A VARIATIONAL FORMULATION FOR DIRAC OPERATORS IN BOUNDED DOMAINS. APPLICATIONS TO SPECTRAL GEOMETRIC INEQUALITIES.

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Abstract. We investigate spectral features of the Dirac operator with infinite mass boundary conditions in a smooth bounded domain of $\mathbb{R}^2$. Motivated by spectral geometric inequalities, we prove a non-linear variational formulation to characterize its principal eigenvalue. This characterization turns out to be very robust and allows for a simple proof of a Szegö type inequality as well as a new reformulation of a Faber-Krahn type inequality for this operator. The paper is complemented with strong numerical evidences supporting the existence of a Faber-Krahn type inequality.

Contents

1. Introduction 2
  1.1. Motivations and state of the art 2
  1.2. Structure of the paper 5

2. Preliminaries 5
  2.1. Sobolev spaces on $\partial\Omega$ 5
  2.2. Periodic pseudo-differential operators 6
  2.3. Cauchy singular integral operators 7

3. Maximal Wirtinger operators 7

4. Bergman and Hardy spaces on $\Omega$ 10
  4.1. Potential theory of the Wirtinger derivatives 11
  4.2. Explicit description of the Bergman and Hardy spaces 13
  4.3. Explicit description of the domain of the maximal Wirtinger operators 15

5. Variational characterization of the principal eigenvalue 16
  5.1. The quadratic form $q_{E}^{\Omega}$ and its associated self-adjoint operator $H_{E}^{\Omega}$ 17
  5.2. Concavity of the first min-max level 19
  5.3. Proof of the variational principle 21

6. Geometric upper bounds on the spectral gap 22
  6.1. A simple upper bound 22
  6.2. A sharp upper bound 24

7. About the Faber-Krahn conjecture 25
  7.1. A new conjecture 26
  7.2. Link with the Bossel-Daners inequality 27

8. Numerics 27
  8.1. Numerical Methods 28
  8.2. Numerical Results 29

Acknowledgments 33

References 33
1. Introduction

1.1. Motivations and state of the art. In the past few years there has been a growing interest in the study of Dirac operators among the mathematical physics community; the main reason being that low-energy electrons in a single-layered sheet of graphene are driven by an effective hamiltonian being a two-dimensional massless Dirac operator.

Various mathematical studies have been undertaken, starting with a rigorous mathematical derivation of such hamiltonians, see e.g. [20] for the effective hamiltonian derivation or [3, 8, 30, 37] for the justification of the so-called infinite mass boundary conditions. Many properties of such operators have been investigated as their self-adjointness in bounded domains with specified boundary conditions or coupled with the so-called $\delta$-interactions, see [9, 11]. Let us also mention recent works on spectral properties and asymptotics of Dirac-type operators in specific asymptotic regimes (see [4, 23]).

In this work, we are interested in finding geometrical bounds on the eigenvalues of one of the simplest Dirac operator relevant in physics: the two-dimensional massless Dirac operator with infinite mass boundary conditions.

To set the stage, let $\Omega \subset \mathbb{R}^2$ be a $C^\infty$ simply connected domain and let $n = (n_1, n_2)^\top$ be the outward pointing normal field on $\partial \Omega$. The Dirac operator with infinite mass boundary conditions in $L^2(\Omega, \mathbb{C}^2)$ is defined as

$$D_\Omega := \begin{pmatrix} 0 & -2i\partial_\bar{z} \\ -2i\partial_z & 0 \end{pmatrix},$$

$$\text{dom}(D_\Omega) := \left\{ u = (u_1, u_2)^\top \in H^1(\Omega, \mathbb{C}^2) : u_2 = i nu_1 \text{ on } \partial \Omega \right\},$$

where we have set $n := n_1 + i n_2$ and with the Wirtinger operators defined as usual by

$$\partial_z = \frac{1}{2}(\partial_1 - i \partial_2), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i \partial_2).$$

The Dirac operator with infinite mass boundary conditions $D_\Omega$ is known to be self-adjoint (see [11, Thm. 1.1.]), moreover its spectrum is symmetric with respect to the origin and constituted of eigenvalues of finite multiplicity satisfying

$$\cdots \leq -E_k(\Omega) \leq \cdots \leq -E_1(\Omega) < 0 < E_1(\Omega) \leq \cdots \leq E_k(\Omega) \leq \cdots .$$

In the recent paper [12], the following geometrical lower bound is obtained

$$E_1(\Omega) \geq \sqrt{\frac{2\pi}{|\Omega|}},$$

where $|\Omega|$ denotes the area of the domain $\Omega$. However, this lower bound is never attained among Euclidean domains and by analogy with the famous Faber-Krahn inequality [19, 26], a natural conjecture for the optimal lower-bound is the following.

Conjecture 1. There holds

$$E_1(\Omega) \geq \sqrt{\frac{\pi}{|\Omega|}} E_1(\mathbb{D}),$$

where $\mathbb{D}$ is the unit disk. There is equality in the above inequality if and only if $\Omega$ is a disk.

Remark 2. As explained in [12, Remark 2] (see also [28, Appendix]), the eigenstructure of the unit disk is explicit. Indeed, $E_1(\mathbb{D}) \approx 1.435 \ldots$ is the first non-negative root of the equation $J_0(E) = J_1(E)$ where $J_0$ and $J_1$ are the Bessel functions of the first kind of order 0 and of order 1, respectively. Moreover, an associated
eigenfunction is given for \((x_1, x_2) \in \mathbb{D}\) by
\[
\left(\frac{J_0(|x|)}{(1 \pm i \frac{x_2}{x_1}) J_1(|x|)}\right).
\]

Conjecture 1 motivated part of this paper and is still an open question. However, in Section 8 we provide strong numerical evidences supporting it and in Section 7 we show how Conjecture 1 is intimately connected to the famous Bossel-Daners inequality for the Robin Laplacian (see [14, 16]).

The quest for a geometrical upper-bound has also attracted attention recently as for instance in [28]. In this work, the given geometrical upper-bound is sharp in the sense that it is an equality if and only if the considered domain is a disk. Nevertheless, this upper-bound depends in a complicated fashion of different geometrical parameters and may be hard to compute in practice.

Let us also mention that similar questions are dealt with in the differential geometry literature for lower bounds and upper bounds for Dirac operators on spin-manifolds (see for instance [1, 6, 7, 33]).

One of the main result of this paper is the following theorem which gives a geometrical upper-bound in terms of simple geometric quantities:

\[
E_1(\Omega) \leq \frac{|\Omega|}{\pi r_i^2 + |\Omega|} E_1(\mathbb{D}),
\]

with equality if and only if \(\Omega\) is a disk.

The proof is by combining a new variational characterization of \(E_1(\Omega)\), inspired by min-max techniques for operators with gaps introduced in [17] and the classical proof of Szegő about the eigenvalues of membranes of fixed area [38].

It turns out this new variational characterization is of interest by itself because it also allows for numerical simulations and we believe that it could be an adequate starting point to prove Conjecture 1 as discussed further on in Section 7. To introduce it, consider the quadratic form
\[
q_{E,0}(u) := 4 \int_{\Omega} |\partial_x u|^2 dx - E^2 \int_{\Omega} |u|^2 dx + E \int_{\partial \Omega} |u|^2 ds, \quad \text{dom}(q_{E,0}) := C^\infty(\Omega, \mathbb{C}).
\]

For \(E > 0\), \(q_{E,0}\) is bounded below with dense domain and we consider \(q_{E}^\Omega\) the closure in \(L^2(\Omega)\) of \(q_{E,0}\). Then, we define the first min-max level
\[
\mu^\Omega(E) := \inf_{u \in \text{dom}(q_{E}^\Omega) \setminus \{0\}} \frac{4 \int_{\Omega} |\partial_x u|^2 dx - E^2 \int_{\Omega} |u|^2 dx + E \int_{\partial \Omega} |u|^2 ds}{\int_{\Omega} |u|^2 dx}.
\]

The second main result of this paper is the following non-linear variational characterization of \(E_1(\Omega)\).

**Theorem 4.** \(E > 0\) is the first non-negative eigenvalue of \(D^\Omega\) if and only if \(\mu^\Omega(E) = 0\).

The advantage of the quadratic form \(q_{E}^\Omega\) is two-fold. First, functions in the considered variational space are now scalar valued and, second, the infinite mass boundary conditions do not appear in the variational formulation. However, the first drawback is that \(\text{dom}(q_{E}^\Omega)\) contains the Hardy space \(H^1_{0}(\Omega)\), constituted of holomorphic functions with traces in \(L^2(\partial \Omega)\). In particular, \(\text{dom}(q_{E}^\Omega)\) is not a usual Sobolev space and a special care is needed in order to prove Theorem 4.

In particular, it asks for a precise description of the domain \(\text{dom}(q_{E}^\Omega)\) as well as
the domain of the associated self-adjoint operator via Kato’s first representation theorem (see [25, Chap. VI, Thm. 2.1]). It is done using convolution operators reminiscent of what is done in [5, 31], elliptic regularity properties of the maximal Wirtinger operators as well as using Cauchy singular integral operators on \(\partial \Omega\), seen as periodic pseudo-differential operators.

Theorem 4 is reminiscent of [17, 18], where a similar strategy is used to deal with the Dirac-Coulomb operator. To our knowledge, this is the first time this idea is extended to boundary value problems and now, we describe its heuristic.

Let \((u, v)^\top \in \text{dom } (D_{\Omega})\) be an eigenfunction associated with the eigenvalue \(E > 0\). In \(\Omega\), the eigenvalue equation reads

\[
-2i \partial_z v = Eu, \quad -2i \partial \bar{z} u = Ev.
\]

If we assume that this identity is true up to the boundary \(\partial \Omega\), we obtain the following boundary condition for \(u\):

\[
\bar{\mu} \partial_z u + \frac{E}{2} u = 0 \text{ on } \partial \Omega.
\]

Now, Equation (4) gives

\[
-4 \partial_z \partial_{\bar{z}} u = E^2 u \text{ in } \Omega.
\]

Hence, a weak formulation is obtained taking the scalar product by \(u\), integrating by parts and taking into account the boundary condition (5). This formally gives

\[
q_{\Omega}^E(u) = 0 \text{ and this is the reason for introducing the quadratic form } q_{\Omega}^E \text{ in (2)}.
\]

Let us add two remarks. The first one explains that (5)-(6) can be recast into a non-linear eigenvalue problem for a Laplace operator with oblique boundary conditions. The second remark, explains how Theorem 4 could be extended to handle the next eigenvalues.

**Remark 5.** Note that (6) is an eigenvalue equation for the Laplace operator and reads \(-\Delta u = E^2 u\). The boundary condition (5) is a relation between the normal derivative, the tangential derivative and the value of the function on \(\partial \Omega\). If we let \(t\) be the tangent field on \(\partial \Omega\) such that \((\bar{n}, t)\) is a direct frame, the problem can be re-interpreted as an oblique problem

\[
\begin{aligned}
-\Delta u &= E^2 u \text{ in } \Omega, \\
\partial_n u + i \partial_t u + Eu &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]

where \(\partial_n\) and \(\partial_t\) are the normal and tangential derivatives, respectively.

Note that Problem (7) is non-linear because the parameter \(E > 0\) appears both in the eigenvalue equation and in the boundary condition.

**Remark 6.** For \(j \geq 1\), one can consider the \(j\)-th min-max level of \(q_{\Omega}^E\) defined as

\[
\mu_j^D(E) := \inf_{E \subseteq \text{dom } (D_{\Omega})} \sup_{\dim P = j} \frac{4 \int_\Omega |\partial_z u|^2 dx - E^2 \int_\Omega |u|^2 dx + E \int_{\partial \Omega} |u|^2 ds}{\int_\Omega |u|^2 dx}.
\]

As in [17], Theorem 4 could be extended as follows: \(E > 0\) is the \(j\)-th non-negative eigenvalue of \(D_{\Omega}\) if and only if \(\mu_j^D(E) = 0\). We do not discuss it here because we are concerned only with the principal eigenvalue \(E_1(\Omega)\).

Finally, let us comment the hypothesis on \(\Omega\). First, one would like to lower the smoothness hypothesis to be able to handle, for instance, Lipschitz domains. This is a natural question but there is no reason for the Dirac operator with infinite mass boundary to be self-adjoint on such a domain \(\text{dom } (D_{\Omega})\) (see the case of polygonal domains in [27]). Moreover, as part of the proof relies on pseudo-differential techniques, we prefer to keep the \(C^\infty\) smoothness assumption on \(\partial \Omega\) because it allows for a more efficient treatment of singular integral operators on the boundary.
Second, the simply connectedness assumption may be an unnecessary hypothesis for Theorem 4 to hold. Nevertheless, we are not able to drop it in Theorem 3 because the proof relies on the Riemann mapping theorem to build an admissible test function for $q_E^\Omega$.

1.2. Structure of the paper. In Section 2, we gather several results on Sobolev spaces on $\partial \Omega$, periodic pseudo-differential operators on $\partial \Omega$ and deduce various mapping properties of the Cauchy singular integral operators.

Section 3 contains a description of the domain of the maximal Wirtinger operators. In particular, we discuss the existence of a trace operator for functions belonging to these domains and state a fundamental elliptic regularity result.

Section 4 deals with the description of the Bergman and Hardy spaces on $\Omega$ thanks to integral operators. This is done by introducing the Szegö projectors on the Sobolev spaces on the boundary $H^s(\partial \Omega)$ ($s \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$). As a byproduct of this analysis we are able to describe explicitly the domains of the maximal Wirtinger operators.

Theorem 4 is proved in Section 5. We start by describing the domain of the quadratic form $q^\Omega_E$ in terms of the first-order Sobolev space $H^1(\Omega)$ and the Hardy space on $\Omega$. Then, the analysis is pushed forward to study the domain of the self-adjoint operator associated with $q^\Omega_E$ via Kato’s first representation theorem (see [25, Chap. VI, Thm. 2.1]). Combining these tools, we prove Theorem 4.

Then, we apply Theorem 4 in Section 6 to prove Theorem 3. The proof is by adapting the well-known proof of Szegö [38] to our setting, constructing an adequate test function for the new variational formulation.

In Section 7, we show that Conjecture 1 can be reformulated and that it is related to the famous Bossel-Daners inequality.

We conclude in Section 8 illustrating by numerical experiments the validity of Conjecture 1 and several theoretical results discussed all along the paper.

2. Preliminaries

2.1. Sobolev spaces on $\partial \Omega$. In the following, $\mathbb{T}$ is the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, $\mathcal{D}(\mathbb{T}) = C^\infty(\mathbb{T})$ is the space periodic smooth functions on the torus $\mathbb{T}$ and $\mathcal{D}'(\mathbb{T})$ the space of periodic distributions on the torus $\mathbb{T}$. Let $f \in \mathcal{D}(\mathbb{T})'$ we define its Fourier coefficients using the duality pairing by

$$\hat{f}^s(n) := \langle f, e_{-n} \rangle_{\mathcal{D}(\mathbb{T})', \mathcal{D}(\mathbb{T})}, \quad e_n := t \in \mathbb{T} \mapsto e^{2\pi i nt}.$$  

For $s \in \mathbb{R}$, the Sobolev space of order $s$ on $\mathbb{T}$ is defined as

$$H^s(\mathbb{T}) := \{ f \in \mathcal{D}(\mathbb{T})' : \sum_{n = -\infty}^{+\infty} (1 + |n|^{2s})|\hat{f}(n)|^2 < +\infty \}.$$

Set $\ell := |\partial \Omega|$ and let $\gamma : \mathbb{R}/[0, \ell] \to \partial \Omega$ be a smooth arc-length parametrization of $\partial \Omega$. Consider the map

$$U^* : \mathcal{D}(\mathbb{T}) \to \mathcal{D}(\partial \Omega), \quad (U^* g)(x) := \ell^{-1} g(\ell^{-1} \gamma^{-1}(x)), \quad x \in \partial \Omega,$$

where we have set $\mathcal{D}(\partial \Omega) := C^\infty(\partial \Omega)$. We define the map $U : \mathcal{D}(\partial \Omega)' \to \mathcal{D}(\mathbb{T})'$ as

$$(U f, g)_{\mathcal{D}(\partial \Omega)', \mathcal{D}(\mathbb{T})} := \langle f, U^* g \rangle_{\mathcal{D}(\partial \Omega)', \mathcal{D}(\partial \Omega)}.$$  

The Sobolev space of order $s \in \mathbb{R}$ on $\partial \Omega$ is defined as

$$H^s(\partial \Omega) := \{ f \in \mathcal{D}(\partial \Omega)' : U f \in H^s(\mathbb{T}) \}.$$
2.2. Periodic pseudo-differential operators. Let us start by defining periodic pseudo-differential operators on $\mathbb{T}$.

