Electrostatic self-interaction in the spacetime of a global monopole with finite core

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Abstract

In this paper, we calculate the induced electrostatic self-energy and self-force for an electrically charged particle placed at rest in the spacetime of a global monopole admitting a general spherically symmetric inner structure to it. In order to develop this analysis we calculate the three-dimensional Green function associated with this physical system. We explicitly show that for points outside the monopole’s core the self-energy presents two distinct contributions. The first is induced by the non-trivial topology of the global monopole considered as a point-like object. The second is a correction induced by the non-vanishing inner structure attributed to it. As an illustration of the general procedure the flower-pot model for the region inside the monopole is considered. In this application, it is also possible to find the electrostatic self-energy for points in the region inside the monopole. In the geometry of the global monopole with the positive solid angle deficit, we show that for the flower-pot model the electrostatic self-force is repulsive with respect to the core surface for both exterior and interior regions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is well known that different types of topological objects may have been formed by the vacuum phase transition in the early universe after Planck time [1, 2]. These include domain walls, cosmic strings and monopoles. Global monopoles are heavy topological objects formed in the phase transition of a system composed by a self-coupling iso-triplet scalar field $\Phi^a$ whose original global $O(3)$ symmetry is spontaneously broken to $U(1)$. The scalar matter field plays the role of an order parameter which outside the monopole’s core acquires a non-vanishing
The global monopole was first introduced by Sokolov and Starobinsky [3] and the gravitational effects of the global monopole have been analysed by Barriola and Vilenkin [4]. It has been shown that for points far away from the monopole’s centre the corresponding geometry can be described by the line element

\[ ds^2 = dt^2 - dr^2 - \alpha^2 r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]

(1)

with the parameter \( \alpha^2 = 1 - 8\pi G \eta^2 \) determined by the energy scale \( \eta \) where the global symmetry is spontaneously broken. It is of interest to note that the effective metric produced in superfluid \(^3\)He–A by a monopole is described by line element (1) with the negative angle deficit, \( \alpha > 1 \), which corresponds to the negative mass of the topological object [5].

Many of treatments in the investigation of physical effects around a global monopole deal mainly with the case of the idealized point-like monopole geometry described by line element (1) for all values of the radial coordinate. However, the realistic global monopole has a characteristic core radius determined by the symmetry braking scale at which the monopole is formed. The calculation of the metric tensor in the region inside the global monopole would require the knowledge of the behaviour of the energy–momentum tensor associated with the scalar field \( \Phi^a \), which on the other hand requires the knowledge of the components of the metric tensor, providing, in this way, a non-solvable integral equation [6]. In this paper we shall not go into the details of this calculation. Instead, we shall consider a simplified model described by two sets of the metric tensor for two distinct regions, continuous at a spherical shell of radius \( a \). In the exterior region corresponding to \( r > a \), the line element is given by (1), while in the interior region, \( r < a \), the geometry is described by the static spherically symmetric line element

\[ ds^2 = u^2(r) \, dt^2 - v^2(r) \, dr^2 - w^2(r) (d\theta^2 + \sin^2 \theta \, d\phi^2), \]

(2)

At the boundary of the core the functions \( u(r), v(r), w(r) \) satisfy the conditions

\[ u(a) = v(a) = 1, \quad w(a) = \alpha a. \]

(3)

By introducing a new radial coordinate \( \tilde{r} = w(r) \) with the core centre at \( \tilde{r} = 0 \), the angular part of the line element (2) is written in the standard Minkowskian form. With this coordinate, in general, we will obtain non-standard angular part in the exterior line element.

Many years ago, Linet [7] and Smith [8], independently, have shown that an electrically charged particle placed at rest in the spacetime of an idealized cosmic string becomes subjected to a repulsive self-interaction. This self-interaction is a consequence of the distortion of the particle’s fields caused by the planar angle deficit associated with the conical geometry. Also it was shown in [9] that linear electric or magnetic sources in the spacetime of a cosmic string parallel to the latter become subject to induced self-interactions. More recently the problem of the induced electrostatic self-energy in the spacetime of a thick cosmic string has been considered in [10]. Analogously to what happens in the cosmic string spacetime, a point-like electrically charged particle placed at rest in the spacetime of an idealized global monopole also becomes subjected to a repulsive self-interaction [11]. In the present paper we shall continue in this line of investigation. We shall consider the induced electrostatic self-energy and self-force associated with a point-like charged particle placed at rest in the spacetime of a global monopole with a finite core described by line element (2). The corresponding results specify the conditions under which we can ignore the details of the interior structure and approximate the effect of the global monopole by the idealized model. The analysis of quantum vacuum effects for a scalar field in the model under consideration has been developed in [12]. It was shown that these effects are composed by the sum of a point-like monopole and core-induced parts. Moreover, adopting a specific model for the monopole’s core, the
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flower-pot one, explicit calculations for the vacuum polarization effects in the exterior and interior regions have been done.

