Existence of blow-up solutions for a class of elliptic system with convection term

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Abstract

The present paper concerns with the existence of solutions for a class of elliptic systems involving nonlinearities of the Keller-Osserman type and combined with the convection terms. Firstly, we establish a result involving sub and super-solution for a class of elliptic system whose nonlinearity can depend of the gradient of the solution. This result permits to study the existence of blow-up solution for a large class of systems.

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1 Introduction

In this article, we study the existence of solutions for the following class of elliptic system with convection term

\[
\begin{aligned}
\Delta u + b_1(x)|\nabla u|^{q_1} &= F_u(x,u,v) \quad \text{in } \Omega, \\
\Delta v + b_2(x)|\nabla v|^{q_2} &= F_v(x,u,v) \quad \text{in } \Omega, \\
\end{aligned}
\]

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary or $\Omega = \mathbb{R}^N$, $0 < q_i \leq 2$, $b_i : \overline{\Omega} \to \mathbb{R}^+$ ($i = 1, 2$) are continuous functions and $F : \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1$ function verifying some technical condition, which are mentioned later on.

For the case where $\Omega$ is a bounded domain, the system will be studied under three different types of boundary conditions:

- **Finite Case:** Both components $(u, v)$ bounded on $\partial \Omega$, that is,

\[
\begin{aligned}
&u = \alpha \quad \text{on } \partial \Omega, \\
v = \beta \quad \text{on } \partial \Omega, \\
\end{aligned}
\]

with $\alpha, \beta \in (0, +\infty)$.

- **Infinite Case:** Both components blowing up simultaneously on $\partial \Omega$, that is,

\[
\begin{aligned}
&u = +\infty \quad \text{on } \partial \Omega, \\
v = +\infty \quad \text{on } \partial \Omega, \\
\end{aligned}
\]

where $u = +\infty$ on $\partial \Omega$ and $v = +\infty$ on $\partial \Omega$ should be understood as $u(x) \to +\infty$ and $v(x) \to +\infty$ as $\text{dist}(x, \partial \Omega) \to 0$.

- **Semifinite Case:** One of the components bounded while the other one blows up on $\partial \Omega$, that is,

\[
\begin{aligned}
&u = +\infty \quad \text{on } \partial \Omega, \\
v = \beta \quad \text{on } \partial \Omega, \\
\end{aligned}
\]
or
\[
\begin{cases}
  u = \alpha & \text{on } \partial \Omega, \\
  v = +\infty & \text{on } \partial \Omega.
\end{cases}
\]

(A solution \((u, v) \in C^2(\Omega) \times C^2(\Omega)\) of the system \((S)\) is called a blow-up solution if the condition \((I)\) holds and semifinite blow-up solution when \((SF1)\) or \((SF2)\) holds.

For the case \(\Omega = \mathbb{R}^N\), we consider the following class of elliptic systems
\[
\begin{cases}
  \Delta u + b_1(x)|\nabla u|^{q_1} = F_u(x, u, v) & \text{in } \mathbb{R}^N, \\
  \Delta v + b_2(x)|\nabla v|^{q_2} = F_v(x, u, v) & \text{in } \mathbb{R}^N, \\
  u, v > 0 & \text{in } \mathbb{R}^N.
\end{cases}
\]

(LS)

Associated with this class of systems, our main result is concerned with the existence of entire large solutions, that is, solutions \((u, v)\) satisfying \(u(x) \to +\infty\) and \(v(x) \to +\infty\) as \(|x| \to +\infty\).

The scalar case associated with system \((S)\), namely
\[
\Delta u + b(x)|\nabla u|^q = F_u(x, u, v) \text{ in } \Omega,
\]
has been considered by several authors. We would like to mention the papers of Alarcón, García-Melian & Quass [1], Bandle & Giarrusso [8], Bandle & Marcus [7], Covei [9], Filippucci, Pucci & Rigoli [11], García & Melian [12], García-Melian & Rossi [13], Ghergu,Niculescu & Radulescu [14], Holanda [16], Lair [19], Lair & Wood [20], Mohammed [26], Keller [18], Osserman [27], Mi & Liu [25], and references therein.

