A Dirac Fermion Hierarchy of Composite Fermi Liquids

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Composite Fermi liquids (CFLs) are gapless states that can occur at certain Landau level fillings \( \nu \). They were first explained by the celebrated work of Halperin, Lee, and Read (HLR) [1] as Fermi seas of electromagnetic flux attached composite fermions. While its success in explaining CFL’s metallic feature, it is not obvious how HLR theory is consistent with the particle hole (PH) symmetry [2,3], which is an exact symmetry in a half filled Landau level if Landau level mixing is negligible. Recently an emergent Dirac fermion (DF) theory was proposed by Son [5] at \( \nu = 1/2 \),

\[
\mathcal{L} = i\bar{\psi} \gamma^\mu (\partial_\mu - ia_\mu) \psi - \frac{1}{2} \frac{1}{2\pi} adA + \frac{1}{2} \frac{1}{2\pi} AdA. \tag{1}
\]

where \( \psi \) is the DF field, \( \gamma^\mu \) is the Gamma matrix, \( a_\mu \) and \( A_\mu \) are internal and external gauge fields respectively. We set \( e = \hbar = 1 \), and magnetic length \( l_B^2 = B \) where \( B \) is the external magnetic field. Greek and Latin letters are used to label space-time coordinates \( (t, x, y) \) and pure spatial coordinates respectively. \( \epsilon^{xy} = 1, \epsilon^{yx} = -1 \) is the antisymmetric symbol. Higher order terms in the action are omitted for simplicity. DF theory is explicitly PH symmetric, because PH transformations acting on the physical electrons in a way akin to time reversal on DFs.

At half filling, DF theory predicts a \( \pi \) Berry phase, which in fact is a Fermi sea center \( \pi \) Berry curvature singularity due to the two-component spinor nature of DF, associated with transporting a single composite fermion [DF] along the Fermi sea. This \( \pi \) Berry phase and Berry curvature have been observed from numerics [6,9]. It is worth to mention that the relation between Hall conductivity and Fermi sea Berry phase \( \Phi_{FS} \) is general: as first noticed by Haldane [10] in the context of anomalous Hall effect, the non-quantized part of Hall conductivity is determined by \( \Phi_{FS} \),

\[
\sigma_H = \frac{e^2}{h} \Phi_{FS}, \quad \Phi_{FS} = 2\pi \nu. \tag{2}
\]

In principle, CFLs can occur as long as the HLR flux attached particles are fermions; whether or not they occur depends on the details of interaction. When the underlying physical particles are fermions, the filling fractions of CFLs can be grouped into two classes: \( \nu = 1/2m \) whose Fermi seas are formed by composite fermions [we call them as fCFLs], and \( \nu = 1 - 1/2m \) with Fermi seas formed by composite holes [anti-fCFLs]. We have used and will keep using \( m \) to denote a positive integer throughout the paper. In our definition, fCFLs are equivalent [11] to anti-fCFLs at the PH symmetric point \( m = 1 \): both of them are described by Son’s DF theory.

The purpose of this work is to propose a low energy effective theory for fCFLs and anti-fCFLs at all filling fractions that they can occur. This theory generalizes Son’s DF theory by attaching each DF with \( \pm |m - 1| \) internal flux quanta. PH conjugate states are realized by attaching same amount but an opposite direction fluxes: positive for fCFLs and negative for anti-fCFLs. As a result at long wavelength, CFLs for fermions can be considered as descending from the \( \nu = 1/2 \) PH symmetric states, in analogy to Jain’s hierarchy for incompressible quantum Hall fluids which take integer quantum Hall effects as parent states. We provide theoretical analysis as well as numerical evidence to support this idea.

