Analysis and simulation of a variational stabilization for the Helmholtz equation with noisy Cauchy data

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Abstract
This article considers a Cauchy problem of Helmholtz equations whose solution is well known to be exponentially unstable with respect to the inputs. In the framework of variational quasi-reversibility method, a Fourier truncation is applied to appropriately perturb the underlying problem, which allows us to obtain a stable approximate solution. The corresponding approximate problem is of a hyperbolic equation, which is also a crucial aspect of this approach. Error estimates between the approximate and true solutions are derived with respect to the noise level. From this analysis, the Lipschitz stability with respect to the noise level follows. Some numerical examples are provided to see how our numerical algorithm works well.

Keywords Stabilization · Helmholtz equation · Ill-posed problems · Convergence · Error estimates

Mathematics Subject Classification 65J05 · 65J20 · 35K92

1 Introduction

1.1 Statement of the Cauchy problem

In this work, we are concerned with the reconstruction problem of electromagnetic field from its knowledge on a part of boundary of the physical region \( \Omega \). Here, \( \Omega =\)
(0, 1) \times (0, 1) is our computational domain of interest, but it can be extended easily to 
(0, a_1) \times (0, a_2), where a_1, a_2 are two positive numbers. Often, the propagation of 
the electromagnetic wave field is governed by the system of the Maxwell’s equations for 
the electric field \( E = E(x, y, t) \) and the magnetic field \( B = B(x, y, t) \). Considering 
\( \Omega \) as a homogeneous medium in a region with free currents and charges, this system 
can be reduced to the classical wave equations, cf. [1],

\[
\frac{\partial^2 E}{\partial t^2} - c^2 \Delta E = 0, \quad \frac{\partial^2 B}{\partial t^2} - c^2 \Delta B = 0 \quad \text{in } \Omega \times (0, T),
\]

(1)

where \( c > 0 \) is the speed of light and \( T > 0 \) is the travel time. Consider the frequency 
\( \omega > 0 \). For \( i = \sqrt{-1} \), we take \( E(x, y, t) = e^{i\omega t} E(x, y) \) and \( B(x, y, t) = e^{i\omega t} B(x, y) \).
Then, setting \( k = \omega / c > 0 \), it follows from system (1) that

\[
\Delta E(x, y) + k^2 E(x, y) = 0, \quad \Delta B(x, y) + k^2 B(x, y) = 0,
\]

(2)

which form a system of Helmholtz equations. Since system (2) is uncoupled and linear with respect to each component of \( E(x, y) \) and \( B(x, y) \), it is pertinent to solve the following model:

\[
\Delta u(x, y) + k^2 u(x, y) = 0 \quad \text{in } \Omega.
\]

(3)

Note that in (2), \( E(x, y) \) and \( B(x, y) \) are complex-valued components, but it is sufficient to find a real-valued function \( u = u(x, y) \) in (3). Physically, fields vanish far from the axes and thus, we can assume that the electromagnetic field vanishes on the sides \( \{ y = 0 \} \) and \( \{ y = 1 \} \) of the computational domain \( \Omega \). Mathematically, we consider

\[
u(x, 0) = u(x, 1) = 0 \quad \text{for } x \in (0, 1).
\]

(4)

On the other hand, we assume to measure the electromagnetic Cauchy data at \( x = 0 \),

\[
u(0, y) = u_0(y), \quad u_x(0, y) = u_1(y) \quad \text{for } y \in (0, 1).
\]

(5)

Cf. [2], we remark that the second data (i.e. the Neumann data at \( x = 0 \)) in (5) can be reduced to the zero boundary condition. In fact, let \( U = U(x, y) \) be a solution to the following system:

\[
\begin{aligned}
\Delta U(x, y) + k^2 U(x, y) &= 0 \quad \text{in } \Omega, \\
U(x, 0) &= U(x, 1) = 0 \quad \text{for } x \in (0, 1), \\
U_x(0, y) &= u_1(y), \quad U(1, y) = 0 \quad \text{for } y \in (0, 1).
\end{aligned}
\]

(6)

Next, consider \( V = V(x, y) \) as a solution to the following system:

\[
\begin{aligned}
\Delta V(x, y) + k^2 V(x, y) &= 0 \quad \text{in } \Omega, \\
V(x, 0) &= V(x, 1) = 0 \quad \text{for } x \in (0, 1), \\
V(0, y) &= u_0(y) - U(0, y), \quad V_x(0, y) = 0 \quad \text{for } y \in (0, 1).
\end{aligned}
\]

(7)
With (6) and (7), it is clear that the solution $u$ to system (3)–(5) can be computed via $u = U + V$. By [2, Lemma 1], we know that system (6) is well-posed with $U$ in $H^2(\Omega)$ when $u_1 \in L^2(0, 1)$ and thus, $U(0, y)$ exists in $H^2(0, 1)$ by the embedding $H^2(\Omega) \subset C([0, 1]; H^2(0, 1))$. Henceforth, from (7), instead of working on the Cauchy data (5) we can assume that $u_1 = 0$ in (5) in our analysis below. Details of the well-posedness of (6) are found in the Appendix.

Combining (3), (4) and (5) with $u_1 = 0$ forms our Cauchy problem for the Helmholtz equation. In this scenario, we want to reconstruct the whole wave field in $\Omega$ and especially, the field at the boundary $x = 1$.

**Remark 1** Cf. the appendix of [3], if the incident electric wave field has only one non-zero component, then the propagation of this component in a heterogeneous medium is governed equally well by a single Helmholtz equation. In other words, the Helmholtz equation may play an equal role as the Maxwell’s system when the medium is no longer homogeneous as assumed above.

### 1.2 Historical remarks and our goals

The Cauchy problem for Helmholtz equations (as well as elliptic equations) is well-studied in the Inverse and Ill-posed Problems community. Suffering from the Hadamard instability, this problem is severely ill-posed as the degree of ill-posedness is infinite; see [4] for distinctive classes of ill-posed problems based on the degree of ill-posedness. To overcome the natural instability, there are many researches devoted to regularization of such a Cauchy problem. Those are essentially spectral-based and optimization-based methods. The existing literature on these two types of methods is huge. The spectral regularization method and its variants rely on suitable perturbation of the unbounded kernel involved in the explicit presentation of solution. The kernel can be stabilized by the perturbation of the original PDE or by the direct perturbation inside the kernel. The former perturbation may lead to the so-called PDE-based regularization method. The reader can be referred to the following fundamental works [5–8] and references cited therein for an overview of the spectral regularization method. The optimization-based regularization method is based on the construction of Tikhonov-like cost functionals involving the (strict) convexity; cf. e.g. [9–11]. The obtained minimizer is proved to approximate the true solution in a stable manner. It is necessary to mention here the works [12, 13], where a Carleman weight is appropriately applied to “convexify” the energy functional logarithmically. Lastly, we wish to mention that the Cauchy problem posed in unbounded domains has also been considered in, e.g., [14, 15].

We would like to study in this work a modified quasi-reversibility (QR) method, which is a PDE-based approach. The QR method was originally mentioned in the monograph by Lattès and Lions; see [16] by perturbing the unbounded operator - the main cause of the instability. The modified QR method under investigation has been commenced in the pioneering work [17], where the authors established two operators along with their conditional estimates to guarantee the strong convergence of the scheme solving quasi-linear parabolic equations backwards in time. The key ingredient of the method is that using a suitable perturbation, the ill-posed problem turns to be
a forward-like problem in which we can prove its conditional well-posedness. It is, on the other hand, certain that numerical solutions for forward problems are well-studied nowadays. Recently, this modified QR method has been applied to the Cauchy problem for the Laplace system in [18]. In this regard, we regularize the Cauchy-Laplace problem by the corresponding initial-value hyperbolic problem. Using the same idea, in the present work, we verify the applicability of this method to solve the Cauchy problem for the Helmholtz equation. By the possible involvement of large frequencies \( k \), we, however, remark that the perturbation should be chosen appropriately. Accordingly, we focus ourselves on specific perturbation and stabilized operators.

### 1.3 Organization of the paper

The paper is organized as follows. In Sect. 2, we recall some preliminaries concerning the ill-posedness of the Cauchy problem and how we derive the modified QR scheme from the original PDE. We also specify the perturbation operator and the corresponding stabilized operator in our regularized problem. Conditional estimates of these operators are deduced accordingly. Then, we analyze the conditional well-posedness of the regularized problem and the strong convergence of the scheme in Sect. 3. The Lipschitz stability of the scheme also follows. In Sect. 4, we investigate the corresponding iterative scheme. For the numerical part, various examples are provided in Sect. 5 to corroborate our theoretical analysis. Finally, we close the paper by some concluding remarks in Sect. 6.

### 2 Preliminaries

Let \( A \) be either a Banach space or a Hilbert space. We call \( A' \) the dual space of \( A \). For a certain Banach space \( A \), \( \| \cdot \|_A \) stands for the \( A \)-norm. When \( A \) is Hilbert, we define the \( A \)-norm of \( u \) as \( \| u \|^2_A = \langle u, u \rangle_A \), where \( \langle \cdot, \cdot \rangle_A \) is the corresponding inner product. Throughout the paper, we will use \( \langle \cdot, \cdot \rangle \) to indicate either the scalar product in \( L^2(0, 1) \) or the dual pairing of a continuous linear functional and an element of a function space. We thereby denote by \( \| \cdot \| \) the \( L^2(0, 1) \)-norm.

The Cauchy problem of Helmholtz equation is well known to be unstable with respect to any small perturbation of the data. Based on the zero Dirichlet boundary condition (4), the Laplace operator \( -\partial^2 / \partial y^2 \) is non-negative. According to the standard result for the Dirichlet eigenvalue problem, there exists an orthonormal basis \( \{ \phi_j \} \) of \( L^2(0, 1) \cap C^\infty [0, 1] \) and \( -d^2 / dy^2 \phi_j (y) = \mu_j \phi_j (y) \). The Dirichlet eigenvalues \( \mu_j \) in this case form an infinite sequence such that \( 0 \leq \mu_0 < \mu_1 < \mu_2 < \ldots \), and \( \lim_{j \to \infty} \mu_j = \infty \). It follows from (3) and (5) that we obtain the following initial-value differential system:

\[
\begin{aligned}
\frac{d^2}{dx^2} \{ u(x, \cdot), \phi_j \} - \lambda_{j,k} \{ u(x, \cdot), \phi_j \} &= 0, \\
\{ u(0, \cdot), \phi_j \} &= \{ u_0, \phi_j \}, \\
\frac{d}{dx} \{ u(0, \cdot), \phi_j \} &= 0.
\end{aligned}
\] (8)
In (8), \( \lambda_{j,k} = \mu_j - k^2 \). By this way, we solve system (8) in each of the following set of Fourier frequencies:

\[
A_1 := \{ j \in \mathbb{N} : \lambda_{j,k} > 0 \}, \quad A_2 := \{ j \in \mathbb{N} : \lambda_{j,k} = 0 \}, \quad A_3 := \{ j \in \mathbb{N} : \lambda_{j,k} < 0 \}.
\]

(9)

It is also straightforward to see that \( \phi_j(y) = \sqrt{2} \sin(j \pi y) \), \( \mu_j = j^2 \pi^2 \). In addition, \( \{ \phi_j / \sqrt{\mu_j} \}_{j \in \mathbb{N}^*} \) is an orthonormal basis of \( L^2(0, 1) \). Therefore, it holds that

\[
\| u_y \|^2 = \sum_{j \in \mathbb{N}^*} \left| \left\langle \frac{\phi_j'}{\sqrt{\mu_j}} \right\rangle \right|^2 = \sum_{j \in \mathbb{N}^*} \left| \left\langle \frac{\phi_j''}{\sqrt{\mu_j}} \right\rangle \right|^2 = \sum_{j \in \mathbb{N}^*} \mu_j \left| \langle u, \phi_j \rangle \right|^2.
\]

(10)

**Theorem 1** *The Fourier coefficient \( \langle u(x, \cdot), \phi_j \rangle \) has the form:*

\[
\langle u(x, \cdot), \phi_j \rangle = \begin{cases} 
\cosh \left( \sqrt{\lambda_{j,k}} x \right) \langle u_0, \phi_j \rangle & \text{in } A_1, \\
\cos \left( \sqrt{-\lambda_{j,k}} x \right) \langle u_0, \phi_j \rangle & \text{in } A_3.
\end{cases}
\]

(11)

**Proof** Proof of the theorem can be proceeded as in [2]. In \( A_1 \), solving system (8) gives

\[
\langle u(x, \cdot), \phi_j \rangle = C_1 e^{x \sqrt{\lambda_{j,k}}} + C_2 e^{-x \sqrt{\lambda_{j,k}}}.
\]

(12)

Therefore, when \( x = 0 \), it yields

\[
\langle u_0, \phi_j \rangle = \langle u(0, \cdot), \phi_j \rangle = C_1 + C_2.
\]

(13)

On the other hand, we compute that

\[
\frac{d}{dx} \left\langle u(x, \cdot), \phi_j \right\rangle = \sqrt{\lambda_{j,k}} \left( C_1 e^{x \sqrt{\lambda_{j,k}}} - C_2 e^{-x \sqrt{\lambda_{j,k}}} \right).
\]

When \( x = 0 \), we arrive at

\[
0 = \frac{d}{dx} \left\langle u(0, \cdot), \phi_j \right\rangle = \sqrt{\lambda_{j,k}} \left( C_1 - C_2 \right)
\]

(14)

Combining (13) and (14), we have \( C_1 = C_2 = \frac{1}{2} \langle u_0, \phi_j \rangle \). By back-substitution of these \( C_1 \) and \( C_2 \) into (12), the Fourier coefficient of \( u \) is formulated by

\[
\langle u(x, \cdot), \phi_j \rangle = \cosh \left( \sqrt{\lambda_{j,k}} x \right) \langle u_0, \phi_j \rangle.
\]

(15)

In \( A_3 \), we do the same way and obtain \( \langle u(x, \cdot), \phi_j \rangle = \cos \left( \sqrt{-\lambda_{j,k}} x \right) \langle u_0, \phi_j \rangle \). This completes the proof of the theorem.
Now, we show a very important relation of these Fourier frequencies in the following theorem.

