The Demand Query Model for Bipartite Matching

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Abstract

We introduce a “concrete complexity” model for studying algorithms for matching in bipartite graphs. The model is based on the “demand query” model used for combinatorial auctions. Most (but not all) known algorithms for bipartite matching seem to be translatable into this model including exact, approximate, sequential, parallel, and online ones.

A perfect matching in a bipartite graph can be found in this model with $O(n^{3/2})$ demand queries (in a bipartite graph with $n$ vertices on each side) and our main open problem is to either improve the upper bound or prove a lower bound. An improved upper bound could yield “normal” algorithms whose running time is better than the fastest ones known, while a lower bound would rule out a faster algorithm for bipartite matching from within a large class of algorithms.

Our main result is a lower bound for finding an approximately maximum size matching in parallel: A deterministic algorithm that runs in $n^{o(1)}$ rounds, where each round can make at most $n^{1.99}$ demand queries cannot find a matching whose size is within $n^{o(1)}$ factor of the maximum. This is in contrast to randomized algorithms that can find a matching whose size is 99% of the maximum in $O(\log n)$ rounds, each making $n$ demand queries.

1 Introduction

In the (unweighted, bipartite) matching problem, a bipartite graph with $n$ left vertices and $n$ right vertices is given, and the problem is to find a maximum-size matching: a set of edges of the graph of largest cardinality such that no two of which share a vertex. This problem has numerous applications and has been studied extensively in a variety of computational models including sequential, parallel, online, and approximate. Many extensions of the problem (e.g. to weighted or to non-bipartite graphs) have been studied as well.

The fastest known deterministic algorithm for the problem \cite{5} is close to half a century old and runs in time $O(n^{5/2})$. It is a long standing open problem whether this running time may be improved, but a faster randomized algorithm whose running time is $n^{\omega}$ (where $\omega = 2.3...$ is the matrix multiplication exponent) was given in \cite{16}. Another long standing open problem is whether there exists a deterministic NC algorithm (a parallel algorithm running in poly-logarithmic time using a polynomial number of processors) for finding a maximum matching. Randomized such algorithms were given in \cite{5,17}.

While an enormous amount of algorithmic work was done on many variants of the matching problem, there is currently no hope for proving hardness results: we simply lack any tools that can prove lower bounds for general algorithms. One approach for progress is to define a \textit{concrete model}
of computation – one that is strong enough to capture many of the known algorithms – and try to understand the complexity in that model. This is our approach here.

1.1 The Demand Query Model

Our computational model is inspired by an economic point of view. We consider the right vertices of our bipartite graph to be items and the left vertices to be agents, each who is interested in acquiring exactly a single item from some subset of items – those that are adjacent to it in the graph. In a general economic setting, a “demand query” (see e.g., [3]) asks an economic agent the following type of question: suppose that each item \( j \), from some set of items, could be purchased for price \( p_j \) – which set of items would you demand to purchase? In our setting, we assume that our agents are “unit demand”, i.e. desire a single item, and as our graphs are unweighted our agent will simply choose the least expensive item. Thus in our setting we consider the following type of query on a bipartite graph.

**Definition 1.** A Demand Query accepts a left vertex \( v \) and an order on the right vertices \( (u_1...u_n) \) and returns the index of the first vertex \( i \) in the order such that there is an edge \( (v,u_i) \) in the graph, or 0 to denote that none exists.\(^1\)

Desiring a concrete model, the only cost of an algorithm that we will count is the number of demand queries required. (It will turn out that none of our upper bounds have any other significant algorithmic costs.) In particular every problem in this model has an upper bound of \( O(n^2) \) as a single demand query can certainly tell us whether an edge \( (v,u) \) exists in the graph.

We first observe that many of the classic algorithms for matching can be “implemented” in this model, including augmenting paths methods and primal-dual auction-like algorithms [2, 6]. These can give us the following basic upper bound:

**Theorem 1.** A maximum size matching in a bipartite graph with \( n \) left and \( n \) right vertices can be found using \( O(n^{3/2}) \) demand queries.

