Universality for Moving Stripes: A Hydrodynamic Theory of Polar Active Smectics

Leiming Chen
College of Science, The China University of Mining and Technology, Xuzhou Jiangsu, 200116, P. R. China

John Toner
Department of Physics and Institute of Theoretical Science, University of Oregon, Eugene, OR 97403
(Dated: September 30, 2018)

We present a hydrodynamic theory of polar active smectics, for systems both with and without number conservation. For the latter, we find quasi long-ranged smectic order in $d = 2$ and long-ranged smectic order in $d = 3$. In $d = 2$ there is a Kosterlitz-Thouless type phase transition from the smectic phase to the ordered fluid phase driven by increasing the noise strength. For the number conserving case, we find that giant number fluctuations are greatly suppressed by the smectic order; that smectic order is long-ranged in $d = 3$; and that nonlinear effects become important in $d = 2$.

PACS numbers: 05.65.+b, 64.70.qj, 87.18.Gh

Active matter \cite{1} can exhibit a richer variety of ordered phases than for equilibrium systems, since symmetry differences which have little or no effect in equilibrium can have radical effects on active systems. For example, in equilibrium, systems with “polar” orientational order (e.g., ferromagnets) and those with apolar orientational order (i.e., nematics)\cite{2} have identical scaling of their orientational fluctuations. In contrast, the active polar order of a coherently moving flock\cite{3} fluctuates far less than that in active apolar orientationally ordered systems (i.e., “active nematics”\cite{4}).

In this paper we formulate the hydrodynamic theory of polar active smectics, by which we mean systems that spontaneously form uniformly spaced moving layers, and find that they differ considerably from their apolar (i.e., non-moving) analogs\cite{5}. Examples of such “active polar smectics” include propagating waves in chemical reaction-diffusion systems\cite{6}, and “flocks”\cite{3} of active particles forming uniformly spaced parallel liquid-like layers (density waves). Layers are ubiquitous\cite{7} that spontaneously form uniformly spaced moving layers (i.e., “active nematics”\cite{4}).

Active smectics C\cite{8} of flocking, although they may not look like layers (density waves). Layers are ubiquitous\cite{7} as active polar smectics” include propagating waves in chemical reaction-diffusion systems\cite{6}, and “flocks”\cite{3} of active particles forming uniformly spaced parallel liquid-like layers (density waves). Layers are ubiquitous\cite{7} that spontaneously form uniformly spaced moving layers (i.e., “active nematics”\cite{4}).

More specifically, in $d = 2$

$$\langle \psi^* (\vec{r}, t) \psi(\vec{r}', t) \rangle \propto |\vec{r} - \vec{r}'|^{-\eta},$$

where $\psi$ is the complex smectic order parameter, defined, as in equilibrium smectics \cite{9}, via:

$$\rho(\vec{r}, t) \equiv \rho_0 + \psi(\vec{r}, t)e^{i\gamma q z} + c.c.$$ ,

where $\rho$ is the number density (or, more generally, the spatially modulated scalar field in the problem, such as chemical concentration in reaction-diffusion systems), $\rho_0$ its mean, and $q_0 \equiv 2\pi/a$, with $a$ the distance between neighboring layers. Here we’ve defined the average plane of the layers as the $\perp$ plane, and the normal to this plane as the $z$ axis. Equations (1) and (2) imply quasi-sharp Bragg peaks in light or X-ray scattering (or, equivalently, the numerical Fourier transform of density correlations); that is:

$$I_n(q) \propto |\rho(q, t)|^2 \propto (q_z - q_0)^2 + \gamma q_z^2 + \delta q^2 \propto \delta q^2$$ ,

where $\gamma$ is an $O(1)$ constant, $n$ is an integer denoting the order of the Bragg peak, and $\eta$ is non-universal (i.e., it varies from system to system).

For a finite system, this divergence is cut off for $|\delta q| \sim 1/L$, where $L$ is the spatial linear extent of the system, and $\delta q \equiv \vec{q} - q_0 \hat{z}$. This implies the $n$th peak will have a finite height which scales with $L$ like $L^{2-n-\eta}$. In $d = 3$, the long-ranged nature of the smectic order implies sharp (i.e., $\delta$-function) Bragg peaks. In a finite system, the height of these peaks scales linearly with system volume $L^3$.