**Definition 7.** A linear operator $H$ on $C^\infty(\mathbb{T})$ is a periodic pseudo-differential operator on $\mathbb{T}$ if there exists $h : \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ such that:

1. for all $n \in \mathbb{Z}$, $h(\cdot, n) \in C^\infty(\mathbb{T})$,
2. $H$ acts as $Hf = \sum_{n \in \mathbb{Z}} h(\cdot, n) \hat{f}(n)e_n$,
3. there exists $\alpha \in \mathbb{R}$ such that for all $p, q \in \mathbb{N}_0$ there exists $c_{p,q} > 0$ such that there holds

$$\left| \left( \frac{d^p}{dt^p} (\omega^q h) \right)(t, n) \right| \leq c_{p,q}(1 + |n|)^\alpha - q,$$

where the operator $\omega$ is defined for all $(t, n) \in \mathbb{T} \times \mathbb{Z}$ by $(\omega h)(t, n) := h(t, n + 1) - h(t, n)$.

$\alpha$ is called the order of the pseudo-differential operator $H$. The set of pseudo-differential operators of order $\alpha$ on $\mathbb{T}$ is denoted $\Psi^\alpha$ and we define

$$\Psi^{-\infty} := \bigcap_{\alpha \in \mathbb{R}} \Psi^\alpha.$$

**Example 8.** For further use, we introduce the example of multiplication operators. Consider $H : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ defined as

$$(Hf)(t) := h(t)f(t), \quad h \in C^\infty(\mathbb{T}).$$

Decomposing in Fourier series, one immediately obtains

$$(Hf) = \sum_{n \in \mathbb{Z}} h \hat{f}(n)e_n.$$

There holds $\omega^q h = 0$ for all $q \geq 1$ and, as $h \in C^\infty(\mathbb{T})$, for all $t \in \mathbb{T}$ we obtain

$$\left| \left( \frac{d^p h}{dt^p} \right)(t) \right| \leq c_p, \quad \text{for some } c_p > 0$$

and we get $H \in \Psi^0$.

Using the map $U$ defined in [8], we define periodic pseudo-differential operators on $\partial\Omega$ as follows.

**Definition 9.** A linear operator $H$ on $C^\infty(\partial\Omega)$ is a periodic pseudo-differential operator on $\partial\Omega$ of order $\alpha \in \mathbb{R}$ if the operator $H_0 := UH \cdot \cdot U^{-1} \in \Psi^\alpha$. The set of pseudo differential operators on $\partial\Omega$ of order $\alpha$ is denoted $\Psi^\alpha_\partial\Omega$ and we set

$$\Psi^{-\infty}_\partial\Omega := \bigcap_{\alpha \in \mathbb{R}} \Psi^\alpha_\partial\Omega.$$

We will need the following properties of pseudo-differential operators on $\partial\Omega$. They can be found in [34, §5.8 & 5.9].

**Proposition 10.** Let $s, \alpha, \beta \in \mathbb{R}$ and $H \in \Psi^\alpha_\partial\Omega, G \in \Psi^\beta_\partial\Omega$.

1. $H$ extends uniquely to a bounded linear operator, also denoted $H$, from $H^s(\partial\Omega)$ to $H^{s-\alpha}(\partial\Omega)$.
2. There holds $H + G \in \Psi^{\max(\alpha, \beta)}_\partial\Omega, HG \in \Psi^{\alpha+\beta}_\partial\Omega, \; [H, G] \in \Psi^{\alpha+\beta-1}_\partial\Omega$. 

2.3. Cauchy singular integral operators. For \( f \in C^\infty(\partial \Omega) \), the Cauchy singular integral operator is defined as a principal value by
\[
S_h(f)(z) := \frac{1}{i\pi} \text{p.v.} \int_{\partial \Omega} \frac{f(\xi)}{\xi - z} \, d\xi, \quad z \in \partial \Omega.
\]
We define its anti-holomorphic counterpart as
\[
S_{ah}(f)(z) := \frac{1}{i\pi} \text{p.v.} \int_{\partial \Omega} \frac{f(\xi)}{\xi - z} \, d\xi, \quad z \in \partial \Omega.
\]
It turns out \( S_h \) and \( S_{ah} \) are periodic pseudo-differential operators on \( \partial \Omega \). This is the purpose of the following proposition.

**Proposition 11.** The linear maps \( S_h \) and \( S_{ah} \) are periodic pseudo-differential operators of order 0. In particular, they are bounded linear operators from \( H^s(\partial \Omega) \) onto itself for all \( s \in \mathbb{R} \).

**Proof.** This is proved in \[13\] Prop 2.9.\] where the operators \( S_h \) and \( S_{ah} \) are denoted \( C_{\Sigma} \) and \( -C_{\Sigma}^0 \) respectively (with \( \Sigma := \partial \Omega \)). \( \square \)

We will also need the following property.

**Proposition 12.** Let \( H_n \) be the multiplication operator by the normal \( n \) in \( C^\infty(\partial \Omega) \). There holds:

1. \( H_n \) is a periodic pseudo-differential operator of order 0.
2. Let \( \sharp \in \{h, ah\} \) we have \([H_n, S]\) \( \in \Psi^{-1}_{\partial \Omega} \).
3. There holds \( S_{ah} + S_h \in \Psi^{-\infty}_{\partial \Omega} \).

**Proof.** Point (1) is proved remarking that the operator \( UH_nU^{-1} \) is a multiplication operator in \( T \). Thanks to Example 8 we know that \( UH_nU^{-1} \in \Psi^0 \) hence by definition we get \( H_n \in \Psi^0_{\partial \Omega} \).

Let us deal with Point (2). Let \( \sharp \in \{h, ah\} \), by Proposition 11, \( S \) \( \in \Psi^0_{\partial \Omega} \) and by Point (1) \( H_n \in \Psi^0_{\partial \Omega} \). Hence, by (2) Proposition 10 we obtain Point (2).

Finally, we prove Point (3). By \[13\] Proposition 2.9.\] there exists \( L \in \Psi^0_{\partial \Omega} \) and \( R_1, R_2 \in \Psi^{-\infty}_{\partial \Omega} \) such that
\[
S_h = L + R_1, \quad S_{ah} = -L + R_2.
\]
Hence, \( S_h + S_{ah} = R_1 + R_2 \in \Psi^{-\infty}_{\partial \Omega} \) by (2) Proposition 10. \( \square \)

3. Maximal Wirtinger operators

In this section we describe elemental properties of the maximal Wirtinger operators defined as
\[
\partial u = \partial_x u, \quad \text{dom}(\partial) := \{ u \in L^2(\Omega) : \partial_x u \in L^2(\Omega) \},
\]
\[
\partial_{ah} u = \partial_x u, \quad \text{dom}(\partial_{ah}) := \{ u \in L^2(\Omega) : \partial_x u \in L^2(\Omega) \}.
\]

For \( \sharp \in \{h, ah\} \), consider the operator norms \( \| \cdot \|_\sharp \) defined as
\[
\| u \|_\sharp := \| \partial u \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)}, \quad u \in \text{dom}(\partial_\sharp).
\]
In particular, \( \text{dom}(\partial_\sharp) \) endowed with the scalar product defined for \( u, v \in \text{dom}(\partial_\sharp) \) by
\[
(u, v)_\sharp = (\partial_\sharp u, \partial_\sharp v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)}
\]
is a Hilbert space.

The first lemma is obtained by a simple integration by parts.
Lemma 14. The following identities hold.\

\[ H^1(\mathbb{R}^2) = \{ f \in L^2(\mathbb{R}^2) : \partial_x f \in L^2(\mathbb{R}^2) \} = \{ f \in L^2(\mathbb{R}^2) : \partial_y f \in L^2(\mathbb{R}^2) \} \]

Proof. Let \( f \in C_0^\infty(\mathbb{R}^2) \). Integrating by parts several times we obtain:

\[
\| \nabla f \|_{L^2(\mathbb{R}^2)}^2 = \langle f, -\Delta f \rangle_{L^2(\mathbb{R}^2)} = 4\langle f, -\partial_x \partial_x f \rangle_{L^2(\mathbb{R}^2)} = 4\| \partial_x f \|_{L^2(\mathbb{R}^2)}^2
\]

As \( C_0^\infty(\mathbb{R}^2) \) is dense in \( H^1(\mathbb{R}^2) \), we obtain the expected result. \( \square \)

The next lemma is a density result.

Lemma 13. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain.

The space \( C_0^\infty(\Omega) := C_0^\infty(\Omega, \mathbb{C}) \) is dense in \( \text{dom}(\partial_\Omega) \).

Proof. Let \( u \in \text{dom}(\partial_\Omega) \) and assume that for all \( \varphi \in C_0^\infty(\Omega) \) there holds

\[
0 = \langle u, \varphi \rangle_{\Omega} = (\partial_\Omega u, \partial_\Omega \varphi)_{\Omega} + (u, \varphi)_{\Omega}.
\]

In particular, if \( \varphi \in C_0^\infty(\Omega) \), we obtain \(-\Delta u = -4u \) first in \( D(\Omega)' \) then in \( L^2(\Omega) \).

Define \( v = \partial_\Omega u \) and denote by \( v_0 \) its extension to the whole \( \mathbb{R}^2 \) by \( 0 \). For \( \varphi \in C_0^\infty(\mathbb{R}^2) \) there holds

\[
\langle \partial_\Omega v_0, \varphi \rangle_{D'(\mathbb{R}^2), D(\mathbb{R}^2)} = -\langle v_0, \overline{\partial_\Omega \varphi} \rangle_{D'(\mathbb{R}^2), D(\mathbb{R}^2)}
\]

where \( u_0 \) denotes the extension by zero of \( u \) to the whole \( \mathbb{R}^2 \). It gives \( \partial_\Omega v_0 = u_0 \in L^2(\mathbb{R}^2) \). By Lemma 13, \( v_0 \) is in \( H^1(\mathbb{R}^2) \) and by [15, Prop. IX.18.] we get \( v \in H^1_0(\Omega) \). Remark that in \( D'(\Omega) \), there holds \( \partial_\Omega \partial_\Omega v = v \). Indeed, we have

\[
\partial_\Omega \partial_\Omega v = \partial_\Omega \partial_\Omega \partial_\Omega u = \partial_\Omega u = v.
\]

In particular this identity also holds true in \( L^2(\Omega) \). Now, pick a sequence \( v_n \in C_0^\infty(\Omega) \) converging to \( v \) in the \( H^1(\Omega) \)-norm. There holds

\[
\langle v, v_n \rangle_{L^2(\Omega)} = \langle \partial_\Omega \partial_\Omega v, v_n \rangle_{L^2(\Omega)} = -\langle \partial_\Omega v, \partial_\Omega v_n \rangle_{D'(\Omega), D(\Omega)}
\]

Letting \( n \to +\infty \) one obtains \( \| v \|_{L^2(\Omega)}^2 = -\| \partial_\Omega v \|_{L^2(\Omega)}^2 \) which implies \( v = 0 \). In \( D'(\Omega) \) we have \( \partial_\Omega v = \partial_\Omega \partial_\Omega u = u \). As \( v = 0, u = 0 \) which concludes the proof for \( v = \partial_\Omega \partial_\Omega u = \partial_\Omega u = v \). \( \square \)

In order to describe precisely the domains \( \text{dom}(\partial_\Omega) \) we need to prove the existence of traces on \( \partial \Omega \) for functions in \( \text{dom}(\partial_\Omega) \). To this aim, define the following Dirichlet trace operators

\[
\Gamma^+ : H^1(\Omega) \to H^\frac{1}{2}(\partial \Omega), \quad \Gamma^- : H^1_{loc}(\mathbb{R}^2 \setminus \overline{\Omega}) \to H^\frac{1}{2}(\partial \Omega).
\]

These linear operators are known to be bounded (see [29, Thm. 3.37]) and there exists continuous extension operators such that for \( f \in H^\frac{1}{2}(\partial \Omega) \) there holds

\[
E^+ f \in H^1(\Omega), \quad E^- f \in H^1(\mathbb{R}^2 \setminus \overline{\Omega}) \quad \text{and} \quad \Gamma^+ E^+ f = f.
\]

Actually, the operator \( \Gamma^+ \) can be extended to functions in \( \text{dom}(\partial_\Omega) \) (\( \partial_\Omega \in \{ h, ah \} \)). This is the purpose of the following proposition.
Lemma 15. Let $\varepsilon \in \{h, ah\}$. The operator $\Gamma^+$ defined in (9) extends into a linear bounded operator between $\text{dom}(\partial_1)$ and $H^{-\frac{1}{2}}(\partial \Omega)$.

Proof. Let $(v_n)_{n \in \mathbb{N}} \in C^\infty(\overline{\Omega})^3$ be a sequence that converges to $v$ in the $\| \cdot \|_h$-norm when $n \to +\infty$. Let us prove that $(\Gamma^+ v_n)_{n \in \mathbb{N}}$ has a limit in $H^{-\frac{1}{2}}(\partial \Omega)$. First recall the integration by part formula

$$\frac{1}{2}(\Gamma^+ u, \nabla \Gamma^+ w)_{L^2(\Omega)} = \langle \partial_z u, w \rangle_{L^2(\Omega)} + \langle u, \partial_z w \rangle_{L^2(\Omega)}$$

valid for any $u, w \in H^1(\Omega)$. Second, pick $f \in H^{\frac{1}{2}}(\partial \Omega)$ and consider $w = E^+(nf) \in H^1(\Omega)$. Remark that the definition of $\Gamma^+$ extends into a linear bounded operator between $\overline{\Omega}$ and $\partial \Omega$ thus converges to an element $\Gamma^+ v := g \in H^{-\frac{1}{2}}(\partial \Omega)$. Remark that the definition of $\Gamma^+ v$ does not depend on the chosen sequence $(v_n)_{n \in \mathbb{N}}$ and that we have

$$\|\Gamma^+ v_n\|_{H^{-\frac{1}{2}}(\partial \Omega)} \leq 4c_{\Omega} \|v_n\|_h$$

which implies, when $n \to +\infty$, that $\Gamma^+$ is bounded from $\text{dom}(\partial_1)$ to $H^{-\frac{1}{2}}(\partial \Omega)$. The proof for $\text{dom}(\partial_{ah})$ is handled similarly.

Remark 16. If one picks $R > 0$ such that $\overline{\Omega} \subset B(0, R) := \{ x \in \mathbb{R}^2 : \|x\| < R \}$, one can prove that for $\ast \in \{z, \overline{\Omega}\}$, $\Gamma^-$ extends into a linear bounded operator between the space $\{ u \in L^2(B(0, R) \setminus \Omega) : \partial_\ast u \in L^2(B(0, R) \setminus \Omega) \}$ and $H^{\frac{1}{2}}(\partial \Omega)$. The proof goes along the same lines as the one of Lemma 15 using an extension operator $E^- : H^{\frac{1}{2}}(\partial \Omega) \to H^1(B(0, R) \setminus \Omega)$ constructed such that for all $f \in H^{\frac{1}{2}}(\partial \Omega)$, $E^-(f) |_{\partial \Omega \setminus B(0, R)} = 0$.

