ON FREE CONFORMAL AND VERTEX ALGEBRAS

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Vertex algebras and conformal algebras have recently attracted a lot of attention due to their connections with physics and Moonshine representations of the Monster. See, for example, [6], [10], [15], [17], [19].

In this paper we describe bases of free conformal and free vertex algebras (as introduced in [6], see also [20]).

All linear spaces are over a field $k$ of characteristic 0. Throughout this paper $\mathbb{Z}^+$ will stand for the set of non-negative integers.

In §1 and §2 we give a review of conformal and vertex algebra theory. All statements in these sections are either in [9], [15], [16], [17], [18], [20] or easily follow from results therein. In §3 we investigate free conformal and vertex algebras.

1. Conformal algebras

1.1. Definition of conformal algebras. We first recall some basic definitions and constructions, see [15], [17], [18], [20]. The main object of investigation is defined as follows:

Definition 1.1. A conformal algebra is a linear space $C$ endowed with a linear operator $D : C \to C$ and a sequence of bilinear products $\odot : C \otimes C \to C$, $n \in \mathbb{Z}^+$, such that for any $a, b \in C$ one has

(i) (locality) There is a non-negative integer $N = N(a,b)$ such that $a \odot_n b = 0$ for any $n \geq N$;

(ii) $D(a \odot b) = (Da) \odot b + a \odot (Db)$;

(iii) $(Da) \odot b = -na^{n-1}b$.

1.2. Spaces of power series. Now let us discuss the main motivation for the Definition 1.1. We closely follow [14] and [18].

1.2.1. Circle products. Let $A$ be an algebra. Consider the space of power series $A[[z,z^{-1}]]$. We will write series $a \in A[[z,z^{-1}]]$ in the form

$$a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}, \quad a(n) \in A.$$  

On $A[[z,z^{-1}]]$ there is an infinite sequence of bilinear products $\odot_n$, $n \in \mathbb{Z}^+$, given by

$$(a \odot_n b)(z) = \text{Res}_w (a(w)b(z)(z-w)^n).$$  

Explicitly, for a pair of series $a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$ and $b(z) = \sum_{n \in \mathbb{Z}} b(n)z^{-n-1}$ we have

$$(a \odot_n b)(z) = \sum_m (a \odot_n b)(m)z^{-m-1},$$

where

$$(a \odot_n b)(m) = \sum_{s=0}^n (-1)^s \binom{n}{s} a(n-s)b(m+s).$$  

There is also the linear derivation $D = d/dz : A[[z,z^{-1}]] \to A[[z,z^{-1}]]$. Easy to see $D$ and $\odot$ satisfy conditions (ii) and (iii) of Definition 1.1.

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We can consider formula (1.2) as a system of linear equations with unknowns \(a(k)b(l), \ k \in \mathbb{Z}_+, \ l \in \mathbb{Z}\). This system is triangular, and its unique solution is given by

\[
a(k)b(l) = \sum_{s=0}^{k} \binom{k}{s} (a \odot b)(k + l - s).
\]  

(1.3)

**Remark.** The term “circle products” appears in [18], where the product “\(\odot\)” is denoted by “\(\circ_n\)”. In [17] this product is denoted by “\((n)\)”.

1.2.2. **Locality.** Next we define a very important property of power series, which makes them form a conformal algebra. Let again \(A\) be an algebra.

**Definition 1.2.** (See [1], [15], [17], [18], [20]) A series \(a \in A[[z, z^{-1}]]\) is called **local of order** \(N\) to \(b \in A[[z, z^{-1}]]\) for some \(N \in \mathbb{Z}_+\) if

\[
a(w)b(z) (z - w)^N = 0.
\]

(1.4)

If \(a\) is local to \(b\) and \(b\) is local to \(a\) then we say that \(a\) and \(b\) are **mutually local**.

**Remark.** In [18] and [20] the property (1.4) is called **quantum commutativity**.

Note that (1.4) implies that for every \(n \geq N\) one has \(a \odot b = 0\). We will denote the order of locality by \(N(a, b)\), i.e.

\[
N(a, b) = \min\{n \in \mathbb{Z}_+ \mid \forall k \geq n, a \odot b = 0\}.
\]

Note also that if \(A\) is a commutative or skew-commutative algebra, e.g. a Lie algebra, then locality is a symmetric relation. In this case we say “\(a\) and \(b\) are local” instead of “mutually local”.

Let \(a(z) = \sum_{m \in \mathbb{Z}} a(m)z^{-m-1}\) and \(b(z) = \sum_{n \in \mathbb{Z}} b(n)z^{-n-1}\) be some series, then the locality condition (1.4) reads

\[
\sum_s (-1)^s \binom{N}{s} a(m-s)b(n+s) = 0 \quad \text{for any } n, m \in \mathbb{Z}.
\]

(1.5)

The locality condition (1.4) is known to be equivalent to formula

\[
a(m)b(n) = \sum_{s=0}^{N(a, b) - 1} \binom{m}{s} (a \odot b)(m + n - s).
\]

(1.6)

The following statement is a trivial consequence of the definitions.

**Proposition 1.1.** Let \(A\) be an algebra and let \(S \subset A[[z, z^{-1}]]\) be a space of pairwise mutually local power series, which is closed under all the circle products and \(\partial\). Then \(S\) is a conformal algebra.

One can prove (see, for example, [17]) that such families exhaust all conformal algebras.

Finally, we state here a trivial property of local series:

**Lemma 1.1.** Let \(a, b \in A[[z, z^{-1}]]\) be a pair of formal power series and assume \(a\) is local to \(b\). Then each of the series \(a, Da, za\) is local to each of \(b, Db, zb\).

1.3. **Construction of the coefficient algebra of a conformal algebra.** Given a conformal algebra \(C\) we can build its **coefficient algebra** \(\text{Coeff}\ C\) in the following way. For each integer \(n\) take a linear space \(\hat{A}(n)\) isomorphic to \(C\). Let \(\hat{A} = \bigoplus_{n \in \mathbb{Z}} \hat{A}(n)\). For an element \(a \in C\) we will denote the corresponding element in \(\hat{A}(n)\) by \(a(n)\). Let \(E \subset \hat{A}\) be the subspace spanned by all elements of the form

\[
(Da)(n) + na(n-1) \quad \text{for any } a \in C, \ n \in \mathbb{Z}.
\]

(1.7)

The underlying linear space of \(\text{Coeff}\ C\) is \(\hat{A}/E\). By abuse of notations we will denote the image of \(a(n) \in \hat{A}\) in \(\text{Coeff}\ C\) again by \(a(n)\). The following proposition defines the product on \(\text{Coeff}\ C\).

**Proposition 1.2.** Formula (1.6) unambiguously defines a bilinear product on \(\text{Coeff}\ C\).
Clearly \( 1.4.2 \) defines a product on \( \hat{A} \). To show that the product is well defined on \( \text{Coeff } C \) it is enough to check only that

\[
(Da)(m)b(n) = -ma(m-1)b(n) \quad \text{and} \quad a(m)\,(Db)(n) = -na(m)b(n-1),
\]

which is a straightforward calculation.

1.4. Examples of conformal algebras.

1.4.1. Differential algebras. Take a pair \((A, \delta)\), where \( A \) is an associative algebra, and \( \delta : A \to A \) is a locally nilpotent derivation:

\[
\delta(ab) = \delta(a)b + a\delta(b), \quad \delta^n(a) = 0 \quad \text{for } n \gg 0.
\]

Consider the ring \( A[\delta, \delta^{-1}] \). Its elements are polynomials of the form \( \sum_{i\in\mathbb{Z}} a_i\delta^i \), where only finite number of \( a_i \in A \) are nonzero. Here we put \( a\delta^{-n} = a(\delta^{-1})^n \) and \( a\delta^0 = a \). The multiplication is defined by the formula

\[
a\delta^k \cdot b\delta^l = \sum_{i\geq 0} \binom{k}{i} a\delta^i(b)\delta^{k+l-i}.
\]

It is easy to check that \( A[\delta, \delta^{-1}] \) is a well-defined associative algebra. In fact, \( A[\delta, \delta^{-1}] \) is the Ore localization of the ring of differential operators \( A[\delta] \). If in addition \( A \) has an identity element 1, then \( \delta(1) = 0 \) and \( \delta^{-1} = \delta^{-1} \delta = 1 \).

For \( a \in A \) denote \( \bar{a} = \sum_{n\in\mathbb{Z}} a\delta^n z^{-n-1} \in A[\delta, \delta^{-1}][[z, z^{-1}]] \).

One easily checks that for any \( a, b \in A \), \( \bar{a} \) and \( \bar{b} \) are local and

\[
\bar{a} \ast \bar{b} = a\delta^n(b).
\]

So by Lemma \( 1.2 \) and Proposition \( 1.3 \) series \( \{\bar{a} \mid a \in A\} \subset A[\delta, \delta^{-1}][[z, z^{-1}]] \) generate an (associative) conformal algebra, see \( 1.4.3 \).

