Rational consumer decisions in a peak time rebate program

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Abstract

A rational behavior of a consumer is analyzed when the user participates in a Peak Time Rebate (PTR) mechanism, which is a demand response (DR) incentive program based on a baseline. A multi-stage stochastic programming is proposed from the demand side in order to understand the rational decisions. The consumer preferences are modeled as a risk-averse function under additive uncertainty. The user chooses the optimal consumption profile to maximize his economic benefits for each period. The stochastic optimization problem is solved backward in time. A particular situation is developed when the System Operator (SO) uses consumption of the previous interval as the household-specific baseline for the DR program. It is found that a rational consumer alters the baseline in order to increase the well-being when there is an economic incentive. As results, whether the incentive is lower than the retail price, the user shifts his load requirement to the baseline setting period. On the other hand, if the incentive is greater than the regular energy price, the optimal decision is that the user spends the maximum possible energy in the baseline setting period and reduces the consumption at the PTR time. This consumer behavior produces more energy consumption in total considering all periods. In addition, the user with high uncertainty level in his energy pattern should spend less energy than a predictable consumer when the incentive is lower than the retail price.

Keywords: Demand Response, Peak Time Rebate, Stochastic Programming, Baseline.

1. Introduction

In the smart grid concept, DR is a mechanism implemented by SO to equilibrate the load with power generation by modifying consumption. The main purpose of this kind of program is to curtail load at the peak demand times for maintaining the security of the transmission assets, avoiding to exceed the limit capacity of generators and preventing power outages. Therefore, DR is one of

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the most crucial parts of the future smart grid due to the main objectives of the DR is peak clipping, valley filling and load shifting on the power profile. An important question in DR program design is how to improve the demand profile, namely, to control the noncritical loads at the residential, commercial and industrial levels for matching supply and demand. For instance, DR program might motivate changes in electricity usage by changing the price of electricity or giving an incentive payment.

There are several DR programs implemented as part of strategies to reduce peak power (because the demand trend is growing). In are shown a complete summary regarding mathematical models, pricing methods, optimization formulation and future extensions. The common approach is time-varying pricing (TVP), which charge more money for energy use during peak periods. In TVP program, the consumer does not have a significant incentive to curtail the consumption, just the energy is more expensive at certain hours. Others mechanisms have been implemented where the user behavior is modified through economic incentives, therefore, many utilities have employed a change in the residential electricity rate structure. For instance, Time-of-use (TOU) program, where the day is divided into adjoining blocks of hours. The price of energy varies between blocks, but not within blocks; Critical peak pricing (CPP), is related to TOU, unlike that it is only applied to a small number of event days; in Real-time pricing (RTP), the price varies hourly according to the real-time market cost of delivering electricity; Direct load control, remote control of flexible loads; Emergency demand reduction, users receive incentive by diminishing energy consumption during emergency events; PTR, where customers receive electricity bill rebates by not consuming (relative to a previously established, household-specific baseline) during peak periods, which it is the mechanism studied in this paper; and many other mechanisms.

Moreover, in literature, there are some of more theoretical DR programs such as in by using a smart grid technology, the authors shows a DR program where a load device could offer retail users coupon incentives to induce DR for a future period in anticipation of intermittent generation. In a cooperative dynamic DR under different market architecture is proposed to evaluate the welfare impacts and the efficiency-risk. In addition, devise schemes for scheduling DR in a deregulated environment. The authors create a new market concept trough a pool-based market-clearing strategy. In a real-time DR algorithm is developed. Furthermore, it is possible to find DR program based on the game theory such as to ensure that users tell the truth in relation to their reduced power consumption employing a Vicrey-Clarke-Grove mechanism.

In this paper, a rational consumer behavior is studied when he is enrolled in
a particular DR incentive program based on baseline or counterfactual model called Peak Time Rebate (PTR). In the economic sense, rational behavior means that the users maximize their profits given the mechanism of demand energy reduction. This rebate is calculated using a baseline for each user which is estimated from past energy consumption. In real life, the PTR program has shown to be an inefficient DR mechanism to improve the demand profile because it allows that some users deliberately increase consumption during baseline-setting times \cite{26, 19, 12}. Such consumer behavior of altering the baseline is formulated as a stochastic optimization problem to understand how the users take their decisions of consumption when they are participating in PTR program. While the user intuitively makes decisions according to the operation of the mechanism, in this work, a mathematical model of consumer choice is proposed in order to find solutions to the aforementioned inefficiency of the PTR mechanism.

In this work, the optimal strategy of a user that participates in a PTR program is studied in order to earn the highest economic profit under uncertain decisions. The contribution is described as follows:

- The optimal decision problem is posed in general form taking into account several previous periods of setting-time in a PTR program. The purposed solution is solved backward in time to find the optimal choice for consumers where consumer uncertainty is modeled as a random variable. In addition, the choice of the SO is modeled as a binary random variable, namely, for indicating whether the user is called for participating in PTR mechanism.

- A closed form solution of a PTR program is derived for two periods. The previous consumption is assumed as the baseline and the user is always called to participate in the PTR program. The results show that the consumer alters the baseline when the incentive exists in the DR program. Some numerical examples are presented.

The article is organized as follows. Section II describes the preliminary setting. In Section III, the general problem formulation of the PTR program is developed. Section IV, the mathematical solution for two periods given the optimization problem is explained. Section V, the simulation results are shown. Conclusions are presented in Section VI.

2. Setting

This section presents the notation and assumptions for developing the model. An individual consumer or aggregated demand (a group of users with the same or similar preferences) is considered for this DR model. The decision maker’s preferences are specified by giving utility function $G(q_t; \theta_t)$, where $q_t$ is the consumption at time $t$ and $\theta_t$ is a particular realization of random variable $\Theta$. The randomness $\Theta$ are external factors that influence the energy requirements of the consumer. The randomness in the utility function is modeled as an additive load requirement, that is, $G(q_t; \theta_t) = G(q_t - \theta_t)$. $\Theta$ is assumed to have
2.1 Utility function and rebate description

a probability density function $f_{\Theta}(\theta_t)$ with limited support $[\underline{\theta}, \bar{\theta}]$ and mean zero. The motivation to choose such additive randomness is that an external event, such as a cold wave, will drive the user to increase his energy consumption until he obtains the same comfort than without the event. Then, given a price, the effect of the random event is to shift the equilibrium point to the left in this situation.

The consumer is assumed with risk-averse behavior. Individuals will usually choose with lower risk, therefore, $G(\cdot)$ is concave [22]. This behavior reflects the assumption that marginal utility diminishes as wealth increases. Also, $G(\cdot)$ is considered smooth, positive and nondecreasing.

A competitive electricity market (consumers are price-takers) is assumed. Thus, the energy price $p$ is given and constant since the utility company set an invariable price to the users during the certain period. Then, the following definitions are stated.

**Definition 2.1.** The energy total cost is $\pi(q_t) = pq_t$.

**Definition 2.2.** The payoff function is defined as $U_t(q_t, \theta_t) = G(q_t - \theta_t) - \pi(q_t)$, which indicates the user benefit of consuming $q$ energy during the interval $t$.

