Abstract. In [14] we pointed out the correspondence between a result of Shelah in model theory, i.e. a theory is unstable if and only if it has IP or SOP, and the well known compactness theorem of Eberlein and ˇSmulian in functional analysis. In this paper, we relate a natural Banach space $V$ to a formula $\phi(x,y)$, and show that $\phi$ is stable (resp NIP, NSOP) if and only if $V$ is reflexive (resp Rosenthal, weakly sequentially complete) Banach space. Also, we present a proof of the Eberlein-Šmulian theorem by a model theoretic approach using Ramsey theorems which is illustrative to show some correspondences between model theory and Banach space theory.

Keywords: Eberlein-Šmulian theorem, Ramsey theorem, angelic space, weak sequential compactness, weak sequential completeness

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Contents

1 Introduction 2

2 Eberlain-Šmulian and Ramsey theorems 2
2.1 Pointwise convergence 3
2.2 Weak topology 6

3 Definability of types 6
3.1 Pták’s lemma and stability 7

4 Model theory and Banach space theory 9
4.1 Type space 10
4.2 Banach space for a formula 11
4.3 Stability and reflexivity 11
4.4 NIP and Rosenthal Banach spaces 12
4.5 NSOP and weak sequential completeness 13

A Stability and Eberlein compacta 13

B Angelicity of $C(X)$ 14
1 Introduction

In [15] Shelah introduced the independence property (IP) for 0-1 valued formulas and defined the strict order property as complementary to the independence property: a theory is unstable iff it has IP or SOP. On the other hand, a well known fact in functional analysis, the Eberlein-Šmulian theorem, states that a subset of a Banach space is not weakly precompact iff it has a sequence without any weak Cauchy subsequence or it has a weak Cauchy sequence with no weak limit. In fact, Shelah’s result corresponds to the Eberlein-Šmulian theorem. This was noticed in [14] and some various forms of definability of types for NIP models were proved.

The relation between model theory and Banach space theory is rather deep (e.g. see [2] and its references). So it would be desirable to have clearly understood channels of communication between technical complexity that both fields have attained in the last thirty years, so that techniques from one field might become useful in the other.

In this paper we propose to give a proof of the Eberlein-Šmulian theorem with a model theoretic point of view and investigate correspondence between model theory and Banach space theory. We relate a natural Banach space $V$ to a formula $\phi(x, y)$, and show that $\phi$ is stable (resp NIP, NSOP) if and only if $V$ is reflexive (resp Rosenthal, weakly sequentially complete) Banach space.

However, we believe that the main goal of this paper, if it is achieved, is to show that the correlation between model theory and Banach space theory is much more than what is known so far, and there are many connections between model theoretic classification and Banach space classification which can be studied in future works.

It is worth recalling another lines of research. In [9] and [17] the relationship between NIP and Rosenthal’s dichotomy were noticed in the contexts of $\aleph_0$-categorical structures in continuous logic and classical first order setting, respectively. The relationship between NIP in integral logic and Talagrand’s stability was studied in [13]. The above correspondence was noticed in [14] and some various forms of definability of types for NIP models were proved.

This paper is organized as follows: In the next section we present a proof of the Eberlein-Šmulian theorem and study its connection to the theorem of definability of types. In the third section, we investigate some correspondences between model theory and Banach space theory.

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2 Eberlein-Šmulian and Ramsey theorems

The Eberlein-Šmulian theorem states that the following three versions of precompactness are equivalent for the weak topology of a Banach space. Let $A$ be a subset of a topological space $X$, then
(i) The set $A$ is \textit{precompact} if its closure is compact.

(ii) The set $A$ is \textit{sequentially precompact} if each sequence of elements of $A$ has a subsequence converging to an element of $X$.

(iii) The set $A$ is \textit{countably precompact} if each sequence of elements of $A$ has a cluster point in $X$.

To prove this theorem, first we give a proof for the Banach space $C(X)$ which $X$ is a compact topological space. Then the general case follows easily from this case (see Remark 2.8 below). It is clear that (ii) implies (iii). We prove that (i) implies (ii) using a stronger version of Ramsey’s theorem (see Theorem 2.3 below). The proof of (iii) ⇒ (i) uses the theorem of definability of types and a combinatorial lemma.

\subsection{Pointwise convergence}

First we review some notions and results for the topology of pointwise convergence. If $X$ is any set and $A$ a subset of $\mathbb{R}^X$, then the topology of \textit{pointwise convergence} on $A$ is that inherited from the usual product topology of $\mathbb{R}^X$; that is, the coarsest topology on $A$ for which the map $f \mapsto f(x) : A \to \mathbb{R}$ is continuous for every $x \in X$. A typical neighborhood of a function $f$ is determined by a finite subset $\{x_1, \ldots, x_n\}$ of $X$ and $\epsilon > 0$ as follows:

$$U_f(x_1, \ldots, x_n; \epsilon) = \{g \in \mathbb{R}^X : |f(x_i) - g(x_i)| < \epsilon \text{ for } i \leq n\}.$$

\textbf{Assumption 2.1} In this paper (countable, sequential) precompactness means (countable, sequential) precompactness with respect to the topology of pointwise convergence. Also, we will say that a sequence or net is convergent if it is convergent for the topology of pointwise convergence. Otherwise, we explicitly state that what is our desired topology; e.g. weak, weak* or norm topology.

Now, recall the following standard fact from functional analysis. In fact, it is a topological presentation of stability in model theory (see \cite{14}). (See the appendix for its proof.)

\textbf{Fact 2.2 (Grothendieck’s criterion)} Let $X$ be a compact topological space. Then the following are equivalent for a norm-bounded subset $A \subseteq C(X)$:

(i) The set $A$ is precompact in $C(X)$.

