All bipartite entangled states display some hidden nonlocality

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We show that a violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality can be demonstrated in a certain kind of Bell experiment if and only if the state is entangled. Our protocol allows local filtering measurements and involves shared ancilla states that do not themselves violate CHSH. Our result follows from two main steps. We first provide a simple characterization of the states that violate the CHSH-inequality after local filtering operations in terms of witness-like operators. Second, we prove that for each entangled state $\sigma$, there exists another state $\rho$ not violating CHSH, such that $\rho \otimes \sigma$ violates CHSH. Hence, in this scenario, $\sigma$ cannot be substituted by classical correlations without changing the statistics of the experiment; we say that $\sigma$ is not simulable by classical correlations and our result is that entanglement is equivalent to non-simulability.

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In 1964 Bell ruled out the possibility that a local-realistic theory could reproduce all the experimental predictions given by Quantum Mechanics [1]. In a local-realistic theory the outcomes of measurements are determined in advance by unknown (or hidden) variables, and do not depend on the choice of measurements made by distant observers. Bell’s theorem states that the quantum mechanical probabilities for outcomes of measurements distributed in space cannot, in general, be replicated in any local-realistic theory. This fact is demonstrated for particular states and measurements by the violation of a Bell inequality. Entanglement is in some way responsible for this phenomenon since entangled states are required to demonstrate the violation of Bell inequalities. However since it has been known for some time that there are entangled states that do not violate any Bell inequality [2, 3], the precise relationship between entanglement and Bell inequality violation has remained poorly understood.

The definition of entangled state is made in terms of the physical resources needed for the preparation of the state: a multipartite state is said to be entangled if it cannot be prepared from classical correlations using local quantum operations [2]. But this definition tells us nothing about the “behavior” of the state. For example, does the state violate a Bell inequality, or is it useful in some quantum protocol such as teleportation?

It is known that every pure entangled state violates a Bell inequality [4, 5] and that no separable state does [2], but the situation gets more complicated for mixed entangled states. There are bipartite mixed states that, though being entangled, possess a local hidden variables model (LHVM) whenever measurements are made on a single copy of the state (see for example [2, 3]). But some of these states violate Bell inequalities if, prior to the measurement, the state is processed by local operations and classical communication (LOCC) [6, 7]. Moreover, by jointly measuring more than one copy of these states after some LOCC preprocessing, it was shown that an even larger set of entangled states could be detected through their violation of a Bell inequality [8].

Generalizing this idea one can get a strong test of the nonlocality “hidden” in a state by combining local filtering operations and collective measurements: Perform joint local filtering operations on an arbitrarily large number of copies of the state and then a Bell inequality test on the resulting state. If the resulting probabilities violate a Bell inequality, we say that the original state violates this inequality asymptotically. In Ref. [9] it is shown that a bipartite state violates the Clauser-Horne-Shimony-Holt (CHSH) inequality [10] asymptotically if, and only if, it is distillable. This result suggests that undistillable entangled states may admit a LHVM description even when experiments are performed on many copies of the state.

Given these negative results, it seems necessary to allow still more general protocols for the nonlocality “hidden” in arbitrary entangled states to manifest itself. One natural possibility is to allow joint processing with auxiliary states (that do not themselves violate the Bell inequality) rather than just with more copies of the state in question. This idea has been fruitful to show that useful entanglement can be extracted from all non-separable states [11] and in this Letter we use it to show that there is indeed a kind of hidden nonlocality that is possessed by all entangled states.

In order to investigate the possibilities of this more general kind of hidden nonlocality we introduce the concept of a simulable state. We say that a bipartite state $\sigma$ is simulable by classical correlations, or just simulable, if in any protocol (possibly involving other resources
such as shared quantum states) two separated parties sharing classical correlations instead of \( \sigma \) can obtain the same statistics for the outcomes of the protocol. In this sense, simulable states have a completely classical “behavior”. Of course we are most interested in this Letter in the case where the protocol concerned is a test of nonlocality. Clearly, all separable states are simulable. A possible way to simulate a separable state is by just preparing it from classical correlations [2].

