CHARACTERISTIC CLASSES FOR TC STRUCTURES

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Abstract. In this article we study the construction of characteristic classes for principal $G$-bundles equipped with an additional structure called transitionally commutative structure (TC structure). These structures classify, up to homotopy, possible trivializations of a principal $G$-bundle, such that the induced cocycle have functions that commute in the intersections of their domains. We focus mainly on the cases where the structural group $G$ equals SU(n), U(n) or Sp(n). Our approach is an algebraic-geometric construction that relies on the so called power maps defined on the space $B_{\text{com}}G$, the classifying space for commutativity in the group $G$.

1. Introduction

Suppose that $G$ is a Lie group and consider the set of $n$-tuples with commuting elements which can be identified with $\text{Hom}(\mathbb{Z}^n, G)$. Adem, Cohen and Torres-Giese (see [ACT]) showed that $\{\text{Hom}(\mathbb{Z}^n, G)\}_{n \geq 0}$ can be endowed with a simplicial structure whose geometric realization is denoted by $B_{\text{com}}G$. Additionally, consider a principle $G$-bundle over a compact Hausdorff space $M$, with a classifying function $g: M \to BG$ and trivializations with associated cocycle $\{\rho_{ij}\}$. Suppose then that the cocycles commute with each other in the intersection of their domains, i.e. $\rho_{ij} \cdot \rho_{jk} = \rho_{jk} \rho_{ij}$. Adem and Gomez showed in [AG] that the previous commutativity condition holds if only if there is a lifting, up to homotopy of the classifying map $g$; that is a commutative diagram up to homotopy as the one shown below,

$$
\begin{array}{ccc}
M & \xrightarrow{g} & BG \\
\downarrow f & & \downarrow i \\
B_{\text{com}}G & \xrightarrow{} & \\
\end{array}
$$

The existence of such lifting is what we call a transitionally commutative (TC) structure on a principal $G$-bundle. Where we say that two TC structures $f_1, f_2: M \to B_{\text{com}}G$ are equivalent if the functions are homotopic. TC structures are meant to classify the different ways a principle bundle can have commutative cocycles, up to homotopy.

The interest in studying the spaces $\text{Hom}(\mathbb{Z}^n, G)$ arises from the study of moduli spaces of flat bundles, which are important for Quantum field theories such as the Yang-Mills and Chern- Simons theories. In particular, when the base space is the torus $(S^1)^n$ and the structural group is a compact Lie group $G$, the moduli

The author was sponsored by the Colombian Ministry of Sciences (previously Colciencias) under the public sponsorship act 647 of 2014 for national doctoral programs. Where the results presented here are part of his PhD thesis, supervised by José Manuel Gómez Guerra at the National University of Colombia at Medellín.
spaces of flat bundles can be identified with $\text{Hom} (\mathbb{Z}^n, G) / G$, where $G$ acts under conjugation.

Mathematically speaking, the theory of commuting tuples is interesting in its own right. For example, Adem and Gomez defined in [AG] the commutative $K$-theory of a finite CW-complex $X$ to be $K_{\text{com}} X := Gr (\text{Vect}_{\text{com}} (X))$, where $Gr$ denotes the Grothendieck construction and $\text{Vect}_{\text{com}} (X)$ is the set of equivalence classes of vector bundles over $X$ with commuting cocycles. Later on Adem, Gómez, Lind and Tillman introduce the notion of $q$-nilpotent $K$-theory of a CW-complex $X$ for any $q \geq 2$, which extends the notion of commutative $K$-theory defined by Adem and Gomez, and show that it is represented by $\mathbb{Z} \times B(q, U)$, where $B(q, U)$ is the $q$-th term of a filtration of the infinite loop space $BU$. (See [AGLT].)

In this article we define and develop characteristic classes for TC structures and mainly, we develop an algebraic-geometric method to use Chern-Weil theory to compute them. To start we will see that there is a one to one correspondence between characteristic classes and elements of $H^* (B_{\text{com}} G, \mathbb{R})$. We then use the description of $H^* (B_{\text{com}} G, \mathbb{R})$ given in [AG] for which we exhibit a set of algebraic generators. Then we show how we can use Chern-Weil theory to compute the characteristic classes associated to those generators. We do this for $G$ equal to either $U(n)$, $SU(n)$ or $\text{Sp}(n)$ for the following reasons.

In general the spaces $\text{Hom} (\mathbb{Z}^n, G)$ are not path connected, so the simplicial construction can be reduced to consider the path connected components containing the identity tuple $(1,1,\ldots,1)$. These connected components are denoted by $\text{Hom} (\mathbb{Z}^n, G)_1$, and the geometric realization of them is denoted by $B_{\text{com}} G_1$. Adem and Gomez showed in Proposition 7.1 of [AG] that the cohomology with real coefficients of $B_{\text{com}} G_1$ is isomorphic to

$$ (H^* (BT, \mathbb{R}) \otimes H^* (BT, \mathbb{R}))^W / J , $$

where $T \subset G$ is a maximal torus, $W$ is the Weil group acting diagonally, and $J$ is the ideal generated by elements of the form $p(x) \otimes 1$ with $p(x)$ an $W$-invariant polynomial of positive degree.

Additionally, Adem and Cohen showed in Corollary 2.4 of [AC] that $\text{Hom} (\mathbb{Z}^n, G)$ is path connected when $G$ is either $U(n)$, $SU(n)$ or $\text{Sp}(n)$. For them then Expression (1.1) describes the cohomology of all $B_{\text{com}} G$. To obtain the generators of this cohomology, we first consider the natural inclusion $\text{Hom} (\mathbb{Z}^n, G) \subseteq \mathbb{G}$. The inclusion induces a simplicial map between the simplicial structure of $\{ \text{Hom} (\mathbb{Z}^n, G) \}_{n \geq 0}$ and the bar construction for the classifying space of $G$, $BG$. This in turn gives rise to a map

$$ \iota : H^* (BG, \mathbb{R}) \to H^* (B_{\text{com}} G, \mathbb{R}) . $$

Secondly, we need to consider the assignments

$$ \text{Hom} (\mathbb{Z}^n, G) \to \text{Hom} (\mathbb{Z}^n, G) $$

$$ (g_1, \ldots, g_n) \mapsto (g_1^k, \ldots, g_n^k) . $$

These assignments give rise to simplicial maps, allowing us to obtain the power maps

$$ \Phi^k : H^* (B_{\text{com}} G, \mathbb{R}) \to H^* (B_{\text{com}} G, \mathbb{R}) $$

when $k \in \mathbb{Z}$. By using characterizations of the cohomology rings as well as the effect of these maps on them, we then use the particularities of the action of the Weil group for $U(n)$, $SU(n)$ and $\text{Sp}(n)$ to show that
Theorem. For $G$ equal to $U(n)$, $SU(n)$ or $Sp(n)$ then $H^*(B_{com}G)$ is generated as an algebra by

$$\{ \Phi^k(\text{Im}) \mid k \in \mathbb{Z} \setminus \{0\} \}.$$ 

In order to compute the TC characteristic class associated to a generator of the form $\Phi^k(\varepsilon(s)) \in H^*(B_{com}G, \mathbb{R})$, $s \in H^*(BG, \mathbb{R})$ and $k \in \mathbb{Z} \setminus \{0\}$, we develop another construction. For a TC structure $f : M \to B_{com}G$ over a principal $G$-bundle $E \to M$, we construct a family of principal $G$-bundles $E^k \to M$, called the $k$-th associated bundles. Then we prove that if $\Omega_k$ is the curvature of $E^k$ we have the equality

$$f^*(\Phi^k(\varepsilon(s))) = s(\Omega_k) \in H^*(M, \mathbb{R}),$$

where $s(\Omega_k)$ is the characteristic class of $E^k \to M$ associated to $s$, which is computed using Chern-Weil theory. This lets us obtain our main result:

Theorem. (Chern-Weil theory for TC structures) Consider $\varepsilon \in [M, B_{com}G]$ an equivalence class with an underlying smooth vector bundle $E \to M$, and structure group $U(n)$, $SU(n)$ or $Sp(n)$. Also let $\Omega_k$ be the curvature of $E^k$, the $k$-th associated bundle of $E$. Then every TC characteristic class can be obtained as a linear combination of products of the form

$$s_1(\Omega_{k_1}) \cdot s_2(\Omega_{k_2}) \cdots s_m(\Omega_{k_m}) \in H^*(M, \mathbb{R}),$$

where $s_i \in H^*(BG)$ and $k_i \in \mathbb{Z}$. Each $s_i(\Omega_{k_i})$ is the characteristic class of the vector bundle $E^k \to M$ computed using its curvature.

The outline of this article is as follows: in Section 2 we explain the construction of $B_{com}G$ and define TC structures, TC characteristic classes, power maps and $k$-th associated bundles. We also show how these concepts relate to each other. In Section 3 we show the effect of the power maps in the cohomology with real coefficients of $B_{com}G$. In Section 4 we obtain the generators for $H^*(B_{com}G, \mathbb{R})$ for $G$ equal to $U(n)$, $SU(n)$ or $Sp(n)$. In Section 5 we develop the Chern-Weil theory for TC characteristic classes. Finally in Section 6 we show an example of a computation of a TC characteristic class using a TC structure developed by Ramras and Villareal (see [RV]).

2. TC STRUCTURES AND $B_{com}G$

In this section we introduce all the basic concepts we are using that are related to commuting tuples in a Lie Group.

2.1. Simplicial construction for $B_{com}G$: We first start with a basic description of the simplicial structure used to define the space $B_{com}G$. This will allow us to obtain the generators for its cohomology with real coefficients.

Let us define a simplicial space whose $n$-th level is given by $\text{Hom}(\mathbb{Z}^n, G)$, which is the subspace of $G^n$ consisting of all commuting $n$-tuples. This is $(g_1, \ldots, g_n)$ such that $g_ig_j = g_jg_i$ for every $1 \leq i, j \leq n$. Its face maps $\delta_i : \text{Hom}(\mathbb{Z}^n, G) \to \text{Hom}(\mathbb{Z}^{n-1}, G)$ are given by

$$\delta_i(g_1, \ldots, g_n) := \begin{cases} 
(g_2, \ldots, g_n) & i = 0, \\
(g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_n) & 1 \leq i \leq n-1, \\
(g_1, \ldots, g_{n-1}) & i = n.
\end{cases}$$
and the degeneracy maps $s_i: \text{Hom}(\mathbb{Z}^n, G) \to \text{Hom}(\mathbb{Z}^{n+1}, G)$ are given by
$$s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n).$$

It is routine to see that these maps satisfy the simplicial identities.

**Definition 1.** The space $B_{\text{com}}G$ is defined as the fat realization of the simplicial space $\{\text{Hom}(\mathbb{Z}^n, G)\}_{n \geq 0}$, that is
$$B_{\text{com}}G := \|\text{Hom}(\mathbb{Z}^\bullet, G)\|.$$ It is also important to mention that for this construction the fat realization is homotopy equivalent to the geometrical realization as the simplicial space $\text{Hom}(\mathbb{Z}^\bullet, G)$ is proper. (See the appendix of [AC].)

In general, $\text{Hom}(\mathbb{Z}^m, G)$ is not connected. The path connected component of $\text{Hom}(\mathbb{Z}^m, G)$ containing the element $(1, 1, \ldots, 1)$ is denoted by $\text{Hom}(\mathbb{Z}^m, G)_1$. We can restrict the face and degeneracy maps to obtain a simplicial space $\text{Hom}(\mathbb{Z}^\bullet, G)_1$ whose fat realization is denoted by $B_{\text{com}}G_1$. However Adem and Cohen showed in Corollary 2.4 of [AC] that $\text{Hom}(\mathbb{Z}^m, G)$ is path connected when $G$ is either $U(n)$, $SU(n)$ or $Sp(n)$. So for these groups we have the equality $B_{\text{com}}G_1 = B_{\text{com}}G.$

2.2. **Power Maps:** For each $k \in \mathbb{Z}$ we define maps
$$\Phi^k_k: \text{Hom}(\mathbb{Z}^m, G) \to \text{Hom}(\mathbb{Z}^m, G)$$
$$(g_1, \ldots, g_m) \mapsto (g_1^k, \ldots, g_m^k).$$
These maps are well defined since the power of commuting elements is still commutative. Commutativity is needed here in order for them to induce simplicial maps. By this we mean maps commuting with the face and degeneracy maps. More precisely we need the equality
$$(g_i g_{i+1})^k = g_i^k g_{i+1}^k$$
to hold. Thus, only for commuting tuples we guarantee the existence of the $k$-th power map $\Phi^k_b: B_{\text{com}}G \to B_{\text{com}}G$. In the general bar construction for $G$, the power maps do not necessarily induce simplicial maps.

2.3. **TC structures:** Consider a principal $G$-bundle $\pi: E \to M$ over a compact Hausdorff space $M$. This implies that $M$ has an open cover $\mathcal{U} := \{U_i\}_{i=1}^m$ and trivializations $\varphi_i: \pi^{-1}(U_i) \to U_i \times G$. By considering the second component of the composition
$$\varphi_j \circ \varphi_i^{-1}: (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G$$
we obtain the cocycles $\rho_{ij}: U_i \cap U_j \to G$, which satisfy that
$$\varphi_j \circ \varphi_i^{-1}(x, g) = (x, \rho_{ij}(x) \cdot g),$$
for every $x \in U_i \cap U_j$ and $g \in G$. Assume this cover is a good cover and consider the simplicial construction of the nerve of the cover:
$$\mathcal{N}(\mathcal{U})_n = \bigsqcup \{U_{i_0} \cap U_{i_1} \cdots \cap U_{i_n}\}.$$

\[1\]It is worth recalling that up to equivalence the cocycles characterize a principle bundle.
Take $\mathcal{N}(\mathcal{U}) := \| \mathcal{N}(\mathcal{U}) \|$. Since $\mathcal{U}$ is a good cover, $M$ and $\mathcal{N}(\mathcal{U})$ are homotopy equivalent (See [Hatcher], Corollary 4G.3). This guarantees a bijection $[\mathcal{N}(\mathcal{U}), Y] \cong [M, Y]$ for any space $Y$.

Recall that if we consider the bar construction of $BG$ we have in every level the set of tuples, $G^l$. Then we have a simplicial function $g_n : \mathcal{N}(\mathcal{U})_n \rightarrow G^n$ given by $g_n(x) := (\rho_{i_0i_1}(x), \rho_{i_2i_3}(x), \ldots, \rho_{i_{l-1}i_l}(x))$.

This induces a function $g : \mathcal{N}(\mathcal{U}) \rightarrow BG$, which, up to homotopy, defines the classifying function $g : M \rightarrow BG$.

Now suppose that for the principal $G$-bundle $\pi : E \rightarrow M$ there is a trivialization inducing cocycles that commute with each other. That is that for $x \in U_i \cap U_j \cap U_k$ we have

$$\rho_{ik}(x) \rho_{kj}(x) = \rho_{kj}(x) \rho_{ik}(x).$$

Then we can define $f_n : \mathcal{N}(\mathcal{U})_n \rightarrow \text{Hom}(\mathbb{Z}^n, G)$ given by $f_n(x) := (\rho_{i_0i_1}(x), \rho_{i_2i_3}(x), \ldots, \rho_{i_{l-1}i_l}(x))$.

We have a commuting diagram

$$\begin{array}{ccc}
\mathcal{N}(\mathcal{U})_n & \xrightarrow{f_n} & \text{Hom}(\mathbb{Z}^n, G) \\
& \searrow g_n & \downarrow \\
& & G^n
\end{array}$$

where the vertical is the inclusion. This in turn leads to a diagram commuting up to homotopy

$$\begin{array}{ccc}
M & \xrightarrow{f} & B_{\text{com}}G \\
& \searrow g & \downarrow \\
& & BG,
\end{array}$$

where the vertical map is the inclusion.

