Characterizations of the Hardy Space $H^1$
and BMO

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Abstract
We describe the spaces $H^1(R)$ and BMO($R$) in terms of their closely related, simpler dyadic and two-sided counterparts. As a result of these characterizations we establish when a bounded linear operator defined on dyadic or two-sided $H^1(R)$ into a Banach space has a continuous extension to $H^1(R)$ and when a bounded linear operator that maps a Banach space into dyadic or two-sided BMO($R$) actually maps continuously into BMO($R$).

1 Introduction
In this paper we seek to elucidate the role simple atoms, such as the Haar system, play in the theory of the Hardy space $H^1(R) = H^1$. It becomes quickly apparent that the Haar system, or more generally dyadic atoms, do not suffice to span $H^1$. However, the fact that arbitrary atoms can be written as the sum of at most three atoms, two dyadic and a special atom, makes it possible to gain a greater insight into the structure of $H^1$ and its dual, BMO. We pass now to describe the specific results.

By the Hardy space $H^1(R) = H^1$ we mean the collection of those integrable functions which admit an atomic decomposition in terms of $L^\infty$ atoms. Recall that a compactly supported function $a$ with vanishing integral is an $L^\infty$ atom, or plainly an atom, with defining interval $I$, if

$$ \text{supp}(a) \subseteq I, \quad |a(x)| \leq \frac{1}{|I|}, \quad \text{and} \quad \int_I a(x) \, dx = 0. $$

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$H^1$ is then the Banach space consisting of those $f$’s such that

$$H^1 = \left\{ f = \sum_{j=1}^{\infty} \lambda_j a_j : \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\},$$

where the convergence is in the sense of distributions as well as in $L^1$, the $a_j$’s are $L^\infty$ atoms, and the atomic norm is given by

$$\|f\|_{H^1} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$

For these, and all other well-known basic facts used throughout the article, see [11], [19] and [20].

$H^1_d(R) = H^1_d$, or dyadic $H^1$, is obtained by restricting the defining intervals to be dyadic. Clearly $\|f\|_{H^1} \leq \|f\|_{H^1_d}$, and the inclusion $H^1_d \subset H^1$ is strict. As for $H^1_d$, the closure of $H^1_d$ in $H^1$, it turns out to be the space

$$H^A = \left\{ f \in H^1 : \int_{0}^{\infty} f(x) \, dx = 0 \right\}.$$

More to the point: if $\{H_I\}$ denotes the $H^1$ normalized Haar system indexed by the dyadic intervals $I$ of $R$, then $\overline{\text{span}}\{H_I\}$, the closed span of the Haar system in $H^1$, is also $H^A$.

$H^A$ is not a convenient space – for instance, $f \in H^A$ does not imply that $f \chi_{[0,\infty)}$ belongs to $H^1$ – and we are led to introduce two-sided $H^1$, or $H^1_{2a}(R) = H^1_{2a}$. This is the space of $H^1$ functions with atomic decompositions in terms of atoms whose defining interval lies on either side of the origin. Endowed with the atomic norm, $H^1_{2a}$ is a proper subspace of $H^A$ continuously included in $H^1$. Also, if $f \in H^1_d$, $\|f\|_{H^1_{2a}} \leq \|f\|_{H^1_d}$, and $\overline{\text{span}}\{H_I\}$, the closed span of the Haar system in $H^1_{2a}$, is $H^1_{2a}$, see [18].

As for the atoms themselves, we note that each atom can be written as the sum of at most three atoms, two dyadic and a special atom, see [9]. This allows us to identify $H^1$ as the sum of $H^1_d$ and a space generated by special atoms and to establish the boundedness of linear operators from $H^1$ taking values in a Banach space $X$ given that they map $H^1_d$ into $X$. Similar results hold for $H^A$ and $H^1_{2a}$.

The space of functions of bounded mean oscillation, $\text{BMO}(R) = \text{BMO}$, consists of those locally integrable functions $\varphi$ with

$$\|\varphi\|_* = \sup_I \frac{1}{|I|} \int_I |\varphi(x) - \varphi_I| \, dx < \infty,$$
where $\varphi_I = \frac{1}{|I|} \int_I \varphi(y) \, dy$ is the average of $\varphi$ over $I$, and the sup is taken over all bounded intervals $I$. Modulo constants, $(\text{BMO}, \| \cdot \|_*)$ is a Banach space.

$\text{BMO}_d(R) = \text{BMO}_d$, or dyadic BMO, is defined by restricting the intervals above to be dyadic. The sup is now denoted by $\| \cdot \|_{*,d}$, and modulo two constants (one for each side of the origin), $(\text{BMO}_d, \| \cdot \|_{*,d})$ also becomes a Banach space.

Finally, when the above sup is restricted to those $I$’s that lie on either side of the origin, the sup is denoted $\| \cdot \|_{2s}$, and we have two-sided BMO or $\text{BMO}_{2s}(R) = \text{BMO}_{2s}$. Modulo two constants, $(\text{BMO}_{2s}, \| \cdot \|_{2s})$ becomes a Banach space.

$\text{BMO}$ is the dual of $H^1$, $\text{BMO}_d$ is the dual of $H^1_d$, and below note that $\text{BMO}_{2s}$ is the dual of $H^1_{2s}$ and identify the dual of $H^A$. Each of the decompositions of $H^1$ suggests a characterization of BMO in terms of these dual spaces. These new characterizations of BMO in turn allow us to work in dyadic and dyadic-like settings, and provide us with an effective way to pass from $\text{BMO}_d$, and $\text{BMO}_{2s}$, to BMO. We also establish when a bounded linear operator that maps a Banach space $X$ into $\text{BMO}_d$ actually maps $X$ into BMO.

