Components and Cycles of Random Mappings

STEVEN FINCH

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Abstract. Each connected component of a mapping \( \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) contains a unique cycle. The largest such component can be studied probabilistically via either a delay differential equation or an inverse Laplace transform. The longest such cycle likewise admits two approaches: we find an (apparently new) density formula for its length. Implications of a constraint – that exactly one component exists – are also examined. For instance, the mean length of the longest cycle is \( (0.7824...)\sqrt{n} \) in general, but for the special case, it is \( (0.7978...)\sqrt{n} \), a difference of less than 2%.

Two delay differential equations (DDEs) shall be helpful:

\[ x \rho'(x) + \rho(x - 1) = 0 \text{ for } x > 1, \quad \rho(x) = 1 \text{ for } 0 \leq x \leq 1 \]

where \( \rho \) is Dickman’s function \[1, 2, 3\], and

\[ x \sigma'(x) + \frac{1}{2} \sigma(x) + \frac{1}{2} \sigma(x - 1) = 0 \text{ for } x > 1, \quad \sigma(x) = 1/\sqrt{x} \text{ for } 0 < x \leq 1 \]

where \( \sigma \) could justifiably be called Watterson’s function \[4, 5, 6\]. It is understood that \( \rho(x) = 0 = \sigma(x) \) for \( x < 0 \). One-sided Laplace transforms will also play a role; for example, we have

\[ \mathcal{L}[\rho(\xi)] = \frac{\exp(-E(\eta))}{\eta}, \quad \mathcal{L}[\sigma(\xi)] = \frac{\exp(-\frac{1}{2}E(\eta))}{\sqrt{\eta/\pi}} \quad \eta \in \mathbb{C} \setminus (-\infty, 0] \]

or equivalently

\[ \mathcal{L}^{-1}\left[\frac{\exp(-E(\eta))}{\eta}\right] = \rho(\xi), \quad \mathcal{L}^{-1}\left[\frac{\exp(-\frac{1}{2}E(\eta))}{\sqrt{\eta/\pi}}\right] = \sigma(\xi) \]

where

\[ E(x) = \int_{x}^{\infty} \frac{e^{-t}}{t} dt = -\text{Ei}(-x), \quad x > 0 \]

is the exponential integral \[7, 8, 9, 10\].

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1. Stepanov

For introductory purposes, let us examine solely one-to-one mappings \( \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\} \), i.e., permutations on \( n \) symbols. Let \( \Lambda \) denote the length of the longest cycle in an \( n \)-permutation, chosen uniformly at random. We have

\[
\lim_{n \to \infty} P\{\Lambda \leq a n\} = \rho\left(\frac{1}{a}\right), \quad 0 < a \leq 1
\]

where, from the DDE,

\[
\rho(x) = \begin{cases} 
1 - \ln(x) & \text{if } 1 < x \leq 2, \\
1 - \frac{\pi^2}{12} - \ln(x) + \frac{1}{2} \ln(x)^2 + Li_2\left(\frac{1}{x}\right) & \text{if } 2 < x \leq 3.
\end{cases}
\]

Thus, for example,

\[
\lim_{n \to \infty} P\left\{\frac{1}{3} < \frac{\Lambda}{n} \leq \frac{1}{2}\right\} = \rho(2) - \rho(3) = \frac{\pi^2}{12} - \ln(2) + \ln(3) - \frac{1}{2} \ln(3)^2 - Li_2\left(\frac{1}{3}\right) \\
= 0.258244431148.
\]

Let us now explore a less-familiar approach \[11\]. Define a function \( h(\xi) \) to be equal to 0 for \( \xi < 1 \) and \( \frac{1}{\xi} \) for \( \xi \geq 1 \). Using the series expansion for \( \exp\left(-E(\eta)\right) \) in terms of \( E(\eta) \), we have

\[
\mathcal{L}^{-1}\left[\exp\left(-E(\eta)\right)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mathcal{L}^{-1}\left[\frac{E(\eta)^k}{\eta}\right]
\]

where the power \( E(\eta)^k \) is the Laplace transform of the \( k \)-fold self-convolution \( h_k(\xi) \) of \( h(\xi) \). On the one hand, formulas

\[
\mathcal{L}^{-1}\left[\frac{1}{\eta}\right] = 1, \quad \mathcal{L}^{-1}\left[\frac{E(\eta)}{\eta}\right] = \begin{cases} 
\ln(\xi) & \text{if } \xi \geq 1, \\
0 & \text{if } 0 \leq \xi < 1
\end{cases}
\]

