A two-sided estimate for the Gaussian noise stability deficit

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Abstract

The Gaussian noise-stability of a set $A \subset \mathbb{R}^n$ is defined by

$$S_\rho(A) = \mathbb{P}(X \in A \& Y \in A)$$

where $X, Y$ are standard jointly Gaussian vectors satisfying $\mathbb{E}[X_i Y_j] = \delta_{ij} \rho$. Borell’s inequality states that for all $0 < \rho < 1$, among all sets $A \subset \mathbb{R}^n$ with a given Gaussian measure, the quantity $S_\rho(A)$ is maximized when $A$ is a half-space.

We give a novel short proof of this fact, based on stochastic calculus. Moreover, we prove an almost tight, two-sided, dimension-free robustness estimate for this inequality: by introducing a new metric to measure the distance between the set $A$ and its corresponding half-space $H$ (namely the distance between the two centroids), we show that the deficit $S_\rho(H) - S_\rho(A)$ can be controlled from both below and above by essentially the same function of the distance, up to logarithmic factors.

As a consequence, we also establish the conjectured exponent in the robustness estimate proven by Mossel-Neeman, which uses the total-variation distance as a metric. In the limit $\rho \to 1$, we obtain an improved dimension-free robustness bound for the Gaussian isoperimetric inequality. Our estimates are also valid for a more general version of stability where more than two correlated vectors are considered.

1 Introduction

The topic of this paper is the isoperimetric inequality in Gauss space and an extension of this inequality, referred to as the Gaussian noise stability inequality. The Gauss space is the Euclidean space $\mathbb{R}^n$ equipped with the standard Gaussian measure $\gamma^n$ defined by

$$\gamma^n(A) = \frac{1}{(2\pi)^{n/2}} \int_A e^{-|x|^2/2} dx.$$ 

In the following, we will often abbreviate $\gamma = \gamma^n$. The Isoperimetric inequality, initially proved independently by Sudakov-Tsirelson ([ST]) and Borell ([Bor1]) states that among all subsets of $\mathbb{R}^n$ of a given Gaussian measure, the sets minimizing the Gaussian surface area (defined as the integral of the Gaussian density with respect to the $(n - 1)$-dimensional Hausdorff measure on the boundary of the set) are half-spaces. More recent proofs based on probabilistic, analytic, geometric and discrete methods can be found in [BL, Bo, L1, BM, Eh]. The case equality of has been settled in [CK] which further extends the methods introduced in [Bo].

An extension of this inequality due to C. Borell ([Bor2]), who introduced the notion of Gaussian noise stability, states that half-spaces are the most stable sets. The usual definition of stability of a set is the probability that two standard Gaussian vectors with a given correlation
both lie in the set. For a more precise definition, let $X$, $Y$ and $Y'$ be independent standard Gaussian vectors in $\mathbb{R}^n$ and let $0 \leq \rho \leq 1$. We define the *Gaussian noise stability* with parameter $\rho$ of a measurable set $A \subset \mathbb{R}^n$ by

$$S_\rho(A) = \mathbb{P}\left(\sqrt{\rho}X + \sqrt{1 - \rho}Y \in A \text{ and } \sqrt{\rho}X + \sqrt{1 - \rho}Y' \in A\right).$$

Borell’s theorem states that if $A, H \subset \mathbb{R}^n$ are such that $H$ is a half-space and $\gamma(A) = \gamma(H)$ then $S_\rho(H) \geq S_\rho(A)$ for all $0 \leq \rho \leq 1$. It turns out that this result admits diverse applications in numerous fields such as approximation theory, high-dimensional phenomena and rearrangement inequalities, and recently found also some surprising applications in discrete analysis and game theory (see [MN] and references therein). Alternative proofs of Borell’s inequality were given by Isaksson-Mossel and Kindler-O’Donnell [IM, KO] and recently Mossel and Neeman ([MN]) found a semi-group proof which also settles the equality case.

Both the Gaussian isoperimetric inequality and the noise stability inequality admit so-called *robustness* estimates. As the original inequalities only claim that the minimum (or maximum) of a certain quantity is attained on half-spaces, a robustness estimate also quantifies the deficit in these inequalities in terms of the distance, under a certain metric, of the set from being a half-space. For example, one may try to prove that if $A, H \subset \mathbb{R}^n$ satisfy $\gamma(H) = \gamma(A)$ and if the Gaussian surface area of $A$ differs from that of $H$ by a small number $\delta > 0$, then there necessarily exists a half-space whose total-variation distance to the set $A$ is smaller than some function of $\delta$ and $\gamma(A)$ (and maybe of the dimension) which goes to zero as $\delta \to 0$. A robustness estimate of the noise-stability inequality will do the same where the deficit $\delta = S_\rho(H) - S_\rho(A)$ is considered (and in this case, the distance to the half-space can also be a function of $\rho$).

The first robustness estimate for the Gaussian isoperimetric inequality was proven by Cianchi-Fusco-Maggi-Pratelli in [CFMP], and is based on a geometric approach. Mossel and Neeman found a different proof based on a more analytic approach, and provided a dimension-free estimate. In the more recent paper [MN], Mossel and Neeman managed to prove a robustness result for the Gaussian noise-stability and in the limit case $\rho \to 1$ they also attain an improvement of their isoperimetric robustness. The metric used in all of these estimates is the total-variation distance between the Gaussian measure restricted to the set and to the corresponding half-space.

We would also like to mention a recent result of Bobkov, Gozlan, Roberto and Samson [BGRS] who prove a somewhat-related robust logarithmic-Sobolev inequality, in which the deficit is estimated by transport and information-theoretic distances.

Another possible extension of the noise stability inequality, first explicitly introduced by E. Mossel is the following: for a number $q > 1$ and for $0 < \rho < 1$, we refer to the following as the $q$-stability of $A$, defined

$$S_\rho^q(A) = \mathbb{E}\left[\mathbb{P}(\sqrt{\rho}X + \sqrt{1 - \rho}Y \in A \mid X)^q\right]$$

where $X$ and $Y$ are independent standard Gaussian vectors. Evidently, $S_\rho(A) = S_\rho^2(A)$. When $q$ is an integer, this quantity can be thought of as the probability of $q$ Gaussian vectors whose mutual correlation is $\rho$ to all be inside $A$. It was initially proven by Isaksson and Mossel ([IM]) that half-spaces are the maximizers of this expression when constraining on the Gaussian volume. The equality case and a robustness bound for this extension has been established by J. Neeman ([N]).

This note has a few objectives. First, we present a novel proof of the Gaussian noise stability inequality, based on stochastic calculus. As a consequence, in the limit case we derive a new proof of the Gaussian isoperimetric inequality. Our proof is relatively short, and also provides
a very simple argument for establishing the equality case. Moreover, our proof is also valid for the more general $q$-stability defined above.

Next, we introduce a new metric $\varepsilon(A)$ to measure the distance between the set $A$ and its closest-possible half-space, say $H$. This metric, defined roughly as the distance between the corresponding centroids, turns out to be rather natural in this context: We prove that up to constants that depend only on $\rho$ and on $\gamma(A)$, the deficit $S_{\rho}(H) - S_{\rho}(A)$ can be bounded from both below and above by the same power of $\varepsilon(A)$, with only a logarithmic correction. The lower bound of the estimate we obtain is valid also for the more-general $q$-stability. As a corollary, we improve the dimension-free robustness result of [MN] which uses the total variation metric, getting the best possible exponent, up to a logarithmic factor.

Our estimate has an optimal dependence on the parameter $\rho$. Thanks to this fact we are able, as a limit case, to derive a robustness estimate for the Gaussian isoperimetric inequality, thus obtaining a dimension-free estimate with an optimal exponent (again, up to a logarithmic term).

Let us begin with some definitions, towards the formulation of our results. For a measurable $A \subset \mathbb{R}^n$ whose Gaussian centroid is not the origin, we define

$$v(A) = \int_A x \, d\gamma(x)$$

(1)

(otherwise, if the above integral is zero, we arbitrarily take $v(A) = e_1$ for some fixed vector $e_1 \neq 0$). Let $H(A)$ be the half-space of the form

$$H(A) = \{x; \langle x, v(A) \rangle \geq \alpha\}$$

(2)

where $\alpha$ is chosen such that $\gamma(H(A)) = \gamma(A)$. In other words, $H(A)$ is the half-space whose Gaussian measure is the same as that of $A$ and whose Gaussian center of mass is the closest possible to the Gaussian center of mass of $A$.

In our first theorem, the inequality is due to C. Borell and the characterization of the equality case (in the case $q = 2$) is due to Mossel-Ne’eman. Our main contribution here is giving a rather short and simple proof based on new methods.

**Theorem 1.** *For all measurable subsets $A \subset \mathbb{R}^n$, for all $0 \leq \rho < 1$ and $q > 1$, one has

$$S_{\rho}^q(H(A)) \geq S_{\rho}^q(A).$$

Moreover, equality holds if and only if the symmetric difference between $A$ and $H(A)$ has measure zero.*

We move on to the robustness estimate. Before we can formulate it, we need a few more definitions. Define, for all measurable $B \subset \mathbb{R}^n$,

$$q(B) = \left| \int_B x \, d\gamma(x) \right|.$$ 

We measure the distance between a set and its corresponding half-space using the following metric,

$$\varepsilon(A) := q(H(A))^2 - q(A)^2.$$ 

In other words, up to a factor which depends on the Gaussian measure of $A$, the metric we consider is the distance between the centroid of $A$, and that of the half-space closest to it (we will see below that this quantity is always non-negative).

Our two-sided robustness estimate, the main theorem of this note, reads
**Theorem 2.** For every $q > 1$ and $0 < s < 1$, there exist constants $C_s, c_s, c_{s,q} > 0$ such that the following holds: Let $0 < \rho < 1$ and let $A \subset \mathbb{R}^n$ be a measurable set satisfying $\varepsilon(A) < e^{-1/\rho}$ and $0 < \gamma(A) < 1$. Then

$$c_{\gamma(A)} \varepsilon(A) \log(\varepsilon(A))^{-1} \sqrt{1 - \rho} \leq S_\rho(H(A)) - S_\rho(A) \leq \frac{C_{\gamma(A)}}{\sqrt{1 - \rho}} \varepsilon(A)$$

and

$$c_{\gamma(A),q} \varepsilon(A) \log(\varepsilon(A))^{-1} \sqrt{1 - \rho} \leq S_\rho^n(H(A)) - S_\rho^n(A). \quad (4)$$

The reader is referred to the next subsection for a discussion about the sharpness of the this result and about possible extensions.

Arguably, for many explicit examples of sets $A$, the quantity $\varepsilon(A)$ is significantly easier to calculate than the actual noise stability of $A$, as it only depends on the density of the projection of the set onto a one-dimensional subspace, hence the calculation boils down to computing one-dimensional integrals. In light of the above theorem, one can obtain an approximation of the latter by the former.

As a first corollary to this theorem, we also get a robustness estimate in terms of the total-variation metric between a set and its corresponding half-space. For two sets $A, B$ we denote by $A \Delta B$ the symmetric difference between two sets, and define

$$\delta(A) := \gamma(A \Delta H(A))$$

We get,

**Corollary 3.** For every $q > 1$ and $0 < s < 1$, there exists a constant $c_{s,q} > 0$ such that the following holds: Let $0 < \rho < 1$ and let $A \subset \mathbb{R}^n$ be a measurable set satisfying $\delta(A) < e^{-1/\rho}$ and $0 < \gamma(A) < 1$. Then

$$c_{\gamma(A),q} \delta(A)^2 \log(\delta(A))^{-1} \sqrt{1 - \rho} \leq S_\rho^n(H(A)) - S_\rho^n(A). \quad (5)$$

A second corollary to the robustness estimate for the noise stability is the limit case for $\rho \to 1$, namely a robustness estimate for the Gaussian isoperimetric inequality. For a measurable set $A \subset \mathbb{R}^n$, define the Gaussian surface area of $A$ by

$$O^n(A) := \lim_{\varepsilon \to 0} e^{-1/\gamma(A_\varepsilon \setminus A)}$$

where

$$A_\varepsilon := \{x \in \mathbb{R}^n; \exists y \in A \text{ such that } |x - y| \leq \varepsilon\}$$

is the $\varepsilon$-extension of $A$.

The next result is a direct corollary of theorem 2 following a method introduced by M. Ledoux, about which the author learned from [MN]. By plugging the result of the theorem and of the above corollary into Ledoux’s method, described in [L2] (in the discussion following proposition 8.5) we immediately get the next corollary.

**Corollary 4.** There exists a universal constant $c > 0$ and, for every $0 < s < 1$, a constant $c_s > 0$ such that the following holds: for all measurable $A \subset \mathbb{R}^n$ with $\varepsilon(A) < c$, one has

$$O^n(A) - O^n(H(A)) \geq c_{\gamma(A)} \varepsilon(A) \log(\varepsilon(A))^{-1} \quad (6)$$

and for all measurable $A \subset \mathbb{R}^n$ such that $\delta(A) < c$, one has

$$O^n(A) - O^n(H(A)) \geq c_{\gamma(A)} \delta(A)^2 \log(\delta(A))^{-1}.$$
Remark 5. By combining the upper and lower bounds of theorem we may also get non-trivial relations between the quantities $S_\rho(A)$ at different values of $\rho$, as well as bounds on the noise stability of a set in terms of its surface area. As an example, if $A \subset \mathbb{R}^n$ and $0 < \rho < 1$ are such that $\varepsilon(A) < e^{-1/\rho}$, then by combining (6) with the upper bound in (3) one gets

$$\mathcal{O}^n(A) - \mathcal{O}^n(H(A)) \geq c_{\gamma(A)} \sqrt{1 - \rho S_\rho(A)} \left| \log \left( S_\rho(A) \sqrt{1 - \rho} \right) \right|^{-1}.$$ 

where $c_{\gamma(A)} > 0$ depends only on $\gamma(A)$.

The structure of this note is as follows. In section 2 we prove theorem and in section 2 we prove theorem and corollary Section 3 is an appendix in which we tie up some loose ends.

In the rest of the note, we use the following notation. The constants $C, C', c, c'$ will denote positive universal constants whose values may change between appearances in different formulae. We define $\gamma^k : \mathbb{R}^k \to \mathbb{R}$ the density of the standard Gaussian measure on $\mathbb{R}^k$ and by slight abuse of notation we define $\gamma^k(A)$ to be the Gaussian measure of the set $A \subset \mathbb{R}^k$. We abbreviate $\gamma = \gamma^n$, with $n$ being a fixed dimension all through the note. For two sets $A, B$ we define by $A \Delta B$ the symmetric difference between them. For a positive semi-definite symmetric matrix $A$, we denote its largest eigenvalue by $\|A\|_{o.p.}$. For any matrix $A$, we denote the sum of its diagonal entries by $Tr(A)$, and by $\|A\|^2_{HS}$ we denote the sum of the eigenvalues of the matrix $A^T A$. For a random vector $X$, we denote $\text{Cov}(X) = E[X \otimes X] - E[X] \otimes E[X]$, the covariance matrix of $X$. Finally, for a continuous time stochastic process $X_t$ adapted to a filtration $\mathcal{F}_t$, we denote by $[X]_t$ the quadratic variation of $X_t$ between time 0 and $t$. For a pair of continuous time stochastic processes $X_t, Y_t$, the quadratic covariation will be denoted by $[X, Y]_t$. By $dX_t$ we denote the Itô differential of $X_t$, which we understand as a pair of predictable processes $(\sigma_t, \mu_t)$ such that $X_t$ satisfies stochastic differential equation $dX_t = \sigma_t dB_t + \mu_t dt$ where $B_t$ is a Brownian motion. We also denote

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-x^2/2} dx$$

the Gaussian cumulative distribution function, and write $\Psi(s) = \Phi^{-1}(s)$.

