Sub-exponential algorithm for 2D frustration-free spin systems with gapped subsystems

Nilin Abrahamsen

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA, USA

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Abstract

We show that in the setting of the subvolume law of [Anshu, Arad, Gosset ’19] for 2D locally gapped frustration-free spin systems there exists a randomized classical algorithm which computes the ground states in sub-exponential time. The running time cannot be improved to polynomial unless SAT can be solved in randomized polynomial time, as even the special case of classical constraint satisfaction problems on the 2D grid is known to be NP-hard.

1 Introduction

In a recent breakthrough [AAG19] by Anshu, Arad, and Gosset in the theory of local Hamiltonians it was shown that a subvolume law holds for 2D frustration-free spin system satisfying a local gap condition. This result represented significant progress towards understanding the area law conjecture in 2D.

Entanglement is often viewed as a measure of the complexity of a quantum system. It is therefore an enticing question whether entanglement bounds can lead to algorithmic improvements for the problem of computing ground states. In one dimension the constant entanglement bound of gapped systems [Has07, ALV12, AKLV13] was used to produce polynomial-time algorithms [LVV15, ALVV17].

A special case of locally gapped Hamiltonians is that of a classical constraint satisfaction problem, i.e., when all local interactions are diagonal in the standard basis. Even classical CSPs are NP-hard (thus NP-complete) on a 2D grid [Lic82, KR92], but they can be solved in time $2^{O(\sqrt{n})}$ by a sweep across the grid, enumerating the possible variable assignments at the boundary of the swept region. The closest analogue for a quantum system would try to use boundary contractions as a replacement for variables at the boundary as in the first algorithms for 1D ground states [LVV15]. However, enumeration over an $\epsilon$-net of boundary contractions would lead to a time complexity exponential is the entanglement rank or doubly exponential in the entanglement entropy. In the 2D case such an approach would at best yield a running time $2^{2^{O(\sqrt{n})}}$ even if an area law were known, much worse than the exponential time required to directly diagonalize the Hamiltonian.

In the context of spin chains [ALVV17] introduced an algorithm which exponentially improved the dependence on the spectral gap by implementing an approximate ground space projector (AGSP). In contrast, previous literature had used AGSPs only to prove the 1D area law which was in turn used as a black box to bound the dimension of boundary contractions. We will adapt the AGSP
constructed in the proof [AAG19] of the subvolume law so that it can be implemented in this way, thus achieving a subexponential algorithm. We expect that sub-exponential time complexity is the best one can hope for in the setting of 2D frustration-free locally-gapped Hamiltonians due to the NP-hardness of the classical special case.

1.1 Setting

We consider a system of \( n = wh \) qudits, each with local dimension \( d \). The spins are arranged in a \( w \times h \) lattice with vertex set \( \mathbb{Z}^2 \cap ([1, w] \times [1, h]) \), and we consider a frustration-free local Hamiltonian \( H = \sum_i H_i \) with local interactions \( 0 \preceq H_i \preceq I \), each of which involves only qudits on the grid within a constant diameter. This definition allows for plaquette interactions, hexagonal grids, etc.

Assume without loss of generality that \( h \leq w \), which implies \( h \leq \sqrt{n} \). For a set of vertices \( S \) let \( H_S = \sum H_i \) be the sum of interactions involving only spins in \( S \).

**Definition 1.1 ([AAG19]).** The local gap of \( H \) is \( \gamma := \min_B \tilde{\gamma}(H_B) \) where the minimum is over all rectangles \( B = ([a, b] \times [c, d]) \cap \mathbb{Z}^2 \) and \( \tilde{\gamma}(H_B) = \min(\text{spec } H_B \setminus 0) \).

The local gap condition posits that \( \gamma = \Omega(1) \).

In this setting [AAG19] proved that the entanglement entropy of a unique ground state across a vertical cut is bounded by \( O\left(h^5/3 \gamma^{-5/6} \log(h d/\gamma)^{7/3}\right) \). The use of the local gap assumption in [AAG19] is motivated by finite-size criteria [Kna88, GM16, Lem19, LSW19]. In the context of such criteria a Hamiltonian is gapped precisely because it is locally gapped.

1.2 Results

Let \( Z = \ker H \) be the space of ground states of \( H \), and let the local dimension \( d \) be constant. We allow a degenerate ground space and let \( D \) be a bound on the degeneracy \( D = \dim(Z) \).

**Theorem 1.2.** Let \( H \) be a frustration-free Hamiltonian with local gap \( \gamma \) on the \( w \times h \) lattice with \( n = wh \) qudits. Let \( D \) be a bound on the degeneracy and \( \delta \) an accuracy parameter. There exists a probabilistic algorithm with time complexity

\[
(D/\delta)^{O(1)} 2^{\tilde{O}\left((n/\gamma)^{5/6}\right)}.
\]

which on input \( H \) and parameters \( \gamma, D, \delta \) outputs an MPS (of bond dimension bounded by (1)) representing a subspace \( \tilde{Z} \preceq H \) such that \( \tilde{Z} \approx \delta Z \) with probability at least \( 1/2 \).

The error probability in theorem 1.2 is easily reduced by repetition. Indeed, given outputs \( \{\tilde{Z}_i\} \) the final output can be taken as a \( \tilde{Z}_i \) of maximal dimension subject to \( \|H|\tilde{z}_i\| \leq \delta \) where \( H|\tilde{z}_i\) is the two-sided restriction of \( H \) to \( \tilde{z}_i \).

