ELLIPITC GENUS OF SINGULAR ALGEBRAIC VARIETIES
AND QUOTIENTS

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Abstract. We discuss the basic properties of various versions of two variable elliptic genus with special attention to the equivariant elliptic genus. The main applications are to the elliptic genera attached to non-compact GITs, including the elliptic genera of Witten’s phases on $N = 2$ theories.

1. Preface

This paper provides an overview of the main results on complex elliptic genus while its second part focuses on equivariant elliptic genus and contains additional details regarding treatment of elliptic genera of phases of $N = 2$ theories given in [40]. The first two sections give chronological reviews of highlights of development of elliptic genus as well as relations to other problems since its introduction in 1980s (section 2) and recalls the key definitions along with the properties related to (complex, two variable) elliptic genera (cf. section 3). Then we describe the equivariant elliptic genera using approach to equivariant cohomology given in [20]. It gives a fast way to derive basic properties of equivariant elliptic genus obtained in [52] from non-equivariant version given in [9]. The final sections review the properties of elliptic genera of Witten phases of $N = 2$ theories (cf. [56]), following [40], but also makes explicit specializations of elliptic genus to $\chi_y$-genus and the Euler characteristics, providing in Landau-Ginzburg instance new links between elliptic genus and invariant of singularities. Example 5.1 gives direct calculation of $\chi_y$-genus of LG phase (and can be a starting point for reader interested on singularity theory) while the last section obtains it as a specialization of elliptic genus. Two appendices record well known information on basics of theta-functions and quasi-Jacobi forms introduced in [39].

2. Introduction

Elliptic genus appeared in the middle of 80s in the works of topologists and physicists. In mathematics, it was viewed as either index of an operator on a graded infinite dimensional vector bundle with finite dimensional graded components (cf. [2]) or (via Riemann-Roch) as a combination of characteristic numbers (cf. [3]). Motivation was the problem of finding rigid genera of differentiable Spin-manifolds endowed with a circle action (cf. [37], [38]) extending Atiyah-Hirzebruch rigidity of $A$-genus. In physics, elliptic genera appeared as indices of certain Dirac-like operators in free loop space associated with Spin-manifolds and also in connection with anomaly cancellations (cf. volume [38], [54] for overview of the first results and references therein e.g. [46]).

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Initial versions of elliptic genus were given in the context of differentiable manifolds but complex versions of elliptic genus were proposed by F. Hirzebruch (cf. [28], see also [30], [31]) and by E. Witten (cf. [38]) soon after. A different type of elliptic genus, associated with $C^\infty$-manifolds with vanishing first Pontryagin class was proposed by Witten (cf. [38]). It lead to important connections with homotopy theory and elliptic cohomology (cf. [32], [1]).

This first period culminated with the proof of Witten’s rigidity conjecture by R. Bott and C. Taubes (cf. [10]). In complex case rigidity has been proven by Hirzebruch and in [36] (for somewhat different but closely related versions of complex elliptic genus). Further study of rigidity was done in [41]. Much of material from this first period is summarized in Hirzebruch’s book [29] to which we refer a reader.

Elliptic genus, as an invariant of superconformal field theory (SCFT) (or of a representation of a superconformal algebra), was considered about the same time (cf. [55] and references to earlier works there). In the case of sigma models associated with a manifolds, an invariant of SCFT becomes an invariant of underlying manifold. There are, however, other backgrounds with which one can associate SCFT and obtain the invariants of such a background. Such notable examples are the SCFTs which are minimal models and Landau-Ginzburg models. The latter are associated with weighted homogeneous polynomials with isolated singularities. Elliptic genus, as a character of representation of superconformal algebra in complex setting (i.e. $N=2$ SCFT) was given in [34] following [55].

Mathematical version of the elliptic genus as an invariant of SCFT was presented in [42] where the authors constructed the chiral deRham complex of a manifold and the associated vertex operator algebra (VOA). The characters of vertex operator algebras, which are close relatives of elliptic genus of SCFT’s, were considered from the very beginning of the study of VOAs ([5], [17]) though mathematical study of analogs of $N=2$ superconformal algebras still is not well developed at the moment.

In the late 90’s, the focus of mathematical study shifted to elliptic genera of singular varieties ([50]). On one hand, it was motivated by Goresky-MacPherson problem (cf. [50], [25]) of determining the Chern numbers (rather than Chern classes, which being homological, are not a part of multiplicative structure and do not determine the Chern numbers) of singular varieties admitting a small resolution of singularities, which are independent of a resolution. The intersection homology groups, introduced in early 80s, do possess this property, cf. [25]).

On the other hand, mirror symmetry is inherently interwoven with singular varieties: in its very first example of smooth quintic in $\mathbb{P}^4$ the mirror partner is the orbifold which is the global quotient of such quintic by the action of abelian group of order 125 and exponent 5. The relation between SCFTs, which is a physics definition of mirror symmetry, implies the following relation between the elliptic genera of mirror partners (cf. [57])

\begin{equation}
Ell(X) = (-1)^{\dim X} Ell(X')
\end{equation}

Mathematically, the relations similar to (1) but involving Hodge numbers, Gromov-Witten invariants, derived categories etc. serve as either a definition or as a test of mirror symmetry.

The orbifold elliptic genus was proposed in the context of Landau-Ginzburg models by Witten (cf. [55]). In the context of manifolds (i.e. sigma-models) the elliptic genus of orbifolds apparently was understood in physics terms already right after introduction of orbifold euler characteristic (cf. [15], [30]). Mathematical
definition of the orbifold elliptic genus was given in [8]. This paper also contains
a definition of elliptic genus for certain class of singular varieties, including the
orbifolds, in terms of resolutions of singularities. The relation between both notions
of elliptic genus of orbifolds, which is the so called McKay correspondence for elliptic
genus, had been proven in [9]. It has as a very special case the numerical relations
which in dimension 2, are consequences of the relation between representations of
finite subgroups of $SL_2(\mathbb{C})$ and resolutions of quotients of $\mathbb{C}^2$ due to J.McKay (cf.
[43]).

The identity (1) for the hypersurfaces in toric varieties corresponding to the dual
polyhedra (Batyrev’s mirror symmetry, cf.[3]) was shown in [7]. One of the major
application of orbifold elliptic genus to the elliptic genera of Hilbert schemes of
K3-surfaces was given in [14]. A vast generalization in the context orbifold elliptic
genera of symmetric products was given in [8].

It is interesting to compare elliptic genus with the other invariants of smooth and
singular varieties appearing in the context of mirror symmetry: Hodge numbers,
Gromov-Witten invariants, and derived or Fukaya categories (cf. [35]). The similar
issues, as those mentioned above in the context of elliptic genus, e.g. search for
extension of original definitions from smooth to singular varieties, behavior in mirror
correspondence, MacKay correspondence etc. appeared in the study of all these
invariants. However, results for one type rarely imply the results for others. For
example, it is convenient to organize the Hodge numbers of smooth projective
varieties into E-function: $E(u,v) = \sum h^{p,q}u^pv^q$. It has the Hirzebruch’s genus (cf.
[27]) $\chi_y = \sum \chi(\Omega^p)y^p = \sum_{p,q} (-1)^q h^{p,q}y^p$ as specialization $y = u,v = -1$. The
elliptic genus is related to $\chi_y$-genus via: $\chi_{-y}(X) = \lim_{q \to 0} Ell(X)$. However,
neither $E$-function or $Ell(X)$ determine each other (cf. [7]). Both invariants factors
different universal rings of classes of manifolds: the $K$-group of varieties, in the case of
elliptic genus. In fact, $E(u,v)$ is a homomorphism $K_{\mathbb{C}}(Var) \to \mathbb{Z}[u,v]$ while the
elliptic genus is a homomorphism from the cobordism ring $\Omega^U \to \mathbb{C}[a,b,c,d]$ to a
certain polynomial algebra of functions on the product of $H \times \mathbb{C}$ of upper and the
whole complex plane respectively. Neither, appears to have an extension with good
properties to a bigger ring (cf. however [33]).

An attempt to extend mathematical treatment of elliptic genus to a wider con-
text, which includes the elliptic genera of singular varieties and Landau-Ginzburg
models was made in [40]. More specifically, for certain geometric invariants theory
(GIT) quotients one can define elliptic genus such that for the quotients considered
by Witten in [56] and corresponding to Calabi Yau or Landau Ginzburg models they
reproduce respectively elliptic genera of Calabi Yau manifolds considered in math-
ematics and physics literature and the elliptic genera of Landau Ginzburg models
considered in physics. The approach of [40] is based on use of equivariant elliptic
genus (in mathematics literature equivariant elliptic genus of compact varieties was
considered by R.Waelder [52]). In particular it implies LG/CY correspondence for
elliptic genus as a consequence of equivariant McKay correspondence. In the fol-
lowing sections we spell out some of the details about elliptic genus of such GIT
quotients.
3. Review of previous work

3.1. Complex manifolds. The two variable elliptic genus, which is the subject of this paper, can be defined as the holomorphic Euler characteristic of a bi-graded bundle associated with the manifold. More precisely, given a vector bundle $F$ on a complex manifold $X$, one associates with it the Poincare series $\Lambda_{F} = \sum \Lambda^{i}(F)t^{i}, S_{i}(F) = \sum Sym^{i}(F)t^{i}$ in the ring of polynomials in formal variable $t$ with coefficients in the semi-ring generated by vector bundles. With these notations, the elliptic genus is given by Fourier expansion with coefficients of monomials $q^{i}y^{j}$ being the holomorphic Euler characteristics of bi-graded components of the infinite tensor product of graded bundles with $q^{i}y^{j}$ providing the bi-grading (cf. [30], [31], [50], [7]):

$$E_{\text{Ell}}(X) = y^{\dim X} \chi(X, \otimes_{n\geq 1} \left( \Lambda_{-y^{n-1}q^{n}} \otimes \Lambda_{-y^{-1}q} T_{X} \otimes S_{q} \otimes \Omega_{X}^{1} \otimes S_{q} T_{X} \right) \otimes K_{X}^{-k}).$$

(here $T_{X}, \Omega_{X}^{1}, K_{X}$ are respectively the tangent, cotangent and canonical bundles of $X$ and $k$ is a constant).

Riemann-Roch theorem implies that (2) is a linear combination of Chern numbers defined as follows. Evaluation of (2) for a compact complex manifold provides a homomorphism of cobordism ring $\Omega^{U}$ of almost complex manifolds (cf. [18]). The target of this homomorphism is a ring of holomorphic functions on $\mathbb{H} \times \mathbb{C}$ where $\mathbb{H}$ is the upper half-plane if one interprets the formal variable in (2) as $q = e^{2\pi \sqrt{-1} \tau}, y = e^{2\pi \sqrt{-1} z}, \tau \in H, z \in \mathbb{C}$. Hirzebruch’s formalism (cf. [27]) implies that any such a homomorphism $\phi: \Omega^{U} \rightarrow R$ with values in a commutative ring $R$ (i.e. a $R$-valued genus) can be specified by a formal power series $Q(x) \in R[[x]]$ so that $\phi(X) = \prod Q(x_{i})[X]$ is evaluation of the product series at the Chern roots $x_{i}$ on the fundamental class $[X] \in H_{\dim X}(X)$ of $X$ (the Chern roots $x_{i}$ satisfy $\prod (1 + x_{i}) = c(X)$ where $c(X) \in H^{*}(X)$ is the total Chern class of $X$). In the case of (2), $Q(x)$ is the Taylor series in variables $x$ of the function $\frac{z^{1-k} \phi(\frac{x \sqrt{-1} z}{2\pi \sqrt{-1}}, \tau)}{\theta(\frac{x \sqrt{-1} z}{2\pi \sqrt{-1}}, \tau)}$ and for $k = 0$ one has

$$E_{\text{Ell}}(X) = \prod \pi_{i} \frac{2\theta\left(\frac{x \sqrt{-1} (z, \tau)}{2\pi \sqrt{-1}}, \tau\right)}{\theta\left(\frac{x \sqrt{-1} (z, \tau)}{2\pi \sqrt{-1}}, \tau\right)} [X]$$

The holomorphic functions, which are elliptic genera of manifolds have important modularity properties. If $c_{1}(X) = 0$ then $E_{\text{Ell}}(X)$ is Jacobi form for semidirect product of $SL_{2}(\mathbb{Z})$ and $\mathbb{Z}^{2}$ (the Jacobi group) i.e. obeys the following transformation laws:

$$\phi\left(\frac{at + b}{ct + d}, \frac{z}{ct + d}\right) = (ct + d)^{k} e^{2\pi \sqrt{-1} (\frac{at + b}{ct + d})^{2}} \phi(z, \tau)$$

$$\phi(\tau, z + \lambda \tau + \mu) = (-1)^{2(\lambda + \mu)} e^{2\pi \sqrt{-1} (\frac{\lambda^{2} \tau + 2 \lambda z}{2\pi \sqrt{-1}})} \phi(\tau, z)$$

$a, b, c, d, \lambda, \mu \in \mathbb{Z}, ad - bc = 1$.

\[\text{\footnote{Hohn [31] uses } y \rightarrow -y.\]
Here $k, t \in \mathbb{Z}$ are weight and index respectively of the (weak) Jacobi form $\phi(z, \tau)$. For Calabi Yau manifold of dimension $d$, $Ell(X)$ given by (2) or (3) is Jacobi form of weight zero and index $\frac{d}{2}$.

Without Calabi Yau condition (3) is a quasi-Jacobi form in the sense of [39] (cf. also Appendix II below). It follows (cf. [39] theorem 2.12) that elliptic genera of almost complex manifolds are polynomials in $\hat{\theta}$ which, for $k = 0$, is different from (3) by a factor which is $(e^{-\pi \sqrt{-1}z}X(-z, \tau))^2$. Let

\[ E_n(z, \tau) = \sum_{a, b \in \mathbb{Z}} \left( \frac{1}{z + a\tau + b} \right)^n \quad n \in \mathbb{Z}, n \geq 1 \]

(with appropriate choice of summation order for $n = 1, 2$ cf. [39] and $e_2(\tau)$ is the one variable Eisenstein series.

