AUTOMORPHISM GROUPS OF DESIGNS WITH \( \lambda = 1 \)

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Abstract. If \( G \) is a finite group and \( k = q > 2 \) or \( k = q + 1 \) for a prime power \( q \) then, for infinitely many integers \( v \), there is a \( 2-(v,k,1) \)-design \( D \) for which \( \text{Aut}D \cong G \).

For each prime power \( q > 7 \) there is a design \( D \) having the parameters of the point-line design of \( \text{PG}(3,q) \) and for which \( \text{Aut}D = 1 \).

1. Introduction

Starting with Frucht's theorem on graphs [Fr], there have been many papers proving that any finite group is isomorphic to the full automorphism group of some specific type of combinatorial object. Babai surveyed this topic and stated that he had proved that \( 2 \)-designs with \( \lambda = 1 \) are such objects when \( k = q > 2 \) or \( k = q + 1 \) for a prime power \( q \) ([Ba2], mentioned in [Ba1, p. 8]). (The case of Steiner triple systems was handled in [Me].) The purpose of this note is to provide a proof of Babai's result.

Theorem 1.1. Let \( G \) be a finite group and \( q \) a prime power.

(i) There are infinitely many integers \( v \) such that there is a \( 2-(v,q+1,1) \)-design \( D \) for which \( \text{Aut}D \cong G \).

(ii) If \( q > 2 \) then there are infinitely many integers \( v \) such that there is a \( 2-(v,q,1) \)-design \( D \) for which \( \text{Aut}D \cong G \).

Our proof mimics [DK, Sec. 5] and [Kai, Sec. 4], but the present situation is much simpler. We modify a relatively small number of subspaces of a projective or affine space in such a way that the projective or affine space can be recovered from the resulting design by elementary geometric arguments. Further geometric arguments determine the automorphism group.

It is not at all clear how to determine automorphism groups of designs with large parameters. Presumably "most" designs having the same parameters as the point-line design \( \text{PG}_1(3,q) \) of \( \text{PG}(3,q) \) have full automorphism group of order 1. A variation on part of our argument produces the following weak result:

Theorem 1.2. If \( q > 7 \) is a prime power then there is a design \( D \) having the same parameters as \( \text{PG}_1(3,q) \) and for which \( \text{Aut}D = 1 \).

Notation: We use standard permutation group notation, such as \( x^\pi \) for the image of a point \( x \) under a map \( \pi \) and \( gh = h^{-1}gh \) for conjugation. The group of automorphisms of a projective space \( X = \text{PG}(V) \) defined by a vector space \( V \) is denoted by \( \text{PGL}(V) = \text{PGL}(X) \); this is induced by the group of invertible semilinear transformations on \( V \). Also \( \text{ATL}(V) \) denotes the group of automorphisms of the affine space \( \text{AG}(V) \) defined by \( V \).

\(^1\)This theorem was proved before I knew of Babai's result.
2. A SIMPLE PROJECTIVE CONSTRUCTION

Let $G$ be a finite group. For suitable $n$ let $\Gamma$ be a simple, undirected, connected graph on $\{1, \ldots, n\}$ such that $\text{Aut}\Gamma \cong G$ and $G$ acts semiregularly on the vertices (as in [14]). All sufficiently large $n$ satisfying these conditions; we always assume that $n \geq 6$.

Let $K = F_q' \subset F = F_q^s$, and let $\theta$ generate $F^\times$. Let $V_F$ be an $n$-dimensional vector space over $F$, with basis $v_1, \ldots, v_n$. View $G$ as acting on $V_F$, permuting $\{v_1, \ldots, v_n\}$ as it does $\{1, \ldots, n\}$. View $V_F$ as a vector space $V$ over $K$. We will modify the point-line design $\text{PG}_1(V)$ of $P = \text{PG}(V)$.

We will use two nonisomorphic designs $\Delta_1, \Delta_2$ whose parameters are those of $\text{PG}_1(3, q)$ but are not isomorphic to that design, chosen so that $\text{Aut}\Delta_1$ fixes a point (Proposition 3.4).

Our design $D$ has the set $\mathcal{P}$ of points of $P$ as its set of points. Most blocks of $D$ are lines of $P$, with the following exceptions involving flats $Fv$, $0 \neq v \in V$. For orbit representatives $i$ and $ij$ of $G$ on the vertices and edges of $\Gamma$,

(I) replace the lines of the flat $Fv_i$ by the blocks of $\Delta_1$ (in any way), and then apply all $g \in G$ to these sets of blocks in order to obtain the blocks in $(Fv_i)^g$, $g \in G$, and

(II) replace the lines of the flat $F(v_i + \theta v_j)$ by the blocks of $\Delta_2$ (in any way), and then apply all $g \in G$ to these sets of blocks in order to obtain the blocks in $F(v_i + \theta v_j)^g$, $g \in G$.

The semiregularity of $\text{Aut}\Gamma$ guarantees that these replacements are well-defined and $G$-invariant. It is trivial to see that $D$ is a design having the same parameters as $\text{PG}_1(V)$. We can view $G$ as a subgroup of both $\text{Aut}D$ and $\text{PGL}(V)$. We emphasize that the flats in (I) and (II) cover a tiny portion of the underlying projective space: there are many flats that we have not altered.

