Independent Sets in Direct Products of Vertex-transitive Graphs

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Abstract

The direct product \( G \times H \) of graphs \( G \) and \( H \) is defined by:

\[
V(G \times H) = V(G) \times V(H)
\]

and

\[
E(G \times H) = \{ [(u_1, v_1), (u_2, v_2)]: (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H) \}.
\]

In this paper, we will prove that the equality

\[
\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}
\]

holds for all vertex-transitive graphs \( G \) and \( H \), which provides an affirmative answer to a problem posed by Tardif (Discrete Math. 185 (1998) 193-200). Furthermore, the structure of all maximum independent sets of \( G \times H \) are determined.

Key words: direct product; primitivity; independence number; vertex-transitive

MSC: 05D05, 06A07

1 Introduction

Let \( G \) and \( H \) be two graphs. The direct product \( G \times H \) of \( G \) and \( H \) is defined by

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\[ V(G \times H) = V(G) \times V(H) \]

and

\[ E(G \times H) = \{ [(u_1, v_1), (u_2, v_2)] : (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H) \} \].

It is easy to see this product is commutative and associative, and the product of more than two graphs is well-defined. For a graph \( G \), the products \( G^n = G \times G \times \cdots \times G \) is called the \( n \)-th powers of \( G \).

An interesting problem is the independence number of \( G \times H \). It is clear that if \( I \) is an independent set of \( G \) or \( H \), then the preimage of \( I \) under projections is an independent set of \( G \times H \), and so \( \alpha(G \times H) \geq \max\{ \alpha(G)|H|, \alpha(H)|G| \} \). It is natural to ask whether the equality holds or not. In general, the equality does not hold for non-vertex-transitive graphs (see [13]). So Tardif [17] posed the following problem.

**Problem 1.1** (Tardif [17]) Does the equality

\[ \alpha(G \times H) = \max\{ \alpha(G)|H|, \alpha(H)|G| \} \]

hold for all vertex-transitive graphs \( G \) and \( H \)?

Furthermore, it immediately raises another interesting problem:

**Problem 1.2** When \( \alpha(G \times H) = \max\{ \alpha(G)|H|, \alpha(H)|G| \} \), is every maximum independent set of \( G \times H \) the preimage of an independent set of one factor under projections?

If the answer is yes, we then say the direct product \( G \times H \) is MIS-normal (maximum-independent-set-normal). Furthermore, the direct products \( G_1 \times G_2 \times \cdots \times G_n \) is said to be MIS-normal if every maximum independent set of it is the preimage of an independent set of one factor under projections.

About these two problems, there are some progresses have been made for some very special vertex-transitive graphs.

Let \( n, r \) and \( t \) be three integers with \( n \geq r \geq t \geq 1 \). The graph \( K(t, r, n) \) is defined by: whose vertices set is the set of all \( r \)-element subsets of \( [n] = \{1, 2, \ldots, n\} \), and \( A \) and \( B \) of which are adjacent if and only if \( |A \cap B| < t \). If \( n \geq 2r \), then \( K(1, r, n) \) is the well-known Kneser graph. The classical Erdős-
Ko-Rado Theorem [8] states that \( \alpha(K(1, r, n)) = \binom{n-1}{r-1} \) (where \( n \geq 2r \)), and Frankl [9] first investigated the independence number of the direct products of Kneser graphs. Subsequently, Ahlswede, Aydinian and Khachatrian investigated the general case [2].

**Theorem 1.3** Let \( n_i \geq r_i \geq t_i \) for \( i = 1, 2, \ldots, k \).

(i) (Frankl [9]) if \( t_1 = \cdots = t_k = 1 \) and \( \frac{n_i}{r_i} \geq \frac{1}{2} \) for \( i = 1, 2, \ldots, k \), then

\[
\alpha \left( \prod_{1 \leq i \leq k} K(1, r_i, n_i) \right) = \max \left\{ \frac{r_1}{n_1}, \frac{r_2}{n_2}, \ldots, \frac{r_k}{n_k} \right\} \prod_{1 \leq i \leq k} |K(1, r_i, n_i)|.
\]

