The Lie derivative of spinor fields: theory and applications

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Abstract

Starting from the general concept of a Lie derivative of an arbitrary differentiable map, we develop a systematic theory of Lie differentiation in the framework of reductive $G$-structures $P$ on a principal bundle $Q$. It is shown that these structures admit a canonical decomposition of the pull-back vector bundle $i_P^*(TQ) \equiv P \times_Q TQ$ over $P$. For classical $G$-structures, i.e. reductive $G$-subbundles of the linear frame bundle, such a decomposition defines an infinitesimal canonical lift. This lift extends to a prolongation $\Gamma$-structure on $P$. In this general geometric framework the concept of a Lie derivative of spinor fields is reviewed. On specializing to the case of the Kosmann lift, we recover Kosmann’s original definition. We also show that in the case of a reductive $G$-structure one can introduce a “reductive Lie derivative” with respect to a certain class of generalized infinitesimal automorphisms, and, as an interesting by-product, prove a result due to Bourguignon and Gauduchon in a more general manner. Next, we give a new characterization as well as a generalization of the Killing equation, and propose a geometric reinterpretation of Penrose’s Lie derivative of “spinor fields”. Finally, we present an important application of the theory of the Lie derivative of spinor fields to the calculus of variations.

Introduction

It is perhaps surprising that, despite its paramount importance for applications, the concept of Lie differentiation of spinor fields has nonetheless eluded a general geometric formulation for a very long time.

The first correct definition of a Lie derivative of spinor fields is due Lichnerowicz [28], although with respect to infinitesimal isometries (or “Killing vector fields”) only (cf. §6). His definition was later generalized by Kosmann [26] to include all infinitesimal transformations.

Although Kosmann’s is, in many senses, the most “natural” definition of a Lie derivative of spinor fields one can possibly think of, and has been more

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or less consciously adopted by several authors [4, 39, 20, 32], it is important to stress that (i) it is an ad hoc definition and (ii) is by no means the only definition of a Lie derivative of spinor fields one can possibly give.

A geometric justification of Kosmann’s formula was first proposed by Bourguignon and Gauduchon [2] (see also [4]), but, as we already explained in [15] and shall make clear later on, although their definition coincides with Kosmann’s on spinor fields, it does not in general. In particular, it automatically preserves the metric, which makes it unsuitable in many situations of physical interest.

As for the second point, this is intimately related to the fact that spinors, unlike, e.g., tensor fields, are not sections of a natural bundle (cf. §3). Indeed, although there is always a unique definition of a Lie derivative of a “natural object” with respect to a given vector field $\xi$ on the base manifold since there is always a unique lift functorially induced by $\xi$, there is no such thing for a section of a non natural bundle. Nevertheless, in [7] it was shown that a particular canonical (not natural) lift does play a privileged role in the context of spinor fields, and a geometric explanation of Kosmann’s definition was finally given.

In [15] we showed that this type of lifts actually exists for a whole class of objects, and, to this end, introduced the concept of a reductive $G$-structure.

This paper consists of an expanded version of [15] and also features a number of relevant applications. Its structure is as follows: in §1 preliminary notions on principal bundles are recalled for the main purpose of fixing our notation; in §2 the concept of a reductive $G$-structure and its main properties are introduced; in §3 a constructive approach to gauge-natural bundles is proposed together with a number of relevant examples; in §4 split structures on principal bundles are considered and the notion of a generalized Kosmann lift is defined. In §5 the general theory of Lie derivatives is applied to the context of reductive $G$-structures, allowing us to analyse the concept of the Lie derivative of spinor fields in all its different flavours from the most general point of view. §6 is devoted to introducing the $G$-Killing condition, a generalization of the well-known Killing equation, whereas in §7 we propose a geometric reinterpretation of Penrose’s Lie derivative of “spinor fields” in the light of the general theory of Lie derivatives developed herein. Finally, in §8 we present an important application of the theory of the Lie derivative of spinor fields to the calculus of variations.

1 Principal bundles

Let $M$ be a manifold and $G$ a Lie group. A principal (fibre) bundle $P$ over $M$ with structure group $G$ is obtained by attaching a copy of $G$ to each point of $M$, i.e. by giving a $G$-manifold $P$, on which $G$ acts on the right and which satisfies the following conditions:

(i) The (right) action $r \colon P \times G \to P$ of $G$ on $P$ is free, i.e. $u \cdot a \equiv r^a u := r(u, a) = u, u \in P$, implies $a = e, e$ being the unit element of $G$.

(ii) $M = P/G$ is the quotient space of $P$ by the equivalence relation induced by $G$, i.e. $M$ is the space of orbits. Moreover, the canonical projection $\pi : P \to M$ is smooth.
(iii) $P$ is locally trivial, i.e. $P$ is locally a product $U \times G$, where $U$ is an open set in $M$. More precisely, there exists a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times G$ such that $\Phi(u) = (\pi(u), f(u))$, where the mapping $f : \pi^{-1}(U) \to G$ is $G$-equivariant, i.e. $f(u \cdot a) = f(u) \cdot a$ for all $u \in \pi^{-1}(U)$, $a \in G$.

A principal bundle will be denoted by $(P, M, \pi; G)$, $P(M, G)$, $\pi : P \to M$ or simply $P$, according to the particular context. $P$ is called the bundle (or total) space, $M$ the base, $G$ the structure group, and $\pi$ the projection. The closed submanifold $\pi^{-1}(x)$, $x \in M$, will be called the fibre over $x$. For any point $u \in P$, we have $\pi^{-1}(x) = u \cdot G$, where $\pi(u) = x$, and $u \cdot G$ will be called the fibre through $u$. Every fibre is diffeomorphic to $G$, but such a diffeomorphism depends on the chosen trivialization.

Given a manifold $M$ and a Lie group $G$, the product manifold $M \times G$ is a principal bundle over $M$ with projection $pr_1 : M \times G \to M$ and structure group $G$, the action being given by $(x, a) \cdot b = (x, a \cdot b)$. The manifold $M \times G$ is called a trivial principal bundle.

A homomorphism of a principal bundle $P'(M', G')$ into another principal bundle $P(M, G)$ consists of a differentiable mapping $\Phi : P' \to P$ and a Lie group homomorphism $f : G' \to G$ such that $\Phi(u' \cdot a') = \Phi(u') \cdot f(a')$ for all $u' \in P'$, $a' \in G'$. Hence, $\Phi$ maps fibres into fibres and induces a differentiable mapping $\varphi : M' \to M$ by $\varphi(x') = \pi(\Phi(u'))$, $u'$ being an arbitrary point over $x'$. A homomorphism $\Phi : P' \to P$ is called an embedding if $\varphi : M' \to M$ is an embedding and $f : G' \to G$ is injective. In such a case, we can identify $P'$ with $\Phi(P')$, $G'$ with $f(G')$ and $M'$ with $\varphi(M')$, and $P'$ is said to be a subbundle of $P$. If $M' = M$ and $\varphi = \text{id}_M$, $P'$ is called a reduced subbundle or a reduction of $P$, and we also say that $G$ “reduces” to the subgroup $G'$.

A homomorphism $\Phi : P' \to P$ is called an isomorphism if there exists a homomorphism of principal bundles $\Psi : P \to P'$ such that $\Psi \circ \Phi = \text{id}_{P'}$ and $\Phi \circ \Psi = \text{id}_P$.

2. **Reductive $G$-structures and their prolongations**

**Definition 2.1.** Let $H$ be a Lie group and $G$ a Lie subgroup of $H$. Denote by $\mathfrak{h}$ the Lie algebra of $H$ and by $\mathfrak{g}$ the Lie algebra of $G$. We shall say that $G$ is a reductive Lie subgroup of $H$ if there exists a direct sum decomposition

$$\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is an $\text{Ad}_G$-invariant vector subspace of $\mathfrak{h}$, i.e. $\text{Ad}_a(\mathfrak{m}) \subset \mathfrak{m}$ for all $a \in G$ (which means that the $\text{Ad}_G$ representation of $G$ in $\mathfrak{h}$ is reducible into a direct sum decomposition of two $\text{Ad}_G$-invariant vector spaces: cf. [23], p. 83).

**Remark 2.2.** A Lie algebra $\mathfrak{h}$ and a Lie subalgebra $\mathfrak{g}$ satisfying these properties form a so-called reductive pair (cf. [4], p. 103). Moreover, $\text{Ad}_G(\mathfrak{m}) \subset \mathfrak{m}$ implies $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$, and, conversely, if $G$ is connected, $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$ implies $\text{Ad}_G(\mathfrak{m}) \subset \mathfrak{m}$ [24, p. 190].

**Example 2.3.** Consider a subgroup $G \subset H$ and suppose that an $\text{Ad}_G$-invariant metric $K$ can be assigned on the Lie algebra $\mathfrak{h}$ (e.g., if $H$ is a semisimple Lie group, $K$ could be the Cartan-Killing form: indeed, this form is $\text{Ad}_H$-invariant.
and, in particular, also $\text{Ad}_G$-invariant). Set

$$m := g^\perp \equiv \{ v \in h \mid K(v, u) = 0 \ \forall u \in g \}.$$ 

Obviously, $h$ can be decomposed as the direct sum $h = g \oplus m$ and it is easy to show that, under the assumption of $\text{Ad}_G$-invariance of $K$, the vector subspace $m$ is also $\text{Ad}_G$-invariant.

**Example 2.4 (The unimodular group).** The unimodular group $\text{SL}(m, \mathbb{R})$ is an example of a reductive Lie subgroup of $\text{GL}(m, \mathbb{R})$. To see this, first recall that its Lie algebra $\mathfrak{sl}(m, \mathbb{R})$ is formed by all $m \times m$ traceless matrices. If $M$ is any matrix in $\mathfrak{gl}(m, \mathbb{R})$, the following decomposition holds:

$$M = U + \frac{1}{m} \text{tr}(M)I,$$

where $I := \text{id}_{\mathfrak{gl}(m, \mathbb{R})}$ and $U$ is traceless. Indeed,

$$\text{tr}(U) = \text{tr}(M) - \frac{1}{m} \text{tr}(M) \text{tr}(I) = 0.$$ 

Accordingly, the Lie algebra $\mathfrak{gl}(m, \mathbb{R})$ can be decomposed as follows:

$$\mathfrak{gl}(m, \mathbb{R}) = \mathfrak{sl}(m, \mathbb{R}) \oplus \mathbb{R}I.$$ 

In this case, $m$ is the set of all real multiples of $I$, which is obviously adjoint-invariant under $\text{SL}(m, \mathbb{R})$. Indeed, if $S$ is an arbitrary element of $\text{SL}(m, \mathbb{R})$, for any $a \in \mathbb{R}$ one has

$$\text{Ad}_S(al) \equiv S(al)S^{-1} = alSS^{-1} = al.$$ 

This proves that $\mathbb{R}I \cong \mathbb{R}$ is adjoint-invariant under $\text{SL}(m, \mathbb{R})$, and $\text{SL}(m, \mathbb{R})$ is a reductive Lie subgroup of $\text{GL}(m, \mathbb{R})$.

Given the importance of the following example for the future developments of the theory, we shall state it as

**Proposition 2.5.** The (pseudo-) orthogonal group $\text{SO}(p, q)$, $p + q = m$, is a reductive Lie subgroup of $\text{GL}(m, \mathbb{R})$.