These reflect the fact that the flats in (I) and (II) cover a tiny portion of the underlying projective or affine space. We will distinguish between the lines of $P$ and the blocks of $D$, even when the blocks happen to be lines. A subspace of $D$ is a set of points that contains the block joining any pair of its points. A hyperplane of $D$ is a proper subspace that meets every block.

We will obtain enough 3-spaces of $P$ from $D$ in order to reconstruct all lines of $P$ as subsets of $\mathcal{P}$. In fact, all of the results in this paper depend on reconstructing all lines of an underlying projective or affine space using a design we have described.

Proof of Theorem 1.1(i). Define a Flat of $D$ to be a subspace $X$ of $D$ of size $q^3 + q^2 + q + 1$. Each Flat $X$ not of the form (I) or (II) is the set of points of a 3-space of $P$. Namely, the blocks inside $X$ not inside Flats in (I) or (II) are lines of $P$. If $B \subset X$ is such a block and $x \in X - B$ is not in any Flat in (I) or (II), then the blocks through $x$ meeting $B$ are lines and cover a plane $E$ of $P$, so $E \subset X$. Choose $y \in X - E$ not in any Flat in (I) or (II), and join the points of $E$ to $y$ in order to generate a 3-space of $P$ inside $X$.

Since every line is inside some Flat, this recovers all lines of $P$ not inside Flats in (I) or (II). The latter Flats are distinguished in $D$ by not being isomorphic to projective spaces.

We use these distinguished Flats to define quasi-Flats: sets $X$ of $q^3 + q^2 + q + 1$ points of $D$ such that (a) there is exactly one associated Flat $Fv$ in (I) or (II)
for which $X_0 := Fv \cap X \neq 0$, (b) $X_0$ is a set of $q + 1$ points but is not a block, and (c) any 2 points $x, y \in X$, $x \notin X_0$, are in a block lying in $X$. As above, a quasi-Flat $X$ is the set of points of a 3-space of $V$. Namely, let $B$ be a block in $X$ and let $x \in X - (B \cup X_0)$. The blocks joining $x$ to the points of $B$ are lines covering a plane $E$ of $P$. Then $E \subset X$. Now use $y \in X - (E \cup X_0)$ as before.

Use all intersections $X \cap Fv$ of a quasi-Flat $X$ and its associated Flat $Fv$ to replace all blocks meeting such intersections at least twice. This recovers the lines of $P$ inside the Flats occurring in (I) or (II). Thus, we have recovered all lines of $P$ from $D$ in a natural manner, and hence also $P$, $V$, $PG_1(V)$ and $PTL(V)$, so that $AutD$ is induced by a subgroup of $PTL(V)$.

We next determine the $F$-structure of $V$ using $D$. We claim that the subgroup of $PGL(V)$ fixing each Flat in (I) or (II) arises from scalar multiplications by members of $F^*$. Clearly such scalar multiplications behave this way. Let $h \in GL(V)$ behave as stated. Let $h \in GL(V)$ induce $h$. Since $V_F = \oplus_i Fv_i$, $h: x v_i \mapsto (x A_i) v_i$ for $x \in F$ and a $4 \times 4$ invertible matrix $A_i$ over $K$. If $ij$ is an edge of $\Gamma$ and $x \in F$, then $(x(v_i + \theta v_j)) = (x A_i v_i + ((x \theta) A_j) v_j$ is in $F(v_i + \theta v_j)$, so $(x A_i) \theta = (x \theta) A_j$. Then also $(x A_i) \theta = (x \theta) A_i$, so $(x \theta \theta) A_i = ((x \theta) A_i) \theta = (x A_i) \theta \theta$, and $A_i$ commutes with multiplication by $\theta^2$. By Schur’s Lemma, $A_i: x \mapsto a_i x$ for some $a_i \in F^*$. Then $x a_i \theta = x \theta a_i$, so $a_i = a_j$. Since $\Gamma$ is connected, all $a_i$ are equal, proving our claim.

In particular, the field $F$ and the $F$-space $V_F$ can be reconstructed from $D$. Since $AutD$ normalizes $F^*$ by the preceding paragraph, $AutD \leq PTL(V_F)$. We know that $G$ is inside both $AutD$ and $PGL(V)$. Since the Flats in (II) correspond to (ordered) edges of $\Gamma$, $AutD$ induces $Aut \Gamma \cong G$ on the collection of Flats in (I).

Let $h \in AutD \leq PTL(V_F)$. Multiply $h$ by an element of $G$ in order to have $h$ fix all $Fv_i$. Let $h \in GL(V_F)$ induce $h$, with associated field automorphism $\sigma$. For each $i$ we have $v_i^h = a_i v_i$ for some $a_i \in F^*$. Let $ij$ be an edge of $\Gamma$ and write $b = a_j/a_i$. As above, $F(v_i + \theta v_j)^h = F(a_i v_i + \theta^\sigma a_j v_j) = F(v_i + \theta^\sigma b v_j)$ and $F(\theta v_i + v_j)^h = F(\theta^\sigma a_i v_i + a_j v_j) = F(v_i + \theta^{-\sigma} b v_j)$ both have type (II), so $\theta^\sigma b = \theta^\pm 1$ and $\theta^{-\sigma} b = \theta^{\mp 1}$. Then $b^2 = 1$, $\theta^\sigma = \pm \theta^\pm 1$, and hence $\sigma = 1$ and $b = 1$ since $\theta$ generates $F^*$. Since $\Gamma$ is connected, all $a_i$ are equal.

