Entanglement of a cavity field interacting with a superconducting charge qubit

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1. Introduction

The Josephson junctions making up superconducting circuits are currently considered one of the most experimentally advanced solid state qubits [1]. The behavior of these circuits has been tested at the level of a single qubit [2–6] and also for a pair of qubits [7–11]. The first quantitative experimental study of an entangled pair of coupled superconducting qubits was recently reported [12]. A recent review of the latest developments in some of the leading approaches for a numbers of physical systems as candidates for quantum computation is found in Ref. [13] and the current status and characteristics of different types of qubits are reported in Ref. [14]. In terms of some applications in the fundamentals of quantum mechanics, quantum state engineering has also been proposed as a candidate for use in quantum information processing [15–17], quantum cryptography [18], quantum computing [19], and quantum teleportation [20]. The main ingredient of the applications of quantum computing and teleportation is conditional quantum dynamics, in which the coherent evolution of a subsystem depends on the state of another one and a measurement made upon one of them not only gives information about the other, but also provides its manipulation. Schemes in cavity quantum electrodynamics (QED) have also been proposed for realizing quantum logic gates [21] and teleportation [22,23]. The ion trap is also a good system for quantum information processing [24]. Quantum logic gates have been demonstrated in cavity QED [25] and ion trap [26] experiments. On the other hand, quantum teleportation has been explained by using optical systems [27]. The characterization of a typical Cooper pair box (CPB) is detailed in Ref. [28]; it consists of a superconducting island, several micrometers long and sub-micrometers wide, that is coupled via two sub-micrometer
size Josephson tunnel junctions to a much larger superconducting reservoir, constructed in the gap between the center conductor and the ground plane of the resonator, at an antinode of the field [28].

The aim of this contribution is to introduce an exact solution of the Heisenberg equation in the case of a high-$Q$ cavity for the general interaction of a single-mode microwave cavity field with a superconducting charge qubit. For this reason we devote Sect. 2 to introducing the Hamiltonian model and the time-dependent dynamical operators. Section 3 is devoted to discussing the population inversion, while the linear entropy and consequently the entanglement is discussed in Sect. 4. Meanwhile, we consider in Sects. 5 and 6 the phenomenon of squeezing, with the main concentration on variance and entropy squeezing, respectively. Our conclusion is given in Sect. 7.

2. The physical model and its solution

We consider a system consisting of a two-state flux qubit in interaction within a cavity. The Hamiltonian we adopt in the present paper can be described by:

$$\hat{H} = \frac{\omega}{\hbar} \hat{a}^\dagger \hat{a} + E_z \hat{\sigma}_z - E_J \exp \left( \frac{i\pi}{\Phi_0} \left( \hat{I} \Phi_C + \eta \hat{a} + \eta^* \hat{a}^\dagger \right) \right) \hat{\sigma}_+$$

$$- E_J \exp \left( -\frac{i\pi}{\Phi_0} \left( \hat{I} \Phi_C + \eta \hat{a} + \eta^* \hat{a}^\dagger \right) \right) \hat{\sigma}_-, \quad (1)$$

which is a modification of that presented in Ref. [29], where $\omega$ is the frequency of the cavity field. Here, $a$ and $a^\dagger$ are annihilation and creation operators of the cavity and $I$ is an identity operator, $E_J$ is the coupling constant of the interaction between the qubit and the cavity field. The operators $\hat{\sigma}_z$ and $\hat{\sigma}_\pm$ are defined by $\hat{\sigma}_z = \langle e | e \rangle - | g \rangle \langle g |$ and $\hat{\sigma}_+ = \langle e | g \rangle \langle e |$, $\hat{\sigma}_- = | g \rangle \langle e |$, where $| e \rangle$ and $| g \rangle$ are the excited and ground states of the two-level system, respectively. $E_z$ is the qubit charging energy given by $E_z = -2E_{ch}(1 - 2n_g)$ and depends on the gate charge $n_g$. It should be noted that, in most of the previous work, the researchers restricted their work to the case in which the gate voltage ranges near the degeneracy point $n_g = 1/2$. However, in the present work we avoid any restriction on the gate charge $n_g$. This means that we generalize some of the previous work and consequently more light can be shed on the behavior of the system. The quantity $E_{ch}$ is a single-electron charging energy defined by $E_{ch} = e^2 / (2C_g + 2C_J)$, where $C_J$ and $C_g$ are the Josephson junction and the gate capacities. The third and fourth terms in Eq. (1) are the nonlinear charge qubit–photon interaction that contains the parameter $\Phi_C$ that is generated by a classical applied magnetic field, while $\Phi_0$ is the quantum flux. The parameter $\eta$ has units of magnetic flux; its absolute value represents the strength of the quantum flux inside the qubit and can be expressed as [30]

$$|\eta| = S \sqrt{\frac{\omega \hbar}{\epsilon_0 V C^2}} \cos \left( \frac{2\pi z_0}{\lambda} \right), \quad (2)$$

which shows that $|\eta|$ depends on the surface area $S$ defined by the contour of the SQUID and its position $z_0$, while $\lambda$ and $V$ are the wavelength and the volume of cavity, respectively. If the light field is not so strong, then we can only keep the first order of $\pi \eta / \Phi_0$ and safely neglect all higher orders. In this case the Hamiltonian (1) takes the form

$$\hat{H} = \frac{\omega}{\hbar} \hat{a}^\dagger \hat{a} + E_z \hat{\sigma}_z - E_J (\hat{\sigma}_x \cos \beta + \hat{\sigma}_y \sin \beta)$$

$$- \frac{i\pi E_J}{\Phi_0} (\hat{\sigma}_y \cos \beta - \hat{\sigma}_x \sin \beta)(\eta \hat{a} + \eta^* \hat{a}^\dagger), \quad \beta = \left( \frac{\pi \Phi_C}{\Phi_0} \right). \quad (3)$$

