Asymptotically Exact Solution of the Fredrickson-Andersen Model

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(Dated: February 19, 2019)

The Fredrickson-Andersen (FA) model—a kinetically constrained lattice model—displays an ergodic to non-ergodic transition with a slow two-step relaxation of dynamical correlation functions close to the transition point. We derive an asymptotically exact solution for the dynamical occupation correlation function of the FA model on the Bethe lattice by identifying an exact expression for its memory kernel. The exact solution fulfills a scaling relation between critical exponents and allows to calculate the exponents explicitly. In addition, we propose an approximate dynamics that describes numerical data away from the critical point over many decades in time.

Slow relaxation is not restricted to molecular fluids dominated by pairwise interactions. On the contrary, systems abound where the effective dynamics is facilitated by the number of neighbors in a favorable state exceeding a threshold. Dynamic facilitation applies to opinion dynamics [1], voter models [2, 3], and infection spreading [4] but has also been used to understand the low temperature phase of magnetic alloys [5–7], granular compaction [8], and the jamming transition [9–10]. Most prominently it lies at the heart of the dynamic facilitation picture [11–16] of the glass transition [17–20]. The k-core decomposition can be framed as a dynamic facilitation problem [24] yielding, e.g., insight into the resilience of social network data sets against de-anonymization [25].

A paradigmatic example of a kinetically constraint model [15–26] implementing dynamic facilitation is the Fredrickson-Andersen (FA) model which is defined on a lattice with sites $i = 1, \ldots, N$ decorated with occupation numbers $n_i \in \{0, 1\}$. The Hamiltonian $\mathcal{H} = \mu \sum_i n_i$ is trivial and $\mu > 0$ favors the empty lattice. A site $i$ may, however, change its state if it has at least $f$ empty nearest neighbors [12–27]. Bootstrap percolation [6–25] is concerned with the ground state of the FA model which is kinetically reachable from an initial condition with an occupation probability $p$. For $p = 1$, clearly the occupation probability in the ground state $q = 1$, whereas for $p \approx 0$ an empty ground state, $q = 0$, can be reached almost surely. The question arises, if there is a nontrivial concentration, $p_c$, for the emergence of an infinite occupied cluster in the ground state, $q > 0$. For $f = 1$ and for arbitrary $f$ on hypercubic lattices $\mathbb{Z}^d$ it has been shown that $p_c = 1$ [27–31]. Bootstrap percolation on the Bethe lattice and on random graphs, however, feature a transition at a finite $p_c < 1$ [6, 22, 33].

At finite temperatures $T > 0$ [34] we equip the FA model with transition rates that satisfy detailed balance. Without constraints, the Hamiltonian $\mathcal{H}$ would entail an equilibrium mean occupation $\langle n_i \rangle = 1/(1 + e^{\mu/\beta})$. This still holds under the constrained dynamics as long as $\langle n_i \rangle < p_c$; however, for $\langle n_i \rangle > p_c$, the dynamics is restricted to the sites that are not permanently constrained by the frozen percolating cluster [35]. For $\langle n_i \rangle \neq p_c$, numerical simulations of the FA model [26–39] show a two-step relaxation of time-correlation functions, $\phi(t)$, with a fast relaxation to a plateau value, $\phi(t) \simeq q > 0$, followed by a second relaxation, $\phi(t) \rightarrow 0$, on a time scale that diverges towards $p_c$. A two-step relaxation with a divergent relaxation time is one of the experimental fingerprints of the glass transition [17] and motivated the FA model as an effective description of the glass transition. Close to the plateau, $|\phi(t) - q| \ll 1$, the relaxation is generically well described by power laws [40],

$$\phi(t) \sim \begin{cases} t^{-a} & \text{for } \phi(t) > q, \\ -t^b & \text{for } \phi(t) < q. \end{cases} \quad (1)$$