**Theorem 2** Taking into account set $A_1$, the Fourier coefficient of $u$ satisfies the following relation:

$$
\lambda_{j,k} e^{(1-x)\sqrt{\lambda_{j,k}}} \left( u(x, \cdot), \phi_j \right) + \frac{1}{\sqrt{\lambda_{j,k}}} \left( u_x(x, \cdot), \phi_j \right) = \lambda_{j,k} \left( u(1, \cdot), \phi_j \right) + \sqrt{\lambda_{j,k}} \left( u_x(1, \cdot), \phi_j \right). 
$$

**Proof** From (15), we can compute that

$$
\frac{1}{\sqrt{\lambda_{j,k}}} \left( u_x(x, \cdot), \phi_j \right) = \sinh \left( \sqrt{\lambda_{j,k}} x \right) \left( u_0, \phi_j \right).
$$

Thus, we take $x = 1$ in (17) and in (15) and then combine the resulting formulations to obtain (16). We complete the proof of the theorem. $\square$

Practically, the data $u_0$ in (5) always contain noise of measurement. Therefore, we assume to have $u_0^\varepsilon \in H^1(0, 1)$ as the noisy data such that for $\varepsilon \in (0, 1)$,

$$
\|u_0^\varepsilon - u_0\|_{H^1(0, 1)} \leq \varepsilon.
$$

By Theorem 1, our Cauchy problem is exponentially unstable in $A_1$ due to the natural growth of the hyperbolic cosine function. Any small perturbation of the initial data $u_0$ may cause a huge error when computing solution $u$ of the Cauchy problem. In this work, we then adapt our recent modified quasi-reversibility method (cf. [18] for elliptic operators and [17] for parabolic operators) to solve our system (3)–(5). To do so, we rewrite (3) as

$$
u_{xx} - u_{yy} + 2u_{yy} + 2k^2u = k^2u.
$$

We then perturb (19) by a linear mapping $Q$ and take $P = Q + 2\partial^2/\partial y^2 + 2k^2$. Henceforth, we arrive at

$$
u_{xx} - u_{xy} + Pu = k^2u \quad \text{in } \Omega.
$$

It is worth mentioning that together with the boundary condition (4) and the Cauchy data (5) with measurement $u_0^\varepsilon$, (20) forms a system of linear wave equation. Herewith, $x$ becomes a parametric time variable. Since the noise level $\varepsilon$ is involved, we then seek a sequence of $\{u^\varepsilon\}_{\varepsilon > 0}$ satisfying the following system:

$$
\begin{cases}
    u_{xx}^\varepsilon - u_{yy}^\varepsilon + Pu^\varepsilon = k^2u^\varepsilon & \text{in } \Omega, \\
    u^\varepsilon(x, 0) = u^\varepsilon(x, 1) = 0 & \text{for } x \in (0, 1), \\
    u^\varepsilon(0, y) = u_0^\varepsilon(y), \quad u_x^\varepsilon(0, y) = 0 & \text{for } y \in (0, 1).
\end{cases}
$$
Cf. [17, 18], Q is called *perturbation* as it is to “absorb” high Fourier frequencies in the Laplace operator, and P is called *stabilized operator* as it only contains large enough Fourier frequencies serving for the convergence of the scheme. Let \( \gamma > 1 \). Consider \( B := \{ j \in A_1 : \lambda_{j,k} > \log^2(\gamma) \} \). We choose the following truncation operator:

\[
Q_1 u(x, \cdot) = 2 \sum_{j \in B} \lambda_{j,k} \langle u(x, \cdot), \phi_j \rangle \phi_j + 2 \sum_{j \in A_3} \lambda_{j,k} \langle u(x, \cdot), \phi_j \rangle \phi_j
\]

\[
:= Q_{11} u(x, \cdot) + Q_2 u(x, \cdot).
\]

(22)

As to the corresponding stabilized operator P, we find that

\[
P_1 u(x, \cdot) = 2 \sum_{j \in B \cup A_3} \lambda_{j,k} \langle u(x, \cdot), \phi_j \rangle \phi_j - 2 \sum_{j \in \mathbb{N}} \mu_j \langle u(x, \cdot), \phi_j \rangle \phi_j
\]

\[
+ 2k^2 \sum_{j \in \mathbb{N}} \langle u(x, \cdot), \phi_j \rangle \phi_j
\]

\[
= 2 \sum_{j \in B \cup A_3} (\lambda_{j,k} - \mu_j) \langle u(x, \cdot), \phi_j \rangle \phi_j
\]

\[
- 2 \sum_{j \in \mathbb{N} \setminus (B \cup A_3)} \mu_j \langle u(x, \cdot), \phi_j \rangle \phi_j + 2k^2 \sum_{j \in \mathbb{N}} \langle u(x, \cdot), \phi_j \rangle \phi_j
\]

\[
= -2 \sum_{j \in \mathbb{N} \setminus (B \cup A_3)} \lambda_{j,k} \langle u(x, \cdot), \phi_j \rangle \phi_j.
\]

In view of Parseval’s identity, we now estimate that

\[
\|Q_1 u(x, \cdot)\|^2 = 4 \sum_{j \in B} e^{-2\sqrt{\lambda_{j,k}}} \lambda_{j,k}^2 e^{2\sqrt{\lambda_{j,k}}} \|\langle u(x, \cdot), \phi_j \rangle\|^2
\]

\[
\leq 4\gamma^{-2} \sum_{j \in B} \lambda_{j,k}^2 e^{2\sqrt{\lambda_{j,k}}} \|\langle u(x, \cdot), \phi_j \rangle\|^2.
\]

(23)

By using (16) obtained in Theorem 2, we have

\[
\sup_{x \in [0,1]} \sum_{j \in B} \lambda_{j,k}^2 e^{2\sqrt{\lambda_{j,k}}} \|\langle u(x, \cdot), \phi_j \rangle\|^2
\]

\[
\leq \sup_{x \in [0,1]} \left[ \sum_{j \in B} \lambda_{j,k}^2 e^{2(1-x)} \sqrt{\lambda_{j,k}} \left( \| u(x, x, \cdot \rangle, \phi_j \| + \frac{1}{\sqrt{\lambda_{j,k}}} \| u_x(x, x, \cdot \rangle, \phi_j \|^2 \right) \right]
\]

\[
\leq \sum_{j \in B} (\lambda_{j,k} \langle u(1, \cdot), \phi_j \rangle + \sqrt{\lambda_{j,k}} \langle u_x(1, \cdot), \phi_j \rangle)^2 \leq 2 \| u(1, \cdot) \|_{H^2(0,1)}^2
\]

\[
+ 2 \| u_x(1, \cdot) \|_{H^1(0,1)}^2.
\]
by means of \( [u(x, \cdot), \phi_j] \mid [u_x(x, \cdot), \phi_j] \geq 0 \); cf. (15) and (17). Now we estimate \( Q_2u \) as follows. Observe that if \( \log(\gamma) \geq k \), then \( \mu_i - k^2 \geq k^2 - \mu_j > 0 \) for \( i \in B \) and \( j \in A_3 \). This means that \( \lambda_{i,k} \geq |\lambda_{j,k}| \) for \( i \in B \) and \( j \in A_3 \). Therefore, we estimate that

\[
\|Q_2u(x, \cdot)\|^2 = 4 \sum_{j \in A_3} |\lambda_{j,k}|^2 \|[u(x, \cdot), \phi_j]\|^2 \leq 4 \sum_{j \in B} |\lambda_{j,k}|^2 \|[u(x, \cdot), \phi_j]\|^2 = \|Q_1u(x, \cdot)\|^2. \tag{24}
\]

Henceforth, we can assume that the true solution satisfies \( u(1, \cdot) \in H^2(0, 1) \) and \( u_x(1, \cdot) \in H^1(0, 1) \) to gain the strong convergence of the scheme. Note now that \( P \) is computable, which is relevant to our numerical simulation, compared to many other modified kernel regularization methods. Moreover, since in \( \mathbb{N} \setminus (B \cup A_3) \) it holds that \( 0 \leq \lambda_{j,k} \leq \log^2(\gamma) \), we, according to Parseval’s identity and using (10), get that

\[
\|Pu(x, \cdot)\|^2 \leq 4 \log^2(\gamma) \|u_x(x, \cdot)\|^2. \tag{25}
\]

**Remark 2** In Sect. 3 below, we will prove that the approximate solution \( u^\varepsilon \) approaches \( u \) under an appropriate choice of \( \gamma \) dependent of the noise level \( \varepsilon \). Complying with that, we below denote our operators by \( Q_\varepsilon \) and \( P_\varepsilon \) in lieu of, as above, \( Q \) and \( P \), respectively.

### 3 Analysis of the regularization scheme

We now formulate theorems for the weak solvability of system (21) and convergence analysis of the corresponding regularization scheme. When doing so, we provide the definition of weak solution as follows.

**Definition 1** (Weak solution) For each \( \varepsilon > 0 \), a function \( u^\varepsilon : [0, 1] \to H^1_0(0, 1) \) is said to be a weak solution to system (21) if
- \( u^\varepsilon \in C([0, 1]; H^1_0(0, 1)), \partial_x u^\varepsilon \in C([0, 1]; L^2(0, 1)), \partial^2_{x^2} u^\varepsilon \in L^2(0, 1; (H^1(0, 1))') \);
- For every test function \( \psi \in H^1_0(0, 1) \), it holds that
  \[
  \left\langle \frac{\partial^2 u^\varepsilon}{\partial x^2}, \psi \right\rangle + \left\langle \frac{\partial u^\varepsilon}{\partial y}, \frac{\partial \psi}{\partial y} \right\rangle + \left\langle P_\varepsilon u^\varepsilon, \psi \right\rangle = k^2 \langle u^\varepsilon, \psi \rangle \quad \text{for a.e. in } (0, 1); \tag{26}
  \]
- \( u^\varepsilon(0, y) = u^\varepsilon_0 \in H^1(0, 1), \partial_x u^\varepsilon(0, y) = 0 \).

**Theorem 3** (Existence and uniqueness of a weak regularized solution) For each \( \varepsilon > 0 \), system (21) admits a unique weak solution in the sense of Definition 1. Moreover, it holds that \( u^\varepsilon \in C([0, 1]; H^1_0(0, 1)) \) and \( \partial_x u^\varepsilon \in C([0, 1]; L^2(0, 1)) \).
Proof To prove this theorem, we employ the standard Galerkin approximation. Consider the $n$-dimensional of $H^1_0(0, 1)$ generated by $\phi_0, \phi_1, \ldots, \phi_n$. For each $n \in \mathbb{N}$, we take into account the following Galerkin projection for approximation of (21):

$$u_n^\varepsilon(x, y) = \sum_{j=0}^{n} U_{jn}^\varepsilon(x)\phi_j(y).$$  \hfill (27)

This function $u_n^\varepsilon$ is hereby the solution of the following approximate equation:

$$\left\langle \frac{\partial^2 u_n^\varepsilon}{\partial x^2}, \psi \right\rangle + \left\langle \frac{\partial u_n^\varepsilon}{\partial y}, \frac{\partial \psi}{\partial y} \right\rangle + \left\langle P_{\varepsilon} u_n^\varepsilon, \psi \right\rangle = k^2 \left\langle u_n^\varepsilon, \psi \right\rangle$$  \hfill (28)

This Galerkin equation is endowed with the initial data $\partial_x u_n^\varepsilon(0, y) = 0$ and

$$u_n^\varepsilon(0, y) = \sum_{j=0}^{n} (U_0^\varepsilon)_{jn} \phi_j(y) \xrightarrow{\text{strongly in } H^1(0, 1)} u_0^\varepsilon$$  \hfill (29)

Now, let $\psi = \phi_j$, where recall that $\{\phi_j\}$ is the orthonormal basis of $L^2(0, 1)$. Then functions $U_{jn}^\varepsilon$ are solutions to the Cauchy problem for the system of $n$ vectorial ordinary differential equations:

$$\begin{cases}
\begin{align*}
\frac{d^2}{dx^2} U_{jn}^\varepsilon + (\mu_j - k^2) U_{jn}^\varepsilon + \sum_{i=0}^{n} U_{jn}^\varepsilon \langle P_{\varepsilon} \phi_i, \phi_j \rangle &= 0, \\
U_{jn}^\varepsilon(0) &= (U_0^\varepsilon)_{jn}, \\
\frac{d}{dx} U_{jn}^\varepsilon(0) &= 0.
\end{align*}
\end{cases}$$  \hfill (30)

For any $n \in \mathbb{N}$, we put $Z_{jn}^\varepsilon = dU_{jn}^\varepsilon / dx$. It then deduces from (30) that

$$\frac{d}{dx} \begin{bmatrix} U_{jn}^\varepsilon \\ Z_{jn}^\varepsilon \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k^2 - \mu_j & 0 \end{bmatrix} \begin{bmatrix} U_{jn}^\varepsilon \\ Z_{jn}^\varepsilon \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{i=0}^{n} U_{jn}^\varepsilon \langle P_{\varepsilon} \phi_i, \phi_j \rangle \end{bmatrix}.$$  

Let $z_{jn}^\varepsilon = [U_{jn}^\varepsilon, Z_{jn}^\varepsilon]^T$. Then solving the above closed-form initial-value differential problem, we obtain the following integral equation:

$$z_{jn}^\varepsilon(x) = z_{jn}^\varepsilon(0) + A_j^k \int_0^x z_{jn}^\varepsilon(s)ds + \int_0^x F_j(z^\varepsilon)(s)ds.$$  \hfill (31)

In (31), we denote by

$$A_j^k = \begin{bmatrix} 0 & 1 \\ k^2 - \mu_j & 0 \end{bmatrix}, \quad F_j(z^\varepsilon) = \begin{bmatrix} 0 \\ \sum_{i=0}^{n} U_{jn}^\varepsilon \langle P_{\varepsilon} \phi_i, \phi_j \rangle \end{bmatrix}. \quad \Box$$
We now define \( z^e = \{z_{0n}^e, z_{1n}^e, \ldots, z_{nn}^e\} \in \mathbb{R}^{2(n+1)} \) and denote \( H_j[z^e] \) by the right-hand side of (31). This results in the fixed-point form \( z^e(x) = H[z^e](x) \) where \( H[z^e] = [H_0(z^e), H_1(z^e), \ldots, H_n(z^e)] \). Define the norm of \( Y = C([0, 1]; \mathbb{R}^{2(n+1)}) \) as

\[
\|c\|_Y = \sup_{x \in [0, 1]} \sum_{j=0}^{n} |c_j(x)|, \quad c(x) = [c_0(x), c_1(x), \ldots, c_n(x)] \in \mathbb{R}^{2(n+1)}.
\] (32)

We claim that there exists \( m_0 \in \mathbb{N}^* \) such that the operator \( H^{m_0} := H[H^{m_0-1}] : Y \to Y \) is a contraction mapping. Indeed, by induction we can prove that

\[
|H^m_j[z_1^e](x) - H^m_j[z_2^e](x)| \leq \left[ \sqrt{1 + (k^2 - \mu_j)^2} + 2 \log(y) \right] \frac{x^m}{m!} \|z_1^e - z_2^e\|_Y
\] (33)

for \( m \in \mathbb{N}^* \) and for any \( z_1^e, z_2^e \in Y \). Observe that the inductive hypothesis is true when \( m = 1 \). In particular, in view of the fact that

\[
\{P_{x, \phi_i, \phi_j}\} = \begin{cases} -2\lambda_{i,k} & \text{if } i = j \in \mathbb{N}\setminus(B \cup A_3), \\ 0 & \text{elsewhere,} \end{cases}
\]

we can estimate that

\[
|H_j[z_1^e](x) - H_j[z_2^e](x)| \leq \int_0^x \left( \sqrt{1 + (k^2 - \mu_j)^2} |z_1^e(s) - z_2^e(s)| \right) ds 
+ \sum_{i=0}^{n} |U_{1i}^e - U_{2i}^e| \left| \{P_{x, \phi_i, \phi_j}\} \right| ds 
\leq \int_0^x \left( \sqrt{1 + (k^2 - \mu_j)^2} |z_1^e(s) - z_2^e(s)| \right) ds 
+ \sum_{i=0}^{n} 2 |\lambda_{i,k}| \left| U_{1i}^e - U_{2i}^e \right| ds 
\leq \left[ \sqrt{1 + (k^2 - \mu_j)^2} + 2 \log(y) \right] x \|z_1^e - z_2^e\|_Y. \) (34)

For \( m = m_0 > 1 \), we assume that

\[
|H_j^{m_0}[z_1^e](x) - H_j^{m_0}[z_2^e](x)| \leq \left[ \sqrt{1 + (k^2 - \mu_j)^2} + 2 \log(y) \right] \frac{x^{m_0}}{m_0!} \|z_1^e - z_2^e\|_Y.
\]

We then want to prove that (33) also holds true for \( m = m_0 + 1 \). Using the same token as in (34), we estimate that

\[
|H_j^{m_0+1}[z_1^e](x) - H_j^{m_0+1}[z_2^e](x)|
\]
\[
\begin{align*}
&\leq \int_0^x \left[1 + (k^2 - \mu_j)^2 + 2 \log (\gamma) \right] |H_j^{m_0}[z_1^e](s) - H_j^{m_0}[z_2^e](s)| ds \\
&\leq \int_0^x \left[1 + (k^2 - \mu_j)^2 + 2 \log (\gamma) \right] m_0^{m_0 + 1} \frac{s^{m_0}}{m_0!} \|z_1^e - z_2^e\| \gamma ds \\
&= \left[1 + (k^2 - \mu_j)^2 + 2 \log (\gamma) \right] m_0^{m_0 + 1} \frac{x^{m_0 + 1}}{(m_0 + 1)!} \|z_1^e - z_2^e\| \gamma.
\end{align*}
\]

Henceforth, our inductive hypothesis (33) is true for any \( m \in \mathbb{N} \). It also leads to the following estimate in the norm of \( Y \):

\[
\| H^m[z_1^e] - H^m[z_2^e] \| \leq \sum_{j=0}^{n} \left[1 + (k^2 - \mu_j)^2 + 2 \log (\gamma) \right] \frac{x^m}{m!} \|z_1^e - z_2^e\| \gamma.
\]

Since the following limit holds true

\[
\lim_{m \to \infty} \sum_{j=0}^{n} \left[1 + (k^2 - \mu_j)^2 + 2 \log (\gamma) \right] \frac{x^m}{m!} = 0
\]

we then can find \( m_0 \in \mathbb{N} \) sufficiently large such that

\[
\sum_{j=0}^{n} \left[1 + (k^2 - \mu_j)^2 + 2 \log (\gamma) \right] \frac{x^m}{m!} \leq 1.
\]

This clearly indicates the existence of a constant \( K \in [0, 1) \) satisfies

\[
\| H^{m_0}[z_1^e] - H^{m_0}[z_2^e] \| \leq K \|z_1^e - z_2^e\| \gamma.
\]

In other words, \( H^{m_0} \) is a contraction mapping from \( Y \) onto itself. By the Banach fixed-point theorem, there exists a unique \( z^e \in Y \) such that \( H^{m_0}[z^e] = z^e \). As \( H^{m_0}[H[z^e]] = H[H^{m_0}[z^e]] = H[z^e] \), the integral equation \( z^e = H[z^e] \) admits a unique solution in \( Y \). Hence, this results in the existence and uniqueness of \( U_{jn}^e \in C^1([0, 1]) \) solutions to system (30) for any fixed \( n \in \mathbb{N} \).

