Counting Dominating Sets of Graphs

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Abstract

Counting dominating sets in a graph $G$ is closely related to the neighborhood complex of $G$. We exploit this relation to prove that the number of dominating sets $d(G)$ of a graph is determined by the number of complete bipartite subgraphs of its complement. More precisely, we state the following. Let $G$ be a simple graph of order $n$ such that its complement $\overline{G}$ has exactly $a(G)$ subgraphs isomorphic to $K_{2p,2q}$ and exactly $b(G)$ subgraphs isomorphic to $K_{2p+1,2q+1}$. Then

$$d(G) = 2^n - 1 + 2[a(G) - b(G)].$$

We also show some new relations between the domination polynomial and the neighborhood polynomial of a graph.

1 Introduction

Counting dominating sets in graphs offers a multitude of relations to other graphical enumeration problems. The number of dominating sets in a graph is related to the counting of bipartite subgraphs, vertex subsets with respect to a given cardinality of the intersection of their neighborhoods, induced subgraphs such that all their components have even order [KPT14], and forests of external activity zero [Dod+15]. In this paper, we show that already the number of complete bipartite subgraphs is sufficient to determine the number of dominating sets of a graph. In addition, we obtain a new proof for the known fact [BCS09] that the number of dominating sets of any finite graph is odd.

Let $G = (V,E)$ be a simple undirected graph with vertex set $V$ and edge set $E$. The open neighborhood of a vertex $v$ of $G$ is denoted by $N(v)$ or $N_G(v)$. It is the set of all vertices of $G$ that are adjacent to $v$. The closed neighborhood of $v$ is defined by $N_G[v] = N_G(v) \cup \{v\}$. We generalize the
neighborhood definitions to vertex subsets $W \subseteq V$:

$$N_G(W) = \bigcup_{w \in W} N_G(w) \setminus W,$$

$$N_G[W] = N_G(W) \cup W.$$  

The *edge boundary* $\partial W$ of a vertex subset $W$ of $G$ is

$$\partial W = \{\{u, v\} \mid u \in W \text{ and } v \in V \setminus W\},$$

i.e., the set of all edges of $G$ with exactly one end vertex in $W$. Throughout this paper, we denote by $n$ the number of vertices and by $m$ the number of edges of $G$.

A *dominating set* of $G = (V, E)$ is a vertex subset $W \subseteq V$ with $N[W] = V$. We denote the number of dominating sets of size $k$ in $G$ by $d_k(G)$. The family of all dominating sets of $G$ is denoted by $D(G)$. The *domination polynomial* $D(G, x)$ is the ordinary generating function for the number of dominating sets of $G$:

$$D(G, x) = \sum_{k=0}^{n} d_k(G)x^k = \sum_{W \in D(G)} x^{|W|}.$$  

This polynomial has been introduced in [AL00]; it has been further investigated in [AAP10; Dod+15; Kot+12; KPT14]. The number of dominating sets of $G$ is $d(G) = D(G, 1)$.

# 2 Alternating Sums of Neighborhood Polynomials

The *neighborhood complex* $\mathcal{N}(G)$ of $G$ is the family of all subsets of open neighborhoods of vertices of $G$:

$$\mathcal{N}(G) = \{X \mid \exists v \in V : X \subseteq N(v)\}.$$  

If $G$ has no isolated vertex, then $\{v\} \in \mathcal{N}(G)$ for any $v \in V$. The neighborhood complex is a lower set in the Boolean lattice $2^V$, which means that the relations $X \in \mathcal{N}(G)$ and $Y \subseteq X$ imply $Y \in \mathcal{N}(G)$. Maximal elements in $\mathcal{N}(G)$ are open neighborhoods of vertices. We define for any $k \in \mathbb{N}$, $n_k(G) = |\{X \mid X \in \mathcal{N}(G), |X| = k\}|$. The *neighborhood polynomial* of $G$, introduced in [BN08], is

$$N(G, x) = \sum_{k=0}^{n} n_k(G)x^k = \sum_{W \in \mathcal{N}(G)} x^{|W|}.$$  

If $G'$ is a graph obtained from $G$ by adding some isolated vertices, then $\mathcal{N}(G') = \mathcal{N}(G)$ and hence $N(G', x) = N(G, x)$.  