The scenario that we consider is the typical Bell-like experiment, where two parties share a bipartite system and perform local measurements on it. Alice chooses between the observables \( x = 0, 1 \) and obtains the outcomes \( a = 0, 1 \), and analogously for Bob, \( y \) and \( b \). All the relevant experimental information is contained in the joint probability distribution for the outcomes conditioned on the choice of observables \( P(a, b|x, y) \). It is convenient to define the correlation functions

\[
C_{xy} = P(a = b|x, y) - P(a \neq b|x, y). \tag{1}
\]

By local relabeling of \( (a, b, x, y) \) it is always possible to make \( C_{00}, C_{01}, C_{10} \geq 0 \). With this convention, the distribution \( P(a, b|x, y) \) admits a LHVM if [12], and only if, it satisfies the CHSH-inequality [10]

\[
C_{00} + C_{01} + C_{10} - C_{11} \leq 2. \tag{2}
\]

Let us characterize the set of bipartite states that violate the CHSH-inequality after preprocessing.

**Definition.** Denote by \( \mathcal{C} \) the set of bipartite states that do not violate the CHSH-inequality, even after stochastic local operations without communication.

By stochastic we mean that the operation can fail, and we do not care about the probability of failure, as long as it is strictly smaller than one. Up to normalization, these operations allow the transformations

\[
\rho \to \Omega(\rho) = \sum_i (A_i \otimes B_i) \rho (A_i \otimes B_i)^\dagger, \tag{3}
\]

where \( A_i \) and \( B_i \) are, respectively, Kraus operators acting on the first and second system. This class of maps is known as the *separable maps*.

In Ref. [9] it is shown that the states in \( \mathcal{C} \) do not violate CHSH even after stochastic local operations with communication. So the exact nature of the local operations allowed in the definition of \( \mathcal{C} \) is not important. Clearly, states that do not violate the CHSH-inequality asymptotically are in \( \mathcal{C} \). Thus, \( \mathcal{C} \) contains all undistillable states [9]. We are now able to state precisely the central result of this Letter.

**Theorem.** A bipartite state \( \sigma \) is entangled if, and only if, there exists a state \( \rho \in \mathcal{C} \) such that \( \rho \otimes \sigma \) is not in \( \mathcal{C} \).

The consequences of this theorem are dramatic. If \( \rho \) belongs to \( \mathcal{C} \), no matter how much additional classical correlation (which can always be represented by a separable state \( \eta_{\text{sep}} \)) we supply to it, the result \( \rho \otimes \eta_{\text{sep}} \) is still in \( \mathcal{C} \). Contrary, the state \( \rho \otimes \sigma \) is not in \( \mathcal{C} \) even if both \( \rho \) and \( \sigma \) are in \( \mathcal{C} \). The violation of CHSH manifests the qualitatively different behavior between \( \rho \otimes \sigma \) and \( \rho \otimes \eta_{\text{sep}} \), where \( \eta_{\text{sep}} \) is any separable state, and \( \sigma \) is any entangled state. Summarizing, for each entangled state \( \sigma \) there exists a protocol (which also involves the auxiliary state \( \rho \) associated with the theorem) in which \( \sigma \) cannot be substituted by an arbitrarily large amount of classical correlations without changing the result: *Entangled states are the ones that cannot be simulated by classical correlations.*

The proof of the above theorem has two main ingredients. Firstly we note that \( \mathcal{C} \) is a convex set and provide a characterization of \( \mathcal{C} \) in terms of witness-like operators that detect CHSH-violation. Secondly we use convexity arguments similar to those in Ref. [11] to prove by contradiction that there exists some \( \rho \in \mathcal{C} \) such that one of these witnesses may be constructed for \( \rho \otimes \sigma \) whenever \( \sigma \) is entangled. To carry this argument through we require a characterization of the separable completely positive maps between Bell diagonal states that will be explained further elsewhere [13]. Firstly we describe the witnesses for CHSH-violation.

**Lemma 1.** A bipartite state \( \rho \) acting on \( \mathcal{H}_A \otimes \mathcal{H}_B \) belongs to \( \mathcal{C} \) if, and only if, it satisfies

\[
\text{tr}[\rho (A \otimes B) H_0 (A \otimes B)^\dagger] \geq 0, \tag{4}
\]

for all matrices of the form \( A: \mathbb{C}^2 \to \mathcal{H}_A, B: \mathbb{C}^2 \to \mathcal{H}_B \) and all numbers \( \theta \in [0, \pi/4] \), where

\[
H_0 \equiv I \otimes I - \cos \theta \sigma_x \otimes \sigma_x - \sin \theta \sigma_z \otimes \sigma_z, \tag{5}
\]

\( I \) being the \( 2 \times 2 \) identity matrix and \( \{\sigma_i\}_{i=x,y,z} \) the Pauli matrices.

