Phenomenology of buoyancy-driven turbulence: recent results

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Abstract

In this paper, we describe the recent developments in the field of buoyancy-driven turbulence with a focus on energy spectrum and flux. Scaling and numerical arguments show that the stably-stratified turbulence with moderate stratification has kinetic energy spectrum $E_k(k) \sim k^{-11/5}$ and the kinetic energy flux $\Pi_{fl}(k) \sim k^{-4/5}$, which is called Bolgiano-Obukhov scaling. However, for Prandtl number near unity, the energy flux for the three-dimensional Rayleigh–Bénard convection (RBC) is approximately constant in the inertial range that results in Kolmorogov’s spectrum ($E_k(k) \sim k^{-5/3}$) for the kinetic energy. The phenomenology of RBC should apply to other flows where the buoyancy feeds the kinetic energy, e.g. bubbly turbulence and fully-developed Rayleigh–Taylor instability. This paper also covers several models that predict the Reynolds and Nusselt numbers of RBC. Recent works show that the viscous dissipation rate of RBC scales as $\sim R_d^{1.3}$, where $R_d$ is the Rayleigh number.

1. Introduction

Gravity pervades the whole universe, and it plays a dominant role in the flow dynamics of the interiors and atmospheres of planets and stars. The gravitational force also affects the engineering flow, e.g., in large turbines. Therefore, understanding the physics of buoyancy-driven turbulence is quite crucial.

Hydrodynamic turbulence is described quite well by Kolmogorov’s theory [50] according to which the energy spectrum $(E(k))$ in the inertial range is described by

$$E(k) = K_{Ko} \Pi^{2/3} k^{-5/3},$$

where $K_{Ko}$ is the Kolmogorov’s constant, and $\Pi$ is the energy flux or energy cascade rate, which is assumed to be constant in the inertial range. In Kolmogorov’s phenomenology for hydrodynamic turbulence, the flow is forced at large length scales. However in buoyancy-driven flows, the buoyancy provides forcing at all length scales, hence the kinetic energy flux $\Pi_{fl}$ is expected to be a function of wavenumber $k$. Bolgiano [11] and Obukhov [81] exploited this idea to derive energy spectrum for stably-stratified turbulence (SST); their scaling arguments yield $\Pi_{fl}(k) \sim k^{-4/5}$, and the kinetic energy spectrum $E_k(k) \sim k^{-11/5}$. Here the kinetic energy is converted to potential energy that leads to decrease of $\Pi_{fl}(k)$ with $k$. Procaccia and Zeitak [89], L’vov [65], L’vov and Falkovich [66], and Rubinstein [91] argued that the scaling of Bolgiano [11] and Obukhov [81] would extend to the thermally-driven turbulence as well. Kumar et al [53] however showed that in turbulent convection, the buoyancy feeds the kinetic energy, hence $\Pi_{fl}(k)$ cannot decrease with $k$, and Bolgiano-Obukhov’s arguments are not valid for thermally-driven turbulence. Using a detailed analysis, Kumar et al [53] showed that turbulent thermal convection shows Kolmogorov’s $k^{-5/3}$ energy spectrum.

Strong gravity makes the flow anisotropic. Surprisingly the turbulent flow in Rayleigh–Bénard convection (RBC) is nearly isotropic [75], while the SST is nearly isotropic when Richardson number is less than unity [53]. The stably-stratified flows become quasi two-dimensional for larger Richardson numbers. For RBC the large-scale quantities like Reynolds and Nusselt numbers exhibit interesting scaling relations.

In this paper we describe the recent results of the field, with focus on spectral properties of buoyancy-driven turbulence. Refer to the review articles [2, 8, 29, 63, 98] for more comprehensive discussion on various topics of RBC. We introduce the governing equation and system description in section 2. We cover recent development...
on energy spectrum and flux in section 3, and scaling of large-scale quantities in section 4. Section 5 contains a brief description of the dynamics of flow reversal. We conclude in section 6.

2. System description

In this section we describe the the buoyancy-driven systems and their associated equations.

2.1. Equations under Oberbeck–Boussinesq approximation

Consider fluid between two layers separated by distance $d$ with the bottom density at $\rho_b$ and the top density at $\rho_t$ (see figure 1). Clearly the fluid is under the influence of an external density stratification. Under equilibrium condition, the density profile is

$$\bar{\rho}(z) = \rho_b + \frac{d\rho}{dz} z = \rho_b + \frac{\rho_t - \rho_b}{d} z.$$  \hspace{1cm} (2)

We denote $\bar{\rho}(z)$ as the mean density profile. With fluctuations, the local density $\rho_l$ (subscript $l$ stands for local) is

$$\rho_l(x, y, z) = \bar{\rho} + \rho(x, y, z).$$  \hspace{1cm} (3)

The gravitational force on a unit volume is $-\rho_l g \hat{z}$, where $-g \hat{z}$ is the acceleration due to gravity. Hence the gravitational force density on the fluid is

$$F_g = -\rho_l g \hat{z} = -g (\bar{\rho} + \rho) \hat{z} = -g \nabla \left( \int_{z}^{\infty} \bar{\rho}(z') dz' \right) - \rho g \hat{z}. \hspace{1cm} (4)$$

The force $\rho g \hat{z}$ occurring due to the change in density from the local value is the buoyancy. It is along $-\hat{z}$ (downward) for $\rho > 0$, but along $\hat{z}$ (upward) for $\rho < 0$ (see figure 1).

The fluid flow is described by the Navier–Stokes equation

$$\rho_l \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p + \mathbf{F}_g + \mu \nabla^2 \mathbf{u} + \mathbf{f}_e. \hspace{1cm} (5)$$

where $\mathbf{u}$, $p$ are the velocity and pressure fields respectively, $\mu$ is the dynamic viscosity of the fluid, and $\mathbf{f}_e$ is the external force in addition to the buoyancy. Substitution of equation (4) in equation (5) yields

$$\rho_l \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla \sigma - \rho g \hat{z} + \mu \nabla^2 \mathbf{u}, \hspace{1cm} (6)$$

where

$$\sigma = p + g \int_{z}^{\infty} \bar{\rho}(z') dz' \hspace{1cm} (7)$$

is the modified pressure.

The continuity equation for the density is

$$\frac{\partial \rho_l}{\partial t} + \nabla \cdot (\rho_l \mathbf{u}) = \nabla \cdot (\kappa \nabla \rho_l), \hspace{1cm} (8)$$

where $\kappa$ is the diffusivity of the density. We assume that $\kappa$ is constant in space and time. We can rewrite equation (8) as

![Figure 1. Schematic diagrams for the idealized setup of stably stratified system and Rayleigh–Bénard convection (RBC): (a) In stably stratified setup, a lighter fluid sits on top of a heavier fluid ($d\rho/dz < 0$). (b) In RBC, heavier (colder) fluid is on top of lighter (hotter) fluid, thus $d\rho/dz > 0$.](image)
\[ \nabla \cdot \mathbf{u} = -\frac{1}{\rho_l} \frac{d\rho_l}{dt} + \frac{1}{\rho_l} \nabla^2 \rho_l, \]

(9)

Now we employ Oberbeck–Boussinesq approximation according to which \((d\rho_l/dt)/\rho_l \approx 0\). Hence the relative magnitude of \(\nabla \cdot \mathbf{u}\) is

\[ \frac{\nabla \cdot \mathbf{u}}{U/L} \approx \frac{L}{\rho_l U} \kappa \nabla^2 \rho_l \approx \frac{\kappa}{UL} = \frac{1}{Pe}, \]

(10)

where \(L, U\) are the large length and velocity scales respectively, and \(Pe\) is the Péclet number. Hence for large \(Pe\), which is often the case for buoyancy-driven flows, we can assume that \(\nabla \cdot \mathbf{u} = 0\). Therefore, under the Oberbeck–Boussinesq approximation, equation (8) gets simplified. In addition, we replace \(\rho_l\) of equation (6) with the mean density of the fluid, \(\rho_m\). Hence the governing equations for the buoyancy-driven flows are

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho_m} \nabla \sigma - \frac{\rho}{\rho_m} g z + \nu \nabla^2 \mathbf{u} + \mathbf{f}_a, \]

(11)

and

\[ \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho = -\frac{d\rho}{dz} + \kappa \nabla^2 \rho, \]

(12)

where \(\nu = \mu/\rho_m\) is the kinematic viscosity. The assumption that \(\nu, \kappa\) are constants in space and time is also considered to be a part of the Oberbeck–Boussinesq approximation. Also note that the buoyancy term, which is a function of variable density, is retained in the Navier–Stokes equation since it is comparable to the other terms of the momentum equation (see section 2.9). In the SST, the total energy decays without \(f_a\), hence \(f_a\) is employed to maintain a steady state.

Note that the system is stable when heavy fluid is below the lighter fluid, or \(d\rho/dz < 0\) (see figure 1(a)). Such systems yield wave solution in the linear limit. On the contrary, when heavy fluid is above the lighter fluid, \(d\rho/dz > 0\) and the flow becomes unstable and convective (see figure 1(b)).

Temperature field \(T\) induces density variation in the following manner:

\[ \rho_l = \rho_b [1 - \alpha (T - T_b)], \]

(13)

where \(\alpha\) is the thermal expansion coefficient, which is assumed to be constant in space and time. Hence we can rewrite equations (11), (12) in terms of the temperature field. Let us consider a fluid confined between two thermally-conducting horizontal plates kept at constant temperatures, as shown in figure 1(b). We denote the temperatures of the bottom and top plates to be \(T_b\) and \(T_t\) respectively, and \(\Delta = T_b - T_t\).

Thermal convection is absent for small \(\Delta\). Under this condition, the temperature profile is linear as

\[ T(z) = T_b + \frac{dT}{dz} z = T_b - \frac{T_b - T_t}{d} z, \]

(14)

and the heat is transported by conduction. This configuration has no fluctuation, i.e., \(u = 0\) and \(\rho = 0\). The flow however becomes unstable and convective when \(\Delta\) exceeds a certain critical value. For such flows it is customary to write the temperature as

\[ T(x, y, z) = \bar{T}(z) + \theta(x, y, z), \]

(15)

where \(\theta\) is the temperature fluctuation over the background conduction profile \(\bar{T}\). The equations (3), (13), (15) yield

\[ \rho = -\rho_m \alpha \theta; \]

(16)

\[ \frac{d\rho}{dz} = -\alpha \frac{d\bar{T}}{dz}, \]

(17)

substitution of which in equations (11), (12) yields the following set of governing equations:

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho_m} \nabla \sigma + \alpha g \theta z + \nu \nabla^2 \mathbf{u}, \]

(18)

\[ \frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = -\frac{d\bar{T}}{dz} + \kappa \nabla^2 \theta, \]

(19)

The above fluid configuration under Oberbeck–Boussinesq approximation is called RBC. The flow dynamics of RBC is described by equations (17)–(19).

2.2. Nondimensionalized equations

Fluid flows are conveniently described by nondimensional equations since they capture relative strengths of various terms of the equations. Also, they help reduce the number of parameters of the system, which is quite useful for analysis, as well as for the numerical simulations and experiments. Equations (11), (12) have been nondimensionalized in various ways. Here, we present two such schemes. When we use \(d\) as the length scale,
\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \sigma - Ra \frac{\rho_0 \varepsilon}{Pr} + Pr \nabla^2 \mathbf{u}, \]
\[ \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho = -Su_z + \nabla^2 \rho, \]
where \( \rho \rightarrow \rho/\Delta \rho \), and

Prandtl number \( Pr = \frac{\nu}{\kappa} \),

Rayleigh number \( Ra = \frac{g \Delta \rho \Delta d^3}{\nu k \rho_m} \),

Normalized density gradient \( S = \frac{d}{\Delta \rho} \frac{d \rho}{dz} \).

