Axial momentum for the relativistic Majorana particle

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Abstract

The Hilbert space of states of the relativistic Majorana particle consists of normalizable bispinors with real components, and the usual momentum operator $-i\nabla$ can not be defined in this space. For this reason, we introduce the axial momentum operator, $-i\gamma_5\nabla$ as a new observable for this particle. In the Heisenberg picture, the axial momentum contains a component which oscillates with the amplitude proportional to $m/E$, where $E$ is the energy and $m$ the mass of the particle. The presence of the oscillations discriminates between the massive and massless Majorana particle. We show how the eigenvectors of the axial momentum, called the axial plane waves, can be used as a basis for obtaining the general solution of the evolution equation, also in the case of free Majorana field. Here a novel feature is a coupling of modes with the opposite momenta, again present only in the case of massive particle or field.

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1 Introduction

Recently a lot of attention has been given to the Majorana particles and fields \[1\], both in the context of physics of neutrinos, see, e.g., \[2\], \[3\], and in condensed matter physics, \[4\], \[5\]. The theoretical framework used in these investigations is relativistic quantum field theory or quantum statistical mechanics, respectively. Historically, relativistic quantum field theory had been preceded by relativistic quantum mechanics, which still continues to be one of the pillars of the perturbative approach to scattering amplitudes in quantum field theory. Of course, the most popular examples of relativistic quantum mechanics suffer from such well-known problems as the presence of energies unbounded from below (in the case of Dirac particle), or negative probability densities (for a scalar particle). Nevertheless, such quantum mechanics can be very useful. Apart from the above mentioned background for the perturbative expansion, on should mention also its importance for approximate description of many physical phenomena. The importance of the Dirac equation for atomic physics is a good illustration of this point.

In our paper we address two interesting problems of the relativistic quantum mechanics of a single free Majorana particle, which hitherto, to the best of our knowledge, have not been discussed in literature. The first problem is that in the Majorana case the standard momentum $\hat{p} = -i\nabla$ has to be replaced with some other operator. The second problem arises from the fact that the generator of time evolution, i.e., the Hamiltonian, is not Hermitian. Therefore, according to the standard rules of quantum mechanics, it can not be accepted as the energy observable. Thus, we need a new energy operator. Moreover, it turns out that the eigenvectors of the Hamiltonian can not be taken as a basis in the pertinent Hilbert space. Recall that traditionally eigenvectors of the Hamiltonian form the basis in which we write the general solution of the wave equation, irrespectively whether we consider the quantum mechanics or a free Majorana field. Therefore, such general solution has to be written in another basis. Below we offer solutions to all these problems. On the whole, we present a self-contained formulation of the relativistic quantum mechanics of the Majorana particle based on real bispinors.

Let us stress that the same formalism is relevant also for the free Majorana field. Nevertheless, for the sake of clarity, below we use only the language of quantum mechanics.

Let us describe the problems mentioned above and proposed solutions in more detail. By definition, the Majorana bispinors $\text{\dag}$ are invariant with respect to the


\[1\] For definitions, and a comparison with the Dirac and Weyl bispinors, see, e.g., \[6\].
charge conjugation operator $C$. In the Majorana representation for the Dirac matrices $\gamma^\mu$, in which all these matrices are purely imaginary, the charge conjugation is reduced just to the complex conjugation. Therefore, in this representation all four components of the Majorana bispinor are real numbers. (Let us note in passing that this fact is used in [7] to draw an analogy with the Boltzmann equation.) The pertinent Hilbert space $\mathcal{H}$ consists of all such real normalizable bispinors. The ordinary momentum operator $\hat{p} = -i\nabla$ turns real bispinors into complex ones, therefore it is not an operator in $\mathcal{H}$. The problem persists if we use a non-Majorana representation for the Dirac matrices, because the momentum operator does not commute with the charge conjugation $C$ in any representation. Below we introduce a new operator, called by us the axial momentum and denoted as $\hat{p}_5$, with the purpose to replace the ordinary momentum operator $\hat{p}$, and we examine its properties. We find that there is an intriguing difference between massless and massive Majorana particles. In the latter case, the direction of the axial momentum in the Heisenberg picture oscillates in time, see formulas (6) and (10) below. The amplitude of the oscillating component is of the order $m/E$, where $m$ is the rest mass of the particle and $E$ its energy. In the case of neutrino with the mass of the order $1eV$ and the energy $1MeV$ this amplitude is rather small, of the order $10^{-6}$. As the energy observable we adopt the operator $\hat{E} = \sqrt{m^2 + \hat{p}_5^2}$ which does not have negative eigenvalues.

