PRIMITIVE STABLE REPRESENTATIONS OF FREE KLEINIAN GROUPS

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Abstract. In this paper, we give a complete criterion for a discrete faithful representation $\rho : F_n \to \text{PSL}(2, \mathbb{C})$ to be primitive stable. This will answer Minsky’s conjectures about geometric conditions on $\mathbb{H}^3/\rho(F_n)$ regarding the primitive stability of $\rho$.

1. Introduction

Let $F$ be a non-abelian free group of rank $n$. For any group $G$, the automorphism group $\text{Aut}(F)$ acts on $\text{Hom}(F, G) = G^n$ by precomposition. This action projects down to the action of the outer automorphism group $\text{Out}(F)$ on the character variety $\mathcal{X}(F, G)$ which is defined as the geometric quotient of $\text{Hom}(F, G)$ by inner automorphisms of $G$. When $G$ is PSL(2, C), Minsky studied a dynamical decomposition of $\mathcal{X}(F, G)$ with respect to the $\text{Out}(F)$-action. Here a dynamical decomposition means decomposing $\mathcal{X}(F, G)$ in terms of proper discontinuity and ergodicity of the action. See [24] for more information about this decomposition. Minsky defined the set $\mathcal{PS}(F)$ of primitive stable characters, and his main results are as follows ([29]).

1. $\mathcal{PS}(F)$ is an open subset of $\mathcal{X}(F, \text{PSL}(2, \mathbb{C}))$, and $\text{Out}(F)$ acts on $\mathcal{PS}(F)$ properly discontinuously.
2. $\mathcal{PS}(F)$ is strictly larger than the set of Schottky characters.
3. For every proper free factor $A$ of $F$ and a primitive stable representation $\rho$, the restriction $\rho|_A$ is Schottky.

Since the set of Schottky characters is known to be the interior of the set of discrete faithful characters by Sullivan [41], it follows that the dynamical decomposition of $\mathcal{X}(F, G)$ is different from the well-known geometric decomposition of $\mathcal{X}(F, G)$ in terms of being discrete faithful and having dense image.

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When a primitive stable representation $\rho$ is discrete and faithful, $\mathbb{H}^3/\rho(F)$ becomes a hyperbolic 3-manifold which we call a hyperbolic handlebody and our main interest is finding a geometric condition on $\mathbb{H}^3/\rho(F)$ under which $\rho$ becomes primitive stable. In the same paper [29], Minsky conjectured that

1. Every discrete faithful representation of $F$ without parabolics is primitive stable.
2. A discrete faithful representation of $F$ is primitive stable if and only if every component of its ending lamination is blocking.

We shall see that a disc-busting minimal lamination is blocking. For definitions, see section 2. In this paper, we shall prove the first conjecture and also give an answer to the second one.

**Theorem 1.1.** If $\rho$ is a discrete faithful representation of $F$ without parabolics then $\rho$ is primitive stable.

In this case, the ending lamination of $\mathbb{H}^3/\rho(F)$ is necessarily connected and in the Masur domain.

**Theorem 1.2.** Let $\rho$ be a discrete, faithful and geometrically infinite representation with parabolics such that the non-cuspidal part $M_0$ of $M = \mathbb{H}^3/\rho(F)$ is the union of the relative compact core $H$ and finitely many end neighbourhoods $E_i$ facing $S_i \subset \partial H$. Then the representation $\rho$ is primitive stable if and only if every parabolic curve is disc-busting, and every geometrically infinite end $E_i$ has the ending lamination $\lambda_i$ which is disc-busting on $\partial H$.

Theorem 1.1 and the sufficiency part of Theorem 1.2 were announced in [14]. The idea of proof is following Minsky’s construction of a primitive stable representation which is not Schottky, and the main new ingredient comes from the recent result of Mj about Cannon-Thurston maps of free Kleinian groups (see [37]).

2. Preliminaries

2.1. Primitive stability. Let us recall that $F$ is a non-abelian free group of rank $n$. Let $\vee S^1$ be a bouquet of $n$ oriented circles, whose fundamental group is $F$ with a fixed generating set $X = \{x_1, \cdots, x_n\}$. Then its universal cover $\widetilde{\vee S^1}$ can be identified with the Cayley graph $\Gamma_F$ of $F$ with respect to $X$, which is a tree with the canonical word metric. Following [5], the space of oriented lines $\mathcal{B}(F)$ in $\Gamma_F$ can be identified with $(\partial_\infty F \times \partial_\infty F) \setminus \Delta$ where $\partial_\infty F$ denotes the Gromov boundary of the tree $\Gamma_F$ and $\Delta$ denotes the diagonal. The free group $F$ acts diagonally on $\mathcal{B}(F)$ as the covering transformations. We denote the quotient space of $\mathcal{B}(F)$ under this action by $\mathcal{B}(F)$, and call each element of $\mathcal{B}(F)$ also a line.

For $w \in F$, if we let $\overline{w}$ be the bi-infinite periodic word determined by concatenating infinitely many copies of $w$, then it defines an $F$-invariant family of lines in $\mathcal{B}(F)$ and these lines are projected to a line in $\mathcal{B}(F)$ which
can be identified with \( w \) modulo shift or the conjugacy class \([w]\). An element of \( F \) is called primitive if it can be a member of a free generating set and we let \( \mathcal{P}(F) \) denote the subset of \( \mathcal{B}(F) \) consisting of \( w \) for conjugacy classes \([w]\) of primitive elements, which is \( \text{Out}(F) \)-invariant.

Given a representation \( \rho : F \to \text{PSL}(2, \mathbb{C}) \) and a base point \( o \in \mathbb{H}^3 \), there is a unique \( \rho \)-equivariant map \( \tau_{\rho,o} : \Gamma_F \to \mathbb{H}^3 \) sending the origin \( e \) of \( \Gamma_F \) to \( o \) and taking each edge to a geodesic segment [12]. A representation \( \rho : F \to \text{PSL}(2, \mathbb{C}) \) is primitive stable if there are constants \( K, \delta \) such that \( \tau_{\rho,o} \) takes all lines (in \( \tilde{\mathcal{B}}(F) \)) corresponding to \( \mathcal{P}(F) \) to \((K, \delta)\)-quasi-geodesics in \( \mathbb{H}^3 \). This definition is independent of the choice of the base point \( o \in \mathbb{H}^3 \), which we can easily see by changing \( \delta \). Since the primitive stability is also invariant under conjugacy, for simplifying arguments for checking primitive stability, we shall define a unique element of \( \tilde{\mathcal{B}}(F) \) corresponding to a cyclically reduced \( w \in F \) as follows. Since \( \Gamma_F \) is a tree, there exists a unique oriented line \( \tilde{w} \) on \( \Gamma_F \) passing through all the \( w^k(e) \) for \( k \in \mathbb{Z} \), where we regard \( w \) as a covering transformation. Then clearly, the broken geodesic image \( \tau_{\rho,o}(\tilde{w}) \) passes through the base point \( o \).

### 2.2. Whitehead lemma

We refer to [7, 42] for the basic theory of geodesic and measured laminations. The space of measured laminations on a hyperbolic surface is a completion of weighted simple geodesics, to which the geometric intersection number continuously extends. Recall that the Masur domain [25] of a handlebody \( H \) consists of projective classes of measured laminations which have positive intersection number with every non-empty limit of weighted meridians of \( H \). A measured lamination \( \lambda \) (or a simple closed curve) is said to be disc-busting if there exists \( \eta > 0 \) such that for any essential disc \( A \), \( i(\partial A, \lambda) > \eta \). Otherwise \( \lambda \) is called disc-dodging.

A lamination \( \lambda \) is called doubly incompressible if \( i(\partial A, \lambda) > \eta \) for any essential disc or essential annulus \( A \). The set of projective doubly incompressible measured laminations is strictly bigger than the Masur domain (see [23] and [17]).