**Proof.** Let $\eta$ denote the standard metric of signature $(p, q)$, with $p + q = m$, on $\mathbb{R}^m \equiv \mathbb{R}^{p,q}$ and $M$ be any matrix in $\mathfrak{gl}(m, \mathbb{R})$. Denote by $M^\top$ the adjoint (“transpose”) of $M$ with respect to $\eta$, defined by requiring $\eta(M^\top v, v') = \eta(v, Mv')$ for all $v, v' \in \mathbb{R}^m$. Of course, any traceless matrix can be (uniquely) written as the sum of an antisymmetric matrix and a symmetric traceless matrix. Therefore,

$$\mathfrak{so}(p, q) = \mathfrak{so}(p, q) \oplus V,$$

$\mathfrak{so}(p, q)$ denoting the Lie algebra of the (pseudo-) orthogonal group $\text{SO}(p, q)$ for $\eta$, formed by all matrices $A$ in $\mathfrak{gl}(m, \mathbb{R})$ such that $A^\top = -A$, and $V$ the vector space of all matrices $V$ in $\mathfrak{sl}(m, \mathbb{R})$ such that $V^\top = V$. Now, let $O$ be any element of $\text{SO}(p, q)$ and set $V' := \text{Ad}_O V \equiv O V O^{-1}$ for any $V \in V$. We have

$$V'^\top = (O V O^{-1})^\top = V'.$$
because $V^T = V$ and $O^{-1} = O^T$. Moreover,
\[ \text{tr}(V') = \text{tr}(O) \text{tr}(V) \text{tr}(O^{-1}) = 0 \]
since $V$ is traceless. So, $V'$ is in $V$, thereby proving that $V$ is adjoint-invariant under $\text{SO}(p, q)$. Therefore, $\text{SO}(p, q)$ is a reductive Lie subgroup of $\text{SL}(m, \mathbb{R})$ and, hence, also a reductive Lie subgroup of $\text{GL}(m, \mathbb{R})$ by virtue of Example 2.4.

**Definition 2.6.** A reductive $G$-structure on a principal bundle $Q(M, H)$ is a principal subbundle $P(M, G)$ of $Q(M, H)$ such that $G$ is a reductive Lie subgroup of $H$.

Now, since later on we shall consider the case of spinor fields, it is convenient to give the following general

**Definition 2.7.** Let $P(M, G)$ be a principal bundle and $\rho: \Gamma \to G$ a central homomorphism of a Lie group $\Gamma$ onto $G$, i.e. such that its kernel is discrete and contained in the centre of $\Gamma$ [18] (see also [19]). A $\Gamma$-structure on $P(M, G)$ is a principal bundle map $\zeta: \tilde{P} \to P$ which is equivariant under the right actions of the structure groups, i.e.
\[ \zeta(\tilde{u} \cdot \alpha) = \zeta(\tilde{u}) \cdot \rho(\alpha) \]
for all $\tilde{u} \in \tilde{P}$ and $\alpha \in \Gamma$.

Equivalently, we have the following commutative diagrams
\[
\begin{array}{ccc}
\tilde{P} & \xrightarrow{\zeta} & P \\
\downarrow{\pi} & & \downarrow{\pi} \\
M & \xrightarrow{} & P
\end{array}
\quad
\begin{array}{ccc}
\tilde{P} & \xrightarrow{\tilde{r}^o} & \tilde{P} \\
\downarrow{\zeta} & & \downarrow{\zeta} \\
P & \xrightarrow{r^o} & P
\end{array}
\]
$r^o$ and $\tilde{r}^o$ denoting the right action on $P$ and $\tilde{P}$, respectively (see [8]). This means that, for $\tilde{u} \in \tilde{P}$, both $\tilde{u}$ and $\zeta(\tilde{u})$ lie over the same point, and $\zeta$, restricted to any fibre, is a “copy” of $\rho$, i.e. it is equivalent to it. The existence condition for a $\Gamma$-structure on $P$ can be formulated in terms of Čech cohomology [19, 18, 27].

**Remark 2.8.** The bundle map $\zeta: \tilde{P} \to P$ is a covering space since its kernel is discrete.

Recall now that for any principal bundle $(P, M, \pi, G)$ a (principal) automorphism of $P$ is a diffeomorphism $\Phi: P \to P$ such that $\Phi(u \cdot a) = \Phi(u) \cdot a$ for every $u \in P$, $a \in G$. Each $\Phi$ induces a unique diffeomorphism $\varphi: M \to M$ such that $\pi \circ \Phi = \varphi \circ \pi$. Accordingly, we shall denote by $\text{Aut}(P)$ the group of all principal automorphisms of $P$. Assume that a vector field $\Xi$ on $P$ generates a local 1-parameter group $\{\Phi_t\}$. Then, $\Xi$ is $G$-invariant if and only if $\Phi_t$ is an automorphism of $P$ for every $t \in \mathbb{R}$. Accordingly, we denote by $X_G(P)$ the Lie algebra of $G$-invariant vector fields on $P$.

Now recall that, given a fibred manifold $\pi: B \to M$, a projectable vector field on $B$ over a vector field $\xi$ on $M$ is a vector field $\Xi$ on $B$ such that $T\pi \circ \Xi = \xi \circ \pi$. It follows
Proposition 2.9. Let $P(M, G)$ be a principal bundle. Then, every $G$-invariant vector field $\Xi$ on $P$ is projectable over a unique vector field $\xi$ on the base manifold $M$.

Proposition 2.10. Let $\zeta: \tilde{P} \rightarrow P$ be a $\Gamma$-structure on $P(M, G)$. Then, every $G$-invariant vector field $\Xi$ on $P$ admits a unique ($\Gamma$-invariant) lift $\tilde{\Xi}$ onto $\tilde{P}$.

Proof. Consider a $G$-invariant vector field $\Xi$, its flow being denoted by $\{\Phi_t\}$. For each $t \in \mathbb{R}$, $\Phi_t$ is an automorphism of $P$. Moreover, $\zeta: \tilde{P} \rightarrow P$ being a covering space, it is possible to lift $\Phi_t$ to a (unique) bundle map $\tilde{\Phi}_t: \tilde{P} \rightarrow \tilde{P}$ in the following way. For any point $\tilde{u} \in \tilde{P}$, consider the (unique) point $\zeta(\tilde{u}) = u$. From the theory of covering spaces it follows that, for the curve $\gamma_u: \mathbb{R} \rightarrow P$ based at $u$, that is $\gamma_u(0) = u$, and defined by $\gamma_u(t) := \Phi_t(u)$, there exists a unique curve $\tilde{\gamma}_u: \mathbb{R} \rightarrow \tilde{P}$ based at $\tilde{u}$ such that $\zeta \circ \tilde{\gamma}_u = \gamma_u$. It is possible to define a principal bundle map $\tilde{\Phi}_t: \tilde{P} \rightarrow \tilde{P}$ covering $\Phi_t$ by setting $\tilde{\Phi}_t(\tilde{u}) := \tilde{\gamma}_u(t)$. The 1-parameter group of automorphisms $\{\tilde{\Phi}_t\}$ of $\tilde{P}$ defines a vector field $\tilde{\Xi}(\tilde{u}) := \frac{\partial}{\partial t}[\tilde{\Phi}_t(\tilde{u})]|_{t=0}$ for all $\tilde{u} \in \tilde{P}$.

Proposition 2.11. Let $\zeta: \tilde{P} \rightarrow P$ be a $\Gamma$-structure on $P(M, G)$. Then, every $G$-invariant vector field $\Xi$ on $P$ is projectable over a unique $G$-invariant vector field $\tilde{\Xi}$ on $\tilde{P}$.

Proof. Consider a $\Gamma$-invariant vector field $\tilde{\Xi}$ on $\tilde{P}$. Denote its flow by $\{\Phi_t\}$. Each $\Phi_t$ induces a unique automorphism $\Phi_t: P \rightarrow P$ such that $\zeta \circ \Phi_t = \Phi_t \circ \zeta$ and, hence, a unique vector field $\Xi$ on $P$ given by $\Xi(u) := \frac{\partial}{\partial t}[\Phi_t(u)]|_{t=0}$ for all $u \in P$.

Corollary 2.12. Let $\zeta: \tilde{P} \rightarrow P$ be a $\Gamma$-structure on $P(M, G)$. There is a bijection between $G$-invariant vector fields on $P$ and $\Gamma$-invariant vector fields on $\tilde{P}$.

3 Gauge-natural bundles

In this section we shall introduce the category of gauge-natural bundles [6, 25] and give a number of relevant examples. Geometrically, gauge-natural bundles possess a very rich structure, which generalizes the classical one of natural bundles. From the physical point of view, this framework enables one to treat at the same time, under a unifying formalism, natural field theories such as general relativity, gauge theories, as well as bosonic and fermionic matter field theories (cf. [8, 10, 16, 29]).

The material presented in this section is standard: for further detail see, e.g., [25, §15 and Chapter XII] or [11, §5.1–2].

Definition 3.1. Let $j^k_\ell f$ denote the $\ell$-th order jet prolongation of a map $f$ evaluated at a point $p$ (cf., e.g., [25]). The set

$$\{ j^k_\ell \alpha \mid \alpha: \mathbb{R}^m \rightarrow \mathbb{R}^m, \alpha(0) = 0, \text{ locally invertible} \}$$

equipped with the jet composition $j^k_\ell \alpha \circ j^k_\ell \alpha' := j^k_\ell (\alpha \circ \alpha')$ is a Lie group called the $k$-th differential group and denoted by $G^k_m$.

For $k = 1$ we have, of course, the identification $G^1_m \cong \text{GL}(m, \mathbb{R})$. 
Definition 3.2. Let $M$ be an $m$-dimensional manifold. The principal bundle over $M$ with group $G^k_m$ is called the $k$-th order frame bundle over $M$ and will be denoted by $L^kM$.

For $k = 1$ we have, of course, the identification $L^1M \cong LM$, where $LM$ is the usual (principal) bundle of linear frames over $M$ (cf., e.g., [23]).

Definition 3.3. Let $G$ be a Lie group. Then, the space of $(m,h)$-velocities of $G$ is defined as

$$T^h_m G := \{ j^h_0 a \mid a : \mathbb{R}^m \to G \}.$$ 

Thus, $T^h_m G$ denotes the set of $h$-jets with source at the origin $0 \in \mathbb{R}^m$ and target in $G$. It is a subset of the manifold $J^h(\mathbb{R}^m, G)$ of $r$-jets with source in $\mathbb{R}^m$ and target in $G$. The set $J^h(\mathbb{R}^m, G)$ is a fibre bundle over $\mathbb{R}^m$ with respect to the canonical jet prolongation of $J^h(\mathbb{R}^m, G)$ on $\mathbb{R}^m$, and $T^h_m G$ is its fibre over $0 \in \mathbb{R}^m$. Moreover, the set $T^h_m G$ can be given the structure of a Lie group. Indeed, let $S, T \in T^h_m G$ be any elements. We define a (smooth) multiplication in $T^h_m G$ by:

\[
\begin{align*}
T^h_m \mu : T^h_m G \times T^h_m G &\to T^h_m G \\
T^h_m \mu : (S = j^h_0 a, T = j^h_0 b) &\mapsto S \cdot T := j^h_0 (a \cdot b)
\end{align*}
\]

where $(a \cdot b)(x) := a(x) \cdot b(x) \equiv \mu(a(x), b(x))$ is the group multiplication in $G$. The mapping $(S, T) \mapsto S \cdot T$ is associative; moreover, the element $j^h_0 e$, $e$ denoting both the unit element in $G$ and the constant mapping from $\mathbb{R}^m$ to $e$, is the unit element of $T^h_m G$, and $j^h_0 a^{-1}$, where $a^{-1}(x) := (a(x))^{-1}$ (the inversion being taken in the group $G$), is the inverse of $j^h_0 a$.

Definition 3.4. Consider a principal bundle $P(M, G)$. Let $k$ and $h$ be two natural numbers such that $k \geq h$. Then, by the $(k, h)$-principal prolongation of $P$ we shall mean the bundle

$$W^{k,h}P := L^kM \times_M J^hP,$$

where $L^kM$ is the $k$-th order frame bundle of $M$ and $J^hP$ denotes the $h$-th order jet prolongation of $P$. A point of $W^{k,h}P$ is of the form $(j^k_0 \epsilon, j^h_0 \sigma)$, where $\epsilon : \mathbb{R}^m \to M$ is locally invertible and such that $\epsilon(0) = x$, and $\sigma : M \to P$ is a local section around the point $x \in M$.