**Definition 2.3.** Given $G(\cdot)$, $\theta_t$ and $p$, the rational behavior of the consumer that maximizes the payoff function $U_t(q_t, \theta_t)$ is

$$q_t^* (\theta_t) = \bar{q} + \theta_t$$

this result is found by solving the optimization problem

$$q_t^* = \max_{q_t \in [0, q_{max}]} U_t(q_t, \theta_t) = G(q_t - \theta_t) - \pi(q_t)$$

where $q_{max}$ is the maximum allowable consumption value, $\bar{q}$ is the optimal solution to the previous condition when $\theta_t = 0$.

2.1 Utility function and rebate description

Under assumption that $G(\cdot)$ is a smooth and concave function, the utility function can be approximated by a second order polynomial around $\bar{q}$. Therefore, a quadratic function is considered, where the user utility is zero whether his consumption is zero and saturates after achieving the maximum of the quadratic form, i.e.,

$$G(q_t) = \begin{cases} 
-\frac{q_t}{2\gamma} (q_t - \bar{q})^2 + p (q_t - \bar{q}) + k & 0 \leq q_t \leq \bar{q} + \frac{E}{\gamma} \\
-\frac{q_t^2}{2\gamma} + \frac{\bar{q}^2}{2} + k & q_t > \bar{q} + \frac{E}{\gamma}
\end{cases}$$

The saturated part is motivated due to the fact that the agent has a limited well-being with respect to his energy consumption.
2.2 Rebate definition

Definition 2.4. Under additive uncertainty and using the previous consideration (1), the utility function can be written as follows:

\[ G(q_t - \theta_t) = \begin{cases} 
-\frac{1}{2} (q_t - q_t^*)^2 + p (q_t - q_t^*) + k & 0 \leq q_t \leq q_t^* + \frac{L}{\gamma} \\
-\frac{p^2}{2\gamma} + \frac{p^2}{2} + k & q_t > q_t^* + \frac{L}{\gamma} 
\end{cases} \]  
(2)

where \( \gamma \) and \( k \) are constant. In particular, \( \gamma \) depicts consumer preferences and \( k \) is settled when \( G(q_t - \theta_t) = 0 \) if \( q_t - \theta_t = 0 \). A similar approach to model a utility function is found in [18]. A further discussion about \( \gamma \) can be reviewed in [2].

Note that \( \gamma \) is in dollar or any other currency divided by energy units squared, therefore, this parameter could be interpreted as the marginal utility that the consumer has as decision-maker into the electricity market. The first order approximation of \( \frac{\partial G(q_t - \theta_t)}{\partial q_t} \) when \( 0 \leq q_t \leq q_t^* + \frac{L}{\gamma} \) around \( q_t^* \) is

\[ \frac{\partial G(q_t - \theta_t)}{q_t} = p - \gamma (q_t - q_t^*) \]

where \(-\gamma\) is the second derivative of \( G(\cdot) \) when \( 0 \leq q_t \leq q_t^* + \frac{L}{\gamma} \).

2.2 Rebate definition

Basically, DR programs request customers to curtail demand in response to a price signal or economic incentive. Typically the invitation to reduce demand is made for a specific time period. There are three main concepts:

Definition 2.5. Baseline: The amount of energy the user would have consumed in the absence of a request to reduce (counterfactual model) [4]. This quantity cannot be measured, then it is estimated from the previous consumption of the agent, i.e., the baseline takes into account \( q_{t-1}, \ldots, q_{t-n} \). Where \( n \) defines the historical consumer behavior, i.e., \( n \) corresponds to the periods taken into account within the baseline function.

\[ \text{Baseline} = b(q_{t-1}, \ldots, q_{t-n}) \]  
(3)

Definition 2.6. Actual Use \( (q_t) \): The amount of energy the customer actually consumes during the event period.

Definition 2.7. Load Reduction \( (\Delta_t (b(\cdot), q_t)) \): The difference between the baseline and the actual use.

\[ b - q_t = \Delta_t (b(q_{t-1}, \ldots, q_{t-n}), q_t) \]

In PTR programs, the rebate is only received if there is an energy reduction. Otherwise, the user does not get any incentive or penalty (see fig. [1]). Mathematically,
Definition 2.8. Let $p_2$ the rebate price received by the user due to energy reduction in peak periods. The PTR incentive $\pi_2$ is

$$\pi_2(b(q_{t-1}, \ldots, q_{t-n}), q_t) = \begin{cases} p_2(\Delta_t(b(q_{t-1}, \ldots, q_{t-n}), q_t)) = p_2(b(q_{t-1}, \ldots, q_{t-n}) - q_t) & q_t < b \\ 0 & q_t \geq b \end{cases}$$

The consumer payoff function when he is enrolled in a PTR program is

$$U_t(q_t, \theta_t, b(q_{t-1}, \ldots, q_{t-n})) = G(q_t - \theta_t) - \pi(q_t) + r\pi_2(b(q_{t-1}, \ldots, q_{t-n}), q_t)$$

where $r$ is a particular realization of a binary random variable $R$ representing whether the consumer is called to participate in the program according to what the SO decides.

3. General problem formulation

The theories of Von-Neumann and Morgenstern are employed here to model decision-making under uncertainty. That is, the agent is assumed to behave as if he maximizes the expected value of the payoff function according to his actions and possible consequences. The reader can found more information in [23, 22, 16]. Subsequently, the consumer problem is to find the optimal decision when he is going to participate in a PTR program in order to increase his personal well-being and economic profit. In addition, the optimization formulae must include all the possible stochastic scenarios given the uncertainty of the variable $\theta$.

The general problem formulation from the demand side is:

$$\max_{q_t, \ldots, q_{t-n} \in [0, q_{\text{max}}]} E[U_{t-n}(q_{t-n}, \theta_{t-n}) + \ldots + U_{t-1}(q_{t-1}, \theta_{t-1}) + U_t(q_t, \theta_t, b(q_{t-1}, \ldots, q_{t-n}))]$$

where $E[\cdot]$ is the expectation operator. The optimization problem takes $n - 1$ previous decisions to determine the best choice for all periods including the choice at the time $t$. Notice that the rebate price is only received at the period $t$ (present time), namely, the payoff function at the time $t$ is given by (4).
It is important to claim that the baseline could be estimated using several techniques according to the energy policies of each country or state. In [3], some methods for baseline calculation are found. The proposed solution for [5] is to formulate \( n \)-stages optimization problems solved backward in time. At stage \( i \), the realization of \( \theta_i \) is known. The stochastic programming algorithm,

1. \( q^o_t(q_{t-1}, ..., q_{t-n}; \theta_t) = \arg\max_{q_t \in [0,q_{\text{max}}]} U_t(q_t, \theta_t, b(q_{t-1}, ..., q_{t-n})) \)
2. \( q^o_{t-1}(q_{t-2}, ..., q_{t-n}; \theta_{t-1}) = \arg\max_{q_{t-1} \in [0,q_{\text{max}}]} U_{t-1}(q_{t-1}, \theta_{t-1}) + E[U_t(q^o_t(q_{t-1}, ..., q_{t-n}); \theta_{t-1}, b(q_{t-1}, ..., q_{t-n}))] \)

\( \vdots \)

\( n. \) \( q^o_{t-n}(\theta_{t-n}) = \arg\max_{q_{t-n} \in [0,q_{\text{max}}]} U_{t-n}(q_{t-n}, \theta_{t-n}) + E[U_{t-(n-1)}(q^o_{t-(n-1); \theta_{t-n}) + ... + U_{t-1}(q^o_{t-1}; \theta_{t-1}) + U_1(q^o_t; \theta_t, b(q^o_t, q^o_{t-1}, q^o_{t-2}, ..., q_{t-n}))] \)

It is vital to highlight that each power consumption period considered in this algorithm has similar features of consumption, i.e., consumer preferences and energy costs are the same in each period. For instance, the period between 7 and 8 pm for a week.

