THE ADDITION OF THE LOWER LEVEL TO SPECTRUMS OF MATRIX AND SCALAR COMPONENTS OF D=2 SUSY HAMILTONIAN

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ABSTRACT. Supersymmetrical quantum–mechanical system is consider in the case of d=2. The problem of addition of the lower level to spectrums of matrix and scalar components of d=2 SUSY Hamiltonian is investigated. It is shown that in the case, the level E=0 may be degenerate. The multi–dimensional scalar Hamiltonians with energy spectra coinciding up to finite number of discrete levels are constructed.

1. Introduction

Supersymmetric quantum mechanics realizes the quantum description of systems with double degeneracy of energy levels. When d=1, the supersymmetry has absorbed in itself the one–dimensional factorization method being intrinsically connected with the Darboux transformation (DT) [1], [2]. The DT group together two Hamiltonians \( h_0 \) and \( h_1 \) with equivalent spectra:

\[
h_0 = q^+ q + E_0, \quad h_1 = q q^+ + E_0, \quad q \equiv \frac{d}{dx} - (ln \varphi)' ,
\]

where \( q^+ \) is the Hermitian conjugate with \( q \) operator, \( \varphi \) is solution of equation \( h_0 \varphi = E_0 \varphi \) (support function). It easy to see that \( h_0 \) and \( h_1 \) are intertwined by \( q \) and \( q^+ \)

\[
qh_0 = h_1 q, \quad h_0 q^+ = q^+ h_1 ,
\]

and therefore

\[
\psi^{(1)} = q \psi, \quad \psi = q^+ \psi^{(1)} ,
\]

if

\[
h_0 \psi = E_1 \psi , \quad h_1 \psi^{(1)} = E_1 \psi^{(1)} ,
\]

and \( \varphi^{(1)} = \varphi^{-1} \). For the brevity, the normalizing multipliers in (3) are omitted.

DT give us the way to construct one–dimensional potentials with arbitrary preassigned discrete spectrums. For example if support function \( \varphi(E_0; x) \) is wave function (w.f.) of the ground state \( h_0 \), then the discrete spectrum of \( h_1 \) is coinciding with the spectrum of \( h_0 \) without lower level \( E_0 \) [3]. It is possible to add the level \( E_0 \) (missing in spectrum of \( h_0 \)) to spectrum of \( h_1 \). To this end it is sufficient to exploit such \( \varphi \) that:

\[
\varphi \rightarrow +\infty , \quad x \rightarrow \pm \infty ,
\]

and \( \varphi \) is positive definite function for all values \( x \). I is convenient to choose \( \varphi \) as:

\[
\varphi = \lambda \varphi_+ + (1 - \lambda)\varphi_- ,
\]

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where $\varphi_+$ and $\varphi_-$ are positive definite functions with the following asymptotic behavior:

$$
\varphi_{\pm} \to \begin{cases} 
+\infty, & \text{for } x \to \pm\infty \\
0, & \text{for } x \to \mp\infty,
\end{cases}
$$

(6)

and $\lambda$ is real parameter lying in interval $[0,1]$. If $0 < \lambda < 1$, then the level $E_0$ is the lower of spectrum $h_1$. If $\lambda = 0$ or $\lambda = 1$, the level $E_0$ is missing in spectrums of $h_0$ and $h_1$ and spectrums of these Hamiltonians are coinciding with each other.

All this relates to the one-dimensional supersymmetric quantum mechanics based on the following commutation relations:

$$
[Q, H] = [Q^+, H] = 0, \quad \{Q, Q^+\} = H,
$$

(7)

where

$$
Q = q\sigma_+ + \partial_x, \quad Q^+ = q^+\sigma_- - \partial_x, \quad H = \text{diag}(h_0 - E_0, h_1 - E_0),
$$

(8)

and $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$, $\sigma_{1,2}$ are Pauli matrices.

Let’s remind that $q$ and $q^+$ are – bosonic and $\sigma_-, \sigma_+$ – fermionic operators of creation-annihilation. If the spectrums of $h_0$ and $h_1$ differ by the only level, then (7) corresponds to the exact supersymmetry. If the level $E_0$ is absent at spectra of both operators - the supersymmetry is said broken. It is easy to see that the level $E_0$ cannot simultaneously be presented at the spectrum of $h_0$ and at spectrum of $h_1$. It means that if the lowest level at the spectrum of the supersymmetric hamiltonian is zero, it is degenerate.