1.2 From the Demand Query Model to General Algorithms

When converting an algorithm in the demand query model into a “real algorithm” that does not have a demand query as a primitive, but rather must implement it over a normal computation model, each demand query can be trivially implemented in \( O(n) \) time by going over the edges of the queried vertex and finding the one with minimum rank in the list. Using this \( O(n^{3/2}) \) bound with this simulation matches the currently best \( O(n^{5/2}) \) algorithm for bipartite matching [8].\(^2\) While we generally focus on the case of dense graphs, we note that for sparse graphs with \( m << n^2 \) edges, one may also obtain the improved known upper bound of \( O(mn^{1/2}) \), as a demand query to vertex \( v \) can in fact be simulated in \( O(d_v) \) time, where \( d_v \) is the degree of \( v \) in the graph.\(^3\) We also find that the complexity of finding an approximately maximal matching, one whose size is at least \( (1 - \epsilon) \) of the maximal in the demand query model is only \( O(n/\epsilon) \). Again, with the direct

\(^1\)For compatibility with the notion of demand queries, we only allow \( v \) to be a left vertex. In the graph context it may be natural to consider also variants of our model where \( v \) can be any vertex in the graph. It turns out that our main lower bound applies to this stronger model as well.

\(^2\)One may certainly attempt to obtain faster simulations of the demand query model by normal algorithms using, say, clever data structures, but we have not been able to do so in general.

\(^3\) Getting the \( O(mn^{1/2}) \) total running time is slightly non-trivial since this requires that high-degree vertices are not queried too often by the simulated algorithm. In case that they are, some data structure work will be needed to even things out in an amortized sense.
simulation of demand queries by “normal” algorithms, this implies the (known) $O(n^2)$ algorithm (linear in the input size) for finding a matching of size at least 99% of maximal.

While most algorithms for bipartite matching seem to be translatable into this model, a known algorithmic technique that does not seem to fall under this model is an algebraic one relying on matrix multiplication. A simple example is solving the decision problem of whether a perfect matching exists using randomization to test whether the symbolic determinant of the graph is identically zero [14] which can be done in time $O(n^\omega)$ (where $\omega = 2.3...$ is the matrix multiplication exponent). A randomized algorithm with the same running time that actually finds a maximum matching is given in [16]. These algorithms are all randomized and it is not clear whether deterministic algorithms can match this performance. Another known technique that does not seem to fall in this model is using interior point methods, which in [15] was used for giving an $O(m^{10/7})$ algorithm, where $m$ is the total number of edges in the graph, which beats the $O(n^{5/2})$ bound for very sparse graphs.

1.3 Lower Bounds?

A lower bound of $\Omega(n)$ for bipartite matching in the demand query model is trivial, even for the approximate version of the problem. Our main open problem is to close the gap between this trivial lower bound and the $O(n^{3/2})$ upper bound: either improve the upper bound, which would likely imply faster normal algorithms, or prove a lower bound, which would rule out faster algorithms from a rather wide class of algorithms.

Open Problem 1: Does there exists a faster algorithm for finding a maximal matching in the demand query model? (I.e. one that uses $o(n^{3/2})$ queries, or perhaps even $O(n)$ queries?)

A hint that an answer may be nontrivial is that the nondeterministic (certificate) complexity of maximum matching in this model is only $\Theta(n)$: a maximum matching can be given by listing the $n$ edges in it, where each edge can be certified by a single demand query, and, by Hall’s theorem, the maximality of said matching of size $k$ can be certified by exhibiting a set vertices (left and right) of total size $2n - k$ with no edges between them, which can be certified using one demand query for each left vertex in the set.

1.4 Related Models

The demand query model lies between two well studied models: Boolean decision trees [5] and communication complexity [12]. In the Boolean decision tree model the allowed queries are only to individual edges, i.e. may ask whether an edge $(v,u)$ exists in the graph. In the communication complexity model the left vertices are treated as agents that each “knows” the set of edges connected to it and considers the amount of communication needed between these agents. This would be equivalent to a decision tree model that allows arbitrary queries about the set of egdes adjacent to any single left vertex.

In the decision tree model, it is not difficult to see that the complexity of even determining whether a perfect matching exists is $\Theta(n^2)$, even non-deterministically (for certifying none existence). In the communication complexity model what is known is identical to what was described

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4 The distinction between the decision problem and the search problem is not significant for anything in this paper though: all upper bounds actually find a matching, and the lower bound applies even for the decision variant.

