SCATTERING AND BLOWUP FOR \( L^2 \)-SUPERCRITICAL AND \( \dot{H}^2 \)-SUBLTICRITICAL BIHARMONIC NLS WITH POTENTIALS

QING GUO, HUA WANG AND XIAOHUA YAO

Abstract. We mainly consider the focusing biharmonic Schrödinger equation with a large radial repulsive potential \( V(x) \):

\[
\begin{cases}
    iu_t + (\Delta^2 + V)u - |u|^{p-1}u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\
    u(0,x) = u_0(x) \in H^2(\mathbb{R}^N),
\end{cases}
\]

If \( N > 8, 1 + \frac{4}{N} < p < 1 + \frac{8}{N} \) (i.e. the \( L^2 \)-supercritical and \( \dot{H}^2 \)-subcritical case), and \( \langle x \rangle^\beta (|V(x)| + |\nabla V(x)|) \in L^\infty \) for some \( \beta > N + 4 \), then we firstly prove a global well-posedness and scattering result for the radial data \( u_0 \in H^2(\mathbb{R}^N) \) which satisfies that

\[
M(u_0)^{\frac{2-p}{p-1}} E(u_0) < M(Q)^{\frac{2-p}{p-1}} E_0(Q) \quad \text{and} \quad \|u_0\|_{L^2}^{\frac{2-p}{p-1}} \|H^\frac{1}{2} u_0\|_{L^2} < \|Q\|_{L^2}^{\frac{2-p}{p-1}} \|\Delta Q\|_{L^2},
\]

where \( x_i = \frac{2}{p-1} \frac{1}{\sqrt{N}} \in (0,2) \), \( H = \Delta^2 + V \) and \( Q \) is the ground state of \( \Delta^2 Q + (2 - x_i) Q - |Q|^{p-1} Q = 0 \).

We crucially establish full Strichartz estimates and smoothing estimates of linear flow with a large potential \( V \), which are fundamental to our scattering results.

Finally, based on the method introduced in [2, T. Boulenger, E. Lenzmann, Blow up for biharmonic NLS, Ann. Sci. Éc. Norm. Supér., 50(2017), 503-544], we also prove a blow-up result for a class of potential \( V \) and the radial data \( u_0 \in H^2(\mathbb{R}^N) \) satisfying that

\[
M(u_0)^{\frac{2-p}{p-1}} E(u_0) < M(Q)^{\frac{2-p}{p-1}} E_0(Q) \quad \text{and} \quad \|u_0\|_{L^2}^{\frac{2-p}{p-1}} \|H^\frac{1}{2} u_0\|_{L^2} > \|Q\|_{L^2}^{\frac{2-p}{p-1}} \|\Delta Q\|_{L^2}.
\]

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1. Introduction

In this paper, we consider the biharmonic NLS with a potential (BNLS)

\begin{equation}
\begin{cases}
  iu_t + Hu + \lambda|u|^{p-1}u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
  u(0, x) = u_0(x) \in H^2(\mathbb{R}^N),
\end{cases}
\end{equation}

where \( u : I \times \mathbb{R}^N \to \mathbb{C} \) is a complex-valued function, \( H = H_0 + V, H_0 = \Delta^2, V : \mathbb{R}^N \to \mathbb{R}, \lambda = \pm 1 \) and \( 1 < p < \infty \). The defocusing regime corresponds to the case \( \lambda = +1 \), and the focusing regime to the case \( \lambda = -1 \). The biharmonic Schrödinger equation has been introduced by Karpman [20] and Karpman and Shagalor [21] to take into account the role of small fourth order dispersion terms in the propagation of intense laser beams in a bulk medium with kerr nonlinearity. The equation (1.1) has two important conservation laws in the energy space \( H^2(\mathbb{R}^N) \): The mass is defined by

\begin{equation}
M(u) = \int_{\mathbb{R}^N} |u(x)|^2 \, dx,
\end{equation}

and the energy is defined by

\begin{equation}
E(u) = E_V(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u(x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx + \frac{\lambda}{p+1} \int_{\mathbb{R}^N} |u(x)|^{p+1} \, dx.
\end{equation}

When \( V \) vanishes, we replace \( E(u) \) by \( E_0(u) \). Moreover, you can easily see that the equation (1.1) without potentials is invariant under the scaling transformation \( u(x, t) \to \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t) \), which also leaves the norm of the homogeneous Sobolev space \( H^s(\mathbb{R}^N) \) invariant, where \( s_c = \frac{N}{2} - \frac{4}{p-1} \). So we call that the equation (1.1) is energy subcritical for \( n \leq 4 \) or \( p < 1 + \frac{4}{N-4} \) when \( n \geq 5 \), which correspond to \( s_c < 2 \). Energy-criticality appears with the power \( p = 1 + \frac{4}{N-4} \), corresponding to \( s_c = 2 \), and mass-criticality with power \( p = 1 + \frac{8}{N} \) when \( s_c = 0 \).

Let’s recall some progress on the global well-posedness and scattering to (1.1) when \( V = 0 \). Fibich, Ilan and Papanicolaou [12] describe various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. Segata in [37] proved scattering for the cubic nonlinearity in \( \mathbb{R} \); while in higher dimensions \( 5 \leq N \leq 8 \), the scattering results in \( H^2(\mathbb{R}^N) \) were obtained by Pausader in [32], which was extended by Miao, Xu and Zhao in [30] to a low regularity space \( H^r(\mathbb{R}^N) \) with some \( s < 2 \) for \( 5 \leq N \leq 7 \). Global well-posedness and scattering for the energy critical case were considered by Miao, Xu and Zhao in [28], [29] and Pausader in [31] and [33]. In [34], Pausader and Shao proved that scattering for the mass-critical fourth-order Schrödinger equation holds true in \( L^2(\mathbb{R}^N) \) in high dimensions \( N \geq 5 \). As for the mass-supercritical and energy-subcritical case, that is with the power \( 1 + \frac{s_c}{2} < p < 1 + \frac{s_c}{4} (N \geq 5) \), the scattering results for the defocusing case (\( \lambda = +1 \)) in the energy space could be obtained using the argument in Lin and Strauss [27] as discussed in [33], also in [5]. The same results were established in [35] for low dimensions \( 1 \leq N \leq 4 \) and \( 1 + \frac{s_c}{2} < p < \infty \).

While for the corresponding focusing case (\( \lambda = -1 \)), the first author [14] recently obtained a mass-supercritical and energy-subcritical scattering result with radial initial data for all dimensions. Note that when \( \lambda = -1 \), one cannot hope to get a similar global result as in [33]. Indeed, the existence of a nontrivial solution of the elliptic equation

\begin{equation}
\Delta^2 Q + (2 - s_c)Q - |Q|^{p-1}Q = 0
\end{equation}
which we refer to as the ground state \( Q \in H^2(\mathbb{R}^N) \), can be obtained by similar method to that used in [2]. We then conclude that solitary waves \( u(x, t) = e^{i(2-\frac{N}{2})t}Q(x) \) do not scatter. One can refer to [12] for some similar results. The first author obtained the following result of scattering for the solution of (1.1) with \( V = 0 \) and radial data, which would complement the recent analysis on blowup theory by Boulenger and Lenzmann [2].

**Theorem 1.1.** (See [2, 14]) Assume that \( V = 0 \), \( \lambda = -1 \), \( 1 + \frac{2}{N} < p < 1 + \frac{8}{N} \) (when \( 2 \leq N \leq 4 \), \( 1 + \frac{8}{N} < p < \infty \)). Let \( u_0 \in H^2(\mathbb{R}^N) \) be radial and \( u \in C(I; H^2(\mathbb{R}^N)) \) be the corresponding solution to (1.1) with maximal forward time interval of existence \( I \subset \mathbb{R} \). Then

(i) If

\[
M(u_0) \frac{2}{N} E_0(u_0) < M(Q) \frac{2}{N} E_0(Q),
\]

and

\[
\|u_0\|_{L^2(\mathbb{R}^N)} \|\Delta u_0\|_{L^2(\mathbb{R}^N)} < \|Q\|_{L^2(\mathbb{R}^N)} \|\Delta Q\|_{L^2(\mathbb{R}^N)},
\]

where \( Q \) is the solution of (1.4), then \( I = (-\infty, +\infty) \), and \( u \) scatters in \( H^2(\mathbb{R}^N) \). That is, there exists \( \phi_\ast \in H^2(\mathbb{R}^N) \) such that \( \lim_{t \to \pm \infty} \|u(t) - e^{i\theta \hat{h}} \phi_\ast\|_{H^2(\mathbb{R}^N)} = 0 \).

(ii) Either if \( E_0(u_0) < 0 \) or if \( E_0(u_0) \geq 0 \), assume that (1.5) and

\[
\|u_0\|_{L^2(\mathbb{R}^N)} \|\Delta u_0\|_{L^2(\mathbb{R}^N)} > \|Q\|_{L^2(\mathbb{R}^N)} \|\Delta Q\|_{L^2(\mathbb{R}^N)}
\]

hold, then the solution \( u \in C([0, T); H^2(\mathbb{R}^N)) \) of (1.1) blows up in finite time, i.e., there exists some \( 0 < T < +\infty \) such that \( \lim_{t \uparrow T} \|\Delta u(t)\|_{L^2} = +\infty \).

Motivated by these works, we naturally hope to extend Theorem 1.1 above in Q. Guo [14] and Boulenger and Lenzmann [2] to the case with a potential \( V \), that is, to get the scattering and blow-up results in the energy space for the focusing BNLS\(_V\) (1.1). For the end, however, there are several crucial obstacles due to the existence of potential \( V \).

Firstly, we need to establish the Strichartz estimates of linear group \( e^{it\Delta^2} \), which are fundamental to the nonlinear equation BNLS\(_V\) (1.1). Recall that in the free biharmonic operator \( \Delta^2 \), Ben-Artzi, Koch and Saut [1] had proven the following sharp kernel estimate,

\[
|D^\alpha I_0(t, x)| \leq C|t|^{-\frac{N+|\alpha|}{2}} \left(1 + |t|^{-1/4}|x|\right)^{|\alpha|-N/2}, \quad t \neq 0, \quad x \in \mathbb{R}^N,
\]

where \( I_0(t, x) \) is the kernel of \( e^{it\Delta^2} \). The above estimate implies the \( L^1 \to L^{\infty} \)-estimate of \( e^{it\Delta^2} \), namely

\[
\|D^\alpha e^{it\Delta^2}\|_{L^1(\mathbb{R}^N) \to L^{\infty}(\mathbb{R}^N)} \leq C|t|^{-\frac{N+|\alpha|}{2}}, \quad t \neq 0, \quad |\alpha| \leq N.
\]

Hence the endpoint Strichartz estimates for the free group \( e^{it\Delta^2} \) can be established by using the \( L^1 \to L^{\infty} \) estimate (1.10) and Keel-Tao arguments (See [23]). For instance, by (1.10) we can establish that for any \( S\)-admissible pairs \((q, r)\) and \((a, b)\), and any \( s \geq 0 \),

\[
\|\nabla^s u\|_{L^2(I; L^r)} \leq C \left(\|\nabla^s t^{\frac{2}{r} - 2} u_0\|_{L^2} + \|\nabla^s t^{\frac{2}{r} - 2} h\|_{L^2(I; L^r)}\right),
\]

and so on, where \( u \) is the solution given by

\[
u(t) = e^{it\Delta^2}u_0 + \int_0^t e^{i(t-s)\Delta^2}h(s)ds.
\]
Indeed, one can see more refined Strichartz estimates with regularity in Section 2 below. For the fourth-order Schrödinger operator \( H = \Delta^2 + V \), it is much difficult to establish the similar kernel estimate (1.9) for \( e^{itH} \), and hard to prove the \( L^1 \to L^\infty \)-estimate (1.10). In order to obtain Strichartz estimates of \( e^{itH} \), we will use Jensen-Kato decay estimate and local decay estimate of \( H \) to overcome the difficulties caused by the potential \( V \).

Very recently, Feng, Soffer and Yao [9] have firstly established the following Jensen-Kato type decay estimate of the fourth order Schrödinger operator \( H = \Delta^2 + V \) (see Lemma 2.2 below):

\[
\| |(x)^{-\sigma} e^{-itP_{ac}}| \|_{L^2(R^N), L^2(R^N)} \leq C(t)^{-N/4}, \quad t \in \mathbb{R}, \quad \sigma > N/2 + 2,
\]

under the assumptions that \( (x)^{\beta} V(x) \in L^\infty(R^N) \) for some large \( \beta > 0 \), and \( H \) has no positive embedded eigenvalues and 0 is not an eigenvalue nor resonance of \( H \). Here \( P_{ac} \) denotes the projection onto the absolutely continuous spectrum space of \( H \), which removes the eigenstates and is necessary to dispersive estimate of \( e^{itH} \). We remark that Jensen-Kato type estimates is original in Jensen and Kato’s famous work [17] for Schrödinger operator \( -\Delta + V \), which is later plays key roles in many important problems, such as \( L^p \)-decay estimates of Schrödinger operator in [19], Soliton stability of NLS in [4], and so on.

In this paper, we will used Kato-Jensen estimates (1.13) to establish several useful Strichartz estimates and smoothing estimates for the linear solution \( e^{itH} \) of (1.1), under the helps of some further conditions on \( V \) and the restriction of dimension \( N \). Here, we do not attempt to express these specific Strichartz estimates with potential. One can see Proposition 2.3, Proposition 2.5 and Proposition 2.6 in Section 2 below. Finally, we mention that some Strichartz type estimates obtained here are independent of the scaling \( V(x) \to V_r = \frac{1}{r^2} V(\frac{x}{r}) \) for any given \( r > 0 \), which is very important to establish linear profile decomposition with a potential (see Proposition 6.3 below).

Our first scattering result in this paper can be stated as follows:

**Theorem 1.2.** Let \( V \) be a radial real \( C^1 \)-function of \( \mathbb{R}^N \) satisfying that \( x \cdot \nabla V \leq 0 \) and

\[
|V(x)| + |\nabla V(x)| \leq C(1 + |x|)^{-\beta}
\]

for some \( \beta > N + 4 \). Suppose that \( \lambda = -1, N > 8, 1 + \frac{8}{N} < p < 1 + \frac{8}{N-4}, \ u_0 \in H^2(\mathbb{R}^N) \) is radial and \( u \in C(I; H^2(\mathbb{R}^N)) \) is the corresponding solution to (1.1) with maximal forward time interval of existence \( I \subset \mathbb{R} \). If

\[
M(u_0) \xrightarrow{\text{w}} E(u_0) < M(Q) \xrightarrow{\text{w}} E_0(Q),
\]

and

\[
\|u_0\|_{L^2(\mathbb{R}^N)} \|H^{1/2} u_0\|_{L^2(\mathbb{R}^N)} < \|Q\|_{L^2(\mathbb{R}^N)} \|\Delta Q\|_{L^2(\mathbb{R}^N)}
\]

where \( Q \) is the solution of (1.4) and \( H = \Delta^2 + V \), then \( I = (-\infty, +\infty) \), and \( u \) scatters in \( H^2(\mathbb{R}^N) \). That is, there exist \( \phi_\pm \in H^2(\mathbb{R}^N) \) such that

\[
\lim_{t \to \pm\infty} \|u(t) - e^{itH} \phi_\pm\|_{H^2(\mathbb{R}^N)} = 0.
\]

Some comments on the conditions on \( V \) and results of Theorem 1.2 as follows:

**Remark 1.3.** There exist a great number of potentials \( V \) satisfying the conditions in Theorem 1.2 above. For instance, simply, for any \( \sigma > N/2 + 2 \), we can take

\[
V(x) = \frac{C}{(1 + |x|^2)^\sigma}, \quad C > 0.
\]
The radial requirement of $V$ comes from the focusing case ($\lambda = -1$), which can be removed in the following defocusing case ($\lambda = 1$). The decay index $\beta$ and restriction of dimension $N$ are technical and at present not optimal, which surely can be improved further. The repulsive condition $(x \cdot \nabla)V(x) \leq 0$ plays an important role in the spectrum of $H = \Delta^2 + V$ and Morawetz estimates in this paper. In particular, the repulsive condition can be used to show that $H$ has no any eigenvalue in $\mathbb{R}$, see Section 2 below.

**Remark 1.4.** Note that both $(x \cdot \nabla)V(x) \leq 0$ and $\lim_{x \to \infty} V(x) = 0$, imply $V(x) \geq 0$. In fact, it can be easily concluded by the following integral

$$V(x) = -\int^\infty_1 \frac{d}{ds}(V(sx))ds \geq 0, \quad x \neq 0,$$

where $\frac{d}{ds}(V(sx)) = \frac{1}{s}(sx \cdot \nabla)V(sx) \leq 0$. Thus, $H = \Delta^2 + V$ is a nonnegative self-adjoint operator, and

$$\|H^\dagger u_0\|_{L^2}^2 = \langle Hu_0, u_0 \rangle = \int_{\mathbb{R}^N} |\Delta u_0|^2 dx + \int_{\mathbb{R}^N} V|u_0|^2 dx.$$  

In particular, if $0 \leq V \in L^\frac{N}{2}(\mathbb{R}^N), N > 4$, then we have

$$\|H^\dagger f\|_{L^2} \sim \int_{\mathbb{R}^N} |\Delta u_0|^2 dx = \|\Delta u_0\|_{L^2}^2.$$  

Indeed, since $V \geq 0$, clearly $\|\Delta u_0\|_{L^2}^2 \leq \|H^\dagger u_0\|_{L^2}^2$. On the other hand, since $V \in L^\frac{N}{2}$, it follows from Hölder inequality and Sobolev embedding that

$$\|H^\dagger u_0\|_{L^2}^2 \leq \|\Delta u_0\|_{L^2}^2 + \|V\|_{L^\frac{N}{2}} \|u_0\|_{L^\infty}^2 \leq \|\Delta u_0\|_{L^2}^2.$$  

Using the Morawetz estimates, Feng, the second and third authors [10] considered the small potential $V$ when $N \geq 7$ for the defocusing BNLS$_V$ (1.1) with non-radial initial data. Based on the new Strichartz estimates with large potentials, in this paper we can extend the result in [10] to the large potential case in $N > 8$ (As the proof is almost the same as the one in [10], it will be omitted here).

