Two-parameter deformed supersymmetric oscillators with $\text{SU}_{q_1/q_2}(n \mid m)$-covariance

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Abstract

A two-parameter deformed superoscillator system with $SU_{q_1/q_2}(n \mid m)$-covariance is presented and used to construct a two-parameter deformed $N = 2$ SUSY algebra. The Fock space representation of the algebra is discussed and the deformed Hamiltonian for such generalized superoscillators is obtained.
1 Introduction

A great deal of effort has recently been spent to the study of many aspects of quantum groups and their associated algebras, which are specific deformations of the usual Lie groups and Lie algebras with some deformation parameter $q^\hbar$. Many of such studies can be mentioned within a wide spectrum of research of theoretical physics such as noncommutative geometry\[5\], two-dimensional conformal field theories\[6\], quantum mechanics\[7\].

After the realization of intimate relationship between $q$-deformed bosonic as well as fermionic oscillators and quantum groups (and their algebras)\[8]-[10], these relations have subsequently been extended to two-parameter deformed versions of such oscillator algebras\[11] and quantum group structures\[12]. Meanwhile, several deformed forms of superalgebras and supergroups have extensively been constructed by a natural association of one or two-parameter deformed bosonic and fermionic oscillator algebras\[13]-[15].

As is well known that the ordinary $N = 2$ SUSY algebra developed by Witten\[16] combines undeformed bosons with undeformed fermions. This algebra has the following form:

$$\{Q, Q^*\} = H, \quad Q^2 = (Q^*)^2 = 0, \quad [H, Q] = [H, Q^*] = 0, \quad (1)$$

where $Q$ and $Q^*$ are odd generators called supercharges, $H$ is an even generator called Hamiltonian. These generators are assumed to be well defined on the relevant Hilbert space. After this $N = 2$ SUSY algebra has been constructed, several $q$-deformed versions of this algebra have been proposed by using either $q$-deformed boson operators or $q$-deformed fermionic operators\[17]-[20]. These studies have been done by mutually commuting $q$-deformed bosonic and $q$-deformed fermionic oscillator variables. Recently, such studies have also been extended to $q$-oscillator systems covariant under some quantum supergroup transformations\[13]-[21].

The present paper investigates an interesting generalization for the $N = 2$ SUSY algebra. In our generalization, the superoscillators system is accomplished by two independent real deformation parameters $(q_1, q_2)$ and has a covariance under the two-parameter deformed quantum supergroup $SU_{q_1/q_2}(n | m)$. Moreover, our generalization gives mutually non-commuting two-parameter deformed bosons and fermions.

We should also mention that another example of such a mutually non-commuting bosons and fermions property with $sl_q(n | n)$-covariance was recently introduced using the one-parameter deformed oscillator variables by Chung\[21]. On the other hand, different algebraic forms of SUSY structure such as fractional supersymmetry and parasupersymmetry have been studied by taking deformation parameter $q$ being a root of unity\[22]-[23].

In this paper, our aim is to construct a two-parameter deformed $N = 2$ SUSY algebra by using the $SU_{q_1/q_2}(n | m)$-covariant $(q_1, q_2)$-deformed bosonic and $(q_1, q_2)$-deformed fermionic oscillator system. However, our superoscillator algebra construction serves as a generalization related to the studies on the
deformation of the conventional \( N = 2 \) SUSY algebra.

The paper is organized as follows: In section 2, the two-parameter deformed superoscillator algebra and its \( SU_{q_1/q_2}(1 \mid 1) \)-covariance are shown. In section 3, we construct the \((q_1, q_2)\)-deformed SUSY quantum mechanics for \( SU_{q_1/q_2}(n \mid m) \)-covariant \((q_1, q_2)\)-deformed boson and \((q_1, q_2)\)-deformed fermion system. In section 4, we analyze the Fock space representation of the \((q_1, q_2)\)-deformed \( N = 2 \) SUSY algebra and obtain the deformed Hamiltonian of the two-parameter deformed superoscillator system. Finally, we give our conclusions in section 5.

