Homogeneous isotropic cosmological models with pseudoscalar torsion function in Poincare gauge theory of gravity and accelerating Universe

A.V. Minkevich\textsuperscript{1,2}, A.S. Garkun\textsuperscript{1} and V.I. Kudin\textsuperscript{3}

\textsuperscript{1}Department of Theoretical Physics, Belarussian State University,
\textsuperscript{2}Department of Physics and Computer Methods, Warmia and Mazury University in Olsztyn, Poland,
\textsuperscript{3}Department of Technical Physics, Belarussian National Technic University

Abstract. The ”dark energy” problem is investigated in the framework of the Poincare gauge theory of gravity in 4-dimensional Riemann-Cartan space-time. By using general expression for gravitational Lagrangian homogeneous isotropic cosmological models with pseudoscalar torsion function are built and investigated. It is shown that by certain restrictions on indefinite parameters of gravitational Lagrangian the cosmological equations at asymptotics contain effective cosmological constant and can lead to observable acceleration of cosmological expansion. This effect has geometrical nature and is connected with space-time torsion.

1 Introduction

Cosmological researches fulfilled during several last decades were fruitful for theoretical and observational cosmology. The creation of inflationary paradigm, which permits to solve a number of problems of standard cosmological scenario, in particular, to explain the homogeneity and isotropy of the Universe at initial stages of cosmological expansion, is important achievement of the early Universe cosmology [1]. The discovery of the acceleration of cosmological expansion at present epoch is the most principal achievement of observational cosmology [2]. By using Friedmann cosmological equations in order to explain accelerating cosmological expansion, the notion of dark energy (or quintessence) was introduced in cosmology. According to obtained estimations, approximately 70% energy in our Universe is connected with some hypothetical form of gravitating matter — “dark energy” — of unknown nature. Recently many investigations devoted to dark energy problem were carried out (see review [3]). According to widely known opinion, the dark energy is associated with cosmological term. If the cosmological term is connected with the vacuum energy density, it is necessary to explain, why it has the value close to critical energy density at present epoch.

The present paper is devoted to investigation of the “dark energy” problem in the framework of the Poincare gauge theory of gravity (PGTG), which is natural generalization of Einsteinian general relativity theory (GR) by applying the gauge approach to theory of gravitational interaction. According to PGTG the physical space-time possesses the structure of Riemann-Cartan continuum with curvature and torsion. By using gravitational Lagrangian
of PGTG in general form including both a scalar curvature and various invariants quadratic in the curvature and torsion tensors (see below), isotropic cosmology was built and investigated in a number of papers (see [4–6] and references herein). As it was shown, the PGTG permits to solve the problem of cosmological singularity of GR: all solutions of generalized cosmological Friedmann equations (GCFE) for homogeneous isotropic cosmological models (HICM) deduced in the framework of PGTG are regular in metrics, Hubble parameter, its time derivative, if gravitating matter satisfies certain physically acceptable condition at extremely high energy densities and pressures [4]. It is a consequence of gravitational repulsion effect, which takes place at extreme conditions and is connected with space-time torsion essentially [6]. Properties of discussed HICM in PGTG coincide practically with that of GR at sufficiently small energy densities, which are much less in comparison with limiting (maximum) energy density for such models. By including cosmological term of corresponding value to the GCFE, we can obtain regular cosmological solutions with observable accelerating expansion stage. However, like GR, the problem of dark energy in such theory is not solved.

From geometrical point of view, the structure of HICM in PGTG can be more complicated in comparison with models describing by GCFE. Really, in the case of homogeneous isotropic models the torsion tensor \( S^\lambda_{\mu\nu} = -S^\lambda_{\nu\mu} \) can have the following non-vanishing components [7, 8]: \( S_{10}^1 = S_{20}^2 = S_{30}^3 = S_1(t), S_{123} = S_{231} = S_{312} = S_2(t) \frac{R^3r^2}{\sqrt{1 - kr^2}} \sin \theta \), where \( S_1 \) and \( S_2 \) are two functions of time, spatial spherical coordinates are used. The functions \( S_1 \) and \( S_2 \) have different properties with respect to transformations of spatial inversions, namely, the function \( S_2(t) \) has pseudoscalar character. The GCFE follow from gravitational equations of PGTG for HICM together with \( S_2 = 0 \). Obtained physical consequences of GCFE have principal character. However, it is necessary to note that gravitational equations of PGTG for HICM have also other solution with non-vanishing function \( S_2 \). The HICM with two torsion functions are studied below in the frame of PGTG in connection with the dark energy problem. In Section 2 gravitational equations of PGTG for HICM with two torsion functions are obtained and investigated. In Section 3 the asymptotics of cosmological solutions for such models is analyzed.