**Theorem 1.5.** Let $\lambda = +1$, $N > 8$, $1 + \frac{\beta}{N} < p < 1 + \frac{8}{N-4}$ and $V$ be a real $C^1$-function of $\mathbb{R}^N$ satisfying that $x \cdot \nabla V \leq 0$ and

$$|V(x)| + |\nabla V(x)| \leq C(1 + |x|)^{-\beta}$$

for some $\beta > N + 4$. Assume that $u_0 \in H^2(\mathbb{R}^N)$ and $u \in C(I; H^2(\mathbb{R}^N))$ be the corresponding solution to (1.1) with maximal forward time interval of existence $I \subset \mathbb{R}$. Then $I = (-\infty, +\infty)$, and $u$ scatters in $H^2(\mathbb{R}^N)$.

Finally, we turn to state our blow-up result. Note that Boulenger and Lenzmann in [2] have utilized the (localized) Riesz bivariance to get blow-up for biharmonic NLS, then we can apply their method to the Biharmonic NLS equation with certain large potential $V$. For the end, in the following blowup result, we assume that $V$ is a nonnegative radial real $C^1$-function of $\mathbb{R}^N$ satisfying $|\nabla V(x)| \leq C(1 + |x|)^{-1}$, and set

$$W(x) := 4V(x) + x \cdot \nabla V(x) = W_+(x) - W_-(x),$$

where $W_+(x)$ denote the positive part and negative part of $W(x)$.
Theorem 1.6. Suppose that \( \lambda = -1 \), \( N \geq 5 \), \( 1 + \frac{2}{N} < p < 1 + \frac{N}{N-4} \), \( u_0 \in H^{\frac{N}{2}}(\mathbb{R}^N) \) is radial and \( u \in C(I; H^2(\mathbb{R}^N)) \) is the corresponding solution to (1.1) with maximal forward time interval of existence \( I \subset \mathbb{R} \). Let \( W \geq 0 \) or \( ||W_-||_{\mathcal{L}^q} \) be sufficiently small. If \( u_0 \) furthermore satisfies one of the following two conditions: (i) \( E(u_0) < 0 \); and (ii) \( E(u_0) \geq 0 \),

\[
M(u_0)^{\frac{2}{N-4}} E(u_0) < M(Q)^{\frac{2}{N-4}} E_0(Q),
\]

and

\[
||u_0||_{L^q(\mathbb{R}^N)} ||H^\frac{N}{2} u_0||_{L^2(\mathbb{R}^N)} > ||Q||_{L^q(\mathbb{R}^N)} ||\Delta Q||_{L^2(\mathbb{R}^N)},
\]

where \( Q \) is the solution of (1.4) and \( H = \Delta^2 + V \), then the solution \( u \in C([0, T); H^2(\mathbb{R}^N)) \) of (1.1) blows up in finite time, i.e., there exists some \( 0 < T < +\infty \) such that \( \lim_{t \to T} ||\Delta u(t)||_{L^2} = +\infty \).

Remark 1.7. That \( ||W_-||_{\mathcal{L}^q} \) is sufficiently small, means that there exists some constant \( \delta > 0 \) such that \( ||W_-||_{\mathcal{L}^q} \leq \delta \). Clearly, if \( V \) is suitably small, then the whole \( W \) is small. Nevertheless, the smallness of \( V \) is not absolutely necessary to the blowup result above. Combining with Theorem 1.6, we remark that the condition

\[
||u_0||_{L^q(\mathbb{R}^N)} ||H^\frac{N}{2} u_0||_{L^2(\mathbb{R}^N)} < ||Q||_{L^q(\mathbb{R}^N)} ||\Delta Q||_{L^2(\mathbb{R}^N)},
\]

is sharp for scattering result in Theorem 1.2.

In the sequel, we only consider the case \( \lambda = -1 \). This present paper is organized as follows. We fix notations at the end of Section 1. In Section 2, We establish some Strichartz type estimates, upon which we obtain linear scattering. In Section 3, we establish local theory, the small data scattering and the perturbation theory. The variational structure of the ground state of an elliptic problem is given in Section 4. In Section 5 we prove a dichotomy proposition of global well-posedness versus blowing up, which yields the comparability of the total energy and the gradient. The concentration compactness principle is used in Section 6 to give a critical element, which yields a contradiction through a virial-type estimate in Section 7, concluding the proof of Theorem 1.2. In Section 8, we prove the blow-up results, based on the argument of Boulenger and Lenzmann [2].

Notations:

we fix notations used throughout the paper. In what follows, we write \( A \leq B \) to signify that there exists a constant \( C \) such that \( A \leq CB \). And we denote \( A \sim B \) when \( A \leq B \leq A \).

Let \( L^q = L^q(\mathbb{R}^N) \) be the usual Lebesgue spaces, and \( L^q_t L^r_x \) or \( L^q(I, L^r_x) \) be the space of measurable functions from an interval \( I \subset \mathbb{R} \) to \( L^r_x \) whose \( L^q_t L^r_x \) norm \( || \cdot ||_{L^q_t L^r_x} \) is finite, where

\[
||u||_{L^q_t L^r_x} = \left( \int_I ||u(t)||^q_{L^r_x} \, dt \right)^{\frac{1}{q}}.
\]

When \( I = \mathbb{R} \) or \( I = [0, T] \), we may use \( L^q_t L^r_x \) or \( L^q(I, L^r_x) \) instead of \( L^q_t L^r_x \), respectively. In particular, when \( q = r \), we may simply write them as \( L^q_t L^q_x \) or \( L^q(I, L^q_x) \), respectively.

Moreover, the Fourier transform on \( \mathbb{R}^N \) is defined by \( \hat{f}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} f(x) \, dx \). For \( s \in \mathbb{R} \) and \( \sigma \in \mathbb{R} \), define the inhomogeneous weighted Sobolev space by

\[
H^s_\sigma(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N) : ||(x)^{\sigma} (i\nabla)^s f||_{L^2(\mathbb{R}^N)} < \infty \}.
\]
and the homogeneous weighted Sobolev space by
\[ H^s_0(\mathbb{R}^N) = \{ f \in S'(\mathbb{R}^N) : \|\nabla |\nabla|^s f\|_{L^2(\mathbb{R}^N)} < \infty \}, \]
where \( S'(\mathbb{R}^N) \) denotes the space of tempered distributions and \( \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \). When \( \sigma = 0 \), \( H^s(\mathbb{R}^N) (H^s_0(\mathbb{R}^N)) \) denotes the space \( H^s_0(\mathbb{R}^N) (H^s(\mathbb{R}^N)) \) and when \( s = 0 \), \( L^2(\mathbb{R}^N) \) denotes \( H^0(\mathbb{R}^N) \).

Given \( p \geq 1 \), let \( p' \) be the conjugate of \( p \), that is \( \frac{1}{p} + \frac{1}{p'} = 1 \).

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2. Strichartz Type Estimates Associated with \( H = \Delta^2 + V \)

We start in this section with recalling the Strichartz estimates of linear bi-harmonic Schrödinger equations with \( V = 0 \). We say a pair \((q, r)\) is Schrödinger admissible, or \( S \)-admissible for short, if \( 2 \leq q, r \leq \infty \), \((q, r, N) \neq (2, \infty, 2)\) and
\[ \frac{2}{q} + \frac{N}{r} = \frac{N}{2}. \]
Also, we use the terminology that a pair \((q, r)\) is biharmonic admissible, or \( B \)-admissible for short, if \( 2 \leq q, r \leq \infty \), \((q, r, N) \neq (2, \infty, 4)\) and
\[ \frac{4}{q} + \frac{N}{r} = \frac{N}{2}. \]
We define the Strichartz norm by
\[ \|u\|_{S(\mathbb{R}^N)} := \sup_{(q, r): B\text{-admissible}} \|u\|_{L^q(I, L^r)} \]
and its dual norm by
\[ \|u\|_{S'(\mathbb{R}^N)} := \inf_{(q, r): B\text{-admissible}} \|u\|_{L^q'(I, L^r')} \]
The Strichartz estimates are stated as follows (see e.g. [33]):

Lemma 2.1. Let \( u \in C(I, H^{-4}(\mathbb{R}^N)) \) be a solution of
\[ u(t) = e^{i\Delta^2} u_0 + i \int_0^t e^{i(t-s)\Delta^2} h(s)ds. \]
Then we have
\[ \|u\|_{S(\mathbb{R}^N)} \leq C \left( \|u_0\|_{L^2} + \|h\|_{S'(\mathbb{R}^N)} \right). \]
More generally, for any \( S \)-admissible pairs \((q, r)\) and \((a, b)\), and any \( s \geq 0 \),
\[ |||\nabla^a u|||_{L^b(I, L^r)} \leq C \left( |||\nabla^{a-2} u_0|||_{L^2} + |||\nabla^{a-2} h|||_{L^{r'}(I, L')} \right). \]

Note that from Sobolev embedding inequality, the estimate (2.3) implies the estimate (2.2). Thus a direct consequence of (2.3) and the Sobolev’s inequality is that, if \( u \in C(I, H^{-4}(\mathbb{R}^N)) \) be a solution of (2.1) with \( u_0 \in H^2 \) and \( \nabla h \in L^2(I, L^{\frac{2N}{N+2}}) \), then \( u \in C(I, H^2(\mathbb{R}^N)) \) and for any \( B \)-admissible pairs \((q, r)\),
\[ \|\Delta u\|_{L^2(I, L^r)} \leq C \left( \|\Delta u_0\|_{L^2} + \|\nabla h\|_{L^2(I, L^{\frac{2N}{N+2}})} \right). \]
A key feature of (2.4) is that the second derivative of \( u \) is estimated using only one derivative of the forcing term \( h \). The same argument gives

\[
\|\nabla u\|_{L^2(I; L^2)} \leq C \left( \|\nabla u_0\|_{L^2} + \|h\|_{L^2(I; L^2)} \right).
\]

If \( u_0 \in H^s(\mathbb{R}^N) \), then we also can establish a \( H^s \)-version of the Strichartz inequality (2.2). More precisely, we introduce that a pair \((q, r)\) is \( H^s \)-biharmonic admissible and denote it by \((q, r) \in \Lambda_s \) if \( 0 \leq s < 2 \) and

\[
\frac{4}{q} + \frac{N}{r} = \frac{N}{2} - s, \quad \frac{2N}{N - 2s} \leq r \leq \frac{2N}{N - 4}.
\]

Correspondingly, we call the pair \((q', r')\) dual \( H^s \)-biharmonic admissible, denoted by \((q', r') \in \Lambda'_s\), if \((q, r) \in \Lambda_s\), and \((q', r')\) is the conjugate exponent pair of \((q, r)\). In particular, \((q, r) \in \Lambda_0\) is just a \( B \)-admissible pair, which is always denoted by \((q, r) \in \Lambda_B\).

We also define the exotic Strichartz norm by

\[
\|u\|_{S(I; H^s; I')} := \sup_{(q, r) \in \Lambda_s} \|u\|_{L^q(I; L^r)},
\]

and its dual norm by

\[
\|u\|_{S^*(I'; H^{-s}; I')} := \inf_{(q, r) \in \Lambda'_s} \|u\|_{L^{q'}(I'; L^{r'})}.
\]

Now we can infer the following \( \dot{H}^s \)-Strichartz estimates on \( I = [0, T] \):

\[
\|u\|_{S(I; \dot{H}^s; I')} = \left\| e^{it\Delta} u_0 + i \int_0^t e^{i(t-s)\Delta} h(s, x) ds \right\|_{S(I; \dot{H}^s)} \leq C \left( \|u_0\|_{\dot{H}^s} + \|h\|_{S(I; \dot{H}^s)} \right).
\]

If the time interval \( I \) is not specified, we take \( I = \mathbb{R} \). We also refer to [13, 22, 23, 33, 40, 41] for more discussion on the homogeneous and inhomogeneous type Strichartz estimates.

Next, we need to establish the Strichartz type estimates corresponding to (2.2)-(2.6) for solutions of inhomogeneous linear biharmonic Schrödinger equation with potential \( V \):

\[
\begin{cases}
\dot{u} + (\Delta^2 + V)u + h = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
u(0, x) = u_0(x) \in H^s(\mathbb{R}^N),
\end{cases}
\]

where \( N \geq 5 \). For the purpose, we now recall the following Local decay estimate, Jensen-Kato estimate and Strichartz type estimate established by Feng-Soffer-Yao [9].

**Lemma 2.2.** Let \( N \geq 5 \), \( V \) satisfy that \( (\chi^\beta V)(x) \in L^\infty(\mathbb{R}^N) \) for some \( \beta > N + 4 \). And assume that operator \( H = \Delta^2 + V \) has no positive embedded eigenvalues and 0 is a regular point for \( H \). If \( u \) be the solution of the initial value problem (2.7), then the following estimates hold:

\[
\int_{\mathbb{R}} \| e^{-itH} P_{ac} u_0 \|_{L^2(\mathbb{R}^N)} dt \leq C \| u_0 \|_{L^2(\mathbb{R}^N)}^2, \quad \text{for} \quad \sigma > 1/2,
\]

\[
\| e^{-itH} P_{ac} u_0 \|_{L^2(\mathbb{R}^N)} \leq C (t)^{-N/4} \| u_0 \|_{L^2(\mathbb{R}^N)} \quad \text{for} \quad \sigma > N/2 + 2,
\]

\[
\| P_{ac} u \|_{S(I; L^2)} \leq C \left( \| u_0 \|_{L^2} + \| h \|_{S(I; L^2)} \right),
\]

where \( L^\sigma_\nu(\mathbb{R}^d) \) is the weighted \( L^2 \)-function space, \( P_{ac} \) is the projection on the absolutely continuous spectrum of \( H \), and \( S(I^2, I); S(I, I^2) \) are the Strichartz norm and its dual norm in Proposition 2.1, respectively.
We say that a resonance occurs at zero for $H = \Delta^2 + V$, provided there is a distributional solution $u$ of the equation $(\Delta^2 + V)u = 0$ such that for any $s > 4 - N/2$, $u \in L^2_s(\mathbb{R}^N) \setminus L^2(\mathbb{R}^N)$. We call that zero is a regular point of $H$, which means that 0 is neither eigenvalue nor resonance. Moreover, when $N > 8$, $H = \Delta^2 + V$ has no zero resonance (see [9, Remark 2.8]).

Now, based on the local decay estimates and Kato-Jensen estimates of Lemma 2.2 above, by putting the further repulsive condition on $V$ and restriction on dimension $N$, we will further establish $\dot{H}^s$-Strichartz estimates with potential and global smoothing estimate. For the purpose, we will use the following conditions:

\begin{enumerate}
\item $V$ is a real $C^1$-function of $\mathbb{R}^N$ satisfying that $x \cdot \nabla V \leq 0$ and $|V(x)| + |\nabla V| \leq C(1 + |x|)^{-\beta}$ for some $\beta > N + 4$.
\item $0$ is not a resonance of $H = \Delta^2 + V$.
\end{enumerate}

It was well-known from Virial’s argument that the repulsive condition $x \cdot \nabla V \leq 0$ makes that the operator $H = \Delta^2 + V$ has no any eigenvalue in $\mathbb{R}$ (i.e. $\sigma_p(H) = \emptyset$), e.g. see Reed and Simon [36, Theorem XIII.59] for Schrödinger operator, and Feng [11] for general operators $P(D) + V$.

Hence it follows from the assumptions on $V$ and dimension $N$ in Theorem 1.2 that the operator $H = \Delta^2 + V$ has no any eigenvalues (i.e. $\sigma(H) = \sigma_{ac}(H) = [0, \infty)$), and zero is not a resonance. Thus, $P_{ac}$ is an identity operator, and the above estimates (2.8)-(2.10) still hold true for $u$ in place of $P_{ac}u$.

**Proposition 2.3.** Let $0 \leq s < 2$, $N > 4 + 2s$, $H = \Delta^2 + V$ and $V$ satisfy the conditions $(\mathcal{C}_1)$-$(\mathcal{C}_2)$. If $u$ be the solution of the initial value problem (2.7), then we have

$$
\|u\|_{S(\dot{H}^s, L^2)} = \left\|e^{itH} u_0 + i \int_0^t e^{i(t-s)H} h(s) ds \right\|_{S(\dot{H}^s, L^2)} \leq C \left( \|u_0\|_{\dot{H}^s} + \|h\|_{S'(\dot{H}^{-s}, L^2)} \right),
$$

where $S(\dot{H}^s, L^2)$ and $S'(\dot{H}^s, L^2)$ are the exotic Strichartz norm and its dual norm in (2.6).

**Proof.** The solution $u$ to the problem (2.7) can be expressed as

$$
u(t,x) = e^{itH} u_0 + i \int_0^t e^{i(t-s)H} h(s)ds \doteq u_1(t,x) + u_2(t,x),
$$

where $u_1$ and $u_2$ may also be expressed respectively as

$$
u_1(t,x) = e^{it\Delta^2} u_0 + i \int_0^t e^{i(t-s)\Delta^2} V u_1(s) ds
$$

and

$$
u_2(t,x) = i \int_0^t e^{i(t-s)\Delta^2} V u_2(s) ds + i \int_0^t e^{i(t-s)\Delta^2} h(s) ds.
$$

We first use the Jensen-Kato type decay estimate (2.9) to control $u_1(t,x)$.