Proposed action.—Due to the lack of PH symmetry when \( \nu \neq 1/2 \), a Chern-Simons (CS) term and a mass term \(-K_{\phi} \phi \partial_\phi \mu \psi \phi \) can be included into the DF action Eqn. [1], where \( K_\phi \) and \( M \) are the CS level and the DF mass respectively. Differentiating this action with respect to \( A_0 \), we see that the field strength of the emergent gauge field \( B_\psi = e^{ab} \partial_0 a_b \) is set by the physical electron density \( \rho_\alpha = \frac{e^2}{4\pi a_0} \),

\[
\frac{B_\psi}{2\pi} = -2(\rho_\alpha - \frac{1}{2} \frac{B_A}{2\pi}) = (1 - 2\nu) \frac{B_A}{2\pi}. \tag{3}
\]

Taking variations with respect to \( a_0 \), the DF density \( \rho_\phi = \psi \gamma^0 \bar{\psi} \) is set by,

\[
\rho_\phi = \frac{1}{2} \frac{B_A}{2\pi} + k \frac{B_\psi}{2\pi}. \tag{4}
\]

Several constraints are to be imposed for a candidate theory to describe Fermi liquid. First, as a necessary condition, the DFs should eventually perceive zero net magnetic fields. Second, it has been conjectured [5,6,12] that
CFLs satisfy Luttinger theorem, which is a fundamental property of Landau Fermi liquid stating that Fermi sea volume remains invariant under weak interactions. We assume Luttinger theorem for CFLs and will provide supporting numerical evidence later on. Third, the Fermi sea of an fCFL and its PH partner anti-fCFL must be of the same size [11,14]: actually Fermi sea size is expected to be determined by the minor charge carrier motivated by the experimental observation that Fermi wave vector is determined by composite fermions at $\nu < 1/2$ and by composite holes if otherwise [15]. Fermi sea in DF theory is made up of DFs. Hence according to Luttinger theorem the DF density must be the following,

$$\rho_\psi = \frac{1}{2m} \frac{1}{2\pi l_B^2}, \quad \nu = \begin{cases} 1/2m, \\ 1 - 1/2m. \end{cases}$$

(5)

This, together with the PH symmetry constraint in a half filled Landau level, shows that the CS level $K$ has a unique solution,

$$K = \begin{bmatrix} -\frac{1}{2}, & \nu = 1/2m, \\ 0, & \nu = 1/2, \\ +\frac{1}{2}, & \nu = 1 - 1/2m. \end{bmatrix}$$

(6)

We see that at $\nu \neq 1/2$, DF theory receives a half level CS term, which makes sense since a theory describing the Fermi sea and low energy excitations must have the irrelevant modes integrated out. Due to the lack of PH symmetry, a mass gap $M$ exists separating the DF’s conduction and valance band. It is well known that integrating out the inertial valance band produces a half level CS term $\frac{\text{sgn}(M) \pi}{4} da$.

At $m \neq 1$, the filling $\nu_\psi$ of DFs is easily derived,

$$\nu_\psi = \frac{\rho_\psi}{B_\psi/2\pi} = \begin{cases} \frac{1}{2m-1}, & \nu = 1/2m, \\ \frac{1}{2m-2}, & \nu = 1 - 1/2m. \end{cases}$$

(7)

which is again an even denominator number, indicating that upon a statistics preserving flux attachment transformation, the DFs would perceive no magnetic fields on average hence a Fermi liquid is possible to form. Note that PH conjugation is realized by bounding same size [11–14]; actually Fermi sea size is expected to be determined by composite holes if otherwise [15]. Fermi sea in DF theory is made up of DFs. Hence according to Luttinger theorem the DF density must be the following,

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which is again an even denominator number, indicating that upon a statistics preserving flux attachment transformation, the DFs would perceive no magnetic fields on average hence a Fermi liquid is possible to form. Note that PH conjugation is realized by bounding same amount but an opposite direction of fluxes. We call the effective theory proposed here flux-attached DF theory. Effective theories are illustrated in FIG. 1.