![Figure 1: MPS representation of \( \tilde{Z} \).](image)

In the MPS of theorem 1.2 each physical index represents a column of qudits. The dependence on \( D \) is consistent with the generalization of the subvolume law of [AAG19] to subexponential degeneracy which follows from [Abr20]. The bound of theorem 1.2 can be strengthened when \( w \) and \( h \) are not proportional by replacing \( n \) with \( \min(w, h)^2 \leq n \). We give the statement at the end of section 4.4.
It is important to ask whether the output of theorem 1.2 can be used to compute the expectation values of local observables. In fact we may modify the algorithm with a post-processing step which prepares a list of all such expectation values on the ground states.

**Corollary 1.3 (Post-processing).** Let $S \subset (\{\sigma_i\}_{i=1}^d)^n$ be the set of Pauli observables which act nontrivially on at most $k \leq n^{5/6}$ spins. The algorithm of theorem 1.2 can be modified to output a 3-dimensional table $T$ such that, for some basis $\{|z_i\rangle\}$ for $Z$, $|T_{\sigma ij} - \langle z_i | \sigma | z_j \rangle| \leq \delta$ for each $\sigma \in S$ and $i,j = 1,\ldots,D$ with probability at least $1/2$. The time complexity of the modified algorithm is still $O(1)$.

**Proof.** The modified algorithm runs the algorithm of theorem 1.2 and then contracts the resulting MPS to compute each entry of $T$. It is well-known [Vid03, PKS+19] that contracting the MPS is polynomial in the bond dimension and linear in $n$. Moreover, the number of entries of $T$ is $D^2 (n^k / \delta)$, so we can absorb the time complexity in $O(1)$.

### 1.3 NP-hardness

To illustrate hardness of the problem in theorem 1.2 in the most informative way we should show hardness in as restrictive a special case as possible. We therefore consider the case when $H$ is a satisfiable classical 3SAT-formula and moreover the degeneracy is $D = 1$, i.e., the satisfying assignment is promised to be unique. Then the local gap is $\gamma = 1$, and the satisfying assignment can be found by computing the 1-local observables using corollary 1.3 to constant accuracy $\delta$.

**Lemma 1.4 ([Lic82, VV85]).** Let $A$ be the set of 3SAT instances on a 2D grid and let $uA \subset A$ be the set of such instances with exactly 1 satisfying assignment. Suppose there exists a polynomial-time algorithm which given an instance from $uA$ outputs the satisfying assignment with probability 1/2. Then NP equals RP (randomized polynomial time).

**Proof.** SAT is parsimoniously reducible to 3SAT [Koz92] which itself is parsimoniously reducible to rectilinear planar 3SAT [Lic82, KR92, Dem14] (All reductions mentioned are polynomial-time). A rectilinear planar 3SAT instance is easily embedded in the 2D grid with 3-local constraints. So there exists a parsimonious reduction $g$ which takes SAT instances to $A$ and unique SAT instances to $uA$.

By the Valiant-Vazirani theorem [VV85] there exists a randomized reduction $f$ from SAT to unique SAT. Since $g$ preserves uniqueness of solutions $g \circ f$ gives a randomized reduction from SAT to the problem of computing the solution to an instance of $uA$.

It follows that the running time $(D/\delta)^{O(1)} 2^{O((n/\gamma)^5/6)}$ of theorem 1.2 and corollary 1.3 cannot be improved from sub-exponential to polynomial in $n/\gamma$ unless $NP = RP$.

**Corollary 1.5.** Suppose there exists a probabilistic algorithm which approximates the 1-local expectation values as in corollary 1.3 in time $F(D/\delta)p(n/\gamma)$ for some arbitrary function $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and some polynomial $p$. Then $NP = RP$.

**Proof.** For inputs from $uA$ we can use parameters $D = 1$, $\gamma = 1$, and $\delta = .9$ so that the time complexity is $O(p(n))$. From the approximate expectation values $\langle Z_v \rangle$ output by the algorithm we write down the satisfying assignment $b$ such that $(-1)^{bv} = \text{sign}(\langle Z_v \rangle)$ for each vertex $v$ in the grid.

The same argument implies that if the running time in corollary 1.3 can be improved to quasi-polynomial then all NP problems can be solved in randomized quasi-polynomial time.
2 Preliminaries

We write the Hilbert space of the spin system as \( \mathcal{H} \). Given a subspace \( Z \preceq \mathcal{H} \) let \( P_Z \) be the projection onto \( Z \). Let \( \mathcal{S}(\mathcal{H}) \) be the sphere of unit vectors in \( \mathcal{H} \). \( \mathcal{B}(\mathcal{H}) \) is the space of linear operators on \( \mathcal{H} \) and \( I \in \mathcal{B}(\mathcal{H}) \) the identity.

Definition 2.1.

- Let \( Z, Y \preceq \mathcal{H} \) be subspaces. \( Y \) is \( \mu \)-overlapping onto \( Z \) (\( Y \succeq_{\mu} Z \)) if \( \|P_Z|y\|^2 \geq \mu \) for all \( |y\rangle \in \mathcal{S}(Y) \). Writing \( \mu = 1 - \delta \) one also says that \( Y \) is \( \delta \)-viable for \( Z \). Two subspaces are \( \delta \)-close (\( \simeq_{\delta} \)) if each is \( \delta \)-viable for the other.

- \([\text{ALVV17}]\) Given \( Z \preceq \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), a subspace \( V \preceq \mathcal{H}_1 \) is (left) \( \delta \)-viable for \( Z \) iff \( V \otimes \mathcal{H}_2 \) is \( \delta \)-viable for \( Z \).

Definition 2.2 (\([\text{AL12}]\)). A \( \Delta \)-AGSP (approximate ground space projector) for Hamiltonian \( \mathcal{H} \) with ground space \( Z \) is an operator of the form \( K = 1_Z \otimes K_Z \), where \( K_Z \in \mathcal{B}(\mathcal{Z}^2) \) and \( \|K_Z\| \leq \sqrt{\Delta} \).

