ON THE CRITICAL ONE-COMPONENT VELOCITY REGULARITY CRITERIA TO 3-D INCOMPRESSIBLE MHD SYSTEM

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Abstract. Let \((u, b)\) be a smooth enough solution of 3-D incompressible MHD system. We prove that if \((u, b)\) blows up at a finite time \(T^*\), then for any \(p \in [4, \infty[\), there holds
\[
\int_0^{T^*} \left( \|u(t')\|_p^p + \|b(t')\|_p^p \right) dt' = \infty.
\]
We remark that all these quantities are in the critical regularity of the MHD system.

1. Introduction

In this work, we investigate necessary conditions for the breakdown of regular solutions to the following 3-D incompressible Magnetohydrodynamics (MHD in short) system

\begin{equation}
\begin{cases}
\partial_t u + u \cdot \nabla u - b \cdot \nabla b + \nabla p = \Delta u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\
\partial_t b + u \cdot \nabla b - b \cdot \nabla u = \Delta b, \\
\text{div} u = \text{div} b = 0, \\
u|_{t=0} = u_0, b|_{t=0} = b_0,
\end{cases}
\end{equation}

(1.1)

where \(u, p\) denote the velocity and scalar pressure of the fluid respectively, and \(b\) denotes the magnetic field.

When the initial magnetic field \(b_0\) is identically zero, the system (1.1) reduces to the classical Navier-Stokes equations, the global regularity of which is still one of the biggest open questions in the field of mathematical fluid mechanics. Of course, the analogous problem for the MHD system remains just as difficult due to the coupling with the magnetic field.

This system has two major basic features. First of all, the total kinetic energy is conserved for smooth enough solutions of (1.1)

\begin{equation}
\frac{1}{2} \left( \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \right) + \int_0^t \left( \|\nabla u(t')\|_{L^2}^2 + \|\nabla b(t')\|_{L^2}^2 \right) dt' = \frac{1}{2} \left( \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \right).
\end{equation}

(1.2)

The second basic feature is the scaling invariance. Indeed, if \((u, b, p)\) is a solution of (1.1) on \([0, T] \times \mathbb{R}^3\), then \((u, b, p)_\lambda\) defined by

\begin{equation}
(u, b, p)_\lambda(t, x) \overset{\text{def}}{=} \left( \lambda u(\lambda^2 t, \lambda x), \lambda b(\lambda^2 t, \lambda x), \lambda^2 p(\lambda^2 t, \lambda x) \right)
\end{equation}

(1.3)

is also a solution of (1.1) on \([0, \lambda^{-2} T] \times \mathbb{R}^3\). This leads to the notion of critical regularity corresponding to the System (1.1).

Before Proceeding, let us set

\begin{equation}
\Omega \overset{\text{def}}{=} \nabla \times u, \quad j \overset{\text{def}}{=} \nabla \times b, \quad \omega \overset{\text{def}}{=} \Omega \cdot e^3, \quad d \overset{\text{def}}{=} j \cdot e^3 \quad \text{with} \quad e^3 = (0, 0, 1).
\end{equation}

(1.4)

Motivated by the critical one component criteria in [6] by Chemin and Zhang for the 3-D classical Navier-Stokes system, Yamazaki [9] proved the following regularity criteria for the System (1.1):
Theorem 1.1. Let \( \Omega_0, j_0 \in L^2_0(R^3) \). Then the MHD system (1.1) has a unique solution \((u, b)\) on \([0, T^\ast]\) such that \(u, b \in C([0, T^\ast]; \dot{H}^{\frac{3}{2}}(R^3)) \cap L^2_{loc}([0, T^\ast]; \dot{H}^{\frac{3}{2}}(R^3))\) and
\[
\sup_{t \in [0, T]} \left( \|\Omega(t)\|_{L^2}^3 + \|j(t)\|_{L^2}^3 \right) + \int_0^T \int_{R^3} \left( \|\nabla(\Omega + j)\|^2 \Omega + j - \frac{1}{2} + \|\nabla(\Omega - j)\|^2 |\Omega - j|^{-\frac{1}{2}} \right) dx dt' < \infty,
\]
(1.5)
for any \( T < T^\ast \). Moreover, for \( p \in ]4, 6[\), \( p_1 > 9, p_2 > \frac{9}{2} \), we denote
\[
\|b\|_{SC_{p, p_1, p_2}} \overset{\text{def}}{=} \|b\|^p_{\dot{H}^{\frac{3}{2} - \frac{1}{p}}} + \|b\|^r_{L^p} + \|\nabla b\|^r_{L^p}, \quad \text{with} \quad \frac{3}{p_1} + \frac{2}{r_1} = 1, \frac{3}{p_2} + \frac{2}{r_2} = 2.
\]
If \( T^\ast < \infty \), then
\[
\int_0^{T^\ast} \left( \|u^3(t)\|^p_{\dot{H}^{\frac{3}{2} + \frac{1}{p}}} + \|b(t')\|^p_{SC_{p, p_1, p_2}} \right) dt' = \infty.
\]
(1.6)
It is easy to check that when \( T^\ast = \infty \), the quantity (1.6) is scaling invariant under the scaling transformation (1.3).

The main result in [6] states that if \( u \) is a Fujita-Kato type solution to the classical Navier-Stokes system on \([0, T^\ast]\) and if \( T^\ast < \infty \), then (1.6) holds for \( p \in ]4, 6[\) with \( b = 0 \). Very recently, this result was extended by Chemin, Zhang and Zhang in [7] for \( p \in ]4, \infty[\). Corresponding to [7], the purpose of this work is to extend \( p \) in Theorem 1.1 to be in \([4, \infty[\) and to get rid of the terms \( \|b\|_{L^p}^r + \|\nabla b\|_{L^p}^r \) in (1.6) by using the symmetric structure of the MHD system (1.1). One may check [9] and the references therein for the other types of regularity criteria for the MHD system (see [3] for instance).

In all that follows, we consider initial data \((u_0, b_0)\) with \( \Omega_0, j_0 \in L^2_0(R^3) \) so that Theorem 1.1 always holds. We shall concentrate on the proof of the extended regularity criterion. In order to do so, let us recall the following family of spaces from [7].

Definition 1.1. For \( r \in [3/2, 2] \), we denote by \( V^r \) the space of divergence free vector fields with the vorticity of which belongs to \( L^2 \cap L^r \).

Let us remark that, if we denote
\[
\alpha(r) \overset{\text{def}}{=} \frac{1}{r} - \frac{1}{2},
\]
the dual Sobolev embedding \( L^r \hookrightarrow \dot{H}^{-3\alpha(r)} \) together with Biot-Savart’s law implies that \( V^r \) is a subspace of \( \dot{H}^{\frac{1}{2}} \cap \dot{H}^{1 - 3\alpha(r)} \).

Our main result states as follows:

Theorem 1.2. Let us consider initial data \( u_0, b_0 \in V^2 \). If the lifespan \( T^\ast \) of the unique maximal solution \((u, b)\) given by Theorem 1.1 is finite, then for any \( p \in ]4, \infty[\), we have
\[
\int_0^{T^\ast} \left( \|u^3(t')\|^p_{\dot{H}^{\frac{3}{2} + \frac{1}{p}}} + \|b(t')\|^p_{SC_{p, p_1, p_2}} \right) dt' = \infty.
\]
(1.8)
Let us complete this introduction by the notations we shall use in the whole text.

Let \( A, B \) be two operators, we denote \([A; B] = AB - BA\), the commutator between \( A \) and \( B \). For \( a \lesssim b \), we mean that there is a uniform constant \( C \), which may be different on different lines, such that \( a \leq Cb \), and \( a \sim b \) means that both \( a \lesssim b \) and \( b \lesssim a \) hold. \( C \) stands for some universal positive constant which may change from line to line and \( C_0 \) denotes a.
positive constant depending on the initial data only. For a Banach space \( B \), we shall use the shorthand \( L^p_t(B) \) for \( L^p([0, t]; B) \).

2. Scheme of the Proof and the Organization of the Paper.

In fact, we shall prove the following more general version of Theorem 1.2:

**Theorem 2.1.** Let \( r \in [3/2, 2[ \) and \( u_0, b_0 \in \mathcal{V}^r \). If the lifespan \( T^* \) of the unique maximal solution \((u, b)\) given by Theorem 1.1 is finite, then for any \( p \in ]4, \frac{2r}{2-r}[, \) we have

\[
\int_0^{T^*} \left( \|u^3(t')\|_{\dot{H}^{\frac{1}{2}+\frac{r}{4}}}^p + \|b(t')\|_{\dot{H}^{\frac{1}{2}+\frac{r}{4}}}^p \right) dt' = \infty.
\]

The main idea of the proof here basically follow from [6, 7, 9]. We first recall some important definitions and notations. Let

\[
\nabla_h \text{ def } = (-\partial_2, \partial_1), \quad \text{and} \quad \Delta_h \text{ def } = \partial_1^2 + \partial_2^2,
\]

for any \( f = (f^h, f^3) \) with \( \text{div} f = 0 \), we write

\[
f^h = f^h_{\text{curl}} + f^h_{\text{div}} \quad \text{with} \quad f^h_{\text{curl}} \text{ def } = \nabla_h^{-1} \Delta_h^{-1} (\nabla \times f) \cdot e^3, \quad f^h_{\text{div}} \text{ def } = -\nabla_h \Delta_h^{-1} \partial_3 f^3.
\]

This is sort of Hodge decomposition for the horizontal variables, and we emphasize that this is a key identity to be used frequently in what follows. Moreover, because of the operator \( \nabla_h \Delta_h^{-1} \), it is naturally to measure horizontal derivatives and vertical derivatives differently. This leads to the following definition of the anisotropic Sobolev spaces.

**Definition 2.1** (Definition 2.1 of [6, 7]). For \((s, s')\) in \( \mathbb{R}^2 \), \( \dot{H}^{s,s'} \) denotes the space of tempered distribution a such that

\[
\|a\|_{\dot{H}^{s,s'}}^2 \text{ def } = \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_3|^{2s'} |\hat{a}(\xi)|^2 d\xi < \infty \quad \text{with} \quad \xi_h = (\xi_1, \xi_2).
\]

For \( \alpha(r) \) given by (1.7) and \( \theta \in ]0, 3\alpha(r)[ \), we denote \( \mathcal{H}^{\theta,r} \text{ def } = \dot{H}^{3\alpha(r) + \theta, -\theta} \).

Then it follows from (2.7) of [7] that

\[
\|\partial_3 u^3\|_{\mathcal{H}^{\theta,r}} \lesssim \|u\|_{\dot{H}^{1-3\alpha(r)}}.
\]

To use the space efficiently in the proof, we need to rely on anisotropic Littlewood-Paley theory and also anisotropic Besov spaces. These will be done in the following section.

The first step to prove Theorem 2.1 is the following proposition:

**Proposition 2.1.** Under the hypothesis of Theorem 2.1, for any \( p \in ]4, \frac{2r}{2-r}[ \), \( \theta \in ]0, \alpha(r)[ \), a constant \( C \) exists such that for any \( t < T^* \), we have

\[
\frac{1}{r} (\| (\Gamma_+)_{\xi}^2(t) \|_{L^2}^2 + \| (\Gamma_-)_{\xi}^2(t) \|_{L^2}^2 ) \\
+ \left( \frac{2(r-1)}{r^2} \int_0^t (\| \nabla (\Gamma_+)_{\xi}(t') \|_{L^2}^2 + \| \nabla (\Gamma_-)_{\xi}(t') \|_{L^2}^2 ) dt' \right) \\
\lesssim \left( \frac{2}{r} (\| \omega_0 \|_{L^r} + \| d_0 \|_{L^r} ) \\
+ \left( \int_0^t \| \nabla \partial_3 V_+(t') \|_{\mathcal{H}^{\theta,r}}^2 + \| \nabla \partial_3 V_-(t') \|_{\mathcal{H}^{\theta,r}}^2 dt' \right)^{\frac{1}{2}} \right) \cdot \mathcal{E}(t).
\]
Here and in all that follows, we always denote
\[ \Gamma_+ \overset{\text{def}}{=} \omega + d, \quad \Gamma_- \overset{\text{def}}{=} \omega - d, \quad V_+ \overset{\text{def}}{=} u^2 + b^2, \quad V_- \overset{\text{def}}{=} u^2 - b^2 \quad \text{and} \]
\[ a_\alpha \overset{\text{def}}{=} \frac{a}{|a|^\alpha}, \quad \mathcal{E}(t) \overset{\text{def}}{=} \exp\left( C \int_0^t \left( \|u^3(t')\|_H^{\frac{3}{2}+\frac{3}{p}} + \|b(t')\|_H^{\frac{1}{2}+\frac{3}{p}} \right) dt' \right). \]
for scalar function \( a \) and \( \alpha \in [0, 1] \).

To prove this proposition, we need to use the structures of the equations for \( \omega \) and \( d \), namely (4.1). The quadratic terms \( u_{\text{curl}} \cdot \nabla \omega \) and \( u_{\text{curl}} \cdot \nabla d \) look dangerous. As in [6, 7, 9], a way to get rid of it is to use an energy type estimate and the divergence-free condition. Here shall we perform an \( L^r \) energy estimate for \( \omega \) and \( d \) based on the following lemma.

