Irreducible Modules for the Quantum Affine Algebra $U_q(\hat{sl}_2)$ and its Borel Subalgebra $U_q(\hat{sl}_2)^{\geq 0}$

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Abstract

Let $U_q(\hat{sl}_2)^{\geq 0}$ denote the Borel subalgebra of the quantum affine algebra $U_q(\hat{sl}_2)$. We show that the following hold for any choice of scalars $\varepsilon_0, \varepsilon_1$ from the set $\{1, -1\}$.

(i) Let $V$ be a finite-dimensional irreducible $U_q(\hat{sl}_2)^{\geq 0}$-module of type $(\varepsilon_0, \varepsilon_1)$. Then the action of $U_q(\hat{sl}_2)^{\geq 0}$ on $V$ extends uniquely to an action of $U_q(\hat{sl}_2)$ on $V$. The resulting $U_q(\hat{sl}_2)$-module structure on $V$ is irreducible and of type $(\varepsilon_0, \varepsilon_1)$.

(ii) Let $V$ be a finite-dimensional irreducible $U_q(\hat{sl}_2)$-module of type $(\varepsilon_0, \varepsilon_1)$. When the $U_q(\hat{sl}_2)$-action is restricted to $U_q(\hat{sl}_2)^{\geq 0}$, the resulting $U_q(\hat{sl}_2)^{\geq 0}$-module structure on $V$ is irreducible and of type $(\varepsilon_0, \varepsilon_1)$.

1 The quantum affine algebra $U_q(\hat{sl}_2)$

The affine Kac-Moody Lie algebra $\hat{sl}_2$ has played an essential role in diverse areas of mathematics and physics. Elements of $\hat{sl}_2$ can be represented as vertex operators, which are certain generating functions that appear in the dual resonance models of particle physics (see [15] and [8]). The algebra $\hat{sl}_2$ also features prominently in the study of Knizhnik-Zamolodchikov equations

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and conformal field theory (see for example, [2] and [5]). Our main object of interest is a $q$-analogue of $\hat{\mathfrak{sl}}_2$, the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$, which also has a representation by vertex operators [7] and has many important connections with quantum field theory and symmetric functions, in particular with Kostka-Foulkes polynomials ([10], [9]). In this paper, we focus on the finite-dimensional irreducible modules of $U_q(\hat{\mathfrak{sl}}_2)$. These modules have been classified up to isomorphism by V. Chari and A. Pressley ([1]). Our aim here is to relate them to the finite-dimensional irreducible modules of the Borel subalgebra $U_q(\hat{\mathfrak{sl}}_2)_{\geq 0}$.

Throughout the paper $\mathbb{F}$ will denote an algebraically closed field. We fix a nonzero scalar $q \in \mathbb{F}$ that is not a root of unity and adopt the following notation:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 0, 1, \ldots$$ (1.1)

**Definition 1.2** The quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ is the unital associative $\mathbb{F}$-algebra with generators $e_i^\pm$, $K_i^\pm$, $i \in \{0, 1\}$ which satisfy the following relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1,$$  \hspace{1cm} (1.3)

$$K_0 K_1 = K_1 K_0,$$ \hspace{1cm} (1.4)

$$K_i e_i^\pm K_i^{-1} = q^{\pm 2} e_i^\pm,$$ \hspace{1cm} (1.5)

$$K_i e_j^\pm K_i^{-1} = q^{\mp 2} e_j^\pm, \quad i \neq j,$$ \hspace{1cm} (1.6)

$$[e_i^+, e_i^-] = \frac{K_i - K_i^{-1}}{q - q^{-1}},$$ \hspace{1cm} (1.7)

$$[e_0^+, e_1^+] = 0,$$ \hspace{1cm} (1.8)

$$(e_i^\pm)^3 e_i^\pm - [3](e_i^\pm)^2 e_i^\pm + [3] e_i^\pm (e_i^\pm)^2 - e_j^\pm (e_i^\pm)^3 = 0, \quad i \neq j.$$ \hspace{1cm} (1.9)

We call $e_i^\pm$, $K_i^\pm$, $i \in \{0, 1\}$ the Chevalley generators for $U_q(\hat{\mathfrak{sl}}_2)$ and refer to (1.3) as the $q$-Serre relations. We denote by $U_q(\hat{\mathfrak{sl}}_2)_{\geq 0}$ the subalgebra of $U_q(\hat{\mathfrak{sl}}_2)$ generated by the elements $e_i^\pm$, $K_i^\pm$, $i \in \{0, 1\}$. We call $U_q(\hat{\mathfrak{sl}}_2)_{\geq 0}$ the Borel subalgebra of $U_q(\hat{\mathfrak{sl}}_2)$, because of its similarity to the universal enveloping algebra of the standard Borel subalgebra of a finite-dimensional simple Lie algebra over the complex numbers.

It is apparent from the definitions that in $U_q(\hat{\mathfrak{sl}}_2)$ or $U_q(\hat{\mathfrak{sl}}_2)_{\geq 0}$, $K_0 K_1$ is central and so by Schur’s Lemma must act as a scalar on any finite-dimensional irreducible module. Therefore, finite-dimensional irreducible
modules of the Borel subalgebra are closely related to the finite-dimensional irreducible modules of the following algebra.

**Definition 1.10** The algebra $U_{\geq 0}$ is the unital associative $F$-algebra with generators $R, L, K^{\pm 1}$, which satisfy the defining relations:

\[ KK^{-1} = K^{-1}K = 1, \]
\[ KRK^{-1} = q^{2}R, \]
\[ KLK^{-1} = q^{-2}L, \]
\[ R^{3}L - [3]R^{2}LR + [3]RLR^{2} - LR^{3} = 0, \]
\[ L^{3}R - [3]L^{2}RL + [3]LRL^{2} - RL^{3} = 0. \]

Our first goal is to explain the exact relationship between the finite-dimensional irreducible $U_{q}(\hat{\mathfrak{sl}}_{2})$-modules and the finite-dimensional irreducible $U_{\geq 0}$-modules. In order to state our results precisely, it is necessary to make a few comments.

Let $V$ denote a finite-dimensional irreducible $U_{q}(\hat{\mathfrak{sl}}_{2})$-module. Then the actions of $K_{0}$ and $K_{1}$ on $V$ are semisimple [1, Prop. 3.2]. Furthermore (also by [1, Prop. 3.2]), there exists an integer $d \geq 0$ and scalars $\varepsilon_{0}, \varepsilon_{1}$ chosen from $\{1, -1\}$ such that

(i) the set of distinct eigenvalues of $K_{0}$ on $V$ is $\{\varepsilon_{0}q^{2i-d} \mid 0 \leq i \leq d\}$; and

(ii) $K_{0}K_{1} - \varepsilon_{0}\varepsilon_{1}I$ vanishes on $V$.

We call the ordered pair $(\varepsilon_{0}, \varepsilon_{1})$ the type of $V$.

Now let $V$ denote a finite-dimensional irreducible $U_{\geq 0}$-module. As we will see in Section 2, the action of $K$ on $V$ is semisimple. Moreover, there exists an integer $d \geq 0$ and a nonzero scalar $\alpha \in F$ such that the set of distinct eigenvalues of $K$ on $V$ is $\{\alpha q^{2i-d} \mid 0 \leq i \leq d\}$. We refer to $\alpha$ as the type of $V$.

Our main results concerning $U_{q}(\hat{\mathfrak{sl}}_{2})$ and $U_{\geq 0}$ are contained in the following two theorems.

**Theorem 1.16** Let $V$ denote a finite-dimensional irreducible $U_{\geq 0}$-module of type $\alpha$. Assume $\varepsilon_{0}, \varepsilon_{1}$ are scalars in $\{1, -1\}$. Then there exists a unique $U_{q}(\hat{\mathfrak{sl}}_{2})$-module structure on $V$ such that the operators $e_{0}^{+} - R$, $e_{1}^{+} - L$, $K_{0}^{\pm 1} - \varepsilon_{0}\alpha^{\pm 1}K_{0}^{\pm 1}$, and $K_{1}^{\pm 1} - \varepsilon_{1}\alpha^{\pm 1}K_{1}^{\pm 1}$ vanish on $V$. This $U_{q}(\hat{\mathfrak{sl}}_{2})$-module structure is irreducible and of type $(\varepsilon_{0}, \varepsilon_{1})$.  

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Theorem 1.17 Let $V$ be a finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}}_2)$-module, and assume $(\varepsilon_0, \varepsilon_1)$ is its type. Let $\alpha$ denote a nonzero scalar in $\mathbb{F}$. Then there exists a unique $U^\geq_0$-module structure on $V$ such that the operators $e^+_0 - R$, $e^+_1 - L$, $K^\pm_0 - \varepsilon_0 \alpha^{\mp 1} K^\pm_1$, and $K^\pm_1 - \varepsilon_1 \alpha^{\pm 1} K^\mp_1$ vanish on $V$. This $U^\geq_0$-module structure is irreducible and of type $\alpha$.

Remark 1.18 Let $\alpha$ be a nonzero scalar in $\mathbb{F}$, and let $\varepsilon_0, \varepsilon_1$ denote scalars in $\{1, -1\}$. Combining Theorem 1.16 and Theorem 1.17 we obtain a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $U^\geq_0$-modules of type $\alpha$;

(ii) the isomorphism classes of finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}}_2)$-modules of type $(\varepsilon_0, \varepsilon_1)$.