In \([\text{Abr20}]\) the author gave the following error reduction bound. In particular the post-AGSP error is bounded by \( \delta' \leq \Delta - \delta/\mu \leq \Delta/\mu \):

Lemma 2.3 (\([\text{Abr20}]\)). Suppose \( V \preceq \mathcal{H} \) covers \( Z \) with overlap \( \mu \), and let \( K \) be a \( \Delta \)-AGSP for \( Z \). Then \( K \mathcal{V} \) is \( \delta' \)-viable for \( Z \) where \( \frac{\delta}{\mu'} \leq \Delta \mu \) for \( \mu = 1 - \delta \) and \( \mu' = 1 - \delta' \).

2.1 Complexity-reducing procedures of \([\text{ALVV17}]\)

Our algorithm will make use of the same main steps as the algorithm for ground states of spin chains in \([\text{ALVV17}]\): AGSP to improve the overlap, and random sampling and bond trimming to reduce complexity. We now review the complexity reduction methods.

Lemma 2.4 (\([\text{ALVV17}]\) lemma 5). Let \( Y \preceq \mathcal{H}_L \) be left \( \mu \)-overlapping onto \( Z \preceq \mathcal{H}_L \otimes \mathcal{H}_R \) and let \( Y = \text{dim}(Y) \), \( D = \text{dim}(Z) \). Let \( V \subset Y \) be a Haar-uniform random subspace with dimension \( V \). Then with probability \( 1 - \eta \), \( V \) is left \( \nu \)-overlapping onto \( Z \) where \( \nu = \frac{\sqrt{y}}{8\pi} \cdot \mu \) and \( \eta = (1 + 2\nu^{-1/2})^D e^{-V/16} \).

Consider a multipartite space \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_w \) and a subspace \( Y \preceq \mathcal{H} \). Given integers \( a, b \) let \( \mathcal{H}_{[a,b]} = \mathcal{H}_{a} \otimes \mathcal{H}_{a+1} \otimes \cdots \otimes \mathcal{H}_b \).

The second complexity-reduction method of \([\text{ALVV17}]\) trims the bonds in an MPS \( \mathcal{H} \). While we will only use a simplified version of this trimming procedure we nevertheless state it for completeness. The following definition is equivalent with \([\text{ALVV17}]\) definition 4 for \( \xi = \sqrt{\nu} \) (here restated in terms of the projector \( P_Y \) as opposed to its purification).

Definition 2.5 (\([\text{ALVV17}]\) definition 4). Given a subspace \( Y \preceq \mathcal{H} \) let \( \rho_{[1,i]}^Y = \text{tr}_{[i+1,w]}(P_Y) \) be the reduced density matrices for \( i = 1, \ldots, w \). Define \( P_{[1,i]} \) inductively as the spectral projection \( P_{[1,i]} = \mathbb{I}_{[i,\infty)}(\hat{P}_{[1,i-1]}|\rho_{[1,i]}^Y|\hat{P}_{[1,i-1]}^\dagger) \) where \( \hat{P}_{[1,i-1]} \) is the extension \( P_{[1,i-1]} \otimes I_i \) and \( \mathbb{I} \) denotes an indicator function. Then the trimmed \( Y \) of \([\text{ALVV17}]\) is the image \( \hat{P}_{[1,1]} \hat{P}_{[1,2]} \cdots \hat{P}_{[1,w-1]}(Y) \) where \( \hat{P}_{[1,i]} = P_{[1,i]} \otimes I_{[i+1,w]} \).

2.2 Simple trimming and analysis

To simplify our analysis we do not directly use the trimming method of definition 2.5 but instead trim in a modular way. More precisely we iterate the bipartite case of definition 2.5\(^1\):

\(^1\text{We used a similar modular trimming in [Abr19] but with a more complicated analysis} \)
Definition 2.6. Given $\mathcal{Y} \subseteq \mathcal{H}_{AB}$ and $\varepsilon > 0$ introduce the projection $P_A = 1_{[\varepsilon,\infty)}(\rho_A^Y)$ where $\rho_A^Y = \text{tr}_B(P_Y)$ is the reduced density matrix and $1$ denotes an indicator function. Then $\text{trim}_\varepsilon^A(\mathcal{Y})$ is the image $[P_A \otimes 1_B](\mathcal{Y})$.

The trimmed subspace is contained in $\mathcal{V} \otimes \mathcal{H}_B$ where $\mathcal{V} = P_A(\mathcal{H}_A)$ where Markov’s inequality gives the bound $\dim(\mathcal{V}) = \text{rank } P_A \leq \dim(\mathcal{Y})/\varepsilon$ since $\text{tr}(\rho_A^Y) = \dim(\mathcal{Y})$.

Definition 2.7. Given a subspace $\mathcal{Y} \subseteq \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_j$, define

$$\text{Trim}_\varepsilon(\mathcal{Y}) = \text{trim}_1^\varepsilon \circ \text{trim}_2^\varepsilon \circ \cdots \circ \text{trim}_j^\varepsilon(\mathcal{Y}).$$

Lemma 2.8. Let $\mathcal{Z} \subseteq \mathcal{H}_{ABC}$ and let $\mathcal{Y} \subseteq \mathcal{H}_A$ be $\delta$-viable for $\mathcal{Z}$. If there exists $\mathcal{V} \subseteq \mathcal{H}_A$ with $\dim(\mathcal{V}) = \varepsilon$ which is $\alpha$-viable for $\mathcal{Z}$, then $\mathcal{Y}_\varepsilon = \text{trim}_\varepsilon^A(\mathcal{Y})$ is $\delta'$-viable for $\mathcal{Z}$ where $\delta' = \delta + \sqrt{\varepsilon}V + \sqrt{\alpha}$.

