FINITE SIZE SCALING OF PROBABILITY DISTRIBUTIONS IN
SU(2) LATTICE GAUGE THEORY AND $\phi^4$ FIELD THEORY

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ABSTRACT

For a system near a second order phase transition, the probability distribution
for the order parameter can be given a finite size scaling form. This fact is
used to compare the finite temperature phase transition for the Wilson lines in
$d = 3+1$ SU(2) lattice gauge theory with the phase transition in $d = 3$ $\phi^4$ field
theory. I exhibit the finite size scaled probability distributions in the form of
a function of two variables (the reduced ‘temperature’ and the magnetization)
for both models. The two surfaces look identical, and an analysis of the errors
also suggests that they are the same. This strengthens the idea that the SU(2)
effective line theory is in the Ising universality class. I argue for the wider
application of the method used here.

1. Overview

Some time ago, Svetitsky and Yaffe\(^3\) conjectured that the phase transition un-
dergone by the Wilson lines in finite temperature SU(2) lattice gauge theory should be
in the same universality class as the corresponding Ising model. In particular the 3+1
dimensional SU(2) theory should have the same exponents and so forth as the three-
dimensional Ising model. The basis for this conjecture is no more certain than noticing
that the dimensionality and the symmetry of the Wilson line theory are the same as
those of the Ising model, convincing oneself that the interactions in the effective line
model are short range, and hoping that under renormalization group transformations
the two theories tend to the same fixed point in the appropriate space of possible
actions.

To test this idea, a considerable amount of work has been done both using various
analytic approximations and numerically. However, to date, work has mainly concen-
trated on verifying that the critical exponents of the two models are the same. The
critical exponents of the three-dimensional Ising model are now known fairly precisely, and measurements of the corresponding quantities for SU(2) are in reasonably good agreement. However, there are many other quantities which should be constant across a universality class. For instance, there are a number of amplitude ratios which can be measured. There are also a variety of functions which are universal up to some rescalings. A typical example would be the finite size scaling form of the susceptibility. Some attention has been paid to these issues, but the agreement between the models is only partially satisfactory. It appears that the small lattices to which we are currently limited in simulating the gauge theory forbid a very convincing comparison of amplitude ratios. The purpose of this paper is to address any consequent uncertainty as to the universality class of the gauge theory.

Some time ago it was noted by Bruce and by Binder that, in the vicinity of a second order phase transition, the whole probability distribution for the order parameter can be given a finite size form. The usual finite size forms of the susceptibility, magnetization, etc. then appear as various moments of this function. In this paper I shall consider the finite size form of the probability distribution at varying temperatures. Thus the object forms a surface; the probability is a function of the scaled magnetization and of the scaled temperature. This surface should be universal, and hence it is possible to compare the phase transitions in two different models by comparing their probability surfaces. It should be noted that the form of the surface does depend on the boundary conditions, though not the details of the action. In the past this has only been done to the limited extent of comparing the probability distributions right at the critical point. Here I expand on the technique and consider the full surface in a finite region around the phase transition.

I will show the form of this surface for SU(2) based on existing Monte Carlo data. I have taken Monte Carlo data for the $\phi^4$ scalar model, which should also be in the same universality class, and I will exhibit the surface for that model, and argue that the two are the same within the appropriate errors. Visually they are strikingly similar. This provides further evidence that SU(2) lattice gauge theory really is in the same universality class as the Ising model and $\phi^4$ field theory. For readers who prefer to look at the pictures before they read the equations, the comparisons are in Figures 4 through 6.

As to the organisation of the paper: Section 2 introduces notation and definitions; Section 3 contains the facts about the finite size scaling surface that are needed in this paper; Section 4 provides renormalization group arguments for the universality of the scaling surface; Section 5 briefly discusses the generation of the data for $\phi^4$; Section 6 consists of the comparison of the surface for the two models, and Section 7 concludes.

*As the author rapidly tires of writing (or reading) ‘finite size scaling’, this phrase will often be abbreviated to just ‘finite size’ in what follows.
2. Definitions

In the case of $\phi^4$ field theory, we take an $N^3$ lattice with fields $\phi_i$ living at each site. We posit the action to be

$$S_\phi = \frac{\alpha}{2} \sum_{<ij>} (\phi_i - \phi_j)^2 - \frac{r}{2!} \sum_i \phi_i^2 + \frac{u}{4!} \sum_i \phi_i^4.$$  \hspace{1cm} (1)

Note the sign of the second term. We will be mainly interested in the absolute magnetization $\phi$, which is defined by

$$\phi = \frac{1}{N^3} \left| \sum_i \phi_i \right|.$$  \hspace{1cm} (2)

This quantity is defined at every instant, or perhaps at every sweep in a Monte Carlo simulation. Its thermal average is denoted by $M_\phi = \langle \phi \rangle$. We can also define a corresponding susceptibility $X_\phi$ by

$$X_\phi = N^3 (\langle \phi^2 \rangle - \langle \phi \rangle^2).$$  \hspace{1cm} (3)

In the case of the SU(2) gauge theory, we have an $N^3 \times N_\tau$ lattice with gauge variables $U_i$ living on each link. The Euclidean action is the normal Wilson one:

$$S_U = \frac{4}{g^2} \sum_\square (1 - \frac{1}{2} \text{Tr} U_\square),$$  \hspace{1cm} (4)

where $U_\square$ is the usual plaquette variable

$$U_\square = \prod_{l \in \square} U_l.$$  \hspace{1cm} (5)

The physical temperature in this theory is given by

$$T = \frac{1}{aN_\tau}.$$  \hspace{1cm} (6)

We require that this temperature is finite (ie. $N_\tau < N$), but otherwise we shall not be concerned with it.

