A WEIGHTED OSTROWSKI TYPE INEQUALITY FOR $L_1[a,b]$
AND APPLICATIONS

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Abstract. The aim of this paper is to obtain some generalized weighted Ostrowski inequalities for differentiable mappings. Some well known inequalities can be derived as special cases of the inequalities obtained here. In addition, perturbed mid-point inequality and perturbed trapezoid inequality are also obtained. The inequalities obtained here have direct applications in Numerical Integration, Probability Theory, Information Theory and Integral Operator Theory. Some of these applications are discussed.

1. Introduction

Inequalities appear in most of the domains of Mathematics and has applications in numerical integration, probability theory, information theory and integral operator theory. Inequalities as a field came into prominence with the publication of a book by Hardy, Littlewood and Polya [6] in 1934. In 1938, Ostrowski [8] discovered a useful inequality, which is known after his name as Ostrowski inequality. In many practical investigation, it is necessary to bound one quantity by another. This classical Ostrowski inequality is very useful for this purpose. Beckenbach and Bellman [2] and Mitrinović [12] highlighted the importance of inequalities in their respective publications.

More recently new inequalities of Ostrowski type were presented Dragomir and Wang [5] in 1997 and Dragomir and Rassias [4] in 2002. The weighted version of Ostrowski inequality was first presented in 1983 by Pecarić and Savić [9]. In 2003, Roumeliotis [11] did some improvement in the weighted version of Ostrowski-Grüss type inequalities. In [10] and [7], Qayyum and Hussain discussed the weighted version of Ostrowski-Grüss type inequalities. The tools that are used in this paper are weighted Peano kernel approach which is the classical and extensively used approach in developing Ostrowski integral inequalities. The results presented in this paper are very general in nature. The inequalities proved by Dragomir et al [5], Barnett et al [1] and Cerone et al [3] are special cases of the inequalities developed here.

Ostrowski [8] proved the classical integral inequality which is stated here without proof.

Theorem 1. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$, whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on $(a,b)$, i.e. $\|f'\|_{\infty} = \ldots$

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\sup_{t \in [a, b]} |f'(t)| < \infty \quad \text{then}

\begin{align}
(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| & \leq \left[ \frac{1}{4} + \frac{(x - a + b/2)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \\
& \quad \text{for all } x \in [a, b]. \quad \text{The constant } \frac{1}{4} \text{ is sharp in the sense that it can not be replaced by a smaller one.}
\end{align}

Dragomir and Wang [5] proved (1.1) for \( f' \in L_1 [a, b] \), as follows:

\textbf{Theorem 2.} Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping in \( I^o \) and \( a, b \in I^o \) with \( a < b \). If \( f' \in L_1 [a, b] \), then the inequality holds

\begin{align}
(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| & \leq \left[ \frac{1}{2} + \frac{|x - a + b|}{b-a} \right] \|f'\|_1 \\
& \quad \text{for all } x \in [a, b].
\end{align}

They also pointed out some applications of (1.2) in Numerical Integration as well as for special means.

Barnett et al. [1] proved out an inequality of Ostrowski type for twice differentiable mappings which is in terms of the \( \| . \|_1 \) norm of the second derivative \( f'' \) and apply it in numerical integration and for some special means.

The following inequality of Ostrowski’s type for mappings which are twice differentiable, holds [3].

\textbf{Theorem 3.} Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous on \( [a, b] \) and twice differentiable in \( (a, b) \) and \( f'' \in L_1 (a, b) \). Then the inequality obtained

\begin{align}
(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \left( x - \frac{a+b}{2} \right) f' (x) \right| & \leq \frac{1}{2 (b-a)} \left( \left| x - \frac{a+b}{2} \right| + \frac{1}{2} (b-a) \right)^2 \| f'' \|_1 \\
& \quad \text{for all } x \in [a, b].
\end{align}

J. Roumeliotis [4], presented product inequalities and weighted quadrature. The weighted inequality was also obtained in Lebesgue spaces involving first derivative of the function, which is given by

\begin{align}
(1.2) \quad \left| \frac{1}{b-a} \int_a^b w(t) f(t) \, dt - m (a,b) f (x) \right| & \leq \frac{1}{2} \left[ m (a,b) + | m (a,x) - m (x,b) | \right] \| f'' \|_1
\end{align}

Motivated and inspired by the work of the above mentioned renowned mathematicians, we will establish a new inequality by using weight function , which will be better and generalized than those developed in [1 – 4]. Some other interesting inequalities are also presented as special cases. In the last, we presented applications for some special means and in numerical integration.
2. Main Results

In order to prove our main result we first give the following essential definition. We assume that the weight function (or density) \( w : (a, b) \rightarrow [0, \infty) \) to be non-negative and integrable over its entire domain and

\[
\int_a^b w(t) dt < \infty.
\]

The domain of \( w \) may be finite or infinite and may vanish at the boundary point. We denote the moment

\[
m(a, b) = \int_a^b w(t) dt.
\]

We now give our main result.