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Theorem 1  The neighborhood polynomial of any simple graph $G = (V, E)$ satisfies

$$N(G, x) = \sum_{v \in V} (1 + x)^{\deg v} - \sum_{W \subseteq V, |W| \geq 2} (-1)^{|W|}(1 + x)^{\bigcap_{w \in W} N(w)}.$$ 

Proof. First we rewrite the statement of the theorem as

$$N(G, x) = \sum_{\emptyset \neq W \subseteq V} (-1)^{|W|+1}(1 + x)^{\bigcap_{w \in W} N(w)}, \quad (1)$$

which is correct as $\deg v = |N(v)|$ for any $v \in V$. The neighborhood polynomial is given by

$$N(G, x) = \sum_{W \in \mathcal{N}(G)} x^{|W|} = \sum_{W \in \bigcup_{v \in V} 2^\mathcal{N}(v)} x^{|W|}. \quad (2)$$

Comparing Equations (1) and (2), we obtain

$$\sum_{W \in \bigcup_{v \in V} 2^\mathcal{N}(v)} x^{|W|} = \sum_{\emptyset \neq W \subseteq V} (-1)^{|W|+1}(1 + x)^{\bigcap_{w \in W} N(w)}. \quad (3)$$

According to the principle of inclusion–exclusion, we have

$$\left| \bigcup_{v \in V} 2^{\mathcal{N}(v)} \right| = \sum_{\emptyset \neq W \subseteq V} (-1)^{|W|+1} \left| \bigcap_{w \in W} 2^{\mathcal{N}(w)} \right|. \quad (4)$$

Equation (4) is just the counting version (or generating function expression) of this principle. $\blacksquare$

Theorem 2  Let $G = (V, E)$ be a simple graph with $E \neq \emptyset$ that is not the disjoint union of a complete bipartite graph and an empty (that is edgeless) graph. Then

$$\sum_{F \subseteq E} (-1)^{|F|}N(G - F, x) = 0.$$

Proof. Let $W \in \mathcal{N}(G)$ be a vertex subset subset that is contained in the neighborhood complex of $G$. We define the family of edge subsets

$$\mathcal{E}(W) = \{F \mid F \subseteq E \text{ and } W \in \mathcal{N}(V, F)\}.$$ 

Using the definition of the neighborhood polynomial, we obtain

$$\sum_{F \subseteq E} (-1)^{|F|}N(G - F, x) = \sum_{F \subseteq E} (-1)^{|V| - |F|}N((V, F), x)$$

$$= \sum_{F \subseteq E} \sum_{W \in \mathcal{N}((V, F))} (-1)^{|V| - |F|} x^{|W|}$$

$$= \sum_{W \in \mathcal{N}(G)} x^{|W|} \sum_{F \in \mathcal{E}(W)} (-1)^{|V| - |F|}.$$
We will show that the inner sum vanishes, which provides the statement of the theorem. Assume that $E \setminus \partial W \neq \emptyset$. Then for any edge $e \in E \setminus \partial W$ and any edge subset $A \subseteq E$ the relation $A \setminus \{e\} \in \mathcal{E}(W)$ is satisfied if and only if $A \cup \{e\} \in \mathcal{E}(W)$, which yields
\[
\sum_{F \in \mathcal{E}(W)} (-1)^{m - |F|} = 0.
\] (4)

If $E = \partial W$, then $G$ is bipartite and $W$ is one partition set; we denote the second one by $Z$. Since $W \in \mathcal{N}(G)$, there is a vertex $z \in Z$ with $\mathcal{N}(z) = W$. There exists a vertex $z' \in Z$ with $\emptyset \neq \mathcal{N}(z') \subset W$ (“$\subset$” meaning proper subset), otherwise $G$ would be a disjoint union of a complete bipartite graph and an empty graph, the latter one possibly being the null graph. Now let $e$ be an edge incident to $z'$. Then for any set $A \in \mathcal{E}(W)$ that does not contain $e$, the set $A \cup \{e\}$ is also in $\mathcal{E}(W)$ and vice versa, $A \cup \{e\} \in \mathcal{E}(W)$ implies $A \in \mathcal{E}(W)$. The consequence is again that Equation (4) is satisfied, which completes the proof. 