Remark 1.19 As V. Chari and A. Pressley [1] have shown, each finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}}_2)$-module has a realization as a tensor product of evaluation modules. In our proofs below, we never have occasion to invoke this realization. In fact, our arguments are quite elementary and require only linear algebra. It follows from our work and the results of [1] that all of the finite-dimensional irreducible modules for $U^\geq_0$ and for $U_q(\hat{\mathfrak{sl}}_2)^\geq_0$ can be obtained from tensor products of evaluation modules of $U_q(\hat{\mathfrak{sl}}_2)$.

The plan for the paper is as follows. In Section 2, we state some preliminaries concerning $U_q(\hat{\mathfrak{sl}}_2)$-modules and $U^\geq_0$-modules. Sections 3–11 are devoted to proving Theorem 1.16. In Section 12, we prove Theorem 1.17 and in Section 13, we discuss irreducible modules for the Borel subalgebra $U_q(\hat{\mathfrak{sl}}_2)^\geq_0$.

The proof of Theorem 1.16 is an adaptation of a construction which T. Ito and the second author used to get $U_q(\hat{\mathfrak{sl}}_2)$-actions from a certain type of tridiagonal pair [22]. Indeed, the original motivation for our work came from the study of tridiagonal pairs ([10], [11]) and the closely related Leonard pairs ([17], [13], [19], [20], [21], [22], [23]). A Leonard pair is a pair of semisimple linear transformations on a finite-dimensional vector space, each of which acts tridiagonally on an eigenbasis for the other [23, Defn. 1.1]. There is a close connection between Leonard pairs and the orthogonal polynomials that make up the terminating branch of the Askey scheme ([14], [20], [23, Appendix A]). A tridiagonal pair is a mild generalization of a Leonard pair [10, Defn. 1.1]. See [3], [2] for related topics.
2 Preliminaries

In this section, we present some background material on irreducible modules for $U_q(\widehat{\mathfrak{sl}_2})$ and $U_{\geq 0}$. Towards this purpose, we adopt the following conventions. Assume $V$ is a nonzero finite-dimensional vector space over $\mathbb{F}$. Let $d$ denote a nonnegative integer. By a decomposition of $V$ of diameter $d$, we mean a sequence $U_0, U_1, \ldots, U_d$ of nonzero subspaces of $V$ such that

$$V = U_0 \oplus U_1 \oplus \cdots \oplus U_d.$$ 

Note we do not assume that the spaces $U_0, U_1, \ldots, U_d$ have dimension 1. For notational convenience we set $U_{-1} := 0$ and $U_{d+1} := 0$.

**Lemma 2.1** [1, Prop. 3.2] Let $V$ denote a finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}_2})$-module. Then there exist scalars $\varepsilon_0, \varepsilon_1$ in $\{1, -1\}$ and a decomposition $U_0, \ldots, U_d$ of $V$ such that

$$(K_0 - \varepsilon_0 q^{2i-d} I) U_i = 0 \quad \text{and} \quad (K_1 - \varepsilon_1 q^{d-2i} I) U_i = 0 \quad (2.2)$$

hold for all $i = 0, 1, \ldots, d$. The sequence $\varepsilon_0, \varepsilon_1; U_0, \ldots, U_d$ is unique. Moreover, for $0 \leq i \leq d$ we have

$$e_0^+ U_i \subseteq U_{i+1}, \quad e_1^- U_i \subseteq U_{i+1}, \quad (2.3)$$

$$e_0^- U_i \subseteq U_{i-1}, \quad e_1^+ U_i \subseteq U_{i-1}. \quad (2.4)$$

**Remark 2.5** If $\mathbb{F}$ has characteristic 2, then in Lemma 2.1 we view $\{1, -1\}$ as having a single element.

**Definition 2.6** The ordered pair $(\varepsilon_0, \varepsilon_1)$ in Lemma 2.1 is the type of $V$ and $d$ is the diameter of $V$. The sequence $U_0, \ldots, U_d$ is the weight space decomposition of $V$ (relative to $K_0$ and $K_1$).

**Lemma 2.7** [1, Prop. 3.3] For any choice of scalars $\varepsilon_0, \varepsilon_1$ from $\{1, -1\}$, there exists an $\mathbb{F}$-algebra automorphism of $U_q(\widehat{\mathfrak{sl}_2})$ such that

$$K_i \rightarrow \varepsilon_i K_i, \quad e_i^+ \rightarrow \varepsilon_i^+ e_i^+, \quad e_i^- \rightarrow \varepsilon_i e_i^-$$

for $i \in \{0, 1\}$.

**Remark 2.8** Given a finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}_2})$-module, we can alter its type to any other type by applying an automorphism as in Lemma 2.7.
The next lemma is reminiscent of Lemma 2.1.

**Lemma 2.9** Let $V$ be a finite-dimensional irreducible $U^{\geq 0}$-module. Then there exist a nonzero $\alpha \in \mathbb{F}$ and a decomposition $U_0, \ldots, U_d$ of $V$ such that

$$(K - \alpha q^{2i-d} I)U_i = 0 \quad (0 \leq i \leq d).$$

(2.10)

The sequence $\alpha; U_0, \ldots, U_d$ is unique. Moreover

$$RU_i \subseteq U_{i+1} \quad (0 \leq i \leq d),$$

(2.11)

$$LU_i \subseteq U_{i-1} \quad (0 \leq i \leq d).$$

(2.12)

**Proof:** For $\theta \in \mathbb{F}$, let $V^{(\theta)} = \{ v \in V \mid Kv = \theta v \}$. Using (1.12), (1.13) we find that

$$RV^{(\theta)} \subseteq V(q^2 \theta), \quad LV^{(\theta)} \subseteq V(q^{-2}\theta).$$

(2.13)

Since $\mathbb{F}$ is algebraically closed, $K$ has an eigenvalue in $\mathbb{F}$, so $V^{(\theta)} \neq 0$ for some $\theta \in \mathbb{F}$. Observe $\theta \neq 0$ since $K$ is invertible on $V$. The scalars $\theta, q^{-2}\theta, q^{-4}\theta, \ldots$ are mutually distinct since $q$ is not a root of unity, and not all of them can be eigenvalues of $K$ on $V$. Consequently, there is a nonzero $\zeta \in \mathbb{F}$ such that $V^{(\zeta)} \neq 0$ and $V^{(q^{-2}\zeta)} = 0$. There exists an integer $d \geq 0$ such that $V^{(q^{2d}\zeta)}$ is nonzero for $0 \leq i \leq d$ and zero for $i = d + 1$. Set $U_i = V^{(q^{2i}\zeta)}$ for $0 \leq i \leq d$ and define $U_{-1} = 0, U_{d+1} = 0$. Note that

$$(K - q^{2i}\zeta I)U_i = 0 \quad (0 \leq i \leq d).$$

(2.14)

Line (2.10) is an easy consequence of (2.14) by taking $\alpha := \zeta q^d$. Observe that $\alpha \neq 0$. Equations (2.11) and (2.12) follow from (2.13). We claim $U_0, \ldots, U_d$ is a decomposition of $V$. From the construction, each of the spaces $U_0, \ldots, U_d$ is nonzero. Lines (2.10)–(2.12) imply $\sum_{i=0}^d U_i$ is invariant under $R, L, K^{\pm 1}$. Since $\sum_{i=0}^d U_i$ is nonzero and $V$ is an irreducible $U^{\geq 0}$-module, it must be that $V = \sum_{i=0}^d U_i$. The sum is direct, since $U_0, \ldots, U_d$ are eigenspaces for $K$ corresponding to distinct eigenvalues. Therefore, $U_0, \ldots, U_d$ is a decomposition of $V$. It is clear that the sequence $\alpha; U_0, \ldots, U_d$ is unique. $\Box$

**Definition 2.15** In Lemma 2.9 $\alpha$ is said to be the type and $d$ the diameter of $V$. The sequence $U_0, \ldots, U_d$ is the weight space decomposition of $V$ (relative to $K$).
Lemma 2.16 For each nonzero $\alpha \in \mathbb{F}$, there exists an $\mathbb{F}$-algebra automorphism of $U_{\geq 0}$ such that

$$K \to \alpha K, \quad R \to R, \quad L \to L.$$ 

Proof: This is immediate from Definition 1.10 \hfill \square

Remark 2.17 Given a finite-dimensional irreducible $U_{\geq 0}$-module, we can change its type to any other type by applying an automorphism from Lemma 2.16

3 An outline of the proof for Theorem 1.16

Our proof of Theorem 1.16 will consume most of the paper from Section 4 to Section 11. Here we sketch an overview of the argument.