The second problem considered by us stems from the fact that the Dirac equation for the massive Majorana particle is equivalent to an evolution equation with a Hamiltonian which is real and antisymmetric, but it is not Hermitian. Such a wave equation in quantum mechanics is not standard one\textsuperscript{2}. Therefore, time evolution of wave functions should be carefully examined, in particular, conservation of the norm. We check that the norm remains constant in time. Furthermore, the eigenvectors of the Hamiltonian have complex components, hence they can not be used as a basis in the Hilbert space consisting of real bispinors. Instead, we use the basis of axial plane waves which are the common eigenvectors of the axial momentum and of the new energy operator. Next, we find the general explicit solution of the wave equation. Surprisingly, in the massive case the time evolution mixes the modes with the opposite values of the axial momentum. The usual unitary factor $\exp(iEt)$ is replaced by two $SO(4)$ matrices of the form $\exp(K_{\pm}Et)$, see formulas (13).

It is worth mentioning that in the presented below quantum mechanics of the Majorana particle we do not need any complex numbers. They appear in our

\textsuperscript{2}But it is not new, see, e.g., [8] and references therein, for other non-Hermitian Hamiltonians.
formulas, but their presence is superficial – the only reason for it is that we wish to adhere to the standard notation with the Dirac matrices. To some extent, the role of the imaginary unit \( i \) is played by the real antisymmetric matrix \( i\gamma_5 \), which has the property \((i\gamma_5)^2 = -I\). We think that such explicit example of relativistic quantum mechanics devoid of complex numbers can be interesting on its own right.

The plan of our paper is as follows. In the next section we introduce the axial momentum, as well as other observables for the Majorana particle, including the energy. The time evolution of the axial momentum in the Heisenberg picture is investigated in Section 3. Section 4 is devoted to the general solution of the wave equation. Section 5 contains a summary and several remarks.

2 The axial momentum and other observables

We adhere to the standard axiomatics of quantum mechanics. In particular, observables are represented by linear, Hermitian operators in a Hilbert space \( \mathcal{H} \). In our case, elements of \( \mathcal{H} \) are real bispinors \( \psi(x) = (\psi^\alpha(x)) \), \( \alpha = 1, 2, 3, 4 \), and the scalar product has the form

\[
\langle \psi_1 | \psi_2 \rangle = \int d^3 x \, \psi_1^T(x) \psi_2(x),
\]

where \( T \) denotes the matrix transposition. The bispinors are regarded as one-column matrices. An observable \( \hat{O} \) obeys, in particular, the condition \( \langle \psi_1 | \hat{O} \psi_2 \rangle = \langle \hat{O} \psi_1 | \psi_2 \rangle \) for all bispinors \( \psi_1, \psi_2 \) from the domain of \( \hat{O} \). Furthermore, the operator \( \hat{O} \) should be real, that is, the bispinor \( \hat{O} \psi \) should be real like \( \psi \). This last condition has the far reaching consequence: it eliminates the standard momentum operator \( \hat{p} = -i \nabla \). If we just remove the imaginary unit \( i \) to make it real, the \( -\nabla \) operator is not Hermitian.