Adem and Gomez proved in Theorem 2.2 of [AG] that if there is a lifting up to homotopy of the classifying function of a principal $G$-bundle, then there exists a trivialization with commuting cocycles for that bundle. That is, that the existence of a homotopy lifting for the classifying function is a necessary and sufficient condition for the existence of commuting cocycles for the principal $G$-bundle. This allow us to define the following.

**Definition 2.** Given a space $M$ and a principal $G$-bundle with classifying function $f : M \rightarrow BG$, a **TC structure** over $M$ is a function $g : M \rightarrow B_{\text{com}}G$ such that

$$\begin{array}{ccc}
M & \xrightarrow{f} & B_{\text{com}}G \\
& \searrow g & \downarrow \\
& & BG,
\end{array}$$

commutes up to homotopy. We say that two TC structures $f_1, f_2 : M \rightarrow B_{\text{com}}G$ are equivalent if the functions are homotopic.
At this point is important to remark that given a principal bundle there can be several different TC structures over it. That is, there can exist functions \( g : M \to BG \) and \( f_1, f_2 : M \to B\text{com}G \) such that there are homotopies \( \iota \circ f_1 \cong g \) and \( \iota \circ f_2 \cong g \) but where \( f_1 \) is not homotopic to \( f_2 \). At the end of this article we exhibit an example with a homotopy trivial \( g : S^4 \to B\text{SU}(2) \) with a non homotopy trivial lifting \( G : S^4 \to B\text{comSU}(2) \).

The assignment \( \text{Top} \to [-, B\text{com}G] \) is a contravariant functor: given a continuous function \( h : M \to N \) we can consider its pullback
\[
 h^* : [N, B\text{com}G] \to [M, B\text{com}G] \\
 [f] \mapsto [f \circ h],
\]
where by \([f]\) we mean the homotopy class of the function \( f \). From this we define

**Definition 3.** A characteristic class for TC structures or **TC characteristic class** is a natural transformation \( \eta : [-, B\text{com}G] \to H^*(-, \mathbb{R}) \). Here \([-, B\text{com}G]\) is the functor assigning to a space the set of homotopy classes of functions from the space to \( B\text{com}G \) and \( H^*-\mathbb{R} \) is the functor of cohomology with real coefficients.

**Proposition 4.** There is a one to one correspondence between the TC characteristic classes and elements of \( H^* (B\text{com}G, \mathbb{R}) \).

**Proof.** Consider a TC characteristic class \( \eta \), and define
\[
c_\eta := \eta (B\text{com}G)([\text{Id}_{B\text{com}G}]) \in H^* (B\text{com}G, \mathbb{R})
\]
where \( \text{Id}_{B\text{com}G} \) is the identity on \( B\text{com}G \). We want to see that the assignment \( \theta : \eta \mapsto c_\eta \) is a one to one and onto.

First, consider a continuous function \( f : M \to B\text{com}G \). By naturality of \( \eta \) we have a commuting diagram
\[
\begin{array}{ccc}
[B\text{com}G, B\text{com}G] & \xrightarrow{\eta (B\text{com}G)} & H^* (B\text{com}G, \mathbb{R}) \\
\downarrow f^* & & \downarrow f^* \\
[M, B\text{com}G] & \xrightarrow{\eta (M)} & H^* (M, \mathbb{R})
\end{array}
\]
where we use \( f^* \) to distinguish the pullback of the functor \([-, B\text{com}G]\) from the pullback from cohomology. The commutativity of the previous diagram means that
\[
f^* (c_\eta) = \eta (M) (f^{**} ([\text{Id}_{B\text{com}G}])).
\]
But since
\[
f^{**} ([\text{Id}_{B\text{com}G}]) = [\text{Id}_{B\text{com}G} \circ f] = [f],
\]
we can conclude that
\[
(2.1) \quad f^* (c_\eta) = \eta (M) ([f]) \in H^* (M, \mathbb{R}).
\]

Now, to see that \( \theta : \eta \mapsto c_\eta \) is surjective, take \( c \in H^* (B\text{com}G, \mathbb{R}) \) and define
\[
\eta_c (M) : [M, B\text{com}G] \to H^* (M, \mathbb{R}) \\
[f] \mapsto f^* (c).
\]
This can be seen to be a well defined natural transformation thanks to the properties of cohomology, so \( \eta_c \) is a TC characteristic class. Now, by definition and Equation 2.1 it follows that
\[
c_{\eta_c} = \eta_c (B_{\text{com}}G) ([\text{Id}_{B_{\text{com}}G}]) = \text{Id}_{B_{\text{com}}G}^* (c) = c,
\]
which implies that \( \theta : \eta \mapsto c_{\eta} \) sends \( \eta_c \) into \( c \).

On the other hand to prove injectivity, take \( c_{\eta} \) and consider \( \eta_{c_{\eta}} \) as defined before.

For any \( f : M \to B_{\text{com}}G \) it follows that
\[
\eta_{c_{\eta}} (M) ([f]) = f^* (c_{\eta}) = \eta (M) ([f]),
\]
where the first equality is true by definition, and the second thanks to Equation 2.1. The equality \( \eta_{c_{\eta}} (M) ([f]) = \eta (M) ([f]) \) means that \( \eta_{c_{\eta}} = \eta \), so if \( \omega \) is another TC characteristic class such that \( c_{\eta} = c_{\omega} \), then
\[
n = \eta_{c_{\eta}} = \eta_{c_{\omega}} = \omega.
\]
That is, \( \theta : \eta \mapsto c_{\eta} \) is injective. \( \square \)

2.4. The \( k \)-th associated bundles: Once again let \( \{U_\alpha\}_{\alpha \in J} \) be an open cover of a space \( M \) such that there are trivializations with a cocycle \( \{\rho_{ij} : U_i \cap U_j \to G\} \), such that if \( x \in U_i \cap U_j \cap U_l \) then
\[
\rho_{ij} (x) \rho_{lj} (x) = \rho_{lj} (x) \rho_{lk} (x).
\]
These transition functions satisfy the cocycle condition as well, that is,
\[
\rho_{ij} (x) = \rho_{il} (x) \rho_{lj} (x).
\]
In particular these two properties imply that for \( k \in \mathbb{Z} \) we have
\[
\rho_{ij} (x)^k = (\rho_{il} (x) \rho_{lj} (x))^k = \rho_{il} (x)^k \rho_{lj} (x)^k.
\]
This tell us that the collection of functions \( \rho_{ij}^k : U_i \cap U_j \to G \) defined as
\[
\rho_{ij}^k (x) := \rho_{ij} (x)^k
\]
also satisfy the cocycle condition. Then for each \( k \in \mathbb{Z} \) we can construct a principal bundle \( p_k : E^k \to M \) with trivializations over \( \{U_i\}_{i \in I} \) with cocycle \( \{\rho_{ij}^k\} \). We call it the \( k \)-th associated bundle of \( E \). Here \( E^k \) is obtained as the quotient space
\[
\left( \bigcup_{i \in I} U_i \times G \right) \sslash \sim,
\]
where for \( x \in U_i \) and \( y \in U_j \), \( (x, g) \sim (y, h) \) if only if \( x = y \) and \( \rho_{ij}^k (x) \cdot g = h \).

**Proposition 5.** (Classifying functions for \( k \)-th associated bundles) If \( f : M \to B_{\text{com}}G \) defines a TC structure over a principal \( G \)-bundle, and \( f^k : M \to B_{\text{com}}G \) is the corresponding lifting over the \( k \)-th associated bundle, then the following map diagram commutes
\[
\begin{array}{ccc}
M & \xrightarrow{f} & B_{\text{com}}G \\
\downarrow{f^k} & & \downarrow{\Phi^k} \\
B_{\text{com}}G & & B_{\text{com}}G.
\end{array}
\]
Where \( \Phi^k : B_{\text{com}}G \to B_{\text{com}}G \) are the power maps.
Proof. As it was explained before, to obtain the classifying functions for the $k$-th associated bundle $p(k): E^k \to M$ we need to consider a simplicial map $f^k_l: N(U) \to \text{Hom}(Z^l, G)$. The components of this function are given by the transition functions: if $x \in U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_l+1}$, we take

$$f^k_l(x) = \left( \rho^k_{i_0 i_1}(x), \ldots, \rho^k_{i_l-1 i_l}(x) \right) = \left( \rho_{i_0 i_1}(x)^k, \ldots, \rho_{i_l-1 i_l}(x)^k \right).$$

This can be rewritten using the power functions as

$$f^k_l = \Phi^k_l \circ f_l.$$

The desired result is obtained after passing to the geometric realization. □

3. Power maps and cohomology of $B_{\text{com}}G$

In this section we go through the reasoning behind the computation of

$$H^*(B_{\text{com}}G_1, \mathbb{R})$$

made in [AG] to obtain the effect of the power maps on cohomology. Thus we track such effect in the main steps of the computation. To make notation simpler, we assume $B_{\text{com}}G_1 = B_{\text{com}}G$, which is true when $G$ is either $U(n)$, $SU(n)$ or $Sp(n)$, as mentioned before. We also fix a maximal torus $T \subseteq G$ with Weyl group $W$ and we write $H^*(Y)$ to refer to the cohomology of $Y$ with real coefficients.

In Section 7 of [AG] it is proved that for a maximal torus $T$ of $G$ with Weyl group $W$ we have

$$H^*(B_{\text{com}}G) \cong (H^*(BT) \otimes H^*(BT))^W / J,$$

where $J$ is the ideal generated by the see

$$\{ f(x) \otimes 1 \in H^*(BT) \otimes H^*(BT) \mid f \text{ of positive degree } \text{ polynomial and } n \cdot f(x) = f(x) \text{ for all } n \in W \}$$

and $W$ acts on $H^*(BT) \otimes H^*(BT)$ diagonally.

In order to reach the description of the induced power maps $\Phi^k$ on cohomology, we need to consider some auxiliary maps that are used in [AG]. In this process we will see what is their relationship with the power maps. First, since all the tuples of $T^m$ have commuting elements, we can consider the power maps for the torus $\psi^k: H^*(BT) \to H^*(BT)$. This is the map induced in the $m$-level the by the power maps,

$$\Phi^k_m: \text{Hom}(Z^m, T) \to \text{Hom}(Z^m, T)$$

$$(g_1, \ldots, g_m) \mapsto \left( g_1^k, \ldots, g_m^k \right).$$

Also consider

$$\varphi_m: G \times T^m \to \text{Hom}(Z^m, G)$$

$$(g, t_1, \ldots, t_n) \mapsto \left( gt_1g^{-1}, \ldots, gt_ng^{-1} \right).$$

Because $\text{Hom}(Z^m, G)$ is path connected, an $m$-tuple $(g_1, \ldots, g_m)$ has commuting elements if and only if there is a maximal tori containing all $g_i$ (see Lemma 4.2 of [Baird]). Then, since every maximal tori is conjugated to $T$, the previous map is
surjective. We also have an action of the normalizer of $T$ in $G$, $N_G(T)$, on $G \times T^m$, where for $\eta \in N_G(T)$ we have

$$\eta \cdot (g, t_1, \ldots, t_m) = (g\eta^{-1}, \eta t_1\eta^{-1}, \ldots, \eta t_m\eta^{-1}).$$

On the other hand, consider the flag variety $G/T$. It is easy to verify that the maps $\varphi_m$ factor through the product $G/T \times T^m$ giving us a commutative diagram

$$
\begin{array}{ccc}
G \times T^m & \xrightarrow{\varphi_m} & \text{Hom}(\mathbb{Z}^m, G) \\
\downarrow & & \downarrow \\
G/T \times T^m & & \\
\end{array}
$$

such that the diagonal map is also surjective. We call it $\varphi_m$ as well. This family of maps give rise to a simplicial map

$$\varphi_* : G/T_\bullet \times T^\bullet \rightarrow \text{Hom}(\mathbb{Z}^\bullet, G).$$

Here $G/T_\bullet$ is the trivial simplicial space with $G/T$ on every level, and $T^\bullet$ is the simplicial space obtained by the bar construction for the classifying space applied to $T$.

Furthermore using representatives of the Weyl group $[\eta] \in W = N_G(T)/T$, we have a well defined action on $G/T \times T^m$ given by

$$[\eta] \cdot ([g], t_1, \ldots, t_m) = ([g\eta^{-1}], \eta t_1\eta^{-1}, \ldots, \eta t_m\eta^{-1}).$$

It is easy to see that this action makes $\varphi_m$ $W$-invariant. Also we can construct a simplicial space, $G/T \times_W T^\bullet$, having the space of orbits $G/T \times T^m$ on the $m$-th level. Where the simplicial structure is hered from $G/T_\bullet \times T^\bullet$, giving us a simplicial map $\pi_* : G/T_\bullet \times T^\bullet \rightarrow G/T \times_W T^\bullet$ where on each level we have the natural quotient map. Then we have a commuting diagram

$$
\begin{array}{ccc}
G/T_\bullet \times T^\bullet & \xrightarrow{\varphi_*} & \text{Hom}(\mathbb{Z}^\bullet, G) \\
\downarrow & & \downarrow \\
G/T \times_W T^\bullet & & \\
\end{array}
$$

where $\bar{\varphi}_m : G/T \times_W T^m \rightarrow \text{Hom}(\mathbb{Z}^m, G)$ is the induced map. Finally, we have maps

$$P_m^k : G/T \times T^m \rightarrow G/T \times T^m$$

$$([g], t_1, \ldots, t_m) \mapsto ([g], t_1^k, \ldots, t_m^k).$$

By direct computation it can be seen that these maps are compatible with the simplicial structure. They are also $W$-equivariant, that is

$$[\eta] \cdot P_m^k (g, t_1, \ldots, t_m) = P_m^k ([\eta] \cdot (g, t_1, \ldots, t_m)).$$

This is true since, $(t\eta^{-1})^k = \eta^k \eta^{-1}$ for $t \in T$.

From this point we are showing that the arguments given in [AG] are natural. This allow us to include the power maps in their conclusions.
Proposition 6. Suppose $G$ is a compact connected Lie group such that

$$\text{Hom}(\mathbb{Z}^m, G)$$

is path connected for every non negative integer $m$. Then for the cohomology with real coefficients we have a commutative diagram

$$(3.1) \quad H^\ast(\text{Hom}(\mathbb{Z}^m, G)) \xrightarrow{\varphi^\ast_m} H^\ast(G/T \times T^m)^W$$

$$\xrightarrow{(\Phi^k_m)^\ast} H^\ast(\text{Hom}(\mathbb{Z}^m, G)) \xrightarrow{\varphi^\ast_m} H^\ast(G/T \times T^m)^W.$$ 

where the horizontal maps are isomorphisms.

