Effects of Time Delay in Feedback Control of Linear Quantum Systems

K. Nishio, K. Kashima and J. Imura
Graduate School of Information Science and Engineering,
Tokyo Institute of Technology, Tokyo 152-8552, Japan.
(Dated: November 27, 2008)

We investigate feedback control of linear quantum systems subject to feedback-loop time delays. In particular, we examine the relation between the potentially achievable control performance and the time delays, and provide theoretical guidelines for the future experimental setup in two physical systems, which are typical in this research field. The evaluation criterion for the analysis is given by the optimal control performance formula, the derivation of which is from the classical control theoretic results about the input-output delay systems.

PACS numbers: 87.19.lr, 02.30.Yy, 02.30.Ks, 03.65.Ta, 03.65.Yz

I. INTRODUCTION

For reliable realization of quantum feedback control, it is indispensable to take into consideration some real-world limitations, such as incomplete knowledge of the physical systems and poor performance of the control devices. Various efforts on these issues have been undertaken in these few years, see e.g., [1, 2, 3, 4] for the controller devices, are extremely serious, since their effect may completely lose the benefit of feedback control [3, 4, 7]. To avoid the time delays, one can think to use the Markovian feedback control, in which the measurement results are directly fed back [8, 9]. However, while these experimental simplifications have been extensively studied, theoretical ways to evaluate the effect of the time delays have not been proposed so far.

In this paper, we investigate the effect of the time delays on the control performance, which is defined in terms of the cost function optimized by feedback control. This investigation provides theoretical guidelines for the feedback control experiment. As the controlled object, the linear quantum systems are considered. In order to prepare the tool for the analysis, we first consider the optimal LQG control problem subject to the constant time delay. The optimal controller is obtained via the existing results in the classical control theory [10]. Further, these results allow us to obtain the formula for the optimal value of the cost.

The obtained formula enables us to examine the relation between the optimal control performance and the time delay both in an analytical and a numerical ways. Then, the intrinsic stability of the systems is dominant for the performance degradation effect. If the system is stable, the degradation effect converges to some value in the large time delay limit. Otherwise, the performance monotonically deteriorates as the delay length becomes larger. Based on this fact, we perform the analysis stated above for several physical systems that possess different stability properties. In addition to the controller design, we examine the relationship between the measurement apparatus and the best achievable performance. Based on this, we propose a detector parameter tuning policy for feedback control of the time-delayed systems.

This paper is organized as follows. Linear quantum control systems are introduced in the next section. In Section III, we state the control problem for dealing with the time delay issue, and provide its optimal solution. In Section IV, we investigate the effect of the time delay in quantum feedback control based on two typical examples possessing different stability properties. Section V concludes the paper.

We use the following notation. For a matrix $A = (a_{ij})$, $A^T$, $A^!$ and $A^*$ are defined by $A^T = (a_{ji})$, $A^! = (a_{ji}^!)$ and $A^* = (a_{ij}^*)$, respectively, where the matrix element $a_{ij}$ may be an operator and $a_{ij}^*$ denotes its adjoint. The symbols Re$(A)$ and Im$(A)$ denote the real and imaginary parts of $A$, respectively, i.e., $\text{Re}(A) = (A + A^*)/2$ and $\text{Im}(A) = (A - A^*)/2i$. All the rules above are applied to any rectangular matrix.

