**STRONG LAW OF LARGE NUMBERS FOR UPPER SET-VALUED AND FUZZY-SET VALUED PROBABILITY**

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**Abstract.** In this paper, we introduce the concepts of upper-lower set-valued probabilities and related upper-lower expectations for random variables. With a new concept of independence for random variables, we show a strong law of large numbers for upper-lower set-valued probabilities. Furthermore, we extend those concepts and theorem to the case of fuzzy-set.

1. **Introduction.** It is well-known that the classic strong laws of large numbers (SLLNs) as fundamental limit theorems in probability theory play an important role in the development of probability theory and its application. The additivity of point-valued probability and expectation is the key in the proofs of these classic SLLNs. However, such additivity assumption is not adaptable in many areas of application, such as mathematical economics, statistics, quantum mechanics and finance, because many uncertain phenomena can not be well modeled and interpreted by additive probability and additive expectation. Therefore, a multitude of scholars have extended SLLNs to nonlinear probabilities mainly by two methods.

On one hand, considering the situation that there are instances in Bayesian estimation when the prior probability is not known precisely, L. DeRobertis and J. A. Hartigan [7] suggest using an interval of measures rather than a single prior, and extend the Bayes theorem in this setting. Consider the set-valued probability space $(\Omega, \mathcal{F}, \Pi)$, where $\Pi : \mathcal{F} \rightarrow \mathcal{P}(\mathbb{R})$ is a set-valued function satisfying (i) $\Pi(A) \neq \emptyset$ for every $A \in \mathcal{F}$ and (ii) $\Pi(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pi(A_i)$ for every disjoint family $\{A_i\}_{i=1}^{\infty}$ with $A_i \in \mathcal{F}$. Then the concept which seems to be useful in such situations is the expectation of random variables with respect to a set-valued measure (see [1] and [6]). M. L. Puri and D. A. Ralescu [19] define the expectation $E_{\Pi}$ of random variable $X$ with respect to set-valued measure $\Pi$ by $E_{\Pi}[X] = \int_{\Omega} X d\Pi = \left\{ \int_{\Omega} XdQ : Q \text{ is a selector of } \Pi \right\}$. Then under the set-valued probability space $(\Omega, \mathcal{F}, \Pi)$, assuming that $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables such that $\Pi \ll P$ where $P$ is a probability measure, M. L. Puri and D. A. Ralescu [19] prove a SLLN that

$$\lim_{n \to \infty} d \left( \frac{\sum_{i=1}^{n} X_i}{n}, E_{\Pi}[X_1] \right) = 0, \quad a.s \quad w.r.t. \quad \Pi.$$
Subsequently, D. A. Ralescu [22] describes the concepts of fuzzy probability measure $\mu$ and the expectation $E_\mu$ of random variable $X$ in this framework and proves a SLLN:

$$\lim_{n \to \infty} D \left( \frac{\sum_{i=1}^{n} X_i}{n}, E_\mu[X_1] \right) = 0, \quad a.s. \quad w.r.t \quad \mu,$$

where $D(x, M)$ is the distance from a point $x$ to a fuzzy set $M$ defined by $D(x, M) = \int_{0}^{1} d(x, L_\alpha(M))d\alpha$.

On the other hand, numerous papers use non-additive/imprecise point-valued probabilities or nonlinear expectations (for example Choquet expectation, G-expectation, g-expectation) to describe and interpret uncertainty in mathematical economics, statistics and finance, and develop SLLNs under different frameworks. For a given set $\mathcal{P}$ of probability measure on $(\Omega, \mathcal{F})$, we define a pair $(\mathbb{V}, v)$ of non-additive probabilities by $\mathbb{V}(A) := \sup_{P \in \mathcal{P}} P(A)$ and $v(A) := \inf_{P \in \mathcal{P}} P(A)$ for any $A \in \mathcal{F}$. The pair of so-called upper-lower expectation $(\mathbb{E}, \mathbb{E})$ with respect to $\mathcal{P}$ is defined by $\mathbb{E}[X] := \sup_{P \in \mathcal{P}} E_P[X]$ and $\mathbb{E}[X] := \inf_{P \in \mathcal{P}} E_P[X]$. Here and in the sequel, $E_P$ denotes the classical expectation under probability $P$. The corresponding upper and lower Choquet expectation are denoted by $(\mathbb{C}_\mathbb{V}, \mathbb{C}_v)$.

Considering a sequence $\{X_i\}_{i=1}^{\infty}$ of independent and identical distributed random variables for non-additive probabilities, some general results for SLLNs under non-additivity probabilities are developed by G. Cooman and E. Miranda [5], L. Epstein and D. Schneider [8], F. Maccheroni and M. Marinacci [10] and M. Marinacci [11]. They indicate that any cluster point of empirical averages lies between the upper Choquet expectation $\mathbb{C}_\mathbb{V}[X_1]$ and the lower Choquet expectation $\mathbb{C}_v[X_1]$ with probability one under non-additive probability $v$.

Moreover, because of the fact that $\mathbb{C}_v[X] \leq \mathbb{E}[X] \leq \mathbb{C}[X] \leq \mathbb{C}_\mathbb{V}[X]$, the gap between the Choquet expectations $\mathbb{C}_\mathbb{V}[X_1]$ and $\mathbb{C}_v[X_1]$ is bigger than that of the upper-lower expectations $\mathbb{E}[X_1]$ and $\mathbb{E}[X_1]$. Z. Chen et al. [4] derive a more precise SLLN

$$v \left( \omega \in \Omega : \mathbb{E}[X_1] \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \leq \mathbb{E}[X_1] \right) = 1,$$

with the notion of independence for random variables under upper expectation and without identical distribution assumption.

Motivated by the two different methods of extending classic SLLNs, we wonder that could we get a new SLLN when consider a given set of set-valued probabilities. Fortunately, if the concepts of upper set-valued probability $\Gamma$, the expectation $E_\Gamma$ of random variable on $(\Omega, \mathcal{F}, \Gamma)$, independence of random variables can be defined appropriately, then applying the SLLN in [4], we obtain the SLLN under $(\Omega, \mathcal{F}, \Gamma)$

$$\lim_{n \to \infty} d \left( \frac{\sum_{i=1}^{n} X_i}{n}, E_\Gamma[X_1] - E_\Gamma[X_1] \right) = 0 \quad a.s. \quad w.r.t \quad \Gamma.$$

That is any cluster point of empirical averages $\frac{\sum_{i=1}^{n} X_i}{n}$ lies in the set $E_\Gamma[X_1] - E_\Gamma[X_1]$ almost surely with respect to the upper set-valued measure $\Gamma$.

Furthermore, applying [22], we can extend our result to upper fuzzy-set valued probability $\mathcal{U}$. For fuzzy set $M$ and the corresponding membership function $m_M$ on $\mathbb{R}$, $m_M(x) = 1$ is denoted by $x \in M$. Therefore, we derive the SLLN in the
following form

\[ 1 \in \mathcal{U} \left( \lim_{n \to \infty} D \left( \frac{\sum_{i=1}^{n} X_i}{n}, E_{\mathcal{U}}[X_1^+] - E_{\mathcal{U}}[X_1^-] \right) = 0 \right). \]

The paper is organized as follow: In Section 2, we recall some basic concepts and related lemmas about set-valued probability and upper-lower probabilities which will be useful in this paper. Then we define the upper-lower set-valued probabilities and expectations in this setting. In Section 3, we prove a strong law of large numbers under the framework of upper set-valued probability. In Section 4, we extend the main result of Section 3 to the case of upper-lower fuzzy-set valued valued probabilities.

2. Preliminary. Let \( \Omega \) be a set, \( \mathcal{F} \) be a \( \sigma \)-algebra on \( \Omega \), and \( \mathcal{P}(\mathbb{R}) \) denote all the subsets of the Euclidean space \( \mathbb{R} \). A set-valued measure is a function \( \Pi : \mathcal{F} \to \mathcal{P}(\mathbb{R}) \) such that:

(i). \( \Pi(A) \neq \emptyset \) for every \( A \in \mathcal{F} \);

(ii). \( \Pi(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pi(A_i) \) for every disjoint family \( \{A_i\}_{i=1}^{\infty} \), with \( A_i \in \mathcal{F} \).

Here the infinite sum of subsets of \( \mathbb{R} \) is defined by \( \sum_{i=1}^{\infty} A_i = \{ \sum_{i=1}^{\infty} a_i | a_i \in A_i, \text{and} \sum_{i=1}^{\infty} |a_i| < \infty \} \).

An atom of the set-valued measure \( \Pi \) is an event \( A \in \mathcal{F} \) with \( \Pi(A) \neq \{0\} \) such that \( A_1 \subseteq A \) implies \( \Pi(A_1) = \{0\} \) or \( \Pi(A \setminus A_1) = \{0\} \). A set-valued measure with no atoms is called nonatomic.

A selector \( P \) of \( \Pi \) is a point-valued measure \( P : \mathcal{F} \to \mathbb{R} \) such that \( P(A) \in \Pi(A) \) for every \( A \in \mathcal{F} \).

The following lemma due to [1] is fundamental in the framework of this paper.