**Proof.** To start off, we recall that, without preprocessing, a two-qubit state \( \rho \) violates the CHSH inequality iff \( \mu_1^2 + \mu_2^2 > 1 \), where \( \mu_1 \) and \( \mu_2 \) are the two largest singular values of the \( 3 \times 3 \) real matrix

\[
R_{ij} = \text{tr}[\rho \sigma_i \otimes \sigma_j],
\]

with indices \( i, j = x, y, z \) [14, 15]. Equivalently, \( (\mu_1, \mu_2) \) derived from \( \rho \) must lie outside the unit circle \( \mu_1^2 + \mu_2^2 = 1 \), which is true if and only if there exists \( \theta \in [0, 2\pi] \) such that

\[
\mu_1 \cos \theta + \mu_2 \sin \theta > 1. \tag{6}
\]

Now, it is also well known that by appropriate local unitary transformations \( U, V \), it is always possible to arrive at a local basis such that \( R \) is diagonal with \( \mu_1 = R_{xx} \) and \( \mu_2 = R_{zz} \). From the definition of \( R \) it follows that

\[
\mu_1 \cos \theta = \text{tr}[(U \otimes V) \rho (U \otimes V)^\dagger (\cos \theta \sigma_z \otimes \sigma_z)], \tag{7}
\]
with the expression for $\mu_2 \sin \theta$ involving obvious modifications. Since singular values are non-negative, it thus follows that if $\rho$ violates CHSH then there exist $U, V \in SU(2)$, $\theta \in [0, \pi/4]$ such that

$$\text{tr} [\rho (U \otimes V) \hat{H}_0 (U \otimes V)] < 0.$$ \hspace{1cm} (8)

On the other hand suppose that there exists some $(U, V, \theta)$ satisfying (8). Thus we have $R_{xx} \cos \theta + R_{xz} \sin \theta > 1$. If we assume $\mu_1 \geq \mu_2$, the inequalities $|R_{xz}|, |R_{zz}| \leq \mu_1 \leq 1$ follow from the definition of singular values and the well known fact that all singular values of $R$ are less than one. Since $0 \leq \theta \leq \pi/4$ both $R_{xx}$ and $R_{xz}$ must be positive and since $\cos \theta \geq \sin \theta$ we may assume without loss of generality that $R_{xx} \geq R_{xz}$. The singular values of $R$ obey the inequality $|R_{xx} + R_{xz}| \leq \mu_1 + \mu_2$ [16] and as a result we find $\mu_1 \cos \theta + \mu_2 \sin \theta > 1$ so $\rho$ violates the CHSH-inequality. Thus $\rho$ violates CHSH iff (8) holds.

Let us come back to the question of CHSH violation after local filtering operations. Assume that $\rho$ violates CHSH after stochastic local operations. Let us show that it must violate (4) for some $(A, B, \theta)$. In Ref. [9] it is proven that, if a state violates CHSH then it can be transformed by stochastic local operations into a two-qubit state which also violates CHSH. Therefore, there must exist a separable map $\Omega$ with two-qubit output, such that the state $\Omega(\rho)$ satisfies condition (8) for some $(U, V, \theta)$, denote them by $(U_0, V_0, \theta_0)$. Clearly, if $\Omega(\rho)$ satisfies (8) there must exist at least one value of $i$ in the Kraus decomposition of (3) such that $(A_i \otimes B_i) \rho (A_i \otimes B_i)^\dagger$ also satisfies (8). This implies that $\rho$ violates (4) for $A = A_i U_0^\dagger$, $B = B_i V_0^\dagger$ and $\theta = \theta_0$, which proves one direction of the lemma, let us show the other.

Assume that $\rho$ violates (4) for $(A_0, B_0, \theta_0)$. It is straightforward to see that $\rho$ violates CHSH after stochastic LOCC. Consider operator that transforms $\rho$ into $(A_0 \otimes B_0) \rho (A_0 \otimes B_0)^\dagger$. By assumption, the final state satisfies (8) with $U = V = \mathbb{I}$ and $\theta = \theta_0$, which implies that it violates CHSH. $\square$

The above characterization is interesting on its own. Here we use it to prove our main result.

Proof of the Theorem. If $\sigma$ is separable then, $\rho \in C$ implies $\rho \otimes \sigma \in C$. This is so because the preprocessing by LOCC of $\rho$, before the Bell experiment, can include the preparation of the state $\sigma$. Let us prove the other direction of the theorem.

