CONFLUENCE OF HYPERGEOMETRIC FUNCTIONS AND INTEGRABLE
HYDRODYNAMIC TYPE SYSTEMS

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Abstract. It is known that a large class of integrable hydrodynamic type systems can be con-
structed through the Lauricella function, a generalization of the classical Gauss hypergeometric
function. In this paper, we construct novel class of integrable hydrodynamic type systems which
govern the dynamics of critical points of confluent Lauricella type functions defined on finite di-
imensional Grassmannian \(Gr(2, n)\), the set of \(2 \times n\) matrices of rank two. Those confluent functions
satisfy certain degenerate Euler-Poisson-Darboux equations. It is also shown that in general, hy-
drodynamic type system associated to the confluent Lauricella function is given by an integrable
and non-diagonalizable quasi-linear system of a Jordan matrix form. The cases of Grassmannian
\(Gr(2, 5)\) for two component systems and \(Gr(2, 6)\) for three component systems are considered in
details.

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1. Introduction

Systems of quasi-linear partial differential equations of the first order, in particular, the hydro-
dynamic type systems have attracted a considerable interest during last decades due to the rich
variety of mathematical structures associated with and numerous applications in physics (see e.g.
[27, 24, 26, 7, 28]). Recently it was observed [15] that a large class of diagonalizable hydrodynamic
type systems can be viewed also as equations governing the dynamics of critical points for functions
obeying the linear Darboux system for the function \(\Psi(x)\) of \(x = (x_1, \ldots, x_n)\).

\[
\frac{\partial^2 \Psi}{\partial x_i \partial x_k} = A_{ik} \frac{\partial \Psi}{\partial x_i} + A_{ki} \frac{\partial \Psi}{\partial x_k}, \quad i \neq k, i, k = 1, \ldots, n
\]

where the functions \(A_{ik}(x)\) obey the nonlinear Darboux system and values of \(x_i\) at the critical points
of \(\Psi\) are Riemann invariants (see also section 2). The simplest system (1.1), namely, the system of
Euler-Poisson-Darboux (EPD) equations

\[
\frac{\partial^2 \Psi}{\partial x_i \partial x_k} = \frac{1}{x_i - x_k} \left( \epsilon_k \frac{\partial \Psi}{\partial x_i} - \epsilon_i \frac{\partial \Psi}{\partial x_k} \right), \quad i \neq k, i, k = 1, \ldots, n,
\]
with arbitrary constants \((\epsilon_1, \ldots, \epsilon_n)\), provides us the so-called \(\epsilon\)-systems \[22\], the dispersionless coupled KdV equations \[16\] etc. Those integrable hydrodynamic type systems are expressed by the Riemann invariant form for \(u = (u_1, \ldots, u_n)\),

\[
\frac{\partial u_i}{\partial t_k} = \lambda^k_i(u) \frac{\partial u_i}{\partial t_0} \quad \text{for} \quad k = 1, 2, \ldots .
\]

At \(k = 1\), this system gives the generalized \(\epsilon\)-system with \(\lambda^1_i = u_i + \sum_{j=1}^{n} \epsilon_j u_j\) \[22\]. Here the variables \(u(t_1, t_2, \ldots)\) are given by the critical point of a family of generalized hypergeometric functions \[15\] (see also section \[2\]). It was also shown in \[15\] that the EPD equations allow to build highly nontrivial solutions of the Darboux system \[1.1\] associated, for instance, with the multi-phase Whitham equations for KdV and NLS equations \[4, 7\]. This fact demonstrates an importance of the EPD equations \[1.2\] for the approach proposed in \[15\]. In different contexts, the hypergeometric functions and their generalizations also appear in the study of integrable hydrodynamic type systems (see e.g. \[23, 21\]).

The system \[1.2\] plays a fundamental role also in apparently completely different subject. It is the central system of equations in the theory of multi-dimensional hypergeometric functions created by Appell \((n = 2)\) \[11\] and Lauricella \[17\] (arbitrary \(n\), pages 140-143). Theory of such functions has been then generalized \[8, 11, 10, 9\] and various associated problems have been studied (see e.g. \[3, 18, 25, 2, 12\]). The demonstration that finite-dimensional Grassmannians are highly appropriate for the study of these generalized hypergeometric functions was one of the important observations within this development \[8, 11, 10, 9, 2\]. In particular, it was shown \[10, 12\] that within this setting the classical confluence process for the Gauss hypergeometric function as shown in the scheme,

\[
\text{Gauss} \rightarrow \text{Kummer} \quad \downarrow \quad \text{Hermite} \quad \uparrow \quad \text{Airy}
\]

\[
\downarrow \quad \text{Bessel} \quad \uparrow
\]

can be straightforwardly extend to multi-dimensional hypergeometric functions (see section \[3\]).

General multi-dimensional hypergeometric functions are particular solutions of the EPD system \[1.2\]. The corresponding integrable hydrodynamic type systems have been constructed in the paper \[15\] (see, in particular, the equation \((58)\) for which the characteristic speeds are \(\lambda_i = \frac{1}{\epsilon_i} \frac{\partial F_{D}}{\partial u_i}\) with the Lauricella hypergeometric function \(F_D(u_1, \ldots, u_n)\)). So the natural question arises how the confluence process as shown above affects the hydrodynamic type systems associated with the corresponding confluent Lauricella type functions.

This problem is addressed in the present paper. First, we discuss the confluence process for Lauricella functions on the Grassmannians \(\text{Gr}(2, n + 3)\) following \[10, 12\]. At the top cell these Lauricella functions are solutions of the EPD system \[1.2\] with arbitrary \(\epsilon_1, \ldots, \epsilon_n\) and Lauricella function has singular points \(\left\{0, 1, \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}, \infty\right\}\). Confluence means that one or several regular singular points \(x^*_i \to \infty\) in a way that, for instance, \(x_i = \delta x^*_i, \epsilon_i = \frac{\delta}{\epsilon} \epsilon^*_i\) with \(\delta \to 0\) so that the product \(x_i \epsilon_i\) remains finite. In the most degenerate case when all points \(\frac{1}{x_i} \to \infty\) the corresponding system of linear PDEs is given in \[10\],

\[
\frac{\partial \Phi}{\partial x^*_i} = \frac{\partial^2 \Phi}{\partial x^*_i \partial x^*_j}, \quad \text{for all} \quad i + j = l
\]

which is the degenerated GKZ system \[8, 11, 10, 9\]. In the intermediate cases the basic system of linear PDEs is the mixture of degenerated EPD equations (specific Darboux equations \[1.1\] and equations \[1.3\]). It is important that during the confluence process the number of independent variables \(x^*_i\) remains the same.
It was shown that the dynamics of the critical points \( \mathbf{u} = (u_1, \ldots, u_n) \) of the families of Lauricella type functions are governed by integrable hierarchies of hydrodynamic type systems (see e.g. [15], and section 2). In this paper, we extend these results to confluent Lauricella type functions. In particular, the deformations of the critical points of functions obeying the system (1.4) are described by the non-diagonalizable systems (having strongly non-strict hyperbolicity),

\[
\frac{\partial \mathbf{u}}{\partial t_k} = A_k(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial t_0} \quad \text{for} \quad k = 1, 2, \ldots,
\]

where \( \mathbf{u} = (u_1, \ldots, u_n) \), and in particular, at \( k = 1 \), we have an \( n \times n \) Jordan block form,

\[
A_1(\mathbf{u}) = \begin{pmatrix}
  u_1 & 1 & 0 & \cdots & 0 \\
  0 & u_1 & 1 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & u_1 & 1 \\
  0 & \cdots & 0 & 0 & u_1
\end{pmatrix}.
\]

Change of variables \( x_i \) in (1.2) together with an appropriate limit \( \delta \to 0 \) transforms the \( n \)-dimensional EPD system into the equations (1.4) and the generalized \( \epsilon \)-systems (1.3) into the system (1.5).

Functions \( \mathbf{u}(t) \) in (1.5) are not Riemann invariants and the system (1.4) are not of the Darboux form (1.1). So, the system (1.4) is the evidence that the critical points approach considered in [15] is applicable to a wider, than the Darboux system (1.1), class of systems of linear PDEs. The system (1.5) is not tractable via Tsarev’s generalized hodograph equations. Nevertheless, it is integrable in the sense that it has infinite set of commuting flows, and the corresponding hodograph equations are of the matrix type discussed in [13] (see also section 6).

Within different approaches the integrable nondiagonalizable systems of hydrodynamic type have been studied earlier in the papers [5, 6, 19, 20]. However, in contrast to the systems (1.4) with (1.6) whose eigenvalues are degenerate and nondiagonalizable coefficient matrix, all the systems considered there are strictly hyperbolic one having real and distinct eigenvalues.

The paper is organized as follows. In section 2, Lauricella functions and the corresponding hydrodynamic type systems are reviewed. In section 3, we present the generalized hypergeometric functions on Grassmannian \( \text{Gr}(2, n+3) \) and their confluence based on the papers [8, 11, 9, 12]. Each confluence can be parametrized by a partition of the number \( n+3 = i_1 + \cdots + i_m \), denoted by \( n+3 = (i_1, \ldots, i_m) \). In section 4, we give an explicit method to construct the hydrodynamic type systems with two components (\( n = 2 \)) corresponding to the generalized Lauricella functions defined on \( \text{Gr}(2, 5) \). The hydrodynamic system associated with the confluence given by the partition \( 5 = (5) \), the most degenerate case, is shown to be a strongly non-strict hyperbolic system (see (1.5)). In section 5, we construct the hydrodynamic type systems for \( \text{Gr}(2, 6) \). In this case, there are three types of hydrodynamic type systems with three components. The coefficient matrix of the system is either diagonal, a \( 3 \times 3 \) Jordan block or a mixed one having a \( 2 \times 2 \) Jordan block. In particular, we discuss the details of the mixed type. In section 6, we discuss the most degenerate confluent case, i.e. \( n+3 = (n+3) \), and construct the corresponding hydrodynamic type systems given by (1.5). We also provide the hodograph type solutions which are given in matrix form.