By \( U_{jn}^e \in C^1([0, 1]) \), we have \( \partial_x u_n^e \in C([0, 1]; S_n) \). Multiply both sides of (28) by \( e^{-rx} \) and then put \( v_n^e = e^{-rx} u_n^e \). Therefore, we obtain the Galerkin equation for \( v_n^e \) as follows:

\[
\begin{align*}
&\left\langle \frac{\partial^2 v_n^e}{\partial x^2}, \psi \right\rangle + \left\langle \frac{\partial v_n^e}{\partial y}, \frac{\partial \psi}{\partial y} \right\rangle + 2r \left\langle \frac{\partial v_n^e}{\partial x}, \psi \right\rangle + \left\langle P_s v_n^e, \psi \right\rangle \\
&= (k^2 - r^2) \left\langle v_n^e, \psi \right\rangle \quad \text{for } \psi \in S_n \text{ and a.e. in } (0, 1).
\end{align*}
\]

Thus, we choose \( \psi = \partial_x v_n^e \) in (35) and \( r > k \) to get that

\[
\frac{d}{dx} \left[ \left\| \partial_x v_n^e(x, \cdot) \right\|^2 + \left\| \partial_y v_n^e(x, \cdot) \right\|^2 + (r^2 - k^2) \left\| v_n^e(x, \cdot) \right\|^2 \right]
\]
By integrating the estimate (36) with respect to \( v_\varepsilon \), we arrive at

\[
\begin{align*}
    &\|\partial_x v_\varepsilon^n(x, \cdot)\|^2 + \|\partial_y v_\varepsilon^n(x, \cdot)\|^2 + (r^2 - k^2)\|v_\varepsilon^n(x, \cdot)\|^2 \\
    \leq& \|\partial_x v_\varepsilon^n(0, \cdot)\|^2 + \|\partial_y v_\varepsilon^n(0, \cdot)\|^2 + (r^2 - k^2)\|v_\varepsilon^n(0, \cdot)\|^2 \\
    &+ 2 \log(\gamma) \int_0^x \left(\|\partial_y v_\varepsilon^n(s, \cdot)\|^2 + \|\partial_x v_\varepsilon^n(s, \cdot)\|^2\right) ds.
\end{align*}
\]

By using Gronwall’s inequality, we thus get

\[
\begin{align*}
    \|\partial_x v_\varepsilon^n(x, \cdot)\|^2 + \|\partial_y v_\varepsilon^n(x, \cdot)\|^2 + (r^2 - k^2)\|v_\varepsilon^n(x, \cdot)\|^2 \\
    \leq& \left(\|\partial_x v_\varepsilon^n(0, \cdot)\|^2 + \|\partial_y v_\varepsilon^n(0, \cdot)\|^2 + (r^2 - k^2)\|v_\varepsilon^n(0, \cdot)\|^2\right) \gamma^{2x}.
\end{align*}
\]

Since \( v_\varepsilon^n(0, \cdot) = u_\varepsilon^n(0, \cdot) \) and \( \partial_x v_\varepsilon^n(0, \cdot) = -r v_\varepsilon^n(0, \cdot) + \partial_x u_\varepsilon^n(0, \cdot) \), there exists a constant \( C > 0 \) independent of \( n \) such that

\[
\|\partial_x v_\varepsilon^n(0, \cdot)\|^2 + \|\partial_y v_\varepsilon^n(0, \cdot)\|^2 + (r^2 - k^2)\|v_\varepsilon^n(0, \cdot)\|^2 \leq C; \text{ cf. (29) and (18).}
\]

Therefore, for any \( n \in \mathbb{N} \) we obtain

\[
\begin{align*}
    v_\varepsilon^n &\text{ is bounded in } L^\infty \left(0, 1; H^1(0, 1)\right), \\
    \partial_x v_\varepsilon^n &\text{ is bounded in } L^\infty \left(0, 1; L^2(0, 1)\right).
\end{align*}
\]

By the Banach-Alaoglu theorem, we can extract a subsequence of \( v_\varepsilon^n \) (which we still denote by \( \{v_\varepsilon^n\}_{n \in \mathbb{N}} \)) such that for each \( \varepsilon > 0 \),

\[
\begin{align*}
    v_\varepsilon^n &\rightharpoonup v^\varepsilon \text{ weakly-* in } L^\infty \left(0, 1; H^1(0, 1)\right), \\
    \partial_x v_\varepsilon^n &\rightharpoonup \partial_x v^\varepsilon \text{ weakly-* in } L^\infty \left(0, 1; L^2(0, 1)\right).
\end{align*}
\]

Let \( S^\perp_n \) is a closed subspace of \( H^1_0(0, 1) \) such that \( H^1_0(0, 1) = S_n \oplus S^\perp_n \). For all \( \psi \in H^1_0(0, 1) \), we can write \( \psi \) of the form \( \psi = \psi_n + \psi^\perp_n \) where \( \psi_n \in S_n \) and \( \psi^\perp_n \in S^\perp_n \). Using the Galerkin Eq. (35), we can show that \( \partial^2_{xx} v_\varepsilon^n \in L^2(0, 1; S_n) \). In particular, for \( \psi_n \in S_n \), we have

\[
\langle \partial^2_{xx} v_\varepsilon^n(x, \cdot), \psi_n \rangle = -\langle \partial_y v_\varepsilon^n(x, \cdot), \partial_y \psi_n \rangle - 2r \langle \partial_x v_\varepsilon^n(x, \cdot), \psi_n \rangle \\
- \{P_n v_\varepsilon^n(x, \cdot), \psi_n\} + (k^2 - r^2) \{v_\varepsilon^n(x, \cdot), \psi_n\}.
\]
Using Cauchy-Schwarz’s inequality and the fact that \( \| \psi_n \|_{H^1_0(0,1)} \leq \| \psi \|_{H^1_0(0,1)} \) with \( \psi = \psi_n + \psi_n^\perp \), we have

\[
\begin{align*}
\left| \partial_y v_n^\epsilon(x, \cdot, \partial_y \psi_n) \right| & \leq \left\| \partial_y v_n^\epsilon(x, \cdot) \right\| \left\| \partial_y \psi_n \right\| \leq \left\| \partial_y v_n^\epsilon(x, \cdot) \right\| \| \psi \|_{H^1_0(0,1)}, \\
\left| \partial_y v_n^\epsilon(x, \cdot, \psi_n) \right| & \leq \left\| \partial_y v_n^\epsilon(x, \cdot) \right\| \| \psi_n \| \leq \left\| \partial_y v_n^\epsilon(x, \cdot) \right\| \| \psi \|_{H^1_0(0,1)}, \\
\left| \{ \partial_y v_n^\epsilon(x, \cdot, \psi_n) \} \right| & \leq \left\| \{ \partial_y v_n^\epsilon(x, \cdot) \} \right\| \| \psi_n \| \leq 2 \log (\gamma) \left\| \partial_y v_n^\epsilon(x, \cdot) \right\| \| \psi \|_{H^1_0(0,1)}, \\
\left| \{ \psi_n \} \right| & \leq \left\| \psi_n \right\| \leq \left\| \psi \right\|_{H^1_0(0,1)}.
\end{align*}
\]

Thus, combining the above four estimates (38), (39), (40) and (41), we get

\[
\begin{align*}
\left\| \partial_x^2 v_n^\epsilon(x, \cdot) \right\|_{H^{-1}(0,1)} & = \sup_{\psi \in H^1(0,1) \setminus \{0\}} \frac{\left\langle \partial_x^2 v_n^\epsilon(x, \cdot), \psi \right\rangle}{\| \psi \|_{H^1_0(0,1)}} \\
& = \sup_{\psi \in H^1(0,1) \setminus \{0\}} -\left\langle \partial_x v_n^\epsilon(x, \cdot), \partial_y \psi_n \right\rangle - 2r \left\langle \partial_x v_n^\epsilon(x, \cdot), \psi_n \right\rangle - \left\langle \{ \partial_x v_n^\epsilon(x, \cdot) \} \right\rangle \psi_n + (k^2 - r^2) \left\langle \{ \psi_n \} \right\rangle \psi_n \\
& \leq C \left( \left\| \partial_x v_n^\epsilon(x, \cdot) \right\| + \left\| \partial_y v_n^\epsilon(x, \cdot) \right\| + (r^2 - k^2) \left\| \psi_n \right\| \right).
\end{align*}
\]

Henceforth, we can find a constant \( C > 0 \) independent of \( n \) to bound the \( H^{-1} \) norm of \( \partial_x^2 v_n^\epsilon \) in the following manner:

\[
\begin{align*}
\left\| \partial_x^2 v_n^\epsilon(x, \cdot) \right\|_{H^{-1}(0,1)} & = \sup_{\psi \in H^1(0,1) \setminus \{0\}} \frac{\left\langle \partial_x^2 v_n^\epsilon(x, \cdot), \psi \right\rangle}{\| \psi \|_{H^1_0(0,1)}} \\
& = \sup_{\psi \in H^1(0,1) \setminus \{0\}} -\left\langle \partial_x v_n^\epsilon(x, \cdot), \partial_y \psi_n \right\rangle - 2r \left\langle \partial_x v_n^\epsilon(x, \cdot), \psi_n \right\rangle - \left\langle \{ \partial_x v_n^\epsilon(x, \cdot) \} \right\rangle \psi_n + (k^2 - r^2) \left\langle \{ \psi_n \} \right\rangle \psi_n \\
& \leq C \left( \left\| \partial_x v_n^\epsilon(x, \cdot) \right\| + \left\| v_n^\epsilon(x, \cdot) \right\|_{H^1(0,1)} \right).
\end{align*}
\]

Square the above estimate, integrate the resulting estimate with respect to \( x \) and then apply (37). By the Banach-Alaoglu theorem, we can choose a subsequence of \( v_n^\epsilon \) so that

\[
\partial_x^2 v_n^\epsilon \rightharpoonup \partial_x^2 v^\epsilon \text{ weakly in } L^2 \left( 0, 1; H^{-1}(0, 1) \right).
\]

Now, we combine the above weak-star and weak limits to conclude that the limit function \( v^\epsilon \) satisfies

\[
\begin{align*}
v^\epsilon & \in L^\infty \left( 0, 1; H_0^1(0, 1) \right), \quad \partial_x v^\epsilon \in L^\infty \left( 0, 1; L^2(0, 1) \right), \\
\partial_x^2 v^\epsilon & \in L^2 \left( 0, 1, H^{-1}(0, 1) \right).
\end{align*}
\]
which, by back-substitution $u_n^\varepsilon = e^\varepsilon x v_n^\varepsilon$, leads to
\[
    u^\varepsilon \in L^\infty \left( 0, 1; H_0^1(0, 1) \right), \quad \partial_x u^\varepsilon \in L^\infty \left( 0, 1; L^2(0, 1) \right),
\]
\[
    \partial_{xx}^2 u^\varepsilon \in L^2 \left( 0, 1, H^{-1}(0, 1) \right).
\]  \tag{44}

Note that the first and second properties in (44) are obtained directly from (43). Meanwhile, the last property in (44) can be deduced from (28) using the same token as in (42). Moreover, using the Aubin-Lions lemma and the Rellich-Kondrachov embedding theorem $H^1_0(0, 1) \subset L^2(0, 1)$ for the first and second properties in (44), we find that
\[
    u_n^\varepsilon \to u^\varepsilon \text{ strongly in } L^2 \left( 0, 1; H_0^1(0, 1) \right). \tag{45}
\]

Now, we multiply both sides of the Galerkin Eq. (28) by an $x$-dependent test function $\tilde{w} \in C_c^\infty (0, 1)$, then by integrate the resulting equation with respect to $x$ to get
\[
\int_0^1 \left( \partial_{xx}^2 u_n^\varepsilon, v \right) dx + \int_0^1 \left( \partial u_n^\varepsilon, \partial_x v \right) dx + \int_0^1 \left( P_x u_n^\varepsilon, v \right) dx = k^2 \int_0^1 \left( u_n^\varepsilon, v \right) dx.
\]

where we have denoted by $v = v(x, y) = \tilde{w}(x)\psi(y)$ for $\psi \in \mathbb{S}_n$. Henceforth, we pass the limit of this equation as $n \to \infty$ and obtain
\[
\int_0^1 \left( \partial_{xx}^2 u^\varepsilon, v \right) dx + \int_0^1 \left( \partial u^\varepsilon, \partial_x v \right) dx + \int_0^1 \left( P_x u^\varepsilon, v \right) dx = k^2 \int_0^1 \left( u^\varepsilon, v \right) dx. \tag{46}
\]

The convergence of the second, third and fourth terms in the limit Eq. (46) is deduced using (45). We remark that the limit Eq. (46) holds for $v = \tilde{w}\psi$ with $\psi \in H_0^1(0, 1)$. In addition, since $\tilde{w} \in C_c^\infty (0, 1)$ is arbitrary, our function $u^\varepsilon$ obtained from approximate solutions $u_n^\varepsilon$ satisfies the weak formulation (28) for every test function $\psi \in H_0^1(0, 1)$. Besides, exploiting the Aubin-Lions lemma and the Gelfand triple $H^1_0(0, 1) \subset L^2(0, 1) \subset H^{-1}(0, 1)$, (44) gives
\[
    u^\varepsilon \in C \left( [0, 1]; H_0^1(0, 1) \right), \quad \partial_x u^\varepsilon \in C \left( [0, 1], L^2(0, 1) \right). \tag{47}
\]

Next, we verify the initial data by the following arguments. We take an arbitrary $x$-dependent function $\kappa \in C^1([0, 1])$ satisfying $\kappa(0) = 1$ and $\kappa(1) = 0$. By the second argument in (44), we have
\[
\int_0^1 \left( \partial_x u_n^\varepsilon, \psi \right) \kappa(x) dx \to \int_0^1 \left( \partial_x u^\varepsilon, \psi \right) \kappa(x) dx \text{ for } \psi \in L^2(0, 1).
\]
Then using integration by parts, we arrive at
\[-\langle u'_n(0), \psi \rangle \kappa(0) - \int_0^1 \langle u''_n, \psi \rangle \kappa x \, dx \rightarrow -\langle u^e(0), \psi \rangle \kappa(0) - \int_0^1 \langle u^e, \psi \rangle \kappa x \, dx.
\]

Henceforth, by the first argument in (44), we obtain the limit \( \langle u'_n(0), \psi \rangle \rightarrow \langle u^e(0), \psi \rangle \) for all \( \psi \in H^1_0(0, 1) \). By the strong \( H^1 \) convergence of \( u^e_n \) designated in (29), we obtain \( \langle u'_n(0), \psi \rangle \rightarrow \langle u^e_0, \psi \rangle \) for all \( \psi \in H^1(0, 1) \). By the uniqueness of limit, it holds true that \( \langle u^e(0), \psi \rangle = \langle u^e_0, \psi \rangle \) for all \( \psi \in H^1(0, 1) \). Thus, \( u^e(0) = u^e_0 \) for a.e. in \( (0, 1) \).

