Screening of persistent currents in mesoscopic metal rings

Axel Völker and Peter Kopietz
Institut für Theoretische Physik der Universität Göttingen,
Bunsenstraße 9, 37073 Göttingen, Germany
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Abstract

The effect of the Coulomb-interaction on persistent currents in disordered mesoscopic metal rings threaded by a magnetic flux $\phi$ is studied numerically. We use the simplest form of self-consistent Hartree theory, where the spatial variations of the self-consistent Hartree potential are ignored. In this approximation the self-consistent Hartree energies are simply obtained by diagonalizing the non-interacting system via the Lanczos method and then calculating the (disorder-dependent) particle number on the ring self-consistently. In the diffusive regime we find that the variance of the total particle number is strongly reduced, in agreement with the prediction of the random-phase approximation. On the other hand, the variance of the number of energy levels in a small interval below the Fermi energy is not affected by the Coulomb interaction. We argue that this implies that the experimentally observed enhancement of the persistent current is due to long-range Coulomb interactions.

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I. INTRODUCTION

The usual diagrammatic approach [1] to disordered metals is based on the existence of two small parameters: the smallness of \((k_F\ell)^{-1}\) (where \(k_F\) is the Fermi wave-vector and \(\ell\) is the elastic mean free path) justifies the perturbative treatment of the disorder potential, and the smallness of the parameter \(\kappa/k_F\) at high densities (where \(\kappa\) is the Thomas-Fermi screening wave-vector) justifies the so-called random-phase approximation (RPA), which resums the most divergent diagrams in the expansion of the Coulomb potential. This leads to the screening of the long-range tail of the Coulomb potential, so that the electron-electron interaction can be taken into account via an effective short-range interaction of the Hubbard-type.

In Ref. [2] this standard diagrammatic approach was used to calculate the effect of electron-electron interactions on the average persistent current in mesoscopic metal rings threaded by a magnetic flux \(\phi\) [3,4]. Surprisingly, the average current was found to be almost two orders of magnitude smaller than the experimental result by Lévy et al. [5]. This experiment and subsequent experiments by other groups [6,7] have stimulated many recent theoretical works. Nevertheless, till now a generally accepted theoretical explanation of the experimental data [5,6] has not been found, perhaps because the standard perturbative approach is not applicable in the persistent current problem. This has several reasons: First of all, the experiments [5,6] are performed at constant particle number \(N\), so that the persistent current should be calculated from the flux-derivative of the canonical free energy \(F(N,\phi)\),

\[
I(N,\phi) = -\frac{e}{\hbar} \frac{\partial F(N,\phi)}{\partial \phi}, \quad (1)
\]

where \(\phi = \phi/\phi_0\) is the flux measured in units of the flux quantum \(\phi_0 = hc/e\). On the other hand, with the standard machinery of many-body theory one calculates the current \(I(\mu,\phi)\) at constant chemical potential \(\mu\), which is related to the grand canonical potential \(\Omega(\mu,\phi)\) via

\[
I(\mu,\phi) = -\frac{e}{\hbar} \frac{\partial \Omega(\mu,\phi)}{\partial \phi}. \quad (2)
\]

Several authors [8–11] have noticed that for non-interacting electrons there exist striking differences between the average currents \(\overline{I(N,\phi)}\) and \(\overline{I(\mu,\phi)}\) (the overline denotes averaging over the disorder). In particular, for non-interacting electrons in the diffusive regime \(\overline{I(\mu,\phi)}\) is exponentially small, whereas \(\overline{I(N,\phi)}\) is not. The question whether these qualitative differences persist even in the presence of electron-electron interactions has not yet been investigated.

We would like to point out two more possible reasons for the failure of conventional perturbative many-body techniques in the persistent current problem. The first is the existence of two different purely geometric length scales in mesoscopic rings, namely the circumference \(L\), and the characteristic thickness \(L_\perp\) of the rings. Thus, diagrams which are usually neglected because they involve higher orders in \((k_F\ell)^{-1}\) may become important because they are geometrically enhanced by additional powers of \(L/L_\perp \gg 1\). This has first been pointed
out by Béal-Monod and Montambaux \[12\]. Finally, it is questionable whether the conventional RPA is valid for persistent currents. Note that the flux-dependent parts of $F(N, \varphi)$ and $\Omega(\mu, \varphi)$ are much smaller than the flux-independent parts. The RPA has been developed to re-sum the leading (flux-independent) terms in the expansion of $\Omega(\mu, \varphi)$ in powers of the Coulomb interaction \[13\]. However, it is not at all clear whether such a resummation of a formally divergent series remains valid for the sub-leading flux-dependent part of $\Omega(\mu, \varphi)$. We shall come back to this point in Sec.III.

