Configurational entropy in the O(3)-sigma model

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Abstract

Using a spherically symmetric ansatz, we show that the Chern-Simons O(3) - sigma model with a logarithmic potential admits topological solutions. This result is quite interesting since the Gausson-type logarithmic potential only predicted topological solutions in (1 + 1)D models. To accomplish our goal, the Bogomol’nyi-Prasad-Sommerfield (BPS) method is used, to saturate the energy and obtain the BPS equations. Next, we show by numerical method the graphical results of the topological fields, as well as, the magnetic field behavior that generates a flux given by $\Phi_{flux} = -Q/\kappa$ and the energy density of the structures of vortices. On the other hand, we evaluate the measure of the differential configurational complexity (DCC) of the topological structures, by considering the energy density of the vortex. This analysis is important because it will provide us with information about the possible phase transitions associated with the localized structures and it shows that our model only supports one phase transition.

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I. INTRODUCTION

The non-linear sigma model is constructed by scalar fields that are mapped in a target space where they are coordinates. Using a target space as a Riemannian manifold, the gauging of the target space’s isometries was introduced by Alvarez-Gaumme and D. Freedman in 1981 [1]. Subsequently works consolidated the method [2–5]. The initial motivation was to apply in conformal field theory and superstring models. Some years later, topological solitons were obtained through a breaking of the scale invariance by the gauging of a U(1) subgroup of the O(3) sigma model implemented by B. Schroers [6]. In that work, a Maxwell term and a suitable scalar potential were included in order to construct a Bogomol’nyi type field theory. Soon after, P. Ghosh and S. Ghosh included a Chern-Simons term in the model [7]. Several works in the following years treat the subject under many point of views [8–10]. Recently, the gauged O(3) sigma model was readdressing in the context of Lorentz symmetry breaking models [11, 12] and in several approach of new kinds of Bogomol’nyi-Prasad-Sommerfield (BPS) solutions [13, 14].

It is interesting to mention here that the gauging of a symmetry is also used in the treatment of the so-called Skyrme model in order to breaking a scale invariance. Indeed, since the Skyrme model is just a non-linear sigma model generalized to be invariant under the group $SO(3)$, a gauging of an $U(1)$ subgroup of $SO(3)$, generates a coupling of scalar field to a gauge field [15]. The result is a (3+1)D Skyrme-Maxwell model, which is a rich physical model predicting the proton-neutron mass difference [16]. The Skyrme model has a (2+1)D version which is called baby-Skyrme model [17].

Topological and non-topological solutions can be find using the already famous BPS approach [18]. This method consists of a series of inequalities for solutions of partial differential equations that depend on the homotopic class of solutions in the spatial infinite. In the context of topological defects, the BPS limit is represented by a saturation limit of the energy of the model [18, 19]. It is important to mention that when energy saturation occurs, the equations of motion are known as BPS equations and the order of these equations is reduced from second to first order [18].

In the context of BPS solutions, several works assume the dynamics of the gauge field governed by a Chern-Simons term [7, 20–24]. These systems become self-dual for a suitable choice of potential [25, 26], so their topological solutions [27], as well as, non-topological
solutions [28] will be present in the model.

In this work, we will introduce for the first time a logarithmic potential in the $O(3)$-sigma model context. It is known that a logarithmic potential generates solitonic solutions in bidimensional models with logarithmic nonlinearity [29]. Recently, it has been observed that the logarithmic potential can generate ring-like vortices with intense magnetic flux when the Higgs field is coupled to the gauge field [30]. But, the first works on relativistic theories are due to G. Rosen [31, 32], and to Bialynicki-Birula and Mycielski (BBM) in the context of Schrödinger theories [33]. For their Gaussian shape, BBM called gaussons the soliton-like solutions of the Schrödinger equation with logarithmic nonlinearity. Inspired by them, we called the Chern-Simons $O(3)$-sigma model with logarithmic potential of Chern-Simons-Gausson $O(3)$-sigma model.