We complete the existence result for system (21).

Now, let \( u^e_1 \) and \( u^e_2 \) be two weak solutions to system (21) that we have obtained in the above part. Consider \( d^e = e^{-rx} (u^e_1 - u^e_2) \). Similar to (35), \( d^e \) satisfies the following wave equation:
\[
\left( \frac{\partial^2 d^e}{\partial x^2}, \psi \right) + \left( \frac{\partial d^e}{\partial y}, \frac{\partial \psi}{\partial y} \right) + 2r \left( \frac{\partial d^e}{\partial x}, \psi \right) + \left( P_x d^e, \psi \right) = (k^2 - r^2) \langle d^e, \psi \rangle
\]
for \( \psi \in H^1_0(0, 1) \). (48)

Taking in (48) \( \psi = \partial_x d^e \), we follow the same process of getting (37). Thus, we derive that
\[
\left\| \partial_x d^e(x, \cdot) \right\|^2 + \left\| \partial_y d^e(x, \cdot) \right\|^2 + (r^2 - k^2) \left\| d^e(x, \cdot) \right\|^2 \\
\leq \left( \left\| \partial_x d^e(0, \cdot) \right\|^2 + \left\| \partial_y d^e(0, \cdot) \right\|^2 + (r^2 - k^2) \left\| d^e(0, \cdot) \right\|^2 \right) e^{2k}.
\]
(49)

Since \( u^e_1 \) and \( u^e_2 \) have the same boundary and initial data, we find that
\[
d^e(0, y) = u^e_1(0, y) - u^e_2(0, y) = 0, \\
\partial_x d^e(0, y) = -r d^e(0, y) + \partial_x u^e_1(0, y) - \partial_x u^e_2(0, y) = 0, \\
\partial_y d^e(0, y) = \partial_y u^e_1(0, y) - \partial_y u^e_2(0, y) = 0.
\]

This shows that the left-hand side of (49) is non-positive for \( r > k \), which indicates the uniqueness result for system (21). Hence, we complete the proof of the theorem.

\[\square\]

It is worth mentioning that the weight \( e^{-rx} \) is employed in the proof of Theorem 3. Commonly, this is called Carleman weight, playing a vital role not only in prove the existence and uniqueness results, but also in convergence estimates of regularization schemes for inverse and ill-posed problems. This is manifested in the present PDE-approach as well as its variant for ill-posed parabolic problems; cf. [17, 18]). In the so-called convexification method, which is a Tikhonov-like regularization technique, the Carleman weight is used to "convexify" nonlinear cost functionals to obtain a unique minimizer; cf. e.g. [19–22]. The use of the smooth weight \( e^{-rx} \) in the present work is based on the following reasons. First, it maximizes the presence of initial data since the weight is exponentially decreasing. Second, it helps to control large
stability estimate of the stabilized operator (i.e. the term \( \log(\gamma) \) as \( \gamma \to \infty \)) as well as the presence of terms involving \( k \) that negatively affect the energy estimates. Below, we continue to apply the Carleman weight to prove the distance between regularized solution \( u^\varepsilon \) and true solution \( u \) in Theorem 4. Then, convergence results follow.

**Theorem 4** (Rigorous mixed error estimates) Let \( u \in C([0, 1]; H^2(0, 1)) \cap C^1([0, 1]; H^1(0, 1)) \) be a unique solution of the Cauchy problem (3)–(5). Let \( M > 0 \) independent of \( \varepsilon \) and \( k \) be such that the true solution satisfies \( \|u\|_{C([0,1];H^2(0,1)) \cap C^1([0,1];H^1(0,1))} \leq M \). Let \( u^\varepsilon \) be a unique weak solution of system (21) as defined in Definition 1 and analyzed in Theorem 3. Assume that \( \log(\gamma) \geq k \) holds true. Then, the following mixed \( L^2-H^1 \) error estimates hold true for any \( \varepsilon > 0 \):

\[
\begin{align*}
\|u^\varepsilon(x, \cdot) - u(x, \cdot)\|^2 & \leq \left[ \frac{(\rho^2 + 1) e^2}{\rho^2 - k^2} + \varepsilon^2 + \frac{(1 - e^{-2\rho_x}) \rho^{-1} M^2 \gamma^{-2}}{8k (\rho^2 - k^2)} \right] \gamma^{2x} e^{2\rho_x}, \\
\|u_x^\varepsilon(x, \cdot) - u_x(x, \cdot)\|^2 & \leq \left[ \frac{(\rho^2 + 1) e^2}{\rho^2 - k^2} + \varepsilon^2 \left( \rho^2 - k^2 \right) + \frac{(1 - e^{-2\rho_x}) \rho^{-1} M^2 \gamma^{-2}}{8k} \right] \gamma^{2x} e^{2\rho_x}, \\
\|u_x^\varepsilon(x, \cdot) - u_x(x, \cdot)\|^2 & \leq \left[ \frac{(\rho^2 + 1) e^2}{\rho^2 - k^2} + \varepsilon^2 \left( \rho^2 - k^2 \right) + \frac{(1 - e^{-2\rho_x}) \rho^{-1} M^2 \gamma^{-2}}{8k} \right] \gamma^{2x} e^{2\rho_x} \left[ e^{\rho_x} + \frac{\rho}{\rho^2 - k^2} \right]^2.
\end{align*}
\]

**Proof** Let \( w = e^{-\rho x} (u^\varepsilon - u) \) where \( \rho > 0 \) is a constant needed to be chosen latter. From (19) and (21), we are capable of computing the difference equation. In particular, the difference function \( w \) satisfies the following damped wave equation:

\[
w_{xx} - w_{yy} + 2\rho w_x + (\rho^2 - k^2)w = -P_{e}w - e^{-\rho x} Q_{e} u.
\]

This equation is associated with the Dirichlet boundary condition and the initial conditions

\[
\begin{align*}
w(x, 0) &= w(x, 1) = 0, & \text{for } x \in [0, 1], \\
w(0, y) &= u_0^\varepsilon(y) - u_0(0), & \partial_x w(0, y) = -\rho w(0, y) & \text{for } x \in [0, 1].
\end{align*}
\]

Multiplying both sides of (53) by \( w_x \) and integrating the resulting equation with respect to \( y \) from 0 to 1, we have

\[
\begin{align*}
\frac{d}{dx} \|w_x(x, \cdot)\|^2 + \frac{d}{dx} \|w_y(x, \cdot)\|^2 + 4\rho \|w_x(x, \cdot)\|^2 + (\rho^2 - k^2) \frac{d}{dx} \|w(x, \cdot)\|^2 & = -2 \langle P_{e}w(x, \cdot), w_x(x, \cdot) \rangle - 2e^{-\rho x} \langle Q_{e}u(x, \cdot), w_x(x, \cdot) \rangle,
\end{align*}
\]
or equivalently,

\[
\frac{1}{\rho^2 - k^2} \frac{d}{dx} \left( \|w_x(x, \cdot)\|^2 + \|w_y(x, \cdot)\|^2 \right) + \frac{4\rho}{\rho^2 - k^2} \|w_x(x, \cdot)\|^2 + \frac{d}{dx} \|w(x, \cdot)\|^2 \\
= -\frac{2}{\rho^2 - k^2} \langle P_\varepsilon w(x, \cdot), w_x(x, \cdot) \rangle - \frac{2e^{-\rho x}}{\rho^2 - k^2} \langle Q_\varepsilon u(x, \cdot), w_x(x, \cdot) \rangle. 
\]

(56)

Now, we use the energy estimates of the perturbing and stabilized operators derived in (23), (24), (25). Then applying the Cauchy–Schwarz inequality, we estimate two terms in the right-hand side of (56) as follows:

\[
\frac{2}{\rho^2 - k^2} |\langle P_\varepsilon w(x, \cdot), w_x(x, \cdot) \rangle| \\
\leq \frac{2}{\rho^2 - k^2} \|P_\varepsilon w(x, \cdot)\| \|w_x(x, \cdot)\| \\
\leq \frac{4 \log(\gamma)}{\rho^2 - k^2} \|w_y(x, \cdot)\| \|w_x(x, \cdot)\|, \\
\frac{2e^{-\rho x}}{\rho^2 - k^2} |\langle Q_\varepsilon u(x, \cdot), w_x(x, \cdot) \rangle| \\
\leq \frac{2e^{-\rho x}}{\rho^2 - k^2} \|Q_\varepsilon u(x, \cdot)\| \|w_x(x, \cdot)\| \\
\leq \frac{e^{-2\rho x} M^2 \gamma^{-2}}{4k (\rho^2 - k^2)} + \frac{4k \|w_x(x, \cdot)\|^2}{\rho^2 - k^2}. 
\]

Integrating (53) with respect to \(x\) from 0 to \(\xi\), then the left-hand side of (56) is bounded from above by

\[
\frac{1}{\rho^2 - k^2} \left( \|w_x(\xi, \cdot)\|^2 + \|w_y(\xi, \cdot)\|^2 \right) + \|w(\xi, \cdot)\|^2 \\
\leq \frac{1}{\rho^2 - k^2} \left( \|w_x(0, \cdot)\|^2 + \|w_y(0, \cdot)\|^2 \right) + \|w(0, \cdot)\|^2 \\
+ \int_0^\xi \left[ \frac{2 \log(\gamma)}{\rho^2 - k^2} \|w_y(x, \cdot)\|^2 + \frac{2 \log(\gamma)}{\rho^2 - k^2} \|w_x(x, \cdot)\|^2 + \frac{e^{-2\rho x} M^2 \gamma^{-2}}{4k (\rho^2 - k^2)} \right] dx \\
+ \int_0^\xi \frac{4k - 4\rho}{\rho^2 - k^2} \|w_x(x, \cdot)\|^2 dx. 
\]

(57)

In (57), we choose \(\rho\) arbitrarily such that \(\rho > k\). Furthermore, at \(x = 0\), the difference function \(w\) and its gradients are bounded, according to (18), by

\[
\frac{1}{\rho^2 - k^2} \left( \|w_x(0, \cdot)\|^2 + \|w_y(0, \cdot)\|^2 \right) + \|w(0, \cdot)\|^2
\]
\[
\begin{align*}
\leq & \frac{1}{\rho^2 - k^2} \left( \rho^2 \| u_0^\epsilon - u_0 \|^2 + \| \partial_y u_0^\epsilon - \partial_y u_0 \|^2 \right) \\
+ & \| u_0^\epsilon - u_0 \|^2 \leq \frac{\rho^2 + 1}{\rho^2 - k^2} \epsilon^2 + \epsilon^2.
\end{align*}
\]

Henceforth, we continue to estimate the left-hand side of (57) as follows:

\[
\begin{align*}
\frac{1}{\rho^2 - k^2} \left( \| w_x(\xi, \cdot) \|^2 + \| w_y(\xi, \cdot) \|^2 \right) + \| w(\xi, \cdot) \|^2 & \\
\leq & \frac{\rho^2 + 1}{\rho^2 - k^2} \epsilon^2 + \epsilon^2 + \frac{1 - e^{-2\rho\xi}}{8k (\rho^2 - k^2)} \rho^{-1} M^2 \gamma^{-2} \left( \| w_x(x, \cdot) \|^2 + \| w_y(x, \cdot) \|^2 \right) + \| w(x, \cdot) \|^2 dx.
\end{align*}
\]

Thus, using Gronwall’s inequality we obtain

\[
\begin{align*}
\frac{1}{\rho^2 - k^2} \left( \| w_x(\xi, \cdot) \|^2 + \| w_y(\xi, \cdot) \|^2 \right) + \| w(\xi, \cdot) \|^2 & \\
\leq & \left[ \frac{\rho^2 + 1}{\rho^2 - k^2} \epsilon^2 + \epsilon^2 + \frac{1 - e^{-2\rho\xi}}{8k (\rho^2 - k^2)} \rho^{-1} M^2 \gamma^{-2} \right] \gamma^{2\xi}.
\end{align*}
\]

We are now in a great position to deduce the error estimate by back-substitution \( w = e^{-\rho x} (u^\epsilon - u) \). Dropping the gradient terms in the left-hand side of (59), we arrive at

\[
\| u^\epsilon(\xi, \cdot) - u(\xi, \cdot) \|^2 \leq \left[ \frac{\rho^2 + 1}{\rho^2 - k^2} \epsilon^2 + \epsilon^2 + \frac{1 - e^{-2\rho\xi}}{8k (\rho^2 - k^2)} \rho^{-1} M^2 \gamma^{-2} \right] \gamma^{2\xi} e^{2\rho\xi}.
\]

Similarly, dropping the first and third terms in the left-hand side of (59), we get

\[
\| u_y^\epsilon(\xi, \cdot) - u_y(\xi, \cdot) \|^2 \leq \left[ \left( \rho^2 + 1 \right) \epsilon^2 + \epsilon^2 \left( \rho^2 - k^2 \right) + \frac{1 - e^{-2\rho\xi}}{8k (\rho^2 - k^2)} \rho^{-1} M^2 \gamma^{-2} \gamma^{2\xi} e^{2\rho\xi} \right].
\]

In view of the fact that \( w_x = -\rho w + e^{-\rho x} (u_x^\epsilon - u_x) \), we find that

\[
e^{-\rho\xi} \| u_x^\epsilon(\xi, \cdot) - u_x(\xi, \cdot) \| - \rho \| w(\xi, \cdot) \| \leq \| w_x(\xi, \cdot) \| \leq \left[ \left( \rho^2 + 1 \right) \epsilon^2 + \epsilon^2 \left( \rho^2 - k^2 \right) \right].
\]
\[ + \left( 1 - e^{-2\rho_\varepsilon} \right) \frac{\rho^{-1} M^2 \gamma^{-2}}{8k} \right]^{1/2} \gamma^{\varepsilon} e^{\rho_\varepsilon} \].

Combining this with (60) leads to

\[
\| u^\varepsilon_x (\xi, \cdot) - u_x (\xi, \cdot) \| \leq \left[ \left( \rho^2 + 1 \right) \varepsilon^2 + \varepsilon^2 \left( \rho^2 - k^2 \right) \right]
\]

\[
+ \left( 1 - e^{-2\rho_\varepsilon} \right) \frac{\rho^{-1} M^2 \gamma^{-2}}{8k} \right]^{1/2} \gamma^{\varepsilon} e^{2\rho_\varepsilon} + \rho \| u^\varepsilon (\xi, \cdot) - u (\xi, \cdot) \|
\]

\[
\leq \left[ \left( \rho^2 + 1 \right) \varepsilon^2 + \varepsilon^2 \left( \rho^2 - k^2 \right) \right]
\]

\[
+ \left( 1 - e^{-2\rho_\varepsilon} \right) \frac{\rho^{-1} M^2 \gamma^{-2}}{8k} \right]^{1/2} \gamma^{\varepsilon} e^{2\rho_\varepsilon} \left[ e^{\rho_\varepsilon} + \frac{\rho}{\rho^2 - k^2} \right].
\]

This is equivalent to

\[
\| u^\varepsilon_x (\xi, \cdot) - u_x (\xi, \cdot) \|^2 \leq \left[ \left( \rho^2 + 1 \right) \varepsilon^2 + \varepsilon^2 \left( \rho^2 - k^2 \right) \right]
\]

\[
+ \left( 1 - e^{-2\rho_\varepsilon} \right) \frac{\rho^{-1} M^2 \gamma^{-2}}{8k} \right]^{1/2} \gamma^{2\varepsilon} e^{2\rho_\varepsilon} \left[ e^{\rho_\varepsilon} + \frac{\rho}{\rho^2 - k^2} \right]^2.
\]

Hence, we complete the proof of the theorem. \(\square\)

As a consequence of the above theorem, one can obtain the Lipschitz stability of \(u^\varepsilon\). Indeed, let \(v^\varepsilon\) be a solution to the same system as (21), i.e.,

\[
\begin{align*}
& v^\varepsilon_{xx} - v^\varepsilon_{yy} + P_\varepsilon v^\varepsilon = k^2 v^\varepsilon & & \text{in } \Omega, \\
& v^\varepsilon(x, 0) = v^\varepsilon(x, 1) = 0 & & \text{for } x \in (0, 1), \\
& v^\varepsilon(0, y) = v^\varepsilon_0(y), \quad v^\varepsilon_x(0, y) = 0 & & \text{for } y \in (0, 1).
\end{align*}
\]

Then, proceeding as in the proof of Theorem 4 (without the presence of \(Q_\varepsilon\)), we can prove the following Lipschitz stability estimates:

\[
\| u^\varepsilon(x, \cdot) - v^\varepsilon(x, \cdot) \|^2 \leq \left[ \rho^2 \| u^\varepsilon_0 - v^\varepsilon_0 \|^2 + \| \partial_y u^\varepsilon_0 - \partial_y v^\varepsilon_0 \|^2 + \| u^\varepsilon_0 - v^\varepsilon_0 \|^2 \right] \gamma^{2x} e^{2\rho_\varepsilon},
\]

\[
\| u^\varepsilon_y(x, \cdot) - v^\varepsilon_y(x, \cdot) \|^2.
\]
The estimates are valid for any \( \gamma \) satisfying \( \log(\gamma) \geq k \). Now, to prove the convergence of \( u^\varepsilon \) toward \( u \), we below rely on a suitable choice of \( \gamma \). In this regard, \( \gamma \) is appropriately dependent of the noise level \( \varepsilon \). Since Theorem 4 below is a direct consequence of Theorem 4, its proof is omitted.

