POLYTIME REDUCTIONS OF AF-ALGEBRAIC PROBLEMS

DANIELE MUNDICI

Abstract. We assess the computational complexity of several decision problems concerning (Murray-von Neumann) equivalence classes of projections of AF-algebras whose Elliott classifier is lattice-ordered. We construct polytime reductions among many of these problems.

1. Introduction

Let $\mathfrak{A}$ be an AF-algebra, [3]. The partial addition $+$ in Elliott’s local semigroup $[9]$ of $\mathfrak{A}$ is uniquely extendible to a total operation that preserves all algebraic and order properties of $+$ iff the Murray-von Neumann order $\preceq$ of $\mathfrak{A}$ is a lattice, for short, $\mathfrak{A}$ is an AF-$\ell$-algebra. See Theorem 1.1. AF-$\ell$-algebras have a preeminent role in the literature on AF-algebras. The Elliott classifier $E(\mathfrak{A})$ of any AF-$\ell$-algebra $\mathfrak{A}$ is a MV-algebra, i.e., a Lindenbaum algebra in $\mathbb{L}_\infty$ logic [6, 19]. All countable MV-algebras arise as $E(\mathfrak{A})$ for some AF-$\ell$-algebra $\mathfrak{A}$. Since $\mathfrak{A}$ is a quotient of the universal AF-$\ell$-algebra $\mathfrak{M}$ of [16 §8], and $E(\mathfrak{M})$ is the free MV-algebra over countably many generators, each $\mathbb{L}_\infty$-formula $\phi$ naturally codes an equivalence class $[\phi]$ of projections in $\mathfrak{A}$.

We equip the Elliott monoid $E(\mathfrak{A})$ of every AF-$\ell$-algebra $\mathfrak{A}$ with a partial order $\phi \preceq \psi$ intuitively meaning that all projections in $\phi$ are “less eccentric” than those in $\psi$. We prove that $p$ is central iff its (always Murray-von Neumann) equivalence class $[p]$ is $\preceq$-minimal iff it is a (Freudenthal) characteristic element of $K_0(\mathfrak{A})$ iff $[p] \cup [1-p] = 0$. Among others, we consider the following decision problems: (a) $\phi \preceq \psi$, (b) $\phi \preceq \psi$, (c) $\phi \preceq \psi$, (d) $\phi \preceq ? 0$, (e) Is $\phi$ central? We prove that problems (a)-(d) for $\mathfrak{A}$ have the same computational complexity, up to a polytime reduction. If $\mathfrak{A}$ is simple, or if $\mathfrak{A}$ has no quotient isomorphic to $C_0$, then also problem (e) for $\mathfrak{A}$ has the same complexity as (a)-(d).

The complexity of each problem (a)-(e) is polytime for many relevant AF-algebras in the literature, including the Behncke-Leptin algebras $A_{m,n}$ and every Effros-Shen algebra $F_\theta$, for $\theta \in [0,1] \setminus \mathbb{Q}$ a real algebraic integer, or for $\theta = 1/e$. For every AF-$\ell$-algebra $\mathfrak{A}$ we let $\text{prim}(\mathfrak{A})$ denote the set of primitive (always closed and two-sided) ideals of $\mathfrak{A}$ equipped with the Jacobson topology. By [3 §3.8], an ideal $\mathfrak{P}$ of $\mathfrak{A}$ is primitive if it is prime, in the sense that whenever ideals $\mathfrak{P}_1, \mathfrak{P}_2$ of $\mathfrak{A}$ satisfy $\mathfrak{P}_1 \cap \mathfrak{P}_2 = \mathfrak{P}$ then either $\mathfrak{P}_1 = \mathfrak{P}$ or $\mathfrak{P}_2 = \mathfrak{P}$.

For any MV-algebra $A$ we let $\text{Spec}(A)$ denote the space of prime ideals of $A$ endowed with the Zariski (hull-kernel) topology, ([19 4.14]). As shown in [6 §1.2.14],

$$\bigcap \text{Spec}(A) = \{0\}. \quad (1)$$

The basic properties of AF-$\ell$-algebras are summarized by the following two results:

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Theorem 1.1. Let $\mathfrak{A}$ be an AF-algebra.

(i) [20, Theorem 1] Elliott’s partial addition $+$ in $L(\mathfrak{A})$ has at most one extension to an associative, commutative, monotone operation $\oplus: L(\mathfrak{A})^2 \rightarrow L(\mathfrak{A})$ such that for each projection $p \in \mathfrak{A}$, $[1_\mathfrak{A} - p]$ is the smallest $[q] \in L(\mathfrak{A})$ satisfying $[p] \oplus [q] = [1_\mathfrak{A}]$. The semigroup $(S(\mathfrak{A}), \oplus)$ expanding $L(\mathfrak{A})$ exists iff $\mathfrak{A}$ is an AF-$\ell$-algebra.