On the other hand, we can find some applications of the logarithmic potential in several models [34–36]. Indeed, when we consider electric charge as a self-consistent configuration of an electromagnetic field interacting with a physical vacuum, the nonlinearity could be extremely important since the system is effectively described as a logarithmic quantum Bose liquid [37]. Also, models with the so called gausson logarithmic term were used to describe theories of quantum gravity [38], as well as quantum effects in nonlinear quantum theory [39].

We introduced, throughout the work, for the first time the discussion of the differential configurational complexity (DCC) of BPS vortices in an Abelian model. DCC is a variant of configurational entropy (CE) that appeared in Ref. [40], underpinned by information theory of Claude E. Shannon [41]. From a quantum-mechanical point of view, Shannon’s entropy is interested in giving us information on the probability of a particle evolving from one quantum state to another [42]. CE and its variants were reintroduced with the proposal to give us information about a stability measure applied to a localized structure. The applications of this theory are vast. For example, the CE can be used to investigate stable Q-ball solutions [43, 44] at the Chandrasekhar limit for white dwarfs; in the study of the non-equilibrium dynamics of spontaneous symmetry breaking [44]; in the study of Bose-Einstein condensates [45], and in braneworlds to investigate multi-kink type field configurations [46].

The other key objective of this work is to investigate the existence of stationary solutions and calculate the DCC in the $O(3)$-sigma model coupled to the Chern-Simons (CS) field and subject to a kind of nonpolynomial potential. Using the DCC, we investigated the possible
existence of phase transitions that can create new topological structures in the model.

This work is organized as follows. In the Sec. II, we review the gauged O(3) sigma model with Chern-Simons term. In Sec. III, is devoted to the introduction and definition of the Chern-Simons-Gausson O(3)-sigma model. Also we investigated here their stationary solutions using a numerical approach. Then, in Sec. IV, we investigate the DCC for analyze the possible phase transitions that create new topological structures in model. Finally, in Sec. V, we present the conclusions and final considerations obtained throughout the work.

II. THE GAUGED O(3)-SIGMA MODEL WITH CHERN-SIMONS TERM

We started our investigation considering a Lagrangian density in flat space-time in $(2 + 1)D$ as proposed in Refs. [7, 29], given by

$$\mathcal{L} = \frac{1}{2} D_\mu \Phi \cdot D^\mu \Phi + \frac{\kappa}{4} \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} - V(\phi_3).$$  \hspace{1cm} (1)

where $V(\phi_3)$ is the potential and the component $\phi_3$ is the field configuration responsible for the spontaneous breaking of the symmetry of the model.

The $O(3)$-sigma model is well known in the specialized literature as the result of a mapping of two unit spheres, denoted by $S^2$ [48]. It is important to mention that due to this mapping the field $\Phi$ of the $O(3)$-sigma model, must respect the constraint

$$\Phi \cdot \Phi = 1 \rightarrow \phi_1^2 + \phi_2^2 + \phi_3^2 = 1.$$  \hspace{1cm} (2)

Due to the choice of a potential of the type $V(\phi_3)$, the Lagrangian must be invariant to an isorotation of the preference axis $\hat{n}_3$, i. e., $\hat{n}_3 = (0, 0, 1)$.

We must define the covariant derivative of the model as

$$D_\mu \Phi = \partial_\mu \Phi + A_\mu \hat{n}_3 \times \Phi.$$  \hspace{1cm} (3)

The $U(1)$ nature of the model can be seen by the following identity

$$D_\mu \Phi \cdot D^\mu \Phi = |(\partial_\mu + iA_\mu)(\phi_1 + i\phi_2)|^2 + \partial_\mu \phi_3 \partial^\mu \phi_3.$$  \hspace{1cm} (4)

We will assume that our metric signature will be $\eta_{\mu\nu} = \text{diag}(+,-,-)$ with $\varepsilon^{012} = 1$; $\mu, \nu = 0, 1, 2$ and $i, j = 1, 2$. 