**Lemma 3** Let $k, r$ be two positive integers and $\{E_1, \ldots, E_r\}$ a partition of a set $E$ with exactly $r$ blocks of size $k$. Define a family of subsets of $E$ by $M_{kr} = (2^{E_1 \setminus \{E_1\}}) \times \cdots \times (2^{E_r \setminus \{E_r\}})$. Then
\[
\sum_{A \in M_{kr}} (-1)^{|A|} = (-1)^{(k-1)r}.
\]

**Proof.** First consider the case $r = 1$. As $E_1 \neq \emptyset$ we have
\[
\sum_{A \subseteq E_1} (-1)^{|A|} = 0
\]
and consequently
\[
\sum_{A \in M_{k_1}} (-1)^{|A|} = \sum_{A \subseteq E_1} (-1)^{|A|} = (-1)^{k-1}.
\]

For $r > 1$, the calculation of the sum
\[
\sum_{A \in M_{kr}} (-1)^{|A|} = \sum_{A_1 \in 2^{E_1 \setminus \{E_1\}}} \cdots \sum_{A_r \in 2^{E_r \setminus \{E_r\}}} (-1)^{|A_1 \cup \cdots \cup A_r|}
\]
\[
= \sum_{A_1 \in 2^{E_1 \setminus \{E_1\}}} (-1)^{|A_1|} \sum_{A_r \in 2^{E_r \setminus \{E_r\}}} (-1)^{|A_r|}
\]
\[
= (-1)^{(k-1)r}
\]
yields the proof of the statement. 

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Lemma 4 Let \( k, r \) be two positive integers and \( \{E_1, \ldots, E_r\} \) a partition of a set \( E \) with exactly \( r \) blocks of size \( k \). Define a family of subsets of \( E \) by

\[
F(k, r) = \{ A \mid A \subseteq E \text{ and } \exists j \in \{1, \ldots, r\} : E_j \subseteq A \}.
\]

Then

\[
\sum_{A \in F(k, r)} (-1)^{|A|} = (-1)^{kr-r+1}.
\]

Proof. Define \( E' = E \setminus E_r \),

\[
\mathcal{M}' = \{ A \cup E_r \mid A \subseteq E' \text{ and } \exists i \in \{1, \ldots, r-1\} : E_i \subseteq A \},
\]

\[
\mathcal{M}'' = \{ A \cup B \mid A \subseteq E', B \subseteq E_r \text{ and } \exists i \in \{1, \ldots, r-1\} : E_i \subseteq A \}.
\]

We can rewrite the definition of \( \mathcal{M}' \) as

\[
\mathcal{M}' = \{ A \cup E_r \mid A \in 2^{E_1} \setminus \{E_1\} \times \cdots \times 2^{E_{r-1}} \setminus \{E_{r-1}\} \},
\]

which shows that, according to Lemma \( \Box \) each set of \( \mathcal{M}' \) is a disjoint union of a set of \( \mathcal{M}_{k,r-1} \) and \( E_r \). By Lemma \( \Box \) we obtain

\[
\sum_{A \in \mathcal{M}'} (-1)^{|A|} = \sum_{A' \in \mathcal{M}_{k,r-1}} (-1)^{|A'|+|E_r|} = (-1)^{(k-1)(r-1)+k} = (-1)^{kr-r+1}.
\]

Observe that the set \( F(k, r) \) defined in Lemma \( \Box \) can be represented as the disjoint union

\[
F(k, r) = \mathcal{M}' \cup \mathcal{M}'',
\]

which implies

\[
\sum_{A \in F(k, r)} (-1)^{|A|} = \sum_{A \in \mathcal{M}'} (-1)^{|A|} + \sum_{A \in \mathcal{M}''} (-1)^{|A|}.
\]

In order to complete the proof, we show that the second sum vanishes. This follows from

\[
\mathcal{M}'' = 2^{E_r} \times \{ A \mid A \subseteq E' \text{ and } \exists i \in \{1, \ldots, r-1\} : E_i \subseteq A \}
\]

and therefore

\[
\sum_{A \in \mathcal{M}''} (-1)^{|A|} = \sum_{A' \subseteq E_r} (-1)^{|A'|} \sum_{A'' \in \mathcal{M}_{k,r-1}} (-1)^{|A''|} = 0,
\]

since the first sum at the right-hand side equals zero. \( \blacksquare \)

Theorem 5 Let \( G = (V, E) = K_{p,q} \) be a complete bipartite graph with \( p + q \) vertices. Then

\[
\sum_{F \subseteq E} (-1)^{|F|} N(G - F, x) = (-1)^{q-1} x^p + (-1)^{p-1} x^q.
\]
Proof. We use the presentation of the neighborhood polynomial as in the proof of Theorem 2

\[
\sum_{F \subseteq E} (-1)^{|F|} N(G-F, x) = \sum_{W \in N(G)} x^{|W|} \sum_{F \in E(W)} (-1)^{m-|F|}
\]

