ON WEAK FRAÏSSÉ LIMITS

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Abstract. Using the natural action of $S_\infty$ we show that a countable hereditary class $\mathcal{C}$ of finitely generated structures has the joint embedding property (JEP) and the weak amalgamation property (WAP) if and only if there is a structure $M$ whose isomorphism type is comeager in the space of all countable, infinitely generated structures with age in $\mathcal{C}$. In this case, $M$ is the weak Fraïssé limit of $\mathcal{C}$.

This applies in particular to countable structures with generic automorphisms and recovers a result by Kechris and Rosendal [Proc. Lond. Math. Soc., 2007].

1. Introduction

After writing the note [2] showing that the class of universal-homogeneous structures is comeager in some appropriate space of structures, we noticed that the methods from the paper of Kechris and Rosendal [3] easily yield a stronger and more general result applying to weak Fraïssé classes of finitely generated structures.

Thus, the aim of the present note is to prove that for an arbitrary hereditary class $\mathcal{C}$ of finitely generated (not necessarily finite) structures, the space of all countable, infinitely generated structures with age in $\mathcal{C}$ contains a comeager subset of structures isomorphic to some countable structure $M$ if and only if $\mathcal{C}$ has the weak Fraïssé property and $M$ is the weak Fraïssé limit of $\mathcal{C}$. As a corollary, this proves the existence and uniqueness of the weak Fraïssé limit of a weak Fraïssé class. We also explain how this in turn reproves the result by Kechris and Rosendal [3] on comeager conjugacy classes of automorphisms of such structures in the slightly more general setting of finitely generated structures.

Similar results for classes of finite structures were obtained by Kruckman in his PhD thesis [4] using Banach–Mazur games on posets.

2. Background and the main result

2.1. Weak Fraïssé limits. Let $L$ be a countable language. For background on model theory we refer to [6]. Here and in what follows, the word ‘structure’ always means ‘$L$-structure’.

Definition 2.1. Let $\mathcal{C}$ be a countable class of (isomorphism types of) finitely generated structures and assume that $\mathcal{C}$ is hereditary, that is closed under taking finitely generated substructures. We say that $\mathcal{C}$ is a weak Fraïssé class if $\mathcal{C}$ satisfies

(WAP) (Weak Amalgamation Property) For any $A \in \mathcal{C}$ there is some $B \in \mathcal{C}$ and embedding $e : A \to B$ such that for all $B_1, B_2 \in \mathcal{C}$ and embeddings $h_i$ of $B$ into $B_i, i = 1, 2$, there is some $C \in \mathcal{C}$ and embeddings $g_i$ of $B_i$ into $C, i = 1, 2$, such that $g_1 \circ h_1 \circ e = g_2 \circ h_2 \circ e$. 

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(JEP) (Joint Embedding Property) For any \( A, B \in C \) there is some \( D \in C \) containing \( A \) and \( B \) as a substructure.

The class \( C \) is called a Fraïssé class if in condition (WAP) we can choose \( B = A \) and \( e = id \). The notion of Weak Amalgamation was first studied by Ivanov in [1].

We call \( C \) unbounded if any \( A \in C \) can be embedded as a proper substructure into some \( B \in C \).

For a structure \( M \), the age of \( M \), \( \text{age}(M) \), is the class of isomorphism types of finitely generated substructures of \( M \).

**Definition 2.2.** Let \( C \) be a countable hereditary class of finitely generated structures and let \( M \) be a countable structure.

1. \((C\text{-Universality})\) We say that \( M \) is \( C \)-universal if \( \text{age}(M) = C \).
2. \((\text{Weak } C\text{-Homogeneity})\) We say that \( M \) is weakly \( C \)-homogeneous if for any structure \( A \in C \) there is some \( B \in C \) and embedding \( e : A \to B \) such that for any substructures \( B_1, B_2 \subset M \) with \( (A_1, B_1), (A_2, B_2) \) isomorphic to \( (e(A), B) \) any isomorphism \( \varphi : (A_1, B_1) \to (A_2, B_2) \) extends to an automorphism \( f \) of \( M \) with \( \varphi|_{A_1} = f|_{A_1} \).
3. \((\text{Weak } C\text{-Saturation})\) For any structure \( A \in C \) there is some \( B \in C \) and embedding \( e : A \to B \) such that for any \( D \in C \) containing \( B \) and any embedding \( h : B \to M \) there is an embedding of \( D \) into \( M \) extending \( h \circ e \) on \( A \).