Let \( \Delta = \{\delta_1, \cdots, \delta_n\} \) be a system of compressing discs on a handlebody \( H \) along which one can cut \( H \) into a 3-ball. We call such a system a cut system. A free generating set of \( \pi_1(H) = F \) is dual to such a system. The Whitehead graph \( Wh(\lambda, \Delta) \) (of \( \lambda \) with respect to \( \Delta \)) is defined as follows. First, we isotope \( \Delta \) so that its boundary consists of closed geodesics with respect to a fixed hyperbolic metric. (This removes all inessential intersections between \( \lambda \) and \( \partial \Delta \).) By cutting \( \partial H \) along \( \Delta \), we get a planar surface with \( 2n \) boundary components. We label the two discs coming from cutting \( H \) along \( \delta_i \) as \( \delta_i^+, \delta_i^- \) so that the oriented loop corresponding to a generator of \( \pi_1(H) \) passes from \( \delta_i^- \) to \( \delta_i^+ \). Then \( \partial \delta_i^+ \) and \( \partial \delta_i^- \) are identified on \( \partial H \). The vertices of the graph \( Wh(\lambda, \Delta) \) are the boundary components of \( \Delta \), and two vertices are connected by an edge if and only if the corresponding boundary components are joined by an arc of \( \lambda \setminus \Delta \). Note that we can regard the edges of \( Wh(\lambda, \Delta) \)
as embedded in $\partial H \setminus \Delta$, and we can replace the vertices with small circles $\partial \delta_i^\pm$.

A geodesic lamination $\lambda$ is said to be in **tight position** with respect to a cut system $\Delta$ (or in more general, to a disjoint union of compressing discs) if there are no arcs $\alpha$ disjoint from $\lambda$ with interior disjoint from $\delta$, and $\beta$ on $\partial \Delta$ with $\partial \alpha = \partial \beta$, which are homotopic in $H$ fixing the endpoints but not homotopic on $\partial H$. For a meridian or a system of meridians $m$ on $H$, a **$m$-wave** is an arc on $\partial H$ with endpoints on $m$ which is homotopic in $H$ relative to endpoints, but not in $\partial H$ to a subarc of $m$. A tight lamination has no waves with respect to $\partial \Delta$. Otal showed the following in Proposition 3.10 of his thesis [39].

**Lemma 2.1.** Suppose that a geodesic lamination $\lambda$ is in a tight position to a cut system $\Delta$. Then $Wh(\lambda, \Delta)$ is connected and has no cut points.

For disc-busting measured laminations, we can show the following.

**Lemma 2.2.** Let $\lambda$ be a disc-busting measured lamination. Then there is a generating set which is dual to a cut system $\Delta$ such that $\lambda$ is in tight position to $\Delta$, and hence $Wh(\lambda, \Delta)$ is connected and has no cut points.

**Proof.** Let $\Delta$ be any cut system of $H$. Suppose that $\lambda$ is not in tight position with respect to $\Delta$. Then there are arcs $\alpha$ disjoint from $\lambda$ with interior disjoint from $\Delta$ and $\beta$ on a component $D$ of $\Delta$ with $\partial \alpha = \partial \beta$ such that $\alpha$ and $\beta$ are homotopic in $H$ fixing the endpoints but are not on $\partial H$. Let $\gamma$ be $\alpha \cup \beta$. Since $\alpha$ is homotopic to $\beta$ in $H$ fixing the endpoints, $\gamma$ bounds a compressing disc $D'$. Since $\lambda$ was assumed to be disc-busting, $i(\lambda, D') > \eta$. On the other hand, $(\partial D \setminus \beta) \cup \alpha$ is also a meridian, and bounds a compressing disc $D''$. Since $\beta$ is disjoint from $\lambda$, by replacing $D$ with $D''$, the intersection number with $\lambda$ is reduced by more than $\eta$. We now define a new cut system $\Delta'$ to be $(\Delta \setminus \{D\}) \cup \{D''\}$. As was observed above we have $i(\lambda, \Delta') < i(\lambda, \Delta) - \eta$. Since $i(\lambda, \Delta)$ is finite and any meridian has intersection number more than $\eta$ with $\lambda$, this process must terminate in finite steps, and we reach a cut system with respect to which $\lambda$ is in tight position. □

Originally in [43, 44], Whitehead considered $Wh(A, X)$ for some finite set $A \subset F$ and a generating set $X$ to check the separability of $A$. Here, $A$ is said to be **separable** if there exists a free decomposition $F = F_1 \ast F_2$ such that for every element $a \in A$, it is conjugate into one of $F_i$. The vertices of $Wh(A, X)$ are $\{ \pm x | x \in X \}$ and for any $a \in A$, two vertices $x$ and $y$ are connected by an edge if and only if $xy^{-1}$ appears in $a$ or in a cyclic permutation of $a$. If $g$ is primitive, then it is separable. Due to the following lemma [43, 44], $Wh(g, X)$ for a primitive element $g$, is either disconnected, or has a cut point for any generating set $X$.

**Lemma 2.3.** (Whitehead) Let $g$ be a cyclically reduced word in a free group $F$, and let $X$ be a fixed generating set. If $Wh(g, X)$ is connected and has no cut point, then $g$ is not separable, hence in particular not primitive.
A word $g$ is called blocking if there exists $n$ such that $g^n$ is not a subword of a cyclically reduced primitive word. Likewise, a lamination $\lambda$ on $\partial H$ is called blocking with respect to a generating system if $\lambda$ is in tight position with respect to the cut system dual to the generating system, and there exists some $k$ such that every length $k$ subword of the infinite word determined by a leaf of $\lambda$ does not appear as a subword in a cyclically reduced primitive word. Recall that a primitive word is a member of a generating system of $F$.

**Corollary 2.4.** A disc-busting minimal lamination $\lambda$ on the boundary of a handlebody is blocking with respect to some generating set.

**Proof.** Consider a generating system dual to a cut system given in Lemma 2.2. Suppose $\lambda$ is not blocking. Then for all $k$, there is a subword of length $k$ of the infinite word determined by $\lambda$ which is a subword of a cyclically reduced primitive word. Taking a sufficiently large $k$, the Whitehead graph of the length $k$ subword is equal to $\text{Wh}(\lambda, \Delta)$. Lemmata 2.1 and 2.3 give us a contradiction. □

When we consider Cannon-Thurston maps, we shall need to deal with both geodesics on $\partial H$ and those in the Cayley graph $\Gamma_F$. For that, we shall reinterpret the lemmata above for geodesics in the Cayley graph.

Let $\widetilde{\Delta}$ be the universal cover of the handlebody $H$. For any fixed Riemannian metric on $H$, its pull back to $\widetilde{\Delta} \Delta$ makes $\widetilde{\Delta}$ quasi-isometric to the Cayley graph $\Gamma_F$ (for any generator system). Therefore in particular $\widetilde{\Delta}$ is Gromov hyperbolic, and it can be compactified by adding the boundary at infinity $\partial_\infty F$.

Let $\Delta$ be a cut system for $H$. Then the preimage $\widetilde{\Delta}$ of $\Delta$ in $\widetilde{\Delta}$ cuts $\widetilde{\Delta}$ into balls, and each of its components separates $\widetilde{\Delta}$ into two. A point in $\partial_\infty F$ corresponds to a sequence of distinct components (discs) $\{D_1, D_2, \ldots\}$ in $\widetilde{\Delta}$ such that all of $D_{i+1}, D_{i+2}, \ldots$ lie on the same side of $D_i$. For two such sequences of discs $\{D_i\}$ and $\{D'_i\}$, they represent the same point at infinity if and only if for each $D_i$ there is $j_0$ such that all the discs $D'_i, D'_{i+1}, \ldots$ lie on the same side of $D_i$ as $D_{i+1}$, and conversely for each $D'_j$ there is $i_0$ such that all the discs $D_{i_0}, D_{i_0+1}, \ldots$ lie on the same side of $D'_j$ as $D'_{j+1}$.