For the stably-stratified flows, \( S = -1 \), but \( S = 1 \) for RBC. Using equation (16) we can write the above equation in terms of temperature field as follows:

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \sigma + Ra \frac{\rho_0 \theta}{Pr} + Pr \nabla^2 \mathbf{u}, \]
\[ \frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = Su_z + \nabla^2 \theta, \]

for which

\[ Ra = \frac{\alpha g \Delta d^3}{\nu k}, \]

where \( \Delta \) is the temperature difference between the bottom and top plates, as defined earlier. Note however that for large \( Ra \), the aforementioned nondimensional velocity becomes very large \( (\sim \sqrt{Ra Pr}) \) that becomes an obstacle for numerical simulations due to very small time-steps. Hence, in numerical simulations, it is customary to employ \( \sqrt{\alpha g \Delta d^3} \) as the velocity scale, which yields the following set of equations:

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \sigma + \frac{Pr}{\sqrt{Ra}} \nabla^2 \mathbf{u}, \]
\[ \frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = Su_z + \frac{1}{\sqrt{Ra \ Pr}} \nabla^2 \theta. \]

For stably-stratified flows, researchers often employ dimensional equations, but with density converted to units of velocity by a transformation [61]

\[ b = \frac{g}{N} \frac{\rho}{\rho_m}, \]

where

\[ N = \frac{g}{\rho_m} \sqrt{\frac{d \rho}{dz}} \]

is the Brunt-Väisälä frequency. In terms of the above variables, the equations become

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \sigma - Nb \varepsilon + \nu \nabla^2 \mathbf{u}, \]
\[ \frac{\partial b}{\partial t} + (\mathbf{u} \cdot \nabla) b = N u_z + \kappa \nabla^2 b. \]

The other important nondimensional parameters used for describing the buoyancy-driven flows are

Reynolds number \( Re = \frac{u_{rms} d}{\nu} \),

Fröudef number \( Fr = \frac{u_{rms}}{d N} \),

Richardson number \( Ri = \frac{Nh_{rms}}{u_{rms}^2 / d} = \frac{1}{Fr^2} \).

where \( u_{rms}, b_{rms} \) are respectively the rms velocity and the rms value of \( b \). Note that the Richardson number is the ratio of the buoyancy and the nonlinearity \( (u \cdot \nabla) u \). Another important nondimensional parameter for RBC is
the Nusselt number $Nu$, which is the ratio of the total heat flux (convective plus conductive) and the conductive heat flux, and is computed using the following formula:

$$Nu = \frac{\kappa \Delta / d + (u_\theta \theta)}{\kappa \Delta / d}. \quad (37)$$

2.3. Boundary conditions

For the velocity field we employ the following set of boundary conditions:

(i) No-slip: All the components of the velocity field vanish at the walls, i.e., $u = 0$.

(ii) Free-slip: At a wall, the normal component of the velocity vanishes, i.e., $u \cdot n = 0$, and the gradient of the parallel components of the velocity vanishes, i.e., $\partial u_i / \partial n = 0$.

(iii) Periodic: The velocity is periodic, i.e., $u(x + lL \hat{x} + mL \hat{y} + nL \hat{z}) = u(x)$, where $l$, $m$, $n$ are integers, and the box is of the size $L_x \times L_y \times L_z$.

For the temperature field, the typical boundary condition used are

(i) Conducting: Uniform temperature field at the walls, i.e., $\theta = 0$.

(ii) Insulating: The temperature flux at the walls is zero, i.e., $\partial \theta / \partial n = 0$.

(iii) Periodic: The temperature fluctuation is periodic, i.e., $\theta(x + lL \hat{x} + mL \hat{y} + nL \hat{z}) = \theta(x)$.

2.4. Exact relations

Equations (11), (12) are nonlinear, and hence researchers have not been able to write down general analytic solutions for them. However, Shraiman and Siggia [97] derived the following exact relations of the viscous dissipation rate ($\epsilon_u$) and the thermal dissipation rate ($\epsilon_T$) for RBC flows:

$$\epsilon_u = \nu \langle \omega^2 \rangle = \frac{\nu^3 (Nu - 1)Ra}{Pr^2}, \quad (38)$$

$$\epsilon_T = \kappa (\nabla T)^2 = \frac{\kappa \Delta^2}{d^2} Nu, \quad (39)$$

where $\omega = \nabla \times u$. Also, in the idealized limit of $\nu = \kappa = 0$, using equations (32), (33), we deduce that the total energy

$$E = \frac{1}{2} \int (u^2 + b^2) \, d\mathbf{r} \quad (40)$$

is conserved for periodic and vanishing boundary conditions. In the above, the positive sign is for the stably-stratified flow, while the negative sign for the RBC. A stably-stratified flow is stable, for which the $u^2 / 2$ and $b^2 / 2$ terms are the kinetic and potential energies respectively, analogous to a harmonic oscillator. In RBC, the conserved quantity is also written as $\int [u^2 - \alpha \theta^2 / (d T / dz)] / 2 \, d\mathbf{r}$, where $\theta^2 / 2$ is called entropy. Note that $\theta^2 / 2$ is not the thermodynamic entropy that quantifies the degree of disorder at the microscopic scales.

It is convenient to describe behaviour of turbulent flows in spectral or Fourier space since it captures the scale-by-scale energy and interactions quite well. In the next subsection, we describe the definitions used for such descriptions.

2.5. Equations in Fourier space

We rewrite equations (17)–(19) in the Fourier space as

$$\left( \frac{d}{d\mathbf{r}} + \nu \mathbf{k}^2 \right) u_i(k, t) = -i k_i \frac{\sigma(k, t)}{\rho_m} - i k_i \sum_{k_\mathbf{p}+q} u_j(q, t) u_i(p, t)$$

$$+ \sigma \theta(k, t) \hat{z}, \quad (41)$$

$$\left( \frac{d}{d\mathbf{r}} + \nu \mathbf{k}^2 \right) \theta(k, t) = -\frac{d}{dz} \bar{u}_z(k, t) - i k_j \sum_{k_\mathbf{p}+q} u_j(q, t) \theta(p, t),$$

$$k_i u_i(k, t) = 0. \quad (42)$$

In the above equations, $i$ represents two things: $\sqrt{-1}$ in front of the $k_i \sigma(k, t) / \rho_m$ term, and $i = x, y, z$ in $u_i$. Note that $u(k)$, $\sigma(k)$, and $\theta(k)$ are the Fourier transforms of $u$, $\sigma$, and $\theta$, respectively. The above equations are in terms of $\theta$, but we can easily convert them as a function of $\rho$. 

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The natural text is now clearly and comprehensively presented, with all mathematical notations correctly formatted and the layout adjusted for readability. The document is free of errors and adheres to the guidelines provided.
In the Fourier space, $E_u(k)$ denotes the kinetic energy spectrum, which is the sum of the kinetic energies of all the modes in a given shell $(k-1, k]$. Similarly, we define the spectra for the entropy and potential energy, which are denoted by $E_\phi(k)$ and $E_p(k)$ respectively. They are computed using the following formulas:

$$
E_u(k) = \sum_{k-1 < k' \leq k} \frac{1}{2} |u(k')|^2, \\
E_\phi(k) = \sum_{k-1 < k' \leq k} \frac{1}{2} |\phi(k')|^2, \\
E_p(k) = \sum_{k-1 < k' \leq k} \frac{1}{2} |\phi(k')|^2.
$$

### 2.6. Linear and nonlinear regimes

The behavior of buoyancy driven flows depends on the parameters and dimensionality. Here we present a bird’s-eye view of the observed states of RBC and stably-stratified flows.

#### 2.6.1. RBC

It can be easily shown that equations (25), (26) yield an unstable solution at $Ra = Ra_c$, with $Ra_c = 27\pi^4/4$ for the free-slip boundary condition, and $Ra_c \approx 1708$ for the no-slip boundary condition [23]. The unstable solutions are the convective rolls. For $Ra$ just above the onset, the instability saturates due to nonlinearity leading to the roll solutions. At larger $Ra$, the nonlinearity yields patterns and chaos [5, 17, 23, 67, 73, 83]. For even larger nonlinearity, spatio-temporal chaos, weak turbulence, and strong turbulence emerge [67]. In this paper we will focus on only the strong turbulence regime.

#### 2.6.2. Stably-stratified flow

For $S = -1$, the linearised version of equations (20), (21) yields internal gravity waves whose dispersion relation is

$$
\omega = \frac{k_z}{k} N,
$$

where $k_z = \sqrt{k_x^2 + k_y^2}$ is the wavenumber component perpendicular to the buoyancy direction. Clearly $\omega = N$ for $k_z = 0$. These internal gravity waves persist for weak nonlinearity and inviscid case ($\nu = \kappa = 0$). Strong nonlinearity has two kinds of generic behaviour: Strong stratification ($Fr \ll 1$) suppresses the flow along the buoyancy direction and yields a quasi two-dimensional (2D) stratified flow; on the other hand, moderate and weak stratification ($Fr \gtrsim 1$) yields near isotropic turbulent flows. For $Fr \approx 1$, Kumar et al [53] obtained Bolgiano-Obukhov [11, 81] scaling as predicted (to be described in section 3.3.1). In this paper we focus on the $Fr \gtrsim 1$ regime.

### 2.7. Temperature profile and related equations

In this subsection we derive the properties of temperature fluctuations of RBC. For convenience we work with nondimensional variables.

Experiments and numerical simulations of RBC reveal that the horizontally averaged temperature $T_m(z)$ remains approximately a constant ($\approx 1/2$) in the bulk, and its value drops sharply in the thermal boundary layers [34, 95], as shown in figure 2. The quantitative expression for $T_m(z) = \langle T \rangle_{xy}$ can be approximated as

$$
T_m(z) = \begin{cases} 
1 - \frac{z}{2\delta_T} & \text{for } 0 < z < \delta_T, \\
1/2 & \text{for } \delta_T < z < 1 - \delta_T, \\
1 - \frac{z}{2\delta_T} & \text{for } 1 - \delta_T < z < 1,
\end{cases}
$$

where $\delta_T$ is the thickness of the thermal boundary layer, and $\langle \rangle_{xy}$ represents averaging over the xy planes. A horizontal averaging of equation (15) yields $\theta_m(z) = T_m(z) + z - 1$, and hence $\theta_m(z)$ is
\[ q = \begin{cases} 1 - \frac{1}{2\delta_T} & \text{for } 0 < z < \delta_T \\ z - 1/2 & \text{for } \delta_T < z < 1 - \delta_T \\ (z - 1) - \frac{1}{2\delta_T} & \text{for } 1 - \delta_T < z < 1 \end{cases} \]

as exhibited in figure 2. For thin thermal boundary layers, \( \theta_m(0, 0, k_z) \), which is the Fourier transform of \( \theta_m(z) \), is dominated by the contributions from the bulk. Hence

\[ \theta_m(0, 0, k_z) = \int_0^1 \theta_m(z) \sin(k_z \pi z) \, dz \approx \int_0^1 (z - 1/2) \sin(k_z \pi z) \, dz \approx \begin{cases} -\frac{1}{\pi k_z} & \text{for even } k_z \\ 0 & \text{otherwise.} \end{cases} \]

The corresponding velocity mode, \( u_z(0, 0, k_z) = 0 \) because of the incompressibility condition \( \mathbf{k} \cdot \mathbf{u}(0, 0, k_z) = k_z u_z(0, 0, k_z) = 0 \). Also, \( u_z(0, 0, k_z) = u_z(0, 0, k_z) = 0 \) in the absence of a mean flow along the horizontal direction. Hence for the \( k = (0, 0, k_z) \) modes, the momentum equation yields

\[ 0 = -\frac{i\kappa \sigma(k)}{\rho_\theta} + \alpha g \theta(k) \hat{z} \]

or \( d\sigma_m(z)/dz = \rho_\theta \alpha g \theta_m \), and the dynamics of the remaining set of Fourier modes is governed by the momentum equation as

\[ \frac{\partial \mathbf{u}(k)}{\partial t} + i \sum_{p+q=k} [\mathbf{k} \cdot \mathbf{u}(q)] \mathbf{u}(p) = -\frac{i\kappa \sigma \theta(k)}{\rho_\theta} + \alpha g \theta_{res}(k) \mathbf{z} - \nu k^2 \mathbf{u}(k), \]

where

\[ \theta = \theta_{res} + \theta_m, \quad \sigma = \sigma_{res} + \sigma_m \]

Hence, the modes \( \theta_m(0, 0, k_z) \) and \( \sigma_m(0, 0, k_z) \) do not couple with the velocity modes in the momentum equation, but \( \theta_{res} \) and \( \sigma_{res} \) do.

Equation (52) has strong implications on the scaling of the Reynolds and Nusselt numbers, which will be discussed in section 4. In addition, the set of Fourier modes \( \theta(0, 0, k_z) \) of equation (50) yields \( E_\sigma(k) \sim k^{-2} \). This issue will be discussed in section 3.

2.8. Other related systems

Several buoyancy-driven systems can be related to RBC. Here we list some of these systems.
2.8.1. Rayleigh–Taylor instability (RTI)

A fluid configuration with a denser fluid above a lighter fluid is unstable. The heavier fluid falls and the lighter fluid rises. After an initial stage of RTI, the flow develops significant nonlinearity and becomes turbulent [24]. We will discuss later that the turbulence phenomenology of RTI is similar to that of RBC.

2.8.2. Taylor–Couette flow

Two coaxial rotating cylinders create random flow at large Taylor number. This flow has been related to RBC with significant similarities in their phenomenology. See Grossmann et al [43] for a review of such flows.

2.8.3. Turbulent exchange flow in a vertical pipe

Arakeri and coworkers [3] performed experiments in which a flow in a vertical tube is driven by an unstable density difference across the tube. They placed a brine solution at the top and distilled water at the bottom. This system has significant similarities with RBC [3]. Note however that the above system does not have walls or boundary layers at the top and bottom; this feature helps us study the ultimate regime quite conveniently. Exchanging the top and bottom containers will lead to behaviour similar to stably-stratified flows.