At this work we consider the case of two-dimension supersymmetric quantum mechanics. The explicit form of operators that satisfy the algebraic relations (7) at $d = 2$ is determined by the expressions:

$$
Q = \begin{pmatrix}
0 & 0 & 0 & 0 \\
q_1 & 0 & 0 & 0 \\
q_2 & 0 & 0 & 0 \\
0 & -q_1 & 0 & 0
\end{pmatrix},
$$

$$
Q^+ = \begin{pmatrix}
0 & q_1^+ & q_2^+ & 0 \\
0 & 0 & 0 & q_2^+ \\
0 & 0 & 0 & -q_1^+ \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

(8)

$$
H = \text{diag}(h_0 - E_0, \tilde{h}_{lm} - 2\delta_{lm}E_0, h_1 - E_0),
$$

(9)

where

$$
h_0 = q_m^+q_m + E_0, \quad h_1 = q_m^+q_m + E_0, \quad \tilde{h}_{lm} \equiv h_{lm} + H_{lm} - E_0\delta_{lm},
$$

(10)

by the way

$$
h_{lm} = q_lq_m^+ + E_0\delta_{lm}, \quad H_{lm} = p_mp_l^+ + E_0\delta_{lm},
$$

(11)

$q_l = \partial_l - \partial_l(\ln \varphi)$, $p_l = \varepsilon_{lk}q_k^+$, $\varepsilon_{lk}$ is the antisymmetric tensor $\partial_l \equiv \partial/\partial x^l$, with indices $l=1,2$ and the sum by repeated indices is implied.

In contrast to $d=1$, there is no any association between spectra $h_0$ and $h_1$ in general case. For the illustration of the last statement let us consider The Coulomb potential (the example is taken from the article [3]) $u = -\alpha r^{-1}$, $\alpha > 0$. It is easy to verify that $u^{(1)} = +\alpha r^{-1}$.

At the example, ad hoc, the hamiltonian $h_1$ has no discrete spectrum at all whereas at $d=1$ the spectrum of $h_1$ should be obtained by the deleting of the lowest level from the spectrum of $h_0$. The assertion should clear that when $d > 1$ we have no formulas expressing wave functions of $h_1$ via WF of $h_0$, that could be similar to one-dimensional case. The existence of such formulas means the existence of connections between spectra $h_0$ and $h_1$. However it is not forbidden the existence of potentials of a special form that allow the connections.
between spectra. Moreover there could be expressions that connect wave functions of the corresponding hamiltonians \( h_0, 1 \) that do not relay to a physical spectrum. As we shall see both possibilities allow a realization.

The general coupling between the spectra exist for pairs \( h_0, h_{lm} \) and \( h_1, H_{lm} \). Really, taking into account that \( h_1 \) may be represented as \( h_1 = p_m^+ p_m + E_0 \), it is easy to verify the validity of the intertwine relations:

\[
q_l h_0 = h_{lm} q_m, \quad p_l h_1 = H_{lm} p_m,
\]

\[
h_0 q_l^+ = q_m^+ h_{ml}, \quad h_1 p_l^+ = p_m^+ H_{ml}, \tag{12}
\]

That means the presence of such coupling. By the same formulas the operator \( \tilde{h}_{lm} \) is inter(laced)twined with \( h_0 \) and \( h_1 \). Its spectrum coincides with the spectrums of the scalar hamiltonians excluding may be the level \( E_0 \).

At the works \([3, 4, 5]\) it was studied the supersymmetry defined by operators (8)–(9), with assumption that \( \varphi \) is a wave function of the basic state of the hamiltonian \( h_0 \). As it was shown at the cited papers such a choice of \( \varphi \) leads to the assertion that the level \( E_0 \) is absent at the physical parts of spectra of \( \tilde{h}_{lm} \) and \( h_1 \), or to unbroken supersymmetry. In this paper we study the inverse problem – the addition of the level \( E_0 \), that is absent at spectrum of \( h_0 \) to the spectrums of both other operators. In this part the general approach should be developed that give a possibility to realize this procedure. At the Sec. 3 we consider the explicit example of potentials (scalar and matrix) with the additional level. It will be shown that the resulting supersymmetric hamiltonian possess a twice degenerated level with \( E = 0 \) – that we did not met for \( d=1 \) case in general and for \( d=2 \) within the ”level deleting” case. At the last section we describe briefly the algorithm of \( d=2 \) widen supersymmetry model construction.