**Theorem 4.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\) and satisfy the condition \( \theta \leq f' \leq \Phi \), \( x \in (a, b) \). Then we have the inequality

\[
\left| f(x) - \frac{1}{m(a, b)} w(x) (b - a) \left( x - \frac{a + b}{2} \right) f'(x) - \frac{1}{m(a, b)} \int_a^b f(t) w(t) dt \right|
\]

\[
\leq \frac{1}{2m^2(a, b)} w(x) \left( \frac{1}{2} (b - a)^2 + 2 \left( x - \frac{a + b}{2} \right)^2 \right)
\]

\[
\times \left( \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right) \| f'' \|_{w, 1}
\]

\[(2.1)\]

for all \( x \in [a, b] \).

**Proof.** Let us define the mapping \( P(., .) : [a, b] \rightarrow \mathbb{R} \) given by

\[
P(x, t) = \begin{cases} 
\int_t^x w(u) du & \text{if } t \in [a, x] \\
\int_x^b w(u) du & \text{if } t \in (x, b].
\end{cases}
\]

Integrating by parts, we have

\[(2.2)\]

\[P(x, t) f'(t) dt = \int_a^b f(x) m(a, b) - \int_a^b f(t) w(t) dt.
\]

Applying the identity (2.2) for \( f'(.) \), we get

\[f'(t) = \frac{1}{m(a, b)} \int_a^b P(t, s) f''(s) ds + \frac{1}{m(a, b)} \int_a^b f'(s) w(s) ds.
\]

Applying \( f'(t) \) in the right membership of (2.2), we have

\[
f(x) = \frac{1}{m^2(a, b)} \int_a^b \int_a^b P(x, t) P(t, s) f''(s) ds dt
\]

\[
+ \frac{1}{m^2(a, b)} \int_a^b P(x, t) dt \int_a^b f'(s) w(s) ds dt + \frac{1}{m(a, b)} \int_a^b f(t) w(t) dt.
\]

\[(2.2)\]

Since

\[
\int_a^b P(x, t) dt = w(x) (b - a) \left( x - \frac{a + b}{2} \right)
\]

and

\[
\int_a^b f'(s) w(s) ds = f'(x) m(a, b).
\]
From (2.3) therefore we obtain
\[ f(x) = \frac{1}{m(a,b)} w(x) (b-a) \left( x - \frac{a+b}{2} \right) f'(x) + \frac{1}{m(a,b)} \int_a^b f(t) w(t) dt \]
\[ (2.3) \]
\[ + \frac{1}{m^2(a,b)} \int_a^b \int_a^b P(x,t) P(t,s) f''(s) ds dt. \]
Now
\[ \int_a^b |P(t,s)| ds = \frac{1}{2w(t)} \left[ (t - a)^2 + (t - b)^2 \right], \]
\[ \int_a^b |P(x,t)| \left[ \frac{w(t)}{2} \left( (t - a)^2 + (b - t)^2 \right) |f''(s)| ds \right] dt \leq \frac{1}{2} w(x) \left( (x - a)^2 + (b - x)^2 \right) \max \{ t - a, b - t \} \| f'' \|_{w,1}. \]
From (2.4), we have
\[ \left| f(x) - \frac{1}{m(a,b)} w(x) (b-a) \left( x - \frac{a+b}{2} \right) f'(x) - \frac{1}{m(a,b)} \int_a^b f(t) w(t) dt \right| \]
\[ (2.4) \leq \frac{1}{2m^2(a,b)} w(x) \left( (x - a)^2 + (b - x)^2 \right) \max \{ t - a, b - t \} \| f'' \|_{w,1}. \]
Using
\[ \max \{ t - a, b - t \} = \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \]
in (2.5), we get our desired result. \( \square \)

**Remark 1.** For \( w(t) = 1 \), the inequality (2.1) gives
\[ \left| f(x) - \left( x - \frac{a+b}{2} \right) f'(x) - \frac{1}{(b-a)} \int_a^b f(t) dt \right| \]
\[ \leq \frac{1}{2(b-a)^2} \left( \frac{1}{2} (b-a)^2 + 2 \left( x - \frac{a+b}{2} \right)^2 \right) \]
\[ \times \left( \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right) \| f'' \|_1 \]
which is similar to Barnett’s result proved in [1].

