EXTREMAL SEQUENCES FOR THE BELLMAN FUNCTION OF
THE DYADIC MAXIMAL OPERATOR IN RELATION WITH
KOLMOGOROV’S INEQUALITY

ELEFtherios N. NIKOLIDAKIS

Abstract: We give a characterization of the extremal sequences for the Bellman function of the dyadic maximal operator in relation with Kolmogorov's inequality. In fact we prove that they behave approximately like eigenfunctions of this operator for a specific eigenvalue. For this approach we use the one introduced in [], where the respective Bellman function has been precisely evaluated.

1. Introduction

The dyadic maximal operator is defined by

\[ M_d \phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(u)| \, du : x \in Q, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\} \]

for every \( \phi \in L^1_{\text{loc}}(\mathbb{R}^n) \) where \( | \cdot | \) is the Lesbesgue measure on \( \mathbb{R}^n \) and the dyadic cubes are those formed by the grids \( 2^{-N} \mathbb{Z}^n, N = 0, 1, 2, \ldots \).

As it is well known it satisfies the following weak type \( (1,1) \) inequality:

\[ |\{ x \in \mathbb{R}^n : M_d \phi(x) > \lambda \}| \leq \frac{1}{\lambda} \int_{\{ M_d \phi > \lambda \}} |\phi(u)| \, du, \]

for every \( \phi \in L^1(\mathbb{R}^n) \) and every \( \lambda > 0 \), form which follows in view of Kolmogorov’s inequality, the following \( L^q \)-inequality:

\[ \int_E |M_d \phi(u)|^q \, du \leq \frac{1}{1 - q} |E|^{1-q} \| \phi \|_1^q. \]

for every \( q \) with \( 0 < q < 1 \), every \( \phi \in L^1(\mathbb{R}^n) \) and every measurable subset of \( \mathbb{R}^n, E \), with finite measure.

It is easy to see that the weak type inequality \( (1.2) \) is best possible, while \( (1.3) \) is sharp as can be seen in [].

An approach for studying such maximal operators is to find certain refinements of inequalities satisfied by it such as \( (1.2) \) and \( (1.3) \).

Concerning \( (1.3) \) an important function has been introduced which is the following:

\[ B^Q_Q(f, h) = \sup \left\{ \frac{1}{|Q|} \int_Q (M_d \phi)^q : \phi \geq 0, Av_Q(\phi) = f, Av_Q(\phi^q) = h \right\} \]

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MSC Number: 42B25
E-mail address: lefteris@math.uoc.gr
where $Q$ is a fixed dyadic cube on $\mathbb{R}^n$, $\phi \in L^p(Q)$, $0 < h \leq f^q$, and for
\[
g \in L^1(Q) : Av_Q(g) = \frac{1}{|Q|} \int_Q |g(u)|du.
\]
This is the main Bellman function of two variables associated to the dyadic maximal operator.

The function given (1.4) has been precisely computed in [2]. In fact more general Bellman functions have been evaluated there, by using an effective linearization for the dyadic maximal operator on an adequate set of functions (called $T$-good, see definition below) which includes the step functions over dyadic cubes in the case of $q^a\mathbb{R}^n$.

Actually this has been done in a more general setting of a non-atomic probability measure space $(X, \mu)$ where the dyadic sets are now given in a family of sets $T$ (called tree) which satisfies similar conditions to those that are satisfied by the dyadic cubes on $[0,1]^n$. Then the associated dyadic maximal operator $M_T$ is defined by
\[
M_T\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi|d\mu : x \in I \in T \right\}
\]
Then the Bellman function of two variables for $M_T$, associated to Kolmogorov’s inequality is given by
\[
B_q(f, h) = \sup \left\{ \int_X (M_T\phi)^q d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = h \right\}
\]
where $0 < h \leq f^q$.

In [2] (1.6) has been found to be equal to
\[
B_q(f, h) = h\omega_q(f^q/h) \text{ where } \omega_q : [1, +\infty) \rightarrow [1, +\infty)
\]
is defined by $\omega_q(f) = (H_q^{-1}(z))^q$, and
\[
H_q(z) = (1 - q)z^q + qz^{q-1}, \text{ } z \geq 1
\]
In this paper we study those sequences $(\phi_n)_n$ that are extremal for the Bellman function (1.6).

That is $\phi_n : (X, \mu) \rightarrow \mathbb{R}^+$ must satisfy
\[
\int_X \phi_n d\mu = f, \int_X \phi_n^q d\mu = h \quad \text{and} \quad \lim_n \int_X (M_T\phi_n)^q d\mu = h\omega_q\left(\frac{f^q}{h}\right).
\]
Additionally we suppose that $\phi_n$ is $T$-good, for every $n \in \mathbb{N}$ (see definition of the beginning of Section ). In [2] it is proved that every such sequence must satisfy a selfsimilar property, that is for every $I \in T$
\[
\lim_n \frac{1}{\mu(I)} \int_I \phi_n d\mu = f \quad \text{and} \quad \lim_n \frac{1}{\mu(I)} \int_I \phi_n^q d\mu = h.
\]
This gives that in all interesting cases (including $\mathbb{R}^n$, where the tree $T$ differentiates $L^1(X, \mu)$) that extremal functions do not exist for the Bellman function.
That is for every $\phi \in L^1(X, \mu)$, $\phi \geq 0$, with $\int_X \phi d\mu = f$ and $\int_X \phi^q d\mu = h$ then $\int_X (M_T \phi)^q d\mu < h \omega_q(f^q/h)$.

In this paper we characterize these extremal sequences by proving the following: **Theorem:** Let $(\phi_n)_n$ be a sequence of $T$-good functions, such that $\int_X \phi_n d\mu = f$ and $\int_X \phi_n^q d\mu = h$. Then $(\phi_n)_n$ is extremal for (1.6) if and only if

$$\lim_n \int_X |M_T \phi_n - c \phi_n|^q d\mu = 0,$$

where $c = \omega_q(f^q/h)^{1/q}$.

Obviously, if the above limit is zero, then $(\phi_n)_n$ is extremal so one must give attention to the opposite direction.

We need also to say that the study of the extremal sequences for the case $p > 1$ (Bellman function of two variables with respect to $L^p$-norms) has been given in [3], inspired by [1].

2. Preliminaries

Let $(X, \mu)$ be a non-atomic probability measure space. We give the following from [1].

**Definition 2.1.** A set $T$ of measurable subsets of $X$ will be called a tree if the following are satisfied

i) $X \in T$ and for every $I \in T$, $\mu(I) > 0$.

ii) For every $I \in T$ there corresponds a finite or countable subset $C(I)$ of $T$ containing at least two elements such that

   a) the elements of $C(I)$ are pairwise disjoint subsets of $I$

   b) $I = \bigcup C(I)$.

iii) $T = \bigcup_{m \geq 0} T_{(m)}$, where $T_{(0)} = \{X\}$ and $T_{(m+1)} = \bigcup_{I \in T_{(m)}} C(I)$.

iv) The following holds

$$\lim_{m \to \infty} \sup_{I \in T_{(m)}} \mu(I) = 0.$$  

We state now the following lemma given in [1].

