Resurgent aspects of applied exponential asymptotics

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\textbf{Abstract}
In many physical problems, it is important to capture exponentially small effects that lie beyond-all-orders of an algebraic asymptotic expansion; when collected, the full asymptotic expansion is known as a trans-series. Applied exponential asymptotics has been enormously successful in developing practical tools for studying the leading exponentials of a trans-series expansion, typically for singularly perturbed nonlinear differential or integral equations. Separately to applied exponential asymptotics, there exists a related line of research known as Écalle’s theory of resurgence, which, via Borel resummation, describes the connection between trans-series and a certain class of holomorphic functions known as resurgence functions. Most applications and examples of Écalle’s resurgence theory focus mainly on nonparametric asymptotic expansions (i.e., differential equations without a parameter). The relationships between these latter areas with applied exponential asymptotics have not been thoroughly examined—largely due to differences in language and emphasis. In this work, we establish these connections as an alternative framework to the factorial-over-power ansatz procedure in applied exponential asymptotics and clarify a number of key points.
of aspects of applied exponential asymptotic methodology, including Van Dyke’s rule and the universality of factorial-over-power ansatzes. We provide a number of useful tools for probing more pathological problems in exponential asymptotics and establish a framework for future applications to nonlinear and multidimensional problems in the physical sciences.

KEYWORDS
asymptotics beyond-all-orders, Borel summation, exact WKB analysis, exponential asymptotics, resurgence, Stokes phenomena

1 | INTRODUCTION

Exact solutions to interacting physical systems are rare. This sparsity persists in nature across a wide range of energy scales, from strongly interacting gauge and string theories, through to quantum mechanics, and to classical effective field theories, such as the Navier–Stokes equations. Unless a model enjoys some additional symmetry, then typically closed-form expressions for generic physical observables cannot be found. Instead, we often turn to studying observables at isolated points in some parameter space where nondimensional parameters are small (or large). Typically, we suppose that a model will be exactly soluble at some trivial point \( \epsilon = 0 \), and we look to extrapolate observables, say \( y \), across the whole space by seeking an expansion in a small parameter,

\[
y \sim y_0 + y_1 \epsilon + y_2 \epsilon^2 + \cdots
\]

One may hope that such expansions may be patched together in some sense to cover the entire parameter space.

The method of seeking an expansion of physical quantities in small parameters has proved very powerful (see, e.g., classic references by Bender and Orszag\(^1\) or Hinch\(^2\)). Unfortunately, one often runs into a problem with such a perturbative approach. In many cases, the series (1) will have a zero radius of convergence. While such series can be tremendously accurate with few terms\(^1\), eventually, the series will diverge, and one is faced with the apparent impossibility of recovering an analytic extrapolation in the sense hoped-for above. The physical mechanisms for such divergence are diverse. For example, in quantum field theory, a heuristic is the factorial growth of Feynman diagrams\(^3\) or Dyson’s\(^4\) instability argument for the zero radius of convergence for series (1) arising in quantum electrodynamics. In applied mathematics and classical physics, expansions such as (1) may arise as singular perturbations, where, for example, later terms in \( y \) depend on derivatives of earlier terms, and factorial divergence is born in a more prescriptive way (see, e.g., problems in the works of Segur et al.\(^5\) and Boyd\(^6\)).

However, all is not lost as it was soon realized that divergent perturbative series contain much more information than first appears. In particular, an argument, principally due to Berry\(^7\) and Dingle,\(^8\) shows that when one truncates (1) at the optimal point before it begins to diverge, one
may uncover hidden nonperturbative contributions to the series that are exponentially small, for example, $e^{-\chi/\epsilon}$. As $\epsilon$ varies in the complex plane, terms such as these are smoothly switched-on across so-called Stokes lines; although they begin exponentially small, eventually they may come to dominate the asymptotics in other sectors of $\mathbb{C}$; thus, we find nonperturbative physics hiding in the late (divergent) perturbative coefficients. Resummation methods allow one to study these nonanalytic additions by assigning an analytic value to $y(\epsilon)$. Borel resummation, in particular, realizes $y(\epsilon)$ as an asymptotic expansion of an integral of an auxiliary function, the Borel transform. The aforementioned exponentially small terms related to divergence are then associated with singularities of the Borel transform. Resummation methods have enjoyed enormous success in the analysis of differential equations with singular points and, by now, there is significant literature on Borel resummation in theoretical physics; in many such cases, the Borel singularities are physically meaningful, and correspond to nonperturbative semiclassical contributions to the path integral (e.g., instantons).

In the seminal work of Écalle, it was shown that the perturbative expansion and all its nonperturbative corrections (known as the trans-series) and resummation techniques can be phrased in the language of complex analysis. Écalle elucidates a beautiful correspondence between formal trans-series and resurgent functions:

$$\sum_{i} e^{-\chi_{i}/\epsilon} y^{(i)}(\epsilon) \leftrightarrow \text{Resurgent functions, } y_B(w). \quad (2)$$

Under this correspondence, the notion that perturbative series know about their own nonperturbative completion (the left-hand side, above) is, in fact, simply analytic continuation through an alternative viewpoint (the right-hand side). The theory of hyperasymptotics extends least term truncation approaches and mirrors the more formal resurgence theory for beyond-all-orders asymptotics. In addition, hyperterminants allow one to derive rigorous error bounds and assign analytic meaning to trans-series expansions. We refer the reader to the works of Olde Daalhuis, Bennett et al. for a discussion of integral representations of hyperterminants and their computation, and also Bennett et al. for a thorough study of computing error bounds using such hyperterminant representations. In brief, hyperasymptotics is the theory of optimally truncating tails (and tails-of-tails ad. infinitum.) of divergent series, and then Borel resumming the resultant remainders in order to obtain hyperterminant expressions. One can relate resurgence and hyperasymptotics by recovering hyperterminant expressions through a careful remainder analysis of the integral formula for the inverse Borel transform of a divergent series.

### 1.1 Goals of this work and connections to applied exponential asymptotics

We have three main goals for this work. First, we shall apply some aspects of resurgence to the practical study of singularly perturbed linear ordinary differential equations. Typical examples include:

$$\epsilon y'(z) + G(z)y(z) = \epsilon H(z), \quad (3a)$$

$$\epsilon^2 y''(z) + \epsilon P(z)y'(z) + Q(z)y(z) = F(z), \quad (3b)$$
with \( z \in \mathbb{C} \), the parameter \( \epsilon \) considered small, and we will consider only certain (natural) boundary conditions on \( y \). The coefficient functions \( G, H, P, Q \), and \( F \) will have isolated singularities. A perturbative expansion of a solution to such an equation takes the general form

\[
y(z; \epsilon) = y^{(0)}(z; \epsilon) + \sum_{i} e^{-\chi(z)/\epsilon} y^{(i)}(z; \epsilon),
\]

where the \( y^{(i)}(z; \epsilon) \) are divergent asymptotic expansions themselves in \( \epsilon \). The above is thus a (finite) trans-series\(^1\) in a small parameter, \( \epsilon \), and involves an additional holomorphic variable, \( z \in \mathbb{C} \). Although (3) are relatively simple differential equations, we emphasize that resurgence theory is better developed for nonparametric differential equations and hence to expansions such as (1). We refer the reader to Delabaere and Pham\(^{17}\) for an overview of resurgence theory as it applies to exact WKB (or Liouville–Green) analysis, and also related applied works by, for example, Howls\(^{18}\) (on trans-series for boundary-value problems) and Byatt-Smith\(^{19}\) (on the Borel transform). We believe that the approach of working entirely in the Borel plane (including a Borel plane perspective on inner–outer matching) for such singularly perturbed problems is not as well appreciated, especially in connection with applied exponential asymptotics problems.

Singularly perturbed differential equations such as (3) are standard in applied mathematics, where problems may also take more obscure forms, including nonlinear equations, integro-differential equations, partial differential equations, and boundary-value problems (see examples in the classic references of Bender and Orszag\(^{1}\) and Hinch\(^{2}\)). In many such problems, the emphasis is more to derive leading-order exponentially-small estimates—perhaps only the first approximation to, say \( e^{-\chi(z)/\epsilon} y^{(1)} \) in (4). Surprisingly, these exponentially small effects can dictate a number of key properties of the associated physical problem; applications in classical physics have included modeling of dendritic crystal growth,\(^{20}\) Saffman–Taylor viscous fingering,\(^{21–25}\) water waves,\(^{26–28}\) transition to turbulence,\(^{29}\) vortex reconnection,\(^{30}\) pattern formation,\(^{31}\) woodpile chain nanoptera,\(^{32}\) and many others. Reviews can be found in the works of Boyd\(^{6}\) and Segur et al.\(^{5}\) While the development of exponential asymptotics for such problems has been enormously successful, in some sense, the focus has been on the trans-series side of the correspondence (2)—that is, manipulation and analysis of the divergent series.

In the methodology of Chapman et al.,\(^{33}\) for instance, one begins by studying the early terms of the asymptotic expansion, say \( y_0(z) \) or \( y_1(z) \), and noting that these early orders often contain singularities in \( \mathbb{C}_z \) (poles or branch points). By the singular nature of the differential equation, subsequent terms, say \( y_n \), depend on differentiation of the previous terms. By the linearity of this procedure, no new singularities are introduced beyond those that appear in the early terms. Thus, further differentiation of the early terms can lead to factorial-over-power divergence as \( n \to \infty \) and one posits that the late terms of \( y^{(0)}(z) \) in (4) satisfy

\[
y_n^{(0)}(z) \sim \frac{A(z)\Gamma(n+\gamma)}{\chi(z)^{n+\gamma}} \quad \text{as } n \to \infty.
\]

The components such as \( A, \gamma, \) and \( \chi \) are derived using matched asymptotics in the complex \( \mathbb{C}_z \) plane, and optimal truncation and Stokes smoothing is applied to relate the above divergence to the leading-order trans-series correction, involving \( e^{-\chi(z)/\epsilon} \). Such complex-plane asymptotics have

\(^1\)Throughout we use the term trans-series even when the number of exponential terms is finite.
been a standard approach because the seminal work of Kruskal and Segur\textsuperscript{20} and other similar applications can be found in the compendium by Segur et al.\textsuperscript{5}

Though the above approach is powerful, recent research has revealed an increasing number of problems that stretch or challenge this conventional methodology of applied exponential asymptotics. These include, for instance, problems involving coalescing singularities,\textsuperscript{34} interacting Stokes lines,\textsuperscript{35} partial differential equations,\textsuperscript{36–38} and higher order Stokes phenomena.\textsuperscript{39} Such pathologies indicate a need for new approaches and techniques. Our second goal in this work is to develop analogs of the factorial-over-power ansatz using aspects of resurgence; we shall argue that this provides some powerful intuition to the rich geometric structure of such problems. The approach has the potential to allow the creation of new model problems in singular perturbation theory that exhibit exponential asymptotic effects—and for which physical intuition may be lacking. This is crucial, for instance, in applications to partial differential equations where techniques remain severely limited.

Our final goal is to provide important links between the disparate communities studying beyond-all-orders asymptotics. Unsurprisingly, due to both the ubiquity of perturbation theory in the physical sciences and the rich mathematical structure of resurgence, there is a wide range of researchers working in the closely related, but often disparate, fields of exponential asymptotics, singular perturbation theory, Borel resummation, and resurgence across the spectrum of applied mathematics, geometry/quantum field theory, and analysis.

This present work grew out of the recent program on “Applicable resurgent asymptotics” (Isaac Newton Institute, University of Cambridge, 2021–22), and we hope that parts of the paper serve as a useful review to bridge the language barrier between such different communities. Indeed, our principle aim is to initiate steps toward unifying the various approaches, while reviewing and making more accessible some aspects of resurgence theory to the applied exponential asymptotics community.

1.2 Outline and summary of results

We first provide a brief review of applied exponential asymptotics in Section 2. Then, in Section 3 and Section 4, we review the background and terminology of resurgence, trans-series, and Borel transforms. In Section 5, we turn our attention to holomorphic trans-series that arise as solutions to singularly perturbed linear ordinary differential equations. In applied exponential asymptotics, the celebrated “factorial-over-power” ansatz method (briefly outlined in Section 2) determines the (leading order) components of the trans-series side of (2). We shall explain this ansatz in terms of the holomorphic side of the correspondence (i.e., in terms of a singularity ansatz in the Borel plane), and thereby extend the method to determine all-orders of the trans-series on the left-hand side of (2).

In this work, we emphasize the following perspective: from the view of the Borel plane, singularly perturbed problems are comprised of two parts: an “operator” part (outer) and an “initial data” part (inner). In terms of the operator part, we follow many of the same ideas as the communities studying the exact WKB method (cf. the monographs by Aoki et al.\textsuperscript{40} and Honda et al.\textsuperscript{41}). In particular, we review how in the Borel plane, singularly perturbed problems yield partial differential equations, $\mathcal{P}_B y_B = 0$, for the parametric resurgent function on the right-hand side of the correspondence (2). We show that such partial differential equations yield ordinary differential equations for the holomorphic components of the trans-series (or equivalently $z$-dependent coefficients in the expansions near singularities in the Borel plane).
However, the operator part does not provide initial conditions for these equations—in this way, singular perturbation theory furnishes us with a “blank template” trans-series. The initial data (inner) part of the methodology fills in the blanks. In our work, we show that, by rescaling near physical singularities $\Gamma_z$, we obtain an inner Borel ODE operator, $\mathcal{P}^{\text{inner}}_B$, describing a constant (0-dimensional) trans-series problem of the type originally studied by Écalle. Upon translating back to “physical space,” inner equations may be ordinary differential equations with irregular singular points. This problem is “difficult” but solving this resurgent connection problem and finding the (infinitely many) associated Stokes’ constants yields initial conditions at $\Gamma_z$ for the infinitely many ordinary differential equations for trans-series components—thereby solving the original singularly perturbed problem.

Finally, in Section 6, we conclude with some examples, pathologies, and future directions. Turning back to holomorphic trans-series more generally, we shall discuss how modifications of the “factorial-over-power” ansatz of traditional exponential asymptotics are necessary if singularities in the Borel plane differ from power-law singularities. Along the way, we give an argument that bypasses, and generalizes, the usual optimal truncation and Stokes switching approach found in the applied mathematics literature. Instead, we highlight how asymptotics of power series coefficients may be related to Hankel integrals, which subsequently determine leading-order exponentially small terms. The perspective elucidated here sheds light on how the matched asymptotics techniques of Chapman et al. may be generalized.

2 A REVIEW OF THE METHODOLOGY OF APPLIED EXPONENTIAL ASYMPTOTICS

We begin by reviewing the methodology of Chapman et al. as applied to the linear exemplar equation (3a). Further expository treatments can be found in the works of, for example, Mortimer and Boyd. Crucially, it should be noted that the methodology of Chapman et al. applies to considerably more general nonlinear ordinary and partial differential equations; nevertheless, (3a) is enough illustrate the key ideas. Consider:

$$\varepsilon y'(z) + y(z) = \frac{\varepsilon}{z},$$

with $y \to 0$ as $z \to -\infty$. The solution of the above is related to the exponential integral. The task is to derive the exponentially small corrections of (6).

Step 1. Establishing late-order divergence
First, we seek an asymptotic formal power series solution for small $\varepsilon$. Substituting $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ into (3) yields:

$$y_0 = 0, \quad y_1 = \frac{1}{z}, \quad \text{and} \quad y_n + y'_{n-1} = 0, \quad n \geq 2.$$  

We observe the singularity in $y_1$ at $z = 0$. In this work, we refer to such points as physical singularities, associated with the set $\Gamma_z$. By the singularly perturbed nature of the differential equation, higher order terms involve derivatives of the previous orders; hence, singularities in the analytic continuation of the early terms will often increase in strength and lead to eventual divergence of the asymptotic expansion. For such a simple linear example, the general form of $y_n$ can be found
by induction (in fact, it is $y_n = (n-1)!/z^n = \Gamma(n)/z^n$). Below, we describe the general procedure applicable to more general problems. To derive the form of $y_n$ as $n \to \infty$, we may substitute the factorial-over-ansatz expression (5) into the general $O(\epsilon^n)$ expression in (7). This gives, after some rearrangement,

$$(-\chi' + 1) + \frac{A'}{A} \chi \frac{\Gamma(n + \gamma - 1)}{\Gamma(n + \gamma)} = 0.$$  \hfill (8)

The ratios of gamma functions are expanded as $n \to \infty$, and the above yields at leading order an equation for the singulant function, $\chi$, and at next order, an amplitude equation for the prefactor, $A$. We obtain that $\chi(z) = z + \text{const.}$ and $A(z) = \Lambda$, constant. Since the divergence is due to the point $z = 0$, we impose $\chi(0) = 0$ and hence $\chi = z$. The conclusion is that, as $n \to \infty$, the divergence follows

$$y_n \sim \frac{\Lambda \Gamma(n + \gamma)}{\chi^{n+\gamma}}, \quad \text{where} \quad \chi = z,$$

with the constants $\Lambda$ and $\gamma$ to be determined.

**Step 2. Inner analysis and matching**

To determine the constants $\gamma$ and $\Lambda$, the solution of (6) is studied in the so-called inner region, near the singularity $z = 0$. First, since $y_1 = 1/z$, then in order for the form (9) to be compatible with this early order, this requires $\gamma = 0$. Second, to determine $\Lambda$, we rescale $z = \epsilon s$ and write the inner solution as, $y(z) = Y(s)$. The differential equation is then

$$Y''(s) + Y(s) = \frac{1}{s}.$$  \hfill (10)

We may then match the inner solution with the outer solution by expanding $Y = \sum_{n=0}^{\infty} U_n / s^n$, valid in the limit $s \to \infty$. Matching (7) yields $U_0 = 0, U_1 = 1$, and the recurrence relation $U_n = (n-1)U_{n-1}$ for $n \geq 2$. Solving the recurrence relation thus reveals $U_n = (n-1)! = \Gamma(n)$. If the general $n$th-order term (9) is written in inner variables, we obtain $\epsilon^n y_n \sim \Lambda \Gamma(n)$ and therefore $\Lambda = 1$.