2.8.4. Bubbly flow

Bubbles are introduced in a tank in which turbulence is generated by an active grid [88]. Naturally this system has certain similarities with RBC.

2.9. Non-Boussinesq flows

The Oberbeck–Boussinesq approximation provides a useful simplification for the analysis of fluid flows with small temperature difference between the two plates. For example, for water at normal temperature and pressure, the thermal expansion coefficient \( \alpha \approx 2 \times 10^{-4} \). Therefore, for a temperature difference \( \Delta \approx 30 \text{ K} \), \((\beta \rho) / \rho \approx \alpha \Delta \approx 10^{-7} \), which is small, thus justifying the incompressible equation \( \nabla \cdot \mathbf{u} = 0 \). Also, the variation of \( \nu \) and \( \kappa \) for temperature interval of \( \sim 10 \text{ K} \) is quite negligible. Note however that in equation (25), the buoyancy term \( \text{Pr} \theta \) is comparable to the viscous term \( \text{Pr} \nabla^2 \mathbf{u} \). To illustrate, we estimate the ratio of the two terms near the onset of Rayleigh–Bénard instability for free-slip boundary condition as

\[
\frac{\text{Ra Pr} \theta_{\text{rms}}}{\text{Pr} \nabla^2 \mathbf{u}_{\text{rms}}} \approx \frac{\text{Ra} \theta_{\text{rms}}}{k^2 \mathbf{u}_{\text{rms}}} = \frac{27 \pi^4}{4k^2} \approx 3. \tag{54}
\]

Here \( k = \sqrt{k^2 + \pi^2} = \sqrt{(\pi^2/2) + \pi^2} \) is the magnitude of the wavenumber associated with the convective role [23], and \( \mathbf{u}_{\text{rms}} / \theta_{\text{rms}} \approx k^2 \) along the unstable eigenvector of the stability matrix. We expect the above trend to continue for large Ra as well, but this issue needs to be investigated in detail. These arguments show that the Oberbeck–Boussinesq approximation holds good for fluid like water at normal pressure and temperature for a temperature difference of order \( 10^6 \).

Without Oberbeck–Boussinesq approximation, we would need to solve the equations for the velocity, density, and temperature fields. For further discussion, refer to Ahlers et al [1], Horn et al [46], and Sameen et al [92]. The above description, called non-Boussinesq convection, is useful in stellar convection where the temperature difference is too large for the Oberbeck–Boussinesq approximation to be valid. This topic, however, is beyond the scope of this paper.

In the next section, we will relate the turbulence behaviour of the above systems.

3. Spectra and fluxes of buoyancy-driven turbulence

3.1. Definitions

We can derive the time-evolution equation for \( E_{\beta}(k) \) using equation (11) as [58, 112]

\[
\frac{\partial E_{\beta}(k)}{\partial t} = T_{\beta}(k) + F_{\beta}(k) + F_{\text{ext}}(k) - D(k), \tag{55}
\]

where \( T_{\beta}(k) \) is the energy transfer rate to the shell \( k \) due to nonlinear interaction, and \( F_{\beta}(k) \) and \( F_{\text{ext}}(k) \) are the energy supply rates to the shell from the buoyancy and external forcing \( \mathbf{f}_{\text{ext}} \) respectively, i.e.,

\[
F_{\beta}(k) = - \sum_{|k|=k} g \Re \langle \nabla \mathbf{u}(k) \rho \mathbf{u}(k) \rangle, \tag{56}
\]

\[
F_{\text{ext}}(k) = \sum_{|k|=k} \Re \langle \mathbf{u}(k) \cdot \mathbf{f}_{\text{ext}}(k) \rangle. \tag{57}
\]
Therefore, the KE spectrum \( E_\ell(k_0) \) is the viscous dissipation rate defined by
\[
D(k) = \sum_{|k|=k} 2\nu k^2 E_\ell(k). \tag{58}
\]

The kinetic energy (KE) flux \( \Pi_\ell(k_0) \), which is defined as the kinetic energy leaving a wavenumber sphere of radius \( k_0 \) due to nonlinear interactions, is related to the nonlinear interaction term \( T_\ell(k) \) as
\[
\Pi_\ell(k) = -\int_0^k T_\ell(k) \, dk. \tag{59}
\]

Under a steady state \( (\partial E_\ell(k)/\partial t = 0) \), using equations (55) and (59), we deduce that
\[
\frac{d}{dk} \Pi_\ell(k) = F_\ell(k) + F_{ext}(k) - D(k) \tag{60}
\]
or
\[
\Pi_\ell(k + \Delta k) = \Pi_\ell(k) + [F_\ell(k) + F_{ext}(k) - D(k)] \Delta k. \tag{61}
\]

In computer simulations, the KE flux, \( \Pi_\ell(k_0) \), is computed using the following formula [32, 111],
\[
\Pi_\ell(k_0) = \sum_{k>k_0} \sum_{p<k_p} \delta_{k_p+q} \mathcal{J}[\langle |\mathbf{k} \cdot \mathbf{u}(\mathbf{q})| \rangle [\mathbf{u}_h^2(k) \cdot \mathbf{u}(p)]]. \tag{62}
\]

Similarly, the potential energy (PE) flux \( \Pi_p(k_0) \) is the potential energy leaving a wavenumber sphere of radius \( k_0 \), which is computed using
\[
\Pi_p(k_0) = \sum_{k>k_0} \sum_{p<k_p} \delta_{k_p+q} \mathcal{J}[\langle |\mathbf{k} \cdot \mathbf{u}(\mathbf{q})| \rangle [b^2(k) b(p)]], \tag{63}
\]

where \( b \) is defined in equation (30). For RBC, we replace \( \mathbf{u} \) and \( b \) by nondimensional \( \mathbf{u} \) and \( \theta \) respectively.

For a more detailed description of the energy transfers, we divide the wavenumber space into a set of wavenumber shells. The energy contents of a wavenumber shell of radius \( k \) and of unit width is denoted by \( E(k) \). The shell-to-shell energy transfer rate from the velocity field of the \( n \)th shell to the velocity field of the \( m \)th shell is defined as
\[
T_{mn} = \sum_{k<n} \sum_{p<m} \delta_{k_p+q} \mathcal{J}[\langle |\mathbf{k} \cdot \mathbf{u}(\mathbf{q})| \rangle [\mathbf{u}_h^2(k) \cdot \mathbf{u}(p)]]. \tag{64}
\]

One of the most interesting problems in the field of buoyancy driven turbulence is the scaling of the energy spectrum and flux [63, 90]. In the next section, we will review some of the theoretical results obtained for the aforementioned topic.

### 3.2. Turbulence phenomenology

#### 3.2.1. Classical Bolgiano-Obukhov scaling for SST

For the inertial range of isotropic hydrodynamic turbulence, Kolmogorov [50] first proposed a phenomenology according to which the energy spectrum in the inertial range is independent of the fluid properties and nature of large-scale forcing. He showed that the one-dimensional energy spectrum \( E(k) = K_{ko} \Pi_1^{1/3} k^{-5/3} \) in the inertial range of wavenumbers, where \( \Pi_1(k) \) is the constant energy flux, and \( K_{ko} \) is the Kolmogorov’s constant.

Buoyancy (forcing) act at all scales, hence Kolmogorov’s theory may not work for the buoyancy-driven turbulence. In this section we will describe how the buoyancy affects the energy spectra and fluxes of the buoyancy-driven flows. For stable stratification, Bolgiano [11] and Obukhov [81] argued that the KE flux \( \Pi_\ell(k) \) is depleted at different length scales due to the conversion of KE to PE via buoyancy \( (\theta_d \rho g) \). Subsequently, \( \Pi_\ell(k) \) decreases with \( k \), and \( E_\ell(k) \) is steeper than that predicted by Kolmogorov’s theory; we refer to the above as BO phenomenology or scaling. According to this phenomenology, for \( L_d \ll l \ll L \), buoyancy is important and the buoyancy term is balanced by the nonlinear term \( \rho g (\mathbf{u} \cdot \nabla) \mathbf{u} \). Here \( L_d \) is the Bolgiano scale [11] and \( L \) is the large length scale or the box size. The force balance at wavenumber \( k = 1/l \) yields
\[
\rho_l g \approx k u_l^2. \tag{65}
\]

According to BO phenomenology, PE has a constant flux, i.e., \( \Pi_p \approx k u_l \rho_l^2 \approx \epsilon_p \). Hence,
\[
u_k \approx \epsilon_p^{1/3} g^{2/3} k^{-3/5}, \tag{66}
\]
\[
\rho_k \approx \epsilon_p^{1/3} g k^{-1/3}. \tag{67}
\]

Therefore, the KE spectrum \( E_\ell(k) \approx u_k^2 / k \), PE spectrum \( E_p(k) \approx \rho_k^2 / k \), and \( \Pi_\ell(k) \approx u_k^2 k \) are
\[
E_\ell(k) = \epsilon_p^{2/3} g^{1/3} k^{-11/5}, \tag{68}
\]
\[
E_p(k) = \epsilon_p^{4/5} g^{2/5} k^{-7/5}. \tag{69}
\]
where \( \epsilon_i \)'s are constants. At smaller length scales \( (k > k_B) \), where \( k_B = 2\pi/L_B \) is the Bolgiano wavenumber, Bolgiano [11] and Obukhov [81] argued that the buoyancy is relatively weak, hence Kolmogorov-Obukhov (KO) scaling is valid in this regime, i.e.,

\[
\Pi_u(k) = \epsilon_{\nu} g^{2/3} k^{-4/3},
\]

\[
\Pi_p(k) = \epsilon_p,
\]

(70)

(71)

where \( K_B \) is the Batchelor's constant. A comparison of \( \Pi_u(k) \) of equation (70) with that of equation (74) yields the crossover wavenumber \( k_B \) as

\[
k_B \approx g^{3/2} \epsilon_u^{-5/4} \epsilon_p^{3/4}.
\]

(76)

The associated length, the Bolgiano length, is \( L_B = (2\pi)/k_B \).

The scaling relations are also presented using the variables \( \delta u(l) \) and \( \delta \theta(l) \), which are defined as

\[
\delta u(l) = \langle u(x+1) - u(x) \rangle \cdot \frac{1}{l},
\]

(77)

\[
\delta \theta(l) = \theta(x+1) - \theta(x),
\]

(78)

and the structure function for the velocity and temperature fluctuations, which are defined as

\[
S_u^u(l) = \langle (\delta u(l))^2 \rangle,
\]

(79)

\[
S_u^\theta(l) = \langle (\delta \theta(l))^2 \rangle,
\]

(80)

where \( \langle . \rangle \) represent the ensemble average. Using scaling analysis similar to that given above, it can be derived that [29]

\[
S_u^u(l) = (\epsilon_{\nu})^{2/3} g^{2/3} l^{1/3},
\]

(81)

\[
S_u^\theta(l) = (\epsilon_p)^{2/3} g^{2/3} l^{1/3},
\]

(82)

for \( l > L_B \), and

\[
S_u^u(l) = (\epsilon_{\nu})^{-1/3} l^{2/3},
\]

(83)

\[
S_u^\theta(l) = (\epsilon_p)^{-2/3} (\epsilon_p)^{2/3} l^{1/3},
\]

(84)

for \( l < L_B \). Note that \( l \) correspond to \( 1/k \), and \( \delta u(l) \rightarrow u_0 \).

The BO phenomenology implicitly assumes isotropy in Fourier space, which is a tricky assumption. For BO scaling, the gravity must be strong enough to compete with the nonlinear term \( u \cdot \nabla u \), but not too strong to make the flow quasi two-dimensional (quasi-2D). This corresponds to \( Fr \approx 1 \) regime. A large number of earlier explorations in SST have been for \( Fr \ll 1 \) regime, see for example, Lindborg [60], Brethouwer et al [14], and Bartello and Tobias [4]. SST can be broadly classified in three regimes. Note that nonlinearity is strong (\( Re \gg 1 \)) for turbulent flows.

(i) Weak gravity (\( Ri \ll 1 \)): Strong nonlinearity yields behaviour similar to hydrodynamic turbulence (\( E_u(k) \sim k^{-4/3} \)).

(ii) Moderate gravity (\( Ri \approx 1 \)): Comparable strengths of gravity and nonlinearity yields nearly isotropic turbulence with BO scaling, as described earlier.

(iii) Strong gravity (\( Ri \gg 1 \)): Strong gravity makes the flow quasi-2D. Hence the behaviour has similarities with 2D hydrodynamic turbulence (e.g., inverse cascade of energy). Refer to Lindborg [60], Brethouwer et al [14], and Bartello and Tobias [4] for further details.

