TORUS INVARIANT CURVES

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Abstract. Using the language of T-varieties, we study torus invariant curves on a complete normal variety $X$ with an effective codimension-one torus action. In the same way that the $T$-invariant Weil divisors on $X$ are sums of “vertical” divisors and “horizontal” divisors, so too is each $T$-invariant curve a sum of “vertical” curves and “horizontal” curves. We give combinatorial formulas that calculate the intersection between $T$-invariant divisors and $T$-invariant curves, and generalize the celebrated toric cone theorem to the case of complete complexity-one $T$-varieties.

1. Introduction

A $T$-variety is a normal complex algebraic variety with an effective action of an algebraic torus. This definition matches the definition of a toric variety, except that the dimension of the torus may be less than the dimension of the variety on which it acts. In particular, any normal algebraic variety is a $T$-variety when endowed with the trivial action of $(\mathbb{C}^*)^0$. We therefore can’t expect to prove much about general $T$-varieties; we usually restrict our attention to complexity-one $T$-varieties, where the dimension of the torus is exactly one less than the dimension of the variety. In this paper, we study the $T$-invariant curves of a complete complexity-one $T$-variety, find formulas for their intersection with $T$-invariant divisors (using the theory of $T$-invariant divisors developed by Petersen and Süss in [PS]), and prove that the numerical equivalence classes of these curves generate the Mori cone of the $T$-variety.

We review the basics of $T$-varieties in Section 2. Informally speaking, a complexity-one $T$-variety is encoded by a family (parametrized by a projective curve $Y$) of polyhedral subdivisions of a vector space, all with the same tailfan. In Section 3, we describe two kinds of $T$-invariant curves in a $T$-variety, vertical curves and horizontal curves. The vertical curves correspond to walls (codimension-one strata) of one of these polyhedral subdivisions, while the horizontal curves correspond to certain maximal-dimensional cones of the tailfan. We give formulas that calculate the intersection of these curves with a $T$-invariant divisor using the language of Cartier support functions from [PS].

In Section 4, we generalize the toric cone theorem, which states that the Mori cone of a toric variety is generated as a cone by the classes of $T$-invariant curves corresponding to the walls of its fan. In our generalization, we show that the Mori cone of a complete complexity-one $T$-variety is generated as a cone by the classes of a finite collection of vertical curves and horizontal curves. We end the paper with examples in Section 5.

2. Primer on $T$-varieties

In this section, we review the basic notation and construction of $T$-varieties. The presentation favors brevity over pedagogy; we encourage any reader unfamiliar with $T$-varieties to read the excellent survey article [A] for a friendlier exposition to this beautiful topic.

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2.1. **Notation.** Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus, and $M, N$ be the character lattice of $T$ and the lattice of 1-parameter subgroups of $T$ respectively. These lattices embed in the vector spaces

$$
N_\mathbb{Q} := \mathbb{Q} \otimes N \quad \quad \quad M_\mathbb{Q} := \mathbb{Q} \otimes M
$$

and are dual to one another. In classic toric geometry, one studies the correspondence between cones (and fans) in $N_\mathbb{Q}$ and the toric varieties encoded by these combinatorial data. Analogously, we study $T$-varieties through the correspondence between combinatorial gadgets called $p$-divisors (and divisorial fans) and the $T$-varieties they encode. Informally speaking, a $p$-divisor is a Cartier divisor on a semiprojective variety $Y$ with polyhedral coefficients; a divisorial fan is a collection of $p$-divisors whose polyhedral coefficients “fit together” in a suitable way. To make formal these definitions, we begin by discussing monoids of polyhedra.

Let $\sigma$ be a pointed cone in $N_\mathbb{Q}$, and $\sigma^\vee \subseteq M_\mathbb{Q}$ its dual. The set $\text{Pol}_\mathbb{Q}^+(N, \sigma)$ of all polyhedra in $N_\mathbb{Q}$ having $\sigma$ as its tailcone (with the convention that $\emptyset \in \text{Pol}_\mathbb{Q}^+(N, \sigma)$) is a monoid under Minkowski addition with identity element $\sigma$. Any nonempty $\Delta \in \text{Pol}_\mathbb{Q}^+(N, \sigma)$ defines a map

$$
h_\Delta : \sigma^\vee \to \mathbb{Q}
$$

$$
u \mapsto \min_{v \in \Delta} \langle v, u \rangle
$$

called the *support function* of $\Delta$. A nonempty $\Delta \in \text{Pol}_\mathbb{Q}^+(N, \sigma)$ also defines a *normal quasifan* $\mathcal{N}(\Delta)$ in $M_\mathbb{Q}$ consisting of a cone $\lambda_F$ for each face $F$ of $\Delta$ defined by

$$
\lambda_F = \{ u \in \sigma^\vee \mid \langle u, v \rangle = h_\Delta(u) \ \forall v \in F \}.
$$

The figure below shows an example of a polyhedron and its normal quasifan.

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**Figure 1. A polyhedron and its normal quasifan**

**Proposition 2.1.** ([AH03], Lemma 1.4 and Proposition 1.5) The support function $h_\Delta$ is a well-defined map whose regions of linearity are the maximal cones of $\mathcal{N}(\Delta)$. Moreover, any function in $\text{Hom}(\sigma^\vee, \mathbb{Q})$ whose regions of linearity define a quasifan can be realized as $h_\Delta$ for some $\Delta$.

Let $\text{Pol}_\mathbb{Q}(N, \sigma)$ be the Grothendieck group of $\text{Pol}_\mathbb{Q}^+(N, \sigma)$. Let $Y$ be a semiprojective variety, with $\text{CaDiv}(Y)$ its group of Cartier divisors. An element

$$
\mathcal{D} \in \text{Pol}_\mathbb{Q}(N, \sigma) \otimes \mathbb{Z} \text{CaDiv}(Y)
$$

is a *polyhedral divisor* with tailcone $\sigma$ if it has a representative of the form $\mathcal{D} = \sum \mathcal{D}_P \otimes P$ for some $\mathcal{D}_P \in \text{Pol}_\mathbb{Q}^+(N, \sigma)$ and $P$ prime. We will describe a procedure for constructing an affine $T$-variety

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1In this paper, when a picture of $N_\mathbb{Q}$ is juxtaposed with a picture of $M_\mathbb{Q}$, the reader may assume that the bases for these vector spaces have been chosen so that the pairing between them is the standard dot product.