In this paper, we present hydrodynamic type systems having one Jordan block in the coefficient matrix. This class of non-diagonalizable hydrodynamic type systems has not been discussed in the context of integrable systems. In a future study, we will extend the systems to have arbitrary number of Jordan blocks, and discuss the systems from the confluent process of the generalized hypergeometric type functions on Grassmannian \( \text{Gr}(k, n) \).
2. Lauricella Functions and Hydrodynamic Type Systems

We start with the following differential, called the Lauricella differential \[ [18], \]
\[
\eta(x; z) dz = \prod_{j=1}^{n} (1 - x_j z)^{-\epsilon_j} \, dz,
\]
where \( x = (x_1, \ldots, x_n) \) with \( x_i \in \mathbb{C} \) and each \( \epsilon_j \in \mathbb{C} \) is a fixed constant.

An important property of the Lauricella differential is that \( \eta(x; z) \) satisfies the EPD system \([12]\),
\[
\frac{\partial^2 \eta}{\partial x_i \partial x_j} = \frac{1}{x_i - x_j} \left( \epsilon_i \frac{\partial \eta}{\partial x_i} - \epsilon_j \frac{\partial \eta}{\partial x_j} \right).
\]
Such solution can be found easily by assuming the solution to be in the separation of variables.

2.1. Hydrodynamic type systems associated to the Lauricella-type functions. Here we explain how we construct hydrodynamic systems from the Lauricella function \([21]\) based on the method proposed in \([15]\).

First we expand the Lauricella function \( \eta(x; z) \) in terms of \( z \),
\[
\eta(x; z) = \sum_{k=0}^{\infty} F^k(x) z^k \quad \text{with} \quad F^0(x) = 1.
\]
Here the first few \( F^k(x) \) are given by
\[
F^1 = \sum_{j=1}^{n} \epsilon_j x_j, \quad F^2 = \sum_{j=1}^{n} \frac{\epsilon_j(\epsilon_j + 1)}{2} x_j^2 + \sum_{i<j} \epsilon_i \epsilon_j x_i x_j, \quad F^3 = \sum_{j=1}^{n} \frac{\epsilon_j(\epsilon_j + 1)(\epsilon_j + 2)}{3!} x_j^3 + \sum_{i<j} \frac{\epsilon_i \epsilon_j}{2} \left( (\epsilon_i + 1)x_i + (\epsilon_j + 1)x_j \right) x_i x_j + \sum_{i<j<l} \epsilon_i \epsilon_j \epsilon_l x_i x_j x_l.
\]
Notice that each \( F^k(x) \) is a solution of the EPD system \([12]\).

Then we define a function, called the generating function of hydrodynamic type system,
\[
\Phi(t; x) := \sum_{k=0}^{m} t_k F^{k+1}(x),
\]
for arbitrary \( m > n \). Since each \( F^k(x) \) satisfies the EPD system, the function \( \Phi(t; x) \) as a function of \( x \) with the parameters \( t \) also satisfies the EPD system.

The critical point \( u = (u_1, \ldots, u_n) \) of \( \Phi(t; x) \) with respect to \( x \) is given by
\[
\frac{\partial \Phi}{\partial x_j} \bigg|_{x=u} = \sum_{k=0}^{m} t_k \frac{\partial F^{k+1}}{\partial x_j} \bigg|_{x=u} = 0 \quad \text{for} \quad j = 1, \ldots, n.
\]
which define \( u \) as the functions of \( t \), i.e. \( u = u(t) \). Let us simply write \( \frac{\partial \Phi}{\partial x_j} \bigg|_{x=u} = \frac{\partial \Phi}{\partial u_j} \) and \( \frac{\partial F^k}{\partial x_j} \bigg|_{x=u} = \frac{\partial F^k}{\partial u_j} \). Then taking the derivative of the critical equations with respect to \( t_i \), we have
\[
\frac{\partial F^{i+1}}{\partial u_j} + \sum_{k=1}^{m} t_k \frac{\partial^2 F^{i+1}}{\partial u_j \partial u_l} \frac{\partial u_l}{\partial t_i} = \frac{\partial F^{i+1}}{\partial u_j} + \sum_{l=1}^{n} \frac{\partial^2 \Phi}{\partial x_j \partial x_l} \bigg|_{x=u} \frac{\partial u_l}{\partial t_i} = 0.
\]
The EPD system \([12]\) implies
\[
\frac{\partial^2 \Phi}{\partial x_j \partial x_l} \bigg|_{x=u} = \frac{1}{u_j - u_l} \left( \epsilon_l \frac{\partial \Phi}{\partial u_j} - \epsilon_j \frac{\partial \Phi}{\partial u_l} \right) = 0 \quad \text{when} \quad j \neq l.
\]
Then we have
\[
\frac{\partial F^{k+1}}{\partial u_j} + \frac{\partial^2 \Phi}{\partial x_j^2} \bigg|_{x=u} \frac{\partial u_j}{\partial t_k} = 0
\]
which then gives a hierarchy of hydrodynamic type systems in the Riemann invariant form for \(u = u(t)\),
\[
\frac{\partial u_j}{\partial t_k} = \lambda_j^k(u) \frac{\partial u_j}{\partial t_0}, \quad \left\{ \begin{array}{l} j = 1, \ldots, n \\ k = 1, \ldots, m. \end{array} \right.
\]
Here \(\lambda_j^k(u)\) is the characteristic speed defined by
\[
\lambda_j^k(u) = \frac{\partial F^{k+1}}{\partial u_j} / \frac{\partial F^1}{\partial u_j} = \frac{1}{\epsilon_j} \frac{\partial F^{k+1}}{\partial u_j},
\]
(note \(\lambda_j^0 = 1\)). The system (2.4) with \(k = 1\) is nothing but the generalized \(\epsilon\)-system [22], that is, we have
\[
\lambda_j^1(u) = u_j + F^1(u) = u_j + \sum_{i=1}^n \epsilon_i u_i.
\]
For example, we have the following hydrodynamic type system for \(n = 2\),
\[
\frac{\partial u_1}{\partial t_1} = ((1 + \epsilon_1)u_1 + \epsilon_2 u_2) \frac{\partial u_1}{\partial t_0}, \\
\frac{\partial u_2}{\partial t_1} = (\epsilon_1 u_1 + (1 + \epsilon_2) u_2) \frac{\partial u_2}{\partial t_0}
\]
In general, the characteristic speed for the \(t_k\)-th flow is given by
\[
\lambda_j^k(u) = \sum_{l+m=k} u_j^l F^m(u).
\]
One has an infinite family of hydrodynamic type systems with all possible values of \((\epsilon_1, \ldots, \epsilon_n)\). In particular, for \(n = 2\), the following cases are well known: (1) for \(\epsilon_1 = \epsilon_2 = \frac{1}{2}\) one has the dispersionless NLS system and its higher counterparts, (2) for \(\epsilon_1 = \epsilon_2 = -\frac{1}{2}\) it is the dispersionless Toda lattice, (3) for \(\epsilon_1 = -\epsilon_2 = -\frac{1}{2}\) it is a mixed dispersions NLS-Toda equation ((49) in [15]), (4) for \(\epsilon_1 = \epsilon_2 = \frac{1}{6}\) one has the dispersionless Boussinesq equation.

Remark 2.1. The compatibility of the system (2.4), \(\frac{\partial^2 u_j}{\partial x_i \partial t_k} = \frac{\partial^2 u_j}{\partial x_l \partial t_k}\), is given by
\[
\frac{\partial \lambda_j^k}{\partial u_i} \frac{\partial \lambda_i^k}{\lambda_i^k - \lambda_j^k} = \frac{\partial \lambda_j^k}{\partial u_i} \frac{\partial \lambda_i^k}{\lambda_i^k - \lambda_j^k} \quad \text{for} \quad i \neq j.
\]
Using the formula of \(\lambda_j^k\) in (2.5) and the EPD equation for \(F^k\), the left hand side of the compatibility equation becomes
\[
\frac{\partial \lambda_j^k}{\partial u_i} \frac{\partial \lambda_j^k}{\lambda_j^k - \lambda_i^k} = \frac{\epsilon_i}{u_i - u_j}.
\]
Since the right hand side does not depend on the index \(k\) for \(t_k\), we have the compatibility among the different equations in the hierarchy (2.4).

Remark 2.2. One should note that the critical equations (2.3) provide the hodograph solutions of the system (2.4), i.e.
\[
t_0 + \sum_{k=1}^m t_k \lambda_j^k(u) = 0 \quad \text{for} \quad j = 1, \ldots, n.
\]
As we will show in the later sections, the hodograph form will be extended to have a matrix form [13] which is directly obtained from the critical equations of the corresponding function $\Phi(t; x)$ for our new class of hydrodynamic type systems.

3. Lauricella Type Functions and Their Confluences

The Lauricella function defined by
\[
F(x) = \int_{\Delta} \eta(x; z) dz = \int_{\Delta} \prod_{j=1}^{n} (1 - x_j z)^{-\epsilon_j} dz,
\]
was introduced as a multivariable extension of the Gauss hypergeometric function,
\[
F(\alpha, \beta, \gamma; x) = \int_{0}^{1} z^{\alpha-1}(1 - z)^{-\epsilon_{\alpha}} \cdot \gamma^{-1}(1 - xz)^{-\beta} dz,
\]
which is the case $n = 3$ with $x_1 = 0$, $x_2 = 1$ and $x_3 = x$. The Gauss differential $\eta(0, 1, x; z)dz$ has the regular singular points $\{0, 1, \frac{1}{x}, \infty\}$. It is then well-known that the Gauss hypergeometric function is reduced to the Kummer confluent hypergeometric function after the confluence of the singular points $\frac{1}{x}$ and $\infty$. Furthermore, the Kummer function can be reduced to either the Hermite-Weber function or Bessel function by taking further confluences. The confluences are parametrized by the partitions of the number $n + 3 = 4$.

Let $(i_1, \ldots, i_m)$ denote the partition $n + 3 = i_1 + \cdots + i_m$ with $i_1 \geq i_2 \geq \ldots \geq i_m$. Each number $i_j$ represents the number of confluences at the corresponding singular point for the Gauss (Lauricella) differential. For $n = 3$, i.e. the case of Gauss hypergeometric function, we have $4 = (1, 1, 1)$. Then the partition $4 = (2, 1, 1)$ corresponds to the Kummer function, $4 = (2, 2)$ to the Bessel function, $4 = (3, 1)$ to the Hermite-Weber function, and $4 = (4)$ to the Airy function as illustrated in the diagram below.

\[
\begin{array}{ccc}
\text{Gauss} & \rightarrow & \text{Kummer} \\
(1, 1, 1) & \rightarrow & (2, 1, 1) \\
\text{Hermite} & \rightarrow & \text{Airy} \\
\text{Bessel} & \rightarrow & (4)
\end{array}
\]

In this section, we extend those confluence processes for the Lauricella function. For the general references, we recommend the following papers, which are most relevant to our study, [8, 11, 10, 9, 2, 12].