**Step 3. Stokes smoothing**

We have therefore completely determined the late-order representation (9). The importance of the divergent representation is revealed in the following argument. The divergent series is optimally truncated and the remainder sought:

$$y = \sum_{n=0}^{N-1} \epsilon^n y_n + R_N.$$

This yields

$$\epsilon R_N + R_N = -\epsilon^N y_{N-1} = \epsilon^N y_N.$$  \hfill (11)

Note that in the limit $\epsilon \to 0$, the optimal truncation point, $N \to \infty$. Thus, the above equation establishes a relationship between the remainder, $R_N$, and the late-order behavior, $y_N$ as $N \to \infty$. It can
be verified that the optimal truncation point is $N \sim |z|/\epsilon$. Substituting this into (11) and using the divergent form (9) allows the remainder to be derived.

By analyzing the equation for the remainder, we may reach two primary conclusions. The first is that the remainder is exponentially small everywhere in the complex $z$-plane except near the critical line,

$$\mathfrak{I} \chi = 0 \quad \text{and} \quad \Re \chi \geq 0,$$

where recall $\chi = z$, and thus, the above corresponds to the positive real axis. The second conclusion is reached by rescaling (11) and then integrating $R_N$ about the line (12). If this procedure is carried out, then we would observe that the remainder smoothly, but sharply, varies in value within a boundary layer of thickness $O(\sqrt{\epsilon})$ about $\Re \chi \geq 0$. This jump is

$$\left[ R_N \right] \sim \frac{2\pi i}{\epsilon^{\gamma}} e^{-z/\epsilon} = 2\pi i e^{-z/\epsilon}.$$  

(13)

This switching-on of exponentially small terms is called the Stokes phenomenon; it occurs across Stokes lines (here $z \geq 0$) satisfying the criterion (12) associated with Dingle. The boundary-layer analysis leading to (13) is attributed to Berry and referred to as “Stokes line smoothing.” The fact that the remainder, in (13), is expressed in those very same quantities as the divergence, given in (9), is one of the remarkable facts of exponential asymptotics (and resurgence).

Above, we have summarized the crucial steps of the applied exponential asymptotics methodology. Many of these concepts, from factorial-over-power divergence, matched asymptotics, and Stokes lines and Stokes phenomena will be properly reintroduced in a different manner in the forthcoming sections.

**Remark.** It is important to note that in studying (6), we have taken essentially the simplest possible example of an ordinary differential equation exhibiting exponential asymptotics. The power of methodologies such as the one in Chapman et al.33 is that the same steps outlined above can be applied, in great part, to much more challenging nonlinear ordinary and partial differential equations. We briefly highlight some differences when more challenging differential equations are studied: (i) multiple singularities in $z \in \mathbb{C}$ and divergences are possible; (ii) the prediction of the factorial-over-power divergence (9) and solution of the components $\chi, Q, \gamma$, is nontrivial; (iii) the recurrence relation(s) involved in the inner analysis cannot be solved in closed form; (iv) matching of inner-and-outer expansions is nontrivial (cf. Van Dyke’s rule in Section 5.4); and (v) additional higher order correction terms are difficult to determine, and so on. There are many beautiful and complex problems that are intriguing extensions of the above themes.

## 3 | BACKGROUND

We now review the background material necessary to understand certain resurgent aspects of singular perturbation theory. We begin with a reframing of some well-known properties of holomorphic functions with discrete singularities. Then, we review the method of Borel resummation and elucidate the correspondence between trans-series and resurgent functions. Our aim in this work is to make the tools and approaches of resurgence more accessible to the applied
mathematics community. For further details on some parts of this section, we recommend the excellent reviews by Aniceto et al.\textsuperscript{10} and Dorigoni.\textsuperscript{45}

3.1 Coefficients of holomorphic functions and power series

Consider a convergent power series

\[ f(x) = f_0 + f_1 x + f_2 x^2 + \cdots \]  

(14)

thought of as a holomorphic germ defined near 0. The uniqueness of analytic continuation suggests that the coefficients \( \{f_n\} \) must necessarily contain essential information about the nature and location of singularities of \( f \). The following discussion elucidates how the singularity data are encoded and may be read from the asymptotics of \( f_n \). Although this is an elementary fact of complex analysis, we present the results in a way that will be convenient for later application to Stokes phenomena.

Suppose that the analytic continuation of \( f(x) \) has a discrete set of branch point singularities at \( x = \chi_1, \chi_2, \chi_3, \ldots \in \mathbb{C} \). Under suitable regularity conditions at infinity, the coefficients \( f_n \) of the germ of \( f \) may be expressed as an integral

\[ f_n = \frac{1}{2\pi i} \sum_{\chi \in \mathcal{H}_\chi} \int_{\mathcal{C}_\chi} d\chi \ e^{-\omega n} f(e^\omega), \]  

(15)

where the sum is taken over the singularities, \( \chi = \chi_j \) for \( j = 1, 2, \ldots \), and \( -\mathcal{H}_\chi \) denotes the image in the \( \omega \)-plane of a Hankel contour that encircles each respective singularity (traversed in the clockwise direction as shown in Figure 1). This integral is a rewriting of Cauchy’s integral formula

\[ f_n = \frac{1}{2\pi i} \oint_{S_1^r} \frac{dx}{x^{n+1}} f(x), \]  

(16)

where the radius of the circular \( S_1^r \) contour is smaller than the absolute value of any of the singularities of \( f \). Applying the conformal transformation \( x = e^\omega \), we may then deform the contour to a sum over Hankel contours as illustrated in Figure 1.

The above discussion demonstrates the relationship between the coefficients of a holomorphic germ and the singularities in the function’s analytic continuation. A version of the classic Darboux theorem\textsuperscript{2}, notably described by Dingle\textsuperscript{8}, may be seen to follow from (15). Let us suppose that the analytic continuation of \( f \) has a single singularity at \( x = \chi \). Suppose further that at this point, the singularity takes the form of a power law:

\[ f(x) = (\chi - x)^{-\alpha} \left( a_0 + (\chi - x)a_1 + (\chi - x)^2 a_2 + \cdots \right) + \text{regular terms} \]  

(17)

with \( \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \). Since \( x = \chi \) is the only singularity, then both the regular terms and \( h(x) \) are entire. We may evaluate the integral (15) at large \( n \) with (17) to determine the asymptotics of the

\textsuperscript{2}As written by Dingle “By a theorem of Darboux (1878), late terms in a Taylor series originate from the singularity in the function expanded which lies closest to the origin of expansion”\textsuperscript{[8, p. 4].}
FIGURE 1  The power series coefficients, $f_n$, of (14) can be expressed as a sum of Hankel integrals, $H_x$, around each singularity. This is shown via subplots (A, C, E) for an example with three singularities at $x = \chi_1, \chi_2,$ and $\chi_3$. From the transformation $x = e^w = e^{\Re w} e^{i\Im w}$, the coefficients can then be expressed in terms of Laplace-type integrals (15). The correspondence of points in $x$ to points in $w$ can be visualized via the cylindrical geometry shown in subplots (B, D, F). Together, this illustrates the sequence of necessary deformation to the Hankel contour in either coordinate system.
coefficients. After a change of variables and scaling \((e^w = \chi e^{s/n})\), one may use results from the Appendix to evaluate the integral at large \(n\), term-by-term, giving
\[
f_n = \frac{n^{\alpha-1}}{\chi^{n+\alpha}} \left( \frac{a_0}{\Gamma(\alpha)} + \frac{1}{n} \left( \frac{a_0\alpha(\alpha-1)}{2\Gamma(\alpha)} + \frac{\chi a_1}{\Gamma(\alpha-1)} \right) + \cdots \right).
\]

For singularities of the type (17), the above may be alternatively written, with \(n \to \infty\), as:
\[
f_n = \frac{\Gamma(n+\alpha)}{\chi^{n+\alpha}\Gamma(n+1)} \left( \frac{a_0}{\Gamma(\alpha)} + \frac{1}{(n+\alpha-1)\Gamma(\alpha-1)} \frac{\chi a_1}{\Gamma(\alpha-1)} + \cdots \right),
\]
which follows from the binomial theorem. In the context of applied singular perturbation theory, the leading factorial-over-power form in (19) is convenient for application because the use of the gamma function aids manipulations; indeed, it is the form that is preferred by many practitioners (e.g., Chapman et al. 33).

Now suppose that \(f\) has additional power-law singularities at \(x = \chi_1, \chi_2, \ldots\) and near each \(f\) can be written as a local expansion,
\[
f(x) = (\chi_i - x)^{-\alpha_i} (a_0^{(i)} + a_1^{(i)} (\chi_i - x) + \cdots) + \text{regular terms}, \quad \text{for } i = 1, 2, 3, \ldots
\]
In this case, the regular parts have finite radii of convergence, and consequently, the large-\(n\) term-by-term evaluation of the integral (15), similar to the writing of (19), is only asymptotically valid. Indeed, in general, we obtain a divergent trans-series in the small parameter \(1/n\) for the coefficients \(f_n\), where more distant singularities are suppressed by an exponentially small prefactor, \(\chi^{-n-\alpha} = \exp(-(n-\alpha) \log \chi)\). This leads to the expansion in \(1/n\) of the coefficients
\[
f_n \sim \sum_i \frac{1}{\chi_i^{n+\alpha_i}} \frac{\Gamma(n+\alpha_i)}{\Gamma(n+1)} \left( \frac{a_0^{(i)}}{\Gamma(\alpha_i)} + \frac{1}{(n+\alpha_i-1)\Gamma(\alpha_i-1)} \frac{a_1^{(i)}}{\Gamma(\alpha_i-1)} + \cdots \right).
\]

This section may be summarized as follows. The coefficients of an expansion about any given singularity can be found via the large-\(n\) asymptotics of the coefficients about the origin. This may seem relatively unsurprising when posed in the language of complex analysis, however, as we shall review in the following sections, these same results can be translated to facts about divergent asymptotic series via Borel resummation. In this context, we will note the striking physical resurgence principle that divergent perturbative series “know” about their own nonperturbative corrections.

We conclude this section with a result closely related to the above discussion. The resultant expression will be related to the analysis of the so-called Stokes phenomenon to follow. Suppose that \(f\) has a singularity at \(x = \chi\) in its analytic continuation. Consider the following integral:
\[
I_f(\varepsilon) := \int_{H_\chi} dw e^{-w/\varepsilon} f(w) = \int_{H_\chi} dw e^{-w/\varepsilon}(\chi - w)^{-\alpha}(a_0 + a_1(\chi - w) + \cdots),
\]
where $H_{\chi}$ is the Hankel contour centered on $\chi$ (traversed anticlockwise). We may evaluate this integral term-by-term (using properties of the gamma function reviewed in the Appendix) as

$$I_f(\epsilon) = \left( -\frac{2\pi i}{e^\alpha - 1} \right) e^{-\chi/\epsilon} \left( a_0 + (\alpha - 1)a_1 \epsilon + (\alpha - 1)(\alpha - 2)a_2 \epsilon^2 + \cdots \right), \quad (23)$$

for sufficiently small $\epsilon$. The integral (23) is closely related to (15), and later, we see that the asymptotic expansion in $\epsilon$ of the Hankel integral, $I_f(\epsilon)$, appearing in (23) is very closely related to the asymptotic expansion in $1/n$ of the power-series coefficients of $f$, which appears in (19).

### 3.2 Asymptotics and trans-series for nonparametric functions

In this introductory section, we are concerned with (nonparametric) algebraic asymptotic expansions of the form

$$y(\epsilon) = y_0 + y_1 \epsilon + y_2 \epsilon^2 + \cdots \quad (24)$$

with zero radius of convergence in $\mathbb{C}_\epsilon$. We further restrict to so-called *Gevrey-1 series* that satisfy the bound $|y_n| \leq AB^n n!$ for some constants $A$ and $B$ and refer to such sequences loosely as factorially divergent. We consider such asymptotic series as an element of the formal (Gevrey-1) power series algebra denoted as $\mathbb{C}_1[[\epsilon]]$, where multiplication is pointwise power series multiplication. We extend this algebra with exponential symbols and write asymptotic sequences formally as

$$y(\epsilon) = \left[ y_0^{(0)} + y_1^{(0)} \epsilon + y_2^{(0)} \epsilon^2 + \cdots \right] + \left[ e^{-\chi_1/\epsilon} \epsilon^{-\alpha_1} (y_0^{(1)} + y_1^{(1)} \epsilon + \cdots) \right] + \left[ e^{-\chi_2/\epsilon} \epsilon^{-\alpha_2} (y_0^{(2)} + y_1^{(2)} \epsilon + \cdots) \right] + \cdots \quad (25)$$

with $\alpha_i, \chi_i, y_i^{(j)} \in \mathbb{C}$. Formal extended asymptotic sequences of this form are known as log-free, height-1 trans-series. Trans-series are closed under many operations; in particular, they are an exponentially closed ordered differential field, which we denote by $\mathbb{C}_1[[\epsilon]]$. We refer the reader to the work of Edgar for a thorough pedagogical review of trans-series, though we will make little use of the full machinery in the present work. We refer to trans-series of the form (25) as *constant* trans-series, in contrast to the series studied later where the components $\chi_i$ and $y_i^{(j)}$ will be functions of an additional holomorphic variable $z \in \mathbb{C}_z$.

**Remark.** Note that the power-series coefficient expansion of a holomorphic function, $f$, seen in, for example, (21) yields such a trans-series with $1/n$ playing the role of $\epsilon$.

### Parametric versus nonparametric

Note that throughout we refer to expansions of $f(x)$ as $x \to 0$ or $x \to \infty$ as *nonparametric asymptotic expansions*. The addition of extra holomorphic dependence, that is, $f(x; z)$ is thus regarded as a *parametric expansion* with $z$ as the parameter. This vocabulary is in contrast to the language

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3Note that the power series part of $I_f(\epsilon)$ will have a zero radius of convergence in $\epsilon$ if there is an additional singularity besides $w = \chi$ in the analytic continuation of $f$. 

---
sometimes used in the applied asymptotics literature. Here, one often studies the solution of, say, (3) with \( y = y(z; \varepsilon) \) and \( \varepsilon \) small. Now \( \varepsilon \) is referred to as the parameter and \( z \) is allowed to freely vary.

### 3.3 The Borel transform

The Borel transform, \( B \), may be viewed as a way of associating an asymptotic series or trans-series to a holomorphic germ. For the moment, let us consider the case of the asymptotic expansion (24) with \( y_0 \equiv 0 \) (we shall reintroduce the constant part in the following section). We define the Borel transform, \( B : \varepsilon C_1[[\varepsilon]] \to \mathcal{O}_0(\mathbb{C}) \), by

\[
B : (y_1, y_2, y_3, \ldots) \mapsto \frac{y_1}{0!} + \frac{y_2}{1!}w + \frac{y_3}{2!}w^2 + \cdots + \frac{y_n}{(n-1)!}w^{n-1} + \cdots \quad (26)
\]

Informally, the Borel transform \( B \) divides out the factorial divergence of an asymptotic sequence. Notice that the Gevrey-1 condition of the expansion (24) ensures that the germ (26), obtained in the above way, will have a nonzero (but not necessarily infinite) radius of convergence. In a minor abuse of notation, we shall write

\[
B[y](w) \equiv y_B(w),
\]

to denote the holomorphic function obtained by the analytic continuation of the germ generated from \( y(\varepsilon) \). Subsequently, we may form a Riemann surface, \( \Sigma \), associated with \( y_B(w) \). The function \( y_B \) will often contain singularities, and we denote the set of singular points in the analytic continuation of \( y_B \) by \( \Gamma_w \).

The map \( B \) may be extended to account for \( y_0 \) as follows. First, we define product on Borel space

\[
f_B \star g_B(w) = \int_0^w ds f_B(s)g_B(w-s). \quad (27)
\]

The integral here encodes the power-series convolution product. We then define a formal \( \delta \)-function that satisfies \( \delta \star f_B = f_B \). The Borel transform may then be extended to all asymptotic series, \( C_1[[\varepsilon]] \), by defining, in addition to (26) the rule,

\[
B : y_0 \mapsto y_0 \delta. \quad (28)
\]

On a practical level, this means that performing the Borel resummation procedure discussed below, the constant, \( y_0 \), term of a formal asymptotic series is left untouched by the integral transform (one can alternatively subtract \( y_0 \) from the series in consideration and study the remainder). We refer the reader to the excellent works of Nikolaev\(^{47}\) and Nemes\(^{48}\) for more thorough reviews of rigorous aspects of Borel resummation and Gevrey-1 asymptotics.

**Resurgent functions**

One of the great achievements of Écalle\(^{11,49}\) (see also the more recent excellent reviews by Sauzin\(^{50,51}\)) is to construct an algebra of functions known as *resurgent functions*, defined on the Riemann surface, \( \Sigma \).
A resurgent function is a holomorphic function that arises as the Borel transform of a Gevery-1 asymptotic series and admits endless analytic continuation. Informally, this means that it must be possible to analytically continue $y_B$ along any ray from the origin by avoiding a discrete set of singularities. In this work, we exclude logarithmic singularities and we focus mainly on the case that the analytic continuation contains only algebraic singularities (except in Section 6 where we consider particular examples with more pathological singularities). This class of singularities is ubiquitous in typical applied exponential asymptotics problems.

