Correlation functions of minimal models coupled to two dimensional quantum supergravity

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ABSTRACT

We compute the three point functions of Neveu–Schwarz primary fields of the minimal models on the sphere when coupled to supergravity in two dimensions. The results show that the three point correlation functions are determined by the scaling dimensions of the fields, as in the bosonic case.

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1. Introduction

Currently, there are several approaches to two dimensional quantum gravity including the light–cone gauge approach [1], matrix models [2], topological field theory [3] and conformal matter coupled to Liouville theory [4,5,6]. However, in the case of two dimensional supergravity, no satisfactory matrix model formulation has been constructed. In the topological field theory approach, even though some progress has been made, correlation functions are yet to be computed [7,8].

One advantage the super Liouville approach has over the others is that it is straightforward, at least conceptually, to generalize the facts established in the bosonic case to the supersymmetric case. The three point functions were computed in the bosonic case using the area operator as a screening charge and then using the results from conformal field theory [9]. A crucial ingredient in their method is a formal continuation of an identity shown on the integers to non–integer values, which was justified using a semi–classical analysis in [10].

In this work, we shall compute the genus zero three point functions of minimal models [11,12] coupled to two dimensional supergravity using the Liouville theory approach [6]. The computation is performed for general Neveu-Schwarz primary fields in the $N = 1$ minimal models, including the non–unitary ones. In fact, the results may be stated simply; the three point functions on the sphere are determined by the scaling dimensions of the operators, as in the bosonic case. The Liouville approach is the most natural one from the field theoretical point of view. We first compute the correlation function for a fixed metric and integrate over the metric, modulo symmetries. Weyl invariance, which was present in the classical theory, is destroyed by the conformal anomaly, so that the integration over the metric corresponds to the integration over the moduli space of Riemann surfaces and the scale factor of the metric using the Liouville measure at a fixed fiducial metric. The correlation functions for the fixed metric are none other than the conformal field theory correlation functions. The conformal field theory three point functions on the sphere for the minimal models have been computed in [13] for the supersymmetric case. Another strong point of the Liouville approach is that the matter sector has been actively studied and is reasonably well understood from the works trying to classify conformal field theory. In particular, the so called minimal models and their supersymmetric generalizations have been classified [11,12,14,15,16].
2. Minimal models coupled to supergravity

In this section, we compute the three point functions of the minimal models coupled to two dimensional supergravity. The calculation parallels that of the bosonic case in [9,10]. We will work in superspace and follow the conventions of [17]. We shall formulate the super Liouville approach to two dimensional quantum supergravity along the lines of [6]. A general correlation function on a genus \(p\) surface is given by (genus zero and one are special due to the existence of continuous conformal automorphisms and will be explained below.)

\[
\left\langle \prod_{j=1}^{N} \Psi_j \right\rangle = \int_{sM_p} dm \int_{Dh} \Phi e^{-S_{SL}} \prod_{j=1}^{N} \int d^2 z_j e^{\beta(h_j)\Phi(z_j)} \left\langle \prod_{j=1}^{N} \Psi_{\text{matter}}(z_j) \right\rangle _{\hat{E}} \tag{2.1}
\]

where \(sM_p\) denotes the moduli space of genus \(p\) super Riemann surfaces and the super Liouville action \(S_{SL}\) is

\[
S_{SL} = \frac{1}{4\pi} \int \hat{E} \left[ \frac{1}{2} D_{a} \Phi D^{a} \Phi - \kappa \hat{R} \Phi + \mu e^{\alpha} \Phi \right], \tag{2.2}
\]

The parameters in the super Liouville action may be determined, by the conditions that the matter part combined with the gravitational part is conformal invariant and that \(\int \exp(\beta(h)\Phi)\Psi_{\text{matter}}\) should have conformal dimension \((0,0)\), to be

\[
\kappa = \sqrt{\frac{9 - \hat{c}}{2}}, \quad \beta(h) = -\sqrt{9 - \hat{c} + \sqrt{1 - \hat{c} + 16h}} \frac{1}{2\sqrt{2}}, \quad \alpha = \beta(h_{11}). \tag{2.3}
\]