Recently one of us \[14,15\] has proposed that the flux-dependent part of the average Hartree energy in a mesoscopic ring is only weakly screened, and showed that in this way the large magnitude of the experimentally observed currents \[5\] can be easily explained. The arguments given in Refs. \[14,15\] were simple but powerful, and made use of the exact Hohenberg-Kohn theorem of density-functional theory \[16\]. Although Ref. \[14\] has been criticized \[17,18\], all arguments put forward against Ref. \[14\] where essentially based on the RPA. The possibility that the RPA might not be a good approximation for the calculation of the persistent current \[19,20\] was ignored. Because even in the absence of disorder it is very difficult to calculate the dielectric function beyond the RPA, numerical methods seem to be the only controlled way to settle the issue. So far all numerical studies of two- and three-dimensional systems in the diffusive regime have found that the long-range part of the Coulomb interaction indeed strongly enhances the average persistent current \[21–24\]. Among the numerical methods, the exact diagonalization of the many-body Hamiltonian \[21,22\] is free of any approximations. Unfortunately, it is very difficult to study systems with more than approximately 10 electrons with this method. This is certainly too small to address the screening problem. Calculations based on the self-consistent Hartree-Fock approximation have been pushed to electron numbers of the order of $10^2$. Although these calculations do not produce the exact solution of the many-body system, they are non-perturbative by virtue of the self-consistency condition, and are therefore more reliable than any low-order diagrammatic calculation.

Guided by the evidence that the enhancement of the persistent current is closely related to the physics of screening, we shall in this work examine the screening problem numerically. As a starting point, we follow Ref. \[14\] and use the approximate mapping from the average canonical current onto an effective grand canonical average,

$$I(N, \varphi) - I(\mu^*, \varphi) \approx -\frac{e}{\hbar} \frac{1}{2\kappa(\mu^*, \varphi)} \frac{\partial}{\partial \varphi} \Sigma^2(\mu^*, \varphi),$$

where $I(\mu^*, \varphi)$ is the average grand canonical current at chemical potential $\mu^*$, and

$$\kappa(\mu, \varphi) = \left( \frac{\partial N(\mu, \varphi)}{\partial \mu} \right)_\varphi$$

is the average compressibility. $\Sigma^2(\mu^*, \varphi)$ is the variance of the particle number $N(\mu^*, \varphi)$ in a grand canonical ensemble

$$\Sigma^2(\mu^*, \varphi) = N^2(\mu^*, \varphi) - [N(\mu^*, \varphi)]^2.$$

The value of $\mu^*$ should be chosen such that the disorder- and flux-averaged particle number in the corresponding grand canonical ensemble agrees with the given particle number $N$ in the original canonical ensemble, i.e.
Note that for non-interacting electrons in the diffusive regime $I(\mu^*, \varphi)$ is exponentially small \[25\]. The leading interaction contribution to $I(\mu^*, \varphi)$ has been calculated by Ambegaokar and Eckern [2]. In this work we would like to focus on the effect of electron-electron interactions on the difference between the canonical and the grand canonical current, i.e. the term on the right-hand side of Eq. 3. Recently Berkovits and Avishai [22] studied this term via exact diagonalizations of small systems of electrons interacting with Coulomb forces. However, with exact diagonalizations it is impossible to reach system sizes where the condition $\ell \ll L \ll \xi$ is realized. Here $\ell$ is the elastic mean free path, $L$ is the circumference of the ring, and $\xi$ is the localization length. This condition defines the diffusive regime, which is relevant for the experiments of Refs. [5,6]. In this work we shall therefore take a different approach, and evaluate the right-hand side of Eq. 3 within the simplest possible approximation which still contains the physics of screening, namely the simplified capacitance model [15,20,26]. In this model the spatial dependence of the self-consistent Hartree potential is ignored, so that it simply renormalizes the chemical potential. Nevertheless, diagrammatically this model still contains the usual infinite series of bubble diagrams which lead to the RPA-picture of screening. But the model contains infinitely many other non-RPA diagrams, so that the numerical solution of the self-consistent Hartree theory for this model allows us to check the accuracy of the RPA. Note that the simplified capacitance model is easily solved numerically by diagonalizing the \textit{non-interacting} system with the help of the Lanczos method, and then imposing a simple self-consistency loop for the particle number. The most difficult step in this procedure is the exact diagonalization, so that in this way we can reach the same system sizes as without interactions.