The paper is organized as follows. Section two is devoted to $H^1$, in section three we discuss BMO, and we conclude the paper in section four by pointing out how these results can be extended to $\mathbb{R}^n$, $n > 1$.

2 The Hardy Space $H^1(R)$

The Haar System

A dyadic interval $I$ is an interval of the special form

$$ I = I_{n,k} = [k2^n, (k + 1)2^n), $$

where $k$ and $n$ are arbitrary integers, positive, negative or 0. Note that $I = I_L \cup I_R$, where the left half $I_L$ and the right half $I_R$ of $I$ are also dyadic.

For each dyadic interval $I$, the $H^1$ normalized Haar function $H_I$ is given by

$$ H_I(x) = \frac{1}{|I|} \chi_{I_L}(x) - \frac{1}{|I|} \chi_{I_R}(x). $$
Finally, throughout the paper $b$ will denote the function
\[ b(x) = \frac{1}{2} \left[ \chi(-1,0)(x) - \chi(0,1)(x) \right]. \]

To describe how $H^1_d$ fits in $H^1$, let $a$ be an atom. If the origin is not an interior point of the defining interval of $a$, $a$ is a multiple of a dyadic atom. On the other hand, if the origin is interior to the defining interval of $a$, let
\[ a(x) = \left[ a(x) + 2 \left( \int_0^\infty a(y) \, dy \right) b(x) \right] - 2 \left( \int_0^\infty a(y) \, dy \right) b(x). \]
Since $a$ has vanishing integral the first function above is a linear combination of two dyadic atoms with defining intervals on opposite sides of the origin, and the second is a multiple of the fixed function $b$. Thus $H^1_d$ is of codimension one in $H^1$, and since $H^1_d \subset H^A$, actually $H^1_d = H^A$.

To show that the same is true for the closed span of the Haar system in $H^1$, we make use of the following observation. Its proof is left to the reader.

**Lemma 2.1.** Suppose a locally integrable function $\varphi$ satisfies
\[ \int_R H_I(x) \varphi(x) \, dx = 0 \quad \text{for all dyadic intervals } I. \]
Then for some constants $c, d,$
\[ \varphi(x) = d \chi(-\infty,0)(x) + c \chi(0,\infty)(x). \]

We then have,

**Theorem 2.1.** The closed span of the Haar system in $H^1$ is $H^A$.

**Proof.** It suffices to show that $\mathfrak{s}^d \{ H_I \}$, the closed span of the Haar system in $H^1_d$, is $H^1_d$. Let $L$ be a bounded linear functional on $H^1_d$ that vanishes on the $H_I$’s. Then there is $\varphi \in \text{BMO}_d(R)$ with $\| \varphi \|_* \sim \| L \|$ such that for compactly supported $f \in H^1_d$, $L(f) = \int_R f(x) \varphi(x) \, dx$. Now, since $L$ vanishes on the $H_I$’s, by Lemma 2.1, $\varphi(x) = d \chi(-\infty,0)(x) + c \chi(0,\infty)(x)$, and consequently for those $f$’s
\[ L(f) = d \int_{-\infty}^0 f(x) \, dx + c \int_0^\infty f(x) \, dx = 0. \]
Thus $L$ is the zero functional and we have finished. ■

The reader will have no difficulty in establishing the quantitative version of Theorem 2.1 in terms of $d(f, H^A)$, the distance of $f$ in $H^1$ to $H^A$.

**Proposition 2.1.** Let $f \in H^1$. Then $d(f, H^A) \sim \left| \int_0^\infty f(x) \, dx \right|$. 

4
Structure of Atoms

Since dyadic atoms do not suffice to represent $H^1$ functions, we consider how far apart dyadic atoms are from arbitrary atoms. The answer is that an arbitrary $H^1$ atom can be expressed as a sum of at most three atoms, two dyadic and a special atom, see [9].

Lemma 2.2. Let $a$ be an $H^1$ atom. Then there are at most three atoms $a_L, a_R, b_{n,k}$, such that

1. $a_L$ and $a_R$ are dyadic atoms.
2. For some integers $n, k$,
   \[ b_{n,k}(x) = \frac{1}{2^{n+1}} \left[ \chi_{[(k-1)2^n,k 2^n]}(x) - \chi_{[k 2^n,(k+1)2^n]}(x) \right]. \]
3. $a = c_1 a_L + c_2 a_R + c_3 b_{n,k}$, where $|c_1|, |c_2|, |c_3| \leq 4$.

Proof. Let $I$ be the defining interval for $a$, and let $n$ be the integer such that $2^{n-1} \leq |I| < 2^n$ and $k$ the integer such that $I \subset [k 2^n, (k+1)2^n]$. Set now $a_L$ equal to

\[
    a_L(x) = \begin{cases} 
        \frac{1}{4} \left( a(x) - \frac{1}{2^n} \int_{[(k-1)2^n,k 2^n]} a(y) \, dy \right), & x \in [(k-1)2^n,k 2^n), \\
        0, & \text{otherwise.} 
    \end{cases}
\]

Since $a$ is an atom with defining interval $I$ it readily follows that

\[ \|a_L\|_\infty \leq \frac{1}{4} \left( \frac{1}{|I|} + \frac{1}{|I|} \right) \leq \frac{1}{2 |I|} \leq 2^{-n}. \]

Furthermore, since $a_L$ is supported in $[(k-1)2^n, k 2^n]$ and has integral 0, $a_L$ is a dyadic atom.