Inverse Borel transform and Borel resummation

We have seen how we may associate a holomorphic function, $y_B(w)$, to a divergent series, $y(\epsilon)$, via the Borel transform. We now review the definition of the inverse Borel transform, the asymptotic evaluation of which returns a divergent trans-series.

Let $l_\vartheta = [0, \infty) e^{i \vartheta}$ be a ray in the complex $w$-plane emanating from the origin in the direction $\vartheta$. The inverse Borel transform, $B^{-1}$, defined in the direction $\vartheta$, is given by

$$Y_\vartheta(\epsilon) := B^{-1}[y_B](\epsilon) = \int_{l_\vartheta} dw \ e^{-w/\epsilon} y_B(w).$$  \hfill (29)

We say that $y(\epsilon)$ is Borel summable in the direction $\vartheta$ if this integral converges to a holomorphic function.

Suppose that $y_B$ is Borel summable in the direction $\vartheta = 0$. From the definition of the gamma function via (A1), we may compute the asymptotics of $Y(\epsilon) := Y_{\vartheta=0}$ to be

$$Y(\epsilon) = \epsilon \left( \frac{y_1}{1!} \right) + \epsilon^2 \left( \frac{y_2}{2!} \right) + 2! \epsilon^3 \left( \frac{y_3}{3!} \right) + \cdots,$$  \hfill (30)

which may be compared with $y(\epsilon) - y_0$ in (24). Hence, we may regard the expansion of the inverse transform (29), via the gamma function, as reintroducing the factorial divergence from $y_B$ to $y$. This procedure of taking a divergent series, $y(\epsilon)$, associating with it a holomorphic germ (or power series), $y_B$, with nice analytic continuation properties, and then defining a holomorphic function, $Y(\epsilon)$, with the same asymptotics as $y - y_0$ is known as Borel resummation. In Figure 2, we illustrate the procedure and relationship between divergent series, the Borel transform, and the inverse Borel transform.

Replacement rules for $y_B$

Above, we have explained the definition of the Borel resummation procedure of a given divergent series. Suppose now that $y(\epsilon)$ arises as the perturbative solution to a differential equation (which
may include, e.g., linear, nonlinear, or difference terms). Rather than solving the equation directly in physical space $C_\varepsilon$, we may seek a solution of the form

$$y(\varepsilon) = y_0 + \int_{(0,\infty)} dw \ e^{-w/\varepsilon} y_B(w), \quad (31)$$

so now the Borel transform, $B y = y_B(w)$, is determined without explicit manipulation of the perturbative expansion for $y$. To determine $y_B$, the differential equation in $y$ is transformed to a differential equation for $y_B$. Notice that (31) is the inverse Laplace transform of $y_B(w)$ with respect to $\eta = 1/\varepsilon$, and hence, the following is familiar from standard Laplace transform theory. Note that the contour of integration of (31) assumes $\varepsilon > 0$ and may need to be deformed if $\varepsilon$ takes negative or complex values (see next section).

**Replacement rules**

Given a differential equation for $y = y(\varepsilon)$, a governing equation for $y_B$ is derived using the following observations:

$$B \left[ \frac{1}{\varepsilon} y \right] = y_B'(w), \quad B \left[ \varepsilon \frac{d y}{d \varepsilon} \right] = w y_B(w), \quad B[y(\varepsilon)^2] = y_B \star y_B(w). \quad (32)$$

We shall refer to the above relations as “replacement rules,” and these yield the governing equation for the Borel transform $y_B(w)$. These follow directly from the definition of the inverse Borel transform (29). For instance, we write $\frac{1}{\varepsilon} y = \int dw \ e^{-w/\varepsilon}(y_B/\varepsilon)$, and integrate by parts to derive the first rule of (32).

Momentarily, we will also discuss the Borel transform of difference equations. In this case, it is easier to write expressions in terms of $\varepsilon = 1/\eta$ with $y(\varepsilon) = y(1/\eta) \equiv f(\eta)$. Then, for instance, one may verify that

$$B[f(\eta + 1)] = e^{-w} y_B(w). \quad (33)$$

### 3.4 The Stokes phenomenon

We shall now review the concept of the Stokes phenomenon (in $\varepsilon$). When computing the inverse Borel transform, (29), there may be directions $l_\delta$ along which the integrand, $y_B(w)$, is singular. If one varies the value of $\varepsilon$ in the complex plane, then the convergence of the integral requires us to adjust the contour in $C_w$ with arg $\varepsilon$. As this occurs, if the contour $l_\delta$ reaches a singularity of $y_B$, then, in order to keep the integration class constant, we must pick up a contribution from a Hankel contour around the singularity. This procedure is shown in Figure 3(A) where for one value of $\varepsilon$ the contour is $l_{\delta_-}$ and after the analytic continuation of $\varepsilon$, the contour must be $l_{\delta_+}$. Then, for values of $\varepsilon$ past this threshold, one must add the contour shown in Figure 3(B) to the contour $l_{\delta_+}$.

Suppose that $y_B$ has a singularity at $w = \chi$ with arg $\chi = \delta$, then we see that

$$Y_{\delta_-}(\varepsilon) - Y_{\delta_+}(\varepsilon) = \int_{H_{\chi}} dw \ e^{-w/\varepsilon} y_B(w), \quad (34)$$
FIGURE 3  (A) A singularity lies at \( w = \chi \). As \( \epsilon \) varies in \( \mathbb{C}_\epsilon \), it is necessary to adjust the integration contour \( \gamma \). Then an initial integration path along \( \gamma^- \) is equal to a new integration along \( \gamma^+ \), but with an additional Hankel contour shown in (B).

with \( H_\chi \) traversed as in Figure 3. Suppose further that the singularity in \( y_B \) is of power-law form so that \( y_B \sim a_0 (\chi - w)^{-\alpha} \) with \( \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0} \). Then the integral (34) may be evaluated using (23) to observe a jump of

\[
\frac{2\pi i}{\epsilon^{\alpha-1}} \left( \frac{a_0}{\Gamma(\alpha)} \right) e^{-\chi/\epsilon}.
\]  (35)

The above thus occurs when \( \text{arg} \epsilon = \text{arg} \chi \) and the critical line in \( \mathbb{C}_\epsilon \) is then the Stokes line. The corresponding jump in the asymptotics as described by Equation (35) is called the Stokes phenomenon. In particular, we see that across the Stokes line, the asymptotics of \( y_0 + Y(\epsilon) \) jump\(^4\) according to

\[
y_0 + Y(\epsilon) \to \left[ y_0 + \epsilon y_1 + \cdots \right] + \frac{1}{\epsilon^{\alpha-1}} e^{-\chi/\epsilon} (c_0 + \cdots),
\]  (36)

for the constant \( c_0 \) specifically given by the prefactors in (35). In general, one may have multiple singularities on \( \mathbb{C}_w \), and the corresponding collection of Stokes lines in \( \mathbb{C}_\epsilon \) form a Stokes graph. An example follows.

Example 1. A canonical example of a linear first-order equation with an irregular singular point is Euler’s equation, given by

\[
\epsilon^2 \frac{dy}{d\epsilon} + y = \epsilon,
\]  (37)

with the boundary condition \( y(\epsilon) \to 0 \) as \( \epsilon \to \infty \). The equation has an irregular singular point at \( \epsilon = 0 \), and so, the power series solution is only asymptotic in a sectorial domain in \( \mathbb{C}_\epsilon \) with corner

\(^4\) Berry\(^7\) shows that this transition is smooth by evaluating the asymptotics of the integral within a different, boundary layer, regime. This is known as Berry smoothing.
FIGURE 4  Stokes lines for Euler equation (37). The asymptotic expansions on either side of the Stokes line differ by an exponentially small term, $2\pi i e^{1/\epsilon}$.

At $\epsilon = 0$. Initially, we shall consider (37) as defined along the positive real $\epsilon$-axis, and then consider the solution as $\epsilon$ varies.

Using the replacement rules (32), we Borel transform (37). This yields $y_B = 1/(w + 1)$, which has singular set $\Gamma_w = \{-1\}$ and is Borel summable everywhere except along the negative real axis. We may then recover a locally holomorphic solution to the ODE by substituting $y_B$ into the integral representation (31). In general, for $\epsilon > 0$, the contour $\gamma$ may be composed of $\gamma_0 = (0, \infty)$ and the additional loop, $\gamma_1$, around $\Gamma = \{-1\}$. We may write the general solution as

$$y(\epsilon) = \int_{\gamma_0 = (0, \infty)} \frac{dw}{1 + w} e^{-w/\epsilon} + c \int_{\gamma_1} \frac{dw}{1 + w} e^{-w/\epsilon}.$$ (38)

Imposing the boundary condition of $y \to 0$ as $\epsilon \to \infty$, then we find that $c = 0$. Now, we may analytically continue the integral onto $\mathbb{C}_\epsilon \setminus \mathbb{R}_{\epsilon < 0}$ by adjusting the contour appropriately; then as $\epsilon$ is varied, the contour varies according to $\gamma_0 = e^{i \arg(\epsilon)}(0, \infty)$. Then as $\epsilon$ is analytically continued, anticlockwise, in the upper-half plane to cross the negative real axis, we see that the solution jumps according to

$$y_+ - y_- = \int_{\mathcal{H}_{-1}} \frac{dw}{1 + w} e^{-w/\epsilon},$$ (39)

where the negative and positive signs in $y_{\pm}$ correspond to approaching along $\Im\epsilon > 0$ and $\Im\epsilon < 0$, respectively. The Hankel contour, in this case, picks up the residue of the simple pole at $w = -1$, so asymptotically,

$$y_-(\epsilon) = \epsilon - \epsilon^2 + 2! \epsilon^3 - 3! \epsilon^4 + \cdots,$$

$$y_+(\epsilon) = \epsilon - \epsilon^2 + 2! \epsilon^3 - 3! \epsilon^4 + \cdots + 2\pi i e^{1/\epsilon},$$ (40)

and the asymptotic expansions on either side of the Stokes line $l = (-\infty, 0)$ differ by an exponentially small term. The relevant Stokes graph is then as shown in Figure 4.
3.5 Late terms

In applied exponential asymptotics approaches, it is common to analyze the late terms of perturbative solutions to differential equations.\(^5,6,33\) Let us discuss a surprising relationship between the trans-series (in \(1/n\)) structure of late terms and the trans-series structure (in \(\epsilon\)) of a divergent series \(y(\epsilon)\). We first reinterpret the result (15) in Section 3.1. We may expand \(y_B\) about the origin, writing

\[
y_B(w) = c_0 + c_1 w + c_2 w^2 + \cdots + c_n w^n + \cdots.
\]

(41)

By Equation 15, we can then write the \(n\)th coefficient of \(y_B\) as

\[
c_n = \frac{1}{2\pi i} \sum \chi \int \frac{dw}{\mathcal{H}_\chi} w^{-n} y_B(e^w) \equiv \hat{\mathcal{B}}^{-1}[y_B(e^w)](n).
\]

(42)

Above we have defined an inversion operator, \(\hat{\mathcal{B}}\), which is analogous to the usual Borel inversion, but with a conformally mapped integration variable and different contours. The above sheds light on an association between \(\epsilon\)-expansions (captured by the Borel transform of \(y(\epsilon)\) in \(\epsilon\)) and \(1/n\)-expansions (captured by modified Borel operator above in \(n\)). This link between expansions in \(n\) and expansions in \(\epsilon\) is essentially a consequence of the resurgence properties discussed following Equation 15. We demonstrate this idea with an example.

**Example 2.** Consider the following differential equation for \(y = y(\epsilon)\):

\[
2\epsilon^3 \frac{d^2 y}{d\epsilon^2} - 2\epsilon(3 + \epsilon) \frac{dy}{d\epsilon} + \left(9 + \frac{4}{\epsilon}\right)y = 4.
\]

(43)

Computing a few terms of the power series, with \(y = \sum \epsilon^n y_n\) reveals \(y = 0 + \epsilon - (3/4)\epsilon^2 - (1/16)\epsilon^3 + \cdots\). If we were to proceed to higher order, we would discover that \(y_n\) grows factorially.

Use of the replacement rules (32) confirms that the above corresponds to the following equation for the Borel transform \(y_B(w)\) of \(y(\epsilon)\):

\[
2(1 - w)(2 - w)y_B^\prime(w) - (2w - 3)y_B(w) = 0,
\]

(44)

where we impose \(y_B(0) = y_1/0! = 1\). We then seek a power series solution for the Borel transform, \(y_B(w) = c_0 + c_1 w + c_2 w^2 + \cdots\) in (44). This yields the recurrence relation

\[
4(n + 1)c_{n+1} + (3 - 6n)c_n + 2(n - 2)c_{n-1} = 0, \quad n \geq 2,
\]

(45)

with initial conditions \(c_{-1} = 0\) and \(c_0 = 1\) that follow from the formal power series solution to (43). We may now take the (modified) Borel transform of \(c_n\), as in (38). From the discrete rule (33), the Borel transform of the coefficients, \(c_B(\bar{w}) = \hat{\mathcal{B}}[c_n](\bar{w})\), solves

\[
c_B^\prime(\bar{w})(4e^{-\bar{w}} - 6 + 2e^{\bar{w}}) + c_B(\bar{w})(3 - 2e^{\bar{w}}) = 0.
\]

(46)

This may be integrated to

\[
c_B(\bar{w}) \propto \sqrt{(e^{\bar{w}} - 2)(e^{\bar{w}} - 1)}.
\]

(47)
On the other hand, we may solve the first-order differential equation (44) for $y_B$ directly. This yields

$$y_B(w) = \frac{1}{\sqrt{2}} \sqrt{(w - 2)(w - 1)} = 1 - \frac{(3/4)}{1!}w - \frac{(1/16)}{2!}w^2 + \cdots. \quad (48)$$

Notice the proportional equivalence of $c_B$ (47) and $y_B$ (48), under the conformal map $w \mapsto e^w$. This indeed relates series expansions in $\epsilon$ of $y$ (involving $y_B$) and series expansions in $1/n$ of $y_n$ (involving $c_B$).

To summarize: suppose that $y(\epsilon)$ is a perturbative solution to a differential equation, to understand the trans-series structure of $y(\epsilon)$, it is equivalent to work with the $1/n$ trans-series structure of the late terms in the expansion of $y(\epsilon)$.

## 4 | PARAMETRIC BOREL RESUMMATION

In the previous section, we reviewed (nonparametric) asymptotic sequences, their Borel transform, and the associated resurgent properties. The corresponding theory for parametric asymptotic sequences of the form

$$y(z; \epsilon) = y_0(z) + \epsilon y_1(z) + \epsilon^2 y_2(z) + \cdots \quad (49)$$

where now $z \in \mathbb{C}$ may be developed similarly but now such sequences exhibit new features due to the possibility of distinguished limits involving $z$ and $\epsilon$. In this section, we shall review the application of the Borel transform to sequences such as (49), and discuss in addition the notion of boundary layers in parametric trans-series.

Analogously to Section 3.2, we have a notion of Gevrey-1 asymptotics for (49) whereby we require that $y(z; \epsilon)$ is Gevrey-1 in $\epsilon$ for each $z \in \mathbb{C}_z$ away from a discrete set of singularities. We denote the algebra of such sequences by $\mathbb{C}_1[[\epsilon]](z)$. We further restrict to resurgent sequences (i.e., for fixed $z \in \mathbb{C}_z$, we suppose that the resulting asymptotic sequence is resurgent in the sense of Section 3.3). Then, following (26), given a series of the form (49), we may define a parametric Borel transform, where $z \in \mathbb{C}_z$ is a (holomorphic) parameter:

$$\mathbb{B} : (y_1(z), y_2(z), \ldots) \rightarrow y_B(w, z) := \sum_{n=0}^{\infty} \frac{y_{n+1}(z)}{n!} w^n. \quad (50)$$

The Borel transform, $y_B(w, z)$, may now be considered a function on (possible covers of) $\mathbb{C}_z \times \mathbb{C}_w$. Note that $w \in \mathbb{C}_w$ is the distinguished variable here, since we will integrate it out to obtain a perturbative series. For fixed $z \in \mathbb{C}_z$, we may consider the analytic continuation of $y_B(w, z)$ which we again suppose has power-law singularities at $w = \chi_i(z)$ with local powers $\alpha_i$. In this work, we denote this set of singularities in the Borel plane by $\Gamma_w(z)$.

We may similarly define a holomorphic function that recovers the asymptotics of $y(z; \epsilon)$ by the inverse Borel transform (cf. (29)):

$$y(z; \epsilon) = y_0(z) + \int_0^\infty dw e^{-w/\epsilon} y_B(w, z). \quad (51)$$
We note that any parametric resurgent function \( y_B(w, z) \) will define a (asymptotic) parametric series via this relation.

### 4.1 Stokes lines in parametric series

Stokes lines now arise in a subtly different way to the Stokes lines discussed in Section 3.4. We now consider a fixed contour \( \gamma = (0, \infty) \) for the inverse transform (51) and consider varying \( z \in \mathbb{C}_z \).

Along the following locus in \( \mathbb{C}_z \):

\[
 l_\chi = \{ z : \Im \chi(z) = 0, \Re \chi(z) > 0 \},
\]

the contour, \( \gamma \), intersects a singularity in \( \Gamma_w(z) \) and, in order to keep the integration class constant, the contour is deformed to include a Hankel contour about the singularity (this is the Stokes phenomenon). The union of all such \( l_\chi \) defines a Stokes graph \( l = \bigcup l_\chi \) on \( \mathbb{C}_z \) as opposed to a Stokes graph on \( \mathbb{C}_\epsilon \) as in Section 3.4. Crossing such a locus thus adds exponential terms to the trans-series expansion.