This paper focuses on the way to solve [3] for finding a closed form result for the consumer decision. In the next section, The stochastic optimization problem is solved for two periods. Furthermore, it is assumed that the user is always called to participate in PTR program, henceforth \( r = 1 \) is considered.

4. Problem formulation for two periods

A single previous period \( t - 1 \) is assumed to estimate the baseline in eq. (3). Then, the baseline is \( b(q_{t-1}) = q_{t-1} \). In this regard, the problem formulation is:

\[
\max_{q_t, q_{t-1} \in [0,q_{\text{max}}]} E[U_{t-1}(q_{t-1}, \theta_{t-1}) + U_t(q_t, \theta_t, b(q_{t-1}))]
\]

(6)

First, the agent maximizes the energy consumption at the "present" time \( t \), given that the realization of \( \theta_t \) and the value of \( q_{t-1} \) are known.

\[ q^o_t(q_{t-1};\theta_t) = \arg\max_{q_t \in [0,q_{\text{max}}]} U_t(q_t, \theta_t, b(q_{t-1})) \]

Second, the decision-maker determines the best consumption for the baseline setting period, knowing the rational choice \( q^o_t \) for the future. The realization of \( \theta_{t-1} \) is given and the user faces uncertainty in \( \theta_t \) only, i.e.,

\[ q^o_{t-1}(\theta_{t-1}) = \arg\max_{q_{t-1} \in [0,q_{\text{max}}]} U_{t-1}(q_{t-1}, \theta_{t-1}) + E[U_t(q^o_t; \theta_t, b(q_{t-1}))] \]

4.1. First-stage stochastic programming

The following result presents the solution \( q^o_t \) to the first-stage stochastic optimization at the time \( t \).

**Theorem 4.1.** The optimal consumption \( q^o_t \) of a user participating in a PTR program (i.e. the solution of the first-stage stochastic programming), given \( G(\cdot) \) in [4] and \( U_t(\cdot) \) in [4], is:
4.1 First-stage stochastic programming

\[ q_t^* (q_{t-1}; \theta_t) = \begin{cases} 
q_t^* & r = 1 \text{ and } q_{t-1} - \frac{q_t^*}{2} < \theta_t \leq \frac{\theta}{2} \\
q_t^* - \frac{p_2}{\gamma} & r = 1 \text{ and } \frac{q_t^*}{2} - \gamma < \theta_t \leq q_{t-1} - \frac{q_t^*}{2} + \frac{p_2}{\gamma} \\
0 & r = 1 \text{ and } \frac{p_2}{\gamma} \leq \theta_t \leq \frac{q_t^*}{2} - \gamma \\
q_t^* & r = 0 
\end{cases} \]

Note that when the SO calls the user (r = 1), strategy A means that the user decides rationally to spend \( q_t^* \) of energy (That is, he does not reduce energy consumption), strategy B depicts whether the consumer chooses to diminish the demand to \( q_t^* - \frac{p_2}{\gamma} \) and finally, strategy C is when the best decision is to consume zero energy.

According to theorem 4.1, note that the best decision depends on the realization of \( \theta_t \). Then, the user chooses a strategy at the time \( t \) given his actual demand. The proof is shown in appendix A.

**Corollary 4.2.** The expected value of the load \( q_t^* \) is:

\[ E[q_t^* (q_{t-1}; \theta_t)] = \begin{cases} 
\tau & r = 1 \text{ and } p_2 \leq 2\gamma (\tau - q_{t-1}) \\
\tau - \frac{p_2}{\gamma} & r = 1 \text{ and } 2\gamma (\tau - q_{t-1}) < p_2 \leq \tau \gamma \\
0 & r = 1 \text{ and } \tau \gamma < p_2 \\
\tau & r = 0 
\end{cases} \]

The expected value of consumer payoff \( E[U_t (q_t^*, \theta_t, b(q_{t-1}))] \) is found assuming a continuous uniform distribution function \( f_{\Theta}(\theta_t) \). Since \( r = 1 \), from theorem 4.1, the consumer has three available strategies according to the realization of uncertainty \( \theta_t \). In addition, the random variable is symmetric with respect to zero. Whether strategies A, B and C are feasible according to the parameters \( \tau, \theta_t, p_2, \gamma, \tau \) and the variable \( q_{t-1} \) then these strategies is within the probability density function of \( \theta_t \) which is shown in fig. 2.

Looking in detail the intervals of \( \theta_t \) that define strategy C, these depend on constant values, whereas the intervals for strategies A and B depend on the optimization variable \( q_{t-1} \). Therefore, the probabilistic events change with \( q_{t-1} \). For instance, strategy A has zero probability when \( q_{t-1} > \tau + \frac{p_2}{2\gamma} \).

![Figure 2: User optimal strategies within \( \theta_t \) probability density function.](image)

**Corollary 4.3.** The expected value of the payoff in t, \( E[U_t (q_t^*, \theta_t, b(q_{t-1}))] \), depends on the probabilities of available strategies. Therefore, a construction by cases is employed to solve \( E[U_t (q_t^*, \theta_t, b(q_{t-1}))] \). There are three main cases:
4.2 Second-stage stochastic programming

case 1: \( \theta > \frac{p_{2}}{\gamma} - \gamma \), strategy C does not exist. Then, \( E \left[ U_{t} (q^{*}_{t}, \theta, b(q_{t-1})) \right] \) depends on the value of \( q_{t-1} \). Therefore,

\[
E \left[ U_{t} (q^{*}_{t}, \theta, b(q_{t-1})) : \theta > \frac{p_{2}}{\gamma} - \gamma \right] = \left\{ \begin{array}{ll}
E_{A} = \int_{0}^{\gamma} U_{t} (q^{*}_{t}) f_{\theta}(\theta) d\theta & q_{t-1} \in \left[ 0, \gamma + \theta - \frac{p_{2}}{2\gamma} \right] \\
E_{AB} = \int_{0}^{q_{t-1} - \gamma} \int_{0}^{\gamma} U_{t} (q^{*}_{t}) f_{\theta}(\theta) d\theta d\theta & q_{t-1} \in \left[ \gamma + \theta - \frac{p_{2}}{2\gamma}, \gamma + \theta - \frac{p_{2}}{2\gamma} \right] \\
E_{B} = \int_{q_{t-1} - \gamma}^{q_{t-1}} U_{t} \left( q^{*}_{t} - \frac{p_{2}}{\gamma} \right) f_{\theta}(\theta) d\theta & q_{t-1} \in \left[ \gamma + \theta - \frac{p_{2}}{2\gamma}, q_{\text{max}} \right] \\
\end{array} \right.
\]