Similarly, set $a_R$ equal to

\[
    a_R(x) = \begin{cases} 
        \frac{1}{4} \left( a(x) - \frac{1}{2^n} \int_{[k 2^n,(k+1)2^n]} a(y) \, dy \right), & x \in [k 2^n, (k+1)2^n), \\
        0, & \text{otherwise.} 
    \end{cases}
\]

$a_R$ is supported in $[k 2^n, (k+1)2^n]$, $\|a_R\|_\infty \leq 2^{-n}$, and has integral 0, so $a_R$ is also a dyadic atom.
Finally put

\[ b_{n,k}(x) = \frac{1}{2^{n+1}} \left[ \chi_{[(k-1)2^n,k2^n]}(x) - \chi_{[k2^n,(k+1)2^n]}(x) \right]. \]

Since \( a \) has vanishing integral and \( I \subset [(k - 1)2^n, (k + 1)2^n] \) it is also true that

\[ -\int_{[(k-1)2^n,k2^n]} a(y) \, dy = \int_{[k2^n,(k+1)2^n]} a(y) \, dy, \]

and consequently, since

\[ a(x) - 4a_L(x) - 4a_R(x) \]

is equal to

\[
\begin{cases}
\frac{1}{2^n} \int_{[(k-1)2^n,k2^n]} a(y) \, dy, & x \in [(k-1)2^n,k2^n], \\
\frac{1}{2^n} \int_{[k2^n,(k+1)2^n]} a(y) \, dy, & x \in [k2^n,(k+1)2^n],
\end{cases}
\]

we have

\[ a(x) = 4a_L(x) + 4a_R(x) + 2 \left( \int_{[(k-1)2^n,k2^n]} a(y) \, dy \right) b_{n,k}(x). \]

Thus the conclusion follows in this case with \( c_1 = c_2 = 4 \) and

\[ |c_3| \leq 2 \int_{[(k-1)2^n,k2^n]} |a(y)| \, dy \leq 2 \frac{2^n}{|I|} \leq 4. \]

\section*{H as the sum of Banach Spaces}

As a first application of the decomposition of atoms we will show that \( H \) can be written as the sum of various Banach spaces. We have already seen that \( H = H^A + \text{sp}\{b\} \) as linear spaces. In fact, the reader will have no difficulty in verifying that actually \( H = H^A + \text{sp}\{h\} \), where \( h \) is an arbitrary function in \( H \setminus H^A \), and \( \|f\|_{H^1} \sim \|f\|_{H^A + \text{sp}\{h\}} \).

Furthermore, for integers \( n, k \), let \( b_{n,k} \) denote the dyadic dilations and integer translations of \( b \), i.e., the collection of atoms given by

\[ b_{n,k}(x) = \frac{1}{2^{n+1}} \left[ \chi_{[(k-1)2^n,k2^n]}(x) - \chi_{[k2^n,(k+1)2^n]}(x) \right]. \]

Note that the special atoms \( b_{n,k} \) are multiples of dyadic atoms if \( k \) is odd, but not if \( k \) is even. Also, if \( k \neq 0 \), the support of \( b_{n,k} \) lies on one side of the origin.
Let $H^1_\delta(R) = H^1_\delta$ be the space which consists of the linear combinations

$$H^1_\delta = \{ \varphi = \sum_{1}^{\infty} \lambda_j b_{n_j,k_j} : \sum_{1}^{\infty} |\lambda_j| < \infty \}.$$ 

Endowed with the atomic norm

$$\|\varphi\|_{H^1_\delta} = \inf \left\{ \sum_{1}^{\infty} |\lambda_j| : \varphi = \sum_{1}^{\infty} \lambda_j b_{n_j,k_j} \right\},$$

$(H^1_\delta, \| \cdot \|_{H^1_\delta})$ is a Banach space. Observe that if $f \in H^1_\delta$, then $f \in H^1$ and $\|f\|_{H^1} \leq \|f\|_{H^1_\delta}$.

Similarly, when $k = 0$ we denote the resulting space $H^1_{\delta,0}$. It is clear that if $f \in H^1_{\delta,0}$, then $f \in H^1_\delta$ and $\|f\|_{H^1} \leq \|f\|_{H^1_{\delta,0}}$. $H^1_{\delta}$ and $H^1_{\delta,0}$ are the spaces of special atoms alluded to above, see [7].

From Lemma 2.2 it readily follows that

**Proposition 2.2.** $H^1 = H^1_{\delta} + H^1_{\delta,0}$, and $\|f\|_{H^1} \sim \|f\|_{H^1_{\delta} + H^1_{\delta,0}}$.

The meaning of this decomposition is the following. The Haar system, or more generally the dyadic atoms, divide the line in two regions, $(-\infty, 0]$ and $[0, \infty)$. To allow for the information carried by a dyadic interval to be transmitted to an adjacent dyadic interval, they must be connected. The $b_{n,0}$’s channel information across the origin and the remaining $b_{n,k}$’s connect adjacent dyadic intervals that are not subintervals of the same dyadic interval.