We predict that with increasing noise the active smectic phase undergoes a dynamical phase transition into the fluid, polar ordered phase treated in much prior work\cite{8}. This transition is in the equilibrium XY universality class\cite{10}, which in $d = 2$ is of the the Kosterlitz-Thouless type\cite{11}. The phase diagram in the parameter space of our model is illustrated in Fig. 2.

In the case where the number of the particles is conserved, we find long-ranged smectic order in $d = 3$. In
$d = 2$, the linearized version of the full hydrodynamic theory predicts quasi-long-ranged smectic order; however, there are marginal non-linearities which may invalidate this conclusion. We’ll investigate this in a future publication.\[13\]

In neither case are there giant number fluctuations in $d = 3$; nor are there any in $d = 2$ in the Malthusian case. The linearized hydrodynamic theory predicts none in $d = 2$ for the number conserving case either; but the aforementioned non-linear terms could change this as well.

We’ll now outline the derivation of these results, starting with the case without number conservation. Then the only important hydrodynamic variable is the displacement $\langle u(r,t)\rangle$ of the layers along $z$, which is proportional to the phase of the smectic order parameter $\psi(r,t)$:

$$\psi(r,t) = |\psi_0| e^{-iq_0 u(r,t)} \rangle \langle 9, 11\rangle.$$  

Symmetry considerations (specifically, translation and rotation invariance) require that $u$’s equation of motion, to lowest order in a gradient expansion, take the form:

$$\partial_t u = v_0 - 2\lambda_\perp \partial_z u + (\nu_\perp \nabla^2 u + \nu_\perp \partial_z u) u$$

$$+ \lambda_\perp (\partial_z u)^2 + \lambda_\perp |\nabla u|^2 + f,$$

where $f$ is a Gaussian, zero-norm, white noise with variance $\langle f(r,t)f(r',t') \rangle = 2\Delta \delta^d(r-r')\delta(t-t')$. Rotation invariance forces the coefficient of the $\partial_z u$ term to be exactly $-2$ times that of the $|\nabla u|^2$ term, because only the combination $\partial_z u - \frac{1}{2} |\nabla u|^2$ is unchanged by a uniform rotation of the smectic layers.\[8\]

The term with coefficient $\nu_\perp$ in (4) is forbidden by rotation-invariance of the free energy in equilibrium. It is, however, permitted here\[11\] simply because rotation-invariance at the level of the equation of motion, which is all one can demand in an active system, does not rule it out. Its physical content is that layer curvature produces a local vectorial asymmetry which must modify the directed motion of the layers as this is a driven system.

In contrast to apolar active smectics\[3\], Eq. (4) is not invariant under the simultaneous transformation $u \rightarrow -u$, $z \rightarrow -z$, due to the lack (by definition) of up-down symmetry in polar smectics.

To simplify Eq. (4), we introduce another field variable:

$$u' = u - v_0 t$$

and another coordinate system $t' = t, z' = -2\lambda_\perp t, \vec{r}_\perp' = \vec{r}_\perp$. In terms of these Eq. (4) becomes:

$$\partial_{t'} u' = \nu_\perp \nabla^2 u' + \nu_\perp \partial_z u' + \lambda_\perp |\nabla u'|^2 + \lambda_\perp (\partial_z u')^2$$

$$+ f,$$

where $\lambda_\perp, \nu_\perp$ must be positive for the smectic state to be dynamically stable. Either sign of $\lambda_\perp$ can be stable; indeed, their signs need not be the same.

Eq. (6) has exactly the same form as the anisotropic KPZ equation.\[19\] However, there is a crucial difference. The original KPZ equation describes the hydrodynamics of crystal growth, and the hydrodynamic variable is $h$, the height of a $d$ dimensional surface. Clearly, states with different heights $h$ are always physically distinguishable. However, for smectics, the state is periodic in $u'$ with period $a$, the spacing between neighboring smectic layers. This allows for the existence of topologically stable dislocations, which can unbind, thereby “melting” (i.e., disordering) the smectic, in analogy to such “dislocation mediated melting” in a variety of translationally ordered equilibrium systems.\[12, 21\]

In $d = 2$, this is the aforementioned Kosterlitz-Thouless phase transition, which is absent in the anisotropic KPZ equation.