This paper is organized as follows. Writing the Maxwell equations in the spherically symmetric spacetime, in section 2 we calculate the three-dimensional Green function for points outside and inside the monopole’s core. As a consequence, we provide a general expression for the electrostatic self-energy and the related self-force. We shall see that in the exterior region the corresponding expressions are composed of two parts. The first ones are induced by the global monopole considered as a point-like object, while the second parts are induced by the non-vanishing inner structure attributed to it. As an illustration of the general results obtained, in section 3 we consider the flower-pot model for the region inside the core. In this model, we explicitly calculate the self-energy in exterior and interior regions and describe its behaviour in various asymptotic regions of the parameters. In section 5, we present our conclusions and more relevant remarks.

2. Self-energy outside the monopole core

The main objective of this paper is to evaluate the electrostatic self-energy and the self-force for a point-like charged particle at rest, induced by the spacetime geometry associated with a global monopole with the core of finite radius. We will assume that in the region inside the monopole core the geometry is described by line element (2), and in the exterior region we have the standard line element (1) with the solid angle deficit $4\pi(1 - \alpha^2)$. For the covariant components of the electromagnetic 4-vector potential, $A_i$, from the Maxwell equations we have

$$\partial_i \left[ \sqrt{-g} g^{im} g^{kn} (\partial_m A_n - \partial_n A_m) \right] = -4\pi \sqrt{-g} j^i,$$

(4)

where $j^i$ is the 4-vector electric current density. For a point-like particle at rest with coordinates $r_0 = (r_0, \theta_0, \phi_0)$, in the coordinate system corresponding to the line element (2), the static 4-vector current and potential read $j^i = (j^0, 0, 0, 0)$ and $A_n = (A_0, 0, 0, 0)$. The only non-trivial component of (4) is the $i = 0$ one with

$$j^0(x) = q \frac{\delta(r - r_0)}{\sqrt{-g}},$$

(5)

where $q$ is the charge of the particle. So, in the spherically symmetric spacetime defined by (2), the differential equation obeyed by $A_0$ reads

$$\partial_r \left( \frac{u^2}{uv} \partial_r A_0 \right) - \frac{v}{u} \hat{L}^2 A_0 = -\frac{4\pi q}{\sin \theta} \delta(r - r_0),$$

(6)

with $\hat{L}$ being the operator of the angular momentum. The solution of this equation can be written in terms of the Green function associated with the differential operator defined by the left-hand side as follows:

$$A_0(r) = 4\pi q G(r, r_0),$$

(7)

with the equation for the Green function

$$\left[ \partial_r \left( \frac{u^2}{uv} \partial_r \right) - \frac{v}{u} \hat{L}^2 \right] G(r, r_0) = -\frac{\delta(r - r_0)}{\sin \theta} \delta(\theta - \theta_0) \delta(\phi - \phi_0).$$

(8)

For the electrostatic self-force on a charged test particle held stationary outside a Schwarzschild black hole see [13].
Having the electrostatic self-potential for the charge we can evaluate the corresponding self-force by using the standard formula

$$f_{el}^i(r_0) = q g_{ik} F_{km} u^m = q g_{ik} \frac{\partial}{\partial r} A_0 |_{r=r_0} = 4\pi q^2 \frac{g_{ik}}{u} \lim_{r \to r_0} [\partial \frac{\partial}{\partial r} G(r, r_0)].$$

(9)

An alternative way to obtain the self-force is to consider first the electrostatic self-energy given by [7, 8]

$$U_{el}(r_0) = \frac{q A_0(r_0)}{2} = 2\pi q^2 \lim_{r \to r_0} G(r, r_0),$$

(10)

and then to derive the force on the base of the formula

$$f_{el}^i(r_0) = \frac{g_{ik}}{u} \frac{\partial}{\partial r} U_{el}(r_0).$$

(11)

In accordance to the Synge’s theorem, formulae (9) and (11) lead to the same result for the self-force.