2 \( SU_{q_1/q_2}(1 \mid 1) \) -covariant two-parameter deformed superoscillator algebra

In this section, we consider a system of one \((q_1, q_2)\) -deformed bosonic and one \((q_1, q_2)\)-deformed fermionic oscillators, and show their covariance under the quantum supergroup \( SU_{q_1/q_2}(1 \mid 1) \). For this aim, we introduce the following \((q_1, q_2)\)-deformed superoscillator algebra:

\[
AB = \frac{q_1}{q_2} BA, \quad (2)
\]
\[
AB^* = q_1 q_2 B^* A, \quad (3)
\]
\[
AA^* - q_1^2 A^* A = q_2^{2(N_b + N_f)}, \quad (4)
\]
\[
B^2 = (B^*)^2 = 0, \quad (5)
\]
\[
BB^* + q_2^2 B^* B = q_2^{2(N_b + N_f)} + (q_1^2 - q_2^2) A^* A = q_1^{2N_b} q_2^{2N_f}, \quad (6)
\]

where \( A, A^* \) and \( B, B^* \) are the deformed bosonic and fermionic annihilation and creation operators, respectively. By using the tensor product, the annihilation operators which satisfy the above algebraic relations can be written as

\[
A = a \otimes q_2^{N_f}, \quad B = q_1^{N_b} \otimes c
\]

where \( a \) is the deformed bosonic annihilation operator and \( c \) is the deformed fermionic annihilation operator. \( q_1, q_2 \in \mathbb{R}^+ \). \( N_b \) and \( N_f \) are the bosonic and fermionic number operators, respectively. It is important to notice that, in this consideration, the two-parameter deformed bosonic and fermionic oscillators do not commute with each other (Eqs. (2) and (3)), and the system still satisfies the Pauli exclusion principle (Eq. (5)). Eq. (4) gives deformed bosonic commutation relation whereas Eq. (6) gives the deformed fermionic anticommutation one.

By considering the limit \( q_2 = 1 \), one can easily realize that above algebraic relations take the \( SU_q(1 \mid 1) \) covariant form\[13\]. It is helpful to remember this limiting case when one tries to find the transformation matrix \( T \) which leaves
invariant the system defined in Eqs. (2)-(6). In the light of these facts, let us consider the following transformation [21]:

\[
\begin{pmatrix}
A' \\
B'
\end{pmatrix} = T \begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
a & \beta \\
-\beta^*^{-1}a^*^{-1} & (a^*)^{-1}
\end{pmatrix} \begin{pmatrix}
A \\
B
\end{pmatrix},
\]

(7)

where \(T\) is the 2 \(\times\) 2 quantum super-matrix with two even \((a, a^*)\) and two odd \((\beta, \beta^*)\) generators. Since these generators satisfy the following algebraic relations:

\[
a\beta = \frac{q_1}{q_2} \beta a,
\]

\[
a\beta^* = \frac{q_1}{q_2} \beta^* a,
\]

\[
\beta^2 = (\beta^*)^2 = 0, \quad \beta\beta^* + \frac{q_1^2}{q_2^2} \beta^* \beta = 0,
\]

\[
aa^* + \beta\beta^* = 1, \quad aa^* - a^*a = \left(\frac{q_1^2}{q_2^2} - 1\right) \beta^* \beta.
\]

the matrix \(T\) is an element of \(SU_{q_1/q_2}(1 \mid 1)\). It is noticed that although the relations involving the oscillator creation and annihilation operators depend on the two deformation parameters \(q_1\) and \(q_2\), the relations containing the matrix elements of \(T\) effectively involve a single parameter \(r = q_1/q_2\). This is different from Celik’s study [24] where the quantum matrix which leaves invariant another two-parameter superoscillator algebra also depends on the two deformation parameters separately.