2 Homogeneous isotropic models with two torsion functions in PGTG

At first let us mention some general relations of the PGTG. Gravitational field is described in the frame of PGTG by means of the orthonormalized tetrad \( h^i_{\mu} \) and anholonomic Lorentz connection \( A^{ik}_{\mu} \) (tetrad and holonomic indices are denoted by latin and greek respectively); corresponding field strengths are torsion \( S^i_{\mu\nu} \) and curvature \( F^{ik}_{\mu\nu} \) tensors defined as

\[
S^i_{\mu\nu} = \partial_{[\nu} h^i_{\mu]} - h_{k[\mu} A^{ik}_{\nu]},
\]

\[
F^{ik}_{\mu\nu} = 2\partial_{[\mu} A^{ik}_{\nu]} + 2A^{il}_{[\mu} A^{k}_{\nu]}.
\]
We will consider the PGTG based on the following general form of gravitational Lagrangian
\[
\mathcal{L}_G = h\left[f_0 F + F^{\alpha\beta\mu\nu} (f_1 F_{\alpha\beta\mu\nu} + f_2 F_{\alpha\mu\beta\nu} + f_3 F_{\mu\alpha\nu\beta}) + F^{\mu\nu} (f_4 F_{\mu\nu} + f_5 F_{\nu\mu}) + f_6 F^2 + S^\alpha_{\mu\nu} (a_1 S_{\alpha\mu\nu} + a_2 S_{\nu\alpha\mu}) + a_3 S^{\alpha\alpha\nu\beta}\right],
\]
where \( h = \text{det}(h^i_\mu), F_{\mu\nu} = F^\alpha_{\mu\alpha\nu}, F = F^\mu_{\mu}, f_i (i = 1, 2, \ldots, 6), a_k (k = 1, 2, 3) \) are indefinite parameters, \( f_0 = (16\pi G)^{-1} \), \( G \) is Newton’s gravitational constant. Gravitational equations of PGTG obtained from the action integral \( I = \int (\mathcal{L}_g + \mathcal{L}_m) \, d^4x \), where \( \mathcal{L}_m \) is the Lagrangian of matter, contain the system of 16+24 equations corresponding to gravitational variables \( h^i_\mu \) and \( A^{ik}_\mu \).

Any homogeneous isotropic gravitating system in PGTG is characterized in general case by three functions of time: the scale factor of Robertson-Walker metrics \( R \) and two torsion functions \( S_1 \) and \( S_2 \). Below by using spherical coordinate system, the tetrad is taken in diagonal form. Then the curvature tensor has the following non-vanishing tetrad components denoted by means of the sign \( \hat{~} \):
\[
\begin{align*}
F^{00}_{00} &= F^{0i}_{00} = F^{00}_{00} \equiv A_1, & F^{i2}_{12} &= F^{i3}_{13} = F^{23}_{23} \equiv A_2, \\
F^{02}_{02} &= F^{02}_{0i} = F^{0i}_{02} \equiv A_3, & F^{32}_{32} &= F^{30}_{30} = F^{20}_{20} \equiv A_4,
\end{align*}
\]
with
\[
\begin{align*}
A_1 &= \dot{H} + H^2 - 2HS_1 - 2\dot{S}_1, \\
A_2 &= \frac{k}{R^2} + (H - 2S_1)^2 - S_2, \\
A_3 &= 2 (H - 2S_1) S_2, \\
A_4 &= \dot{S}_2 + HS_2,
\end{align*}
\]
where \( H = \dot{R}/R \) is the Hubble parameter and a dot denotes the differentiation with respect to time.