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It is to be mentioned that the effect of the bias parameter \( (E_z \neq 0) \) in the ultra-strong coupling regime on the possibility of preparing nonclassical states (see Ref. \[31\]) and on the sudden transition from an uncorrelated state to an increasingly correlated one have recently been studied; for more details see Ref. \[32\].

Now let us introduce the following operators:

\[
\hat{S}_x = (\hat{\sigma}_x \cos \beta + \hat{\sigma}_y \sin \beta),
\]
\[
\hat{S}_y = (\hat{\sigma}_y \cos \beta - \hat{\sigma}_x \sin \beta),
\]
\[
\hat{S}_z = \hat{\sigma}_z, \tag{4}
\]

with the property \([\hat{S}_x, \hat{S}_y] = [\hat{\sigma}_x, \hat{\sigma}_y]\). The rotated operators \(\hat{S}_z, \hat{S}_x, \) and \(\hat{S}_y\) satisfy the commutation relations

\[
\begin{align*}
[\hat{S}_x, \hat{S}_y] &= 2i\hat{S}_z, & [\hat{S}_y, \hat{S}_z] &= 2i\hat{S}_x, & [\hat{S}_z, \hat{S}_x] &= 2i\hat{S}_y. \tag{5}
\end{align*}
\]

Because of the last line of Eq. (4), the eigenstates \( |e\rangle \) and \( |g\rangle \) are the eigenstates of \(\hat{S}_z\). The use of \(\hat{S}\) operators leads us to write the Hamiltonian (3) in the form

\[
\frac{\hat{H}}{\hbar} = \omega \hat{a}^\dagger \hat{a} + E_z \hat{S}_z - E_J \hat{S}_x - \frac{\pi E_J}{\Phi_0} \hat{S}_y (\eta \hat{a} + \eta^* \hat{a}^\dagger). \tag{6}
\]

In order to handle the above Hamiltonian we apply another rotation as follows:

\[
\begin{align*}
\hat{Q}_x &= \hat{S}_x \cos 2\xi - \hat{S}_z \sin 2\xi, & \hat{Q}_z &= \hat{S}_z \cos 2\xi + \hat{S}_x \sin 2\xi, & \hat{Q}_y &= \hat{S}_y,
\end{align*}
\]

where \(\xi = \frac{1}{2} \tan^{-1} \left( \frac{E_J}{E_z} \right)\).

In this case we have to introduce the states \( |+\rangle \) and \( |−\rangle \) as the excited and ground states corresponding to the \(\hat{Q}\) operators, respectively. The connection between the states \( |e\rangle, |g\rangle \) and the eigenstates \( |\pm\rangle \) of the operator \(\hat{Q}_z\) is given by:

\[
|+\rangle = [\cos \xi |e\rangle + \sin \xi |g\rangle], \quad |−\rangle = [\cos \xi |g\rangle - \sin \xi |e\rangle]. \tag{8}
\]

The rotating operators \(\hat{Q}_z\) and \(\hat{Q}_\pm\) are defined to satisfy the following properties:

\[
[\hat{Q}_+, \hat{Q}_-] = \hat{Q}_z, \quad [\hat{Q}_z, \hat{Q}_\pm] = \pm 2 \hat{Q}_\pm. \tag{9}
\]

where

\[
\hat{Q}_+ = |+\rangle \langle -|, \quad \hat{Q}_- = |−\rangle \langle +|, \quad \hat{Q}_z = [|+\rangle \langle +| + |−\rangle \langle −|]. \tag{10}
\]

The operators \(\hat{Q}_+, \hat{Q}_-, \) and \(\hat{Q}_z\) are the raising and lowering operators and the inversion operator in the bases \( |+\rangle, |−\rangle \). This in fact would give us the advantage of transforming the Hamiltonian (6) into the form

\[
\hat{H} = \omega \hat{a}^\dagger \hat{a} + \Omega_0 \hat{Q}_z - \frac{\pi E_J}{\Phi_0} \cos^2 \xi (\eta \hat{a} \hat{Q}_+ + \eta^* \hat{a}^\dagger \hat{Q}_-), \tag{11}
\]

where \(\Omega_0 = \sqrt{E_z^2 + E_J^2}\), and the coupling depends on the rotation angle \(\xi\), which is different from zero for \(n_\pi \neq \frac{1}{2}\).
Now we turn our attention to finding the dynamical operators of the present system. This can be achieved if we use, e.g., the Heisenberg equations of motion:

\[
\frac{d\hat{O}}{dt} = \frac{1}{i\hbar}[\hat{O}, \hat{H}] + \frac{\partial\hat{O}}{\partial t},
\]