II. THE SIMPLEST FORM OF SELF-CONSISTENT HARTREE THEORY

To calculate the effect of electron-electron interactions on the persistent current, let us use the self-consistent Hartree approximation to take the Coulomb interactions between all charges (i.e. electrons and positively charged ions) into account. Numerically it is convenient to use a lattice model. The Hartree energies $E_\alpha(\varphi)$ and wave-functions $\psi_\alpha(r)$ are obtained by solving the Schrödinger equation

\begin{equation}
- \sum_{\mu=x,y,z} \left[ t_\mu \psi_\alpha(r + a_\mu) + t_\mu^* \psi_\alpha(r - a_\mu) \right] + [U(r) + U_H(r)] \psi_\alpha(r) = E_\alpha(\varphi) \psi_\alpha(r), \tag{7}
\end{equation}

where $r$ labels the sites of a three-dimensional cubic lattice with primitive vectors $a_x, a_y, a_z$. To model the Aharonov-Bohm flux $\phi$ through the ring, we call the azimuthal direction the $x$-direction and choose $t_x = t e^{i \frac{2\pi x}{N_x}}$, $t_y = t_z = t$. Here $N_x$ is the number of lattice sites in the $x$-direction. For convenience, from now on all energies will be measured in units of the hopping energy $t$. As usual, disorder is introduced via random potentials $U(r)$, which are assumed to be independent random variables with zero average and uniform distribution in the interval $[-w/2, w/2]$. The self-consistency is imposed by requiring that the Hartree potential $U_H(r)$ satisfies

\begin{equation}
\int_0^1 d\varphi N(\mu^*, \varphi) = N. \tag{6}
\end{equation}
\[ U_H(r) = \sum_{r' \neq r} \frac{e^2}{|r - r'|} [n(r') - n_+(r')] , \tag{8} \]

with \( n(r) = \sum_\alpha |\psi_\alpha(r)|^2 \). The prime indicates that the \( \alpha \)-sum is in a canonical ensemble over the lowest \( N \) solutions of Eq.7, and in a grand canonical ensemble over all \( \alpha \) with \( E_\alpha < \mu \), where \( \mu \) is the chemical potential. The number density \( n_+(r) \) of the positive background charge is assumed to be fixed. Given the self-consistent solutions of Eqs.7 and 8, the canonical persistent current can be calculated from Eq.1 with \[ F(N, \varphi) = \frac{1}{N} \sum_{\alpha=1}^{N} E_\alpha(\varphi) - \frac{1}{2} \sum_{r} n(r) U_H(r) , \tag{9} \]

and the grand canonical current is obtained from Eq.2 with \( \Omega(\mu, \varphi) = F(N(\mu, \varphi), \varphi) - \mu N(\mu, \varphi) \), where the function \( N(\mu, \varphi) \) is now explicitly given by \[ N(\mu, \varphi) = \sum_\alpha \Theta(\mu - E_\alpha(\varphi)) . \tag{10} \]

In order to obtain the average current, we should solve Eq.7 self-consistently for many realizations of disorder and then average the result. Of course, this is a very difficult problem, which is beyond the scope of this work. Instead, we shall use Eq.3 to calculate the difference between the canonical- and the grand canonical current approximately.

To reduce the numerical work even further without losing the physics of screening, let us replace the self-consistent Hartree potential by its spatial average, \( U_H(r) \rightarrow \frac{1}{N_L} \sum_r U_H(r) \), where \( N_L = \sum_r \) is the number of lattice sites. In this approximation \( U_H \) is simply given by

\[ U_H \approx \frac{e^2(N - N_+)}{C_0} , \quad \frac{1}{C_0} = \frac{1}{N_L(N_L - 1)} \sum_{r, r' \neq r} \frac{e^2}{|r - r'|} , \tag{11} \]

