Asymptotic scaling corrections in QCD with Wilson fermions from the 3-loop average plaquette

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Abstract

We calculate the 3-loop perturbative expansion of the average plaquette in lattice QCD with $N_f$ massive Wilson fermions and gauge group $SU(N)$. The corrections to asymptotic scaling in the corresponding energy scheme are also evaluated. We have also improved the accuracy of the already known pure gluonic results at 2 and 3 loops.
1 Introduction and calculation

The lattice numerical calculation of the matrix element of any operator $A$ with mass dimension $d$ yields $a^d A$, where $a$ is the lattice spacing, up to scaling violating contributions of order $O(a^{d+1})$. The calculation of $a$ as a function of the coupling constant can be improved if an effective scheme is used [1, 2]. A nice example of such improvement has been shown to occur in pure Yang-Mills theories [3] and is known to work rather well in other scalar theories (see for example [4, 5] for the 2D spin models).

In this paper we calculate the 3-loop internal energy in full QCD with Wilson fermions on the lattice. The 3-loop beta function for this theory has been recently calculated [6, 7] and we can compute the corresponding corrections to asymptotic scaling in both the standard and effective schemes.

We will use the Wilson action $S$ with $N_f$ flavours and gauge group $SU(N)$. This action is

$$S = S_W + S_f,$$

$$S_W = \beta \sum_\Box E_W(\Box),$$

$$S_f = \sum_x E_f(x)$$

with

$$E_W(\Box) = 1 - \frac{1}{N} \text{Re Tr} (\Box),$$

$$E_f(x) = \sum_{\text{flavours}} a^4 \left[ \left( m + \frac{4r}{a} \right) \bar{\psi}_x \psi_x - \frac{1}{2a} \sum_\mu \left( \bar{\psi}_{x+\hat{\mu}} (r + \gamma_\mu) U_\mu(x) \psi_x + \bar{\psi}_x (r - \gamma_\mu) U_\mu(x) \psi_{x+\hat{\mu}} \right) \right].$$

$\beta$ is the coupling on the lattice $\beta \equiv 2N/g^2$ and $r$ the Wilson parameter. $\Box$ stands for the plaquette and $U_\mu(x) \equiv \exp (iagA_\mu(x))$ for the link at the site $x$ pointing towards $x + \hat{\mu}$. We will parametrize the results in terms of $N$, $N_f$ and the couple $(\kappa, r)$ where $\kappa$ is the usual hopping parameter

$$\kappa = \frac{1}{8r + 2am}. \quad (3)$$
The average of $E_f$ can be straightforwardly computed to all orders by rescaling the fermionic action $S_f$ by a factor $\epsilon$ under the fermionic path integral $Z^f$

$$Z^f(\epsilon) \equiv \int \mathcal{D}\overline{\psi}(x)\mathcal{D}\psi(x) \exp(-\epsilon S_f) = \epsilon^{4NN_f} Z^f(\epsilon = 1) \quad (4)$$

and by using

$$\langle E_f \rangle = -\frac{\partial}{\partial \epsilon} \left( \frac{\ln Z^f(\epsilon)}{V} \right)_{\epsilon=1} = -4NN_f. \quad (5)$$

On the other hand, the average of $E_W$ in presence of the action Eq.(2) is calculated in perturbation theory

$$\langle E_W \rangle = c_1 g^2 + c_2 g^4 + c_3 g^6 + \cdots \quad (6)$$

The $n$-loop coefficient can be written as $c_n = c_n^g + c_n^f$ where $c_n^g$ is the pure Yang-Mills contribution, known since ref. [8, 9] up to 3-loops, and $c_n^f$ is the fermionic contribution. To calculate $c_n^f$ we will first compute the free energy $-(\ln Z)/V$ up to 3 loops, $Z$ being the full partition function

$$Z \equiv \int \mathcal{D}U_\mu(x)\mathcal{D}\overline{\psi}(x)\mathcal{D}\psi(x) \exp(-S). \quad (7)$$

The average of $E_W$ is then extracted as follows

$$\langle E_W \rangle = -\frac{1}{6} \frac{\partial}{\partial \beta} \left( \ln Z \right). \quad (8)$$

The Feynman diagrams necessary to calculate the fermion contribution to the free energy up to 3 loops are shown in Fig. 1. We have used the Feynman gauge. The involved algebra of the lattice perturbation theory was carried out by making use of the computer code developed by us [10].