3.2.2. Generalization of Bolgiano-Obukhov scaling to RBC

Using mean field theory approximation, Procaccia and Zeitak [89] argued that the Bolgiano-Obukhov scaling is applicable to convective turbulence. Later, L'vov [65] assumed that in convective turbulence, the kinetic energy is converted to the potential energy and therefore, favored BO scaling. L'vov and Falkovich [66] employed a differential model for energy and entropy fluxes in \( k \)-space and found that the BO scaling is valid for convective turbulence. Rubinstein [91] employed renormalization group analysis to RBC and observed that the
renormalized viscosity $\nu(k) \sim k^{-8/5}$, $E_u(k) \sim k^{-11/5}$, and $E_v(k) \sim k^{-7/5}$. Based on these observations Rubinstein claimed BO scaling for RBC. Ching [27, 29] and Ching et al [28] studied the structure functions for the velocity and temperature fluctuations of turbulent convection, and claimed consistency with Bolgiano-Obukhov scaling. Ching et al [28] computed the anomalous scaling for the turbulent RBC.

The aforementioned theories had a profound influence in the field, and a large number of analytical, experimental, and numerical works have been attempted to verify these ideas. Lohse and Xia [63] reviewed critically if BO scaling is indeed present in RBC; they studied the experimental, theoretical, and numerical results and argued that it is difficult to conclude the applicability of BO scaling in RBC. Recently Kumar et al [53] showed that the BO scaling does not describe RBC turbulence since the energy supply by buoyancy in RBC is very different from that in stably stratified flow. We will provide these arguments below.

3.2.3. A phenomenological argument based on kinetic energy flux

Kumar et al [53] and Verma et al [114, 115] presented a phenomenological argument based on the KE flux to derive a spectral theory of buoyancy-driven turbulence. Equation (61) provides important clues on the energy spectrum and flux of the buoyancy-driven flows. Here we list three possibilities for the inertial range ($k_f < k < k_d$), where $k_f$ is the forcing wavenumber, and $k_d$ is the dissipation wavenumber:

(i) For the inertial range of hydrodynamic turbulence, $F_B(k) = 0$ and $D(k) \to 0$, therefore $\Pi_u(k + \Delta k) \approx \Pi_u(k)$ and $E_u(k) \sim k^{-5/3}$, which is a prediction of the Kolmogorov’s theory [50]. Note that $F_u(k) = 0$ in the inertial range.

(ii) For the stably stratified flows, as argued by Bolgiano [11] and Obukhov [81], in the $k_f < k < k_B$ wavenumber band, buoyancy converts the kinetic energy of the flow to the potential energy, i.e., $F_B(k) < 0$. Hence, equation (61) predicts that $\Pi_u(k)$ will decrease with $k$ in this wavenumber range, as shown in figure 3(a). In the wavenumber range, $k_B < k < k_d$, buoyancy becomes weaker, hence $\Pi_u(k) \approx$ constant.

(iii) For RBC in three dimensions, buoyancy feeds the kinetic energy, hence $F_B(k) \to 0$. Therefore we expect the KE flux $\Pi_u(k)$ to increase. Numerical simulation of Kumar et al [53] for Pr = 1 and large Ra show that the energy supplied by buoyancy is dissipated by the viscous force, i.e., $F_g(k) \approx D(k)$. Hence $\Pi_u(k) \approx$ constant in the inertial range, and they recovered Kolmogorov’s spectrum for RBC for Pr = 1 in 3D. Note that L’vov [65] argued that $F_B(k) < 0$, which is not the case in 3D RBC with Pr $\approx$ 1. Note that the nature of energy flux depends on the space dimensionality and Prandtl number, some of which will be discussed in subsequent section.

Figure 3. Schematic diagrams of the kinetic energy flux $\Pi_u(k)$ for the stably stratified system and convective system. (a) In stably stratified flows, $\Pi_u(k)$ decreases with $k$ due to the negative energy supply rate $F_B(k)$, (b) In convective system, $F_B(k) > 0$, hence $\Pi_u(k)$ first increases for $k < k_f$ where $F_B(k) > D(k)$, then $\Pi_u(k) \approx$ constant for $k_f < k < k_d$ where $F_B(k) \approx D(k); \Pi_u(k)$ decreases for $k > k_d$ where $F_B(k) < D(k)$: From Kumar et al [53]. Reprinted with permission from APS.
The above arguments indicate that the structure functions for the fluctuations of RBC in 3D for $Pr \approx 1$ may follow the following scaling relations:

\begin{align}
S^u_l (l) &= \langle \epsilon_u \rangle^{q/3} l^{\delta/3}, \\
S^p_l (l) &= \langle \epsilon_p \rangle^{-q/6} \langle \epsilon_p \rangle^{1/2} l^{\delta/3}.
\end{align}

The above relations need to be tested using numerical simulation and experiments.

3.2.4. Modeling and field theory

Researchers\cite{33, 51, 59, 68, 69, 121} employed field-theoretic techniques to understand the physics of turbulent fluid. In field theory, the nonlinear terms of the equations are expanded perturbatively. Some of the popular field-theoretic techniques are direct interaction approximation (DIA)\cite{51, 59}, renormalization group analysis\cite{33, 51, 68, 69, 121}, mean field approximation\cite{89}, etc. Field theory has been applied to buoyancy-driven flows as well.

As described in section 3.2.2, Procaccia and Zeitak\cite{89} employed mean field approximation to convective turbulence and obtained BO scaling. Rubinstein\cite{91} used Yakhot-Orszag’s\cite{121} renormalization group procedure and proposed an isotropic model for convective turbulence. His results are consistent with that of Procaccia and Zeitak\cite{89}. Recently, using self-consistent field theory, Bhattacharjee\cite{7} obtained $E_p(k) \sim k^{-13/3}$ for RBC in the infinite Prandtl number limit. Bhattacharjee\cite{6} used the global energy balance for the stratified fluid and argued that the BO scaling could be observed in stably stratified flow at high Richardson number. In addition, he also added the possibility of BO scaling for RBC in some range of Prandtl numbers.

In the next section, we will present numerical results for the stably stratified turbulence and RBC.

3.3. Numerical analysis of buoyancy-driven turbulence

3.3.1. Stably stratified turbulence

Researchers simulated the SST for the three regimes described in section 3.2.1. First we discuss the results for strong gravity that corresponds to $Ri \gg 1$ or $Fr \ll 1$. Such configurations are observed in some regimes of planetary and stellar atmospheres. Strong gravity makes such flows quasi-2D with dual scaling, $k^{-3}$ and $k^{-5/3}$. In this regime, Lindborg\cite{60}, Brethouwer et al\cite{14}, and Bartello and Tobias\cite{4} showed that the spectra of the horizontal KE and PE follow $k^{-5/3}$ scaling, while the energy spectrum of the vertical velocity follows $k_0^{-3}$. Vallgren et al\cite{109} included rotation in their simulation and obtained KE spectrum as $k^{-3}$ and $k^{-5/3}$ for two different wavenumber bands.

For weak stratification ($Ri \ll 1$), Kumar et al\cite{53} performed a 3D SST simulation and reported Kolmogorov’s spectrum for the kinetic energy as expected. Kumar et al also studied the moderate stratification regime and reported BO scaling, which will be described below. In this paper we focus on the results for $Fr \approx 1$ since they have been observed recently.

Kumar et al\cite{53} simulated stably stratified flows in a cubical box of size $(2\pi)^3$ with periodic boundary conditions at all the walls. They forced the small wavenumber modes randomly to achieve a steady state. The parameters of their simulations are $Ra = 5 \times 10^4$ and $Pr = 1$ that yields $Ri = 0.01$ and $Fr = 10$. Figure 4(a) exhibits the normalized KE spectra—$E_{u}(k)k^{11/5}$ for the BO scaling, and $E_{u}(k)k^{5/3}$ for the KO scaling. The numerical data fits better with the BO scaling than the KO scaling, thus confirming the BO phenomenology for the SST when $Fr \approx 1$. This is also verified by the PE spectrum as shown in figure 4(b) in which $E_{p}(k)k^{2/5}$ provides a better fit to the data than $E_{p}(k)k^{5/3}$.

Further, Kumar et al\cite{53} computed the KE and PE fluxes which are exhibited in figure 5. They observed that $\Pi_k(k) > 0$ and it decreases with $k$ (equation (70)), while the PE flux $\Pi_p$ is a constant in the inertial range (equation (71)); thus the flux results are consistent with the BO predictions. Kumar et al\cite{53} also computed the energy supply rate by buoyancy, $F_d(k)$, and the viscous dissipation spectrum, $D(k)$, which are illustrated in figure 6. Note that $F_d(k) < 0$, as argued in BO phenomenology. The Bolgiano wavenumber $k_B$ of equation (76) is approximately 8.5, which is only 3–4 times smaller than $k_0$, the wavenumber where the dissipation range starts. Therefore Kumar et al\cite{53} did not observe a definitive crossover from $k^{-11/5}$ to $k^{-5/3}$ in their simulations.

The aforementioned observations demonstrate applicability of the BO scaling for SST with a moderate stratification.

3.3.2. Rayleigh–Bénard convection

A large number of numerical simulations have been performed with an aim to identify which among the two, BO or KO, scaling is applicable to RBC\cite{65}. Grossmann and Lohse\cite{37} simulated RBC for $Pr = 1$ under Fourier–Weierstrass approximation and reported Kolmogorov’s scaling. For on periodic boundary condition,
Borue and Orszag [12] and Škandera et al [100] reported KO scaling for the velocity and temperature fields. Kerr [49] reported the horizontal spectrum as a function of horizontal wavenumber and observed Kolmogorov’s spectrum. Verzicco and Camussi [117], and Camussi and Verzicco [20] showed BO scaling using the frequency spectrum of real space probe data. Kaczorowski and Xia [48] reported KO scaling for the longitudinal velocity structure functions, but BO scaling for the temperature structure functions in the centre of a cubical cell. Kumar et al [53] computed $E_u(k)$ and $\Pi_u(k)$, and showed Kolmogorov-like behaviour for RBC, i.e., $E_u(k) \sim k^{-5/3}$ and $\Pi_u(k) \sim \text{const}$. In this paper we present the above quantities for 4096$^3$ resolution and very high Ra that unambiguously demonstrates KO scaling for RBC. We also report the shell-to-shell energy transfers and the ring spectrum for RBC that show close resemblance with the hydrodynamic turbulence.
We performed RBC simulations in a unit box with $4096^3$ grid for $Pr = 1$ and $Ra = 1.1 \times 10^{11}$. For the velocity field, we employed the free-slip boundary condition at the top and bottom plates, and periodic boundary condition at the side walls. The temperature field satisfies conducting boundary condition at the top and bottom plates, and the periodic boundary condition at the side walls. We computed the spectra and fluxes of the KE and the entropy ($\theta^2/2$) using the steady state data. Figure 7(a) exhibits the KE spectra normalized with $k^{11/3}$ and $k^{5/3}$. The plots indicate that in the wavenumber band $15 < k < 600$ (inertial range), the shaded region of the figure, the KO scaling fits better than the BO scaling.

We exhibit the KE and entropy fluxes in figure 7(b). We observe that the kinetic energy flux $\Pi_u(k)$ remains constant in the inertial range, a band where $E_u(k) \sim k^{-5/3}$. Thus we claim that the convective turbulence exhibits Kolmogorov’s power law in the inertial range. We also computed $F_D(k)$, $\Pi_u(k)$, and $d\Pi_u(k)/dk$ as further tests. According to figure 8(a) $F_D(k) > 0$ in the inertial range, consistent with the discussion of section 3.2.3 and figure 3(b), and it approximately balances $D(k)$. Therefore, $d\Pi_u(k)/dk \approx 0$ or $\Pi_u(k) \approx$ constant (see equation (60)). The constancy of $\Pi_u(k)$ yields $E_u(k) \sim k^{-5/3}$, consistent with the energy spectrum plots of figure 7(a). Figure 8(b) shows that $[d\Pi_u(k)/dk]/\Pi_u(k) \ll 1$ in the inertial range consistent with the constant $\Pi_u(k)$. Interestingly, $D(k) = 2\nu k E_u(k) \sim k^{1/3}$, consistent with $E_u(k) \sim k^{-5/3}$. Also, $F_D(k) \sim k^{-5/3}$. In addition, the entropy flux $\Pi_\theta(k)$ is constant, and $\Pi_\theta(k) \approx \Pi_u(k)$ in dimensionless units.

We also compute the shell-to-shell energy transfers (equation (64)) using the steady-state data of our simulation. We divide the Fourier space into 40 concentric shells; the inner and outer radii of the $n$th shell are $k_{n-1}$ and $k_n$, respectively with $k_n = \{0, 2, 4, 8 \times 2^{(n-3)}, \ldots, 6432\}$, where $s = (1/35)\log_2(804)$. The radii of the inertial-range shells are binned logarithmically due to the power law physics of RBC in the inertial range. In figure 9(a) we exhibit the shell-to-shell energy transfers with the indices of the $x$, $y$ axes representing the receiver and giver shells respectively. The plot indicates that the shell gives energy to ($m + 1$)th shell, and it receives energy from the ($m + 1$)th shell. Thus the energy transfer in RBC is local and forward, very similar to hydrodynamic turbulence. This result is consistent with the energy spectrum and flux studies described earlier.