A complementary description of the glass transition, independent of the dynamic facilitation picture, is provided by the number of neighbors in a favorable state exceeding $\mu > 0$ favors the empty lattice. A site $i$ may, however, change its state if it has at least $f$ empty nearest neighbors [12–27]. Bootstrap percolation [6–25] is concerned with the ground state of the FA model which is kinetically reachable from an initial condition with an occupation probability $p$. For $p = 1$, clearly the occupation probability in the ground state $q = 1$, whereas for $p \approx 0$ an empty ground state, $q = 0$, can be reached almost surely. The question arises, if there is a nontrivial concentration, $p_c$, for the emergence of an infinite occupied cluster in the ground state, $q > 0$. For $f = 1$ and for arbitrary $f$ on hypercubic lattices $\mathbb{Z}^d$ it has been shown that $p_c = 1$ [27–31]. Bootstrap percolation on the Bethe lattice and on random graphs, however, feature a transition at a finite $p_c < 1$ [6, 22, 33].

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$$\phi(t) \sim \begin{cases} t^{-a} & \text{for } \phi(t) > q, \\ -t^b & \text{for } \phi(t) < q. \end{cases} \quad (1)$$

Arrows denote the orientation, filled (open) vertices denote occupied (empty) sites, and blue vertices are frozen while green may change. See Table I for values of the critical exponents.

**FIG. 1.** Generic occupation correlation $\phi(t)$ (blue) of the oriented FA model (OFA) close to the critical point, $\sigma \ll 1$, compared to the asymptotic scaling function $G_{\sigma}(t) \simeq \phi(t) - q_c$ [Eq. (8), red] and the critical laws $t^{-a}$ and $-t^b$ [Eq. (1), black]. The plateau $q_c$ (horizontal dashed line) is crossed, on a divergent timescale $t_\sigma \sim \sigma^{-\delta}$ and $G_{\sigma}(t)$ provides a faithful description in a divergent time window $\tau_0 \ll t \ll t'_c \sim \sigma^{-\gamma}$ beyond the microscopic relaxation time $\tau_0$, followed by exponential relaxation on a timescale $\tau_0 \sim t'_c$. Inset: Section of the OFA for coordination $k = 3$, and facilitation $f = 2$. Crossed arrows denote the orientation, filled (open) vertices denote occupied (empty) sites, and blue vertices are frozen while green may change.
by mode-coupling theory (mct) [41,43] which starts from the formally exact equation of motion

\[ \tau_0 \dot{\phi}(t) + \phi(t) + \int_0^t dt' m(t-t') \dot{\phi}(t') = 0, \tag{2} \]

where the dot denotes the time derivative and \( \tau_0^{-1} \) is the short time relaxation rate. The eponymous mode-coupling approximation (mca) expresses the unknown memory kernel \( m(t) \) by a polynomial in \( \phi(t) \). Standard MCT predicts a scaling relation,

\[ \Gamma^2(1-a) = \Gamma^2(1+b) \frac{\Gamma(1+2a)}{\Gamma(1+2b)}, \tag{3} \]

between the exponents in Eq. (1) involving the Euler Gamma-function. MCAs have been attempted for the FA model [44,48] starting with Fredrickson and Andersen [12] but were of limited success. In particular, MCT for the FA model has a tendency to predict spurious transitions [15]. Also other approaches did not capture the slow relaxation [49].

Recent numerical evidence, however has shown that despite these reservations, the scaling relation [4] seems to be verified in the FA model on the Bethe lattice modeled as a random regular graph (rrg) [36,39,50]. Proof for this surprising discovery is highly desired [27,51] but missing so far.

In this letter we derive an asymptotically exact solution of Eq. (2) for the FA model on the Bethe lattice. We show that Eq. (1) constitutes the lowest order in a series expansion of this solution and that the scaling relation (3) holds exactly. Encouraged by these results we propose an approximate, regularized memory kernel valid for all times. Comparing with numerical data far away from the critical point, we are able to describe the two step relaxation of the FA model over many decades in time.