One can instead consider \( A[\delta, \delta^{-1}] \) to be a Lie algebra, with respect to the commutator \([p, q] = pq - qp\). If two series \( \bar{a} \) and \( \bar{b} \) are local in the associative sense they are local in the Lie sense too. One computes also

\[
\bar{a} \ast \bar{b} = a\delta^n(b) - \sum_{s \geq 0} (-1)^{n+s} \frac{1}{s!} \partial^s (b\delta^{n+s}(a))^{-}, \tag{1.9}
\]

where \( \partial = d/dz \). Note that in \( 1.4.4 \) the circle products are defined by

\[
(\bar{a} \circ \bar{b})(m) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} [a\delta^{n-s}, b\delta^{m+s}]. \tag{1.10}
\]

Again, it follows that \( \{\bar{a} \mid a \in A\} \subset A[\delta, \delta^{-1}][[z, z^{-1}]] \) generate a (Lie) conformal algebra, see \( 1.4.3 \).

An important special case is when there is an element \( \nu \in A \) such that \( \delta(\nu) = 1 \). Then \( \bar{v} = \sum_{n} \nu\delta^n z^{-n-1} \in A[\delta, \delta^{-1}][[z, z^{-1}]] \) generates with respect to the product \( 1.10 \) a (centerless) Virasoro conformal algebra. It satisfies the following relations:

\[
\bar{v} \circ \bar{v} = \partial \bar{v}, \quad \bar{v} \ast \bar{v} = 2\bar{v},
\]

and the rest of the products are 0.

1.4.2. Loop algebras. Let \( \mathfrak{g} \) be a Lie algebra over an algebraically closed field \( \mathbb{k} \), and let \( \sigma : \mathfrak{g} \to \mathfrak{g} \) be an automorphism of finite order, \( \sigma^p = \text{id} \). Then \( \mathfrak{g} \) is decomposed into a direct sum of eigenspaces of \( \sigma \):

\[
\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}/p\mathbb{Z}} \mathfrak{g}_k, \quad \sigma \big|_{\mathfrak{g}_k} = e^{2\pi ik/p}.
\]

Define twisted loop algebra \( \mathfrak{g} \subset \mathfrak{g} \otimes \mathbb{k}[t, t^{-1}] \) by

\[
\mathfrak{g} = \left\{ \sum_{j} a_j t^j \mid a_j \in \mathfrak{g}_{j \mod p} \right\}.
\]

The Lie product in \( \mathfrak{g} \) is given by \([a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n}\). If \( p = 1 \), then \( \mathfrak{g} = \mathfrak{g} \otimes \mathbb{k}[t, t^{-1}] \), of course.
Now for any \( a \in \mathfrak{g}_k, \ 0 \leq k < p \), define
\[
\tilde{a} = \sum_{j \in \mathbb{Z}} a t^{pj+k} z^{-j-1} \in \mathfrak{g}[[z, z^{-1}]].
\]

Easy to see that any two \( \tilde{a}, \tilde{b} \) are local with \( N(\tilde{a}, \tilde{b}) = 1 \) and if \( a \in \mathfrak{g}_k \) and \( b \in \mathfrak{g}_l \) we have
\[
\tilde{a} \otimes \tilde{b} = \begin{cases} 
[a, b] & \text{if } k + l < p \\
\tilde{z}(a, b) & \text{if } k + l \geq p.
\end{cases}
\]

As in \([4,4.1]\), we conclude that \( \{\tilde{a} \mid a \in \mathfrak{g}\} \subset \mathfrak{g}[[z, z^{-1}]] \) generate a (Lie) conformal algebra. Again, see \([4.1]\) for the definition of varieties of conformal algebras.

1.5. More on coefficient algebras. Let \( C \) be a conformal algebra and let \( A = \text{Coeff} \ C \). Define
\[
A_+ = \text{Span}\{a(n) \mid a \in C, \ n \geq 0\}, \\
A_- = \text{Span}\{a(n) \mid a \in C, \ n < 0\}, \\
A(n) = \text{Span}\{a(n) \mid a \in C\}.
\]

Define also for each \( n \in \mathbb{Z} \) linear maps \( \phi(n) : C \rightarrow A(n) \) by \( a \mapsto a(n) \), and let \( \phi = \sum_{n \in \mathbb{Z}} \phi(n) z^{-n-1} : C \rightarrow A[[z, z^{-1}]] \) so that \( \phi a = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \).

Here we summarize some general properties of conformal algebras and their coefficient algebras.

**Proposition 1.3.** (a) \( A = A_- \oplus A_+ \) is a direct sum of subalgebras.
(b) \( A_+ \) and \( A_- \) are filtered algebras with filtrations given by
\[
A(0) \subseteq A(1) \subseteq \cdots \subseteq A_+, \quad A_- = A(-1) \supseteq A(-2) \supseteq \cdots
\]
\[
\bigcup_{n \geq 0} A(n) = A_+, \quad \bigcap_{n < 0} A(n) = 0.
\]
(c) \( \ker \phi(n) = \begin{cases} 
D^{n+1}C + \bigcup_{k \geq 1} \ker D^k & \text{if } n \geq 0 \\
\ker D^{-n-1} & \text{if } n < 0.
\end{cases} \)

In particular, \( \phi(-1) \) is injective.
(d) The map \( \phi : C \rightarrow A[[z, z^{-1}]] \), given by \( a \mapsto \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \) is an injective homomorphism of conformal algebras, i.e. it preserves the circle products and agrees with the derivation:
\[
\phi(a \otimes b) = \phi(a) \otimes \phi(b), \quad \phi(Da) = D\phi(a). \quad (1.11)
\]
(e) The map \( \phi : C \rightarrow A[[z, z^{-1}]] \) has the following universal property: for any homomorphism \( \psi : C \rightarrow B[[z, z^{-1}]] \) of \( C \) to an algebra of formal power series, there is the unique algebra homomorphism \( \rho : A \rightarrow B \) such that the corresponding diagram commutes:
\[
A[[z, z^{-1}]] \xrightarrow{\phi} B[[z, z^{-1}]] \\
\xrightarrow{\psi} C
\]
(f) The formula \( D(a(n)) = -na(n-1) \) defines a derivation \( D : A \rightarrow A \), such that \( DA_- \subset A_- \), \( DA_+ \subset A_+ \).

**Proof.** From formula \([1.6]\) for the product in \( A \) it easily follows that \( A_+ \) and \( A_- \) are indeed subalgebras. Also none of the linear identities \([1.7]\) contain both generators with negative and non-negative index. This proves (a). Similar arguments establish also (b).

Now we prove that \( \ker \phi(n) \) is included in the right-hand side of (c). Take some \( a \in C, \ a \neq 0 \), and assume that \( a(n) = 0 \). Then \( a(n) \in \tilde{A} \) is a linear combination of identities \([1.7]\) (see \([1.2]\)) so we must have in \( \tilde{A} \)
\[
a(n) = \sum_{k = k_{\text{min}}}^{k_{\text{max}}} \lambda_k ((Da_k)(k) + ka_k(k-1)).
\]
We can assume that $\lambda_k \neq 0$ for all $k_{\min} \leq k \leq k_{\max}$ and that $a_k \neq 0$ for $k = k_{\min}$ and for $k = k_{\max}$.

Assume also that $\lambda_k = 0$ if $k > k_{\max}$ or $k < k_{\min}$.

Comparing terms with index $k$ for $k_{\min} \leq k \leq k_{\max}$, we get

$$\delta_{kn} a = \lambda_k Da_k + \lambda_{k+1}(k+1)a_{k+1}.$$  

(1.12)

Taking in (1.12) $k = k_{\min} - 1$ we see that there are two cases: either (1) $k_{\min} = 0$ and $n \geq 0$ or (2) $n + 1 = k_{\min} \neq 0$.

Case (1): Taking in (1.12) $k = 0, \ldots, n - 1$ we get that $a_k \in D\kappa^k C$ for $0 \leq k \leq n$. Now we have two subcases: $k_{\max} > n$ and $k_{\min} \leq n$.

If $k_{\max} > n$ we substitute in (1.12) $k = k_{\max}, k_{\max} - 1, \ldots, n + 1$ and get that $D^{k_{\max} - k+1}a_k = 0$. Now take $k = n$ in (1.12) and get that $a \in D^{n+1}C + \text{Ker} D^{k_{\max} - n}$.

If $k_{\min} \leq n$ we have $\lambda_{n+1} = 0$, and hence substitution $k = n$ in (1.12) gives $a \in D^{n+1}C$.

Case (2): Here we again have two subcases: $n \geq 0$ or $n < 0$.

If $n \geq 0$ then as in the previous case, we get $D^{k_{\max} - k+1}a_k = 0$ for $n + 1 \leq k \leq k_{\max}$. Now taking $k = n$ in (1.12) we get $a \in \text{Ker} D^{k_{\max} - n}$.

Finally, if $n < 0$ then, since $\lambda_n = 0$, we have $a = \lambda_{n+1}(n+1)a_{n+1}$. Then we substitute $k = n + 1, n + 2, \ldots$ into (1.12) until for some $k_0 \leq -1$ we get $\lambda_{k_0+1}(k_0 + 1)a_{k_0+1} = 0$. It follows that $D^{k_0 - k+1}a_k = 0$ for $n + 1 \leq k \leq k_0$, therefore $a \in \text{Ker} D^{k_0 - n} \subset \text{Ker} D^{n-1}$. This proves one inclusion in (c). It also follows that $\text{Ker} \phi(-1) = 0$.

Next we show that $\phi$ is a homomorphism of conformal algebras, that is, formulas (1.11) hold. For the first identity we have

$$\phi(Da) = \sum_{n \in \mathbb{Z}} (Da)(n)z^{-n-1} = \sum_{n \in \mathbb{Z}} (-n)a(n-1)z^{-n-1} = d \sum_{n} a(n)z^{-n-1}.$$

The second identity reads

$$(a \otimes b)(m) = \sum_{s} (-1)^{s} {n \choose s} a(n-s)b(m+s),$$

which is precisely the formula (1.2).