Proof. Introduce projectors $P_+ = 1_{[\varepsilon,\infty)}(\rho_A^Y)$ and $P_- = 1_{[0,\varepsilon)}(\rho_A^Y)$ on $\mathcal{H}_A$. Denote extensions of operators and subspaces as $\tilde{P} = P \otimes 1_{BC}$ and $\mathcal{Y} = \mathcal{Y} \otimes \mathcal{H}_C$.

Given any $|z\rangle \in \mathcal{S}(\mathcal{Z})$ pick $|y\rangle \in \mathcal{S}(\mathcal{Y})$ satisfying $|z\rangle Y \geq 1$. Let $|y'\rangle = \tilde{P}_+ |y\rangle$ so that $|y'\rangle \in \mathcal{Y}_\varepsilon$ and $||y'\rangle|| \leq 1$. Then,

$$\langle z|y\rangle - \langle z|y'\rangle = \langle z|P_- |y\rangle = \langle z|\tilde{P}_+ \tilde{P}_- |y\rangle + \langle z|\tilde{P}_+ \tilde{P}_- |y\rangle. \quad (2)$$

Bound the first term on the RHS by

$$\|\tilde{P}_+ \tilde{P}_- |y\rangle\| = \text{tr}(P_- \text{tr}_BC(|y\rangle \langle y|)) \leq \sqrt{\text{tr}(P_- \text{tr}_BC(|y\rangle \langle y|))} \leq \sqrt{\varepsilon V.$$}

since $\|P_- \rho_A^Y\| \leq \varepsilon$ and rank $P_Y = V$. Bound the second term on the RHS of (2) by $\|\tilde{P}_+ \tilde{P}_- |z\rangle\| \leq \sqrt{\alpha}$. By (2), $\langle z|y'\rangle \geq \langle z|y\rangle - \sqrt{\varepsilon V} - \sqrt{\alpha}$. \qed

Corollary 2.9. Suppose $\mathcal{Z} \subseteq \mathcal{H}_{1 \ldots w}$ is such that for each $i$ there exists a $\alpha$-viable space $\mathcal{V}_{[1,i]} \subseteq \mathcal{H}_{[1,i]}$ for $\mathcal{Z}$ with $\dim(\mathcal{V}_{[1,i]}) \leq V$. If $\mathcal{Y} \subseteq \mathcal{H}_{[1,j]}$ is $\delta$-viable for $\mathcal{Z}$ then $\text{Trim}_\varepsilon^j \mathcal{Y}$ is $\delta'$-viable for $\mathcal{Z}$ where $\delta' = \delta + w(\varepsilon V + \alpha)$.

3 Algorithm given an implementable AGSP

Consider a multipartite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_w$ and a $\Delta$-AGSP $K \in \mathcal{B}(\mathcal{H})$ represented as a matrix product operator (MPO) with bond dimension $R$. If the bond dimension of the MPO satisfies an appropriate bound, say subexponential, then we call $K$ an implementable AGSP. We first describe the algorithm of theorem 1.2 generically given $K$ before we return to the construction of such an implementable AGSP in section 4.

Let $d = \max\{\dim(\mathcal{H}_1), \ldots, \dim(\mathcal{H}_w)\}$ and let $D$ be an upper bound on the degeneracy $\dim(\mathcal{Z})$. For each $i$ let $K_{[1,i]}$ be the subspace of operators acting on $\mathcal{H}_{[1,i]}$ which is encoded by the left part of the MPO for $K$ where the cut bond is left open. Then $\dim(K_{[1,i]}) \leq R$. $K_{[1,i]}$ is a $\Delta$-PAP in the terminology of [Abr19].

The following algorithm is based on the tools of [ALVV17] for ground states of spin chains. However the algorithm given here is simpler in that it avoids the use of an inner loop.

In the last line of algorithm 1 we use the notation $A|\mathcal{Y} = \Gamma A \Gamma^\dagger$ where $\Gamma : \mathcal{H} \to \mathcal{Y}$ is the (surjective) projection onto $\mathcal{Y} \subseteq \mathcal{H}$ and $\Gamma^\dagger : \mathcal{Y} \to \mathcal{H}$ the inclusion map.

3.1 Analysis of algorithm 1

To bound the error from trimming with corollary 2.9 we need an entanglement off-the-rack bound [Abr20].
Lemma 2.4 succeeds, by the induction hypothesis.

Proof. Applying [Abr20] lemma 4.5 to \( K^5 \) there exists a \( \mu = \frac{1}{8D} \)-overlapping subspace \( V_0 \) with dimension \( O(D \log R) \). Let \( V = K^pV_0 \) where \( p = \lceil \log \Delta(\alpha \mu) \rceil \). By lemma 2.3 the viability error of \( K^pV_0 \) is at most \( \Delta p / \mu \leq \alpha \). We bound the dimension using \( p < 1 + \log R(\frac{1}{\alpha \mu}) = 6 + \log R(32/\alpha) \) which implies \( \dim(V) \leq \frac{22}{5} R^6 D \log R \). \( \square \)

Corollary 3.2. Suppose \( R \Delta \leq 1/2 \). Given \( \delta \) there exists a choice \( \varepsilon = \frac{1}{8} \left( \frac{5}{32} \right)^{O(1)} \) such that \( \text{trim}_\varepsilon \) increases the viability error by at most \( \delta \).