Since $h$ fixes $Fv_1$ it induces an automorphism of the portion of $D$ contained in the Flat $Fv_1$. By (I) and our condition on $\Delta_1$, $h$ fixes a point of $Fv_1$. Then $a_1 \in K$, so that $h = 1$ on $\mathcal{F}$ and $AutD \cong G$. □

Section 7 contains further properties of $D$.

3. A SIMPLER PROJECTIVE CONSTRUCTION

We need a fairly weak result (Proposition 3.3) concerning designs with the parameters of $PG_1(3, q)$. We know of two constructions for designs having those parameters, due to Skolem [Wi] p. 268] and Lonerger [Le]. However, isomorphism questions seem difficult using their descriptions. Instead, we will use a method that imitates [Sh, Ka] (but which was hinted at by Skolem’s idea).

Consider a hyperplane $X$ of $P = PG(d, q)$, $d \geq 3$; we identify $P$ with $PG_1(d, q)$. Let $\pi$ be any permutation of the points of $X$. Define a geometry $P_\pi$ as follows:
- the set $\mathcal{F}$ of points is the set of points of $P$, and
- the lines of $P$ not in $X$, and
- the sets $L^\pi$ for lines $L \subset X$. 

AUTOMORPHISM GROUPS OF DESIGNS WITH $\lambda = 1$
Once again it is trivial to see that $P_\pi$ is a design having the same parameters as $P$. Note that $\pi$ has nothing to do with the incidences between points and the blocks not in $X$.

We have a hyperplane $X$ of $P_\pi$ such that the blocks of $P_\pi$ not in $X$ are lines of a projective space $P$ for which $\mathfrak{P}$ is the set of points. The lines of this projective space can be recovered from $P_\pi$ and $X$ in a natural manner. Namely, take 2 intersecting blocks each meeting $X$ in a point, use blocks to join points not in $X$ on one of these blocks to the points of the other block, and iterate, in order to obtain $q^2 + q$ blocks and $q^2 + q + 1$ points of a plane $E$ of $P$ (this process did not produce the line $X \cap E$). Then include each such set $X \cap E$ in order to obtain all lines of the original projective space $P$.

The symbol $X$ is ambiguous: it will always mean either a set of points or a subspace of the underlying projective space (as in the next result). It will not refer to $X$ together with a different set of lines produced by a permutation $\pi$.

**Proposition 3.1.** The designs $P_\pi$ and $P_{\pi'}$ are isomorphic by an isomorphism sending $X$ to itself if and only if $\pi$ and $\pi'$ are in the same $\text{PGL}(X) \times \text{PGL}(X)$ double coset in $\text{Sym}(X)$.

Moreover, the pointwise stabilizer of $X$ in $\text{Aut}P_\pi$ is transitive on the points outside of $X$, and the stabilizer $(\text{Aut}P_\pi)_X$ of $X$ induces $\text{PGL}(X) \cap \text{PGL}(X)^\pi$ on $X$.

**Proof.** Let $g : P_\pi \to P_{\pi'}$ be such an isomorphism. We just saw that $P$ is naturally reconstructible from either design. It follows that $g$ is a collineation of $P$; its restriction $\bar{g}$ to $X$ is in $\text{PGL}(X)$.

If $L \subseteq X$ is a line of $P$ then $g$ sends the block $L^\pi \subseteq X$ of $P_\pi$ to a block $L^{\pi g} \subseteq X$ of $P_{\pi'}$. Then $L^{\pi g \pi'^{-1}}$ is a line of $P$, so that $\pi g \pi'^{-1}$ is a permutation of the points of the hyperplane $X$ of $P$ sending lines to lines, and hence is an element $h \in \text{PGL}(X)$. Thus, $\pi$ and $\pi'$ are in the same $\text{PGL}(X) \times \text{PGL}(X)$ double coset.

Conversely, if $\pi$ and $\pi'$ are in the same $\text{PGL}(X) \times \text{PGL}(X)$ double coset let $\bar{g}, h \in \text{PGL}(X)$ with $\pi \bar{g} \pi'^{-1} = h$. Extend $\bar{g}$ to $g \in \text{Aut}P$ in any way. We claim that $g$ is an isomorphism $P_\pi \to P_{\pi'}$. It preserves incidences between blocks not in $X$ and points of $P$ since those incidences have nothing to do with $\pi$ and $\pi'$. Consider an incidence $x \in B \subseteq X$ for a block $B$ of $P_\pi$. Then $B = L^\pi$ for a line $L \subseteq X$, so $x^g \in B^g = B^h = L^{\bar{g}^h} = (L^h)^\pi$, which is a block of $P_{\pi'}$, as required.