Proof. Under this setting, Theorem 3.3 of [Baird] is applied to conclude that we have the following natural isomorphisms

$$(3.2) \quad H^\ast(\text{Hom}(\mathbb{Z}^m, G)) \xrightarrow{(\varphi^\ast_m)^\ast} H^\ast(G/T \times W T^m) \xrightarrow{\pi^\ast} H^\ast(G/T \times T^m)^W.$$ 

Now let us see how the power maps are related to this constructions so far. We have maps

$$P^k_m : G/T \times T^m \to G/T \times T^m$$

$$([g], t_1, \ldots, t_m) \mapsto ([g], t_1^k, \ldots, t_m^k).$$

By direct computation it can be seen that these maps are compatible with the simplicial structure. They are also $W$-equivariant, that is

$$[\eta] \cdot P^k_m (g, t_1, \ldots, t_m) = P^k_m ([\eta] \cdot (g, t_1, \ldots, t_m)).$$

This is true since, $(\eta t_1^{-1})^k = \eta t_1^{-1}$ for $t \in T$. Thus, they induced a well define map $\bar{P}^k_m : G/T \times W T^m \to G/T \times W T^m$, and we get the following commuting diagram

$$\text{H}^\ast(G/T \times W T^m) \xrightarrow{\pi^\ast} \text{H}^\ast(G/T \times T^m)$$

$$\xrightarrow{(\bar{P}^k_m)^\ast} \text{H}^\ast(G/T \times W T^m) \xrightarrow{\pi^\ast} \text{H}^\ast(G/T \times T^m).$$

We know that the homomorphism

$$\pi^\ast : \text{H}^\ast(G/T \times W T^m) \to \text{H}^\ast(G/T \times T^m)$$

actually has image equal to $\text{H}^\ast(G/T \times T^m)^W$, since

$$\text{H}^\ast(G/T \times W T^m) \xrightarrow{\pi^\ast} \text{H}^\ast(G/T \times T^m)^W.$$ 

Thus, we actually have the diagram

$$\text{H}^\ast(G/T \times W T^m) \xrightarrow{\pi^\ast} \text{H}^\ast(G/T \times T^m)^W$$

$$\xrightarrow{(\bar{P}^k_m)^\ast} \text{H}^\ast(G/T \times W T^m) \xrightarrow{\pi^\ast} \text{H}^\ast(G/T \times T_m)^W.$$
where the horizontal maps are isomorphism. This implies that \((P^k_m)^*\) preserves \(W\)-invariance:

\[
(P^k_m)^* \left( H^* (G/T \times T^m)^W \right) \subseteq H^* (G/T \times T^m)^W.
\]

Also, by direct computation from the definitions and the fact that \((gtg^{-1})^k = gt^k g^{-1}\), it follows that \(\varphi_m \circ P^k_m = \Phi^k_m \circ \varphi_m\) holds. And since \((P^k_m)^*\) preserves \(W\)-invariance, we obtain the following commuting diagram:

\[
\begin{array}{ccc}
H^* (\text{Hom}(\mathbb{Z}^m, G)) & \xrightarrow{\varphi_m^*} & H^* (G/T \times T^m)^W \\
\downarrow{(P^k_m)^*} & & \downarrow{(P^k_m)^*} \\
H^* (\text{Hom}(\mathbb{Z}^m, G)) & \xrightarrow{\varphi_m^*} & H^* (G/T \times T^m)^W.
\end{array}
\]

Here the horizontal maps are isomorphism as they can be factored by the isomorphisms \(\bar{\varphi}_m^*: H^* (\text{Hom}(\mathbb{Z}^m, G)) \rightarrow H^* (G/T \times T^m)^W\) and \(\pi^*: H^* (G/T \times T^m) \rightarrow H^* (G/T \times T^m)^W\).

\(\square\)

Next, thanks to the naturality in Theorems 5.15 and 1.19 of [Dupont] it can be concluded that

**Proposition 7.** Let \(X_\bullet\) and \(Y_\bullet\) be two simplicial spaces with a simplicial map \(f: X_\bullet \rightarrow Y_\bullet\). Suppose also that there is a finite group \(K\) with an action on every level \(X_q\) compatible with the simplicial structure, such that there is an isomorphism \(H^p (C^* (X_q))^K \equiv H^p (C^* (Y_q))\) induce on every level by the maps of \(f\). Then there is natural isomorphism

\[
\|f\|^*: H^* (\|Y\|) \rightarrow H^* (\|X\|)^K,
\]

where \(\|X\|\) and \(\|Y\|\) are the fat realizations.

**Proposition 8.** Suppose \(G\) is a compact connected Lie group such that

\[
\text{Hom}(\mathbb{Z}^m, G)
\]

is path connected for every non negative integer \(m\). Then for the cohomology with real coefficients we have a commutative diagram

\[
\begin{array}{ccc}
H^* (B_{\text{com}} G) & \xrightarrow{\pi^*} & H^* (\| (G/T)_\bullet \times BT_\bullet \|)^W \\
\downarrow{\Phi^k} & & \downarrow{\Phi^k} \\
H^* (B_{\text{com}} G) & \xrightarrow{\pi^*} & H^* (\| (G/T)_\bullet \times BT_\bullet \|)^W,
\end{array}
\]

where the horizontal maps are isomorphisms. Here we are abusing notation by using the same names for the power map and its induce map on cohomology.

**Proof.** Because of Proposition [6] the conditions of Proposition [7] can be applied to conclude that \(\pi^*: H^* (B_{\text{com}} G) \rightarrow H^* (\| (G/T)_\bullet \times BT_\bullet \|)^W\) is an isomorphism. Then Diagram [5.1] implies that Diagram [3.3] commutes. \(\square\)
**Proposition 9.** Suppose $G$ is a compact connected Lie group such that $\text{Hom}(\mathbb{Z}^m, G)$ is path connected for every non-negative integer $m$. Then for the cohomology with real coefficients we have a commutative diagram

\[
\begin{array}{ccc}
H^*(B_{\text{com}}G) & \longrightarrow & (H^*(BT) \otimes H^*(BT))^W / J \\
\phi^k & & \downarrow \text{Id} \otimes \psi^k \\
H^*(B_{\text{com}}G) & \longrightarrow & (H^*(BT) \otimes H^*(BT))^W / J,
\end{array}
\]

where the horizontal maps are the same isomorphism given above, and $\Phi^k$ are the power maps on cohomology.

**Proof.** In the Diagram 3.3, we can consider the naturality on the Kunneth formulas and the fact that the realization of the simplicial product are naturally isomorphic to the product of the realizations of each of the simplicial spaces involved (see Theorem 14.3 of [May]). Then we obtain the commuting diagram

\[
\begin{array}{ccc}
H^*(B_{\text{com}}G) & \longrightarrow & (H^*(G/T) \otimes H^*(BT))^W \\
\phi^k & & \downarrow \text{Id} \otimes \psi^k \\
H^*(B_{\text{com}}G) & \longrightarrow & (H^*(G/T) \otimes H^*(BT))^W,
\end{array}
\]

where the horizontal maps are still isomorphisms.

Next in the proof of Proposition 7.1 of [AG], they replace $H^*(G/T)$ giving natural arguments, which allow us to obtain our conclusion. \(\square\)

The last result is important since it tells us that in order to obtain the effect of power maps on cohomology of $B_{\text{com}}G$, we need to understand their effect when the Lie group is a torus, $T = (S^1)^n$.

**Theorem 10.** Consider the $k$-th power map

\[
\psi^k : T \to T
\]

\[(t_1, \ldots, t_n) \mapsto (t_1^k, \ldots, t_n^k).\]

Then by identifying $H^*(BT) \cong \mathbb{R}[x_1, \ldots, x_n]$, the induced $k$-th power map is characterized by

\[
\begin{array}{c}
\mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n] \\
x_i \mapsto kx_i.
\end{array}
\]

**Proof.** On a circle the $k$-th power of its elements induces the multiplication by $k$ on the fundamental group: if

\[
S^1 = \{z \in \mathbb{C} \mid |z| = 1\}
\]

then the $k$-th power map is given by

\[
\eta : S^1 \to S^1
\]

\[z \mapsto z^k\]

\[2\text{When } \text{Hom}(\mathbb{Z}^m, G) \text{ is path connected for every } m.\]
which is known to be a map of degree \( k \). This means that if identify \( \pi_1 (S^1) \cong \mathbb{Z} \) then the \( k \)-th power maps induces multiplication by \( k \) on the fundamental group.

Consider the projections

\[ p_i : (S^1)^n \rightarrow S^1 \]
\[ (z_1, \ldots, z_n) \mapsto z_i. \]

It is well known that the map \( q : \pi_1 ((S^1)^n) \rightarrow \pi_1 (S^1)^n \) given by

\[ q ([\alpha]) := ([p_1 \circ \alpha], \ldots, [p_n \circ \alpha]) \]

is an isomorphism, since \( S^1 \) is path connected. Since the power map \( \psi^k : T \rightarrow T \) considers the \( k \)-th power component wise, it follows that we have a commutative diagram

\[
\begin{array}{ccc}
\pi_1 (T) & \xrightarrow{\psi^k} & \pi_1 (T) \\
\downarrow q & & \downarrow q \\
\pi_1 (S^1)^n & \xrightarrow{\Pi \eta} & \pi_1 (S^1)^n.
\end{array}
\]

Since the horizontal maps are isomorphisms, we have the characterization

\[ \psi^k : \pi_1 (T) \rightarrow \pi_1 (T) \]
\[ \alpha \mapsto k\alpha \]

where we see \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), and \( k\alpha = (k\alpha_1, \ldots, k\alpha_n) \).

Now let us consider the fiber sequence of the classifying space of the torus

\[ T \rightarrow ET \rightarrow BT. \]

This induces an exact sequence on homotopy groups

\[ \cdots \rightarrow \pi_m (ET) \rightarrow \pi_m (BT) \rightarrow \pi_{m-1} (T) \rightarrow \cdots \rightarrow \pi_1 (ET) \rightarrow \pi_1 (BT) \rightarrow \pi_0 (T) \rightarrow \pi_0 (ET) \rightarrow \pi_0 (BT) \rightarrow 0 \]

but since \( ET \) is contractible, we get an isomorphism \( \pi_m (BT) \rightarrow \pi_{m-1} (T) \). In particular we get

\[ \pi_m (BT) = \begin{cases} \mathbb{Z}^n & m = 2, \\ 0 & \text{otherwise}. \end{cases} \]

Since the exact sequence is natural, we get that the power map on \( BT \) induces the multiplication by \( k \) on the second homotopy group. Furthermore since \( BT \) is simply connected, by Hurewicz’s theorem we get that \( H_2 (BT, \mathbb{Z}) \cong \pi_2 (BT) \), and once again because of naturality the effect on the second homology is multiplication by \( k \).

We now apply the universal coefficients theorem to get that

\[ H^2 (BT) \cong \text{Hom} (H_2 (BT, \mathbb{Z}), \mathbb{R}) \cong \mathbb{R}^n. \]

Naturality allow us to conclude that the effect of the \( k \)-th power map is once again multiplication by \( k \). Finally it is known that the real cohomology of \( BT \) is the polynomial ring \( \mathbb{R} [x_1, \ldots, x_n] \) where \( x_i \in H^2 (BT) \) for \( 1 \leq i \leq n \) (see [Dupont], Proposition 8.11). Since we know that the effect of the \( k \) power map is multiplication by \( k \) on the \( x_i \), this determines the effect on the whole cohomology ring. \( \square \)

As corollary of Proposition 9 and Theorem 10 we obtain the following:
Theorem 11. Identify the real cohomology ring of an $n$-torus with $\mathbb{R}[x_1, \ldots, x_n]$, and suppose that $G$ is a Lie group such that

$$\text{Hom}(\mathbb{Z}^m, G) = \text{Hom}(\mathbb{Z}^m, G)_1$$

for every $m$, then

$$H^*(B_{\text{com}}G) \cong (\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^W / J.$$ 

Here $J$ is the ideal generated by the invariant polynomials of positive degree on the $x_i$ under the action of the Weyl group, $W$. Further, the power maps $\Phi_k : H^*(B_{\text{com}}G, F) \rightarrow H^*(B_{\text{com}}G, F)$ are induced by the homomorphism characterized by sending $x_i \mapsto x_i$ and $y_i \mapsto ky_i$ for every $1 \leq i \leq n$.

4. Generators of $H^*(B_{\text{com}}G, \mathbb{R})$ for $G = U(n), \text{Sp}(n)$ and $SU(n)$

Here we are going to examine the cases of the Lie groups $G = U(n), \text{Sp}(n)$ and $SU(n)$.

From this we see that possible differences between the different cases depend entirely on the Weyl group and its action on the cohomology of $BT$. For this we first need to establish some facts and definitions.

If $B_{\text{com}}G_1 = B_{\text{com}}G$ we know that

$$H^*(B_{\text{com}}G, \mathbb{R}) \cong (\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^W / J.$$ 

While in general it is known that for a compact and connected Lie group $G$

$$H^*(BG, \mathbb{R}) \cong H^*(BT, \mathbb{R})^W \cong P[t]^W,$$

where $W$ is its Weil group. Its action is induced by adjunction. That is, if $t$ is the Lie algebra of $T$, $P[t]$ is the polynomial algebra of $t$. An element $[n] \in W \cong N_G(T)/T$ has a well defined action given by adjunction, $\text{ad}(n) : t \rightarrow t$. This in turn induces an action of $W$ on $P[t]$. Even further, if $n$ is the dimension of $t$, $P[t]$ can be identified with $\mathbb{R}[z_1, \ldots, z_n]$, and under such identification, we have an action of $W$ on the latter.

There is a natural inclusion $B_{\text{com}}G \hookrightarrow BG$, inducing a map

$$\iota : H^*(BG, \mathbb{R}) \rightarrow H^*(B_{\text{com}}G, \mathbb{R}).$$

In terms of the previous identifications, $\iota$ is induced by the homomorphism ([Gritschacher, Corollary A.2.])

$$\mathbb{R}[z_1, \ldots, z_n] \rightarrow \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n]$$

$$z_i \mapsto x_i + y_i.$$ 

Additionally we saw in the previous section that the power maps,

$$\Phi_k : H^*(B_{\text{com}}G, \mathbb{R}) \rightarrow H^*(B_{\text{com}}G, \mathbb{R})$$

are induced by the map characterized by sending $x_i \mapsto x_i$ and $y_i \mapsto ky_i$ for every $1 \leq i \leq n$. 

Definition 12. We call the subalgebra generated by \( \{ \Phi^k (\text{Im} \, \iota) \mid k \in \mathbb{Z} \setminus \{0\} \} \subset H^* (B_{\text{com}} G, \mathbb{R}) \) by

\[
S := \langle \Phi^k (\text{Im} \, \iota) \mid k \in \mathbb{Z} \setminus \{0\} \rangle.
\]

On this section we use the previous maps to see that if \( G = U (n), \text{Sp} (n) \) and \( SU (n) \) then \( S \) is all of \( H^* (B_{\text{com}} G, \mathbb{R}) \). We do this by dealing with the explicit descriptions of their actions and the specific Weyl groups on each case.

Before dealing with each individual case, it is worth proving the following

Lemma 13. The subalgebra \( S \) is closed under the power maps.

Proof. This is true since \( \Phi^k \) is a \( \mathbb{R} \)-homomorphism of algebras as well because of the equality \( \Phi^k \circ \Phi^l = \Phi^{kl} \). This implies that for \( q_j \in \mathbb{R} [z_1, \ldots, z_n], \alpha_j \in \mathbb{R} \)

\[
\Phi^k \left( \sum_{l=1}^{s} \alpha_j \Phi^{k_j} \circ \iota (q_j) \right) = \sum_{l=1}^{s} \alpha_j \Phi^{kk_j} \circ \iota (q_j) \in S.
\]

Next we explore individually the particular cases of \( G \) equal to \( U (n), \text{SU} (n) \) and \( \text{Sp} (n) \) to show that \( S = H^* (B_{\text{com}} G, \mathbb{R}) \).

