Some new Hardy-Hilbert-type inequalities with multiparameters

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Abstract: The purpose of this paper is to build some new Hardy-Hilbert-type inequalities with multiparameters and their equivalent forms and variants, which generalize some existing results. Similarly, the corresponding Hardy-Hilbert-type integral inequalities are also given.

Keywords: Hardy–Hilbert-type inequality; Hölder’s inequality; Beta function

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1. Introduction

Let $a_n, b_n \geq 0, p > 1, 1/p + 1/q = 1$. If $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left[ \sum_{n=1}^{\infty} a_n^p \right]^{1/p} \left[ \sum_{n=1}^{\infty} b_n^q \right]^{1/q},$$

(1.1)

where the constant $\pi/\sin(\pi/p)$ is the best possible. The inequality (1.1) can be called as the well known Hardy-Hilbert’s inequality [1]. An equivalent form of inequality (1.1) is presented as follows.

$$\sum_{n=1}^{\infty} \left[ \sum_{m=1}^{\infty} \frac{a_m}{m+n} \right]^p < \left[ \frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=1}^{\infty} a_n^p,$$

(1.2)

where the constant $[\pi/\sin(\pi/p)]^p$ is also the best possible. In connection with applications in analysis, their generalizations and variants have received considerable interest recent years [2–12]. By introducing some parameters, Yang [13] obtained a generalization of Hardy-Hilbert’s integral inequality with a best constant factor that involves the beta function. In the paper [14], Das and Sahoo considered a generalization of multiple Hardy-Hilbert’s inequality with the best constant factor. Sroysang [15] established a generalization on the kinds of Hardy-Hilbert’s integral inequality with the weight homogeneous function.
By introducing a parameter, the following extension of (1.1) was obtained by Yang [3]. Let \( a_n, b_n \geq 0, p > 1, 1/p + 1/q = 1 \). If \( 0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p \), then

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^p + n^q} < \frac{\pi}{\lambda \sin(\pi/p)} \left( \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p \right)^{1/p} \left( \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q \right)^{1/q},
\]  

(1.3)

where the constant \( \pi/\lambda \sin(\pi/p) \) is the best possible.

Nextly, Sun [8] gave an extension of (1.3) as follows. Let \( a_n, b_n \geq 0, p > 1, 1/p + 1/q = 1, 0 < \lambda < \min(p, q), c > 0 \). If \( 0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p \), then

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m^p + n^q)^{1/2}} < c_{\lambda, p} \left[ \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p \right]^{1/p} \left[ \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q \right]^{1/q},
\]  

(1.4)

where the constant \( c_{\lambda, p} = (1/c)B(\lambda/(c p), \lambda/(c q)) \) is the best possible, \( B(\bullet, \bullet) \) denotes the beta function.

Finally, by introducing two parameters, Xu [9] presented another extension of (1.3) as follows. Let \( a_n, b_n \geq 0, p > 1, 1/p + 1/q = 1, 0 < \lambda_1 < \lambda_2 < q \). If \( 0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda_1)} a_n^p \), then

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^{1/2} + n^{1/2}} < \frac{\pi}{\lambda_1^{1/2} \lambda_2^{1/2} \sin(\pi/p)} \left[ \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda_1)} a_n^p \right]^{1/p} \left[ \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda_2)} b_n^q \right]^{1/q},
\]  

(1.5)

where the constant \( \pi/(\lambda_1^{1/2} \lambda_2^{1/2} \sin(\pi/p)) \) is the best possible.

Recently, some Hilbert-type integral inequalities with quasi-homogeneous integral kernels and multiple functions were established [16]. By means of the technique of real analysis and the weight functions, Xin [17] obtained an equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane and the best constant factor related to the beta function. By using the weight function and the technique of real analysis, Liu [18, 19] established some multi-parameter Hilbert-type integral inequalities with the hybrid kernel and the best constant factors, respectively. By using weight functions and introducing parameters, Chen and Yang [20] presented a reverse Hardy-Hilbert-type integral inequality involving one derivative function and the beta function. Based on the theory of operators, Hong et al. [21] obtained a necessary and sufficient condition and the best constant factor for the Hilbert-type multiple integral inequality with kernel. Liao et al. [22] investigated a new half-discrete Hilbert-type inequality involving the variable upper limit integral and partial sums and proved the equivalent conditions of the best possible constant factor related to several parameters.

In the paper, motivated by the mentioned references above, we will obtain a Hardy-Hilbert-type inequality with multiparameters, which can be see as a new generalization of (1.3)–(1.5). And its equivalent form and variant are given. Furthermore, their integral forms are also presented.

2. New Hardy-Hilbert-type inequalities with multiparameters for double series

For convenience, we assume always that \( B(\bullet, \bullet) \) represents the beta function throughout the paper.
**Theorem 2.1.** Let $a_n, b_n \geq 0$, and $p > 1$, $1/p + 1/q = 1$, $0 < \lambda_1 < p$, $0 < \lambda_2 < q$, $0 < c \leq \lambda$. If $0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\frac{\lambda_1}{\lambda_2})} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\frac{\lambda_2}{\lambda_1})} b_n^q < \infty$, then

$$
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_n}{(m^{\lambda_1} + n^{\lambda_2})^\tau} < c_{\lambda, p} \left[ \sum_{n=1}^{\infty} n^{(p-1)(1-\frac{\lambda_1}{\lambda_2})} a_n^p \right]^{1/p} \left[ \sum_{n=1}^{\infty} n^{(q-1)(1-\frac{\lambda_2}{\lambda_1})} b_n^q \right]^{1/q},
$$

where the constant $c_{\lambda, p} = (1/\lambda_1^{(1/q)} \lambda_2^{1/p})B(c/(\lambda p), c/(\lambda q))$ is the best possible.

**Remark 2.1.** The above inequality presented in (2.1) concretely produces some known Hardy-Hilbert type inequalities based on different settings of the parameters $\lambda_1, \lambda_2$, and $\lambda$.

(A1) If $\lambda_1 = \lambda_2 = \lambda = c$, then inequality (2.1) reduces to the inequality (1.3) given in Yang [3].

(A2) If $\lambda_1 = \lambda_2 = \lambda$, then inequality (2.1) develops into the inequality (1.4) presented in Sun [8].

(A3) If $\lambda = c$, then inequality (2.1) converts into the inequality (1.5) given in Xu [9].

The idea of proof of Theorem 2.1 comes similarly from [3] and [9]. To prove the Theorem 2.1, we need some lemmas in the same way, which are new generalizations of some lemmas given in [8, 9].

**Lemma 2.1.** Let $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2 > 0$, $0 < c \leq \lambda$. Define the weight function $\omega_{\lambda_1, \lambda_2}(y, p) = \int_{0}^{\infty} \left(1/(x^{\lambda_1} + y^{\lambda_2}c^{1 \lambda_2}) \right)^{1/(\lambda p)} x^{c-1}dx$, $y \in (0, \infty)$ and weight coefficient $\tilde{\omega}_{\lambda_1, \lambda_2}(n, p) = \sum_{m=1}^{\infty} \left(1/(m^{\lambda_1} + n^{\lambda_2}c^{1 \lambda_2}) \right)^{1/(\lambda p)} (n^{\lambda_2}/m^{\lambda_1}c^{1 \lambda_2}) m^{\lambda_1/\lambda_2 - 1}$.

Then (a) $\omega_{\lambda_1, \lambda_2}(y, p) = (1/\lambda_1)B(c/(\lambda_1 p), c/(\lambda_2 q))$, and (b) If $0 < \lambda_1 < p$, and $n \in N$, then $\tilde{\omega}_{\lambda_1, \lambda_2}(n, p) < \omega_{\lambda_1, \lambda_2}(n, p)$ holds.

