A PURE HYDRODYNAMIC INSTABILITY IN SHEAR FLOWS AND ITS APPLICATION TO ASTROPHysical ACCRETION DISKS

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ABSTRACT

We provide a possible resolution for the century-old problem of hydrodynamic shear flows, which are apparently stable in linear analysis but shown to be turbulent in astrophysically observed data and experiments. This mismatch is noticed in a variety of systems, from laboratory to astrophysical flows. There are so many uncontrollable attempts made so far to resolve this mismatch, beginning with the early work of Kelvin, Rayleigh, and Reynolds toward the end of the nineteenth century. Here we show that the presence of stochastic noise, whose inevitable presence should not be neglected in the stability analysis of shear flows, leads to pure hydrodynamic linear instability therein. This explains the origin of turbulence, which has been observed/interpreted in astrophysical accretion disks, laboratory experiments, and direct numerical simulations. This is, to the best of our knowledge, the first solution to the long-standing problem of hydrodynamic instability of Rayleigh-stable flows.

Key words: accretion, accretion disks – hydrodynamics – instabilities – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

The astrophysically ubiquitous Keplerian accretion disks should be unstable and turbulent in order to explain observed data, but they are remarkably Rayleigh stable. They are found in active galactic nuclei, around newly formed stars, etc. (see, e.g., Pringle 1981). The main puzzle of accreting material in disks is its inadequacy of molecular viscosity to transport them toward the central object. Thus, the idea of turbulence and, hence, turbulent viscosity has been proposed. There is a similar issue in certain shear flows, e.g., plane Couette flow, which are shown to be linearly stable for any Reynolds number (Re) but in the laboratory could be turbulent for Re as low as 350. Therefore, linear perturbation cannot induce the turbulent viscosity to transport matter inward and angular momentum outward, in the Keplerian disks. Note that the issue of linear instability of the Couette–Taylor flow (when accretion disks are the subset of it) is a century-old problem.

Although in the presence of vertical shear and/or stratification Keplerian flow may reveal Rayleigh–Taylor-type instability (e.g., Nelson et al. 2013; Stoll & Kley 2014; Barker & Latter 2015; Lin & Youdin 2015; Richard et al. 2016; Stoll & Kley 2016; Umurhan et al. 2016), convective overstability (Klahr & Hubbard 2014; Lyra 2014), and the Zombie Vortex Instability (Marcus et al. 2013, 2015), we intend here to solve the classic century-old problem of the origin of linear instability with the exponential growth of perturbation in purely hydrodynamical Rayleigh-stable flows with only radial shear. The convective overstability does not correspond to an indefinitely growing mode, and it has some saturation (Latter 2016). In addition, the Zombie Vortex Instability is not sufficient to transport angular momentum significantly in a small domain of study. In fact, all of them could exhibit only a smaller Shakura–Sunyaev viscosity parameter (Shakura & Sunyaev 1973), \( \alpha_{\text{SS}} < 10^{-3} \), than that generally required to explain observation. The robustness of our work is that it can explain the turbulent behavior of any kind of Rayleigh-stable shear flows, starting from the laboratory to astrophysical flows.

While many realistic nonmagnetized and Keplerian flows could be stratified in both the vertical and radial directions of the disks, it is perhaps impossible to prove that all the nonmagnetized accretion disks have a significant amount of vertical shear and/or stratification to sustain the above-mentioned instabilities. Note that indeed many accretion disks are geometrically thin. Moreover, the laboratory Taylor-Couette flows have no vertical shear and/or stratification.

In 1991, with the application of magnetorotational instability (MRI; Velikhov 1959; Chandrasekhar 1960) to Keplerian disks, Balbus & Hawley (1991) showed that an initial weak magnetic field can lead to the perturbations growing exponentially. Within a few rotation times, such exponential growth could reveal the onset of turbulence. However, for charge neutral flows MRI should not work. Note also that for flows having strong magnetic fields, where the magnetic field is tightly coupled with the flow, MRI is not expected to work (e.g., Nath & Mukhopadhyay 2015).

It is a long-standing controversy (see, e.g., Dauchot & Daviaud 1995; Richard & Zahn 1999; Gu et al. 2000; Kim & Ostriker 1999; Rudiger & Zhang 2001; Klahr & Bodenheimer 2003; Yecko 2004; Afshordi et al. 2005; Dubrulle et al. 2005a, 2005b; Mukhopadhyay et al. 2005, 2011; Mahajan & Krishan 2008; Mukhopadhyay & Chattopadhyay 2013) whether the matter in Rayleigh-stable astrophysical disks is stable or unstable. The answer has profound significance for our understanding of how stars and planets form. It is argued, however, that some types of Rayleigh-stable flows certainly can be destabilized (Bottin & Chaté 1998; Avila et al. 2011; Balbus 2011; Barkley et al. 2015). Based on “shearing sheet” approximation, without (Balbus et al. 1996; Hawley et al. 1999) and with (Lesur & Longaretti 2005) explicit viscosity, some authors attempted to tackle the issue of turbulence in hot accretion disks. However, other authors argued for limitations in this work (Pumir 1996; Fromang & Papaloizou 2007). Based on the simulations including explicit viscosity, the authors could achieve \( Re \approx 4 \times 10^4 \) and concluded that Keplerian-like flows could exhibit very weak turbulence in the absence of a magnetic field. Nevertheless, the
recent experimental results by Paoletti et al. (2012) clearly argued for a significant level of transport from hydrodynamics alone. Moreover, the results from direct numerical simulations (Avila 2012) and exploration of transient amplification, in otherwise linearly stable flows, with and without noise (e.g., Trefethen et al. 1993; Mukhopadhyay et al. 2005; Cantwell et al. 2010), also argued for (plausible) hydrodynamic instability and turbulence at low Re. Interestingly, accretion disks have huge Re (≥10^5) (Mukhopadhyay 2013), prompting the belief that they are hydrodynamically unstable.