where \( \hat{O} \) is any operator. Straightforward calculations lead us to four coupled differential equations:

\[
\frac{d\hat{a}}{dt} = -i\omega - \frac{\pi E_J}{\Phi_0} \cos^2 \xi \hat{a}^+ \hat{Q}_- - \frac{\pi E_J}{\Phi_0} \cos^2 \xi \hat{a} \hat{Q}_z, \\
\frac{d\hat{Q}_z}{dt} = \frac{2\pi E_J}{\Phi_0} \cos^2 \xi (\hat{Q}_- \eta^* \hat{a}^+ - \eta \hat{a} \hat{Q}_+), \\
\frac{d\hat{n}}{dt} = -\frac{\pi E_J}{\Phi_0} \cos^2 \xi (\hat{Q}_- \eta^* \hat{a}^+ - \eta \hat{a} \hat{Q}_+),
\]

with their Hermitian conjugate. Note that \( \hat{n} = \hat{a}^+ \hat{a} \) is the photon operator and, due to the properties of the \( \hat{Q}_{\pm z} \) operators, we are able to define the operators

\[
\hat{C} = \frac{\Delta}{2} \hat{Q}_z - \frac{\pi E_J}{\Phi_0} \cos^2 \xi (\eta \hat{a} \hat{Q}_+ + \hat{Q}_- \eta^* \hat{a}^+), \quad \hat{N} = (\hat{n} + \frac{1}{2} \hat{Q}_z), \quad \Delta = (2\Omega_0 - \omega),
\]

which commute with each other where \([\hat{C}, \hat{N}] = 0\), and also commute with the Hamiltonian, Eq. (11); consequently, they are constants of motion.

This leads us to write the time evolution operator \( \hat{U}(t) \) in the form

\[
\hat{U}(t) = \exp(-i\hat{H}t) = \exp[-i\omega\hat{N}t] \exp[-i\hat{C}t].
\]

After straightforward calculations we find that

\[
\exp[-i\omega\hat{N}t] = \begin{pmatrix}
\exp[-i\omega(\hat{a}^+ \hat{a} + \frac{1}{2}t)] & 0 \\
0 & \exp[-i\omega(\hat{a}^+ \hat{a} - \frac{1}{2}t)]
\end{pmatrix}, \\
\exp[-i\hat{C}t] = \begin{pmatrix}
\hat{F}_{11}(t) & \hat{F}_{12}(t) \\
\hat{F}_{21}(t) & \hat{F}_{22}(t)
\end{pmatrix}
\]

where

\[
\hat{F}_{11}(t) = \left(\cos \hat{\mu}_1 t - \frac{i\Delta}{\hat{\mu}_1} \sin \hat{\mu}_1 t\right), \quad \hat{F}_{12}(t) = -i\frac{\sin \hat{\mu}_1 t}{\hat{\mu}_1} \hat{a}^+ \\
\hat{F}_{22}(t) = \left(\cos \hat{\mu}_2 t + \frac{i\Delta}{\hat{\mu}_2} \sin \hat{\mu}_2 t\right), \quad \hat{F}_{21}(t) = -i\frac{\sin \hat{\mu}_2 t}{\hat{\mu}_2} \hat{a}^+
\]

where \( \hat{\mu}_1 \) and \( \hat{\mu}_2 \) are given by

\[
\hat{\mu}_1 = \sqrt{\Delta^2 + \lambda^2 (\hat{a}^+ \hat{a} + 1)}, \quad \hat{\mu}_2 = \sqrt{\Delta^2 + \lambda^2 (\hat{a}^+ \hat{a} + 1)}, \quad \lambda = \frac{\pi |\eta| E_J}{\Phi_0} \cos^2 \xi.
\]

Suppose we consider that the field is initially in the coherent state \( |\psi\rangle_f \), such that

\[
|\psi\rangle_f = \sum_{n=0}^{\infty} R_n |n\rangle, \quad R_n = \exp\left(-\frac{|\alpha|^2}{2}\right) \frac{\alpha^n}{\sqrt{n!}},
\]

where \( \alpha \) is the coherent complex parameter. Provided we assume the qubit to be initially in the qubit coherent state \( |\psi\rangle_a \),

\[
|\psi\rangle_a = \cos \theta |e\rangle + e^{i\phi} \sin \theta |g\rangle,
\]
where $\theta$ is the coherence angle and $\phi$ is a relative phase angle; therefore, after we apply the rotating state given by Eq. (8), the initial state takes the form

$$|\psi\rangle_a = (\cos \theta \cos \xi + e^{i\phi} \sin \theta \sin \xi)|+\rangle - (\cos \theta \sin \xi - e^{i\phi} \sin \theta \cos \xi)|-\rangle$$

(20)

where $|+\rangle$ and $|-\rangle$ are the eigenstates of $\hat{Q}_z$. Then the wave function for the system at $t = 0$ is now given by $|\psi(0)\rangle = |\psi\rangle_f \otimes |\psi\rangle_a$. Now we are in a position to obtain all the information related to the present system. In terms of $|\pm\rangle$ states, the wave function at $t > 0$ is given by the expression

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} R_n [\hat{F}(n,t)|n, +\rangle + \hat{G}(n,t)|n, -\rangle].$$