**Proof.** (a) Set $t = x^{\lambda_1}/y^{\lambda_2}$. Then

$$
\omega_{\lambda_1, \lambda_2}(y, p) = \int_{0}^{\infty} \frac{1}{(x^{\lambda_1} + y^{\lambda_2}c^{1 \lambda_2})} \left(1/(x^{\lambda_1} + y^{\lambda_2}c^{1 \lambda_2}) \right)^{c-1}dx = \frac{1}{\lambda_1} \int_{0}^{\infty} \frac{1}{(1 + t)^{\tau}} dt = \frac{1}{\lambda_1} B\left(\frac{c}{\lambda p}, \frac{c}{\lambda_2 q}\right).
$$

(b) Since $1 - (c\lambda_1/\lambda) + (c\lambda_2/\lambda_2) > 0$, fixed $\lambda_1, \lambda_2$, and $n \in N$, the function $f(x) = (1/(x^{\lambda_1} + n^{\lambda_2}c^{1 \lambda_2})(n^{\lambda_2}/m^{\lambda_1}c^{1 \lambda_2}) m^{\lambda_1/\lambda_2 - 1}$ is strict dececing in $(0, \infty)$. Then

$$
f(m) < \int_{m-1}^{m} f(x)dx \Rightarrow \sum_{m=1}^{\infty} f(m) < \int_{0}^{\infty} f(x)dx.
$$

So we have $\tilde{\omega}_{\lambda_1, \lambda_2}(n, p) \leq \omega_{\lambda_1, \lambda_2}(n, p)$. This completes the proof of Lemma 2.1.

Similar to Lemma 2.1, we can introduce the following lemma.

**Lemma 2.2.** Let $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2 > 0$, $0 < c \leq \lambda$. Define the weight function $\tilde{\omega}_{\lambda_1, \lambda_2}(x, p) = \int_{0}^{\infty} \left(1/(x^{\lambda_1} + y^{\lambda_2}c^{1 \lambda_2})(x^{\lambda_1}/y^{\lambda_2}c^{1 \lambda_2}) \right)^{c-1}dy$, $x \in (0, \infty)$ and weight coefficient $\tilde{\omega}_{\lambda_1, \lambda_2}(m, p) = \sum_{n=1}^{\infty} \left(1/(m^{\lambda_1} + n^{\lambda_2}c^{1 \lambda_2}) \right)^{1/(\lambda p)} (m^{\lambda_1}/n^{\lambda_2}c^{1 \lambda_2}) m^{\lambda_1/\lambda_2 - 1}$.

Then (a) $\tilde{\omega}_{\lambda_1, \lambda_2}(x, p) = (1/\lambda_1)B(c/(\lambda_1 p), c/(\lambda_2 q))$, and (b) If $0 < \lambda_2 < q$, and $m \in N$, then $\tilde{\omega}_{\lambda_1, \lambda_2}(m, p) < \tilde{\omega}_{\lambda_1, \lambda_2}(m, p)$ holds.

**Lemma 2.3.** Let $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2 > 0$, $0 < c \leq \lambda$, $\varepsilon \in (0, q/(2p))$. Then

$$
\int_{1}^{\infty} \frac{1}{x^{1+\frac{\lambda_1}{\lambda}} \varepsilon} \int_{0}^{\frac{1}{\varepsilon}} \frac{1}{(1+u)^{\tau}} u^{\varepsilon(1-\frac{\lambda}{p})-1} du dx = O(1), \quad \varepsilon \to 0^+.
$$

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Proof. It follows from $\varepsilon \in (0, q/(2p))$ that $1/p - \varepsilon/q > 1/(2p)$. Then we have

$$0 < \int_1^\infty \frac{1}{x^{1+\varepsilon/p}} \int_0^{x^{1/p}} \frac{1}{(1+u)^{2}} u^{1/(1-\varepsilon/p)-1} du dx < \int_1^\infty \frac{1}{x} \int_0^{x^{1/p}} \frac{1}{(1+u)^{2}} u^{\varepsilon/(1-\varepsilon/p)-1} du dx$$

which implies (2.2). This finishes the proof of Lemma 2.3.

Nextly, we will give the proof of Theorem 2.1.

Proof. By using the Hölder’s inequality and Lemmas 2.1 and 2.2, we can capture

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m^{1/n} + n^{1/m})^2} \leq \sum_{n=1}^\infty \sum_{m=1}^\infty \left[ \frac{a_m}{(m^{1/n} + n^{1/m})^{2/q}} \left( \frac{m^{1/n}}{n^{1/q}} \right)^{1/p} \right]^{1/q} \left[ \frac{b_n}{(m^{1/n} + n^{1/m})^{2/q}} \left( \frac{n^{1/m}}{m^{1/q}} \right)^{1/p} \right]^{1/q}$$

$$= \left[ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{(m^{1/n} + n^{1/m})^2} \left( \frac{m^{1/n}}{n^{1/q}} \right) \left( \frac{n^{1/m}}{m^{1/q}} \right) \right]^{1/q} \left[ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{(m^{1/n} + n^{1/m})^2} \right]^{1/q}$$

which implies (2.1).

Now we prove that the constant $c_{\alpha, p}$ is the best possible. Suppose that the constant $c_{\alpha, p}$ in inequality (2.1) is not the best possible, then there exists positive $k < c_{\alpha, p}$ such that $k$ replaces with $c_{\alpha, p}$ in inequality (2.1) holds all the same. Especially, for $\varepsilon \in (0, q/(2p))$, let $\hat{a}_n = (1/n)^{1/(1+\varepsilon/p)}$ and $\hat{b}_n = (1/n)^{1/(1+\varepsilon/q)}$ for $n \in N$, then we obtain

$$\sum_{n=1}^\infty n^{p(1-\varepsilon/q)} \hat{a}_n^{1/p} \leq \sum_{n=1}^\infty n^{1+\varepsilon/q}$$