Proof. By corollary 2.9 it suffices to verify the existence of \( \alpha \)-viable subspaces of dimension \( V \) such that \( w(\sqrt{\varepsilon}V + \sqrt{\alpha}) \leq \delta \). Let \( \alpha = \left( \frac{1}{32} \right)^2 \). By lemma 3.1 we can take \( V \leq \left( w/\delta \right)^2 DR^{O(1)} \). Then pick \( \varepsilon = \frac{1}{8} \left( \frac{5}{32} \right)^2 \). \( \square \)

Lemma 3.3. Given \( 0 < \delta \leq 1/2 \) suppose \( R \Delta \leq \frac{\delta}{1-2\delta} \). Then there exists a choice \( V = \Theta(D \log(Rd) + \log w) \) and \( \varepsilon = \frac{1}{8} \left( \frac{5}{32} \right)^{O(1)} \) such that with probability at least \( 1/2 \) each \( V \) is \( \delta \)-viable for \( Z \) in algorithm 1.

Proof. At the beginning of the \( i \)th iteration, \( \dim(V_{i, i-1}) \leq \dim(K_{i, i-1}) \leq RV \). \( V_{i, i-1} \) then has dimension at most \( Y = dRV \).

Let \( \nu = \frac{1}{16Rd} \). The error probability of lemma 2.4 is bounded by \( \eta = (9/\nu)^{D/2} Ye^{-V/16} \) (cf. appendix B.2 of [Abr19]). Pick \( V \) such that

\[
V = 16 \log V \geq 8D \log(144Rd) + 16 \log(2dRw).
\]

Then \( \eta \leq (9/\nu)^{D/2} Ye^{-V/16} \left( \log(16Rd) + \log(2dRw) \right) = \frac{1}{16Rd} \). By a union bound lemma 2.4 succeeds at each iteration with probability at least \( 1/2 \). We perform an induction within this event.

Induction step. By the induction hypothesis \( V_{i, i-1} \) is \( 1/2 \)-viable for \( Z \). As lemma 2.4 succeeds, \( V \) has left overlap \( \nu = \frac{1}{16Rd} \) onto \( Z \). By lemma 2.3 \( K_{i, i} V \) is \( \delta/2 \)-viable for \( Z \) since \( \Delta/\nu = 16dR \Delta \leq \delta/2 \). By corollary 3.2 the trimming increases the error only by \( \delta/2 \), so \( V_{i, i} \) is \( \delta \)-viable for \( Z \). \( \square \)

Having shown that \( V_{i, i} \) is \( \delta \)-viable for \( Z \) it remains to analyze the restriction on the last line of algorithm 1.

Lemma 3.4. If \( \mathcal{V} = \mathcal{V}_{i, i} \) is \( \delta \)-viable for \( Z \) then the output of algorithm 1 is \( 2\delta \)-close to \( Z \).

Proof. We show more precisely that \( \tilde{Z} \) is \( \delta/\bar{\gamma} \)-close to \( Z \) where \( \bar{\gamma} = 1 - \Delta \geq 1/2 \). By the symmetry lemma of [Abr19, Abr20] it suffices to show that

1. \( Z \) is \( \delta/\bar{\gamma} \)-viable for \( \tilde{Z} \) and
2. \( \dim(\tilde{Z}) \geq \dim(Z) \).
1. By definition $\tilde{Z} \preceq Y$ is such that $H|\tilde{Z} \preceq \delta$. Since $K$ is a $\Delta$-AGSP we can write $\tilde{H} = 1 - K^\dagger K = 0_2 \oplus \tilde{H}_{Z'}$ where $\tilde{\gamma} \leq \tilde{H}_{Z'}$. So $\tilde{\gamma} P_Z P_{Z'} \preceq P_{Z'} \tilde{H} P_{Z'} \preceq \delta P_Z$, which implies that $Z$ is $\delta/\tilde{\gamma}$-viable for $\tilde{Z}$.

2. Since $Y$ is $\delta$-viable for $Z$, lemma 2.9 of [Abr20] implies that $Z' := P_Y Z \approx_{\delta} Z$. Therefore $P_Z \tilde{H} P_{Z'} \preceq P_Z P_{Z'} P_{Z'} \preceq \delta$. So $Z'$ is a subspace of $Y$ where $\tilde{H}$ has energy at most $\delta$ which implies $\dim(Z') \leq \dim(\tilde{Z})$. Item 2 follows since $\dim(Z) = \dim(Z')$.

**Corollary 3.5.** Given $0 < \delta_{\text{goal}} < 1$ suppose $R \Delta \leq \frac{\delta_{\text{goal}}}{\theta}$. There exists a choice of parameters $V, \varepsilon, \delta$ such that with probability at least $1/2$ the output $\tilde{Z}$ of algorithm 1 satisfies $\tilde{Z} \approx_{\delta_{\text{goal}}} Z$ and such that the time complexity (and bond dimension of the output) is polynomial in $D \frac{R \delta}{\varepsilon}$.

**Proof.** By lemmas 3.3 and 3.4 we can take $V = \Theta(D \log(R \varepsilon) + \log w)$, $\varepsilon = \frac{1}{D}(\frac{\varepsilon}{\theta})^{O(1)}$, and $\delta = \delta_{\text{goal}}/2$.

Since $\dim(K_{[1,i]} V_{[1,i]}) \leq RV$ the bond dimension of the trimmed space $V_{[1,i]}$ is bounded by $RV/\varepsilon$ in each iteration. This bounds the bond dimension of $V_{[1,i-1]} \otimes \mathcal{H}_i$ at the beginning of each iteration by $d RV/\varepsilon$, and the same bound holds for the bond dimension of $V_{[1,i]}$. So the largest bond dimension encountered throughout the algorithm, that of $K_{[1,i]} V_{[1,i]}$ before trimming, is bounded by $d R^2 V/\varepsilon = (D Rw/\delta)^{O(1)} \cdot \log d$.