We also have

**Proposition 2.3.** $H^1 = H^1_{2s} + H^1_{\delta,0}$, and $\|f\|_{H^1} \sim \|f\|_{H^1_{2s} + H^1_{\delta,0}}$.

This characterization allows us to identify the range of the projection mapping $P$ of $H^1$ functions $f$ into $Pf = \chi_{[0,\infty)} f$. It is isomorphic to $H^1_o$, the odd functions of $H^1$, and consists of those functions in $L^1(R^+)$ whose Telyakovskii transform also belongs to $L^1(R^+)$, see [10].

The reader will have no difficulty of verifying the following observation, which is useful when considering mappings $T : H^1 \to X$.

**Proposition 2.4.** Let $B = B_0 + B_1$, where $B_0, B_1$ are Banach spaces, and assume $T$ is a linear operator that maps $B_0$ continuously into a Banach
space \( X \) with norm \( \| T \|_0 \). Then \( T : B \to X \) with norm \( \| T \| \) if and only if \( T : B_1 \to X \) is bounded with norm \( \| T \|_1 \) and

\[
\| T \| = \max ( \| T \|_0 , \| T \|_1 ).
\]

We now apply Proposition 2.4 to Hardy type operators in the setting \( H^1 = H^1_{2s} + H^1_{\delta,0} \). For \( 0 \leq \varepsilon \leq 1 \) let \( \tau_\varepsilon \) be given by

\[
\tau_\varepsilon f(x) = \frac{1}{|x|^{1-\varepsilon}} \int_{-x}^{x} f(y) dy, \quad x \neq 0 .
\]

We set \( 1/p = 1 - \varepsilon , 1 \leq p \leq \infty \), and consider when \( \tau_\varepsilon f \) is in \( X = L^p(R) \). Since \( \tau_\varepsilon b_{n,0} = 0 \) for \( b_{n,0} \in H^1_{\delta,0} \), the continuity on \( H^1 \) is equivalent to that on \( H^1_{2s} \). The case \( \varepsilon = 1 \) is trivial and merely states \( \| \tau_1 f \|_\infty \leq \| f \|_{H^1} \).

For the remaining cases we begin by observing that for a two-sided atom \( a \) with defining interval \( I \), \( \tau_\varepsilon a \) is also supported in \( I \), and \( \left( \int_R |\tau_\varepsilon a(x)|^p dx \right)^{1/p} \leq \| \ln | \cdot | \|^{1/p}_{*2s} \). Let now \( f \in H^1_{2s} \) have the atomic decomposition \( f(x) = \sum_j \lambda_j a_j \). Since the convergence also takes place in \( L^1 \), it readily follows that \( \tau_\varepsilon f(x) = \sum_j \lambda_j \tau_\varepsilon a_j(x) \), and thus by Minkowski’s inequality, upon taking the infimum over all possible decompositions of \( f \), we get

\[
\left( \int_R |\tau_\varepsilon f(x)|^p dx \right)^{1/p} \leq \| \ln | \cdot | \|^{1/p}_{*2s} \| f \|_{H^1_{2s}}.
\]

In short, \( \tau_\varepsilon \) maps \( H^1 \) into \( L^p(R) \) with norm \( \leq \| \ln | \cdot | \|^{1/p}_{*2s} \). A similar reasoning applies to the more general operators of Hardy type discussed in [11], which include the Fourier transform.

Sublinear operators may be treated in a similar fashion. Consider, for instance, \( \widetilde{M}_{\varepsilon,d} \), the maximal operator on \( H^1_d \) given by

\[
\widetilde{M}_{\varepsilon,d} f(x) = \sup_{x \in I} \frac{1}{|I|^{1-\varepsilon}} \left| \int_I f(y) dy \right| ,
\]

where \( 0 \leq \varepsilon < 1 \), and \( I \) varies over the collection of dyadic intervals containing \( x \), see [3]. If \( a \) is a dyadic atom with defining interval \( I \),

\[
\widetilde{M}_{\varepsilon,d} a(x) \leq \frac{1}{|I|^{1-\varepsilon}} \chi_I(x),
\]

and consequently

\[
\| \widetilde{M}_{\varepsilon,d} a \|_p \leq c , \quad 0 < \frac{1}{p} = 1 - \varepsilon .
\]
Thus $\tilde{M}_{\varepsilon,d}$ is uniformly bounded on atoms and since it satisfies an appropriate $\sigma$-sublinearity condition, it can be extended continuously as a mapping from $H^1_d$ into $L^p$. However, since $\tilde{M}_{\varepsilon,d} b(x) \sim \min(1, |x|^{\varepsilon-1})$, $\tilde{M}_{\varepsilon,d}$ only maps the special atoms into $\text{wk}-L^p$. So, $\tilde{M}_{\varepsilon,d}$ is bounded on $H^1_d$ but not on $H^1$.

On the other hand, the truncated version of this maximal operator is better behaved. For a positive integer $N$, let $\tilde{M}^N_{\varepsilon,d} f(x) = \sup_{x \in I} \frac{1}{|I|^{1-\varepsilon}} \left| \int_I f(y) \, dy \right|$, where $0 \leq \varepsilon < 1$, and $I$ varies over the collection of dyadic intervals containing $x$ of size $2^{-N} \leq |I| \leq 2^N$. Then $\tilde{M}^N_{\varepsilon,d}$ also maps the $b_{n,k}$’s into $L^p$, and this maximal function maps $H^1$ into $L^p$.