(ii) (Ahlswede, Aydinian and Khachatrian [2])

\[
\alpha \left( \prod_{1 \leq i \leq k} K(t_i, r_i, n_i) \right) = \max \left\{ \frac{\alpha(K(t_i, r_i, n_i))}{|K(t_i, r_i, n_i)|} : 1 \leq i \leq k \right\} \prod_{1 \leq i \leq k} |K(t_i, r_i, n_i)|.
\]

The circular graph \( \text{Circ}(r, n) \) (\( n \geq 2r \)) is defined by:

\[
\text{V}(\text{Circ}(r, n)) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}
\]

and

\[
\text{E}(\text{Circ}(r, n)) = \{(i, j) : |i - j| \in \{r, r + 1, \ldots, n - r\}\}.
\]

It is well known that \( \alpha(\text{Circ}(r, n)) = r \). Mario and Juan [16] determined the independence number of the direct products of circular graphs.

**Theorem 1.4** (Mario and Juan [16]) Let \( n_i \geq 2r_i \) for \( i = 1, 2, \ldots, k \). Then

\[
\alpha \left( \prod_{1 \leq i \leq k} \text{Circ}(r_i, n_i) \right) = \max \left\{ \frac{r_1}{n_1}, \frac{r_2}{n_2}, \ldots, \frac{r_k}{n_k} \right\} \prod_{1 \leq i \leq k} n_i.
\]

For positive integers \( n \), let \( S_n \) denote the permutation group on \([n]\). Two permutations \( f \) and \( g \) are said to be intersecting if there exists an \( i \in [n] \) such that \( f(i) = g(i) \). We define a graph on \( S_n \) as that two permutations are adjacent if and only if they are not intersecting. For brevity, this graph is also denoted by \( S_n \). Deza and Frankl [7] first obtained that \( \alpha(S_n) = (n - 1)! \). Cameron and Ku [6] proved that each maximum independent set of \( S_n \) is a coset of the stabilizer of a point, to which Larose and Malvenuto [14], Wang and Zhang [18] and Godsil and Meagher [10] gave alternative proofs, respectively. Recently, Cheng and Wong [11] further investigated the independence number and the MIS-normality of the direct products of \( S_n \).
Theorem 1.5 (Cheng and Wong[11]) Let \(2 \leq n_1 = \cdots = n_p < n_{p+1} \leq \cdots, n_q, 1 \leq p \leq q\). Then

\[
\alpha(S_{n_1} \times S_{n_2} \times \cdots \times S_{n_q}) = (n_1 - 1)! \prod_{2\leq i \leq q} n_i!,
\]

and the direct products \(S_{n_1} \times S_{n_2} \times \cdots \times S_{n_q}\) is MIS-normal except for the following cases:

(i) \(n_1 = \cdots = n_p < n_{p+1} = 3 \leq n_{p+2} \leq \cdots \leq n_q\);
(ii) \(n_1 = n_2 = 3 \leq n_3 \leq \cdots \leq n_q\);
(iii) \(n_1 = n_2 = n_3 \leq n_4 \leq \cdots \leq n_q\).

In [15], Larose and Tardif investigated the relationship between projectivity and the structure of maximum independent sets in powers of some vertex-transitive graphs, and obtained the MIS-normality of the powers of Kneser graphs and circular graphs.

Theorem 1.6 (Larose and Tardif [15]) Let \(n\) and \(r\) be two positive integers. If \(n > 2r\), then both \(K^k(1, r, n)\) and \(Circ^k(r, n)\) are MIS-normal for all positive integer \(k\).

Besides the above results, Larose and Tardif [15] prove that if \(G\) is vertex-transitive, then \(\alpha(G^n) = \alpha(G)|V(G)|^{n-1}\) for all \(n > 1\). They also ask whether or not \(G^n\) is MIS-normal if \(G^2\) is MIS-normal. Recently, Ku and Mcmillan [12] gave an affirmative answer to this problem, and we solved this problem in a more general setting [20].

In this paper we shall solve both Problem 1.1 and Problem 1.2. To state our results we need to introduce some notations and notions.