**Model.**—We consider the oriented FA model (OFA) with facilitation parameter \( f \geq 2 \) on the Bethe lattice. To be precise, we define the Bethe lattice [52] as the infinite \( k \)-ary rooted tree [53], \( 2 \leq f < k \). In line with Sellitto’s numerical work [36,38] we assume Metropolis dynamics with transition rates \( w(n_i \rightarrow 1-n_i) = \exp[(n_i-1)/T]C_f(K_i) \). Here \( K_i \) denotes the set of children of site \( i \) and \( C_f : K_i \rightarrow \{0,1\} \) implements the kinetic constraint [54] (cf. Fig. 1).

For simplicity we aim to describe the relaxation to equilibrium from a well defined initial condition. Assume the initial \( n_i(0) \) are drawn from a Bernoulli distribution with \( p = 1/(1+e^{-1/T}) > 1/2 \) [55]. To assure ergodicity, we limit our discussion to \( p < p_c \).

**Percolation Transition.**—Recall that the probability, \( q \), that a site is occupied in the ground state can be given implicitly as [6]

\[ q = pQ(q) := \sum_{i=0}^{f-1} \binom{k}{i} q^i(1-q)^{f-i}. \tag{4} \]

Note that as \( p > q \), the largest real solution of Eq. (4) is physically relevant. Trivially, \( q = 0 \) is always a solution of Eq. (4). The critical probability \( p_c \) locates a bifurcation to additional solutions. Generically, Eq. (4) displays a fold bifurcation (Arnold’s type \( A_2 \) [56]) with a finite \( q_c \equiv q(p_c) > 0 \) and close to the transition \( q(p > p_c) = q_c \sim \sqrt{p - p_c} \).

**Equation of Motion.**—We wish to describe the single site occupation correlation function

\[ \phi(t) = \langle n_s(0)n_s(t) \rangle, \tag{5} \]

where \( n_s \) denotes the occupation number of an arbitrary but fixed site \( s \) and the average \( \langle \cdot \rangle \) is taken with respect to the initial distribution. Note that \( \phi(t) \) is normalized such that \( \phi(0) = p \) and \( \phi(t \rightarrow \infty) = q \).

For \( p < p_c \), the OFA is a Markov process obeying detailed balance. Hence standard techniques allow to give the time evolution of the distribution function in terms of an effective Hamiltonian \( H \) [57]. Applying a Mori projector, \( n_s \langle n_s^2 \rangle^{-1} n_s \), and rewriting the memory kernel in terms of its irreducible counterpart, \( m(t) \), yields Eq. (2) [58]. The rate \( \tau_0^{-1} \) can be calculated explicitly [59]. The memory kernel, however, is only known formally.

**Critical Dynamics.**—It is instructive to rewrite Eq. (2) in the Laplace domain, \( \hat{\phi}(z) = LT[\phi(t)] \) [60]. For \( z \ll \tau_0 \) one finds

\[ \frac{z \hat{\phi}(z)}{p + z \hat{\phi}(z)} = z \hat{m}(z). \tag{6} \]

In particular \( q/(p-q) = m(t \rightarrow \infty) \). Comparing this with Eq. (4) we arrive at our central result: Asymptotically the memory kernel of the OFA on the Bethe lattice is given exactly as

\[ m(t \rightarrow \infty) \equiv m(p,q) = pQ(q)/(p-q). \tag{7} \]

Sufficiently close to the critical point, \( \sigma := (p_c - p)/p_c \ll 1 \), we expect a growing window in time, centered around a diverging time scale \( t_\sigma \) where \( \phi(t) = q_c + G_\sigma(t/t_\sigma) \) such that \( G_\sigma(t) \) is small, \( |G_\sigma(t/t_\sigma)| \ll 1 \).
and slowly varying, $|\zeta \hat{G}_\sigma(\zeta)| \ll 1$, where $\zeta := z t_\sigma$. To this end we expand Eq. (6) around $q_c$ to lowest order in $G_\sigma$\cite{61},

$$\lambda \Delta t [G^2_\sigma(\zeta) + \zeta \hat{G}^2_\sigma(\zeta)] = -\sigma/\zeta p_c,$$  \hspace{1cm} (8)

where $\lambda := 1 + (p_c - q_c)^3 G^2_\sigma \Delta m(p_c, q_c)/p_c < 1$\cite{62}. As $\lambda$ can be calculated exactly for the OPA, the same holds for $G_\sigma(t)$. Eq. (8) can be solved by standard numerical techniques (cf. Fig. 1) but more information can be gained analytically.

