Anisotropic fluid for a set of non-diagonal tetrads in 
\( f(T) \) gravity

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Abstract

We consider \( f(T) \) gravity for a Weitzenbock’s spherically symmetric and static spacetime, where the metric is projected in the tangent space to the manifold, for a set of non-diagonal tetrads. The matter content is coupled through the energy momentum tensor of an anisotropic fluid, generating various classes of new black hole and wormhole solutions. One of these classes is that of cold black holes. We also perform the reconstruction scheme of the algebraic function \( f(T) \) for two cases where the radial pressure is proportional to \( f(T) \) and its first derivative.

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1 Introduction

A special attention is now attached to the so-called Teleparallel Theory (TT). This is a geometric theory which possesses only torsion, without curvature, for characterising the gravitational interaction between matter fields. The TT, which is dynamically equivalent to the General Relativity (GR) \([2]\),

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possesses as fundamental field the tetrads ones with which the Weitzenbock’s connection is generated [1]. The contraction of the Weitzenbock’s connection through a linear combination defines the action of the theory.

With the progress of the measurements about the evolution of the universe, as the expansion and the acceleration, the dark matter and dark energy, various proposals for modifying the GR are being tested. As the unifications theories, for the scales of low energies, it appears in the effective actions besides the Ricci scalar, the terms $R^2, R^\mu_\nu R_\mu^\nu$ and $R^\mu_\nu R_\mu^\nu R^\alpha_\beta$ and the proposal of modified gravity that agrees with the cosmological and astrophysical data is $f(R)$ theory [3, 4]. The main problem that one faces with this theory is that the equation of motion is of order 4, being more complicated than the GR for any analysis. Since the GR possesses the TT as analogous, it has been thought the so-called $f(T)$ gravity, $T$ being the torsion scalar, which would be the analogous to the generalizes GR, the $f(R)$ gravity. The $f(T)$ gravity is the generalization of the TT as we shall see later. Note also that the $f(T)$ theory is free of the curvature, i.e., it is defined from the Weitzenbock’s connection. However, it has been shown recently that this theory breaks the invariance of the local Lorentz transformations [5]. Other recent problem is that the $f(T)$ gravity appears to be dependent on the used frame, i.e., it is not covariant [5, 6].

The $f(T)$ gravity appears in Cosmology as the source driving the inflation [7]. Also, it has been used as a theory that reproduces the acceleration of the universe without the necessity of introducing the dark energy [8, 9, 10]. The contributions of the cosmological perturbations of this theory have been studied [11]. In gravitation, the first model of black hole which has been analysed was the BTZ [12]. Recently, also in the framework of $f(T)$ gravity, several black hole and wormhole solutions with spherical symmetry have been found [13, 14, 15]. Moreover, other analysis about various themes are being investigated in the context of $f(T)$ gravity [21].

In this paper, we study some solutions obtained by fixing the spherical symmetries and the staticity of the metric, as well as the more usual methods of GR [16]. Introducing an anisotropic matter content, we obtain new black hole and traversable wormhole solution for the $f(T)$ gravity, considering a set of non-diagonal tetrads.

The paper is organized as follows. In Section 2, we present a brief revision of the fundamental concepts of the Weitzenbok’s geometry, the action of the $f(T)$ gravity and the equations of motion. In Section 3, we fix the symmetries of the geometry and present the equations of the energy density, the radial and tangential pressures. The Section 4 is devoted for obtaining new solutions in the $f(T)$ gravity. In Section 5, we present a summary of the reconstruction method for the static case of the $f(T)$ theory and reconstruct two simplest cases linked with the radial pressure. The conclusion and perspectives are presented in the Section 6.
2 The field equations from \( f(T) \) theory

The mathematical concept of the \( f(T) \) gravity is based on the Weitzenbock’s geometry and there exists some excellent works in this way [2, 17, 18]. Our convention and nomenclature are the following: the Latin index describe the elements of the tangent space to the manifold (spacetime), while the Greek ones describe the elements of the spacetime. For a general spacetimes metric, we can define the line element as

\[
dS^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{1}
\]