**Theorem 5** (Interior convergence estimates) *Under the assumptions of Theorem 4, if we choose \( \gamma = \varepsilon^{-\alpha} \) for \( \alpha \in (0, 1] \) and \( \varepsilon \leq e^{-k/\alpha} \), then the following H"{o}lder rates of convergence hold true:*

\[
\| u^\varepsilon(x, \cdot) - u(x, \cdot) \|^2 \leq \left[ \frac{(\rho^2 + 1) \varepsilon^{2(1-\alpha x)}}{\rho^2 - k^2} + \varepsilon^{2(1-\alpha x)} + \frac{(1 - e^{-2\rho x}) \rho^{-1} M^2 e^{2\alpha(1-x)}}{8k (\rho^2 - k^2)} \right] e^{2\rho x},
\]

\[
\| u^\varepsilon_y(x, \cdot) - u_y(x, \cdot) \|^2 \leq \left[ (\rho^2 + 1) \varepsilon^{2(1-\alpha x)} + (\rho^2 - k^2) \varepsilon^{2(1-\alpha x)} + \frac{(1 - e^{-2\rho x}) \rho^{-1} M^2 e^{2\alpha(1-x)}}{8k} \right] e^{2\rho x},
\]

\[
\| u^\varepsilon_x(x, \cdot) - u_x(x, \cdot) \|^2 \leq \left[ (\rho^2 + 1) \varepsilon^{2(1-\alpha x)} + (\rho^2 - k^2) \varepsilon^{2(1-\alpha x)} + \frac{(1 - e^{-2\rho x}) \rho^{-1} M^2 e^{2\alpha(1-x)}}{8k} \right] e^{2\rho x}
\]

\[
\times \left[ e^{\rho x} + \frac{\rho}{\rho^2 - k^2} \right]^2.
\]

It is straightforward that regardless of the choice of \( \gamma \), we do not have the convergence at \( x = 1 \) due to the term \( \gamma^{2-2x} \) in (50)–(52). The same can be manifested in (61)–(63). For the interior points \( x \in (0, 1) \), we obtain the H"{o}lder rate of convergence in mixed \( L^2-H^1 \) norms. We also remark that even though convergence result (61) is point wise in the frequency \( k \), the corresponding uniform-in-\( k \) estimate can be obtained by a suitable choice of \( \rho \). Observing the exponential growth (in \( \rho \)) of the inverse Carleman weight, \( \rho \) should be close to \( k \) to “optimize” that growth. However, the closeness should also ensure the \( L^2 \) convergence in \( \varepsilon \). Thus, one possibility is taking \( \rho = k \log (\log (e^{-\beta})) \) for \( \beta \in (0, 1) \) and \( \varepsilon \) being sufficiently small such that \( \log (\log (e^{-\beta})) \geq \sqrt{2} \). By this way, \( 1 < \rho^2 \leq 2(\rho^2 - k^2) \) and thus, it follows from \( e^{2\alpha(1-x)} \geq e^{2(1-\alpha x)} \) that

\[
\| u^\varepsilon(x, \cdot) - u(x, \cdot) \|^2 \leq C \varepsilon^{2\alpha(1-x)} \log^{2k_x}(e^{-\beta}),
\]

where \( C > 0 \) is a constant depending only on \( M \).
Theorem 6 (Boundary convergence estimates) Under the assumptions of Theorem 4, we can always find \( x_\varepsilon \in (0, 1) \) such that \( \lim_{\varepsilon \to 0} x_\varepsilon = 0 \) and \( u^\varepsilon (1 - x_\varepsilon, \cdot) \) approximates well \( u(1, \cdot) \). In particular, we can find a constant \( C(\rho, k, M) > 0 \) depending on \( \rho, k, M \) such that the following logarithmic convergence estimate holds true:

\[
\| u^\varepsilon (1 - x_\varepsilon, \cdot) - u(1, \cdot) \| \leq \frac{C(\rho, k, M)}{1 + \sqrt{1 + 4 \log (\varepsilon^{-\alpha})}}.
\]

Moreover, for \( \varepsilon \) being sufficiently small such that \( \log (\log (\varepsilon^{-\beta})) \geq \sqrt{2} \), the corresponding uniform-in-\( k \) error estimate can be rigorously obtained in the following form:

\[
\| u^\varepsilon (1 - x_\varepsilon, \cdot) - u(1, \cdot) \| \leq \frac{C(M)}{1 - k \log (\log (\varepsilon^{-\beta})) + \sqrt{(1 - k \log (\log (\varepsilon^{-\beta})))^2 + 4k \log (\log (\varepsilon^{-\beta})) + 4 \log (\varepsilon^{-\alpha})}}.
\]

where \( C(M) > 0 \) is a constant depending only on \( M \).

**Proof** We, for brevity, can find some constant \( C(\rho, k, M) > 0 \) such that for \( x \in (0, 1) \),

\[
\| u^\varepsilon (x, \cdot) - u(x, \cdot) \| \leq C^2 (\rho, k, M) \varepsilon^{2\alpha(1-x)},
\]
deduced from (61). Thereby, using the triangle inequality, we get

\[
\| u^\varepsilon (1 - x_\varepsilon, \cdot) - u(1, \cdot) \| \leq \| u^\varepsilon (1 - x_\varepsilon, \cdot) - u(1 - x_\varepsilon, \cdot) \| + \| u(1 - x_\varepsilon, \cdot) - u(1, \cdot) \| \\
\leq C(\rho, k, M) \varepsilon^{\alpha x_\varepsilon} + x_\varepsilon \| u_x \|_{C([0,1]; L^2(0,1))}.
\]

Therefore, to prove the target estimate (65), we seek the infimum \( \frac{1}{2} \inf_{x_\varepsilon > 0} (\varepsilon^{\alpha x_\varepsilon} + x_\varepsilon) \). When doing so, we solve the following algebraic equation:

\[
\varepsilon^{\alpha x_\varepsilon} = x_\varepsilon,
\]
expecting that \( x_\varepsilon \in (0, 1) \) is sufficiently small. In terms of \( x_\varepsilon \), we see that the left-hand side of (67) is decreasing, while the right-hand side grows linearly. Thus, for every \( \varepsilon > 0 \), there exists a unique solution \( x_\varepsilon \in (0, 1) \) to (67). Taking now the logarithm on both sides of (67) and using the standard inequality \( \log(a) > 1 - a^{-1} \) for any \( a > 0 \), we arrive at the following quadratic inequality:

\[
\alpha \log (\varepsilon) x_\varepsilon^2 - x_\varepsilon + 1 > 0.
\]
Since the discriminant is positive, i.e. $1 + 4 \log (e^{-\alpha}) > 0$, and $\log(e) < 1$, we find that

$$x_\varepsilon \in \left( \frac{1 + \sqrt{1 + 4 \log (e^{-\alpha})}}{2\alpha \log (\varepsilon)}, \frac{1 - \sqrt{1 + 4 \log (e^{-\alpha})}}{2\alpha \log (\varepsilon)} \right).$$

By the rationalizing technique, it is clear that $x_\varepsilon \to 0$ as $\varepsilon \to 0$. In particular, we have

$$\lim_{\varepsilon \to 0} \left( \frac{1 - \sqrt{1 + 4 \log (e^{-\alpha})}}{2\alpha \log (\varepsilon)} \right) = \lim_{\varepsilon \to 0} \frac{-4 \log (e^{-\alpha})}{2\alpha \log (\varepsilon) \left( 1 + \sqrt{1 + 4 \log (e^{-\alpha})} \right)} = \lim_{\varepsilon \to 0} \frac{2}{1 + \sqrt{1 + 4 \log (e^{-\alpha})}} = 0,$$

and similarly,

$$\lim_{\varepsilon \to 0} \left( \frac{1 + \sqrt{1 + 4 \log (e^{-\alpha})}}{2\alpha \log (\varepsilon)} \right) = \lim_{\varepsilon \to 0} \frac{-4 \log (e^{-\alpha})}{2\alpha \log (\varepsilon) \left( 1 - \sqrt{1 + 4 \log (e^{-\alpha})} \right)} = \lim_{\varepsilon \to 0} \frac{2}{1 - \sqrt{1 + 4 \log (e^{-\alpha})}} = 0.$$

Henceforth, we obtain the following error estimate point-wise in $k$:

$$\| u^\varepsilon (1 - x_\varepsilon, \cdot) - u (1, \cdot) \| \leq \frac{2(C(\rho, k, M) + M)}{1 + \sqrt{1 + 4 \log (e^{-\alpha})}}.$$

We can prove the uniform-in-$k$ error estimate (66) using the same vein. To do so, we rely on the estimate we have briefly analyzed in (64). In this case, we have

$$\| u^\varepsilon (1 - x_\varepsilon, \cdot) - u (1, \cdot) \| \leq C \varepsilon \alpha x_\varepsilon \log^{k(1 - x_\varepsilon)} (e^{-\beta}) + x_\varepsilon \| u_x \|_{C([0,1];L^2(0,1))}.$$

Therefore, we study the infimum $\frac{1}{2} \inf_{x_\varepsilon > 0} (\varepsilon^{\alpha x_\varepsilon} \log^{k(1 - x_\varepsilon)} (e^{-\beta}) + x_\varepsilon)$ by solving the following algebraic equation: $\varepsilon^{\alpha x_\varepsilon} \log^{k(1 - x_\varepsilon)} (e^{-\beta}) = x_\varepsilon$. Taking the logarithm on
both sides of this equation and then using the logarithmic inequality \( \log(a) > 1 - a^{-1} \) for any \( a > 0 \), we obtain
\[
\alpha x_\varepsilon \log(\varepsilon) + k (1 - x_\varepsilon) \log(\log(\varepsilon^{-\beta})) = \log(x_\varepsilon) > 1 - \frac{1}{x_\varepsilon},
\]
or equivalently,
\[
\left[ \alpha \log(\varepsilon) - k \log(\log(\varepsilon^{-\beta})) \right] x_\varepsilon^2 - (1 - k \log(\log(\varepsilon^{-\beta}))) x_\varepsilon + 1 > 0. \tag{69}
\]
In view of the facts that \( \alpha \log(\varepsilon) - k \log(\log(\varepsilon^{-\beta})) < 0 \) and
\[
D_\varepsilon := \left( 1 - k \log(\log(\varepsilon^{-\beta})) \right)^2 - 4 \left[ \alpha \log(\varepsilon) - k \log(\log(\varepsilon^{-\beta})) \right] = \left( 1 - k \log(\log(\varepsilon^{-\beta})) \right)^2 + 4 k \log(\log(\varepsilon^{-\beta})) + 4 \log(\varepsilon^{-\alpha}) > 0,
\]
the above quadratic inequality (69) admits the following solution:
\[
x_\varepsilon \in \left( \frac{1 - k \log(\log(\varepsilon^{-\beta})) - \sqrt{D_\varepsilon}}{2(\alpha \log(\varepsilon) - k \log(\log(\varepsilon^{-\beta})))}, \frac{1 - k \log(\log(\varepsilon^{-\beta})) + \sqrt{D_\varepsilon}}{2(\alpha \log(\varepsilon) - k \log(\log(\varepsilon^{-\beta})))} \right).
\]
By the rationalizing technique, we show the zero limit of \( x_\varepsilon \) as follows:
\[
\lim_{\varepsilon \to 0} \left[ \frac{1 - k \log(\log(\varepsilon^{-\beta})) - \sqrt{D_\varepsilon}}{2(\alpha \log(\varepsilon) - k \log(\log(\varepsilon^{-\beta})))} \right] = \lim_{\varepsilon \to 0} \frac{\left[ 1 - k \log(\log(\varepsilon^{-\beta})) \right]^2 - D_\varepsilon}{2(\alpha \log(\varepsilon) - k \log(\log(\varepsilon^{-\beta}))) \left( 1 - k \log(\log(\varepsilon^{-\beta})) + \sqrt{D_\varepsilon} \right)} = \lim_{\varepsilon \to 0} \frac{2}{1 - k \log(\log(\varepsilon^{-\beta})) + \sqrt{D_\varepsilon}} = 0.
\]
Hence, it follows from (68) that
\[
\| u^\varepsilon (1 - x_\varepsilon, \cdot) - u (1, \cdot) \| \leq \frac{2C + 2M}{1 - k \log(\log(\varepsilon^{-\beta})) + \sqrt{D_\varepsilon}}.
\]
Therefore, we complete the proof of the theorem. \( \square \)

4 Iterative scheme

In the previous section, we have studied the strong convergence of \( u^\varepsilon \) toward the exact solution \( u \). Observe that by the choice of the perturbation and stabilization in our regularization problem (21) is not really computable, albeit the problem is linear in terms of its solution and the series is truncated appropriately in \( \varepsilon \). For each \( \varepsilon > 0 \), we
construct an iterative sequence \( \{u^{\varepsilon,q}\}_{q\in\mathbb{N}} \) to approximate \( u^\varepsilon \) of (21) in the following sense.

\[
\begin{aligned}
&\begin{cases}
 u^{\varepsilon,q+1}_{xx} - u^{\varepsilon,q+1}_{yy} + P_{\varepsilon} u^{\varepsilon,q+1} = k^2 u^{\varepsilon,q} & \text{in } \Omega, \\
 u^{\varepsilon,q+1}(x, 0) = u^{\varepsilon,q+1}(x, 1) = 0 & \text{for } x \in (0, 1), \\
 u^{\varepsilon,q+1}(0, y) = u^{\varepsilon}_0(y), & u^{\varepsilon,q+1}_x(0, y) = 0 & \text{for } y \in (0, 1).
\end{cases}
\end{aligned}
\]

In the above iteration scheme, we choose the initial guess \( u^{\varepsilon,0} \) (i.e. \( q = 0 \)) is chosen to be \( u^0(y) \). We choose this initial guess because it is a unique function that contains close information of our sought \( u^\varepsilon \) under stabilization. Even though proposing this iterative scheme can be a curse of dimensionality, our previous work [18] shows numerically that we only need a very small amount of iteration steps (about \( q = 2 \)) to obtain a fine approximation.

For every \( \varepsilon \), we can divide the interval \([0, 1]\) of \( x \) into many finite subintervals. Our convergence result below shows that the mesh-width in \( x \) should be dependent of the noise level \( \varepsilon \) for local approximation of the regularized solution \( u^\varepsilon \). It is sufficient to analyze the convergence of the linearization in a subinterval \([0, \bar{x}] \subset [0, 1]\) since we can repeat the linearization procedure in every subinterval. Below, we prove the strong convergence of the scheme in a suitable topology involving the space \( \Upsilon_{\Delta} = C([0, \bar{x}] ; L^2(0, 1)) \).