Flux attachment induces electrical charge to the DFs, making the flux-attached DF theory not in LLL [16]: same problem appears in HLR theory at all fillings. Nevertheless, following the same spirit of vortex metal theory [17], it is straightforward in the final action Eqn. 8 to see that the charge neutrality condition is restored at low energy [near Fermi surface] as a consequence of Fermi sea Berry phase. In spite of not being a LLL theory, the effect of LLL projection is well known: as a fundamental property of a dipolar electron in magnetic field, the dipole vector $d$ is perpendicular and strength proportional to the kinetic momentum vector $k$; LLL projection, generally speaking, shifts the flux attachment center away from the electron’s location to create a dipole. In summary, as a result of flux attachment and LLL projection, the dipolar DF has dipole-momentum locking, in addition to the intrinsic spin-orbit locking that Son’s DF has.

The dipole-momentum and spin-orbit locking have a nontrivial impact on the Berry phase associated with transporting a composite fermion [dipolar DF] in the momentum space $[\mathbf{k}$-space]. The Berry curvature distribution is predicted to be: (I) $\mathbf{k}$-space uniform except at the Fermi sea center point $\mathbf{k} = 0$, (II) where there is an additional $\pi$ Berry phase. The reasoning goes as follows.

The dipole-momentum locking provides a nature mapping from the real space to the $\mathbf{k}$-space, and therefore (I) is a manifestation of the real space Aharonov-Bohm effect. The Berry phase from (I) is purely geometrical, independent on the emergent DF mass $M$, and area proportional for a homogeneous state when the finite-size induced density fluctuation is negligible. The contribution to the Fermi sea Berry phase $\Phi_{FS}$ from (I) should be $2\pi \nu - \pi$ in accordance with the fact that it is zero at half filling. On the other hand, (II) originates from the spin-orbit locking effect of a massless spin-half DF. Although the sign of the emergent mass $\text{sgn}(M)$ turns out to be important [as it distinguishes ICFLs and anti-fCFLs], we argue its absolute value is small $|M| \approx 0$ in the thermodynamic limit: finite mass tilts the DF’s spin away from the 2D plane, hence $|M|$ represents how much the Fermi sea Berry phase $\Phi_{FS}$ deviates from $2\pi \nu$. We thus postulate that even at $\nu \neq 1/2$ there is a $\pi$ Berry phase at Fermi sea center, which we emphasis is not protected by symmetry but instead constrained to take this value by the Fermi sea Berry phase.

The flux attachment singular gauge transformation can be unambiguously carried out to cancel $B_\psi$ on average,
after integrating out the infinitely deep valence band Dirac sea. To summarize, the resulting theory describing CFLs of fermions at \( \nu = 1/2m \) and \( 1 - 1/2m \) is,

\[
\mathcal{L} = i\psi^\dagger \gamma^\mu (\partial_\mu - ia^\mu_i) \psi' - \frac{1}{2m} \frac{\pi}{2} a^\mu dA + \frac{\text{sgn}(M)}{2m} \frac{1}{4\pi} a^\mu' da' + \left[ \frac{1}{2} - \frac{\text{sgn}(M)m - 1}{m} \right] \frac{1}{4\pi} AdA + \mathcal{M} \psi' \psi'.
\]

where the flux-attached DF \( \psi' \) field perceives no net magnetic fields \( \langle a^a \partial a^b_i / 2\pi \rangle = 0 \). The sign of the DF mass distinguishes fCFLs and anti-fCFLs: \( \text{sgn}(M) = +1 \) for fCFLs at \( \nu = 1/2m \), and \( \text{sgn}(M) = -1 \) for anti-fCFLs at \( \nu = 1 - 1/2m \). Note that the coefficient in front of \( \frac{1}{4\pi} AdA \) is precisely the Hall conductivity. While \( \text{sgn}(M) \) is crucial, we conjecture that due to the Fermi sea Berry phase, \( |\mathcal{M}| \to 0 \) in the thermodynamic limit. Higher order terms including interaction terms and Maxwell terms are omitted for simplicity. We then provide numerical evidence to support Eqn. \( \text{(8)} \). We will see that the Berry curvatures obtained from HLR motivated wave functions agree with the prediction out of the flux-attached DF picture, strongly suggesting that Eqn. \( \text{(8)} \) is equivalent to HLR at infrared. Further testings of Eqn. \( \text{(8)} \) can be obtained from studying response functions, which we decide to show somewhere else.