Remark 17. Pick $u \in \text{dom}(\partial_{ah})$ and $w \in H^1(\Omega)$. Note that by definition, the following Green’s Formula holds

$$\langle \partial_z u, w \rangle_{L^2(\Omega)} = -\langle u, \partial_z w \rangle_{L^2(\Omega)} + \frac{1}{2}(\nabla \Gamma^+ u, \Gamma^+ w)_{H^{-\frac{1}{2}}(\partial \Omega), H^{\frac{1}{2}}(\partial \Omega)}. \quad (10)$$

The following elliptic regularity result is rather well known (see the analogous statement [11, Lemma 2.4.]).

Lemma 18. Let $\varepsilon \in \{h, ah\}$ and $u \in \text{dom}(\partial_\varepsilon)$. If $\Gamma^+ u \in H^{\frac{1}{2}}(\partial \Omega)$ then $u \in H^1(\Omega)$.

Proof. Let $u \in \text{dom}(\partial_\varepsilon)$ be such that $\Gamma^+ u \in H^{\frac{1}{2}}(\partial \Omega)$ and set $v = u - E^+(\Gamma^+ u)$. Then, $\Gamma^+ v = 0$ and if $v \in H^1(\Omega)$ the result is proved. If $v_n \in C^\infty(\Omega)$ is a sequence converging to $v$ in the $\| \cdot \|_h$-norm there holds $\Gamma^+ v_n \to 0$ in $H^{-\frac{1}{2}}(\partial \Omega)$ by Lemma 15. In particular, it gives for any $w \in H^1(\Omega)$

$$\langle v, \partial_z w \rangle_{L^2(\Omega)} = \lim_{n \to +\infty} \left( -\langle \partial_z v_n, w \rangle_{L^2(\Omega)} + \frac{1}{2}(\Gamma^+ v_n, \nabla \Gamma^+ w)_{L^2(\partial \Omega)} \right)$$

$$= -\langle \partial_z v, w \rangle_{L^2(\Omega)}.$$
Let \( v_0 \) (resp. \( h_0 \)) be the extension of \( v \) (resp. \( h := \partial_z v \)) by zero to the whole \( \mathbb{R}^2 \). If \( \varphi \in C^\infty_0(\mathbb{R}^2) \), there holds
\[
-\langle h_0, \varphi \rangle_{D'(\mathbb{R}^2), D(\mathbb{R}^2)} = -\langle h, \varphi \rangle_{L^2(\Omega)} = \langle v, \partial_z \varphi \rangle_{L^2(\Omega)} = \langle v_0, \partial_z \varphi \rangle_{D'(\mathbb{R}^2), D(\mathbb{R}^2)} = -\langle \partial_z v_0, \varphi \rangle_{D'(\mathbb{R}^2), D(\mathbb{R}^2)}.
\]
Thus \( \partial_z v_0 = h_0 \in L^2(\mathbb{R}^2) \) and by Lemma 13, \( v_0 \in H^1(\mathbb{R}^2) \) and \( v \in H^1_0(\Omega) \). The proof for \( u \in \text{dom}(\partial_{\bar{z}}) \) is handled similarly.

4. Bergman and Hardy spaces on \( \Omega \)

We introduce \( A^2_\mathbb{H}(\Omega) \) and \( A^2_{\text{ah}}(\Omega) \) the holomorphic and anti-holomorphic Bergman spaces on \( \Omega \), respectively. They are defined as
\[
A^2_\mathbb{H}(\Omega) := \{ u \in H^1(\Omega) : u \in L^2(\Omega) \}, \quad A^2_{\text{ah}}(\Omega) := \{ u : \pi \in A^2_\mathbb{H}(\Omega) \},
\]
where \( H^1(\Omega) \) denotes the space of holomorphic functions in \( \Omega \). The holomorphic and anti-holomorphic Hardy spaces, denoted \( H^2_\mathbb{H}(\Omega) \) and \( H^2_{\text{ah}}(\Omega) \), respectively, are defined as
\[
H^2_\mathbb{H}(\Omega) := \{ u \in A^2_\mathbb{H}(\Omega) : u \in L^2(\partial\Omega) \}, \quad H^2_{\text{ah}}(\Omega) := \{ u : u \in H^2(\Omega) \}.
\]
This section aims to describe explicitly the Bergman and Hardy spaces on \( \Omega \) in terms of Cauchy integrals and Szegö projectors that we define now.

For \( f \in C^\infty(\partial\Omega) \) consider the Cauchy integrals defined for \( z \in \mathbb{C} \setminus \partial\Omega \) by
\[
\Phi_h(f)(z) := \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi, \quad \Phi_{\text{ah}}(f)(z) := -\frac{1}{2\pi} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi.
\]
It is well-known (see [34, §4.1.2.]) that \( \Phi_h(f) \) (resp. \( \Phi_{\text{ah}}(f) \)) defines a holomorphic function (resp. anti-holomorphic function) in \( \mathbb{R}^2 \setminus \partial\Omega \).

The well-known Plemelj-Sokhotski formula (see [34, Thm. 4.1.1]) states that for \( f \in C^\infty(\partial\Omega) \) the functions \( \Phi_h(f) \) and \( \Phi_{\text{ah}}(f) \) have an interior and an exterior Dirichlet trace, denoted respectively \( \gamma^\circ_0 \) and \( \gamma^\circ_0 \), such that:
\[
\gamma^\circ_0 \Phi_h(f) = \pm \frac{1}{2} f + \frac{1}{2} S_h f, \quad \gamma^\circ_0 \Phi_{\text{ah}}(f) = \pm \frac{1}{2} f + \frac{1}{2} S_{\text{ah}} f.
\]
(12)

Let \( \xi \in \{ h, \text{ah} \} \), note that by [10, Theorem 3.1.], for \( f \in C^\infty(\partial\Omega) \) we know that \( \Phi_\xi(f)(\Omega) \in C^\infty(\overline{\Omega}) \) as well as \( \Phi_\xi(f)(\Omega) \in C^\infty(\mathbb{R}^2 \setminus \Omega) \). In particular, the traces \( \gamma^\circ_0 \Phi_\xi(f) \) coincide with \( \Gamma \Phi_\xi(f) \), where \( \Gamma \) are the trace operators defined in Lemma 13 and Remark 16.

**Definition 19.** We define the Szegö projectors in \( C^\infty(\partial\Omega) \) by
\[
\Pi^\pm_h := \pm \Gamma^\pm \Phi_h, \quad \Pi^\pm_{\text{ah}} := \pm \Gamma^\pm \Phi_{\text{ah}}.
\]
(13)

**Proposition 20.** Let \( s \in \mathbb{R} \) and \( \xi \in \{ h, \text{ah} \} \). The Szegö projectors \( \Pi^\pm_\xi \) extend uniquely into bounded linear operators from \( H^s(\partial\Omega) \) onto itself. Moreover, \( \Pi^\pm_\xi \) are projectors and \( \Pi^+_\xi + \Pi^-_\xi = 1 \).

**Proof.** Remark that for \( \xi \in \{ h, \text{ah} \} \) and \( f \in C^\infty(\partial\Omega) \), there holds
\[
\Pi^\pm_\xi f = \frac{1}{2} f \pm \frac{1}{2} S_\xi f.
\]
By Proposition 11, \( \Pi^\pm_\xi \) extends into a bounded linear operator from \( H^s(\partial\Omega) \) onto itself for all \( s \in \mathbb{R} \).
Let $s \in \mathbb{R}$ and $f \in H^s(\partial \Omega)$. A fundamental fact is that $S_2^2 f = f$ (see [34, Eqn. (4.10)]), in particular it implies that $S_{ah}^2 f = f$. Hence, we obtain
\[
(P_{\pm}^\sharp)^2 = (\frac{1}{2} \pm \frac{1}{2} S_2^2) (\frac{1}{2} \pm \frac{1}{2} S_2^2)
\]
\[
= \frac{1}{4} S_2^2 \pm \frac{1}{2} S_2^2
\]
Hence $P_{\pm}^\sharp$ are projectors and one easily checks that $P_{\pm}^\sharp + P_{\mp}^\sharp = 1$. □

The main goal of this section is to prove the following description of the Bergman and Hardy spaces. As we will see further on in Proposition 22, this description relies on an extension of the operators $\Phi^\sharp$ to Sobolev spaces on the boundary $\partial \Omega$ ($\sharp \in \{h, ah\}$).

**Theorem 21.** Let $\sharp \in \{h, ah\}$. The Bergman spaces satisfy
\[
A^2_\sharp(\Omega) = \{\Phi^\sharp(f) : f \in H^{-\frac{1}{2}}(\partial \Omega), P_{\mp}^\sharp f = 0\}.
\]
The Hardy spaces satisfy
\[
H^2_\sharp(\Omega) = \{\Phi^\sharp(f) : f \in L^2(\partial \Omega), P_{\mp}^\sharp f = 0\}.
\]

**4.1. Potential theory of the Wirtinger derivatives.** In this paragraph we prove the following proposition.

**Proposition 22.** Let $\sharp \in \{h, ah\}$ and $s \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$. The operator $\Phi^\sharp$ extends uniquely into a bounded operator from $H^s(\partial \Omega)$ to $H^{s+\frac{1}{2}}(\Omega)$ also denoted $\Phi^\sharp$.

In order to prove Proposition 22 we will need a few lemma. Let us start by defining fundamental solutions of the Wirtinger operators $\partial_h$ and $\partial_{ah}$:
\[
\varphi_h(x) = \frac{1}{\pi(x_1 + ix_2)}, \quad \varphi_{ah}(x) = \frac{1}{\pi(x_1 - ix_2)}.
\]

**Lemma 23.** Let $\sharp \in \{h, ah\}$. The linear map
\[
N_\sharp : u \in L^2(\Omega) \mapsto \varphi_\sharp * u_0
\]
is bounded from $L^2(\Omega)$ to $H^1_{loc}(\mathbb{R}^2)$. Here $u_0$ denotes the extension of $u$ by zero to the whole $\mathbb{R}^2$.

**Proof.** Let us prove it for $\sharp = h$ the proof for $\sharp = ah$ being similar. In the space of distributions $D'(\mathbb{R}^2)$, there holds
\[
\partial_{\sharp} \varphi_h = \delta_0,
\]
where $\delta_0$ is the delta-Dirac distribution.

Now, for $u$ in the Schwartz space $S(\mathbb{R}^2)$ recall that the Fourier transform of $u$ is defined as
\[
\hat{u}(k) := \int_{\mathbb{R}^2} f(x) e^{-2\pi i (x,k) x} dx, \quad \text{for all } k \in \mathbb{R}^2
\]
and $\hat{u} \in S'(\mathbb{R}^2)$. The Fourier transform extends to the space of tempered distribution $S'(\mathbb{R}^2)$ and as $\delta_0 \in S'(\mathbb{R}^2)$, the Fourier transform of (14) yields
\[
\hat{\varphi}_h(k) = \frac{1}{\pi i(k_1 + ik_2)}, \quad k = (k_1, k_2) \in \mathbb{R}^2 \setminus \{(0,0)\}.
\]
Let $K$ be a compact subset of $\mathbb{R}^2$ and take $u \in L^2(\Omega)$. We extend $u$ by zero to $\mathbb{R}^2$ and denote this extension $u_0 \in L^2(\mathbb{R}^2)$. 

$$\|\varphi_h * u_0\|_{H^1(K)}^2 \leq \|\varphi_h * u_0\|_{L^2(K)}^2 + \frac{1}{\pi} \int_{\mathbb{R}^2} |k|^2 |\varphi_h(u_0)(k)|^2 dk$$

$$= \|\varphi_h * u_0\|_{L^2(K)}^2 + \frac{1}{\pi^2} \int_{\mathbb{R}^2} |u_0(k)|^2 dk$$

$$= \|\varphi_h * u_0\|_{L^2(K)}^2 + \frac{1}{\pi^2} \|u\|_{L^2(\Omega)}^2$$

Now, let $R > 0$ be such that $K \subset \{ x \in \mathbb{R}^2 : |x| < R \}$ and $\overline{\Omega} \subset \{ x \in \mathbb{R}^2 : |x| < R \}$. Consider a cut-off function $\chi \in C_c^\infty([0, +\infty))$ such that $\chi(\rho) = 1$ whenever $0 \leq \rho < 2R$, $\chi(\rho) = 0$ whenever $\rho > 3R$.

Define the function $u_\chi$ as

$$u_\chi(x) := \int_{\mathbb{R}^2} \chi(|x-y|) \varphi_h(x-y) u_0(y) dy.$$ 

As defined, $u_\chi|_K = (\varphi_h * u_0)|_K$. Hence, we get

$$\|\varphi_h * u_0\|_{L^2(K)} = \|u_\chi\|_{L^2(\mathbb{R}^2)} \leq \|u_\chi\|_{L^2(\Omega)} \leq \|\chi(|\cdot|)\varphi_h\|_{L^1(\mathbb{R}^2)} \|u\|_{L^2(\Omega)}$$

where we have used Young’s inequality because $\chi(|\cdot|)\varphi_h \in L^1(\mathbb{R}^2)$. Indeed, there holds

$$\|\chi(|\cdot|)\varphi_h\|_{L^1(\mathbb{R}^2)} \leq \frac{1}{\pi} \int_{B(0,3R)} \frac{1}{|x|} dx = 6R$$

In particular, there exists $c_K > 0$, such that

$$\|\varphi_h * u_0\|_{H^1(K)} \leq c_K \|u\|_{L^2(\Omega)}$$

Hence, for any compact $K \subset \mathbb{R}^2$, $N_1$ is a bounded linear operator from $L^2(\Omega)$ to $H^1(K)$ and the proposition is proved.

Next, we recall that the Dirichlet trace on $\partial \Omega$ of a function in $H^1_{loc}(\mathbb{R}^2)$ can be defined as

$$\Gamma : H^1_{loc}(\mathbb{R}^2) \to H^{\frac{1}{2}}(\partial \Omega)$$

and is a bounded linear operator from $H^1_{loc}(\mathbb{R}^2)$ to $H^{\frac{1}{2}}(\partial \Omega)$ (see [23] Thm. 3.37)

Moreover, for $s \in [0, \ell]$, we introduce $t(s) := \gamma_1(s) + i\gamma_2(s)$ the expression of the tangent vector in the complex plane at the point $\gamma_1(s) + i\gamma_2(s)$.