3.1. The Generalized Hypergeometric Functions. To explain the confluence for the Lauricella function, we first introduce the generalized (Aomoto-Gel’fand) hypergeometric function on the Grassmannian $\text{Gr}(2, n + 3)$, which generalizes the Lauricella function [8, 9, 2]. Let $\zeta$ be a point of Grassmannian $\text{Gr}(2, n + 3)$. That is, $\zeta$ can be expressed by a $2 \times (n + 3)$ matrix of rank 2, denoted by $\zeta \in M_{2 \times (n + 3)}(\mathbb{C})$. Recall that $\text{Gr}(2, n + 3)$ is given by $\text{Gr}(2, n + 3) \cong \text{GL}_2(\mathbb{C}) \backslash M_{2 \times (n + 3)}(\mathbb{C})$. Then the generalized hypergeometric function is defined by
\[
F(\zeta; \mu) = \int_{\Delta} \chi(\tau; \mu) \omega_\tau \quad \text{with} \quad \omega_\tau := \tau_0 d\tau_1 - \tau_1 d\tau_0
\]
where $\tau = (\tau_0 : \tau_1) \in \mathbb{CP}^1$, and $\Delta$ is a path on $\mathbb{CP}^1$. The function $\chi(\tau; \mu)$ is the character of the centralizer of a regular element of $\mathfrak{gl}_{n+3}(\mathbb{C})$ with the weight $\mu = (\mu_0, \ldots, \mu_n)$. Each regular element $A \in \mathfrak{gl}_{n+3}$ can be expressed as a Jordan matrix associated to the partition $n + 3 = (i_1, \ldots, i_m)$,
\[
A_{(i_1, \ldots, i_m)} := A_{i_1} + A_{i_2} + \cdots + A_{i_m}.
\]
where $A_{i_k}$ is the $i_k \times i_k$ Jordan block with an eigenvalue $a_{i_k}$, i.e.
\[
A_{i_k} := \begin{pmatrix}
a_{i_k} & 1 & 0 & \cdots & 0 \\
0 & a_{i_k} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1 \\
0 & 0 & \cdots & \cdots & a_{i_k}
\end{pmatrix} = a_{i_k}I_{i_k} + \Lambda_{i_k},
\]
where $I_{i_k}$ is the $i_k \times i_k$ identity matrix, and $\Lambda_{i_k}$ is the nilpotent matrix having 1’s in the upper super-diagonal. Here one should have $a_{i_j} \neq a_{i_k}$ if $i_j \neq i_k$. The partition $(i_1, \ldots, i_m)$ gives a parametrization of the corresponding centralizer which we denote
\[
H_{(i_1, \ldots, i_m)} := \{ h \in \text{GL}_{n+3}(\mathbb{C}) : hA_{(i_1, \ldots, i_m)} = A_{(i_1, \ldots, i_m)}h \}.
\]
The $H_{(i_1, \ldots, i_m)}$ can be expressed by
\[
H_{(i_1, \ldots, i_m)} = H_{i_1} \oplus H_{i_2} \oplus \cdots \oplus H_{i_m},
\]
where $H_k$ is given by the set of matrices with the form,
\[
\sum_{j=0}^{k-1} h_j \Lambda_k^j = \begin{pmatrix} h_0 & h_1 & \cdots & h_{k-1} \\
0 & h_0 & \cdots & h_{k-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & h_0 \end{pmatrix},
\]
where $h_0 \in \mathbb{C}^\times$ and $h_j \in \mathbb{C}$ for $j = 1, \ldots, k - 1$. In particular, $H_{(1, \ldots, 1)}$ consists of the diagonal matrices with distinct diagonal elements. One should note that the dimension of the group $H_{(i_1, \ldots, i_m)}$ is $n + 3$ which is just the number of parameters in the group.

Following the paper [12], we construct a group character $\chi$ for each $H_k$ (more precisely, its universal cover $\tilde{H}_k$, and $\chi : \tilde{H}_k \to \mathbb{C}^\times$) as follows. First introduce a function $\theta_j(h)$ on $H_k$ for $j = 0, 1, \ldots, k - 1$,
\[
\log \left( h_0 + \sum_{j=1}^{k-1} h_j \Lambda_k^j \right) = \log h_0 + \sum_{j=1}^{k-1} \theta_j(h)\Lambda_k^j.
\]
First four terms of $\theta_j(h)$ are given by
\[
\begin{align*}
\theta_1(h) &= \frac{h_1}{h_0}, \\
\theta_2(h) &= \frac{h_2}{h_0} - \frac{1}{2} \left( \frac{h_1}{h_0} \right)^2, \\
\theta_3(h) &= \frac{h_3}{h_0} - \frac{h_1 h_2}{h_0^2} + \frac{1}{3} \left( \frac{h_1}{h_0} \right)^3, \\
\theta_4(h) &= \frac{h_4}{h_0} - \frac{h_1 h_3}{h_0^2} - \frac{1}{2} \left( \frac{h_2}{h_0} \right)^2 + \frac{h_1^2 h_2}{h_0^3} - \frac{1}{4} \left( \frac{h_1}{h_0} \right)^4.
\end{align*}
\]
Then the character $\chi(h; \mu)$ with $\mu = (\mu_0, \mu_1, \ldots, \mu_{k-1}) \in \mathbb{C}^k$ is defined by
\[
\chi(h_0, h_1, \ldots, h_{k-1}; \mu) = h_0^{\mu_0} \exp \left( \sum_{j=1}^{k-1} \mu_j \theta_j(h) \right).
\]
For the centralizer $H_{(i_1, \ldots, i_m)}$, we have the group character defined by
\[
(3.2) \quad \chi(h; \mu) = \prod_{k=1}^{m} \chi_{i_k}(h^{(k)}; \mu^{(k)}) = \prod_{k=1}^{m} \left( h_0^{(k)} \right)^{\mu_0^{(k)}} \exp \left( \sum_{j=1}^{i_k-1} \mu_j^{(k)} \theta_j(h^{(k)}) \right).
\]
where \( h \) is assigned as \( h = (h_0^{(1)}, \ldots, h_{i_1-1}^{(1)}, h_0^{(2)}, \ldots, h_{i_2-1}^{(2)}, \ldots, h_0^{(m)}, \ldots, h_{i_m-1}^{(m)}) \). In the case of \( H_{(1,\ldots,1)} \), we have
\[
\chi(h;\mu) = \prod_{k=1}^n \left(\frac{h_0^{(k)}}{h_0^{(k)}}\right)^{\mu_0^{(k)}}.
\]

Note that the parameters \( h_j^{(k)} \) for \( k = 1, \ldots, m \) are given by
\[
h_0^{(k)} \in \mathbb{C}^\times \quad \text{and} \quad h_j^{(k)} \in \mathbb{C} \quad \text{for} \quad j = 1, \ldots, i_k - 1.
\]

We now consider an action on the subset \( Z \subset M_{2\times(n+3)}(\mathbb{C}) \) whose \( 2 \times 2 \) minors are all nonzero (i.e. \( Z \) can be identified with the top cell of the Grassmannian \( \text{Gr}(2, n+3) \)),
\[
\text{GL}_2(\mathbb{C}) \times H_{(i_1,\ldots,i_m)} : Z \longrightarrow M_{2\times(n+3)}(\mathbb{C})
\]
\[
(g,h) \quad \mapsto \quad \zeta \quad \mapsto \quad g\zeta h
\]

Note here that the action \( \text{GL}_2(\mathbb{C}) \) from the left gives a canonical form of the point in \( \text{Gr}(2, n+3) \).

Then the space of matrices obtained by the image of this action may be expressed in the form,
\[
Z_{i_1,\ldots,i_m} = \left\{ \zeta = (\zeta^{(1)}, \ldots, \zeta^{(m)}) \in M_{2\times(n+3)} : \zeta^{(j)} = (\zeta_0^{(j)}, \ldots, \zeta_{i_j-1}^{(j)}) \in M_{2\times i_j}, j = 1, \ldots, m \right\}
\]
with \( \det(\zeta_0^{(j)}, \zeta_1^{(j)}) \neq 0 \), \( \det(\zeta_0^{(i)}, \zeta_0^{(j)}) \neq 0 \).

One should note that the dimension of \( Z_{(i_1,\ldots,i_m)} \) is \( n \), i.e.
\[
\dim \text{Gr}(2, n+3) - (\dim H_{(i_1,\ldots,i_m)} - 1) = 2(n+1) - (n+2) = n.
\]

Then we define the generalized Lauricella function \( \eta(x; z) \) by
\[
(3.3) \quad \eta(x; z) dz = \chi(\tau\zeta; \mu) \omega_\tau
\]
where \( x = x(\zeta), z = \frac{\tau_0}{\tau_1}, \tau = (\tau_0 : \tau_1) \in \mathbb{CP}^1 \) and \( \zeta \in Z_{i_1,\ldots,i_m} \).