Let us say two discs $D$ and $D'$ in $\widetilde{\Delta}$ are adjacent if they lie on the boundary of the same ball obtained by cutting $\widetilde{H}$ along $\widetilde{\Delta}$. For any point at infinity $p \in \partial_\infty F$, we can choose a sequence $\{D_i\}$ as above representing $p$ such that $D_i$ and $D_{i+1}$ are adjacent for every $i$. We call such a sequence maximal. For two maximal sequences $\{D_i\}$ and $\{D'_j\}$, they represent the same point at infinity if and only if they are eventually the same, i.e., there are $i_0$ and $j_0$ such that $D_{i_0+k} = D'_{j_0+k}$ for every $k \in \mathbb{N}$.

Let $\Gamma_F$ be a Cayley graph of $F$ with respect to the generator system dual to $\Delta$. Since $\mathcal{V}S^1$ is a spine of $H$, the Cayley graph $\Gamma_F$ is contained in $\widetilde{H}$, and there is a quasi-isometric deformation retraction $r : \widetilde{H} \to \Gamma_F$ which projects
each disc of $\tilde{\Delta}$ to a point. Now, for any geodesic $l$ in $\Gamma_F$ which can be either a segment or a ray or a line, we can define the Whitehead graph $Wh(l, \Delta)$ in the same way as for laminations on $\partial H$ as follows.

We consider the discs in $\tilde{\Delta}$ which $l$ intersects and array them in the order in which $l$ intersects them, to get a sequence $\{D_i\}$. This sequence is maximal in the sense defined above. Note that each disc $D_i$ in $\tilde{\Delta}$ projects to a disc $\delta_i$ in $\Delta$. We construct a Whitehead graph $Wh(l, \Delta)$ by letting the vertices be as before and connecting two vertices corresponding to $\delta_i$ to $\delta_{i+1}$ for each $i$, with signs in accordance with the way $l$ connects $D_i$ and $D_{i+1}$.

**Lemma 2.5.** Let $\lambda$ be a minimal lamination on $\partial H$ which is in tight position with respect to $\Delta$. Let $l$ be a lift of a leaf of $\lambda$ to $\tilde{H}$, and let $l^*$ be a geodesic in $\Gamma_F$ which has the same endpoints at infinity as $r(l)$ where $r : \tilde{H} \rightarrow \Gamma_F$ is the quasi-isometric deformation retraction. Then we have $Wh(l^*, \Delta) = Wh(\lambda, \Delta)$.

**Proof.** Since $\lambda$ is minimal, every leaf is dense in $\lambda$. Thus, if an arc in $\lambda \setminus \Delta$ connects two discs in some homotopy class of arcs (fixing the endpoints), then so does the projection of $l$ to $S$. Therefore $l$ passes through discs of $\tilde{\Delta}$ in such a way that every pair of successive intersections corresponds to an edge of $Wh(\lambda, \Delta)$ and every edge in $Wh(\lambda, \Delta)$ is realised by a pair of successive intersections. Therefore, we have only to show that $l^*$ and $l$ intersect the same discs in $\tilde{\Delta}$ in the same order. Since $l^*$ is a geodesic in the tree, and $l$ and $l^*$ have the same endpoints at infinity, $l$ must intersect all the discs in $\tilde{\Delta}$ that $l^*$ intersects. On the other hand $l$ can have an intersection with a disc $D$ in $\tilde{\Delta}$ which is disjoint from $l^*$ only when it intersects $D$ twice in the opposite directions. This implies that $l$ contains a wave and thus $l$ is not in tight position with respect to $\Delta$, contradicting our assumption. Thus we have completed the proof. \hfill \Box

We say that a geodesic ray or a line $k$ in $\Gamma_F$ is **asymptotic** to a geodesic lamination $\lambda$ if there is a lift $\tilde{l}$ of a leaf of $\lambda$ to $\tilde{H}$ which shares at least one of the endpoints at infinity with $k$.

**Lemma 2.6.** Suppose that a geodesic $k$ in $\Gamma_F$ is asymptotic to a minimal geodesic lamination $\lambda$ which is in tight position with respect to $\Delta$. Then we have $Wh(k, \Delta) \supset Wh(\lambda, \Delta)$.

**Proof.** Let $l$ be a lift of a leaf of $\lambda$ which shares an endpoint at infinity with $k$, and $l^*$ the geodesic in $\Gamma_F$ having the same endpoints at infinity as $l$. By the proof of the previous lemma, we see that $l^*$ intersects the same discs in $\tilde{\Delta}$ in the same order as $l$. Let $\{D_i\}$ be a sequence of discs in $\tilde{\Delta}$ arrayed in the same order as $l^*$ intersects them. Since $k$ shares an endpoint with $l$, hence with $l^*$, there is $i_0$ such that $k$ intersects $\{D_{i_0+1}, D_{i_0+2}, \ldots \}$ in this order. Now, since $\lambda$ is minimal, each leaf of $\lambda$ is recurrent. Let $l'$ be a sub-leaf of $l^*$ starting from the intersection of $l$ with $D_{i_0}$. Then we have...
Since $k$ intersects $\{D_{i_0}, D_{i_0+1}, \ldots \}$ in this order, we have $Wh(k, \Delta) \supset Wh(l', \Delta)$. This completes the proof. □

2.3. Hyperbolic 3-manifold. We shall mainly concentrate on hyperbolic handlebodies which can be represented as $M = H^3/\rho(F)$ for a discrete faithful representation $\rho$. If $M$ is geometrically finite without parabolics, then $\rho$ is just a Schottky representation by Maskit [26] and if $M$ is geometrically infinite without parabolics, then its compact core is a compact handlebody $H$ and the end $M \setminus int(H)$ is homeomorphic to $\partial H \times [0, \infty)$ by the Tameness theorem [1, 6]. In this case, the ending lamination on $\partial H$ is connected, filling, and contained in the Masur domain of $\partial H$ by Canary [9]. When $\rho(F)$ has parabolics, they all have to be contained in rank-1 maximal parabolic groups, and a relative compact core is also homeomorphic to a compact handlebody $H$ which meets each rank-1 closed cusp neighbourhood along a single annulus. The complement of these annuli in $\partial H$ consists of several components $S_i$ which may be either compressible or incompressible. Each end neighbourhood $E_i = S_i \times [0, \infty)$ facing $S_i$ may be geometrically finite or otherwise be geometrically infinite, and in the latter case it has the ending lamination $\lambda_i$. The union of a relative compact core and the closed cusp neighbourhoods is called a augmented Scott core and will be denoted by $H'$.

3. Cannon-Thurston map

Started in the pioneering work of Cannon and Thurston [8] for closed 3-manifolds fibring over a circle, Cannon-Thurston maps have been generalised in several ways by Bowditch [2], Klarreich [19], McMullen [27], Minsky [28] and Mj [31, 32, 33, 34, 35, 36] (see also [30], [40]). Mj proved the existence of Cannon-Thurston maps for Kleinian surface groups in [34], and described the points identified by Cannon-Thurston maps in [35] (see [38] for the case of punctured surfaces). Recently he also proved the existence of Cannon-Thurston maps for arbitrary Kleinian groups and described the points identified by the maps in [37]. We shall make use of his result in the case of free Kleinian groups as a main ingredient for the proof of our main results.