After grouping the diagrams in several infrared-finite sets, we calculated the resulting finite integrals on finite lattice-sizes $L$ and then the results were extrapolated to infinite size. The extrapolating function was of the type [11, 12]

$$a_0 + \sum_{i\leq2j, j=1,2,\cdots} a_{ij} \frac{(\ln L)^i}{L^{2j}}. \quad (9)$$
We used a broad spectrum of such functional forms and analyzed the quality of each extrapolation to assign a weight to each one, to finally produce a reliable estimate of the systematic error. The criterion to judge the quality of the extrapolation was based on the accuracy of the fitted functional form to reproduce known results at finite lattice-sizes. The different functional forms were obtained by truncating the series in Eq. (9) at different values of \( j \) and assuming vanishing coefficients \( a_{ij} \) for some \( i \) and \( j \).

Recall [8, 9] that the pure gluonic contributions are (presented here with improved accuracy)

\[
c^g_1 = \frac{N^2 - 1}{8N},
\]

\[
c^g_2 = (N^2 - 1) \left( 0.0051069297 - \frac{1}{128N^2} \right),
\]

\[
c^g_3 = (N^2 - 1) \left( \frac{0.0023152583(50)}{N^3} - \frac{0.002265487(17)}{N} + 0.000794223(19)N \right).
\]

The 2-loop coefficient can be written in closed form as

\[
(N^2 - 1) \left( \frac{1}{384} + \frac{P_1}{24} - \frac{P_2}{12} - \frac{Q_1}{24} - \frac{Q_2}{288} - \frac{1}{128N^2} \right),
\]

where \( P_1, Q_1 \) and \( Q_2 \) are finite integrals defined and evaluated in [13].

In tables I and II we show \( c^f_2 \) and \( c^f_3 \) for several pairs \((\kappa, r)\). They are parametrized in terms of four constants \( h_2, h_{30}, h_{31} \) and \( h_{32} \) as follows

\[
c^f_1 = 0,
\]

\[
c^f_2 = (N^2 - 1) h_2 \frac{N_f}{N},
\]

\[
c^f_3 = (N^2 - 1) \left( h_{30} N_f + h_{31} \frac{N_f}{N^2} + h_{32} \frac{N_f^2}{N} \right).
\]

In table III we show the result for \( c_2 = c^g_2 + c^f_2 \) and \( c_3 = c^g_3 + c^f_3 \) for \( r = 1 \), \( N_f = 3 \) and \( N = 2 \). In table IV the result for \( r = 1, N_f = 3 \) and \( N = 3 \) is shown.
2 Corrections to asymptotic scaling

The beta function in QCD with $N_f$ Wilson fermions can be written as

$$\beta^L(g) \equiv -a \frac{dg}{da}|_{g_R, \mu} = -b_0g^3 - b_1g^5 - b_2g^7 - \cdots$$  \hspace{1cm} (13)

The non-universal 3-loop coefficient $b_2$ has been recently calculated in ref. [6], (see also [7]). In terms of the bare coupling $g$, the lattice spacing $a$ approaches the continuum limit as

$$a\Lambda_L = \exp \left( -\frac{1}{2b_0g^2} \right) \left( b_0g^2 \right)^{-b_1/2b_0^2} \left( 1 + q g^2 + \cdots \right),$$  \hspace{1cm} (14)

where $q$ is the 3-loop correction to asymptotic scaling

$$q \equiv \frac{b_1^2 - b_2b_0}{2b_0^3}.$$  \hspace{1cm} (15)