**Lemma 2.1** (Lemma 3.1 of [6]). Let \( r \in [1, 2[ \) and \( a_0 \) a function in \( L^r \). Let us consider a function \( f \) in \( L^1_{t, \text{loc}}(\mathbb{R}^+; \mathbb{R}^r) \) and \( v \) a divergence free vector field in \( L^2_{t, \text{loc}}(\mathbb{R}^+; L^\infty) \). If a solves
\[
\begin{cases}
\partial_t a - \Delta a + v \cdot \nabla a = f \\
a|_{t=0} = a_0,
\end{cases}
\]

then \( |a|^\frac{r}{2} \) belongs to \( L^\infty_{t, \text{loc}}(\mathbb{R}^+; L^2) \cap L^2_{t, \text{loc}}(\mathbb{R}^+; H^1) \) and
\[
\frac{1}{r} \int_{\mathbb{R}^3} |a(t, x)|^r dx + (r - 1) \int_0^t \int_{\mathbb{R}^3} |\nabla a(t', x)|^2 |a(t', x)|^{r-2} dx dt' 
= \frac{1}{r} \int_{\mathbb{R}^3} |a_0(x)|^r dx + \int_0^t \int_{\mathbb{R}^3} f(t', x) a(t', x) |a(t', x)|^{r-2} dx dt'.
\]

The proof of Proposition 2.1 is the purpose of the fourth section.

We remark that for the MHD system (1.1), additional difficulty arises in the estimate of \( \|\nabla \partial_3 V_+\|_{L^2_t(H^{\theta, r})} + \|\nabla \partial_3 V_-\|_{L^2_t(H^{\theta, r})} \) due to the appearance of terms like \( 2(\partial_1 u^h - \partial_3 u^h, \partial_1 u^h) \) in right-hand side of (4.1). This is the purpose of the next proposition.

**Proposition 2.2.** Under the hypothesis of Theorem 2.1, for any \( p \in \left] 4, \frac{2r}{r-\theta} \right[ \), \( \theta \in \left] 3\alpha(r) - \frac{2}{p}, \alpha(r) \right[ \), a constant \( C \) exists such that for any \( t < T^* \), we have
\[
\|\nabla \partial_3 V_+(t)\|_{L^2_t(H^{\theta, r})}^2 + \|\nabla \partial_3 V_-(t)\|_{L^2_t(H^{\theta, r})}^2 + \int_0^t \left( \|\nabla \partial_3 V_+(t')\|_{H^{\theta, r}}^2 + \|\nabla \partial_3 V_-(t')\|_{H^{\theta, r}}^2 \right) dt' 
\leq \left( \|\Omega_0\|_{L^r}^2 + \|j_0\|_{L^r}^2 + \int_0^t \left( \|u^3\|_{H^{\frac{3}{2}+\frac{3}{p}}}^2 + \|b^3\|_{\frac{1}{2}+\frac{3}{p}}^2 \right) \right) \times \right.
\]
\[
\left. \left( \|\nabla (\Gamma_+)\|_{L^2}^{2(2\alpha(r)+\frac{1}{p})} + \|\nabla (\Gamma_-)\|_{L^2}^{2(1-\frac{1}{p})} \right) + \left( \|u^3\|_{H^{\frac{3}{2}+\frac{3}{p}}}^2 + \|b^3\|_{\frac{1}{2}+\frac{3}{p}}^2 \right) \left( \|\nabla (\Gamma_+)\|_{L^2}^{2(1-\frac{1}{p})} + \|\nabla (\Gamma_-)\|_{L^2}^{2(1-\frac{1}{p})} \right) \right) \frac{1}{\mathcal{E}(t)}. \]

The proof of Proposition 2.2 is the purpose of the fifth section.

Finally we close the estimates by the following proposition:

**Proposition 2.3.** Under the hypothesis of Theorem 2.1, for any \( p \in \left] 4, \frac{2r}{r-\theta} \right[ \), \( \theta \in \left] 3\alpha(r) - \frac{2}{p}, \alpha(r) \right[ \), a constant \( C \) exists such that for any \( t < T^* \), we have
\[
\|\nabla (\Gamma_+)\|_{L^2_t(L^r)}^{2(1+2\alpha(r))} + \|\nabla (\Gamma_-)\|_{L^2_t(L^r)}^{2(1+2\alpha(r))} + \|\nabla (\Gamma_+)\|_{L^2_t(L^r)}^{2(1+2\alpha(r))} 
\leq \left( \|\Omega_0\|_{L^r}^{r(1+2\alpha(r))} + \|j_0\|_{L^r}^{r(1+2\alpha(r))} \right) \exp(C \mathcal{E}(t)).
\]
\[ \| \partial_3 V_+(t) \|^2_{L^{\theta_1},r} + \| \partial_3 V_-(t) \|^2_{L^{\theta_1},r} \]
\[ + \int_0^t (\| \nabla \partial_3 V_+(t') \|^2_{L^{\theta_1},r} + \| \nabla \partial_3 V_-(t') \|^2_{L^{\theta_1},r}) \, dt \]
\[ \leq (\| \Omega_0 \|^2_{L^r} + \| j_0 \|^2_{L^r}) \exp (CE(t)). \]

The proof of Proposition 2.3 is the purpose of the sixth section.

Now we have controls on the quantities
\[ \sup_{t \in [0,T^*]} \| (\Gamma_+)(t) \|_{L^r}, \int_0^{T^*} \| \nabla (\Gamma_+)(t) \|^2_{L^2} \, dt', \int_0^{T^*} \| \nabla \partial_3 V_+ \|^2_{L^{\theta_1},r} \, dt' \]
\[ \sup_{t \in [0,T^*]} \| (\Gamma_-)(t) \|_{L^r}, \int_0^{T^*} \| \nabla (\Gamma_-)(t) \|^2_{L^2} \, dt', \int_0^{T^*} \| \nabla \partial_3 V_- \|^2_{L^{\theta_1},r} \, dt'. \]

We want to prove that all the above quantities prevent the solution of (1.1) from blowing up. The details will be presented in the last section.

3. Preliminaries

In this section, we first recall some basic facts on anisotropic Littlewood-Paley theory from [1, 4, 8], and then we collect some interesting estimates from [6, 7] that will be used later on.

3.1. Basic facts on Littlewood-Paley theory. Let \( \mathcal{C} \) define \( \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \} \) and \( \mathcal{B} \) define \( \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3} \} \). There exist two radial functions \( \chi \in \mathcal{D}(\mathcal{B}) \) and \( \varphi \in \mathcal{D}(\mathcal{C}) \) such that
\[ \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \ \forall \xi \in \mathbb{R}^d \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \ \forall \xi \in \mathbb{R}^d \setminus \{0\}. \]

For every \( a \in \mathcal{S}'(\mathbb{R}^3) \), we recall the dyadic operator for both isentropic and anisotropic version
\[ \Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j} |\xi|) \hat{a}), \quad \dot{S}_j a = \mathcal{F}^{-1}(\chi(2^{-j} |\xi|) \hat{a}), \]
\[ \Delta_k^a a = \mathcal{F}^{-1}(\varphi(2^{-k} |\xi_h|) \hat{a}), \quad \dot{S}_k^a a = \mathcal{F}^{-1}(\chi(2^{-k} |\xi_h|) \hat{a}), \]
\[ \Delta_j^a a = \mathcal{F}^{-1}(\varphi(2^{-j} |\xi_3|) \hat{a}), \quad \dot{S}_j^a a = \mathcal{F}^{-1}(\chi(2^{-j} |\xi_3|) \hat{a}), \]
where \( \xi_h = (\xi_1, \xi_2) \), \( \mathcal{F} a \) and \( \hat{a} \) denote the Fourier transform of \( a \).

Moreover, it is easy to verify that for any \( u \) in \( \mathcal{S}_h' \), which means that \( u \) belongs to \( \mathcal{S}' \) and satisfies \( \lim_{j \to -\infty} \| \dot{S}_j u \|_{L^\infty} = 0 \), there holds \( u = \sum_{j \in \mathbb{Z}} \Delta_j u \).

Let us recall the homogeneous isentropic Besov space from [1].

Definition 3.1. Let \( 1 \leq p, r \leq +\infty \) and \( s \in \mathbb{R} \). For any \( u \) in \( \mathcal{S}_{h}^s(\mathbb{R}^3) \), we set
\[ \| u \|_{\dot{B}^s_{p,r}} \overset{\text{def}}{=} \| (2^j s) \Delta_j u \|_{L^r}(\mathbb{Z}). \]

- For \( s < \frac{3}{p} \) (or \( s = \frac{3}{p} \) if \( r = 1 \)), we define \( \dot{B}^s_{p,r}(\mathbb{R}^3) \) define \( \{ u \in \mathcal{S}_{h}^s(\mathbb{R}^3) : \| u \|_{\dot{B}^s_{p,r}} < \infty \} \).
- If there exists some positive integer \( k \) such that \( \frac{3}{p} + k \leq s < \frac{3}{p} + k + 1 \) (or \( s = \frac{3}{p} + k + 1 \) if \( r = 1 \)), then we define \( \dot{B}^s_{p,r}(\mathbb{R}^3) \) as the subset of distributions \( u \) in \( \mathcal{S}_{h}^s(\mathbb{R}^3) \) such that \( \partial_3 u \) belongs to \( \dot{B}^{s-k-1}_{p,r} \) whenever \( |\beta| = k + 1 \).
We remark that in particular, $B_{2,2}^{s_2}$ coincides with the classical homogeneous Sobolev space $H^s$. Similarly, we can also define the homogeneous anisotropic Besov space.

**Definition 3.2.** Let us define the homogeneous anisotropic Besov space $(\tilde{B}_{p,q}^{s_1})_{h}(\tilde{B}_{p,q}^{s_2})_{v}$ as the subspace of distributions $u$ in $S'_h(\mathbb{R}^d)$ such that

$$
\|u\|_{(\tilde{B}_{p,q}^{s_1})_{h}(\tilde{B}_{p,q}^{s_2})_{v}} \overset{def}{=} \left( \sum_{k \in \mathbb{Z}} 2^{q_1k} \left( \sum_{\ell \in \mathbb{Z}} 2^{q_2\ell s_2} \|\Delta_k^h \Delta_\ell^v u\|_{L_p^s}^{q_1/2} \right)^{2/q_1} \right)^{1/2}
$$

is finite.

We remark that in particular, $(\tilde{B}_{2,2}^{s_1})_{h}(\tilde{B}_{2,2}^{s_2})_{v}$ coincides with the homogeneous anisotropic Sobolev space $H^{s_1,s_2}$, and thus the space $(\tilde{B}_{2,2}^{-3\alpha(r)+\theta})_{h}(\tilde{B}_{2,2}^{-\theta})_{v}$ is the space $\mathcal{H}^{\theta,r}$ given by Definition 2.1. Let us also remark that in the case when $q_1$ is different from $q_2$, the order of summation is important.

By virtue of the above definitions, one has

**Lemma 3.1** (Lemma 4.3 of [6]). For any $s > 0$ and any $\theta \in ]0, s[$, we have

$$
(3.2) \quad \|f\|_{(\tilde{B}_{p,q}^{s_1})_{h}(\tilde{B}_{p,q}^{s_2})_{v}} \lesssim \|f\|_{\tilde{B}_{p,q}^{s}}.
$$

We also recall the following Bernstein type lemmas:

**Lemma 3.2** (Isentropic version, see [1]). Let $C$ be an annulus and $B$ a ball of $\mathbb{R}^3$. Then for any nonnegative integer $N$, and $1 \leq p \leq q \leq \infty$, we have

$$
\text{Supp } \hat{a} \subset \lambda C \implies \|D^N a\|_{L^q} \overset{def}{=} \sup_{|\alpha| = N} \|\partial^\alpha a\|_{L^q} \lesssim \lambda^{N+3(\frac{1}{p} - \frac{1}{q})} \|a\|_{L^p},
$$

$$
\text{Supp } \hat{a} \subset \lambda C \implies \|D^N a\|_{L^p} \lesssim \|D^N a\|_{L^p} \lesssim \lambda^N \|a\|_{L^p}.
$$

**Lemma 3.3** (Anisotropic version, see [4, 8]). Let $C_h$ (resp. $C_v$) be an annulus of $\mathbb{R}^3_h$ (resp. $\mathbb{R}^3_v$), and $B_h$ (resp. $B_v$) a ball of $\mathbb{R}^3_h$ (resp. $\mathbb{R}^3_v$). Then for any nonnegative integer $N$, and $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$, we have

$$
\text{Supp } \hat{a} \subset \lambda C_h \implies \|\partial_\alpha^3 a\|_{L^3_{h}} \lesssim \lambda^{N+3(\frac{1}{p_1} - \frac{1}{q_1})} \|a\|_{L^3_{h}},
$$

$$
\text{Supp } \hat{a} \subset \lambda C_v \implies \|\partial_\alpha^3 a\|_{L^3_{v}} \lesssim \lambda^{N+3(\frac{1}{p_1} - \frac{1}{q_1})} \|a\|_{L^3_{v}},
$$

$$
\text{Supp } \hat{a} \subset \lambda C_h \implies \|a\|_{L^3_{h}} \lesssim \lambda^{-N} \|a\|_{L^3_{h}},
$$

As a corollary of Lemma 3.3, for any $1 \leq p_2 \leq p_1 \leq \infty$, we have

$$
\|a\|_{(\tilde{B}_{p_1,q_1}^{s_1})_{h}(\tilde{B}_{p_2,q_2}^{s_2})_{v}} \lesssim \|a\|_{(\tilde{B}_{p_1,q_1}^{s_1})_{h}(\tilde{B}_{p_2,q_2}^{s_2})_{v}}.
$$

### 3.2. Some technical inequalities.

For the convenience of the readers, we recall some inequalities from [6, 7] that will be used in what follows.