For a graph \(G\), let \(I(G)\) denote the set of all maximum independent sets of \(G\). Given a subset \(A\) of \(V(G)\), we define

\[
N_G(A) = \{b \in V(G) : (a, b) \in E(G) \text{ for some } a \in A\}
\]

\[
N_G[A] = N_G(A) \cup A \text{ and } \bar{N}_G[A] = V(G) - N_G[A].
\]

If \(G\) is clear from the context, for simplicity, we will omit the index \(G\).

In [20], by the so-called “No-Homomorphism” lemma of Albertson and Collins [1] we proved the following result.
Proposition 1.7 (20) Let $G$ be a vertex-transitive graph. Then, for every independent set $A$ of $G$, $\frac{|A|}{|N_G[A]|} \leq \frac{\alpha(G)}{|V(G)|}$. Equality implies that $|S \cap N_G[A]| = |A|$ for every $S \in I(G)$, and in particular $A \subseteq S$ for some $S \in I(G)$.

An independent set $A$ in $G$ is said to be imprimitive if $|A| < \alpha(G)$ and $\frac{|A|}{|V(G)|} = \frac{\alpha(G)}{|V(G)|}$. And $G$ is called IS-imprimitive if $G$ has an imprimitive independent set. In any other cases, $G$ is called IS-primitive. From definition we see that a disconnected vertex-transitive graph $G$ is IS-imprimitive and hence an IS-primitive vertex-transitive graph $G$ is connected.

The following Theorem is the main result of this paper.

Theorem 1.8 Let $G$ and $H$ be two vertex-transitive graphs with $\frac{\alpha(G)}{|G|} \geq \frac{\alpha(H)}{|H|}$. Then

$$\alpha(G \times H) = \alpha(G)|H|,$$

and either:

(i) $G \times H$ is MIS-normal, or
(ii) $\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}$ and one of them is IS-imprimitive, or
(iii) $\frac{\alpha(G)}{|G|} > \frac{\alpha(H)}{|H|}$ and $H$ is disconnected.

We leave the proof of Theorem 1.8 to the next section, while in Section 3, we discuss the MIS-normality of the direct products of more than two vertex-transitive graphs.

2 Proof of Theorem 1.8

Let $S$ be a maximum independent set of $G \times H$. Then $|S| \geq \alpha(G)|H| \geq |G|\alpha(H)$. We now prove $\alpha(G \times H) \leq \alpha(G)|H|$. We refer [19] for details.
In the language of cross-intersecting families, Borg \cite{3,4,5} introduce a decomposition of \( X_a \) as follows.

\[
X_a^* = \{x \in X_a : N_H(x) \cap X_a = \emptyset\},
\]

\[
X_a' = \{x \in X_a : N_H(x) \cap X_a \neq \emptyset\}
\]

and

\[
X' = \bigcup_{a \in V(G)} X_a'.
\]

Clearly, \( X_a^* \) is an independent set of \( H \) for every \( a \in V(G) \), and \(|S| = \sum_{a \in V(G)} |X_a|\). Here, the empty set is regarded as an independent set.

We list all distinct \( X_a^* \)'s as \( Y_1, Y_2, \ldots, Y_k \), and define

\[
B_i = \{a \in V(G) : X_a^* = Y_i\}, \quad i = 1, 2, \ldots, k.
\]

We then obtain a partition of \( V(G) \) as \( V(G) = B_1 \cup B_2 \cup \cdots \cup B_k \). Then

\[
|S| = \sum_{a \in V(G)} |X_a| = \sum_{a \in V(G)} (|X_a^*| + |X_a'|) = \sum_{i=1}^k \sum_{a \in B_i} |X_a^*| + \sum_{a \in V(G)} |X_a'|
\]

\[
= \sum_{i=1}^k |Y_i||B_i| + \sum_{x \in X'} |A_x|,
\]

where

\[
A_x = \{a \in V(G) : x \in X_a'\}.
\]

For every pair \( a, b \in V(G) \), it is easy to verify that \((a, b) \notin E(G)\) if \( X_a' \cap X_b' \neq \emptyset\). Therefore, \( A_x \) is an independent set of \( G \). By Proposition \ref{Prop_1}, we have that

\[
|A_x| \leq \frac{\alpha(G)}{|V(G)|}|N_G[A_x]|,
\]

and equality holds if and only if \(|A_x| = 0\), or \(|A_x| = \alpha(G)\), or \( A_x \) is an imprimitive independent set of \( G \).

