On q-series and continued fractions

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Abstract: In this paper we have established interesting results involving continued fraction. Special cases of the result established herein have also been discussed.

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1. Introduction, notations and definitions

Continued fractions have been playing a very important role in number theory and classical analysis ever since the times of Euler and Gauss. Generalized hypergeometric series, both ordinary and basic, have been a very significant tool in the derivation of continued fraction representations. Bhargava and Adiga (1984), Bhargava, Adiga, and Somashekara (1987), Denis and Singh (2000, 2011), Srivastava, Singh, and Singh (2015) and many others have established a good number of results involving q-series and continued fractions. One is also referred to see the papers Cao and Srivastava (2013), Choi and Srivastava (2014), Luo and Srivastava (2011), Srivastava (2011), Srivasatava and Choi (2012) and Srivastava and Choudhary (2015) for useful and interesting similar results. In what follows, we shall use the following usual notations and definitions. Some interesting applications in the direction of quantum calculus can be seen in Mishra, Khatri, Mishra, and Deepmala (2013), Mishra, Khan, Khatri, and Mishra (2013), Mishra, Shrama, and Mishra (2016), Gairola, Deepmala, and Mishra (2016a, 2016b), and Singh, Gairola, and Deepmala (2016).

Let

\[ [a; q]_n = (a; q)_n = \begin{cases} (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}) & \text{if } n \geq 1; \\ 1 & \text{if } n = 0. \end{cases} \]

Following (Srivastava and Karlsson, 1985, 272 p. 347) a basic hypergeometric series is defined as,
\( r \Phi_n [ a_1, a_2, \ldots, a_n; q; z ] \)
\[
= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_n; q)_n z^n}{(q, b_1, b_2, \ldots, b_n; q)_n} (-1)^n q^{n(n-1)/2} \]^{1+s-r}

where \(|q| < 1\). When \(1 + s > r\), series converges for \(|z| < \infty\) and \(1 + s = r\), it converges in the unit circle \(|z| < 1\).

An expression of the form
\[
a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_n \:\quad \frac{1}{b_1 + b_2 + b_3 + \ldots + b_n}\]

is said to be a finite continued fraction and as \(n \to \infty\), it is called an infinite continued fraction. It is also represented by,
\[
\frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ldots}}}
\]

### 1.1. Main results

In this section we shall establish our main result:

\[
\sum_{n=0}^{\infty} \frac{(b;q)_n z^n}{(c;q)_n} = \sum_{n=0}^{\infty} \frac{(b;q)_n (bzq/c;q)_n c^n z^n (1-bzq^{2n}) q^{n^2-n}}{(c;q)_n (z;q)_{n+1}}
\]

\[
= \frac{1}{1-(1-q)c^{-1}} \frac{z(1-b) z(1-q)(b-c) zq(1-bq)(1-c) zq^3(1-q^3)(b-cq) (1-cq^3) - (1-cq^5) - (1-cq^7) - \ldots}{zq^3(1-q^3)(b-cq^3) (1-cq^7) - \ldots}
\]

**Proof of (1.1)**

In order to prove (1.1), let us consider the ratio,

\[
\sum_{n=0}^{\infty} \frac{(a_1a_2a_3 \ldots a_n; q)_n z^n}{(q, a_1, a_2, \ldots, a_n; q)_n} = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{(a_1 a_2 a_3 \ldots a_n; q)_n}{(q, a_1, a_2, \ldots, a_n; q)_n} (1-q^n)}
\]

\[
= \frac{1}{1 - \frac{\sum_{n=1}^{\infty} \frac{(a_1 a_2 a_3 \ldots a_n; q)_n}{(q, a_1, a_2, \ldots, a_n; q)_n} (1-q^n)}{\sum_{n=1}^{\infty} \frac{(a_1 a_2 a_3 \ldots a_n; q)_n}{(q, a_1, a_2, \ldots, a_n; q)_n} (1-q^n)}}
\]

On simplifications (1.2) gives

\[
\frac{zq}{1 - \frac{zq}{1 - \frac{zq}{1 - \frac{zq}{1 - \ldots}}}}
\]

where \(|q| < 1\).
Equation (1.3)

\[ 1 - \frac{1}{az(1-b)/(1-c)} - \frac{1}{z(b-c)(1-aq)/(1-c)(1-cq)} - \frac{1}{zq(a-c)(1-bq)/ (1-cq)(1-cq^2) zq(b-cq)(1-aq^2)/(1-cq^2)(1-cq^3)} - \frac{1}{zq^2(a-cq)(1-bq^2)/(1-cq^3)(1-cq^4) 1-\ldots} \]

Iterating this process, we get

\[
\sum_{n=0}^{\infty} \frac{(aqq_n/b_q)z^n}{(cq_q/c_q)_n} = \frac{1}{1-\frac{az(1-b)/(1-c)}{1-\frac{z(b-c)(1-aq)/(1-c)(1-cq)}{1-\frac{zq(a-c)(1-bq)/(1-cq)(1-cq^2) zq(b-cq)(1-aq^2)/(1-cq^2)(1-cq^3)}{1-\frac{zq^2(a-cq)(1-bq^2)/(1-cq^3)(1-cq^4) }{1-\ldots}}}}
\]