We show here that linearly perturbed, apparently Rayleigh-stable flows driven stochastically can be made unstable even in the absence of any magnetic field. We also discuss why stochastic noise is inevitable in such flows. They exist in the flows under consideration inherently. We develop our theory following the seminal concept based on fluctuating hydrodynamics of randomly stirred fluid, pioneered by Forster et al. (1977), De Dominicis & Martin (1979), and Nelson et al. (2013), which, however, was never applied in the context of accretion flows or other shear flows. This work provides a new path of linear hydrodynamic instability of shear flows, which will have vast applications from accretion disks to laboratory flows, for the first time.

The plan of the paper is the following. In the next section, we introduce equations describing the system under consideration. Then Section 3 describes the evolution of various perturbations in stochastically driven hydrodynamic flows. Subsequently, we discuss the relevance of white noise in the context of shear flows in Section 4. Finally, we summarize with conclusions in Section 5. In the Appendix, we demonstrate in detail the generation of white noise from random walk, particularly in the present context.

2. EQUATIONS DESCRIBING PERTURBED ROTATING SHEAR FLOWS IN THE PRESENCE OF NOISE

The linearized Navier–Stokes equation in the presence of background plane shear (0, −x, 0) and angular velocity Ω ∝ r−α, with r being the distance from the center of the system, in a small section approximated as incompressible flow with −1/2 ≤ x ≤ 1/2, has already been established (Mukhopadhyay & Chattopadhyay 2013). Here, any length is expressed in units of the size L of the system in the x-direction, the time in units of Ω−1, the velocity in qQL (1 < q < 2), and other variables are expressed accordingly (see, e.g., Mukhopadhyay et al. 2005, 2011; Mukhopadhyay & Chattopadhyay 2013, for detailed description of the choice of coordinates in a small section). Hence, in dimensionless units, the linearized Navier–Stokes equation and continuity equation (for an incompressible flow) can be recast into the well-known Orr–Sommerfeld and Squire equations, but in the presence of stochastic noise and Coriolis force (Mukhopadhyay & Chattopadhyay 2013), given by

\[
\left( \frac{\partial}{\partial t} - x \frac{\partial}{\partial y} \right) \nabla^2 u + \frac{2}{q} \frac{\partial \zeta}{\partial z} = \frac{1}{Re} \nabla^4 u + \eta_1(x, t),
\]

\[
\left( \frac{\partial}{\partial t} - x \frac{\partial}{\partial y} \right) \zeta + \frac{\partial u}{\partial z} \left( 1 - \frac{2}{q} \right) = \frac{1}{Re} \nabla^2 \zeta + \eta_2(x, t),
\]

where \( \eta_{1,2} \) are the components of noise arising in the linearized system due to stochastic forcing such that \( \langle \eta_1(x, t) \eta_1(x', t') \rangle = D_1(x) \delta^3(x - x') \delta(t - t') \delta_{ij} \) (Nelson et al. 2013), where \( D_i(x) \) is a constant for white noise and \( i, j = 1, 2; u \) is the x-component of velocity perturbation vector and \( \zeta \) the x-component of vorticity perturbation vector.

Now, we can resort to a Fourier series expansion of \( u, \zeta \) and \( \eta_i \) as

\[
A(x, t) = \int \tilde{A}_{k,\omega} e^{i(k x - \omega t)} d^3k d\omega,
\]

where \( A \) can be any one of \( u, \zeta \) and \( \eta_i; k \) and \( \omega \) are the wavevector and frequency, respectively, in the Fourier space such that \( \mathbf{k} = (k_x, k_y, k_z) \) and \( |k| \).}

3. EVOLUTION OF PERTURBATION IN STOCHASTICALLY DRIVEN HYDRODYNAMIC ACCRETION FLOWS

Writing down Equations (1) and (2) in Fourier space by using Equation (3), and taking an ensemble average, we obtain the equations involving the evolution of mean values of perturbations in the presence of noise:

\[
2\pi k_x k^2 \frac{\partial \langle \hat{u}_{k,\omega} \rangle}{\partial k_x} = \left( i \omega k^2 - 4q k_x k_y - \frac{k^4}{Re} \right) \langle \hat{u}_{k,\omega} \rangle + \frac{2ikz}{q} \langle \hat{\zeta}_{k,\omega} \rangle - m\delta(k) \delta(\omega),
\]

\[
2\pi k_y \frac{\partial \langle \hat{\zeta}_{k,\omega} \rangle}{\partial k_x} = -ik_z \left( 1 - \frac{2}{q} \right) \langle \hat{u}_{k,\omega} \rangle + \left( i \omega - \frac{k^2}{Re} \right) \langle \hat{\zeta}_{k,\omega} \rangle + m\delta(k) \delta(\omega),
\]

where Fourier transformations of \( \eta_{1,2} \) are basically \( \delta(k) \delta(\omega) \) multiplied by a random number, and on ensemble average it appears to be a constant \( m \), which is the mean value of the white noise (we get \( m = 0 \) and \( m \neq 0 \) when the drift coefficient of the Brownian motion or Wiener process corresponding to the white noise is zero and nonzero, respectively; see the Appendix for details), and \( \langle \hat{u}_{k,\omega} \rangle, \langle \hat{\zeta}_{k,\omega} \rangle \) are the Fourier transforms of \( \langle u \rangle \) and \( \langle \zeta \rangle \), which are the mean or the ensemble averaged values of \( u \) and \( \zeta \), respectively.