where \( N = \sum_r n(r) \) is the total number of electrons on the ring, and \( N_+ = \sum_r n_+(r) \) is the total number of positively charged ions. Note that \( C_0 \) is nothing but the classical capacitance of the ring. In Fourier space the approximation means that we neglect the momentum transfer through all Coulomb lines. Because for a thin ring (with \( L_\perp \ll L \)) the long-wavelength Fourier components of the Coulomb potential depend only logarithmically on the wave-vector, we expect that in this case the approximation is sufficient for a qualitative estimate of the importance of Coulomb interactions. Of course, in a canonical ensemble Eq.11 is a trivial flux-independent constant, so that the interaction-contribution to the persistent current vanishes. For this reason only the non-zero Fourier components of the Coulomb-potential were retained in Ref. At long wavelengths these can be expressed in terms of generalized capacitances \( C_k \), describing long-wavelength charge fluctuations which do not change the total electron number. On the other hand, in a grand canonical ensemble the particle number \( N(\mu, \varphi) \) is a very complicated flux-dependent random variable, which has to be determined self-consistently by solving Eq.10. By making the approximation on the right-hand side of Eq.3, we can estimate the interaction contribution to the canonical
persistent current in a way that is very well suited for numerical calculations. For the simplified capacitance model Eq.10 reads

\[ N(\mu, \varphi) = \sum_{\alpha} \Theta(\tilde{\mu} - \epsilon_{\alpha}(\varphi) - \frac{e^2 N(\mu, \varphi)}{C_0}) \quad , \quad \tilde{\mu} = \mu + \frac{e^2 N}{C_0} \ , \]  

where the energies \( \epsilon_{\alpha}(\varphi) \) are the solutions of Eq.7 for \( U_H = 0 \). Eq.12 can be viewed as the simplest form of self-consistent Hartree theory.

To appreciate the non-perturbative nature of Eq.12 let us calculate the average compressibility \( \kappa(\mu, \varphi) \) defined in Eq.4. Therefore we simply differentiate both sides of Eq.12 with respect to \( \mu \) (taking into account that \( \partial N_+/\partial \mu = 0 \), because the positive background charge is fixed), solve for \( \partial N/\partial \mu \), and finally average. This yields

\[ \kappa(\mu, \varphi) = \rho(\mu, \varphi) \left[ 1 + \frac{e^2}{C_0} \rho(\mu, \varphi) \right]^{-1} \]  

where the self-consistent density of states is given by

\[ \rho(\epsilon, \varphi) = \sum_{\alpha} \delta(\epsilon - \epsilon_{\alpha}(\varphi) - \frac{e^2(N(\mu, \varphi) - N_+)}{C_0}) \ . \]  

Ignoring random fluctuations of the density of states, we may factorize the average,

\[ \rho(\mu, \varphi) \left[ 1 + \frac{e^2}{C_0} \rho(\mu, \varphi) \right]^{-1} \rightarrow \rho(\mu, \varphi) \left[ 1 + \frac{e^2}{C_0} \rho(\mu, \varphi) \right]^{-1} \ . \]  

In the diffusive regime it is also reasonable to neglect interaction contributions to the average density of states [26], so that

\[ \overline{\rho(\mu, \varphi)} \rightarrow \rho_0(\mu, \varphi) \equiv \sum_{\alpha} \delta(\mu - \epsilon_{\alpha}(\varphi)) = \frac{1}{\Delta} \ , \]  

where \( \Delta \) is the average level spacing at the Fermi energy in the absence of interactions. Hence we obtain

\[ \kappa(\mu, \varphi) = Z_{RPA} \Delta^{-1} \quad , \quad Z_{RPA} = \frac{1}{1 + \frac{e^2}{\epsilon_0 \Delta}} \ . \]  

Note that \( \Delta^{-1} \) is the average compressibility without interactions. Because \( e^2/(C_0 \Delta) \gg 1 \) in the experimentally relevant parameter regime, the RPA renormalization factor \( Z_{RPA} \) is small compared with unity. Hence, the interactions lead to a drastic reduction of the average compressibility. A similar result has also been obtained by Vignale within a density functional approach [27]. A strong reduction of the compressibility has also been observed in the exact diagonalization study by Berkovits and Avishai [22]. The origin for this effect is intuitively obvious, because physically \( \kappa^{-1} \) is the energy associated with the addition of an electron to the system. In the absence of interactions this is simply the average level spacing, but with Coulomb interactions one has to pay the charging energy \( e^2/C_0 \) for adding
an electron. Hence, the inverse compressibility is pushed from $\Delta$ to the much larger energy $e^2/C_0$. We would like to stress that in the conventional diagrammatic approach to the many-body problem for infinite translationally invariant systems the Hartree diagram responsible for this renormalization is usually ignored, because it is assumed to be canceled exactly by the positive background charge [28]. Substituting Eq.17 into Eq.3, we obtain

$$I(N, \varphi) - I(\mu^*, \varphi) \approx -\frac{e}{h} \frac{\Delta}{2Z_{\text{RPA}}} \frac{\partial}{\partial \varphi} \Sigma^2(\mu^*, \varphi)$$

$$\approx -\frac{e^2}{2C_0} \frac{\partial}{\partial \varphi} \Sigma^2(\mu^*, \varphi) ,$$