This metric can be projected in the tangent space to the manifold, using the representation of the tetrad matrix, where the line element is

\[
dS^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ij}\theta^i \theta^j, \tag{2}
\]

\[
dx^\mu = e^i_\mu \theta^i, \quad \theta^i = e^i_\mu dx^\mu, \tag{3}
\]

where \( \eta_{ij} = \text{diag}[1, -1, -1, -1] \) and \( e^\mu_i e^i_\nu = \delta^\mu_\nu \) or \( e^\mu_i e^i_\mu = \delta^i_j \). The square root of the metric determinant is given by \( \sqrt{-g} = \det[e^i_\mu] = e \). Now, we describe the spacetime through the tetrad matrix and then define the Weitzenbock’s connection as

\[
\Gamma^\alpha_{\mu\nu} = e^i_\alpha \partial^i e^j_{\mu} = -e^i_\mu \partial^i e^\alpha_{\nu}. \tag{4}
\]

From the definition of connection (4), the spacetime possesses an identically null curvature, having only the torsion contribution and its related quantities for this geometry. Due to the fact that the antisymmetric part of the connection does not vanish, we can define directly from the components of the connection, the torsion tensor, whose components are given by

\[
T^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = e^\alpha_i \left( \partial_\mu e^i_\nu - \partial_\nu e^i_\mu \right). \tag{5}
\]

Through the torsion tensor, we can define two important tensors in this geometry: the contorsion and the tensor \( S \), whose components can be written as

\[
K^\mu^\nu_\alpha = -\frac{1}{2} (T^\mu^\nu_\alpha - T^\nu^\mu_\alpha - T^\alpha_{\mu\nu}) \tag{6},
\]

\[
S^\alpha_{\mu\nu} = \frac{1}{2} \left( K^\mu^\nu_\alpha + \delta^\mu_\alpha T^\beta^\nu_\beta - \delta^\nu^\beta T^\beta^\mu_\alpha \right). \tag{7}
\]

We are now able to define easily the scalar that makes up the action of \( f(T) \) gravity, i.e. the torsion scalar \( T \). Through (5)-(7), we define the scalar torsion scalar as

\[
T = T^\alpha_{\mu\nu}S^\alpha_{\mu\nu}. \tag{8}
\]

Following the same idea of the GR and \( f(R) \) theory for the coupling of the geometry with the matter part, we define the action of the \( f(T) \) gravity as

\[
S[e^\mu_i, \Phi_A] = \int d^4x \left[ \frac{1}{16\pi} f(T) + \mathcal{L}_{\text{Matter}}(\Phi_A) \right], \tag{9}
\]
where we used the units $G = c = 1$ and the $\Phi_A$ are the matter fields. Considering the action (9) as a functional of the fields $e_i^\mu$ and $\Phi_A$, and vanishing the variation of the functional with respect to the field $e_i^\nu$, i.e. the principle of minimum action, one obtains the following equation of motion [5]

$$S_{\nu\rho} \partial_\rho T_{\mu\nu} + \left[ e^{-1} e_i^\mu \partial_\mu (ee_i^\alpha S_{\alpha\nu}) + T^\alpha_{\lambda\mu} S_{\alpha\nu} \right] f_T + \frac{1}{4} \delta_\nu^\mu f = 4\pi T_\mu^\nu ,$$

(10)

where $T_{\mu\nu}$ is the energy momentum tensor, $f_T = df(T)/dT$ and $f_{TT} = d^2 f(T)/dT^2$. If we consider $f(T) = a_1 T + a_0$, the TT is recovered with a cosmological constant.

We now introduce the matter content as being described by an anisotropic fluid, whose energy-momentum components are given by

$$T_{\mu\nu} = (\rho + p_t) u_\mu u_\nu - p_t \delta_\mu^\nu + (p_r - p_t) v_\mu v_\nu ,$$

(11)

where $u_\mu$ is the four-velocity, $v^\mu$ the unitary space-like vector in the radial direction, $\rho$ the energy density, $p_t$ the pressure in the direction of $v^\mu$ (tangential pressure) and $p_r$ the pressure orthogonal to $v_\mu$ (tangential pressure). Since we are assuming an anisotropic spherically symmetric matter, one has $p_r \neq p_t$, such that their equality corresponds to an isotropic fluid sphere.