Simple power counting shows that the nonlinear terms in Eq. (6) are irrelevant in $d = 3$; hence, the linear theory is valid. A straightforward calculation then shows that $\langle |u'(q,t)|^2 \rangle \propto 1/q^2$ for all directions of wavevector $q$. This in turn implies that the real space fluctuation $\langle |u'(r,t)|^2 \rangle$ is finite as system size $L \rightarrow \infty$, which implies long-ranged smectic order.\[22\]

In $d = 2$ the nonlinear terms in Eq. (6) become marginal, and a dynamical renormalization group (RG) analysis is needed. This has already been done for the $d = 2$ crystal growth problem\[15\]; the resulting RG recursion relations are:

$$\frac{d\nu_\perp}{d\ell} = \left[ z - 2 + \frac{g}{32\pi} \right] \nu_\perp,$$

$$\frac{d\lambda_\perp z}{d\ell} = \left[ \lambda + z - 2 \right] \lambda_\perp z,$$

$$\frac{d\lambda_\perp}{d\ell} = \left[ -2\chi + z - 2 + \frac{g}{64\pi} \right] \lambda_\perp,$$

$$\frac{d\nu_\perp}{d\ell} = -\frac{g}{32\pi} \nu_\perp \nu_\perp \left( 1 - \Gamma^2 \right),$$

where $\chi$ and $z$ are the rescaling exponents of $u'$ and $t$, (i.e., $u' \rightarrow u'e^{\chi t}, t' \rightarrow t' e^{zt}$), $\Gamma \equiv \frac{\lambda_\perp \nu_\perp}{\nu_\perp \lambda_\perp}, g \equiv \frac{\lambda_\perp^2}{\nu_\perp^2 \lambda_\perp^2}$, and we have chosen to rescale lengths isotropically. Eqs. (7-10) imply a set of closed flow equations in $\Gamma - g$ space:

$$\frac{d\Gamma}{d\ell} = \frac{\Gamma g}{32\pi} \left( 1 - \Gamma^2 \right),$$

$$\frac{dg}{d\ell} = \frac{g^2}{32\pi} \left( \Gamma^2 + 4\Gamma - 1 \right).$$

The ratio of these two equations yields a separable ODE for $\Gamma$ as a function of $g$. Inserting the resultant solution for $\Gamma(g)$ into (12) yields an integrable equation for $g(l)$; inserting that $g(l)$ into (11) yields a solvable equation for $\Gamma(l)$. We thereby find:

$$\frac{\Gamma}{(1+\Gamma)^2} = \frac{\Gamma g_0}{1+\Gamma} \right( \frac{1 - \Gamma_0}{1+\Gamma_0} \right)^2 \ell + \frac{\Gamma_0}{(1+\Gamma_0)^2},$$

$$g = \frac{\Gamma g_0}{\Gamma} \left( \frac{1 - \Gamma_0}{1+\Gamma_0} \right)^2 \left( \frac{1 + \Gamma}{1+\Gamma_0} \right)^2,$$

where $g_0$ and $\Gamma_0$ denote respectively the “bare” values of $g$ and $\Gamma$ (i.e., their values at $\ell = 0$). This solution implies a stable fixed point for $\Gamma_0 < 0$: as $\ell \rightarrow \infty$, $\Gamma(\ell) \rightarrow -1$, \[14\]
1 + \Gamma(\ell) \propto 1/\sqrt{\ell}, and g(\ell) \propto 1/\ell. The RG flow loci in $\Gamma$-$g$ space are illustrated in Fig. 1.

We will now use the trajectory integral matching method\textsuperscript{[22]} to compute $\langle |u'(\vec{q}, t)|^2 \rangle$, which determines the presence or absence of smectic order\textsuperscript{[22]} We restricted this calculation to the case $\Gamma > 0$, since, as we’ll see, only then is a stable smectic phase possible. Performing this standard procedure, we obtain

$$\langle |u'(\vec{q}, t)|^2 \rangle = \frac{\Delta(\ell^*)}{\nu_\perp(\ell^*)} \frac{e^{2\chi \ell^*}}{q_\perp^2 + \nu_\perp(\ell^*) \frac{q_\perp^2}{\ell^*}}$$

(15)

where $\ell^* \equiv \ln \Lambda/q$, with $\Lambda$ the ultraviolet cutoff. For small $q (\ll \Lambda)$, $\ell^* \gg 1$; in this limit, we find

$$\Delta(\ell^*) = \exp \left[ -2\chi(\ell^* - \ell_1) + C \int_{\ell_1}^{\ell^*} \frac{d\ell'}{\ell'} \right] \times \left( \frac{\Delta(\ell_1)}{\nu_\perp(\ell_1)} \right)$$

(16)

using Eqns. (13) and (14), if we cared.

