Analytic solutions for coupled linear perturbations

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ABSTRACT

Analytic solutions for the evolution of cosmological linear density perturbations in the baryonic gas and collisionless dark matter are derived. The solutions are expressed in a closed form in terms of elementary functions, for arbitrary baryonic mass fraction. They are obtained assuming $\Omega = 1$ and a time-independent comoving Jeans wavenumber, $k_j$. By working with a time variable $\tau = \ln(t^{1/3})$, the evolution of the perturbations is described by linear differential equations with constant coefficients. The new equations are then solved by means of Laplace transformation, assuming that the gas and dark matter trace the same density field before a sudden heating epoch. In a dark matter-dominated Universe, the ratio of baryonic to dark matter density perturbation decays with time roughly as $\exp(-5\tau/4) \propto t^{-5/6}$ to the limiting value $1/[1 + (k/k_j)^2]$. For wavenumbers $k > k_j/24$, the decay is accompanied by oscillations with a period $8\pi/\sqrt{24}(k/k_j)^2 - 1$ in $\tau$. In comparison, as $\tau$ increases in a baryonic matter-dominated Universe, the ratio approaches $1 - (k/k_j)^2$ for $k \leq k_j$, and zero otherwise.

Key words: gravitation – intergalactic medium – cosmology: theory – dark matter.

1 INTRODUCTION

Several methods (e.g. Croft et al. 1998; Gnedin 1998; Nusser & Haehnelt 1999) have recently been proposed for extracting information on the mass density field from the Lyman-$\alpha$ forest. The underlying physical picture behind these methods is that the absorbing neutral hydrogen in the low-density intergalactic medium (IGM) is tightly related to the mass density field on scales larger than the Jeans length. Below the Jeans length, gas pressure segregates the baryons from the total mass fluctuations. On scales near the Jeans length, the evolution of the baryonic perturbation can affect estimates of the clustering amplitude from observations of the Lyman-$\alpha$ forest.

Hydrodynamical simulations (Petitjean, Mäcket & Katz 1995; Zhang, Anninos & Norman 1995; Hernquist et al. 1996; Miralda-Escude et al. 1996; Theuns et al. 1998) and semi-analytic models (e.g. Bi, Börner & Chu 1992) of the IGM have been successful in explaining observations of the forest. Despite the success of the simulations, it is usually difficult to use them to study in detail the evolution of the gas below the Jeans length (Theuns et al. 1998). The equations governing the evolution of baryons and dark matter in the nonlinear regime are extremely difficult to solve, even for special configurations like spherical collapse. Fortunately, since most of the IGM is of moderate density, linear analysis can be a suitable tool for understanding the evolution of the baryons (Gnedin & Hui 1998). Here we derive analytic solutions to the linear equations in a flat universe without a cosmological constant. Although the linear equations can readily be numerically integrated under a variety of conditions (Gnedin & Hui 1998), analytic treatment offers better understanding of the equations. Further, the paucity of analytic solutions makes their pursuit worthwhile, even if tedious at times. The analytic solutions we derive here are subject to the condition that the baryonic and dark matter trace the same density and velocity fields before a sudden reionization epoch. After reionization the temperature of the IGM is assumed to be inversely proportional to the scalefactor, so that the comoving Jeans length is constant. The solutions exhibit a variety of features similar to the behaviour of coupled perturbations in the photon and matter fluids at the last scattering surface (e.g. Kodama & Sasaki 1986; Hu & Sugiyama 1996).

The paper is organized as follows. In Section 2 we cast the equations in the form of linear differential equations with constant coefficients. In Section 3 we present the solutions to these equations for several cases. We conclude in Section 4.

2 THE LINEAR EQUATIONS

Let $\delta_b(t,k)$ and $\delta_x(t,k)$ be, respectively, the Fourier modes of baryonic and dark matter density fluctuations. Let also $f_x$ and $f_b = 1 - f_x$ be the mean mass fractions of these two types of matter. We will restrict the analysis to perturbations in a flat universe without a cosmological constant. The linear equations governing the evolution of $\delta_b$ and $\delta_x$ are (e.g. Bi et al. 1992; Padmanabhan 1993; Gnedin & Hui 1998)

$$\frac{d^2\delta_b}{dt^2} + 2H\frac{d\delta_b}{dt} = \frac{3}{2} H^2 (f_x \delta_x + f_b \delta_b),$$

$$\frac{d^2\delta_x}{dt^2} + 2H\frac{d\delta_x}{dt} = \frac{3}{2} H^2 (f_x \delta_x + f_b \delta_b) - \frac{3}{2} H^2 \left( \frac{k}{k_j} \right)^2 \delta_b. \quad (1)$$
3 THE SOLUTIONS