Let us now introduce parametric dependence into the nonparametric theory of Section 3.1. We restrict to the case of the Borel transform, \( y_B \), with a power-law singularity at \( w = \chi(z) \) with power \( -\alpha \chi \). Then, from the inverse Borel transform (51), we see that across a Stokes line, \( l_\chi \), the following term enters the trans-series:

\[
 \int_{\pm \mathcal{H}_\chi} dw \, e^{-w/\epsilon} (\chi(z) - w)^{-\alpha \chi} (a_0^\chi(z) + (\chi(z) - w)a_1^\chi(z) + \cdots) \]

\[
 = \mp \frac{2\pi i}{\epsilon^{\alpha \chi - 1}} \left( a_0^\chi(z) \frac{\Gamma(\alpha \chi)}{\Gamma(\alpha \chi - 1)} + \epsilon a_1^\chi(z) \frac{\Gamma(\alpha \chi)}{\Gamma(\alpha \chi - 1)} + \cdots \right),
\]

where \( \mathcal{H}_\chi \) denotes a Hankel contour around \( w = \chi \) (anticlockwise) and we note that switching the integral and summation typically gives an asymptotic expansion on the right-hand side. This can be compared with (23). Notice that the sign choice, \( \pm \), is related to the direction in which the Stokes line is crossed. We will typically only consider the jump in value below.

The leading-order exponentially small correction, under the assumption of a power-law singularity at \( \chi(z) \) in the Borel plane, may be arrived at more succinctly as follows. Suppose that a perturbative sequence (49) has late-term asymptotics:

\[
y_{n+1}(z) \sim \frac{a(z)}{\Gamma(\alpha \chi)} n! n^{\alpha - 1} \chi(z)^n, \quad \text{as } n \to \infty.
\]

(54a)

Then, to leading order in \( \epsilon \), the trans-series expansions of the Borel sum of the \( y_n(z) \) differs by the following term\(^6\) across the Stokes line associated with \( \chi(z) \):

\[
 \sim \frac{2\pi i}{\epsilon^{\alpha \chi - 1}} \frac{a(z)}{\Gamma(\alpha \chi)} e^{-\chi(z)/\epsilon}.
\]

(54b)

\(^5\)The particular choice \( \gamma = (0, \infty) \) is a consequence of the fact that we typically consider physical problems where \( \epsilon \) is real—otherwise the perturbative part of the contour may be appropriately rotated.

\(^6\)Note that like the analogous jump (13), this only indicates a jump in magnitude; there is a sign consideration dependent on the direction in which the Stokes line is crossed.
This follows since the perturbative terms, $y_n(z)$, of a sequence are related to the germ of the Borel transform by (50). One may then compare the leading term with $n \to \infty$ in Equation (18), and the leading term with $\epsilon \to 0$ in Equation (23).

**Remark.** Notice that in addition, this can be placed in the form preferred by some practitioners \(^{27,33}\) (cf. Section 2). If the late terms follow

$$y_n(z) \sim A(z) \frac{\Gamma(n + \gamma)}{\chi(z)^{n+\gamma}}, \quad \text{as } n \to \infty,$$

(55a)

then across Stokes lines, the following term switches on:

$$\sim \frac{2\pi i}{\epsilon \gamma} A(z) e^{-\chi(z)/\epsilon}.$$

(55b)

Results (54) and (55) are related via Stirling’s approximation, with $\alpha = \gamma + 1$ and $A(z) = a(z)/\Gamma(\alpha)$.

**Remark.** It should be noted that, unlike often seen presentations of Berry’s Stokes smoothing argument in the applied exponential asymptotics literature, the above result does not require the specification of a singularly perturbed differential equation (or an analogous problem) yielding $y$—it requires only the late-term specification of the asymptotic sequence.

We now discuss a mild generalization of the above discussion to go beyond power-law singularities in the Borel plane, and explain the relationship between exponentially small terms and coefficient asymptotics at leading order.

Intuitively, the Hankel integrals that arise from crossing Stokes lines are associated with Figure 1(A), whereas the coefficient integrals are associated with the cylinder of Figure 1(B). Thus, roughly speaking, the trans-series structure is the same up to this conformal map between the plane and the cylinder. In more detail, suppose that $y_B(w, z)$ is a Borel transform with a singularity at $w = \chi(z)$. Crossing the Stokes line associated with $\chi$ picks up the following asymptotic contribution:

$$\int_{\pm H_\chi} dw \ e^{-w/\epsilon} y_B(w, z) = \left[ \epsilon \int_{\pm H_0} ds \ e^{-s} y_B(\chi + s\epsilon) \right] e^{-\chi/\epsilon}.$$

(56)

Above and in the following, we have suppressed the $z$-dependence to simplify notation. On the right-hand side, we have considered a changed variable via $w = \chi + s\epsilon$. On the other hand, the Hankel integral formula (15) for the coefficients of $y_B$ gives

$$(y_B)_n = \frac{1}{2\pi i} \int_{-H_{\text{reg}}} dw \ e^{-n w} y_B(e^w).$$

(57)

If we consider another change of variable $w = \log \chi + \tilde{s}/n$ and expand the above to leading order in $1/n$, we obtain:

$$(y_B)_n = \frac{1}{n \chi^n} \left[ \frac{1}{2\pi i} \int_{-H_0} d\tilde{s} \ e^{-\tilde{s}} y_B\left( \chi + \frac{\tilde{s}}{n} \chi + \cdots \right) \right].$$

(58)
Now we compare the expressions (56) and (58) to leading order in \( \varepsilon \) and \( 1/n \). Let us first define a function \( g(n, z) \) in terms of the leading behavior in the above square-bracketed quantity. Recalling the relationship between \( (y_B)_n \) and the coefficients of the original perturbative series, we have

\[
(y_B)_n \sim \frac{g(n, z)}{n^\chi n} = \frac{y_{n+1}}{n!}.
\] (59)

This now establishes a leading-order asymptotic relationship between the late-term divergence and the term that enters the trans-series across the Stokes line associated with \( \chi \). Thus, if the late terms obey:

\[
y_{n+1}(z) \sim n! \frac{g(n, z)}{\chi(z)^n} \frac{1}{n!},
\] (60a)

then the following jump (Stokes switching) occurs:

\[
\sim 2\pi i \frac{g(\chi(z)/\varepsilon, z)}{1/\varepsilon} e^{-\chi(z)/\varepsilon}.
\] (60b)

The intuitive idea sketched above is now realized concretely by observing that in the exponentially small term, \( 1/n \) is switched with \( \varepsilon \) and in each case is related by the exponential conformal map.

We note that, in practice, one does not need to understand the nature of the Borel singularity (which may be transcendental and difficult to analyze) in order to understand the more general leading-order Stokes switching. In applications, one may consider fitting \( g(n, z) \) to a numerical analysis of the late term behavior of \( y_n(z) \) and immediately deduce the exponentially small contributions via the above relation—we propose this as a useful method in applications where a closed form of the late term asymptotics is not available.

In Section 6, we discuss how the typical factorial-over-power assumption\(^7\) may be violated in simple examples by introducing more general \( g(n, z) \) functions and discussing the implication for singularities in the Borel plane. We now conclude this section with a simple concrete example of a parametric resurgent function. Many of the examples of parametric resurgent functions that we discuss in this work have finite-order pole singularities (as opposed to branch cuts) in the Borel plane. Such examples exclude some interesting resurgence phenomena,\(^8\) but we focus on these simple examples in order to give a clear demonstration of the essential features of the holomorphic inner–outer matching procedure to follow.

**Example 3** (Stokes lines from parametric trans-series). Consider the parametric holomorphic function,

\[
y_B(w, z) = \frac{1}{(w - z)(w - (z^2 - 1))}.
\] (61)

This has a singular set in the Borel plane given by \( \Gamma_w(z) = \{ \chi_1(z) = z, \chi_2(z) = z^2 - 1 \} \). We may recover an asymptotic series from \( y_B(w, z) \) via the inverse Borel transform,

---

\(^7\) The case \( g(n, z) = (n/\chi(z))^a \).

\(^8\) For example, Écalle’s bridge equation is trivial in this case and the so-called higher-order Stokes phenomenon\(^52\) can also not occur.
that is,
\begin{equation}
y(z; \epsilon) = \int_{(0, \infty)} d\omega \, e^{-\omega/\epsilon} y_B(\omega, z) = \epsilon \frac{1}{z(z^2 - 1)} + \frac{z^2 + z - 1}{z^2(z^2 - 1)^2} \epsilon^2 + \cdots \tag{62}
\end{equation}
and we see that the singularities in the physical plane, \( \mathbb{C}_z \), are given by \( \Gamma_z = \{-1, 0, 1\} \). The Stokes graph \( l = l_1 \cup l_2 \) is given by the locus,
\begin{align*}
l_1 &= \{ z : \Im z = 0, \Re z \geq 0 \}, \\
l_2 &= \{ z : \Im (z^2 - 1) = 0, \Re (z^2 - 1) \geq 0 \}. \tag{63a/b}
\end{align*}

Then, by computing the local expansions around \( \omega = \chi_1(z) \) and \( \omega = \chi_2(z) \), we may deduce nonperturbative corrections around the saddle points to arbitrarily high order in \( \epsilon \). For example, crossing the Stokes line associated with \( \chi_1 \) gives a leading-order jump of size
\begin{equation}
\sim \frac{2\pi i}{(-z^2 + z + 1)} e^{-z/\epsilon}. \tag{64}
\end{equation}

4.2 Boundary layers in parametric trans-series

In contrast to the Borel resummation of constant (nonparametric) trans-series, holomorphic trans-series of the form (71a) often exhibit the complication of boundary layers, or distinguished limits in both \( \epsilon \) and \( z \). Informally, when one has a perturbative series with an additional holomorphic parameter, \( z \), say \( y(z, \epsilon) = y_0(z) + \epsilon y_1(z) + \cdots \), the asymptotic expansion may not only “reorder” as \( n \to \infty \), but may also “reorder” as \( z \) tends to certain critical values. The region near such critical points where the function exhibits large gradients typically shrinks as \( \epsilon \to 0 \) and is called the boundary layer. This had led to the outer- and inner-region analyses in Section 2. We can also discuss a version of this notion on the complex analytic side of the correspondence in (2)—in brief, we may think about boundary layers as occurring near points in \( \mathbb{C}_z \) where the radius of convergence of the Borel germ, at \( w = 0 \), shrinks to zero. We begin with a motivating example.

**Example 4** (Boundary layers in the Borel plane). Consider the parametric function on \( \mathbb{C}_w \times \mathbb{C}_z \) given by
\begin{equation}
y_B(w, z) = \frac{1}{w - (z^2 - 1)}. \tag{65}
\end{equation}

We write the expansion of (65) around \( w = 0 \), and then take the inverse Borel transform to give an asymptotic series
\begin{equation}
y(z; \epsilon) = \epsilon \frac{1}{1 - z^2} - \epsilon^2 \frac{1}{(1 - z^2)^2} + \cdots. \tag{66}
\end{equation}

From (65), note that the Borel singularities are given by
\begin{equation}
\Gamma_w(z) = \{ w : w = \chi(z) = z^2 - 1 \} \subset \mathbb{C}_w, \tag{67}
\end{equation}
and the singularities in the physical plane $C_z$ of the low-order perturbative terms in (66) are denoted

$$\Gamma_z = \{z_* : z_* = \pm 1\} \subset C_z.$$  \hspace{1cm} (68)

We note that the function $y_B(w, z)$ itself is not singular at the points $\Gamma_z$ for all $w$; rather the pole at $w = \chi(z)$ moves to the origin in $C_w$ rendering the power series around $w = 0$ ill-defined.

One may approach the point $(w = 0, z = z_*) \in C_w \times C_z$ along two different directions. First, as above, we may expand the function near $w = 0$ and then send $z$ to $z_*$. Alternatively, we may think about $y_B(w, z)$ as a function on $C_z$ with parametric dependence on $w \in C_w$ and consider a locally convergent expansion. For example, around $z_* = 1$, we have from (65),

$$y_B(w, z) = \frac{1}{w} + \frac{2(z - 1)}{w^2} + \frac{(4 + w)(z - 1)^2}{w^3} + \cdots$$  \hspace{1cm} (69)

We see that at $z_* = 1$, we have a simple pole at $w = 0$. The nature of this singularity is obscured in the asymptotic expansion (66). In Section 5.3, we discuss this phenomena further by introducing holomorphic inner variables.

Motivated by the above example, we say that a *boundary layer* of a holomorphic parametric function forms near a point $z_* \in C_z$ where the corresponding $y_B(w, z)$ (thought as a function on $C_w$ with parametric dependence on $z \in C_z$) develops a singularity at $w = 0$. We denote this set by $\Gamma_z \subset C_z$, and throughout the work, we assume that it is discrete.

Note that if $y_B(w, z)$ is a parametric resurgent function with a singularity at $w = \chi(z)$ lying on the same Riemann sheet as $w = 0$, then, whenever $z_*$ is such that $\chi(z_*) = 0$, we expect $z_*$ to correspond to a singularity in some $y_n$ at finite $n$. Conversely, suppose that some early order, $y_n$, is singular at $z = z_*$. Then, we expect that there is a singularity, $w = \chi(z)$, on the same sheet as $w = 0$; this singularity moves to $w = 0$ as $z \to z_*$. Hence, in many cases, boundary layers of the perturbative germ coincide with zeros of $\chi$. In fact, there may be additional boundary layers $\Gamma^\chi_z \subset C_z$ associated with each $\chi$. The case discussed above, singularities of the perturbative germ $y_n(z)$, defines $\Gamma_z = \Gamma^0_z$ but germs around $w = \chi(z) \in \Gamma_w$ may have $\Gamma^\chi_z \neq \Gamma_z$—these correspond to singular points of early $a^\chi(z)$ in the local Borel expansions. In this work, we focus only on inner–outer matching to the perturbative germ, so consider $\Gamma_z$ only.

**Remark.** It is also an interesting question to understand the complex analytic formalism of how one sets a boundary condition for a differential equation at a boundary layer. It is expected that in this case, one may obtain an additional “constant” resurgent trans-series problems for the coefficients $c_\chi$ in the singularly perturbed trans-series:

$$y(z; \varepsilon) = c_\chi \sum_\chi e^{-\chi(z)/\varepsilon} y^\chi(z; \varepsilon).$$  \hspace{1cm} (70)

Indeed, the work of Howls\(^{18}\) finds such a structure. In this work, we avoid this complication by always assuming that the Borel transform, with the perturbative contour $\gamma = (0, \infty)$, satisfies the boundary condition.
5 | SINGULAR PERTURBATION THEORY AND DIFFERENTIAL EQUATIONS

The previous two sections reviewed the resurgent properties of asymptotic sequences studied in isolation. We now primarily study sequences generated by singularly perturbed ordinary differential equations. We consider divergent trans-series of the general form

\[ y(z; \varepsilon) = y^0(z; \varepsilon) + \sum_{\chi} e^{-\chi(z)/\varepsilon} y^\chi(z; \varepsilon). \]  

(71a)

We refer to the perturbative germ or base series as

\[ y^0(z; \varepsilon) \equiv y_0(z) + \varepsilon y_1(z) + \varepsilon^2 y_2(z) + \cdots + \varepsilon^n y_n(z) + \cdots, \]  

(71b)

while we refer to the fluctuations\(^9\) around the saddles \((e^{-\chi(z)/\varepsilon})\) as

\[ y^\chi(z; \varepsilon) \equiv e^{-\alpha_{\chi}} \left[ y_0^\chi(z) + \varepsilon y_1^\chi(z) + \varepsilon^2 y_2^\chi(z) + \cdots + \varepsilon^n y_n^\chi(z) + \cdots \right]. \]  

(71c)

for constant \(\alpha_{\chi}\). The summation in (71a) is taken over all the \(\chi(z)\) associated with singularities, \(\Gamma_w\), in the Borel plane. Note that we choose to distinguish the base series, with \(\chi = 0\), since many of our later examples will involve a solution approximated to leading order by an algebraic expansion in \(\varepsilon\).

The kind of examples studied in this work is singularly perturbed inhomogeneous ordinary differential equations. The terminology singularly perturbed arises because the solutions to such equations with \(\varepsilon \to 0\) are qualitatively different to solutions at \(\varepsilon = 0\). Naive asymptotic expansions to such equations yield perturbative series of the form, for example, \(y(z; \varepsilon) = y_0(z) + \varepsilon y_1(z) + \varepsilon^2 y_2(z) + \cdots\), and hence, we are concerned with the parametric expansions given by (71a) rather than the constant-coefficient expansions of the type (24).

In this section, our goal is to develop the right-hand side of the correspondence (2) and discuss how problems in applied exponential asymptotics can be studied via the parametric complex function, \(y_B(w; z)\), on the Borel plane. The basis of this approach relies upon the behavior of \(y_B\) near power-law singularities, \(w = \chi\), where we may write locally

\[ y_B(w, z) = (\chi(z) - w)^{-\alpha_{\chi}} (a_0^\chi(z) + (\chi(z) - w)a_1^\chi(z) + \cdots) + \text{regular terms}. \]  

(72)

Then, given the above knowledge of the local expansion of \(y_B(w, z)\), one may apply the inverse Borel transform (51) and determine, via (23) the relationship between the components \(\{\alpha_{\chi}, a_i^\chi(z)\}\) and the fluctuations, \(y^\chi(z)\), in the trans-series expansion (71a).