On the other hand, we have

$$\frac{1}{\sigma} = \int_1^{\infty} \frac{1}{x^{1+\sigma}} dx < \sum_{n=1}^\infty \frac{1}{n^{1+\sigma}} = 1 + \sum_{n=2}^\infty \frac{1}{n^{1+\sigma}} < 1 + \int_1^{\infty} \frac{1}{x^{1+\sigma}} dx = 1 + \frac{1}{\sigma}(\sigma > 0).$$
It follows from (2.4) and (2.5) that we observe
\[
\left[ \sum_{n=1}^{\infty} n^{p-1} (1 - \frac{1}{n^2}) u_n \right]^{1/p} \left[ \sum_{n=1}^{\infty} n^{q-1} (1 - \frac{1}{n^2}) \hat{b}_n^q \right]^{1/q} = \left[ \frac{\lambda}{c\lambda_1} + O(1) \right]^{1/p} \left[ \frac{\lambda}{c\lambda_2} + O(1) \right]^{1/q} = \frac{\lambda}{c\lambda_1} (1 + o(1)) \left[ \frac{\lambda}{c\lambda_2} (1 + o(1)) \right]^{1/q} (1 + o(1)), \tag{2.6}
\]
as \varepsilon \to 0^+$. According to the assumption, we have
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\hat{a}_m \hat{b}_n}{(m^{41} + n^{42})^{25}} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m^{41} + n^{42})^2} \left( \frac{1}{m} \right) \left( \frac{1}{n} \right) \frac{1}{q} \frac{1}{p} \frac{1}{1+\frac{1}{p} - \frac{1}{q} - 1} dx dy
\]
\[
= \frac{1}{\lambda_2} \int_{1}^{\infty} \frac{1}{x^{1+\epsilon q}} \int_{0}^{\infty} \frac{1}{1+u} \int_{1}^{\infty} \frac{1}{1+u} \int_{0}^{\frac{1}{x^{1+\epsilon q}}} \frac{u^{1+\epsilon q}}{1+u} du dx dy
\]
where in the third equality we set that $u = y^{1/2} x^{1/2}$. At the same time, when \(\varepsilon\) tends to $0^+$, we have
\[
\lim_{\varepsilon \to 0^+} B(\frac{c}{\lambda q}, \frac{c}{\lambda p}) = B(\frac{c}{\lambda q}, \frac{c}{\lambda p}),
\]
\[
\int_{0}^{\infty} \frac{1}{1+u} u^{1+\epsilon q} du = B\left(\frac{c}{\lambda q}, \frac{c}{\lambda p}\right) = B\left(\frac{c}{\lambda q}, \frac{c}{\lambda p}\right) + o(1),
\]
which implies that if \(\varepsilon\) tends to $0^+$, the following equations holds true
\[
\frac{1}{\lambda_2} \int_{1}^{\infty} \frac{1}{x^{1+\epsilon q}} \int_{0}^{\infty} \frac{1}{1+u} \int_{1}^{\frac{1}{x^{1+\epsilon q}}} \frac{u^{1+\epsilon q}}{1+u} du dx dy = \frac{\lambda}{c\lambda_1 \lambda_2} B\left(\frac{c}{\lambda p}, \frac{c}{\lambda q}\right) + o(1).
\]
It follows from Lemma 2.3 that when \(\varepsilon\) tends to $0^+$, we get
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\hat{a}_m \hat{b}_n) \frac{1}{(m^{41} + n^{42})^{25}} = \frac{\lambda}{c\lambda_1 \lambda_2} B\left(\frac{c}{\lambda p}, \frac{c}{\lambda q}\right) + o(1) - O(1)
\]
\[
= \frac{\lambda}{c\lambda_1 \lambda_2} \frac{1}{c\lambda_1 \lambda_2} \left( \frac{1}{1+o(1)} \right) = \frac{\lambda}{c\lambda_1 \lambda_2} \frac{1}{c\lambda_1 \lambda_2} [1 + o(1)]. \tag{2.7}
\]
Due to the equations (2.6) and (2.7), we can obtain
\[
\frac{\lambda}{c\lambda_1^{1/p} \lambda_2^{1/q} e} c_{\lambda,p} [1 + o(1)] < \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\hat{a}_m \hat{b}_n}{(m^{41} + n^{42})^{25}} < \frac{\lambda}{c\lambda_1^{1/p} \lambda_2^{1/q} e} k(1 + o(1)),
\]
as \(\varepsilon\) tends to $0^+$. We have $c_{\lambda,p} < k$, in contradiction with supposition. So the constant $c_{\lambda,p}$ in inequality (2.1) is the best possible. This finishes the proof of Theorem 2.1. \(\square\)
Theorem 2.2. Let $a_m \geq 0$, and $p > 1$, $1/p + 1/q = 1$, $0 < \lambda_1 < p$, $0 < \lambda_2 < q$, $0 < c \leq \lambda$. $c_{\lambda,p} = (1/\lambda_1^{1/q} \lambda_2^{1/p}) B(c/\lambda p), c/(\lambda q))$. If $0 < \sum_{m=1}^{\infty} m^{(p-1)(1-\lambda_1)} a_m^p < \infty$, then

$$0 < \sum_{n=1}^{\infty} n^{c_{\lambda,p}} \left( \sum_{m=1}^{\infty} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^\frac{\lambda}{2}} \right)^p < c_{\lambda,p} \sum_{m=1}^{\infty} m^{(p-1)(1-\lambda_1)} a_m^p, \quad (2.8)$$

where the constant $c_{\lambda,p}$ is the best possible. Moreover, inequality (2.8) is equivalent to inequality (2.1).

Proof. There exists $k_0 \in N$, such that $\sum_{m=1}^{k} (a_m/(m^{\lambda_1} + n^{\lambda_2})^{\lambda/2}) > 0$ for $k > k_0$. Setting $b_n(k) = n^{c_{\lambda,p}-1} \sum_{m=1}^{k} (a_m/(m^{\lambda_1} + n^{\lambda_2})^{\lambda/2})^{p-1}$ for $k > k_0$. Thanks to (2.1), then we get

$$0 < \sum_{n=1}^{k} n^{(q-1)(1-c_{\lambda,p})} b_n^q(k) = \sum_{n=1}^{k} n^{c_{\lambda,p}-1} \left( \sum_{m=1}^{k} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^{\lambda/2}} \right)^p = \sum_{n=1}^{k} \sum_{m=1}^{k} \frac{a_m b_n(k)}{(m^{\lambda_1} + n^{\lambda_2})^{\lambda/2}} < c_{\lambda,p} \left[ \sum_{n=1}^{k} n^{(q-1)(1-c_{\lambda,p})} a_n^p \right]^{1/p} \left[ \sum_{n=1}^{k} n^{(q-1)(1-c_{\lambda,p})} b_n^q(k) \right]^{1/q},$$

which imply the following inequality

$$\sum_{n=1}^{k} n^{(q-1)(1-c_{\lambda,p})} b_n^q(k) < c_{\lambda,p} \sum_{n=1}^{k} n^{(p-1)(1-c_{\lambda,p})} a_n^p.$$ 

Letting $k$ tends to $\infty$, we have $0 < \sum_{n=1}^{\infty} n^{(p-1)(1-c_{\lambda,p})} a_n^p < \infty$. Then we obtain

$$0 < \sum_{n=1}^{\infty} n^{c_{\lambda,p}-1} \left( \sum_{m=1}^{\infty} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^{\lambda/2}} \right)^p < c_{\lambda,p} \sum_{m=1}^{\infty} m^{(p-1)(1-\lambda_1)} a_m^p.$$ 

Therefore, inequality (2.8) holds from inequality (2.1).

On the other hand, if inequality (2.8) holds, by applying the Hölder’s inequality, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m^{\lambda_1} + n^{\lambda_2})^{\lambda/2}} = \sum_{n=1}^{\infty} n^{c_{\lambda,p}-1} \sum_{m=1}^{\infty} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^{\lambda/2}} \left[ n^{1-c_{\lambda,p}} b_n \right]^{1/p} \left[ \sum_{n=1}^{\infty} n^{(q-1)(1-c_{\lambda,p})} b_n^q \right]^{1/q} < c_{\lambda,p} \left[ \sum_{n=1}^{\infty} n^{(p-1)(1-c_{\lambda,p})} a_n^p \right]^{1/p} \left[ \sum_{n=1}^{\infty} n^{(q-1)(1-c_{\lambda,p})} b_n^q \right]^{1/q},$$

which means that inequality (2.1) holds. Since the constant $c_{\lambda,p}$ in inequality (2.1) is the best possible, and inequality (2.1) is the equivalent to inequality (2.8), the constant $c_{\lambda,p}$ in inequality (2.8) is also the best possible. This completes the proof of Theorem 2.2. □

Now we present a variant of Theorem 2.1 as follows.
Theorem 2.3. Let \( a_m, b_n \geq 0 \), and \( p > 1 \), \( 1/p + 1/q = 1 \), \( 0 < \lambda_1 < p \), \( 0 < \lambda_2 < q \), \( 0 < c \leq \lambda \). If
\[
0 < \sum_{n=2}^{\infty} (\ln n^{1/(p-1)(1-\frac{1}{c})n^{p-1}} d_n^p < \infty, \quad \text{and} \quad 0 < \sum_{n=2}^{\infty} (\ln n^{1/(q-1)(1-\frac{1}{c})q^{q-1}} b_n^q < \infty, \}
\]
then
\[
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{(\ln m^{1/(n^{1/(c-1)})})^\gamma} < c_{1,p} \left[ \sum_{n=2}^{\infty} (\ln n^{1/(p-1)(1-\frac{1}{c})n^{p-1}} d_n^p \right]^{1/p} \left[ \sum_{n=2}^{\infty} (\ln n^{1/(q-1)(1-\frac{1}{c})q^{q-1}} b_n^q \right]^{1/q},
\]
where the constant \( c_{1,p} = (1/\lambda_1^{1/(1-p)} \lambda_2^{1/(p-q)} B(c/(\lambda p), c/(\lambda q)) \) is the best possible.

