Exact three-body local correlations for excited states of the 1D Bose gas

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We derive an exact analytic expression for the three-body local correlations in the Lieb–Liniger model of 1D Bose gas with contact repulsion. The local three-body correlations control the thermalization and particle loss rates in the presence of terms which break integrability, as is realized in the case of 1D ultracold bosons. Our result is valid not only at finite temperature but also for a large class of non-thermal excited states in the thermodynamic limit. We present finite temperature calculations in the presence of external harmonic confinement within local density approximation, and for a highly excited state that resembles an experimentally realized configuration.

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When ultracold bosons are confined to move in only one dimension (1D), they provide a very clean realization of a seminal exactly solvable model introduced by Lieb and Liniger (LL) [4]. Being an integrable model, it has very special dynamics showing almost no relaxation in experiments [3].

This fact stimulated lots of theoretical interest to understand the thermalization of isolated 1D systems and the role of integrability as well as its breaking in this process [5–9]. In particular, it has been shown [8,10], that virtual excitations of bosons to higher transverse modes of a confining potential result in a weak three-body local interaction that violates integrability of the many-body problem. Thus it is important to understand three-body local correlations in the absence of integrability breaking terms first. Such correlations have also been measured recently using analysis of particle losses [11,12], density fluctuation statistics [13], time-of-flight correlation statistics [14], and scanning electron microscopy [15]. They provide a very sensitive test of coherence, and e.g., for Bose–Einstein condensates they increase by a factor of 6 = 3! if the temperature is raised to be much larger than the condensation temperature [16,17]. In spite of the LL model being integrable, analytical calculation of its correlation functions is notoriously hard [18]. Two-body local correlations in equilibrium can be simply obtained using the Hellmann–Feynman theorem and exact thermodynamics, and show excellent agreement with experiments [19]. Three body local correlations were analytically calculated only at zero temperature in a remarkable tour de force [20], as well as numerically in Ref. [21].

In this Letter, we exactly evaluate three-body local correlations in the thermodynamic limit for a large class of excited states which can be described by density matrices diagonal in the energy representation. In particular, we apply our method at finite temperatures and for highly excited states similar to the ones created in experiments [3], and we take into account external harmonic confinement within local density approximation (LDA). We note that local two-body correlations in 1D play the role of the “contact” introduced by S. Tan [22,23]. Similarly, three-body local correlations correspond to a three-body contact which is being actively explored [24]. Our exact results provide an important benchmark for such theories, as well as for numerical methods for simulating field theories in 1D [25].

The model.— The LL model describes a system of identical bosons in 1D interacting via a Dirac-delta potential. The Hamiltonian in second quantized formulation is given by

\[
H = \int_0^L dx \frac{\hbar^2}{2m} \left( \partial_x \psi^\dagger \partial_x \psi + c \psi^\dagger \psi^\dagger \psi \psi \right),
\]

where \(c > 0\) in the repulsive regime we wish to study, and \(m\) is the atomic mass. The dimensionless coupling constant is given by \(\gamma = c/n\), where \(n = N/L\) is the density of the gas. We will express temperature \(T\) in dimensionless units \(\tau = T/T_D\), where \(T_D = \hbar^2 n^2/(2mk_B)\) is the quantum degeneracy temperature.

The exact thermodynamics of the model can be obtained via Bethe Ansatz [4,15]. Each eigenstate of the system with \(N\) particles on a ring of circumference \(L\) is characterized by a distinct set of quantum numbers \(\{I_j\}\) that are integers (half-integers) for \(N\) odd (even). The wave function can be expressed in terms of \(N\) quasimomenta \(\{p_j\}\) that satisfy a set of algebraic equations

\[
Lp_j + \sum_{k=1}^N \theta(p_j - p_k) = 2\pi I_j,
\]

where \(\theta(p) = 2 \arctan(p/c)\). The wave function is identically zero if any two of the \(\{I_j\}\) coincide, which is reminiscent of the Pauli principle for fermions. In the Tonks–Girardeau (TG) limit \(c \to \infty\), \(\{I_j\}\) correspond to the quantum numbers of occupied single-particle states of free fermions.

In the thermodynamic limit, if one wants to consider a mixed state diagonal in the energy basis, this is achieved by introducing a filling fraction \(0 < f_I < 1\) in the space of quantum numbers, which plays a role similar to the occupation number of free fermions. All results of the present Letter are valid for \(f_I\) which have a finite thermodynamic limit at constant \(I/N\); the limiting function
should be piecewise continuous and normalized. For calculations, it is more convenient to define a function $f(p)$ in terms of the quasimomenta: denoting by $\rho(p)$ the maximal allowed density of quasimomenta in the vicinity of $p$, the quasimomenta density for a mixed state is given by $f(p)\rho(p)$.

Since all quasimomenta are coupled to each other by Eq. (2), the density $\rho(p)$ is not independent of $f(p)$: it satisfies the integral equation and normalization condition

$$\rho(p) = \frac{1}{2\pi} + \int \frac{dp'}{2\pi} f(p') \varphi(p - p') \rho(p'),$$

$$n = \int dp f(p) \rho(p),$$

with the kernel $\varphi(p) = 2c/(p^2 + c^2)$. In thermal equilibrium, $f(p)$ has to satisfy a set of nonlinear integral equations [20, 27], but our results will be valid for more general $f(p)$.

**Local correlations.**— The local $k$-body correlation functions are defined as

$$g_k(\gamma, \tau) = \frac{\langle \psi^\dagger_k(x) \psi^k(x) \rangle}{n^k}.$$  

The first two of them are relatively easy to calculate: $g_1 = 1$, while $g_2$ in equilibrium is given by the Hellmann–Feynman theorem [19, 27].