Now we can prove the other inclusion in (c). If $a \in \text{Ker} D^k C$, then $\phi a$ is a solution of differential equation $\partial_z^k \phi a(z) = 0$, hence $\phi a$ is a polynomial of degree at most $k-1$, therefore $\phi(n)a = 0$ for $n \geq 0$ and for $n < -k$. If $a \in D^k C$, then $\phi(n)a = 0$ for $0 \leq n \leq k-1$, by induction and (1.7).

Statement (e) is clear, since identities (1.7) hold for any homomorphism $\psi : C \rightarrow \mathbb{B}[z, z^{-1}]$.

Finally, the formula $D(a(n)) = -na(n-1)$ defines a derivative of $\widehat{A}$. So in order to prove (f) we have to show that $D$ agrees with the identities (1.7). This is indeed the case:

$$D((Da)(n) + na(n-1)) = -n((Da)(n-1) + (n-1)a(n-2)).$$

\qed

1.6. Varieties of conformal algebras. Consider now a variety of algebras $\mathfrak{A}$ (see [8], [13]).

**Definition 1.3.** A conformal algebra $C$ is a $\mathfrak{A}$-conformal algebra if $\text{Coeff} C$ lies in the variety $\mathfrak{A}$.

The identities in $\mathfrak{A}$-conformal algebras are all the circle-products identities $R$ such that for any integer $m$, $R(m)$ becomes an $\mathfrak{A}$-algebra identity after substitution of (1.2) for every circle product in $R$. Conversely, given a classical algebra identity $r$ we can substitute (1.4) for all products in $r$ and get an identity of $\mathfrak{A}$-conformal algebras. This way we get a correspondence between classical and conformal identities. See the next section for examples.

Combining Proposition 1.1 and (d) of Proposition 1.3, we get the following well-known fact:

**Proposition 1.4.** $\mathfrak{A}$-conformal algebras are exhausted (up to isomorphism) by conformal algebras of formal power series $S \subset A[z, z^{-1}]$ for $\mathfrak{A}$-algebras $A$. 

\qed
1.7. **Associative and Lie conformal algebras.** The following theorem gives the explicit correspondence between conformal and classical algebras in some important cases.

**Theorem 1.1** (See [16]). Let $C$ be a conformal algebra and $A = \text{Coeff } C$ its coefficient algebra.

(a) $A$ is associative if and only if the following identity holds in $C$:

$$
(a \triangleleft b) \triangleleft c = \sum_{s=0}^{n} (-1)^s \binom{n}{s} a \triangleleft (b \triangleleft (c \triangleleft (c + s) c)).
$$

(b) The Jacoby identity $[[a, b], c] = [a, [b, c]] - [b, [a, c]]$ in $A$ is equivalent to the following conformal Jacoby identity in $C$:

$$
(a \triangleleft b) \triangleleft c = \sum_{s=0}^{n} (-1)^s \binom{n}{s} (a \triangleleft (b \triangleleft (c \triangleleft c)) - b \triangleleft (a \triangleleft (c \triangleleft c)) - b \triangleleft (c \triangleleft (a \triangleleft c)).
$$

(c) The skew-commutativity identity $[a, b] = -[b, a]$ in $A$ corresponds to the quasisymmetry identity:

$$
a \triangleleft b = \sum_{s \geq 0} (-1)^{n+s+1} \frac{1}{s!} D^s(b \triangleleft (c \triangleleft (c + s)) a).
$$

(d) The commutativity of $A$ is equivalent to

$$
a \triangleleft b = \sum_{s \geq 0} (-1)^{n+s} \frac{1}{s!} D^s(b \triangleleft (c \triangleleft (c + s)) a).
$$

The identities (1.13), (1.14) and (1.15) immediately imply the following

**Corollary 1.1.** Let $C$ be a Lie conformal or an associative conformal algebra, and $A = \text{Coeff } C$ its coefficient algebra. Then $C$ is an $A_+$-module with the action given by $a(n)c = a \odot c$ for $a, c \in C$, $n \in \mathbb{Z}_+$. Moreover, this action agrees with the derivations on $A_+$ and $C$: $(Da(n))c = [D, a(n)]c$.

From now on we will deal only with associative or Lie conformal algebras.

1.8. **Dong’s lemma.** We end this section by stating a very important property of formal power series over associative or Lie algebras. This property allows to construct conformal algebras by taking a collection of generating series.

**Lemma 1.2.** Let $A$ be an associative or a Lie algebra, and let $a, b, c \in A[[z, z^{-1}]]$ be three formal power series. Assume that they are pairwise mutually local. Then for all $n \in \mathbb{Z}_+$, $a \odot b$ and $c$ are mutually local. Moreover, in the Lie algebra case,

$$
N(a \odot b, c) = N(c, a \odot b) \leq N(a, b) + N(b, c) + N(c, a) - n - 1,
$$

and in the associative case

$$
N(a \odot b, c) \leq N(b, c), \quad N(c, a \odot b) \leq N(c, a) + N(a, b) - n - 1.
$$

2. **Vertex algebras**

2.1. **Fields.** Let now $V$ be a vector space over $\mathbb{k}$. Denote by $gl(V)$ the Lie algebra of all $\mathbb{k}$-linear operators on $V$. Consider the space $F(V) \subset gl(V)[[z, z^{-1}]]$ of fields on $V$, given by

$$
F(V) = \left\{ \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \mid \forall v \in V, a(n)v = 0 \text{ for } n \gg 0 \right\}.
$$

For $a(z) \in F(V)$ denote

$$
a_-(z) = \sum_{n < 0} a(n) z^{-n-1}, \quad a_+(z) = \sum_{n \geq 0} a(n) z^{-n-1}.
$$

Denote also by $\mathbb{1} = \mathbb{1}_{F(V)} \in F(V)$ the identity operator, such that $\mathbb{1}(-1) = \text{Id}_V$, all other coefficients are 0.
Remark. In [18] and [20] the elements of $F(V)$ are called quantum operators on $V$.

We view $gl(V)$ as a Lie algebra, and [1,2,1] gives a collection of products $\otimes$, $n \in \mathbb{Z}_+$, on $F(V)$. Now in addition to these products we introduce products $\odot$ for $n < 0$. Define first $\ominus$ by

$$a(z) \ominus b(z) = a_-(z)b(z) + b(z)a_+(z). \quad (2.1)$$

Note that the products in $\ominus$ make sense, since for any $v \in V$ we have $a(n)v = b(n)v = 0$ for $n \gg 0$. The $-1$-st product is also known as the normally ordered product (or Wick product) and is usually denoted by $:a(z)b(z):$.

Next, for any $n < 0$ set

$$a(z) \odot b(z) = \frac{1}{(-n-1)!} : (D^{-n-1}a(z))b(z):, \quad (2.2)$$

where $D = \frac{d}{dz}$. Taking $b = 1$ we get

$$a \ominus 1 = a, \quad a \ominus 2 = Da. \quad (2.3)$$

Easy to see that

$$1 \ominus a = \delta_{-1,n}a.$$

We have the following explicit formula for the circle products: if $(a \ominus b)(z) = \sum_m (a \ominus b)(m)z^{-n-1},$ then

$$(a \ominus b)(m) = \sum_{s \leq n} (-1)^{s+n} \binom{n}{n-s} a(s)b(m+n-s)$$

$$- \sum_{s \geq n} (-1)^{s+n} \binom{n}{s} b(m+n-s)a(s). \quad (2.4)$$

Note that if $n > 0$ then $\ominus$ becomes

$$(a \ominus b)(m) = \sum_{s \geq 0} (-1)^{s+n} \binom{n}{s} [a(s), b(m+n-s)],$$

which is precisely formula (1.2) for Lie algebras.

It is easy to see, that $D$ is a derivation of all the circle products:

$$D(a \ominus b) = Da \ominus b + a \ominus Db. \quad (2.5)$$

Note also that the Dong’ Lemma 1.2 remains valid for negative $n$ and the estimate (1.17) still holds.

2.2. Definition of vertex algebras. Instead of giving a formal definition of vertex algebra in spirit of Definition 1.1, we present a description of these algebras similar to Proposition 1.4. For a more abstract approach see e.g. [4, 17, 18] or [20].

Definition 2.1. A vertex algebra is a subspace $S \subset F(V)$ of fields over a vector space $V$ such that

(i) Any two fields $a, b \in S$ are local (in the Lie sense).
(ii) $S$ is closed under all the circle products $\otimes$, $n \in \mathbb{Z}$, given by (2.4).
(iii) $1 \in S$.

Note that from (2.3) it follows that a vertex algebra is closed under the derivation $D = d/dz$.

Let $S \subset F(V)$ be a vertex algebra. We introduce the left action map $Y : S \to F(S)$ defined by

$$Y(a) = \sum_{n \in \mathbb{Z}} (a \otimes \cdot ) \zeta^{-n-1}. \quad (2.6)$$

Clearly, $Y(1_S) = 1_{F(S)}$.