According to Eq. (7), one can easily write the transformed annihilation and creation operators as

\[
A' = a \otimes A + \beta \otimes B,
\]

\[
B' = -(a^*)^{-1} \beta^* (a^*)^{-1} \otimes A + (a^*)^{-1} \otimes B,
\]

\[
(A')^* = a^* \otimes A^* - \beta^* \otimes B^*,
\]

\[
(B')^* = -a^{-1} \beta a^{-1} \otimes A^* + a^{-1} \otimes B^*.
\]

In order to see all relations in Eqs. (2)-(6) remain unchanged for transformed operators, we should also use braided tensor product rule which can be shown as

\[
(x \otimes f_1)(f_2 \otimes y) = -x f_2 \otimes f_1 y,
\]

where \(f_1\) and \(f_2\) denote fermionic operators.

With the generalization of the above system to the \((n + m)\)-dimensional case, one can write the bosonic and fermionic generators as

\[
A_i = \prod_{k=1}^{i-1} q_1^{(N_k)_k} a_i \prod_{k=i+1}^{n} q_2^{(N_k)_k} \prod_{k=1}^{m} q_2^{(N_f)_k},
\]
\[ B_j = \prod_{k=1}^{n} q_1^{(N_k)_k} \prod_{k=1}^{j-1} (-q_1)^{(N_j)_{k}} c_j \prod_{k=j+1}^{m} q_2^{(N_j)_k}, \]

respectively. Here \( i \) stands for 1 to \( n \) whereas \( j \) stands for 1 to \( m \). These generators satisfy the following algebraic relations:

\[ A_i A_j = \frac{q_1}{q_2} A_j A_i, \quad i < j, \quad (10) \]

\[ A_i A^*_j = q_1 q_2 A^*_j A_i, \quad i \neq j, \quad (11) \]

\[ A_i A^*_i - q_1^2 A^*_i A_i = q_2^{2N} + (q_1^2 - q_2^2) \sum_{j=1}^{i-1} A^*_j A_j, \quad (12) \]

\[ A_i B_k = \frac{q_1}{q_2} B_k A_i, \quad (13) \]

\[ A_i B^*_k = q_1 q_2 B^*_k A_i, \quad (14) \]

\[ B_k B_l = -\frac{q_1}{q_2} B_l B_k, \quad k < l, \quad (15) \]

\[ B_k B^*_l = -q_1 q_2 B^*_l B_k, \quad k \neq l, \quad (16) \]

\[ B^2_k = (B^*_k)^2 = 0, \quad (17) \]

\[ B_k B^*_k + q_2^2 B^*_k B_k = q_2^{2N} + (q_1^2 - q_2^2) \sum_{i=1}^{n} A^*_i A_i + (q_1^2 - q_2^2) \sum_{l=1}^{k-1} B^*_l B_l, \quad (18) \]

such that they are invariant under \( SU_{q_1/q_2}(n | m) \) transformation. It is noted that in Eqs. (12) and (18), \( N \) represents the total number operator which is nothing but the addition of the fermionic and bosonic number operators. It is clear that, in the limit \( q_2 = 1 \), above system reduce the one parameter deformed \( SU_{q_1}(n | m) \)-covariant supersymmetric algebra\(^{13}\).

In the next section, the above deformed supersymmetric algebra will be used to construct a two-parameter deformed \( N = 2 \) SUSY algebra.

3 The construction of a two-parameter deformed \( N = 2 \) SUSY algebra for \( SU_{q_1/q_2}(n | m) \)-covariant \((q_1, q_2)\)-deformed bosons and fermions

In this section, we construct the \((q_1, q_2)\)-deformed SUSY quantum mechanics for \( n \) \((q_1, q_2)\)-deformed bosonic and \( m \) \((q_1, q_2)\)-deformed fermionic oscillators covariant under the quantum supergroup \( SU_{q_1/q_2}(n | m) \). For the sake of simplicity, we begin with the \( n = m = 1 \) case. We have the following deformed supercharges constructed from above deformed supersoscillators variables:

\[ Q = A^* B, \quad Q^* = B^* A, \quad (19) \]
where the operators \( A, B \) satisfy the commutation relations given in Eqs. (2)-(6). The odd generators \( Q, Q^* \) satisfy the nilpotency condition:

\[
Q^2 = (Q^*)^2 = 0,
\]

which can be obtained from Eqs. (3), (5). From all considerations above, we have the following two-parameter deformed \( N = 2 \) SUSY algebra with \( SU_{q_1/q_2}(1 \mid 1) \)-covariance:

\[
\{Q, Q^*\}_{q_1} = \tilde{H} = q_1^{-2} \left\{ q_2^{2N} \left[ A^* A + \frac{q_2}{q_1} B^* B \right] + (q_1^2 - q_2^2)(A^* A)^2 \right\},
\]

\[
\left[ Q, \tilde{H} \right]_{q_2/q_1} = 0,
\]

\[
\left[ Q^*, \tilde{H} \right]_{q_1/q_2} = 0,
\]

\[
Q^2 = (Q^*)^2 = 0,
\]

where \( N = N_b + N_f \), and also \( \{A, B\}_r = AB + rBA \) and \( [A, B]_r = AB - rBA \). The deformed Hamiltonian \( \tilde{H} \) in Eq. (21) gives a two-parameter generalization of the Hamiltonian for the supersymmetric oscillator in quantum mechanics.

The algebra constructed in Eqs. (21)-(24) has some interesting limiting cases: In the limit \( q_2 = 1 \), we find the one-parameter deformed \( N = 2 \) SUSY algebra [18],[21]. The conventional \( N = 2 \) SUSY algebra in Eq. (1) can be recovered in the limit \( q_1 = q_2 = 1 \).

For arbitrary indices of deformed boson and fermion variables, we have the following 2\((n + m)\) supercharges:

\[
Q_i = A_i^* B_i, \quad Q_i^* = B_i^* A_i.
\]

These supercharges are also nilpotent:

\[
Q_i^2 = (Q_i^*)^2 = 0,
\]

which can be obtained from Eqs. (14) and (17). Thus the generalized two-parameter deformed \( N = 2 \) SUSY algebra for \( SU_{q_1/q_2}(n \mid m) \)-covariant \((q_1, q_2)\)-deformed bosonic and \((q_1, q_2)\)-deformed fermionic oscillators is constructed by the following deformed commutation and anti-commutation relations:

\[
\{Q_i, Q_j\} = 0,
\]

\[
\{Q_i, Q_j^*\}_{q_2^2/q_1^2} = 0,
\]

\[
\{Q_i, Q_j^*\}_{q_1^2} = \tilde{H}_i = q_1^{-2} \left\{ q_2^{2N} \left[ A_i^* A_i + \frac{q_2}{q_1} B_i^* B_i \right] + (q_1^2 - q_2^2)(\frac{q_2}{q_1})^2 B_i^* B_i \left( \sum_{j=1}^{i-1} A_j^* A_j \right) \right\} + q_1^{-2} \left\{ (q_1^2 - q_2^2)(A_i^* A_i) \left[ \sum_{j=1}^{n} (A_j^* A_j) + \sum_{j=1}^{i-1} (B_j^* B_j) \right] \right\},
\]

where the operators \( A, B \) satisfy the commutation relations given in Eqs. (2)-(6). The odd generators \( Q, Q^* \) satisfy the nilpotency condition:

\[
Q^2 = (Q^*)^2 = 0,
\]
\[ \left[ Q_j, \tilde{H}_i \right]_{q_j^2/q_i^2} = 0, \quad (30) \]
\[ \left[ Q_i, \tilde{H}_i \right]_{q_j^2/q_i^2} = 0, \quad (31) \]

where \( N = N_b + N_f \). It is important to note that one can recover the one-parameter deformed \( N = 2 \) SUSY algebra with \( SU_{q_1}(n \mid m) \)-covariance in the limit \( q_2 = 1 \).