Remark 2.2. The above inequality presented in (2.9) concretely produces some known Hardy-Hilbert type inequalities based on different settings of the parameters \( \lambda_1, \lambda_2 \) and \( \lambda \).

(B1) If \( \lambda_1 = \lambda_2 = \lambda = c = 1 \), Theorem 2.3 reduces to Theorem 2.1 given by Yang [4].

(B2) If \( \lambda = c \), then Theorem 2.3 is converted into Theorem 4 obtained by Xu [9].

(B3) If \( \lambda_1 = \lambda_2 = \lambda = c = 1 \) and \( a_m, b_n \) are replaced by \( a_m/m, b_n/n \), respectively, then the inequality (2.9) is developed into the Muholland’s inequality presented in [10].

When \( a_m, b_n \) are replaced by \( a_m/m, b_n/n \) in inequality (2.9), respectively, it is easy to obtain that inequality (2.9) is equivalent to the following new variant of Muholland’s inequality.

\[
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{mn(\ln m^{1/(n^{1/(c-1)})})^\gamma} < c_{1,p} \left[ \sum_{n=2}^{\infty} (\ln n^{1/(p-1)(1-\frac{1}{c})n^{p-1}} d_n^p \right]^{1/p} \left[ \sum_{n=2}^{\infty} (\ln n^{1/(q-1)(1-\frac{1}{c})q^{q-1}} b_n^q \right]^{1/q},
\]
where \( 0 < \sum_{n=2}^{\infty} (\ln n^{1/(p-1)(1-\frac{1}{c})n^{p-1}} d_n^p < \infty, \quad 0 < \sum_{n=2}^{\infty} (\ln n^{1/(q-1)(1-\frac{1}{c})q^{q-1}} b_n^q < \infty, \)
and the constant \( c_{1,p} = (1/\lambda_1^{1/(1-p)} \lambda_2^{1/(p-q)} B(c/(\lambda p), c/(\lambda q)) \) is also the best possible.

The method of proof of Theorem 2.3 comes from [4, 9]. To prove Theorem 2.3, we need the following two lemmas, which are new generalizations of Lemma 3 and Lemma 4 in [9].

Lemma 2.4. Let \( p > 1 \), \( 1/p + 1/q = 1 \), \( \lambda_1, \lambda_2 > 0 \), \( 0 < c \leq \lambda \). Define the weight function \( \omega'_{\lambda_1, \lambda_2}(y, p) = \int_1^{\infty} (1/(\ln x^{1/(y^{1/(c-1)})}))^\gamma (\ln x^{1/(y^{1/(c-1)})})^{\gamma-1} x^{-1} dx, \ y \in (0, \infty), \) and weight coefficient \( \tilde{\omega'}_{\lambda_1, \lambda_2}(n, p) = \sum_{n=2}^{\infty} (1/(\ln n^{1/(n^{1/(c-1)})}))^\gamma (\ln n^{1/(n^{1/(c-1)})})^{\gamma-1} n^{-1} \). Then (a) \( \omega'_{\lambda_1, \lambda_2}(y, p) = (1/\lambda_1) B(c/(\lambda p), c/(\lambda q)) \), and (b) if \( n \in N \) and \( n \geq 2 \), then \( \omega'_{\lambda_1, \lambda_2}(n, p) \leq \omega'_{\lambda_1, \lambda_2}(n, p) \) holds.

Proof. (a) Set \( t = \ln x^{1/(\ln y^{1/(c-1)})} \). Then
\[
\omega'_{\lambda_1, \lambda_2}(y, p) = \int_1^{\infty} \frac{1}{(\ln x^{1/(\ln y^{1/(c-1)})})^\gamma} (\ln x^{1/(\ln y^{1/(c-1)})})^{\gamma-1} x^{-1} dx = \frac{1}{\lambda_1} \int_0^{\infty} \frac{1}{(1+t)^\gamma} t^{\gamma-1} dt = \frac{1}{\lambda_1} B(c/(\lambda p), c/(\lambda q)).
\]

(b) For fixed \( \lambda_1, \lambda_2 \), and \( 2 \leq n \in N \), the function \( f(x) = (1/(\ln x^{1/(n^{1/(c-1)})}))^\gamma (\ln x^{1/(n^{1/(c-1)})})^{\gamma-1} x^{-1} \) is strictly decreasing in \((1, \infty)\). Then \( f(m) < \int_m^{\infty} f(x) dx \). Furthermore, we can obtain \( \sum_{m=1}^{\infty} f(m) < \int_0^{\infty} f(x) dx \). So we have \( \omega'_{\lambda_1, \lambda_2}(n, s) < \omega'_{\lambda_1, \lambda_2}(n, s) \). \( \square \)

Similar to Lemma 2.4, we can get the following lemma.

Lemma 2.5. Let \( p > 1 \), \( 1/p + 1/q = 1 \), \( \lambda_1, \lambda_2 > 0 \), \( 0 < c \leq \lambda \). Define the weight function \( \omega'_{\lambda_1, \lambda_2}(x, p) = \int_1^{\infty} (1/(\ln x^{1/(y^{1/(c-1)})}))^\gamma (\ln x^{1/(y^{1/(c-1)})})^{\gamma-1} y^{-1} dy, \ x \in (0, \infty), \) and weight coefficient \( \tilde{\omega'}_{\lambda_1, \lambda_2}(m, p) = \sum_{m=2}^{\infty} (1/(\ln m^{1/(n^{1/(c-1)})}))^\gamma (\ln m^{1/(n^{1/(c-1)})})^{\gamma-1} n^{-1} \). Then (a) \( \omega'_{\lambda_1, \lambda_2}(x, p) = (1/\lambda_2) B(c/(\lambda p), c/(\lambda q)) \), and (b) if \( m \in N \), and \( m \geq 2 \) then \( \omega'_{\lambda_1, \lambda_2}(m, p) < \omega'_{\lambda_1, \lambda_2}(m, p) \) holds.
Nextly, we will give the proof of Theorem 2.1.

**Proof.** By using the Hölder’s inequality and Lemmas 2.4 and 2.5, we have

\[
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{(\ln m^{1/4} n^{1/5})^2} = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \left[ \frac{a_m}{(\ln m^{1/4} n^{1/5})^2} \left( \frac{\ln m^{1/4}}{\ln n^{1/5}} \right)^{\frac{1}{p}} \left( \frac{\ln n^{1/5}}{\ln m^{1/4}} \right)^{\frac{1}{q}} \right] \left( \frac{\ln n^{1/5}}{\ln m^{1/4}} \right)^{\frac{1}{p} - 1} \left( \frac{\ln m^{1/4}}{\ln n^{1/5}} \right)^{\frac{1}{q} - 1} \frac{m^{1/2}}{n^{1/2}}
\]

\[
\leq \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \left[ \frac{b_n}{(\ln m^{1/4} n^{1/5})^2} \left( \frac{\ln m^{1/4}}{\ln n^{1/5}} \right)^{\frac{1}{p}} \left( \frac{\ln n^{1/5}}{\ln m^{1/4}} \right)^{\frac{1}{q}} \right] \left( \frac{\ln m^{1/4}}{\ln n^{1/5}} \right)^{\frac{1}{p} - 1} \left( \frac{\ln n^{1/5}}{\ln m^{1/4}} \right)^{\frac{1}{q} - 1} \frac{1}{n} \frac{1}{m}
\]