Unlike $J^hP$, $W^{k,h}P$ is a principal bundle over $M$ whose structure group is $W^k_m G := G^k_m \times T^h_m G$.

$W^k_m G$ is called the $(m;k,h)$-principal prolongation of $G$. The group multiplication on $W^k_m G$ is defined by the following rule:

$$(j^k_0 \alpha, j^h_0 a) \odot (j^k_0 \beta, j^h_0 b) := \left( j^k_0 (\alpha \circ \beta), j^h_0 ((a \circ \beta) \cdot b) \right),$$

denoting the group multiplication in $G$. The right action of $W^k_m G$ on $W^{k,h}P$ is then defined by:

$$(j^k_0 \epsilon, j^h_0 \sigma) \odot (j^k_0 \alpha, j^h_0 a) := \left( j^k_0 (\epsilon \circ \alpha), j^h_0 (\sigma \cdot (a \circ \alpha^{-1} \circ \epsilon^{-1})) \right),$$

denoting now the canonical right action of $G$ on $P$. 
Definition 3.5. Let $\Phi: P \to P$ be an automorphism over a diffeomorphism $\varphi: M \to M$. We define an automorphism of $W^{k,h}P$ associated with $\Phi$ by

$$
\begin{aligned}
W^{k,h}\Phi & : W^{k,h}P \to W^{k,h}P \\
W^{k,h}\Phi & : (j^k_0, j^h_x \sigma) \mapsto (j^k_0(\varphi \circ \epsilon), j^h_x(\Phi \circ \sigma \circ \varphi^{-1}))
\end{aligned}
$$

(3.2)

Proposition 3.6. The bundle morphism $W^{k,h}\Phi$ preserves the right action, thereby being a principal automorphism.

By virtue of (3.1) and (3.2) $W^{k,h}$ turns out to be a functor from the category of principal $G$-bundles over $m$-dimensional manifolds and local isomorphisms to the category of principal $W^{k,h}_m G$-bundles [25]. Now, let $P_\lambda := W^{k,h}P \times_\lambda F$ be a fibre bundle associated with $P(M,G)$ via an action $\lambda$ of $W^{k,h}_m G$ on a manifold $F$. There exists canonical representation of the automorphisms of $P$ induced by (3.2). Indeed, if $\Phi: P \to P$ is an automorphism over a diffeomorphism $\varphi: M \to M$, then we can define the corresponding induced automorphism $\Phi_\lambda$ as

$$
\begin{aligned}
\Phi_\lambda & : P_\lambda \to P_\lambda \\
\Phi_\lambda & : [u, f]_\lambda \mapsto [W^{k,h}\Phi(u), f]_\lambda
\end{aligned}
$$

(3.3)

which is well-defined since it turns out to be independent of the representative $(u, f)$, $u \in P$, $f \in F$. This construction yields a functor $\cdot_\lambda$ from the category of principal $G$-bundles to the category of fibred manifolds and fibre-respecting mappings.

Definition 3.7. A gauge-natural bundle of order $(k, h)$ over $M$ associated with $P(M,G)$ is any such functor.

If we now restrict attention to the case $G = \{e\}$ and $h = 0$, we can recover the classical notion of natural bundles over $M$. In particular, we have the following

Definition 3.8. Let $\varphi: M \to M$ be a diffeomorphism. We define an automorphism of $L^kM$ associated with $\varphi$, called its natural lift, by

$$
\begin{aligned}
L^k\varphi & : L^kM \to L^kM \\
L^k\varphi & : j^k_0 \epsilon \mapsto j^k_0(\varphi \circ \epsilon)
\end{aligned}
$$

Then, $L^k$ turns out to be a functor from the category of $m$-dimensional manifolds and local diffeomorphisms to the category of principal $G^k_m$-bundles. Now, given any fibre bundle associated with $L^kM$ and any diffeomorphism on $M$, we can define a corresponding induced automorphism in the usual fashion. This construction yields a functor from the category of $m$-dimensional manifolds to the category of fibred manifolds.

Definition 3.9. A natural bundle of order $k$ over $M$ is any such functor.

We shall now give some important examples of (gauge-) natural bundles.

Example 3.10 (Bundle of tensor densities). A first fundamental example of a natural bundle is given, of course, by the bundle $w^r T^s M$ of tensor densities of weight $w$ over an $m$-dimensional manifold $M$. Indeed, $w^r T^s M$ is a vector
bundle associated with $L^1M$ via the following left action of $G^1_m \cong W^{1,0}_m \{e\}$ on the vector space $T^*_s(\mathbb{R}^m)$:

$$
\begin{align*}
\lambda: G^1_m \times T^*_s(\mathbb{R}^m) &\to T^*_s(\mathbb{R}^m) \\
\lambda: (\alpha^i_k, a^B_i, w^A) &\mapsto (\alpha^i_k, a^B_i, w^A) \circ (\det \alpha)^{-w} 
\end{align*}
$$

the tilde over a symbol denoting matrix inversion. For $w = 0$ we recover the bundle of tensor fields over $M$. This is a definition of $^{w}T^*_sM$ which is appropriate for physical applications, where one usually considers only those (active) transformations of tensor fields that are naturally induced by some transformations on the base manifold. Somewhat more unconventionally, though, we can regard $^{w}T^*_sM$ as a gauge-natural vector bundle associated with $W^{0,0}(LM)$. Of course, the two bundles under consideration are the same as objects, but their morphisms are different.

**Example 3.11 (Bundle of principal connections).** Let $P(M, G)$ be a principal bundle, and $(A_a)^A_B$ the coordinate expression of the adjoint representation of $G$. Set $A := (\mathbb{R}^m)^* \otimes \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of $G$, and consider the action

$$
\begin{align*}
\ell: W^{1,1}_m \times \mathcal{A} &\to \mathcal{A} \\
\ell: ((\alpha^j_k, a^B_j, w^A), A_a) &\mapsto (A_a)^A_B (w^B_j - a^B_j) \tilde{\alpha}^j_k, 
\end{align*}
$$

where $(a^A, a^B)$ denote natural coordinates on $T^*_sG$: a generic element $r_0 f \in T^*_sG$ is represented by $a = f(0) \in G$, i.e. $a^A = f^A(0)$, and $a^B_i = (\partial_i (a^{-1} \cdot f(x)))_{x=0}^B$. It is immediate to realize that the sections of $W^{1,1}_m \times \mathcal{A}$ are in 1-1 correspondence with the principal connections on $P$. A section of $W^{1,1}_m \times \mathcal{A}$ will be called a $G$-connection. Clearly, $W^{1,1}_m \times \mathcal{A}$ is a gauge-natural affine bundle of order $(1, 1)$.

**Example 3.12 (Bundle of $G$-invariant vector fields).** Let $\mathcal{V} := \mathbb{R}^m \oplus \mathfrak{g}$, $\mathfrak{g}$ denoting the Lie algebra of $G$, and consider the following action:

$$
\begin{align*}
\lambda: W^{1,1}_m \times \mathcal{V} &\to \mathcal{V} \\
\lambda: ((\alpha^j_k, a^B_j, w^A), (v^i, w^A)) &\mapsto (\alpha^i_j v^j, (A_a)^A_B (w^B_j + a^B_j v^j)) \cdot 
\end{align*}
$$

Obviously, $W^{1,1}_m \times \mathcal{V} \cong TP/G$, its sections thus representing $G$-invariant vector fields on $P$.

**Example 3.13 (Bundle of vertical $G$-invariant vector fields).** Take $\mathfrak{g}$ as the standard fibre and consider the following action:

$$
\begin{align*}
\lambda: W^{1,1}_m \times \mathfrak{g} &\to \mathfrak{g} \\
\lambda: ((\alpha^j_k, a^B_j, w^A), (v^i, w^A)) &\mapsto (A_a)^A_B (w^B_j + a^B_j v^j) \cdot 
\end{align*}
$$

It is easy to realize that $W^{1,1}_m \times \mathfrak{g} \cong VP/G \cong (P \times \mathfrak{g})/G$, the bundle of vertical $G$-invariant vector fields on $P$. Of course, in this example, we see that $\mathfrak{g}$ is already a $G$-manifold and so $(P \times \mathfrak{g})/G$ is a gauge-natural bundle of order $(0, 0)$, i.e. a (vector) bundle associated with $W^{0,0}_m \cong P$. In other words, giving action (3.6) amounts to regarding the original $G$-manifold $\mathfrak{g}$ as a $W^{1,1}_m G$-manifold via the canonical projection of Lie groups $W^{1,1}_m G \to G$. It is also meaningful to think of action (3.6) as setting $v^i = 0$ in (3.5), and hence one sees that the first jet contribution, i.e. $a^A$, disappears.
4 Split structures on principal bundles

It is known that, given a principal bundle \( P(M,G) \), a \textit{principal connection} on \( P \) may be viewed as a fibre \( G \)-equivariant projection \( \Phi: TP \to VP \), i.e. as a 1-form in \( \Omega^1(P,TP) \) such that \( \Phi \circ \Phi = \Phi \) and \( \text{im} \Phi = VP \). Here, “\( G \)-equivariant” means that \( Tr^a \circ \Phi = \Phi \circ Tr^a \) for all \( a \in G \).

Then, \( HP := \ker \Phi \) is a constant-rank vector subbundle of \( TP \), called the \textit{horizontal bundle}. We have a decomposition \( TP = HP \oplus VP \) and \( T_uP = H_uP \oplus V_uP \) for all \( u \in P \). The projection \( \Phi \) is called the \textit{vertical projection} and the projection \( \chi := \text{id}_{TP} - \Phi \), which is also \( G \)-equivariant and satisfies \( \chi \circ \chi = \chi \) and \( \text{im} \chi = \ker \Phi \), is called the \textit{horizontal projection}.

This is, of course, a well-known example of a “split structure” on a principal bundle. We shall now give the following general definition, due—for pseudo-Riemannian manifolds—to a number of authors [37, 38, 3, 17, 12] and more generally to Gladush and Konoplya [14].

**Definition 4.1** ([15]). An \( r \)-\textit{split structure} on a principal bundle \( P(M,G) \) is a system of \( r \) fibre \( G \)-equivariant linear operators \( \{ \Phi^i \in \Omega^1(P,TP) \} \), \( i = 1,2,\ldots,r \), of constant rank with the properties:

\[
\Phi^i \cdot \Phi^j = \delta^{ij} \Phi^j, \quad \sum_{i=1}^r \Phi^i = \text{id}_{TP}. \tag{4.1}
\]

We introduce the notations:

\[
\Sigma^i_u := \text{im} \Phi^i_u, \quad n_i := \dim \Sigma^i_u, \tag{4.2}
\]

where \( \text{im} \Phi^i_u \) is the image of the operator \( \Phi^i_u \) at a point \( u \) of \( P \), i.e. \( \Sigma^i_u = \{ v \in T_uP \mid \Phi^i_u \circ v = v \} \). Owing to the constancy of the rank of the operators \( \{ \Phi^i \} \), the numbers \( \{ n_i \} \) do not depend on the point \( u \) of \( P \). It follows from the very definition of an \( r \)-split structure that we have a \( G \)-equivariant decomposition of the tangent space:

\[
T_uP = \bigoplus_{i=1}^r \Sigma^i_u, \quad \dim T_uP = \sum_{i=1}^r n_i.
\]

Obviously, the bundle \( TP \) is also decomposed into \( r \) vector subbundles \( \{ \Sigma^i \} \) so that

\[
TP = \bigoplus_{i=1}^r \Sigma^i, \quad \Sigma^i = \bigcup_{u \in P} \Sigma^i_u. \tag{4.3}
\]

**Remark 4.2.** In general, the \( r \) vector subbundles \( \{ \Sigma^i \to P \} \) are \textit{anholonomic}, i.e. non-integrable, and are not vector subbundles of \( VP \). For a principal connection, i.e. for the case \( TP = HP \oplus VP \), the subbundle \( VP \) is integrable.