The Bianchi identities in this case are reduced to two following relations:
\[
\begin{align*}
\dot{A}_2 + 2H (A_2 - A_1) + 4S_1 A_1 + 2S_2 A_4 &= 0, \\
\dot{A}_3 + 2H (A_3 - A_4) + 4S_1 A_4 - 2S_2 A_1 &= 0.
\end{align*}
\tag{3}
\]
By using the gravitational Lagrangian (1) and the Bianchi identities (3), we obtain the following system of gravitational equations for HICM
\[
\begin{align*}
a (H - S_1) S_1 - 2bS_2^2 - 2f_0 A_2 + 4f (A_1^2 - A_2^2) + 2q_2 (A_3^2 - A_4^2) &= -\rho, \\
a \left(\dot{S}_1 + 2HS_1 - S_1^2\right) - 2bS_2^2 - 2f_0 (2A_1 + A_2) - 4f (A_1^2 - A_2^2) - 2q_2 (A_3^2 - A_4^2) &= p, \\
f \left[\left(\dot{A}_1 + \dot{A}_2\right) + 4S_1 (A_1 + A_2)\right] + q_2 S_2 A_3 + (2f - q_1) S_2 A_4 + \left(f_0 + \frac{a}{8}\right) S_1 &= 0, \\
q_2 \left[\left(\dot{A}_3 + \dot{A}_4\right) + 4S_1 (A_3 + A_4)\right] - 4f S_2 A_2 - 2 (q_1 + q_2) S_2 A_1 - (f_0 - b) S_2 &= 0,
\end{align*}
\tag{4-7}
\]
where
\[ a = 2a_1 + a_2 + 3a_3, \quad b = a_2 - a_1, \]
\[ f = f_1 + \frac{f_2}{2} + f_3 + f_4 + f_5 + 3f_6, \]
\[ q_1 = f_2 - 2f_3 + f_4 + f_5 + 6f_6, \quad q_2 = 2f_1 - f_2, \]

\( \rho \) is the energy density, \( p \) is the pressure and the average of spin distribution of gravitating matter is supposed to be equal to zero. Equations (4)–(5) do not contain high derivatives for the scale factor \( R \), if \( a = 0 \), moreover, equations (6)–(7) take more symmetric form, if \( 2f = q_1 + q_2 \). Then the system of gravitational equations for HICM take the following form:

\[ -2b S_2^2 - 2f_0 A_2 + 4f (A_1^2 - A_2^2) + 2q_2 (A_3^2 - A_4^2) = -\frac{1}{3} \rho, \tag{8} \]
\[ -2b S_2^2 - 2f_0 (2A_1 + A_2) - 4f (A_1^2 - A_2^2) - 2q_2 (A_3^2 - A_4^2) = p, \tag{9} \]
\[ f \left[ (\dot{A}_1 + \dot{A}_2) + 4S_1 (A_1 + A_2) \right] + q_2 S_2 (A_3 + A_4) + f_0 S_1 = 0, \tag{10} \]
\[ q_2 \left[ (\dot{A}_3 + \dot{A}_4) + 4S_1 (A_3 + A_4) \right] - 4f S_2 (A_1 + A_2) - (f_0 - b) S_2 = 0. \tag{11} \]

The system of equations (8)–(11) together with definition of curvature functions (2) is the base of our investigation of HICM below. Note also that the conservation law for spinless matter has usual form:

\[ \dot{\rho} + 3H (\rho + p) = 0. \tag{12} \]

From (8)–(9) follows that

\[ A_1 + A_2 = \frac{1}{12f_0} (\rho - 3p) - \frac{b}{f_0} S_2^2. \tag{13} \]

By using (13) and the formula following from definition (2) of curvature functions

\[ A_3^2 - A_4^2 = 4A_2 S_2^2 - 4 \left( \frac{k}{R^2} - S_2^2 \right) S_2^2 - \left( \dot{S}_2 + HS_2 \right)^2, \]

we find from gravitational equations (8)–(9) the following expressions for \( A_1 \) and \( A_2 \):

\[ A_1 = -\frac{1}{12f_0 Z} \left[ \rho + 3p - \frac{\alpha}{2} (\rho - 3p - 12bS_2^2)^2 \right] - \frac{\alpha \varepsilon f_0}{Z} (\rho - 3p - 12bS_2^2) S_2^2 \]
\[ + \frac{3 \alpha \varepsilon f_0}{Z} \left[ (HS_2 + \dot{S}_2)^2 + 4 \left( \frac{k}{R^2} - S_2^2 \right) S_2^2 \right], \tag{14} \]
\[ A_2 = \frac{1}{6f_0 Z} \left[ \rho - 6bS_2^2 + \frac{\alpha}{4} (\rho - 3p - 12bS_2^2)^2 \right] \]
\[ - \frac{3 \alpha \varepsilon f_0}{Z} \left[ (HS_2 + \dot{S}_2)^2 + 4 \left( \frac{k}{R^2} - S_2^2 \right) S_2^2 \right], \tag{15} \]