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1.1 Discussion

Before we move on to the proofs, we would like to discuss the optimality of our estimates and suggest possible future research.

First, consider the robustness inequality (3). It is easy to see that the dependence of the upper bound on $\varepsilon(A)$ is tight: for $\varepsilon \geq 0$ define

$$A_{\varepsilon} = (-\infty, \Psi(1/2 - \varepsilon)] \cup [\Psi(3/4), \Psi(3/4 + \varepsilon)].$$
where $\Psi(x) = \Phi^{-1}(x)$ is the inverse Gaussian cumulative distribution function. We claim that this set demonstrates that the upper bound is tight. It is easy to check that $\varepsilon(A) \sim \varepsilon$ and that if $X, Y$ are Gaussian variables whose correlation is $0.01 < \rho < 0.99$ then

$$\mathbb{P}(Y \in (-\infty, \Psi(1/2 - \varepsilon)) \mid X \in [\Psi(1/2 - \varepsilon), 0]) > (1 + c)\mathbb{P}(Y \in (-\infty, \Psi(1/2 - \varepsilon)) \mid X \in [\Psi(3/4, \Psi(3/4 + \varepsilon)]) \geq (1 + c)c'$$

for all $\varepsilon < 1/4$ where $c, c' > 0$ are constants which do not depend on $\varepsilon$. Therefore

$$\mathcal{S}_\rho(H(A)) = \mathcal{S}_\rho((-\infty, \Psi(1/2 - \varepsilon)]) + 2\varepsilon\mathbb{P}(Y \in (-\infty, \Psi(1/2 - \varepsilon)) \mid X \in [\Psi(1/2 - \varepsilon), 0]) + o(\varepsilon)$$

and

$$\mathcal{S}_\rho(A) = \mathcal{S}_\rho((-\infty, \Psi(1/2 - \varepsilon)]) + 2\varepsilon\mathbb{P}(Y \in (-\infty, \Psi(1/2 - \varepsilon)) \mid X \in [\Psi(3/4, \Psi(3/4 + \varepsilon)]) + o(\varepsilon)$$

so

$$\mathcal{S}_\rho(H(A)) - \mathcal{S}_\rho(A) > 2c'\varepsilon + o(\varepsilon).$$

This shows that the dependence of the upper bound on $\varepsilon$ cannot be improved. One could hope that the logarithmic term in the lower bound is fully removed thus obtaining a tight bound. Alas, if we define

$$A_\varepsilon = (-\infty, \Psi(1/2 - \varepsilon)] \cup [\Psi(1 - \varepsilon), \infty)$$

then it is not hard to see that $\varepsilon(A) \sim \varepsilon \sqrt{\log(\varepsilon)}$. On the other hand, we have

$$\mathcal{S}_\rho(H(A_\varepsilon)) \leq \mathcal{S}_\rho((-\infty, \Psi(1/2 - \varepsilon)]) + O(\varepsilon)$$

and

$$\mathcal{S}_\rho(A_\varepsilon) \geq \mathcal{S}_\rho((-\infty, 1/2 - \varepsilon]).$$

It follows that

$$\mathcal{S}_\rho(H(A_\varepsilon)) - \mathcal{S}_\rho(A_\varepsilon) = O(\varepsilon)$$

so at least a term of the order $\sqrt{\log \varepsilon(A)}$ is necessary.

It seems from the proofs that this type of “large deviation” phenomenon might be the only reason for which the logarithmic term is needed. We would like to formulate a conjecture suggesting that a slightly perturbed metric could provide a tight bound. To define this metric we write $v = v(H(A)) / |v(H(A))|$ (where $v(\cdot)$ is defined in equation (119)), and let $\mu$ and $\nu$ be the push-forward under the map $x \rightarrow \langle v, x \rangle$ of the Gaussian measure restricted to the sets $A$ and $H(A)$ respectively. Denote by $f(x)$ and $g(x)$ the corresponding densities of $\mu$ and $\nu$ with respect to $\gamma^1$.

Inspired by equation (119) below, we define

$$\tilde{\varepsilon}_\rho(A) = \int_{\mathbb{R}} \Phi\left(\frac{\rho x - \alpha}{\sqrt{1 - \rho^2}}\right) (g(x) - f(x)) d\gamma^1(x).$$

**Conjecture 6.** For every $0 < s < 1$, there exist constants $C_s, c_s > 0$ such that the following holds: Let $0 < \rho < 1$ and let $A \subset \mathbb{R}^n$ be a measurable set satisfying $\varepsilon(A) < e^{-1/\rho}$ and $0 < \gamma^n(A) < 1$. Then

$$C_{\gamma(A)} \tilde{\varepsilon}_\rho(A) \geq \mathcal{S}_\rho(H(A)) - \mathcal{S}_\rho(A) \geq c_{\gamma(A)} \tilde{\varepsilon}_\rho(A)(1 - \rho).$$

(7)

In particular, the expression $\mathcal{S}_\rho(H(A)) - \mathcal{S}_\rho(A)$ is equivalent, up to constants depending only on $\rho$ and $\gamma(A)$, to an expression depending only on the marginal of the set $A$ on the direction $v$. 


Finally, let us discuss the optimality of corollary 3. We claim that the exponent 2 of the expression $\delta(A)^2$ appearing in equation (3) cannot be improved. Consider the example

$$B_\varepsilon = (-\infty, \Psi(1/2 - \varepsilon)] \cup [\Psi(1/2 + \varepsilon), \Psi(1/2 + 2\varepsilon)].$$

It is easy to verify that $\varepsilon(B) \sim \varepsilon^2$ while $\delta(B) = \varepsilon$. It follows from the upper bound in theorem 2 that as $\varepsilon \to 0$, the dependence of the deficit on $\delta(A)$ is correct, maybe up to the logarithmic factor. We conjecture that the logarithmic factor in this corollary can be removed.

2 Proof of theorem 1

This section is dedicated to the proof of theorem 1. We begin with a few definitions.

First denote

$$\gamma_{v,\sigma}(x) = \frac{1}{\sigma^n(2\pi)^{n/2}} \exp \left( -\frac{1}{2\sigma^2} |x - v|^2 \right)$$

the density of the Gaussian centered at $v$ with covariance matrix $\sigma^2 \text{Id}$, and abbreviate $\gamma(x) = \gamma_{0,1}(x)$. Let $X$ be a standard Gaussian random vector in $\mathbb{R}^n$.

Given a measurable set $A \subset \mathbb{R}^n$, our goal is to analyse the quantity

$$S^q_\rho(A) = \mathbb{E}_X \left[ \left( \int_A \gamma_{\sqrt{\rho}X,\sqrt{1-\rho}}(x) dx \right)^q \right].$$

Instead of considering the vector $X$, let $W_t$ be a standard Brownian motion in $\mathbb{R}^n$, adapted to a filtration $\mathcal{F}_t$. Clearly $\sqrt{\rho}X$ has the same distribution of $W_\rho$, and therefore

$$S^q_\rho(A) = \mathbb{E} \left[ \left( \int_A \gamma_{W_\rho,\sqrt{1-\rho}}(x) dx \right)^q \right].$$

The main idea of the proof is to consider the process

$$S_t = \mathbb{P}(W_1 \in A | \mathcal{F}_t) = \int_A \gamma_{W_\rho,\sqrt{1-\rho}}(x) dx,$$

so that our quantity of interest becomes

$$S^q_\rho(A) = \mathbb{E}[S^q_t].$$

This process is a continuous martingale (we will see this fact more clearly later on), therefore this quantity can be analysed by calculating the differential $dS^q_t$. The following lemma will be helpful to us in calculating Itô differentials related to the process $S_t$.

Lemma 7. Denote

$$F_t(x) = \gamma_{W_\rho,\sqrt{1-\rho}}(x).$$

For every $x \in \mathbb{R}^n$ the process $F_t(x)$ is a martingale satisfying the stochastic differential equation

$$F_0(x) = \gamma(x), \quad dF_t(x) = (1-t)^{-1} F_t(x)(x - W_t, dW_t). \quad (8)$$

The proof, which is a straightforward calculation, is postponed to the appendix.
Remark 8. Equation (8) could be seen as a stochastic evolution equation on the space of Gaussian densities. Equations of a similar nature, where the initial density is an arbitrary function seem to be rather useful tool for proving concentration inequalities, as demonstrated in [E1, EL, E2].

By the notation of the lemma, we have

\[ S_t = \int_A F_t(x) \, dx. \]

We can now calculate, using this lemma

\[ dS_t = d \int_A F_t(x) \, dx = \int_A dF_t(x) \, dx = (1 - t)^{-1} \int_A \left\langle x - W_t, dW_t \right\rangle F_t(x) \, dx = \]

\[ \frac{(1 - t)^{-1}}{(2\pi(1-t))^{(n/2)}} \int_A \left\langle x - W_t, dW_t \right\rangle \exp \left( -\frac{1}{2(1-t)}|x - W_t|^2 \right) \, dx = \]

(substituting \( y = \frac{x - W_t}{\sqrt{1 - t}} \))

\[ (1 - t)^{-1/2} \left\langle \int_{A_t} y \gamma(y) \, dy, dW_t \right\rangle. \]

Recall that for all measurable \( B \subset \mathbb{R}^n \), we define

\[ q(B) = \left| \int_B x \gamma(x) \, dx \right| \]

and define

\[ A_t = \frac{A - W_t}{\sqrt{1 - t}}. \]

With this notation, we get

\[ S_t = \int_{A_t} \gamma(x) \, dx = \gamma(A_t), \quad (10) \]

\[ dS_t = (1 - t)^{-1/2} \left\langle \int_{A_t} x \gamma(x) \, dx, dW_t \right\rangle, \quad (11) \]

and

\[ d[S]_t = (1 - t)^{-1} q^2(A_t) \, dt \]

where \([S]_t\) denotes the quadratic variation of the process \( S_t \).

In particular, we see that \( S_t \) is an Itô process, so thanks to Itô’s formula,

\[ dS_t^q = q S_t^{q-1} dS_t + \frac{1}{2} q(q - 1) S_t^{q-2} d[S]_t. \]

Since \( S_t \) is a martingale we have

\[ S_t^q(A) = \mathbb{E}[S_t^q] = S_0^q + \frac{1}{2} q(q - 1) \mathbb{E} \left[ \int_0^t S_t^{q-2} d[S]_t \right] \]

(13)
and in particular, $S_{\rho}(A) = \gamma(A)^2 + \mathbb{E}[S]_{\rho}$.

Our goal is to compare $S_{\rho}^q(A)$ with $S_{\rho}^q(H(A))$. To that end, we want to define the process $Q_t$ to be an analogous process to $S_t$ where the initial set $A$ is replaced by its corresponding half-space $H(A)$. In other words, we define

$$Q_t = \mathbb{P}\left(\tilde{W}_1 \in H(A) \mid \tilde{F}_t\right) = \int_{H(A)} \gamma_{\tilde{W}_t, \sqrt{1-t}}(x)dx, \tag{14}$$

where $\tilde{W}_t$ is a standard Brownian motion adapted with to a filtration $\tilde{F}_t$ (at this point we consider the processes $W_t, \tilde{W}_t$ as two processes which live on different probability spaces). In analogy with (13) we have

$$S_{\rho}^q(H(A)) = \mathbb{E}[Q_{\rho}^q] = Q_0^q + \frac{1}{2}q(q-1)\mathbb{E}\left[\int_0^\rho Q_{\rho}^{q-2}d[Q]_t\right] \tag{15}$$

and the proof is reduced to showing that,

$$\mathbb{E}\left[\int_0^\rho Q_{\rho}^{p-2}d[Q]_t\right] \geq \mathbb{E}\left[\int_0^\rho S_{\rho}^{p-2}d[S]_t\right]. \tag{16}$$

with equality only if $\gamma(A \Delta H(A)) = 0$.

By slight abuse of notation, we also define a function $q : [0, 1] \rightarrow \mathbb{R}$ by

$$q(s) = -\int_{-\infty}^{\Phi^{-1}(s)} x\,d\gamma^1(x)$$

(where $\Phi(\cdot)$ is the standard Gaussian cumulative distribution function) so that $q(\gamma(B)) = q(H(B))$ for all measurable $B \subset \mathbb{R}^n$.

Clearly, we will have $Q_0 = \gamma(H(A)) = S_0$. Moreover, observe that if $B$ is a half-space then $q(B) = q(\gamma(B))$, and therefore in exactly the same manner we derived equations (10) and (12), we have

$$Q_t = \gamma\left(\frac{H(A) - \tilde{W}_t}{\sqrt{1-t}}\right)$$

and

$$d[Q]_t = (1-t)^{-1}q\left(\frac{H(A) - \tilde{W}_t}{\sqrt{1-t}}\right)^2 dt = (1-t)^{-1}q(Q_t)^2 dt. \tag{17}$$

Recall that $S_t$ and $Q_t$ are martingales. According to the Dambis / Dubins-Schwartz theorem, there exists a standard Brownian motion $B(t)$ such that

$$S_t = B([S]_t), \forall 0 \leq t < 1$$

and another Brownian motion $\tilde{B}(t)$ such that

$$Q_t = \tilde{B}([Q]_t), \forall 0 \leq t < 1.$$ 

Remark that those equations also imply the assumption $B(0) = \tilde{B}(0) = S_0$. Note that both $Q_t$ and $S_t$ are defined up to time 1 which is exactly the time where these processes reach the
value 0 or 1. Therefore, both $B(T)$ and $\tilde{B}(T)$ are Brownian motions defined up to the first time they reach $\{0, 1\}$. Since both these Brownian motions have the same distribution, we may 
**couple** between the two so that they are defined on the same probability space using the simple assumption

$$B(T) = \tilde{B}(T)$$

(18)

for all values of $T$ in which they are defined. We also define

$$T_f = \sup\{T; 0 < B(T) < 1\}$$

so that $B(T)$ is defined in the interval $0 \leq T < T_f$, and equation (18) holds for all such $T$.

Define $T_1(t) = [S]_t$ and $T_2(t) = [Q]_t$, and let $\tau_1, \tau_2$ be their respective inverse functions. Equation (12) written differently is

$$\frac{d}{dt}[S]_t = (1 - t)^{-1}q^2(A_t)_t,$$

(19)

which gives

$$T'_1(\tau_1(T)) = (1 - \tau_1(T))^{-1}q(A_{\tau_1(T)})^2, \ \forall 1 \leq T < T_f$$

and, by the inverse function formula,

$$\tau'_1(T) = (1 - \tau_1(T))q(A_{\tau_1(T)})^{-2}, \ \forall 1 \leq T < T_f.$$  

(20)

Similarly, by (17) and by the fact that $Q_{\tau_2(T)} = B(T)$,

$$T'_2(\tau_2(T)) = (1 - \tau_2(T))q(B(T))^{-2}, \ \forall 1 \leq T < T_f$$

(21)

Finally, define

$$\omega_1(T) = -\log(1 - \tau_1(T)), \ \omega_2(T) = -\log(1 - \tau_2(T)).$$

So by (20) and (21), we have

$$\omega_1(T)' = q(A_{\tau_1(T)})^{-2}, \ \omega_2(T)' = q(B(T))^{-2}$$

(22)

for all $0 \leq T < T_f$.

The following claim will provide the only inequality in the proof of the theorem. Its proof is very simple, and we postpone it to the end of the section.

**Claim 9.** For all measurable $B \subset \mathbb{R}^n$, one has

$$q(B) \leq q(\gamma(B)) = q(H(B)).$$

with equality if and only if $\gamma(B \Delta H(B)) = 0$.