Denote by $\Delta_\delta$ and $\Delta_\theta$ the Laplace transforms of $\delta_\delta$ and $\delta_\theta$, respectively. By taking the Laplace transform of (3), we can obtain relations between $\Delta_\delta$ and $\Delta_\theta$. The initial conditions are contained in the Laplace transforms of the first and second derivatives of the densities. So, first, we have to specify in mathematical terms our choice of the initial conditions. For simplicity of notation we fix the initial conditions at $\tau = 1$, assuming that before that time the temperature of the baryonic fluid is zero, i.e., $\kappa = 0$. The initial conditions are fixed by the values of $\delta_\delta$ and $\delta_\theta$, and their first derivatives at $\tau = 1$. Before $\tau = 1$, we have $\delta_\delta = \delta_\theta = \exp(\tau)$, ignoring the decaying mode and setting arbitrarily $\delta_\delta(\tau = 1) = 1$. The first derivatives of $\delta_\delta$ and $\delta_\theta$ are therefore equal to unity at $\tau = 1$. This fixes the initial conditions necessary for solving (3). Although we will present solutions satisfying only these initial conditions, we will, for completeness, write the Laplace transformation of (3) for $\delta_\delta = \alpha \delta_\delta$ and $\delta_\theta/d\tau = \alpha d\delta_\theta/d\tau$, at $\tau = 1$. Then the Laplace transformation of (3) yields

$$
\begin{align*}
(s^2 + 2)\Delta_\delta &= \frac{3}{2}(f_\delta \Delta_\delta + f_\theta \Delta_\theta) + s + \frac{3}{2}, \\
(s^2 + 2)\Delta_\theta &= \frac{3}{2}(f_\delta \Delta_\theta + f_\theta \Delta_\delta) - \frac{3}{2} \kappa^2 \Delta_\delta + \alpha s + \frac{3}{2} \alpha,
\end{align*}
$$

where we have used (5) to computed the transforms of the first and second derivatives of $\delta_\delta$ and $\delta_\theta$. For $f_\delta = 1$, the first of these equations yields

$$
\Delta_\delta = \frac{1}{s^2 - 1},
$$

which is the Laplace transform of $\exp(\tau)$. If we take $\alpha = (1 + \kappa^2)^{-1}$ and substitute (9) in the second equation of (8), we get

$$
\Delta_\theta = \frac{1}{s^2 - 1} \times \frac{1}{1 + \kappa^2} = \frac{\Delta_\delta}{1 + \kappa^2},
$$

which leads to the well-known solution $\Delta_\theta = \delta_\delta/(1 + \kappa^2)$. Subsequently we will present solutions only for $\alpha = 1$. In this case, equations (8) yield

$$
\Delta_\delta = \frac{(s^2 + 1)(\kappa^2 + s^2 + 1)}{(s^2 + 1)(\kappa^2 + s^2 + 1) - 2f_\delta \kappa^2},
$$

and

$$
\Delta_\theta = \frac{s^2 + \frac{1}{2}s}{\frac{1}{2} \kappa^2 + s^2 + \frac{1}{2}s} \Delta_\delta.
$$

Before solving these equations for any value of $f_\delta$ in the range $0–1$, it is instructive to examine the solutions for the special values $f_\delta = 1$ and $0$.

3.1 Case I: $f_\delta = 1$

In this case, $\Delta_\delta = (s - 1)^{-1}$ and equation (12) can be written in the form

$$
\Delta_\theta = \frac{s^2 + \frac{1}{2}s}{(s - 1)(s - 1)/(s - s_s)},
$$

where $s_s$ are the roots of $3\kappa^2/2 + s^2 + s/2$. They are given by $s_s = -\frac{1}{4}(1 \pm \chi)$; $\chi^2 = 1 - 24\kappa^2$.