(ii) For every sequences $\{f_n\} \subseteq A$ and $\{x_n\} \subseteq X$, we have

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m),$$

whenever both limits exist.
Here, we will present a form of Ramsey’s theorem which we use in this paper. For this, we give some notations. Let \([N]\) denote all infinite subsequences of \(\mathbb{N}\). For \(n \in \mathbb{N}\), \([N]^n\) denotes all finite subsequences of \(\mathbb{N}\) of length \(n\). If \(M \in [N]\) we use similar notation, \([M]\) and \([M]^n\) to denote all subsequences (or all length \(n\) subsequences) of \(M\). Suppose we can color, using colors \(R\) and \(B\), all infinite subsequences of \(\mathbb{N}\). Thus \(M \in [\mathbb{N}]\) implies \(M \in R\) or \(M \in B\). The following Ramsey’s theorem shows that if the set \(R\) has a “good” property then there exists \(M \in [\mathbb{N}]\) so that either for all \(N \in [M]\), \(N \in R\) or for all \(N \in [M]\), \(N \in B\). To be more precise we topologize \([\mathbb{N}]\) by the product topology; thus a basic open set in \([\mathbb{N}]\) is of the form \(O(n_1, \ldots, n_k) = \{M = (m_i) \in [\mathbb{N}] : m_i = n_i \text{ for } i \leq k\}\) where \(n_1 < \cdots < n_k\) is arbitrary.

**Theorem 2.3 (Ramsey theorem)** Assume that \(A \subseteq [\mathbb{N}]\) is Borel. Then there is an \(N \in [\mathbb{N}]\) so that either \([N] \subseteq A\) or \([N] \cap A = \emptyset\).

The above theorem is due to Galvin and Prikry in [7].

**Definition 2.4** A sequence \(\{f_n\}\) of real valued functions on a set \(X\) is said to be independent if there exist real numbers \(s < r\) such that

\[
\bigcap_{n \in P} f_n^{-1}(\mathbb{R}_+ \cup \{0\}, s) \cap \bigcap_{n \in M} f_n^{-1}(r, \mathbb{R}_+) \neq \emptyset
\]

for all finite disjoint subsets \(P, M\) of \(\mathbb{N}\).

**Lemma 2.5 (Rosenthal)** Let \(X\) be a compact space and \(F \subseteq C(X)\) a bounded subset. Then the following conditions are equivalent:

(i) \(F\) does not contain an independent subsequence.

(ii) Each sequence in \(F\) has a convergent subsequence in \(\mathbb{R}^X\).

**Proof.** (i) \(\Rightarrow\) (ii): Suppose that \(\{f_n\} \subseteq F\) has no convergent subsequence. Therefore, there are rational numbers \(r > s\) such that for all infinite subset \(M \subseteq \mathbb{N}\) there exists \(x \in X\) so that \(x\) belongs to infinitely many \(A_n = f_n^{-1}((-\infty, s])\)’s, \(n \in M\) and to infinitely many \(B_n = f_n^{-1}(r, \infty)\)’s, \(n \in M\). (Indeed, if this were not true, let \((r_i, s_i)_{i=1}^{\infty}\) (with \(r_i > s_i\)) be dense in \(\mathbb{R}^2\) and inductively choose \(M_i+1 \subseteq [M_i]\) so that the above conditions fail. If \(M\) is a diagonal of \(M_i\)’s then \(\{f_n\}_{n \in M}\) is convergent.) Now, let

\[
\mathcal{A} = \{L = (x_i)_{i=1}^{\infty} \in X^\mathbb{N} : \bigcap_{i=1}^k A_{l_{2i-1}} \cap \bigcap_{i=1}^k B_{l_{2i}} \neq \emptyset \text{ for all } k \in \mathbb{N}\}.
\]

Then \(\mathcal{A}\) is closed, because \(\mathcal{A} = \bigcap_{k=1}^{\infty} \mathcal{A}_k\) where \(\mathcal{A}_k = \{L = (x_i)_{i=1}^{\infty} \in X^\mathbb{N} : \bigcap_{i=1}^k A_{l_{2i-1}} \cap \bigcap_{i=1}^k B_{l_{2i}} \neq \emptyset\}\). (We note that \(\mathcal{A}_k\)’s are closed.) By Ramsey theorem 2.3
observation, there is some infinite subset \( L \subseteq \mathbb{N} \) such that \([L] \subseteq A\). Clearly \( \{f_{i2i}\}_{i<\omega} \) is independent.

(ii) \( \Rightarrow \) (i) is an easy exercise. (Indeed, if \( \{f_n\} \) was independent and \( r > s \) were witness, let \( \{f_{n_k}\} \) be a convergent subsequence of it, and let \( M \subseteq \omega \) be infinite and coinfinit in \( \{n_k : k < \omega\} \). Let \( M_n \) be the initial segment of \( M \) where \( |M_n| = n \), and let \( x_M \) be a cluster point of \( \{x_{M_n} : n < \omega\} \) where for all \( n, f_{n_k}(x_{M_n}) > r \) if \( k \in M_n \) and \( f_{n_k}(x_{M_n}) < s \) if \( k \in \{1, \ldots, \max(M_n)\} \setminus M_n \). Then \( f_{n_k}(x_M) \) converges to two different numbers, a contradiction.) \( \square \)

Haskell P. Rosenthal [16] used the above lemma for proving his famous \( \ell_1 \) theorem: a sequence in a Banach space is either ‘good’ (it has a subsequence which is weakly Cauchy) or ‘bad’ (it contains an isomorphic copy of \( \ell_1 \)). We will shortly discuss this topic.

We continue the discussion of some of the topics. The following lemma is a generalization of a model theoretic fact, i.e. IP implies OP.

