Vortex condensation and black hole in the matrix model of 2-d string theory

Vladimir A. Kazakov

Abstract. We review recent results in the matrix model approach to the 2-d noncritical string theory compactified in time, in the phase of condensation of the world sheet vortices (above the Berezinski-Kosterlitz-Thouless phase transition) \[1, 2, 3\]. This phase is known to describe strings on the 2-d black hole background, due to the conjecture of V.Fateev, A. and Al.Zamolodchikov. The corresponding matrix model has an integrable Toda structure which allows to compute many interesting physical quantities, such as string partition functions of various genera and the 1- and 2-point correlators of vorticities of arbitrary charges. (Talk presented at Strings 2001, Mumbai, India)

1. Introduction

String theory formulated as a CFT on the world sheet can be perturbed, as any CFT, by marginal operators. The RG flows in the target space corresponding to these perturbations can bring the theory from the initial flat target space to the fixed points corresponding to curved backgrounds, such as black holes. Whereas in critical strings the task of describing such a flow is usually very difficult it can be feasible in the context of a “toy” noncritical string theory in two target space dimensions.

An example of such nontrivial flow was given in \[4\] who considered the so called Sine-Liouville (SL) theory with the Lagrangian

\[
L = \frac{1}{4\pi} \left[(\partial x)^2 + (\partial \phi)^2 - 2Q \hat{R} \phi + \lambda e^{b\phi} \cos R(x_L - x_R)\right].
\]

The central charge of this theory is \(c = 2 + 6Q^2\), the compactification radius of the bosonic field \(x = x_L + x_R\) fixed as \(R^2 = 2 + Q^{-2}\) and \(b = -1/Q\) for the Sine-Liouville perturbation to be marginal. The perturbation here is just the sum of two lowest winding modes (with winding charges \(\pm 1\)). For \(\lambda = 0\) this is a linear sigma model describing the simplest compactified version of the non-critical 2-d string theory. \[\] When one turns on \(\lambda\) the Sine-Liouville perturbation drives the theory to a new IR fixed point in the \((x, \phi)\) target space characterized by a high density of windings. As was conjectured by \[4\] this fixed point corresponds precisely to the

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Strictly speaking the Lagrangian \[1\] describes the usual non-critical \(c_M = 1\) 2-d string theory only for \(R = 3/2\) when \(c = 26\), unless one changes the matter field content.
level $k$ WZNW CFT on the coset $SL(2,C)/SU(2) \times U(1)$ with the central charge $c = \frac{3k}{k-2} - 1$. For $k = 9/4$ we have $c = 26$ and this sigma model describes the 2-d string theory in dilatonic Euclidean black hole (“cigar”) background.

Already superficially, both theories have many similar features: their asymptotic background at $\phi \to \infty$ is just a cylinder of the radius $R = \sqrt{k}$ in $(X, \phi)$ coordinates, so the asymptotic states are the same. In the Sine-Liouville theory they are defined by the vertex operators:

$$V_{j,n_1,n_2} = \exp[ip_L x_L + ip_R x_R + 2Q j \phi]$$

with $P_{L,R} = R^{-1}n_1 + Rn_2$, where $n_1$ and $n_2$ are arbitrary integers. These operators have the conformal dimensions:

$$\Delta_{j,n_1,n_2} = -\frac{j(j+1)}{R^2} + \frac{1}{4} \left(R^{-1}n_1 + Rn_2\right)^2$$

These dimensions coincide with the dimensions of the corresponding operators on the coset model side.

It was also noticed by [4] that both theories have coinciding 2- and 3-point correlators of these operators (some of them were already calculated in [5] and [6]).

Qualitatively, the reason for the same physics in these two theories lies in the fact that both have a “wall” in the target space which prevents the propagation of the states beyond a certain point along the axis of the cylinder: in the cigar model this wall corresponds just to the tip of the cigar, whereas in the theory (1.1) it is the SL potential which plays the role of the “wall”. The theory is strongly coupled in the region $\phi < 0$, and although the SL potential is penetrable classically (due to the cos oscillations) it appears to be unpenetrable due to the quantum effects.

The conjectured equivalence is an example of the strong-weak duality: at large $k$ the curvature of the cigar becomes small, thus the theory is weakly coupled, whereas the SL wall becomes steeper and the interaction is concentrated in the strong coupling region near this wall.

A strong piece of evidence of this duality was brought in the recent paper [7] where the equivalence was proven for the $N = 2$ supersymmetric analogues of these models and appeared to be an example of the mirror symmetry.

We conclude that although the straightforward proof of the conjecture of [4] is still missing in the bosonic case we have enough of evidence to view the SL theory (1.1) as an alternative model for the description of the dilatonic black hole (at least at $R = 3/2$).