Recall that $\gamma(A_t) = S_t$, so $B(T) = \gamma(A_{\tau_1(T)})$. The above claim shows that

$$q(A_{\tau_1(T)}) \leq q(B(T)).$$

Consequently,

$$\omega_1'(T) - \omega_2'(T) = q(A_{\tau_1(T)})^{-2} - q(B(T))^{-2} \geq 0, \ \forall 0 \leq T < T_f,$$

(23)
which implies, by definition, that \( \tau_1(T) \geq \tau_2(T) \) for all \( 1 \leq T < T_f \). In other words

\[
[S]_t \leq [Q]_t, \quad \forall 0 \leq t \leq 1.
\] (24)

By changing variables \( T = [Q]_t \), the left hand side of equation (16) becomes

\[
\mathbb{E} \left[ \int_0^\rho Q_t^{q-2} d[Q]_t \right] = \mathbb{E} \left[ \int_0^{[Q]_\rho} B(T)^{q-2} dT \right]
\]

and by substituting \( T = [S]_t \), the right hand side becomes

\[
\mathbb{E} \left[ \int_0^\rho S_t^{q-2} d[S]_t \right] = \mathbb{E} \left[ \int_0^{[S]_\rho} B(T)^{q-2} dT \right].
\]

Equation (16) shows us that our goal is to prove that

\[
\mathbb{E} \left[ \int_0^{[Q]_\rho} B(T)^{q-2} dT \right] \geq \mathbb{E} \left[ \int_0^{[S]_\rho} B(T)^{q-2} dT \right].
\] (25)

Since both integrands are positive and in light of (24), the proof of the inequality complete.

To analyse the equality case, we note that almost surely, for all \( T < T_f \), we have \( 0 < B(T) < 1 \). This is true since the process \( B(T) = S_{\tau_1(T)} \) is a Brownian motion defined up to time \( [S]_1 \) so if \( B(T) \) reaches one of the endpoints of the interval \([0, 1]\) before time \([S]_1 \) it contradicts the fact that \( 0 \leq S_t \leq 1 \) for all \( 0 \leq t \leq 1 \). It follows that there could only be an equality in formula (25) if \([S]_\rho = [Q]_\rho\), which implies that for almost all \( 0 \leq t \leq \rho \), there is an equality in equation (23). But according to claim 9, the only case in which there can be equality in equation (23) is if \( \gamma(A_t \Delta H(A_t)) = 0 \) which in turn implies that \( \gamma(A \Delta H(A)) = 0 \). The equality case is thus also established.

We still have to prove claim 9. It will be useful to first prove the following more general fact

**Lemma 10.** Let \( m : \mathbb{R} \to [0, 1] \) be a measurable function. One has,

\[
\left| \int_\mathbb{R} x m(x) d\gamma^1(x) \right| \leq q \left( \int_\mathbb{R} m(x) d\gamma^1(x) \right)
\]

and there is an equality in the above if and only if \( m(x) \) is of either the form \( 1_{x \leq \alpha} \) or of the form \( 1_{x \geq \alpha} \) for almost every \( x \).

**Proof.** Without loss of generality, we may assume that \( \int_\mathbb{R} x m(x) d\gamma^1(x) \geq 0 \) (otherwise replace \( m(x) \) by \( m(-x) \)). Let \( h(x) \) be the function of the form \( h(x) = 1_{x \geq \alpha} \) where \( \alpha \) is chosen such that \( \int h(x) d\gamma^1(x) = \int m(x) d\gamma^1(x) \). The claim of the lemma boils down to showing that

\[
\int_\mathbb{R} x(h(x) - m(x)) d\gamma^1(x) \geq 0
\] (26)

with equality if and only if \( m(x) = h(x) \) almost surely. Indeed, we have

\[
\int_\mathbb{R} x(h(x) - m(x)) d\gamma^1(x) = \int_\mathbb{R} (x - \alpha)(h(x) - m(x)) d\gamma^1(x)
\]
now, by definition of \( h(x) \) and by the fact that \( 0 \leq m(x) \leq 1 \) for all \( x \), the function \( h(x) - m(x) \) has the same sign as \( (x - \alpha) \), which means that the right hand side of the above equation is non-negative, and is zero if and only if \( h(x) = m(x) \) almost surely.

\[ \square \]

**Proof of claim** Let \( B \subset \mathbb{R}^n \). Define,

\[ \theta = \frac{\int_B x \gamma(x) dx}{\int_B x^2 \gamma(x) dx} \]

(if the denominator is zero then there’s nothing to prove). Let \( \mu \) be the push-forward of the restriction of \( \gamma \) to \( B \) under the map \( x \rightarrow \langle \theta, x \rangle \). Define \( m(x) = \frac{d\mu}{d\gamma}(x) \). Since \( \gamma(x) \) is a product measure,

\[ q(B) = \left| \int_B \langle \theta, x \rangle \gamma(x) dx \right| = \left| \int_{\mathbb{R}} x m(x) \gamma(x) dx \right| \]

An application of lemma (10) finishes the proof.

\[ \square \]

### 3 The robustness estimate

The goal of this section is to prove theorem 2 and corollary 3.

Let us briefly describe the steps of our proof. Our starting point is equations (13) and (15), according to which we have

\[ S^q_\rho(H(A)) - S^q_\rho(A) = \mathbb{E}[Q^q_\rho] - \mathbb{E}[S^q_\rho] = \]

\[ \frac{1}{2} q(q - 1) \left( \mathbb{E} \left[ \int_0^\rho Q^{q-2}_t d[Q]_t \right] - \mathbb{E} \left[ \int_0^\rho S^{q-2}_t d[S]_t \right] \right) \]

As described above, we couple the processes \( S_t \) and \( Q_t \) using the equations

\[ S_t = B([S]_t), \quad Q_t = B([Q]_t) \]

where \( B(T) \) is a Brownian motion satisfying \( B(0) = S_0 = Q_0 \), and defined up to the time it reaches \( \{0, 1\} \). With this coupling, the above equation becomes

\[ S^q_\rho(H(A)) - S^q_\rho(A) = \frac{1}{2} q(q - 1) \mathbb{E} \left[ \int_0^{[Q]_\rho} B(T)^{q-2} dT - \int_0^{[S]_\rho} B(T)^{q-2} dT \right] = \quad (27) \]

\[ \frac{1}{2} q(q - 1) \mathbb{E} \left[ \int_{[S]_\rho}^{[Q]_\rho} B(T)^{q-2} dT \right] . \]

As in the previous section, we define

\[ T_1(t) = [S]_t, \quad T_2(t) = [Q]_t \]

and denote by \( \tau_1, \tau_2 \) their corresponding inverse functions. We also define

\[ \omega_1(T) = -\log(1 - \tau_1(T)), \quad \omega_2(T) = -\log(1 - \tau_2(T)) . \]
We have, as in (22),
\[ \omega_1(T)' = q(A_{\tau_1(T)})^{-2}, \quad \omega_2(T)' = q(B(T))^{-2}. \] (28)
and by claim 9
\[ \omega_1'(T) - \omega_2'(T) = q(A_{\tau_1(T)})^{-2} - q(B(T))^{-2} \geq 0, \quad \forall 0 \leq T < T_f. \] (29)
Finally, it will also be convenient to define the stopping times
\[ \Theta_1 = \min\{T; \omega_1(T) = - \log(1 - \rho)\} = T_1(\rho), \quad \Theta_2 = \min\{T; \omega_2(T) = - \log(1 - \rho)\} = T_2(\rho). \]
So equation (27) becomes
\[ S^2_\rho(H(A)) - S^2_\rho(A) = \frac{1}{2}q(q - 1)\mathbb{E} \left[ \int_{\Theta_1}^{\Theta_2} B(T)^{q-2}dT \right]. \] (30)
Our goal is to show that the right hand side of the above quantity is not too small. For that, we would like to show two things: (i) That the expectation \( \Theta_2 - \Theta_1 \) is quite large and (ii) that \( B(T) \) is not too close to zero when we reach \( \Theta_1 \) and thus the integrand will be non-negligible in the (rather large) interval \([\Theta_1, \Theta_2]\).

We will first roughly show that up to time \( \Theta_2 \), the \( B(T) \) is bounded away from zero and from one with a probability that only depends on \( B(0) \) and on \( \rho \) (this is done partly in lemma 11 and partly in lemma 17 below). This ensures that the integrand in the above formula is not too small, and hence it will be enough to prove (i).

The main step in the proof of (i) will be to show that
\[ \mathbb{P}(\omega_1(T_0) - \omega_2(T_0) \geq \delta) \geq p \] (31)
with \( \delta \) and \( \rho \) being as large as possible and \( T_0 \leq \Theta_1 \). This roughly means that the process \( Q_t \) is "lagged" with respect to the process \( S_t \) (when considering the above coupling) meaning that in the future, when the process \( S_t \) stops (i.e., when \( T = \Theta_1 \)), the process \( Q_t \) will still have some time left (until \( T = \Theta_2 \)) in order to accumulate some quadratic variation. In other words, in order to use this fact to control the difference \( \Theta_2 - \Theta_1 \), we use the fact that \( \int_{\Theta_1}^{\Theta_2} \omega_2(T)dT \geq \delta \), and invoke (28) in order to get a lower bound for the integrand. This is done in lemma 15 below.

In order to prove an equation of the form (31), we will define
\[ \epsilon_t = q^2(S_t) - q^2(A_t). \] (32)
Note that \( \epsilon_0 = \epsilon(A) \). We will use formula (28), which tells us that
\[ \omega_1'(T) - \omega_2'(T) = q^{-2}(A_{\tau_1(T)}) - q^{-2}(S_{\tau_1(T)}) \geq c\epsilon_{\tau_1(T)} \] (33)
hence difference \( \omega_1'(T) - \omega_2'(T) \) is controlled by the quantity \( \epsilon_{\tau_1(T)} \). Thanks to this, in order to prove that the difference \( \omega_1(\Theta_1) - \omega_2(\Theta_1) \) is quite large, it will be enough to prove that with a non-negligible probability, one has
\[ \epsilon_t > c\epsilon_0, \quad \forall 0 < t < \alpha \] (34)
where \( c, \alpha \) are not too small. If this is true, we can integrate equation (33) and deduce that for all \( t \geq \alpha \) one has \( w_1(\alpha) - w_2(\alpha) \geq c\epsilon_0\alpha \). Once we have this, we can finally ensure that
\( \omega_1(\Theta_1) - \omega_2(\Theta_1) = \delta \) where \( \delta \) is not too small, with a non-negligible probability. This is eventually done in lemma 14 below.

The only fact we will still have to explain is why an estimate of the form (34) holds (which will be proven in lemma 12 below). This is, in fact, the most difficult step of the proof and the two consequent subsections are dedicated to it. The idea of its proof is to calculate the Itô differential of the process \( \epsilon_t \) (which turns out to be an Itô process) using the formula (8) and bound it in terms of \( S_t \) and \( \epsilon_t \) itself. An entire subsection is dedicated to the calculation of this differential, and another subsection is dedicated to bounding its drift and quadratic variation, which boils down to bounding the Hilbert-Schmidt norm of a certain matrix. The outcome of these two subsections is concluded in proposition 13 below. Finally, the upper bound for the deficit is proven in subsection 3.3.

We are finally ready to begin the proof. We start by defining a stopping time,

\[
T = \min\{T; B(T)(1 - B(T)) \leq B(0)(1 - B(0))/2\}
\]

The following simple lemma shows that we can expect the quantity \( S_t(1 - S_t) \) to remain bounded away from zero with a non-negligible probability.

**Lemma 11.** There exists a universal constant \( c_1 > 0 \) such that

\[
P\left( \tau_1(T) \geq \frac{1}{2} \right) \geq c_1 S_0(1 - S_0)
\]  

**Proof.** First, it will be useful to notice that, according to (17),

\[
T'_2(t) = (1 - t)^{-1}q(Q_t)^2 \leq (1 - t)^{-1}q(1/2)^2 = \frac{1}{2\pi}(1 - t)^{-1}.
\]

Therefore, using (24),

\[
T_1(t) \leq T_2(t) \leq -\frac{1}{2\pi} \log(1 - t).
\]  

Define

\[
U = \min\{T; B(T)(1 - B(T)) \leq B(0)(1 - B(0))/2 \text{ or } B(T) = 1/2\}.
\]

Assume for now that \( B(0) \leq 1/2 \). Let \( \beta \) be the solution to the equation

\[
\beta(1 - \beta) = B(0)(1 - B(0))/2
\]

satisfying \( \beta < 1/2 \). It is easy to verify that

\[
\beta \leq B(0)/2.
\]

Since \( B(T) \) is a martingale, the optional stopping theorem implies that

\[
P(B(U) = 1/2) = \frac{B(0) - \beta}{1/2 - \beta} \geq B(0).
\]

In a completely similar manner, when \( B(0) > 1/2 \) one has \( P(B(U) = 1/2) \geq 1 - B(0) \), and we conclude that

\[
P(B(U) = 1/2) \geq B(0)(1 - B(0)).
\]
Equation (36) teaches us that $T_1(1/2) \leq \frac{1}{2\pi} \log 2$. Clearly, there exists a constant $c > 0$ such that a Brownian motion starting at $1/2$ at time $T_0$ remains inside the interval $[1/4, 3/4]$ by time $T_0 + \frac{1}{2\pi} \log 2$ with probability at least $c$. So, by remarking that

$$B(T)(1 - B(T)) \leq B(0)(1 - B(0))/2 \Rightarrow B(T)(1 - B(T)) \leq 1/8 \Rightarrow B(T) \notin [1/4, 3/4],$$

we learn that

$$P \left( B(T) > B(0)(1 - B(0))/2, \forall U \leq T \leq U + \frac{1}{2\pi} \log 2 \mid B(U) = 1/2 \right) > c.$$

Combining this fact with (37) gives

$$P(T \geq T_1(1/2)) \geq cB(0)(1 - B(0)) = cS_0(1 - S_0).$$

The proof is complete.

For a number $\delta > 0$, define the event

$$F_\delta = \{ |\epsilon_t - \epsilon_0| \leq \epsilon_0/2, \forall 0 \leq t \leq \delta |\log \epsilon_0|^{-1}S_0^2(1 - S_0)^2 \}$$

where $\epsilon_t$ is defined in equation (32).

**Lemma 12.** There exists a universal constant $c_2 > 0$ such that whenever $\epsilon_0 < 1/2$,

$$P \left( \tau_1(T) \geq \frac{1}{2} \text{ and } F_{c_2} \text{ holds} \right) \geq c_2S_0(1 - S_0) \quad (38)$$

The main ingredient of this lemma will be the following proposition, to the proof of which we dedicate the next two subsections. The point of the proposition is that $\epsilon_t$ does not move too much provided that it is small and that $S_t$ is bounded away from 0 and 1.

