$ (78), the HJ equation (26) for $V^+_1$ is equal to

$$\max\{c - V^+_1, g - z - V^+_1, V^+_{1,t} - \mathcal{H}^+_1(a,b)\} = 0.$$
Algorithm 1: Computing the solution $V_1^\pm$ or $V_1$ to the HJ equations for Problem 1 in the zero-sum game and optimal control settings. This algorithm deals with the four HJ equations: (26), (43), (48), and (75).

1: **Input:** the temporal discretization: \( \{ t_0 = 0, \ldots, t_K = T \} \), the spatial discretization: \( \{(x_0, z_0), \ldots, (x_N, z_N)\} \subset \mathbb{R}^n \times \mathbb{R}^n \)
2: **Output:** $V_1^\pm$ (or $V_1$)
3: $V_1^\pm(t_k, x_i, z_i) \leftarrow \max\{c(T, x_i), g(T, x_i) - z_i\}, \forall i$
4: for $k \in \{K - 1, \ldots, 0\}$ do
5: **case** solving the HJ equations (26) or (48)
6: $V_1^\pm(t_k, x_i, z_i) \leftarrow V_1^\pm(t_k+1, x_i, z_i) - \Delta_k \hat{H}^\pm(\phi^+_x, \phi^-_x, \phi^+_z, \phi^-_z)$, $\forall i$
7: $V_1^\pm(t_k, x_i, z_i) \leftarrow \max\{c(t_k, x_i), g(t_k, x_i) - z_i\}, \forall i$
8: end **case**
9: **case** solving the HJ equations (43) or (52)
10: $V_1^\pm(t_k, x_i, z_i) \leftarrow V_1^\pm(t_k+1, x_i, z_i) - \Delta_k \min\{0, \hat{H}^\pm(\phi^+_x, \phi^-_x, \phi^+_z, \phi^-_z)\}$, $\forall i$
11: $V_1^\pm(t_k, x_i, z_i) \leftarrow \max\{c(t_k, x_i), g(t_k, x_i) - z_i\}, \forall i$
end **for**

Algorithm 2: Computing the solution $V_2^\pm$ or $V_2$ to the HJ equations for Problem 2 in the zero-sum game and optimal control settings. This algorithm deals with the four HJ equations: (58), (68), (72), and (75).

1: **Input:** the temporal discretization: \( \{ t_0 = 0, \ldots, t_K = T \} \), the spatial discretization: \( \{(x_0, z_0), \ldots, (x_N, z_N)\} \subset \mathbb{R}^n \times \mathbb{R}^n \)
2: **Output:** $V_2^\pm$ (or $V_2$)
3: $V_2^\pm(t_k, x_i, z_i) \leftarrow \max\{c(T, x_i), g(T, x_i) - z_i\}, \forall i$
4: for $k \in \{K - 1, \ldots, 0\}$ do
5: **case** solving the HJ equations (58) or (72)
6: $V_2^\pm(t_k, x_i, z_i) \leftarrow V_2^\pm(t_k+1, x_i, z_i) - \Delta_k \hat{H}^\pm(\phi^+_x, \phi^-_x, \phi^+_z, \phi^-_z)$, $\forall i$
7: $V_2^\pm(t_k, x_i, z_i) \leftarrow \min\{g(t_k, x_i) - z_i, 0\}, \forall i$
8: $V_2^\pm(t_k, x_i, z_i) \leftarrow \max\{c(t_k, x_i), g(t_k, x_i) - z_i\}, \forall i$
9: **case** solving the HJ equations (68) or (75)
10: $V_2^\pm(t_k, x_i, z_i) \leftarrow V_2^\pm(t_k+1, x_i, z_i) - \Delta_k \max\{0, \hat{H}^\pm(\phi^+_x, \phi^-_x, \phi^+_z, \phi^-_z)\}$, $\forall i$
11: $V_2^\pm(t_k, x_i, z_i) \leftarrow \max\{c(t_k, x_i), g(t_k, x_i) - z_i\}, \forall i$
end **for**

VII. Example

This section provides an example for Problem 1. For Problem 2, an example for Problem 3 can be found in [6]. In this example, we solve a zero-sum game for two ponds, as shown in Figure 1. This example is motivated by the water system in [13].

Precipitation on pond 1 increases the water level of pond 1, and pond 1 (player A) wants to minimize the highest water level in the time horizon (1 s) by controlling amount of outflow to pond 2. We assume that the water level increasing rate on pond 1 due to the precipitation is unknown but bounded from 0 and 10 m/s. The precipitation is considered as player B. These numbers and units can be easily changed to realistic problems. We determine an optimal control strategy in pond 1.
the worst behavior of player B.