Investigating the equations of motion of the model we will have

\[ J^\mu = -\Phi \times D^\mu \Phi, \]  

where the local current is given by \( J^\mu = -j^\mu \cdot \hat{n}_3 \) and

\[ j^\mu = \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}. \]  

The \( j^0 \) component is known as Gauss’ law. In this way, the field configurations carry a magnetic flux and a non-zero charge given by \( Q = -\kappa \Phi_{\text{flux}}. \)

Similarly, the equation of motion for the scalar field will be

\[ D_\mu D^\mu \Phi = -\frac{\partial V}{\partial \Phi}. \]  

Therefore, we can write

\[ D_\mu J^\mu = \Phi \times \frac{\partial V}{\partial \Phi}. \]  

Considering that the \( T_{00} \) component of the energy-moment tensor represents the energy density of the model, we have

\[ T_{00} = \mathcal{E} = \frac{1}{2} (D_1 \Phi)^2 + \frac{1}{2} (D_2 \Phi)^2 + \frac{\kappa^2 F_{12}^2}{2(1 - \phi_3^2)} + V, \]  

with the energy that describes the configurations of the vortices being the integral in all space of the previous expression, namely,

\[ E = \int d^2 x \mathcal{E}. \]  

### III. THE CHERN-SIMONS O(3)-SIGMA MODEL WITH LOGARITHMIC POTENTIAL

We are interested in the study of static field configurations at the BPS bound, that is, vortex configurations. For this purpose, let us rewrite the energy density of the model as follows

\[ \mathcal{E} = \frac{1}{2} (D_i \Phi \pm \varepsilon_{ij} \Phi \times D_j \Phi)^2 + \frac{\kappa^2}{2(1 - \phi_3^2)} \left[ F_{12} \mp \sqrt{2(1 - \phi_3^2) V} \right]^2 \pm 4\pi Q_0. \]  

Now, we define the topological charge of the model as

\[ Q_\mu = \frac{1}{8\pi} \varepsilon_{\mu\nu\lambda} \left[ \Phi \cdot D^\nu \Phi \times D^\lambda \Phi + F^{\nu\lambda} \sqrt{\frac{2\kappa^2 V}{(1 - \phi_3^2)}} \right]. \]
In this way, we will have that the energy density of the model is reduced to

$$\mathcal{E} = \frac{1}{2} (D_i \Phi \pm \varepsilon_{ij} \Phi \times D_j \Phi)^2 + \frac{\kappa^2}{2(1 - \phi_3^2)} \left[ F_{12} \mp \sqrt{\frac{2(1 - \phi_3^2)V}{\kappa^2}} \right]^2 \pm 4\pi Q_0. \quad (13)$$

Considering that the energy of the model is the integration of $T_{00}$ in all space, we will therefore have

$$E = \frac{1}{2} \int d^2x \left\{ (D_i \Phi \pm \varepsilon_{ij} \Phi \times D_j \Phi)^2 + \frac{\kappa^2}{2(1 - \phi_3^2)} \left[ F_{12} \mp \sqrt{\frac{2(1 - \phi_3^2)V}{\kappa^2}} \right]^2 \right\} \pm 4\pi \int d^2x Q_0. \quad (14)$$

We define that BPS energy is

$$E = \pm 4\pi \int d^2x Q_0. \quad (15)$$

Note that the energy associated to vortices is limited, i.e.,

$$E \geq E_{BPS}. \quad (16)$$

At the bound of saturation of the energy, the static fields configurations obey the first order Bogomol’nyi equations, given by:

$$D_i \Phi = \mp \varepsilon_{ij} \Phi \times D_j \Phi \quad (17)$$

and

$$F_{12} = \pm \sqrt{(1 - \phi_3^2)\phi_3^2 \ln \left( \frac{\phi_3^2}{\vartheta^2} \right)}. \quad (18)$$

Note that we can propose the following potential,

$$V(\phi_3) = \frac{\kappa^2}{2} \phi_3^2 \ln \left( \frac{\phi_3^2}{\vartheta^2} \right), \quad (19)$$

in order to fulfill the model’s equations of motion as well as their BPS first-order equations (17) and (18).