Given a discrete faithful representation $\rho : F \to PSL(2, \mathbb{C})$ without parabolics, the Cannon-Thurston map is a continuous extension of $\tau_{\rho,0} : \Gamma \to H^3$ to $\tilde{\tau}_{\rho,0} : \tilde{\Gamma} \to H^3 \cup \mathbb{C}$ where $\mathbb{C}$ is the ideal boundary of $H^3$ and $\tilde{\Gamma}$ is the Gromov compactification of $\Gamma$. The boundary of $H$ is a covering of $S$, which we denote by $S_F$. We fix a hyperbolic metric on $S$ and pull it back to $S_F$. As was explained in [2.2] the Gromov boundary of both $S_F$ and $\tilde{\Gamma}$ is identified with $\partial_\infty F$. Mj’s theorem for free Kleinian groups without parabolics is as follows.

Theorem 3.1. Given a discrete faithful representation $\rho : F \to PSL(2, \mathbb{C})$ without parabolics, let $\Gamma_F$ be the Cayley graph of $F$ and $i : \Gamma_F \to H^3$ the natural identification of $\Gamma_F$ with its image under $\tau_{\rho,0}$ for a chosen base point
$o \in \mathbb{H}^3$. Let $\lambda$ be the ending lamination of $\rho(F)$ and $\tilde{\lambda}$ its preimage in $S_F$. Then $i$ extends continuously to a map $\tilde{i} : \tilde{\Gamma_F} \to \mathbb{H} \cup \tilde{C}$. If we let $\partial i$ denote the restriction of $i$ to the Gromov boundary $\partial_\infty F$, then $\partial i(a) = \partial i(b)$ if and only if $a, b$ are either ideal endpoints of a leaf of $\tilde{\lambda}$, or ideal vertices of one of the complementary ideal polygons of $\tilde{\lambda}$, where $\partial_\infty F$ is regarded as the Gromov boundary of $S_F$.

Recall that $M := \mathbb{H}^3/\rho(F)$ can be decomposed into $M = H \cup E$ where $H$ is a genus $n$ compact handlebody and $E$ is homeomorphic to $\partial H \times [0, \infty)$. We use the convention that $\partial H$ and $\partial H \times \{0\}$ are identified and denote $\partial H$ as $S$. When the end $E$ is geometrically infinite, we have an ending lamination $\lambda$ on $S$. By Lemma 2.2, there is a cut system $\Delta$ with respect to which $\lambda$ is in tight position. The lamination is realised as a geodesic lamination uniquely once we fix a hyperbolic metric $S$. As was explained in §2.2, each leaf of $\lambda$ can be lifted to a geodesic $l$ in $\partial H$ (with the pulled-back hyperbolic metric) and its projection by the retraction $r$ forms a quasi-geodesic in $\Gamma_F$. We denote the geodesic on $\Gamma_F$ with the same endpoints at infinity as $r(l)$ by $l^*$ as before. Then $Wh(l^*, \Delta) = Wh(\lambda, \Delta)$ by Lemma 2.4.

By Lemma 2.6 for any geodesic in $\Gamma_F$ sharing an endpoint with $l^*$, we have $Wh(k, \Delta) \supset Wh(\lambda, \Delta)$. If we connect two endpoints $a, b$ in $\partial_\infty F$ such that $\partial i(a) = \partial i(b)$, then by Theorem 2.1 the geodesic on $\Gamma_F$ connecting $a$ with $b$ is asymptotic to a leaf of $\tilde{\lambda}$. Therefore, we get the following corollary.

**Corollary 3.2.** Suppose that $\rho$ has no parabolics. Let $\Delta$ be a cut system with respect to which $\lambda$ is in tight position. Given $a, b \in \partial_\infty F$ which are identified by $\partial i$, if we let the geodesic on $\Gamma_F$ joining $a, b$ be $k$, then $Wh(k, \Delta)$ contains $Wh(\lambda, \Delta)$, and is connected and has no cut point.

Now we discuss the case of a free Kleinian group with parabolics. Recall that when $\rho(F)$ has parabolics, $M = \mathbb{H}^3/\rho(F)$ has a relative compact core $H$ of the non-cuspidal part $M_0$ which intersects each cusp neighbourhood along an annulus whose core curve we call a parabolic curve. Each frontier component $S_i$ of $H$ in $M_0$ faces an end neighbourhood $E_i$ with ending lamination $\lambda_i$ which is a filling minimal lamination on $S_i$. Now we consider the union $\Lambda$ of all the ending laminations $\lambda_i$ and all the parabolic curves. Then $\Lambda$ itself is a geodesic lamination on $S$. Suppose, as in the setting of Theorem 1.2, that all the parabolic curves and the $\lambda_i$ are disc-busting. Then $\Lambda$ is also disc-busting. By Lemma 2.1, we can find a cut system $\Delta'$ with respect to which $\Lambda$ is in tight position. We lift $\Lambda$ to a geodesic lamination $\tilde{\Lambda}$ on $S_F$. Since $\Lambda$ is in tight position with respect to $\Delta'$, each $\lambda_i$ has no waves although $\lambda_i$ may not be tight with respect to $\Delta'$. Thus no leaf of $\tilde{\Lambda}$ can intersect a component of the preimage of $\Delta'$ twice and each leaf of $\tilde{\Lambda}$ connects two distinct points at infinity on $\partial_\infty F$. We introduce a relation $\mathcal{R}$ for points in $\partial_\infty F$ such that $a \mathcal{R} b$ if and only if either $a$ and $b$ are the endpoints of a leaf of $\tilde{\Lambda}$ or ideal vertices of a complementary region of $\tilde{\Lambda}$. In contrast to the case of groups without parabolics, there are complementary
regions of \( \tilde{\Lambda} \) which are ideal polygons with infinitely many sides. These are exactly regions touching lifts of parabolic curves, which are isolated leaves. The relation \( \mathcal{R} \) may not be transitive. We define \( \tilde{\mathcal{R}} \) to be the transitive closure of \( \mathcal{R} \), which is an equivalence relation.

Mj’s theorem about the identified points of a Cannon-Thurston map can be adapted to our case as follows.

**Theorem 3.3.** Let \( \rho : F \to \text{PSL}(2, \mathbb{C}) \) be a discrete faithful representation with parabolics, and let \( M = \mathbb{H}^3/\rho(F) = \mathbb{H}' \cup \cup E_i \) as above. Then the natural identification \( i \) of \( \Gamma_F \) with its image under \( \tau_{\rho,o} \) extends continuously to a map \( \hat{i} : \Gamma_F \cup \partial_{\infty} F \to \mathbb{H}' \cup \hat{\mathbb{C}} \). Then for \( a, b \in \partial_{\infty} F \), \( \partial i(a) = \partial i(b) \) if and only if \( a \tilde{\mathcal{R}} b \).

Assume that \( \partial i(a) = \partial i(b) \) for \( a, b \in \partial_{\infty} F \). Let \( k \) be a geodesic connecting \( a \) and \( b \) on \( \Gamma_F \). Then \( k \) is asymptotic to some leaf \( l \) of \( \tilde{\Lambda} \) by our definition of \( \mathcal{R} \). Let \( \lambda \) be a component of \( \Lambda \) containing the projection of \( l \) and take \( \Delta \) to be the disc system such that \( Wh(\lambda, \Delta) \) is connected and has no cut point by Lemma 2.1. Since two Cayley graphs coming from two disc systems \( \Delta' \) and \( \Delta \) are quasi-isometric, if we let \( k' \) be the geodesic in the Cayley graph with respect to \( \Delta \) connecting \( a \) and \( b \), then \( k' \) is asymptotic to \( l \). By the same argument as in the case without parabolics, Corollary 3.2 holds also in this case.

**Corollary 3.4.** Suppose that \( \rho(F) \) has parabolics. Given \( a, b \in \partial_{\infty} F \) which are identified by \( \partial i \), if we let the geodesic on \( \Gamma_F \) joining \( a, b \) be \( k' \), then \( Wh(k', \Delta) \) contains \( Wh(\lambda, \Delta) \) for some component \( \lambda \) of \( \Lambda \) and a cut system \( \Delta \) with respect to which \( \lambda \) is in tight position. \( Wh(k', \Delta) \) is connected and has no cut point for the disc system \( \Delta \).