Suppose \( x \in N_H[Y_i] = N_H(Y_i) \cup Y_i \). If \( x \in N_H(Y_i) \), then there exists \( y \in Y_i \) such that \((x, y) \in E(H)\) and \{\((a, x), (b, y)\)\} \( \subset S \) for any \( b \in B_i \) and \( a \in A_x \), hence \((a, b) \notin E(G)\) since \( S \) is an independent set; if \( x \in Y_i \), then for each \( a \in A_x \), there is \( z \in X_a \) with \((x, z) \in E(H)\) and \{\((a, z), (b, x)\)\} \( \subset S \), yielding \((a, b) \notin E(G)\). Thus proving that \( B_i \subseteq \tilde{N}_G[A_x] \) if \( x \in N_H[Y_i] \). From this it follows that

\[
\sum_{i: x \in N_H[Y_i]} |B_i| \leq |\tilde{N}_G[A_x]| = |V(G)| - |N_G[A_x]|,
\]
i.e.,
\[ |N_G[A_x]| \leq |V(G)| - \sum_{i: x \in N_H[Y_i]} |B_i| = \sum_{i: x \in N_H[Y_i]} |B_i|. \tag{3} \]

Note that
\[ X' \subseteq \bigcup_{i=1}^{k} \bar{N}_H[Y_i]. \tag{4} \]

Together with (2), (3) and (4), we then obtain that
\[
\sum_{x \in X'} |A_x| \leq \frac{\alpha(G)}{|V(G)|} \sum_{x \in X' : i: x \in N_H[Y_i]} |B_i|
\leq \frac{\alpha(G)}{|V(G)|} \sum_{i=1}^{k} \sum_{x \in \bar{N}_H[Y_i]} |B_i| = \frac{\alpha(G)}{|V(G)|} \sum_{i=1}^{k} |B_i||\bar{N}_H[Y_i]|. \tag{5}
\]

Combining (2) and (5) gives that
\[
|S| = \sum_{i=1}^{k} |Y_i||B_i| + \sum_{x \in X'} |A_x|
\leq \sum_{i=1}^{k} |Y_i||B_i| + \frac{\alpha(G)}{|V(G)|} \sum_{i=1}^{k} |B_i||\bar{N}_H[Y_i]|
= \sum_{i=1}^{k} |B_i| \left( \frac{\alpha(G)}{|V(G)|} |H| + |Y_i| - \frac{\alpha(G)}{|V(G)|} |N_H[Y_i]| \right)
= \alpha(G)|H| + \sum_{i=1}^{k} |B_i| \left( |Y_i| - \frac{\alpha(G)}{|V(G)|} |N_H[Y_i]| \right)
\leq \alpha(G)|H|.
\]

The last inequality follows from that
\[
|Y_i| - \frac{\alpha(G)}{|G|} |N_H[Y_i]| \leq |Y_i| - \frac{\alpha(H)}{|V(H)|} |N_H[Y_i]| \leq 0, \tag{6}
\]
by Proposition 1.7.

The maximum of $|S|$ implies that $|S| = \alpha(G)|H|$, from which it follows that equalities (2), (3), (4) and (6) hold. Also, from Proposition 1.7 equality (6) means that either $Y_i = \emptyset$, or $\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|V(H)|}$ and $Y_i$ is either imprimitive or a maximum independent set of $H$ for $i = 1, 2, \ldots, k$.