In the water system, we have two states: $x_1$ and $x_2$ represent the water level of pond 1 and 2. The state trajectories are solving the following dynamics:

$$
\begin{align*}
\dot{x}_1(s) &= \beta(s) - \sqrt{2g} x_1(s) \alpha(s), \\
\dot{x}_2(s) &= 0.5 \sqrt{2g} x_1(s) \alpha(s) - 0.5 x_2(s),
\end{align*}
$$

(91)

where $\alpha(s) \in [0, 1]$, $\beta(s) \in [0, 10]$, and $g$ is the gravitational constant: 9.81 m/s$^2$. In the dynamics for $x_1$, the first term $\beta$ is by the precipitation (player B), and the second term $\sqrt{2g} x_1 \alpha$ is the water level decreasing rate by pond 1 (player A). The term $\sqrt{2g} x_1$ is by Bernoulli’s equation, and pond 1 controls the area of outflows ($\alpha$) between 0 and 1. We set the bottom area of pond 2 is twice bigger than pond 1, thus the dynamics for $x_2$ contains $0.5 \sqrt{2g} x_1 \alpha$. Also, we assume that pond 2’s water is used for drinking water, which causes a decreasing rate $0.5 x_2$.

The dynamics (91) is not Lipschitz at $x_1 = 0$. To avoid this, we approximate $\sqrt{2g} x_1$ with a sinusoidal function $4.82 \sin(1.17 x_1)$ if $x_1$ is less than 1. This sinusoidal-approximate function has the same value and first derivative at $x_1 = 1$: $\sqrt{2g} \approx 4.82 \sin(1.17)$ and $\sqrt{y^2} \approx 4.82 * 1.17 * \cos(1.17)$. The approximated (Lipschitz) dynamics are

$$
\begin{align*}
\dot{x}_1(s) &= \beta(s) - \left\{ \begin{array}{ll} \\
2g x_1(s) \alpha(s), & x_1(s) \geq 1, \\
4.82 \sin(1.17 x_1(s)) \alpha(s), & x_1(s) < 1,
\end{array} \right. \\
\dot{x}_2(s) &= 0.5 \sqrt{2g} x_1(s) \alpha(s), \\
&= 2.41 \sin(1.17 x_1(s)) \alpha(s), \\
&= 2.41 \sin(1.17 x_1(s)) \alpha(s) - 0.5 x_2(s),
\end{align*}
$$

(92)

We solve the two zero-sum games: the upper value function is

$$
\begin{align*}
\varphi^+(0, x_1, x_2) &= \min_{\delta \in \Delta(0)} \max_{\alpha \in \mathcal{A}(0)} \max_{\tau \in [0, 1]} x_1(\tau), \\
&\text{subject to } \max\{|x_1(s) - 7.5| - 7.5, |x_2(s) - 3| - 2\} \leq 0,
\end{align*}
$$

(93)

(94)

where $\mathcal{A}(0) = \{0, 1\} \to A \mid \|a\|_{L_\infty(0, 1)} < \infty$, $\mathcal{B}(0) = \{0, 1\} \to B \mid \|\beta\|_{L_\infty(0, 1)} < \infty$, $\Delta(0)$ is a set of non-anticipative strategies for player B (pond 2) as in (7), and $(x_1, x_2)$ solves (92) for $(\alpha, \delta(\alpha))$; and

the lower value function is

$$
\begin{align*}
\varphi^-_1(0, x_1, x_2) &= \min_{x_2 \in \mathcal{B}(0)} \max_{\alpha \in \mathcal{A}(0)} \max_{\tau \in [0, 1]} x_1(\tau), \\
&\text{subject to } \max\{|x_1(s) - 7.5| - 7.5, |x_2(s) - 3| - 2\} \leq 0,
\end{align*}
$$

(95)

(96)

where $\Gamma(0)$ is a set of non-anticipative strategies for player A (pond 1) as in (8), and $(x_1, x_2)$ solves (92) for $(\gamma(\beta), \beta)$. The state constraint implies that the water level of pond 1 has to be between 0 and 15 m and the one of pond 2 is between 1 and 5 m. In these games, pond 1 (player A) wants to minimize the worst water level of pond 1 in the time horizon while satisfying the state constraint for preventing flood in pond 1 and 2.

We will solve the HJ equation (26) for $V^+_1$ corresponding to $\varphi^+_1$ (94) or (96). We have the Hamiltonian

$$
\begin{align*}
\bar{H}^+(t, x, z, p, q) &= \min_{a \in A} \max_{b \in B} -p_1 b + 0.5 p_2 x_2 \\
&\quad + \left\{ \begin{array}{ll} \\
(p_1 - 0.5 p_2) \sqrt{2g} x_1 & \text{if } x_1 \geq 1 \\
(p_1 - 0.5 p_2) 4.82 \sin(1.17 x_1) a & \text{if } x_1 < 1
\end{array} \right.
\end{align*}
$$

(97)

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $p = (p_1, p_2) \in \mathbb{R}^2$. (97) implies

$$
V^+_1 = V^-_1 = V^+_2 = V^-_2 = \varphi^-_1 = \varphi^+_1 = \varphi^+.
$$

We use $V^\pm$ to denote the same value functions $V^+_1$ and $V^-_1$.