In formulae (9) and (10) the limit is divergent. To obtain a finite and well-defined result for the self-force, we should apply some renormalization procedure for the Green function. The procedure that we shall adopt is the standard one (see, for instance, [14]): we subtract from the Green function the terms in the corresponding DeWitt–Schwinger adiabatic expansion which are divergent in the coincidence limit. So, we define the renormalized Green function as

$$G_{ren}(r, r_0) = G(r, r_0) - G_{DS}(r, r_0).$$

(12)

In this way the renormalized self-energy, $U_{el, ren}(r_0)$, and self-force, $f_{el, ren}(r_0)$, are obtained by formulae (9) and (10) with the replacement $G(r, r_0) \to G_{ren}(r, r_0)$. Note that here the subtraction of the divergent part of the Green function corresponds to the renormalization of the particle mass.

Taking into account the spherical symmetry of the problem, we may present the Green function as the expansion

$$G(r, r_0) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_l(r, r_0) Y^m_l(\theta, \phi) Y^m_l(\theta_0, \phi_0),$$

(13)

with $Y^m_l(\theta, \phi)$ being the ordinary spherical harmonics. Substituting (13) into (8) and using the well-known closure relation for the spherical harmonics, we arrive at the differential equation for the radial function:

$$\left[ \frac{d}{dr} \left( \frac{w^2}{u} \frac{d}{dr} \right) - \frac{v}{u} l(l+1) \right] g_l(r, r_0) = -\delta(r - r_0).$$

(14)

As the functions $u(r)$, $v(r)$, $w(r)$ are continuous at $r = a$, from (14) it follows that the function $g_l(r, r_0)$ and its first radial derivative are also continuous at this point. The function $g_l(r, r_0)$ is continuous for $r = r_0$ as well. The junction of the first radial derivative at $r = r_0$ is obtained by the integration of (14) about this point:

$$\frac{dg_l(r, r_0)}{dr} \bigg|_{r=r_0^+} - \frac{dg_l(r, r_0)}{dr} \bigg|_{r=r_0^-} = -\frac{u(r_0)v(r_0)}{w^2(r_0)}.$$

(15)

In the region inside the core, we denote by $R_{1l}(r)$ and $R_{2l}(r)$ the linearly independent solutions of the homogeneous equation corresponding to (14). We shall assume that the function $R_{1l}(r)$ is regular at the core centre $r = r_c$ and that the solutions are normalized by the Wronskian relation

$$R_{1l}(r) R^{*}_{2l}(r) - R^{*}_{1l}(r) R_{2l}(r) = -\frac{u(r)v(r)}{w^2(r)}.$$

(16)
In the region outside the core the linearly independent solutions to the corresponding homogeneous equation are the functions \( r^{\lambda_1} \) and \( r^{\lambda_2} \), where

\[
\lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2\alpha} \sqrt{\alpha^2 + 4l(l + 1)}.
\]

(17)

Now, we can write \( g_i(r, r_0) \) as a function of the radial coordinate \( r \) in the separate regions \([r_c, \min(r_0, a)), (\min(r_0, a), \max(r_0, a)) \) and \((\max(r_0, a), \infty) \) as a linear combination of the above-mentioned solutions with arbitrary coefficients. The requirement of the regularity at the core centre and at infinity reduces the number of these coefficients to four. They are determined by the continuity condition at the monopole’s core boundary and by the matching conditions at \( r = r_0 \). In this way we find the following expressions:

\[
g_i(r, r_0) = \frac{(ar_0)^{\lambda_1} R_{lj}(r)}{\alpha^2[aR_{lj}(a) - \lambda_2 R_{lj}(a)]}, \quad \text{for} \quad r \leq a,
\]

(18)

\[
g_i(r, r_0) = \frac{r^\lambda r^\lambda}{\alpha^2(\lambda_1 - \lambda_2)} \left[ 1 - \left( \frac{a}{r} \right)^{\lambda_1 - \lambda_2} D_{lj}(a) \right], \quad \text{for} \quad r \geq a,
\]

(19)

in the case \( r_0 > a \), and

\[
g_i(r, r_0) = R_{lj}(r_<) R_{lj}(r) - R_{lj}(r_0) R_{lj}(r) D_{lj}(a), \quad \text{for} \quad r \leq a,
\]

(20)

\[
g_i(r, r_0) = \frac{a^\lambda r^\lambda R_{lj}(r_0)}{\alpha^2[aR_{lj}(a) - \lambda_2 R_{lj}(a)]}, \quad \text{for} \quad r \geq a,
\]

(21)

in the case \( r_0 < a \). In these formulae, \( r_< = \min(r, r_0) \) and \( r_> = \max(r, r_0) \), and we have used the notation

\[
D_{lj}(a) = \frac{aR_{lj}'(a) - \lambda_j R_{lj}(a)}{aR_{lj}'(a) - \lambda_2 R_{lj}(a)}, \quad j = 1, 2.
\]