Let $V$ denote a finite-dimensional irreducible $U_{\geq 0}$-module of type $\alpha$. For any choice of $\varepsilon_0, \varepsilon_1$ from $\{1, -1\}$, we begin the construction of the $U_q(\hat{sl}_2)$-action on $V$ by requiring that the operators $e_0^+ - R, e_1^+ - L, K_0^{\pm 1} - \varepsilon_0 \alpha^{\pm 1} K^\pm 1, K_1^{\pm 1} - \varepsilon_1 \alpha^{\pm 1} K_1^{\pm 1}$ vanish on $V$. This gives the actions of the elements $e_0^+, e_1^+, K_0^{\pm 1}, K_1^{\pm 1}$ on $V$. We define the actions of $e_0^-, e_1^-$ on $V$ as follows. First we prove that $K + R$ and $K^{-1} + L$ act semisimply on $V$. Then we show that the set of distinct eigenvalues of $K + R$ (resp. $K^{-1} + L$) on $V$ is $\{\alpha q^{2i-d} \mid 0 \leq i \leq d\}$ (resp. $\{\alpha^{-1} q^{d-2i} \mid 0 \leq i \leq d\}$), where $d$ is the diameter of $V$. For $0 \leq i \leq d$, we let $V_i$ (resp. $V_i^*$) denote the eigenspace of $K + R$ (resp. of $K^{-1} + L$) on $V$ associated with the eigenvalue $\alpha q^{2i-d}$ (resp. $\alpha^{-1} q^{d-2i}$). Then $V_0, \ldots, V_d$ (resp. $V_0^*, \ldots, V_d^*$) is a decomposition of $V$. To motivate what comes next, we mention that the weight space decomposition $U_0, \ldots, U_d$ of $V$ satisfies

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d) \quad (0 \leq i \leq d).$$

For $0 \leq i \leq d$, we define

$$W_i = (V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i}),$$

$$W_i^* = (V_{d-i}^* + \cdots + V_d^*) \cap (V_i + \cdots + V_d).$$

We argue that both $W_0, \ldots, W_d$ and $W_0^*, \ldots, W_d^*$ are decompositions of $V$. Therefore, there exist linear transformations $B : V \to V$ (resp. $B^* : V \to V$) such that for $0 \leq i \leq d$, $W_i$ (resp. $W_i^*$) is an eigenspace for $B$ (resp. $B^*$) with eigenvalue $q^{2i-d}$ (resp. $q^{d-2i}$). We let $e_0^-$ (resp. $e_1^-$) act on $V$ as $\alpha^{-1} I - K^{-1} B$
times $\varepsilon_0 q^{-1} (q^{-1})^{-2}$ (resp. $\alpha I - KB^*$ times $\varepsilon_1 q^{-1} (q^{-1})^{-2}$). We display some relations that are satisfied by $B, B^*$, and the generators of $U^{\geq 0}$. Using these relations, we argue that the above actions of $e^{\pm}_0, e^{\pm}_1, K^{\pm 1}_0, K^{\pm 1}_1$ satisfy the defining relations for $U_q(\hat{sl}_2)$. In this way, we obtain the required action of $U_q(\hat{sl}_2)$ on $V$.

4 The elements $A$ and $A^*$

As we proceed with our investigation of $U^{\geq 0}$, we find it convenient to work with the elements $K + R$ and $K^{-1} + L$ instead of $R$ and $L$. Hence we are led to the following definition.

**Definition 4.1** Let $A$ and $A^*$ denote the following elements of $U^{\geq 0}$:

$$A = K + R, \quad A^* = K^{-1} + L. \quad (4.2)$$

We observe $A, A^*, K^{\pm 1}$ form a generating set for $U^{\geq 0}$.

**Lemma 4.3** The following relations hold in $U^{\geq 0}$:

$$\frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} = 1, \quad (4.4)$$

$$\frac{qKA^* - q^{-1}A^*K}{q - q^{-1}} = 1. \quad (4.5)$$

**Proof:** To verify (4.4), substitute $A = K + R$ and simplify the result using (1.12). Relation (4.5) can be verified similarly using (1.13).

** Remark 4.6** Each of the equations (4.4) and (4.5) is essentially an instance of the $q$-Weyl relation, which is the defining relation of the $q$-Weyl algebra (see [9], for example). A presentation of $U_q(\hat{sl}_2)$ which involves the $q$-Weyl relations and the $q$-Serre relations can be found in [12].

**Lemma 4.7** The elements $A, A^*$ in Definition 4.1 satisfy these relations:

$$A^3 A^* - [3] A^2 A^* A + [3] A A^* A^2 - A^* A^3 = 0, \quad (4.8)$$

$$A^{*3} A - [3] A^{*2} A A^* + [3] A^* A A^{*2} - A A^{*3} = 0. \quad (4.9)$$

**Proof:** To verify (4.8), substitute $A = K + R$ and $A^* = K^{-1} + L$, and simplify the result using (1.12)–(1.14). Line (4.9) can be checked in the same way.
Lemma 4.10 Let $V$ be a finite-dimensional irreducible $U^\geq 0$-module with type $\alpha$ and weight space decomposition $U_0, \ldots, U_d$. Then for $0 \leq i \leq d$ we have
\begin{align}
(A - \alpha q^{2i-d} I)U_i &\subseteq U_{i+1}, \\
(A^* - \alpha^{-1} q^{d-2i} I)U_i &\subseteq U_{i-1}.
\end{align}

Proof: Relation (4.11) follows directly from (2.10), (2.11), and the fact that $A = K + R$. Line (4.12) is a consequence of (2.10), (2.12), and $A^* = K^{-1} + L$. □

Lemma 4.13 Let $V$ denote a finite-dimensional irreducible $U^\geq 0$-module of type $\alpha$ and diameter $d$. Then the elements $A$ and $A^*$ in Definition 4.1 act semisimply on $V$. The set of distinct eigenvalues for $A$ on $V$ is $\{\alpha q^{2i-d} \mid 0 \leq i \leq d\}$. The set of distinct eigenvalues for $A^*$ on $V$ is $\{\alpha^{-1} q^{d-2i} \mid 0 \leq i \leq d\}$.

Proof: It is apparent from (4.11) that the product $\prod_{i=0}^d (A - \alpha q^{2i-d} I)$ vanishes on $V$. Since the scalars $\alpha q^{2i-d}$ ($0 \leq i \leq d$) are mutually distinct, we find $A$ is semisimple on $V$. It is clear from (4.11) that the complete set of distinct eigenvalues of $A$ on $V$ is $\{\alpha q^{2i-d} \mid 0 \leq i \leq d\}$. Our assertions concerning $A^*$ follow similarly. □

5 The eigenspace decompositions for $A$ and $A^*$

Definition 5.1 Assume $V$ is a finite-dimensional irreducible $U^\geq 0$-module. Referring to Lemma 4.13 we let $V_i$ (resp. $V_i^*$) denote the eigenspace of $A$ (resp. $A^*$) on $V$ corresponding to the eigenvalue $\alpha q^{2i-d}$ (resp. $\alpha^{-1} q^{d-2i}$) for $0 \leq i \leq d$. Then $V_0, \ldots, V_d$ and $V_0^*, \ldots, V_d^*$ are each decompositions of $V$.

Lemma 5.2 Let $V$ denote a finite-dimensional irreducible $U^\geq 0$-module with weight space decomposition $U_0, \ldots, U_d$. Suppose $V_0, \ldots, V_d$ and $V_0^*, \ldots, V_d^*$ are the decompositions in Definition 5.1 Then for $0 \leq i \leq d$ we have
\begin{align}
(i) &\quad U_i + \cdots + U_d = V_i + \cdots + V_d, \\
(ii) &\quad U_0 + \cdots + U_i = V_0^* + \cdots + V_i^*, \\
(iii) &\quad U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).
\end{align}
Proof: (i) Set \( X_i = \sum_{j=i}^d U_j \) and \( X_i' = \sum_{j=i}^d V_j \). We show \( X_i = X_i' \). Assume \( \alpha \) is the type of \( V \), and let \( T_i := \prod_{j=i}^d (A - \alpha q^{2j-d}I) \). Then \( X_i' = \{ v \in V \mid T_i v = 0 \} \), and \( T_i X_i = 0 \) by (4.11) so \( X_i \subseteq X_i' \). Now let \( S_i := \prod_{j=i}^{i-1} (A - \alpha q^{2j-d}I) \). Observe that \( S_i V = X_i' \), and \( S_i V \subseteq X_i \) by (4.11) so \( X_i' \subseteq X_i \). We conclude that \( X_i = X_i' \) holds as required.

(ii) Mimic the proof of (i).

(iii) Combine parts (i) and (ii) above.

Lemma 5.3 Let \( V \) be a finite-dimensional irreducible \( U^{\geq 0} \)-module of type \( \alpha \) and diameter \( d \). Assume \( V_0, \ldots, V_d \) and \( V_0^*, \ldots, V_d^* \) are the decompositions in Definition 5.1. Then for \( 0 \leq i \leq d \) we have

(i) \((K^{-1} - \alpha^{-1} q^{d-2i}I)V_i \subseteq V_{i+1}\),

(ii) \((K - \alpha q^{2i-d}I)V_i \subseteq V_{i+1} + \cdots + V_d\),

(iii) \((K - \alpha q^{2i-d}I)V_i^* \subseteq V_{i-1}^*\),

(iv) \((K^{-1} - \alpha^{-1} q^{d-2i}I)V_i^* \subseteq V_0^* + \cdots + V_{i-1}^*\).

Proof: (i) Recall that for \( 0 \leq i \leq d \), \( V_i \) is an eigenspace for \( A \) with corresponding eigenvalue \( \alpha q^{2i-d} \). Therefore it suffices to show that

\[(A - \alpha q^{2i+2-d}I)(K^{-1} - \alpha^{-1} q^{d-2i}I)\]

vanishes on \( V_i \) for \( 0 \leq i \leq d \). Since \( A - \alpha q^{2i-d}I \) vanishes on \( V_i \), so does

\[(K^{-1} - \alpha^{-1} q^{d-2i-2}I)(A - \alpha q^{2i-d}I).\]

Using (4.4) we see that

\[qK^{-1}A - q^{-1}AK^{-1} - (q - q^{-1})I\]

is 0 on \( V_i \). Subtracting (5.5) from \( q^{-1} \) times (5.5) we find that (5.4) vanishes on \( V_i \). Hence \((K^{-1} - \alpha^{-1} q^{d-2i}I)V_i \subseteq V_{i+1}\) for \( 0 \leq i \leq d \).