Now, by an appeal of (Jones & Thoren, 1980, (2.3.14), p. 33) we have

\[
\sum_{n=0}^{\infty} \frac{(aqq_n/b_q)z^n}{(cq_q/c_q)_n} = \frac{1}{1-\frac{az(1-b)}{1-\frac{z(b-c)(1-aq)}{1-\frac{zq(a-c)(1-bq) zq(b-cq)(1-aq^2)}{1-\frac{zq^2(a-cq)(1-bq^2)}{1-\frac{zq^3(b-cq^3)(1-aq^4)}{1-\ldots}}}}}}
\]

Finally, taking \(a = 1\) in (1.5) and using Rogers-Fine identity (Andrews & Berndt, 2005, (9.1.1), p. 223) we get (1.1).

2. Special cases

In this section we shall discuss some special cases of the results (1.1) and (1.5).

(i) Putting \(z/b\) for \(z\) in (1.1) and then taking \(b \to \infty\) and \(c = 0\) in it we get,

\[
\sum_{n=0}^{\infty} (-)^n q^{n(n-1)/2} z^n = \frac{1}{1+\frac{z(1-q) zq^2 zq(1-q^2) zq^4}{1-\ldots}}
\]

(ii) If we take \(z = aq\) in (2.1) we have

\[
\sum_{n=0}^{\infty} (-)^n q^{n(n+1)/2} a^n = \frac{1}{1+\frac{aq aq(1-q) aq^3 aq^2(1-q^2) aq^5}{1-\ldots}}
\]

which is a known result (Andrews & Berndt, 2005, (6.2.29), p. 152).
(iii) Taking \(a = -1\) in (2.2) we get,

\[
\sum_{n=0}^{\infty} q^{n+1} = \sum_{n=0}^{\infty} q^{2n+1}(1 + q^{2n+1})
\]

\[
= q \frac{1}{1 - \frac{q}{1 + \frac{q}{1 + \frac{q}{1 + \ldots}}}}.
\]  

(2.3)

where \(\Psi(q) = \sum_{n=0}^{\infty} q^{n+1} \) is Ramanujan's theta function. Also

\[
\psi(q) = \sum_{n=0}^{\infty} q^{n+1} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}.
\]

Thus, from (2.3) we have,

\[
\Psi(q) = \sum_{n=0}^{\infty} q^{n+1} = \sum_{n=0}^{\infty} q^{2n+1}(1 + q^{2n+1}) = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}
\]

\[
= q \frac{1}{1 - \frac{q}{1 + \frac{q}{1 + \frac{q}{1 + \ldots}}}}.
\]  

(2.4)

(iv) Replacing \(q\) by \(q^2\) in (2.1) and then taking \(z = -aq\) in it we find,

\[
\sum_{n=0}^{\infty} q^n q^{n^2} = \frac{1}{1 - \frac{aq}{1 + \frac{aq}{1 + \frac{aq}{1 + \ldots}}}}
\]  

(2.5)

(v) Taking \(c = bq\) in (1.1) we find,

\[
(1 - b) \sum_{n=0}^{\infty} \frac{z^n}{(1 - bq^n)} = (1 - b) \sum_{n=0}^{\infty} \frac{b^n z^n q^{n^2} (1 - bq^{2n})}{(1 - bq^n)(1 - q^{2n})}
\]

\[
= \frac{z(1 - b)}{1 - (1 - b)q^2 - (1 - b)^2q^4 - (1 - b)^3q^6 - (1 - b)^4q^8 - (1 - b)^5q^{10} - \ldots}.
\]  

(2.6)

(vi) Replacing \(q\) by \(q^2\) and then taking \(z = q^2\) and \(b = q^4\) in (2.6) we have,

\[
(1 - q^4) \sum_{n=0}^{\infty} \frac{q^{n^2}}{1 - q^{n^2}} = (1 - q^4) \sum_{n=0}^{\infty} \frac{q^{n^2+i}q^{n^2} + q^{n^2}q^{n^2+i}}{(1 - q^{n^2})(1 - q^{n^2+i})}
\]

\[
= \frac{1}{1 - (1 - q^2)^{-2} - (1 - q^{5})^{-2} - (1 - q^{10})^{-2} - (1 - q^{15})^{-2} - \ldots}.
\]  

(2.7)

(vii) For \(j = i\), (2.7) yields,

\[
(1 - q^4) \sum_{n=0}^{\infty} \frac{q^{n^2+i}}{1 - q^{n^2+i}} = (1 - q^4) \sum_{n=0}^{\infty} \frac{q^{n^2+i+q^{n^2+i}}}{(1 - q^{n^2+i})}
\]

\[
= \frac{1}{1 - (1 - q^2)^2 - (1 - q^{5})^2 - (1 - q^{10})^2 - (1 - q^{15})^2 - \ldots}.
\]  

(2.8)

which gives the continued fraction representation of a Lambert series and is believed to be new.