**The duals of $H^1_{2s}$ and $H^A$**

An argument along by now familiar ideas allows us to identify the dual of $H^1_{2s}$ as $\text{BMO}_{2s}$. Indeed, we have

**Theorem 2.2.** $\text{BMO}_{2s}$ is the dual of $H^1_{2s}$. More precisely, for every $\varphi$ in $\text{BMO}_{2s}$, the functional $L_\varphi$ defined initially for bounded compactly supported functions $f \in H^1_{2s}$ by the integral $L_\varphi(f) = \int f(x) \varphi(x) \, dx$ has a bounded extension to $H^1_{2s}$ with norm less than or equal to $c \| \varphi \|_{*,2s}$.

Moreover, for any functional $L \in (H^1_{2s})^*$, there is $\varphi \in \text{BMO}_{2s}$, with norm $\| \varphi \|_{*,2s} \sim \| L \|$, such that $L(f) = L_\varphi(f) = \int f(x) \varphi(x) \, dx$ for every compactly supported bounded $f \in H^1_{2s}$.

We consider the dual of $H^A$ next. Let $\varphi, \psi \in \text{BMO}$. We say that $\varphi \sim_2 \psi$ if for some constants $c, d$,

$$\varphi(x) - \psi(x) = d \chi_{(-\infty,0]}(x) + c \chi_{[0,\infty)}(x) \quad \text{a.e.}$$

Clearly $\sim_2$ is an equivalent relation, and the norm for the element $\Phi \in B = \text{BMO}/\sim_2$ with representative $\varphi \in \text{BMO}$ is given by

$$\| \Phi \|_B = \inf \{ \| \psi \|_* : \varphi \sim_2 \psi \} .$$

Now, since $(H^A)^*$ is isometrically isomorphic to

$$(H^1)^*/(H^A)^\perp = \text{BMO}/(H^A)^\perp,$$
we have \((H^A)^* = B\).

This identification allows us to distinguish between \(H^1_{2s}\) and \(H^A\). Indeed, since \(\chi(x,0,\infty)(x)\ln x\) is in \(\text{BMO}_{2s}\) but not in \(B\), the inclusion \(H^1_{2s} \subset H^A\) is strict. We can also exhibit \(f \in H^A \setminus H^1_{2s}\). For \(n \geq 1\), let \(f_n(x) = 2^n \chi(2^n,2^{n+1})(x)\), and with \(L = \sum_{n=1}^{\infty} n^{-2}\), let \(f(x)\) be the odd extension of the function

\[L \chi(0,1)(x) - \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x), \quad x > 0.\]

Then \(f \in H^A\), and \(\|f\|_{H^1} \leq 6L\). On the other hand, since

\[\int_0^\infty f_n(x) \ln x \, dx = -n \ln 2 + (2 \ln 2 - 1),\]

it readily follows that \(\int_0^\infty f(x) \ln x \, dx = \infty\), and \(f \notin H^1_{2s}\). In other words, \(f \in H^A\) and its projection \( Pf \) is an integrable function with vanishing integral supported in \([0,\infty)\), but \( Pf \notin H^1\). However, if \(g \in H^A\) vanishes for \(x < 0\), then \(P g \in H^1_{2s}\). For, if \(g\) has an atomic decomposition \(g(x) = \sum_j \lambda_j a_j(x)\), it also has the two-sided decomposition

\[g(x) = \sum_j 4 \lambda_j \frac{1}{4}[a_j(x) + a_j(-x)]\chi(0,\infty)(x).\]

### 3 Characterizations of BMO

#### From \(\text{BMO}_d\) to \(\text{BMO}\)

When restricted to linear functionals, Proposition 2.4 suggests different characterizations of \(\text{BMO}\). We discuss the dyadic case first. Given a \(\text{BMO}\) function \(\varphi\), consider the bounded linear functional on \(H^1\) induced by \(\varphi\). When acting on individual atoms, two conditions, one for dyadic atoms and the other for special atoms, must be satisfied for this functional to be bounded. The condition on the dyadic atoms suggests that \(\varphi \in \text{BMO}_d\), whereas the condition on the \(b_n,k\)'s, restates that the integral of \(\varphi\) is in the Zygmund class.

This motivates the following definition. For a locally integrable function \(\varphi\) let

\[A(\varphi) = \sup_{n,k} \frac{1}{2^{n+1}} \left| \int_{[k2^n,(k+1)2^n]} \varphi(x) \, dx - \int_{[k2^n,(k+1)2^n]} \varphi(x) \, dx \right|,\]
and put

$$\Lambda = \{ \varphi \in \text{BMO}_d : A(\varphi) < \infty \}, \quad \| \varphi \|_\Lambda = \max \left( \| \varphi \|_{*,d}, A(\varphi) \right).$$

Our next result describes how to pass from BMO$_d$ to BMO.

**Theorem 3.1.** BMO = \Lambda. More precisely, if \varphi \in BMO, then \varphi \in \Lambda and \| \varphi \|_\Lambda \leq \| \varphi \|_*$. Also, if \varphi \in \Lambda, then \varphi \in BMO and \| \varphi \|_* \leq c \| \varphi \|_\Lambda$.