2Because $\sigma$ (not $\emptyset$) is the identity element of $\text{Pol}_\mathbb{Q}(N, \sigma)$, the summation notation in this sentence implies that only finitely many of the polyhedral coefficients $\mathcal{D}_P$ differ from $\sigma$. 

from a certain kind of polyhedral divisor (called a $p$-divisor); this construction will involve taking the spectrum of the global sections of a sheaf of rings defined over a subset of $Y$. This subset, called the locus of $D$, is

$$\text{Loc}(D) := Y \setminus \cup_{D_P = \emptyset} P.$$  

The evaluation of $D$ at $u \in M \cap \sigma^\vee$ is the $\mathbb{Q}$-Cartier divisor

$$D(u) := \sum_{D_P \neq \emptyset} h_{D_P}(u)|_{\text{Loc}(D)}.$$  

We say that $D$ is a $p$-divisor if $D(u)$ is semiample for all $u \in \sigma^\vee$ and big for all $u$ in the interior of $\sigma^\vee$. The direct sum of the sheaves defined by the evaluations $D(u)$ is an $M$-graded sheaf of rings

$$\mathcal{O}(D) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\text{Loc}(D)}(D(u)) \chi^u$$

over $\text{Loc}(D)$. There are two different $T$-varieties encoded by the $p$-divisor $D$

$$\widetilde{TV}(D) := \text{Spec}_{\text{Loc}(D)} \mathcal{O}(D)$$

and

$$TV(D) := \text{Spec} \Gamma(\text{Loc}(D), \mathcal{O}(D))$$

where the torus action is given by the $M$-grading on $\mathcal{O}(D)$. All affine $T$-varieties can be constructed this way.

**Theorem 2.2.** ([AH03, Corollary 8.14]) Every normal affine variety with an effective torus action can be realized as $TV(D)$ for some $p$-divisor $D$

Similar to the way that a fan of a non-affine toric variety can be obtained by “gluing together” the cones constituting an affine cover, so too can a non-affine $T$-variety be encoded by “gluing together” the $p$-divisors constituting an affine cover. To make formal these concepts, we first define the intersection of two $p$-divisors $D, D'$ on $Y$ as the $p$-divisor

$$D \cap D' := \sum(D_P \cap D'_P) \otimes P.$$  

We say that $D'$ is a face of $D$ if $D'_P \subseteq D_P$ for each $P$ and the induced map $TV(D') \to TV(D)$ is an open embedding. A finite set $S$ of $p$-divisors on $Y$ is a divisorial fan if for any $D, D' \in S, D \cap D'$ is an element of $S$ and is a face of both $D$ and $D'$. We define $TV(S)$ and $\widetilde{TV}(S)$ to be the $T$-varieties obtained by gluing together the $T$-varieties $\{TV(D)\}_{D \in S}$ and $\{\widetilde{TV}(D)\}_{D \in S}$ according to these face relations. This process is detailed in [AH03].

### 2.2. Geometry of $TV(S)$ and $\widetilde{TV}(S)$

Because $\widetilde{TV}(D)$ is defined as the relative spectrum of a sheaf of rings on $\text{Loc}(D)$, there is a natural projection map $\pi : \widetilde{TV}(D) \to \text{Loc}(D) \subseteq Y$. Because $TV(D)$ is defined as the spectrum of the global sections of the structure sheaf on $\widetilde{TV}(D)$, we also have a natural map $p : \widetilde{TV}(D) \to \Gamma(\widetilde{TV}(D), \mathcal{O}_{\widetilde{TV}(D)}) \cong TV(D)$. Given a divisorial fan $S$, the maps $\pi, p$ corresponding to the different $D \in S$ glue into maps

![Diagram](attachment:image.png)

In this subsection, we describe the fibers of $p$ and $\pi$. In particular, we will notice that for $y \in Y$, the reduced fiber $\pi^{-1}(y)$ is a union of irreducible toric varieties, and that the contraction map $p$ identifies

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3Some authors define a “$\mathbb{Q}$-Cartier” divisor to be a Weil divisor with a Cartier multiple. Our $\mathbb{Q}$-Cartier divisors are elements of $\mathbb{Q} \otimes \text{Div}(Y)$ having a Cartier multiple (so may have rational coefficients). The pedantic reader is invited to replace all instances of “$\mathbb{Q}$-Cartier divisor” in this paper with “$\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.”
certain disjoint torus orbits of $\widehat{TV}(S)$. Many of these results simplify when $TV(S)$ is a complexity-one $T$-variety; because this is the only case we will need for later sections, we will henceforth assume that $Y$ is a projective curve. The reader interested in higher-complexity $T$-varieties should read [A] for the more general results.

In [P], the author describes the reduced fibers of $\pi$ using the language of *dappled toric bouquets*. We begin by reviewing this language.

**Definition 2.3.** The *fan ring* of a quasifan $\Lambda$ in $\mathbb{M}_\mathbb{Q}$ is

$$\mathbb{C}[\Lambda] := \bigoplus_{u \in |\Lambda| \cap \mathbb{M}} \mathbb{C} \chi^u$$

with multiplication defined by

$$\chi^u \chi^v = \begin{cases} \chi^{u+v} & \text{if } u, v \in \lambda \text{ for some cone } \lambda \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

For a nonempty $\Delta \in \text{Pol}^+(N, \sigma)$ and a cone $\lambda_F$ of its inner normal quasifan $N(\Delta)$, let

$$M_{\lambda_F} := \{ u \in \lambda_F \cap M \mid h_\Delta(u) \in \mathbb{Z} \}.$$  

**Remark 2.4.** In other papers, $M_{\lambda_F}$ is defined differently: when $\Delta \otimes [P]$ appears as a summand in a $p$-divisor, the elements $u \in M_{\lambda_F}$ are required to satisfy the condition that $h_\Delta(u)[P]$ is locally principal at $P$. In the complexity-one case, this condition coincides with our condition that $h_\Delta(u) \in \mathbb{Z}$.

Finally, let $S_\Delta \subseteq |\Lambda(\Delta)| \cap M$ consist of those $u$ such that $S_\Delta \cap \lambda_F = M_{\lambda_F}$ for every cone $\lambda_F \in N(\Delta)$. $S_\Delta$ can be thought of as a conewise-varying sublattice of $M$. The figure below shows an example of $S_\Delta$ for a given $\Delta$; the elements of $S_\Delta \subseteq M$ are in bold.

**Figure 2.** $S_\Delta$ is a conewise-varying sublattice of $M$

**Definition 2.5.** The *dappled fan ring* of $\Delta$ is the following subring of $\mathbb{C}[N(\Delta)]$

$$\mathbb{C}[N(\Delta), S_\Delta] := \bigoplus_{u \in S_\Delta} \mathbb{C} \chi^u$$

**Definition 2.6.** The *dappled toric bouquet* encoded by $\Delta$ is the variety $TB(\Delta) := \text{Spec}(\mathbb{C}[N(\Delta), S_\Delta])$. Given a polyhedral complex $\Sigma = \{ \Delta \}$ in $\mathbb{N}_\mathbb{Q}$, the dappled toric bouquet encoded by $\Sigma$ is the variety $TB(\Sigma)$ obtained by gluing the $\{ TB(\Delta) \}_{\Delta \in \Sigma}$ according to the face relations among the polyhedra.
Observe that $TB(\Delta)$ and $TB(\Sigma)$ have a natural torus action induced by the $M$-grading of the dappled fan rings. For a $T$-variety $TV(S)$ over $Y$ and a point $y \in Y$, the polyhedra $\{D_y\}_{D \in S}$ fit together into a polyhedral complex $S_y$ of $N_Q$.