\(\hat{c}\) denotes the amount of matter coupled to supergravity measured in the units of scalar superfields. We shall specialize to the case of \(N = 1\) minimal model in the Neveu–Schwarz sector. Then

\[
\hat{c} = 1 - \frac{8}{q(q + 2)}, \quad h_{r' r} = \frac{(qr' - (q + 2)r)^2 - 4}{8q(q + 2)}, \\
\kappa = \frac{2(q + 1)}{\sqrt{q(q + 2)}}, \quad \beta(h_{r' r}) = \frac{-2(q + 1) + |qr' - (q + 2)r|}{2\sqrt{q(q + 2)}}, \quad \alpha = -\sqrt{\frac{q}{(q + 2)}}. \tag{2.4}
\]

Integer \(q\) corresponds to the unitary minimal models and \(q\) is rational for general non–unitary models. We should point out that the method of screening charges which we use to compute correlation functions is valid even for irrational \(q, r'_j, r_j\) at tree level as long as the sum of the external charges is integer. However, for generic values of \(q\), it is not known how to construct interacting theories.
Integrating over the constant mode of $\Phi$,

$$
\int D_{\hat{E}} \Phi e^{-S_{SL}} \prod_{j=1}^{N} e^{\beta(h_j)\Phi(z_j)} = \left( \frac{\mu}{2\pi} \right)^s \alpha^{-1} \Gamma(-s) \int D_{\hat{E}}' \Phi e^{-S'_{SL}} \left( \int \hat{E} e^{\alpha\Phi} \right)^s \prod_{j=1}^{N} e^{\beta(h_j)\Phi(z_j)}
$$

where $D'_{\hat{E}} \Phi$ denotes the integration over the modes orthogonal to the constant mode and

$$
S'_{SL} \equiv \frac{1}{4\pi} \int \hat{E} \left[ \frac{1}{2} D\alpha \Phi D\alpha \Phi - \kappa \hat{R} \Phi \right], \quad s = -\frac{\kappa}{\alpha} (1 - p) - \sum_{j=1}^{N} \frac{\beta(h_j)}{\alpha}
$$

Let us now specialize to correlation functions on the sphere. The sphere has no moduli but has a group of conformal automorphisms that act on the surface, $\text{Osp}(2|1; \mathbb{C})$ of complex dimension 3|2. Therefore, we integrate over the location of the operators, $z_j$, and then divide by the volume of the group of conformal automorphisms. This corresponds to integrating with the measure $|z_1 - z_2| |z_2 - z_3| |z_3 - z_1| d^2 \nabla$ after combining the gravitational part with the matter part of the correlation function. Here, $\nabla, \nabla$ are invariants under $\text{Osp}(2|1; \mathbb{C})$.

We will concentrate the curvature of the sphere at infinity, $\infty|0$ and work with the flat zweibein on the plane.

$$
\int D_{\hat{E}} \Phi e^{-S_{SL}} \prod_{j=1}^{3} e^{\beta(h_j)\Phi(z_j)} = \left( \frac{\mu}{2\pi} \right)^s \frac{\Gamma(-s)}{\alpha} \prod_{i<j}^{3} |z_i - z_j|^{-2\beta(h_i)\beta(h_j)}

\times \int \prod_{j=1}^{s} d^2 w_j \prod_{i=1}^{3} |z_i - w_j|^{-2\alpha \beta(h_i)} \prod_{i<j}^{s} |w_i - w_j|^{-2\alpha^2}.
$$

The general integral of this type may be deduced from the work of [13], possibly up to a normalization factor for the screening charges, which may be absorbed into the normalization.