3.1. Evolution of Vertical Perturbations

Now let us take the trial solutions, \( \langle u \rangle, \langle \zeta \rangle = u_0, \zeta_0 \exp(i(\alpha x - \beta z)) \), where \( u_0, \zeta_0 \) are the constant, in general complex, amplitudes of perturbation and \( \alpha = 0, 0, \alpha \) is a vertical wavevector (one should not confuse this \( \alpha \) with the Shakura–Sunyaev viscosity parameter). A vertical wavevector is chosen since it will be unaffected by shear (Balbus & Hawley 1991). This gives \( \langle \hat{u}_{k,\omega} \rangle, \langle \hat{\zeta}_{k,\omega} \rangle = u_0, \zeta_0 \delta(\alpha - k) \delta(\beta - \omega) \) (using Equation (3)). Substituting these trial solutions in Equations (4) and (5), integrating with respect to \( k \) and \( \omega \), we obtain

\[
\left( i\beta\alpha^2 - \frac{\alpha^4}{Re} \right) u_0 + 2i\alpha \zeta_0 - m = 0,
\]

\[
-\alpha(1 - \frac{2}{q})u_0 + \left( i\beta - \frac{\alpha^2}{Re} \right) \zeta_0 + m = 0.
\]
increases for any \( \xi = 1.5; \) however, we obtain almost the same results for other admissible values of \( \xi \).

3.1.1. Case I

Now eliminating \( m \) and assuming \( \xi_0 = iu_0 \), we obtain the dispersion relation
\[
(i\beta\alpha^2 - \frac{\alpha^4}{Re}) - \frac{2\alpha}{q} = i\alpha\left(1 - \frac{2}{q}\right) + \left(\beta + \frac{i\alpha^2}{Re}\right).
\] (8)

If we find any pair of \( \alpha \) and \( \beta \) satisfying Equation (8) for which the imaginary part of \( \beta \) is positive, then we can say that the mean value of perturbation is unstable. Equation (8) is the hydrodynamic counterpart of the dispersion relation obtained due to MRI (Balbus & Hawley 1991), leading to the avenue of pure hydrodynamic instability. For \( m = 0 \), from Equations (6) and (7), either \( u_0 \) and \( \xi_0 \) both turn out to be zero or there is no instability for nontrivial \( u_0 \) and \( \xi_0 \). Overall, \( m = 0 \) gives rise to stable solutions like the zero magnetic field for MRI.

Figure 1 shows the ranges of \( \alpha \) giving rise to linear instability. It is easy to understand that similar results could be obtained with the choice of unequal ensemble averages of white noise in Equations (6) and (7) and a more general phase difference between \( \xi_0 \) and \( u_0 \).

3.1.2. Case II

Now for a given \( u_0 \) and \( m \), after eliminating \( \xi_0 \) from Equations (6) and (7), we obtain a dispersion relation between \( \alpha \) and \( \beta \) as
\[
\alpha^2\beta^2 + i\beta\left(2\alpha^2 + \frac{m}{u_0}\right)
+ \left(\frac{2im\alpha}{u_0} - \frac{4\alpha^2}{q^2} + \frac{2\alpha^2}{q} - \frac{m\alpha^2}{u_0Re} - \frac{\alpha^6}{Re^2}\right) = 0,
\] (9)

which is second order in \( \beta \) and hence has two roots \( \beta_1 \) and \( \beta_2 \). If we find any pair of \( \alpha \) and \( \beta \) for which the imaginary part of \( \beta \) is positive, then we can say that the mean value of perturbation is unstable. For \( m = 0 \) in Equation (9), there is no instability, like the zero magnetic field for MRI.

In Figure 2, for \( m/u_0 = 10^8 \) and different values of \( Re \) above a certain value, we show that for Keplerian flows there are modes for which the mean values of perturbation are unstable. If the amplitude of perturbations decreases, the value of \( m/u_0 \) increases for any fixed nonzero \( m \), leading to a larger range of \( \alpha \) for instability. However, for \( m = 0 \), i.e., for the white noise with zero mean (which also corresponds to the hydrodynamic accretion flows without any noise), we obtain no such unstable modes. While modes are stable for smaller \( Re \), with the increase of \( Re \) they become unstable, the range of \( \alpha \) giving rise to instability increases with increasing \( Re \), and for \( Re \to \infty \) unstable modes arise all the way up to \( |\alpha| \to \infty \).

3.1.3. Case III

Now we assume, for simplicity and without loss of much generality, \( u_0 = \xi_0 = A_0 \). Then expressing \( \alpha \) in terms of \( m/A_0 \) from Equations (6) and (7), by means of a cubic equation, given by
\[
i\left(1 - \frac{2}{q}\right)\alpha^3 - \frac{m}{A_0}\alpha^2 + \frac{2i}{q}\alpha - \frac{m}{A_0} = 0
\] (10)

and supplemented by Equation (7), we obtain three roots of \( \beta \). Figure 3 shows that the first solution of \( \beta (\beta_1) \) exhibits unstable modes for any \( m/A_0 > 0 \) (however small the magnitude may be), which is also independent of \( Re \) (however, \( \beta_1 \) and \( \beta_3 \) need not be \( Re \) independent). Therefore, if we have any stochastic forcing with an arbitrarily small but fixed nonzero value of \( m \) (drift coefficient), we will always have unstable mean perturbation modes since \( A_0 \) can be made arbitrarily small.