Part (ii) follows from (i) above, while (iii) can be obtained by an argument similar to the proof of (i). Finally, (iv) is a consequence of (iii). \( \square \)

Lemma 5.7 Let \( V \) be a finite-dimensional irreducible \( U^{\geq 0} \)-module. Let the decompositions \( V_0, \ldots, V_d \) and \( V_0^*, \ldots, V_d^* \) be as in Definition 5.1. Then for \( 0 \leq i \leq d \) we have
(i) \( A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}, \)

(ii) \( AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*. \)

Proof: (i) Let \( \alpha \) denote the type of \( V \). Recall that for \( 0 \leq i \leq d \), \( V_i \) is the eigenspace for \( A \) corresponding to the eigenvalue \( \alpha q^{2i-d} \). Therefore it suffices to show

\[
(A - \alpha q^{2i-2-d}I) (A - \alpha q^{2i-d}I) (A - \alpha q^{2i+2-d}I) A^*V_i = 0
\]

for \( 0 \leq i \leq d \). For \( v \in V_i \) we have

\[
0 = (A^3 A^* - [3]A^2 A^* A + [3]AA^* A^2 - A^* A^3) v \quad \text{(by (4.8))}
\]

\[
= (A^3 A^* - [3]A^2 A^* \alpha q^{2i-d} + [3]AA^* \alpha^2 q^{4i-2d} - A^* \alpha^3 q^{6i-3d}) v
\]

\[
= (A - \alpha q^{2i-2-d}I) (A - \alpha q^{2i-d}I) (A - \alpha q^{2i+2-d}I) A^*v,
\]

which gives the desired result.

(ii) This can be argued analogously. \( \square \)

6 Yet two more decompositions

Definition 6.1 Let \( V \) denote a finite-dimensional irreducible \( U_{\geq 0} \)-module. Assume \( V_0, \ldots, V_d \) and \( V_0^*, \ldots, V_d^* \) are the decompositions from Definition 5.1. For \( 0 \leq i \leq d \), we define

\[
W_i = (V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i}),
\]

(6.2)

\[
W_i^* = (V_{d-i}^* + \cdots + V_d^*) \cap (V_i + \cdots + V_d).
\]

(6.3)

Our next goal is to show that the spaces \( W_0, \ldots, W_d \) and \( W_0^*, \ldots, W_d^* \) in Definition 6.1 afford decompositions of \( V \). Towards this purpose, the following definition will be useful.

Definition 6.4 Let \( V \) be a finite-dimensional irreducible \( U_{\geq 0} \)-module. Assume \( V_0, \ldots, V_d \) and \( V_0^*, \ldots, V_d^* \) are the decompositions in Definition 5.1. Set

\[
W(i,j) = \left( \sum_{h=0}^{i} V_h^* \right) \cap \left( \sum_{k=0}^{j} V_k \right)
\]

(6.5)

for all integers \( i, j \). We interpret the left sum in (6.5) to be 0 (resp. \( V \)) if \( i < 0 \) (resp. \( i > d \)). Likewise the right sum in (6.5) is interpreted to be 0 (resp. \( V \)) if \( j < 0 \) (resp. \( j > d \)).
Example 6.6 With reference to Definition 6.4, the following hold.

(1) \( W(i, d) = V_0^* + V_1^* + \cdots + V_i^* \quad (0 \leq i \leq d) \).

(2) \( W(d, j) = V_0 + V_1 + \cdots + V_j \quad (0 \leq j \leq d) \).

Proof: To obtain (1), set \( j = d \) in (6.5) and recall that \( \sum_{k=0}^d V_k = V \); for (2), use \( i = d \) and \( \sum_{h=0}^d V_h^* = V \). \( \square \)

Lemma 6.7 Let \( V \) denote a finite-dimensional irreducible \( U^{\geq 0} \)-module with type \( \alpha \) and diameter \( d \). Then for the spaces \( W(i, j) \) in Definition 6.4, we have

(i) \( (A - \alpha q^{2j - d} I) W(i, j) \subseteq W(i + 1, j - 1) \),

(ii) \( (A^* - \alpha^{-1} q^{d - 2i} I) W(i, j) \subseteq W(i - 1, j + 1) \),

(iii) \( (K^{-1} - \alpha^{-1} q^{d - 2i} I) W(i, j) \subseteq W(i - 1, j + 1) \),

(iv) \( (K - \alpha q^{2i - d} I) W(i, j) \subseteq \sum_{h=1}^i W(i - h, j + h) \),

for \( 0 \leq i, j \leq d \).

Proof: (i) Using Lemma 5.7(ii), we find that

\[
(A - \alpha q^{2j - d} I) \sum_{h=0}^i V_h^* \subseteq \sum_{h=0}^{i+1} V_h^*.
\] (6.8)

Because each \( V_k \) is an eigenspace for \( A \) with eigenvalue \( \alpha q^{2k - d} \), we have

\[
(A - \alpha q^{2j - d} I) \sum_{k=0}^j V_k = \sum_{k=0}^{j-1} V_k.
\] (6.9)

Evaluating \( (A - \alpha q^{2j - d} I) W(i, j) \) using (6.5)–(6.9), we see that it is contained in \( W(i + 1, j - 1) \).

(ii) This part can be demonstrated using the relations

\[
(A^* - \alpha^{-1} q^{d - 2i} I) \sum_{h=0}^i V_h^* = \sum_{h=0}^{i-1} V_h^*,
\] (6.10)

\[
(A^* - \alpha^{-1} q^{d - 2i} I) \sum_{k=0}^j V_k \subseteq \sum_{k=0}^{j+1} V_k
\] (6.11)
in conjunction with \((6.5)\).

(iii) From Lemma 5.3 (iv) and Lemma 5.3 (i), we find that

\[
(K^{-1} - \alpha^{-1} q^{d-2i}) \sum_{h=0}^{i} V_h^* \subseteq \sum_{h=0}^{i-1} V_h^*, \tag{6.12}
\]

\[
(K^{-1} - \alpha^{-1} q^{d-2j}) \sum_{k=0}^{j} V_k \subseteq \sum_{k=0}^{j+1} V_k. \tag{6.13}
\]

Evaluating \((K^{-1} - \alpha^{-1} q^{d-2i}) W(i, j)\) using \((6.5)\), \((6.12)\), and \((6.13)\), we see that it is contained in \(W(i-1, j+1)\).

(iv) This assertion follows from (iii).

\[\blacksquare\]

**Lemma 6.14** For the spaces \(W(i, j)\) in Definition 6.4

\[W(i, j) = 0 \text{ if } i + j < d, \quad (0 \leq i, j \leq d). \tag{6.15}\]

**Proof:** Lemma 6.7 implies that for \(0 \leq r < d\) the sum

\[W := W(0, r) + W(1, r - 1) + \cdots + W(r, 0) \tag{6.16}\]

is invariant under \(A, A^*, \) and \(K^{\pm 1}\). Since \(A, A^*, K^{\pm 1}\) is a generating set for \(U^{\geq 0}\), we find that \(W\) is a \(U^{\geq 0}\)-submodule of \(V\). Because \(V\) is an irreducible \(U^{\geq 0}\)-module, we have \(W = 0\) or \(W = V\). By \((6.5)\), each term in \((6.16)\) is contained in

\[V_0 + V_1 + \cdots + V_r, \tag{6.17}\]

so \(W\) is contained in \((6.17)\). The containment of \(W\) in \(V\) is proper since \(r < d\). Thus \(W = 0\) and \((6.15)\) follows.

\[\blacksquare\]

**Lemma 6.18** Let \(V\) denote a finite-dimensional irreducible \(U^{\geq 0}\)-module. Then the sequence \(W_0, \ldots, W_d\) from Definition 6.4 is a decomposition of \(V\).

**Proof:** First we argue that \(\sum_{i=0}^{d} W_i = V\). Comparing \((6.2)\) and \((6.5)\), we find that \(W_i = W(i, d-i)\) for \(0 \leq i \leq d\); by this and Lemma 6.7, we have that \(\sum_{i=0}^{d} W_i\) is invariant under \(A, A^*, K^{\pm 1}\). Because \(A, A^*, K^{\pm 1}\) generate \(U^{\geq 0}\), \(\sum_{i=0}^{d} W_i\) must be 0 or \(V\). Observe that \(\sum_{i=0}^{d} W_i\) contains \(W_0 = V_0^* \neq 0\), so \(\sum_{i=0}^{d} W_i = V\). To show that the sum \(\sum_{i=0}^{d} W_i\) is direct, we prove that

\[(W_0 + W_1 + \cdots + W_{i-1}) \cap W_i = 0\]

for \(1 \leq i \leq d\). From the construction,

\[W_j \subseteq V_0^* + V_1^* + \cdots + V_{i-1}^* \]

for \(1 \leq i \leq d\).
for \(0 \leq j \leq i - 1\), and

\[ W_i \subseteq V_0 + V_1 + \cdots + V_{d-i}. \]