The time evolution of the real bispinors is governed by the Dirac equation,

\[
i\gamma^\mu \partial_\mu \psi - m \psi = 0,
\]

where the Dirac matrices \( \gamma^\mu \) are all purely imaginary, and \((\gamma^0)^T = -\gamma^0\), \((\gamma^k)^T = \gamma^k\), \((\gamma^0 \gamma^k)^T = \gamma^0 \gamma^k\), \( k = 1, 2, 3 \). A concrete choice for these matrices is made below formula (2). The Dirac equation can be rewritten as

\[
\partial_\mu \psi = \hat{h} \psi,
\]

(1)
\[ \hat{h} = -\gamma^0 \gamma^k \partial_k - \text{i} m \gamma^0. \]

This operator is real, but it is not Hermitian. Nevertheless, the scalar product is constant in time,

\[ \partial_t \int d^3x \psi^T_1(x, t) \psi_2(x, t) = -\int d^3x \partial_k (\psi^T_1(x, t) \gamma^0 \gamma^k \psi_2(x, t)) = 0 \]

if \( \psi_1(x, t), \psi_2(x, t) \) obey Eq. (1) and vanish sufficiently quickly at the spatial infinity.

Hint about the form of the momentum observable comes from classical field theory. It is shown in [9] that in the massless case \((m = 0)\), Eq. (1) is equivalent to the Euler-Lagrange equation obtained from the Lagrangian

\[ \mathcal{L} = -\text{i} \bar{\psi} \gamma^5 \gamma^\mu \partial_\mu \psi, \]

where \( \bar{\psi} = \psi^T \gamma^0 \), and \( \gamma_5 = \text{i} \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). The \( \gamma_5 \) matrix is purely imaginary, \( \gamma_5^2 = \mathbb{I}, \gamma_5^T = -\gamma_5 \). Because \( \psi \) is real, the presence of the \( \gamma_5 \) matrix is crucial, otherwise \( \mathcal{L} \) would be a total divergence yielding no evolution equation. The Noether theorem applied to this Lagrangian yields the conserved 4-momentum \( P^\mu \) of the classical Majorana field \( \psi \) with real (not Grassmannian) components, namely

\[ P^0 = -\text{i} \int d^3x \psi^T \gamma^0 \gamma^k \gamma_5 \partial_k \psi, \quad P^k = -\text{i} \int d^3x \psi^T \gamma_5 \partial_k \psi. \]

These formulas look like expectation values of the operators

\[ \hat{E}_0 = \gamma^0 \gamma^k \hat{p}_5^k, \quad \hat{p}_5^k = -\text{i} \gamma_5 \partial_k, \]

which are Hermitian, real, and they commute with each other. The presence of \( \gamma_5 \) suggests the name for \( \hat{p}_5 \) present in \( \hat{p} \): the axial momentum. In spite of the fact that the Lagrangian \( \mathcal{L} \) is for the massless Majorana field, we consider these operators also when \( m \neq 0 \).

The real matrix \( \text{i} \gamma_5 \) present in \( \hat{p}_5 \) has the property \( (\text{i} \gamma_5)^2 = -\mathbb{I} \). Therefore it may be regarded as a matrix replacement for the imaginary unit \( \text{i} \) in \( \hat{p} = -\text{i} \nabla \). In this vein, equation (1) with \( m = 0 \) can be written in the equivalent form

\[ \text{i} \gamma_5 \partial_t \psi = \hat{E}_0 \psi, \]
which is the Schroedinger equation with \( i \) replaced by the matrix \( i\gamma_5 \). Therefore, in the massless case \( \hat{E}_0 \) can be regarded as the Hamiltonian. Because it is Hermitian, one may adopt it as the energy operator, analogously as in the case of Dirac particle. As is well-known, such identification leads to unphysical spectrum with unbounded from below negative energies. In the massive case, instead of \( \hat{E}_0 \) we have \( i\gamma_5\hat{h} \). This last operator is not Hermitian (unless \( m = 0 \)), hence we have to seek another energy operator. The good candidate is \( \hat{E} = \sqrt{m^2 + \hat{p}_5^2} = \sqrt{m^2 - \Delta} \), which is the correct universal formula for the energy of arbitrary single, free relativistic particle, \( \Delta \) denotes the Laplacian. This operator commutes with the Hamiltonian \( \hat{h} \), hence its expectation values are constant in time. Moreover, \( \hat{E}^2 - \hat{p}_5^2 = m^2 \), as expected for any single, free relativistic particle with the rest mass \( m \) on the basis of Poincaré invariance in quantum field theory. The spectrum of \( \hat{E} \) is positive. There is no reason to introduce negative energies for the free Majorana particle when we do not equate the energy operator with the generator of time evolution \( \hat{h} \) (the Hamiltonian). The energy operator \( \hat{E} \) does not play any important role in our considerations because it is a simple function of \( \hat{p}_5 \).