**Lemma 2.1.** For every $I \in T$ and every $a \in (0,1)$ there exists a subfamily $F(I) \subseteq T$ consisting of pairwise disjoint subsets of $I$ such that

$$\mu\left( \bigcup_{J \in F(I)} J \right) = \sum_{J \in F(I)} \mu(J) = (1-a)\mu(I).$$  

Now given a tree $T$ we define the maximal operator associated to it as follows
\[
\mathcal{M}_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi|d\mu : x \in I \in T \right\}, \quad \text{for every } \phi \in L^1(X, \mu).
\]

From [2] we recall the following

**Theorem 2.1.** The following holds:
\[
\sup \left\{ (\mathcal{M}_T \phi)^q d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = h \right\} = h \omega_q(f^q/h),
\]
for every $0 < h \leq f^q$. 

At last we give the following

**Definition 2.2.** Let $(\phi_n)_n$ be a sequence of $\mu$-measurable non-negative functions defined on $X$, $q \in (0, 1)$ and $0 < h \leq f^q$. Then $(\phi_n)$ is called extremal if the following hold:
\[
\int_X \phi_n d\mu = f, \quad \int_X \phi_n^q d\mu = h \quad \text{for every } n \in \mathbb{N}, \quad \text{and}
\]
\[
\lim_n \int_X (\mathcal{M}_T \phi_n)^q d\mu = h \omega_q(f^q/h).
\]

3. Characterization of the extremal sequences

For the proof of the Theorem 2.1 an effective linearization for the operator $\mathcal{M}_T$ was introduced valid for certain functions $\phi$. We describe it.

For $\phi \in L^1(X, \mu)$ nonnegative function and $I \in T$ we define $Av_I(\phi) = \frac{1}{\mu(I)} \int_I \phi d\mu$.

We will say that $\phi$ is $T$-good if the set
\[
\mathcal{A}_\phi = \{ x \in X : \mathcal{M}_T \phi(x) > Av_I(\phi) \quad \text{for all } I \in T \quad \text{such that } x \in I \}
\]
has $\mu$-measure zero.

Let now $\phi$ be $T$-good and $x \in X \setminus \mathcal{A}_\phi$.

We define $I_\phi(x)$ to be the largest in the nonempty set
\[
\{ I \in T : x \in I, \mathcal{M}_T \phi(x) = Av_I(\phi) \}.
\]

Now given $I \in T$ let
\[
A(\phi, I) = \{ x \in X \setminus \mathcal{A}_\phi : I_\phi(x) = I \} \subseteq I \quad \text{and}
\]
\[
S_\phi = \{ I \in T : \mu(A(\phi, I)) > 0 \} \cup \{ X \}.
\]

Obviously, $\mathcal{M}_T \phi = \sum_{I \in S_\phi} Av_I(\phi) J_{A(\phi, I)}$, $\mu$-a.e. where $J_E$ is the characteristic function of $E$.

We define also the following correspondence $I \to I^*$ by: $I^*$ is the smallest element of $\{ J \in S_\phi : I \subset J \}$. It is defined for every $I \in S_\phi$ except $X$. It is obvious that the
A(\phi, I)'s are pairwise disjoint and that \( \mu\left( \bigcup_{I \in S_\phi} (A(\phi, I)) \right) = 0 \), so that \( \bigcup_{I \in S_\phi} A(\phi, I) \approx X \), where by \( A \approx B \) we mean that \( \mu(A \setminus B) = \mu(B \setminus A) = 0 \).

Now the following is true, obtained by [2].

Lemma 3.1. Let \( \phi \) be \( T \)-good

i) If \( I, J \in S_\phi \) then either \( A(\phi, J) \cap I = \emptyset \) or \( J \subseteq I \).

ii) If \( I \in S_\phi \) then there exists \( J \in C(I) \) such that \( J \notin S_\phi \).

iii) For every \( I \in S_\phi \) we have that

\[
I \approx \bigcup_{J \in S_\phi \atop J \subseteq I} A(\phi, J).
\]

iv) For every \( I \in S_\phi \) we have that

\[
A(\phi, I) = I \setminus \bigcup_{J \in S_\phi \atop J \neq I} J, \text{ so that } \mu(A(\phi, I)) = \mu(I) - \sum_{J \in S_\phi \atop J \neq I} \mu(J).
\]

From the above we see that

\[
\text{Av}_I(\phi) = \frac{1}{\mu(I)} \sum_{J \in S_\phi \atop J \subseteq I} \int_{A(\phi, J)} \phi \, d\mu =: y_I, \text{ for } I \in S_\phi.
\]

We define also \( x_I := \frac{1}{a_I A(\phi, I)} \int \phi \, d\mu \), where \( a_I = \mu(A(\phi, I)) \).

We prove now the following:

Theorem 3.1. Let \( \phi \) be \( T \)-good function such that \( \int_X \phi \, d\mu = f \). Let also \( B = \{I_i\}_j \) be a family of pairwise disjoint elements of \( S_\phi \), which is maximal on \( S_\phi \), under \( \subseteq \) relation. That is \( I \in S_\phi \Rightarrow I \cap (\cup I_i) \neq \emptyset \). Then the following inequality holds

\[
\int_{X \setminus \bigcup I_i} (M_T \phi)^q \, d\mu \leq \frac{1}{(1 - q)^\beta} \left[ (\beta + 1)^q \left( f^q - \sum |I_i|^q y_{I_i}^q \right) - (\beta + 1)^q \int_{X \setminus \bigcup I_i} \phi^q \, d\mu \right]
\]

for every \( \beta > 0 \), where \( y_{I_i} = \text{Av}_{I_i}(\phi) \).

Proof. We follow [2].