### 3.2. Examples for \( \text{Gr}(2,5) \)
We here compute all the generalized Lauricella differentials \( \chi(\tau\zeta; \mu) \omega_\tau = \eta(x; z) dz \) for \( \text{Gr}(2,5) \) i.e. \( n = 2 \). We have 7 cases:

1. With the partition \( 5 = (1,1,1,1,1) \), we have
   \[
   \zeta = \begin{pmatrix} 1 & 0 & 1 & 1 & -1 & -x_1 & -x_2 \\ 0 & 1 & -1 & -x_1 & 1 & \end{pmatrix}
   \]
   with \( x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 - x_2) \neq 0 \)
   \[
   \chi(\tau\zeta; \mu) \omega_\tau = \tau_0^{\mu_0^{(1)}} \tau_1^{\mu_1^{(1)}} (\tau_0 - \tau_1)^{\mu_0^{(2)}} (\tau_0 - x_1 \tau_1)^{\mu_0^{(3)}} (\tau_0 - x_2 \tau_1)^{\mu_0^{(4)}} (\tau_0 - x_2 z)^{\mu_0^{(5)}} \omega_\tau
   \]
   \[
   = z^{\mu_0^{(2)}} (1 - z)^{\mu_0^{(3)}} (1 - x_1 z)^{\mu_0^{(4)}} (1 - x_2 z)^{\mu_0^{(5)}} dz = \eta(x; z) dz.
   \]
   where \( z = \tau_1/\tau_0 \), and we have used \( \sum_{j=0}^4 \mu_0^{(j)} = -2 \). Notice that all the minors of this matrix is nonzero, i.e. \( \zeta \) gives a point on the top cell of \( \text{Gr}(2,5) \). Setting \( \mu_0^{(4)} = -\epsilon_1, \mu_0^{(5)} = -\epsilon_2, \mu_0^{(2)} = -\epsilon_3, \mu_0^{(3)} = -\epsilon_4 \) with \( x_3 = 0 \) and \( x_4 = 1 \), we have the Lauricella differential for \( n = 4 \) with regular singular points \( \{0, 1, \frac{1}{x_1}, \frac{1}{x_2}, \infty\} \), i.e.
   \[
   \eta(x; z) = z^{-\epsilon_3}(1 - z)^{-\epsilon_4}(1 - x_1 z)^{-\epsilon_1}(1 - x_2 z)^{-\epsilon_2}.
   \]
(2) With the partition $5 = (2, 1, 1, 1)$, we have
\[
\zeta = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & x_1 & 1 & -1 & -x_2 \end{pmatrix}
\]
with $x_1 x_2 (x_2 - 1) \neq 0$
\[
\chi(\tau \zeta; \mu) \omega_\tau = \tau_0^{(1)} \tau_1^{(2)} e^{\mu_1^{(1)} x_1 \frac{x_1}{\tau_0} + \mu_2^{(2)} (\tau_0 - x_2 \tau_1)} \mu_0^{(3) \omega_\tau}
\]
\[
= e^{\mu_1^{(1)} x_1 z \mu_0^{(2)} (1 - z)} \mu_0^{(3) (1 - x_2 z)} d \zeta = \eta(x; z) d z.
\]
Note that the minors are nonzero except one with the indices $(2, 3)$, i.e. this is a point of a cell with co-dimension one. Now the singular points are $\{0, 1, \frac{1}{x_2}, \infty\}$, and in particular, the point $z = \infty$ is an irregular singular point due to the confluence of the regular singular points $z = \frac{1}{x_1}$ and $z = \infty$ of the Lauricella differential in (1). The generalized Lauricella function for this case is given by
\[
\eta(x; z) = z^{-\varepsilon_3} (1 - z)^{-\varepsilon_4} (1 - x_2 z)^{-\varepsilon_2} e^{\varepsilon_1 x_1}.
\]

(3) With $5 = (2, 2, 1)$, we have
\[
\zeta = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & x_1 & 1 & 0 & -x_2 \end{pmatrix}
\]
with $x_1 x_2 \neq 0$
\[
\chi(\tau \zeta; \mu) \omega_\tau = \tau_0^{(1)} \tau_1^{(2)} e^{\mu_1^{(1)} x_1 \frac{x_1}{\tau_0} + \mu_2^{(2)} (\tau_0 - x_2 \tau_1)} \mu_0^{(3) \omega_\tau}
\]
\[
= e^{\mu_1^{(1)} x_1 z \mu_0^{(2)} (1 - z)} \mu_0^{(3) (1 - x_2 z)} d \zeta = \eta(x; z) d z.
\]
Note that two minors with the index sets $(2, 3)$ and $(1, 4)$ are zero, i.e. this is a point of a cell with co-dimension two. The generalized Lauricella function is then given by
\[
\eta(x; z) = z^{-\varepsilon_3} (1 - x_2 z)^{-\varepsilon_2} e^{\varepsilon_1 x_1 z \mu_0^{(4)} + \varepsilon_4}.
\]

(4) With $5 = (3, 1, 1)$, we have
\[
\zeta = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & x_1 & 1 & -x_2 \end{pmatrix}
\]
with $x_1 x_2 \neq 0$
\[
\chi(\tau \zeta; \mu) \omega_\tau = \tau_0^{(1)} \tau_1^{(2)} e^{\mu_1^{(1)} x_1 \frac{x_1}{\tau_0} + \mu_2^{(2)} (\tau_0 - x_2 \tau_1)} \mu_0^{(3) \omega_\tau}
\]
\[
= e^{\mu_1^{(1)} x_1 z \mu_0^{(2)} (1 - z)} \mu_0^{(3) (1 - x_2 z)} d \zeta = \eta(x; z) d z.
\]
Note that two minors with the index sets $(2, 3)$, $(2, 4)$ and $(3, 4)$ are zero. Here the vanishing minor with $(3, 4)$ is a consequence of the Plücker relation, and the $\zeta$ is a point of a cell with co-dimension two. The generalized Lauricella function is then given by
\[
\eta(x; z) = z^{-\varepsilon_3} (1 - x_2 z)^{-\varepsilon_2} e^{\varepsilon_1 x_1 z \mu_0^{(4)} + \varepsilon_4 z}.
\]

(5) With $5 = (3, 2)$, we have
\[
\zeta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_1 & 1 & x_2 \end{pmatrix}
\]
with $x_1 x_2 \neq 0$
\[
\chi(\tau \zeta; \mu) \omega_\tau = \tau_0^{(1)} \tau_1^{(2)} e^{\mu_1^{(1)} x_1 \frac{x_1}{\tau_0} + \mu_2^{(2)} (\tau_0 - x_2 \tau_1)} \mu_0^{(3) \omega_\tau}
\]
\[
= z^{\mu_0^{(3)}} e^{\mu_1^{(1)} z \mu_2^{(2)} (x_1 z - \frac{1}{x_2} z^2) + \mu_2^{(2)} x_2 z} d \zeta = \eta(x; z) d z.
\]
Note that the minors with the index sets \((2, 3), (2, 4)\) and \((2, 5)\) are zero. Other vanishing minors are obtained by the Plücker relations, and the \(ζ\) is a point of a cell with co-dimension three. The generalized Lauricella function is then given by

\[
\eta(\mathbf{x}; z) = z^{-\varepsilon_3} e^{\varepsilon_1 (x_1 z - \frac{1}{2} z^2) + \varepsilon_2 x_2 z + \varepsilon_4 z}.
\]

(6) With \(5 = (4, 1)\), we have

\[
ζ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x_1 & -x_2 \end{pmatrix} \quad \text{with} \quad x_1 x_2 \neq 0
\]

\[
\chi(τζ; μ) ω_τ = \frac{e^{μ(1)_{0}}}{τ_0^{μ(1)_{0}}} μ^{(1)_{0}} μ^{(1)_{0}} + μ^{(1)_{0}} \left( -\frac{1}{2} \left( \frac{1}{τ_0} \right)^2 \right) + μ^{(1)_{0}} \left( -x_1 \frac{1}{τ_0} + \frac{1}{2} \left( \frac{1}{τ_0} \right)^2 \right) \left( τ_0 - x_2 τ_1 \right) μ^{(2)_{0}} ω_τ
\]

\[
= e^{μ(1)_{0}} z^{-μ(1)_{0}} \frac{1}{2} z^2 - μ^{(1)_{0}} \left( x_1 z - \frac{1}{2} z^2 \right) \left( 1 - x_2 z \right) μ^{(2)_{0}} dz = η(\mathbf{x}; z) dz.
\]

Note that the minors with the index sets \((1, 3), (2, 3), (2, 4), (3, 4)\) and \((3, 5)\) are zero. Here the vanishing minor with \((3, 4)\) is a consequence of the Plücker relation, and the \(ζ\) is a point of a cell with co-dimension three. The generalized Lauricella function is then given by

\[
η(\mathbf{x}; z) = (1 - x_2 z)^{-ε_2} e^{ε_1 (x_1 z - \frac{1}{2} z^2) + ε_3 z + ε_4 z^2}.
\]

(7) With \(5 = (5)\), we have

\[
ζ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x_1 & -x_2 \end{pmatrix} \quad \text{with} \quad x_1 x_2 \neq 0
\]

\[
\chi(τζ; μ) ω_τ = \frac{e^{μ(1)_{0}}}{τ_0^{μ(1)_{0}}} μ^{(1)_{0}} μ^{(1)_{0}} + μ^{(1)_{0}} \left( -x_1 \frac{1}{τ_0} + \frac{1}{2} \left( \frac{1}{τ_0} \right)^2 \right) + μ^{(1)_{0}} \left( -x_2 \frac{1}{τ_0} + x_1 \left( \frac{1}{τ_0} \right)^2 - \frac{1}{4} \left( \frac{1}{τ_0} \right)^4 \right) \left( τ_0 - x_2 τ_1 \right) μ^{(2)_{0}} ω_τ
\]

\[
= e^{μ(1)_{0}} z^{-μ(1)_{0}} \frac{1}{2} z^2 - μ^{(1)_{0}} \left( x_1 z - \frac{1}{2} z^2 \right) + μ^{(1)_{0}} \left( -x_2 x_1 z^2 - \frac{1}{4} z^4 \right) dz = η(\mathbf{x}; z) dz.
\]

Note that the minors with the index sets \((1, 3), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5)\) and \((4, 5)\) are zero. This \(ζ\) is a point of a cell with co-dimension four. The generalized Lauricella function is then given by

\[
η(\mathbf{x}; z) = e^{ε_1 (x_1 z - \frac{1}{2} z^2) + ε_2 (x_2 z - x_1 z^2 - \frac{1}{4} z^4) + ε_3 z + ε_4 z^2}.
\]

3.3. The most degenerate case for \(\text{Gr}(2, n + 3)\). In the case of \(\text{Gr}(2, n + 3)\), the character \(χ\) associated to the partition \(n + 3 = (n + 3)\) can be calculated as follows. A canonical form of \(ζ \in \text{Gr}(2, n + 3)/H_{(n+3)}\) is expressed by

\[
ζ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & -x_1 & -x_2 & \cdots & -x_n \end{pmatrix} \quad \text{with} \quad \prod_{j=1}^{n} x_n \neq 0.
\]