**Corollary 1.** Under the assumptions of Theorem 4 and choosing \( x = \frac{a+b}{2} \), we have the perturbed midpoint inequality
\[ \left| f \left( \frac{a+b}{2} \right) - \frac{1}{m(a,b)} \int_a^b f(t) w(t) dt \right| \]
\[ (2.6) \leq \frac{1}{8m^2(a,b)} w(x) (b-a)^3 \| f'' \|_{w,1}. \]

**Proof.** This follows from inequality (2.1). \( \square \)
Corollary 2. Under the assumptions of Theorem 4, we have the perturbed trapezoidal inequality
\[
\frac{f(a) + f(b)}{2} - \frac{1}{m(a, b)} \int_a^b f(t)w(t)dt + \frac{1}{m(a, b)} \frac{(b-a)^2}{4} (w(a)f''(a) - w(b)f''(b)) \\
\leq \frac{1}{4m^2(a, b)} (b-a)^3 [w(a) + w(b)] \|f''\|_{w,1}.
\]

Proof. Put \(x = a\) and \(x = b\) in (2.1), summing up the obtained inequalities, using the triangle inequality and dividing by 2, we get the required inequality. \(\Box\)

Remark 2. The result given in (2.8) is different from the comparable results available in [4].

Remark 3. We can get the best estimation from the inequality (2.1), only when \(x = \frac{a+b}{2}\), this yields the inequality (2.7). It shows that mid point estimation is better than the trapezoid estimation.

3. Application for some special means

We may now apply inequality (2.1), to deduce some inequalities for special means by the use of particular mappings as follows:

Remark 4. Consider \(f(x) = \sqrt{x} \ln x\), \(x \in [a, b] \subset (0, \infty)\) and
\(w(x) = \frac{1}{\sqrt{x}}\).

The inequality (2.1) therefore gives
\[
\sqrt{x} \ln x \left( b - a \right) \ln I(a, b) \\
- \frac{1}{8 \left( \sqrt{b} - \sqrt{a} \right)^2} \frac{1}{\sqrt{x}} \left( \frac{1}{2} (b-a)^2 + 2 (x-A)^2 \right) \\
\times \left( \frac{1}{2} (b-a) + |x-A| \right) \left( \frac{b-a}{4ab} \left( 1 - \frac{\ln b^a - \ln a^b}{b-a} \right) \right).
\]

Choosing \(x = A\) in (3.1), we get
\[
\sqrt{A} \ln A - \frac{1}{2 \left( \sqrt{b} - \sqrt{a} \right)} (b-a) \ln I(a, b) \\
\leq \frac{1}{128ab \left( \sqrt{b} - \sqrt{a} \right)^2} \frac{1}{\sqrt{x}} (b-a)^4 \left( 1 - \frac{\ln b^a - \ln a^b}{b-a} \right).
\]

Remark 5. Consider \(f(x) = \frac{1}{x} \sqrt{x}\), \(x \in [a, b] \subset [1, \infty)\) and
\(w(x) = \frac{1}{\sqrt{x}}\).
The inequality (2.1) therefore gives
\[
\left| \frac{1}{2} \sqrt{\frac{1}{x}} + \frac{1}{4(\sqrt{b} - \sqrt{a}) x^2} (b - a) (x - A) \right| - \frac{1}{2(\sqrt{b} - \sqrt{a})} (b - a) L_{-1}^1 \right| \leq \frac{1}{8 \left( \sqrt{b} - \sqrt{a} \right)^2} \frac{1}{\sqrt{x}} \left( \frac{1}{2} (b - a)^2 + 2 (x - A)^2 \right)
\]
(3.3)
\[
\times \frac{3}{8} \left( \frac{1}{2} (b - a) + |x - A| \right) \left( \frac{b^2 - a^2}{a^2 b^2} \right).
\]
Choosing \( x = A \) in (3.3), we get
\[
\left| \frac{1}{A} \sqrt{A} - \frac{1}{2 \left( \sqrt{b} - \sqrt{a} \right)} (b - a) L_{-1}^1 \right| \leq \frac{3}{256 \left( \sqrt{b} - \sqrt{a} \right)^2} \frac{1}{\sqrt{A}} (b - a)^3 \left( \frac{b^2 - a^2}{a^2 b^2} \right).
\]
(3.4)

Remark 6. Consider \( f(x) = x^p \sqrt{x}, \ x \in [a, b] \ f : (0, \infty) \to R \), where \( p \in R \setminus \{-1, 0\} \) then for \( a < b \),

\[
w(x) = \frac{1}{\sqrt{x}}
\]