4.1. Generators of \( H^* (B_{\text{com}} U (n), \mathbb{R}) \): For this case recall that the Weil group of \( U (n) \) is isomorphic to the symmetric group \( S_n \). By the previous section we know that

\[
H^* (B_{\text{com}} U (n), \mathbb{R}) = (\mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n])^{S_n} / J
\]

where \( S_n \) acts diagonally on the tensor product, permuting the variables of each factor. \( J \) is the ideal generated by the symmetric polynomials of positive degree on the \( x_i \). It is also known that

\[
H^* (BU (n), \mathbb{R}) = (\mathbb{R} [x_1, \ldots, x_n])^{S_n},
\]

where the action is once again by permuting variables. \( H^* (BU (n), \mathbb{R}) \) is generated by the power polynomials

\[
p_m := z_1^m + z_2^m + \cdots + z_n^m,
\]

which are clearly invariant under the action of \( S_n \). These polynomials have their counterparts on two variables polynomials in the form of

\[
P_{a,b} (n) := x_1^a y_1^b + x_2^a y_2^b + \cdots + x_n^a y_n^b,
\]

where \( 1 \leq a + b \leq n \). These generate the algebra \( (\mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n])^{S_n} \) (See [Vaccarino], Theorem 1). Thus to prove that \( S \) is all of \( H^* (B_{\text{com}} U (n), \mathbb{R}) \) it is enough to see that the multisymmetric polynomials (modulo \( J \)) are in fact in \( S \). To see it, we first need a couple of lemmas.

Lemma 14. For every \( n \in \mathbb{N} \) and \( 1 \leq a + b \leq n \) with \( a, b \geq 0 \) we have

\[
\Phi^k (P_{a,b} (n)) = k^b P_{a,b} (n).
\]
Proof. Since \( \Phi^k \) is a homomorphism of algebras, we have
\[
\Phi^k (P_{a,b} (n)) = \Phi^k (x_1^a y_1^b + x_2^a y_2^b + \cdots + x_n^a y_n^b)
= \sum_{i=1}^{n} \Phi^k (x_i^a y_i^b).
\]
But we have
\[
\Phi^k (x_i^a y_i^b) = \Phi^k (x_i)^a \Phi^k (y_i)^b = k^b x_i^a y_i^b.
\]
Where the last equality is true since we already saw that \( \Phi^k (x_i) = x_i \) and \( \Phi^k (x_i) = k y_i \) for every \( 1 \leq i \leq n \).

To prove the goal of this subsection, we illustrate explicitly the cases \( n = 2 \) and \( n = 3 \).

- Suppose first that \( n = 2 \).
  
  We want to show that the following multisymmetric polynomials are indeed in \( S \):
  
  - \( P_{0,1} (2) = y_1 + y_2 \),
  - \( P_{1,1} (2) = x_1 y_1 + x_2 y_2 \) and
  - \( P_{0,2} (2) = y_1^2 + y_2^2 \).

  We ignore \( P_{1,0} (2) = x_1 + x_2 \) since this is zero modulo \( J \). For this first observe that

  \( \iota (z_1 + z_2) = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = P_{1,0} (2) + P_{0,1} (2) \)

  clearly belongs to \( S \). Since \( P_{1,0} (2) = 0 \mod J \) we are done. For \( P_{1,1} (2) \) and \( P_{0,2} (2) \) notice that the total degree (the sum of the power of each term) is 2, thus we have to consider \( \iota (p_2) \):

  \[ \iota (z_1^2 + z_2^2) = (x_1 + y_1)^2 + (x_2 + y_2)^2 = (x_1^2 + x_2^2) + 2 (x_1 y_1 + x_2 y_2) + (y_1^2 + y_2^2) . \]

  This can be rewritten as

  \[ \iota (z_1^2 + z_2^2) = P_{1,0} (2) + 2P_{1,1} (2) + P_{0,2} (2) . \]

  Then we consider

  \[ \Phi^{-1} (\iota (z_1^2 + z_2^2)) = (x_1^2 + x_2^2) - 2 (x_1 y_1 + x_2 y_2) + (y_1^2 + y_2^2) \]

giving us that

  \[ \iota (z_1^2 + z_2^2) + \Phi^{-1} (\iota (z_1^2 + z_2^2)) = 2 (y_1^2 + y_2^2) \mod J \]

  meaning that \( P_{0,2} (2) \in S \), since \( S \) is a subalgebra closed under power maps.

  Finally modulo \( J \) we get

  \[ P_{1,1} (2) = \frac{\iota (z_1^2 + z_2^2) - P_{0,2} (2)}{2} \in S \]

  which finishes the proof for \( n = 2 \).

- Suppose now that \( n = 3 \).
  
  The arguments used in the case \( n = 2 \) can be used to obtain the first two of the next equalities, where once again they are taken to be modulo \( J \).
(1) \( P_{0,1}(3) = \iota (z_1 + z_2 + z_3) \).

(2) \( P_{0,2}(3) = \frac{1}{3} \left( \iota \left( z_1^3 + z_2^3 + z_3^3 \right) + \Phi^{-1} \left( \iota \left( z_1^2 + z_2^2 + z_3^2 \right) \right) \right) \).

(3) \( P_{1,1}(3) = \frac{1}{6} \left( \iota \left( z_1^2 + z_2^2 + z_3^2 \right) - P_{0,2} \right) \).

We are left to obtain \( P_{a,b}(3) \) such that \( a + b = 3 \). For this we can reorder to see that

\[
\iota \left( z_1^3 + z_2^3 + z_3^3 \right) = (x_1 + y_1)^3 + (x_2 + y_2)^3 + (x_3 + y_3)^3
= (x_1^3 + x_2^3 + x_3^3) + 3(x_1^2y_1 + x_2^2y_2 + x_3^2y_3)
+ 3(x_1y_1^2 + x_2y_2^2 + x_3y_3^2) + (y_1^3 + y_2^3 + y_3^3)
\]

which amounts to

\[
\iota \left( z_1^3 + z_2^3 + z_3^3 \right) = 3P_{2,1} + 3P_{1,2} + P_{0,3} \mod J.
\]

We use the power maps to get that

\[
\Phi^{-1} \left( \iota \left( z_1^3 + z_2^3 + z_3^3 \right) \right) = -3P_{2,1} + 3P_{1,2} - P_{0,3} \mod J.
\]

By adding the last two equalities we get

\[
P_{1,2} \mod J = \frac{1}{6} \left( \Phi^{-1} \left( \iota \left( z_1^3 + z_2^3 + z_3^3 \right) \right) + \iota \left( z_1^3 + z_2^3 + z_3^3 \right) \right) \in S.
\]

Thus we have \( \iota \left( z_1^3 + z_2^3 + z_3^3 \right) - 3P_{1,2} \mod J \in S \), and by closure under power maps we obtain

\[
8P_{0,3} \mod J = \Phi^2 \left( \iota \left( z_1^3 + z_2^3 + z_3^3 \right) - 3P_{1,2} \right) - 6 \left( \iota \left( z_1^3 + z_2^3 + z_3^3 \right) - 3P_{1,2} \right) \in S
\]

from which we conclude that \( P_{0,3} \mod J \in S \). We finally have

\[
P_{2,1} = \frac{1}{3} \left( \iota \left( z_1^3 + z_2^3 + z_3^3 \right) - 3P_{1,2} - P_{0,3} \right) \mod J
\]

which finishes the case \( n = 3 \).

In the previous two examples we see that for non negative numbers \( a \) and \( b \), we proved that \( P_{a,b}(n) \) belongs to \( S \) using induction on the value \( a + b \). This was done in such a way that the induction process did not depend on \( n \). These arguments can be generalized more methodically to obtain.

**Theorem 15.** The algebra \( H^* \left( B_{com}U(n) ; \mathbb{R} \right) \) is equal to the subalgebra

\[
S := \left\{ \Phi^k \left( \text{Im} \iota \right) \mid k \in \mathbb{Z} \setminus \{0\} \right\}.
\]

**Proof.** For this proof we will be working modulo \( J \). Also, for an arbitrary \( n \) consider a fixed \( m \in \{1,2,\ldots,n\} \). Now take

\[
p_m := z_1^m + z_2^m + \cdots + z_n^m.
\]

An easy reordering gives us

\[
\iota (p_m) = \left( \sum_{i=1}^{n} (x_i + y_i)^m \right) = \sum_{i=1}^{n} \sum_{j=0}^{m} \binom{m}{j} x_i^{m-j} y_i^j
= \sum_{j=0}^{m} \binom{m}{j} P_{m-j,j} (n) = \sum_{j=1}^{m} \binom{m}{j} P_{m-j,j} (n),
\]

(4.1)
where the last equality holds because we are working modulo \( J \). From this point we will use the power maps \( \Phi^k \) to obtain the various \( P_{m-j,j} (n) \). First we use the following recursion to get first \( P_{0,m} (n) \) from \( \Phi^1 \). Let \( A_0 := \iota (p_m) \),

\[
A_1 := \Phi^2 (A_0) - 2A_0 = \sum_{j=2}^{m} (2^j - 2) \binom{m}{j} P_{m-j,j} (n)
\]

and

\[
A_2 := \Phi^3 (A_1) - 3^2 A_1 = \sum_{j=3}^{m} (2^j - 2) (3^j - 3^2) \binom{m}{j} P_{m-j,j} (n).
\]

In general for \( 1 \leq k \leq m - 1 \) we define

\[
A_k := \Phi^{k+1} (A_{k-1}) - (k + 1)^k A_{k-1}.
\]

Notice that every \( A_k \) has non zero coefficients only for \( P_{m-j,j} (n) \) for \( k+1 \leq j \leq m \). Since \( A_0 \in S \) by definition and every \( A_k \) is defined in terms of the power maps and \( A_{k-1} \), induction implies that \( A_k \in S \) for every \( 1 \leq k \leq m - 1 \). Some easy calculations allow us to obtain that

\[
P_{0,m} (n) = \left( \prod_{k=2}^{m} (k^m - k^{k-1}) \right)^{-1} A_{m-1} \in S.
\]

And thus we obtain that

\[
\iota (p_m) - P_{0,m} (n) = \sum_{j=1}^{m-1} \binom{m}{j} P_{m-j,j} (n) \in S.
\]

Then we can apply a new recursion to conclude that \( P_{1,m-1} (n) \in S \). By continuing with this backwards recursion we obtain that \( P_{a,b} (n) \in S \) for all positive \( a, b \) such that \( a + b = m \). Since we picked \( m \in \{1, 2, \ldots, n\} \) arbitrarily, this finishes the proof. \( \square \)

4.2. Generators of \( H^* (B_{com}SU (n) ; \mathbb{R}) \): To obtain that

\[
H^* (B_{com}SU (n) ; \mathbb{R}) = \left\langle \Phi^k (1m) \mid k \in \mathbb{Z} \setminus \{0\} \right\rangle,
\]

we use a different presentation of \( H^* (BT, \mathbb{R}) \). A maximal torus of \( SU (n) \) is the set of diagonal matrices with entries in \( S^1 \subseteq \mathbb{C} \), such that their product equals one. Under such presentation it is routine to show that

\[
H^* (BT, \mathbb{R}) \cong (\mathbb{R} [z_1, \ldots, z_n] / \langle z_1 + \cdots + z_n \rangle)
\]

where the Weyl group is then \( S_n \) acting by permutation. This implies that

\[
H^* (BSU (n) ; \mathbb{R}) \cong (\mathbb{R} [z_1, \ldots, z_n] / \langle z_1 + \cdots + z_n \rangle)^{S_n},
\]

but since \( p_1 := z_1 + \cdots + z_n \) is already invariant, the previous ring is isomorphic to

\[
H^* (BSU (n) ; \mathbb{R}) \cong \mathbb{R} [z_1, \ldots, z_n]^{S_n} / \langle z_1 + \cdots + z_n \rangle.
\]

Since \( \mathbb{R} [z_1, \ldots, z_n]^{S_n} \) is itself a polynomial algebra on \( p_i = z_1^i + \cdots + z_n^i \) for \( 1 \leq i \leq n \), (see [Humphrey], Chapter 3.5: Chevalley’s Theorem), we finally get that

\[
H^* (BSU (n) ; \mathbb{R}) \cong \mathbb{R} [p_1, \ldots, p_n] / \langle p_1 \rangle \cong \mathbb{R} [p_2, \ldots, p_n].
\]

We will use this to conclude the following
Theorem 16. The real cohomology of $\tilde{B}_{\text{com}}SU(n)$ can be given by

$$H^*(\tilde{B}_{\text{com}}SU(n), \mathbb{R}) \cong (\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^{S_n}/\tilde{J},$$

where $\tilde{J}$ is the ideal generated by $x_1^i + \cdots + x_n^i, 1 \leq i \leq n$ and $y_1^i + \cdots + y_n^i$.

Proof. We saw in Theorem 9 that

$$H^*(B_{\text{com}}SU(n), \mathbb{R}) \cong (H^*(BT) \otimes H^*(BT))^{S_n}/J,$$

where $J$ is the ideal generated by the $S_n$-invariants on the first component. Now, for convenience, let us call

$$\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n],$$

$$\mathbb{R}[y] := \mathbb{R}[y_1, \ldots, y_n],$$

$$f(x) = x_1 + \cdots + x_n$$

and

$$f(y) = y_1 + \cdots + y_n$$

The previous reasoning then gives us

$$H^*(\tilde{B}_{\text{com}}SU(n), \mathbb{R}) \cong (\mathbb{R}[x]/\langle f(x) \rangle \otimes \mathbb{R}[y]/\langle f(y) \rangle)^{S_n}/J.$$ 

Notice that this is well defined since the $S_n$-invariance of $x_1 + \cdots + x_2$ and $y_1 + \cdots + y_2$ allow us to have a well define action of $S_n$ on

$$R := \mathbb{R}[x]/\langle f(x) \rangle \otimes \mathbb{R}[y]/\langle f(y) \rangle.$$ 

Consider first the map

$$p : \mathbb{R}[x] \otimes \mathbb{R}[y] \to R,$$

which is induced by the projection

$$\mathbb{R}[x] \times \mathbb{R}[y] \to \mathbb{R}[x]/\langle f(x) \rangle \times \mathbb{R}[y]/\langle f(y) \rangle.$$ 

The map $p$ is naturally $S_n$-equivariant, thus it induces a map

$$\tilde{p} : (\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^{S_n} \to R^{S_n}.$$ 

Also, since $p$ is surjective, and the action is diagonal, we have that $\tilde{p}$ is also onto. We can further consider the composition with the quotient by $J$ to obtain a surjective map

$$q : (\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^{S_n} \to (R^{S_n})/J.$$ 

It is easy to see that the kernel of this map is what we called $\tilde{J}$, so the result follows. \qed

Even further, since the map

$$\mathbb{R}[z_1, \ldots, z_n] \to \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n]$$

$$z_i \mapsto x_i + y_i,$$

induces the map $\iota : H^*(BSU(n), \mathbb{R}) \to H^*(\tilde{B}_{\text{com}}SU(n), \mathbb{R})$, we still have the same characterization under the identifications given above. That is, $\iota$ can be seen as the map

$$(\mathbb{R}[z_1, \ldots, z_n]/\langle z_1 + \cdots + z_n \rangle)^{S_n} \to (\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^{S_n}/\tilde{J}$$

induce by $z_i \mapsto x_i + y_i$. The $k$-th power map on

$$(\mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n])^{S_n}/\tilde{J}$$
is also still induce by the assignment \( x_i \mapsto x_i \) and \( y_i \mapsto ky_i \). Thus, with slight changes we can still apply the arguments given in the proof of Theorem 15 to obtain the main result.

**Theorem 17.** The algebra \( H^* (B_{\text{com}} SU(n); \mathbb{R}) \) is equal to the subalgebra
\[
S := \langle \Phi^k (\text{Im} \mu ) \mid k \in \mathbb{Z} \setminus \{0\}\rangle,
\]
where \( \Phi^k \) is the \( k \)-th power map.