**Proposition 2.7.** [P, Prop 1.29] Let $S$ be a $p$-divisor on the smooth projective curve $Y$. The reduced fiber $\pi^{-1}(y)$ of $\pi : TV(S) \rightarrow Y$ is equivariantly isomorphic to $TB(S_y)$.

Motivated by Proposition 2.7 to study the geometry of non-affine toric bouquets, we construct a fan for each vertex of a polyhedral subdivision $\Sigma$ of $N_Q$: the toric varieties they encode will be precisely the irreducible components of $TB(\Sigma)$. For a vertex $v \in \Sigma$, define the lattice

$$M_v = \{u \in M \mid \langle u, v \rangle \in \mathbb{Z}\}$$

Because $M_v$ is a sublattice of $M$, $N$ is a sublattice of $N_v := M_v^\vee \subseteq N_Q$. Let $i_v : N_Q \rightarrow (N_v)_Q$ be the map induced by this inclusion. As $\Delta$ ranges over all polyhedra in $\Sigma$ containing $v$, the cones $i_v(Q_{\geq 0} \cdot (\Delta - v))$ form a fan $F_v$ in $(N_v)_Q$. For any cone $\sigma = i_v(Q_{\geq 0} \cdot (\Delta - v))$ of $F_v$, the semigroup $\sigma^\vee \cap N_v^\vee$ is isomorphic to the semigroup $\lambda_\Delta \cap S_\Delta$. Because this isomorphism commutes with the gluing data induced by the face relations, we have the following description of the irreducible components of $TB(\Sigma)$.

**Proposition 2.8.** The irreducible components of $TB(\Sigma)$ are equivariantly isomorphic to the toric varieties $\{TV(F_v)\}$ where the set ranges over the vertices $v$ of $\Sigma$.

For example, the polyhedral complex in Figure 3 encodes a toric bouquet with three irreducible toric components. We have drawn the lattices $N_v$ not as a square grid, but in a way that the sublattice $N \subseteq N_v$ (in bold) is a square grid so that the angles between the polyhedra are preserved. In the example, one fan encodes $\mathbb{P}^2$ and the other fans encode weighted projective spaces.

![Figure 3. Components of a toric bouquet](image-url)
many \( y \), the polyhedral subdivisions \( S_y \) differ from \( \text{tail}(S) \) for only finitely many \( y \). By Proposition \( \text{2.8} \), the fiber of \( \pi \) over \( y \in Y \) is equal to \( TV(\text{tail}(S)) \) for all but finitely many \( y \) and specializes to a (possibly non-reduced) union of toric varieties at these finitely many points.

By the discussion above, the familiar orbit-cone correspondence for toric varieties translates into a correspondence between \( T \)-orbits in \( TV(S) \) and pairs \( (y, F) \) where \( y \in Y \) and \( F \in S_y \). To understand \( TV(S) \), we will describe how the map \( p \) identifies certain of these orbits in different fibers. We first consider the case of an affine \( T \)-variety. For a \( p \)-divisor \( D \) with tailcone \( \sigma \) and a \( u \in \sigma^\vee \cap M \), the semiample divisor \( D(u) \) defines a map

\[
\xi_u : \text{Loc}(D) \to \text{Proj} \left( \bigoplus_{k \geq 0} \Gamma(\text{Loc}(D), D(\lambda u)) \right).
\]

**Theorem 2.9.** ([AH03, Theorem 10.1]) The map \( p : \overline{TV(D)} \to TV(D) \) induces a surjection

\[
\{ (y, F) : y \in Y, F \text{ is a face of } D_y \} \to \{ T \text{-orbits in } TV(D) \}
\]

that identifies the orbits corresponding to \( (y, F) \) and \( (y', F') \) iff \( \lambda_F = \lambda_{F'} \subseteq M_Q \) and \( \xi_u(y) = \xi_u(y') \) for some (equivalently, for any) \( u \in \text{relint}(\lambda_F) \).

In the non-affine case, the gluing maps among \( \{ TV(D) \}_{D \in S} \) are prescribed by the face relations between the \( p \)-divisors, which identifies precisely those \( T \)-orbits in \( TV(D) \) and \( TV(D') \) corresponding to the faces \( \{ (y, D_y \cap D'_y) \}_{y \in Y} \).

### 3. \( T \)-invariant curves and intersection theory

In this section, we study the intersection theory of complete complexity-one \( T \)-varieties over a projective curve \( Y \). For the rest of the paper, all \( T \)-varieties are complete, complexity-one \( T \)-varieties over a projective curve \( Y \). The “completeness” condition translates into the combinatorial requirement that \( |S_y| = N_Q \) for all \( y \). Motivated by the correspondence between \( T \)-invariant Cartier divisors and Cartier support functions introduced in [PS], we define the notion of a \( Q \)-Cartier support function to encode \( Q \)-Cartier torus invariant divisors. We will describe two kinds of \( T \)-invariant curves — vertical curves and horizontal curves — then give formulas that compute the intersection of these curves with a \( T \)-invariant \( Q \)-Cartier divisor.

**Definition 3.1.** Given a nontrivial \( \Delta \in \text{Pol}^+_Q(N, \sigma) \) and an affine \( \varphi : \Delta \to Q \), the linear part of \( \varphi \) is the function

\[
\text{lin} \varphi : \sigma \to \mathbb{Q}
\]

\[
n \mapsto \varphi(p + n) - \varphi(p)
\]

where \( p \) is any point in \( \Delta \). If \( \langle \sigma \rangle \subseteq N_Q \) is the subspace spanned by \( \sigma \), the function \( \text{lin} \varphi \) extends uniquely to a linear function \( \langle \sigma \rangle \to \mathbb{Q} \), which will also be written \( \text{lin} \varphi \) without risk of confusion.

**Definition 3.2.** Let \( S \) be the divisorial fan of a complexity-one \( T \)-variety over \( Y \). A \( Q \)-Cartier support function is a collection of affine functions

\[
\{ h_{D_y} : |D_y| \to \mathbb{Q} \}_{D \in S}
\]

with rational slope and rational translation such that

1. For a fixed \( y \in Y \), the functions \( \{ h_{D_y} \} \) define a continuous function \( h_y : |S_y| \to \mathbb{Q} \).
   That is, \( h_{D_y} \) and \( h_{D'_y} \) agree on \( D_y \cap D'_y \) for \( D, D' \in S \).
2. For each \( D \in S \) with complete locus, there exists \( u \in M, f \in K(Y) \) and \( N \in \mathbb{Z}_{>0} \) such that \( Nh_{D,y}(v) = -\text{ord}_y(f) - \langle u, v \rangle \) for all \( y \in Y \) and all \( v \in N_Q \).
(3) If $\mathcal{D}_y, \mathcal{D}'_y$ have the same tailcone, then $\text{lin} h_{\mathcal{D}, y} = \text{lin} h_{\mathcal{D}', y}$.
(4) For a fixed $\mathcal{D}$, $h_{\mathcal{D}, y}$ differs from $\text{lin} h_{\mathcal{D}, y}$ for only finitely many $y$.