One can easily check that the operator \( \hat{E}_0 \) is proportional to the standard helicity operator \( \hat{\lambda} = \hat{S}\hat{p}/|\hat{p}| \), namely

\[
\hat{E}_0 = 2|\hat{p}|\hat{\lambda},
\]

where \( \hat{p} = -i\nabla, \hat{S}^j = i\epsilon_{jkl}[\gamma^k, \gamma^l]/8 \) are the standard spin matrices, and \( |\hat{p}| = \sqrt{\hat{p}^2} \). Notice that \( \hat{p}^2 = -\nabla^2 = \hat{p}_5^2 \), hence \( |\hat{p}_5| = |\hat{p}| \). The operator \( \hat{\lambda} \) is real, as opposed to \( \hat{S} \) and \( \hat{p} \). Thus, \( \hat{E}_0 \) is to be associated with the helicity rather than with the energy. Notice that the helicity operator can be written in the form \( \hat{\lambda} = \hat{\Sigma}\hat{p}_5/|\hat{p}_5| \), where \( \hat{\Sigma} = \gamma_5\hat{S} \) is the correct spin matrix (real and Hermitian) in our formalism, see the Section 5.

The axial momentum can be regarded as the generator of the spatial translations. This can be inferred from the fact the axial momentum is related to the Noether charge corresponding to the translations, as described above. One can also see this more directly. Let us start from the well-known formula for the translations which uses the standard momentum \( \hat{p} = -i\nabla \),

\[
\exp(i\hat{a}\hat{p}) \psi(x, t) = \exp(a\nabla) \psi(x, t) = \psi(x + a, t).
\]

We see from this formula that the translations are in fact generated by the operator \( \nabla \). Because this operator is not Hermitian, one usually adds the coefficient \(-i\) to \( \nabla \) in order to produce a Hermitian operator (i.e., \( \hat{p} \)), and also the factor \( i \) in the
first exponent for correctness of the formula. In the Majorana case we simply use the real matrix \( i\gamma_5 \) instead of \( i \). Then, the formula for translations reads

\[
\exp(i\gamma_5 a \hat{p}_5) \psi(x, t) = \exp(a\nabla) \psi(x, t) = \psi(x + a, t).
\]

The exponential operator on the l.h.s. is, of course, unitary. The generators of rotations \( \hat{\Sigma} = \gamma_5 \hat{S} \) are obtained in analogous manner.

Normalized eigenvectors of the axial momentum, which we call the axial plane waves, are defined by the conditions

\[
\hat{p}_5 \psi_p(x) = p \psi_p(x), \quad \int d^3x \psi_p^T(x) \psi_q(x) = \delta(p - q).
\]

They have the form

\[
\psi_p(x) = (2\pi)^{-3/2} \exp(i\gamma_5 px) v, \tag{2}
\]

where \( v \) an arbitrary real, constant, normalized \((v^T v = 1)\) bispinor, and

\[
\exp(i\gamma_5 px) = \cos(px)I + i\gamma_5 \sin(px).
\]

In order to find the common eigenvectors of \( \hat{p} \) and \( \hat{E}_0 \), we choose for the Dirac matrices

\[
\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^1 = i \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \quad \gamma^2 = i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \gamma^3 = -i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}.
\]