**Lemma 3.4** (Lemma 3.1 of [7]). For $r$ in $[3/2, 2]$, we have

$$
(3.3) \quad \|\nabla a\|_{L^r} \lesssim \|\nabla a\|_{L^2}^{3/2} \|a\|_{L^2}^{1/2}.
$$

Moreover, for $s$ in $[-3\alpha(r), 1 - \alpha(r)]$, we have

$$
(3.4) \quad \|a\|_{\tilde{H}^s} \lesssim \|\nabla a\|_{L^2}^{3\alpha(r)+s} \|a\|_{L^2}^{1-\alpha(r)-s}.
$$
**Lemma 3.5** (Proposition 3.1 of [7]). Let $u$ be a divergence-free vector field. For $\theta \in [0, 3\alpha(r)]$ and $\beta \in [0, 1/2]$, we have

\[
\|u^h\|_{(B_{2,1}^{1-3\alpha(r) - \beta})_0} \lesssim \|\omega_2\|_{L^2}^{2\alpha(r) + \beta} + \|\nabla \omega_2\|_{L^2}^{1-\beta} + \|\partial_3 u^3\|_{\mathcal{H}^{\theta, r}} + \|\nabla \partial_3 u^3\|_{\mathcal{H}^{\theta, r}}^{1-\beta}.
\]

It is easy to observe that the proof of Lemma 5.2 in [7] implies the following inequality:

**Lemma 3.6.** Let $f = (f^1, f^2, f^3), g = (g^1, g^2, g^3)$ and $f^h = (f^1, f^2)$. Then for any $p$ in $]4, \frac{2p}{2-p}[\text{ and any } \theta \in ]3\alpha(r) - 2/p, \alpha(r)[\text{, we have}

\[
\|(f^h \cdot \nabla_h \partial_3 g^3)\|_{\mathcal{H}^{\theta, r}} \lesssim \|g^3\|_{H^{\frac{\theta + \frac{3}{2}}{2}}(\Omega)} \left( \|\nabla_h f^h\|_{H^{\frac{\theta - \frac{3}{2}}{2} - 3\alpha(r) + \theta + \frac{1}{2} - \theta}^2} \right)
\]

\[
+ \|\partial_3 g^3\|_{H^{\frac{\theta - \frac{3}{2}}{2} - 3\alpha(r) + \theta + \frac{1}{2} - \theta}} + \|f^h\|_{(B_{2,1}^{1-3\alpha(r) - \frac{3}{2}})_{\theta}} \|\nabla \partial_3 g^3\|_{\mathcal{H}^{\theta, r}}.
\]

4. PROOF OF PROPOSITION 2.1

The purpose of this section is to present the proof of Proposition 2.1. Note that $\omega = \partial_1 u^2 - \partial_2 u^1$, $d = \partial_1 b^2 - \partial_2 b^1$, then it follows from (1.1) that

\[
\partial_t \omega + u \cdot \nabla \omega - b \cdot \nabla d - \Delta \omega = \partial_1 u \cdot \nabla u^2 - \partial_2 u \cdot \nabla u^1 - \partial_1 b \cdot \nabla b^2 + \partial_2 b \cdot \nabla b^1,
\]

\[
\partial_t d + u \cdot \nabla d - b \cdot \nabla - \Delta d = \partial_1 u \cdot \nabla b^2 - \partial_2 u \cdot \nabla b^1 - \partial_1 b \cdot \nabla u^2 + \partial_2 b \cdot \nabla u^1,
\]

from which, and div $u = \text{div} b = 0$, we deduce

\[
\begin{cases}
\partial_t \omega + u \cdot \nabla \omega - b \cdot \nabla d - \Delta \omega = (\partial_3 u^3 \omega + \partial_2 u^3 \partial_3 u^1 - \partial_1 u^3 \partial_3 u^2)
\end{cases}
\]

\[
- (\partial_3 b^3 d + \partial_2 b^3 \partial_3 b^1 - \partial_1 b^3 \partial_3 b^2),
\]

\[
\begin{cases}
\partial_t d + u \cdot \nabla d - b \cdot \nabla - \Delta d = \partial_3 u^3 d - \partial_3 b^3 \omega - \partial_1 u^3 \partial_3 b^2 + \partial_2 u^3 \partial_3 b^1
\end{cases}
\]

\[
+ \partial_1 b^3 \partial_3 u^2 - \partial_2 b^3 \partial_3 u^1 + 2(\partial_1 b^h \cdot \partial_2 u^h - \partial_2 b^h \cdot \partial_1 u^h).
\]

Summing up these two equations gives

\[
\partial_1 \Gamma_+ + u \cdot \nabla \Gamma_+ - b \cdot \nabla \Gamma_+ - \Delta \Gamma_+ = \partial_3 \nabla - \Gamma_+
\]

\[
- \partial_1 V_\pm \partial_3 (u^2 + b^2) + \partial_2 V_\pm \partial_3 (u^1 + b^1) + 2(\partial_1 b^h \cdot \partial_2 u^h - \partial_2 b^h \cdot \partial_1 u^h).
\]

Since div $u = \text{div} b = 0$, we get, by applying Lemma 2.1, that

\[
\frac{1}{r} \|(\Gamma_+)/(r)\|_{L^2}^2 + \frac{4(r - 1)}{r^2} \int_0^t \|\nabla (\Gamma_+)/(r)\|_{L^2} dt' = \frac{1}{r} \|(\omega_0 + d_0)\|_{L^r}^r + \sum_{i=1}^3 I_i,
\]

where

\[
I_1 = \int_0^t \int \partial_3 V_- \nabla \cdot (\Gamma_+)_{r-1} dx dt',
\]

\[
I_2 = \int_0^t \int (\partial_1 V_\pm \partial_3 (u^2 + b^2) + \partial_2 V_\pm \partial_3 (u^1 + b^1))(\Gamma_+)_{r-1} dx dt',
\]

\[
I_3 = 2\int_0^t \int (\partial_1 b^h \cdot \partial_2 u^h - \partial_2 b^h \cdot \partial_1 u^h)(\Gamma_+)_{r-1} dx dt'.
\]
We first get, by using integrating by parts, that
\[
|I_1| \leq r \int_0^t \int |V_-| \| \partial_3 \Gamma_+ + |\Gamma_+|^{r-1} dx dt'
\]
\[
= r \int_0^t \int |V_-| \| \partial_3 \Gamma_+ + |(\Gamma_+)|^\frac{2}{r} dx dt'
\]
\[
\leq r \int_0^t \left( \| u^2 \|_{L^{\frac{2p}{p-2}}} + \| b^3 \|_{L^{\frac{2p}{p-2}}} \right) \| \partial_3 \Gamma_+ + |(\Gamma_+)|^\frac{2}{r} dx dt',
\]
where \( r' \) denotes the conjugate index of \( r \) so that \( \frac{1}{r} + \frac{1}{r'} = 1 \). As \( p \in ]4, \frac{2p}{r}[ \), we have that \( r' \cdot \frac{p-2}{2p} \in ]0, 1[ \), then Sobolev embedding and interpolation inequality imply that
\[
\| (\Gamma_+)_r \|_{L^{\frac{2p}{p-2}+r}} \leq \| (\Gamma_+)_r \|_{L^{\frac{2p}{p-2}}} \leq \| (\Gamma_+)_r \|_{L^{\frac{2p}{p-2}+r}} \| \nabla (\Gamma_+)_r \|_{L^{\frac{2p}{p-2}+r}},
\]
from which, \( H^{\frac{1}{2}+\frac{p}{r}}(\mathbb{R}^3) \rightarrow L^{\frac{2p}{p-2}}(\mathbb{R}^3) \) and (3.3) of Lemma 3.4, we infer
\[
|I_1| \leq \int_0^t \left( \| u^2 \|_{H^{\frac{1}{2}+\frac{p}{r}}} + \| b^3 \|_{H^{\frac{1}{2}+\frac{p}{r}}} \right) \| (\Gamma_+)_r \|_{L^{\frac{2}{r}}} \| \nabla (\Gamma_+)_r \|_{L^{\frac{2}{r}}} dt'.
\]
Applying Young’s inequality, we obtain
\[
(4.5) \quad |I_1| \leq \frac{r-1}{r^2} \int_0^t \| \nabla (\Gamma_+)_r \|_{L^2}^2 dt' + C \int_0^t \left( \| u^2 \|_{H^{\frac{1}{2}+\frac{p}{r}}} + \| b^3 \|_{H^{\frac{1}{2}+\frac{p}{r}}} \right) \| (\Gamma_+)_r \|_{L^2}^2 dt'.
\]

In order to deal with \( I_2 \) and \( I_3 \), we need the following lemma:

**Lemma 4.1** (Lemma 4.1 of [7]). Let \( \theta \in ]0, \alpha(r), r \in ]r'/4, 1[ \), and \( s = \frac{1}{2} + 1 - \frac{2p}{r} \). Then
\[
\left| \int_{\mathbb{R}^3} \partial_h \Delta_h^{-1} f \cdot \partial_h g \cdot h_{r-1} dx \right| \leq \min \left\{ \| f \|_{L^r}, \| f \|_{H^{\theta,r}} \right\} \| g \|_{H^s} \| h^\frac{s}{2} \|_{H^s}.
\]

Next, we estimate \( I_2 \). We first write by (2.2)
\[
(4.7) \quad I_2 = I_{2,1} + I_{2,2},
\]
where
\[
I_{2,1} \overset{\text{def}}{=} - \int_0^t \int \left( \partial_1 V_- \partial_3 \partial_1 \Delta_h^{-1} \Gamma_+ + \partial_2 V_- \partial_3 \partial_2 \Delta_h^{-1} \Gamma_+ \right) \| \Gamma_+ \|_{r-1} dx dt',
\]
\[
I_{2,2} \overset{\text{def}}{=} \int_0^t \int \left( \partial_1 V_- \partial_3 \Delta_h^{-1} \partial_3^2 V_- - \partial_2 V_- \partial_3 \Delta_h^{-1} \partial_3^2 V_+ \right) \| \Gamma_+ \|_{r-1} dx dt'.
\]

Applying Lemma 4.1 with \( f = \partial_3 \Gamma_+, g = V_- \), \( h = \Gamma_+ \), Gagliardo-Nirenberg inequality and (3.3), we get
\[
|I_{2,1}| \leq \int_0^t \| \partial_3 \Gamma_+ \|_{L^r} \| V_- \|_{H^{\frac{1}{2}+\frac{p}{r}}} \| (\Gamma_+)_r \|_{L^2}^\frac{2}{r} \| \nabla (\Gamma_+)_r \|_{L^2} dt'
\]
\[
\leq \int_0^t \| \nabla (\Gamma_+)_r \|_{L^2} \| (\Gamma_+)_r \|_{L^2}^\frac{2}{2-1} \| \nabla (\Gamma_+)_r \|_{L^2}^\frac{2}{2-1} \| \nabla (\Gamma_+)_r \|_{L^2}^\frac{2}{2-1} \| \nabla (\Gamma_+)_r \|_{L^2} \| (\Gamma_+)_r \|_{L^2}^\frac{2}{2-1} dt'
\]
\[
\leq \int_0^t \| V_- \|_{H^{\frac{1}{2}+\frac{p}{r}}} \| (\Gamma_+)_r \|_{L^2}^\frac{2}{2-1} \| \nabla (\Gamma_+)_r \|_{L^2} \| (\Gamma_+)_r \|_{L^2}^\frac{2}{2-1} dt'.
\]
Choosing \( \sigma = \frac{(p-2)r'}{3p} \), which is between \( \frac{r'}{4} \) and 1 since \( p \in ]4, \frac{2r'}{r-2}[ \), gives

\[
|I_{2,1}| \lesssim \int_0^t \left( \|u^3\|_{H^{\sigma + \frac{r'}{2} + \frac{3}{2}}} + \|b^3\|_{H^{\sigma + \frac{r'}{2} + \frac{3}{2}}} \right) \|\nabla (\Gamma_+)^{\frac{3}{2}}\|_{L^2}^2 \|\nabla (\Gamma_+)^{\frac{1}{2}}\|_{L^2}^{2(1-\frac{r'}{2})} dt'.
\]