5 E.g. for distinguishing between a maximum matching of any positive size and zero size, i.e. whether the graph is empty.

6 In principle, as the demand query model formally allows “non-uniform” algorithms, this simulation by normal algorithms is not a formal theorem, but we do not know of any example where this non-uniformity is used.
above in the demand query model, and one may consider the demand query model as an interme-
diate step towards understanding the complexity in the communication complexity model.

A model that turns out to be equivalent, up to log factors, to the demand query model is a
model that allows “OR” queries on the edges of a single left vertex. I.e. an “OR” query specifies a
left vertex \( v \) and a set \( S \) of right vertices and asks whether there exists an edge between \( v \) and some
vertex in \( S \). This binary query is clearly weaker than the demand query, but it is not difficult to
use binary search to simulate demand queries using \( O(\log n) \) “OR” queries. It will be convenient
to prove our lower bound in the binary “OR” query model which them implies the lower bound in
the demand query model.

A slightly stronger model (which is incomparable to the communication complexity model, but
generally seems rather weaker) would allow asking about the “OR” of an arbitrary set of edges,
not just those connected to a single vertex. I.e. such a query can present an arbitrary subset of \( n^2 \)
possible edges and ask whether at least one of them exists in the graph. It is known \[13\] that such
a model is equivalent, up to log factors, to the following complexity measure in Boolean decision
trees: the maximal number of 1-answers on the path to any leaf of the tree.

An even stronger model would allow asking for an OR of a subset of edges and their negations
(e.g. “is either \((u_1,v_1)\) an edge in the graph OR \((u_2,v_2)\) not an edge?”). It can be shown that this
model captures, up to a log factor, the logarithm of minimum Boolean decision tree size. We do
not know any improved bounds in these models and proving an improved upper bound in one of
them is an open problem as well.

1.5 Parallel Algorithms in the Demand Query Model

Up to this point we have introduced our general model, its motivation, and the main open problem.
We now move onto the issue of the parallel complexity in this model, which is our main technical
result.

A parallel algorithm in the demand query model proceeds in rounds, where at each round
multiple demand queries can be made. The point is that the identity of all the queries within
a single round must all be determined only based on the outcomes of queries from the previous
rounds. The two basic parameters in a parallel algorithm are thus the number of rounds and the
(maximum allowed) number of queries per round. Any sequential algorithm that makes a total
of \( q \) queries can be viewed as a parallel algorithm with \( q \) rounds, each of 1 query. Every parallel
algorithm with \( r \) rounds of \( q \) queries each, can be trivially converted to a sequential algorithm with
\( r \cdot q \) queries, but the converse is not necessarily true.

Any problem (on graphs with with \( n^2 \) possible edges) can be trivially solved with a single round
of \( n^2 \) queries. The question that we ask is whether a maximum matching in a bipartite graph (or
at least an approximation thereof) can be found with “few” rounds or “few” queries each, say is
\( n^{o(1)} \) rounds of \( n^{1+o(1)} \) demand queries per round. (Recall that the trivial lower bound on the total
number of queries is \( \Omega(n) \).) In \[4\] a randomized algorithm was presented that operates within fully
within this model and within \( O(\log n/\epsilon^2) \) rounds, each of \( n \) demand queries (a single query per
left vertex) outputs, with high probability, a matching whose size is at least \((1 - \epsilon)\) fraction of the
maximum matching.

**Open Problem 2:** Does there exists a randomized parallel algorithm that uses “few” (even
\( O(\log n) \)) rounds each with “few” (even \( O(n) \)) demand queries and outputs, with high probability,
a perfect matching if one exists.
1.6 Main Result: A Lower Bound for Deterministic Parallel Algorithms

Our main result is a lower bound showing that randomization is crucial here: no deterministic algorithm that uses “few” rounds of “few” queries can get a decent approximation to the maximum matching size. In fact our lower bounds show that the algorithms cannot even distinguish between graphs with a perfect matching and those with a small maximum matching size. We provide a general tradeoff between the number of rounds, the number of queries per round, and the level of approximation possible which in particular implies:

**Instance of main theorem with a specific choice of parameters:** Deterministic algorithms with $n^{1/7}$-rounds each using $n^{8/7}$ demand queries cannot approximate the maximum matching size within a factor of $n^{1/7}$.