(22)

First, let us consider the case when the charge is situated outside the monopole’s core \((r_0 > a)\). Substituting the function \((19)\) into \((13)\), we see that the Green function is presented in the form of the sum

\[
G(r, r_0) = G_m(r, r_0) + G_c(r, r_0),
\]

(23)

where

\[
G_m(r, r_0) = \frac{1}{4\pi a r_>} \sum_{l=0}^{\infty} \frac{2l + 1}{\sqrt{\alpha^2 + 4(l + 1)}} \frac{r_<^{\lambda_j} P_l(\cos \gamma)}{r_0^{\lambda_j}},
\]

(24)

is the Green function for the geometry of a point-like global monopole, and the term

\[
G_c(r, r_0) = -\frac{1}{4\pi a} \sum_{l=0}^{\infty} \frac{(2l + 1)D_{lj}(a)}{\sqrt{\alpha^2 + 4(l + 1)} (r_0)^{\lambda_j} P_l(\cos \gamma)}
\]

(25)

is induced by non-trivial structure of the core. In formulae \((24)\) and \((25)\), \( \gamma \) is the angle between the directions \((\theta, \varphi)\) and \((\theta_0, \varphi_0)\), and \( P_l(\gamma) \) represents the Legendre polynomials. It can be seen that \((24)\) coincides, up to the redefinition of the radial variable \( r \to \alpha r \), with the expression found in [11], for the case of a point-like global monopole spacetime. The part \((25)\) depends on the structure of the core through the radial function \( R_{lj}(r) \).

As we have already mentioned, the induced self-energy is obtained from the renormalized Green function taking the coincidence limit. We can observe that for points with \( r > a \), the core-induced term \((25)\) is finite in the coincidence limit and the divergence appears in the
point-like monopole part only. So, in order to provide a well-defined finite value to (10), we have to renormalize Green function \( G_m(\mathbf{r}, \mathbf{r}_0) \) only:

\[
G_{\text{ren}}(r_0, r_0) = G_{m, \text{ren}}(r_0, r_0) + G_\perp(r_0, r_0),
\]

(26)

As explained before, to find \( G_{m, \text{ren}}(r_0, r_0) \), we subtract from (24) the terms in the corresponding DeWitt–Schwinger adiabatic expansion which are divergent in the coincidence limit:

\[
G_{m, \text{ren}}(r_0, r_0) = \lim_{r \to r_0} \left[ G_m(\mathbf{r}, \mathbf{r}_0) - G_{m, \text{DS}}^{(\text{div})}(\mathbf{r}, \mathbf{r}_0) \right].
\]

(27)

The part \( G_{m, \text{DS}}^{(\text{div})}(\mathbf{r}, \mathbf{r}_0) \) is found from the general formula given, for instance, in [14], specifying the parameters for the problem under consideration. It can be seen that here the first term in the DeWitt–Schwinger expansion contributes only to the divergent part. For simplicity, taking the separation of the points along the radial direction only (\( \gamma = 0 \)), we find

\[
G_{m, \text{DS}}^{(\text{div})}(r, r_0) = \frac{1}{4\pi|r - r_0|}.
\]

(28)

Now, by using formulae (24) and (28), one obtains

\[
G_{m, \text{ren}}(r_0, r_0) = \frac{1}{4\pi r_0} \lim_{t \to 1} \left[ \frac{1}{\alpha} \sum_{l=0}^{\infty} \frac{2l + 1}{\sqrt{\alpha^2 + 4l(l + 1)}} t^{l+1} - \frac{1}{1 - t} \right],
\]

(29)

where \( t = r \to r_0 \). To evaluate the limit on the right, we note that

\[
\lim_{t \to 1} \left( \frac{1}{\alpha} \sum_{l=0}^{\infty} t^{l(\alpha+1)/2a - 1/2} - \frac{1}{1 - t} \right) = 0.
\]

(30)

On the basis of this relation, replacing in (29) \( 1/(1 - t) \) by the first term in the brackets in (30), we find

\[
G_{m, \text{ren}}(r_0, r_0) = \frac{S(\alpha)}{4\pi r_0},
\]

(31)

where we have introduced the notation

\[
S(\alpha) = \frac{1}{\alpha} \sum_{l=0}^{\infty} \left[ \frac{2l + 1}{\sqrt{\alpha^2 + 4l(l + 1)}} - 1 \right].
\]

(32)