**Lemma 2.6 (IP \( \Rightarrow \) OP)** Let \( X \) be compact and \( A \subseteq C(X) \) be precompact (in \( C(X) \)). Then every sequence in \( A \) has a convergent subsequence in \( \mathbb{R}^X \), and so in \( C(X) \).

**Proof.** Suppose, if possible, that the sequence \( \{f_n\} \subseteq A \) has no convergent subsequence. Thus, by Lemma 2.6 \( \{f_n\} \) is independent, i.e. there are \( r > s \) such that for all finite disjoint \( P, M \subseteq \mathbb{N} \), we have

\[
\left\{ x \in X : \left( \bigwedge_{i \in P} f_i(x) \leq s \right) \land \left( \bigwedge_{i \in M} f_i(x) \geq r \right) \right\} \neq \emptyset.
\]

Since \( X \) is compact, in the definition of independent sequence, one can assume that \( P \subseteq \mathbb{N} \) is infinite and \( M = \mathbb{N} \setminus P \). Now, a straightforward adaptation of a classical result in model theory, i.e. IP implies OP, shows that there are subsequences of \( \{f_n\} \) (still denoted by \( \{f_n\} \) and \( \{x_m\} \) in \( X \) such that \( \lim_{m} \lim_{n} f_n(x_m) \geq r > s \geq \lim_{n} \lim_{m} f_n(x_m) \). (Indeed, suppose that \( (x_P)_{P \subseteq \omega} \) witness IP. Given \( i < \omega \), let \( P_i : \omega \to 2 \) such that \( P_i(j) = 0 \) if and only if \( i < j \). Then we have \( f_i(x_P) \leq s \) iff \( P_j(i) = 1 \) iff \( i < j \), and \( f_i(x_P) \geq r \) iff \( P_j(i) = 0 \) iff \( i \geq j \). Take \( x_m = x_{p_m} \).) This is a contradiction by Fact 2.2. Moreover, since \( A \) is precompact, the limit of every convergent sequence is continuous. \( \square \)

Now, we are ready to give a proof of the Eberlain-Šmulian for the topology of pointwise convergence on \( C(X) \).

**Theorem 2.7 (Eberlain-Šmulian for \( C_p(X) \))** Suppose that \( C(X) \) is the Banach space of all continuous real-valued functions of on a compact space \( X \) and \( A \subseteq C(X) \) is norm-bounded. Then for the topology of pointwise convergence the following are equivalent.

(i) The set \( A \) is precompact.

(ii) The set \( A \) is sequentially precompact.

(iii) The set \( A \) is countably precompact.
Proof. (i) ⇒ (ii) is just Lemma 2.6

(iii) ⇒ (i): Suppose that \( A \subseteq C(X) \) is countably precompact and norm-bounded. Suppose that \( f_n \in A \) and \( x_n \in X \) form two sequences and the limits \( \lim_n \lim_m f_n(x_m) \) and \( \lim_m \lim_n f_n(x_m) \) exist. Let \( f \) in \( C(X) \) and \( x \) in \( X \) be cluster points of \( \{f_n\} \) and \( \{x_m\} \). Thus,
\[
\lim_n \lim_m f_n(x_m) = \lim_m f_n(x) = f(x) = \lim_m f(x_m) = \lim_m \lim_n f_n(x_m).
\]
By Fact 2.2, \( A \) is precompact. \( \square \)

This is the end of the story if one replaces the ‘topology of pointwise convergence’ with the ‘weak topology’. But, it needs some works:

2.2 Weak topology

Recall that for a normed space \( U \), the topology generated by \( U^* \) is known as the weak topology on \( U \). The next remark states that every normed space with its weak topology lives inside a space of the form \( C(X) \), with the topology of pointwise convergence, where \( X \) is a compact space.

Remark 2.8 For an arbitrary normed space \( U \), write \( X \) for the unit ball of the dual space \( U^* \), with its weak* topology. Then \( X \) is compact by Alaoglu’s theorem and the natural map \( u \mapsto \hat{u} : U \to \mathbb{R}^X \), defined by setting \( \hat{u}(x) = x(u) \) for \( x \in X \) and \( u \in U \), is a homeomorphism between \( U \), with its weak topology, and its image \( \hat{U} \) in \( C(X) \), with the topology of pointwise convergence.

If we show that the direction (i) ⇒ (ii) in Theorem 2.7 holds for any subspace \( Y \) of \( C(X) \), i.e. every precompact subset of subspace \( Y \) is sequentially precompact in \( Y \), then by Remark 2.8 for every Banach space (or even normed space), the weak precompactness implies weak sequential precompactness. Indeed, if \( A \subseteq Y \) be precompact in \( Y \), then \( A \) is countably precompact in \( Y \), and so is countably precompact in \( C(X) \). Therefore, if \( \{f_n\} \subseteq A \), by the direction (i) ⇒ (ii) for \( C(X) \), there exists \( \{g_n\} \subseteq \{f_n\} \) and \( f \in C(X) \) such that \( g_n \to f \). Since \( A \) is countably precompact in \( Y \), the cluster point of \( \{g_n\} \), i.e. \( f \), is in \( Y \). Thus the direction (i) ⇒ (ii) holds for any subspace \( Y \subseteq C(X) \).

In the next section we present a proof of the direction (iii) ⇒ (i) using the theorem of definability of types and a well known combinatorial result.

3 Definability of types

For convex subsets, one can show that the direction (iii) ⇒ (i) is a consequence of a well known fact in model theory, that the theorem of definability of types. This theorem says that for a formula \( \phi(x,y) \) stable in a model \( M \), and every type \( p \in S_{\phi}(M) \) there is a sequence \( \psi_n(y) \) of the convex combinations of \( \phi(a,y) \)’s, \( a \in M \), such that \( \psi_n(y) \) is uniformly convergent to a (continuous) function \( \psi(y) \) where \( \phi(x,b)^p = \psi(b) \) for all \( b \in M \).
The following proof is a straightforward translation of the proof of definability of types for continuous logic, as can be found in [3, Appendix B]:

**Fact 3.1 (Definability of types)** Let $X$ be a compact space and $A \subseteq C(X)$ norm-bounded and countably precompact. Then every point of the closure of $A$ is a uniform limit of a sequence in the convex hull of $A$, denoted by $\text{conv}(A)$.