**Proposition 4.3.** An \textit{equivariant decomposition} of \( TP \) into \( r \) vector subbundles \( \{ \Sigma^i \} \) as given by (4.3), with \( T_u \sigma^a(\Sigma^i_u) = \Sigma^i_{u \cdot a} \), induces a system of \( r \) fibre \( G \)-equivariant linear operators \( \{ \Phi^i \in \Omega^1(P,TP) \} \) of constant rank satisfying properties (4.1) and (4.2).

**Proposition 4.4.** Given an \( r \)-split structure on a principal bundle \( P(M,G) \), every \( G \)-invariant vector field \( \Xi \) on \( P \) splits into \( r \) invariant vector fields \( \{ \Xi^i \} \) such that \( \Xi = \Xi_1 \oplus \cdots \oplus \Xi_r \) and \( \Xi^i(u) \in \Sigma^i_u \) for all \( u \in P \) and \( i = 1,2,\ldots,r \).
Remark 4.5. The vector fields \( \{ \Xi_i \} \) are compatible with the \( \{ \Sigma^i \} \), i.e. they are sections \( \{ \Xi_i : P \to \Sigma^i \} \) of the vector bundles \( \{ \Sigma^i \to P \} \).

Corollary 4.6. Let \( P(M,G) \) be a reductive \( G \)-structure on a principal bundle \( Q(M,H) \) and let \( i_P : P \to Q \) be the canonical embedding. Then, any given \( r \)-split structure on \( Q(M,H) \) induces an \( r \)-split structure restricted to \( P(M,G) \), i.e. an equivariant decomposition of \( i_P^* (TQ) \equiv P \times_Q TQ = \{ (u,v) \in P \times TQ \mid ip(u) = \tau_Q (v) \} \) such that \( i_P^* (TQ) = i_P^* (\Sigma^1) \oplus \cdots \oplus i_P^* (\Sigma^r) \), and any \( H \)-invariant vector field \( \Xi \) on \( Q \) restricted to \( P \) splits into \( r \) \( G \)-invariant sections of the pull-back bundles \( \{ i_P^* (\Sigma^i) \equiv P \times_Q \Sigma^i \} \), i.e. \( \Xi = \Xi_1 \oplus \cdots \oplus \Xi_r \) with \( \Xi_i (u) \in \left( i_P^* (\Sigma^i) \right)_u \) for all \( u \in P \) and \( i \in \{ 1, 2, \ldots , r \} \).

Remark 4.7. Note that the pull-back \( i_P^* \) is a natural operation, i.e. it respects the splitting \( i_P^* (TQ) = i_P^* (\Sigma^1) \oplus \cdots \oplus i_P^* (\Sigma^r) \). In other words, the pull-back of a splitting for \( Q \) is a splitting of the pull-backs for \( P \). Furthermore, although the vector fields \( \{ \Xi_i \} \) are \( G \)-invariant sections of their respective pull-back bundles, they are \( H \)-invariant if regarded as vector fields on the corresponding subsets of \( Q \).

In §3 we saw that \( W^{k,h} P \) is a principal bundle over \( M \). Consider in particular \( W^{1,1} P \), the \((1,1)\)-principal prolongation of \( P \). The fibred manifold \( W^{1,1} P \to M \) coincides with the fibred product \( W^{1,1} P := L^1 M \times_M J^3 P \) over \( M \). We have two canonical principal bundle morphisms \( \text{pr}_1 : W^{1,1} P \to L^1 M \) and \( \text{pr}_2 : W^{1,1} P \to P \). In particular, \( \text{pr}_2 : W^{1,1} P \to P \) is a \( G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m \)-principal bundle, \( G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m \) being the kernel of \( W^{1,1} G \to G \). Indeed, recall from group theory that, if \( f : G \to G' \) is a group epimorphism, then \( G' \cong G/\ker f \). Therefore, if \( \pi_{0,0}^1 \) denotes the canonical projection from \( W^{1,1} G \) to \( G \), then \( G \cong W^{1,1} G/\ker \pi_{0,0}^1 \).

Hence, recalling the definition of a principal bundle (§1), we have:

\[
W^{1,1} P/W^{1,1} G = M = P/G = P/(W^{1,1} G/\ker \pi_{0,0}^1),
\]

from which we deduce that \( \text{pr}_2 : W^{1,1} P \to P \) is a \( (\ker \pi_{0,0}^1) \)-principal bundle. It remains to show that \( \ker \pi_{0,0}^1 \cong G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m \), but this is obvious if we consider that \( \ker \pi_{0,0}^1 \) is coordinatized by \((\alpha^k, e^B, a^c)\) (cf., e.g., Example 3.11).

The following lemma recognizes \( \tau_P : TP \to P \) as a vector bundle associated with the principal bundle \( W^{1,1} P \to P \).

Lemma 4.8. The vector bundle \( \tau_P : TP \to P \) is isomorphic to the vector bundle \( T^{1,1} P := (W^{1,1} P \times V)/(G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m) \) over \( P \), where \( V := \mathbb{R}^m \oplus \mathfrak{g} \) is the left \( G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m \)-manifold with action given by:

\[
\begin{align*}
\tau : & G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m \times V \to V \\
\tau : & (\alpha^k, e^B, a^c), (v^i, w^A) \mapsto (\alpha^j v^j, w^A + a^A v^A).
\end{align*}
\]  

(4.4)

Proof. It is easy to show that the tangent bundle \( TG \) of a Lie group \( G \) is again a Lie group, and, if \( P(M,G) \) is a principal bundle, so is \( TP(TM,TG) \) (cf., e.g., [25, §10]). Now, the canonical right action \( r \) on \( P \) induces a canonical right action on \( TP \) simply given by \( Tr \). It is then easy to realize that the space of orbits \( TP/G \), regarded as vector bundle over \( M \), is canonically isomorphic to the bundle of \( G \)-invariant vector fields on \( P \). Hence, taking Example 3.12 into account, we have:

\[
TP/(W^{1,1} G/G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m) \cong TP/G \cong (W^{1,1} P \times V)/W^{1,1} G,
\]
from which it follows that $\tau_P : TP \to P$ is a [gauge-natural vector] bundle [of order $(0,0)$] associated with $pr_2 : W^{1,1}P \to P$. Action (4.4) is nothing but action (3.5) restricted to $G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m$.

\[\text{Lemma 4.9.} \quad \text{VP} \to P \text{ is a trivial vector bundle associated with } W^{1,1}P \to P.\]

\[\text{Proof.} \quad \text{We already know that } \text{VP} \to P \text{ is a trivial vector bundle. To see that it is associated with } W^{1,1}P \to P, \text{ we follow the same argument as before, this time taking into account Example 3.13. We then have:} \]

\[\text{VP}/(W_m^{1,1}G/G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m) \cong VP/G \cong (W^{1,1}P \times \mathfrak{g})/W_m^{1,1}G,\]

whence the result follows. \[\square\]

\[\text{Lemma 4.10.} \quad \text{Let } P(M,G) \text{ be a reductive } G \text{-structure on a principal bundle } Q(M,H) \text{ and } i_P : P \to Q \text{ the canonical embedding. Then, } i_P^*(TQ) = P \times Q TQ \text{ is a vector bundle over } P \text{ associated with } W^{1,1}P \to P.\]

\[\text{Proof.} \quad \text{It follows immediately from Lemma 4.8 once one realizes that } i_P^*(TQ) \text{ is by definition a vector bundle over } P \text{ with fibre } \mathbb{R}^m \oplus \mathfrak{h} \text{ and the same structure group as } TP \to P \text{ (see also Figure 1 below).} \quad \square\]

From the above lemmas it follows that another important example of a split structure on a principal bundle is given by the following.

\[\text{Theorem 4.11.} \quad \text{Let } P(M,G) \text{ be a reductive } G \text{-structure on a principal bundle } Q(M,H) \text{ and let } i_P : P \to Q \text{ be the canonical embedding. Then, there exists a canonical decomposition of } i_P^*(TQ) \to P \text{ such that} \]

\[i_P^*(TQ) = TP \oplus \mathcal{M}(P),\]

\[\text{i.e. at each } u \in P \text{ one has} \]

\[T_u Q = T_u P \oplus \mathcal{M}_u,\]

\[\mathcal{M}_u \text{ being the fibre over } u \text{ of the subbundle } \mathcal{M}(P) \to P \text{ of } i_P^*(VQ) \to P. \text{ The bundle } \mathcal{M}(P) \text{ is defined as } \mathcal{M}(P) := (W^{1,1}P \times \mathfrak{m})/(G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m), \text{ where } \mathfrak{m} \text{ is the (trivial left) } G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m \text{-manifold.}\]

\[\text{Proof.} \quad \text{From Lemma 4.10 and the fact that } G \text{ is a reductive Lie subgroup of } H \text{ (Definition 2.1) it follows that} \]

\[i_P^*(TQ) \cong (W^{1,1}P \times (\mathbb{R}^m \oplus \mathfrak{h}))/(G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m) \]

\[= (W^{1,1}P \times (\mathbb{R}^m \oplus \mathfrak{g}))/((G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m) \oplus_p (W^{1,1}P \times \mathfrak{m})/(G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m)) \]

\[\cong TP \oplus_p \mathcal{M}(P).\]

The trivial $G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m$-manifold $\mathfrak{m}$ corresponds to action (3.6) of Example 3.13 with $W_m^{1,1}G$ restricted to $G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m$, and $\mathfrak{g}$ restricted to $\mathfrak{m}$. Of course, since the group $G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m$ acts trivially on $\mathfrak{m}$, it follows that $\mathcal{M}(P)$ is trivial, i.e. isomorphic to $P \times \mathfrak{m}$, because $W^{1,1}P/(G_m^1 \times \mathfrak{g} \otimes \mathbb{R}^m) \cong P$. \[\square\]

From the above theorem two corollaries follow, which are of prime importance for the concepts of a Lie derivative we shall introduce in the next section.
**Corollary 4.12.** Let $P(M,G)$ and $Q(M,H)$ be as in the previous theorem. The restriction $\Xi|_P$ of an $H$-invariant vector field $\Xi$ on $Q$ to $P$ splits into a $G$-invariant vector field $\Xi_K$ on $P$, called the generalized Kosmann vector field associated with $\Xi$, and a “transverse” vector field $\Xi_G$, called the generalized von Göden vector field associated with $\Xi$.

The situation is schematically depicted in Figure 1 [Q is represented as a straight line, and $P$ as the half-line stretching to the mark; $TQ$ is represented as a parallelepiped over $Q$, $i_P^*(TQ)$ as the part of it corresponding to $P$, whereas $TP$ is the face of $i_P^*(TQ)$ facing the reader].

**Corollary 4.13.** Let $P(M,G)$ be a classical $G$-structure, i.e. a reductive $G$-structure on the bundle $LM$ of linear frames over $M$. The restriction $L_\xi|_P$ to $P \to M$ of the natural lift $L_\xi$ onto $LM$ of a vector field $\xi$ on $M$ splits into a $G$-invariant vector field on $P$ called the generalized Kosmann lift of $\xi$ and denoted simply by $\xi_K$, and a “transverse” vector field called the von Göden lift of $\xi$ and denoted by $\xi_G$.

**Remark 4.14.** The last corollary still holds if, instead of $LM$, one considers the $k$-th order frame bundle $L^kM$ and hence a classical $G$-structure of order $k$, i.e. a reductive $G$-subbundle $P$ of $L^kM$. Note also that the Kosmann lift $\xi \mapsto \xi_K$ is not a Lie algebra homomorphism, although $\xi_K$ is a $G$-invariant vector field and projects on $\xi$. See [30, §2.5] for further detail.

