BOND PRICING FORMULAS FOR MARKOV-MODULATED AFFINE TERM STRUCTURE MODELS

MARIANITO R. RODRIGO
School of Mathematics and Applied Statistics
University of Wollongong
Wollongong, New South Wales, Australia

ROGEMAR S. MAMON*
Department of Statistical and Actuarial Sciences
University of Western Ontario
London, Ontario, Canada
and
Division of Physical Sciences and Mathematics
University of the Philippines Visayas
Miag-ao, Iloilo, Philippines

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Abstract. This article provides new developments in characterizing the class of regime-switching exponential affine interest rate processes in the context of pricing a zero-coupon bond. A finite-state Markov chain in continuous time dictates the random switching of time-dependent parameters of such processes. We present exact and approximate bond pricing formulas by solving a system of partial differential equations and minimizing an error functional. The bond price expression exhibits a representation that shows how it is explicitly impacted by the rate matrix and the time-dependent coefficient functions of the short rate models. We validate the bond pricing formulas numerically by examining a regime-switching Vasicek model.

1. Introduction. The modeling of interest rate evolution is of vital importance in finance and economics. An interest rate represents the cost of borrowing and lending, and it is crucial in the valuation of assets and in the pricing and hedging of fixed income instruments. As argued in Elliott and Wilson [15], it is also a primary macroeconomic indicator as a government directs its central bank, through monetary policy decisions, to set a target rate.

This paper investigates a broad class of interest rate models called regime-switching affine term structure models (ATSMs). This particular class of models possesses enormous flexibility in accurately replicating the dynamic changes in the economic and financial environment. Salient properties that must be captured by an interest rate model, which we stress, include mean reversion, stochasticity, and

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* Corresponding author: Rogemar S. Mamon, Department of Statistical and Actuarial Sciences, University of Western Ontario, London, Ontario, Canada. E-mail: rmamon@stats.uwo.ca.
random switches of economic regimes. Stochasticity comes from the normal perturbations and scaled by a volatility factor, while the model's switching feature acts as a mechanism to handle extreme movements at a random moment. The regime-switching capability enables the model to adequately describe the interest rate's distribution that constantly changes in shape characteristics through its moments. In essence, regime-switching models are alternatives to multi-factor and jump models as well as to non-parametric approaches.

The simple form of our modeling framework has its origins from the parametric assumptions for the short rate dynamics in continuous time and the form of the bond price discussed in Duffie and Kan [6]. It was shown in [6] that if the price of a zero-coupon bond has an exponential affine form, then the underlying short rate has a linear/Gaussian form or square-root/affine form; conversely, if a solution exists to some Riccati equation involving the bond's required maturity, as implied by the linear/Gaussian or square-root/affine form of the short rate, then the bond price has an exponential affine form. Elliott and van der Hoek (2001) made use of a technique based on stochastic flows and forward measures to construct explicitly the bond price from linear ordinary differential equations (ODEs), showing that Riccati equations are not necessary. An ATSM is a model, given by its stochastic differential equation, that has affine drift and variance in its short rate dynamics. A pedagogical exposition of these characterization results for the equivalence between ATSMs and exponential affine bond prices are delineated in Mamon [27]. Further theoretical developments on affine processes and their connections to finance, especially highlighting links to interest rate modeling and bond pricing, are elaborated in the works of Duffie, et al. [5], Filipović [18], Cuchiero, et al. [4], and Eksi and Filipović [7].

The balance between the capacity to capture complicated dynamics and the tractability of the bond pricing solution offered by affine regime-switching models renders them ideal for adoption in practice and further theoretical investigation given their rich mathematical structures. For ATSMs with no regime switching, Rodrigo and Mamon [28] established a unifying methodology that leads to the explicit bond pricing representation when the short rate has ATSM specifications. Practical implementation in the industry requires the ability to calibrate this type of models to current market prices, and this was addressed by the same authors in [29].