case 2: \( \theta \leq \frac{p_{2}}{\gamma} - \gamma < \theta \), strategy C has positive probability. Therefore,

\[
E \left[ U_{t} (q^{*}_{t}, \theta, b(q_{t-1})) \right] \text{ is given by:}
\[
E \left[ U_{t} (q^{*}_{t}, \theta, b(q_{t-1})) : \theta \leq \frac{p_{2}}{\gamma} - \gamma < \theta \right] = \left\{ \begin{array}{ll}
E_{A} = \int_{0}^{\gamma} U_{t} (q^{*}_{t}) f_{\theta}(\theta) d\theta & q_{t-1} \in \left[ 0, \gamma + \theta - \frac{p_{2}}{2\gamma} \right] \\
E_{AC} = \int_{q_{t-1} - \gamma}^{q_{t-1}} \int_{0}^{\gamma} U_{t} (q^{*}_{t}) f_{\theta}(\theta) d\theta d\theta & q_{t-1} \in \left[ \gamma + \theta - \frac{p_{2}}{2\gamma}, \gamma + \theta - \frac{p_{2}}{2\gamma} \right] \\
E_{ABC} = \int_{q_{t-1} - \gamma}^{q_{t-1}} \int_{0}^{\gamma} U_{t} \left( q^{*}_{t} - \frac{p_{2}}{\gamma} \right) f_{\theta}(\theta) d\theta d\theta & q_{t-1} \in \left[ \gamma + \theta - \frac{p_{2}}{2\gamma}, \gamma + \theta - \frac{p_{2}}{2\gamma} \right] \\
E_{B} = \int_{q_{t-1} - \gamma}^{q_{t-1}} U_{t} \left( q^{*}_{t} - \frac{p_{2}}{\gamma} \right) f_{\theta}(\theta) d\theta & q_{t-1} \in \left[ \gamma + \theta - \frac{p_{2}}{2\gamma}, q_{\text{max}} \right] \\
\end{array} \right.
\]

case 3: \( \theta \geq \gamma \), a priori, strategy C has probability one. However, the main point is \( q_{t-1} - \gamma + \frac{p_{2}}{\gamma} \) then it could be exist other strategies different from C. Thus, \( E \left[ U_{t} (q^{*}_{t}, \theta, b(q_{t-1})) \right] \) is given by:

\[
E \left[ U_{t} (q^{*}_{t}, \theta, b(q_{t-1})) : \theta \geq \gamma \right] = \left\{ \begin{array}{ll}
E_{A} = \int_{0}^{\gamma} U_{t} (q^{*}_{t}) f_{\theta}(\theta) d\theta & q_{t-1} \in \left[ 0, \gamma + \theta - \frac{p_{2}}{2\gamma} \right] \\
E_{AC} = \int_{q_{t-1} - \gamma}^{q_{t-1}} \int_{0}^{\gamma} U_{t} (q^{*}_{t}) f_{\theta}(\theta) d\theta d\theta & q_{t-1} \in \left[ \gamma + \theta - \frac{p_{2}}{2\gamma}, \gamma + \theta - \frac{p_{2}}{2\gamma} \right] \\
E_{ABC} = \int_{q_{t-1} - \gamma}^{q_{t-1}} \int_{0}^{\gamma} U_{t} \left( q^{*}_{t} - \frac{p_{2}}{\gamma} \right) f_{\theta}(\theta) d\theta d\theta & q_{t-1} \in \left[ \gamma + \theta - \frac{p_{2}}{2\gamma}, \gamma + \theta - \frac{p_{2}}{2\gamma} \right] \\
E_{B} = \int_{q_{t-1} - \gamma}^{q_{t-1}} U_{t} \left( q^{*}_{t} - \frac{p_{2}}{\gamma} \right) f_{\theta}(\theta) d\theta & q_{t-1} \in \left[ \gamma + \theta - \frac{p_{2}}{2\gamma}, q_{\text{max}} \right] \\
\end{array} \right.
\]

Note that the expected value \( E \left[ U_{t} (q^{*}_{t}, \theta, q_{t-1}) \right] \) is a piecewise function that depends on the value of \( q_{t-1} \).

4.2 Second-stage stochastic programming

For the second-stage, the rational choice for \( q^{*}_{t} \) is known and the realization of \( \theta_{t-1} \) is given. Then \( q^{*}_{t-1} \) is found by using the result of theorem 4.1. The optimization problem is:
\[ q_{t-1}^o(\theta_{t-1}) = \arg\max_{q_{t-1} \geq 0} G(q_{t-1} - \theta_{t-1}) - p q_{t-1} + E[G(q_{t}^o - \theta_t) - p q_{t}^o + r S_2(q_{t-1}, q_{t}^o)] \]

The mathematical solution of (7) is developed in the following three theorems for each case mentioned in the corollary 4.3 and the proofs are found in appendixes B, C, and D.

**Theorem 4.4.** Given \( \theta > \frac{p_2}{\gamma} - \bar{\theta} \) (case 1) and \( \bar{\theta} + \frac{p}{\gamma} > \bar{\theta} + \frac{p_2}{2 \gamma} \), then the optimal solution \( q_{t-1}^o \) for (7) is:

\[
E[q_{t-1}^o(\theta_{t-1})] = \begin{cases} 
\bar{\theta} - \frac{p_2}{2 \gamma} + \frac{2p_2 \theta}{2 \gamma^2 - p_2} & 0 \leq p_2 < \frac{2 \gamma}{3} \theta \\
\bar{\theta} + \frac{p_2}{\gamma} & \frac{2 \gamma}{3} \theta \leq p_2 < p \\
q_{max} & p \leq p_2 < \gamma(\bar{\theta} + \bar{\theta}) 
\end{cases}
\]

**Theorem 4.5.** Given \( \bar{\theta} \leq \frac{p_2}{\gamma} - \bar{\theta} < \bar{\theta} \) (case 2) and \( \bar{\theta} + \frac{p}{\gamma} > \bar{\theta} + \frac{p_2}{2 \gamma} \), then the optimal solution \( q_{t-1}^o \) for (7) is:

\[
E[q_{t-1}^o(\theta_{t-1})] = \begin{cases} 
\bar{\theta} - \frac{p_2}{2 \gamma} + \frac{2p_2 \bar{\theta}}{2 \gamma^2 - p_2} & \gamma(\bar{\theta} + \bar{\theta}) \leq p_2 < \frac{2 \gamma}{3} \theta \\
\bar{\theta} + \frac{p_2}{\gamma} & \frac{2 \gamma}{3} \theta \leq p_2 < p \\
q_{max} & p < p_2 \leq \gamma(\bar{\theta} + \bar{\theta}) 
\end{cases}
\]

**Theorem 4.6.** Given \( \frac{p_2}{\gamma} - \bar{\theta} \geq \bar{\theta} \) (case 3) and \( \bar{\theta} + \frac{p}{\gamma} > \bar{\theta} + \frac{p_2}{2 \gamma} \), then the optimal solution \( q_{t-1}^o \) for (7) is:

\[
E[q_{t-1}^o(\theta_{t-1})] = \begin{cases} 
\bar{\theta} + \frac{p_2}{\gamma} & \gamma(\bar{\theta} + \bar{\theta}) \leq p_2 < p \\
q_{max} & p_2 > p 
\end{cases}
\]

