Representations of the Quantum Algebra $su_q(1,1)$ and Discrete $q$-Ultraspherical Polynomials

Valentyna GROZA

National Aviation University, 1 Komarov Ave., Kyiv, 03058 Ukraine
E-mail: groza@i.com.ua

Received September 16, 2005, in final form November 09, 2005; Published online November 15, 2005

Abstract. We derive orthogonality relations for discrete $q$-ultraspherical polynomials and their duals by means of operators of representations of the quantum algebra $su_q(1,1)$. Spectra and eigenfunctions of these operators are found explicitly. These eigenfunctions, when normalized, form an orthonormal basis in the representation space.

Key words: Quantum algebra $su_q(1,1)$; representations; discrete $q$-ultraspherical polynomials

2000 Mathematics Subject Classification: 17B37; 33D45

1 Representations of $su_q(1,1)$ with lowest weights

The aim of this paper is to study orthogonality relations for the discrete $q$-ultraspherical polynomials and their duals by means of operators of representations of the quantum algebra $su_q(1,1)$. Throughout the sequel we always assume that $q$ is a fixed positive number such that $q < 1$. We use (without additional explanation) notations of the theory of special functions and the standard $q$-analysis (see, for example, [1]).

The quantum algebra $su_q(1,1)$ is defined as an associative algebra, generated by the elements $J_+, J_-, q^{J_0}$ and $q^{-J_0}$, subject to the defining relations

$$
q^{J_0}q^{-J_0} = q^{-J_0}q^{J_0} = 1,
$$
$$
q^{J_0} J_\pm q^{-J_0} = q^{\pm 1} J_\pm,
$$
$$
[J_-, J_+] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}},
$$

and the involution relations $(q^{J_0})^* = q^{J_0}$ and $J_+^* = J_-$. (We have replaced $J_-$ by $-J_-$ in the common definition of the algebra $U_q(sl_2)$; see [2, Chapter 3].)

We are interested in representations of $su_q(1,1)$ with lowest weights. These irreducible representations are denoted by $T_{l}^+$, where $l$ is a lowest weight, which can be any complex number (see, for example, [3]). They act on the Hilbert space $\mathcal{H}$ with the orthonormal basis $|n\rangle$, $n = 0, 1, 2, \ldots$. The representation $T_{l}^+$ can be given in the basis $|n\rangle$, $n = 0, 1, 2, \ldots$, by the formulas

$$
q^{\pm J_0} |n\rangle = q^{\pm (l+n)} |n\rangle,
$$
$$
J_+ |n\rangle = \frac{q^{-(n+l-1/2)/2}}{1-q} \sqrt{(1-q^{n+1})(1-q^{2l+n})} |n+1\rangle,
$$
$$
J_- |n\rangle = \frac{q^{-(n+l-3/2)/2}}{1-q} \sqrt{(1-q^{n})(1-q^{2l+n-1})} |n-1\rangle.
$$
For positive values of $l$ the representations $T_l^+$ are $*$-representations. For studying discrete $q$-ultraspherical polynomials we use the representations $T_l^+$ for which $q^{2l-1} = -a$, $a > 0$. They are not $*$-representations. But we shall use operators of these representations which are symmetric or self-adjoint. Note that $q^l$ is a pure imaginary number.

## 2 Discrete $q$-ultraspherical polynomials and their duals

There are two types of discrete $q$-ultraspherical polynomials [4]. The first type, denoted as $C_n^{(a)}(x; q)$, $a > 0$, is a particular case of the well-known big $q$-Jacobi polynomials. For this reason, we do not consider them in this paper. The second type of discrete $q$-ultraspherical polynomials, denoted as $\tilde{C}_n^{(a)}(x; q)$, $a > 0$, is given by the formula

$$
\tilde{C}_n^{(a)}(x; q) = (-i)^n C_n^{(-a)}(ix; q) = (-i)^n 3\phi_2 \left( q^{-n}, -a q^{n+1}, ix; i\sqrt{aq}, -i\sqrt{aq}; q, q \right).
$$

(Here and everywhere below under $\sqrt{a}$, $a > 0$, we understand a positive value of the root.)