In the next section, we will make some considerations for the manifold symmetries in order to obtain simplifications in the equations of motion and the specific solutions of these symmetries.

3 Spherically symmetric geometry

We consider from the beginning the tetrads matrix as the fundamental fields of the $f(T)$ theory. Now, in the same way that the frames were constructed for the TT theory in [22] and [23], our tetrad ansatz is elaborated fixing the degree of freedom as follows $e_0^\mu = u^\mu$, where $u^\mu$ is the four velocity of an observer in free fall, and $e_1^\mu, e_2^\mu$ and $e_3^\mu$ are oriented along the unitary vectors in the Cartesian directions $x$, $y$ and $z$, resulting in the matrix

$$\{e_i^\mu\} = \begin{pmatrix}
 e^{a/2} & 0 & 0 & 0 \\
 0 & e^{b/2} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
 0 & e^{b/2} \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
 0 & e^{b/2} \cos \theta & -r \sin \theta & 0
\end{pmatrix} ,
$$

(12)

for spherical and static symmetries. Using the relations (2) and (3), one can write the components of the metric, through the line element of spherically symmetric and static spacetimes as

$$dS^2 = e^{a(r)} dt^2 - e^{b(r)} dr^2 - r^2 \left[ d\theta^2 + \sin^2(\theta) d\phi^2 \right] .
$$

(13)

This choice of tetrad matrices is not unique, because the aim of letting the line element invariant under local Lorentz transformations, is to obtain the form (2). Other choices have been performed with
diagonal matrices, as in references [14, 15]. Using (12), one can obtain \( e = \det [e^\mu_\nu] = e^{(a+b)/2}r^2 \sin(\theta) \), and with (4)-(8) we determine the torsion scalar in terms of \( r \)

\[
T(r) = \frac{2e^{-b}}{r^2} \left( e^{b/2} - 1 \right) \left( e^{b/2} - 1 - ra' \right),
\]

where the prime ('') denotes the derivative with respect to the radial coordinate \( r \). One can now re-write the equations of motion (10) for an anisotropic fluid as

\[
4\pi\rho = \frac{f}{4} - \left( \frac{T}{2} - \frac{1}{r^2} - e^{-b/r^2} (1 - rb') \right) \frac{f_T}{2} - \frac{e^{-b/2}}{r} \left( e^{-b/2} - 1 \right) (f_T)',
\]

\[
4\pi p_r = \left( \frac{T}{2} + \frac{e^{-b}}{r^2} (1 + ra') - \frac{1}{r^2} \right) \frac{f_T}{2} - \frac{f}{4},
\]

\[
4\pi p_t = \frac{e^{-b}}{2} \left( \frac{a'}{2} + \frac{1}{r} - \frac{e^{b/2}}{r} \right) (f_T)' + \left[ \frac{T}{2} + e^{-b} \left( \frac{a''}{4} + \frac{1}{2r} \right) (a' - b') \right] \frac{f_T}{2} - \frac{f}{4},
\]

where \( p_r \) and \( p_t \) are the radial and tangential pressures respectively. In the case where there does not exist a set of non-diagonal tetrads, as in [14, 15], we get an off-diagonal equation (component \( \theta - r \)) which imposed to the algebraic function \( f(T) \) to be a linear function of \( T \), or the torsion scalar had to be a constant, with a free \( f(T) \). Now, in this present paper, with the choice of a set of non-diagonal tetrads, we do not get such constraint equation, but rather, \( f(T) \) may assume an arbitrary functional form. In the next section, we will determine new solutions for the \( f(T) \) theory making some consideration about the matter component \( p_r(r) \).