3.1.4. Plane Couette Flow and Negative \( m \)

Figures 2 and 3 demonstrate instability for positive \( m \) and real \( \alpha \). However, negative \( m \) with the appropriate choice of \( \alpha \) (real or complex) could also lead to instability for Keplerian flows (the same is true for positive \( m \) and complex \( \alpha \)). For plane Couette flow, however, in order to demonstrate instability, either \( m \) has to be negative with real \( \alpha \) or \( \alpha \) has to be complex with positive \( m \). From Equation (9), for \( q \to \infty \) (i.e., plane Couette flow), we obtain the corresponding two possible dispersion relations as
\[
\beta = -i\frac{\alpha^2}{Re}, -i\frac{Re}{\alpha^2}m/u_0 + \alpha^4.
\] (11)

The second solution will lead to the instability for a negative \( m \) satisfying \( |m|/u_0 > \alpha^2/Re \). Note that for real \( \alpha \), there are always appropriate values of \( m \) leading to instability in both Keplerian and plane Couette flows. For negative \( m \), the Keplerian flows remain unstable up to \( |\alpha| \to 0 \), as shown in Figure 4, unlike the positive \( m \) cases.

3.2. Evolution of Perturbations with Spherical Modes

In this section we show that there are also other perturbation modes that are linearly unstable. Here we show this for spherical modes as an example. However, such perturbation modes might be taken only under the assumption that they do not get distorted much due to shear, which may not be completely correct (Afshordi et al. 2005; Mukhopadhyay et al. 2005; Nath & Mukhopadhyay 2015). Writing down Equations (4) and (5) for spherical waves (i.e., \( \kappa_x = \kappa_y = k_z = k/\sqrt{3} \)), we obtain the equations involving the evolution
As the mean values of perturbations in the presence of noise:

\[ 2\pi k^3 \frac{\partial \langle \tilde{u} \rangle_{k,\omega}}{\partial k} = \left( i \omega k^2 - \frac{4\pi k^2}{3} - \frac{k^4}{Re} \right) \langle \tilde{u} \rangle_{k,\omega} \]

\[ + \frac{2ik}{\sqrt{3}q} \langle \tilde{\zeta} \rangle_{k,\omega} + m \delta(k) \delta(\omega), \quad (12) \]

Substituting the trial solutions for \( \langle u \rangle \) and \( \langle \zeta \rangle \), as described above Equation (6) but replacing vertical \( \alpha \) by spherical \( \alpha \), in

\[ 2\pi k \frac{\partial \langle \tilde{\zeta} \rangle_{k,\omega}}{\partial k} = -\frac{ik}{\sqrt{3}q} \left( 1 - \frac{2}{q} \right) \langle \tilde{u} \rangle_{k,\omega} \]

\[ + \left( i\omega - \frac{k^2}{Re} \right) \langle \tilde{\zeta} \rangle_{k,\omega} + m \delta(k) \delta(\omega). \quad (13) \]
Equations (12) and (13), and integrating with respect to $k$ and \( \omega \), we obtain

\[
6\pi \alpha^2 u_0 = \left( i\beta \alpha^2 - \frac{4\pi\alpha^2}{3} - \frac{\alpha^3}{Re} \right) u_0 + \frac{2i\alpha}{\sqrt{3} q} \zeta_0 + m, \tag{14}
\]

\[
2\pi \zeta_0 = -\frac{i\alpha}{\sqrt{3}} \left( 1 - \frac{2}{q} \right) u_0 + \left( i\beta - \frac{\alpha^2}{Re} \right) \zeta_0 + m. \tag{15}
\]

### 3.2.1. Case I

Now eliminating \( m \) from Equations (14) and (15) and assuming \( \zeta_0 = i\zeta_0 \), we obtain the dispersion relation

\[
6\pi \alpha^2 - \left( i\beta \alpha^2 - \frac{4\pi\alpha^2}{3} - \frac{\alpha^3}{Re} \right) + \frac{2\alpha^2}{\sqrt{3} q} = 2n + \frac{i\alpha}{\sqrt{3}} \left( 1 - \frac{2}{q} \right) + \left( i\beta - \frac{\alpha^2}{Re} \right). \tag{16}
\]

Figure 5 shows the ranges of \( \alpha \) giving rise to linear instability. It is easy to understand that similar results could be obtained with the choice of unequal ensemble averages of white noise in Equations (14) and (15) and a more general phase difference between \( \zeta_0 \) and \( u_0 \).

### 3.2.2. Case II

For a given \( u_0 \) and \( m \), after eliminating \( \zeta_0 \) from Equations (14) and (15), we obtain a dispersion relation between \( \alpha \) and \( \beta \) for spherical perturbation:

\[
\alpha^2 \beta^2 + i\beta \left( \frac{2\alpha^3}{Re} + \frac{8\pi\alpha^2}{3} + \frac{m}{u_0} \right) + \frac{2}{3q} \left( 1 - \frac{2}{q} \right) \alpha^2 \\
- \left( 2\pi + \frac{\alpha^2}{Re} \right) \left( \frac{2\pi\alpha^2}{3} + \frac{\alpha^3}{Re} \right) \\
- \left( 2\pi + \frac{\alpha^2}{Re} - \frac{2i\alpha}{\sqrt{3} q} \right) \frac{m}{u_0} = 0. \tag{17}
\]

In Figure 6, for \( m/u_0 = 10^5 \) and different values of \( \text{Re} \), we show that for Keplerian flows, there are several spherical modes, for which the mean values of perturbations grow exponentially, as Figure 2 shows for vertical perturbations.