This potential is known as Gausson potential. This term was created by Bialynicki-Birula and Mycielski [33], in order to nominate the soliton-like solutions of the Schrödinger equation with logarithmic nonlinearity, due their Gaussian shape. Clearly, the parameters $\kappa$ and $\vartheta$ adjust the dimension of the model and the factor $\kappa^2/2$ is conveniently applied to get the BPS bound. Also note from Fig. (1) that the parameter $\vartheta$ is associated with the vacuum state of the model.
Due to the constraint of the $O(3)$-sigma model, we choose a spherical symmetry for the variable field. This means that

$$\phi_1 = \sin f(r) \cos N\theta;$$
$$\phi_2 = \sin f(r) \cos N\theta;$$
$$\phi_3 = \cos f(r). \tag{20}$$

Meanwhile, for gauge field we assume that

$$A(\rho, \theta) = -\frac{Na(r)}{\kappa r} \hat{e}_{\theta}. \tag{21}$$

Considering the self-dual potential (19), the ansatz for the scalar field (20) and the ansatz for the gauge field (21), we recast Bogomol’nyi’s equations as

$$f'(r) = \pm N \frac{a + 1}{r} \sin f(r), \tag{22}$$

and

$$a'(r) = \pm \frac{r}{N} \sqrt{\cos f(r)^2 \sin f(r)^2 \ln \left( \frac{\cos f(r)^2}{\partial^2} \right)}. \tag{23}$$

1. Asymptotic analysis

From now on, we are interested in investigating the topological vortex structures of the model. In other words, we will investigate the solutions of the equations (22) and (23). To
ensure that the variable field has no singularity at the origin, we consider that the field near the origin has the form
\[ f(0) = n\pi, \quad n \in \mathbb{N}. \] (24)

For this behavior of the variable field \( f(r) \) near the origin, the gauge field must assumes the behavior \( a(0) \to 0 \). It is important to mention that the solutions of the variable field are symmetric under \( f(r) = 2\pi \).

We are interested in investigating topological solutions, i.e., solutions that have a non-zero and finite energy configuration. For this, we are interested in field configurations that are restricted to the conditions
\[ f(0) = 0 \quad \text{and} \quad f(\pi) = \pi. \] (25)

Considering the condition \( f(0) = 0 \), it is very useful to use the transformation \( f(r) = \pi + \xi(r) \). We use the negative sign of the Bogomol’nyi equations and initially it is assumed a positive vorticity of the model, i.e., \( N > 0 \). Without losing generality, being \( \xi(r) \ll 1 \), the model assume solutions of the type
\[ \xi(r) = A_0 r^N. \] (26)

Consequently, the approximate solution of the gauge field is
\[ a(r) \simeq -\frac{A_0}{N(N+2)(N+3)} r^{N+3} \ln\left(\frac{1}{\theta}\right) + O(r), \] (27)

where \( O(r) \) are the high order terms of \( r \).

On the other hand, if we consider that \( f(0) = 0 \) and \( N < 0 \), we can assume that
\[ f(r) = \bar{B}_0 r^{-N}. \] (28)

In this case, the solution for the gauge field is
\[ a(r) \simeq -\frac{\bar{B}_0}{N(2-N)} r^{2-N} \ln\left(\frac{1}{\theta}\right) + O(r). \] (29)

Similarly, at infinity, there are two asymptotic behaviors in Bogomol’nyi’s equations. First, when \( f(\infty) = \pi \), we can again write \( f(r) = \pi + \xi(r) \) and obtain the localized energy solutions with \( a(\infty) = \eta_1 \). That way, we can assume
\[ \xi(r) = C_\infty r^{N(1-\eta_1)}. \] (30)
As a result, we get that
\[ a(r) \simeq - \frac{C_\infty}{[N(1 - \eta_1) + 2]} r^{[N(1 - \eta_1) + 2]} \sqrt{\ln \left( \frac{1}{\vartheta^2} \right)} - \eta_1. \] (31)

Finally, we analyze the conditions:
\[ f(\infty) = 0 \quad \text{and} \quad a(\infty) = \eta_2. \] (32)

Therefore, with \( N < 0 \), we have
\[ f(r) = \tilde{C}_\infty r^{N(1 + \eta_2)} \] (33)

Consequently, we obtain the result
\[ a(r) = - \frac{\tilde{C}_\infty}{N[N(1 + \eta_2) + 2]} r^{[N(1 + \eta_2) + 2]} \sqrt{\ln \left( \frac{1}{\vartheta^2} \right)} + \eta_2. \] (34)

The parameters \( \eta_1 \) and \( \eta_2 \) indicate whether the solutions are topological or non-topological.