For the final assertion, the pointwise stabilizer of $X$ in $\text{Aut}P$ is in $\text{Aut}P_\pi$ by the definition of $P_\pi$. We have seen that the group induced on $X$ by $\text{Aut}P_\pi$ corresponds to pairs $(\bar{g}, h) \in \text{PGL}(X) \times \text{PGL}(X)$ satisfying $\pi \bar{g} \pi'^{-1} = h$. \hfill $\square$

Note that there are many extensions $g$ of $\bar{g}$ since the designs $P_\pi$ have large groups of automorphisms inducing the identity on $X$. Double cosets arise naturally in this type of result; compare [Ka, Theorem 4.4].

Let $v_1 = (q^d - 1)/(q - 1)$.

**Corollary 3.2.** There are at least $v_d!/v_{d+1}|\text{PGL}(d, q)|^2$ pairwise nonisomorphic designs having the same parameters as $P$.

**Proof.** Fix $\pi$ in the proposition. There are at most $v_{d+1}$ hyperplanes $Y$ of $P_\pi$ (as in [JT, Theorem 2.2]). By the proposition there are then at most $|\text{PGL}(X)|^2$ choices for $\pi'$ such that $P_\pi \cong P_{\pi'}$ by an isomorphism sending $Y$ to $X$. Since there are $v_d!$ choices for $\pi$ we obtain the stated lower bound. \hfill $\square$
Remark 3.3. We describe a useful trick. A transposition $\sigma$ and a 3-cycle $\tau$ are in different $\text{PTL}(d, q)$, $\text{PTL}(d, q)$ double cosets in $S_N$, $N = (q^d - 1)/(q - 1)$, if $d \geq 3$. For, if $\sigma q = h\tau$ with $g, h \in \text{PTL}(d, q)$ then $g^{-1}h = \sigma\tau^{-1} \in \text{PTL}(d, q)$ fixes at least $N - 5$ points, and hence is 1 since $d \geq 3$, whereas $\sigma$ and $\tau$ are not conjugate in $S_N$.

**Proposition 3.4.** For any $q$ there are two designs having the parameters of $P = \text{PG}(3, q)$ and not isomorphic to one another or to $P$, for one of which the automorphism group fixes a point.

**Proof.** If $q = 2$ then there are even such designs with trivial automorphism group [CCW]. (Undoubtedly such designs exist for all $q$; cf. Theorem 1.2.)

Assume that $q > 2$. The preceding proposition and remark provide us with two nonisomorphic designs. It remains to deal with the final assertion constructively.

Let $L$ be a line of $X$ and $x \in X - L$. Let $\pi$ be a $(q + 2)$-cycle on $L \cup \{x\}$. We will show that $P_\pi$ behaves as stated.

We claim that $X$ is the only hyperplane of $P_\pi$. For, the blocks in another hyperplane $Y$ all meet $X$, so $X \cap Y$ is a block inside $X$, hence has the form $M^\pi$ for some line $M$ of $X$. All blocks in $Y$ not contained in $X$ are lines of $P$, so $Y$ is a plane of $P$ and $X \cap Y$ is a line of $P$. However, the definition of $\pi$ shows that $X$ has no line $M$ such that $M^\pi$ is also a line $X \cap Y$ of $X$, which proves our claim.

By Proposition 3.1 $\text{Aut}P_\pi$ induces $\text{PTL}(X) \cap \text{PTL}(X)^\pi$ on $X$. Let $\pi g\pi^{-1} = h$ for $g, h \in \text{PTL}(X)$. Then $\pi^{-1}g^{-1}h = \pi^3\pi^{-1}$ is a collineation of $X$ that moves at most $2(q + 2)$ points of $X$ and hence fixes at least $q^2 + q + 1 - 2(q + 2) > q + \sqrt{q} + 1$ points if $q > 3$. By elementary (semi)linear algebra, the only such collineation is 1, so that $\pi g = h$ commutes with $\pi$ and hence fixes $L \cup \{x\}$ and then also $x$, as required. Similarly, if $q = 3$ then $\pi^3\pi^{-1}$ fixes at least 4 points; we have seen that we may assume that $\pi^3\pi^{-1} \neq 1$, so it fixes a line pointwise. However, it is easy to check that, since $\pi^3$ has support $L^3 \cup \{x^3\}$, no line is fixed pointwise by $\pi^3\pi^{-1}$.

**Remark 3.5.** By excluding the possibilities $q \leq 8$ and $q$ prime in the previous section we could have used nondesarguesian projective planes (and $[F : K] = 3$).

4. **Order 1**

**Proof of Theorem 1.2** We use essentially the same method as in the preceding section but move closer to Skolem’s idea [W] p. 268. Start with a line $L_0$ of $P = \text{PG}(3, q)$ and the planes $X_1, \ldots, X_{q+1}$ containing it. Let $L_0, M_{i1}, M_{i2}$ be lines forming a triangle in $X_i$. For $i = 1, \ldots, q + 1$, let $A_i$ be a set of $i - 1$ points of $X_i - (L_0 \cup M_{i1} \cup M_{i2})$ such that each line contains at most $q - 3$ of them.