(18)

where the second line is valid for $e^2/C_0 \gg \Delta$. The crucial question is now whether the flux-dependent part of the variance $\Sigma^2(\mu^*, \varphi)$ is reduced by the interactions or not. Note that naive application of the standard RPA [17,27] leads to the replacement $\Sigma^2(\mu^*, \varphi) \rightarrow Z_{\text{RPA}}^2 \Sigma_0^2(\mu^*, \varphi)$, where $\Sigma_0^2(\mu^*, \varphi)$ is the variance in the absence of interactions (see Eq.20 below). However, as already mentioned, the flux-dependent part of $\Sigma^2(\mu^*, \varphi)$ is much smaller than the flux-independent part, and it is not at all clear whether it is screened in exactly the same way as the flux-independent part. Because even without disorder it is very difficult to calculate corrections to the RPA in a systematic way [13], we shall in this work study this problem numerically.

III. NUMERICAL RESULTS

We have calculated the eigenvalues $E_\alpha(\varphi) = \epsilon_\alpha(\varphi) + \frac{\Delta}{\epsilon_\alpha}(N(\mu, \varphi) - N_+)$ of Eq.7 numerically for fixed realizations of the random potential $U(\mathbf{r})$ on finite lattices. The eigenvalues $\epsilon_\alpha(\varphi)$ without the Hartree term were calculated with the Lanczos method [29,30]. To obtain the energy shift due to the Hartree potential, the self-consistency equation 12 for the particle number for a given realization of the random potential was then solved via a simple Newton procedure. It turns out that at zero temperature this equation does not always have a unique solution, because by definition $N$ must be an integer. This problem is easily avoided by working at finite temperature $T$, and then extrapolating for $T \rightarrow 0$. Because for $T > 0$ the step function in Eq.12 is smoothed out into the Fermi function, there exists always a unique solution of Eq.12. This is easy to see from the fact that the left-hand side of Eq.12 is a monotonically increasing function of the particle number, while the right-hand side is monotonically decreasing.

Most theoretical calculations in disordered systems are based on the assumption that in the energy window of interest the energy-dependence of the average density of states can be ignored. On the other hand, in our finite system the average density of states is non-zero only in a finite interval, and exhibits a broad maximum at zero energy, as shown in Fig.1. As we have recently pointed out [31], for non-interacting electrons the universal weak localization effects related to the Cooperon pole are completely washed out if the energy-dependence of the average density of states becomes significant in the energy interval under consideration. Hence, in order to compare our numerical calculations with theoretical predictions for models with constant density of states, we shall retain only the central part of the spectrum, and discard all energies outside an interval $[\mu - E_0, \mu]$. From Fig.1 it is obvious that for our set
of parameters $E_0 = 2$ is a good choice. A similar procedure has also been adopted in Ref. [32]. Note that in this way we formally replace the density of states of our original tight binding model by the density of states with sharp edges shown in Fig.4. The total particle number is still defined as in Eq.12, except that now the $\alpha$-sum is over energies satisfying $\epsilon_\alpha \in [\mu - E_0, \mu]$. The great advantage of this construction is that the average density of states is practically constant in the entire regime where it is non-zero, so that non-universal effects related to the energy-dependence of the density of states should be clearly visible at the band edges.

The drastic effect of the Coulomb interaction on the statistics of the number of electrons in the system is clearly seen in Fig.2, were we plot the probability distribution $P(N)$ of the particle number for different values of $e^2/C_0$. The narrowing of the distribution for increasing interaction parameter can be viewed as a screening effect. The usual RPA prediction for the second moment $\Sigma^2(\mu, \varphi) = (N(\mu, \varphi) - N_+)^2$ of the distribution $P(N)$ around $N_+$ is easily obtained from Eq.12 by expanding the right-hand side to first order

$$N - N_+ = N_0 - N_+ - \frac{e^2}{C_0} \rho_0(\mu, \varphi)(N - N_+) + O((N - N_+)^2) \; ,$$

where $N_0 = \sum_\alpha \Theta(\mu - \epsilon_\alpha)$ is the particle number in the absence of interactions, and $\rho_0(\mu, \varphi)$ is the non-interacting density of states, see Eq.16. Solving Eq.13 for $N - N_+$, squaring, averaging, and ignoring random fluctuations of the density of states (see Eq.15), we obtain for the second moment