On the other hand, the usual Liouville theory at the central charge $c = 26$ is known to be very effectively described and studied in the matrix quantum mechanics (MQM) formulation of the noncritical 2-d ($c_M = 1$) string theory. The SL perturbation amounts to introducing the vortices of the charge $\pm 1$ on the world sheet of the string (similar to the vortices in the usual Sine-Gordon model). Since we know how to implement vortices on the MQM language [8, 9, 10] we can formulate and study the strings in the 2-d black hole background by means of powerful matrix methods.
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2. Black hole as a perturbation of \( c_M = 1 \) string

Let us turn to the noncritical 2-d bosonic string perturbed by the SL term, described by the Lagrangian

\[
L = \frac{1}{4\pi} \left[ (\partial x)^2 + (\partial \phi)^2 - 4\hat{R}\phi + \mu \phi e^{-2\phi} + \lambda e^{(R-2)\phi} \cos R(x_L - x_R) \right]
\]

Here the parameters are chosen in such a way that the total central charge is always \( c_{\text{tot}} = 26 \) and the SL term and the cosmological term are marginal to enforce the conformal invariance, whereas the compactification radius \( R \) can be arbitrary. Note that at \( R = 3/2 \) and zero cosmological constant \( \mu = 0 \) we have again the SL theory (1.1) at the black hole point \( Q = 2 \). Hence the theory (2.1) is just a different (comparing to the WZNW or the SL models) deformation from the conventional 2-d black hole with respect to the parameters \( \mu \) and \( R \).

From the dependence of the Lagrangian (2.1) on the zero mode of the Liouville field \( \phi \) we can immediately read off the target space scaling of physical quantities, such as the SL coupling \( \lambda \) (the fugacity of vortices) and the string coupling \( g_s \), with \( \mu \):

\[
\lambda \sim \mu^\frac{2-8}{R} , \quad g_s \sim 1/\mu.
\]

From the matrix model results \[11\] we know that the free energy for \( \lambda = 0 \) has the following expansion in genera:

\[
F(\mu, \lambda = 0) = \log Z_{\mu}[0] = \frac{1}{2\pi} \int_{-\infty}^{\infty} dE \sum_{k=0}^{\infty} \frac{k + \frac{1}{2}}{E^2 + (k + \frac{1}{2})^2} \log(1 + e^{-2\pi R(\mu - E)}) = -\frac{1}{2R} \mu^2 \log \mu - \frac{1}{24} (R + \frac{1}{R}) \mu + R \sum_{h=2}^{\infty} \frac{1}{2^{(h-1)}} g_h(R) + O(e^{-2\pi \mu}),
\]

where we introduced the polynomials in \( \frac{1}{R} \)

\[
g_h(R) = (2h - 3)! \frac{1}{2^{2h}} \sum_{n=0}^{h} \left( \frac{1}{R} \right)^n \frac{2^{2(h-n)} - 2^{2n} - 2(B_{2(h-n)} B_{2n})}{[2(h-n)]!![2n]!!},
\]

and \( B_m \) are Bernoulli numbers. The partition function (2.3) has a T-duality symmetry, \( R \to \frac{1}{R} \), \( \mu \to R\mu \). This symmetry will be broken in the presence of vortices (i.e. for \( \lambda \neq 0 \)).

The first two terms of \( 1/\mu \) expansion in (2.3) corresponding to the sphere and torus partition functions of the 2-d string theory are known as well from the direct calculations in the conformal \( \sigma \)-model (2.1) (see, for example, \[11\]).

Using this result and the scaling (2.2) we can represent the genus expansion for the free energy (“string partition function”) \( F(\lambda, \mu) = \sum_{h=0}^{\infty} F_h(\lambda, \mu) \) in 2-d string theory already for \( \lambda \neq 0 \) in the form:

\[
F_0(\lambda, \mu) = -\frac{R}{2} \mu^2 \log \mu + \mu^2 A_0(z),
\]

\[
F_1(\lambda, \mu) = -\frac{1}{24} (R + \frac{1}{R}) \log \mu + A_1(z), \quad F_h(\lambda, \mu) = \mu^{2-2h} A_h(z),
\]

where \( z = \sqrt{R - 1}\mu \frac{1}{\sqrt{R}} \) and is, according to the scaling (2.2), a dimensionless parameter. It is clear that \( A_h(0) = f_h(R) \) from (2.3).
The black hole limit of (2.5) corresponds to \( z \to \infty \). In this limit it is rather the SL term in (2.1) and not the cosmological term which governs the size of the worldsheet. It means that the gas of vortices generated by the SL interaction becomes dense. So the creation of the black hole is closely related to the condensation of vortices. It is known \([8, 9, 10]\) that vortices start condensing for \( R < \frac{2}{\sqrt{N}} \), i.e. for the temperatures above the Berezinski-Kosterlitz-Thouless (BKT) phase transition \( (R_{BKT} = 2) \). The 2-d black hole point \( (R = 3/2) \) lies already in the phase of condensing vortices.

In order to be able to study the black hole physics we have to find the values of coefficients \( A_h(z) \) in (2.5), or at least their limiting values \( A_h(\infty) \). For that we will use the matrix model techniques and the integrability properties of the corresponding MQM.