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* We shall use the notations $z \equiv z|\theta_z, \quad w \equiv w|\theta_w, \quad z - w \equiv z - w - \theta_z \theta_w, \quad \text{etc.} \ nabla$ is defined as

$$
\nabla \equiv \frac{\theta_{z_1} (z_2 - z_3) + \theta_{z_2} (z_3 - z_1) + \theta_{z_3} (z_1 - z_2) + \theta_{z_1} \theta_{z_2} \theta_{z_3}}{[(z_1 - z_2) (z_2 - z_3) (z_3 - z_1)]^{1/2}}
$$
of the external fields. (Below, \( \langle x \rangle \) denotes the fractional part of \( x \), i.e. \( \langle x \rangle \equiv x - [x] \).)

\[
\mathcal{I}_{n,n'}(a_j', a_j, \rho, \rho; z_j) \equiv \frac{1}{n!n'} \int \prod_{i=1}^{n'} \sum_{j=1}^{3} d^2 w_i \prod_{j=1}^{n} \prod_{j=i+1}^{n} |z_j - w_i|^2 \int \prod_{j=1}^{n} \prod_{j=i+1}^{n} |z_j - w_i|^2 \]

\[
\times \prod_{i<i'} \prod_{j<i} |w_i' - w_j'|^{4 \rho} \prod_{i<i'} \prod_{j<i} |w_i - w_j|^{4 \rho} \prod_{i=1}^{n'} \prod_{j=1}^{n} |w_i' - w_j|^2
\]

\[
= \left( \frac{\pi}{2} \right)^{n+n'} (2\rho)^{-2m - 4(2n'/2) + n-n'} \Delta^{-n'}(\rho' + 1/2) \Delta^{-n}(\rho + 1/2) \prod_{i<j}^{3} |z_i - z_j|^{2\delta_{ij}}
\]

\[
\times \prod_{i=1}^{n'} \Delta(-n/2 + (i+n)/2 + i\rho') \prod_{i=0}^{n'-1} \prod_{j=1}^{3} \Delta(1-n/2 -(i+n)/2 + a_j' + i\rho')
\]

\[
\times \prod_{i=1}^{n} \Delta((i/2) + i\rho) \prod_{i=0}^{n-1} \prod_{j=1}^{3} \Delta(1 - (i/2) + a_j + i\rho) \times \begin{cases} 
1 & n+n' \in 2\mathbb{Z} \\
\nabla\nabla(2\rho')^{-1} & n, n'-1 \in 2\mathbb{Z} \\
\nabla\nabla(2\rho)^{-1} & n-1, n' \in 2\mathbb{Z}
\end{cases}
\]

where \( \Delta(x) \equiv \Gamma(x)/\Gamma(1-x) \), \( \delta_{ij} \equiv n'a_1 + na_2 + na_2 - (n'a_3 + na_3) \) and the other \( \delta_{ij} \)'s may be obtained using cyclic permutations. The above formula is valid when the integral has no weight with respect to \( w_j \), i.e.,

\[
\sum a_j' + 2\rho'(n' - 1) - (n - 1) = 0, \quad \sum a_j + 2\rho(n - 1) - (n' - 1) = 0 \tag{2.9}
\]

This condition corresponds just to charge conservation in the free field theory and guarantees the covariance of the correlation function under conformal transformations with respect to \( z_j \).

Using this integral formula for \( (n', n) = (0, s) \), we find

\[
\int D_E \Phi e^{-S_{SL}} \prod_{j=1}^{3} e^{\beta(h_j)} \Phi(z_j) = \mu^s 2^{-3s} \alpha^{-1} \Gamma(-s) \Gamma(s + 1) \Delta^{-s}(-\rho' + 1/2) \prod_{i<j}^{3} |z_i - z_j|^{-2\gamma_{ij}}
\]

\[
\times \prod_{(x', x) = (0, 0)}^{s} \prod_{(r_j', r_j')} \Delta(|2x'\rho' - x|/2 + (i/2) - i\rho') \times \begin{cases} 
1 \\
\nabla\nabla(2\rho')^{-1} \end{cases} \quad s \text{ even}
\]

\[
\begin{cases} 
\nabla\nabla(2\rho)^{-1} \end{cases} \quad s \text{ odd}
\]