**Proposition 13.** There exists a universal constant $C > 0$ such that the following holds: There exist two predictable processes $\alpha_t \in \mathbb{R}^n$ and $\beta_t \in \mathbb{R}$ satisfying

$$d\epsilon_t = \langle \alpha_t, dW_t \rangle + \beta_t dt$$

and such that the following bounds hold,

(i) For all $0 \leq t \leq 1$,

$$|\alpha_t| \leq (1 - t)^{-1/2} \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t \sqrt{|\log \epsilon_t|}.$$  

(ii) For all $0 \leq t \leq 1$,

$$|\beta_t| \leq (1 - t)^{-1} \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t \sqrt{|\log \epsilon_t|}.$$  

**Proof of lemma 12.** Define the stopping time

$$u = \min\{t; |\epsilon_t - \epsilon_0| \geq \epsilon_0/2\} \wedge 1.$$  

By the notation of proposition 13 we have

$$d\epsilon_t = \langle \alpha_t, dW_t \rangle + \beta_t dt.$$
By definition for all \( t \leq \tau_1(T) \), one has
\[
\frac{1}{S_t(1 - S_t)} \leq \frac{2}{S_0(1 - S_0)}.
\] (39)

Consequently, according to part (i) of the above proposition, we have
\[
|\alpha_t| \leq \frac{C_0}{S_0^3(1 - S_0)^3} S_0^2 \log |\beta_0|, \quad \forall 0 \leq t \leq \tau_1(T) \wedge u \wedge 1/2.
\] (40)

for a universal constant \( C_0 > 0 \) and according to part (ii) of the proposition,
\[
|\beta_t| \leq \frac{C_1}{S_0^3(1 - S_0)^3} S_0^2 \log |\beta_0|, \quad \forall 0 \leq t \leq \tau_1(T) \wedge u \wedge 1/2
\] (41)

for a universal constant \( C_1 > 0 \). Fix a constant \( \delta > 0 \) and define
\[
t_0 = \delta \log |\beta_0|^{-1} S_0^2 (1 - S_0)^7 \wedge \tau_1(T) \wedge u \wedge \frac{1}{2},
\]

Equation (41) and the fact that \( \epsilon_0 < 1/2 \) imply
\[
\left| \int_0^{t_0} \beta_t dt \right| \leq C_3 \delta \epsilon_0
\] (42)

for some universal constant \( C_3 > 0 \). Using the triangle inequality gives
\[
\mathbb{P}\left( |\epsilon_{t_0} - \epsilon_0| \geq \epsilon_0/2 \right) \leq \mathbb{P}\left( \left| \int_0^{t_0} \langle \alpha_t, dW_t \rangle \right| \geq \epsilon_0/2 \right) \leq \mathbb{P}\left( \left| \int_0^{t_0} \langle \alpha_t, dW_t \rangle \right| \geq \epsilon_0/2 - C_3 \delta \epsilon_0 \right).
\]

By assuming that \( \delta \) is a small enough universal constant, we can assert that \( \epsilon_0/2 - C_3 \delta \epsilon_0 \geq \epsilon_0/4 \), and obtain
\[
\mathbb{P}\left( |\epsilon_{t_0} - \epsilon_0| \geq \epsilon_0/2 \right) \leq \mathbb{P}\left( \left| \int_0^{t_0} \langle \alpha_t, dW_t \rangle \right| \geq \epsilon_0/4 \right).
\] (43)

To estimate the right hand side, we use equation (40) to get
\[
[\epsilon]_{t_0} = \int_0^{t_0} |\alpha_t|^2 dt \leq C_0 \delta S_0 (1 - S_0) \epsilon_0^2.
\] (44)

By Itô’s formula,
\[
\mathbb{E}\left( \left( \int_0^{t_0} \langle \alpha_t, dW_t \rangle \right)^2 \right) \leq C_0 S_0 (1 - S_0) \delta \epsilon_0^2.
\]

So, by Chebyshev’s inequality,
\[
\mathbb{P}\left( \left| \int_0^{t_0} \langle \alpha_t, dW_t \rangle \right| > \epsilon_0/4 \right) < 16 C_0 S_0 (1 - S_0) \delta.
\]

Combining this with (43) finally gives
\[
\mathbb{P}\left( |\epsilon_{t_0} - \epsilon_0| \geq \epsilon_0/2 \right) \leq 16 C_0 S_0 (1 - S_0) \delta.
\]
This shows that there exists a universal constant $c_2 > 0$ such that if $\delta \leq c_2$, then
\[
\mathbb{P} \left( |\epsilon_{t_0} - \epsilon_0| \geq \epsilon_0/2 \right) < c_1 S_0(1 - S_0)/2.
\]
where $c_1$ is the constant from equation (35). In other words, by definition of $t_0$ and $u$ and by the continuity of $\epsilon_t$,
\[
\mathbb{P}(t_0 = u) \leq c_1 S_0(1 - S_0)/2. \tag{45}
\]
Define
\[
\alpha = c_2|\log \epsilon_0|^{-1} S_0^7(1 - S_0)^7.
\]
The assumption that $\epsilon_0 < 1/2$ can ensure (by taking $c_2$ to be small enough) that $\alpha \wedge 1/2 = \alpha$, this means that
\[
t_0 < u \Rightarrow t_0 = \alpha \wedge \tau_1(T)
\]
and by definition, if $t_0 = \tau_1(T)$ it means that $\tau_1(T) \leq \frac{1}{2}$ so equation (45) becomes
\[
\mathbb{P}(|\epsilon_t - \epsilon_0| \leq \epsilon_0/2, \ \forall 0 \leq t \leq \alpha \text{ or } \tau_1(T) < 1/2) > 1 - c_1 S_0(1 - S_0)/2.
\]
Using a union bound with the result of lemma [11] finishes the proof. \qed

From this point on, we denote $\alpha = c_2|\log \epsilon_0|^{-1} S_0^7(1 - S_0)^7$ where $c_2$ is the constant which appears in equation (38) and define
\[
G = F_{c_2} \cap \left\{ \tau_1(T) \geq \frac{1}{2} \right\} = \{ |\epsilon_t - \epsilon_0| \leq \epsilon_0/2, \ \forall 0 \leq t \leq \alpha \} \cap \left\{ \tau_1(T) \geq \frac{1}{2} \right\}. \tag{46}
\]
According to the previous lemma, we have
\[
\mathbb{P}(G) \geq c_2 S_0(1 - S_0).
\]
Next, we show:

**Lemma 14.** There exists a universal constant $c > 0$ such that
\[
G \text{ holds } \Rightarrow \omega_1(T_1(\alpha)) - \omega_2(T_1(\alpha)) \geq \alpha \epsilon_0/2. \tag{47}
\]

**Proof.** We start with recalling formula (29). According to this formula, we have
\[
(\omega_1 - \omega_2)'(T) = q(A_{\tau_1(T)})^{-2} - q(S_{\tau_1(T)})^{-2}. \tag{48}
\]
Moreover, according to formula (12)
\[
T_1'(t) = \frac{d}{dt}[S(t)] = (1 - t)^{-1} q(A_t)^2.
\]
By the chain rule, we get
\[
\frac{d}{dt}(\omega_1 - \omega_2)(T_1(t)) = (1 - t)^{-1} \left( q(A_t)^{-2} - q(S_t)^{-2} \right) q(A_t)^2 = (1 - t)^{-1} \left( 1 - \frac{q(A_t)^2}{q(S_t)^2} \right). \tag{49}
\]

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Next, we observe that the function $q(\cdot)$ is bounded from above by $q(1/2) < 1$. It follows that for all $0 < t < 1$,

$$
\epsilon_t = q(S_t)^2 - q(A_t)^2 = q(S_t)^2 \left(1 - \frac{q(A_t)^2}{q(S_t)^2}\right) \leq 1 - \frac{q(A_t)^2}{q(S_t)^2}.
$$

The two last equations yield

$$
\frac{d}{dt}(\omega_1(T_1(t)) - \omega_2(T_1(t))) \geq \epsilon_t, \quad \forall 0 \leq t \leq 1.
$$

Under the assumption that $G$ holds, by integrating both sides, we get

$$
\omega_1(T_1(\alpha)) - \omega_2(T_1(\alpha)) \geq \int_0^\alpha \epsilon_t dt \geq \alpha \epsilon_0 / 2.
$$

The lemma is complete.

The next lemma helps us take advantage of the deficit $\omega_1(T) - \omega_2(T)$ in order to give a lower bound for the right hand side of equation (30).

**Lemma 15.** Let $0 < \delta < 1$. Suppose that at a certain time $0 \leq t_0 \leq \rho$, one has

$$
\omega_1(T_1(t_0)) - \omega_2(T_1(t_0)) \geq \delta.
$$

Then,

$$
\mathbb{E} \left[ \int_{\Theta^2} B(T)^q - 2 dT \Big| \mathcal{F}_{t_0} \right] \geq \frac{c \delta}{2^q} \mathbb{E} \left[ Q_{t_1}^{q+1}(1 - Q_{t_1})^{q+1} \Big| \mathcal{F}_{t_0} \right]. \tag{50}
$$

where $t_1$ is defined by the equation

$$
- \log(1 - t_1) = - \log(1 - \rho) - \delta
$$

and $c$ is a positive universal constant.

Before we prove the lemma, we will need the estimate

**Lemma 16.** There exist universal constants $c, C > 0$ such that for all $0 < s < 1$,

$$
cs(1 - s) \leq q(s) \leq Cs(1 - s) \sqrt{|\log(s(1-s))|} \tag{52}
$$

Moreover, the function $q(s)/s$ is decreasing and one has

$$
q(s) \leq Cs \sqrt{|\log s|}, \quad \forall 0 < s < 1 \tag{53}
$$

The elementary and technical proof of this lemma is postponed to the appendix.

**Proof of lemma 15** Define $\tilde{\Theta} = T_2(t_1)$. By definition, we have

$$
\omega_2(\tilde{\Theta}) = \omega_2(T_2(t_1)) = - \log(1 - t_1) = - \log(1 - \rho) - \delta. \tag{54}
$$

Thanks to equation (29) and by the assumption of the lemma we know that for all $T_1(t_0) \leq T < T_f$, one has

$$
\omega_2(T) \leq \omega_1(T) - \delta.
$$
In particular, since \( t_0 < \rho \), we may take \( T = T_1(\rho) = \Theta_1 \) in the previous equation, which gives
\[
\omega_2(\Theta_1) \leq \omega_1(\Theta_1) - \delta = -\log(1 - \rho) - \delta.
\] (55)

We conclude from equations (54) and (55) that
\[
\Theta_1 \leq \tilde{\Theta} \leq \Theta_2.
\] (56)

Moreover, since \( \omega_2(\Theta_2) = -\log(1 - \rho) \), equation (54) gives
\[
\omega_2(\Theta_2) - \omega_2(\tilde{\Theta}) = \delta.
\]
This equation written differently is just
\[
\int_{\Theta}^{\Theta_2} \omega_2'(T)dT = \delta
\]
and an application of formula (28) yields
\[
\int_{\Theta}^{\Theta_2} q(B(T))^{-2}dT = \delta.
\]
Consequently,
\[
(\Theta_2 - \tilde{\Theta}) \max_{\tilde{\Theta} \leq T \leq \Theta_2} q(B(T))^{-2} \geq \delta
\]
or in other words,
\[
\Theta_2 - \tilde{\Theta} \geq \delta \min_{\tilde{\Theta} \leq T \leq \Theta_2} q(B(T))^2.
\]
Since \( q(s) < 1 \) for all \( s \in [0, 1] \) and by the assumption \( \delta < 1 \) we get that
\[
\Theta_2 - \tilde{\Theta} \geq \delta \min_{\tilde{\Theta} \leq T \leq \Theta_2} q(B(T))^2.
\]
(here, in case that \( \tilde{\Theta} + 1 > T_f \), we define \( \min_{\tilde{\Theta} \leq T \leq \tilde{\Theta} + 1} q(B(T))^2 = 0 \)). With the estimate (52), this formula becomes
\[
\Theta_2 - \tilde{\Theta} \geq c\delta \min_{\tilde{\Theta} \leq T \leq \tilde{\Theta} + 1} B(T)^2(1 - B(T))^2
\]
for a universal constant \( c > 0 \). This implies that for all \( q > 1 \),
\[
\int_{\Theta}^{\Theta_2} B(T)^q dT \geq c\delta \min_{\tilde{\Theta} \leq T \leq \tilde{\Theta} + 1} B(T)^q(1 - B(T))^q.
\]
Now, since the expression in the integral is non-negative and by (56), we may integrate on the larger interval \( \Theta_1 < T < \Theta_2 \) and finally get
\[
\int_{\Theta_1}^{\Theta_2} B(T)^q dT \geq c\delta \min_{\tilde{\Theta} \leq T \leq \tilde{\Theta} + 1} B(T)^q(1 - B(T))^q.
\] (57)

Next, we would like to bound from below the probability that the right hand side is not too small. Define the stopping time
\[
U = \min \left\{ T \geq \tilde{\Theta}; B(T) = \frac{1}{2} \text{ or } B(T)(1 - B(T)) = B(\tilde{\Theta})(1 - B(\tilde{\Theta}))/2 \right\}.
\]
Since $B(T)$ is a martingale, in complete analogy with the derivation of equation (37), we get using an optional stopping argument
\[
\mathbb{P}(B(U) = 1/2 \mid B(\tilde{\Theta})) \geq B(\tilde{\Theta})(1 - B(\tilde{\Theta})).
\] (58)
and since there exists a constant $c_1 > 0$ such that a Brownian motion starting at $1/2$ does exist the interval $[1/4, 3/4]$ by time $1$ with probability at least $c_1$, we get
\[
\mathbb{P}\left(\min_{\tilde{\Theta} \leq T \leq \tilde{\Theta} + 1} B(T)^q(1 - B(T))^q \geq \frac{B(\tilde{\Theta})^q(1 - B(\tilde{\Theta}))^q}{2^q} \mid B(\tilde{\Theta})\right) > c_1B(\tilde{\Theta})(1 - B(\tilde{\Theta})).
\]
Combined with the previous inequality this becomes
\[
\mathbb{P}\left(\int_{\Theta_1}^{\Theta_2} B(T)^{q-2}dT \geq c\delta \frac{B(\tilde{\Theta})^q(1 - B(\tilde{\Theta}))^q}{2^q} \mid B(\tilde{\Theta})\right) \geq c_1B(\tilde{\Theta})(1 - B(\tilde{\Theta})).
\]
Together with equation (57) and with the assumption $\delta < 1$, we get
\[
\mathbb{E}\left[\int_{\Theta_1}^{\Theta_2} B(T)^{q-2}dT \mid \mathcal{F}_{t_0}\right] \geq c\delta \frac{B(\tilde{\Theta})^q(1 - B(\tilde{\Theta}))^q}{2^q} \mathbb{E}\left[B(\tilde{\Theta})^{q+1}(1 - B(\tilde{\Theta}))^{q+1} \mid \mathcal{F}_{t_0}\right] =
\]
\[
\frac{c\delta}{2^q} \mathbb{E}\left[Q_{t_1}^{q+1}(1 - Q_{t_1})^{q+1} \mid \mathcal{F}_{t_0}\right].
\]
This proves equation (50) and the proof is complete. \hfill \Box

Before we can finally prove the theorem, the only ingredient we need is a bound the right hand side of formula (50), provided in the next lemma. Roughly speaking, this lemma ensures us that when $\Theta_1$ is reached then $B(T)$ is bounded away from $0$ and from $1$ with a large enough probability.

**Lemma 17.** There exists a universal constant $c > 0$ such that for any number $q > 1$ and for all $t_0, t_1$ such that $0 \leq t_0 \leq \min(1/2, t_1)$,
\[
\mathbb{E}[Q_{t_1}^{q+1}(1 - Q_{t_1})^{q+1} \mid \mathcal{F}_{t_0}] \geq c^q S_{t_0}^{q+2}(1 - S_{t_0})^{q+2} \sqrt{1 - t_1}.
\]

**Proof.** Define a stopping time
\[
u = \min \{t \geq \tau_2(T_1(t_0)); \ Q_t(1 - Q_t) = S_{t_0}(1 - S_{t_0})/2 \text{ or } Q_t = 1/2\}.
\]
Since $Q_t$ is a martingale, and since $Q_{\tau_2(T_1(t_0))} = S_{t_0}$, we can use the optional stopping theorem with a similar argument as the one preceding equation (37) to get
\[
\mathbb{P}(Q_u = 1/2 \mid \mathcal{F}_{t_0}) \geq S_{t_0}(1 - S_{t_0}).
\] (59)
We claim that it is enough to show that if $Q_u = 1/2$ and $u < t_1$, then
\[
\mathbb{P}(Q_{t_1} \in [1/4, 3/4] \mid \mathcal{F}_u) > c\sqrt{1 - t_1}
\] (60)
for a universal constant $c > 0$, where $\tilde{F}_u$ is the $\sigma$-algebra generated by the Brownian motion $W_t$ stopped at time $\tau_1(T_2(u))$. Indeed, define the event $E = \{Q_u = 1/2\} \cup \{t_1 < u\}$. By the above equation and by the definition of $u$ we have, under the assumption that (60) holds,

$$\mathbb{E}[Q_{t_1}^{q+1}(1 - Q_{t_1})^{q+1}1_E|\mathcal{F}_{t_0}] \geq \mathbb{P}(t_1 < u|\mathcal{F}_{t_0})S_0^{q+1}(1 - S_0)^{q+1}2^{-q-1} + \mathbb{P}(Q_u = 1/2 & t_1 \geq u|\mathcal{F}_{t_0})(1/4)^{2q+4}c\sqrt{1 - t_1} \geq \mathbb{P}(Q_u = 1/2|\mathcal{F}_{t_0})c_1^{q+1}S_0^{q+1}(1 - S_0)^{q+1}\sqrt{1 - t_1} \geq c_1^{q+1}S_0^{q+2}(1 - S_0)^{q+2}\sqrt{1 - t_1},$$

for a universal constant $c_1 > 0$, which would finish the proof. We turn to prove formula (60).