are well-known. On the other hand,

\[
\mathcal{L}^{-1}\left[\frac{E(\eta)^2}{\eta}\right] = \begin{cases} 
-\frac{\pi^2}{6} + \ln(\xi)^2 + 2 Li_2\left(\frac{1}{\xi}\right) & \text{if } \xi \geq 2, \\
0 & \text{if } 0 \leq \xi < 2.
\end{cases}
\]

does not appear in \[12\]. Since \( h_k(\xi) = 0 \) for \( 0 \leq \xi < k \), only terms \( k = 0, 1, \ldots, \lfloor \xi \rfloor \) of the series need to be summed. Consequently, for \( 1/3 < a \leq 1/2 \),

\[
\mathcal{L}^{-1}\left[\frac{\exp\left(-E(\eta)\right)}{\eta}\right]_{\xi \rightarrow 1/a} = 1 - \ln\left(\frac{1}{a}\right) + \frac{1}{2} \left[-\frac{\pi^2}{6} + \ln(a)^2 + 2 Li_2\left(a\right)\right]
\]

which is consistent with before.
2. Mutafchiev

Let us now remove the restriction that mappings be one-to-one. Let \( \Lambda \) denote the length of the largest component in an \( n \)-mapping, chosen uniformly at random. We have

\[
\lim_{n \to \infty} \mathbb{P}\{ \Lambda \leq a n\} = \frac{1}{\sqrt{a}} \sigma \left( \frac{1}{a} \right), \quad 0 < a \leq 1
\]

where, from the DDE,

\[
\sqrt{x} \sigma(x) = 1 - \frac{1}{2} \ln \left( \frac{1 + \sqrt{1 - \frac{1}{x}}}{1 - \sqrt{1 - \frac{1}{x}}} \right) \quad \text{if } 1 < x \leq 2.
\]

For reasons of simplicity, change our example domain from \([1/3, 1/2]\) to \([1/2, 2/3]\). Hence

\[
\lim_{n \to \infty} \mathbb{P}\left\{ \frac{1}{2} < \frac{\Lambda}{n} \leq \frac{2}{3} \right\} = \sqrt{3} \frac{2}{3} \sigma \left( \frac{3}{2} \right) - \sqrt{2} \sigma(2)
\]

\[
= -\frac{1}{2} \ln \left( \frac{1 + \sqrt{1 - \frac{2}{3}}}{1 - \sqrt{1 - \frac{2}{3}}} \right) + \frac{1}{2} \ln \left( \frac{1 + \sqrt{1 - \frac{2}{3}}}{1 - \sqrt{1 - \frac{2}{3}}} \right)
\]

\[
= 0.222894638557\ldots
\]

Let us again explore the less-familiar approach [13]. Define \( h(\xi) \) as previously. From

\[
\sqrt{\pi} \xi \mathcal{L}^{-1} \left[ \exp \left( -\frac{1}{2} \frac{E(\eta)}{\sqrt{\eta}} \right) \right] = \sqrt{\pi} \xi \sum_{k=0}^\infty \frac{(-1)^k}{2^k k!} \mathcal{L}^{-1} \left[ \frac{E(\eta)^k}{\sqrt{\eta}} \right]
\]

we recognize two well-known formulas:

\[
\mathcal{L}^{-1} \left[ \frac{1}{\sqrt{\eta}} \right] = \frac{1}{\sqrt{\pi} \xi}, \quad \mathcal{L}^{-1} \left[ \frac{E(\eta)}{\sqrt{\eta}} \right] = \left\{ \begin{array}{ll}
\frac{2}{\sqrt{\pi} \xi} \arctanh \left( \sqrt{1 - \frac{1}{\xi}} \right) & \text{if } \xi \geq 1, \\
0 & \text{if } 0 \leq \xi < 1.
\end{array} \right.
\]

Since \( h_k(\xi) = 0 \) for \( 0 \leq \xi < k \), only terms \( k = 0, 1, \ldots, \lfloor \xi \rfloor \) of the series need to be summed. Consequently, for \( 1/2 < a \leq 1 \),

\[
\sqrt{\frac{\pi}{a}} \mathcal{L}^{-1} \left[ \frac{\exp \left( -\frac{1}{2} \frac{E(\eta)}{\sqrt{\eta}} \right)}{\sqrt{\eta}} \right]_{\xi \to 1/a} = 1 - \arctanh \left( \sqrt{1 - a} \right) = 1 - \frac{1}{2} \ln \left( \frac{1 + \sqrt{1 - a}}{1 - \sqrt{1 - a}} \right)
\]

which again is consistent with before.
As an aside, had we kept the domain as \([1/3, 1/2]\), then numerics are possible:

\[
\lim_{n \to \infty} \mathbb{P}\left\{\frac{1}{3} < \frac{\Lambda}{n} \leq \frac{1}{2}\right\} = \int_{1/3}^{1/2} \frac{1}{2t^{3/2}} \sigma\left(\frac{1-t}{t}\right) \, dt
\]

\[
= \frac{1}{2} \int_{1/3}^{1/2} \frac{1}{t\sqrt{1-t}} \left[ 1 - \frac{1}{2} \ln\left(\frac{1+\sqrt{1-t}}{1-\sqrt{1-t}}\right)\right] \, dt
\]

\[
= 0.110414874191\ldots
\]

but symbolics seem unlikely. A closed-form expression for the inverse Laplace transform of \(E(\eta)^2/\sqrt{\eta}\) also remains open.