**Lemma 2.1.** (1) If \( \Pi \) is a bounded, nonatomic set-valued measure, then \( \Pi(A) \) is convex for every \( A \in \mathcal{F} \);

(2) If \( \Pi \) is a bounded set-valued measure, then for every \( A \in \mathcal{F} \) and \( x \in \Pi(A) \), there exists a selector \( P \) of \( \Pi \) such that \( P(A) = x \).

**Definition 2.2.** A nonatomic closed set-valued probability on \( (\Omega, \mathcal{F}) \) is a nonatomic set-valued measure \( \Pi : \mathcal{F} \to \mathcal{P}([0,1]) \) such that \( 1 \in \Pi(\Omega) \) and \( \Pi(\Omega) \) is closed.

Obviously, for any \( A \in \mathcal{F} \), \( \Pi(A) \) is convex.

In what follows we only consider the nonatomic closed set-valued probability. It follows that for such measures, \( \Pi(\emptyset) = \{0\} \).

**Definition 2.3.** Let \( \{\Pi_n\}_{n=1}^{\infty} \) be a sequence of nonatomic closed set-valued probabilities. Define two set-valued functions from \( \mathcal{F} \) to \( \mathcal{P}([0,1]) \) by \( \Gamma(\cdot) = \bigcup_{n=1}^{\infty} \Pi_n(\cdot) \) and \( \Gamma(\cdot) = \bigcap_{n=1}^{\infty} \Pi_n(\cdot) \) respectively. \( \Gamma, \Gamma \) is called a pair of upper-lower set-valued probabilities if for any \( A \in \mathcal{F} \), \( \bigcup_{n=1}^{\infty} \Pi_n(A) \) is a convex set and \( \bigcap_{n=1}^{\infty} \Pi_n(A) \) is a nonempty convex set.

Since the set-valued probabilities \( \{\Pi_n\}_{n=1}^{\infty} \) we considered in this paper are all nonatomic closed set-valued probabilities, the values of those mappings are convex sets by Lemma 2.1 (1). Then our request that for any \( A \in \mathcal{F} \), \( \bigcup_{n=1}^{\infty} \Pi_n(A) \) is a convex set and \( \bigcap_{n=1}^{\infty} \Pi_n(A) \) is a nonempty convex set is corresponding to the classic set-valued probabilities. We will construct a pair of upper-lower set-valued probabilities to illustrate that our definition is natural and reasonable.

**Example 2.4.** Suppose that \( P_1 \) is a measure and \( P_2 \) is a probability such that \( P_1(A) \leq P_2(A) \) for any \( A \in \mathcal{F} \). Then define a sequence of nonatomic closed set
values probabilities by
\[
\Pi_n(A) = \left[ \frac{n}{n+1} P_1(A), \frac{n}{n+1} P_2(A) + \frac{1}{n+1} \right], \quad \text{for } n \geq 1.
\]
Then for any \( A \in \mathcal{F} \), we have
\[
\Gamma(A) = \bigcup_{n=1}^{\infty} \Pi_n(A) = \left[ \frac{1}{2} P_1(A), \frac{P_2(A) + 1}{2} \right], \quad \Gamma(\Omega) = \bigcap_{n=1}^{\infty} \Pi_n(\Omega) = [P_1(\Omega), P_2(\Omega)].
\]
Therefore, \((\Gamma, \Gamma)\) is a pair of upper-lower set-valued probabilities that defined in Definition 2.3.

**Proposition 2.5.** If \( \Gamma \) is an upper set-valued probability, then
1. \( \Gamma(\mathcal{A}) \neq \emptyset \) for every \( A \in \mathcal{F} \);
2. \( 1 \in \Gamma(\Omega) \);
3. \( \Gamma(\bigcup_{i=1}^{\infty} A_i) \subseteq \sum_{i=1}^{\infty} \Gamma(A_i) \), where \( A_i \in \mathcal{F} \) and \( A_i \cap A_j = \emptyset \) for any \( i \neq j \).

**Proof.** Due to the definition of \( \Gamma \), where \( \Gamma(\mathcal{A}) = \bigcup_{n=1}^{\infty} \Pi_n(\mathcal{A}) \), the first and second assertions can be easily proved. Therefore we only prove the third assertion.

Assume \( \{A_i\}_{i=1}^{\infty} \) to be a family of disjointed sets of \( \mathcal{F} \). Then by the additivity of each \( \Pi_n \),
\[
\Gamma\left( \bigcup_{i=1}^{\infty} A_i \right) = \bigcup_{n=1}^{\infty} \Pi_n \left( \bigcup_{i=1}^{\infty} A_i \right) = \bigcup_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} \Pi_n(A_i) \right).
\]
Therefore, for any \( x \in \Gamma(\bigcup_{i=1}^{\infty} A_i) \), there exists \( n_0 \) such that \( x \in \sum_{i=1}^{\infty} \Pi_{n_0}(A_i) \). By the definition of addition of sets, there is a sequence \( \{y_i\}_{i=1}^{\infty} \) such that \( x = \sum_{i=1}^{\infty} y_i \) and \( y_i \in \Pi_{n_0}(A_i) \) for every \( i \). Obviously, \( y_i \in \bigcup_{n=1}^{\infty} \Pi_n(A_i) \) for every \( i \). Consequently, \( y_i \in \Gamma(A_i) \) for every \( i \). Again by the definition of addition of sets, \( x = \sum_{i=1}^{\infty} y_i \in \sum_{i=1}^{\infty} \Gamma(A_i) \). Therefore, by the arbitrariness of \( x \), \( \Gamma(\bigcup_{i=1}^{\infty} A_i) \subseteq \sum_{i=1}^{\infty} \Gamma(A_i) \). \( \square \)

An upper set-valued probability space is a triple \((\Omega, \mathcal{F}, \Gamma)\), where \( \Gamma \) is an upper set-valued probability.

**Definition 2.6.** An event \( A \) holds almost surely with respect to \( \Gamma(\text{a.s. w.r.t. } \Gamma) \), if \( \Gamma(A^c) = \{0\} \).

In order to state and prove a law of large numbers, we need to define the concept of expectation. In the case of set-valued measure, this concept first appeared in [19]. Let \( X : \Omega \to \mathbb{R} \) be a random variable on \( \Omega \) and \( \Pi : \mathcal{F} \to \mathcal{P}([0,1]) \) be a set-valued probability. Without loss of generality, assume that \( \Pi \) is absolutely continuous with respect to some probability measure \( P \), denoted by \( \Pi \ll P \) (i.e. \( P(A) = 0 \) implies \( \Pi(A) = \{0\} \)). If \( EP[|X|] = \int_{\Omega} |X|dP < \infty \), the expectation of \( X \) with respect to \( \Pi \) is defined as
\[
E_{\Pi}[X] = \int_{\Omega} Xd\Pi = \left\{ \int_{\Omega} XdQ : Q \text{ is a selector of } \Pi \right\}.
\]
Now we will extend the concept of expectation under the upper set-valued probability space \((\Omega, \mathcal{F}, \Gamma)\).

**Definition 2.7.** Let \((\Omega, \mathcal{F}, \Gamma)\) be an upper set-valued probability space and \( \Gamma(\cdot) \) be the lower set-valued probability corresponding to \( \Gamma(\cdot) \). Let \( X : \Omega \to \mathbb{R} \) be a real
random variable. Then the upper expectation $E_{\Gamma}[X]$ and lower expectation $E_{\Lambda}[X]$ are defined respectively by

$$E_{\Gamma}[X] = \text{closure} \left\{ \bigcup_{n=1}^{\infty} E_{\Pi_n}[X] \right\},$$

$$E_{\Lambda}[X] = \bigcap_{n=1}^{\infty} E_{\Pi_n}[X].$$

**Proposition 2.8.** Assume that $X$ and $Y$ are nonnegative random variables, then the upper expectation has following properties:

1. $E_{\Gamma}[\lambda X] = \lambda E_{\Gamma}[X]$, $\forall \lambda \geq 0$;
2. $E_{\Gamma}[X + Y] \subseteq E_{\Gamma}[X] + E_{\Gamma}[Y]$;
3. $E_{\Gamma}[X - Y] \subseteq E_{\Gamma}[X] - E_{\Gamma}[Y]$.

**Remark 2.9.** The subtraction of two bounded sets $A$ and $B$ is defined by $A - B = \{a - b | a \in A, b \in B\}$.

**Proposition 2.10.** Assume that $\Gamma_1$ and $\Gamma_2$ are two upper set-valued probabilities such that $\Gamma_1(A) \subseteq \Gamma_2(A)$ for any $A \in \mathcal{F}$, then for any nonnegative random variables $X$ and $Y$, we have

1. $E_{\Gamma_1}[X] \subseteq E_{\Gamma_2}[X]$;
2. $E_{\Gamma_1}[X] + E_{\Gamma_1}[Y] \subseteq E_{\Gamma_2}[X] + E_{\Gamma_2}[Y]$;
3. $E_{\Gamma_1}[X] - E_{\Gamma_1}[Y] \subseteq E_{\Gamma_2}[X] - E_{\Gamma_2}[Y]$.

The above propositions are important in the proof of our main theorem in Section 3. To prove them, we need the following lemmas.

**Lemma 2.11.** ([19] Theorem 4.1) Let $\Pi : \mathcal{F} \rightarrow \mathcal{P}([0,1])$ be a nonatomic closed set-valued probability. Then $\Pi(A) = [P_1(A), P_2(A)]$ for every $A \in \mathcal{F}$, where $P_1$ is a measure and $P_2$ is a probability such that $P_1(A) \leq P_2(A)$ for any $A \in \mathcal{F}$.