**Proof.** Since $A$ is countably precompact in $C(X)$, it is stable in the sense of model theory, i.e. the condition (ii) in Fact 2.2 holds. With out lose of generality we can assume that all functions in $A$ are $[0,1]$-valued. Let $f \in A$. We claim that for any $\epsilon > 0$, there is a finite sequence $(f_i : i < n_\epsilon)$ in $A$ such that for all $x, y \in X$, if for all $i < n_\epsilon$, $|f_i(x) - f_i(y)| \leq \epsilon$ then $|f(x) - f(y)| \leq 3\epsilon$. If not, by induction on $n$ one can find $f_n \in A$, $x_n, y_n \in X$ as follows. At each step, there are by assumption $x_n, y_n \in X$ such that $|f_n(x_n) - f_n(y_n)| \leq \epsilon$ for all $i \leq n$, and yet $|f(x_n) - f(y_n)| > 3\epsilon$. Once these choices are made, since $f \in A$ we may therefore find $f_n \in A$ such that $|f_n(x_n) - f_n(y_n)| > 3\epsilon$ for all $i \leq n$. When the construction is complete, we have $\lim_n \lim_i |f_i(x_n) - f_i(y_n)| < \epsilon < 3\epsilon \leq \lim_n \lim_i |f_i(x_n) - f_i(y_n)|$, a contradiction. Now, we define increasing function $g_\epsilon : [0,1]^n \rightarrow [0,1]$ by $g_\epsilon(u) = \sup \{f(x) : x \in X \text{ and } f_i(x) \leq u_i \text{ for all } i < n_\epsilon \}$. Clearly we can find a continuous increasing function $h_\epsilon$ such that $g_\epsilon(\bar{u}) \leq h_\epsilon(\bar{u}) \leq g_\epsilon(\bar{u} + \epsilon)$. Thus, $|f(x) - h_\epsilon(f_i(x) : i < n_\epsilon)| \leq 3\epsilon$. As $\epsilon$ is arbitrary, for $\epsilon = \frac{1}{n}$, there is a continuous function $f_n(x) = h_\epsilon(f_i(x) : i < n_\epsilon)$ such that $|f(x) - f_n(x)| < 3\epsilon$ for all $x \in X$, and so $f_n \rightarrow f$ uniformly.

**The direction (iii) \Rightarrow (i) for convex sets.** Let $Y$ be any subset of $C(X)$. If a convex set $A \subseteq Y$ is countably precompact in $Y$, then $A$ is countably precompact in $C(X)$, so $\overline{A}$, the closure of $A$ in $C(X)$, is compact (see Theorem 2.7). Now if $x \in \overline{A}$, by the theorem of definability of types (Fact 3.1), there is a sequence $(x_n)$ in $A$ converging to $x$; but $(x_n)$ must have a cluster point in $Y$, and (because the topology is Hausdorff) this cluster point can only be $x$. Accordingly $\overline{A} \subseteq Y$ and is the closure of $A$ in $Y$. Thus $A$ is precompact in $Y$. Therefore, by Remark 2.8 above, a convex subset $A$ (of a normed space $Y$) is weakly precompact if it is weakly countably precompact.

### 3.1 Pták’s lemma and stability

By a combinatorial result due to Pták’s, one can show that the theorem of definability of types implies the direction (iii) \Rightarrow (i) of the main theorem. First we need some definitions.

A convex mean on $N$ is a function $\mu : N \rightarrow [0,1]$ such that (1) $\sum_{i=0}^{\infty} \mu(i) = 1$, (2) $\text{supp}(\mu) = \{i : \mu(i) > 0\}$ is finite. For $F \subseteq N$, let $\mu(F) = \sum_{i \in F} \mu(i)$. If $B \subseteq N$, then $M_B$ will denote the set of all convex means $\mu$ on $N$ such that $\text{supp}(\mu) \subseteq B$. Let $F$ be a collection of finite subsets of $N$. We denote $M_B(F, \epsilon) = \{\mu \in M_B : \forall F \in F \, \mu(F) < \epsilon\}$. Then

**Fact 3.2 (Pták’s lemma)** The two following are equivalent:

(i) There exists a strictly increasing sequence $A_1 \subset A_2 \subset \ldots$ of finite subsets of $A$, and a sequence $F_n \in F$ such that $F_n \subseteq A_n$ for all $n$. 
(ii) There exists an infinite subset $B$ of $A$ and an $\epsilon > 0$ such that $M_B(F, \epsilon) = \emptyset$.

Proof. See [15], page 327. \hfill \Box

Fact 3.3 If a bounded subset $A$ of $C(X)$ has interchangeable double limits property, then so does $\text{conv}(A)$.

Proof. Use Pták’s lemma. The proof is a straightforward translation of (4) in [15], page 328, for the topology of pointwise convergence. \hfill \Box

Corollary 3.4 (The direction (iii) $\Rightarrow$ (i)) If $A$ is countably weakly precompact, then $A$ is weakly precompact.