The Wilson line observables $L_i$ are obtained by tracing over all the timelike links at a fixed spatial position $i$:

$$L_i = \frac{1}{2} \text{Tr} \prod_{\tau=1}^{N_\tau} U_{i,\tau}.$$  \hspace{1cm} (7)

These are defined in such a way as to be real variables taking values in $[-1, 1]$, and hence one can think of them as though they were spins. We can then define an absolute instantaneous average of these quantities:

$$L = \frac{1}{N^3} \left| \sum_i L_i \right|.$$  \hspace{1cm} (8)
By analogy with the $\phi^4$ case we also define the thermal expectation value $M_L = \langle L \rangle$, and a susceptibility
\[ \chi_L = N^3 (\langle L^2 \rangle - \langle L \rangle^2). \] (9)

For the remainder of this paper, we will take no interest in the $U_i$ variables but will instead focus entirely on the $L_i$ variables, treating them as the site variables in some effective three-dimensional theory of which we do not quite know the action. It is known, however, that this theory undergoes a second order phase transition at some critical value of the coupling $g_c$. Therefore we define a reduced ‘temperature’ by
\[ t_L = \frac{4/g^2 - 4/g_c^2}{4/g_c^2}. \] (10)

This quantity is distinct from the physical temperature of Equation (3). Similarly, the $\phi^4$ theory has a critical point at some $r_c$. We define its reduced ‘temperature’ as
\[ t_\phi = \frac{r - r_c}{r_c}. \] (11)

For convenience, in the rest of this paper we will refer to these variables as reduced temperatures (without quotes).

Now, with our definitions, both models are in their broken (magnetized) phase when $t$ is positive, and their unbroken phase when $t$ is negative. We expect the normal amplitude-exponent behaviour of the infinite system in the critical region:
\[ M = M_0 + t^\beta \] (12)

for $t$ small and positive, while
\[ \chi = \chi_\pm |t|^{-\gamma} \] (13)

for $t$ small and positive ($\chi_+$), or negative ($\chi_-\phi$). We also take the usual definition for the correlation length exponent $\nu$, so that in the critical region
\[ \xi = \xi_\pm |t|^{-\nu}. \] (14)

Here $M$, $\chi$ and $t$ without subscripts are doing duty for both models discussed in this work. Potentially the exponents for the two models could be different. Research to date provides moderately convincing evidence that they are the same. The logic of this paper is to assume that the exponents are the same and then investigate whether the other quantities that will be introduced match as we expect.

3. Nature of the Finite Size Scaling Surface

If we look at the $\phi$ measured on one sweep of the lattice in a simulation, it will not typically be equal to its thermal average $M_\phi$, but rather will be drawn from some probability distribution $P_\phi(N, t_\phi, \phi)$ which has $M_\phi$ as its expectation value and $\chi_\phi/N^3$ as its variance. This last is obvious from the definition of $\chi_\phi$, Equation (3).
We can study this probability distribution by taking a sufficiently large collection of measurements of $\phi$ and making a normalised histogram of them. As the notation for $P_\phi$ indicates, the form of this histogram depends not only on the reduced temperature $t_\phi$, but also on the size $N$ of the lattice.

The first thing we shall need is to define a finite size scaling version of these histograms. We take the ansatz

$$P_\phi = N^{\beta/\nu} Q_\phi(t_\phi N^{1/\nu}, \phi N^{\beta/\nu}).$$  \hspace{1cm} (15)

An algorithm to obtain $Q_\phi$ by simulations is then to construct the histograms $P_\phi$ for a range of different values of $N$ and $t_\phi$, scale their ordinates by $N^{-\beta/\nu}$, scale their second abscissae by $N^{\beta/\nu}$, rearrange them along the $t_\phi N^{1/\nu}$ axis, and then interpolate between them in some appropriate manner. The result is a surface $Q(z_\phi, \tilde{\phi})$ (or at any rate an approximation to it). Here we have introduced

$$z_\phi = t_\phi N^{1/\nu},$$  \hspace{1cm} (16)

and

$$\tilde{\phi} = \phi N^{\beta/\nu}$$  \hspace{1cm} (17)

for convenience.

Of course, we can follow exactly the same procedure for the SU(2) case, constructing the probability distributions $P_L(N, t_L, L)$ and then taking

$$P_L = N^{\beta/\nu} Q_L(z_L, \tilde{L}),$$  \hspace{1cm} (18)

where

$$z_L = t_L N^{1/\nu}$$  \hspace{1cm} (19)

and

$$\tilde{L} = L N^{\beta/\nu}.$$  \hspace{1cm} (20)

We shall refer to these surfaces $Q$ as finite size scaling surfaces.