Here we report the results for $k = 2$ and $k = 3$ for general $f(p)$, which can be written in terms of functions $h_m(p)$ ($m = 1, 2$) satisfying the following integral equations:

$$h_m(p) = p^m + \int \frac{dp'}{2\pi} f(p') \varphi(p - p') h_m(p').$$

In the case of $g_2$ the final formula is

$$g_2(\gamma, \tau) = \frac{2\gamma^2}{c^3} \int \frac{dp}{2\pi} f(p) \left[ 2\pi p \rho(p) p^2 - h_1(p) \right],$$

which agrees with the result of the Hellmann–Feynman theorem for thermal equilibrium [27], but is more general. Similarly, for $k = 3$ the final expression is given by

$$g_3(\gamma, \tau) = \frac{\gamma^3}{c^4} \int \frac{dp}{2\pi} f(p) \left[ (p^4 + c^2 p^2) 2\pi \rho(p) - (4p^2 + (1 + 2/\gamma)c^2) p h_1(p) + 3p^2 h_2(p) \right] + \frac{2\gamma^3}{c^4} \left( \int dp f(p) \rho(p) \right)^2.$$  

In the case when $f(p)$ is even, in equilibrium for example, the last term in Eq. (7) is zero because the integrand is odd in $p$. Both Eq. (6) and Eq. (7) are Galilean invariant expressions [27]. In the following we will first consider an equilibrium case and then will proceed to highly excited states.

In Fig. 1 we plot the result in thermal equilibrium for fixed $\tau$ as a function of the coupling $\gamma$. In particular, at zero temperature our result agrees with that of Ref. [20], up to the precision of the numerical evaluation of both expressions, $\approx 10^{-3}$. The behavior of $g_3$ is qualitatively similar to that of $g_2$ analyzed in Ref. [19] and it distinguishes three different physical regimes: (a) $\gamma \gtrsim \max(1, \sqrt{\tau})$, strong coupling (TG) regime, $g_3 \ll 1$; (b) $\tau^2 \lesssim \gamma \lesssim 1$, quasicondensate regime, $g_3 \approx 1$; (c) $\gamma \lesssim \min(\tau^2, \sqrt{\tau})$, decoherent regime, $g_3 \gtrsim 6$. In the inset of Fig. 1 the large $\gamma$ asymptotics are plotted together with the analytic forms of Ref. [19]: $g_3 \sim 16\tau^6/(15\gamma^6)$ for $\tau = 0$ and $g_3 \sim 9\tau^3/\gamma^6$ for $\tau^2 \gg \tau \gg 1$.

**Harmonic traps.**— Next we turn to the experimentally more realistic case of atoms confined in a waveguide with a harmonic longitudinal potential. The 1D regime is reached if $\mu, k_B T \ll \hbar \omega_z$, where $\mu$ is the chemical potential and $\omega_z$ is the transverse oscillator frequency [25]. If the density profile in the trap varies smoothly, the correlations can be calculated by combining our exact results with LDA [29]. The relevant properties of the gas can be characterized by the LL coupling $\gamma_0$ and the temperature parameter $\tau_0$ at the center of the trap. In Fig. 2 we plot the three-body correlator $g_3(\gamma_0, \tau_0)$ at the trap center and the averaged average, $g_3(\gamma_0, \tau_0) = \int dx \psi^\dagger_3(x) \psi^3(x) / \left( \int dx n^3(x) \right)$, against the dimensionless temperature $\tau_0$ for different fixed values of $\gamma_0$. Similarly to the results of Ref. [29] for the case of $g_2$, we find that unless the coupling $\gamma_0$ is very small, $g_3(\gamma_0, \tau_0) \approx \overline{g}_3(\gamma_0, \tau_0)$ at any temperature.

The curves in Fig. 2 are related to the observed change in time of the particle loss rate in Ref. [12]. With increasing temperature, the three-body correlations grow according to our result, which leads to a higher probability of inelastic three-particle processes which in turn
raises further the temperature of the gas. This positive feedback causes a non-trivial dependence of the particle loss on time, and a detailed analysis of heating mechanisms is needed to describe the time dependences of the loss rates.

Three-body correlations for highly excited states.— Since Eq. (8) is valid for general distributions \( f(p) \), we can use our results (6) and (7) in situations where the system is neither in equilibrium nor is in its ground state. We will illustrate this by considering a state which is motivated by the experiment of Kinoshita et al. [3], where each atom was put in a momentum superposition state, after which the two clouds performed many oscillations without observable thermalization. The state created in the experiment is not an eigenstate, and the harmonic trap might play an important role. However, let us consider here a simple “caricature” eigenstate which might capture the behavior of \( g_3 \), and hence the role of integrability breaking, in these experiments. This state is characterized by an \( f(p) \) consisting of two disjoint rectangular “Fermi steps” symmetric with respect to \( p = 0 \) at zero temperature: \( f(p) = \theta(p^2 - p_2^2) - \theta(p^2 - p_1^2) \) with \( p_2 > p_1 > 0 \) (see inset of Fig. 3). We are interested in the dependence of \( g_{3ex}^\infty(\gamma, p_1) \) on the “inner Fermi quasimomentum” \( p_1 \). If \( p_1 \) is fixed then the “outer Fermi quasimomentum”, \( p_2 \), is determined from the normalization (3). The momentum kick in Ref. [3] corresponds to \( p_1/c \) of order one. In Fig. 3 we plot \( g_{3ex}^\infty(\gamma, p_1) \) for fixed values of \( \gamma \) as a function of \( p_1 \). We find that the correlations grow with the momentum of the kick and they can become greater than 1. For large \( p_1 \), the quasimomentum distributions of left and right goers become approximately independent of each other. However, to obtain the correct limit as \( p_1 \rightarrow \infty \) one needs to take into account deviations of \( \theta(2p_1/c \gg 1) \) in Eq. (3) from \( \pi \). This results in \( g_{3ex}^\infty(\gamma, \infty) = [g_3(2\gamma) + 9g_3(2\gamma)]/4 \); in particular, \( g_{3ex}^\infty(\gamma \rightarrow 0, \infty) = 5/2 \). Similarly, \( g_{3ex}^\infty(\gamma \rightarrow 0, \infty) = [g_2(2\gamma) + 2]/2, \) and \( g_{3ex}^\infty(\gamma \rightarrow 0, \infty) = 3/2 \).