The inequality (2.1) therefore gives
\[
\left| x^p \sqrt{x} - \frac{1}{2(\sqrt{b} - \sqrt{a})} x \right| \left( b - a \right) (x - A) (p + \frac{1}{2} x^p \right) \left| - \frac{1}{2(\sqrt{b} - \sqrt{a})} (b - a) L_p^1 \right| \leq \frac{1}{8 \left( \sqrt{b} - \sqrt{a} \right)^2} \frac{1}{\sqrt{x}} \left( \frac{1}{2} (b - a)^2 + 2 (x - A)^2 \right)
\]
(3.5)
\[
\times \left( \frac{1}{2} (b - a) + |x - A| \right) \left( \frac{p^2 - \frac{1}{4}}{p - 1} \right) \left( b^{p-1} - a^{p-1} \right).
\]
Choosing \( x = A \) in (3.5), we get
\[
\left| A^p \sqrt{A} - \frac{1}{2 \left( \sqrt{b} - \sqrt{a} \right)} (b - a) L_p^1 \right| \leq \frac{1}{32 \left( \sqrt{b} - \sqrt{a} \right)^2} \frac{1}{\sqrt{A}} (b - a)^3 \left( \frac{p^2 - \frac{1}{4}}{p - 1} \right) \left( b^{p-1} - a^{p-1} \right).
\]
(3.6)

4. An Application to Numerical integration

Let \( I_n : a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \) be a division of the interval \([a, b]\) and \( \xi = (\xi_0, \xi_1, \ldots, \xi_{n-1}) \), a sequence of intermediate points \( \xi_i \in [x_i, x_{i+1}] \) \((i = 0, 1, \ldots, n - 1) \). Consider the perturbed Riemann sum defined by

\[
A = \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f(\xi_i) - \sum_{i=0}^{n-1} w(\xi_i) h_i \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) f(\xi_i)
\]
(4.1)
Theorem 5. Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\), such that \( f' : (a, b) \rightarrow \mathbb{R} \) is bounded on \((a, b)\) and assume that \( \gamma \leq f' \leq \Gamma \) for all \( x \in (a, b) \). \( f'' : (a, b) \rightarrow \mathbb{R} \) belongs to \( L_1(a, b) \), i.e.

\[
\| f'' \|_{w, 1} := \int_a^b |w(t)f(t)| \, dt < \infty.
\]

we have

\[
(4.2) \quad \int_a^b f(t)w(t)\,dt = A(f, I, w, \xi) + R(f, I, w, \xi),
\]

where the remainder \( R \) satisfies the estimation

\[
|R(f, I, w, \xi)| \leq \frac{\| f'' \|_{w, 1}}{2m(x_i, x_{i+1})} w(\xi_i) \left( \frac{1}{2} (h_i)^2 + 2 \left( \frac{\xi_i - x_i + x_{i+1}}{2} \right)^2 \right)
\]

\[
\times \left( \frac{1}{2} (h_i) + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right)
\]

for any choice \( \xi \) of the intermediate points.

Proof. Apply Theorem 4 on the interval \([x_i, x_{i+1}]\), \( \xi_i \in [x_i, x_{i+1}] \), where \( h_i = x_{i+1} - x_i \) (\( i = 1, 2, 3, ..., n - 1 \)), to get

\[
\left| m(x_i, x_{i+1}) f(\xi_i) - \int_{x_i}^{x_{i+1}} f(t)w(t)\,dt - (\xi_i - \frac{x_i + x_{i+1}}{2}) w(\xi_i) (x_{i+1} - x_i) f'(\xi_i) \right|
\]

\[
\leq \frac{\| f'' \|_{w, 1}}{2m(x_i, x_{i+1})} w(\xi_i) \left( \frac{1}{2} (x_{i+1} - x_i)^2 + 2 \left( \frac{\xi_i - x_i + x_{i+1}}{2} \right)^2 \right)
\]

\[
\times \left( \frac{1}{2} (x_{i+1} - x_i) + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right)
\]

for all \( \xi_i \in [x_i, x_{i+1}] \) and \( i \in \{0, 1, ..., n - 1\} \). Summing the above two inequalities over \( i \) from 0 to \( n - 1 \) and using the generalized triangular inequality, we get the desired estimation. \( \square \)

5. Conclusions

We established weighted Ostrowski type inequality for bounded differentiable mappings which generalizes the previous inequalities developed and discussed in \([1, 3, 5] \) and \([8]\). Perturbed midpoint and trapezoid inequalities are obtained. Some closely new results are also given. This inequality is extended to account for applications in some special means and numerical integration to show its applicability towards obtaining direct relationship of these means. These generalized inequalities will also be useful for the researchers working in the field of the approximation theory, applied mathematics, probability theory, stochastic and numerical analysis to solve their problems in engineering and in practical life.
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