Then we have

\[
τζ = (τ_0, τ_1, 0, -x_1 τ_1, -x_2 τ_1, \ldots, -x_n τ_1)
\]

and the corresponding character is

\[
χ(τζ, μ) = τ_0^{μ(0)} \exp \left( \sum_{i=1}^{n+2} μ_i \theta_i(\mathbf{x}; z) \right) \quad \text{with} \quad z = \frac{τ_1}{τ_0},
\]

where the sum in the exponent is given by

\[
\sum_{i=1}^{n+2} μ_i \theta_i(\mathbf{x}; z) = \sum_{m=1}^{n} (-1)^m \left( \sum_{l=1}^{m-1} μ_{l+m+1} x_1 \right) z^m + φ(z),
\]
Here \( \varphi(z) \) is the part depending only on \( z \). We introduce the variables,

\[
y_m := (-1)^m \sum_{i=1}^{n-m+1} \mu_{i+m+1} x_i,
\]

that is, we have simply

\[
\sum_{i=1}^{n+2} \mu_i \theta_i(x; z) = \xi(y; z) + \varphi(z) \quad \text{with} \quad \xi(y; z) := \sum_{m=1}^{n} y_m z^m.
\]

Then choosing \( \mu_0 = -2 \), we have

\[
(3.4) \quad \eta(y; z) \, dz = \chi(\tau \zeta, \mu) \omega_\tau = f(z) \exp \left( \sum_{m=1}^{n} y_m z^m \right) \, dz.
\]

4. Hydrodynamic systems associated to confluent Lauricella functions on \( \text{Gr}(2,5) \)

As explained in the previous section, the number of free variables in \( Z_{(i_1, \ldots, i_m)} \) is given by \( n \).

For example, the case \( n = 1 \) has a single variable \( x = x_1 \), and the Lauricella differential given by

\[
\eta(x; z) \, dz = z^{-r_1}(1-z)^{-r_2}(1-xz)^{-c_3} \, dz
\]

gives the Gauss hypergeometric function,

\[
F_{\text{Gauss}}(x) = \int_{\Delta} z^{-r_1}(1-z)^{-r_2}(1-xz)^{-c_3} \, dz.
\]

The points \( \{0, 1, \frac{1}{x}, \infty\} \) are regular singular points of the hypergeometric equation.

The confluence from Gauss to Kummer can be directly obtained from the Gauss hypergeometric function by taking the limit \( \delta \to 0 \) with

\[
x = \delta y \quad \text{and} \quad \epsilon_3 = \frac{1}{\delta}.
\]

The limit then gives

\[
\lim_{\delta \to 0} (1-xz)^{-\epsilon_3} = \lim_{\delta \to 0} (1-\delta yz)^{-\frac{1}{\delta}} = e^{yz}.
\]

The corresponding hypergeometric function, called the Kummer confluent hypergeometric function, is given by

\[
F_{\text{Kummer}}(y) = \int_{\Delta} z^{-r_1}(1-z)^{-r_2} e^{yz} \, dz.
\]

In this limiting process, the point \( z = \infty \) is now an irregular singular point. The partition \( (2, 1, 1) \) can be considered as the multiplicity of the singular points \( (\infty, 0, 1) \) of the system \( (2.1) \). That is, the limit gives \( \frac{1}{x} \to \infty \).

4.1. Confluent hydrodynamic systems. Example 3.2 in the previous section shows that there are, in fact, only three different confluent Lauricella-type functions. They are of the following form:

1. For the cases with the partitions \( 5 = (i_1, \ldots, i_p) \) with \( 2 \leq p \leq 4 \), except \( 5 = (3, 2) \), the Lauricella type function has the form,

\[
\eta_1(x, y; z) = f(z)(1-xz)^{-\epsilon_3} e^{yz}
\]

with some function \( f(z) \).

2. For the case with \( 5 = (3, 2) \), we have

\[
\eta_2(x, y; z) = g(z)e^{(y_1+y_2)z}
\]

with a function \( g(z) \).

3. This corresponds to the most degenerate case with \( 5 = (5) \), and the Lauricella-type function is

\[
\eta_3(x, y; z) = h(z)e^{y_1z+y_2z^2},
\]

with some \( h(z) \).
The following subsections, we construct hydrodynamic type system associated to those two cases.

4.1.1. Case (1). The Lauricella-type function \( \eta_1(x, y; z) \) can be also obtained directly from the Lauricella function (2.1) by taking the limit \( \delta \to 0 \) with

\[
x_1 = x, \quad x_2 = \delta y, \quad \epsilon_1 = \epsilon, \quad \epsilon_2 = \frac{1}{\delta}.
\]

The corresponding limit of the EPD equation (1.2) for \( \eta_1(x, y; z) \) is

\[
(4.1) \quad x \frac{\partial^2 \eta_1}{\partial x \partial y} = \frac{\partial \eta_1}{\partial x} - \epsilon \frac{\partial \eta_1}{\partial y}
\]

which is not of EPD type. The function \( F(x, y) = \int_{\Delta} f(z) \eta_1(x, y; z) \, dz \) with a particular choice of \( f(z) \) describes the confluence of Appell’s function \( F_D \) to Kummer’s confluent hypergeometric function (see subsection 8.7 in [12]).

Expanding the function \( \eta_1(x, y; z) \) with \( f(z) = 1 \) in terms of \( z \) gives

\[
\eta_1(x, y; z) = \sum_{j=0}^{\infty} F_{k+1}^1(x, y) z^j,
\]

where \( (\epsilon, i) := \epsilon(\epsilon + 1) \cdots (\epsilon + i - 1) \). The first few terms of \( F_{k+1}^1(x, y) \) are given by

\[
F_1^1 = \epsilon x + y, \quad F_2^1 = \frac{\epsilon(\epsilon + 1)}{2} x^2 + \epsilon xy + \frac{1}{2} y^2.
\]

We now define the generating function \( \Phi_1(t; x, y) \),

\[
\Phi_1(t; x, y) := \sum_{k=0}^{m} t_k F_{k+1}^1(x, y).
\]

Note here that \( \Phi_1(t; x, y) \) also satisfies the equation (4.1), the confluent EPD equation. The critical point at \( (x, y) = (u_1, u_2) \) is defined by

\[
\frac{\partial \Phi_1}{\partial x} \bigg|_{(u_1, u_2)} = \frac{\partial \Phi_1}{\partial y} \bigg|_{(u_1, u_2)} = 0,
\]

which then give a deformation of the critical point \( u = u(t) \). One should also note from (4.1) that at the critical point, we have

\[
\frac{\partial^2 \Phi_1}{\partial x \partial y} \bigg|_{(u_1, u_2)} = 0.
\]

Then taking the derivatives of the critical equations with respect to \( t_k \), we have

\[
\frac{\partial F_{k+1}^1}{\partial u_i} + \Phi_{1,i} \frac{\partial u_i}{\partial t_k} = 0 \quad \text{for} \quad i = 1, 2,
\]

where \( \Phi_{1,1} := \frac{\partial^2 \Phi_1}{\partial x^2} \bigg|_{(u_1, u_2)} \) and \( \Phi_{1,2} := \frac{\partial^2 \Phi_1}{\partial y^2} \bigg|_{(u_1, u_2)} \). This system then gives the hydrodynamic type equations for \( u = (u_1, u_2) \),

\[
\frac{\partial u_i}{\partial t_k} = \lambda_k^i \frac{\partial u_i}{\partial x} \quad \text{for} \quad k = 1, \ldots, m.,
\]

where the characteristic velocities are given by

\[
\lambda_1^i = \frac{1}{\epsilon} \frac{\partial F_{k+1}^1}{\partial x} \bigg|_{(u_1, u_2)} \quad \text{and} \quad \lambda_2^k = \frac{\partial F_{k+1}^1}{\partial y} \bigg|_{(u_1, u_2)}.
\]

That is, the functions \( u_1 \) and \( u_2 \) are Riemann invariants for this confluent system.
Using the formula of $F^k$ above, one can show that
\[
\frac{1}{\epsilon} \frac{\partial F^{k+1}_1}{\partial x} = \sum_{i+j=k+1} x^i F^j_1 \quad \text{and} \quad \frac{\partial F^{k+1}_1}{\partial y} = F^k_1.
\]
The first two members with $k=1,2$ of this hierarchy are
\[
\frac{\partial u_1}{\partial t_1} = ((1+\epsilon)u_1 + u_2) \frac{\partial u_1}{\partial t_0},
\]
\[
\frac{\partial u_2}{\partial t_1} = (\epsilon u_1 + u_2) \frac{\partial u_2}{\partial t_0}
\]
and
\[
\frac{\partial u_1}{\partial t_2} = \left( \frac{1}{2} (\epsilon + 1)(\epsilon + 2)u_1^2 + (\epsilon + 1)u_1u_2 + \frac{1}{2} u_2^2 \right) \frac{\partial u_1}{\partial t_0},
\]
\[
\frac{\partial u_2}{\partial t_2} = \left( \frac{1}{2} (\epsilon + 1)u_1^2 + \epsilon u_1u_2 + \frac{1}{2} u_2^2 \right) \frac{\partial u_2}{\partial t_0}.
\]
The commutativity of the hierarchy is immediate consequence of the fact that all $\eta(x,y;z)$ obey the equation \eqref{confluent equation}.

At $\epsilon = \frac{1}{2}$ and $\epsilon = -\frac{1}{2}$ the above systems give us the first confluence limits of the dispersionless NLS and dispersionless Toda equations, respectively.

4.1.2. Case (2). The Lauricella type function $\eta_2(y_1,y_2;z)$ can be obtained by the limits $\delta \to 0$ with
\[
x_1 = \delta y_1, \quad x_2 = \delta y_2, \quad \epsilon_1 = \epsilon_2 = \frac{1}{\delta}.
\]
Then the EPD equation \eqref{epd equation} becomes simply
\[
0 = \frac{\partial \eta_2}{\partial y_1} - \frac{\partial \eta_2}{\partial y_2}.
\]
This means that we have essentially one variable $y := y_1 + y_2$, and the corresponding hydrodynamic type system is just the Burgers-Hoph equations \cite{14}. That is, we have the generating function,
\[
\Phi_2(t;y) := \sum_{k=0}^m t_k F^{k+1}_2(v) \quad \text{with} \quad F^k_2 := \frac{1}{k!} w^k,
\]
where $F^k_2$ is the coefficient of the expansion of $\eta_2(y_1,y_2;z) = \sum_{k=0}^\infty F^k_2(y) z^k$. Then the dynamics of the critical point $\frac{\partial \Phi_2}{\partial y}|_{y=v} = 0$ is given by the Burgers-Hoph hierarchy,
\[
\frac{\partial v}{\partial t_k} = F^k_2(v) \frac{\partial v}{\partial t_0} = \frac{v^k}{k!} \frac{\partial v}{\partial t_0}.
\]
Thus we have a \textit{reducible} system with just one variable.