**Example 4.15 (Kosmann lift).** A fundamental example of a $G$-structure on a manifold $M$ is given, of course, by the bundle $SO(M,g)$ of its (pseudo-) orthonormal frames with respect to a metric $g$ of signature $(p,q)$, where $p+q = m \equiv \dim M$. $SO(M,g)$ is a principal bundle (over $M$) with structure group $G = SO(p,q)^c$. Now, recall that the natural lift of a vector field $\xi$ onto $LM$ is defined as

$$L_\xi := \left. \frac{\partial}{\partial t} L_1 \varphi_t \right|_{t=0},$$
{\varphi_i}$ denoting the flow of $\xi$. If $(\rho_a^b)$ denotes a (local) basis of right $GL(m, \mathbb{R})$-invariant vector fields on $LM$ reading $(\rho_a^b = u_a^\nu \partial / \partial u_c^a)$ in some local chart $(x^\mu, u_a^b)$ and $(e_a =: e_a^\mu \partial / \partial \mu)$ is a local section of $LM$, then $L\xi$ has the local expression

$$L\xi_a = \xi^a e_a + (L\xi)^a_b \rho_a^b,$$

where $\xi =: \xi^a e_a$ and

$$(L\xi)^a_b := \xi^\rho (\partial_a \xi^\sigma e_b^\nu - \xi^\nu \partial_b e_a^\rho).$$

If we now let $(e_a)$ and $(x^\mu, u_a^b)$ denote a local section and a local chart of $SO(M, g)$, respectively, then the generalized Kosmann lift $\xi_K$ on $SO(M, g)$ of a vector field $\xi$ on $M$, simply called its Kosmann lift [7], locally reads

$$\xi_K = \xi^a e_a + (L\xi)^a_b A^{ab},$$

where $(A^{ab})$ is a basis of right $SO(p, q)^c$-invariant vector fields on $SO(M, g)$ locally reading $(A^{ab} = \eta^{[a} b^{c]} d^c A^d)$, $(L\xi)^a_b := \eta_{ac}(L\xi)^c_b$, and $(\eta_{ac})$ denote the components of the standard “Minkowski” metric of signature $(p, q)$.

Now, combining Proposition 2.10 and Theorem 4.11 yields the following result, which, in particular, will enable us to extend the concept of a Kosmann lift to the important context of spinor fields.

**Corollary 4.16.** Let $\zeta : \tilde{P} \to P$ be a $\Gamma$-structure over a classical $G$-structure $P(M, G)$. Then, the generalized Kosmann lift $\xi_K$ of a vector field $\xi$ on $M$ lifts to a unique ($\Gamma$-invariant) vector field $\tilde{\xi}_K$ on $\tilde{P}$, which projects on $\xi_K$.

## 5 Lie derivatives on reductive $G$-structures

The general theory of Lie derivatives stems from Trautman’s seminal paper [36]. Here, we mainly follow the notation and conventions of [25, §47].

**Definition 5.1.** Let $M$ and $N$ be two manifolds and $f : M \to N$ a map between them. By a vector field along $f$ we shall mean a map $Z : M \to TN$ such that $\tau_N \circ Z = f$, $\tau_N : TN \to N$ denoting the canonical tangent bundle projection.

**Definition 5.2.** Let $M$, $N$ and $f$ be as above, and let $X$ and $Y$ be two vector fields on $M$ and $N$, respectively. Then, by the generalized Lie derivative $\hat{L}_{(X, Y)} f$ of $f$ with respect to $X$ and $Y$ we shall mean the vector field along $f$ given by

$$\hat{L}_{(X, Y)} f := T f \circ X - Y \circ f.$$

If $\{\varphi_t\}$ and $\{\Phi_t\}$ denote the flows of $X$ and $Y$, respectively, then one readily verifies that

$$\hat{L}_{(X, Y)} f = \left. \frac{\partial}{\partial t} (\Phi_{-t} \circ f \circ \varphi_t) \right|_{t=0}.$$

An important specialization of Definition 5.2 is given by the following

**Definition 5.3.** Let $\pi : B \to M$ be a fibred manifold, $\sigma : M \to B$ a section of $B$, and $\Xi$ a projectable vector field on $B$ over a vector field $\xi$ on $M$. Then, by the generalized Lie derivative $\hat{L}_{\Xi} \sigma$ of $\sigma$ with respect to $\Xi$ we shall mean the map

$$\hat{L}_{\Xi} \sigma := \hat{L}_{(\xi, \Xi)} \sigma : M \to VB. \quad (5.1)$$
The Lie derivative of spinor fields: theory and applications

(It is easy to realize that \( \hat{T}_\Xi \sigma \equiv T\sigma \circ \xi - \Xi \circ \sigma \) takes indeed values in the vertical tangent bundle simply by applying \( T\pi \) to it and remembering that \( \Xi \) is projectable.)

Now recall that a fibred manifold \( \pi: B \to M \) admits a **vertical splitting** if there exists a linear bundle isomorphism (covering the identity of \( B \)) \( \alpha_B: VB \to B \times_M B \), where \( \hat{\pi}: \hat{B} \to M \) is a vector bundle. In particular, a vector bundle \( \pi: E \to M \) admits a **canonical** vertical splitting \( \alpha_E: VE \to E \times_M E \). Indeed, if \( \hat{\tau}_E: VE \to E \) denotes the (canonical) tangent bundle projection restricted to \( VE \), \( y \) is a point in \( E \) such that \( y = \hat{\tau}_E(v) \) for a given \( v \in VE \), and \( \gamma: \mathbb{R} \to E_y \equiv \pi^{-1}(\pi(y)) \) is a curve such that \( \gamma(0) = y \) and \( \frac{\partial}{\partial t}\gamma = v \), then \( \alpha_E \) is given by \( \alpha_E(v) := (y, w) \), where \( w := \lim_{t \to 0} \frac{1}{t}(\gamma(t) - \gamma(0)) \). Analogously, an affine bundle \( \pi: A \to M \) modelled on a vector bundle \( \hat{\pi}: \hat{A} \to M \) admits a canonical vertical splitting \( \alpha_A: VA \to A \times_M \hat{A} \), defined—mutatis mutandis—exactly as before, except for the fact that now \( \gamma(t) - \gamma(0) \) is a vector field on \( P \) projecting on a vector field \( \xi \) at \( t = 0 \) in the classical sense, one can re-express \( \hat{T}_\Xi \sigma \) in the form

\[
\hat{T}_\Xi \sigma(x) = \lim_{t \to 0} \frac{1}{t} (\Phi_{-t} \circ \sigma \circ \varphi_t(x) - \sigma(x)).
\]

Now, we can specialize Definition 5.3 to the case of gauge-natural bundles in a straightforward manner.

**Definition 5.6.** Let \( P_\lambda \) be a gauge-natural bundle associated with some principal bundle \( P(M,G) \), \( \Xi \) a \( G \)-invariant vector field on \( P \) projecting on a vector field \( \xi \) on \( M \), and \( \sigma: M \to P_\lambda \) a section of \( P_\lambda \). Then, by the **generalized (gauge-natural) Lie derivative of \( \sigma \)** with respect to \( \Xi \) we shall mean the map

\[
\hat{T}_\Xi \sigma: M \to VP_\lambda, \quad \hat{T}_\Xi \sigma := T\sigma \circ \xi - \Xi \circ \sigma,
\]

where \( \Xi_\lambda \) is the generator of the 1-parameter group \( \{ (\Phi_t)_\lambda \} \) of automorphisms of \( P_\lambda \) functorially induced by the flow \( \{ \Phi_t \} \) of \( \Xi \) [cf. (3.3)]. Equivalently,

\[
\hat{T}_\Xi \sigma = \frac{\partial}{\partial t} ((\Phi_{-t})_\lambda \circ \sigma \circ \varphi_t)|_{t=0},
\]

\( \{ \varphi_t \} \) denoting the flow of \( \xi \).

As usual, whenever \( P_\lambda \) admits a canonical vertical splitting, we shall write \( \hat{T}_\Xi \sigma: M \to P_\lambda := \mathcal{P}_\lambda \) for the corresponding restricted Lie derivative.

Furthermore, for each \( \Gamma \)-structure \( \zeta: \hat{P} \to P \) on \( P \), we shall simply write \( \hat{T}_\Xi \sigma := \hat{T}_\Xi \sigma: M \to \hat{P}_\lambda, \hat{P}_\lambda \) denoting a gauge-natural bundle associated with \( \hat{P} \) (admitting a canonical vertical splitting) and \( \hat{\sigma}: M \to \hat{P}_\lambda \) one of its sections,
since $\Xi$ admits a unique ($\Gamma$-invariant) lift $\tilde{\Xi}$ onto $\tilde{P}$ (cf. Proposition 2.10). We stress that Definition 5.6 is the conceptually natural generalization of the classical notion of a Lie derivative [40], to which it suitably reduces when applied to natural objects and, hence, notably, to tensor fields and tensor densities. In this case, $\Xi = L\xi$, and we shall simply write $\tilde{L}_\xi [L_{\xi}]$ for $\tilde{L}_\xi [L_{\xi}]$, as customary.

Of course, we can now further specialize to the case of classical (reductive) $G$-structures and, in particular, give the following

**Definition 5.7.** Let $P_\lambda$ be a gauge-natural bundle associated with some classical $G$-structure $P(M,G)$, $\xi_K$ the generalized Kosmann lift (on $P$) of a vector field $\xi$ on $M$, and $\sigma: M \to P_\lambda$ a section of $P_\lambda$. Then, by the generalized Lie derivative $\tilde{L}_{\xi_K} \sigma$ of $\sigma$ with respect to $\xi_K$ we shall mean the generalized Lie derivative of $\sigma$ with respect to $\xi_K$ in the sense of Definition 5.6.

Consistently, we shall simply write $L_{\xi_K} \sigma: M \to \tilde{P}_\lambda$ for the corresponding restricted Lie derivative, whenever defined, and $L_{\xi_K} \tilde{\sigma} := L_{\xi_K} \tilde{\sigma}: M \to \tilde{P}_\lambda$ for the (restricted) Lie derivative of a section $\sigma$ of a gauge-natural bundle $\tilde{P}_\lambda$ associated with some principal prolongation of a $\Gamma$-structure $\zeta: \tilde{P} \to P$ (and admitting a canonical vertical splitting), which makes sense since $\xi_K$ admits a unique ($\Gamma$-invariant) lift $\tilde{\xi}_K$ onto $\tilde{P}$ (cf. Corollary 4.16).

**Example 5.8 (Lie derivative of spinor fields. 1).** In Example 4.15 we mentioned that a fundamental example of a $\Gamma$-structure on a manifold $M$ is given by the bundle $SO(M,g)$ of its (pseudo-) orthonormal frames. An equally fundamental example of a $\Gamma$-structure on $SO(M,g)$ is given by the corresponding spin bundle $Spin(M,g)$ with structure group $\Gamma = Spin(p,q)^e$. Now, it is obvious that spinor fields can be regarded as sections of a suitable gauge-natural bundle over $M$. Indeed, if $\lambda$ is the linear representation of $Spin(p,q)^e$ on the vector space $\mathbb{C}^m$ induced by a given choice of $\gamma$ matrices, then the associated vector bundle $S(M) := Spin(M,g) \times_{\lambda} \mathbb{C}^m$ is a gauge-natural bundle of order $(0,0)$ whose sections represent spinor fields (or, more precisely, spin-vector fields). Therefore, in spite of what is sometimes believed, a Lie derivative of spinors (in the sense of Definition 5.6) always exists, no matter what the vector field $\xi$ on $M$ is. Locally, such a Lie derivative reads

$$L_{\Xi} \psi = \xi^a e_a \psi + \frac{1}{4} \Xi_{ab} \gamma^a \gamma^b \psi$$

for any spinor field $\psi$, $(\Xi_{ab} = \Xi_{[ab]})$ denoting the components of an $SO(p,q)^e$-invariant vector field $\Xi = \xi^a e_a + \Xi_{ab} A^{ab}$ on $SO(M,g)$. $\xi := \xi^a e_a$, and $e_a \psi$ the Pfaff derivative of $\psi$ along the local section $(e_a =: e^a_{\mu} \partial_{\mu})$ of $SO(M,g)$ induced by some local section of $Spin(M,g)$. This is the most general notion of a (gauge-natural) Lie derivative of spinor fields and the appropriate one for most situations of physical interest (cf. [16, 29]): the generality of $\Xi$ might be disturbing, but is the unavoidable indication that $S(M)$ is not a natural bundle.