2. The level addition

Let \( u = u(x, y) \) be a n integrable potential i.e. it is supposed that we able to solve the Schrödinger equation \( h_0 \psi = E \psi \) explicitly for any spectral parameter value \( E \), \( h_0 = -\Delta + u \). Unlike the one-dimensional case the potential

\[
u^{(1)} = u - 2\Delta \ln \varphi, \tag{13}
\]

(where \( \varphi \) is support function, is not integrable. Suppose that the spectral parameter value \( E_0 \) lies below the basic state energy of the hamiltonian \( h_0 \). The following question is important for us: how the support function \( \varphi \) should be chosen that the level \( E_0 \) appear at the physical part of the spectrums of \( h_1 \) and \( h_{lm} \)?

For a scalar hamiltonian the reply for this question is not difficult. Really it is easy to verify that the function \( \varphi^{-1} \) satisfy the equation

\[
h_1^{-1} \varphi = E_0^{-1} \varphi, \tag{14}
\]

therefore it is enough to choose \( \varphi \) as positive function for all \( x \) and \( y \) that grows exponentially in all directions in plane. The situation coincide literally with one-dimensional case (if one don’t consider the exited levels).

For the matrix hamiltonian it is necessary the separate consideration First of all we note that if the function \( \psi \) is the second solution of Schrödinger equation with \( E_0 \), than the function

\[
\tilde{\psi}_m = q_m \psi, \tag{15}
\]
satisfy the equation
\[ \tilde{h}_{lm} \tilde{\psi} = E_0 \tilde{\psi}. \]  
(16)

Show now that for rapidly decreasing \( \tilde{\psi} \) the representation (15) is not only sufficient but necessary as well.

For the beginning we prove that the level \( E_0 \) belong to the spectrum of \( \tilde{h}_{lm} \), iff the respecting is normable wave function \( \tilde{\psi}_m \) satisfy the condition:

\[ h_{lm} \tilde{\psi}_m = H_{lm} \tilde{\psi}_m = E_0 \tilde{\psi}_m. \]  
(17)

Really let exists the function \( \tilde{\psi}_m \) such that

\[ \tilde{h}_{lm} \tilde{\psi}_m = E_0 \tilde{\psi}_m, \quad (\tilde{\psi}_m, \tilde{\psi}_m) = 1. \]  
(18)

Define the functions \( \rho_m \) and \( \sigma_m \) by equalities

\[ \rho_m \equiv h_{lm} \tilde{\psi}_m, \quad \sigma_m \equiv H_{lm} \tilde{\psi}_m. \]  
(19)

From (10)–(11) it follows that \( \sigma + \rho = 2E_0 \tilde{\psi} \) (indices omitted), i.e.

\[ (\rho + \sigma, \rho + \sigma) = 4E_0^2 \]  
(20)

if \( \rho \) and \( \sigma \) - are normable. Otherwise one may check that

\[ h_{mk} H_{kl} = H_{mk} h_{kl} = E_0 \tilde{h}_{ml} \]  
(21).

From (21) it follows

\[ h_{lm} \sigma_m = H_{lm} \rho_m = E_0^2 \tilde{\psi}_l. \]  
(22)

Hence

\[ (\tilde{\psi}_m, h_{lm} \sigma_l) = (h_{lm} \tilde{\psi}_m, \sigma_l) = (\rho_m, \sigma_m) = E_0^2. \]  
(23)

Combining with (20) we obtain \( (\rho - \sigma, \rho - \sigma) = 0 \), therefore \( \sigma_m = \rho_m = E_0 \tilde{\psi}_m \). Finally from (19) we go to the equation (17) ■.

Thus for the level \( E_0 \) lie at physical spectrum \( \tilde{h}_{lm} \), it is necessary to find a normable solution of (17). Let \( \tilde{\psi}_m \) is such the function . Acting for it by \( \tilde{h}_{lm} \), we get the equation

\[ q_m^+ \tilde{\psi}_m = p_m^+ \tilde{\psi}_m = 0. \]  
(24)

It means that there exist two functions \( \psi \) and \( \psi^{(1)} \), such that

\[ \tilde{\psi}_m = q_m \psi = p_m \psi^{(1)} \]  
(25)

and that do satisfy the equations

\[ h_0 \psi = E_0 \psi, \quad h_1 \psi^{(1)} = E_0 \psi^{(1)}. \]  
(26)

Solving (25) with respect to \( \psi^{(1)} \), we get the importany relation that couple \( \psi \) and \( \psi^{(1)} \):

\[ \psi^{(1)} = \frac{1}{\varphi} \int dx_k \varepsilon_{km} (\varphi \partial_m \psi - \psi \partial_m \varphi), \]  
(27)
that is known as Moutard transformation [6]. It remains to note that from the established connections between $\tilde{\psi}_m$, $\rho_m$ and $\sigma_m$, the formula (15) obviously follows.