The red curvature in Figure 2 (a) shows the zero-level set of $V^+_1(0, x_1, x_2)$, numerically computed by Algorithm 1. This algorithm is programmed by utilizing the level set toolbox [8] and the helperOC toolbox [14] in Matlab. and this simulation is carried out on a laptop with a 2.8 GHz Quad-Core i7 CPU and 16 GB RAM. Each of $x_1$, $x_2$, and $\gamma$ axis has 81 discretization points, and the time interval [0, 1] is discretized with 201 points. The computation time for $V^+_1$ is 237 s. In Figure 2 (a), the value of $V^+_1$ inside of the red curvature is negative, on the other hand, the value outside of the curvature is positive.

This example is time-invariant, so both HJ equations in (26) and (43) can be utilized. In this example, we solve the HJ equation (26).

Lemma 1 describes how to compute $\varphi^+_1$ from the zero-level set of $V^+_1$, which is illustrated in Figure 2. Figure 2 (a) illustrates the intersection of the zero-level set of $V^+_1$ and each z-level plane, and Figure 2 (b) shows these intersections in the state space, $(x_1, x_2)$: the z-level sets on the zero-level set of $V^+_1$. As illustrated in Figure 2 (b), the lower z-level is achieved in the smaller region in $(x_1, x_2)$. In this example, as z-level is increasing, the inner area of the z-level set on
unique $\theta_1$ (player A) to satisfy the state constraint, which implies that with $(z, x)$ the subzero-level set of $V_1^\pm$ is shown in this figure. The value of $V_1^\pm$ inside of the curvature is negative, but the value is positive outside. The blue planes are $z$-level planes of 4, 6, 8, 10, 12, 14, and 16. (b) The $z$-level sets are shown in $(x_1, x_2)$-space. The $z$-level sets for $z \geq 15$ are the same (the outer curvature). The $z$-level sets also show $\theta_1^\pm$ by Lemma 1. For example, for (1.60, 2.85) on the $z$-level set of 4, $\theta_1^\pm$ is 4. On the other hand, consider (0.11, 2) on multiple $z$-level sets from 4.22 to any greater levels, for which $\theta_1^\pm$ is the minimum $z$-level that contains the point: $\theta_1^\pm(0.05, 2) = 4.22$.

The subzero-level set of $V_1^\pm$ is increasing and also converging at the $z$-level of 15, which is the outer curvature in Figure 2 (b). For $(x_1, x_2)$ outside of the outer curvature indicated with $z \geq 15$, there is no control signal or strategy for pond 1 (player A) to satisfy the state constraint, which implies that $\theta_1^\pm(0, x_1, x_2)$ is infinity. On the other hand, for $(x_1, x_2)$ on a unique $z$-level set, the $z$-level is equal to $\theta_1^\pm$. For example, the $z$-level set of 6 is the only $z$-level set passing through (2.6, 2). In this case, $\theta_1^\pm(0, 2.6, 2) = 6$. On the other hand, for $(x_1, x_2)$ on multiple $z$-level sets, the minimum value of $z$-level is $\theta_1^\pm$. For example, (0.05, 2) is on the $z$-level sets of any number greater than or equal to 4.5. In this case, $\theta_1^\pm(0, 0.05, 2)$ is 4.5 since $\theta_1^\pm$ is the minimum $z$-level that contains the point (0.05, 2).

Using the value function $V_1^\pm$ and $\theta_1^\pm$, the method presented in Section VII provides a state trajectory and an optimal control and strategy for the two players (pond 1 and the precipitation). Among multiple solutions for optimal control and strategy presented in Remark 4, we choose

$$a_* \in \operatorname{arg\ max} \min_{a \in A} \min_{b \in B} \hat{H}_1^+(a, b),$$

$$b_* \in \operatorname{arg\ max} \min_{b \in B} \min_{a \in A} \hat{H}_1^+(a, b),$$

(99)

which satisfies the eight equations (82) to (86) since $\hat{H}_1^+ = \hat{H}_1^−$ and $\hat{H}_1^− = \hat{H}_1^−$, where $\hat{H}_1^\pm$ is equal to $D_2V_1^\pm+f+D_2V_1^\pm L$ as defined in (78).

Figure 3 shows state trajectories for two different initial states: $(x_1, x_2) = (10, 4)$ and $(2, 1.1)$. As shown in Figure 3 (a), for the initial state $(10, 4), x_2$ hits the boundary of the state constraint: $x_2(1) = 5, x_1$ is maximized at $t = 1$. Since the initial water levels of the two ponds are high, the precipitation (player B) tries to increase the water level of pond 1 for all time, but player A tries to balance the water levels of the two ponds. On the other hand, for the initial state $(2, 1.1)$, Figure 3 (b) shows that $x_1$ strictly satisfies the state constraint $[1, 5], x_1$ is maximized at $t = 0.015$ and increasing for the later time. Since the initial water levels of the two ponds are low, the precipitation (player B) tries to violate the state constraint by not increasing the water level of pond 1. However, player A tries to balance the two ponds’ water level so that all ponds have more water than the minimum levels.