**Numerical studies.**—In the following, we examine the CFL model wave functions (MWFs) at \( \nu = 1/4 \) as a case study. Supporting evidence for Luttinger theory and plots of Berry curvature are to be provided. The MWFs were proposed based on the ideas of HLR’s flux attachment [18, 19]. Key ingredients of the MWFs at \( \nu = 1/2m \) include flux attachment represented by the Jastrow factor and the LLL projection \( P_{\text{LLL}} \) operator,

\[
|\Psi_{\text{CF}}(\{k_i\})\rangle = P_{\text{LLL}} \{ \det e^{ik_i \cdot z_i} R^N \prod_{i<j} (z_i - z_j)^{2m} \}. \tag{9}
\]

where \( \{k_i\} \) are distinct and clustered to form a compact Fermi sea. Holomorphic determinant MWFs are obtained after approximating \( P_{\text{LLL}} \) by creating dipoles \( \{d_i\} \), in accordance with the dipole-momentum locking mentioned. They are [20],

\[
\Psi_{\frac{1}{2}m} = \prod_{i<j} \sigma^{2m-2n}(z_i - z_j) \prod_{k} \sigma(z - \alpha_k) \prod_{i} e^{-\frac{1}{2} \alpha_k i \cdot \alpha_k i}, \tag{10}
\]

where \( \alpha_k \) are center of mass zeros, \( \sigma(z) \) is the modified Weierstrass sigma function [21], \( \alpha \) is a free parameter, i.e. changing \( \alpha \) only rescales the MWF [22]. \( n \) represents a scheme of flux attachment: \( 2n \) out of the total \( 2m \) flux quanta are shifted from electron’s position to form a dipole. Such as momentum quantization, dipoles \( \{d_i\} \) are quantized by periodic boundary condition [23, 24] to take discrete values \( d \in \{L/(nN)\} \) where \( L \) is the \( 2D \) periodic lattice defining the torus.

**We adopt the lattice Monte Carlo method [8] to study the Berry phase.** We only consider MWFs with \( 2m > 2n \) because otherwise the MWFs are 0/0 indeterminate forms when multiple particles collide onto the same lattice site. The analytical values of these indeterminate forms are well defined, but numerically they are hard to evaluate. For this reason, there are two candidate MWFs at \( \nu = 1/4 \): \( \Psi_{\frac{1}{4}n=1} \) and \( \Psi_{\frac{1}{4}n=2} \). They are found to have large overlaps with each other for all dipole configurations, e.g. \( \langle \Psi_{\frac{1}{4}n=1} \mid \Psi_{\frac{1}{4}n=2} \rangle \geq 97\% \) for \( N = 69 \) particles. This means that observables computed from either of them are almost identical.

In a half filled LLL, Coulomb interaction low energy states were found to have a remarkably large overlap [7] with the cluster-like ansatz of Eqn. \( \text{(10)} \). At \( \nu = 1/4 \), projecting a first quantized MWF into second quantized numerical basis to compute overlap is not easy for large system sizes. Instead, as shown in FIG. 2, we present the energy spectrum of LLL Coulomb interaction and the variational energy of MWF for \( N = 10 \) electrons on a square torus. The variational energies of MWFs and exact diagonalization energies are close, but slightly worse compared to one half states. As pointed out in [19], at half filling in the two LLLs, varying short range part of the interaction induces a first-order phase transition from striped phase to a strongly paired Moore-Read state, followed by a possible crossover to a weak pairing phase. The exact diagonalization states we obtained at Coulomb point at one quarter, presumably, are weakly paired states; tuning \( v_{1,3} \) pseudo-potentials might help improve the overlaps. Further investigating the phase transition and the critical point at one quarter is one interesting future direction. Note that the MWF’s energy