ATSMs with the addition of a regime-switching attribute were studied in Elliott and Mamon [11] and Elliott and Siu [12] in reference to finding closed-form solutions to the bond price under cases when some but not all model parameters are switching regimes. For an accessible implementation of estimating the parameters under a Markov chain driven regime-switching model, using historical yield rates as proxy for the short rate observations, see [39]; such implementation is relevant for the measurement and management of risks in a portfolio whose value is largely determined by the level and movement of the term structure. A dynamic estimation, designed to recover switching parameters and the state of the Markov chain, is via the hidden Markov model filtering technique; see for example, Elliott, et al. [10] for continuous-time filters, and Erlwein and Mamon [16] for discrete-time filters, as well as Xi and Mamon [35] for the filters involving a higher-order Markov chain in the Hull-White setting. Filters for the parameter estimation of a multivariate Hull-White setting were devised in Tenyakov, et al. [31]. The issue of choosing the appropriate martingale measures, akin to valuing bonds under the Vasicek and
CIR models with regime switching, is dealt with in Elliott, et al. [13]. Recently, regime-switching affine models also started to penetrate the research area of annuity and longevity product valuation within the aim of jointly modeling interest and mortality risks (e.g., [19, 20]). We note that the Markov switching of regimes in a model can alternatively be formulated via a marked point process (see Last and Brandt [26] for the pertinent notion). There is a relatively narrow literature on this alternative formulation, but examples include Criego and Swishchuk [3], Landén [25], and Wu and Zeng [34].

From the economic modeling viewpoint, the continued relevance of ATSMs, during and after the worst financial crisis of the post-war period, is elaborated in Beacco et al. [1]. Amidst the current fragile global economy and vulnerable financial system, ATSMs meet challenges arising from low interest rates. What was once a major criticism of ATSMs (i.e., the property of yielding negative values) became its utmost strength under the present financial climate. With the specification of a stochastic discount factor, ATSMs enable the fair pricing of financial assets over time. The discretisation of continuous-time ATSMs is carried out more often for the purpose of model estimation using historical market observations. Moreover, as econometric methods are almost always developed in discrete time, econometric advances could be integrated directly into a discrete-time setting.

Research involving ATSMs especially in the application and dynamic parameter estimation areas has dominated the literature in the last few years. The discrete-time implementation of continuous-time filters in a mean-reverting ATSM was illustrated in Grimm et al. [22] to examine the yield curve movements of three EU countries. ATSM estimation was also pursued in Gonon and Teichmann [21] by approximate filters that entail the solution of a system of generalised Riccati differential equations, and the efficiency of their proposed method was validated numerically on the CIR and Wishart processes. Hlouskova and Sögnér [23] employed the generalized method of moments combined with a multi-start search and quasi-Bayesian technique to obtain reliable ATSM parameter estimates and inference results. The option valuation problem is tackled by Zhu et al. [40] through the incorporation of robust filtered estimates under the framework of Markovian regime-switching interest rate and volatility.

This paper can be regarded as an extension of [12], but covers a number of significant generalizations in the following respects:

(i) Only the Vasicek and Cox-Ingersoll-Ross (CIR) cases were considered in [12], whereas we are considering the extensive class of ATSMs. Moreover, parameters in our modeling setup are also time dependent.

(ii) The method used in [12] for the Vasicek case does not work for the CIR case; as a result, a more complicated derivation using the forward measure arose for the CIR model. The feasibility of our method encompasses a broader ATSM class under which the Vasicek and CIR models are special cases.

(iii) Our method relies on solving partial differential equations (PDEs) while that of [12] relies on a probabilistic approach.

(iv) Our expression for the bond price has an explicit decomposition (see (3.7) below), indicating the extent of the contributions coming from the rate matrix along with the coefficients from the drift and variance of the short rate factors.