From now on $\sigma$ is an arbitrary entangled state acting on $H = \mathcal{H}_A \otimes \mathcal{H}_B$. Let us show that there always exists an ancilla state $\rho \in C$ such that $\rho \otimes \sigma \notin C$. Fix $\rho$ to act on the bipartite Hilbert space $[\mathcal{H}_A \otimes \mathcal{H}_{A'}] \otimes [\mathcal{H}_{B'} \otimes \mathcal{H}_{B''}]$, where $\mathcal{H}_{A'} = \mathcal{H}_A$, $\mathcal{H}_{B'} = \mathcal{H}_B$ and $\mathcal{H}_{A''} = \mathcal{H}_{B''} = \mathbb{C}^2$ (see Fig. 1).

Our aim is to prove that the state $\rho \otimes \sigma$ violates (4) for some choice of $A$, $B$, and $\theta$. In particular, let

$$A = |\Phi_{AA'} \rangle \otimes I_{A''}, \quad B = |\Phi_{BB'} \rangle \otimes I_{B''}, \quad \theta = \pi/4,$$

where $|\Phi_{AA'} \rangle$ is the maximally-entangled state between the spaces $\mathcal{H}_A$ and $\mathcal{H}_{A'}$ (which have the same dimension), and $I_{A''}$ is the identity matrix acting on $\mathbb{C}^2$ (analogously for Bob). A little calculation shows that for any $\rho$

$$\text{tr} [\rho (A \otimes B) H_{\pi/4} (A \otimes B)^\dagger] = \nu \text{tr} [\rho (\sigma^T \otimes H_{\pi/4})]$$

where $\nu$ is a positive constant and $\sigma^T$ stands for the transpose of $\sigma$. The inequality (4) with $\theta = \pi/4$, $A = B = B$ becomes

$$\text{tr} [\rho (\sigma^T \otimes H_{\pi/4})] < 0.$$ \hspace{1cm} (9)

For convenience, in the rest of the proof we allow $\rho$ to be unnormalized. The only constraints on the matrices $\rho \in C$ are: positive semi-definiteness ($\rho \in S^+$), and satisfiability of all the inequalities (4) in Lemma 1. $C$ is now a convex cone, and its dual cone is defined as

$$C^* = \{X : \text{tr} [\rho X] \geq 0, \forall \rho \in C\},$$ \hspace{1cm} (10)

where $X$ are Hermitian matrices. Farkas’ Lemma [17] states that all matrices in $C^*$ can be written as non-negative linear combinations of matrices $P \in S^+$ and matrices $(A \otimes B) H_{\theta} (A \otimes B)\dagger$ with $A : \mathbb{C}^2 \to \mathcal{H}_A \otimes \mathcal{H}_{A''}$ and $B : \mathbb{C}^2 \to \mathcal{H}_{B'} \otimes \mathcal{H}_{B''}$.

FIG. 1: Schematic diagram illustrating the local filtering operations $A$ and $B$ involved in our protocol. The solid box on top is a schematic representation of the state $\sigma$ whereas that on the bottom is for the ancilla state $\rho$. Left and right dashed boxes, respectively, enclose the subsystems possessed by the two experimenters $A$ and $B$.

We now show that there always exists $\rho \in C$ satisfying (9) by supposing otherwise and arriving at a contradiction. Suppose that for all $\rho \in C$ the inequality $\text{tr} [\rho (\sigma^T \otimes H_{\pi/4})] \geq 0$ holds, and thus the matrix $\sigma^T \otimes H_{\pi/4}$ belongs to $C^*$. Applying Farkas’ Lemma [17] we can write

$$\sigma^T \otimes H_{\pi/4} = \int dx (A_x \otimes B_x) H_{\theta} (A_x \otimes B_x)^\dagger + \int dy P_y,$$ \hspace{1cm} (11)
which is equivalent to
\[
\sigma^T \otimes H_{\pi/4} - \int dx \Omega_x (H_{\theta_0}) \geq 0 , \tag{12}
\]
where each \( \Omega_x \) is a separable map (3). We prove in Lemma 2 that (12) requires that \( \sigma \) is separable, which gives the desired contradiction. Thus the result is proven. \( \square \)

In order to arrive at a contradiction from (12) it is necessary to use the constraint that the maps \( \Omega_x \) are separable. The problem of characterizing the separable maps is hard in general since it maps onto the separability problem for bipartite states. However it turns out only to be necessary to determine the set of separable maps that take Bell diagonal states to Bell diagonal states and this can be done exactly [13]. This characterization may be used to prove the following lemma and thereby our theorem.

