Thermal equilibrium of black di-rings

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Abstract. We show some physical properties of the systems of regular black di-ring (the regular gravitational systems with two $S^1$-rotational black rings arranged in concentric way in five-dimensional asymptotically flat spacetimes). Especially the existence of isothermal systems of black di-ring are shown, in which both isothermality and isorotation between the inner black ring and the outer black ring are realized. We also give some properties of the thermodynamic black di-ring including discussion about thermodynamic stabilities of the system.

1. Introduction
Since the groundbreaking discovery of $S^1$-rotating black rings by Emparan and Reall [1], several non-trivial black hole systems have been obtained and used to clarify peculiar features of the higher dimensional gravity. Especially in five dimensions, solitonic solution-generation methods invented in the legendary era from the 1970s to the 1980s have been proved to be greatly powerful and applied to systematic reconstruction of known solutions including the Myers-Perry black holes [2], the $S^1$-rotating black rings [3, 4] and also to generation of new solutions: $S^2$-rotating black rings [5, 6, 7] and doubly rotating black rings [8], for example. As further trial, hunting new solutions have been attempted ambitiously to obtain novel black holes that have never been expected in four dimensions, like black hole systems with multi-horizons [9, 10, 11, 12, 13] or a topologically non-trivial horizon [14, 15].

The solutions of black di-ring among them correspond to concentric configurations composed of two independently $S^1$-rotating black rings. The authors first discovered the regular di-rings by using the solitonic method similar to the Bäcklund transformation [10]. Successively, Evslin and Krishman constructed another di-ring solution-set [11]. They used the inverse scattering method that was modified by Pomeransky to treat the higher dimensional case (hereafter abbreviated to PISM) [2]. However, because of the complexity of their expressions further investigation of the physics of the di-ring systems remained to be done until quite recently.

In the recent work we confirmed the equivalence of these two di-ring solution sets, and tried to do some investigation of the physics of di-ring systems [16]. So in this place we shall show some properties of regular black di-rings, especially consider the thermodynamic regular black di-ring systems (the states in which both isothermality and isorotation between the inner black ring and the outer black ring are realized). For one of the main purposes to study the higher dimensional gravity is to clarify the phase structure of thermal states of higher dimensional black objects and classify the properties into the universal ones independent of the dimensional number of spacetimes and the peculiar ones depending on the dimensional number. To accomplish this task the study of thermodynamic multihorizon systems is clearly necessary because the multihorizon
systems appear naturally in higher dimensions. The thermodynamic black di-ring system may serve as the starting point for the study of thermodynamic multihorizon systems.

For the systems of the black di-rings, we consider five dimensional spacetimes with three commuting Killing vector fields: a time-like Killing vector field and two axial-Killing vector fields. We assume further that one of the axial-Killing vector fields is orthogonal to the other. So the line-elements adopted here can be reduced to

$$ds^2 = G_{tt}(dt)^2 + 2G_{t\psi}dt d\psi + G_{\psi\psi}(d\psi)^2 + G_{\phi\phi}(d\phi)^2 + e^{2\nu}(d\rho^2 + dz^2) ,$$

where the metric coefficients are the functions of $(\rho, z)$ and $\det G = -\rho^2$ is imposed. Based on this metric form, we can construct regular di-ring systems using both the solitonic methods mentioned above. It is noteworthy that the original representation of di-ring solutions which was first found by us can be directly obtained by PISM [16].

Here we omit the explanation of the solution-generating methods, the expressions of the metric form and physical quantities of the di-ring solutions and also the detail of how to confirm the equivalence of the two different solution-sets of the black di-rings (one was generated by the authors and the other was by Evslin and Krishnan). In section 2 we show the existence of regular thermodynamic black di-ring systems, and also comment some properties of the thermodynamic black di-rings. In section 3 we give the discussion about thermodynamic stability (instability) of the systems.