The polynomials $\tilde{C}_n^{(a)}(x; q)$ satisfy the recurrence relation

$$
x \tilde{C}_n^{(a)}(x; q) = A_n \tilde{C}_{n+1}^{(a)}(x; q) + C_n \tilde{C}_{n-1}^{(a)}(x; q),
$$

where

$$
A_n = \frac{1 + aq^{n+1}}{1 + aq^{2n+1}}, \quad C_n = A_n - 1 = \frac{aq^{n+1}(1 - q^n)}{1 + aq^{2n+1}}.
$$

Note that $A_n \geq 1$ and, hence, coefficients in the recurrence relation satisfy the conditions $A_n C_{n+1} > 0$ of Favard’s characterization theorem for $n = 0, 1, 2, \ldots$. This means that these polynomials are orthogonal with respect to a positive measure. Orthogonality relation for them is derived in [4]. We give here an approach to this orthogonality by means of operators of representations $T_l^+$ of $su_q(1, 1)$.

Dual to the polynomials $C_n^{(a)}(x; q)$ are the polynomials $D_n^{(a)}(\mu(x; a)|q)$, where $\mu(x; a) = q^{-x} + aq^{x+1}$. These polynomials are a particular case of the dual big $q$-Jacobi polynomials, studied in [5], and we do not consider them. Dual to the polynomials $\tilde{C}_n^{(a)}(x; q)$ are the polynomials

$$
\tilde{D}_n^{(a)}(\mu(x; -a)|q) := 3\phi_2 \left( q^{-x}, -a q^{x+1}, q^{-n}; i\sqrt{aq}, -i\sqrt{aq}; q, -q^{n+1} \right).
$$

For $a > 0$ these polynomials satisfy the conditions of Favard’s theorem and, therefore, they are orthogonal. We derive an orthogonality relation for them by means of operators of representations $T_l^+$ of $su_q(1, 1)$.

## 3 Representation operators $I$ and $J$

Let $T_l^+$ be the irreducible representation of $su_q(1, 1)$ with lowest weight $l$ such that $q^{2l-1} = -a$, $a > 0$ (note that $a$ can take any positive value). We consider the operator

$$
I := \alpha q^{l_0/4} (J_+ A + AJ_-) q^{l_0/4}
$$

of the representation $T_l^+$, where $\alpha = (a^2/q)^{1/2}(1 - q)$ and

$$
A = \frac{q^{-l_0/2+2}/2 \sqrt{(1 - a^2 q^{l_0-1+1})(1 + aq^{l_0-l_0+1})}}{\sqrt{(1 + a^2 q^{2l_0-2l_0+1})(1 + a^2 q^{2l_0-2l_0+2})(1 + a^2 q^{2l_0-2l_0+3})}}.
$$
We have the following formula for the symmetric operator $I$:

$$I\ket{n} = a_n\ket{n+1} + a_{n-1}\ket{n-1},$$  \hspace{1cm} (5)

$$a_{n-1} = \left(a^2q^{n+1}\right)^{1/2} \frac{(1-q^n)(1+a^2q^n)}{(1+a^2q^{2n-1})(1+a^2q^{2n+1})}^{1/2}.$$  

The operator $I$ is bounded. We assume that it is defined on the whole representation space $\mathcal{H}$. This means that $I$ is a self-adjoint operator. Actually, $I$ is a Hilbert–Schmidt operator since $a_{n+1}/a_n \to q^{1/2}$ when $n \to \infty$. Thus, a spectrum of $I$ is simple (since it is representable by a Jacobi matrix with $a_n \neq 0$), discrete and have a single accumulation point at 0 (see Chapter VII).