Then by using Young’s inequality, we get

\[
|I_{2,1}| \leq \frac{r-\frac{1}{4}}{4r^2} \int_0^t \|\nabla (\Gamma_+)^{\frac{3}{2}}\|_{L^2}^2 + C \int_0^t \left( \|u^3\|_{H^{\sigma + \frac{r'}{2} + \frac{3}{2}}}^p + \|b^3\|_{H^{\sigma + \frac{r'}{2} + \frac{3}{2}}}^p \right) \|\nabla (\Gamma_+)^{\frac{1}{2}}\|_{L^2}^{2} dt'.
\]

Similarly by applying Lemma 4.1 with \( f = \partial_3^2 V_+, g = V_-, h = \Gamma_+ \), and \( \sigma = \frac{(p-2)r'}{3p} \), we get

\[
|I_{2,2}| \lesssim \int_0^t \|\partial_3^2 V_+\|_{H^\sigma} \left( \|u^3\|_{H^{\frac{r}{2} + \frac{3}{2} + \frac{r'}{2}}} + \|b^3\|_{H^{\frac{r}{2} + \frac{3}{2} + \frac{r'}{2}}} \right) \|\nabla (\Gamma_+)^{\frac{1}{2}}\|_{L^2}^{2} dt'.
\]

As we have \( \frac{1}{2} + \alpha(r) + \left( \frac{1}{2} - \alpha(r) \right) + \left( \frac{1}{2} - \frac{1}{2} \right) = 1 \), applying Hölder’s inequality ensures that

\[
|I_{2,2}| \lesssim \left( \int_0^t \|\partial_3^2 V_+\|_{H^\sigma}^\frac{1}{2} dt' \right)^\frac{1}{2} \left( \int_0^t \left( \|u^3\|_{H^{\frac{r}{2} + \frac{3}{2} + \frac{r'}{2}}} + \|b^3\|_{H^{\frac{r}{2} + \frac{3}{2} + \frac{r'}{2}}} \right) dt' \right)^{\alpha(r)}
\]

\[
\times \left( \int_0^t \|\nabla (\Gamma_+)^{\frac{1}{2}}\|_{L^2}^\frac{1}{2} dt' \right)^{\frac{1}{2} - \frac{1}{2}}.\]

Then applying Young’s inequality leads to

\[
|I_{2,2}| \leq \frac{r-\frac{1}{4}}{4r^2} \int_0^t \|\nabla (\Gamma_+)^{\frac{3}{2}}\|_{L^2}^2 dt' + C \int_0^t \left( \|u^3\|_{H^{\frac{r}{2} + \frac{3}{2} + \frac{r'}{2}}}^{2} + \|b^3\|_{H^{\frac{r}{2} + \frac{3}{2} + \frac{r'}{2}}}^{2} \right) \|\nabla (\Gamma_+)^{\frac{1}{2}}\|_{L^2}^{2} dt'.
\]

Combining (4.7)-(4.9), we obtain

\[
|I_2| \leq \frac{r-\frac{1}{4}}{2r^2} \int_0^t \|\nabla (\Gamma_+)^{\frac{3}{2}}\|_{L^2}^2 + \|\nabla (\Gamma_-)^{\frac{3}{2}}\|_{L^2}^2 dt'
\]

\[
+ C \int_0^t \left( \|u^3\|_{H^{\frac{r}{2} + \frac{3}{2} + \frac{r'}{2}}} + \|b^3\|_{H^{\frac{r}{2} + \frac{3}{2} + \frac{r'}{2}}} \right) \left( \|\nabla (\Gamma_+)^{\frac{1}{2}}\|_{L^2}^2 + \|\nabla (\Gamma_-)^{\frac{1}{2}}\|_{L^2}^2 \right) dt'.
\]

We now turn to the last term \( I_3 \). Use (2.2) once again, we write

\[
I_3 = 2 \int_0^t \int \left( \partial_t b^h \cdot \partial_2 (\nabla_h^4 \partial_h^{-1} \omega - \nabla_h \Delta_h^{-1} \partial_3 u^3) 
\right.
\]

\[
- \partial_2 b^h \cdot \partial_1 (\nabla_h^4 \partial_h^{-1} \omega - \nabla_h \Delta_h^{-1} \partial_3 u^3) (\Gamma_+)^{\frac{1}{2}}_{-1} dx dt'.
\]
By virtue of Lemma 4.1, with \( f = \nabla_h \omega \), \( g = b^h \), \( h = \Gamma_+ \), and \( \sigma = \frac{(p-2)\nu'}{2p} \), we get

\[
\left| \int_0^t \int \partial_1 b^h \cdot \partial_2 \nabla_h \Delta_h^{-1} \omega (\Gamma_+ r)^{-1} dx dt' \right| \leq \int_0^t \| \nabla_h \omega \|_{L^r} \| b^h \|_{H^\frac{1}{2} + 2 (\frac{1}{2} - \frac{p}{r})} \| (\Gamma_+) \frac{2}{p} \|_{H^\sigma} dt'
\leq \int_0^t \left( \| \nabla_h \Gamma_+ \|_{L^r} + \| \nabla_h \Gamma_- \|_{L^r} \right) \| b^h \|_{H^\frac{1}{2} + 2 (\frac{1}{2} - \frac{p}{r})} \| (\Gamma_+) \frac{2}{p} \|_{H^\sigma} dt'
\leq \int_0^t \| \nabla_h (\Gamma_+) \|_{L^r} \| (\Gamma_+) \frac{2}{p} \|_{L^2} \left( \| \nabla_h (\Gamma_-) \|_{L^2} \right) \| \nabla (\Gamma_+) \|_{L^2} \| \nabla (\Gamma_-) \|_{L^2} \right)^{2 (1 - \frac{1}{p})} dt',
\]

where we have used the fact that \( \omega = \frac{1}{2} (\Gamma_+ + \Gamma_-) \). The same estimate holds for \( \int_0^t \int \partial_2 b^h \cdot \partial_1 \nabla_h \Delta_h^{-1} \omega (\Gamma_+ r)^{-1} dx dt' \).

Along the same line, applying Lemma 4.1 with \( f = \nabla_h \partial_3 u^3 \), \( g = b^h \), \( h = \Gamma_+ \), \( \sigma = \frac{(p-2)\nu'}{2p} \), and the fact that \( u^3 = \frac{1}{2} (V_+ + V_-) \), yield

\[
\left| \int_0^t \int \partial_1 b^h \cdot \partial_2 \nabla_h \Delta_h^{-1} \partial_3 u^3 (\Gamma_+ r)^{-1} dx dt' \right| \leq \int_0^t \| \nabla_h \partial_3 u^3 \|_{H^\theta, r} \| b^h \|_{H^\frac{1}{2} + \frac{2p}{r}} \| (\Gamma_+) \frac{2}{p} \|_{H^\sigma} dt'
\leq \int_0^t \| \nabla_h \partial_3 u^3 \|_{H^\theta, r} \| b^h \|_{H^\frac{1}{2} + \frac{2p}{r}} \| (\Gamma_+) \frac{2}{p} \|_{H^\sigma} dt'
\leq \left( \int_0^t \| \nabla_h \partial_3 V_+ \|_{L^2}^2 + \| \nabla_h \partial_3 V_- \|_{H^\theta, r}^2 \right)^{\frac{1}{2}} \left( \int_0^t \| b^h \|_{H^\frac{1}{2} + \frac{2p}{r}}^p \right)^{\frac{1}{2}} \left( \int_0^t \| \nabla (\Gamma_+) \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left( \int_0^t \| \nabla (\Gamma_-) \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \cdot \frac{1}{2} - \frac{1}{p},
\]

The same estimate holds for \( \int_0^t \int \partial_2 b^h \cdot \partial_1 \nabla_h \Delta_h^{-1} \partial_3 u^3 (\Gamma_+ r)^{-1} dx dt' \).

Therefore by applying Young’s inequality, we obtain

\[
|I_3| \leq \frac{\tau - 1}{2r^2} \int_0^t \| \nabla (\Gamma_+) \|_{L^2}^2 + \| \nabla (\Gamma_-) \|_{L^2}^2 \right) dt'
(4.12)
\]

\[
\begin{aligned}
&+ C \int_0^t \| b^h \|_{H^\frac{1}{2} + \frac{2p}{r}}^p \left( \| (\Gamma_+) \|_{L^2}^2 + \| (\Gamma_-) \|_{L^2}^2 \right) dt'
&+ C \left( \int_0^t \| b^h \|_{H^\frac{1}{2} + \frac{2p}{r}}^p \right)^{1 - \frac{r}{2}} \left( \int_0^t \| \nabla_h \partial_3 V_+ \|_{H^\theta, r}^2 + \| \nabla_h \partial_3 V_- \|_{H^\theta, r}^2 \right)^{\frac{1}{2}}.
\end{aligned}
\]

Summing up (4.3)-(4.5), (4.10) and (4.12) leads to

\[
\begin{aligned}
&\frac{1}{r} \| (\Gamma_+) \|_{L^2}^2 + \frac{2(r - 1)}{r^2} \int_0^t \| \nabla (\Gamma_+) \|_{L^2}^2 - \frac{r - 1}{2r^2} \int_0^t \| \nabla (\Gamma_-) \|_{L^2}^2 dt'
&\leq \frac{1}{r} \left( \| \omega_0 \|_{L^r} + \| d_0 \|_{L^r} \right) + C \int_0^t \left( \| (\Gamma_+) \|_{L^2}^2 + \| (\Gamma_-) \|_{L^2}^2 \right) \left( \| u^3 \|_{H^\frac{1}{2} + \frac{2p}{r}}^p + \| b^h \|_{H^\frac{1}{2} + \frac{2p}{r}}^p \right) dt'
&+ C \left( \int_0^t \| \nabla \partial_3 V_+ \|_{H^\theta, r}^2 + \| \nabla \partial_3 V_- \|_{H^\theta, r}^2 \right)^{\frac{1}{2}} \left( \int_0^t \| u^3 \|_{H^\frac{1}{2} + \frac{2p}{r}}^p + \| b^h \|_{H^\frac{1}{2} + \frac{2p}{r}}^p \right)^{1 - \frac{r}{2}}.
\end{aligned}
\]
Along the same line, we can get a similar estimate for \((\Gamma_-)^{\frac{3}{2}}\), namely

\[
\frac{1}{r} \|(\Gamma_-)^{\frac{3}{2}}(t)\|^2_{L^2} + \frac{2(r - 1)}{r^2} \int_0^t \|\nabla (\Gamma_-)^{\frac{3}{2}}\|^2_{L^2} dt - \frac{r - 1}{2r^2} \int_0^t \|\nabla (\Gamma_+)^{\frac{3}{2}}\|^2_{L^2} dt' 
\]

\[
(4.14) \leq \frac{1}{r}(\|\omega_0\|^2_{L^r} + \|d_0\|^2_{L^r}) + C \int_0^t (\|\nabla (\Gamma_+)^{\frac{3}{2}}\|^2_{L^2} + \|\nabla (\Gamma_-)^{\frac{3}{2}}\|^2_{L^2}) (\|u^3\|^p_{H^{\frac{1}{2} + \frac{2}{p}}} + \|b\|^p_{H^{\frac{1}{2} + \frac{2}{p}}}) dt' 
\]

\[
+ C \left( \int_0^t (\|\nabla \partial_3 V_+\|^2_{H^{\frac{1}{2} + \frac{2}{p}}} + \|\nabla \partial_3 V_-\|^2_{H^{\frac{1}{2} + \frac{2}{p}}}) dt' \right)^{\frac{\gamma}{2}} \left( \int_0^t (\|u^3\|^p_{H^{\frac{1}{2} + \frac{2}{p}}} + \|b\|^p_{H^{\frac{1}{2} + \frac{2}{p}}}) dt' \right)^{1 - \frac{\gamma}{2}}.
\]

Summing up (4.13) and (4.14) and then using Gronwall’s inequality gives rise to

\[
(4.15) \lesssim \left( \frac{2}{r}(\|\omega_0\|^2_{L^r} + \|d_0\|^2_{L^r}) + \left( \int_0^t (\|\nabla \partial_3 V_+\|^2_{H^{\frac{1}{2} + \frac{2}{p}}} + \|\nabla \partial_3 V_-\|^2_{H^{\frac{1}{2} + \frac{2}{p}}}) dt' \right)^{\frac{\gamma}{2}} \right) \exp \left( C \int_0^t (\|u^3\|^p_{H^{\frac{1}{2} + \frac{2}{p}}} + \|b\|^p_{H^{\frac{1}{2} + \frac{2}{p}}}) dt' \right).
\]

This completes the proof of Proposition 2.1, once we notice the elementary inequality

\[ x^\gamma e^{c_1 x} \lesssim e^{c_2 x}, \quad \forall \gamma \geq 0, \quad x \geq 0. \]

5. PROOF OF PROPOSITION 2.2

Applying \(\partial_3\) on the third components of (1.1), we obtain

\[
\partial_3 \partial_3 u^3 - \Delta \partial_3 u^3 = - \partial_3 u \cdot \nabla u^3 - (u \cdot \nabla) \partial_3 u^3 + \partial_3 b \cdot \nabla b^3 + (b \cdot \nabla) \partial_3 b^3 
\]

\[- \partial_3^2 (-\Delta)^{-1} \sum_{\ell,m=1}^{3} (\partial_\ell u^m \partial_m u^\ell - \partial_\ell b^m \partial_m b^\ell), \]

\[
\partial_3 \partial_3 b^3 - \Delta \partial_3 b^3 = - \partial_3 u \cdot \nabla b^3 - (u \cdot \nabla) \partial_3 b^3 + \partial_3 b \cdot \nabla u^3 + (b \cdot \nabla) \partial_3 u^3.
\]