(v) We have approximate analytical and numerical bond pricing solutions, as well as illustrative examples, while this numerical aspect was not the emphasis of [12].
The structure of this paper is as follows. In Section 2, we present the bond pricing problem formulation under a regime-switching setting governed by a finite-state Markov chain in continuous time. Section 3 details the main results underscoring the bond valuation formula. The focus of Section 4 is on the numerical and analytical approximations for the bond price, while that of Section 5 is on the illustration of such algorithms for the proposed approximations. We conclude in Section 6.

2. Problem formulation. We consider a continuous-time financial market with a finite time horizon $[0, T]$, where $T > 0$. Suppose that there is a fixed underlying probability space $(\Omega, \mathcal{F}, P)$ under which all processes are defined, including a Wiener process and a Markov chain. The probability space $(\Omega, \mathcal{F}, P)$ is assumed to be equipped with a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ satisfying the usual right-continuity and $P$-completeness conditions. We also assume that $P$ is a risk-neutral measure.

To describe the evolution of the economic state over time, we use a continuous-time, $n$-state, observable Markov chain $\{X_t : t \in [0, T]\}$. Without loss of generality, we may identify the state space of $X_t$ at any time $t$ to be the set $\{u_1, \ldots, u_n\}$ of canonical unit vectors in $\mathbb{R}^n$. Note that here, when necessary for the matrix operations to be defined, we identify an $n$-tuple in $\mathbb{R}^n$ with an $n \times 1$ matrix. Let $Q(t) = (q_{i,j}(t))_{i,j=1}^n$ be the $n \times n$ rate matrix of $X_t$, where $q_{i,j}(t)$ is the transition intensity of $X_t$ from state $u_i$ to state $u_j$ at time $t$. Then Elliott [8] and Elliott et al. [9] gave a semimartingale decomposition for $X_t$, namely,

$$X_t = X_0 + \int_0^t Q(s)X_s \, ds + M_t,$$

where $\{M_t : t \in [0, T]\}$ is an $\mathbb{R}^n$-valued martingale with respect to the natural filtration generated by $\{X_t : t \in [0, T]\}$ under $P$.

We assume that the dynamics for the short rate process $\{r_t : t \in [0, T]\}$ are described by

$$dr_t = a(t, r_t, X_t) \, dt + b(t, r_t, X_t) \, dW_t,$$

where $\{W_t : t \in [0, T]\}$ is a Wiener process under $P$. For simplicity, we take the Markov chain and the Wiener process to be independent under $P$. Within a time-dependent ATSM framework, it is assumed that the deterministic functions $a = a(t, r, X)$ and $b = b(t, r, X)$, where $t \in [0, T]$, $r \in (0, \infty)$, and $X \in \mathbb{R}^n$, are such that

$$a(t, r, X) = a_0(t, X) + a_1(t, X)r, \quad b(t, r, X)^2 = b_0(t, X) + b_1(t, X)r \quad (2.1)$$

for suitable deterministic functions $a_k = a_k(t, X)$ and $b_k = b_k(t, X)$ with $k = 0, 1$.

The Markovian regime-switching property is captured by postulating that

$$a_k(t, X_t) = \langle \tilde{a}_k(t), X_t \rangle, \quad b_k(t, X_t) = \langle \tilde{b}_k(t), X_t \rangle, \quad k = 0, 1, \quad (2.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$ and $\tilde{a}_k(t), \tilde{b}_k(t) \in \mathbb{R}^n$ are known for all $t \in [0, T]$. In the time-independent ATSM case, $\tilde{a}_k$ and $\tilde{b}_k$ are known constant vectors in $\mathbb{R}^n$.