**Parametric replacement rules**

Linear differential equations may be transferred to operators on functions on Borel space using the following consequence of the definition of the inverse Borel transform in (51): the operators

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\(^9\)This terminology is motivated by analogy with theoretical physics. In general, the semi-classical evaluation of path integrals organizes as a sum over perturbative and nonperturbative sectors. Nonperturbative contributions to this calculation include saddle point field configurations of the path integral. Perturbative contributions around these saddle points correspond to loop fluctuations in the relevant quantum field. The resurgent structure may, however, be complicated in quantum field theory where not all trans-series divergences can be straightforwardly associated with semiclassical saddle points due to the existence of renormalons.
\[
\frac{d}{dz} \text{ and multiplication by } \varepsilon \text{ become}
\]
\[
\frac{d}{dz} \rightarrow \partial_z \quad \text{and} \quad \varepsilon^{-1} \rightarrow \partial_w,
\]
(73)
so that e.g. \( B[\frac{dy}{dz}] = \partial_z y_B \) and \( B[\frac{1}{\varepsilon} y(z)] = \partial_w y_B \).

In this work, we shall primarily focus on the case of singularly perturbed linear inhomogeneous \( N \)-th-order differential equations for \( y = y(z; \varepsilon) \) given by \( \mathcal{P} y = F(z) \). As a result of the above transformation rules, the operator is mapped as follows:
\[
\mathcal{P} = \sum_{i=0}^{N} \varepsilon^i P_i(z) \frac{d^i}{dz^i} \rightarrow \mathcal{P}_B = \sum_{i=0}^{N} P_i(z) \partial_z^i \partial_w^{N-i}.
\]
(74)

**Example 5** (First-order example of \( \mathcal{P} \) and \( \mathcal{P}_B \)). Using the above rule, we may write the correspondence between the following first-order operator and its Borel analog:
\[
\mathcal{P} = \varepsilon \frac{d}{dz} + G(z) \rightarrow \mathcal{P}_B = \partial_z + G(z) \partial_w.
\]
(75)

If we solve \( \mathcal{P} y = F(z) = \varepsilon H(z) \) with an inhomogeneous term, then it follows from Definition 4.2 that, in the Borel plane, we are led to solve \( \mathcal{P}_B y_B = 0 \) together with the “initial data” \( y_B(w = 0, z) = H(z)/G(z) \). We return to the general solution to such equations in Section 6.2.

Note that in contrast to Section 3, we now study the Borel transform, \( y_B \), given by a *partial* differential equation (PDE), \( \mathcal{P}_B y_B = 0 \). In the context of the exact WKB analysis of Schrödinger equations,\(^{53}\) the resultant partial differential equations have been studied in great detail. For example, the work of Takei et. al.\(^{54}\) argues that the singular locations, say \( w = \chi \), may be determined by the study of propagation of singularities of (74). It can be shown that the so-called *microlocal property* of such operators explains the type and location of singularities.\(^{40,55}\)

### 5.1 A singularity ansatz in the Borel plane

In our work, we shall take a slightly different approach and explain the Borel-plane analog of the factorial-over-power late terms ansatz approaches of applied exponential asymptotics\(^{33}\): we shall posit a singularity ansatz for \( y_B \), to be satisfied via its governing partial differential equation. In this case, one can show that the bicharacteristic curves studied in the exact WKB approach yield effective ordinary differential satisfied by the singulant \( \chi(z) \).

Given the base series (71b), the factorial-over-power approach of applied exponential asymptotics posits (typically only the leading term of) the following ansatz for the large \( n \) asymptotics:
\[
y_{n+1}(z) = \frac{\Gamma(n + \alpha_\chi)}{\chi(z)^{n+\alpha_\chi}} \left( a_0^\chi(z) + \frac{1}{n + \alpha_\chi - 1} a_1^\chi(z) + \cdots \right) \quad \text{as } n \rightarrow \infty.
\]
(76)
Instead of the above, we may study the problem directly in the Borel plane, where we use assumption that \( y_B \) exhibits a power-law singularity at \( w = \chi \). Hence, we posit the ansatz
\[
y_B(w, z) = (\chi(z) - w)^{-\alpha \chi}(a_0^\chi(z) + a_1^\chi(z)(\chi(z) - w) + \cdots),
\]
and then seek to solve for the components, \( \chi, \alpha, \) and \( a_0^\chi \) and consequently derive the trans-series \((71a)\). We demonstrate the Borel-plane ansatz by example.

**Example 6** (Borel-plane ansatz for a second-order ODE). Key illustrative examples in this work will be singularly perturbed inhomogeneous second-order linear ODEs of the form
\[
\mathcal{P}y = \varepsilon^2 y''(z) + \varepsilon P(z)y'(z) + Q(z)y(z) = F(z),
\]
with \( P(z), Q(z), \) and \( F(z) \) meromorphic functions on \( \mathbb{C}z \). For concreteness, we assume that the boundary conditions for \( y \) are determined by the inverse Borel transform with \( \gamma_0 = (0, \infty) \). However, note that the boundary conditions are not relevant to the following analysis in determining the Stokes lines and associated Stokes jumps. Expanding \( y = y_0 + \varepsilon y_1 + \cdots \), we obtain from \((78)\) the first two orders,
\[
y_0(z) = \frac{F(z)}{Q(z)} \quad \text{and} \quad y_1(z) = -\frac{P(z)}{Q(z)} y_0'(z),
\]
whose singular boundary layer set, \( \Gamma_z \), on \( \mathbb{C}z \) thus consists of poles of \( F(z) \) and \( P(z) \), and zeroes of \( Q(z) \). In Borel space, \( \mathbb{C}w \), the relevant operator is
\[
\mathcal{P}_B = \partial_2^3 + P(z)\partial_z\partial_w + Q(z)\partial_1^2.
\]
Now we consider a singularity ansatz for \( y_B \) of the form of \((77)\). Seeking a solution to \( \mathcal{P}_B y_B = 0 \) at leading order with \( w \) near \( \chi(z) \), we obtain an algebraic equation for \( \chi'(z) \):
\[
(\chi')^2 - P(z)\chi' + Q(z) = 0,
\]
and hence, two first-order differential equations for the singulant, \( \chi(z) \). Note that the multivalued nature of \( \chi(z) \) may be viewed as a result of treating \( \mathbb{C}w \) and \( \mathbb{C}z \) differently—this is in contrast to the singularity propagation approach popular in exact WKB analysis.

To proceed, it is convenient to make the affine transformation \( v = \chi(z) - w \) to move the singularity to the origin. In this new variable, we obtain a new effective Borel PDE centered on \( \chi(z) \) given by
\[
\mathcal{P}_B = -(P - 2\chi')\partial_z\partial_v + \chi''\partial_v + \partial_z^2.
\]
Using \((77)\) and comparing orders in \( v \) of \( \mathcal{P}_B y_B = 0 \) yields a set of recurrence relations for the coefficients, \( a_n^\chi(z) = a_n(z) \), with \( n = 0, 1, 2, \ldots \) given by
\[
(P - 2\chi')a_0'(z) - \chi''a_0(z) = 0,
\]
\[
(n - \alpha)(P - 2\chi')a_n'(z) - \chi''(n - \alpha)a_n(z) - a_{n-1}'(z) = 0, \quad \text{for } n > 0.
\]
Note that there is an apparent absence of initial conditions for the differential equations satisfied by the \( a_n(z) \), that is, (83a) and (83b). In the following section, we will find initial conditions at singular points \( z_\ast \in \Gamma_z \) using an inner–outer matching procedure in the complex plane. Also, note that if \( P \) and \( Q \) in (83a) are now holomorphic on \( \mathbb{C} \) then by the linearity of the differential equation, \( a_0(z) \) has possible singular points only at points \( z \in \mathbb{C}_z \) where \( (P - 2\chi')/\chi'' \) is zero. These singularities will, in general, be distinct from those in \( \Gamma_z \).

**Example 7.** Consider (78) with \( P = 2 \) and \( Q = 1 - z \) so that the singularity locations satisfy \( \chi' = 1 \pm z^{1/2} \) from (81). Assuming \( \alpha \) is not an integer (which depends on the choice of \( F \)), we find from (83b), in the case of \( \chi = \chi_+ \):

\[
a_0(z) = a_0^{\chi_+}(z) = \frac{c_0}{z^{1/4}} \quad \text{and} \quad a_1(z) = a_1^{\chi_+}(z) = \frac{c_1}{z^{1/4}} + \frac{5c_0}{48(1 - \alpha)z^{7/4}},
\]

for (Stokes) constants \( c_0, c_1 \), and so forth. In accordance with the discussion in Example 6, the coefficients in (84) are singular at the origin in \( \mathbb{C}_z \) only. The singularities define a new set (in this case one) of boundary layers in the physical plane \( \Gamma_\chi_z = \{0\} \subset \mathbb{C}_z \) associated with \( \chi_+ \in \Gamma_\chi_z \).

**Example 8.** Let us consider again the general second-order singularly perturbed linear ODE

\[
\mathcal{P}y = \varepsilon^2 y'' + \varepsilon P(z)y' + Q(z)y,
\]

with associated linear Borel operator

\[
\mathcal{P}_B y_B = \delta_z^2 + P(z)\delta_z \delta_w + Q(z)\delta_w^2.
\]

We now explain how the singularity ansatz (or equivalently the factorial-over-power ansatz) method for obtaining the singulant \( \chi(z) \) is related to the exact WKB bicharacteristic method. It is argued in Ref. [56] that singularities of \( \mathcal{P}_B \) propagate along bicharacteristics in \( T^*(\mathbb{C}_z \times \mathbb{C}_w) \). These are curves defined by Hamiltonian flow (with respect to the natural complex symplectic structure on the cotangent space) defined by the symbol \( \sigma \) of \( \mathcal{P}_B \). Introducing coordinates \((z, w)\) for \( \mathbb{C}_z \times \mathbb{C}_w \) and \((\xi, \eta)\) for the respective cotangent directions, then we see that in the second-order example, the symbol is given by

\[
\sigma = \xi^2 + P(z)\xi \eta + Q(z)\eta^2,
\]

and the associated Hamilton equations are given by

\[
\dot{z} = \delta_z \sigma = 2\xi + P(z)\eta, \quad \dot{w} = \delta_w \sigma = P(z)\xi + 2Q(z)\eta,
\]

\[
\dot{\xi} = -\delta_z \sigma = P'(z)\xi \eta + Q'(z)\eta^2, \quad \dot{\eta} = -\delta_w \sigma = 0,
\]

together with the “conserved energy” condition \( \sigma = 0 \). In the above equations, the dot superscript denotes differentiation with respect to a “time” parameter \( t \in \mathbb{C} \) so that solutions are parametric curves in \( T^*(\mathbb{C}_z \times \mathbb{C}_w) \). If we now seek the projection of solutions to \( \mathbb{C}_z \times \mathbb{C}_w \) in terms of \( w = \chi(z) \), then we find

\[
\chi'(z) = \frac{dw}{dz} = \frac{\dot{w}}{\dot{z}} = \frac{P(z)\xi + 2Q(z)}{2\xi + P(z)},
\]
where we have used the fact that we are free to set $\eta = 1$. Using the relation $\sigma = 0$, we may verify that

$$\chi'(z)^2 - P(z)\chi'(z) + Q(z) = 0,$$

and we recover the singulant equation (81).

**Remarks on the singularity ansatz procedure**

To review, the coefficients $a_i^\chi(z)$ that appear in the singularity ansatz of $y_B$ in (77), may be viewed from three equivalent perspectives.

1. By definition, they represent the series coefficients when the Borel transform, $y_B$, is expanded about singularities, $w = \chi(z)$, in $\Gamma_w$.
2. They also appear in the $1/n$ expansion of coefficients, $y_n$, of the base series (71b) (shown in (76)).
3. Finally, by Borel inversion and the Hankel contour formula (23), the coefficients $a_i^\chi$ are directly related to the trans-series for the saddle fluctuations, $y^\chi$, in (71c).

These are consequences of the definition of the Borel transform and the argument of Section 3 that relates the $1/n$ trans-series of the late terms to the corresponding $\varepsilon$ trans-series. Note that this structure persists to “higher singularities” in the sense that the same conclusions may be drawn from the rescaled expansion around a singularity $\chi(z)$ (i.e., consider the $a_i^\chi(z)$ as defining a new Borel germ) and the next nearest singularity $\tilde{\chi}(z)$. That is to say, the late terms of $y^\chi(z)$ may again be factorially divergent.

In the next subsection, we will explain how the typical methodology of applied exponential asymptotics (Section 2) may be mapped to the Borel plane. The main idea is as follows. If the perturbative series, $y^0(z)$ or $y^\chi(z)$ in (49) arise as solutions to a singularly perturbed differential equation, then many of the features of the trans-series are fixed by the operator form of $\mathcal{P}$ (or $\mathcal{P}_B$). As we have seen via Example 6, $\mathcal{P}_B$ imposes certain ordinary differential equations on the components $\chi(z)$ and the associated $a_i^\chi(z)$ in (72). The initial conditions of these ODEs are not known a priori. Thus, we may interpret $\mathcal{P}$ or $\mathcal{P}_B$ as yielding “blank template” trans-series that must now be populated with constants or initial data. The late-terms ansatz methods of, for example, Chapman et al., 33 Kruskal and Segur 20 explain how these (essentially Stokes) constants may be fixed using methods of matched asymptotics.

### 5.2 Matching to the origin in the Borel plane (determining $\chi, \alpha$)

The next part of the ansatz procedure is to identify the initial data involved in determining $\chi(z)$ (see, e.g., (81)) and also the constant $\alpha$, which appears in (77) and determines the nature of the singularity at $w = \chi(z)$. In the applied exponential asymptotics approach (Section 2), these steps are often associated with ensuring that the ansatz (77) is “consistent with the low-order perturbative expansion.”

Let us suppose that $w = \chi(z)$ lies on the same branch as our perturbative germ at $w = 0$. From the general arguments of Section 4.2, we know that $\chi(z)$ must be zero at singularities of the low-order terms $y_1(z), y_2(z), ..., $ this fixes initial conditions for $\chi(z)$. Namely, let us consider a
singulant, $\chi$, with

$$\chi(z^*) = 0 \quad \text{at} \quad z^* \in \Gamma_z. \quad (92)$$

We now work in a neighborhood of a point $z^* \in \Gamma_z$ so that $\chi(z)$ becomes the nearest singularity to $w = 0$. The setup for this subsection is illustrated in Figure 5.

Recall that the local expansion of $y_B(w, z)$ with a power-law singularity, $w = \chi(z)$, takes the form (77). In accordance with the arguments of Section 4.2, we may consider $y_B$ at $w = 0$ and compare with the perturbative solution near the origin

$$y_B(w = 0, z) = (\chi(z))^{-\alpha} a_0(z) + \cdots = y_1(z). \quad (93)$$

The power-law singularity ansatz in the Borel plane is then consistent if we may expand locally about the point $z^* \in \Gamma_z$ to determine $\alpha$. This is a basic assumption of the applied exponential asymptotics literature\(^{33}\) where the above expansion is interpreted as ensuring that the late-orders ansatz in (76) at large $n \to \infty$ is consistent with early asymptotic orders. In particular, setting $n = 0$ in the late terms ansatz (76), we find

$$y_1(z) = (\chi(z))^{-\alpha} a_0(z) + \cdots, \quad (94)$$

which gives an equivalent consistency condition to (93). However, in the present work, this constraint is interpreted as a patching condition for local holomorphic expansions in the Borel plane.

**Example 9.** Suppose $\chi(z)$ has a zero of order $\gamma$ at a point $z^* \in \Gamma_z$. Let us then expand

$$\chi(z) = X_1(z - z^*)^\gamma + \cdots \quad (95a)$$

$$y_1(z) = d_1(z - z^*)^{-\delta} + \cdots \quad (95b)$$

$$a_i(z) = a_i(z - z^*)^{\delta_i} + \cdots \quad (95c)$$
for the singulant, leading perturbative term, and saddle coefficient in (77), respectively. Above, $X_1$, $d_1$, and $a_i$ are nonzero constants. Note, for example, from the discussion on second-order differential equations above that when $P$ is constant, then $a_i(z)$ for $i = 1, 2, \ldots$ are nonsingular at roots of $\chi(z)$ and so $\beta_i = 0$.\(^{10}\) Now, comparing the singularity expansion of $y_B$ to the perturbative germ, we find

$$y_B(0, z) = c(z - z_\star)^{-\alpha + \beta + \cdots} = d_1(z - z_\star)^{-\delta + \cdots},$$

and hence, we find that $\alpha = (\beta + \delta)/\gamma$. In general, the constant prefactor, $c$, may acquire infinitely many contributions from the $a_i$ constants in the saddle fluctuations, and so, these remain undetermined.

**Example 10.** Let us supplement Example 7 with an inhomogeneous term

$$F(z) = \frac{1}{(1 - z)(2 - z)},$$

and seek a solution to $\mathcal{P}y = F$ or equivalently $\mathcal{P}_B y_B = 0$ with the initial data $y_B(w = 0, z) = y_1(z)$ in the Borel plane. We have

$$y_1(z) = \frac{2(-5 + 3z)}{(-2 + z)^2(-1 + z)^4}.$$

We focus on the singularity $z_\star = 1$ and the corresponding singulant, $\chi_+ = z + z^{3/2} - 2$. From (95) and (84), we find $\gamma = 1, \delta = 4, \text{ and } \beta = 0$. Hence, via Example 9, we thus find $\alpha = (\beta + \delta)/\gamma = 4, \text{ and therefore, an order 4 pole at } w = \chi_+ \text{ in the Borel plane.}$

**Example 11.** We may generalize the singulant and matching to $N$th-order inhomogeneous linear differential equation, $\mathcal{P}y = F(z)$, with the associated Borel operator given by (74). Now, instead of (81), $\chi'$ satisfies the algebraic equation

$$\sum_{i=0}^N P_i(z)(\chi')^i = 0.$$

The forcing term, $F(z)$, plays a key role in determining the initial perturbative terms, $y_0(z), y_1(z), y_2(z), \ldots$. We see that there are two distinct types of singularities for forced singularly perturbed differential equations. First, those associated with poles $\{z_1, \ldots, z_f\}$ of $F(z)$ (which we assume throughout is meromorphic)—in that case, we obtain multiple singularities satisfying $\chi(z_i) = 0$ for $i = 1, \ldots, f$. And second, there are those singularities associated with zeros of $P_i(z)$, denoted as $\{z_1, \ldots, z_p\}$. In the latter case, $\chi'(z) = 0$ at these points and they may be interpreted as points where $\chi^{-1}(z)$ is multivalued. We shall see both these two types of singularities in the first-order example of Section 6.