$$\Sigma^2(\mu, \varphi) = Z_{\text{RPA}}^2 \Sigma_0^2(\mu, \varphi) \; ,$$

where $Z_{\text{RPA}}$ is given in Eq.17, and $\Sigma_0^2(\mu, \varphi) = (N_0(\mu, \varphi) - N_+)^2$. The obvious question is now whether Eq.20 is accurate or not. Note that this expression is based on the expansion 19, which is a priori uncontrolled, because it is not clear whether the effect of the higher order terms that have been ignored is important or not. Furthermore, it is by no means clear whether the flux-dependent part of $\Sigma^2(\mu, \varphi)$ is modified in exactly the same way by the interaction as the dominant flux-independent part. To investigate this point, let us expand $\Sigma^2(\mu, \varphi)$ in a Fourier series. Because $N(\mu, \varphi)$ is an even periodic function of $\varphi$ with a fundamental period of unity, we may expand

$$\Sigma^2(\mu, \varphi) = \frac{P^{(0)}}{2} + \sum_{n=1}^{\infty} P^{(n)} \cos(2\pi n \varphi) \; .$$

Denoting by $P_0^{(n)}$ the corresponding Fourier components in the absence of interactions, the RPA predicts that $P^{(n)}/P_0^{(n)} = Z_{\text{RPA}}^2$ for all $n$. As shown in Fig.3, for the zeroth and the first two even Fourier components ($n = 2m = 0, 2, 4$) this prediction is in excellent quantitative agreement with our numerical self-consistent solution of Eq.12. The odd components ($n = 2m + 1$) are numerically found to be much smaller than the even ones [31], but are screened in a similar fashion. We conclude that for our simplified capacitance model the RPA is an excellent approximation, so that the interaction actually reduces the current in Eq.18 by a factor of $Z_{\text{RPA}} \ll 1$. In this case the difference between the canonical and the grand canonical current is completely negligible compared with the leading contribution to the average grand canonical current $\bar{I}(\mu^*, \varphi)$ that has been considered by Ambegaokar and Eckern [3].
However, the screening behavior is completely changed if we exclude the energies in the vicinity of the band edges. Such a procedure can be motivated physically as follows: It is well-known that in an interacting many-body system propagating quasi-particles exist only for energies sufficiently close to the Fermi energy. Moreover, the flux-dependence of physical observables should be due to quasi-particles that can coherently propagate around the ring, and in this way probe the sensitivity to twists in the boundary conditions. Hence, we expect that in a realistic interacting many-body system \( \frac{\partial \Sigma^2(\mu, \varphi)}{\partial \varphi} \) is essentially determined by weakly damped states with energies in a small interval of width \( E_{\text{in}} \) below the Fermi energy. Here \( E_{\text{in}} \) is some inelastic cutoff energy that is small compared with the Fermi energy. In other words, inelastic processes (which are completely ignored in the Hartree and the Hartree-Fock approximation) will naturally restrict the states contributing to the flux-dependent part of \( \Sigma^2(E, \varphi) \) to those with energies in the interval \([\mu - E_{\text{in}}, \mu]\). The theory developed in Refs. [14,15] is based on the existence of such a cutoff energy.

To study the contributions from the part of the spectrum close to the Fermi energy to \( \Sigma^2(\mu, \varphi) \), let us therefore introduce the auxiliary quantity

\[
\sigma^2(E, \mu, \varphi) = \int_{\mu - E}^{\mu} d\epsilon \int_{\mu - E}^{\mu} d\epsilon' K_2(\epsilon, \epsilon', \varphi),
\]

(22)

where the self-consistent density of states \( \rho(\epsilon, \varphi) \) is given in Eq. 14. Note that \( \sigma^2(E, \mu, \varphi) \) is the variance of the number of energy levels in an interval of width \( E \) below the Fermi energy, and that by construction \( \Sigma^2(\mu, \varphi) = \lim_{E \to \infty} \sigma^2(E, \mu, \varphi) \). In fact, given our truncated density of states shown in Fig. 1, we have \( \Sigma^2(\mu, \varphi) = \sigma^2(E, \mu, \varphi) \) for all \( E > E_0 \). It is again convenient to expand \( \sigma^2(E, \mu, \varphi) \) in a Fourier series,

\[
\sigma^2(E, \mu, \varphi) = \frac{\sigma_0^2(E)}{2} + \sum_{n=1}^{\infty} p(n) \cos(2\pi n \varphi).
\]