(21)

where $\hat{F}(n,t)$ and $\hat{G}(n,t)$ are given by

$$\hat{F}(n,t) = e^{-i\omega(\hat{a}^\dagger \hat{a}+1)t} \{F_{11}(t)(\cos \theta \cos \xi + e^{i\phi} \sin \theta \sin \xi)$$

$$- F_{12}(t)(\cos \theta \sin \xi - e^{i\phi} \sin \theta \cos \xi)\},$$

$$\hat{G}(n,t) = e^{-i\omega(\hat{a}^\dagger \hat{a}-1)t} \{F_{22}(t)(e^{i\phi} \sin \theta \cos \xi - \cos \theta \sin \xi)$$

$$+ F_{21}(t)(\cos \theta \cos \xi + e^{i\phi} \sin \theta \sin \xi)\}.$$  

(22)

Since our main task in this communication is to examine the effect of the external field using the available experimental data, we shall use Eqs. (21) and (22) to calculate the expectation values for the dynamical operators $\hat{Q}_z$ and $\hat{Q}_{x,y}$, from which we shall be able to find $\hat{\sigma}_z$ and $\hat{\sigma}_{x,y}$, where

$$\hat{\sigma}_x(t) = (\hat{Q}_x \cos 2\xi - \hat{Q}_z \sin 2\xi) \cos \beta - \hat{Q}_y \sin \beta,$$

$$\hat{\sigma}_y(t) = (\hat{Q}_x \cos 2\xi - \hat{Q}_z \sin 2\xi) \sin \beta + \hat{Q}_y \cos \beta,$$

$$\hat{\sigma}_z(t) = \hat{Q}_z \cos 2\xi + \hat{Q}_x \sin 2\xi,$$

$\beta = \frac{\pi \Phi_c}{\Phi_0}.$

(23)

Note that, in contrast to the operators $\hat{\sigma}_x(t)$ and $\hat{\sigma}_y(t)$, the operator $\hat{\sigma}_z(t)$ does not depend on the flux; this can be realized from the above equation.

In the forthcoming discussion we assume that the SC Cooper-pair box is made from aluminum, with a BCS energy gap of $\sim 2.4$ K (about 50 GHz) [30]; the charge energy $E_{ch}$ and the Josephson energy $E_J$ are $4E_{ch}/h = 149$ GHz and $2E_J/h = 13.0$ GHz, respectively [30]. The frequency of the cavity field is taken as 40 GHz, corresponding to a wavelength $\sim 0.75$ cm. The above numbers show that the SC energy gap has the largest energy, so the quasi particle excitation on the island can be well suppressed at low temperatures, e.g., 20 mK. The SQUID area is assumed to be about $50 \mu$m $\times$ $50 \mu$m; thus, the absolute value $|\lambda|$ of the qubit–photon coupling constant is about $\lambda = 4 \times 10^6$ rad s$^{-1}$ and the detuning parameter $\Delta = 9 \times 10^6$ rad s$^{-1}$. Note that the gate charge is experimentally reported as $n_g = 0.634$; with these values the angle $\xi$ is estimated to be about 35$^\circ$. This means that the result would not change qualitatively if one had frequencies in the range 5–10 GHz. Using the above data together with the expectation values of the dynamical operators we discuss some statistical properties of the system. Therefore, we start with population inversion, followed by linear entropy and variance squeezing as well as entropy squeezing.

3. Population inversion

Qubit state inversion can be considered as the simplest important quantity in any quantum system that includes qubits. It is defined as the difference between the probabilities of finding the qubit in
the excited state and in the ground state. However, for the present system the qubit state inversion can be obtained if we use Eq. (21) and consequently we have to find \( \langle \hat{Q}_x(t) \rangle, \langle \hat{Q}_y(t) \rangle, \) and \( \langle \hat{Q}_z(t) \rangle. \) If we assume that the parameter \( \alpha \) is real, then after some calculations we have

\[
\langle \hat{Q}_x(t) \rangle = -\frac{1}{2} \sum_{n=0}^{\infty} |R_n|^2 \left\{ \Delta \left( \cos \mu_1t \cos \mu_2t + \cos \mu_2t \sin \mu_1t \right) \cos \omega t \\
+ \left( \cos \mu_1t \cos \mu_2t - \Delta \delta \sin \mu_1t \sin \mu_2t \right) \sin \omega t \right\} \sin 2\delta
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} R_{n-1} R_{n+1} \sqrt{v_1 v_2} \left( \sin \mu_1t \sin \mu_2t \cos \omega t \right) \sin 2\delta
\]

\[
- \sum_{n=0}^{\infty} R_n R_{n+1} \sqrt{v_1} \left( \cos \mu_2t \sin \omega t + \Delta \delta \sin \mu_1t \cos \omega t \right) \sin \mu_1t \sin^2 \delta
\]

\[
+ \sum_{n=1}^{\infty} R_n R_{n-1} \sqrt{v_2} \left( \cos \mu_1t \sin \omega t + \Delta \delta \sin \mu_2t \cos \omega t \right) \sin \mu_2t \cos^2 \delta
\]  