Derivation of Eqs. (6) and (7).— In Ref. [31] a novel method was proposed to calculate the \( g_k \) correlators based on the observation that the LL model can be viewed as the combined non-relativistic, weak coupling limit of the sinh–Gordon model. The resulting formula reads as

\[
g_k = \sum_{s=\pm} \sum_{k=\pm} 1 \int \frac{dP_j}{2\pi} f(P_j) \frac{\gamma}{F_s^k(p_1, \ldots, p_s)},
\]

where the form factors \( F_s^k(p_1, \ldots, p_s) \) are the infinite volume single-particle diagonal matrix elements of the operator \( \psi_k\dagger\psi^k \), which can be obtained from known sinh–Gordon form factors [32, 33]. These series were investigated previously by truncating them after the first few terms [31]. Here we resum these series to all orders obtaining closed analytical expressions for the local correlations.

It has been proven in Ref. [30] that

\[
F_s^{(1)} = \frac{1}{c^3} \sum_p \varphi(p_{12})\varphi(p_{23})\cdots\varphi(p_{s-1,s}),
\]

\[
F_s^{(2)} = \frac{1}{c^3} \sum_p \varphi(p_{12})\varphi(p_{23})\cdots\varphi(p_{s-1,s}) p_1^2 s,
\]

and based on evaluations performed in Mathematica for the first few \( F_s^{(3)} \) we conjecture

\[
F_s^{(3)} = \frac{1}{c^3} \sum_p \varphi(p_{12})\varphi(p_{23})\cdots\varphi(p_{s-1,s}) \times \frac{1}{2} p_{1,s} [p_{3,s}^3 - (p_{12}^3 + p_{23}^3 + \cdots + p_{s-1,s}^3)].
\]
where \( p_{ij} = p_i - p_j \) and \( \sum_{\rho} \) denotes a sum over all permutations of \( \{ p_{ij} \} \). Below we will illustrate how series \[8\] can be analytically resummed for \( g_2 \), and details of similar calculations for \( g_3 \) are presented in EPAPS [27].

We will use abbreviations \( \Delta p = dp/(2\pi) f(p) \) and \( \varphi_{ij} = \varphi(p_{ij}) \). Using the symmetries of the integrand in Eq. [8], we have for \( c^3 g_2(\gamma, \tau)/(2\gamma^2) \)

\[
\int_1^\infty \sum_{s=2}^\infty \int_{p_1} \cdots \int_{p_s} \varphi_{12} \cdots \varphi_{s-1,s}(p_1 - p_s)^2 =
\int_{p_1} \left[ \int_{p_2} \varphi_{12} + \int_{p_3} \varphi_{12} \varphi_{23} + \cdots \right] -
\int_{p_1} \left[ \int_{p_2} \varphi_{12} p_2 + \int_{p_3} \varphi_{12} \varphi_{23} p_3 + \cdots \right] =
\int_{p_1} \frac{1}{p_1^2} [2\pi f(p_1) - 1] - \int_{p_1} \left[ h_1(p_1) - 1 \right],
\]

where the terms in the first square bracket coincide with the iterative solution of the integral equation [24]. Similarly, comparison of the terms in the second bracket with the iterative solution of Eq. [9] leads to \( h_1(p_1) - 1 \). Now the second terms in the parentheses cancel each other and we obtain Eq. [6].

In summary, we derived an exact formula for the local three-body correlation in a paradigmatic system, the 1D Lieb-Liniger Bose gas. Given that exact expressions for correlation functions are scarce even in integrable models, we emphasize the analytic nature of our result. Our non-perturbative formula is valid at any temperature for\( g_2 \), and details of similar calculations for \( g_3 \) are presented in EPAPS [27].
Supplementary Material for EPAPS

Exact three-body local correlations for excited states of the 1D Bose gas

THERMAL EQUILIBRIUM

In thermal equilibrium \[26\], the filling fraction can be written as \( f(p) = (1 + e^{\varepsilon(p)})^{-1} \), where the pseudo-energy \( \varepsilon(p) \) satisfies the nonlinear integral equation

\[
\varepsilon(p) = -\frac{\mu}{k_B T} + \frac{\hbar^2 p^2}{2mk_B T} - \int \frac{dp'}{2\pi} \varphi(p-p') \log \left( 1 + e^{-\varepsilon(p')} \right), \tag{11}
\]

and the free energy is given by

\[
F = L \left( \mu n - k_B T \int \frac{dp}{2\pi} \log(1 + e^{-\varepsilon(p)}) \right). \tag{12}
\]

Eq. (11) is coupled to the equations for the density

\[
\rho(p) = \frac{1}{2\pi} + \int \frac{dp'}{2\pi} f(p') \varphi(p-p') \rho(p'), \tag{13a}
\]

\[
n = \int dp f(p) \rho(p), \tag{13b}
\]

At \( T = 0 \), Eqs. (13) and (11) decouple and \( 1/(1 + e^{\varepsilon(p)}) \) becomes a Fermi step function \( f(p) = \theta(p^2 - p_F^2) \) where \( p_F \) is determined from \( \int_{-p_F}^{p_F} dp \rho(p) = n \).