Using (2.6) and Hölder inequality successively yields that

$$
\|u_1\|_{S(\dot{H}^s, L^2)} \lesssim \left( \|u_0\|_{\dot{H}^s} + \|V u_1\|_{L^2_t L^{\infty}_x} \right)^{\frac{1}{2}}
$$

$$
\lesssim \left( \|u_0\|_{\dot{H}^s} + \|\langle \cdot \rangle^s V\|_{L^{\infty}_x} \right)^{\frac{1}{2}} \|\langle \cdot \rangle^{-s} u_1\|_{L^2_t L^2_x}.
$$
Now we aim to show that

\begin{equation}
\|(x)^{-\sigma}u_1\|_{L^2_x L^2_t} \lesssim \|u_0\|_{H^s}.
\end{equation}

Consider the Cauchy problem

\begin{equation}
\begin{aligned}
& i\dot{\phi}_t = -\Delta^2 \phi - h(t) = -H\phi + V\phi - h(t), \\
& u(0, x) = u_0(x) \in \dot{H}^s(\mathbb{R}^N).
\end{aligned}
\end{equation}

Then Duhamel formula for the solution \(\phi\) gives

\begin{equation}
\phi(t) = e^{it\Delta^2}u_0 = e^{itH}u_0 - i \int_0^t e^{i(t-s)H}V\phi(s)ds = u_1(t) - i \int_0^t e^{i(t-s)H}V\phi(s)ds,
\end{equation}

Hence

\begin{equation}
u_1(t) = \phi(t) + i \int_0^t e^{i(t-s)H}V\phi(s)ds = e^{it\Delta^2}u_0 + i \int_0^t e^{i(t-s)H}V\phi(s)ds.
\end{equation}

By Hölder inequality, (2.2) and Sobolev embedding estimates, we have

\begin{equation}
\begin{aligned}
\|\langle x\rangle^{-\sigma}\phi\|_{L^2_x L^2_t} &= \|\langle x\rangle^{-\sigma}e^{itH}u_0\|_{L^2_x L^2_t} \\
& \lesssim \|\langle x\rangle^{-\sigma}\|_{L^\infty_x} \|e^{itH}u_0\|_{L^2_x L^2_t} \\
& \lesssim \|u_0\|_{H^s}.
\end{aligned}
\end{equation}

The second term can be estimated by the Jensen-Kato type decay estimates (2.9) and Young inequality:

\begin{equation}
\begin{aligned}
\|\langle x\rangle^{-\sigma}\int_0^t e^{i(t-s)H}V\phi(s)ds\|_{L^2_x L^2_t} & \lesssim \int_0^t \|\langle x\rangle^{-\sigma}e^{i(t-s)H}V\phi(s)\|_{L^2_x L^2_t} ds \\
& \lesssim \int_0^t (t-s)^{-\frac{\sigma}{2}} \|\langle x\rangle^{2\sigma}V\phi(s)\|_{L^2_x L^2_t} ds \\
& \lesssim \|\langle x\rangle^{2\sigma}V\phi(s)\|_{L^2_x L^2_t} \lesssim \|\langle x\rangle^{-\sigma}\phi\|_{L^2_x L^2_t} \\
& \lesssim \|u_0\|_{H^s}.
\end{aligned}
\end{equation}

Putting (2.20) and (2.21) together gives the desired (2.16).

Next, by using the same argument we will show that

\begin{equation}
\|u_2\|_{S(H^{s}, J)} \lesssim \|h\|_{S'(H^{-s}, J)}.
\end{equation}

Indeed, using (2.6) gives that

\begin{equation}
\|u_2\|_{S(H^{s}, J)} \leq C \left( \|h\|_{S'(H^{-s}, J)} + \|Vu_2\|_{L^2_x L^2_t} \right)
\end{equation}

\begin{equation}
\leq C \left( \|h\|_{S'(H^{-s}, J)} + \|\langle x\rangle^{2\sigma}V\|_{L^\infty_x} \|\langle x\rangle^{-\sigma}u_2\|_{L^2_x L^2_t} \right).
\end{equation}

Now we aim to show that

\begin{equation}
\|\langle x\rangle^{-\sigma}u_2\|_{L^2_x L^2_t} \lesssim \|h\|_{S'(H^{-s}, J)}.
\end{equation}

Consider the Cauchy problem

\begin{equation}
\begin{aligned}
i\dot{\phi}_t = -\Delta^2 \phi - h(t) = -H\phi + V\phi - h(t), \\
0(0, x) = 0.
\end{aligned}
\end{equation}
Then Duhamel formula for the solution $\phi$ reads
\begin{equation}
\phi(t) = i \int_0^t e^{i\theta(t-s)F_0} h(s) ds - i \int_0^t e^{i\theta(t-s)F_0} V\phi(s) ds + i \int_0^t e^{i\theta(t-s)F_0} h(s) ds = -i \int_0^t e^{i\theta(t-s)F_0} V\phi(s) ds + u_2(t),
\end{equation}

Hence
\begin{equation}
(2.27) \quad u_2(t) = \phi(t) + i \int_0^t e^{i\theta(t-s)F_0} V\phi(s) ds = i \int_0^t e^{i\theta(t-s)F_0} h(s) ds + i \int_0^t e^{i\theta(t-s)F_0} V\phi(s) ds.
\end{equation}

By Hölder inequality and (2.6), we have
\begin{align}
\|\langle x \rangle^{-\sigma} \phi \|_{L^2_x L^2_t} &= \|\langle x \rangle^{-\sigma} \int_0^t e^{i\theta(t-s)F_0} h(s) ds \|_{L^2_x L^2_t} \\
&\leq \|\langle x \rangle^{-\sigma} \|_{L^2_x L^2_t} \| \int_0^t e^{i\theta(t-s)F_0} h(s) ds \|_{L^2_x L^2_t} \\
&\leq \|\|\langle x \rangle^{-\sigma} \|_{L^2_x L^2_t} \| \int_0^t e^{i\theta(t-s)F_0} h(s) ds \|_{L^2_x L^2_t}.
\end{align}

The other term can be estimated by the Jensen-Kato type decay estimates (2.9) and Young inequality:
\begin{align}
\|\langle x \rangle^{-\sigma} \int_0^t e^{i\theta(t-s)F_0} V\phi(s) ds \|_{L^2_x L^2_t} &\leq \|\|\langle x \rangle^{-\sigma} e^{i\theta(t-s)F_0} V\phi(s) \|_{L^2_x L^2_t} ds \|_{L^2_t} \\
&\leq \|\int_0^t (t-s)^{-\frac{\sigma}{2}} \|\langle x \rangle^{-\sigma} V\phi(s) \|_{L^2_x L^2_t} ds \|_{L^2_t} \\
&\leq \|\int_0^t \langle x \rangle^{-\sigma} V\phi(s) \|_{L^2_x L^2_t} \| \langle x \rangle^{-\sigma} \phi \|_{L^2_x L^2_t} \\
&\leq \|h\|_{S^{\cdot}H^{-\frac{\sigma}{2},J}}.
\end{align}

Putting (2.28) and (2.29) together gives (2.24). Collecting (2.15), (2.16) and (2.22) completes the proof of (2.11). \hfill \Box

Remark 2.4. Note that the constant $C$ in the estimate (2.11) is dependent of the scaling $V(x) \mapsto V = \frac{1}{V} V(\frac{x}{r})$, which is fundamental to the linear profile decomposition Proposition 6.3. Indeed, let $C = C(V) > 0$ be the sharp constant for (2.11). Then for any $(q, r) \in \Lambda_\sigma$ and $(\bar{q}, \bar{r}) \in \Lambda_{-\sigma}$, we have
\begin{align}
&\| e^{i\theta(t-s)F_0} f(\frac{x}{r}) \|_{L^2_x L^2_t} + \| e^{i\theta(t-s)(H_0 + V)} f(\frac{x}{r}) \|_{L^2_x L^2_t} \\
&= \| e^{i\theta(t-s)F_0} f(\frac{x}{r}) \|_{L^2_x L^2_t} + \| e^{i\theta(t-s)(\Delta + V)} f(\frac{x}{r}) \|_{L^2_x L^2_t} \\
&= C(V) \left( \| f(\frac{x}{r}) \|_{L^2_x L^2_t} + \| f(\frac{x}{r}) \|_{L^2_x L^2_t} \right)
\end{align}

As $f$ are arbitrarily chosen, we have $C(V_0) \leq C(V)$ for all $r > 0$. On the other hand, since $V(x) = r^2 V_0(rx)$ for any $r > 0$, by symmetry we can conclude that $C(V_0) = C(V)$ for all $r > 0.$
Proposition 2.5. Let $N > 6$ and $V$ satisfy the conditions $(\mathcal{C}_1)-(\mathcal{C}_2)$. Then for any given $B$-admissible pair $(q, r)$, the solution $u$ to (2.7) satisfies the inequality
\[(2.31) \quad ||\Delta u||_{L^2(\mathbb{R}^+)} \leq C(||u_0||_{H^r} + ||(\nabla)h||_{L^2(\mathbb{R}^+)}).\]

Proof. The solution $u$ to (2.7) can be expressed as
\[(2.32) \quad u(t, x) = e^{it\Delta}u_0 + i \int_0^te^{i(t-s)\Delta}Vu(s)ds + i \int_0^te^{i(t-s)\Delta}h(s)ds.\]

Using (2.4) yields
\[(2.33) \quad ||\Delta u||_{L^2} \leq ||\Delta u_0||_{L^2} + ||\nabla h||_{L^2} \frac{2}{2} + ||\nabla(Vu)||_{L^2} \frac{2}{2},\]
where
\[(2.34) \quad ||\nabla(Vu)||_{L^2} \frac{2}{2} \leq ||V(\nabla u)||_{L^2} \frac{2}{2} + ||(\nabla V)u||_{L^2} \frac{2}{2} \leq ||\langle x \rangle^r \nabla V||_{L^2} ||\langle x \rangle^{-r} u||_{L^2} + ||V||_{L^2} ||\nabla u||_{L^2} \frac{2}{2}.\]

Since
\[(2.35) \quad ||\nabla u||_{L^2} \frac{2}{2} \leq ||\nabla u_0||_{L^2} + ||h||_{L^2} \frac{2}{2} + ||Vu||_{L^2} \frac{2}{2} \leq ||\nabla u_0||_{L^2} + ||h||_{L^2} \frac{2}{2} + ||\langle x \rangle^r V||_{L^2} ||\langle x \rangle^{-r} u||_{L^2},\]
where we have used the estimate (2.5) in the first inequality. It follows from (2.33)-(2.35) that it suffices to control $||\langle x \rangle^{-r} u||_{L^2}$. We note that using (2.16) and (2.24) with $s = 1$ in Proposition 2.3 yields that
\[(2.36) \quad ||\langle x \rangle^{-r} u||_{L^2} \leq ||\nabla u_0||_{L^2} + ||h||_{L^2} \frac{2}{2}.\]

Thus putting (2.33)-(2.36) together gives (2.31). \qed

By the estimate (2.3), when take $s = 0$ and $q = 2$, we have
\[\left||\nabla e^{it\Delta}u_0\right||_{L^2} \frac{2}{2} \leq ||u_0||_{L^2},\]
which by a dual argument deduces the following smoothing estimate:
\[(2.37) \quad \left||\Delta \int_\mathbb{R} e^{-is\Delta}h(s)ds\right||_{L^2} \leq ||h||_{L^2} \frac{2}{2}.\]

Similarly, to get the scattering result in our paper, we need the following estimate with $H$ in place of $\Delta^2$ inside the integral, which can be also used to establish the linear scattering (see Proposition 2.10) as follows.

Proposition 2.6. Let $N > 4$, $H_0 = \Delta^2$, $H = H_0 + V$ and $V$ satisfy the conditions $(\mathcal{C}_1)-(\mathcal{C}_2)$, then we have the smoothness estimate
\[(2.38) \quad \left||H_0^\frac{7}{2} \int_\mathbb{R} e^{-i\tau H}h(s)ds\right||_{L^2} \leq ||h||_{L^2} \frac{2}{2}.\]

To get (2.38), we need to show that the fractional power associated with $H$ is bounded by the one associated with $H_0$ in $L^1$-norm, because it compensate the non-commutativity between $H_0^\frac{7}{2}$ and $e^{itH}$.
Lemma 2.7. Let $N > 4$, $0 \leq V \in L^\infty(\mathbb{R}^N)$, $H_0 = \Delta^2$ and $H = H_0 + V$. Then for $s \in [0, 4]$ and
\[
\frac{2N}{N+4} < r < \min \left\{ \frac{8N}{4(N-4) - (N-12)s}, \frac{2N}{N-4} \right\},
\]
(2.39)
\[\|H^2 f\|_{L^r} \leq C_r \|H_0^2 f\|_{L^r} \sim \|\nabla^2 f\|_{L^r}.\]
In particular, taking $s = 1$ and $\frac{2N}{N+4} \leq r \leq \frac{2N}{N-2}$, we have
\[
\|H^2 f\|_{L^r} \leq C_N \|H_0^2 f\|_{L^r} \sim \|\nabla f\|_{L^r}.
\]
(2.40)

Proof. By Fourier transform we can define the power of $H_0$ as follows
\[\widehat{H_0 f}(\xi) = |\xi|^2 \widehat{f}(\xi), \ z \in \mathbb{C} \]
where $\widehat{f}$ denotes the Fourier transform of $f$. When $z = iy$, it was well-known from Mihlin’s multiplier theorem that the imaginary power operator $H_0^{-s}$ are bounded on $L^q$ for all $1 < q < \infty$.

For the nonnegative self-adjoint operator $H = \Delta^2 + V$, $H^s$ can be defined by the functional calculus
\[
H^s = \int_0^\infty \lambda^s dE_H(\lambda).
\]
(2.41)

Since the fourth order Schrödinger semigroup $e^{-itH}$ satisfies the following $(p, q)$ off-diagonal estimates (See [6, Section 2]):
\[
\|\chi_{B(x,t^{1/4})} e^{-itH} \chi_{B(y,t^{1/4})}\|_{L^p \to L^q} \leq C_T^{\frac{N}{4} + \frac{1}{p} - \frac{1}{q}} e^{-\frac{t}{2} \frac{|y - x|^2}{4}},
\]
(2.42)
for $\frac{2N}{N+4} \leq p \leq q \leq \frac{2N}{N-4}$ and $N > 4$, where $B(x, t^{1/4})$ is the ball centered at $x$ with radius $t^{1/4}$ and $\chi_{B(x, t^{1/4})}$ is the characteristic function of $B(x, t^{1/4})$, it follows from [3, Theorem 1.2] that the imaginary power operator $H^{-s}$ are bounded on $L^q$ for all $\frac{2N}{N+4} < q < \frac{2N}{N-4}$.

Define a family of operators $T_z$ as follows:
\[
T_z = H^s H_0^{-z}, \quad z = x + iy \in \mathbb{C}, \ 0 \leq x \leq 1.
\]
(2.43)

Obviously, for $z = iy$ and all $\frac{2N}{N+4} < q < \frac{2N}{N-4}$,
\[
\|T_z\|_{L^p \to L^q} \leq (1 + |y|)^{2\alpha}, \ \alpha \geq [N/2] + 1.
\]
(2.44)

Now turn to $z = 1 + iy$. For $1 < p < \frac{N}{2}$, by using Hölder inequality and Sobolev embedding theorem we get that
\[
\|H f\|_{L^p} \leq \|H_0 f\|_{L^p} + \|V f\|_{L^p} \leq \|H_0 f\|_{L^p} + \|V\|_{L^\infty} \|H_0 f\|_{L^p} \leq \|H_0 f\|_{L^p},
\]
(2.45)

which implies that for all $\frac{2N}{N+4} < q < \min\{\frac{2N}{N-4}, \frac{N}{4}\}$
\[
\|T_{1+iy}\|_{L^p \to L^q} = \|H^s T_1 H_0^{-s}\|_{L^p \to L^q} \leq (1 + |y|)^{2\alpha}, \ \alpha \geq [N/2] + 1.
\]
(2.46)

By collecting (2.44) and (2.46), it follows from Stein-Weiss interpolation that real number $\theta \in [0, 1]$,
\[
\|T_\theta\|_{L^p \to L^q} \leq 1,
\]
(2.47)
for all
\[
\frac{2N}{N + 4} < r < \min \left\{ \frac{2N}{(N - 4) - (N - 12)}, \frac{2N}{N - 4} \right\}.
\]
Let \( s = 4\theta \), then for \( s \in [0, 4] \), we obtain the following desired estimates
\[
\|H^{r}f\|_{L^q} \leq \|H_{0}^{r}f\|_{L^q} \sim \|\nabla f\|_{L^q}.
\]
In particular, when \( s = 1 \) and \( \frac{2N}{N - 2} \leq r \leq \frac{2N}{N - 2} \), we have
\[
\|H^{r}f\|_{L^q} \leq C_{N} \|H_{0}^{r}f\|_{L^q} \sim \|\nabla f\|_{L^q}.
\]

\[\textbf{Remark 2.8.}\quad \text{In the} (p,q)\text{-off diagonal estimate (2.42) of } e^{-r(N^2 + V)}, \text{if} (p,q) = (1, \infty), \text{then the estimate (2.42) is equivalent to the following Gaussian kernel estimate:}
\]
\[
|e^{-r(N^2)}(x,y)| \leq C_{r} e^{-r|\nabla x|^2}, \quad t > 0
\]
for some constants \( C, c > 0 \). It was well-known that the semigroup \( e^{-rN^2} \) satisfies the pointwise estimate (2.49). So it would be desirable to obtain the Gaussian kernel estimate (2.49) for \( e^{-r(N^2 + V)} \) in the case \( V \geq 0 \), which actually implies the \( (p,q)\)-off diagonal estimate (2.42) for all \( 1 \leq p \leq q \leq \infty \). However, we remark that the higher order semigroup \( e^{-rN^2} \) is not positivity-preserving and also not a contractive one on \( L^p(\mathbb{R}^N) \) \( (p \neq 2) \) (see e.g.,[26]), which becomes difficult to use famous Trotter formula to establish the pointwise estimate (2.49) for \( e^{-r(N^2 + V)} \). As for more studies about the higher order kernel estimates, one can see [7] and references therein.