Then,

\[
\gamma_5 = i \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}.
\]

Here \( \sigma_k \) are the Pauli matrices, and \( \sigma_0 \) is the \( 2 \times 2 \) unit matrix. The condition

\[
\hat{E}_0 \psi_p(x) = E_0 \psi_p(x)
\]

leads to the matrix equation for the bispinor \( v \)

\[
\gamma^0 \gamma^k p^k v = E_0 v,
\]

well-known in the context of the Dirac equation. The eigenvalues \( E_0 = \pm |p| \), which correspond to the helicities \( \lambda = \pm 1/2 \), are double degenerate. In the case \( E_0 = +|p| \) we find

\[
v_1^{(+)}(p) = \frac{1}{\sqrt{2|p|(|p| - p^2)}} \begin{pmatrix} -p^3 \\ p^2 - |p| \\ p^1 \\ 0 \end{pmatrix}, \quad v_2^{(+)}(p) = i\gamma_5 v_1^{(+)}(p), \quad \tag{3}
\]
and for \( E_0 = -|\mathbf{p}| \)

\[
v_1^{(-)}(\mathbf{p}) = i\gamma^0 v_1^{(+)}(\mathbf{p}), \quad v_2^{(-)}(\mathbf{p}) = i\gamma_5 v_1^{(-)}(\mathbf{p}) = -\gamma^0 v_1^{(+)}(\mathbf{p}). \tag{4}
\]

These bispinors are real, and orthonormal

\[
(v_j^{(\epsilon)}(\mathbf{p}) v_k^{(\epsilon')}(\mathbf{p})) = \delta_{\epsilon\epsilon'}\delta_{jk},
\]

where \( \epsilon, \epsilon' = +, - \), and \( j, k = 1, 2 \). Their concrete form depends on the choice for the Dirac matrices \( \gamma^\mu \), of course. Notice that the complete set (3), (4) of bispinors is generated from \( v_1^{(+)} \) by acting with the antisymmetric real matrices \( i\gamma_5, i\gamma^0 \).

### 3 Time evolution of the observables

Formal solution of Eq. (1) has the form

\[
|t\rangle = \exp(\hat{\mathbf{h}} t) |t_0\rangle,
\]

where \( |t_0\rangle \) denotes the initial state. Because \( \hat{\mathbf{h}}^T = -\hat{\mathbf{h}} \), the operator \( \exp(\hat{\mathbf{h}} t) \) is orthogonal one. Time dependent expectation value of an observable \( \hat{\mathcal{O}} \) has the form

\[
\langle t|\hat{\mathcal{O}}|t\rangle = \langle t_0|\exp(-t\hat{\mathbf{h}}) \hat{\mathcal{O}} \exp(t\hat{\mathbf{h}})|t_0\rangle.
\]

Therefore, the Heisenberg picture version of \( \hat{\mathcal{O}} \) has the form

\[
\hat{\mathcal{O}}(t) = \exp(-t\hat{\mathbf{h}}) \hat{\mathcal{O}} \exp(t\hat{\mathbf{h}}).
\]

It follows that

\[
\frac{d\hat{\mathcal{O}}(t)}{dt} = -\left[ \hat{\mathbf{h}}, \hat{\mathcal{O}}(t) \right] + (\partial_t \hat{\mathcal{O}})(t), \tag{5}
\]

where the last term on the r.h.s. is present only if \( \hat{\mathcal{O}} \) has an explicit time dependence in the Schrödinger picture – in the present paper we do not consider such operators.

The observables \( \hat{E}_0, \hat{\mathbf{p}}^2, |\hat{\mathbf{p}}| \) and \( \hat{\lambda} \) commute with \( \hat{\mathbf{h}} \), hence they are constant in time. The axial momentum is constant in time only in the massless case.