The function \( S(\alpha) \) is positive (negative) for \( \alpha < 1 \) (\( \alpha > 1 \)) and, hence, the corresponding self-force is repulsive (attractive). Developing a series expansion in the parameter \( \eta_0^2 = 1 - \alpha^2 \), we can see that

\[
\alpha S(\alpha) = \sum_{n=1}^{\infty} \frac{(\pi \eta_0)^{2n}}{2(n!)^2} |B_{2n}| (1 - 2^{-2n}),
\]

where \( B_n \) are the Bernoulli numbers. The leading term in the expression on the right is \( \pi^2 (1 - \alpha^2)/16 \). For large values \( \alpha \) the main contribution into the series in (32) comes from large values \( l \). Replacing the summation by the integration we can see that in the limit \( \alpha \to \infty \) the function \( S(\alpha) \) tends to the limiting value \(-1/2\). For small values \( \alpha, \alpha \ll 1 \), the main contribution comes from the term \( l = 0 \) and one has \( S(\alpha) \approx 1/\alpha^2 \).

Combining formulae (10), (25) and (31), for the renormalized electrostatic self-energy we get

\[
U_{\text{el,ren}}(r_0) = \frac{q^2 S(\alpha)}{2r_0} - \frac{q^2}{2\alpha r_0} \sum_{l=0}^{\infty} \frac{(2l + 1) D_l(a)}{\sqrt{\alpha^2 + 4l(l + 1)}} \left( \frac{\alpha}{r_0} \right)^{l+1} \frac{1}{\sqrt{l+1}/l!}.
\]

(33)
The second term of the renormalized self-energy provides a convergent series for \( r_0 > a \). Here the dependence of the self-energy on the core structure appears through the function \( D_{1l}(a) \). For large distances from the core, \( r_0 \gg a \), the main contribution into the core-induced part comes from the term \( l = 0 \) and one has

\[
U_{el,\text{ren}}(r_0) \approx \frac{q^2}{2r_0} \left[ S(a) - \frac{a D_{10}(a)}{a^2 r_0} \right].
\]  

The self-force is obtained from (33) by using formula (11):

\[
f_{el,\text{ren}}(r_0) = \frac{q^2}{2a^2 r_0} \sum_{l=0}^{\infty} (2l + 1) D_{1l}(a) \left( \frac{a}{r_0} \right) \sqrt{1 + 4l(l+1)/a^2}.
\]

In accordance with the symmetry of the problem, the self-force has only a radial component. We can see that the same result for the self-force is obtained on the basis of formula (9).

Now we turn to the case when the charge is inside the core, \( r_0 < a \). The corresponding Green function is obtained from (20) and is written in the form

\[
G(r, r_0) = G_0(r, r_0) + G_a(r, r_0),
\]

where

\[
G_0(r, r_0) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) R_{1l}(r_<) R_{2l}(r_>) P_l(\cos \gamma)
\]

is the Green function for the background geometry described by the line element (2) for all values \( r_c \leq r < \infty \), and the term

\[
G_a(r, r_0) = -\frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) R_{1l}(r_0) R_{2l}(r) D_{1l}(a) P_l(\cos \gamma)
\]

is due to the global monopole geometry in the region \( r > a \). For the points away from the core boundary the latter is finite in the coincidence limit. The self-energy for the charge inside the core is written in the form

\[
U_{el,\text{ren}}(r_0) = 2\pi q^2 G_{0,\text{ren}}(r_0, r_0) - \frac{q^2}{2} \sum_{l=0}^{\infty} (2l+1) D_{2l}(a) R_{1l}^2(r_0),
\]

where

\[
G_{0,\text{ren}}(r_0, r_0) = \lim_{r \to r_0} \left[ G_0(r, r_0) - G_{0,\text{div}}(r, r_0) \right].
\]

The only contribution in the divergent part of the Green function comes from the first term of the DeWitt–Schwinger expansion. Note that near the centre of the core one has \( R_{1l}(r_0) \propto (r_0 - r_c)^l \) and the main contribution into the second term on the right of (39) comes from the term with \( l = 0 \). Substituting the self-energy given by (39) into formula (11), we obtain the self-force for the charge inside the monopole core.