Theorems 4.4, 4.5, and 4.6 present the optimal consumption \( q_{t-1} \) given the solutions of theorem 4.1. For theorem 4.4, the result is rightfull for incentives less than \( \gamma(\bar{\theta} + \bar{\theta}) \), which means that strategy \( C \) does not exist. In addition, the saturation part of the consumer (see equation (2)) is \( \bar{\theta} + \bar{\theta} - \frac{p_2}{2 \gamma} < \bar{\theta} + \frac{p}{\gamma} < q_{max} \), namely, when \( E[U_t(q_{t}^o, \theta_{t-1}, b(q_{t-1})) : 0 > \frac{p_2}{\gamma} - \bar{\theta}] \) is strategy \( B \), specifically, \( q_{t-1} \in \left[ \bar{\theta} + \bar{\theta} - \frac{p_2}{2 \gamma}, q_{max} \right] \). Whether the user has low uncertainty, the theorem 4.4 is employed for estimating optimal decision at the time \( t-1 \). Note that if \( 0 \leq p_2 < \frac{2 \gamma}{3} \theta \) the solutions is decreasing with respect to \( p_2 \), therefore, the situation when the incentive is too small, it is risky to increase the energy consumption at the baseline setting period. Nonetheless, this event is not common owing to the incentive is equal or greater than retail price. Next, whether \( \frac{2 \gamma}{3} \theta \leq p_2 < p \) then the optimal strategies is to increase \( \bar{\theta} + \frac{p_2}{\gamma} \). Finally, if the incentive is greater than \( p \) then the optimal choice is to increase the energy consumption as much as possible. Moreover, The meaning of theorem 4.3 is the same as the theorem 4.4. However, the consumer uncertainty is larger and the incentive limit is given by \( \gamma(\bar{\theta} + \bar{\theta}) < p_2 < \gamma(\bar{\theta} + \bar{\theta}) \). Finally, theorem 4.6 is valid for \( p_2 \geq \gamma(\bar{\theta} + \bar{\theta}) \) and \( p > \gamma(\bar{\theta} + \bar{\theta}) \). Note that there are only two solutions.
that depend on incentive $p_2$. The uncertainty is greater than the previous two theorems. In general, the saturation of consumer preferences causes that the user wastes energy.

5. Numerical examples

In this section, simulation results are presented to illustrate the optimal behavior of a user when he is participating in a PTR program. The utility function for this example is

$$G(q_t - \theta_t) = \begin{cases} 
-\frac{\gamma}{2} (q_t - q_t^*)^2 + p (q_t - q_t^*) + \frac{\gamma}{2} \theta_t^2 + p\theta_t & 0 \leq q_t \leq \overline{q} + \frac{\theta}{\gamma} + \theta_t \\
\frac{p}{p} q_t + \frac{n}{p} \gamma + \frac{\theta}{\gamma} & q_t > \overline{q} + \frac{\theta}{\gamma} + \theta_t
\end{cases}$$

The retail price is $p = 0.26\$/kWh (based on peak summer rate in 10/1/16 by Pacific Gas and Electric Company in San Francisco, California), deterministic baseline $\overline{q} = 8$ kWh and the curvature $\gamma = 0.05$. Randomness $\theta_t$ for each period has been created as a uniform random variable with zero mean and symmetric support. A Monte Carlo simulation is performed with 10000 realizations of $\theta_t$ for each value of $q_{t-1}$.

The flowchart in fig. 3 shows the conditions to determine the case that the user faces for choosing his decision according to the value of $q_{t-1}$, in order to solve the expected value in (7). This flowchart is derived from fig. 2 and corollary 4.3 by analyzing when strategies have positive probability.
5.1 Incentives analysis

In this subsection, the effect of the incentive $p_2$ on the load and user utility at time $t$ given the baseline $q_{t-1}$ is studied by changing the reward $p_2$. For this analysis, $\theta_t \sim \text{unif} [-0.25\bar{q}, 0.25\bar{q}]$ and $q_{\max} = 20\text{KWh}$ are assumed. In fig. 4 are shown three different situations that depend on the incentive value. The first column, the reward $p_2$ is presented for each situation. The second one, the plot of energy consumptions at the time $t$ versus the consumption at the time $t-1$ are shown according to the incentive. Third column, the expected value of profit function based on the decisions at time $t-1$.

First, the event when the incentive is lower than retail price, i.e., $p_2 < p$ is evaluated. For $p_2 = 0.158$/KWh, the optimal solution is to increase energy consumption at the period $t-1$, close to $q_{t-1} = \bar{q} + \theta_{t-1} + \frac{p_2}{\gamma} = 11\text{KWh}$ and
reduce energy consumption at time $t$ to $q_t^o = \overline{\gamma} + \theta_t - \frac{\theta_{t-1}}{\gamma} = 5 \text{kWh}$. Next, when the incentive and the retail price are the same $p_2 = p$. The rational user consumes at the past time whatever value comprising in $q_{t-1} = [11 \text{kWh}, 20 \text{kWh}]$, then, taking into account the worst event, the user consumes $q_{t-1} = 20 \text{kWh}$, therefore, the optimal consumption at the time $t$, it is $q_t^o = \overline{\gamma} + \theta_t - \frac{\theta_{t-1}}{\gamma} = 2.79 \text{kWh}$. Finally, the situation when the incentive is greater than retail price, namely, $p_2 > p$ is assessed. For $p_2 = 0.45$\$/kWh, the optimal behavior is to consume as much energy as possible, $q_{t-1} = q_{max}$, irrespective of parameters $\gamma$, $\overline{\gamma}$, $\overline{\theta}$ and $\overline{\theta}$. If the maximum value is $q_{t-1} = 20 \text{kWh}$ then, he would consume zero energy $q_t = 0 \text{kWh}$ at the time $t$ in order to get the maximum profit. These behaviors are predicted by corollary 4.2 and theorem 4.4.

| $p_2$ (\$/kWh) | Expected profit ($) | $q_{t-1}^o$ (kWh) | $q_t^o$ (kWh) | $q_{t-1}^o + q_t^o$ (kWh) |
|----------------|--------------------|------------------|---------------|------------------------|
| 0              | 3.2                | 8                | 8             | 16                     |
| 0.15           | 3.65               | 11               | 5             | 16                     |
| 0.26           | 4.55               | 20               | 2.79          | 22.79                  |
| 0.45           | 8.13               | 20               | 0             | 20                     |

Table 1: Comparison of optimal user strategies under different incentives.

The previous results are summarized in table 1. Whether the user is not called or the incentive is zero then the trivial solution is not to alter his behavior. On the other hand, when the price incentive is higher than zero but lower than the retail price ($p_2 = 0.15$), the user is induced to raise his consumption to alter
the baseline and get the highest economic benefits by reducing the consumption at time $t$, getting a profit of $3.65. Likewise, whether the agent gets an incentive equal or greater than the retail price then he alters his consumption up to the maximum possible load to maximize the profit to $4.55 or $8.13 according to the incentive, consuming much more energy than in the previous situations. This alteration of the baseline causes economic inefficiency to the SO. A mechanism design should be designed in order to manage properly the signal $r$ to face this problem when the DR program is based on baseline method.