Therefore

\[
(W_0 + W_1 + \cdots + W_{i-1}) \cap W_i \\
\subseteq (V_0^* + V_1^* + \cdots + V_{i-1}^*) \cap (V_0 + V_1 + \cdots + V_{d-i}) \\
= W(i - 1, d - i) \\
= 0,
\]

in view of Lemma 6.14. We have now shown that the sum \(\sum_{i=0}^{d} W_i\) is direct. Next we argue that \(W_i \neq 0\) for \(0 \leq i \leq d\). We have \(W_0 = V_0^* \neq 0\) and \(W_d = V_0 \neq 0\). Suppose there exists an integer \(i\) \((1 \leq i \leq d - 1)\) such that \(W_i = 0\). By Lemma 6.7 the sum \(\sum_{h=0}^{i-1} W_i\) is a \(U^{\geq 0}\)-submodule of \(V\) since it is invariant under each of \(A, A^*, K\). Therefore \(\sum_{h=0}^{i-1} W_i = 0\) or \(\sum_{h=0}^{i-1} W_i = V\). But \(\sum_{h=0}^{i-1} W_i \neq V\) since \(i \geq 1\) and \(W_0 \neq 0\); and \(\sum_{h=0}^{i-1} W_i \neq V\) since \(i - 1 < d\) and \(W_d \neq 0\). We have reached a contradiction. Consequently, \(W_i \neq 0\) for \(0 \leq i \leq d\). Thus, the sequence \(W_0, \ldots, W_d\) is a decomposition of \(V\). \(\square\)

**Lemma 6.19** Let \(V\) denote a finite-dimensional irreducible \(U^{\geq 0}\)-module. Then the sequence \(W_0^*, \ldots, W_d^*\) from Definition 6.1 is a decomposition of \(V\).

**Proof:** Imitate the proof of Lemma 6.18 \(\square\)

We record a few helpful facts for later use.

**Lemma 6.20** Assume \(V\) is a finite-dimensional irreducible \(U^{\geq 0}\)-module of type \(\alpha\) and diameter \(d\). Then for the decomposition \(W_0, \ldots, W_d\) in Definition 6.1, the following hold for \(0 \leq i \leq d\).

(i) \((A - \alpha q^{d-2i})W_i \subseteq W_{i+1}\).

(ii) \((A^* - \alpha^{-1} q^{d-2i})W_i \subseteq W_{i-1}\).

(iii) \((K - \alpha^{-1} q^{d-2i})W_i \subseteq W_{i-1}\).

(iv) \((K - \alpha q^{2i-d})W_i \subseteq W_0 + \cdots + W_{i-1}\).

**Proof:** Set \(j = d - i\) and \(W(i, d - i) = W_i\) in Lemma 6.7 \(\square\)
Lemma 6.21  Let $V$ denote a finite-dimensional irreducible $U \geq 0$-module of type $\alpha$ and diameter $d$. Then for the decomposition $W^*_0, \ldots, W^*_d$ in Definition 6.1, the following hold for $0 \leq i \leq d$.

(i) $(A - \alpha q^{2i-d}) W^*_i \subseteq W^*_{i+1}.$
(ii) $(A^* - \alpha^{-1} q^{2i-d}) W^*_i \subseteq W^*_{i-1}.$
(iii) $(K - \alpha q^{2i-d}) W^*_i \subseteq W^*_{i+1}.$
(iv) $(K^{-1} - \alpha^{-1} q^{d-2i}) W^*_i \subseteq W^*_{i+1} + \cdots + W^*_d.$

Lemma 6.22  Assume $V$ is a finite-dimensional irreducible $U \geq 0$-module. Let the decompositions $V_0, \ldots, V_d$ and $V^*_0, \ldots, V^*_d$ be as in Definition 5.1. Let the decompositions $W_0, \ldots, W_d$ and $W^*_0, \ldots, W^*_d$ be as in Definition 6.1. Then the following hold for $0 \leq i \leq d$.

(i) $W_i + \cdots + W_d = V_0 + \cdots + V_{d-i}.$
(ii) $W_0 + \cdots + W_i = V^*_0 + \cdots + V^*_i.$
(iii) $W^*_i + \cdots + W^*_d = V_i + \cdots + V_d.$
(iv) $W^*_0 + \cdots + W^*_i = V^*_{d-i} + \cdots + V^*_d.$

Proof: (i) Set $X_i := \sum_{j=0}^d W_j$ and $X'_i := \sum_{j=0}^{d-i} V_j$. We show $X_i = X'_i$. Let $\alpha$ denote the type of $V$ and set $T_i := \prod_{j=0}^{d-i} (A - \alpha q^{2j-d} I)$. Then $X'_i = \{ v \in V \mid T_i v = 0 \}$, and $T_i X_i = 0$ by Lemma 6.21 (i), so $X_i \subseteq X'_i$. Now let $S_i := \prod_{j=d-i+1}^d (A - \alpha q^{2j-d} I)$. Observe that $S_i V = X'_i$, and $S_i V \subseteq X_i$ by Lemma 6.21 (i), so $X'_i \subseteq X_i$. We conclude $X_i = X'_i$ and the result follows.

(ii)–(iv) These equations can be verified in a similar fashion.  

7 The linear transformations $B$ and $B^*$

Definition 7.1  Assume $V$ is a finite-dimensional irreducible $U \geq 0$-module. Let the decompositions $W_0, \ldots, W_d$ and $W^*_0, \ldots, W^*_d$ be as in Definition 6.1.

1. Let $B : V \to V$ denote the linear transformation such that for $0 \leq i \leq d$, $W_i$ is an eigenspace of $B$ with eigenvalue $q^{2i-d}$.

2. Let $B^* : V \to V$ denote the linear transformation such that for $0 \leq i \leq d$, $W^*_i$ is an eigenspace of $B^*$ with eigenvalue $q^{d-2i}$.
Next we show that $A, A^*, B, B^*$ satisfy $q$-Weyl relations.

Lemma 7.2 Assume $V$ is a finite-dimensional irreducible $U_{\geq 0}$-module of type $\alpha$. For $A, A^*$ from Definition 4.1 and $B, B^*$ from Definition 7.1,

\[
\frac{qAB - q^{-1}BA}{q - q^{-1}} = \alpha I, \tag{7.3}
\]

\[
\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} = \alpha^{-1} I, \tag{7.4}
\]

\[
\frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} = \alpha^{-1} I, \tag{7.5}
\]

\[
\frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} = \alpha I. \tag{7.6}
\]

Proof: Let the decomposition $W_0, \ldots, W_d$ be as in Lemma 6.18. To prove (7.3), we show that $qAB - q^{-1}BA - \alpha(q - q^{-1}) I$ is 0 on $W_i$ for $0 \leq i \leq d$. Since $B - q^{2i-d} I$ vanishes on $W_i$ by Definition 7.1, so does $A - \alpha q^{d-2i-2} I (B - q^{2i-d} I)$.

By Lemma 6.20 (i),

\[
(B - q^{2i+2-d} I)(A - \alpha q^{d-2i} I) \tag{7.8}
\]

vanishes on $W_i$. Subtracting $q^{-1}$ times (7.8) from $q$ times (7.7) we find that $qAB - q^{-1}BA - \alpha(q - q^{-1}) I$ is 0 on $W_i$. Relation (7.3) follows. Lines (7.4)–(7.6) can be proved in a similar manner. □

8 The action of $B$ and $B^*$ on the decompositions

In this section we describe how the maps $B, B^*$ act on our five decompositions.

Lemma 8.1 Let $V$ denote a finite-dimensional irreducible $U_{\geq 0}$-module with weight space decomposition $U_0, \ldots, U_d$. Assume that the decompositions $V_0, \ldots, V_d$ and $V_0^*, \ldots, V_d^*$ are as in Definition 5.1. Let the decompositions $W_0, \ldots, W_d$ and $W_0^*, \ldots, W_d^*$ be as in Definition 6.1. Assume $B, B^*$ are as in Definition 7.1. Then the following hold for $0 \leq i \leq d$.

(i) $(B - q^{d-2i} I)V_i \subseteq V_{i-1}$ and $(B - q^{2i-d} I)V_i^* \subseteq V_{i-1}^*$.  

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(ii) \((B^* - q^{d-2i})V_i \subseteq V_{i+1}\) and \((B^* - q^{2i-d})V_i^* \subseteq V_{i+1}^*\).

(iii) \((B - q^{2i-d})U_i \subseteq U_{i-1}\).

(iv) \((B^* - q^{d-2i})U_i \subseteq U_{i+1}\).

(v) \(BW_i^* \subseteq W_{i-1}^* + W_i^* + W_{i+1}^*\).

(vi) \(B^*W_i \subseteq W_{i-1} + W_i + W_{i+1}\).

Proof: (i), (ii) Let \(\alpha\) denote the type of \(V_i\). Recall that for \(0 \leq i \leq d\), \(V_i\) is the eigenspace for \(A\) with eigenvalue \(\alpha q^{2i-d}\). To obtain the first half of (i) it is sufficient to show that

\[
(A - \alpha q^{2i-2-d}I)(B - q^{d-2i}I)
\]

vanishes on \(V_i\) for \(0 \leq i \leq d\). Since \(A - \alpha q^{2i-d}I\) vanishes on \(V_i\), so does

\[
(B - q^{d-2i+2}I)(A - \alpha q^{2i-d}I)
\]

Equation (8.3) implies that

\[
qAB - q^{-1}BA - \alpha(q - q^{-1})I
\]

is 0 on \(V_i\). Adding \(q\) times (8.4), we find that (8.2) vanishes on \(V_i\). Consequently, \((B - q^{d-2i}I)V_i \subseteq V_{i-1}\) for \(0 \leq i \leq d\).