For example, the elliptic genus of a complex surface of degree $d$ in $\mathbb{P}^3$ can be calculated as

\[ (E_2^2 \left( \frac{1}{2} d^2 - 4d + 8 \right) + (E_2 - e_2)(\frac{d^2}{2} - 2d)(\theta(z, \tau) - \theta(0, \tau))^2 \]

In particular for K3-surface, i.e. the case $d = 4$, one obtains

\[ 24(E_2 - e_2)(\theta(z, \tau) - \theta(0, \tau))^2 \]

The elliptic genus considered in [7] is given by (3) or (2) with $k = 0$. Up to a factor depending only on dimension (cf. [7], Prop.2.3), it coincides with the elliptic genus considered in [36], [50], [31]. The latter two works use Weierstrass $\sigma$-function (cf. [77], in Appendix I) writing the characteristic series as follows. Let

\[ \Upsilon(x, \tau) = e^{-\frac{G_2(\tau)x^2}{2}} e^{-\frac{x}{2}} \sigma(x, \tau) = e^{-\frac{x}{2}} 2 \sinh(x) \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2} \]

(here $G_2(\tau) = -\frac{1}{2\pi} + \sum_{n \geq 1} (\sum d(n) q^n)$ cf. [8], [3], Appendix I). Then

\[ Q(x) = e^{kx} \Upsilon(x, \tau) \Upsilon(-x, \tau) \]

(which, for $k = 0$, is differ from [3] by a factor which is $(e^{-\pi \sqrt{-1}z}X(-z, \tau))^2 \dim X$).

Hirzebruch-Witten elliptic genus of an almost complex manifold corresponds to the characteristic series:

\[ \exp \left( \frac{\pi}{N} \alpha \right) \Upsilon(x - z, \tau) \]

Specialization $q = 0$ of (8) yields the holomorphic euler characteristic $\chi_g(X, K_X)$ and specialization of (2) to $z = \frac{2\pi \sqrt{-1}(1 + b)}{N}$, up to a factor depending on dimension gives Hirzebruch-Witten elliptic genus [3] (7). As [3], this is invariant of almost complex manifolds but it has the following modular property: if $c_1(X) = 1$ different normalizations in [3] used in some papers, may lead to a different weight and index.

3D.Zagier pointed out that, at least for some of these functions, the term quasi-elliptic would be more appropriate.

4i.e. $E_2(0, \tau)$ with omitted summand corresponding to $a = b = 0$.

5these papers use instead of $\Upsilon$ the notation $\phi(x, \tau)$ which a little different than the one used in [29], notations here and below are the same as in [29].
0 mod $N$ then Hirzebruch-Witten elliptic genus is a modular form for the subgroup $\Gamma_0(N)$ of the modular group. If $N = 2$ in [S] then the Hirzebruch-Witten genus depends on Pontryagin (rather than Chern) classes of $X$ only and is an invariant of $C^\infty$-manifolds which is modular (for $\Gamma_0(2)$) if manifold is Spin. This is the first instance of elliptic genus which appeared in mathematics literature and is due to Ochanine-Landweber-Stong (cf. [38]).

3.2. Orbifold elliptic genus. The elliptic genus of orbifolds which are global quotients was defined in [S] as follows (this definition was extended to arbitrary orbifolds in [18]).

Let $X$ be a smooth projective variety and let $\Gamma$ be a finite group of its automorphisms. For an element $g \in \Gamma$ let $X^g$ denote its fixed point set. For a connected component $\bar{X}^g$ of $X^g$ we consider the decomposition into the eigenspaces of $g$ for the restriction $TX|_{\bar{X}^g}$ of the tangent bundle of $X$ on $\bar{X}^g$. We represent each eigenvalue of $g$ acting on this restriction, in the form $\exp(2\pi i \lambda(g))$ where $0 \leq \lambda(g) < 1$ and denote the eigenbundle corresponding to this eigenvalue as $V_{\lambda(g)}$. In particular $V_0$ is the tangent bundle to $\bar{X}^g$ and $TX|_{\bar{X}^g} = V_0 \oplus (\oplus_{\lambda(g) \neq 0} V_{\lambda(g)})$. We also denote by $F(g, \bar{X}^g) = \sum \lambda(g)$, “the fermionic shift” corresponding to the component $\bar{X}^g$.

Then we let

$$V_{h, X^h \subseteq X} := \otimes_{k=0}^1 (\Lambda_{gq^{-1}}^* V_0 \otimes \Lambda_{g-1}^* V_0 \otimes \sum_{\lambda(h)} V_{\lambda(h)} \otimes \sum_{\lambda(h)} V_{\lambda(h)} \otimes \sum_{\lambda(h)} V_{\lambda(h)} \otimes \sum_{\lambda(h)} V_{\lambda(h)} \otimes \sum_{\lambda(h)} V_{\lambda(h)})$$

With these notations one defines the orbifold elliptic genus as:

$$Ell_{orb}(X, \Gamma; y, q) := \sum_{\{h\} \in Conj(\Gamma), X^h} y^{dim X/2} \sum_{g \in \Gamma(h)} L(g, V_{h, X^h \subseteq X}) \frac{1}{C(h)}$$

where $Conj(\Gamma)$ is the set of conjugacy classes in $\Gamma$, $C(h) \subseteq \Gamma$ is the centralizer of $h \in \Gamma$ and $L(g, V)$ is the holomorphic Lefschetz number of $g$ with coefficients in a holomorphic $g$-bundle $V$ i.e. $L(g, V) = \sum (-1)^i tr(g, H^i(X^g, V))$. Equivalent form of (10) is

$$Ell(X, \Gamma; y, q) := \sum_{\{h\} \in Conj(\Gamma), X^h \subseteq X} y^{dim X/2} \sum_{g \in Conj(\Gamma), X^h \subseteq X} L(g, V_{h, X^h \subseteq X}) \chi(H^*(V_{h, X^h \subseteq X})^C(h))$$

where $\chi(H^*(V_{h, X^h \subseteq X})^C(h))$ is the alternating sum of the dimensions of $C(h)$-invariant subspaces of the cohomology of bundles $V_{h, X^h \subseteq X}$.

Atiyah-Bott holomorphic Lefschetz formula (cf. [2]), allows to rewrite (10) as follows. For a pair of commuting elements $g, h \in \Gamma$, let $X^{g,h} = X^g \cap X^h$ denotes the set of points in $X$ fixed by both $g$ and $h$. Then the expression of (10) in terms of characteristic classes is:

$$\frac{1}{|\Gamma|} \sum_{g, h, gh = hg} (\prod_{\lambda(g) = \lambda(h) = 0} x^{\lambda}) \prod_{\lambda} \frac{\theta(x \lambda)}{\theta(\frac{x \lambda}{2\pi i})} \lambda(g) - \tau \lambda(h) - z \frac{e^{2\pi i z \lambda(h)}}{e^{2\pi i z \lambda(h)}} [X^{g,h}]$$

where the products are taken over all Chern roots $x_\lambda$ (counted with their multiplicities) of the eigenbundles $V_\lambda$ corresponding to the logarithms $\lambda$ of the characters of abelian subgroup of $\Gamma$ generated by $g, h$. The term in this sum corresponding to the pair in which both $g, h$ are the identities coincides with the elliptic genus of $X$. We shall call this term the “trivial” sector of the orbifold elliptic genus. It is
such that $\hat{Q}$ is called Gorenstein if a Weil $K$- Kawamata log-terminal singularities) if there exist a resolution of singularities of degree differential form, is Cartier. A

$$f \in \mathbb{Q}$$

Definition 3.2. A normal variety $X$ is called Gorenstein if a Weil $Q$-divisor, which is a multiple of the divisor of the top degree differential form, is Cartier. A $\mathbb{Q}$-Gorenstein variety is call klt (i.e. having Kawamata log-terminal singularities) if there exist a resolution of singularities $\hat{X} \rightarrow X$ such that coefficients of decomposition $K_{\hat{X}} = f^*(K_X) + \sum \alpha_k E_k$ satisfy $\alpha_k > -1$.

To define the elliptic genus of singular varieties with singularities as in 3.2 one first defines the elliptic genus of pairs $X, E$ where $X$ is smooth and projective and $E$ is a $\mathbb{Q}$-divisor on $X$ i.e. $E = \sum \alpha_k E_k$ is a formal sum such that components $E_k$ are smooth divisors on $X$ intersecting transversally. Moreover, one assumes that $\alpha_k > -1$ for all $k$. In this situation one defines the cohomology class, called the elliptic class of pair $(X, E)$:

$$\mathcal{E}_{\text{LL}}(X, E) = \prod_i \frac{x_i^\theta(\frac{\pi}{2\sqrt{1-x_i}}, -z, \tau)\theta'(0, \tau)}{\theta(\frac{\pi}{2\sqrt{1-x_i}}, \tau)\theta(-z, \tau)} \prod_k \frac{\theta(\frac{\pi}{2\sqrt{1-\alpha_k+1}}, z, \tau)}{\theta(\frac{\pi}{2\sqrt{1-\alpha_k+1}}, -z, \tau)} \theta(-\alpha_k, \tau)$$

The elliptic genus of pair then is evaluation of the elliptic class on the fundamental class of $X$: $\text{Ell}(X, E) = \mathcal{E}_{\text{LL}}(X, E)[X]$.

The fundamental property of the elliptic class of pairs, allowing to define the elliptic genus of a singular variety as the elliptic genus of pair consisting of a resolution and certain divisor on the latter, is the compatibility in the blowups:

$$f_*(\mathcal{E}_{\text{LL}}(\hat{X}, \hat{E})) = \mathcal{E}_{\text{LL}}(X, E)$$

6in $H^*(X, \mathbb{Q})$ tensored with a ring of functions in $z, \tau$ appearing in expanding (13) in $x$. The ring of quasi-Jacobi forms described in Appendix II can be used. Often below we shall by abuse of terminology tell that we consider elliptic class in cohomology (or Chow groups) meaning in fact that this class is in the extended in such a way cohomology (or Chow theory)
In particular $\text{Ell} (\hat{X}, \hat{E}) = \text{Ell} (X, E)$.

**Corollary 3.4.** Let $X$ be a $\mathbb{Q}$-Gorenstein projective variety with at most klt singularities. Let $X \to X$ be a resolution of its singularities and $E = \sum \alpha_k \hat{E}_k$ be a normal crossing divisor on $\hat{X}$ such that $f^*(K_X) = K_{\hat{X}} + \hat{E}$. Then $\text{Ell} (\hat{X}, \hat{E})$ depends only on $X$ i.e. is independent of a choice of $(\hat{X}, \hat{E})$ (and called (singular) elliptic genus of $X$). It will be denoted $\text{Ell}_{\text{sing}} (X)$.

The fundamental relation between singular and orbifold elliptic genera is given by the so called MacKay correspondence for elliptic genus:

**Theorem 3.5.** Let $X$ be a smooth projective variety on which a group $\Gamma$ acts effectively via biholomorphic transformations. Let $\mu : X \to X/\Gamma$ be the quotient map. Assume that $\mu$ does not have ramification divisors i.e. fixed points of elements of $\Gamma$ have codimension greater than one. Then

$$\text{ELL}_{\text{orb}} (X, \Gamma; z, \tau) = \left( \frac{2\pi \sqrt{-1} \theta(-z, \tau)}{\theta(0)} \right)^{\text{dim} X} \text{ELL}_{\text{sing}} (X/\Gamma, z, \tau)$$

In particular, orbifold elliptic genus coincides with the elliptic genus of any crepant resolution (if such exist).

We refer to [9] theorem 5.3 for a more general statement in the category of Kawamata log-terminal pairs and for the case of quotients maps admitting ramification divisors.

An immediate corollary is reinterpretation of the series in theorem 3.1 in case when $\text{dim} X = 2$, as the generating series of elliptic genera of Hilbert schemes:

**Corollary 3.6.** Let $X$ be a smooth projective surface and $\text{Ell} (X) = \sum_{m,l} q^m y^l$. Then

$$\sum_{p^n} \text{Ell} (\text{Hilb}_n, q, y) = \prod_{i=1}^{\infty} \prod_{m,l} \frac{1}{1 - p^i q^m y^l}^{c_{(m,l)}}$$

Indeed, in the case of the surfaces the morphism $\text{Hilb}_n (X) \to X^n / \Sigma_n$ is a smooth crepant resolution (i.e. $\alpha_k = 0$ in definition 3.2).

4. Equivariant elliptic genus

In this section we discuss an equivariant version of the elliptic genus. In particular we shall describe equivariant analog of push forward formula (i.e. theorem 3.3) for elliptic class, equivariant McKay correspondence, equivariant localization and push forward properties of the contributions of compact components of fixed point sets into elliptic class. Our approach is based on equivariant intersection theory as developed in [20] (cf. also [50]). It allows to derive equivariant results from their non-equivariant counterparts, already discussed in section 3.3 applied in appropriately formulated context. As in [20] and [9], instead of ordinary cohomology, we work in Chow theory, but a reader of course can interpret all statements as those in ordinary cohomology.

4.1. Equivariant intersection theory. We start with working in the category of quasi-projective normal varieties (over $\mathbb{C}$) with various assumptions on singularities such as $\mathbb{Q}$-Gorenstein and klt conditions (cf. section 3.3). We also assume that a reductive algebraic group $G$, $\text{dim} G = g$ acts on such $X$ via a linearized action. The latter means that an ample line bundle $L$ is presented on $X$ together with a
G-action on the total space of L such that bundle projection on X is equivariant (cf. [44]). We shall refer to [20] Section 6 for precise conditions on the action which assure that constructions, needed for equivariant intersection theory to run, will work.

Let \( V, \dim V = l \) be a representation of \( G \) and \( U \subset V \) is an open set such that \( G \) acts on \( U \) freely and \( \text{codim} V/U \) is sufficiently large. Then \( U/G \) is smooth and for a given \( n \), the Chow groups \( A^n(U/G) = A_{l-g-n}(U/G) \) are well defined for \( l >> n \), and so are the products among them for all \( n' < n \). The Chow ring \( A^*(BG) \) is defined as the graded ring having \( A^n(U/G) \) for \( n << l \) as its graded components: again, these are independent of \( l \) as long as \( l \) is large enough.

Since \( G \) acts freely on \( U \), the diagonal \( G \)-action on \( X \times U \) is free as well, the quotient space \( X_G = (X \times U)/G \) does exist and equivariant Chow group \( A_G^*(X) \) can be defined as the usual Chow group \( A_{n+g}(X_G) \). Again, it is independent of \( V, U \) as long as \( \text{codim} V \setminus U \) is sufficiently large (cf. [20] Prop.-Def). The intuition behind such choice of indices is that in the case when \( X \) is smooth, projective and the quotient \( U/G \) is compact, one has \( \dim X_G = \dim X + l - g \) and \( A_{l+g}(X_G) = A^{\dim X - l}(X_G) \) by Poincare duality.

Let \( E \) be an equivariant \( G \)-bundle on a quasi-projective variety with action of \( G \) i.e. the total space of \( E \) is endowed with \( G \)-action such that projection \( E \to X \) is \( G \)-equivariant. Then \( E_G = (E \times U)/G \to X_G \) is a vector bundle on \( X_G \) and equivariant Chern class \( c_E^G \in A^G_*(X) \) is the Chern class of the vector bundle \( E_G \) on \( X_G \). As in non-equivariant case, one associates with an equivariant bundle the (equivariant) Chern roots \( x_i^G \in A_*(X_G) \).