For instance, Lair [19] showed the existence of solutions of the problem
\[
\Delta u = r(x) h(u(x)) \text{ for } x \in \Omega \subseteq \mathbb{R}^N, N \geq 3
\]
where the function \(h : [0, +\infty) \to [0, \infty)\) satisfies the \(F\)-condition:

- \(h \in C^1([0, \infty))\), \(h(0) = 0\), \(h'(t) \geq 0 \forall t \in [0, \infty)\),

- \(h(t) > 0 \forall t \in (0, \infty)\)

and the well known Keller-Osserman condition ( [18], [27] ), that is,
\[
\int_1^\infty \frac{1}{H(t)^{1/2}} dt < \infty, \quad H(t) := \int_0^t h(s) ds, \quad (1.1)
\]
and the function \( r : \Omega \to (0, \infty) \) satisfies the \( \mathcal{P} \)-condition:

\[
\in \mathcal{C}^0,\alpha (\Omega), \quad \text{if } \Omega \text{ is a bounded domain}
\]
or

\[
\in \mathcal{C}^0,\alpha (\mathbb{R}^N), \quad \text{if } \Omega = \mathbb{R}^N.
\]

In [20], Lair & Wood proved the existence of non-negative solutions of the problem

\[
\Delta u + |\nabla u|^q = r(x) u^\gamma \quad \text{in } \Omega
\]

with \( r \in \mathcal{P}, \ q \in (0,2] \) and \( \gamma > \max \{1,q\} \).

Later, Ghergu,Niculescu & Radulescu [14] considered the equation

\[
\Delta u + q(x)|\nabla u|^a = r(x) h(u) \quad \text{in } \Omega,
\]

assuming \( a \in (0,2], \ h \in \mathcal{F} \) and

\[
H(t)/t^{2/a} \to 0 \quad \text{as } t \to \infty,
\]

for some suitable functions \( q, \ r \) of the class \( \mathcal{P} \). An important common point among the above papers is the fact that they assume that nonlinearity is monotone.

Recently, Alves & Holanda in [3], combining variational method with the existence of sub and super-solution, obtained solutions with boundary conditions (F), (I) and (SF), for the system of the form

\[
\Delta U = \nabla F(x,U) \quad \text{in } \Omega
\]

in which \( U := (u,v) \),

\[
\nabla F := \begin{cases} 
(F_u(x,u,v), F_v(x,u,v)) & \text{if } \Omega \subset \mathbb{R}^N \\
(a_1(x) F_u(x,u,v), a_2(x) F_v(x,u,v)) & \text{if } \Omega = \mathbb{R}^N,
\end{cases}
\]

and for suitable functions which are not necessarily monotone, but for a particular class of systems of the form (S), where \( b_i = 0 \) for \( i = 1,2 \).

The motivation to study system (S) comes from the study of the chemical, physical, biological and economical phenomena, see [8,10] for details. Also such systems can model phenomena from the study of ecological prey-predator models, in that context we refer to Leung [23,24].
Throughout this article, we assume that \( b_i \in \mathcal{P} \) and \( F_u, F_v \) are locally Hölder continuous with exponent \( \alpha \in (0,1) \), verifying the following additional condition:

There are \( a_i, a_i^2 \in \mathcal{P} \) \( (i = 1, 2) \) and \( f_i, g \in \mathcal{F} \), satisfying

\[
F_i(x, t, s) \geq a_1(x)f_1(t) \forall x \in \overline{\Omega}, \ t, s > 0 \tag{1.3}
\]
\[
F_s(x, t, s) \geq a_2(x)f_2(s) \forall x \in \overline{\Omega}, \ t, s > 0 \tag{1.4}
\]
and

\[
g(t) > \max_{i=1,2} \left\{ \frac{F_i(x, t, t)}{a_i^2(x)} \right\} \forall x \in \overline{\Omega}, \ t > 0. \tag{1.5}
\]

A simple example of nonlinearity \( F \) satisfying the above assumptions is

\[
F(x, u, v) = c_1(x)u^\rho + c_2(x)u^\sigma v^\gamma + c_3(x)v^\theta
\]

where \( \rho, \theta > 2, \sigma + \gamma > 2 \) with \( \sigma, \gamma < 2 \) and \( c_i \ (i = 1, 2, 3) \) are some suitable functions.