Other couplings can be defined. A popular effective coupling in terms of the plaquette energy is [1, 2]

$$g_{EW}^2 \equiv \frac{1}{c_1} \langle E_W \rangle_{MC},$$  \hspace{1cm} (16)

where $\langle \cdot \rangle_{MC}$ indicates Monte Carlo average. In terms of $g_{EW}$, the approach of the lattice spacing to the continuum limit is written as in Eq.(14) with $q_{EW}$ instead of $q$. This 3-loop correction to asymptotic scaling is

$$q_{EW} = q - \frac{b_0c_3 - b_1c_2}{2c_1b_0^2}. \hspace{1cm} (17)$$

Moreover the lattice Lambda parameters are related by the equation

$$\Lambda_{EW} = \exp \left( \frac{c_2}{2b_0c_1} \right) \Lambda_L.$$  \hspace{1cm} (18)

We give the coefficients $q_{EW}$ for $SU(2)$ in table V and for $SU(3)$ in table VI. In both cases we show the result for $r = 1$ and for several choices of $\kappa$. These results must be compared with the value of $q$ in the standard bare coupling scheme. This is [3]

$$SU(2) \rightarrow \begin{array}{ll}
q(N_f = 1) &= 0.12617(4) \\
q(N_f = 3) &= 0.2547(1)
\end{array},$$

$$SU(3) \rightarrow \begin{array}{ll}
q(N_f = 1) &= 0.23956(4) \\
q(N_f = 3) &= 0.3681(2)
\end{array}.$$  \hspace{1cm} (19)
(notice that in the standard bare coupling scheme, $q$ does not depend on the fermion mass [6]). The 3-loop correction to asymptotic scaling for $N_f = 3$, $N = 3$ is $\sim 37\%$ in the standard scheme and $\sim 18 - 21\%$ in the $\langle E_W \rangle$ scheme. Apparently this improvement is not as dramatic as it was for the pure gluonic case [3] (for $SU(3)$ $q \sim 19\%$ in the standard scheme and $q \sim 1\%$ in the energy scheme). The improvement therefore seems to be more efficient in the quenched case. This fact can also be seen from tables V and VI where the approach to the quenched case (either $N_f \rightarrow 0$ or $\kappa \rightarrow 0$) is accompanied by the lowering of $q_{E_W}$. However, the only relevant test for the improvement in asymptotic scaling from the standard to some energy scheme is the practical use of the effective scheme. For example, the 2D spin models [4, 5] show a definite improvement in spite of the behaviour of the 3-loop coefficient $q$ which for the $O(3)$ models passes from $q = -0.09138$ in the standard scheme to $0.1694$ in the energy scheme.

3 Discussion

We have calculated the free energy up to 3 loops in QCD with Wilson fermions. We have given the result of the internal energy average $\langle E_W \rangle$ as a function of the number of fermions $N_f$, their masses and the Wilson parameter $r$. These expansions have been used to study the 3-loop corrections to asymptotic scaling in this theory. We have found that at 3 loops the corresponding energy scheme [4, 5] provides a moderate improvement with respect to the standard scheme. However, only the practical use of this scheme in particular problems will reveal how useful it is.

We can numerically compute the set of coefficients $c_f^n$ and $q$ also for other choices of $(\kappa, r)$, $N_f$ and $N$.

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Figure caption

Figure 1. Feynman diagrams contributing to the fermionic sector of $\ln Z/V$ at 2 and 3 loops. Curly, arrowed and dashed lines indicate gluons, fermions and ghosts respectively.

Table captions

Table I. 2-loop fermion contribution to the internal energy $\langle E_W \rangle$. See text for notation.

Table II. 3-loop fermion contribution to the internal energy $\langle E_W \rangle$. See text for notation.

Table III. Coefficients $c_2$ and $c_3$ for SU(2) and $N_f = 3$ at $r = 1$.

Table IV. Coefficients $c_2$ and $c_3$ for SU(3) and $N_f = 3$ at $r = 1$.

Table V. 3-loop correction to asymptotic scaling in SU(2) with $r = 1$ in the energy scheme $\langle E_W \rangle$.