The structure \( M \) is called \( C \)-homogeneous (\( C \)-saturated, respectively) if we can choose \( A = B \) and \( e = id \).

A countable structure \( M \) is called a (weak) Fraïssé limit for \( C \) if it is \( C \)-universal and (weakly) \( C \)-homogeneous.

We will show that a countable hereditary unbounded class \( C \) of finitely generated structures has a weak Fraïssé limit if and only if \( C \) satisfies (JEP) and (WAP).

### 2.2. The space of structures and the action of \( S_{\infty} \)

Let \( C \) be a countable hereditary class of finitely generated structures and let \( S \) be the set of all structures \( M \) defined on the universe \( \omega \) (or any other fixed countable universe) with the following two properties:

1. The age of \( M \) is contained in \( C \) and
2. \( M \) is not finitely generated.

**Metric on \( S \):** We endow \( S \) with the following metric: For two different structures \( M, N \in S \) we put \( d(M, N) = 2^{-k} \) where \( k \in \{0, 1, \ldots \} \) is minimal such that the substructures of \( M \) and \( N \) generated by the elements \( 0, 1, \ldots, k \) are different, i.e. either these substructures have different universes as subsets of \( \omega \) or their universes agree, but some constant, relation or function differs on the universe. Note that with this metric, \( S \) becomes a complete (ultra)metric space. Consequently, the Baire category theorem holds in \( S \).

**Topology on \( S \):** For a finitely generated structure \( B \) whose isomorphism type is contained in \( C \) and such that \( \text{dom } B \subset \omega \), let \( \mathcal{O}_B \) be the set of all structures \( A \in S \) whose restriction to \( \text{dom } B \) coincides with \( B \). It is easy to see that these sets (after discarding the empty ones) form a basis of some topology \( \mathcal{T} \) on \( S \), see [2], and that \( \mathcal{T} \) is the topology induced by the metric \( d \) on \( S \).

Proposition 2.3. The space \((S, T)\) is separable if and only if every structure from \(\mathcal{C}\) that can be embedded into a countable, infinitely generated structure, is finite.

Proof. Let \((S, T)\) be separable with a countable dense set \(\{X_1, X_2, \ldots\}\). Assume, by contraposition, that there is an infinite structure \(A_0 \in \mathcal{C}\) that can be embedded into some countable infinitely generated structure. Taking some infinite set \(U \subset \omega\) as a universe, we can construct a structure \(A(U)\) isomorphic to \(A_0\). The open sets of the form \(O_{A(U)}\) are non-empty, hence any such set contains some point of the form \(X_i\). Since there are uncountably many possible \(U\)'s, there is at least one \(X_i\) contained in uncountably many sets of the form \(O_{A(U)}\). Hence, \(X_i\) has uncountably many finitely generated substructures with pairwise different universes. But this is a contradiction. Conversely, if all finitely generated substructures of any \(M \in S\) are finite, then there are only countably many non-empty basic open sets of the form \(O_B\). By choosing a point in any such set, we obtain a countable dense subset of \(S\), thus proving separability. □

Action of \(S_\infty\) on \(S\): Consider \(S_\infty\), the permutation group on \(\omega\), as a Polish group with a neighbourhood basis of the identity consisting of pointwise stabilizers of finite sets. There is a natural continuous action of \(G = S_\infty\) on \(S\) just given by the action \(g \in G\) on the domain \(\omega\) of a structure and transferring the structure to the image. Under this action, the orbit of a structure \(M \in S\) is the class of structures isomorphic to \(M\) and the stabilizer \(G_M\) consists of all automorphisms of \(M\).

Proposition 2.4. ([3], Prop. 3.2) Let \(G\) be a closed subgroup of \(S_\infty\) acting continuously on a topological space \(X\) and let \(x \in X\). Then the following are equivalent:

1. The orbit of \(x\) under \(G\) is non-meager.
2. For any open subgroup \(V\) of \(G\) the orbit of \(x\) under \(V\) is somewhere dense.

Proof. \(\neg(2) \Rightarrow \neg(1)\): Suppose that the orbit of \(x\) under some open subgroup \(V \leq G\) is nowhere dense. Since \(V\) has countable index in \(G\) and the orbit of \(x\) under each coset \(gV\) is also nowhere dense, the orbit of \(x\) under \(G\) is meager.