(24)

and

\[
\langle \hat{Q}_y(t) \rangle = \frac{1}{2} \sum_{n=0}^{\infty} |R_n|^2 \left\{ \left( \cos \mu_1t \cos \mu_2t - \Delta \delta \sin \mu_1t \sin \mu_2t \right) \cos \omega t \\
- \Delta \delta \left( \cos \mu_1t \frac{\sin \mu_2t}{\mu_2} + \cos \mu_2t \frac{\sin \mu_1t}{\mu_1} \right) \sin \omega t \right\} \sin 2\delta
\]

\[
+ \frac{1}{2} \sum_{n=1}^{\infty} R_{n-1} R_{n+1} \sqrt{v_1 v_2} \left( \sin \mu_1t \sin \mu_2t \cos \omega t \right) \sin 2\delta
\]

\[
+ \sum_{n=0}^{\infty} R_n R_{n+1} \sqrt{v_1} \left( \cos \mu_2t \cos \omega t - \Delta \delta \sin \mu_1t \sin \omega t \right) \sin \mu_1t \sin^2 \delta
\]

\[
- \sum_{n=0}^{\infty} R_n R_{n-1} \sqrt{v_2} \left( \cos \mu_1t \cos \omega t - \Delta \delta \sin \mu_2t \sin \omega t \right) \sin \mu_2t \cos^2 \delta
\]  

(25)

while

\[
\langle \hat{Q}_z(t) \rangle = \cos 2\delta + 2 \sum_{n=0}^{\infty} \left\{ |R_n|^2 \left( v_2 \left( \frac{\sin \mu_2t}{\mu_2} \right)^2 \sin^2 \delta - v_1 \left( \frac{\sin \mu_1t}{\mu_1} \right)^2 \cos^2 \delta \right) \right\}
\]

\[
+ \sin 2\delta \sum_{n=0}^{\infty} R_n R_{n+1} \Delta \sqrt{v_1} \left( \frac{\sin \mu_1t}{\mu_1} \right)^2
\]

\[
+ \sin 2\delta \sum_{n=1}^{\infty} R_n R_{n-1} \Delta \sqrt{v_2} \left( \frac{\sin \mu_2t}{\mu_2} \right)^2
\]  

(26)

and where we have dropped the qubit state phase angle and defined

\[
v_1 = \lambda^2 (n + 1), \quad v_2 = \lambda^2 n, \quad \delta = (\theta - \xi).
\]

(27)

To discuss the population inversion we have fixed the value \( \alpha \) of the coherent state of the field at \( \alpha = 5 \). We have also fixed the value of the coupling parameter \( \lambda = 4 \times 10^6 \text{ rad s}^{-1} \) [30] related to
Fig. 1. The time evolution of population inversion as a function of the scaled time $10^6 t$, where the field is prepared in the coherent state for the fixed parameters $\alpha = 5, \lambda = 4 \times 10^6 \text{ rad s}^{-1}$ and for $\Delta = 9 \times 10^6 \text{ rad s}^{-1}$ but for different values of $\theta$. (a) $\theta = 0$, (b) $\theta = 2\xi$, (c) $\theta = 4\xi$, and (d) $\theta = 6\xi$.

the junction coupling parameter $E_j$ and considered the field frequency to be unity. In the meantime, we consider the value of the detuning parameter $\Delta = 9 \times 10^6 \text{ rad s}^{-1}$. In this context we examine the behavior of the qubit state inversion as a result of changing the value of the coherence angle $\theta$. For $\theta = 0$, the qubit in its excited state corresponds to a superposition state in the framework of the rotated angle $\xi$. In our numerical investigations the phenomenon of collapses and revivals that is detected in the case of the coherent state in the Jaynes–Cummings model is observed when we plot the qubit inversion against the scaled time $10^6 t$. As one can see from Fig. 1(a), the first revival period occurs after the onset of the interaction where the function fluctuates above 0.5 and just below −0.5. Meanwhile, the function in its first period of collapse decreases its value in an inclined way to intersect with the horizontal line of the time. In the second period of revival the function decreases its amplitude compared with the first period. Also, the time increases as the amplitude of the revival decreases but interference takes place. When we increase the value of the qubit coherence angle $\theta = 2\xi$, the function is shifted up, fluctuates above the time axis, and decreases its amplitude; see Fig. 1(b). Although the function displays the same shape as in the previous two cases, however, the situation is different when we consider the case in which $\theta = 4\xi$. In this case, we can observe that there is a reduction in the amplitude of the second period of the revival and the function increases its amplitude in the third period. Also, the function in the first revival period fluctuates just below 0 and above 0.5: this behavior is in contrast with the previous two cases; see Fig. 1(c). Finally, we have examined the case in which $\theta = 6\xi$, where the maximum value of the population inversion does not reach 0.4. Note that the first period of the revival fluctuates above 0.2 and below 0.4, while
the minimum value of the amplitude of the second period approaches $-0.05$. This behavior is not repeated in the other revival periods for the considered time; see Fig. 1(d). The effect of the term $\cos(\omega t)$ is shown clearly in this case, especially during the collapse period.