As discussed in Section VI, there are some numerical issues in Algorithm 1. First, we observe that (99) provides a bang-bang control, thus the state trajectories are not smooth as shown in Figure 3. This happens due to frequent sign change of the gradient along the time horizon. Second, the numerical error on $V_1^\pm$ causes inaccurate $\theta_1^\pm$ by Lemma 1, which could potentially cause unsafety even though the violation of the state constraint might be smaller for the smaller grid size. In practice, we suggest having a safety margin to the state constraint: for example, use $c(s, x(s)) + \epsilon \leq 0$ for small $\epsilon > 0$ instead of $c(s, x(s)) \leq 0$. 

Fig. 3. State trajectories by applying an optimal control signal and strategy for two players (pond 1 and the precipitation) where the initial states are (a) $(x_1, x_2) = (10, 4)$ and (b) $(x_1, x_2) = (2, 1.1)$. 
VIII. CONCLUSION AND FUTURE WORK

This paper presented four HJ equations for the two classes of state-constrained zero-sum games where the terminal time is a variable to be determined and the stage cost is non-zero. For each class of problems, two HJ equations have presented: one for time-varying version, and the other for the time-invariant version. This paper also analyzed the optimal control and strategy for each player using the gradient of the viscosity solution to the HJ equations, and also presented a numerical algorithm to compute the viscosity solution. As a practical example, a 2D water system demonstrates one of the presented version. This paper also analyzed the optimal control and each class of problems, two HJ equations have presented: one

B. Proof of Lemma 3

Proof. Consider \((x, z)\) solving (17) for any \((\alpha, \beta)\), and a small \(h > 0\). (18) implies

\[
J_1(t, x, z, \alpha, \beta) = \max \left\{ \max_{s \in [t, t + h]} c(s, x(s)), \max_{s \in [t + h, T]} g(s, x(s)) - z(s), \max_{s \in [t + h, T]} c(s, x(s)) \right\}. 
\]

(102)

(i) For all \(\alpha \in A(t)\) and \(\delta \in \Delta(t)\), there exists \(\alpha_1 \in A(t), \delta_1 \in \Delta(t), \alpha_2 \in A(t + h), \delta_2 \in \Delta(t + h)\) such that

\[
\alpha(s) = \begin{cases} 
\alpha_1(s), & s \in [t, t + h], \\
\alpha_2(s), & s \in (t + h, T], 
\end{cases} 
\]

\[
\delta(s) = \begin{cases} 
\delta_1(s), & s \in [t, t + h], \\
\delta_2(s), & s \in (t + h, T]. 
\end{cases} 
\]

(103)

(104)

Then, we have

\[
V_1^+(t, x, z) = \sup_{\delta_1 \in \Delta(t)} \inf_{\alpha_1 \in A(t)} \max_{s \in [t, t + h]} c(s, x(s)), \max_{s \in [t + h, T]} g(s, x(s)) - z(s), \sup_{\alpha_2 \in A(t + h)} \inf_{\delta_2 \in \Delta(t + h)} \max_{s \in [t + h, T]} \left\{ \max_{s \in [t + h, T]} c(s, x(s)), \max_{s \in [t, t + h]} g(s, x(s)) - z(s) \right\}. 
\]

(105)

The last equality is deduced by combining (102) and that the first two terms of \(V_1^+\) (max, \(s \in [t, t + h]\)) are independent of \((\alpha_2, \delta_2)\). (105) concludes (24).

(ii) The proof for (25) is similar to (i).

C. Proof of Theorem

Proof. (i) At \(t = T\), the definition of \(V_1^+\) (13) and (14) implies (29).

(ii) For \(U \in C^\infty([0, T] \times \mathbb{R}^n \times \mathbb{R})\) such that \(V_1^+ - U\) has a local maximum at \((t_0, x_0, z_0) \in (0, T) \times \mathbb{R}^n \times \mathbb{R}\), there exists a threshold such that \(V_1^+ - U\) is locally maximized at \((t_0, x_0, z_0) = 0\), we will prove

\[
\max_{s \in [t, t + h]} \left\{ c(t_0, x_0) - U_0, g(t_0, x_0), z_0 - U_0, D_x U_0, D_z U_0 \right\} \geq 0. 
\]

(106)

where \(U_0 = U(t_0, x_0, z_0), U_{t_0} = U_0(t_0, x_0, z_0), D_x U_0 = D_x U(t_0, x_0, z_0)\) and \(D_z U_0 = D_z U(t_0, x_0, z_0)\).

Suppose not. There exists \(\theta > 0, a_1 \in A\) such that

\[
c(t, x) - U_0 < -\theta, \quad g(t, x) - z - U_0 < -\theta, 
\]

(107)

\[
U_1(t, x, z) + D_x U(t, x, z) \cdot f(t, x, a_1, b) - D_z U(t, x, z) L(t, x, a_1, b) \leq -\theta 
\]

(108)

for all \(b \in B\) and all points \((t, x, z)\) sufficiently close to \((t_0, x_0, z_0): |t - t_0| + |x - x_0| + |z - z_0| < h_1\) for small enough \(h_1 > 0\). Consider state trajectories \(x\) and \(z\) solving (17) for \(a_1 \equiv a_1, t = t_0, x = x_0, z = z_0,\) and any \(\beta \in B(t_0)\). By Assumption 2 there exists a small \(h\) such...
that \( \| x(s) - x_0 \| + |z(s) - z_0| < h_1 - h \) (s \( \in [t_0, t_0 + h] \)), then,
\[
c(s(x, s)) - U_0 < -\theta, \quad g(s(x, s)) - z(s) - U_0 < -\theta, \quad (109)
\]
\[
U_i(s(x, s), z(s)) + D_x U(s(x, s), z(s)) \cdot f(s(x, s), a_1, \beta(s)) + D_z U(s(x, s), z(s)) L(s(x, s), a_1, \beta(s)) \leq -\theta \quad (110)
\]
for all \( s \in [t_0, t_0 + h] \) and \( \beta \in \mathcal{B}(t_0) \).