\(\neg(1) \Rightarrow \neg(2)\): Conversely, assume that the orbit of \(x\) under \(G\) is meager which means that it is contained in the countable union of closed nowhere dense sets \(F_n, n \in \omega\). Then the sets \(K_n = \{g \in G : g(x) \in F_n\}\) are closed in \(G\) and their union is all of \(G\). At least one of the \(K_n\) has non-empty interior, so contains a translate of some open subgroup \(V\). Thus the orbit of \(x\) under (all translates of) \(V\) is nowhere dense. □

2.3. Main result. Recall that a subset of a topological space is called comeager if its complement is meager or, equivalently, if the set can be represented as a countable intersection of subsets with dense interior.

Theorem 2.5. Let \(\mathcal{C}\) be a countable hereditary unbounded class of finitely generated structures and let \((S, T)\) be the topological space of all countable structures on \(\omega\) with age contained in \(\mathcal{C}\) and which are not themselves finitely generated. The topology \(T\) is given by the metric \(d\) as above. Then the following are equivalent:

\[\text{An example due to Márk Poór shows that the assumption that } \mathcal{C} \text{ be unbounded is necessary.}\]
The class $\mathcal{C}$ has a weak Fraïssé limit $M \in \mathcal{S}$.  
(2) There is a $\mathcal{C}$-universal and weakly $\mathcal{C}$-saturated structure $M \in \mathcal{S}$.  
(3) The $S_\infty$-orbit of some structure $M \in \mathcal{S}$ is comeager in $\mathcal{S}$.  
(4) For some structure $M \in \mathcal{S}$ the set of structures isomorphic to $M$ is comeager in $\mathcal{S}$.  
(5) The class $\mathcal{C}$ satisfies (JEP) and (WAP).  
(6) There is a $\mathcal{C}$-universal weakly $\mathcal{C}$-saturated structure $M \in \mathcal{S}$ such that (*) holds:  

(*) any finitely generated substructure $A$ of $M$ is contained in a finitely generated substructure $B \subset M$ (not necessarily the one given by weak $\mathcal{C}$-saturation) such that any $D \in \mathcal{C}$ containing a copy of $B$ embeds over $A$ into $M$.  

It follows in particular, that a weak Fraïssé limit, if it exists, is unique up to isomorphism.  

**Proof.** (1) $\Rightarrow$ (2): Let $M \in \mathcal{S}$ be a weak Fraïssé limit of $\mathcal{C}$. We have to show that $M$ is weakly $\mathcal{C}$-saturated. Let $A \subseteq B \in \mathcal{C}$ be as given by weak $\mathcal{C}$-homogeneity. Assume that $(A_0, B_0)$ is an isomorphic copy of $(A, B)$ in $M$. Let $D \in \mathcal{C}$ contain an isomorphic copy of $B$. By $\mathcal{C}$-universality, $M$ contains a substructure $D_0$ isomorphic to $D$. By $\mathcal{C}$-homogeneity, any isomorphism between the copy of $A$ inside $D_0$ and $A_0$ can be extended to an automorphism of $M$, yielding an embedding of $D$ over $A_0$ into $M$ as required.  

(2) $\Rightarrow$ (3): Let $M \in \mathcal{S}$ be $\mathcal{C}$-universal and weakly $\mathcal{C}$-saturated. Then $age(M) = \mathcal{C}$ and thus the $S_\infty$-orbit of $M$ is dense. It now suffices to verify that the $S_\infty$-orbit of $M$ is non-meager: since the action of $S_\infty$ on the metric space $\mathcal{S}$ is continuous, it then follows from [3], Thm. 4.4 that the orbit is $G_\delta$ and hence comeager (given that it is dense). To see that the $S_\infty$-orbit of $M$ is non-meager, it suffices by Proposition 2.4 to see that for any open subgroup $V \leq S_\infty$ the $V$-orbit of $M$ is somewhere dense. We may assume that there is a finite set $A_0$ such that $V = G_{A_0}$ is the pointwise stabilizer of $A_0$. Let $A \subset M$ be the substructure generated by $A_0$ and let $B \in \mathcal{C}$ be such that $B$ contains an isomorphic copy of $A$ and such that any extension of $B$ can be embedded into $M$ by weak $\mathcal{C}$-saturation. Now since $B \in age(M)$ we may assume that $B \subset M$. Let $A'$ denote the isomorphic copy of $A$ inside $B$ with $A'_0$ denoting the finite set corresponding to $A_0$. Then $V' = G_{A'_0}$ is conjugate to $G_{A_0}$ in $S_\infty$ and it clearly suffices to prove the claim for $V'$ and $A'$. We claim that the $V'$-orbit of $M$ is dense in $O_B$: By $\mathcal{C}$-saturation of $M$, any extension $D$ of $B$ can be embedded over $A'$ into $M$. This says that for some $g \in V'$ we have $g(M) \in O_D$, which is enough.  