A $\mathbb{Q}$-Cartier support function is called a Cartier support function if each $h_{\mathcal{D}, y}$ has integral slope and integral translation and $N = 1$ in condition (2). We write $\text{CaSF}(\mathcal{S})$ and $\mathbb{Q}\text{CaSF}(\mathcal{S})$ to denote the abelian group (under standard addition of functions) of Cartier support functions and $\mathbb{Q}$-Cartier support functions respectively.

For any $T$-invariant Cartier divisor $D$ on $TV(\mathcal{S})$ and any $p$-divisor $\mathcal{D} \in \mathcal{S}$, we can always find an open cover $\{U_i\}$ of $Y$ for which there exists Cartier data for $D|_{TV(\mathcal{D})}$ of the form $(TV(\mathcal{D}|_{U_i}), f_i\chi^u)$ (see proof of [PS], Prop 3.10 for details). These Cartier data define functions

$$\{h_{\mathcal{D}, y}(v) = -\text{ord}_y(f_i) - (u_i, v)\}_{y \in U_i}$$

which agree on the overlaps of the $U_i$ to define $h_{\mathcal{D}, y}$ for all $y$. In this way, we can define a Cartier support function for any Cartier divisor on $TV(\mathcal{S})$.

**Proposition 3.3.** ([PS], Prop 3.10) Let $T - \text{CaDiv}(\mathcal{S})$ denote the group of $T$-invariant Cartier divisors on $TV(\mathcal{S})$. This association of a Cartier support function to a $T$-invariant Cartier divisor defines an isomorphism of groups

$$T - \text{CaDiv}(\mathcal{S}) \cong \text{CaSF}(\mathcal{S})$$

If $\{h_{\mathcal{D}, y}\}$ is the Cartier support function for $ND$, where $N > 0$ and $D$ is a $T$-invariant $\mathbb{Q}$-Cartier divisor, then $\{N^{-1}h_{\mathcal{D}, y}\}$ is a $\mathbb{Q}$-Cartier support function. In this way, we can associate a $\mathbb{Q}$-Cartier support function to any $T$-invariant $\mathbb{Q}$-Cartier divisor on $TV(\mathcal{S})$. The following is an immediate corollary of Proposition 3.3.

**Corollary 3.4.** Let $T - \mathbb{Q}\text{CaDiv}(\mathcal{S})$ denote the group of $T$-invariant $\mathbb{Q}$-Cartier divisors on $TV(\mathcal{S})$. Then the association described above is an isomorphism of groups

$$T - \mathbb{Q}\text{CaDiv}(\mathcal{S}) \cong \mathbb{Q}\text{CaSF}(\mathcal{S})$$

### 3.1. Vertical Curves

Toward the goal of describing the intersection theory of a $T$-variety, we study its $T$-invariant curves. We start with vertical curves, which are images (under $p$) of a $T$-invariant curve contained in a single fiber of $\pi$.

Recall from Proposition 2.8 that for $y \in Y$, the reduced fiber $\pi^{-1}(y)$ has as its irreducible components a toric variety for each vertex $v$ of $\mathcal{S}_y$. A toric variety has a $T$-invariant curve corresponding to each wall of its fan (by taking the closure of the corresponding torus orbit). Translating this fact into the context of toric bouquets, we call a codimension-one element of a polyhedral complex a wall if it can be realized as the intersection of two top-dimensional polyhedra; there is a $T$-invariant curve in a toric bouquet for each wall of the corresponding polyhedral complex. In this section, we study the curves in $TV(\mathcal{S})$ and $TV(\mathcal{S})$ corresponding to these $T$-invariant curves.

Fix a $T$-variety $TV(\mathcal{S})$ and a point $y \in Y$. Let $\tau \in \mathcal{S}_y$ be a wall of the polyhedral complex $\mathcal{S}_y$, let $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ be two $p$-divisors for which $\tau = \mathcal{D} \cap \mathcal{D}'$, let $\lambda_{\tau, \mathcal{D}} \subseteq M_\mathbb{Q}$ be the cone in $N(\mathcal{D}_y)$ dual to $\tau$, and let $u_{\tau, \mathcal{D}}$ be the semigroup generator of $\lambda_{\tau, \mathcal{D}}$. As usual, unwieldy notation obscures a simple picture: if $\mathcal{D}$ and $\mathcal{D}'$ have polyhedral coefficients over $y$ as shown in Figure 4 (the horizontal plane in a single orthant at a height of $2/3$), then the sublattice $\mathbb{Z} : u_{\tau, \mathcal{D}} = \lambda_{\tau, \mathcal{D}} \cup \lambda_{\tau, \mathcal{D}'}$ consists of the bold elements of the vertical axis of $M \cong \mathbb{Z}^3$ shown on the right.

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4A wall of a fan is a codimension-one cone that can be realized as the intersection of two top-dimensional cones.
Figure 4. The sublattice $\mathbb{Z} \cdot u_{\tau, D}$ corresponding to the wall $\tau = D_y \cap D'_y$

If $g \in K(Y)$ is Cartier data for $D(u_{\tau, D})$ in some neighborhood of $y$, the maps

$$\Gamma(\text{Loc}(D), \mathcal{O}(D)) \to \mathbb{C}[z]$$

$f_{\chi}^u \mapsto \begin{cases} 0 & u \notin M_{\lambda, D} \\ (g^k f)(y) z^k & u = ku_{\tau, D} \end{cases}$

$\Gamma(\text{Loc}(D'), \mathcal{O}(D')) \to \mathbb{C}[z^{-1}]$

$f_{\chi}^u \mapsto \begin{cases} 0 & u \notin M_{\lambda, D'} \\ (g^{-k} f)(y) z^{-k} & u = -ku_{\tau, D} \end{cases}$

The image of which we will call the vertical curve $C_{\tau, y}$.

**Proposition 3.5.** The vertical curve $C_{\tau, y}$ is the image under $p$ of the closure of the torus orbit in $TB(S_y) \subseteq \widetilde{TV}(S)$ corresponding to the wall $\tau$.

**Proof.** For any affine open $U \subseteq Y$ containing $y$, Map 2 factors

$$\mathbb{P}^1 \to TV(S)$$

the image of which we will call the vertical curve $C_{\tau, y}$.