\[
= \left[ \sum_{n=2}^{\infty} \frac{1}{(\ln m^{1/4} n^{1/5})^2} \left( \frac{\ln m^{1/4}}{\ln n^{1/5}} \right)^{\frac{1}{p}} \left( \frac{\ln n^{1/5}}{\ln m^{1/4}} \right)^{\frac{1}{q}} \right] \left[ \sum_{n=2}^{\infty} \frac{1}{(\ln m^{1/4} n^{1/5})^2} \left( \frac{\ln m^{1/4}}{\ln n^{1/5}} \right)^{\frac{1}{p} - 1} \left( \frac{\ln n^{1/5}}{\ln m^{1/4}} \right)^{\frac{1}{q} - 1} \frac{1}{n} \frac{1}{m} \right]^{1/p} 
\]

\[
= c_{d,n} \left[ \sum_{n=2}^{\infty} (\ln m^{1/4})^{(p-1)(1/5)} m^{p-1} a_m \right]^{1/p} \left[ \sum_{n=2}^{\infty} (\ln n^{1/5})^{(q-1)(1/5)} n^{q-1} b_n \right]^{1/q},
\]

which implies (2.9). Now we prove that the constant \(c_{d,n}\) is the best possible. Suppose that the constant \(c_{d,n}\) in inequality (2.9) is not the best possible, then there exists positive \(k < c_{d,n}\), such that \(k\) replaces with \(c_{d,n}\), inequality (2.9) holds all the same. Especially, for \(\varepsilon \in (0, q/(2p))\), set \(\delta_n = 1/(n(\ln n^{1/4})^{(1-\varepsilon)/p+1+\varepsilon/(3p)})\) and \(\delta_n = 1/(n(\ln n^{1/5})^{(1-\varepsilon)/p+1+\varepsilon/(3p)})\) for \(n \in N\), then we have

\[
\sum_{n=2}^{\infty} (\ln n^{1/4})^{(p-1)(1/5)} n^{p-1} \delta_n^p = \sum_{n=2}^{\infty} \frac{1}{n(\ln n^{1/4})^{1+\varepsilon}} \quad \text{and} \quad \sum_{n=2}^{\infty} (\ln n^{1/5})^{(q-1)(1/5)} n^{q-1} \delta_n^q = \sum_{n=2}^{\infty} \frac{1}{n(\ln n^{1/5})^{1+\varepsilon}}.
\]

As \(\sigma\) tends to 0+, we have

\[
\frac{1}{\lambda^{1+\varepsilon}} = \int_\varepsilon^\infty \frac{1}{x(\ln x^{1+\varepsilon})} dx < \sum_{n=2}^{\infty} \frac{1}{n(\ln n^{1+\varepsilon})} = \frac{1}{2(\ln 2^{1+\varepsilon})} + \frac{1}{3(\ln 3^{1+\varepsilon})} + \sum_{n=4}^{\infty} \frac{1}{n(\ln n^{1+\varepsilon})} < O(1) + \int_\varepsilon^\infty \frac{1}{x(\ln x^{1+\varepsilon})} dx = O(1) + \frac{1}{\lambda^{1+\varepsilon}},
\]

which implies that as \(\varepsilon\) tends to 0+, we obtain
\[
\left[\sum_{n=2}^{\infty} (\ln n)^{p-1} (n^{-\frac{1}{2}} n_{p-1})^{\frac{1}{p}} \right] \left[\sum_{n=2}^{\infty} (\ln n)^{q-1} (n^{-\frac{1}{2}} n_{q-1})^{\frac{1}{q}} \right]^{1/\lambda} + O(1) = \left[\frac{1}{c_{\lambda_1}} \left(1 + e^{c_{\lambda_1}}\right)\right]^{1/\lambda} + O(1).
\]

From the above inequalities, we can obtain

\[
\text{where in the third equality we set that } u = e^{c_{\lambda_1}}.\]

From the assumptions, we have

\[
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{\ln(m^{\lambda_1} n^{\lambda_2})^2} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} = \sum_{n=2}^{\infty} \int_{e}^{\infty} \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy = \frac{1}{\lambda_2} \int_{e}^{\infty} \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy = \frac{1}{\lambda_2} \int_{e}^{\infty} \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy.
\]

where in the third equality we set that \( u = \ln y^{\lambda_2}/ \ln x^{\lambda_1} \). Since

\[
0 < \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy = \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy = 0, \quad \epsilon \to 0^+.
\]

From the above inequalities, we can obtain

\[
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{\ln(m^{\lambda_1} n^{\lambda_2})^2} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} = \frac{1}{\lambda_2} \int_{e}^{\infty} \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy = \frac{1}{\lambda_2} \int_{e}^{\infty} \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy.
\]

which implies

\[
0 < \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy = 0, \quad \epsilon \to 0^+.
\]

From the above inequalities, we can obtain

\[
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{\ln(m^{\lambda_1} n^{\lambda_2})^2} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} = \frac{1}{\lambda_2} \int_{e}^{\infty} \int_{e}^{\infty} \frac{1}{x(\ln x)^{1+\frac{c_{\lambda_1}}{2}}} \lambda \left(1 + e^{c_{\lambda_1}}\right)^{1/\lambda} \left(1 + u\right)^{1+\frac{c_{\lambda_2}}{2}} dxdy = 0, \quad \epsilon \to 0^+.
\]

According to (2.10) and (2.11), then we give

\[
\frac{\lambda}{\lambda_1^{1+\frac{c_{\lambda_1}}{2}} \lambda_2^{1+\frac{c_{\lambda_2}}{2}}} < \frac{\lambda}{\lambda_2^{1+\frac{c_{\lambda_2}}{2}} \lambda_1^{1+\frac{c_{\lambda_1}}{2}}} k(1 + o(1)), \quad \epsilon \to 0^+.
\]
which implies \(c_{1,p} < k\), in contradiction with supposition. So the constant \(c_{1,p}\) in inequality (2.9) is the best possible. This completes the proof of Theorem 2.3.

**Theorem 2.4.** Let \(a_n, b_n \geq 0\), and \(p > 1, 1/p + 1/q = 1\), \(0 < \lambda_1 < p, 0 < \lambda_2 < q, 0 < c \leq \lambda\), \(c_{1,p} = (1/\lambda_1^{1/p} \lambda_2^{1/q}) B(c/(\lambda p), c/(\lambda q))\). If \(0 < \sum_{m=2}^{\infty} (\ln n^{1/(p-1)(1-\gamma)} n^{p-1} a_m < \infty\), then

\[
0 < \sum_{n=2}^{\infty} \frac{1}{n(\ln n^{x_k})^{1/2}} \left[ \sum_{m=2}^{\infty} \frac{a_m}{(\ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right]^{1/p} < c_{1,p} \sum_{m=2}^{\infty} \left( \ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right),
\]

where the constant \(c_{1,p}\) is the best possible. This completes the proof of Theorem 2.4.