II. LINEAR QUANTUM SYSTEM

Consider a quantum system which interacts with a vacuum electromagnetic field through the system operator

$$c = Cx,$$  
(1)

where $x = [q, p]^T$ and $C = [c_1, c_2] \in \mathbb{C}^{1 \times 2}$. When the system Hamiltonian is denoted by $H$, this interaction is described by a unitary operator $U_t$ obeying the following quantum stochastic differential equation called the Hudson-Parthasarathy equation [11]:

$$dU_t = \left[\left( -iH - \frac{1}{2} c^! c \right) dt + c dB_t^! - c^! dB_t \right] U_t,$$  
(2)

where $U_0$ is the identity operator. The field operators $B_t^1$ and $B_t$ are the creation and annihilation operator processes, which satisfy the following quantum Itô rule:

$$d B_t^1 dB_t = dt, \quad d B_t dB_t = dB_t^! dB_t = dB_t^1 dB_t^! = 0.$$  
(3)

Further, suppose that the system is trapped in a harmonic potential, and that a linear potential is an input
to the system. The system Hamiltonian $H_t$ at time $t$ is given by

$$H_t = \frac{1}{2} x^T G x - x^T \Sigma B u_t$$

(4)

where $u_t \in \mathbb{R}$ is the control input at time $t$, the system parameters $G \in \mathbb{R}^{2 \times 2}$ and $B \in \mathbb{R}^2$ are a symmetric matrix and a column vector, and $\Sigma$ is given by

$$\Sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$  

Then, by defining $x_t = [q_t, p_t]^T = [U_t q U^T_t, U_t p U^T_t]^T$ and by using the commutation relation $[q, p] = i$ and the quantum Itô formula, we obtain the following linear equation:

$$dx_t = Ax_t dt + Bu_t dt + i\Sigma(C^T dB_t^1 - C dB_t^1),$$

where $A := \Sigma[G + \text{Im}(C^T C)]$. Measurement processes are described as follows. Suppose that the field observable $e^{-i\phi} B_t^1 + e^{i\phi} B_t^1$ is measured by the perfect homodyne detector, where $\phi \in [0, 2\pi]$ denotes the detector parameter that the experimenter can change [12]. Then, the output signal $y_t$ is obtained by

$$y_t = U^T_t(e^{-i\phi} B_t + e^{i\phi} B_t^1) U_t.$$  

(6)

The simple calculation yields the infinitesimal increment of the observable $y_t$ as follows:

$$dy_t = (e^{-i\phi} C + e^{i\phi} C^*) x_t dt + e^{-i\phi} dB_t + e^{i\phi} dB_t^1.$$  

(7)

In the following section, we refer to [3] and (7) as the system dynamics and the output equation, respectively.

### III. OPTIMAL FEEDBACK CONTROL

#### A. Input-output delay system

As stated in the introduction, the effect of time delays is significant in feedback control of quantum systems. Those delays are mainly originated from the computational time for a controller and the transition delay of signals. Thus, they should be modelled practically as input-output delays in the feedback loop, i.e., at time $t$, the signal $u_{t-h}$ works as a control input for the system and the information $\{y_s\}_{s \leq t-h}$ is available in the controller, where we assume that $h_1$ and $h_2$ are constants. Without loss of generality, when we consider the optimal control problem for such a system, the total delay time can be simply put together into one input (or output) delay. Then, the system dynamics are modified as follows:

$$dx_t = Ax_t dt + Bu_{t-h} dt + i\Sigma(C^T dB_t^1 - C dB_t^1).$$

(8)

Here, the real constant $h$ denotes the total time delay in the feedback loop, i.e., $h = h_1 + h_2$. Note here that $u_t$ should be determined by $\{y_s\}_{s \leq t}$.

#### B. Optimal control performance

We consider the optimal control problem for the system described by (7) and (8). The following system expression is convenient for exploiting results in the classical control theory. Let us define a quantum noise vector

$$w_t := \begin{bmatrix} e^{-i\phi} B_t + e^{i\phi} B_t^1 \\ -iB_t + iB_t^1 \end{bmatrix}.$$  

(9)

It is shown that the quantum noise vector satisfies the following properties:

$$\langle w_t \rangle = 0,$$

$$dw_t dw_t^\dagger = \begin{cases} F_\phi dt, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases}$$