3. Matrix quantum mechanics and vortices on the worldsheet

To describe the gas of vortices on the worldsheets of the 2-d string we will use the MQM defined by the partition function

\[
Z_N(\Omega) = \int_{M(2\pi R = \Omega, M(0))/\Omega} D^N M(x) e^{-\text{tr} \int_0^{2\pi R} dx \left[ \frac{1}{4} (\partial_x M)^2 + V(M) \right]} \tag{3.1}
\]

where \( M(x) \) is a Hermitian \( N \times N \) matrix field depending on the compact “time” \( x \), \( \Omega \in SU(N) \) is a twist matrix and the matrix potential can be chosen, for example, as a cubic polynomial: \( V(M) = \frac{1}{2} M^2 - \frac{g}{3\sqrt{N}} \). Due to the periodic boundary condition the theory is compactified on the time circle of the length \( 2\pi R \), as the coordinate \( x = x_L + x_R \) of the model (2.1).

The planar Feynman graphs generated by the MQM can be viewed as discretized world sheets of the \( c = 1 \) noncritical string theory \([12]\). The time \( x \) represents one of the two target space coordinates of the theory (another one - the Liouville coordinate - being hidden into the matrix structure of the theory (see \([25, 26]\)). In the compactified version of the model the vortex excitations can appear on the discretized worldsheets, in analogy with the model of planar rotators on a regular lattice used in statistical mechanics to describe the BKT phenomenon. The propagators (corresponding to the double index lines) look as follows

\[
x_v^i \ x_v^j = \sum_{m=\infty}^{\infty} \exp(-|x_v - x_v' + 2\pi R m|) (\Omega^m)^{ij}(\Omega^{-m})^{kl}, \tag{3.2}
\]

where \( x_v \) is a coordinate of the graph vertex \( v \) on the \( x \)-circle and \( m \) is the number of windings of the propagator around the circle (in a given term in the sum).

Applying the Feynman rules for the double-lined graphs we obtain the following formal expression for the free energy \( F_N(\Omega) = \log Z_N(\Omega) \) as a sum over graphs \( G_n^{(h)} \) and their genera \( h \) and sizes \( n \) (the number of cubic vertices in the graph):

\[
F_N(\Omega) = \sum_{h=0}^{\infty} N^{2-2h} \sum_{n=2}^{\infty} g^n \sum_{G_n^{(h)}} \prod_{f \in G} \frac{\text{tr} \Omega^w}{N} \sum_{\{m_{\nu \nu'}\}} \prod_{\nu \in G} d x_v \prod_{\nu \nu'} \exp\left(-\left(|x_v - x_v' + 2\pi R m_{\nu \nu'}|\right)\right), \tag{3.3}
\]

where \( w_f = \sum_{\ell \in \partial f} m_\ell \) is the vorticity flowing through the face \( f \) of a graph.
The partition sum on any fixed graph \( G^{(h)} \) (including the sum over \( m \)’s and integrals over \( x_v \)’s with the last exponential factor) resembles very much the Villain model \([13]\). Originally the Villain model was formulated on a regular square lattice and one had a square of the expression in the exponent. However, the most important feature to retain here is the periodicity in \( x_v \) variables, so both models should be in the same class of universality (different of course for regular and random graphs). We conclude that the MQM (3.1) describes the gas of vortices living on faces of dynamical lattices. The weights \( tr\Omega_n^m \) play the role of fugacities of vortices with different charges \( w_f \).

The continuum limit of the model corresponds, as usually, to tuning \( g \to g_c \) where the characteristic graphs become very big and play the role of continuous world sheets of the Polyakov string. The sum over graphs simulates the sum over 2-d metrics, and the integrals over \( x_v \) represent the functional integral over the bosonic field.

In the CFT language, we can represent the analogous model (at least in the weak coupling regime) by the Lagrangian

\[
L = \frac{1}{4\pi}[(\partial x)^2 + (\partial \phi)^2 - 4\hat{R}\phi + \mu \phi e^{-2\phi} + \sum_{n\neq 0} t_n e^{(ln|R-2|)\phi} e^{ln(R(x L-x R))}].
\]

where \( t_n \sim tr\Omega^n \) up to some subtleties discussed below. The Lagrangian (3.4) generalizes (3.3) to the perturbations by vortices of arbitrary charge. However, as we see from the Liouville field dependence of the \( n \)’th term in (3.4), only the vortices of charge \( \pm 1 \) are relevant in the target space in the interval \( 1 < R < 2 \) which we will be interested in.