**Theorem 7** (Convergence of linearization) Under the assumptions of Theorem 4, the approximate solution \( u^{\varepsilon,q} \) defined in (70) is strongly convergent in \( \Upsilon_{\Delta} \). Moreover, for each \( \varepsilon > 0 \), there exists a sufficiently small \( \eta_{\varepsilon} \in (0, 1) \) such that for \( \sigma \geq 1 \),

\[
\| u^{\varepsilon,q}_y - u^\varepsilon_y \|_{\Upsilon_{\Delta}} + \sigma \log (\varepsilon^{-\alpha}) \| u^{\varepsilon,q} - u^\varepsilon \|_{\Upsilon_{\Delta}} \\
\leq \frac{\eta_{\varepsilon}^q}{1 - \eta_{\varepsilon}} \left( \| u^{\varepsilon,1}_y - u^{\varepsilon,0}_y \|_{\Upsilon_{\Delta}} + k \| u^{\varepsilon,1} - u^{\varepsilon,0} \|_{\Upsilon_{\Delta}} \right), \\
\| u^{\varepsilon,q}_x - u^\varepsilon_x \|_{\Upsilon_{\Delta}} \leq \frac{\eta_{\varepsilon}^q}{1 - \eta_{\varepsilon}} \left( \| u^{\varepsilon,1}_y - u^{\varepsilon,0}_y \|_{\Upsilon_{\Delta}} + k \| u^{\varepsilon,1} - u^{\varepsilon,0} \|_{\Upsilon_{\Delta}} \right).
\]

**Proof** We proceed as in the proof of Theorem 4, scrutinizing energy estimates under the Carleman weight of the form \( e^{-\kappa x} \). Let \( W^{q+1} = e^{-\kappa x} (u^{\varepsilon,q+1} - u^{\varepsilon,q}) \) where \( \kappa > 0 \) is a constant chosen later. Thus, \( W^{q+1} \) satisfies the following system:

\[
\begin{aligned}
&\begin{cases}
 W^{q+1}_{xx} - W^{q+1}_{yy} + 2\kappa W^{q+1}_x + \kappa^2 W^{q+1} = -P_{\varepsilon} W^q + k^2 W^q & \text{in } \Omega \\
 W^{q+1}(x, 0) = W^{q+1}(x, 1) = 0 & \text{for } x \in (0, 1), \\
 W^{q+1}_x(0, y) = 0, & W^{q+1}_x(0, y) = 0 & \text{for } y \in (0, 1).
\end{cases}
\end{aligned}
\]

Multiplying the difference equation by \( W^{k+1}_x \) and then integrating the resulting equation from 0 to 1, we find that

\[
\frac{d}{dx} \left\| W^{q+1}_x (x, \cdot) \right\|^2 + \frac{d}{dx} \left\| W^{q+1}_y (x, \cdot) \right\|^2 + \kappa^2 \frac{d}{dx} \left\| W^{q+1} (x, \cdot) \right\|^2
\]
\[ = -2 \left( \mathbf{P}_e W^q, W_x^{q+1} \right) + 2k^2 \left( W^q, W_x^{q+1} \right) - 4\kappa \left\| W_x^{q+1}(x, \cdot) \right\|^2. \]

Integrating the above equation from 0 to \( x \) and choosing \( \kappa = \sigma \log(\gamma) \geq k \) for \( \sigma \geq 1 \), we estimate that

\[
\left\| W_x^{q+1}(x, \cdot) \right\|^2 + \left\| W_y^{q+1}(x, \cdot) \right\|^2 + \kappa^2 \left\| W^{q+1}(x, \cdot) \right\|^2 \\
\leq \int_0^\gamma \left[ \log(\gamma) \left\| W_y^q(s, \cdot) \right\|^2 + k^2 \left\| W^q(s, \cdot) \right\|^2 \\
+ (4\log(\gamma) + 1) \left\| W_x^{q+1}(s, \cdot) \right\|^2 - 4\kappa \left\| W_x^{q+1}(s, \cdot) \right\|^2 \right] ds \\
\leq \bar{\pi} \left( \log(\gamma) \left\| W_y^q \right\|_{T_\pi}^2 + k^2 \left\| W^q \right\|_{T_\pi}^2 \right) e^{\bar{\pi}}. \tag{71} \]

Dropping the first term on the left-hand side of (71), we, after back-substitution, get

\[
\left\| u_y^{e,q+1}(x, \cdot) - u_y^{e,q}(x, \cdot) \right\|^2 + \kappa^2 \left\| u^{e,q+1}(x, \cdot) - u^{e,q}(x, \cdot) \right\|^2 \\
\leq \bar{\pi} \log(\gamma) e^{\bar{\pi}} \left( \left\| u_y^{e,q} - u_y^{e,q-1} \right\|_{T_\pi}^2 + k \left\| u^{e,q} - u^{e,q-1} \right\|_{T_\pi}^2 \right) e^{2\kappa x}. \]

This leads to

\[
\left\| u_y^{e,q+1} - u_y^{e,q} \right\|_{T_\pi}^2 \leq \bar{\pi} \log(\gamma) e^{\bar{\pi}} \left( 4\kappa x \right) \left( \left\| u_y^{e,q} - u_y^{e,q-1} \right\|_{T_\pi}^2 + k \left\| u^{e,q} - u^{e,q-1} \right\|_{T_\pi}^2 \right). \tag{72} \]

Next, by the standard inequality \( (a - b)^2 \geq \frac{1}{4}a^2 - b^2 \) for all \( a, b \in \mathbb{R} \), we have

\[
\frac{1}{2} \left\| u_x^{e,q+1}(x, \cdot) - u_x^{e,q}(x, \cdot) \right\|^2 \leq \kappa^2 \left\| u^{e,q+1}(x, \cdot) - u^{e,q}(x, \cdot) \right\|^2 \\
\leq \left\| u_x^{e,q+1}(x, \cdot) - u_x^{e,q}(x, \cdot) \right\|^2 \leq e^{2\kappa x} \left\| W_x^{q+1}(x, \cdot) \right\|^2 \\
\leq \gamma^{2\sigma x} \bar{\pi} \left( \log(\gamma) \left\| u_y^{e,q} - u_y^{e,q-1} \right\|_{T_\pi}^2 + k^2 \left\| u^{e,q} - u^{e,q-1} \right\|_{T_\pi}^2 \right) e^{\bar{\pi}}. \tag{73} \]
where we have used dropping the second and third terms on the left-hand side of (71). It now follows from (73) and (71) that

$$
\left\| u^{e,q+1} - u^{e,q} \right\|_{\gamma_T}^2 \leq 2\kappa^2 \left\| u^{e,q+1} - u^{e,q} \right\|_{\gamma_T}^2 + 2\gamma 2\sigma_\beta \left( \log(\gamma) \right) \left\| u^{e,q} - u^{e,q-1} \right\|_{\gamma_T}^2 + k^2 \left\| u^{e,q} - u^{e,q-1} \right\|_{\gamma_T}^2 e^\gamma
$$

$$
\leq 4\gamma e^\gamma 2\sigma_\beta \left( \log(\gamma) \right) \left\| u^{e,q} - u^{e,q-1} \right\|_{\gamma_T}^2 + k \left\| u^{e,q} - u^{e,q-1} \right\|_{\gamma_T}^2 \leq 4\gamma e^\gamma 2\sigma_\beta \log(\gamma) \left( \left\| u^{e,q} - u^{e,q-1} \right\|_{\gamma_T}^2 + k \left\| u^{e,q} - u^{e,q-1} \right\|_{\gamma_T}^2 \right). \tag{74}
$$

Therefore, we choose $\bar{\gamma}$ small enough such that

$$
\eta^2_{\gamma} := 4\gamma e^\gamma 2\sigma_\beta \log(\gamma) < 1. \tag{75}
$$

Then, using the Minkowski inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for $a, b \in \mathbb{R}$ and triangle inequality, we deduce from (72) that for $l \geq 1$,

$$
\left\| u^{e,q+l} - u^{e,q} \right\|_{\gamma_T} \leq \sum_{j=1}^l \left( \left\| u^{e,q+j} - u^{e,q+j-1} \right\|_{\gamma_T} + \kappa \left\| u^{e,q+j} - u^{e,q+j-1} \right\|_{\gamma_T} \right)
$$

$$
\leq \sum_{j=1}^l \eta^l_{\gamma} \left( \left\| u^{e,1} - u^{e,0} \right\|_{\gamma_T} + k \left\| u^{e,1} - u^{e,0} \right\|_{\gamma_T} \right)
$$

$$
= \frac{\eta^l_{\gamma}}{1 - \eta^l_{\gamma}} \left( \left\| u^{e,1} - u^{e,0} \right\|_{\gamma_T} + k \left\| u^{e,1} - u^{e,0} \right\|_{\gamma_T} \right).
$$

Henceforth, $\{u^{e,q}\}_{q\in\mathbb{N}}$ and $\{u^{e,q}\}_{q\in\mathbb{N}}$ are Cauchy sequences in $\gamma_T$, respectively. Thus, there exists uniquely $u^e \in \gamma_T$ such that $u^{e,q} \to u^e$ strongly in $\gamma_T$ as $q \to \infty$. Similarly, we obtain a unique $u^y \in \gamma_T$ such that $u^{y,q} \to u^y$ strongly in $\gamma_T$ as $q \to \infty$. By (74), we also obtain that $\{u^{e,q}\}_{q\in\mathbb{N}}$ is a Cauchy sequence in $\gamma_T$ and thus, there exists a unique limit $u^x$ that converges strongly to $u^{e,q}$ in $\gamma_T$. Moreover, taking $l \to \infty$ we have

$$
\left\| u^{y,q} - u^e \right\|_{\gamma_T} + \kappa \left\| u^{e,q} - u^e \right\|_{\gamma_T} \leq \frac{\eta^q_{\gamma}}{1 - \eta^q_{\gamma}} \left( \left\| u^{e,1} - u^{e,0} \right\|_{\gamma_T} + k \left\| u^{e,1} - u^{e,0} \right\|_{\gamma_T} \right),
$$

$$
\left\| u^{y,q} - u^e \right\|_{\gamma_T} \leq \frac{\eta^q_{\gamma}}{1 - \eta^q_{\gamma}} \left( \left\| u^{e,1} - u^{e,0} \right\|_{\gamma_T} + k \left\| u^{e,1} - u^{e,0} \right\|_{\gamma_T} \right).
$$
In addition, we obtain the strong convergence (as \( q \to \infty \)) \( P_\varepsilon u^{\varepsilon, q} \) in the following manner:

\[
\| P_\varepsilon u^{\varepsilon, q} - P_\varepsilon u^\varepsilon \|_{Y_\tau} \leq 2 \log (\gamma) \left( \left\| u^{\varepsilon, q}_{\cdot \cdot} - u^\varepsilon_{\cdot \cdot} \right\|_{Y_\tau} + k \left\| u^{\varepsilon, 1}_{\cdot \cdot} - u^{\varepsilon, 0}_{\cdot \cdot} \right\|_{Y_\tau} \right).
\]

Hence, the limit \( u^\varepsilon \in Y_\tau \) found above is the solution of the regularized system (21) in the subinterval \([0, \overline{x}]\). We complete the proof of the theorem.

\( \square \)

**Remark 3** It is not hard to see that we do not really need to linearize the term \( k^2 u \) on the right-hand side of (21), while the convergence is still guaranteed from the theoretical standpoint. However, numerical observations show that linearization of the term \( k^2 u \) give better numerical results. This mainly explains why we choose the current linearization procedure.

### 5 Numerical examples

#### 5.1 Finite difference settings

Given \( M, N \in \mathbb{N} \), we consider uniform grids of mesh-points \( x_m = (m - 1) \Delta x \), \( y_n = (n - 1) \Delta y \) for \( 1 \leq m \leq M + 1, 1 \leq n \leq N + 1 \) with \( \Delta x, \Delta y \) being the mesh-widths in \( x \) and \( y \), respectively. For any function \( u(x, y) \), we denote by \( u_{m,n} \approx u(x_m, y_n) \) the corresponding discrete function. To generate the data, we apply the central finite difference method (FDM) to solve the Helmholtz Eq. (3) with the Dirichlet boundary conditions imposed on four sides of \( \Omega = (0, 1)^2 \), viz.

\[
u(0, y) = u_0(y), \quad u(1, y) = g(y), \quad u(x, 0) = 0, \quad u(x, 1) = 0. \tag{76}
\]

In our numerical performance of the stabilization scheme below, we do not choose the true solution of the Helmholtz Eq. (3). Instead, we choose its boundary data \( u_0, g \) in (76) so that our choice is more flexible. This is relevant because (3) with full data (76) is a well-posed problem and the central FDM is well known to be stable and convergent with respect to the refinement of \( x \) and \( y \). In this circumstance, one can consider the discrete function \( u_{m,n} \) obtained from that well-posed problem as a reliable true solution. The Neumann data \( u_1 \) in (5) can be generated using the fact that

\[
u_0(y_n) \approx u(x_2, y_n) - u_1(y_n) \Delta x. \tag{77}
\]

The same FDM is applied when we solve \( U(x, y) \) of system (6). For ease of presentation, we only detail below this FDM for \( U(x, y) \), while the scheme for \( u \) can be established in the same manner. The center approximation for partial derivatives
with respect to \( x \) and \( y \) is given by

\[
U_{xx}(x_m, y_n) \approx \frac{U_{m+1,n} - 2U_{m,n} + U_{m-1,n}}{(\Delta x)^2},
\]

\[
U_{yy}(x_m, y_n) \approx \frac{U_{m,n+1} - 2U_{m,n} + U_{m,n-1}}{(\Delta y)^2},
\]

(78)

Thus, the PDE in (6) is discretized as follows:

\[
\frac{U_{m+1,n} - 2U_{m,n} + U_{m-1,n}}{(\Delta x)^2} + \frac{U_{m,n+1} - 2U_{m,n} + U_{m,n-1}}{(\Delta y)^2} + k^2 U_{m,n} = 0.
\]

Put \( r = \Delta x / \Delta y \). We obtain

\[
U_{m+1,n} + U_{m-1,n} + r^2 U_{m,n+1} + \left[ (k \Delta x)^2 - 2 - 2r^2 \right] U_{m,n} + r^2 U_{m,n-1} = 0.
\]

(79)

Denote the unknown \( U_m = (U_{m,2}, U_{m,3}, U_{m,4}, \ldots, U_{m,N})^T \). We rewrite (79) in the following matrix form:

\[
\begin{bmatrix}
K_1 I_{N-1} & 0 & \ldots & 0 \\
I_{N-1} K_2 & I_{N-1} & \ldots & 0 \\
0 & I_{N-1} & K_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & K_2
\end{bmatrix}
\begin{bmatrix}
U_2 \\
U_3 \\
U_4 \\
\vdots \\
U_M
\end{bmatrix}
= \begin{bmatrix}
F_2 \\
F_3 \\
F_4 \\
\vdots \\
F_M
\end{bmatrix},
\]

where \( I_{N-1} \in \mathbb{M}^{(N-1) \times (N-1)} \) stands for the identity matrix, the block matrices \( K_1, K_2 \in \mathbb{M}^{(N-1) \times (N-1)} \) are defined with \( T_k = (k \Delta x)^2 - 2 - 2r^2 \), as follows:

\[
K_1 = \begin{bmatrix}
T_k + 1 & r^2 & 0 & \ldots & 0 \\
r^2 & T_k + 1 & r^2 & \ldots & 0 \\
0 & r^2 & T_k + 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & T_k + 1
\end{bmatrix},
\]

\[
K_2 = \begin{bmatrix}
T_k r^2 & 0 & \ldots & 0 \\
r^2 & T_k r^2 & 0 & \ldots & 0 \\
0 & r^2 & T_k & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & T_k
\end{bmatrix},
\]

and vectors \( F_m \) are denoted by

\[
F_2 = \Delta x (u_1(y_2), u_1(y_3), u_1(y_4), \ldots, u_1(y_N))^T,
\]

\[
F_m = (0, 0, 0, \ldots, 0)^T, \quad 3 \leq m \leq M.
\]