To define equivariant elliptic class, we note that the map \( \pi \) induced by projection on the second factor:

\[
X_G = (X \times U)/G \xrightarrow{\pi} U/G = BG
\]

is a locally trivial fibration with the fiber \( X \).

**Definition 4.1.** Let \( X \) be a smooth projective variety with an action of algebraic group \( G \). The equivariant elliptic genus of \( (X, G) \) is the push forward of the equivariant elliptic class i.e. the class \([3]\) where \( x_i \) are the equivariant Chern roots of the tangent bundle of \( X \) with its natural \( G \)-structure:

\[
Eli^G(X) = \pi_*([\mathbb{L}(LL(X_G))]) \in A_*(BG) \otimes \text{Jac}
\]

where \( \pi : X_G \to BG \) is induced by projection of \( X \times U \) on the second factor and \( \text{Jac} \) is the ring of quasi-Jacobi forms i.e. the ring of functions on \( \mathbb{C} \times H \) generated by coefficients of Taylor expansion in \( x \) of a factor in the product \([3]\) (cf. Appendix II)\(^7\).

By equivariant Riemann Roch theorem, one can interpret \([19]\) as the character decomposition of holomorphic euler characteristic of the \( G \)-equivariant bundle \([3]\) where \( T_X, \Omega^1 \) endowed with natural \( G \)-structure (cf. [22]).

In the case when \( G \) is a torus \( T \) (affine connected commutative algebraic group) of dimension \( r \), the equivariant elliptic class in \( A^*(BT, \text{Jac}) \) can be viewed as an element of the ring of polynomials in \( r \) variables with coefficients in the ring of quasi-Jacobi forms (cf. Appendix II).

\(^7\) as in [14] one can use any ring of functions containing the coefficients of expansion of elliptic genera of manifolds in Chern classes.
4.2. **Equivariant localization.** Let $T$ be a torus acting algebraically on a smooth quasiprojective scheme $X$. Let $\hat{T}$ be the group of characters of $T$. An identification $T = \mathbb{C}^*$ induces the identification of $\hat{T}$ with a free abelian group generated by character $t_1, \ldots, t_r \in \hat{T}$ (such that $t_i(z_1, \ldots, z_r) = z_i \in \mathbb{C}^*$). $T$-equivariant Chow ring of a point, i.e. $\mathbb{A}^*(BT)$, as was already mentioned, is isomorphic to the symmetric algebra of free abelian group $\hat{T}$. More generally, if $T$ acts trivially on $X$ then $\mathbb{A}^2_T(X) = A_*(X) \otimes \text{Sym}(T)$ (here $\text{Sym}(T)$ is the symmetric algebra with generators $t_1, \ldots, t_r$; cf. [21]). For details of the following we refer to [21].

**Theorem 4.2.** Let $R_T = \text{Sym}(\hat{T})$, $\Omega_T = (R_T^+)^{-1}R_T$ where $R_T^+$ is the semigroup of elements of positive degree and $i : X_T \rightarrow X$ be the embedding of the fixed point set. Then

\[(20) \quad i_* : A_*(X_T) \otimes \Omega_T \rightarrow A^T_*(X) \otimes \Omega_T
\]
is an isomorphism.

If $Y$ and $X$ are smooth, $j : Y \rightarrow X$ is a regular embedding of codimension $d$, $N$ is the normal bundle of $Y$ in $X$ and $\alpha \in A_*(Y)$, one has the self intersection formula $j^*j_*(\alpha) = c_d(N) \cap \alpha$ (cf. Sect. 6.3, Cor. 6.3 [24]). If $F$ is fixed point set of a torus $T$ acting on a smooth scheme $X$, then $F$ is smooth and self intersection formula applied to $i_F : F \times U/G \rightarrow X \times U/G$ implies $i^*_F i_*(\alpha) = c_d^T(N) \cap \alpha$. This results in an explicite localization isomorphism:

\[(21) \quad A^T_*(X) \otimes \Omega_T \rightarrow A_*(X_T) \otimes \Omega_T : \quad \beta \rightarrow \frac{i^*_F(\beta)}{c_d^T(N)}
\]

(here $c_d^T(N)$ denotes the equivariant Chern class of the normal bundle to the fixed point set).

4.3. **Push forward of equivariant elliptic class and equivariant McKay correspondence.** Above approach to equivariant intersection theory allows to deduce directly the equivariant counterparts of the key results about elliptic genus: the push forward formula of elliptic class and the McKay correspondence. A different derivation of these properties was given in [52].

Let $\bar{X}$ be a smooth projective variety with a biholomorphic action of a torus $T$. Let $E = \sum \alpha_i E_i$ be a normal crossings divisor on $\bar{X}$ such that all irreducible components $E_i$ are $T$-invariant. Then (in notations of section [14]) $(E_i \times U)/T$ is a divisor on $(\bar{X} \times U)/T$ and hence the classes $e_i^T \in A^T_*(\bar{X})$ are well defined. Using [14] we obtain the equivariant elliptic class $E \mathcal{L} \mathcal{L}_T(\bar{X}, E) \in A^T_*(\bar{X}, Q\text{Jac})$.

**Theorem 4.3.** (Push forward formula.) Let $\bar{X}$ be a smooth projective variety with a torus $T$ acting on $\bar{X}$ via biregular automorphisms. Let $E$ be a $T$-invariant normal crossings divisor and $Z$ a smooth $T$-invariant submanifold of $\bar{X}$ transversal to all irreducible components of $E$. Let $\phi : \bar{X}' \rightarrow \bar{X}$ be the blow up of $\bar{X}$ with center at $Z$ and let $E'$ be the divisor on $\bar{X}'$ such that $\phi^*(K_{\bar{X}} + E) = K_{\bar{X}', E'}$. Then the action of $T$ on $\bar{X} \setminus Z$ extends to the action on $\bar{X}'$ leaving $E'$ invariant and

\[(22) \quad \phi_* (E \mathcal{L} \mathcal{L}_T(\bar{X}', E')) = E \mathcal{L} \mathcal{L}_T(\bar{X}, E)
\]

where on the left one has the equivariant elliptic class for the action on $\bar{X}'$ induced by the action of $T$ on $\bar{X} \setminus Z$. 

Proof. Let \( \pi : \hat{X}_T \rightarrow BT \) be a locally trivial fibration defined by the action of \( T \) and a representation of \( T \) as in section 4.1 (recall that \( BT = U/T \) is the quotient space of a Zariski open set \( U \) in the representation space with sufficiently large codimension of the complement to \( U \)). Since \( Z \) and \( E_i \) are \( T \)-invariant, one has embedding of fibrations \( Z_T \rightarrow \hat{X}_T, (E_i)_{\hat{T}} \rightarrow \hat{X}_T \) of subvarieties of \( \hat{X}_T \) corresponding to \( Z \) and \( E_i \) compatible with projections on \( T \). Let \( \hat{X}'_T = (\hat{X}' \times U)/T \) and \( \phi_T : \hat{X}'_T \rightarrow \hat{X}_T \) be induced morphism. \( \hat{X}'_T = (\hat{X}' \times U)/T \) can be identified with the blow up of \( \hat{X}_T \) along \( Z_T \). This can be seen for example from a local description of blow up as in [51] Def. 3.23. Moreover, \( E_T = \sum \alpha_i(E_i)_{\hat{T}} \), the multiplicity of \( (E_i)_{\hat{T}} \) along \( Z_T \) is the same as multiplicity \( \beta_i \) of \( E_i \) along \( Z \) and codimension of \( Z_T \) in \( \hat{X}_T \) coincides with the codimension of \( Z \) in \( \hat{X} \). It follows that \( (E'_i)_{\hat{T}} \), which irreducible components are the proper preimages of \( (E_i)_{\hat{T}} \), and the exceptional locus of \( \phi_T \) all have the same multiplicities as do the corresponding components in \( E' \) (cf. [8] p.327 and also theorem 3.3). Therefore \( \phi^*_T(K_{\hat{X}_T} + E_T) = K_{\hat{X}'_T} + E_T \). Now Theorem 3.5 in [9] immediately implies Theorem 4.3. \qed

As in non-equivariant case, push forward formula (22) shows that the following definition is independent of resolution it uses.

**Definition 4.4. Equivariant singular elliptic class.** Let \( X \) be a \( \mathbb{Q} \)-Gorenstein projective variety with at most klt singularities on which a torus \( T \) acts by regular automorphisms. Let \( f : \hat{X} \rightarrow X \) be an equivariant resolution of its singularities and \( \hat{E} = \sum \alpha_k \hat{E}_k \) be a normal crossing divisor on \( \hat{X} \) such that \( f^*(K_{\hat{X}}) = K_{\hat{X}} + \hat{E} \). Equivariant singular elliptic class is defined as

\[
\mathcal{E}LL^T_{\text{sing}}(X) = f_* (\mathcal{E}LL^T(\hat{X}, \hat{E}))
\]

(it is independent of a choice of equivariant resolution). Equivariant singular elliptic genus is the push forward of \( \mathcal{E}LL^T_{\text{sing}}(X) \) to the Chow ring of the point (cf. Def. [19]).

In the case when the singular variety is an orbifold with an action of a torus one has equivariant version of orbifold elliptic class related to just described equivariant singular elliptic class.

**Theorem 4.5.** (Equivariant version of McKay correspondence) Let \( X \) be a smooth projective variety with a torus \( T \) acting on \( X \) via biregular automorphisms. Let \( \Gamma \) be a finite group which action commute with the action of \( T \). Then, for any pair \( (g, h) \in \Gamma \) of commuting elements, the fixed point locus \( X^{g,h} \) is \( T \)-invariant, the class obtained by replacing in elliptic class appearing in (17) the ordinary Chern roots of the bundles \( V_\lambda \) by the equivariant Chern roots of these bundle with natural \( T \)-structure, and called the **equivariant orbifold class** of \((X, T, \Gamma)\), satisfies the following push forward formula.\(^8\) If \( \psi : X \rightarrow X/\Gamma \) is the quotient morphism, then

\[
\psi_* (\mathcal{E}LL^T_{\text{orb}}(X, \Gamma)) = \mathcal{E}LL^T_{\text{sing}}(X/\Gamma)
\]

**Proof.** This follows from corresponding results in [9] as in the proof of theorem 4.3. Since the actions of \( T \) and \( \Gamma \) commute, the torus \( T \) acts on \( X \), \( g, h \in \Gamma \), the action

\(^8\)Here we consider the full elliptic class i.e. for each commuting pair \( g, h \) one takes the cap product of class obtained by expansion of \( \theta \)-functions with the fundamental class \([X^{g,h}]\). This cup product is an element of equivariat Chow ring of \( X^{g,h} \). The push forward of this cap product to the Chow ring of a point gives the equivariant orbifold elliptic genus and is an element in the ring of formal power series in characters of \( T \).
of $\Gamma$ on $X$ induces the action of $X_T = (X \times U)/T$ via action on the first factor and the fixed point set of $\gamma \in \Gamma$ is $X_T^\gamma$. Hence $\mathcal{E}LL^T_{\text{orb}}(X, \Gamma) = \mathcal{E}LL_{\text{orb}}(X_T, \Gamma)$. Now the theorem follows from the theorem 5.3 in [9] applied to the action of $\Gamma$ on $X_T$.

4.4. Push forward of contributions of components of fixed point set. The localization map $[20]$ allows to associate with a fixed component $F$ of an action of a torus an invariant constructed using contribution of $F$ into equivariant elliptic class of $X$. In the case when $X$ is a smooth projective variety the sum over all fixed components of these contributions evaluated on corresponding fundamental classes of the components coincides with the equivariant elliptic genus of $X$ (cf. [2]). In the case when $X$ is only quasi-projective but a component $F$ is compact, the corresponding contribution is well defined and though by itself it does not have a geometric interpretation, this contribution does play the key role in definitions of next section. Here we shall describe the push forward property of contributions of compact components and its generalization to the orbifold case.

Definition 4.6. (Local contribution of a component of fixed point set: smooth case.) Let $X$ be a smooth quasi-projective variety, $T$ as above and let $E$ denotes a normal crossing divisor with $T$-invariant irreducible components. Let $F$ be a component of the fixed point set. Assume that $F$ is compact and let $\iota_F : F \to X$ denotes its embedding. Let $c_{\text{codim}F}(N_F)$ be the equivariant Chern class of the normal bundle of $F$ in $X$. Then the local contribution of $F$ into equivariant elliptic genus of the pair $(X, E)$ is the class

$$\mathcal{E}LL(X, E) = \frac{i_F^* \mathcal{E}LL(X, E)}{c_{\text{codim}F}(N_F)} \in A_*(F)\{\{q, y\}\} \otimes \Omega$$

Theorem 4.7. (Push forward for local contribution of equivariant elliptic genus) Let $X$ be a smooth quasi-projective variety with action of a torus $T$ and let $F \subset X$ be a component of the fixed point set which is compact. Denote by $\phi : X' \to X$ $T$-equivariant blow up with $T$-invariant center $Z \subset F$ and let $Z' = \bigcup_{F' \in \text{Irr}(T)} F'$ be the union of submanifolds $F'$ from the set $\text{Irr}(T)$ of irreducible components of the fixed point set $X'$ of the action of $T$ on $X'$ mapped by $\phi$ onto $F$. Let $E$ be $T$-invariant normal crossing divisor all component of which are transversal to $Z$ and $E'$ be the divisor on $X'$ such that $\phi^*(K_X + E) = K_{X'} + E'$. Then

$$\phi_* \left( \sum_{F' \in \text{Irr}(T)} \frac{i_{F'}^* \mathcal{E}LL^T(X', E')}{c_{\text{codim}F'\subset X'}(N_{F'}/X')} \right) = \frac{i_{F'}^* \mathcal{E}LL^T(X, E)}{c_{\text{codim}F\subset X}(N_{F}/X)}$$

Proof. Let $\bar{X}$ be a compactification of $X$ and $\bar{X}'$ be the blow up of $\bar{X}$ at $Z \subset X \subset \bar{X}$. Let $T = \bigcup_{F' \in \text{Irr}(T)} F'$ (resp. $\bar{T} = \bigcup_{F' \in \text{Irr}(\bar{T})} F'$) be the submanifold of $\bar{X}'$ of fixed points of action of $T$ on $\bar{X}'$ (resp. $\bar{X}$) and $i_{T'} : T' \to \bar{X}'$ (resp. $i_{\bar{T}} : \bar{T} \to \bar{X}$) be their embeddings. The push forward formula of theorem 4.3 can be rewritten as:

$$i_{T'}(i_{T'}^{-1} \phi_* i_{T'}) \mathcal{E}LL^T(X', E') = \mathcal{E}LL^T(X, E)$$