4 Constructing an implementable AGSP

The sub-exponentially implementable AGSP will be a straightforward modification of the AGSP $K(m, t, k)$ defined by Anshu, Arad, and Gosset to prove the subvolume law [AAG19]. We begin by recalling this AGSP, which we refer to as the subvolume law-AGSP.

4.1 The subvolume law-AGSP of [AAG19]

Let $t$ and $m$ be integer parameters. Define the narrow bands $B_i = (3it - 2t, 3it + 2t] \times [0, 2t]$ for $i = 0, 1, \ldots, \frac{N}{2t} + O(1)$. These are vertical bands of width $4t$ (except $B_0$) such that two neighboring bands have an overlap of width $t$.

Let $Q_i$ be the ground space projection for $H_{B_i}$. The AGSP of [AAG19] is based on the $t$-coarse grained detectability lemma operator [AALV09], $DL(t) = Q_{\text{odd}} Q_{\text{even}}$ where $Q_{\text{odd}} = \prod_i Q_i$ and $Q_{\text{even}} = \prod_i Q_i$. The AGSP construction replaces some factors $Q_i$ in $DL(t)$ with a polynomial in subsystem Hamiltonians $H_{B_i}$ to control the entanglement rank.

**Inner polynomial approximation.** Based on the Chebyshev polynomials, [AAG19] constructs step polynomials Step$(\cdot)$ of degree $\Theta(\sqrt{t}/\gamma)$ such that Step$(0) = 1$ and Step$([\frac{\gamma}{C t^4}, 1]) \subset [-\frac{1}{20}, \frac{1}{20}]$ where $C/4 = O(1)$ is a bound on the number of interactions involving a single qudit (so $\|H_{B_i}\| \leq C t$). Then $Q_i = \text{Step}(\frac{\gamma}{C t^4} H_{B_i})$ is an approximation to $Q_i$. More precisely, considering an eigenbasis for $H_{B_i}$ it is clear that $Q_i Q_i = Q_i$ and $\|Q_i Q_i - Q_i\| = \|Q_i - Q_i\| < 1/20$ where $Q_i = 1 - Q_i$.

**Outer polynomial approximation.** [AAG19] cleverly combine the (approximate) projections $Q_i$ on narrow bands using the robust AND polynomial $p_{\text{AND}}$ [She12], an $m$-variate polynomial of degree $O(m)$ with the property that $p_{\text{AND}}(\overline{1}) = 1$ where $\overline{1}$ is the tuple of $m$ ones, and $|p_{\text{AND}}(\overline{x})| \leq e^{-m}$ for all $\overline{x} \in ([-\frac{1}{20}, \frac{1}{20}] \cup \{1\})^m$ such that $\overline{x} \neq \overline{1}$.

Given a set $Z$ of indices $x_1 < \ldots < x_m$ such that $B_{x_1}, \ldots, B_{x_m}$ are disjoint, let $P(Z) = p_{\text{AND}}(Q_{x_1}, \ldots, Q_{x_m})$. 
Definition 4.1 ([AAG19]). Given some integer \(c\) representing the vertical cut a horizontal position \(3ct\), let \(\Xi = (c - m, c + m] \cap \mathbb{N}_{\text{odd}}\) be a set of \(m\) odd indices around \(c\) and let \(Q_{\text{rest}} = \prod_{i \notin \Xi} Q_i\) (with even-index \(Q_i\) on the right). Then the subvolume law-AGSP of [AAG19] is \(K(m, t, k) := (\tilde{P}(\Xi)Q_{\text{rest}})^k\).

\[ Q_{\text{rest}} \hspace{1cm} \tilde{P}(\Xi) \hspace{1cm} (\text{Diagram}) \]

Figure 2: The operator \(K(m = 5, t, k = 1)\) of [AAG19]. Short line segments represent coarse-grained projectors \(Q_i\) on narrow (width 4\(t\)) bands \(B_i\). Wavy line segments indicate inner approximations \(\hat{Q}_i\).

4.2 The implementable AGSP \(\tilde{K}\)

We now modify the AGSP \(K(m, t, k)\) such that we can simultaneously control the entanglement across every vertical cut. We denote the resulting AGSP as \(\tilde{K}(m, t, k)\) or simply \(\tilde{K}\), suppressing the dependence on the parameters. Define the wide bands \(B_j = (6(j - 1)mt, 6jmt] \times [1, h] \cap \mathbb{Z}^2\) for \(j = 1, 2, \ldots, w \sim \frac{w}{6mt}\). These are disjoint vertical bands of width 6\(mt\).

Definition 4.2. Let \(\Xi_i = (2(j - 1)m, 2jm) \cap \mathbb{N}_{\text{odd}}\) be the set of odd indices \(i\) such that the narrow band \(B_i\) is contained in \(B_j\). Define the implementable AGSP as

\[ \tilde{K}(m, t, k) = (\tilde{P}Q_{\text{even}})^k \quad \text{where} \quad \tilde{P} = \bigotimes_{i=1}^w \tilde{P}(\Xi_j). \]

\[ \tilde{P}(\Xi_0) \hspace{1cm} \tilde{P}(\Xi_1) \hspace{1cm} \tilde{P}(\Xi_2) \hspace{1cm} \tilde{P}(\Xi_3) \hspace{1cm} \tilde{P}(\Xi_4) \]

\[ Q_{\text{even}} \quad B_0 \quad \hdots \quad B_3 \quad B_4 \]

Figure 3: The modified operator \(\tilde{K}(m, t, 1)\).

4.3 Properties of \(\tilde{K}\) adapted from [AAG19]

The entanglement bound [AAG19] theorem 5.1 of the subvolume law-AGSP holds across every cut of the implementable AGSP.