Proof. This is a consequence of the above fact, the theorem of definability of types and Remark 2.8. Indeed, let $Y$ be any convex subset of $C(X)$. If an arbitrary subset $A \subseteq Y$ is countably precompact in $Y$, then $A$ is countably precompact in $C(X)$. By Fact 3.3, $\text{conv}(A)$ is countably precompact in $C(X)$, so $\overline{\text{conv}(A)}$, the closure of $\text{conv}(A)$ in $C(X)$, is compact (see Theorem 2.7). Now by the theorem of definability of types (Fact 3.1 above), it is easy to verify that $\text{conv}(A) \subseteq Y$ and is the closure of $\text{conv}(A)$ in $Y$ (see the paragraph after Fact 3.1). Thus $\text{conv}(A)$ is precompact in $Y$, so $A$ is compact in $Y$. Therefore, by Remark 2.8 above, an arbitrary subset $A$ (of a normed space $Y$) is weakly precompact if it is weakly countably precompact. \hfill \Box

Fact 3.3 leads us to a new characterization of local stability inside a model. Assume that $M$ is a structure, $\phi(x,y)$ a formula. Let

$$\text{conv}(\phi) = \{ \psi(x,\bar{b}) : \psi(x,\bar{b}) \text{ is a convex combination of}$$

(at most finitely many) formulas $\phi(x,b), b \in M \} \}.

Corollary 3.5 Assume that $\phi(x,y)$ and $M$ are as above. Then the following are equivalent.

(i) The formula $\phi$ is stable in $M$.

(ii) If $a_n \in M$ and $\psi(x,\bar{b}_n) \in \text{conv}(\phi)$ form two sequences we have

$$\lim_n \lim_m \psi(a_m,\bar{b}_n) = \lim_m \lim_n \psi(a_m,\bar{b}_n),$$

whenever both limits exist.

By Mazur’s lemma (see below), the theorem of definability of types is a consequence of Theorem 2.7 above.

Remark 3.6 (Mazur Lemma) If $(f_n)$ is a bounded sequence of continuous functions on $X$ which converges to a continuous function $f$, there exists a sequence $g_n \in \text{conv}((f_k)_{k \geq n})$ which uniformly converges to $f$. (Here $\text{conv}((h_k))$ denotes the set of convex combinations of the $h_k$’s.) Therefore $f$ can be written as a uniform limit of continuous functions of the form $\frac{1}{n} \sum_{i<n} f_i$. Standard proofs of Mazur’s lemma use the Hahn-Banach theorem and Lebesgue’s Dominated Convergence Theorem. Pták’s lemma gives a different proof of Mazur’s result (see [19], page 14).
Thus, a formula $\phi(x,y)$ is stable in a model $M$ iff for every type $p \in S_\phi(M)$ there is a sequence $\phi(a_n,y)$, $a_n \in M$, (pointwise) converging to a continuous function $\psi(y)$ such that $\phi(x,b)^p = \psi(b)$ for all $b \in M$. (Indeed, note that $C(X)$ is a Fréchet-Urysohn space (see Fact [B.2]).)

4 Model theory and Banach space theory

In this short section we continue the discussion of some the topics raised above.

Recall that a Banach space $X$ is reflexive if a certain natural isometry of $X$ into $X^{**}$ is onto. This mapping is $\hat{\cdot}: X \to X^{**}$ given by $\hat{x}(x^*) = x^*(x)$.

Now, we analyze the weak sequential compactness. Obviously, a Banach space $X$ is weakly sequentially compact if the following conditions hold:

(a) every bounded sequence $(x_n)$ of $X$ has a weak Cauchy subsequence, (i.e. there is $(y_n) \subseteq (x_n)$ such that for all $x^* \in X^*$ the sequence $(x^*(y_n))_{n=1}^\infty$ is a convergent sequence of reals, so $y_n \to x^*$ weak* in $X^{**}$ for some $x^* \in X^*$), and

(b) every weak Cauchy sequence $(x_n)$ of $X$ has a weak limit (i.e. if $\hat{x}_n \to x^*$ weak* in $X^{**}$ then $x^* \in \hat{X}$).

It is easy to check that the condition (a) corresponds to NIP and the condition (b) corresponds to NSOP in model theory (see below). In functional analysis, the condition (a) is called the weak* sequential compactness of $\hat{X}$ (short W*S-compactness), and the condition (b) is called the weak sequential completeness (short WS-completeness). Clearly, a weakly sequentially compact set is weakly* sequentially compact, but the converse fails. Indeed, the sequence $y_n = (1, \ldots, 1, 0, \ldots)$ form a weakly Cauchy sequence in $c_0$ without weak limit.

On the other hand, the Fréchet-Urysohn property of the space $C(X)$ corresponds to the definability of types in stable theories: Let $\phi$ be a formula, $M$ a model of a stable theory in continuous logic, and $S_\phi(M)$ the space of all complete $\phi$-types on $M$ (see [2]). A type $p$ in $S_\phi(M)$ is a point in the closure of realized types in $M$, thus if $a_\alpha \in M$ and $tp(a_\alpha/M) \to p$, then there exists a continuous function $\psi$ such that $\phi(a_\alpha,y) \to \psi(y)$ pointwise ($\psi$ is continuous because the theory is stable, see [14]). Now, by the Fréchet-Urysohn property of $C(S_\phi(M))$, there is a sequence $(a_n) \subseteq M$ such that $\lim_n \phi(a_n,y) = \psi(y)$, i.e. $p$ is definable by $\psi$.

A standard fact in functional analysis is that a Banach space $X$ is reflexive if and only if $B_X = \{x \in X : \|x\| \leq 1\}$ is weakly compact. Thus by the Eberlain-Šmulian theorem, $X$ is reflexive if and only if the conditions (a) and (b) above hold for $A = B_X$. This and the above observations show that stability in model theory corresponds to reflexivity in functional analysis. Thus, one can say that ‘first order logic is angelic.’ To summarize:
Logic | Analysis
--- | ---
Definability of types | Fréchet-Urysohn property
NIP | W*S-compactness
NSOP | WS-completeness
Shelah’s theorem | Eberlein-Smulian theorem

In the next subsection we study these connections more exactly.