Adding these two equations gives

\[
\partial_3 \partial_3 V_+ - \Delta \partial_3 V_+ = - \partial_3 u \cdot \nabla V_+ + \partial_3 b \cdot \nabla V_+ - u \cdot \nabla \partial_3 V_+ 
\]

\[
+ b \cdot \nabla \partial_3 V_+ - \partial_3^2 (-\Delta)^{-1} \sum_{\ell,m=1}^{3} (\partial_\ell u^m \partial_m u^\ell - \partial_\ell b^m \partial_m b^\ell).
\]

(5.1)

We write

\[
- \partial_3 u \cdot \nabla V_+ + \partial_3 b \cdot \nabla V_+ = - \sum_{\ell=1}^{2} \partial_3 (u^\ell - b^\ell) \partial_\ell V_+ - (\partial_3 u^3)^2 + (\partial_3 b^3)^2,
\]

and

\[
\sum_{\ell,m=1}^{3} (\partial_\ell u^m \partial_m u^\ell - \partial_\ell b^m \partial_m b^\ell) = \sum_{\ell,m=1}^{2} (\partial_\ell u^m \partial_m u^\ell - \partial_\ell b^m \partial_m b^\ell) + 2 \sum_{\ell=1}^{2} (\partial_3 u^\ell \partial_\ell u^3 - \partial_3 b^\ell \partial_\ell b^3) + (\partial_3 u^3)^2 - (\partial_3 b^3)^2.
\]
Then we take the $\mathcal{H}^{0,r}$ inner product of (5.1) with $\partial_3 V_+$ to obtain

$$\frac{1}{2} \frac{d}{dt} \| \partial_3 V_+(t) \|_{\mathcal{H}^{0,r}}^2 + \| \nabla \partial_3 V_+ \|_{\mathcal{H}^{0,r}}^2 = - \sum_{i=1}^5 II_i,$$

with

$$II_1 \overset{\text{def}}{=} ((Id + \partial_3^2 (-\Delta)^{-1})(\partial_3 u^3 - (\partial_3 b^3)^2) | \partial_3 V_+ )_{\mathcal{H}^{0,r}},$$

$$II_2 \overset{\text{def}}{=} (\partial_3^2 (-\Delta)^{-1} \sum_{\ell,m=1}^2 \partial_\ell u^m \partial_m u^\ell - \partial_\ell b^m \partial_m b^\ell | \partial_3 V_+ )_{\mathcal{H}^{0,r}},$$

$$II_3 \overset{\text{def}}{=} \sum_{\ell=1}^2 (\partial_3 (u^\ell - b^\ell) \partial_\ell V_+ | \partial_3 V_+ )_{\mathcal{H}^{0,r}},$$

$$II_4 \overset{\text{def}}{=} (2 \partial_3^2 (-\Delta)^{-1} \sum_{\ell=1}^2 (\partial_3 u^\ell \partial_\ell u^3 - \partial_3 b^\ell \partial_\ell b^3) | \partial_3 V_+ )_{\mathcal{H}^{0,r}},$$

$$II_5 \overset{\text{def}}{=} ((u - b) \cdot \nabla \partial_3 V_+ | \partial_3 V_+ )_{\mathcal{H}^{0,r}}.$$

Let us first recall the following lemma from [7].

**Lemma 5.1** (Lemma 5.1 of [7]). *Let $A$ be a bounded Fourier multiplier. If $p$ and $\theta$ satisfy*

$$0 < \theta < \frac{1}{2} - \frac{1}{p},$$

*then we have*

$$\| (A(D)(fg) | \partial_3 h^3 )_{\mathcal{H}^{0,r}} \| \lesssim \| f \|_{\mathcal{H}^{\frac{1}{2} - 3\alpha(r) + \theta, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}} \| g \|_{\mathcal{H}^{\frac{1}{2} - 3\alpha(r) + \theta, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}} \| h^3 \|_{\mathcal{H}^{\frac{1}{2} - \frac{2\alpha(r)}{3}, \frac{1}{2}}}. $$

Noting that $p > 4, r > \frac{4}{3}$, we have $\frac{1}{p} + \frac{1}{r} < 1$, and hence $\theta < \alpha(r) < \frac{1}{2} - \frac{1}{p}$, i.e. the condition (5.3) is satisfied under the assumption of Proposition 2.2. Because $\partial_3^2 (-\Delta)^{-1}$ is a bounded Fourier multiplier, applying Lemma 5.1 with $f = \partial_3 V_+, g = \partial_3 V_-, h = u + b$, gives

$$|II_1| \lesssim \| V_+ \|_{\mathcal{H}^{\frac{1}{2} + \frac{2\alpha(r)}{3}, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \| \partial_3 V_+ \|_{\mathcal{H}^{\frac{1}{2} - 3\alpha(r) + \theta, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \| \partial_3 V_- \|_{\mathcal{H}^{\frac{1}{2} - 3\alpha(r) + \theta, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}}.$$

While a direct calculation from Definition 2.1 gives

$$\| a \|^2_{\mathcal{H}^{\frac{1}{2} - 3\alpha(r) + \theta, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \lesssim \int_{\mathbb{R}^3} |\hat{a}(\xi)|^2 \left( \| \xi \|_{\mathcal{H}^{0}} \right)^2 \| \xi_h \|_{\mathcal{H}^{2(-3\alpha(r) + \theta)}} \| \xi_3 \|_{\mathcal{H}^{2(-3\alpha(r) + \theta)}}^{-2\theta} d\xi$$

$$\lesssim \left( \int_{\mathbb{R}^3} |\hat{a}(\xi)|^2 \| \xi_h \|_{\mathcal{H}^{2(-3\alpha(r) + \theta)}} \| \xi_3 \|_{\mathcal{H}^{2(-3\alpha(r) + \theta)}}^{-2\theta} d\xi \right)^{\frac{1}{p}}$$

$$\times \left( \int_{\mathbb{R}^3} |\hat{a}(\xi)|^2 \| \xi \|_{\mathcal{H}^{2(-3\alpha(r) + \theta)}} \| \xi_3 \|_{\mathcal{H}^{2(-3\alpha(r) + \theta)}}^{-2\theta} d\xi \right)^{\frac{1}{p}},$$

that is

$$\| a \|^2_{\mathcal{H}^{\frac{1}{2} - 3\alpha(r) + \theta, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \lesssim \| a \|_{\mathcal{H}^{0, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \| \nabla a \|_{\mathcal{H}^{0, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}}.$$

Applying (5.5) to $\partial_3 V_\pm$ in (5.4) gives

$$|II_1| \lesssim \| V_+ \|_{\mathcal{H}^{\frac{1}{2} + \frac{2\alpha(r)}{3}, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \| \partial_3 V_+ \|_{\mathcal{H}^{0, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \| \nabla \partial_3 V_+ \|_{\mathcal{H}^{0, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \| \partial_3 V_- \|_{\mathcal{H}^{0, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}} \| \nabla \partial_3 V_- \|_{\mathcal{H}^{0, \frac{5}{6} - \frac{2\alpha(r)}{3}, \frac{1}{2}}}.$$
then Young’s inequality and mean inequality ensure
\begin{equation}
|II_1| \leq \frac{1}{20} \left( \| \nabla \partial_3 V_+ \|_{L^2}^2 + \| \nabla \partial_3 V_- \|_{L^2}^2 \right) + C \| V_+ \|_{H^{1/2, \frac{1}{p}}} \left( \| \partial_3 V_+ \|_{H^{1/2, \frac{1}{p}}}^2 + \| \partial_3 V_- \|_{H^{1/2, \frac{1}{p}}}^2 \right).
\end{equation}

For \(II_2\) term, one key point is to write it in a symmetric form, namely
\begin{equation}
II_2 = \left( \partial_3^2 (-\Delta)^{-1} \sum_{\ell, m=1}^2 \frac{1}{2} \left( \partial_\ell (u^{m} + b^m) \partial_m (u^\ell - b^\ell) + \partial_\ell (u^{m} - b^m) \partial_m (u^\ell + b^\ell) \right) \| \partial_3 V_+ \right)_{H^{1/2, \frac{1}{p}}}.
\end{equation}

Applying the Hodge decomposition for the horizontal variables to \(u^\ell \pm b^\ell\), and noting both \(\partial_3^2 (-\Delta)^{-1}\) and \(\partial_3^2 \Delta_3^{-1}\) are bounded Fourier multipliers, then Lemma 5.1 ensures that
\begin{align*}
|II_2| \leq & \| V_+ \|_{H^{1/2, \frac{1}{p}}} \left( \| \partial_3 V_+ \|_{H^{1/2, \frac{1}{p}}} \| \nabla \partial_3 V_+ \|_{L^2} + \| (\Gamma_+) \|_{H^{1/2, \frac{1}{p}}} \| \nabla (\Gamma_+) \|_{L^2} \right) \\
& \times \left( \| \partial_3 V_- \|_{H^{1/2, \frac{1}{p}}} \left( \| \nabla \partial_3 V_- \|_{L^2} + \| (\Gamma_-) \|_{H^{1/2, \frac{1}{p}}} \right) \right).
\end{align*}

Yet it follows from Lemma 3.1 and Lemma 3.4 that for any function \(a\)
\begin{equation}
\|a\|_{H^{1/2, \frac{1}{p}}} \leq \|a\|_{L^2} \lesssim \left\| a \right\|_{L^2} \lesssim \left\| a \right\|_{L^2},
\end{equation}
where \(p'\) denotes the conjugate index of \(p\). Then applying (5.7) to \(\Gamma_+\), (5.5) to \(\partial_3 V_-\) gives
\begin{align*}
|II_2| \leq & \| V_+ \|_{H^{1/2, \frac{1}{p}}} \left( \| \partial_3 V_+ \|_{H^{1/2, \frac{1}{p}}} \| \nabla \partial_3 V_+ \|_{L^2} + \| (\Gamma_+) \|_{H^{1/2, \frac{1}{p}}} \| \nabla (\Gamma_+) \|_{L^2} \right) \\
& \times \left( \| \partial_3 V_- \|_{H^{1/2, \frac{1}{p}}} \left( \| \nabla \partial_3 V_- \|_{L^2} + \| (\Gamma_-) \|_{H^{1/2, \frac{1}{p}}} \right) \right),
\end{align*}
which implies that
\begin{equation}
|II_2| \leq \frac{1}{20} \left( \| \nabla \partial_3 V_+ \|_{H^{1/2, \frac{1}{p}}}^2 + \| \nabla \partial_3 V_- \|_{H^{1/2, \frac{1}{p}}}^2 \right) + C \| V_+ \|_{H^{1/2, \frac{1}{p}}} \left( \| \partial_3 V_+ \|_{H^{1/2, \frac{1}{p}}}^2 + \| \partial_3 V_- \|_{H^{1/2, \frac{1}{p}}}^2 \right) \\
+ C \| V_+ \|_{H^{1/2, \frac{1}{p}}} \left( \| (\Gamma_+) \|_{H^{1/2, \frac{1}{p}}}^{2(1/2 + 2\alpha)} + \| (\Gamma_-) \|_{H^{1/2, \frac{1}{p}}}^{2(1/2 + 2\alpha)} \right).
\end{equation}