Lastly, in fig. 5 are shown the optimal consumption at the time $t - 1$ and $t$, the net consumption $(q_t + q_{t-1})$ and the expected value of consumer profit versus the incentive payment $p_2$. The optimal decision at the setting time is to increase the consumption as the incentive is raising, namely, the user alters the baseline in order to improve his profit. Note that whether $p_2 > 0.26$/kWh the energy expenditure is saturated to $q_{max} = 20$kWh. In addition, the rational choice at the period $t$ is to diminish the energy consumption to receive the benefits of participating in the PTR program. For $p_2 > 0.4$/kWh, the consumed energy goes to zero. Besides, whether $p_2 \in [0, 0.26]$ $$/kWh, the net consumption is less or equal to 16kWh, that is, the user shifts his energy consumption. On the other hand, for $p_2 > 0.26$/kWh, the user spends more energy that he needs, taking into account all periods. Finally, the expected value of consumer profit is an increasing function, thus, the incentive payment improve the consumer benefits. However, the PTR mechanism is favorable for the SO as long as $p_2 < p$ because the consumer is shifting his energy consumption. In other situations, the incentive goes against with the objectives of a DR program.

5.2. Uncertainty variation

In this part, user behaviors for different uncertainty levels are analyzed. The realization of uniform random variable $\theta_t$ is settled with four different supports in order to assess the uncertainty. These supports are proposed as percentages of the deterministic baseline $q_t$. For this survey are considered the following percentages: 10%, 30%, 50% and 90%. In fig. 6 is compared the optimal decision $q_t$ for all the stated uncertainties. Note that the optimal choice at time $t$ does not depend on the uncertainty level owing to in this period. Moreover, in fig. 7 is shown the rational choices at the period $t - 1$. For $p_2 \in [0, 0.26]$, the user with high uncertainty (e.g. with 50%) should spend less energy than a predictable consumer (e.g. with 10%) since his consumption is unknown then he reduces his consumption for facing this variation and pursuing the benefits of the PTR program. Whether the incentive is greater than the retail price, hence, all decisions are saturated. Furthermore, A similar behavior is found whether the net consumption is analyzed (see fig. 8). It is vital to restate that the consumption variation is perceived when the incentive is lower than the retail price. Lastly, In fig. 9 is presented the expected value of consumer profit. The expected profits are the same for all percentages because each situation has the same preferences. In brief, the uncertainty affects low payments of incentive, therefore, the user does not have certainty related to his consumption pattern.
5.2 Uncertainty variation

under this conditions, then, his best strategy is to be cautious and spend less energy than a predictable consumer.

Finally, in fig 10 is shown a thermal graph of the net optimal consumption according to the incentive price $p_2$ and uncertainty variation $\theta_t$ as a plot summary. An important threshold is when the incentive is equal to the retail price, i.e., $p_2 = 0.26$/kWh. Even more, the maximum consumption is detected when $p_2$ is just slightly higher than $p_2$, rising around 22 kWh represented by a yellow color. In this situation, the optimal consumption does not change with the uncertainty level. In addition, the rational consumption decreases for incentives between 0.26$/kWh to 0.4$/kWh. For higher incentives, the net consumption remains constant in 20kWh. On the other hand, when the incentive is lower than the retail price, the optimal consumption depends on uncertainty variation. If a user is not sure of his demand then the optimal choice is to consume less energy than a predictable consumer. In particular, this non-linear pattern.
is depicted by variations in blue tones of fig 10. Furthermore, the maximum energy consumption is 16 kWh for $p_2$ lower than $p$, therefore, a rational consumer shifts or reduces his load requirement under this incentive conditions.

Figure 10: Thermal graph of optimal decisions given the incentive and uncertainty variation.

6. Conclusions

In this paper was analyzed the rational behavior of a consumer that participates in a PTR program within an electricity market. The problem was addressed using a stochastic programming algorithm. A closed-form solution was found for a two-periods framework. The previous consumption was taken as the baseline and it was assumed that the user is always called to participate in the PTR program. The formulation allowed linking the consumer decisions among different consumption periods. Furthermore, uncertainty in load requirements was considered and coupled through conditional expectation.

It was found that a rational user changes his consumption pattern in order to alter the baseline construction and increase his well-being. Whether the incentive is lower than the regular energy price, the user’s best strategy is to shift the energy consumption from the DR event to the baseline settling period. Otherwise, whether the incentive is greater than the retail price then the consumer maximizes his profits consuming as much energy as possible during the baseline setting period, harming the system reliability. In addition, the effect of uncertainty in the consumer energy requirement was analyzed. It was found that the best decision for a consumer with high uncertainty is to spend less energy than a predictable user.
PTR programs aim to induce users to reduce their energy consumption during a peak event. However, the analysis of the proposed model showed that in most cases, users shift or increase their demand in order to maximize their profits. Only those consumers with high levels of uncertainty reduce their consumption when the incentive is lower than the retail price. Therefore, it was found that a PTR program is not suitable if the SO is seeking a net reduction of energy consumption on the demand side.

For future works, a mechanism design for the demand side would be a significant improvement for DR programs based on baseline methods. This mechanism should include a participation condition or what should be the minimum incentive in order to motivate energy reduction or shifting, this property is known as individual rationality constraint. Furthermore, incentive-compatible, budget balance and efficiency should be evaluated for this kind of incentive-based demand response programs. Moreover, the key solution is associated with controlling properly the user participation. Therefore, it could be interesting to discuss what kind of technology is required to follow the consumer behavior under this program in order to ensure system efficiency.

Acknowledgements

J. Vuelvas received a doctoral scholarship from COLCIENCIAS (Call 647-2014). This work has been partially supported by COLCIENCIAS (Grant 1203-609-4538, Acceso Universal a la Electricidad).

Appendix A. Proof of the Theorem 4.1

Proof. The optimization problem is analyzed by intervals according to the established setting. Then the global maximum is found.

Strategy A1: \( r = 1 \) (Called), \( q^2_t \geq q_{t-1} \) (Non-participant) and \( 0 \leq q^2_t \leq q^*_t + \frac{p}{\gamma} \) (G non-saturated)

\[
[q^2_t] = \arg\max_{q_t \in [0, q^*_t + \frac{p}{\gamma}]} -\frac{\gamma}{2} (q_t - q^*_t)^2 + p (q_t - q^*_t) + k - pq_t
\]

The first-order optimality condition yields to

\[
q_t^0 = q_t^* = \frac{q}{\gamma} + \theta_t \quad (A.1)
\]

Strategy A2: \( r = 1 \) (Called), \( q^2_t \geq q_{t-1} \) (Non-participant) and \( q^2_t > q^*_t + \frac{p}{\gamma} \) (G saturated).

\[
[q^2_t] = \arg\max_{q_t \in [q^*_t + \frac{p}{\gamma}, q_{max}]} -\frac{p^2}{2\gamma} + \frac{p^2}{\gamma} + k - pq_t
\]

This function is unbounded below. The corner solution is

\[
q_t^2 = q^*_t + \frac{p}{\gamma} \quad (A.2)
\]
Comparing the optimal solutions (A.1) and (A.2), the optimal strategy is (A.1) when the user is called but does not participate in DR.