The next thing to note is that the surface $Q(z, \tilde{\phi})$ is a universal quantity up to rescalings of its two arguments. In other words, all models in a particular universality class should have a $Q$ surface of the same shape. The next section of this paper will be devoted to justifying from a renormalization group approach that these claims are valid.

It is straightforward to check that the truth of (15), the finite size form for the probability distributions, implies the usual finite size relation for the magnetization $\mathcal{M}$

$$\mathcal{M} = N^{-\beta/\nu} Q_M(z)$$  \hspace{1cm} (21)

and a similar equation for the susceptibility $\mathcal{X}$. It also follows that the universality of $Q$ guarantees the universality of the finite size forms of these quantities.
4. Theoretical arguments

Here we will justify, on the basis of real space renormalization group arguments (block spins and so on), the statements made in the previous section. Really, the reasons why the probability distributions have a finite size form are exactly the same as the reasons why anything else has a finite size form. However, for completeness, we shall go through it. The standard textbook treatments of the real space renormalization group are gestured at in a few sentences and then we turn to finite size scaling per se in somewhat more detail.

We have our lattice with fields $\phi_i$ on a lattice of size $N^d$. We suppose that we have divided the lattice up into some blocks of size $b$ on a side. The block fields are averages of the $\phi_i$ in the block, except that, if we wish our block spin transformation to have a non-trivial fixed point, we have to rescale them by some non-trivial factor as well. The correct definition to do this is

$$\psi_k = \frac{1}{b^{d-\beta/\nu}} \sum_{i \in k} \phi_i. \quad (22)$$

Now we imagine integrating out the original $\phi_i$ variables to find the effective theory for these block fields $\psi_k$. The result is some different action from the one that we had before. We also have to add a constant to the action to keep the energy scale from running away but this will not concern us. As usual, we suppose that this renormalization group transformation is represented by some operator on the space of all possible actions for theories with the same symmetry and dimensionality. We further suppose that the transformation has some fixed point, that the transformation equations are analytic at that fixed point, and that everything that matters happens either in the region in which the transformation equations can safely be linearized (when the eigenvectors of the linearized transformation are the scaling fields), or at least in a region outside that, in which the scaling fields have become non-linear but still respond to RG transformations in the same way. We assume that in the linear region the reduced temperature is one of the scaling fields, and its associated exponent is $1/\nu$. Thus in the region we shall assume we are in, the effect of a RG transformation on the reduced temperature is to change

$$t \rightarrow b^{1/\nu}t. \quad (23)$$

Strictly speaking, after enough RG transformations, we will be driven far enough along the direction of this scaling field that it will no longer be safe to approximate it just by $t$, but it will be some more complex function of $t$ and other parameters. We will just call it the $t$-field, whatever it might be.

We are interested in probability distributions for the average value of the fields. Now the average over the $\psi_k$ is exactly the average over the $\phi_i$ except for the factor of $b^{-\beta/\nu}$ that appears in the denominator in Equation (22). We can describe this by saying that under a renormalization group transformation

$$\phi \rightarrow b^{\beta/\nu} \phi. \quad (24)$$
We also note at this point that in equation (22) we have arranged matters so that the scale of the \( \phi_i \) is fixed under RG transformations. Now, for our two models to end up at the same fixed point action after many RG transformations we must ensure that the scale of the fields in both models is the same. Thus we must rescale one or the other of them. However, since the two sets of fields enter into actions which are of superficially quite different forms, we cannot calculate this non-universal scale factor and must leave it arbitrary in our analysis.

The main assumption of RG derivations of finite size scaling\(^4\) is that the RG transformation mostly preserves the locality of the action. That is, if the original action only involves a few local couplings, then only a few couplings at finite distances will be important in the renormalized action also. This means, and this is the crucial point, that the form of the RG equations (for one set of couplings in terms of the other) will be the same in the finite system as in the infinite system. The only way this can go wrong is if the finite system is so small that it cannot hold all the couplings needed. This places an upper limit on the number of times we can apply a RG transformation to a theory on a finite lattice.

So, we now have all the assumptions at hand and can begin the argument proper. Imagine our two theories at different points in the space of all possible actions, \( S_1 \) and \( S_2 \). We suppose that we apply the maximum feasible number of RG transformations to them, which will be \( n \) say, so that the resulting lattices are the same minimal size, \( N_{\text{min}} \), and have actions \( S'_1 \) and \( S'_2 \). We assume that the original lattices were very large, and the original theories were at their critical points. Hence \( n \) is large, and \( S'_1 \) and \( S'_2 \) are very close to the fixed point action \( S_f \). Since these three theories are very similar (ie. all their parameters are very similar), and since they are living on a small finite lattice (of the minimal size to fit the necessary couplings on), they all have an extremely similar probability distribution for the average value of their fields. Since the average value of these fields is exactly the average value of the original fields, except for a known scaling factor, this distribution will give us the shape function we need for the surface at \( t=0 \). Suppose this distribution is \( P_f(\psi_n) \). In this instance (24) becomes

\[
\psi_n = b^{n\beta/\nu} \phi,
\]

so

\[
P(N, 0, \phi) = b^{n\beta/\nu} P_f(b^{n\beta/\nu} \phi).
\]

The \( b^{n\beta/\nu} \) factor in front comes from keeping the measure straight (or making sure \( P \) and \( P_f \) are both normalised). We can rewrite this as

\[
P(N, 0, \phi) = (N/N_{\text{min}})^{\beta/\nu} P_f((N/N_{\text{min}})^{\beta/\nu} \phi).
\]

This is clearly of the form of the finite size surface (13) (restricted to \( t = 0 \)). The same form applies to both models equally (except for the arbitrary scale factor mentioned earlier). It is clear, however, that since the origin of this shape is a theory on a small
lattice, the details of the form are likely to be dependent on the boundary conditions (as has proved to be the case).