Since \( V_1^+ - U \) has a local maximum at \((t_0, x_0, z_0)\),
\[
V_1^+(t_0 + h, x(t_0 + h), z(t_0 + h)) - V_1^+(t_0, x_0, z_0)
\leq U(t_0 + h, x(t_0 + h), z(t_0 + h)) - U(t_0, x_0, z_0)
= \int_{t_0}^{t_0 + h} U_i(s(x, x), z(s)) \nonumber
+ D_x U(s(x, x), z(s)) \cdot f(s(x, x), a_1, \alpha_1[s])
+ D_z U(s(x, x), z(s)) L(s(x, x), a_1, \alpha_1[s]) ds \leq -\theta h \quad (111)
\]
for all \( \delta \in \Delta(t_0) \), according to \[109\]. Lemma \[2\] implies
\[
V_1^+(t_0, x_0, z_0) \leq \sup_{\delta \in \Delta(t_0)} \max \left\{ \max_{s \in [t_0, t_0 + h]} c(s(x, s)), \quad \sup_{s \in [t_0, t_0 + h]} \max_{s \in [t_0, t_0 + h]} g(s(x, x)) - z(s), \quad V_1^+(t_0 + h, x(t_0 + h), z(t_0 + h)) \right\} 
\]
\[
(112)
\]
By subtracting \( U_0 \) on the both sides in \[112\] and then applying \[109\] and \[111\], we have
\[
0 \leq \max \{-\theta, -\theta, -\theta h\} < 0,
\]
which is contradiction. Thus, \[106\] is proved.

(iii) For \( U \in C^\infty([0, T] \times \mathbb{R}^n \times \mathbb{R}) \) such that \( V_1^+ - U \) has a local minimum at \((t_0, x_0, z_0) \in (0, T] \times \mathbb{R}^n \times \mathbb{R} \) and \( V_i^+ - U(t_0, x_0, z_0) = 0 \), we will prove
\[
\max \{ c(t_0, x_0) - U_0, g(t_0, x_0) - z_0 - U_0, \quad U_0 - H^+(t_0, x_0, z_0, D_x U_0, D_z U_0) \} \leq 0,
\]
\[
(114)
\]
Since \( J_1(t_0, x_0, z_0, \alpha) \) \[15\] is greater than the value at \( \tau = t_0 \),
\[
J_1(t_0, x_0, z_0, \alpha, \delta[\alpha]) \geq \max \{ c(x_0, x_0), g(t_0, t_0) - z_0 \},
\]
\[
(115)
\]
for all \( \alpha \in \mathcal{A}(t_0), \delta \in \Delta(t_0) \). By subtracting \( U_0 \) on the both sides, and taking the supremum over \( \delta \) and the infimum over \( \alpha \), sequentially, on the both side, we have
\[
0 \geq \max \{ c(x_0, x_0) - U_0, g(t_0, t_0) - z_0 - U_0 \}. \quad (116)
\]
The rest of the proof is to show
\[
U_0 - H^+(t_0, x_0, z_0, D_x U_0, D_z U_0) \leq 0. \quad (117)
\]
Suppose not. For some \( \theta > 0 \),
\[
U_i(t, x, z) + \max_{b \in B} D_x U(t, x, z) \cdot f(t, x, a, b)
- D_z U(t, x, z) L(t, x, a, b) \geq \theta \quad (118)
\]
for all \( a \in A \) and all points \( (t, x, z) \) sufficiently close to \((t_0, x_0, z_0) : |t - t_0| + \| x - x_0 \| + |z - z_0| < h_1 \) for small enough \( h_1 > 0 \). Consider state trajectories \( x_1 \) and \( z_1 \) solving \[17\] for any \( \alpha \in \mathcal{A}(t_0), \beta = \delta[\alpha] \), where
\[
d_1[\alpha](s) \in \arg \max_{b \in B} D_x U(s, x_1(s), z_1(s)) \cdot f(s, x_1(s), \alpha(s), b)
- D_z U(s, x_1(s), z_1(s)) L(s, x_1(s), \alpha(s), b), \quad (119)
\]
\[
t = t_0, \quad x = x_0, \quad z = z_0. \quad \text{Since there exists a small } h > 0 \quad \text{such that} \quad \| x_1(s) - x_0 \| + |z_1(s) - z_0| < h_1 - h \quad \text{for} \quad \text{all } \| t - t_0 \| \quad \text{small enough} \quad h_1 > 0. \quad \text{Consider state trajectories} \quad x_1 \quad \text{and} \quad z_1 \quad \text{solving} \quad (109) \quad \text{and} \quad (111), \quad \text{we have}
\]
\[
U(t_0 + h, x(t_0 + h), z(t_0 + h)) - U(t_0, x_0, z_0) \quad (121)
\]
Since \[121\] holds for all \( \alpha \in \mathcal{A}(t_0) \) and \( \delta \in \Delta(t_0) \),
\[
\sup_{\delta \in \Delta(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} U(t_0 + h, x(t_0 + h), z(t_0 + h)) \quad (122)
\]
\[
U(t_0 + h, x(t_0 + h), z(t_0 + h)) - U(t_0, x_0, z_0) \quad (123)
\]
\[
\text{according to} \quad (123). \quad \text{However, Lemma} \quad (2) \quad \text{implies}
\]
\[
\inf_{\delta \in \Delta(t_0)} \sup_{\alpha \in \mathcal{A}(t_0)} V_1^+(t_0 + h, x(t_0 + h), z(t_0 + h)) \quad (124)
\]
\[
\text{which contradicts} \quad (123).
\]