**Lemma 24.** The dual adjoints of $(t\Gamma N_h)$ and $(\overline{\Omega} N_{ah})$, denoted $(t\Gamma N_h)'$ and $(\overline{\Omega} N_{ah})'$ respectively, are bounded linear maps from $H^{-\frac{1}{2}}(\partial \Omega)$ to $L^2(\Omega)$. Moreover if $f \in C^\infty(\partial \Omega)$, in $L^2(\Omega)$ there holds:

$$\Phi_{ah}(f) = \frac{i}{2} (t\Gamma N_h)'(f), \quad \Phi_h(f) = -\frac{i}{2} (\overline{\Omega} N_{ah})'(f).$$

**Proof.** Thanks to Lemma [23] and the mapping properties of $\Gamma$ we know that $\Gamma N_2$ is a bounded linear map from $L^2(\Omega)$ to $H^{\frac{1}{2}}(\partial \Omega)$ (for $\mathfrak{z} \in \{ h, ah \}$). As $\Omega$ is smooth, $t \in C^\infty(\partial \Omega)$ and $\overline{\mathfrak{f}} \in C^\infty(\partial \Omega)$. In particular the multiplication operators by $t$ and $\overline{\mathfrak{f}}$ are bounded and invertible in $H^{\frac{1}{2}}(\Omega)$. Hence, their dual adjoints satisfy the expected mapping property.
Now, pick $f \in C^\infty(\partial\Omega)$ and $v \in L^2(\Omega)$. Denoting by $v_0$ the extension of $v$ by zero to the whole $\mathbb{R}^2$ and using Fubini's theorem, there holds
\[
\langle (t\Gamma N_h)'f, v \rangle_{L^2(\Omega)} = \langle f, t\Gamma N_hv \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = \langle f, t\Gamma N_hv \rangle_{L^2(\Omega)}
\]
\[
= \int_{x \in \partial\Omega} \int_{y \in \mathbb{R}^2} \frac{f(x)v_0(y)t(x)}{\pi((x_1 - iy_2) - (y_1 - iy_2))} dyds(x)
\]
\[
= \int_{y \in \mathbb{R}^2} \frac{1}{\pi} \left( \int_{s=0}^t \frac{f(\gamma(s))(\gamma'_1(s) - iy\gamma'_2(s))}{\gamma_1(s) - iy\gamma_2(s)} ds \right) v_0(y)dy
\]
\[
= \int_{y \in \mathbb{R}^2} \frac{1}{\pi} \int_{\xi \in \partial\Omega} \frac{f(\xi)}{\xi - (y_1 - iy_2)} dyds(\xi) v_0(y)dy
\]
\[
= (-2i\Phi_{ah}(f), v)_{L^2(\Omega)}.
\]

The proof for $(\Gamma N_{ah})'$ goes along the same lines, which concludes the proof of this lemma.

For further use, we still denote $\Phi_{ah}$ and $\Phi_h$ the operators $\frac{1}{2}(t\Gamma N_h)'$ and $-\frac{1}{2}(\Gamma N_{ah})'$. Now, for $z \in \{h, ah\}$, when considering the operators
\[
\Phi_z : (C^\infty(\partial\Omega), \| \cdot \|_{H^{-\frac{1}{2}}(\partial\Omega)}) \rightarrow (\text{dom}(\partial_z), \| \cdot \|_z)
\]
they are bounded operators because for any $f \in C^\infty(\partial\Omega)$, $\Phi_z(f)$ and $\Phi_{ah}(f)$ are holomorphic and anti-holomorphic in $\Omega$, respectively. The density of $C^\infty(\partial\Omega)$ in $H^{-\frac{1}{2}}(\partial\Omega)$ yields for each operator a unique extension to $H^{-\frac{1}{2}}(\partial\Omega)$ which coincide with the previous one. In particular, for any $f \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi_z(f) \in \text{dom}(\partial_z)$ and $\partial_z\Phi_z(f) = 0$.

Now, we have collected all the tools to prove Proposition 22.

**Proof of Proposition 22**. For $s = -\frac{1}{2}$, Proposition 22 holds true, because of Lemma 24 and the density of $C^\infty(\partial\Omega)$ in $H^{-\frac{1}{2}}(\partial\Omega)$. Let us prove it for $s = \frac{1}{2}$. Remark that $\Phi_z(f) \in \text{dom}(\partial_z)$ if $f \in H^{\frac{1}{2}}(\partial\Omega)$ we also have $\Gamma^+\Phi_z(f) = \Pi_z^+f \in H^{\frac{1}{2}}(\partial\Omega)$ by Proposition 20. Hence, by Lemma 18 $\Phi_z(f) \in H^1(\Omega)$.

Let us use the closed graph theorem and take a sequence of functions $f_n \in H^{\frac{1}{2}}(\partial\Omega)$ such that $f_n \rightarrow f$ in the $H^{\frac{1}{2}}(\partial\Omega)$-norm. Assume also that $\Phi_z(f_n) \rightarrow u \in H^1(\Omega)$ where the convergence holds in the $H^1(\Omega)$-norm.

Because of the continuous embedding of $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$, $f_n \rightarrow f$ also in the $H^{-\frac{1}{2}}(\partial\Omega)$-norm. In particular, by Proposition 22 for $s = -\frac{1}{2}$, $\Phi_z(f_n) \rightarrow \Phi_z(f)$ in $L^2(\Omega)$. Consequently, the equality $u = \Phi_z(f)$ holds not only in $L^2(\Omega)$ but also in $H^1(\Omega)$ and by the closed graph theorem, $\Phi_z$ is a continuous linear map between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^1(\Omega)$.

The result for $s = 0$ holds by (real) interpolation theory (see [35] Prop. 2.1.62, & Prop. 2.3.11. & Prop. 2.4.3.).

**4.2. Explicit description of the Bergman and Hardy spaces**. Let us prove Theorem 21 starting with the following proposition concerning the Bergman spaces.

**Proposition 25.** Let $z \in \{h, ah\}$. There holds:
\[
\mathcal{A}_z^2(\Omega) = \{ \Phi_z(f) : f \in H^{-\frac{1}{2}}(\partial\Omega) \text{ such that } \Pi_z^\perp f = 0 \}, \quad z \in \{h, ah\}.
\]
Moreover, for all $f \in H^{-\frac{1}{2}}(\partial\Omega)$ there holds
\[
\Phi_z(f) = \Phi_z(\Pi_z^\perp f).
\]
Proof. Denote $\mathcal{E}_h$ the set on the right-hand side of (15). We prove it for $\xi = h$, the proof for $\xi = ah$ being similar.

Inclusion $A_h^0(\Omega)$. Let $u = \Phi_h(f) \in \mathcal{E}_h$, with $f \in H^{-\frac{1}{2}}(\partial \Omega)$ such that $\Pi_h f = 0$. By Proposition 22, $\Phi_h$ maps $H^{-\frac{1}{2}}(\partial \Omega)$ to $L^2(\Omega)$ thus $u \in L^2(\Omega)$. Moreover, there holds $\partial_z u = 0$ which implies that $u \in A_h^0(\Omega)$.

Inclusion $A_h^2(\Omega) \subset E_h$. For $u \in C^\infty(\overline{\Omega})$, $x \in \Omega$ and $\varepsilon > 0$ sufficiently small there holds

$$0 = \frac{1}{\pi} \int_{\partial \Omega(x,\varepsilon)} \partial_z \left( \frac{1}{x_1 + i x_2} - (y_1 + iy_2) \right) u(y) dy$$

$$= -\frac{1}{\pi} \int_{\partial \Omega(x,\varepsilon)} \partial_z u(y) dy$$

$$+ \frac{1}{2\pi} \int_{\partial \Omega(x,\varepsilon)} \frac{u(y)}{(x_1 + i x_2) - (y_1 + iy_2)} ds(y)$$

$$= -A + B + C.$$ 

However, we have

$$C = \frac{1}{2\pi} \int_0^{2\pi} u(x + \varepsilon (\cos t, \sin t)) dt \longrightarrow u(x), \text{ when } \varepsilon \to 0.$$ 

By definition, if $\gamma : [0, \ell] \to \partial \Omega$ is a smooth arc-length parametrization of $\partial \Omega$ there holds

$$B = -\frac{1}{2\pi} \int_0^{\ell} \frac{u(\gamma(t))}{(x_1 + i x_2) - (\gamma_1(t) + i \gamma_2(t)) (\gamma_1'(t) + i \gamma_2'(t)) dt}$$

$$= -\frac{1}{2\pi} \int_{\partial \Omega} \frac{u(\xi)}{\xi - (x_1 + i x_2)} d\xi = -\Phi_h(\Gamma^+ u)(x).$$

In particular, we obtain

$$\Phi_h(\Gamma^+ u)(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + \varepsilon(\cos t, \sin t)) dt$$

$$- \frac{1}{\pi} \int_{\partial \Omega(x,\varepsilon)} \frac{\partial_z u(y)}{(x_1 + i x_2) - (y_1 + iy_2)} dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(x + \varepsilon(\cos t, \sin t)) dt$$

$$- \frac{1}{\pi} \int_{\mathbb{C}^2 \setminus B(0,\varepsilon)} \frac{1}{(x_1 + i x_2) - (y_1 + iy_2)} (\partial_z u(y))_{\Omega}(y) dy.$$ 

(16)

Note that the linear form on $C_0^\infty(\mathbb{R}^2)$ defined by

$$p.v. \left( \frac{1}{x_1 + i x_2} \right) := \varphi \in C_0^\infty(\mathbb{R}^2) \mapsto \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \frac{1}{x_1 + i x_2} \varphi(x) dx \in \mathbb{C}$$

belongs to $\mathcal{D}'(\mathbb{R}^2)$. Remark that $(\partial_z u)_{\Omega} \in \mathcal{D}'(\mathbb{R}^2)$ and has compact support. Hence, $p.v.(\frac{1}{x_1 + i x_2}) \ast (\partial_z u)_{\Omega} \in \mathcal{D}'(\mathbb{R}^2)$ and taking the duality pairing with $\varphi \in C_0^\infty(\Omega)$ in (16) and $\varepsilon \to 0$ we get

$$\langle \Phi_h(\Gamma^+ u) - u, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \frac{1}{\pi} (p.v. \left( \frac{1}{x_1 + i x_2} \right) \ast (\partial_z u)_{\Omega}, \varphi)_{\mathcal{D}'(\mathbb{R}^2), \mathcal{D}(\mathbb{R}^2)}.$$ 

(17)

Now, remark that $A_h^2(\Omega) \subset \text{dom}(\partial_h)$ and pick a sequence of $C^\infty(\overline{\Omega})$ functions $(v_n)_{n \in \mathbb{N}}$ which converges to $v \in A_h^2(\Omega)$ in the norm of $\text{dom}(\partial_h)$ when $n \to +\infty$. 

In particular, \((v_n)_{n \in \mathbb{N}}\) converges to \(v\) and \((\partial_z v_n)_{n \in \mathbb{N}}\) converges to 0 when \(n \to +\infty\) in \(\mathcal{D}'(\mathbb{R}^2)\). Using (17) for \(u = v_n\) and letting \(n \to +\infty\) we obtain that in \(\mathcal{D}'(\Omega)\) there holds \(v = \Phi_h(\Gamma^+ v)\) where we have used the continuity of \(\Phi_h \circ \Gamma^+ : \text{dom}(\partial \xi) \to L^2(\Omega)\), and the continuity of the convolution in \(\mathcal{D}'(\mathbb{R}^2)\). Now, remark that we also have \(v = \Phi_h(\Gamma^+ v)\) in \(A^2_h(\Omega)\) and taking the trace \(\Gamma^+\) on both side of this identity we get
\[
\Pi^+_h \Gamma^+ v = \Gamma^+ v
\]
which implies \(v = \Phi_h(\Pi^+_h \Gamma^+ v)\) and proves the other inclusion. 

We are now in a good position to prove Theorem 21.

Proof of Theorem 21. Proposition 25 is precisely the first statement of Theorem 21 thus, the only thing left to prove is the statement for the Hardy spaces. Now, recall that for \(\xi \in \{h, ah\}\), we have defined the Hardy spaces in (11) and that we want to prove
\[
\mathcal{H}^2_{\mathcal{E}_h}(\Omega) = \{\Phi_{\mathcal{E}}(f) : f \in L^2(\partial \Omega), \Pi^+_\mathcal{E} f = 0\}.
\]
Let \(\mathcal{E}_h\) be the set on the right-hand side, we prove both inclusions.

Inclusion \(\mathcal{H}^2_{\mathcal{E}_h}(\Omega) \subset \mathcal{H}^2_{\mathcal{E}_h}(\Omega)\). Let \(u = \Phi_{\mathcal{E}}(f) \in \mathcal{E}_h\), by definition \(u \in \mathcal{A}^2_h\) et \(\Gamma^+ u = f \in L^2(\partial \Omega) \subset H^{-\frac{1}{2}}(\partial \Omega)\) which proves this inclusion.

Inclusion \(\mathcal{H}^2_{\mathcal{E}_h}(\Omega) \subset \mathcal{E}_h\). Let \(u \in \mathcal{H}^2_{\mathcal{E}_h}(\Omega)\). We know that in particular \(u = \Phi_{\mathcal{E}}(f)\) for some \(f \in H^{-\frac{1}{2}}(\partial \Omega)\) such that \(\Pi^+_\mathcal{E} f = 0\). But we have \(\Gamma^+ u = f \in L^2(\partial \Omega)\) which proves this inclusion and concludes the proof. 

4.3. Explicit description of the domain of the maximal Wirtinger operators. In this paragraph, we prove the following description of the domains of the maximal Wirtinger operators introduced in Section 3. This description involves the Bergman spaces introduced in the beginning of Section 4.

Proposition 26. Let \(\xi \in \{h, ah\}\). The following direct sum decomposition holds:
\[
\text{dom}(\partial \xi) = \{u \in H^\xi(\Omega) : \Pi^+_\mathcal{E} \Gamma^+ u = 0\} + \mathcal{A}^2_h(\Omega).
\]

For \(\xi \in \{h, ah\}\), the range of the trace operator \(\Gamma^+ : \text{dom}(\partial \xi) \to H^{-\frac{1}{2}}(\partial \Omega)\) is of crucial importance to prove Proposition 26. We describe its range now, thanks to the Szegö projectors introduced in (13) but first, we prove a regularization result.

Lemma 27. Let \(\xi \in \{h, ah\}\). The operator \(\Pi^+_\mathcal{E} \circ \Gamma^+\) is a bounded linear operator from \(\text{dom}(\partial \xi)\) to \(H^\xi(\partial \Omega)\).

Proof. Let \(u \in \text{dom}(\partial \xi)\) and \(u_n \in C^\infty(\Omega)\) be a sequence converging to \(u\) in the \(\| \cdot \|_\kappa\)-norm when \(n \to +\infty\). Pick \(f \in C^\infty(\partial \Omega)\), an integration by parts yields:
\[
\langle \Gamma^+ u_n, \Pi^+_\mathcal{E} f \rangle_{L^2(\partial \Omega)} = 2 \langle \partial \xi u_n, \Phi_{\mathcal{E}} (f) \rangle_{L^2(\Omega)}.
\]
It gives
\[
|\langle \Gamma^+ u_n, \Pi^+_\mathcal{E} f \rangle_{L^2(\partial \Omega)}| \leq 2 c \| u_n \|_h \| f \|_{H^{-\frac{1}{2}}(\partial \Omega)},
\]
for some \(c > 0\), where we have used Lemma 15 and Proposition 22. As in \(L^2(\partial \Omega)\) there holds \(S^*_n = -S_n\) we get
\[
(\Pi^+_\mathcal{E})^* = \Pi^+_\mathcal{E} n.
\]
In particular, there holds
\[
|\langle \Gamma^+ u_n, \Pi^+_\mathcal{E} f \rangle_{L^2(\partial \Omega)}| = |\langle (\Pi^+_\mathcal{E} n, \mathcal{E}^* f)_{L^2(\partial \Omega)} | \leq 2 c \| u_n \|_h \| f \|_{H^{-\frac{1}{2}}(\partial \Omega)}.
\]
Letting $n \to +\infty$, we get $\Pi_n^+ n \Gamma^+ u \in H^{\frac{3}{2}}(\partial \Omega)$ and that $\Pi_n^+ \circ H_n \circ \Gamma^+$ is a linear bounded map from $H^{-\frac{3}{2}}(\partial \Omega)$ to $H^{\frac{3}{2}}(\partial \Omega)$. However, there holds

$$\Pi_n^+ \Gamma^+ u = \left( n \Pi_n^+ n - n(\Pi_n^+, n) \right) \Gamma^+ u = n \Pi_n^+ n \Gamma^+ u + n[S_h, n] \Gamma^+ u.$$ 

By Proposition 12, $[S_h, n] \in \Psi^{-1}_{\theta(\Omega)}$ hence, it is a bounded operator from $H^{-\frac{3}{2}}(\partial \Omega)$ to $H^{\frac{3}{2}}(\partial \Omega)$. Finally, as the multiplication operator by $n$ is bounded in $H^{\frac{3}{2}}(\partial \Omega)$ we obtain the expected result.