(10) (11)

where $\langle \cdot \rangle$ denotes the expectation and $F$ is the non-negative Hermitian matrix given by

$$F_\phi := \begin{bmatrix} 1 & ie^{-i\phi} \\ -ie^{i\phi} & 1 \end{bmatrix}.$$  

Also, we define the matrix $S_\phi := \frac{1}{2}(F_\phi + F_\phi^\dagger)$. By substituting the terms of the field observables $B_t$, $B_t^1$ with the noise vector $w_t$, we obtain the following equations:

$$dx_t = Ax_t dt + B_1 dw_t + B_2 u_{t-h} dt,$$

$$z_t = C_1 x_t + D_{12} u_{t-h},$$

$$dy_t = C_2 x_t dt + D_{21} dw_t.$$  

(12)

Here, $z_t$ is an additional output signal defined to evaluate the system performance, and $C_1 \in \mathbb{R}^{2 \times 2}$ and $D_{12} \in \mathbb{R}^2$ are matrices freely tunable in controller design. The other system matrices are defined as follows:

$$B_1 := \Sigma \text{Im}(C^T \left[ \begin{array}{cc} 2 \exp(-i\phi) & 2i \\ 1 + \exp(-i2\phi) & 1 + \exp(-i2\phi) \end{array} \right]),$$

$$B_2 := B,$$

$$C_2 := e^{-i\phi} C + e^{i\phi} C^*,$$

$$D_{21} := \begin{bmatrix} 1 & 0 \end{bmatrix}.$$  

As depicted in Fig. [1] we investigate the feedback loop consisting of the system and a controller implemented by classical devices, such as analogue or digital circuits. Then, the optimal control problem is stated as follows.

**Problem 1** For the linear quantum system [12], find the causal, linear and time-invariant control law $u : \{y_s\}_{s \leq t} \rightarrow u_t$ that minimizes the cost functional

$$J := \lim_{t \rightarrow \infty} \langle z_t^\dagger z_t \rangle,$$

(13)

and determine the minimum value of $J$.

We make the following assumption, which is standard in the classical control theory; see [13] for the details.

**Assumption 1**
(A, B₂) is stabilizable and (A, C₂) is detectable.
2. For any ζ ∈ ℜ,

\[
\begin{bmatrix}
  A - jζI & B₂ \\
  C₁ & D₁₂
\end{bmatrix}
\begin{bmatrix}
  A - jζI & B₁ \\
  C₂ & D₂₁
\end{bmatrix}
\]

are row- and column-full rank, respectively.
3. E₁ := D₁₂D₁₂ and E₂, φ := D₂₁SφD₂₁ are non-singular.

As in the other results for linear quantum systems, the solution of Problem 1 can be obtained by slightly modifying the derivation of the classical result in [10]. Here, we only provide the minimum value of the cost J, which is of importance for the later discussion. For the specific form of the optimal controller, see the appendix.

**Theorem 1** Consider Problem 1 with Assumption 1. Let X, Y be the solutions of the matrix Riccati equations XA + AᵀX + C₁ᵀC₁ − FᵀE₁F = 0 and Y Aᵀ + AY + B₁SφB₁ᵀ − LE₂, φLᵀ = 0 with F := −E₁⁻¹(B₂X + D₁₂C₁) and L := −(YC₂ᵀ + B₁S₂D₂₁)E₂, φ⁻¹ such that A + B₂F, A + LC₂ are stable. Then, the optimal value of the cost functional J is given by

\[
J_{opt} = J_{φ} + \int_{0}^{h} (F e^{A τ} L)^2 dτ,
\]  

(14)

where \(J_{φ} := \text{tr}(B₁S₂B₁ᵀX) + \text{tr}(FᵀE₁FY)\) is the optimal value of J when \(h = 0\).

Note that the existence of the Riccati solutions X and Y follows from Assumption 1.