![FIG. 2: Variational energies [red dots for \( \Psi_{\frac{1}{4}n=1} \), blue dots for \( \Psi_{\frac{1}{4}n=2} \) and exactly diagonalized Coulomb energies [dashed lines] as a function of many body momentum \( (K_1, K_2) \) for \( N = 10 \) electrons for the \( \nu = 1/4 \) filled LLL on a square torus. Energies are plotted in units of \( e^2/\varepsilon_B \). For each \( K_1, K_2 \) is chosen to match the momentum of the lowest energy state. Due to inversion symmetry, only \( K_1 \in [0, 5] \) are plotted. The reason why the comparison in the \( (5, 0) \) sector is not good can be found in the body.](image-url)
is off the exact lowest energy at the \((5, 0)\) sector. Similar issue \[25\] is found at half filling. We thus believe that it is due to the difficulty of finding a compact Fermi sea in the \((5, 0)\) sector for 10 dipoles on a square torus.

We then turn to the density-density correlation functions. As a hallmark of CFLs, there are peaks in the guiding center structure factor \(S(k)\) and the peak positions are tied to \(k = 2k_F\) where \(k_F\) is the Fermi wave vector. This provides a way to identify \(k_F\) numerically. In FIG. 3 we investigate Luttinger theorem for the CFLs by computing \(S(k)\) of the \(\nu = 1/4\) MWF. The measured \(k_F\) agreed with the value predicted from Luttinger theorem on square Fermi seas, suggesting that Luttinger theorem applies to CFLs [5, 6, 12] and thus justifies part of the logic flow above Eqn. [5].

We finally turn to the numerical investigation of the Berry curvature as shown in FIG. 4. Since only MWFs with compact dipole configurations are identified as CFLs [4, 9], we neither take off composite fermions deep inside the Fermi sea nor excite composite fermions far away from the Fermi surface. Instead, Berry phases were computed on paths close to the Fermi surface, after which the Berry curvatures were mapped out by a linear regression. We leave in the appendix technical details and more examples including \(N = 69\) Fermi seas. The Berry curvature was found to be uniform on the Fermi sea except at the center \(k = 0\) where an additional \(\pi\) strength was observed, in agreement with the prediction from flux attached DF theory. Same results were found even for bosonic \(\nu = 1/3\) states. From wave function point of view, the Berry curvature feature is possible to be explained as follows [26]: the determinant and the Jastrow factor are implementing respectively the \(\pi\) Berry phase and the \(U(1)\) gauge field. We believe that the Berry curvature feature we observed on \(\nu = 1/4\) model states applies to other filling fractions as well since MWFs at different fillings differ only by a Jastrow factor. While the MWFs are essentially transcriptions of HLR field theory hence represent the CFL phase, the Berry curvature is not unveiled until numerical calculation is carried out. Therefore, we conjecture that flux attached DF theory, which not only describes Fermi liquids of the correct Hall conductivity but also satisfies Luttinger theorem, and most importantly captures the Berry curvature information in a much more transparent way, is an equivalent formulation of the LLL problem as HLR.

Conclusion.—We derived a flux-attached DF theory for CFLs of fermions at all Landau level fillings that can occur. We found the sign of DF mass plays a crucial role, and conjecture its absolute value is negligible in the thermodynamical limit. We numerically tested Luttinger theorem and computed the Berry phase of moving a single composite fermion near the Fermi surface. The Berry curvature distribution is found to be uniform except at the Fermi sea center where an additional \(\pi\) phase is observed, in agreement with the prediction from flux attached DF theory. We conjecture flux-attached DF theory is equivalent to HLR theory in the LLL. Future directions include, for instance, looking for: (i) Comparison of flux attached DF theory and HLR theory from e.g. random phase approximation calculations; (ii) Nature of composite fermions in CFLs of bosons; (iii) Influence of flux attachment on particle vortex duality [27, 28]; (iii) Phase transition and critical points at \(\nu = 1/4\) [19].