Not before to enumerate our results about the considered system we wish to say that if the nonlinearities are not necessarily non-decreasing it is known that the problem of uniqueness of solution is not so easy even we refer to the scalar case treated in various references. But, for some particular cases of nonlinearities we can see that many authors appeals to the asymptotic behavior of the solution in order to prove the uniqueness of explosive solutions for both scalar and systems cases. In this article we will restrict our research only to the problem of existence of solutions. The uniqueness problem becomes more delicate topic included in our future goals.

Our first result related to the problem \((P)\) is the following:

**Theorem 1.1** Suppose that \( \Omega \) is a bounded domain, \( b_i \in \mathcal{P} \) and \((1.3)-(1.5)\) hold. Then:

i) Problem \((P)\) admits positive solution with the boundary condition \((F)\).

ii) Problem \((P)\) admits positive solution with the boundary condition \((I)\).

iii) Problem \((P)\) admits positive solution with the boundary condition \((SF1)\) or \((SF2)\).
Our next result is related to existence of entire large solution for system (S) for the case where $\Omega = \mathbb{R}^N$. For expressing the next result, we assume that functions $a_i^2$ ($i = 1, 2$) belongs to $\mathcal{P}$ and that the problem

$$-\Delta z(x) = \sum_{i=1}^{2} a_i^2(x) \quad \text{for } x \in \mathbb{R}^N, \quad z(x) \to 0 \text{ as } |x| \to \infty$$  \hspace{0.5cm} (1.6)

has a $C^2$ supersolution.

**Theorem 1.2** Assume that (1.3)-(1.6) hold. Then system (P) has an entire large solution.

Before to conclude this introduction, we would like to say that Theorems 1.1 and 1.2 complete the study made in [3], in the sense that, in that paper the authors considered only the case where $b_i = 0$ ($i = 1, 2$). Moreover, we would like to detach that the authors does know any result involving sub and supersolution that can be used for system (S). To overcome this difficulty, we prove in Section 2 a result that allows us to apply sub and supersolution for (S).

## 2 An auxiliary system

In this section, we will work with an auxiliary system associated with (S). In what follows, fixed $R > 0$, we denote by $\xi_R : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous functions verifying

$$\xi_R(t) = t \text{ for } t \in [0, R] \text{ and } \xi(t) = R \text{ for } t \geq R.$$  

Using this function, we consider the system

$$\begin{cases}
\Delta u + b_1(x)\xi_R(|\nabla u|^q) = F_u(x, u, v) \text{ in } \Omega, \\
\Delta v + b_2(x)\xi_R(|\nabla v|^q) = F_v(x, u, v) \text{ in } \Omega, \\
u = \alpha, v = \beta \text{ on } \partial \Omega.
\end{cases} \hspace{0.5cm} (AS)_R$$

Without loss of generality, we will consider that $\alpha = \beta = 0$. Our result main related to $(AS)_R$ is the following:
Theorem 2.1 Assume that there are \((u, v), (\bar{u}, \bar{v}) \in (C^2(\Omega) \cap L^\infty(\Omega))^2\) verifying:
\[
    u \leq \bar{u} \text{ and } v \leq \bar{v} \text{ in } \Omega,
\]
\[
    u \leq \alpha, v \leq \beta \text{ and } \bar{u} \geq \alpha, \bar{v} \geq \beta \text{ on } \partial \Omega
\]
and that,
\[
    \begin{cases}
    \Delta u + b_1(x) \xi_R(|\nabla u|^{q_1}) \geq F_u(x, u, v) \text{ in } \Omega, \\
    \Delta v + b_2(x) \xi_R(|\nabla v|^{q_2}) \geq F_v(x, u, v) \text{ in } \Omega,
    \end{cases}
\]
and
\[
    \begin{cases}
    \Delta \bar{u} + b_1(x) \xi_R(|\nabla \bar{u}|^{q_1}) \leq F_u(x, \bar{u}, \bar{v}) \text{ in } \Omega, \\
    \Delta \bar{v} + b_2(x) \xi_R(|\nabla \bar{v}|^{q_2}) \leq F_v(x, \bar{u}, \bar{v}) \text{ in } \Omega.
    \end{cases}
\]
Then, there is \((u, v) \in (H^1(\Omega))^2\) such that
\[
    u \leq u \leq \bar{u} \text{ and } v \leq v \leq \bar{v} \text{ in } \Omega
\]
and
\[
    \begin{cases}
    \Delta u + b_1(x) \xi_R(|\nabla u|^{q_1}) = F_u(x, u, v) \text{ in } \Omega, \\
    \Delta v + b_2(x) \xi_R(|\nabla v|^{q_2}) = F_v(x, u, v) \text{ in } \Omega, \\
    u = \alpha, v = \beta \text{ on } \partial \Omega.
    \end{cases}
\]