**IV. EFFECT OF FEEDBACK DELAY**

In the experiment of the feedback control, it is of importance to reduce the time delay by carefully setting up the experimental devices and achieve the best performance possible [6, 7]. However, some quantity of the time delay remains in practice. In this section, we investigate how the time delay deteriorates the optimal control performance by using the formula (14). In addition to the algorithm in the controller, we have the tunable parameter in the measurement apparatus. Thus, we do the analysis taking the detector parameter tuning into consideration. It should be noted that the optimal measurement technique was first introduced by Wiseman and Doherty [14]. Their technique is only for the delay-free systems, i.e., the optimization of the value of \(J_{φ}^{opt}\).

First of all, notice that the performance degradation effect is mainly determined by the exponential term in (14). This means that the degradation is largely related to the system’s intrinsic stability, i.e., the eigenvalues of the matrix A. Thus, it is obvious from the exponential growth of \(J_{φ}^{opt}\) that the unstable system easily deteriorates as the time delay length increases, and that their control is significantly difficult.

On the other hand, the remaining two classes, i.e., stable and marginally stable systems, are relatively insensitive to the time delay and it is worth to analyze them in detail. In order to provide some guidelines for the experiments, we analyze the two physical systems that frequently appear in the context of quantum feedback control. In the following, we choose the matrices C₁ and D₁₂ as

\[
C₁ = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad D₁₂ = \begin{bmatrix} 1 \end{bmatrix}.
\]

**Stable system** - Consider a damped cavity with an on-threshold parametric down converter. The system Hamiltonian and the coupling operator are given by

\[
H₁ = \frac{γ}{2}(qp + pq) - u₁ - \delta q, \quad c = δ(q + ip),
\]

(15)

where \(γ > 0\) and \(δ > 0\) are constant parameters. If they satisfy \(γ < δ²\), the system is stable. In this case, clearly, \(J_{φ}^{opt}\) converges as \(h → \infty\) since the real part of every eigenvalue of A is negative. When we choose the parameter as \(γ = 1/2\) and \(δ = 1\), the optimal performance curves with the different detector parameters φ are given by Fig. 2. It is certainly confirmed that the performance degradation converges in the limit of \(h → \infty\) for any detector parameters. The dashed line depicts the value of J when there is no control input field, i.e., \(u₁ ≡ 0\) for any \(t ≥ 0\). We can see from the figure that even in the large delay limit, the appropriate measurement strategy significantly enhances the control performance compared to the uncontrolled case.

In order to examine the effective detector parameter tuning, let us look at Figs. 2 and 3. These show that the optimal detector parameter hardly fluctuates over every delay length. In fact, we can see from Fig. 2 that the difference between two performance curves around the optimal one, i.e., the curve corresponding to \(φ = 1.98\), is very small. Further, it is shown that the optimal measurement strategy in the delay-free case is perfectly the same as that in the large delay case.
Theorem 2 Consider Problem \([15]\) with \(H_t\) and \(c\) defined by \([15]\). Let \(\phi_h^{\text{opt}}\) denote the detector parameter that minimizes \(J_h^{\text{opt}}\) for the fixed delay length \(h\). Then, the following relation holds:
\[
\lim_{h \to \infty} \phi_h^{\text{opt}} = \phi_0^{\text{opt}}. \tag{16}
\]