The case $u \in \text{dom}(\partial_n h)$ is handled similarly.

We are now in a good position to describe the range of the trace operator $\Gamma^+$.

**Corollary 28.** Let $\sharp \in \{h, ah\}$. There holds

$$\text{ran}(\Gamma^+) = \{ f \in H^{-\frac{3}{2}}(\partial \Omega) : \Pi_2^+ f \in H^{\frac{3}{2}}(\partial \Omega) \}.$$ 

**Proof.** Let us start by proving the reverse inclusion. Let $f$ be in the set on right-hand side, there holds $f = \Pi_2^+ f + \Pi_2^- f$. We know that there exists an extension operator $E^+$ from $H^{\frac{3}{2}}(\partial \Omega)$ to $H^{1}(\Omega)$ such that $\Gamma^+ E^+ g = g$ for all $g \in H^{\frac{3}{2}}(\partial \Omega)$. Now, if $\Pi_2^- f \in H^{\frac{3}{2}}(\partial \Omega)$, we set

$$u := \Phi_2(\Pi_2^+ f) + E^+(\Pi_2^- f).$$

It is easily seen that $u \in \text{dom}(\partial_2)$ and $\Gamma^+ u = \Pi_2^+ f + \Pi_2^- f = f$.

Now, let us prove the direct inclusion and pick $f \in \text{ran}(\Gamma^+)$. We know that there exists $u \in \text{dom}(\partial_2)$ such that $f = \Gamma^+ u$. In particular, by Lemma 27, we know that $\Pi_2^- f = \Pi_2^- \Gamma^+ u \in H^{\frac{3}{2}}(\partial \Omega)$ which concludes the proof.

We are now able to prove Proposition 26.

**Proof of Proposition 26.** First, let us prove that the sum is direct. Let $v = \Phi_2(f) = u$ with $\Pi_2^+ f = 0$ and $\Pi_2^- \Gamma^+ u = 0$. Then, taking the traces we obtain:

$$\Gamma^+ v = \Pi_2^+ f = \Pi_2^- \Gamma^+ u,$$

which implies $f = \Gamma^+ u = 0$. Consequently, $v = \Phi_2(f) = 0$.

Second, let us pick $v \in \text{dom}(\partial_2)$. There holds

$$v = \Phi_2(\Pi_2^+ \Gamma^+ v) + v - \Phi_2(\Pi_2^+ \Gamma^+ v).$$

However, remark that $u := v - \Phi_2(\Pi_2^+ \Gamma^+ v) \in \text{dom}(\partial_2)$ and satisfies $\Gamma^+ u = \Pi_2^- \Gamma^+ v \in H^{\frac{3}{2}}(\partial \Omega)$ by Lemma 27. Hence, by Lemma 18, we obtain $u \in H^{1}(\Omega)$ and $\Gamma^+ u \in \ker \Pi_2^+ = \text{ran} \Pi_2^-$, which concludes the proof.

5. VARIATIONAL CHARACTERIZATION OF THE PRINCIPAL EIGENVALUE

The aim of this section is to prove Theorem 3. In 5.1 we describe precisely the domains $\text{dom}(q^0)'$ and $\text{dom}(H^0)$, where $H^0$ is the unique self-adjoint operator associated with $q^0$ via Kato’s first representation theorem. In 5.2 we investigate the behavior of the map $E \in [0, +\infty) \mapsto \mu_\Omega(E)$. Finally, in 5.3 we prove Theorem 4.
5.1. The quadratic form $q_E^\Omega$ and its associated self-adjoint operator $H_E^\Omega$.  
For $E > 0$, recall that $q_E^\Omega$ is defined in \[2\] on the domain consisting of the closure of the $C^\infty(\Omega)$ functions with respect to the norm of the quadratic form
\[
N_E^\Omega(u) := \sqrt{\|\partial_2 u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + E\|u\|_{L^2(\partial\Omega)}^2}.
\]
Remark that as defined, $q_E^\Omega$ is a closed, densely defined and bounded below quadratic form, by Kato’s first representation theorem (see \[25\], Chap. VI, Thm. 2.1]), $q_E^\Omega$ is associated with a unique self-adjoint operator $H_E^\Omega$ acting in $L^2(\Omega)$ satisfying $\text{dom}(H_E^\Omega) \subset \text{dom}(q_E^\Omega)$. 

In this paragraph, we describe properties of the domains $\text{dom}(q_E^\Omega)$ and $\text{dom}(H_E^\Omega)$ and start with the domain of the quadratic form $q_E^\Omega$.

**Proposition 29.** Let $E > 0$. The form domain $\text{dom}(q_E^\Omega)$ admits the following direct sum decomposition
\[
\text{dom}(q_E^\Omega) = \{ u \in H^1(\Omega) : \Pi_n^+ \Gamma^+ u = 0 \} + H_E^2(\Omega).
\]
Moreover, $\text{dom}(q_E^\Omega)$ is continuously embedded in $H^\frac{1}{2}(\Omega)$. 

**Proof.** Set $E = \{ u \in H^1(\Omega) : \Pi_n^+ \Gamma^+ u = 0 \} + H_E^2(\Omega)$ and remark that the sum is direct by the same arguments as in the proof of Proposition \[26\]. We prove the set equality by proving both inclusions.

**Inclusion $E \subset \text{dom}(q_E^\Omega).** Let $v := u + \Phi_h(f) \in E$ and take $(u_n)_{n \in \mathbb{N}}$ and $(f_n)_{n \in \mathbb{N}}$ two sequences of functions such that for all $n \in \mathbb{N}$ $u_n \in C^\infty(\Omega)$, $f_n \in C^\infty(\partial\Omega)$; and
\[
\text{when } n \to +\infty \text{ there holds } \|u_n - u\|_{H^1(\Omega)} \to 0, \quad \|f_n - f\|_{L^2(\partial\Omega)} \to 0.
\]
By \[10\] Theorem 3.1., we have $v_n := u_n + \Phi_h(f_n) \in C^\infty(\Omega)$ and for $E > 0$, there exists $C > 0$ such that there holds
\[
q_E^\Omega(v - v_n) + (E^2 + 1)\|v - v_n\|_{L^2(\Omega)}^2 = \|\partial_2 (u - u_n)\|_{L^2(\Omega)}^2 + E\|\Gamma^+ (u - u_n) + \Pi_n^+ (f - f_n)\|_{L^2(\partial\Omega)}^2 + \|\Pi_n^+ f\|_{L^2(\partial\Omega)}^2 + \|u - u_n\|_{L^2(\partial\Omega)}^2 + \|f - f_n\|_{L^2(\partial\Omega)}^2 \leq C \left( \|u - u_n\|_{H^1(\Omega)} + \|f - f_n\|_{L^2(\partial\Omega)} \right),
\]
where we have used the mapping properties of $\Phi_h$, $\Gamma^+$, $\Pi_n^+$ and the continuity of the embedding of $L^2(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$. Letting $n \to +\infty$, we obtain that $v \in \text{dom}(q_E^\Omega)$ and this inclusion is proved.

**Inclusion $\text{dom}(q_E^\Omega) \subset E$.** For all $u \in C^\infty(\Omega)$, there holds
\[
q_E^\Omega(u) + (E^2 + 1)\|u\|_{L^2(\Omega)}^2 \geq \|u\|_{L^2(\Omega)}^2.
\]
In particular, the closure of $C^\infty(\Omega)$ for the norm $N_E^\Omega$ is included in $\text{dom}(\partial_h)$. It rewrites $\text{dom}(q_E^\Omega) \subset \text{dom}(\partial_h)$ and by Proposition \[26\], any $v \in \text{dom}(q_E^\Omega)$ writes $v = u + \Phi_h(f)$, for some $u \in H^1(\Omega)$ with $\Pi_n^+ \Gamma^+ u = 0$ and some $f \in H^{-\frac{1}{2}}(\partial\Omega)$ with $\Pi_n^+ f = 0$. Now, if $v_n \in C^\infty(\Omega)$ converges to $v \in \text{dom}(q_E^\Omega)$ in the norm of the quadratic form, we have
\[
\|\Gamma^+ v - \Gamma^+ v_n\|_{L^2(\partial\Omega)} \leq E^{-1} q_E^\Omega(v - v_n) \to 0, \quad n \to +\infty.
\]
In particular $\Gamma^+ v = \Gamma^+ u + f \in L^2(\partial\Omega)$ and as $\Gamma^+ u \in H^\frac{1}{2}(\partial\Omega)$ we get $f \in L^2(\partial\Omega)$ which concludes the proof of this inclusion.
Let us consider the inclusion map
\[ \mathcal{I} := \text{dom}(q^0_E) \to H^\frac{1}{2}(\Omega), \quad (\mathcal{I}u) = u. \]
By Proposition\textsuperscript{22} for \( s = 0 \), this map is well-defined. Consider \( v_n := u_n + \Phi_h(f_n) \in \text{dom}(q^0_E) \) which converges to \( v \) in the norm of the quadratic form \( q^0_E \) and assume that \( v_n \to w \) in the \( H^\frac{1}{2}(\Omega) \)-norm. In particular, as \( v \in \text{dom}(q^0_E) \), there holds \( v = u + \Phi_h(f) \) for some \( u \in H^1(\Omega) \) and \( f \in H^\frac{1}{2}(\partial\Omega) \) as in the definition of \( \mathcal{E} \). In particular, in \( D'(\Omega) \) we obtain
\[ u + \Phi_h(f) = w \]
and as both terms belong to \( H^\frac{1}{2}(\Omega) \), the closed graph theorem gives that \( \mathcal{I} \) is continuous. \( \square \)

Because of the compact embedding of \( H^\frac{1}{2}(\Omega) \) into \( L^2(\Omega) \), an immediate corollary of Proposition\textsuperscript{29} reads as follows.

**Corollary 30.** Let \( E > 0 \), the operator \( H^0_E \) has compact resolvent and its spectrum consists of a non-decreasing sequence of eigenvalues denoted \( (\mu^0_j(E))_{j \geq 1} \). Moreover, there holds
\[ \mu^0_j(E) = \inf_{F \subset \text{dom}(q^0_E)} \sup_{\dim F = j} \frac{4 \int_\Omega |\partial_z u|^2 dx - E^2 \int_\Omega |u|^2 dx + E \int_{\partial\Omega} |u|^2 ds}{\int_\Omega |u|^2 dx}. \]

**Remark 31.** For \( E = 0 \), the counterpart of Proposition\textsuperscript{29} would read
\[ \text{dom}(q^0_0) = \{ u \in H^1(\Omega) : \Pi^+_n u = 0 \} + A^0_E(\Omega). \]
In particular, note that \( \text{dom}(q^0_0) \) can not be included in any Sobolev space \( H^s(\Omega) \), \( (s > 0) \). Indeed, for any Bergman function \( u \in A^0_E(\Omega) \), there holds \( q^0_0(u) = 0 \) which implies that for all \( j \geq 1 \) we have \( \mu^0_j(0) = 0 \). Thus 0 is an eigenvalue of \( H^0_0 \) of infinite multiplicity which would not be possible if we had \( \text{dom}(q^0_0) \subset H^s(\Omega) \) because of the compact embedding of \( H^s(\Omega) \) in \( L^2(\Omega) \). This phenomena is reminiscent of what happens for the Dirac operator with zig-zag boundary conditions as discussed in \[30]\.

We conclude this paragraph by a description of the domain of the operator \( H^0_E \).

**Proposition 32.** Let \( E > 0 \), there holds:
\[ \text{dom}(H^0_E) = \{ u \in H^1(\Omega) : \partial_z u \in H^1(\Omega) \text{ and } \partial_z u + \frac{E}{2} u = 0 \text{ on } \partial\Omega \}. \]

**Proof.** Let \( \mathcal{E} \) denote the set in the right-hand side of Proposition\textsuperscript{32} The proof is performed proving both inclusions.\[
\text{Inclusion dom}(H^0_E) \subset \mathcal{E}. \quad \text{Let } u \in \text{dom}(H^0_E) \text{ and } v \in C_0^\infty(\Omega), \text{ there holds }
\langle H^0_E u, \overline{v} \rangle_{D'(\Omega), D(\Omega)} = \langle H^0_E u, v \rangle_{L^2(\Omega)} = q^0_E[u, v]
= 4(\partial_z u, \partial_z \overline{v})_{D'(\Omega), D(\Omega)} - E^2(u, \overline{v})_{D'(\Omega), D(\Omega)}
= \langle (\Delta - E^2) u, \overline{v} \rangle_{D'(\Omega), D(\Omega)},
\]
where \( q^0_E[\cdot, \cdot] \) denotes the sesquilinear form associated with the quadratic form \( q^0_E \). Hence, in \( L^2(\Omega) \), there holds \( H^0_E u = (\Delta - E^2) u \). Remark that if \( u \in \text{dom}(H^0_E) \)
then \( \partial_z u \in \text{dom}(\partial_{ah}) \), in particular, by Green’s Formula (10), for all \( v \in C^\infty(\overline{\Omega}) \) we get:

\[
(H^2_E u, v)_{L^2(\Omega)} = -4(\partial_z (\partial_z u), v)_{L^2(\Omega)} - E^2(u, v)_{L^2(\Omega)} = 4(\partial_z u, \partial_z v)_{L^2(\Omega)} - E^2(u, v)_{L^2(\Omega)} - 2\langle \mathbf{m}^+ + \partial_z u, \Gamma^+ v \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = q^2_E[u, v] - 2\langle \mathbf{m}^+ + \partial_z u + Eu, \Gamma^+ v \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}.
\]

As \( v \in \text{dom}(q^2_E) \) we necessarily have \( 2\langle \mathbf{m}^+ + \partial_z u + Eu, \Gamma^+ v \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = 0 \). As this is true for all \( v \in C^\infty(\overline{\Omega}) \) we obtain

\[
2\mathbf{m}^+ + \partial_z u + Eu = 0, \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega). \tag{18}
\]

Taking the Szegö projectors in (18) we obtain

\[
(\Gamma^+ \partial_z u) + \frac{n}{2} \Gamma^+ u = 0 \iff \begin{cases} 
\Pi_{ah}^+(\Gamma^+ (\partial_z u)) + \frac{E}{2} \Pi_{ah}^+ n \Gamma^+ u = 0 \\
\Pi_{ah}^- (\Gamma^+ (\partial_z u)) + \frac{E}{2} \Pi_{ah}^- n \Gamma^+ u = 0
\end{cases}
\]

Nevertheless, there holds

\[
\Pi_{ah}^- = \Pi_{ah}^+ = \frac{1}{2}(S_{ah} + S_h), \quad \Pi_{ah}^+ = \Pi_{ah}^- = \frac{1}{2}(S_{ah} + S_h).
\]

In particular, we get

\[
\Pi_{ah}^- (\Gamma^+ (\partial_z u)) = -\frac{E}{2} \Pi_{ah}^- (n \Gamma^+ u) = -\frac{E}{2} \left( n \Pi_{ah}^+ \Gamma^+ u + [\Pi_{ah}^+, n] \Gamma^+ u - \frac{1}{2}(S_{ah} + S_h) (n \Gamma^+ u) \right)
\]

\[
= -\frac{E}{2} \left( n \Pi_{ah}^+ \Gamma^+ u + [S_h, n] \Gamma^+ u - \frac{1}{2}(S_{ah} + S_h) (n \Gamma^+ u) \right).
\]

It rewrites

\[
\Pi_{ah}^- \Gamma^+ u = -\Pi \left( \frac{2}{E} \Pi_{ah}^+ (\partial_z u) + [S_h, n] \Gamma^+ u - \frac{1}{2}(S_{ah} + S_h) (n \Gamma^+ u) \right).
\]

Remark that the right-hand side belongs to \( H^\frac{1}{2}(\partial\Omega) \). This holds for the first term because of Lemma 27 and for the last two terms because of Proposition 12. As \( \Pi^- \Gamma^+ u \in H^\frac{1}{2}(\partial\Omega) \) by Lemma 27 we get \( \Gamma^+ u = \Pi^+ \Gamma^+ u + \Pi^- \Gamma^+ u \in H^\frac{1}{2}(\partial\Omega) \) thus, by Lemma 27 we have \( u \in H^1(\Omega) \). In particular \( \Pi_{ah}^+ (\Gamma^+ (\partial_z u)) = -\frac{E}{2} \Pi_{ah}^- (n \Gamma^+ u) \in H^\frac{1}{2}(\partial\Omega) \) and as \( \Pi_{ah}^- \Gamma^+ (\partial_z u) \in H^\frac{1}{2}(\partial\Omega) \) by Lemma 27 we obtain \( \Gamma^+ \partial_z u = \Pi_{ah}^+ \Gamma^+ (\partial_z u) + \Pi_{ah}^- \Gamma^+ (\partial_z u) \in H^\frac{1}{2}(\partial\Omega) \) and by Lemma 18 we obtain \( \partial_z u \in H^1(\Omega) \). It concludes the proof of this inclusion.

