Chiral symmetry breaking, entanglement, and the nucleon spin decomposition

Silas R. Beane and Peter Ehlers

1 Department of Physics, University of Washington, Seattle, WA 98195-1560, USA

(Dated: May 10, 2019 - 0:31)

The nucleon is naturally viewed as a bipartite system of valence spin —defined by its non-vanishing chiral charge—and non-valence or sea spin. The sea spin can be traced over to give a reduced density matrix, and it is shown that the resulting entanglement entropy acts as an order parameter of chiral symmetry breaking in the nucleon. In the large-$N_c$ limit, the entanglement entropy vanishes and the valence spin accounts for all of the nucleon spin, while in the limit of maximal entanglement entropy, the nucleon loses memory of the valence spin and consequently has spin dominated by the sea. The nucleon state vector in the chiral basis, fit to low-energy data, gives a valence spin content consistent with experiment and lattice QCD determinations, and has large entanglement entropy.

Introduction. An important goal of present-day nuclear science is the development of a qualitative and quantitative understanding of the structure of the proton directly from the underlying QCD interactions. For instance, the decomposition of the spin of the proton into components that have a well-defined interpretation in terms of the fundamental quark and gluon degrees of freedom of QCD is a primary goal [1]. QCD reveals that the proton is a complicated many-body quantum system, and therefore the breakdown of its spin content is highly complex and requires intrinsically non-perturbative methods to unravel. Recent work in lattice QCD addresses the nucleon spin decomposition in a quantitative manner with controlled uncertainties [2–4]. However, qualitative features of this decomposition, such as the suppression of the nucleon's valence spin content, remain enigmatic, and call out for an explanation grounded in QCD.

The complex decomposition of the nucleon spin is a striking signature of strong entanglement among the nucleon's constituents. How does one characterize nucleon entanglement in a strongly-coupled quantum field theory like QCD? One way is to partition the nucleon state vector into a bipartite system, trace over one of the subspaces, and obtain an entanglement entropy. This partitioning has been done in momentum space in the context of high-energy scattering [5], and in position space in the context of deep-inelastic scattering [6]. These partitionings rely on being in a regime of large momentum transfer where non-perturbative aspects of QCD are subsumed into parton distribution functions. Many interesting findings and potential experimental signatures are found in these studies; a common qualitative conclusion is that in high energy processes the nucleon constituents decohere, giving rise to a maximal entanglement entropy given by the logarithm of the number of gluons in the nucleon, which grows exponentially with energy in a known manner [7, 8].

In this letter, it is argued that spin entanglement between valence and non-valence spin degrees of freedom provides valuable insight into the nucleon spin decomposition, and, more generally, into the nature of chiral symmetry breaking in QCD. As a first step toward addressing spin entanglement in the nucleon, special care must be taken to define a relativistic nucleon state vector which represents the internal degrees of freedom of the nucleon in a manner that is independent of kinematics. This is achieved through the use of light-cone coordinates; that is, using null-planes as quantization surfaces. In the Fock-space basis, which emerges naturally in light-cone coordinates, the nucleon state vector of definite helicity is built out of elements labeled by the number of fundamental QCD constituents, which we will refer to in this letter as partons. The nucleon state vector can also be expressed in a chiral basis, which makes use of a fundamental property of the null-plane quark fields: states of definite helicity transform irreducibly with respect to the chiral symmetry group. The chiral basis therefore suggests a natural bipartite Hilbert space description of the nucleon state vector: the valence space is by definition the space which carries non-vanishing chiral charge, while the non-valence space, or parton sea, carries spin, but no chiral charge. The entanglement entropy between these two subspaces drives chiral symmetry breaking in the nucleon and provides both a qualitative and quantitative explanation of why the valence spin content of the nucleon is so small.