Note that the integral over $\ell'$ in this expression converges as $\ell \to \infty$; hence, $\frac{\Delta(\ell^*)}{\nu_\perp(\ell^*)} \to -\frac{\lambda_\perp(\ell^*)}{\lambda_\perp(\ell^*)} = \frac{\lambda_\perp^0}{\lambda_\perp^0}$, the last equality following because the ratio of the $\lambda$’s does not renormalize, as can be seen from the recursion relation\textsuperscript{[8]} for the $\lambda$’s. Thus we have, finally,

$$\langle |u'(\vec{q}, t)|^2 \rangle = \frac{C'}{q_\perp^2 + \frac{\lambda_\perp^0}{\lambda_\perp^0} q_\perp^2}$$

(17)

which clearly scales as $1/q_\perp^2$ for all directions of wavevector $\vec{q}$. This scaling implies “logarithmic roughness” of the smectic layers in $d = 2$: that is,

$$\langle |u'(\vec{r}, t) - u'(\vec{r}', t)|^2 \rangle = C' \frac{\lambda_\perp^0}{\lambda_\perp^0} \ln \left[ \frac{(r_\perp - r_\perp')^2}{a^2} + \left( \frac{z - z'}{a^2} \right) \right]$$

(19)

which implies Eq. (1) via\textsuperscript{[22]} with $\eta = C'q_\perp^2 \sqrt{\lambda_\perp^0/\lambda_\perp^0} / 4\pi$; i.e., quasi-long-ranged smectic order\textsuperscript{[14]} This is the first real demonstration
that the roughness of the anisotropic KPZ equation is only logarithmic in $d = 2$; earlier arguments did not address the possibility that the integral in equation (10) could fail to converge as $\ell \to \infty$, thereby changing the scaling from that predicted by the linearized theory (as happens in $d = 3$ equilibrium smectics [21]).

When there is no number conservation, as in the model we are considering here, we will show later that, in both $d = 2$ and $d = 3$, the spatial Fourier transform of the number density $\rho$ is given by

$$\langle |\rho(\vec{q}, t)|^2 \rangle = C_1 q_2^2 \langle |u(\vec{q}, t)|^2 \rangle + C_2, \quad (21)$$

where $C_1$ and $C_2$ are constants. Since this remains finite as $q \to 0$, there are no giant number fluctuations [22].

Now we discuss the stability of the smectic phase in $d = 2$. Doing an anisotropic rescaling $r''_{\perp} = r'_{\perp}, \ z'' = \sqrt{\nu_{\perp}/\nu_{\parallel}} z'$, expressing $u'$ in terms of $\theta = 2\pi u'/a$, and ignoring the nonlinear terms, we can write Eq. (6) as

$$\partial_{\nu} \theta = \nu_{\perp} \nabla^2 \theta + f', \quad (22)$$

where the statistics of $f'$ are given by

$$\langle f'(\vec{r}'', t') f'(\vec{0}, 0) \rangle = \kappa \nu_{\perp} \delta(\vec{r}'') \delta(t'). \quad (23)$$

with $\kappa = \Delta (2\pi/a)^2/\sqrt{\nu_{\perp} \nu_{\parallel}}$. Eq. (22) is identical to the simplest relaxational equation of motion for an equilibrium XY model with $\theta$ being the angle of the magnetization, and $\nu_{\perp} = 2k_B T$. This mapping implies a dislocation unbinding phase transition [21] in active smectics in $d = 2$, when $\kappa = \pi$. For $\kappa > \pi$, the system is in the ordered fluid phase; for $\kappa < \pi$, the system is in the smectic phase.