Suppose that the price $P(t, T)$ of a zero-coupon bond at time $t$ and with maturity $T$ is represented as

$$P(t, T) = v^T(t, r_t, X_t),$$

where the deterministic function $v^T = v^T(t, r, X)$ satisfies some PDE. Define

$$v^T_j(t, r) = v^T(t, r, u_j), \quad j = 1, 2, \ldots, n$$
and

\[ V(t, r) = \begin{pmatrix} v_1^T(t, r) & v_2^T(t, r) & \cdots & v_n^T(t, r) \end{pmatrix}^\dagger, \]

where \((\cdot)^\dagger\) denote the matrix transpose. Then \(v^T\) satisfies the PDE

\[
\frac{\partial v^T}{\partial t} - rv^T + a(t, r, X) \frac{\partial v^T}{\partial r} + \frac{1}{2} b(t, r, X)^2 \frac{\partial^2 v^T}{\partial r^2} + \langle V, QX \rangle = 0 \tag{2.3}
\]

with the final condition \(v^T(T, r, X) = 1\). For simplicity, we assume that \(Q\) is independent of \(t\). Evaluating (2.3) at each \(X = u_j\) for \(j = 1, 2, \ldots, n\), we have

\[
\frac{\partial v_j^T}{\partial t} - rv_j^T + a(t, r, u_j) \frac{\partial v_j^T}{\partial r} + \frac{1}{2} b(t, r, u_j)^2 \frac{\partial^2 v_j^T}{\partial r^2} + \langle V, Qu_j \rangle = 0. \tag{2.4}
\]

Recalling (2.2), we define

\[ a_{k,j}(t) = \langle \tilde{a}_k(t), u_j \rangle, \quad b_{k,j}(t) = \langle \tilde{b}_k(t), u_j \rangle, \quad j = 1, 2, \ldots, n, \quad k = 0, 1. \]

Thus, using (2.1), we can write

\[
a(t, r, u_j) = a_0(t, u_j) + a_1(t, u_j)r = \langle \tilde{a}_0(t), u_j \rangle + \langle \tilde{a}_1(t), u_j \rangle r
\]

and

\[
b(t, r, u_j)^2 = b_0(t, u_j) + b_1(t, u_j)r = \langle \tilde{b}_0(t), u_j \rangle + \langle \tilde{b}_1(t), u_j \rangle r
\]

for \(j = 1, 2, \ldots, n\). Moreover,

\[ Qu_j = \begin{pmatrix} q_{1,j} & q_{2,j} & \cdots & q_{n,j} \end{pmatrix}^\dagger \]

and

\[ \langle V, Qu_j \rangle = q_{1,j}v_1^T + q_{2,j}v_2^T + \cdots + q_{n,j}v_n^T, \quad j = 1, 2, \ldots, n. \]

Therefore (2.4) generates a system of \(n\) PDEs of the form

\[
\frac{\partial v_j^T}{\partial t} - rv_j^T + [a_{0,j}(t) + a_{1,j}(t)r] \frac{\partial v_j^T}{\partial r} + \frac{1}{2} [b_{0,j}(t) + b_{1,j}(t)r] \frac{\partial^2 v_j^T}{\partial r^2} + \langle V, Qv_j \rangle = 0, \tag{2.5}
\]

for \(j = 1, 2, \ldots, n\). It is convenient to use matrix notation. Introducing the diagonal matrices

\[
A_k(t) = \text{diag}(a_{k,1}(t), a_{k,2}(t), \ldots, a_{k,n}(t)),
\]

\[
B_k(t) = \text{diag}(b_{k,1}(t), b_{k,2}(t), \ldots, b_{k,n}(t)), \tag{2.6}
\]

we can express (2.5) in matrix form as

\[
\frac{\partial v}{\partial t} - rv + [A_0(t) + rA_1(t)] \frac{\partial v}{\partial r} + \frac{1}{2} [B_0(t) + rB_1(t)] \frac{\partial^2 v}{\partial r^2} + Q^T v = 0, \tag{2.7}
\]

where 0 here is the \(n \times 1\) zero matrix, subject to the terminal condition

\[ v(T, r) = v_T, \quad v_T = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^\dagger. \tag{2.8} \]