A relativistic state vector. In non-relativistic quantum mechanics, the state vector of a many-body system describes the internal degrees of freedom in a manner that is independent of the choice of reference frame. This is because the underlying kinematics is governed by the Galilean group, and Galilean boosts do not depend on the interaction. By contrast, in a relativistic theory of quantum mechanics, the spacetime symmetry group is the Poincaré group. In general, the Poincaré boost operator depends on the interaction, unless a foliation of spacetime is chosen such that the boost operator is non-dynamical [9]. This can be realized if null-planes are chosen as initial quantization surfaces [9–11]. In this choice of light-cone coordinates, the energy and the two transverse components of spin are dynamical, while the boosts, the momenta and the longitudinal component of

null-planes.
spin—the helicity—are kinematical. In an arbitrary reference frame, the energy and transverse spin which act on the internal degrees of freedom are described by the (Hamiltonian) operators $M^2$ and $M_j$ ($r = 1, 2$), respectively, which commute with boosts and momenta, and together with $J_3$, satisfy the algebra of the Poincaré group [11]. An eigenstate of momentum and helicity, describing a nucleon, $N$, can be expressed as

$$|N, \Lambda; p^+, p_\perp \rangle = |N, \Lambda \rangle \otimes |p^+, p_\perp \rangle,$$

where $p^+$ and $p_\perp$ are the longitudinal and transverse components of the nucleon momenta, respectively, and $\Lambda$ is the total helicity, the eigenvalue of $J_3$. The direct product on the right indicates that the part of the state which describes the internal degrees of freedom can be separated completely from the kinematics. Achieving this separation is essential as otherwise there is no starting point for a description of the state vector of a nucleon which represents the internal degrees of freedom. In principle, specification of the operators $M^2$ and $M_j$, which act on the state $|N, \Lambda \rangle$, followed by diagonalization completely solves the dynamics. While this is intractable in QCD, the symmetry properties of these operators can be exploited to powerful effect, as will be seen below.

**Fock-space expansion and the chiral basis.** The separation of dynamics and kinematics described above allows the Fock-basis expansion of the nucleon state, in which the state vector is given by a (infinite) sum of contributions labeled by the QCD field content 1. The contributions involving the fewest numbers of partons have been worked out in Refs. [13, 16–18]. The dynamical light-cone quark field $\Psi_+$ is

$$\Psi_+(x) = \sum_{\lambda,|\mu\rangle} \int \frac{dk^+d^2k_\perp}{2k^+(2\pi)^3} \left\{ b_\lambda(k^+, k_\perp) u_\lambda(k, \lambda) e^{-ik \cdot x} + d_\lambda^c(k^+, k_\perp) v_\lambda(k, \lambda) e^{ik \cdot x} \right\},$$

where $u_\lambda, v_\lambda$ are Dirac wavefunctions, $b_\lambda(k^+, k_\perp)$ destroys a quark and $d_\lambda^c(k^+, k_\perp)$ creates an antiquark, and the flavor and color indices have been suppressed. With standard normalization, the simplest Fock component in the proton state with $\Lambda = \frac{1}{2}$ is then

$$|u_+ u_1 d_1, \frac{1}{2}, 0 \rangle = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \varphi(\kappa_1, \kappa_2, \kappa_3) \frac{\delta^{abc}}{\sqrt{6}}\left( u^a(x, k_1) b_1^b(x_2, k_2) b_1^c(x_3, k_3) - (u \leftrightarrow d) \right) |0\rangle,$$

where $x_i$ is the longitudinal momentum of the quark in units of the proton longitudinal momentum, the shorthand, $\kappa_i \equiv (x_i, k_\perp)$, has been used, the integrations are over all constituent momenta, and $\varphi(\kappa_1, \kappa_2, \kappa_3)$ is the (Fourier transform of the) light-cone wavefunction for this particular Fock component. Here the states are labeled by the QCD field content, the total valence quark light-cone helicity, and all other sources of helicity (including orbital components), respectively. Therefore, for instance, a Fock component of the helicity-$\frac{1}{2}$ proton with, in addition to the valence quarks, one gluon and no source of orbital angular momentum could be labeled as $|u_1 u_2 d_1 g, -\frac{1}{2}, 1\rangle$. Note that in Eq. 3, the proton is in the special frame with $p^+ = 1$ and $p_\perp = 0$. The Fock component in a boosted frame is obtained by simply re-labeling the momenta of the field creation operators, while leaving the wavefunction, which carries the internal information, unchanged, in accord with the claim made above that the internal degrees of freedom are cleanly separated from the kinematics. The difficulty with the Fock expansion of the nucleon state vector is that there is no small parameter in QCD to indicate which components should dominate out of the infinite space of possible contributions (the large-$N_c$ approximation is an exception as we will argue below).