Lemma 4.3 (By proof of [AAG19] theorem 5.1). Let \(m, t, k\) be at most polynomial in \(h/\gamma\). Then the Schmidt rank of \(\tilde{K}\) across any vertical cut is at most

\[ R = (hd/\gamma)^{O(mnth + k\gamma^{-1/2}\sqrt{h/t})}. \]

[AAG19] theorem 4.1 bounded the shrinking factor of the subvolume law-AGSP by \((e^{-m} + 2e^{-\Omega(t\sqrt{\gamma})})^{2k}\). By a similar argument one has:

Lemma 4.4. \(\tilde{K}\) is an AGSP with shrinking factor \(\Delta = (w'e^{-m} + 2e^{-\Omega(t\sqrt{\gamma})})^{2k}\), where \(w' \leq w\) is the number of wide bands.

Corollary 4.5. Suppose \(h = n^{\Omega(1)}\). For any \(\delta > 0\) there exists a choice of parameters \(m, t, k\) such that \(\tilde{K}(m, t, k)\) is a \(\Delta\)-AGSP with entanglement rank most \(R\) across each vertical cut, \(R\Delta \leq \delta\), and \(R = \delta^{-1} \exp \left[ O\left(h^2 \gamma^{-2} \log^5 \left(\frac{\Delta}{\delta}^2\right)\right)\right]\).

The proofs of lemmas 4.3 and 4.4 and corollary 4.5, adapted from [AAG19], are given in appendix A.1.
4.4 MPO for the implementable AGSP

We represent $\tilde{K}(m, t, k)$ by an MPO with $w$ tensors, each corresponding to a vertical column of qudits. Lemma 4.3 gives the existence of such an MPO with bond dimension $R$. However, we need not only for such an MPO to exist, but also for the MPO representation to be computable in subexponential time. Fortunately, this turns out to be easy:

**Lemma 4.6.** An MPO for $\tilde{K}(m, t, k)$ with a local tensor for each column of qudits can be constructed in time $(\text{hd}/\gamma)^{O(mth + k\gamma^{-1/2} \sqrt{h/t})}$.

**Proof.** We begin by constructing a coarser MPO $T$ for $\tilde{K}(m, t, k)$ with bond dimension $R$ where each physical index represents operators on a wide band $B_j$.

To construct $T$ begin by constructing explicit matrices for the operators $\tilde{P}(\Xi_j)$ in time $w d^{O(mth)}$. Since $\tilde{P}$ is product across $H_{B_1} \otimes \cdots \otimes H_{B_w}$, we get an MPO for $\tilde{K}(m, t, 1)$ with bond dimension $d^{O(th)}$ (from $Q_{\text{even}}$).

By lemma 4.3 the operators $\tilde{K}(m, t, k')$ with $k' \leq k$ satisfy a uniform bound $R$ on their entanglement rank across any vertical cut. Then, for $k' = 2, 4, \ldots, k$ (assuming $k \in 2N$ for simplicity), alternate between the following two steps:

1. Squaring $\tilde{K}(m, t, k') \leftarrow \tilde{K}(m, t, k' - 1)^2$.
2. Trim the bonds of the MPO for $\kappa^{k'}$ to its entanglement rank.

This concludes the construction of $T$. Finally replace each local tensor on $B_j$ with an MPO with bond dimension $R^2 \dim(H_{B_j})$.

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**Figure 4:** The coarse matrix product operator $T$ for the sub-exponentially implementable AGSP $\tilde{K}$.

We conclude by applying algorithm 1 to the implementable AGSP $\tilde{K}$.

**Corollary (Theorem 1.2).** Let $H$ be a frustration-free Hamiltonian with local gap $\gamma$ on the $w \times h$ lattice with $n = wh$ qudits. Suppose $w/h$ is at most polynomial in $n$. Let $D$ be a bound on the degeneracy and $\delta$ an accuracy parameter. Then there exists a probabilistic algorithm with time complexity $(D \delta)^{O(1)} \exp \left[ O \left( \frac{h^3 \gamma^{-2} \log \frac{\delta}{\sqrt{\gamma}}}{d^h} \right) \right]$ which outputs an MPS representing a subspace $\tilde{Z} \preceq H$ such that $\tilde{Z} \approx Z$ with probability at least $1/2$.

**Proof.** Let $H_i = H_{\{i\} \times [1, h]}$ for $i = 1, \ldots, w$ and let $d_i = \dim(H_r) = d^h$. Corollary 4.5 gives parameters such that $\tilde{K}$ is a $\Delta$-AGSP with an MPO of bond dimension $R$ such that $R \Delta \leq \frac{4}{\Delta^2}$ and $R = 6d^h \exp \left[ O \left( \frac{h^3 \gamma^{-2} \log \frac{\delta}{\sqrt{\gamma}}}{d^h} \right) \right]$, and we can absorb the factor $64d^h$. Apply corollary 3.5 to $\tilde{K}$. The time complexity is $(D \text{tr} d^h)^{O(1)}$ where we can again absorb $d_i$ in $R$, and we can absorb $w$ since $h = n^{O(1)}$.


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Appendix A

A.1 Proofs of properties of $\tilde{K}$ adapted from [AAG19]

Proof of lemma 4.3 (entanglement rank). It suffices to show the bound on the antentanglement rank across a cut through the middle of a wide band $B_j$. Indeed, any cut differs from some such cut by at most $O(mth)$ sites which can contribute only $d^{O(mth)}$ to the entanglement rank.