Now using description of the inverse of $i_*$ given in [21] and $\phi_\bullet \gamma_{\bar{T}} = i_{T'}^{-1} \phi_* i_{T'}$, we obtain

$$\phi_* \frac{i_{T'}^* \mathcal{E}LL^T(X', E')}{c_{\text{top}}(N_{\bar{T}/\bar{X}'})} = \frac{i_{T'}^* \mathcal{E}LL^T(X, E)}{c_{\text{top}}(N_{\bar{T}/\bar{X}})}$$

9the ring in this formula can be taken to be $A_*(F, Q, \text{Jac}) \otimes \Omega$
Fixed point set $\mathcal{F}'$ (resp. $\mathcal{F}$) is a disjoint union of smooth irreducible components and hence $A^*(\mathcal{F}') = \oplus_{F' \in Irr(\mathcal{F}')} A^*(F')$ (similar direct sum decomposition for $\mathcal{F}$) where summation is over the set $Irr(\mathcal{F}')$ of irreducible components of $\mathcal{F}'$ (resp. $\mathcal{F}$). The split is given by projections $i_{F'}^* : A^*(\mathcal{F}') \rightarrow A^*(F')$ (resp. $i_F^* : A^*(\mathcal{F}) \rightarrow A^*(F)$) where $i_{F'} : F' \rightarrow \mathcal{F}'$ is embedding of an irreducible component into the disjoint union (and the same for $F$). The map $\phi|_{\mathcal{F}'}$ respects the above direct sum decomposition with $\phi|_{\mathcal{F}'}^{-1}(A^*(F)) = \oplus_{F' \in Irr(\mathcal{F}') \cap F'} A^*(F')$. This implies (20). □

4.5. Contributions of components of fixed point set into orbifold elliptic genus. Let $X$ be a smooth quasi-projective variety, let $T$ be a torus acting on $X$ effectively and let $\Gamma$ is a finite group acting upon $X$, (all actions are via biholomorphic automorphisms). We shall assume that the action of $\Gamma$ commutes with the action of $T$ i.e. for all $t \in T, \gamma \in \Gamma$ and any $x \in X$ one has $\gamma t \cdot x = t \cdot \gamma x, \gamma, t \in Aut(X)$. This implies that $\Gamma$ leaves invariant the fixed point set $X^T$ of the torus $T$, each fixed point set $X^g, g \in \Gamma$ is $T$-invariant and that $T$ acts on the quotient $X/\Gamma$. We denote by $T^{eff}$ the quotient of $T$ which acts effectively on $X/\Gamma$.

If $F$ is a connected component of $X^T$ and $F^\gamma$ is a component of the fixed point set of an element $\gamma \in \Gamma$ acting upon $F$ then restriction of cotangent (or tangent) bundle $\Omega^1_X$ of $X$ on $F^\gamma$ has a canonical structure of an equivariant $T$-bundle. If $V$ is an eigenbundle of this $T$-action on $\Omega^1_X|F^\gamma$, then, since we assume that actions of $\Gamma$ and $T$ commute, $V$ is invariant under the action of $\gamma$ as well.

If $rk V = 1$ then, as in section 3.2 for $\gamma \in \Gamma$ we let $\lambda(\gamma)$ denote the logarithm $\frac{1}{2\pi i} \log \in [0, 1)$ of the value on $\gamma$ of the character of action on $V$ of the subgroup $< \gamma >$ of $\Gamma$ generated by $\gamma$. We assign the subscript $\lambda$ to such a line bundle $V$, put $x_\lambda = c_1(V_\lambda) \in A^2_t(F^\gamma)$ and count the class $x_\lambda$ with multiplicity equal to the multiplicity of the character $\gamma \rightarrow exp(2\pi i \lambda(\gamma))$ in the bundle $\Omega^1_X|F^\gamma$. Similar collection of equivariant Chern classes arises from the normal bundles to the fixed point sets $F \cap F^h$ of pairs $g, h$ commuting elements in $\Gamma$.

Definition 4.8. Let $F \subset X$ be a connected compact component of the fixed point set of an action of $T$ and let for a commuting pair $g, h \in \Gamma$, $F^g_h$ denotes submanifold of $F$ consisting of the points fixed by both $g$ and $h$. We associate with a connected component of $F^g_h$ and a rank one $T$-eigenbundle $V$ of $\Omega^1_X|F^g_h$, the characteristic class in the ring $A^*_T(X, \mathbb{C})[[g, h]]$ given by:

$$\Phi^{T}_{F^g_h}(x^T, g, h, z, \tau, \Gamma) = \frac{\theta(x^T_{2\pi i} + \lambda(g) - \tau \lambda(h) - z)}{\theta(x^T_{2\pi i} + \lambda(g) - \tau \lambda(h))} e^{2\pi i z \lambda(h)}$$

where $x^T$ is equivariant Chern class of $V$.

Below we also denote by $Cong(\Gamma)$ the set of conjugacy classes of $\Gamma$, $C(g), g \in \Gamma$ will denote the centralizer of $g$, $\Lambda$ be the set of $(g, h)$-eigenbundles of tangent bundle to $X$ restricted to $F^g_h$ and $\Lambda_{F^g_h}$ will be the collection of $(g, h)$-eigenbundles of $N_{F^g_h, C}F$ such that $\lambda(g) = \lambda(h) = 0$.

Definition 4.9. The contribution of $F \in X^T$ into $T$-equivariant orbifold elliptic genus of $(X, \Gamma)$ is the sum:

$$\mathcal{EL}(X, \Gamma, u, z, \tau) =$$

$$\sum_{(g) \in Cong(\Gamma)} \frac{1}{|C(g)|} \sum_{h \in C(g)} ( \prod_{\lambda \in \Lambda_{F^g_h}} x_\lambda ) \prod_{\lambda \in \Lambda} \Phi^{T}_{F^g_h}(x_\lambda, g, h, z, \tau, \Gamma)[F^g_h]$$
where $N_{F^g,h} \subset F$ is the normal bundle to $F^g,h$ in $F$ and all equivariant Chern classes expressed in terms of the characters of $T^{eff}$.

The motivation of this definition is the following. Orbifold elliptic genus \((1,2)\) is a sum over pairs of commuting elements in $\Gamma$ of classes in the Chow ring (which are combinations of Chern classes $x_{\lambda}$ of bundles $V_{\lambda}$ in (10)) evaluated on the fundamental class of $X^g,h$ (cf. proof of theorem 4.3 in [8]). In the case when $X$ is projective, the localization formula (cf.(21)) applied to the equivariant version of the orbifold elliptic genus replaces each summand in (12) by the sum over components $F$ of pullbacks to $F^g,h = F \cap X^g,h$ classes (12) divided by the equivariant top Chern class of the normal bundle to $F^g,h$ in $X^g,h$. Definition 4.9 is the sum over $(g,h)$ of contribution from one individual component $F$.

Example 4.10. Trivial sector of contribution described in Definition 4.9 for $\chi_y$-genus. Specialization to the case $q = 0$ of the term corresponding to pair $g = h = 1$ (i.e. the trivial sector) gives the following local contribution of component $F$ of fixed point set of action of $T$ on $X$ into $\Gamma$-orbifold $\chi_y$-genus:

\[
\chi_y(X, \Gamma, g = h = 1)_{T^{eff}} = \prod_{\lambda} \left( y^{-\frac{1}{2}} x_{\lambda(T)} \frac{1 - ye^{-x_{\lambda(N)}}}{1 - e^{-x_{\lambda(N)}}} \right) \cdot \prod_{\lambda} \left( y^{-\frac{1}{2}} \frac{1 - ye^{-x_{\lambda(N)}}}{1 - e^{-x_{\lambda(N)}}} \right)
\]

where $x_{\lambda(T)}$ are the Chern roots of the tangent bundle to $F$ (appearing in the first product) and $x_{\lambda(N)}$ are the equivariant Chern roots of the normal bundle to $F$ (contributing to the second factor in (31)). Indeed, in the sector $g = h = 1$ in (30), we have only one term which is specialization of class $\Phi$ given in Definition 4.8.

The contributions into orbifold elliptic genus corresponding to compact components of fixed point set described in Def. 4.9 satisfy the following, localized at $F$, McKay correspondence proof of which can be obtained in the same way as the proof of theorem 4.7. More general case, providing local equivariant version for pairs as in [9] can be obtained similarly.

Theorem 4.11. Let $X$ be a smooth quasi-projective variety, $T$ a torus and $\Gamma$ a finite group both acting on $X$ via biholomorphic automorphisms so that their actions commute i.e. $\gamma \cdot tv = t \cdot \gamma v, \gamma, t \in T, v \in X$. Let $\phi : \tilde{X} \to X/\Gamma$ be a crepant resolution of singularities of the quotient $X/\Gamma$ (if it exist.) As above, denote by $T^{eff}$ the quotient of $T$ by the finite group which acts effectively on $X/\Gamma$. Let $F \subset X/\Gamma$ be a component of the fixed point set of $T^{eff}$ and $F'$ be the collection of components of the fixed point set of $T$ such that $\phi(F') \subset F, F' \in F'$. Then

\[
\sum_{F' \in F'} \mathcal{E}LL_{F'}^{T^{eff}}(\tilde{X}) = \mathcal{E}LL_{T^{eff}}^{T^{eff}}(X, \Gamma)
\]

In the next section, we consider explicit examples of calculations of contributions of fixed components of $\mathbb{C}^*$-actions on the GIT quotients by the actions of tori on bundles over quasi-projective varieties. They will provide ample illustration to the theorem 4.11.

5. Elliptic genus of phases.

This section discusses applications of the local contributions of compact components of the fixed point sets introduced in previous section in the special case when action of $T = \mathbb{C}^*$ takes place on a GIT quotient of the total space of a vector bundle.
by an action of reductive group. This action of $T$ is canonical in the sense that it is induced from the action of $T$ on the total space of vector bundle by dilations $v \to t \cdot v, t \in \mathbb{C}^*$. This is an extension of the framework of examples considered by Witten in [56]. Following this work, in [40] we called our GIT quotient phases we well. We also attached to such framework an elliptic genus and describe its orbifoldization when additional symmetries are present. We show that this extends well known elliptic genera of Landau Ginzburg and $\sigma$-models.

5.1. Phases. We will start with a very special example of a phase considered by Witten ([54]) in which we calculate contribution of component of the fixed point set not into elliptic genus but rather into $\chi_y$-genus (which is the limit $q \to 0$ of the elliptic genus). “Advantage” of $\chi_y$-genus of course is that this is a Laurent polynomial, rather than a more general holomorphic function. In this example we work with $\chi_y$-genus directly, i.e. perform localization of $\chi_y$-genus rather than elliptic genus. Already this calculation in the case of Landau-Ginzburg phase results in Arnold-Steenbrink spectrum of weighted homogeneous singularity, providing interpretation of the latter using equivariant cohomology.

Example 5.1. Let $w_1, \ldots, w_n, D$ be collection of positive integers. Consider $G = \mathbb{C}^*$-action on $\mathbb{C} \times \mathbb{C}^n$ given by

$$
\lambda(s, z_1, \ldots, z_n) = (\lambda^D s, \lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n)
$$

The quotient of the subset in $\mathbb{C} \times \mathbb{C}^n$ given by $s \neq 0$ is the orbifold $W/\mu_D$ where $W$ is a vector space, $\dim W = n$ and $\mu_D$ is the group of roots of unity of degree $D$ acting via multiplication on $W$. The group $T^{eff} = \mathbb{C}^*$ acts on $\mathbb{C} \times \mathbb{C}^n$ via $t(s, z_1, \ldots, z_n) \to (ts, z_1, \ldots, z_n)$ and this action induces effective action of $T^{eff}$ on $W/\mu_D$. The effective $\mathbb{C}^*$-action on $W$, which induces this action of $T^{eff}$ on $W/\mu_D$, is multiplication by $r \in \mathbb{C}^*$ where $r^{-D} = t$ (and $r \exp(2\pi \sqrt{-1} l/D), l \in \mathbb{Z}$ induce the same automorphisms of $W/\mu_D$) i.e. $W$ is acted upon by $T = \mathbb{C}^*$ which the $D$-fold cover of the group $T^{eff}$ acting on $W/\mu_D$. In particular infinitesimal characters of the normal bundle at the fixed point $\mathbb{0}$ of the action of $T$ on $W$ (i.e. the origin) in terms of the characters of $T^{eff}$ are $x_i = -\frac{w_i u}{D}$, where $u$ is the infinitesimal character of $T$.

It follows from (31) that the trivial sector of the local contribution of $\mathbb{0}$ into orbifold $\chi_y$-genus is given by:

$$
\prod_i y^{\frac{1}{2}} \left( \frac{w_i u}{D} \right) \left( 1 - ye^{-\frac{w_i u}{D}} \right) \cdot \frac{1}{\sqrt{w_i u}}
$$

For special value of $u$ given by $u = 2\pi \sqrt{-1} z$ one obtains:

$$
\prod_i y^{\frac{1}{2}} \frac{1 - y^{(1 - \frac{w_i}{D})}}{1 - y^{-\frac{w_i}{D}}} = \prod_i y^{\frac{1}{2}} \frac{y^\frac{w_i}{D} - y}{y^\frac{w_i}{D} - 1}
$$

which coincides with generating function of spectrum as calculated in [47] (its definition reminded in Prop. 6.4).

Now we consider general case for which Example 5.1 is an illustration.

Definition 5.2. (cf. [40]) Let $E$ be the total space of a vector bundle $E$ on a smooth quasi-projective manifold $X$. Let $G$ be a reductive algebraic group acting by biholomorphic automorphism on $E$. Let $\kappa$ be a linearization of this $G$-action
satisfying the conditions of Prop.3.1 in [40]. Phase of $G$-action on $E$ corresponding to linearization $\kappa$ is the GIT quotient $E//_\kappa G = E^{ss}/G$ endowed with the $\mathbb{C}^*$-action induced by $\mathbb{C}^*$-action given by dilations $t(v) = t \cdot v, t \in \mathbb{C}^*, v \in E$.

A phase is called Landau-Ginzburg if this GIT quotient is an orbifold biholomorphic to a quotient of $\mathbb{C}^n$ by a finite subgroup of $GL_n(\mathbb{C})$.

A phase is called $\sigma$-model (resp. Calabi Yau) if this GIT quotient is biholomorphic to the total space of a vector bundle (resp. the canonical bundle) over a compact orbifold.

Change of linearization $\kappa$ of $G$-action on $E$ may result in a birationally equivalent GIT-quotient. More specifically, if $NS^G(E)$ denotes the equivariant Neron-Severi group (in the case when $E$ is an affine space this is just the group $Char$ of characters of $G$), then there is a partition of $NS^G(E) \otimes \mathbb{Q}$ into a union of cones such that GIT-quotients are birational for linearizations within a cone and $E//_\kappa G$ acquires change when $\kappa$ belongs to the boundary of a cone or is moving into adjacent one. For general discussion of changes of GIT-quotients we refer to [40] and to [40] for particular case of the total spaces of bundles as in Definition 5.2.