**Proof.** The goal is to show that the minimal values of \(J_\phi^{\text{opt}}\) and \(J_{\infty,\phi}^{\text{opt}}\) are achieved with the same detector parameter value. Firstly, we compute the value of \(\partial J_\phi^{\text{opt}} / \partial \phi\). Note that
\[
J_\phi^{\text{opt}} = \text{tr}(B_1 S_\phi B_1^T X) + \text{tr}(F^T E_1 F Y)
\]
\[
= \delta^2(x_{11} + x_{22}) + \frac{1}{2}(x_{11} + 1)^2 y_{11}
\]
\[
+ (x_{12} + 1)(x_{22} + 1)y_{12} + \frac{1}{2}(x_{22} + 1)^2 y_{22},
\]
where
\[
X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix}.
\]
Then, since \(X\) is independent of \(\phi\), we obtain
\[
\frac{\partial J_\phi^{\text{opt}}}{\partial \phi} = \frac{1}{2}(x_{11} + 1)^2 \left( \frac{\partial y_{11}}{\partial \phi} \right)
\]
\[
+ (x_{12} + 1)(x_{22} + 1) \left( \frac{\partial y_{12}}{\partial \phi} \right)
\]
\[
+ \frac{1}{2}(x_{22} + 1)^2 \left( \frac{\partial y_{22}}{\partial \phi} \right). \tag{17}
\]
Besides, the relation
\[
\frac{\partial}{\partial \phi}(YA^T + Ay + B_1 S_\phi B_1^T - LE_2,\phi L^T) = 0
\]
provides us with
\[
\frac{\partial}{\partial \phi} \int_0^h (Fe^{\tau L})^2 d\tau
\]
\[
= \int_0^h Fe^{\tau L} \left( \frac{\partial Y}{\partial \phi} A^T + \frac{\partial Y^T}{\partial \phi} A \right) e^{\tau L} F^T d\tau
\]
\[
= \int_0^h Fe^{\tau L} \left( \frac{\partial Y}{\partial \phi} A^T + \frac{\partial Y^T}{\partial \phi} A \right) e^{\tau L} F^T d\tau
\]
\[
= -\frac{1}{2} \left\{ \frac{(x_{12} + 1)^2 \partial y_{11}}{\partial \phi} + (x_{12} + 1)(x_{22} + 1) \frac{\partial y_{12}}{\partial \phi} \right. \\
+ \left. \frac{(x_{22} + 1)^2 \partial y_{22}}{\partial \phi} + (x_{12} + 1) \frac{\partial y_{11}}{\partial \phi} \right\}
\]
\[
+ \frac{(x_{12} + 1)(x_{22} + 1)}{\partial \phi} e^{-2(\delta^2 - \gamma)h} \frac{\partial y_{11}}{\partial \phi}
\]
\[
+ \frac{(x_{22} + 1)}{\partial \phi} e^{-2\delta^2 h} \left( \frac{\partial y_{12}}{\partial \phi} \right)
\]
\[
+ \frac{(x_{22} + 1)^2}{4} e^{-2(\delta^2 + r)h} \left( \frac{\partial y_{22}}{\partial \phi} \right),
\]
where we used
\[
\frac{\partial}{\partial \phi}(B_1 S_\phi B_1^T) = 0, \quad LE_2,\phi L^T = LL^T.
\]

Thus, with the attention to \([17]\), the following equation is obtained:
\[
\frac{\partial J_{h,\phi}^{\text{opt}}}{\partial \phi} = \frac{\partial J_\phi^{\text{opt}}}{\partial \phi} + \frac{\partial}{\partial \phi} \int_0^h (Fe^{\tau L})^2 d\tau
\]
\[
= \frac{1}{2} \frac{\partial J_\phi^{\text{opt}}}{\partial \phi} + \frac{(x_{12} + 1)^2}{4} e^{-2(\delta^2 - \gamma)h} \left( \frac{\partial y_{11}}{\partial \phi} \right)
\]
\[
+ \frac{(x_{22} + 1)(x_{22} + 1)}{2} e^{-2\delta^2 h} \left( \frac{\partial y_{12}}{\partial \phi} \right)
\]
\[
+ \frac{(x_{22} + 1)^2}{4} e^{-2(\delta^2 + r)h} \left( \frac{\partial y_{22}}{\partial \phi} \right).
\]

Hence, we obtain
\[
\frac{\partial J_{\infty,\phi}^{\text{opt}}}{\partial \phi} = \lim_{h \to \infty} \frac{\partial J_{h,\phi}^{\text{opt}}}{\partial \phi} = \frac{1}{2} \frac{\partial J_\phi^{\text{opt}}}{\partial \phi},
\]
where the first equality follows from the pointwise convergence of \(J_{h,\phi}^{\text{opt}}\) and the uniform convergence of \(\partial J_{h,\phi}^{\text{opt}} / \partial \phi\) on the domain of \(\phi\). This completes the proof. \(\blacksquare\)

From the discussion above, we can conclude that Wise- man’s measurement strategy (the optimal tuning for delay-free case) is valid for the stable delay systems in that \(\phi_h^{\text{opt}} \approx \phi_0^{\text{opt}}\) for any \(h \geq 0\).