**Proof.** There exists \(k_0 \in \mathbb{N}\) such that \(\sum_{m=2}^{k} a_m/(\ln m^{1/(p-1)(1-\gamma)}) > 0\) for \(k > k_0\). Setting \(b_n(k) = 1/(\ln n^{x_k})^{1/2} \left( \sum_{m=2}^{\infty} a_m/(\ln m^{1/(p-1)(1-\gamma)}) \right)^{1/p-1}\) for \(k > k_0\), then

\[
0 < \sum_{n=2}^{\infty} \frac{1}{n(\ln n^{x_k})^{1/2}} \left[ \sum_{m=2}^{\infty} \frac{a_m}{(\ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right]^{1/p} = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_m(k)}{(\ln m^{1/(p-1)(1-\gamma)})} < c_{1,p} \sum_{m=2}^{\infty} \left( \ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right)^{1/q},
\]

which implies that the following inequality holds

\[
\sum_{n=2}^{\infty} \frac{1}{n(\ln n^{x_k})^{1/2}} \left[ \sum_{m=2}^{\infty} \frac{a_m}{(\ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right]^{1/p} < c_{1,p} \sum_{m=2}^{\infty} \left( \ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right),
\]

As \(k\) tends to \(\infty\), we have

\[
0 < \sum_{n=2}^{\infty} \frac{1}{n(\ln n^{x_k})^{1/2}} \left[ \sum_{m=2}^{\infty} \frac{a_m}{(\ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right]^{1/p} < c_{1,p} \sum_{m=2}^{\infty} \left( \ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right),
\]

which implies inequality (2.12). If inequality (2.12) holds, by applying the Hölder’s inequality, then

\[
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{(\ln m^{1/(p-1)(1-\gamma)})^{1/2}} = \sum_{n=2}^{\infty} \left[ \sum_{m=2}^{\infty} \frac{a_m}{(\ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right]^{1/p} \left[ \sum_{n=2}^{\infty} \left( \ln m^{1/(p-1)(1-\gamma)} m^{p-1} a_m \right)^{1/q},
\]

which implies inequality (2.12), thanks to the inequality (2.12), this means the inequality (2.9) holds. Since the inequalities (2.9) and (2.12) are equivalent, then \(c_{1,p}\) is also the best possible. This completes the proof of Theorem 2.4.

**3. New corresponding Hardy-Hilbert-type integral inequalities**

In this section, we will give new double integral forms of double series inequalities.

**Theorem 3.1.** Let \(f(x), g(x) \geq 0\), \(p > 1\), \(1/p + 1/q = 1\), \(\lambda_1, \lambda_2, \lambda, c > 0\). \(c_{1,p} = (1/(\lambda_1^{1/p} \lambda_2^{1/q}) B(c/(\lambda p), c/(\lambda q))\). If \(0 < \int_{0}^{\infty} x^{p-1(1-\gamma)} f^p(x) dx < \infty, 0 < \int_{0}^{\infty} y^{q-1(1-\gamma)} g^q(y) dy < \infty\), then

\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{(x^\lambda + y^\lambda)^{2}} dxdy < c_{1,p} \left[ \int_{0}^{\infty} x^{p-1(1-\gamma)} f^p(x) dx \right]^{1/p} \left[ \int_{0}^{\infty} y^{q-1(1-\gamma)} g^q(y) dy \right]^{1/q},
\]

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where the inequalities (3.1) and (3.2) are also the best possible. Moreover, the inequalities (3.1) and (3.2) are equivalent.

Proof. By means of Hölder’s inequality and Lemmas 2.1 and 2.2, we have

\[
\int_0^\infty \int_0^\infty f(x)g(y) \frac{1}{(x^{\lambda_1} + y^{\lambda_2})^\gamma} \, dx \, dy = \int_0^\infty \int_0^\infty f(x) g(y) \left( \frac{1}{x^{\lambda_1}} + \frac{1}{y^{\lambda_2}} \right)^{\gamma/p} \, dx \, dy \leq \left[ \int_0^\infty \int_0^\infty f^p(x) \left( \frac{1}{x^{\lambda_1}} + \frac{1}{y^{\lambda_2}} \right)^{\gamma/p} \, dx \, dy \right]^{1/p} \left[ \int_0^\infty \int_0^\infty g^q(y) \left( \frac{1}{x^{\lambda_1}} + \frac{1}{y^{\lambda_2}} \right)^{\gamma/q} \, dx \, dy \right]^{1/q} = \left[ \int_0^\infty f^p(x) \, dx \int_0^\infty g^q(y) \, dy \right]^{1/p} \left[ \int_0^\infty \omega(x, q)^{p-1(1-\varepsilon)} f^p(x) \, dx \right]^{1/p} \left[ \int_0^\infty \omega^q(y, t)^{q-1(1-\varepsilon)} g^q(y) \, dy \right]^{1/q} = c_{t, p} \left[ \int_0^\infty x^{p-1(1-\varepsilon)} f^p(x) \, dx \right]^{1/p} \left[ \int_0^\infty y^{q-1(1-\varepsilon)} g^q(y) \, dy \right]^{1/q}.
\]

According to the hypotheses, it is easy to see that the equality in the above the second inequality is not possible. Now we prove that \( x < c_{t, p} \) is the best possible. If the constant \( c_{t, p} \) is not the best possible, then there exists \( k < c_{t, p} \) such that \( k \) replaces with \( c_{t, p} \), inequality (3.1) holds all the same. Especially, for \( \varepsilon \in (0, q/(2p)) \), we set

\[
\tilde{f}(x) = \begin{cases} \frac{1}{x^{\lambda_1} + y^{\lambda_2}(1-\varepsilon)} & x \in [0, 1), \\ 0, & x \in [1, \infty), \end{cases} \quad \text{and} \quad \tilde{g}(x) = \begin{cases} \frac{1}{x^{\lambda_1} + y^{\lambda_2}(1-\varepsilon)} & x \in [0, 1), \\ 0, & x \in [1, \infty). \end{cases}
\]

Based on the course of the proof of Theorem 2.1, we have

\[
\int_0^\infty \int_0^\infty \frac{\tilde{f}(x) \tilde{g}(y)}{(x^{\lambda_1} + y^{\lambda_2})^\gamma} \, dx \, dy < \frac{\lambda}{c_{t, p}^{1/p} \lambda_2^{1/q} \varepsilon} c_{t, p}(1 + o(1)) < \int_0^\infty \int_0^\infty \tilde{f}(x) \tilde{g}(y) \, dx \, dy < \frac{\lambda}{c_{t, p}^{1/p} \lambda_2^{1/q} \varepsilon} k(1 + o(1)),
\]

as \( \varepsilon \) tends to 0. We have \( c_{t, p} < k \), in contradiction with supposition. So the constant \( c_{t, p} \) in the inequality (3.1) is the best possible.

There exists \( t_0 > 0 \) such that \( \int_0^T f(x)/(x^{\lambda_1} + y^{\lambda_2})^\gamma \, dx > 0 \) for \( T > t_0 \), setting \( g(y, T) = \gamma^{\varepsilon/(1-\varepsilon)} \left( \int_0^T f(x)/(x^{\lambda_1} + y^{\lambda_2})^\gamma \, dx \right) \). Thanks to the inequality (3.1), thus we get

\[
\int_0^T y^{q-1(1-\varepsilon)} g^q(y, T) \, dy = \int_0^T y^{\varepsilon/(1-\varepsilon)} \left[ \int_0^T f(x)/(x^{\lambda_1} + y^{\lambda_2})^\gamma \, dx \right] \, dy = \int_0^T \int_0^T f(x) g(y, T)/(x^{\lambda_1} + y^{\lambda_2})^\gamma \, dx \, dy.
\]
≤ c_{A,p} \left[ \int_0^T x^{(p-1)(1-\frac{c_1}{p})} f^p(x)dx \right]^{1/p} \left[ \int_0^T y^{(q-1)(1-\frac{c_1}{q})} g^q(y, T)dy \right]^{1/q},

which implies that as $T$ tends to $\infty$, then we have

$$\int_0^\infty y^{c_2^{1/p}-1} \left[ \int_0^\infty \frac{f(x)}{(x^{1/\lambda} + y^{1/\lambda})^{\frac{c_2}{p}}} dx \right]^p dy \leq c_{A,p} \int_0^\infty x^{(p-1)(1-\frac{c_1}{p})} f^p(x)dx,$$

which means the inequality (3.2) holds. If inequality (3.2) holds, by using the Hölder’s inequality, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^{1/\lambda} + y^{1/\lambda})^{\frac{c_2}{p}}} dxdy = \int_0^\infty \left[ \int_0^\infty \frac{f(x)}{(x^{1/\lambda} + y^{1/\lambda})^{\frac{c_2}{p}}} dx \right] ^{1/p} \left[ \int_0^\infty \frac{g(y)}{(y^{1/\lambda} + x^{1/\lambda})^{\frac{c_2}{q}}} dy \right] ^{1/q} \leq \left[ \int_0^\infty \int_0^\infty \frac{f(x)}{(x^{1/\lambda} + y^{1/\lambda})^{\frac{c_2}{p}}} dx \right] ^p \left[ \int_0^\infty \frac{g(y)}{(y^{1/\lambda} + x^{1/\lambda})^{\frac{c_2}{q}}} dy \right] ^q \leq \left[ \int_0^\infty x^{(p-1)(1-\frac{c_1}{p})} f^p(x)dx \right] ^{1/p} \left[ \int_0^\infty \frac{g(y)}{(y^{1/\lambda} + x^{1/\lambda})^{\frac{c_2}{q}}} dy \right] ^q,$$

which implies that the inequality (3.1) holds. Therefore, inequalities (3.1) and (3.2) are equivalent. It is easy to see that the constant $c_{A,p}$ in the inequality (3.2) is also the best possible. □

Now we present a new variant of Theorem 3.1 as follows.