4 Fock space representation of the two-parameter deformed \( N = 2 \) SUSY algebra with \( SU_{q_1/q_2}(n \mid m) \)-covariance

We now discuss the Fock space representation of the two-parameter deformed \( N = 2 \) SUSY algebra with \( SU_{q_1/q_2}(n \mid m) \)-covariance. We first consider the simplest case with \( n = m = 1 \). The bosonic and fermionic number operators \( N_b \) and \( N_f \) satisfy the following commutation relations:

\[ [A, N_b] = A, \quad [A^*, N_b] = -A^*, \]
\[ [B, N_f] = B, \quad [B^*, N_f] = -B^*. \quad (32) \]

We introduce the Fock basis \( |n_b, n_f\rangle \) and the number operators also satisfy the following relations:

\[ N_b |n_b, n_f\rangle = n_b |n_b, n_f\rangle, \quad n_b = 0, 1, 2, \ldots, \]
\[ N_f |n_b, n_f\rangle = n_f |n_b, n_f\rangle, \quad n_f = 0, 1, \]
\[ N |n_b, n_f\rangle = (n_b + n_f) |n_b, n_f\rangle = n |n_b, n_f\rangle, \quad (33) \]

The representations of the operators \( A, A^*, B, B^* \) are

\[ A |n_b, n_f\rangle = q_2^{n_f} \sqrt{|n_b|} |n_b - 1, n_f\rangle, \quad (34) \]
\[ A^* |n_b, n_f\rangle = q_2^{n_f} \sqrt{|n_b + 1|} |n_b + 1, n_f\rangle, \quad (35) \]
\[ B |n_b, 0\rangle = 0, \quad B |n_b, 1\rangle = q_1^{n_b} |n_b, 0\rangle, \quad (36) \]
\[ B^* |n_b, 1\rangle = 0, \quad B^* |n_b, 0\rangle = q_1^{n_b} |n_b, 1\rangle, \quad (37) \]

where

\[ A^* A = [N_b] q_2^{2N_f} = \left( q_2^{2N_b} - q_1^{2N_b} \right) q_2^{2N_f}, \quad (38) \]
\[ B^* B = N_f q_1^{2N_b}, \quad (39) \]

which can be deduced from Eqs. (4) and (6). Acting the supercharges on the Fock basis \( |n_b, n_f\rangle \), we have

\[ Q |n_b, 0\rangle = 0, \quad Q^* |n_b, 1\rangle = 0, \quad (40) \]
\[ Q |n_b, 1\rangle = q_1^n \sqrt{|n_b + 1|} |n_b + 1, 0\rangle, \]
\[ Q^* |n_b, 0\rangle = q_1^{n_b-1} \sqrt{|n_b|} |n_b - 1, 1\rangle. \]
The energy eigenvalues for the deformed Hamiltonian in Eq. (21) are
\[ H |n_b, 1\rangle = \left(\frac{q_2}{q_1}\right)^4 q_1^{2n_b} |n_b + 1\rangle |n_b, 1\rangle, \]
\[ H |n_b, 0\rangle = q_1^{2(n_b-1)} |n_b\rangle |n_b, 0\rangle. \]