Solving (79) allows us to find a numerical solution of \( U(x, y) \) to system (6). Thereby, it follows that an approximation of \( U(0, y) \) can be obtained for the Dirichlet data in (7). To solve for \( V(x, y) \) in (7) numerically, we accordingly apply the iterative
scheme investigated in Sect. 4. That means we construct a sequence of \( \{V^{e,q}\}_{q \in \mathbb{N}} \) satisfying

\[
\begin{cases}
V^{e,q+1}_{xx}(x_m, y_n) - V^{e,q+1}_{yy}(x_m, y_n) + \mathbf{P}V^{e,q} - k^2 V^{e,q} = 0 & \text{in } \Omega, \\
V^{e,q+1}(x, 0) = V^{e,q+1}(x, 1) = 0 & \text{for } x \in (0, 1), \\
V^{e,q+1}(0, y) = u_0^e(y) - U(0, y), \quad V^{e,q+1}_x(0, y) = 0 & \text{for } y \in (0, 1).
\end{cases}
\tag{80}
\]

In (80), we recall that

\[
\mathbf{P}u(x, \cdot) = -2 \sum_{j \in \mathbb{N} \setminus (B \cup A)} \lambda_{j,k} [u(x, \cdot), \phi_j] \phi_j
\]

and the initial guess \( V^{e,0} \) (i.e. \( q = 0 \)) is chosen to be \( u_0^e(y) - U(0, y) \). As mentioned in Sect. 4, we choose this initial guess because it is a unique function that contains many information of our sought \( V^{e} \) under stabilization. Let \( V^{e,q}(x_i, y_j) \approx V_{i,j}^{e,q} \), and the same difference operators in (78) are applied to the PDE of (80). It yields that

\[
V^{e,q+1}_{xx}(x_m, y_n) \approx \frac{V^{e,q+1}_{m+1,n} - 2V^{e,q+1}_{m,n} + V^{e,q+1}_{m-1,n}}{(\Delta x)^2},
\]

\[
V^{e,q+1}_{yy}(x_m, y_n) \approx \frac{V^{e,q+1}_{m,n+1} - 2V^{e,q+1}_{m,n} + V^{e,q+1}_{m,n-1}}{(\Delta y)^2}.
\]

Combining these with the standard Riemann sum approximating the inner product in \( \mathbf{P} \), we seek \( V_{m,n}^{e,q} \) satisfying the following approximate equation:

\[
\frac{V^{e,q+1}_{m+1,n} - 2V^{e,q+1}_{m,n} + V^{e,q+1}_{m-1,n}}{(\Delta x)^2} - 2\Delta y \sum_{j \in \mathbb{N} \setminus (B \cup A)} (\mu_j - k^2) \sum_{l=1}^{N+1} V_{m,j}^{e,q} \phi_j(y_l) \phi_j(y_n) = k^2 V_{m,n}^{e,q}.
\]

Recall that \( r = \Delta x / \Delta y \). We get

\[
V^{e,q+1}_{m+1,n} = r^2 V^{e,q+1}_{m,n+1} + (2 - 2r^2) V^{e,q+1}_{m,n} + r^2 V^{e,q+1}_{m,n-1} - V^{e,q+1}_{m-1,n}
\]

\[
+ 2(\Delta x)^2 \Delta y \sum_{j \in \mathbb{N} \setminus (B \cup A)} (\mu_j - k^2) \sum_{l=1}^{N+1} V_{m,j}^{e,q} \phi_j(y_l) \phi_j(y_n) + k^2 (\Delta x)^2 V_{m,n}^{e,q}.
\]
Let \( V_{m}^{ε,q} = \left( V_{m,2}^{ε,q}, V_{m,3}^{ε,q}, V_{m,4}^{ε,q}, \ldots, V_{m,N}^{ε,q} \right)^{T} \). The above equation can be rewritten in the following matrix form:

\[
V_{m+1}^{ε,q+1} = KV_{m}^{ε,q+1} - V_{m-1}^{ε,q+1} + f(V_{m}^{ε,q}) + k^{2}(Δx)^{2}V_{m}^{ε,q},
\]

where we have denoted by

\[
K = \begin{bmatrix}
2 - 2r^{2} & r^{2} & 0 & \cdots & 0 \\
\text{2 - 2r}^{2} & r^{2} & 0 & \cdots & 0 \\
0 & \text{2 - 2r}^{2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 2 - 2r^{2}
\end{bmatrix},
\]

\[
f(V_{m}^{ε,q}) = \begin{bmatrix}
f(V_{m}^{ε,q}) (y_{2}) \\
f(V_{m}^{ε,q}) (y_{3}) \\
f(V_{m}^{ε,q}) (y_{4}) \\
\vdots \\
f(V_{m}^{ε,q}) (y_{N})
\end{bmatrix}.
\]

Herewith, elements in \( f(V_{m}^{ε,q}) \) are understood as

\[
f(V_{m}^{ε,q}) (y_{n}) = 2(Δx)^{2}Δy \sum_{\frac{k^{2}}{π^{2}} ≤ j^{2} ≤ \frac{k^{2} + \log^{2}(y)}{π^{2}}} \left( μ_{j} - k^{2} \right) \sum_{l=1}^{N+1} V_{m,l}^{ε,q}ϕ_{j}(y_{l})ϕ_{j}(y_{n})
\]

\[
= 2(Δx)^{2}Δy \sum_{\frac{k^{2}}{π^{2}} ≤ j^{2} ≤ \frac{k^{2} + \log^{2}(y)}{π^{2}}} \left( μ_{j} - k^{2} \right) \begin{bmatrix}
ϕ_{j}(y_{2}) \\
ϕ_{j}(y_{3}) \\
ϕ_{j}(y_{4}) \\
\vdots \\
ϕ_{j}(y_{N})
\end{bmatrix}^{T} \begin{bmatrix}
V_{m,2}^{ε,q} \\
V_{m,3}^{ε,q} \\
V_{m,4}^{ε,q} \\
\vdots \\
V_{m,N}^{ε,q}
\end{bmatrix} ϕ_{j}(y_{n}).
\]

After having \( V^{ε,q} \) from (81), we obtain an approximation of \( u^{ε} \) via \( u_{m,n}^{ε} = U_{m,n} + V_{m,n}^{ε,q} \). As to the measured data \( u_{0}^{ε} \) in (80), we apply the additive noise in the following sense: \( u_{0}^{ε}(y) = u_{0}(y) + ε \text{rand}(y) \), where rand is a uniformly distributed random number such that \( \text{max}_{y \in [0,1]} |\text{rand}(y)| ≤ 1/(2N) \). At the discretization level, the gradient of \( u_{0}^{ε} \) is then approximated by

\[
\partial_{y}u_{0}^{ε}(y_{n}) ≈ \frac{u_{0}^{ε}(y_{n+1}) - u_{0}^{ε}(y_{n})}{Δy} ≈ \partial_{y}u_{0}(y_{n}) + ε N (\text{rand}(y_{n+1}) - \text{rand}(y_{n})).
\]

Therefore, we can see that assumption (18) is fulfilled. The (local) convergence of the linearization scheme for (80) has been studied in Sect. 4. In this regard, we, according to (75), condition that

\[
η^{2} = 4Δxe^{Δxε}e^{-2αΔx} \log (e^{-α}) < 1
\]

indicating \( \sigma = 1 \) is taken. Henceforth, a suitable fine mesh for variable \( x \) should be applied. Below, we fix \( α = 1 \) and \( N = 40 \) when enjoying the numerical performance of
the QR scheme for different noise levels. Moreover, we choose \( q = 1 \) in our iterative procedure for the QR scheme. The choice of \( M \) will be specified in each example since cf. (83), it depends on values of \( \varepsilon \). Also, for simplicity, we take \( g(y) = 0 \) for all examples below, while varying \( u_0(y) \) in (76). Last but not least, we below consider the following relative error:

\[
E = \sqrt{\sum_{m=0}^{M} \sum_{n=0}^{N} |u_{m,n}^\varepsilon - u_{m,n}|^2} \times 100%.
\]

5.2 Numerical performance for variable noise levels

Example 1: Low frequency

We begin this section by a numerical example with a low frequency profile. In this test, we particularly choose \( k = 5 \) and

\[
u_0(y) = -e^{-2(0.5^4 + (y-0.5)^4)} + 0.5^4 + (y - 0.5)^4.
\]

Such \( k \) is suitable in the context of landmine detection; cf. e.g. [20]. In this low frequency profile, we observe numerically that the scheme works well with intermediate noise levels. Therefore, in this test the numerical results are taken into account with \( \varepsilon = 10^{-1} \) and \( 10^{-2} \). Note that cf. (83), \( \eta_\varepsilon^2 \) increases when \( \varepsilon \) decreases, and it decreases when \( M \) becomes larger. Henceforth, for our comparison purpose, to keep \( \eta_\varepsilon \) unchanged when decreasing \( \varepsilon \), we need different values of \( M \). In particular, when \( \varepsilon = 10^{-1} \), we take \( M = 40 \), which gives \( \eta_\varepsilon^2 \approx 0.26 \). When \( \varepsilon = 10^{-2} \), we choose \( M = 80 \).

Depicted in Fig. 1 are the graphical illustrations of the true solution and its reconstructed with intermediate noise (\( \varepsilon = 10^{-1} \)) and small noise (\( \varepsilon = 10^{-2} \)). When \( \varepsilon \) is smaller, the computed solution is very close to the true one in terms of the value and, furthermore, the shape and location of the yellow circular protrusion; see Fig. 1a and c. On the other hand, the relative error reduces from 34.703% for \( \varepsilon = 10^{-1} \) to 3.481% for \( \varepsilon = 10^{-2} \), which shows that the regularized solution obtained from solving (80) approximates well the true solution in this low frequency profile.

Example 2: Intermediate frequency

In this test, we take into account an intermediate frequency problem with \( k = 15 \). We choose that

\[
u_0(y) = \frac{1}{0.1 + 0.1(y - 0.5)^2}.
\]

We verify the numerical performance of the iterative QR scheme with \( \varepsilon = 10^{-1} \) and \( \varepsilon = 10^{-2} \). For each \( \varepsilon \), we use the same parameters as taken in Example 1. Similar to
the previous example, we observe numerically that the scheme reconstructs well the inclusions inside of the computational domain. The true solution and the reconstructed ones with $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-2}$ are reported in Fig. 2. Graphically, all yellow and blue inclusions are visible in Fig. 2b when the reconstruction is proceeded with $\varepsilon = 10^{-1}$ – an intermediate noise. Their locations are also quite accurate, while only the values should be improved. Taking $\varepsilon$ smaller ($\varepsilon = 10^{-2}$), we can see the values in Fig. 2c are very close to the true ones in Fig. 2a. We also report that the relative error in this test reduces from 30.614% (for $\varepsilon = 10^{-1}$) to 3.205% (for $\varepsilon = 10^{-2}$).
Example 3: High frequency

In this test, we consider a high frequency problem with \( k = 50 \) and

\[
  u_0(y) = \frac{-\sin\left(7\sqrt{0.001 + (y - 0.5)^2}\right)}{7\sqrt{1 + (y - 0.5)^2}}.
\]

High frequency problems are usually challenging. Our numerical results for the well-posed problem (6) of \( U \) report that \( M \) should be large enough for better resolution. In the regularized problem (80), this also corresponds to choosing smaller values of \( \varepsilon \). Thus, in this test, we report our numerical results with \( \varepsilon = 10^{-2} \) and \( \varepsilon = 10^{-4} \). When \( \varepsilon = 10^{-2} \), we illustrate the reconstructed solution with \( M = 80 \). When \( \varepsilon = 10^{-4} \), we take \( M = 160 \). Doing so ensures the same value of \( \eta_2^2 \) discussed in the previous example.

Similar to Examples 1 and 2, we can see the reconstruction becomes better when \( \varepsilon \) decreases, in this case, from \( 10^{-2} \) to \( 10^{-4} \); see Fig. 3. Especially, when \( \varepsilon = 10^{-4} \), the computed solution, cf. Figure 3c, shows exactly the same shape and location of all yellow bands in the true solution (Fig. 3). As can be seen from Fig. 3b for \( \varepsilon = 10^{-2} \), those bands are not even visible, and the value of the computed solution still undergoes the blow-up phenomenon due to the natural Hadamard instability. This graphical observation is not captured well in Example 1; see Fig. 1b and c. This also explains why regularization of high frequency problems is rather challenging. We finally report that the relative error \( E \) in this case reduces significantly from 1687.3\% to 7.212\%, when \( \varepsilon \) decreases from \( 10^{-2} \) to \( 10^{-4} \).

Example 4: Extremely high frequency

In this last numerical example, we would like to see the performance of the scheme with a very large frequency. In particular, we choose \( k = 150 \) and

\[
  u_0(y) = 50 \sin(2\pi y) \cos(4\pi y).
\]
Similar to Example 3, we observe numerically that the scheme works with small noise levels. In this test, we verify the scheme when \( \varepsilon = 10^{-2} \) and \( 10^{-4} \), and the same parameters are taken as in Example 3. In Fig. 4b, our reconstruction when \( \varepsilon = 10^{-2} \) shows a slight accuracy in terms of shape and location of yellow bands within \( x \in (0, 0.2) \). When being far away from the “initial” data \( u_0 \), the reconstructed solution is pretty much inaccurate. This causes a huge relative error of 70.731%. When \( \varepsilon \) is down to \( \varepsilon = 10^{-4} \), this error reduces substantially to 1.153%. The fine accuracy of the computed solution with \( \varepsilon = 10^{-4} \) can also be seen in Fig. 4c, compared with the true one in Fig. 4a. Note that as compared to the previous examples reconstructing a few inclusions, this last example indicates the efficiency of the method. It is, in fact, challenging if one wants to reconstruct several inclusions.

**Remark 4** The linearized model (80) is applied to seek a stable approximation for system (7). Notice that the standard boundary compatibility conditions for system (7) are \( u_0(0) = U(0, 0) \) and \( u_0(1) = U(0, 1) \). These conditions are spontaneously fulfilled in our approximation procedure. Indeed, it can be manifested in our programming environment that, cf. (77), \( u_0(y_n) = u(x_2, y_n) - u_1(y_n) \Delta x = u(x_2, y_n) - U(x, y_n) \Delta x \). At the same time, we have \( -U_x(0, y_n) \Delta x = U(0, y_n) - U(x_2, y_n) \) in the forward solver for \( U \). Henceforth, it holds true that

\[
u_0(y_n) = u(x_2, y_n) - U(x_2, y_n) + U(0, y_n).
\]

In view of the fact that \( u(x_2, y_n) = 0, U(x_2, y_n) = 0 \) at the endpoints \( y = 0, 1 \), we, thus, get the boundary compatibility conditions.

### 6 Conclusions and further research

In this work, we have presented a new pair of perturbation and stabilized operators in a novel variational quasi-reversibility (QR) framework to solve a Cauchy problem of Helmholtz equations. Conditional weak well-posedness and strong convergence rate of the method have been studied thoroughly. Our numerical experiments show that the method delivers reliable, accurate solutions in relatively coarse mesh grids. When the frequency is not too large, the scheme only needs intermediate noise levels to reconstruct the inclusion’s shape well in the domain of interest.

By the method’s inception, our concentration is the stabilization design while only considering the simple rectangle domain for explicit eigen-elements. Certainly, when it comes to arbitrarily complex domains in \( y \), the complication of the eigenpairs solver is augmented. Nevertheless, there are distinctive numerical methods, nowadays, that can approximate well the eigenvalue problem posed in certain complicated geometries; cf. e.g. [23] for irregular convex domains whose boundary are described by \( C^1 \) Jordan curves, [24] for typical perforated domains, and [25] for arbitrary convex/non-convex geometries. Needless to say, the built-in PDE Toolbox in MATLAB currently appears to be a fundamental means to provide accurate eigen-elements in many complex circumstances.
We also remark that when the test function $\psi$ in (29) is chosen to be the eigenfunction $\phi_j$, the weak form reduces to the conventional Fourier series. By the wide choice of $\psi$, our developing method is not necessarily a variant of the Fourier series that wholly consists in using global bases. Our weak formulation being used here can be adapted well to the populated finite element framework that use local bases to gain localized features in two- and three-dimensional domains.