3. Purdom & Williams

We change the subject to cycles, i.e., loops in the functional graph. Let \(\Lambda\) denote the length of the longest cycle in an \(n\)-mapping, chosen uniformly at random. Our goal is to find \(\lim_{n \to \infty} \mathbb{P}\{\Lambda \leq b\sqrt{n}\}\). While no relevant DDE is yet known, there is an associated inverse Laplace transform

\[
\sqrt{\frac{\pi}{b}} \mathcal{L}^{-1}\left[\frac{\exp\left(-E\left(\sqrt{2b}\eta\right)\right)}{\sqrt{\eta}}\right]_{\xi \to 1/b} = \sqrt{\frac{\pi}{b}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mathcal{L}^{-1}\left[\frac{E\left(\sqrt{2b}\eta\right)^k}{\sqrt{\eta}}\right]_{\xi \to 1/b}
\]

due to Mutafchiev [13]. Unlike previously, \(0 < b < \infty\) holds (rather than \(0 < a < 1\)) and \(E\left(\sqrt{2b}\eta\right)\) is the Laplace transform of \(\frac{1}{\pi i} \text{erfc}\left(\sqrt{\frac{b}{2\pi}\xi}\right)\). Self-convolutions of this function do not enjoy the same vanishing properties as those for \(h(\xi)\). Truncating the infinite series, although it is convergent, will unfortunately lead to non-zero error.

An alternative expression

\[
\sqrt{2\pi} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \exp\left(-E(b\zeta) + \frac{\zeta^2}{2}\right) \, d\zeta
\]

is due to Stepanov [11]. Letting \(\eta = \frac{b\zeta^2}{2}\), i.e., \(\zeta = \sqrt{\frac{2\eta}{b}}\) and \(d\zeta = \frac{dt}{\sqrt{2b\eta}}\), demonstrates that this complex contour integral is equal to the preceding inverse Laplace transform.

There is, thankfully, a different approach available. If the number \(N\) of cyclic points in a random mapping is fixed, then as Kolchin [14] wrote, “... these \(N\) cyclic points... form a random permutation”. This suggests multiplying [15] the conditional probability of \(\Lambda\) given \(N\):

\[
\mathbb{P}\{\Lambda \leq \lambda\sqrt{n} \mid N = \nu\sqrt{n}\} = \rho\left(\frac{\nu}{\lambda}\right)
\]
by [16, 17] the limiting density (as \( n \to \infty \)) of \( N \):

\[
\nu \exp \left( -\frac{\nu^2}{2} \right) \quad \text{(Rayleigh)}
\]

and then differentiating (with respect to \( \lambda \)) to obtain the joint density of \((\Lambda, N)\):

\[
f(\lambda, \nu) = \nu \exp \left( -\frac{\nu^2}{2} \right) \rho \left( \frac{\nu - \lambda}{\lambda} \right)
= \nu \lambda \exp \left( -\frac{\nu^2}{2} \right) \rho \left( \frac{\nu - \lambda}{\lambda} \right)
= \nu \lambda \exp \left( -\frac{\nu^2}{2} \right) \rho \left( \frac{\nu - \lambda}{\lambda} \right)
\]

where \( 0 < \lambda < \nu < \infty \). We have not seen this formula in the literature: it is apparently new. For \( r \geq 2 \), let \( \Lambda_r \) denote the length of the \( r \)th longest cycle in an \( n \)-mapping, chosen uniformly at random. If the permutation has no \( r \)th cycle, then its \( r \)th longest cycle is defined to have length 0. Define the \( r \)th generalized Dickman function \( \rho_r(\xi) \) to satisfy

\[
\xi \rho'_r(\xi) + \rho_r(\xi - 1) = \rho_{r-1}(\xi - 1) \quad \text{for} \quad \xi > 1, \quad \rho_r(\xi) = 1 \quad \text{for} \quad 0 \leq \xi \leq 1
\]

where \( \rho_1 = \rho \). By the same argument, for \( r \geq 2 \),

\[
f_r(\lambda, \nu) = \nu \lambda \exp \left( -\frac{\nu^2}{2} \right) \left[ \rho_r \left( \frac{\nu - \lambda}{\lambda} \right) - \rho_{r-1} \left( \frac{\nu - \lambda}{\lambda} \right) \right].
\]