(2.10)

We defined \( \rho' \equiv \alpha^2/2 \), \( \gamma_{12} \equiv h_1 + h_2 - h_3 - 1/2 \) and so on. In this formula and in the expression for the correlation function of the matter fields, there are factors that become ill-defined as \( r_j, r_j' \) approach integers. Below, we shall keep \( r_j, r_j' \) to be non-integer and take the limit of \( r_j, r_j' \) going to integers after combining the gravitational and the matter part
of the correlation function. At that point, the expressions will be well defined. It is also possible to derive formulas that are well defined for integer values of \( r_j, r'_j \) as in the bosonic case [9,10]. However it makes the calculation less transparent and since it is not necessary, we shall not do so here.

The three point function on the sphere, \( \left\langle \prod_{j=1}^{3} \Psi_{j}^{\text{matter}}(z_j) \right\rangle_{\hat{E}} \), may be computed using the method analogous to that of [13,18]. An arbitrary primary field in the Kac table \( \{ \Psi_{r,r'}^{\text{matter}}, 1 \leq r' \leq q + 2, 1 \leq r \leq q \} \) may be expressed using a free field \( \Phi' \) as

\[
\Psi_{r,r'}^{\text{matter}} = e^{i\alpha_{r'}\Phi'}, \quad \alpha_{r'} = \frac{1}{2} \left[ (1-r')\alpha_+ + (1-r)\alpha_- \right], \quad \alpha_- = \alpha = -\frac{1}{\alpha_+} = -\sqrt{\frac{q}{q+2}}
\]

with the background charge \( 2\alpha_0 \) at infinity (\( 2\alpha_0 \equiv \alpha_+ + \alpha_- \)). \( \Phi' \) is governed by the action

\[
S_{\Phi'} = \frac{1}{4\pi} \int \hat{E} \left[ \frac{1}{2} D_{\alpha} \Phi' D^\alpha \Phi' + i\alpha_0 \hat{R} \Phi' + e^{i\alpha_- \Phi'} + e^{i\alpha_+ \Phi'} \right]
\]

Then the three point function is

\[
\left\langle \prod_{j=1}^{3} \Psi_{j}^{\text{matter}}(z_j) \right\rangle_{\hat{E}} = \int D_{\Phi'} e^{-S_{\Phi'}} \prod_{j=1}^{3} e^{i\alpha_{r'_j} \Phi'(z_j)} \frac{1}{n'!n!} \left( \int e^{i\alpha_- \Phi'} \right)^{n'} \left( \int e^{i\alpha_+ \Phi'} \right)^{n}
\]

\[
= \prod_{i<j} \left| z_i - z_j \right|^{2\alpha_{r'_i} \alpha_{r'_j}} \mathcal{J}_{n'n!}(\alpha_- \alpha_{r'_j}, \alpha_+ \alpha_{r'_j}, \rho, \rho'; z_j)
\]

Here,

\[
2n' \equiv \sum_{j=1}^{3} r'_j - 1, \quad 2n \equiv \sum_{j=1}^{3} r_j - 1, \quad \rho' = \alpha_-^2 / 2, \quad \rho = \alpha_+^2 / 2
\]

We combine the gravitational part and the matter part of the correlation function, (2.10), (2.13), and then integrate over the locations of the operators \( \Psi_j \) and divide by the volume of the conformal automorphisms of the sphere to obtain the following expression for the three

\[
* \text{We normalize the scalar fields } \Phi, \Phi' \text{ according to the standard convention, } \Phi(z)\Phi(z') = -\ln |z - z'|^2 + O(1) \text{ and likewise for } \Phi'. \text{ This differs from the normalization of [18] by a factor of } \sqrt{2}.
\]
point function.