The second half of (i) and the relations in (ii) can be established similarly.

(iii) We have

\[
(B - q^{2i-d}I)U_i \subseteq (B - q^{2i-d}I)(U_0 + \cdots + U_i)
\]

\[
= (B - q^{2i-d}I)(V_0^* + \cdots + V_i^*) \quad \text{(by Lemma 5.2(ii))}
\]

\[
\subseteq V_0^* + \cdots + V_{i-1}^* \quad \text{(by Lemma 8.1(i))}
\]

\[
= U_0 + \cdots + U_{i-1} \quad \text{(by Lemma 5.2(ii)),}
\]

and also

\[
(B - q^{2i-d}I)U_i \subseteq (B - q^{2i-d}I)(U_i + \cdots + U_d)
\]

\[
= (B - q^{2i-d}I)(V_i + \cdots + V_d) \quad \text{(by Lemma 5.2(i))}
\]

\[
\subseteq V_{i-1} + \cdots + V_d \quad \text{(by Lemma 8.1(i))}
\]

\[
= U_{i-1} + \cdots + U_d \quad \text{(by Lemma 5.2(i)).}
\]

Combining these observations we find \((B - q^{2i-d}I)U_i \subseteq U_{i-1}\).

(iv) To obtain this part, imitate the argument for (iii).
(v) We have
\[
BW_i^* \subseteq B(W_0^* + \cdots + W_i^*) \\
= B(V_i^* + \cdots + V_d^*) \quad \text{(by Lemma 6.22(iv))} \\
\subseteq V_{d-i-1}^* + \cdots + V_d^* \quad \text{(by Lemma 8.1(i))} \\
= W_0^* + \cdots + W_{i+1}^* \quad \text{(by Lemma 6.22(iv))},
\]
and also
\[
BW_i^* \subseteq B(W_i^* + \cdots + W_d^*) \\
= B(V_i + \cdots + V_d) \quad \text{(by Lemma 6.22(iii))} \\
\subseteq V_{i-1} + \cdots + V_d \quad \text{(by Lemma 8.1(i))} \\
= W_{i-1}^* + \cdots + W_d^* \quad \text{(by Lemma 6.22(iii))}.
\]
Together these relations imply \( BW_i^* \subseteq W_{i-1}^* + W_i^* + W_{i+1}^* \).

(vi) This argument is identical to (v). \(\Box\)

9 Some relations involving \( B, B^*, K^{\pm 1} \)

In this section we show \( B, B^*, K^{\pm 1} \) satisfy \( q \)-Weyl relations.

**Lemma 9.1** Let \( V \) be a finite-dimensional irreducible \( U^{\geq 0} \)-module of type \( \alpha \). Assume \( B, B^* \) are as in Definition 7.1. Then both
\[
qBK^{-1} - q^{-1}K^{-1}B = \alpha^{-1}I, \quad (9.2)
\]
\[
qB^*K - q^{-1}KB^* = \alpha I. \quad (9.3)
\]

**Proof:** Let \( U_0, \ldots, U_d \) denote the weight space decomposition for \( V \). Recall that for \( 0 \leq i \leq d \), \( U_i \) is an eigenspace for \( K \) with eigenvalue \( \alpha q^{2i-d} \). To obtain (9.2), we show \( qBK^{-1} - q^{-1}K^{-1}B - \alpha^{-1}(q - q^{-1})I \) vanishes on \( U_i \) for \( 0 \leq i \leq d \). Observe that \( K^{-1} - \alpha^{-1}q^{-2i}I \) vanishes on \( U_i \) so
\[
(B - q^{2i-d-2}I)(K^{-1} - \alpha^{-1}q^{d-2i}I)
\]
is 0 on \( U_i \). From Lemma 8.1 (iii) we see that \( (B - q^{2i-d}I)U_i \subseteq U_{i-1} \). Therefore
\[
(K^{-1} - \alpha^{-1}q^{-2i+2}I)(B - q^{2i-d}I)
\]
vanishes on \( U_i \). Subtracting \( q^{-1} \) times (9.5) from \( q \) times (9.4) we find \( qBK^{-1} - q^{-1}K^{-1}B - \alpha^{-1}(q - q^{-1})I \) vanishes on \( U_i \). Equation (9.2) follows, and relation (9.3) is proved similarly. \(\Box\)
10 The $q$-Serre relations

Next we show that the elements $B, B^*$ from Definition 7.1 satisfy the $q$-Serre relations.

**Theorem 10.1** Let $V$ be a finite-dimensional irreducible $U^{\geq 0}$-module. Then the transformations $B, B^*$ in Definition 7.1 satisfy the relations

$$B^3 B^* - [3]B^2 B^* B + [3]BB^* B^2 - B^* B^3 = 0,$$

(10.2)

$$B^* B - [3]B^* B^* + [3]B^* BB^* - BB^* B^3 = 0.$$  

(10.3)

**Proof:** Let the decomposition $W_0, \ldots, W_d$ be as in Definition 6.1. Recall that for $0 \leq i \leq d$, $W_i$ is an eigenspace for $B$ with eigenvalue $q^{2i-d}$. In order to prove (10.2) we show that the transformation $\Psi := B^3 B^* - [3]B^2 B^* B + [3]BB^* B^2 - B^* B^3$ is 0 on $W_i$ for $0 \leq i \leq d$. Let $i$ be given and pick $v \in W_i$. Observe that $B^* v \in W_{i-1} + W_i + W_{i+1}$ by Lemma 8.1(vi). Next note that $(B - q^{2i-2-d} I)W_{i-1} = 0, (B - q^{2i-d} I)W_i = 0,$ and $(B - q^{2i+2-d} I)W_{i+1} = 0$. By these comments

$$(B - q^{2i-2-d} I)(B - q^{2i-d} I)(B - q^{2i+2-d} I)B^* v = 0.$$

Therefore,

$$\Psi v = (B^3 B^* - [3]B^2 B^* B + [3]BB^* B^2 - B^* B^3)v$$

$$= (B^3 B^* - [3]B^2 B^* q^{2i-d} + [3]BB^* q^{4i-2d} - B^* q^{6i-3d})v$$

$$= (B - q^{2i-2-d} I)(B - q^{2i-d} I)(B - q^{2i+2-d} I)B^* v$$

$$= 0.$$

We have now shown $\Psi W_i = 0$ for $0 \leq i \leq d$. Consequently $\Psi = 0$ and (10.2) follows. Line (10.3) can be proved similarly. □

11 The proof of Theorem 1.16

This section is devoted to a proof of Theorem 1.16.

**Definition 11.1** Let $V$ be a finite-dimensional irreducible $U^{\geq 0}$-module of type $\alpha$. Using the transformations $B, B^*$ in Definition 7.1, we introduce linear transformations $r : V \rightarrow V$ and $l : V \rightarrow V$ as follows:

$$r = \frac{\alpha I - KB^*}{q(q-q^{-1})^2},$$

$$l = \frac{\alpha^{-1} I - K^{-1} B}{q(q-q^{-1})^2}.$$
**Lemma 11.2** With reference to Definition 11.1, we have

\[ B = \alpha^{-1}K - q(q^{-1})^2 Kl, \]
\[ B^* = \alpha K^{-1} - q(q^{-1})^2 K^{-1} r. \]

**Theorem 11.3** Let \( V \) denote a finite-dimensional irreducible \( U^{\geq 0} \)-module of type \( \alpha \). Then the generators \( R, L, K^{\pm 1} \) of \( U^{\geq 0} \), together with \( r, l \) from Definition 11.1, satisfy the following relations on \( V \):

\[
\begin{align*}
KK^{-1} &= K^{-1}K = 1, \quad (11.4) \\
KRK^{-1} &= q^2 R, \quad KLR^{-1} = q^{-2} L, \quad (11.5) \\
KrK^{-1} &= q^2 r, \quad KlK^{-1} = q^{-2} l, \quad (11.6) \\
\alpha - K - K^{-1} &= -\alpha - K^{-1} K, \quad (11.7) \\
lL &= Ll, \quad rR = Rr, \quad (11.8) \\
0 &= R^3 L - [3] R^2 LR + [3] RLR^2 - LR^3, \quad (11.9) \\
0 &= L^3 R - [3] L^2 RL + [3] LRL^2 - RL^3, \quad (11.10) \\
0 &= r^3 l - [3] r^2 lr + [3] rlr^2 - lr^3, \quad (11.11) \\
0 &= l^3 r - [3] l^2 rl + [3] lrl^2 - rl^3. \quad (11.12)
\end{align*}
\]

**Proof:** The relations in (11.4), (11.5) are defining relations (1.11)–(1.13) of \( U^{\geq 0} \). To obtain (11.6), evaluate each of (9.2), (9.3) using Lemma 11.2. For (11.7), (11.8), use Definition 4.1, Lemma 11.2 and equations (11.5) and (11.6) to evaluate (7.3)–(7.6). Lines (11.9), (11.10) are just defining relations (1.14), (1.15) respectively. Finally, to demonstrate (11.11) and (11.12), substitute the expressions in Lemma 11.2 into (10.2), (10.3), and apply relations (11.5), (11.6). \( \square \)