In order to get an estimate regarding the distribution of $Q_{t_1}$, we recall the original definition of the process $Q_t$ in equation (14):

$$Q_t = \int_{H(A)} \gamma_{\tilde{W}_t, \sqrt{1-t}}(x)dx$$

where $\tilde{W}_t$ is a Brownian motion. According to this equation,

$$\mathbb{P}(Q_{t_1} \in [1/4, 3/4] | \tilde{F}_u) = \mathbb{P}\left(\int_{H(A)} \gamma_{\tilde{W}_{t_1}, \sqrt{1-t_1}}(x)dx \in [1/4, 3/4] | \tilde{F}_u\right).$$

The above formula clearly does not change if we project both $H(A)$ and $\tilde{W}_t$ on the direction $v(H(A))$. Therefore, we may assume that $H(A) = [\alpha, \infty)$ for some $\alpha \in \mathbb{R}$. It is easy to check that

$$|\tilde{W}_{t_1} - \alpha| < 0.1\sqrt{1-t_1} \Rightarrow \int_{H(A)} \gamma_{\tilde{W}_{t_1}, \sqrt{1-t_1}}(x)dx \in [1/4, 3/4].$$

Therefore, it is enough to show that whenever $\tilde{W}_u = \alpha$ and $u < t_1$,

$$\mathbb{P}\left(|\tilde{W}_{t_1} - \alpha| < 0.1\sqrt{1-t_1} | \tilde{W}_u\right) > c_2\sqrt{1-t_1} \tag{61}$$

for a universal constant $c > 0$. Noting that the assumption $Q_u = 1/2$ implies that $\tilde{W}_u = \alpha$, and therefore the above is just a trivial fact about the Gaussian distribution. The lemma is complete. \qed

We are finally ready to prove our robustness estimate. The proof is just a combination of the lemmas in this section.

**Proof of theorem 2** Define

$$\alpha = c_2|\log \epsilon_0|^{-1}S_0^7(1 - S_0)^7$$

and

$$G = \{|\epsilon_t - \epsilon_0| \leq \epsilon_0/2, \forall 0 \leq t \leq \alpha\} \cap \left\{\tau_1(T) \geq \frac{1}{2}\right\}.$$ 

as in equation (46) above. According to lemma 12 we have

$$\mathbb{P}(G) \geq c_2S_0(1 - S_0). \tag{62}$$
According to lemma [14], we know that

\[ G \text{ holds } \Rightarrow \omega_1(T_1(\alpha)) - \omega_2(T_1(\alpha)) \geq \alpha \epsilon_0/2. \]

Together with the legitimate assumption that \( c_2 < 1 \), it is easy to verify that the assumption \( \epsilon(A) \leq e^{-1/\rho} \) guarantees that \( \rho \geq \alpha \). Thus, we can invoke lemma [15] with \( t_0 = \alpha \) and \( \delta = \epsilon_0 \alpha/2 \) to get

\[
\mathbb{E} \left[ \int_{\Theta_1} B(T) q^{-2} dT \Bigg| G \right] \geq 2^{-q} c' \epsilon_0 \alpha \mathbb{E} \left[ Q_{t_1}^{q+1} (1 - Q_{t_1})^{q+1} \Bigg| G \right] \tag{63}
\]

where \( t_1 \) is defined in equation (51). Now, it follows from equation (29) that \( \Theta_2 = \Theta_1 \) almost surely. Using this along with equations (30) and (62) gives

\[
S_{\alpha}^q(H(A)) - S_{\alpha}^q(A) \geq \frac{1}{2} q(q - 1) \mathbb{E} \left[ \int_{\Theta_1} B(T) q^{-2} dT \Bigg| G \right] P(G) \geq \tag{64}
\]

\[
c'^{2q} (q - 1) \epsilon_0 \alpha S_0 (1 - S_0) \mathbb{E} \left[ Q_{t_1}^{q+1} (1 - Q_{t_1})^{q+1} \Bigg| G \right].
\]

Recall that, by definition of \( T \), whenever \( G \) holds one has

\[
S_{\alpha} (1 - S_{\alpha}) \geq S_0 (1 - S_0)/2
\]

so by invoking lemma [17] with \( t_0 = \alpha \) (and \( t_1 \) as we have already defined above), we get

\[
\mathbb{E} \left[ Q_{t_1}^{q+1} (1 - Q_{t_1})^{q+1} \Bigg| F_{\alpha} \right] \geq c'^{q+2} S_{\alpha}^{q+2} (1 - S_{\alpha})^{q+2} \sqrt{1 - t_1}.
\]

Remark that whenever \( G \) holds, one has by the definition of \( T \), \( S_{\alpha} (1 - S_{\alpha}) \geq S_1 (1 - S_0)/2 \). Therefore,

\[
\mathbb{E} \left[ Q_{t_1}^{q+1} (1 - Q_{t_1})^{q+1} \Bigg| G \right] \geq (c/2)^{q+2} S_0^{q+2} (1 - S_0)^{q+3} \sqrt{1 - t_1}.
\]

Combining the last formula with (64) gives

\[
S_{\alpha}^q(H(A)) - S_{\alpha}^q(A) \geq c'^q (q - 1) \epsilon_0 \alpha S_0^{q+3} (1 - S_0)^{q+3} \sqrt{1 - t_1}.
\]

Using the definition of \( \alpha \), this gives

\[
S_{\alpha}^q(H(A)) - S_{\alpha}^q(A) \geq c'^q (q - 1) \epsilon_0 \log \epsilon_0 |\log \epsilon(A)|^{-1} S_0^{q+11} (1 - S_0)^{q+11} \sqrt{1 - t_1} =
\]

\[
c'^q (q - 1) \epsilon(\alpha) |\log \epsilon(A)|^{-1} \gamma(A)^{q+11} (1 - \gamma(A))^{q+11} \sqrt{1 - t_1}.
\]

The proof of the lower bound is complete. The upper bound is proven in subsection 3.3. \( \square \)

Once theorem [2] is established, the proof of corollary [3] is reduced to a simple upper bound to \( \delta(A)^2 \) in terms of \( \epsilon(A) \).

Proof of corollary [3] Suppose that \( \gamma(A \Delta H(A)) = \delta \). Observe first that suffices to show that

\[
q(H(A)) - q(A) \geq c \delta^2 \tag{65}
\]

for a universal constant \( c > 0 \). Indeed, this assumption combined with the bound (52) would attain

\[
\epsilon(A) = q(H(A))^2 - q(A)^2 = (q(H(A)) - q(A))(q(H(A)) + q(A)) \geq \]

\[
c_1 \gamma(A)(1 - \gamma(A)) \delta^2
\]

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and plugging this fact into equations (3) and (4) would finish the proof.

We turn to prove (65) which is, in some sense, a quantitative version of claim 9. Define

\[
\theta = \frac{\int_{H(A)} x \gamma(x) \, dx}{\int_{H(A)} x \gamma(x) \, dx}.
\]

Denote \( \mu = \gamma|_A \), the restriction of the Gaussian measure to \( A \), and let \( \tilde{\mu} \) be the push-forward of \( \mu \) under the map \( x \to \langle \theta, x \rangle \). We have by definition

\[
q(A) = \left| \int_{\mathbb{R}^n} \langle \theta, x \rangle \, d\mu(x) \right| = \left| \int_{\mathbb{R}} x \, d\tilde{\mu}(x) \right|. \tag{66}
\]

Clearly, the measure \( \tilde{\mu} \) is absolutely continuous with respect to \( \gamma^1(x) \), and we may define \( m(x) = \frac{d\tilde{\mu}}{d\gamma^1}(x) \).

The choice of the direction of \( \theta \) determines that \( \int_{\mathbb{R}} x m(x) d\gamma^1(x) \geq 0 \). Let \( h(x) \) be the function of the form \( h(x) = 1_{x \geq \alpha} \) where \( \alpha \) is chosen such that \( \int h(x) d\gamma^1(x) = \int m(x) d\gamma^1(x) \). By definition

\[
q(H(A)) = \int_{\mathbb{R}} x h(x) d\gamma^1(x).
\]

The proof is reduced to showing

\[
\int_{\mathbb{R}} x(h(x) - m(x)) d\gamma^1(x) \geq c\delta^2. \tag{67}
\]

We write

\[
\int_{\mathbb{R}} x(h(x) - m(x)) d\gamma^1(x) = \int_{\mathbb{R}} (x - \alpha)(h(x) - m(x)) d\gamma^1(x).
\]

Now, by definition of \( h(x) \) and by the fact that \( 0 \leq m(x) \leq 1 \) for all \( x \), the function \( h(x) - m(x) \) has the same sign as \( (x - \alpha) \), so it is enough to prove that

\[
\int_{\mathbb{R}} |x - \alpha| g(x) d\gamma^1(x) \geq c\delta^2 \tag{68}
\]

where \( g(x) = |h(x) - m(x)| \gamma^1(x) \) and \( c > 0 \) is a universal constant. Thanks to the fact that

\[
\langle x, \theta \rangle \geq 0, \quad \forall x \in H(A) \setminus A
\]

and

\[
\langle x, \theta \rangle \leq 0, \quad \forall x \in A \setminus H(A)
\]

we learn that

\[
\int_{\mathbb{R}} g(x) \, dx = \gamma(A \Delta H(A)) = \delta.
\]

Finally, since \( \gamma^1(x) \leq 1 \) for all \( x \), we have \( g(x) \leq 1 \), and by Markov’s inequality

\[
\int_{\{ |x - \alpha| \geq \delta/4 \}} g(x) \, dx \geq \delta/2.
\]

Since \( g \) is non-negative, we get

\[
\int_{\mathbb{R}} |x - \alpha| g(x) \, dx \geq \delta^2/8.
\]

Equation (68) is proven and the corollary is established. \( \square \)
3.1 Calculation of the differential

This entire subsection, which is the first step in the proof of proposition 13, is dedicated to the calculation of the differential of the process

\[ \epsilon_t = q(S_t)^2 - q(A_t)^2. \]

It will merely be a straightforward calculation for which we recruit the formula (8) to our service. Before we begin the calculation, we introduce a few definitions and recall some facts from section 2.

Our starting point is formula (9), which reads

\[ dS_t = (1 - t)^{-1} \left\langle \int_A (x - W_t) F_t(x) dx, dW_t \right\rangle. \]

Define,

\[ V_t = \int_A (x - W_t) F_t(x) dx \]

so

\[ dS_t = (1 - t)^{-1} \langle V_t, dW_t \rangle \]

and

\[ d[S]_t = (1 - t)^{-2} |V_t|^2 dt \quad (69) \]

It will also be convenient to define the linear map

\[ L_t(x) := \frac{x - W_t}{\sqrt{1 - t}} \]

so that \( L_t \) pushes forward the measure whose density is \( F_t(x) \) to the standard Gaussian measure. Also note that \( A_t = L_t A \). By substituting \( y = L_t x \), we have

\[ d[S]_t = (1 - t)^{-2} \left| \int_A (x - W_t) F_t(x) dx \right|^2 dt = (1 - t)^{-1} \left| \int_{A_t} yd\gamma(y) \right|^2 \]

so together with equation (69), we have

\[ (1 - t)^{-1} q(A_t)^2 = (1 - t)^{-2} |V_t|^2. \]

This encourages us to define

\[ U_t = \frac{V_t}{\sqrt{1 - t}}, \quad u_t = \frac{U_t}{|U_t|} \]

so that

\[ U_t = \int_{A_t} x d\gamma(x), \quad q(A_t) = |U_t|. \quad (70) \]

So far, we have established that

\[ \epsilon_t = q(S_t)^2 - |U_t|^2 = q(S_t)^2 - \frac{|V_t|^2}{1 - t}. \quad (71) \]
We are finally ready to begin differentiating, and we start with the second term. We first calculate

\[ dV_t = d \int_A (x - W_t) F_t(x) \, dx = \]

\[ - \int_A dW_t F_t(x) dx + \int_A (x - W_t) dF_t(x) dx - \int_A d[W, F(x)]_t dx = \]

\[ - \int_A F_t(x) dW_t + (1 - t)^{-1} \int_A (x - W_t)(x - W_t, dW_t) F_t(x) dx - \]

\[ (1 - t)^{-1} \int_A (x - W_t) F_t(x) dx dt = \]

(substituting \( x \rightarrow L_t(x) \) in the first and second terms)

\[-\gamma(A_t) dW_t + \left( \int_A x \otimes x d\gamma(x) \right) dW_t - (1 - t)^{-1} V_t dt = \]

\[ B_t dW_t - (1 - t)^{-1} V_t dt \]

where

\[ B_t = \int_A (x \otimes x - \text{Id}) \gamma(x) dx. \]  

(72)

Next, we have

\[ d \left( \frac{|V_t|^2}{1 - t} \right) = \frac{-2|V_t|^2 dt}{1 - t} + 2\langle V_t, B_t dW_t \rangle + \|B_t\|_{HS}^2 dt. \]

where the last term stands for the squared Hilbert-Schmidt norm of \( B_t \). And so

\[ d \left( \frac{|V_t|^2}{1 - t} \right) = -(1 - t)^{-2}|V_t|^2 dt + 2(1 - t)^{-1} \langle B_t V_t, dW_t \rangle + (1 - t)^{-1} \|B_t\|_{HS}^2 dt. \]  

(73)

In other words,

\[ d|U_t|^2 = -(1 - t)^{-1}|U_t|^2 dt + 2(1 - t)^{-1/2} \langle B_t U_t, dW_t \rangle |U_t| + (1 - t)^{-1} \|B_t\|_{HS}^2 dt. \]  

(74)

Our next goal is to calculate the differential of the term

\[ q(S_t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Psi(S_t)} xe^{-x^2/2} dx \]

where \( \Psi(s) = \Phi^{-1}(s) \) is the inverse Gaussian cumulative distribution function. First, we calculate the derivatives of the function \( q(\cdot) \):

\[ q'(s) = -\frac{1}{\sqrt{2\pi}} \Psi'(s) \Psi(s) \exp(-\Psi(s)^2/2) = \]

\[ -\frac{1}{\sqrt{2\pi}} \Phi'(\Psi(s)) \Psi(s) \exp(-\Psi(s)^2/2) = \]

\[ -e^{\Psi(s)^2/2} e^{-\Psi(s)^2/2} \Psi(s) = -\Psi(s). \]

Also

\[ q''(s) = -\Psi'(s) = -\frac{1}{\Phi'(\Psi(s))} = -\frac{1}{\sqrt{2\pi}} e^{\Psi(s)^2/2}. \]  

(76)
Indeed, by integration by parts

\[ dq(S_t) = q'(S_t) dS_t + \frac{1}{2} q''(S_t) d[S]_t = -\Psi(S_t) dS_t - (1 - t)^{-2} \sqrt{\pi/2} e^{\Psi(S_t)^2/2} |V_t|^2 dt. \]

Next, we observe the identity,

\[ -\Psi(x) = q(x)^{-1} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Psi(x)} (s^2 - 1) e^{-s^2/2} ds. \quad (77) \]