(ii) [9] Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be AF-$\ell$-algebras. For each $j = 1, 2$ let $\oplus_j$ be the extension of Elliott’s addition given by (i). Then the semigroups $(S(\mathfrak{A}_1), \oplus_1)$ and $(S(\mathfrak{A}_2), \oplus_2)$ are isomorphic iff so are $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

(iii) [20, Theorem 2, Proposition 2.2] $(S(\mathfrak{A}), \oplus)$ has the richer structure of a monoid $(E(\mathfrak{A}), 0^*, \oplus)$ with an involution $[p]^* = [1_\mathfrak{A} - p]$. The Murray-von Neumann lattice-order of equivalence classes of projections $[p], [q] \in E(\mathfrak{A})$ is definable by the involutive monoidal operations of $E(\mathfrak{A})$ upon setting $[p] \vee [q] = ([p]^* \oplus [q])^* \oplus [q]$ and $[p] \wedge [q] = ([p]^* \vee [q]^*)^*$. We say that $E(\mathfrak{A})$ is the Elliott monoid of $\mathfrak{A}$.

(iv) [10] Theorem 3.9] The map $\mathfrak{A} \mapsto (E(\mathfrak{A}), 0^*, \oplus)$ is a bijective correspondence between isomorphism classes of AF-$\ell$-algebras and isomorphism classes of countable abelian monoids with a unary operation $^*$ satisfying the equations: $x^* = x$, $0^* = 0^*$, and $(x^* + y)^* \oplus y = (y^* \oplus x)^* \oplus x$. The involutive monoids defined by these equations are known as MV-algebras, [5, 6, 19]. Let $\Gamma$ be the categorical equivalence between unital $\ell$-groups and MV-algebras defined in [16, §3]. Then $(E(\mathfrak{A}), 0^*, \oplus)$ is isomorphic to $\Gamma(K_0(\mathfrak{A}), K_0(\mathfrak{A})^+, [1_\mathfrak{A}])$.

(v) [7, 8, 11] For any AF-$\ell$-algebra $\mathfrak{A}$ the dimension group
\[ K_0(\mathfrak{A}) = (K_0(\mathfrak{A}), K_0(\mathfrak{A})^+, [1_\mathfrak{A}]) \]
is a countable lattice-ordered abelian group with a distinguished strong order unit (for short, a unital $\ell$-group). All countable unital $\ell$-groups arise in this way. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be AF-$\ell$-algebras. Then $K_0(\mathfrak{A}_1)$ and $K_0(\mathfrak{A}_2)$ are isomorphic unital $\ell$-groups iff $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic.

Theorem 1.2. In any AF-$\ell$-algebra $\mathfrak{A}$ we have:

(i) The map $\eta: \mathfrak{I} \mapsto K_0(\mathfrak{I}) \cap E(\mathfrak{A})$ is an isomorphism of the lattice of ideals of $\mathfrak{A}$ onto the lattice of ideals of $E(\mathfrak{A})$. Primitive ideals of $\mathfrak{A}$ correspond via $\eta$ to prime ideals of $E(\mathfrak{A})$.

(ii) In particular, $\eta$ is a homeomorphism of $\text{prim}(\mathfrak{A})$ onto $\text{Spec}(E(\mathfrak{A}))$.

(iii) Suppose $\mathfrak{I}$ is an ideal of the countable MV-algebra $A$. Let the AF-$\ell$-algebra $\mathfrak{A}$ be defined by $E(\mathfrak{A}) = A$, as in Theorem 1.1(iv). Let the ideal $\mathfrak{I}$ of $\mathfrak{A}$ be defined by $\eta(\mathfrak{I}) = \mathfrak{I}$. Then $\mathfrak{A}/\mathfrak{I}$ is isomorphic to $E(\mathfrak{A}/\mathfrak{I})$.

(iv) For every ideal $\mathfrak{I}$ of $\mathfrak{A}$, the map $[p/\mathfrak{I}] \mapsto [p]/\eta(\mathfrak{I})$, (with $p$ ranging over all projections in $\mathfrak{A}$), is an isomorphism of $E(\mathfrak{A}/\mathfrak{I})$ onto $E(\mathfrak{A}/\eta(\mathfrak{I}))$. In particular, for every $\mathfrak{P} \in \text{prim}(\mathfrak{A})$ the MV-algebra $E(\mathfrak{A}/\mathfrak{P})$ is totally ordered and $\mathfrak{A}/\mathfrak{P}$ has comparability of projections in the sense of Murray-von Neumann.

(v) The map $\mathfrak{I} \mapsto \mathfrak{I} \cap \Gamma(K_0(\mathfrak{A}))$ is an isomorphism of the lattice of ideals of $K_0(\mathfrak{A})$ (i.e., kernels of unit preserving $\ell$-homomorphisms of $K_0(\mathfrak{A})$) onto the lattice of ideals of $E(\mathfrak{A})$. Further, \[ \frac{\Gamma(K_0(\mathfrak{A}))}{\mathfrak{I}} \cong \frac{\Gamma(K_0(\mathfrak{A}))}{\Gamma(\mathfrak{I})} \]

\textit{Proof.} (i) From [7, Proposition IV.5.1] and [11, p.196, 21H] one gets an isomorphism of the lattice of ideals of $\mathfrak{A}$ onto the lattice of ideals of the unital $\ell$-group $K_0(\mathfrak{A})$. The preservation properties of $\Gamma$, [6, Theorems 7.2.2, 7.2.4], then yield the
desired isomorphism. The second statement immediately follows from [4, Theorem 3.8] and the above mentioned characterization of prime ideals in MV-algebras, ([19 Proposition 4.13]).

(ii) follows from (i), by definition of the topologies of \( \text{prim}(\mathfrak{A}) \) and \( \text{Spec}(E(\mathfrak{A})) \).