For each $i$ define $\pi_i \in \text{Sym}(X_i)$ to be a $(q + 1)$-cycle on $L_0$ (these need not be the same as $i$ varies) and a $([2q - 1] + i - 1)$-cycle $\sigma_i$ on $(M_{i1} \cup M_{i2} \cup A_i) - L_0$ such that $(M_{i1} \cap L_0)^{\sigma_i} \neq M_{i2} \cap L_0$, $(M_{i2} \cap L_0)^{\sigma_i} \neq M_{i1} \cap L_0$, $(M_{i1} - L_0)^{\sigma_i} \neq M_{i2} - L_0$ and $(M_{i2} - L_0)^{\sigma_i} \neq M_{i1} - L_0$.

These permutations have the following properties:

(i) the cycles $\sigma_i$ have different lengths $\neq q + 1$;
(ii) the support of $\sigma_i$ contains $q$ points of precisely 2 lines $\neq L_0$;
(iii) no nontrivial collineation of $X_i$ is the identity on the support of $\sigma_i$ (since that support spans $X_i$); and
(iv) if $L \neq L_0$ is a line of $X_i$ and $x \in L$, then $(L - x)^{\pi_i}$ is not contained in a line of $X_i$.  


Define a geometry \( \mathbf{P} \), as follows:
the set \( \mathcal{Q} \) of points is the set of points of \( \mathbf{P} \), and
blocks are of two sorts:
the lines of \( \mathbf{P} \) not in any \( X_i \), and
the sets \( L^\pi_i \) for lines \( L \subset X_i \).

Once again it is easy to see that \( \mathbf{P} \) is a design having the same parameters as \( \mathbf{P} \).

We have hyperplanes \( X_i \) of \( \mathbf{P} \), containing a block \( L_0 \) of \( \mathbf{P} \), such that the blocks
of \( \mathbf{P} \) in no \( X_i \) are the lines of a (projective space) \( \mathbf{P} \) for which \( \mathcal{Q} \) is the set of points.
Let \( \mathbf{P}(L_0) \) denote the substructure of \( \mathbf{P} \) whose points are those of \( \mathcal{Q}-L_0 \) and whose
lines are those of \( \mathbf{P} \) contained in no \( X_i \). Clearly, this is also a substructure of \( \mathbf{P} \).

The nonempty intersections of the lines of \( \mathbf{P} \) with \( \mathcal{Q}-L_0 \) can be recovered from
\( \mathbf{P}(L_0) \), and hence from \( \mathbf{P}_* \) and \( L_0 \). For, take 2 intersecting lines of \( \mathbf{P} \) in no \( X_i \),
use lines to join pairs of points of these lines that are in no \( X_i \), and iterate in
order to obtain the points not in \( L_0 \) of a plane \( E \) of \( \mathbf{P} \) and the lines of \( E \) not containing \( x := E \cap L_0 \).
The remaining line of the affine plane \( X_j - L_0 \) is the set
\((E \cap X_j) - L_0 \). (It is also easy to reconstruct a copy of \( \mathbf{P} \) from \( \mathbf{P}(L_0) \) by introducing
"ideal points", but there is not enough information available to identify these ideal
points with points of \( L_0 \).)

It follows that \( \text{Aut}(\mathbf{P}(L_0)) \) and \( \text{Aut}(\mathbf{P})_{L_0} \) induce the same group on \( \mathcal{Q} - L_0 \).

We claim that the only hyperplanes of \( \mathbf{P}_* \) are the \( X_i \). For, consider another
hyperplane \( X \), so \( |X| = q^2 + q + 1 \). Then \( B_1 := X \cap X_1 \) is a block of \( \mathbf{P} \), and
hence \( B_1 = L^\pi_i \) for some line \( L \neq L_0 \) of \( X_1 \). Every block in \( X \) other than \( X \cap X_1 \),
1 \( \leq i \leq q + 1 \), is a line of \( \mathbf{P} \). By considering the intersections of these lines we see
that \( X - x \) is contained in a plane \( E \) of \( \mathbf{P} \), where \( x := X \cap L \). If \( x \neq y \in B_1 \) then
all blocks \( \neq B_1 \) in \( X \) on \( y \) are lines of \( E \), so \( B_1 - x \) is contained in the remaining
line of \( E \) on \( y \). Since \( B_1 - x = (L - x^\pi_i^-1)^\pi_i \), this contradicts (iv).

Thus, any \( g \in \text{Aut}(\mathbf{P}_*) \) permutes the hyperplanes \( X_i \); given \( X_i \) there is an \( X_j \)
such that \( X^g = X_j \). We proceed as in the proof of Proposition \( \ref{prop:3.1} \), using the fact
that \( g \) agrees on \( \mathcal{Q} - L_0 \) with an element of \( \text{Aut}(\mathbf{P})_{L_0} \).