(iv) The proof for the viscosity solution \( V_i^- \) is similar to (ii) and (iii) for \( V_i^+ \). Also, the uniqueness follows from the uniqueness theorems for viscosity solutions, Theorem 4.2 in [15], and the extension of Theorem 1 in [16]. \[\blacksquare\]

**D. Proof of Lemma 3**

**Proof.** Set \( V_1^+ \) and \( V_1^- \) be the right hand terms in \[40\] and \[41\], respectively. \( V_1^+ \) are \( V_i^- \) are defined in \[13\] and \[14\], respectively.

(i) In this proof, we utilize the following properties in [3], [17], presented as below.

Denote a pseudo-time operator \( \sigma_\mu : [t, T] \to [t, T] \) for a given \( \mu \in \mathcal{M}(t) \) (defined in \[31\]) and the corresponding inverse operator:
\[
\sigma_\mu(s) = \int_t^s \mu(\tau) d\tau + t; \quad (125)
\]
\[
\sigma_\mu^{-1}(s) := \min \{ \tau \text{ subject to } \sigma_\mu(\tau) = s \}. \quad (126)
\]
Then,
\[
\sigma_\mu(\sigma_\mu^{-1}(s)) = s, \quad s \in [t, \sigma_\mu(T)], \quad (127)
\]
\[
\sigma_\mu^{-1}(\sigma_\mu(s)) = s, \quad s \in \text{Range}(\sigma_\mu^{-1}), \quad (128)
\]

where \( \text{Range}(\sigma^{-1}_\mu) := \{ \sigma^{-1}_\mu(s) \mid s \in [t, \sigma_\mu(T)] \} \).

Consider two state trajectories: \((x, z)\) solving \((17)\) for \((\tilde{\alpha}(\sigma^{-1}_\mu(\cdot)), \tilde{\beta}(\sigma^{-1}_\mu(\cdot)))\) for \(s \in [t, \sigma_\mu(T); (\tilde{x}, \tilde{z})\) solving \((39)\) for \((\tilde{\alpha}, \tilde{\beta}, \mu),\) and \(x(t) = \tilde{x}(t) = x\). Then,

\[
x(\sigma_\mu(s)) = \tilde{x}(s), \quad s \in [t, T], \tag{129}
g(\sigma_\mu(T)) - z(\sigma_\mu(T)) = g(\tilde{x}(T)) - \tilde{z}(T). \tag{130}
\]

\((129)\) is according to Lemma 4 in \cite{9}, and \((130)\) is derived by combining two lemmas (Lemma 4 and 6) in \cite{9}.

(ii) \(V^*_1(t, x, z) \geq V^*_1(t, x, z)\)

For small \(\epsilon > 0\), there exists \(\delta_1 \in \Delta(t)\) such that

\[
V^*_1(t, x, z) - \epsilon \leq \inf \max_{\tau \in [t, T]} \max_{s \in [t, \tau]} \left\{ \max \left\{ c(x_1(s)), g(x_1(\tau)) - z_1(\tau) \right\} \right\}, \tag{131}
\]

where \((x_1, z_1)\) solves \((17)\) for \((\alpha, \delta_1[\alpha])\). Denote \(\tau_*(\alpha)\) is the maximizer of the right hand term in \((131)\) for each \(\alpha \in A(t):\)

\[
\tau_*(\alpha) := \arg \max_{\tau \in [t, T]} \max_{s \in [t, \tau]} \left\{ c(x_1(s)), g(x_1(\tau)) - z_1(\tau) \right\}. \tag{132}
\]

Define a particular strategy \(\nu_{A,1}(t, z)\):

\[
\nu_{A,1}[\alpha](s) := \begin{cases} 
1, & s \in [t, \tau_*(\alpha)], \\
0, & s \in (\tau_*(\alpha), T]. 
\end{cases} \tag{133}
\]