4.1.3. Case (3). We will discuss the general case of the most degenerate confluence with the single partition $n = (n)$ in the next section. Here we give some details of the most degenerate case for $\text{Gr}(2,5)$. First we observe that the confluence process which takes the Lauricella function $\eta(x_1,x_2;z) = (1-x_1 z)^{-\epsilon_1} (1-x_2 z)^{-\epsilon_2}$ to the function $\eta_3(y_1,y_2;z) = e^{y_1 z + y_2 z^2}$ corresponds to the limit $\delta \to 0$ with
\[
(4.2) \quad x_1 = \delta \sqrt{y_2} + \frac{1}{2} \delta^2 y_1, \quad x_2 = -\delta \sqrt{y_2} + \frac{1}{2} \delta^2 y_1, \quad \epsilon_1 = \epsilon_2 = \frac{1}{\delta^2}.
\]
This transformation explicitly shows that the limit gives the confluence of two singular points $\frac{1}{x_1}, \frac{1}{x_2} \to \infty$. This formula will be generalized to the case with $n$ variables in section 6.
The transformation (1.2) converts the EPD equation (1.2) with two variables into the heat equation

\[ \frac{\partial \eta_3}{\partial y_2} = \frac{\partial^2 \eta_3}{\partial^2 y_1}, \]

which in contrast to the previously considered cases contains second order derivative instead of the mixed derivative. Such equations have appeared in the paper [10] in connection with the multidimensional analogs of Airy function.

We now construct a hydrodynamic-type system which describes the deformations of critical points of a generating function \( \Phi_3 \) associated to \( \eta_3(y_1, y_2; z) \). First we expand the Lauricella-type function \( \eta_3(y_1, y_2; z) \),

\[ \eta_3(y_1, y_2; z) = e^{y_1 z + y_2 z^2} = \sum_{k=0}^{\infty} F^k_3(y_1, y_2) z^k, \]

where \( F^k_3(y_1, y_2) \) is given by the elementary Schur polynomial \( p_k(y_1, y_2) \), i.e.

\[ F^k_3(y_1, y_2) = p_k(y_1, y_2) := \sum_{j_1 + 2j_2 = k} y_1^{j_1} y_2^{j_2}. \]

Notice that these polynomial satisfies

\[ \frac{\partial p_k}{\partial y_1} = p_{k-1}, \quad \frac{\partial p_k}{\partial y_2} = p_{k-2} \]

where \( p_0 = 1 \) and \( p_n = 0 \) for \( n < 0 \).

Then we define the generating function \( \Phi_3(t; y_1, y_2) \) as

\[ \Phi_3(t; y_1, y_2) = \sum_{k=0}^{m} t_k F^k_3(y_1, y_2) = \sum_{k=0}^{m} t_k p_k(y_1, y_2). \]

The equations for the critical point at \( (y_1, y_2) = (u_1, u_2) \) of \( \Phi_3(t; y_1, y_2) \) are given by

\[ \frac{\partial \Phi_3}{\partial y_1} \bigg|_{(u_1, u_2)} = \sum_{k=0}^{m} t_k p_k (u_1, u_2) = 0, \quad \text{and} \quad \frac{\partial \Phi_3}{\partial y_2} \bigg|_{(u_1, u_2)} = \sum_{k=1}^{m} t_k p_{k-1}(u_1, u_2) = 0. \]

Now taking the derivatives with respect to \( t_k \), we have

\[ p_k + \Phi_{1,2} \frac{\partial u_2}{\partial t_k} = 0, \quad p_{k-1} + \Phi_{2,1} \frac{\partial u_1}{\partial t_k} + \Phi_{2,2} \frac{\partial u_2}{\partial t_k} = 0, \]

where we denote

\[ \Phi_{1,2} = \Phi_{2,1} = \sum_{k=2}^{m} t_k p_{k-2}, \quad \Phi_{2,2} := \sum_{k=3}^{m} t_k p_{k-3}. \]

Together with the equations for \( k = 0 \), one can eliminate \( \Phi_{3,i,j} \) and obtains the following hydrodynamic type equations for \( u = (u_1, u_2) \),

\[ \frac{\partial u}{\partial t_k} = \Lambda_k \frac{\partial u}{\partial t_0} \quad \text{with} \quad \Lambda_k := \begin{pmatrix} p_k & p_{k-1} \\ 0 & p_k \end{pmatrix} \]

The first two systems are given by

\[ \frac{\partial}{\partial t_1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} \frac{\partial}{\partial t_0} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \]

\[ \frac{\partial}{\partial t_2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} u_1^2 + w_2 \\ 0 & \frac{1}{2} u_1^2 + w_2 \end{pmatrix} \frac{\partial}{\partial t_0} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \]

It is a direct check that all these flows commute. So one has an integrable hierarchy of the hydrodynamic type systems. We like to emphasize that the variables \( (u_1, u_2) \) are not Riemann invariants in
contrast to other systems considered above. This fact is the consequence of the form of the equation for \( q_2 \) which contains the second order derivative \( \frac{\partial^2 q_2}{\partial y_2^2} \).

The critical equations (4.4) are hodograph equations for these systems. They have a matrix form discussed in the paper [13]. That is, we have

\[
\sum_{k=0}^{m} t_k A_k = 0.
\]

In the section 6, we will discuss the general case of the most degenerate confluent system.

5. CONFLUENT HYDRODYNAMIC TYPE SYSTEMS OF MIXED CASE FOR Gr(2, 6)

There are three different and irreducible types of the generalized Lauricella functions for Gr(2, 6), and they are

\[
\begin{align*}
\eta^1_3(x_1, x_2, y_1, y_2; z) &= (1 - x_1 z)^{-\epsilon_1} (1 - x_2 z)^{-\epsilon_2} e^{y_1 z}, \\
\eta^2_3(x_1, x_2, y_1, y_2; z) &= (1 - x_1 z)^{-\epsilon_1} e^{y_1 z + y_2 z^2}, \\
\eta^3_3(y_1, y_2, y_3; z) &= e^{y_1 z + y_2 z^2 + y_3 z^3}.
\end{align*}
\]

The hydrodynamic type system corresponding to the first one is given by a Riemann invariant form (diagonalizable case). The system corresponding to the second one will be discussed in section 6 for arbitrary \( n \). Here we consider the second case which is given by the confluence for Gr(2, 6) with the partition 6 = (5, 1).

In the similar transformation as (4.2) for the case (2) of Gr(2, 5), one can obtain \( \eta^2_3(x_1, x_2, y_1, y_2; z) \) from the Lauricella function \( \eta(x_1, x_2, x_3) = \prod_{i=1}^{5} (1 - x_i z)^{-\epsilon_i} \) in the limit \( \delta \to 0 \) with \( x_1 = x_2, \epsilon_1 = \epsilon \) and

\[
x_2 = \delta \sqrt{y_2} + \delta^2 \frac{y_1}{2}, \quad x_3 = -\delta \sqrt{y_2} + \delta^2 \frac{y_1}{2}, \quad \epsilon_2 = \epsilon_3 = \frac{1}{\delta^2}.
\]

In this limit, the EPD system (4.2) becomes

\[
(5.1) \quad \frac{\partial \eta^2_3}{\partial y_2} = \frac{\partial^2 \eta^2_3}{\partial y_1^2}, \quad \frac{x \partial^2 \eta^2_3}{x \partial y_1} = \frac{\partial \eta^2_3}{\partial x} - \frac{\partial \eta^2_3}{\partial y_1}, \quad \frac{x^2 \partial^2 \eta^2_3}{x \partial y_2} = \frac{\partial \eta^2_3}{\partial x} - \frac{\partial \eta^2_3}{\partial y_1} - \epsilon x \frac{\partial \eta^2_3}{\partial y_2}.
\]

Note here that the last equation can be derived from the first two equations.

Expanding the \( \eta^2_3 \) function in terms of \( z \), we have

\[
\eta^2_3 = \left( \sum_{k=0}^{\infty} p_k(y_1, y_2) z^k \right) \left( \sum_{j=0}^{\infty} q_j(x) z^j \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} p_{n-k}(y_1, y_2) q_k(x) \right) z^n,
\]

where \( p_k \) is the elementary Schur polynomial, and

\[
q_j(x) = \frac{\epsilon^j (\epsilon + 1) \cdots (\epsilon + j - 1)}{j!} x^j \quad \text{for} \quad j = 1, 2, \ldots,
\]

with \( q_0 = 1 \). Then we define

\[
(5.2) \quad \Phi(t; x_1, y_1, y_2) = \sum_{k=0}^{\infty} t_k F^{k+1}(x, y_1, y_2) \quad \text{with} \quad F^k = \sum_{j=0}^{k} p_{k-j}(y_1, y_2) q_j(x).
\]

where \( F^0 = 1 \). (For \( x = 0 \), this is just the case of Gr(2, 5) with the partition 5 = (5).) Note the identities,

\[
\frac{\partial F^{k+1}}{\partial y_1} = F^k, \quad \frac{\partial F^{k+1}}{\partial y_2} = F^{k-1} \quad \text{and} \quad \frac{\partial F^{k+1}}{\partial x} = \epsilon \sum_{j=0}^{k} F^{k-j} x^j.
\]
The last equation can be shown by the induction, i.e.
\[
\sum_{k=1}^{n} (\epsilon + k - 1)p_{n-k}q_{k-1} = \epsilon \sum_{k=0}^{n-1} p_{n-1-k}q_k + \sum_{k=2}^{n} (k - 1)p_{n-k}q_{k-1},
\]
with \((k - 1)q_{k-1} = (\epsilon + k - 2)xq_{k-2} = \epsilon xq_{k-2} + (k - 2)xq_{k-2}\).