In the massive case,

\[
\left[ \hat{\mathbf{h}}, \hat{p}_5^k \right] = -2im \gamma^0 \hat{p}_5^k,
\]

hence

\[
\frac{d\hat{p}_5^k(t)}{dt} = 2im \gamma^0(t) \hat{p}_5^k(t),
\]

8
where
\[ \gamma^0(t) = \exp(-\hat{t}\hat{h})\gamma^0 \exp(\hat{t}\hat{h}). \]
Because \([\hat{h}, \nabla] = 0\), \(\hat{p}^k_5(t)\) can be written in the form
\[ \hat{p}^k_5(t) = -i\gamma^5(t)\partial_k, \]
where \(\gamma^5(t)\) obeys the equation
\[ \frac{d\gamma^5(t)}{dt} = 2im\gamma^0(t)\gamma^5(t). \]
The operator \(\gamma^0(t)\gamma^5(t)\) present on the r.h.s. of Eq. (7) obeys the following equation
\[ \frac{d(\gamma^0(t)\gamma^5(t))}{dt} = 2\gamma^5(t)\gamma^k(t)\partial_k + 2im\gamma^5(t), \]
and
\[ \frac{d^2(\gamma^0(t)\gamma^5(t))}{dt^2} = 4(\partial_k\partial_k - m^2)\gamma^0(t)\gamma^5(t). \]
The latter equation has the following solution
\[ \gamma^0(t)\gamma^5(t) = \cos(2\hat{E}t)\gamma^0\gamma^5 - i(\hat{E}_0\gamma^0 - m\gamma^5)\hat{E}^{-1} \sin(2\hat{E}t), \]
where \(\hat{E} = \sqrt{m^2 - \partial_k\partial_k}\) is constant in time. The solution (9) obeys the required initial conditions at \(t = 0\):
\[ \gamma^0(t)\gamma^5(t) \bigg|_{t=0} = \gamma^0\gamma^5, \quad \left. \frac{d(\gamma^0(t)\gamma^5(t))}{dt} \right|_{t=0} = 2\gamma^5\gamma^k\partial_k + 2im\gamma^5, \]
where the second condition follows from Eq. (8). Next, we insert solution (9) on the r.h.s. of Eq. (7) and integrate for \(\gamma^5(t)\),
\[ \gamma^5(t) = \gamma^5 + im\hat{E}^{-1}\gamma^0\gamma^5 \left[ \sin(2\hat{E}t) + \hat{J}(1 - \cos(2\hat{E}t)) \right], \]
where \(\hat{J} = \hat{h}/\hat{E}\). Notice that \(\hat{J}^2 = -I\). Therefore, the two oscillating terms present on the r.h.s. of formula (10) are of the same order \(m/\hat{E}\). The absolute value of the axial momentum, \(|\hat{p}| = \sqrt{\hat{p}^2} = \sqrt{-\nabla^2}\), remains constant in time.

One may ask whether the axial momentum is the best replacement for the ordinary momentum, because it is not constant in time in the case \(m \neq 0\). We have searched for other possibilities, without satisfactory results. In particular, we have considered the operators
\[ \hat{p}^k_m = \hat{p}^k_5 - m\gamma^5\gamma^k, \]
with \(k = 1, 2, 3\). They are real, Hermitian, they commute with \(\hat{h}\), and for \(m = 0\) they coincide with the axial momentum. Unfortunately, they do not commute with each other.
4 Time evolution of the Majorana bispinor

The axial plane waves can be used in a general solution of the evolution equation (1). Such solution is written as a superposition of the axial plane waves (2), which appear in place of the ordinary plane waves. Thus, the expansion of $\psi$ into the axial plane waves has the form

$$\psi(x, t) = (2\pi)^{-3/2} \sum_{\alpha=1}^{2} \int d^3 p \, e^{i\gamma_5 px} \left( v_\alpha^+(p)c_\alpha(p, t) + v_\alpha^-(p)d_\alpha(p, t) \right).$$  \hspace{1cm} (11)