Assuming QCD with two massless flavors, and therefore $SU(2)_L \otimes SU(2)_R$ chiral symmetry, the associated light-cone charges are straightforward to construct [12]. With these charges, it is found that the light-cone quark fields of definite helicity transform irreducibly with respect to the chiral group $SU(2)_L \otimes SU(2)_R$:

$$\Psi_{+R} = \psi_{+R}^\dagger \in (1, 2), \quad \Psi_{+L} = \psi_{+L} \in (2, 1)$$

where $(R_L, R_R)$ labels the $SU(2)_L \otimes SU(2)_R$ content and $R_{LR}$ are $SU(2)_{L,R}$ representations.

Hence in the chiral basis, the most general helicity-$\frac{1}{2}$ nucleon state, $|N, \frac{1}{2}\rangle$, with three valence quarks will, in general, be a linear combination of the six classes 3 of states: $|(2, 1), \frac{1}{2}, 0\rangle, |(2, 3, 2), \frac{1}{2}, 0\rangle, |(1, 2), -\frac{1}{2}, 1\rangle, |(3, 2, 2), -\frac{3}{2}, 1\rangle, |(1, 2), \frac{3}{2}, -1\rangle, |(2, 1), -\frac{3}{2}, 2\rangle$, where the states have been labeled as

$$|(R_L, R_R), \lambda, \varphi_3\rangle \equiv (R_L, R_R), \lambda \rangle \otimes |\varphi_3\rangle,$$

where $R$ is an $SU(2)$-isospin representation in the product $R_R \otimes R_L$. The total helicity operator is divided into a valence spin operator, $\hat{S}_3$, and an operator, $\hat{P}_3$, that counts all other sources of spin:

$$J_3 = \hat{S}_3 \otimes 1 + 1 \otimes \hat{P}_3,$$

so that $\hat{S}_3 |\lambda\rangle = \lambda |\lambda\rangle$, $\hat{P}_3 |\varphi_3\rangle = \varphi_3 |\varphi_3\rangle$ and the total helicity is $\Lambda = \lambda \pm \varphi_3$. In QCD, the gauge-invariant operator $\hat{S}_3$ is, up to a factor of two, given by the anomalous $U(1)_A$ chiral charge [12]. This decomposition is therefore gauge invariant, but scale dependent.

---

1 The conventions for the light-cone coordinates and momenta are defined and described in Ref. [12, 13].

2 Clear explanations of the QCD Fock basis are found in Refs. [14, 15].

3 The states with $R_L \leftrightarrow R_R$ are contained in $|N, -\frac{1}{2}\rangle$. 

1\footnote{The conventions for the light-cone coordinates and momenta are defined and described in Ref. [12, 13].}

2\footnote{Clear explanations of the QCD Fock basis are found in Refs. [14, 15].}
be led by (quasi)free massless partons, each of which is labeled by $i$. The ground state should be described in terms of some valence degrees of freedom (red dots) and a finite number of sea partons (black dots), where the blue rings indicate that the partons and their interactions are dressed by the renormalization-group evolution.