**Proof of Proposition 2.6.** By Remark 1.4, Hölder inequality and (2.40), we have
\[
\left\| H_{0}^{r} \int_{\mathbb{R}} e^{-i\alpha H} h(s) ds \right\|_{L^2_{x}} \leq \left\| H_{0}^{r} \int_{\mathbb{R}} e^{-i\alpha H} h(s) ds \right\|_{L^2_{x}}
\]
\[
= \sup_{\|g\|_{L^2_{x}}} \left\{ \int_{\mathbb{R}} e^{-i\alpha H} H_{0}^{r} h(s) ds, g(x) \right\}
\]
\[
= \sup_{\|g\|_{L^2_{x}}} \int_{\mathbb{R}} \left\langle H_{0}^{r} h(s), e^{i\alpha H} H_{0}^{r} g \right\rangle ds
\]
\[
\leq \|H_{0}^{r} h\|_{L^{2}_{x}} \sup_{\|g\|_{L^2_{x}}} \|e^{i\alpha H} H_{0}^{r} g\|_{L^{2}_{x}} \|g\|_{L^2_{x}}
\]
\[
\leq \|\nabla h\|_{L^{2}_{x}} \|g\|_{L^2_{x}} \sup_{\|g\|_{L^2_{x}}} \|\nabla e^{i\alpha H} g\|_{L^{2}_{x}} \|g\|_{L^2_{x}}.
\]
(2.50)

Now it suffices to prove that
\[
\left\| \nabla e^{i\alpha H} g \right\|_{L^{2}_{x}} \|g\|_{L^2_{x}} \leq \|g\|_{L^2_{x}}.
\]
Indeed, let \( u(t,x) = e^{i\alpha H} g \), then it can be expressed as
\[
u(t) = e^{i\Delta^2} g + i \int_{0}^{t} e^{i(t-s)\Delta^2} (Vu(s)) ds,
\]
and using (2.3) with \( s = 1 \) and \( (q, r) = (2, \frac{2N}{N-2}) \), Sobolev embedding, Hölder inequality and Sobolev embedding again leads to
\[
\left\| \nabla |u| \right\|_{L^\frac{2N}{N-2}L^\infty} \leq \left\| \nabla |e^{i\theta H} g| \right\|_{L^\frac{2N}{N-2}L^\infty} + \left\| \nabla \int_0^t e^{i(t-s)H} V u(s) ds \right\|_{L^\frac{2N}{N-2}L^\infty}
\leq \|g\|_{L^2} + \|\nabla^{-1}(Vu)\|_{L^\frac{2N}{N-2}L^\infty}
\leq \|g\|_{L^2} + \|Vu\|_{L^\frac{2N}{N-2}L^\infty}
\leq \|g\|_{L^2} + \|(x)^{-\sigma} V\|_{L^\frac{2N}{N-2}L^\infty}(x)^{-\sigma} u\|_{L^2}
\leq \|g\|_{L^2},
\]
where in the last step we have used local decay estimate (2.8). \( \square \)

**Remark 2.9.** Just as we said in Remark 2.4, the constant \( c \) for the estimate (2.38) is dependent of the scaling \( V(x) \mapsto V_r = \frac{1}{r^2} V(\frac{x}{r}) \), which is also an important fact for the linear profile decomposition Proposition 6.3. Indeed, Let \( c = c(V) > 0 \) be the sharp constant for (2.38). Then we have
\[
\|\Delta \int \int e^{-i(tH_0 + V)H}(\frac{x}{r}, \frac{y}{r})\|_{L^2} \leq \|\Delta(\int \int e^{-i\frac{t}{r^2}(H_0 + V)H}(\frac{S}{r^2}, \frac{x}{r}) ds)\|_{L^2} = r^{2}\|\Delta \int \int e^{-iH}(h(s) ds)\|_{L^2}
\leq c(V) r^{2}\|\nabla h\|_{L^\frac{2N}{N-2}}
\]
As \( r \) and \( h \) are arbitrarily chosen, we find that \( c(V_r) = c(V) \) for all \( r > 0 \).

As a simple application of Proposition 2.6, we shall establish the following linear scattering.

**Proposition 2.10.** Let \( V \) satisfy the conditions (C1)-(C2) and \( N > 8 \). Then

(i) For any given \( \phi \in L^2(\mathbb{R}^N) \), there exist \( \phi^+ \) such that
\[
\lim_{t \to \pm \infty} \|e^{itH_0} \phi - e^{itH} \phi^+\|_{L^2(\mathbb{R}^N)} = 0.
\]

(ii) For any given \( \phi \in H^2(\mathbb{R}^N) \), there exist \( \phi^+ \) such that
\[
\lim_{t \to \pm \infty} \|e^{itH_0} \phi - e^{itH} \phi^+\|_{H^2(\mathbb{R}^N)} = 0.
\]

**Proof.** (i) Let \( u(t) = e^{itH_0} \phi \), then it can be also expressed as
\[
u(t) = e^{itH} \phi - i \int_0^t e^{i(t-s)H}(Vu(s)) ds.
\]
Applying Strichartz estimates (2.11) with \( s = 0 \) yields that for any \( t_1, t_2 \in \mathbb{R} \),
\[
\|e^{-it_2H} e^{it_1H_0} \phi - e^{-it_2H} e^{it_1H_0} \phi\|_{L^2} \leq \|e^{-it_2H} u(t_1) - e^{-it_2H} u(t_2)\|_{L^2}
\leq \left\| \int_{t_1}^{t_2} e^{-itH}(Vu(s)) ds \right\|_{L^\frac{2N}{N-2}} \leq \|Vu(t_1)\|_{L^\frac{2N}{N-2}} \|\mu\|_{L^\frac{2N}{N-2}} \leq \|V\|_{L^\frac{2N}{N-2}} \|u\|_{L^\frac{2N}{N-2}} \to 0
\]
Using Strichartz estimates (2.2), it is easy to find that
\[
\|u\|_{L^\frac{2N}{N-2}} \to 0
\]
as \( t_1, t_2 \to \pm \infty \), which implies that the limit of \( e^{-itH} e^{itH_0} \phi \) exists in \( L^2 \) as \( t \) tends to \( \pm \infty \), and it is namely \( \phi^+ \) we need to find.
In fact, repeating the process of (2.57) yields that
\[ \|e^{it\mathcal{H}}\phi - e^{it\mathcal{H}}\phi^2\|_{L^2} = \|e^{-it\mathcal{H}}e^{it\mathcal{H}}\phi - \phi^2\|_{L^2} \]
\[ = \left\| \int_{t_{-\infty}}^{t_{+\infty}} e^{-it\mathcal{H}}(Vu(s))d\sigma \right\|_{L^2} \]
\[ \leq \|Vu(t)\|_{L^2(-\infty,\infty)} \to 0 \]  
(2.59)
as \( t \) tends to \( \pm \infty \).

(ii) Applying \( V \), H"older inequality, Sobolev embedding and Strichartz estimate (2.2) in turn gives
\[ \|e^{-it\mathcal{H}}e^{it\mathcal{H}}\phi - e^{-it\mathcal{H}}e^{it\mathcal{H}}\phi^2\|_{H^s} \leq \|H^{\frac{1}{2}}(e^{-it\mathcal{H}}e^{it\mathcal{H}}\phi - e^{-it\mathcal{H}}e^{it\mathcal{H}}\phi^2)\|_{L^2} \]
\[ = \left\| \int_{t_1}^{t_2} e^{-it\mathcal{H}}(Vu(s))d\sigma \right\|_{L^2} \]
\[ \leq \|V\|_{L^2} \left( \|\nabla u\|_{L^2_{t_1\to t_2}} + \|\nabla \nabla V\|_{L^2} \right) \]
\[ \leq \left( \|V\|_{L^2} + \|\nabla \nabla V\|_{L^2} \right) \|\Delta u\|_{L^2_{t_1\to t_2}} \to 0 \]  
(2.60)
as \( t_1, t_2 \to \pm \infty \), where in the last inequality we have used the generalized Hardy equality \((N > 8)\), which can be stated as follows (e.g., see Theorem B* in Stein-Weiss [39]): Let \( 1 < p < \infty \), \( 0 \leq s < \frac{N}{p} \), then we have
\[ \|x^{-s}f\|_{L^p(\mathbb{R}^N)} \leq \|\nabla^s f\|_{L^p(\mathbb{R}^N)}. \]

Therefore, it follows from (2.57) and (2.60) that the limit of \( e^{-it\mathcal{H}}e^{it\mathcal{H}}\phi \) exists in \( H^s \) as \( t \) tends to \( \pm \infty \), which is denoted by \( \phi^\pm \). Repeating the process of (i) gives our desired result (2.55). \( \square \)

3. Local wellposedness theory and scattering criterion

Once established Strichartz type estimates Proposition 2.3, Proposition 2.5 and Proposition 2.6, then in this section we will apply them to obtain local well-posedness result, small data theory, finite \( \mathcal{S}(H^k) \) norm condition on scattering and perturbation lemma for BNLSV (1.1), whose proofs are similar to the case without potential. Let’s first look at local well-posedness.

Lemma 3.1. Let \( V \), \( p \) and \( N \) satisfy the assumptions of Theorem 1.2. Then the problem BNLSV is locally well-posed in \( H^2(\mathbb{R}^N) \).

Proof. For \( \mathcal{M} = c\|u_0\|_{H^2} \), we define a map as
\[ \Phi(u) = e^{i\mathcal{H}u_0} - i \int_0^t e^{it-s\mathcal{H}}[u]^{p-1}u(s)ds, \]
(3.1)
and a complete metric space as
\[ B_M = \{ u \in C(I, H^2) : \|x\Delta u\|_{L^2(I; J)} \leq 2M \} \]
(3.2)
with the metric \( d(u, v) = \|u - v\|_{L^2(I; J)} \), where \( I = [0, T] \).

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Proposition 3.2. Let $V$ and $N$ satisfy the assumptions of Theorem 1.2, and assume $u_0 \in H^2(\mathbb{R}^N)$, $t_0 \in I$ an interval of $\mathbb{R}$. Then there exists $\delta_{sd} > 0$ such that if

\[
\|e^{itH}u_0\|_{S^{H^\infty,2}} \leq \delta_{sd},
\]

there exists a unique solution $u \in C(I, H^2(\mathbb{R}^N))$ of (1.1) with initial data $u_0$. Moreover, the solution has conserved mass and energy, and satisfies

\[
\|u\|_{S^{H^\infty,2}} \leq 2\delta_{sd}, \quad \|u\|_{L^\infty(I, H^2)} \leq c\|u_0\|_{H^2}.
\]

Proof. For $\delta = \delta_{sd}$ and $M = c\|u_0\|_{H^2}$, we define a map as

\[
\Phi(u) = e^{it_0H}u_0 + i \int_{t_0}^t e^{i(s-t)H}u|^{p-1}_0 u(s) ds,
\]

and a set as

\[
B_{M,\delta} = \{ v \in C(I, H^2) : \|v\|_{S^{H^\infty,2}} \leq 2\delta, \quad \|\langle \Delta \rangle v\|_{L^\infty(I, L^2)} \leq 2M \}
\]
equipped with the $S(H^\infty, I)$ norm. Then from the Strichartz estimates (2.11) and (2.31), using the Sobolev embedding and Hölder inequality, we have for any $u \in B_{M,\delta}$,

\[
\|\Phi(u)\|_{S^{H^\infty,2}} \leq \delta + c\|u\|_{S^{H^\infty,2}}^p,
\]

and

\[
\|\langle \Delta \rangle \Phi(u)\|_{L^\infty(I, L^2)} \leq c\|\langle \Delta \rangle u\|_2 + c\|u\|_{L^{2p-1}(I, L^{\infty(2p-2)})} \|\langle \nabla \rangle u\|_{L^\infty(I, L^\infty(2p-2))}.
\]
\[ + c \|u\|^{p-1}_{L^{p-1}(I; L^\infty)} \|u\|_{L^p(I; L^\infty)} \leq c \|\Delta u\|_2 + c \|u\|^{p-1}_{S(H^{1,p}, \ell)} \|\Delta u\|_{L^p(I; L^2)}. \]  

(3.12)

Moreover, for any \(u, v \in B_{M, \delta}\),

\[ \|\Phi(u) - \Phi(v)\|_{S(H^{1,p}, \ell)} \leq c \left(\|u\|^{p-1}_{S(H^{1,p}, \ell)} + \|v\|^{p-1}_{S(H^{1,p}, \ell)}\right) \|u - v\|_{S(H^{1,p}, \ell)}. \]

(3.13)

From a standard argument, we can obtain that if \(\delta\) is sufficiently small, the map \(u \mapsto \Phi(u)\) is a contraction mapping principle gives a unique solution \(u\) in \(B_{M, \delta}\) satisfying (3.8).

Now we turn to use a similar argument as in [33] to establish the following scattering result, which can be combined with Proposition 3.2 to get a scattering result of small data.

**Proposition 3.3.** Let \(V, p\) and \(N\) satisfy the assumptions of Theorem 1.2. If \(u(t) \in C(\mathbb{R}, H^2(\mathbb{R}^N))\) be a solution of (1.1) such that \(\sup_{t \in \mathbb{R}} \|u(t)\|_{H^2} < \infty\). If \(\|u\|_{S(H^{1,p})} < \infty\), then \(u(t)\) scatters in \(H^2(\mathbb{R}^N)\). That is, there exists \(\phi^\pm \in H^2(\mathbb{R}^N)\) such that

\[ \lim_{t \to \pm \infty} \|u(t) - e^{itH} \phi^\pm\|_{H^2(\mathbb{R}^N)} = 0. \]

**Proof.** We claim that

\[ \phi^\pm := u_0 - i \int_{0}^{\pm \infty} e^{-istH} (|u|^{p-1} u)(s) ds \]

exist in \(H^2\). Indeed, using \(V \geq 0\), Proposition 2.3 and Proposition 2.6 gives

\[ \left\| \int_{t_1}^{t_2} e^{-istH} (|u|^{p-1} u)(s) ds \right\|_{H^2} \leq \left\| H^2_0 \int_{t_1}^{t_2} e^{-istH} (|u|^{p-1} u)(s) ds \right\|_{L^2} \]

\[ + \left\| \int_{t_1}^{t_2} e^{-istH} (|u|^{p-1} u)(s) ds \right\|_{L^2} \leq \left\| \nabla (|u|^{p-1} u) \right\|_{L^2_{t_1} L^\infty_{x}} + \left\| |u|^{p-1} u \right\|_{L^2_{t_1} L^\infty_{x}} \]

\[ \leq \left\| |u|^{p-1} \right\|_{L^2_{t_1} L^{\infty}_{x}} \left\| \nabla u \right\|_{L^2_{t_1} L^\infty_{x}} \]

\[ + \left\| |u|^{p-1} \right\|_{L^2_{t_1} L^{\infty}_{x}} \left\| \nabla u \right\|_{L^2_{t_1} L^{\infty}_{x}} \]

(3.15)

as \(t_1, t_2\) tend to \(\pm \infty\).

Hence, \(\phi^\pm\) is well defined. Then, using (3.14) and repeating the above estimates again, we obtain that

\[ \left\| u(t) - e^{itH} \phi^\pm \right\|_{H^2} = \left\| \int_{0}^{\pm \infty} e^{-istH} (|u|^{p-1} u)(s) ds \right\|_{H^2} \]

\[ \leq \|u\|^{p-1}_{S(H^{1,p}, \ell; \infty)} \sup_{t \in \mathbb{R}} \|u(t)\|_{H^2} \to 0, \]

(3.16)

as \(t\) tends to \(\pm \infty\).
Finally, we state a useful perturbation lemma, whose proof shall be omitted, since it is similar to that for [14].