### 3.2.3. Case III

In Figure 7, we show how spherical perturbation modes in Keplerian flows vary with \( m/A_0 \). This is very similar to what is shown in Figure 3 for vertical perturbations, except that the modes are stable for a very small but nonzero \( m/A_0 \), while for vertical perturbation the modes remain unstable for \( m/A_0 \rightarrow 0 \).

### 3.2.4. Plane Couette Flow and Negative \( m \)

For plane Couette flows, making \( q \rightarrow \infty \) in Equation (17), we obtain the corresponding dispersion relation:

\[
\beta = -i\frac{2\pi \text{Re} + \alpha^2}{\text{Re}} - i\frac{3\text{Re}m/u_0 + 2\pi\text{Re}^2 + 3\alpha^4}{3\text{Re}^2}. \tag{18}
\]

While the first root always corresponds to the stable mode for a real \( \alpha \), the second one will lead to the unstable solution for a negative \( m \) satisfying \( |m|/u_0 > 2\pi\alpha^2/3 + \alpha^4/\text{Re} \). Figure 8 shows that for spherical perturbations, the Keplerian flows remain unstable up to \( |\alpha| \rightarrow 0 \), as shown in Figure 4 for vertical perturbation cases.

### 4. RELEVANCE OF WHITE NOISE IN THE CONTEXT OF SHEAR FLOWS

Now we shall discuss how relevant the white noise is and how likely it is to be present in shear flows. The Rayleigh-stable flows under consideration have a background shear profile, with some molecular viscosity, however small that may be, and hence some drag (e.g., in protoplanetary disks, it could be due to the drag between gas and solid particles). For plane Couette flow, such shear is driven in the fluids by moving the boundary walls by externally applied force. If the external force (cause) is switched off, the shearing motion (effect) dies out. Similarly, in accretion disks, the central gravitational force plays the role of driving force (cause) producing differential velocity (shear) in the flow. Hence, by the fluctuation–dissipation theorem of statistical mechanics (see, e.g., Miyazaki & Bedeaux 1995; Lukić et al. 2005), there must be some thermal fluctuations in such flows, with some temperature, however low it may be, and that cause the fluid particles to have Brownian motion. Therefore, the time variation (derivative) of this Brownian motion, which is defined as white noise, plays the role of the extra stochastic forcing term in the Orr–Sommerfeld equations (Equations (1) and (2)) which are present generically, in particular when perturbation is considered.

Now, due to the presence of background shear in some preferential direction, it is very likely for the fluid particles to have Brownian motion with nonzero drift, however small it may be. The detailed technical description of generation of white noise (with zero and nonzero mean) from Brownian motion has been included in the Appendix. Therefore, if \( X(t) \) is the random displacement variable of a Brownian motion with drift coefficient \( m \), its probability density function \( P(X(t)) \) can be written as

\[
P(X(t)) = \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ -\frac{(X(t) - mt)^2}{2\sigma^2} \right], \tag{19}
\]

where \( \sigma \sqrt{t} \) is the standard deviation of the distribution and \( t \) the time. Taking the stochastic time derivative of \( X(t) \), we obtain the white-noise process, which we denote by \( \eta(t) = \dot{X}(t) \). Since the stochastic variable \( X(t) \) is not differentiable in the
usual sense, we consider a finite-difference approximation of \( \eta(t) \) using a time interval of width \( \Delta t \) as
\[
\eta_{\Delta t}(t) = \frac{X(t + \Delta t) - X(t)}{\Delta t}.
\] (20)

Therefore, the presence of infinitesimal molecular viscosity (and shear), which is always there, would be enough just to give rise to a nonzero (infinitesimal) temperature, leading to thermal noise, which can do the rest of the job of governing instability. Note that a very tiny mean noise strength, due to tiny asymmetry in the system, is enough to lead to linear instability, as demonstrated in previous sections. Here, the externally applied force (for plane Couette) or the force arising due to the presence of a strongly gravitating object (accretion disk) introduces the asymmetry in the system, just like, e.g., the Brownian ratchets, which have several applications in soft condensed matter and biology (see, e.g., van Oudenaarden & Boxer 1999). The measure of asymmetry and drag determines the value of \( m \), which furthermore controls the growth rate of perturbation. The corresponding power spectrum appears to be
almost flat/constant (for ideal white noise it is purely flat). Although in our chosen shearing box the azimuthal direction is assumed to be periodic, every such small box always encounters drag and hence thermal fluctuation, which assures the presence of nonzero mean noise. As a result, every such sharing box reveals exponential growth of perturbation.

5. DISCUSSION AND CONCLUSIONS

We have shown that linearly perturbed, hydrodynamic, apparently Rayleigh-stable rotating shear flows, including accretion disks, and plane Couette flow, driven stochastically, can indeed be unstable, since the averaged values of the perturbations grow exponentially. Due to background shear and hence drag, thermal fluctuations arise in these flows that induce Brownian motion of the fluid particles and hence stochastic forcing by white noise. Therefore, the accretion flows, in particular due to perturbation, are inevitably driven by white noise, which cannot be neglected. It is indeed shown in experiments that the stochastic details decide whether turbulence will spread or eventually decay (Avila et al. 2011), which furthermore argues for the determining factor played by stochastic forcing, which we demonstrate here for the first time. Since the forcing term in this system is a random variable, the solutions of the perturbations $u(x, t)$, $\zeta(x, t)$ are also random variables and hence have some distributions whose averaged values are investigated. Hence, we have shown that even in the absence of a magnetic field, accretion disks can be made unstable and plausibly turbulent if they are driven by stochastic noise, which is very likely to be present in the disks due to thermal fluctuations. In fact, we argue that neglecting the stochastic noise in accretion flows and any other shear flows is vastly an inappropriate assumption. This is because some shear is always there (because those are always driven externally by definition), which leads to some temperature fluctuations. In other words, there is no such thing as a perfectly laminar flow.