Let \( S = S_\phi \), \( a_I = \mu(A(\phi, I)) \), \( \rho_I = \frac{a_I}{\mu(I)} \in (0, 1) \) except possibly for \( I = X \), and

\[
y_I = \text{Av}_I(\phi) = \frac{1}{\mu(I)} \sum_{J \in S : J \subseteq I} a_J x_J, \text{ for every } I \in S.
\]

It is easy to see in view of Lemma 3.1 iv) that

\[
y_I \mu(I) = \sum_{J \in S : J \neq I} y_J \mu(J) + a_I x_I,
\]
and so, using the concavity of the function $t \rightarrow r^q$, we have for any $I \in S$,

$$[y_I \mu(I)]^q = \left( \sum_{J \in S: J^* = I} y_J \mu(J) + a_I x_I \right)^q$$

$$= \left( \sum_{J \in S: J^* = I} \tau_I \mu(J) \frac{y_J}{\tau_I} + \sigma_I a_I \frac{x_I}{\sigma_I} \right)^q$$

$$\geq \sum_{J \in S: J^* = I} \tau_I \mu(J) \left( \frac{y_J}{\tau_I} \right)^q + \sigma_I a_I \left( \frac{x_I}{\sigma_I} \right)^q,$$

(3.1)

where the $\tau_I, \sigma_I > 0$ satisfy

$$\tau_I (\mu(I) - a_I) + \sigma_I a_I = \sum_{J \in S: J^* = I} \tau_I \mu(J) + \sigma_I a_I = 1.$$

We now fix $\beta > 0$ and let

$$\sigma_I = ((\beta + 1) \mu(I) - \beta a_I)^{-1}, \quad \tau_I = (\beta + 1) \sigma_I$$

which satisfy the above relation, so we get by dividing with $\sigma_I^{-q}$ that

$$((\beta + 1) \mu(I) - \beta a_I)^{1-q} [y_I \mu(I)]^q \geq \sum_{J \in S: J^* = I} ((\beta + 1)^{1-q} \mu(J) y_J^q + a_I x_I^q,$$

(3.2)

However,

$$x_I^q = \left( \frac{1}{a_I} \int_{A(\phi, I)} \phi d\mu \right)^q \geq \frac{1}{a_I} \int_{A(\phi, I)} \phi^q d\mu.$$

(3.3)

We sum now (3.2) over all $I \in S$ such that $I \supseteq I_j$ for some $j$ and we obtain

$$\sum_{I \supseteq \text{piece}(B)} ((\beta + 1) \mu(I) - \beta a_I)^{1-q} [y_I \mu(I)]^q \geq \sum_{I \supseteq \text{piece}(B) \neq X} ((\beta + 1)^{1-q} \mu(I) y_I^q$$

$$+ \sum_j ((\beta + 1)^{1-q} \mu(I_j) y_{I_j}^q + a_I x_I^q,$$

(3.4)

The first two sums are produced in (3.1) because of the maximality of $(I_j)$. Here we note also that by $I \supseteq \text{piece}(B)$ we mean that $I \supseteq I_j$ for some $j$. (3.4) now gives:

$$\sum_{I \supseteq \text{piece}(B)} ((\beta + 1)^{1-q} \mu(I) y_I^q - \sum_{I \supseteq \text{piece}(B)} ((\beta + 1) \mu(I) - \beta a_I)^{1-q} [y_I \mu(I)]^q$$

$$\leq ((\beta + 1)^{1-q} y_2^q - \int_{X \setminus \cup I_j} \phi^q d\mu - \sum_j ((\beta + 1)^{1-q} \mu(I_j) y_{I_j}^q,$$

(3.5)
in view of Holder’s inequality \(3.3\). Thus \(3.5\) gives

\[
\sum_{I \supseteq \text{piece}(B)} ((\beta + 1)^{1-q} \mu(I) - ((\beta + 1) \mu(I) - \beta a_I)^{1-q} \mu(I)^q) y_I^q
\]

\[
\leq (\beta + 1)^{1-q} \left( f^q - \sum \mu(I_j) y_{I_j}^q \right) - \int_{X \setminus \cup I_j} \phi^q d\mu,
\]

(3.6)

On the other side we have that

\[
\frac{1}{\mu(I)} [(\beta + 1)^{1-q} \mu(I) - ((\beta + 1) \mu(I) - \beta a_I)^{1-q} \mu(I)^q]
\]

\[
= (\beta + 1)^{1-q} - ((\beta + 1) - \beta \rho_I)^{1-q} \geq (1 - q)(\beta + 1)^{-q} \beta \rho_I
\]

(3.7)

where the inequality on \(3.7\) comes from the differentiation mean value the sum from calculus.

From the last two inequalities we conclude

\[
(1 - q)(\beta + 1)^{-q} \beta \sum_{I \supseteq \text{piece}(B)} a_I y_I^q \leq (\beta + 1)^{1-q} \left( f^q - \sum \mu(I_j) y_{I_j}^q \right) - \int_{X \setminus \cup I_j} \phi^q d\mu.
\]

(3.8)

Now it is easy to see that

\[
\sum_{I \supseteq \text{piece}(B)} a_I y_I^q = \int_{X \setminus \cup I_j} (M_T \phi)^q d\mu,
\]

so \(3.8\) becomes

\[
\int_{X \setminus \cup I_j} (M_T \phi)^q d\mu \leq \frac{1}{(1 - q)\beta} \left[ (\beta + 1)^{1-q} \left( f^q - \sum \mu(I_j) y_{I_j}^q \right) - (\beta + 1)^q \int_{X \setminus \cup I_j} \phi^q d\mu \right]
\]

for any fixed \(\beta > 0\), and \(\phi : T\)-good.

Thus our Theorem is proved. \(\square\)

We have now the following generalization of Theorem 3.1.

**Theorem 3.2.** Let \(\phi\) be \(T\)-good and \(A = \{I_j\}\) be a pairwise disjoint family of elements of \(S_\phi\). Then for every \(\beta > 0\) we have that:

\[
\int_{\cup I_j} (M_T \phi)^q d\mu \leq \frac{1}{(1 - q)\beta} \left[ (\beta + 1)^{1-q} \sum \mu(I_j) y_{I_j}^q - (\beta + 1)^q \int_{\cup I_j} \phi^q d\mu \right].
\]

**Proof.** We use the technique mentioned above in Theorem 3.1 by summing inequality \(3.2\) up to all \(I \in S_\phi\) with \(I \subseteq I_j\) for any \(j\). The rest details are easy to be verified. \(\square\)

We have now the following generalization of Theorem 3.1.
Corollary 3.1. Let $\phi$ be $T$-good and $A = \{I_j\}$ be a pairwise disjoint family of elements of $S_\phi$. Then for every $\beta > 0$
\[
\int_{X \setminus \bigcup I_j} (M_T \phi)^q d\mu \leq \frac{1}{(1-q)\beta} \left[ (\beta + 1) \left( \sum_{I \in A_n} \mu(I) y^q_{I,n} \right) - (\beta + 1)^q \int_{X \setminus \bigcup I_j} \phi^q d\mu \right].
\]

Proof. We choose a pairwise disjoint family $(J_i)_i \subseteq S_\phi$ such that the union $A \cup B$ is maximal under $\subseteq$ relation in $S_\phi$ and $I_j \cap J_i = \phi \forall i, \forall j$. Then we apply Theorem 3.1 for $A \cup B$, and Theorem 3.2 for $B$. We sum the two inequalities, and conclude Corollary 3.1. □

We have now the following

Theorem 3.3. Let $(\phi_n)_n$ be an extremal sequence consisting of $T$-good functions. Consider for every $n \in \mathbb{N}$ a pairwise disjoint family $A_n = \{I^n_j\}$ of elements of $S_{\phi_n}$ such that the following limit exists
\[
\lim_n \sum_{I \in A_n} \mu(I) y^q_{I,n}, \quad \text{where} \quad y_{I,n} = \text{Av}_I(\phi_n), \quad I \in A_n.
\]
Then
\[
\lim_n \int_{\bigcup A_n} (M_T \phi_n)^q d\mu = \omega_q(f^q/h) \lim_n \int_{A_n} \phi^q_n d\mu
\]
meaning that if one of the limits on the above relation exists then the other also does and we have the stated equality.