4. Primitive stable representations of a free group

4.1. Free groups without parabolics. In this section, we shall prove the first of our theorems. The overall argument is as follows. Assuming that the representation is not primitive stable, there exists a sequence of primitive elements \( \{w_n\} \) such that \( \{\tau_{\rho,o}(\tilde{w}_n)\} \) is not a family of uniform quasi-geodesics, where \( \tilde{w}_n \) is a line in \( \Gamma_F \) corresponding to \( w_n \). After passing to a subsequence, \( \tilde{w}_n \) converges uniformly on every compact set to a bi-infinite line \( \tilde{w}_\infty \) in the Cayley graph with two distinct end points in the Gromov boundary. Then we shall show that the endpoints of \( \tau_{\rho,o}(\tilde{w}_\infty) \) are the same point in \( \partial \mathbb{H}^3 = \hat{\mathbb{C}} \). Applying Mj’s result cited as Theorem 3.1 above, we shall conclude that the endpoints of \( \tilde{w}_\infty \) are either the endpoints of a lift of a leaf of the ending lamination of \( M = \mathbb{H}^3/\rho(F) \) or ideal endpoints of a complementary polygon. Finally we shall apply Lemma 2.3 to draw a contradiction.

**Proof of Theorem 1.1.** Suppose that \( \rho \) is not primitive stable. Then there exists a sequence \( \{w_n\} \) of cyclically reduced primitive words such that the
$\tau_{\rho, o}(\bar{w}_n)$ are not uniform quasi-geodesics for every lift $\bar{w}_n$ of $w_n$ to $\Gamma_F$. Recall that $M = \mathbb{H}^3/\rho(F)$ is homeomorphic to $H \cup (\partial H \times [0, \infty))$ where $\partial H$ is identified with $\partial F \times \{0\}$.

If a lift of an arbitrary closed geodesic in $M$ is contained in a uniformly thickened neighbourhood of $\tau_{\rho, o}(\Gamma_F)$ then $\rho$ has to be primitive stable. This is an observation made by Minsky in proving that every Schottky representation is primitive stable in Lemma 3.2 of [29].

Let $\gamma_{w_n}$ be the geodesic in $\mathbb{H}^3$ joining the endpoints at infinity of $\tau_{\rho, o}(\bar{w}_n)$. Since $\rho$ is not primitive stable, there exists a sequence of positive numbers $\{\epsilon_n\}$ such that $\gamma_{w_n}$ is not contained in the $\epsilon_n$-neighbourhood of the core graph $\tau_{\rho, o}(\Gamma_F)$, where $\epsilon_n \to \infty$ as was explained above. Since $\gamma_{w_n}$ is not contained in the $\epsilon_n$-neighbourhood of $\tau_{\rho, o}(\Gamma_F)$ in $\mathbb{H}^3$, neither is it in the $\epsilon_n$-neighbourhood of $\tau_{\rho, o}(\bar{w}_n)$. Since the distance function is convex in $\mathbb{H}^3$ (see for example [12]), we can choose a vertex of $\tau_{\rho, o}(\bar{w}_n)$ whose minimal distance from $\gamma_{w_n}$ is larger than $\epsilon_n$. Moreover, we can shift the words $w_n$ so that the specified vertex is the base point $o$ as follows.

A vertex on $\tau_{\rho, o}(\bar{w}_n)$ is expressed as $\rho(w_n^iv_n)o$ where $w_n = g_1g_2\ldots g_k$ and $v_n = g_1\ldots g_l$ for $l < k$ and $i \in \mathbb{Z}$. Assume that $d_{\mathbb{H}^3}(\rho(w_n^iv_n)o, \gamma_{w_n}) > \epsilon_n$. Then, noting that $\gamma_{w_n}$ is the axis of the loxodromic isometry $\rho(w_n)$, we get

$$d_{\mathbb{H}^3}(\rho(w_n^iv_n)o, \gamma_{w_n}) = d_{\mathbb{H}^3}(\rho(v_n)o, \gamma_{w_n}) = d_{\mathbb{H}^3}(\rho(v_n)^{-1}\gamma_{w_n}),$$

and

$$\rho(v_n)^{-1}\gamma_{w_n} = \gamma_{w_n^{-1}w_nv_n}.$$ 

Since $v_n^{-1}w_nv_n$ is a shifted word of $w_n$, it is also cyclically reduced and primitive. We also have

$$d_{\mathbb{H}^3}(o, \gamma_{w_n^{-1}w_nv_n}) > \epsilon_n.$$ 

Therefore, $\gamma_{w_n^{-1}w_nv_n}$ has to leave every compact subset in $\mathbb{H}^3$ as $n \to \infty$, and $\{v_n^{-1}w_nv_n\}$ is a desired sequence.

Now, we have obtained a new sequence $\{w'_n\}$ such that $d_{\mathbb{H}^3}(o, \gamma_{w'_n})$ goes to $\infty$ as $n \to \infty$, which means that the spherical distance between the endpoints of $\gamma_{w'_n}$ goes to 0, so that the two endpoints of $\tau_{\rho, o}(\bar{w}'_n)$ in $\partial \mathbb{H}^3$ converge to the same point as $n \to \infty$.

**Lemma 4.1.** In $\Gamma_F$, after passing to a subsequence, $\{\bar{w}'_n\}$ converges uniformly on every compact set to a bi-infinite geodesic $\bar{w}_\infty$ with distinct end points and such that for some cut system $\Delta$, we have $\text{Wh}(\bar{w}_\infty, \Delta) \supset \text{Wh}(\lambda, \Delta)$, where $\lambda$ is the ending lamination of $\rho(F)$.

**Proof.** Since $F$ has a set of finite number of generators $X$, after passing to a subsequence, we may assume that all $w'_n$ have $x_i$ and $x_j$ in $\{x^\pm| x \in X\}$ as their first and last letter respectively for fixed $i, j$. Note that $x_j$ cannot be $x_i^{-1}$ because $w'_n$ is cyclically reduced. Then the geodesic $\bar{w}'_n$ has endpoints at infinity in the regions determined by $x_i$ and $x_j^{-1}$. Hence, passing to a subsequence, $\bar{w}'_n$ has a limit geodesic $\bar{w}_\infty$ with distinct endpoints $a, b$ at
infinity in the regions determined by $x_i$ and $x_j^{-1}$. Since the endpoints of $\tau_{\rho,o}(\tilde{w}_n')$ in $\partial H^3$ converge to the same point, $a$ and $b$ are identified under the Cannon-Thurston map.

□

Now returning to the main proof, $Wh(\lambda, \Delta)$ is connected and has no cut point with respect to $\Delta$ by Lemma 3.2 and from $Wh(w_\infty, \Delta) \supset Wh(\lambda, \Delta)$, we can see that the same is true for $Wh(w'_n, \Delta)$ for large $n$. On the other hand for any primitive word $w'_n$, this is not possible by Lemma 2.3. □

4.2. Free groups with parabolics. Recall that when $\rho : F \to \text{PSL}(2, \mathbb{C})$ has parabolics, $M = H^3/\rho(F) = H' \cup \cup E_i$ where $H'$ is the augmented scott core and $E_i$ is an end neighbourhood facing the relative compact core $H$ along $S_i \subset \partial H$. The $S_i$ are glued to each other by parabolic loci $A_i$ whose core curves are denoted by $c_i$. When $E_i$ is geometrically infinite, it has ending lamination $\lambda_i$ which is disc-busting by assumption.