**Theorem 11.13** Let \( V \) be a finite-dimensional irreducible \( U^{\geq 0} \)-module of type \( \alpha \). Assume the maps \( r, l \) are as in Definition 11.1 and let \( \varepsilon_0, \varepsilon_1 \) denote scalars in \( \{1, -1\} \). Then \( V \) supports an irreducible \( U_q(\widehat{sl}_2) \)-module structure of type \( (\varepsilon_0, \varepsilon_1) \) for which the Chevalley generators act as follows:

| generator | \( e_0^+ \) | \( e_1^+ \) | \( e_0^- \) | \( e_1^- \) | \( K_0 \) | \( K_1 \) | \( K_0^{-1} \) | \( K_1^{-1} \) |
|-----------|---------|---------|---------|---------|--------|--------|--------|--------|
| action on \( V \) | \( R \) | \( L \) | \( \varepsilon_0 l \) | \( \varepsilon_1 r \) | \( \varepsilon_0 \alpha^{-1} K \) | \( \varepsilon_1 \alpha K^{-1} \) | \( \varepsilon_0 \alpha K^{-1} \) | \( \varepsilon_1 \alpha^{-1} K \) |

**Proof:** To see that the above action on \( V \) determines a \( U_q(\widehat{sl}_2) \)-module, compare equations (11.3)–(11.2) with the defining relations for \( U_q(\widehat{sl}_2) \) in 20
Therefore, the first equation in (11.8), combined with the fact that $\varepsilon as $V W \neq 0$ on $V$ is irreducible as a $V$-module. Assume $l, r$ are as in Definition 1.13. To show $e_0^-$ acts on $V$ as $\varepsilon_0 l$, we set $W = \{ v \in V \mid (e_0^- - \varepsilon_0 l)v = 0 \}$, and argue that $W = V$. For this, it suffices to prove that $W \neq 0$, and $W$ is invariant under the operators $R, L, K^{\pm 1}$. Using the right-hand equation in (11.6), we find that $lU_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. In particular, $lU_0 = 0$. Also $e_0^- U_0 = 0$ by (2.4) so $e_0^- - \varepsilon_0 l$ vanishes on $U_0$. Therefore $U_0 \subseteq W$, so $W \neq 0$. By (1.7) (with $i = 0$), the second relation in (1.7), and the fact that both $e_0^- - R$ and $K_0 - \varepsilon_0 \alpha^{-1} K$ vanish on $V$, we deduce that the commutator $[e_0^- - \varepsilon_0 l, R]$ vanishes on $V$. This implies $RW \subseteq W$. By (1.8) we have $[e_0^+, e_1^-] = 0$.

Therefore, the first equation in (11.8), combined with the fact that $e_1^- - L$ vanishes on $V$ implies that $[e_0^- - \varepsilon_0 l, L]$ vanishes on $V$. Using this, we find that $LW \subseteq W$. Observe $K_0 e_0^- = q^{-2} e_0^- K_0$ by (1.20) and $K_0 - \varepsilon_0 \alpha^{-1} K$ vanishes on $V$, so $K e_0^- - q^{-2} e_0^- K$ is 0 on $V$. Combining this with the second equation in (11.6) we determine that $K(e_0^- - \varepsilon_0 l)$ and $q^{-2}(e_0^- - \varepsilon_0 l) K$ agree on $V$. Thus, $KW \subseteq W$ and then $K^{-1} W \subseteq W$. We have now shown that $W \neq 0$ and $W$ is invariant under each of $R, L, K^{\pm 1}$. Therefore $W = V$ since $V$ is irreducible as a $U^{\geq 0}$-module. We conclude $(e_0^- - \varepsilon_0 l)V = 0$ so $e_0^-$ acts on $V$ as $\varepsilon_0 l$. By a similar argument we find $e_1^-$ acts on $V$ as $\varepsilon_1 r$. Consequently, $e_0^-, e_1^-$ must act on $V$ according to the table of Theorem 1.16. Hence, the given $U_q(\mathfrak{sl}_2)$-module structure is unique. We already showed that this $U_q(\mathfrak{sl}_2)$-module structure is irreducible and it clearly has type $(\varepsilon_0, \varepsilon_1)$. \hfill \end{proof}
12 The proof of Theorem 1.17

This section is devoted to a proof of Theorem 1.17. We begin with a few comments about the quantum algebra $U_q(sl_2)$ and its modules.

**Definition 12.1** [13, p. 122] The quantum algebra $U_q(sl_2)$ is the unital associative $F$-algebra with generators $e^\pm, k^\pm 1$ which satisfy the following relations:

\[
\begin{align*}
kk^{-1} &= k^{-1}k = 1, \\
ke^\pm k^{-1} &= q^\pm 2 e^\pm, \\
[e^+, e^-] &= \frac{k - k^{-1}}{q - q^{-1}}.
\end{align*}
\]

**Lemma 12.2** [13, p. 128] If $V$ is a finite-dimensional irreducible $U_q(sl_2)$-module, then there exist $\varepsilon \in \{1, -1\}$ and a basis $v_0, v_1, \ldots, v_d$ for $V$ such that $kv_i = \varepsilon q^{2i-d}v_i$ for $0 \leq i \leq d$, $e^+ v_i = [i + 1]v_{i+1}$ for $0 \leq i \leq d - 1$, $e^+ v_d = 0$, $e^- v_i = \varepsilon[d - i + 1]v_{i-1}$ for $1 \leq i \leq d$, and $e^- v_0 = 0$.

The proof of the next lemma is straightforward.

**Lemma 12.3** Let $V$ denote a finite-dimensional irreducible $U_q(\hat{sl}_2)$-module. Then for $i \in \{0, 1\}$, there exists a unique $U_q(sl_2)$-module structure on $V$ such that $e^\pm - e^\pm_i$ and $k^\pm 1 - K^\pm 1_i$ vanish on $V$.

In our proof of Theorem 1.17 we will use the following facts concerning $U_q(\hat{sl}_2)$-modules.

**Lemma 12.4** Let $V$ be a finite-dimensional irreducible $U_q(\hat{sl}_2)$-module, and let $U_0, \ldots, U_d$ denote the corresponding weight space decomposition relative to $K_0, K_1$. Then the following hold for $0 \leq j \leq d/2$.

(i) The restriction of $(e^+_0)^{d-2j}$ to $U_j$ is an isomorphism of vector spaces from $U_j$ to $U_{d-j}$.

(ii) For all $v \in U_j$, $(e^+_0)^{d-2j+1}v = 0$ if and only if $e^-_0 v = 0$.

(iii) The restriction of $(e^+_1)^{d-2j}$ to $U_{d-j}$ is an isomorphism of vector spaces from $U_{d-j}$ to $U_j$.

(iv) For all $v \in U_{d-j}$, $(e^-_1)^{d-2j+1}v = 0$ if and only if $e^-_1 v = 0$. 

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Proof: For (i) and (ii), we view $V$ as a $U_q(\mathfrak{sl}_2)$-module via Lemma 12.2 (with $i = 0$). As a $U_q(\mathfrak{sl}_2)$-module, $V$ is a direct sum of irreducible $U_q(\mathfrak{sl}_2)$-modules (see for example, [13, p. 144]). Let $W$ denote one of the irreducible $U_q(\mathfrak{sl}_2)$-module summands. Applying Lemma 12.2 to $W$, we find there exists an integer $r$ ($0 \leq r \leq d/2$) such that for $0 \leq i \leq d$, $W \cap U_i$ is one-dimensional if $r \leq i \leq d - r$ and is zero otherwise. Moreover, $e_0^+ (W \cap U_i) = W \cap U_{i+1}$ for $r \leq i \leq d - r - 1$, $e_0^- (W \cap U_{d-r}) = 0$, $e_0^- (W \cap U_i) = W \cap U_{i-1}$ for $r + 1 \leq i \leq d - r$, and $e_0^- (W \cap U_i) = 0$. Results (i), (ii) follow. The statements in (iii) and (iv) can be shown in exactly the same way. \hfill \Box

Proof of Theorem 1.17: Let $V$ be a finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}}_2)$-module of type $(\varepsilon_0, \varepsilon_1)$. We first prove that the desired $U^\geq_0$-module structure on $V$ exists. Let $R, L, K^{\pm 1}$ act on $V$ as $e_0^+, e_1^+, \varepsilon_0 \alpha^{\pm 1} K_0^{\pm 1}$ respectively. Then using the defining relations for $U_q(\hat{\mathfrak{sl}}_2)$ in Definition 12.2, it is easy to see that $R, L, K^{\pm 1}$ satisfy (1.11)–(1.15), and therefore induce a $U^\geq_0$-module structure on $V$. From the construction, the transformations $e_0^+ - R, e_1^+ - L, K_0^{\pm 1} - \varepsilon_0 \alpha^{\pm 1} K^{\pm 1}$ vanish on $V$. Since $K_0^{\pm 1} - \varepsilon_0 \alpha^{\pm 1} K^{\pm 1}$ vanish on $V$, and since $K_0 K_1 - \varepsilon_0 \varepsilon_1 I$ also vanishes on $V$ by Lemma 2.1 we find that $K_1^{\pm 1} - \varepsilon_1 \alpha^{\pm 1} K^{\pm 1}$ vanish on $V$. We have now shown the desired $U^\geq_0$-module structure exists, and it is clear this $U^\geq_0$-module structure is unique. Next we show the $U^\geq_0$-module structure is irreducible. Let $W$ denote an irreducible $U^\geq_0$-submodule of $V$. Then $W$ is invariant under the actions of $e_0^+, e_1^+, K^{\pm 1}$, and $K_1^{\pm 1}$. We argue that $e_0^- W \subseteq W$ and $e_1^- W \subseteq W$. To demonstrate the first of these assertions, it will be enough to show that $\tilde{W} = \{ w \in W \mid e_0^- w \in W \}$ is nonzero and invariant under each of $R, L, K^{\pm 1}$. It follows from (1.7) (with $i = 0$) and the fact that $e_0^+ - R$ and $\varepsilon_0 K^{\pm 1} - \alpha^{\pm 1} K^{\pm 1}$ are 0 on $V$ that $R \tilde{W} \subseteq \tilde{W}$. Also $[e_0^-, e_1^+] = 0$ by (1.8) and $e_1^+ - L$ vanishes on $V$, so that $L \tilde{W} \subseteq \tilde{W}$. By (1.5) we have $K_0 e_0^- K_0^{-1} = q^{-2} e_0^-$. By this and since $K_0 - \varepsilon_0 \alpha^{-1} K$ is 0 on $V$, we find that $K e_0^- - q^{-2} e_0^- K$ vanishes on $V$. Consequently, $K \tilde{W} \subseteq \tilde{W}$, and then $K^{-1} \tilde{W} \subseteq \tilde{W}$ holds as well.