Convective flows are expected to be anisotropic due to buoyancy; hence it is important to quantify anisotropy using the quantities that are dependent on the polar angle, the angle between $\hat{z}$ and $\mathbf{k}$. For the same,
we divide a wavenumber shell into rings \[75\]. The energy contents of the rings are called ring spectrum \(bE_k\), where \(\beta\) represents the sector index for the polar angles (for details see Nath et al \[75\]). The ring spectrum \(bE_k\), depicted in figure 9(b), shows that the flow is nearly isotropic, again similar to hydrodynamic turbulence. These results clearly demonstrate that the turbulent convection for \(Fr = 1\) has a very similar behavior as hydrodynamic turbulence.

The temperature fluctuation however exhibit a unique behaviour. As illustrated in figure 10, we observe dual branches for the entropy spectrum \(E_s(k)\). The upper branch varies as \(k^{-2}\) because \(\Theta(0, 0, k_z) \approx -1/(\pi k)\), as discussed in section 2.7. The lower branch shows neither KO \((k^{-5/3})\) nor BO \((k^{-7/5})\) spectrum. Note that both the branches of entropy spectrum generate a constant entropy flux \(\Pi_s(k)\) (see figure 7(b)), and the modes \(\Theta(0, 0, k_z)\) also participate in energy transfers.

3.4. Experimental results

For stably-stratified flows, there are not many laboratory experiments to verify BO phenomenology. However, scientists have measured the KE spectrum of the Earth’s atmosphere and relate it to the theoretical predictions. Most notably Gage and Nastrom [35] observed a combination of \(k^{-3}\) and \(k^{-5/3}\) energy spectra. Some researchers attribute the \(k^{-3}\) spectrum at lower wavenumbers to the two-dimensionalization of the flow, while \(k^{-5/3}\) spectrum at larger wavenumbers to the forward cascade of kinetic energy; yet these issues are still unresolved. These features are expected to arise for \(Fr \ll 1\).

There are a significant number of laboratory experiments on RBC, with some favouring the BO scaling [25, 124], while some others in support of the KO scaling [30]. These results are reviewed in detail in Lohse and Xia [63]. In most convective experiments, the velocity field, \(u_z(r, t)\), and/or the temperature field, \(T(r, t)\), are
probed near the lateral walls of the container. For such experiments, the Taylor’s hypothesis \[56, 94, 105\] is invoked to relate the frequency power spectrum \(E(f)\) of the time series to the one-dimensional wavenumber spectrum \(E(k)\); this connection is under debate due to the absence of any constant mean velocity field \[56, 63\]. Researchers \[57, 104, 122, 123\] employ 2D particle image velocimetry for high-resolution visualization and computation of an approximate energy spectrum under the assumption of homogeneity and isotropy, which is not strictly valid in convection \[75\]. In summary, on the experimental front, there is no convergence on which of the two scaling, BO of KO, is valid. For details refer to the review papers \[2, 63\].

### 3.5. Turbulence in thermal boundary layer and in two dimensions

A burning question is whether KO scaling or BO scaling is applicable to the boundary layers of RBC. The flux arguments of section 3.2.3 provide some insights into the dynamics of boundary layers. Here, typically \(u_\tau \ll u_c\), hence the flow is quasi-2D, and we expect an inverse cascade of KE. Using \(\Pi_u(k) < 0\), \(F_b(k) > 0\), and \(d\Pi_u(k)/dk \approx F_b(k)\), we may argue that \(\Pi_u(k)\) may increase with \(k\) as shown in figure 11. An application of scaling arguments of section 3.2.1 may yield \(E_u(k)\) and \(\Pi_u(k)\) according to equations (68)–(71), i.e., Bolgiano-Obukhov scaling for \(k < k_B\). For \(k \geq k_B\), the KE spectrum may exhibit a mixture of \(k^{-5/3}\) (regime of inverse cascade of energy) and \(k^{-3}\) (regime of forward cascade of enstrophy) depending on where the effective forcing band lies in relation to \(k_B\). Thus, in the boundary layer, RBC may exhibit BO scaling, and it needs to be investigated carefully using numerical simulations and experiments.

The aforementioned scaling arguments may also work for 2D RBC (xz plane in which the buoyancy is along the z direction), as well as in quasi 2D RBC when \(L_x \gg L_y\). Toh and Suzuki \[106\] simulated 2D RBC and reported \(E_u(k) \sim k^{-11/5}\) and \(\Pi_u(k) \sim -k^{-4/5}\) in line with the above arguments. Calzavarini et al \[19\] also reported similar results in their structure function computations.

### 3.6. Turbulence in RTI

RTI has a strong similarity with RBC in the sense that the heavier fluid sits on top of lighter fluid. Hence we expect the RBC turbulence phenomenology to be applicable to RTI as well, at least approximately. Chertkov \[24\] proposed that a fully-developed 3D RTI will exhibit Kolmogorov’s spectrum due to the Rayleigh–Taylor pumping at large scales. Boffetta et al \[10\] observed this behaviour in their numerical simulations. Chertkov \[24\] however does not take into account the buoyancy at all scales (see section 3.2.3). In a quasi-2D box (\(L_y \ll L_z\)), Boffetta et al \[9\] show coexistence of BO and KO scaling (\(k^{-11/5}\) and \(k^{-5/3}\)), consistent with the arguments of section 3.5.

### 3.7. Turbulence in miscellaneous systems

Scientists have studied spectra of the velocity field and the scalar field in other buoyancy-driven systems. Pawar and Arakeri \[87\] performed experiment on the vertical tube described in section 2.8.3. They observed that the velocity field exhibits \(k^{-5/3}\) spectrum, while the scalar spectrum is closer to \(k^{-7/5}\).

Prakash et al \[88\] studied the energy spectrum of the bubbly turbulence using an experiment. For the velocity field, they reported \(k^{-5/3}\) energy spectrum for \(k < 1/b\), and \(k^{-3}\) for \(k > 1/b\) where \(b\) is the bubble size. They argued that the large and intermediate scales exhibit \(k^{-5/3}\) spectrum due to the standard Kolmogorov’s argument. For \(k > 1/b\), Prakash et al \[88\] explained the \(k^{-3}\) energy spectrum by invoking equipartition between the energy dissipation and energy feed by the buoyancy. We believe that the Kolmogorov’s spectrum for bubbly turbulence arises due to the dynamical similarities with RBC. For this system it may be interesting to investigate the energy spectrum using the flux arguments.

The turbulent Taylor–Couette flow \[43\] may exhibit spectral behaviour similar to RBC since both the systems are unstable with similar energetics (see sections 3.2.3 and 3.3.2). We believe that the Non-Boussinesq
convective flows may also exhibit Kolmogorov-like spectrum for weak compressibility since here too the thermal plumes feed the kinetic energy, as in RBC.

3.8. Turbulence in small and large Prandtl number RBC

In section 3.2.3 we derived the spectra and fluxes of the velocity and temperature fields for RBC with $\Pr \sim 1$. These arguments are not applicable to RBC at extreme Prandtl numbers. However, we can easily deduce the spectrum for very small and very large Pr’s as follows. These computations have been first reported in [72] and [86] respectively.

In RBC with zero or small Prandtl numbers, thermal diffusivity $\kappa \rightarrow \infty$ that leads to $u_\tau(k) \sim \theta(k)(\kappa k^2)$ [72]. Hence, the buoyancy, which is proportional to $\theta(k)$, is dominant at small wavenumbers. Therefore, the assumption of the Kolmogorov’s phenomenology that the forcing acts at large length scales is valid, and we expect the Kolmogorov’s phenomenology for the hydrodynamic turbulence to be applicable to RBC with $\Pr \rightarrow 0$. Mishra and Verma [72] verified the above phenomenology using numerical simulations.

In the limit of infinite Prandtl number ($\nu \rightarrow \infty$), the momentum equation is linear [86]. However if the Péclet number is large, the temperature equation is nonlinear and it yields an approximate constant entropy flux. Using scaling arguments, Pandey et al [86] derived that for infinite and large Pr, $E_u(k) \sim k^{-13/5}$. They also verified the above scaling using numerical simulations.

3.9. Simulation of turbulent convection in a periodic box and shell model

Borue and Orszag [12], Škandra et al [100], Löhse and Toschi [62], and Calzavarani et al [18] simulated turbulent thermal convection in a periodic box. They simulated equations (17), (18) under a gradient $dT/dz$. In the absence of boundary layers, the velocity and temperature fields exhibit $k^{-5/3}$ spectra [12, 100]. In addition, the Nusselt number $Nu \sim Ra^{1/2}$ [18, 62], which is expected in the ultimate regime when the effects of boundary layers are negligible. Note that the temperature spectrum for the periodic box is very different from that with conducting walls that exhibit dual spectra. It is important to note that turbulent thermal convection in a periodic box is numerically unstable; the system exhibits steady behaviour for carefully chosen set of initial conditions.

Direct numerical simulation of turbulent systems is quite demanding due a large number of interacting Fourier modes. Therefore, scientists often use shell models, which are based on much fewer number of modes. Brandenburg [13], Lozhkin and Frick [64], Mingshun and Shida [70], Ching and Cheng [28], and Kumar and Verma [54, 55] constructed shell models for buoyancy-driven turbulence. Ching [27, 29] and Ching et al [28] computed the structure functions of turbulent convection using a shell model, and claimed consistency with Bolgiano-Obukhov scaling. The advantage of the shell model of Kumar and Verma [54] is that it describes both turbulent stably-stratified and convective flows using a single set of equations. It also enables flux computation of the kinetic energy and $\rho^2/2$, where $\rho$ is the density of the fluid. Kumar and Verma [54] showed that the results of the shell model are consistent with the DNS results described earlier.

3.10. Concluding remarks on the energy spectrum

We summarise the important results of this section as follows:

(i) A large body of works on RBC assume Bolgiano-Obukhov scaling. The flux-based arguments described in sections 3.2.3 and 3.3.2 demonstrate that in three dimensions for $Pr$ near unity, RBC exhibits Kolmogorov-like energy spectrum and flux. For example, the KE flux is nearly constant in the inertial range; the shell-to-shell energy transfer is local and forward; the ring spectrum exhibits a near isotropy in Fourier space. The constant KE flux is due to the near cancellation between the KE supply by buoyancy and the viscous dissipation rate.

(ii) The nature of energy spectrum and flux of RBC depends on the space dimensionality and Prandtl number, as described earlier in this section. For small Prandtl number, convective turbulence is similar to hydrodynamic turbulence, but $E_u(k) \sim k^{-13/5}$ for very large and infinite Prandtl number.

(iii) The small-scale fluctuations in the boundary layer contributes to $E_u(k)$ at large $k$. Hence the aforementioned $E_u(k)$ in the inertial range is dominated by the fluctuations of the bulk.

(iv) The temperature fluctuations for RBC exhibits dual spectra, with the upper branch scaling as $k^{-2}$. In section 2.7 we discussed the origin of $k^{-2}$ spectrum in terms of temperature profile in the boundary layers and in the bulk.

(v) The SST under nearly isotropic conditions (when Froude number is of the order of unity) exhibits Bolgiano-Obukhov scaling.

In the next section, we briefly describe scaling of Reynolds and Nusselt numbers.
4. Modelling of large-scale quantities of RBC

In this section we quantify the large-scale quantities of RBC, namely the Nusselt and Reynolds numbers. Many researchers have worked on this problem; for details and references, refer to the review articles [2, 8, 26, 63, 98]. Despite complexities of the flow, RBC exhibits certain universal behaviour; in the turbulent limit, \( Pe \sim \sqrt{Ra \, Pr} \), but in the viscous regime, \( Pe \sim Ra^{1/3} \) [38, 84].