### 4.3. Generators of \( H^* (B_{\text{com}} Sp(n); \mathbb{R}) \):
In this section \( \mathbb{Z}_2 \) will denote the multiplicative group \( \{-1,1\} \).

The Weyl group \( W \) of the simplectic group \( Sp(n) \) is isomorphic to the semidirect product \( \mathbb{Z}_2^n \rtimes S_n \), where \( \sigma \in S_n \) acts on \( (a_1, \ldots, a_n) \in \mathbb{Z}_2^n \) by
\[
\sigma \cdot (a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)}).
\]
Under these identifications, if
\[
f \in \mathbb{R} [x_1, \ldots, x_n] \cong H^*(T)
\]
and
\[
g = ((a_1, \ldots, a_n), \sigma) \in \mathbb{Z}_2^n \rtimes S_n
\]
we have
\[
g \cdot f (x_1, \ldots, x_n) = f (a_1x_{\sigma(1)}, \ldots, a_nx_{\sigma(n)}).
\]
Recall that
\[
H^* (B_{\text{com}} Sp(n); \mathbb{R}) \cong (\mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n])^W / J
\]
where \( W \) acts diagonally: for \( n \in W \) and \( p(x) \otimes q(y) \in \mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n] \) we have
\[
n \cdot (p(x) \otimes q(y)) := (n \cdot p(x)) \otimes (n \cdot q(y)).
\]
\( J \) is the ideal generated by the symmetric polynomials on the variables \( x_i^2 \). For brevity, let us call \( R := (\mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n])^W \) the signed multisymmetric polynomials.

Once again we want to see that \( S := \langle \Phi^k (\text{Im} \mu ) \mid k \in \mathbb{Z} \setminus \{0\}\rangle \) is equal to all of \( H^* (B_{\text{com}} Sp(n); \mathbb{R}) \). For this let us see first that the set
\[
\{ P_{a,b} (n) \mid a, b \geq 0 \text{ and } a + b \in 2\mathbb{Z} \}
\]
generates all of the signed multisymmetric polynomials as an algebra. This will allow us to use the same arguments used in the case of \( U(n) \) to obtain that
\( S = H^* (B_{\text{com}} Sp(n); \mathbb{R}) \). We need the following lemmas, where the first has a straightforward proof.

**Lemma 18.** Let \( \mu : \mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n] \rightarrow R \) be the operator defined as
\[
\mu (f) = \frac{1}{|W|} \sum_{g \in W} g \cdot f.
\]
This is a well defined \( \mathbb{R} \)-linear map, where \( |W| \) is the cardinality of the Weyl group. We call this operator the symmetrization operator.

**Lemma 19.** If \( f \in R \) and \( h \in \mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n] \), then \( \mu (f) = f \) and \( \mu (fh) = f \cdot \mu (h) \).
Lemma 21. If a pair of multi indices \( I, J \) denote \( I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_n) \in \mathbb{N}^n \) (including zero as a natural number) and let us denote
\[
x^I y^J : = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} y_1^{j_1} \cdots y_n^{j_n}.
\]

Definition 20. We say a pair of multi indices \( (I, J) \in \mathbb{N}^n \times \mathbb{N}^n \) is odd if there if \( 1 \leq k \leq n \) such that \( i_k + j_k \) is odd. Such a pair is even if it is not odd.

Lemma 21. If a pair of multi indices \( (I, J) \) is odd, then \( \mu(x^I y^J) = 0 \).

Proof. Let \( (I, J) = ((i_1, \ldots, i_n), (j_1, \ldots, j_n)) \) and let’s assume \( i_k + j_k \) is odd. Let
\[
h_k := \left( \begin{array}{c} \ell_1, \ldots, \ell_n \\ 1, \ldots, 1 \\ 1, \ldots, 1 \end{array} \right) \in W,
\]
where \( e \) is the identity permutation. Denote by \( H \subseteq W \) the subgroup generated by \( h_k \) and the partition by right cosets \( \{Hg_1, \ldots, Hg_n\} \) of \( W \). Since \( h_k \) has order 2
\[
W = \{g_1, \ldots, g_m\} \cup \{h_kg_1, \ldots, h_kg_m\}
\]
and thus
\[
\mu(x^I y^J) = \frac{1}{|W|} \sum_{l=1}^{m} (g_l x^I y^J + h_k (g_l x^I y^J)).
\]
Notice that in general if \( g = ((a_1, \ldots, a_n), \sigma) \), then since \( i_k + j_k \) is odd we get
\[
h_k (g \cdot x^I y^J) = h_k \left( a_1^{i_1+j_1} \cdots a_n^{i_n+j_n} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(n)}^{i_n} y_{\sigma(1)}^{j_1} \cdots y_{\sigma(n)}^{j_n} \right)
\]
\[
= (-1)^{i_1+j_1} a_1^{i_1+j_1} \cdots a_n^{i_n+j_n} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(n)}^{i_n} y_{\sigma(1)}^{j_1} \cdots y_{\sigma(n)}^{j_n}
\]
\[
= -a_1^{i_1+j_1} \cdots a_n^{i_n+j_n} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(n)}^{i_n} y_{\sigma(1)}^{j_1} \cdots y_{\sigma(n)}^{j_n}
\]
\[
= -g \cdot x^I y^J.
\]
This implies that
\[
\mu(x^I y^J) = \frac{1}{|W|} \sum_{l=1}^{m} (g_l x^I y^J - g_l x^I y^J) = 0.
\]
Theorem 22. If a polynomial is signed multisymmetric then its monomials have all even multi indices.

Proof. An element $f \in \mathbb{R}[x_1, \ldots, x_n] \otimes \mathbb{R}[y_1, \ldots, y_n]$ can be uniquely written as

$$f = c_0 + \sum_{k=1}^{m} c_k x^{I_k} y^{J_k}.$$ 

Where $c_0 \in \mathbb{R}$ and for $k > 0$, $c_k \in \mathbb{R} \setminus \{0\}$, $I_k$ and $J_k$ are multi indices of $n$ variables, not all of them zero. If $f$ is signed multisymmetric,

$$f = \mu(f) = c_0 + \sum_{k=1}^{m} c_k \mu(x^{I_k} y^{J_k}).$$

These two last expressions for $f$ imply that

$$(4.2) \quad \sum_{k=1}^{m} c_k x^{I_k} y^{J_k} = \sum_{k=1}^{m} c_k \mu(x^{I_k} y^{J_k}).$$

But by the previous lemma, we know that if $(I_t, J_t)$ is odd for a given $t$, then $\mu(x^{I_t} y^{J_t}) = 0$. Since $\mu(x^{I_k} y^{J_k})$ is itself a sum of monomials, the expression

$$\sum_{k=1}^{m} c_k \mu(x^{I_k} y^{J_k})$$

must have only monomials with an even set of multi indices. Since all the coefficients in

$$\sum_{k=1}^{m} c_k x^{I_k} y^{J_k}$$

are non zero, the last equality and the uniqueness of the expression for non zero coefficients of a polynomial, allow us to conclude that $(I_k, J_k)$ is even for every $1 \leq k \leq m$. \hfill \Box

In particular this proof allows us to obtain

Corollary 23. Every signed multisymmetric polynomial can be written in the form

$$f = c_0 + \sum_{k=1}^{m} c_k \mu(x^{I_k} y^{J_k}),$$

where $(I_k, J_k)$ is even for every $1 \leq k \leq m$.

If a multisymmetric polynomial has monomials with even multi indices, such polynomial is signed symmetric, meaning that is invariant under the action of elements of the form $((a_1, \ldots, a_n), e) \in W$. In particular we can now conclude:

Theorem 24. A multisymmetric polynomial is signed symmetric if only if all its monomials have even multi indices.

This result grant us the framework to obtain generators for the algebra

$$H^*(B_{com}Sp(n); \mathbb{R}).$$

Recall that multisymmetric are generated by the power polynomials

$$P_{a,b} := \sum_{i=1}^{n} x_i^{a_i} y_i^{b_i}.$$
On the other hand, due to the last result we know \( P_{a,b} \) is signed multi symmetric if and only if \( a + b \) is even. Let’s see that they in fact generate all of the signed multisymmetric polynomials.

**Theorem 25.** \( (\mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n])^{\mathbb{Z}_2 \times S_n} \) is generated as an algebra by the set

\[ G := \left\{ P_{a,b} := \sum_{i=1}^{n} x_i^a y_i^b \mid 0 \leq a, b \text{ and } a + b \in 2\mathbb{Z} \right\}. \]

**Proof.** By Corollary 23 is enough to show that for even multi indices \((I, J)\),

\[ \mu(x^I y^J) \in \text{gen} G. \]

To see this, note that any permutation of the set of indices have the same symmetrization. This is, for \( k_1, \ldots, k_p, \in \{1, \ldots, n\} \) all mutually different, \( p \leq n \), we have

\[ \mu(x_{k_1}^{i_1} \cdots x_{k_p}^{i_p} y_{k_1}^{j_1} \cdots y_{k_p}^{j_p}) = \mu(x_1^{i_1} \cdots x_p^{i_p} y_1^{j_1} \cdots y_p^{j_p}). \]

So it is enough to show that

\[ \mu(x_1^{i_1} \cdots x_p^{i_p} y_1^{j_1} \cdots y_p^{j_p}) \in \text{gen} G, \]

where of course \( i_k + j_k \) is even for every \( 1 \leq k \leq p \). We do it using induction on \( p \). The cases \( p = 1 \) is immediate, since in this case \( \mu(x^I y^J) \) is a scalar multiple of even power polynomials of the form \( P_{a,0}, P_{0,b} \) or \( P_{a,b} \).

Next, assume we know \( \mu(x_1^{i_1} \cdots x_p^{i_p} y_1^{j_1} \cdots y_p^{j_p}) \in \text{gen} G \) for \( 1 \leq p \leq k \). By reordering we have

\[ \mu(x_1^{i_1} y_1^{j_1}) \mu(x_2^{i_2} \cdots x_{k+1}^{i_{k+1}} y_2^{j_2} \cdots y_{k+1}^{j_{k+1}}) = c \mu(x_1^{i_1} \cdots x_{k+1}^{i_{k+1}} y_1^{j_1} \cdots y_{k+1}^{j_{k+1}}) + \Theta, \]

with

\[ \Theta = \sum_{r=2}^{k+1} c_r \mu(x_2^{i_2} \cdots x_r^{i_r+i_1} \cdots x_{k+1}^{i_{k+1}} y_2^{j_2} \cdots y_r^{j_1+i_1} \cdots y_{k+1}^{j_{k+1}}), \]

where \( c_2, \ldots, c_{k+1} \) are integers, and \( c \) is a non zero integer. This implies that

\[ \mu(x_1^{i_1} \cdots x_{k+1}^{i_{k+1}} y_1^{j_1} \cdots y_{k+1}^{j_{k+1}}) = \frac{1}{c} \left( \mu(x_1^{i_1} y_1^{j_1}) \mu(x_2^{i_2} \cdots x_k^{i_k} y_2^{j_2} \cdots y_k^{j_k}) - \sum_{r=2}^{k+1} c_r \mu(x_2^{i_2} \cdots x_r^{i_r+i_1} \cdots x_k^{i_k} y_2^{j_2} \cdots y_r^{j_1+i_1} \cdots y_k^{j_k}) \right). \]

By the induction hypothesis all of the terms in the right are in \( \text{gen} G \), which implies that

\[ \mu(x_1^{i_1} \cdots x_{k+1}^{i_{k+1}} y_1^{j_1} \cdots y_{k+1}^{j_{k+1}}) \]

belongs to \( \text{gen} G \).

With the last result at hand we can imitate the reasoning in the proof of Theorem 15 to obtain the main result of this part.

**Theorem 26.** The algebra \( H^* (B_{com} \text{Sp} (n) ; \mathbb{R}) \) is equal to the subalgebra

\[ S := \langle \Phi^k (\text{Im} u) \mid k \in \mathbb{Z} \setminus \{0\} \rangle. \]
Where $\Phi^k$ are the power maps and $\iota : H^* (BSp (n) ; \mathbb{R}) \to H^* (B_{com}Sp (n) ; \mathbb{R})$ is the map induced by the homomorphism

$$\mathbb{R} [z_1, \ldots, z_n] \to \mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n]$$

$$z_i \mapsto x_i + y_i.$$

**Proof.** Take once again

$$p^m = z_1^m + z_2^m + \cdots + z_n^m$$

for $m$ even. We also work modulo $J$, the ideal generated by the $x_i^2$. Recall that

$$\iota (p^m) = \sum_{j=1}^{m} \binom{m}{j} P_{m-j,j} (n).$$

Since $(m-j) + j = m$, all of the power polynomials $P_{m-j,j} (n)$ are even. Now we use recursion to get first $P_{0,m} (n)$ from the last equality: for this we name $A_0 := \iota (p_m)$, then we take

$$A_1 := \Phi^2 (A_0) - 2A_0 = \sum_{j=2}^{m} \left( \binom{m}{j} \right) P_{m-j,j} (n)$$

and

$$A_2 := \Phi^3 (A_1) - 3^2 A_1 = \sum_{j=3}^{m} \left( \binom{m}{j} \right) P_{m-j,j} (n).$$

In general for $1 \leq k \leq m - 1$ we define

$$A_k := \Phi^{k+1} (A_{k-1}) - (k+1)^k A_{k-1}.$$

Notice that every $A_k$ has non zero coefficients only for $P_{m-j,j} (n)$ for $k+1 \leq j \leq m$. Since $A_0 \in S$ by definition and every $A_k$ is defined in terms of the power maps and $A_{k-1}$, induction implies that $A_k \in S$ for every $1 \leq k \leq m - 1$. In particular we have

$$P_{0,m} (n) = \left( \prod_{k=2}^{m} (k^m - k^{k-1}) \right)^{-1} A_{m-1} \in S.$$

We now can apply a similar procedure to

$$\iota (p_m) - P_{0,m} (n) = \sum_{j=1}^{m-1} \binom{m}{j} P_{m-j,j} (n) \in S$$

to conclude that if $m = 2k$, and $P_{a,b}$ is such that $a + b = m$ then $P_{a,b} \in S$. \qed

## 5. Chern-Weil theory for TC structures

In this section we develop characteristic classes for TC structures. Our central goal is to obtain characteristic classes for TC structures using Chern-Weil theory. Specifically, we will develop this for TC structures over vector bundles whose structural group is either $U (n)$ or $SU (n)$. Thus, by $G$ we will mean one of these groups.

Recall that we have the $k$-th power maps $\Phi^k : H^* (B_{com}G) \to H^* (B_{com}G)$, and a natural inclusion $\iota : H^* (BG) \to H^* (B_{com}G)$. We already proved that

**Theorem 27.** For $G$ equal to $U (n)$ or $SU (n)$, then

$$H^* (B_{com}G) = \text{gen} \{ \Phi^k \circ \iota (c) \mid c \in H^* (BG), k \in \mathbb{Z} \setminus \{0\} \}.$$
This means that given a class in $H^* (B_{\text{com}} G)$, it can be written as a sum of finite products of elements of the form $\Phi^k \circ \iota (c)$, $c \in H^* (BG), k \in \mathbb{Z} \setminus \{0\}$. Now we continue with a construction that allows us to use Chern-Weil theory to compute characteristic classes associated to the previous classes.