**4. Integration over the twist angles**

As we will see later, it is advantageous instead of fixing the twist variable \( \Omega \) to integrate over it with a special weight:

\[
Z_N[\lambda] = \int [D\Omega]_{SU(N)} \exp \left( \sum_{n\in\mathbb{Z}} \lambda_n tr\Omega^n \right) Z_N(\Omega).
\]

In this way we change the original matrix variable \( \Omega \) to the infinite set of new variables \( \lambda_n \), \( n \in \mathbb{Z} \). We see from (4.1) that under the \( U(1) \) shift \( \Omega \to e^{i\alpha} \Omega \) the couplings \( \lambda_n \) transform as \( \lambda_n \to e^{-i\alpha n} \lambda_n \), i.e. in the same way as the fugacity of a vorticity \( \frac{tr\Omega^n}{N} \) in a face. This suggests that the couplings \( \lambda_n \) play the same role in the MQM formalism as the couplings \( t_n \) in the Lagrangian (3.4), i.e. \( \lambda_n \sim t_n \) in the continuous limit. A more precise statement is based on the observation made in [14] who noticed that for large enough \( N \) and \( \lambda_n \sim N \)’s the integration rules for the unitary matrices can be written as

\[
\int D\Omega_{SU(N)} e^{\sum_{n\neq 0} \lambda_n tr\Omega^n} = e^{\sum_{n\neq 0} n\lambda_n} + O(e^{-N}).
\]

So the variables \( tr\Omega^n \) look like independent Gaussian variables in this approximation.

If we substitute (3.3) into (4.1) and start integration over \( \Omega \) according to the rules (4.3) we will get 3 kinds of contributions: first, the terms in the exponent in (4.1) will couple to each other and give a trivial overall factor equal to the r.h.s. of (4.2); second, the \( tr\Omega^n \) will couple to \( \frac{tr\Omega^n}{N} \) (at \( w_f = -n \)) which will result
in the substitution $\text{tr} \Omega w \rightarrow \lambda w$, thus giving to $\lambda$ the meaning of the fugacity of vortices of charge $n$; and third, $\text{tr} \Omega w$ in (3.3) will couple to each other. This last contribution is not yet very well understood (see [15] for the details). On the one hand, one can give some arguments that it is inessential in the double scaling limit considered below; on the other hand, it gives some new configurations of planar graphs (various remote faces can glue together in this way) which might be important to the right combinatorics of the discretized “worldsheets”.

We conclude this section by stating that the partition function (4.1) is a good candidate for the matrix model description of the 2-d string theory with vortices on the world sheet. The couplings $t_n$ of vortex operators of charge $n$ are proportional to $\lambda^n$.

5. Double scaling limit

The MQM formulated in the previous section is a complicated theory. It would be hard to try to perform the integrals over $M(x)$ or over $\Omega$ directly in (3.1). On the other hand, we can significantly simplify the model by going to the double scaling limit which means that we are not interested any more in the detailed structure of the planar graphs but rather want to concentrate ourselves on very big graphs giving a good approximation to the smooth worldsheets. In the matrix action it corresponds to the fact [16] that the only relevant part of the potential in (3.1) is concentrated around the maximum. If one shifts $M \rightarrow M + 1/g$ the quadratic term reverses the sign and the cubic term plays the role of a steep wall at a distance $\sim \sqrt{N}$ from the top of the quadratic maximum. Thus we obtain instead of (3.1) a MQM model of the “inverted oscillator”, i.e. with the upside-down quadratic potential:

$$Z_N(O) = \int_{M(2\pi R) = M(0) \Omega} D^{N^2} M(x) e^{-\text{tr} \int_0^{2\pi R} dx \left[ \frac{1}{2} (\partial_x M)^2 - \frac{1}{2} M^4 \right]}.$$  

We can calculate the functional integral over $M(x)$ by formally continuing (5.1) analytically to the usual stable harmonic oscillator and than changing $R \rightarrow iR$ in the final answer. One can do the integration by the mode expansion on the circle separately for each matrix element $M_{kj}(x)$ (they are completely decoupled before the integration over $\Omega$) with the twisted boundary condition $M_{kj}(2\pi R) = z_j z_k M_{kj}(0)$ ($k, j$ are fixed), where we took, without a loss of generality, the twist matrix in the diagonal form $\Omega = \text{diag}(z_1, z_2, \ldots, z_N)$.

It is also natural in the double scaling to do the grand canonical transformation of (4.1) passing from fixed $N$ to fixed conjugated variable - chemical potential $\mu$. The final formula for the partition function of the model in terms of the integral over twist angles (taken with the Haar measure for the Cartan subgroup $\frac{1}{N!} \int \prod_{k=1}^N \frac{dz_k}{2\pi i z_k} \prod_{m>j} |x_m - x_j|^2 \cdots$) looks as follows

$$Z[\mu, \lambda] \equiv e^{F(\mu, [\lambda])} = \sum_{N=1}^\infty \frac{e^{-2\pi R \mu N}}{N!} \int \prod_{k=1}^N \frac{dz_k}{2\pi i z_k} \left( e^{i\pi R} - e^{-i\pi R} \right) \prod_{j \neq j'} \frac{e^{u(z_j)}}{e^{i\pi R} z_j - e^{-i\pi R} z_j},$$

where $u(z) = \sum \lambda_n z^n$.