Define

\[
G_{r,h} = \frac{1}{h!(r - 1)!} \int_0^\infty x^{h-1} \exp \left( -x \right) \frac{1}{n} E(x)^{r-1} \exp \left( -E(x) - x \right) \, dx
\]

(in this paper, rank \( r = 1, 2, 3 \) or 4; height \( h = 1 \) or 2). Purdom & Williams [18], building on Shepp & Lloyd [19], discovered asymptotic formulas for moments \( E(\Lambda_r^h) \). We can easily verify their findings:

\[
\lim_{n \to \infty} \frac{E(\Lambda_r)}{\sqrt{n}} = \sqrt{\frac{\pi}{2}} G_{r,1} = \begin{cases} 
0.78248160099165661501... & \text{if } r = 1, \\
0.26267067265131265469... & \text{if } r = 2, \\
0.11068781528281010827... & \text{if } r = 3, \\
0.05056118481134243184... & \text{if } r = 4;
\end{cases}
\]

\[
\lim_{n \to \infty} \frac{V(\Lambda_r)}{n} = 2G_{r,2} - \frac{\pi}{2} G_{r,1} = \begin{cases} 
0.241111407342881901748... & \text{if } r = 1, \\
0.04395998473216610374... & \text{if } r = 2, \\
0.01235552055537805858... & \text{if } r = 3, \\
0.00380619224804518754... & \text{if } r = 4.
\end{cases}
\]
The mode of \( \Lambda_1 \) occurs when
\[
0 = \frac{d}{d\lambda} \int_{\lambda}^{\infty} f(\lambda, \nu) d\nu = -f(\lambda, \lambda) + \int_{\lambda}^{\infty} \frac{\partial f}{\partial \lambda}(\lambda, \nu) d\nu
\]
\[
= -\exp\left(\frac{-\lambda^2}{2}\right) + \int_{\lambda}^{\infty} \nu \exp\left(\frac{-\nu^2}{2}\right) \frac{\partial}{\partial \lambda} \left[ \frac{1}{\lambda} \rho \left( \frac{\nu}{\lambda} - 1 \right) \right] d\nu;
\]
the inner derivative becomes
\[
-\frac{1}{\lambda^2} \rho \left( \frac{\nu}{\lambda} - 1 \right) + \frac{1}{\lambda} \rho' \left( \frac{\nu}{\lambda} - 1 \right) \left( \frac{-\nu}{\lambda^2} \right) = -\frac{1}{\lambda^2} \rho \left( \frac{\nu}{\lambda} - 1 \right) - \frac{1}{\lambda} \frac{\rho (\frac{\nu}{\lambda} - 1)}{\frac{\nu}{\lambda} - 1} \left( \frac{-\nu}{\lambda^2} \right)
\]
\[
= -\frac{1}{\lambda^2} \rho \left( \frac{\nu - \lambda}{\lambda} \right) + \frac{\nu}{\lambda^2 (\nu - \lambda)} \rho \left( \frac{\nu - 2\lambda}{\lambda} \right);
\]
solving the equation
\[
\exp\left(\frac{-\lambda^2}{2}\right) = \frac{1}{\lambda^2} \int_{\lambda}^{\infty} \left[ -\rho \left( \frac{\nu - \lambda}{\lambda} \right) + \frac{\nu}{\nu - \lambda} \rho \left( \frac{\nu - 2\lambda}{\lambda} \right) \right] \nu \exp\left(\frac{-\nu^2}{2}\right) d\nu
\]
yields 0.4809... as the mode. The median 0.6842... of \( \Lambda_1 \) arises simply from
\[
\frac{1}{2} = \int_{\lambda}^{\infty} \int_{\mu}^{\infty} f(\mu, \nu) d\nu d\mu.
\]

For \( \Lambda_2 \), the mode is 0; we did not pursue the median. Another new asymptotic result is the cross-correlation between \( \Lambda_r \) and \( N \):
\[
\lim_{n \to \infty} \frac{\mathbb{E}(\Lambda_r N) - \mathbb{E}(\Lambda_r) \mathbb{E}(N)}{\sqrt{\mathbb{V}(\Lambda_r)} \sqrt{\mathbb{V}(N)} = \frac{\sqrt{2 - \frac{\pi}{2} G_{r,1}}}{\sqrt{2G_{r,2} - \frac{\pi}{2} G^2_{r,1}}}
\]
\[
= \begin{cases} 
0.83298010... & \text{if } r = 1, \\
0.65486924... & \text{if } r = 2, \\
0.52094617... & \text{if } r = 3, \\
0.42505712... & \text{if } r = 4.
\end{cases}
\]

Using formulas in [10, 20, 21], it is possible to similarly compute the cross-correlation between \( \Lambda_r \) and \( \Lambda_s \) where \( r < s \).