4 New solutions for a set of non-diagonal tetrads

In this section, we will study two simplest cases for a set of non-diagonal tetrads. The first is when the radial pressure \( p_r \), in (16), is identically null. This has been done originally in GR by Florides [16] and used later by Boehmer et al [19], and in \( f(T) \) gravity, the same condition has been used for the cases of diagonal tetrads [14, 15]. We can classify these two cases:

1. Considering \( p_r = 0 \) in (16), we get

\[
f(T) = 2f_T(T) \left\{ \frac{T}{2} + \frac{1}{r^2} \left[ e^{-b}(1 + ra') - 1 \right] \right\}.
\]

We can explain two main cases here:

(a) Considering \( f(T) = \exp[a_1T] = \sum_{n=0}^{\infty} \frac{(a_1T)^n}{n!} \) in (18), one obtains

\[
T(r) = \frac{1}{a_1} - \frac{2}{r^2} \left[ e^{-b}(1 + ra') - 1 \right].
\]

Choosing the quasi-global coordinate \( a(r) = -b(r) \) and equating (14) with (19), one gets

\[
e^{a(r)} = e^{-b(r)} = \frac{-3 + r(6a_1 + r^2) + 2\sqrt{3} \sqrt{3a_1^2 r^2 + a_1 r^4 - 3a_1 r}}{12a_1 r}.
\]
This is a black hole solution whose the horizon is given by \( r_H = \sqrt{3} \), for \( a_1 < 0 \). The torsion scalar (19) is given by

\[
T(r) = \frac{-6\sqrt{3}a_1^2 r + 3(2a_1 + r^2)\sqrt{a_1 r(3a_1 r + r^3 - 3)} - \sqrt{3}a_1 (4r^3 - 3)}{6a_1 r^2 \sqrt{a_1 r(3a_1 r + r^3 - 3)}} .
\]  

(b) Considering

\[
T(r) = \frac{2}{r^2} \left[ e^{-b}(1 + ra') - 1 \right] ,
\]

the equation (18) yields

\[
f(T) = \sqrt{\frac{T}{T_0}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n!)}{(1 - 2n)(n!)^2 (4^n)} \left( \frac{T - T_0}{T_0} \right)^n .
\]  

When we impose

\[
e^a(r) = \left( 1 - \frac{r_H}{r} \right)^k ,
\]

with \( k \geq 2 \), equating (14) with (22), we obtain

\[
e^{b(r)} = \left[ \frac{2r + (k - 2)r_H}{2(r - r_H)} \right]^2 , \quad T(r) = \frac{-2k^2 r_H^2}{r^2[2r + (k - 2)r_H]^2} .
\]

This is the first class of cold black holes solutions\(^4\) in \( f(T) \) gravity. The degenerated event horizon of order \( k \) is obtained in \( r = r_H \).

We can still look at the limit where \( T << T_0 \) in (23), which, at the second order, leads to

\[
f(T) = \sqrt{\frac{T}{T_0}} \approx a_0 + a_1 T + a_2 T^2 ,
\]

where \( a_0 = 3/8 \), \( a_1 = 3/(4T_0) \) and \( a_2 = -1/(8T_0^2) \). In this limit, the theory becomes the TT one plus a quadratic term in the torsion scalar \( T \).

(c) With the choice

\[
e^{b(r)} = \frac{r_0}{r} ,
\]

and equating (14) with (22), one gets

\[
a(r) = -4 \sqrt{\frac{T_0}{r} - 2 \ln r} .
\]

This solution is completely different from that obtained in [15], taking the same condition of coordinate, but for a set of diagonal tetrads.