GIT-quotients are often singular but we will be interested in the cases when they are biholomorphic to global quotients of a smooth manifold which we call uniformization of a global quotient.

**Definition 5.3.** A smooth quasi-projective variety $\bar{X}$ together with an action of a finite group $\Gamma$ is called an uniformization of a phase $E//_\kappa G$ if

1. there exist a biholomorphic isomorphism $E//_\kappa G \to \bar{X}/\Gamma$
2. there is an action of 1-dimensional complex torus $T$ on $X$, a finite degree covering map $\pi : T \to T^{eff}$ of 1-dimensional torus $T^{eff}$ acting on $E//_\kappa G$ via dilations (cf. Def. 5.2) such that the quotient map $\phi : \bar{X} \to \bar{X}/\Gamma = E//_\kappa G$ is equivariant i.e. $\phi(t \cdot x) = \pi(t) \cdot \phi(x), t \in T$.

The following is an illustration to Definitions 5.2 and 5.3 with example borrowed from [50].

**Example 5.4.** Quotient in Example 5.1 is a special case of the quotients considered in Definition 5.2 with $X = \mathbb{C}^n$, $E = \mathbb{C}^{n+1}$ being the total space of the trivial line bundle and $G = \mathbb{C}^*$ acting on $E$ via $\mathbb{C}^*$. In this case $Char(\mathbb{C}^*) \otimes \mathbb{Q} = \mathbb{Q}$, there are two cones and for a pair of linearizations $\kappa_1, \kappa_2$ from distinct cones, the corresponding semi-stable loci are:

$$(\mathbb{C}^{n+1})^{ss}_{\kappa_1} = \mathbb{C} \times (\mathbb{C}^n - 0) \subset \mathbb{C}^{n+1} \quad (\mathbb{C}^{n+1})^{ss}_{\kappa_2} = (\mathbb{C} - 0) \times \mathbb{C}^n \subset \mathbb{C}^{n+1}$$

In the simplest case, when $w_1 = 1, D = n$, the corresponding GIT quotients are respectively the total space $[\mathcal{O}_{\mathbb{P}^{n-1}}(-n)]$ of the canonical bundle over $\mathbb{P}^{n-1}$ and the quotient $W/\mu_n, W = \mathbb{C}^n$ by the group of roots of unity of degree $n$ acting diagonally. As was mentioned in discussion of Example 5.1 the dilations $t \cdot (s, z_1, ..., z_n) = (ts, z_1, ..., z_n)$ induce on $\mathbb{C}^n/\mu_n$ the action $t \cdot [(z_1, ..., z_n) \mod \mu_n] = t(1, z_1, ..., z_n) \mod \mathbb{C}^n = (t, z_1, ..., z_n) \mod \mathbb{C}^n = (t^{- \frac{1}{n}}z_1, ..., t^{- \frac{1}{n}}z_n) \mod \mu_n$. This action is effective on the quotient $W/\mu_n$. Denote by $\pi : \lambda \to t = \lambda^{-n}$ the (cyclic) covering map of one-dimensional tori $\mathbb{C}^*_\lambda \to \mathbb{C}^*_t$ and let $\phi : \mathbb{C}^n \to \mathbb{C}^n/\mu_n$ be the quotient map. Assume that $\mathbb{C}^*_\lambda$ is acting on $\mathbb{C}^n$ via multiplication of coordinates by $\lambda$.

\footnote{which implies that the $\mathbb{C}^*$-action by dilations is well defined on the GIT quotient}
and \( \mathbb{C}^*_\mu \) acts on \( \mathbb{C}^n/\mu_n \) as above. Then \( \phi(t) = t^i \phi(v) \) and therefore we have an uniformization in the sense of Definition 5.3. Here we have a LG phase. The quotient which is the total space \( O_{\mathbb{P}^n-1}(-n) \) is the \( \sigma \)-model (in fact CY) phase. Here GIT quotient is smooth, dilations on \( \mathbb{C} \times \mathbb{C}^n \) induce on \( O_{\mathbb{P}^n-1}(-n) \) the multiplication by elements of \( \mathbb{C}^* \) which is an effective action and does not require uniformization.

5.2. **Elliptic genus of a phase.** Next we shall define elliptic genus of a phase for which the fixed point set of \( \mathbb{C}^*_\mu \)-action induced by dilations has a compact component.

**Definition 5.5. (Elliptic genus of a phase)** Let \( X,G,\mathcal{E},\kappa \) be as in Def. 5.2. Assume that \( \mathcal{E}/\kappa G \) admits uniformization \( \widetilde{\mathcal{E}}/\kappa G \) i.e. \( \mathcal{E}/\kappa G/\Gamma = \widetilde{\mathcal{E}}/\kappa G \) for an action of a finite group \( \Gamma \) and that one has the action of \( T = \mathbb{C}^* \) on \( \mathcal{E}/\kappa G \) such that the quotient map \( \mathcal{E}/\kappa G \rightarrow \mathcal{E}/\kappa G \) is equivariant for the \( \mathbb{C}^* \)-action on \( \mathcal{E}/\kappa G \) induced by dilations on \( \mathcal{E} \). Let \( F \subset \mathcal{E}/\kappa G \) be a compact component of fixed point set of \( T \)-action on \( \mathcal{E}/\kappa G \). Consider the local contribution of the component \( F \) into \( T \)-equivariant orbifold elliptic genus

\[
Ell_{\text{orb}}^T (\mathcal{E}/\kappa G, \Gamma, u, z, \tau)
\]

given by Definition 5.3 where \( u \) is an infinitesimal character of the action of maximal, effectively acting quotient \( T^{\text{eff}} \). Then the elliptic genus of the phase \((X,G,\mathcal{E},\kappa)\) relative to the component \( F \), denoted as \( Ell(\mathcal{E}/\kappa G, F, z, \tau) \), is defined as the restriction of the local contribution \((37)\) on the diagonal \( u = z \in \mathbb{C} \times \mathbb{C} \times \mathcal{E} \):

\[
Ell(\mathcal{E}/\kappa G, F, z, \tau) = Ell_{\text{orb}}^T (\mathcal{E}/\kappa G, \Gamma, F, z, z, \tau)
\]

More generally, the same definition can be used in the cases when \( \mathcal{E}/\kappa G \) has Kawamata log-terminal singularities and when \( Ell(\mathcal{E}/\kappa G, \Gamma) \) is well defined as the orbifold elliptic genus of a pair obtained via a resolution of singularities and taking into account the divisor determined by the discrepancies of the resolution (cf. [5]).

In the next theorem we shall describe a class of phase transitions in which one can apply equivariant McKay correspondence to obtain invariance of elliptic genus in such transitions.

**Theorem 5.6. (Invariance of elliptic genus in phase transitions.)** Let \( \mathcal{E}/\kappa_1 G = X_1 = \bar{X}_1/\Gamma, \mathcal{E}/\kappa_2 G = X_2 = \bar{X}_2/\Gamma, X_1, X_2, \Gamma \) are as in 5.3. Assume that \( \psi : X_1 \rightarrow X_2 \) is a K-equivalence i.e. \( \psi^*(K_{X_2}) = K_{X_1} \). Then

\[
\sum_{F_i} Ell(L/\kappa_1, F_i) = Ell(L/\kappa_2, F)
\]

where \( F_i \) is collection of fixed point sets which \( \psi \) takes into \( F \).

5.3. **Quotients of phases by the action of a finite group.** Constructions of mirror symmetry in toric or weighted homogeneous case (cf. [3] and [4]) suggest to consider orbifoldization of phases with respect to finite groups. Even the very first construction of mirror symmetric of Calabi Yau quintic in \( \mathbb{P}^4 \) (cf. [11]) was obtained via orbifoldization. The orbifoldization of elliptic genus of Calabi Yau and Landau-Ginzburg models was proposed in [4], [34]. Here we discuss orbifoldization of arbitrary phases including hybrid ones.
Let $X$ be quasi-projective manifold with an action for a reductive group $G$ and let $\Gamma$ be a finite subgroup of the group of biregular automorphisms of $X$ which normalizes $G$ i.e. for any $\gamma \in \Gamma, g \in G$ one has $\gamma g \gamma^{-1} \in G$. We say that $\Gamma$ normalizes a linearization $\kappa$ of $G$-action on $X$ if action of $\Gamma$ of $X$ lifts to the action on the total space of the ample line bundle $L_\kappa$ underlying $\kappa$ so that this lift normalizes the action of $G$ on the total space of $L_\kappa$. This assumption implies that $\Gamma$ acts on the semi-stable locus

$$X^{ss} = \{x \in X|\exists \gamma \in \Gamma(X, L_\kappa^\otimes m)^G, s(x) \neq 0\}$$

Here the action of either $G$ or $\Gamma$ on $\Gamma(X, L_\kappa^\otimes m)$ is given by $(g)(x) = gs(g^{-1}x)$ ($g$ is an element of either $G$ or $\Gamma$). Indeed, $(\gamma \cdot s)(\gamma(x)) = \gamma s(x) \neq 0$ if $s(x) \neq 0$. The action of $\Gamma$ on $X^{ss}$ in turn defines its action on $X/\kappa\Gamma$.

First we shall consider orbifoldization of elliptic genus (i.e. defining the elliptic genus of the corresponding orbifold) in the case when GIT quotient $E/\kappa\Gamma$ is smooth.

**Definition 5.7.** (Orbifoldization of smooth phases). Let $X, E, G, \kappa$ be as in Definition 5.2, $\Gamma$ be a finite group of automorphisms of bundle $E \to X$ normalizing linearization $\kappa$ and $E/\kappa\Gamma$ be the phase corresponding to $X, E, \kappa$ endowed with the action of $\Gamma$ induced from the action on $G$-semistable locus in $E$ corresponding to $\kappa$. If $E/\kappa\Gamma$ is smooth and $F$ is a compact component the fixed point set of the $\mathbb{C}^*$ action on $E/\kappa\Gamma$ induced by dilations $\lambda(v) = \lambda \cdot v, v \in E, \lambda \in \mathbb{C}^*$ then the $\Gamma$-orbifoldized elliptic genus of this phase corresponding to $F$ is the contribution (4.3) of component $F$ into $\mathbb{C}^*$-equivariant $\Gamma$-orbifold elliptic genus of $E/\kappa\Gamma$.

More generally, in the case when $(E/\kappa\Gamma)$ is an orbifold, assume further that it is a global quotient admitting as uniformization $(Y, \Gamma, T)$ in the sense of Definition 5.3 and that there is a finite group $\Delta$ of automorphisms of $Y$, containing $\Gamma$ as a normal subgroup, with action of $\Delta$ commuting with the action of $T$. We want to describe the $\Delta/\Gamma$-orbifold elliptic genus attached to $E/\kappa\Gamma = Y/\Gamma$ for the action induced by the action of $\Delta$.

Let $F_Y$ be the preimage in uniformization of a component of the fixed point set $F \subset (E/\kappa\Gamma)$. Then the $\Delta$-orbifoldized contribution of $F$ is the sum over all connected components $Q$ in $F_Y$ of $\Delta$-orbifoldized contributions of components $Q$ into equivariant elliptic genus of $Y$ as described in Definition 4.3. More precisely, let $Q_{g,h}$ be fixed point set of pair of commuting elements $g, h \in \Delta$ acting on $Q$, $V_\lambda \subset T_Y|_{Q_{g,h}}$ be the eigenbundle of the subgroup $g, h \Delta$ generated by $(g, h)$, $\Lambda$ is the full set of such eigenbundles in $T_Y|_{Q_{g,h}}$, $\Lambda_{g,h} = \{\lambda\} \subset \Lambda$ is the set of eigenbundles in the normal bundle to $Q_{g,h}$ in $Q$ such that $\lambda(g) = \lambda(h) = 0$. Since we assume that the actions of $\Delta$ and $T$ commute, bundles $V_\lambda$ are the eigenbundles of $T$ as well. Let $x_\lambda^T$ be $T$-equivariant Chern classes of $V_\lambda$ written in terms of the characters of $T^{eff}$, which is the quotient of $T$ acting effectively on the orbifold $E/\kappa\Gamma = Y/\Gamma$.

**Definition 5.8.** $\Delta$-orbifoldized contribution of component $Q$ into equivariant elliptic genus of $E/\kappa/\Gamma$ is given as follows:

$$\frac{1}{|\Delta|} \sum_{g,h = bh, g,h \in \Delta} \prod_{\lambda \in \Lambda_{Q_{g,h}}} x_\lambda \prod_{\lambda \in \Lambda_Q} \Phi_{Q_{g,h}}^{eff}(x_\lambda, g, h, z, \tau, \Delta)[Q_{g,h}]$$

In particular $\Delta$ acts on the quotient $Y/\Gamma$.
where
\[
\Phi_{g,h}^{T_{eff}} (x^T, g, h, z, \tau, \Delta) = \frac{\theta(\frac{x^T}{2\pi i} + \lambda(g) - \tau \lambda(h) - z)}{\theta(\frac{x^T}{2\pi i} + \lambda(g) - \tau \lambda(h))} e^{2\pi i \lambda(h)}
\]

The next final section contains examples showing how these definitions yield the invariants of Calabi Yau and Landau-Ginzburg models which already appeared in the literature as well as explicit examples of some hybrid models.

6. Calculations of elliptic genera of phases and their specializations.

6.1. Elliptic case: weighted projective spaces and LG models. The following is continuation of examples 5.1 and 5.4 giving explicit form of elliptic genera of corresponding phases and their specializations. We shall start with the case of GIT quotient from Example 5.4 i.e. Example 5.1 with \( w_i = 1, D = n \).

Proposition 6.1. Consider \( \mathbb{C}^* \)-action on \( \mathbb{C} \times \mathbb{C}^n \) given by:
\[
\lambda(s, z_1, ..., z_n) = (\lambda^{-n} s, \lambda z_1, ..., \lambda z_n)
\]
There are two GIT quotients corresponding to linearizations \( \psi(\lambda) = \lambda^r \) with \( r > 0 \) (called \( \sigma \)-model phase) biholomorphic to the total space \( \mathcal{O}_{\mathbb{P}^n-1}(-n) \) of canonical bundle of \( \mathbb{P}^n-1 \) and \( r < 0 \) (called Landau-Ginzburg phase) biholomorphic to \( \mathbb{C}^n/\mu_n \).