(3) $\Leftrightarrow$ (4): clear.  

(3) $\Rightarrow$ (5): Let $M \in \mathcal{S}$ be an element whose orbit under $G = S_\infty$ is comeager. Since $\mathcal{S}$ is a Baire space, a comeager set is dense and thus we see that $age(M) = \mathcal{C}$ and hence (JEP) holds. It remains to prove that $\mathcal{C}$ satisfies (WAP). Let $A \in \mathcal{C}$. Then $M$ contains a substructure isomorphic to $A$ and we may assume that $A \subset M$. Let $A_0 \subset A$ be a finite set generating $A$. Then the pointwise stabilizer of $A_0$ in $G$ is an open subgroup $G_{A_0}$ and by Proposition 2.4 the orbit of $M$ under $G_{A_0}$ is dense in some basic open set $O_B$ for some finitely generated structure $B$. Thus, $M' := f(M) \in O_B$ for some $f \in G_{A_0}$. Let $A' = f(A)$ be the structure (with domain contained in $\omega$) obtained by transferring $A$ by $f$. Then, $M' \in O_B \cap O_{A'}$ meaning that $B$ and $A'$ are substructures of $M'$. Let $D$ be the substructure of $M'$ generated by $A'$ and $B$. Then, $O_D \subset O_{A'} \cap O_B$. To prove (WAP), let $E, F \in \mathcal{C}$ be structures containing $D$ (and assume that the domains of $E$ and $F$ are embedded into $\omega$). Since the orbit of $M$ (and hence $M'$) under $G_{A_0}$ is dense in $O_D$, there exist $g, h \in G_{A_0}$ such
that \( g(M') \in O_E \) and \( h(M') \in O_F \). Observe that both \( g \) and \( h \) equal to the identity map on \( A_0 \), hence \( g|_{A'} = h|_{A'} = id \). Hence the substructure of \( M' \) generated by \( g^{-1}(E) \cup h^{-1}(F) \subset M' \) is the required weak amalgam over \( A' \).

(5) \( \Rightarrow \) (6): Assume that \( \mathcal{C} \) has (JEP) and (WAP). We will construct a \( \mathcal{C} \)-universal and weakly \( \mathcal{C} \)-saturated structure satisfying (*). Choose an enumeration \( (C_i)_{i \in \omega} \) of all isomorphism types in \( \mathcal{C} \). We construct \( M \) as the union of an ascending chain

\[
M_0 \subset M_1 \subset \cdots \subset M
\]
of elements of \( \mathcal{C} \). Suppose that \( M_i \) is already constructed. If \( i = 2n \) is even, let \( A_{i+1} \) be the structure embedding both \( M_i \) and \( C_n \) given by (JEP) and let \( M_{i+1} \in \mathcal{C} \) be the structure given by (WAP) for \( A_{i+1} \).

We use the odd steps to ensure that the limit \( M \) is weakly \( \mathcal{C} \)-saturated. For \( i = 2n + 1 \), let \( A \in \mathcal{C} \) and let \( B \in \mathcal{C} \) be as guaranteed by (WAP) where we assume that \( A \subset B \). Let \( D \in \mathcal{C} \) and assume we are given two embeddings \( f_0 : B \rightarrow M \) and \( f_1 : B \rightarrow D \). Then we let \( M_{i+1} \in \mathcal{C} \) be the structure guaranteed to exist by (WAP).

Assume now that we have \( A \subset B, D \subset \mathcal{C} \) and embeddings \( f_0 : B \rightarrow D \) and \( f_1 : B \rightarrow M \). Since \( B \) is finitely generated, the image of \( f_0 \) will be contained in some \( M_j \). Thus, in order to guarantee the weak \( \mathcal{C} \)-saturation of \( M \), we have to ensure during the construction of the \( M_i \) that eventually, for some odd \( i \geq j \), the embeddings \( f_0 : B \rightarrow M_i \) and \( f_1 : B \rightarrow D \) were used in the construction of \( M_{i+1} \). This can be done since for each \( j \) there are \( \omega \) many isomorphisms \(-\) at most countably many possibilities. Thus there exists an embedding \( g_1 : D \rightarrow M_{i+1} \) with \( f_0|_A = g_1 \circ f_1|_A \).