Proof. Here, we will use a result due to Alves & Moussaoui [4, Theorem 2.1]. First of all, we observe that without loss of generality we can assume that \(\alpha = \beta = 0\). Setting the functions \(H, G : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}\) given by
\[
    H(x, s, t, \eta, \zeta) = -F_s(x, s, t) + b_1(x) \xi_R(|\eta|)
\]
and
\[
    G(x, s, t, \eta, \zeta) = -F_t(x, s, t) + b_2(x) \xi_R(|\zeta|),
\]
we observe that they are continuous functions and given \(T, S > 0\), there exists \(C = C(R) > 0\) such that
\[
    |H(x, s, t, \eta, \zeta)|, |G(x, s, t, \eta, \zeta)| \leq C, \ \forall (x, s, t, \eta, \zeta) \in \Omega \times [0, T] \times [0, S] \times \mathbb{R}^N \times \mathbb{R}^N,
\]
finishing the proof of Theorem 2.1. \(\square\)
Corollary 2.1 On the hypotheses of Theorem 2.1, if the $L^\infty(\Omega)$ norms of the pairs $(u, v)$, $(\bar{u}, \bar{v})$ are independent of $R$, for $R$ large enough, then there is $R^* > 0$ such that if $R > R^*$, the solution $(u, v)$ given by Theorem 2.1 verifies

$$\max_{x \in \Omega} |\nabla u(x)|, \max_{x \in \Omega} |\nabla v(x)| < \min\{R^{1 \over n}, R^{1 \over m}\}.$$ 

Thus, $(u, v)$ is a solution of the original system $(S)$.

Proof. A first point that we should mention is the fact that by Elliptic Regularity,

$$u, v \in W^{2,p}(\Omega) \quad \forall p \in [1, +\infty),$$

because $\xi_R \in L^\infty([0, +\infty))$, $u, v \in L^\infty(\Omega)$ and $F_u, F_v$ are continuous functions. From now on, we will fix $p$ such that

$$W^{2,p}(\Omega) \hookrightarrow C^{1,\alpha}(\Omega) \quad (2.1)$$

is a continuous embedding. Now, we observe that $u$ is a solution of the problem

$$\Delta u - u = B_R(x)(1 + |\nabla u|^2),$$

where

$$B_R(x) = -u + F_u(x, u, v) - b_1(x)\xi_R(|\nabla u|^q) \over 1 + |\nabla u|^2.$$ 

Once that

$$\xi_R(t) \leq t \quad \forall t \geq 0,$$

a direct computation shows that there is $C^* > 0$, independent of $R > R^*$, such that

$$|B_R(x)| \leq C^* \quad \forall x \in \Omega,$$

leading to

$$\|B_R\|_\infty \leq C^* \quad \forall R > R^*. \quad (2.2)$$

By using a result due to Amann & Crandall [5, Lemma 4], there is an increasing function $\gamma_0 : [0, +\infty) \to [0, \infty)$, depending only of $\Omega$, $p$ and $N$, and satisfying

$$\|u\|_{W^{2,p}(\Omega)} \leq \gamma_0(\|B_R\|_\infty).$$

Combining the last inequality with (2.1) and (2.2),

$$\|u\|_{C^{1,\alpha}(\Omega)} \leq C\gamma_0(C^*),$$
for some $C > 0$. Fixing

$$K = C \gamma_0 (C^*)$$

we derive that

$$\left| \frac{\partial u(x)}{\partial x_i} \right| \leq K \quad \forall x \in \overline{\Omega} \text{ and } i = 1, 2, ..., N.$$  

Thereby,

$$|\nabla u(x)| \leq NK \quad \forall x \in \overline{\Omega},$$

implying that

$$\max_{x \in \Omega} |\nabla u(x)| \leq NK.$$  

Fixing $R_1^* = (NK)^{\eta_1}$, it follows that

$$\max_{x \in \Omega} |\nabla u(x)| \leq (R_1^*)^{\frac{1}{\eta_1}}.$$  

By a similar argument, we get $R_2^* > 0$ verifying

$$\max_{x \in \Omega} |\nabla v(x)| \leq (R_2^*)^{\frac{1}{\eta_2}}.$$  

Now, the corollary follows setting $R^* = \max \{ R_1^*, R_2^* \}$.  