\(^4\)Solutions that possess identically null Hawking temperature.
2. Let us consider here the case where the radial pressure is simply a function of the radial coordinate \( r \) and the algebraic function \( f(T) \) is chosen as

\[
f(T) = a_2 T^2 + a_1 T + a_0 .
\]  

(29)

In this case, substituting (29) into (16) and taking the coordinate condition [15]

\[
a(r) = \ln \left( \frac{T_0}{r} \right)
\]  

(30)

we obtain

\[
T(r) = \frac{2}{r^2} \left[ 1 \pm \sqrt{1 - \frac{a_1}{2a_2} r^2 + \frac{r^4}{4a_2} (a_0 + 16\pi p_r(r))} \right].
\]  

(31)

Equating (31) with (14) and taking into account (30), we get the solution

\[
dS^2 = \frac{r_0}{r} dt^2 - \left[ 1 - \frac{r^2}{2} T(r) \right]^{-2} dr^2 - r^2 d\Omega^2,
\]  

(32)

where \( T(r) \) is given in (31) and the condition

\[
1 - \frac{a_1}{2a_2} r^2 + \frac{r^4}{4a_2} (a_0 + 16\pi p_r(r)) \geq 0,
\]  

(33)

has to be satisfied for \( T(r) \) being a real function. This is a new class of traversable wormhole solutions. We can observe this by following the same process as in [14]. This line element (13) can be put in the form

\[
dS^2 = e^{a(r)} dt^2 - dl^2 - r^2(l)d\Omega^2,
\]  

(34)

where \( a(r) \) is denoted redshift function, and through the redefinition \( \beta(r) = r \left[ 1 - e^{-b(r)} \right] \), with \( b(r) \) being the metric function given in (13), \( \beta(r) \) is called shape function. Therefore, the conditions of existence of a traversable wormhole are: a) the function \( r(l) \) must possess a minimum value \( r_1 \) for \( r \), which imposes \( d^2 r(l)/dl^2 > 0 \); b) \( \beta(r_1) = r_1 \); c) \( a(r_1) \) has a finite value; and finally d) \( d\beta(r)/dr |_{r=r_1} \leq 1 \).

In this case, \( \beta(r) = r[1 - (1 - r^2T(r)/2)^2] \). Hence, the conditions of a traversible wormhole are applied on the torsion scalar \( T(r) \), in (31). The condition a), which can be explained in \( d^2 r/l^2 = [\beta(r)-r\beta'(r)]/2r^2 = -r[1-r^2T(r)/2][T(r)+rT'(r)/2] > 0 \), leads to \( T(r)+rT'(r)/2 < 0 \); the condition b), yields \( T(r_1) = 2/r_1^2 \); the condition c) is always satisfied, as we can see through (32); and finally, the condition d) leads to \( 1 - r^2T(r)/2 \geq 2r^2[T(r)+rT'(r)/2], \) which is always true when a) is satisfied.

Therefore, we must choose the torsion scalar, which will be defined from the radial pressure \( p_r(r) \), in (31), such that it satisfies the conditions a)-d) and (33), for obtaining traversable wormhole.
solutions. Let us consider first a class of solutions coming from the following example, 

\[ T(r) = 2r_1^{m-2}/r^m, \]

with \( r_1 > 0 \) and \( m \geq 3 \), which is obtained for 

\[ p_r(r) = \frac{[2a_1r^{2m}r_1^4 - a_0r^{2(m+1)}r_1^4 - 8a_2r^2r_1^{2m}]}{16\pi r_1^2r^{2(m+1)}}, \]

which satisfies (33). The condition a) is satisfied, since 

\[ \frac{d^2r}{dl^2} = (r_1^{m-2}/r^{m-1})[1 - (r_1/r)^m] > 0; \]

the conditions b) and c) are also directly satisfied; and d) is satisfied through a).

5 Reconstruction in static f(T) theory

A method widely used in cosmology in order to obtain the algebraic form of the gravitational part of the action is the so-called reconstruction scheme. This method stems from the introduction of an auxiliary field for the reconstruction of the algebraic function of the main action, as in the case of the theory \( f(R) \) for example [20].