**Lemma 2.12.** Let $(\Gamma, \Pi)$ be a pair of upper-lower set-valued probabilities. Then for any $A \in \mathcal{F}$, $\text{closure}(\Gamma(A)) = [\inf_{n \geq 1} P_{n1}(A), \sup_{n \geq 1} P_{n2}(A)]$ and $\text{closure}(\Lambda(A)) = [\inf_{n \geq 1} P_{n1}(A), \sup_{n \geq 1} P_{n2}(A)]$, where $\{P_{n1}\}_{n=1}^{\infty}$ is a sequence of point-valued measures and $\{P_{n2}\}_{n=1}^{\infty}$ is a sequence of probabilities such that $P_{n1}(A) \leq P_{n2}(A)$ for any $A \in \mathcal{F}$.

**Proof.** Let $\Gamma(\cdot) = \bigcup_{n=1}^{\infty} \Pi_n(\cdot)$ and $\Lambda(\cdot) = \bigcap_{n=1}^{\infty} \Pi_n(\cdot)$, where $\{\Pi_n\}_{n=1}^{\infty}$ is a sequence of nonatomic closed set-valued probability. By Lemma 2.11, there exist a measure $P_{n1}$ and a probability $P_{n2}$, such that for any fixed $n$, $P_{n1}(A) \leq P_{n2}(A)$ and $\Pi_n(A) = [P_{n1}(A), P_{n2}(A)]$ for any $A \in \mathcal{F}$. Therefore, by Definition 2.3, we have

$$\text{closure}(\Gamma(A)) = \text{closure}\left(\bigcup_{n=1}^{\infty} \Pi_n(A)\right) = \text{closure}\left(\bigcup_{n=1}^{\infty} [P_{n1}(A), P_{n2}(A)]\right)$$

$$= [\inf_{n \geq 1} P_{n1}(A), \sup_{n \geq 1} P_{n2}(A)].$$

Similarly, we can prove $\text{closure}(\Lambda(A)) = [\sup_{n \geq 1} P_{n1}(A), \inf_{n \geq 1} P_{n2}(A)].$

**Lemma 2.13.** Let $\Pi : \mathcal{F} \rightarrow \mathcal{P}([0,1])$ be a nonatomic closed set-valued probability. Then for any nonnegative random variable $X$, $E_{\Pi}[X] = [E_{P_1}[X], E_{P_2}[X]]$, where $P_1$ and $P_2$ denote the same measure and probability deduced in Lemma 2.11.
Proof. By Lemma 2.11, $\Pi(A) = [P_1(A), P_2(A)]$ for any $A \in \mathcal{F}$. Since $X$ is nonnegative and

$$E_\Pi[X] = \{ \int_\Omega XdQ : Q \text{ is a selector of } \Pi \},$$

it is obvious that $E_{P_1}[X]$ is the lower bound of $E_\Pi[X]$ and $E_{P_2}[X]$ is the upper bound of $E_\Pi[X]$, which indicates $E_\Pi[X] \subseteq [E_{P_1}[X], E_{P_2}[X]]$. Consequently, we only need to prove $E_\Pi[X] \supseteq [E_{P_1}[X], E_{P_2}[X]]$, that is, for any $x \in \int_\Omega XdP_1, \int_\Omega XdP_2$, there exists a selector $P_0$ of $\Pi$ such that $x = \int_\Omega XdP_0$. For any $x \in \int_\Omega XdP_1, \int_\Omega XdP_2$, there exists $\lambda \in [0,1]$ such that $x = \lambda \int_\Omega XdP_1 + (1 - \lambda) \int_\Omega XdP_2$. Define $P_0 := \lambda P_1 + (1 - \lambda) P_2$ where $\lambda \in [0,1]$, then $P_0(A) \in [P_1(A), P_2(A)]$ for any $A \in \mathcal{F}$, which implies $P_0$ is a selector of $\Pi$. Therefore, $E_\Pi[X] = [E_{P_1}[X], E_{P_2}[X]]$. \hfill $\blacksquare$

**Lemma 2.14.** Let $(\Gamma, \Xi)$ be a pair of upper-lower set-valued probabilities. Then for any nonnegative random variable $X$, $E_\Gamma[X] = \inf_{n \geq 1} E_{P_{n_1}}[X], \sup_{n \geq 1} E_{P_{n_2}}[X]$ and $E_\Xi[X] = \sup_{n \geq 1} E_{P_{n_2}}[X], \inf_{n \geq 1} E_{P_{n_2}}[X]$, where $\{P_{n_1}\}_{n=1}^\infty$ and $\{P_{n_2}\}_{n=1}^\infty$ denote the same sequences of measures and probabilities deduced in Lemma 2.12.

Proof. We will prove that for any nonnegative random variable $X$,

$$E_\Gamma[X] = \text{closure} \left\{ \bigcup_{n=1}^\infty E_{P_{n_1}}[X] \right\} = \text{closure} \left\{ \bigcup_{n=1}^\infty [E_{P_{n_1}}[X], E_{P_{n_2}}[X]] \right\}$$

$$= \inf_{n \geq 1} E_{P_{n_1}}[X], \sup_{n \geq 1} E_{P_{n_2}}[X].$$

It is obvious that $\inf_{n \geq 1} E_{P_{n_1}}[X]$ and $\sup_{n \geq 1} E_{P_{n_2}}[X]$ are the lower and upper bound of $E_\Gamma[X]$ respectively. Thus, we only need to show that $\bigcup_{n=1}^\infty [E_{P_{n_1}}[X], E_{P_{n_2}}[X]]$ is a convex set. In the following, we will prove this assertion in two steps.

**Step 1.** Assume $X$ to be a simple nonnegative random variable, that is $X = \sum_{i=1}^n a_i I_{A_i}$, where $\bigcup_{i=1}^n A_i = \Omega, A_i \cap A_j = \emptyset$ for any $i \neq j$, and $a_i \geq 0$ for $i = 1, 2, \ldots, n$. Now we will prove that for any $j, k \in \mathbb{N}^+$,

$$[E_{P_{j_1}}[X], E_{P_{j_2}}[X]] \bigcap [E_{P_{k_1}}[X], E_{P_{k_2}}[X]] \neq \emptyset.$$

Without loss of generality, we assume that $E_{P_{j_2}}[X] \leq E_{P_{k_2}}[X]$. We have

$$E_{P_{j_2}}[X] = \sum_{i=1}^n a_i P_{j_2}(A_i) = \sum_{i' \in I_1} a_{i'} P_{j_2}(A_{i'}) + \sum_{i'' \in I_2} a_{i''} P_{j_2}(A_{i''}),$$

$$E_{P_{k_2}}[X] = \sum_{i=1}^n a_i P_{k_2}(A_i) = \sum_{i' \in I_1} a_{i'} P_{k_2}(A_{i'}) + \sum_{i'' \in I_2} a_{i''} P_{k_2}(A_{i''}),$$

where $I_1$ and $I_2$ are index sets such that for any $i' \in I_1$ we have $P_{j_2}(A_{i'}) \geq P_{k_2}(A_{i'})$ and for any $i'' \in I_2$, we have $P_{j_2}(A_{i''}) < P_{k_2}(A_{i''})$. It is obvious that $I_1 \cup I_2 = \{1, 2, \ldots, n\}$ and $I_1 \cap I_2 = \emptyset$.

For any $i' \in I_1$, $P_{k_1}(A_{i'}) \leq P_{k_2}(A_{i'}) \leq P_{j_2}(A_{i'})$. Therefore, $\sum_{i' \in I_1} a_{i'} P_{k_1}(A_{i'}) \leq \sum_{i' \in I_1} a_{i'} P_{j_2}(A_{i'})$. On the other hand, by Lemma 2.12, we get for any $A \in \mathcal{F}$, $[P_{k_1}(A), P_{k_2}(A)] \bigcap [P_{j_1}(A), P_{j_2}(A)] \neq \emptyset$. Then for any $i'' \in I_2$,

$$[P_{k_1}(A_{i''}), P_{k_2}(A_{i''})] \bigcap [P_{j_1}(A_{i''}), P_{j_2}(A_{i''})] \neq \emptyset.$$
Since $P_{j2}(A_{\nu}) < P_{k2}(A_{\nu})$, we get $P_{k1}(A_{\nu}) \leq P_{j2}(A_{\nu})$. Therefore,
\[
\sum_{i'' \in I_{2}} a_{i''} P_{k1}(A_{i''}) \leq \sum_{i'' \in I_{2}} a_{i''} P_{j2}(A_{i''}).
\]
Consequently, we have
\[
E_{P_{k1}}[X] = \sum_{i=1}^{n} a_{i} P_{k1}(A_{i}) = \sum_{i' \in I_{1}} a_{i'} P_{k1}(A_{i'}) + \sum_{i'' \in I_{2}} a_{i''} P_{k1}(A_{i''})
\leq \sum_{i' \in I_{1}} a_{i'} P_{j2}(A_{i'}) + \sum_{i'' \in I_{2}} a_{i''} P_{j2}(A_{i''}) = E_{P_{j2}}[X].
\]
Since $E_{P_{k1}}[X] \leq E_{P_{j1}}[X] \leq E_{P_{k2}}[X]$, we have
\[
[E_{P_{j1}}[X], E_{P_{j2}}[X]] \cap [E_{P_{k1}}[X], E_{P_{k2}}[X]] \neq \emptyset.
\]
By the arbitrariness of $j$ and $k$, we have that $\{(E_{P_{k1}}[X], E_{P_{k2}}[X])\}_{n=1}^{\infty}$ is pairwise intersected. Therefore, we get that $\bigcup_{n=1}^{\infty} [E_{P_{k1}}[X], E_{P_{k2}}[X]]$ is a convex set.