Given \( i \), let \( \bar{g} : X_i - L_0 \to X_j - L_0 \) denote the restriction of \( g \) to the affine plane
\( X_i - L_0 \). Also let \( \bar{\pi}_i \) denote the restriction of \( \pi_i \) to that affine plane. If \( L \) is a line of \( X_i \)
then \( g \) sends the block \( L^\pi_i \subset X_i \) of \( \mathbf{P} \) to a block \( L^{\pi_i}g \subset X_j \) of \( \mathbf{P}_* \). Then
\( L^{\bar{\pi}_i} \bar{g} \bar{\pi}_j ^{-1} \subset X_j \) is a line of \( \mathbf{P}_* \), so that \( \bar{\pi}_i \bar{g} \bar{\pi}_j ^{-1} \) maps \( X_i - L_0 \) to \( X_j - L_0 \), sending
affine lines to affine lines, and hence is an isomorphism \( \bar{h} \) between the affine planes
\( X_i - L_0 \) and \( X_j - L_0 \).

Now \( \bar{g}^{-1}h = \bar{\pi}_i \bar{g} \bar{\pi}_j ^{-1} \) is a collineation of the affine plane \( X_j - L_0 \) fixing at least
\( q^2 - 2([2q - 1] + q) > q \) points since \( q > 7 \), and hence is 1. Then \( \bar{\pi}_i \bar{g} = \bar{\pi}_j \), so \( i = j \)
by (i). Now \( \bar{g} \) commutes with \( \bar{\pi}_i \) and hence conjugates its nontrivial cycle \( \sigma_i \) to itself.
By (ii), \( \bar{g} \) fixes the intersection of the 2 affine lines contained in the support
of \( \sigma_i \), and hence fixes each point in the cycle \( \sigma_i \). Then \( \bar{g} = 1 \) on \( X_i - L_0 \) by (iii).
Thus, \( g = 1 \) on each hyperplane \( X_i \) of \( \mathbf{P}_* \), and hence \( g = 1 \).

5. A simple affine construction

We now consider Theorem \ref{thm:1.1}(ii). The proof is similar to that of Theorem \ref{thm:1.1}(i).
Since that result handles all \( q \leq 5 \) we will assume that \( q \geq 7 \).

Let \( G \) and \( \Gamma \) be as in Section \ref{sec:2} This time \( K = F_q \subset F = F_{q^2} \); once again \( \theta \)
generates \( F^* \). Let \( V_F \) be an \( n \)-dimensional vector space over \( F \), with basis \( v_1, \ldots, v_n \).
View \( V_F \) as a vector space \( V \) over \( K \). We will modify the point-line design \( AG_1(V) \)
of $A = AG(V)$, using nonisomorphic designs $Δ_1, Δ_2$ whose parameters are those of $AG_1(3, q)$ but are not isomorphic to that design, chosen so that $\text{Aut} Δ_1$ fixes a point (Proposition 6.2).

Our design $D$ has $V$ as its set of points. Most blocks of $D$ are lines of $A$, with exceptions involving the flats $F_v, 0 \neq v \in V$, in Section 2(I, II). As before, each design $AG_1(F_v)$ or $AG_1(F(v_i + \theta v_j))$ is replaced by a copy of $Δ_1$ or $Δ_2$. Since different flats meet only in a single point, the modifications made inside different flats are unrelated. Once again it is trivial to check that this produces a design $D$ with the desired parameters such that $G \leq \text{Aut} D$.

**Proof of Theorem 1.1(ii).** Define a Flat of $D$ to be a subspace $X$ of size $q^3$. Each Flat not in (I) or (II) arises from a 3-space of $A$. Namely, the blocks inside $X$ not inside Flats in (I) or (II) are lines of $A$. If $B \subset X$ is such a block and $x \in X - B$ is not in any Flat in (I) or (II), then the blocks through $x$ meeting $B$ are lines and cover a plane $E$ of $A$ except for the line of $E$ through $x$ parallel to $B$. Varying $x$ among these points of $E$ we obtain $E \subset X$. Now let $y \in X - E$ and join the points of $E$ to $y$, and vary $E \subset X$, in order to generate a 3-space inside $X$.

Since every line is inside some Flat, this also recovers all lines of $A$ not inside a Flat in (I) or (II). The latter Flats are distinguished in $A$ by not being isomorphic to affine spaces, so $\text{Aut} D$ fixes the intersection 0 of those Flats.

We use these distinguished Flats to define quasi-Flats: sets $X$ of $q^3$ points of $D$ such that (a) there is exactly one associated Flat $F_v$ in (I) or (II) for which $X_0 := F_v \cap X \neq \emptyset$, (b) $X_0$ is a set of $q$ points but is not a block, and (c) any 2 points $x, y \in X, x \notin X_0$, are in a block lying in $X$. As above, a quasi-Flat $X$ is the set of points of a 3-space of $A$. Namely, let $B$ be a block in $X$ and let $x \in X - (B \cup X_0)$. The blocks joining $x$ to the points of $B$ are lines covering a plane $E$ of $A$ except for the line of $E$ through $x$ parallel to $B$. Now vary $x$ in order to obtain $E \subset X$, and then use $y \in X - (E \cup X_0)$ as before.

Use all intersections $X \cap F_v$ of a quasi-Flat $X$ and its associated Flat $F_v$ to replace all blocks meeting such intersections at least twice. This recovers all lines of $A$ inside the Flats in (I) or (II).