1. The trivial sector of elliptic genus of Landau Ginzburg phase is given by
\[
(-1)^n \left( \frac{\theta(z(1 - \frac{1}{n}))}{\theta(\frac{z}{n})} \right)^n
\]
2. The elliptic genus of Landau-Ginzburg phase is given by:
\[
\frac{1}{n} \sum_{0 \leq a, b < n} \left( \frac{-\theta((1 - \frac{1}{n}) z + \frac{(a-b)z}{n})}{\theta(z + \frac{(a-b)z}{n})} \right)^n e^{2\pi i k z}
\]
3. The elliptic genus of \( \sigma \)-model phase is given by:
\[
\left( x^{\frac{n}{2\pi i}} z \right)^{n-1} \left( \frac{\theta\left(\frac{nx}{2\pi i}\right)}{\theta\left(\frac{nx}{2\pi i} - z\right)} \right)^n [\mathbb{P}^{n-1}]
\]
and coincides with the elliptic genus of smooth hypersurface of degree \( n \) in \( \mathbb{P}^{n-1} \).
4. (LG-CY correspondence) The elliptic genera (44) and (43) of \( \sigma \) and LG models respectively coincide.

Proof. Calculation of GIT quotients was already made in Example 5.4. The uniformization is given by \( W \rightarrow W/\mu_n \) with \( \mathbb{C}^* \)-action given by dilations of \( W \). The normal bundle of the fixed point, i.e. the origin \( \emptyset \) is direct sum of lines with equivariant Chern class being \( \frac{u}{n} \) where \( u \) is the infinitesimal character of \( \mathbb{C}^* \) acting effectively on \( W/\mu_n \). Hence contribution of the origin \( \mathcal{E}LL_{\emptyset}^C(W, \mu_n) \) into equivariant elliptic genus is given by
\[
\frac{1}{n} \sum_{0 \leq a, b < n} \left( \frac{-\theta\left(\frac{nx}{2\pi i} - z + \frac{(a-b)z}{n}\right)}{\theta\left(\frac{nx}{2\pi i} + \frac{(a-b)z}{n}\right)} \right)^n e^{2\pi i k z}
\]
which for \( u = 2\pi i z \) gives (43). For \( a = b = 1 \) one obtains (42).

In the case of \( \sigma \)-model, the \( \mathbb{C}^* \)-action is the action via dilations on the fibers of the total space of \( \mathcal{O}_{\mathbb{P}^n-1}(-n) \). The tangent bundle, of this total space \([\mathcal{O}_{\mathbb{P}^n-1}(-n)]\),

\[
\Phi_{g,h}^{T_{eff}} (x^T, g, h, z, \tau, \Delta) = \frac{\theta(\frac{x^T}{2\pi i} + \lambda(g) - \tau \lambda(h) - z)}{\theta(\frac{x^T}{2\pi i} + \lambda(g) - \tau \lambda(h))} e^{2\pi i \lambda(h)}
\]
restricted to the fixed point set, i.e. the zero section, get contributions from the tangent bundle to $\mathbb{P}^{n-1}$ and from line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$. The equivariant Chern polynomial of the tangent bundle to $\mathbb{P}^{n-1}$ is $(1 + x)^n$ and the equivariant Chern class of $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ is $\frac{-nx}{2\pi i}$. Hence the contribution of the fixed point set is:

$$\left(x \frac{\theta(\frac{nx}{2\pi i})}{\theta(\frac{x}{2\pi i})}\right)^{n-1} \left(x \frac{\theta\left(-\frac{nx}{2\pi i} + \frac{n}{2\pi i}ight)}{\theta\left(-\frac{x}{2\pi i} + \frac{1}{2\pi i}\right)}\right)[\mathbb{P}^{n-1}]$$

Since $\theta(z)$ is an odd function, for $u = 2\pi iz$ we obtain \[14\]. Since the Chern roots of a hypersurface $V_{n-2}$ of degree $n$ in $\mathbb{P}^{n-1}$ are found from relation $c(V_{n-2}) = \frac{(u+x)^n}{(1+ux)}|_{V_{n-2}}$, it follows that elliptic genus of hypersurface is given by

$$\left(x \frac{\theta(\frac{x}{2\pi i})}{\theta(\frac{x}{2\pi i})}\right)^{n-1} \left(x \frac{\theta\left(-\frac{x}{2\pi i} - \frac{x}{2\pi i}\right)}{n \cdot \theta\left(-\frac{x}{2\pi i} - \frac{x}{2\pi i}\right)}\right)[V_{n-2}]$$

The latter coincides with \[17\] since $[V_{n-2}] = nx \cap [\mathbb{P}^{n-1}]$.

The LG/CY correspondence follows from McKay correspondence since contraction $[O_{\mathbb{P}^{n-1}}(-n)] \to \mathbb{C}^n/\mu_n$ is a crepant morphism. \[\square\]

In the case when the action in Example 5.1 has arbitrary weights we obtain the following:

**Proposition 6.2.** Consider $\mathbb{C}^*$-action on $\mathbb{C} \times \mathbb{C}^n$ with weights $w_1, \ldots, w_n$ ($w_i \in \mathbb{Z}$ pairwise relatively prime) and degree $D \in \mathbb{Z}_{>0}$ given by \[33\]:

$$\lambda(s, z_1, \ldots, z_n) = (\lambda \cdot D, s, \lambda^w z_1, \ldots, \lambda^w z_n)$$

There are two GIT quotients corresponding to linearizations $\psi(\lambda) = \lambda^r$ with $r > 0$ (called $\sigma$-model phase) and $r < 0$ (called Landau-Ginzburg phase) respectively.

1. The trivial sector of elliptic genus of Landau-Ginzburg phase is

$$\prod_{\lambda} \frac{\theta(\frac{w_z}{D} - z)}{\theta(\frac{w_z}{D} - z)}$$

2. The elliptic genus of Landau-Ginzburg phase is given by:

$$\frac{1}{D} \sum_{0 \leq a < b < D} \prod_{i=1}^{a(n-1)} \theta\left(\frac{a}{D} - 1\right) + \frac{w_i(a-b\tau)}{D} \right) e^{2\pi \sqrt{-1} \frac{b w_i}{D}}$$

3. Let $\Gamma = \mu_{w_1} \times \ldots \times \mu_{w_n}$ be group of product of roots of unity acting coordinatewise on $\mathbb{P}^{n-1}$. Then with $x \in H^2(\mathbb{P}^{n-1}, \mathbb{Z})$ being the positive generator and with notations used in \[32\] the elliptic genus of $\sigma$-model phase is given by

$$\left(\prod_{g = h g, C} \pi_{\lambda(g) = \lambda(h)} = 0 \right)^2 \prod_{\lambda} \theta\left(\frac{x\lambda + \lambda(g) - \tau \lambda(h) - z}{2\pi i}\right) \theta\left(\frac{Dx}{2\pi i} - \frac{a-b\tau}{D}\right) e^{2\pi iz \lambda(h)} e^{2\pi i z \lambda(h)} e^{2\pi i z \lambda(h)} e^{2\pi i z \lambda(h)}$$

(sum is taken over connected components $C$ of the fixed point sets of pairs $g, h$).

4. (LG-CY correspondence) If $\sum w_i = D$ then the elliptic genus of LG model is equal to the orbifold elliptic genus of the hypersurface of degree $D$ in the weighted projective space $\mathbb{P}(w_1, \ldots, w_n)$ i.e. the $\Gamma$-orbifoldzed elliptic genus of hypersurface of degree $D$ in $\mathbb{P}^{n-1}$ invariant under the action of the group $\Gamma$.

**Proof.** Semistable loci corresponding to two linearizations of $\mathbb{C}^*$ action \[33\] are $s \neq 0$ and $\sum_i |z_i^2| \neq 0$. The quotient of the first locus is the quotient of $\mathbb{C}^n$ by the
action of \( \mu_D \) and gives the Landau-Ginzburg phase. Parts 1 and 2 follows directly from Definition 5.5 using uniformization \( W \) as used in (5.1).

The quotient of the second locus has projection onto \( \mathbb{C}^n \setminus 0 / \mathbb{C}^* \) with action on \( \mathbb{C}^n \setminus 0 \) being the restriction of the action (33). Hence this GIT quotient can be identified with the orbifold bundle over weighted projective space. Using its presentation as the quotient of the total space of \( \mathcal{O}_{\mathbb{P}^n}(-D) \) by the action of \( \mu_{w_1} \times \cdots \times \mu_{w_n} \), we obtain an uniformization of this phase. \( c_f \) of the normal bundle to the fixed point set in uniformization is \(-Dx + u\) where \( u \) is the infinitesimal character and the claim follows from Definition 5.5.

**Remark 6.3.** Though without Calabi-Yau condition the equality of elliptic genus of LG model and \( \sigma \)-model fails, McKay correspondence for pairs (cf. [9]) still provides an expression for elliptic genus of LG model as the elliptic genus of a pair.

### 6.2. Specialization of elliptic genus \( q \to 0 \)

Proposition 6.2 has as immediate consequence the following relation between the spectrum of weighted homogeneous singularities and \( \chi_y \) genus of corresponding hypersurfaces.

**Proposition 6.4.**

1. (Trivial sector of LG models) Specialization \( q \to 0 \) of elliptic genus of LG phase corresponding to the action (33) is given by

   \[
   \lim_{q \to 0} ELL(LG) = y^{-n} \prod_{j=1}^{n} \frac{y^{\omega_{jD}} - y}{y^{\omega_j} - 1}
   \]

   (where \( y = \exp(2\pi i z) \)).

2. (Relation between trivial sector of LG model and the spectrum) Let \( \{q_l\}, q_l \in \mathbb{Q} \) be the Steenbrink spectrum of isolated singularity of a weighted homogeneous polynomial \( s = f(z) \) with weights \( w_i \) and degree \( D \), i.e. \( q_l \) is the collection of \( \mu \), where \( \mu \) is the Milnor number of \( f \), rational numbers \( q_l \) such that \( \exp(2\pi i q_l) \) is an eigenvalue of the monodromy acting on the graded component \( \text{Gr}_F(H^{n-1}(X_{\infty,f})) \) of the Hodge filtration of the limit mixed Hodge structure on the cohomology of the Milnor fiber of \( f = 0 \) (with multiplicity of \( q_l \) being equal to the dimension of the eigenspace). Here \( p \) is such that the integer part \( \lfloor q_l \rfloor \) is equal to \( n - p - 1 \) (resp. \( n - p \)) if \( \exp(2\pi i q_l) \neq 1 \) (resp. \( \exp(2\pi i q_l) = 1 \)). Let

   \[
   \Xi(y) = y^{-n} \sum_{l=1}^{n} y^{q_l}
   \]

   Then

   \[
   \lim_{q \to 0} ELL(LG)(y) = (-1)^n \Xi(y)
   \]

3. (Orbifoldized-\( \chi_y \) genus of LG model) Specialization of elliptic genus of Landau-Ginzburg model is given by:

\[
\lim_{q \to 0} ELL(LG)(y) = (-1)^n \Xi(y)
\]

where \( \omega_D = e^{2\pi i D} \).

---

\(^{12}\) i.e. a polynomial \( f(z_1, ..., z_n) \) such that \( sf(z) \) is invariant for the action (33)
4. In the case $w_i = 1, D = n$ (i.e. Calabi Yau condition is satisfied) the specialization $q = 0$ has the form:

\begin{equation}
\frac{1}{n} y^{-\frac{n}{2}} \sum_{k=0}^{n-1} \left( \frac{1 - y^{-\frac{1}{n}} e^{2\pi i \frac{k}{n}}}{1 - y^{-\frac{1}{n}} e^{2\pi i \frac{k}{n}}} \right)^n + \sum_{b=1}^{n-1} y^b
\end{equation}

Proof. Trivial sector of $\chi_y$-genus of LG model was already derived directly in Example 5.1. Now we shall obtain it as $q \to 0$ limit of the trivial sector of elliptic genus given in Part 1 of Prop. 6.2. Indeed, Part 1 of proposition 6.2 follows from:

\begin{equation}
\lim_{\tau \to i \infty} \frac{\theta(u(z, \tau) - z, \tau)}{\theta(u(z, \tau), \tau)} e^{2\pi \sqrt{-1} c z} = y^{-\frac{1}{2}} \frac{1 - ye^{-2\pi \sqrt{-1} u(z,0)}}{1 - e^{-2\pi \sqrt{-1} u(z,0)}} y^c
\end{equation}

where $u(z, \tau)$ is a linear in $z$ function and, as above, $y = e^{2\pi \sqrt{-1}}$. (53) implies that the factor corresponding to $w_i D$ in (45) has $y^{-\frac{1}{2}} y^b$ while each factor in summand with $b = 0$ becomes $y^{-\frac{1}{2}} y^b$. Applying (53) to (46) one obtains:

\begin{equation}
\frac{1}{D} y^{-\frac{n}{2}} \sum_{0 \leq a < D} \left( \prod_{y^{w_j}} - y^{-\omega a w_j} \right) + \sum_{1 \leq b < D} y^b
\end{equation}

This implies 3 while 4 follows from it immediately. \hfill \Box

6.3. Specialization $q \to 0, y = 1$. Such specialization leads to numerical invariants of phases.

**Corollary 6.5.**

1. Specialization $q = 0, y = 1$ of untwisted section of LG model is given by

\begin{equation}
\mathcal{EL}(LG)(q = 0, y = 1) = \prod_j (1 - \frac{D}{w_j})
\end{equation}

i.e. up to sign coincides with the Milnor number of the weighted homogeneous singularity with weights $w_1, ..., w_n$ and degree $D$.

2. Specialization $q = 0, y = 1$ of elliptic genus of LG model in the case 4 of Prop. 6.4 gives the orbifoldized euler characteristic of LG model \[\text{13}\].

\begin{equation}
\frac{1}{D} [(1 - D)^2 + D^2 - 1]
\end{equation}

and coincides with the euler characteristic of smooth hypersurface of degree $n$ in $\mathbb{P}^{n-1}$ (LG/CY correspondence for euler characteristic, recall that for $w_i = 1$ CY condition is $n = D$).

Proof. Contributions of either trivial or remaining sectors follow from (52) and

\begin{equation}
\lim_{y \to 1} \frac{1 - y^{\frac{1}{n}} e^{2\pi i \frac{k}{n}}}{1 - y^{-\frac{1}{n}} e^{2\pi i \frac{k}{n}}} = \begin{cases} (1 - D) & k = 0 \\ 1 & k \neq 0 \end{cases}
\end{equation}

13 or "orbifoldized Milnor number"
In fact specialization of Prop 6.4 part 4 gives \( \frac{1}{n}(1 - D)^n + (D - 1) + D(D - 1) \), with the first and second summands corresponding to the first summand in the bracket with \( k = 0 \) and \( k \geq 1 \) respectively (since for \( k > 0 \) each factor in the product is equal to 1). The claim about matching the euler characteristic of LG model and smooth hypersurface can be seen directly, i.e. without use of McKay correspondence as in 4 in Prop. 6.2 using the following formula (cf. [27]) for the euler characteristic of a smooth \((N-2)\)-dimensional hypersurface of degree \( D \):

\[
e(V^D_{N-2}) = \frac{(1 - D)^N + ND - 1}{D}
\]

\( \square \)

6.4. Orbifoldization of phases by the action of finite groups. In this section we illustrate the orbifoldization of elliptic genus of phases as defined in section 5.3.