5.1. **Chern-Weil theory for TC structures**: For the rest of this section let $\varepsilon \in [M, B_{\text{com}} G]$ be an equivalence class with an underlying smooth vector bundle $E \to M$ and structure group $U (n)$ or $SU (n)$. For an element $p \in H^* (B_{\text{com}} G, \mathbb{R})$ we denote by $p (\varepsilon) \in H^* (M)$ the value of the TC characteristic class on the TC equivalence class $\varepsilon$. Also, recall that via the Chern-Weil isomorphism, if $g$ is the Lie algebra of $G$, then $H^* (BG) \cong I (g)$. Here $I (g)$ is the subalgebra of invariant polynomials under conjugation of the polynomial algebra of $g$. Under this identification, every characteristic class for vector bundles (having $G$ as its structure group) can be identified with a polynomial $c \in I (g)$.

Now recall that for a smooth vector bundle $F \to M$ with curvature $\Omega$, the value on $F$ of the characteristic class associated to $c$ is equal to $c (\Omega) \in H^* (M)$. Under these terms, we are now able to compute the TC characteristic classes associated to the set of generators of $H^* (B_{\text{com}} G), \{ \Phi^k \circ \iota (c) \mid 1 \leq i \leq n, k \in \mathbb{Z} \setminus \{0\} \}$. Here, we take $\iota$ to be a map from $I (g)$ to $H^* (B_{\text{com}} G)$.

**Theorem 28.** Consider $\varepsilon \in [M, B_{\text{com}} G]$ an equivalence class with an underlying smooth vector bundle $E \to M$, and structure group $U (n)$ or $SU (n)$. Also let $\Omega_k$ be the curvature of $E^k$, the $k$-th associated bundle of $E$. Then for $c \in I (g)$ and $p = \Phi^k \circ \iota (c) \in H^* (B_{\text{com}} G)$, the TC characteristic class $p (\varepsilon)$ has same class in $H^* (M)$ as the characteristic class for vector bundles $c (E^k)$. This implies that

$$p (\varepsilon) = c (\Omega_k) \in H^* (M).$$

**Proof.** This is straight forward. First, by Theorem 5 we know that if $f$ and $f_k$ the the classifying functions for TC structures over $E \to M$ and $E^k \to M$, respectively, then there is the following commuting diagram

$$
\begin{array}{ccc}
H^* (B_{\text{com}} G) & \xrightarrow{f^*} & H^* (M) \\
\Phi^k & \downarrow & \\
H^* (B_{\text{com}} G). & \leftarrow & \\
\end{array}
$$

This means that for $c \in H^* (BG)$ we have the identity $f^* (\Phi^k \circ \iota (c)) = f_k^* (\iota (c))$ in $H^* (M)$.

In turn, since the composition $f_k^* \circ \iota$ is a classifying function for the vector bundle $E^k \to M$, we can apply the Chern-Weil isomorphism. That is, we can consider the curvature $\Omega_k$ of $E_k$ to obtain that

$$f_k^* (\iota (c)) = c (\Omega_k).$$

The conclusion of the theorem then follows by transitivity. \qed

**Theorem 29.** (Chern-Weil theory for TC structures) Consider $\varepsilon \in [M, B_{\text{com}} G]$ an equivalence class with an underlying smooth vector bundle $E \to M$, and structure group $U (n)$ or $SU (n)$. Also let $\Omega_k$ be the curvature of $E^k$, the $k$-th associated
bundle of \( E \). Then every TC characteristic class can be obtained as a linear combination of products of the form

\[
s_1 (\Omega_{k_1}) \cdot s_2 (\Omega_{k_2}) \cdots s_m (\Omega_{k_m}) \in H^* (M),
\]

where \( s_i \in H^* (BG) \) and \( k_i \in \mathbb{Z} \). Each \( s_i (\Omega_{k_i}) \) is the characteristic class of the vector bundle \( E^k \to M \) computed using its curvature.

**Proof.** Recall that if we set \( s \) as the subalgebra of \( H^* (B_{\text{com}}G) \) generated by

\[
\{ \Phi^k \circ \iota (s) \mid 1 \leq i \leq n, k \in \mathbb{Z} \setminus \{0\}, s \in H^* (BG) \}
\]

then we have \( S = H^* (B_{\text{com}}G) \). Thus, every element of \( H^* (B_{\text{com}}G) \) can be written as a linear combination of products of the form

\[
\Phi^{k_1} (\iota (s_1)) \cdot \Phi^{k_2} (\iota (s_2)) \cdots \Phi^{k_m} (\iota (s_m)).
\]

Then we can apply the previous theorem to obtain

\[
\Phi^{k_i} (\iota (s_i)) = s_i (\Omega_{k_i}).
\]

\[ \square \]

As suggested by the name of the theorem, we are now able to compute TC characteristic classes by using Chern-Weil theory. This is done in a three steps process for a class in \( s \in H^* (B_{\text{com}}G) \) and a TC structure \( \xi \) over a vector bundle \( E \to M \): first we need to decompose \( s \) in terms of the generators in

\[
\{ \Phi^k \circ \iota (c) \mid 1 \leq i \leq n, k \in \mathbb{Z} \setminus \{0\} \}.
\]

Secondly, for each of the generators \( \Phi^k \circ \iota (c) \) in the decomposition of \( s \) we use the curvature of the \( k \)-th associated bundle, \( \Omega_k \), to compute the characteristic class associated to it, \( c (\Omega_k) \in H^* (M) \) (this class is equal to the TC class given by \( (\Phi^k \circ \iota (c)) (\xi) \)). Finally we replace the values of each \( (\Phi^k \circ \iota (c)) (\xi) \) to obtain \( s (\xi) \in H^* (M) \).

Recall from Chapter 3 that when \( G \) is equal to \( U(n) \), then

\[
H^* (B_{\text{com}}G, \mathbb{R}) \cong (\mathbb{R} [x_1, \ldots, x_n] \otimes \mathbb{R} [y_1, \ldots, y_n])^{S_n} / J
\]

where \( S_n \) acts by permutation on their indexes and \( J \) is the ideal generated by the invariant polynomials of positive degree on the \( x_i \). When \( G \) is \( SU(n) \) is the same description for \( H^* (B_{\text{com}}G, \mathbb{R}) \) except \( J \) is generated by the invariant polynomials of positive degree on \( x_i \) and the polynomial \( y_1 + \cdots + y_n \).

We also have the identifications

\[
H^* (BU(n), \mathbb{R}) \cong \mathbb{R} [z_1, \ldots, z_n]^{S_n}
\]

and

\[
H^* (BSU(n), \mathbb{R}) \cong \mathbb{R} [z_1, \ldots, z_n]^{S_n} / \langle z_1 + \cdots + z_n \rangle.
\]

Then we have that the polynomials

\[
p_i = z_1^i + \cdots + z_n^i \in \mathbb{R} [z_1, \ldots, z_n]
\]

generated all of \( H^* (BG, \mathbb{R}) \), when \( G \) is \( U(n) \) or \( SU(n) \). Even further for \( a, b \in \mathbb{N} \cup \{0\} \) such that \( 0 < a + b \) then

\[
P_{a,b} (n) := \sum_{i=1}^{n} x^a y^b \mod J.
\]
generated all of $H^* (B_{\text{com}} G, \mathbb{R})$ as an algebra. We also saw in the proof of Theorem 15 there every $P_{a,b} (n)$ can be obtain, via a recursive procedure, as a linear combination of elements of the form $\Phi^k (\iota (p_i))$. With that recursive procedure and the previous theorem, we can compute the TC characteristic classes corresponding to each $P_{a,b} (n)$.

Recall that another set of generators for $H^* (B G, \mathbb{R})$, when $G$ is $U (n)$ or $SU (n)$ is given by the polynomials $\sigma_i$, characterized by the equation
\[
\det (I - tX) = 1 + t\sigma_1 (X) + t^2 \sigma_2 (X) + \cdots + t^n \sigma_n (X).
\]
These generators are more commonly used instead of the $p_i$, as $\sigma_i$ are used in the definition of Chern classes.

Example 30. We saw previously that for $G = U (3)$ we have the equalities
\[
\begin{align*}
(1) & \quad y_1 + y_2 + y_3 = \iota (z_1 + z_2 + z_3), \\
(2) & \quad y_1^2 + y_2^2 + y_3^2 = \frac{1}{2} (\iota (z_1^2 + z_2^2 + z_3^2) + \Phi^{-1} (\iota (z_1^2 + z_2^2 + z_3^2))), \\
(3) & \quad x_1 y_1 + x_2 y_2 + x_3 y_3 = \frac{1}{4} (\iota (z_1^2 + z_2^2 + z_3^2) - \Phi^{-1} (\iota (z_1^2 + z_2^2 + z_3^2))).
\end{align*}
\]
Now consider a TC structure $\xi$ with underlying vector bundle $E \to M$, with curvature $\Omega$ and $\Omega_k$ is the curvature of the $k$-th associated bundle. Now since we have that $p_1 = \sigma_1$ and that
\[
p_2 = \sigma_1^2 - 2\sigma_2
\]
we obtain that
\[
\begin{align*}
(1) & \quad (y_1 + y_2 + y_3) (\xi) = \sigma_1 (\Omega), \\
(2) & \quad (y_1^2 + y_2^2 + y_3^2) (\xi) = \frac{1}{2} (\sigma_1 (\Omega)^2 + \sigma_1 (\Omega^{-1})^2) - (\sigma_2 (\Omega) + \sigma_2 (\Omega^{-1})), \\
(3) & \quad (x_1 y_1 + x_2 y_2 + x_3 y_3) (\xi) = \frac{1}{4} \left( \sigma_1 (\Omega)^2 - \sigma_1 (\Omega^{-1})^2 \right) \\
& \quad + \frac{1}{2} (\sigma_2 (\Omega^{-1}) - \sigma_2 (\Omega)).
\end{align*}
\]
For $G = U (n)$ we know that $H^* (B G)$ is a polynomial algebra generated by the Chern classes $c_i$, $1 \leq i \leq n$. Thus it follows that $S$ is generated by the set $\{ \Phi^k \circ \iota (c_i) \mid 1 \leq i \leq n, k \in \mathbb{Z} \setminus \{ 0 \} \}$.

Definition 31. We call the classes of the form $c_i^k := \Phi^k \circ \iota (c_i) \in H^* (B_{\text{com}} U (n))$ the **TC Chern classes**. Also, for a TC structure $\varepsilon$ with underlying vector bundle $E \to M$ we call
\[
c_i^k (\varepsilon) := f^* (c_i^k) \in H^* (M)
\]
the TC $(i,k)$-Chern class. Here $f : M \to B_{\text{com}} U (n)$ is the classifying function of the TC structure.

From the previous theorem we have the immediate following corollary:

Corollary 32. Let $E \to M$ by the underlying bundle of a TC structure structure $\varepsilon$, and let $\Omega_k$ be the curvature of the $k$-th associated bundle. Then $c_i^k (\varepsilon) = c_i (\Omega_k)$.

It is immediate from our results that
\[
\{ c_i^k \mid k \in \mathbb{Z}, i \in \mathbb{N} \}
generates all of $B_{\text{com}}U(n)$ as an algebra. That is, every class in $H^*\left(B_{\text{com}}U(n)\right)$ can be written in the form

$$s = \sum_{j=1}^{m} \alpha_j C_j$$

where $\alpha_j \in \mathbb{R}$ and

$$C_j = \prod_{t=1}^{m_j} c_{i_{j,t}}^{k_{j,t}},$$

where $k_{j,t} \in \mathbb{Z}$ and $i_{j,t} \in \mathbb{N}$. Then it follows that if $\xi$ is a TC structure with underlying vector bundle $E \to M$, with curvature its $\Omega$ and $\Omega_k$ the curvature of the $k$-th associated bundle, then

$$s(\xi) = \sum_{j=1}^{m} \alpha_j \left( \prod_{t=1}^{m_j} c_{i_{j,t}}^{\Omega_{k_{j,t}}} \right) \in H^*(M).$$

At this point it is worth mentioning that $H^*\left(B_{\text{com}}G\right)$ is in general not a polynomial algebra. For example when $G = U(n)$, the TC Chern classes are not algebraically independent. However, the relationships governing them are rather complicated. As such, their values in a given TC structure can vary significantly. We see an example of this in the next chapter.

**Remark 33.** The concepts developed in this section can also be applied to vector bundles on the quaternions. In this case, the structural group is the simplectic group $\text{Sp}(n)$. The main ideas we needed to developed TC characteristic classes also hold for this group. As we saw in the previous section we also have power maps on cohomology, and $H^*\left(B_{\text{com}}\text{Sp}(n)\right)$ is also generated by as an algebra by $\{ \Phi^k \circ \iota(\xi) \mid 1 \leq i \leq n, k \in \mathbb{Z} \setminus \{0\} \}$. Where again $\iota : H^*\left(B\text{Sp}(n)\right, \mathbb{R}\right) \to H^*\left(B_{\text{com}}\text{Sp}(n)\right, \mathbb{R}\right)$ is induced by the natural inclusion $B_{\text{com}}\text{Sp}(n) \to B\text{Sp}(n)$. Also, since $\text{Sp}(n)$ is a compact group, the Chern-Weil homomorphism is in fact an isomorphism. Thus, most of the ideas we used through out this section apply to this case as well.

### 6. Example

In this final section we exhibit explicit calculations of examples using Chern-Weil theory to compute TC characteristic classes. In particular we show there is a TC structure $\xi$ over a 4 sphere such that $c_i(\xi) = 0$ for every $i \in \mathbb{N}$ while $c_2^{-1}(\xi) \neq 0$. This shows that a TC Chern class $c^n_i$ does not necessarily determines another TC Chern class $c^m_i$, if $m \neq n$. This confirms that the underlying vector bundle of a TC structure does not determine completely the TC structure.

For this we develop the computations that allow us to obtain Chern classes in terms of clutching functions, with $SU(2)$ as the structural group. This treatment is based on the concepts presented in [Morita], Chapter 5.

#### 6.1. Connection for a vector bundle with a two sets cover with trivializations:

Let $\pi : E \to M$ denote a smooth vector bundle over $\mathbb{C}$ of dimension $n$, with
$M$ a closed manifold. Assume we can find an open cover $\{U_1, U_2\}$ of $M$ together with trivializations

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$$

$$e \mapsto (\pi(e), h_i(e)) .$$

Suppose these trivializations have structure group a Lie group of matrices $G$. This is, we have a function $\rho : U_1 \cap U_2 \to G \subseteq GL_n(\mathbb{C})$ characterized by

$$\varphi_2 \circ \varphi_1^{-1} : U_1 \cap U_2 \times \mathbb{C}^n \to U_1 \cap U_2 \times \mathbb{C}^n$$

$$(x,v) \mapsto (x,\rho(x)v) .$$

These trivializations induce smooth sections

$$s_{ij} : U_i \to \pi^{-1}(U_i)$$

$$x \mapsto \varphi_j^{-1}(x, e_j) ,$$

where $e_j$ is the $j$-th vector of the standard basis of $\mathbb{C}^n$, and $i = 1, 2$. This setting implies that for $x \in U$, the set $\{s_{i1}(x), s_{i2}(x), \ldots, s_{in}(x)\} \subset \pi^{-1}(x)$ is a basis.

Under these conditions, for a point $x \in U_1 \cap U_2$ it follows that

$$(6.1)\quad s_{1k}(x) = \sum_{l=1}^{n} \rho_{lk}(x) s_{2l}(x)$$

where we take $\rho = [\rho_{lk}]_{k,l=1}^{n}$.