Evidently this mechanism works for magnetized shear flows as well, because thermal fluctuations are available there also. For example, a background field of the order of unity with $B / N_0 \sim 10^8$ can easily lead to unstable modes of perturbation for $\alpha \gtrsim 5$ in the limit of very large Re and $m$, which is the case in accretion disks. In the future, we will report this result in detail. Indeed, earlier we studied stochastically driven magnetized flows and showed them to be plausibly unstable and turbulent by calculating the correlation functions of perturbations (Nath et al. 2013; Nath & Chattopadhyay 2014). Hence, the pure hydrodynamic instability explored here is generic. This is, to the best of our knowledge, the first solution to the century-old problem of hydrodynamic instability of apparently Rayleigh-stable flows. In due course, one has to investigate how exactly the required value of stochastic forcing strength could arise in real systems and whether the growth rates of unstable modes could adequately explain data. In certain cases, only high Re reveals instability, which might be difficult to achieve in laboratory experiments and numerical simulations as of now.

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APPENDIX

GENERATION OF WHITE NOISE FROM RANDOM WALK VIA BROWNIAN MOTION

We have assumed here that the white noise has a nonzero mean value.

The term white noise is ambiguous. To shed light on this matter, here we point out the two definitions of white noise. Brown (1983) defines it as

“... a stationary random process having a constant spectral density function.”

Papoulis (1991) defines it as follows:

“We shall say that a process $\nu(t)$ is white noise if $\nu(t_i)$ and $\nu(t_j)$ are uncorrelated for every $t_i$ and $t_j \neq t_i$; $C(t_i, t_j) = 0$, $t_i \neq t_j$.”

The following subsections explore the implications of each definition with respect to the mean of the resulting process.

A.1. White Noise as a Stochastic Process with Constant Power Spectral Density (Brown’s Definition)

Let $X(t)$ be an ergodic stochastic process with the property that it has a constant power spectral density, i.e.,

$$\Phi_{XX}(\omega) = \alpha,$$  \hspace{1cm} (21)

where $\Phi_{XX}(\omega)$ is the power spectral density of the random variable $X(t)$ and $\alpha$ is a constant. Then the corresponding autocorrelation function for the process is

$$E[X(t)X(t + \tau)] = \Phi_{XX}(\tau) = \alpha \delta(\tau),$$  \hspace{1cm} (22)

by taking inverse Fourier transform of $\Phi_{XX}(\omega)$, where $E[\cdot]$ denotes the expectation value. Now let us assume that $X(t)$ is a zero mean white-noise process and $Y(t) = X(t) + m$ is a nonzero mean process. Then

$$\Phi_{YY}(\tau) = E[Y(t)Y(t + \tau)] = E[(X(t) + m) \times (X(t + \tau) + m)] = \alpha \delta(\tau) + m^2.$$  \hspace{1cm} (23)

Therefore,

$$\Phi_{YY}(\omega) = \alpha + 2\pi m^2 \delta(\omega),$$  \hspace{1cm} (24)

which is not constant; thus, $Y(t)$ violates the requirement of the white-noise process by this definition.

A.2. White Noise as an Uncorrelated Stochastic Process (Papoulis’s Definition)

Representing Papoulis’s definition of white noise in our notations, we can write that a stochastic process $X(t)$ is called a

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A white-noise process if any two distinct random variables of this stochastic process are independent and uncorrelated, i.e., the autocovariance function $C(X(t), X(t + \tau)) = 0$ when $\tau \neq 0$. In mathematical notation,

$$C(X(t), X(t + \tau)) = E[(X(t) - m_t)(X(t + \tau) - m_{t+\tau})] = E[X(t)E[X(t + \tau)] - m_t m_{t+\tau} = m_t m_{t+\tau} - m_t m_{t+\tau} = 0,$$  \hspace{1cm} (25)

where $m_t$ and $m_{t+\tau}$ are the corresponding mean values of the random variables $X(t)$ and $X(t + \tau)$, respectively. We can write the second equality in Equation (25) since $X(t)$, for different values of $t$, are independent random variables by definition. Thus, it is not necessary for a white-noise process always to have a zero mean, from Papoulis’s definition. That is, a stochastic process having nonzero mean can be a white-noise process according to this definition. In the present work, we have used Papoulis’s definition of white noise, which can indeed have a nonzero mean. Now let us explain why we have chosen Papoulis’s definition over Brown’s definition.

A.3. Why Ideal White Noise, Having a Constant Power Spectral Density, Is Impossible in Reality

Let us consider a signal $f(t)$ with constant power spectral density $\Phi_f(\omega)$. That is,

$$\Phi_f(\omega) = E[f(t)f(t + \tau)], \hspace{1cm} (26)$$

and the Fourier transform of $\Phi_f(\tau)$ is $\Phi_f(\omega) \equiv \text{a constant.}$ Now, Parseval’s theorem tells us that

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\phi(\omega)|^2 d\omega, \hspace{1cm} (27)$$

where $\phi(\omega)$ is the Fourier transform of $f(t)$. Since $\Phi_f(\omega)$ and consequently $|\phi(\omega)|^2$ has a constant positive value according to Brown’s definition of white noise, Equation (27) tells us that the total power of the signal is infinity (see, e.g., Gardiner 1985; Kliemann & Namachchivaya 1995; Kuo 1996; Poor 2013; Zhong 2006). In mathematical terminology, the energy norm of the signal $f(t)$ is infinity, and hence the function $f(t)$ is not $L^2$-integrable. Therefore, driving a system by a stochastic noise with constant power spectral density is the same as injecting an infinite amount of energy into the system, which is unphysical (Gardiner 1985).