If we wish nonetheless to remove such a generality, we must choose some canonical (not natural) lift of $\xi$ onto $SO(M,g)$. The conceptually (not mathematically) most “natural” choice is perhaps given by the Kosmann lift (recall Example 6 and use Corollary 4.16). The ensuing Lie derivative locally reads

$$L_{\xi} \psi = \xi^a e_a \psi + \frac{1}{4} (L_{\xi})_{[ab]} \gamma^a \gamma^b \psi.$$  \hfill (5.5)
Of course, if ‘\(\nabla\)’ denotes the covariant derivative operator associated with the Levi-Civita (or Riemannian) connection with respect to \(g\), the previous expression can be recast into the form

\[
£_{\xi} K \psi = \xi^a \nabla_a \psi - \frac{1}{4} \nabla_{[a} \xi_{b]} \gamma^a \gamma^b \psi, \tag{5.5'}
\]

which reproduces exactly Kosmann’s definition [26] (see [7] for further details and a more thorough discussion). We stress that, although in this case its local expression would be identical with (5.5), this is not the “metric Lie derivative” introduced by Bourguignon and Gauduchon in [2]. To convince oneself of this it is enough to take the Lie derivative of the metric \(g\), which is a section of the natural bundle \(\bigwedge^2 T^*M\), ‘\(\bigwedge\)’ denoting the symmetrized tensor product. Since the (restricted) Lie derivative \(£_{\xi} K\) in the sense of Definition 5.7 must reduce to the ordinary one on natural objects, it holds that

\[
£_{\xi} g = £_{\xi} K g.
\]

On the other hand, if \(£_{\xi} K\) coincided with the operator \(£_{\xi} g\) defined by Bourguignon and Gauduchon, the right-hand side of the above identity should equal zero [2, Proposition 15], thereby implying that \(\xi\) is a Killing vector field (cf. §6), contrary to the fact that \(\xi\) is completely arbitrary. Indeed, in order to recover Bourguignon and Gauduchon’s definition, another concept of a Lie derivative must be introduced.

We shall start by recalling two classical definitions [22].

**Definition 5.9.** Let \(P(M, G)\) be a (classical) \(G\)-structure. Let \(\varphi\) be a diffeomorphism of \(M\) onto itself and \(L^1 \varphi\) its natural lift onto \(LM\). If \(L^1 \varphi\) maps \(P\) onto itself, i.e. if \(L^1 \varphi(P) \subseteq P\), then \(\varphi\) is called an automorphism of the \(G\)-structure \(P\).

**Definition 5.10.** Let \(P(M, G)\) be a \(G\)-structure. A vector field \(\xi\) on \(M\) is called an infinitesimal automorphism of the \(G\)-structure \(P\) if it generates a local 1-parameter group of automorphisms of \(P\).

We can now generalize these concepts to the framework of reductive \(G\)-structures as follows.

**Definition 5.11.** Let \(P(M, G)\) be a reductive \(G\)-structure on a principal bundle \(Q(M, H)\) and \(\Phi\) a principal automorphism of \(Q\). If \(\Phi\) maps \(P\) onto itself, i.e. if \(\Phi(P) \subseteq P\), then \(\Phi\) is called a generalized automorphism of the reductive \(G\)-structure \(P\).

Of course, each element of \(\text{Aut}(P)\), i.e. each principal automorphism of \(P\), is by definition a generalized automorphism of the reductive \(G\)-structure \(P\). Analogously, we have

**Definition 5.12.** Let \(P(M, G)\) be a reductive \(G\)-structure on a principal bundle \(Q(M, H)\). An \(H\)-invariant vector field \(\Xi\) on \(Q\) is called a generalized infinitesimal automorphism of the reductive \(G\)-structure \(P\) if it generates a local 1-parameter group of generalized automorphisms of \(P\).
Of course, each element of \( \mathfrak{X}_G(P) \), i.e. each \( G \)-invariant vector field on \( P \), is by definition a generalized infinitesimal automorphism of the reductive \( G \)-structure \( P \).

Now, along the lines of [24, Proposition X.1.1] it is easy to prove

**Proposition 5.13.** Let \( P(M, G) \) be a reductive \( G \)-structure on a principal bundle \( Q(M, H) \). An \( H \)-invariant vector field \( \Xi \) on \( Q \) is a generalized infinitesimal automorphism of the reductive \( G \)-structure \( P \) if and only if \( \Xi \) is tangent to \( P \) at each point of \( P \).

We then have the following important

**Lemma 5.14.** Let \( P(M, G) \) be a reductive \( G \)-structure on a principal bundle \( Q(M, H) \) and \( \Xi \) a generalized infinitesimal automorphism of the reductive \( G \)-structure \( P \). Then, the flow \( \{ \Phi_t \} \) of \( \Xi \), it being \( H \)-invariant, induces on each gauge-natural bundle \( Q_\lambda \) associated with \( Q \) a 1-parameter group \( \{ (\Phi_t)_\lambda \} \) of global automorphisms.

**Proof.** Since \( \Xi \) is by assumption a generalized infinitesimal automorphism, it is by definition an \( H \)-invariant vector field on \( Q \). Therefore, its flow \( \{ \Phi_t \} \) is a 1-parameter group of \( H \)-equivariant maps on \( Q \). Then, if \( Q_\lambda = W^{k,h}Q \times_\lambda F \), we set

\[
(\Phi_t)_\lambda([u, f]) := [W^{k,h} \Phi_t(u), f],
\]

\( u \in Q, f \in F \), and are back to the situation of formula (3.3).

**Corollary 5.15.** Let \( P(M, G) \) and \( Q(M, H) \) be as in the previous lemma, and let \( \Xi \) be an \( H \)-invariant vector field on \( Q \). Then, the flow \( \{ (\Phi_K)_t \} \) of the generalized Kosmann vector field \( \Xi_K \) associated with \( \Xi \) induces on each gauge-natural bundle \( Q_\lambda \) associated with \( Q \) a 1-parameter group \( \{ ((\Phi_K)_t)_\lambda \} \) of global automorphisms.

**Proof.** Recall that, although the generalized Kosmann vector field \( \Xi_K \) is a \( G \)-invariant vector field on \( P \), it is \( H \)-invariant if regarded as a vector field on the corresponding subset of \( Q \) (cf. Remark 4.7 and Corollary 4.12). Therefore, its flow \( \{ (\Phi_K)_t \} \) is a 1-parameter group of \( H \)-equivariant automorphisms on the subset \( P \) of \( Q \).

We now want to define a 1-parameter group of automorphisms \( \{ ((\Phi_K)_t)_\lambda \} \) of \( Q_\lambda = W^{k,h}Q \times_\lambda F \). Let \( [u, f]_\lambda \in Q_\lambda, u \in Q \) and \( f \in F \), and let \( u_1 \) be a point in \( P \) such that \( \pi(u_1) = \pi(u), \pi : Q \to M \) denoting the canonical projection. There exists a unique \( a_1 \in H \) such that \( u = u_1 \cdot a_1 \). Set then

\[
((\Phi_K)_t)_\lambda([u, f]) := [W^{k,h} (\Phi_K)_t(u_1), a_1 \cdot f].
\]

We must show that, given another point \( u_2 \in P \) such that \( u = u_2 \cdot a_2 \) for some (unique) \( a_2 \in H \), we have

\[
[W^{k,h} (\Phi_K)_t(u_1), a_1 \cdot f] = [W^{k,h} (\Phi_K)_t(u_2), a_2 \cdot f].
\]

Indeed, since the action of \( H \) is free and transitive on the fibres, from \( u = u_1 \cdot a_1 \) and \( u = u_2 \cdot a_2 \) it follows that \( a_1 = a \cdot a_2 \) or \( a = a_1 \cdot (a_2)^{-1} \) or \( a_2 = a^{-1} \cdot a_1 \).
But then
\[ [W^{k,h}(\Phi_K)\xi(u_2), a_2 \cdot f]_\lambda = [W^{k,h}(\Phi_K)\xi(u_1 \cdot a), a^{-1} \cdot a_1 \cdot f]_\lambda \]
\[ = [W^{k,h}(\Phi_K)\xi(u_1) \odot W^{k,h}_m a, a^{-1} \cdot a_1 \cdot f]_\lambda \]
\[ = [W^{k,h}(\Phi_K)\xi(u_1), a_1 \cdot f]_\lambda, \]

as claimed. It is then easy to see that the so-defined \((\Phi_K)_\xi\) does not depend on the chosen representative. \(\square\)

By virtue of the previous corollary, we can now give the following

**Definition 5.16.** Let \(P(M, G)\) be a reductive \(G\)-structure on a principal bundle \(Q(M, H)\), \(G \neq \{e\}\), and \(\Xi\) an \(H\)-invariant vector field on \(Q\) projecting on a vector field \(\xi\) on \(M\). Let \(Q_\lambda\) be a gauge-natural bundle associated with \(Q\) and \(\sigma: M \to Q_\lambda\) a section of \(Q_\lambda\). Then, by the generalized \(G\)-reductive Lie derivative of \(\sigma\) with respect to \(\Xi\) we shall mean the map

\[ \tilde{\mathcal{L}}^G_\xi \sigma := \frac{\partial}{\partial t}\left(((\Phi_K)_\xi - t) \circ \sigma \circ \varphi_t\right)|_{t=0}, \]

\(\{\varphi_t\}\) denoting the flow of \(\xi\).

The corresponding notions of a restricted Lie derivative and a (generalized or restricted) Lie derivative on an associated \(\Gamma\)-structure can be defined in the usual way.

**Remark 5.17.** Of course, since \(\Xi_K\) is by definition a \(G\)-invariant vector field on \(P\), Definition 5.16 makes sense also when \(\sigma\) is a section of a gauge-natural bundle \(P_\lambda\) associated with \(P\), for which one does not even need Corollary 5.15.

**Remark 5.18.** When \(Q = P\) (and \(H = G\)), \(\Xi_K\) is just \(\Xi\), and we recover the notion of a (generalized) Lie derivative in the sense of Definition 5.6, but, as \(G\) is required not to equal the trivial group \(\{e\}\), \(Q_\lambda\) is never allowed to be a (purely) natural bundle.

By its very definition, the (restricted) \(G\)-reductive Lie derivative does not reduce, in general, to the ordinary (natural) Lie derivative on fibre bundles associated with \(L^M\). This fact makes it unsuitable in all those situations where one needs a unique operator which reproduce “standard results” when applied to “standard objects”.

In other words, \(\mathcal{L}^G_\Xi\) is defined with respect to some pre-assigned (generalized) symmetries. We shall make this statement explicit in Proposition 5.20 below, which provides a generalization of a well-known classical result.