### 4.1 Type space

We assume that the reader is familiar with continuous logic (see [2]). Of course, we study real-valued formulas instead of $[0,1]$-valued formulas. One can assign bounds to formulas and retain compactness theorem in a local way again.

Suppose that $L$ is an arbitrary language. Let $M$ be an $L$-structure, $A \subseteq M$ and $T_A = Th(M,a)_{a \in A}$. Let $p(x)$ be a set of $L(A)$-statements in free variable $x$. We shall say that $p(x)$ is a type over $A$ if $p(x) \cup T_A$ is satisfiable. A complete type over $A$ is a maximal type over $A$. The collection of all such types over $A$ is denoted by $S^M(A)$, or simply by $S(A)$ if the context makes the theory $T_A$ clear. The type of $a$ in $M$ over $A$, denoted by $tp^M(a/A)$, is the set of all $L(A)$-statements satisfied in $M$ by $a$. If $\phi(x,y)$ is a formula, a $\phi$-type over $A$ is a maximal consistent set of formulas of the form $\phi(x,a) \geq r$, for $a \in A$ and $r \in \mathbb{R}$. The set of $\phi$-types over $A$ is denoted by $S^\phi(A)$.

We now give a characterization of complete types in terms of functional analysis. Let $\mathcal{L}_A$ be the family of all interpretations $\phi^M$ in $M$ where $\phi$ is an $L(A)$-formula with a free variable $x$. Then $\mathcal{L}_A$ is an Archimedean Riesz space of measurable functions on $M$ (see [5]). Let $\sigma_A(M)$ be the set of Riesz homomorphisms $I : \mathcal{L}_A \rightarrow \mathbb{R}$ such that $I(1) = 1$. The set $\sigma_A(M)$ is called the spectrum of $T_A$. Note that $\sigma_A(M)$ is a weak* compact subset of $\mathcal{L}_A^*$. The next proposition shows that a complete type can be coded by a Riesz homomorphism and gives a characterization of complete types. In fact, by Kakutani representation theorem, the map $S^M(A) \rightarrow \sigma_A(M)$, defined by $p \mapsto I_p$ where $I_p(\phi^M) = r$ if $\phi(x) = r$ is in $p$, is a bijection.

**Proposition 4.1** Suppose that $M$, $A$ and $T_A$ are as above.

1. The map $S^M(A) \rightarrow \sigma_A(M)$ defined by $p \mapsto I_p$ is bijective.

2. $p \in S^M(A)$ if and only if there is an elementary extension $N$ of $M$ and $a \in N$ such that $p = tp^N(a/A)$.

We equip $S^M(A) = \sigma_A(M)$ with the related topology induced from $\mathcal{L}_A^*$. Therefore, $S^M(A)$ is a compact and Hausdorff space. For any complete type $p$ and formula $\phi$, we let $\phi(p) = I_p(\phi^M)$. It is easy to verify that the topology on $S^M(A)$ is the weakest topology in which all the functions $p \mapsto \phi(p)$ are continuous. This topology sometimes called the logic topology. The same things are true for $S^\phi(A)$.
It is easy to check that for each \( I \in \sigma_M(M), \|I\| = 1 \) and \( I \) is a positive and multiplicative, i.e. \( I(f \times g) = I(f) \times I(g) \) for all \( f, g \in L_M \). (Recall that \( I \) is positive if \( I(f) \geq 0 \) for all \( f \geq 0 \).) Also, \( \sigma_M(M) \) is the set of extreme points of the state space \( S = \{ I \in L_M^* : I(1) = 1 \) and \( I \) is positive \}. \( S \) also called the space of Keisler types. (By the Krein-Milman theorem, \( S = \overline{\text{conv}}^* (\sigma_M(M)) \), i.e. it is the (weak) closure of convex hull of its extreme points.) Then \( \sigma_M(M) \subset S \subset S^1 = \{ I \in L_M^* : \|I\| = 1 \} \subset B_{L_M^*} = \{ I \in L_M^* : \|I\| \leq 1 \} \).

Since every member of \( L_M^* \) is expressible as the difference of two positive linear functionals, \( \sigma_M(M) \) determines the set \( B_{L_M^*} \) (and hence the Banach space \( L_M^* \)). Thus, space of types can be equipped with a natural norm space structure, and we can study this Banach space (i.e. \( L_M^* \)) instead of the types space. The weak* topology of this Banach space on the space of types is the logic topology, and we have a natural linear structure on the space of types, i.e. for all types \( p, q \), the addition \( p + q \) is well defined, also \( rp \) is well defined for each real number \( r \) (Indeed, \( p + q := I_p + I_q \) and \( rp := rI_p \)). Of course, \( p + q \) is not necessary a (classical) type, but it is easier to study the Banach space determined by types.

### 4.2 Banach space for a formula

Let \( M \) be an \( L \)-structure, \( \phi(x, y) : M \times M \to \mathbb{R} \) a formula (we identify formulas with real-valued functions defined on models).

Let \( S_\phi(M) \) be the space of complete \( \phi \)-types over \( M \) and set \( A = \{ \phi(x, a), -\phi(x, a) \in C(S_\phi(M)) : a \in M \} \). The (closed) convex hull of \( A \), denoted by \( \overline{\text{conv}}(A) \), is the intersection of all (closed) convex sets that contain \( A \). \( \overline{\text{conv}}(A) \) is convex and closed, and \( \|f\| \leq \|\phi\| \) for all \( f \in \overline{\text{conv}}(A) \). We claim that \( B = \overline{\text{conv}}(A) \) is the unit ball of a Banach space. Set \( V = \bigcup_{\lambda > 0} \lambda B \). It is easy to verify that \( V \) is a Banach space with the normalized norm and \( B \) is its unit ball. This space will be called the space of linear \( \phi \)-definable relations. One can give an explicit description of it:

\[
V = \left( \left\{ \sum_{i=1}^{n} r_i \phi(x, a_i) : a_i \in M, r_i \in \mathbb{R}, n \in \mathbb{N} \right\} ; \|\| \cdot \|\| \right).
\]

where \( \|\| \cdot \|\| \) is the normalized norm.