We will deal with the case $\chi^2 = 0$ at the end of this subsection. For $\chi^2 \neq 0$, all three poles of $\Delta_\theta$ are simple, and so, by (7), its inverse

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The first term is the solution given in the previous section for $\alpha = 1/(1 + \kappa^2)$. The maximum value $\chi^2$ attains is unity, so the second and third terms are always decaying. If $\chi^2 < 0$, then (15) gives the solution

$$
\delta_b = \exp(\tau) + \frac{1}{1 + \kappa^2} \times \frac{\kappa^2}{1 + \kappa^2} [(s - 1) \exp(s - \tau) - (s + 1) \exp(s + \tau)].
$$

For $\chi^2 > 0$, the roots $s_{\pm}$ are real and the result is

$$
\delta_b = \exp(\tau) + \frac{1}{2\chi} \times \frac{\kappa^2}{1 + \kappa^2} [(\chi - 5) \exp(-\frac{\chi}{4}\tau) + (\chi + 5) \exp(\frac{\chi}{4}\tau)] \exp(-\frac{\tau}{4}).
$$

The first term is the solution given in the previous section for $\alpha = 1/(1 + \kappa^2)$. The maximum value $\chi^2$ attains is unity, so the second and third terms are always decaying. If $\chi^2 < 0$, then (15) gives the solution

$$
\delta_b = \exp(\tau) + \frac{1}{1 + \kappa^2} \times \frac{\kappa^2}{1 + \kappa^2} \times [5 \sin(\frac{\chi}{4}\tau) + \chi \cos(\frac{\chi}{4}\tau)] \exp(-\frac{\tau}{4}).
$$

where $\chi$ is the imaginary part of $\chi$. The solution shows an oscillatory behaviour with a period of $16\pi/\chi$. The envelope of these oscillations decays as $\exp(-\tau/4) \propto \tau^{-1/6}$.

We deal now with the case $\chi^2 = 0$, which occurs for $\kappa^2 = 1/24$. Here special care is needed, because $s_+ = s_- = -1/4$. However, the contribution of the pole at $s = -1$ to the Bromwich integral remains unchanged, and the contribution of the second-order pole at $s = -1/4$ is simply the first derivative of $\exp(s\tau)(s + 1/4)^2 \delta_b$ at $s = -1/4$. The result is

$$
\delta_b = \frac{24}{25} \exp(\tau) + \frac{1}{20} (\tau + \frac{4}{5}) \exp(-\frac{\tau}{4}).
$$

The first term on the left is the familiar $\delta_b/(1 + \kappa^2)$ evaluated at $\kappa^2 = 1/24$. The expression can also be derived by taking the limit $\chi^2 \to 0$ in either (16) or (17).

In the limit $\tau \to \infty$, the ratio $\delta_b/\delta_s$ is $1/(1 + \kappa^2)$.

### 3.2 Case II: $f_a = 0$

This is equivalent to ignoring the gravity of the dark matter. Of course, here only the behaviour of the perturbation in the baryons is relevant, since the dark matter plays no role. However, for the sake of completeness and comparison with other situations, we will solve for the dark matter fluctuations as well.

We first find the solution for $\delta_s$. If $f_a = 0$, we can express (11) in terms of $s_{\pm}$, the roots of $3\kappa^2/2 + s^2 + s/2 - 3/2$, as

$$
\Delta_s = \left(\frac{s + \frac{1}{2}}{s + \frac{1}{2}}\right)^{\sqrt{3\kappa^2/2 + s^2 + s/2 - 3/2}}.
$$

where

$$
s_{\pm} = -\frac{1}{2} (1 \pm \chi); \quad \chi^2 = 25 - 24\kappa^2
$$

If $\kappa \neq 1$, then the function $\Delta_s$ has three simple poles at $s = 0$, $s_-$, and $s_+$. So, for $\kappa \neq 1$ and $\chi^2 > 0$, the Bromwich integral yields

$$
\delta_s = \frac{\kappa^2}{1 - \kappa^2} \left[ 2 \exp(-\frac{\tau}{4}) - \tau \right] + \frac{1}{2\chi} \times \frac{\kappa^2}{1 - \kappa^2} \times [(\chi - 5) \exp(-\frac{\chi}{4}\tau) + (\chi + 5) \exp(\frac{\chi}{4}\tau)] \exp(-\frac{\tau}{4}).
$$