**Proof.** It is clear that \| \varphi \|_\Lambda \leq c \| \varphi \|_*$. Conversely, assume that \varphi \in \Lambda and observe that for an atom \( a = c_1a_L + c_2a_R + c_3b_{n,k}, \) we have

$$\left| \int_R a(x) \varphi(x) \, dx \right| 
\leq 4 \left| \int_R a_L(x) \varphi(x) \, dx \right| + 4 \left| \int_R a_R(x) \varphi(x) \, dx \right| + 4 \left| \int_R b_{n,k}(x) \varphi(x) \, dx \right|$$

$$\leq 4 \| \varphi \|_{*,d} + 4 \| \varphi \|_{*,d} + 4 A(\varphi) \leq 12 \| \varphi \|_\Lambda.$$
Finally, since
\[ \| \varphi \|_* = \sup_{f \in H^1, \| f \|_{H^1} \leq 1} \left| \int_R f(x) \varphi(x) \, dx \right|, \]
where \( f \) is compactly supported and bounded, \( \| \varphi \|_* \leq c \| \varphi \|_{\Lambda} \). ■

As an application of the above characterization, the following holds.

**Proposition 3.1.** Let \( T \) be a continuous linear operator defined on a Banach space \( X \) which assumes values in \( \text{BMO}_d \) with norm \( \| T \|_d \). Then \( T \) maps \( X \) continuously into BMO if and only if for all integers \( n, k \),
\[ \frac{1}{2^{n+1}} \int_{[(k-1)2^n,k2^n]} Tf(x) \, dx - \int_{[k2^n,(k+1)2^n]} Tf(x) \, dx \leq M \| f \|_X, \]
and \( \| T \| = \max (\| T \|_d, M) \).

**Shifted BMO**

The process of averaging the translates of dyadic BMO functions leads to BMO, and is an important tool in obtaining results in BMO once they are known to be true in its dyadic counterpart, \( \text{BMO}_d \), see [12]. It is also known that BMO can be obtained as the intersection of \( \text{BMO}_d \) and one of its shifted counterparts, cf. [16]. These results motivate the observations in this section.

Given a dyadic interval \( I = [(k-1)2^n,k2^n] \) of length \( 2^n \), we call the interval \( I' = [(k-1)2^n + 2^{n-1}, k2^n + 2^{n-1}] \) the shifted interval of \( I \) by its half-length. Clearly \( |I'| = |I| \), and \( I' = [(2k-1)2^{n-1}, (2k+1)2^{n-1}] \) is not dyadic.

Let \( J = \{ J_{n,k} \} \) be the collection of all dyadic shifted by their half-length, \( J_{n,k} = [(k-1)2^n, (k+1)2^n] \), all integers \( n, k \), and let \( \text{BMO}_{d^*} \) be the space consisting of those locally integrable functions \( \varphi \) such that
\[ \| \varphi \|_{*,d^*} = \sup_{n,k} \frac{1}{|J_{n,k}|} \int_{J_{n,k}} |\varphi(x) - \varphi_{J_{n,k}}| \, dx < \infty. \]

We then have

**Theorem 3.2.** \( \text{BMO} = \text{BMO}_d \cap \text{BMO}_{d^*} \).
Proof. It is obvious that if \( \varphi \in \text{BMO} \), then \( \| \varphi \|_{*,d^*} = \| \varphi \|_* \). Conversely, it suffices to show that \( \varphi \in \Lambda \). Since \( \varphi \in \text{BMO}_d \) it is enough to show that \( A(\varphi) < \infty \). For integers \( n, k \), consider

\[
\frac{1}{2^{n+1}} \left| \int_{[(k-1)2^n,k2^n]} \varphi(x) \, dx - \int_{[k2^n,(k+1)2^n]} \varphi(x) \, dx \right|
\]

\[
= \frac{1}{2^{n+1}} \left| \int_{[(k-1)2^n,k2^n]} \left[ \varphi(x) - \varphi_{J_n,k} \right] \, dx - \int_{[k2^n,(k+1)2^n]} \left[ \varphi(x) - \varphi_{J_n,k} \right] \, dx \right|
\]

\[
\leq \frac{1}{2^{n+1}} \int_{[(k-1)2^n,(k+1)2^n]} \left| \varphi(x) - \varphi_{J_n,k} \right| \, dx \leq \| \varphi \|_{*,d^*},
\]

which implies that \( A(\varphi) \leq \| \varphi \|_{*,d^*} \), and we are done. \( \blacksquare \)

Further Characterizations of BMO

We further describe BMO in terms of the duals of the various spaces describing \( H^1 \). From \( H^1 = H^A + \text{sp}\{h\} \) it follows that \( \text{BMO} = B \cap (\text{sp}\{h\})^* \), where \( h \in H^1 \) satisfies \( \int_{[0,\infty)} h(y) \, dy \neq 0 \). To fix ideas we pick \( h = -b \) and introduce the equivalence relation \( \sim_b \) in BMO as follows. We say that \( \varphi \sim_b \psi \) if \( \varphi - \psi = \eta \) for some \( \eta \in \text{BMO} \) with \( \int_{\mathbb{R}} \eta(y) \, b(y) \, dy = 0 \). We endow these equivalence classes, which we denote by \( B_b \), with the quotient norm, and observe that the norm in BMO is equivalent to the norm in \( B \cap B_b \). It is possible, however, to work with a simpler expression.