Before we start the proof of our second theorem, we remark that in the case when $\rho$ has parabolics, the manifold may not be primitive stable even if every proper free factor is Schottky. The following example due to Minsky shows this. Let $M$ be a handlebody of even genus. Then $M$ is homeomorphic to $\Sigma \times I$ where $\Sigma$ is a genus $g$ surface with one boundary component. Consider a discrete faithful representation $\rho : \pi_1(M) = F_{2g} \to \text{PSL}(2, \mathbb{C})$ such that the boundary curve of $\Sigma$ corresponds to a parabolic element and at least one end is degenerate. Then we can see that $\rho$ is not primitive stable using the following argument which is also due to Minsky.

First, we can see that the covering of $M$ corresponding to a free factor of $F_{2g}$ is convex cocompact by using Canary’s covering theorem [10]. This shows that if we restrict $\rho$ to a proper free factor, then the representation is Schottky. Noting that every non-peripheral non-separating simple closed curve on $\Sigma$ is primitive, suppose that $\{p_i\}$ is a sequence of such primitive simple closed curves converging to the ending lamination of $M$, whose geodesic representatives exit the end. If we suppose $\rho$ is primitive stable, then a line passing through the identity element determined by the conjugacy class of $p_i$ in the Cayley graph is mapped to a uniform $(K, d)$-geodesic in $H^3$ passing through a fixed point $o$ since the $p_i$ are primitive. Then the geodesic lines homotopic to its image by $\tau_{\rho,o}$ cannot be far from $o$, hence their images in $M$ are near the projection of $o$. This contradicts to the fact that they exit the end.

We note that in this example the ending lamination, which we denote by $\lambda$, is not disc-busting. In fact, $\lambda$ is contained in the Hausdorff limit of meridians in the form of $a_i \times \{0\} \cup \partial a_i \times I \cup a_i \times \{1\}$, where the $a_i$ are essential arcs on $\Sigma$ whose Hausdorff limit contains $\lambda$.

Proof of Theorem 1.2(sufficiency). If there is no ending lamination $\lambda_i$, then by the same argument as the proof of Theorem 4.1 in Minsky [29], we see that $\rho$ is primitive stable.
If there is at least one ending lamination, then we consider Λ as before, and repeating the same argument as in the case without parabolics replacing Corollary 3.2 with Corollary 3.4, we complete the proof. □

By Minsky’s result that the restriction of a primitive stable representation to a proper free factor of the free group is Schottky, we immediately get

**Corollary 4.2.** Let $M = \mathbb{H}^3/\rho(F)$ as in Theorem 1.1 or 1.2. If $F_n = A * B$ into two nontrivial free factors, then the covering manifold corresponding to $A$ or $B$ is convex cocompact, i.e., Schottky.

Also as a corollary of Theorems 1.1 and 1.2, we get

**Corollary 4.3.** Let $M = \mathbb{H}^3/\rho(F)$ as in Theorem 1.1 or 1.2. Then there is a compact set in $M$ containing all the closed geodesics representing primitive elements.

**Proof.** This follows from the definition of primitive stability. If ρ is primitive stable, there are uniform $K, \epsilon$ such that for any primitive word $w_n$, the line $τ_{ρ,o}(\tilde{w}_n)$ is $(K, \epsilon)$-quasi-geodesic. This implies that there is a uniform $L$ such that $τ_{ρ,o}(\tilde{w}_n)$ and the geodesic line connecting their ideal endpoints are within Hausdorff distance $L$. By projecting them down to $M$, we see that any primitive closed geodesic stays in the $L$-neighbourhood of the projection of $τ_{ρ,o}(Γ_F)$, which is compact. □

5. **Necessary condition**

In this section, we shall prove the necessity part of Theorem 1.2. We shall first show the following lemma giving a necessary condition about parabolic curves.

**Lemma 5.1.** If there is a parabolic curve on $∂H$ which is disjoint from a meridian, then $ρ$ is not primitive stable.

**Proof.** Suppose that $c$ is a parabolic curve which is disjoint from a meridian $m$. If $m$ is separating, then it bounds a separating compressing disc $D$ which gives rise to a free-product decomposition of $F = π_1(H)$. One of the free factors which contains an element corresponding to $c$ is not Schottky. By Minsky’s result in [29] stated as (3) in our Introduction, this shows that $ρ$ is not primitive stable. For the case when $m$ is non-separating, we need the following lemma.

**Lemma 5.2.** Suppose that $c$ is simple closed curve on $∂H$ which is disjoint from a non-separating meridian $m$. Then there is a simple closed curve $d$ on $∂H$ which is homotopic to $c$ in $H$ and is disjoint from a separating meridian $m'$.

**Proof.** Let $D$ be a compressing disc bounded by $m$, which is disjoint from $c$. Since $m$ is non-separating, $∂H \setminus m$ is connected. If $c$ does not separate the two open ends of $∂H \setminus m$, there is a simple closed curve $e$ on $∂H \setminus c$ with
i(e, m) = 1. If c separates the two open ends of \( \partial H \setminus \partial D \), we take e with 
\[ i(e, c) = i(e, m) = 1. \]
Let \( N \subset H \) be a regular neighbourhood of \( D \cup e \). Then 
\[ \Delta := \text{Fr} N \] 
is a separating compressing disc.

If e is disjoint from c, then c is disjoint from \( \Delta \) and we are done by setting 
\[ d = c \text{ and } m' = \partial \Delta. \] 
Otherwise we take a component \( k \) of \( e \setminus (c \cup m) \), which is an open arc connecting c and m, and consider a regular neighbourhood 
\[ N' \subset H \] of \( c \cup k \cup D \) in H. Its frontier \( \text{Fr} N' \) is the union of a compressing disc 
isotopic to D and an essential annulus A. One component of \( \partial A \) is isotopic to c on \( \partial H \), and the other component is homotopic to c in H and is disjoint from \( \Delta \) (hence from \( m' \)). Setting d to be the latter component of \( \partial A \) and 
\[ m' = \partial \Delta, \] we complete the proof. 

This reduces the case when m is non-separating to that when m is separating, and we have completed the proof of Lemma 5.1. 

For a given hyperbolic structure on \( \partial H \), we denote by \( S(\mu) \) the unique minimal compact subsurface of \( \partial H \) with geodesic boundaries containing a geodesic lamination \( \mu \) in the following sense: there is a well defined (up to isotopy) minimal essential subsurface that contains \( \mu \) and that surface is homotopic to a unique surface with geodesic boundary. Notice that \( S(\mu) \) may not be embedded since two of its boundary components may coincide, but \( S(\mu) \setminus \partial S(\mu) \) is embedded. For example if there is only one simple closed curve c that is disjoint from \( \mu \) then \( S(\mu) \) is obtained by taking the metric completion of \( \partial H - c \) and mapping it to \( \partial H \) in the obvious way, thus it has two boundary components which are both mapped to c.

Next we shall deal with the ending laminations. Thanks to the following lemma, this will mostly consist in applying the previous corollary to parabolic curves which lie on the boundary of the minimal supporting surface of ending laminations.

**Lemma 5.3.** Let \( \lambda \) be a disc-dodging ending lamination of \( \rho \). Then either a component of \( \partial S(\lambda) \) is disc-dodging or \( H \) is homeomorphic to an I-bundle in such a way that \( S(\lambda) \) is a component of the corresponding \( \partial I \)-bundle.

**Proof.** Suppose that \( \lambda \) is a disc-dodging ending lamination. Since \( \lambda \) is an ending lamination, it is minimal and is not a simple closed curve. If a component of \( \partial S(\lambda) \) is disc-dodging, we are done. Therefore, we may assume that every component of \( \partial S(\lambda) \) is disc-busting, hence in particular that every component of \( \partial H \setminus \partial S(\lambda) \) is incompressible.

