HYERS-ULAM STABILITY OF JENSEN FUNCTIONAL EQUATION ON AMENABLE SEMIGROUPS

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Abstract. In this paper, we give a proof of the Hyers-Ulam stability of the Jensen functional equation

\[ f(xy) + f(x\sigma(y)) = 2f(x), \quad x, y \in G, \]

where \( G \) is an amenable semigroup and \( \sigma \) is an involution of \( G \).

1. Introduction

In 1940, Ulam [17] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: Given a group \( G_1 \), a metric group \((G_2, d)\), a number \( \epsilon > 0 \) and a mapping \( f : G_1 \rightarrow G_2 \) which satisfies \( d(f(xy), f(x)f(y)) < \epsilon \) for all \( x, y \in G_1 \), does there exist an homomorphism \( g : G_1 \rightarrow G_2 \) and a constant \( k > 0 \), depending only on \( G_1 \) and \( G_2 \) such that \( d(f(x), g(x)) < k\epsilon \) for all \( x \in G_1 \)?

In 1941, Hyers [10] considered the case of approximately additive mappings under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces. Rassias [16] provided a generalization of the Hyers’ Theorem for linear mappings, by allowing the Cauchy difference to be unbounded.

Beginning around the year 1980, several results for the Hyers-Ulam-Rassias stability of very many functional equations have been proved by several researchers. For more detailed, we can refer for example to [5, 7, 8, 11, 12, 15, 19].

Let \( G \) be a semigroup with neutral element \( e \). Let \( \sigma \) be an involution of \( G \), which means that \( \sigma(xy) = \sigma(y)\sigma(x) \) and \( \sigma(\sigma(x)) = x \) for all \( x, y \in G \).

We say that \( f : G \rightarrow \mathbb{C} \) satisfies the Jensen functional equation if

\[ f(xy) + f(x\sigma(y)) = 2f(x) \tag{1.1} \]

for all \( x, y \in G \).

The Jensen functional equation (1.1) takes the form

\[ f(xy) + f(xy^{-1}) = 2f(x) \tag{1.2} \]
for all \(x, y \in G\), when \(\sigma(x) = x^{-1}\) and \(G\) is a group.

The stability in the sense of Hyers-Ulam of the Jensen equations (1.1) and (1.2) has been studied by various authors when \(G\) is an abelian group or a vector space. The interested reader can be referred to Jung [13], Kim [14] and Bouikhalene et al. [1, 2].

Recently, Faiziev and Sahoo [4] have proved the Hyers-Ulam stability of equation (1.2) on some noncommutative groups such as metabelian groups and \(T(2, K)\), where \(K\) is an arbitrary commutative field with characteristic different from two.

In this paper, motivated by the ideas of Forti, Sikorska [6] and Yang [18], we give a proof of the Hyers-Ulam stability of the Jensen functional equation (1.1), under the condition that \(G\) is an amenable semigroup.

2. HYERS-ULAM STABILITY OF EQUATION (1.1) ON AMENABLE SEMIGROUPS

In this section we investigate the Hyers-Ulam stability of equation (1.1) on an amenable semigroup \(G\).

We recall that a semigroup \(G\) is said to be amenable if there exists an invariant mean on the space of the bounded complex functions defined on \(G\). We refer to [9] for the definition and properties of invariant means.

The main result of the present paper is the following.

**Theorem 2.1.** Let \(G\) be an amenable semigroup with neutral element \(e\). Let \(f : G \to \mathbb{C}\) be a function satisfying the following inequality

\[
|f(xy) + f(x\sigma(y)) - 2f(x)| \leq \delta
\]

for all \(x, y \in G\) and for some nonnegative \(\delta\). Then there exists a unique solution \(g\) of the Jensen equation (1.1) such that

\[
|f(x) - g(x) - f(e)| \leq 3\delta
\]

for all \(x \in G\).

First, we recall the following lemma which is a generalization of the useful lemma obtained by Forti and Sikorska in [6]. The proof was given by the authors in [3].

**Lemma 2.2.** Let \(G\) be a semigroup and \(B\) be a Banach space. Let \(f : G \to B\) be a function for which there exists a solution \(g\) of the Drygas functional equation

\[
g(yx) + g(\sigma(y)x) = 2g(x) + g(y) + g(\sigma(y)), \quad x, y \in G
\]
such that \( \| f(x) - g(x) \| \leq M \), for all \( x \in G \) and for some \( M \geq 0 \).