It is worth mentioning that the revival times can be estimated, as in Refs. [33–37], and consequently the revival times for a coherent state can be written as $t_R = 2\pi \sqrt{|\alpha|^2 + \Delta^2/\lambda^2}$. In the meantime, the atomic inversion displays a sinusoidal behavior and the difference between the present system and that of the Jaynes–Cummings model is due to the linear combination between $\langle \hat{Q}_z \rangle$ and $\langle \hat{Q}_x \rangle$, as has been reported before; see Ref. [38]. Also, we report that, as the time develops, the phenomenon of collapse diminishes. It is to be noted that the phenomenon of coherent trapping, in which the qubit energy is trapped in a state and no exchange occurs with the field, is not shown here because of the inclusion of the detuning parameter. Finally, we point out that the irregular behavior of the inversion is due to the dependence of the Rabi frequencies on the photon number and its distribution [39].

4. Linear entropy

Linear entropy is an important tool that can be used to discuss the degree of entanglement. To study the linear entropy we must calculate the reduced atomic density matrix of the qubit:

$$\hat{\rho}_a = Tr_f (|\psi(t)\rangle_a \langle \psi(t)|_a),$$

where $|\psi(t)\rangle$ is the wave function for both atom and field.

The qubit is said to be in a pure state if the condition $Tr(\rho_a^2) = 1$ is satisfied and in a mixed state if $Tr(\rho_a^2) < 1$. Therefore, the maximum value of the mixed state corresponds to $Tr(\rho_a^2) = 0.5$. The linear entropy is in general a simpler quantity to measure than the von Neumann entropy as there is no need for diagonalization, which is a very useful operational measure of the purity. The mathematical expression for linear entropy is given by

$$P(t) = \left\{ 1 - \left[ \langle \hat{Q}_x(t) \rangle^2 + \langle \hat{Q}_y(t) \rangle^2 + \langle \hat{Q}_z(t) \rangle^2 \right] \right\}$$

where $\langle \hat{Q}_x(t) \rangle$, $\langle \hat{Q}_y(t) \rangle$, and $\langle \hat{Q}_z(t) \rangle$ are given by Eqs. (24), (25), and (26), respectively. This means that the parameter $\beta = \pi \Phi_c/\Phi_0$ has no effect.

In our discussion of linear entropy we consider the same data as we used for atomic inversion. For the case in which $\theta = 0$, the function displays irregular fluctuations with interference between the patterns. Also, we can observe a decrease in its minimum as the time develops. Furthermore, the function shows a strong entanglement and reaches its maximum after a considerable period of time; see Fig. 2(a). When we consider $\theta = 2\xi$, the function displays similar behavior to that of the previous case. However, it decreases its minimum (compared with the previous case), showing weak entanglement, but it approaches its maximum as the time increases; see Fig. 2(b). More rapid fluctuations can be seen for the case in which $\theta = 4\xi$, in addition to an increase in the periods of strong entanglement, where the function nearly approaches the maximum value of the mixed state. We also noted that the function increases its minimum (compared with the previous case); see Fig. 2(c). A different observation can be made when we examine the case in which $\theta = 6\xi$. In this case the function reduces its maximum and never reaches the value 0.5; it also increases its minimum in addition to a decrease in its amplitude. In the meantime, it fluctuates within a small range of its maxima; see Fig. 2(d). Finally we would like to point out that, in all previous cases, the atom goes to a complete mixture of the states right after the onset of the interaction, where the complete collapse regions of the atomic population occur. The next complete collapse period takes place at $\lambda t_R = 2\pi \sqrt{n + \Delta^2/\lambda^2}$, which
occurs in the middle of the collapse region of the qubit inversion; see Figs. 2(a), 1(a). Furthermore, irregular revivals of the atomic inversion take place not only at the first revival period of the inversion but also at $\lambda t_R = 2k\pi \sqrt{n + \Delta^2/\lambda^2}$ ($k$ is an integer). Meanwhile, the linear entropy never attains a 0 value (i.e., disentanglement) when the qubit is either in its upper or lower state (i.e., a pure state), while strong fluctuations in the entanglement occur when the inversion is in the revival periods.

5. Variance squeezing

Squeezing is another important nonclassical phenomenon that is worth considering. Therefore, we devote the present section to considering and discussing the behavior of the variance as well as the entropy squeezing related to the present system. To reach our goal, we calculate the quadrature variances for the qubit operators by employing the Heisenberg uncertainty principle. As is well known, the Heisenberg uncertainty relation cannot give us sufficient information on qubit squeezing for some cases. However, it can be used as a general criterion for the squeezing in terms of the information entropy of a two-level qubit, as in the JCM. For this reason we shall consider both variance and entropy squeezing [40–43].

The uncertainty relation for a two-level qubit characterized by the Pauli operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ is given by

$$\Delta \hat{\sigma}_x \Delta \hat{\sigma}_y \geq \frac{1}{2} |\langle \hat{\sigma}_z \rangle|,$$

(30)
Fig. 3. The time evolution of the variance squeezing as a function of the scaled time $10^6 t$, where the field is prepared in the coherent state for the fixed parameter $\alpha = 5$ and for $\lambda = 4 \times 10^6$ rad s$^{-1}$, $\Delta = 9 \times 10^6$ rad s$^{-1}$, and for different values of $\theta$. (a) $\theta = 0$, (b) $\theta = 2\xi$, (c) $\theta = 4\xi$, and (d) $\theta = 6\xi$.