Since \( \lambda \) is disc-dodging, there exists a sequence of meridians \( m_i \) such that 
\[ i(m_i, \lambda) \rightarrow 0. \]
We take a subsequence so that \( \{m_i\} \) converges in the Hausdorff topology to a geodesic lamination \( \mu \) on \( \partial H \). By Casson’s criterion (see Casson-Long [11], Otal [39], and [22, Theorem B1]), \( \mu \) contains a homoclinic leaf \( h \). By [20, Lemma 8], we see that \( S(\overline{h}) \) contains a meridian, where \( \overline{h} \) denotes the closure of \( h \). Since every component of \( \partial H \setminus \partial S(\lambda) \) is incompressible, this shows that \( h \) crosses \( \partial S(\lambda) \). In particular, we have 
\[ h \cap S(\lambda) \neq \emptyset. \]
Since \( i(m_i, \lambda) \to 0 \) and \( h \) is contained in the Hausdorff limit of \( m_i \), the leaf \( h \) does not intersect \( \lambda \) transversely. It follows that each component of \( h \cap S(\lambda) \) is asymptotic to a leaf of \( \lambda \) on \( \partial H \) since \( S(\lambda) \setminus \lambda \) contains no essential proper arcs. In particular, each component of \( h \cap S(\lambda) \) is a half-leaf of \( h \). This shows that \( h \cap S(\lambda) \) consists of either only one half-leaf \( h^+ \) or two half-leaves \( h^+, h^- \). In the former case, \( h \setminus \partial S(\lambda) \) has another half-leaf entirely contained in \( \partial H \setminus \text{Int} S(\lambda) \), which we also denote by \( h^- \). Let \( \tilde{h} \subset \tilde{H} \) be a lift of \( h \) and let \( \tilde{h}^\pm \subset \tilde{h} \) be lifts of \( h^\pm \) contained in \( \tilde{h} \). Since every component of \( \partial H \setminus \partial S(\lambda) \) is incompressible, \( \tilde{h}^\pm \) have well-defined endpoints \( \xi^\pm \in \partial_\infty \tilde{H} = \partial_\infty F \). Since \( \tilde{h} \) is homoclinic, we have \( \xi^+ = \xi^- \), which we denote by \( \xi \), and \( h \cup \xi \) is a Jordan curve separating \( \partial \tilde{H} \cup \partial_\infty \tilde{H} \) into two topological discs. We need now to consider two cases depending on whether \( \tilde{h}^+ \) and \( \tilde{h}^- \) are asymptotic on \( \partial \tilde{H} \) or not. Next we shall show that since we are assuming that \( \partial H \setminus \partial S(\lambda) \) is incompressible, the case when \( \tilde{h}^+ \) and \( \tilde{h}^- \) are asymptotic on \( \partial \tilde{H} \) cannot happen.

**Claim 5.4.** The half-geodesics \( \tilde{h}^+ \) and \( \tilde{h}^- \) are not asymptotic on \( \partial \tilde{H} \)

**Proof.** Seeking a contradiction, assume that \( \tilde{h}^+ \) and \( \tilde{h}^- \) are asymptotic on \( \partial \tilde{H} \). Then there is a sequence of geodesic arcs \( \tilde{k}_n \subset \partial \tilde{H} \) joining \( \tilde{h}^+ \) to \( \tilde{h}^- \) such that \( \tilde{k}_n \cap \tilde{h} = \partial \tilde{k}_n \) and the length of \( \tilde{k}_n \) goes to 0 (for the pull-back of some fixed hyperbolic metric on \( \partial H \)). Let \( k_n \) be the projection of \( \tilde{k}_n \) to \( \partial H \). Since \( \lambda \) is an ending lamination, we can put a transverse measure on \( \lambda \) with full support and regard \( \lambda \) as a measured lamination. Then we see that \( \int k_n \, d\lambda \to 0 \), where \( d\lambda \) denotes the transverse measure of \( \lambda \), as follows. Suppose, seeking a contradiction, that there exists \( \epsilon > 0 \) such that \( \int k_n \, d\lambda > \epsilon \). Since \( \partial H \) is compact, passing to a subsequence, the arcs \( k_n \) converge to a point \( p \) on \( \partial H \), and any transverse arc passing through \( p \) has to have measure greater than \( \epsilon \). This contradicts the fact that \( \lambda \) is minimal and is not a simple closed curve.

Since \( h \) is disjoint from \( \lambda \), the arcs \( k_n \) are homotopic without their endpoints passing through \( \lambda \), and hence \( \int k_n \, d\lambda \) does not depend on \( n \). It follows that \( \int k_n \, d\lambda = 0 \) and that \( k_n \) is disjoint from \( \lambda \) for every \( n \). By shortening \( h^\pm \) if necessary, we may assume that \( \partial \tilde{h}^\pm \) lie on \( \partial \tilde{k}_1 \). Then \( \tilde{d} = \tilde{k}_1 \cup (\tilde{h} \setminus \tilde{h}^\pm) \) bounds a disc in \( \tilde{H} \) which is disjoint from the preimage of \( \lambda \) in \( \tilde{H} \) since \( \tilde{H} \) is simply connected. Hence the projection \( d \) of \( \tilde{d} \) to \( \partial H \) bounds a (possibly not embedded) disc which is disjoint from \( \lambda \). It follows then from Dehn's lemma that there is a compressing disc for \( H \) which is disjoint from \( \lambda \). Hence each component of \( \partial S(\lambda) \) is also disc-dodging. This contradicts the assumption that every component of \( \partial S(\lambda) \) is disc-busting. \( \square \)

As we have seen, \( \tilde{h}^+ \) and \( \tilde{h}^- \) have the same endpoints. We shall follow the ideas of [OtP], as explained in [22, §2.3], to construct an essential \( I \)-bundle \( W \subset H \) such that \( S(\lambda) \) is a component of the corresponding \( \partial I \)-bundle.
Claim 5.5. There is an essential I-bundle \( W \subset H \) such that \( S(\lambda) \) is a component of the corresponding \( \partial I \)-bundle.

Proof. Let us fix a convex cocompact representation \( \rho : F_n \to \text{PSL}(2, \mathbb{C}) \), and a homeomorphism from \( H \) to \( \mathcal{N}(\rho) \). The surface \( \partial H \) is endowed with the induced hyperbolic structure. Recall that \( S(\lambda) \) is a surface with geodesic boundary containing \( \lambda \) and that two different boundary components may coincide. We denote by \( S^- \) the component of \( \partial H \setminus \partial S(\lambda) \) containing \( h^- \) and set \( S^+ = S(\lambda) \setminus \partial S(\lambda) \). We may have \( S^+ = S^- \).

Let \( \tilde{S}^\pm \subset \partial \tilde{H} \) be the closure of the lift of \( S^\pm \) containing \( \tilde{h}^\pm \). Since \( \partial H - \partial S(\lambda) \) is incompressible, the closure of \( \tilde{S}^\pm \) in \( \partial \tilde{H} \cup \partial_\infty \tilde{H} \) is a disc (see [22, Lemma 2.4] for example). In particular, since \( \tilde{h}^+ \) and \( \tilde{h}^- \) have the same endpoint, if \( \tilde{S}^+ = \tilde{S}^- \), \( \tilde{h}^+ \) and \( \tilde{h}^- \) are asymptotic on \( \tilde{S}^+ \) and hence on \( \partial \tilde{H} \).

Thus, by Claim 5.4, we have \( \tilde{S}^+ \neq \tilde{S}^- \).