**Marginally stable system** - The next system is a single particle trapped in the harmonic potential and coupled to the probe field via the position operator. The system Hamiltonian and the coupling operator are given by
\[
H_t = \frac{1}{2} m \omega^2 q^2 + \frac{1}{2} m p^2 - u_{t-q} q, \quad c = q, \tag{18}
\]
where \(m\) and \(\omega\) are the mass of the particle and the angular frequency of the harmonic potential, respectively. For this system, the shape of the optimal performance curves is analytically calculated.
Theorem 3 Consider Problem II with $H_t$ and $c$ defined by \((18)\). Then there exist constants $A$, $B$ and $\theta$ such that the best achievable performance is given by

$$J_{h,\phi}^{opt} = J_{\phi}^{opt} + Ah + B \sin(\omega h + \theta). \quad (19)$$

Moreover, $A$ and $B$ are independent of the choice of $\phi$. 

Proof. Notice that the Riccati solution $Y$ is dependent on the parameter $\phi$. To make this dependence explicit, we write $Y$ as $Y_\phi$ and, similarly, $L$ as $L_\phi$ throughout the proof. Then, by Theorem I, the best achievable performance is given by

$$J_{\phi}^{opt} = \text{tr} \left( B_1 S_\phi B_1^T X \right) + \text{tr} \left( F^T E_1 F Y_\phi \right).$$

On the other hand, direct computation yields

$$F e^{A T} L_\phi = \sqrt{\left\{ l_1^2 + \left( \frac{l_2}{m \omega} \right)^2 \right\} \left\{ f_1^2 + (m \omega f_2)^2 \right\} \sin(\omega T + \theta)},$$

where $F = [f_1, f_2]$, $L_\phi = [l_1, l_2]^T$ and $\theta_\phi$ satisfies

$$\tan \theta = \frac{m \omega (f_1 l_1 + f_2 l_2)}{f_1 l_2 - (m \omega)^2 f_2 l_1}.$$

By combining this with \((20)\), we obtain the first claim. It should be emphasized that $\phi$ contributes to $A$ and $B$ only through

$$l_1^2 + \left( \frac{l_2}{m \omega} \right)^2. \quad (21)$$

Hence, it is sufficient to show the second claim that \((21)\) does not depend on $\phi$. When defining

$$Y_\phi = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix},$$

simple calculation yields

$$(4 \cos^2 \phi) y_{12}^2 + (2 m \omega^2 - 4 \sin 2 \phi) y_{12} + 4 \sin^2 \phi - 1 = 0,$$

$$(2 m \cos^2 \phi) y_{11}^2 - y_{12} = 0.$$

Then, we obtain the following:

$$l_1^2 + \left( \frac{l_2}{m \omega} \right)^2$$

$$= (4 \cos^2 \phi) y_{11}^2 + \frac{4 (\cos \phi y_{12} - \sin \phi)^2}{m^2 \omega^2}$$

$$= \frac{1}{m^2 \omega^2} \left\{ (4 \cos^2 \phi) y_{12}^2 + (2 m \omega^2 - 4 \sin 2 \phi) y_{12} + 4 \sin^2 \phi \right\}$$

$$= \frac{1}{m^2 \omega^2}.$$

This completes the proof. \(\blacksquare\)

Roughly speaking, the first statement says that $J_{h,\phi}^{opt}$ increases linearly with the oscillation as the delay length $h$ becomes large. This is a natural result of the fact that the matrix $A$ has only pure imaginary eigenvalues. On the other hand, the second statement gives us a nontrivial insight: the growth rate $A$ and the oscillation amplitude $B$ are independent of the detector parameter $\phi$.