DIMENSIONLESS VARIABLES

It is useful to introduce dimensionless quasimomenta \( q = p/c \), and to change slightly the notation for the densities: \( \rho(c q) \to \rho(q) \), and similarly for \( f(q) \) and \( \varepsilon(q) \). Then the equations for the density become

\[
\rho(q) = \frac{1}{2\pi} + \int \frac{dq'}{2\pi} f(q') \varphi(q-q') \rho(q'), \tag{14a}
\]

\[
\frac{1}{\gamma} = \int dq f(q) \rho(q), \tag{14b}
\]

where the kernel is \( \varphi(q) = 2/(q^2 + 1) \). In thermal equilibrium, \( f(q) = (1 + e^{\varepsilon(q)})^{-1} \), and the equation for \( \varepsilon(q) \) is

\[
\varepsilon(q) = -\alpha + \frac{q^2 \gamma^2}{\tau} - \int \frac{dq'}{2\pi} \varphi(q-q') \log \left( 1 + e^{-\varepsilon(q')} \right), \tag{15}
\]

The dimensionless chemical potential \( \alpha = \mu/k_B T \) needs to be determined by the self-consistent solution of Eqs. (14),(15). Once \( \varepsilon(q) \) is found, the free energy \( F \) is given by

\[
F = L \tau \frac{\hbar^2 n^3}{2m} \left( \alpha - \gamma \int \frac{dq}{2\pi} \log(1 + e^{-\varepsilon(q)}) \right). \tag{16}
\]

The dimensionless form of the functions \( h_m \) are defined as

\[
h_m(q) = q^m + \int \frac{dq'}{2\pi} f(q') \varphi(q-q') h_m(q'). \tag{17}
\]

HELLMANN–FEYNMAN THEOREM

For completeness, in this section we derive a formula which was established earlier in Ref. [31] based on the Hellmann–Feynman theorem. This theorem states that

\[
\frac{2}{L} \left\langle \frac{\partial H}{\partial g} \right\rangle = \langle \psi^\dagger \psi^\dagger \psi \psi \rangle = 2 \frac{d}{dg} \left( \frac{F}{L} \right), \tag{18}
\]
where the free energy can be calculated from Eq. (16). In dimensionless variables,
\[
g_2(\gamma, \tau) = \tau \frac{d}{d\gamma} \left( \alpha - \gamma \int \frac{dq}{2\pi} \log(1 + e^{-\varepsilon(q)}) \right).
\]
Now we recast it in a form which does not exhibit explicit derivatives. We will need the derivative of Eq. (15) with respect to \( \gamma \):
\[
\varepsilon'(q) \equiv \frac{d\varepsilon(q)}{d\gamma} = -\alpha' + 2q^2 \gamma + \int \frac{dq}{2\pi} \varphi(q - \bar{q}) \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}}.
\]
We start by writing
\[
g_2(\gamma, \tau) = \tau \alpha' - \int \frac{dq}{2\pi} \log(1 + e^{-\varepsilon(q)}) + \tau \gamma \int \frac{dq}{2\pi} \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}},
\]
where the prime denotes derivative with respect to \( \gamma \). The key step is substituting for \( 1/2\pi \) in the last integral the rest of Eq. (14a):
\[
\int \frac{dq}{2\pi} \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}} = \int dq \left( \rho(q) - \int \frac{dq}{2\pi} \varphi(q - \bar{q}) \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}} \right) \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}} =
\int dq \rho(q) \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}} - \int \frac{dq}{2\pi} \varphi(q - \bar{q}) \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}}.
\]
In the last step we used the fact that the kernel is an even function. Now we use Eq. (20) to express the convolution term and continue with the equalities as
\[
\int \frac{dq}{2\pi} \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}} = \int dq \rho(q) \frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)}} - \int \frac{dq}{2\pi} \varphi(q - \bar{q}) \left( \varepsilon'(q) + \alpha' - \frac{2q^2 \gamma}{\tau} \right) =
-\frac{\alpha'}{\gamma} + 2\frac{\gamma}{\tau} \int d\bar{q} \frac{\rho(q)}{1 + e^{\varepsilon(q)}} \bar{q}^2,
\]
where the terms with \( \varepsilon' \) dropped out and we used Eq. (14b). Plugging this result into Eq. (21) even the \( \alpha' \) terms cancel and we finally arrive at
\[
g_2(\gamma, \tau) = 2\gamma^2 \int dq \frac{\rho(q)}{1 + e^{\varepsilon(q)}} q^2 - \tau \int dq \frac{1}{2\pi} \log(1 + e^{-\varepsilon(q)}).
\]
We can rewrite the second integral using partial integration:
\[
g_2(\gamma, \tau) = 2\gamma^2 \int dq \frac{\rho(q)}{1 + e^{\varepsilon(q)}} q^2 - \tau \int dq \frac{1}{2\pi} q \frac{1}{1 + e^{\varepsilon(q)}} \frac{d\varepsilon(q)}{dq}.
\]
Differentiating Eq. (15) with respect to \( q \) we obtain
\[
\frac{d\varepsilon(q)}{dq} = \frac{2\gamma^2}{\tau} q + \int dq \frac{1}{2\pi} \frac{1}{1 + e^{\varepsilon(q)}} \varphi(q - \bar{q}) \frac{d\varepsilon(q)}{dq},
\]
which shows that \( d\varepsilon(q)/dq \) satisfies the same integral equation as \( 2\gamma^2/\tau h_1(q) \) (see Eq. (17)). Thus expression (25) is clearly the special case of the general result
\[
g_2(\gamma, \tau) = 2\gamma^2 \int dq f(q) \rho(q) q^2 - 2\gamma^2 \int dq \frac{1}{2\pi} f(q) h_1(q) q,
\]
for equilibrium when \( f(q) = 1/(1 + e^{\varepsilon(q)}) \).