2. Numerical result

From this moment on, we turn our attention to the numerical study of the dynamics of the static scalar fields that describe the \( O(3) \)-sigma model, as well as the dynamics of the Chern-Simons field that is coupled to the scalar fields. In order to achieve our purpose, we consider both fields described by respective eqs. (22) and (22). In this way, assuming the condition of topological boundary (25) and performing a brief interpolation in order to find field configuration with positive-defined and finite energy, we obtain the result shown in fig. (2).

On the other hand, with the help of the equation (23), we are able to obtain the graphic behavior of the Chern-Simons field associated to vortices structures and represented in the figure (3).

With the numerical solutions of the variable field and the gauge field, we can easily describe the behavior of some quantities of the model, namely: energy density, magnetic field and magnetic flux associated to the vortices. To obtain the result of the BPS energy of the model we consider the numerical solutions presented in the previous figures and remember that the energy density at the BPS bound is given by equations (9) and (10). In this way, we obtain the result presented in fig. (4).
Using the expression (18), we obtain the magnetic field associated with the vortices structures. Clearly from the numerical results presented, we can realized kink-like topological vortex structures. We can see that this structure has a finite and positive-defined energy density with a very significant intensity. This result is due to the fact that the vortices of the model interact with the Chern-Simons field generating a flux due to the magnetic induction shown in fig. (5).

**IV. DCC OF VORTEX**

The CE (and its variants) is a quantity that is related to the measure of the informational complexity of a localized field configuration and can be expressed by a Fourier transform of
the energy density, as we can see in Refs. [46, 49–51]. It is important to mention that this quantity has been studied intensively in several cosmological scenarios [52–54]. With this in mind, we propose to study of the DCC to understand the structures of vortex and the possible existence of phase transition of the model discussed above.

From Refs. ([40, 49–51]), we define the concept of differential entropy as

\[
S = \int \rho(\omega) \ln[\rho(\omega)] \, d\omega. \tag{35}
\]

It is worth noting that the DCC is not invariant under a change of coordinates. This is a consequence of the fact that the probability density transforms to a scalar density under the coordinate transformations \( x \to \tilde{x} \), see Ref. [55].

Consider an energy density \( \mathcal{E}(\mathbf{r}) \) of the field, located in space. The decomposition into
wave modes in d-dimensional space is provided by the Fourier transform:

$$\mathcal{F}(\omega) = (2\pi)^{-(d/2)} \int \mathcal{E}(\mathbf{r}) e^{-i\mathbf{\omega} \cdot \mathbf{r}} \, d^d \mathbf{r}. \quad (36)$$

The contribution of wave mode to a $|\omega|_*$ is the modal fraction,

$$f(\omega) = \frac{|\hat{\mathcal{F}}(\omega)|^2}{|\mathcal{F}(\omega_*)|^2} \quad (37)$$

The DCC is defined as

$$S_C = -\int f(\omega) \ln[f(\omega)] \, d^d\omega, \quad (38)$$

note that the above integral is positive definite since $g(\omega) \leq 1$.

Due to the symmetry of the problem, it is convenient to write the DCC in spherical symmetry, for this let us remember that the hyperspheric Fourier transform is given by

$$\mathcal{F} = \omega^{1-\frac{d}{2}} \int_0^\infty \mathcal{E}(r) J_{\frac{d}{2}-1}(\omega r) r^{d/2} \, dr, \quad (39)$$

with $\mathcal{E}(r)$ the energy density of the vortex and $J_\nu$ the Bessel function.

Using (39), we have that the DCC is

$$S_C = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty \omega^{d-1} f(\omega) \ln[f(\omega)] \, d\omega, \quad (40)$$

where the entropic density $\rho(\omega)$ is the integrand of the DCC.

In the context of the braneworlds, the above quantity is able to describe new topological structures, e. g. multi-kink solutions which are the fruits of multiples phase transition [46]. In the context in which we are applying the study of this quantity, we want to understand whether the vortex solutions discussed above are unique and whether our model nonpolynomial admit such multiple transitions.