Inclusion \( E \subset \text{dom}(H^2_E) \). Pick \( u \in E \). One easily sees that \( (-\Delta - E^2) u \in L^2(\Omega) \), moreover for all \( v \in \text{dom}(q^2_E) \), there holds

\[
q^2_E[u, v] = \langle (-\Delta - E^2) u, v \rangle_{L^2(\Omega)}.
\]

By definition of \( H^2_E \) it implies \( u \in \text{dom}(H^2_E) \) and \( H^2_E u = (-\Delta - E^2) u \).

\[ \square \]

5.2. Concavity of the first min-max level. In this paragraph we investigate the behavior of the first min-max level \( \mu^3(E) \) with respect to the spectral parameter \( E > 0 \). This behavior is illustrated in Figure 3 for various domains \( \Omega \).

**Proposition 33.** The map \( \mu^3 : E \geq 0 \mapsto \mu^3(E) \) verifies the following properties.

1. \( \mu^3 \) is a continuous and concave function on \( \mathbb{R}_+ \).
2. We have \( \mu^3(0) = 0 \) and there exists \( E_*^\Omega > 0 \) such that for all \( E \in (0, E_*^\Omega) \) there holds \( \mu^3(E) > 0 \).
3. Let \( 0 < E_1 < E_2 \), there holds

\[
\mu^3(E_2) \leq \frac{E_2}{E_1} \mu^3(E_1) - E_2(E_2 - E_1).
\]
In particular, if \( \mu^\Omega(E_1) = 0 \) (resp. \( \mu^\Omega(E_2) = 0 \)) there holds \( \mu^\Omega(E_2) < 0 \) (resp. \( \mu^\Omega(E_1) > 0 \)).

**Proof.** As for all \( u \in \text{dom}(q^\Omega_E) \) the function \( (E \geq 0 \rightarrow q^\Omega_E(u)) \) is a continuous and concave, so is \( (E \geq 0 \rightarrow \mu^\Omega(E)) \) and Point (1) is proved.

Regarding Point (2), one observes that for all \( u \in \text{dom}(q^\Omega_E) \) there holds \( q^\Omega_0(u) \geq 0 \) and in particular \( \mu^\Omega(0) \geq 0 \). Now, for any \( f \in L^2(\partial \Omega) \) we have \( \Phi_h(f) \in \text{dom}(q^\Omega_E) \) and \( q^\Omega_0(u) = 0 \) because \( \Phi_h(f) \) is holomorphic in \( \Omega \). Consequently, there holds \( \mu^\Omega(0) = 0 \).

To prove the second part of Point (2), let \( u \in \text{dom}(q^\Omega_E) \) and remark that

\[
q^\Omega_E(u) = (4 - E)\|\partial_\nu u\|^2_{L^2(\Omega)} - E^2\|u\|^2_{L^2(\Omega)} + E\Omega(u) \tag{19}
\]

where the quadratic form \( \Omega \) is defined as

\[
\Omega(u) = \|\partial_\nu u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial \Omega)} , \quad \text{dom}(\Omega) = \text{dom}(q^\Omega_E).
\]

Now, remark that \( \Omega \geq 0 \) thus, by Kato’s first representation theorem, there exists a unique self-adjoint operator \( \mathcal{H} \) such that \( \text{dom}(\mathcal{H}) \subset \text{dom}(\Omega) \) and its spectrum is a sequence of non-decreasing eigenvalues because \( \text{dom}(q^\Omega_E) = \text{dom}(q^\Omega_E) \) is compactly embedded into \( L^2(\Omega) \). Let \( \lambda^\Omega_1 \) be its smallest eigenvalue, we already know by the min-max principle that \( \lambda^\Omega_1 \geq 0 \). Moreover, if \( \lambda^\Omega_1 = 0 \), for an associated eigenfunction \( u \), we obtain \( \Omega(u) = 0 \) which implies that \( \partial_\nu u = 0 \) hence \( u \) is holomorphic with trace in \( L^2(\partial \Omega) \). Consequently, \( u \) belongs to \( H^2_0(\Omega) \) and \( u = \Phi_h(f) \) for some \( f \in L^2(\partial \Omega) \) such that \( \Gamma^+u = f \). However, as \( \Omega(u) = 0 \), we also obtain \( \Gamma^+u = f = 0 \) which yields \( u = 0 \) which is not possible because \( u \) is an eigenfunction. It implies that \( \lambda^\Omega_1 > 0 \) and using the min-max principle in (19), we get for all \( u \in \text{dom}(q^\Omega_E) \):

\[
q^\Omega_E(u) \geq (4 - E)\|\partial_\nu u\|^2_{L^2(\Omega)} - E^2\|u\|^2_{L^2(\Omega)} + E\lambda^\Omega_1\|u\|^2_{L^2(\Omega)}.
\]

In particular, if \( E < 4 \) we obtain

\[
q^\Omega_E(u) \geq E(\lambda_1^\Omega - E)\|u\|^2_{L^2(\Omega)}
\]

and the min-max principle yields

\[
\mu^\Omega(E) \geq E(\lambda_1^\Omega - E).
\]

Thus, setting \( E^\Omega := \min(4, \lambda_1^\Omega) \), for all \( E \in (0, E^\Omega) \), we have \( \mu^\Omega(E) > 0 \).

Let us prove Point (3). Let \( u \in \text{dom}(q^\Omega_E) \) and \( 0 < E_1 < E_2 \). There holds

\[
q^\Omega_{E_1}(u) = q^\Omega_{E_1}(u) - (E_2^2 - E_1^2) \int_{\partial \Omega} |u|^2 \, dx + (E_2 - E_1) \int_{\partial \Omega} |u|^2 \, ds. \tag{20}
\]

Now, pick \( u_1 \) a normalized eigenfunction of \( H^2_{E_1} \) associated with the eigenvalue \( \mu^\Omega(E_1) \). We have \( q^\Omega_{E_1}(u_1) = \mu^\Omega(E_1) \) which implies

\[
\int_{\partial \Omega} |u_1|^2 \, ds \leq \frac{1}{E_1} \left( 4 \int_{\partial \Omega} |\partial_\nu u|^2 \, dx + E_1 \int_{\partial \Omega} |u|^2 \, ds \right) = \frac{1}{E_1} (q^\Omega_{E_1}(u_1) + E_1^2) \leq \frac{E_2^2 + \mu^\Omega(E_1)}{E_1}.
\]

Thus, evaluating (20) with \( u = u_1 \) we obtain

\[
q^\Omega_{E_1}(u_1) \leq \mu^\Omega(E_1) - (E_2^2 - E_1^2) + \frac{E_2 - E_1}{E_1} (E_1^2 + \mu^\Omega(E_1)).
\]

The min-max principle finally gives the sought inequality

\[
\mu^\Omega(E_2) \leq \mu^\Omega(E_1) - (E_2^2 - E_1^2) + \frac{E_2 - E_1}{E_1} (E_1^2 + \mu^\Omega(E_1))
\]

\[
= \frac{E_2}{E_1} \mu^\Omega(E_1) - E_2(E_2 - E_1).
\]

Now, assume that \( \mu^\Omega(E_1) = 0 \). It yields

\[
\mu^\Omega(E_2) \leq -E_2(E_2 - E_1) < 0.
\]
Similarly, if $\mu^\Omega(E_2) = 0$ we get
\[
0 < E_1(E_2 - E_1) \leq \mu^\Omega(E_1).
\]
5.3. **Proof of the variational principle.** In our way to prove Theorem 4 we will need the following two propositions.

**Proposition 34.** Let $E > 0$ be such that $\mu^\Omega(E) = 0$ then $E \in Sp_{dis}(D^\Omega)$.

**Proof.** Let $E > 0$ be such that $\mu^\Omega(E) = 0$ and consider a normalized associated eigenfunction $v \in \text{dom}(H_0^2)$. Set $u = (u_1, u_2)^\top = (v, -\frac{2i}{E} \partial_z v)^\top$, by Proposition 32 $u \in H^1(\Omega, \mathbb{C}^2)$ and as $v \in \text{dom}(H_0^2)$, in $H^*(\partial\Omega)$ there holds
\[
\Gamma^+(\partial_z v) + \frac{E}{2} \Gamma^+ v = 0 \iff -2E^{-1} : \Gamma^+(\partial_z v) = \text{Im} \Gamma^+ u \iff \Gamma^+ u_2 = \text{Im} \Gamma^+ u_1.
\]
Hence, $(u_1, u_2)^\top \in \text{dom}(D^\Omega)$ and there holds
\[
D^\Omega(u_1, u_2)^\top = \begin{pmatrix} 0 & -2i \partial_z \\ -2i \partial_z & 0 \end{pmatrix} (u_1, u_2)^\top = (-2i \partial_z u_2, -2i \partial_z u_1)^\top = (-\frac{1}{E} \Delta u, Eu_2)^\top = E(u_1, u_2)^\top.
\]
Hence, $E \in Sp_{dis}(D^\Omega)$ and it concludes the proof of Proposition 34. \qed

**Proposition 35.** Let $E \in Sp_{dis}(D^\Omega) \cap \mathbb{R}^*_+$ then $\mu^\Omega(E) \leq 0$.

**Proof.** Let $E \in Sp_{dis}(D^\Omega) \cap \mathbb{R}^*_+$ and pick $u = (u_1, u_2)^\top \in \text{dom}(D^\Omega)$ a normalized eigenfunction of $D^\Omega$ associated with $E$. We have
\[
\begin{cases}
D^\Omega u = Eu & \text{in } \Omega, \\
u_2 = \text{im} u_1 & \text{on } \partial\Omega.
\end{cases}
\]
In particular, we have $-2i \partial_z u_1 = Eu_2$ and $\partial_z u_1 \in H^1(\Omega)$. It yields
\[
Eu_1 = -2i \partial_z u_2 = -\frac{4}{E} \partial_z \partial_z u_1.
\]
Taking the scalar product with respect to $u_1$ on both side of the previous equation we get
\[
E^2 \int_\Omega |u_1|^2 dx = -4 \int_\Omega (\partial_z \partial_z u_1) \overline{u_1} dx = 4 \int_\Omega |\partial_z u_1|^2 dx - 2 \int_{\partial\Omega} \overline{u_1} (\partial_z u_1) dx ds. \tag{21}
\]
Now, remark that on $\partial\Omega$, we have
\[
-\frac{2i}{E} \partial_z u_1 = u_2 = \text{im} u_1
\]
which implies that on $\partial\Omega$
\[
2 \text{im} \partial_z u_1 + Eu_1 = 0.
\]
Hence, (21) becomes
\[
E^2 \int_\Omega |u_1|^2 = 4 \int_\Omega |\partial_z u_1|^2 dx + E \int_{\partial\Omega} |u_1|^2 ds
\]
which reads $\sigma^\Omega_E(u_1) = 0$ thus, the min-max principle gives $\mu^\Omega(E) \leq 0$. \qed

Now, we have all the tools to prove Theorem 4. The proof is performed proving each implication.
Proof of Theorem 4. By Proposition 35 we have $\mu^2(E_1(\Omega)) \leq 0$. Assume that $\mu^2(E_1(\Omega)) < 0$, by Proposition 33 we know that there exists $0 < E < E_1(\Omega)$ such that $\mu^2(E) = 0$ which, by Proposition 34 implies $E \in Sp_{dis}(D^\Omega)$. It is not possible because, by definition of $E_1(\Omega)$, $E \geq E_1(\Omega)$ consequently, we obtain $\mu^2(E_1(\Omega)) = 0$.

Let $E > 0$ be such that $\mu^2(E) = 0$. By Proposition 34 $E \in Sp_{dis}(D^\Omega)$ and necessarily $E \geq E_1(\Omega)$. If $E > E_1(\Omega)$, by Proposition 33 we obtain $\mu^2(E_1(\Omega)) > 0$ but by Proposition 35 we necessarily have $\mu^2(E_1(\Omega)) \leq 0$ which implies that necessarily there holds $E = E_1(\Omega)$. \hfill \qed

6. Geometric upper bounds on the spectral gap

The goal of this section is to prove Theorem 3 and this is discussed in §6.2. But first, in §6.1, we give a simple geometric upper bound on the spectral gap which illustrates how Theorem 4 can be used.

6.1. A simple upper bound. An immediate consequence of Theorem 4 reads as follows.

Proposition 36. Let $\Omega \subset \mathbb{R}^2$ be $C^\infty$ and simply connected. There holds

$$E_1(\Omega) \leq \frac{|\partial \Omega|}{|\Omega|}.$$

There is no reason for the above upper bound to be attained among Euclidean domains. However, the bound brings into play simple geometric quantities: the perimeter and the area of $\Omega$.

Proof. Let $E > 0$ and $u \equiv 1$ the function constant to 1 in $\Omega$. As $u \in \text{dom}(q^\Omega_E)$, by the min-max principle we obtain

$$\mu^2(E) \leq \frac{q^\Omega_E(u)}{|u|_{L^2(\Omega)}} = E\left(\frac{|\partial \Omega|}{|\Omega|} - E\right).$$

So in $E_{\text{crit}} := \frac{|\partial \Omega|}{|\Omega|}$ we get $\mu^2(E_{\text{crit}}) \leq 0$ and by Proposition 33 we know that $E_1(\Omega) \leq E_{\text{crit}} = \frac{|\partial \Omega|}{|\Omega|}$.

\hfill \qed

6.2. A sharp upper bound. It turns out Theorem 3 is a consequence of the following result.

Theorem 37. Let $\Omega \subset \mathbb{R}^2$ be a $C^\infty$ simply connected domain. There holds

$$E_1(\Omega) \leq \frac{|\partial \Omega| + \sqrt{|\partial \Omega|^2 + 8\pi E_1(D)(E_1(D) - 1)(\pi r_i^2 + |\Omega|)}}{2(\pi r_i^2 + |\Omega|)}$$

with equality if and only if $\Omega$ is a disk.

Now, we have all the tools to prove Theorem 3.

Proof of Theorem 3. Using that $\pi r_i^2 \leq |\Omega|$ and the isoperimetric inequality we obtain $4\pi^2 r_i^2 \leq 4\pi|\Omega| \leq |\partial \Omega|^2$. It gives

$$|\partial \Omega|^2 + 8\pi E_1(D)(E_1(D) - 1)(\pi r_i^2 + |\Omega|) \leq |\partial \Omega|^2(2E_1(D) - 1)^2.$$

Note that in the above inequalities, we have equality if and only if $\Omega$ is a disk and combining this bound with the one of Theorem 37 we get Theorem 3. \hfill \qed
In the rest of this section we focus on proving Theorem 37 and assume, without loss of generality, the following.