At the critical point, $\sigma \equiv 0$, Eq. (8) is solved by $G_\sigma(t) \sim t^x$, provided $\lambda = \Gamma^2(1+x)/\Gamma(1+2x)$\cite{61}. Asymptotically, $G_\sigma(t \to \infty) \sim t^{-a}$, the smallest negative $x \equiv -a < 0$ will dominate. Away from the critical point, for finite $\sigma$, $G_\sigma(t) = \sqrt{\sigma} \tilde{g}(t)$ acquires a square-root dependence on $\sigma$. Eq. (8) still admits power law solutions, $g(t) \sim t^x$, iff the left hand side dominates over the right hand side. For the approach to the plateau, $t \to t_\sigma$, $g(t/t_\sigma) \sim (t/t_\sigma)^{-a}$, as long as $(t/t_\sigma)^{2a} \ll 1$. Matching $G_\sigma \tilde{g}(t/t_\sigma) = G_\sigma(t)$ for $t \to \infty$, yields, $t_\sigma \sim \sigma^{-b}$, where $\delta = 1/2a$\cite{61}. For $\lambda > 1/2$ and times $t > t_\sigma$, the decay away from the plateau is governed by the smallest positive $x \equiv b$, $g(t/t_\sigma) \sim - (t/t_\sigma)^b$, as soon as $g^2(t/t_\sigma) \gg 1$, i.e., dependent on $b < 1/2$ ($b > 1/2$) for times $(t/t_\sigma)^{2b} \gg 1$ ($t/t_\sigma \gg 1$).

For long times the validity of this law is limited by the slowly varying condition, $|G_\sigma(\zeta)| \sim \sqrt{\zeta} \ll 1$, i.e., for times $t \ll t_\sigma \sim \sigma^{-\gamma}$, where $\gamma := (1/2a) + (1/2b)$\cite{61}.

The above constitutes a precise statement of Eq. (1) for the OPA and proves that the scaling relation (3) holds exactly. We summarize the (numerically) exact results we obtain for the simplest model, $k = 3$ and $f = 2$ in Tab. 1.

Asymptotic Relaxation.— The asymptotic relaxation to zero, $\phi(t \to \infty) = 0$, on times $t/t_\sigma \gg 1$ is governed by a scaling function, $\phi(t \to \infty) = \tilde{\phi}(t/t_\sigma)$\cite{61}. For the OPA it has been shown that $\phi(t) \propto \exp(-t/t_\sigma)$ where $t_\sigma \sim \sigma^{-\gamma'}$ and the exponent $\gamma' \geq 2$ could only be bounded from below\cite{43}. In terms of the scaling function we find $\tau_{\phi} \sim t_\sigma$ and in particular the preceding analysis determines the exponent $\gamma' \equiv \gamma$ compatible with the bound.

Persistence Function.— The persistence function $\psi(t)$ yields the fraction of sites that have not changed their state since $t = 0$. Its asymptotic value, $\psi_{\infty} = \psi(t \to \infty)$, does not only include the persistently occupied sites but also a fraction of the empty sites that are permanently frozen, $\psi_{\infty}(q) = pQ(p) + (1-p)Q(p) = q/p$. Close to the plateau, $\psi_{\infty} = q_c/p_c$, we can expand

$$\psi(t \sim t_\sigma) \approx \psi_{\infty} + \delta t \psi_{\infty}(q_c) |\phi(t) - q_c|,$$  \hspace{1cm} (9)

i.e., $\psi(t) \simeq \psi_{\infty} + \sqrt{\sigma} \tilde{g}(t)/p_c$. In particular, the persistence function is governed by the same critical exponents and master function $g(t)$ that apply to $\phi(t)$.