**Step 2.** Assume $X$ to be a nonnegative random variable. Construct a sequence of simple nonnegative random variables:
\[
X_{m} = \sum_{i=0}^{m} \frac{t}{2m} I_{\{\frac{i}{2m} \leq X < \frac{i+1}{2m}\}} + m I_{\{X > m\}}.
\]
Then we get $X_{m} \leq X$, $X_{m} \uparrow X$ and $E_{P_{j_1}}[X] = \lim_{m \to \infty} E_{P_{j_1}}[X_{m}]$ for $i = 1, 2$. By the same method in Step 1, we only need to prove that for any $j, k \in \mathbb{N}^{+}$, $[E_{P_{j_1}}[X], E_{P_{j_2}}[X]] \cap [E_{P_{k1}}[X], E_{P_{k2}}[X]] \neq \emptyset$, that is
\[
\left[ \lim_{m \to \infty} E_{P_{j_1}}[X_{m}], \lim_{m \to \infty} E_{P_{j_2}}[X_{m}] \right] \cap \left[ \lim_{m \to \infty} E_{P_{k1}}[X_{m}], \lim_{m \to \infty} E_{P_{k2}}[X_{m}] \right] \neq \emptyset.
\]
Without loss of generality, we assume that $\lim_{m \to \infty} E_{P_{k2}}[X_{m}] \leq \lim_{m \to \infty} E_{P_{j_2}}[X_{m}]$. Then for any $\epsilon > 0$, there exists a positive integer $M$, such that for any $m > M$, $E_{P_{k_2}}[X_{m}] < E_{P_{j_2}}[X] + \epsilon$. By Step 1, for any $m$,
\[
[E_{P_{k1}}[X_{m}], E_{P_{k2}}[X_{m}]] \cap [E_{P_{j_1}}[X_{m}], E_{P_{j_2}}[X_{m}]] \neq \emptyset.
\]
So it is obvious that
\[
[E_{P_{k1}}[X_{m}], E_{P_{k2}}[X_{m}]] \cap [E_{P_{j_1}}[X_{m}] - \epsilon, E_{P_{j_2}}[X_{m}] + \epsilon] \neq \emptyset.
\]
Then we get $E_{k_2}[X_{m}] \geq E_{j_1}[X_{m}] - \epsilon$. Since $\epsilon$ can be arbitrary small, we achieve that $\lim_{m \to \infty} E_{P_{k_2}}[X_{m}] \geq \lim_{m \to \infty} E_{P_{j_1}}[X_{m}]$. Consequently, combined with the assumption that $\lim_{m \to \infty} E_{P_{k_2}}[X_{m}] \leq \lim_{m \to \infty} E_{P_{j_2}}[X_{m}]$, we get
\[
[E_{P_{j_1}}[X], E_{P_{j_2}}[X]] \cap [E_{P_{k1}}[X], E_{P_{k2}}[X]] \neq \emptyset
\]
for any $j, k \in \mathbb{N}^{+}$. This proof is complete. \(\square\)

Now, we will prove Proposition 2.8 and Proposition 2.10.
Proof. We will only prove the second and third assertions in Proposition 2.8 and the third assertion in Proposition 2.10, since others can be easily proved.

**Proposition 2.8 (2).** By Lemma 2.14, since $X$ and $Y$ are nonnegative random variables, there exist a sequence of measures $(P_{n_1})_{n=1}^\infty$ and a sequence of probabilities $(P_{n_2})_{n=1}^\infty$ such that $E_{\Gamma}[X + Y] = \inf_{n \geq 1} E_{P_{n_1}}[X + Y]$, $\sup_{n \geq 1} E_{P_{n_2}}[X + Y]$ and

\[
E_{\Gamma}[X] + E_{\Gamma}[Y] = \left( \inf_{n \geq 1} E_{P_{n_1}}[X], \sup_{n \geq 1} E_{P_{n_2}}[X] \right) + \left( \inf_{n \geq 1} E_{P_{n_1}}[Y], \sup_{n \geq 1} E_{P_{n_2}}[Y] \right) \\
= \left( \inf_{n \geq 1} E_{P_{n_1}}[X] + \inf_{n \geq 1} E_{P_{n_1}}[Y], \sup_{n \geq 1} E_{P_{n_2}}[X] + \sup_{n \geq 1} E_{P_{n_2}}[Y] \right).
\]

Since

\[
\sup_{n \geq 1} E_{P_{n_2}}[X + Y] \leq \sup_{n \geq 1} E_{P_{n_2}}[X] + \sup_{n \geq 1} E_{P_{n_2}}[Y]
\]

and

\[
\inf_{n \geq 1} E_{P_{n_1}}[X + Y] \geq \inf_{n \geq 1} E_{P_{n_1}}[X] + \inf_{n \geq 1} E_{P_{n_1}}[Y],
\]

we obtain $E_{\Gamma}[X + Y] \subseteq E_{\Gamma}[X] + E_{\Gamma}[Y]$.

**Proposition 2.8 (3).** By Lemma 2.14, since $X$ and $Y$ are nonnegative random variables, we have $E_{\Gamma}[X - Y] = \inf_{n \geq 1} E_{P_{n_1}}[X - Y]$, $\sup_{n \geq 1} E_{P_{n_2}}[X - Y]$ and

\[
E_{\Gamma}[X] - E_{\Gamma}[Y] = \left( \inf_{n \geq 1} E_{P_{n_1}}[X], \sup_{n \geq 1} E_{P_{n_2}}[X] \right) - \left( \inf_{n \geq 1} E_{P_{n_1}}[Y], \sup_{n \geq 1} E_{P_{n_2}}[Y] \right) \\
= \left( \inf_{n \geq 1} E_{P_{n_1}}[X] - \sup_{n \geq 1} E_{P_{n_2}}[Y], \sup_{n \geq 1} E_{P_{n_2}}[X] - \inf_{n \geq 1} E_{P_{n_1}}[Y] \right).
\]

Since

\[
\sup_{n \geq 1} E_{P_{n_2}}[X - Y] \leq \sup_{n \geq 1} E_{P_{n_2}}[X] - \inf_{n \geq 1} E_{P_{n_2}}[Y] \leq \sup_{n \geq 1} E_{P_{n_2}}[X] - \inf_{n \geq 1} E_{P_{n_1}}[Y]
\]

and

\[
\inf_{n \geq 1} E_{P_{n_1}}[X - Y] \geq \inf_{n \geq 1} E_{P_{n_1}}[X] - \sup_{n \geq 1} E_{P_{n_1}}[Y] \geq \inf_{n \geq 1} E_{P_{n_1}}[X] - \sup_{n \geq 1} E_{P_{n_2}}[Y],
\]

we obtain $E_{\Gamma}[X - Y] \subseteq E_{\Gamma}[X] - E_{\Gamma}[Y]$.

**Proposition 2.10 (3).** By Lemma 2.12, we have $\Gamma_1(A) = [\inf_{n \geq 1} P_{n_1}(A), \sup_{n \geq 1} P_{n_2}(A)]$ and $\Gamma_2(A) = [\inf_{n \geq 1} Q_{n_1}(A), \sup_{n \geq 1} Q_{n_2}(A)]$, where $(P_{n_1})_{n=1}^\infty$ and $(Q_{n_1})_{n=1}^\infty$ are two sequences of measures and $(P_{n_2})_{n=1}^\infty$ and $(Q_{n_2})_{n=1}^\infty$ are two sequences of probabilities. Since $\Gamma_1(A) \subseteq \Gamma_2(A)$ for any $A \in \mathcal{F}$, we have $\inf_{n \geq 1} P_{n_1}(A) \geq \inf_{n \geq 1} Q_{n_1}(A)$ and $\sup_{n \geq 1} P_{n_2}(A) \leq \sup_{n \geq 1} Q_{n_2}(A)$. Then for nonnegative random variables $X$ and $Y$, we have

\[
E_{\Gamma_1}[X] - E_{\Gamma_1}[Y] = \left( \inf_{n \geq 1} E_{P_{n_1}}[X], \sup_{n \geq 1} E_{P_{n_2}}[X] \right) - \left( \inf_{n \geq 1} E_{P_{n_1}}[Y], \sup_{n \geq 1} E_{P_{n_2}}[Y] \right) \\
= \left( \inf_{n \geq 1} E_{P_{n_1}}[X] - \sup_{n \geq 1} E_{P_{n_2}}[Y], \sup_{n \geq 1} E_{P_{n_2}}[X] - \inf_{n \geq 1} E_{P_{n_1}}[Y] \right),
\]

and

\[
E_{\Gamma_2}[X] - E_{\Gamma_2}[Y] = \left( \inf_{n \geq 1} E_{Q_{n_1}}[X], \sup_{n \geq 1} E_{Q_{n_2}}[X] \right) - \left( \inf_{n \geq 1} E_{Q_{n_1}}[Y], \sup_{n \geq 1} E_{Q_{n_2}}[Y] \right) \\
= \left( \inf_{n \geq 1} E_{Q_{n_1}}[X] - \sup_{n \geq 1} E_{Q_{n_2}}[Y], \sup_{n \geq 1} E_{Q_{n_2}}[X] - \inf_{n \geq 1} E_{Q_{n_1}}[Y] \right).
\]

Since

\[
\inf_{n \geq 1} E_{P_{n_1}}[X] - \sup_{n \geq 1} E_{P_{n_2}}[Y] \geq \inf_{n \geq 1} E_{Q_{n_1}}[X] - \sup_{n \geq 1} E_{Q_{n_2}}[Y]
\]
and \[
\sup_{n \geq 1} E_{P_{n_2}}[X] - \inf_{n \geq 1} E_{P_{n_1}}[Y] \leq \sup_{n \geq 1} E_{Q_{n_2}}[X] - \inf_{n \geq 1} E_{Q_{n_1}}[Y],
\]
we have
\[
E_{P_1}[X] - E_{P_1}[Y] \leq E_{P_2}[X] - E_{P_2}[Y].
\]

Next, we will introduce the concepts of upper-lower probability and upper-lower expectation, which will be useful to prove the main theorem in next section.