Then

\[
g(x) = \lim_{n \to +\infty} 2^{-2n} \left\{ f^e(x^{2^n}) + \frac{1}{2} \sum_{k=1}^{n} 2^{k-1} \left[ f^e((x^{2^{n-k}} \sigma(x)^{2^{k-1}})^{2^{n-k}}) + f^e((x^{2^{n-k}} x^{2^{n-k}})^{2^{k-1}}) \right] \right\}
+ 2^{-n} \left\{ f^o(x^{2^n}) + \frac{1}{2} \sum_{k=1}^{n} \left[ f^e((x^{2^{k-1}} \sigma(x)^{2^{k-1}})^{2^{n-k}}) - f^e((x^{2^{k-1}} x^{2^{k-1}})^{2^{n-k}}) \right] \right\},
\]

where \( f^e(x) = \frac{f(x) + f(\sigma(x))}{2} \), \( f^o(x) = \frac{f(x) - f(\sigma(x))}{2} \) are the even and odd part of \( f \).

**Lemma 2.3.** Let \( G \) be semigroup with neutral element \( e \) and \( B \) a complex Banach space. Assume that \( f: G \to B \) satisfies the inequality

\[
| f(xy) + f(x\sigma(y)) - 2f(x) | \leq \delta
\]

for all \( x, y \in G \) and for some \( \delta \geq 0 \). Then the limit

\[
g(x) = \lim_{n \to \infty} 2^{-n} f(x^{2^n})
\]

exists for all \( x \in G \) and satisfies

\[
| f(x) - g(x) - f(e) | \leq \frac{3\delta}{2} \quad \text{and} \quad g(x^2) = 2g(x)
\]

for all \( x \in G \).

The function \( g \) with the condition \((2.7)\) is unique.

**Proof.** We define a function \( h: G \to \mathbb{C} \) by \( h(x) = f(x) - f(e) \). First, by using the inequality \((2.5)\) we obtain

\[
| h(xy) + h(x\sigma(y)) - 2h(x) | \leq \delta
\]

for all \( x, y \in G \). Letting \( x = e \) in \((2.8)\), we get

\[
| h^e(y) | \leq \frac{\delta}{2}
\]

for all \( y \in G \). Similarly, we can put \( y = x \) in \((2.8)\) to obtain

\[
| h(x^2) + h(x\sigma(x)) - 2h(x) | \leq \delta
\]

for all \( x \in G \). Since \( h(x\sigma(x)) = h^e(x\sigma(x)) \), so from \((2.9)\) and \((2.10)\), we have

\[
| h(x^2) - 2h(x) | \leq \frac{3\delta}{2}
\]

for all \( x \in G \). Now, by applying some approach used in [4], we get the rest of the proof. \( \square \)
Proof of Theorem 2.1. In the proof we use some ideas from Yang [18] and Forti, Sikorska [6].

Setting $x = e$ in (2.1) we have

$$|f^e(y) - f(e)| \leq \frac{\delta}{2}$$

for all $y \in G$.

The inequalities (2.1), (2.12) and the triangle inequality gives

$$|f(xy) + f(yx) - 2f(x) - 2f(y) + 2f(e)|$$

$$\leq |f(xy) + f(x\sigma(y)) - 2f(x)| + |f(yx) + f(y\sigma(x)) - 2f(y)|$$

$$+ |2f(e) - f(x\sigma(y)) - f(y\sigma(x))|$$

$$\leq 3\delta.$$

Hence, from (2.1), (2.12) and (2.13) we get

$$|f(xy) + f(\sigma(y)x) - 2f(x)| \leq |f(xy) + f(xy) - 2f(y) - 2f(x) + 2f(e)|$$

$$+ |f(\sigma(y)x) + f(x\sigma(y)) - 2f(\sigma(y)) - 2f(x) + 2f(e)|$$

$$+ |f(xy) - f(x\sigma(y)) + 2f(x)| + |f(yx) + f(\sigma(y)) - 4f(e)| \leq 9\delta$$

Now, from (2.1) and (2.14) we obtain

$$|f(xy) - f(\sigma(x)\sigma(y)) + f(y\sigma(x)) - f(x\sigma(y)) - 2f(y) - f(\sigma(y))|$$

$$\leq |f(xy) + f(y\sigma(x)) - 2f(y)| + |f(x\sigma(y)) + f(\sigma(x)\sigma(y)) - 2f(\sigma(y))|$$

$$\leq 10\delta.$$ 

Consequently, we get

$$|f^0(xy) + f^0(y\sigma(x)) - 2f^0(y)| \leq 5\delta$$

for all $x, y \in G$. So, for fixed $y \in G$, the functions $x \mapsto f^0(xy) - f^0(x\sigma(y))$ and $x \mapsto f^0(xy) + f^0(x\sigma(y)) - 2f^0(x)$ are bounded on $G$. Furthermore,

$$m\{f^0_{\sigma(y)\sigma(z)} + f^0_{\sigma(y)z} - 2f^0_{\sigma(y)}\} = m\{(f^0_{\sigma(z)} + f^0_z - 2f^0)_{\sigma(y)}\}$$

$$= m\{f^0_{\sigma(z)} + f^0_z - 2f^0\},$$

where $m$ is an invariant mean on $G$.