On the other hand, for any real valued functions \(a\) and \(b\), and any couple \((\alpha, \beta) \in \mathbb{R}^2\), applying Hölder’s inequality gives
\begin{equation}
|\langle a, b \rangle|_{H^{1/2, \frac{1}{p}}} = \left| \int_{\mathbb{R}^3} (|\xi_h| - 6\alpha + 2\beta - \beta |\xi_3| + 2\alpha |\xi_3|) |\xi_3|^{-\beta - 2\theta} \hat{a}(\xi)| \xi_3|^{\alpha} |\xi_3|^{\beta} \hat{\xi}(\xi) \xi d\xi \right| \\
\leq \|a\|_{H^{1/2, 6\alpha - 2\beta - 2\theta}} \|b\|_{H^{1/2, \alpha - 2\theta}}.
\end{equation}
Note that \(4 < p < \frac{2\theta}{2-\beta}\) and \(3\alpha(r) - \frac{2}{p} < \theta < \alpha(r)\), we have \(\frac{2}{p} + 3\alpha(r) - \theta \in [0, 1]\), and hence
\begin{align*}
|\xi_h|^{2(1 - 6\alpha + 2\beta - 2\theta)} |\xi_3|^{2(\alpha + 3\alpha - 2\theta)} = & \left( |\xi_h|^{2(1 - 6\alpha + 2\beta - 2\theta)} |\xi_3|^{2(\alpha + 3\alpha - 2\theta)} \right) \\
\leq & \left( |\xi_h|^{2(1 - 6\alpha + 2\beta - 2\theta)} |\xi_3|^{2(\alpha + 3\alpha - 2\theta)} \right) \\
\leq & \left( |\xi_h|^{2(1 - 6\alpha + 2\beta - 2\theta)} |\xi_3|^{2(\alpha + 3\alpha - 2\theta)} \right) \\
\leq & \left( |\xi_h|^{2(1 - 6\alpha + 2\beta - 2\theta)} |\xi_3|^{2(\alpha + 3\alpha - 2\theta)} \right),
\end{align*}
which implies for any function \(a\)
\begin{equation}
|\langle a, b \rangle|_{H^{1/2, 6\alpha - 2\beta - 2\theta}} \leq \left( \int_{\mathbb{R}^3} |\xi_h|^{2(1 - 6\alpha + 2\beta - 2\theta)} |\xi_3|^{2(\alpha + 3\alpha - 2\theta)} \hat{a}(\xi)| \xi d\xi \right) \\
\|\nabla a\|^2_{H^{1/2, \frac{1}{p}}},
\end{equation}
Along the same line, one has
\begin{equation}
\|a\|_{H^{1/2, 6\alpha - 2\beta - 2\theta}} \leq \left( \int_{\mathbb{R}^3} |\xi_h|^{2(1 - 6\alpha + 2\beta - 2\theta)} |\xi_3|^{2(\alpha + 3\alpha - 2\theta)} \hat{a}(\xi)| \xi d\xi \right) \\
\|\nabla a\|^2_{H^{1/2, \frac{1}{p}}}. 
\end{equation}
In order to estimate $II_3$, we apply Bony’s decomposition in the vertical variable to write
\[
\partial_3 (u^\ell - b^\ell) \partial_\ell V_+ = (T^\nu + \bar{T}^\nu + R^\nu)(\partial_3 (u^\ell - b^\ell), \partial_\ell V_+).
\]
Applying (5.9) with $\alpha = 1 - 6\alpha(r) - \frac{2}{p} + 2\theta$, $\beta = \frac{2}{p} + 3\alpha(r) - 2\theta$, the law of product (see Lemma 4.5 of [6] for example) and (5.10) ensures that

\[
\left| \left( \sum_{\ell=1}^{2} (T^\nu + \bar{T}^\nu)(\partial_3 (u^\ell - b^\ell), \partial_\ell V_+) \right) \right|_{H^{\theta, r}} 
\lesssim \sum_{\ell=1}^{2} \| (T^\nu + \bar{T}^\nu)(\partial_3 (u^\ell - b^\ell), \partial_\ell V_+) \|_{H^{\frac{2}{2} - 1, -3\alpha(r)} - \frac{2}{p}} \| \partial_3 V_+ \|_{H^{1 - 6\alpha(r) - \frac{2}{p} + 2\theta, \frac{2}{p} + 3\alpha(r) - 2\theta}} 
\tag{5.12}
\]

Applying (5.9) with $\alpha = 0$, $\beta = -\frac{1}{2} + \frac{2}{p}$, the law of product, and (5.11) ensures that

\[
\left| \left( \sum_{\ell=1}^{2} R^\nu (\partial_3 (u^\ell - b^\ell), \partial_\ell V_+) \right) \right|_{H^{\theta, r}} 
\lesssim \sum_{\ell=1}^{2} \| R^\nu (\partial_3 (u^\ell - b^\ell), \partial_\ell V_+) \|_{H^{-6\alpha(r) + 2\theta, \frac{1}{2} - 2\theta} - \frac{2}{p} + 2\theta} \| \partial_3 V_+ \|_{H^{0, \frac{1}{2} + \frac{2}{p}}} 
\tag{5.13}
\]

Therefore, by virtue of Lemma 3.5 with $\beta = \frac{2}{p}$, inequalities (5.12) and (5.13) ensure that

\[
|II_3| \lesssim \| V_+ \|_{H^{\frac{1}{2} + \frac{2}{p}}} \| \nabla \partial_3 V_+ \|_{H^{\theta, r}} 
\times \left( \| (\Gamma - \frac{1}{2}) \|_{L^2}^{2(\alpha(r) + \frac{1}{p})} \| \nabla (\Gamma - \frac{1}{2}) \|_{L^2}^{1 - \frac{2}{p}} + \| \partial_3 V_+ \|_{H^{\theta, r}}^{\frac{2}{p}} \| \nabla \partial_3 V_+ \|_{H^{\theta, r}}^{1 - \frac{2}{p}} \right) .
\]

Applying Young’s inequality and mean inequality yields

\[
|II_3| \leq \frac{1}{20} \left( \| \nabla \partial_3 V_+ \|_{H^{\theta, r}}^{2} + \| \nabla \partial_3 V_- \|_{H^{\theta, r}}^{2} \right) + C \| V_+ \|_{H^{\frac{1}{2} + \frac{2}{p}}} \| \partial_3 V_- \|_{H^{\theta, r}}^{2}
\tag{5.14}
\]

The term $II_4$ can be handled as above. Indeed we first rewrite it in a symmetric form

\[
II_4 = \left( 2\partial_3^2(-\Delta)^{-1} \sum_{\ell=1}^{2} (\partial_3 (u^\ell + b^\ell) \partial_\ell V_- + \partial_3 (u^\ell - b^\ell) \partial_\ell V_+) \right) \| \partial_3 V_+ \|_{H^{\theta, r}}.
\]
Then it follows from the estimate of $I_{23}$ that

$$|I_{23}| \lesssim (\|V_+\|_{H^{\frac{1}{2}}+\frac{1}{p}} + \|V_-\|_{H^{\frac{1}{2}}+\frac{1}{p}})(\|\nabla \delta_3 V_+\|_{\dot{H}^{\sigma,r}} + \|\nabla \delta_3 V_-\|_{\dot{H}^{\sigma,r}})$$

$$\times \left( (\|\Gamma_+\|_{L^2}^{\frac{2(\alpha(r)+1)}{p}}) \|\nabla (\Gamma_+)^{-\frac{1}{2}}\|_{L^2}^{\frac{1-\frac{2}{p}}{\gamma}} + \|\partial_3 V_+\|_{\dot{H}^{\sigma,r}} \|\nabla \partial_3 V_+\|_{\dot{H}^{\sigma,r}} \right)$$

$$+ (\|\Gamma_-\|_{L^2}^{\frac{2(\alpha(r)+1)}{p}}) \|\nabla (\Gamma_-)^{-\frac{1}{2}}\|_{L^2}^{\frac{1-\frac{2}{p}}{\gamma}} + \|\partial_3 V_-\|_{\dot{H}^{\sigma,r}} \|\nabla \partial_3 V_-\|_{\dot{H}^{\sigma,r}} \right).$$

Applying Young’s inequality yields

$$|I_{23}| \lesssim \frac{1}{20} (\|\nabla \delta_3 V_+\|_{\dot{H}^{\sigma,r}}^2 + \|\nabla \delta_3 V_-\|_{\dot{H}^{\sigma,r}}^2) + C(\|V_+\|_{H^{\frac{1}{2}}+\frac{1}{p}}^2 + \|V_-\|_{H^{\frac{1}{2}}+\frac{1}{p}}^2)$$

$$\times \left( (\|\Gamma_+\|_{L^2}^{\frac{4(\alpha(r)+1)}{p}}) \|\nabla (\Gamma_+)^{-\frac{1}{2}}\|_{L^2}^{\frac{2(1-\frac{2}{p})}{\gamma}} + (\|\Gamma_-\|_{L^2}^{\frac{4(\alpha(r)+1)}{p}}) \|\nabla (\Gamma_-)^{-\frac{1}{2}}\|_{L^2}^{\frac{2(1-\frac{2}{p})}{\gamma}} \right)$$

$$+ C(\|V_+\|_{H^{\frac{1}{2}}+\frac{1}{p}}^p + \|V_-\|_{H^{\frac{1}{2}}+\frac{1}{p}}^p) \|\partial_3 V_+\|_{\dot{H}^{\sigma,r}}^2 + \|\partial_3 V_-\|_{\dot{H}^{\sigma,r}}^2).$$

Finally, let us turn to the estimate of $I_{5}$. We first decompose it as

$$I_{5} = (\langle u^{h} - b^{h} \rangle \cdot \nabla \partial_{3} V_{+}|\partial_{3} V_{+}\rangle_{\dot{H}^{\sigma,r}} + \langle V_{-} \cdot \partial_{3}^{2} V_{+}|\partial_{3} V_{+}\rangle_{\dot{H}^{\sigma,r}}) \overset{\text{def}}{=} I_{5,1} + I_{5,2}.$$  

Applying Lemma 3.6 gives

$$|I_{5,1}| \lesssim \left( \left( \|\nabla h(u^{h} - b^{h})\|_{H^{\frac{1}{2} - 3\alpha (r) + \frac{\sigma}{2} + \frac{1}{2} - \theta}}^{2} + \|\partial_3 V_+\|_{\dot{H}^{\frac{1}{2} - 3\alpha (r) + \frac{\sigma}{2} + \frac{1}{2} - \theta}}^{2} \right) \|\nabla \partial_3 V_+\|_{\dot{H}^{\sigma,r}} + \|\partial_3 V_+\|_{\dot{H}^{\sigma,r}} \right).$$

While using Hodge decomposition (2.2), and then (5.5),(5.7), we have

$$\|\nabla h(u^{h} - b^{h})\|_{H^{\frac{1}{2} - 3\alpha (r) + \frac{\sigma}{2} + \frac{1}{2} - \theta}}^{2} \lesssim \|\Gamma_{-}\|_{L^2}^{\frac{2}{\gamma}} \|\nabla (\Gamma_{-})^{-\frac{1}{2}}\|_{L^2}^{\frac{2}{p}} + \|\partial_3 V_-\|_{\dot{H}^{\sigma,r}} \|\nabla \partial_3 V_-\|_{\dot{H}^{\sigma,r}} + \|\partial_3 V_-\|_{\dot{H}^{\sigma,r}} \|\nabla \partial_3 V_-\|_{\dot{H}^{\sigma,r}}.$$  

Inserting this estimate and (3.5) with $\beta = \frac{2}{p}$ into (5.17) yields

$$|I_{5,1}| \lesssim \|V_+\|_{H^{\frac{1}{2} + \frac{1}{p}}} \left( \|\Gamma_{-}\|_{L^2}^{\frac{2(\alpha(r)+1)}{p}} \|\nabla \partial_3 V_+\|_{\dot{H}^{\sigma,r}} + \|\partial_3 V_-\|_{\dot{H}^{\sigma,r}} \|\nabla \partial_3 V_-\|_{\dot{H}^{\sigma,r}} \right)$$

$$\times \left( (\|\Gamma_{-}\|_{L^2}^{\frac{2(\alpha(r)+1)}{p}}) \|\nabla (\Gamma_{-})^{-\frac{1}{2}}\|_{L^2}^{\frac{1-\frac{2}{p}}{\gamma}} + \|\partial_3 V_-\|_{\dot{H}^{\sigma,r}} \|\nabla \partial_3 V_-\|_{\dot{H}^{\sigma,r}} \right).$$

In order to estimate $I_{5,2}$, we need the following lemma:

**Lemma 5.2** ((95) of [9]). Let $s_{1} < 1, s_{2} < 1, s_{1} + s_{2} > 0$ and $0 < \sigma < 1$. Then we have

$$\|f g\|_{H^{s_{1} + s_{2} - 1, \sigma - \frac{1}{2}}} \lesssim \|f\|_{(B_{s_{1} + s_{2} - 1, 1}^{-1})_{\eta}} \|g\|_{H^{s_{2}, \sigma - \frac{1}{2}}}.$$  

This lemma together with (5.9) and Lemma 3.1 ensure that

$$|I_{5,2}| \lesssim \|V_{-}\|_{H^{\frac{1}{2} + \frac{1}{p}}} \|\partial_3^{2} V_{+}\|_{\dot{H}^{\sigma,r}} + \|\partial_3 V_{+}\|_{\dot{H}^{\sigma,r}} \|\partial_3 V_{+}\|_{\dot{H}^{\sigma,r}} + \|\partial_3 V_{+}\|_{\dot{H}^{\sigma,r}}$$

$$\lesssim \|V_{-}\|_{H^{\frac{1}{2} + \frac{1}{p}}} \|\partial_3^{2} V_{+}\|_{\dot{H}^{\sigma,r}} + \|\partial_3 V_{+}\|_{\dot{H}^{\sigma,r}} \|\partial_3 V_{+}\|_{\dot{H}^{\sigma,r}}.$$
This along with the interpolation, which claims that for any function \(a\)

\[
\|a\|^2_{H^{1-3\alpha(r)-\frac{2}{p}+\theta,-\theta}} \leq \left( \int_{\mathbb{R}^3} |\xi|^6 |\xi|^2 |\tilde{a}(\xi)|^2 d\xi \right)^{\frac{2}{p}}
\]