Beyond tightening our tradeoffs, a natural challenge is to extend our lower bound to the stronger communication complexity model. Our lower bound uses a direct adversary technique which does not seem to extend to the communication model.

**Open Problem 3:** Does there exist a deterministic communication protocol between the $n$ agents (left vertices) that uses “few” (even $O(\log n)$) rounds of communication where in each round each player sends “few” (even $O(1)$) bits of communication and outputs, with high probability, a perfect matching if one exists.

Improving upon [7,1], in [4] a nearly logarithmic lower bound on the number of rounds was proven if each round allows, say, $\text{polylog}(n)$ bits of communication per player. This bound is exponentially lower than the query bound that we obtain, and unlike our bound, applies also to randomized protocols. A gap between deterministic and randomized simultaneous communication protocols was exhibited in [7].

2 Some algorithms

2.1 Warmup: The Greedy Online Algorithm

We start by looking at the simplest greedy algorithm that also has the advantage of being “online” [10]: we can have the left vertices come one after another in an online fashion, and as each vertex arrives we immediately match him to an arbitrary yet unmatched right vertex. This can be done using a single demand query per vertex: the query will place all the unmatched vertices (in an arbitrary order) before all the matched ones. This is known to produce a matching whose size is at least $1/2$ of the maximum size. A randomized variant fixes a random order on the right vertices and allocates the first unmatched vertex rather in this order than an arbitrary one (still a single demand query per vertex, but now with the fixed random order on the yet unmatched vertices), and it is known that this produces, in expectation, a $(1 - 1/e)$-approximation to the maximum matching [10].

**Proposition 1.** There is a deterministic online algorithm for maximum matching that uses 1-demand query per vertex and provides a $1/2$-approximately maximum matching. There is a randomized online algorithm for maximum matching that uses 1-demand query per vertex and provides a $(1-1/e)$-approximately maximum matching.
2.2 The Ascending Auction Algorithm

We now present the “ascending auction” algorithm of [6, 2] which is naturally described in the demand-query model, and may be viewed as motivating this model. It will be beneficial to present the algorithm as an approximation algorithm that is parametrized by the desired approximation ratio.

**Theorem 2.** For every parameter $0 < \epsilon < 1$, there exists an algorithm that uses $O(n/\epsilon)$ demand queries and produces a matching whose size is at least $(1-\epsilon)$ fraction of the maximum size matching.

Here is the algorithm essentially due to [6]:

1. For each left vertex $u$ initialize “prices” $p_u = 0$.
2. Initialize the matching $M = \emptyset$.
3. Initialize the set of “TBD” vertices $A$ to be the set of all right vertices.
4. While $A$ is not empty:
   a. Pick an arbitrary vertex $v \in A$ and remove it from $A$.
   b. Ask a demand query on $v$ with the order on $u$’s induced by increasing values of $p_u$ (lowest price first, and ties broken arbitrarily), and let $u$ be the answer to that query.
   c. If $p_u < 1$:
      i. If for some vertex $v'$ we have that $(v', u) \in M$, insert $v'$ into $A$ and remove $(v', u)$ from $M$.
      ii. Insert $(v, u)$ into $M$.
      iii. Increase $p_u$ by $\epsilon$.
5. Output $M$

For completeness, the correctness of this algorithm is sketched below. (For ease of exposition let us assume that $\epsilon = 1/t$ for some $t$.)

**Proof.** The invariant that we keep is that for every left vertex $v \notin A$ we have one of the following two cases: either (1) for all existing edges $(v, u)$ we have that $p_u = 1$ or (2) for some edge $(v, u)$ we have that there exists a unique right vertex $u$ such that $(v, u) \in M$ and for all other edges $(v, w)$ it holds that $p_w \geq p_u - \epsilon$.