Note that \( V \) is a subspace of \( C(S_\phi(M)) \). Recall that for an infinite compact Hausdorff \( X \), the space \( C(X) \) is not reflexive, nor is it weakly complete. So, if \( V \) is a lattice (or algebra), then it is not reflexive, nor is it weakly complete (since, in this case, \( V \) is isomorphic to \( C(X) \) for some compact Hausdorff space \( X \)).

### 4.3 Stability and reflexivity

A formula \( \phi : M \times M \to \mathbb{R} \) has the order property if there exist sequences \( (a_m) \) and \( (b_n) \) in \( M \) such that

\[
\lim_{m} \lim_{n} \phi(a_m, b_n) \neq \lim_{n} \lim_{m} \phi(a_m, b_n)
\]

11
We say that $\phi$ has the double limit property (DLP) if it has not has the order property.

**Definition 4.2** We say that $\phi(x, y)$ is unstable if either $\phi$ or $-\phi$ has the order property. We call $\phi$ stable if $\phi$ is not unstable.

The following is a well known result in functional analysis:

**Fact 4.3** A Banach space $X$ is reflexive iff its unite ball is weakly compact.

If $\phi$ is stable then the set $A$ (see above) is weakly precompact in $C(S_\phi(M))$ by Grothendiek’s criterion. (Note that the collection of types realized in $M$ is dense in $S_\phi(M)$.) By Pták’s lemma, the convex hull of $A$, $\text{conv}(A)$, is weakly precompact, and therefore the closed convex hull of $A$, i.e. $B$, is so. (Note that for convex sets, weakly closed = uniform closed.)

**Corollary 4.4** Assume that $\phi(x, y)$, $M$, $B$ and $V$ are as above. Then the following are equivalent:

(i) $\phi$ is stable on $M$.

(ii) $B$ is weakly compact.

(iii) The Banach space $V$ is reflexive.

Recall that for an infinite compact Hausdorff space $X$, the space $C(X)$ is not reflexive.

### 4.4 NIP and Rosenthal Banach spaces

We say that a formula $\phi(x, y)$ is NIP on a model $M$ if every sequence of the set $A = \{\phi(x, a), -\phi(x, a) \in C(S_\phi(M)) : a \in M\}$ is not independent in the sense of Definition 2.4

**Definition 4.5** ([8], 2.10) A Banach space $X$ is said to be **Rosenthal** if it does not contain an isomorphic copy of $\ell_1$.

**Fact 4.6** (H.P. Rosenthal) For a Banach space $X$ the following either $X$ is Rosenthal or it has a sequence with no weak Cauchy subsequence.

If $\phi(x, y)$ is NIP on $M$ then the set $A$ is not an independent family, and $\text{conv}(A)$ is nor (see [4], page 878). (Note that the collection $M_0$ of types realized in $M$ is dense in $S_\phi(M)$ and $A$ is not independent iff the family $\{f|_{M_0} : f \in A\}$ is not independent (see [8], Lemma 7.9.) So, its (uniform) closed convex hull, $\text{conv}(A)$, is not independent. Now, the above fact and Lemma 2.5 imply that:

**Corollary 4.7** Assume that $\phi(x, y)$, $M$, and $V$ are as above. Then the following are equivalent:

(i) $\phi$ is NIP on $M$.

(ii) $V$ is Rosenthal Banach space.

(iii) Every bounded sequence of $V$ has a weak Cauchy subsequence.
4.5 NSOP and weak sequential completeness

Let $M(=U)$ be a monster model (of theory $T$) and $\phi(x,y)$ a formula.

**Corollary 4.8** Assume that $\phi(x,y)$, $M$, and $V$ are as above. Then the following are equivalent:

(i) $\phi$ is NSOP (on $M$).

(ii) Every weak Cauchy sequence of $V$ has a weak limit (in $V$).

**Proof.** Use the above corollaries and the Eberlein-Šmulian Theorem. \(\square\)

Note that for a compact Hausdorff space $X$, the space $C(X)$ contains an isomorphic copy of $c_0$, and so $C(X)$ is not weakly sequentially complete (see [1], Proposition 4.3.11).

In [10] Jose Iovino pointed out the correspondence between stability and reflexivity. He showed that a formula $\phi(x,y)$ is stable iff $\phi$ is the pairing map on the unite ball of $E \times E^*$, where $E$ is a reflexive Banach space. In this paper, we gave a ‘concrete and explicit’ description of the Banach space $V$, such that it is reflexive iff $\phi$ is stable. This space is uniquely determined by $\phi$ and the formula $\phi$ is completely coded by $V$. The value of $\phi$ is exactly determined by the evaluation map $\langle \cdot, \cdot \rangle \colon V \times V^* \to \mathbb{R}$ defined by $\langle f, I \rangle = I(f)$.

At the end of this paper - of not the story - we note that the basic ideas came from model theory, in the other word, techniques from one field became useful in the other. One might therefore hope to obtain other connections between stability theory and Banach space theory. We will continue this way in a future work.