Now we must deal with \( t \neq 0 \) (but still in the critical region). If we start off with our \( S_1 \) and \( S_2 \) and start applying RG transformations then they will initially follow trajectories roughly toward the fixed point. However, \( t \) is being increased with every transformation (ie. it is relevant), so after our \( n \) transformations we are very close to the line which has all scaling fields zero except for the \( t \)-field. This gets us to some theory on that line for the minimal size lattice. That gives us our shape function. We then just pull it back to our original theories. The shape function will be analytic in \( t \) since it comes from a theory on a finite lattice (of size \( N_{\text{min}} \)). The algebra is similar to that for \( \phi \) above, and so we get

\[
P(N, t, \phi) = (N/N_{\text{min}})^{\beta/\nu} Q((N/N_{\text{min}})^{\beta/\nu} \phi, (N/N_{\text{min}})^{1/\nu} t),
\]

which is the same as (15) after we absorb \( N_{\text{min}} \) into the definition of \( Q \). Where did the arbitrary rescaling of \( t \) between the models come from? Well we made no effort to define our reduced temperature variables, (10) and (11), in a comparable way (not that we could easily have done so), so they will not be the same thing in terms of the \( t \) in some systematic coordinatisation of the space of actions.

We could go on and do the same for the magnetic field \( h \), and we would get a finite size scaling hyper-surface. However, quite besides the affront to the English language, the author was unable to locate any good four-dimensional plotting software. Hence all relevant variables besides \( t \) are taken as zero in this paper.

5. Monte Carlo data

To make use of the above ideas I needed to take data for the probability distributions for another model in the same universality class as SU(2), data for which was already available. For other purposes, I had developed a multigrid Monte Carlo code for \( \phi^4 \) field theory in three dimensions, so I made my comparison with that model. The correctness of the code was checked by tests in the Gaussian limit and by comparison with other published results for the susceptibility.

All the data reported in this paper were taken at \( u = 50 \) and \( \alpha = 1 \). Data were taken on \( 8^3 \), \( 16^3 \), and \( 32^3 \) lattices with periodic boundary conditions. A summary of the data set is given in Table I. The critical point was determined by a finite size comparison of the peaks of the susceptibility to be at \( r = 4.2485 \). The exponents were taken to be the Ising values. Specifically, based on the discussion in Liu and Fisher, I took \( \nu = 0.63 \), and \( \beta = 0.33 \). All the histograms were then rescaled as suggested in Section II and plotted on a single graph. The results are shown in Figure I. However, it is hard to visualise data plotted in such a manner, and still harder to compare two different sets of data. For that reason, I fitted an appropriate smooth function to the data. For numerical convenience in the fitting process it is easier to switch to using a log of the finite size form of the histograms \( Q(z, \tilde{\phi}) \) introduced earlier, and so I looked at

\[
S_Q(z, \tilde{\phi}) = -\ln Q(z, \tilde{\phi}).
\]
The form used to fit this was then

\[ g(z, \phi) = -\frac{a(z)\phi^2}{2!} + \frac{b(z)\phi^4}{4!} + \frac{c(z)\phi^6}{6!} - d(z), \]  

(30)

where

\[ a(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3, \]  

(31)

\[ b(z) = b_0 + b_1 z + b_2 z^2, \]  

(32)

\[ c(z) = c_0 + c_1 z + c_2 z^2, \]  

(33)

\[ d(z) = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4. \]  

(34)

Here \( a_0 \) through \( d_4 \) are parameters determined by minimizing the \( \chi^2 \) of the fit. The form (30) through (34) is somewhat inspired by perturbative considerations \(^\text{17}\); however, in this paper I make no attempt to connect with \( \epsilon \)-expansion work. It is also the first few terms of the most general power series possible given the global symmetry (which dictates the use of only even powers of \( \phi \)). My criterion for selecting it is simply that it provides a fairly good fit to the data.

The process of fitting is complicated by the lack of any simple means to do an error analysis. There are two main problems. The first is that the fluctuations in the histograms from the values they would have given an infinite data set are correlated across neighbouring bins (i.e., different values of \( \phi \)). The second is that there are systematic errors between the histograms from different lattice sizes used in the surface, probably due to corrections to finite size scaling (presumably both from irrelevant operators and from analytic effects).