4. **Rényi**

A mapping is said to be **connected** (or **indecomposable**) if it possesses exactly one component. This is a rare event, in the sense that
\[
\mathbb{P}\{M = 1\} \sim \sqrt{\frac{\pi}{2n}} \quad \text{as } n \to \infty
\]
where $M$ counts the components. Let $\Lambda$ denote the length of the unique cycle in a connected $n$-mapping, chosen uniformly at random. Our goal is to find $\lim_{n \to \infty} \mathbb{P} \{ \Lambda \leq b\sqrt{n} \}$ as before, but the circumstances are vastly simpler. Rényi [22] proved that the limiting density (as $n \to \infty$) of $\Lambda$ is

$$\sqrt{\frac{2}{\pi}} \exp \left( \frac{-\lambda^2}{2} \right)$$

(half-normal)

for $0 < \lambda < \infty$, which implies immediately that

$$\lim_{n \to \infty} \frac{\mathbb{E}(\Lambda)}{\sqrt{n}} = \sqrt{\frac{2}{\pi}} = 0.79788456080286535587...,$$

$$\lim_{n \to \infty} \frac{\mathbb{V}(\Lambda)}{n} = 1 - \frac{2}{\pi} = 0.36338022763241865692... .$$

It is surprising that arbitrary mappings and connected mappings differ so little here ($0.7824...$ versus $0.7978...$). We might have expected that uniqueness would carry more influence.

Of course, $N = \Lambda$ when there is just one component. Allowing instead $m$ components, where $m \geq 2$ is a fixed integer, rarity persists [23, 24]

$$\mathbb{P} \{ M = m \} \sim \frac{1}{2^{m-1} (m-1)!} \sqrt{\frac{\pi}{2n}} \ln(n)^{m-1} \text{ as } n \to \infty$$

but the asymptotic values $\sqrt{2/\pi}$ and $1 - 2/\pi$ for $\mathbb{E}(N)$ and $\mathbb{V}(N)$ evidently do not change. Section 6 contains more on this issue.

5. Pavlov

For arbitrary mappings, the expected number of components [25, 26] is $\sim \frac{1}{2} \ln(n)$. If our constraint from Section 4 loosens so that $m \to \infty$ but so slowly that $m / \ln(n) \to 0$, then Rényi’s formula still applies, as proved by Pavlov [24]. This leads us to a set of conjectural results comparable to those in Section 3.

Let $\Lambda$ be the longest cycle length and $N$ be the cyclic points total. As before, a conditional probability coupled with the limiting density (as $n \to \infty$) of $N$:

$$\sqrt{\frac{2}{\pi}} \exp \left( \frac{-\nu^2}{2} \right)$$

suffice to give the joint density of $(\Lambda, N)$:

$$f(\lambda, \nu) = \sqrt{\frac{2}{\pi}} \frac{1}{\lambda} \exp \left( \frac{-\nu^2}{2} \right) \rho \left( \frac{\nu - \lambda}{\lambda} \right)$$
where $0 < \lambda < \nu < \infty$. Further,
\[
f_r(\lambda, \nu) = \sqrt{\frac{2}{\pi}} \frac{1}{\lambda} \exp \left( -\frac{\nu^2}{2} \right) \left[ \rho_r \left( \frac{\nu - \lambda}{\lambda} \right) - \rho_{r-1} \left( \frac{\nu - \lambda}{\lambda} \right) \right]
\]
for $r \geq 2$. Moments are
\[
\lim_{n \to \infty} \frac{\mathbb{E}(\Lambda_r)}{\sqrt{n}} = \sqrt{\frac{2}{\pi}} G_{r,1} = \begin{cases} 
0.49814325870512904597... & \text{if } r = 1, \\
0.1672134383091813637... & \text{if } r = 2, \\
0.07046605176920746245... & \text{if } r = 3, \\
0.0321882499652303019... & \text{if } r = 4; 
\end{cases}
\]
\[
\lim_{n \to \infty} \frac{\mathbb{V}(\Lambda_r)}{n} = G_{r,2} - \frac{2}{\pi} G_{r,1}^2 = \begin{cases} 
0.17854905846627743895... & \text{if } r = 1, \\
0.02851495566901143371... & \text{if } r = 2, \\
0.00732819205178914862... & \text{if } r = 3, \\
0.00217522939296169629... & \text{if } r = 4.
\end{cases}
\]
The mode of $\Lambda_1$ occurs at 0; the median at 0.3903... For $\Lambda_2$, we did not pursue the median. The cross-correlation between $\Lambda_r$ and $N$ is
\[
\lim_{n \to \infty} \frac{\mathbb{E}(\Lambda_r N) - \mathbb{E}(\Lambda_r) \mathbb{E}(N)}{\sqrt{\mathbb{V}(\Lambda_r)} \sqrt{\mathbb{V}(N)}} = \sqrt{1 - \frac{2}{\pi}} G_{r,1} = \begin{cases} 
0.89066843... & \text{if } r = 1, \\
0.74816251... & \text{if } r = 2, \\
0.62190221... & \text{if } r = 3, \\
0.52141727... & \text{if } r = 4.
\end{cases}
\]
and, again, it is possible to compute the cross-correlation between $\Lambda_r$ and $\Lambda_s$ where $r < s$.