(24)

In the absence of interactions and for energies \( E \) small compared with \( \hbar/\tau \) (where \( \tau \) is the elastic lifetime) an approximate expression for the Fourier coefficients \( p_0(n) \) (where the subscript 0 indicates that interactions are neglected) can be obtained from the Feynman diagrams with two Cooperons and two Diffusons given by Altshuler and Shklovskii [33]. In this approximation one obtains \( p_0(2m+1)(E) = 0 \), and the even Fourier components are given by

\[
p_0(2m)(E) = \frac{2}{\pi^2m} \left\{ \exp\left[-m\tilde{\Gamma}^{1/2}\right] - \exp\left[-\frac{m}{\sqrt{2}} \left( \sqrt{\tilde{E}^2 + \tilde{\Gamma}^2 + \tilde{\Gamma}} \right)^{1/2} \right] \right\}.
\]

(25)

The flux average is

\[
p_0^{(0)}(E) = \frac{2}{\pi^2} \left[ \frac{1}{\sqrt{2}} \left( \sqrt{\tilde{E}^2 + \tilde{\Gamma}^2 + \tilde{\Gamma}} \right)^{1/2} - \tilde{\Gamma}^{1/2} \right] + \sum_{m=1}^{\infty} p_0(2m)(E).
\]

(26)
Here $\tilde{E} = E/E_c$, and $\tilde{\Gamma} = \Gamma/E_c$, where $E_c$ is the Thouless energy and $\Gamma$ is some cutoff energy that has been introduced into the non-interacting theory by hand \cite{9}. For non-interacting electrons on a quasi-one dimensional ring (where diffusion is only possible along the circumference) we have recently confirmed Eqs.25 and 26 numerically \cite{31}. Here we would like to investigate whether in the presence of long-range Coulomb interactions these expressions are strongly reduced by screening effects. Note that according to Altland and Gefen \cite{18} there should be no qualitative differences between the screening of fluctuations of the *spectral* density and the *total* density; if this would be correct, then the variance $\sigma^2(E, \mu, \varphi)$ of the number of energy levels in an interval $E$ below the Fermi energy should be screened in precisely the same way as the variance $\Sigma^2(\mu, \varphi)$ of the total particle number (which is strongly affected by screening, see Fig.3). We are now in the position to settle this controversy numerically within the simplified capacitance model. We would like to emphasize that, in spite of its simplicity, this model contains the physics of screening.

In Fig.4 we show our numerical results for the zeroth Fourier component $p^{(0)}(E)$ as function of $E$ for different values of $e^2/(C_0 \Delta)$. Corresponding results for the first even Fourier component $p^{(2)}(E)$ are shown in Fig.5. Evidently, in the entire energy interval where the average density of states is approximately constant, the flux average of $\sigma^2(E, \mu, \varphi)$ and the dominant flux-dependent contribution are only weakly affected by the Coulomb interaction. Moreover, in Fig.5 we also show the Altshuler-Shklovskii prediction 25 for $p^{(2)}(E)$. The agreement with the numerical results for the simplified capacitance model clearly demonstrates that for small energies it is indeed allowed to ignore screening corrections to $\sigma^2(E, \mu, \varphi)$. Thus, our numerical results support the arguments put forward by one of us in Refs. 14,15,19,20, and strongly disagree with the statement of Altland and Gefen \cite{18} that the fluctuations of the spectral density is screened in the same way as the total density. From Figs.4 and 5 it is also obvious that the RPA screening of the fluctuation of the total particle number is recovered as soon as the interval $E$ includes the band edge $E_0$. The sharp drop for $p^{(0)}(E)$ and $p^{(2)}(E)$ for $E \approx E_0$ is in agreement with the RPA result for the Fourier components $P^{(n)}$ of the variance of the total particle number shown in Fig.3. Of course, to obtain the physical persistent current, fluctuations on all energy scales should be included, so that the simplified capacitance model does not show any enhancement of the average persistent current due to long-range Coulomb interactions. However, as explained in Refs. 13,31, we expect that in a realistic many body system the damping of quasi-particles which are not close to the Fermi surface will eliminate the contribution from high-energy states to the *flux-dependent* part of $\Sigma^2(\mu, \varphi)$. In other words, for a realistic interacting many-body system we expect

$$\frac{\partial \Sigma^2(\mu, \varphi)}{\partial \varphi} \approx \frac{\partial \sigma^2(E, \mu, \varphi)}{\partial \varphi}, \text{ for all } E \gtrsim E_{in}, \tag{27}$$

where $E_{in}$ is the unknown inelastic cutoff energy mentioned above. Note that in Ref. 11 Altshuler, Gefen, and Imry implicitly seem to make a similar assumption 34.