To verify that $\tilde{W} \neq 0$, let $U_0, \ldots, U_d$ denote the weight space decomposition for the $U_q(\mathfrak{sl}_2)$-module $V$ relative to $K_0, K_1$. As $W$ is invariant under $K_0$ and $K_0 K_1$ acts as $\varepsilon_0 \varepsilon_1 I$ on $V$, it must be that $W = \sum_{i=0}^d (W \cap U_i)$. Since $W \neq 0$, there exists an integer $i$ ($0 \leq i \leq d$) such that $W \cap U_i \neq 0$. Define $r = \min \{ i \mid 0 \leq i \leq d, W \cap U_i \neq 0 \} \text{ and } s = \max \{ i \mid 0 \leq i \leq d, W \cap U_i \neq 0 \}$. We prove that $r + s = d$. Suppose for the moment that $r + s < d$. As $r \leq s$, we must have $r < d/2$. Then for any nonzero $v \in W \cap U_r$, $(e_0^+)^{d-2r} v$ is contained in $W \cap U_{d-r}$ which is 0 because $d - r > s$, so $(e_0^+)^{d-2r} v = 0$,
contradicting Lemma 12.4(i). Next assume \( r + s > d \). Then since \( r \leq s \), we have \( s > d/2 \). For any nonzero \( v \in W \cap U_s \), \((e_1^+)^{2s-d}v = 0\) is contained in \( W \cap U_{d-s} = 0 \), so \((e_1^+)^{2s-d}v = 0\), which contradicts Lemma 12.4(iii). Thus, \( r + s = d \) must hold, and from \( r \leq s \), we deduce that \( r \leq d/2 \). Let \( v \) denote a nonzero vector in \( W \cap U_r \). Observe \((e_0^+)d-2r+1v\) is contained in \( W \cap U_{d-r+1} \), and \( W \cap U_{d-r+1} = 0 \), so \((e_0^+)d-2r+1v = 0\). Applying Lemma 12.4(ii) to the \( U_q(\hat{\mathfrak{sl}}_2)\)-module \( V \), we obtain that \( e_0^-v = 0 \). Therefore, \( v \) is a nonzero element of \( \tilde{W} \). We have now shown \( \tilde{W} \) is nonzero and invariant under each of the operators \( R, L, K^{\pm 1} \). Consequently, by the irreducibility of \( W \) as a \( U^{\geq 0} \)-module, \( \tilde{W} = W \) must hold. Therefore \( e_0^-W \subseteq W \). In just the same fashion, \( e_1^-W \subseteq W \), so that \( W \) is a \( U_q(\hat{\mathfrak{sl}}_2) \)-submodule of \( V \). By construction, \( W \neq 0 \) so \( W = V \). We conclude that the \( U^{\geq 0} \)-module structure on \( V \) is irreducible. It is routine to show the \( U^{\geq 0} \)-module structure on \( V \) has type \( \alpha \).

\[ \square \]

13 Irreducible \( U_q(\hat{\mathfrak{sl}}_2)^{\geq 0} \)-modules

In this section we compare the finite-dimensional irreducible \( U_q(\hat{\mathfrak{sl}}_2) \)-modules with the finite-dimensional irreducible \( U_q(\hat{\mathfrak{sl}}_2)^{\geq 0} \)-modules.

Let \( V \) be a finite-dimensional irreducible module for \( U_q(\hat{\mathfrak{sl}}_2)^{\geq 0} \). The central element \( K_0K_1 \) must act as some scalar \( \gamma \) times the identity map on \( V \). Arguing as in Lemma 12.4(iii) we see that there exists a nonzero scalar \( \alpha \in \mathbb{F} \) and a decomposition \( U_0, \ldots, U_d \) of \( V \) such that \((K_0 - \alpha q^{2i-d})U_i = 0\) for \( 0 \leq i \leq d \). Then \((K_1 - \gamma \alpha^{-1} q^{d-2i})U_i = 0\) for \( 0 \leq i \leq d \), so each of the spaces \( U_0, \ldots, U_d \) is a common eigenspace for \( K_0, K_1 \). Setting \( \beta = \gamma \alpha^{-1} \), we say \( V \) has type \((\alpha, \beta)\) and diameter \( d \).

Assume \( V \) is a finite-dimensional irreducible \( U_q(\hat{\mathfrak{sl}}_2)^{\geq 0} \)-module of type \((\alpha, \beta)\). Then \( V \) remains irreducible when regarded as a module for the subalgebra of \( U_q(\hat{\mathfrak{sl}}_2)^{\geq 0} \) generated by \( K_0^{\pm 1}, e_0^+ \) and \( e_1^+ \). Thus, \( V \) admits the structure of an irreducible \( U^{\geq 0} \)-module of type \( \alpha \). By Theorem 12.4(f) for any choice of scalars \( \varepsilon_0, \varepsilon_1 \) from \( \{1, -1\} \), there exists a unique \( U_q(\mathfrak{sl}_2) \)-module structure on \( V \) such that the operators \( e_0^+ - R, e_1^+ - L, K_0^{\pm 1} - \varepsilon_0 \alpha^{\mp 1} K^{\pm 1}, \) and \( K_1^{\pm 1} - \varepsilon_1 \alpha^{\mp 1} K^{\mp 1} \) vanish on \( V \). This \( U_q(\mathfrak{sl}_2) \)-module structure is irreducible and of type \((\varepsilon_0, \varepsilon_1)\). When this \( U_q(\mathfrak{sl}_2) \)-module structure on \( V \) is then restricted to \( U_q(\hat{\mathfrak{sl}}_2)^{\geq 0} \), we will recover the original \( U_q(\hat{\mathfrak{sl}}_2)^{\geq 0} \)-structure on \( V \), provided \( \alpha = \varepsilon_0 \) and \( \beta = \varepsilon_1 \).

Next suppose that \( V \) is a finite-dimensional irreducible module for \( U_q(\hat{\mathfrak{sl}}_2) \).
We claim that $V$ remains irreducible as a module for the subalgebra $U_q(\hat{\mathfrak{sl}_2})_{\geq 0}$. To see this, let $W$ denote a nonzero $U_q(\hat{\mathfrak{sl}_2})_{\geq 0}$-submodule of $V$. We show $W = V$. By its definition, $W$ is invariant under each of $e^+_0, e^+_1, K^+_0, K^+_1$.

By Theorem 1.17, for any nonzero scalar $\alpha$ in $\mathbb{F}$, there is the structure of a $U_{\geq 0}$-module on $V$ such that the operators $e^+_0 - R$, $e^+_1 - L$, $K^+_0 - \epsilon_0 \alpha^{-1} K^+_1$, and $K^+_1 - \epsilon_1 \alpha^{+1} K^+_1$ vanish on $V$. From this we see that $W$ is invariant under each of $R, L, K^+_\pm$, and is therefore a $U_{\geq 0}$-submodule of $V$. But the $U_{\geq 0}$-module structure on $V$ is irreducible by Theorem 1.17, so $W = V$ and our claim is proved. Note that if $V$ has type $(\epsilon_0, \epsilon_1)$ as a module for $U_q(\hat{\mathfrak{sl}_2})$, then $V$ has type $(\epsilon_0, \epsilon_1)$ as a module for $U_q(\hat{\mathfrak{sl}_2})_{\geq 0}$.

Let us summarize these findings in our final result.

**Theorem 13.1** For any scalars $\epsilon_0, \epsilon_1$ taken from the set $\{1, -1\}$ the following hold.

(i) Let $V$ be a finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}_2})_{\geq 0}$-module of type $(\epsilon_0, \epsilon_1)$. Then the action of $U_q(\hat{\mathfrak{sl}_2})_{\geq 0}$ on $V$ extends uniquely to an action of $U_q(\hat{\mathfrak{sl}_2})$ on $V$. The resulting $U_q(\hat{\mathfrak{sl}_2})$-module structure on $V$ is irreducible and of type $(\epsilon_0, \epsilon_1)$.

(ii) Let $V$ be a finite-dimensional irreducible $U_q(\hat{\mathfrak{sl}_2})$-module of type $(\epsilon_0, \epsilon_1)$. When the $U_q(\hat{\mathfrak{sl}_2})$-action is restricted to $U_q(\hat{\mathfrak{sl}_2})_{\geq 0}$, the resulting $U_q(\hat{\mathfrak{sl}_2})_{\geq 0}$-module structure on $V$ is irreducible and of type $(\epsilon_0, \epsilon_1)$.

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