\[
\left\langle \prod_{j=1}^{3} \Psi_j \right\rangle = \mu^2 n'^{-3} n^{-3} n'^{n'} + n \Gamma(-s) \Gamma(s + 1)(2\rho)^{n'^{-4} + n^{-4} + n^{-1} + 2}
\]

\[
\times \Delta^{-n'} + (\rho + 1/2) \Delta^{-n} \rho + 1/2 \prod_{(x', x) = (0, 0)} \prod_{(r'_j, r_j)} \left\{ \prod_{i=1}^{s} \Delta(2x' \rho' - \rho)/2 + \langle i/2 \rangle - \langle \rho' \rangle \right\}
\]

\[
\times \prod_{i=1}^{n'} \Delta((x - 2x' \rho' - n)/2 + \langle (i + n)/2 \rangle + \langle \rho' \rangle) \prod_{i=1}^{n} \Delta((x' - 2x \rho)/2 + \langle i/2 \rangle + \langle \rho \rangle) \]

(2.15)

Integration over \( \nabla, \nabla \) requires that \( n' + n + s \) be odd.

We recall that in minimal models, the representation of primary fields in the Kac table is doubly degenerate, namely, the fields \( \Psi_{r', r''}^{matter}, \Psi_{q+2-r', q-r''}^{matter} \) represent the same field. However, the conformal field theory three point functions are not invariant when this interchange is applied to one of the fields in the correlation function but is invariant when the operation is applied to two of the fields. (Up to normalization factors that may be absorbed into the normalization of the external fields.) Since \( r_j - 2r'_j \rho' \) just changes sign under the operation \((r', r) \leftrightarrow (q + 2 - r', q - r)\), we may assume that \( r_j - 2r'_j \rho' \) are all of the same sign for the fields in the three point function without any loss in generality.

When \( r_j - 2r'_j \rho' \geq 0 \) for \( j = 1, 2, 3 \), from (2.4) and (2.6), we obtain

\[
2\rho' = \frac{n}{n' + s + 1}
\]

(2.16)

As shown in the Appendix, the following formula holds when \( n + n' + s \) is odd.

\[
\prod_{i=1}^{s} \Delta(i/2 + \langle i/2 \rangle - \langle \rho' \rangle) = (2\rho)^{(n+1-2y)(n'+s+1)/2-n/2+\langle n/2 \rangle} \Delta^{-1}(y/2)
\]

\[
\times \prod_{i=1}^{n} \Delta^{-1}(\langle i/2 \rangle + (-y + i)\rho) \prod_{i=1}^{n'} \Delta^{-1}(y - n)/2 + \langle (i + n)/2 \rangle + \langle \rho' \rangle)
\]

(2.17)

Using this formula for \( y = 0, r_j - 2r'_j \rho' \) and from (2.15), we obtain the formula for the three point function as

\[
\left\langle \prod_{j=1}^{3} \Psi_j \right\rangle = \mu^2 n'^{-3} n^{-3} n'^{n'} + n \Gamma(-s) \Gamma(s + 1)(2\rho)^{s+3/2} \Delta^{-n' + s}(\rho' + 1/2) \Delta^{-n}(\rho + 1/2)
\]

\[
\times \prod_{j=1}^{3} \Delta^{-1}((r_j - 2r'_j \rho')/2)
\]

(2.18)

A sublety needs to be mentioned here; as in the bosonic case [9], the expression here contains
a factor $\Gamma(-s)\Gamma(s+1)S(s)$ which is of indeterminate form 0/0 when $s$ is integer. This factor is taken to be one. This formal procedure is justified by the semi–classical analysis which establishes the identity as $s$ goes to infinity in the complex $s$ plane [10].

Rescaling the external fields as

$$
\Psi_j \mapsto \text{const.} \times 2^{-2r_j^\prime + (-1/2 + 3\rho)r_j\pi} (r_j^\prime + r_j)/2 (2\rho)^{pr_j - r_j^\prime/2} \Delta^{pr_j - r_j^\prime}(\rho^\prime + 1/2)
$$

the three point function reduces just to

$$
\left\langle \prod_{j=1}^{3} \Psi_j \right\rangle = \mu^s
$$

The constant factor in the formula (2.19) denotes a factor that does not depend on the external indices $\{(r_j^\prime, r_j)\}$.