**Lemma 2.** Let \( \Omega_\theta : [\mathbb{C}^2] \otimes [\mathbb{C}^2] \to [\mathcal{H}_A \otimes \mathbb{C}^2] \otimes [\mathcal{H}_B \otimes \mathbb{C}^2] \) be a family of maps, separable with respect to the partition denoted by the brackets. Let \( \mu \) be a unitary, positive semi-definite matrix acting on \( [\mathcal{H}_A] \otimes [\mathcal{H}_B] \) such that
\[
\mu^T \otimes H_{\pi/4} - \int dx \Omega_x (H_{\theta_0}) \geq 0 , \tag{13}
\]
where \( H_{\theta_0} \) is defined in (5), then \( \mu \) has to be separable.

**Proof.** Now, let us characterize the solutions \( \Omega_x \) of (13). The Bell basis is defined as
\[
\{ \Phi^1_1 \} = 2^{-1/2} (|00⟩ ± |11⟩) , \tag{14}
\]
\[
\{ \Phi^1_2 \} = 2^{-1/2} (|01⟩ ± |10⟩) . \tag{15}
\]
The matrices \( H_{\theta_0} \) are diagonal in this basis, \( H_{\theta_0} = \sum_{i=1}^4 N_{\theta}^i \Pi_i \), where \( \Pi_i \equiv \{ \Phi^1_i \} \Phi^1_i \) are the Bell projectors and \( N_{\theta}^i \) are the components of the vector
\[
N_\theta = \begin{bmatrix} 1 - \cos \theta - \sin \theta \\ 1 + \cos \theta - \sin \theta \\ 1 - \cos \theta + \sin \theta \\ 1 + \cos \theta + \sin \theta \end{bmatrix} . \tag{16}
\]
For each value of \( x \) define the sixteen matrices
\[
\omega^{ij}_x \equiv \text{tr}_{A'} G_{x'} \{ (I \otimes \Pi_i) \Omega_x (\Pi_j) \} , \tag{17}
\]
for \( i, j = 1, 2, 3, 4 \), where the identity matrix \( I \) acts on \( \mathcal{H}_{A'} \otimes \mathcal{H}_{B'} \) and \( \Pi_i \) acts on \( \mathcal{H}_{A'} \otimes \mathcal{H}_{B'} \). Each \( \omega^{ij}_x \) is the result of a physical operation, and hence positive. One can see \( \omega^{ij}_x \) as the Jamiołkowski state corresponding to the map \( \Omega_x \), after “twirling” the input and output subsystems with the group of unitaries that leaves Bell-diagonal states invariant. Projecting the left hand side of (13) onto the four Bell projectors \( \Pi_i \), and taking the relevant partial trace, we get
\[
\mu^T N_{\pi/4}^i - \int dx \sum_{j=1}^4 \omega^{ij}_x N_{\theta_0}^j \geq 0 , \tag{18}
\]
for \( i = 1, 2, 3, 4 \). Denote by \( M_x \) the \( 4 \times 4 \) matrix with components \( M_x^{ij} = \text{tr}[\omega^{ij}_x] \). Performing the trace on the left hand side of (18) we obtain the four inequalities
\[
N_{\pi/4} - \int dx M_x \cdot N_{\theta_0} \geq 0 , \tag{19}
\]
where \( 0 \) is the 4-dimensional null vector, and the symbols \( \cdot \) and \( \geq \) mean, respectively, standard matrix multiplication and component-wise inequality. Consider the set of matrices \( M \) that are generated by tracing the left-hand side of (17) when \( \Omega_x \) is any separable map. The characterization of this set of matrices is obtained in Ref. [13], and goes as follows. Denote by \( D \) the set of \( 4 \times 4 \) doubly-stochastic matrices, that is, the convex hull of the permutation matrices [16]. Denote by \( G \) the convex hull of all matrices obtained when independently permuting the rows and columns of
\[
G_0 \equiv \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{20}
\]
It is shown in Ref. [13] that, any matrix \( M \) as defined above can be written as
\[
M = pD + qG , \tag{21}
\]
where \( D \in D, G \in G, \) and \( p, q \geq 0 \). Then, any solution of (19) can be specified by giving \( (\theta_x, p_x, q_x, d_x, g_x) \). Using the fact that \( G \cdot N_\theta \geq 0 \) for all \( \theta \) and \( G \in G \), all solutions of (19) must satisfy
\[
N_{\pi/4} - \int dx p_x D_x \cdot N_{\theta_0} \geq 0 . \tag{22}
\]
Recall that this component-wise inequality entails four inequalities. Adding them together we obtain the condition \( \int dx p_x \leq 1 . \) Now, denote by \( N \) the set of all vectors obtained by permuting the components of \( N_\theta \) (16) when \( \theta \) runs through \([0, \pi/4] \). The convex hull of \( N \) (conv \( N \)) is precisely the set of vectors that can be written as the right-hand side of (22) under the constraint \( \int dx p_x \leq 1 . \) The first inequality of (22) is
\[
1 - \sqrt{2} \geq N^1 , \tag{23}
\]
for \( N^1 \) is the first component of \( N \) in \( \text{conv} \, N \). All vectors \( N \in \text{conv} \, N \) satisfy the inequality \( 1 - \sqrt{2} \leq N^1 \), and only \( N_{\pi/4} \) saturates it. Hence, the only possible value for the right-hand side of (22) is \( N_{\pi/4} \). Substituting this into (19), and again using (21), we obtain \( \int dx q_x G_x \cdot N_{\theta_0} \geq 0 \). But as said above, \( G \cdot N_\theta \geq 0 \) for all \( \theta \) and \( G \in G \), which implies that for any solution \( \int dx q_x G_x \cdot N_{\theta_0} = 0 \). Therefore, inequality (19) becomes \( N_{\pi/4} - M_0 \cdot N_{\pi/4} \geq 0 \), where \( M_0 \) is a doubly-stochastic matrix such that \( M_0 \cdot N_{\pi/4} = N_{\pi/4} \). Due
to the form of $N_{\pi/4}$, doubly-stochastic matrices that satisfy this equality must have the form of