A.4. Generation of Brownian Motion (with Zero and Nonzero Drift) from Random Walk

In this section, we outline the derivation of white noise starting from the random walk, via Brownian motion. Figure 9 shows an array of positions $ja$ where $j = 0, \pm 1, \pm 2$, etc., and $a$ is the spacing between points. At each interval of time, $\tau$, a hop is made with probability $p$ to the right and $q = 1 - p$ to the left. The distribution of $r$, of hops to the right, in $N$ steps is given by the Bernoulli distribution

$$P_N(r) = \frac{N!}{r!(N-r)!} p^r q^{N-r}. \hspace{1cm} (28)$$

The first moment (mean) and the second moment (variance) of the Bernoulli distribution in Equation (28) are given by

$$\langle r \rangle = Np, \hspace{1cm} \langle (\Delta r)^2 \rangle = Npq. \hspace{1cm} (29)$$

A particle that started at 0 and took $r$ steps to the right and $N - r$ steps to the left arrives at the position

$$n = r - (N - r) = 2r - N, \hspace{1cm} (30)$$

with mean value

$$\langle n \rangle = N(2p - 1) = N(p - q). \hspace{1cm} (31)$$

Notice that if $p = q = 1/2$, or equal probability to jump to the right or the left, the average position after $N$ steps will remain 0. The second moment about the mean is given by

$$\langle (\Delta n)^2 \rangle = 4 \langle (\Delta r)^2 \rangle = 4Npq. \hspace{1cm} (32)$$

Therefore, from the central limit theorem, the limiting distribution after many steps is Gaussian, with the first and second moments just obtained (in Equations (31) and (32)), given by

$$P_N(n) = \frac{1}{\sqrt{2\pi(4Npq)}} \exp\left\{ -\frac{(n - N(p - q))^2}{8Npq} \right\}. \hspace{1cm} (33)$$

If we introduce the position and time variables by the relations

$$x = na, \hspace{1cm} N = t/\tau, \hspace{1cm} (34)$$

the moments of $x$ are given by

$$\langle x \rangle = N(p - q)a = (p - q)at/\tau, \hspace{1cm} \langle (\Delta x)^2 \rangle = 4Npaq^2 = \frac{4pq^2 \tau}{\tau} = 2D \tau. \hspace{1cm} (35)$$

The factor of 2 in the definition of diffusion coefficient $D$ is appropriate for one dimension and would be replaced by $2d$ if we consider the random walk in a space of dimension $d$. Thus, the distribution moves with a “drift” velocity

$$v = (p - q)a/\tau \hspace{1cm} (36)$$

and spreads with a diffusion coefficient defined by

$$D = \frac{2pq^2}{\tau}. \hspace{1cm} (37)$$

Thus, the probability distribution of the displacement $x$ of a particle under this random walk is

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x - vt)^2}{2\sigma^2} \right\}, \hspace{1cm} (38)$$

where $\sigma = \sqrt{2D}$. A stochastic process in which the random variables $X(t)$s are stationary and independent and have distribution as in Equation (38) is called a Brownian motion or Wiener process. It is very clear from Equation (36) that, when $p = q = 1/2$, the drift velocity is 0, which means that if some random walk is fully symmetric without any bias, then only we...
obtain the zero drift velocity of the corresponding Brownian motion (which is known as standard Brownian motion in literature). However, if some process has any asymmetry (for example, hydrodynamic flows with shear in a particular direction, bulk hydrodynamic flows, flows under gravity, etc.), the random walk of particles in that process will have some bias (i.e., \( p \neq q \)), which eventually introduces a Brownian motion with nonzero drift velocity.

**A.5. White Noise from Brownian Motion**

If we take the stochastic time derivative of a Brownian motion or Wiener process, we obtain a white-noise process. If \( X(t) \) is the random displacement variable of a Brownian motion with drift velocity \( m \), its probability density function \( P(X(t)) \) can be written as (using Equation (38))

\[
P(X(t)) = \frac{1}{\sqrt{2\pi\sigma t}} \exp\left[-\frac{(X(t) - mt)^2}{2\sigma^2 t}\right].
\]

(39)

where \( \sigma \sqrt{t} \) is the standard deviation of the distribution and \( t \) the time. Taking the stochastic time derivative of \( X(t) \), we obtain the white-noise process, which we denote by \( \eta(t) = \dot{X}(t) \). Since the stochastic variable \( X(t) \) is not differentiable in the usual sense, we consider a finite-difference approximation of \( \eta(t) \) using a time interval of width \( \Delta t \) as

\[
\eta_{\Delta t}(t) = \frac{X(t + \Delta t) - X(t)}{\Delta t}.
\]

(40)

Since the stochastic random variables \( X(t) \) corresponding to a Brownian motion process are stationary and independent, from Equations (39) and (40) we obtain that the white-noise process has mean/averaged value \( m \) and variance \( \sigma^2/\Delta t + 2\sigma^2t/\Delta t^2 \).

As \( \Delta t \to 0 \), the variance \( \sigma^2/\Delta t + 2\sigma^2t/\Delta t^2 \to \infty \), and this white noise tends to the ideal white noise having a constant power spectral density (Kuo 1996). However, since Brownian motion is not differentiable anywhere, the ideal white noise does not exist, as also explained above from the energy norm point of view.