(i) \(0 \in \Omega\) is such that \(r_i = \max_{x \in \partial \Omega} |x|\),
(ii) \(f : \mathbb{D} \to \Omega\) is a conformal map such that \(f(0) = 0\) and we write
\[
f(z) = \sum_{n \geq 1} c_n z^n,
\]
where \((c_n)_{n \geq 1}\) is a sequence of complex numbers.

Before going through the proof of Theorem 37, we gather in the following paragraph some known properties linking the geometry of \(\Omega\) with the conformal map \(f\).

6.2.1. Preliminaries. The next proposition can be found in [32, §3.10.2] and relates the area of \(\Omega\) with the conformal map \(f\).

**Proposition 38** (Area formula). There holds
\[
|\Omega| = \pi \sum_{n \geq 1} n|c_n|^2.
\]

The second proposition is a consequence of the Schwarz lemma (see Koebe’s estimate in [22, Chap. I, Thm. 4.3]). It gives a relation between the first coefficient \(c_1\) of the conformal map \(f\) and the inradius \(r_i\).

**Proposition 39** (Koebe’s estimate). There holds
\[
|f'(0)| = |c_1| \geq r_i.
\]

Finally, the last geometric relation between the conformal map \(f\) and the geometry of \(\Omega\) we need to prove Theorem 37 is that the perimeter \(|\partial \Omega|\) of \(\Omega\) can be expressed as
\[
|\partial \Omega| = \int_0^{2\pi} |f'(e^{i\theta})| d\theta. \tag{22}
\]
(22) is a simple consequence of the fact that \(f|_{\partial \mathbb{D}}\) is a parametrization of \(\partial \Omega\).

6.2.2. Proof of the upper bound on the spectral gap. To prove Theorem 37, we construct an adequate test function for \(q_{E}^{\Omega}\) transplanting the eigenfunction of the unit disk \(\mathbb{D}\) in the domain \(\Omega\) thanks to the conformal map \(f\). We obtain an upper bound on \(\mu^\Omega(E)\) which is a second order polynomial in the spectral parameter \(E > 0\) and with coefficients depending on the geometry of \(\Omega\). It translates into an optimization problem for the spectral parameter \(E > 0\) that we solve in the last step of the proof.

**Proof of Theorem 37.** Let us go through all the steps of the proof.

Step 1. Let us denote by \(J_0\) (resp. \(J_1\)) the Bessel function of the first kind of order 0 (resp. of order 1). For \(x \in \mathbb{D}\), consider \(u_0(x) = J_0(E_1(\mathbb{D})|x|)\) \(\in H^1(\mathbb{D}) \subset \text{dom}(q_{E_1(\mathbb{D})}^{\mathbb{D}})\). As explained in Remark 2, \(u(x) = (u_0(x), i\frac{2\pi}{|x|} J_1(E_1(\mathbb{D})|x|))^\top\) is an eigenfunction of \(D^2\) associated with \(E_1(\mathbb{D})\). Theorem 4 implies
\[
0 = q_{E_1(\mathbb{D})}^{\mathbb{D}}(u_0) = 2\pi E_1(\mathbb{D})^2 \int_0^1 J_1(E_1(\mathbb{D})r)^2 r dr - 2\pi E_1(\mathbb{D})^2 \int_0^1 J_0(E_1(\mathbb{D})r)^2 r dr + 2\pi E_1(\mathbb{D}) J_0(E_1(\mathbb{D}))^2. \tag{23}
\]
Step 2. For $x = (x_1, x_2) \in \Omega$, consider $v_0(x_1, x_2) = u_0(f^{-1}(x_1 + ix_2)) \in H^1(\Omega) \subset \text{dom}(\tilde{\varphi}_0^2)$. By the min-max principle, there holds

$$
\mu^\Omega (E) \leq \frac{\varphi^2_0 (v_0)}{\|v_0\|_{L^2(\Omega)}} = \frac{\|\nabla v_0\|^2_{L^2(\Omega)} + E \|v_0\|^2_{L^2(\partial \Omega)}}{\|v_0\|^2_{L^2(\Omega)}} - E^2,
$$

where we have used Parseval identity.

Step 3. Now, as $f$ is a conformal map, we know that

$$
\|\nabla v_0\|_{L^2(\Omega)} = \|\nabla u_0\|_{L^2(\Omega)} = 2\pi E_1(\mathbb{D}) \int_0^1 J_1(E_1(\mathbb{D}) r)^2 r dr.
$$

Using (22), we obtain

$$
\|v_0\|^2_{L^2(\partial \Omega)} = \int_0^{2\pi} |v_0(f(e^{i\theta}))|^2 |f'(e^{i\theta})|^2 |d\theta = J_0(E_1(\mathbb{D}))^2 |\partial \Omega|.
$$

Finally, the last integral reads

$$
\|v_0\|^2_{L^2(\partial \Omega)} = \int_0^{2\pi} \int_0^1 |u_0(r)|^2 |f'(r e^{i\theta})|^2 r dr d\theta
= \int_0^1 |u_0(r)|^2 \left( \sum_{n \geq 1} c_n r^n e^{i(n-1)\theta} \right)^2 r dr
= 2\pi \sum_{n \geq 1} n |c_n|^2 M_n,
$$

where for $n \geq 1$, $M_n := n \int_0^1 J_0(E_1(\mathbb{D}) r)^2 r^{2n-1} dr$,

Step 4. Taking into account (25), (26) and (27), (24) becomes

$$
\mu^\Omega (E) \leq 2\pi E_1(\mathbb{D}) \frac{\int_0^1 J_1(E_1(\mathbb{D}) r)^2 r dr}{2\pi \sum_{n \geq 1} n |c_n|^2 M_n} - E^2
+ E \frac{J_0(E_1(\mathbb{D}))^2 |\partial \Omega|}{2\pi \sum_{n \geq 1} n |c_n|^2 M_n}.
$$

Let us find a lower bound on the sequence $(M_n)_{n \geq 1}$. Using first an integration by parts we find

$$
M_n = \frac{1}{2} J_0(E_1(\mathbb{D}))^2 + \frac{E_1(\mathbb{D})}{2} \int_0^1 J_0(E_1(\mathbb{D}) r) J_1(E_1(\mathbb{D}) r) r^{2n-1} dr.
$$

In particular, for $n = 1$ it gives

$$
M_1 = \int_0^1 J_0(E_1(\mathbb{D}) r)^2 r dr = J_0(E_1(\mathbb{D}))^2
= E_1(\mathbb{D}) \int_0^1 J_0(E_1(\mathbb{D}) r) J_1(E_1(\mathbb{D}) r) r^2 dr.
$$

Now, for $n \geq 1$, one notices that $h_1 := \left( r \mapsto (J_0 J_1)(E_1(\mathbb{D}) r)^2 \right)$ and $h_2 := \left( r \mapsto r^{2n-2} \right)$ are non-decreasing functions on $[0, 1]$ and by Chebyshev’s inequality for non-decreasing functions, we obtain

$$
M_n \geq \frac{1}{2} M_1 + \frac{1}{2} M_1 \int_0^1 r^{2n-2} dr = \frac{n}{2n-1} M_1.
$$
In particular, we have

$$2\pi \sum_{n \geq 1} n |c_n|^2 M_n \geq J_0(E_1(\mathbb{D}))^2 \left( 2\pi |c_1|^2 + 2\pi \sum_{n \geq 2} \frac{n^2}{2n - 1} |c_n|^2 \right)$$

$$\geq J_0(E_1(\mathbb{D}))^2 \left( 2\pi |c_1|^2 + \pi \sum_{n \geq 2} n |c_n|^2 \right)$$

$$= J_0(E_1(\mathbb{D}))^2 (\pi |c_1|^2 + |\Omega|)$$

$$\geq J_0(E_1(\mathbb{D}))^2 (\pi |r_1|^2 + |\Omega|),$$

where we have used Proposition 38 and Proposition 39. Remark that in the first two inequalities above we have equality if and only if $c_n = 0$ for all $n \geq 2$. Similarly, in the last equality, we have equality if and only if $|c_1| = r_1$. In particular there is equality in the above inequalities if and only if $f(z) = c_1 z$ and $\Omega$ is a disk centered in 0 of radius $r_1$.

Combining (23) and (30) in (28), we obtain

$$\mu^0(E) \leq -E^2 + \frac{2\pi E_1(\mathbb{D})^2 \int_0^1 J_0(E_1(\mathbb{D})r)^2 r dr + J_0(E_1(\mathbb{D}))^2 \left( E|\partial \Omega| - 2\pi E_1(\mathbb{D}) \right)}{J_0(E_1(\mathbb{D}))^2 (\pi r_1^2 + |\Omega|)}.$$

Using (29), we obtain

$$\mu^0(E) \leq -E^2 + \frac{2\pi E_1(\mathbb{D})^2 + (E|\partial \Omega| - 2\pi E_1(\mathbb{D}))}{\pi r_1^2 + |\Omega|}$$

$$= \frac{(2\pi E_1(\mathbb{D})^2 - (\pi r_1^2 + |\Omega|)E^2) + (E|\partial \Omega| - 2\pi E_1(\mathbb{D}))}{\pi r_1^2 + |\Omega|}$$

$$= \frac{P(E)}{\pi r_1^2 + |\Omega|}, \quad P(E) := -E^2(\pi r_1^2 + |\Omega|) + E|\partial \Omega| + 2\pi E_1(\mathbb{D})(E_1(\mathbb{D}) - 1).$$

Step 5. Remark that by (1), there holds $E_1(\mathbb{D}) - 1 \geq \sqrt{2} - 1 > 0$. In particular, the discriminant of $P$ satisfies

$$\delta(P) := |\partial \Omega|^2 + 8\pi E_1(\mathbb{D})(E_1(\mathbb{D}) - 1)(\pi r_1^2 + |\Omega|) > 0.$$ 

Thus, $P$ has two real roots and as $P(0) > 0$, the only positive root is

$$E_{\text{crit}} := \frac{|\partial \Omega| + \sqrt{|\partial \Omega|^2 + 8\pi E_1(\mathbb{D})(E_1(\mathbb{D}) - 1)(\pi r_1^2 + |\Omega|)}}{2(\pi r_1^2 + |\Omega|)}.$$

One obtains $\mu^0(E_{\text{crit}}) \leq \frac{P(E_{\text{crit}})}{\pi r_1^2 + |\Omega|} = 0$ and by Proposition 33 and Theorem 4 we get

$$E_1(\mathbb{D}) \leq E_{\text{crit}}$$

which is precisely Theorem 37.

7. About the Faber-Krahn conjecture

In this section we discuss how the variational formulation established in Theorem 4 can be used to investigate Conjecture 1. §7.1 deals with a new Faber-Krahn type conjecture for the operator $H_\Omega^D$ introduced in §5.1 and how this new conjecture is related to Conjecture 1. In §7.2 we discuss how the well-known Bossel-Daners inequality for the Robin Laplacian is linked to Conjecture 1 (see [14, 16]).
7.1. A new conjecture. Let us introduce a new Faber-Krahn type conjecture for \( \mu^\Omega(E) \), the first eigenvalue of \( H^\Omega_E \).

**Conjecture 40.** Let \( \Omega \subset \mathbb{R}^2 \) be \( C^\infty \) and simply connected. For all \( E > 0 \), there holds

\[
\mu^\Omega(E) \geq \frac{\pi}{|\Omega|} \mu^\Omega \left( \sqrt{\frac{|\Omega|}{\pi}} E \right).
\]

Moreover, there is equality in the above inequality if and only if \( \Omega \) is a disk.

It turns out Conjecture 40 is equivalent to Conjecture 41 and this is what we prove in the rest of this paragraph.

**Proof.** First, remark that a simple scaling argument gives, for all \( E > 0 \), that

\[
\sqrt{\frac{\pi}{|\Omega|}} E_1(\mathbb{D}) = E_1(\rho \mathbb{D}), \quad \mu^{\rho \mathbb{D}}(E) = \frac{\pi}{|\Omega|} \mu^\Omega \left( \sqrt{\frac{|\Omega|}{\pi}} E \right) \text{ where } \rho := \sqrt{\frac{|\Omega|}{\pi}}.
\]

Second, assume that Conjecture 41 holds true. If \( \Omega \) is a disk, there holds \( \mu^\Omega(E) = \mu^{\rho \mathbb{D}}(E) \) so now, we assume that \( \Omega \) is not a disk. Let us prove that for all \( E > 0 \) there holds

\[
\mu^\Omega(E) > \mu^{\rho \mathbb{D}}(E).
\]

Let us reason by *reduction ad absurdum* and assume there exists \( E_* > 0 \) such that \( \mu^\Omega(E_*) \leq \mu^{\rho \mathbb{D}}(E_*) \).

**Case** \( E_* < E_1(\rho \mathbb{D}) \). By hypothesis and Proposition 33 there holds

\[
\mu^\Omega(E_*) \leq \mu^{\rho \mathbb{D}}(E_*) \leq \frac{E_1(\rho \mathbb{D})}{E_*} \mu^{\rho \mathbb{D}}(E_1(\rho \mathbb{D})) - E_1(\rho \mathbb{D})(E_1(\rho \mathbb{D}) - E_*).
\]

In particular, \( \mu^\Omega(E_*) < 0 \) which implies \( E_* > E_1(\Omega) \). However, if Conjecture 40 holds true we obtain \( E_* > E_1(\Omega) > E_1(\rho \mathbb{D}) \) which contradicts our hypothesis.

**Case** \( E_1(\rho \mathbb{D}) < E_* \leq E_1(\Omega) \). By hypothesis and Proposition 33 there holds

\[
0 \leq \mu^\Omega(E_*) \leq \mu^{\rho \mathbb{D}}(E_*) \leq 0,
\]

which contradicts our hypothesis because we obtain \( E_* = E_1(\Omega) = E_1(\rho \mathbb{D}) \) but we have assumed that \( \Omega \) is not a disk thus, this equality can not hold if Conjecture 41 holds true.

**Case** \( E_* > E_1(\Omega) \). By hypothesis and Proposition 33 there holds

\[
0 = \mu^\Omega(E_1(\Omega)) \leq \frac{E_*}{E_1(\Omega)} \mu^\Omega(E_*) - E_1(\Omega)(E_* - E_1(\Omega))
\]

In particular, we obtain \( \mu^{\rho \mathbb{D}}(E_*) \geq E_1(\Omega)(E_* - E_1(\Omega)) > 0 \). Hence, \( E_* < E_1(\rho \mathbb{D}) \) which contradicts Conjecture 41.

Consequently, we have proved that if Conjecture 40 holds true so does Conjecture 41.

Finally, let us assume that Conjecture 40 holds true. If \( \Omega \) is a disk, we obtain that for all \( E > 0 \), \( \mu^\Omega(E) = \mu^{\rho \mathbb{D}}(E) \). In particular, in \( E = E_1(\Omega) \) we get \( \mu^{\rho \mathbb{D}}(E_1(\Omega)) = 0 \) and \( E_1(\rho \mathbb{D}) = E_1(\Omega) \).