### 3. Flower-pot model

As we have mentioned in the introduction, it is not possible to provide a closed expression to the metric tensor in the region inside the global monopole. A few years ago, Harari and Lousto [15] proposed a simplified model for the monopole where the region inside the core is described by the de Sitter geometry. The vacuum polarization effects associated with a massless scalar field in the region outside the core of this model have been investigated in [16].
As to the cosmic string spacetime, in the literature two different models have been adopted to describe the geometry inside core: the ballpoint-pen model proposed independently by Gott and Hiscock [17], which corresponds to replacing the conical singularity at the string axis by a constant curvature spacetime in the interior region, and the flower-pot model [18], where the curvature is concentrated on a ring and the spacetime inside the string is flat. Adopting the latter model for a global monopole, in [12] we were able to provide exact expressions for the vacuum polarization effects associated with a massive scalar field in both exterior and interior regions of the monopole spacetime. So, motivated by this result we decided, as an illustration of the general results described above, to consider this model in the present analysis of the zero curvature condition one finds from the continuity condition for the function \( w(r) \) at the core boundary which gives \( \text{const} = (\alpha - 1)a \). Hence, in the flower-pot model the interior line element has the form

\[
\text{d}s^2 = \text{d}r^2 - [r + (\alpha - 1)a]^2(d^2\theta + \sin^2\theta \text{ d}^2\psi). \tag{41}
\]

In terms of the radial coordinate \( r \) the origin is located at \( r = r_c = (1 - \alpha)a \). Defining \( \tilde{r} = r + (\alpha - 1)a \), the line element takes the standard Minkowskian form. From the Israel matching conditions for the metric tensors corresponding to (1) and (41), we find the nonzero components of the corresponding surface energy–momentum tensor located on the bounding surface \( r = a \) [12]:

\[
0 = 2\tau_0, \quad 2\tau_2 = 2\tau_3 = \frac{1}{4\pi} \frac{1 - 1}{\alpha}. \tag{42}
\]

Note that the surface energy density is positive for \( \alpha < 1 \).

Now in the flower-pot model we can express the renormalized Green function in the region outside the monopole core by taking into account that in the interior region we have the linearly independent solutions

\[
R_{\ell}(r) = \tilde{r}^{\ell}, \quad R_{\ell+1}(r) = \tilde{r}^{l-1}/(2l + 1). \tag{43}
\]

So, from formula (33), the self-energy in the exterior region reads

\[
U_{\text{el,ren}}(r_0) = \frac{q^2 S(\alpha)}{2r_0} + \frac{2q^2(1 - \alpha)}{ar_0} \sum_{l=0}^{\infty} \frac{l(2l + 1)}{\sqrt{a^2 + 4l(l + 1)}} \frac{(a/r_0)\sqrt{4l(l + 1)/a^2}}{\sqrt{\alpha^2 + 4l(l + 1) + \alpha + 2l(l + 1)}}. \tag{44}
\]

The second term on the right-hand side of this formula is positive for \( \alpha < 1 \) and negative for \( \alpha > 1 \). Combining this with the properties of the function \( S(\alpha) \) discussed in the previous section, we conclude that the electrostatic self-energy is positive for \( \alpha < 1 \) and negative for \( \alpha > 1 \). The corresponding self-force is directly found from (35) and is repulsive in the first case and attractive in the second one. For large distances from the monopole core the main contribution into the core-induced part comes from the \( \ell = 1 \) term (note that the \( \ell = 0 \) term vanishes) and we have

\[
U_{\text{el,ren}}(r_0) \approx \frac{q^2}{2r_0} \left[ S(\alpha) + \frac{12(1 - \alpha)}{\alpha\sqrt{\alpha^2 + 8} + 8} \frac{(a/r_0) \sqrt{1 + 8/a^2}}{\sqrt{\alpha^2 + 8 + \alpha + 2^2}} \right], \quad a \ll r_0. \tag{45}
\]

In this limit the effects induced due to the finite core are relatively suppressed by the factor \((a/r_0)\sqrt{1 + 8/a^2}\). The core-induced part in (44) diverges at the core boundary, \( r_0 = a \). The surface divergences in the coincidence limit of the Green function in field theories on background of manifolds with boundaries are well investigated in the literature related to the Casimir effect. Noting that for points near the boundary the main contribution into (44)
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Figure 1. Electrostatic self-energy in the flower-pot model for a charge outside the monopole core as a function of the monopole parameter $\alpha$ and rescaled radial coordinate $r_0/a$.

comes from large values $l$, to the leading order we find

$$U_{el,\, \text{ren}}(r_0) \approx q^2 \alpha - \frac{1}{8\alpha a} \ln[1 - (a/r_0)^{1/\alpha}],$$

and the self-energy is dominated by the core-induced part.