Example 6.6. Consider the \( \sigma \)-model phase corresponding to the action \((33)\) with \( w_i = 1, i = 1, ..., n \) and linearization with semistable locus \( C \times (\mathbb{C}^n \setminus 0) \). The GIT quotient \( C \times \mathbb{C}^n \setminus 0/\mu C^* \) is the total space of the line bundle denoted as \( \mathcal{O}_{\mathbb{P}^n-1}(-D) \). Let \( \Gamma \subset SL_n(\mathbb{C}) \) be a finite subgroup which we consider as acting on \( E = C \times \mathbb{C}^n \) via \( (s,v) = (s,\gamma \cdot v), \gamma \in \Gamma \). The orbifoldization of contribution of the only fixed component of \( C^* \) action by dilations, which is the zero section of \( \mathcal{O}_{\mathbb{P}^n-1}(-D) \), is given by the same formula as \((17)\) but in which \( \Gamma \) is an arbitrary subgroup of \( SL_n(\mathbb{C}) \) viewed as acting on the total space of bundle \( \mathcal{O}_{\mathbb{P}^n-1}(-D) \). As in the proof of part 3 Prop. 6.2 we see that orbifoldization of the \( \sigma \)-model phase is the \( \Gamma \)-orbifoldized elliptic genus of the hypersurface of degree \( D \) in \( \mathbb{P}^{n-1} \).

Example 6.7. Next we shall consider the \( \Gamma \)-quotients of LG models in the sense of section 6.4.

In this section we illustrate the orbifoldization of phases as defined in section 5.3. First let us look at LG model corresponding to the case \( w_1 = .... = w_n = D = 1 \) and its orbifoldization by the cyclic group \( \Gamma = \mu_D \) generated by the exponential grading operator \( J_W = (......,exp(2\pi\sqrt{-1}\gamma_i),......) \). The GIT quotient corresponding to this LG phase is \( C^n \) i.e. we have orbifoldization of smooth phase and elliptic genus of such orbifoldization coincides with the elliptic genus of LG models with \( C^* \)-action \((33)\) as is specified in Definition 5.7.

Example 6.8. Now we shall look at orbifoldization of arbitrary LG phase. Let \( \Delta \subset SL_n(\mathbb{C}) \) be a finite subgroup containing exponential grading operator \( J_W = (......,exp(2\pi\sqrt{-1}\gamma_i),......) \) and such that \( J_W \) belongs to its center. These conditions imply that one can use as uniformization of LG phase with \( \Delta \)-action, the space \( S' \) such that for cyclic group \( \Gamma = \mu_D \) generated by \( J_W \) one has \( W/\Gamma = C \times \mathbb{C}^n/\mu C^* \).

Now Def. 5.8 yields the following expression for orbifoldized LG phase:

\[
\frac{1}{|\Delta|} \sum_{g,h \in \Delta, gh = hg} \prod \frac{\theta((w_i - 1)z + \lambda(g) - \lambda(h)\tau)}{\theta(\lambda(g) - \lambda(h)\tau)} e^{2\pi i \lambda(h)z}
\]

The specialization \( q \rightarrow 0 \) of orbifoldized phases goes as follows:

Proposition 6.9. With notations as above, the elliptic genus of LG phase orbifoldized by a group \( \Delta \) for \( q \rightarrow 0 \) specializes to

\[
\sum_{\{h\} \in \text{Conj}(\Delta), X^h} y^{\sum_{\lambda, \lambda(h)\neq 0} \left(-\frac{1}{2} + \lambda(h)\right)} \frac{1}{|C(g)|} \sum_{g \in \text{Cent}(\Delta(h)) \lambda, \lambda(h) = 1} y^{-\frac{1}{\tau} - y^{1 - \frac{1}{\tau}} \exp(2\pi i \lambda(g))} \left(1 - y^{-\frac{1}{\tau}} \exp(2\pi i \lambda(g))\right)
\]

\footnote{for discussion of the origins of this condition see [6], Corollary 2.3.5}
(here $X^h$ is the maximal subspace of $\mathbb{C}^n$ fixed by a representative of a conjugacy class). In the case when $\Delta$ is abelian one has:

$$\frac{1}{|G|} \sum_{\{h\} \in \Gamma, X^h} y^\sum_{\lambda, \lambda(h) \neq 0}(-\frac{1}{2} + \lambda(h)) \prod_{\lambda, \lambda(h) = 1} \frac{y^{\frac{1}{2}} - y^{(\frac{1}{n})} - \frac{1}{2} \exp(2\pi i \lambda(g))}{1 - y^{(\frac{1}{n})}\exp(2\pi i \lambda(g))}$$

Remark 6.10. The expression (60) coincides with the one given in [4] and expression (61) coincides with the one given in Theorem 6 in [19].

Proof. The term $\Phi^T(x, g, h, z, \tau, \Gamma)$ for $q \to 0$ has as limit:

$$y^{\frac{1}{2} + \lambda(h)} \text{ if } \lambda(h) \neq 0 \text{ resp. } y^{\frac{1}{2}} \frac{1 - ye^{x + 2\pi i \lambda(g)}}{1 - e^{x + 2\pi i \lambda(g)}}$$

For action corresponding to the weighted homogeneous polynomials with weights $w_i$ and degree $D$ the equivariant Chern class of action of $\mathbb{C}^*$ is $\frac{\Phi^T}{\partial \tau}$. This implies the proposition. \hfill \Box

6.5. Hybrid models. Here we shall consider types of phases which are neither $\sigma$-models or LG, called hybrid models (cf. [56], [13]).

6.5.1. Complete intersection. Sigma models corresponding to Calabi Yau complete intersections have hybrid counterparts rather than LG phases appearing in the case of hypersurfaces. See [13] for alternative treatment of complete intersections via hybrid models.

Definition 6.11. Phases of complete intersection. Consider the $\mathbb{C}^*$-action on $\mathbb{C}^r \times \mathbb{C}^n$ given by:

$$\lambda(p_1, ..., p_r, z_1, ..., z_n) = (\lambda^{-q_1}p_1, ..., \lambda^{-q_r}p_r, \lambda z_1, ..., \lambda z_n)$$

One of the GIT quotients, $Q_1$, is the total space of the bundle $\oplus \mathcal{O}_{\mathbb{P}^n}(-q_i)$ (corresponding to a linearization in one of the cones in $\text{Char}(\mathbb{C}^*) \otimes \mathbb{Q}$) having as semistable locus $\mathbb{C}^r(p_1, ..., p_r) \times (\mathbb{C}^n(z_1, ..., z_n) \setminus 0)$. For linearizations in the second cone the semistable locus is $(\mathbb{C}^r \setminus 0) \times \mathbb{C}^n$. The corresponding GIT quotient $Q_2$ is the $\mu_D$-quotient of the total space of the direct sum of $n$ copies of line bundles over weighted projective space $\mathcal{O}_{\mathbb{P}^n(q_1, ..., q_r)}(-1)^{\oplus n}/\mu_D$ where $D = gcd(q_1, ..., q_r)$ and $\mu_D$ is the group of roots of unity of degree $D$ acting diagonally on the fibers of this direct sum.

In the first case, the contribution into equivariant elliptic genus of the component of the fixed point set is given by

$$\frac{x\theta(x/2\pi \sqrt{-1} - z, \tau)}{\theta(x/2\pi \sqrt{-1}, \tau)} \prod_{i=1}^r \frac{x\theta((\frac{-q_i x}{2\pi \sqrt{-1}} + u - z, \tau)}{\theta(\frac{q_i x}{2\pi \sqrt{-1} + u}, \tau))} \frac{x \cdot \theta((\frac{x}{2\pi \sqrt{-1}} - z, \tau)}{\theta((\frac{x}{2\pi \sqrt{-1}}, \tau))}$$

where $u$ is the infinitesimal generator of the equivariant cohomolgy $H^*_\mathbb{C}^r(p)$ of a point. For $u = z$ one obtains the elliptic genus of smooth complete intersection of hypersurfaces of degree $q_1, ..., q_r$ in $\mathbb{P}^{n-1}$.

Now let us calculate he elliptic genus in the second case (when one has a hybrid model cf. [56]). The GIT quotient $((\mathbb{C}^r \setminus 0) \times \mathbb{C}^n)/\mathbb{C}^* = \mathcal{O}_{\mathbb{P}^n(q_1, ..., q_r)}(-1)^{\oplus n}/\mu_D$ is a fiber space with the orbifold $\mathbb{C}^n/\mu_D$ as a fiber and its base being the weighted projective space with the orbifold structure given by viewing $\mathbb{P}^{r-1}(q_1, ..., q_r)$ as a quotient of $\mathbb{P}^{r-1}$ by the action of abelian group $\Gamma = \oplus_i \mu_{q_i}$. The uniformization can be obtained by taking quotient of the total space $[\mathcal{O}_{\mathbb{P}^{r-1}}(-1)]$ of split vector
bundle on \( \mathbb{P}^{r-1} \) by the action of \( \Gamma \) such that projection on \( \mathbb{P}^{n-1} \) is compatible with \( \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}/\Gamma = \mathbb{P}(q_1, ..., q_r) \). The fixed point set of the action of \( C^* \) on the GIT-quotient induced by action \((t_1, ..., t_r, z_1, ..., z_n) \to (t_1, ..., t_r, z_1, ..., z_n)\) is \( \mathbb{P}^{r-1}(q_1, ..., q_r) \) and for induced \( C^* \)-action on \( |\mathcal{O}_{\mathbb{P}^{r-1}}(1)| \) it is the zero section of this bundle. Hence for each pairs \((g, h)\) of elements of \( \Gamma \), contribution of \( C^* \)-fixed point to the summand of orbifold elliptic genus \( \mathcal{E}LL_{\mathbb{P}^{r-1}}^{\mathbb{P}^{r-1}}(|\mathcal{O}_{\mathbb{P}^{r-1}}(1)|, \Gamma) \) corresponding to \((g, h)\) will have two factors. One is coming from restriction of the tangent bundle

\[
T_{|\mathcal{O}_{\mathbb{P}^{r-1}}(1)|}|_{\mathbb{P}^{r-1}, g, h}
\]

to the subspace of \( \mathbb{P}^{r-1} \) fixed by both \( g, h \). The latter coincides with \( T_{\mathbb{P}^{r-1}}|_{\mathbb{P}^{r-1}, g, h} \). This contribution is the summand \( \mathcal{E}LL_{\mathbb{P}^{r-1}}(\mathbb{P}^{r-1}, \Gamma)^{g, h, C} \) of elliptic class corresponding to pair \( g, h \) and connected component \( C \) of their fixed point set since \( \mathbb{P}^{r-1} \) is the fixed point set of \( C^* \)-action. The quotient \( T_{|\mathcal{O}_{\mathbb{P}^{r-1}}(1)|}|_{\mathbb{P}^{r-1}, g, h}/T_{\mathbb{P}^{r-1}}|_{\mathbb{P}^{r-1}, g, h} \) is just \( \mathcal{O}_{\mathbb{P}^{r-1}}(1)^n|_{\mathbb{P}^{r-1}, g, h} \). The total space of this bundle acted upon by the group \(< g, h > \) considered as the automorphisms group of \( |\mathcal{O}_{\mathbb{P}^{r-1}}(1)^n| \). It also support the \( C^* \)-action by dilatation. The corresponding equivariant contribution of this part of \( T_{|\mathcal{O}_{\mathbb{P}^{r-1}}(1)|}|_{\mathbb{P}^{r-1}} \) over connected component \( C \) of \( \mathbb{P}^{r-1}, g, h \) is

\[
\left(\frac{\theta(\frac{x}{2\pi i} + \frac{y}{\beta} - z + \lambda(g) - \tau(h))}{\theta(\frac{x}{2\pi i} + \frac{y}{\beta} + \lambda(g) - \tau(h))}\right)^n e^{2\pi i \lambda(h) z} \cdot e^{2\pi i \lambda(h) z}
\]

where \( \lambda \) is the character of \( < g, h > \) acting on this eigenbundle over the connected component \( C \) (term \( \frac{1}{\beta} \) reflects that contribution written in terms of character of \( C^*/\mu_D \) acting effectively on the fibers). The resulting elliptic genus of hybrid model hence can described as

\[
\frac{1}{|\Gamma|} \sum_{g, h, C} \mathcal{E}LL_{\mathbb{P}^{r-1}}(\mathbb{P}^{r-1}, \Gamma)^{g, h, C} \cdot \left(\frac{\theta(\frac{x}{2\pi i} + \frac{1}{\beta} - 1) z + \lambda(g) - \lambda(h) \tau}{\theta(\frac{x}{2\pi i} + \frac{1}{\beta} + \lambda(g) - \lambda(h) \tau)}\right)^n [C]
\]

(sum over connected components \( C \subset \mathbb{P}^{r-1}, g, h \) of the fixed point sets of pairs \( g, h \)). Note that this expression in the case \( r = 1 \) becomes the elliptic genus of LG-model since \( x = 0, \Gamma = \mu_D \) and for \( g = e^{2\pi i \frac{p}{\beta}}, h = e^{2\pi i \frac{q}{\beta}} \) one has \( \lambda_{g, h}(g) = \frac{\alpha}{\beta}, \lambda_{g, h}(h) = \frac{\beta}{\beta} \).