To make some effort to control this, I did the fits using error bars for the data generated in two different ways. In the first case I just gave every bin in every histogram an error bar of equal size, arbitrarily taken to be one, and then minimised the resulting \( \chi^2 \) to obtain a fit. In the second case I obtained error bars by fitting to each individual histogram (prior to taking the logarithm) a form derived from the fit surface (30) by taking its exponential and fixing \( z \) (the value of which has no effect in this instance, being absorbed into the \( a_0 \)...\( d_4 \) variables). These fits all had very reasonable \( \chi^2 \) per degree of freedom. Then I derived the error bars for each bin by assuming its value was a Poisson variable with a mean given by the value of the fit at that \( \phi \). (This was actually repeated until fit and error bars were self consistent). Then those error bars were used as the input to a full fit of (30) to all the data at once (now with minus the logarithm taken). This second procedure has the advantage that the resultant \( \chi^2 \) is a meaningful number with the usual interpretation: the \( \chi^2 \) per degree of freedom is supposed to be one (assuming the errors are independent and Gaussian). All analyses presented here were done with both procedures and the conclusions were not substantially different. Although this is a somewhat crude method, I feel it is unlikely that any feature of the treatment which appears the same in two such different procedures will be misrepresented due to the defects of the error analysis. Graphs and numbers
presented in this paper are based on the second method (which did work somewhat better than the first), and the actual parameters obtained are, for this $\phi^4$ case, given in Table 3. The full data sets and error bars are available on request from the author.

The results of this procedure are displayed in Figure 2. At the risk of insulting the reader’s intelligence I remind him or her once again that the original histograms have by this point been rescaled by some factors of the lattice size and have had minus their logarithm taken. Both fit and data are shown there, and readers with good eyesight will be able to ascertain that the surface is generally plausible as a fit to the data. There are some problems, particularly for large $z$ and modest $\tilde{\phi}$. The $\chi^2$ per degree of freedom for the whole fit is 1.69. Clearly then there are systematic failures of the data to conform to this surface. Nonetheless, I feel that these are really rather modest, particularly given that the smallest lattices used were only $8^3$, and that Figure 2 does provide fairly good evidence both for the existence of a finite size scaling surface $S_Q$, and for the efficacy of the polynomial (30) as a fit to it.

6. Comparison of models

For the case of the SU(2) lattice gauge theory, data is available for $N_\tau = 4$ and the work of computing the critical point and so forth has already been done. I simply rescaled the histograms and fit the surface of form (30) in the same manner as described in the previous section for the $\phi^4$ model. The coefficients are in Table 4. The resulting picture is in Figure 3. Again, the data appear to be making a reasonable effort to conform to a smooth surface. The reader has only to look at the untouched histograms in Figure 1 of the paper by Engels et al in which the SU(2) data originally appeared to realise that, by contrast, they certainly do not form a single surface in the unscaled variables ($t_L$ and $L$ rather than $z_L$ and $\tilde{L}$).

The value of $\chi^2$ per degree of freedom for this fit of all the SU(2) data is 3.54. Clearly there are more significant systematic deviations from the data in this case than in the $\phi^4$ case. I do not have a complete understanding of why this should be. It may be simply that the corrections to scaling are larger for SU(2). An alternative explanation is that the SU(2) data has better statistics, allowing systematic histogram to histogram variations to stand out of the random errors more clearly. This motivates the following slightly speculative digression on practical matters. The run lengths are given as $N_{run}$ in Tables 1 and 2. The SU(2) numbers are indeed much larger. On the face of things, though, they are not directly comparable as the algorithms used to obtain the data were not the same. The SU(2) data were taken with a local heat bath algorithm, while the $\phi^4$ data were obtained with a multigrid Metropolis algorithm. However, in practice this may make little difference. This is because measuring a histogram requires knowing many higher moments of the distribution to get the smooth shape right, not just the magnetization. In a simulation, my intuition is that these higher moments decorrelate much faster than the low moments (that is their auto-correlation time is much lower). This is because a small change in the order parameter measurement typically involves a large change in high powers of the order parameter. Alternatively, one can think of the
high moments of the distribution as coming from summing over Green’s functions with many external legs which are very susceptible to fluctuations. Hence, major reductions in the autocorrelation time for the magnetization may involve throwing away intermediate measurements of higher moments which were already effectively independent. To take the simplest case, if the different algorithms thereby become effectively comparable and the only thing that matters is the number of measurements, then the SU(2) runs have about two or three times the statistics of the $\phi^4$ runs. If the systematic errors were about the same size in both cases, this would mean the SU(2) fit would have a $\chi^2$ per degree of freedom about two or three times as large, which is consistent with the observed difference. These considerations also tie in with my observations (space considerations forbid extensive reporting of such details) that bin to bin correlations in the histograms are actually very slight, even where magnetization auto-correlation times are large (this is why the individual histograms, as opposed to all of them together, could be fit with a statistically satisfactory $\chi^2$ even though the error bar estimate had assumed independence). I do not presently understand whether these arguments imply that reducing critical slowing down in the magnetization measurements is never useful in the context of finite size surfaces, or if they only mean that lattice sizes must be that much larger before it becomes useful. Other workers have also observed an apparent absence of critical slowing down for the probability distributions.

Certainly, non-local algorithms that have lower auto-correlation times but need more cpu time should be used with caution in this context.