Indeed, by integration by parts

\[ \int_{-\infty}^{\Psi(x)} s^2 e^{-s^2/2} ds = -\Psi(x) e^{-\Psi(x)^2/2} + \int_{-\infty}^{\Psi(x)} e^{-s^2/2} ds = -\Psi(x) e^{-\Psi(x)^2/2} + \sqrt{2\pi} x, \]

so

\[ \int_{-\infty}^{\Psi(x)} (s^2 - 1) e^{-s^2/2} ds = -\Psi(x) e^{-\Psi(x)^2/2}. \]

Moreover,

\[ q(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Psi(x)} s e^{-s^2/2} ds = \frac{1}{\sqrt{2\pi}} e^{-\Psi(x)^2/2}. \quad (78) \]

Combining these two equalities yields (77). Plugging (77) and (78) into the formula for \( dq(S_t) \) above gives,

\[ dq(S_t) = q(S_t)^{-1} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Psi(S_t)} (s^2 - 1) e^{-s^2/2} ds \right) dS_t - \frac{1}{2} (1 - t)^{-2} q(S_t)^{-1} |V_t|^2 dt = q(S_t)^{-1} \left( \xi(S_t) dS_t - \frac{1}{2} (1 - t)^{-2} |V_t|^2 dt \right) \]

where

\[ \xi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Psi(x)} (s^2 - 1) e^{-s^2/2} ds. \]

We continue calculating

\[ dq^2(S_t) = 2q(S_t) dq(S_t) + d[q(S)]_t = 2\xi(S_t) dS_t - (1 - t)^{-2} |V_t|^2 dt + (1 - t)^{-2} \xi(S_t)^2 \frac{|V_t|^2}{q(S_t)^2} dt. \quad (79) \]

The reader may note the similarity between \( \xi(S_t) \) and the matrix \( B_t \) which encourages us to define

\[ \tilde{B}_t := \int_{H(A_t)} (x \otimes x - \text{Id}) \ d\gamma(x). \]

It is straightforward to verify that for all \( v \in \mathbb{R}^n \) one has \( \tilde{B}_t v = \xi(S_t) u_t (v, u_t) \), which gives

\[ (1 - t)^{-1/2} \left\langle \tilde{B}_t u_t, dW_t \right\rangle |U_t| = (1 - t)^{-1/2} \xi(S_t) \langle U_t, dW_t \rangle = \xi(S_t) dS_t. \]
Formula (79) becomes
\[ d \left( q(S_t)^2 \right) = 2(1 - t)^{-1/2} \left( \langle \tilde{B}_t u_t, dW_t \rangle |U_t| - (1 - t)^{-1}|U_t|^2dt + (1 - t)^{-1}||\tilde{B}_t||_{HS}^2 \frac{|U_t|^2}{q(S_t)^2}dt \]

Combining the last equation with (71) and (74) finally gives
\[ de_t = d \left( q(S_t)^2 - |U_t|^2 \right) = 2(1 - t)^{-1/2}|U_t| \left( \langle \tilde{B}_t - B_t \rangle u_t, dW_t \rangle - (1 - t)^{-1}||B_t||_{HS}^2dt + (1 - t)^{-1}||\tilde{B}_t||_{HS}^2 \frac{|U_t|^2}{q(S_t)^2}dt. \]

### 3.2 Bounding the differential

This subsection is dedicated to bounding the right hand side of equation (80) in terms of \( \epsilon_t \) and \( S_t \), thus proving proposition [13].

The proof of this bound will be carried out in three main lemmas, each of which uses a different idea. A glance at formula (80) shows that, in order for the differential of \( \epsilon_t \) to be small (in the sense of both drift and quadratic variation) one should show that the matrices \( B_t \) and \( \tilde{B}_t \) are quite close to each other in a certain sense.

Recall that \( \tilde{B}_t \) is a rank-one matrix of the form \( \alpha u_t \otimes u_t \) for a constant \( \alpha \in \mathbb{R} \). We should therefore expect that the matrix \( \tilde{B} \) is close to such a rank-one matrix.

Define \( E = \text{sp}\{u_t\} \) and let \( P_E, P_{E^\bot} \) be the orthogonal projections onto \( E \) and \( E^\bot \), respectively. Our first lemma (lemma [18] below) is of one-dimensional nature, and will provide a bound for \( \|P_E(B - \tilde{B})P_E\|_{HS} \). Next, lemma [19] will essentially be the only place in this note where the high dimension plays a role, and will give a bound for \( \|P_{E^\bot}BP_{E^\bot}\|_{HS} \). Finally, in lemma [22] which is of a two-dimensional nature, we give a bound for \( \|P_EBP_E\|_{HS} \).

In all the proofs of this section, the time \( t \) will be fixed, so the reader may consider the set \( A_t \) as an arbitrary fixed set. For convenience, we repeat a few definitions which will be used intensively in our proofs. First of all,

\[ B_t = \int_{A_t} (x \otimes x - \text{Id}) \gamma^n(x)dx. \]

and

\[ \tilde{B}_t = \int_{H(A_t)} (x \otimes x - \text{Id}) d\gamma^n(x) = \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\Psi(S_t)} (s^2 - 1)e^{-s^2/2}ds \right) u_t \otimes u_t \]

where

\[ S_t = \gamma^n(A_t) = \gamma^n(H(A_t)) \]

and

\[ u_t = \frac{\int_{H(A_t)} xd\gamma^n(x)}{\left| \int_{H(A_t)} xd\gamma^n(x) \right|}. \]

Moreover,

\[ \epsilon_t = q(S_t)^2 - q(A_t)^2 = \left| \int_{H(A_t)} xd\gamma^n(x) \right|^2 - \left| \int_{A_t} xd\gamma^n(x) \right|^2. \] (81)

Finally, it will be useful to recall that

\[ H(A(t)) = \{ x; \langle x, u_t \rangle \geq -\Psi(S_t) \}. \]

We begin with:
Lemma 18. For all \(0 \leq t \leq 1\), one has
\[ \left| \left\langle \left( B_t - \tilde{B}_t \right) u_t, u_t \right\rangle \right| \leq \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t \sqrt{\log \epsilon_t}. \]  \hfill (82)
for a universal constant \(C > 0\).

Proof. Let \(f : \mathbb{R} \to [0, 1]\) be the unique continuous function satisfying, for all measurable subsets \(W \subset \mathbb{R}\),
\[ \int_W f(x) d\gamma^1(x) = \int_{A_t} 1_{\{x, u_t \in W\}}(x) d\gamma^n(x) \]
hence \(f\) is the density with respect to the Gaussian measure of the marginal on \(sp\{u_t\}\) of the standard Gaussian measure restricted to the set \(A_t\), and similarly define \(h(x)\) by
\[ \int_W h(x) d\gamma^1(x) = \int_{H(A_t)} 1_{\{x, u_t \in W\}}(x) d\gamma^n(x). \]
By definition, we get
\[ \int_{\mathbb{R}} f(x) d\gamma^1(x) = \int_{\mathbb{R}} h(x) d\gamma^1(x) = S_t \]  \hfill (83)
and
\[ \left\langle B_t u_t, u_t \right\rangle = \int_{\mathbb{R}} (x^2 - 1) f(x) d\gamma^1(x), \quad \left\langle \tilde{B}_t u_t, u_t \right\rangle = \int_{\mathbb{R}} (x^2 - 1) h(x) d\gamma^1(x). \]
Next, define
\[ g(x) = h(x) - f(x). \]
Equation (83) teaches us that \(\int g(x) d\gamma^1(x) = 0\), which gives
\[ \left\langle \left( B_t - \tilde{B}_t \right) u_t, u_t \right\rangle = \int_{\mathbb{R}} (x^2 - 1) g(x) d\gamma^1(x) = \int_{\mathbb{R}} x^2 g(x) d\gamma^1(x). \]  \hfill (84)
We claim that in order to complete the proof, it is enough to show that
\[ \int_{\mathbb{R}} x^2 g(x) d\gamma^1(x) \leq \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t \sqrt{\log \epsilon_t} \]  \hfill (85)
for a universal constant \(C > 0\). Indeed, observe that for all \(B \subset \mathbb{R}^n\), one has \(\varepsilon(B) = \varepsilon(B^C)\). Consequently, the right hand side of formula (82) remains invariant if we replace \(A_t\) by \(A_t^C\). Therefore, if the left hand side of the above equation is negative, we may replace \(g(x)\) with \(-g(-x)\) which corresponds to replacing \(A_t\) by \(A_t^C\) and continue as usual.

Denote \(\delta = \int x g(x) d\gamma^1(x)\). We have
\[ \delta = \int_{H(A_t)} \langle x, u_t \rangle d\gamma(x) - \int_{A_t} \langle x, u_t \rangle d\gamma(x) = q(S_t) - q(A_t) = \frac{\epsilon_t}{q(S_t) + q(A_t)}. \]
which, together with the bound (52) and the fact that \(q(s) < 1\) for all \(0 < s < 1\) implies that
\[ \epsilon_t/2 \leq \delta \leq \epsilon_t/q(S_t) \leq \frac{C}{S_t(1 - S_t)} \epsilon_t \]  \hfill (86)
for a universal constant \(C > 0\). Define
\[ p = 100 \sqrt{\log \delta}. \]
The fact that \( \int_\mathbb{R} (x^2 + 1) \, d\gamma^1(x) < \infty \) implies that the left hand side of (82) is always smaller than a universal constant, therefore we remark that if \( \varepsilon_t \geq S_t^2 (1 - S_t)^2 \) then this formula holds trivially and there is nothing to prove. Consequently, we may assume that \( \delta \leq S_t(1 - S_t) \). A well-known estimate about the Gaussian distribution is

\[
|\Psi(S_t)| \leq 10 \sqrt{\log((S_t)(1 - S_t))} \leq 10 \sqrt{\log \delta}.
\]

And therefore,

\[
p \geq 10 |\Psi(S_t)|. \tag{87}
\]

Define \( L = -\Psi(S_t) \) so that \( h(x) = 1_{x \geq L} \). Clearly, \( g(x) \geq 0 \) for \( x > L \) and \( g(x) \leq 0 \) for \( x < L \). We have

\[
\int_\mathbb{R} x^2 g(x) \, d\gamma^1(x) = \int_\mathbb{R} (x - L)^2 g(x) \, d\gamma^1(x) + 2L \delta \leq (88)
\]

and also

\[
\int_L^\infty (x - L) g(x) \, d\gamma^1(x) \leq \delta. \tag{89}
\]

First, we estimate

\[
\int_p^\infty (x - L)^2 g(x) \, d\gamma^1(x) \leq \int_p^\infty (x - L)^2 \, d\gamma^1(x) \leq (88)
\]

(according to equation (87))

\[
\int_p^\infty (x + p)^2 \, d\gamma^1(x) \leq 4 \int_p^\infty x^2 \, d\gamma^1(x) = \frac{4}{\sqrt{2\pi}} \int_p^\infty x^2 e^{-x^2/2} \, dx =
\]

(integration by parts)

\[
4pe^{-p^2/2} + 4(1 - \Phi(p)) \leq 10pe^{-p^2/2} \leq \delta^2.
\]

(where we use the legitimate assumption that \( \delta \) is smaller than some universal constant, justified above). On the other hand, using (87) and (89), we have

\[
\int_L^p (x - L)^2 g(x) \, d\gamma^1(x) \leq (p - L) \int_L^p (x - L) g(x) \, d\gamma^1(x) \leq 2p \delta.
\]

The last two equations with (88) give

\[
\int_\mathbb{R} x^2 g(x) \, d\gamma^1(x) \leq 5p \delta = 500 \delta \sqrt{\log \delta}.
\]

Equation (86) now tells us that

\[
\int_\mathbb{R} x^2 g(x) \, d\gamma^1(x) \leq \frac{2C}{S_t(1 - S_t) \varepsilon_t} \sqrt{\log \varepsilon_t}.
\]

Thus, equation (85) holds and the proof is complete. \( \Box \)

Recall that we denote by \( P_{E^\perp} \) the orthogonal projection onto \( E^\perp = \text{sp}\{u_t\}^\perp \). Next, we would like to prove
Lemma 19. For all $0 < t < 1$,
\[
\|P_{E^\perp} B_t P_{E^\perp}\|_{HS}^2 \leq \frac{C}{S_t^2 (1 - S_t)^2 t^\epsilon}
\]
where $C > 0$ is a universal constant.

Before we prove this lemma, we first need

Lemma 20. Let $0 < h \leq 1$ and let $f: \mathbb{R}^{n-1} \rightarrow [0, 1]$ be such that
\[
\int_{\mathbb{R}^{n-1}} f(x) d\gamma^{n-1}(x) = h.
\]
Then
\[
\int_{\mathbb{R}^{n-1}} \frac{f(x)}{h} \log \left( \frac{f(x)}{h} \right) d\gamma^{n-1}(x) \leq h^{-2} \int_{\mathbb{R}^{n-1}} (q(h) - q(f(x))) d\gamma^{n-1}(x).
\]

For this lemma we will need the following fact, whose simple yet technical proof is postponed to the appendix.

Fact 21. For all $0 \leq h, s \leq 1$,
\[
-\frac{4}{h^2 (1 - h)^2} (s - h)^2 \leq q(s) - q(h) - q'(h)(s - h) \leq -(s - h)^2.
\]
where $C > 0$ is a universal constant.

Proof of lemma 20. An application of fact 21 gives
\[
\int_{\mathbb{R}^{n-1}} (q(h) - q(f(x))) d\gamma^{n-1}(x) = \int_{\mathbb{R}^{n-1}} (q(h) - q(f(x)) + q'(h)(f(x) - h)) d\gamma(x) \geq \int_{\mathbb{R}^{n-1}} (f(x) - h)^2 d\gamma^{n-1}(x) = h^2 \int_{\mathbb{R}^{n-1}} \left( \frac{f(x)}{h} - 1 \right)^2 d\gamma^{n-1}(x).
\]
(90)

On the other hand, it is easy to check that one has,
\[
s \log s - (s - 1) \leq (s - 1)^2
\]
for all $s \geq 0$. Consequently,
\[
\int_{\mathbb{R}^{n-1}} \frac{f(x)}{h} \log \left( \frac{f(x)}{h} \right) d\gamma^{n-1}(x) = \int_{\mathbb{R}^{n-1}} \left( \frac{f(x)}{h} \log \left( \frac{f(x)}{h} \right) - \left( \frac{f(x)}{h} - 1 \right) \right) d\gamma^{n-1}(x) \leq \int_{\mathbb{R}^{n-1}} \left( \frac{f(x)}{h} - 1 \right)^2 d\gamma^{n-1}(x).
\]
(91)
A combination of (90) and (91) finishes the proof. \qed
Proof of lemma 19} We begin with the observation that

$$B_t = \int_{A_t} (x \otimes x - \text{Id}) \, d\gamma^1(x) = - \int_{A_t^c} (x \otimes x - \text{Id}) \, d\gamma^1(x).$$

Moreover \(q(A_t) = q(A_t^c).\) Thanks to this, the statement of the lemma remains invariant when replacing that set \(A_t\) with the set \(A_t^c.\) Consequently, it is legitimate to make the assumption

$$S_t \geq \frac{1}{2}. \tag{92}$$

Let \(\mu\) be the measure \(\gamma^n|_{A_t}\), hence the Gaussian measure restricted to the set \(A_t\). Define by \(\tilde{\mu}\) and \(\tilde{\gamma}\) the push-forward under \(P_{E^\perp}\) of the measures \(\mu\) and \(\gamma^n\) respectively. Define the function \(f : E^\perp \to \mathbb{R}\) by,

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{(y+E) \cap A_t} \exp \left(- \frac{1}{2} \langle x, u_t \rangle^2 / 2 \right) \, dx$$