\section{Proof of Theorem 1.1}

We begin the proof of Theorem 1.1 by Finite Case.

\textbf{Case 1: Finite case}

In what follows, we fix $M > \max \{ \alpha, \beta \}$, $m < \min \{ \alpha, \beta \}$ and denote by $\psi \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ the unique positive solution of the problem

$$\begin{cases}
\Delta \psi = \sum_{i=1}^{2} a_i^2(x) g(\psi) \text{ in } \Omega, \\
\psi > 0 \text{ in } \Omega, \\
\psi = m \text{ on } \partial \Omega,
\end{cases}$$

which exists by a result found in [19, Proposition 1]. The pairs $(u, v) = (\psi(x), \psi(x))$ and $(\overline{u}, \overline{v}) = (M, M)$,
satisfy the hypotheses of Corollary 2.1. Thus, the system

\[
\begin{aligned}
\Delta u + b_1(x)|\nabla u|^{q_1} &= F_u(x, u, v), \quad \text{in } \Omega \\
\Delta v + b_2(x)|\nabla v|^{q_2} &= F_v(x, u, v), \quad \text{in } \Omega \\
u = \alpha, v = \beta & \quad \text{on } \partial \Omega,
\end{aligned}
\]

has a solution. Moreover, by elliptic regularity, we must have \( u, v \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \). \( \square \)

**Case 2: Infinite case.**

In this case, we denote by \((u_n, v_n)\) the solution of the system

\[
\begin{aligned}
\Delta u + b_1(x)|\nabla u|^{q_1} &= F_u(x, u, v), \quad \text{in } \Omega \\
\Delta v + b_2(x)|\nabla v|^{q_2} &= F_v(x, u, v), \quad \text{in } \Omega \\
u = v = n & \quad \text{on } \partial \Omega,
\end{aligned}
\]

which exists by finite case. We remark that \((u_n, v_n)\) can be chosen satisfying the inequality

\[
u_n \leq u_{n+1} \text{ and } v_n \leq v_{n+1} \quad \forall n \in \mathbb{N}.
\]

Indeed, note that \((u_1, v_1)\) satisfies

\[
\begin{aligned}
\Delta u + b_1(x)|\nabla u|^{q_1} &= F_u(x, u, v), \quad \text{in } \Omega \\
\Delta v + b_2(x)|\nabla v|^{q_2} &= F_v(x, u, v), \quad \text{in } \Omega \\
u_1 = v_1 = 1 & \quad \text{on } \partial \Omega,
\end{aligned}
\]

for all

\[R > \max \{ \max_{x \in \Omega} |\nabla u_1(x)|^{q_1}, \max_{x \in \Omega} |\nabla v_1(x)|^{q_2} \},\]