**Theorem 3.2.** Let $f(x), g(x) \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1, \lambda_2, \alpha > 0, \lambda_{A,p} = (1/(\lambda_1^{1/q} \lambda_2^{1/p}))B(c/(\lambda p), c/(\lambda q))$. If $0 < \int_0^\infty (\ln x^{1/\lambda})^{(p-1)(1-\frac{c_1}{p})} x^{p-1} f^p(x)dx < \infty$ and $0 < \int_0^\infty (\ln y^{1/\lambda})^{(q-1)(1-\frac{c_1}{q})} y^{q-1} g^q(y)dy < \infty$, then

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln x^{1/\lambda} y^{1/\lambda})^{\frac{c_2}{p}}} dxdy \leq c_{A,p} \left[ \int_1^\infty (\ln x^{1/\lambda})^{(p-1)(1-\frac{c_1}{p})} x^{p-1} f^p(x)dx \right] ^{1/p} \left[ \int_1^\infty (\ln y^{1/\lambda})^{(q-1)(1-\frac{c_1}{q})} y^{q-1} g^q(y)dy \right] ^{1/q}, \quad (3.3)$$

$$\int_1^\infty \frac{1}{y^{(\ln y^{1/\lambda})^{1-\frac{c_1}{q}}}} \left[ \int_1^\infty \frac{f(x)}{(\ln x^{1/\lambda} y^{1/\lambda})^{\frac{c_2}{p}}} dx \right] ^p dy \leq c_{A,p} \left[ \int_1^\infty (\ln x^{1/\lambda})^{(p-1)(1-\frac{c_1}{p})} x^{p-1} f^p(x)dx \right] ^{1/p} \left[ \int_1^\infty (\ln y^{1/\lambda})^{(q-1)(1-\frac{c_1}{q})} y^{q-1} g^q(y)dy \right] ^{1/q}, \quad (3.4)$$

where the constant factors in the inequalities (3.3) and (3.4) are also the best possible. Furthermore, inequalities (3.3) and (3.4) are equivalent.

**Proof.** By applying the Hölder’s inequality and Lemmas 2.4 and 2.5, we have

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln x^{1/\lambda} y^{1/\lambda})^{\frac{c_2}{p}}} dxdy = \int_1^\infty \int_1^\infty \left[ \frac{f(x)}{(\ln x^{1/\lambda} y^{1/\lambda})^{\frac{c_2}{p}}} \left( \frac{\ln x^{1/\lambda}}{\ln y^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{\ln y^{1/\lambda}}{\ln x^{1/\lambda}} \right)^{\frac{c_2}{p}} \right] dx dy \leq \left[ \int_1^\infty \int_1^\infty \frac{f^p(x)}{(\ln x^{1/\lambda} y^{1/\lambda})^{\frac{c_2}{p}}} \left( \frac{\ln x^{1/\lambda}}{\ln y^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{\ln y^{1/\lambda}}{\ln x^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{1}{x^{\frac{c_2}{p}}} \right)^{\frac{1}{p}} \right] \left[ \int_1^\infty \int_1^\infty \frac{g^q(y)}{(\ln x^{1/\lambda} y^{1/\lambda})^{\frac{c_2}{p}}} \left( \frac{\ln x^{1/\lambda}}{\ln y^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{\ln y^{1/\lambda}}{\ln x^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{1}{y^{\frac{c_2}{q}}} \right)^{\frac{1}{q}} \right] \leq \left[ \int_1^\infty \int_1^\infty \frac{f^p(x)}{(\ln x^{1/\lambda} y^{1/\lambda})^{\frac{c_2}{p}}} \left( \frac{\ln x^{1/\lambda}}{\ln y^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{\ln y^{1/\lambda}}{\ln x^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{1}{x^{\frac{c_2}{p}}} \right)^{\frac{1}{p}} \right] \left[ \int_1^\infty \int_1^\infty \frac{g^q(y)}{(\ln x^{1/\lambda} y^{1/\lambda})^{\frac{c_2}{p}}} \left( \frac{\ln x^{1/\lambda}}{\ln y^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{\ln y^{1/\lambda}}{\ln x^{1/\lambda}} \right)^{\frac{c_2}{p}} \left( \frac{1}{y^{\frac{c_2}{q}}} \right)^{\frac{1}{q}} \right].$$
which implies the inequality (3.3). According to the hypotheses, it is easy to see that the equality in the above second inequality is not possible. Now we prove that the constant \( c_{\lambda,p} \) is the best possible. Suppose that the constant \( c_{\lambda,p} \) in inequality (3.3) is not the best possible, then exists positive \( k < c_{\lambda,p} \) such that when \( k \) replaces with \( c_{\lambda,p} \), the inequality (3.3) holds all the same. Especially, for \( \varepsilon \in (0, q(2p)) \), we set

\[
\hat{f}(x) = \begin{cases} \frac{1}{x(\ln x^1)^{1-\frac{1}{q}}}, & x \in [0, 1), \\ 0, & x \in [1, \infty), \end{cases} \quad \text{and} \quad \hat{g}(x) = \begin{cases} \frac{1}{x(\ln x^1)^{1-\frac{1}{q}}}, & x \in [0, 1), \\ 0, & x \in [1, \infty). \end{cases}
\]

Due to the course of the proof of Theorem 2.3, we have

\[
\frac{\lambda}{c_{\lambda^1[1+p+\varepsilon]/2,\lambda^2/q} c_{\lambda,p}}(1 + o(1)) < \frac{1}{\int_1^\infty \frac{f(x)\hat{g}(y)}{(\ln x^1)^{1-\frac{1}{q}}}} dx dy < \frac{\lambda}{c_{\lambda^1[1+p+\varepsilon]/2,\lambda^2/q} c_{\lambda,p}} k(1 + o(1)),
\]

as \( \varepsilon \) tends to 0. We have \( c_{\lambda,p} < k \), in contradiction with supposition. So the constant \( c_{\lambda,p} \) in inequality (3.3) is the best possible.