Strategy B1: $r = 1$ (Called), $q_t^o < q_{t-1}$ (Participant), and $0 \leq q_t^o \leq q_t^* + \frac{\theta}{\gamma}$ (G non-saturated).

$$[q_t^o] = \arg\max_{q_t \in [0, q_t^* + \frac{\theta}{\gamma}]} - \frac{\gamma}{2} (q_t - q_t^*)^2 + p (q_t - q_t^*) + k - pq_t + p_2 (q_{t-1} - q_t)$$

The first-order optimality condition yields to

$$q_t^o = q_t^* - \frac{p_2}{\gamma} = \bar{q} + \theta_t - \frac{p_2}{\gamma} \quad (A.3)$$

Strategy B2: $r = 1$ (Called), $q_t^o < q_{t-1}$ (Participant) and $q_t^o > q_t^* + \frac{\theta}{\gamma}$ (Saturated).

$$[q_t^o] = \arg\max_{q_t \in [q_t^* + \frac{\theta}{\gamma}, q_{\text{max}}]} - \frac{p^2}{2\gamma} + \frac{p^2}{\gamma} + k - pq_t + p_2 (q_{t-1} - q_t)$$

This function is unbounded below. The corner solution is

$$q_t^o = q_t^* + \frac{p}{\gamma} \quad (A.4)$$

Comparing the optimal solutions (A.3) and (A.4), the optimal strategy is (A.3) when the user is called and participates in DR.

Strategy C: Importantly, the incentive $p_2$ can be so high to drive (A.3) negative values. As there is no sense in a negative consumption, the problem is limited to $[0, q_{\text{max}}]$, then

$$q_t^o = 0 \text{ if } \theta_t \leq \frac{p_2}{\gamma} - \bar{q} \quad (A.5)$$

Strategy D1: $r = 0$ (Non-called) and $0 \leq q_t^o \leq q_t^* + \frac{\theta}{\gamma}$ (Non-saturated)

$$[q_t^o] = \arg\max_{q_t \in [0, q_t^* + \frac{\theta}{\gamma}]} - \frac{\gamma}{2} (q_t - q_t^*)^2 + p (q_t - q_t^*) + k - pq_t$$

The first-order optimality condition yields to

$$q_t^o = q_t^* = \bar{q} + \theta_t \quad (A.6)$$

Strategy D2: $r = 0$ (Non-called) and $q_t^o > q_t^* + \frac{\theta}{\gamma}$ (saturated)

$$[q_t^o] = \arg\max_{q_t \in [q_t^* + \frac{\theta}{\gamma}, q_{\text{max}}]} - \frac{p^2}{2\gamma} + \frac{p^2}{\gamma} + k - pq_t$$

This function is unbounded below. The corner solution is

$$q_t^o = q_t^* + \frac{p}{\gamma} \quad (A.7)$$
Comparing (A.6) and (A.7), the optimal strategy when the user is not called is (A.6).

Note that, when called \((r = 1)\), the user decides to participate (Strategy B) when \(q_t^R < q_t - 1\), i.e., \(\theta_t < q_t - 1 - \frac{p_2}{\gamma}\). While the user does not participate (Strategy A) when \(q_t^R > q_t - 1\), i.e., \(\theta_t > q_t - 1 - \frac{p_2}{\gamma}\). Then, for any realization of the additive uncertainty \(\theta_t\) within the interval \(q_t - 1 - \frac{p_2}{\gamma} < \theta_t < q_t - 1 - \frac{0}{\gamma} + \frac{p_2}{\gamma}\), there are two local maxima.

In order to find the global solution, the payoff in strategies A and B are compared. The critical value of \(\theta_t\) that provides the same payoff in both strategies is:

\[
U\left(\theta_t + \frac{p_2}{\gamma}, q_t - 1\right) = U\left(\theta_t, q_t - 1\right)
\]  \hspace{1cm} (A.8)

Solving for \(\theta_t\),

\[
\theta_t = q_t - 1 - \frac{p_2}{\gamma}
\]  \hspace{1cm} (A.9)

Eq. (A.9) gives the limit of the uncertain load when the user commutes from strategy A to strategy B.

Organizing by intervals the results (A.6), (A.1), (A.3) and (A.5), the solution is given by theorem 4.1.

Appendix B. Proof of the Theorem 4.4

Proof. Let \(r = 1\), i.e., the user is always called to participate in the PTR program. Under the assumption (see the corollary 1.3 or the fig. 3) that \(\bar{\theta} > \frac{p_2}{\gamma} - \bar{q}\), namely, strategy C does not exist in the density probability function \(f_0(\theta_t)\) (See fig. 2 with zero probability for strategy C). Also, it is assumed that \(\overline{\theta} = -\theta\) and \(\overline{\theta} > \frac{p_2}{\gamma} - \bar{q}\) and the parameters \(\bar{q}\) and \(p_2\) are positives.

Subsequently, the net payoff function for all periods is given by figure B.11. Notice that the intervals are given when the conditions \(q_{t-1} - \bar{q} + \frac{p_2}{\gamma}\) is equal to \(\bar{\theta}\) and \(\bar{\theta}\). Besides, the saturation part according the utility function (equation 2) is assumed between \(\bar{q} + \theta - \frac{p_2}{\gamma} \leq \bar{q} + \frac{p_2}{\gamma} \leq q_{\max}\). This assumption about the saturation part is motivated due to \(\bar{\theta}\) is relatively small when the user has not too much uncertainty. Thus, an optimization problem is formulated by intervals according to strategies that are feasible. Then, the maximum global is found comparing all the local maxima.
The first local maximum is found when strategy A is feasible.

\[
\begin{align*}
\left[ q^*_{t-1} \right] &= \arg \max_{q_{t-1}} G(q_{t-1} - \theta_{t-1}) - pq_{t-1} + E_A \\
&\quad \text{s.t. } 0 \leq q_{t-1} \leq \bar{q} + \frac{p_2}{2\gamma}
\end{align*}
\]

The Karush Kuhn Tucker conditions for the above formulations are:

\[
\begin{align*}
\frac{\partial}{\partial q_{t-1}} (G(q_{t-1} - \theta_{t-1}) - pq_{t-1} + E_A) + \mu_1 - \mu_2 &= 0 \\
0 &\leq q_{t-1} \perp \mu_1 \geq 0 \\
q_{t-1} &\leq \bar{q} + \bar{\theta} - \frac{p_2}{2\gamma} \perp \mu_2 \geq 0
\end{align*}
\]

Being the \( E[\theta_{t-1}] = 0 \), it is found that:

\[
E\left[ q^*_{t-1}(\theta_{t-1}) \mid \theta > \frac{p_2}{\gamma} - \bar{q} \text{ and } q_{t-1} \in \left[ 0, \bar{q} + \bar{\theta} - \frac{p_2}{2\gamma} \right] \right] = \left\{ \begin{array}{ll}
0 & \theta \geq 0 \\
\bar{q} + \frac{p_2}{2\gamma} & \theta > 0 \text{ and } p_2 < 2\gamma \theta \\
p_2 & \theta \geq 0 \text{ and } p_2 \geq 2\gamma \theta
\end{array} \right.
\]