To find eigenvectors $\psi_\lambda$ of the operator $I$, $I\psi_\lambda = \lambda\psi_\lambda$, we set

$$\psi_\lambda = \sum_{n} \beta_n(\lambda)\ket{n}.$$  

Acting by $I$ upon both sides of this relation, one derives

$$\sum_{n} \beta_n(\lambda)(a_n\ket{n+1} + a_{n-1}\ket{n-1}) = \lambda \sum_{n} \beta_n(\lambda)\ket{n},$$  

where $a_n$ are the same as in (5). Collecting in this identity factors at $\ket{n}$ with fixed $n$, we obtain the recurrence relation for the coefficients $\beta_n(\lambda)$: $a_n\beta_{n+1}(\lambda) + a_{n-1}\beta_{n-1}(\lambda) = \lambda\beta_n(\lambda)$. Making the substitution

$$\beta_n(\lambda) = \left[(a^2q;q)_n(1+a^2q^{2n+1})/(q;q)_n(1+a^2q)a^{2n}\right]^{1/2} q^{-n(n+3)/4} \beta'_n(\lambda)$$

we reduce this relation to the following one

$$A_n\beta'_{n+1}(\lambda) + C_n\beta'_{n-1}(\lambda) = \lambda\beta'_n(\lambda),$$

where

$$A_n = \frac{1+a^2q^{n+1}}{1+a^2q^{2n+1}}, \quad C_n = \frac{a^2q^{n+1}(1-q^n)}{1+a^2q^{2n+1}}.$$  

It is the recurrence relation for the discrete $q$-ultraspherical polynomials $\tilde{C}_n^{(a^2)}(\lambda;q)$. Therefore, $\beta'_n(\lambda) = \tilde{C}_n^{(a^2)}(\lambda;q)$ and

$$\beta_n(\lambda) = \left(\frac{(-a^2q;q)_n(1+a^2q^{2n+1})}{(q;q)_n(1+a^2q)a^{2n}}\right)^{1/2} q^{-n(n+3)/4} \tilde{C}_n^{(a^2)}(\lambda;q).$$ \hspace{1cm} (6)

For the eigenfunctions $\psi_\lambda(x)$ we have the expansion

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} \left(\frac{(-a^2q;q)_n(1+a^2q^{2n+1})}{(q;q)_n(1+a^2q)a^{2n}}\right)^{1/2} q^{-n(n+3)/4} \tilde{C}_n^{(a^2)}(\lambda;q)|n\rangle.$$ \hspace{1cm} (7)

Since a spectrum of the operator $I$ is discrete, only a discrete set of these functions belongs to the Hilbert space $\mathcal{H}$ and this discrete set determines the spectrum of $I$.

We intend to study the spectrum of $I$. It can be done by using the operator

$$J := q^{-J_0+1} - a^2 q^{J_0-l+1}.$$
In order to determine how this operator acts upon the eigenvectors $\psi_\lambda$, one can use the $q$-difference equation

\begin{align*}
(q^{-n} - a^2 q^{n+1})\tilde{C}_n^{(a^2)}(\lambda; q) &= -a^2 q \lambda^{-2}(\lambda^2 + 1) \tilde{C}_n^{(a^2)}(q\lambda; q) \\
&+ \lambda^{-2} a^2 q (1 + q) \tilde{C}_n^{(a^2)}(\lambda; q) + \lambda^{-2} (\lambda^2 - a^2 q^2) \tilde{C}_n^{(a^2)}(q^{-1}\lambda; q)
\end{align*}

for the discrete $q$-polynomials polynomials. Multiply both sides of (8) by $d_n |n\rangle$, where $d_n$ are the coefficients of $\tilde{C}_n^{(a^2)}(\lambda; q)$ in the expression (9), for the coefficients $\beta_n(\lambda)$, and sum up over $n$. Taking into account formula (7) and the fact that $J |n\rangle = (q^{-n} - a^2 q^{n+1}) |n\rangle$, one obtains the relation

\[ J \psi_\lambda = -a^2 q \lambda^{-2}(\lambda^2 + 1)\psi_\lambda + \lambda^{-2} a^2 q (1 + q) \psi_\lambda + \lambda^{-2} (\lambda^2 - a^2 q^2) \psi_{q^{-1}\lambda} \]

which is used below.