For a given set \( \mathcal{P} \) of point-valued probability measures on \( (\Omega, \mathcal{F}) \), we define a pair \((\mathcal{V}, v)\) of upper-lower probabilities by
\[
\mathcal{V}(A) := \sup_{P \in \mathcal{P}} P(A), \quad v(A) := \inf_{P \in \mathcal{P}} P(A), \quad \forall A \in \mathcal{F}.
\]

It is easy to check that \( \mathcal{V}(\emptyset) = 0, \mathcal{V}(\Omega) = 1 \), and \( \mathcal{V} \) satisfies the following properties:
(i) \( \mathcal{V}(\emptyset) = 0, \mathcal{V}(\Omega) = 1 \);
(ii) \( \mathcal{V}(A) \leq \mathcal{V}(B) \), whenever \( A \subseteq B \) and \( A, B \in \mathcal{F} \);
(iii) \( \mathcal{V}(A_n) \uparrow \mathcal{V}(A) \), if \( A_n \uparrow A \), where \( A_n, A \in \mathcal{F} \).

**Definition 2.15.** An event \( A \) holds almost surely with respect to \( \mathcal{V}(\text{a.s. w.r.t.} \mathcal{V}) \), if \( \mathcal{V}(A^c) = 0 \).

Now we define the upper expectation \( \mathbb{E}[\cdot] \) and the lower expectation \( \mathcal{E}[\cdot] \) on \( (\Omega, \mathcal{F}) \) generated by \( \mathcal{P} \). For each \( \mathcal{F} \)-measurable real random variable \( X \) such that \( E_P[X] \) exists for each \( P \in \mathcal{P} \), define
\[
\mathbb{E}[X] := \sup_{P \in \mathcal{P}} E_P[X], \quad \mathcal{E}[X] := \sup_{P \in \mathcal{P}} E_P[X].
\]

It is easy to check that \( \mathcal{E}[X] = -\mathbb{E}[-X] \) and \( \mathbb{E}[\cdot] \) is a sub-linear expectation, in other words, \( \mathbb{E}[\cdot] \) satisfies the following properties (a)-(d): for all random variables \( X \) and \( Y \),
(a) Monotonicity: \( X \geq Y \) implies \( \mathbb{E}[X] \geq \mathbb{E}[Y] \);
(b) Constant preserving: \( \mathbb{E}[c] = c, \forall c \in \mathbb{R} \);
(c) Positive homogeneity: \( \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0 \);
(d) Sub-additivity: \( \mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y] \).

**Definition 2.16.** Let \( X_1, X_2, \ldots, X_{n+1} \) be real measurable random variables on \( (\Omega, \mathcal{F}) \). \( X_{n+1} \) is said to be independent of \( (X_1, X_2, \ldots, X_n) \) under \( \mathbb{E}[\cdot] \), if for each nonnegative measurable function \( \varphi_i(\cdot) \) on \( \mathbb{R} \) with \( \mathbb{E}[\varphi_i(X_i)] < \infty, i = 1, \ldots, n+1 \), we have
\[
\mathbb{E}[\varphi_1(X_1) \cdot \varphi_2(X_2) \cdots \varphi_{n+1}(X_{n+1})] = \mathbb{E}[\varphi_1(X_1) \cdot \varphi_2(X_2) \cdots \varphi_n(X_n)]\mathbb{E}[\varphi_{n+1}(X_{n+1})]
\]
\( \{X_n\}_{n=1}^{\infty} \) is said to be a sequence of independent random variables under \( \mathbb{E}[\cdot] \) if \( X_{n+1} \) is independent of \( (X_1, X_2, \ldots, X_n) \) for each \( n \geq 1 \).

**Lemma 2.17.** ([4] Theorem 3.3) Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of independent random variables under \( \mathbb{E}[\cdot] \). Suppose \( \sup_{n \geq 1} \mathbb{E}[|X_i|^1+\alpha] < \infty \) for some \( \alpha > 0 \). And \( \mathbb{E}[X_i] = \mu, \mathcal{E}[X_i] \equiv \mu. \) Set \( S_n := \sum_{i=1}^{n} X_i. \) Then
\[
\mathcal{V}\left(\left\{\liminf_{n \to \infty} \frac{S_n}{n} < \mu\right\} \cup \left\{\limsup_{n \to \infty} \frac{S_n}{n} > \mu\right\}\right) = 0,
\]
and

\[ v \left( \mu \leq \liminf_{n \to \infty} \frac{S_n}{n} \leq \limsup_{n \to \infty} \frac{S_n}{n} \leq \overline{\mu} \right) = 1. \]

3. Main results. Let \((\Omega, \mathcal{F}, \Gamma)\) be an upper set-valued probability space. Then we define the concept of independent random variables with respect to \(E\Gamma\) as the extension of Definition 2.16.

**Definition 3.1.** Let \(X_1, X_2, \ldots, X_{n+1}\) be real measurable random variables on \((\Omega, \mathcal{F})\). \(X_{n+1}\) is said to be independent of \((X_1, X_2, \ldots, X_n)\) under \(E\Gamma[\cdot]\), if for each nonnegative measurable function \(\varphi(\cdot)\) on \(\mathbb{R}\) such that \(\{E\Gamma[\varphi_i(X_i)]\}_{i=1}^{n+1}\) are bounded sets, we have

\[ E\Gamma[\varphi_1(X_1) \cdot \varphi_2(X_2) \cdots \varphi_{n+1}(X_{n+1})] = E\Gamma[\varphi_1(X_1)] \cdot E\Gamma[\varphi_2(X_2)] \cdots E\Gamma[\varphi_{n+1}(X_{n+1})], \]

where the product of sets \(A\) and \(B\) is defined by \(AB = \{ab: a \in A, b \in B\}\).

\(\{X_n\}_{n=1}^{\infty}\) is said to be a sequence of independent random variables under \(E\Gamma[\cdot]\) if \(X_{n+1}\) is independent of \((X_1, X_2, \ldots, X_n)\) for each \(n \geq 1\).

**Remark 3.2.** This kind of independence is asymmetric and directional. Namely, \(X\) is independent of \(Y\) does not imply automatically that \(Y\) is independent of \(X\).

Finally, we need another notation: if \(x \in \mathbb{R}\), and \(A \subseteq \mathbb{R}\), then

\[ d(x, A) = \inf_{y \in A} |x - y|. \]

**Theorem 3.3.** Let \(\{X_i\}_{i=1}^{\infty}\) be a sequence of independent random variables under \(E\Gamma[\cdot]\). Suppose for some \(\alpha > 0\), \(\{E\Gamma[|X_i|^{1+\alpha}]\}_{i=1}^{\infty}\) are all bounded sets and \(E\Gamma[X_i^+] = E\Gamma[X_1^+], E\Gamma[X_i^-] = E\Gamma[X_1^-], E\Gamma[X_i^+] = E\Gamma[X_1^+], E\Gamma[X_i^-] = E\Gamma[X_1^-]\) for any \(i \geq 1\). Then

\[ \lim_{n \to \infty} d\left( \frac{\sum_{i=1}^{n} X_i}{n}, E\Gamma[X_1^+] - E\Gamma[X_1^-] \right) = 0 \]

almost surely with respect to \(\Gamma\).