Consider a nucleon on the initial hyperplane $x^+ = 0$ and with longitudinal momentum $p^+$. In a frame with vanishing transverse momentum. On the initial time slice, a nucleon is defined to be a collection of (quasi)free massless partons, each of which is labeled by $i$ and carries a longitudinal momentum fraction $x_i \equiv k_{i\perp}^+ / p^+$ and a transverse momentum $k_{i\perp}$ so that the nucleon energy is

$$M^2 = \sum_i k_{i\perp}^2 / x_i + \ldots$$  \hfill (7)

In null-plane quantization small $x_i$ means high energy, and therefore integrating out high-energy physics is equivalent to integrating out low-$x_i$ partons (so-called wee partons). Assuming, for simplicity, that interactions are local in $x_i$ (nearest-neighbor), and integrating out shells, one first integrates out all partons with $x_i < x_{e1}$. Integrating out these small $x_i$ partons result in “dressed” partons that have new interactions with the “frozen partons” that are represented by effective Hamiltonians with a hierarchy of interactions governed by the small parameter $x_{e1} / x_i$ [25, 26]. Schematically, the energy takes the form

$$M^2 = M^2_{x_i > x_{e1}} + M^2_{x_i = x_{e1}} + \ldots$$  \hfill (8)

where the dots represent the frozen degrees of freedom with $x_i < x_{e1}$. The first term represents the active, dynamical partons, while the second term corresponds to the interaction of these partons with the frozen sea. As non-perturbative physics in null-plane quantization is carried by the low-$x_i$ partons, this second term carries the chiral-symmetry breaking contribution to the energy; i.e. the part of the energy that carries non-vanishing chiral charge and transforms as $(2, 2)$ with respect to the chiral group: $M^2_{22}$. One can then further integrate out the partons with $x_i < x_{e2}$, etc. Finally this procedure results in a description of the ground state that involves a minimal number of partons —presumably the valence partons —interacting with a few sea partons. This procedure of integrating out low-$x_i$ partons is illustrated in Fig. 1. The resulting effective Hamiltonians are complicated, because, as noted above, all of the vacuum physics is low-$x_i$ physics and therefore the RG evolution necessarily accounts for the non-perturbative physics of QCD. The practical consequence of this simple RG argument is that the nucleon ground state can be described by a nucleon state vector, with a finite number of states of definite chiral charge, whose detailed form can be fit to experimental data. Crucially, the truncation of the chiral basis is scale dependent as it depends on the energy cutoff, $x_c$.

Spin entanglement defined. In the chiral basis, the nucleon state vector decomposes into a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, illustrated in Fig. 2, in which the valence helicity lives in $\mathcal{H}_A$, and all other sources of helicity live in $\mathcal{H}_B$. If measuring the valence helicity content of the nucleon, the sea is traced over to give the reduced density matrix of the valence helicity,

$$\rho_A = \text{tr}_B (| N, \Lambda \rangle \langle N, \Lambda |),$$  \hfill (9)

and the entanglement entropy is

$$S_N = -\text{tr}_A (\rho_A \log \rho_A).$$  \hfill (10)
The nucleon loses all memory of the spin of its valence quarks. The minimal realistic representation in this system is shown to drive chiral symmetry breaking. In the large-system consisting of the chiral spin due to the valence quarks and the remaining spin content, the chiral decomposition of the nucleon null-plane wavefunction implies a natural bipartite basis of dimension given by \( N \), allowing a formal solution of the null-plane Hamiltonians.

\[
\hat{H} = \sum_{n} \sum_{j} d_{ij} |(\mathcal{R}_L, \mathcal{R}_R)\rangle \langle \mathcal{R}_L, \mathcal{R}_R| \otimes |p_{3}\rangle_j \quad (11)
\]

where \( n_{\chi} (n_s) \) is the number of valence (non-valence) states. Generally, it is expected that \( n_s \gg n_{\chi} \), as is the case in the Fock expansion where components with arbitrary numbers of sea partons can contribute, however, the renormalization group argument given above suggests that many of the components are high-energy degrees of freedom that can be integrated out of the ground-state vector under consideration. In any event, the Schmidt decomposition theorem reveals that via a basis change, the state vector can be expressed in a basis of dimension given by \( n = \min\{n_{\chi}, n_s\} \). It is convenient to view the nucleon state vector as a perturbation of the large-\( N_c \) result,

\[
|N, \frac{1}{2}\rangle = |(2, 3)_2, \frac{1}{2}, 0\rangle = |(2, 3)_2, \frac{1}{2}\rangle \otimes |0\rangle \quad (12)
\]