\section{Spectrum of $I$ and orthogonality of discrete $q$-ultraspherical polynomials}

Let us analyse a form of the spectrum of $I$ by using the representations $T^+_1$ of the algebra $su_q(1,1)$ and the method of paper [7]. If $\lambda$ is a spectral point of $I$, then (as it is easy to see from (9)) a successive action by the operator $J$ upon the eigenvector $\psi_\lambda$ leads to the vectors $\psi_{q^{-m}\lambda}$, $m = 0, \pm 1, \pm 2, \ldots$. However, since $I$ is a Hilbert–Schmidt operator, not all these points can belong to the spectrum of $I$, since $q^{-m}\lambda \to \infty$ when $m \to \infty$ if $\lambda \neq 0$. This means that the coefficient $\lambda^{-2}(\lambda^2 - a^2 q^2)$ at $\psi_{q^{-1}\lambda}$ in (9) must vanish for some eigenvalue $\lambda$. There are two such values of $\lambda$: $\lambda = aq$ and $\lambda = -aq$. Let us show that both of these points are spectral points of $I$. We have

\[ \tilde{C}_n^{(a^2)}(aq; q) = 2\phi_1(q^{-n}, a^2 q^{n+1}; -aq; q, q) = a^2 q^{n+1}. \]

Likewise,

\[ \tilde{C}_n^{(a^2)}(-aq; q) = a^2 q^{n+1}. \]

Hence, for the scalar product $\langle \psi_{aq}, \psi_{aq} \rangle$ in $\mathcal{H}$ we have the expression

\begin{align*}
\sum_{n=0}^{\infty} \frac{(1 + a^2 q^{2n+1})(-a^2 q; q)_n}{(1 + a^2 q)(q; q)_n a^{2n} q^{(n+3)/2}} \tilde{C}_n^{(a^2)}(aq; q)^2 &= \frac{(-a^2 q^2, -q; q)_\infty}{(-a^2 q^2; q^2)_\infty} \\
&= \frac{(-a^2 q^3, q^2)_\infty}{(q; q^2)_\infty} < \infty.
\end{align*}

Similarly,

\[ \langle \psi_{-aq}, \psi_{-aq} \rangle = (-a^2 q^2, -1; q)_\infty/(-a^2 q^2; q^2)_\infty < \infty. \]

Thus, the values $\lambda = aq$ and $\lambda = -aq$ are the spectral points of $I$.

Let us find other spectral points of the operator $I$. Setting $\lambda = aq$ in (9), we see that the operator $J$ transforms $\psi_{aq}$ into a linear combination of the vectors $\psi_{aq^2}$ and $\psi_{aq}$. We have to show that $\psi_{aq^2}$ belongs to the Hilbert space $\mathcal{H}$, that is, that

\[ \langle \psi_{aq^2}, \psi_{aq^2} \rangle = \sum_{n=0}^{\infty} \frac{(1 + a^2 q^{2n+1})(-a^2 q; q)_n}{(1 + a^2 q)(q; q)_n a^{2n} q^{-n(n+3)/2}} \tilde{C}_n^{(a^2)}(aq^2; q)^2 < \infty. \]
It is made in the same way as in the case of big $q$-Jacobi polynomials in paper [1]. The above inequality shows that $\psi_{aq^2}$ is an eigenvector of $I$ and the point $aq^2$ belongs to the spectrum of $I$. Setting $\lambda = aq^2$ in (2) and acting similarly, one obtains that $\hat{\psi}_{aq^3}$ is an eigenvector of $I$ and the point $aq^3$ belongs to the spectrum of $I$. Repeating this procedure, one sees that $\psi_{aq^n}, n = 1, 2, \ldots,$ are eigenvectors of $I$ and the set $aq^n, n = 1, 2, \ldots,$ belongs to the spectrum of $I$. Likewise, one concludes that $\psi_{-aq^n}, n = 1, 2, \ldots,$ are eigenvectors of $I$ and the set $-aq^n, n = 1, 2, \ldots,$ belongs to the spectrum of $I$. Let us show that the operator $I$ has no other spectral points.