The integrals in this formula look very divergent because of the poles in the last product. Hence we will employ a strategy different from the direct calculations of
these integrals and rather will obtain a set of differential equations on this partition function.

6. The partition function as a \( \tau \)-function of Toda integrable hierarchy

It is easy to recognize in (5.2) a particular case of the \( \tau \)-function of Toda integrable hierarchy \( \tau_l[t] \) defined on the fermionic vacuum of charge \( l \), as it was done in \[17\]. For that it is sufficient to compare it with the general solitonic solution of this hierarchy given, for example, in \[18\]. The relation looks as follows:

\[
Z(\mu - il, [\lambda]) = e^{\sum_n n t_n - n \tau_l[t]}
\]

where \( \lambda_n = 2i \sin(\pi n/R) t_n \). Note that the dependence on \( l \) is absorbed into the imaginary part of \( \mu \).

The Toda \( \tau \)-function satisfies a set of Hirota bilinear equations which can be encoded into the identity

\[
\begin{align*}
\int_{C_0} \frac{dz}{2\pi i} z^{l-1} & \exp \left( \sum_{n>0} (t_n - t_n') z^n \right) \tau_l(t - \tilde{\zeta}^+) \tau_l(t' + \tilde{\zeta}^+) = \\
\int_{C_0} \frac{dz}{2\pi i} z^{l-1} & \exp \left( \sum_{n<0} (t_n - t_n') z^{-n} \right) \tau_{l+1}(t - \tilde{\zeta}^-) \tau_{l-1}(t' + \tilde{\zeta}^-) = 
\end{align*}
\]

where \( \tilde{\zeta}^+ = (\ldots, 0, 0, z^{-1}, z^{-2}/2, z^{-3}/3, \ldots) \), \( \tilde{\zeta}^- = (\ldots, z^3/3, z^2/2, z, 0, 0, \ldots) \). Expanding in \( y_n = t_n' - t_n \) we obtain an infinite hierarchy of soliton partial differential equations. The lowest, Toda equation, which is the most important for us, is obtained as the coefficient in front of \( y_{-1} \):

\[
\partial \tau_l \partial_{-1} \tau_l - \partial \tau_l \partial_{-1} \tau_l + \tau_{l+1} \tau_{l-1} = 0.
\]

Written directly in physical variables \( \mu, \lambda \pm 1 \) for the free energy \( F(\mu, [\lambda]) \) it looks as

\[
\frac{\partial}{\partial \lambda_{+1}} \frac{\partial}{\partial \lambda_{-1}} F(\lambda, \mu) + \exp [F(\lambda, \mu + i) + F(\lambda, \mu - i) - 2F(\lambda, \mu)] = 1.
\]

This equation describes the dynamics of vortices of the lowest charges \( n = \pm 1 \) and hence is appropriate for studying the 2-d string with the Sine-Liouville type interaction \[21\]. We will put for the moment \( \lambda_n = 0 \), \( n = \pm 2, \pm 3, \ldots \) and denote \( \lambda_{\pm 1} = e^{\pm i\alpha} \lambda \). Note also that due to the vortex charge neutrality of the system the free energy does not depend on the phase \( \alpha \) and is only a function of two variables \( \mu \) and \( \lambda \) (and a parameter \( R \)) which we denote just as \( F(\mu, \lambda) \). It satisfies the eq.

\[
\frac{1}{4} \lambda^{-1} \partial \lambda \partial \lambda F(\mu, \lambda) + \exp [F(\mu + i, \lambda) + F(\mu - i, \lambda) - 2F(\mu, \lambda)] = 1.
\]

This equation should be supplied by an appropriate boundary condition. It is known \[13\] that in the double scaling limit \( F(\mu, \lambda = 0) \) as defined in (5.2) is given by the Gross-Klebanov formula \[23\] with the cosmological coupling \( \mu \) expressed in the matrix model through the chemical potential of the matrix eigenvalues which appear to be free fermion coordinates in the upside down matrix potential in this case. So we can take (5.3) as a boundary condition. Another necessary boundary condition stems from the fact that \( F(\mu, \lambda) \) has a regular \( \lambda^2 \) expansion at \( \lambda = 0 \) since it is just an expansion in the number of vortex-antivortex pairs.

Let us note here that the eq. (6.4) has zero modes for any coefficients \( C_n, D_n \):

\[
\Delta F = \sum_{n=0} \left( C_n + D_n \log \lambda \right) e^{-2\pi \mu n}.
\]
It means that nonperturbatively this equation does not fix the whole solution. The similar \( O(e^{-2\pi\mu}) \) terms are present in the boundary condition (2.3), together with the terms \( O(e^{-2\pi\mu R}) \) following from the T-duality of (2.3).