which is a product state \( (n = 1) \) and therefore has vanishing entanglement entropy. As the matrix element of the symmetry-breaking energy, \( M_{22} \), vanishes between these states and turns on only when the nucleon state vector ceases to be a product state, it is clear that the spin entanglement defined here is intimately related to chiral symmetry breaking and, consequently, to the nucleon spin decomposition. Evidently, chiral symmetry is broken in the nucleon if and only if \( n > 1 \). Therefore, some measure of entanglement is acting as an order parameter of chiral symmetry breaking. This claim is considered in generality in Ref. [27]. Here a simple two-component model which captures the essence of the idea is explored.

**An illustrative model.** Consider the simplest nucleon state vector that extends the large-\( N_c \) result to include chiral symmetry breaking:

\[
|N, \frac{1}{2}\rangle = \sin \psi |(1, 2), -\frac{1}{2}, 1\rangle + \cos \psi |(2, 3)_2, \frac{1}{2}, 0\rangle. \quad (13)
\]

The state orthogonal to the nucleon in this model may be viewed as a collective of isodoublet excited states of both parities. The vector space spans the reducible chiral representation \((1, 2) \oplus (2, 3)\). The large-\( N_c \) limit is recovered as \( \psi \rightarrow 0 \), and the matrix element of the symmetry-breaking Hamiltonian scales as

\[
\langle N, \frac{1}{2} | M_{22}^2 | N, \frac{1}{2} \rangle \sim \sin 2\psi. \quad (14)
\]

The valence or intrinsic helicity contribution is

\[
\langle N, \frac{1}{2} | S_3 | N, \frac{1}{2} \rangle \equiv \frac{1}{2} \Delta \Sigma = \frac{1}{2} \cos 2\psi. \quad (15)
\]

Tracing over the sea gives the density matrix for the valence content

\[
\rho_A = \sin^2 \psi |(1, 2)\rangle \langle 1, 2| + \cos^2 \psi |(2, 3)_2\rangle \langle 2, 3)_2| \quad (16)
\]

from which follows the entanglement entropy

\[
S_N(\psi) = -\sin^2 \psi \log (\sin^2 \psi) - \cos^2 \psi \log (\cos^2 \psi). \quad (17)
\]

The probability amplitude for a spin-flip interaction is

\[
\sin \psi = \langle (1, 2), -\frac{1}{2}, 1 | N, \frac{1}{2} \rangle. \quad (18)
\]

This amplitude depends not on the parton momentum fractions, which are integrated over in the state vector, but rather on the cutoff on the parton momentum fractions, \( x_\epsilon \), as suggested by the RG arguments given above. A simple way of modeling this amplitude is to neglect sea quarks and assume that there are interactions between one of the valence quarks—which carries longitudinal momentum fraction \( x_\epsilon \), and a sea of gluons [28]. Taking \( \mathcal{G}(x_\epsilon) \) to be the density of the sea of gluons relative to valence quarks and \( \mathcal{P}(x_\epsilon) \) to be the probability of a spin-flip interaction between the valence quark and the sea, then \( \frac{1}{2} \mathcal{P}(x_\epsilon) \mathcal{G}(x_\epsilon) + 1 \) is the number of spins aligned in the initial direction of the valence quark’s spin and \( \frac{1}{2} \mathcal{P}(x_\epsilon) \mathcal{G}(x_\epsilon) \) is the number of spins in the opposite direction. Assuming a purely statistical distribution of the spins among interacting partons gives the probability of a spin-flip interaction

\[
\sin^2 \psi = \frac{\frac{1}{2} \mathcal{P}(x_\epsilon) \mathcal{G}(x_\epsilon)}{1 + \mathcal{P}(x_\epsilon) \mathcal{G}(x_\epsilon)}. \quad (19)
\]

Note that if there is no spin-flip interaction, then this amplitude vanishes and the large-\( N_c \) result is recovered.