**IV. CONCLUSIONS**

In this work we have presented numerical results for the simplified capacitance model, which describes the effect of long-range Coulomb interaction on the average canonical per-
sistent current. The model can be considered as the simplest form of self-consistent Hartree theory, which still contains the non-trivial physics of screening. We have confirmed one of the fundamental assumptions of Ref. [14], namely that the fluctuations of the spectral density for sufficiently small energies are not screened. Of course, within the approximations inherent in our model all eigen-energies (including those far away from the Fermi energy) correspond to quasi-particles with infinite lifetime, so that the physical persistent current is eventually screened once the energies at the band edges are taken into account. However, in physically more realistic models we expect that the damping of the quasi-particles will automatically eliminate non-universal effects related to the band edges. Given the fact that the low-energy fluctuations are not screened, the universal current proposed in Ref. [14] directly follows.

The problem of verifying by explicit calculation that the damping of quasi-particles restricts the flux-dependent part of the spectrum to a small interval of energies close to the Fermi energy remains open. We believe that this problem is closely related to the rather obscure inelastic cutoff $\Gamma$, which is usually introduced phenomenologically into the Cooperon and Diffuson propagators of the non-interacting theory. For a recent microscopic calculation of $\Gamma$ see Ref. [35]. More generally, the inelastic cutoff should be momentum- and frequency-dependent, $\Gamma(q, \omega)$. It is tempting to speculate that the energy scale where $\Gamma(0, \omega)$ begins to deviate significantly from its zero-energy limit $\Gamma(0, 0)$ is proportional to the cutoff energy $E_{\text{in}}$ mentioned above (see Eq.27).

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FIG. 1. Non-interacting average density of states of a $20 \times 5 \times 5$ system with $w = 4$. For our numerical calculations we discarded all energies outside the interval $[\mu - E_0, \mu]$ with $\mu = 0, E_0 = 2$ (see the dashed lines) in order to obtain a model with approximately constant density of states.
FIG. 2. Probability distribution $P(N)$ of the particle number in the simplified capacitance model on a $20 \times 5 \times 5$ lattice for different values of $e^2/(C_0\Delta)$, with $\Delta^{-1} = \rho_0(\mu) = 67$. (a) $e^2/(C_0\Delta) = 0$; (b) $e^2/(C_0\Delta) = \Delta^{-1}/100 = 0.67$; (c) $e^2/(C_0\Delta) = 3\Delta^{-1}/100 = 2.01$; (d) $e^2/(C_0\Delta) = \Delta^{-1}/20 = 3.35$. Here and in all subsequent figures we have chosen $w = 4$, $\mu = 0$ and $N_+ = 130.2$. In this case $N_+$ agrees with the average particle number, so that on average the ensemble of rings is not charged.
FIG. 3. Ratio of the even Fourier components $P^{(2m)}/P_0^{(2m)}$ (see Eq.21) of the particle number variance for different values of $m$ as function of $e^2/(C_0\Delta)$. The solid line is the RPA prediction $Z_{\text{RPA}}^2$. 
FIG. 4. Zeroth Fourier component $p^{(0)}(E)$ of the variance $\sigma^2(E, \mu, \varphi)$ as function of $E/E_c$ with $E_c = 1/180$ and $\Delta^{-1} = 67$ for different values of $e^2/(C_0\Delta)$. (a) $e^2/(C_0\Delta) = 0$; (b) $e^2/(C_0\Delta) = 0.67$; (c) $e^2/(C_0\Delta) = 6.7$. The band edge is at $E_0/E_c = 360$. 
FIG. 5. Second Fourier component $p^{(2)}(E)$ of the variance $\sigma^2(E, \mu, \varphi)$ as function of $E/E_c$ for different values of $e^2/(C_0 \Delta)$ with $\Delta^{-1} = 67$. (a) $e^2/(C_0 \Delta) = 0$; (b) $e^2/(C_0 \Delta) = 0.67$; (c) $e^2/(C_0 \Delta) = 6.7$. The solid line is the Altshuler-Shklovskii prediction with $E_c = 1/180$ and $\tilde{\Gamma} = 3.85$. 