There exists \( t_0 > 0 \) such that \( \int_1^T f(x)/(\ln x^1)^{1-\frac{1}{q}} dx > 0 \) for \( T > t_0 \). Setting

\[
g(y, T) = (1/(y(\ln y^1)^{1-c/\lambda}))(\int_1^T f(x)/(\ln x^1)^{1-\frac{1}{q}} dx)^{p-1} \text{ for } T > t_0,
\]

then

\[
\int_1^T (\ln y^1)^{(q-1)(1-\frac{1}{q})} y^{q-1} g(y, T) dy = \int_1^T \int_1^T \frac{f(x)g(y, T)}{(\ln x^1)^{1-\frac{1}{q}}} dx dy \\
< c_{\lambda,p} \left[ \int_1^T (\ln x^1)^{(p-1)(1-\frac{1}{q})} x^{p-1} f^p(x) dx \right]^{1/p} \left[ \int_1^T (\ln y^1)^{(q-1)(1-\frac{1}{q})} y^{q-1} g^q(y, T) dy \right]^{1/q},
\]

which implies that as \( T \) tends to \( \infty \), then we have

\[
\int_1^\infty \frac{f(x)g(y)}{(\ln x^1)^{1-\frac{1}{q}}} dx dy < c_{\lambda,p} \int_1^\infty (\ln x^1)^{(p-1)(1-\frac{1}{q})} x^{p-1} f^p(x) dx,
\]

which implies the inequality (3.4). If inequality (3.4) holds, by H"older’s inequality, then

\[
\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln x^1)^{1-\frac{1}{q}}} dx dy = \int_1^\infty \int_1^\infty \frac{1}{y(\ln y^1)^{1-\frac{1}{q}}} \frac{f(x)}{(\ln x^1)^{1-\frac{1}{q}}} dx dy \left[ \frac{1}{y(\ln y^1)^{1-\frac{1}{q}}} g(y) \right] dy \\
< \left[ \int_1^\infty \frac{1}{y(\ln y^1)^{1-\frac{1}{q}}} \left( \int_1^\infty \frac{f(x)}{(\ln x^1)^{1-\frac{1}{q}}} dx \right)^{p/q} dy \right]^{p/p} \left[ \int_1^\infty (\ln y^1)^{(q-1)(1-\frac{1}{q})} y^{q-1} g^q(y) dy \right]^{1/q},
\]

which means the inequality (3.3) holds. Since the inequalities (3.3) and (3.4) are equivalent, then \( c_{\lambda,p} \) is also the best possible. The proof of Theorem 3.2 is completed.

\[\square\]
Remark 3.1. It is easy to see that inequality (3.3) is equivalent to the following new Mullholand’s inequality associated integral.

\[
\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{xy(\ln x^{\lambda_1}y^{\lambda_2})^2} \, dx \, dy < c_{t,p} \left[ \int_1^\infty (\ln x^{\lambda_1})^{(p-1)(1-\frac{1}{p})} x^{-1} f^p(x) \, dx \right]^{1/p} \\
\times \left[ \int_1^\infty (\ln y^{\lambda_2})^{(q-1)(1-\frac{1}{q})} y^{-1} g^q(y) \, dy \right]^{1/q},
\]

where \(0 < \int_1^\infty (\ln x^{\lambda_1})^{(p-1)(1-\frac{1}{p})} x^{-1} f^p(x) \, dx < \infty, 0 < \int_1^\infty (\ln y^{\lambda_2})^{(q-1)(1-\frac{1}{q})} y^{-1} g^q(y) \, dy < \infty, \) and the constant factor \(c_{t,p}\) is also the best possible.

Remark 3.2. In the paper [16], by using the quasi-homogeneous integral kernels, Cao et al. obtained some Hilbert-type integral inequalities involving multiple functions with the best constant factors. In this paper, we not only give some Hilbert-type integral inequalities with multiparameters, but also obtain some Hilbert-type inequalities with multiparameters for double series. Therefore, our results are different from Cao et al. [16].

4. Conclusions

In this paper, we have established a new Hardy-Hilbert-type inequality with multiparameters. Furthermore, its equivalent forms and variants, which generalize some existing results, have been also presented. Finally, the corresponding Hardy-Hilbert-type integral inequalities have been obtained.

Conflict of interest

The authors declare that there is no conflict of interests.

References

1. G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, London: Cambridge University Press, 1952.
2. B. C. Yang, L. Debnath, On a new generalization of Hardy-Hilbert’s inequality and its applications, *J. Math. Anal. Appl.*, 233 (1999), 484–497. doi: 10.1006/jmaa.1999.6295.
3. B. C. Yang, On an extension of Hardy-Hilbert’s inequality, *Chin. Ann. Math.*, 23 (2002), 247–254.
4. B. C. Yang, On a new inequality similar to Hardy-Hilbert’s inequality, *Math. Inequal. Appl.*, 6 (2003), 37–44. doi: 10.7153/mia-06-04.
5. M. Z. Gao, B. C. Yang, On the extended Hilbert’s inequality, *Proc. Amer. Math. Soc.*, 126 (1998), 751–759. doi: 10.1090/S0002-9939-98-04444-X.
6. Y. J. Li, B. He, On inequalities of Hilbert’s type, *B. Aust. Math. Soc.*, 76 (2007), 1–13. doi: 10.1017/S0004972700039423.
7. J. Kuang, Note on an extensions of Hardy-Hilbert’s inequality, *J. Math. Anal. Appl.*, 235 (1999), 608–614. doi: 10.1006/jmaa.1999.6373.
8. B. J. Sun, on an extension of Hardy-Hilbert’s inequality with a best possible constant factor, *Adv. Math. China*, 36 (2007), 39–46.
9. J. S. Xu, Hardy-Hilbert’s inequalities with two parameters, *Adv. Math. China*, 36 (2007), 189–202.

10. H. P. Muholland, Some theorems on Dirichlet series with positive coefficients and related integrals, *Proc. London Math. Soc.*, 29 (1929), 281–292. doi: 10.1112/plms/s2-29.1.281.

11. B. C. Yang, On Hilbert’s integral inequality, *J. Math. Anal. Appl.*, 220 (1998), 778–785. doi: 10.1006/jmaa.1997.5877.

12. B. C. Yang, A dual Hardy-Hilbert’s inequality and generalizations, *Adv. Math. China*, 35 (2006), 102–108.

13. B. C. Yang, On Hardy-Hilbert’s integral inequality, *J. Math. Anal. Appl.*, 261 (2001), 295–306. doi: 10.1006/jmaa.2001.7525.

14. N. Das, S. Sahoo, A generalization of multiple Hardy-Hilbert’s integral inequality, *J. Math. Inequal.*, 3 (2009), 139–154. doi: 10.7153/jmi-03-14.

15. B. Sroysang, A generalization on some new types of Hardy-Hilbert’s integral inequalities, *J. Funct. Space*, 2013 (2013), 925464. doi: 10.1155/2013/925464.

16. J. F. Cao, B. He, Y. Hong, B. C. Yang, Equivalent conditions and applications of a class of Hilbert-type integral inequalities involving multiple functions with quasi-homogeneous kernels, *J. Inequal. Appl.*, 2018 (2018), 206. doi: 10.1186/s13660-018-1797-5.

17. D. M. Xin, B. C. Yang, A. Z. Wang, Equivalent property of a Hilbert-Type integral inequality related to the beta function in the whole plane, *J. Funct. Space*, 2018 (2018), 2691816. doi: 10.1155/2018/2691816.

18. Q. Liu, A Hilbert-type integral inequality under configuring free power and its applications, *J. Inequal. Appl.*, 2019 (2019), 91. doi: 10.1186/s13660-019-2039-1.

19. Q. Liu, On a mixed Kernel Hilbert-type integral inequality and its operator expressions with norm, *Math. Meth. Appl. Sci.*, 44 (2021), 593–604. doi: 10.1002/mma.6766.

20. Q. Chen, B. C. Yang, A reverse Hardy-Hilbert-type integral inequality involving one derivative function, *J. Inequal. Appl.*, 2020 (2020), 259. doi: 10.1186/s13660-020-02528-0.

21. Y. Hong, J. Q. Liao, B. C. Yang, Q. Chen, A class of Hilbert-type multiple integral inequalities with the kernel of generalized homogeneous function and its applications, *J. Inequal. Appl.*, 2020 (2020), 140. doi: 10.1186/s13660-020-02401-0.

22. J. Q. Liao, S. H. Wu, B. C. Yang, On a new half-discrete Hilbert-type inequality involving the variable upper limit integral and partial sums, *Mathematics*, 8 (2020), 229. doi: 10.3390/math8020229.