Proof. We apply Theorem 3.2 and Corollary 3.1 for $A_n$. Then we have the following inequalities
\[
\int_{\bigcup A_n} (M_T \phi_n)^q d\mu \leq \frac{1}{(1-q)\beta} \left[ (\beta + 1) \sum_{I \in A_n} \mu(I) y^q_{I,n} - (\beta + 1)^q \int_{\bigcup A_n} \phi^q_n \right],
\]
\[
\int_{X \setminus \bigcup A_n} (M_T \phi_n)^q d\mu \leq \frac{1}{(1-q)\beta} \left[ (\beta + 1) \left( f^q - \sum_{I \in A_n} \mu(I) y^q_{I,n} \right) - (\beta + 1)^q \int_{X \setminus \bigcup A_n} \phi^q_n \right].
\]

Summing (3.9) and (3.10) we conclude that
\[
\int_X (M_T \phi_n)^q d\mu \leq \frac{1}{(1-q)\beta} \left[ (\beta + 1) f^q - (\beta + 1)^q h \right], \quad \text{for any} \quad \beta > 0.
\]
But the last inequality is just (3.18) of [2] which for $\beta = \omega_q(g^q/h)^{1/q} - 1$ gives equality in the limit since $\phi_n$ is extremal. We set now $\ell_n = \sum_{I \in A_n} \mu(I) y^q_{I,n}$, and we suppose that $\lim \ell_n$ exists and equals $\ell \in \mathbb{R}^+$.

Because of the fact that we have equality in (3.11) and because of the inequalities stated in (3.9) and (3.10), we must have equality on both of them in the limit, for the value of $\beta$ that was mentioned above, supposing that $\ell_n \to \ell$. 
So from (3.9) we obtain

\[ \lim_n \int_{\cup A_n} (M_T \phi_n)^q d\mu \leq \frac{1}{(1-q)\beta} \left[ (\beta + 1)\ell - (\beta + 1)^q \lim_n \int_{\cup A_n} \phi_n^q d\mu \right], \]

with equality for \( \beta = \omega_q(f^q/h)^{1/q} - 1 \).

But as in [2], the right hand side of (3.12) is minimized for

\[ \beta = \omega_q \left( \ell / \lim_n \int_{\cup A_n} \phi_n^q \right)^{1/q} - 1. \]

Thus, we must have that

\[ \frac{f^q}{h} = \lim_n \frac{\ell}{\int_{\cup A_n} \phi_n^q} \Rightarrow \ell = \frac{f^q}{h} \lim_n \int_{\cup A_n} \phi_n^q. \]

Thus, from the opposite equality (3.12) we must have that

\[ \lim_n \int_{\cup A_n} (M_T \phi_n)^q d\mu = c_0 \lim_n \int_{\cup A_n} \phi_n^q, \]

with

\[ c_0 = \frac{1}{(1-q)\beta} \left[ (\beta + 1)\frac{f^q}{h} - (\beta + 1)^q \right], \quad \text{for} \quad \beta = \omega_q(f^q/h)^{1/q} - 1. \]

We prove now that \( c_0 = \omega_q(f^q/h) \). This is true since it is equivalent to

\[ \frac{f^q}{h} = (\beta + 1)^{q-1} [q + (\beta + 1)(1-q)] \Leftrightarrow q(\beta + 1)^{q-1} + (1-q)(\beta + 1)^q = f^q/h \]

which is true because of the definition of \( \omega_q(z) \), for \( z \geq 1 \).

Thus, Theorem 3.3 is proved. \( \square \)

We give now some notation

Let \( \phi \) be \( T \)-good. For each \( I \in S_\phi \) we consider the set \( A_I = A(\phi, I) \) is a union of elements of \( T \), because of the definition of tree \( T \) and Lemma 3.1 iv). Using now Lemma 2.1 we construct for each \( a \in (0,1) \) a pairwise disjoint family \( A_\phi^I \) of elements of \( T \) and subset

\[ \sum_{J \in A_\phi^I} \mu(J) = a\mu(A_I). \]
We define the following function $g_{\phi} : X \to \mathbb{R}^+$ in the following way. For each $I \in S_\phi$ we define:

\begin{equation}
\begin{aligned}
g_{\phi} &:= \gamma_I^\phi, \text{ on } A_I^I \\
&:= 0, \quad \text{on } A_I \setminus \cup A_I^I
\end{aligned}
\end{equation}

such that

\begin{equation}
\begin{aligned}
\int_{A_I} g_{\phi} d\mu &= \gamma_I^\phi \int_{A_I} \phi d\mu \quad \text{and} \\
\int_{A_I} g_{\phi}^q d\mu &= (\gamma_I^\phi)^q \int_{A_I} \phi^q d\mu,
\end{aligned}
\end{equation}

where $\gamma_I^\phi = \mu(\cup A_I^I) = a \mu(A_I)$.

It is easy to see that such chases of $\gamma_I^\phi$ and $\gamma_I^\phi$, for every $I \in S_\phi$ are possible.

In fact (3.16) give

\[
\gamma_I^\phi = \left[ \frac{\int_{A_I} \phi d\mu}{\int_{A_I} \phi^q d\mu} \right]^{1/(q-1)} \leq \mu(A_I), \quad \text{by Holder’s inequality}
\]

so we just need to set

\[
a = \frac{1}{\mu(A_I)} \left[ \frac{\int_{A_I} \phi d\mu}{\int_{A_I} \phi^q d\mu} \right]^{1/(q-1)}.
\]

Then, if $A_I^I$ is such that (3.14) is satisfied for this $a$, by setting $\gamma_I^\phi = a \mu(A_I)$ and

\[
e^\phi = \frac{A_I^I}{\gamma_I^\phi},
\]

we have that (3.16) are valid.

Since $\bigcup_{I \in S_\phi} A_I \approx X g_{\phi}$ is well defined on $X$.