We can briefly present this method as follows. Considering the algebraic function

\[ f(T) = P(\varphi)T + Q(\varphi), \tag{35} \]

the functional variation of the action (9), with respect to \( \varphi \), is given by

\[ \frac{\delta S}{\delta \varphi} = \frac{e}{16\pi} \left[ \frac{dP}{d\varphi} T + \frac{dQ}{d\varphi} \right] = 0. \tag{36} \]

Solving this equation, we get \( \varphi \equiv \varphi(T) \), then, \( f(T) = P[\varphi(T)]T + Q[\varphi(T)] \). Hence, we have the following identities

\[ f_T(T) = P + \left( \frac{dP}{d\varphi} T + \frac{dQ}{d\varphi} \right) \frac{d\varphi}{dT} = \frac{P[\varphi(T)]}{T}, \tag{37} \]

\[ f_{TT}(T) = \frac{dP[\varphi(T)]}{dT}. \tag{38} \]

Having in hand the equations (35), (37) and (38), and substituting into (15)-(17), we get

\[ 4\pi\rho = -\frac{e^{-b/2}}{r} \left( e^{-b/2} - 1 \right) \frac{dP}{dT} + \frac{Q}{4} + \frac{P}{2r^2} \left[ 1 + e^{-b}(1-rb') \right], \tag{39} \]

\[ 4\pi p_r = \frac{P}{2r^2} \left[ -1 + e^{-b}(1+ra') \right] - \frac{Q}{4}, \tag{40} \]

\[ 4\pi p_t = \frac{e^{-b}}{2} \left( \frac{\alpha'}{\alpha} + \frac{1}{r} - \frac{e^{-b/2}}{r} \right) \frac{dP}{dT} + P\frac{e^{-b}}{4} \left[ a'' + \left( \frac{1}{r} + \frac{\alpha'}{\alpha} \right)(\alpha' - b') \right] - \frac{Q}{4}. \tag{41} \]

Since \( \varphi \) is an arbitrary field, the reconstruction can be performed directly choosing \( \varphi = r \). This method may be used for re-obtaining or reconstructing, when the inversion \( r \equiv r(T) \) is possible, the \( f(T) \) theory in the static case.

Let us draw up to cases here:
1. For the case where \( f(T) \) is a linear algebraic function, one has the condition \( P \in \mathbb{R} \), due to (38).

Considering now the condition (30), and that the radial pressure (40) obeys

\[ p_r = g_1 f(T) , \]  

(42)

where \( g_1 \in \mathbb{R} \), we get

\[ Q = - \left( 2 \frac{P}{16\pi g_1 + 1} \right) \frac{P}{r^2} - \left( \frac{16\pi g_1}{16\pi g_1 + 1} \right) PT , \]

(43)

\[ f(T) = \left( \frac{1}{16\pi g_1 + 1} \right) PT - \left( \frac{2}{16\pi g_1 + 1} \right) \frac{P}{r^2} . \]

(44)

Differentiating (44) with respect to \( T \), equating it to \( P \) and integrating, we obtain the relation

\[ r(T) = \left[ 8\pi g_1 (T_0 - T) \right]^{-1/2} , \]

(45)

where \( T_0 \) is an integration real constant. Substituting (45) in (44), the algebraic function \( f(T) \) is then constructed as

\[ f(T) = PT + \left( -\frac{16\pi g_1 PT_0}{16\pi g_1 + 1} \right) . \]

(46)

We see that \( Q \) is also a constant, given by the second term of (46). Substituting (14) into (45), one gets the following solution

\[ dS^2 = \frac{r_0}{r} dt^2 - \left( \frac{16\pi g_1 + 1}{16\pi g_1} - \frac{T_0}{2} r^2 \right)^{-2} dr^2 - r^2 d\Omega^2 . \]

(47)

This is a new traversable wormhole solution. Following the same process of the solution (32), we obtain \( \beta(r) = r [1 - (k - T_0 r^2/2)] \), with \( k = (16\pi g_1 + 1)/16\pi g_1 \), and then, the minimum of \( r \) is given by \( r_1 = \sqrt{2k/T_0} \). The condition a), which in this case is given by \( d^2r/dt^2 = -T_0 r' (k - T_0 r^2/2) \), is satisfied for \( T_0 < 0 \) and \(-1/16\pi < g_1 < 0 \). The conditions b) and c) are directly satisfied, and d) is also satisfied for \( \beta'(r_1) = 1 \).