In the original dimensionful variables one also has to take care of the coupling constant dependence of the kernel. It is interesting to note that the analogous derivation for the \( T = 0 \) case is more subtle because then one has to differentiate the boundary of integration explicitly.
SUMMING UP THE SERIES \[ \text{FOR} \ k = 3 \]

Every permutation in the form factors contains all the momenta and not only the first and the last few of them, which seems to render the procedure used in the main text for \( k = 2 \) unfeasible. However, this difficulty can be overcome by rewriting the form factors exploiting their special structure:

\[
F^{(3)}_s = \frac{1}{2} \sum_p \varphi(q_{12})\varphi(q_{23}) \cdots \varphi(q_{s-1,s}) \left[ q_{1,s}^4 - q_{1,s} \sum_{d=1}^{s-1} q_{d,d+1} \left( \frac{2}{\varphi(q_{d,d+1})} - 1 \right) \right] = \\
\frac{1}{2} \sum_p \left\{ \varphi(q_{12})\varphi(q_{23}) \cdots \varphi(q_{s-1,s}) \left( q_{1,s}^4 + q_{1,s}^2 \right) - 2q_{1,s} \sum_{d=1}^{s-1} \varphi(q_{12}) \cdots \varphi(q_{d-1,d})\varphi(q_{d+1,d+2}) \cdots \varphi(q_{s-1,s})q_{d,d+1} \right\}, \quad (28)
\]

where in the last term the kernel \( \varphi(q_{d,d+1}) \) is missing. We used here the explicit form of the kernel and \( \sum_{d=1}^{s-1} q_{d,d+1} = q_{1,s} \). In the last term in the second line the chain of kernels is broken and the polynomial only depends on momenta at the ends of the resulting chains, which allows us to use the same technique as before.

**First term**

Let us start with expanding the polynomial part

\[
q_{1,s}^4 + q_{1,s}^2 = (q_1^4 + q_1^2) - (4q_1^3 + q_1)q_s + 3q_1^2q_s^2 + (q_3^4 + q_3^2) - (4q_3^3 + q_3)q_1 + 3q_3^2q_1^2. \quad (29)
\]

The second line of the expansion can be obtained from the first line by the transformation \( q_1 \leftrightarrow q_s \), that is, by a reflection of the indices. Since the kernel \( \varphi(q) \) is an even function, the form factor is invariant under this transformation (actually, every single term in the permutation sum is invariant). We are thus allowed to focus on the terms in the first line, taking the second line into account by multiplying the result by \( 2 \) which cancels the overall \( 1/2 \).

- **Terms containing** \( q_1 \) **only.**
  The subseries corresponding to the terms \( \left( q_1^4 + q_1^2 \right) \) is

\[
\int \tilde{d}q_1 \left( q_1^4 + q_1^2 \right) \left( \int \tilde{d}q_2 \int \tilde{d}q_3 \varphi(q_{12})\varphi(q_{23}) + \int \tilde{d}q_2 \int \tilde{d}q_3 \varphi(q_{12})\varphi(q_{23})\varphi(q_{34}) + \cdots \right) = \\
\int \tilde{d}q_1 \left( q_1^4 + q_1^2 \right) \left[ 2\pi \rho(q_1) - 1 - \int \tilde{d}q_2 \varphi(q_{12}) \right], \quad (31)
\]

where we used Eq. \[14a\].

- **Terms containing** \( q_1, q_s \).
  For the term \( -\left( 4q_1^3 + q_1 \right)q_s \) we have the series

\[
- \int \tilde{d}q_1 \left( 4q_1^3 + q_1 \right) \left( \int \tilde{d}q_2 \int \tilde{d}q_3 \varphi(q_{12})\varphi(q_{23})q_3 + \int \tilde{d}q_2 \int \tilde{d}q_3 \varphi(q_{12})\varphi(q_{23})\varphi(q_{34})q_4 + \cdots \right) = \\
- \int \tilde{d}q_1 \left( 4q_1^3 + q_1 \right) \left[ h_1(q_1) - q_1 - \int \tilde{d}q_2 \varphi(q_{12})q_2 \right], \quad (32)
\]

where in the last step we used Eq. \[17\].