One intriguing complication involves the subtle assumption at the start of this Section 5.2, which was that generic singularity at $w = \chi(z)$ is assumed to lie on the same branch as the perturbative germ at $w = 0$. Indeed, there are situations where $\chi(z)$ may not lie on this Riemann sheet,

\(^{10}\) In general, one needs a singular differential operator acting here to ensure that $\beta$ does not increase with $i$—the procedure is not necessarily consistent for an arbitrary trans-series.
and the above argument of directly matching to \( w = 0 \) is then invalid. The equation studied in Examples 7 and 10 is studied in the Appendix of the work of Trinh and Chapman\(^7\); there, it is suggested that the equation exhibits “higher order Stokes phenomena.” In this case, the trans-series contribution \( y^{\chi}(z; \varepsilon) \) has again a factorial divergence in its perturbative expansion. Loosely speaking, this new singularity \( \tilde{\chi}(z) \) lives on a higher sheet above \( w = 0 \) (at least for some regions in \( \mathbb{C}_z \)) and must be matched to the singularity \( \chi(z) \) instead.

### 5.3 Inner–outer matching of trans-series (determining \( a^\chi_i \))

In Section 5.2, we explained an ansatz to determine the location, \( w = \chi(z) \), and nature, \( \alpha \), of singularities of \( y_B(w, z) \). Further, when \( y_B(w, z) \) is expanded about \( w = \chi(z) \) and the ansatz (77) is substituted into the Borel PDE, \( \mathcal{P} y_B = 0 \), we obtain a set of differential equations for the power series coefficients, \( a^\chi_i(z) \). In this sense, we now have a “blank template” trans-series

\[
y(z; \varepsilon) = y^{(0)}(z) + \sum \chi e^{-\chi(z)/\varepsilon} y^{\chi}(z; \varepsilon).
\]

with the \( z \)-dependent coefficients in the perturbative expansions of the \( y^{\chi}(z; \varepsilon) \) essentially coinciding with the \( a^\chi(z) \) functions in the expansion of \( y_B \) about \( w = \chi \) (cf. (23)).

We now require initial data on the coefficients, \( a^\chi_i(z) \). It turns out that for singularly perturbed differential equations, we may reduce this problem of obtaining initial data for \( a^\chi_i \) to a “constant” resurgent problem of the type discussed in Section 3. This reduction to an expansion of a nonparametric function occurs due to the coupling of \( w \) and \( z \) in the inner region.

In this section, we focus more on the \( \mathbb{C}_z \)-plane, which, up to this point, has largely played an auxiliary role. Recall that we have a discrete set of distinguished points (with associated boundary layers) \( z^*_e \in \Gamma_z \subset \mathbb{C}_z \) that correspond to singularities of the early terms in the perturbative germ, \( y^{(0)}(z) \), in (100)—this is where we will set initial data. Expressing \( y_B(w, z) \) in a new set of variables that move singularities in \( \mathbb{C}_w \) to the unit disk (so that, in particular, \( z \)-dependence is lost for these singularities) reveals a constant resurgent problem of the type discussed in Section 3. We therefore extend the correspondence (2) to include the method of matched asymptotics on the left-hand side and a version solely in the Borel-plane on the right-hand side.

We note again that it is possible that expansions around \( \Gamma_w \) have additional singularities \( \tilde{\chi} \in \Gamma_w' \) and additional boundary layers \( \Gamma^\chi_w \) associated with singularities of low orders of \( a^\chi(z) \) in the same way as above. In this case, one must match between \( \Gamma^\chi_w \) and \( \Gamma^\chi_z \)—and so on \( \textit{ad infinitum} \). The idea is the same but we focus, for notational simplicity, on the “first level” of inner–outer matching in this work and further consider in detail only the sheet of the Borel plane connected to the perturbative germ.

**Setup and notation**

Suppose \( y_B(w, z) \) is a parametric resurgent function with singularities at \( w = \chi_1(z), \chi_2(z), \ldots \). Pick a particular singularity, \( w = \chi(z) \), and let \( z^*_e \) be an element of \( \Gamma_z \) with \( \chi(z^*_e) = 0 \). We define a new inner variable

\[
s = \frac{w}{\chi(z)}.
\]

so that \( y_B(s, z) \) is singular at the points \( s = 1 \) and \( s = \chi_i(z)/\chi(z) \) for \( i \geq 1 \). By the discussion in Section 4.2, whenever \( z = z^*_e \) is such that \( \chi(z^*_e) = 0 \), then we note that the early terms, \( y_i(z) \), are
singular for some $i$; consequently, the asymptotic expansion in $y(z; \varepsilon)$ breaks down due to the presence of a boundary layer. The inner variable (101) satisfies $s \to \infty$ as $z \to z_\star$.

In what follows, we shall primarily consider expanding $y_B$ about different points. When expanded about a given singularity in the Borel plane, that is, $s = w/\chi = 1$, we have

$$y_B(w, z) = (1 - s)^{-\alpha} \sum_{j=0}^{\infty} \tilde{a}_j(z)(1 - s)^j,$$

(102)

where the set of functions $\tilde{a}(z)$ are related to the functions $a(z)$ in the ansatz (77) by

$$\tilde{a}_n(z) = (\chi(z))^{-\alpha + n} a_n(z).$$

(103)

On the other hand, near a singularity in the physical plane where the associated $\chi(z_\star) = 0$, we shall assume an expansion of the form

$$y_B(w, z) = (z - z_\star)^{-\beta} \sum_{i=0}^{\infty} \varphi_i(s)(z - z_\star)^i,$$

(104)

and the coefficients $\varphi_i$ are hence introduced as above. We notice that the coefficients, $\varphi_i(s)$, essentially dictate the behaviour of $y_B$ near the singularities $z = z_\star$. In the language of differential equations (to follow), we shall often refer to $\varphi_i$ as the inner solutions.

Note that we may obtain initial values of such inner solutions, $\varphi_i(0), \varphi'_i(0), \varphi''_i(0), \ldots$ by comparing with the low-order perturbative terms. That is, from the expansion of $y_B$ about $w = 0$, we may write

$$y_0(z) + [\chi(z)s] y_1(z) + [\chi(z)s]^2 (2!) y_2(z) + \cdots = Z^{-\beta} \left[(\varphi_0(0) + Z\varphi_1(0) + Z^2\varphi_2(0) \cdots ) + s(\varphi'_0(0) + Z\varphi'_1(0) + \cdots ) + \cdots \right],$$

(105)

with $Z \equiv z - z_\star$. Then we may expand the early perturbative terms on the left-hand side about the relevant point in $\Gamma_z$. The setup is illustrated in Figure 6 and we now discuss the complex analytic matching with an example.

**Example 12.** Consider the function on $\mathbb{C}_w \times \mathbb{C}_z$ given by

$$y_B(w, z) = \frac{1}{w - \chi_1(z)} + \frac{1}{w - \chi_2(z)},$$

(106)

where $y_B(w, z)$ has simple poles at $\chi_1(z) = z$ and $\chi_2(z) = z^2 + 1$. We remind the reader that typically the goal is, in the case of perturbative solutions to differential equations, to reconstruct an unknown $y_B(w, z)$ from the local expansion $w = 0$; here we have rare knowledge of the exact $y_B(w, z)$. The perturbative data for this problem may be obtained by expanding the Borel transform around $w = 0$

$$y_B(w, z) = y_0(z) + w y_1(z) + \cdots$$

(107)

from which we may read off the asymptotic series

$$y(z; \varepsilon) = -\varepsilon \left(\frac{1}{z^2 + 1} + \frac{1}{z} \right) + \varepsilon^2 \left(\frac{1}{z^2 + 1} + \frac{1}{(z^2 + 1)^2} \right) + \cdots$$

(108)
From the above, we have singular points $z_\ast \in \Gamma_z = \{0, \pm i\}$. Crucially, notice that if we were to examine the analytic expression (106), those points $z_\ast$ are largely unremarkable—$w = 0$ is simply a poor choice of expansion point near $z_\ast$. Let us now define new inner variables,

$$s_1 = \frac{w}{\chi_1(z)} \quad \text{and} \quad s_2 = \frac{w}{\chi_2(z)}. \quad (109)$$

These variables send zeros of $\chi_i(z)$ to $s_1, s_2 \to \infty$. Note that $y_B$ is singular at the points $s_1 = 1$ and $s_2 = 1$. Further, from the discussion in Section 4.2, we know that when either $s_1 = 0$ or $s_2 = 0$ then $y_B(w, z)$ will be singular at zeros of $\chi_1$ or $\chi_2$. We now compare two dual expansions, thought of as local expansions on $C_z$ or $C_s$. We focus on the singularity $\chi_1$ to illustrate the point. First, note that in the new variable $s_1$,

$$y_B(s_1, z) = \frac{1/z}{s_1 - 1} + \frac{1/z}{s_1 - z^2 + 1}. \quad (110)$$

The first expansion, about $s_1 = 1$, is thus

$$y_B(s_1, z) = (1 - s_1)^{-1} \left[ \frac{1}{z(z^2 + z - 1)} + (1 - s_1) \frac{1}{(z^2 + z - 1)^2} \right. \nonumber
+ \left. \frac{z}{(z^2 + z - 1)^2} (1 - s_1)^2 + \cdots \right]. \quad (111)$$

Typically, the task is to obtain the coefficient functions in this expansion, or at least their leading-order (near $\Gamma_z$) constants, since after a rescaling by factors of $\chi$, these will be the sought-after coefficients, $a_i(z)$, discussed in the previous sections.
Alternatively we may expand $y_B$ around a zero of $\chi_1(z)$ (in this case $z = 0$), giving
\begin{equation}
y_B(s_1, z) = \frac{1}{z} \left( (1 - s_1)^{-1} + z s_1 (1 - s_1)^{-1} + z^2 (1 + s_1) + z^3 \frac{s_1 (2 - s_1)}{s_1 - 1} + \cdots \right) .
\end{equation}
(112)

Again, the coefficients here are typically unknowns, so let us write according to (104):
\begin{equation}
y_B(s_1, z) = \frac{1}{z} \left( \varphi_0(s_1) + z \varphi_1(s_1) + z^2 \varphi_2(s_1) + z^3 \varphi_3(s_1) + \cdots \right).
\end{equation}
(113)

Now, expanding at $s_1 = 0$ allows us to determine “initial conditions” in each $\varphi_i(s)$. For example, from (112), we have $\varphi_0(0) = -1, \varphi_0'(0) = 1,$ and so forth, and also $\varphi_1(0) = -1, \varphi_1''(0) = 0,$ and so forth.

In the context of singularly perturbed ODEs, $\varphi_0(s), \varphi_1(s), \ldots$ will be determined via differential equations. In that case, we may obtain the coefficients, $\tilde{a}_i(z)$, via expansions about $z = z_*$; an example of this appears in Section 6.1.

**Remark** (Nested boundary layers). There is a subtle assumption in the above discussion whereby, for a given $\chi(z)$ with $\chi(z^*) = 0$ for $z_* \in \Gamma_z$, it was possible to obtain initial data for the inner $\varphi_i(s)$ at $s = 0$. It is possible to construct examples where this is not the case, and the inner–outer matching procedure becomes more involved.

Consider, for example, three simple poles in the Borel plane at $w = \chi_i$ with $\chi(z^*) = 0$ for $z_* \in \Gamma_z$, together with a perturbative germ, $y = y_0(z) + \varepsilon y_1(z) + \varepsilon^2 y_2(z) + \cdots$. Notice that all three singularities coalesce to the origin, $w = 0$, as $z \to 0$. We then rescale according to the inner variable,
\begin{equation}
s_i = \frac{w}{\chi_i(z)},
\end{equation}
for some choice of $i$. For the chosen $i$, the corresponding singularity will be at $s_i = 1$ of the inner $C_{s_i}$ plane, whereas the other two will move to either zero or infinity (or both) as $z \to 0$. This configuration is illustrated in Figure 7.

The complication is that if one were to work, for example, in the inner variable $s_1$, then the inner equation for $\varphi(s_1)$ will have a singular point at $s_1 = 0$, and we cannot determine the initial data from the perturbative germ in the way described above. The resolution is to first work with the inner variable, $s_3$, so that the singularities associated with $\chi_2$ and $\chi_3$ move to infinity of the $C_{s_3}$ plane. We may then determine the solution near $w = \chi_3(z)$ since now, initial data for $\varphi(s_3)$ are determined by the perturbative germ at $s_3 = 0$. We may then consider the inner expansion for $\chi_2$ where now the initial data for the $\varphi(s_2)$ equations are determined from the singularity expansion associated with $\chi_3$—this singularity now moves to the origin in the $s_2$ variable. In this way, the inner–outer matching procedure may be iterated recursively. In this case, one needs to pay careful attention to the higher physical singularities $\Gamma^{\chi}_z$.

The lesson is that, when viewed purely from the physical plane, $C_z$, the application of exponential asymptotics for solving such problems with nested boundary layers can be difficult. Here, the above method is manifest in the Borel plane.

**Second-order linear ODEs**
Our discussions regarding the connections between the expansions of $y_B$, either about singularities in the Borel plane (i.e., $s = 1$) or about singularities in the physical plane (i.e., $z = z_*$) should
This illustrates an example of three singularities, $\chi_1, \chi_2,$ and $\chi_3$ sharing a common $z_\star \in \Gamma_z$.

(Left) As $z \to z_\star$, the singularities coalesce at the origin in $C_w$.
(Right) The compactified Borel planes, drawn on the Riemann sphere, expressed in each Borel inner variable, $s$, are shown near $z_\star$.

now be considered from the context of differential equations, rather than explicit resurgent functions. We now return to the second-order linear ODE of Example 6, and hence consider $\mathcal{P}$ from (78) and $\mathcal{P}_B$ from (80). As previously explained, inhomogeneous terms for the physical operator yield initial data for the Borel PDE. For a given singularity, $\chi \in \Gamma_w(z)$, we may consider changing variables in the Borel PDE to the inner variable $s = w/\chi(z)$. In this variable, the various relevant differential operators become

\[
\partial_z^2 = s^2 \left(\frac{\chi'}{\chi}\right)^2 \partial_s^2 + s \left(\frac{2(\chi')^2 - \chi''\chi}{\chi^2}\right) \partial_s - 2s \frac{\chi'}{\chi} \partial_s \partial_s + \partial_z^2,
\]

\[
\partial_w \partial_z = -\frac{\chi'}{\chi^2} \partial_s - s \frac{\chi'}{\chi^2} \partial_s^2 + \frac{1}{\chi} \partial_s \partial_z.
\]

The idea of the inner analysis is the following. Without loss of generality, suppose that $\chi$ has an algebraic zero of order $\gamma$ at $z = z_\star = 0$ so that we may write locally $\chi(z) = \chi_0 z^\gamma$. Then, from the fact $(\chi')^2 - P(\chi') + Q = 0$, we have that $P$ and $Q$ must have zeroes of (at most) orders $\gamma - 1$ and $2\gamma - 2$, respectively. We thus write them similarly to the leading term as

\[
P(z) = P_0 z^{\gamma - 1} + \cdots \quad \text{and} \quad Q(z) = Q_0 z^{2\gamma - 2} + \cdots
\]

The consequence is that in the new variables, the Borel PDE is homogeneous\(^{11}\) in factors of $z$ and $\partial_z$. We apply the operator with the inner variable scaling on the expansion (104), and this couples $\varphi_i(s), \varphi'_i(s),$ and $\varphi''_i(s)$ at the same order in $z$ after expanding $\chi(z)$ and the coefficients $P(z)$ and $Q(z)$. We call the resulting equations satisfied by the $\varphi_i(s)$ the Borel inner equations. For example,

\(^{11}\) If one assigns a multiplicative weight $w = 1$ to $z$ and $w = -1$ to $\partial_s$, then every term has weight $w = -2$.\]
to the lowest order, we find
\[ \begin{align*}
\frac{\partial^2 z}{\partial s^2} & = \gamma^2 s^2 z^{-2} \partial^2 z + sz^{-2} \gamma (\gamma + 1) \partial s - 2s \gamma z^{-1} \partial s + \partial^2 z, \\
\partial_w \partial_z & = \chi_0^{-1} (\epsilon z^{-1} \partial s - sy z^{-1} \partial^2 s + z^{-\gamma} \partial s \partial_z).
\end{align*} \tag{117} \]

Acting with these new local operators on \( \mathcal{P}_B \), we obtain the leading inner ODE operator
\[ \begin{align*}
\mathcal{P}^{\text{In.},0}_B & \equiv \left( \chi_0^2 \gamma^2 s^2 - \chi_0 P_0 \gamma s + Q_0 \right) \frac{d^2}{ds^2} + \chi_0 \left( \chi_0 \gamma (\gamma + 1 + 2\beta) s - P_0 (\gamma + \beta) \right) \frac{d}{ds} + \chi_0^2 \beta (\beta + 1). \tag{118} \end{align*} \]

Higher order inner equations can be obtained similarly: one replaces \( \beta \to \beta + k \) for the \( k \)th inner equation, and replaces \( P_0, Q_0 \) and \( \chi_0 \) accordingly. We remind the reader that the inner equation depends on a particular choice of \( \chi \), with a particular singularity, \( z^* \), and an associated \( \beta \). The discussion at the beginning of this section explains how to obtain initial values for this differential equation—namely, the initial data may be read off from the perturbative initial values \( y_0(z) \) and \( y_1(z) \). We note that setting \( s = 1 \), we find for the leading coefficient in (118),
\[ \chi_0^2 \gamma^2 - \chi_0 P_0 \gamma + Q_0 = 0, \tag{119} \]
where this is zero because of the algebraic equation satisfied by \( \chi' \). Hence, as expected, the ODE has a leading singular coefficient at \( s = 1 \).