Let then \(K\) be a tensor over the vector space \(\mathbb{R}^m\) (i.e., an element of the tensor algebra over \(\mathbb{R}^m\)) and \(G\) the group of linear transformations of \(\mathbb{R}^m\) leaving \(K\) invariant. Recall that each reduction of the structure group \(GL(m, \mathbb{R})\) to \(G\) gives rise to a tensor field \(K\) on \(M\). Indeed, we may regard each \(u \in LM\) as a linear isomorphism of \(\mathbb{R}^m\) onto \(T_x M\), where \(x = \pi(u)\) and \(\pi: LM \to M\) denotes, as usual, the canonical projection. Now, if \(P(M, G)\) is a \(G\)-structure, at each point \(x\) of \(M\) we can choose a frame \(u\) belonging to \(P\) such that \(\pi(u) = x\). Since \(u\) is a linear isomorphism of \(\mathbb{R}^m\) onto the tangent space \(T_x M\), it induces an isomorphism of the tensor algebra over \(\mathbb{R}^m\) onto the tensor algebra over \(T_x M\).
Then $K_x$ is the image of $K$ under this isomorphism. The invariance of $K$ by $G$ implies that $K_x$ is defined independent of the choice of $u$ in $\pi^{-1}(x)$. Then, we have the following classical result [22].

**Proposition 5.19.** Let $K$ be a tensor over the vector space $\mathbb{R}^m$ and $G$ the group of linear transformations of $\mathbb{R}^m$ leaving $K$ invariant. Let $P$ be a $G$-structure on $M$ and $K$ the tensor field on $M$ defined by $K$ and $P$. Then

(i) a diffeomorphism $\varphi : M \to M$ is an automorphism of the $G$-structure $P$ iff $\varphi$ leaves $K$ invariant;

(ii) a vector field $\xi$ on $M$ is an infinitesimal automorphism of $P$ iff $\mathcal{L}_\xi K = 0$.

An analogous result for generalized automorphisms of $P$ follows.

**Proposition 5.20.** In the same hypotheses of the previous proposition\(^1\),

(i) an automorphism $\Phi : LM \to LM$ is a generalized automorphism of the $G$-structure $P$ iff $\Phi$ leaves $K$ invariant;

(ii) a $\text{GL}(m, \mathbb{R})$-invariant vector field $\Xi$ on $LM$ is an infinitesimal generalized automorphism of $P$ iff $\mathcal{L}_\Xi K = 0$;

(iii) $\mathcal{L}_\Xi K \equiv 0$ for any $\text{GL}(m, \mathbb{R})$-invariant vector field $\Xi$ on $LM$.

**Proof.** First, note that, here, $K$ is regarded as a section of a gauge-natural, not simply natural, bundle over $M$ (cf. Example 3.10). Then, since $K$ is $G$-invariant, an automorphism $\Phi : LM \to LM$ will leave $K$ unchanged if and only if it maps $P$ onto itself and is $G$-equivariant on $P$, i.e. iff it is a generalized automorphism of $P$, whence (i) follows. Part (ii) is just the infinitesimal version of (i), whereas (iii) follows from (ii) and Definition 5.16 since $\Xi_K$ is by definition a $G$-invariant vector field on $P$ and hence, in particular, a generalized automorphism of $P$. The choice $\Xi = L\xi$ reproduces Kobayashi’s classical result, which can therefore be stated in a (purely) natural setting, as in Proposition 5.19. \square

**Corollary 5.21.** Let $\Xi$ be a $\text{GL}(m, \mathbb{R})$-invariant vector field on $LM$, and let $g$ be a metric tensor on $M$ of signature $(p, q)$. Then, $\mathcal{L}\Xi_{\text{SO}(p,q)}^* g \equiv 0$.

**Proof.** It follows immediately from Proposition 5.20(iii). \square

The last corollary suggests that Bourguignon and Gauduchon’s metric Lie derivative might be a particular instance of a reductive Lie derivative. This is precisely the case, as explained in the following fundamental

**Example 5.22 (Lie derivative of spinor fields. II).** We know that the Kosmann lift $\xi_K$ onto $\text{SO}(M, g)$ of a vector field $\xi$ on $M$ is an $\text{SO}(p, q)^e$-invariant vector field on $\text{SO}(M, g)$, and hence its lift $\xi_K$ onto $\text{Spin}(M, g)$ is a Spin$(p, q)^e$-invariant vector field. As the spinor bundle $S(M)$ is a vector bundle associated with $\text{Spin}(M, g)$, the $\text{SO}(p, q)^e$-reductive Lie derivative $\mathcal{L}\xi_{\text{SO}(p,q)^e}^* \psi$ of a spinor field $\psi$ coincides with $\mathcal{L}\xi_K^* \psi$, i.e. locally with expression (5.5) or (5.5’)\(^1\). Indeed, in this case we have, with an obvious notation, $P = \text{SO}(M, g)$, $G = \text{SO}(p, q)^e$, $\tilde{P} = \text{Spin}(M, g)$ and $\tilde{P}_\lambda = S(M)$.

\(^{1}\)Here and in the sequel we shall always assume that all given (classical) $G$-structures are reductive (cf. Corollary 4.13).
The Lie derivative of spinor fields: theory and applications

For $L_{L^e}^{SO(p,q)} g$ a similar remark to the one above for $L_{L^e}^G K$ applies and therefore, if $g = g_{\mu\nu} \, dx^\mu \wedge dx^\nu$ in some natural chart, we have the local expression

$$L_{L^e}^{SO(p,q)} g_{\mu\nu} \equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2g_{\rho(\mu}(\xi_{\nu)}\rho)$$

$$\equiv \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho(\mu} \partial_{\nu)} \xi^\rho - \delta^\rho_{(\mu} \delta^\sigma_{\nu)} \partial_\rho \xi^\sigma - \xi^\rho \delta^\rho_{(\mu} \partial_{\nu)} g_{\sigma)}$$

$$\equiv 0$$

$$\equiv L_{L^e}^{SO(p,q)} g_{\mu\nu},$$

quite different from the usual (natural) Lie derivative

$$L_{L^e}^g \xi_{\mu\nu} \equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2g_{\rho(\mu}(\xi_{\nu)}\rho)$$

$$\equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2g_{\rho(\mu} \partial_{\nu)} \xi^\rho$$

$$\equiv 2\nabla_\mu \xi_\nu$$

$$\equiv L_{\xi}^{K} g_{\mu\nu}.$$ 

Hence, we can identify Bourguignon and Gauduchon’s metric Lie derivative $L_{L^e}^g$ with $L_{L^e}^{SO(p,q)} g_{\mu\nu}$, not $L_{\xi}^{K}.$

6 The G-Killing condition

Let $M$ be an $m$-dimensional manifold equipped with a (pseudo-) Riemannian metric $g$ of signature $(p, q)$. It is well known that the condition for a vector field $\xi$ on $M$ to be Killing (with respect to $g$) is (cf., e.g., [40])

$$L_{\xi} g \equiv L_{L^e} g = 0,$$ 

(6.1)

where we are implicitly regarding $g$ as a natural object, as it is usually the case in physics. On the other hand, for $\Xi = L_{\xi}$ Corollary 5.21 gives

$$L_{L^e}^{SO(p,q)} g \equiv 0.$$

Hence, recalling Definition 5.16 (and Example 4.15), we notice that Killing condition (6.1) can be rephrased as

$$L_{\xi}^{SO(M,g)} = \xi_{K} \equiv (L_{\xi})_{K}. $$

What is more, in this form, the Killing equation lends itself to a straightforward generalization,

$$\Xi|_P \equiv \Xi_{K},$$

(6.2)

where $\Xi|_P$ is the restriction of an $H$-invariant vector field $\Xi$ on a principal bundle $Q(M,H)$ to any reductive $G$-structure $P$ on $Q(M,H)$. We shall call equation (6.2) the G-Killing condition.

In particular, when $\Xi = L_{\xi}$ and $G = CSO(p,q)^e := SO(p,q)^e \times \mathbb{R}^+,$ we have

$$L_{\xi}^{S\xi (M,g)} = \xi_P := (L_{\xi})_{K},$$

(6.3)

where $(L_{\xi})_{K}$ is the appropriate generalized Kosmann lift of $\xi$ on $SO(M,g)$.

We shall call $\xi_P$ the Penrose lift of $\xi$ for reasons which will become apparent in the next section. It is easy to see that condition (6.3) is equivalent to

$$L_{\xi} g = \frac{2}{m} \text{tr} (\nabla g \xi),$$

(6.4)
where ‘\(\nabla\)’ denotes the covariant derivative operator corresponding to the Levi-Civita connection associated with \(g\), i.e. to \(\xi\) being conformal Killing.

Finally, note that for a GL\((m, \mathbb{R})\)-structure (which is just a way to refer to \(\text{LM} \) generically, i.e. without necessarily regarding it as a natural bundle) and \(\Xi = L\xi\) condition (6.2) reads

\[
L\xi = (L\xi)_K \equiv L\xi,
\]

since, here, the generalized Kosmann lift of the natural lift \(L\xi\) of \(\xi\) is—trivially—\(L\xi\) itself. In other words, in this case, condition (6.2) amounts to no condition at all, as one might have reasonably expected.

### 7 Penrose’s Lie derivative of “spinor fields”

We shall now give a reinterpretation of Penrose and Rindler’s [35] definition of a Lie derivative of spinor fields in the light of the general theory of Lie derivatives. This definition has become quite popular among the physics community despite its being restricted to infinitesimal conformal isometries, and was already thoroughly analysed by Delaney [5]. Although his analysis is correct for all practical purposes, Delaney fails to understand the true reasons of the aforementioned restriction because these are, crucially, of a functorial nature, and only a functorial analysis can unveil them.

Throughout this section we shall assume \(m \equiv \dim M = 4\), and ‘\(\nabla\)’ will denote the covariant derivative operator corresponding to the Levi-Civita connection associated with the given metric \(g\). Also, we shall use a covariantized form for all our Lie derivatives so that our formulae can be reinterpreted in an abstract index fashion, if so desired \([\text{cf.}, \text{ e.g., } (5.5')]\). We refer the reader to [21, 34] for the basics of the 2-spinor formalism.

Penrose and Rindler’s Lie derivative \(\mathcal{L}_\xi^P\psi\) of a 2-spinor field \(\phi\) (locally) reads

\[
\mathcal{L}_\xi^P\phi^A = \xi^a\nabla_a\phi^A - \left(\frac{1}{2} \nabla_{BA'}\xi^{A'A'} + \frac{1}{4} \nabla_c\xi^c\delta^{AA'}_{B}\right)\phi^B,
\]

(cf. [35, §6.6]) or, in 4-spinor formalism,

\[
\mathcal{L}_\xi^P\psi = \xi^a\nabla_a\psi - \left(\frac{1}{4} \nabla_{(a}\xi_{b)}\gamma^{a}\gamma^{b} + \frac{1}{4} \nabla_c\xi^c\right)\psi
\]

for some suitable 4-spinor \(\psi\), and, in the authors’ formulation, only holds if \(\xi\) is conformal Killing. From \(7.1'\) we immediately note that the (particular) lift of \(\xi\) with respect to which the Lie derivative is taken is not \(\text{SO}(1,3)^e\)-invariant, but rather \(\text{CSO}(1,3)^e\)-invariant, i.e., strictly speaking \(\mathcal{L}_\xi^P\psi\) is not a Lie derivative of a spinor field, but of a conformal spinor field. Furthermore, it is easy to realize that the given lift is actually the local expression of the Penrose lift defined in the previous section.

Now, the Penrose lift \(\xi_P\) of a vector field \(\xi\) is just a particular instance of a generalized Kosmann lift, and one should be able to take the Lie derivative of a conformal spinor field with respect to \(\xi_P\) no matter what \(\xi\) is (cf. Definition 5.7). The fact that formula (7.1) above is stated to hold only for conformal Killing vector fields then suggests that an additional condition has been (tacitly) imposed. From the discussion in §6 we deduce that this condition must be the \(G\)-Killing condition [with \(\Xi = L\xi\) and \(G = \text{CSO}(1,3)^e\)].
This is actually the impression one gets from [21], where from the 2-dimensionality of the (“unprimed”) 2-spinor bundle $\bar{2}S(M)$ it is deduced that, for all Lie derivatives $L_\Xi$ of spinor fields, the following should hold

$$L_\Xi g_{ab} = L_\Xi (\varepsilon_{AB} \varepsilon_{A'B'}) = (\lambda + \bar{\lambda}) g_{ab}. \quad (7.2)$$

This is true: actually, in the strictly spinorial case $\lambda = 0$ and in the conformal one $\lambda + \bar{\lambda} = 2/m \operatorname{tr}(\nabla \xi)$, but this means treating $g_{ab} \equiv \varepsilon_{AB} \varepsilon_{A'B'}$ as a non natural object. If we want $g$ to represent the usual space-time metric (and hence transform naturally) and still make some sense of (7.2), the only way we can do this is by imposing the $G$-Killing condition [with $G = \operatorname{SO}(1,3)^c$ or $G = \operatorname{CSO}(1,3)^c$], which establishes a link between the natural lift $L_\xi$ and its appropriate generalized Kosmann lifts (cf. §6).