because

\[
\xi_R(|\nabla u_1|^{q_1}(x)) = |\nabla u_1(x)|^{q_1} \quad \text{and} \quad \xi_R(|\nabla v_1(x)|^{q_2}) = |\nabla v_1(x)|^{q_2} \quad \forall x \in \overline{\Omega}.
\]
Applying Corollary 2.1 with $(u, v) = (u_1, v_1)$ and $(u, v) = (2, 2)$, there exists a solution $(u_2, v_2)$ of
\[
\begin{align*}
\Delta u + b_1(x)|\nabla u|^{q_1} &= F_u(x, u, v) \text{ in } \Omega, \\
\Delta v + b_2(x)|\nabla v|^{q_2} &= F_v(x, u, v) \text{ in } \Omega, \\
u_1 = v_2 &= 2 \text{ on } \partial \Omega,
\end{align*}
\]
satisfying
\[u_1 \leq u_2 \text{ and } v_1 \leq v_2 \leq M_1 := 2 \text{ in } \overline{\Omega}.
\]
Repeating the above argument, of an iterative way, for each $M_n = n + 1$; $n = 1, 2, \ldots$, the pair $(u_n, v_n)$ satisfies
\[
\begin{align*}
\Delta u_n + b_1(x)|\nabla u_n|^{q_1} &= F_u(x, u_n, v_n) \text{ in } \Omega, \\
\Delta v_n + b_2(x)|\nabla v_n|^{q_2} &= F_v(x, u_n, v_n) \text{ in } \Omega, \\
u_n &= v_n \leq n + 1 \text{ on } \partial \Omega.
\end{align*}
\]
Applying again Corollary 2.1 with $(u, v) = (u_n, v_n)$ and $(u, v) = (n+1, n+1)$, we get a solution $(u_{n+1}, v_{n+1})$ of
\[
\begin{align*}
\Delta u + b_1(x)|\nabla u|^{q_1} &= F_u(x, u, v) \text{ in } \Omega, \\
\Delta v + b_2(x)|\nabla v|^{q_2} &= F_v(x, u, v) \text{ in } \Omega, \\
u = v &= n + 1 \text{ on } \partial \Omega,
\end{align*}
\]
with
\[u_n \leq u_{n+1} \text{ and } v_n \leq v_{n+1} \leq M_n := n + 1 \text{ in } \overline{\Omega}.
\]
Once that sequences $(u_n)$ and $(v_n)$ are nondecreasing, there are functions $u, v : \Omega \to \mathbb{R}$ verifying
\[u_n(x) \to u(x) \text{ and } v_n(x) \to v(x) \text{ in } \Omega.
\]
From now on, we denote by $\bar{u}$ and $\bar{v}$ the solutions of the problems
\[
\begin{align*}
\Delta u + \|b_1\|_\infty |\nabla u|^{q_1} &= \min_{x \in \overline{\Omega}} a_1(x) f_1(u) \text{ in } \Omega, \\
u > 0 \text{ in } \Omega, \\
u &= +\infty \text{ on } \partial \Omega,
\end{align*}
\]
and
\[
\begin{cases}
\Delta v + \|b_2\|_{\infty} |\nabla v|^{q_2} = \min_{x \in \Omega} a_2(x) f_2(v) \text{ in } \Omega, \\
v > 0 \text{ in } \Omega, \\
v = +\infty \text{ on } \partial \Omega,
\end{cases}
\]
which exist from a result due to Bandle & Giarrusso [6]. We claim that
\[(i) \ u_n \leq \tilde{u} \text{ and } (ii) \ v_n \leq \tilde{v} \text{ in } \overline{\Omega} \ \forall n \geq 1. \tag{3.4}\]
Indeed, suppose by contradiction that (3.4)(i) does not hold. Then, there exists \(x_0 \in \Omega\) such that
\[u_n(x_0) > \tilde{u}(x_0) \text{ in } \Omega \text{ for some } n \geq 1.\]
Since
\[
\lim_{d(x, \partial \Omega) \to 0} [u_n(x) - \tilde{u}(x)] = -\infty,
\]
we deduce that \(\max_{x \in \Omega}(u_n - \tilde{u})(x)\) is achieved in \(\Omega\), for example at \(x_1 \in \Omega\). This form,
\[|\nabla u_n(x_1)|^{q_1} = |\nabla \tilde{u}(x_1)|^{q_1}\]
and
\[0 \geq \Delta (u_n - \tilde{u})(x_1) = a_1(x_1) f_1(u_n) - b_1(x_1) |\nabla u_n|^{q_1} - \min_{x \in \Omega} a_1(x_1) f_1(\tilde{u}) + \|b_1\|_{\infty} |\nabla \tilde{u}|^{q_1} > 0,
\]
obtaining a contradiction. Therefore, (3.4)(i) holds. The same argument works to prove (3.4)(ii).