where $\Delta \hat{\sigma}_i = \sqrt{(\langle \hat{\sigma}_i^2 \rangle - \langle \hat{\sigma}_i \rangle^2)} \forall i = x, y$. In the meantime, fluctuations in the component $\hat{\sigma}_i$ of the qubit dipole $V_i = \left( \Delta \hat{\sigma}_i - \sqrt{\frac{|\langle \hat{\sigma}_z \rangle|}{2}} \right)$ are said to be squeezed if $V_i$ satisfies the condition

$$V_i = \left( \sqrt{1 - \langle \hat{\sigma}_i \rangle^2} - \sqrt{\frac{|\langle \hat{\sigma}_z \rangle|}{2}} \right) < 0 \quad \forall i = x, y. \quad (31)$$

To discuss the variance squeezing, we plot in Fig. 3 the function $V_x(t)$ against the scaled time $10^6 t$. In our numerical examination we have used the same value of the parameter involved as in the previous sections. For the qubit angles $\theta = 0$, the squeezing phenomenon is observed in both quadrature variances $V_x(t)$ and $V_y(t)$. In this case the squeezing occurs first in the quadrature $V_x(t)$ and consequently in the quadrature $V_y(t)$ for just a short period of time. However, the squeezing becomes more pronounced later in $V_x(t)$ after a certain period of time; see Fig. 3(a). On the other hand, when we consider $\theta = 2\xi$, the squeezing only occurs in the second quadrature $V_y(t)$ and is absent from the quadrature $V_x(t)$ during the considered period of time; see Fig. 3(b). For $\theta = 4\xi$ the squeezing is observed once in the quadrature $V_x(t)$ just for a short period of time. However, it is observed twice in the second quadrature $V_y(t)$, where the amount of squeezing in this quadrature is greater than the amount observed in the first quadrature; see Fig. 3(c). When we consider the case $\theta = 6\xi$, squeezing occurs after the onset of the interaction in the quadrature $V_x(t)$, as for the cases $\theta = 0$ and $4\xi$. In the meantime, there is a small amount of squeezing observed during the period of time considered. On the other hand, squeezing is also realized twice in the second quadrature $V_y(t)$,
which also shows regular oscillations; see Fig. 3(d). In conclusion, variance squeezing is observed in both quadratures for all the considered cases except for $\theta = 2\xi$. Also, the phenomenon of squeezing is observed within a short period of time and this is noted during our computational examination for the phenomenon. All this is displayed in a very short period elapsing after the onset of the interaction.

6. Entropy squeezing

As a second example of squeezing, we shall consider entropy squeezing. As is well known, in an even $N$-dimensional Hilbert space, the non-degenerate eigenvalues with the optimal entropic uncertainty relation for sets of $N + 1$ complementary observables have been investigated [44,45]. This is described by the inequality

$$\sum_{i=1}^{N+1} H(\sigma_i) \geq N \ln \left( \frac{\sqrt{N(N+2)}}{2} \right) + \ln \left( 1 + \frac{N}{2} \right).$$

(32)

where $H(\sigma_i)$ represents the information entropy of the variable $\hat{\sigma}_i$. For a two-level qubit, with $N = 2$, one can use the reduced qubit density operator $\hat{\rho}(t)$ to obtain the information entropies of the qubit operators $\hat{\sigma}_i, i = x, y, z$, which are connected with the operators $\hat{Q}_i, i = x, y, z$ given by Eq. (23), where

$$\hat{Q}_x = (|+\rangle \langle +| - |\rangle \langle -|), \quad \hat{Q}_y = i(|+\rangle \langle +| + |\rangle \langle -|).$$

(33)

while $\hat{Q}_z$ is defined by Eq. (10).

For the present case, we find that

$$H(\sigma_x) = \ln \left( \frac{1 - 2\text{Re}(\rho_{+,-}(t))}{1 + 2\text{Re}(\rho_{+,-}(t))} \right) + \text{Re}(\rho_{+,-}(t)) \ln \left( \frac{1 - 2\text{Re}(\rho_{+,-}(t))}{1 + 2\text{Re}(\rho_{+,-}(t))} \right),$$

(34)

and

$$H(\sigma_y) = \ln \left( \frac{1 - 2\text{Im}(\rho_{+,-}(t))}{1 + 2\text{Im}(\rho_{+,-}(t))} \right) + \text{Im}(\rho_{+,-}(t)) \ln \left( \frac{1 - 2\text{Im}(\rho_{+,-}(t))}{1 + 2\text{Im}(\rho_{+,-}(t))} \right),$$

(35)

while

$$H(\sigma_z) = -\rho_{+,+}(t) \ln \rho_{+,+}(t) - \rho_{-,+}(t) \ln \rho_{-,+}(t),$$

(36)

where the quantities $\rho_{++,}, \rho_{+,-}, \rho_{-,-}$, and $\rho_{--}$ are given from the relations

$$\rho_{ij}(t) = \langle i | \hat{\rho}(t) | j \rangle, \quad i, j = +, -, \mu.$$

(37)

Note that, for a two-level qubit, where $N = 2$, we have

$$1 \leq \exp(H(\hat{\sigma}_i)) \leq 4$$

(38)

and, hence, the information of the operators $\hat{\sigma}_x, \hat{\sigma}_y$, and $\hat{\sigma}_z$ will satisfy the inequality

$$H(\hat{\sigma}_x) + H(\hat{\sigma}_y) + H(\hat{\sigma}_z) \geq 2 \ln 2.$$ 

(39)