The vectors $\psi_{aq^n}$ and $\psi_{-aq^n}, n = 1, 2, \ldots,$ are linearly independent elements of the representation space $\mathcal{H}$. Suppose that $aq^n$ and $-aq^n, n = 1, 2, \ldots,$ constitute the whole spectrum of $I$. Then the set of vectors $\psi_{aq^n}$ and $\psi_{-aq^n}, n = 1, 2, \ldots,$ is a basis of $\mathcal{H}$. Introducing the notations $\Xi_n := \psi_{aq^{n+1}}$ and $\Xi'_n := \psi_{-aq^{n+1}}, n = 0, 1, 2, \ldots,$ we find from (9) that

$$J \Xi_n = -q^{-2n-1}(1 + a^2q^{2(n+1)}) \Xi_{n+1} + d_n \Xi_n - q^{-2n}(1 - q^{2n}) \Xi_{n-1},$$

$$J \Xi'_n = -q^{-2n-1}(1 + a^2q^{2(n+1)}) \Xi'_{n+1} + d_n \Xi'_n - q^{-2n}(1 - q^{2n}) \Xi'_{n-1},$$

where $d_n = q^{-2n-1}(1 + q)$.

As we see, the matrix of the operator $J$ in the basis $\Xi_n, \Xi'_n, n = 0, 1, 2, \ldots,$ is not symmetric, although in the initial basis $\{n\}, n = 0, 1, 2, \ldots,$ it was symmetric. The reason is that the matrix $M := (a_{mn} a'_{m'n'})$ with entries

$$a_{mn} := \beta_m(aq^{n+1}), \quad a'_{m'n'} := \beta_{m'}(-aq^{n'+1}), \quad m, n, m', n' = 0, 1, 2, \ldots,$$

where $\beta_m(dq^{n+1}), d = \pm a$, are coefficients in the expansion

$$\psi_{aq^{n+1}} = \sum_m \beta_m(dq^{n+1}) |n\rangle,$$

is not unitary. (This matrix $M$ is formed by adding the columns of the matrix $(a'_{m'n'})$ to the columns of the matrix $(a_{mn})$ from the right.) It maps the basis $\{n\}$ into the basis $\{\psi_{aq^{n+1}}, \psi_{-aq^{n+1}}\}$ in the representation space. The nonunitarity of the matrix $M$ is equivalent to the statement that the basis $\Xi_n, \Xi'_n, n = 0, 1, 2, \ldots,$ is not normalized. In order to normalize it we have to multiply $\Xi_n$ by appropriate numbers $c_n$ and $\Xi'_n$ by numbers $c'_n$. Let

$$\hat{\Xi}_n = c_n \Xi_n, \quad \hat{\Xi'}_n = c'_n \Xi'_n, \quad n = 0, 1, 2, \ldots,$$

be a normalized basis. Then the operator $J$ is symmetric in this basis and has the form

$$J \hat{\Xi}_n = -c_{n+1}^{-1} c_n q^{-2n-1}(1 + a^2q^{2(n+1)}) \hat{\Xi}_{n+1} + d_n \hat{\Xi}_n - c_{n-1}^{-1} c_n q^{-2n}(1 - q^{2n}) \hat{\Xi}_{n-1},$$

$$J \hat{\Xi'}_n = -c_{n+1}^{-1} c'_n q^{-2n-1}(1 + a^2q^{2(n+1)}) \hat{\Xi'}_{n+1} + d_n \hat{\Xi'}_n - c'_{n-1}^{-1} c'_n q^{-2n}(1 - q^{2n}) \hat{\Xi'}_{n-1}.$$

The symmetricity of the matrix of the operator $J$ in the basis $\{\hat{\Xi}_n, \hat{\Xi'}_n\}$ means that for coefficients $c_n$ we have the relation

$$c_{n+1}^{-1} c_n q^{-2n-1}(1 + a^2q^{2(n+1)}) = c_{n-1}^{-1} c_n q^{-2n-2}(1 - q^{2(n+1)}).$$

The relation for $c'_n$ coincides with this relation. Thus,

$$\frac{c_n}{c_{n-1}} = \frac{c'_n}{c'_{n-1}} = \sqrt{\frac{q(1 + a^2q^{2n})}{1 - q^{2n}}}.$$