**Lemma 3.4.** Let \( V, p \) and \( N \) satisfy the assumptions of Theorem 1.2. Then for any given \( A \), there exist \( \epsilon_0 = \epsilon_0(A, n, p) \) and \( \epsilon = c(A) \) such that for any \( \epsilon \leq \epsilon_0 \), any interval \( I = (T_1, T_2) \subset \mathbb{R} \) and any \( \tilde{u} = \tilde{u}(x, t) \in H^2 \) satisfying

\[
\tilde{u}_t + H \tilde{u} - |\tilde{u}|^{p-1} \tilde{u} = e,
\]

if for some \((q, r) \in \Lambda_{-c, q} \)

\[
\| \tilde{u} \|_{S(H^{q, r})} \leq A, \quad \| e \|_{L^r(I; L^q')} \leq \epsilon
\]

and

\[
\| e^{H(t-\Delta)}(u(t_0) - \tilde{u}(t_0)) \|_{S(H^{q, r})} \leq \epsilon,
\]

then the solution \( u \in C(I; H^2) \) of (1.1) satisfying

\[
\| u - \tilde{u} \|_{S(H^{q, r})} \leq c(A) \epsilon.
\]

**4. Sharp Gagliardo-Nirenberg Inequality**

In this section, under the assumptions of Theorem 1.2, we will find a minimizing sequence of the nonlinear functional

\[
J_V(u) = \frac{\| u \|_{L^{p+1}(\mathbb{R}^n)}^{p+1} - \frac{N(p-1)}{4(p+1)} \| \Delta u \|_{L^2}^2}{\| u \|_{L^{p+1}}^{p+1}} + \int_{\mathbb{R}^n} V |u|^2 \, dx.
\]

(4.1)

It’s known from [14, 42] that for \( V = 0 \), \( J_0(u) \) attains its minimum \( J_0 \) at \( u = Q(x) \geq 0 \), which solves the equation (1.4), and

\[
J_0 = J_0(Q) = \frac{\| Q \|_{L^{p+1}}^2 - \frac{N(p-1)}{4(p+1)} \| \Delta Q \|_{L^2}^2}{\| Q \|_{L^{p+1}}^{p+1}}.
\]

(4.2)

which together with the identities

\[
\| \Delta Q \|_{L^2}^2 = \frac{N(p-1)}{4(p+1)} \| Q \|_{L^{p+1}}^2, \quad \| Q \|_{L^{p+1}}^2 = \frac{p-1}{2(p+1)} \| Q \|_{L^{p+1}}^2, \quad E_0(Q) = \frac{N(p-1)}{8(p+1)} \| Q \|_{L^{p+1}}^2,
\]

implies that the best constant of the Gagliardo-Nirenberg inequality

\[
\| u \|_{L^{p+1}}^{p+1} \leq C_{GN} \| u \|_{L^{p+1}}^{p+1} - \frac{N(p-1)}{4(p+1)} \| \Delta u \|_{L^2}^2
\]

(4.3)

is

\[
C_{GN} = \frac{1}{J_0} = \frac{4(p+1)}{N(p-1)} \frac{1}{\| Q \|_{L^{p+1}}^{p+1} - \frac{N(p-1)}{4(p+1)} \| \Delta Q \|_{L^2}^2}.
\]

(4.4)

**Lemma 4.1.** If \( V \geq 0 \), then \( \{ Q(\cdot - n) \}_{n=1}^\infty \) is a minimizing sequence for \( J_V(u) \).

**Proof.** it follows from (4.2), (4.4) and (4.5) that

\[
J_0(Q) \leq J_0(u).
\]

(4.6)

On the one hand,

\[
\lim_{n \to \infty} J_V(u(\cdot - n)) = J_0(Q).
\]

(4.7)
where we used the inequality
\[(4.8) \quad \int_{\mathbb{R}^n} V(x) Q(x-n)^2 \, dx \leq \|V\|_{L^{\frac{2n}{n-p}}} \|Q\|_{L^{\frac{2n}{n-p}}}^2 \leq \|V\|_{L^{\frac{2n}{n-p}}} \|\Delta Q\|_{L^2}^2\]

On the other hand, for \( V \geq 0 \), it is easy to see that
\[(4.9) \quad J_0(u) \leq J_V(u)\]

Putting (4.6), (4.7) and (4.9) together yields that
\[(4.10) \quad \lim_{t \to +\infty} J_V(Q(-n)) \leq J_V(u).\]

Thus, we get our desired result. \( \square \)

**Remark 4.2.** It follows from lemma 3.1 that \( J_0(Q) = \lim_{r \to +\infty} J_V(Q(-n)) \leq J_V(u) \) holds for any \( u \), which implies that the following sharp inequality holds:
\[(4.11) \quad \|u\|_{L^{p+1}} \leq C_{GN}\|u\|_{L^2}^{p+1-\frac{mp+1}{p}} \|H^\nu u\|_{L^2}^{\frac{mp+1}{p}},\]

where \( C_{GN} \) is the same Gagliardo-Nirenberg constant (4.5).

5. **Criteria for global well-posedness**

In this section we first give a criteria for global well-posedness, but we omit its proof, since it is similar to that of Theorem 4.1 in [14]. Indeed, it suffices to use the previous section’s result (4.11) and replace \( \Delta \) by \( H^\nu \) in the proof.

**Theorem 5.1.** Let \( V, p \) and \( N \) the assumptions of Theorem 1.2 hold, \( u_0 \in H^2(\mathbb{R}^N) \) and \( I = (T_-, T_+) \) be the maximal time interval of existence of \( v(t) \) solving (1.1). Suppose that
\[(5.1) \quad M(u)^{\frac{2}{2-n}} E(u) < M(Q)^{\frac{2}{2-n}} E_0(Q).\]

If (5.1) holds and
\[(5.2) \quad \|u_0\|_{L^2}^{\frac{2}{2-n}} \|H^\nu u_0\|_{L^2} < \|Q\|_{L^2}^{\frac{2}{2-n}} \|\Delta Q\|_{L^2},\]

then \( I = (-\infty, +\infty) \), i.e., the solution exists globally in time, and for all time \( t \in \mathbb{R} \),
\[(5.3) \quad \|u(t)\|_{L^2}^{\frac{2}{2-n}} \|H^\nu u(t)\|_{L^2} < \|Q\|_{L^2}^{\frac{2}{2-n}} \|\Delta Q\|_{L^2}.\]

If (5.1) holds and
\[(5.4) \quad \|u_0\|_{L^2}^{\frac{2}{2-n}} \|H^\nu u_0\|_{L^2} > \|Q\|_{L^2}^{\frac{2}{2-n}} \|\Delta Q\|_{L^2},\]

then for \( t \in I, \)
\[(5.5) \quad \|u(t)\|_{L^2}^{\frac{2}{2-n}} \|H^\nu u(t)\|_{L^2} > \|Q\|_{L^2}^{\frac{2}{2-n}} \|\Delta Q\|_{L^2}.\]

**Remark 5.2.** From the proof of Theorem 5.1, we conclude that if the condition (5.2) holds, then there exists \( \delta > 0 \) such that \( M(u)^{\frac{2}{2-n}} E(u) < (1-\delta)M(Q)^{\frac{2}{2-n}} E_0(Q) \), and thus there exists \( \delta_0 = \delta_0(\delta) \) such that \( \|u(t)\|_{L^2}^{\frac{2}{2-n}} \|H^\nu u(t)\|_{L^2} < (1-\delta_0(\delta))Q_{L^2}^{\frac{2}{2-n}} \|\Delta Q\|_{L^2}. \)

The next two lemmas provide some additional properties for the solution \( u \) under the hypotheses (5.1) and (5.2) of Theorem 5.1. These lemmas will be needed in the proof of Theorem 1.2 through a virial-type estimate, which will be established in the last two sections.
Lemma 5.3. In the situation of Theorem 5.1, take $\delta > 0$ such that $M(u_0) \overset{2-p}{\rightarrow} E(u_0) < (1 - \delta)M(Q) \overset{2-p}{\rightarrow} E_0(Q)$. If $u$ is a solution of the problem (1.1) with initial data $u_0$, then there exists $C_\delta > 0$ such that for all $t \in \mathbb{R}$,

\begin{equation}
\|\Delta u\|_{L^2}^2 - \frac{N(p-1)}{4(p+1)} \|u\|_{L^p}^{p+1} \geq C_\delta \|\Delta u\|_{L^2}^2.
\end{equation}

Proof. By Remark 5.2, there exists $\delta_0 = \delta_0(\delta)$ such that

\begin{equation}
\|u(t)\|_{L^2}^{2-p} \|H^1 u(t)\|_{L^2} \leq (1 - \delta_0)\|Q\|_{L^2}^{2-p} \|\Delta Q\|_{L^2}.
\end{equation}

Since $V$ is nonnegative, it is obvious that

\begin{equation}
\|\Delta u\|_{L^2} \leq \|H^1 u\|_{L^2},
\end{equation}

which combined with (5.7) yields that

\begin{equation}
\|u(t)\|_{L^2}^{2-p} \|\Delta u(t)\|_{L^2} \leq (1 - \delta_0)\|Q\|_{L^2}^{2-p} \|\Delta Q\|_{L^2}.
\end{equation}

The remaining proof is the same as that for Lemma 4.2 in [14].

The following lemma is about the comparability of the gradient and the total energy, and we omit its proof as well, since we only replace $\Delta$ by $H^1$ in the proof of Lemma 4.3 in [14].

Lemma 5.4. In the situation of Theorem 5.1, we have

\begin{equation}
\frac{N(p-1) - 8}{2N(p-1)}\|H^1 u(t)\|_{L^2}^2 \leq E(u) \leq \frac{1}{2}\|H^1 u(t)\|_{L^2}^2.
\end{equation}

Finally, we give the result about existence of wave operators, which will be used to established the scattering theory.

Proposition 5.5. Under the assumptions of Theorem 1.2, and suppose $\psi \in H^2(\mathbb{R}^N)$ and

\begin{equation}
\frac{1}{2}\|\psi\|_{L^2}^{2-4p+4} \|H^1 \psi\|_{L^2} \leq E_0(Q)M(Q) \overset{2-p}{\rightarrow}.
\end{equation}

Then there exists $v_0 \in H^2(\mathbb{R}^N)$ such that the solution $v$ of (1.1) with initial data $v_0$ obeys the assumptions (5.1) and (5.2) and satisfies

\begin{equation}
\lim_{t \to \pm \infty} \|v(t) - e^{itH}\psi\|_{H^2(\mathbb{R}^N)} = 0.
\end{equation}

Proof. Similar to the proof of the small data scattering theory Proposition 3.2, we can solve the integral equation

\begin{equation}
v(t) = e^{itH}\psi + i \int_{t}^{\infty} e^{i(t-s)H} |v|^{p-1} v(s) ds
\end{equation}

for $t \geq T$ with $T$ large. In fact, there exists some large $T$ such that $\|e^{itH}\psi\|_{L^2(\mathbb{R}^N)} \leq \delta_{sd}$, where $\delta_{sd}$ is defined by Proposition 3.2. Then, the same arguments as used in Proposition 3.2 give a solution $v \in C([T, \infty), H^2)$ of (5.13). Moreover, we also have

\begin{equation}
\|v\|_{L^p(T, \infty); H^2} \leq 2\delta_{sd}, \quad \text{and} \quad \|v\|_{L^p([T, \infty); H^2)} \leq c\|v_0\|_{H^2}.
\end{equation}
Thus by Proposition 2.3, Proposition 2.6,
\begin{align*}
\|v - e^{iHT} \psi^+\|_{L^2_{T,\infty} \mathbb{H}^2} & \leq \left\| \int_t^\infty e^{i(t-s)H} |v|^{p-1} v(s) ds \right\|_{L^2_{T,\infty} \mathbb{H}^2} \\
& \leq H_0^2 \int_t^\infty \left\| e^{i(t-s)H} (|v|^{p-1} v)(s) \right\|_{L^2_{T,\infty} \mathbb{H}^2} + \left\| \int_t^\infty e^{i(t-s)H} (|v|^{p-1} v)(s) ds \right\|_{L^2_{T,\infty} \mathbb{H}^2},
\end{align*}

(5.15)

which implies \( v(t) - e^{iHT} \psi^+ \to 0 \) in \( H^2_0(\mathbb{R}^N) \) as \( t \to \infty \). Thus by Sobolev embedding, we obtain that \( v(t) - e^{iHT} \psi^+ \to 0 \) in \( L^{p+1}_t(\mathbb{R}^N) \) as \( t \to \infty \), which implies that \( \lim_{t \to \infty} E(v(t)) = \lim_{t \to \infty} E(e^{iHT} \psi^+) \). Thus, in view of (5.11), we obtain that
\begin{align*}
M(v(T))^{\frac{2}{p-1}} E(v(T)) &= \lim_{t \to \infty} M(v(t))^{\frac{2}{p-1}} E(v(t)) \\
&= \lim_{t \to \infty} M(e^{iHT} \psi^+)^{\frac{2}{p-1}} E(e^{iHT} \psi^+) \\
&= \lim_{t \to \infty} \| \psi^+ \|_{L^2_{T,\infty}}^{\frac{2}{p-1}} \left( \frac{1}{2} \| H^2 \psi^+ \|_{L^2}^2 + \frac{1}{p+1} \| e^{iHT} \psi^+ \|_{L^{p+1}_t(\mathbb{R}^N)}^{p+1} \right),
\end{align*}

(5.17)

Moreover, we note that
\begin{align*}
\lim_{t \to \infty} \|v(t)\|_{L^2_{T,\infty}}^{\frac{2}{p-1}} \left( \| H^2_0 v(t) \|_{L^2_{T,\infty}}^2 + \| e^{iHT} \psi^+ \|_{L^2_{T,\infty}}^2 \right) &= \| \psi^+ \|_{L^2_{T,\infty}}^{\frac{2}{p-1}} \left( \| H^2 \psi^+ \|_{L^2}^2 + 2 E_0(\mathbb{Q}) M(\mathbb{Q})^{\frac{-2}{p-1}} \right) \\
&= \frac{N(p-1) - 8}{N(p-1)} \| \mathbb{Q} \|_{L^2_{T,\infty}}^{\frac{2}{p-1}} \| \Delta \mathbb{Q} \|_{L^2_{T,\infty}}^2 < \| \mathbb{Q} \|_{L^2_{T,\infty}}^{\frac{2}{p-1}} \| \Delta \mathbb{Q} \|_{L^2_{T,\infty}}^2.
\end{align*}

(5.18)

Hence, for sufficiently large \( T \), \( v(T) \) satisfies (5.1) and (5.2), which implies that \( v(t) \) is a global solution in \( H^2_0(\mathbb{R}^N) \). Thus, we can evolve \( v(t) \) from \( T \) back to the initial time 0. By the same way, we can show (5.12) for negative time.

\[ \square \]

6. Existence and compactness of a critical element

**Definition 6.1.** We say that \( SC(u_0) \) holds if for \( u_0 \in H^2_0(\mathbb{R}^N) \) satisfying
\[ \|u_0\|_{L^2_1} \| H^2_0 u_0 \|_{L^2_1} < \| \mathbb{Q} \|_{L^2_{T,\infty}}^{\frac{2}{p-1}} \| \Delta \mathbb{Q} \|_{L^2_{T,\infty}} \]

and
\[ E(u_0) M(u_0)^{\frac{2}{p-1}} < E_0(\mathbb{Q}) M(\mathbb{Q})^{\frac{-2}{p-1}}, \]

the corresponding solution \( u \) of (1.1) with the maximal interval of existence \( I = (-\infty, +\infty) \) satisfies
\[ \|u\|_{S(\mathcal{H}^s)} < +\infty. \]
We first claim that there exists $\delta > 0$ such that if
\[ E(u)M(u)\frac{2}{p} < \delta, \quad \|u_0\|_{L^2} \|H^{\frac{3}{2}}u_0\|_{L^2} < \|Q\|_{L^6} \|\Delta Q\|_{L^2}, \]
then (6.1) holds. In fact, by Proposition 2.3, the norm equivalence Remark 1.4 and (5.10), we have
\[ \|e^{itH}u_0\|_{L^6}^{\frac{2}{p}} \leq \|\nabla u_0\|_{L^2}^{\frac{2}{p}} \leq \|u_0\|_{L^2}^{\frac{4}{p-1}} \|\Delta u_0\|_{L^2}^{\frac{2}{p-1}} \sim \|u_0\|_{L^2}^{\frac{4}{p-1}} \|H^{\frac{3}{2}}u_0\|_{L^2}^{\frac{2}{p-1}} \sim E(u_0)M(u_0)\frac{2}{p}. \]
Hence, it follows from Proposition 3.2 and Proposition 3.3 that (6.1) holds for all sufficiently small $\delta > 0$.

Now for each $\delta > 0$, we define the set $S_\delta$ to be the collection of all such initial data in $H^2$:
\[ S_\delta = \{u_0 \in H^2(\mathbb{R}^N) : E(u)M(u)\frac{2}{p} < \delta \text{ and } \|u_0\|_{L^2} \|H^{\frac{3}{2}}u_0\|_{L^2} < \|Q\|_{L^6} \|\Delta Q\|_{L^2} \}. \]
We also define
\[ (M^\frac{2}{p-1} E)_c = \sup \{ \delta : u_0 \in S_\delta \Rightarrow SC(u_0) \text{ holds} \}. \]
If $(M^\frac{2}{p-1} E)_c = M(Q)^\frac{2}{p} E_0(Q)$, then we are done. Thus we assume now
\[ (M^\frac{2}{p-1} E)_c < M(Q)^\frac{2}{p} E_0(Q). \]
Our goal in this section is to show the existence of an $H^2(\mathbb{R}^N)$ solution $u_c$ of (1.1) with the initial data $u_{c,0}$ such that
\[ \|u_{c,0}\|_{L^6}^{\frac{2}{p-1}} \|H^{\frac{3}{2}}u_{c,0}\|_{L^2} < \|Q\|_{L^6}^{\frac{2}{p}} \|\Delta Q\|_{L^2}, \]
\[ M(u_c)^\frac{2}{p-1} E(u_c) = (M^\frac{2}{p-1} E)_c \]
and $SC(u_{c,0})$ does not hold. Moreover, we show that if $\|u_c\|_{L^6(\mathbb{R}^N)} = \infty$, then $K = \{u_c(x, t) \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^N)$.

Prior to fulfilling our main task, we first establish the decay property for the semigroup $e^{itH}\phi$ in $L^{p+1}$, where $1 < p < 1 + \frac{8}{N-4}$ and $\phi \in H^2(\mathbb{R}^N)$. It was well-known that $L^1-L^n$ estimates of $e^{itH}\phi$ can imply the decay property. However, as far as we know, there are not our required dispersive estimate at present. Hence, we shall give a detailed proof of the decay property.

**Lemma 6.2.** $\lim_{t \to \infty} \|e^{itH}\phi\|_{L^{p+1}} = 0$ for $1 < p < 1 + \frac{8}{N-4}$ and $\phi \in H^2(\mathbb{R}^N)$.

**Proof.** In view of the Strichartz estimates,
\[ \|e^{itH}\phi\|_{L^{p+1}} \leq \|\phi\|_{L^2}, \]
it suffices to prove that $\lim_{t \to \infty} \|e^{itH}\phi\|_{L^{p+1}}$ exists. To this end, we need to show that the map $e^{itH} : t \mapsto L^{p+1}$ is uniformly bounded and uniformly continuous. Uniformly boundedness can be followed from Sobolev embedding and the equivalence norm Remark 1.4, that is, for $1 < p < 1 + \frac{8}{N-4}$,
\[ \|e^{itH}\phi\|_{L^{p+1}} \leq \|e^{itH}\phi\|_{H^s} \leq \|\phi\|_{H^s}. \]
On the other hand, for any $t_1, t_2 \in \mathbb{R}$, applying Gagliardo-Nirenberg inequality and the equivalence norm gives
\[ \|e^{it_1H}\phi - e^{it_2H}\phi\|_{L^{p+1}} \leq \|e^{it_1H}\phi - e^{it_2H}\phi\|_{L^2}^{\frac{N(p-1)}{Np+1}} \|\Delta (e^{it_1H}\phi - e^{it_2H}\phi)\|_{L^2}^{\frac{N(p-1)}{Np+1}}. \]
Proposition 6.3. Let $V$, $p$ and $N$ satisfy the assumptions of Theorem 1.2, $\phi_n(x)$ be radial and uniformly bounded in $H^2(\mathbb{R}^N)$, and $r_n = 1$, $r_n \to 0$ or $r_n \to \infty$. Then for each $M$ there exists a subsequence of $\phi_n$, which is denoted by itself, such that the following statements hold.