**Proposition 3.2.** For a locally integrable function \( \varphi \), let

\[
A_b(\varphi) = \left| \int_{\mathbb{R}} \varphi(y) \, b(y) \, dy \right|.
\]

Let \( \varphi \in \text{BMO} \) be the representative of \( \Phi \in B \). Then

\[
\| \varphi \|_* \sim \max(\| \Phi \|_B, A_b(\varphi)).
\]

**Proof.** If \( \varphi \in \text{BMO} \), then clearly \( \| \Phi \|_B \leq \| \varphi \|_* \). Also, \( |A_b(\varphi)| \leq \| \varphi \|_* \| b \|_{H^1} \leq \| \varphi \|_* \).

As for the other inequality, we have \( \| \varphi \|_* \leq c \max(\| \Phi \|_B, \| \Phi \|_{B_b}) \). Let \( A = \int_{\mathbb{R}} \varphi(y) \, b(y) \, dy \), put \( \psi(x) = 2A b(x) \), and observe that \( \psi \in L^\infty \) and \( \varphi(y) \sim_b \psi \). Then \( \| \Phi \|_{B_b} \leq \| \psi \|_* \leq \| \psi \|_{\infty} \leq |A| = A_b(\varphi) \), and we have finished. \( \blacksquare \)
We also have the following result for BMO$_{2s}$.

**Proposition 3.3.** A function $\varphi \in \text{BMO}$ if and only if $\varphi \in \text{BMO}_{2s}$ and

$$A_0(\varphi) = \sup_n \left( \frac{1}{2^{n+1}} \left| \int_{[-2^n,0]} \varphi(y) \, dy - \int_{[0,2^n]} \varphi(y) \, dy \right| \right) < \infty.$$  

Moreover, there is a constant $c$ such that

$$\|\varphi\|_* \sim \max(\|\varphi\|_{*,2s}, A_0(\varphi)).$$

We leave the verification of this fact to the reader, and point out an interesting consequence, see [1].

**Proposition 3.4.** Suppose $\varphi \in \text{BMO}_{2s}$ is supported in $[0,\infty)$.

1. The even extension $\varphi_e$ of $\varphi$ belongs to BMO, and $\|\varphi_e\|_* = \|\varphi\|_{*,2s}$.

2. The odd extension $\varphi_o$ of $\varphi$ belongs to BMO if and only if

$$\sup_n \frac{1}{2^n} \left| \int_{[0,2^n]} \varphi(y) \, dy \right| < \infty,$$

and in this case

$$\|\varphi_o\|_* \sim \|\varphi\|_{*,2s} + \sup_n \frac{1}{2^n} \left| \int_{[0,2^n]} \varphi(y) \, dy \right|.$$

As an illustration of the use of the above results we will consider the $T(1)$ Theorem, which establishes the continuity in $L^2$ of a standard CZO operator essentially under two kinds of assumptions, the weak boundedness property and the $T(1), T^*(1) \text{ BMO}$ assumption. Indeed, we have, see [6],

**$T(1)$ Theorem.** Suppose $T$ is a standard CZO that satisfies

(WBP) For every interval $I$, $|\langle T \chi_I, \chi_I \rangle| \leq c |I|.$

(BMO condition) $\|T(1)\|_* + \|T^*(1)\|_* \leq c.$

Then $T$ is a continuous mapping in $L^2$.

In applications it is of interest to state the BMO condition in a form that is easily verified. For instance, in the dyadic setting, the following two conditions may be assumed instead,
\[(1_d) \| T(1) \|_{s,d} + \| T^*(1) \|_{s,d} \leq c. \]

\[(2_d) \text{ For all integers } n, k, |\langle T(b_{n,k}), 1 \rangle| + |\langle T^*(b_{n,k}), 1 \rangle| \leq c. \]

Then clearly \(T(1), T^*(1) \in \text{BMO},\) and the \(T(1)\) Theorem obtains.

Similarly, in the two-sided setting, the following two conditions may be used instead of the BMO assumption,

\[(1_s) \| T(1) \|_{s,2s} + \| T^*(1) \|_{s,2s} \leq c. \]

\[(2_s) \text{ For all integers } n, |\langle T(b_{n,0}), 1 \rangle| + |\langle T^*(b_{n,0}), 1 \rangle| \leq c. \]

Two particular instances of this last observation come to mind. Let \(T\) be a CZO with WBP that satisfies \((1_s)\). Also, assume that for any interval \(I = I_{n,0}\), \(T(\chi_I)\) is supported in \(I\), and similarly for \(T^*\). Now, since

\[\langle T(b_{n,0}), 1 \rangle = \frac{1}{2^{n+1}} \left[ \langle T\chi_{[-2^n,0]}, \chi_{[-2^n,0]} \rangle - \langle T\chi_{[0,2^n]}, \chi_{[0,2^n]} \rangle \right],\]

by WBP, \(|\langle T(b_{n,0}), 1 \rangle| \leq c.\) The estimate for \(T^*\) is obtained in a similar fashion, and therefore \((2_s)\) also holds. Thus \(T\) is bounded in \(L^2\).

Finally, when the kernel of \(T\) is even, or odd, in \(x\) and \(y\), \(T(1)\) and \(T^*(1)\) are even, or odd, respectively. Now, by Proposition 3.4, if \(T(1)\) is even and \(T(1) \chi_{[0,\infty)}\) is in \(\text{BMO}_{2s}\), then \(T(1) \in \text{BMO};\) similarly for \(T^*(1)\). On the other hand, if \(T(1), T^*(1)\) are odd and \(T(1) \chi_{[0,\infty)}, T^*(1) \chi_{[0,\infty)}\) are in \(\text{BMO}_{2s}\), we also require that

\[\sup_n \frac{1}{2^n} \left| \int_{[0,2^n]} T(1)(y) \, dy \right|, \sup_n \frac{1}{2^n} \left| \int_{[0,2^n]} T^*(1)(y) \, dy \right| \leq c.\]

Under these assumptions \((1_s)\) holds and together with \((2_s)\) obtain the continuity of \(T\) in \(L^2\).