Now we turn to the investigation of the core-induced part in asymptotic regions of the parameter $\alpha$. For large values $\alpha$ we replace the summation over $l$ by the integration and to the leading order we find

$$U_{el,\, \text{ren}}(r_0) \approx \frac{q^2}{4r_0} \left[ 1 + \int_1^\infty dx \frac{\sqrt{x^2 - 1} - x + 1}{\sqrt{x^2 - 1} + x + 1} \left( \frac{a}{r_0} \right)^x \right], \quad \alpha \gg 1. \quad (47)$$

Hence, in the limit $\alpha \to \infty$ the renormalized self-force tends to the finite limiting value. For small values $\alpha$, the main contribution into the core-induced part of the self-energy comes from the mode $l = 1$ and this part is exponentially suppressed by the factor $\exp \left[ 2\sqrt{2} \ln(a/r_0)/\alpha \right]/\alpha$.

We recall that in this limit the point-like monopole part behaves like $1/\alpha^2$ and, hence, it strongly dominates. In figure 1, we have plotted the electrostatic self-energy in the flower-pot model for a charge outside the core versus the parameter $\alpha$ and the radial coordinate of the charge.

Now we turn to the investigation of the self-energy in the flower-pot model for the particle inside the monopole core. Substituting the functions (43) into formulae (37) and (38), for the corresponding Green functions in the interior region one finds

$$G_0(r, r_0) = \frac{1}{4\pi |r - r_0|}, \quad (48)$$

$$G_\alpha(r, r_0) = \frac{1}{4\pi a \alpha} \sum_{l=0}^\infty \frac{2l + 2 - \alpha - \sqrt{\alpha^2 + 4l(l + 1)}}{2l + \alpha + \sqrt{\alpha^2 + 4l(l + 1)}} \left( \frac{\tilde{r}_0 \alpha}{a} \right)^l P_l(\cos \gamma). \quad (49)$$

Because in the flower-pot model the geometry in the region inside the monopole is a Minkowski one, we have $G_{D,\, \text{DS}}^{(0)}(r, r_0) = G_0(r, r_0)$ and, hence, $G_{0,\, \text{ren}}(r_0, r_0) = 0$. Finally, the electrostatic self-energy in the region inside the monopole core reads

$$U_{el,\, \text{ren}}(r_0) = 2\pi q^2 G_{\text{ren}}(r_0, r_0) = \frac{q^2}{2\alpha a} \sum_{l=0}^\infty \frac{2l + 2 - \alpha - \sqrt{\alpha^2 + 4l(l + 1)}}{2l + \alpha + \sqrt{\alpha^2 + 4l(l + 1)}} \left( \frac{\tilde{r}_0 \alpha}{a} \right)^l. \quad (50)$$
As in the case of the exterior region, this self-energy is positive for \( \alpha < 1 \) and negative for \( \alpha > 1 \). The corresponding self-force is easily found from relation (11) and is repulsive with respect to the boundary of the monopole core in the first case and attractive in the second case.

Near the core centre the main contribution into the self-energy comes from the lowest modes and one has

\[
U_{\text{el},\text{ren}}(r_0) \approx \frac{q^2}{2\alpha a} \left[ \frac{1 - \alpha}{\alpha} + \frac{4 - \alpha - \sqrt{\alpha^2 + 8}}{2 + \alpha + \sqrt{\alpha^2 + 8}} \left( \frac{r_0}{\alpha a} \right)^2 \right].
\]

(51)

As in the exterior case, on the core surface the self-energy given by (50) diverges. Under the condition \( 1/\ln(\alpha a/r_0) \gg \alpha \), the leading term in the corresponding asymptotic expansion is given by the formula

\[
U_{\text{el},\text{ren}}(r_0) \approx \frac{q^2}{8\alpha a} (1 - \ln \left( \frac{r_0}{\alpha a} \right)).
\]

(52)

For large values \( \alpha \), assuming that the ratio \( r_0/\alpha a \) is fixed and \( \alpha \gg 1/\ln(\alpha a/r_0) \) (note that \( \alpha a \) is the core radius for an internal Minkowskian observer), from (50) we find

\[
U_{\text{el},\text{ren}}(r_0) \approx -\frac{q^2}{2\alpha a} \frac{1}{1 - (r_0/\alpha a)^2}.
\]

(53)

For small values \( \alpha \) with the fixed value of the ratio \( r_0/\alpha a \), the leading term in the self-energy is obtained substituting \( \alpha = 0 \) into the fraction of the expression under the summation sign in (50). In figure 2, we have presented the dependence of the electrostatic self-energy in the flower-pot model for a charge inside the core as a function on \( \alpha \) and \( r_0/\alpha a \).