6.5.2. Hypersurfaces in the products of projective spaces. This material is discussed in \[56\], Section 5.5. Consider the action of \( C^* \times C^* \) on \( \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^m \) given by

\[
(\lambda, \mu)(p, x_1, ..., x_n, y_1, ..., y_m) = (\lambda^{-n} \mu^{-m} p, \lambda x_1, ..., \lambda x_n, \mu y_1, ..., \mu y_m)
\]

There are 3 cones in \( Char((C^*)^2) \otimes \mathbb{Q} \) corresponding to linearizations with constant GIT with semistable loci respectively:

\[
\{ \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^m \}_{ss} = \begin{cases}
\mathbb{C} \times (\mathbb{C}^n \setminus 0) \times (\mathbb{C}^m \setminus 0), & \text{Calabi Yau phase} \\
\mathbb{C}^* \times (\mathbb{C}^n \setminus 0) \times (\mathbb{C}^m), & \text{hybrid phase} \\
\mathbb{C}^* \times (\mathbb{C}^n) \times (\mathbb{C}^m \setminus 0), & \text{hybrid phase}
\end{cases}
\]
with the GIT quotients being respectively:

\[
\begin{align*}
\{ p_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(-n) \otimes p_2^* \mathcal{O}_{\mathbb{P}^{m-1}}(-m) \}, \\
\{ \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(m) \} / \mu_m, \\
\{ \oplus \mathcal{O}_{\mathbb{P}^{m-1}}(-m) \} / \mu_n.
\end{align*}
\]

(69)

The respective elliptic genera are:

\[
\begin{align*}
\left( \frac{e^{\frac{\pi}{2}i(z - \frac{1}{\tau})}}{\eta(z - \frac{1}{\tau})} \right)^m \left( \frac{e^{\frac{\pi}{2}i(z - \frac{1}{\tau})}}{\eta(z - \frac{1}{\tau})} \right)^n \left( \frac{e^{\frac{\pi}{2}i(z - \frac{1}{\tau})}}{\eta(z - \frac{1}{\tau})} \right)^{\frac{p^m}{m!}} [\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}] \\
\frac{1}{m!} \sum_{0 \leq a, b < m} \left( \frac{\theta(-m + a + b + 1) + \frac{m - 1}{2} + \frac{a}{2} + \frac{b}{2} - \tau}{\theta(\frac{a}{2} + \frac{b}{2} - \tau)} \right) \frac{e^{\frac{\pi}{2}i(z - \frac{1}{\tau})}}{\eta(z - \frac{1}{\tau})} m \left[ \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \right] \\
\frac{1}{m!} \sum_{0 \leq a, b < m} \left( \frac{\theta(-m + a + b + 1) + \frac{m - 1}{2} + \frac{a}{2} + \frac{b}{2} - \tau}{\theta(\frac{a}{2} + \frac{b}{2} - \tau)} \right) \frac{e^{\frac{\pi}{2}i(z - \frac{1}{\tau})}}{\eta(z - \frac{1}{\tau})} n \left[ \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \right]
\end{align*}
\]

(70)

The expression in the upper row represent the elliptic genus of Calabi Yau hypersurface of bidegree \((n, m)\) in \(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}\).

7. **Appendix I: Theta functions**

Jacobi theta function \(\theta(z, \tau), z \in \mathbb{C}, \tau \in \mathbb{H}\) is entire function on \(\mathbb{C} \times \mathbb{H}\) where \(\mathbb{H}\) is the upper half plane defined as the product:

\[
\theta(z, \tau) = q^{\frac{1}{2}}(2 \sin \pi z) \prod_{l=1}^{\infty} (1 - q^l) \prod_{l=1}^{\infty} (1 - q^l e^{2\pi i z})(1 - q^l e^{-2\pi i z})
\]

(71)

where \(q = e^{2\pi i \tau}\).

Its transformation law is as follows:

\[
\theta(z + 1, \tau) = -\theta(z, \tau), \quad \theta(z + \tau, \tau) = e^{-2\pi i \tau} \theta(z, \tau)
\]

(72)

The derivative \(\theta'(0, \tau)\) appears in expansion \(\theta(z, \tau) = \theta'(0, \tau)z + \theta''(0, \tau)z^2 + \ldots\)

and satisfies:

\[
\theta'(0, \tau) = \eta^3(\tau), \quad \text{where} \quad \eta(\tau) = q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)
\]

(73)

(Dedekind’s) \(\eta(\tau)\)-function transforms as follows:

\[
\eta(-\frac{1}{\tau}) = \left( \frac{\tau}{1} \right)^{\frac{1}{2}} \eta(\tau)
\]

(74)

It follows that

\[
\frac{\theta(z, -\frac{1}{\tau})}{\theta'(0, -\frac{1}{\tau})} = \frac{e^{\frac{\pi}{2}i(z - \frac{1}{\tau})}}{\tau} \frac{\theta(z, \tau)}{\theta'(0, \tau)}
\]

(75)

Let

\[
\Upsilon(x, \tau) = (1 - e^{-x}) \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2}
\]

(76)

and

\[
\Phi(x, \tau) = e^{\frac{\pi}{2}i} \Upsilon(x, \tau) = (e^{\frac{\pi}{2}i} - e^{-\frac{\pi}{2}i}) \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2}
\]

\(15\theta_1(z, \tau)\) or \(\theta_{1,1}(z, \tau)\) are other common notations
Subsection

Definition 8.1. (cf. [23], [39])

\[ \Phi(x, \tau) = 2 \sinh \left( \frac{x}{2} \right) \prod_{n=1}^{\infty} \frac{(1 - q^n e^x)(1 - q^n e^{-x})}{(1 - q^n)^2} \]

(cf. [29] p.117) i.e.

\[ \Phi(x, \tau) = \frac{i \theta \left( \frac{x}{2\eta^3}, \tau \right)}{\eta^3(\tau)} \]

(cf. [7] p.461).

Weierstrass \( \sigma \)-function is defined by

\[ \sigma(z, \tau) = z \prod_{\omega \neq 0, \omega \in \mathbb{Z} + \mathbb{Z} \tau} (1 - \frac{z}{\omega}) e^{\frac{z}{\omega} + \frac{1}{2}(\frac{z}{\omega})^2} \]

(cf. [12] p.52) which can be used to describe \( \Phi(z, \tau) \) where \( z = \frac{x}{2 \pi \sqrt{-1}} \) (cf. [29] p.145, Corollary 5.3):

\[ \Phi(z, \tau) = \exp(4 \pi^2 G_2(\tau) z^2) \sigma(z, \tau) = \exp(-\frac{e_2(\tau)}{2} z^2) \sigma(z, \tau) \]

Here the quasi-modular forms \( G_2(\tau) \) and \( e_2(\tau) \) are given by

\[ G_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty} (\sum_{d|n} d) q^n = -\frac{1}{8 \pi^2} e_2(\tau) \] where \( e_2(\tau) = \sum_{n,m,n \neq 0} \frac{1}{(m+n\tau)^2} \)

We also consider the following product expansion (cf. [53] Ch.4 sect.3):

\[ \phi(z, \tau) = x \prod_{t \in W}(1 - \frac{z}{w}) \]

(refer to (77)), product is taken over the elements \( w \) of the lattice \( W = \{1, \tau\} \), subscripts \( e \) designates Eisenstein ordering of factors and \( t \) indicates omitting \((0,0) \in W \). \( \phi(z, \tau) \) admits the following product formula in \( q \) (cf. (15) ibid)

\[ \phi(z, \tau) = \frac{1}{2 \pi \sqrt{-1}} \left( e^{\pi \sqrt{-1} z} - e^{-\pi \sqrt{-1} z} \right) \prod_{n \geq 1} (1 - q^n e^{2 \pi \sqrt{-1} z})(1 - q^n e^{-2 \pi \sqrt{-1} z}) \]

\[ \prod_{n \geq 1} (1 - q^n)^2 \]

\[ = \frac{1}{2 \pi \sqrt{-1}} \frac{\Phi(x, \tau)}{2 \pi \eta^3(\tau)} = \frac{\theta(z, \tau)}{\eta^3(\tau)} \]

8. Appendix: Quasi-Jacobi forms

Recall the following:

**Definition 8.1.** (cf. [23], [39])

Meromorphic Jacobi form of index \( t \in \frac{1}{2} \mathbb{Z} \) and weight \( k \) for a finite index subgroup of the Jacobi group \( \Gamma_t' = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \) is defined as a meromorphic in elliptic variable \( z \) function \( \chi \) on \( \mathbb{H} \times \mathbb{C} \) having expansion

\[ \sum \gamma_{n,r} q^n \zeta^r \] in \( q = \exp(2 \pi \sqrt{-1} \tau) \) and satisfying the following functional equations:

\[ \chi \left( \frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d} \right) = (c \tau + d)^k e^{\frac{2 \pi i c z^2}{c \tau + d}} \chi(\tau, z) \]

16 In [7], Hirzebruch’s \( T(z, \tau) \)-function is denoted as \( \Phi(z, \tau) \); Notation \( \Phi(z, \tau) \) is the one used in appendix to [29] Cor.5.3;p.145.

17 i.e. in terms of \( x = 2 \pi \sqrt{-1} z \) for which the lattice is \( 2 \pi \sqrt{-1}(\mathbb{Z} + \mathbb{Z} \tau) \) one has \( \Phi(x, \tau) = \sigma(x, \tau) \exp(-G_2(\tau) x^2) \). Ref. [29], [28] use this notation while we selected traditional notations (in particular consistent with [53]).
\( \chi(\tau, z + \lambda \tau + \mu) = (-1)^{2(\lambda + \mu)}e^{-2\pi i(\lambda^2 \tau + 2\lambda z)} \chi(\tau, z) \)

for all elements \([ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0 ] \) and \([ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (a, b) \) in \( \Gamma \).

A meromorphic Jacobi form is called a weak Jacobi form if
a) it is holomorphic in \( \mathbb{H} \times \mathbb{C} \) and
b) it has Fourier expansion \( \sum c_{n,r} q^n z^r \) in \( q = exp(2\pi \sqrt{-1} \tau) \) in which \( n \geq 0 \).

The functional equation (83) implies that Fourier coefficients \( c_{n,r} \) depend on \( r \mod 2m \) and \( \Delta = 4nm - r^2 \) (the discriminant). A weak Jacobi form is called Jacobi form (resp. cusp form) if the coefficients \( c_{n,r} \) with \( \Delta < 0 \) (resp. \( \Delta \leq 0 \)) are vanishing.

Remark 8.2. Presentation (2) provides Fourier expansion of elliptic genus having non-negative powers of \( q \) (i.e. yields a weak Jacobi form) while powers of \( y \) can be negative.

The algebra of Jacobi forms is the bi-graded algebra \( J = \bigoplus J_{t,k} \) and the algebra of Jacobi forms of index zero is the sub-algebra \( J_0 = \bigoplus J_{0,k} \subset J \).

We shall need below the following real analytic functions:

\( \lambda(z, \tau) = \frac{z - \bar{z}}{\tau - \bar{\tau}}, \quad \mu(\tau) = \frac{1}{\tau - \bar{\tau}} \)

Their transformation properties are as follows:

\( \lambda(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) = (c\tau + d)\lambda(z, \tau) - 2icz \)
\( \lambda(z + m\tau + n, \tau) = \lambda(z, \tau) + m \)

\( \mu(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^2 \mu(\tau) - 2ic(c\tau + d) \)

Definition 8.3. Almost meromorphic Jacobi form of weight \( k \), index zero and a depth \( (s, t) \) is a (real) meromorphic function in \( \mathbb{C}\{q^+ z \} [z^{-1}, \lambda, \mu] \), with \( \lambda, \mu \) given by (85), i.e. polynomial in \( \lambda, \mu \) with complex meromorphic functions as coefficients which
a) satisfies the functional equations in Definition 8.1 of Jacobi forms of weight \( k \) and index zero and
b) which has degree at most \( s \) in \( \lambda \) and at most \( t \) in \( \mu \).

Quasi-Jacobi form of weight \( k \), index zero and depth \( (s, t) \) is the term of bi-degree \( (0, 0) \) in \( \lambda, \mu \) of an almost meromorphic Jacobi form of weight \( k \) and depth \( (s, t) \).

Algebra of quasi-Jacobi forms is bi-graded filtered algebra generated by filtered algebra of quasi-Jacobi forms and algebra of Jacobi forms (which have depth \( (0, 0) \) and have trivial filtration).

Example 8.4. 1. Two variable Eisenstein series (cf. [53], [39]). Consider the following, meromorphic in \( z \) functions

\( E_a(z, \tau) = \sum_{a, b \in \mathbb{Z}^2} \left( \frac{1}{z + a\tau + b} \right)^n \quad n \in \mathbb{Z}, n \geq 1 \)

18mentioning that this condition on Fourier expansion applicable in holomorphic case only and the restriction \( n \geq 0 \) were inadvertently omitted in [39].
These series are absolutely convergent for \( n \geq 3 \) and yields meromorphic Jacobi forms of weight \( n \) and index 0. For \( n = 1, 2 \) one obtains meromorphic function using Eisenstein summation (cf. \[53\]) which are quasi-Jacobi forms of index 0, weight \( n = 1, 2 \) and depth \((1, 0)\) for \( n = 1 \) and \((0, 1)\) for \( n = 2 \) (cf. \[39\]). \( E_2 - e_2 \) is Jacobi form (here \( e_2(\tau) \) is quasi-modular form which is the one variable Eisenstein series).

The products

\[
\hat{E}_n(z, \tau) = E_n(z, \tau) \left( \frac{\theta(z, \tau)}{\theta'(0, \tau)} \right)^n \quad (n \neq 2) \quad \hat{E}_2 = (E_2(z, \tau) - e_2(\tau)) \left( \frac{\theta(z, \tau)}{\theta'(0, \tau)} \right)^2
\]

are holomorphic quasi-Jacobi forms (Jacobi forms for \( n \geq 2 \)).

The structure of the algebra of quasi-Jacobi forms generated by forms \((88)\) is as follows.

**Theorem 8.5.** The algebra \( QJac_{0, *} \) (or simply \( QJac \)) of quasi-Jacobi forms of weight zero and index \( d, d \in \mathbb{Z} \geq 1 \) is polynomial algebra with generators \( \hat{E}_n, n = 1, 2, 3, 4 \). The algebra \( Jac_{0, *} \) of Jacobi forms of weight zero and index \( d \) (or \( Jac \)) is polynomial algebra in three generators \( \hat{E}_2, \hat{E}_3, \hat{E}_4 \).

The algebra \( QJac \) is isomorphic to the algebra of complex cobordisms \( \Omega^C \) modulo the ideal \( I \) generated by \( X_1 - X_2 \) where \( X_1, X_2 \) are \( K \)-equivalent. The algebra \( Jac \) is isomorphic to the algebra \( \Omega^{SU} \) of complex cobordisms of manifolds with trivial first Chern class modulo the ideal \( I \cap \Omega^{SU} \).

**Remark 8.6.**
1. Different generators of the algebra \( Jac \) are described in \[26\].
2. Term “quasi-Jacobi forms” used in \[45\] in a slightly more narrow sense than in \[39\] and above, where author apparently was unaware of \[39\]. Quasi-Jacobi forms considered in \[45\] belong to the algebra generated by the function:

\[
\frac{\theta(z, \tau)}{\eta^{3}(\tau)}, \frac{\partial \log \left( \frac{\theta(z, \tau)}{\eta(z, \tau)} \right)}{\partial z}, e_2(\tau), e_4(\tau), \varphi(z, \tau), y \frac{d \varphi(z, \tau)}{dy}
\]

\((y = \exp(2\pi \sqrt{-1} z))\) are in the algebra of meromorphic quasi-Jacobi forms as defined in \[83\] (cf. also \[39\]). Indeed \( \frac{\partial \theta(z, \tau)}{\partial z} = E_1(z, \tau) \) (it follows from Appendix I, also cf. \[53\] ch. IV, sect. 3 (15)) and also \( \varphi(z, \tau) = E_2 - e_2, y \frac{d \varphi(z, \tau)}{dy} = -2E_3(z, \tau) \) and modular functions are clearly part of the algebra described in Def. \[83\].

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