(iii) We have an exact sequence \( 0 \to J \to \mathfrak{A} \to \mathfrak{A}/J \to 0 \). Correspondingly ([7 IV.15] or [3 Corollary 9.2]), we have an exact sequence \( 0 \to K_0(J) \to K_0(\mathfrak{A}) \to K_0(\mathfrak{A}/J) \to 0 \), whence the unital \( \ell \)-groups \( K_0(\mathfrak{A}/J) \) and \( K_0(\mathfrak{A})/K_0(\mathfrak{A}) \) are isomorphic. The preservation properties of \( \Gamma \) under quotients [6 Theorem 7.2.4], together with Theorem 1.1(iv)-(v) yield

\[
E(\frac{\mathfrak{A}}{J}) \cong \Gamma\left(\frac{K_0(\mathfrak{A})}{K_0(J)}\right) \cong \frac{\Gamma(K_0(\mathfrak{A}))}{\Gamma(K_0(J))} \cong \frac{E(\mathfrak{A})}{\eta(J)} \cong A/1.
\]

(iv) Combine (i) and (iii) with the preservation properties of \( K_0 \) for exact sequences and the preservation properties of \( \Gamma \) under quotients. The MV-algebra \( E(\mathfrak{A}/\mathfrak{P}) \) is totally ordered. As a matter of fact, by (ii), \( \eta(\mathfrak{P}) \) belongs to \( \text{Spec}(E(\mathfrak{A})) \) whenever \( \mathfrak{P} \) belongs to \( \text{prim}(\mathfrak{A}) \). By Theorem 1.1(iv), \( \mathfrak{A}/\mathfrak{P} \) has comparability of projections.

(v) This again follows from [6 Theorems 7.2.2, 7.2.4].

2. Characterizing central projections in AF-\( \ell \)-algebras

Following [12 p.22], by a characteristic element in a partially ordered abelian group \( G \) with order unit \( u \), we mean an element \( c \) with \( 0 \leq c \leq u \) such that the greatest lower bound of the set \( \{c, u-c\} \) exists and equals \( 0 \). In the framework of real vector lattices, with \( u \) any arbitrary positive element, \( c \) is known as a “component of \( u^* \), [11 p.13], [13 p.284]. This notion goes back to Freudenthal, [10 p. 643, (5.2)].

In every MV-algebra \( A \) one defines \( a \circ b = (a^* \oplus b^*)^* \) and \( \text{dist}(a,b) = (a \circ b^*) \oplus (b \circ a^*) \). Chang [5] writes \( x \cdot y \) instead of \( x \circ y \), and \( d(x,y) \) instead of \( \text{dist}(x,y) \). The latter is known as Chang’s distance function. See [3 p.8 and 1.2.4] and [5 p.477]. Further, let us use the notation \( B(A) \) for the set of idempotent elements of the MV-algebra \( A \),

\[
B(A) = \{a \in A \mid a \oplus a = a\}.
\]

As observed by Chang [5 Theorems 1.16-1.17], \( B(A) \) is a subalgebra of \( A \) which turns out to be a boolean algebra. By an MV-chain we mean an MV-algebra \( D \) whose underlying order is total, [6 p.10].

The binary relation “\( p \) is closer than \( q \) to the center”. Let \( \mathfrak{P} \) be a primitive ideal of an AF-\( \ell \)-algebra \( \mathfrak{A} \). By Theorem 1.2 the quotient AF-\( \ell \)-algebra \( \mathfrak{A}/\mathfrak{P} \) has comparability of projections. Thus for any projections \( p, q \in \mathfrak{A} \) we have the following three mutually incompatible cases:

\[
p/\mathfrak{P} \prec q/\mathfrak{P}, \ p/\mathfrak{P} \succ q/\mathfrak{P}, \ p/\mathfrak{P} \sim q/\mathfrak{P}.
\]

In particular, we have incompatible cases:

\[
p/\mathfrak{P} \prec p^*/\mathfrak{P}, \ p/\mathfrak{P} \succ p^*/\mathfrak{P}, \ p/\mathfrak{P} \sim p^*/\mathfrak{P}.
\]

The proof of the following result will appear elsewhere:

**Theorem 2.1** (Characterization of central projections). Suppose \( \mathfrak{A} \) is an AF-\( \ell \)-algebra. In view of Theorem 1.1(iv)-(v), let us identify \( E(\mathfrak{A}) \) with the unit interval \( \Gamma(K_0(\mathfrak{A})) \) of the unital \( \ell \)-group \( K_0(\mathfrak{A}) \).

(i) For every projection \( p \) of \( \mathfrak{A} \) the following conditions are equivalent:

\[ p/\mathfrak{P} \in \{0,1\}, \ (the \ trivial \ elements \ of \ \mathfrak{A}/\mathfrak{P}) \ for \ all \ \mathfrak{P} \in \text{prim}(\mathfrak{A}). \]
(ii) \([p] \in \mathcal{B}(E(\mathfrak{A}))\).
(iii) \(p\) is central in \(\mathfrak{A}\).
(iv) \([p]\) is a characteristic element of \(K_0(\mathfrak{A})\).