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APPENDIX: MORE NUMERICAL DETAILS ASSOCIATED WITH BERRY CURVATURE

In this appendix, we provide more details about how we obtained the Berry curvature distributions. We focus on Fermi seas with $N = 37$ and 69 dipoles on a square torus as shown in FIG. 5. The red lines represent the Fermi sea boundary, along with transporting a composite fermion has $2\pi v$ Berry phase within Monte Carlo precision. The labels $\phi$ represent the Berry phase associated with the corresponding grids, i.e. the discretized Berry curvatures. For completeness, in this appendix, we also include numerical results for $v = 1/3$ CFL states, whose underlying physical particles are bosons.

Within the composite Fermi liquid (CFL) phase, low energy exact diagonalization states can be identified with model wave functions whenever model wave functions have clustered dipole configurations. Model wave functions with non-compact Fermi seas are hard to be identified with a single exact diagonalization state. For this reason, we did not consider moving a composite hole deep inside the Fermi sea to measure $\phi$, nor exciting a composite fermion far away. Instead, we only consider the Berry phase $\Phi$ associated with transporting a single composite particle anti-clock-wisely along paths $\Gamma$ that are near Fermi surface.

We adopted the lattice Monte Carlo method [8, 21, 29] to compute the many body Berry phase $\tilde{\Phi}$ which is defined as,

$$\tilde{\Phi} = \text{Tr} \prod_{\Gamma} \langle \Psi(\mathbf{K}_{\Gamma}) \mid \rho(\mathbf{K}_{\Gamma} - \mathbf{K}_{i}) \rangle \rangle \mid \Psi(\mathbf{K}_{i}) \rangle,$$

where $\langle \Psi(\mathbf{K}) \rangle$ is a many body model wave function, $|D\rangle$ is the amplitude. The $\mathbf{K}$ is the many body momentum which essentially is the sum of all single dipole momentums. The $\rho(q) = \sum_{i}^{N} e^{i \mathbf{q} \cdot \mathbf{R}_{i}}$, is the density operator projected into the lowest Landau level (LLL) and satisfies the Girvin-MacDonald-Plazman algebra [30]. The $\mathbf{R}_{i}$ are non-commutative guiding center coordinates $[\mathbf{R}_{i}, \mathbf{R}_{j}] = -i\epsilon^{a}l_{B}^{2}$ where $i,j$ label electrons, $a, b$ label spacial directions, $l_{B} = \sqrt{\hbar / eB}$ is the magnetic length, $\epsilon^{xy} = -eB_{x}$ is 1. The many body Berry phase has a path dependent phase $(i)^{N_{+} - N_{-}}$ where $N_{+}/ N_{-}$ are anti-clock-wisely/ clock-wisely step numbers.

The $\Phi$ is an area weighted sum of the discretized curvature $\phi$. The paths $\Gamma$ we took and the phase $\Phi$ are listed in TABLE II. We found an empirical formula for the Berry
phase as follows,

$$\Phi_T = \eta \cdot \pi + (2\pi \nu - \pi) \cdot \frac{A_T}{A_{FS}}. \tag{13}$$

where $\eta$ is the winding number of the path $\Gamma$ relative to the Fermi sea center. $A_T, A_{FS}$ are momentum space area enclosed by $\Gamma$ and the whole Fermi sea area respectively. The importance of Eqn. (13) has already been emphasized in the main text: it implies a uniform Berry curvature distribution and an additional $\pi$ Berry phase at Fermi sea center. In FIG. 6 and FIG. 7 we compare the $\Phi_T$ value computed from Monte Carlo and the value computed from assuming Eqn. (13) for $N = 37$ and $N = 69$ Fermi sea respectively to illustrate that Eqn. (13) is true.