7. Partition functions for fixed genera

We are ready now to calculate \( F(\mu, \lambda) \), at least perturbatively in powers of \( g_s \sim 1/\mu \). The finite difference operator in (6.4) will be understood as an expansion in powers of \( \partial_\mu \) as well. For example, in the spherical limit the eq.(6.4) becomes

\[
\frac{1}{4} \lambda^{-1} \partial_\lambda \lambda \partial_\lambda F(\mu, \lambda) + \exp \left[ -\partial_\mu^2 F(\mu, \lambda) \right] = 1. \tag{7.1}
\]

Plugging the expansion (2.5) into (6.4) and analyzing it order by order in \( 1/\mu \) we obtain the following results for the individual genera:

(i) Genus 0

\[
\partial_\mu^2 F_0(\mu, \lambda) = -\frac{2R}{2-R}(\log(\lambda\sqrt{R-1}) + X_0(y)) \tag{7.2}
\]

where \( X_0(y) \) satisfies the equation

\[
y = e^{-\frac{1}{4R}X_0} - e^{\frac{1}{4R}X_0}. \tag{7.3}
\]

Using this solution we can expand \( F_0(\mu, \lambda) \) in powers of \( \lambda^2 \mu^2 \), thus encountering the black hole limit of our model:

\[
F_0(\lambda, \mu) = -\frac{(2-R)^2}{R-1} \frac{\lambda^4}{\lambda^{2R}} \tag{7.4}
\]

(ii) Genus 1

Proceeding with the \( 1/\mu \) expansion in (6.4) and (2.3) we obtain the following torus partition function:

\[
F_1(\lambda, \mu) = \frac{R+R^{-1}}{24} \left( R^{-1}X_0(y) - \frac{2}{2-R} \log(\lambda) \right) - \frac{1}{24} \log \left( 1 - (R-1)e^{\frac{2\pi R}{R}X_0(y)} \right) \tag{7.5}
\]

(iii) Equation for any genus \( h \geq 2 \)

Note that for \( R < 1 \) the singularity defining the convergence radius of this series in terms of \( z \) variable becomes real; as was explained in \([20]\) this critical point corresponds to the trivial \( c_M = 0 \) in the IR regime. However it is not the case for the \( 1 < R < 2 \) interval where we will encounter a new “black hole” phase.
In the same way one can obtain a linear equation relating the genus $h$ partition function $F_h(\lambda, \mu) = \tilde{\lambda}^{4h} f_h(y)$ to the partition functions of all lower genera:

$$ (y \partial_y + 2h - 2)^2 f_h - e^{-X_0(y)} \partial_y^2 f_h = H(f_0, \ldots, f_{h-1}). $$

(7.7)

where $H(f_0, \ldots, f_{h-1})$ is a known function (see [1] for the details). The l.h.s. of this equation expressed in terms of the variable $\xi = e^{-\frac{1}{\lambda} X_0(y)}$ represents a linear 2-nd order differential operator with polynomial coefficients.

8. Black hole limit and thermodynamics

As was explained in the previous sections, the standard 2-d black hole should correspond to $\mu = 0$ and $R = 3/2$. We will impose the first condition but leave the radius $R$ arbitrary, to see the general properties of the “black hole” phase $1 < R < 2$ and to be able to discuss the temperature $(T = \frac{1}{2\pi R})$ dependence of the string partition function. For $\mu = 0$ we have from the results of the previous section the following genus expansion for the free energy

$$ F(\lambda, \mu = 0) = -\frac{1}{4} (2 - R)^2 \frac{\lambda^{4h}}{R - 1} - \frac{R + R^{-1}}{48} \log \left( \tilde{\lambda}^{4h} \right) + \sum_{h=2}^{\infty} \frac{\lambda^{4h}}{R - 1} f_h(0) \quad (8.1) $$

It would be very interesting to find the universal coefficients of expansion $f_h(0)$. For that we have to solve the equations (7.7) genus by genus, using as an input the constants $g_h(R) = \lim_{y \to \infty} y^{2h-2} f_h(y)$. Note that the coefficient of the first term in the r.h.s. is also universal, as a consequence of our approach based on the Toda equation.

For the proper black hole case we set $R = 3/2$ and notice that according to the hypothesis of [4] we have the following relation between the black hole mass $M$ and the SL coupling $\lambda$ (in appropriate units): $M = \tilde{\lambda}^8$. It means that the genus expansion of the free energy (8.1) takes the form of the inverse mass expansion

$$ F(\lambda, \mu = 0) = -\frac{1}{8} M - \frac{13}{288} \log M + \sum_{h=2}^{\infty} M^{1-h} f_h(0) \quad (8.2) $$

The fact that the sphere partition function given by the first term is nonzero seems to contradict the old result obtained from the effective dilaton gravity action for the 2-d string theory (see [2] and references therein). The Euclidean space-time action evaluated on the black hole background is divergent due to linear dilaton vacuum contribution, and to extract its universal finite part (corresponding to the dropped non-universal terms regular in $\lambda$ in the MQM free energy) one has to fix a subtraction procedure. The thermodynamic approach of [21] based on the leading $\alpha'$ order background (see also [2] for its refinement based on the exact black hole background) consists in subtracting the vacuum contribution for fixed values of temperature $T$ and dilaton charge at the "wall". It gives $S = M/T$ for the entropy and zero value for the free energy $F$. It was suggested in [2] that in order to establish the correspondence with a non-vanishing matrix model result for $F$ one may need an alternative reparametrization-invariant subtraction procedure using analogy with non-critical string theory (i.e. replacing the spatial coordinate by the dilaton field). The subtraction of the dilaton divergence then produces a finite value for the free energy.