By assumption, the closure of \( \tilde{S}^+ \) and \( \tilde{S}^- \) intersect. We consider in \( \tilde{S}^\pm \) the convex hull \( \tilde{C}^\pm \) of \( S^\pm \). It follows from [22, Affirmation 2.5] that the projection \( C^\pm \) of \( \tilde{C}^\pm \) to \( \partial H \) is a compact surface (or a simple closed geodesic, see [21, Affirmation 2.1.4]). If we denote by \( \Gamma^\pm \subset \rho(F_n) \) the stabiliser of \( \tilde{S}^\pm \), then it is now easy to see that \( \Gamma^+ \cap \Gamma^- \) is exactly the stabiliser of \( \tilde{C}^+ \) and \( \tilde{C}^- \) (in particular, they have the same stabiliser). Any element \( g \in \Gamma^+ \cap \Gamma^- \) corresponds to closed curves \( g^\pm \subset C^\pm \). Then \( g^+ \) and \( g^- \) are homotopic and thus bounds an annulus which is not necessarily embedded. It follows then from the characteristic submanifold theory in [13, 16] that \( g^\pm \) lie in the boundary of a characteristic \( I \)-pair relative to \( S^+ \cup S^- \). Since this applies to any element \( g \) of \( \Gamma^+ \cap \Gamma^- \), we see that \( C^\pm \) lie in the boundary of a characteristic \( I \)-pair \( W \) relative to \( S^+ \cup S^- \).

Let us go back to the case which we are considering here, namely \( \tilde{S}^\pm \) are closures of lifts of components of \( \partial H \setminus \partial S(\lambda) \). Since \( \xi^+ = \xi^- \), then \( h^+ \) eventually lies in \( C^+ \) and \( h^- \) eventually lies in \( C^- \). It follows that \( C^+ \supset S(\lambda) \) and that \( C^- \) is a the closure of a component of \( \partial H \setminus \partial S(\lambda) \). This is possible only if the boundary of \( W \) is \( S^+ \cup S^- \).

Thus we have finally proved that an essential \( I \)-bundle \( W \subset H \) such that \( S(\lambda) \) is a component of the corresponding \( \partial I \)-bundle.

It will follow from the next lemma that all the components of \( \partial W \) are boundary-parallel.

Lemma 5.6. For any essential annulus \( A \) in \( H \), there is a meridian which is disjoint from \( A \).

Proof. What we want to show is that there is a compressing disc for \( H \) disjoint from \( A \). Let \( D \) be a compressing disc for \( H \). We isotope \( D \) so that there is no inessential intersection between \( D \) and \( A \). If \( D \cap A = \emptyset \), then we are done. Suppose not. We consider an arc \( k \) in \( D \cap A \) which is outermost in \( D \) and cuts off a semi-disc \( \Delta \) from \( D \). If \( k \) connects the same component of \( \partial A \), then it cuts off a disc \( \Delta' \) from \( A \) and \( \Delta \cup \Delta' \) is a compressing disc.
which can be isotoped off $A$. If $k$ connects two components of $\partial A$, then we can boundary-compress $A$ along $\Delta$, and get a compressing disc disjoint from $A$. □

Notice that since any small neighbourhood of a Möbius band embedded in $H$ contains an annulus, this lemma also shows that any essential Möbius band has a meridian from which it is disjoint.

By assumption, each component of $\partial H \cap \partial W$ is disc-busting. Hence by Lemma 5.6, no component of $\partial W \setminus \partial H$ is essential. Since they are all incompressible, they must be boundary parallel. It follows that $H$ is obtained by gluing solid tori along the component of $\partial W \setminus \partial H$. Let $T$ be such a solid torus, which is glued to $\partial W \setminus \partial H$ along an annulus $A \subset \partial T$. Since $A$ is boundary-parallel, a core curve of $A$ is homotopic to a core curve of $T$. It follows that $W$ can be changed by a homotopy so that we have $W = H$. This concludes the proof of Lemma 5.3. □

Now we shall turn Lemma 5.3 into the following lemma which will be used to prove the necessity part of Theorem 1.2

**Lemma 5.7.** Let $\lambda$ be a disc-dodging ending lamination, then either a component of $\partial S(\lambda)$ is disc-dodging or $\lambda$ is a Hausdorff limit of a sequence of disc-dodging simple closed curves on $\partial H$.

**Proof.** Starting with Theorem 1.2, we only need to show that when $H$ is a homeomorphic to an $I$-bundle in such a way that $S(\lambda)$ is a component of the corresponding $\partial I$-bundle, then $\lambda$ is a Hausdorff limit of a sequence of disc-dodging simple closed curves on $\partial H$. We shall construct a sequence of annuli and use Lemma 5.6 to prove the following.

**Claim 5.8.** Let $c \subset S(\lambda)$ be a simple closed curve. Then $c$ is disc-dodging.

**Proof.** Let us start with the case when $W$ is the trivial $I$-bundle $S(\lambda) \times I$. Taking the union of the fibres above $c$, we get an essential annulus. It follows then from Lemma 5.6 that $c$ is disc-dodging.

In the other case when $W$ is a twisted $I$-bundle, it is doubly covered by the trivial $I$-bundle $S(\lambda) \times I$. The two lifts of $c$ to $S(\lambda) \times I$ are $c' = c \times \{0\}$ and $c'' = c \times \{1\}$. In particular $c' \cup c''$ bounds an essential annulus. By Lemma 5.6, there is an essential disc $D'$ which is disjoint from $c'$ and $c''$. Then $\partial D'$ projects down in $W$ to an essential curve which is disjoint from $c_i$ and is contractible in $H$. In particular $\partial H \setminus c_i$ is compressible and $c_i$ is disc-dodging. □

Take any sequence of simple closed curves $c_i$ that converges in the Hausdorff topology to $\lambda$. For $i$ large enough, $c_i \subset S(\lambda)$. By Claim 5.8, $c_i$ is disc-dodging for $i$ large enough. This concludes the proof of Lemma 5.7. □

**Proof of Theorem 1.2 (necessity).** By Lemma 5.1 if $\rho$ is primitive stable, then any parabolic curve is disc-busting. Therefore we have only to show
that every ending lamination is disc-busting. Suppose, seeking a contradiction, that there is a disc-dodging ending lamination \( \lambda \) whereas \( \rho \) is primitive stable. Since all component of \( \partial S(\lambda) \) are parabolics, they are disc-busting. Therefore, there is a sequence of disc-dodging simple closed curves \( c_i \) whose Hausdorff limit is \( \lambda \) by Lemma 5.7.

Since \( c_i \) is disc-dodging, there is a meridian \( m_i \) disjoint from \( c_i \). Suppose first that \( m_i \) is separating. Let \( D_i \) be a compressing disc bounded by \( m_i \). Then \( D_i \) separates \( H \) into two handlebodies \( H_1 \) and \( H_2 \) one of which, say \( H_2 \), contains \( c_i \). We take a primitive closed curve \( d_i \) contained in \( H_1 \). We fix some arcs connecting a basepoint to \( c_i \) and \( d_i \), and regard them as elements in \( \pi_1(H) = F \). We can then consider an element of \( F = \pi_1(H) \) represented as \( d_i c_i^{n_i} \), which is also primitive. By choosing a sufficiently large \( n_i \) for each \( i \), we can make the closed geodesic \( c_i^* \) in \( \mathbb{H}^3/\rho(F) \) representing \( d_i c_i^{n_i} \) pass a very thin neighbourhood of the closed geodesic representing \( c_i \), which we denote by \( c_i^* \). Since \( c_i \) converges to the ending lamination \( \lambda \), the closed geodesic \( c_i^* \) tends to an end. This shows that the closed geodesics \( c_i^* \) representing primitive classes do not stay in a compact set. This implies that \( \rho \) is not primitive stable by the proof of Corollary 4.3.

In the case when \( m_i \) is non-separating, by Lemma 5.2, we can find a simple closed curve \( c'_i \) on \( \partial H \) homotopic to \( c_i \) in \( H \) which is disjoint from a separating meridian \( m'_i \). Since the closed geodesic in \( \mathbb{H}^3/\rho(F) \) representing \( c_i \) also represents \( c'_i \), the argument above works by replacing \( c_i \) with \( c'_i \). Thus in either case, we get a contradiction. \( \Box \)

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