We state here the following characterizing property of $Y$ (see [17] or [20]):

Proposition 2.1. The left action map $Y : S \to F(S)$ is an isomorphism of vertex algebras, i.e. $Y(S) \subset F(S)$ is a vertex algebra and

$$Y(a \otimes b) = Y(a) \otimes Y(b), \quad Y(1_S) = 1_{F(S)}. \quad (2.7)$$
2.3. **Enveloping vertex algebras of a Lie conformal algebra.** Let $C$ be a Lie conformal algebra and $L = \text{Coeff} C$ its coefficient Lie algebra.

**Definition 2.2.** (See [14], [17])

(a) An $L$-module $M$ is called *restricted* if for any $a \in C$ and $v \in M$ there is some integer $N$ such that for any $n \geq N$ one has $a(n)v = 0$.

(b) An $L$-module $M$ is called a *highest weight* module if it is generated over $L$ by a single element $m \in M$ such that $L_+ m = 0$. In this case $m$ is called the *highest weight vector*

Clearly any submodule and any factor-module of a restricted module are restricted.

Let $M$ be a restricted $L$-module. Then the representation $\rho : L \to gl(M)$ could be extended to the map $\rho : L[[z, z^{-1}]] \to F(M)$ which combined with the canonical embedding $\phi : C \to L[[z, z^{-1}]]$ (see (d) of Proposition 1.3) gives conformal algebra homomorphism $\psi : C \to F(M)$. Then $\psi(C) \subset F(M)$ consists of pairwise local fields, and by Dong’s Lemma [2], $\psi(C)$ together with $\mathbb{1} \in F(M)$ generates a vertex algebra $S_M \subset F(M)$.

The following proposition is well-known, see e.g. [1].

**Proposition 2.2.** (a) The vertex algebra $S = S_M$ has a structure of a highest weight module over $L$ with the highest weight vector $\mathbb{1}$. The action is given by

$$a(n)\beta = \psi(a) \otimes \beta, \quad a \in C, \; n \in \mathbb{Z}, \; \beta \in S_M.$$  

Moreover this action agrees with the derivations:

$$(Da(n))\beta = [D, a(n)]\beta.$$  

(b) Any $L$-submodule of $S$ is a vertex algebra ideal. If $M_1$ and $M_2$ are two restricted $L$-modules, $S_1 = S_{M_1}$, $S_2 = S_{M_2}$, and $\mu : S_1 \to S_2$ is an $L$-module homomorphism such that $\mu(\mathbb{1}) = \mathbb{1}$, then $\mu$ is a vertex algebra homomorphism.

2.4. **Universal enveloping vertex algebras.** Now we build a universal highest weight module $V$ over $L$, which is often referred to as *Verma module*. Take the 1-dimensional trivial $L_+$-module $\mathbb{1}_V$, generated by an element $\mathbb{1}_V$. Then let

$$V = \text{Ind}^L_{L_+} \mathbb{1}_V = U(L) \otimes_{U(L_+)} \mathbb{1}_V \cong U(L)/U(L)L_+.$$  

It is easy to see that $V$ is a restricted module and hence we get an enveloping vertex algebra $S = S_V \subset F(V)$ and an homomorphism $\psi : C \to S$. Clearly, $\psi$ is injective, since $\rho : L \to gl(V)$ is injective.

**Theorem 2.1.** (a) The map $\chi : S \to V$ given by $a \mapsto a(-1)\mathbb{1}_V$ is an $L$-module isomorphism, and $\chi(\mathbb{1}_S) = \mathbb{1}_V$.

(b) $S$ is the universal enveloping vertex algebra of $C$ in the following sense: If $\mu : C \to U$ is another homomorphism of $C$ to a vertex algebra $U$, then there is the unique map $\tilde{\mu} : S \to U$ which makes up the following commutative triangle:

$$S \xrightarrow{\mu} U \quad \psi \circ \tilde{\mu} \quad U$$

From now on we identify $V$ and $S_V$ via $\chi$ and write $V = V(C)$ for the universal enveloping vertex algebra of a Lie conformal algebra $C$ and $\mathbb{1}_S = \mathbb{1}_V = \mathbb{1}$. The embedding $\psi : C \to V = U(L)/U(L)L_+$ is then given by $a \mapsto a(-1)\mathbb{1}$. By (c) of Proposition 1.3, the map $\phi(-1) : C \to L_-$, defined by $a \mapsto a(-1)$, is an isomorphism of linear spaces. Therefore, the image of $C$ in $V$ is equal to $\psi(C) = L_- \mathbb{1} = L \mathbb{1} \subset V$. 

From [23] and [23] it follows that $Y$ also agrees with $D$:

$$Y(Da) = \partial_Y(a) = [D, Y(a)].$$
3. Free conformal algebras

3.1. Definition of free conformal and free vertex algebras. Let $\mathcal{B}$ be a set of symbols. Consider a function $N : \mathcal{B} \times \mathcal{B} \to \mathbb{Z}_+$, which will be called a locality function.

Let $\mathfrak{A}$ be a variety of algebras. In all the applications $\mathfrak{A}$ will be either Lie or associative algebras. Consider the category $\mathfrak{Conf}(N)$ of $\mathfrak{A}$-conformal algebras (see §1) generated by the set $\mathcal{B}$ such that in any conformal algebra $C \in \mathfrak{Conf}(N)$ one has

$$a \odot b = 0 \quad \forall a, b \in \mathcal{B} \quad \forall n \geq N(a, b).$$

By abuse of notations we will not make a distinction between $\mathcal{B}$ and its image in a conformal algebra $C \in \mathfrak{Conf}(N)$.

The morphisms of $\mathfrak{Conf}(N)$ are, naturally, conformal algebra homomorphisms $f : C \to C'$ such that $f(a) = a$ for any $a \in \mathcal{B}$.

We claim that $\mathfrak{Conf}(N)$ has the universal object, a conformal algebra $C = C(N)$, such that for any other $C' \in \mathfrak{Conf}(N)$ there is the unique morphism $f : C \to C'$. We call $C(N)$ a free conformal algebra, corresponding to the locality function $N$.

In order to build $C(N)$, we first build the corresponding coefficient algebra $A = \text{Coeff} C$ (see §1.3).

Let $A \in \mathfrak{A}$ be the algebra presented by the set of generators

$$X = \{b(n) \mid b \in \mathcal{B}, \ n \in \mathbb{Z}\}$$

with relations

$$\left\{ \sum_{s} (-1)^{s} \left( N(b, a) \atop s \right) b(n - s)a(m + s) = 0 \mid a, b \in \mathcal{B}, \ m, n \in \mathbb{Z} \right\}. \quad (3.2)$$

For any $b \in \mathcal{B}$ let $\bar{b} = \sum b(n)z^{-n-1} \in A[[z, z^{-1}]]$. From (3.1) follows that any two $\bar{a}$ and $\bar{b}$ are mutually local, therefore by Dong’s Lemma 1.6 they generate a conformal algebra $C \subset A[[z, z^{-1}]]$.

Proposition 3.1. (a) $A = \text{Coeff} C$.

(b) The conformal algebra $C$ is the free conformal algebra corresponding to the locality function $N$.

Proof. (a) Clearly, there is a surjective homomorphism $A \twoheadrightarrow \text{Coeff} C$, since relations (3.2) must hold in $\text{Coeff} C$. Now the claim follows from the universal property of $\text{Coeff} C$ (see (e) of Proposition 1.3).

(b) Take another algebra $C' \in \mathfrak{Conf}(N)$, and let $A' = \text{Coeff} C'$. Obviously, there is an algebra homomorphism $f : A \to A'$ such that $f(b(n)) = b(n)$ for any $b \in \mathcal{B}$ and $n \in \mathbb{Z}$. It could be extended to a map $f : A[[z, z^{-1}]] \to A'[[z, z^{-1}]]$. Now it is easy to see that the restriction $f|_C$ gives the desired conformal algebra homomorphism $C \to C'$:

$$\begin{array}{ccc}
A[[z, z^{-1}]] & \xrightarrow{f} & A'[[z, z^{-1}]] \\
\uparrow & & \uparrow \\
C & \xrightarrow{f} & C'
\end{array}$$

Indeed, due to formula (1.2), $f$ preserves the circle products, and, since $\partial$ is a derivation of the products, and $f(\partial \bar{a}) = \partial f(\bar{a})$, for $a \in C$ one has $f(\partial \phi) = \partial f(\phi)$ for any $\phi \in C$.

In case when $\mathfrak{A}$ is the variety of Lie algebras, we may consider the universal vertex enveloping algebra $V(C)$ of a free Lie conformal algebra $C = C(N)$. In accordance with Theorem 2.4 we call $V(C)$ a free vertex algebra.

Though the construction of a free conformal and vertex algebras makes sense for an arbitrary locality function $N : \mathcal{B} \times \mathcal{B} \to \mathbb{Z}_+$, results of §§3.4–3.7 are valid only for the case when $N$ is constant.
3.2. The positive subalgebra of $\text{Coeff } C(N)$. Let again $C = C(N)$ be a free conformal algebra corresponding to a locality function $N : B \times B \to \mathbb{Z}_+$, $B$ being an alphabet, and let $A = \text{Coeff } C$. Recall that by Proposition 3.3(a) we have the decomposition $A = A_+ \oplus A_-$ of the coefficient algebra into the direct sum of two subalgebras. Denote $X_i = \{b(n) \mid b \in B, n \geq i\} \subseteq X$.