(here \(dx\) stands for the 1-dimensional Hausdorff measure on \(y + E\)). One can verify that for \(y \in E^\perp\), this function satisfies

$$\frac{d\tilde{\mu}}{d\tilde{\gamma}}(y) = f(y).$$

Thus \(\int_{E^\perp} f(y) \, d\tilde{\gamma}(y) = \gamma(A_t) = S_t\). An application of lemma 20 and of equation (92) now gives

$$\int_{E^\perp} f(y) \log \left( \frac{f(y)}{S_t} \right) \, d\tilde{\gamma}(y) \leq 4 \left( q(S_t) - \int_{E^\perp} q(f(y)) \, d\tilde{\gamma}(y) \right). \tag{93}$$

Now, it follows from claim 9 that for all \(y \in E^\perp,\)

$$\frac{1}{\sqrt{2\pi}} \left| \int_{(y+E) \cap A_t} \langle x, u_t \rangle \exp \left(- \frac{1}{2} \langle x, u_t \rangle^2 / 2 \right) \, dx \right| \leq q \left( \frac{1}{\sqrt{2\pi}} \int_{(y+E) \cap A_t} \exp \left(- \frac{1}{2} \langle x, u_t \rangle^2 / 2 \right) \, dx \right) = q(f(y)).$$

Integrating this inequality over \(E^\perp\) with respect to \(\tilde{\gamma}\) and using equation (81) gives

$$\int_{E^\perp} q(f(y)) \, d\tilde{\gamma}(y) \geq q(A_t)$$

Combining this with equation (93) gives

$$\int_{E^\perp} f(y) \log \left( \frac{f(y)}{S_t} \right) \, d\tilde{\gamma}(y) \leq 4(q(S_t) - q(A_t)) \leq \frac{4}{q(S_t)} \epsilon_t \leq \frac{C}{S_t (1 - S_t) \epsilon_t} \tag{94}$$

for a universal constant \(C > 0\) (in the last inequality we used formula (52)). The above equation allows us to use Talagrand’s transportation-entropy inequality (11) which teaches us that there exists a function \(T : E^\perp \to E^\perp\) which pushes forward the measure \(\tilde{\gamma}\) to the measure \(S_t^{-1} \tilde{\mu}\) and such that,

$$\int_{E^\perp} |T(y) - y|^2 \, d\tilde{\gamma}(y) \leq \frac{2C}{S_t (1 - S_t) \epsilon_t}.\$$

Denote \(D = P_{E^\perp} B_t P_{E^\perp}\). Let \(X\) be a random vector whose law is \(\tilde{\gamma}\), then by definition

$$D = S_t (\text{Cov}(T(X)) - \text{Cov}(X))$$
where \( \text{Cov}(Y) := \mathbb{E}[(Y - \mathbb{E}[Y]) \otimes (Y - \mathbb{E}[Y])] \) is the covariance matrix of a vector \( Y \) (here, we use the fact that \( \int_{A_t} P_{E \perp x} d\gamma(x) = 0 \)). Let \( e_1, ..., e_{n-1} \) be an orthogonal basis of \( E \perp \) which diagonalizes \( D \). We have, for all \( 1 \leq i \leq n - 1 \),

\[
\text{Var}(\langle X + T(X), e_i \rangle) \leq 8 \text{Var}(\langle X, e_i \rangle) + 2 \text{Var}(\langle X - T(X), e_i \rangle) \leq 8 + 2 \epsilon_t \leq C' \epsilon_t \leq C' S_t (1 - S_t) \epsilon_t,
\]

for a universal constant \( C' > 0 \). We calculate,

\[
\|D\|_{HS}^2 = S_t^2 \sum_{i=1}^{n} (\text{Var}(\langle T(X), e_i \rangle) - \text{Var}(\langle X, e_i \rangle))^2 = \]

\[
S_t^2 \sum_{i=1}^{n} \text{Cov}(\langle T(X) - X, e_i \rangle, \langle T(X) + X, e_i \rangle)^2 \leq \]

\[
S_t^2 \sum_{i=1}^{n} \text{Var}(\langle T(X) - X, e_i \rangle) \text{Var}(\langle T(X) + X, e_i \rangle) \leq \frac{C}{S_t(1 - S_t)} \mathbb{E}|T(X) - X|^2 \leq \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t,
\]

and we are done. \( \square \)

Our last lemma concerns with the part of the matrix \( B \) which is ”off-diagonal” with respect to \( E, E \perp \).

**Lemma 22.** For any \( 0 \leq t < 1 \) and for any unit vector \( v \) with \( v \perp u_t \), one has

\[
|\langle v, B_t u_t \rangle| \leq \frac{C}{S_t^3(1 - S_t)^3} \epsilon_t \sqrt{\log \epsilon_t}
\]

where \( C > 0 \) is a universal constant.

**Proof.** Let \( \mu \) be the measure whose density is \( \gamma^n|_{A_t} \). Denote \( F = \text{sp}\{v, u_t\} \), and let \( \tilde{\mu} \) and \( \tilde{\gamma} \) be the push-forward of \( \mu \) and \( \gamma \) under the orthogonal projection onto \( F \) respectively. Define,

\[
f(x, y) = \frac{d\tilde{\mu}}{d\tilde{\gamma}}(xu_t + yv), \quad \forall (x, y) \in \mathbb{R}^2.
\]

By definition, we have

\[
\langle v, B_t u_t \rangle = \int_{A_t} \langle x, v \rangle \langle x, u_t \rangle d\gamma(x) = \int_{\mathbb{R}^2} xy f(x, y) d\gamma^2(x,y).
\]

so our objective is to bound the right hand side of the above equation. For every \( y \in \mathbb{R} \), we write

\[
g(y) = \int_{\mathbb{R}} f(x, y) d\gamma^1(x),
\]

the density of the marginal of \( f(x, y) \) onto the \( y \) coordinate with respect to the Gaussian measure, and

\[
Q(y) = \int_{\mathbb{R}} xf(x, y) d\gamma^1(x).
\]
Equation (95) becomes
\[ (v, B_t u_t) = \int_{\mathbb{R}} yQ(y) d\gamma^1(y). \] (96)

By equation (81), we know that
\[ q(A_t) = \left| \int_{A_t} \langle x, u_t \rangle d\gamma(x) \right| = \left| \int_{\mathbb{R}^2} x f(x, y) d\gamma^2(x, y) \right| = \left| \int_{\mathbb{R}} Q(y) d\gamma^1(y) \right|. \]

So, by the definition of \( \epsilon_t \),
\[ q^2(S_t) - \left| \int_{\mathbb{R}} Q(y) d\gamma^1(y) \right|^2 = \epsilon_t. \] (97)

Since \(|f(x, y)| \leq 1\) for all \((x, y) \in \mathbb{R}^2\) and by lemma [10] we have
\[ |Q(y)| \leq q(g(y)), \forall y \in \mathbb{R}. \] (98)

The two above equations give
\[ q^2(S_t) - \left( \int_{\mathbb{R}} q(g(y)) d\gamma^1(y) \right)^2 \leq \epsilon_t, \]
and therefore
\[ q(S_t) - \int_{\mathbb{R}} q(g(y)) d\gamma^1(y) \leq \epsilon_t / q(S_t). \]

Next, observe that \( \int_{\mathbb{R}} g(y) d\gamma = S_t \). Using fact [21] this yields
\[ q(S_t) - \int_{\mathbb{R}} q(g(y)) d\gamma^1(y) = - \int_{\mathbb{R}} (g(y)) - q(S_t) - q'(S_t)(g(y) - S_t)) d\gamma^1(y) \geq \int_{\mathbb{R}} (g(y) - S_t)^2 d\gamma^1(y), \] (99)

so
\[ \int_{\mathbb{R}} (g(y) - S_t)^2 d\gamma^1(y) \leq \epsilon_t / q(S_t). \] (100)

Next, by definition of the vector \( u_t \), we know that the center of mass of \( \tilde{\mu} \) is orthogonal to \( v \), which implies that \( \int_{\mathbb{R}} y g(y) d\gamma^1(y) = 0 \). This gives,
\[ \int_{\mathbb{R}} yg(g(y)) d\gamma^1(y) = \int_{\mathbb{R}} y(q(g(y)) - q(S_t) - q'(S_t)(g(y) - S_t)) d\gamma^1(y) \] (101)

and, by fact [21]
\[ \left| \int_{\mathbb{R}} yg(g(y)) d\gamma^1(y) \right| \leq \frac{4}{S_t^2(1 - S_t)^2} \int_{\mathbb{R}} |y|(g(y) - S_t)^2 d\gamma^1(y). \] (102)

We claim that the last equation combined with (100) gives,
\[ \left| \int_{\mathbb{R}} yg(g(y)) d\gamma^1(y) \right| \leq \frac{4}{S_t^2(1 - S_t)^2} q(\epsilon_t) / q(S_t). \] (103)

Indeed, observe that for all \( y \), the quantity \((g(y) - S_t)^2\) is smaller than 1. We invoke lemma [10] with \( m(y) = (g(y) - S_t)^2 \) and use the bound (100) to get
\[ \int_{\mathbb{R}} |y|(g(y) - S_t)^2 d\gamma^1(y) = \int_{-\infty}^{\infty} y(g(y) - S_t)^2 d\gamma^1(y) - \int_{-\infty}^{0} y(g(y) - S_t)^2 d\gamma^1(y) \leq \]

33
\[
2 \int_{-\infty}^{\Psi(\epsilon_t/q(S_t))} |y| d\gamma(y) = 2q(\epsilon_t/q(S_t)).
\]

In the last equality, we have used the legitimate assumption that \(\epsilon_t < \frac{1}{2}\). Equation (103) now follows from the sub-linearity of \(q(\cdot)\) suggested by equation (75).

Next, another application of claim 9 on the set \(\{(x, y); x \leq \Psi(g(y))\}\) teaches us that

\[
q(S_t) - \int_{\mathbb{R}} q(g(y)) d\gamma(y) \geq 0.
\]

Combining this fact with (97) suggests that

\[
0 \leq \left( \int_{\mathbb{R}} q(g(y)) d\gamma(y) \right)^2 - \left( \int_{\mathbb{R}} Q(y) d\gamma(y) \right)^2 \leq \epsilon_t.
\] (104)

Next, we note that the assumption

\[
\epsilon_t \leq q(S_t)/2
\] (105)

is a legitimate one. Indeed, one has \(\int_{\mathbb{R}^2} |xy| d\gamma(x, y) < \infty\) which, thanks to equation (95) teaches us that

\[
\langle v, B_t u_t \rangle \leq C_1
\]

for some universal constant \(C_1 > 0\). The estimate (52) ensures us that if \(\epsilon_t \geq q(S_t)/2\) then the quantity \(q(\epsilon_t)/S_t(1 - S_t)\) is larger than a universal constant, which would imply the result of the lemma, so the assumption can be made. Using assumption (105) with equation (104) yields

\[
\int_{\mathbb{R}} (q(g(y)) - Q(y)) d\gamma(y) \leq \frac{\epsilon_t}{\int_{\mathbb{R}} q(g(y)) d\gamma(y)} \leq 2\epsilon_t/q(S_t).
\]

According to (95) and since \(q(s) \leq q(1/2)\) for \(0 < s < 1\), we have \(0 \leq q(g(y)) - Q(y) \leq 1\). Thus, in a similar way that (102) implied (103), the above equation implies

\[
\left| \int_{\mathbb{R}} y(q(g(y)) - Q(y)) d\gamma(y) \right| \leq 4q(\epsilon_t)/q(S_t).
\]

Finally, combine the above equation with (96) and (103) to get

\[
|\langle v, B_t u_t \rangle| = \left| \int_{\mathbb{R}} yQ(y) d\gamma(y) \right| \leq
\]
\[
\left| \int_{\mathbb{R}} y(Q(y) - q(g(y)) d\gamma(y) \right| + \left| \int_{\mathbb{R}} yq(g(y)) d\gamma(y) \right| \leq
\]
\[
\left( \frac{4}{S_t^2(1 - S_t)^2} + 4 \right) q(\epsilon_t)/q(S_t).
\]

Using the estimate (52) completes the proof.

We are now ready to prove the main proposition of the section.
**Proof of proposition 13.** The proof is just a combination of the lemmas in the section together with equation (80), which we rewrite the equation for the convenience of the reader:

\[
d\epsilon_t = d \left( q(S_t)^2 - |U_t|^2 \right) =
\]

\[
2(1 - t)^{-1/2} |U_t| \left\langle \left( \tilde{B}_t - B_t \right) u_t, dW_t \right\rangle - (1 - t)^{-1} \|B_t\|_{\text{HS}}^2 dt + (1 - t)^{-1} \|\tilde{B}_t\|_{\text{HS}}^2 \frac{|U_t|^2}{q(S_t)^2} dt.
\]

Denote,

\[
\alpha_t = 2(1 - t)^{-1/2} |U_t| \left( \tilde{B}_t - B_t \right) u_t
\]

and

\[
\beta_t = (1 - t)^{-1} \left( \|\tilde{B}_t\|_{\text{HS}}^2 \frac{|U_t|^2}{q(S_t)^2} - \|B_t\|_{\text{HS}}^2 \right)
\]

so that

\[
d\epsilon_t = \langle \alpha_t, dW_t \rangle + \beta_t dt.
\]

Since \( |U_t| \leq q(1/2) \), in order to prove part (i) of the proposition it suffices to show that

\[
\left| \left( B_t - \tilde{B}_t \right) u_t \right| < \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t \sqrt{\log \epsilon_t}
\]

for a universal constant \( C > 0 \). By the triangle inequality,

\[
\left| \left( B_t - \tilde{B}_t \right) u_t \right| \leq \left| \langle \left( B_t - \tilde{B}_t \right) u_t, u_t \rangle \right| + \max_{\|v\| = 1} |\langle v, B_t u_t \rangle|.
\]

A combination of lemmas 18 and 22 establishes (107). In order to prove part (ii) of the proposition, we write

\[
\|B\|_{\text{HS}}^2 = \|PE_{\perp}BtPE_{\perp}\|_{\text{HS}}^2 + \max_{\|v\| = 1} |\langle v, B_t u_t \rangle|^2 + \|\langle B_t u_t, u_t \rangle\|^2.
\]

Also, by definition of the rank-one matrix \( \tilde{B}_t \), we have

\[
\|\tilde{B}_t\|_{\text{HS}}^2 = \left| \langle \tilde{B}_t u_t, u_t \rangle \right|^2.
\]

These two facts combined and the triangle inequality give

\[
(1 - t)|\beta_t| = \left| \left( \|\tilde{B}\|_{\text{HS}}^2 - \|B\|_{\text{HS}}^2 \right) + \|\tilde{B}\|_{\text{HS}}^2 \left( \frac{|U_t|^2}{q(S_t)^2} - 1 \right) \right| \leq \frac{1}{q(S_t)^2} \left| \langle \tilde{B}_t u_t, u_t \rangle \right|^2 \epsilon_t
\]

\[
\|PE_{\perp}B_t PE_{\perp}\|_{\text{HS}}^2 + \max_{\|v\| = 1} |\langle v, B_t u_t \rangle|^2 + \left| \langle \left( B_t - \tilde{B}_t \right) u_t, u_t \rangle \right| \left| \langle \left( B_t + \tilde{B}_t \right) u_t, u_t \rangle \right| + \frac{1}{q(S_t)^2} \left| \langle \tilde{B}_t u_t, u_t \rangle \right|^2 \epsilon_t
\]

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We turn to estimate each term separately. First we remark that by the triangle inequality
\[ B_t^2 \leq \left( \int_{\mathbb{R}^n} x \otimes x d\gamma(x) + \text{Id} \right)^2 \leq 4\text{Id}. \] (109)
Therefore,
\[ |\langle B_t u_t, u_t \rangle| \leq 2 \]
and, analogously,
\[ |\langle \tilde{B}_t u_t, u_t \rangle| \leq 2. \] (110)
These two equations together with lemma 18 give
\[ \| (B_t - \tilde{B}_t) u_t, u_t \|_2 \leq \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t \sqrt{\log \epsilon_t}. \] (111)
Equation (109) also teaches us that \[ |\langle v, B_t u_t \rangle| \leq 4 |\langle v, B_t u_t \rangle|. \] We now use lemma 22 and lemma 19 which together give
\[ \| P_{E \perp} B_t P_{E \perp} \|_{HS}^2 + \max_{v \perp u_t, \|v\| = 1} |\langle v, B_t u_t \rangle|^2 \leq \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t \sqrt{\log \epsilon_t}. \] (112)
Another application of (110) together with equation (52) gives
\[ \frac{1}{q(S_t)^2} \left| \left\langle \tilde{B}_t u_t, u_t \right\rangle \right|^2 \epsilon_t \leq \frac{C}{S_t^2(1 - S_t)^2} \epsilon_t. \] (113)
Finally, plugging the estimates (111), (112) and (113) into (108) gives
\[ |\beta_t| \leq (1 - t)^{-1} \frac{C}{S_t^3(1 - S_t)^3} \epsilon_t \sqrt{\log \epsilon_t} \]
and the proof is complete.