We now discuss a generic case for the Fock space representation of the two-parameter deformed superoscillator algebra generators \( A_i, A_i^* \) and \( B_i, B_i^* \). We introduce the Fock basis \( |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle \) as follows:
\[ (N_b)_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = (n_b)_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle, \quad (n_b)_i = 0, 1, 2, ..., \]
\[ (N_f)_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = (n_f)_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle, \quad (n_f)_i = 0, 1, \]
\[ N_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = (N_b + N_f)_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = n_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle, \]
where we have used the abbreviation for the Fock basis
\[ |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = |(n_b)_1, (n_b)_2, ..., (n_b)_n; (n_f)_1, (n_f)_2, ..., (n_f)_m\rangle. \]
The representations of the operators \( A_i, A_i^* \) are
\[ A_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = q_1^{\sum_{k=1}^{i-1}(n_b)_k} q_2^{\sum_{k=i+1}^{n}(n_b)_k} q_2^{\sum_{k=1}^{i-1}(n_f)_k} \sqrt{|\{n_b\}_i|} |\{\bar{n}_b\} - 1, \{\bar{n}_f\}\rangle, \]
\[ A_i^* |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = q_1^{\sum_{k=1}^{i-1}(n_b)_k} q_2^{\sum_{k=i+1}^{n}(n_b)_k} q_2^{\sum_{k=1}^{i-1}(n_f)_k} \sqrt{|\{n_b\}_i + 1|} |\{\bar{n}_b\} + 1, \{\bar{n}_f\}\rangle, \]
where
\[ A_i^* A_i = \left(\frac{q_1}{q_2}\right)^2 \sum_{k=1}^{i-1}(n_b)_k \left(\frac{q_1}{q_2}\right)^2 \sum_{k=i+1}^{n}(n_b)_k \left(\frac{q_1}{q_2}\right)^2 \sum_{k=i+1}^{n}(n_f)_k - q_1^{\sum_{k=1}^{i-1}(n_f)_k}, \]
and \( |\{n_b\}_i\rangle \) is defined by Eq. (38).

For the fermionic sector, we have the following number operator for a generic case:
\[ B_i^* B_i = (n_f)_i \left(\frac{q_1}{q_2}\right)^2 \sum_{k=1}^{i-1}(n_b)_k \left(\frac{q_1}{q_2}\right)^2 \sum_{k=i+1}^{n}(n_f)_k - q_1^{\sum_{k=1}^{i-1}(n_f)_k}, \]
which can be deduced from the defining relation of \( B_i \) in section 2. Therefore, the representations of the operators \( B_i, B_i^* \) are
\[ B_i |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = \begin{cases} 0 & \text{if } (n_f)_i = 0, \\
 q_1^{\sum_{k=1}^{i}(n_b)_k} q_2^{2\sum_{k=i+1}^{n}(n_f)_k} (-q_1) \sum_{k=1}^{i-1}(n_f)_k |\{\bar{n}_b\}, \{\bar{n}_f\} - 1\rangle & \text{if } (n_f)_i = 1, \end{cases} \]
\[ B_i^* |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = \begin{cases} 0 & \text{if } (n_f)_i = 1, \\
 q_1^{\sum_{k=1}^{i}(n_b)_k} q_2^{2\sum_{k=i+1}^{n}(n_f)_k} (-q_1) \sum_{k=1}^{i-1}(n_f)_k |\{\bar{n}_b\}, \{\bar{n}_f\} + 1\rangle & \text{if } (n_f)_i = 0, \end{cases} \]
where
\[ |\{\bar{n}_b\}, \{\bar{n}_f\}\rangle = |(n_b)_1, ..., (n_b)_n; (n_f)_1, ..., (n_f)_i - 1, ..., (n_f)_m\rangle. \]
5 Conclusions

In this paper, we defined a two-parameter deformed superoscillator algebra with $SU_{q_1/q_2}(n \mid m)$-covariance. By means of such generalized superoscillator system, we constructed a two-parameter deformed $N = 2$ SUSY algebra covariant under the quantum supergroup $SU_{q_1/q_2}(n \mid m)$. We explicitly studied for the case of one $(q_1, q_2)$-deformed boson and one $(q_1, q_2)$-deformed fermion system with $SU_{q_1/q_2}(1 \mid 1)$-covariance. For this system, we particularly discussed the Fock space properties and found the energy eigenvalues for the deformed Hamiltonian in terms of two deformation parameters. The two-parameter deformed $N = 2$ SUSY algebra constructed here has some important limiting cases: The one-parameter deformed $N = 2$ SUSY algebra\[21\] can be recovered in the limit $q_2 = 1$. The limit $q_1 = q_2 = q$ gives the $SU(n \mid m)$-covariant one-parameter deformed $N = 2$ SUSY algebra constructed from the $q$-deformed bosonic and fermionic Newton oscillators\[20\]. The conventional $N = 2$ SUSY algebra in Eq. (1) can be obtained in the limit $q_1 = q_2 = 1$.

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