4 Final remarks

We sketch now the extension of the results to higher dimensions. To avoid technicalities we restrict ourselves to the case \(n = 2\), but stress that appropriate versions remain valid for arbitrary \(n\). Also, since the proofs follow along similar lines to the case \(n = 1\), they will be omitted.
The two-dimensional Haar system is generated by the integer translations and dyadic dilations of the three basic orthogonal functions

\[
\Psi_1(x, y) = H(x) \chi_{[0,1]}(y), \quad \Psi_2(x, y) = \chi_{[0,1]}(x) H(y), \\
\Psi_3(x, y) = H(x) H(y),
\]

cf. [21]. More precisely, the functions

\[
\Psi_{1,n,k,l}(x, y) = 2^n \Psi_1(2^n x - k, 2^n y - l), \quad \Psi_{2,n,k,l}(x, y) = 2^n \Psi_2(2^n x - k, 2^n y - l), \\
\Psi_{3,n,k,l}(x, y) = 2^n \Psi_3(2^n x - k, 2^n y - l),
\]

for arbitrary integers \( n, k, l \), generate the two-dimensional Haar system. In three dimensions seven basic functions are required.

We then have

**Theorem 4.1.** The closed span of the two-dimensional Haar system in \( H^1(R^2) \) is the subspace \( H^A(R^2) \) of \( H^1(R^2) \) which consists of those functions that have 0 integral on each quadrant.

Clearly there is some redundancy in this statement. Since functions in \( H^1(R^2) \) have 0 integral, it suffices to require that the functions in question have 0 integral in any three quadrants.

Let \( Q_1, Q_2, Q_3, Q_4 \) denote the four quadrants of \( R^2 \). It is not hard to see that the functions \( \varphi \) with the property that \( \int \int f \varphi = 0 \) for all \( f \in \text{sp}\{\Psi_{j,n,k}\} \) are of the form \( \varphi = \sum c_i \chi_{Q_i} \), and this suggests how close \( H^A(R^2) \) is to \( H^1(R^2) \). In fact, we have,

\[
H^1(R^2) = \overline{\text{sp}\{\Psi_{j,n,k}, b_1, b_2, b_3\}},
\]

where \( b_1(x, y) = b(x) \chi_{[0,1]}(y), \ b_2(x, y) = \chi_{[0,1]}(x)b(y), \) and \( b_3(x, y) = b(x) \chi_{[-1,0]}(y) \).

As for arbitrary atoms \( a \) in \( H^1(R^2) \), they can be expressed as a sum of at most five atoms, four dyadic and a special atom. More precisely, if we denote \( Q_{n,k,m,l} = I_{n,k} \times I_{m,l} \), then

**Lemma 4.1.** Let \( a \) be an \( H^1(R^2) \) atom. Then there are at most five atoms \( a_1, a_2, a_3, a_4, b_{n,k,m,l} \), such that

i. The \( a_i \)'s are dyadic atoms.
ii. For some integers \( n, k, m, l \), \( b_{n,k,m,l}(x) \) is equal to

\[
\frac{1}{2^{n+m+2}} \left[ k_1 \chi_{Q_{n,k,m,l}} + k_2 \chi_{Q_{n,k-1,m,l}} + k_3 \chi_{Q_{n,k,m,l-1}} + k_4 \chi_{Q_{n,k-1,m,l-1}} \right],
\]

where \( |k_i| \leq c \), an absolute constant independent of \( a \), and \( \sum_1^4 k_i = 0 \).

iii. \( a = \sum_1^4 c_ia_i + c_5 b_{n,k,m,l} \), where \( |c_i| \leq c \), an absolute constant independent of \( a \).

With this decomposition of individual atoms available, we can describe \( H^1(R^2) \) in various ways. For instance, if \( H^1_d(R^2) \) denotes the dyadic Hardy space, and \( H^1_\delta(R^2) \) denotes the subspace of \( H^1(R^2) \) spanned by the \( b_{n,k,m,l}'s \), then \( H^1(R^2) = H^1_d(R^2) + H^1_\delta(R^2) \) in the sense of sum of Banach spaces. Thus the continuity of a linear operator acting on \( H^1(R^2) \) can be characterized in terms of the continuity of its restrictions to \( H^1_d(R^2) \) and \( H^1_\delta(R^2) \).

Also, decompositions of \( H^1(R^2) \) lead to characterizations of BMO(\( R^2 \)). More precisely,

**Lemma 4.2.** A locally integrable \( \varphi \in \text{BMO}(R^2) \) if and only if \( \varphi \) belongs to \( \text{BMO}_d(R^2) \) and for a constant \( c \),

\[
\left| \int \int_{R^2} \varphi(x, y) b_{n,k,m,l}(x, y) \, dx \, dy \right| \leq c, \quad \text{all } n, k, m, l.
\]

Finally, as a consequence of this result one can write a version of the \( T(1) \) theorem in \( R^2 \) under dyadic-like assumptions.

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