Lemma 3.1. The subalgebra $A_+ \subset A$ is isomorphic to the algebra $\hat{A}_+$ presented by the set of generators $X_0$ and those of relations (3.3) which contain only elements of $X_0$:

\[
\left\{ \sum_s (-1)^s \left( N(b,a) \begin{array}{c}s \\ b(n-s)a(m+s) = 0 \end{array} \right) \right\}.
\]

Proof. Clearly, there is a surjective homomorphism $\varphi : \hat{A}_+ \to A_+$ which maps $X_0$ to itself. We prove that $\varphi$ is in fact an isomorphism. We proceed in four steps.

Step 1. First we prove that $A_+ \subset A$. Indeed, we have $X_0 \subset A$. On the other hand, $A_+ \supset X_0$ is spanned by elements of the form $a(m)$, where $m \geq 0$ and $a \in C$ is a circle-product monomial in $B$. By induction on the length of $a$ it is enough to check that if $a = a_1 \otimes a_2$, then $a(m)$ is in the subalgebra, generated by $X_0$, which follows from (3.3).

Step 2. Let $\hat{\tau} : \hat{A}_+ \to \hat{A}_+$ be the homomorphism, which acts on the generators $X_0$ by $a(n) \mapsto a(n+1)$, so that $\hat{\tau}(A_+) = \text{subalgebra of } \hat{A}_+$. We claim that $\hat{\tau}$ is injective, and therefore $\hat{\tau}(A_+) \cong A_+$. Indeed, $\hat{\tau}$ acts on the free associative algebra $k(X_0)$. Assume that for some $p \in \hat{A}_+$ we have $\hat{\tau}(p) = 0$. Take any preimage $P \in k(X_0)$ of $p$. Then we have $\hat{\tau}(P) = \sum_i \xi_i R_i$, where $\xi_i \in k(X_0)$ and $R_i$ are relations (3.3), such that in all $\xi_i$ and $R_i$ appear only indexes greater or equal to 1. But then $P$ itself must be of the form $\sum_i \xi_i R_i$, where "" stands for decreasing all indexes by 1, hence $p = 0$.

Step 3. Next we claim that there is an automorphism $\tau$ of the algebra $A$ which acts on the generators $X$ by the shift $a(n) \mapsto a(n+1)$. Indeed, relations (3.2) are invariant under the shift, and clearly, $\tau$ is invertible. For any integer $n$ denote $A_n = \tau^n A_+$. We have $A_n \cong A_+ = A_0$ for every $n$.

Step 4. Now for each integer $n$ take a copy $A_n$ of $A_+$. Let $\hat{\tau}_n : \hat{A}_n \to \hat{A}_{n-1}$ be the isomorphism of $\hat{A}_+$ onto $\hat{\tau}(A_+)$, built in Step 1. Let $\hat{\tau}$ be the limit of all these $\hat{\tau}_n$ with respect to the maps $\varphi_n$. We identify generators of $\hat{A}_+$ with the set $X_n$. It is easy to see that $\varphi : \hat{A}_0 \to A_0$ extends to the homomorphism $\varphi : \hat{A} \to A$, such that $\varphi(A_n) = A_n$, and $\varphi|_X = \text{id}$. Now we observe that all the defining relations (3.3) of $A$ hold in $\hat{A}$, hence there is an inverse map $\varphi^{-1} : A \to \hat{A}$, and therefore $\varphi$ is an isomorphism.

\[\square\]

3.3. The Diamond Lemma. For the future purposes we need a digression on the Diamond Lemma for associative algebras. We closely follow [3], but use more modern terminology.

Let $X$ be some alphabet and $K$ be some commutative ring. Consider the free associative algebra $K\langle X \rangle$ of non-commutative polynomials with coefficients in $K$. Denote by $X^*$ the set of words in $X$, i.e. the free semigroup with 1 generated by $X$.

A rule on $K\langle X \rangle$ is a pair $\rho = (w, f)$, consisting of a word $w \in X^*$ and a polynomial $f \in K\langle X \rangle$. The left-hand side $w$ is called the principal part of rule $\rho$. We will denote $w = \bar{w}$.

Let $\mathcal{R}$ be a collection of rules on $K\langle X \rangle$. For a rule $\rho = (w, f) \in \mathcal{R}$ and a pair of words $u, v \in X^*$ consider the $K$-linear endomorphism $r_{uv} : K\langle X \rangle \to K\langle X \rangle$, which fixes all words in $X^*$ except for $uvu$, and sends the latter to $ufv$.

A rule $\rho = (w, f)$ is said to be applicable to a word $v \in X^*$ if $w$ is a subword of $v$, i.e. $v = v'uv''$. The result of application of $\rho$ to $v$ is, naturally, $r_{uv'}(v) = v'fv''$. If $p \in K\langle X \rangle$ is a polynomial which involves a word $v$, such that a rule $\rho$ is applicable to $v$, then we say that $\rho$ is applicable to $p$.

A polynomial $p \in K\langle X \rangle$ is called terminal if no rule from $\mathcal{R}$ is applicable to $p$, that is, no term of $p$ is of the form $uv^p$ for $p \in \mathcal{R}$.

Define a binary relation $\rightarrow$ on $K\langle X \rangle$ in the following way: Set $p \rightarrow q$ if and only if there is a finite sequence of rules $\rho_1, \ldots, \rho_n \in \mathcal{R}$, and a pair of sequences of words $u_i, v_i \in X^*$ such that $q = r_{u_n, \rho_n, v_n} \cdots r_{u_1, \rho_1, v_1}(p)$.
Definition 3.1. (a) A set or rules $R$ is a rewriting system on $K\langle X \rangle$ if there are no infinite sequences of the form

$$p_1 \rightarrow p_2 \rightarrow \ldots,$$

i.e. any polynomial $p \in K\langle X \rangle$ can be modified only finitely many times by rules from $R$.

(b) A rewriting system is confluent if for any polynomial $p \in K\langle X \rangle$ there is the unique terminal polynomial $t$ such that $p \rightarrow t$.

Any rule $\rho = (w, f) \in R$ gives rise to an identity $w - f \in K\langle X \rangle$. Let $I(R) \subset K\langle X \rangle$ is the two-sided ideal generated by all such identities.

Let $v_1, v_2 \in X^*$ be a pair of words. A word $w \in X^*$ is called composition of $v_1$ and $v_2$ if $w = w'u w''$, $v_1 = w'u$, $v_2 = u w''$ and $u \neq 0$.

Finally, take a word $v \in X^*$. Let us call it an ambiguity if there are two rules $\rho, \sigma \in R$ such that either $v$ is a composition of $\bar{\rho}$ and $\bar{\sigma}$ or if $v = \bar{\rho}$ and $\bar{\sigma}$ is a subword of $\bar{\rho}$.

Now we can state the Lemma.

Lemma 3.2 (Diamond Lemma). (a) A rewriting system $R$ is confluent if and only if all terminal monomials form a basis of $K\langle X \rangle/I(R)$.

(b) A rewriting system is confluent if and only if it is confluent on all the ambiguities, that is, for any ambiguity $v \in X^*$ there is the unique terminal $t \in K\langle X \rangle$ such that $v \rightarrow t$.

Remark. Statement (a) appears in [21]. A variant of Lemma 3.2 appears in [3] and [4]. It was also known to Shirshov (see [25]). The name “Diamond” is due to the following graphical description of the confluence property, see [21]. Let $R$ be a rewriting system in sense of Definition 3.1 (a), and let “$\rightarrow$” be defined as above. Assume $p, q_1, q_2 \in K\langle X \rangle$ are such that $p \rightarrow q_1$ and $p \rightarrow q_2$. Then there is some $t \in K\langle X \rangle$ such that $q_1 \rightarrow t$ and $q_2 \rightarrow t$:

G. Bergman in [3] uses the existence of a semigroup order with descending chain condition on the set of words $X^*$. Though in our case there is an order on the set $\langle X \rangle$, this order does not satisfy the descending chain condition, so we slightly modify the argument in [3].

Proof of Lemma 3.2. (a) Assume that the rewriting system $R$ is confluent. Define a map $r : K\langle X \rangle \rightarrow K\langle X \rangle$ by taking $r(p)$ to be the unique terminal monomial such that $p \rightarrow r(p)$. The crucial observation is that $r$ is a $K$-linear endomorphism of $K\langle X \rangle$. So if $p = \sum_i \xi_i u_i(w_i - f_i)v_i \in I(R)$, $\xi_i \in K$, $u_i, v_i \in X^*$, $(w_i, f_i) \in R$, then $r(p) = \sum_i \xi_i r(u_i(w_i - f_i)v_i) = 0$, therefore the terminal monomials are linearly independent modulo $I(R)$.

From the other side, if $R$ is not confluent, then there are a polynomial $p \in K\langle X \rangle$ and terminals $q_1, q_2 \in K\langle X \rangle$ such that $p \rightarrow q_1$, $p \rightarrow q_2$ and $q_1 \neq q_2$, and then $q_1 - q_2 \in I(R)$.

(b) Take a polynomial $p \in K\langle X \rangle$. We prove that there is the unique terminal $t$ such that $p \rightarrow t$ by induction on the number $n(p) = \{q \mid p \rightarrow q\}$. The condition (a) of Definition 3.1 ensures that $n(p)$ is always finite.