**Proof.** Since \(\{X_i^+\}_{i=1}^{\infty}\) is a sequence of nonnegative random variables, then by Lemma 2.14, we obtain that \(E\Gamma[X_i^+] = [\inf_{n \geq 1} E_{P_{n1}}[X_i^+], \sup_{n \geq 1} E_{P_{n2}}[X_i^+]]\) and \(E\Gamma[X_i^-] = [\sup_{n \geq 1} E_{P_{n1}}[X_i^-], \inf_{n \geq 1} E_{P_{n2}}[X_i^-]]\), where \(\{P_{n1}\}_{n=1}^{\infty}\) is a sequence of measures and \(\{P_{n2}\}_{n=1}^{\infty}\) is a sequence of probabilities such that for any \(A \in \mathcal{F}\), \(P_{n1}(A) \leq P_{n2}(A)\). Since \(\inf_{n \geq 1} E_{P_{n1}}[X_i^+] \leq \inf_{n \geq 1} E_{P_{n2}}[X_i^+]\), it is obviously that

\[ [\inf_{n \geq 1} E_{P_{n1}}[X_i^+], \sup_{n \geq 1} E_{P_{n2}}[X_i^+]] \subseteq E\Gamma[X_i^+]. \]

Regard \(\mathcal{D} = \{P_{n2}\}_{n=1}^{\infty}\) as a family of probabilities. Denote \(E[X_i^+] = \sup_{n \geq 1} E_{P_{n1}}[X_i^+]\) and \(E[X_i^-] = \inf_{n \geq 1} E_{P_{n2}}[X_i^-]\). Then \([E[X_i^+], E[X_i^-]] \subseteq E\Gamma[X_i^+].\)

Since \(E\Gamma[X_i^+] = E\Gamma[X_1^+]\) and \(\sup_{n \geq 1} E_{P_{n2}}[X_i^+]\) is the upper bound of \(E\Gamma[X_i^+]\), we can get \(\sup_{n \geq 1} E_{P_{n2}}[X_i^+] = \sup_{n \geq 1} E_{P_{n2}}[X_i^-], \) that is \(E[X_i^+] = E[X_i^-]\) for any \(i \geq 1\). We denote \(E[X_i^+] \equiv \overline{\mu}\). Similarly, \(E\Gamma[X_i^+] = E\Gamma[X_1^+]\) and \(\inf_{n \geq 1} E_{P_{n2}}[X_i^+]\) is the upper bound of \(E\Gamma[X_i^+]\). We get \(\inf_{n \geq 1} E_{P_{n2}}[X_i^+] = \inf_{n \geq 1} E_{P_{n2}}[X_i^-], \) that is \(E[X_i^+] = E[X_i^-]\) for any \(i \geq 1\). We denote \(E[X_i^+] \equiv \underline{\mu}.\) Therefore, \([\underline{\mu}, \overline{\mu}] \subseteq E\Gamma[X_i^+].\)

Next, we will show that \(\{X_i\}_{i=1}^{\infty}\) are independent random variables under \(E\Gamma\) implies \(\{X_i^+\}_{i=1}^{\infty}\) are independent random variables under \(\mathbb{E}\). Define \(g(x) = \max(x, 0)\).
If $\varphi_i$ is a nonnegative measurable function, then $(\varphi_i \circ g)(\cdot) = \varphi_i(g(\cdot))$ is also a nonnegative measurable function. Then by Definition 3.1,

\[
E_1[\varphi(X^+_1) \varphi_2(X^+_2) \cdots \varphi_{n+1}(X^+_n)] = E_1[\varphi_1(g(X_1)) \varphi_2(g(X_2)) \cdots \varphi_{n+1}(g(X_{n+1}))]
\]

Similarly, we get

\[
E_1[\varphi_1(g(X_1)) \varphi_2(g(X_2)) \cdots \varphi_{n+1}(g(X_{n+1}))]
\]

Finally, from $\Gamma$ is the so-called upper expectation introduced in Section 2.

\[
\{\varphi_1 \cdots \varphi_n \} \cup \{\varphi_{n+1} \cdots \varphi_m \}
\]

Therefore, $(X^+_i)_{i=1}^\infty$ is a sequence of independent random variables under $E_1$. Since $\varphi_1(X^+_1) \cdot \varphi_2(X^+_2) \cdots \varphi_m(X^+_m)$ is nonnegative for any $m \geq 1$, we get

\[
E_1[\varphi(X^+_1) \cdots \varphi_m(X^+_m)] = \inf_{n \geq 1} E_{P_{n1}}[\varphi_1(X^+_1) \cdots \varphi_m(X^+_m)], \sup_{n \geq 1} E_{P_{n2}}[\varphi_1(X^+_1) \cdots \varphi_m(X^+_m)].
\]

Therefore, $E[\varphi_1(X^+_1) \cdot \varphi_2(X^+_2) \cdots \varphi_m(X^+_m)]$ is the upper bound of $E_1[\varphi_1(X^+_1) \cdot \varphi_2(X^+_2) \cdots \varphi_m(X^+_m)]$ for any $m \geq 1$. Consequently,

\[
E_1[\varphi_1(X^+_1) \cdots \varphi_{n+1}(X^+_{n+1})] = E_1[\varphi_1(X^+_1) \cdots \varphi_n(X^+_n)E_1[\varphi_{n+1}(X^+_{n+1})]
\]

implies

\[
E[\varphi_1(X^+_1) \cdots \varphi_{n+1}(X^+_{n+1})] = E[\varphi_1(X^+_1) \cdots \varphi_n(X^+_n)E[\varphi_{n+1}(X^+_{n+1})].
\]

Therefore, $(X^+_i)_{i=1}^\infty$ is a sequence of independent random variables under $E$, where $E$ is the so-called upper expectation introduced in Section 2.

Define $V(A) = \sup_{n \geq 1} P_{n2}(A)$ and $\nu(A) = \inf_{n \geq 1} P_{n2}(A)$ for any $A \in \mathcal{F}$. Then $(V, \nu)$ is the pair of upper-lower probability corresponding to upper-lower expectation $(E, \varepsilon)$. Obviously, $\Gamma \ll V$, that is if $V(A) = 0$, then $\Gamma(A) = \{0\}$.

Now, applying Lemma 2.17, we get

\[
V \left( \left\{ \liminf_{n \to \infty} \frac{\sum^n_{i=1} X^+_i}{n} < \mu \right\} \bigcup \left\{ \limsup_{n \to \infty} \frac{\sum^n_{i=1} X^+_i}{n} > \nu \right\} \right) = 0.
\]

That is

\[
\mu \leq \liminf_{n \to \infty} \frac{\sum^n_{i=1} X^+_i}{n} \leq \limsup_{n \to \infty} \frac{\sum^n_{i=1} X^+_i}{n} \leq \nu \quad a.s. \ w.r.t. \ V.
\]

Therefore,

\[
\lim_{n \to \infty} d(\frac{\sum^n_{i=1} X^+_i}{n}, [\mu, \nu]) = 0 \quad a.s. \ w.r.t. \ V.
\]

Since $[\mu, \nu] \subset E_1 X^+_1$,

\[
\lim_{n \to \infty} d(\frac{\sum^n_{i=1} X^+_i}{n}, E_1 X^+_1) = 0 \quad a.s. \ w.r.t. \ V.
\]

Finally, from $\Gamma \ll V$, we get

\[
\lim_{n \to \infty} d(\frac{\sum^n_{i=1} X^+_i}{n}, E_1 X^+_1) = 0 \quad a.s. \ w.r.t. \ \Gamma.
\]

Similarly, we get

\[
\lim_{n \to \infty} d(\frac{\sum^n_{i=1} X^-_i}{n}, E_1 X^-_1) = 0 \quad a.s. \ w.r.t. \ \Gamma.
\]
Therefore,
\[
\lim_{n \to \infty} d\left(\sum_{i=1}^{n} \frac{X_i^+}{n}, \sum_{i=1}^{n} \frac{X_i^-}{n}, E_{\Gamma}[X_i^+] - E_{\Gamma}[X_i^-]\right) = 0 \quad \text{a.s. w.r.t. } \Gamma,
\]
i.e.
\[
\lim_{n \to \infty} d\left(\sum_{i=1}^{n} \frac{X_i}{n}, E_{\Gamma}[X_i^+] - E_{\Gamma}[X_i^-]\right) = 0 \quad \text{a.s. w.r.t. } \Gamma.
\]

4. Law of large numbers under upper fuzzy-set probability. D. Ralescu has come up with the SLLN with respect to a fuzzy-set valued probability as an extension of the SLLN with respect to set-valued probability. Similar to the method of D. Ralescu, we will extend Theorem 3.3 to the case of upper fuzzy-set valued probability.

At the beginning, we will introduce some concepts (see [22]) of fuzzy-set valued probability briefly.

Fuzzy sets are sets whose elements have degrees of membership. Denote the membership function of fuzzy set \(M\) by \(m_M\), \(m_M(x) : \mathbb{R} \to [0, 1]\). In this sequel, we only consider the space \(F_C(\mathbb{R})\) of fuzzy sets \(M\) such that \(L_\alpha(M) = \{x \in \mathbb{R} \mid m_M(x) \geq \alpha\}\) is compact, convex, and nonempty for every \(\alpha \in (0, 1]\), and \(\text{supp}(M) = \text{closure}\{x \in \mathbb{R} \mid m_M(x) > 0\}\) is compact. The distance considered in \(F_C(\mathbb{R})\) extends the Hausdorff distance as follow:

\[
d(M,N) = \sup_{0<\alpha\leq 1} d_H(L_\alpha(M), L_\alpha(N)), \quad M,N \in F_C(\mathbb{R}),
\]
where \(d_H(A,B) = \max\{\sup_{a\in A} \inf_{b\in B} |a-b|, \sup_{b\in B} \inf_{a\in A} |a-b|\}\) is the Hausdorff distance between sets \(A\) and \(B\).

It is known that \((F_C(\mathbb{R}), d)\) is complete (see[20]). The operations of fuzzy-sets are defined through their corresponding membership functions as follow.