Differentiating \(\Phi\) with respect to \(y_1, y_2\) and \(x\), we have
\[
\frac{\partial \Phi}{\partial y_1} = \sum_{j=0}^{m} t_j F^j, \quad \frac{\partial \Phi}{\partial y_2} = \sum_{j=1}^{m} t_j F^{j-1},
\]
\[
\frac{\partial \Phi}{\partial x} = \epsilon \sum_{k=0}^{m} t_k \left( \sum_{i=0}^{k} F^{k-l} x_i \right) = \epsilon \frac{\partial \Phi}{\partial y_1} + \epsilon x \frac{\partial \Phi}{\partial y_2} + \epsilon x^2 \sum_{j=2}^{m} t_j \left( \sum_{i=2}^{j} F^{j-l} x_i \right)
\]
Then at the critical point \((x, y_1, y_2) = (u, v_1, v_2)\) of \(\Phi\), we have
\[
\sum_{j=0}^{m} t_j F^j = 0, \quad \sum_{j=1}^{m} t_j F^{j-1} = 0, \quad \sum_{j=2}^{m} t_j \left( \sum_{i=2}^{j} F^{j-l} x_i \right) = 0.
\]

One should note that the \(\Phi(t; x, y_1, y_2)\) satisfies the system \((5.1)\) given by the confluence limit of the EPD system \((1.2)\).

Now taking the derivatives of \((5.3)\) with respect to \(t_k\), we have
\[
F^k + \Phi_{1,2} \frac{\partial v_2}{\partial t_k} = 0, \quad F^{k-1} + \Phi_{2,1} \frac{\partial v_1}{\partial t_k} + \Phi_{2,2} \frac{\partial v_2}{\partial t_k} = 0, \quad G^k + \Phi_{0,0} \frac{\partial u}{\partial t_k} = 0
\]
where we define \(\Phi_{i,j} = \frac{\partial^2 \Phi}{\partial y_i \partial y_j} \big|_{(u, v_1, v_2)}\), \(\Phi_{0,j} := \frac{\partial^2 \Phi}{\partial y_j} \big|_{(u, v_1, v_2)}\), and
\[
G^k := \sum_{j=0}^{k} F^{k-j} u^j.
\]

Here we have used the fact that \(\Phi_{1,1} = \Phi_{0,1} = \Phi_{0,2} = 0\), since the corresponding second derivatives of \(\Phi\) are expressed by the linear combinations of the first derivatives, see \((5.1)\).

Let us define the following matrix,
\[
M = \begin{pmatrix}
\Phi_{0,0} & 0 & 0 \\
0 & \Phi_{1,2} & \Phi_{2,2} \\
0 & 0 & \Phi_{1,2}
\end{pmatrix}
\]
Then \((5.3)\) gives the following matrix equation for \(u = (u, v_1, v_2)^T\),
\[
M \frac{\partial u}{\partial t_k} = - (G^k, F^{k-1}, F^k)^T.
\]

Now we have the following proposition.

**Proposition 5.1.** The critical point at \((x, y_1, y_2) = (u, v_1, v_2)\) of the generating function \(\Phi(t; x, y_1, y_2)\) in \((5.2)\) satisfies the hydrodynamic type systems,

\[
\frac{\partial u}{\partial t_k} = A_k \frac{\partial u}{\partial x}, \quad \text{with} \quad A_k = \begin{pmatrix}
G^k & 0 & 0 \\
0 & F^k & 0 \\
0 & 0 & F^k
\end{pmatrix}
\]

**Proof.** For \(k = 0\), \((5.5)\) gives
\[
M \frac{\partial u}{\partial t_0} = -(1, 0, 1)^T.
\]
Then note that the right hand side of \((5.5)\) is given by
\[
(G^k, F^{k-1}, F^k)^T = A_k (1, 0, 1)^T.
\]
Now eliminating the matrix \( M \) from those \( t_0 \) and \( t_k \) equations, and noting that the matrices \( A_k \) commute with \( M \), we obtain the system.

For example, the first flow of the hierarchy in this proposition is given by

\[
\frac{\partial}{\partial t_1} \begin{pmatrix} u \\ v_1 \\ v_2 \\ \end{pmatrix} = \begin{pmatrix} 2u + v_1 & 0 & 0 \\ 0 & u + v_1 & 1 \\ 0 & 0 & u + v_1 \end{pmatrix} \frac{\partial}{\partial t_0} \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix}.
\]

6. Confluent hydrodynamic type systems for the most degenerate cases

As shown in example 6.3, the confluent Lauricella type function (the character) is given by the exponential form,

\[
\eta_*(y; z) = \exp \left( \sum_{i=1}^{n} y_i z^i \right) = \sum_{i=0}^{\infty} p_i(y) z^i,
\]

This Lauricella function can be directly obtained from the original Lauricella function \( \eta(x; z) = \prod_{j=1}^{n} (1 - x_j z)^{-\epsilon_j} \) by the following limit \( \delta \to 0 \): First take all \( \epsilon_j = \frac{1}{4\pi} \). We then note

\[
\prod_{j=1}^{n} (1 - x_j z) = \sum_{i=0}^{\infty} (-1)^i \sigma_i(x) z^i \quad \text{with} \quad \sigma_i(x) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{j_1} \cdots x_{j_i}.
\]

Now consider the \( n \)-th degree polynomial of \( x \),

\[
x^n = \sum_{i=0}^{n-1} \delta^n y_{n-i} z^i \quad \text{i.e.} \quad \delta^n y_i = (-1)^{i+1} \sigma_i(x),
\]

This means that we have the following equation which we take the limit \( \delta \to 0 \),

\[
\prod_{j=1}^{n} (1 - x_j z)^{-\epsilon_j} = \left( 1 - \delta^n \sum_{i=1}^{n} y_i z^i \right)^{-\frac{1}{\delta^n}} \rightarrow \exp \left( \sum_{i=1}^{n} y_i z^i \right).
\]

To find those \( x_i \)'s which are the roots of \( (6.2) \), we look for the root in a series form,

\[
x_i = \sum_{j=1}^{n} \delta^j a_{i,j} + \mathcal{O}(\delta^{n+1}) \quad \text{for} \quad i = 1, \ldots, n.
\]

Then it is easy to see that the expansion is well defined and one can find the coefficients \( a_{i,j} \) uniquely in the perturbation expansion. For example, in the case of \( n = 4 \), we have

\[
a_{i,1} = \omega_4^i y_4^4, \quad a_{i,2} = \frac{y_3}{4a_{i,1}^2}, \quad a_{i,3} = -\frac{y_3^2}{32a_{i,1}^5} + \frac{y_2}{4a_{i,1}}, \quad a_{i,4} = \frac{y_1}{4}.
\]

where \( \omega_4 = \exp(\frac{2\pi \sqrt{-1}}{4}) \), the fourth root of unity. Then the transformations for the variables \( x_i \) are given by

\[
x_i = \delta^{i} \omega_4^i y_4 + \delta^{2} \omega_4^{2i} \frac{y_2 y_4}{4} - \frac{1}{\delta} \cdot \frac{y_3}{32} \cdot \delta^{3} \omega_4^{3i} \frac{y_2 y_4}{4} + \delta^{4} \omega_4^{4i} \frac{y_1}{4} + \mathcal{O}(\delta^5).
\]

One should note that the higher order terms of \( \mathcal{O}(\delta^{n+1}) \) in the series \( (6.3) \) will vanish in the limit \( \delta \to 0 \), that is, we can use the truncated form of the transformations \( x_i \) up to \( \mathcal{O}(\delta^n) \) (e.g. see \( (4.2) \) for the case of \( n = 2 \)).
Remark 6.1. We would like to mention that the expression of \( x_i \)'s in (6.3) has some freedom, and for example, one can also find \( x_i \)'s in a triangular form, i.e. \( a_{ij} = 0 \) for \( j > i \). For example, at \( n = 2 \), we have
\[
x_1 = \delta y_2^{\frac{1}{2}}, \quad x_2 = -\delta y_2^{\frac{1}{2}} + \delta^2 y_1,
\]
and at \( n = 3 \), we have
\[
x_1 = \delta y_3^{\frac{1}{3}}, \quad x_2 = \delta \omega_3 y_3^{\frac{1}{3}} + \delta^2 q y_2 y_3^{-\frac{1}{3}}, \quad x_3 = \delta \omega_3^2 y_3^{\frac{1}{3}} - \delta^2 q y_2 y_3^{-\frac{1}{3}} + \delta^3 y_1,
\]
where \( \omega_3 = \exp\left(\frac{2\pi i}{3}\right) \) and \( q = \omega_3/(\omega_3^2 - 1) \). Such transformation will be discussed in more details elsewhere.

The coefficients \( p_i(y) \) in (6.1) are sometime referred to as the elementary Schur polynomials which are given by
\[
p_i(y) = \sum_{j_1 + 2j_2 + \cdots + nj_n = i} \frac{y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n}}{j_1! j_2! \cdots j_n!}
\]
The first few polynomials are
\[
p_0 = 1, \quad p_1 = y_1, \quad p_2 = y_2 + \frac{1}{2} y_1^2, \quad p_3 = y_3 + y_1 y_2 + \frac{1}{3} y_1^3.
\]
Notice that \( p_i(y) \) satisfy
\[
\frac{\partial p_i}{\partial y_j} = p_{i-j} \quad \text{with} \quad p_m = 0 \quad \text{if} \quad m < 0.
\]

Now we define the function,
\[
\Phi(t; y) := \sum_{k=0}^{\infty} t_k F^{k+1}(y) \quad \text{where} \quad F^k(y) = p_k(y).
\]
Then we have the following Lemma on the equations for \( \Phi \) which can be considered as a degenerate EPD system.

Lemma 6.2. For each \( r \), we have
\[
\frac{\partial^2 \Phi}{\partial y_i \partial y_j} = \frac{\partial^2 \Phi}{\partial y_i \partial y_j} \quad \text{for any} \quad i + j = r.
\]

Now we consider the critical point \( y = u \) of \( \Phi(t; y) \), i.e.
\[
\frac{\partial \Phi}{\partial y_i} \bigg|_{y=u} = 0 \quad \text{for} \quad i = 1, \ldots, n.
\]

We write
\[
\Phi_i := \frac{\partial \Phi}{\partial y_i} \bigg|_{y=u}, \quad \Phi_{i,j} := \frac{\partial^2 \Phi}{\partial y_i \partial y_j} \bigg|_{y=u}.
\]
Note that from Lemma 6.2, we have
\[
\Phi_{i,j} = \Phi_{i+j} = 0 \quad \text{if} \quad i + j \leq n.
\]
Then differentiating (6.4) with respect to \( t_k \) for \( k = 0, 1, \ldots, n \), we have the following Lemma.