Now let $\{f_1, f_2\}$ be a partition of the unity subordinated to $\{U_1, U_2\}$, as well as the trivial connections over each $U_i$, $\nabla^i$. We can now define the connection

$$\nabla_X s := f_1 \nabla_X^1 s + f_2 \nabla_X^2 s .$$

This means that for a vector field $X$ and a section $s$, we consider their restriction to $U_i$ in order to evaluate $\nabla_X^i$. That is, we need first to consider the decomposition of $s$ in terms of the basis $\{s_{i1}, \ldots, s_{in}\}$, which means that there are smooth functions $\alpha^i_j : U_i \to \mathbb{C}$ such that for $x \in U_i$

$$s \mid_{U_i} (x) = \sum_{j=1}^{n} \alpha^i_j (x) s_{ij}(x) .$$

Then, applying the product rule and the definition, we have

$$\nabla_X s := \sum_{ij} f_i X (\alpha^i_j) s_{ij} .$$

Recall that with $n$-linearly independent sections $\{s_1, \ldots, s_n\}$ we have the local expressions for both the connection and the curvature, $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(E)$. There exists 1-forms $\omega_{ij}$ and two forms $\Omega_{ij}$ such that we can write

$$\nabla_X s_i = \sum_i \omega_{ij} (X) s_j$$

and

$$R(X,Y) (s_i) = \sum_j \Omega_{ij} (X,Y) s_j$$

which gives rise to the local connection and curvature matrices

$$\omega := [\omega_{ij}] \text{ and } \Omega := [\Omega_{ij}] .$$
These local forms are related to the transition function in the following way. From Equality 6.1 we get that
\[ \nabla_X s_{1k} = \sum_{l=1}^{n} f_2 X (\rho_{lk} (x)) s_{2l} (x). \]

From differential geometry we know that for a function \( f : M \to \mathbb{R} \), \( X (f) = df (X) \) holds, where \( d \) is the external derivation. Thus we get the expression
\[ \nabla_X s_{1k} = \sum_{l=1}^{n} f_2 d (\rho_{lk} (x)) (X) s_{2l} (x), \]
which allow us to write
\[ \nabla s_{1k} = \sum_{l=1}^{n} f_2 d (\rho_{lk} (x)) s_{2l} (x). \]

By the properties of cocycles we also know that
\[ s_{2l} (x) = \sum_{t=1}^{n} \rho_{tl}^{-1} (x) s_{1t} (x), \]
where by \( \rho_{tl}^{-1} \) we mean the components of the matrix \( \rho^{-1} \). Thus, we can write
\[ \nabla s_{1k} = \sum_{l=1}^{n} \left( f_2 \left( \sum_{t=1}^{n} \rho_{tl}^{-1} (x) d (\rho_{lk} (x)) \right) \right) s_{1t}. \]

By comparing this expression with the local form, we conclude that
\[ (6.2) \quad \omega^1 = f_2 \rho^{-1} d \rho. \]

Our next step is to obtain the local form of the curvature. For this we use the structural equation (see Morita Theorem 5.21.)
\[ \Omega^i = d \omega^i + \omega^i \wedge \omega^i. \]

Consider the equality \( \rho^{-1} \rho = I \). An application of the product rule allow us to write:
\[ 0 = dI = d (\rho^{-1}) \rho + \rho^{-1} d \rho. \]

This in turn implies that
\[ d (\rho^{-1}) \rho = -\rho^{-1} d \rho \Rightarrow d (\rho^{-1}) = -\rho^{-1} (d \rho) \rho^{-1}. \]

Since \( dd = 0 \), we obtain \( d (\rho^{-1} d \rho) = d (\rho^{-1}) \wedge d (\rho) \), which allow us to conclude that
\[ d \omega^1 = ((df_2) \rho^{-1} d \rho - f_2 \rho^{-1} d \rho \wedge \rho^{-1} d \rho). \]

On the other hand
\[ \omega^1 \wedge \omega^1 = (f_2 \rho^{-1} d \rho \wedge (f_2 \rho^{-1} d \rho) = f_2^2 \rho^{-1} d \rho \wedge \rho^{-1} d \rho, \]
which finally gives us
\[ (6.3) \quad \Omega^1 = ((df_2) \rho^{-1} d \rho - f_2 \rho^{-1} d \rho \wedge \rho^{-1} d \rho) + f_2^2 \rho^{-1} d \rho \wedge \rho^{-1} d \rho. \]

Observe that in a point \( x \notin U_1 \cap U_2 \), \( \Omega^1 \) is zero since the closure of the support of \( f_2 \) is contained in \( U_2 \). Similarly, an analogue formula can be deduce for the local form of the curvature in \( U_2 \), and deduced that it is also zero outside \( U_1 \cap U_2 \). Thus, we can conclude that
Suppose that we have a vector bundle $E \to M$ be a smooth vector bundle with $\{U_1, U_2\}$ an open cover of $M$, both having trivializations of $E$, $\varphi_1$ and $\varphi_2$, respectively. Let $\{f_1, f_2\}$ be a partition of unity associated to $\{U_1, U_2\}$, respectively. If $\rho$ is the transition function associated to $\varphi_2 \circ \varphi_1^{-1}$, then the curvature $\Omega_k$ of the $k$-th associated bundle is given by

$$(\Omega_k)_x = \begin{cases} (\Omega^k_1)_x & x \in U_1 \cap U_2, \\ 0 & x \notin U_1 \cap U_2. \end{cases}$$

Where

$$(6.4) \quad \Omega^k_1 = (df_2) \rho^{-k} d(\rho^k) + (f^2_2 - f_2) \rho^{-k} d(\rho^k) \wedge \rho^{-k} d(\rho^k)$$

is the local expression on $U_1$.

**Proof.** Since the $k$-th associated vector bundle has the same cover associated to its TC structure, with transition functions equal to $\rho^k$, the previous discussion provides a proof of the theorem. \hfill \square

It is worth mentioning that it is possible to deduce a similar formula to (6.3) for a arbitrary number of sets in an open cover, but we will not need this.

### 6.2. Second Chern class for clutching functions with values on $SU(2)$

Suppose that we have a vector bundle $p : E \to M$ in such a way that we can find an open cover $\{U_1, U_2\}$ of $M$ together with a transition function $\rho : U_1 \cap U_2 \to SU(2)$. First, we are going to compute the determinant of the curvature form in terms of the components of the matrices in $SU(2)$,

$$SU(2) := \left\{ \left[ \begin{array}{cc} z & -\bar{w} \\ w & \bar{z} \end{array} \right] | |z|^2 + |w|^2 = 1 \right\}.$$ 

So let us take

$$\rho = \left[ \begin{array}{cc} z & -\bar{w} \\ w & \bar{z} \end{array} \right],$$

for which we want to compute the curvature

$$\Omega = (df_2) \rho^{-1} d(\rho) + (f^2_2 - f_2) \rho^{-1} d(\rho) \wedge \rho^{-1} d(\rho).$$

Since $z\bar{z} + w\bar{w} = 1$, we get differentiating that

$$0 = (\bar{z}dz + \bar{w}dw) + (zd\bar{z} + wdz\bar{w}) \Rightarrow zd\bar{z} + wdz\bar{w} = -(zd\bar{z} + wdz\bar{w})$$

and so we have

$$\tau := \rho^{-1}d\rho = \left[ \begin{array}{cc} \bar{z}dz + \bar{w}dw & \bar{w}dz - \bar{z}dw \\ -wdz + zdw & (\bar{z}dz + \bar{w}dw) \end{array} \right].$$

Now take $\theta := \rho^{-1}d\rho \wedge \rho^{-1}d\rho$. Using that $\tau_{22} = -\tau_{11}$ and $|z|^2 + |w|^2 = 1$ we get that

$$\theta = \left[ \begin{array}{cc} (\bar{w}dz - \bar{z}dw) \wedge (-wdz + zdw) & 2\bar{d}z \wedge \bar{d}w \\ -2dz \wedge dw & (\bar{w}dz - \bar{z}dw) \wedge (-wdz + zdw) \end{array} \right].$$

which is the same as expressing it as

$$\theta = \left[ \begin{array}{cc} \tau_{12} \wedge \tau_{21} & \theta_{12} \\ \theta_{21} & -\tau_{12} \wedge \tau_{21} \end{array} \right].$$

Now, by making $f := f_2$ and $g := (f - 1)$ we may express the curvature as

$$\Omega = df\tau + g\theta = \left[ \begin{array}{cc} df\tau_{11} + g\tau_{12} \wedge \tau_{21} & df\tau_{12} + g\theta_{12} \\ df\tau_{21} + g\theta_{21} & -(df\tau_{11} + g\tau_{12} \wedge \tau_{21}) \end{array} \right].$$
and its determinant is then given by
\[
\det(\Omega) = -(df \tau_{11} + g \tau_{12} \wedge \tau_{21}) \wedge (df \tau_{11} + g \tau_{12} \wedge \tau_{21}) \\
- (df \tau_{21} + g \theta_{21}) \wedge (df \tau_{12} + g \theta_{12}).
\]

In order to reduce this expression, we recall that the wedge product of a one form with itself is zero. Also, one forms commute with two forms, so we get:
\[
\det(\Omega) = -g^2 \theta_{12} \wedge \theta_{21} - gdf \wedge (\tau_{12} \wedge \theta_{21} + \tau_{21} \wedge \theta_{12} + 2\tau_{11} \wedge \tau_{12} \wedge \tau_{21}).
\]

By recalling that \(\tau_{11} \wedge \tau_{12} = dz \wedge dw\) we get:
\[
\tau_{11} \wedge \tau_{12} \wedge \tau_{21} = -(zd\bar{z}d\bar{w} + zd\bar{z}d\bar{w})
\]
and
\[
\tau_{21} \wedge \theta_{12} = -2(zd\bar{z}d\bar{w} + zd\bar{z}d\bar{w}).
\]
Now take
\[
A := \tau_{12} \wedge \theta_{21} + \tau_{21} \wedge \theta_{12} + 2\tau_{11} \wedge \tau_{12} \wedge \tau_{21}
\]
then
\[
A = 2\left((zd\bar{z}d\bar{w} + zd\bar{z}d\bar{w}) - (zd\bar{z}d\bar{w} + zd\bar{z}d\bar{w})
- (zd\bar{z}d\bar{w} + zd\bar{z}d\bar{w})
\right)
\]
which gives us
\[
A = 2(zd\bar{z}d\bar{w} + zd\bar{z}d\bar{w} - 2(zd\bar{z}d\bar{w} + zd\bar{z}d\bar{w}))
\]
which we can now replace to have
\[(6.5) \quad \det(\Omega) = 4 (f_2 - 1)^2 f_2^2 zd\bar{z}d\bar{w} - (f_2 - 1) f_2 df_2 \wedge A.\]

Now we are going to use this formula to find the second Chern class in terms of a smooth Clutching function \(\varphi : S^3 \to SU(2)\) (see [Hatcher II, Chapter 1]). Consider the sets
\[
S^4 = \{x = (x_1, \ldots, x_5) \in \mathbb{R}^5 \mid \|x\| = 1\},
\]
\[
D_+ = \{(x_1, \ldots, x_5) \in S^4 \mid x_5 \geq 0\},
\]
\[
D_- = \{(x_1, \ldots, x_5) \in S^4 \mid x_5 \leq 0\}
\]
and the open set
\[
V = \{(x_1, \ldots, x_5) \in S^4 \mid -1/3 < x_5 < 1/3\}.
\]
Also let
\[
U_1 := D_+ \cup V \text{ and } U_2 := D_- \cup V
\]
and identify \(S^3\) with the equator \(\{(x_1, \ldots, x_5) \in S^4 \mid x_5 = 0\}\).

Using "bump" functions we can obtain a partition of the unity \(f_1, f_2 : S^4 \to [0, 1]\) such that they depend only on the "height" \(x_5\) and \(f_1|_{U_1 \setminus V} = 1\). Also the clutching function \(\varphi : S^3 \to SU(2)\) can be composed with a smooth "perpendicular" retraction of \(V\) to \(S^3\), to obtain a transition function \(\rho : V \to SU(2)\) independent of \(x_5\).

Under these conditions is clear that
\[
\bullet \ df_2 = \frac{\partial f_2}{\partial r} dr, \text{ and}
\]
• If 
\[\rho = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}\]
any four form depending on \(z, \bar{z}, w\) and \(\bar{w}\) is zero, since these functions depend only on three variables.

We are in position to apply the previous results to obtain that 
\[\det(\Omega) = 4(1 - f_2)^2 f_2^2 dzd\bar{z}dwd\bar{w} - (1 - f_2) f_2 df_2 \wedge A.\]

Where 
\[A = 2(\bar{z}dzd\bar{w} + \bar{w}zd\bar{z}d\bar{w} + \bar{w}zd\bar{z}d\bar{w}).\]
However, by construction we have that 
\[dzd\bar{z}dwd\bar{w} = 0\]
and so
\[\int_{S^4} \det(\Omega) = \left(\int_{-1}^{1} (1 - f_2) f_2 \frac{df_2}{dr}\right) dr \int_{S^3} A.\]

First, notice that by construction it follows that
\[\int_{-1}^{1} (1 - f_2) f_2 \frac{df_2}{dr} dr = -\frac{1}{6}.\]

Finally since the second Chern class in this case is the determinant of the curvature times \(\left(\frac{1}{2\pi}\right)^2\), we get

**Proposition 35.** The second Chern class associated to a clutching function \(\varphi : S^3 \to SU(2)\) is given by
\[c_2 = \frac{1}{24\pi^2} \int_{S^3} A.\]

Here \(A\) is a 3-form given by
\[2(\bar{z}dzd\bar{w} + \bar{w}zd\bar{z}d\bar{w} - 2(zd\bar{z}d\bar{w} + wzd\bar{z}d\bar{w}))\]
and the functions \(z, w : S^3 \to SU(2)\) are determined by the clutching function,
\[\varphi = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}.\]

6.3. **A non trivial TC structure over a trivial vector bundle.** It is already known that there are trivial vector bundles with non trivial TC structures over them. In this section we are going to use such a structure to show that:

**Theorem 36.** There exists a TC structure 
\[\xi = \{E \to S^4, \{U_1, U_2, U_3\}, \rho_{ij} : U_i \cap U_j \to SU(2)\}\]
such that \(E \to S^4\) is a trivial bundle, and such that \(c_2^{-1}(\xi) = -1\), implying that the TC structure is non trivial.

This in particular highlights how the TC characteristic classes depends on the TC structure and not on the equivalence class of their underlying bundle.

Now, to prove this theorem we are based on the construction made by D. Ramras and B. Villareal ([RV], Chapter 3). In what follows, we first define the vector bundle by defining an open cover on \(S^4\) and transition functions on them. This defines a TC structure 
\[\xi = \{E \to S^4, \{U_1, U_2, U_3\}, \rho_{ij} : U_i \cap U_j \to SU(2)\}\].

Then by considering the \((-1)\)-powers of these transition functions we also obtain the \((-1)\)-th associated bundle, \(E^{-1}\).
Next we are going to use Lemma 3.1 of [RV] to show that both vector bundles obtained can be described, up to isomorphism, by a given clutching functions. Then on one hand by showing that the clutching function associated to $E$ is trivial, we conclude that $E$ is trivial. On the other hand, we use the clutching function associated to $E^{-1}$ together with the formulas of the previous sections, to conclude that $c_2^{-1}(\xi) = -1$.

We outline how their initial construction can be made in the smooth category, which allows us to reduce the problem of computing the Chern class by using Clutching functions.

We are constructing a TC structure on a vector bundle defined over $S^4$ in terms of a triple open cover $\{U_1, U_2, U_3\}$ and transition functions between them. These transition functions themselves will be described in terms of two functions $\rho_1, \rho_2 : D_3 \to SU(2)$, where $D_3$ is the 3-dimensional closed disk of radius 1.

For this, take $V = D_3 \cap U_1$.

Now suppose that the functions $\rho_1, \rho_2 : D_3 \to SU(2)$ are smooth functions such that:

- They are independent of the radius in $D_3$ in $V$.
- They are commutative in the closure of $V$.