When \( \Omega \) is not a disk, for all \( E > 0 \) there holds \( \mu^{\rho \mathbb{D}}(E) < \mu^\Omega(E) \). In \( E = E_1(\Omega) \), we obtain \( \mu^{\rho \mathbb{D}}(E_1(\Omega)) < 0 \) and by Proposition 33, we obtain \( E_1(\rho \mathbb{D}) < E_1(\Omega) \) which is precisely Conjecture 41. \( \square \)
7.2. Link with the Bossel-Daners inequality. The first eigenvalue of the Robin Laplacian with positive parameter $E > 0$ in the domain $\Omega$, denoted $\lambda_{\text{Rob}}^\Omega (E)$, is given by the variational characterization
\[
\lambda_{\text{Rob}}^\Omega (E) := \inf_{u \in C^\infty (\Omega) \setminus \{0\}} \frac{\|\nabla u\|_{L^2(\Omega)}^2 + E \int_{\partial \Omega} |u|^2 ds}{\|u\|_{L^2(\Omega)}^2}
\]
and the Bossel-Daners inequality states that
\[
\lambda_{\text{Rob}}^\Omega (E) \geq \frac{\pi}{|\Omega|} \lambda_{\text{Rob}}^\Omega \left( \sqrt{\frac{|\Omega|}{\pi}} E \right),
\]
with equality if and only if $\Omega$ is a disk. Note that the structure of (31) is similar to that of Conjecture 10 and it turns out they are intimately connected. This is the purpose of the following proposition.

**Proposition 41.** Conjecture 1 implies the Bossel-Daners inequality (31).

**Proof.** As Conjecture 1 is equivalent to Conjecture 10 as discussed in §7.1, we can assume that Conjecture 10 holds. Let us start by remarking that for all $E > 0$, if $u \in \text{dom}(H^2_{\text{Rob}})$ is a normalized eigenfunction associated with $\mu^\Omega (E)$ then $u$ can be picked real-valued. Hence, we get
\[
\mu^\Omega (E) = \inf_{v \in C^\infty (\Omega, \mathbb{R})} \frac{\|\nabla v\|_{L^2(\Omega)}^2 - E^2 \|v\|_{L^2(\Omega)}^2 + \int_{\partial \Omega} |v|^2 ds}{\|v\|_{L^2(\Omega)}^2} = \lambda_{\text{Rob}}^\Omega (E) - E^2.
\]

Now, we remark that for any domain $\Omega$ there holds
\[
\lambda_{\text{Rob}}^\Omega (E) - E^2 = \inf_{v \in C^\infty (\Omega, \mathbb{R}) \setminus \{0\}} \frac{\|\nabla v\|_{L^2(\Omega)}^2 - E^2 \|v\|_{L^2(\Omega)}^2 + \int_{\partial \Omega} |v|^2 ds}{\|v\|_{L^2(\Omega)}^2}
\]
\[
= \inf_{v \in C^\infty (\Omega, \mathbb{R}) \setminus \{0\}} \frac{4 \|\partial_x v\|_{L^2(\Omega)}^2 - E^2 \|v\|_{L^2(\Omega)}^2 + \int_{\partial \Omega} |v|^2 ds}{\|v\|_{L^2(\Omega)}^2} \geq \inf_{v \in \text{dom}(H^2_{\text{Rob}}) \setminus \{0\}} \mu^\Omega (E).
\]

Hence, using (32) and (33), we get
\[
\lambda_{\text{Rob}}^\Omega (E) - E^2 \geq \mu^\Omega (E) \geq \frac{\pi}{|\Omega|} \mu^\Omega \left( \sqrt{\frac{|\Omega|}{\pi}} E \right) = \frac{\pi}{|\Omega|} \lambda_{\text{Rob}}^\Omega \left( \sqrt{\frac{|\Omega|}{\pi}} E \right) - E^2.
\]

If $\Omega$ is a disk, all the above inequalities are equalities. Else, we obtain
\[
\lambda_{\text{Rob}}^\Omega (E) \geq \frac{\pi}{|\Omega|} \lambda_{\text{Rob}}^\Omega \left( \sqrt{\frac{|\Omega|}{\pi}} E \right),
\]
which is precisely the Bossel-Daners inequality (31).

8. Numerics

The goal of this section is to illustrate numerically some theoretical results discussed in the previous sections and to support the validity of Conjecture 1.

In §8.1 we discuss the two numerical schemes we have employed in §8.2 in order to study the principal eigenvalue of the Dirac operator with infinite mass boundary conditions in various domains $\Omega$. We also discuss the structure of the associated eigenfunctions.
8.1. Numerical Methods. In this paragraph we present a brief description of the numerical methods that we use in this work.

We have implemented two different numerical approaches, respectively to calculate the eigenvalues of the Dirac operator with infinite mass boundary conditions, directly from the formulation of the eigenvalue problem and to solve the minimization problem associated with the non-linear variational characterization [5], defining \( \mu^\Omega(E) \).

The eigenvalues of the Dirac operator with infinite mass boundary conditions are calculated using a numerical method based on Radial Basis Functions (RBF) (see eg. [24, 21]). We have chosen a set of RBF centers \( y_1, \ldots, y_N \in \mathbb{R}^2 \), for some \( N \in \mathbb{N} \), which are generated by a node repel algorithm (see [2] for details). The eigenfunction \( u = (u_1, u_2)^T \) is defined in \( H^1(\Omega, \mathbb{C}^2) \) and we use the notation \( u_1 = v_1 + iv_2 \), \( v_2 = v_2 + iv_2 \), where \( v_1, w_1 \) and \( v_2, w_2 \) are the real and imaginary parts of \( v_1 \) and \( v_2 \), respectively. The RBF numerical approximation for each of these functions is defined by

\[
\begin{align*}
v_1(x) &= \sum_{j=1}^N \alpha_j^{(1)} \phi_j(x), \quad w_1(x) = \sum_{j=1}^N \beta_j^{(1)} \phi_j(x), \\
v_2(x) &= \sum_{j=1}^N \alpha_j^{(2)} \phi_j(x), \quad w_2(x) = \sum_{j=1}^N \beta_j^{(2)} \phi_j(x),
\end{align*}
\]

where \( \phi_j(x) = \phi(|x - y_j|) \), for some function \( \phi : \mathbb{R}_0^+ \to \mathbb{R} \). Several RBF functions can be considered (eg. [21, 2]), but in this work we consider the multiquadric one \( \phi(r) = \sqrt{1 + (r/\epsilon)^2} \), for some \( \epsilon > 0 \).

The eigenvalue problem for the Dirac operator with infinite mass boundary conditions can be written as

\[
\begin{align*}
&\frac{\partial v_2}{\partial x_2} + \frac{\partial w_2}{\partial x_1} + i \left( -\frac{\partial v_2}{\partial x_1} - \frac{\partial w_2}{\partial x_2} \right) = E \left( v_1 + iv_1 \right) \quad \text{in } \Omega \\
&\frac{\partial v_1}{\partial x_1} + \frac{\partial w_1}{\partial x_2} + i \left( -\frac{\partial v_1}{\partial x_2} - \frac{\partial w_1}{\partial x_1} \right) = E \left( v_2 + iv_2 \right) \quad \text{in } \Omega \\
&(v_2 + iv_2) = i(n_1 + in_2)(v_1 + iv_1) \quad \text{on } \partial \Omega
\end{align*}
\]

and splitting in real and imaginary parts we have

\[
\begin{align*}
&\frac{\partial v_2}{\partial x_2} + \frac{\partial w_2}{\partial x_1} = Ev_1 \quad \text{in } \Omega \\
&\frac{\partial v_1}{\partial x_1} + \frac{\partial w_1}{\partial x_2} = Ev_2 \quad \text{in } \Omega \\
&v_2 = -n_1w_1 - n_2v_1 \quad \text{on } \partial \Omega \\
w_2 = n_1v_1 - n_2w_1 \quad \text{on } \partial \Omega
\end{align*}
\]

These equations are imposed at a discrete set of interior and boundary points. We consider \( M^{\Omega_1} \in \mathbb{N} \) points \( p_1, \ldots, p_{M^{\Omega_1}} \) uniformly distributed on \( \partial \Omega \) and \( M^{\Omega} \in \mathbb{N} \) points \( q_1, \ldots, q_{M^{\Omega}} \) located at a grid defined on \( \Omega \). Then, we calculate the matrices

\[
M^{\Omega} = \begin{bmatrix} \phi_1(q_1) & \cdots & \phi_N(q_1) \\ \vdots & \ddots & \vdots \\ \phi_1(q_{M^{\Omega}}) & \cdots & \phi_N(q_{M^{\Omega}}) \end{bmatrix}, \quad M_1^{\Omega} = \begin{bmatrix} \partial_1 \phi_1(q_1) & \cdots & \partial_1 \phi_N(q_1) \\ \vdots & \ddots & \vdots \\ \partial_1 \phi_1(q_{M^{\Omega}}) & \cdots & \partial_1 \phi_N(q_{M^{\Omega}}) \end{bmatrix},
\]

\[
M_2^{\Omega} = \begin{bmatrix} \partial_2 \phi_1(q_1) & \cdots & \partial_2 \phi_N(q_1) \\ \vdots & \ddots & \vdots \\ \partial_2 \phi_1(q_{M^{\Omega}}) & \cdots & \partial_2 \phi_N(q_{M^{\Omega}}) \end{bmatrix}, \quad M^{\Omega_1} = \begin{bmatrix} \phi_1(p_1) & \cdots & \phi_N(p_1) \\ \vdots & \ddots & \vdots \\ \phi_1(p_{M^{\Omega_1}}) & \cdots & \phi_N(p_{M^{\Omega_1}}) \end{bmatrix},
\]

and

\[
M_1^{\Omega_1} = \begin{bmatrix} n_1(p_1) \phi_1(p_1) & \cdots & n_1(p_1) \phi_N(p_1) \\ \vdots & \ddots & \vdots \\ n_1(p_{M^{\Omega_1}}) \phi_1(p_{M^{\Omega_1}}) & \cdots & n_1(p_{M^{\Omega_1}}) \phi_N(p_{M^{\Omega_1}}) \end{bmatrix},
\]
and we have $N$ of the Faber-Krahn type inequality stated in Conjecture 1 shall hold for the Dirac principal eigenvalue is minimized for the domain which also minimizes the perimeter is minimized by the ball. Thus, these numerical results suggest that

$$M_{2}^{\beta N} = \begin{bmatrix}
 n_{2}(p_{1})\phi_{1}(p_{1}) & \cdots & n_{2}(p_{1})\phi_{N}(p_{1}) \\
 \vdots & \ddots & \vdots \\
 n_{2}(p_{M_{2}})\phi_{1}(p_{M_{2}}) & \cdots & n_{2}(p_{M_{2}})\phi_{N}(p_{M_{2}})
\end{bmatrix}$$

Taking into account the definitions of the RBF linear combinations (34), the numerical approximations for the eigenvalues are the values $E$ for which we have nonzero solutions of the overdetermined system of linear equations

$$0 = \begin{bmatrix}
 0 & 0 & -M_{2}^{\Omega} & M_{2}^{\Omega} \\
 0 & 0 & -M_{1}^{\Omega} & M_{1}^{\Omega} \\
 0 & -M_{1}^{\Omega} & 0 & 0 \\
 0 & M_{1}^{\Omega} & 0 & M_{1}^{\Omega}
\end{bmatrix} - E \begin{bmatrix}
 M_{1}^{\Omega} & 0 & 0 & 0 \\
 0 & M_{2}^{\Omega} & 0 & 0 \\
 0 & 0 & M_{2}^{\Omega} & 0 \\
 0 & 0 & 0 & M_{2}^{\Omega}
\end{bmatrix} \begin{bmatrix}
 \alpha^{(1)} \\
 \beta^{(1)} \\
 \alpha^{(2)} \\
 \beta^{(2)}
\end{bmatrix}$$

The numerical solution of the minimization problem associated to the non-linear variational characterization is obtained directly from (3), defining the function

$$F(\alpha^{(1)}, \ldots, \alpha^{(N)}, \beta^{(1)}, \ldots, \beta^{(N)}) = \frac{4 \int_{\Omega} |\partial_{x} u_{1}|^{2} dx - E^{2} \int_{\Omega} |u_{1}|^{2} dx + E \int_{\partial \Omega} |u_{1}|^{2} ds}{\int_{\Omega} |u_{1}|^{2} ds}$$

that we minimize by a gradient type method. We refer to [2] for details about the numerical quadratures to approximate the boundary and volume integrals in the definition of $F$.

8.2. Numerical Results. We start by testing our numerical algorithm for the calculation of the eigenvalues of the Dirac operator with infinite mass boundary conditions in the case of the unit disk, for which we know that the principal eigenvalue $E_{1}(\mathbb{D})$ is the smallest non-negative solution of the equation

$$J_{0}(\mu) = J_{1}(\mu)$$

and we have $E_{1}(\mathbb{D}) = 1.434695650819...$ In Table 1 we show the absolute errors of the numerical approximations for the principal eigenvalue $E_{1}(\mathbb{D})$, for several choices of $\epsilon$ and $N$ and show that the numerical method can be highly accurate, even with a moderate value of $N$.

| $N$ | $\epsilon$ = 5 | $\epsilon$ = 10 | $\epsilon$ = 15 |
|-----|----------------|----------------|----------------|
| 242 | $4.45 \times 10^{-4}$ | $8.55 \times 10^{-8}$ | $1.33 \times 10^{-8}$ |
| 323 | $1.30 \times 10^{-5}$ | $2.78 \times 10^{-8}$ | $4.93 \times 10^{-8}$ |
| 402 | $4.92 \times 10^{-6}$ | $9.21 \times 10^{-9}$ | $1.16 \times 10^{-9}$ |

Table 1. Absolute errors of the numerical approximations for the principal eigenvalue $\lambda_{1}(\mathbb{D})$, for several choices of $\epsilon$ and $N$.

We have computed the principal eigenvalue for 2500 domains (with smooth boundary) randomly generated satisfying $|\Omega| = \pi$. The corresponding eigenvalues are plotted in Figure 1 as a function of the perimeter. We observe that the principal eigenvalue is minimized for the domain which also minimizes the perimeter. By the classical isoperimetric inequality it is well know that for fixed area, the perimeter is minimized by the ball. Thus, these numerical results suggest that the Faber-Krahn type inequality stated in Conjecture 1 shall hold for the Dirac operator with infinite mass boundary conditions.

Next, we present some numerical results for the minimization problem associated to the non-linear variational characterization (3). Figure 2 shows three domains (denoted by $\Omega_{1}$, $\Omega_{2}$ and $\Omega_{3}$) verifying $|\Omega_{i}| = \pi$, $(i = 1, 2, 3)$ to illustrate the
numerical results that we gathered. In Figure 3 we plot $\mu_{\Omega_i}(E)$, $i = 1, 2, 3$ together with the curve $\mu_D(E)$. We verify that for all $E > 0$, we have

$$\mu_{\Omega_i}(E) \geq \mu_D(E), \ i = 1, 2, 3$$

which illustrates Conjecture 40.

Finally, Figure 4 shows the absolute value (left plots) and argument (right plots) of a (normalized) eigenfunction associated to the principal eigenvalue of the domains $\Omega_i$, $i = 1, 2, 3$. Remark that the point of maximal modulus seems to be localized at the incenter of $\Omega_i$ which is in line with our choice of test function in the proof of Theorem 3. However, there is absolutely no reason for the associated eigenfunction to be real-valued and this has two consequences. First, Theorem 3 could be improved if one considers an adequate test function in the domain of the operator and not only in the form domain as we do. Second, Conjecture 1 cannot be reduced to the Bossel-Daners inequality because, contrary to the Robin eigenvalue problem, there is a priori no reason for an eigenfunction to have a non-constant argument as illustrated in Figure 4.
Figure 3. Plots of $\mu_{i1}$, $i = 1, 2, 3$, together with the curve $\mu^D$ as a function of the spectral parameter $E > 0$. 
Figure 4. Plots of the absolute value (left plots) and argument (right plots) of the eigenfunction associated to the principal eigenvalue of $\Omega_i$, $i = 1, 2, 3$. 
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