Proposition 3.5. The vertical curve $C_{\tau, y}$ is the image under $p$ of the closure of the torus orbit in $TB(S_y) \subseteq \widetilde{TV}(S)$ corresponding to the wall $\tau$.

**Proof.** For any affine open $U \subseteq Y$ containing $y$, Map 2 factors

$$\mathbb{P}^1 \to \pi^{-1}(U) \subseteq \widetilde{TV}(S) \to TV(S)$$

where $\mathbb{P}^1 \to \pi^{-1}(U)$ is given by

$$\Gamma(U, \mathcal{O}(D)) \to \mathbb{C}[z]$$

$f_{\chi}^u \mapsto \begin{cases} 0 & u \notin M_{\lambda, D} \\ (g^k f)(y) z^k & u = ku_{\tau, D} \end{cases}$

$\Gamma(U, \mathcal{O}(D')) \to \mathbb{C}[z^{-1}]$

$f_{\chi}^u \mapsto \begin{cases} 0 & u \notin M_{\lambda, D'} \\ (g^{-k} f)(y) z^{-k} & u = -ku_{\tau, D} \end{cases}$

Therefore, it suffices to show that the image of $\mathbb{P}^1 \to \pi^{-1}(U)$ is the closure of the torus orbit in $TB(S_y) \subseteq \widetilde{TV}(S)$ corresponding to the wall $\tau$. To do so, we recall some relevant details about the isomorphism between the reduced fibers of $\pi$ and a dappled toric bouquet (see [AH03], Proposition 7.10 for details). This isomorphism is constructed by first choosing a collection of functions $\{g_{\mathcal{D}(u)} \in K(Y)\}_{u \in S_{\Delta}}$ such that, after possibly shrinking $U$,

$$\text{div}(g_{\mathcal{D}(u)})|_U = \mathcal{D}(u)|_U \quad \text{and} \quad g_{\mathcal{D}(u+u')} = g_{\mathcal{D}(u)} g_{\mathcal{D}(u')}$$
Then the isomorphism between the fiber and the dappled toric bouquet is induced by
\begin{equation}
\Gamma(U, O(D)) \to \mathbb{C}[\mathcal{N}(D_y), \mathcal{S}_{D_y}]
\end{equation}
\[ f\chi^u \mapsto \begin{cases} (g_{D(y)}f(y)\chi^u & \text{if } u \in \mathcal{S}_{D_y} \\ 0 & \text{otherwise} \end{cases} \]

(and similarly for \(D'\)). On the other hand, the closure of the torus orbit corresponding to \(\tau\) in the toric bouquet is parametrized by gluing the maps
\begin{align*}
\mathbb{C}[\mathcal{N}(D_y), \mathcal{S}_{D_y}] & \to \mathbb{C}[z] \\
\chi^u & \mapsto \begin{cases} z^k & \text{if } u = ku_{\tau,D}, k \in \mathbb{Z} \\
0 & \text{otherwise} \end{cases} \\
\mathbb{C}[\mathcal{N}(D'_y), \mathcal{S}_{D'_y}] & \to \mathbb{C}[z^{-1}] \\
\chi^u & \mapsto \begin{cases} z^{-k} & \text{if } u = -ku_{\tau,D}, k \in \mathbb{Z} \\
0 & \text{otherwise} \end{cases}
\end{align*}

The composition of Equation (5) and Equation (6) yields Equation (4) proving the claim. \(\square\)

To find a formula that calculates the intersection between \(C_{\tau,y}\) and a \(T\)-invariant Cartier divisor \(D\), we pick Cartier data for \(D\) that includes two sets of the form
\[ \{ (TV(D|_U), f\chi^u), (TV(D'|_{U'}), f'\chi'^u) \} \]
where \(U, U' \subseteq Y\) are open sets containing \(y\). The Cartier support function for \(D\) includes
\[ h_{D,y} = -\text{ord}_y(f) - \langle u, v \rangle \quad \text{and} \quad h_{D',y} = -\text{ord}_y(f') - \langle u', v \rangle. \]

Because \(h_{D,y}\) and \(h_{D',y}\) agree on \(\tau\), it must be the case that
\[ \text{ord}_y(f) = \text{ord}_y(f') + \langle u - u', v \rangle = 0 \]
for all \(v \in \tau\). In particular, \(u - u' \in \mathbb{Q} \cdot u_{\tau,D}\). Moreover, since \(\langle u - u', v \rangle = \text{ord}_y(f') - \text{ord}_y(f) \in \mathbb{Z}\), it must be the case that \(\langle u - u', v \rangle \in \mathbb{Z}\) for \(v \in \tau\). Therefore, \(u - u' = ku_{\tau,D}\) for some \(k \in \mathbb{Z}\), and the quotient of the two Cartier data is \(f\chi^u/f'\chi'^u = (f/f')^{ku_{\tau,D}}\). Under the parametrization of \(C_{\tau,y}\) in Equation (2), this rational function pulls back to \((g^k f/f')(y) z^k\) on \(C_{\tau,y} \cong \mathbb{P}^1\), where \(g\) is Cartier data for \(D(u_{\tau,D})\). Therefore, the degree of the pullback of \(D\) onto \(C_{\tau,y}\) is \(k\). This is precisely \(\mu^{-1}_{\tau}(u - u', n_{\tau,D})\), where \(\mu_{\tau}\) is the index of \(\mathbb{Z} \cdot u_{\tau,D}\) in \(\mathbb{Q} \cdot u_{\tau,D} \cap M\) and \(n_{\tau,D} \in N\) is any representative of the generator of \(N/(u_{\tau,D})\) that pairs positively with \(u_{\tau,D}\) (equivalently, \(n_{\tau,D}\) is any element of \(N\) such that \(\langle n_{\tau,D}, u_{\tau,D} \rangle = \mu_{\tau}\)).

\[ \langle D, C_{\tau,y} \rangle = \mu^{-1}_{\tau}(u - u', n_{\tau,D}) \]
or, using the language of Cartier support functions,
\begin{equation}
\langle D, C_{\tau,y} \rangle = \mu^{-1}_{\tau}(\text{lin}h_{D',y} - \text{lin}h_{D,y})(n_{\tau,D})
\end{equation}

By linearity, the same formula applies when \(D\) is a \(T\)-invariant \(\mathbb{Q}\)-Cartier divisor.