Now we will show that the white noise defined in Equation (40) satisfies Papoulis’s definition of white noise, i.e., the process is an uncorrelated stochastic process. To establish this, let us first note that if \( X(t) \) and \( X(s) \) are two random variables of a Brownian motion with \( s \leq t \), then

\[
C(X(t), X(s)) = E[(X(t) - mt)(X(s) - ms)]
\]

\[
= E[(X(t) - mt)(X(s) - ms) + (X(s) - ms)(X(s) - ms)]
\]

\[
= E[(X(t) - X(s))(s - ms)]
\]

\[
= (X(s) - ms)\left[0 + \sigma^2s = \sigma^2\min\{t, s\}\right]
\]

\[
eq 0 + \sigma^2s = \sigma^2\min\{t, s\}.
\]

(41)

The third equality is possible since \( (X(t) - X(s)) \) and \( X(s) \) are independent random variables for a Brownian motion. Having the result of Equation (41) in hand, we now calculate the autocovariance of white noise. It is very easy to verify that the autocovariance function \( C(X, Y) \) of two random variables \( X \) and \( Y \) is a linear function in both of its arguments.

Therefore,

\[
C(\eta_{\Delta t}(t), \eta_{\Delta t}(s)) = C\left[\frac{X(t + \Delta t) - X(t)}{\Delta t}, \frac{X(s + \Delta t) - X(s)}{\Delta t}\right]
\]

\[
= \frac{1}{\Delta t^2} C(X(t + \Delta t), X(s + \Delta t))
\]

\[
- C(X(t + \Delta t), X(s))
\]

\[
- C(X(t), X(s + \Delta t))
\]

\[
+ C(X(t), X(s)).
\]

(42)

When \( |t - s| \leq \Delta t \), i.e., \( s - \Delta t \leq t \leq s + \Delta t \), then using Equation (41), from Equation (42) we obtain

\[
C(\eta_{\Delta t}(t), \eta_{\Delta t}(s)) = \frac{\sigma^2}{\Delta t^2}(\min\{t, s\}
\]

\[
+ \Delta t - s - t + \min\{t, s\})
\]

\[
= \frac{\sigma^2}{\Delta t}\left[1 - \left(\frac{s + t - 2\min\{t, s\}}{\Delta t}\right)\right]
\]

\[
= \frac{\sigma^2}{\Delta t}\left[1 - \left(\frac{|t - s|}{\Delta t}\right)\right].
\]

(43)

Now let us consider the cases when \( |t - s| \geq \Delta t \), i.e., when \( s + \Delta t \leq t \) or \( t + \Delta t \leq s \). For \( s + \Delta t \leq t \), Equations (41) and (42) imply

\[
C(\eta_{\Delta t}(t), \eta_{\Delta t}(s)) = \frac{\sigma^2}{\Delta t^2}(s + \Delta t - s - (s + \Delta t) + s)
\]

\[
= 0,
\]

(44)

and also for \( t + \Delta t \leq s \),

\[
C(\eta_{\Delta t}(t), \eta_{\Delta t}(s)) = \frac{\sigma^2}{\Delta t^2}(t + \Delta t - (t + \Delta t) - t + t)
\]

\[
= 0.
\]

(45)

Therefore,

\[
C(\eta_{\Delta t}(t), \eta_{\Delta t}(s)) = \frac{\sigma^2}{\Delta t}\left[1 - \left(\frac{|t - s|}{\Delta t}\right)\right], \text{ when } |t - s| \leq \Delta t
\]

\[
= 0, \text{ otherwise},
\]

(46)

i.e., \( \eta_{\Delta t}(t) \) and \( \eta_{\Delta t}(s) \) are uncorrelated. Let us define

\[
t - s = \tau, \text{ and, } \delta_{\Delta t}(\tau) = \frac{1}{\Delta t}\left(1 - \left(\frac{\tau}{\Delta t}\right)\right), \text{ when } |\tau| \leq \Delta t
\]

\[
= 0, \text{ otherwise}.
\]

(47)

Figure 10 shows the variation of \( \delta_{\Delta t}(\tau) \) for two different small values of \( \Delta t \). The function \( \delta_{\Delta t}(\tau) \) defined in Equation (47) is an approximation of the well-known delta function \( \delta(\tau) \), because

\[
\lim_{\Delta t \to 0} \delta_{\Delta t}(\tau) = \delta(\tau),
\]
as is seen from Figure 2. Also the function $\delta_\Delta(\tau)$ satisfies the integral property of the delta function as shown below,

$$
\int_{-\infty}^{\infty} \delta_\Delta(\tau) d\tau = \int_{-\infty}^{\infty} \frac{1}{\Delta t} \left( 1 - \frac{|\tau|}{\Delta t} \right) d\tau = \frac{1}{\Delta t} \int_{-\Delta t}^{\Delta t} \left( 1 + \frac{\tau}{\Delta t} \right) d\tau + \frac{1}{\Delta t} \int_{0}^{\Delta t} \left( 1 - \frac{\tau}{\Delta t} \right) d\tau = 1.
$$

Hence, when $\Delta t \to 0$, from Equation (46) we obtain

$$
C(\eta_\Delta(t), \eta_\Delta(s)) = \sigma^2 \delta(\tau) = \sigma^2 \delta(t - s).
$$

Therefore, the noise with nonzero mean, obtained from the stochastic time derivative of Brownian motion with nonzero drift, is a white-noise process according to Papoulis’s definition and also has the correlators as defined below Equation (2).

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