Thus for the presence of the level $E_0$ in the spectrum of $\tilde{h}_{lm}$ it should be pointed out two normable solutions $\psi$, $\varphi$ of the Schrödinger equation with a potential $u$ and the spectral parameter $E = E_0$, for the function $\tilde{\psi}_m = q_m \psi$ be normable. At the next section we would illustrate this procedure by an example.

3. Potentials with cylindrical symmetry.

Let $\psi$ and $\varphi > 0$ are the solutions described at the end of previous section. For the construction of matrix potentials with the level $E_0$ it is convenient to introduce an auxiliary function

$$f \equiv \frac{\psi}{\varphi},$$

that satisfies the equation

$$\partial_m (\varphi^2 \partial_m f) = 0.$$ (29)

Then

$$\tilde{\psi}_m = \varphi \partial_m f.$$ (30)

Consider the case when the seed potential possess the cylindrical symmetry $u = u(r)$. Integrating (29) and substituting in (30) one get

$$\tilde{\psi}_m = \frac{x_m}{r^2 \varphi}.$$ (31)

The normalizing integral of (31) converge if $\varphi$ grows at infinity as a power function with arbitrary index and at the vicinity of zero it behave as $r^{-k}$, $k > 0$. If one require that the asymptotic behavior of $\varphi$ is determined by the conditions:

$$\varphi \to \left\{ \begin{array}{ll} r^a, & \text{for } x^2 + y^2 \to \infty \\ r^b, & \text{for } x^2 + y^2 \to 0, \end{array} \right.$$ (32)

where $a > 1$, $b < 1$, then the normalizing integral of $\varphi^{-1}$ should converge as well. This means that the level $E_0$ will present at the both spectrums of the operators $h_1$ and $\tilde{h}_{lm}$ simultaneously.

In the reference [3] it was shown that the similar situation cannot take place for the hamiltonians $h_0$ and $\tilde{h}_{lm}$. It is easy to see the difference between these couples. For example using as the support function $1/\varphi$, it is possible to construct a new supersymmetric hamiltonian

$$\hat{H} = \text{diag}(h_1 - E_0, \hat{h}_{lm} - 2\delta_{lm}E_0, h_0 - E_0).$$ (33)

The operator $\hat{h}_{lm}$ differs from $\tilde{h}_{lm}$ by that it is intertwined with $h_1$ not by the operators $p_m$, but the dual ones $q^+_m$. Respectively in its spectrum the level $E_0$ do not exist as for the rest of these operators spectrums coincide. Note that such "equivalent by spectrum" matrix operators were considered in the work [7].

The spectrum of the supersymmetrical Hamiltonian (9), consist of the levels

$$\{E_i - E_0, E_i^{(1)} - E_0\}.$$
where $E_i$, $E_i^{(1)}$ – are the levels of the discrete spectrums parts of $h_0$ and $h_1$ – respectively. Now it is seen that if the condition (32) is satisfied then at the spectrum of (9) there is twice degenerated level $E = 0$, to which the following eigen functions correspond

$$
\Psi_1 = \begin{pmatrix}
0 \\
0 \\
0 \\
1/\varphi
\end{pmatrix}, \\
\Psi_2 = \begin{pmatrix}
0 \\
\varphi \partial_1 f \\
\varphi \partial_2 f \\
0
\end{pmatrix}.
$$

(34)

Use the explicit form of the odd supersymmetric operators (8), prove the validity of the known relations for wave functions for the zero level

$$
Q\Psi_{1,2} = Q^+\Psi_{1,2} = 0.
$$

As an example choose $\varphi = \exp(br)/r^k$, where $b, k > 0$. Such function satisfy the necessary asymptotic (32). As a result we obtain two scalar potentials of the hamiltonians $h_0$ and $h_1$:

$$
u = \frac{k^2}{r^2} - \frac{b(2k - 1)}{r}, \quad u^{(1)} = \frac{k^2}{r^2} - \frac{b(2k + 1)}{r}.
$$

(35)

The additional level corresponds to the energy $E_0 = -b^2$. It may be verified by the note that the potentials are integrated by means of degenerated Hypergeometric functions. The discrete spectrums are determined by the formulas:

$$
E_N = -\frac{b^2(2k \pm 1)^2}{(1 + 2[N + \sqrt{m^2 + k^2}])^2},
$$

(36)

where the sign ”-“ corresponds to $u$, and ”+“ – to $u^{(1)}$, $N$ – is the principal and $m$ – magnetic quantum numbers.