**Example 3.6.** Let \(D, D' \in S\) be \(p\)-divisors that have the slices shown in Figure 4. Suppose that with respect to the standard basis given by the coordinate axes in the picture, a \(T\)-invariant divisor \(D\) has the following Cartier support functions
\[ h_{D,y}(v) = -10 + \langle (9, 4, 17), v \rangle \quad \text{and} \quad h_{D',y}(v) = 0 + \langle (9, 4, 2), v \rangle \]

Then
\[ \langle D, C_{\tau,y} \rangle = 3^{-1}\langle (0, 0, -15), (0, 0, 1) \rangle = -5 \]
3.2. Horizontal Curves. Let $\sigma$ be a full-dimensional cone of tail($S$). Because the $T$-varieties we study are complete, every $S_y$ contains a polyhedron with tailcone $\sigma$. Such a polyhedron corresponds to a fixed point in the fiber $\pi^{-1}(y)$. Taking the union (as $y$ varies) of these fixed points defines a curve $C_\sigma \subseteq TV(S)$. Theorem 2.9 shows that $p$ contracts $C_\sigma$ precisely if there is some $D \in S$ with tailcone and complete locus. In this case, we say that $\sigma$ is marked.

**Definition 3.7.** A cone $\sigma$ of tail($S$) is marked if $\sigma$ is the tailcone of a $p$-divisor $D \in S$ with complete locus.

When $\sigma$ is unmarked, Theorem 2.9 shows that no distinct points of $\tilde{C}_\sigma$ are identified by $p$. Toward the goal of finding an intersection formula for these horizontal curves $C_\sigma := p(\tilde{C}_\sigma)$, we parametrize them. Let $TV(S)$ be a $T$-variety and let $\sigma$ be an unmarked full-dimensional cone of tail($S$). For $D \in S$ with tailcone $\sigma$, we have a map of rings

$$\varphi_D : \Gamma(TV(D), O(D)) \to \Gamma(\text{Loc}(D), O_Y)$$

$$f \chi^u \mapsto \begin{cases} f & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases}$$

Because each $\{\text{Loc}(D) \mid \text{tail}(D) = \sigma\}$ is affine, these glue into a map $s_\sigma : Y \to TV(S)$ where we used the fact that $S$ is complete (so $|S_y| = \mathbb{N}_q$ for all $y$) to deduce that $Y$ is covered by $\{\text{Loc}(D) \mid \text{tail}(D) = \sigma\}$. The map $s_\sigma$ factors through $TV(S)$. By carefully following the isomorphism between the fibers of $\pi$ and the corresponding toric bouquets (as in the proof of Proposition 3.5), we see that the image of $s_\sigma$ indeed equals the horizontal curve $p(\tilde{C}_\sigma)$.

We can use this parametrization to find an intersection formula for $T$-invariant divisors and horizontal curves. Fix a cone $\sigma$ of tail($S$) of full dimension and a $T$-invariant Cartier divisor $D$ with Cartier support function $\{h_{D,Y}\}$. Because $\sigma$ has full dimension, there is a unique $u_\sigma \in M$ and collection of integers $\{a_y \in \mathbb{Z}\}_{y \in Y}$ such that for each $D$ with tailcone $\sigma$ and each $y \in \text{Loc}(D)$,

$$h_{y,D}(v) = -a_y - (u_\sigma, v).$$

We can find Cartier data for $D$ whose open sets and rational functions are of the form

$$(TV(D|_U), f_{D,U} \chi^{u_\sigma})$$

for open sets $U \subseteq Y$. Then $\text{ord}_y(f_{D,U}) = a_y$ for all $D$ with tailcone $\sigma$ and $y \in U$. When $\sigma$ is unmarked, the open sets $U$ appearing in the Cartier data are affine, and the pullback of the transition function $f_{D,U} f_{D,U}^{-1} \chi^0$ onto the curve $C_\sigma \cong Y$ is the function $f_{D,U} f_{D,U}^{-1} \chi^0$, on $U \cap U'$. That is, the functions $f_{D,U}$ appearing in the Cartier data for $D$ are themselves the Cartier data for the pullback of $D$ onto $C_\sigma \cong Y$. As a Weil divisor, the pullback of $D$ onto $C_\sigma$ is $\sum a_y[y]$; we call this divisor $D_\sigma$.

**Definition 3.8.** Given a $\mathbb{Q}$-Cartier support function $\{h_{D,y}\}$, a cone $\sigma$ of full dimension in tail($S$), and a point $y$, there is a unique $a_y \in \mathbb{Z}$ such that for every $D$ with tailcone $\sigma$ and $\text{Loc}(D) \ni y$,

$$h_{D,y} = -a_y - \text{lin}(h_{D,y}).$$

Then define

$$D_\sigma = \sum_{y \in Y} a_y[y]$$

**Remark 3.9.** The definition of $D_\sigma$ makes sense even when $\sigma$ is marked. However, if $D$ has complete locus, then by (PS, Proposition 3.1) every invariant Cartier divisor on $TV(D)$ is principal. It follows that $\deg(D_\sigma) = 0$ for every marked $\sigma$.

**Remark 3.10.** Compare this definition to (PS, Definition 3.26). In our notation, $D_\sigma = -h|_\sigma(0)$.
With this new definition, we can summarize the discussion above with the following equation for the intersection theory of a T-invariant divisor with a horizontal curve.

$$\langle D, C_\sigma \rangle = \deg(D_\sigma)$$

By linearity, the same formula applies when $D$ is a $T$-invariant $\mathbb{Q}$-Cartier divisor.

4. The $T$ Cone Theorem

Given a normal variety $X$, let $Z_1(X)$ be the proper 1-cycles, and define

$$N^1(X) := (\text{CaDiv}(X)/\sim) \otimes_\mathbb{Z} \mathbb{R}$$

$$N_1(X) := (Z_1(X)/\sim) \otimes_\mathbb{Z} \mathbb{R}$$

where $\sim$ denotes numerical equivalence of divisors in the first definition, and numerical equivalence of curves in the second. The vector space $N^1(X)$ contains the cone $\text{Nef}(X)$ generated by classes of nef divisors, and the vector space $N_1(X)$ contains the cone $\text{NE}(X)$ generated by classes of irreducible complete curves. The Mori cone $\mathbf{NE}(X)$ is the closure of $\text{NE}(X)$. With respect to the intersection product, $N_1(X)$ and $N^1(X)$ are dual vector spaces, and the cones $\text{Nef}(X), \mathbf{NE}(X)$ are dual cones.

When $X$ is the toric variety of a fan $\Sigma$, the closure of the torus orbit corresponding to a wall of $\Sigma$ defines an element of $\mathbf{NE}(X)$. The celebrated toric cone theorem ([CLO], Theorem 6.3.20(b)) states that $\mathbf{NE}(X)$ is generated as a cone by these classes. In this section, we prove the corresponding result for $T$-varieties. We continue to assume that all $T$-varieties are complete complexity-one $T$-varieties over a projective curve $Y$.