It is obvious that $\int_X g_{\phi} d\mu = f$, and $\int_X g_{\phi}^q d\mu = h$. It is also easy to see that for every $I \in S_\phi$ it holds

\[
\mu(\{g_{\phi} = 0\} \cap A_I) \geq \mu(\{\phi = 0\} \cap A_I),
\]

and so as a consequence

\[
\mu(\{\phi = 0\}) \leq \mu(\{g_{\phi} = 0\}).
\]

Let now $(\phi_n)_n$ be an extremal sequence, consisting of $T$-good functions and let $g_n = g_{\phi_n}$. We prove the following.

**Lemma 3.1.** For an extremal $(\phi_n)_n$ sequence of $T$-good functions we have that $\lim_n \mu(\{\phi_n = 0\}) = 0$.

Before we proceed to the proof of the above lemma we prove the following.
Lemma 3.2. For an extremal sequence \((\phi_n)_n\) consisting of \(T\)-good functions with respective tree subtree \(S_{\phi_n}\) the following holds

\[
\lim_{n} \sum_{I \in S_{\phi_n}} \left( \frac{\int A(\phi_n,I) \phi d\mu}{a_{I,n}^q} \right)^q = h, \quad \text{where}
\]

\[
a_{I,n} = \mu(A(\phi_n,I)), \quad \text{for every } n \in \mathbb{N}, \quad I \in S_{\phi_n}.
\]

Proof. This is true because in the proof of Lemma 6 in \([2]\) the following inequality was used in order to pass from (3.16) to (3.17) in \([2]\):

\[
\sum a_I x_I^q \geq \int_X \phi^q d\mu = h \quad \text{in view of Holder’s inequality.}
\]

Since \((\phi_n)_n\) is extremal we must have equalities in the limit in all inequalities of the above mentioned Lemma 6 (for the specific value of \(\beta\)) and so by definition of \(x_I, a_I\) we conclude Lemma 3.2 as it is stated above.

We now have

Proof of Lemma 3.1. Let \(\phi\) runs along \((\phi_n)_n\). We prove that

\[
\lim_{\phi} \mu(\{\phi = 0\}) = 0.
\]

For this it is enough to prove that

\[
\lim_{\phi} \mu(\{g_\phi = 0\}) = 0.
\]

This is exactly \(\lim_{\phi} \sum_{I \in S_\phi} (a_I - \gamma_I) = 0\) where \(\gamma_I = \gamma_I^\phi\) as above, and \(a_I = \mu(A(\phi,I))\), for \(I \in S_\phi\).

For \(I \in S_\phi\) we set

\[
P_I = \frac{\int A_I \phi d\mu}{a_I^{q-1}}, \quad A_I = A(\phi,I).
\]

Then, obviously \(\sum_{I \in S_\phi} a_I^{q-1} P_I = h\).

Additionally, \(\sum_{I \in S_\phi} \gamma_I^{q-1} P_I \geq h\) since \(q < 1\) and \(\gamma_I \leq a_I\) for every \(I \in S_p\).

However,

\[
\sum_{I \in S_\phi} \gamma_I^{q-1} P_I = \sum_{I \in S_\phi} \gamma_I^{q-1} \frac{c_I^q \gamma_I}{a_I^q} = \sum_{I \in S_\phi} \frac{(\gamma_I c_I)^q}{a_I^{q-1}}
\]

\[
= \sum_{I \in S_\phi} \frac{\left( \frac{\int A_I \phi}{a_I^q} \right)^q}{\phi} \approx h, \quad \text{by Lemma 3.2}
\]

We define now for any \(R > 0\)

\[
S_{\phi,R} = \bigcup \{A_I : I \in S_\phi, \ P_I < Ra_I^{2-q} \}.
\]
Then for any $I \in S_{\phi}$, such that $P_I < Ra_I^{2-q}$ we have that

\[
\left( \int_{A_I} \phi^q \right) a_I^{1-q} < Ra_I^{2-q} \Rightarrow \\
\int_{A_I} \phi^q < Ra_I \Rightarrow \text{(summing up to all such } I) \\
\int_{S_{\phi,R}} \phi^q < R\mu(S_{\phi,R}).
\]

Additionally, we have that

\[
\left| \sum_{I \in S_{\phi}} a_I^{q-1} P_I - h \right| = \int_{S_{\phi,R}} \phi^q d\mu,
\]

\[
\left| \sum_{I \in S_{\phi}} \gamma_I^{q-1} P_I - h \right| \approx \sum_{I \in S_{\phi}} \gamma_I^{q-1} P_I - \sum_{I \in S_{\phi}} \gamma_I^{q-1} P_I \\
= \sum_{I \in S_{\phi}} \gamma_I^{q-1} P_I = \sum_{I \in S_{\phi}} \gamma_I^{q-1} c_I^{q-1} = \sum_{I \in S_{\phi}} \frac{(\gamma_I c_I)^q}{a_I^{q-1}}
\]

\[
\left( \int_{A_I} \phi \right)^q a_I^{q-1} \approx \int_{S_{\phi,R}} \phi^q d\mu,
\]

the same reasons mentioned in \((3.17)\).

Thus, from \((3.19)\) and \((3.20)\) we conclude that

\[
\limsup_{\phi} \sum_{I \in S_{\phi}} (\gamma_I^{q-1} - a_I^{q-1}) P_I \leq 2 \lim_{\phi} \int_{S_{\phi,R}} \phi^q d\mu,
\]

By Theorem 3.3 now we have, assuming that $\lim_{\phi} \int_{S_{\phi,R}} \phi^q d\mu$ exists, (we can always consider it as given, by passing if necessary to a subsequence), that

\[
\lim_{\phi} \int_{S_{\phi,R}} (\mathcal{M}_T \phi)^q d\mu = \omega_q(f^q/h) \lim_{\phi} \int_{S_{\phi,R}} \phi^q d\mu.
\]

Since $\mathcal{M}_T \phi \geq f$ on , we obtain from \((3.18)\)

\[
f^q \limsup_{\phi} [\mu(S_{\phi,R})] \leq \omega_p(f^q/h) R \limsup_{\phi} [\mu(S_{\phi,R})],
\]

Thus if $R > 0$ is chosen small enough we must have because of \((3.22)\) that

\[
\lim_{\phi} \mu(S_{\phi,R}) = 0.
\]
From (3.18) we conclude now that \( \lim_{\phi} \int_{S_{\phi, R}} \phi d\mu = 0 \), so from (3.21) we see that for this \( R \):

\[
\lim_{\phi} \sum_{I \in S_{\phi}} (\gamma_I^{q-1} - a_I^{q-1})P_I = 0.
\]

We consider now for any \( y > 0 \) the function

\[
\phi_y(x) = \frac{x^{q-1}y^{2-q} - y}{y - x},
\]

defined on \((0, y)\). It is easy to see that \( \lim_{x \to 0^+} \phi_y(x) = +\infty \) and \( \lim_{x \to y^-} \phi_y(x) = 1 - q \).