The constructed potentials are interesting as an example that exhibits a difference between DT in multidimensions and their unidimensional counterpart. Specifically the comparison of spectrums of hamiltonians $h_0$ and $h_1$ shows that the addition of the lowest level shifts all spectrum. If consider the potentials (35), it could be seen that when

$$
k = \frac{(N + 1)^2 - m^2}{2(N + 1)},
$$

the addition of the level $E_0 = -b^2$, do not move the exited level with the number $N$ and fixed $m$. I general the levels of the hamiltonian $h_1$ go down in respect to levels of $h_0$. This displacement is maximal in the lowest part of the well and decrease as $1/N^2$ in the higher part of the spectrum. In turn the spectrum of the supersymmetric Hamiltonian (9) is double degenerated, including the level $E = 0$. Its normable vacuum wave functions are given by the expressions (34), and

$$
\frac{1}{\varphi} = r^k \exp(-br), \quad \partial_m f = \frac{x_m}{(r\varphi)^2}.
$$

(37)

4. Extended supersymmetry

In the previous section we had demonstrated how to built up a two-dimensional Hamiltonian with double degenerated level $E = 0$. One may obtain models with all degenerated levels. Such thing is realized in models with extended symmetry [8], [9]:

$$
\{Q_i, Q_k\} = \delta_{ik} H, \quad [Q_i, H] = 0, \quad i, k = 1, ..., N.
$$

(38)
Two Hamiltonians to be defined: $H_1$ is determined by (9), and $H_2$, differs from it by the permutation of $h_0$ and $h_1$. We need the three operators more: $Q_1$ defined by (8) and

$$Q_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-q_2^+ & 0 & 0 & 0 \\
q_1^+ & 0 & 0 & 0 \\
0 & -q_1^+ & -q_2^+ & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & q_2 & -q_1 & 0 \\
-q_1 & 0 & 0 & q_2^+ \\
-q_2 & 0 & 0 & -q_1^+ \\
0 & -q_1^+ & -q_2^+ & 0
\end{pmatrix}, \quad (39)$$

that satisfy the commutation relations

$$Q_2^+ B + B Q_1^+ = Q_2 B + B Q_1 = BH_1 - H_2 B = 0, \quad (40)$$

$$H_1 = \{Q_1^+, Q_1\} = B^+ B, \quad H_2 = \{Q_2^+, Q_2\} = B B^+. \quad (41)$$

One can verify that the operators:

$$H(2) = diag(H_1, H_2), \quad Q_1(2) = diag(Q_1, Q_2), \quad (42)$$

$$Q_2(2) = \begin{pmatrix}
0 & 0 \\
B & 0
\end{pmatrix}, \quad (43)$$

form the algebra (38) when $N = 2$.

The introduced operators are the construction blocks for the building of extended supersymmetry matrices for any $N$. It is possible to demonstrate that at every step there is an operator factorizing a superHamiltonian as in (41). At the step $N$, the superHamiltonian

$$H(N) = \begin{pmatrix}
B_N^+ B_N & 0 \\
0 & B_N B_N^+
\end{pmatrix}, \quad (44)$$

is factorized by the operators $B_{N+1} = diag(B_N, B_N^+)$. In turn $H(N+1), B_k(N+1), (k \leq N)$ are defined by the substitution $B_N \to B_{N+1}^+$ and the addition of the new operator $Q_{N+1}(N+1)$ with the new matrix structure. At every step the dimension of matrices duplicates therefore the corresponded algebra is realized by matrices $2^{N+1} \times 2^{N+1}$. For example at $N = 4$ there would be four operators $Q_i$ of the dimension $32 \times 32$ and the similar superHamiltonian $H$:

$$H \equiv H(4) = diag(H_1, H_2, H_2, H_1, H_1, H_1, H_1, H_2), \quad (45)$$

$$Q_1(4) = diag(Q_1, Q_2, Q_2, Q_1, Q_2, Q_1, Q_1, Q_2), \quad (46)$$

$$Q_s(4) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B^+ & -B^+ & 0 & 0 & 0 & 0 & 0 \\
B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B^+ & 0 & 0 & -B^+ & 0 & 0 & 0 \\
0 & 0 & B^+ & 0 & -B^+ & 0 & 0 & 0 \\
0 & 0 & 0 & B & 0 & -B & B & 0
\end{bmatrix}, \quad (47)$$

and also $s = 2, 3, 4$. The nonzero elements of the three corresponding operators are over the main diagonal of the matrix (47). If as a basic scalar model one take the potential considered in the previous section it is obvious that all levels including zero one are degenerate with the multiplicity $2^N$. 

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