In terms of the valence quark spin content of the nucleon,

\[
\Delta \Sigma = \cos 2\psi = \frac{1}{\mathcal{P}(x_\epsilon) \mathcal{G}(x_\epsilon) + 1}. \quad (20)
\]

Since the density of the sea, as given by the gluon distribution function, is expected to diverge as a power law with \( x_\epsilon \), it is expected that

\[
\cos 2\psi \xrightarrow{x_\epsilon \to 0} x_\epsilon^\delta \quad (21)
\]

where \( \delta \) is a positive number (assuming BFKL evolution for the gluon distribution function [7, 8], \( \delta \sim 4\alpha_s N_c \log 2/\pi \), where \( \alpha_s \) is the strong coupling constant and \( N_c \) is the number of QCD colors). As the cutoff
is taken to zero and the density of gluons diverges, the nucleon is driven to the maximally entangled state with \( \psi = 45^\circ \), and the nucleon spin is given entirely by the spin of the sea. Of course, this divergence is expected to be cut off by some kind of saturation mechanism [29]. The behavior of the entanglement entropy as a function of chiral symmetry breaking and valence spin content is shown in Fig. 3. The value of the mixing angle can be fit directly to low-energy data. In this model, it is straightforward to find the nucleon and \( \Delta \) axial and vector couplings. For instance, \( |C_{\Delta N}| = 2 \cos \psi \) and \( |g_A| = \frac{1}{2}(4 + \cos 2\psi) \). Fitting \( |C_{\Delta N}| \) to the \( \Delta \)-resonance decay width to pions gives \( \psi = 41 \pm 4^\circ \) which determines, among other things \([27]\), \( g_A = 1.38 \pm 0.05 \), \( \Delta \Sigma = 0.14 \pm 0.13 \) and \( S_N = 0.98 \pm 0.02 \) (in units of the maximum entropy of log 2). This result for the valence spin content is to be compared to \( \Delta \Sigma = 0.36 \pm 0.09 \) taken from the JAM collaboration’s global analysis \([30]\), at a renormalization scale of \( Q^2 = 1 \text{ GeV}^2 \).

**Realistic model-independent state vectors.** In realistic models of the nucleon state vector, the number of states in the sum, Eq. 11, should grow as the RG scale \( x_s \) is reduced, with the mass of the highest excitation in the nucleon’s chiral representation setting the scale of the separation into valence and sea spin. Detailed construction of such realistic models is beyond the scope of this letter but may be found in Ref. [27]. Many of the generic features of the two-component model persist in more realistic state vectors: for instance, the large-\( N_c \) component is always dominant due to the prominent nucleon-\( \Delta \) axial transition, but not so much so that it has much of an effect on the entanglement entropy, which is always near its maximum value\(^4\). The valence spin content in these models is generally small and consistent with the experimental range, as expected in a system where the entanglement entropy is large.

**Conclusion.** Entanglement between the valence and non-valence helicity components of the nucleon state vector drives chiral symmetry breaking in the nucleon. The entanglement entropy therefore acts as an order parameter of chiral symmetry breaking in the nucleon, and provides an explanation for why the valence spin content of the nucleon is small in the following sense. The interaction of the valence chiral charge with the sea, which breaks chiral symmetry, becomes stronger as the density of sea partons increases and the entanglement entropy rises. The separation between valence and non-valence spin is scale dependent, and the renormalization group implies that a state vector with a small number of components should suffice to provide a reasonably accurate description of the ground state. As the highly-energetic sea partons are integrated out, reducing the size of the vector space, the entanglement entropy decreases and the valence spin content of the nucleon increases. Conversely, as more sea partons are integrated in, increasing the size of the vector space, the entanglement entropy increases till it reaches its maximum value where presumably some kind of equilibrium or saturation state is achieved. It is interesting that this pattern of RG evolution is consistent with perturbative QCD evolution \([32]\). Realistic models of the nucleon state vector in the chiral basis which are fit to low-energy observables give values of the valence spin content which are consistent with experiment and with lattice QCD simulations, and possess an entanglement entropy near its maximum value. It would be interesting to access this chiral entanglement entropy in lattice QCD simulations.