This has the advantage of regarding the space-time metric and (the tensorial equivalent of) the composite spinorial object $\varepsilon \otimes \bar{\varepsilon}$ as exactly the same mathematical entity as far as Lie differentiation is concerned, but has the strong drawback of restricting ourselves to infinitesimal [conformal] isometries. The difference between the two objects is made explicit in the formula

$$g_{\mu\nu}(x) = g_{ab}(x) \theta^a_{\mu}(x) \theta^b_{\nu}(x), \quad (7.3)$$

where, here, $(\theta^a_{\mu}(x))$ is not to be understood as the components of the matrix representing the transformation from an anholonomic to a holonomic basis of $T_xM$ (or, which is the same, the components of the local soldering form, pull-back on $M$ of the [global] canonical 1-form on $LM$—cf., e.g., [23]), but as (the components of) a gauge-natural object $\theta$ called a $G$-tetrad, transforming the gauge-natural object $(g_{ab} \equiv \varepsilon_{AB} \varepsilon_{A'B'})$ into the natural object $(g_{\mu\nu})$, and vice versa (cf. [30]). So, unlike soldering form components, $G$-tetrads cannot be suppressed from formulae like (7.3) even in an abstract index type notation\footnote{In an abstract index type notation, Greek and Latin indices would not denote holonomic and anholonomic components, but rather natural and gauge-natural objects, respectively.}, because Lie derivatives are, crucially, category-dependent operators. In this context, the $G$-Killing condition can be rewritten as

$$L_\Xi \theta = 0.$$

To further justify our statement, let us briefly recall the way Penrose and Rindler [35] arrive at formula (7.1). Using the fact that a (complex) bivector field $K$, i.e. a section of $\Lambda^2(TM)^c$, can be represented spinorially as $\kappa \otimes \kappa \otimes \bar{\varepsilon}$, $\kappa$ being a section of $\bar{2}S(M)$ [$TM^c := TM \otimes \mathbb{C} \cong \bar{2}S(M) \otimes \bar{2}S(M)$], they write

$$L_\xi^D (\kappa^A \kappa^B \varepsilon^{A'B'}) = \xi^a \nabla_a (\kappa^A \kappa^B \varepsilon^{A'B'}) - \kappa^D \kappa^B \varepsilon^{D'B'} \nabla_a \xi^a - \kappa^A \kappa^D \varepsilon^{A'D'} \nabla_a \bar{\xi}^b, \quad (7.4)$$

assuming that $\kappa \otimes \kappa \otimes \bar{\varepsilon}$ transforms with the usual natural lift. Note, though, that, in this kind of vector bundle isomorphisms, $TM^c$ (or $TM$) is assumed to be a gauge-natural vector bundle associated with $\operatorname{SO}(M,g)$, not a natural bundle (associated with $LM$). Hence, if we want (7.4) to make any sense at all from the point of view of the general theory of Lie derivatives, we must interpret $L^D_\xi K$ as the particular Lie derivative $L_{L_\xi} K$ on a vector bundle associated with a $\operatorname{GL}(4,\mathbb{R})$-structure, not as the standard (natural) Lie derivative $L_\xi K$ of the
natural object $K$: in any case, $\kappa$ will not transform with $\text{SL}(2, \mathbb{C})$ as a standard spinor. Now, a little algebra [35, p. 102] shows that (7.4) implies

$$\nabla_{(A} \xi_{B')}^{(A' B')} = 0,$$

(7.5)

which is readily seen to be equivalent to conformal Killing equation (6.4). But now recall from §6 that, in the case of a $\text{GL}(4, \mathbb{R})$-structure, $\mathcal{L}_\xi K$ identically satisfies the $G$-Killing equation, and, unlike in the $\text{SO}(p, q)^c$ or the $\text{CSO}(p, q)^c$ case, this did not imply any further restriction, so we might wonder why we ended up with (7.5). Note that, even starting from a general $\text{GL}(4, \mathbb{R})$-invariant vector field $\Xi$ (projecting on $\xi$), i.e.

$$\mathcal{L}_\Xi (\kappa^A \kappa^B \varepsilon^{A'B'}) = \xi^c \nabla_c (\kappa^A \kappa^B \varepsilon^{A'B'}) - \kappa^D \kappa^B \varepsilon^{D'B'} \tilde{\Xi}^a_d - \kappa^A \kappa^D \varepsilon^{A'D'} \tilde{\Xi}^b_d,$$

$\tilde{\Xi}$ denoting the vertical part of $\Xi$ with respect to the Levi-Civita connection $\omega$ associated with $g$ (i.e. $\Xi^a_b = \Xi^a_b + \omega^a_{\mu b} \xi^\mu$), one finds oneself restricted to $\text{cso}(1, 3)$, i.e.

$$\tilde{\Xi}^{(A'B')}_{(AB)} = 0,$$

(7.6)

whereas the same argument applied in the strictly spinorial case [i.e. to an $\text{SO}(1, 3)^c$-invariant vector field] leads to the trivial identity

$$0 = 0,$$

i.e. to no restriction whatsoever.

The reason why (7.5) or, more generally, (7.6) appears is due to the particular vector space we are using to represent these “$\text{GL}(4, \mathbb{R})$-spinors”. Indeed, note that, although we can represent a $\text{GL}(4, \mathbb{R})$-invariant vector field $\Xi$ spinorially, explicitly

$$\Xi^{ab} = \Xi^{(AB)(A'B')} + \frac{1}{2} \Xi^{CC'} \varepsilon_{CC'} \varepsilon^{AB} \varepsilon^{A'B'} + \frac{1}{2} \Xi^{(AB)C'} \varepsilon_{C'} \varepsilon^{A'B'} + \Xi^{C} \varepsilon_{C'} \varepsilon^{A'B'} \varepsilon^{AB},$$

(7.7)

we cannot “move” a section of $\mathbb{S}(M)$ with $\Xi$ because the largest group that can act on a 2-dimensional complex vector space is $\text{GL}(2, \mathbb{C})$ and

$$\dim \mathfrak{gl}(4, \mathbb{R}) = 4 \cdot 4 = 16 \neq 8 = (2 \cdot 2)2 \equiv \dim \mathbb{R} \mathfrak{gl}(2, \mathbb{C}),$$

unlike in the strictly spinorial case, where

$$\dim \mathfrak{so}(1, 3) \equiv \frac{4(4-1)}{2} = 6 = (2 \cdot 2 - 1)2 \equiv \dim \mathbb{R} \mathfrak{sl}(2, \mathbb{C}).$$

If we nevertheless attempt to do so, we get the 9 conditions (7.6).

One could still wonder why we get 9 equations instead of 8. The reason is that, as we can also easily see from (7.7), the irreducible decomposition of $\mathfrak{gl}(4, \mathbb{R})$ is

$$\mathfrak{gl}(4, \mathbb{R}) = \mathfrak{so}(1, 3) \oplus \mathbb{R} \oplus V$$

$V$ being the vector space of all traceless symmetric matrices, and no combination of the dimensions of the terms on the r.h.s. adds up to 8, so that we are left with $\mathfrak{cso}(1, 3) = \mathfrak{so}(1, 3) \oplus \mathbb{R}$. This also explains why it is only possible to determine the real part of $\lambda$ (corresponding to the term $\tilde{\Xi}^c_C$ of the general case) in (7.2), as already observed in [35, p. 102].
8 An application to the calculus of variations

So far, we have seen purely mathematical applications of the general theory of Lie derivatives of spinor fields we developed. One might wonder whether there are any concrete physical situations where such a generality is desirable, if not necessary. An important example is given by the Einstein (-Cartan) -Dirac theory, i.e. the (classical) field theory describing a spin 1/2 massive Fermion field minimally coupled with Einstein’s gravitational field in a curved space-time (possibly allowing for the presence of torsion as a “source” of spin).

Now, it is well known that one of the most powerful tools of (Lagrangian) field theory is the so-called “Noether theorem” [31, 25, 13]. It turns out that, when phrased in modern geometrical terms, this theorem crucially involves the concept of Lie differentiation, and here is where a functorial approach is not only useful, but also intrinsically unavoidable, since—as we mentioned earlier and our discussion clearly shows—Lie derivatives are category-dependent operators.

In our case, then, not only do we have to be in a position to take a Lie derivative of a spinor field with respect to the most general infinitesimal transformation possible, but it also turns out that, when coupled with Dirac fields, Einstein’s general relativity can no longer be regarded as a purely natural theory because, in order to incorporate spinors, one must enlarge the class of morphisms of the theory. In other words, the Einstein (-Cartan) -Dirac theory must be regarded as gauge-natural field theory.

The variables of this theory are a spin- [frame induced SO(1,3)\text{-}] tetrad $\theta$ \text{(cf. [8, 10])}, a spin- [frame induced SO(1,3)\text{-}] connection $\omega$ (independent of $\theta$ in the Einstein-Cartan case—\text{cf. [16, 29]}) and a spinor field $\psi$. We refer the reader to [16, 29] for all detail. Here we shall limit ourselves to mention that not only does our theory of Lie derivatives allow us to perform all the required calculations, which could not be carried out otherwise, but also gives rise to an interesting indeterminacy in the concept of conserved quantities.

This type of indeterminacy, which is ultimately due to the lack of a natural lift on a non [purely] natural bundle, is intrinsic to all gauge-natural field theories admitting a non-trivial superpotential. At the same time, on physical grounds, the [generalized] Kosmann lift seems to play once again a privileged role.

Specifically, the Kosmann lift enables one to reproduce the standard “Komar superpotential” in the Einstein (-Cartan) -Dirac theory [29] and is precisely what is needed to recover the “one-quarter-area law” for the entropy in the triad-affine formulation of the (2 + 1)-dimensional BTZ black hole solution [9]. Also, the generalized Kosmann lift is crucial for achieving the full correspondence between the variation of conserved quantities in Chern-Simons AdS$_3$ gravity and (2 + 1)-General Relativity [1].

An elegant way to solve the above indeterminacy is to require the second variation of the action functional to vanish \textit{as well} [33].

Conclusions

In this paper we have investigated the hoary problem of the Lie derivative of spinor fields from a very general point of view, following a functorial approach. We have done so by relying on three nice geometric constructions: split structures, gauge-natural bundles and the general theory of Lie derivatives.
Such analysis has shown that, although for (purely) natural objects over a manifold $M$ there is a conceptually and mathematically natural definition of a Lie derivative with respect to a vector field $\xi$ on $M$, there is no such thing for more general gauge-natural objects, $\xi$ being necessarily replaced by an $H$-invariant vector field $\Xi$ on some principal bundle $Q(M,H)$.

On a classical $G$-structure $P(M,G)$, though, there is defined a canonical, not natural, lift of $\xi$, which we called its “generalized Kosmann lift”. As we saw, this lift can be easily extended to a prolongation $\Gamma$-structure on $P$.

In this context, we analysed several ad hoc definitions of Lie derivatives of spinor fields given in the past, providing a clear-cut geometric reinterpretation for each one of them and clearly stating their limit of applicability.

Finally, we outlined one of the many physical applications undoubtedly benefitting from the generality of our approach.

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