Table VI. 3-loop correction to asymptotic scaling in SU(3) with $r = 1$ in the energy scheme $\langle E_W \rangle$. 
Table I

| κ   | r   | $h_2 \times 10^3$ |
|-----|-----|-----------------|
| 0.1675 | 0.1 | -0.269079(27)  |
| 0.161  | 0.1 | -0.236105(26)  |
| 0.156  | 0.1 | -0.212480(25)  |
| 0.1675 | 0.5 | -0.502547(23)  |
| 0.161  | 0.5 | -0.434010(22)  |
| 0.156  | 0.5 | -0.385777(21)  |
| 0.1675 | 1   | -0.502547(23)  |
| 0.164  | 1   | -0.434010(22)  |
| 0.16   | 1   | -0.385777(21)  |
| 0.156  | 2   | -6.44013(90)   |
| 0.1675 | 2   | -6.44013(90)   |
| 0.161  | 2   | -6.29318(41)   |
| 0.156  | 2   | -6.16472(95)   |

Table II

| κ    | r   | $h_{30} \times 10^4$ | $h_{31} \times 10^3$ | $h_{32} \times 10^3$ |
|------|-----|----------------------|----------------------|----------------------|
| 0.1675 | 0.1 | -0.0194886(93) | 0.030311(54)  | 0.000585(20)  |
| 0.161  | 0.1 | -0.017144(10) | 0.026733(47)  | 0.000451(18)  |
| 0.156  | 0.1 | -0.015466(11) | 0.024140(41)  | 0.000366(17)  |
| 0.1675 | 0.5 | -0.032691(24) | 0.050837(49)  | 0.002041(15)  |
| 0.161  | 0.5 | -0.025205(15) | 0.039464(51)  | 0.001202(14)  |
| 0.1675 | 1   | -0.361568(64) | 0.427183(82)  | 0.049019(10)  |
| 0.164  | 1   | -0.330133(54) | 0.390773(14)  | 0.043556(15)  |
| 0.16   | 1   | -0.293659(46) | 0.348654(20)  | 0.038177(14)  |
| 0.1575 | 1   | -0.270560(43) | 0.322111(22)  | 0.034772(11)  |
| 0.156  | 1   | -0.256620(43) | 0.306132(22)  | 0.032785(66)  |
| 0.1675 | 2   | -1.4249(32)   | 1.6754(19)    | 0.34483(95)   |
| 0.161  | 2   | -1.3838(44)   | 1.6257(34)    | 0.3297(14)    |
| 0.156  | 2   | -1.3541(29)   | 1.5868(22)    | 0.3166(14)    |
Table III

| $\kappa$ | $c_2 \times 10^3$ | $c_3 \times 10^4$ |
|----------|------------------|------------------|
| 0.1675   | $-1.376291(62)$  | 0.60413(60)      |
| 0.164    | $-0.770912(55)$  | 0.73543(54)      |
| 0.16     | $-0.06648(16)$   | 0.89226(47)      |
| 0.1575   | 0.38045(11)      | 0.99446(44)      |
| 0.156    | 0.650945(61)     | 1.05715(42)      |

Table IV

| $\kappa$ | $c_2 \times 10^3$ | $c_3 \times 10^4$ |
|----------|------------------|------------------|
| 0.1675   | 14.64396(11)     | 7.3440(16)       |
| 0.164    | 15.720192(98)    | 7.8775(14)       |
| 0.16     | 16.97251(29)     | 8.5042(12)       |
| 0.1575   | 17.76705(19)     | 8.9061(12)       |
| 0.156    | 18.24794(11)     | 9.1504(12)       |
### Table V

| $\kappa$ | $q_{EW}(N_f = 1)$ | $q_{EW}(N_f = 3)$  |
|--------|------------------|------------------|
| 0.1675 | 0.05318(5)       | 0.2052(2)        |
| 0.164  | 0.05069(4)       | 0.1954(1)        |
| 0.16   | 0.04782(5)       | 0.1842(2)        |
| 0.1575 | 0.04599(5)       | 0.1770(1)        |
| 0.156  | 0.04489(5)       | 0.1727(1)        |

### Table VI

| $\kappa$ | $q_{EW}(N_f = 1)$ | $q_{EW}(N_f = 3)$  |
|--------|------------------|------------------|
| 0.1675 | 0.06645(4)       | 0.2091(2)        |
| 0.164  | 0.06355(4)       | 0.1989(2)        |
| 0.16   | 0.06021(4)       | 0.1871(1)        |
| 0.1575 | 0.05809(4)       | 0.1796(1)        |
| 0.156  | 0.05681(4)       | 0.1752(2)        |
Fig. 1b