To better visualize the Berry curvature distribution, we did a linear regression and mapped out the $\phi$ for $N = 37$ Fermi sea. The linear relations of $\Phi_T$ and $\phi$ on $N = 37$ Fermi sea are in Eqn. (14). The plots of Berry curvature are seen in the main text.

$$\Phi_{T1} = 2\phi_4 + \phi_7$$
$$\Phi_{T2} = 2\phi_3 + 3\phi_4 + 2\phi_6 + \phi_7 + \phi_8$$
$$\Phi_{T3} = 2\phi_2 + 2\phi_3 + 2\phi_4 + 2\phi_5 + 4\phi_6 + 2\phi_7 + \phi_8$$
$$\Phi_{T4} = 2\phi_3 + 2\phi_6 + \phi_8$$
$$\Phi_{T5} = 2\phi_2 + 2\phi_3 + 2\phi_5 + 4\phi_6 + \phi_7 + \phi_8$$
$$\Phi_{T6} = 4\phi_1 + 4\phi_2 + 4\phi_3 + 4\phi_4$$
$$\Phi_{T7} = \phi_3 + 2\phi_4 + \phi_6 + \frac{1}{2} \phi_5$$
$$\Phi_{T8} = \phi_2 + 3\phi_3 + 4\phi_4 + \phi_5 + \phi_6$$
$$\Phi_{T9} = 4\phi_1 + 6\phi_2 + 2\phi_3 + 2\phi_5 + 4\phi_6 + 2\phi_7 + 2\phi_8 + 2\phi_9$$
$$\Phi_{T10} = 4\phi_1 + 6\phi_2 + \frac{89}{15} \phi_3 + \frac{11}{3} \phi_4 + 2\phi_5 + \frac{41}{15} \phi_6 + \frac{3}{5} \phi_7$$
$$+ \frac{1}{15} \phi_8. \tag{14}$$

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**TABLE I:** Paths $\Gamma$ and phases $\Phi_T$ for $N = 37$ and 69 Fermi sea. Paths consist of the positions of the extra dipole in each step.

| $N = 37$, path $\Gamma$ | $\Phi_T^{\nu=\frac{1}{2}}/\pi$ | $\Phi_T^{\nu=\frac{3}{2}}/\pi$ |
|-------------------------|-------------------------------|-------------------------------|
| $\Gamma_1$ (1, 4) $\rightarrow$ (−1, 4) $\rightarrow$ (−2, 3) $\rightarrow$ (2, 3) $\rightarrow$ (1, 4) | 0.011 ± 0.015 | −0.004 ± 0.021 |
| $\Gamma_2$ (1, 4) $\rightarrow$ (−1, 4) $\rightarrow$ (−3, 2) $\rightarrow$ (3, 2) $\rightarrow$ (1, 4) | −0.006 ± 0.010 | −0.067 ± 0.023 |
| $\Gamma_3$ (1, 4) $\rightarrow$ (−1, 4) $\rightarrow$ (−4, 1) $\rightarrow$ (4, 1) $\rightarrow$ (1, 4) | −0.077 ± 0.013 | −0.150 ± 0.018 |
| $\Gamma_4$ (2, 3) $\rightarrow$ (−2, 3) $\rightarrow$ (−3, 2) $\rightarrow$ (3, 2) $\rightarrow$ (2, 3) | −0.015 ± 0.014 | −0.056 ± 0.023 |
| $\Gamma_5$ (2, 3) $\rightarrow$ (−2, 3) $\rightarrow$ (−4, 1) $\rightarrow$ (4, 1) $\rightarrow$ (2, 3) | −0.081 ± 0.014 | −0.152 ± 0.022 |
| $\Gamma_6$ (4, 1) $\rightarrow$ (−4, 1) $\rightarrow$ (−4, −1) $\rightarrow$ (4, −1) $\rightarrow$ (4, 1) | 0.905 ± 0.017 | 0.832 ± 0.018 |
| $\Gamma_7$ (0, 4) $\rightarrow$ (−4, 0) $\rightarrow$ (−4, −1) $\rightarrow$ (4, −1) $\rightarrow$ (0, 4) | −0.002 ± 0.013 | −0.029 ± 0.019 |
| $\Gamma_8$ (0, 4) $\rightarrow$ (−4, 0) $\rightarrow$ (−4, −2) $\rightarrow$ (2, 4) $\rightarrow$ (0, 4) | −0.046 ± 0.014 | −0.098 ± 0.021 |
| $\Gamma_9$ (2, 4) $\rightarrow$ (−4, −2) $\rightarrow$ (−2, −4) $\rightarrow$ (4, 2) $\rightarrow$ (2, 4) | 0.848 ± 0.018 | 0.741 ± 0.021 |
| $\Gamma_{10}$ (1, 4) $\rightarrow$ (−1, 4) $\rightarrow$ (−4, 1) $\rightarrow$ (4, 1) $\rightarrow$ (1, 4) | 0.800 ± 0.011 | 0.688 ± 0.019 |