(i) For each $1 \leq j \leq M$, there exists (fixed in $n$) a radial profile $\psi_j(x)$ in $H^2(\mathbb{R}^N)$ and a sequence (in $n$) of time shifts $t'_n$, and there exists a sequence (in $n$) of remainders $W^M_n(x)$ in $H^2(\mathbb{R}^N)$ such that

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it'_j H_n} \psi_j(x) + W^M_n(x).$$

(ii) The time sequences have a pairwise divergence property, i.e., for $1 \leq j \neq k \leq M$,

$$\lim_{n \to +\infty} |t'_n - t'_k| = +\infty.$$

(iii) The remainder sequence has the following asymptotic smallness property:

$$\lim_{M \to +\infty} \left( \lim_{n \to +\infty} \|e^{it'H_n} W^M_n \|_{L^2(\mathbb{R}^N)} \right) = 0.$$

(iv) For each fixed $M$, we have the asymptotic Pythagorean expansion as follows

$$\| \phi_n \|^2_{L^2} = \sum_{j=1}^{M} \| \psi_j \|^2_{L^2} + \| W^M_n \|^2_{L^2} + o_n(1),$$

$$\| H_{t_n}^\dagger \phi_n \|^2_{L^2} = \sum_{j=1}^{M} \| H_{t_n}^\dagger \psi_j \|^2_{L^2} + \| H_{t_n}^\dagger W^M_n \|^2_{L^2} + o_n(1),$$

where $o_n(1) \to 0$ as $n \to +\infty$.

Proof. Let’s first consider the case $r_n \to 0$ or $r_n \to \infty$. According to Lemma 5.3 of the first author [14], there exists a subsequence of $\phi_n$, which is still denoted by itself, such that

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it'_j H_n} \psi_j(x) + W^M_n(x).$$
In order to get the form of (6.11), we can rewrite (6.16) as

\[
\phi_n(x) = \sum_{j=1}^{M} e^{-i\epsilon H_n \phi_j(x)} + \overline{W}_n(x),
\]

where

\[
\overline{W}_n(x) = W_n(x) + \sum_{j=1}^{M} \left( e^{-i\epsilon H_n \phi_j(x)} - e^{-i\epsilon \lambda_n \phi_j(x)} \right).
\]

Now we start verifying that (6.17) satisfies the properties (6.12)-(6.15). It's obvious that (6.12) is true, so let's look at (6.13). Applying the formula (2.56) to $e^{i\epsilon H_n} W_n^M$ yields that

\[
\|e^{i\epsilon H_n} W_n^M\|_{L^2(H^\infty)} \leq \left\| e^{i\epsilon H_n} W_n^M \right\|_{L^2(H^\infty)} + \left\| \int_{\mathbb{R}^d} e^{i\epsilon H_n} (V_r e^{i\epsilon H_n} W_n) ds \right\|_{L^2(H^\infty)}
\]

\[
\leq \|V_r e^{i\epsilon H_n} W_n^M\|_{L^2(H^\infty)} + \left\| \int_{\mathbb{R}^d} e^{i\epsilon H_n} (V_r e^{i\epsilon H_n} W_n) ds \right\|_{L^2(H^\infty)}
\]

\[
\leq \|V_r e^{i\epsilon H_n} W_n^M\|_{L^2(H^\infty)} + \left\| \int_{\mathbb{R}^d} e^{i\epsilon H_n} (V_r e^{i\epsilon H_n} W_n) ds \right\|_{L^2(H^\infty)}
\]

\[
= (1 + \|V\|_{L^2}^2) \|e^{i\epsilon H_n} W_n^M\|_{L^2(H^\infty)} \to 0,
\]

as $n \to \infty$ and $M \to \infty$.

Using the same argument to $e^{-i\epsilon \lambda_n \phi_j(x)} - e^{-i\epsilon \lambda_n \phi_j(x)}$, we obtain

\[
\|e^{i\epsilon H_n} (e^{-i\epsilon \lambda_n \phi_j} - e^{-i\epsilon \lambda_n \phi_j})\|_{L^2(H^\infty)}
\]

\[
= \left\| \int_{\mathbb{R}^d} e^{i\epsilon H_n} (V_r e^{i\epsilon H_n} \phi_j) ds \right\|_{L^2(H^\infty)}
\]

\[
\leq \|V_r e^{i\epsilon H_n} \phi_j\|_{L^2(H^\infty)} \to 0,
\]

as $n \to \infty$, where the last step follows from

\[
\|V_r e^{i\epsilon H_n} \phi_j\|_{L^2(H^\infty)} \leq \|V_r\|_{L^2} \|e^{i\epsilon H_n} \phi_j\|_{L^2(H^\infty)} \leq \|V\|_{L^2} \|\phi\|_{H^\infty},
\]

and the condition $r_n \to 0 \text{ or } \infty$. Thus $\overline{W}_n(x)$ in (6.17) satisfies the property (6.13).

To get (6.14), it suffices to prove

\[
\|\overline{W}_n\|_{L^2)^2} = \|W_n\|_{L^2)^2} + o_n(1).
\]

It follows from the expression of $\overline{W}_n(x)$ (6.18) that

\[
\|\overline{W}_n\|_{L^2)^2} = \|W_n\|_{L^2)^2} + 2 \sum_{j=1}^{M} (\sum_{l \neq j} (W_n \cdot e^{-i\epsilon \lambda_n \phi_j} - e^{-i\epsilon \lambda_n \phi_j})
\]

\[
+ 2 \sum_{k \neq j} (e^{-i\epsilon \lambda_n \phi_j} - e^{-i\epsilon \lambda_n \phi_j}, e^{-i\epsilon \lambda_n \phi_j} - e^{-i\epsilon \lambda_n \phi_j})
\]

\[
+ \sum_{j=1}^{M} \|e^{-i\epsilon \lambda_n \phi_j} - e^{-i\epsilon \lambda_n \phi_j}\|_{L^2)^2},
\]

from which, we only need to show that

\[
\|e^{-i\epsilon \lambda_n \phi_j} - e^{-i\epsilon \lambda_n \phi_j}\|_{L^2)^2} \to 0,
\]
as \( n \to \infty \).

In fact,

\[
\|e^{-i(t')H_0}\psi \rangle - e^{-i(t')H_n}\psi \rangle\|_{L^2} \leq \left\| \int_{t'-\epsilon}^{t'} e^{-i(s')H_n}(V_{\epsilon, n} e^{iH_0\psi \rangle}) ds \right\|_{L^2} \\
\leq \|V_{\epsilon, n} e^{iH_0\psi \rangle}\|_{L^2} \rightarrow 0,
\]

(6.25)
as \( n \to \infty \), where the last step follows from

\[
\|V_{\epsilon, n} e^{iH_0\psi \rangle}\|_{L^2} \leq \|V_{\epsilon, n}\|_{L^2} \|e^{iH_0\psi \rangle}\|_{L^2} \leq \|V\|_{L^2} \|\psi \rangle\|_{L^2},
\]

(6.26)
and the condition \( r_n \to 0 \) or \( \infty \). Thus, we complete the proof of (6.14).

Now we turn to (6.15). Since

\[
\|H_{\epsilon, n}^1 f_n\|_{L^2}^2 = \|\Delta f_n\|_{L^2}^2 + (V_{\epsilon, n} f_n)
\]

and

\[
|(V_{\epsilon, n} f_n)| \leq \|V_{\epsilon, n}\|_{L^2} \|f_n\|_{L^2} \leq \|V\|_{L^2} \|\Delta f_n\|_{L^2}^2,
\]

we have

\[
\|H_{\epsilon, n}^1 f_n\|_{L^2}^2 = \|\Delta f_n\|_{L^2}^2 + o_n(1),
\]

(6.27)
provided that \( \|\Delta f_n\|_{L^2} \) is uniformly bounded. Hence, applying (6.27) with \( \phi_n, \phi \) and \( \tilde{W}_n \) and using the asymptotic Pythagorean expansion associated with the free linear propagator Lemma 5.3 in [14], we find that (6.15) can be deduced from the following expression

\[
\|\Delta \tilde{W}_n\|_{L^2}^2 = \|\Delta \tilde{W}_n^M\|_{L^2}^2 + o_n(1).
\]

(6.28)
As in the proof of (6.22), it suffices to prove

\[
\|\Delta(e^{-it'H_0\psi \rangle} - e^{-it'H_n\psi \rangle})\|_{L^2} \to 0,
\]

(6.29)
as \( n \to \infty \). Indeed, using Proposition 2.6, we have

\[
\|\Delta(e^{-it'H_0\psi \rangle} - e^{-it'H_n\psi \rangle})\|_{L^2} = \left\| H_{\epsilon, n}^1 \int_{t'-\epsilon}^{t'} e^{-i(s')H_n}(V_{\epsilon, n} e^{iH_0\psi \rangle}) ds \right\|_{L^2} \\
\leq \left\| \Delta V_n(e^{iH_0\psi \rangle}) \right\|_{L^2} \rightarrow 0,
\]

(6.30)
as \( n \to \infty \), where the last step follows from

\[
\left\| \Delta V_n(e^{iH_0\psi \rangle}) \right\|_{L^2} \leq \left\| \Delta e^{iH_0\psi \rangle} \right\|_{L^2} \leq \left\| \Delta e^{iH_0\psi \rangle} \right\|_{L^2} \leq \left\| \Delta e^{iH_0\psi \rangle} \right\|_{L^2} \leq \left\| \Delta e^{iH_0\psi \rangle} \right\|_{L^2} \leq \left\| \Delta e^{iH_0\psi \rangle} \right\|_{L^2} \leq \left\| \Delta e^{iH_0\psi \rangle} \right\|_{L^2} \leq 0.
\]

(6.31)
Now let’s consider the other case \( r_n = 1 \). Using (6.16) again gives

\[
\phi_n(x) = \sum_{j=1}^{M} e^{-it'H_0\psi \rangle(x)} + W_n(x).
\]

(6.32)
If $t_n \to \infty$, by Proposition 2.10, there exists $\tilde{\psi}^j \in H^2(\mathbb{R}^N)$ such that $\|e^{-it\Delta_H \psi^j} - e^{-it\tilde{H} \tilde{\psi}^j}\|_{H^2} \to 0$. If, on the other hand, $t_n \to 0$, we set $\tilde{\psi}^j = \psi^j$. To sum up, in either case, we obtain a new profile $\tilde{\psi}^j$ for the given $\psi^j$ such that

\begin{equation}
\|e^{-it\Delta_H \psi^j} - e^{-it\tilde{H} \tilde{\psi}^j}\|_{H^2} \to 0, \quad \text{as } n \to +\infty.
\end{equation}

In order to get the form of (6.11), we can rewrite (6.32) as

\begin{equation}
\phi_n(x) = \sum_{j=1}^M e^{-it\Delta_H \tilde{\psi}^j(x)} + \tilde{W}_n^M(x),
\end{equation}

where

\begin{equation}
\tilde{W}_n^M(x) = W_n^M(x) + \sum_{j=1}^M \left( e^{-it\Delta_H \tilde{\psi}^j(x)} - e^{-it\tilde{H} \tilde{\psi}^j(x)} \right).
\end{equation}

Here we only give the proof of (6.13), since all the proofs of (6.13)-(6.15) can be obtained by following the same argument in the case $r_n \to 0$ or $\infty$ and using (6.33). Indeed, (6.19) with $r_n = 1$ is still valid, which yields

\begin{equation}
\lim_{M \to +\infty} \left( \lim_{n \to +\infty} \|e^{itH} W_n^M \|_{L(H^\infty)} \right) = 0.
\end{equation}

And using the Strichartz estimate (2.11) and (6.33), we have

\begin{equation}
\|e^{itH} (e^{-it\Delta_H \tilde{\psi}^j} - e^{-it\tilde{H} \tilde{\psi}^j}) \|_{L(H^\infty)} \lesssim \|e^{-it\Delta_H \tilde{\psi}^j} - e^{-it\tilde{H} \tilde{\psi}^j} \|_{H^2} \to 0,
\end{equation}

as $n \to \infty$, putting (6.36) and (6.37) together gives (6.13), that is,

\begin{equation}
\lim_{M \to +\infty} \left( \lim_{n \to +\infty} \|e^{itH} \tilde{W}_n^M \|_{L(H^\infty)} \right) = 0.
\end{equation}

\begin{remark}
In the linear profile decomposition (6.11), we still have the property, for any $j \geq 1$,

\begin{equation}
W_n^j - e^{-it\Delta_H \psi^j} \to 0 \quad \text{in } H^2(\mathbb{R}^N)
\end{equation}

In fact, when $r_n \to 0$ or $r_n \to \infty$, by (6.18), we have, for any $M \geq 1$,

\begin{equation}
\tilde{W}_n^M(x) = W_n^M(x) + \sum_{j=1}^M \left( e^{-it\Delta_H \psi^j} - e^{-it\tilde{H} \tilde{\psi}^j} \right).
\end{equation}

It follows from (6.24) and (6.29) that, for any $j \geq 1$,

\begin{equation}
e^{-it\Delta_H \psi^j} - e^{-it\tilde{H} \tilde{\psi}^j} \to 0 \quad \text{in } H^2(\mathbb{R}^N),
\end{equation}

which together with the known result $W_n^M - e^{-it\Delta_H \psi^M} \to 0$ in $H^2(\mathbb{R}^N)$ implies that

\begin{equation}
\tilde{W}_n^M - e^{-it\tilde{H} \tilde{\psi}^M} \to 0 \quad \text{in } H^2(\mathbb{R}^N).
\end{equation}

Using (6.41) with $j = M + 1$ again gives

\begin{equation}
\tilde{W}_n^M - e^{-it\tilde{H} \tilde{\psi}^M} \to 0 \quad \text{in } H^2(\mathbb{R}^N),
\end{equation}

which is namely our desired result (6.39). On the other hand, when $r_n = 1$, by (6.35)

\begin{equation}
\tilde{W}_n^M(x) = W_n^M(x) + \sum_{j=1}^M \left( e^{-it\Delta_H \psi^j} - e^{-it\tilde{H} \tilde{\psi}^j} \right).
\end{equation}

\end{remark}
By (6.33), whenever $t_n^j = 0$ or $t_n^j \to \infty$,

\begin{equation}
\label{eq:6.44}
e^{-it_n^j H} \psi^j(x) - e^{-it_n^j \tilde{H}} \tilde{\psi}^j(x) \to 0 \text{ in } H^2(R^N).
\end{equation}

Similarly, we have

\begin{equation}
\label{eq:6.45}
\tilde{W}_n^j - e^{-it_n^j \tilde{H}} \tilde{\psi}^{j+1} \to 0 \text{ in } H^2(R^N).
\end{equation}

Next, we shall use Lemma 6.2, Proposition 6.3 and Remark 6.4 to establish the energy Pythagorean expansion.

**Lemma 6.5.** In the situation of Proposition 6.3, we have

\begin{equation}
\label{eq:6.46}
E_{V_n}(\phi_n) = \sum_{j=1}^M E_{V_n}(e^{-it_n^j H_0} \psi^j) + E_{V_n}(W_n^M) + o_n(1).
\end{equation}

**Proof.** According to (6.14) and (6.15), it suffices to establish for all $M \geq 1$,

\begin{equation}
\label{eq:6.47}
\|\phi_n\|_{p+1}^p = \sum_{j=1}^M \|e^{-it_n^j H_0} \psi^j\|_{p+1}^p + \|W_n^M\|_{p+1}^p + o_n(1).
\end{equation}

In fact, there are only two cases to consider.

Case 1. There exists some $j$ for which $t_n^j$ converges to a finite number, which, without loss of generality, we assume is 0. In this case we will show that $\lim_{n \to \infty} \|W_n^M\|_{p+1} = 0$ for $M > j$, $\lim_{n \to \infty} \|e^{-it_n^j H_0} \psi^k\|_{p+1} = 0$ for all $k \neq j$, and $\lim_{n \to \infty} \|\phi_n\|_{p+1} = \|\psi\|_{p+1}$, which gives (6.47) again.

Case 2. For all $j$, $|t_n^j| \to \infty$. In this case we will show that $\lim_{n \to \infty} \|e^{-it_n^j H_0} \psi^k\|_{p+1} = 0$ for all $k$ and $\lim_{n \to \infty} \|\phi_n\|_{p+1} = \|\psi\|_{p+1}$, which gives (6.47) again.

For Case 1: We infer from Remark 6.4 that $W_n^{j-1} \to \psi^j$. By the compactness of the embedding $H^2_{rad} \hookrightarrow L^{p+1}$, it follows that $W_n^{j-1} \to \psi^j$ strongly in $L^{p+1}$. Let $k \neq j$. Then we get from (6.12) that $|t_n^k| \to \infty$. By Lemma 6.2, we obtain that $\|e^{-it_n^k H_0} \psi^k\|_{p+1} \to 0$. Recalling that

\begin{equation}
\label{eq:6.48}
W_n^{j-1} = \phi_n - e^{-it_n^j H_0} \psi^1 - \ldots - e^{-it_n^{j-1} H_0} \psi^{j-1},
\end{equation}

we conclude that $\phi_n \to \psi^j$ strongly in $L^{p+1}$. Since

\begin{equation}
\label{eq:6.49}
W_n^M = W_n^{j-1} - \psi^j - e^{-it_n^{j+1} H_0} \psi^{j+1} - \ldots - e^{-it_n^M H_0} \psi^M,
\end{equation}

we also conclude that $\lim_{n \to \infty} \|W_n^M\|_{p+1} \to 0$ strongly in $L^{p+1}$, for $M > j$.