The last thing to do is to adjust the scales of one of the plots so that it fits the other as nearly as possible. Two operations can be performed. Firstly, $z$ can be scaled by $z \rightarrow \sigma z$; secondly, $\phi$ can be changed in a similar manner, $\phi \rightarrow \rho \phi$, but the height of the surface must be adjusted down by $\ln \rho$ to preserve the normalisation of the probability distributions from which the surface $S_Q$ is obtained. The choice is arbitrary, but I decided to scale the $\phi^4$ surface to the SU(2) one. These transformations are understood to be acting on the arguments of the fit function $g(z, \tilde{\phi})$. Thus if $\sigma$ is 2, then the portion of the $\phi^4$ surface which did appear on a graph at $z = 3$ now appears at $z = 1.5$.

A rigorous statistical analysis of the situation is not possible because of the uncertain nature of the errors in the data and the uncertainties added by the inevitable imperfections of the fits. I adopted a reasonable method which should be semi-quantitative as to whether the surfaces do match or not. The way in which I did this was to define a figure of merit, $\mathcal{R}(\sigma, \rho)$, for the agreement of the two surfaces, and then minimize this figure. $\mathcal{R}$ was obtained as follows. First, the $(z, \phi)$ plane was divided up into rectangles which were small enough that the surface could be considered approximately linear over them. Those rectangles which (approximately) coincided with the regions where data was available to constrain the fit surface were used; the rest were not.

For each rectangle, the mean square distance of the data in that rectangle from the surface was separately computed for both models. This distance was taken as a measure of the error in the surface over that square. In so far as the discrepancies
between data and surface were due to systematic errors (either in the fit or because of slight corrections to finite size scaling between the histograms), this is a fair measure of the error. If the process were working ideally (huge lattices, many histograms with much data, perfect fit function, small rectangles) then the errors of the data from the fit surface would be entirely random. In such a case my error measure for each rectangle would be an overestimate and should be divided by the square root of the number of data in the rectangle. Based on the small lattices involved, inspection of pictures like Figure 4, and particularly the $\chi^2$ of the fits to the data, I expected the errors to be predominantly of the systematic kind, not random. The two errors for each rectangle were then combined in quadrature to give an expected error for the difference between the surfaces.

Finally, the actual mean square difference of the fit surfaces in each rectangle was computed numerically, and divided by the combined error just described. The figure of merit $R(\sigma, \rho)$ was then the root mean square of the individual figures of merit for all the rectangles. This number should be less than about one everywhere, and could be considerably less than one in the ideal case. It measures the extent to which the surfaces differ by more than the fluctuations in the data. The scalings were then obtained by minimising this function of merit. They were in good agreement with estimates obtained by eyeball scaling of the finite size scaling forms of the susceptibility to each other (about 3% different). The actual values were, for the $z$ scaling $\sigma = 2.631$, and for the $\phi$ scaling $\rho = 1.359$.

Several graphs comparing the SU(2) surface with the rescaled $\phi^4$ surface are in Figures 4-6. I believe the reader will agree with me that the similarity is very striking, particularly in the contour plots, Figure 6. As to a more quantitative understanding of the situation: a density plot of the merit figure defined above is to be found in Figure 7. The numbers on which it is based are in Table 5. The root mean square $R$ value was 0.50. Pretty much all the values for the individual squares are within the reasonable range. I think this makes it clear that the surfaces for the two models are indeed the same, within the errors.

7. Conclusion and Outlook

In this paper I have argued that by taking histograms of the instantaneous magnetization at different reduced temperatures and lattice sizes and rescaling them in the appropriate way we can produce smooth finite size scaling surfaces. I have further argued that these surfaces should be universal up to two rescalings. I have exhibited data for two different models that provide credible evidence that the histograms can indeed be scaled into surfaces. I have further shown that, within reasonable error bars, the two surfaces are the same. This conclusion can be looked at in two ways. If we believe that SU(2) and $\phi^4$ are in the same universality class, then this work can be seen as evidence that the finite size surfaces really are universal quantities. On the other hand, if we accept the renormalization group arguments for this universality, the paper provides further support for the view that the effective line theory for SU(2)
lattice gauge theory is in the $\phi^4$/Ising universality class. The mutual consistency of the views lends credence to both.

Several further lines of enquiry could be pursued. It would be of interest to know what connection could be made with perturbative expansions. Some work has been done to compute the surface of interest here. Indeed an explicit comparison of the probability form at the critical point ($z = 0$) in simulations with an $O(\epsilon)$ result has been done, but no comparison exists yet for the form as it varies with temperature.

There should be a similar surface obtained by finite size scaling histograms of measurements of the internal energy density in a simulation. These would have the average internal energy as their expectation value and their variance would be related to the specific heat. This may be of interest because of the numerically small value of the specific heat exponent $\alpha$. This is very hard to measure accurately in a simulation. It being hard to extract from the data is precisely the condition needed to be able to construct the surface without knowing $\alpha$ well. Thus it may be possible to use this to check universality in the energy variables, where it is difficult to do with an exponent.

It would be of interest to know whether the scheme works as well in other universality classes as it does here. It would be interesting to see how different the pictures look from those in Figure 3.