From our numerical examples 1–4, we additionally find the bounds of the Fourier index in the following manner, cf. (82),

- Example 1 ($k = 5$): $j \in [1.59, 1.75]$ for $\varepsilon = 10^{-1}$, $j \in [1.59, 2.16]$ for $\varepsilon = 10^{-2}$;
- Example 2 ($k = 15$): $j \in [4.77, 4.83]$ for $\varepsilon = 10^{-1}$, $j \in [4.77, 4.99]$ for $\varepsilon = 10^{-2}$;
- Example 3 ($k = 50$): $j \in [15.91, 15.98]$ for $\varepsilon = 10^{-2}$, $j \in [15.91, 16.18]$ for $\varepsilon = 10^{-4}$;
- Example 4 ($k = 150$): $j \in [47.74, 47.76]$ for $\varepsilon = 10^{-2}$, $j \in [47.74, 47.83]$ for $\varepsilon = 10^{-4}$.

Since $j \in \mathbb{N}$, the Fourier series in $P$ vanishes in most of the cases. In such cases, we essentially approximate the ill-posed linear elliptic problem by a well-posed linear wave model. This interesting point does not appear in any Fourier-based approach.

Henceforth, our upcoming work will address finite element analysis and simulation of the method posed in regular and irregular geometries.

**Appendix**

This appendix is devoted to the well-posedness of system (6). Recall that $\lambda_{j,k} = \mu_j - k^2$, where $\mu_j$ are the Dirichlet eigenvalues mentioned in Sect. 2. Similar to (8), if we multiply the PDE in (6) by $\phi_j$, the associated eigenfunction, and then integrate the resulting equation from 0 to 1, we obtain the following differential system:

$$
\begin{align*}
\frac{d^2}{dx^2} \langle U(x, \cdot) , \phi_j \rangle &= \lambda_{j,k} \langle U(x, \cdot) , \phi_j \rangle, \\
\frac{d}{dx} \langle U(0, \cdot) , \phi_j \rangle &= \langle u_1 , \phi_j \rangle, \quad \langle U(1, \cdot) , \phi_j \rangle = 0.
\end{align*}
$$

(C84)

Cf. (9), we will determine the solution to (C84) in $A_1$. Solving the differential equation in (C84), we have the form:

$$
\langle U(x, \cdot) , \phi_j \rangle = C_1 e^x \sqrt{\lambda_{j,k}} + C_2 e^{-x} \sqrt{\lambda_{j,k}}.
$$

(C85)

Taking $x = 1$ in (C85) gives

$$
C_1 e^{\sqrt{\lambda_{j,k}}} + C_2 e^{-\sqrt{\lambda_{j,k}}} = \langle U(1, \cdot) , \phi_j \rangle = 0.
$$

(C86)

On the other hand, we find from (C85) that

$$
\frac{d}{dx} \langle U(x, \cdot) , \phi_j \rangle = \sqrt{\lambda_{j,k}} \left( C_1 e^x \sqrt{\lambda_{j,k}} - C_2 e^{-x} \sqrt{\lambda_{j,k}} \right).
$$
which, by taking $x = 0$, implies that $\langle u_1, \phi_j \rangle = \sqrt{\lambda_{j,k}} (C_1 - C_2)$. This is equivalent to

$$C_1 = \frac{1}{\sqrt{\lambda_{j,k}}} \langle u_1, \phi_j \rangle + C_2. \quad (87)$$

Next, by inserting (87) into (86), we arrive at

$$\left( \frac{1}{\sqrt{\lambda_{j,k}}} \langle u_1, \phi_j \rangle + C_2 \right) e^{\sqrt{\lambda_{j,k}}} + C_2 e^{-\sqrt{\lambda_{j,k}}} = 0.$$

Thus, it yields

$$C_2 = -\frac{e^{\sqrt{\lambda_{j,k}}}}{\sqrt{\lambda_{j,k}} \left( e^{\sqrt{\lambda_{j,k}}} + e^{-\sqrt{\lambda_{j,k}}} \right)} \langle u_1, \phi_j \rangle,$$

and by (87), we have

$$C_1 = \frac{1}{\sqrt{\lambda_{j,k}}} \langle u_1, \phi_j \rangle - \frac{e^{\sqrt{\lambda_{j,k}}}}{\sqrt{\lambda_{j,k}} \left( e^{\sqrt{\lambda_{j,k}}} + e^{-\sqrt{\lambda_{j,k}}} \right)} \langle u_1, \phi_j \rangle = \frac{1}{\sqrt{\lambda_{j,k}}} \langle u_1, \phi_j \rangle \frac{e^{-\sqrt{\lambda_{j,k}}}}{e^{\sqrt{\lambda_{j,k}}} + e^{-\sqrt{\lambda_{j,k}}}}.$$

Henceforth, in $A_1$ the solution to system (85) is of the following form:

$$\langle U(x, \cdot), \phi_j \rangle = \frac{1}{\sqrt{\lambda_{j,k}}} \langle u_1, \phi_j \rangle \frac{e^{(x-1)\sqrt{\lambda_{j,k}}}}{e^{\sqrt{\lambda_{j,k}}} + e^{-\sqrt{\lambda_{j,k}}}} - \frac{e^{(1-x)\sqrt{\lambda_{j,k}}}}{\sqrt{\lambda_{j,k}} \left( e^{\sqrt{\lambda_{j,k}}} + e^{-\sqrt{\lambda_{j,k}}} \right)} \langle u_1, \phi_j \rangle$$

$$= \frac{\langle u_1, \phi_j \rangle}{\sqrt{\lambda_{j,k}}} \frac{\sinh \left( (x - 1) \sqrt{\lambda_{j,k}} \right)}{\cosh \left( \sqrt{\lambda_{j,k}} \right)}.$$

In $A_2$, we seek

$$\langle U(x, \cdot), \phi_j \rangle = C_1 + C_2 x.$$

At $x = 1$, we have $C_1 + C_2 = 0$ or equivalently, $C_1 = -C_2$. At the same time, we compute that $\frac{d}{dx} \langle U(x, \cdot), \phi_j \rangle = C_2$. Therefore, it is straightforward that $C_2 = \langle u_1, \phi_j \rangle$ and $C_1 = -\langle u_1, \phi_j \rangle$. The solution is then expressed as

$$\langle U(x, \cdot), \phi_j \rangle = (x - 1) \langle u_1, \phi_j \rangle. \quad (88)$$

In $A_3$, we seek the solution in the following form:

$$\langle U(x, \cdot), \phi_j \rangle = C_1 \cos \left( x \sqrt{-\lambda_{j,k}} \right) + C_2 \sin \left( x \sqrt{-\lambda_{j,k}} \right).$$
At \( x = 1 \), we deduce that
\[
C_1 \cos (\sqrt{-\lambda_{j,k}}) + C_2 \sin (\sqrt{-\lambda_{j,k}}) = 0.
\]

Taking the derivative both sides of (88), we get
\[
\frac{d}{dx} \langle U(x, \cdot), \phi_j \rangle = -C_1 \sqrt{-\lambda_{j,k}} \sin (x \sqrt{-\lambda_{j,k}}) + C_2 \sqrt{-\lambda_{j,k}} \cos (x \sqrt{-\lambda_{j,k}}).
\]

Then at \( x = 0 \), we find that
\[
C_2 \sqrt{-\lambda_{j,k}} = \langle u_1, \phi_j \rangle.
\]

As long as \( \sqrt{-\lambda_{j,k}} \neq \frac{\pi}{2} + q\pi \) for \( q \in \mathbb{N} \), we obtain
\[
C_1 = -\frac{\langle u_1, \phi_j \rangle}{\sqrt{-\lambda_{j,k}} \cos (\sqrt{-\lambda_{j,k}})}.
\]

Henceforth, the solution to system (85) in this case reads as
\[
\langle U(x, \cdot), \phi_j \rangle = \frac{\langle u_1, \phi_j \rangle}{\sqrt{-\lambda_{j,k}}} \left[ -\frac{\sin (\sqrt{-\lambda_{j,k}})}{\cos (\sqrt{-\lambda_{j,k}})} \cos (x \sqrt{-\lambda_{j,k}}) + \sin (x \sqrt{-\lambda_{j,k}}) \right]
\]
\[
= \frac{\langle u_1, \phi_j \rangle}{\sqrt{-\lambda_{j,k}}} \left[ -\sin (\sqrt{-\lambda_{j,k}}) \cos (x \sqrt{-\lambda_{j,k}}) + \cos (\sqrt{-\lambda_{j,k}}) \sin (x \sqrt{-\lambda_{j,k}}) \frac{\cos (\sqrt{-\lambda_{j,k}})}{\cos (\sqrt{-\lambda_{j,k}})} \right]
\]
\[
= \frac{\langle u_1, \phi_j \rangle}{\sqrt{-\lambda_{j,k}}} \sin ((x - 1) \sqrt{-\lambda_{j,k}}).
\]

To sum up, for a fixed frequency \( k \) satisfying the mild restriction \( k \neq \sqrt{\left( \frac{\pi}{2} + q\pi \right)^2 + \mu_j} \) for any \( q \in \mathbb{N} \) and \( j \in A_3 \), the Fourier coefficients of \( U(x, y) \) solutions to system (6) are determined by
\[
\langle U(x, \cdot), \phi_j \rangle = \begin{cases} 
\frac{\langle u_1, \phi_j \rangle}{\sqrt{-\lambda_{j,k}}} \sinh ((x - 1) \sqrt{-\lambda_{j,k}}), & j \in A_1, \\
\frac{\langle u_1, \phi_j \rangle}{\sqrt{-\lambda_{j,k}}} \cosh (\sqrt{-\lambda_{j,k}}), & j \in A_2, \\
(x - 1) \frac{\langle u_1, \phi_j \rangle}{\sqrt{-\lambda_{j,k}}} \sin ((x - 1) \sqrt{-\lambda_{j,k}}), & j \in A_3.
\end{cases}
\]

This guarantees the existence and uniqueness of a solution to system (6). From the numerical examples in Sect. 5, we have chosen \( k \in \mathbb{N}^* \) which satisfies well the above mild condition. We can also prove the stability of the solution with respect to the input.
For every fixed $k$ satisfying some further conditions. Denote

$$A_{31} := A_3 \cap \left\{ j \in \mathbb{N} : \sqrt{k^2 - \mu_j} \in \bigcup_{q \in \mathbb{N}} \left[ q \pi, (2q + 1) \frac{\pi}{2} \right) \right\},$$

$$A_{32} := A_3 \cap \left\{ j \in \mathbb{N} : \sqrt{k^2 - \mu_j} \in \bigcup_{q \in \mathbb{N}} \left( (2q + 1) \frac{\pi}{2}, (q + 1) \pi \right) \right\}.$$

By Parseval’s identity, we have

$$\|U(x, \cdot)\|^2 = \sum_{j \in \mathbb{N}^*} \|U(x, \cdot) \phi_j\|^2 = \sum_{j \in A_1} \|U(x, \cdot) \phi_j\|^2 + \sum_{j \in A_2} \|U(x, \cdot) \phi_j\|^2$$

$$= \sum_{j \in A_1} \|u_1, \phi_j\|^2 \frac{\sinh^2 \left( (x - 1) \sqrt{\mu_j - k^2} \right)}{(\mu_j - k^2) \cosh^2 \left( \sqrt{\mu_j - k^2} \right)}$$

$$+ \sum_{j \in A_2} \|u_1, \phi_j\|^2 \frac{\sin^2 \left( (x - 1) \sqrt{k^2 - \mu_j} \right)}{(k^2 - \mu_j) \cos^2 \left( \sqrt{k^2 - \mu_j} \right)}$$

$$+ \sum_{j \in A_{31}} \|u_1, \phi_j\|^2 \frac{\sin^2 \left( (x - 1) \sqrt{k^2 - \mu_j} \right)}{(k^2 - \mu_j) \cos^2 \left( \sqrt{k^2 - \mu_j} \right)}$$

In view of the fact that

$$\sinh^2 \left( (x - 1) \sqrt{\mu_j - k^2} \right) \leq \left( \mu_j - k^2 \right) \cosh^2 \left( \sqrt{\mu_j - k^2} \right) \quad \text{for } j \in A_1 \text{ and } \mu_j \geq 1,$$

$$(x - 1)^2 \leq 1 \quad \text{for any } x \in [0, 1],$$

we get

$$\sum_{j \in A_1} \|u_1, \phi_j\|^2 \frac{\sinh^2 \left( (x - 1) \sqrt{\mu_j - k^2} \right)}{(\mu_j - k^2) \cosh^2 \left( \sqrt{\mu_j - k^2} \right)}$$
In $A_3$, we have $\sin^2 \left( (x - 1)\sqrt{k^2 - \mu_j} \right) \leq (x - 1)^2 \left( k^2 - \mu_j \right) \leq k^2 - \mu_j$. Besides, we can always find a $\tau_1 \geq 1$ such that
\[
\tau_1 k \in \bigcup_{q \in \mathbb{N}} \left( q\pi, (2q + 1) \frac{\pi}{2} \right).
\] (91)

Since $\frac{1}{\cos^2(t)}$ is periodic and increasing in $\bigcup_{q \in \mathbb{N}} \left( q\pi, (2q + 1) \frac{\pi}{2} \right)$, we estimate that
\[
\sum_{j \in A_{31}} \left| \left( u_1, \phi_j \right) \right|^2 \frac{\sin^2 \left( (x - 1)\sqrt{k^2 - \mu_j} \right)}{(k^2 - \mu_j) \cos^2 \left( \sqrt{k^2 - \mu_j} \right)} \leq \sum_{j \in A_{31}} \left| \left( u_1, \phi_j \right) \right|^2 \frac{(x - 1)^2}{\cos^2 \left( \sqrt{k^2 - \mu_j} \right)} \leq \frac{1}{\cos^2 (\tau_1 k)} \| u_1 \|^2.
\] (92)

Next, we know that $\frac{1}{\cos^2(t)}$ is decreasing in $\bigcup_{q \in \mathbb{N}} \left( (2q + 1) \frac{\pi}{2}, (q + 1) \pi \right)$. Since $\frac{1}{\cos^2(t)}$ is symmetric with respect to $t = q\pi$ for $q \in \mathbb{N}^*$, there always exists a point $\sqrt{\lambda_{i,j,k}} \geq \tau_2 k$ for some $\tau_2 > 0$ and
\[
\sqrt{\lambda_{i,j,k}} \leq \tau_2 k \quad \text{for any } j \in A_{32}.
\]

Therefore, we estimate that
\[
\sum_{j \in A_{32}} \left| \left( u_1, \phi_j \right) \right|^2 \frac{\sin^2 \left( (x - 1)\sqrt{k^2 - \mu_j} \right)}{(k^2 - \mu_j) \cos^2 \left( \sqrt{k^2 - \mu_j} \right)} \leq \sum_{j \in A_{32}} \left| \left( u_1, \phi_j \right) \right|^2 \frac{(x - 1)^2}{\cos^2 \left( \sqrt{k^2 - \mu_j} \right)} \leq \frac{1}{\cos^2 (\tau_2 k)} \| u_1 \|^2.
\] (93)

Combining (90), (92), (93), we obtain
\[
\| U (x, \cdot) \|^2 \leq \max \left\{ \frac{1}{\cos^2 (\tau_1 k)}, \frac{1}{\cos^2 (\tau_2 k)} \right\} \| u_1 \|^2.
\]

Note that since $k$ is fixed, $|A_3| < \infty$. Thus, the choices of $\tau_1$ and $\tau_2$ are flexible; see Fig. 5.

**Remark 5** The following remarks should be singled out:
The form (89) of the Fourier coefficient of $U(x, y)$ to system (6) can be extended, in the same vein, to a more general bounded domain $(0, a_1) \times (0, a_2)$. Henceforth, an analog of proof for the well-posedness of (6) can be derived.

In our analysis above, we, overall, require that $\sqrt{k^2 - \mu_j} \neq \frac{\pi}{2} + q\pi$ for any $j \in A_3$, and thus, there are various values of $k$ satisfying this condition. This condition extends significantly the one assumed in [2]. In [2], one particularly requires that $k < \pi/2$ in the unit domain we are working on, which is very strict. It is clear that $k = 5, 15, 50, 150$ we have chosen in Sect. 5 satisfy well our condition, but violate $k < \pi/2$ in [2].

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