This means that

$$c_n = C \left( \frac{q^n(-a^2q^2; q^2)_n}{(q^2; q^2)_n} \right)^{1/2}, \quad c'_n = C' \left( \frac{q^n(-a^2q^2; q^2)_n}{(q^2; q^2)_n} \right)^{1/2},$$

where $C$ and $C'$ are some constants.
Therefore, in the expansions

\[ \hat{\xi}_n \equiv \sum_m \hat{a}_{mn} |m\rangle, \quad \hat{\xi}_n(x) \equiv \sum_m \hat{a}'_{mn} |m\rangle \]

the matrix \( \hat{M} \equiv (\hat{a}_{mn} \hat{a}'_{mn}) \) with entries \( \hat{a}_{mn} = c_n \beta_m(aq^n) \) and \( \hat{a}'_{mn} = c_n \beta_m(eq^n) \) is unitary, provided that the constants \( C \) and \( C' \) are appropriately chosen. In order to calculate these constants, one can use the relations \( \sum_{m=0}^{\infty} |a_{mn}|^2 = 1 \) and \( \sum_{m=0}^{\infty} |a'_{mn}|^2 = 1 \) for \( n = 0 \). Then these sums are multiples of the sum in (10), so we find that

\[ C = C' = \left( \frac{(-a^2 q^2; q^2)_\infty}{(-a^2 q^2, -q; q)_\infty} \right)^{1/2} = \left( \frac{(q; q^2)_\infty}{(-a^2 q^2; q^2)_\infty} \right)^{1/2}. \]

The orthogonality of the matrix \( \hat{M} \equiv (\hat{a}_{mn} \hat{a}'_{mn}) \) means that

\[ \sum_m \hat{a}_{mn} \hat{a}_{mn'} = \delta_{nn'}, \quad \sum_m \hat{a}'_{mn} \hat{a}'_{mn'} = \delta_{nn'}, \quad \sum_m \hat{a}_{mn} \hat{a}'_{mn'} = 0, \quad \sum_n (\hat{a}_{mn} \hat{a}_{mn'} + \hat{a}'_{mn} \hat{a}'_{mn'}) = \delta_{nn'}. \tag{11} \]

Substituting the expressions for \( \hat{a}_{mn} \) and \( \hat{a}'_{mn} \) into (12), one obtains the relation

\[
\sum_{n=0}^{\infty} \frac{(-a^2 q^2; q^2)_n q^n}{(q^2; q^2)_n} \left[ \tilde{C}(a^2) \left( aq^{n+1} \right) \tilde{C}'(a^2) \left( aq^{n+1} \right) + \tilde{C}(a^2) \left( -aq^{n+1} \right) \tilde{C}'(a^2) \left( -aq^{n+1} \right) \right] \\
= \frac{(-a^2 q^2; q^2)_\infty}{(q; q^2)_\infty} \frac{(1 + a^2 q) (q; q)_m a^{2m} q^{n(n+3)/2}}{(1 + a^2 q^{2m+1}) (-a^2 q; q)_m} \delta_{nn'}. \tag{13}
\]

This identity must give an orthogonality relation for the discrete \( q \)-ultraspherical polynomials \( \tilde{C}(a^2) (y) \equiv \tilde{C}^{(a^2)}(y; q) \). An only gap, which appears here, is the following. We have assumed that the points \( aq^{n+1} \) and \( -aq^{n+1}, n = 0, 1, 2, \ldots \), exhaust the whole spectrum of the operator \( I \). If the operator \( I \) would have other spectral points \( x_k \), then on the left-hand side of (13) would appear other summands \( \mu_{x_k} \tilde{C}(a^2) (x_k; q) \tilde{C}'(a^2) (x_k; q) \), which correspond to these additional points. Let us show that these additional summands do not appear. For this we set \( m = m' = 0 \) in the relation (13) with the additional summands. This results in the equality

\[
\sum_{n=0}^{\infty} \frac{(aq, -aq; q)_n q^n}{(q, -q; q)_n} \sum_{n=0}^{\infty} \frac{(aq, -aq; q)_n q^n}{(q, -q; q)_n} + \sum_k \mu_{x_k} = 1. \tag{14}
\]

In order to show that \( \sum_k \mu_{x_k} = 0 \), take into account formula (2.10.13) in [1]. By means of this formula it is easy to show that the relation (14) without the summand \( \sum_k \mu_{x_k} \) is true. Therefore, in (14) the sum \( \sum_k \mu_{x_k} \) does really vanish and formula (13) gives an orthogonality relation for the discrete \( q \)-ultraspherical polynomials.