For Case 2: Since

\begin{equation}
\label{eq:6.50}
W_n^M = \phi_n - e^{-it_n^j H_0} \psi^1 - \ldots - e^{-it_n^M H_0} \psi^M,
\end{equation}

and for all $j$, $|t_n^j| \to \infty$, which gives $\lim_{n \to \infty} \|e^{-it_n^j H_0} \psi^j\|_{p+1} = 0$, we conclude that $\phi_n - W_n^M \to 0$ in $L^{p+1}$. Hence, we have $\lim_{n \to \infty} \|\phi_n\|_{p+1} = \lim_{n \to \infty} \|W_n^M\|_{p+1}$. 



**Proposition 6.6.** Under the assumptions of Theorem 1.2, then there exists a radial $u_{c,0}$ in $H^2(R^N)$ with

\begin{equation}
\label{eq:6.51}
M(u_{c,0}) \frac{\psi}{\psi_{0}} \to M(u_{c,0}) \psi_{0} < M(Q) \psi_{0} \to \psi_{0}.
\end{equation}

\begin{equation}
\label{eq:6.52}
\|u_{c,0}\|_{H^2} < \|\psi_{0}\|_{H^2} \psi_{0} < \|\Delta Q\|_{L^2}.
\end{equation}
such that the corresponding solution $u_c$ of (1.1) to the initial data $u_{c,0}$ is global and

$$\|u_c\|_{S(H^c)} = \infty.$$  

**Proof.** By the assumption (6.5) and the definition of $(M^{\frac{4}{4+\epsilon}}E)_c$, we can find a sequence of solutions $u_n(t) = \text{BNLS}_V u_{n,0}$ of (1.1) with initial data $u_{n,0}$ such that

(6.53) \[ M(u_{n,0})^{\frac{4}{4+\epsilon}} E(u_{n,0}) \downarrow (M^{\frac{4}{4+\epsilon}}E)_c. \]

(6.54) \[ \|u_{n,0}\|_{L^2}^2 \|H^2 u_{n,0}\|_{L^2}^2 < ||Q||_{L^2} \|\Delta Q\|_{L^2}, \]

and

(6.55) \[ \|u_{n,0}\|_{S(H^c)} = \infty. \]

Note that it’s not obvious for uniform boundedness of $||u_{n,0}||_{H^2}$ because of shortness of scaling invariance for the equation (1.1). Hence, the first step is to show that $||u_{n,0}||_{H^2}$ is uniformly bounded, which can be obtained from the fact that passing to a subsequence,

(6.56) \[ r_n = ||u_{n,0}||_{L^2}^{-\frac{4}{4+\epsilon}} \sim 1. \]

Indeed, by $V \geq 0$, we have

$$||u_{n,0}||_{H^2}^2 = ||u_{n,0}||_{L^2}^2 + ||\Delta u_{n,0}||_{L^2}^2 \leq ||u_{n,0}||_{L^2}^2 + ||H^2 u_{n,0}||_{L^2}^2 \leq r_n^{-2\epsilon} + ||Q||_{L^2} \|\Delta Q\|_{L^2} 2^{2\epsilon} r_n^{-2\epsilon}. \quad (6.57)$$

Let (6.56) be false, then we may assume that $r_n \to 0$ or $\infty$. Next, we shall apply the linear profile decomposition and the perturbation lemma to get a contradiction. To this end, we define

$$\tilde{u}_n(x,t) = \frac{1}{r_n^{\frac{4}{4+\epsilon}}} u_n\left(\frac{x}{r_n}, \frac{t}{r_n^2}\right),$$

and

$$\tilde{u}_{n,0}(x) = \frac{1}{r_n^{\frac{4}{4+\epsilon}}} u_{n,0}\left(\frac{x}{r_n}\right).$$

Hence, $\tilde{u}_n = \text{BNLS}_V \tilde{u}_{n,0}$, that is, $\tilde{u}_n$ is the solution to the initial value problem

(6.58) \[ \begin{cases} i\partial_t \tilde{u}_n + H_c \tilde{u}_n - |\tilde{u}_n|^{p-1} \tilde{u}_n = 0, \\ \tilde{u}_n(0) = \tilde{u}_{n,0} \end{cases}, \]

and $||\tilde{u}_{n,0}||_{H^2}$ is uniformly bounded, which follows from

$$||\tilde{u}_{n,0}||_{L^2}^2 = r_n^{-2\epsilon} ||u_{n,0}||_{L^2}^2 = 1$$

and

$$||\Delta \tilde{u}_{n,0}||_{L^2}^2 \leq ||H_{r_n}^{\|} \tilde{u}_{n,0}||_{L^2}^2 = r_n^{-2\epsilon} ||H^2 u_{n,0}||_{L^2}^2 \leq ||u_{n,0}||_{L^2}^2 ||H^2 u_{n,0}||_{L^2}^2 < ||Q||_{L^2} \|\Delta Q\|_{L^2}^2.$$  

Therefore, we apply Proposition 6.3 to $\tilde{u}_{n,0}$ to get

(6.59) \[ \tilde{u}_{n,0}(x) = \sum_{j=1}^{M} e^{-i(\xi_j H_n)} \psi_j(x) + W_n^{M}(x). \]
Then by (6.46), we have further

\[ \lim_{n \to \infty} E_{V_n}(e^{-it\hat{H}_0} \psi^j) + \lim_{n \to \infty} E_{V_n}(W_n^M) = \lim_{n \to \infty} E_{V_n}(\tilde{u}_n, 0). \]  

(6.60)

Since also by the profile expansion, we have

\[ 1 = ||\tilde{u}_{n,0}||_{L_x^2}^2 = \sum_{j=1}^{M} ||\psi^j||_{L_x^2}^2 + ||W_n^M||_{L_x^2}^2 + o_n(1), \]

(6.61)

\[ ||H^2_{\tilde{u}, 0}||_{L_x^2}^2 = \sum_{j=1}^{M} ||H^2 \psi^j||_{L_x^2}^2 + ||W_n^M||_{L_x^2}^2 + o_n(1), \]

(6.62)

Since from the proof of Lemma 5.4, each energy in nonnegative and then

\[ \lim_{n \to \infty} E_{V_n}(e^{-it\hat{H}_0} \psi^j) \leq \lim_{n \to \infty} E_{V_n}(\tilde{u}_{n,0}) = \lim_{n \to \infty} M(u_n, 0) \frac{\omega}{2} E(u_n, 0) \]

(6.63)

For a given \( j \), if \( |t^j_n| \to +\infty \), we may assume \( t^j_n \to +\infty \) or \( t^j_n \to -\infty \) up to a subsequence. In this case, by (6.61) and (6.63) with \( V = 0 \), we have

\[ \frac{1}{2} ||\psi^j||_{L_x^2}^2 ||\Delta \psi^j||_{L_x^2} < M(Q) \frac{\omega}{2} E_0(Q). \]

(6.64)

If we denote by \( \text{BNLS}_0(t) \phi \) a solution of (1.1) with \( V = 0 \) and initial data \( \phi \), then we get from the existence of wave operators (Proposition 5.5 with \( V = 0 \) or Proposition 4.4 in [14]) that there exists \( \tilde{\psi}^j \) such that

\[ ||\text{BNLS}_0(-t^j_n) \tilde{\psi}^j - e^{-it\hat{H}_0} \psi^j||_{H^2} \to 0, \quad \text{as} \quad n \to +\infty. \]

(6.65)

If, on the other hand, \( t^j_n = 0 \), we set \( \tilde{\psi}^j = \psi^j \). To sum up, in either case, we obtain a \( \tilde{\psi}^j \) for the given \( \psi^j \) such that (6.65).

In order to use the perturbation theory to get a contradiction, we set \( \psi^j(t) = \text{BNLS}_0(t) \tilde{\psi}^j, \)

\[ v_n(t) = \sum_{j=1}^{M} \psi^j(t - t^j_n), \]

and \( \tilde{v}_n(t) = \text{BNLS}_0 v_n(0) \). We will prove successively the following three claims to get a contradiction.

Claim 1. There exists a large constant \( A_0 \) independent of \( M \) such that there exists \( n_0 = n_0(M) \) such that for \( n \geq n_0, \)

\[ ||\tilde{v}_n||_{S(H^2)} \leq A_0. \]

(6.66)

Indeed, using (6.12) and (6.65), we have that

\[ E_0(v_n(0)) = \sum_{j=1}^{M} E_0(\psi^j(-t^j_n)) + o_n(1) = \sum_{j=1}^{M} E_0(e^{-it\hat{H}_0} \psi^j) + o_n(1) \]

(6.67)

By (6.24), (6.29), the assumption \( r_n \to 0 \) or \( \infty \) and Lemma 6.5, we have

\[ \sum_{j=1}^{M} E_0(e^{-it\hat{H}_0} \psi^j) = \sum_{j=1}^{M} E_{V_n}(e^{-it\hat{H}_0} \psi^j) + o_n(1) \]

\[ \leq E_{V_n}(\tilde{u}_{n,0}) + o_n(1) = r_n^{2n-4} E(u_{n,0}) + o_n(1) \]

(6.68)
Collecting (6.67) and (6.68) gives
\[ E_0(v_n(0)) \leq r_n^{2p-4}E(u_{n,0}) + o_n(1) \]
Similarly, we have
\[ M(v_n(0)) \leq M(\tilde{u}_{n,0}) + o_n(1) = r_n^{2p}M(u_{n,0}) + o_n(1) \]
and
\[ \|\Delta v_n(0)\|_2^2 \leq \|H\tilde{\tau}^\perp \tilde{u}_{n,0}\|_2^2 = r_n^{p-2}\|H\tilde{\tau}^\perp u_{n,0}\|_2^2 \]
Hence, (6.69)-(6.71) imply for large \( n \),
\[ M(v_n(0)) \frac{\|E_0(v_n(0))\|}{\|M(\tilde{u}_{n,0}) + o_n(1)\|} \leq \frac{\|\Delta v_n(0)\|_2^2}{\|M(\tilde{u}_{n,0}) + o_n(1)\|} < M(Q) \frac{\|E_0(Q)\|}{\|\Delta Q\|_2^2} \]
Claim 2. There exists a large constant \( A_1 \) independent of \( M \) such that there exists \( n_1 = n_1(M) \) such that for \( n \geq n_1 \),
\[ \|v_n\|_{S(H^\perp)} \leq A_1. \]
In fact, we note that
\[ i\partial_t v_n + \Delta^2 v_n - |v_n|^{p-1}v_n = e_n, \]
where
\[ e_n = \sum_{j=1}^{M} |v_j(t-t_j)|^{p-1}v_jt_j - |v_j(t-t_j)|^{p-1}v_j(t-t_j) \]
If \( p-1 > 1 \), we estimate
\[ |e_n| \leq c \sum_{k \neq j} |v_j(t-t_j)|^{p-1}v_j(t-t_j)|\|v_k(t-t_k)|^{p-2} + |v_j(t-t_j)|^{p-2}; |v_j(t-t_j)|^{p-2}; \]
while if \( p-1 < 1 \),
\[ |e_n| \leq c \sum_{k \neq j} |v_j(t-t_j)|^{p-1}v_j(t-t_j)|^{p-1}. \]
Since, for \( j \neq k, |v_j(t-t_j)| \to +\infty \), then we obtain that \( \|e_n\|_{S(H^\perp)} \) goes to zero as \( n \to \infty \), which, combined with (6.66) and Lemma 3.4 with \( V = 0 \), gives (6.73).
Claim 3. There exists a large constant \( A_2 \) independent of \( M \) such that there exists \( n_2 = n_2(M) \) such that for \( n \geq n_2 \),
\[ \|\tilde{u}_n\|_{S(H^\perp)} \leq A_2. \]
To see this, we note that
\[ i\partial_t v_n + H_{\tau} v_n - |v_n|^{p-1}v_n = \tilde{e}_n, \]
Let \( \tilde{e}_n = V_{r_n} v_n + e_n \).

We will use the perturbation theory to get (6.78). To this end, we will control two norms, that is,
\[
\|e^{itH_n} (\tilde{u}_{n,0} - v_n(0))\|_{S(H^\infty)} \quad \text{and} \quad \|\tilde{e}_n\|_{S(H^{\infty})}.
\]

From (6.59) and the definition of \( v_n(t) \), we have
\[
\tilde{u}_{n,0} - v_n(0) = W_n^M + \sum_{j=1}^M (e^{-i\frac{M^j}{M} H_n} \psi^j - v^j(-t_n^j)).
\]

Let \( \epsilon_0 = \epsilon_0(A_2, n, p) \) be a small number given in Lemma 3.4. By (6.13), taking \( M \) large enough such that there exists \( n_3 = n_3(M) \) satisfying
\[
\|e^{iH_n} W_n^M \|_{S(H^{\infty})} < \frac{\epsilon_0}{2}
\]
for all \( n \geq n_3 \). Next we turn to the estimate of
\[
\|e^{iH_n} (e^{-i\frac{M^j}{M} H_n} \psi^j - v^j(-t_n^j))\|_{S(H^{\infty})}
\]
for each \( j \). From the triangle inequality, Strichartz estimates, (6.41) and (6.65), it follows that there exists \( n_4 = n_4(M) \) such that for each \( j \) and \( n \geq n_4 \)
\[
\|e^{iH_n} (e^{-i\frac{M^j}{M} H_n} \psi^j - v^j(-t_n^j))\|_{S(H^{\infty})} < \frac{\epsilon_0}{2M}.
\]

From (6.83) and (6.85), it follows that
\[
\|e^{iH_n} (\tilde{u}_{n,0} - v_n(0))\|_{S(H^{\infty})} < \epsilon_0
\]
for all \( n \geq \max\{n_3, n_4\} \).

Similar to the proof of (6.20) and using (6.73), we have that \( \|V_{r_n} v_n\|_{S'(H^{\infty})} \) goes to zero as \( n \to \infty \), which together with \( \lim_{n \to \infty} \|\tilde{e}_n\|_{S'(H^{\infty})} = 0 \) gives
\[
\lim_{n \to \infty} \|\tilde{e}_n\|_{S'(H^{\infty})} = 0.
\]

Applying Lemma 3.4 with (6.86), (6.87) and (6.73), we get (6.78).

By scaling, we have
\[
\|u_n\|_{S(H^{\infty})} = \|\tilde{u}_n\|_{S(H^{\infty})} \leq A_2,
\]
contradicting (6.55). So \( \|u_{n,0}\|_{H^2} \) is uniformly bounded.

The next step is to extract \( u_{n,0} \) from a bounded sequence \( \{u_{n,0}\}_{n=1}^{+\infty} \). We omit the proof because it is similar to the proof of Proposition 5.5 in [14]. Indeed, it suffices to replace \( e^{-iH_0} \) by \( e^{-iH} \) in the proof.

Once we established Proposition 6.6, we can obtain the following results of precompactness and uniform localization of the minimal blow-up solution, the proof of which is standard and we omit here.

**Proposition 6.7.** Let \( u_c \) be as in Proposition 6.6. Then
\[
K = \{u_c(t) \mid t \in \mathbb{R}\} \subset H^2(\mathbb{R}^N)
\]
is precompact in \( H^2(\mathbb{R}^N) \).
Corollary 6.8. Let $V$, $p$ and $N$ satisfy the assumptions of Theorem 1.2. Suppose that $u$ be a solution of (1.1) such that $K = \{u(t)\mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^N)$. Then for each $\epsilon > 0$, there exists $R > 0$ independent of $t$ such that, for any $1 \leq i, j \leq N$,

$$
(6.89) \quad \int_{|x|>R} |\partial_i u(x, t)|^2 + |\partial_j u(x, t)|^2 + |u(x, t)|^2 + |u(x, t)|^{p+1} dx \leq \epsilon.
$$

7. Proof of Theorem 1.2

In this section, we prove the following rigidity statement and finish the proof of Theorem 1.2.

Theorem 7.1. Suppose that $u_0 \in H^2(\mathbb{R}^N)$ is radial,

$$
M(u_0) \overset{\text{def}}{=} E(u_0) < M(\mathbb{Q}) \overset{\text{def}}{=} E_0(\mathbb{Q})
$$

and

$$
\|u_0\|_{L^2} \|H^2 u_0\|_{L^2} < \|\mathbb{Q}\|_{L^2}^{\frac{1}{2}} \|\Delta Q\|_{L^2}^{\frac{1}{2}}.
$$

Let $u$ be the corresponding solution of the equation (1.1) of Theorem 1.2 with initial data $u_0$. If $K_+ = \{u(t) : t \in [0, \infty)\}$ is precompact in $H^2(\mathbb{R}^N)$, then $u_0 \equiv 0$. The same conclusion holds if $K_- = \{u(t) : t \in (-\infty, 0]\}$ is precompact in $H^2(\mathbb{R}^N)$.