**Definition 4.1.** Assume \(M\) and \(N\) are fuzzy-sets, then

1. Addition: \(m_{M+N}(x) = \sup_{x_1+x_2=x} \min(m_M(x_1), m_N(x_2))\);
2. Scalar multiplication:
\[
m_{\lambda M} = \begin{cases} m_M(\frac{x}{\lambda}) & \lambda \neq 0 \\ I_{\{0\}}(x) & \lambda = 0; \end{cases}
\]
3. Product: \(m_{MN}(x) = \sup_{x_1 \cdot x_2 = x} \min(m_M(x_1), m_N(x_2))\);
4. Union: \(m_{M \cup N}(x) = \max(m_M(x), m_N(x))\);
5. Intersection: \(m_{M \cap N}(x) = \min(m_M(x), m_N(x))\).

**Remark 4.2.** Subtraction of two fuzzy sets is defined through the corresponding membership function

\[
m_{M-N}(x) = \sup_{x_1-x_2=x} \min(m_M(x_1), m_N(x_2)),
\]
which can be deduced form (1) and (2).

**Remark 4.3.** If we regard the indicator function \(I_A\) of classic set \(A\) as a membership function, then \(A\) is a special fuzzy set. Furthermore, all the traditional operations of sets satisfy Definition 4.1 by using the indicator function as membership function.
Definition 4.4. ([22]) A fuzzy-set valued probability, \( f \)-probability for short, is a function \( \mu : \mathcal{F} \to \mathcal{F}_{c}(\{0, 1\}) \) such that

1. \( \mu(\emptyset) = \{0\} \);
2. \( \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \), for every disjoint family \( \{A_i\}_{i=1}^{\infty} \);
3. \( m_{\mu}(\Omega)(1) = 1 \), i.e. \( 1 \in \mu(\Omega) \).

The concept of \( f \)-probability is an extension of set-valued probability. Set-valued measures which are related to a \( f \)-probability can be defined as follows: if \( \mu \) is a \( f \)-probability, then define

\[
(\text{supp}\mu)(A) = \text{supp}(\mu(A)), \quad A \in \mathcal{F}
\]

\[
\mu^\alpha(A) = L_\alpha(\mu(A)), \quad \alpha \in (0, 1].
\]

For proofs that \( \text{supp}\mu \) and \( \mu^\alpha \) defined above are set-valued probabilities, see [21].

Definition 4.5. Let \( \{\mu_n\}_{n=1}^{\infty} \) be a sequence of \( f \)-probabilities. \( U(\cdot) = \bigcup_{n=1}^{\infty} \mu_n(\cdot) \) and \( U(\cdot) = \bigcap_{n=1}^{\infty} \mu_n(\cdot) \) are two set-valued functions from \( \mathcal{F} \) to \( \mathcal{F}_{c}(\{0, 1\}) \). \( (U, \bar{U}) \) is called a pair of upper-lower \( f \)-probabilities if for any \( A \in \mathcal{F} \), \( \alpha \in (0, 1] \), \( \bigcap_{n=1}^{\infty} \mu_n^\alpha (A) \) is a convex set and \( \bigcap_{n=1}^{\infty} \mu_n^\alpha (A) \) is a nonempty convex set.

Proposition 4.6. Denote \( U^\alpha(\cdot) = L_\alpha(U(\cdot)) : \mathcal{F} \to \mathcal{P}([0, 1]) \) and \( \bar{U}^\alpha(\cdot) = L_\alpha(\bar{U}(\cdot)) : \mathcal{F} \to \mathcal{P}([0, 1]) \). Then \( (U^\alpha, \bar{U}^\alpha) \) is a pair of upper-lower set-valued probabilities defined in Definition 2.3.

Proof.

\[
U^\alpha(A) = L_\alpha(U(A))
\]

\[
= L_\alpha \left( \bigcup_{n=1}^{\infty} \mu_n(A) \right)
\]

\[
= \{ x \in \mathbb{R} \mid m_{\bigcup_{n=1}^{\infty} \mu_n(A)}(x) \geq \alpha \}
\]

\[
= \{ x \in \mathbb{R} \mid \max_{n \geq 1} (m_{\mu_1(A)}(x), \ldots, m_{\mu_n(A)}(x), \ldots) \geq \alpha \}
\]

\[
= \{ x \in \mathbb{R} \mid m_{\mu_1(A)}(x) < \alpha, \ldots, m_{\mu_n(A)}(x) < \alpha, \ldots \}^c
\]

\[
= \left( \bigcap_{n=1}^{\infty} \{ x \in \mathbb{R} \mid m_{\mu_n(A)}(x) < \alpha \} \right)^c
\]

\[
= \bigcup_{n=1}^{\infty} \{ x \in \mathbb{R} \mid m_{\mu_n(A)}(x) < \alpha \}^c
\]

\[
= \bigcup_{n=1}^{\infty} \mu_n^\alpha (A)
\]

Since \( \{\mu_n^\alpha\}_{n=1}^{\infty} \) is a sequence of nonatomic closed set-valued probabilities and \( \bigcup_{n=1}^{\infty} \mu_n^\alpha (A) \) is a convex set for any \( A \in \mathcal{F} \), by the Definition 2.3, \( U^\alpha(\cdot) \) is an upper set-valued probability. Similarly, we can prove that \( \bar{U}^\alpha(\cdot) \) is a lower set-valued probability corresponding to \( U^\alpha \).

Now, we will give the concept of expectations under upper-lower \( f \)-probabilities.

Definition 4.7. Let \( U(\cdot) = \bigcup_{n=1}^{\infty} \mu_n(\cdot) \) and \( \bar{U}(\cdot) = \bigcap_{n=1}^{\infty} \mu_n(\cdot) \) be upper \( f \)-probability and lower \( f \)-probability respectively, where \( \{\mu_n\}_{n=1}^{\infty} \) is a sequence of \( f \)-probabilities. If for any \( \alpha \in (0, 1] \), \( L_\alpha(\bigcup_{n=1}^{\infty} E_{\mu_n}[X]) \) is closed set, then the fuzzy sets
Proof. To the case of upper and lower expectations of random variable $X$ respectively, that is

$$E_U[X] = \bigcup_{n=1}^{\infty} E_{\mu_n}[X], \quad E_L[X] = \bigcap_{n=1}^{\infty} E_{\mu_n}[X]$$

where $\{E_{\mu_n}[X]\}_{n=1}^{\infty}$ are fuzzy sets such that $L_\alpha(E_{\mu_n}[X]) = E_{\mu_n^\alpha}[X]$ for any $\alpha \in (0, 1]$.

**Remark 4.8.** We require that $\int_\Omega |X|\,d\text{supp}\mu_n < \infty$ for the expectation to exist and the existence and uniqueness of $E_{\mu_n}[X]$ are from [12].

**Proposition 4.9.** For any $\alpha \in (0, 1]$ and any random variable $X$, $L_\alpha(E_U[X]) = E_{U^\alpha}[X]$ and $L_\alpha(E_L[X]) = E_{L^\alpha}[X]$.

*Proof.*

$$L_\alpha(E_U[X]) = L_\alpha \left( \bigcup_{n=1}^{\infty} E_{\mu_n}[X] \right)$$

$$= \{ x \in \mathbb{R} | m_{\bigcup_{n=1}^{\infty} E_{\mu_n}[X]}(x) \geq \alpha \}$$

$$= \{ x \in \mathbb{R} | \max(m_{E_{\mu_n}[X]}(x), \cdots, m_{E_{\mu_n}[X]}(x), \cdots) \geq \alpha \}$$

$$= \{ x \in \mathbb{R} | m_{E_{\mu_n}[X]}(x) < \alpha, \cdots, m_{E_{\mu_n}[X]}(x) < \alpha, \cdots \}^c$$

$$= \left( \bigcap_{n=1}^{\infty} \{ x \in \mathbb{R} | m_{E_{\mu_n}[X]}(x) < \alpha \} \right)^c$$

$$= \bigcup_{n=1}^{\infty} \{ x \in \mathbb{R} | m_{E_{\mu_n}[X]}(x) < \alpha \}^c$$

$$= \bigcup_{n=1}^{\infty} L_\alpha(E_{\mu_n}[X])$$

$$= \bigcup_{n=1}^{\infty} E_{\mu_n^\alpha}[X]$$

$$= E_{\bigcup_{n=1}^{\infty} \mu_n^\alpha}[X]$$

$$= E_{U^\alpha}[X]$$

The penultimate equality is from the assumption that $L_\alpha(E_U[X])$ is a closed set and the last equality is from the definition of expectation of upper $f$-probability.

Similarly, we can achieve $L_\alpha(E_L[X]) = E_{L^\alpha}[X]$. \qed

In the next, we will extend the concept of independence defined in Definition 3.1 to the case of upper $f$-probability $U$.

**Definition 4.10.** Let $X_1, X_2, \cdots, X_{n+1}$ be real measurable random variables on $(\Omega, \mathcal{F})$. $X_{n+1}$ is said to be independent of $(X_1, X_2, \cdots, X_n)$ under $E_U[\cdot]$, if for each nonnegative measurable function $\varphi_i(\cdot)$ on $\mathbb{R}$ with that $\{E_{\text{supp}\mu}[\varphi_i(X_i)]\}_{i=1}^{n+1}$ are bounded sets, we have

$$E_U[\varphi_1(X_1) \cdots \varphi_{n+1}(X_{n+1})] = E_U[\varphi_1(X_1) \cdots \varphi_n(X_n)]E_U[\varphi_{n+1}(X_{n+1})],$$

where the product of fuzzy sets $M$ and $N$ is defined in Definition 4.1.
\[ \{X_n\}_{n=1}^{\infty} \] is said to be a sequence of independent random variables under \( E_{\alpha} \), if \( X_{n+1} \) is independent of \( (X_1, X_2, \cdots, X_n) \) for each \( n \geq 1 \).