(5.21)

\[
\times \left( \int_{\mathbb{R}^3} |\xi|^6 |\xi|^2 |\tilde{a}(\xi)|^2 d\xi \right)^{1-\frac{2}{p}}
\]

\[
= \|a\|_{\mathcal{H}^{0,r}}^\frac{2}{p} \|\nabla_h \tilde{a}\|_{\mathcal{H}^{0,r}}^{2(1-\frac{2}{p})},
\]

events that

\[
|I_{5,2}| \lesssim \|V_+\|_{\mathcal{H}^{0,r}} \|\partial_3 V_+\|_{\mathcal{H}^{0,r}} \|\partial_3 V_-\|_{\mathcal{H}^{0,r}} \|\nabla_3 V_+\|_{\mathcal{H}^{0,r}} \|\nabla_3 V_-\|_{\mathcal{H}^{0,r}}^{1-\frac{2}{p}}
\]

(5.22)

\[
\lesssim \|V_-\|_{\mathcal{H}^{0,r}} \|\partial_3 V_+\|_{\mathcal{H}^{0,r}} \|\nabla_3 V_+\|_{\mathcal{H}^{0,r}} \|\nabla_3 V_-\|_{\mathcal{H}^{0,r}}^{1-\frac{2}{p}},
\]

Hence, by summing up (5.18) and (5.22), we obtain

\[
|I_5| \lesssim \left( \|V_+\|_{\mathcal{H}^{0,r}}^2 + \|V_-\|_{\mathcal{H}^{0,r}}^2 \right) \left( \|\nabla(V_+)\|_{L^2}^{2(\alpha(r)+\frac{1}{p})} + \|\nabla(V_-)\|_{L^2}^{2(\alpha(r)+\frac{1}{p})} \right)
\]

(5.23)

\[
+ \|\partial_3 V_+\|_{\mathcal{H}^{0,r}} \|\nabla_3 V_+\|_{\mathcal{H}^{0,r}} \|\nabla_3 V_-\|_{\mathcal{H}^{0,r}} \|\nabla_3 V_-\|_{\mathcal{H}^{0,r}}^{1-\frac{2}{p}}
\]

Substituting the estimates (5.6), (5.8), (5.14), (5.15) and (5.23) into (5.2) leads to

\[
\frac{1}{2} \frac{d}{dt} \|\partial_3 V_+(t)\|_{\mathcal{H}^{0,r}}^2 + \|\nabla_3 V_+\|_{\mathcal{H}^{0,r}}^2 \leq \frac{1}{4} \|\nabla_3 V_+\|_{\mathcal{H}^{0,r}}^2 + \|\nabla_3 V_-\|_{\mathcal{H}^{0,r}}^2
\]

(5.24)

\[
+ C \left( \|u^3\|_{\mathcal{H}^{0,r}} + \|k^3\|_{\mathcal{H}^{0,r}} \right) \left( \|\nabla(V_+)\|_{L^2}^{2(\alpha(r)+\frac{1}{p})} + \|\nabla(V_-)\|_{L^2}^{2(\alpha(r)+\frac{1}{p})} \right)
\]

\[
\times \left( \|\nabla(V_-)\|_{L^2}^{2(\alpha(r)+\frac{1}{p})} + \|\nabla(V_+)\|_{L^2}^{2(\alpha(r)+\frac{1}{p})} \right)
\]

Exactly along the same line to the derivation of the above inequality, we have

\[
\frac{1}{2} \frac{d}{dt} \|\partial_3 V_-(t)\|_{\mathcal{H}^{0,r}}^2 + \|\nabla_3 V_-\|_{\mathcal{H}^{0,r}}^2 \leq \text{the right hand side of (5.24)}.
\]

(5.25)
Summing up the above two estimates gives rise to
\[
\frac{d}{dt}(\|\partial_3 V_+(t)\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\partial_3 V_-(t)\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2) + \|\nabla \partial_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\nabla \partial_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 \\
\leq C\left(\|u_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^p + \|b_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^p\right) \left(\|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2\right)
\]
(5.26)
\[
+ C\left(\|u_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|b_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2\right) \left(\|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2\right)
\]
Then Gronwall's inequality allows to conclude the proof of Proposition 2.2 by noticing that
\[
\|\partial_3 u_3(0)\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\partial_3 b_3(0)\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 \lesssim \|u(0)\|_{H^{1-3\alpha(r)}}^2 + \|b(0)\|_{H^{1-3\alpha(r)}}^2 \lesssim \|\Omega_0\|_{L^r}^2 + \|j_0\|_{L^r}^2,
\]
by (2.3) and the Sobolev embedding $L^r \hookrightarrow H^{-3\alpha(r)}$.

6. Proof of Proposition 2.3

The purpose of this section is to present the proof of Proposition 2.3. Indeed it follows from Proposition 2.2 that: for any $t \in [0, T]$,
\[
E(T) \cdot \left(\int_0^t \|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2, dt\right)^{\frac{2}{\vartheta}'}
\]
(6.1)
\[
\leq E(T) \cdot \left(\|\Omega_0\|_{L^r}^2 + \|j_0\|_{L^r}^2\right) + III_1(t) + III_2(t),
\]
where
\[
III_1(t) \overset{\text{def}}{=} E(T) \cdot \left(\int_0^t \left(\|u_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|b_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2\right) \left(\|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2\right) \|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r} \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r} dt\right)^{\frac{2}{\vartheta}'}
\]
\[
+ \|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 \left(\|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2\right) \|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r} \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r} dt\right)^{\frac{2}{\vartheta}'}
\]
\[
III_2(t) \overset{\text{def}}{=} E(T) \cdot \left(\int_0^t \left(\|u_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|b_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2\right) \left(\|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2 + \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^2\right) \|\nabla_3 V_+\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r} \|\nabla_3 V_-\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r} dt\right)^{\frac{2}{\vartheta}'}
\]
We emphasize that the constants in $E(t)$ may change from line to line.
Applying Hölder’s inequality gives
\[
\|III_1(t)\| \leq E(T) \left(\int_0^t \|\nabla_3 V_+\|_{L^2}^2 dt\right)^{\frac{2}{\vartheta}'} \left(\int_0^t \|\nabla_3 V_-\|_{L^2}^2 dt\right)^{\frac{2}{\vartheta}'}
\]
\[
\times \left(\int_0^t \left(\|u_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^p + \|b_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^p\right) \|\nabla_3 V_+\|_{L^2} \|\nabla_3 V_-\|_{L^2} dt\right)^{\frac{2}{\vartheta}'}
\]
\[
+ E(T) \left(\int_0^t \|\nabla_3 V_+\|_{L^2}^2 dt\right)^{\frac{2}{\vartheta}'} \left(\int_0^t \|\nabla_3 V_-\|_{L^2}^2 dt\right)^{\frac{2}{\vartheta}'}
\]
\[
\times \left(\int_0^t \left(\|u_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^p + \|b_3\|_{H^{\frac{3}{2} + \frac{1}{p}}^\vartheta, r}^p\right) \|\nabla_3 V_+\|_{L^2} \|\nabla_3 V_-\|_{L^2} dt\right)^{\frac{2}{\vartheta}'}
\]
Thus we deduce from (6.3) that
\[
|III_1(t)| \leq \frac{r-1}{3r^2} \int_0^t \left\| \nabla (\Gamma+)^\frac{r}{2} \right\|^2_{L^2} + \left\| \nabla (\Gamma-)^\frac{r}{2} \right\|^2_{L^2} dt'
\]
(6.2)
\[
+ \mathcal{E}(T) \left( \int_0^t \left( \|u^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} + \|b^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} \right) \left( \|(\Gamma+)^\frac{r}{2}\|_{L^2} + \|(\Gamma-)^\frac{r}{2}\|_{L^2} \right)^2 (1+2\alpha(r)) dt' \right)^\frac{1}{1+2\alpha(r)}.
\]
Similarly, we have
\[
|III_2(t)| \leq \frac{r-1}{3r^2} \int_0^t \left\| \nabla (\Gamma+)^\frac{r}{2} \right\|^2_{L^2} + \left\| \nabla (\Gamma-)^\frac{r}{2} \right\|^2_{L^2} dt'
\]
(6.3)
\[
+ \mathcal{E}(T) \left( \int_0^t \left( \|u^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} + \|b^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} \right) \left( \|(\Gamma+)^\frac{r}{2}\|_{L^2} + \|(\Gamma-)^\frac{r}{2}\|_{L^2} \right)^2 (1+2\alpha(r)) dt' \right)^\frac{1}{1+2\alpha(r)}.
\]
For the last term, we get, by applying Hölder’s inequality, that
\[
\left( \int_0^t \left( \|u^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} + \|b^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} \right) \left( \|(\Gamma+)^\frac{r}{2}\|_{L^2} + \|(\Gamma-)^\frac{r}{2}\|_{L^2} \right)^2 (1+2\alpha(r)) dt' \right)^\frac{1}{1+2\alpha(r)}
\]
\[
\leq \left( \int_0^t \left( \|u^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} + \|b^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} \right) dt' \right)^\frac{1+2\alpha(r)}{1+2\alpha(r)} \left( \int_0^t \left( \|(\Gamma+)^\frac{r}{2}\|_{L^2} + \|(\Gamma-)^\frac{r}{2}\|_{L^2} \right)^2 (1+2\alpha(r)) dt' \right)^\frac{1+2\alpha(r)}{1+2\alpha(r)},
\]
and the definition of $\mathcal{E}(T)$ implies
\[
\mathcal{E}(T) \cdot \left( \int_0^t \left( \|u^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} + \|b^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} \right) dt' \right)^\frac{1+2\alpha(r)}{1+2\alpha(r)} \leq \mathcal{E}(T).
\]
Thus we deduce from (6.3) that
\[
|III_2(t)| \leq \frac{r-1}{3r^2} \int_0^t \left\| \nabla (\Gamma+)^\frac{r}{2} \right\|^2_{L^2} + \left\| \nabla (\Gamma-)^\frac{r}{2} \right\|^2_{L^2} dt'
\]
(6.4)
\[
+ \mathcal{E}(T) \left( \int_0^t \left( \|u^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} + \|b^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} \right) \left( \|(\Gamma+)^\frac{r}{2}\|_{L^2} + \|(\Gamma-)^\frac{r}{2}\|_{L^2} \right)^2 (1+2\alpha(r)) dt' \right)^\frac{1}{1+2\alpha(r)}.
\]
Inserting (6.2) and (6.4) into (6.1) gives, for any $t \in [0, T]$,
\[
\mathcal{E}(T) \left( \int_0^t \left\| \nabla \partial_t V \right\|^2_{H^{\delta+r}} + \left\| \nabla \partial_t V^{-1} \right\|^2_{H^{\delta+r}} dt' \right) \leq \frac{2(r-1)}{3r^2} \int_0^t \left\| \nabla (\Gamma+)^\frac{r}{2} \right\|^2_{L^2} + \left\| \nabla (\Gamma-)^\frac{r}{2} \right\|^2_{L^2} dt'
\]
\[
+ \mathcal{E}(T) \left( \|\Omega_0\|_{L^r} + \|j_0\|_{L^r} \right) + \mathcal{E}(T) \left( \int_0^t \left( \|u^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} + \|b^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} \right) \left( \|(\Gamma+)^\frac{r}{2}\|_{L^2} + \|(\Gamma-)^\frac{r}{2}\|_{L^2} \right)^2 (1+2\alpha(r)) dt' \right)^\frac{1}{1+2\alpha(r)}.
\]
Then inserting the above inequality into the right hand side of (2.4) gives
\[
\frac{1}{r} \left( \|(\Gamma+)^\frac{r}{2}(t)\|_{L^2} + \|(\Gamma-)^\frac{r}{2}(t)\|_{L^2} \right) + \frac{r-1}{r^2} \int_0^t \left\| \nabla (\Gamma+)^\frac{r}{2} \right\|^2_{L^2} + \left\| \nabla (\Gamma-)^\frac{r}{2} \right\|^2_{L^2} dt'
\]
\[
\leq \left( \frac{2}{r} + 1 \right) \mathcal{E}(T) \left( \|\Omega_0\|_{L^r} + \|j_0\|_{L^r} \right)
\]
\[
+ \mathcal{E}(T) \left( \int_0^t \left( \|u^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} + \|b^3\|^p_{H^\frac{3}{2} + \frac{r}{p}} \right) \left( \|(\Gamma+)^\frac{r}{2}\|_{L^2} + \|(\Gamma-)^\frac{r}{2}\|_{L^2} \right)^2 (1+2\alpha(r)) dt' \right)^\frac{1}{1+2\alpha(r)}.
\]
Taking the power $1 + 2p\sigma(r)$ of this inequality and using the elementary inequality
\[(a + b)^\sigma \sim a^\sigma + b^\sigma,\]
for any positive index $\sigma$ and $a, b > 0$, then we obtain for any $t \in [0, T]$,
\begin{align*}
\|[(\Gamma_+)^{1/2}(1 + 2p\sigma(r))]_L^2 + \|[(\Gamma_-)^{1/2}(1 + 2p\sigma(r))]_L^2 & \|^{1 + 2p\sigma(r)}
+ \left(\int_0^t \left(\|\nabla [(\Gamma_+)^{1/2}(1 + 2p\sigma(r))]_L^2 + \|\nabla [(\Gamma_-)^{1/2}(1 + 2p\sigma(r))]_L^2\right) dt\right)^{1 + 2p\sigma(r)}
\leq \mathcal{E}(T) \left(\|\Omega_0\|_{L_p}^{(1 + 2p\sigma(r))} + \|j_0\|_{L_p}^{(1 + 2p\sigma(r))}\right)
+ \mathcal{E}(T) \left(\int_0^t \left(\|u^3\|_{H^{1/2} + \frac{2}{p}}^p + \|b^3\|_{H^{1/2} + \frac{2}{p}}^p\right) \left(\|[(\Gamma_+)^{1/2}(1 + 2p\sigma(r))]_L^2 + \|[(\Gamma_-)^{1/2}(1 + 2p\sigma(r))]_L^2\right) dt\right).
\end{align*}

Then Gronwall's inequality leads to (2.7), which completes the proof of the first part of Proposition 2.3.