(viii) Taking \(c = q\) in (1.1) we get,

\[
\frac{(bz; q)_{\infty}}{(z; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{(bz; q)_{n}(bz; q)_{n} z^n q^{n^2} (1 - bzq^{2n})}{(q; q)_{n}(z; q)_{n+1}}
\]

\[
= \frac{z(1 - b) z(1 - q) + qz(1 - bq)(1 - q) zq(1 - q^2) (b - q^2)}{1 - (1 - q^2) - (1 - q^2) - (1 - q^4) - \ldots}.
\]  

(2.9)
(ix) Replacing $b$ by $q^3$ and then taking $b = z = q$ in (2.9) we get,

$$
\frac{(q^3; q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1- (1- q^3)} \frac{q(1-q)\ q^3(1- q^3)(1- q^3)\ q^4(1- q^3)(1- q^5)}{(1- q^6)(1- q^8)\ ...}.
$$

(2.10)

(x) Replacing $q$ by $q^4$ and then taking $z = q, b = q^2$ in (2.9) we have,

$$
\frac{(q^4; q^4)_\infty}{(q;q^4)_\infty} = \sum_{n=0}^{\infty} \frac{(q^4; q^4)_n (q^4; q^4)_n q^{4n+1}}{(q^4; q^4)_n (q^4; q^4)_{n+1}} (1- q^{8n+3})

= \frac{1}{1- (1- q^4)} \frac{q(1-q^2)\ q^2(1- q^4)(1- q^4)\ q^4(1- q^4)(1- q^6)}{(1- q^8)(1- q^8)\ ...}.
$$

(2.11)

which by an appeal of (Jones & Thoren, 1980, (2.3.14), p. 33) gives

$$
= \frac{1}{1- (1+ q^2)} \frac{q\ q^3\ q^5\ q^7}{(1+ q^2)\ ...},
$$

which is known result (Andrews & Berndt, 2005, (7.1.2), p. 179).

(xi) Taking $b = 0$ and $z = q$ in (2.9) we have,

$$
\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{i+n}}{(q;q)_n (q;q)_{n+1}}

= \frac{1}{1- (q^2)} \frac{q(1-q)\ q^3(1- q^2)}{(1- q^3)+ (1- q^5)+ ...}.
$$

(2.12)

(xii) Again, putting $z/ b$ for $z$ and then taking $b \to \infty$ in (2.9) we get,

$$
\frac{z;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} \frac{(-)^n q^{n(n-1)/2} (z; q)_n q^n (1- z q^{2n})}{(q;q)_n}

= \frac{1}{1+ (1- q)} \frac{z(1-q)\ z q^2(1- q^2)\ z q(1- q^2)}{z (1+ q)+ (1- q^4)+ ...}

= \frac{1}{1+ (1- q)} \frac{z q^2}{z (1+ q)+ (1- q^4)+ ...}.
$$

(2.13)

(xiii) For $Z = -q$, (2.13) yields

$$
\frac{(-q; q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} (-q; q)_n (1+ q^{2n+1})}{(q;q)_n}

= \frac{1}{1- (1- q)+ (1+ q)- (1+ q^3)+ (1- q^5)+ ...}.
$$

(2.14)

In (2.12) and (2.14) we have established the continued fraction representations for partition generating functions.
(xiv) Replacing $z$ by $z/ab$ in (1.5) and then taking $a, b \to \infty$ we find,

$$
\sum_{n=0}^{\infty} \frac{q^n z^n}{(q^n; q^n)_n} = \frac{1}{1 + (1 - c) + (1 - cq) + (1 - cq^2) + (1 - cq^3) + (1 - cq^4) + (1 - cq^5) + \cdots}
$$

(2.15)

(xv) Taking $z = aq$ and $c = 0$ in (2.15) we find,

$$
\sum_{n=0}^{\infty} \frac{q^{n+1} a^n}{(aq^n; q^n)_n} = \frac{1}{1 + 1 + 1 + 1 + 1 + \cdots}
$$

(2.16)

which is generalized Rogers–Ramanujan continued fraction.

(xvi) Taking $z = q$ and $c = q$ in (2.15) and using the special case of (Andrews & Berndt, 2005, (6.2.28), p. 152) we get,

$$
\sum_{n=0}^{\infty} \frac{q^{n+1} a^n}{(aq^n; q^n)_n} = \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2}
$$

(2.17)

$$
\sum_{n=0}^{\infty} \frac{q^n}{(aq^n; q^n)_n} = \frac{1}{1 + (1 - q) + (1 - q^2) + (1 - q^3) + (1 - q^4) + (1 - q^5) + \cdots}
$$

(xvii) For $c = -q$ and $z = q$, (2.15) yields

$$
\sum_{n=0}^{\infty} \frac{q^{n+1} a^n}{(aq^n; q^n)_n} = \frac{1}{1 - (1 + q) - (1 + q^2) - (1 + q^3) - (1 + q^4) - (1 + q^5) - \cdots}
$$

(2.18)

A number of other interesting results can also be scored.

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