We may verify that the inverse Borel transform of the above equation indeed recovers the more familiar “inner equation” in matched asymptotics posed in \( \mathbb{C} \). In principle, one could recover the complete expansions of the singularly perturbed trans-series components, \( a_i(z) \), by expanding all of the \( \varphi_i(s) \) inner solutions.

**Example 13.** The inner equation for the singularly perturbed first-order ODE \( \varepsilon y'(z) + y(z) = 1/z \) is given by
\[ (1 - s) \varphi_0'(s) - \varphi_0(s) = 0. \tag{120} \]
Following (32), one may take the inverse transform to obtain
\[ \frac{dY}{dX} + Y = \frac{1}{X}, \tag{121} \]
where \( Y := B^{-1} \varphi_0 \) is the governing inner ODE in physical space. This is the familiar physical inner equation obtained by the inner rescaling \( X = z/\varepsilon \) in the original ODE.

**Example 14.** Consider the differential equation
\[ \varepsilon^2 y''(z) - y(z) = \varepsilon^2 F(z). \tag{122} \]
In the Borel plane, the corresponding operator is the wave equation
\[ \mathcal{P}_B = \partial_w^2 - \partial_z^2 = (\partial_w + \partial_z)(\partial_w - \partial_z), \tag{123} \]
together with the initial data \( y_B(w = 0, z) = -F(z) \). Since the operator factorizes, we may find \( y_B \) in closed form. This is an interesting toy example that allows us to check a number of aspects of the formalism outlined above. Note however that in this example, the singularities in the Borel plane are doubled, that is, lie at \( w = \pm \chi(z) \), and the resurgence relations of Section 3 need to be modified slightly to include functions with multiple power-law singularities at the same radius.

### 5.4 Van-Dyke’s matching rule in the Borel plane

In the methodology of Chapman et al.,\(^{33} \) there is a crucial step where the heuristic Van Dyke matching rule\(^{58} \) is applied in order to match the inner limit of the divergent outer solution, \( y_n \), with the outer limit of the leading-order inner solution. Van Dyke’s matching rule indicates that, for most problems in matched asymptotics, the \( n \)th term of the outer expansion, written in inner variables, and reexpanded to \( m \) terms, is equal to the \( m \)th-term inner expansion, written in outer variables, and reexpanded to \( n \) terms. Hence, this provides a procedure for which unknown constants in the (outer) terms, \( y_n \), can be determined through analysis of the (inner-region) solution near the singularities \( \Gamma_z \). In our notation, this establishes the “initial-data” of the coefficients \( \alpha^\chi_i \) in (77). We now discuss how this procedure works in the Borel plane. In particular, this interpretation, via the complex-analytic side of correspondence (2), demonstrates that the validity of Van Dyke’s rule is a consequence of resurgent properties.

Consider the leading-order inner solution, \( \varphi_0(s) \), as defined by the expanding \( y_B \) about the physical singularities, \( z = z_* \), from (104). We then further expand about \( s = 0 \), via

\[
\varphi_0(s) = \varphi_0^{(0)} + \varphi_0^{(1)} s + \varphi_0^{(2)} s^2 + \cdots + \varphi_0^{(n)} s^n + \cdots
\]  

(124)

As will be seen in Section 6.1, in applications of differential equations, one often obtains a (constant) recurrence relation for the \( \varphi_0^{(n)} \).

Recall from (23) and the resurgence result of (21) that the \( n \)th coefficient of the power series (124) is related to the series expansion of \( \varphi_0 \) about its singularities. Here, we know by choice of \( s \) in (101) that \( y_B \) has a singularity at \( s = 1 \) and further singularities at \( s = \chi_i/\chi \). If we write in the form of (21), the \( 1/n \) trans-series component of \( \varphi_0^{(n)} \) follows

\[
\varphi_0^{(n)} = \frac{\Gamma(n + \delta)}{\Gamma(n + 1)} \left( C_0 + \frac{1}{n + \delta - 1} C_1 + \cdots \right) + \cdots
\]  

(125)

Above, \( C_i \) are constants and \( \delta \) is equal to the value of \( \alpha = \alpha^\chi \) found in either the singularity ansatz (77) or equivalently the expansion (102). In addition, notice that the above expression relates specifically to the singularity at \( s = 1 \); other singularities will produce exponentially subdominant contributions scaling with \( 1/\chi^n \).

By the discussion around (23), the above expansion at \( O(s^n) \) is related to behavior of \( \varphi_0 \) near \( s = 1 \), that is,

\[
\varphi_0(s) = (1 - s)^{-\delta} (C_0 + C_1 (1 - s) + \cdots).
\]  

(126)

Recall that our task is to determine initial data on \( \tilde{a}_i(z) \), defined from the expansion

\[
y_B = (1 - s)^{-\alpha} (\tilde{a}_0(z) + (1 - s)\tilde{a}_1(z) + (1 - s)^2 \tilde{a}_2(z) + \cdots).
\]  

(127)
FIGURE 8  An illustration of Van Dyke’s rule. The task is to relate the expansion of $y_B$ about $s = 1$ in $y_B = (1 - s)^{-\alpha}(\tilde{a}_0(z) + (1 - s)\tilde{a}_1(z) + \cdots)$ to the expansion about $z = z^*$ in $y_B = (z - z^*)^{-\beta}(\varphi_0(s) + z\varphi_1(s) + \cdots)$. For example, solid lines correspond to the successive late terms of $\varphi_0(s)$ determining the leading terms of $\tilde{a}_i(z)$ in $z$ and dashed lines correspond to the late terms of $\varphi_1(s)$ determining the next-to-leading terms of the $\tilde{a}_i(z)$.

Then, comparing the two previous formulae, we see that $\tilde{a}_i(z) \sim C_i$ as $z \to z^*$. In this way, we may determine the leading-order values of each $\tilde{a}_i(z)$ for $i = 1, 2, \ldots$ near $z = z^*$ using the $O(s^n)$ coefficient of $\varphi_0(s)$. This is the previously referenced Van-Dyke matching rule connection between $n$th-order outer terms and leading-order inner terms, which occurs in applied exponential asymptotics.

Similarly, the subleading inner solutions, $\varphi_i$, $i \geq 1$, determine the subleading (in $z^*$) corrections to $\tilde{a}_i(z)$. An illustration of this idea is given in Figure 8. We note that, as explained in this form, Van Dyke’s rule is seen as a property of asymptotic sequences with holomorphic $z$ dependence—hence does not require connection to differential equations a priori. It does, however, provide for us the sought-after initial data for $\tilde{a}_i(z)$ at $\Gamma_z$ in the case of singularly perturbed problems.

Remark. In practice, if one has a singularly perturbed ODE, it is easier to directly make an ansatz $\varphi_0(s) = (1 - s)^{-\alpha}(\varphi_0^{(0)} + (1 - s)\varphi_0^{(1)} + \cdots)$. However, we present the above result from the perspective of the late terms since this is how one typically proceeds in matched asymptotic problems.

To conclude, if one obtains inner ODEs satisfied by the $\varphi_i(s)$, then one may obtain initial data for the $\tilde{a}_i(z)$ at $\Gamma_z$ by using the holomorphic Van-Dyke method outlined above and all components of the parametric trans-series may be obtained. We illustrate how Van Dyke’s rule is applied in practice in Section 6.1.

6 | FURTHER EXAMPLES

In this section, we provide examples of the methodology presented in Sections 4 and 5.
6.1 A worked second-order example

We begin with a linear second-order singularly perturbed differential equation that, for simplicity, is designed so that the leading singularity in the Borel plane is a pole. We hope that the reader familiar with the applied exponential asymptotics methodology will recognize below the Borel plane analogs of each step of the method. Consider the equation (cf. (78))

\[ \mathcal{P} y = \varepsilon^2 y''(z) - 3z\varepsilon y'(z) + 2z^2 y(z) = z, \]  

(128)

for which the corresponding operator in Borel space is

\[ \mathcal{P}_B = \partial^2_z - 3z\partial_z \partial_w + 2z^2 \partial^2_w. \]  

(129)

Seeking a power series solution to \( \mathcal{P}_B y_B = 0 \) about \( w = 0 \) of the form \( y_B(w, z) = u_0(z) + wu_1(z) + \cdots \), we obtain the recurrence relation, for \( n \geq 2 \),

\[ u_{n-2}(z) - 3z(n-1)u_{n-1}(z) + 2z(n-1)(n-2)u_n(z) = 0. \]  

(130)

As expected, this corresponds (up to the requisite factorial factors) with the perturbative solution to (128). The forcing term of the ODE (128) provides us with a nontrivial power series about \( w = 0 \) with leading terms

\[ u_0(z) = -\frac{3}{4z^3}, \quad u_1(z) = \frac{23}{8z^5}, \ldots \]  

(131)

We thus see that \( \Gamma_z = \{0\} \). We now make the Borel singularity ansatz

\[ y_B(w, z) = (\chi(z) - w)^{-\alpha}(a_0(z) + (\chi(z) - w)a_1(z) + \cdots), \]  

(132)

from which we obtain two singularities:

\[ \chi_1(z) = -\frac{z^2}{2} + c_1, \quad \chi_2(z) = -z^2 + c_2. \]  

(133)

By the discussion in Section 4.2 (namely, initial conditions for \( \chi \) are given by setting \( \chi(z_*) = 0 \) for \( z_* \in \Gamma_z \)), we further have \( c_1 = c_2 = 0 \). Let us focus on the closer singularity \( \chi_1(z) = -z^2/2 \). The singularity ansatz provides us a differential equations for the coefficients, \( a_l(z) \), that is, [cf. (83)],

\[ za_0'(z) - a_0(z) = 0, \]  

(134a)

\[ (\alpha - n)(za_n'(z) + a_n(z)) = 0, \quad \text{for } n \geq 1, \]  

(134b)

so that either we learn nothing\(^{12}\) about \( a_n(z) \) in the case \( \alpha \) is an integer (which is a pole in the Borel plane) or we learn a linear form, \( a_n(z) = a_nz \). In summary, so far we have the local power series expansion

\[ y_B(w, z) = (-\frac{z^2}{2} - w)^{-\alpha_1}(a_0(z) + a_1(z)(-\frac{z^2}{2} - w) + \cdots), \]  

(135)

\(^{12}\)This is not a problem because the regular parts of \( y_B(w, z) \) do not contribute to the perturbative trans-series expansion in this case.
where $\alpha = \alpha_1$ denotes the appropriate power near the singularity $\chi_1$. We now find initial conditions at $z = 0 \in \Gamma_z$ for the $a_i(z)$. Comparing to the perturbative germ at $w = 0$

$$y_B(w, z) = -\frac{3}{4z^3} + \frac{23}{8z^5}w + \cdots$$

(136)

we learn that $\alpha_1 = 2$. We now turn to inner–outer matching in order to determine the initial conditions. The inner variable associated to $\chi_1$ is $s_1 = -w/z^2$, and in the notation of Section 5.3, we note the relevant parameters $\gamma = 2$, $P_0 = -3$, $Q_0 = 2$, and $\beta = 3$. The first inner equation thus reads

$$(s - 1)(s - 2)\varphi''_0(s) + \frac{1}{2}(9s - 15)\varphi'_0(s) + 3\varphi_0(s) = 0,$$

(137)

and Equation (136) tells us that the initial data are

$$\varphi_0(0) = -\frac{3}{4}, \quad \varphi'_0(0) = \frac{23}{16}.\quad (138)$$

We seek a power series solution about the singular point $s = 1$ and find

$$\varphi_0(s) = (1 - s)^{-2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{16}(1 - s)^2 + \cdots\right).$$

(139)

Alternatively (and more closely to the typical matched asymptotic analysis), one may seek a solution of the form $\varphi_0(s) = \sum_{n=0}^{\infty} \varphi_{0,n}s^n$ and solving the resultant (typically numerical) recursion relation to find at leading order

$$\varphi_{0,n} \rightarrow -\frac{\sqrt{2}}{2^n} + \cdots.$$ 

(140)

The general theory of Section 5.1 finally tells us that we now have initial data for the $\tilde{a}_0(z)$ and $\tilde{a}_1(z)$, and, in turn, the $a_0(z)$ and $a_1(z)$ (in this case, these are the only coefficients that are not part of the regular part). However, we note that the $(1 - s)^{-1}$ term of (139) is zero that implies that we have to consider higher inner equations to determine the initial data for $\tilde{a}_1$. Proceeding and solving the differential equations (134b) gives $a_0(z) = a_{0,0}z$, and we may write the inner expansion as

$$y_B(s, z) = (1 - s)^{-2}\left(\frac{4}{z^3}a_{0,0} + \frac{2}{z}a_{1,0} + \cdots\right).$$

(141)

We deduce from the above that $a_{0,0} = -\frac{\sqrt{2}}{8}$. Now we must consider the third inner equation for $\varphi_2(s)$ in order to determine $a_{1,0}$. Following the method of Section 5.3, this equation reads

$$(s - 1)(s - 2)\varphi''_2(s) + \frac{1}{2}(5s - 9)\varphi'_2(s) + \frac{1}{2}\varphi_2(s) = 0.$$ 

(142)

However, comparing with the perturbative data, we learn that $\varphi_2(0) = \varphi'_2(0) = 0$ so that, in fact, $a_{1,0} = 0$. We now have all the components of our local expansion near $\chi_1$:

$$y_B(w, z) = (-z^2/2 - w)^{-2}\left(-\frac{\sqrt{2}}{8}z\right) + \text{reg.}$$ 

(143)
From Section 5.1, this is sufficient to determine the Stokes phenomena to all orders in $\epsilon$. Namely, crossing the Stokes’ line $l_1$ turns on the term

$$\frac{2\pi i}{\epsilon} \frac{\sqrt{2z}}{8} e^{-z^2/(2\epsilon)}.$$

The analysis for the second singularity is similar except one finds a branch point rather than a pole in the Borel plane and the fluctuations do not truncate as above. We illustrate the Stokes lines for this problem in Figure 9.

### 6.1.1 Padé approximants for singular problems

The Borel germ $y_B(w, z)$ about $w = 0$ has a finite radius of convergence up to the nearest $w = \chi(z)$. Padé approximants may be used as a heuristic to study the analytic continuation of the germ of $y_B(w, z)$ from a finite number of perturbative coefficients, and hence, allow a useful glimpse into the whole Borel plane. In general, a Padé approximant is a rational approximation to a locally holomorphic function $f(x)$, which may be computed from the first $N$ coefficients about an analytic point. Given $f(x) = a_0 + a_1 x + \cdots + a_{2N} x^{2N}$, we solve the algebraic system resulting for $\{r_j\}$ and $\{x_j\}$ from equating derivatives at zero on either side of

$$a_0 + a_1 x + \cdots + a_{2N} x^{2N} = \sum_{j=0}^N \frac{r_j}{x - x_j} =: P_N:N-1(x).$$

The unique solution then determines the off-diagonal Padé approximant $P_N:N-1(x)$. While uniform convergence of $P_N:N-1(x)$ to $f(x)$ for large $N$ is far from guaranteed—even pointwise convergence fails in general—Padé approximants often give an unreasonably good picture of the analytic continuation of $f(x)$. We refer to reviews of the state of the art of Padé approximant methods and note here only some heuristics. Returning to the example of Section 6.1, for fixed $z$,

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13 The fact we have a double pole means the asymptotics terminate and there are no further corrections.
we may compute the Padé approximant of the germ

\[ y_B(w, z) = u_0(z) + wu_1(z) + w^2u_2(z) + \cdots \]  

(146)
giving \( z \)-dependent \( \{r_j(z)\} \) and \( \{x_j(z)\} \). It is then interesting to observe the singularities cross the contour as \( z \in \mathbb{C} \) is varied; we present some representative plots for our worked example in Figure 10. We draw particular attention to the coalescing poles that Padé typically assigns to branch cuts—these are associated with the branch point singularity at \( \chi_2(z) = -z^2 \) in the present example.

**Remark.** We note that for any linear singularly perturbed ODE, there is an algorithmic procedure to recover Padé plots around any singularity in the Borel plane. One may first compute the Padé approximant about \( w = 0 \) in the way discussed above. Second, around a singularity \( \chi \), the local coefficients \( a_\chi^X(z) \) may be determined to an arbitrary order in \( (w - \chi) \) by solving the recursive differential equations (83b) and inner equations (118). One may then compute a Padé approximant around each \( \chi \) to see the Borel plane expansion around each singularity. This is a useful heuristic to investigate higher order singularities on different Riemann sheets (for particular regions of \( z \)).