| $N = 69$, path $\Gamma$ | $\Phi_T^{\nu=\frac{1}{2}}/\pi$ | $\Phi_T^{\nu=\frac{3}{2}}/\pi$ |
|-------------------------|-------------------------------|-------------------------------|
| $\Gamma_1$ (2, 5) $\rightarrow$ (−2, 5) $\rightarrow$ (−3, 4) $\rightarrow$ (3, 4) $\rightarrow$ (2, 5) | 0.016 ± 0.026 | −0.002 ± 0.028 |
| $\Gamma_2$ (2, 5) $\rightarrow$ (−2, 5) $\rightarrow$ (−4, 3) $\rightarrow$ (4, 3) $\rightarrow$ (2, 5) | −0.042 ± 0.024 | −0.049 ± 0.033 |
| $\Gamma_3$ (2, 5) $\rightarrow$ (−2, 5) $\rightarrow$ (−5, 2) $\rightarrow$ (5, 2) $\rightarrow$ (2, 5) | −0.036 ± 0.027 | −0.098 ± 0.035 |
| $\Gamma_4$ (5, 2) $\rightarrow$ (−5, 2) $\rightarrow$ (−5, 1) $\rightarrow$ (5, 1) $\rightarrow$ (5, 2) | −0.024 ± 0.022 | −0.074 ± 0.031 |
| $\Gamma_5$ (5, 1) $\rightarrow$ (−5, 1) $\rightarrow$ (−5, −1) $\rightarrow$ (5, −1) $\rightarrow$ (5, 1) | 0.919 ± 0.026 | 0.849 ± 0.028 |
| $\Gamma_6$ (5, 2) $\rightarrow$ (−5, 2) $\rightarrow$ (−5, −2) $\rightarrow$ (5, −2) $\rightarrow$ (5, 2) | 0.820 ± 0.022 | 0.719 ± 0.026 |

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FIG. 5: The Fermi sea and the discretized Berry curvature $\phi$. The left and right Fermi sea has $N = 37$ and 69 dipoles respectively, defined on a square torus. The Berry curvatures related by rotation symmetry and inversion symmetry are not represented. The red dashed line represents the Fermi surface boundary along where transporting a single dipole has $\Phi_T = 2\pi \nu$ Berry phase.
FIG. 6: The Berry phases associated with paths $\Gamma_1, \ldots$, listed in Table I for $N = 37$ Fermi sea computed from $\nu = 1/4$ [upper panel] and $\nu = 1/3$ [lower panel] CFL model wave functions, where the dashed lines are Berry phases according to Eqn. (13) and the red dots stand for Monte Carlo values which can be found in Table I.

FIG. 7: Same as FIG. 6 but for $N = 69$ Fermi sea.