The mean 0.4981... is sharply less than the other means 0.7824... and 0.7978... we have exhibited. Why should this counterintuitive fact be true? The scenario $m/\ln(n) \to 0$ is intermediate to the others. This is why we describe our work here as conjectural.

If instead $m/\ln(n) \to c$ for some constant $0 < c < \infty$, then Pavlov’s [27] density formula is
\[
\frac{2^c \Gamma(c)}{\sqrt{2\pi} \Gamma(2c)} \nu^{2c} \exp \left( -\frac{\nu^2}{2} \right).
\]
This reduces to the density found in [16, 17] when $c = 1/2$.

Given an arbitrary mapping, the deepest cycle is contained within the largest component, whereas the richest component contains the longest cycle. The deepest cycle need not be longest; the richest component need not be largest. What can be said about the probability of either event, or the average size of either structure? Questions about interplay at this level appear to be difficult to answer.

For completeness, we mention [28, 29, 30, 31, 32], which may offer additional insights and paths forward.
6. Flajolet & Odlyzko

Let \( a_{nml} \) denote the number of \( n \)-mappings possessing exactly \( m \) components and exactly \( \ell \) cyclic points, where \( n \geq \ell \geq m \geq 2 \). We have \[33, 34, 35\]

\[
\sum_{n=\ell}^{\infty} \sum_{\ell=m}^{n} \frac{a_{nml}}{n!} x^n y^\ell = \frac{1}{m!} \ln \left( \frac{1}{1 - y \tau(x)} \right)^m
\]

where \( \tau(x) = x \exp(\tau(x)) \) is Cayley’s tree function. The dominant singularity of \( \tau(x) \) is at \( x = e^{-1} \) and

\[
\tau(x) \sim 1 - 2^{1/2} \sqrt{1 - e^x} \quad \text{as} \quad x \to e^{-1}.
\]

Differentiating with respect to \( y \):

\[
\sum_{n=\ell}^{\infty} \sum_{\ell=m}^{n} \frac{\ell a_{nml}}{n!} x^n y^{\ell-1} = \frac{1}{(m-1)!} \frac{\tau(x)}{1 - y \tau(x)} \ln \left( \frac{1}{1 - y \tau(x)} \right)^{m-1}
\]

and setting \( y = 1 \):

\[
\frac{1}{(m-1)!} \frac{\tau(x)}{1 - \tau(x)} \ln \left( \frac{1}{1 - \tau(x)} \right)^{m-1} \sim \frac{1}{(m-1)!} \frac{1}{2^{1/2} \sqrt{1 - e^x}} \ln \left( \frac{1}{2^{1/2} \sqrt{1 - e^x}} \right)^{m-1}
\]

we deduce

\[
\sum_{\ell=m}^{n} \frac{\ell a_{nml}}{n!} \sim \frac{1}{2^{m-1/2}(m-1)!} \left( \frac{1}{e} \right)^n \sqrt{n \ln(n) \Gamma(m-1/2)} \ln(n)^{m-1} e^n
\]

by the singularity analysis theorem of Flajolet & Odlyzko \[36\]. Multiplying both sides by \( n!/n^n \) and using Stirling’s approximation, we obtain

\[
\sum_{\ell=m}^{n} \frac{\ell a_{nml}}{n^n} \sim \frac{\ln(n)^{m-1}}{2^{m-1}(m-1)!}.
\]

From Section 4,

\[
\sum_{\ell=m}^{n} \frac{a_{nml}}{n^n} \sim \frac{\ln(n)^{m-1}}{2^{m-1}(m-1)!} \sqrt{\frac{\pi}{2n}}
\]
and hence, forming the ratio, \( E(N) \sim \sqrt{2n/\pi} \) as \( n \to \infty \).