The effect of the nonlinear terms can be included simply by replacing the bare $\kappa$ with its (finite) renormalized value. This implies that the transition occurs when

$$\lim_{\ell \to \infty} \kappa(\ell) = \pi. \quad (24)$$

Note that Eq. (21) is valid only if the period of the rescaled system on $u'$ is kept fixed at $a$, which requires no $u'$ rescaling (i.e., $\chi = 0$). Using our earlier solution to the recursion relations it is straightforward to show that

$$\lim_{\ell \to \infty} \kappa(\ell) \to -\frac{(1 - \Gamma_0)^2}{4 \Gamma_0} \kappa_0. \quad (25)$$

Using this result in Eq. (24) we obtain the phase boundary in terms of $\kappa_0$ and $\Gamma_0$ for $\Gamma_0 < 0$

$$\kappa_0 = -\frac{4\pi \Gamma_0}{(1 - \Gamma_0)^2}. \quad (26)$$

This phase diagram in the $\kappa_0$-$\Gamma_0$ parameter space is illustrated in Fig. 2.

We now turn to the case where the number of particles is conserved. In this case the fluctuation $\delta \rho = \rho - \rho_0$ of the density $\rho$ about its mean value $\rho_0$ becomes another important hydrodynamical variable. Number conservation implies $\partial_t \delta \rho = -\nabla \cdot \vec{j}$, where $\vec{j}$ is the number density current. Based on symmetry arguments, a gradient expansion of $\vec{j}$, keeping only "relevant" terms is given by $\vec{j} = \vec{j}_L + \vec{j}_{NL}$, where the linear piece

$$\vec{j}_L = -\left[j_0 + v_{\rho} \delta \rho + D_{\perp} \partial_{\perp} \delta \rho + v_{\parallel} \partial_{\parallel} \delta \rho u \right]
\left[c_{\perp} - w \right] \nabla_{\perp} \delta \rho + D_{\perp} \nabla_{\perp} \delta \rho - v_{\parallel} \nabla_{\parallel} \partial \delta \rho u - G_u \delta \rho \nabla_{\parallel} \delta \rho u - \vec{f}_\rho, \quad (27)$$

where $\vec{f}_\rho$ is a Gaussian noise with statistics

$$\langle f_{\rho L}(\vec{r}, t) f_{\rho L}(\vec{0}, 0) \rangle = \left(\Delta_{\perp} \delta_{\perp} + \Delta_{\parallel} \delta_{\parallel} \right) \delta(\vec{r}) \delta(t). \quad (28)$$

while the nonlinear piece is given by

$$\vec{j}_{NL} = -\left[\lambda_{\perp} \nabla_{\perp} \delta \rho \partial_{\perp} \partial_{\parallel} \delta \rho u + \lambda_{\parallel} \nabla_{\parallel} \delta \rho \partial_{\parallel} \partial_{\perp} \delta \rho u + \lambda_{\perp} \nabla_{\perp} \delta \rho \partial_{\parallel} \partial_{\parallel} \delta \rho u + g \nabla_{\perp} \delta \rho \partial_{\parallel} \partial_{\parallel} \delta \rho u + g \nabla_{\parallel} \delta \rho \partial_{\parallel} \partial_{\perp} \delta \rho u \right]. \quad (29)$$

Similar symmetry arguments and gradient expansions give the equation of motion for $u$, again keeping all relevant terms,

$$\partial_t u = v_0 + v_{\parallel} \partial_{\parallel} u + v_{\perp} \partial_{\perp} u + \nu_{\perp} \partial_{\perp}^2 u + \nu_{\parallel} \partial_{\parallel}^2 u + \nu_{\perp} \partial_{\parallel} \partial_{\perp} \delta \rho + \lambda_{\perp} \nabla_{\perp} \delta \rho \partial_{\parallel} \partial_{\parallel} \delta \rho u + g \nabla_{\parallel} \delta \rho \partial_{\parallel} \partial_{\perp} \delta \rho u \right] + g \nabla_{\perp} \delta \rho \partial_{\parallel} \partial_{\parallel} \delta \rho u + g \nabla_{\perp} \delta \rho \partial_{\parallel} \partial_{\perp} \delta \rho u + f_u. \quad (30)$$

where the noise $f_u$ has the same statistics as $f$ in Eq. (4).