We define the transition functions $\rho_{ij} : U_i \cap U_j \to SU(2)$ by

\[ \rho_{ij} := \rho_i \circ r. \]
• \( \rho_{23} := \rho_2 \circ r \).
• \( \rho_{13} := (\rho_1 \circ r) (\rho_2 \circ r) \).

Since \( r \left( U_1 \cap U_2 \cap U_3 \right) \subseteq V \) by construction, the previous cocycles commute with each other in their common domain \( U_1 \cap U_2 \cap U_3 \). This transition functions allow us to construct a smooth vector bundle \( E \to S^4 \), and so we have constructed a TC structure

\[
\xi = \left\{ E \to S^4, \{U_1, U_2, U_3\}, \rho_{ij} : U_i \cap U_j \to SU(2) \right\}.
\]

6.4. Associated clutching functions: Before dealing with the result we need, it is important to highlight the following. Suppose \( E_1 \to M \) and \( E_2 \to M \) are smooth vector bundles with classifying functions \( f_i : M \to BSU(n) \), \( i = 1, 2 \). If there is a (non necessarily continuous) homotopy between \( f_1 \) and \( f_2 \), and there is class \( c \in H^* (BSU(n)) \), it follows that \( f_1^* (c) = f_2^* (c) \in H^* (M) \). Now consider the curvatures \( \Omega_1 \) and \( \Omega_2 \) for \( E_1 \) and \( E_2 \), respectively. By the Chern-Weil isomorphism, we get that \( c(\Omega_1) = f_1^* (c) \) and \( c(\Omega_2) = f_2^* (c) \), and thus \( c(\Omega_1) = c(\Omega_2) \). In particular if there is a continuous (but not smooth) isomorphism of vector bundles between \( E_1 \) and \( E_2 \), their classifying functions will be homotopic and their characteristic classes will coincide.

Now consider the closed sets

\[
C_1 := \left\{ (x_1, \ldots, x_5) \in S^4 \mid x_5 \geq 0 \right\},
\]

\[
C_2 := \left\{ (x_1, \ldots, x_5) \in S^4 \mid x_5 \leq 0, x_4 \geq 0 \right\}
\]

and

\[
C_3 := \left\{ (x_1, \ldots, x_5) \in S^4 \mid x_5 \leq 0, x_4 \leq 0 \right\}.
\]

It is clear that there is a retraction \( r_i : U_i \to C_i \) leaving \( C_i \) fixed, for \( i = 1, 2, 3 \). Notice that by applying on \( U_2 \cap U_3 \) \( r_2 \) first and then \( r_3 \), we obtain a retraction \( r_{23} : U_2 \cap U_3 \to C_2 \cap C_3 \) leaving \( C_2 \cap C_3 \) fixed. For \( U_1 \cap U_2 \) we apply first \( r_2 \) and then \( r_3 \), we obtain a retraction \( r_{12} : U_1 \cap U_2 \to C_1 \cap C_2 \) leaving \( C_1 \cap C_2 \) fixed, and similarly we obtain \( r_{13} : U_1 \cap U_3 \to C_1 \cap C_3 \) leaving \( C_1 \cap C_3 \) fixed. Via this restrictions of \( \rho_{ij} \) we obtain transition functions for the closed cover \( \{C_1, C_2, C_3\} \):

\[
\tilde{\rho}_{ij} : C_i \cap C_j \to SU(2).
\]

This new transition functions are clearly homotopic to \( \rho_{ij} \) via the retractions \( r_{ij} \). Thus, they characterized vector bundles over \( S^4 \) whose classifying functions are homotopic.

Consider the identification \( S^3 \cong \left\{ (x_1, \ldots, x_5) \in S^4 \mid x_5 = 0 \right\} \). This setting allow us to apply Lemma 3.1 of [RV]. There they show that the bundle induced by these three cocycles is isomorphic to the vector bundle with clutching function \( \varphi : S^3 \to SU(2) \) defined for \( x = (x_1, \ldots, x_5) \) by

\[
\varphi (x) := \begin{cases} 
\rho_1 (r (x)) \rho_2 (r (x)) & x_4 \geq 0, \\
\rho_1 (r (x)) \rho_2 (r (x)) & x_4 \leq 0.
\end{cases}
\]

The function \( \varphi \) can clearly be extended continuously to the whole disk \( D_- \), since we defined \( r \) on \( D_- \). This implies that \( \varphi \) is null homotopic, and thus, the vector bundle given by these cocycles is trivial.

Now lets consider the same construction but using the cocycles given by \( \sigma_{ij} = \rho_{ij}^{-1} \). They give rise to the \((-1)\)-th associated bundle by definition. Once again

\[
\tilde{\rho}_{ij} : C_i \cap C_j \to SU(2).
\]
allow us to use Lemma 3.1 of [RV]. We conclude that this bundle can be obtain, up to isomorphisms, by the clutching function given by

\[ \phi(y) := \begin{cases} 
\rho_1^{-1}(r(x)) \rho_2^{-1}(r(x)) & x_4 \geq 0, \\
\rho_2^{-1}(r(x)) \rho_1^{-1}(r(x)) & x_4 \leq 0.
\end{cases} \]

In this case this function cannot be extended continuously to \( D_- \) if \( \rho_1 \) and \( \rho_2 \) do not commute everywhere in \( D_3 \). So \( \phi \) is not necessarily null homotopic.

6.5. **Existence of a non trivial TC structure:** From the previous part, we need to show that it is possible to obtain a non null homotopic clutching function \( \phi \). For this it is enough to display two functions \( \rho_1, \rho_2 : D_3 \to SU(2) \) such that they commute in \( \partial D_3 \equiv S^3 \), giving us a non zero Chern class for the bundle with clutching function \( \phi : S^3 \to SU(2) \).

We can describe \( \phi \) in terms of the northern and southern hemispheres of \( S^3 \). Each of them can be identify with the 3-dimensional disc \( D_3 \). Then we get that

\[ \phi(y) := \begin{cases} 
\rho_1^{-1} \rho_2^{-1} & \text{in } D_+, \\
\rho_2^{-1} \rho_1^{-1} & \text{in } D_-.
\end{cases} \]

For brevity allow us to write the matrices of \( SU(2) \) as

\[ (a, b) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \]

**Proposition 37.** Consider \( D_3 \) under spherical coordinates and take

\[ \rho_1(\alpha, \beta, r) := \begin{cases} 
\left( \sin \left( \frac{\pi}{2} r \right) e^{i\alpha}, \cos \left( \frac{\pi}{2} r \right) \right), & \text{if } 0 \leq \beta \leq \pi/2, \\
\left( \sin (r\beta) e^{i\alpha}, \cos (r\beta) \right) & \text{if } \pi/2 \leq \beta \leq \pi.
\end{cases} \]

\[ \rho_2(\alpha, \beta, r) := \begin{cases} 
\left( -\cos (\pi r) e^{2i\beta}, \sin (\pi r) \right), & \text{if } 0 \leq \beta \leq \pi/2, \\
\left( \cos (\pi r), \sin (\pi r) \right) & \text{if } \pi/2 \leq \beta \leq \pi.
\end{cases} \]

then the second Chern class of \( \phi \) is \( c_2(\phi) = -1 \).

**Proof.** Recalled from the previous section that if we make \( \phi = (z, w) \), the second Chern class of \( \phi \) is then given by

\[ c_2 = \frac{1}{24\pi^2} \int_{S^3} A. \]

Where \( A \) is a three form given by

\[ 2 (\bar{z}d\bar{z}dw\bar{w} + \bar{w}d\bar{z}d\bar{w}) - 2 (zd\bar{z}d\bar{w} + wd\bar{z}d\bar{w}) \).

We can split this integral as

\[ \int_{S^3} A = \int_{D_-} A + \int_{D_+} A. \]

Now, call \( \rho_1^{-1} \rho_2^{-1} = (z_1, w_1) \) and \( \rho_2^{-1} \rho_1^{-1} = (z_2, w_2) \). Because of orientations, we get

\[ \int_{S^3} A = \int_{D_3} A_2 - \int_{D_3} A_1 = \int_{D_3} (A_2 - A_1) \]

where

\[ A_1 = 2 (\bar{z}_1dz_1dw_1d\bar{w}_1 + \bar{w}_1dz_1d\bar{z}_1dw_1 - 2 (z_1d\bar{z}_1dw_1d\bar{w}_1 + w_1dz_1d\bar{z}_1d\bar{w}_1)) \]

and

\[ A_2 = 2 (\bar{z}_2dz_2dw_2d\bar{w}_2 + \bar{w}_2dz_2d\bar{z}_2dw_2 - 2 (z_2d\bar{z}_2dw_2d\bar{w}_2 + w_2dz_2d\bar{z}_2d\bar{w}_2)) \]
$A_2 = 2(\bar{z}_2 d\bar{z}_2 d\bar{w}_2 + \bar{w}_2 d\bar{z}_2 d\bar{w}_2 - 2(\bar{z}_2 d\bar{z}_2 d\bar{w}_2 + w_2 d\bar{z}_2 d\bar{w}_2))$. 

From this we have for $0 \leq \beta \leq \pi/2$ that

$$(z_2, w_2) = \rho_2^{-1} \rho_1^{-1} = \left(-\sin \left(\frac{\pi}{2} r\right) \cos (\pi r) e^{-(\alpha+2\beta)i} - \sin (\pi r) \cos \left(\frac{\pi}{2} r\right)\right)$$. 

$$(z_1, w_1) = \rho_1^{-1} \rho_2^{-1} = \left(-\sin \left(\frac{\pi}{2} r\right) \cos (\pi r) e^{-(\alpha+2\beta)i} - \sin (\pi r) \cos \left(\frac{\pi}{2} r\right)\right)$$. 

while for $\pi/2 \leq \beta \leq \pi$ we have

$$(z_2, w_2) = \rho_2^{-1} \rho_1^{-1} = (\sin (r\beta) \cos (\pi r) e^{-\alpha i} - \sin (\pi r) \cos (r\beta)) - \sin (\pi r) \sin (r\beta) e^{i\alpha} - \cos (\pi r) \cos (r\beta))$$

$$(z_1, w_1) = \rho_1^{-1} \rho_2^{-1} = (\sin (r\beta) \cos (\pi r) e^{-\alpha i} - \sin (\pi r) \cos (r\beta)) - \sin (\pi r) \sin (r\beta) e^{i\alpha} - \cos (\pi r) \cos (r\beta))$$.

Observe that in both cases we have that $z_1 = z_2$ and $w_1 = \bar{w}_2$. Then we have to integrate the form

$$A_2 - A_1 = 4(2z_1 d\bar{z}_1 - \bar{z}_1 dz_1) dw_1 d\bar{w}_1 + 6(\bar{w}_1 dw_1 - \bar{w}_1 d\bar{w}_1) d\bar{z}_1 dz_1$$.

Now consider the decomposition $z_1 = x + yi$ and $w_1 = w = u + vi$, where $x, y, u$ and $v$ are real functions. Then it follows that

$$(2z_1 d\bar{z}_1 - \bar{z}_1 dz_1) = (xdx + ydy) + 3(ydx - xdy)i$$

$$dw_1 d\bar{w}_1 = -2i du \wedge dv$$,

$$(w_1 dw_1 - \bar{w}_1 dw_1) = 2(vdu - udv)i$$

and $dzd\bar{z} = -2idx \wedge dy$. This gives us

$$B_1 = 6(ydx - xdy) du \wedge dv - 2i(xdx + ydy) du \wedge dv$$

and

$$B_2 = 4(vdu - udv) dx \wedge dy$$.

Since we only need to compute the real part of the first form, we consider the form $6(ydx - xdy) du \wedge dv$ instead of all of $B_1$. Then by definition we get

$$ydx - xdy = \left(\frac{\partial x}{\partial \alpha} - x \frac{\partial y}{\partial \alpha}\right) d\alpha + \left(\frac{\partial x}{\partial \beta} - x \frac{\partial x}{\partial \beta}\right) d\beta + \left(y \frac{\partial x}{\partial \alpha} - x \frac{\partial y}{\partial \alpha}\right) dr$$,

$$vdu - udv = \left(v \frac{\partial u}{\partial \alpha} - u \frac{\partial v}{\partial \alpha}\right) d\alpha + \left(v \frac{\partial u}{\partial \beta} - u \frac{\partial x}{\partial \beta}\right) d\beta + \left(v \frac{\partial u}{\partial \alpha} - u \frac{\partial v}{\partial \alpha}\right) dr$$,

$$du \wedge dv = \left(\frac{\partial u}{\partial \alpha} \frac{\partial v}{\partial \beta} - \frac{\partial v}{\partial \alpha} \frac{\partial u}{\partial \beta}\right) d\alpha d\beta + \left(\frac{\partial u}{\partial \alpha} \frac{\partial u}{\partial \alpha} - \frac{\partial v}{\partial \alpha} \frac{\partial v}{\partial \beta}\right) d\alpha dr + \left(\frac{\partial u}{\partial \beta} \frac{\partial u}{\partial \beta} - \frac{\partial v}{\partial \beta} \frac{\partial u}{\partial \beta}\right) d\beta dr$$

and

$$dx \wedge dy = \left(\frac{\partial x}{\partial \alpha} \frac{\partial y}{\partial \beta} - \frac{\partial y}{\partial \alpha} \frac{\partial x}{\partial \beta}\right) d\alpha d\beta + \left(\frac{\partial x}{\partial \alpha} \frac{\partial y}{\partial \beta} - \frac{\partial y}{\partial \alpha} \frac{\partial x}{\partial \beta}\right) d\alpha dr + \left(\frac{\partial x}{\partial \beta} \frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta} \frac{\partial y}{\partial \beta}\right) d\beta dr$$.

Now call $J_1 := (ydx - xdy) du \wedge dv$ and $J_2 := (ydx - xdy) du \wedge dv$. Then we have
\[ J_1 = \left[ \left( y \frac{\partial x}{\partial \alpha} - x \frac{\partial y}{\partial \alpha} \right) \left( \frac{\partial u}{\partial \beta} \frac{\partial v}{\partial r} - \frac{\partial v}{\partial \beta} \frac{\partial u}{\partial r} \right) - \left( y \frac{\partial x}{\partial \beta} - x \frac{\partial y}{\partial \beta} \right) \left( \frac{\partial u}{\partial \alpha} \frac{\partial v}{\partial r} - \frac{\partial v}{\partial \alpha} \frac{\partial u}{\partial r} \right) \right] \, d\alpha \wedge d\beta \wedge dr, \]

\[ J_2 = \left[ \left( v \frac{\partial u}{\partial \alpha} - u \frac{\partial v}{\partial \alpha} \right) \left( \frac{\partial x}{\partial \beta} \frac{\partial y}{\partial r} - \frac{\partial y}{\partial \beta} \frac{\partial x}{\partial r} \right) - \left( v \frac{\partial u}{\partial \beta} - u \frac{\partial x}{\partial \beta} \right) \left( \frac{\partial u}{\partial \alpha} \frac{\partial v}{\partial r} - \frac{\partial v}{\partial \alpha} \frac{\partial u}{\partial r} \right) \right] \, d\alpha \wedge d\beta \wedge dr. \]

Then by replacing we get \( A_2 - A_1 = 24 \left( J_1 + J_2 \right), \) and even further
\[ c_2 (\phi) = \frac{1}{\pi^2} \int \left( J_1 + J_2 \right). \]

Where by using computational software we obtain that \( \int \left( J_1 + J_2 \right) = -\pi^2, \) giving us \( c_2 (\phi) = -1. \)

We conclude that \( c_2^{-1} (\xi) = -1 \) for our TC structure, implying that the TC structure is non trivial. \( \square \)

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