Consider a state trajectory \((\tilde{x}_1, \tilde{z}_1)\) solving \((39)\) for \((\alpha, \delta_1[\alpha], \nu_{A,1}[\alpha])\). Then, we have

\[
(\tilde{x}_1, \tilde{z}_1)(s) = \begin{cases} 
(x_1, z_1)(s), & s \in [t, \tau_*(\alpha)], \\
(x_1, z_1)(\tau_*(\alpha)), & s \in (\tau_*(\alpha), T). 
\end{cases} \tag{134}
\]

Since \(V^*_1\) has the supremum over \((\delta, \nu_A)\)-space operation,

\[
V^*_1(t, x, z) \geq \inf_{\alpha} \max_{s \in [t, T]} \max_{\tau \in [t, T]} \left\{ \max \left\{ c(\tilde{x}_1(s)), g(\tilde{x}_1(T)) - \tilde{z}_1(T) \right\} \right\}
= \inf_{\alpha} \max_{s \in [t, T]} \max_{\tau \in [t, \tau_*(\alpha) \wedge T]} \left\{ c(x_1(s)), g(x_1(\tau)) - z_1(\tau) \right\}
\geq V^*_1(t, x, z) - \epsilon. \tag{135}
\]

The second equality is according to \((134)\), and the third inequality is by \((131)\).

(iii) \(V^*_1(t, x, z) \geq \tilde{V}^*_1(t, x, z)\)

Define \(\mathfrak{A}_\mu : A(t) \to A(t)\) and its pseudo inverse function \(\mathfrak{A}_\mu : A(t) \to A(t)\):

\[
(\mathfrak{A}_\mu(\alpha))(s) := \begin{cases} 
\alpha(\sigma_\mu(s)), & s \in \text{Range}(\sigma_\mu^{-1}), \\
\alpha(\sigma_\mu(s)), & s \in A, \quad s \notin \text{Range}(\sigma_\mu^{-1}). 
\end{cases} \tag{136}
\]

\[
(\mathfrak{A}_\mu(\tilde{\alpha}))(s) := \begin{cases} 
\tilde{\alpha}(\sigma^{-1}_\mu(s)), & s \in [t, \sigma_\mu(T)], \\
\tilde{\alpha}(\sigma^{-1}_\mu(s)), & s \in A, \quad s \notin \text{Range}(\sigma^{-1}_\mu). 
\end{cases} \tag{137}
\]

Also, define \(\mathfrak{D}_\mu : \Delta(t) \to \Delta(t)\) and its pseudo inverse function \(\mathfrak{D}_\mu : \Delta(t) \to \Delta(t)\):

\[
(\mathfrak{D}_\mu(\delta))(\alpha)(s) := \begin{cases} 
\delta[\mathfrak{A}_\mu(\alpha)](\sigma_\mu(s)), & s \in \text{Range}(\sigma_\mu^{-1}), \\
\delta[\mathfrak{A}_\mu(\alpha)](\sigma_\mu(s)), & s \in A, \quad s \notin \text{Range}(\sigma_\mu^{-1}). 
\end{cases} \tag{138}
\]

These definitions satisfy the following properties: for any \(\alpha \in A(t)\) and \(\delta \in \Delta(t)\),

\[
(\mathfrak{D}_\mu(\delta))(\alpha)(s) := \begin{cases} 
\delta[\mathfrak{A}_\mu(\alpha)](\sigma_\mu(s)), & s \in \text{Range}(\sigma_\mu^{-1}), \\
\delta[\mathfrak{A}_\mu(\alpha)](\sigma_\mu(s)), & s \in A, \quad s \notin \text{Range}(\sigma_\mu^{-1}). 
\end{cases} \tag{139}
\]

Consider \((\tilde{x}, \tilde{z})\) solving \((39)\) for \((\tilde{\alpha}, \tilde{\delta}[\tilde{\alpha}], \mu), (x, z)\) solving \((17)\) for \((A_\mu(\tilde{\alpha}), \mathfrak{D}_\mu(\tilde{\delta})), (\mathfrak{A}_\mu(\tilde{\alpha})),\) and \((x_1, z_1)\) solving \((17)\) for \((\alpha, \delta[\alpha])\). Then, we have

\[
sup_{\delta \in \Delta(t)} \inf_{\alpha} \max_{s \in [t, T]} \max_{\tau \in [t, \tau_*(\alpha) \wedge T]} \left\{ c(x_1(s)), g(x_1(\tau)) - z(\sigma_\mu(T)) \right\}
\leq V^*_1(t, x, z). \tag{140}
\]

\[(140)\] is by \((129)\) and \((130)\), and \((145)\) is according to \((142)\) and \((143)\). Since the above inequality holds for all \(\mu\), we substitute \(\nu_{A}[\alpha]\) for \(\mu\) and take the supremum over \(\nu_A\) on the both sides, which concludes \(V^*_1(t, x, z) \leq V^*_1(t, x, z)\).

By (ii) and (iii), we conclude \(V^*_1(t, x, z) = V^*_1(t, x, z)\).