Equation (1) gives the following equations for the real amplitudes $c_\alpha, d_\alpha$:

$$E_p^{-1} \dot{c}_1(p, t) = n^2 c_2(p, t) + n^1 c_1(-p, t) - n^3 c_2(-p, t),$$

$$E_p^{-1} \dot{c}_2(p, t) = -n^2 c_1(p, t) - n^3 c_1(-p, t) - n^1 c_2(-p, t),$$

$$E_p^{-1} \dot{d}_1(p, t) = -n^2 d_2(p, t) + n^1 d_1(-p, t) + n^3 d_2(-p, t),$$

$$E_p^{-1} \dot{d}_2(p, t) = n^2 d_1(p, t) + n^3 d_1(-p, t) - n^1 d_2(-p, t),$$

where the dots stand for $d/dt$, $E_p = \sqrt{p^2 + m^2}$, and

$$n^1 = \frac{m p^1}{E_p \sqrt{(p^1)^2 + (p^3)^2}}, \quad n^2 = \frac{|p|}{E_p}, \quad n^3 = \frac{m p^3}{E_p \sqrt{(p^1)^2 + (p^3)^2}}.$$

The coefficients $n^1, n^2, n^3$ are proportional to the scalar products $v_{\alpha\epsilon}^T v_{\beta\epsilon}$. Note that $n^3 = 1$.

We see that in the case $m \neq 0$ rather unexpected mixing between the modes with the opposite eigenvalues $p$ and $-p$ of the axial momentum appears. Looking back at Eq. (1), the coupling of the modes $p$ and $-p$ appears because $\gamma^0 \exp(i\gamma_5 px) = \exp(-i\gamma_5 px)\gamma^0$. It is present always when $(n^1)^2 + (n^3)^2 > 0$, but it can be rather weak. For example, if we take $m = 1$ eV and $E_p = 1$ MeV, then $\sqrt{(n^1)^2 + (n^3)^2} = m/E_p = 10^{-6}$ and $n^2 = \sqrt{1 - m^2/E_p^2} \approx 1$. On the other hand, the mixing is dominant when $E_p \approx m$.

In order to solve the equations for the amplitudes $c_\alpha, d_\alpha$, we split the amplitudes into the even and odd parts,

$$c_\alpha(p, t) = c_\alpha'(p, t) + c_\alpha''(p, t), \quad d_\alpha(p, t) = d_\alpha'(p, t) + d_\alpha''(p, t),$$
where \( c'_{\alpha}(-p, t) = c'_{\alpha}(p, t) \), \( c''_{\alpha}(-p, t) = -c''_{\alpha}(p, t) \), and similarly for \( d', d'' \).

Such a splitting is unique. Using the notation
\[
\vec{c}(p, t) = \begin{pmatrix} c'_1 \\ c''_1 \\ c'_2 \\ c''_2 \end{pmatrix}, \quad \vec{d}(p, t) = \begin{pmatrix} d'_1 \\ d''_1 \\ d'_2 \\ d''_2 \end{pmatrix}, \quad K_{\pm}(p) = \begin{pmatrix} 0 & -n^1 & \pm n^2 & \pm n^3 \\ n^1 & 0 & \mp n^2 & \pm n^3 \\ \mp n^2 & \pm n^3 & 0 & n^1 \\ \mp n^3 & \pm n^2 & -n^1 & 0 \end{pmatrix},
\]
we rewrite the equations for the amplitudes in the form
\[
\dot{\vec{c}}(p, t) = E_p K_{+}(p) \vec{c}(p, t), \quad \dot{\vec{d}}(p, t) = E_p K_{-}(p) \vec{d}(p, t).
\] (12)

Solutions of these equations have the form
\[
\vec{c}(p, t) = \exp(E_p K_{+}(p) t) \vec{c}(p, 0), \quad \vec{d}(p, t) = \exp(E_p K_{-}(p) t) \vec{d}(p, 0).
\] (13)

The matrices \( K_{\pm} \) are antisymmetric, hence the exponential matrices present in (13) are orthogonal ones – they belong to the \( SO(4) \) group. Furthermore, because \( K^2_{\pm} = -I \), we may write
\[
\exp(E_p K_{\pm}(p) t) = \cos(E_p t) I + \sin(E_p t) K_{\pm}(p).
\] (14)

The fact that the general solution (13) has such a simple form is a nice surprise. Because the axial plane waves are eigenvectors of the operator \( \hat{p}_5 \) which does not commute with \( \hat{h} \) when \( m \neq 0 \), one could expect much more complicated time evolution of the amplitudes.