Differentiating again and setting \( y = 1 \):

\[
\sum_{n=\ell}^\infty \sum_{\ell=m}^n \frac{\ell(\ell-1)a_{nm\ell}}{n!} x^n \sim \frac{1}{(m-1)!} \left( \frac{\tau(x)}{1 - \tau(x)} \right)^2 \ln \left( \frac{1}{1 - \tau(x)} \right)^{m-1}
\]

\[
\sim \frac{1}{(m-1)!} \frac{1}{2(1-e^x)} \ln \left( \frac{1}{1 - \frac{1}{2^{1/2} \sqrt{1 - e^x}}} \right)^{m-1}
\]

\[
\sim \frac{1}{2^{m-1}(m-1)!} \frac{1}{1 - e^x} \ln \left( \frac{1}{1 - e^x} \right)^{m-1}
\]

we deduce

\[
\sum_{\ell=m}^n \frac{\ell(\ell-1)a_{nm\ell}}{n!} \sim \frac{1}{2^{m-1}(m-1)!} \frac{\ln(n)^{m-1}}{e^n}
\]

Multiplying by \( n!/n^n \) and via the preceding, we obtain

\[
\sum_{\ell=m}^n \frac{\ell^2a_{nm\ell}}{n^n} \sim \frac{\ln(n)^{m-1}}{2^{m-1}(m-1)!} \sqrt{\frac{2\pi n}{2}} \sim \frac{\ln(n)^{m-1}}{2^{m-1}(m-1)!} \sqrt{\pi n}
\]

Forming the ratio, \( E(N^2) \sim n \) as \( n \to \infty \) and thus \( V(N) \sim (1 - 2/\pi)n \).

7. Addendum: Divisibility

Let \( m \) be a positive integer. A random variable \( X \) is \textit{m-divisible} if it can be written as \( X = Y_1 + Y_2 + \cdots + Y_m \), where \( Y_1, Y_2, \ldots, Y_m \) are independent and identically distributed. A random variable is \textit{infinitely divisible} if it is \( m \)-divisible for every \( m \). We wish to study the allocation of \( X = N \) cyclic points among a fixed number \( m \) of components, given a constrained random mapping. This would be a matter of determining the inverse Laplace transform of the \( m^{th} \) root of

\[
\mathcal{L} \left[ \sqrt{\frac{2}{\pi}} \exp \left( -\xi^2/2 \right) \right] = \exp \left( \eta^2/2 \right) \operatorname{erfc} \left( \eta/\sqrt{2} \right).
\]

Pavlov’s work \[24, 27\] is crucial here. We confront, however, a surprising theoretical obstacle: the half-normal density is provably \textit{not} infinitely divisible \[37, 38, 39\]. The independence requirement fails, in fact, beginning at \( m = 2 \). Let us offer a plausibility argument supporting this latter assertion.
On the one hand, if $Y_2 = 0$ is fixed, then the density of $Y_1 = N$ clearly approaches $\sqrt{2/\pi}$ as $\xi \to 0^+$, i.e., it is bounded near the origin.

On the other hand, if no condition is placed on $Y_2$, then the density of $Y_1 + Y_2 = N$ is a convolution in the $\xi$-domain, which becomes multiplication in the $\eta$-domain. Starting with [40] [41] [42]

$$\frac{\pi}{\sqrt{2\pi + \pi \eta}} < \sqrt{\frac{\pi}{2}} \exp\left(\frac{\eta^2}{2}\right) \text{erfc}\left(\frac{\eta}{\sqrt{2}}\right) < \frac{\pi}{\sqrt{2\pi + 2\eta}}$$

for all $\eta > 0$, we deduce

$$\sqrt{\frac{1}{1 + \sqrt{\frac{2\eta}{\pi}}}} < \exp\left(\frac{\eta^2}{4}\right) \sqrt{\text{erfc}\left(\frac{\eta}{\sqrt{2}}\right)} < \sqrt{\frac{1}{1 + \sqrt{\frac{2\eta}{\pi}}}}$$

and upper/lower bounds are tight approximations of the center for small/large values of $\eta$. No closed-form expression for $L^{-1}[\text{center}]$ seems to be possible; $L^{-1}[\text{lower bound}]$ and $L^{-1}[\text{upper bound}]$ are

$$\frac{2^{1/4}}{\pi^{3/4} \sqrt{\xi}} \exp\left(-\sqrt{\frac{2}{\pi}} \xi\right) \quad \text{and} \quad \frac{1}{(2\pi)^{1/4} \sqrt{\xi}} \exp\left(-\sqrt{\frac{\pi}{2}} \xi\right)$$

respectively. Both expressions approach infinity as $\xi \to 0^+$, tentatively implying that the density of $Y_1$ is unbounded near the origin. This contrasts with the behavior described earlier, i.e., information about $Y_2$ truly affects how $Y_1$ is distributed. Therefore $Y_1$ and $Y_2$ must be dependent.