Adapt the proof of [AAG19] lemma 5.6 to the modified AGSP $\tilde{K}$ by replacing $Q_{\text{rest}}$ with $Q_{\text{rest}} := (\bigotimes_{j \neq j'} \tilde{P}(\Xi_j))Q_{\text{even}}$. Combining lemma 5.6 with corollary 5.5 (with $N = \frac{frk}{2mt}$ in lemma 5.6) yields the entanglement rank bound

$$R = d^{O(N + mth)}(h/\gamma)^{O(mt + N + k)},$$

where the factor $(h/\gamma)^{O(mt)}$ is from the prefactor of corollary 5.5. $f = \Theta(\sqrt{th/\gamma})$ is the degree of Step and $r = \Theta(m)$ the degree of $p_{\text{AND}}$, so $N = \frac{frk}{2mt} \propto \frac{18}{C}$.

Proof of lemma 4.4 (shrinking factor). By [AAG19] lemma 3.1, $\|DL(t)P_{Z^\perp}\| = 2e^{-\Omega(t\sqrt{\tau})}$. Moreover $\|\tilde{K}(m, t, 1) - DL(t)\| = \|(\tilde{P} - Q_{\text{odd}})Q_{\text{even}}\| \leq \|\tilde{P} - Q_{\text{odd}}\|$. Let $\tilde{P}_j = \tilde{P}(\Xi_j)$ and $P_j = \prod_{i \in \Xi_j} Q_i$ for $j = 1, \ldots, w'$ and write

$$\tilde{P} - Q_{\text{odd}} = \sum_{j=1}^{w'} \left( \bigotimes_{j' < j} \tilde{P}_{j'} \right) \otimes (\tilde{P}_j - P_j) \otimes \left( \bigotimes_{j' > j} P_{j'} \right).$$

By the proof of theorem 4.1 of [AAG19] it holds that $\|\tilde{P}_j - P_j\| \leq e^{-m}$ for each $j$, so $\|\tilde{P} - Q_{\text{odd}}\| \leq w' e^{-m}$ by the triangle inequality. Then,

$$\|\tilde{K}(m, t, 1)P_{Z^\perp}\| \leq \|DL(t)P_{Z^\perp}\| + \|\tilde{P} - Q_{\text{odd}}\| \leq w' e^{-m} + 2e^{-\Omega(t\sqrt{\tau})}.$$

Combining with each $\tilde{P}_j$ act as the identity of $Z$ takes $Z^\perp$ to itself, hence so does $\tilde{K}(m, t, 1)$. So $\tilde{K}(m, t, 1)$ is an $(w' e^{-m} + 2e^{-\Omega(t\sqrt{\tau})})^2$-AGSP and the result follows by raising to the $k$th power.

The choice of parameters for the implementable AGSP is as in [AAG19] for the subvolume law-AGSP. To motivate the relations between the parameters, note that balancing the terms in the shrinking factor bound of theorem 4.4 suggests choosing $m \propto \sqrt{T}$ so that $\Delta = e^{-\Omega(mk)} = e^{-\Omega(k\sqrt{\gamma})}$ if $m \geq 2 \log n$.

Proof of corollary 4.5 (tradeoff). Fix the relation $t \propto \gamma^{-1/2}m$ and let $m \geq 2 \log n$. Bounding the shrinking factor $\Delta$ using lemma 4.4 and the Schmidt rank of $\tilde{K}$ using lemma 4.3 we get $\Delta = e^{-mk/C}$ and

$$R = \left( \frac{hd}{\gamma} \right)^{O(mth + k\gamma^{-1/2} \sqrt{h/m})} \leq \left( \frac{hd}{\gamma} \right)^{C(m^2 \gamma^{-1/2} + k\gamma^{-1/2}) \sqrt{h/m}}.$$
for some large constant $C$. We will ensure that the parameters satisfy

$$C^{-1}mk \geq \log \frac{1}{\delta} + C(m^2 \gamma^{-\frac{1}{2}} h + k \gamma^{-\frac{1}{4}} \sqrt{h/m}) \log \left(\frac{h d}{\gamma}\right).$$

For this it suffices that

$$mk \geq 2C \log \frac{1}{\delta} \vee 4C^2 (m^2 \gamma^{-\frac{1}{2}} h \vee k \gamma^{-\frac{1}{4}} \sqrt{h/m}) \log \left(\frac{h d}{\gamma}\right).$$

(3)

Let $\tilde{C} = 4C^2 \log (hd/\gamma)$ and pick

$$m = \left\lceil \tilde{C} \frac{2}{\gamma} \gamma^{-\frac{1}{2}} \sqrt{\frac{2C}{h}} \log \frac{1}{\delta} \vee 2 \log n \right\rceil, \quad k = \lceil \tilde{C} m \gamma^{-\frac{1}{2}} h \rceil.$$

This choice ensures that $mk$ is larger than each of the two rightmost terms in (3). Moreover, expanding the expression for $k$,

$$mk \geq \tilde{C} m^2 \gamma^{-\frac{1}{2}} h \geq \tilde{C} \left(\gamma^{\frac{1}{4}} \sqrt{\frac{2C}{h}} \log \frac{1}{\delta} \right)^2 \gamma^{-\frac{1}{2}} h = 2C \log \frac{1}{\delta}.$$

So (3) is satisfied, hence $R \Delta \leq \delta$. The bound on $R$ follows from

$$\log R \leq C \left(m^2 \gamma^{-\frac{1}{2}} h + k \gamma^{-\frac{1}{4}} \sqrt{h/m}\right) \log \left(\frac{h d}{\gamma}\right)$$

$$= \log \frac{1}{\delta} + O(\gamma^{-\frac{1}{2}} h \log \frac{1}{\delta} + \gamma^{-\frac{1}{4}} h (\log n)^2 \log \left(\frac{h d}{\gamma}\right)).$$

Since $h \geq (\log n)^3$ we may absorb the last term in the middle term. □

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