(ii) For any \(x, y \in E(\mathfrak{A})\) let us write \(x \subseteq y\) iff for every prime ideal \(p\) of \(E(\mathfrak{A})\)

\((y/p < y'/p \implies x/p \leq y/p)\) and \((y/p > y'/p \implies x/p \geq y/p)\),

with \(\leq\) the underlying total order of \(E(\mathfrak{A})/p\) defined in Theorem \(1.1(iii)\). Then \(\subseteq\)
endows \(E(\mathfrak{A})\) with a partial order relation. Moreover, for every projection \(p\) in \(\mathfrak{A}\),
the equivalent the conditions in (I) are also equivalent to \([p]\) being \(\subseteq\)-minimal in \(E(\mathfrak{A})\).

3. The algorithmic theory of \(\text{AF}\)-algebras

Our standard reference for computability theory is [14]. For \(L_\infty\) we refer to [16].

The syntax of \(\text{TERM}_n\) and \(\text{TERM}_\omega\). The set \(\{0,*, \oplus\}\) of constant and operation
symbols of involutive monoids is enriched by adding countably many variable symbols \(X_1, X_2, \ldots\). Henceforth, the set \(\mathcal{A} = \{0,*, \oplus, X_1, X_2, \ldots, (\cdot)\}\) is our alphabet.
Parentheses are added to construct a non-ambiguous readable syntax. The set \(A^*\) of strings
over \(\mathcal{A}\) is defined by \(A^* = \{s_1, s_2, \ldots, s_l \in A^l | l = 0, 1, 2, \ldots\}\). Let \(n = 1, 2, \ldots\). By a term in the variables 
\(X_1, \ldots, X_n\), we mean a string \(\varphi \in A^*\) obtainable by the following inductive definition:
(*): \(0\) and \(X_1, X_2, \ldots, X_n\) are terms;
(**): if \(\alpha\) and \(\beta\) are terms, then so are \(\alpha^*\) and \((\alpha \oplus \beta)\). We let \(\text{TERM}_n\)
be the set of terms in the variables \(X_1, X_2, \ldots, X_n\). Elements of \(\text{TERM}_n\) are also known
as \(n\)-variable Lukasiewicz formulas. We also let \(\text{TERM}_\omega = \bigcup_n \text{TERM}_n\).

Coding equivalence classes of projections by \(L_\infty\)-formulas. Fix \(n = 1, 2, \ldots\). By McNaughton’s theorem [15],
the coordinate functions \(\{\pi_1, \ldots, \pi_n\}\) are a distinguished free generating set of the free MV-algebra \(M([0, 1]^n) \cong E(\mathfrak{M}_n)\). In view of \(\text{Theorem 1.1(iv)}\), the \(\text{AF}\)-algebra \(\mathfrak{M}_n\) is defined by \(E(\mathfrak{M}_n) = M([0, 1]^n)\).

 Arbitrarily pick representative projections \(p_1, \ldots, p_n \in \mathfrak{M}_n\) such that the equivalence classes \([p_1], \ldots, [p_n] \in E(\mathfrak{M}_n)\) correspond to \(\pi_1, \ldots, \pi_n\) via Elliott’s classification.
We may naturally say that the variable symbol \(X_i\) codes both the coordinate function \(\pi_i \in M(\mathfrak{M}_n)\) and the equivalence class \([p_i] \in E(\mathfrak{M}_n)\). For definiteness, we will say that \(\pi_i\) is the interpretation of \(X_i\) in (the Elliott monoid of) \(\mathfrak{M}_n\), in symbols, \(X_i^{\mathfrak{M}_n} = \pi_i\). For every \(\phi \in \text{TERM}_n\), the interpretation \(\phi^{\mathfrak{M}_n}\) of \(\phi\) in (the Elliott monoid of) \(\mathfrak{M}_n\) is then defined by (*): \(0^{\mathfrak{M}_n}\) is the constant zero function over \([0, 1]^n\) and inductively, (**): \((\psi^*)^{\mathfrak{M}_n} = (\psi^{\mathfrak{M}_n})^*, (\alpha \oplus \beta)^{\mathfrak{M}_n} = (\alpha^{\mathfrak{M}_n} \oplus \beta^{\mathfrak{M}_n})\). We also say that \(\phi\) codes \(\phi^{\mathfrak{M}_n}\) in (the Elliott monoid of) \(\mathfrak{M}_n\). By a traditional abuse of notation, the \(\text{MV}\)-algebraic operation symbols also denote their corresponding operations. More generally, let \(\mathfrak{A} = \mathfrak{M}_n/\mathcal{I}\) be an \(\text{AF}\)-algebra, for some ideal \(\mathcal{I}\) of \(\mathfrak{M}_n\).
Let

\[i = K_0(\mathcal{I}) \cap E(\mathfrak{M}_n).\] (3)

be the ideal of \(M([0, 1]^n)\) corresponding to \(\mathcal{I}\) by Theorem \(1.2(v)\). Via the identification \(E(\mathfrak{M}_n/\mathcal{I}) = M([0, 1]^n)/i\), the interpretation \(\phi^{\mathfrak{M}_n/\mathcal{I}}\) of \(\phi\) in (the Elliott monoid of) \(\mathfrak{A}\) is defined by: \(0^{\mathfrak{A}} = \{0\} \subseteq \mathfrak{A}\), \(X_i^{\mathfrak{A}} = X_i^{\mathfrak{M}_n/\mathcal{I}} = X_i^{\mathfrak{M}_n}/i = \pi_i/i\) and inductively, \((\psi^*)^{\mathfrak{A}} = (\psi^{\mathfrak{M}_n})^*, (\alpha \oplus \beta)^{\mathfrak{A}} = (\alpha^{\mathfrak{M}_n} \oplus \beta^{\mathfrak{M}_n})\). The following identities are easily verified by induction on the syntactical complexity of terms (i.e., the number of symbols occurring in each term), using the unique readability property of the syntax:

\[\psi^{\mathfrak{A}} = (\psi^{\mathfrak{M}_n})^{\mathfrak{M}_n/\mathcal{I}} = (\psi^{\mathfrak{M}_n})^* = (\psi^{\mathfrak{M}_n})^* = (\psi^{\mathfrak{M}_n})^{\mathfrak{M}_n/\mathcal{I}} = (\alpha \oplus \beta)^{\mathfrak{A}} = (\alpha^{\mathfrak{M}_n} \oplus \beta^{\mathfrak{M}_n}) = \]
\[
\alpha m_n \oplus \beta m_n = \alpha^m \ominus \beta^m.\]

We also say that \( \phi \) codes \( \phi^\mathfrak{A} \) in (the Elliott monoid of) \( \mathfrak{A} \).

The interpretation \( \phi^\mathfrak{A} \) of \( \phi \in \text{TERM}_\kappa \) in (the Elliott monoid of) the universal AF\( \ell \)-algebra \( \mathfrak{M} \) and its quotients is similarly defined.

**Decision problems for AF\( \ell \)-algebras.** Fix a cardinal \( \kappa = 1, 2, \ldots, \omega \). Let \( \mathfrak{A} = \mathfrak{M}_\kappa / \mathfrak{I} \) for some ideal \( \mathfrak{I} \) of \( \mathfrak{M}_\kappa \). (Here \( \mathfrak{M}_1 = \mathfrak{M} \).) The word problem \( P_1 \) of \( \mathfrak{A} \) is defined by \( P_1 = \{(\phi, \psi) \in A^* \mid (\phi, \psi) \in \text{TERM}_{2\kappa}^2 \text{ and } \phi^\mathfrak{A} = \psi^\mathfrak{A}\} \). Intuitively, on input strings \( \phi \) and \( \psi \), \( P_1 \) checks if \( \phi \) and \( \psi \) are strings in \( \text{TERM}_\kappa \) coding the same equivalence class of projections of \( \mathfrak{A} \). Likewise, the order problem \( P_2 \) of \( \mathfrak{A} \) is the subset of \( A^* \) given by \( \{(\phi, \psi) \in \text{TERM}_{3\kappa}^2 \mid \phi^\mathfrak{A} \preceq \psi^\mathfrak{A}\} \). Problem \( P_2 \) checks if \( \phi \) codes an equivalence class of projections \( \phi^\mathfrak{A} \) in \( \mathfrak{A} \) that precedes \( \psi^\mathfrak{A} \) in the Murray-von Neumann order \( \preceq \) of projections in \( \mathfrak{A} \). The eccentricity problem \( P_3 = \{(\phi, \psi) \in \text{TERM}_{4\kappa}^2 \mid \phi^\mathfrak{A} \preceq \psi^\mathfrak{A} \} \) checks whether \( \phi \), \( \psi \) code an equivalence class of projections \( \phi^\mathfrak{A} \) in \( \mathfrak{A} \) that precedes \( \psi^\mathfrak{A} \) in the \( \sqsubseteq \)-order of \( \mathfrak{A} \). Further, the zero problem \( P_4 = \{\phi \in \text{TERM}_\kappa \mid \phi^\mathfrak{A} = 0\} \) of \( \mathfrak{A} \) checks if \( \phi^\mathfrak{A} = 0 \). The central projection problem \( P_5 \) of \( \mathfrak{A} \) checks if \( \phi^\mathfrak{A} \) is an equivalence class of central projections in \( \mathfrak{A} \). The nontrivial projection problem \( P_6 \) of \( \mathfrak{A} \) checks if \( \phi^\mathfrak{A} \) different from 0 and 1. The central nontrivial projection problem \( P_7 \) of \( \mathfrak{A} \) checks if \( \phi^\mathfrak{A} \) is an equivalence class of central projections of \( \mathfrak{A} \) other than 0 or 1.

**Polynome problems and (Turing) reductions.** For any formula \( \phi \in \text{TERM}_\kappa \) we let \(|\phi| = \text{the number of occurrences of symbols in } \phi \). Let \( \kappa = 1, \ldots, 7 \). We say that \( P_i \) is decidable in polynomial time if there is a polynomial \( \rho: \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\} \) and a Turing machine \( \mathcal{T} \) with the following property: Over any input string \( \sigma \in A^* \) of \( \mathfrak{A} \), machine \( \mathcal{T} \) decides if \( \sigma \) belongs to \( P_i \), within a number of steps \( \leq \rho(|\sigma|) \). Given problems \( P_i', P'' \subseteq A^* \), a reduction \( \rho \) of \( P_i' \) to \( P'' \) is a map \( \rho: A^* \to A^* \) such that for every string \( \sigma \in A^* \), \( \mathcal{T} \) if \( \rho(\sigma) \in P'' \). When \( \rho \) is computable in polynomial time we say it is a polynomial reduction. Compare with [14] p.211.