The expression for $\delta_b$ is

$$
\delta_b = \frac{1}{2\chi} \left[ (\chi - 5) \exp(-\frac{\chi}{4}\tau) + (\chi + 5) \exp(\frac{\chi}{4}\tau) \right] \exp(-\frac{\tau}{4}).
$$

When $\kappa = 1$, the solution can be found either by taking the limit $\kappa \to 1$ in the previous two expressions or by direct evaluation of the Bromwich integral with two poles of order two at $s = 0$ and $-1/2$. The result is

$$
\delta_s = -27 + 28 \exp(-\frac{\tau}{2}) + 3\tau [3 + 2 \exp(-\frac{\tau}{2})],
$$

and

$$
\delta_b = 3 - 2 \exp(-\frac{\tau}{2}).
$$

This implies that $\delta_s$ grows linearly with $\tau$ at late times, while $\delta_b$ reaches an asymptotic value of 3.

The oscillatory behaviour of $\delta_b$ and $\delta_s$ appears when $\chi^2 < 0$, i.e., for $\kappa^2 > 25/24$. The expressions in this case can be obtained by replacing $\chi$ with $i\chi$ in (21) and (22). For $\kappa^2 > 25/24$, the solution coincides with that given in Padmanabhan (1993).

In the limit $\tau \to \infty$, the ratio $\delta_b/\delta_s$ is $1 - \kappa^2$ for $\kappa \leq 1$, and zero otherwise.

### 3.3 Case III: $0 < f_a < 1$

Again we first derive $\delta_s$. The denominator and numerator in (11) do not have any common roots. Then the poles of $\Delta_s$ are the roots of the denominator. We find these roots as follows. Denote $y(s) = s^2 + s/2 - 3/2$ and equate the denominator to zero to obtain

$$
y^2 + \frac{1}{2} y(1 + \kappa^2) + \frac{9}{8} f_a \kappa^2 = 0,
$$

where we have used $1 - f_a = f_b$. This equation is satisfied for the following values of $y$,

$$
y_{p,m} = -\frac{2}{3} (1 + \kappa^2)(1 \pm \Xi); \quad \Xi^2 = 1 - 4f_b \left(\frac{\kappa^2}{1 + \kappa^2}\right).$$

where the subscripts $p$ and $m$ correspond to the plus and minus sign, respectively. So the roots of the denominator in (11) are the values of $s$ which make $y(s) = y_{p,m}$. Let $s_{p,\pm}$ and $s_{m,\pm}$ be the roots of $y(s) = y_{p,m} = 0$ and $y(s) = y_{m,m} = 0$, respectively. They are given by

$$
s_{m,\pm} = -\frac{1}{2} (1 \pm \chi_m); \quad \chi_m^2 = 25 - 12(1 + \kappa^2)(1 + \Xi),
$$

$$
s_{p,\pm} = -\frac{1}{2} (1 \pm \chi_p); \quad \chi_p^2 = 25 - 12(1 + \kappa^2)(1 - \Xi).
$$

Excluding the values $f_a = 0$ and 1, which we have considered in the previous subsections, we have $0 < \Xi^2 < 1 - f_a$. This ensures that all four roots, $s_{m,\pm}$ and $s_{p,\pm}$, are distinct. Also, $1 < \chi_m^2 < 25$ for any $\kappa$, so the roots $s_{m,\pm}$ are real. On the other hand, $0 < \chi_m^2 < 1$ when $\kappa^2 < 25/(600 - 576f_a)$, and negative otherwise. So $s_{m,\pm}$ can be complex.

For $\chi_m^2 > 0$, the Bromwich integral yields

$$
\delta_s = \frac{24\kappa^2 + \chi_p^2 - 1}{2\kappa_p(\chi_p^2 - \chi_m^2)} \left[ (\chi_p - 5) \exp\left(-\frac{\chi_p}{4}\tau\right) + (\chi_p + 5) \exp\left(\frac{\chi_p}{4}\tau\right) \right] \exp\left(-\frac{\tau}{4}\right),
$$

where the second term on the right-hand side is obtained by interchanging $\chi_p$ and $\chi_m$ in the first term. This result can be extended to $\chi_m^2 < 0$ by writing $\chi_m = i\chi_m$, where $\chi_m$ is real.