### A Stability and Eberlein compacta

**Proof of Fact 2.2.** (i) $\Rightarrow$ (ii): Suppose that $f_n \in A$ and $x_n \in X$ form two sequences and the limits $\lim_n \lim_m f_n(x_m)$ and $\lim_m \lim_n f_n(x_m)$ exist. Let $f \in C(X)$ and $x \in X$ be cluster points of $\{f_n\}$ and $\{x_m\}$. Thus,

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x) = f(x) = \lim_m f(x_m) = \lim_n \lim_m f_n(x_m).$$

(ii) $\Rightarrow$ (i): Since $A$ is bounded (i.e. there is an $r$ such that $|f| \leq r$ for all $f \in A$) and by Tychonoff’s theorem $[-r,r]^X$ is compact, it suffices to show that $\overline{A} \subseteq C(X)$. Let $f \in \overline{A}$. Suppose that $f$ is not continuous at a point $x$ in $X$. Then there is a neighborhood $U$ of $f(x)$ such that each neighborhood of $x$ contains a point $y$ of $X$ with $f(y)$ not belonging to $U$. Take any $f_1$ in $A$; then there is an $x_1$ in $X$ such that $|f_1(x) - f_1(x_1)| < 1$ and $f_1(x_1) \notin U$. Take $f_2$ in $A$ so that $|f_2(x_1) - f_1(x_1)| < 1$ and $f_2(x_1) - f(x_1)| < 1$. Now choose $x_2$ in $X$ such that $|f_i(x) - f_i(x_2)| < 1/2$ ($i = 1, 2$) and $f(x_2) \notin U$. Then take $f_3$ in $A$ so that $|f(x_j) - f_3(x)| < 1/2$ and $|f(x) - f_3(x)| < 1/2$. Proceeding in this way, one obtains sequences $\{f_n\}_n$ and $\{x_m\}_m$ in $A$ and $X$ such that, for each $n$, $|f_i(x) - f_i(x_n)| < 1/n$.
(i = 1, 2, . . . , n), |f(x_j) − f_{n+1}(x_j)| < 1/n (j = 1, 2, . . . , n), |f(x) − f_{n+1}(x)| < 1/n, and f(x_n) \notin U. Then \lim_n \lim_m f_n(x_m) = \lim_n f_n(x) = f(x), and \lim_n f_n(x_m) = f(x_m) \notin U.

Since it is possible to take a subsequence of \{x_m\}_m so that the corresponding subsequence of \{f(x_m)\}_m converges to a point outside of U, the assumption that f is not continuous contradicts the iterated limit condition of (ii).

\[ \square \]

B   Angelicity of C(X)

Definition B.1 (Fremlin) A regular Hausdorff space space is angelic if (i) every countably precompact set is precompact, (ii) the closure of a precompact set is precisely the set of limits of its sequences.

For angelicity of C(X) (where X is a compact Hausdorff space) it suffices to so prove that:

Fact B.2 ([6], 462B) Assume that X is compact and A ⊆ C(X) is bounded and precompact. If g ∈ A, there is a sequence f_n ∈ A such that \lim_n f_n = g.

Proof. For g ∈ A we construct countable sets D ⊆ X, B ⊆ A such that

1. whenever I ⊆ B ∪ \{g\} is finite, ε > 0 and x ∈ X, there is a y ∈ D such that |f(y) − f(x)| ≤ ε for every f ∈ I;
2. whenever J ⊆ D is finite and ε > 0 there is an f ∈ B such that |f(x) − g(x)| ≤ ε for every x ∈ J.

For any finite set I ⊆ \mathbb{R}^X, the set Q_I = \{\{f(x)\}_{f \in I} : x ∈ X\} is a subset of the separable metrizable space \mathbb{R}^I, so is itself separable, and there is a countable dense set D_I ⊆ X such that Q'_I = \{\{f(x)\}_{f \in I} : x ∈ D_I\} is dense in Q_I. Similarly, because g ∈ A, we can choose for any finite set J ⊆ X a sequence \{f_{JI}\}_I in A such that lim_I f_{JI}(x) = g(x) for every x ∈ J.

Now construct \{D_m\}_m, \{B_m\}_m inductively by setting D_m = \bigcup\{D_I : I ⊆ \{g\} \cup \bigcup_{i < m} B_i is finite\}, B_m = \{f_{Jk} : k ∈ \mathbb{N}, J ⊆ \bigcup_{i < m} D_i is finite\}. At the end of the induction, set D = \bigcup_{m \in \mathbb{N}} D_m and B = \bigcup_{m \in \mathbb{N}} B_m. Since the construction clearly ensures that \{D_m\}_m and \{B_m\}_m are non-decreasing sequences of countable sets, D and B are countable, and we shall have D_I ⊆ D and I ⊆ D ∪ \{g\} is finite, while f_{JI} ∈ B whenever i ∈ \mathbb{N} and J ⊆ D is finite. Thus we have suitable sets D and B.

By (2), there must be a sequence \{f_i\}_I in B such that g(x) = lim_I f_i(x) for every x ∈ D. In fact g(y) = lim_I f_i(y) for every y ∈ X. Otherwise, there is an ε > 0 such that J = \{i : |g(y) − f_i(y)| ≥ ε\} is infinite. For each m ∈ \mathbb{N}, I_m = \{f_i : i ≤ m\} is a finite subset of B, so by (1) there is an x_m ∈ D_{I_m} such that |f(x_m) − f(y)| ≤ 2^{-m} for every f ∈ I_m ∪ \{g\}. Then

\[ \lim_i \lim_m f_i(x_m) = \lim_i f_i(y) \neq g(y) = \lim_m g(x_m) = \lim_m \lim_i f_i(x_m). \]

But this is impossible, by Fact 2.2. So g = lim_i f_i.

\[ \square \]
Now, since any subspace of an angelic space is angelic (see [6], 462C(a)), by Remark 2.8 above, every Banach space (or even normed space) is angelic. This leads to another proof of the direction (iii) $\Rightarrow$ (i) of the Eberlain-Šmulian (see Corollary 3.4).

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