The FA Model on Random Regular Graphs.— It is known that bootstrap percolation on the oriented and unoriented Bethe lattice of coordination $k + 1$\cite{53} as well as on random $(k+1)$-regular graphs\cite{52} have the same critical concentration $p_c$. Not much is known regarding the dynamic equivalence. Here we conjecture that due to the bifurcation dominating close to the critical point, the fact that the critical point does not change translates to dynamic equivalence close to $q_c$. For the unoriented FA model the expression for the persistence function, $\tilde{\psi}_{\infty}(q)$, is slightly more involved\cite{43}. Nevertheless, $\partial q \tilde{\psi}_{\infty}(q_c)$ is finite and therefore Eq. (9) applies and $\tilde{\psi}(t)$ is still governed by the critical exponents and scaling function of the OPA.

Simulations\cite{39,60,66} of the FA model are conveniently being performed on RRGs with a finite number of sites $N$. RRGs do not, however, admit an orientation. The effective system size is given by the size of the largest embedded tree $L = O(\log N)$\cite{55} which grows with $N$ but is still small even for $N \sim 2^{24}$. Therefore the existing numerical data is relatively far from the critical point. In addition the FA model is known to display strong finite size effects\cite{60} which, so far, have not been analyzed in detail for RRGs. As a consequence, the empirical critical exponents (Table I) are effective exponents and deviate from the analytical predictions. In the following we propose a memory kernel that allows us to solve Eq. (2) for all times and for appreciable distances $\sigma$ from the critical point relevant to the numerical data.

Approximate Memory Kernel.— To close Eq. (2) we propose to approximate the memory kernel by Eq. (7) for all times as $m(t) \approx \hat{m}(\phi(t)) \equiv m(p, \phi(t))$. Unfortunately this is not viable as $\hat{m}(\phi(t) \to p) \sim |p - \phi(t)|^{-1}$ diverges for small times, $\phi(t) \to 0 \to p$. In order to regularize the memory kernel we assume the divergent term, $|p - \phi(t)|^{-1} = p^{-1} \sum_i M_i |\phi(t)/p|^i$, $M \to \infty$, to be a resummation of many-site interactions. On a finite lat-
tice, the order of interactions should be finite, $M < \infty$. Therefore we propose a regularized approximate memory kernel whose time-dependence is completely determined by $\phi(t)$,

$$m(t) \equiv m[\phi(t)] = Q(\phi(t)) \sum_{i=0}^{M} [\phi(t)/|p|^i]. \quad (10)$$

With this Eq. (2) can be numerically solved for $\phi(t)$ by standard techniques [67].

**Discussion.**—Considering the occupation correlation function $\phi(t)$, Eq. (3), of the OFA on the Bethe lattice, we have identified an explicit expression, Eq. (7), for the long time limit of its memory kernel. Expanding around the bifurcation at $p_c$ that signals the ergodic to non-ergodic transition of the OFA, we find that close to the transition, $\sigma \to 0$, the time evolution of $\phi(t)$ around its plateau value $q_c$ is asymptotically exactly given, $\phi(t) = q_c + \sqrt{\sigma} g(t)$, in terms of a one-parameter scaling function $g(t) \equiv g(t_c)$, Eq. (8). The exponent parameter $\lambda \equiv \lambda(k, f)$ is known explicitly in terms of the lattice coordination $k$ and the facilitation parameter $f$. The properties of Eq. (8), finally, imply Eq. (1) together with the scaling relation (3), for $\lambda > 1/2$, times $\tau_0 \ll t \ll t'_0 \sim \sigma^{-\gamma}$, and not too close to the plateau, $\phi(t) - q_c \gg \sigma$.

The scaling function $G_{\sigma}(t)$, however, goes beyond Eq. (1) as it provides a faithful description of $\phi(t)$ for $\tau_0 \ll t \ll t'_0$, $\lambda > 0$, bounded only by the requirements $|G_{\sigma}(t)|, |\tilde{G}_{\sigma}(\zeta)| \ll 1$ (cf. Fig. 1). On the fast end this could be complemented by ever more sophisticated short-time expansions. On the long-time end, $\phi(t \to \infty) \sim e^{-t/\tau_0}$, with a relaxation time $\tau_0 \sim \sigma^{-\gamma}$.