**Theorem 4.1.** Let $TV(S)$ be an $n$-dimensional $T$-variety, and let $y' \in Y$ be any point for which $S_{y'} = \text{tail}(S)$. Then

$$\mathbf{NE}(TV(S)) = \sum_{\tau \text{ a wall of } S_{y'} \atop \dim(\text{tail}(\tau)) < n-1} \mathbb{R}_{\geq 0}[C_{\tau,y}] + \sum_{\tau \text{ a wall of } \text{tail}(S) \atop \dim(\sigma) = n-1 \atop \sigma \text{ unmarked}} \mathbb{R}_{\geq 0}[C_{\sigma}]$$

For the proof, we review two important facts about divisors on $T$-varieties.

**Proposition 4.2.** Any Cartier divisor $D$ on a $T$-variety $TV(S)$ is linearly equivalent to a $T$-invariant Cartier divisor.

Different authors have different definitions of concavity; to us, a function $\varphi : \mathbb{Q} \to \mathbb{Q}$ is concave if $\varphi(tv + (1-t)w) \geq t\varphi(v) + (1-t)\varphi(w)$ for all $v, w \in \mathbb{Q}$ and all $t \in [0,1]$

**Proposition 4.3.** ([PS], Corollary 3.29) A $T$-invariant Cartier divisor $D \in T - \text{CaDiv}(S)$ with Cartier support function $\{h_{D,y}\}$ is nef iff all $h_y$ are concave and $\deg(D_\sigma) \geq 0$ for every maximal cone $\sigma$ of the tailfan.

Toward our goal of proving Theorem 4.1, we will use Proposition 4.3 to show that a Cartier divisor is nef if it intersects all vertical and horizontal curves nonnegatively. The proof of this fact requires a combinatorial lemma. Given a Cartier support function $\{h_{D,y}\}$ and any $D \in S, y \in Y$ such that $\dim(D_y) = \dim(N_\Sigma)$, define $h_{D,y} : \mathbb{Q} \to \mathbb{Q}$ to be the unique affine function that extends $h_{D,y} : |D_y| \to \mathbb{Q}$.

**Lemma 4.4.** Let $\{h_{D,y}\}$ be a Cartier support function. The following are equivalent

- $h_y : \mathbb{Q} \to \mathbb{Q}$ is concave.
- For every wall $\tau = D_y \cap D'_y$ of $S_y$, there is some $v \in D'_y \setminus D_y$ with $h_{D'''y}(v) \leq h_{D,y}(v)$. 

Proposition 4.5. A Cartier divisor $D \in \text{CaDiv}(TV(S))$ is nef iff $\langle D, C \rangle \geq 0$ for all vertical and horizontal curves $C$.

Proof. The forward direction follows from the definition of nef. To prove the reverse direction, let $D \in \text{CaDiv}(S)$ satisfy the condition that $\langle D, C \rangle \geq 0$ for all vertical and horizontal curves. Replace $D$ with a linearly equivalent $T$-invariant divisor and let $\{h_{D,y}\}$ be its Cartier support function. Let $\tau = D_y \cap D'_y$ be a wall of $S_y$. Fix any $n_{\tau,D} \in N$ with $\langle n_{\tau,D}, u_{\tau,D} \rangle = \mu_{\tau}$ and any $v_{\tau} \in \text{relint}(\tau)$. Then pick $\epsilon > 0$ such that $v := v_\tau + \epsilon n_{\tau,D} \in D_y \setminus D'_y$. Then

$$h_{D,y}(v) = h_{D,y}(v_\tau) + \text{lin} h_{D,y}(\epsilon n_{\tau,D})$$

$$\tilde{h}_{D',y}(v) = h_{D',y}(v_\tau) + \text{lin} h_{D,y}(\epsilon n_{\tau,D}).$$

Because $h_{D,y}$ and $\tilde{h}_{D',y}$ agree on $\tau$,

$$\tilde{h}_{D',y}(v) - h_{D,y}(v) = (\text{lin} h_{D',y} - \text{lin} h_{D,y})(\epsilon n_{\tau,D}) \geq 0$$

where the final inequality comes from applying Equation 4 to the fact that $\langle D, C_{\tau,y} \rangle \geq 0$. Because this holds for all walls in all slices $S_y$, we conclude by Lemma 4.4 that each $h_y$ is concave.

To show that $\deg(D_\sigma) \geq 0$ for every maximal cone $\sigma$ of the tailfan, observe that if $\sigma$ is marked, then $\deg(D_\sigma) = 0$ by Remark 3.9 if $\sigma$ is unmarked, then $\deg(D_\sigma) = \langle D, C_\sigma \rangle \geq 0$.

To put Proposition 4.5 in context, remember that a $T$-variety has infinitely many distinct vertical curves. Indeed, if $\tau$ is a wall of $\text{tail}(S)$, then for every $y \in Y$ there is (by completeness) a vertical curve $C_{\tau',y}$ where $\tau'$ is a wall of $S_y$ with tailcone $\tau$. The next proposition shows that the classes of all such curves lie on a single ray of $N_1(TV(S))$.

Proposition 4.6. Let $\tau = \sigma \cap \sigma'$ be a wall of $\text{tail}(S)$, where $\sigma, \sigma'$ are full dimensional cones of $\text{tail}(S)$. The classes

$$C_\tau = \left\{ [C_{\tau',y}] \mid \tau' = D_y \cap D'_y \text{ for some } D, D' \text{ with } \text{tail}(D) = \sigma, \text{tail}(D') = \sigma' \right\} \subseteq N_1(TV(S))$$

are positive multiples of each other. Specifically, $[C_{\tau_1,y_1}] = \mu_{\tau_1}^{-1} \mu_{\tau_2} [C_{\tau_2,y_2}]$.

Proof. Let $\{h_{D,y}\}$ be the Cartier support function of some $D \in \text{T-CaDiv}(TV(S))$. All $h_{D,y}$ with $\text{tail}(D) = \sigma$ (respectively $\sigma'$) will have the same linear part, say $-u_\sigma \in M_\Theta$ (respectively $-u_{\sigma'} \in M_\Theta$). Then for two classes $[C_{\tau_1,y_1}], [C_{\tau_2,y_2}] \in C_\tau$, Equation 7 calculates the intersections as

$$\langle D, C_{\tau_1,y_1} \rangle = \mu_{\tau_1}^{-1} \langle u_\sigma - u_{\sigma'}, n_{\tau_1,D} \rangle$$

$$\langle D, C_{\tau_2,y_2} \rangle = \mu_{\tau_2}^{-1} \langle u_\sigma - u_{\sigma'}, n_{\tau_2,D} \rangle$$

Since we can choose $n_{\tau_1,D} = n_{\tau_2,D}$, it follows that $\langle D, C_{\tau_1,y_1} \rangle = \mu_{\tau_1}^{-1} \mu_{\tau_2} \langle D, C_{\tau_2,y_2} \rangle$ for all $D$. □

We are finally ready to prove Theorem 4.1. Using the propositions above, the proof is nearly identical to the proof of the toric cone theorem in (CLO, Theorem 6.3.20(b)).