In this context, we refer to Ref. [46], where a review of the probability representation of quantum mechanics is presented, along with a discussion of experimental possibilities to check the uncertainty relations for the position and momentum [47,48], as well as the entropic uncertainty relations [49]. This may open the door for more applications in the near future.
Fig. 4. Time evolution of linear entropy as a function of the scaled time $10^6 t$, where the field is prepared in the coherent state for the fixed parameter $\alpha = 5$ and for $\lambda = 4 \times 10^6 \text{ rad s}^{-1}$, $\Delta = 9 \times 10^6 \text{ rad s}^{-1}$, and for different values of $\theta$. (a) $\theta = 0$, (b) $\theta = 2\xi$, (c) $\theta = 4\xi$, and (d) $\theta = 6\xi$.

In other words, if we define $\delta H(\hat{\sigma}_i) = \exp[H(\hat{\sigma}_i)]$, then the inequality (39) can be written as

$$\delta H(\hat{\sigma}_x)\delta H(\hat{\sigma}_y) \geq \frac{4}{\delta H(\hat{\sigma}_z)}.$$  \hspace{1cm} (40)

Note that the qubit will be in a pure state when $\delta H(\hat{\sigma}_i) = 1$, but when $\delta H(\hat{\sigma}_i) = 2$, the qubit will be in a completely mixed state, since the quantities $\delta H(\hat{\sigma}_x)$ and $\delta H(\hat{\sigma}_y)$ are only measuring the uncertainties of the qubit polarization components $\sigma_x$ and $\sigma_y$, respectively. Therefore, we define the squeezing of the qubit using the inequality (40), which is called entropy squeezing. The fluctuations in component $\hat{\sigma}_i$, $i = x$ or $y$ of the qubit dipole are said to be squeezed entropy if the information entropy $H(\hat{\sigma}_i)$ of $\hat{\sigma}_i$ satisfies the condition

$$E_i(t) = \delta H(\hat{\sigma}_i) - \frac{2}{\sqrt{|\delta H(\hat{\sigma}_z)|}} \leq 0 \quad \forall \ i = x, y.$$  \hspace{1cm} (41)

In what follows we concentrate on discussing and examining the behavior of the entropy squeezing. For this reason, we plot Fig. 4 for the functions $E_i(t)$, $i = x, y$ using the same data as in the previous sections. For the case in which $\theta = 0$, squeezing occurs in the first quadrature $E_x(t)$ for a short period of time after the onset of the interaction to reach its maximum just above $-0.5$; see Fig. 4(a). Meanwhile, squeezing starts to be seen in the second quadrature $E_y(t)$ after a considerable period of time. However, as the time increases, the function $E_y(t)$ decreases its minimum and consequently the phenomenon disappears. When we consider $\theta = 2\xi$, squeezing is only observed in the second quadrature $E_y(t)$. However, as we increase the period of the time, the function increases its minimum and consequently more reduction is seen in the amount of squeezing; see Fig. 4(b). We can also see an
increase in the rapid fluctuations of this quadrature. When we consider $\theta = 4\xi$, squeezing is observed once in the second quadrature $E_y(t)$ after the onset of the interaction, while it is absent from the first quadrature $E_x(t)$; see Fig. 4(c). A similar observation is made for the case in which $\theta = 6\xi$; however, the quadratures $E_x(t)$ and $E_y(t)$ show different behavior to that displayed in Fig. 4(c). In this case, we see the first quadrature decreases its value at several periods of the time considered without, however, observation of any squeezing.

7. Conclusion

In the present paper we have considered the problem of a superconducting qubit coupled to a cavity field. An exact solution for the problem is obtained even for the case in which the gate voltage range is far from the degeneracy point $n_R = 1/2$. This in fact gives us a wide range of flexibility to handle the problem and to obtain more information about the behavior of the system. To reach our result, we have employed a new qubit state operator to overcome any difficulty in obtaining the solution of the wave function. The qubit inversion is discussed for several values of the qubit angle and for fixed values of the other parameters. The system is affected by the qubit charging energy, the junction parameter, and the initial qubit state, as well as the field frequency. The function displays behavior similar to that of the JCM when the system is subjected to an external driving field [38]. Also, we have used the purity to measure the degree of entanglement, in addition to variance squeezing and entropy squeezing. The effect of taking account of moving away from the degeneracy point $n_R = 1/2$ adds new features to the phenomenon of collapses and revivals. Also, the qubit initial coherent state, which can be prepared experimentally, plays a role in this phenomenon as well as the squeezing phenomenon. Furthermore, the parameter $\Phi_0$ may also have effects on the entanglement of the system besides the parameter $E_z$ (bias parameter), which affects any processes. In fact, this parameter is included in the mixing of operators through the angle $\xi$, as is evident in Eq. (23) and, hence, on the phenomena discussed in the previous sections. This means that the existence of the above-mentioned parameters beside $\eta$ in the coupling $\lambda$, with its effect on time scaling, would cause a change in the behavior of the system. Consequently, any change in these parameters would lead to changes in experimental observations.

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