For an arbitrary matching $N$ let us define $util(v, N) = 1 - p_u$ for $(v, u) \in N$ and $util(v, N) = 0$ is $v$ is unmatched in $N$. It is easy to see that our invariant ensures that for any matching $N$ we have that $util(v, N) \leq util(v, M) + \epsilon$ (where $M$ was the matching produced by the algorithm), and in fact for $v$ that is unmatched in $N$ we don’t even lose $\epsilon$: $util(v, N) = 0 \leq util(v, M)$. Let us now sum $util(v, N)$ up over all left vertices $v$: we get $|N|$ times 1 minus $\sum u p_u$ over all $u$ that are matched in $N$ which is bounded from above by $\sum p_u$ over all right vertices $u$. For the case of the matching $M$ found by the algorithm any unmatched right vertex $u$ still has $p_u = 0$ so we get exactly $\sum p_u$ over all left vertices $u$. We thus have $|N| - \sum u p_u \leq \sum v util(v, N) \leq \sum v util(v, M) + |N|\epsilon = |M| - \sum u p_u + |N|\epsilon$. It follows that $|M| \geq (1-\epsilon)|N|$.

This algorithm is quite naturally interpreted as an ascending auction that can be used also in more general scenarios [11] and in our scenario turns out to also has nice incentive properties [6].
In terms of the running time, every iteration of the main loop makes a single demand query and either increases the price $p_u$ of some right vertex by $\epsilon$ or removes a left vertex (forever) from $A$. There can clearly be at most $n$ iterations that remove a vertex and, since the price of any right vertex never increases above 1, there can be at most $n/\epsilon$ price increase iterations, for a total running time of $O(n/\epsilon)$.

In terms of obtaining the perfectly maximum matching, one could take $\epsilon = 1/(n + 1)$ which due to the integrality of the matching size would imply that the algorithm produces the maximum matching. This makes sense as a way of obtaining a “usual” $O(n^3)$-time algorithm for maximum matching (when each demand query is executed in $O(n)$ time), but in our model this would require $O(n^2)$ demand queries which in our model is trivial.

### 2.3 Augmenting Paths

We now present a variant of the directed connectivity problem, which is used as a step (“augmenting path”) in many algorithms for bipartite matching: We are given some mapping from the right vertices to the left vertices $\pi : R \to (L \cup \{\Lambda\})$ and are given a subset $S$ of left vertices. Our goal is to find a directed path $(v_1, u_1, v_2, u_2, \ldots, v_k, u_k)$ such that $v_1 \in S$, $\pi(u_k) = \Lambda$, and for every $1 \leq i < k$ we have that the edge $(v_i, u_i)$ is in the graph and $\pi(u_i) = v_{i+1}$, or say that none exists.

**Lemma 1.** For every $S$ and $\pi$, this problem can be solved with $O(n)$ demand queries.

**Proof.** We will implement breadth first search in the demand query model:

1. Initialize a FIFO Que $Q$ with all left vertices in $S$, and initialize a set of “discarded” left vertices $D = \emptyset$.

2. Repeat until $Q = \emptyset$:

   (a) Let $v$ be the first vertex in $Q$.

   (b) Pick an arbitrary order of the right vertices such that all vertices $u$ with $\pi(u) = \Lambda$ appear before all others and then appear all vertices with $\pi(u) \not\in (D \cup Q)$ and last appear those with $\pi(u) \in (D \cup Q)$, and ask a demand query from $v$ in this order.

   (c) If $\pi(u) = \Lambda$ Then output the path leading to $v$ and then $u$ and halt.

   (d) If $\pi(u) \not\in (D \cup Q)$ Then enqueue $\pi(u)$ into $Q$, attaching to it the path first leading to $v$, then $u$ and then $\pi(u)$.

   (e) If $\pi(u) \in (D \cup Q)$ Then remove $v$ from $Q$ and insert $v$ into $D$.

3. If a path was not found and the loop terminated due to $Q$ being empty, there is no such path.

In each round we make a single demand query and either insert a new vertex into $Q$ (which we can do at most $n$ times) or remove an element from $Q$ (which again we can do at most $n$ times).

**Note:** We could have alternatively simulated depth first search. It is not difficult to prove a matching lower bound for this problem.

This version of the connectivity problem is exactly what is needed for an augmenting path step used in many matching algorithms: A partial matching defines our mapping $\pi$ by having $\pi(u)$ be defined as the vertex that is matched to it in the partial matching (and $\Lambda$ is $u$ is not matched) and defines the set $S$ of unmatched left vertices. As it is known that a matching in a bipartite graph has maximum size if and only it admits no augmenting path, this gives a $O(n)$ algorithm for testing whether a given matching has maximum size.
2.4 An $O(n^{3/2})$-query algorithm

We now have all the ingredients for describing our best algorithm for maximum matching in this model.