Letting

\[ v(t, r) = \Phi(t, r)v_T, \]

we obtain the equivalent final value problem for the \(n \times n\) matrix-valued function \(\Phi = \Phi(t, r)\):

\[
\frac{\partial \Phi}{\partial t} + \frac{1}{2} [B_0(t) + rB_1(t)] \frac{\partial^2 \Phi}{\partial r^2} + [A_0(t) + rA_1(t)] \frac{\partial \Phi}{\partial r} + (Q^T - rI)\Phi = 0, \tag{2.9}
\]
\[ \Phi(T, r) = I, \] (2.10)

where \( I \) and \( O \) are the \( n \times n \) identity and zero matrices, respectively. Due to this equivalence, it suffices to look for the explicit solution of the problem (2.9), (2.10).

3. Main result: Exact and approximate bond pricing formulas. Consider the equivalent final value problems (2.7), (2.8) and (2.9), (2.10). We make the following assumptions:

(i) Let \( y = y(t) \) be a positive smooth function such that

\[ \frac{dy}{dt} = \frac{1}{2} \bar{b}(t)y^2 - \bar{a}(t)y - 1, \quad y(T) = 0 \] (3.1)

for some continuous functions \( \bar{a} = \bar{a}(t) \) and \( \bar{b} = \bar{b}(t) \). Denote by \( Y \) the collection of all such functions.

(ii) Let \( G = G(t; y) \) be the \( n \times n \) matrix-valued function given by

\[ G(t; y) = \frac{1}{2} y(t)^2 [B_1(t) - \bar{b}(t)I] - y(t) [A_1(t) - \bar{a}(t)I]. \] (3.2)

(iii) Define the \( n \times n \) matrix-valued function \( F_1 = F_1(t; y) \) by

\[ F_1(t; y) = \text{diag}(e^{-\int_t^T d_1(s; y) \, ds}, e^{-\int_t^T d_2(s; y) \, ds}, \ldots, e^{-\int_t^T d_n(s; y) \, ds}), \] (3.3)

where

\[ d_j(t; y) = -\frac{1}{2} b_{0,j}(t)y(t)^2 + a_{0,j}(t)y(t), \quad j = 1, 2, \ldots, n. \] (3.4)

Observe that \( F_1(T; y) = I \).

(iv) Let \( F_2 = F_2(t; y) \) be the \( n \times n \) matrix-valued function that satisfies the final value problem

\[ \frac{dF_2}{dt} = -[F_1(t; y)^{-1} Q^T F_1(t; y)] F_2, \quad F_2(T; y) = I. \] (3.5)

(v) Define the functional \( J(y) \), for every \( y \in Y \), by

\[ J(y) = \int_0^T \left( \frac{1}{2} e^{2y(t)^2} \| G(t; y) F_1(t; y) F_2(t; y) \|^2 \right) dt, \] (3.6)

where \( \| \cdot \| \) is some given matrix norm.

Note that

\[ \frac{1}{y(t)^2} \| G(t; y) \|^2 \leq \frac{1}{4} y(t)^3 \| B_1(t) - \bar{b}(t)I \|^2 + y(t) \| A_1(t) - \bar{a}(t)I \| \| B_1(t) - \bar{b}(t)I \| + \| A_1(t) - \bar{a}(t)I \|^2. \]
and therefore

\[
J(y) \leq \int_0^T \frac{1}{y(t)^2} \|G(t; y)F_1(t; y)F_2(t; y)\|^2 \, dt \\
\leq \int_0^T \frac{1}{4} y(t)^3 \|B_1(t) - \tilde{b}(t)I\|^2 \|F_1(t; y)\|^2 \|F_2(t; y)\|^2 \, dt \\
+ \int_0^T y(t) \|A_1(t) - \tilde{a}(t)I\| \|B_1(t) - \tilde{b}(t)I\| \|F_1(t; y)\|^2 \|F_2(t; y)\|^2 \, dt \\
+ \int_0^T \|A_1(t) - \tilde{a}(t)I\|^2 \|F_1(t; y)\|^2 \|F_2(t; y)\|^2 \, dt.
\]

Hence it follows that \( J(y) \) is finite for every \( y \in Y \).