Therefore the unique feasible solution for this situation is:

\[
E\left[ q^*_{t-1}(\theta_{t-1}) \mid \theta > \frac{p_2}{\gamma} - \bar{q} \text{ and } q_{t-1} \in \left[ 0, \bar{q} + \bar{\theta} - \frac{p_2}{2\gamma} \right] \right] = \frac{\bar{q} + \theta}{p_2} \quad p_2 \geq 0
\]

(B.1)

In addition, The sufficient condition is guaranteed, i.e., \( -\gamma < 0 \)

Next, local maxima when strategies A and B are feasible is found solving the following optimization problem:

\[
\begin{align*}
\left[ q^*_{t-1} \right] &= \arg \max_{q_{t-1}} G(q_{t-1} - \theta_{t-1}) - pq_{t-1} + E_{AB} \\
&\quad \text{s.t. } \bar{q} + \bar{\theta} - \frac{p_2}{2\gamma} \leq q_{t-1} \leq \bar{q} + \bar{\theta} - \frac{p_2}{2\gamma}
\end{align*}
\]

A similar analysis using KKT conditions yields the following result:
\[
E\left[ q_{t-1}^\alpha(\theta_{t-1}) \mid q > \frac{p_2}{\gamma} - \overline{q} \text{ and } q_{t-1} \in \left[ \frac{\overline{q} + \frac{p_2}{\gamma} - \frac{p_2}{\gamma'}}{2} \right] \right] = \left\{ \begin{array}{ll}
\frac{\overline{q} + \frac{p_2}{\gamma} - \frac{p_2}{\gamma}}{\overline{q} + \frac{p_2}{\gamma'}} & \text{if } p_2 < \frac{2\overline{q}}{\gamma}
\frac{2\overline{q} \gamma}{p_2} & \text{if } 2\overline{q} \gamma \leq p_2 < \frac{4\overline{q}}{3\gamma}
p_2 \geq \frac{4\overline{q}}{3\gamma} & \text{if } p_2 \geq \frac{4\overline{q}}{3\gamma}
\end{array} \right.
\]

Then as well, \( p_2 \) is positive, resulting
\[
E\left[ q_{t-1}^\alpha(\theta_{t-1}) \mid q > \frac{p_2}{\gamma} - \overline{q} \text{ and } q_{t-1} \in \left[ \frac{\overline{q} + \frac{p_2}{\gamma} - \frac{p_2}{\gamma}}{2} \right] \right] = \left\{ \begin{array}{ll}
\overline{q} + \frac{p_2}{\gamma} & 0 \leq p_2 < \frac{4\overline{q}}{3\gamma}
p_2 \geq \frac{4\overline{q}}{3\gamma} & \text{if } p_2 \geq \frac{4\overline{q}}{3\gamma}
\end{array} \right.
\]

However, the sufficient condition is met when \( p_2 < \gamma (\overline{\theta} - \overline{q}) \). In other circumstances, the solution will be a corner. Given that \( \gamma (\overline{\theta} - \overline{q}) > \frac{2\overline{q}}{3\gamma} \) then the solution is the same.

Finally, the local maximum when strategy \( B \) is feasible.
\[
E_{\left[ q_{t-1}^\alpha(\theta_{t-1}) \mid q > \frac{p_2}{\gamma} - \overline{q} \text{ and } q_{t-1} \in \left[ \frac{\overline{q} + \frac{p_2}{\gamma} - \frac{p_2}{\gamma}}{2} \right] \right] = \left\{ \begin{array}{ll}
\overline{q} + \frac{p_2}{\gamma} & 0 \leq p_2 < \frac{4\overline{q}}{3\gamma}
p_2 \geq \frac{4\overline{q}}{3\gamma} & \text{if } p_2 \geq \frac{4\overline{q}}{3\gamma}
\end{array} \right.
\]

Furthermore, the minimum condition is guaranteed, i.e., \( -\gamma < 0 \).

Lastly, comparing the net payoff at the local maxima given by (B.1), (B.2) and (B.3). The global solution is:
\[
E\left[ q_{t-1}^\alpha(\theta_{t-1}) \mid q > \frac{p_2}{\gamma} - \overline{q} \text{ and } q_{t-1} \in \left[ \frac{\overline{q} + \frac{p_2}{\gamma} - \frac{p_2}{\gamma}}{2} \right] \right] = \left\{ \begin{array}{ll}
\overline{q} + \frac{p_2}{\gamma} & 0 \leq p_2 < \frac{4\overline{q}}{3\gamma}
p_2 \geq \frac{4\overline{q}}{3\gamma} & \text{if } p_2 \geq \frac{4\overline{q}}{3\gamma}
\end{array} \right.
\]

Appendix C. Proof of the theorem 4.5

Proof. The mathematical expression \( \frac{p_2}{\gamma} - \overline{q} \) is located within the limits of the probability density function (see fig. 2). Furthermore, \( \overline{q} + \frac{p_2}{\gamma} > \overline{q} + \frac{p_2}{\gamma} \), i.e., the saturated point is when strategies \( B \) and \( C \) are feasible as it is shown in fig. C.12. Also, let \( \overline{q} \leq \frac{p_2}{\gamma} - \overline{q} < \overline{q} \).
It is uncomplicated to show the following statements

\[
\max_{q_t-1} G(q_{t-1} - \theta_{t-1}) - pq_{t-1} + E_{AB} = \max_{q_t-1} G(q_{t-1} - \theta_{t-1}) - pq_{t-1} + E_{ABC}
\]

\[
\max_{q_t-1} G(q_{t-1} - \theta_{t-1}) - pq_{t-1} + E_B = \max_{q_t-1} G(q_{t-1} - \theta_{t-1}) - pq_{t-1} + E_{BC}
\]

Therefore, the last two zones (\([q_{\max}, \overline{\theta} - \frac{2\gamma}{2\gamma'}\) and \([\overline{\theta} + \frac{\gamma}{\gamma'}, q_{\max} + \frac{\gamma}{\gamma'}]\) from fig B.11 and fig. C.12 have some similarities. Whether the reader follows the same steps of the proof of the theorem 4.4 then the solution for this theorem is:

\[
E\left[ q_{t-1}^* | \frac{\theta}{\gamma} < \frac{p_2}{\gamma} - \overline{\theta} < \frac{\gamma}{\gamma'} \right] = \begin{cases} 
\frac{\gamma}{\gamma'} + \frac{2\gamma^2}{\gamma' \gamma} & \gamma (\overline{\theta} + \gamma) \leq p_2 < \frac{2\gamma}{3}\gamma' \\
q_{\max} & \frac{2\gamma}{3}\gamma' \leq p_2 < \gamma (\overline{\theta} + \gamma) 
\end{cases}
\]
Appendix D. Proof of the theorem 4.6

![Figure D.13: Net payoff function when strategy C is greater than \( \theta \).](image)

This theorem is proved using the same procedure than theorem (4.4) and (4.5).

Fig. D.13 depicts all the zones feasible for this case.

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