Hence, depending on the delay length, the importance of the choice of the measurement apparatus differs. If the delay length is small, the improvable performance level is sensitive to the value of $h$. On the other hand, if the system suffers from the large delay, the apparatus adjustment is not significant, since the improvable performance level is relatively small compared to the value of $J_{h,\phi}^{opt}$. Thus, if the time delay can be made sufficiently small, experimenters have to adjust the detector parameter depending on the resulting delay length. For the illustration, see Fig. 4 for $m = \omega = 1$, which illustrates the optimal performance curve $J_{h,\phi}^{opt}$.
V. CONCLUSION

In this paper, we investigated performance degradation effects due to time delays in optimal control of linear quantum systems. The analysis was performed by the optimal control performance formula from the classical control theory. The obtained remarks are strongly related to the intrinsic stability of the physical systems. In particular, we performed intimate evaluations for two typical systems with different types of stability. These results are expected to give useful guidelines for the future experiments.

APPENDIX A: OPTIMAL FEEDBACK CONTROLLER FOR PROBLEM 1

Consider Problem 1. With the same notation as that in Theorem 1, the optimal feedback controller is given by

\[
d\hat{x}_t = (A + B_2 F + e^{Ah} L C_2 e^{-Ah}) \hat{x}_t dt - e^{Ah} L (dy_t + \pi_t dt) \tag{A1}
\]

\[
u_t = F \hat{x}_t \tag{A2}
\]

\[
d\eta_t = -C_2 e^{-Ah} \hat{x}_t dt + (dy_t + \pi_t dt) \tag{A3}
\]

and the finite-time integration system

\[
\pi_t = C_2 \int_{t-h}^{t} e^{A(t-\tau-h)} B_2 u_\tau d\tau. \tag{A4}
\]

It should be noted that the implementation of this controller involves infinite-dimensional elements. Thus, computers with the finite memory cannot implement it in a precise sense. Fortunately, however, it is known that the approximation method proposed in [15] permits the control with high accuracy.

[1] M. R. James, Phys. Rev. A 69, 032108 (2004).
[2] M. R. James, J. Opt. B: Quantum Semiclass. Opt. 7, 198 (2005).
[3] M. R. James, H. I. Nurdin, and I. R. Petersen, arXiv:quant-ph/0703150v2 (2007).
[4] N. Yamamoto, Phys. Rev. A 74, 032107 (2006).
[5] R. van Handel, J. K. Stockton, and H. Mabuchi, IEEE Trans. Automatic Control 50, 768 (2005).
[6] D. Steck, K. Jacobs, H. Mabuchi, S. Habib, and T. Bhattacharya, Phys. Rev. A 74, 012322 (2006).
[7] J. Stockton, M. Armen, and H. Mabuchi, J. Opt. Soc. Am. B 19, 3019 (2002).
[8] H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett. 70, 548 (1993).
[9] H. M. Wiseman, S. Mancini, and J. Wang, Phys. Rev. A 66, 013807 (2002).
[10] L. Mirkin, IEEE Trans. Automatic Control 48, 543 (2003).
[11] R. L. Hudson and K. R. Parthasarathy, Commun. Math. Phys. 93, 301 (1984).
[12] L. Bouten, R. van Handel, and M. R. James, SIAM J. Control and Optimization 46, 2199 (2007).
[13] K. Zhou, J. C. Doyle, and K. Glover, Robust Optimal Control (Prentice-Hall, 1995).
[14] H. M. Wiseman and A. C. Doherty, Phys. Rev. Lett. 94, 070405 (2005).
[15] L. Mirkin, System and Control Letters 51, 331 (2004).