On the other hand, by using a well known result due to Ladyzenskaya and Ural’treva [22], given \(\Omega_1 \subset \subset \Omega_2 \subset \subset \Omega\), there is \(C > 0\) such that
\[
\max_{x \in \Omega_1} |\nabla u_n| \leq C \max_{x \in \Omega_2} |u_n| \leq C \max_{x \in \Omega_2} |\tilde{u}| = K_5 \ \forall n \in \mathbb{N} \tag{3.5}
\]
and
\[
\max_{x \in \Omega_1} |\nabla v_n| \leq C \max_{x \in \Omega_2} |v_n| \leq C \max_{x \in \Omega_2} |\tilde{v}| = K_6 \ \forall n \in \mathbb{N}. \tag{3.6}
\]
Combining (3.5), (3.6) and Elliptic Regularity, it follows that there are subsequences of \((u_n)\) and \((v_n)\), still denoted by \((u_n)\) and \((v_n)\), such that
\[u_n \to u \text{ and } v_n \to v \text{ in } C^2_{\text{loc}}(\Omega).\]
This fact yields $u, v \in C^2(\Omega)$ and
\[
\begin{cases}
\Delta u + b_1(x) |\nabla u|^{q_1} = F_u(x, u, v) \text{ in } \Omega, \\
\Delta v + b_2(x) |\nabla v|^{q_2} = F_v(x, u, v) \text{ in } \Omega, \\
u, v > 0 \text{ in } \Omega.
\end{cases}
\]
To complete the proof, it suffices to prove that $(u, v)$ blows up at the boundary. Arguing by contradiction, we will assume that $u$ does not blow up at the boundary. Then, there exist $x_0 \in \partial \Omega$ and $(x_k) \subset \Omega$ such that
\[
limit_{k \to \infty} x_k = x_0 \text{ and } \lim_{k \to \infty} u(x_k) = L \in (0, \infty).
\]
In what follows, fix $n > 4L$ and $\delta > 0$ such that $u_n(x) \geq n/2$ for all $x \in \overline{\Omega}_\delta$, where
\[
\overline{\Omega}_\delta = \{x \in \overline{\Omega} | dist(x, \partial \Omega) \leq \delta\}.
\]
Then, for $k$ large enough, $x_k \in \overline{\Omega}_\delta$ and $u_n(x_k) > 2L$. Since
\[
u_n(x_k) \leq u_{n+1}(x_k) \leq \ldots \leq u_{n+j}(x_k) \leq \ldots \leq u(x_k) \forall j,
\]
we have that $u(x_k) \geq 2L$, which is a contradiction. Therefore, $u$ blows up at the boundary. The same approach can be used to prove that $v$ also blows up.

**Case 3: Semifinite case**

Let $(u_n, v_n) \in (C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega}))^2$ be a solution of $(S)_{\alpha, \beta}$ with $\alpha = n$, $n \in \mathbb{N}$ and $\beta$ fixed. As in the previous case, the sequence $(u_n, v_n)$ is bounded on compact subset contained in $\Omega$, implying that there exist functions $u, v$ verifying
\[
u_n \to u \text{ in } C^2(K)
\]
and
\[
u_n \to v \text{ in } C^2(K)
\]
for any compact subset $K \subset \Omega$. Moreover, the arguments used in the previous cases give that $u$ blows up at the boundary, that is,
\[
u(x) = +\infty \text{ on } \partial \Omega.
\]
Related to sequence \((v_n)\), we recall that
\[
\begin{align*}
\Delta v_n + b_2(x)|\nabla v_n|^{q_2} &= F_v(x, u_n, v_n) \quad \text{in } \Omega, \\
v_n &> 0 \quad \text{in } \Omega, \\
v_n &= \beta \quad \text{on } \partial \Omega,
\end{align*}
\]
and
\[
v_n(x) \leq \beta \quad \text{in } \forall x \in \overline{\Omega} \text{ and } n \geq 1.
\]
Passing to the limit as \(n \to +\infty\), we obtain that \(v(x) \leq \beta\) for all \(x \in \Omega\).

**Claim 3.1** Let \(x_0 \in \partial \Omega\) and \((x_k) \subset \Omega\) be a sequence with \(x_k \to x_0\). Then \(v(x_k) \to \beta\) as \(k \to +\infty\).

Indeed, if the limit does not hold, there exist \(\epsilon > 0\) and a subsequence of \((x_k)\), still denoted by itself, such that
\[
x_k \to x_0 \quad \text{and} \quad v(x_k) \leq \beta - \epsilon \quad \forall k \in \mathbb{N}.
\] (3.7)
Since \(v_1 = \beta\) on \(\partial \Omega\), there is \(\delta > 0\) such that
\[
v_1(x) \geq \beta - \frac{\epsilon}{2} \quad \forall x \in \overline{\Omega_\delta}.
\]
Hence, for \(k\) large enough,
\[
x_k \in \overline{\Omega_\delta}
\]
and
\[
v_1(x_k) \geq \beta - \frac{\epsilon}{2} > \beta - \epsilon.
\]
Recalling that \(v_1 \leq v\) in \(\Omega\), it follows that
\[
v(x_k) \geq \beta - \frac{\epsilon}{2} > \beta - \epsilon,
\]
obtaining a contradiction with (3.7).