\[ 4 \]
where $Z \equiv 1 + \alpha [(\rho - 3p) - 12 (b + \varepsilon f_0) S_2^2]$, $\alpha \equiv \frac{f}{f_0}$, $\varepsilon \equiv \frac{q_2}{f_0}$, $\frac{q_2}{f_0} = 3\alpha \varepsilon f_0$. By using the conservation law (12), the formula (13) and the following relation obtained from definition of $A_3$ and $A_4$

$$A_3 + A_4 = \dot{S}_2 + 3HS_2 - 4S_1S_2,$$

we obtain from (10) the following expression for the torsion function $S_1$:

$$S_1 = \frac{3\alpha}{4Z} \left[ 4 (2b - \varepsilon f_0) S_2 \dot{S}_2 - HY \right],$$

where

$$Y \equiv (\rho + p) \left( 3 \frac{dp}{d\rho} - 1 \right) + 12\varepsilon f_0 S_2^2.$$

Then by using formulas (13) and (16) we find from (11) the following differential equation of second order for the torsion function $S_2$:

$$\varepsilon \left[ \ddot{S}_2 + 3H \dot{S}_2 + (3H - 4\dot{S}_1) S_2 + 4S_1S_2 (3H - 4\dot{S}_1) \right]$$

$$- \left[ \frac{1}{3f_0} (\rho - 3p - 12bS_2^2) + \frac{f_0 - b}{f} \right] S_2 = 0. \quad (18)$$

The obtained expressions (14)–(15) for curvature functions $A_2$ and $A_1$ together with their definition (2) give the generalization of cosmological Friedmann equations for HICM:

$$k/R^2 + (H - 2S_1)^2 = \frac{1}{6f_0Z} \left[ \rho + 6 (f_0 Z - b) S_2^2 + \frac{\alpha}{4} (\rho - 3p - 12bS_2^2)^2 \right]$$

$$- \frac{3\alpha f_0}{Z} \left[ (HS_2 + \dot{S}_2)^2 + 4 \left( \frac{k}{R^2} - S_2^2 \right) S_2^2 \right], \quad (19)$$

$$\dot{H} + H^2 - 2HS_1 - 2\dot{S}_1 = -\frac{1}{12f_0Z} \left[ \rho + 3p - \frac{\alpha}{2} (\rho - 3p - 12bS_2^2)^2 \right] - \frac{\alpha \varepsilon}{Z} (\rho - 3p - 12bS_2^2) S_2^2$$

$$+ \frac{3\alpha f_0}{Z} \left[ (HS_2 + \dot{S}_2)^2 + 4 \left( \frac{k}{R^2} - S_2^2 \right) S_2^2 \right]. \quad (20)$$

These equations contain the torsion function $S_1$ determined by (17) and the torsion function $S_2$, satisfying the equation (18). Besides indefinite parameter $\alpha$ determining the scale of extremely high energy densities [4], obtained equations contain additionally two indefinite parameters: $b$ with dimension of parameter $f_0$ and the parameter $\varepsilon$ without dimension. We have to analyze all these equations in order to investigate HICM with pseudoscalar torsion function in the frame of PGTG.
3   Asymptotics of cosmological solutions for HICM with pseudoscalar torsion function

The structure of obtained equations describing HICM with pseudoscalar torsion function is essentially more complicated in comparison with the case of HICM with \( S_2 = 0 \). By using the conservation law (12), the cosmological equation (19) can be written in the following form:

\[
H^2 \left[ \left( Z + \frac{3\alpha}{2} Y \right)^2 + 3\alpha \varepsilon f_0 S_2^2 Z \right] - 6\alpha H \left[ 2 \left( Z + \frac{3\alpha}{2} Y \right) (2b - \varepsilon f_0) S_2 \dot{S}_2 - \varepsilon f_0 S_2 \dot{S}_2 Z \right]
+ 36\alpha^2 (2b - \varepsilon f_0)^2 S_2^2 \dot{S}_2^2 + 3\alpha \varepsilon f_0 \left[ \dot{S}_2^2 + 4 \left( \frac{k}{R^2} - S_2^2 \right) S_2^2 \right] Z + \left( \frac{k}{R^2} - S_2^2 \right) Z^2
= \frac{1}{6f_0} \left[ \rho - 6b S_2^2 + \frac{\alpha}{4} \left( \rho - 3p - 12b S_2^2 \right)^2 \right] Z. \quad (21)
\]