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \eta & \eta & 0 \\ 0 & \eta & 1 - \eta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$  

(24)

where $\eta \in [0, 1]$.

The fact that each of the four components of the left-hand side of (19) is zero, implies that the left-hand side of (18) is traceless for all $i$. The only positive matrix with zero trace is the null matrix, therefore

$$\mu^T N_{\pi/4}^i = \sum_{j=1}^{4} \omega_{0}^{ij} N_{\pi/4}^j, \quad i = 1, 2, 3, 4,$$

(25)

where $\omega_0$ is any $\omega_x$ that gives rise to $M_0$. Using the same argument, the pairs $(i,j)$ for which $M_0^{ij} = 0$ are such that $\omega_{0}^{ij} = 0$. Therefore, by adding the equalities in (25) corresponding to $i = 2, 3$, and using the definition of $\omega_0^j$ in (17), we obtain

$$2 \mu^T = tr_{A''B''}[(\mathbb{I} \otimes \Psi) \Omega_0(\Psi)],$$

(26)

where $\Psi = \Pi_2 + \Pi_3$, and $\Omega_0$ is any $\Omega_x$ that gives rise to $\omega_0$. Using the PPT criterion [18] one can check that the (unnormalized) two-qubit state $\Psi$ is a separable state. Equation (26) implies that $\mu^T$ is the output of a separable map applied to a separable input state, and hence is a separable state as we wanted to prove. □

**Summarizing**, for each entangled state $\sigma$ there exists a protocol (involving the state $\rho$ associated with the theorem) in which $\sigma$ cannot be substituted by an arbitrarily large amount of classical correlations, without changing the result:

**Entangled states are the ones that cannot be simulated by classical correlations.**

This provides us with a new interpretation of entanglement in terms of the behavior of the states, in contrast with the usual definition in terms of the preparation of the states.

Differently, one can be interested in the set of bipartite states $\sigma$ which do not admit a LHVM description in scenarios where no other kind entanglement is present. That is, $\sigma$ may be processed with more copies of itself $\sigma \otimes n$ but never with different entangled states $\rho$. Following [9] we say that a state $\sigma$ violates a Bell inequality asymptotically, if after jointly processing by LOCC a sufficiently large number of copies of a state, the result violates the Bell inequality. In Ref. [9] it is proven that the states violating CHSH asymptotically are the distillable ones. This, together with the results of the present paper, establishes an appealing picture:

entangled $\iff$ non-simulable

distillable $\iff$ asymptotic violation of CHSH

Entangled states are the ones that cannot be generated from classical correlations plus local quantum operations. We show that in the bipartite case, one can equivalently define entangled states as the ones that cannot be simulated by classical correlations.

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