**Definition 4.11.** Assume \( M \) is a fuzzy set, then define the distance between \( x \in \mathbb{R} \) and \( M \) by
\[
D(x, M) = \int_{0}^{1} d(x, L_{\alpha} M) \, d\alpha
\]
where \( d(x, A) \) is the distance between a point \( x \) and a set \( A \).

**Theorem 4.12.** Let \( \{X_i\}_{i=1}^{\infty} \) be a sequence of independent random variables under \( E_{\alpha} \). Suppose that for some \( \beta > 0 \), \( \{E_{\text{supp}([X_i|1+\beta])}\}_{i=1}^{\infty} \) are all bounded sets, and \( E_{\alpha}[X_i^+] = E_{\alpha}[X_i^{-}] \), \( E_{\alpha}[X_i^+] = E_{\alpha}[X_1^+] \), \( E_{\alpha}[X_i^+] = E_{\alpha}[X_1^+] \), for any \( i \geq 1 \). Then
\[
1 \in U \left( \lim_{n \to \infty} D \left( \sum_{i=1}^{n} \frac{X_i}{n}, E_{\alpha}[X_1^+] - E_{\alpha}[X_1^-] \right) = 0 \right).
\]

**Proof.** For any fixed \( \alpha \in (0, 1] \), that \( E_{\text{supp}([X_i|1+\beta])} \) is a bounded set implies \( E_{\text{supp}([X_i|1+\beta])} \) is a bounded set, where \( U^\alpha = L_{\alpha}(U) \) is the upper set-valued probability.

Next, we will show that \( \{X_i\}_{i=1}^{\infty} \) satisfy all the assumptions in Theorem 3.3, provided that \( U^\alpha \) is regarded as the upper set-valued probability. Since \( E_{\alpha}[X_i^+] \) are fuzzy sets, \( E_{\alpha}[X_i^+] = E_{\alpha}[X_i^+] \) implies \( L_{\alpha}(E_{\alpha}[X_i^+]) = L_{\alpha}(E_{\alpha}[X_i^+]) \). Then by Proposition 4.6, for any \( i \geq 1 \), \( E_{\alpha}^\phi[X_i^+] = E_{\alpha}^\phi[X_i^+] \). Similarly, we can prove \( E_{\alpha}^\phi[X_i^-] = E_{\alpha}^\phi[X_i^-] \), \( E_{\alpha}^\phi[X_i^+] = E_{\alpha}^\phi[X_i^+] \) and \( E_{\alpha}^\phi[X_i^-] = E_{\alpha}^\phi[X_i^-] \). Furthermore,
\[
E_{\alpha}[\varphi_1(X_1) \cdots \varphi_{n+1}(X_{n+1})] = E_{\alpha}[\varphi_1(X_1) \cdots \varphi_n(X_n)]E_{\alpha}[\varphi_{n+1}(X_{n+1})]
\]
implies
\[
E_{\alpha}^\phi[\varphi_1(X_1) \cdots \varphi_{n+1}(X_{n+1})] = E_{\alpha}[\varphi_1(X_1)] \cdots \varphi_n(X_n)]E_{\alpha}^\phi[\varphi_{n+1}(X_{n+1})],
\]
which means that \( \{X_i\}_{i=1}^{\infty} \) is a sequence of independent random variables under \( E_{\alpha}^\phi \).

Since \( (U^\alpha, U^\beta) \) is a pair of upper-lower set-valued probabilities, applying Theorem 3.3, we have
\[
\lim_{n \to \infty} d \left( \sum_{i=1}^{n} \frac{X_i}{n}, E_{\alpha}[X_1^+] - E_{\alpha}[X_1^-] \right) = 0, \quad \text{a.s. w.r.t. } U^\alpha
\]
Define a function:
\[
\phi_n(\alpha) = d \left( \sum_{i=1}^{n} \frac{X_i}{n}, E_{\alpha}[X_1^+] - E_{\alpha}[X_1^-] \right)
\]
for \( n = 1, 2, \cdots \) and \( \alpha \in (0, 1] \). Notice that if \( \alpha_1 < \alpha_2 \) then for any \( A \in \mathcal{F} \), \( U^\alpha(A) \supseteq U^{\alpha_2}(A) \). By Proposition 2.10, we have \( E_{\alpha_1}[X_1^+] - E_{\alpha_1}[X_1^-] \supseteq E_{\alpha_2}[X_1^+] - E_{\alpha_2}[X_1^-] \). Therefore,
\[
d \left( \sum_{i=1}^{n} \frac{X_i}{n}, E_{\alpha_1}[X_1^+] - E_{\alpha_1}[X_1^-] \right) \leq d \left( \sum_{i=1}^{n} \frac{X_i}{n}, E_{\alpha_2}[X_1^+] - E_{\alpha_2}[X_1^-] \right)
\]
Thus \( \phi_n(\alpha) \) is increasing with respect to \( \alpha \) and obviously \( \phi_n(\alpha) \leq \phi_n(1) \) for each \( n \geq 1 \).
Since \( U^1 \) is an upper set-valued probability, referring to the proof of Theorem 3.3, there exist an upper probability \( V_1 \) such that
\[
\phi_n(1) = d \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U^{d}}[X_1^+] - E_{U^{d}}[X_1^-] \right) \to 0, \quad \text{a.s. w.r.t. } V_1,
\]
that is
\[
V_1 \left( \lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U^{d}}[X_1^+] - E_{U^{d}}[X_1^-] \right) \neq 0 \right) = 0.
\]
Furthermore, since \( U^\alpha \supseteq U^1 \) for any \( \alpha \in (0, 1) \),
\[
\lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U^{\alpha}}[X_1^+] - E_{U^{\alpha}}[X_1^-] \right) = 0
\]
implies
\[
\lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U^{d}}[X_1^+] - E_{U^{d}}[X_1^-] \right) = 0.
\]
Therefore, for any \( \alpha \in (0, 1) \),
\[
\left\{ \omega \mid \lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i(\omega)}{n}, E_{U^{\alpha}}[X_1^+] - E_{U^{\alpha}}[X_1^-] \right) = 0 \right\}
\subseteq \left\{ \omega \mid \lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i(\omega)}{n}, E_{U^{d}}[X_1^+] - E_{U^{d}}[X_1^-] \right) = 0 \right\},
\]
\[
\left\{ \omega \mid \lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i(\omega)}{n}, E_{U^{d}}[X_1^+] - E_{U^{d}}[X_1^-] \right) \neq 0 \right\}
\supseteq \left\{ \omega \mid \lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i(\omega)}{n}, E_{U^{\alpha}}[X_1^+] - E_{U^{\alpha}}[X_1^-] \right) \neq 0 \right\}.
\]
Consequently, for any \( \alpha \in (0, 1) \),
\[
V_1 \left( \lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U^{\alpha}}[X_1^+] - E_{U^{\alpha}}[X_1^-] \right) \neq 0 \right) = 0,
\]
that is for any \( \alpha \in (0, 1) \),
\[
\lim_{n \to \infty} \phi_n(\alpha) = \lim_{n \to \infty} d \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U^{\alpha}}[X_1^+] - E_{U^{\alpha}}[X_1^-] \right) = 0 \quad \text{a.s. w.r.t. } V_1.
\]
Consequently, \( \phi_n(\alpha) \leq 1 \) when \( n \) is sufficiently large. Since \( \phi_n(\alpha) \to 0 \) a.s. w.r.t \( V_1 \), the Lebesgue dominated convergence theorem can be applied to obtain
\[
\lim_{n \to \infty} \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U^{\alpha}}[X_1^+] - E_{U^{\alpha}}[X_1^-] \right) = \lim_{n \to \infty} \int_0^1 \phi_n(\alpha) d\alpha = 0, \quad \text{a.s. w.r.t. } V_1,
\]
that is
\[
V_1 \left( \lim_{n \to \infty} D \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U}[X_1^+] - E_{U}[X_1^-] \right) \neq 0 \right) = 0
\]
Therefore,
\[
v_1 \left( \lim_{n \to \infty} D \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U}[X_1^+] - E_{U}[X_1^-] \right) \neq 0 \right) = 0.
\]
Subsequently, we get
\[
V_1 \left( \lim_{n \to \infty} D \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U}[X_1^+] - E_{U}[X_1^-] \right) = 0 \right) = 1.
\]
Therefore,
\[
1 \in U^1 \left( \lim_{n \to \infty} D \left( \frac{\sum_{i=1}^n X_i}{n}, E_{U}[X_1^+] - E_{U}[X_1^-] \right) = 0 \right).
\]
Furthermore,

\[
1 \in \mathcal{U} \left( \lim_{n \to \infty} D \left( \frac{\sum_{i=1}^{n} X_i}{n}, E [X_1^+ - E [X_1^-]] \right) = 0 \right).
\]

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