2. Taking the condition (30) and considering that the radial pressure obeys [15]

\[ 4\pi P_r = g_1 f(T) + g_2 f_T(T) , \]

(48)

where \( g_1, g_2 \in \mathbb{R} \), we get from (40) that

\[ Q = - \frac{4P}{4g_1 + 1} \left( g_1 T + g_2 + \frac{1}{r^2} \right) , \quad f(T) = \frac{4P}{4g_1 + 1} \left( \frac{T}{4} - g_2 - \frac{1}{2r^2} \right) . \]

(49)

Differentiating (49) with respect to \( T \) and equating it to \( P \), we get

\[ g_1 P = \frac{dP}{dT} \left( \frac{T}{4} - g_2 - \frac{1}{2r^2} \right) + \frac{P}{r^2} \frac{dr}{dT} . \]

(50)
In order to integrate this equation, we consider first \( P(r) = r^k \), then, (50) becomes

\[
g_1r = \left[ \frac{k}{4} (T - 4g_2) + \frac{1}{r^2} \left( 1 - \frac{k}{2} \right) \right] \frac{dr}{dT} ,
\]

which is separable only for \( k = 2 \), yielding

\[
r(T) = r_1 \left( \frac{T - 4g_2}{T_0 - 4g_2} \right)^{2g_1} ,
\]

where \( T_0 \) and \( r_1 \) are integration real constants. Substituting (52) into (49), the algebraic function \( f(T) \) in then reconstructed as

\[
f(T) = \frac{1}{4g_1 + 1} \left[ r_1^2 (T - 4g_2)^{4g_1 + 1} - 2 \right] .
\]

This algebraic polynomial function \( f(T) \) can be better extended for a particular more simplest case. One can illustrate the particular case in which \( 4g_1 + 1 = 2 \), which leads to

\[
f(T) = a_1 T + (a_2 T^2 + a_0) ,
\]

where \( a_2 = r_1^2 / 2(T_0 - 4g_2) , a_1 = -4g_2 r_1^2 / (T_0 - 4g_2) \) and \( a_0 = -1 \). Then, we see that our reconstruction scheme leads to a function which presents the contributions of higher orders in the torsion scalar, added to the TT term, which in this case is of second and zero orders in (54).

6 Conclusions

The particular choice of the representation of a spherically symmetric and static metric of Weitzenbock’s spacetime can be done in an arbitrary way, when we wish to maintain the invariance under local Lorentz transformations. Therefore, when we choose a set of diagonal tetrads, for a projection in the tangent space to the manifold, the imposition of the function \( f(T) \) being linear in \( T \), or a constant torsion scalar, is recovered. But, as described in this work, choosing a set of non-diagonal tetrads, we can get an arbitrary algebraic function \( f(T) \). Therewith, the second derivative of the function \( f(T) \) appears in the contribution of the energy density and the tangential pressure in (15) and (17).

We shown some examples of new black holes and wormholes solutions. Through the fixation of the null radial pressure and the coordinates conditions \( a(r) = -b(r) \) and (27), we illustrated an example of a black hole, (20), of a cold black hole, in (25), and a solution for comparing the same conditions, with the diagonal case, in (28).

We also imposed to the radial pressure to be just function of the radial coordinate, and \( f(T) \) to be a polynomial function of second order in \( T \). We then obtained, through the coordinate condition (30), a new class of traversable wormhole solutions in (32).

Finally, we made a summary of the reconstruction scheme for our static case of \( f(T) \) gravity. For a particular case of the algebraic function \( f(T) \) being a linear function, we were able to reconstruct it,
considering that it is proportional the radial pressure. This case is that of a new solution of traversable wormhole. Taking into account the condition in which the radial pressure is a linear combination of the algebraic function $f(T)$ and its derivative, we reconstructed $f(T)$ as a polynomial function of the torsion scalar $T$, where we shown the particular case in which $f(T)$ gravity would be the TT plus some terms of second and zero orders of the torsion scalar $T$.

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