It was also proposed in [2] a microscopic estimate for the entropy and energy of the black hole based on the contribution of non-singlet states of the matrix model.
9. One- and two-point correlators from the Toda hierarchy

The Toda integrability properties discovered in our matrix model and summarized in Hirota bilinear relations (6.2) can be used to find the correlators of vortices of arbitrary charges defined as

\[ \tilde{K}_{i_1 \cdots i_n} = \left. \frac{\partial^n}{\partial t_{i_1} \cdots \partial t_{i_n}} \log \tau_0 \right|_{t_{\pm 2} = t_{\pm 3} = \cdots = 0}. \]

with \( \mu \) and \( t_1 = -t_{-1} \) fixed. We will give here the results for the one- and two-point correlators in the spherical (dispersionless) limit computed in [3]. Define the generating functions of the two-point correlators

\[ H_\pm (a, b) = \sum_{m,n=0}^{\infty} a^m b^n \tilde{Y}_{\pm m,n} \]

where we distinguish by the \( \pm \) subscript the cases with vorticities of equal or opposite sign, and of one-point correlators

\[ h(a) = H_\pm (a, 0) = \sum_{n=0}^{\infty} a^n \tilde{Y}_{0,n} \]

with \( \tilde{Y}_{0,0} = 0 \) and

\[ \tilde{Y}_{0,n} = -\frac{1}{|n|} \frac{\partial^2}{\partial t_n \partial \mu} F_0, \quad \tilde{Y}_{n,m} = 2 \frac{1}{|n||m|} \frac{\partial^2}{\partial t_n \partial t_m} F_0. \]

The two-point correlators can be expressed in terms of one-point correlators by direct analysis of Hirota equation (6.2), as was done in [3], or as the correlators of free scalar field used to formulate the Hirota identity (see the Appendix of [1] for the details):

\[ H_+ (a, b) = \log \left[ \frac{4ab}{|a-b|^2} 2^n \left( \frac{1}{2} (h(a) - h(b) + \log \frac{a}{b}) \right) \right] \]

\[ H_- (a, b) = 2 \log \left( 1 - Aabe^{h(a)+h(b)} \right), \]

with \( A = \exp(-\partial_\mu^2 F_0) \).

It is interesting that we can obtain the generating function of one-point correlators \( h(a) \) only by knowing the two point correlators (9.5) and the dependence of the free energy from \( t_{\pm 1} \) given by (7.2)-(7.3). Due to this we can express \( \tilde{Y}_{\pm 1,n} \) through \( \tilde{Y}_{0,n} \) by linear integral-differential operators and write a closed equation on \( h(a) \). The solution of it is given by a simple algebraic equation

\[ e^{h} - z e^{-h} = 1 \]

where \( z = a \frac{\lambda^R}{\sqrt{R}} e^{-\frac{n(R-1)}{2} X_0(y)} \) and \( X_0(y) \) is defined by (7.3). This yields the following explicit formula for the one point correlators

\[ K_n = \frac{(2 - R)n \Gamma(n(R + 1))}{(n + 1)! \Gamma(n(R - 1) + 2) (R - 1)^{n/2}} \times \]

\[ \times \left[ \frac{(n + 1)}{(2 - R)n} e^{-\frac{n(R-1)+2}{2} X_0} - \frac{(n(R - 1) + 1)}{(2 - R)n} e^{-(n+1)\frac{2}{R} X_0} \right]. \]

\(^3\text{We thank I. Kostov for this comment}\)
Note that $K_n \sim \lambda^{\frac{|n|}{2R}} \sim \mu^{|n|/R+1}$ in perfect agreement with the scaling of couplings $t_n \sim \mu^{-\frac{1}{2}|n|R+1}$ in (3.4).

In the “black hole” limit $y \to 0$ the expression in the square brackets equals 1 and the formula simplifies significantly. A rather explicit expressions can be given also for the two point correlators.

These correlators have to be compared (for the black hole case $R = 3/2$) to the correlators calculated in [6, 4, 5]. Due to the problems of identification of operators in the matrix model and CFT formulation it is not yet done.