Additionally

\[
\phi'_y(x) = \frac{(q-1)x^{q-2}y^{3-q} - (q-2)x^{q-1}y^{2-q} - y}{(y-x)^2}, \quad x \in (0, y),
\]

so by setting \( x = \lambda y, \lambda \in (0, 1) \) we see that if \( g \) is defined on \((0, 1)\) by:

\[
g(\lambda) = (q-1)\lambda^{q-2} - (q-2)\lambda^{q-1} - 1,
\]

then

\[
g(\lambda) < 0, \quad \forall \lambda \in (0, 1) \Rightarrow \phi'_y(x) < 0, \quad \forall x \in (0, y) \Rightarrow \phi_y \downarrow \text{ on } (0, y).
\]

Thus,

\[
\phi_y(x) \geq 1 - q, \quad \forall x \in (0, y) \Rightarrow x^{q-1}y^{2-q} - y \geq (1 - q)(y - x), \quad \text{for any } 0 < x < y.
\]

From the above we see that

\[
\lim_{\phi} \sum_{I \in S_{\phi}} (a_I - \gamma_I) = 0.
\]

\[
\Rightarrow 1 - \mu(S_{\phi, R}) - \sum_{I \in S_{\phi}} \gamma_I \phi \approx 0 \Rightarrow (\text{since } \mu(S_{\phi, R}) \Rightarrow 0) \sum_{P_I \geq Ra_I^{2-q}} \gamma_I \phi \approx 1.
\]

But

\[
1 \geq \sum_{I \in S_{\phi}} \gamma_I \geq \sum_{P_I \geq Ra_I^{2-q}} \gamma_I \Rightarrow \lim_{\phi} \sum_{I \in S_{\phi}} \gamma_I = 1 = \sum_{I \in S_{\phi}} a_I \Rightarrow
\]

\[
\sum_{I \in S_{\phi}} (a_I - \gamma_I) \phi \approx 0 \Rightarrow \mu(\{g_\phi = 0\}) \phi \approx 0 \Rightarrow \mu(\{\phi = 0\}) \phi \approx 0
\]

and this is the end of the proof of Lemma 3.1.

Suppose now that \((\phi_n)_n\) is extremal. We remind that for every \( \phi \in \{\phi_n, \ n = 1, 2, \ldots\} \) we have defined \( g_\phi : X \to \mathbb{R}^+ \) by the following

\[
g_\phi(t) := c_I^\phi, \quad t \in A_I^\phi
\]

\[
: = 0, \quad t \in (A_I \setminus A_I^\phi), \quad I \in S_{\phi} \quad \text{with}
\]
\[ \gamma_I = \gamma^\phi_I = \mu(A^I \phi) \quad \text{and} \quad \mu(\{g_\phi = 0\}) = \sum_{I \in S_\phi} (a_I - \gamma_I) \xrightarrow{\phi} 0. \]

\( a_I = \mu(A_I) \), where \( A_I = A(\phi, I) \).

So, if we define \( g_\phi^I : \mathbb{R}^+ \rightarrow \mathbb{R} \) by \( g_\phi^I(t) = c_I^\phi, \ t \in A_I \) for \( I \in S_\phi \) we easily get that:

\[ \lim_{\phi} \int_X g_\phi^I d\mu = f, \quad \lim_{\phi} \int_X (g_\phi^I)^q d\mu = h \quad \text{and} \quad \lim_{\phi} \int_X |g_\phi^I - g_\phi^I| d\mu = 0. \]

Additionally, because of \( \int_{A_I} g_\phi d\mu = \int \phi d\mu, \ I \in S_\phi \), \( I \approx \bigcup_{J \in I} A(\phi, J) \) we have that for every \( I \in S_\phi \)

\[ \text{Av}_I(\phi) = \text{Av}_I(g_\phi). \]

Thus \( M_T g_\phi \geq M_T \phi \) on \( X \Rightarrow \lim_{\phi} \int_X (M_T g_\phi)^q d\mu = h \omega_\phi(f^q/h). \)

Since

\[ \int_X g_\phi d\mu = f \quad \text{and} \quad \int_X g_\phi^I d\mu = h \]

we obtain that \( g_\phi \) is an extremal sequence. We now prove the following lemmas needed for the end of the proof of our characterization of the extremal sequences.

**Lemma 3.3.** With the above notation

\[ \lim_{\phi} \int_X |M_T g_\phi - cg_\phi|^q d\mu = 0. \]

**Proof.** We recall that \( c = \omega_\phi(f^q/h)^{1/q} \). We set for each \( \phi \in \{\phi_n, n = 1, 2, \ldots\} \)

\[ \Delta_\phi = \{t \in X : M_T g_\phi(t) \geq c g_\phi(t)\} \]

We consider now for every \( I \in S_\phi \) the set \( \Delta_\phi \cap A_I \). Since \( g_\phi \) is either \( c_I^\phi \) or 0 on \( A_I \) and \( M_{\Delta_\phi}(t) \geq f > 0, \forall \ t \in X \) we conclude that \( \Delta_\phi \cap A_I = A_I \) or \( \Delta_\phi \cap A_I = A_I \setminus A^I_\phi \).

We remind that \( \sum_{I \in S_\phi} (a_I - \gamma_I) \xrightarrow{\phi} 0 \) or \( \sum_{I \in S_\phi} \mu(A_I \setminus A^I_\phi) \xrightarrow{\phi} 0. \)

Since \( \bigcup_{I \in S_\phi} \approx X \) we have that

\[ \Delta_\phi = \left( \bigcup_{I \in S_{1, \phi}} A_I \right) \cup (E_\phi), \quad \text{where} \quad \mu(E_\phi) \xrightarrow{\phi} 0 \]

and \( S_{1, \phi} \) is a subset of the subtree \( S_\phi \).

According to the same reasons mentioned before we have, by passing if necessary to a subsequence, that

\[ \lim_{\phi} \int_{\bigcup_{I \in S_{1, \phi}} A_I} (M_T \phi)^q d\mu = \omega_\phi(f^q/h) \cdot \lim_{\phi} \int_{\bigcup_{I \in S_{1, \phi}} \phi^q d\mu}, \quad \text{so since} \]

\[ \mu(E_\phi) \rightarrow 0 : \lim_{\phi} \int_{\Delta_\phi} (M_T \phi)^q d\mu = \omega_\phi(f^q/h) \lim_{\phi} \int_{\Delta_\phi} \phi^q d\mu. \]
Because of (3.16), the consequence that 
\[ A_{v_I}(\phi) = A_{v_I}(g_\phi) \]
for every \( I \in S_\phi \) and Lemma 3.1 iii), we obtain that (passing again to a subsequence if necessary) 
\[
(3.23) \quad \lim_{\phi} \int_{\Delta_\phi} (\mathcal{M}_T g_\phi)^q d\mu \geq \omega_q(f^q/h) \lim_{\phi} \int_{\Delta_\phi} g_\phi^q d\mu.
\]
In the same way we have that 
\[
(3.24) \quad \lim_{\phi} \int_{X \setminus D_{e_\phi}} (\mathcal{M}_T g_\phi)^q d\mu \geq \omega_q(f^q/h) \lim_{\phi} \int_{X \setminus \Delta_\phi} g_\phi^q d\mu.
\]
Adding the last two relations one obtain \( \lim_{\phi} \int_X (\mathcal{M}_T \phi)^p d\mu \geq \omega_q(f^q/h)h \), which is in fact equality since \( g_\phi \) is an extremal sequence.

