The Overlap Formalism and Topological Susceptibility on the Lattice

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Recent lattice measurements of the topological susceptibility of $SU(2)$ gauge theory using improved cooling and inverse-blocking are in disagreement. We use the overlap method, which probes the fermionic sector of the theory directly, to help resolve this discrepancy.

Two recent methods\cite{1,2} for measuring the topological susceptibility, $\chi_t = \langle Q^2 \rangle / V$, in pure $SU(2)$ gauge theory using the lattice regularization are considered to be in disagreement. In one method\cite{1}, the gauge fields are smoothed with an improved cooling technique while the topological charge $Q$ is calculated using a lattice discretization. In the other method\cite{3}, inverse-blocking is employed to smooth the lattice configurations, after which the topological charge is calculated using an “algebraic” definition. In both methods, scaling is assumed and a continuum limit for $\chi_t$ obtained, with the value of Ref.\cite{1} being about 20% smaller than that of Ref.\cite{2}. It should be noted that the standard Wilson plaquette action is used in the first method to generate the gauge field configurations on the lattice, while in the second method the gauge field ensemble is generated with a certain fixed-point action. Therefore, in addition to scaling, it is also necessary to assume universality to compare the two numbers above.

The index of the Euclidean chiral Dirac operator on the lattice may be extracted by the overlap method\cite{3} in a clean manner, that is to say, in a way that does not require the smoothing of background gauge fields. As the Atiyah-Singer index theorem\cite{4} equates the index to the topological charge, the overlap method should thus be useful in probing for topological structure. One should keep in mind that the index theorem is a statement about continuum gauge fields and its validity for configurations on the lattice should first be tested. Ref.\cite{5} measures the index distribution of the discretized chiral Dirac operator in pure $SU(2)$ gauge theory using the standard Wilson action at a coupling of $\beta = 2.4$ on a $12^4$ lattice, which enables a direct comparison with the distribution of topological charge in Ref.\cite{1}. The index distribution matches extremely well with the topological charge distribution in accordance with the index theorem. This indicates that we may in fact apply the index theorem to an ensemble of lattice gauge field configurations, even when these configurations themselves are not smooth. Furthermore, the agreement between the index distribution from the overlap method and the distribution of topological charge using improved cooling suggests that overlap techniques might help resolve the discrepancy stated in the first paragraph.

To this end, Anna Hasenfratz provided us with 30 configurations generated on a $12^4$ lattice using the fixed point action of Ref.\cite{2}. In Fig.\cite{1}, we compare the corresponding topological charge distribution as measured by inverse-blocking with the index distribution of the chiral Dirac operator obtained by overlap techniques. Clearly the broader distribution of topological charge arising from inverse-blocking is inconsistent with the index distribution measured by the overlap. Furthermore, if one had estimated the topological susceptibility using the latter, the result would be smaller than that obtained by the inverse-blocking method of Ref.\cite{2}, and possibly closer.
Figure 1. The index distribution measured by the overlap method, $Q_{\text{level}}$, compared against the topological charge distribution obtained by inverse-blocking, $Q_{\text{cycle9}}$, on the same ensemble of configurations. For the latter, 9 cycles of inverse-blocking were employed.

To gain additional insight into the difference between the two distributions in Fig. 1, we must examine the inverse-blocking procedure in more detail.

Inverse-blocking is used by Ref. [2] to smooth the gauge fields so that topological objects can be resolved from the rough thermal backgrounds. An original configuration on the $12^4$ lattice is mapped to a configuration on a $24^4$ lattice by an inverse-blocking transformation, which results in a smoother configuration on the finer lattice. In practice, however, one inverse-blocking step does not sufficiently smooth the configuration, and one must therefore go to even finer lattices. But this is not feasible with present day computers, and the following alternative is employed instead. After the rough configuration on the original lattice is inverse-blocked to the finer lattice, it is subsequently blocked back to the coarser lattice, after which it is expected to remain just as smooth as it was on the finer lattice. If desired, this procedure can be repeated again on the resulting configuration. Successive cycles of inverse-blocking followed by blocking will smooth the configuration a little more each time, until eventually topology can be resolved. To obtain the topological charge distribution of Fig. 1, for example, each configuration of the ensemble was cycled nine times, after which the topological charge was measured using an “algebraic” topological charge operator.

The procedures of Refs. [1] and [2], in which gauge field smoothing is an intricate part of the measurement process, should be contrasted with overlap measurements, which can be performed on the original unsmoothed configurations. The index of the chiral Dirac operator is simply given by the number of level crossings of the Hermitian Hamiltonian $H(\mu) = \gamma_5 W(\kappa)$, where $W(\kappa)$ is the usual Wilson Dirac operator, and the mass parameter $\mu = 4 - \frac{1}{2\kappa}$ can range over all real values. For example, the index distribution of Fig. 1 was determined by counting the number of net level crossings associated with each configuration of the original unsmoothed ensemble. The connection between level crossing and the index is intimately tied to the definition of the chiral determinant, and the details can be found in [3] and [5].

To understand the origin of the discrepancy between the two distributions in Fig. 1, it is useful to focus on individual configurations. Fig. 2 illustrates the spectral flow associated with a specific gauge field of the original unsmoothed ensemble, and we see that the overlap method gives an index equal to one for the chiral Dirac operator. This should be contrasted with the inverse-blocking method, which measures the topology of this configuration to be three after 9 cycles and two after 12 cycles. The overlap method can also be used to measure the index on cycled configurations themselves, which provides additional insight into the effects of cycling. The spectral flow of the 12-cycled configuration is plotted in Fig. 3, which shows a net crossing of two, in agreement with the inverse-blocking measurement on the same configuration. However, the aim is to determine the topology of the original configuration.

It is likely that the overlap measurement on the 9-cycled configuration would also agree with the inverse-blocking result, but the 9-cycled configuration was not readily available and we were unable to perform the measurement.
tion. The success of the inverse-blocking scheme depends on its ability to preserve the topology as the configuration is cycled, and this example shows that this is not always the case.

The overlap method can also be used to obtain the size of topological objects. In fact, one can show that the levels of larger instantons cross at smaller values of \( \mu \), and hence measuring the crossing-point can provide a rough indication of the instanton size. Is it also possible to get a much more precise determination of the instanton size directly from the corresponding eigenvectors near the crossing point. This work is in progress and should reveal even more detailed comparisons of Figs. 2 and 3.

In closing, the overlap method indicates that the cycling procedure of Ref. 2 has not only smoothed the configuration studied above but it has also changed the topology by generating an instanton. If cycling can increase the index, as the example suggests, it is conceivable that this is the cause for the broader distribution in Fig. 1 and an explanation of the disagreement stated at the beginning of this paper.

\( ^5 \)Note that the level crossing in Fig. 3 occurs at a much smaller value of \( \mu \) than in the corresponding Fig. 2 which indeed shows that cycling has smoothed the configuration considerably.

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