If it proves that the scheme does work well elsewhere then it could be quite useful in situations in which a researcher wishes to be sure of the universality class of a model but only has Monte-Carlo data to use. Besides the exponents, it was previously necessary to compare either amplitude ratios of one kind or another, or finite size scaling forms of the magnetization, susceptibility etc. Firstly, all of these quantities are contained in this finite size surface (or in other similar ones). For instance, the susceptibility amplitude ratio comes from how the ‘valley’ at one end of the surface narrows in relation to how the one at the other end of the surface narrows. Secondly, most of these quantities are not very well suited to determination from simulation data. It is hard to determine amplitudes accurately because the region in temperature between where the correlation length becomes long enough to be considered in the critical regime and where finite size effects become pronounced is very small if lattices are only a few tens of spacings on a side. It was just this problem which inspired the author to research the method presented here. Other authors have had similar problems. As to the shape functions of the magnetization or susceptibility, they have the drawback of being uninteresting in the region of the critical point. The magnetization function is close to linear, while the susceptibility function is close to quadratic. Thus, while cross-model comparison of them is not totally unimpressive, it is not as convincing as it might be. Again of course, these functions are just various moments (taken in the $\phi$ direction) of the surface described here. On the other hand, the finite size surface does have pronounced features in the region of the critical point, and therefore makes for a test of universality which is both very visual and quite convincing.

Against this it must be said that the technology for the quantitative comparison of the two surface used here is a little clumsy. The best hope of improving this situation
is to take some account of the leading corrections to scaling. The correction to scaling function and exponent are also universal, and Bruce has had considerable success in matching the critical point probability distribution of two apparently disparate models by making use of this.\footnote{12} The usual problem with invoking corrections to scaling—that they simply provide enough extra parameters that things can be made to fit no matter what—is somewhat offset by the requirement of using a single correction function for both models. Whether this would sufficiently reduce the systematics that a proper statistical treatment of the random errors could be given is open to question.

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Fig. 1. $\phi^4$ histograms for the rescaled field $\phi$, as described in the text, plotted versus $z_\phi$, the finite size scaled reduced temperature. The tendency of the data curves to all lie on a surface is clear.

Fig. 2. $\phi^4$ scaling surface, $S_Q(z_\phi, \phi)$, with associated data. The data is minus the logarithm of the histograms of Figure 1.

Fig. 3. SU(2) scaling surface, $S_Q(z_L, L)$, with associated data. Agreement is generally fairly good except for one tail where the simulation ran up to unusually high field.
Fig. 4. SU(2) scaling surface and rescaled $\phi^4$ scaling surface. The $\phi^4$ surface has been rescaled to match the SU(2) surface, so both are given as functions of $z_L$ and $L$. See the text.

Fig. 5. SU(2) scaling surface and $\phi^4$ scaling surface, a second view. The $\phi^4$ surface has been rescaled to match the SU(2) surface, so both are given as functions of $z_L$ and $L$. See the text.

Table 1. Summary of $\phi^4$ data set.

| $N$ | $r$  | $N_{run}$ | $z_{\phi}$ |
|-----|------|-----------|------------|
| 32  | 4.1500 | 41347    | -5.680     |
| 16  | 4.0000 | 49985    | -4.769     |
| 16  | 4.0500 | 99980    | -3.809     |
| 16  | 4.1000 | 49975    | -2.850     |
| 32  | 4.2000 | 49900    | -2.797     |
| 16  | 4.1250 | 99965    | -2.370     |
| 8   | 3.9000 | 49990    | -2.226     |
| 16  | 4.1500 | 99960    | -1.890     |
| 8   | 4.0000 | 49985    | -1.587     |
| 16  | 4.1750 | 99960    | -1.410     |
| 8   | 4.0500 | 49985    | -1.268     |
| 32  | 4.2300 | 57466    | -1.067     |
| 8   | 4.1000 | 49985    | -0.948     |
| 16  | 4.2000 | 49950    | -0.931     |
| 8   | 4.1500 | 49980    | -0.629     |
| 16  | 4.2250 | 119642   | -0.451     |
| 8   | 4.2000 | 49980    | -0.310     |
| 8   | 4.2500 | 49980    | 0.010      |
| 16  | 4.2500 | 99925    | 0.029      |
| 32  | 4.2500 | 49700    | 0.086      |
| 8   | 4.3000 | 49975    | 0.329      |
| 16  | 4.2750 | 99920    | 0.509      |
| 8   | 4.3500 | 49975    | 0.648      |
| 32  | 4.2600 | 49670    | 0.663      |
| 8   | 4.4000 | 49975    | 0.968      |
| 16  | 4.3000 | 49900    | 0.988      |
| 32  | 4.2700 | 49700    | 1.240      |
| 16  | 4.3250 | 99930    | 1.468      |
| 32  | 4.2800 | 49800    | 1.816      |
| 16  | 4.3500 | 99940    | 1.948      |
| 32  | 4.2900 | 49840    | 2.393      |
| $N$ | $4/g^2$ | $N_{run}$ | $z_L$ |
|-----|---------|-----------|-------|
| 26  | 2.27    | 150000   | -2.185|
| 18  | 2.27    | 100000   | -1.219|
| 26  | 2.29    | 150000   | -0.652|
| 12  | 2.27    | 200000   | -0.640|
| 18  | 2.29    | 300000   | -0.363|
| 8   | 2.27    | 200000   | -0.336|
| 12  | 2.29    | 400000   | -0.191|
| 8   | 2.29    | 200000   | -0.100|
| 8   | 2.30    | 200000   | 0.018 |
| 12  | 2.30    | 300000   | 0.034 |
| 18  | 2.30    | 450000   | 0.064 |
| 26  | 2.30    | 200000   | 0.115 |
| 8   | 2.31    | 400000   | 0.136 |
| 12  | 2.31    | 300000   | 0.258 |
| 18  | 2.31    | 300000   | 0.492 |
| 8   | 2.35    | 200000   | 0.608 |
| 26  | 2.31    | 150000   | 0.882 |