So, we must have equality in both (3.23) and (3.24). Since \( \mathcal{M}_T g_\phi \geq \omega_q(f^q/h)^{1/q} g_\phi \) on \( \Delta_\phi \), we obtain easily from the above relations that 
\[
\lim_{\phi} \int_X |\mathcal{M}_T g_\phi - cg_\phi|^q d\mu = 0
\]
and Lemma 3.3 is now proved in view of the following lemma. \( \square \)

**Lemma 3.4.** Let \( q \in (0, 1) \) and \( g_n, w_n, g, w \) non-negative functions on \( (X, \mu) \) such that 
\( g_n^q = w_n, \ g^q = w \) and \( g_n \geq g \) on \( X \).

Suppose also that \( \int_X w_n d\mu \to \int_X w d\mu \). Then
\[
I = \int_X (g_n - g)^q d\mu \to 0, \quad \text{as} \quad n \to \infty.
\]

**Proof.** We have that 
\[
I = \int_X (w_n^{1/q} - w^{1/q})^q d\mu.
\]
But, for every \( p > 1 \), \( x^p - y^p \leq p(x - y)x^{p-1} \), for every \( x > y > 0 \). Thus, for \( p = 1/q \)
\[
w_n^p - w^p \leq p(w_n - w)w_n^{p-1} \Rightarrow \int_X (w_n^{1/q} - w^{1/q})^q d\mu
\]
\[
\leq \left( \frac{1}{q} \right)^q \cdot \int_X (w_n - w)^q \cdot w_n^{1-q} d\mu
\]
\[
\leq \left( \frac{1}{q} \right)^q \left[ \int_X (w_n - w) \right]^q \cdot \left[ \int_X w_n \right]^{1-q}
\]
in view of the hypothesis of Lemma 3.4 which is now proved. \( \square \)

We proceed now to the next

**Lemma 3.5.** With the above notation
\[
\lim_{\phi} \int_X |g_\phi' - \phi|^q d\mu = 0.
\]
Proof. We are going to use the following inequality \( t + \frac{1 - q}{q} \geq \frac{t^q}{q} \), which holds for every \( t > 0 \) and \( q \in (0, 1) \). We have equality in the above if and only if \( t = 1 \). In view of Lemma 3.4 we just need to prove that

\[
\int_{\phi \geq g_\phi^q} [\phi^q - (g_\phi^q)^q] d\mu \rightarrow 0 \quad \text{and} \quad \int_{\{g_\phi^q > \phi\}} [(g_\phi^q)^q - \phi^q] d\mu \rightarrow 0.
\]

We proceed to this as follows:

We set for every \( I \in S_\phi \)

\[
\Delta_{I, \phi}^{(1)} = \{ g_\phi' \leq \phi \} \cap A(\phi, I),
\]

\[
\Delta_{I, \phi}^{(2)} = \{ \phi < g_\phi' \} \cap A(\phi, I).
\]

We remind that \( g_\phi'(t) := c_{I, \phi} \phi \) for \( t \in A_I = A(\phi, I) \). From the above inequality we see that if \( c_{I, \phi} > 0 \) then

\[
\frac{\phi(x)}{c_{I, \phi}} + \frac{1 - q}{q} \geq \frac{1}{q} \frac{\phi^q(x)}{c_{I, q}^q}, \quad \forall x \in A_I,
\]

so integrating over every \( \Delta_{I, \phi}^{(j)} \), \( j = 1, 2 \) we have that

\[
\frac{1}{c_{I, \phi}} \int_{\Delta_{I, \phi}^{(j)}} \phi d\mu + \frac{1 - q}{q} \mu(\Delta_{I, \phi}^{(j)}) \geq \frac{1}{q} \int_{\Delta_{I, \phi}^{(j)}} \phi^q d\mu \Rightarrow
\]

\[
\sum_{I \in S'_{\phi}} c_{I, \phi}^{q-1} \int_{\Delta_{I, \phi}^{(j)}} \phi d\mu + \frac{1 - q}{q} \sum_{I \in S'_{\phi}} \mu(\Delta_{I, \phi}^{(j)}) c_{I, \phi}^q \geq \frac{1}{q} \int_{\bigcup_{I \in S'_{\phi}} \Delta_{I, \phi}^{(j)}} \phi^q d\mu,
\]

where \( S'_{\phi} = \{ I \in S_\phi : c_{I, \phi} > 0 \} \).

From the definition of \( g_\phi' \) we see that (3.25) gives:

\[
\int_{\bigcup_{I \in S'_{\phi}} \Delta_{I, \phi}^{(j)}} (g_\phi^q)^{q-1} \phi d\mu + \frac{1 - q}{q} \sum_{I \in S'_{\phi}} c_{I, \phi}^q \mu(\Delta_{I, \phi}^{(j)}) \geq \frac{1}{q} \int_{\bigcup_{I \in S'_{\phi}} \Delta_{I, \phi}^{(j)}} \phi^q d\mu, \quad \text{for } j = 1, 2.
\]

But

\[
\sum_{I \in S'_{\phi}} c_{I, \phi}^q \mu(\Delta_{I, \phi}^{(j)}) = \begin{cases} \int_{\{\phi \geq g_\phi^q\}} (g_\phi^q)^q d\mu, & j = 1 \\ \int_{\{\phi < g_\phi^q\}} (g_\phi^q)^q d\mu, & j = 2 \end{cases}
\]

and

\[
\frac{1}{q} \int_{\bigcup_{I \in S'_{\phi}} \Delta_{I, \phi}^{(j)}} \phi^q d\mu = \begin{cases} \frac{1}{q} \int_{\{\phi \geq g_\phi^q\}} \phi^q d\mu, & j = 1 \\ \frac{1}{q} \int_{\{\phi < g_\phi^q\}} \phi^q d\mu, & j = 2, \end{cases}
\]
because if \( c_{I, \phi} = 0 \) for some \( I \in S_\phi \) then \( \phi = 0 \) on the respective \( A_I = A(\phi, I) \), and conversely. Additionally:

\[
\int \bigcup_{I \in S'} \Delta_{j, \phi} (g'_{\phi})^{q-1} \phi \, d\mu = \begin{cases} 
\int_{\{\phi < g'_\phi \leq 0\}} (g'_\phi)^{q-1} \phi \, d\mu, & j = 1 \\
\int_{\{\phi \leq g'_\phi \}} (g'_\phi)^{q-1} \phi \, d\mu, & j = 2
\end{cases}
\]