It is important to mention that the calculate of the CE is not so simple, since our energy density of the O(3)-sigma is given by

$$\mathcal{E} = \frac{1}{2} D_i \Phi \cdot D^i \Phi + \frac{\kappa^2 F_{12}^2}{2(1-\phi_3^2)} + \frac{\kappa^2}{2} \phi_3 \ln \frac{\phi_3^2}{\phi_3^2}, \quad (41)$$

and

$$\Phi = \phi_i(f(r); r) \hat{e}^i, \quad \text{with} \quad i = 1, 2, 3. \quad (42)$$
Aware of the difficulties, we evaluate numerically the modal fraction of the model and show the results in Figs. 6(a) and (b). To obtain the modal fraction we consider the numerical solutions of Eqs. (22) and (23). At first, we noticed that the parameter $\vartheta$, which relate the potential to the magnetic field of the BPS vortex, directly influence modal fraction causing the amplitude of the modal fraction of the vortex to decrease exponentially outside of the structure. By simulation, we also noticed that the amplitude of the first node in figs. 6(a) and (b) corresponds to the solution discussed in the previous section.

![Graphical representation](image)

Figure 6: (a) Modal fraction when $\vartheta$ is constant. (b) Modal fraction when $\kappa$ is constant.

With the numerical solution of modal fraction, we investigate the configurational entropy density (entropic density) of the model. We note that around the core of the vortex $r \approx 0$, the entropic densities are higher both in the case that $\kappa$ is fixed as when $\vartheta$ is fixed. In fact, due to the localized structure it is around $r = 0$ entropic density is high in this region. We note that the parameter $\vartheta$ that controls entropic density is associated to magnetic field, such that when this parameter increases, this magnetic field decreases leading to a decay of the modal fraction (figs. 6) and the entropic density (figs. 7).

Finally, with the result of the DCC for the vortex considered (Fig. 8), it is observed that the $O(3)$-sigma model with the Chern-Simons gauge field and with a nonpolynomial potential type $\phi^4$ admits only one type of solution, i. e., a kink-like solution. This solution is shown in Fig. 2. Unlike the results presented in braneworld scenarios [46], and due to the configurational entropy profile of the $O(3)$-sigma model, we observed that our model does not support multiple phase transitions. We also note that the critical point of the DCC corresponds to the vacuum expected value of the theory.
Figure 7: (a) Configurational entropy density when $\vartheta$ is constant. (b) Configurational entropy density when $\kappa$ is constant.

Figure 8: Configurational entropy of the vortex.

V. CONCLUDING REMARKS

In this work, we investigated topological vortex structures of the $O(3)$-sigma model coupled to the Chern-Simons field and subject to a logarithmic potential. We show that in this case, the topological structures are presented as kink-like solutions due to the spherical symmetry of the problem. We observed that the vortices generated in the model are magnetically charged and have a magnetic flux given by $\Phi_{\text{flux}} = -Q/\kappa$. Finally, it is observed numerically that the vortices generated in the model are quite energetic due to the contribution of the logarithmic potential and the Chern-Simons field.

Once we obtain the BPS solution of the variable field of the sigma model we find the
spatial profile of the energy density of the model. We investigated whether the studied vortices admitted multiple phase transitions. We observed that the DCC assumed a profile type of a directional delta-function centered approximately on the expected value of the vacuum state. Due to this profile of the DCC it is clear that the model does not support multiple phase transitions. In fact, due to the behavior of the DCC and the probability densities, we note that the model supports other solutions, however, these solutions will differ only “how quickly” the fields evolve into a vacuum state.

We also observed that the entropic densities of the model in the region where the structure is located are higher. We conclude, by the DCC that the parameters $\kappa$ and $\vartheta$ directly influence in the vortex structures making it more or less energetic and with a more or less intense field and localized.

Acknowledgment

The authors thank the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), grant No. 308638/2015-8 (CASA), and Coordenação de Aperfeiçoamento do Pessoal de Nível Superior (CAPES) for financial support.

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