By using some computation as the one of (2.14) we get, for every fixed $y \in G$ the function $x \mapsto f^0(xy) + f^0(\sigma(y)x) - 2f^0(x)$ is bounded and

$$m\{z_y f^0 + (\sigma(z)y) f^0 - 2_y f^0\} = m\{y_z f^0 + (\sigma(z) f^0 - 2 f^0)\}$$

$$= m\{z f^0 + (\sigma(z) f^0 - 2 f^0\}$$

Now, define

$$\phi(y) := m\{y f^0 - f^0_{\sigma(y)}\}, \quad y \in G.$$
By using the definition of $\phi$ and $m$ the equalities (2.17) and (2.18), we get

\begin{equation}
\phi(z) + \phi(\sigma(z)y) = m\{z_y f^o - f^o_{\sigma(y)\sigma(z)}\} + m\{\sigma(z)_y f^o - f^o_{\sigma(y)z}\}
\end{equation}

\begin{equation}
= m\{z_y f^o + \sigma(z)_y f^o - 2y f^o\} - m\{f^o_{\sigma(y)\sigma(z)} + f^o_{\sigma(y)z} - 2f^o_{\sigma(y)}\}
\end{equation}

\begin{equation}
+ 2m\{y f^o - f^o_{\sigma(y)}\}
\end{equation}

\begin{equation}
= m\{z f^o + \sigma(z) f^o - 2f^o\} - m\{f^o_{\sigma(z)} + f^o_z - 2f^o\}
\end{equation}

\begin{equation}
+ 2m\{y f^o - f^o_{\sigma(y)}\}
\end{equation}

\begin{equation}
= m\{z f^o - f^o_{\sigma(z)}\} + m\{\sigma(z) f^o - f^o_z\} + 2m\{y f^o - f^o_{\sigma(y)}\}
\end{equation}

which implies that $\phi$ is a solution of the Drygas functional equation (2.3). Furthermore, we have

\begin{equation}
\frac{\phi}{2}(y) - f^o(y) = \frac{1}{2} |\phi(y) - 2f^o(y) - 2f^o(y)\} |
\end{equation}

\begin{equation}
\leq \frac{1}{2} m \sup_{x \in G} |f^o(yx) - f^o(y \sigma(x)) - 2f^o(y)| = \frac{1}{2} \sup_{x \in G} |f^o(yx) + f^o(y \sigma(x)) - 2f^o(y)|
\end{equation}

\begin{equation}
\leq \frac{5}{2} \delta.
\end{equation}

Now, by using Lemma 2.2: the mapping $\frac{\phi}{2}$ is a solution of Drygas functional equation (2.3) and $\frac{\phi}{2} - f^o$ is a bounded mapping, so we have

\begin{equation}
\frac{\phi}{2} = \lim_{n \to +\infty} 2^{-n} f^o(x^{2^n}),
\end{equation}

which implies that $\frac{\phi}{2}$ is odd, so $\frac{\phi}{2}$ is a solution of Jensen functional equation (1.1). Consequently, we have

\begin{equation}
|f(x) - \frac{\phi}{2} - f(e)| = |f^e(x) + f^o(x) - \frac{\phi}{2} - f(e)|
\end{equation}

\begin{equation}
\leq |f^e(x) - f(e)| + |f^o(x) - \frac{\phi}{2}|
\end{equation}

\begin{equation}
\leq \delta + \frac{5\delta}{2} = \frac{\delta}{2} = 3\delta.
\end{equation}

For proving the uniqueness of the obtained solution, we use the following: if $g$ is a solution of equation (1.1), then

\begin{equation}
g(e) = g^e(x) = g(x \sigma(x)); \quad g(x^{2^n}) + (2^n - 1)g(e) = 2^n g(x)
\end{equation}

for all $n \in \mathbb{N}$ and for all $x \in G$.

By using (2.24) and the proof of [Proposition 3, [13]], we get the following result.
Corollary 2.4. Let $G$ be an amenable semigroup with neutral element $e$ and $B$ a Banach space. Let $f: G \rightarrow B$ be a function satisfying the following inequality
\[(2.25) \quad \| f(xy) + f(x\sigma(y)) - 2f(x) \| \leq \delta \]
for all $x, y \in G$ and for some nonnegative $\delta$. Then there exists a unique solution $g$ of the Jensen equation (1.1) such that
\[(2.26) \quad \| f(x) - g(x) - f(e) \| \leq 3\delta \]
for all $x \in G$.

Corollary 2.5. Let $G$ be an amenable semigroup with neutral element $e$ and $B$ a Banach space. Let $f: G \rightarrow B$ be a function satisfying the following inequality
\[(2.27) \quad \| f(xy) + f(xy^{-1}) - 2f(x) \| \leq \delta \]
for all $x, y \in G$ and for some nonnegative $\delta$. Then there exists a unique solution $g$ of the Jensen equation (1.2) such that
\[(2.28) \quad \| f(x) - g(x) - f(e) \| \leq 3\delta \]
for all $x \in G$.

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