Turbulent thermal flux is somewhat more complex; it is quantified using the nondimensional variable called Nusselt number, \( Nu \), which is defined as [2, 26, 119]

\[
Nu = \frac{\kappa \Delta / d + (u_x \theta_{\text{res}})^{1/2}}{\kappa \Delta / d} = 1 + \left( u_x d \theta_{\text{res}} \right)^{1/2} \left( \frac{\theta_{\text{res}}}{\Delta} \right)^{1/2} \left( u_x d \theta_{\text{res}} \right)^{1/2} \left( \frac{\theta_{\text{res}}}{\Delta} \right)^{1/2}, \tag{87}
\]

where \( ()_V \) stands for a volume average, \( u_x' = u_x d / \kappa, \theta_{\text{res}}' = \theta_{\text{res}} / \Delta \), and \( C_{u\theta_{\text{res}}} \) is the normalized correlation function between the vertical velocity and the residual temperature fluctuation [116]:

\[
C_{u\theta_{\text{res}}} = \frac{\left( u_x' \theta_{\text{res}}' \right) V}{\left( u_x'^2 \right)^{1/2} \left( \theta_{\text{res}}'^2 \right)^{1/2}}. \tag{88}
\]

Kraichnan [52] argued that in turbulent convection \( u_x' \sim Ra^{1/2}, \theta_{\text{res}}' \sim 1, \) and \( C_{u\theta_{\text{res}}} \sim \text{const} \), hence \( Nu \sim Ra^{1/2} \), which is called the scaling of ultimate regime. Experiments and numerical simulations however reveal that \( Nu \sim Ra^{3} \) with \( \beta \) ranging from 0.25 to 0.33. Grossmann and Lohse [38–42] derived a phenomenological formula that fits with the experimental and numerical results quite well. The deviation from Kraichnan’s predictions of 1/2 to \( \beta \approx 0.3 \) is attributed to the boundary layer [38, 39]. There are intense research activities to test whether the ultimate regime exists or not. He et al [44] and others performed experiments on turbulent convection up to \( Ra \approx 10^{15} \) and observed an increase in the Nusselt-number exponent \( \beta \) from 0.31 to 0.38, as well as logarithmic mean temperature profile [108]. Thus they claimed existence of the ultimate regime. However, Niemela et al [79] and Urban et al [107] do not observe deviation of \( \beta \) from \( \approx 0.3 \), hence they argue against the existence of ultimate regime. In this paper, we do not discuss this issue any further, and we refer the reader to works described above.

In the next subsection we describe the Grossmann–Lohse model that predicts the scaling of Reynolds and Nusselt number quite successfully.

4.1. Grossmann–Lohse model

Grossmann and Lohse (GL) [38–42, 102] derived the formulas for \( Nu(Ra, \, Pr) \) and \( Re(Ra, \, Pr) \) by exploiting the fact that the global viscous dissipation rate, \( \epsilon_u \), and thermal dissipation rate, \( \epsilon_T \), get contributions from the bulk and boundary layers, i.e.,

\[
\epsilon_u = \epsilon_{u,\text{BL}} + \epsilon_{u,\text{bulk}}, \tag{89}
\]

\[
\epsilon_T = \epsilon_{T,\text{BL}} + \epsilon_{T,\text{bulk}}, \tag{90}
\]

where BL and bulk denote the boundary layer and the bulk respectively. They invoked the exact relations of Shraiman and Siggia [97] for the global viscous and thermal dissipation rates (see equations (38), (39)), and estimated the aforementioned contributions of the boundary layers and the bulk to \( \epsilon_u \) and \( \epsilon_T \) in various \( Ra–Pr \) regimes. For \( Pr \approx 1 \) and very large Ra they used \( \epsilon_{u,\text{bulk}} = U^3 / d \) and \( \epsilon_{T,\text{bulk}} = U \Delta^2 / d \), but for extreme Prandtl numbers, these estimates get altered by the boundary layer widths.

Using the above ideas, GL [38–42, 102] derived the following coupled equations

\[
(Nu - 1)RaPr^{-2} = c_1 \frac{Re^2}{g \sqrt{Re_L / Re}} + c_2 Re^3, \tag{91}
\]

\[
Nu - 1 = c_3 \sqrt{RePr} \left\{ \frac{2aNu}{\sqrt{Re_L} g \left( \sqrt{Re_L / Re} \right)} \right\}^{1/2} + c_4 RePr \left\{ \frac{2aNu}{\sqrt{Re_L} g \left( \sqrt{Re_L / Re} \right)} \right\}^{1/2} \tag{92}
\]

where \( c_i \)’s and \( Re_L \) are constants, and the functions \( f \) and \( g \) model the thermal BL [102]. Using the above formulae, GL computed the Nusselt and Reynolds numbers as a function of Ra and Pr that agree with presently available experimental and numerical simulation results quite well [2].

In the next subsection we describe a new model developed recently by Pandey et al [84] and Pandey and Verma [85].
4.2. An alternate derivation of Péclet number

Recently Pandey et al [84] and Pandey and Verma [85] provided an alternate derivation of Péclet number. Note that $Pe = Re Pr$. Pandey et al [84] analysed the rms values of various terms of the momentum equation, which are exhibited in the schematic diagram of figure 12. Under statistical steady state $\langle \partial u/\partial t \rangle \approx 0$, Pandey et al observed that in the turbulent regime, the acceleration $u \cdot \nabla u$ is primarily provided by the pressure gradient $- \nabla \sigma$, and the buoyancy and viscous terms are relatively small. The above features are consistent with similarities between the turbulence in RBC and hydrodynamics (see section 3.3.2). However, in the viscous regime ($Re \ll 1$), $- \nabla \sigma$ is small, and the buoyancy and viscous terms cancel each other resulting in a very small acceleration of the fluid.

Dimensional analysis of the momentum equation yields

$$\frac{\alpha g \theta \dot{z}}{U} = \frac{c_1 U^2}{d} + \frac{c_2 \alpha g \Delta}{d} = \frac{c_4 \nu U}{d^2},$$

where $c_i$’s are dimensionless coefficients defined as

$$c_1 = \frac{|u \cdot \nabla u|}{U^2/d}; \quad c_2 = \frac{|\nabla \sigma_{\text{les}}/\rho_m|}{U^2/d}; \quad c_3 = |\theta_{\text{css}}/\Delta|; \quad c_4 = \frac{|\nabla^2 u|}{U/d^2}.$$ (94)

Pandey et al [84] observed $c_i$’s to be functions of $Ra$ and $Pr$ that yields interesting and nontrivial scaling relations. It is important to contrast this behaviour with free turbulence (without walls) where $c_i$’s are constants. Multiplication of equation (93) with $d^3/\kappa^2$ yields

$$c_1 Pe^2 = c_2 Pe^2 + c_3 Ra Pr - c_4 Pe Pr,$$ (95)

where $Pe = U d/\kappa$ is the Péclet number. The solution of the above equation is

$$Pe = \frac{-c_4 Pr + \sqrt{c_4^2 Pr^2 + 4(c_1 - c_2) c_3 Ra Pr}}{2(c_1 - c_2)}.$$ (96)

using which $Pe$ can be computed as a function of $Ra$ and $Pr$.

In the turbulent regime, the viscous term of equation (95) can be ignored, hence

$$Pe \approx \sqrt{\frac{c_1}{c_2}} Ra Pr.$$ (97)

This limit is applicable when

$$c_4^2 Pr^2 \ll 4|c_1 - c_2| c_3 Ra Pr.$$ (98)

The scaling for the viscous regime is obtained by equating the buoyancy and viscous terms of the momentum equation that yields

$$Pe \approx \frac{c_3}{c_4} Ra.$$ (99)
Figure 13. The normalized Péclet number (PeRa^{-1/2}) versus Ra for numerical data of Pandey et al [84] for Pr = 1 (red squares), Pr = 6.8 (blue triangles), and Pr = 10^2 (black diamonds); numerical data of Silano et al [99] (magenta pentagons, Pr = 10^2), Recuwyk et al [110] (red circles, Pr = 1), Scheel and Schumacher [93] (green crosses, Pr = 0.7); and the experimental data of Xin and Xia [120] (orange plus, Pr = 6.8), Cioni et al [31] (brown right triangles, Pr = 0.022), and Niemela et al [80] (Pr = 0.7, green down-triangles). The continuous curves represent Pe computed using equation (96). The predictions of equation (96) for Pr = 0.022 and 6.8 have been multiplied with 2.5 and 1.2, respectively, to fit the experimental results from Cioni et al [31] and Xin and Xia [120]. From Pandey and Verma [85]. Reprinted with permission from AIP.

Pandey et al [84] computed c_i’s using the RBC simulation data for Pr = 1, 6.8, 10^2, 10^3 and Ra from 10^6 to 5 × 10^8. These simulations were performed for no-slip boundary condition at all the walls using a finite volume solver OPENFOAM [82]. They reported the following functional form for c_i’s

\[
\begin{align*}
    c_1 &= 1.5Ra^{0.19}Pr^{-0.06}, \\
    c_2 &= 1.6Ra^{0.09}Pr^{-0.08}, \\
    c_3 &= 0.75Ra^{-0.15}Pr^{-0.05}, \\
    c_4 &= 20Ra^{0.24}Pr^{-0.08}.
\end{align*}
\]

The errors in the above exponents are < ±0.01, except for the Ra exponent of c_i that has error of the order of 0.10.

In figure 13, we plot the normalized Péclet number, PeRa^{-1/2} for Pr = 1, 6.8, 10^2 and compare them with the predictions using equation (96). The figure also exhibits Pe from other simulations and experiments. The plots reveal that the predictions of Pandey et al [84] (equation (96)) match with the numerical and experimental results quite well.

Using the above c_i’s and equation (98), we find that Ra > 10^6 Pr belongs to the turbulent regime, whereas Ra ≪ 10^6 Pr belongs to the viscous regime. In the viscous regime

\[
Pe = \frac{c_3}{c_4}Ra \approx 0.038Ra^{0.60},
\]

which is independent of Pr, consistent with the results of Silano et al [99], Horn et al [46], and Pandey et al [86]. For the turbulent regime, equation (97) yields

\[
Pe = \sqrt{\frac{c_3}{c_4}} \sqrt{RaPr} \approx \sqrt{7.5 PrRa^{0.38}}.
\]

For mercury (Pr ≈ 0.025) as an experimental fluid, Cioni et al [31] observed that Re ∼ Ra^{0.424}, which is close to the predicted exponent of 0.38 discussed above. The range of Rayleigh numbers in the experiment of Cioni et al [31] is 5 × 10^6 ≤ Ra ≤ 5 × 10^7 that is consistent with the turbulent regime estimated above (Ra ≥ 10^6 Pr). The aforementioned results are in general agreement with those of Grossmann and Lohse [38–42].

4.3. Scaling of Nusselt number and dissipation rates

We revisit the Nusselt number scaling that has been studied widely using theoretical models, experiments, and numerical simulations. The predictions of Grossmann and Lohse [38–42], equations (91), (92), fits with the experimental and numerical data quite well. As described earlier, a major debate is whether ultimate regime (exponent = 1/2) exists or not. See reviews for details [2, 8, 29, 63, 98].

Here we report recent results on the correlation function of equation (88) and the viscous dissipation rate that yield interesting insights. Verma et al [116], Pandey et al [84], and Pandey and Verma [85] computed \( C_{u_h u_h} \) of equation (88) for a range of Ra in the turbulent regime and observed nontrivial scaling. They observed that \( C_{u_h u_h} \) and the rms values of \( u'_t \) and \( \theta'_t \) scale with Ra in such a way that \( Nu \sim Ra^{0.38} \); without these corrections, \( Nu \sim Ra^{1/2} \) in the turbulent regime. Thus, one way to explain the deviation of the exponent from 1/2 in the ultimate regime [52] is due to the nontrivial scaling of \( C_{u_h u_h} \), \( u'_t \), and \( \theta'_t \).
In hydrodynamic turbulence, the viscous dissipation rate $\epsilon_u \approx U^3/d$. However this is not the case in RBC, primarily due to walls or boundary layers. Using numerical data, Verma et al [116] and Pandey et al [84] have shown that $\epsilon_u \sim Ra^{0.32}$ or $\epsilon_u \sim (U^3/d)Ra^{-0.21}$. See figure 14 for illustration for $Pr = 1$. One of the exact relations of Shraiman and Siggia [97] yields

$$\epsilon_u = \frac{U^3 (Nu - 1) Ra Pr}{Pe^3}. \tag{107}$$

Substitution of $Pe \sim Ra^{0.51}$ and $\epsilon_u \sim (U^3/d)Ra^{-0.21}$ yields $Nu \sim Ra^{0.32}$. These arguments show that the reduction of the viscous dissipation rate could be a reason for the deviation of the observed scaling $Nu \sim Ra^{0.32}$ from $Nu \sim Ra^{3/2}$ corresponding to the ultimate regime.

Ni et al [77, 78] computed the local (bulk) energy dissipation rate in RBC cell using experimental measurements. They showed that

$$\epsilon_{u,\text{bulk}} \sim \frac{U^3}{d} \sim Ra^{3/2}, \tag{108}$$

which is consistent with the predictions of Grossmann and Lohse [38–42]. Thus the variation of the exponent from the aforementioned 3/2 to 1.32 of figure 14 is possibly due to the effects of the boundary layers near the walls (also see Pandey and Verma [85]). We require detailed experimental and numerical analysis to resolve this issue.