When $r_j - 2r_j^\prime \rho' \leq 0$ for $j = 1, 2, 3$, we analogously obtain

$$
2\rho' = \frac{n + 1}{n' - s}
$$

As shown in the Appendix, the following formula holds when $n + n' + s$ is odd.

$$
\prod_{i=1}^{s} \Delta(-y/2 + \langle i/2 \rangle - i\rho') = (2\rho)^{(n-2y)(n'-s)/2-n/2+(n/2)-1} \Delta^{-1}(-y\rho)
$$

$$
\times \prod_{i=1}^{n} \Delta^{-1}(\langle i/2 \rangle + (-y + i)\rho) \prod_{i=1}^{n'} \Delta^{-1}((y - n)/2 + \langle (i + n)/2 \rangle + i\rho')
$$

Again, using this formula for $y = 0, r_j - 2r_j^\prime \rho'$ we obtain the formula for the correlation function. The result is identical to (2.18), except for the replacement

$$
\Delta^{-1}(r_j - 2r_j^\prime \rho')/2 \mapsto \Delta^{-1}((r_j^\prime - 2r_j\rho)/2).
$$

Therefore, the correlation function again reduces to $\mu^s$ after rescaling the fields.

Differentiating with respect to $\mu$ is equivalent to bringing down the area operator:

$$
\frac{\partial}{\partial \mu} \left\langle \prod_{j=1}^{N} \Psi_j \right\rangle = \left\langle 1 \prod_{j=1}^{N} \Psi_j \right\rangle
$$

The three point function which is independent of the normalization of the fields may be
computed using this relation as
\[
\frac{\langle \Psi_1 \Psi_2 \Psi_3 \rangle^2 Z}{\langle \Psi_1 \Psi_1 \rangle \langle \Psi_2 \Psi_2 \rangle \langle \Psi_3 \Psi_3 \rangle} = \prod_{j=1}^{3} \frac{|q r_j' - (q + 2) r_j|}{4(q + 1)(q + 2)}
\]
(2.25)

where \( Z \) is the partition function of the model on the sphere.

3. Summary and Discussion

In this paper, we computed the three point functions on the sphere of Neveu–Schwarz primary fields in \( N = 1 \) minimal matter coupled to supergravity. The result is simple; the three point function reduces to just the power of the chemical potential for the area raised to the power of the scaling dimensions. This general feature is consistent with the expectations from the matrix models and topological models. However, the corresponding (super–)matrix models are yet to be identified, despite some previous effort [8,20]. The computations of correlation functions put quantum supergravity on a more concrete ground, and may provide the basis for identifying the corresponding matrix and topological models.

It should be possible to build higher genus, higher point correlation functions by sewing together genus zero three point functions. Therefore, specifying the spectrum of the model uniquely identifies the model. This important point may be established by computing higher genus or higher point functions. In conformal field theory, modular invariance strongly restricts the spectrum, and in the case of minimal models, the classification is known [15,16]. It is natural to assume that the spectrum of the conformal field theory model is unchanged when coupled to gravity or supergravity. This is also consistent with the fact that the three point functions between general primary fields in the Kac table reduces to the chemical potential raised to the power of the sum of the scaling dimensions. Such is the case in the left–right symmetric (“A–type”) bosonic unitary minimal models coupled to gravity and the corresponding matrix models in this case are Hermitean matrix models. In the non–unitary models, even in the bosonic case, the situation is not clear and it is of interest to find the complete spectrum [19].