The relation (13) and the results of Chapter VII in [6] shows that the spectrum of the operator \( I \) coincides with the set of points \( aq^{n+1}, -aq^{n+1}, n = 0, 1, 2, \ldots \).
5 Dual discrete $q$-ultraspherical polynomials

Now we use the relations \([11]\). They give the orthogonality relation for the set of matrix elements $\tilde{a}_{mn}$ and $\tilde{a}'_{mn}$, considered as functions of $m$. Up to multiplicative factors, they coincide with the functions

$$F_n(x; a^2) := 3\phi_2(x, a^2q/x, aq^{n+1}; iaq, -iaq; q, q), \quad n = 0, 1, 2, \ldots,$$

$$F'_n(x; a^2) := 3\phi_2(x, a^2q/x, -aq^{n+1}; iaq, -iaq; q, q), \quad n = 0, 1, 2, \ldots,$$

considered on the corresponding sets of points.

Applying the relation (III.12) of Appendix III in [1] we express these functions in terms of dual $q$-ultraspherical polynomials. Thus we obtain expressions for $\tilde{a}_{mn}$ and $\tilde{a}'_{mn}$ in terms of these polynomials. Substituting these expressions into the relations (11) we obtain the following orthogonality relation for the polynomials $\tilde{D}_n^{(a^2)}(\mu(m; a^2); q)$:

$$\sum_{m=0}^{\infty} \frac{(1 + a^2q^{2m+1})(-a^2q; q)_m q^{m(m-1)/2} \tilde{D}_n^{(a^2)}(\mu(m; -a^2); q) \tilde{D}'_n^{(a^2)}(\mu(m; -a^2); q)}{(1 + a^2q)(q; q)_m} = \frac{2(-a^2q^3; q^2)_{\infty} (q^2; q^2)_n q^{-n}}{(q; q^2)_{\infty}} (-a^2q^2; q^2)_n \delta_{nn'}.$$

This orthogonality relation coincides with the sum of two orthogonality relations \([9]\) and \([10]\) in \([3]\). The orthogonality measure in \([15]\) is not extremal since it is a sum of two extremal measures.

In order to obtain orthogonality relations \([9]\) and \([10]\) of \([3]\), from the very beginning, instead of operator $I$, we have to consider operators $I_1$ and $I_2$ of the representation $T^+_l$ of the algebra $U_q(su_{1,1})$, which are appropriate for obtaining the orthogonality relations for the sets of polynomials $\tilde{C}_a^{(a)}(\sqrt{aq^{k+1}}; q)$, $k = 0, 1, 2, \ldots$, and $\tilde{C}_a^{(a)}(\sqrt{aq^{k+1}}; q)$, $k = 0, 1, 2, \ldots$, from Section 2 in \([4]\). Then going to the orthogonality relations for dual sets of polynomials in the same way as above, we obtain the extremal orthogonality relations \([9]\) and \([10]\) of \([3]\).

[1] Gasper G., Rahman M., Basic hypergeometric functions, Cambridge, Cambridge University Press, 1990.
[2] Klimyk A., Schmüdgen K., Quantum groups and their representations, Berlin, Springer, 1997.
[3] Burban I.M., Klimyk A.U., Representations of the quantum algebra $U_q(su_{1,1})$, J. Phys. A: Math. Gen., 1993, V.26, 2139–2151.
[4] Atakishiyev N.M., Klimyk A.U., On discrete $q$-ultraspherical polynomials and their duals, J. Math. Anal. Appl., 2005, V.306, N 2, 637–645; math.CA/0403159.
[5] Atakishiyev N.M., Klimyk A.U., On $q$-orthogonal polynomials, dual to little and big $q$-Jacobi polynomials, J. Math. Anal. Appl., 2004, V.294, N 2, 246–257; math.CA/0307250.
[6] Berezanskii Ju.M., Expansions in eigenfunctions of selfadjoint operators, Providence, RI, American Mathematical Society, 1968.
[7] Atakishiyev N.M., Klimyk A.U., Duality of $q$-polynomials, orthogonal on countable sets of points, math.CA/0411249.