2. Existence of regular thermodynamic black di-rings
In this place we show the existence of thermodynamic black di-rings (i.e., systems of regular black di-ring in thermodynamic equilibrium) and their properties. Similar results are also found in [17]. First we must impose the following balance conditions between gravitational attractions and centrifugal forces on the di-ring solution to eliminate conical singularities on the axes:

$$\Delta \phi_L = 2\pi, \quad \Delta \phi_R = 2\pi .$$

Here $\Delta \phi_L$ and $\Delta \phi_R$ are the periodic angles to keep regularity on the outer and inner $\phi$-axes respectively as depicted in Fig. 1.

**Figure 1.** Schematic figure of di-ring. The di-ring has two rotational axes: $\phi$-axis and $\psi$-axis.

**Figure 2.** Schematic figure of thermodynamic di-ring. The outer ring(L) and inner ring(R) have the same horizon temperature and angular velocity.
To ensure that a system of regular black di-ring becomes a thermodynamic equilibrium state, the temperatures ($\tau_L, \tau_R$) and angular velocities ($\omega_L, \omega_R$) of the outer (L) and inner (R) black rings of the system must satisfy the following condition,

$$\tau_L = \tau_R, \quad \omega_L = \omega_R. \quad (3)$$

It should be noticed that in contrast with the case of black Saturn, to construct a thermodynamic black di-ring with two black rings is not so trivial as pointed out in [18]. Actually black rings with the same temperature and angular velocity have the same shape and size [19], so that we must arrange two identical rings into a thermodynamic di-ring keeping the conditions (2) and (3). That is, it seems rather natural that thermodynamic di-rings do not appear in the phase space. If there remains any possibility of thermodynamic di-rings, some non-linearity may play a key role.

Fortunately we can find appropriate four independent moduli-parameters, and essentially solve the above four equations of the conditions (2) and (3) in an analytic way (see [16] for more details). So we can easily confirm the existence of regular thermodynamic black di-rings. In Fig. 3 and Fig. 4 the states of thermodynamic black di-ring appear as a continuous thick curve with a cusp. Other thermodynamic black objects are also shown for comparison.

**Figure 3.** Total area $a_h$ vs squared angular momentum $j^2$. Dotted, thin, dashed, and thick curves correspond to MP black hole (MP-BH), black ring (BR), black Saturn (BS) and black di-ring (BD) respectively.

**Figure 4.** Total area $a_h$ vs squared angular momentum $j^2$. The behavior of the black objects in the thin branch is shown.

The behavior of thermodynamic di-ring is similar to other thermodynamic objects. The phase of the di-ring has a ‘fat ring’ branch and a ‘thin ring’ branch. The state of the di-ring is less dominant than others with respect to entropy (i.e., area $a_h$) in both branches. In the fat branch, the curves of black ring, black Saturn and black di-ring approach that of MP black hole respectively as depicted in Fig. 3. More detailed behavior of the four black objects is shown in the upper right diagram magnified near the extremal point. From the graphs in Fig. 4, in the thin branch the curve of black Saturn immediately asymptotes to that of black ring while the black di-ring acts independently.

3. **Thermal stability (instability) of thermodynamic black di-rings**

Next, as one of important properties of thermodynamic di-rings we discuss whether thermodynamic local stability is realized or not. We follow the approach that was introduced
They found the existence of meta-stable states of black Saturn. To do this, under the condition of fixed mass and angular momentum we shall search for local maxima of the corresponding entropy function, which is a function of appropriate moduli-parameters. If we find that some maximal eigenvalues of the Hessian of the entropy function becomes negative, we can say that meta-stability occurs at this point. The results of the survey for the black di-ring and black Saturn are shown in Fig. 5 and Fig. 6 respectively. Any stability does not appear in the di-ring case, while the window of meta-stability opens in the black Saturn case as pointed in [20].

Figure 5. Max eigenvalue of Hessian of entropy of black di-ring vs one of moduli-parameters ($h_4$)

Figure 6. Max eigenvalue of Hessian of entropy of black Saturn vs one of moduli-parameters ($h_2$)

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