Thus, we have recovered all lines of $A$, and then also $AΓL(V)$, so $\text{Aut} D$ is induced by a subgroup of $AΓL(V)_0 = ΓL(V)$. Recover the field $F$ exactly as in the proof of Theorem 1.1(i). Once again, $\text{Aut} D$ is a subgroup of $ΓL(F)$ that induces $\text{Aut} Γ \cong G$ on the collection of Flats in (I). Finally, by repeating the argument at the end of the proof of Theorem 1.1(i) we find that $\text{Aut} D \cong G$. □

Section 7 contains further properties of $D$.

6. A SIMPLER AFFINE CONSTRUCTION

Consider a plane $X$ of $A = AG(3, q), q > 2$; we identify $A$ with $AG_1(3, q)$. Let $π$ be any permutation of the points of $X$. Define a geometry $A_π$ as follows:

- the set $V$ of points is the set of points of $A$, and
- blocks are of two sorts:
  - the lines of $A$ not in $X$, and
  - the sets $L^π$ for lines $L \subset X$.

Once again it is trivial to see that $A_π$ is a design having the same parameters as $A$. We need the fact that $A_π$ is a resolvable design: each parallel class $\mathcal{C}$ of lines of $A$ produces a parallel class $(\mathcal{C} - (\mathcal{C} \cap X)) \cup (\mathcal{C} \cap X)^π$ of blocks of $A_π$ (where $\mathcal{C} \cap X = \{B \in \mathcal{C} | B \subset X\}$).
Note that \( X \) is an affine hyperplane of \( A_\pi \): a subspace which, if it meets a line in a single point, meets every line in the corresponding parallel class in a point.

As in Section 3 we have an affine hyperplane \( X \) of \( A_\pi \) such that the blocks of \( A_\pi \) not in \( X \) are lines of an affine space for which \( V \) is the set of points. Once again, the lines of this affine space can be recovered from \( A_\pi \) and \( X \) in a natural manner. Namely, take 2 intersecting blocks each meeting \( X \) in a point, use blocks to join points not in \( X \) on one of these blocks to the points of the other block, and iterate, in order to obtain \( q^2 + q - 1 \) blocks and \( q^2 \) points of a plane \( E \) of \( A \) (this process did not produce the line \( X \cap E \)). Now include each missing line in order to obtain all lines of the original affine space \( A \).

**Proposition 6.1.** The designs \( A_\pi \) and \( A_{\pi'} \) are isomorphic by an isomorphism sending \( X \) to itself if and only if \( \pi \) and \( \pi' \) are in the same \( \text{AGL}(X) \), \( \text{AGL}(X) \) double coset in \( \text{Sym}(X) \).

Moreover, the pointwise stabilizer of \( X \) in \( \text{Aut}A_\pi \) is transitive on the points outside of \( X \), and \( \text{Aut}A_\pi \) induces \( \text{AGL}(X) \cap \text{AGL}(X) \) on \( X \).

**Proof.** This is the same as for Proposition [3.1] \( \square \)

**Proposition 6.2.** For any \( q \geq 7 \) there are at least 2 designs having the parameters of \( A = \text{AG}_1(3,q) \), not isomorphic to one another or to \( A \), such that the automorphism group of each fixes a point.

**Proof.** Choose intersecting lines \( L_1, L_2 \) of a plane \( X \) of \( A \), and distinct points \( x_1, x_2 \in X - (L_1 \cup L_2) \). Let \( \pi_i \in \text{Sym}(X) \) denote a \((2q - 1 + i)\)-cycle on \( L_1 \cup L_2 \cup \{x_1, x_2\} \) such that \( L_1^{\pi_i} \neq L_2 \) and \( L_1^{\pi_i} \neq L_1 \) for \( i = 1, 2 \). We will show that the designs \( A_{\pi_i}, i = 1, 2 \), behave as stated. A crucial property of \( \pi_i \) is that \((*) \) there is no line \( M \) of \( X \) such that \( M^{\pi_i} \) is also a line of \( X \). Recall that we have reconstructed \( A \) from \( A_{\pi_i} \), so that any isomorphism \( \text{Aut}A_{\pi_i} \to \text{Aut}A_{\pi_j} \) is in \( \text{Aut}A \).

The crux of the proof of the proposition is the following claim: there is no \( g \in \text{Aut}A_{\pi_i} \) moving \( X \). For if there is such a \( g \) then \( Y = X^g \) is another affine hyperplane of \( A_{\pi_i} \). If \( X \) and \( Y \) meet then the blocks of \( X \) parallel to the blocks of \( X \) on a point of \( X \cap Y \) all meet \( Y \), so \( X \cap Y \) is a block inside \( X \) and so has the form \( M^{\pi_i} \) for some line \( M \) of \( X \). All blocks in \( Y \) not contained in \( X \) are lines of \( A \); by considering the blocks in \( Y \) through pairs of points of \( X \cap Y \) we see that \( Y \) is the set of points of a plane of \( A \). Then \( X \cap Y \) is a line of \( A \), which contradicts \((*)\) since \( X \cap Y = M^{\pi_i} \). Thus, \( X \) and \( Y \) do not meet.