When we employed the word “obstacle” before, it reflected our intention to study order statistics $Z_1 = \min\{Y_1, Y_2\}$ and $Z_2 = \max\{Y_1, Y_2\}$, with a goal of understanding the allocation process (partitioning $N$ cyclic points into two cycles). If $Y_1$ and $Y_2$ were independent with common density $\varphi(y)$, then the joint density of $Z_1 \leq Z_2$ would simply be $2\varphi(z_1)\varphi(z_2)$ for $z_1 \leq z_2$. Dependency renders the analysis more complicated.

As an aside, the Rayleigh density is also not infinitely divisible [39]. The independence requirement again fails beginning at $m = 2$. We argue as before, but less formally. On the one hand, if $Y_2 = 0$ is fixed, then the density of $Y_1$ approaches 0 as $\xi \to 0^+$. On the other hand, if no condition is placed on $Y_2$, then we wish to find the inverse Laplace transform of the square root of

$$\mathcal{L}\left[\xi \exp\left(-\frac{\xi^2}{2}\right)\right] = 1 - \sqrt{\frac{\pi}{2}} \eta \exp\left(\frac{\eta^2}{2}\right) \text{erfc}\left(\frac{\eta}{\sqrt{2}}\right).$$
A remarkably accurate approximation (with error less than 0.5%):

\[
\sqrt{1 - \frac{\pi}{2}} \eta \exp\left(\frac{\eta^2}{2}\right) \text{erfc}\left(\frac{\eta}{\sqrt{2}}\right) \approx \exp\left(\frac{\eta^2}{\pi}\right) \text{erfc}\left(\frac{\eta}{\sqrt{\pi}}\right)
\]

defies easy explanation and yet provides a very helpful estimate:

\[
L^{-1} \left[\exp\left(\frac{\eta^2}{\pi}\right) \text{erfc}\left(\frac{\eta}{\sqrt{\pi}}\right)\right] = \exp\left(-\frac{\pi \xi^2}{4}\right)
\]

which approaches 1 as \( \xi \to 0^+ \). Since \( 0 \neq 1 \), it follows that \( Y_1 \) and \( Y_2 \) must be dependent.

8. Addendum: Fallibility

With the benefit of hindsight, we should have focused not on \( \sigma(x) \), but instead on

\[ \tilde{\sigma}(x) = \sqrt{x} \sigma(x) \]

both here and in \([10]\). The derivative of \( \tilde{\sigma}(1/x) \) is found as follows:

\[
\frac{d}{dx} \tilde{\sigma}\left(\frac{1}{x}\right) = \frac{d}{dx} \left[ \frac{1}{x^{1/2}} \sigma\left(\frac{1}{x}\right) \right] = -\frac{1}{2x^{3/2}} \sigma\left(\frac{1}{x}\right) + \frac{1}{x^{1/2}} \sigma'\left(\frac{1}{x}\right) \left(-\frac{1}{x^2}\right)
\]

but

\[
\sigma'(y) = -\frac{1}{2y} (\sigma(y) + \sigma(y - 1))
\]

therefore

\[
\frac{d}{dx} \tilde{\sigma}\left(\frac{1}{x}\right) = -\frac{1}{2x^{3/2}} \sigma\left(\frac{1}{x}\right) + \frac{1}{x^{1/2}} \left(\frac{x}{2}\right) \left[ \sigma\left(\frac{1}{x}\right) + \sigma\left(\frac{1}{x} - 1\right)\right] \left(-\frac{1}{x^2}\right)
\]

\[= -\frac{1}{2x^{3/2}} \sigma\left(\frac{1}{x}\right) + \frac{1}{2x^{3/2}} \left[ \sigma\left(\frac{1}{x}\right) + \sigma\left(\frac{1}{x} - 1\right)\right] = \frac{1}{2x^{3/2}} \sigma\left(\frac{1-x}{x}\right),
\]

as was to be shown. Finally, \( \rho(x) \) and \( \sigma(x) \) are subsumed by the general DDE \([4, 5, 6]\)

\[x g'(x) + (1 - \theta)g(x) + \theta g(x - 1) = 0 \text{ for } x > 1, \quad g(x) = x^{\theta - 1} \text{ for } 0 < x \leq 1\]

where \( \theta > 0 \) is fixed, and its associated Laplace transform is

\[
\mathcal{L}\{g(x)\} = \frac{\exp(-\theta E(\eta))}{\eta^\theta / \Gamma(\theta)}, \quad \eta \in \mathbb{C} \setminus (-\infty, 0].
\]

It would be good someday to learn, from an interested reader, about possible random mapping-theoretic applications of \( g(x) \) for select \( \theta \notin \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\} \).
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Steven Finch
MIT Sloan School of Management
Cambridge, MA, USA
steven_finch@harvard.edu