We now prove that either $S$ is the preimages of projections of a maximum independent set of $G$ or $H$, or (ii) or (iii) holds. There are two cases to be considered.
Case 1: \( \frac{\alpha(G)}{|G|} > \frac{\alpha(H)}{|H|} \). Then, equality (3) means that \( Y_i = \emptyset \) for all \( i \), and so \( X' = V(H) \) by equality (4). Hence, from equality (2) it follows that \( A_x \) is a maximum independent set of \( G \) for all \( x \in V(H) \). With this assumption we have that for any \( x, y \in V(H) \) with \( (x, y) \in E(H) \), if \( A_x \neq A_y \), there must exist \( a \in A_x \) and \( b \in A_y \) with \( (a, b) \in E(G) \) since both \( A_x \) and \( A_y \) are maximum independent set, so \( [(a, x), (b, y)] \in E(G \times H) \), contradicting \( \{(a, x), (b, y)\} \subset S \). Therefore, \( A_x = A_y \) whenever \( (x, y) \in E(H) \), which implies that \( S \) is the preimage of a maximum independent set of \( G \) under projections if \( H \) is connected.

Case 2: \( \frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|} \). Then, equality (5) means that either \( |Y_i| = 0 \) or \( \alpha(H) \), or \( Y_i \) is an imprimitive independent set of \( H \) for each index \( i \). If \( Y_i \) is an imprimitive independent set of \( H \) for some \( i \), then \( H \) is IS-imprimitive. If \( |Y_i| = \alpha(H) \) for all \( i \), then \( X_a = X_a^* \) is a maximum independent set of \( H \) for all \( a \in V(G) \), and we can prove in the similar way as in Case 1 that \( S \) is the preimage of a maximum independent set of \( H \) under projections if \( G \) is connected. We now suppose that \( |Y_i| = 0 \) for some \( i \). With this assumption, then equality (4) implies \( X' = V(H) \), and then equality (3) means that either \( A_x \) is either imprimitive or a maximum independent set of \( G \) for all \( x \in V(H) \). If the former holds for some \( x \in V(H) \), we have that \( H \) is IS-imprimitive; otherwise, the latter holds for all \( x \in V(H) \), and then we can prove in the similar way as in Case 1 that \( S \) is the preimage of a maximum independent set of \( G \) under projections if \( H \) is connected.

3 Concluding Remark.

Let \( G_1, G_2, \ldots, G_n \) be \( n \) non-empty vertex-transitive graphs, and set \( G = G_1 \times G_2 \times \cdots \times G_n \). From Theorem 1.8 it immediately follows that

\[
\alpha(G) = \alpha(G_1) \prod_{2 \leq i \leq n} |G_i|.
\]

We now discuss the MIS-normality of \( G \). For convenience, we say \( G \) is MIS-normal if \( n = 1 \).

A graph \( H \) is said to be non-empty if \( E(H) \neq \emptyset \). It is well known that if \( H \) is a non-empty vertex-transitive graph, then \( \frac{\alpha(H)}{|H|} \leq \frac{1}{2} \), and equality holds if and only if \( H \) is a bipartite graph.
Without loss of generality we may assume that \( \frac{1}{2} \geq \frac{\alpha(G_1)}{|G_1|} = \cdots = \frac{\alpha(G_n)}{|G_n|} > \frac{\alpha(G_{\ell+1})}{|G_{\ell+1}|} \geq \cdots \geq \frac{\alpha(G_n)}{|G_n|} \), and write \( H_0 = G_1 \times \cdots \times G_{\ell} \) and \( H_i = H_{i-1} \times G_{\ell+1} \) for \( i = 1, \ldots, n \) subject to \( n > \ell \). Then \( G = H_{n-\ell} \) and with \( \frac{\alpha(H_{\ell+1})}{|H_{\ell+1}|} > \frac{\alpha(G_{\ell+1})}{|G_{\ell+1}|} \) for \( i \geq 1 \).

**Proposition 3.1** Suppose \( n > \ell \). Then \( G \) is MIS-normal if and only if \( H_0 \) is MIS-normal and \( G_{\ell+1}, \ldots, G_n \) are all connected.