10. Conclusions

We presented here a matrix model describing the 2-d string theory compactified on the circle of a radius $R$ ($R_{selfdual} < R < R_{KT}$) with a dense gas of vortices on the world sheets. We uncovered the Toda integrable structure of the theory and used the corresponding Toda-Hirota equations to analyze the free energy for various genera, as well as one- and two point correlators of vortices on the sphere. We described the limit of the theory, corresponding (due to the duality of [4]) to the two dimensional string theory on the dilatonic black hole background. We found that the free energy in this limit was finite and proportional to the mass of the black hole.

There are many interesting and mysterious questions related to our approach:

1. The Toda integrable structure found in our MQM approach was never discovered in the corresponding CFT [3,4]. The closest observation on the CFT side was the $W_\infty$ algebra of discrete states leading to the topological description of the 2-d string theory on the self dual radius $R = 1$ [22] giving rise to the Toda description [23]. Our results (and the old conjecture about the Toda structure in the dual version of the 2-d theory [24]) suggest the existence of such an integrability at any $R$ in the corresponding CFT as well.

2. The supersymmetric version of the 2-d string escapes for the moment the similar Toda-like description, although the corresponding supersymmetric CFT is very similar to its bosonic counterpart.

3. The matrix approach is very promising for the clarification of space time physics and of thermodynamics of the 2-d black hole since we know the Hamiltonian of MQM and the origin of a large amount of states responsible for the great entropy of the underlying black hole [4]. However the practical calculations are still missing due to some technical problems. An important step on this way would be the generalization of the effective action of Das-Jevicki-Polchinski to the case of a nonzero vortex density. It would also help to see the target space metric of the problem and to describe the tachyonic scattering.

4. It should be also possible in our matrix approach to get a nonperturbative information about the black hole, trying to analyze the Toda equation (6.3) with the boundary condition (2.3) by the methods different from the genus expansion used until now.

References

4They are due to the “angular” variables of the matrix coordinate excited in higher representations, as opposed to the singlet representation where only the eigenvalues of the matrix are relevant.
[1] V. Kazakov, I. Kostov and D. Kutasov, A Matrix Model for the Two Dimensional Black Hole, hep-th/0101011.

[2] V.A. Kazakov, A.A. Tseytlin, On free energy of 2-d black hole in bosonic string theory, hep-th/0104158.

[3] S. Yu. Alexandrov and V. A. Kazakov, Correlators in 2D string theory with vortex condensation, hep-th/0104094.

[4] V. Fateev, A. Zamolodchikov and Al. Zamolodchikov, unpublished.

[5] P. Baseilhac and V. Fateev, Nucl. Phys. B532 (1998) 567.

[6] J. Teschner, hep-th/9712256, Nucl. Phys. B546 (1999) 390; hep-th/9712258, Nucl. Phys. B546 (1999) 369; hep-th/9906211, Nucl. Phys. B571 (2000) 555.

[7] K. Hori, A. Kapustin, Duality of the Fermionic 2d Black Hole and N=2 Liouville Theory as Mirror Symmetry, hep-th/0104207.

[8] D.Gross and I.Klebanov, Nucl.Phys. B344 (1990) 475.

[9] D.Gross and I.Klebanov, Nucl.Phys. B354 (1990) 459.

[10] D. Boulatov and V. Kazakov, One-Dimensional String Theory with Vortices as Upside-Down Matrix Oscillator, preprint LPTENS 91/24 (1991), Int. J. Mod. Phys. A8 (1993) 809, revised version: hep-th/0012228.

[11] I. Klebanov, String Theory in Two Dimensions proceedings of the ICTP Spring School on String Theory and Quantum Gravity, Trieste, April 1991, hep-th/9108014.

[12] V.A.Kazakov and A.A.Migdal, Nucl.Phyus. B311 (1988) 171.

[13] J.Villain, J.Phys. C 36 (1975) 581.

[14] M. R. Douglas, “Conformal theory techniques in Large N Yang-Mills Theory”, talk at the 1993 Cargèse meeting, hep-th/9311130.

[15] V. Kazakov, I. Kostov and D. Kutasov, Matrix model of the (1+1) dimensional dilatonic black hole in the double scaling limit, JHEP proceedings of TMR Conference "Non-perturbative Quantum Effects 2000", Paris, September 7-13, 2000.

[16] E.Brezin, V.A.Kazakov and Al.B.Zamolodchikov, Nucl.Phys. B338 (1990) 673; D.Gross and N.Miljkovic, Phys.Lett. B238 (1990) 217; F.Ginsparg and J.Zinn-Justin, Phys. Lett. B240 (1990) 333; J.Parisi, Phys.Lett. B238 (1990) 209, 213.

[17] Jens Hoppe, Vladimir Kazakov, Ivan K. Kostov, Dimensionally Reduced SYM4 as Solvable Matrix Quantum Mechanics, Nucl.Phys. B571 (2000) 479-509, hep-th/9907058.

[18] G. Moore, hep-th/9203061.

[19] Antal Jevicki, Developments in 2D String Theory, hep-th/9309115.

Current address: Laboratoire de Physique Théorique de l’Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex05, France

E-mail address: kazakov@physique.ens.fr