For \( N = 1 \) minimal models with even \( q \), the scaling dimensions of the Neveu–Schwarz fields reduce to that of the bosonic minimal model if one restricts the indices of the primary fields to lie in about a quarter of the Kac table. It has even been suggested that the supersymmetric model reduces to the bosonic model where such a truncation of the spectrum occurs [8]. This is possible, but unexpected from our results and also seems to disagree with the one–loop partition function [21]. Also, one might expect that there is a qualitative difference for \( q \) even and \( q \) odd, since the Witten index is zero when \( q \) is odd [8]. We find no qualitative difference between \( q \) even and odd at tree level. Perhaps a promising candidate for the corresponding matrix models are unitary matrix models [22]. They have the feature that the scaling dimensions of some of the operators agree with the Hermitean matrix models, present in \( N = 1 \) minimal matter.
Even though coupling conformal matter to Liouville theory seems like the natural approach from field theory, the computations needed were involved and yet the result simple. It would be interesting to find a method of computation in this or in another approach which obtains the same result in a more straightforward way. If no such scheme is found, Liouville theory might not be the appropriate framework for computing correlation functions of matter coupled to gravity. However, the super–Liouville approach offers the most straightforward generalization from the bosonic case and indeed we note that correlation functions in matter coupled to two dimensional supergravity have not been computed in the other approaches. An important point still to be investigated is how to incorporate Ramond fields, since they break two dimensional supersymmetry.

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APPENDIX

In the paper, the identities, (2.17) and (2.22) were used. Since the proof is essentially the same for both cases, we shall only present the case (2.22), when $2\rho' = (n + 1)/(n' - s)$ and $n' + n + s$ is odd.

Call the left hand side of (2.22) $L(y)$ and right hand side $R(y)$. We shall proceed in two steps;

(i) $L(y)/R(y)$ is periodic in $y$ with period $1 - 2\rho'$.
(ii) $L(y)/R(y)$ goes to 1 as $y$ goes to infinity in the complex $y$ plane.

From

$$
\frac{L(y)}{L(y + 1 - 2\rho')} = (2\rho)^{-s + 2(s/2)} \frac{\Delta(-y/2 + \langle s/2 \rangle - s\rho') \Delta((y + 1)\rho + (s + 1)/2 - \langle s/2 \rangle)}{\Delta(-y/2) \Delta((y + 1)\rho + 1/2)}
$$

$$
\frac{R(y + 1 - 2\rho')} {R(y)} = (2\rho)^{s + 1 - 2(n/2) - 2(n'/2)} \frac{\Delta(-y/2) \Delta(1 - \langle n/2 \rangle + (y - n)\rho)}{\Delta((y - n)/2 + 1 - \langle n/2 \rangle) \Delta(1/2 - (y + 1)\rho)}
$$

$$
\times \frac{\Delta((-y + n)\rho + \langle n/2 \rangle) \Delta((y - n)/2 + 1 - \langle n/2 \rangle - \langle n'/2 \rangle + n'\rho')} {\Delta((-y + n)/2 + \langle n'/2 \rangle) \Delta((y - n)\rho + n'/2 + 1 - \langle n/2 \rangle - \langle n'/2 \rangle)}
$$

(A.1)

Multiplying one formula by the other, it is easy to establish the statement (i) when $n' + n + s$ is odd. To show (ii), we use Stirling’s formula,

$$
\ln \Delta(z) = (z - \frac{1}{2}) \ln(-e^2z^2) - 1 + O(z^{-1})
$$

(A.2)

Then by a straightforward, but tedious calculation, we see that

$$
\ln L(y)/R(y) = (\langle s/2 \rangle + \langle n/2 \rangle + \langle n'/2 \rangle - 4\langle n/2 \rangle \langle n'/2 \rangle - 1/2) \ln(i e^2 y/2) + O(y^{-1})
$$

(A.3)
The right hand side of the equation vanishes when \(n' + n + s\) is odd, so that we show statement (ii) in this case. This in turn completes the derivation of the formula (2.22). The proof of (2.17) is essentially the same.

When \(n\) is odd, (2.22) may also be shown by factoring out two from the numerator and the denominator of \(\rho'\) and using the formulas derived in the bosonic case [9,10]. Likewise, when \(n\) even in (2.17), the formula may again be reduced to the bosonic case.

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