If $n(p) = 0$ then $p$ is a terminal itself and there is nothing to prove. By induction, without loss of generality we can assume that there are at least two different rules $\rho, \sigma \in R$ which are applicable to $p$. It means that there are some words $u, v, x, y \in X^*$ such that $r_{uv}(p) \neq p$, $r_{x\sigma y}(p) \neq p$ and $r_{uv\rho}(p) \neq r_{x\sigma y}(p)$. By induction, both $r_{uv}(p)$ and $r_{x\sigma y}(p)$ are uniquely reduced to terminals, say $r_{uv}(p) \rightarrow t_1$ and $r_{x\sigma y}(p) \rightarrow t_2$. We need to show that $t_1 = t_2$.

Consider two cases: when $\bar{\rho}$ and $\bar{\sigma}$ have common symbols in $p$, and thus $u\bar{\rho}v = x\bar{\sigma}y$ is a word in $p$, and when $\bar{\rho}$ and $\bar{\sigma}$ are disjoint.

In the first case, let $w \in X^*$ be the union of $\bar{\rho}$ and $\bar{\sigma}$ in $p$. Then $w$ is an ambiguity. By assumption, there is the unique terminal $s \in K\langle X \rangle$ such that $w \rightarrow s$. Let $q \in K\langle X \rangle$ be obtained from $p$ by
substituting \( w \) by \( s \). Then we have

\[
\begin{align*}
\rho_{u\rho v}(p) & \rightarrow p \downarrow q \\
r_{x\bar{y}}(p) & \rightarrow r_{x\bar{y}}(p)
\end{align*}
\]

(3.4)

By induction, \( q \) is uniquely reduced to a terminal \( t \), therefore one has \( r_{u\rho v}(p) \rightarrow t \) and \( r_{x\bar{y}}(p) \rightarrow t \).

In the second case note that \( r_{x\bar{y}}r_{u\rho v}(p) = r_{u\rho v}r_{x\bar{y}}(p) \). Denote this polynomial by \( q \). Then relations (3.4) still hold, and we finish by the same argument as in the first case.

### 3.4. Basis of a free vertex algebra

Return to the setup of §3.1. From now on we take the locality function \( N(a, b) = N \) to be constant: \( N(a, b) \equiv N \). Let \( C = C(N) \) be the free Lie conformal algebra and \( L = \text{Coeff} C \) its Lie algebra of coefficients, see Proposition 3.1. In this section we build a basis of the universal enveloping algebra \( U(L) \) of \( L \) and a basis of the free vertex algebra \( V = V(C) \).

We start with endowing \( B \) with an arbitrary linear order. Then define a linear order on the set \( X \) of generators of \( L \), given by (3.1), in the following way:

\[
a(m) < b(n) \iff m < n \quad \text{or} \quad (m = n \quad \text{and} \quad a < b).
\]

(3.5)

On the set \( X^* \) of words in \( X \) introduce the standard lexicographical order: for \( u, v \in X^* \) if \( |u| < |v| \), set \( u < v \); if \( |u| = |v| \) then set \( u < v \) whenever there is some \( 1 \leq i \leq |v| \) such that \( u(i) < v(i) \) and \( u(j) = v(j) \) for all \( 1 \leq j < i \).

In a defining relation from (3.2) the biggest term has form \( b(n)a(m) \) such that \( n - m > N \) or \( (n - m = N \quad \text{and} \quad (b > a \quad \text{or} \quad (b = a \quad \text{and} \quad N \text{ is odd})) \).

Taking it as a principal part we get a rule on \( \langle X \rangle \):

\[
\rho(b(n), a(m)) = \left( b(n)a(m), a(m)b(n) - \sum_{s=1}^{N} (-1)^s \binom{N}{s} [b(n-s), a(m+s)] \right),
\]

(3.7a)

and in case when \( a = b \), \( n - m = N \) and \( N \) is odd,

\[
\rho(a(m+N), a(m)) = \left( a(m+N)a(m), a(m)a(m+N) - \frac{1}{2} \sum_{s=1}^{(N-1)/2} (-1)^s \binom{N}{s} [a(n-s), a(m+s)] \right).
\]

(3.7b)

Denote the set of all such rules by \( \mathcal{R} \):

\[
\mathcal{R} = \{ \rho(b(n), a(m)) \mid (3.6) \text{ holds} \}.
\]

(3.8)

Lemma 3.3. The set of rules \( \mathcal{R} \) is a confluent rewriting system on \( \langle X \rangle \).

We prove this Lemma in §3.5. Here we derive from it and from the Diamond Lemma 3.2 the following theorem.

**Theorem 3.1.**

(a) Let \( C = C(N) \) be the free Lie conformal algebra generated by a linearly ordered set \( B \) corresponding to a constant locality function \( N \). Let \( L = \text{Coeff} C \) be the Lie algebra of coefficients, and let \( U = U(L) \) be its universal enveloping algebra. Then a basis of \( U \) is given by all monomials

\[
a_1(n_1)a_2(n_2) \cdots a_k(n_k), \quad a_i \in B, \quad n_i \in \mathbb{Z},
\]

(3.9)

such that for any \( 1 \leq i < k \) one has

\[
n_i - n_{i+1} \leq \begin{cases} 
N - 1 & \text{if } a_i > a_{i+1} \text{ or } (a_i = a_{i+1} \text{ and } N \text{ is odd}) \\
N & \text{otherwise.}
\end{cases}
\]

(3.10)

(b) A basis of the algebra \( U(L_+) \) is given by all monomials (3.9) satisfying the condition (3.10) and such that all \( n_i \geq 0 \).
(c) Let $V = V(C)$ be the corresponding free vertex algebra. Then a basis of $V$ consists of elements

$$a_1(n_1)a_2(n_2)\cdots a_k(n_k)\mathbb{1}, \quad a_i \in \mathcal{B}, \ n_i \in \mathbb{Z},$$

such that the condition (3.11) holds and, in addition, $n_k < 0$.

Proof. The statement (a) is a direct corollary of Lemma 3.3 and the Diamond Lemma. Because (3.9) is precisely the set of all terminal monomials with respect to $\mathcal{R}$.

(b) follows immediately from Lemma 3.1, since any subset of rules $\mathcal{R}$ is also a confluent rewriting system. Note also that for a rule $\rho$ given by (3.7) if the principal term $\bar{\rho}$ contains only elements from $X_0$ then so does the whole rule $\rho$.

For the proof of (c) recall that $V \cong U/UL_+$ as linear spaces (and even as $L$-modules), where $UL_+$ is the left ideal generated by $L_+$, see §2.4. By Lemma 3.1 this ideal is the linear span of all monomials $a_1(n_1)a_2(n_2)\cdots a_k(n_k)$ such that $n_k \geq 0$. But under the action of the rewriting system $\mathcal{R}$ the index of the rightmost symbol in a word can only increase, hence the linear span of these monomials in $k\langle X \rangle$ is stable under $\mathcal{R}$. It follows that the terminal monomials with a non-negative rightmost index form a basis of $UL_+$. This proves (b). □

3.5. **Proof of Lemma 3.3.** First we prove that the set of rules $\mathcal{R}$, given by (3.8), is a rewriting system on $k\langle X \rangle$. Take a word $u = a_1(m_1)\cdots a_k(m_k) \in X^*$. Let $p \in k\langle X \rangle$ be such that $u \rightarrow p$. Then any word $v$ that appears in $p$ lies in the finite set

$$W_u = \left\{ b_1(n_1)\cdots b_k(n_k) \in X^* \mid n_i \geq \min_{1 \leq j \leq k} \{ m_j \} \text{ and } \sum n_i = \sum m_i \right\},$$

(3.12)

therefore the condition (a) of the Definition 3.1 holds.

Thus we are left to prove that $\mathcal{R}$ is confluent. According to the Diamond Lemma 3.2, it is enough to check that it is confluent on a composition $w = c(k)b(j)a(i)$ of principal parts of a pair of rules $\rho(b(j), a(i)), \rho(c(k), b(j)) \in \mathcal{R}$. Thus it is sufficient to prove the following claim.

**Lemma 3.4.** Let $u = c(k)b(j)a(i) \in X^*$ be a word of length 3. Then $\mathcal{R}$ is confluent on $u$, i.e. there is the unique terminal $r(w) \in k\langle X \rangle$ such that $u \rightarrow r(w)$.

Proof. Assume for simplicity that the three rules $\rho(b(n), a(m)), \rho(c(p), b(n))$ and $\rho(c(p), a(m))$ are of the form (3.7a). The general case is essentially the same, but requires some additional calculations.

Consider the set $W_u$, given by (3.12). We prove that the Lemma holds for all $w \in W_u$ by induction on $w$. If $w$ is sufficiently small then it is a terminal itself. By induction, it is enough to consider $w = c(p)b(n)a(m) \in W_u$ such that $\mathcal{R}$ is applicable to both $b(n)a(m)$ and $c(p)b(n)$. Apply $\rho(b(n), a(m))$ and $\rho(c(p), b(n))$ to $w$ and take the difference of the results:

$$v = b(n)c(p)a(m) - \sum_{s=1}^{N} (-1)^s \binom{N}{s} \left[ c(p-s), b(n+s) \right] a(m)$$

$$- c(p)a(m)b(n) + \sum_{s=1}^{N} (-1)^s \binom{N}{s} c(p) \left[ b(n-s), a(m+s) \right].$$

By induction, $v$ is reduced uniquely to a terminal $t$ and we only have to show that $t = 0$. 