**Theorem 3.** There exists an algorithm for maximum matching in a bipartite graph that uses $O(n^{3/2})$ demand queries.

This is obtained by first running the ascending auction algorithm with $\epsilon = 1/\sqrt{n}$. This requires $O(n^{3/2})$ demand queries and produces a matching whose size is at least $(1 - 1/\sqrt{n})$ times the maximum size. We then run a sequence of augmenting path steps, each of which requires $O(n)$ additional demand queries. Notice that every augmenting path step increases the matching size by 1, and we started with a matching that can be smaller than the maximum matching by at most an additive $O(\sqrt{n})$, at most $\sqrt{n}$ augmenting path steps are needed before the maximum matching is obtained, for a total of at most $O(n^{3/2})$ demand queries.

2.5 A Randomized Parallel Algorithm

In the parallel version of our model we proceed by rounds, where at each round a set of queries is asked, a set that may be determined by the answers to the queries from previous rounds. An $r$-round $q$-query-per-round protocol is one with at most $r$ rounds, where at each round at most $q$ demand queries are made.

**Theorem 4.** *(Dobzinski-Nisan-Oren)* for any $\delta > 0$ there exists a randomized $O(\log n/\delta^2)$-round protocol where at each round there is a single demand query for every left vertex (for a total of $n$ demand queries per round) which returns a matching of size at least $(1 - \delta)$-fraction of the optimal matching.

The algorithm is a randomized parallel variant of the ascending auction algorithm:

1. For each left vertex $u$ initialize “prices” $p_u = 0$.
2. Initialize the matching $M = \emptyset$, and the discarded vertices $D = \emptyset$.
3. Repeat $O(\log n/\delta^2)$ times:

   (a) For each vertex $v \notin D$ that is currently unmatched in $M$, in parallel:

      i. Ask a demand query on $v$ with the order on $u$’s induced by increasing values of $p_u$, with ties broken randomly, and let $u_v$ be the answer to that query.
      ii. If $p_{u_v} > 1$: insert $v$ into $D$.

   (b) For each $u$ such for some $v$ we have that $u = u_v$, increase the price $p_u$ by $\delta$ and pick an arbitrary $v$ such that $u = u_v$ and insert $(v, u)$ into the matching $M$, removing any previous edge matched to $u$, if any.

3 The Lower Bound

It will be more convenient to prove our lower bound with a slightly weaker query, the OR query, which turns out to be essentially equivalent to a demand query, (up to log factors).

**Definition 2.** An OR Query accepts a left vertex $v$ and a subset $S$ of the right vertices and returns whether there exists a vertex $u \in S$ such that $(v, u)$ is an edge in the graph.
While it is clear that a demand query is stronger than an OR query, it turns out that the gap between them is not large:

**Lemma 2.** A demand query can be simulated by \( \log_2(n + 1) \) OR queries.

**Proof.** We simulate each demand query \((v, (u_1, \ldots, u_n))\) by a binary search that uses OR queries: we start by asking whether there is an edge between \(v\) and \(\{u_1, \ldots, u_{n/2}\}\), according to the answer we then either focus on the first half of the variables (asking whether there is an edge to \(\{u_1, \ldots, u_{n/4}\}\)) or the second half (asking whether there is an edge to \(\{u_1, \ldots, u_{3n/4}\}\)), etc.

The overhead in the simulation is clearly optimal since a demand query has \(n\) possible answers. We will proceed by providing our lower bounds in the OR query model, which as we have just shown implies the same lower bounds – with a log factor loss – in the demand query model.

**Theorem 5.** Every deterministic \(r\)-round \(q\)-query (per round) algorithm that uses OR queries cannot distinguish between graphs with a perfect matching (of size \(n\)) and those with a matching of size at most \(\alpha n\) for \(\alpha = r\sqrt{qr\log n}/n\).

**Corollary 1.** Deterministic algorithms with polylog rounds of polylog demand queries per-player can only find matching of size \(O(\sqrt{n})\) in a graph that has a perfect matching.

**Corollary 2.** Deterministic \(r\)-round algorithms require at least \(q = n^2/((r^3\log^4 n))\) demand queries per round in order to find a maximum matching in a bipartite graph or even find a matching whose size is a constant fraction of the maximum size matching.