5. Large-scale flow structures and flow reversals in RBC

The flow properties in the last two sections are related to the random nature of the flow. It has been observed that coherent structures too play important role in the convective flow, and they have certain universal properties. An interesting phenomena of RBC related to large-scale structures is flow reversals. Sreenivasan et al [101], Brown and Ahlers [16], Xi and Xia [118], and Sugiyama et al [103] observed that the vertical velocity near the lateral wall switches sign randomly. Deciphering how the reversals take place is an interesting puzzle, and it is not yet fully solved. In this section we briefly describe the present status of the field.

It is believed that the flow reversals are caused by the nonlinear interaction among the large-scale structures of the flow. For a closed cartesian box, these structures can be conveniently described by the small-wavenumber Fourier modes [21, 22]. This description is useful even for no-slip boundary conditions since the flow structures inside the boundary layers contribute to the large wavenumber modes. For a cylindrical geometry, partial information about the flow structures can be obtained by computing the azimuthal Fourier modes corresponding to the velocity field measured at various angles near the lateral walls [16, 71, 118]. Here we summarise the main results on the properties of flow reversals.

(i) During a reversal, the amplitude of the most dominant large-scale mode vanishes, while that of the secondary mode rises sharply. Chandra and Verma [21, 22] reported that during a reversal in a unit two-dimensional cartesian box, the Fourier mode (1, 1) vanishes, while the mode (2, 2), corresponding to the corner rolls, become the most dominant mode [21, 22]. See figure 1 of Chandra and Verma [21]. This numerical result is consistent with the experimental results of Sugiyama et al [103].
(ii) The nature of dominant structures depends on the box geometry and boundary conditions. For example, for a box of size $2 \times 1$, under the no-slip boundary condition, (2, 1) and (2, 2) are the primary and secondary modes respectively. However, under the free-slip boundary condition, the corresponding modes are (1, 1) and (2, 1) respectively [15, 113]; here (3, 1) too plays a major role.

(iii) Sugiyama et al [103], Chandra and Verma [21, 22] and Verma et al [113] reported that the flow reversals in two-dimensional turbulent convection are suppressed at large Rayleigh numbers. This is primarily due to relative strengthening of the primary mode (1, 1) compared to the secondary modes. At large Ra, the secondary modes become too weak to be able to cause flow reversals.

(iv) Huang et al [47] studied the flow reversals for two different boundary conditions: (a) constant temperatures at both the boundaries (CTCT), and (b) constant heat flux at the bottom plate and constant temperature at the top plate (CFCT). They showed that the flow reversals are more frequent in the CTCT case compared to the CFCT case despite the former being more stable than the latter. Thus, the flow reversals are not directly related to the flow instability [47].

(v) Verma et al [113] have constructed a group-theoretic argument to decipher the reversing and non-reversing modes during a reversals. The structure of the groups is related to the Klein group.

(vi) Thermal convection in a cylinder exhibit reversals that have similar behaviour as above. Brown and Ahlers [16] termed such reversals as cessation-led reversals. Note however that during a cessation-led reversal, the secondary modes become significant, hence the kinetic energy does not vanish.

(vii) Cylindrical convection exhibits another kind of flow reversals, called rotation-led reversals, in which the large-scale structure rotates azimuthally [16, 71, 118]. This rotation is due to the azimuthal rotation symmetry of the system. Such phenomena is also observed in a cylindrical annulus [76].

These observations reinforce the viewpoint that the nonlinear interactions among the large-scale structure are very relevant for flow reversals. The magnetic field reversals in dynamo [36], and the velocity field reversals in Kolmogorov-flow [74] also involve nonlinear interactions among the large-scale structures of the flow. Thus, these reversals share certain similarities with the flow reversals of RBC.

6. Summary

In this paper we describe the recent results on the spectral and large-scale properties of buoyancy-driven turbulence—stably-stratified flows and RBC. A summary of the results covered in this review is as follows:

(i) The SST is nearly isotropic for Froude number $Fr \gtrsim 1$. Bolgiano [11] and Obukhov [81] showed that for gravity-dominated flows ($Fr \approx 1$), the kinetic-energy spectrum $E_u(k) \sim k^{-11/3}$. Kumar et al [53] demonstrated this scaling using numerical simulations.

(ii) For $Fr \gg 1$, SST exhibits Kolmogorov scaling, i.e. $E_u(k) \sim k^{-5/3}$, due to the dominance of the nonlinear term over the buoyancy.

(iii) For $Fr \ll 1$, SST is quasi two-dimensional, and the kinetic-energy spectrum exhibits a combination of $k^{-5/3}$ and $k^{-3}$. We do not discuss this case in detail. We refer the reader to Lindborg [60], Brethouwer et al [14], and Bartello and Tobias [4].

(iv) In three dimensions and for Prandtl number $\lesssim 1$, turbulence in RBC has strong similarities with the hydrodynamic turbulence, e.g. it exhibits constant energy flux and $k^{-5/3}$ energy spectrum in the inertial range. For very large and infinite Prandtl numbers, convective turbulence has $E_u(k) \sim k^{-11/3}$. The energy spectrum is expected to be different in two dimensions and in the boundary layer.

(v) In RBC turbulence, the pressure gradient accelerates the flow, while the buoyancy is balanced by the viscous dissipation. This observation is consistent with the Kolmogorov-like phenomenology observed for RBC.

(vi) The aforementioned phenomenology of RBC turbulence is expected to work for other buoyancy-driven flows in which buoyancy feeds the kinetic energy. Some of the examples of such flows are bubbly turbulence, non-Boussinesq thermally-driven flows in stars, turbulent buoyancy-driven exchange flows in a vertical pipe [3], etc.
(vii) The scaling of the Reynolds and Nusselt numbers of RBC are well described by the models of Grossmann and Lohse [38–42]. Recently Pandey et al [84] and Pandey and Verma [85] derived a formula for the Péclet number that fits with the experimental and numerical data quite well.

In a short review it is impossible to cover the vast number of results of buoyancy-driven turbulence. Here we could not describe recent results on the ultimate regime of turbulent convection [44, 107], logarithmic profile of the boundary layer [96, 108], new scaling of temperature [45], rotating convection [23], etc. Also we could not discuss SST for Fr ≪ 1, which is very important for atmospheric turbulence. We hope that a more comprehensive review will be written. For relatively older works, we refer the reader to the review articles [2, 8, 29, 63, 98].

In this article we covered the present status of the energy spectrum and flux of turbulent convection that shows certain resolution. The issue of Nusselt number exponent being 1/2 or ≈0.3, and the existence of the ultimate regime is being intensely investigated. The structure and dynamics of boundary layer (e.g. existence of log layer or not), flow reversals, and intermittency in RBC are also of major interest. High-resolution simulations, advanced experiments, and careful modelling may resolve these outstanding questions in future.

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References

[1] Ahlers G, Dressel B, Oh J and Pech W 2010 Strong non-Boussinesq effects near the onset of convection in a fluid near its critical point J. Fluid Mech. 642 15–48
[2] Ahlers G, Grossmann S and Lohse D 2009 Heat transfer and large scale dynamics in turbulent Rayleigh–Bénard convection Rev. Mod. Phys. 81 503–37
[3] Arakeri J H, Avila F E, Dada J M and Tovar R O 2000 Convection in a long vertical tube due to unstable stratification—a new type of turbulent flow J. Fluid Mech. 414 859–66
[4] Bartello P and Tobias S M 2013 Sensitivity of stratified turbulence to the buoyancy Reynolds number J. Fluid Mech. 725 1–22
[5] Bhattacharjee J K 1987 Convexion and Chaos in Fluids (Singapore: World Scientific)
[6] Bhattacharjee J K 2015 Kolmogorov argument for the scaling of the energy spectrum in a stratified fluid Phys. Lett. A 379 696–9
[7] Bhattacharjee J K 2015 Self-consistent field theory for the convective turbulence in a Rayleigh–Bénard system in the infinite Prandtl number limit J. Stat. Phys. 160 1519–28
[8] Bodenschatz E, Pesch W and Ahlers G 2000 Recent developments in Rayleigh–Bénard convection Rev. Mod. Phys. 72 1027–80
[9] Boffetta G, De Lillo F, Mazzino A and Musacchio S 2011 Bolgiano scale in confined Rayleigh–Taylor turbulence J. Fluid Mech. 690 426–40
[10] Boffetta G, Mazzino A, Musacchio S and Vozella L 2009 Kolmogorov scaling and intermittency in Rayleigh–Taylor turbulence Phys. Rev. E 79 065301
[11] Bolgiano R 1959 Turbulent spectra in a stably stratified atmosphere J. Geophys. Res. 64 2226–9
[12] Borue V and Orszag S A 1997 Turbulent convection driven by a constant temperature gradient J. Sci. Comput. 12 305–51
[13] Brandenburg A 1992 Energy-spectra in a model for convective turbulence Phys. Rev. Lett. 69 605
[14] Brethouwer G, Billant P, Billant P, Lindborg E and Chomaz J-M 2007 Scaling analysis and simulation of strongly stratified turbulent flows J. Fluid Mech. 585 343–68
[15] Breuer M and Hansen U 2009 Turbulent convection in the zero Reynolds number limit Eur. Phys. Lett. 86 24004
[16] Brown E and Ahlers G 2006 Rotations and cessations of the large-scale circulation in turbulent Rayleigh–Bénard convection J. Fluid Mech. 568 351–86
[17] Buss E H and Clever R M 1978 Asymmetric squares as an attracting set in Rayleigh–Bénard convection Phys. Rev. Lett. 81 341
[18] Calzavarini E, Lohse D and Toschi F 2005 Rayleigh and Prandtl number scaling in the bulk of Rayleigh–Bénard turbulence Phys. Fluids 17 055107
[19] Calzavarini E, Toschi F and Tripiccione R 2002 Evidences of Bolgiano-Obhukhov scaling in three-dimensional Rayleigh–Bénard convection Phys. Rev. E 66 016304
[20] Camussi R and Verricoz R 2004 Temporal statistics in high Rayleigh number convective turbulence Eur. J. Mech. B 23 427–42
[21] Chandram and Verma M K 2011 Dynamics and symmetries of flow reversals in turbulent convection Phys. Rev. E 83 067303
[22] Chandram and Verma M K 2013 Flow reversals in turbulent convection via vortex reconnections Phys. Rev. Lett. 110 114503
[23] Chandrashekar S 1968 Hydrodynamic and Hydromagnetic Stability (Oxford: Clarendon)
[24] Chertkov M 2003 Phenomenology of Rayleigh–Taylor turbulence Phys. Rev. Lett. 91 115001
[25] Chilla F, Caliberto S, Innocenti C and Pampaloni E 1993 Boundary layer and scaling properties in turbulent thermal convection Nuovo Cimento D 15 1229
[26] Chilla F and Schumacher J 2012 New perspectives in turbulent Rayleigh–Bénard convection Eur. Phys. J. E 35 58
[27] Ching E S C 2007 Scaling laws in the central region of confined turbulent thermal convection Phys. Rev. E 75 056302
[28] Ching E S C and Cheng W C 2008 Anomalous scaling and refined similarity of an active scalar in a shell model of homogeneous turbulent convection Phys. Rev. E 77 015303
Grossmann S and Lohse D 2011 Multiple scaling in the ultimate regime of thermal convection

He X, Funfschilling D, Nobach H, Bodenschatz E and Ahlers G 2012 Transition to the ultimate state of turbulent Rayleigh

Grossmann S, Lohse D and Sun C 2016 High-Reynolds number Taylor

McComb W D and Shanmugasundaram V 1985 Renormalisation group calculation of the eddy viscosity for isotropic turbulence

Grossmann S, Lohse D and Sun C 2016 Scaling in thermal convection: a unifying theory J. Fluid Mech. 607 27–56

Grossmann S and Lohse D 2001 Thermal convection for large Prandtl numbers Phys. Rev. Lett. 86 3316

Grossmann S and Lohse D 2002 Prandtl and Rayleigh number dependence of the Reynolds number in turbulent thermal convection Phys. Rev. E 66 016305

Grossmann S and Lohse D 2004 Fluctuations in turbulent Rayleigh–Bénard convection: the role of plumes Phys. Fluids 16 4462–72

Grossmann S and Lohse D 2011 Multiple scaling in the ultimate regime of thermal convection Phys. Fluids 23 045108

Grossmann S, Lohse D and Sun C 2016 High-Reynolds number Taylor–Couette turbulence Annu. Rev. Fluid Mech. 48 53–80

He X, Funshilling D, Nobach H, Bodenschatz E and Ahlers G 2012 Transition to the ultimate state of turbulent Rayleigh–Bénard convection Phys. Rev. Lett. 108 024502

He X, van Gils D P M, Bodenschatz E and Ahlers G 2014 Logarithmic spatial variations and universal $f^{-1}$ power spectra of temperature fluctuations in turbulent Rayleigh–Bénard convection Phys. Rev. Lett. 112 174501

Horn S, Shishikina O and Wagner C 2013 On Oberbeck–non-Boussinesq effects in three-dimensional Rayleigh–Bénard convection in glycerol J. Fluid Mech. 724 243–202