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First we apply the rules \( \rho(b(n), a(m)), \rho(c(p), b(n)) \) and \( \rho(c(p), a(m)) \) to \( v \) several times and get

\[
v \rightarrow - \sum_{s=1}^{N} (-1)^s \binom{N}{s} b(n)[c(p-s), a(m+s)] + b(n)a(m)c(p) \\
- \sum_{s=1}^{N} (-1)^s \binom{N}{s} [c(p-s), b(n+s)]a(m) \\
+ \sum_{s=1}^{N} (-1)^s \binom{N}{s} [c(p-s), a(m+s)]b(n) - a(m)c(p)b(n) \\
+ \sum_{s=1}^{N} (-1)^s \binom{N}{s} c(p)[b(n-s), a(m+s)]
\]

(3.13)

Next we introduce two rules acting on the linear combinations of (formal) commutators: For any \( a(m), b(n), c(p) \in X \) let

\[
\kappa = \left\{ [a(m), [b(n), c(p)]], \left\{ [a(m), b(n)], c(p) \right\} + [b(n), [a(m), c(p)]] \right\}
\]

\[
\lambda = \left\{ [b(n), a(m)], - \sum_{s=1}^{N} (-1)^s \binom{N}{s} [b(n-s), a(m+s)] \right\}
\]

The rule \( \lambda \) is the locality relation, and \( \kappa \) is nothing else but the Jacobi identity. The Lemma will be proved after we show two things:

1) There always exists a finite sequence of applications of the rules \( \kappa \) and \( \lambda \) that reduces (3.13) to 0.
2) All words which appear in the process of reduction in 1) are smaller than the initial word \( u = c(p)b(n)a(m) \) with respect to the order (3.3).

Indeed, assume 1) and 2) hold. Denote the polynomial in (3.13) by \( p_0 \). Let

\[
p_0 \rightarrow p_1 \rightarrow \cdots \rightarrow 0
\]

be the reduction, guaranteed by 1). By 2) and by the induction hypothesis, any two neighboring polynomials \( p_i \rightarrow p_{i+1} \) from this sequence are uniquely \( \mathcal{R} \)-reduced to a terminal, and this terminal must be the same, since either \( p_i \xrightarrow{R} p_{i+1} \) or \( p_{i+1} \xrightarrow{R} p_i \).

Denote the three last terms in (3.13) by \( [\mathbf{a}], [\mathbf{b}] \) and \( [\mathbf{c}] \). In Figure 1 we present a scheme of how \( \kappa \) and \( \lambda \) should be applied in order to reduce (3.13) to 0.

Each box stands for a sum of commutators:

\[
\mathbf{[]} = -[\mathbf{]} = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [c(p-s-t), a(m+t)], b(n+s)],
\]

\[
\mathbf{[k]} = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [c(p-s), [b(n+s-t), a(m+t)]],
\]

\[
\mathbf{[l]} = -[\mathbf{l}] = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [c(p-s), [b(n-t), a(m+s+t)]],
\]
Figure 1. Application of rules $\kappa$ and $\lambda$

\[
\begin{align*}
\Box &= \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [b(n+t), c(p-s-t), a(m+s)], \\
\Delta &= -\Theta = \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [b(n-s+t), c(p-t), a(m+s)], \\
\Theta &= \sum_{s,t=1}^{N} (-1)^{s+t} \binom{N}{s} \binom{N}{t} [b(n-s), [a(m+s+t), c(p-t)]]; \\
\Psi &= -\Psi = \sum_{s,t,r=1}^{N} (-1)^{s+t+r} \binom{N}{s} \binom{N}{t} \binom{N}{r} [b(n+s-t), [a(m+t+r), c(p-s-r)]]; \\
\Xi &= \sum_{s,t,r=1}^{N} (-1)^{s+t+r} \binom{N}{s} \binom{N}{t} \binom{N}{r} [a(m+s+t), [c(p-t-r), b(n-s+r)]], \\
\Xi &= -\Xi = \sum_{s,t,r=1}^{N} (-1)^{s+t+r} \binom{N}{s} \binom{N}{t} \binom{N}{r} [a(m+s+t), [c(p-t-r), b(n-s+r)]],
\end{align*}
\]

One can see that all terminal boxes in the above scheme cancel, so that $\Box + \Delta + \Theta \rightarrow 0$. Claim 2) also holds, since every symbol in every box in Figure 1 is less than $c(p)$.

3.6. Digression on Hall bases. Let again $\mathcal{B}$ be some linearly ordered alphabet, $N \in \mathbb{Z}_+$, $C = C(N)$ the free Lie conformal algebra generated by $\mathcal{B}$ with respect to the constant locality $N$, and $L = \text{Coeff } C(N)$. A basis of the Lie algebra $L$ could be obtained by modifying the construction of a Hall basis of a free Lie algebra, see [22], [23], [24]. Here we review the latter construction. We closely follow [22], except that all the order relations are reversed.

As in §3.3, take an alphabet $X$ and a commutative ring $K$. Let $T(X)$ be the set of all binary trees with leaves from $X$. For typographical reasons we will write the tree $\bar{x}y$ as $(x, y)$. Assume that $T(X)$ is endowed with a linear order such that $(x, y) > \min\{x, y\}$ for any $x, y \in T(X)$.

**Definition 3.2.** A **Hall set** $\mathcal{H} \subset T(X)$ is a subset of all trees $h \in T(X)$ satisfying the following (recursive) properties:

1. If $h = (x, y)$ then $y, x \in \mathcal{H}$ and $x > y$;
2. If $h = (\langle x, y \rangle, z)$ then $z \geq y$, so that $(x, y) > z \geq y$.

In particular, $X \subset \mathcal{H}$. 
Introduce two maps \( \alpha : T(X) \to X^* \) and \( \lambda : T(X) \to K(X) \) in the following recursive way: for \( a \in X \) set \( \alpha(a) = \lambda(a) = a \) and \( \alpha((x,y)) = \alpha(x)\alpha(y), \lambda((x,y)) = [\lambda(x), \lambda(y)] \).

It is a well-known fact (see e.g. \([22]\)) that

(a) \( \lambda(H) \) is a basis of the free Lie algebra generated by \( X \) and

(b) \( \alpha\big|_H \) is injective.

A word \( w \in \alpha(H) \) is called a Hall word.

On the set \( X^* \) of words in \( X \) introduce a (lexicographic) order as follows: if \( u \) is a prefix of \( v \) then \( u > v \), otherwise \( u > v \) whenever for some index \( i \) one has \( u_i > v_i \) and \( u_j = v_j \) for all \( j < i \).

**Definition 3.3.** ([25], [7]) A word \( v \in X^* \) is called Lyndon-Shirshov if it is bigger than all its proper suffixes.

**Proposition 3.2.** (a) There is a Hall set \( H_{LS} \) such that \( \alpha(H_{LS}) \) is the set of all Lyndon-Shirshov words and \( \alpha : T(X) \to X^* \) preserves the order.

(b) For any tree \( h \in H_{LS} \) the biggest term in \( \lambda(h) \) is \( \alpha(h) \).

### 3.7. Basis of the algebra of coefficients of a free Lie conformal algebra.

Here we apply general results from \( \S 3.6 \) to the situation of \( \S 3.7 \).

Recall that starting from a set of symbols \( B \) and a number \( N > 0 \), we build the free conformal algebra \( C = C(N) \) generated by \( B \) such that \( a \otimes b = 0 \) for any two \( a,b \in B \) and \( n \geq N \). Let \( L = \text{Coeff } C \) be the corresponding Lie algebra of coefficients. It is generated by the set \( X = \{ a(n) \mid a \in B, n \in \mathbb{Z} \} \) subject to relations \( (3.2) \).

The set of generators \( X \) is equipped with the linear order defined by \( (3.5) \). We define the order on \( X^* \) as in \( \S 3.4 \). Consider the set of all Lyndon words in \( X^* \) and let \( H = H_{LS} \subset T(X) \) be the corresponding Hall set. Recall that there is a rewriting system \( R \) on \( \mathcal{K}(X) \), given by \( (3.8) \). Define

\[ H_{\text{term}} = \{ h \in H \mid \alpha(h) \text{ is terminal} \}. \]

**Lemma 3.5.** (a) Let \( v_1 \leq \ldots \leq v_n \) be a non-decreasing sequence of terminal Lyndon-Shirshov words. Then their concatenation \( w = v_1 \cdots v_n \in X^* \) is a terminal word.

(b) Each terminal word \( w \in X^* \) can be uniquely represented as a concatenation \( w = v_1 \cdots v_n \), where \( v_1 \leq \ldots \leq v_n \) is a non-decreasing sequence of terminal Lyndon-Shirshov words.

**Proof.** (a) Take two terminal Lyndon-Shirshov words \( v_1 \leq v_2 \). Let \( x \in X \) be the last symbol of \( v_1 \) and \( y \in X \) be the first symbol of \( v_2 \). Then, since a word is less than its prefix and since \( v_1 \) is a Lyndon-Shirshov word, we get

\[ x < v_1 \leq v_2 < y. \]

Therefore, \( xy \) is a terminal, hence \( v_1v_2 \) is a terminal too.

(b) Take a terminal word \( w \in X^* \). Assume it is not Lyndon-Shirshov. Let \( v \) be the maximal among all proper suffixes of \( w \). Then \( v \) is Lyndon-Shirshov, \( v > w \) and \( w = uv \) for some word \( u \). By induction, \( u = v_1 \cdots v_{n-1} \) for a non-decreasing sequence of Lyndon-Shirshov words \( v_1 \leq \ldots \leq v_{n-1} \). We are left to show that \( v \geq v_{n-1} \).