By the even stages, we have \( \mathcal{C} \subset \text{age}(M) \). Conversely, the finitely generated substructures of \( M \) are the substructures of the \( M_i \). Since the \( M_i \) belong to \( \mathcal{C} \), their finitely-generated substructures also belong to \( \mathcal{C} \). Hence \( \text{age}(M) = \mathcal{C} \) as required. Furthermore, since any finitely generated substructure \( A \) is contained in some \( M_i \), the structure \( M_{2i} \) satisfies (*) for \( A \). Note that \( M \) is not finitely generated as otherwise we would have \( M \in \mathcal{C} \) and hence \( \mathcal{C} \) would be bounded.

(6) \( \Rightarrow \) (1): Let \( M \in S \) be \( \mathcal{C} \)-universal and weakly \( \mathcal{C} \)-saturated such that (*) holds. We have to show that \( M \) is weakly \( \mathcal{C} \)-homogeneous. Let \( A_0 \subseteq A_1 \subseteq \mathcal{C} \) be as given by weak \( \mathcal{C} \)-saturation. Assume that \( (B_0, B_1) \) and \( (C_0, C_1) \) are isomorphic copies of \( (A_0, A_1) \) in \( M \) and that \( \varphi_0 : (B_0, B_1) \rightarrow (C_0, C_1) \) is an isomorphism. We want to extend \( \varphi_0|_{B_0} \) to an automorphism of \( M \) by a back-and-forth construction. Suppose we have already constructed \( \varphi_{2i-1} : B_{2i-1} \rightarrow C_{2i-1} \) with \( \varphi_{2i-1}|_{B_0} = \varphi_0 \) such that \( C_{2i-1} \) is an extension of \( C_{2i-2} \) satisfying (*). Let \( B_{2i} \) be the substructure given by (*) containing the substructure of \( M \) generated by \( B_{2i-1} \cup \{i\} \). Then there is a copy \( C_{2i} \) such that there is an isomorphism \( \varphi_{2i} : (B_{2i-1}, B_{2i}) \rightarrow (C_{2i-1}, C_{2i}) \) with \( \varphi_{2i}|_{B_0} = \varphi_0 \). Now let \( C_{2i+1} \) be the structure given by (*) containing the substructure of \( M \) generated by \( C_{2i} \cup \{i\} \) and choose an isomorphic copy \( B_{2i+1} \) of \( C_{2i+1} \) and an isomorphism \( \varphi_{2i+1} : (B_{2i}, B_{2i+1}) \rightarrow (C_{2i}, C_{2i+1}) \) extending \( \varphi_{2i}|_{B_{2i-1}} \).

Thus again \( \varphi_{2i+1}|_{B_0} = \varphi_0 \) and we continue.

\[ \square \]

2.4. Finding automorphisms with comeager conjugacy class. The proof of Theorem 2.5 was motivated by the methods from [3]. We now show how this in turn implies their result on comeager conjugacy classes.
For a Fraïssé class $C$ of finitely generated $L$-structures, let $C_p$ denote the class of all systems of the form $S = \langle A, \psi : B \to C \rangle$ where $A, B, C \in C, B, C \subset A$ and $\psi$ is an isomorphism of $B$ and $C$. An embedding of one system $S = \langle A, \psi : B \to C \rangle$ into another system $T = \langle D, \varphi : E \to F \rangle$ is an embedding $f : A \to D$ such that $f$ embeds $B$ into $E, C$ into $F$ and such that $f \circ \psi \subset \varphi \circ f$.

We now consider comeagerness in the space $S_p$ of structures $(M, h)$ where $M \in S$ and $h \in \text{Aut}(M)$. Then, for any finitely generated substructure $A$ of $M$ and isomorphic finitely generated substructures $B \subset A, C = h(B)$, the system $S = \langle A, h|_B : B \to C \rangle$ is in $C_p$. As before, the space $S_p$ is a complete metric space. The group $\text{Aut}(M)$ is endowed with the usual topology of pointwise convergence.