## 5 Summary and remarks

1. We have presented a self-contained formulation of the relativistic quantum mechanics of the single free Majorana particle. As one of the main results, we have shown that the axial momentum operator \( \hat{p}_5 = -i\gamma_5 \nabla \) is a viable replacement for the ordinary momentum \( \hat{p} = -i\nabla \). In particular, it can be regarded as the generator of spatial translations. Its eigenfunctions – the axial plane waves – provide a new basis for mode decomposition of Majorana bispinors. In this basis, the time evolution is represented by \( SO(4) \) matrices (14), which replace the standard complex exponents \( \exp(\pm iE_p t) \) – this is our second main result. The energy operator \( \hat{E} = \sqrt{m^2 + \hat{p}_5^2} \) differs from the Hamiltonian \( \hat{h} \) (which is not Hermitian),
but it coincides with the well-known formula for the energy of a free relativistic particle.

There are intriguing effects due to nonvanishing rest mass of the particle. First, the direction of the axial momentum oscillates if $m \neq 0$ (the modulus remains constant). One may regard these oscillations as a kind of Zitterbewegung in the momentum space. In the well-known case of the Dirac particle, the Zitterbewegung is present in the position space – the velocity operator $d\mathbf{x}/dt$ is not constant. Second, there is the coupling between the modes with the opposite values of the axial momentum. The strength of these effects is proportional to $m/E_p$, where $E_p = \sqrt{m^2 + \mathbf{p}^2}$ is the energy of the particle.

2. The Dirac equation for the real Majorana bispinor $\psi(x, t)$ of course possesses the Poincaré symmetry, as well as the $P$ and $T$ symmetries. The Poincaré transformations of the wave function have the standard form,

$$\psi'_{L,a}(x) = S(L)\psi(L^{-1}(x - a)),$$

where $x = (t, \mathbf{x})$, $S(L) = \exp(\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]/8)$, and $\omega_{\mu\nu}$ parameterize the proper orthochronous Lorentz group. Starting from this formula, one may obtain in the standard manner the generators of rotations, which are related to the spin observable. The only difference is that the coefficients $i$ used in order to obtain Hermitian matrices in the Dirac case, here should be replaced by $i\gamma_5$, analogously as in the case of momentum.

The $P$ and $T$ transformations have the form, respectively,

$$\psi'_{P}(x, t) = \eta_p i\gamma^0 \psi(-x, t), \quad \psi'_{T}(x, t) = \eta_T \gamma_5 \gamma^0 \psi(x, -t),$$

where $\eta_p = \pm 1$, $\eta_T = \pm 1$ are intrinsic parities of the particle. By definition, the symmetry transformations applied to an arbitrary solution of the Dirac equation yield solutions of the same equation.

3. The results of present work suggest several interesting topics for further research. First, one may delve deeper into the quantum mechanics of the Majorana particle. This includes investigations of evolution of wave packets in external potentials representing interactions with other particles. Furthermore, one could propose concrete procedures for measuring the axial momentum. One can also look for applications in particle physics, and in condensed matter physics.

Second, the axial momentum is the observable that can be used also in the case of Dirac particle, which is described by a complex bispinor. Such a reformulation of the relativistic quantum mechanics of the Dirac particle with the axial
momentum in place of the standard momentum may provide an interesting complementary viewpoint on the Dirac particle.

Last but not least, the mode decomposition (11) with the axial plane waves can be used as the starting point for quantization of the Majorana and the Dirac fields. Here the interesting question is whether the resulting Fock spaces of free particles are equivalent to the standard ones. Work in this direction is in progress.

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