**Corollary 3.** Deterministic \(n^{1/7}\)-round \(n^{8/7}\) queries-per-round algorithms cannot approximate the maximum matching within a factor of \(n^{1/7}\).

**Proof.** We will describe an adversary algorithm for answering the rounds of queries. At every round, our adversary will decide (based on the queries in this round) on the existence or lack of existence of some subset of the edges of the graph, in a way that the answers to all queries in this round are determined by these choices. The edges that were decided to exist in this round will be called ”YES” edges of the round, and those that were decided not to exist will be called ”NO” edges of the round.

For any query to a set \(S\) of edges that contains some edge that was already decided to exist (a ”YES” edge from some previous round), the answer to this query is already determined so the adversary can simply ignore this query in this round (and thus the discussion below can assume that such sets are never queried). Similarly any query to a set \(S\) that includes some subset of ”NO” edges from previous rounds may be considered by the adversary as though it is a query only to the subset of edges that were previously not answered. The rest of the discussion can thus assume without loss of generality that all edges queried in all queries in this round are “new” ones for which the adversary has not committed to an answer yet.

Specifically, for every OR query on a set \(S\) of edges made in this round the adversary will either fix all the edges in \(S\) to be ”NO” edges (which fixes a negative answer to this query), or will fix (at least) one edge in \(S\) to be ”YES” (which fixes a positive answer to this query). The adversary’s goal is to make sure that after fixing all these ”YES” and ”NO” edges in all \(r\) rounds, the underlying graph can still either have a perfect matching or not have any matching of size greater than \(\alpha n\). The adversary will maintain this property by maintaining the following constraints on the ”YES” and ”NO” edges of each round:

1. For every vertex in the graph, strictly less than \(n/(2r)\) edges adjacent to it are ”NO” edges of the round.
2. The set of vertices that are adjacent to any "YES" edge of the round is of size at most $O(\alpha n/r)$.

Given these conditions, the algorithm cannot distinguish between the following two extreme cases: (a) all edges that were not decided in any round in fact do not exist, in which case the maximum possible matching is of size $O(an)$ since every edge in it must be adjacent to a "YES" edge of some round, and there are at most $O(an/r)$ such vertices in every round (b) all edges that were not decided yet do exist in which case a perfect matching exists (which follows from the fact that the degree of every vertex is strictly more that $n/2$ since in each round strictly less than $n/(2r)$ edges adjacent to any vertex were set to be “NO” edges.)

Here is how the adversary makes decisions for a given round that makes $q$ OR queries on sets $S_1...S_q$ (each of which contains only edges that were not answered to be “YES” or “NO” in any previous round). The adversary will handle separately the OR queries on sets that contain at least $\theta = n\sqrt{\log n}/\sqrt{2qr}$ edges (“big queries”) and those of size at most $\theta$ edges (“small queries”).

**Big queries:** the adversary will pick a set of size $O(n \log n/\theta)$ of vertices with the property that it intersects each of the “Big” queries $S_i$ and answer that all edges connected to them (those which were not already decided) exist, i.e are “YES” edges for this round. This suffices for answering all the big queries with a positive answer. The existence of such a set is given by a random construction: choose every vertex at random to be in this set with probability $O(\log n/\theta)$: for a fixed big query the probability that at non of its edges connects to this chosen set is at most $(1 - \log n/\theta)^{\theta} < 1/n^2$ and since there are certainly less than $n^2$ queries, with high probability the chosen set intersects all big queries simultaneously.

**Small queries:** At this point the adversary has already ensured a positive answer to all big queries, and perhaps also to some small queries, so let us focus on the small queries for which the answer is not yet fixed. There are at most $q\theta$ edges in all of the these small queries combined. Since each edge contains two vertices, there can be at most $(2 \cdot q\theta)/(n/(2r)) = 2\sqrt{2qr \log n}$ vertices in the graph that are each adjacent to more than $n/(2r)$ of these edges, which will be called “heavy” vertices. The adversary will answer “YES” to all edges that are adjacent to one of these heavy vertices and “NO” to all other edges. So we only need to show that there are at most $O(\alpha n/r)$ such heavy vertices, which is true since $\alpha n/r = \sqrt{qr \log n}$ as needed.

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