**Proposition 3.1.** Let \( \mathfrak{A} \) be an AF\( \ell \)-algebra whose Elliott monoid is finitely generated. Let \( \mathfrak{I} \) (resp., \( \mathfrak{J} \)) be an ideal of \( \mathfrak{M}_\kappa \) (resp., of \( \mathfrak{M}_\kappa \)) such that \( \mathfrak{A} \equiv \mathfrak{M}_\kappa / \mathfrak{I} \equiv \mathfrak{M}_{\kappa / \mathfrak{I}} \). Let \( \mathfrak{P} \subseteq A^* \) be any problem among \( P_1, \ldots, P_7 \). Then \( \mathfrak{P} \) for \( \mathfrak{M}_\kappa / \mathfrak{I} \) is polytime reducible to \( \mathfrak{P} \) for \( \mathfrak{M}_\kappa / \mathfrak{J} \).

**Proof.** A rational polyhedron \( P \subseteq [0, 1]^n \) is the union of finitely many simplexes \( T_i \subseteq [0, 1]^n \) with rational vertices. Two rational polyhedra \( P \) and \( Q \) are \( \mathbb{Z} \)-homeomorphic if \( Q \) is the image of \( P \) under a piecewise linear homeomorphism \( \eta \) such that all linear pieces of \( \eta \) and \( \eta^{-1} \) have integer coefficients. Theorem 1.2 yields ideals \( i, j \) such that \( E(\mathfrak{A}) \cong \mathcal{M}([0, 1]^n) / i \cong \mathcal{M}([0, 1]^n) / j \). Elliott’s classification, and the \( \mathbb{Z} \)-homeomorphism in [13] 8.7], (coded by an \( m \)-tuple \( \psi_i \) of \( m \)-variable terms) yields a polytime reduction \( \rho: \phi \in \text{TERM}_\kappa \mapsto \phi(\psi_i) \in \text{TERM}_\kappa \) of problem \( P_i \) for \( \mathfrak{M}_\kappa / \mathfrak{I} \) to \( P_i \) for \( \mathfrak{M}_\kappa / \mathfrak{J} \), and a polytime reduction \( \rho'' \) in the opposite direction. □

**Theorem 3.2.** Fix an AF\( \ell \)-algebra \( \mathfrak{A} \) whose Elliott monoid is finitely generated. Then for all \( i, j \in \{1, \ldots, 4\} \) problem \( P_i \) for \( \mathfrak{A} \) is polytime reducible to problem \( P_j \). Thus in particular, \( P_i \) for \( \mathfrak{A} \) is polytime iff so is \( P_j \) for \( \mathfrak{A} \).

**Proof.** Following [13] p.8, for every MV-algebra \( A \) we define the operation \( \odot: A^2 \to A \) by \( x \odot y = x \odot y^* \). By [13] Lemma 1.1.2,

\[
x \leq y \iff x \odot y = 0.
\]
We have just constructed a polytime reduction of the eccentricity problem. By (1), the Elliott monoid obtained by noting that $a \leq b$ if and only if for every prime ideal $p$ of the MV-algebra $E(A)$ the following holds:

\[(b/p < b^*/p \implies a/p \leq b/p) \text{ and } (b/p > b^*/p \implies a/p \geq b/p).\]  

This is equivalent to $b/p \leq b^*/p$ and $a/p \leq b/p$ or $b/p \geq b^*/p$ and $a/p \geq b/p$, which by (1) can be reformulated as

\[(b/p \land b^*/p = 0 = a/p \lor b/p) \text{ or } (b^*/p \lor b/p = 0 = b/p \land a/p).\]  

By definition of the lattice operations in the MV-chain $E(A)/p$, (Theorem [1][ii]), (3)-(6) equivalently state $(\forall i \in \mathbb{N})((b_i/p \land b^*_i/p) \lor (a_i/p \lor b/p)) \land ((b_i^*/p \lor b/p) \land (b_i/p \land a_i/p)) = 0$.

By (1), the Elliott monoid $E(A)$ satisfies: $a \leq b$ iff $((b \land b^*) \lor (a \lor b)) \land ((b \lor b^*) \land (b \land b^*)) = 0$. By definition of $a, b, a \leq b$ iff $(((\beta \land \beta^*) \lor (\alpha \lor \beta)) \land ((\beta^* \land \beta) \lor (\beta \land \alpha)) = 0$.

We have just constructed a polytime reduction of the eccentricity problem $P_3$ for $\mathfrak{A}$ to the zero problem $P_4$ for $\mathfrak{A}$. A converse polytime reduction is immediately obtained by noting that $\phi^\mathfrak{A} = [0]$ iff $\phi^\mathfrak{A} \subseteq [0]$, whence $P_4$ is a subproblem of $P_3$.

A polytime reduction of the order problem $P_2$ to the zero problem $P_4$ for $\mathfrak{A}$ trivially follows by noting that $\phi^\mathfrak{A} \leq \psi^\mathfrak{A}$ if and only if $\phi \lor \psi^\mathfrak{A} = 0$. Conversely, a polytime reduction of $P_4$ to $P_2$ for $\mathfrak{A}$ follows by noting that $\phi^\mathfrak{A} = 0$ iff $\phi^\mathfrak{A} \leq \psi^\mathfrak{A}$.