Proof. (Theorem 4.1) Let $\Gamma$ be the rational polyhedral cone in $\text{NE}(TV(S))$ defined by the right hand side of Equation 9. By definition, $\Gamma$ includes the classes of all horizontal curves; by Proposition 4.3, it also includes the classes of all vertical curves. Therefore, Proposition 4.5 implies that $\Gamma^\vee = \text{Nef}(TV(S))$, so $\Gamma = \Gamma^\vee = \text{NE}(TV(S))$. □
5. Examples

5.1. Example 1. Consider the divisorial fan $S$ shown in Figure 5. $TV(S)$ is the projectivized cotangent bundle of the first Hirzebruch surface. All horizontal divisors in $TV(S)$ are contracted. For each vertical divisor $D_{[y],v}$ and each maximal $p$-divisor $D_i \in S$, we write the Weil divisor $\sum a_y[y]$ and an element $u \in M$ in Table 1 to encode the Cartier support function $\{h_{D_{[y],v}}(w) = -a_y - \langle u, w \rangle\}$ of $D_{[y],v}$. For example, the Cartier support function for $D_{[0],(0,0)}$ includes $h_{D_{[0],\infty}}(v) = 1 - \langle(-2, -1), v\rangle$.

![Diagram of divisorial fan S with horizontal and vertical divisors](image)

**Figure 5.**

| $D_{[0],(0,1)}$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $D_8$ |
|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $D_{[0],(0,0)}$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $D_{[0],(0,1)}$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $D_{[1],(0,0)}$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $D_{[1],(0,1)}$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $D_{[\infty],(0,0)}$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $D_{[\infty],(0,1)}$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |

**Table 1. Torus invariant divisors on TV(S)**

Because every maximal-dimensional cone of $tail(S)$ is marked, $TV(S)$ has no horizontal curves. Let $\tau_{i,j,y}$ be the wall of $S_y$ realized as the intersection between $D_i$ and $D_j$ (if such a wall exists). Using Proposition 1.0, we see that the numerical equivalence class of $C_{\tau_{i,j,y}}$ only depends on $i$ and $j$; to save space, we abbreviate $[C_{\tau_{i,j,y}}]$ as $C_{i,j}$.

As an example of a calculation, consider $C_{1,2}$ and the $T$-invariant divisor $D_{[0],(0,0)}$ with Cartier support function $\{h_{D_{[0],v}}\}$. Using notation from Section 5.1, $n_{\tau, D_2} = (0, 1)$. The relevant linear parts

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*See [PS] for a definition and description of horizontal and vertical divisors.*
of the Cartier support function are \( \text{linh}_{D_1,0} = -(1,-1) \in M \) and \( \text{linh}_{D_2,0} = (0,0) \in M \). The intersection can then be calculated using Equation 7

\[ \langle D_{[0]}(0,0), C_{1,2} \rangle = 1^{-1} \left( (1,1) - (0,0), (0,1) \right) = 1 \]

The complete list of intersections is in Table 2. The canonical divisor is also listed; it can be expressed as a sum of the vertical divisors using the formula from (PS, Theorem 3.21).

| \( D_{[0]}(0,0) \) | \( c_{1,2} \) | \( c_{2,3} \) | \( c_{3,4} \) | \( c_{4,5} \) | \( c_{5,6} \) | \( c_{6,7} \) | \( c_{7,8} \) | \( c_{8,1} \) | \( c_{1,4} \) | \( c_{5,8} \) | \( c_{2,7} \) | \( c_{3,6} \) |
|---------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( D_{[0]}(0,0) \) | 0 | -1 | 0 | 1 | 0 | 0 | 0 | 1 | -1 | 0 | 1 | 1 |
| \( D_{[1]}(0,0) \) | 1 | 0 | 1 | -2 | 1 | 0 | 1 | -2 | 3 | 1 | 0 | 0 |
| \( D_{[\infty]}(0,0) \) | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| \( K_X \) | -2 | 2 | -2 | 0 | -2 | -2 | 0 | -4 | -4 | -4 | -4 | -4 |

| Table 2. Intersections of divisors and curves on TV(\( S \)) |

5.2. Example 2. Let \( \sigma_1, \sigma_2, \sigma_3 \) be the cones

\[
\sigma_1 = \mathbb{Q}_{\geq 0} \cdot (1,0) + \mathbb{Q}_{\geq 0} \cdot (0,1) \\
\sigma_2 = \mathbb{Q}_{\geq 0} \cdot (0,1) + \mathbb{Q}_{\geq 0} \cdot (-1,-1) \\
\sigma_3 = \mathbb{Q}_{\geq 0} \cdot (1,0) + \mathbb{Q}_{\geq 0} \cdot (-1,-1)
\]

and let \( S \) be the divisorial fan on \( \mathbb{P}^1 \) having the following maximal \( p \)-divisors

\[
D_1 = ((2/3,1/2) + \sigma_1)[0] + ((-2/3, -1/2) + \sigma_1)[1] + \emptyset[\infty] \\
D_2 = ((2/3,1/2) + \sigma_2)[0] + ((-2/3, -1/2) + \sigma_2)[1] + ((-1, -1) + \sigma_2)[\infty] \\
D_3 = ((2/3,1/2) + \sigma_3)[0] + ((-2/3, -1/2) + \sigma_3)[1] + ((-1, -1) + \sigma_3)[\infty] \\
D_4 = \emptyset[0] + \emptyset[1] + ((-1, -1) + \sigma_1)[\infty]
\]

![Figure 6. The divisorial fan \( S \)](image)

The \( T \)-variety corresponding to \( S \) is a deformation of \( \mathbb{P}^3 \). The \( T \)-invariant divisors and their intersections with \( T \)-invariant curves are encoded in Table 3 and 4 respectively, using the same notation as in the previous example.
Table 3. Torus invariant divisors on $TV(S)$

| $D_{0,1/2}$ | $D_{2}$ | $D_{3}$ | $D_{4}$ |
|-------------|---------|---------|---------|
| $\{0\}, (2/3, 1/2)$ | $\{1/6\}, 1/4, -1/6, 1/4$ | $\{1/6\}, 1/4, -1/6, 1/4$ | $\{0\}, (0, 0)$ |
| $\{1, -1\}$ | $\{0\}, (0, 0)$ | $\{1, -1\}$ | $\{0\}, (0, 0)$ |

Table 4. Intersections on $TV(S)$

| $D_{0,1/2}$ | $C_{1,2}$ | $C_{1,3}$ | $C_{1,4}$ |
|-------------|---------|---------|---------|
| $\{0\}, (2/3, 1/2)$ | $\{1/6\}$ | $\{1/4\}$ | $\{1/6\}$ |
| $\{0\}, (0, 0)$ | $\{1, -1\}$ | $\{0\}, (0, 0)$ |

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