- **Finally,** in the case of \( 3q_1^2q_s^2 \) we are similarly led to

\[
\int \tilde{d}q_1 3q_1^2 \left( \int \tilde{d}q_2 \int \tilde{d}q_3 \varphi(q_{12})\varphi(q_{23})q_3^2 + \int \tilde{d}q_2 \int \tilde{d}q_3 \varphi(q_{12})\varphi(q_{23})\varphi(q_{34})q_4^2 + \cdots \right) = \\
3 \int \tilde{d}q_1 q_1^2 \left[ h_2(q_1) - q_1^2 - \int \tilde{d}q_2 \varphi(q_{12})q_2^2 \right]. \quad (33)
\]
Combining Eqs. (31-33) we find that several terms cancel each other and we are left with
\[
\int \hat{d}q (q^4 + q^2) 2\pi \rho(q) - \int \hat{d}q (4q^3 + q) h_1(q) + 3 \int \hat{d}q q^2 h_2(q) \\
- \int \hat{d}q_1 \int \hat{d}q_2 \varphi(q_{12}) \left((q_1^4 - 4q_1^3 q_2 + 3q_1^2 q_2^2) + (q_1^2 - q_1 q_2)\right). \tag{34}
\]
Using the symmetry of \(\varphi(q_1 - q_2)\) and then its explicit expression we can write the last term in the nice form
\[
-\frac{1}{2} \int \hat{d}q_1 \int \hat{d}q_2 \varphi(q_{12})(q_{12}^4 + q_{12}^2) = -\frac{1}{2} \int \hat{d}q_1 \int \hat{d}q_2 2q_{12}^2, \tag{35}
\]
so the total contribution from the first term of the form factor is
\[
\int \hat{d}q (q^4 + q^2) 2\pi \rho(q) - \int \hat{d}q (4q^3 + q) h_1(q) + 3 \int \hat{d}q q^2 h_2(q) - \int \hat{d}q_1 \int \hat{d}q_2 q_{12}^2. \tag{36}
\]

**Second term**

The polynomial part is
\[
q_{1,s}q_{d,d+1} = (q_1 q_d - q_1 q_{d+1}) + (q_{d+1} q_s - q_d q_s). \tag{37}
\]
Again, the second two terms give the same contribution as the first two due to the sum over \(d\), so we focus on the first two terms and multiply the result by 2 in the end. Due to the missing kernel, the \(s\)-fold multiple integrals split into the product of a \(d\)-fold and an \((s-d)\)-fold integral.

- \(-2q_1 q_d\).

Taking into account all the prefactors, the subseries corresponding to the first term in the first parenthesis is
\[
-2 \int \hat{d}q_1 q_1^2 \left(\int \hat{d}q_2 \int \hat{d}q_3 \varphi(q_{23}) + \int \hat{d}q_2 \int \hat{d}q_3 \int \hat{d}q_4 \varphi(q_{23}) \varphi(q_{34}) + \ldots\right) - \\
2 \int \hat{d}q_1 \int \hat{d}q_2 \varphi(q_{12}) q_1 q_2 \left(\int \hat{d}q_3 + \int \hat{d}q_3 \int \hat{d}q_4 \varphi(q_{34}) + \ldots\right) - \\
2 \int \hat{d}q_1 \int \hat{d}q_2 \int \hat{d}q_3 \varphi(q_{12}) \varphi(q_{23}) q_1 q_3 \left(\int \hat{d}q_4 + \int \hat{d}q_4 \int \hat{d}q_5 \varphi(q_{45}) + \ldots\right) + \ldots, \tag{38}
\]
where we reshuffled the series: the first line contains the \(d = 1\) terms, the second line contains the \(d = 2\) terms and so on. Let us add and subtract the term 2 \(\int \hat{d}q_1 \int \hat{d}q_2 q_1^2\) from the first line. Then all the parentheses become equal to 1/\(\gamma\) (c.f. Eq. (14b)). This can be factored out leaving another infinite series
\[
-2 \frac{\gamma}{\gamma} \int \hat{d}q_1 q_1 \left(q_1 + \int \hat{d}q_2 \varphi(q_{12}) q_2 + \int \hat{d}q_2 \int \hat{d}q_3 \varphi(q_{12}) \varphi(q_{23}) q_3 + \ldots\right) = -2 \frac{\gamma}{\gamma} \int \hat{d}q_1 q_1 h_1(q_1). \tag{39}
\]
Thus the contribution of the first term in Eq. (37) is
\[
-2 \frac{\gamma}{\gamma} \int \hat{d}q q h_1(q) + 2 \int \hat{d}q_1 \int \hat{d}q_2 q_1^2. \tag{40}
\]

- \(2q_1 q_{d+1}\).

We can proceed in the same way for the second term in the first parenthesis of Eq. (37).
\[
2 \int \hat{d}q_1 q_1 \left(\int \hat{d}q_2 q_2 \int \hat{d}q_3 \varphi(q_{23}) + \int \hat{d}q_2 q_2 \int \hat{d}q_3 \int \hat{d}q_4 \varphi(q_{23}) \varphi(q_{34}) + \ldots\right) + \\
2 \int \hat{d}q_1 q_1 \int \hat{d}q_2 \varphi(q_{12}) \left(\int \hat{d}q_3 q_3 + \int \hat{d}q_3 \int \hat{d}q_4 q_3 \varphi(q_{34}) + \ldots\right) + \\
2 \int \hat{d}q_1 q_1 \int \hat{d}q_2 \int \hat{d}q_3 \varphi(q_{12}) \varphi(q_{23}) \left(\int \hat{d}q_4 q_4 + \int \hat{d}q_4 \int \hat{d}q_5 \varphi(q_{45}) + \ldots\right) + \ldots, \tag{41}
\]
where we reshuffled the series similarly to the previous case. Let us add and subtract now the term \(2 \int \dd q_1 \dd q_2 q_1 q_2\), so that all the parentheses above become equal to \(\int \dd q 2 \pi \rho(q) q\). After factoring these out we are left with the same infinite series, so the contribution of the second term in Eq. (37) is

\[
2 \left( \int \dd q 2 \pi \rho(q) q \right)^2 - 2 \int \dd q_1 \int \dd q_2 q_1 q_2. \tag{42}
\]