Remark 2.6. We could instead consider an expanded language $L_p = L \cup \{f\}$ containing a new symbol $f$ for a partial function. Then $C_p$ denotes the class of all structures $A \in C$ expanded by an interpretation of the symbol $f$ as an isomorphism $f : B \to C$ where $A, B, C \in C$ and $B, C \subset A$. By embedding an $L_p$-structure $A$ into an $L_p$-structure $D$, we may enlarge the domain of the function $f$ inside $A$ (contrary to the usual conventions).

Note that we do not assume the Fraïssé class $C$ to consist of finite structures.

**Theorem 2.7.** Let $C$ be a Fraïssé class of finitely generated structures with Fraïssé limit $M$. Then the following are equivalent:

1. $C_p$ has (JEP) and (WAP).
2. The weak Fraïssé limit of $C_p$ exists in $S_p$ and has comeager isomorphism type in $S_p$.
3. There is $f \in \text{Aut}(M)$ with comeager conjugacy class in $\text{Aut}(M)$.

If any of the above conditions holds, the weak Fraïssé limit of $C_p$ has the form $(M, f)$, where $f \in \text{Aut}(M)$ has a comeager conjugacy class.

**Proof.** (1) $\Rightarrow$ (2): The proof of Theorem 2.5 (5) $\Rightarrow$ (6) shows how to construct the weak Fraïssé limit $(N, g)$. It is left to show that $g$ is a surjective automorphism of $N$. For this it suffices to verify that $g$ is defined on all of $N$ and is surjective. Suppose not, so there is some $a \in N$ where $g$ is not defined. Since $N$ is the weak Fraïssé limit of $C_p$ it is the countable union of systems $S_i = \langle N_i, \varphi_i : B_i \to C_i \rangle$ for $i < \omega$. Let $i$ be minimal such that $g$ is not defined on all of $N_i$. By weak $C_p$-saturation, there is some system $T = \langle D, \varphi : E \to F \rangle$ extending $S_i$ such that any extension of $T$ can be embedded over $S_i$ into $N$. Since $C$ is a Fraïssé class with Fraïssé limit $M$, there is an embedding of $D$ into $M$ and by ultrahomogeneity of $M$, $\varphi$ extends to an automorphism of $\hat{\varphi}$ of $M$. Then $\hat{\varphi}$ is defined on all of $N_i \subset D$. Let $G$ be the substructure of $M$ generated by $D$ and $\hat{\varphi}(D)$. The system $\langle G, \hat{\varphi}|_D : D \to \hat{\varphi}(D) \rangle$ is in $C_p$ and extends $T$, thus embeds over $N_i$ into $N$, showing that $g$ is defined on all of $N$. A similar argument shows that $g$ is surjective.

The proof of Theorem 2.5 (2) $\Rightarrow$ (3) shows that the isomorphism type of $(N, g)$ in $S_p$ is comeager.

(2) $\Rightarrow$ (3): Suppose the class $C_p$ has a weak Fraïssé limit $(N, g)$. To see that $N$ is isomorphic to $M$ it suffices to check that $N$ is weakly $C$-saturated. Then both $M$ and $N$ have comeager isomorphism classes and thus must be isomorphic. Clearly we have $\text{age}(N) = C$.

To see that $N$ is weakly $C$-saturated, let $A \subset N$ be a finitely generated substructure, so
A ∈ C. Consider the system ⟨A, id : E → E⟩ where E is the substructure generated by the emptyset. Note that any partial isomorphism extens id : E → E. By (WAP) for C_p there is some system T = ⟨B, ϕ : C → D⟩ such that any system containing T embeds into N. Now let F ∈ C be a finitely generated structure containing the structure B. Then the system ⟨F, ϕ : C → D⟩ is in C_p and hence embeds into N over A. This shows that we can choose B to witness the weak C-saturation of N.

It is left to prove that the conjugacy class of g is comeager in Aut(N). To see this note that the orbit of (N, g) under S_∞ is comeager in the class S_p. Since g ∈ Aut(N), the orbit of S_∞ under the stabilizer of N is exactly the orbit of g under conjugation in Aut(N). Since Aut(N) is a Polish group, the claim follows from the Kuratowski-Ulam theorem.

(3) ⇒ (1): The same proof as Theorem 2.5 (5) ⇒ (6) and (6) ⇒ (1) shows the required. □

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