Now \( Y \) is a subspace of \( A_{\pi_i} \), whose blocks are not in \( X \) and hence are lines of \( A \). It follows that \( Y \) is a plane of \( A \) parallel to \( X \).

Let \( E \) be a plane of \( A \) not parallel to \( X \). Then \( E \cap X \) and \((E \cap X)^g = E^g \cap Y \) are lines of \( A \). The block of \( A_{\pi_i} \) through 2 points of \( E^g \cap Y \) is a line of \( A \) and hence is \( E^g \cap Y \). Then \( E \cap X = (E^g \cap Y)^{g^{-1}} \) is also a block of \( A_{\pi_i} \) and a line of \( A \). Since \( E \cap X = L_{\pi_i} \) for some line \( L \) of \( X \), this contradicts \((*)\).

For some \( i, j \) suppose that \( g \) is an isomorphism \( A_{\pi_i} \to A_{\pi_j} \). Then \( g \in \text{Aut}A \) and \((\text{Aut}A_{\pi_i})^g = \text{Aut}A_{\pi_j} \) fixes \( X^g \). By Proposition 6.1, \( \text{Aut}A_{\pi_j} = (\text{Aut}A_{\pi_i})_{X} \) moves every block other than \( X \). Thus, \( g \) fixes \( X \). Since \( g \in \text{Aut}A \) it induces a collineation \( \bar{g} \) of \( X \). As usual, \( \pi_i \bar{g} = h \pi_j \) with \( \bar{g}, h \in \text{AGL}(X) \). Then \( \bar{g}^{-1} h = \pi_i^\circ \pi_j^{-1} \) is a collineation of \( X \) that fixes at least \( q^2 - (2q - 1 + i) - (2q - 1 + j) > q \) points as \( q \geq 7 \). Then \( \bar{g} = h \) and \( \pi_i^\circ = \pi_j \). Since \( \pi_i \) and \( \pi_j \) are not conjugate permutations, \( i \neq j \) and \( \bar{g} \) commutes with \( \pi_i \). Then the collineation \( \bar{g} \) fixes \( L_1 \cup L_2 \cup \{x_1, x_i\} \) and hence also \( L_1 \cup L_2 \), as required. \( \square \)
7. Concluding remarks

Remark 7.1. When considering possible consequences of this paper it became clear that additional properties of $D$ should also be mentioned.

(1) Additional properties of the design $D$ in Theorem 1.1(i).
   (a) PG$(3, q)$-connectedness. The following graph is connected: the vertices are the subspaces of $D$ isomorphic to PG$(3, q)$, with two joined when they meet.
   (b) PG$(n-1, q)$ generation. $D$ is generated by its PG$(n-1, q)$ subspaces.
   (c) Every point of $D$ is in a subspace isomorphic to PG$(n-1, q)$ (in fact, many of these).

(2) Additional properties of the design $D$ in Theorem 1.1(ii).
   (a) AG$(3, q)$-connectedness. The following graph is connected: the vertices are the subspaces of $D$ isomorphic to AG$(3, q)$, with two joined when they meet.
   (b) AG$(n, q)$ generation. $D$ is generated by its subspaces isomorphic to AG$(n, q)$.
   (c) Every point of $D$ is in a subspace isomorphic to AG$(n, q)$ (in fact, many of these).

These reflect the fact that the flats in (I) and (II) cover a tiny portion of the underlying projective or affine space. For (1b) we give examples of subspaces of the $K$-space $V$: $(v_1 + \theta v_2, v_1 + v_3 + \theta v_4, \ldots, v_{n-2} + v_{n-1} + \theta v_n, v_1 + v_2 + v_3 + v_4, \theta(v_1 + v_4 + v_5 + v_6))$ for $0 \leq i < q^4 - 1$. These miss all flats in (I), have possibly a point in common with only one possible flat in (II), and their span contains $F v_n$. Now permute the subscripts. Part (1c) holds by using the first of the above $K$-subspaces, and that subspace with $\theta v_2$ replacing $v_1 + \theta v_2$. There are also projective spaces of larger dimension that are subdesigns of $D$.

Remark 7.2. We used $[F: K] = 4$ in the proof of Theorem 1.1(i) and $[F: K] = 3$ in the proof of Theorem 1.1(ii). Larger degree field extensions do not seem to work in our arguments: in the proof of Proposition 3.4 we used the fact that $X \cap Y$ is a subspace of two subspaces, each of which is a projective plane, so that $X \cap Y$ is both a block of $P_\pi$ and a line of $P$. In larger dimensions $X \cap Y$ is still a subspace of both $P_\pi$ and $P$, but this seems to provide no information about its blocks.

Remark 7.3. Each of our designs has the same parameters as some PG$_1(V)$ or AG$_1(V)$. What is needed is a much better type of result, such as: for each finite group $G$ there is an integer $f(|G|)$ such that, if $q$ is a prime power and if $v > f(|G|)$ satisfies the necessary conditions for the existence of a 2-$(v, q + 1, 1)$-design, then there is such a design $D$ for which Aut$D \cong G$.

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