Finally it follows from Proposition 2.2, Hölder’s inequality and (2.7) that
\begin{align*}
\left(\|\partial_3 V_+(t)\|_{H^{\sigma, r}} + \|\partial_3 V_-(t)\|_{H^{\sigma, r}}\right) + \int_0^t \left(\|\nabla \partial_3 V_+\|_{H^{\sigma, r}} + \|\nabla \partial_3 V_-\|_{H^{\sigma, r}}\right) dt
\leq \mathcal{E}(t) \left(\|\Omega_0\|_{L_p}^2 + \|j_0\|_{L_p}^2\right)
+ \left(\|u^3\|_{L_t^p(H^{1/2} + \frac{2}{p})}^2 + \|b^3\|_{L_t^p(H^{1/2} + \frac{2}{p})}^2\right) \left(\|[(\Gamma_+)^{1/2}(1 + 2p\sigma(r))]_L^2 + \|[(\Gamma_-)^{1/2}(1 + 2p\sigma(r))]_L^2\right)
\times \left(\|\nabla [(\Gamma_+)^{1/2}(1 + 2p\sigma(r))]_L^2 + \|\nabla [(\Gamma_-)^{1/2}(1 + 2p\sigma(r))]_L^2\right) + \left(\|u^3\|_{L_t^2(H^{1/2} + \frac{2}{p})}^2 + \|b^3\|_{L_t^2(H^{1/2} + \frac{2}{p})}^2\right)
\times \left(\|[(\Gamma_+)^{1/2}(1 + 2p\sigma(r))]_L^2 + \|[(\Gamma_-)^{1/2}(1 + 2p\sigma(r))]_L^2\right)\left(\|\nabla [(\Gamma_+)^{1/2}(1 + 2p\sigma(r))]_L^2 + \|\nabla [(\Gamma_-)^{1/2}(1 + 2p\sigma(r))]_L^2\right)
\leq \exp(\mathcal{E}(t)) \left(\|\Omega_0\|_{L_p}^2 + \|j_0\|_{L_p}^2\right).
\end{align*}

This completes the proof of Proposition 2.3.

7. Conclusion of the proof of Theorem 2.1

By Proposition 2.3, if we assume
\[(7.1) \int_0^T \|u^3\|_{H^{1/2} + \frac{2}{p}}^p + \|b\|_{H^{1/2} + \frac{2}{p}}^p dt' < \infty,
\]
we know that all the quantities in (2.9) are finite. We want to prove that all the above quantities prevent the solution from blowing up. In order to do so, let us recall the following theorem of anisotropic condition for blow up, which is a generalization of Theorem 2.1 of [6] for the classical Navier-Stokes system:

**Theorem 7.1. (Proposition 4.1 of [9])** Let $u, b \in C([0, T^*]; \dot{H}^{1/2}(\mathbb{R}^3)) \cap L^2([0, T^*]; \dot{H}^{1/2}(\mathbb{R}^3))$ solve the MHD system (1.1). If $T^* < \infty$, then for any $p_{k, l} = \in \ell^\infty, \ k, l \in \{1, 2, 3\}$, one has
\[
\sum_{k, l=1}^3 \int_0^{T^*} \left(\|\partial_3 u^k(t')\|_{B_p^{k,l}} + \|\partial_3 b^k(t')\|_{B_p^{k,l}}\right) dt' = \infty,
\]
where $B_p \overset{\text{def}}{=} \dot{B}^{-2+\frac{2}{p}}_{\infty, \infty}$.
Now let us present the proof of Theorem 2.1. Firstly, for any $p \in [4, \infty]$, 
\[
\max_{1 \leq l \leq 3} \left( \| \partial_l u^3 \|_{B_p} + \| \partial_l b^3 \|_{B_p} \right) \lesssim \sup_{j \in \mathbb{Z}} 2^{j(\frac{3}{2} + \frac{2}{p})} (\| \Delta_j u^3 \|_{L^2} + \| \Delta_j b^3 \|_{L^2}) \lesssim \| u^3 \|_{H^{\frac{3}{2} + \frac{2}{p}}} + \| b^3 \|_{H^{\frac{3}{2} + \frac{2}{p}}}
\]
by Bernstein's inequality, which implies
\[
(7.3) \quad \max_{1 \leq l \leq 3} \int_0^{T^*} \| \partial_l u^3 \|_{B_p}^p + \| \partial_l b^3 \|_{B_p}^p \, dt' \lesssim \int_0^{T^*} \| u^3 \|_{H^{\frac{3}{2} + \frac{2}{p}}}^p + \| b^3 \|_{H^{\frac{3}{2} + \frac{2}{p}}}^p \, dt' \lesssim 1.
\]
Next, using Bernstein’s inequality and the continuity of Riesz transform in $L^p$, $\forall p \in [4, \infty]$, we have
\[
\int_0^{T^*} \| \nabla_h u_{\text{div}}^h \|_{B_p}^p + \| \nabla_h b_{\text{div}}^h \|_{B_p}^p \, dt' \lesssim \int_0^{T^*} \| \nabla_h \nabla_h \Delta_h^{-1} \partial_3 u^3 \|_{B_p}^p + \| \nabla_h \nabla_h \Delta_h^{-1} \partial_3 b^3 \|_{B_p}^p \, dt' \lesssim \int_0^{T^*} \| u^3 \|_{H^{\frac{3}{2} + \frac{2}{p}}}^p + \| b^3 \|_{H^{\frac{3}{2} + \frac{2}{p}}}^p \, dt' \lesssim 1.
\]
The other components of the matrix $\nabla u$ and $\nabla b$ can be estimated with norms which are not of scaling zero, namely norms related to $\omega$ and $d$ which have the scaling of $L^r$ norm as shown in (2.9). To proceed further, we first get for any function $a$
\[
\| \Delta_j a \|_{L^\infty} \lesssim \sum_{k \leq j+1, l \leq j+1} 2^{k\frac{3}{2}} \| \Delta_k^h \Delta_l^v a \|_{L^2} \lesssim \| a \|_{H^{1-3\alpha(r)+\theta,-\theta}} \sum_{k \leq j+1, l \leq j+1} 2^{k(3\alpha(r)-\theta)\frac{3}{2} + \frac{2}{p}} \lesssim 2^{j(\frac{3}{2}+3\alpha(r))} \| a \|_{H^{1-3\alpha(r)+\theta,-\theta}},
\]
because $-(\frac{1}{2} + 3\alpha(r)) = -2 + \frac{3}{2}$, this leads to
\[
(7.5) \quad \| a \|_{B_q(r)} \lesssim \| a \|_{H^{1-3\alpha(r)+\theta,-\theta}},
\]
where $q(r) \overset{\text{def}}{=} \frac{2r'}{r'}$. As $r \in \left[\frac{3}{2}, 2\right]$, $q(r)$ is in $[\frac{3}{2}, 2]$. Applying mean inequality and triangle inequality for the Besov norm, then (7.5), Hölder’s inequality and (2.8), we deduce that
\[
\int_0^{T^*} \| \partial_3 u_{\text{div}}^h \|_{B_q(r)}^{q(r)} + \| \partial_3 b_{\text{div}}^h \|_{B_q(r)}^{q(r)} \, dt' \lesssim \int_0^{T^*} \| \partial_3 (u_{\text{div}}^h + b_{\text{div}}^h) \|_{B_q(r)}^{q(r)} + \| \partial_3 (u_{\text{div}}^h - b_{\text{div}}^h) \|_{B_q(r)}^{q(r)} \, dt' \lesssim \int_0^{T^*} \| \nabla_h \nabla_h \Delta_h^{-1} \partial_3^2 V + \| \partial_3^2 V \|_{B_q(r)}^{q(r)} + \| \nabla_h \nabla_h \Delta_h^{-1} \partial_3^2 V \|_{B_q(r)}^{q(r)} \, dt' \lesssim \int_0^{T^*} \| \partial_3^2 V \|_{L^\infty}^{\frac{2}{r'}} + \| \partial_3^2 V \|_{L^\infty}^{\frac{2}{r'}} \lesssim 1.
\]
Combining (7.7) and (7.9) gives
\[
\|\nabla_h u_h^{\text{curl}}\|_{B^q_{(r)}} + \|\nabla_h b_h^{\text{curl}}\|_{B^q_{(r)}} \lesssim \|\nabla_h (u_{\text{curl}} + b_{\text{curl}})\|_{B^q_{(r)}} + \|\nabla_h (u_{\text{curl}} - b_{\text{curl}})\|_{B^q_{(r)}}
\]
\[
= \|\nabla_h \nabla_h^+ \Delta_h^{-1} \Gamma_{+}\|_{B^q_{(r)}} + \|\nabla_h \nabla_h^+ \Delta_h^{-1} \Gamma_{-}\|_{B^q_{(r)}}
\]
\[
\lesssim \|\partial_h^2 \Delta_h^{-1} \Gamma_{+}\|_{\dot{H}^{1-3\alpha(r)}} + \|\partial_h^2 \Delta_h^{-1} \Gamma_{-}\|_{\dot{H}^{1-3\alpha(r)}}
\]
\[
\lesssim \|\nabla \Gamma_+\|_{\dot{H}^{3\alpha(r)}} + \|\nabla \Gamma_-\|_{\dot{H}^{3\alpha(r)}}
\]
by (7.5), continuity of Riesz transform in $L^p$, $\forall p \in [1, \infty]$ and the Sobolev embedding $L^r \hookrightarrow \dot{H}^{-3\alpha(r)}$. Next, we use anisotropic Bony's decomposition and Bernstein's inequality to get
\[
\|\hat{\Delta}_j \partial_3 (u_{\text{curl}}^{\pm} + b_{\text{curl}}^{\pm})\|_{L^\infty} \lesssim \sum_{k \in j+1, l \in j+1} \|\hat{\Delta}_j \hat{\Delta}_k^{3\alpha} \partial_3 \nabla_h^+ \Delta_h^{-1}(\omega \pm d)\|_{L^\infty}
\]
\[
\lesssim \sum_{k \in j+1, l \in j+1} 2^k \|\partial_3(\omega \pm d)\|_{L^r}
\]
\[
\lesssim 2^j \|\partial_3(\omega \pm d)\|_{L^r}.
\]
Recall $q(r) = \frac{2q}{3}$, we find that $-(\frac{3}{2} - 1) = -2 + \frac{2}{q(r)}$. Thus (7.8) actually leads to
\[
\|\partial_3 (u_{\text{curl}}^{\pm} + b_{\text{curl}}^{\pm})\|_{B^q_{(r)}} \lesssim \|\partial_3(\omega \pm d)\|_{L^r}.
\]
Combining (7.7) and (7.9) gives
\[
\|\nabla u_h^{\text{curl}}\|_{B^q_{(r)}} + \|\nabla b_h^{\text{curl}}\|_{B^q_{(r)}} \lesssim \|\nabla \Gamma_+\|_{L^r} + \|\nabla \Gamma_-\|_{L^r}
\]
\[
\lesssim \|\nabla(\Gamma_+)\|_{L^2} \|\nabla(\Gamma_+)\|_{L^2}^{\frac{q(r)-1}{2}} + \|\nabla(\Gamma_-)\|_{L^2} \|\nabla(\Gamma_-)\|_{L^2}^{\frac{q(r)-1}{2}},
\]
where we used (3.3) in the last step. Then combine (7.10) with (2.7), we get
\[
\int_0^{T^*} \left( \|\nabla u_h^{\text{curl}}(t')\|_{B^q_{(r)}} + \|\nabla b_h^{\text{curl}}(t')\|_{B^q_{(r)}} \right) dt'
\]
\[
\lesssim (T^*)^{(1-\frac{q(r)-1}{2})} \|\nabla(\Gamma_+)\|_{L^\infty([0,T^*];L^2)}^{q(r)-1} \left( \int_0^{T^*} \|\nabla(\Gamma_+)\|_{L^2}^{q(r)-1} dt' \right)^{\frac{q(r)}{2}}
\]
\[
+ (T^*)^{(1-\frac{q(r)-1}{2})} \|\nabla(\Gamma_-)\|_{L^\infty([0,T^*];L^2)}^{q(r)-1} \left( \int_0^{T^*} \|\nabla(\Gamma_-)\|_{L^2}^{q(r)-1} dt' \right)^{\frac{q(r)}{2}} \lesssim 1.
\]
Together with inequalities (7.3), (7.4), (7.6), (7.11) and Theorem 7.1, we conclude the proof of Theorem 2.1.

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