### 6.2 First-order equations and coalescence of singularities

First-order singularly perturbed equations are exactly soluble in the Borel plane.\(^{14}\) Let us write a first-order differential equation as\(^{15}\)

\[ \mathcal{P}y = \varepsilon y'(z) + G(z)y(z) = \varepsilon H(z), \]  

(147)

where \( H(z) \) and \( G(z) \) are assumed to be meromorphic for simplicity. According to the replacement rules (32), the associated partial differential operator in the Borel plane is

\[ \mathcal{P}_B = \partial_z + G(z)\partial_w, \]  

(148)

together with the initial data \( y_B(w = 0, z) = H(z)/G(z) \). The PDE may then be integrated to give

\[ y_B(w, z) = \frac{H(\chi^{-1}(-w + \chi(z)))}{G(\chi^{-1}(-w + \chi(z)))}, \]  

(149)

where \( \chi(z) = \int_{z_0}^{z} ds \ G(s) \). In fact, we may observe that the solution is independent of \( z_0 \).

**Remark.** We note the two types of singularities discussed in Section 5 are manifest here. Those arising from poles of \( H(z) \) and those arising at points where \( G(z) = 0 \), which, by the inverse function theorem, coincide with points where \( \chi(z) \) fails to be invertible.

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\(^{14}\)This is equivalent to the fact that such equations are soluble (using an integrating factor) as an integral.

\(^{15}\)The forcing term may be changed to \( F(z) \) with no \( \varepsilon \) prefactor, but then one must be careful to take the initial data for the Borel PDE as \( y_1(z) \) rather than \( F(z)/G(z) \). As discussed in Section 3.3, the Borel transform then “ignores” \( y_0(z) \). We choose instead here to begin our expansion for \( y(z, \varepsilon) \) at \( \mathcal{O}(\varepsilon) \).
FIGURE 10 Padé approximants for the worked example in Section 6.1. We compute the [20 : 21] off-diagonal Padé with 200 perturbative coefficients in Mathematica for the two cases of (A) \( z = -1/2 + i \) and (B) \( z = 1/2 + i \). Contours correspond to \( \Re y_b \). There are two singularities at
\[ w = \chi_1(z) = -z^2/2 \] (circle) and
\[ w = \chi_2(z) = -z^2 \] (square). As the Stokes line of Figure 9 is crossed, two exponentials switch on due to intersection of \( \chi_{1,2} \) with the Borel integration contour (thick).

**Coalescing singularities**

There are unique complexities in cases where the forcing term, via (147), may depend on \( \epsilon \), and where singularities coalesce as \( \epsilon \to 0 \). For example, let us consider

\[ \epsilon y'(z) + y(z) = \epsilon H(z; \epsilon). \] (150)
According to the replacement rules of (32), in Borel space, the problem reads

\[ \partial_z y_B(w, z) + \partial_w y_B(w, z) = H_B(w, z), \]  

where \( H_B(w; z) \) is the Borel transform of \( H(z; \varepsilon) \). We expand \( H(z; \varepsilon) \) as a power series in \( \varepsilon \):

\[ H(z; \varepsilon) = H_0(z) + H_1(z)\varepsilon + \cdots. \]  

Note that typically the forcing term, \( H(z; \varepsilon) \), thought of as a function of \( \varepsilon \), will, in fact, have a finite radius of convergence. According to the results of Section 3, this means that the Borel transform \( H_B(w, z) \) will be entire. The initial data are \( y_B(w = 0, z) = H_0(z) \), and we may integrate the Borel PDE to give the solution as an explicit integral,

\[ y_B(w, z) = H_0(z - w) + \int_0^w ds \ H_B(s, s + z - w). \]  

**Example 15.** From (150), let us consider a forcing term with two singularities in \( \mathbb{C}_z \) that coalesce to the origin as \( \varepsilon \to 0 \):

\[ H(z; \varepsilon) = \frac{z}{(z + \Delta \varepsilon)(z - \Delta \varepsilon)}, \]  

where \( \Delta \) is an \( O(1) \) real constant. Computing the associated Borel transform, we find

\[ H_B(w, z) = \frac{\Delta}{z^2} \sinh \frac{\Delta w}{z}, \]  

which, as expected, is entire on \( \mathbb{C}_w \). We may now compute the integral solution to the Borel PDE (153) to give

\[ y_B(w, z) = -\frac{\cosh \frac{\Delta w}{z}}{w - z}. \]  

Despite there being two singularities at finite \( \varepsilon \) in the forcing term, there is only one singularity in \( \mathbb{C}_w \) at \( w = z \), in this case a simple pole. The associated single Stokes line is then the real axis. From the Borel inversion formula, we see that upon crossing the Stokes line, the exponentially small contribution

\[ \int_{\mathcal{H}_x} dw \ e^{-w/\varepsilon} \ \frac{\cosh \Delta w}{w - z} = -2\pi i (\cosh \Delta) e^{-z/\varepsilon} \]  

is switched on.

We note that the above Borel-plane procedure can be contrasted to, for example, where applied exponential asymptotics techniques can struggle to develop expansions to such coalescing singularities problems. Here, we see that when viewed via the lens of Borel transforms, the structure of the problem is manifest (namely, the presence of only one Borel singularity in this case).
6.3 Beyond power-law singularities

So far in this work, we have considered holomorphic functions with relatively tame singularities of power-law type (cf. (17)). More exotic singularities may arise as Borel-plane solutions to apparently simple singularly perturbed differential equations. Consider, for example, the equation

\[ \varepsilon y' + e^z y = \varepsilon z, \]  

so that \( G(z) = e^z, \) \( H(z) = z \) and \( \chi(z) = e^z. \) Then, from the expression (149), the solution in the Borel plane is given by

\[ y_B(w, z) = \frac{\log(-w + e^z)}{-w + e^z}. \]  

(159)

We consider initial conditions for (158) such that the inverse Borel transform with the perturbative contour, \( \gamma_0 = (0, \infty) \), solves the equation along the real line:

\[ y(z; \varepsilon) = \int_{(0, \infty)} dw e^{-w/\varepsilon} \left[ \frac{\log(-w + e^z)}{-w + e^z} \right]. \]  

(160)

The asymptotics in different regions of the complex plane \( \mathbb{C}_z \) may then be read off using the procedure discussed in Section 3.3.

Notice that from (159), the Borel transform has a singularity of the form \( y_B \sim \log(-s)/(-s) \) as \( s = w - \chi(z) \to 0. \) Hence, this is not in the class of elementary singularities discussed in (17). However, we may still verify, either using the Cauchy integral result (15), or via direct calculation that the \( n \)th series coefficient as \( n \to \infty \) is given by \( (y_B(z))_n \sim g(n, z)/(n \chi^n) \) where \( g(n, z) = -n \log n/\chi. \) Hence, from (60), the late terms of the original asymptotic expansion diverge in a nontypical factorial-over-power fashion, with

\[ y_n \sim \frac{(-\log n) \Gamma(n)}{\chi(z)^n} = \frac{(-\log n) \Gamma(n)}{(ez/n)^n}. \]  

(161)

The Stokes line, \( l \subset \mathbb{C}_z, \) for this problem is given by

\[ l = \{ z : \Im e^z = 0, \Re e^z > 0 \}, \]  

(162)

and the contribution to the trans-series along this line is given by

\[ \int_{\mathcal{H}_x} dw e^{-w/\varepsilon} \left[ \frac{\log(-w + e^z)}{-w + e^z} \right]. \]  

(163)

We may use the shortcut discussed near (60) to evaluate the exponentially small term to leading order. In the notation of that subsection, in this case, we have \( g(n, z) = -n \log n/\chi \) so that the contribution across \( l \) is (up to a sign)

\[ \sim 2\pi i \log(\varepsilon)e^{-\chi(z)/\varepsilon} = 2\pi i \log(\varepsilon)e^{-ez/\varepsilon}. \]  

(164)

**General singularities**

In the present framework, it is possible to produce first-order differential equations that have prescribed singularities (in the Borel plane) by reverse engineering the solution (149). From this,
TABLE 1 Some example first-order linear differential equations, their associated Borel singularities, late terms, and Stokes contributions. As evidenced by the last example, with unknown entries marked (…), it is possible to derive seemingly innocuous differential equations with unknown late-order properties.

| Borel singularity $\phi(s)$ | Example ODE | Late terms | Stokes contribution |
|----------------------------|-------------|------------|---------------------|
| $\log(s)$ | $\varepsilon y' + e^z y = \varepsilon z e^z$ | $\frac{1}{n [\chi(z)]}$ | $2\pi i e^{-\chi(z)/\varepsilon}$ |
| $\log(s)^2$ | $\varepsilon y' + e^z y = \varepsilon z^2 e^z$ | $\frac{2 \log(n)}{n [\chi(z)]^n}$ | $4\pi i \log(\varepsilon) e^{-\chi(z)/\varepsilon}$ |
| $\frac{\log(s)}{s}$ | $\varepsilon y' + e^z y = \varepsilon z$ | $\frac{\log(n)}{[\chi(z)]}$ | $2\pi i \log(\varepsilon) e^{-\chi(z)/\varepsilon}$ |
| $\text{Ei}^{-1}(s)$ | $\varepsilon y' + z^{-1} e^z y = \varepsilon$ | … | … |

we see that the standard factorial-over-power asymptotics of the form (18) is far from universal— as is often supposed in the applied exponential asymptotics literature. In particular, suppose that one wishes to construct an ODE with the Borel singularity form, $y_B \sim \phi(s)$ near $s = 0$. Then, from the general solution (149), we see that we must solve the functional relation

$$F(s) = \phi\left(\int_s^0 dt \, G(t)\right) G(s),$$

(165)

for $F(s)$ and $G(s)$. Such solutions are not unique, and we give some examples in Table 1. It appears challenging to prove that such singularities form a resurgent algebra in the sense of Section 3.3. For example, it is unclear, at least to the authors, how one would close the algebra under convolution. However, in linear ODEs where there is a relatively simple singularity structure in the Borel plane, this is no barrier to studying the associated Stokes phenomena and examples of singularities beyond the power law form do arise in physical examples.63

In some cases, the asymptotics of the coefficients associated with a singularity are not known. The seemingly innocuous equation in the last row of Table 1 is

$$\varepsilon y' + \frac{e^z}{z} y = \varepsilon.$$  

(166)

The Borel plane solution has singularities coinciding with those of the inverse of the exponential integral function. To the authors’ knowledge, little is known about the asymptotics of the associated coefficients, and therefore, it is not possible at present to determine the Stokes’ switching term for this example. Modifications to the basic factorial over power ansatz (19) have arisen in physical applications—see, for example.34 The modification is explained in the present context as arising due to the presence of a singularity in the Borel plane that is not of power-law form.

6.4 Discussion

The goal of this work has been to establish links between the applied exponential asymptotics procedures of, for example, Chapman et al.33 with the theory of Borel summation and resurgence. We have explained how the application of factorial-over-power ansatzes to determine leading exponential correction can be reinterpreted within the context of the Borel plane. Although the procedures are largely equivalent, the Borel plane methodology provides some new tools for
studying problems in beyond-all-orders asymptotics in practice: the Borel-plane interpretation, for instance, allows for the construction of pathological examples such as nested boundary layers or more general late term divergence. These examples can help to guide intuition for more realistic asymptotics problems found in practice. We conclude with a discussion of some directions for future research.

Singularly perturbed partial differential equations
In the present work, we studied holomorphic trans-series with one additional holomorphic parameter \( z \in \mathbb{C} \):

\[
y(z; \epsilon) = \sum_{\chi} e^{-\chi(z)/\epsilon} y_{\chi}(z; \epsilon),
\]

(167)

we have discussed the correspondence with Borel functions \( y_B(w, z) \) and explained how a differential operator \( \mathcal{P}_B \) yields differential equations (in \( z \)) for the trans-series components. The interplay of singularities \( \Gamma_w(z) \subset \mathbb{C}_w \) in the Borel plane and \( \Gamma_z \subset \mathbb{C}_z \) in the physical plane allowed us to find initial conditions for the trans-series differential equations at points in \( \Gamma_z \) via the process of inner–outer matching.

Much of this perspective neatly lifts to holomorphic trans-series with two parameters \( z_1, z_2 \in \mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \) of the form

\[
y(z_1, z_2; \epsilon) = \sum_{\chi} e^{-\chi(z_1, z_2)/\epsilon} y_{\chi}(z_1, z_2; \epsilon),
\]

(168)

with associated two-parameter Borel germs \( y_B(w; z_1, z_2) \). A linear singularly perturbed PDE \( \mathcal{P} \) similarly yields a linear Borel PDE \( \mathcal{P}_B \) from which now partial differential equations may be derived for the trans-series components \( \chi(z_1, z_2) \) and \( a^\chi(z_1, z_2) \). We now have to define initial data for these equations along curves in \( \mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \).

In the two variable case, the inner–outer matching has more structure because the physical singularity set now has some geometry and defines a set of codimension one curves \( \Gamma_z \subset \mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \). The difference with the one variable case is that we now have to take our coordinate choice on \( \mathbb{C}_w \times \mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \) more seriously. The operator \( \mathcal{P}_B \) yields a set of bicharacteristics now in \( T^*(\mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \times \mathbb{C}_w) \), projecting these to physical space \( \mathbb{C}_{z_1} \times \mathbb{C}_{z_2} \) defines a foliation and gives curves along which singularities propagate (the singulant), we may choose a local coordinate \( q \) adapted to this foliation. For an orthogonal coordinate, we write \( p \) for a local coordinate on a connected component of \( \Gamma_z \). The inner–outer matching is now concerned with local data near \( p = 0 \) with a direction \( q \) surviving (in contrast with the inner–outer matching being local to points in the one parameter case). In this way, we find a single-parameter resurgent problem along \( q \) which we must solve using, for example, the methods of the present work, to determine initial data (as opposed to initial conditions at points) for the trans-series components—again along \( \Gamma_z \). We leave a more thorough exploration of this perspective to future work.

Parametric analytic combinatorics
The world of analytic combinatorics provides a wealth of examples of unusual singularities and late-term asymptotics (we refer the reader to Flajolet and Sedgewick for an excellent review and introduction to the topic). Many of the unusual singularity examples discussed above arise as counts of combinatorial objects known as trees; whereby the perturbative germ gives a generating function and the coefficients of the Borel transform give the exponential generating function.
for these objects. Indeed, note that the coefficients of the Borel germ in each example of Table 1 (when simultaneously expanded about suitable points in $z$) are positive integers so that at least the possibility of counting something is present. In fact, in the context of singular perturbation theory (in the presence of an extra parameter $z$ together with the Borel variable), we find that the dual expansion in $w$ and $z$ often gives a refined (by, e.g., leaves) count of trees that does not appear to have been noted in the analytic combinatorics literature.

Higher-order Stokes phenomena

In contrast to constant resurgent problems, singularly perturbed problems have the additional feature that Stokes lines naively associated to singulants, $\chi(z)$, may not always be “active.” This structure has been explored in a number of examples and perspectives.\textsuperscript{52,65–67}

The phenomena may be explained as follows. Suppose that $y_B(w, z)$ is a germ with Borel singularities, $\Gamma_w(z)$, and physical boundary layers, $\Gamma_z$, at $w = 0$. We have discussed already that it is possible that the fluctuations $a^\chi(z)$ around a singularity, $\chi$, may yield an additional set of physical singular points $\Gamma_z^\chi$. In the applied exponential asymptotics terminology, these extra singularities may “drive divergence” and yield an additional Borel singularity, $w = \tilde{\chi}$, and corresponding term, $e^{-\tilde{\chi}/\varepsilon}$, in the trans-series expansion. This is sometimes referred to as a “second-generation singularity”\textsuperscript{36} and the associated naive Stokes line condition may not always lead to an intersection with the contour $(0, \infty)$ on the principal sheet across the entire Stokes line. It would be interesting to investigate the interplay of higher-order Stokes phenomena and the complex analytic inner–outer matching procedure discussed in the present work.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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APPENDIX A: GAMMA FUNCTION

The gamma function is a meromorphic function that, for \( \Re z > 0 \), is defined by an integral

\[
\Gamma(z) = \int_{0}^{\infty} dx \ e^{-x} x^{z-1}.
\] (A1)

Further properties of the \( \Gamma(z) \) function can be found in, for example, Abramowitz and Stegun,\textsuperscript{68} the DLMF,\textsuperscript{69} and\textsuperscript{70}

Hankel contour integrals play an important role in the present work. Recall the Hankel contour representation of the reciprocal \( \Gamma \) function (eqn (5.9.2) of the DLMF\textsuperscript{69}):

\[
\frac{1}{\Gamma(\alpha)} = -\frac{1}{2\pi i} \int_{\mathcal{H}_0} dt \ (-t)^{-\alpha} e^{-t},
\] (A2)

where \( \alpha \in \mathbb{C} \setminus \mathbb{Z} \leq 0 \). Above, the principal branch of \((-t)^{-\alpha}\) is taken and the Hankel contour, \( \mathcal{H}_0 \), refers to a contour that runs from \( t = \infty + i0^+ \), proceeds anticlockwise around \( t = 0 \), and then to \( t = \infty + i0^- \). An image of \( \mathcal{H}_0 \) can be found in, for example, figure 5.12.2 of the DLMF\textsuperscript{69}. Further details of both (A1) and (A2) can be found in §6 of Abramowitz and Stegun\textsuperscript{68}.

We shall use the following identity from Stirling’s formula\textsuperscript{71}:

\[
\frac{\Gamma(n + \alpha)}{\Gamma(n + 1)} = n^{\alpha-1} \left( 1 + \frac{\alpha(\alpha - 1)}{2n} + O(1/n^2) \right),
\] (A3)

where \( \alpha \in \mathbb{C} \) and \( n \) is a large and positive integer.