Lemma 6.3.
\[
\begin{pmatrix}
\Phi_{n,1} & \Phi_{n,2} & \cdots & \Phi_{n,n} \\
0 & \Phi_{n-1,2} & \cdots & \Phi_{n-1,n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Phi_{1,n}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
= -\begin{pmatrix}
P_{n-1}^k \\
P_{n-2}^k \\
\vdots \\
P_0^k
\end{pmatrix}
\]
where \( P_j^k = \frac{\partial \Phi_j}{\partial y_j} \bigg|_{y=u} = p_{k-j}(u) \).
Notice that each (super) $ith$-diagonal of the coefficient matrix has the same entry $\Phi_{n+i+1}$ which we assume to be nonzero. Then we have the following Proposition.

**Proposition 6.4.** Assume that $\Phi_{i,j} \neq 0$ for $i + j > n$. Then we have

$$\frac{\partial}{\partial t_k} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} F_k^0 & F_k^1 & \cdots & F_k^{n-1} \\ 0 & F_k^0 & \cdots & F_k^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_k^0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

where $F_j^k = p_{k-j}(u)$. Note that $F_j^k = 0$ if $j > k$.

**Proof.** First note that for $k = 0$, we have $F_j^0 = 0$ for $j = 1, \ldots, n-1$ and $F_0^0 = 1$. Then write

$$\begin{pmatrix} F_{k-1}^0 \\ F_{n-2}^k \\ \vdots \\ F_0^k \end{pmatrix} = \begin{pmatrix} F_0^0 & F_0^1 & \cdots & F_0^{n-1} \\ 0 & F_0^0 & \cdots & F_0^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_0^0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Then note that the coefficient matrices in Lemma (6.3) and those in Proposition (6.4) commute. The assumption $\Phi_{n,1} \neq 0$ implies the invertibility of the coefficient matrix $(\Phi_{i,j})$ in Lemma (6.3). This proves the Proposition.

We write the hydrodynamic-type equation in the vector form,

$$(6.5) \quad \frac{\partial \mathbf{u}}{\partial t_k} = \mathbf{A}_k \frac{\partial \mathbf{u}}{\partial x}, \quad \text{for } k = 1, \ldots, m,$$

where $\mathbf{A}_k$ represents the $n \times n$ coefficient matrix in the Proposition. The matrix $\mathbf{A}_1$ is given in (1.5), and $\mathbf{A}_2$ is

$$\mathbf{A}_2 = \begin{pmatrix} u_2 + \frac{1}{2}u_1^2 & u_1 & 1 & \cdots & 0 \\ 0 & u_2 + \frac{1}{2}u_1^2 & u_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & u_2 + \frac{1}{2}u_1^2 & u_1 \\ 0 & \cdots & \cdots & 0 & u_2 + \frac{1}{2}u_1^2 \end{pmatrix}$$

We can also confirm that those systems are compatible.

**Proposition 6.5.** The system of the equations (6.5) is compatible, i.e.

$$\frac{\partial^2 \mathbf{u}}{\partial t_k \partial t_m} = \frac{\partial^2 \mathbf{u}}{\partial t_m \partial t_k}$$

**Proof.** The compatibility conditions give

$$\frac{\partial \mathbf{A}_k}{\partial t_m} - \frac{\partial \mathbf{A}_m}{\partial t_k} + \mathbf{A}_k \frac{\partial \mathbf{A}_m}{\partial x} - \mathbf{A}_m \frac{\partial \mathbf{A}_k}{\partial x} = 0$$

$$\mathbf{A}_k \mathbf{A}_m = \mathbf{A}_m \mathbf{A}_k$$

The commutativity of the second equation is obvious. Noting that the entry $(\mathbf{A}_k)_{i,j}$ is given by $p_{m-j+i}$, we can write the derivative in the vector field form,

$$\frac{\partial \mathbf{A}_k}{\partial t_m} = \left( \sum_{i \leq j} F_{i,j}^m \frac{\partial u_j}{\partial x} \frac{\partial}{\partial u_i} \right) \mathbf{A}_k \quad \text{with} \quad F_{i,j}^m = p_{m-j+i}.$$
The \((r,s)\)-entry of the first equation of the compatibility gives

\[
\sum_{i < j} \left( F^m_{i,j} \frac{\partial u_j}{\partial x} \frac{\partial}{\partial u_i} F^k_{r,s} - F^k_{i,j} \frac{\partial u_j}{\partial x} \frac{\partial}{\partial u_i} F^m_{r,s} + F^m_{i,s} \frac{\partial u_j}{\partial x} \frac{\partial}{\partial u_i} F^k_{r,i} - F^m_{r,i} \frac{\partial u_j}{\partial x} \frac{\partial}{\partial u_i} F^k_{i,s} \right)
\]

Using \(F^m_{i,j} = p_{m-j+i}\), etc, the coefficient of the derivative \(\frac{\partial u_j}{\partial x}\) then gives the following relation among \(p_j(u)\),

\[
\sum_{i=1}^n (p_{m-j+i}p_{k-s+r-i} - p_{k-j+i}p_{m-s+r-i} + p_{k-i+r}p_{m-s+i-j} - p_{m-i+r}p_{k-s+i-j})
\]

where we have \(r \leq i \leq s \leq j\). A direct computation shows that this equation is indeed zero. Note for example, the terms with \(i = r\) and \(i = j\) cancel each other out. \(\square\)

**6.1. Hodograph solution of the system** \((6.5)\). One can write the solution of the system \((6.5)\) is terms of the Schur polynomials. Note that the critical point equation \((6.4)\) can be written in the matrix form (a matrix generalization of the hodograph solution),

\[
(6.6) \quad H(t; u) := \sum_{k=0}^m t_k A_k = 0.
\]

Using this, one can get exact solutions for \(u = (u_1, \ldots, u_n)\) by fixing \(t_i\) for \(i > n\), i.e.

\[
\sum_{k=0}^n t_k A_k(u) + B(u) = 0 \quad \text{where} \quad B(u) = \sum_{k=0}^\infty c_k A_k(u),
\]

with arbitrary constants \(\{a_k : k = 0, 1, \ldots\}\). Notice that the \((1,1)\)-entry of \((6.6)\), say \(h(u, t)\), of the this matrix equation produces all other entries, that is, the \((i, j)\)-entry with \(i \leq j\) is given by

\[
(H(t; u))_{i,j} = \frac{\partial^{j-i} h}{\partial u_{i+t}^{j-i}}(u, t) \quad \text{with} \quad h(t; u) = \sum_{k=0}^\infty t_k p_k(u).
\]

For example, consider the hodograph solution for \(n = 4\), we have

\[
h(t; u) = t_0 + p_1 t_1 + p_2 t_2 + p_3 t_3 + p_4 t_4 = 0.
\]

Note the identity among the symmetric polynomials \(\{h_j = p_j, e_j\}\),

\[
\sum_{i=0}^n (-1)^i e_{n-i} h_i = 0.
\]

The elementary symmetric functions are defined by \(e_j(u) = (-1)^j p_j(-u)\), e.g.

\[
e_1(u) = u_1, \quad e_2(u) = -u_2 + \frac{1}{2} u_1^2, \quad e_3(u) = u_3 - u_1 u_2 + \frac{1}{3} u_1^3.
\]

Then we have

\[
t_i = (-1)^k e_{n-i}(u) \quad \text{for} \quad i = 0, 1, \ldots, n.
\]

More general case, one can consider

\[
h(u, t) = t_0 + p_1 t_1 + p_2 t_2 + p_3 t_3 + p_m = 0.
\]

Then the solution can be represent the Schur polynomials,

\[
t_3 = -S_{(m)} \quad t_2 = S_{(m,1)} \quad t_1 = -S_{(m,1,1)} \quad t_0 = S_{(m,1,1,1)}
\]

where \((i_1, i_2, \ldots, i_m)\) represents the partition of the number \(n = i_1 + \cdots + i_m\) with \(i_1 \geq i_2 \geq \cdots \geq i_m\) (i.e. the Young diagram). The function \(S_{(i_1, \ldots, i_m)}\) is the Schur polynomial associated to the Young diagram \((i_1, \ldots, i_m)\).
Remark 6.6. In a formal limit \( n \to \infty \), the function \( \eta_\ast(y; z) \) in (6.1) has a form,
\[
\eta_\ast(y; z) = \exp \left( \sum_{k=1}^{\infty} y_k z^k \right).
\]
Introduce a finite set of Miwa variables \( \{w_1, w_2, \ldots, w_n\} \) defined by
\[
y_k := \frac{1}{k} \sum_{i=1}^{n} \nu_i w_i^k \quad \text{for some} \quad \nu_i \in \mathbb{C} \quad \text{and} \quad k = 1, 2, \ldots.
\]
which may be considered as a finite reduction of the infinite system of the \( y \)-system. Then the generalized Lauricella function becomes
\[
\eta_\ast(y(w); z) = \prod_{i=1}^{n} (1 - w_i z)^{-\nu_i}.
\]
That is, the most degenerate case of \( \text{Gr}(2, \infty) \) may be considered as the generic one in terms of the Miwa variables.

Remark 6.7. Infinite-component hydrodynamic type systems associated with the function \( \eta_\ast(y; z) \) represent themselves a class of hydrodynamic chains. Indeed, the system (1.3) for \( k=1 \) rewritten as
\[
\frac{\partial u_i}{\partial t_1} = u_i \frac{\partial u_i}{\partial t_0} + \frac{\partial u_{i+1}}{\partial t_0}, \quad \text{for} \quad i = 1, 2, \ldots,
\]
coincides with the strictly positive part of Pavlov’s chain given in [22] (i.e. (37) for \( k = 1, 2, \ldots \) and with \( c_k = -u_k \)). However, in contrast to the chain (37) in [22], all variables in the chain (6.1) are functionally independent. In addition, the finite truncation of the Pavlov’s chain (e.g. \( c_N+k = 0 \) for \( k = 1, 2, \ldots \)) can be achieved exactly by the limit described at the beginning of this section.

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