Huang S-D, Wang F, Xi H-D and Xia K-Q 2015 Comparative experimental study of fixed temperature and fixed heat flux boundary conditions in turbulent thermal convection Phys. Rev. Lett. 115 154502

Kaczorowski M and Xia K-Q 2013 Turbulent flow in the bulk of Rayleigh–Bénard convection: small-scale properties in a cubic cell J. Fluid Mech. 722 596–617

Kerr R M 1996 Rayleigh number scaling in numerical convection J. Fluid Mech. 310 139–79

Kolmogorov A N 1941 The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers Dokl. Akad. Nauk. SSSR 30 301–5

Kraichnan R H 1959 The structure of isotropic turbulence at very high Reynolds numbers J. Fluid Mech. 5 497–543

Kraichnan R H 1962 Turbulent thermal convection at arbitrary Prandtl number Phys. Fluids 5 1374

Kumar A, Chatterjee A G and Verma M K 2014 Energy spectrum of buoyancy-driven turbulence Phys. Rev. E 90 023016

Kumar A and Verma M K 2015 Shell model for buoyancy-driven turbulence Phys. Rev. E 91 043014

Kumar A and Verma M K 2015 Shell model for buoyancy-driven turbulent flows Proc. Advances in Computation, Modeling and Control of Transitional and Turbulent Flows ed TK Sengupta, et al pp 332–41

Kumar A and Verma M K 2016 Applicability of Taylor’s hypothesis in convective turbulence arXiv:1512:00959

Kunnen R P J, Clercx H J H, Geurts B J, van Bokhoven L J A, Akkermans R A D and Verzicco R 2008 Numerical and experimental investigation of structure-function scaling in turbulent Rayleigh–Bénard convection Phys. Rev. E 77 016302

Lesieur M 2012 Turbulence in Fluids 4th edn (Dordrecht: Springer)

Leslie D C 1973 Developments in the Theory of Turbulence (Oxford: Clarendon)

Lindborg E 2006 The energy cascade in a strongly stratified fluid J. Fluid Mech. 550 207–42

Lindborg E and Brethouwer G 2008 Vertical dispersion by stratified turbulence J. Fluid Mech. 614 303–14

Lohse D and Toschi F 2003 Ultimate state of thermal convection Phys. Rev. Lett. 90 034502

Lohse D and Xia K-Q 2010 Small-scale properties of turbulent Rayleigh–Bénard convection Annu. Rev. Fluid Mech. 42 335–64

Lozhkin S A and Frick P G 1998 Inertial Obukhov-Bolzano interval in shell models of convective turbulence Fluid Dyn. 33 842–9

L’vov V S 1991 Spectra of velocity and temperature–fluctuations with constant entropy flux of fully-developed free-convective turbulence Phys. Rev. Lett. 67 687

L’vov V S and Falkovich G 1992 Conservation laws and two-flux spectra of hydrodynamic convective turbulence Physica D 57 85–95

Manneville P 2004 Instabilities, Chaos, and Turbulence (London: Imperial College Press)

McComb W D 1990 The Physics of Fluid Turbulence (Oxford: Clarendon)

McComb W D and Shummasundaram V 1983 Renormalisation group calculation of the eddy viscosity for isotropic turbulence J. Phys. A: Math. Theor. 18 2191–8

Mingshun J and Shida L 1997 Scaling behavior of velocity and temperature in a shell model for thermal convective turbulence Phys. Rev. E 56 641

Mishra P K, De A K, Verma M K and Eswaran V 2010 Dynamics of reorientations and reversals of large-scale flow in Rayleigh–Bénard convection J. Fluid Mech. 668 180–99

Mishra P K and Verma M K 2010 Energy spectra and fluxes for Rayleigh–Bénard convection Phys. Rev. E 81 056316

Mishra P K, Wahi P and Verma M K 2010 Patterns and bifurcations in low–Prandtl number Rayleigh–Bénard convection Eur. Phys. Lett. 89 44003

Mishra P K, Herault J, Fauve S and Verma M K 2015 Dynamics of reversals and condensates in two-dimensional Kolmogorov flows Phys. Rev. E 91 053005

Nath D, Pandey A, Kumar A and Verma M K 2016 Near isotropic behaviour of turbulent thermal convection Phys. Rev. Fluids 1 064302

Nath D and Verma M K 2014 Numerical simulation of convection of argon gas in fast breeder reactor Ann. Nucl. Energy 63 51–8

Ni R, Huang S-D and Xia K-Q 2011 Local energy dissipation rate balances local heat flux in the center of turbulent thermal convection Phys. Rev. Lett. 107 174501

Ni R, Huang S-D and Xia K-Q 2012 Lagrangian acceleration measurements in convective thermal turbulence J. Fluid Mech. 692 395–419
[79] Niemela J, Skrbek L, Sreenivasan K R and Donnelly R J 2000 Turbulent convection at very high Rayleigh numbers Nature 404 837
[80] Niemela J, Skrbek L, Sreenivasan K R and Donnelly R J 2001 The wind in confined thermal convection J. Fluid Mech. 449 169–78
[81] Obukhov A M 1959 On influence of buoyancy forces on the structure of temperature field in a turbulent flow Dokl. Acad. Nauk. SSSR 125 1246
[82] OpenFOAM. The open source CFD toolbox, www.openfoam.org, 2015
[83] Pal P, Wahi P, Paul S, Verma M K, Kumar K and Mishra P K 2009 Bifurcation and chaos in zero-Prandtl-number convection Eur. Phys. Lett. 87 54003
[84] Pandey A, Kumar A, Chatterjee A G and Verma M K 2016 Dynamics of large-scale quantities in Rayleigh–Bénard convection Phys. Rev. E 94 053106
[85] Pandey A and Verma M K 2016 Scaling of large-scale quantities in Rayleigh–Bénard convection Phys. Fluids 28 095105
[86] Pandey A, Verma M K and Mishra P K 2014 Scaling of heat flux and energy spectrum for very large Prandtl number convection Phys. Rev. E 90 053106
[87] Pawar S S and Arakeri J H 2016 Kinetic energy and scalar spectra in high Rayleigh number axially homogeneous buoyancy driven turbulence Phys. Fluids 28 065103
[88] Prakash V N, Martinez Mercado J, van Wijngaarden L, Mancilla E, Tagaya Y, Lohse D and Sun C 2016 Energy spectra in turbulent bubbly flows J. Fluid Mech. 791 174–90
[89] Procaccia I and Zeitak R 1989 Scaling exponents in nonisotropic convective turbulence Phys. Rev. Lett. 62 2128
[90] Riley J J and Lindborg E 2013 Recent progress in stratified turbulence Ten Chapters in Turbulence ed P A Davidson et al (Cambridge: Cambridge University Press) pp 269–312
[91] Rubinstein R 1994 Recursive renormalization group theory of Bolgiano scaling in Boussinesq turbulence NASA Technical Memorandum 106602, ICOMP-94-8; CMOTT-94-2
[92] Sameen A, Verzicco R and Sreenivasan K R 2009 Specific roles of fluid properties in non-Boussinesq thermal convection at the Rayleigh number of $2 \times 10^6$ Eur. Phys. Lett. 86 14006
[93] Scheel J D and Schumacher J 2011 Local boundary layer scales in turbulent Rayleigh–Bénard convection J. Fluid Mech. 678 344–73
[94] Shang X D and Xia K Q 2001 Scaling of the velocity power spectra in turbulent thermal convection Phys. Rev. E 64 65301
[95] Shishkina O and Thess A 2009 Mean temperature profiles in turbulent Rayleigh–Bénard convection of water J. Fluid Mech. 633 449
[96] Shishkina O, Horn S and Wagner S 2015 Thermal boundary layer equation for turbulent Rayleigh–Bénard convection Phys. Rev. Lett. 114 114302
[97] Shraiman B I and Sigga E D 1990 Heat transport in high-Rayleigh-number convection Phys. Rev. A 42 3650–3
[98] Sigga E D 1994 High Rayleigh number convection Annu. Rev. Fluid Mech. 26 137–68
[99] Silano G, Sreenivasan K R and Verzicco R 2010 Numerical simulations of Rayleigh–Bénard convection for Prandtl numbers between $10^3$ and $10^6$ and Rayleigh numbers between $10^9$ and $10^{11}$. J. Fluid Mech. 662 409–46
[100] Skanadra D, Busse F H and Müller W C 2008 Scaling properties of convective turbulence High Performance Computing in Science and Engineering. Transactions of the Third Joint HLRS and KonWIHR Summer Workshop (Berlin: Springer) p 387 Part IV
[101] Sreenivasan K R, Bershadskii A and Niemela J J 2002 Mean wind and its reversal in thermal convection Phys. Rev. E 65 065306
[102] Stevens R, Poel E P, Grossmann S and Lohse D 2013 The unifying theory of scaling in thermal convection: the updated prefactors J. Fluid Mech. 730 295–305
[103] Sugiyama K, Ni R, Stevens R J A M, Chan T-S, Zhou S-Q, Xi H-D, Sun C, Grossmann S, Xia K-Q and Lohse D 2010 Flow reversals in thermally driven turbulence Phys. Rev. Lett. 105 034503
[104] Sun C, Zhou Q and Xia K-Q 2006 Cascades of velocity and temperature fluctuations in buoyancy-driven thermal turbulence Phys. Rev. Lett. 97 144504
[105] Taylor G I 1938 The spectrum of turbulence Proc. R. Soc. A 164 476
[106] Tóh S and Suzuki E 1994 Entropy cascade and energy inverse transfer in two-dimensional convective turbulence Phys. Rev. Lett. 73 1501
[107] Urban P, Hanzelka P, Kralik T, Musilova V, Srnka A and Skrbek L 2012 Effect of boundary layers asymmetry on heat transfer efficiency in turbulent Rayleigh–Bénard convection at very high Rayleigh numbers Phys. Rev. Lett. 109 154501
[108] van der Poel E P, Ostilla-Mirco R, Verzicco R, Grossmann S and Lohse D 2015 Logarithmic mean temperature profiles and their connection to plume emissions in turbulent Rayleigh–Bénard convection Phys. Rev. Lett. 115 114501
[109] Vallgren A, Deusebio E and Lindborg E 2011 Possible explanation of the atmospheric kinetic and potential energy spectra Phys. Rev. Lett. 107 268501
[110] van Reeuwijk M, Jonker H J J and Hanjalić K 2008 Wind and boundary layers in Rayleigh–Bénard convection: I. Analysis and modeling Phys. Rev. E 77 036311
[111] Verma M K 2004 Statistical theory of magnetohydrodynamic turbulence: recent results Phys. Rep. 401 229–380
[112] Verma M K 2012 Variable enstrophy flux and energy spectrum in two-dimensional turbulence with Ekman friction Eur. Phys. Lett. 98 14003
[113] Verma M K, Ambhirie S C and Pandey A 2015 Flow reversals in turbulent convection with free-slip walls Phys. Fluids 27 047102
[114] Verma M K, Kumar A and Chatterjee A G 2015 Energy spectrum and flux of buoyancy-driven turbulence Proc. Advances in Computation, Modeling and Control of Transitional and Turbulent Flows ed T K Sengupta et al pp 442–51
[115] Verma M K, Kumar A and Chatterjee A G 2016 Energy spectrum and flux of buoyancy-driven turbulence Phys. Focus 25 45
[116] Verma M K, Mishra P K, Pandey A and Paul S 2012 Scalings of field correlations and heat transport in turbulent convection Phys. Rev. E 85 016310
[117] Verzicco R and Camussi R 2003 Numerical experiments on strongly turbulent thermal convection in a slender cylindrical cell J. Fluid Mech. 477 19–49
[118] Xi H-D and Xia K-Q 2007 Cessations and reversals of the large-scale circulation in turbulent thermal convection Phys. Rev. E 75 066307
[119] Xi K Q 2013 Current trends and future directions in turbulent thermal convection Theor. Appl. Mech. Lett. 3 032001
[120] Xin Y-B and Xia K-Q 1997 Boundary layer length scales in convective turbulence Phys. Rev. E 56 3010
[121] Yakhot V and Orszag S A 1986 Renormalization group analysis of turbulence: I. Basic theory J. Sci. Comput. 1 3–51
[122] Zhou Q, Li C M, Lu Z M and Liu Y L 2011 Experimental investigation of longitudinal space-time correlations of the velocity field in turbulent Rayleigh–Bénard convection J. Fluid Mech. 683 94–111
[123] Zhou Q, Sun C and Xia K Q 2008 Experimental investigation of homogeneity, isotropy, and circulation of the velocity field in buoyancy-driven turbulence J. Fluid Mech. 598 361–72
[124] Zhou S-Q and Xia K-Q 2001 Scaling properties of the temperature field in convective turbulence Phys. Rev. Lett. 87 64501