A polytime reduction of the word problem $P_1$ to the zero problem $P_4$ for $\mathfrak{A}$ follows from $\phi^\mathfrak{A} = \psi^\mathfrak{A}$ if and only if $\text{dist}(\phi^\mathfrak{A}, \psi^\mathfrak{A}) = 0$ iff $((\phi \lor \psi) \land (\psi \lor \phi))^\mathfrak{A} = 0$. Conversely, a polytime reduction of $P_4$ to $P_1$ for $\mathfrak{A}$ trivially follows from the fact that the former problem is a special case of the latter.

Concerning the central projection problem $P_5$ we have:

**Theorem 3.3.** Arbitrarily fix an $\text{AF}$-algebra $\mathfrak{A}$ whose Elliott monoid $E(\mathfrak{A})$ is finitely generated.

(i) There is a polytime reduction of the central projection problem $P_5$ to the zero problem $P_4$ for $\mathfrak{A}$.

(ii) The converse polytime reduction exists if $\mathfrak{A}$ has no quotient isomorphic to $\mathfrak{C}$, or if $\mathfrak{A}$ is simple.

**Proof.** (i) As a matter of fact, with the notation of (2), we have:

$\phi^\mathfrak{A}$ is the equivalence class of a central projection in $\mathfrak{A}$

iff $\phi^\mathfrak{A}$ belongs to $B(E(\mathfrak{A}))$, by Theorem 2.1

iff $\phi^\mathfrak{A} \land (\phi^\mathfrak{A})^\ast = 0$, by [5] Theorem 1.16, (with $\land$ as in Theorem [2][i][iii])

iff $(\phi \land \phi^\ast)^\mathfrak{A} = 0$.

The desired polytime reduction transforms $\phi$ into $\phi \land \phi^\ast$.

(ii) Let us first assume

$\mathfrak{A}$ has no quotient isomorphic to $\mathfrak{C}$.  

(7)

Pick an $n = 1, 2, \ldots$ and an ideal $\mathfrak{J}$ of $\mathfrak{M}_n$ such that $\mathfrak{A} \cong \mathfrak{M}_n/\mathfrak{J}$. By Proposition [3][i] our proof will not depend on the actual choice of $n$ and $\mathfrak{J}$. Let $A = E(\mathfrak{A})$ and $i$ be the ideal of $E(\mathfrak{M}_n)$ corresponding to $\mathfrak{J}$ by Theorem [1][ii][v], $i = K_0(\mathfrak{J}) \cap E(\mathfrak{M}_n)$. We have $E(\mathfrak{A}) = E(\mathfrak{M}_n/\mathfrak{J}) = E(\mathfrak{M}_n)/i = M([0, 1]^n)/i = A$. Setting $X_i^\mathfrak{A} = \pi_i/i$, $(i = 1, \ldots, n)$, every $\phi \in \text{TERM}_n$ is interpreted as $\phi^\mathfrak{A} = \phi^\mathfrak{M}_n/i$.

To obtain the desired polytime reduction, let us set

$\rho: \phi \mapsto \phi \land \bigwedge_{i=1}^n (X_i \land X_i^\ast)$, for each $\phi \in \text{TERM}_n$.  

(8)
We must prove
\[ \phi^\mathfrak{A} = 0 \iff \left( \phi \land \bigvee_{i=1}^n (X_i \land X_i^*) \right)^\mathfrak{A} \subseteq \{0, 1\} . \] (9)

Indeed, by Theorem 2.1, equation (9) amounts to saying that \( \phi^\mathfrak{A} = 0 \) iff the term \( \rho(\phi) \) given by \( \phi \land \bigvee_{i=1}^n (X_i \land X_i^*) \) codes the equivalence class of a central projection of \( \mathfrak{A} \).

The (\( \Rightarrow \)) direction of (9) is trivial. For the (\( \Leftarrow \)) direction, arguing by way of contradiction, let us assume

\[ \left( \phi \land \bigvee_{i=1}^n (X_i \land X_i^*) \right)^\mathfrak{A} \subseteq \{0, 1\} \quad \text{and} \quad \phi^\mathfrak{A} \neq 0 . \] (10)

By (11), for some prime ideal \( p \) of \( A \) we have
\[ \phi^\mathfrak{A}/p \neq 0 \] in the MV-chain \( A/p \).

On the other hand, every prime ideal \( q \) satisfies \( \phi \land \bigvee_{i=1}^n (X_i \land X_i^*)^\mathfrak{A}/q \subseteq \{0, 1\} \) in the MV-chain \( A/q \). In particular,
\[ \frac{\phi \land \bigvee_{i=1}^n (X_i \land X_i^*)^\mathfrak{A}}{p} \subseteq \{0, 1\} \] in the MV-chain \( A/p \). (12)

Another application of Theorem 1.2 yields a unique primitive ideal \( \mathfrak{P} \) of \( \mathfrak{A} \) such that \( E(\mathfrak{A}/\mathfrak{P}) = E(\mathfrak{A})/p \).

Claim: For each \( i = 1, \ldots, n \) the interpretation \( X_i^\mathfrak{A}/p \) of the variable symbol \( X_i \) in \( \mathfrak{A}/\mathfrak{P} \), as well as in its Elliott monoid \( \mathfrak{A}/p \), is a trivial element 0 or 1.