If \(W\) is not a partition set of \(K_{p,q}\) then there is again an edge that is not contained in \(\partial W\), which can be used to show that the inner sum vanishes. Hence we can assume that \(W\) is one of the two partition sets, say \(|W| = p\). Then the minimal sets in \(E(W)\) are exactly \(q\) disjoint sets of cardinality \(p\) each. We observe that \(E(W)\) has exactly the structure of the set family \(\mathcal{F}(k,r)\) employed in Lemma 4 with \(k = p\) and \(r = q\). From Lemma 4 we obtain

\[
\sum_{A \in \mathcal{F}(p,q)} (-1)^{pq-|A|} = (-1)^{pq-(pq-q+1)} = (-1)^{q-1},
\]

which provides the term \((-1)^{q-1}x^p\) in the theorem, the second one is obtained in the same way. \(\blacksquare\)

For the empty (edge-less) graph \(G\), we have

\[
\sum_{F \subseteq E} (-1)^{|F|} N(G-F, x) = N(G, x) = 1.
\]

In the following statement, we use the notation \(G \approx K_{p,q}\) to indicate that \(G\) is isomorphic to the disjoint union of \(K_{p,q}\) and an empty graph.

**Theorem 6** Define for any graph \(G\)

\[
h(G, x) = \begin{cases} 
(\frac{1}{2})^{q+1}x^p + (\frac{1}{2})^{p+1}x^q, & \text{if } G \approx K_{p,q}, \\
1, & \text{if } G \text{ is empty}, \\
0, & \text{otherwise.}
\end{cases}
\]

*The neighborhood polynomial of any graph \(G\) satisfies

\[
N(G, x) = \sum_{F \subseteq E} h((V, F), x)
= 1 + \sum_{F \subseteq E: (V, F) \approx K_{p,q}} (-1)^{q+1}x^p + (-1)^{p+1}x^q.
\]

**Proof.** The statements of Theorem 2 and Theorem 5 can be combined to

\[
\sum_{F \subseteq E} (-1)^{m-|F|} N((V, F), x) = h(G, x).
\]

Now the statement follows by Möbius inversion. \(\blacksquare\)
3 Relations between Neighborhood and Domination Polynomials

**Theorem 7** Let $G = (V, E)$ be a graph of order $n$ and $\bar{G}$ its complement, then

$$D(G, x) + N(\bar{G}, x) = (1 + x)^n.$$  

**Proof.** The right-hand side of the equation is the ordinary generating function for all subsets of $V$. Therefore it suffices to show that for any graph $G = (V, E)$ the relation

$$D(G) \cup N(\bar{G}) = 2^V$$

is satisfied. Let $W \subseteq V$ be a non-dominating set of $G$. Then there exits a vertex $v \in V$ with $N_G[v] \cap W = \emptyset$. Consequently $v$ is not adjacent to any vertex of $W$ in $G$, which implies that $v$ is adjacent to each vertex of $W$ in $\bar{G}$. We conclude that $W \subseteq N_G(v)$ and hence $W \in N(\bar{G})$.

Now let $W$ be a vertex set with $W \in N(\bar{G})$. Then there is a vertex $v \in V$ with $W \subseteq N_G(v)$, which implies that $N_G[v] \cap W = \emptyset$. We conclude that $W \notin D(G)$. We have shown that any vertex subset of $V$ belongs either to $D(G)$ or to $N(\bar{G})$, which completes the proof. 

**Theorem 8** Let $G$ be a simple graph of order $n$ such that its complement $\bar{G}$ has exactly $a(G)$ subgraphs isomorphic to $K_{2p, 2q}$ and exactly $b(G)$ subgraphs isomorphic to $K_{2p+1, 2q+1}$. Then

$$d(G) = 2^n - 1 + 2[a(G) - b(G)].$$

**Proof.** By Theorem 7 we obtain

$$d(G) = D(G, 1) = 2^n - N(\bar{G}, 1).$$

The substitution of the last term according Theorem 6 results in

$$d(G) = 2^n - 1 - \sum_{F \subseteq E : (V, F) \approx K_{p, q}} (-1)^{p+1} + (-1)^{q+1}.$$  

We observe that the terms of the sum vanish when the parity of $p$ and $q$ differs. A term equals 2 if both $p$ and $q$ are odd, it equals -2 if both are even. 

**Corollary 9 (Brouwer, [BCS09])** The number of dominating sets of any finite graph is odd.
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