If we neglect the non-linear terms in $\vec{j}$ and (30), a straightforward calculation shows that $\langle |u(\vec{q}, t)|^2 \rangle \sim 1/q^2$, which implies quasi-long-ranged smectic order in $d = 2$ and long-ranged order in $d = 3$. We also find that $\langle |\delta \rho(\vec{q}, t)|^2 \rangle$ goes to a finite value as $q \to 0$, which implies no giant number fluctuations in either $d = 2$ or $d = 3$.

Simple power counting shows that the nonlinear terms in $\vec{j}$ and (30) are irrelevant in $d = 3$, in the RG sense. Hence, these linear terms should apply in $d = 3$, at least in systems with sufficiently small non-linearities [21]. In $d = 2$, similar power counting shows that all of the nonlinear terms in $\vec{j}$ and (30) become marginal, and
hence, could potentially change the behavior at long wavelengths.

Supplementing the continuity equation for \( \delta \rho \) with a source term to reflect the tendency of birth and death to restore the local population density to its equilibrium value, and an additional, non-number-conserving noise reflecting statistical fluctuations in the local birth and death rate, as has been done\[11\] for flocks with polar orientational order, enables us to analyze number fluctuations in the Malthusian case as well. Dropping irrelevant terms, we obtain the equation of motion:

\[
\partial_t \delta \rho = \alpha \partial_z u + v^\perp \nabla_x^2 \rho + v^\parallel \partial_x^2 \rho - \delta \rho/\tau + f_{b-d}, \tag{31}
\]

where \( \tau \) is the characteristic relaxation time for density fluctuations to relax away due to birth and death\[11\], the \( \alpha \) term reflects the fact that symmetry allows the local birth and death rate to depend on the local layer spacing, and \( f_{b-d} \) is the aforementioned noise in the birth and death rate. We take \( f_{b-d} \) to be zero mean Gaussian white noise, with statistics \( \langle f_{b-d}(\vec{r}, t) f_{b-d}(\vec{r}', t') \rangle = 2\Delta_{b-d}\delta^3(\vec{r} - \vec{r}') \delta(t - t') \).

Fourier transforming equation (31) in space, and solving the resultant linear stochastic ODE for the correlations of \( \delta \rho \) gives\[27\], to leading order in \( q \), equation (21), with \( C_1 = \tau^2 \alpha^2 \) and \( C_2 = \tau \Delta_{b-d} \).

In conclusion, we have developed the hydrodynamic theory of apolar active smectic in both \( d = 2 \) and \( d = 3 \). We considered both the case in which the number of the particles is conserved, and that in which it is not. We have made various experimental predictions which can be tested. For the number conserving case an RG analysis of the nonlinear terms remains to be done.

We thank P. Romanczuk for invaluable discussions, and the MPIPKS, Dresden, where those discussions took place, for their support (financial and otherwise) and hospitality. JT thanks the U.S. National Science Foundation for their financial support through awards # EF-1137815 and 1006171; LC acknowledges support by the National Science Foundation of China (under Grant No. 11004241) and the Fundamental Research Funds for the Central Universities (under Grant No. 2010LKW09). LC also thanks the China Scholarship Fund for supporting his visit to the University of Oregon, where a portion of this work was done. He is also grateful to the University of Oregon’s Physics Department and Institute for Theoretical Science for their hospitality.