Table 2. Summary of SU(2) data set.

| $a_0$ | 4.702 | $a_0$ | 9.249 |
| $a_1$ | 4.457 | $a_1$ | 20.719 |
| $a_2$ | 0.029 | $a_2$ | -3.266 |
| $b_0$ | 49.298 | $b_0$ | 173.632 |
| $b_1$ | -6.109 | $b_1$ | -60.028 |
| $b_2$ | 1.182 | $b_2$ | -21.462 |
| $c_0$ | 426.811 | $c_0$ | 2336.416 |
| $c_1$ | 41.246 | $c_1$ | 1818.542 |
| $c_2$ | -3.947 | $c_2$ | -735.435 |
| $d_0$ | -0.378 | $d_0$ | -0.122 |
| $d_1$ | -0.938 | $d_1$ | -2.404 |
| $d_2$ | -0.278 | $d_2$ | -1.559 |
| $d_3$ | -0.046 | $d_3$ | -0.547 |
| $d_4$ | -0.003 | $d_4$ | -0.072 |

Table 3. Coefficients of $\phi^4$ surface.  
Table 4. Coefficients of SU(2) surface.

Fig. 6. Contour plots of the SU(2) scaling surface and $\phi^4$ scaling surface. The $\phi^4$ surface has been rescaled to match the SU(2) surface, so both are given as functions of $z_L$ and $L$. The contours run from 7.0 down in steps of 0.5.
| $\phi$ \ $z$ | -2.2 | -2.0 | -1.7 | -1.5 | -1.3 | -1.0 | -0.8 | -0.6 | -0.3 | -0.1 | 0.07 | 0.30 | 0.54 | 0.77 |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.00        | 1.49  | 1.65  | 0.00  | 0.84  | 0.76  | 0.42  | 0.16  | 0.33  | 0.57  | 0.42  | 0.19  | 0.19  | 0.45  | 0.70  |
| 0.09        | 0.60  | 0.46  | 0.00  | 0.42  | 0.58  | 0.25  | 0.11  | 0.36  | 0.59  | 0.36  | 0.16  | 0.18  | 0.44  | 0.61  |
| 0.18        | 0.35  | 0.68  | 0.00  | 0.66  | 0.22  | 0.09  | 0.10  | 0.27  | 0.43  | 0.37  | 0.15  | 0.20  | 0.52  | 0.71  |
| 0.27        | 0.53  | 1.09  | 0.00  | 1.21  | 0.63  | 0.29  | 0.13  | 0.15  | 0.24  | 0.28  | 0.13  | 0.24  | 0.50  | 0.69  |
| 0.36        | 0.44  | 0.82  | 0.00  | 0.85  | 0.55  | 0.24  | 0.09  | 0.09  | 0.09  | 0.13  | 0.06  | 0.18  | 0.43  | 0.75  |
| 0.44        | 0.00  | 0.00  | 0.00  | 0.55  | 0.32  | 0.09  | 0.14  | 0.41  | 0.31  | 0.22  | 0.05  | 0.15  | 0.36  | 0.72  |
| 0.53        | 0.00  | 0.00  | 0.00  | 0.00  | 0.11  | 0.16  | 0.38  | 0.58  | 0.71  | 0.59  | 0.23  | 0.06  | 0.22  | 0.53  |
| 0.62        | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.41  | 0.75  | 0.80  | 0.82  | 0.42  | 0.13  | 0.14  | 0.34  |
| 0.71        | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.69  | 0.69  | 0.85  | 0.33  | 0.17  | 0.28  | 0.26  |       |
| 0.80        | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.69  | 0.57  | 0.31  | 0.13  | 0.28  | 0.42  |       |
| 0.89        | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.24  | 0.07  | 0.24  | 0.43  |       |       |
| 0.98        | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.36  |
| 1.07        | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  |
| 1.16        | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  |

Table 5. Merit ratios used in Fig 7. $\phi/z$ values given here are for the lower left corner of the square shown in the figure (over which the merit ratio figure is an average).

Fig. 7. Figure of merit for comparison of the two surfaces, as a function of $z_L$ and $L$ (described in text - see also Table 5). Black squares represent merit figures of one or worse. Values very close to zero are represented by light colours. White squares mean that no data was available in that square to normalize the discrepancy between the two finite size surfaces.