**Proof.** Since \( \alpha(G) = \alpha(H_0) \prod_{i=\ell+1}^{n} |G_i| \), we have that if \( H_0 \) is not MIS-normal, then \( G \) is not MIS-normal. Furthermore, if \( G_i \) is not connected for some \( i \geq 1 \), writing \( G_i = G_i' \cup G_i'' \), a union of disjoint subgraphs, then, for all \( I_1, I_2 \in I(H_{i-1}) \) with \( I_1 \neq I_2 \), it is clear that \( S = (I_1 \times G_i') \cup (I_2 \times G_i'') \subseteq I(H_i) \), which is not a preimage of any independent set of one factor under projections, i.e., \( H_i \) is not MIS-normal, hence \( G \) is not MIS-normal.

Conversely, suppose \( H_0 \) is MIS-normal, and \( G_{\ell+1} \) is connected for \( i \geq 1 \). Since \( \frac{\alpha(H_{\ell+1})}{|H_{\ell+1}|} > \frac{\alpha(G_{\ell+1})}{|G_{\ell+1}|} \), Theorem 1.8 implies that each maximal-sized independent set is of the form \( S \times G_{\ell+i} \), where \( S \in I(H_{i-1}) \), which means that \( H_i \) is MIS-normal for \( i \geq 1 \). We thus prove that \( G \) is MIS-normal. \( \square \)

We now discuss the case \( n = \ell \), that is, each \( G_i \) has the identical independence ratio. To deal with this case we need a lemma as follows.

**Lemma 3.2** Suppose that \( G \) is a vertex-transitive bipartite graph. Then \( G \) is imprimitive if and only if \( G \) is disconnected.

**Proof.** It is clear that \( G \) is imprimitive if \( G \) is disconnected. On the converse, if \( G \) is imprimitive, then there is an imprimitive independent set \( A \) such that \( \frac{|A|}{|V(G)|} = \frac{\alpha(G)}{|G|} = \frac{1}{2} \). Set \( B = N_G(A) \), \( |B| = |A| \) and \( A \subseteq N_G(B) \) is clearly. If \( N_G(B) \neq A \), then we obtain that \( \sum_{u \in A} d(u) \leq \sum_{v \in B} d(v) \), which induces a contradiction. Hence \( N_G(B) = A \), that is to say \( G \) is disconnected.

**Proposition 3.3** Suppose that \( \frac{\alpha(G_1)}{|G_1|} = \cdots = \frac{\alpha(G_n)}{|G_n|} = \frac{\alpha(G)}{|G|} \). Then \( G \) is MIS-normal if and only if one of the following holds.

(i) \( \frac{\alpha(G)}{|G|} < \frac{1}{2} \) and every \( G_i \) is IS-primitive.

(ii) \( \frac{\alpha(G_1)}{|G_1|} = \frac{1}{2} \), \( n = 2 \) and both \( G_1 \) and \( G_2 \) are connected.

**Proof.** For \( 1 \leq i \leq n \), set \( \hat{G}_i = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n \). Then \( G = \)
\( \hat{G}_i \times G_i \) for \( i = 1, 2, \ldots, n \). If \( G_i \) is imprimitive, letting \( A_i \) be an imprimitive independent set of \( G_i \), for every \( I \in I(\hat{G}_i) \), it is easy to see that \( S = (\hat{G}_i \times A_i) \cup (I \times \overline{N}_{G_i}[A_i]) \in I(G) \), which is not a preimage of any independent set of \( \hat{G}_i \) or \( G_i \) under projections, therefore, \( G \) is not MIS-normal. Conversely, if both \( \hat{G}_i \) and \( G_i \) are IS-primitive, Theorem 1.8 implies that \( G \) is MIS-normal. It remains to check when \( \hat{G}_i \) is IS-primitive. Summing up the above, \( G \) is MIS-normal if and only if both \( \hat{G}_i \) and \( G_i \) are IS-primitive. To complete the proof, it remains to check when \( \hat{G}_i \) is IS-primitive. We distinguish two cases.

Case (i): \( \alpha(\hat{G}_i) < \frac{1}{2} \). In this case, Theorem 2.6 in [20] says that if \( G \) is MIS-normal, then both \( \hat{G}_i \) and \( G_i \) are IS-primitive. The induction implies (i).