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[1] F. Schweitzer, *Brownian agents and active particles: collective dynamics in the natural and social sciences*, Springer series in synergetics (Springer, 2003).
[2] See, e.g., P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics*, Cambridge University Press (Cambridge, U.K.) (1995).
[3] J. Toner and Y. Tu, Phys. Rev. Lett. 75, 4326 (1995); Phys. Rev. E 58, 4828 (1998); J. Toner, Y. Tu, and S. Ramaswamy, Annals of Physics 318, 170 (2005).
[4] S. Ramaswamy, R. Aditi Simha, and J. Toner, Europhys. Lett. 62, 196 (2003).
[5] T. C. Adhyapak, S. Ramaswamy, J. Toner, preprint [arXiv:1204.2708].
[6] S. Jakubith, H. H. Rotermund, W. Engel, A. von Oertzen, and G. Ertl, Phys. Rev. Lett. 65, 3013 (1990).
[7] E. Bertin, M. Droz, G. Gregoire, J. Phys. A: Math. Theor. 42, 445001 (2009); S. Mishra, A. Baskaran, and M. C. Marchetti, Phys. Rev. E 81, 061916 (2010).
[8] T. Vicsek, Phys. Rev. Lett. 75, 1226 (1995); A. Czirok, H. E. Stanley, and T. Vicsek, J. Phys. A 30, 1375 (1997); T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, and O. Shochet, Phys. Rev. Lett. 75, 1226 (1995).
[9] P. G. de Gennes and J. Prost, *The Physics of Liquid Crystals*, 2nd ed. (Clarendon Press, Oxford, 1993).
[10] P. Romanczuk, L. Chen, H. Chate, and J. Toner, unpublished.
[11] J. Toner, Phys. Rev. Lett. 108, 088102 (2012).
[12] J. Toner and D. R. Nelson, Phys. Rev. B 23, 316 (1981).
[13] A. Caille, C.R. Acad. Sci. Ser. 8 274 , 891 (1972); T. C. Lubensky, Phys. Rev. Lett. 29, 206 (1972); P. G. de Gennes, J. Phys. (Paris) 30, C9-65 (1969); L.D. Landau and E. M. Lifshitz, *Statistical Physics*, 2nd ed., (Pergamon, Oxford, 1969), p. 402.
[14] J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973); J. M. Kosterlitz, J. Phys. C 7, 1046 (1974); D. Brewer, ed., Progress in low temperature physics vol. VIIb (North-Holland, Amsterdam, 1978).
[15] L. Chen and J. Toner, unpublished.
[16] In principle, the amplitude \( \phi_0 \) can also fluctuate; however, such fluctuations are non-hydrodynamic; that is, their relaxation rate remains non-zero as \( q \to 0 \). Hence, these fluctuations cannot be ignored in a hydrodynamic treatment.
[17] Such an "active tension" was discussed earlier as an activity-induced tension for a single membrane with active pumps; see [13].
[18] S. Ramaswamy, J. Toner and J. Prost, Phys. Rev. Lett. 84, 3494 (2000).
[19] D. E. Wolf, Phys. Rev. Lett. 67 1783 (1991).
[20] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. 56, 889 (1986).
[21] V. L. Berezinskii, Zh. Eksp. Teor. Fiz. 59, 907 (1970) [Sov. Phys. JETP 32, 493 (1971)]; B. I. Halperin and D. R. Nelson, Phys. Rev. Lett. 41, 121 (1978); D. R. Nelson and B. I. Halperin, Phys. Rev. B 19, 2457 (1979); A. P. Young, ibid. 19, 1855 (1979); J. Toner, Phys. Rev. B 26, 462 (1982).
[22] The presence or absence of smectic order depends on the behavior of the equal-time correlation function \( \langle u(\vec{r}, t) - u(\vec{r}', t) \rangle^2 \). In this problem \( \langle u(\vec{r}, t) - u(\vec{r}', t) \rangle^2 = \langle u(\vec{r}, t) - u(\vec{r}', t) \rangle^2 \) by Eq. (5). To calculate \( \langle u(\vec{r}, t) - u(\vec{r}', t) \rangle^2 \), we only need \( \langle u(\vec{q}, t) \rangle^2 \).
[23] D. R. Nelson, Phys. Rev. B 11, 3504 (1975).
[24] G. Grinstein and R. A. Pelcovits, Phys. Rev. Lett. 47,
For *apolar active fluids*, these are discussed in S. Ramaswamy, R. A. Simha, and J. Toner, Europhys. Lett. 62, 196 (2003).

It is likely that, for sufficiently large non-linear couplings, the $d = 3$ active polar smectic flows under the RG away from the *locally* stable linearized fixed point, to a new, strong coupling fixed point. Indeed, precisely this is believed to happen for the isotropic KPZ equation (see [20]), and the anisotropic KPZ equation (see, e.g., U.C. Tauber and E. Frey, Europhys. Lett. 59, 655 (2002).

Note that $\langle |\rho(\vec{q}, t)|^2 \rangle = \langle |\delta \rho(\vec{q}, t)|^2 \rangle$ for all $\vec{q} \neq \vec{0}$. 

856 (1981).