So that we have the two inequalities:

\[(3.27) \quad \int_{0 < g'_\phi \leq \phi} (g'_\phi)^{q-1} \phi \, d\mu + \frac{1 - q}{q} \int_{g'_\phi \leq \phi} (g'_\phi)^{q} \, d\mu \geq \frac{1}{q} \int_{g'_\phi \leq \phi} \phi^q \, d\mu \quad \text{and} \quad \int_{g'_\phi > \phi} (g'_\phi)^{q-1} \phi \, d\mu \geq \frac{1}{q} \int_{g'_\phi > \phi} \phi^q \, d\mu.
\]

If we sum the above inequalities, we obtain:

\[
\sum_{I \in S'} c_{I, \phi} (g'_\phi)^{q-1} \, d\mu + \frac{1 - q}{q} \int_X (g'_\phi)^{q} \, d\mu \geq \frac{1}{q} \int_X \phi^q \, d\mu
\]

which is equality in view of the facts that

\[
\int_X \phi^q \, d\mu = h, \quad \int_X (g'_\phi)^q \, d\mu \xrightarrow{\phi} h \quad \text{and} \quad \sum_{I \in S'} \gamma_{I, \phi} = \int_X (g'_\phi)^q \, d\mu.
\]

So, we must have equality in both (3.27) and (3.28) above.

As a consequence if we set \( t_\phi = \int_{g'_\phi \leq \phi} \phi \, d\mu, S_\phi = \int_{g'_\phi \leq \phi} (g'_\phi)^q \, d\mu \) we must have that

\[
\int_{\phi \geq g'_\phi > 0} \phi (g'_\phi)^{q-1} + \frac{1 - q}{q} S_\phi \xrightarrow{\phi} \frac{1}{q} t_\phi \quad \text{in the limit}
\]

But as it can be easily seen

\[
\left[ \int_{0 < g'_\phi \leq \phi} \phi (g'_\phi)^{q-1} \, d\mu \right]^q \cdot \left[ \int_{0 < g'_\phi \leq \phi} (g'_\phi)^q \, d\mu \right]^{1-q} \geq \int_{0 < g'_\phi \leq \phi} \phi^q
\]

thus we have that

\[
\frac{t_\phi^{1/q}}{S_\phi^{(1/q)-1}} + \frac{1 - q}{q} S_\phi \leq \frac{1}{q} t_\phi \Rightarrow \left( \frac{t_\phi}{S_\phi} \right)^{1/q} + \frac{1 - q}{q} \leq \frac{1}{q} \left( \frac{t_\phi}{S_\phi} \right).
\]

So by setting \( \left( \frac{t_\phi}{S_\phi} \right)^{1/q} = k_\phi \), we must have that \( k_\phi + \frac{1 - q}{q} = \frac{1}{q} k_\phi^q \) in view of our elementary inequality.

Then \( k_\phi \approx 1 \) and since \( t_\phi, S_\phi \) are bounded

\[
t_\phi - S_\phi \xrightarrow{\phi} 0 \Rightarrow \int_{g'_\phi \leq \phi} [\phi^q - (g'_\phi)^q] \, d\mu \xrightarrow{\phi} 0.
\]
In a similar way we prove that
\[
\int_{\{\phi < g_{\phi}'\}} [(g_{\phi}')^q - (\phi)^q]d\mu \to 0.
\]
So Lemma 3.5 is proved. \(\square\)

We now prove the last Lemma 3.6.

**Lemma 3.6.** With the above notation we have that
\[
\lim_{\phi} \int_X |M_T \phi - c\phi|^q d\mu = 0.
\]

**Proof.** We set
\[
I = \int_X |M_T \phi - c\phi|^q d\mu.
\]
It is true that \((x + y)^q < x^q + y^q\), whenever \(x, y > 0, q \in (0, 1)\). Thus
\[
I \leq \int_X |M_T \phi - M_T g_{\phi}|^q d\mu + \int_X |M_T g_{\phi} - c g_{\phi}|^q d\mu + c^q \int_X |g_{\phi} - \phi|^q d\mu = II_1 + II_2 + II_3.
\]
But \(II_3 = \int_X |g_{\phi} - \phi| d\mu \approx 0\) since Lemma 3.5 is true and by construction it is easily seen that \(\int_X |g_{\phi}' - g_{\phi}|^q d\mu \approx 0\).

The last limit holds by the definition of \(g_{\phi}, g_{\phi}'\) and Lemma 3.1 in the form \(\lim \mu(\{g_{\phi} = 0\}) = 0\).

Additionally, by Lemma 3.3
\[
II_2 = \int_X |M_T g_{\phi} - c g_{\phi}|^q d\mu \approx 0.
\]

We give our attention now to
\[
I_1 = \int_X |M_T \phi - M_T g_{\phi}|^q d\mu.
\]
But as we mentioned before \(M_T g_{\phi} \geq M_T \phi\) in \(X\), thus
\[
I_1 = \int_X (M_T g_{\phi} - M_T \phi)^q d\mu.
\]
Since
\[
\lim_{\phi} \int_X (M_T g_{\phi})^q d\mu = \lim_{\phi} \int_X (M_T \phi)^q d\mu = h \omega_q(f^q/h),
\]
we see immediately from Lemma 3.4 that \(I_1 \to 0\).

It is thus proved the following:

**Theorem 3.4.** Let \((\phi_n)\) be a sequence of non-negative \(T\)-good functions on \((X, \mu)\) such that for every \(n \in \mathbb{N}\), \(\int_X \phi_n d\mu = f\) and \(\int_X \phi_n^q d\mu = h\). Then the following are equivalent
i) \(\phi_n\) is extremal for (1.6)
ii) \(\lim_{n} \int_X |M_T \phi_n - c\phi_n|^q d\mu = 0\), for \(c = \omega_q(f^q/h)^{1/q}\).
Remark 3.1. We need to mention that the above theorem holds true on $\mathbb{R}^n$ without the hypothesis that the sequence $(\phi_n)_n$ consists of $\mathcal{T}$-good functions. This is true since in the case of $\mathbb{R}^n$, where $\mathcal{T}$ is the usual tree of dyadic subcubes of a fixed cube $Q$, the $\mathcal{T}$-good functions are exactly the dyadic step functions on $Q$ which are dense on $L^1(X, \mu)$.

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