Thus the total contribution of the second line of Eq. (28) is

\[
- \frac{2}{\gamma} \int \dd q q h_1(q) + 2 \left( \int \dd q 2 \pi \rho(q) q \right)^2 + \int \dd q_1 \int \dd q_2 (2q_1^2 - 2q_1 q_2). \tag{43}
\]

**The final result: closed expression for \(g_3(\gamma, \tau)\)**

The final result is given by the sum of the partial results (36) and (43). The last terms of these expressions exactly cancel each other because

\[
\int \dd q_1 \int \dd q_2 q_1^2 = \int \dd q_1 \int \dd q_2 (2q_1^2 - 2q_1 q_2), \tag{44}
\]

so we arrive at the dimensionless form of Eq. (7)

\[
\frac{g_3(\gamma, \tau)}{\gamma^3} = \int \dd q (q^4 + q^2) 2 \pi \rho(q) - \int \dd q \left( 4q^2 + 1 + \frac{2}{\gamma} \right) q h_1(q) + 3 \int \dd q q^2 h_2(q) + 2 \left( \int \dd q 2 \pi \rho(q) q \right)^2. \tag{45}
\]

**GALILEAN INVARIANCE**

In this section we show that our expressions (27), (45) are invariant under a Galilean boost, which provides a non-trivial consistency check of our results.

The integral equations after a boost with momentum \(b\) read as

\[
\tilde{h}_m(q) = q^m + \int \frac{\dd q'}{2\pi} f(q' - b) \varphi(q - q') \tilde{h}_m(q'). \tag{46}
\]

Note that as a special case \(h_0(q) = 2 \pi \rho(q)\). After a simultaneous shift in both variables \(q\) and \(q'\) we obtain

\[
\tilde{h}_m(q + b) = (q + b)^m + \int \frac{\dd q'}{2\pi} f(q') \varphi(q - q') \tilde{h}_m(q' + b). \tag{47}
\]

From the iterative solution of these equations it is easy to see that

\[
\begin{align}
\tilde{h}_0(q + b) &= h_0(q), \\
\tilde{h}_1(q + b) &= h_1(q) + b h_0(q), \\
\tilde{h}_2(q + b) &= h_2(q) + 2b h_1(q) + b^2 h_0(q). \tag{48}
\end{align}
\]

Now we can calculate the boosted version of \(g_2\). From Eq. (27)

\[
\frac{\dd g_2}{\gamma^2} = \int \frac{\dd q}{2\pi} f(q - b) \left[ 2 \pi \hat{p}(q) q^2 - f(q - b) \tilde{h}_1(q) q \right]. \tag{49}
\]

Shifting the integration variables and using Eqs. (48) we arrive at

\[
\frac{\dd \tilde{g}_2}{\gamma^2} = \frac{g_2}{\gamma^2} + \int \frac{\dd q}{2\pi} f(q) \left[ h_0(q) (2bq + b^2) - h_1(q) b - h_0(q)(q + b)b \right], \tag{50}
\]

where we used \(h_0(q) = 2 \pi \rho(q)\). Now due to \(\int \dd q f(q) h_0(q) q = \int \dd q f(q) h_1(q)\) the integral can be readily shown to vanish, implying the Galilean invariance of \(g_2\).

The invariance of \(g_3\) can be shown along the same lines.
LOCAL CORRELATIONS IN THE “CARICATURE STATE”

In this section we discuss the derivation of the asymptotic values of the local correlations for an infinite momentum kick. Let us denote the various functions in this excited state by the superscript “ex”. The dimensionless integral equations can be written as

\[
\tag{51}
\hat{h}_m^{ex}(q) = q^m + \left( \frac{\int_{-q_1}^{q_1} dq' + \int_{q_1}^{q_2} dq'}{2\pi} \right) \varphi(q-q') \hat{h}_m^{ex}(q) = q^m + \int_{q_1}^{q_2} \frac{dq'}{2\pi} [\varphi(q-q') + (-1)^m \varphi(q+q')] \hat{h}_m^{ex}(q').
\]

In particular, \( \hat{h}_0^{ex}(q) = 2\pi \rho^{ex}(q) \) with normalization

\[
\int_{-q_1}^{q_2} \frac{dq}{2\pi} \hat{h}_0^{ex}(q) = \frac{1}{2\gamma}.
\]

We want to use as a reference state the ground state solution on the interval \([- (q_2 - q_1)/2, (q_2 - q_1)/2] \equiv [-d, d] \), so we change our variables as

\[
q = k + \frac{q_1 + q_2}{2} = k + a.
\]

This leads to

\[
\hat{h}_m^{ex}(k+a) = \hat{h}_m(k) = (k+a)^m + \int_{-d}^{d} \frac{dk'}{2\pi} \left[ \varphi(k-k') + (-1)^m \varphi(2a + k + k') \right] h_m(k') = (k+a)^m + \int_{-d}^{d} \frac{dk'}{2\pi} \left[ \varphi(k-k') + (-1)^m \varphi(2a + k + k') \right] h_m(k') = \frac{1}{4a^2 \gamma_0} \int_{-d}^{d} \frac{dk}{2\pi} h_m^{b}(k) + \mathcal{O}(a^{-5}),
\]

and \( \int_{-d}^{d} \frac{dk}{2\pi} h_0(k) = 1/(2\gamma) \).