From this equation follows, that cosmological solutions possess extreme points for the scale factor \( R \). In fact, if we put that at \( t = 0 \) the Hubble parameter vanishes \( H = 0 \), then from (21) follows the relation for \( S_2 \), its time derivative and energy density, which has to be valid at a bounce:

\[
36\alpha^2 (2b - \varepsilon f_0)^2 S_{20}^2 \dot{S}_{20}^2 + 3\alpha \varepsilon f_0 \left[ \dot{S}_{20}^2 + 4 \left( \frac{k}{R_{20}^2} - S_{20}^2 \right) S_{20}^2 \right] Z_0 + \left( \frac{k}{R_{20}^2} - S_{20}^2 \right) Z_0^2
= \frac{1}{6f_0} \left[ \rho_0 - 6b S_{20}^2 + \frac{\alpha}{4} \left( \rho_0 - 3p_0 - 12b S_{20}^2 \right)^2 \right] Z_0. \quad (22)
\]

(Values of various quantities at extreme point are denoted by index 0). By using the following equation of state \( p = \rho \) and by choosing some initial values for \( S_{20}, \dot{S}_{20}, \rho_0 \) at a bounce, particular cosmological solution for flat model \( (k = 0) \) presented in Fig. 1 - Fig. 2 was obtained numerically. We see that the pseudoscalar torsion function \( S_2 \) and the Hubble parameter \( H \) have some non-vanishing values at asymptotics. Below we analyze the following question: by what restrictions on indefinite parameters \( b \) and \( \varepsilon \) asymptotical value for the Hubble parameter corresponds to observable accelerating cosmological expansion?

By taking into account that various parameters of HICM have to be small at asymptotics, we see from (18), that if \( \varepsilon \ll 1 \), the pseudoscalar torsion function has the following asymptotics:

\[
S_2^2 = \frac{f_0 (f_0 - b)}{4fb} + \frac{\rho - 3p}{12b}. \quad (23)
\]

Note, that at asymptotics one uses for gravitating matter the equation of state for dust \( p = 0 \); then it follows from (23) that \( b \leq f_0 \) and we obtain at asymptotics: \( Z \rightarrow (b/f_0) \), \( S_1 \rightarrow 0 \). The cosmological equations (19)–(20) at asymptotics take the form of cosmological Friedmann equations with cosmological constant:

\[
\frac{k}{R^2} + H^2 = \frac{1}{6b} \left[ \rho + \frac{3 (f_0 - b)^2}{4f} \right], \quad (24)
\]
From Eqs. (24)–(25) we see, that parameter $b$ has to be very close to $f_0$. The value of $b$ leading to observable acceleration of cosmological expansion depends on the scale of extremely high energy density defined by $\alpha^{-1}$. If we take into account that the value of $\frac{3}{4}(f_0 - b)^2/f = \frac{1}{4}\alpha^{-1}(1 - b/f_0)^2$ has to be equal approximately to $0.7\rho_{cr}$ ($\rho_{cr} = 6f_0H_0^2$, $H_0$ is the value of the Hubble parameter at present epoch), then we obtain that $b = [1 - (2.8\rho_{cr}\alpha)^{1/2}]f_0$. If we suppose that the value of $\alpha^{-1}$ is greater than the energy density for quark-gluon matter, but smaller than the Planckian one, then we can obtain the corresponding estimation for $b$, which is extremely close to $f_0$.

## 4 Conclusion

As it was shown, the presence of pseudo-scalar torsion function in HICM built in the framework of PGTG can lead to effective cosmological constant in asymptotics of cosmological solutions and to observable accelerating cosmological expansion. The effect of acceleration of cosmological expansion in PGTG has the geometrical nature and is connected with spacetime torsion. From the point of view of considered theory hypothetical form of gravitating matter — dark energy or quintessence — is fiction. The further investigation of HICM with pseudoscalar torsion function will be to continued.

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