For otherwise (absurdum hypothesis), say without loss of generality
\[ X_i^\mathfrak{A}/p \notin \{0, 1\} \] whence also \( (X_i^*)^\mathfrak{A}/p \) does not belong to \( \{0, 1\} \). (13)

By (12),
\[ \{0, 1\} \supseteq \frac{\phi \land \bigvee_{i=1}^n (X_i \land X_i^*)^\mathfrak{A}}{p} \]
\[ = \frac{\phi^\mathfrak{A} \land \bigvee_{i=1}^n (X_i \land X_i^*)^\mathfrak{A}}{p} \]
\[ = \frac{\phi^\mathfrak{A} \land \bigvee_{i=1}^n (X_i \land X_i^*)^\mathfrak{A}}{p} . \]

In the MV-chain \( A/p \), from (13) we get
\[ 0 < \frac{(X_1 \land X_1^*)^\mathfrak{A}}{p} < 1 \quad \text{and for each} \quad j = 2, \ldots, n, \quad \frac{(X_j \land X_j^*)^\mathfrak{A}}{p} < 1 . \]

As a consequence, \( k = \frac{n}{n} - 1 \) and for each \( j = 2, \ldots, n \), \( \frac{(X_j \land X_j^*)^\mathfrak{A}}{p} < 1 \) from (11), we obtain
\[ 0 < \frac{\phi^\mathfrak{A} \land \bigvee_{i=1}^n (X_i \land X_i^*)^\mathfrak{A}}{p} < 1 , \] against (10).

Having thus settled our claim, for each \( i = 1, \ldots, n \) we have \( X_i^\mathfrak{A}/p \in \{0, 1\} \) in the MV-chain \( A/p \). Let \( n \) be the only maximal ideal of \( A \) above the prime ideal \( p \), as given by (6) 1.2.12. Then a fortiori, \( \left\{ \frac{X_i^\mathfrak{A}}{n}, \frac{X_i^\mathfrak{A}}{n} \right\} = \{0, 1\} \) in \( \frac{A}{n} \) for each \( i = 1, \ldots, n \). Since the set of elements \( \left\{ X_1^\mathfrak{A}/n, \ldots, X_n^\mathfrak{A}/n \right\} \) generates the simple MV-chain \( A/n \), we obtain \( A/n = \{0, 1\} = \) the set of trivial elements of \( A/n \). In correspondence with the maximal ideal \( n \) of \( A \), Theorem 1.2 (v) yields a maximal ideal \( \mathfrak{N} \) of \( \mathfrak{A} \) such that \( \mathfrak{A}/\mathfrak{N} \cong \mathbb{C} \). This contradicts our standing assumption (7), thus
proving (9). The map $\rho$ defined in (8) is the desired polytime reduction of problem $P_4$ to $P_5$ for $\mathfrak{A}$ whenever $\mathfrak{A}$ has no quotient isomorphic to $C$.

To conclude the proof, there remains to consider the case when $\mathfrak{A}$ is simple. If $\mathfrak{A} \cong C$ then both problems $P_4$ and $P_5$ are polytime, whence they are (trivially) polytime reducible each to the other. On the other hand, if $\mathfrak{A} \ncong C$, the assumed simplicity of $\mathfrak{A}$ ensures that $\mathfrak{A}$ has no quotient isomorphic to $C$. The desired polytime reduction of problem $P_4$ to $P_5$ for $\mathfrak{A}$ is then obtained arguing as in the first part of the proof of (ii). □

Corollary 3.4. Let $\mathfrak{M}$ be the universal AF-algebra of \([16, \S 8]\).

(i) For each $i = 1, \ldots, 5$, problem $P_i$ for $\mathfrak{M}$ is coNP-complete.

(ii) The nontriviality problem $P_6$ is NP-complete.

(iii) If there is a polytime reduction of $P_1$ to $P_6$ then $NP = coNP$.

(iv) The central nontrivial projection problem $P_7$ is (trivially) polytime.

Corollary 3.5. Let $\mathfrak{M}_1$ be the “Farey” AF-algebra of \([17, 3, 18]\).

Problems $P_1, \ldots, P_7$ for $\mathfrak{M}_1$ are decidable in polynomial time.

Corollary 3.6. Problems $P_1, \ldots, P_7$ are decidable in polynomial time for the following AF-algebras:

(i) The Effros-Shen algebra $\mathfrak{F}_\theta$ for $\theta$ a quadratic irrational, or $\theta = 1/e$, or $\theta \in [0, 1] \setminus \mathbb{Q}$ a real algebraic integer.

(ii) The Effros-Shen algebra $\mathfrak{F}_\theta$ for any irrational $\theta \in [0, 1]$ having the following property: There is a real $\kappa > 0$ such that for every $n = 0, 1, \ldots$ the sequence $[a_0, \ldots, a_n]$ of partial quotients of $\theta$ is computable (as a finite list of binary integers) in less than $2^n$ steps.

Corollary 3.7. Problems $P_1, \ldots, P_7$ are decidable in polynomial time for all Behncke-Leptin \([2]\) algebras $A_{m,n}$.

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(D. Mundici) DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE “ULISSE DINI”, UNIVERSITY OF FLORENCE, VIALE MORGAGNI 67/A, I-50134 FLORENCE, ITALY
Email address: daniele.mundici@unifi.it