Up to now we have considered the electrostatic self-interaction for a point-like particle. Similar results with the replacement of the electric charge by the magnetic one can be obtained for a point-like magnetic charge in the global monopole spacetime. In particular, from the results given above it follows that with regard to finite core the composite system of global and magnetic monopoles proposed in [19] can be stable. Note that in the model with a point-like global monopole the corresponding system can have a problem with the stability [20]. Assuming that the gravitational field of the particle can be adequately described by the Newtonian potential in the global monopole spacetime, we can also evaluate the gravitational
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The corresponding self-energy is related to the electrostatic self-energy by the formula

$$U_{gr, ren}(r_0) = -(GM^2/q^2)U_{el, ren}(r_0),$$

where $G$ is the Newton gravitational constant and $M$ is the mass of the particle.

4. Concluding remarks

The objective of this paper was to analyse the induced self-energy and the self-force for a point-like electric charge placed at rest in the spacetime of a global monopole considering a non-trivial inner structure for the core. As it was previously shown [11], for an idealized core of a point-like global monopole, the above quantities present a singular behaviour at the monopole’s position, $r = 0$. In a more realistic model for the global monopole, we should not expect this kind of divergences. So, partly motivated by this idea, we decided to return to this analysis considering a non-vanishing radius to the monopole core. In addition, this investigation enables us to clarify the role of the finite core effects on the induced self-interaction. The latter is a consequence of the distortion on the particle’s electric field caused by the spacetime curvature and topology. For the general spherically symmetric static model of the core with finite thickness we have constructed the corresponding three-dimensional Green function in both exterior and interior regions. In the region outside the core this function is presented as a sum of two distinct contributions. The first one corresponds to the Green function for the geometry of a point-like global monopole, previously investigated in [11], and the second one is induced by the non-trivial structure of the monopole core. The latter is given by formula (25) with the coefficient from (22). This coefficient is determined by the interior radial solution regular at the core centre and describes the influence of the core properties on the physical characteristics in the exterior region. The electrostatic self-energy and self-force are obtained from the Green function in the coincidence limit after the subtraction of the corresponding divergent part. This procedure corresponds to the renormalization of the particle mass. For points very far away from the core the most relevant contribution is given by the first part of the self-energy, while for points near the core surface the most relevant part is represented by the second contribution. For the particle inside the monopole core the electrostatic self-energy is given by formula (39), where the first term on the right-hand side is the self-energy for the background geometry described by the line element (2) for all values $r_c \leq r < \infty$, and the second term is due to the global monopole geometry in the region $r > a$. Having the electrostatic self-energy we can evaluate the corresponding self-force by using formula (11):

$$f_{el, ren}(r_0) = \frac{q^2 r_0}{2\alpha^2} \left\{ S(\alpha) - \sum_{l=0}^{\infty} \frac{2l + 1}{\alpha^2} D_l(a) \left[ \frac{\alpha}{\sqrt{\alpha^2 + 4l(l + 1)}} + 1 \right] \left( a \frac{a}{r_0} \right)^{1+4l(l+1)/\alpha^2} \right\},$$

where $S(\alpha)$ is defined by formula (32).

As an example of the application of the general results, in section 3 we have considered a simple core model with a flat spacetime inside the core, the so-called flower-pot model. In this model, the self-energies in the exterior and interior regions are given by formulae (44) and (50), respectively. The corresponding self-forces are repulsive with respect to the core boundary in the case $\alpha < 1$ and attractive for $\alpha > 1$. In particular, for the first case, the charge placed at the core centre is in a stable equilibrium position. Similar results can be obtained for the case of a point-like magnetic charge. We have investigated the expressions...
for the self-energies in various asymptotic regions of the parameters. In particular, it has been shown that for large values \( \alpha \) the renormalized self-energy in the exterior region tends to a finite limiting value. For small values \( \alpha \), the core-induced part is exponentially suppressed by the factor \( \exp[2\sqrt{2} \ln(\alpha/\alpha_0)/\alpha] \), while the point-like monopole part behaves like \( 1/\alpha^2 \) and, hence, the latter strongly dominates. Although in the flower-pot model we have found a finite value of the self-energy at the monopole’s centre, it presents a logarithmic singular behaviour at the core boundary. In this way the singular behaviour becomes softer than for the point-like monopole case. The corresponding divergences are related to the idealization assuming that the transition range between the interior and exterior geometries has zero thickness. We expect that the results obtained in the present paper will also be valid in a more realistic model at distances from the transition range much greater than the thickness of this range.

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