3.3 The upper bound

The goal of this subsection is to prove the upper bound for the deficit in theorem 2. Namely, we aim to show that for all \(0 < s < 1\) there exists a constant \(c_s\), such that the following holds: for every \(0 < \rho < 1\) and every measurable \(A \subset \mathbb{R}^n\), one has
\[ S_\rho(H(A)) - S_\rho(A) \leq \frac{C(A)}{\sqrt{1 - \rho}} \epsilon(A). \] (114)
We begin defining the operator \(P_\rho\) acting on integrable functions \(f : \mathbb{R}^n \to \mathbb{R}\) by the formula
\[ P_\rho(f)(x) = \int_{\mathbb{R}^n} f \left( \rho x + \sqrt{1 - \rho^2} y \right) d\gamma(y). \]
Under a slightly different parametrization, this is just the Ornstein-Uhlenbeck operator for the Gaussian measure. For a measurable set \(A \subset \mathbb{R}^n\) we abbreviate \(P_\rho(A) = P_\rho(1_A)\). The significance of this definition is the following. We have for two measurable sets \(A, B \subset \mathbb{R}^n\)
\[ \int_A P_\rho(B)(x) d\gamma(x) = \] (115)
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{x \in A} 1_{\sqrt{1-\rho^2} y \in B} d\gamma(y) d\gamma(x) = \mathbb{P}(X \in A & \rho X + \sqrt{1-\rho^2} Y \in B)
\]

where \(X\) and \(Y\) are independent standard Gaussian vectors.

Now, if \(Y'\) is another standard Gaussian random vector independent from \(X, Y\), then it is easy to check that
\[
(\sqrt{\rho} X + \sqrt{1-\rho} Y, \sqrt{\rho} X + \sqrt{1-\rho} Y') \sim (X, \rho X + \sqrt{1-\rho^2} Y).
\]

where the sign \(\sim\) means that both expressions are distributed according to the same law. Consequently, we get by definition that
\[
S_\rho(A) = \int_A P_\rho(A) d\gamma
\]

for every \(A \subset \mathbb{R}^n\) measurable. Moreover, since the right hand side of (115) is invariant under interchanging \(A\) and \(B\), we learn that \(P_\rho\) is a self-adjoint linear operator. Moreover, it is straightforward to check that for all \(f\),
\[
P_\rho(f) = P_{\sqrt{\rho}}(P_{\sqrt{\rho}}(f))
\]

and it follows that \(P_\rho\) is a positive semi-definite operator. Consider the non-negative, symmetric quadratic form
\[
K_\rho(f, g) := \int_{\mathbb{R}^n} f(x) (P_\rho(g)(x)) d\gamma(x).
\]

By formula (116) we have
\[
S_\rho(H(A)) - S_\rho(A) = K_\rho(H(A), H(A)) - K_\rho(A, A) = (K_\rho(H(A), H(A)) - K_\rho(H(A), A)) + (K_\rho(H(A), A) - K_\rho(A, A)).
\]

(117)

We claim that in order to give an upper bound for the deficit, it is enough to estimate the first term in the above equation. Namely, we claim that
\[
S_\rho(H(A)) - S_\rho(A) \leq 2(K_\rho(H(A), H(A)) - K_\rho(H(A), A)).
\]

(118)

Indeed, according to theorem we have
\[
K_\rho(H(A), H(A)) \geq K_\rho(A, A).
\]

By the Cauchy-Schwartz and the arithmetic-geometric inequalities
\[
K_\rho(H(A), A) \leq \sqrt{K_\rho(H(A), H(A))K_\rho(A, A)} \leq \frac{K_\rho(H(A), H(A)) + K_\rho(A, A)}{2},
\]

or in other words
\[
(K_\rho(H(A), H(A)) - K_\rho(H(A), A)) \geq (K_\rho(H(A), A) - K_\rho(A, A)).
\]

Plugging this into (117) gives (118). Let us give an upper bound for the right hand side of (118). We have by definition
\[
I := K_\rho(H(A), H(A)) - K_\rho(H(A), A) =
\]
Let \( v \) be a unit vector, and \( \alpha \in \mathbb{R} \) such that

\[
H(A) = \{ \langle x, v \rangle \geq \alpha \}.
\]

Moreover, let \( \mu \) be the push-forward of the restriction of the standard Gaussian measure to the set \( A \) under the map \( x \rightarrow \langle x, v \rangle \), let \( f(x) \) be the density of \( \mu \) with respect to the standard Gaussian measure and define \( h(x) = 1_{x \geq \alpha} \). Since \( H(A) \) is invariant under translations orthogonal to \( v \), it is clear that

\[
P_\rho(H(A))(x) = P_\rho(h)(\langle x, v \rangle).
\]

With this notation, the above integral becomes

\[
I = \int_{\mathbb{R}} P_\rho(h)(x)(h(x) - f(x))d\gamma^1(x).
\]

We can calculate,

\[
P_\rho(h)(x) = \int_{\mathbb{R}} 1_{\rho x + \sqrt{1-\rho^2}y \geq \alpha} d\gamma(y) = \gamma\left(\left[\frac{\alpha - \rho x}{\sqrt{1-\rho^2}}, \infty\right]\right) = \Phi\left(\frac{\rho x - \alpha}{\sqrt{1-\rho^2}}\right)
\]

and the above integral becomes

\[
I = \int_{\mathbb{R}} \Phi\left(\frac{\rho x - \alpha}{\sqrt{1-\rho^2}}\right)(h(x) - f(x))d\gamma^1(x).
\]

Since \( \gamma(A) = \gamma(H(A)) \), we know that

\[
\int_{\mathbb{R}} (h(x) - f(x))d\gamma^1(x) = 0
\]

and therefore

\[
I = \int_{\mathbb{R}} \left(\Phi\left(\frac{\rho x - \alpha}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{\rho \alpha - \alpha}{\sqrt{1-\rho^2}}\right)\right)(h(x) - f(x))d\gamma^1(x). \tag{119}
\]

Since \( \Phi(x)' \leq 1 \) for all \( x \in \mathbb{R} \), we have

\[
\left| \Phi\left(\frac{\rho x - \alpha}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{\rho \alpha - \alpha}{\sqrt{1-\rho^2}}\right) \right| \leq \frac{\rho}{\sqrt{1-\rho^2}}|x - \alpha|. \tag{120}
\]

Moreover, observe that by definition \( h(x) - f(x) \geq 0 \) for \( x \geq \alpha \) and \( h(x) - f(x) \leq 0 \) for \( x \leq \alpha \). Since \( \Phi(\cdot) \) is an increasing function, it implies that the expression inside the above integral is non-negative. Therefore, we can estimate

\[
I \leq \frac{\rho}{\sqrt{1-\rho^2}} \int_{\mathbb{R}} (x - \alpha)(h(x) - f(x))d\gamma^1(x) = \frac{\rho}{\sqrt{1-\rho^2}}(q(H(A)) - q(A)) \leq \frac{\rho}{\sqrt{1-\rho^2}}\frac{q(H(A))^2 - q(A)^2}{q(H(A))} \leq \frac{1}{1 - \rho} \frac{C_\varepsilon(A)}{\alpha} \frac{q(H(A))}{1 - \gamma(A)}
\]

where the last inequality follows from the bound (52). By plugging this into (118), we get (114) and the upper bound is established.
4 Appendix

In the appendix we fill in a few technical lemmas whose proofs were left out from the note.

Proof of lemma 7 Let $F_t(x)$ be the process satisfying equation (8). We calculate,

$$d \log F_t(x) = \frac{dF_t(x)}{F_t(x)} - \frac{1}{2} \frac{d[F(x)]_t}{F_t(x)^2} =$$

$$(1-t)^{-1} (x - W_t, dW_t) -(1-t)^{-2} \frac{1}{2} |x - W_t|^2 dt =$$

$$\langle x, \frac{dW_t}{1-t} + \frac{W_t}{(1-t)^2} \rangle - \frac{1}{2(1-t)^2} |x|^2 dt - \frac{1}{2} \left( \frac{2\langle W_t, dW_t \rangle}{1-t} + \frac{|W_t|^2 dt}{(1-t)^2} \right) =$$

$$\langle x, d \left( \frac{W_t}{1-t} \right) \rangle - \frac{1}{2(1-t)^2} |x|^2 dt - \frac{1}{2} d \left( \frac{|W_t|^2}{1-t} \right) + \frac{ndt}{2(1-t)}.$$

Integrating this gives,

$$F_t(x) = \gamma(x) \frac{1}{(1-t)^{n/2}} \exp \left( \frac{1}{1-t} \langle W_t, x \rangle - \frac{1}{2} \left( \frac{1}{1-t} - 1 \right) |x|^2 - \frac{1}{2} \frac{|W_t|^2}{1-t} \right) =$$

$$\frac{1}{((1-t)\sqrt{2\pi})^{n/2}} \exp \left( - \frac{1}{2(1-t)} \left( -2\langle x, W_t \rangle + |x|^2 + |W_t|^2 \right) \right) = \gamma_{W_t \sqrt{1-t}}(x).$$

By the uniqueness of the solution to the SDE, the lemma follows.

Proof of fact 21 The upper bound follows immediately from the fact that, according to formula (76) one has $q''(s) < q''(1/2) < -2$ for all $0 < s < 1$. Let us prove the lower bound. By the symmetry of $q(s)$ around $s = 1/2$, we may assume without loss of generality that $h < 1/2$. Define

$$f(s) = q(s) - q(h) - q'(h)(s-h)$$

and

$$g(s) = h^{-2}f(0)(s-h)^2.$$

Note that by definition, the functions $f(0) = g(0), f(h) = g(h) = 0$ and $f'(h) = g'(h) = 0$. Now, according to formula (76), the function $q'(s)$ is convex in $[0, 1/2]$ (here, we use the assumption that $h < 1/2$). Therefore, the function $w(s) = f'(s) - g'(s)$ is also convex in this interval. Now, we know that $w(h) = 0$ and that $\int_0^h w(s) ds = 0$, so from the convexity of $w(s)$ we conclude that there exists $s_0 \in (0, h)$ such that

$$w(h) = 0 \text{ and } w(s)(s - s_0) \leq 0, \forall 0 < s < h$$

(121)

and therefore

$$\int_s^h w(x) dx \leq 0, \forall 0 < s < h.$$

It follows that $g(s) < f(s)$ for all $0 < s < h$. Moreover, since $w(s)$ is convex up to $s = 1/2$, necessarily we have $w(s) > 0$ for $h < s < 1/2$ and it follows that

$$g(s) \leq f(s), \forall 0 \leq s \leq 1/2.$$
Next, we show that $g(s) \leq f(s)$ also for $1/2 < s < 1$, or in other words we will show that
\[ p(s) \leq q(s), \quad \forall 0 < s < 1 \]
where
\[ p(s) = g(s) + q(h) + q'(h)(s - h). \]
Indeed, the fact that $w(s)$ is convex up to $s = 1/2$ and by (121), we know that $w(1/2) > 0$, which means that $p'(1/2) < q'(1/2) = 0$, and therefore the parabola $p(s)$ attains a maximum at some point $b \leq 1/2$ which means that $p(1 - s) \leq p(s)$ for all $s < 1/2$. So by the symmetry of $q(s)$ around $s = 1/2$ we get
\[ q(1 - s) = q(s) \geq p(s) \geq p(1 - s), \quad \forall 0 < s < 1/2. \]
We finally have $f(s) \geq g(s)$ for all $0 \leq s \leq 1$. In order to prove the lower bound, it therefore suffices to show that
\[ -\frac{4}{h^2(1 - h)^2}(s - h)^2 \leq g(s) = h^{-2}f(0)(s - h)^2 = h^{-2}(s - h)^2(-q(h) + hq'(h)) \]
or in other words, using the assumption $h < 1/2$,
\[ 1 \geq q(h) - hq'(h) \]
a combination of (75) with the fact that $q(h) \leq q(1/2) < 1$ finishes the proof.

Proof of lemma 16. We begin with formula (53). By equation (78) we have
\[ q(s) = \frac{e^{-\Psi(s)^2/2}}{\int_{-\infty}^{\infty} e^{-x^2/2} dx}. \]
Denote $y = -\Psi(s)$. Since, by (75), $q'(s)$ is a decreasing function, we may assume that $s < \frac{1}{2}$ and thus $y > 0$. The inequality $\left(y + \frac{1}{y+1}\right)^2 \leq y^2 + 3$ suggests that
\[ \int_{y}^{\infty} e^{-x^2/2} \geq \int_{y}^{y+1/(y+1)} e^{-(y+1/(y+1))^2/2} dx \geq e^{-y^2/2} \]
so
\[ \frac{q(s)}{s} = \frac{e^{-y^2/2}}{\int_{y}^{\infty} e^{-x^2/2} dx} \leq e^{3(y + 1)} = -e^{3(\Psi(s) + 1)} \]
for all $s < 1/2$. But a well known fact about the Gaussian distribution is that for $s < 1/2$
\[ -\Psi(s) \leq C\sqrt{|\log s|} \]
for some a universal constant $C > 0$. Formula (53) follows.

The upper bound of formula (52) now follows immediately from the symmetry of the function $q(s)$ around $s = 1/2$, and we are left with proving the lower bound. Consider the function
\[ h(s) = 4s(1 - s)q(1/2) = \frac{4}{\sqrt{2\pi}} s(1 - s). \]
We know that \( h(s) = q(s) \) for \( s \in \{0, 1/2, 1\} \). Moreover, \( h(s) \) is tangent to \( q(s) \) at \( s = 1/2 \), and lastly, according to formula (76), we see that \( q'(s) \) is a convex function in \( s \in [0, 1/2] \). Consequently, the convex function \( q(s) = q'(s) - h'(s) \) intersects the x-axis exactly once in the interval \((0, 1/2)\), say at the point \( s_0 \) (since it is equal to zero at \( s = 1/2 \) and since its integral on that interval is equal zero). Now, we have

\[
q''(1/2) = -\sqrt{2\pi} > -\frac{8}{\sqrt{2\pi}} = h''(1/2),
\]
which implies that \( q'(1/2) > 0 \). We conclude that \( g(s)(s - s_0) < 0 \) for \( 0 < s < 1/2 \). By the fact that \( q(0) = h(0) \) and \( q(1/2) = h(1/2) \) we know that

\[
\int_0^{1/2} g(s)ds = 0
\]
and therefore

\[
q(s) - h(s) = -\int_s^{1/2} g(x)dx \geq 0, \quad \forall 0 < s < 1/2
\]
so \( q(s) \geq h(s) \) in \( 0 < s < 1/2 \). Since both functions are symmetric around \( s = 1/2 \), we have established that

\[
q(s) \geq h(s) = \frac{4}{\sqrt{2\pi}}s(1-s)
\]
and the upper bound is proven.

\[
\square
\]

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