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Using (12), we similarly obtain $\delta_b$:

$$
\delta_b = \frac{\chi_p^2 - 1}{2\chi_p(\chi_p^2 - \chi_m^2)} \left[ (\chi_p - 5) \exp\left(-\frac{\chi_p}{4}\tau\right) + (\chi_p + 5) \exp\left(\frac{\chi_p}{4}\tau\right) \right] \exp\left(-\frac{\tau}{4}\right) + (\chi_p - \chi_m).
$$

(28)

To visualize these solutions, we show in Fig. 1 the density evolution for various values of $f_x$ and $\kappa$. The dark matter curve for $f_x = 0.9$ is very close to $\exp(\tau)$. For $\kappa = 5$ the baryonic perturbations for both values of $f_x$ show oscillations with similar period and amplitude. This is simply because for high $\kappa$, the evolution is mainly dictated by pressure forces.

In Fig. 2 the solid lines represent the ratio $\delta_b(\kappa)/\delta_1(\kappa)$ as a function of $\kappa$ at different $\tau$ designated in the plot by the redshift, $z$. The initial conditions are satisfied at $z = 6$, and $f_x = 0.9$ was taken. Also plotted, as the dotted line in each panel, is the function $1/(1 + \kappa^2)$, which represents the limiting solution as $\tau \to \infty$. The analytic curves show more oscillations as they get closer to the limiting solution, when the redshift is decreased.

In many applications (e.g. Bi et al. 1992; Bi & Davidsen 1997; Nusser & Haehnelt 2000) the limiting ratio $1/(1 + \kappa^2)$ is often used to filter the mass power spectrum in order to generate density fluctuations in the gas. As was pointed out by Gnedin & Hui (1998), this may lead to a significant bias in statistics of the gas density. As an illustration of this bias, we compute the rms values...

**Figure 1.** Curves of $\delta_b(\tau)$ (solid lines) and $\delta_b(\tau)$ (dotted) for various values of $f_x$ and $\kappa = k/k_J$.

**Figure 2.** Ratios of baryonic to dark matter density as a function of $\kappa$ at different times, indicated by the redshift, $z$, in each panel. The solid lines are the analytic solutions obtained with $f_x = 0.9$ and where the initial conditions are satisfied at $z = 6$. The dotted line in each panel shows $1/(1 + \kappa^2)$, the ratio corresponding to the limiting solution.
of the gas density by filtering a scale-free mass power spectrum of slope $n = -2.5$ is assumed.

of the gas density by filtering a scale-free mass power spectrum of slope $n = -2.5$, with the ratio $(\delta_b/\delta_\xi)^2$ given from the analytic, and the limiting solutions, respectively. Fig. 3 shows the ratio of the former to the latter rms value. As in the previous figure, $f_x = 0.9$ and initial conditions are satisfied at $z = 6$. Using the filter $1/(1 + \kappa^2)$ can seriously underestimate the amplitude of gas density fluctuations. Only when we approach $z = 0$ does the ratio get close to unity.

4 SUMMARY

We have found analytic solutions to the linear equations governing the evolution of baryonic and dark matter under four assumptions. First, the Universe is flat without cosmological constant. Second, sudden reionization of the IGM occurs. Third, the temperature of the low-density IGM drops as $1/a$, so that the comoving Jeans length is time-independent. Fourth, before reionization the IGM is cold, and the baryonic and dark matter trace the same density and velocity fields.

Of these assumptions, only the fourth has a physical basis, at least before any heating has occurred and when the IGM temperature is low. This is also the only assumption for which, if changed, the equations can still be readily solved by Laplace transformation. Unfortunately, relaxing any of the other assumptions complicates the analytic treatment of equations (3) by means of Laplace transformation. For example, suppose that the Jeans wavenumber changes with time according to $a^\beta$. Then the Laplace transformation of the term involving $k_j$ will yield $\Delta_b$ at $(s + \beta)$, while the other terms involve $\Delta_b(s)$.

Yet the solutions can be useful for semi-analytic modelling of the IGM. They offer a convenient improvement over the commonly used filter $1/[1 + (k/k_j)^2]$ for generating gas fluctuations associated with a given mass density field.

The analytic solutions presented here were verified by a comparison with the solutions obtained by numerical integration of equations (1). All numerical and analytic solutions agreed up to the numerical accuracy.

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