From Claim 3.1, we can continuously extend the function \(v\) from \(\Omega\) to \(\overline{\Omega}\), by considering
\[
v(x) = \beta \quad \text{on } \partial \Omega,
\]
concluding this way the proof of the semifinite case. \(\blacksquare\)
3.1 Proof of Theorem 1.2

Firstly, we provide a lower bound for the system \((LS)\). To this end, we consider the function \(w : \mathbb{R}^N \to [0, \infty)\) implicitly defined by

\[
z(x) = \int_{w(x)}^\infty \frac{1}{g(t)} dt \quad x \in \mathbb{R}^N,
\]

where \(z\) was given in (1.6). This is possible due to the fact that Keller-Osserman (1.1) condition gives

\[
\int_1^\infty \frac{1}{g(t)} dt < \infty.
\]

(see [9], [19] for details). Note that \(w \in C^2(\mathbb{R}^N, (0, \infty))\), \(w(x) \to +\infty\) as \(|x| \to \infty\) and

\[
\Delta w(x) \geq \sum_{i=1}^2 a_i^2(x) g(w(x)) \quad \text{for all } x \in \mathbb{R}^N.
\]

Moreover,

\[
\Delta w \geq F_u(x, w, w) \quad \text{in } B_n, \quad w \leq w_n \text{ on } \partial B_n,
\]

and

\[
\Delta w \geq F_v(x, w, w) \quad \text{in } B_n, \quad w \leq w_n \text{ on } \partial B_n.
\]

Using function \(w\), we consider the system

\[
\begin{aligned}
\Delta u + b_1(x) |\nabla u|^{q_1} &= F_u(x, u, v) \quad \text{in } B_n, \\
\Delta v + b_2(x) |\nabla v|^{q_2} &= F_v(x, u, v) \quad \text{in } B_n, \\
u, v &> 0 \quad \text{in } \Omega, \\
u = v = w_n \quad \text{on } \partial B_n,
\end{aligned}
\]

(3.8)

where \(B_n\) is the open ball of radius \(n\) centered at the origin and

\[
w_n = \max_{x \in B_n} w(x).
\]

Applying Theorem 1.1 with \((u, u) = (w, w)\) and \((\overline{u}, \overline{v}) = (w_n, w_n)\), there is a solution \((u_n, v_n) \in [C^2(B_n) \cap C^{1,\alpha}(\overline{B})]^2\) of (3.8). Moreover, we can choose the sequence \((u_n, v_n)\) satisfying

\[
w(x) \leq u_n(x) \leq u_{n+1}(x) \quad \text{for all } x \in \overline{B}_n,
\]
and

\[ w(x) \leq v_n(x) \leq v_{n+1}(x) \text{ for all } x \in \overline{B}_n. \]

From this, there are \( u, v : \mathbb{R}^N \to \mathbb{R} \) such that

\[ u_n(x) \to u(x) \text{ and } v_n(x) \to v(x) \ \forall x \in \mathbb{R}^N. \]

Arguing as in the previous sections, there are subsequences of \((u_n), (v_n)\), still denoted by themself, such that

\[ u_n \to u \text{ and } v_n \to v \text{ in } C^2_{\text{loc}}(\mathbb{R}^N) \]

and

\[ u(x), v(x) \geq w(x) \ \forall x \in \mathbb{R}^N. \]

Consequently, \( u, v \in C^2(\mathbb{R}^N) \) and \((u, v)\) is a solution of

\[
\begin{cases}
\Delta u + b_1(x)|\nabla u|^{q_1} = F_u(x, u, v) \text{ in } \mathbb{R}^N, \\
\Delta v + b_2(x)|\nabla v|^{q_2} = F_v(x, u, v) \text{ in } \mathbb{R}^N, \\
u, v > 0 \text{ in } \mathbb{R}^N, \\
u(x), v(x) \to +\infty \text{ as } |x| \to +\infty,
\end{cases}
\]

showing that \((u, v)\) is a entire large solution for \((LS)\). 

\[ \blacksquare \]

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