We would like to thank S. Brodsky, A. Deur, D.B. Kaplan, N. Klco, Y. Kovchegov, W. Melnitchouk, M.J. Savage and R. Venugopalan for valuable communications. This work was supported in part by the U. S. Department of Energy grant DE-SC001347.

---

\(^4\) By contrast, recent work has found that entanglement in the nuclear force, as measured by the entanglement power, is minimized by the experimental data, and is very near large-\( N_c \) expectations \([31]\).
[557x730]6
[72x730]JETP 45, 199 (1977), [Zh. Eksp. Teor. Fiz.72,377(1977)].
[8] I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. 28, 822 (1978), [Yad. Fiz.28,1597(1978)].
[9] P. A. Dirac, Rev.Mod.Phys. 21, 392 (1949).
[10] J. B. Kogut and D. E. Soper, Phys. Rev. D1, 2901 (1970).
[11] H. Leutwyler and J. Stern, Annals Phys. 112, 94 (1978).
[12] S. R. Beane, Annals Phys. 337, 111 (2013), arXiv:1302.1600 [nucl-th].
[13] A. Belitsky and A. Radyushkin, Phys.Rept. 418, 1 (2005), arXiv:hep-ph/0504030 [hep-ph].
[14] K. Hornbostel, in Workshop on "From Fundamental Fields to Nuclear Phenomena", Boulder, Colorado, September 20-22, (1990) pp. 2–17.
[15] S. J. Brodsky and G. F. de Teramond, Phys. Rev. Lett. 96, 201601 (2006), arXiv:hep-ph/0602252 [hep-ph].
[16] X.-d. Ji, J.-P. Ma, and F. Yuan, Nucl. Phys. B652, 383 (2003), arXiv:hep-ph/0210430 [hep-ph].
[17] X.-d. Ji, J.-P. Ma, and F. Yuan, Phys. Rev. Lett. 90, 241601 (2003), arXiv:hep-ph/0301141 [hep-ph].
[18] X.-d. Ji, J.-P. Ma, and F. Yuan, Eur. Phys. J. C33, 75 (2004), arXiv:hep-ph/0304107 [hep-ph].
[19] S. Weinberg, Phys.Rev. 177, 2604 (1969).
[20] S. Weinberg, Phys.Rev.Lett. 22, 1023 (1969).
[21] S. Weinberg, (1994), arXiv:hep-ph/9412326 [hep-ph].
[22] J.-L. Gervais and B. Sakita, Phys.Rev.Lett. 52, 87 (1984).
[23] R. F. Dashen and A. V. Manohar, Phys.Lett. B315, 425 (1993), arXiv:hep-ph/9307241 [hep-ph].
[24] R. F. Dashen, E. E. Jenkins, and A. V. Manohar, Phys.Rev. D49, 4713 (1994), arXiv:hep-ph/9310379 [hep-ph].
[25] A. Casher and L. Susskind, Phys.Rev. D9, 436 (1974).
[26] L. Susskind and M. Burkardt, (1994), arXiv:hep-ph/9403313 [hep-ph].
[27] S. R. Beane and P. Ehlers, in preparation. (2019).
[28] R. D. Carlitz and J. Kaur, Phys. Rev. Lett. 38, 673 (1977), [Erratum: Phys. Rev. Lett.38,1102(1977)].
[29] Y. V. Kovchegov and E. Levin, Quantum chromodynamics at high energy, Vol. 33 (Cambridge University Press, 2012).
[30] J. J. Ethier, N. Sato, and W. Melnitchouk, Phys. Rev. Lett. 119, 132001 (2017), arXiv:1705.05889 [hep-ph].
[31] S. R. Beane, D. B. Kaplan, N. Kico, and M. J. Savage, Phys. Rev. Lett. 122, 102001 (2019), arXiv:1812.03138 [nucl-th].
[32] D. de Florian and W. Vogelsang, Phys. Rev. D99, 054001 (2019), arXiv:1902.04636 [hep-ph].