Expanding the resolvent of the integral equation in \( a^{-1} \) we obtain the solution

\[
\hat{h}_m(k) = h_m^{b}(k) + \frac{(-1)^m}{2a^2} (1 - \varphi)^{-1} \int_{-d}^{d} \frac{dk'}{2\pi} \left( 1 - \frac{k+k'}{a} + \frac{3(k+k')^2 - 1}{4a^2} \right) h_m^{b}(k') + \frac{1}{4a^4 \gamma_0} \int_{-d}^{d} \frac{dk}{2\pi} h_m^{b}(k) + \mathcal{O}(a^{-5}),
\]

where \( h_m^{b}(k) \) is the solution of the equation obtained by keeping only non-negative powers of \( a \):

\[
\hat{h}_m^{b}(k) = (1 - \varphi)^{-1} \circ (k+a)^m.
\]

These are nothing else but the Galilean boosted functions discussed in the previous section (c.f. Eq. (48b)):

\[
\hat{h}_m^{b}(k) = h_0^{(0)}(k), \quad h_1^{b}(k) = h_1^{(0)}(k) + ah_0^{(0)}(k), \quad h_2^{b}(k) = h_2^{(0)}(k) + 2ah_1^{(0)}(k) + a^2 h_0^{(0)}(k),
\]

where the superscript denotes the ground state solutions on the interval \([-d, d] \). The last term in Eq. (55) is the result of multiple applications of the operator \( (1 - \varphi)^{-1} \), and we introduced the notations

\[
\int_{-d}^{d} \frac{dk}{2\pi} h_0^{(0)}(k) = \frac{1}{\gamma_0}, \quad \int_{-d}^{d} \frac{dk}{2\pi} h_1^{(0)}(k) = e_1, \quad \int_{-d}^{d} \frac{dk}{2\pi} h_2^{(0)}(k) = e_2.
\]

Using Eqs. (57) for \( h_m^{b}(k) \) we find from Eq. (55)

\[
\hat{h}_0(k) = h_0^{(0)}(k) + \frac{1}{2\gamma_0 a^2} h_0^{(0)}(k) - \frac{1}{2\gamma_0 a^2} h_1^{(0)}(k) + \frac{1}{8a^4} \left[ \left( 3c_2 - \frac{1}{\gamma_0} + \frac{2}{\gamma_0} \right) h_0^{(0)}(k) + \frac{3}{\gamma_0} h_2^{(0)}(k) \right] + \mathcal{O}(a^{-5}),
\]

\[
\hat{h}_1(k) = h_1^{(0)}(k) - \frac{1}{2\gamma_0 a^2} h_0^{(0)}(k) + \frac{1}{2\gamma_0 a^2} h_1^{(0)}(k) - \frac{1}{a^3} \left[ \left( \frac{3}{8} e_2 - \frac{1}{2} e_1 - \frac{1}{8\gamma_0} - \frac{1}{4\gamma^2} \right) h_0^{(0)}(k) + h_2^{(0)}(k) \right] + \mathcal{O}(a^{-4}),
\]

\[
\hat{h}_2(k) = h_2^{(0)}(k) + \frac{1}{2\gamma_0} h_0^{(0)}(k) - \frac{1}{2\gamma_0 a^2} h_1^{(0)}(k) + \frac{1}{2\gamma_0} h_2^{(0)}(k) + \frac{3}{4\gamma_0} h_0^{(0)}(k) + \frac{3}{8\gamma_0} h_2^{(0)}(k) + \mathcal{O}(a^{-3}).
\]

Comparing the integral of \( h_0^{(0)}(k) \) and \( h_0(k) \) we can relate \( \gamma \) and \( \gamma_0 \):

\[
\gamma_0 = 2\gamma + \frac{1}{2a^2} + \mathcal{O}(a^{-3}).
\]
Now we are in the position to calculate the local correlations. For $g_2^\text{ex}(\gamma, a \to \infty)$ we only need the leading corrections to $h_0(k)$ and $h_1(k)$ and we obtain

$$
\frac{g_2^\text{ex}(\gamma, a)}{2\gamma^2} = 2 \int_{q_1}^{q_2} \frac{dq}{2\pi} \left[ h_0^\text{ex}(q)q^2 - h_1^\text{ex}(q)q \right] = 2 \int_{-d}^{d} \frac{dk}{2\pi} \left[ h_0(k)(k + a)^2 - h_1(k)(k + a) \right] = \frac{g_2(\gamma)}{2\gamma^2} + \frac{1}{2\gamma^2} + O(a^{-1}), \quad (61)
$$

which gives $g_2^\text{ex}(\gamma, a \to \infty) = g_2(2\gamma)/2 + 1$. For $g_3^\text{ex}(\gamma, a \to \infty)$ we need all the corrections given above. The divergent positive powers of $a$ cancel, and after a lengthy calculation we arrive at the remarkably simple result

$$
g_3^\text{ex}(\gamma, a \to \infty) = \frac{1}{4} g_3(2\gamma) + \frac{9}{4} g_2(2\gamma). \quad (62)
$$

It is interesting to note that as $\gamma \to 0$, the asymptotic values are given by $g_2^\text{ex}(\gamma \to 0, \infty) \to 3/2$ and $g_3^\text{ex}(\gamma \to 0, \infty) \to 5/2$. On the other hand, in the free boson case we recover $g_2^\text{ex}(0, \infty) = 2/2 + 1 = 2$ and $g_3^\text{ex}(0, \infty) = 1/4 \cdot 6 + 9/4 \cdot 2 = 6$. 