Case (ii): \( \frac{\alpha(G_1)}{|G_1|} = \frac{1}{2} \), i.e., every \( G_i \) is bipartite. From Lemma 3.2 it follows that \( \hat{G}_i \) and \( G_i \) is IS-primitive if and only if both \( \hat{G}_i \) and \( G_i \) are connected. However, it is well known that \( \hat{G}_i \) is disconnected if \( n > 2 \), thus proving (ii). \( \square \)

Combining Proposition 3.1 and Proposition 3.3 gives the following theorem.

**Theorem 3.4** Let \( G_1, G_2, \ldots, G_n \) be connected vertex-transitive graphs with \( \frac{1}{2} \geq \frac{\alpha(G_1)}{|G_1|} = \cdots = \frac{\alpha(G_\ell)}{|G_\ell|} \geq \frac{\alpha(G_{\ell+1})}{|G_{\ell+1}|} \geq \cdots \geq \frac{\alpha(G_n)}{|G_n|} \), where \( n \geq 2 \) and \( 1 \leq \ell \leq n \). Then \( G_1 \times G_2 \times \cdots \times G_n \) is MIS-normal if and only if one of the following holds:

(i) \( \frac{\alpha(G_1)}{|G_1|} < \frac{1}{2} \) and \( G_1, G_2, \ldots, G_\ell \) are all IS-primitive whenever \( \ell > 1 \).

(ii) \( \frac{\alpha(G_1)}{|G_1|} = \frac{1}{2} \) and \( \ell \leq 2 \).

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**References**

[1] M.O. Albertson and K.L. Collins, Homomorphisms of 3-chromatic graphs, Discrete Math., 54 (1985) 127-132.

[2] R. Ahlswede, H. Aydinian and L.H. Khachatrian, The Intersection Theorem for Direct Products, European J. Combin., 19 (1998) 649-661.
[3] P. Borg, A short proof of a cross-intersection theorem of Hilton, Discrete Math., 309 (2009) 4750-4753.

[4] P. Borg, Cross-intersecting families of permutations, J. Combin. Theory Ser. A, 117 (2010) 483-487.

[5] P. Borg and I. Leader, Multiple cross-intersecting families of signed sets, J. Combin. Theory Ser. A, 117 (2010) 583-588.

[6] P.J. Cameron and C.Y. Ku, Intersecting families of permutations, European J. Combin., 24 (2003) 881-890.

[7] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Combin. Theory Ser. A, 22 (1977) 352-362.

[8] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser., 2 (12) (1961) 313-318.

[9] P. Frankl, An Erdős-Ko-Rado Theorem for direct products, European J. Combin., 17 (1996) 727-730.

[10] C. Godsil and K. Meagher, A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations, Eurpean J. Combin., 30 (2008) 404-414.

[11] C.Y. Ku and T.W.H. Wong, Intersecting families in the alternating group and direct product of symmetric groups, Electron. J. Combin., 14 (2007).

[12] C.Y. Ku and B.B. Mcmillan, Independent sets of maximal size in tensor powers of vertex-transitive graphs, J. Graph Theory, 60 (2009) 295-301.

[13] P.K. Jha and S. Klavžar, Independence in direct-product graphs, Ars Combin., 50 (1998) 53-60.

[14] B. Larose and C. Malvenuto, Stable sets of maximal size in Kneser-type graphs, European J. Combin., 25 (2004) 657-673.

[15] B. Larose and C. Tardif, Projectivity and independent sets in powers of graph, J. Graph Theory, 40 (2002) 162-171.

[16] V.P. Mario and V. Juan, Independence and coloring properties of direct products of some vertex-transitive graphs, Discrete Math., 306 (2006) 2275-2281.

[17] C. Tardif, Graph products and the chromatic difference sequence of vertex-transitive graphs, Discrete Math., 185 (1998) 193-200.

[18] J. Wang and S.J. Zhang, An Erdős-Ko-Rado-Type Theorem in Coxeter Groups, European J. Combin., 29 (2008) 1112-1115.

[19] J. Wang and H.J. Zhang, Cross-intersecting families and primitivity of symmetric systems, submitted.

[20] H.J. Zhang, Primitivity and independent sets in direct products of vertex-transitive graphs, J. Graph Theory, to appear.