Assume on the contrary that \( v < v_{n-1} \). Then, since \( v > v_{n-1} \), \( v_{n-1} \) must be a prefix of \( v \) so that \( v = v_{n-1}v' \). But then \( v' > v \) which contradicts the Lyndon-Shirshov property of \( v \).

The uniqueness is obvious.

Let \( \varphi : \mathcal{K}(X) \to U(L) \) be the canonical projection with the kernel \( I(R) \).

**Theorem 3.2.** The set \( \varphi(\lambda(H_{\text{term}})) \) is a basis of \( L \).

**Proof.** Let \( s = \{ h_1, \ldots, h_n \} \subset H_{\text{term}} \) be a non-decreasing sequence of terminal Hall trees. Let \( \lambda(s) = \lambda(h_1) \cdots \lambda(h_n) \) \( \in \mathcal{K}(X) \) and \( \alpha(s) = \alpha(h_1) \cdots \alpha(h_n) \) \( \in X^* \).

By the Poincaré-Birkoff-Witt theorem it is sufficient to prove that the set \( \{ \varphi(\lambda(s)) \} \), when \( s \) ranges over all non-decreasing sequences \( s \) of terminal Hall trees, is a basis of \( U(L) \).

By (b) of Proposition \( (3.4) \) \( \lambda(s) = \alpha(s) + O(\alpha(s)) \), where \( O(v) \) stands for a sum of terms which are less than \( v \). Now let \( t(s) \in \mathcal{K}(X) \) be a terminal such that \( \lambda(s) \longrightarrow t(s) \). One can view \( t(s) \) as the
decomposition of $\varphi(\lambda(s))$ in basis \((3.9)\). By Lemma \(3.3\) $\alpha(s)$ is a terminal monomial, hence $t(s)$ has form $t(s) = \alpha(s) + f(s)$ where $f(s)$ is a sum of terms $v \in X^*$ satisfying the following properties:

1. $v$ is terminal and $v < \alpha(s)$;
2. If $v$ contains a symbol $a(n) \in X$ then $a$ appears in $\alpha(s)$ and $n_{\text{min}} \leq n \leq n_{\text{max}}$, where $n_{\text{min}}$ and $n_{\text{max}}$ are respectively minimum and maximum of all indices that appear in $\alpha(s)$.

Indeed, due to Proposition \(3.2\) b) properties 1 and 2 are satisfied by all the terms in $\lambda(s) - \alpha(s)$, and they cannot be broken by an application of the rules $R$.

Property 1 implies that all $t(s)$ and, therefore, $\varphi(\lambda(s))$ are linearly independent. So we are left to show that they span $U(L)$. For that purpose we show that any terminal word $w \in X^*$ can be represented as a linear combination of $t(s)$.

By (b) of Lemma \(3.5\) any terminal word $w$ could be written as $w = \alpha(s)$ for some non-decreasing sequence $s$ of terminal Hall trees. So we can write $w = t(s) - f(s)$. Now do the same with any term $v$ that appears in $f(s)$, and so on. This process should terminate, because every term $v$ that appears during this process must satisfy properties 1 and 2 and there are only finitely many such terms.

\[\square\]

Remark. Alternatively we could use the theorem of L. Bokut’ and P. Malcolmson \[7\].

As in (b) of Theorem \(3.3\), we deduce that all the elements of $\varphi(\lambda(H_{\text{term}}))$ containing only symbols from $X_0$ form a basis of $L_+$.

Note that we have an algorithm for building a basis of the free Lie conformal algebra $C = C(N)$. Let $L = \text{Coeff } C$, $V = V(C)$ and $U = U(L)$. Recall that the image if $C$ in $V$ under the canonical embedding $\psi: C \to V$ is $\psi(C) = L_+ \subseteq U \subseteq V$. So, the algorithm goes as follows: take the basis of $L$ provided by Theorem \(3.2\). Decompose its element in basis \((3.9)\) of the universal enveloping algebra $U(L)$, and then cancel all terms of the form $a_1(n_1) \cdots a_k(n_k)$ where $n_k \geq 0$. What remains, being interpreted as elements of the vertex algebra $V$, form a basis of $\psi(C) \subseteq V$.

3.8. Basis of the algebra of coefficients of a free associative conformal algebra. Let again $B$ be some alphabet, and $N: B \times B \to \mathbb{Z}_+$ be a locality function, not necessarily constant and not necessarily symmetric. By Proposition \(3.1\), the coefficient algebra $A = \text{Coeff } C(N)$ of the free associative conformal algebra $C(N)$ corresponding to the locality function $N$ is presented in terms of generators and relations by the set of generators $X = \{b(n) \mid b \in B, n \in \mathbb{Z}\}$ and relations \((3.2)\).

Theorem 3.3. (a) A basis of the algebra $A_0$ is given by all monomials of the form
\[
a_1(n_1) \cdots a_{l-1}(n_{l-1})a_l(n_l),
\]
where $a_i \in B$ and
\[
-\left[\frac{N_i - 1}{2}\right] \leq n_i \leq \left[\frac{N_i - 1}{2}\right], \quad N_i = N(a_i, a_{i+1}), \quad \text{for } i = 1, \ldots, l - 1.
\]

(b) A basis of the algebra $A_+$ is given by all monomials of the form
\[
a_1(n_1) \cdots a_{l-1}(n_{l-1})a_l(n_l),
\]
where $a_i \in B$ and
\[
0 \leq n_i \leq N_i - 1, \quad N_i = N(a_i, a_{i+1}), \quad \text{for } i = 1, \ldots, l - 1.
\]

Corollary 3.1. Assume that the locality function $N$ is constant. Consider the homogeneous component $A_{k,l}$ of $A$, spanned by all the words of the length $l$ and of the sum of indexes $k$. Then $\dim A_{k,l} = N^{l-1}$.

Proof of Theorem 3.3. (a) Introduce a linear order on $B$, and define an order on the set of generators $X$ by the following rule:
\[
a(m) > b(n) \iff |m| > |n| \text{ or } m = -n > 0 \text{ or } (m = n \text{ and } a > b)
\]
In particular, for some $a \in B$ we have
\[
a(0) < a(-1) < a(1) < a(-2) < a(2) < \ldots.
\]
For any relation $r$ from (3.2) take the biggest term $\bar{r}$ and consider the rule $(\bar{r}, r - \bar{r})$. This way we get a collection of rules

$$
\mathcal{R} = \left\{ \rho_1(b(n), a(m)) \mid a, b \in B, \ n > \left\lfloor \frac{N(b,a)-1}{2} \right\rfloor \right\} \cup \\
\left\{ \rho_2(b(n), a(m)) \mid n < - \left\lfloor \frac{N(b,a)-1}{2} \right\rfloor \right\}
$$

where

$$
\rho_1(b(n), a(m)) = (b(n)a(m), \sum_{s=1}^{N(b,a)} (-1)^{s+1} \left( \binom{N(b,a)}{s} \right) b(n-s)a(m+s)),
$$

$$
\rho_2(b(n), a(m)) = (b(n)a(m), \sum_{s=1}^{N(b,a)} (-1)^{s+1} \left( \binom{N(b,a)}{s} \right) b(n+s)a(m-s)).
$$

By the Diamond Lemma 3.2, we have to prove that these rules form a confluent rewriting system on $\mathbb{k}(X)$. Clearly $\mathcal{R}$ is a rewriting system, since it decreases the order, and each subset of $\mathbb{k}(X)$, containing only finitely many different letters from $B$, has the minimal element, in contrast to the situation of §3.3.

As before, it is enough to check that $\mathcal{R}$ is confluent on any composition $w = c(p)b(n)a(m)$, of the principal parts of rules from $\mathcal{R}$. Consider the set $W = \{ c(k)b(j)a(i) \mid k, j, i \in \mathbb{Z} \} \subset X^*$. We prove by induction on $w \in W$ that $\mathcal{R}$ is confluent on $w$. If $w$ is sufficiently small, then it is terminal. Assume that $w = c(k)b(j)a(i)$ is an ambiguity, for example that $\rho_1(c(p), b(n))$ and $\rho_2(b(n), a(m))$ are both applicable to $w$. Other cases are done in the same way. Let

$$
w_1 = \rho_1(c(p), b(n))(w) = \sum_{s=1}^{N(c,b)} (-1)^s \left( \binom{N(c,b)}{s} \right) c(p-s)b(n+s)a(m),
$$

$$
w_2 = \rho_2(b(n), a(m))(w) = \sum_{t=1}^{N(b,a)} (-1)^t \left( \binom{N(b,a)}{t} \right) c(p)b(n+t)a(m-t).
$$

Applying $\rho_2(b(n+s), a(m))$ for $s = 1, \ldots, N(b,a)$ to $w_1$ gives the same result as we get from applying $\rho_1(c(p), b(n+t))$ for $t = 1, \ldots, N(c,b)$ to $w_2$, namely

$$
\sum_{s,t \geq 1} (-1)^{s+t} \left( \binom{N(c,b)}{s} \binom{N(b,a)}{t} \right) c(p-s)b(n+s+t)a(m-t).
$$

By the induction assumption, $w_1 - w_2$ is uniquely reduced to a terminal, and since all monomials in (3.16) are smaller than $w$, we conclude that this terminal must be 0.

(b) Follows at once from Lemma 3.1.

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