Considering the asymptotic dynamics only, we did not gain information about processes on intermediate time scales. Could we have missed an additional process that will always mask the bifurcation scenario? The answer is no: Any unidentified process must occur on a time scale, $\tau_u$, that remains finite as $\sigma \to 0$. Otherwise it would contribute to Eq. (1). Therefore we can always find a $\sigma_0 > 0$ such that for $\sigma < \sigma_0$, $t_0 \sim \sigma^{-\delta} \gg \tau_0$ and we have a time window which is dominated by the bifurcation.

Given that close to the critical point the persistence function, $\psi(t)$, is governed by the same scaling function $G_{\sigma}(t)$, Eq. (1) provides an asymptotically exact description of the persistence function in a divergent time window before the asymptotic exponential relaxation. The form of Eq. (1) and the scaling relation (3) equally apply to $\psi(t)$ with the qualifications given above. Thereby we confirm the empirical observation of Sellitto [38] and de Candia et al. [39].

To close Eq. (2), we proposed a memory functional, Eq. (10), regularized by a finite length scale $M$ we conjecture to be related to the system size. Formally, Eq. (10) looks like a MCA but let us stress that it was not derived by considering a (physically motivated) coupling of modes, but ultimately from the bifurcation equation (4) of the underlying bootstrap percolation.

To determine the persistence function of the unoriented FA model for all times, we use the knowledge gained so far and interpolate

$$\tilde{\psi}(t) = e^{-t/\tau_0} + \left\{ \tilde{\psi}_\infty + \partial_\phi \tilde{\psi}_\infty(q_e)[\phi(t) - q_e] \right\} \times \left(1 - e^{-t/\tau_0}\right) e^{-t/\tau_0}, \quad (11)$$

where $\tau_0^{-1}$ is the short time relaxation rate of $\tilde{\psi}(t)$. We determine $\phi(t)$ by solving Eq. (2) with the regularized memory kernel [Eq. (10)] and treat $M$ as a fit parameter [68]. Fig. 2 shows excellent agreement between Eq. (11) and the numerical data for all temperatures and over many decades in time.

The success of this approach, derived for the OFA on the Bethe lattice, in describing simulations of the unoriented FA model on RRGs provides reasons to assume that the similarity between the oriented and unoriented FA model extends beyond a common critical point $p_c$ to a universal dynamics close to $p_c$. It is, however, obvious, that simulations much closer to the critical point are needed to challenge the conjectures put forward here and to confirm the critical exponents.

While for sake of brevity we have only presented explicit results for the simplest case, $k = 3, f = 2$, our approach holds for more general coordinations $k > 3$, and facilitation parameters $f \geq 2$ provided $k > f$. The consequences of a tunable $\lambda(k, f)$ will be discussed elsewhere, but let us note that for some combinations $(k, f), \lambda < \lambda(3, 2)$. As a result the exponents $a, b$ increase which may be favorable for simulations.

The signature of a fold bifurcation, $q - q_c \sim \sqrt{\sigma}$, with a finite critical $q_c > 0$ is observed as a hybrid phase transition in a variety of models [69, 71]. A similar analysis to the one introduced here could lead to new insights in those systems as well. Reconciling MCT and replica methods led to many new insights provided by the random first order theory (RFOT) [72]. A deeper analysis of the overlap between MCT and dynamic facilitation theory that has been started here and RFOT and dynamic facilitation [74] is likely to provide additional understanding of the glass transition.

In summary we have provided an asymptotically exact description of the slow relaxation of the oriented Fredrickson-Andersen model on the Bethe lattice close to its critical point, valid over a divergent window in time. We believe our method can be applied to other time correlation functions of the FA and related kinetically constraint models and can provide new insights into the phenomena which can be mapped onto these models.

We are indebted to Wolfgang Götze for posing the questions that led to this work. We thank Mauro Sellitto for sharing his simulation data and acknowledge additional insight from discussions with Thomas Franosch and Thomas Voigtmann. Partial funding was provided.
Fitted $M = 6–23$ increase with $\sigma \to 0$ as expected.

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