(iv) \(V^*_1(t, x) = V^*_1(t, x)\)

Define \(\mathfrak{B}_\mu : B(t) \to B(t)\) and its pseudo inverse function \(\mathfrak{B}_\mu : B(t) \to B(t)\):

\[
(\mathfrak{B}_\mu(\beta))(\mu)(s) := \begin{cases} 
\beta(\sigma_\mu(s)), & s \in \text{Range}(\sigma^{-1}_\mu), \\
\beta(\sigma_\mu(s)), & s \in A, \quad s \notin \text{Range}(\sigma^{-1}_\mu). 
\end{cases} \tag{147}
\]

\[
(\mathfrak{B}_\mu(\tilde{\beta}))(\mu)(s) := \begin{cases} 
\tilde{\beta}(\sigma^{-1}_\mu(s)), & s \in [t, \sigma_\mu(T)], \\
\tilde{\beta}(\sigma^{-1}_\mu(s)), & s \in A, \quad s \notin \text{Range}(\sigma^{-1}_\mu). 
\end{cases} \tag{148}
\]

Also, define \(\mathfrak{C}_\mu : \Gamma(t) \to \Gamma(t),\) where \(\Gamma(t)\) is defined in \((37)\), and its pseudo inverse function \(\mathfrak{C}_\mu : \Gamma(t) \to \Gamma(t)\):

\[
(\mathfrak{C}_\mu(\gamma))(\beta)(\mu)(s) := \begin{cases} 
\gamma[\mathfrak{B}_\mu(\beta)](\sigma_\mu(s)), & s \in \text{Range}(\sigma^{-1}_\mu), \\
\gamma[\mathfrak{B}_\mu(\beta)](\sigma_\mu(s)), & s \in A, \quad s \notin \text{Range}(\sigma^{-1}_\mu). 
\end{cases} \tag{149}
\]

These definitions satisfy the following properties: for any \(\mu \in \)
where

\[
\beta = \mathfrak{B}_\mu(\bar{\beta}) \mid \bar{\beta} \in \mathcal{B}(t),
\]

\[
\gamma = \mathfrak{C}_\mu(\bar{\gamma}) \mid \bar{\gamma} \in \Gamma(t).
\]

Consider \((\bar{x}, \bar{z})\) solving (39) for \((\bar{\gamma}[\tilde{\beta}, \mu], \bar{\beta}, \mu)\), \((x, z)\) solving (17) for \((\mathfrak{C}_\mu(\bar{\gamma})[\mathfrak{B}_\mu(\bar{\beta})], \mathfrak{B}_\mu(\bar{\beta}))\), and \((x_1, z_1)\) solving (17) for \((\gamma[\bar{\beta}, \beta])\).

\[
\tilde{V}_1^-(t, x, z) = \inf_{\gamma \in \Gamma(t)} \sup_{\bar{\beta} \in \mathcal{B}(t), \mu \in \mathcal{M}(t)} \max_{s \in [t, T]} c(\tilde{x}(s)),
\]

\[
= \inf_{\gamma \in \Gamma(t)} \sup_{\bar{\beta} \in \mathcal{B}(t), \mu \in \mathcal{M}(t)} \max_{s \in [t, T]} g(\tilde{x}(T)) - z(T)
\]

\[
\min_{\mathcal{B}(t), \mu \in \mathcal{M}(t)} \max_{s \in [t, T]} c(\tilde{x}(s)),
\]

\[
= \inf_{\gamma \in \Gamma(t)} \sup_{\bar{\beta} \in \mathcal{B}(t), \mu \in \mathcal{M}(t)} \max_{s \in [t, T]} g(\tilde{x}(s)),
\]

\[
\max_{\gamma \in \Gamma(t)} \inf_{\bar{\beta} \in \mathcal{B}(t), \mu \in \mathcal{M}(t)} \min_{\mathcal{B}(t), \mu \in \mathcal{M}(t)} c(\tilde{z}(s)).
\]

Since \(b_d \in [0, 1]\) is non-negative,

\[
\tilde{H}^-_1(x, z, p, q) = \min_{b_d \in [0, 1]} \max_{b \in \mathcal{B}} \min_{x, a, b} -p \cdot f(x, a, b) b_d + qL(x, a, b) b_d.
\]

where \(\tilde{H}^-_1(x, z, p, q) = \min_{b_d \in [0, 1]} \max_{b \in \mathcal{B}} \min_{x, a, b} -p \cdot f(x, a, b) b_d + qL(x, a, b) b_d\).

Since, for all \(a \in A, b \in B\), the term \(-p \cdot f(x, a, b) b_d + qL(x, a, b) b_d\) is minimized at \(b_d = 0\) or 1,

\[
\tilde{H}^-_1(x, z, p, q) = \max_{a \in A} \min_{b \in \mathcal{B}} 0 -p \cdot f(x, a, b) + qL(x, a, b).
\]

Also, 0 does not depend on \(a\), thus, the maximum over \(a\) operation can move into the minimum operation:

\[
\tilde{H}^-_1(x, z, p, q) = \min_{a \in A} \max_{b \in \mathcal{B}} 0 -p \cdot f(x, a, b) + qL(x, a, b).
\]

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