Proof. We first define

$$
(7.1) \quad M_a(t) = 2 \int_{\mathbb{R}^N} \partial_j a \text{Im}(\bar{u}\partial_j u) dx,
$$

where $a \in C_c^\infty(\mathbb{R}^N)$. The direct computation yields (see e.g. Pausader [33])

$$
M_a(t) = 2 \int_{\mathbb{R}^N} \left(2\partial_j u \partial_i \bar{u} \partial_j \partial_i a - \frac{1}{2} \Delta^2 |u|^2 - 4\partial_j u \partial_\bar{i} \bar{u} \partial_j \bar{u} + \Delta^2 |\nabla u|^2 \right) dx
$$

(7.2)

$$
\quad \quad + \frac{2(p-1)}{p+1} \int_{\mathbb{R}^N} \Delta a |u|^{p+1} dx + 2 \int_{\mathbb{R}^N} \nabla a \cdot \nabla |u|^2 dx,
$$

Take a radially symmetric function $\phi \in C_c^\infty$ such that $\phi(x) = |x|^2$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$, and define $a(x) = R^2 \phi(\frac{x}{R})$. By the repulsiveness assumption on the potential $V$, direct computation gives

$$
-M_a'(t) = 16 \int_{\mathbb{R}^N} |\partial_i u|^2 dx - \frac{4n(p-1)}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx - 4 \int_{\mathbb{R}^N} x \cdot \nabla V |u|^2 dx + (\text{Remainder})
$$

(7.3)

$$
\geq 16 \int_{\mathbb{R}^N} |\Delta u|^2 dx - \frac{4n(p-1)}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx + (\text{Remainder}),
$$

where

(7.4)

\begin{align*}
(\text{Remainder}) &= -16 \int_{|x|\geq R} |\partial_i u|^2 dx + 8 \int_{R \leq |x| \leq 2R} (\partial_j \phi)(\frac{x}{R}) \partial_i u \partial_j u dx \\&+ \frac{4n(p-1)}{p+1} \int_{|x|\geq R} |u|^{p+1} dx - \frac{2(p-1)}{p+1} \int_{R \leq |x| \leq 2R} (\Delta \phi)(\frac{x}{R}) |u|^{p+1} dx \\&+ 4 \int_{|x|\geq 2R} x \cdot \nabla V |u|^2 dx - 2 \int_{R \leq |x| \leq 2R} R(\nabla \phi)(\frac{x}{R}) \cdot \nabla V |u|^2 dx \\&- \frac{4}{R^2} \int_{R \leq |x| \leq 2R} \partial_i u \partial_j \bar{u} (\partial_k \phi)(\frac{x}{R}) dx + \frac{1}{R^2} \int_{R \leq |x| \leq 2R} (\Delta^3 \phi)(\frac{x}{R}) |u|^2 dx.
\end{align*}
From Corollary 6.8, we can infer that (Remainder) → 0 as $R \to \infty$ uniformly in $t \in [0, \infty)$. In fact,

\[
\text{(Remainder)} \leq \int_{|x| \leq R} |\partial_t u|^2 \, dx + \int_{|x| \geq R} |u|^{p+1} \, dx + \frac{1}{R^2} \int_{|x| \geq R} |\partial_t u|^2 \, dx
\]

\[
+ \frac{1}{R^4} \int_{|x| \geq R} |u|^2 \, dx + \|x\| \|\nabla V\|_{L^2} \|u\|_{L^{2/p}}^2 \to 0.
\]

(7.5)

Let a positive constant $\delta \in (0, 1)$ be such that $M(u_0) \overset{\sim}{=} E(u_0) < (1 - \delta)M(\infty) = E(\infty)$. By Lemma 5.3, Remark 1.4 and Lemma 5.4, we obtain that there exists some constant $\delta_0 > 0$ such that

\[
4 \int |\Delta u|^2 \, dx - \frac{N(p - 1)}{p + 1} \int |u|^{p+1} \, dx \geq \delta_0 \int_{R^3} |\Delta u_0|^2 \, dx,
\]

which implies by (7.3) and (7.5) that

\[
-M'(t) \geq \delta_0 \int_{R^3} |\Delta u_0|^2 \, dx.
\]

Thus, we have

\[
M_a(0) - M_a(t) \geq \delta_0 t \int_{R^3} |\Delta u_0|^2 \, dx.
\]

On the other hand, by the definition of $M_a(t)$, we should have

\[
|M_a(t)| \leq R\|u\|_{L^2} \|\nabla u\|_{L^2} \leq R\|u\|_{L^2}^2 \|\Delta u\|_{L^2}
\]

\[
\leq R\|u\|_{L^2}^2 \|H^{\frac{1}{2}} u\|_{L^2} \leq R\|Q\|_{H^1}^2,
\]

which is a contradiction for $t$ large unless $u_0 = 0$. \qed

Now, we can finish the proof of Theorem 1.2.

**The Proof of Theorem 1.2.** In view of Proposition 6.7, Theorem 7.1 implies that $u_0$ obtained in Proposition 6.6 cannot exist. Thus, there must holds that $(M^{\sim} E)_c = E_0(\infty) M(\infty)^{\sim}$, which combined with Proposition 3.3 implies Theorem 1.2. \qed

8. **Finite-time blowup**

In this section, we prove the finite-time blowup for radial data in $H^2(\mathbb{R}^N)$, that is, Theorem 1.6.

To this end, we first obtain the localized virial identity using the commutator identities introduced by Boulanger and Lenzmann [2].

Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be a radial function with regularity property $\forall \phi \in L^\infty(\mathbb{R}^N)$ for $1 \leq j \leq 6$ and such that

\[
\phi(r) = \begin{cases} 
\frac{r^2}{2} & r \leq 1, \\
\text{const.} & r \geq 10
\end{cases}
\]

and such that

\[
\phi''(r) \leq 1 \text{ for } r \geq 0.
\]

(8.10)

For $R > 0$ given, we define the rescaled function $\phi_R : \mathbb{R}^N \to \mathbb{R}$ by setting

\[
\phi_R(r) := R^2 \phi\left(\frac{r}{R}\right).
\]

(8.11)
It can be checked that for all $r \geq 0$,
\begin{equation}
1 - \phi'_R(r) \geq 0, \quad 1 - \frac{\phi'_R(r)}{r} \geq 0, \quad N - \Delta \phi_R(r) \geq 0.
\end{equation}
Moreover, we also recall the following properties of $\phi_R$:
\begin{align*}
\nabla \phi_R(r) &= R \phi'(\frac{r}{R}) \frac{x}{|x|} = \begin{cases} \frac{x}{|x|} & r \leq R, \\ 0 & r \geq 10R \end{cases} \\
\|\nabla^j \phi_R\|_{L^\infty} &\leq R^{2-j}, \quad 0 \leq j \leq 6, \\
supp(\nabla^j \phi_R) &\subset \begin{cases} \{ |x| \leq 10R \} & j = 1, 2, \\ \{ |R \leq |x| \leq 10R \} & 3 \leq j \leq 6. \end{cases}
\end{align*}
For $u \in H^2(\mathbb{R}^N)$, we define the localized virial of $u$ to be the quantity
\begin{equation}
\mathcal{M}_R(u) := \langle u, \Gamma_R u \rangle = 2 \text{Im} \int_{\mathbb{R}^N} u \nabla \phi_R \cdot \nabla \bar{u} dx, \quad \Gamma_R := i(\nabla \phi_R \cdot \nabla + \nabla \cdot \nabla \phi_R).
\end{equation}
By the Cauchy-Schwarz inequality, we have $|\mathcal{M}_R(u)| \leq R\|u\|_{L^2}\|\nabla u\|_{L^2}$.

**Lemma 8.1.** Let $N \geq 2$ and $R > 0$. Suppose that $u \in C([0, T); H^2(\mathbb{R}^N))$ is a radial solution of (1.1). Then for any $t \in [0, T)$, we have the differential inequality
\begin{align*}
\frac{d}{dt} \mathcal{M}_R(u(t)) &\leq 2N(p - 1)E(u_0) - ((p - 1)N - 8) \int_{\mathbb{R}^N} |H^2 u|^2 dx \\
&\quad - \int_{\mathbb{R}^N} |u|^2 (2x \cdot \nabla V(x) + 8V(x)) dx \\
&\quad + O\left(R^{-4} + R^{-2}\|\nabla u\|^3_{L^2} + R^{-\frac{N(p-1)}{2}}\|\nabla u\|^2_{L^2} + \|u\|^2_{L^{2(p-1)}(\mathbb{R}^N)} \right).
\end{align*}

**Proof.** We follow the calculating in the proof of [2, Lemma 3.1] and only sketch the steps except those involving the potential function $V$.

**Step 1.** By taking the time derivative and the equation (1.1),
\begin{equation}
\frac{d}{dt} \mathcal{M}_R(u(t)) = \mathcal{A}_1(u(t)) + \mathcal{A}_2(u(t)) + \mathcal{B}(u(t))
\end{equation}
with
\begin{align*}
\mathcal{A}_1(u(t)) &= \langle u(t), [i\Gamma_R, \Delta^2]u(t) \rangle, \quad \mathcal{A}_2(u(t)) = \langle u(t), [i\Gamma_R, V(x)]u(t) \rangle, \quad \mathcal{B}(u(t)) = \langle u(t), [|u|^{p-1}, i\Gamma_R]u(t) \rangle,
\end{align*}
where $[A, B] = AB - BA$.

**Step 2.** Following the proof of (3.13) on page 515 of [2], for the dispersive part $\mathcal{A}_1$, we have
\begin{equation}
[i\Gamma_R, \Delta^2] = 8\partial^2_{kl}(\partial^2_{im} \phi_R) \partial^2_{mk} + 4\partial_k(\partial^2_{kl} V \phi_R) \partial_l + 2\partial_k(\Delta^2 \phi_R) \partial_l + \Delta^3 \phi_R.
\end{equation}
Since for a radial function $f$, and with $r = |x|$,
\begin{align*}
\partial^2_{kl} f &= (\partial^2_{kl} - \frac{x_k x_l}{r^2} \partial_r f) + \frac{x_k x_l}{r^2} \partial_r^2 f, \\
\int_{\mathbb{R}^N} |\Delta u|^2 dx &= \int_{\mathbb{R}^N} \partial^2_{kl} u^2 + \frac{N-1}{r^2} |\partial_r u|^2 dx,
\end{align*}
then we have
\begin{equation}
\mathcal{A}_1(u(t)) \leq 8 \int_{\mathbb{R}^N} |\Delta u|^2 dx + O\left(R^{-4} + R^{-2}\|\nabla u\|^2_{L^2} \right).
\end{equation}
Step 3. By straight calculation,
\[ \mathcal{A}_2(u(t)) = \langle u(t), [i\Gamma_R, V(x)]u(t) \rangle = -2 \int_{\mathbb{R}^N} \nabla \phi_R \cdot \nabla V |u|^2 \, dx. \]

Thus, from the properties of \( \phi_R \) and the decay of \( V \), we get easily that
\[ \mathcal{A}_2(u(t)) = -2 \int_{\mathbb{R}^N} x \cdot \nabla V |u|^2 \, dx + 2 \int_{|x| \geq R} (x \cdot \nabla V - \nabla \phi_R \cdot \nabla V) |u|^2 \, dx \leq -2 \int_{\mathbb{R}^N} x \cdot \nabla V |u|^2 \, dx + C||u||_{L^2(\{x: |x| \geq R\})}^2. \]

Step 4. For the nonlinear term \( \mathcal{B} \), the same calculation as the step 3 on page 516 of [2] gives that
\[ \mathcal{B}(u(t)) = -\frac{2(p - 1)N}{p + 1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx + O\left(R^{-\frac{p+1}{2}+\frac{1}{2}}\right). \]

Finally, we deduce that
\[
\frac{d}{dt} M_R(u(t)) \leq 8 \int_{\mathbb{R}^N} |\Delta u|^2 \, dx - 2 \int_{\mathbb{R}^N} x \cdot \nabla V |u|^2 \, dx - \frac{2(p - 1)N}{p + 1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx + O \left(R^{-2} + R^{-2}||\nabla u||_{L^2}^2 + R^{-\frac{p+1}{2}+\frac{1}{2}}||\nabla u||_{L^2}^{p+1} + ||u||_{L^2(\{x: |x| \geq R\})}^2\right) = 2N(p - 1)E(u_0) - ((p - 1)N - 8) \int_{\mathbb{R}^N} |H^2 u|^2 \, dx - \int_{\mathbb{R}^N} |u|^2(2x \cdot \nabla V(x) + 8V(x)) \, dx + O \left(R^{-2} + R^{-2}||\nabla u||_{L^2}^2 + R^{-\frac{p+1}{2}+\frac{1}{2}}||\nabla u||_{L^2}^{p+1} + ||u||_{L^2(\{x: |x| \geq R\})}^2\right)
\]
and this completes the proof of Lemma 8.1. \( \square \)

In the end, we will proof Theorem 1.6.

**Proof of Theorem 1.6:**

**Case 1:** \( E(u_0) < 0 \).

Setting \( \delta = ((p - 1)N - 8)/2 \), then \( \delta > 0 \) from \( p > 1 + \frac{2}{N} \). From Lemma 8.1, we obtain that
\[
\frac{d}{dt} M_R(u(t)) \leq 2N(p - 1)E(u_0) - 2\delta \int_{\mathbb{R}^N} |H^2 u|^2 \, dx - \int_{\mathbb{R}^N} |u|^2(2x \cdot \nabla V(x) + 8V(x)) \, dx + O \left(R^{-2} + R^{-2}||\nabla u||_{L^2}^2 + R^{-\frac{p+1}{2}+\frac{1}{2}}||\nabla u||_{L^2}^{p+1} + ||u||_{L^2(\{x: |x| \geq R\})}^2\right) \leq 2N(p - 1)E(u_0) - 2\delta \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + ||W_-||_{L^2} \||\Delta u||_{L^2}^2 + O \left(R^{-2} + R^{-2}||\nabla u||_{L^2}^2 + R^{-\frac{p+1}{2}+\frac{1}{2}}||\nabla u||_{L^2}^{p+1} + ||u||_{L^2(\{x: |x| \geq R\})}^2\right) = 2N(p - 1)E(u_0) - 2\delta \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + ||W_-||_{L^2} \||\Delta u||_{L^2}^2 \]
and the previous discussion we deduce the upper bound
\[ + O\left(R^{-4} + R^{-2}\|\Delta u\|_{L^2} + R^{-\frac{(p-1)(p+1)}{2}}\|\Delta u\|_{L^2}^{\frac{p+1}{2}} + \|u\|^2_{L^2(\{x : |x| > R\})}\right)\]
where we use the assumption $2x \cdot \nabla V + (p - 1)NV = W_+ - W_-$ with $W_+ \in L^{\frac{4}{3}}$, the Hölder inequality, the Sobolev embedding and $\|\nabla u\|_{L^2} \leq C(u_0)\|\Delta u\|_{L^2}^{\frac{1}{2}}$.

Since $p - 1 \leq 8$ and $E(u_0) < 0$, we can choose $R$ sufficiently large such that for $t \in [0, T)$,
\[ \frac{d}{dt} M_R(u(t)) \leq 2N(p - 1)E(u_0) - \frac{3\delta}{2} \int_{\mathbb{R}^n} |\Delta u|^2 \, dx + \|W_+\|_{L^2} |\Delta u|_{L^2}^2. \]
And if we suppose $\|W_+\|_{L^2}$ is sufficiently small (e.g. $\|W_+\|_{L^2} < \delta/2$), then it follows that
\[ \frac{d}{dt} M_R(u(t)) \leq 2N(p - 1)E(u_0) - \delta \int_{\mathbb{R}^n} |\Delta u|^2 \, dx, \]
which, combined with the Cauchy-Schwarz inequality $|M_R(u(t))| \leq C(u_0)R|\Delta u|_{L^2}^2$ and by elementary analysis (see the case 1 on page 517 of [2]), gives that $M_R(u(t)) \to -\infty$ as $t \to t_*$. Therefore, $u(t)$ cannot exist for all $t \geq 0$. By blowup alternative for the Energy-subcritical case, this completes the proof of Theorem 1.6.

**Case 2.** $E(u_0) \geq 0$,
\[ M(u_0) \overset{\leq}{\sim} E(u_0) < M(Q) \overset{\leq}{\sim} E_0(Q), \]
and
\[ \|u_0\|_{L^2(\mathbb{R}^n)}^\frac{p+1}{p} \|\nabla^\frac{1}{2} u_0\|_{L^2(\mathbb{R}^n)} > \|Q\|_{L^2(\mathbb{R}^n)}^\frac{p+1}{p} \|\Delta Q\|_{L^2(\mathbb{R}^n)}. \]
In this case, if we take some $\eta > 0$ such that
\[ M(u_0) \overset{\leq}{\sim} E(u_0) < (1 - \eta)M(Q) \overset{\leq}{\sim} E_0(Q), \]
then we actually could obtain (see the case 3 on page 518-519 of [2] or Theorem 4.1 of [14]) that for $\delta = ((p - 1)N - 8)/2$,
\[ 2\delta(1 - \eta)\|\nabla^\frac{1}{2} u\|_{L^2}^2 \geq 2(p - 1)NE(u_0). \]
Therefore, from Lemma 8.1, Remark 1.4 and the previous discussion we deduce the upper bound
\[ \frac{d}{dt} M_R(u(t)) \leq 2N(p - 1)E(u_0) - 2\delta \int_{\mathbb{R}^n} |\nabla^\frac{1}{2} u|^2 \, dx \]
\[ - \int_{\mathbb{R}^n} |u|^2 (2x \cdot \nabla V(x) + NV(x)) \, dx \]
\[ + O\left(R^{-4} + R^{-2}\|\Delta u\|_{L^2} + R^{-\frac{(p-1)(p+1)}{2}}\|\Delta u\|_{L^2}^{\frac{p+1}{2}} + |u|^2_{L^2(\{x : |x| > R\})}\right) \]
\[ \leq -\left(\frac{\delta \eta}{2} + o_\eta(1)\right) \int_{\mathbb{R}^n} |\Delta u|^2 \, dx + o_\eta(1). \]
Hence, by choosing $R > 0$ sufficiently large, we conclude that
\[ \frac{d}{dt} M_R(u(t)) \leq -\frac{\delta \eta}{4} \int_{\mathbb{R}^n} |\Delta u|^2 \, dx. \]
Following case 1, $u(t)$ blows up in finite time, concluding the proof of Theorem 1.6. \qed
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