Since \( J(y) \geq 0 \) for every \( y \in Y \), we know that \( \inf_{y \in Y} J(y) \) exists. If there exists \( y^* \in Y \), with corresponding coefficient functions \( \tilde{a}^* = \tilde{a}^*(t) \) and \( \tilde{b}^* = \tilde{b}^*(t) \) in (3.1), such that

\[
J(y^*) = \inf_{y \in Y} J(y) = 0,
\]

then we claim that

\[
v(t, r; y^*) = e^{-ry^*(t)} F_1(t; y^*)F_2(t; y^*)v_T
\]

is the solution to the final value problem (2.7), (2.8).

**Remark 3.1.** Eq. (3.7) is the regime-switching analogue of the exponential affine bond pricing solution under a standard ATSM framework. Indeed, if there is no regime switching, then \( Q = O \) and (3.5) implies that \( F_2(t; y^*) = I \) for all \( t \in [0, T] \). Thus

\[
v(t, r; y^*) = e^{-ry^*(t)} \text{diag}(e^{-\int_0^t d_1(s;y^*) \, ds}, e^{-\int_0^t d_2(s;y^*) \, ds}, \ldots, e^{-\int_0^t d_n(s;y^*) \, ds})v_T
\]

and each entry of this \( n \times 1 \) matrix has the form of an exponential affine function in the variable \( r \), namely,

\[
e^{-ry^*(t)-\int_0^t d_j(s;y^*) \, ds}, \quad j = 1, 2, \ldots, n.
\]

To prove (3.7), first define the \( n \times n \) matrix-valued function \( R = R(t, r) \) by

\[
R(t, r) = \frac{\partial \Phi}{\partial t}(t, r) + \frac{1}{2} [B_0(t) + rB_1(t)] \frac{\partial^2 \Phi}{\partial r^2}(t, r) + [A_0(t) + rA_1(t)] \frac{\partial \Phi}{\partial r}(t, r) \\
+ (Q^I - rI) \Phi(t, r).
\]

Then \( \Phi \) satisfies the PDE (2.9) if and only if \( R(t, r) = O \) for every \( r > 0 \) and for every \( t \in [0, T] \). Suppose that

\[
\Phi(t, r) = e^{-ry(t)} F(t),
\]

where \( y = y(t) \) is a real-valued function and \( F = F(t) \) is an \( n \times n \) matrix-valued function to be determined such that \( y(T) = 0 \) and \( F(T) = I \). This ensures that \( \Phi(T, r) = I \) and \( v(T, r) = \Phi(T, r) v_T = v_T \). More precisely, let \( y \) satisfy hypothesis (i). Furthermore, let \( F \) satisfy the final value problem for the linear first-order
Substituting into the residual formula (3.8) yields

$$F = \frac{dF}{dt} = \left( -\frac{1}{2}y(t)^2B_0(t) + y(t)A_0(t) - Q^T \right) F, \quad F(T) = I. \quad (3.10)$$

It follows that

$$\frac{\partial \Phi}{\partial t} = e^{-\tau y} \frac{dF}{dt} - r e^{-\tau y} \frac{dy}{dt} F, \quad \frac{\partial \Phi}{\partial r} = -y e^{-\tau y} F, \quad \frac{\partial^2 \Phi}{\partial r^2} = y^2 e^{-\tau y} F.$$ 

Substituting into the residual formula (3.8) yields

$$R = e^{-\tau y} \frac{dF}{dt} - r e^{-\tau y} \frac{dy}{dt} F + \frac{1}{2} (B_0 + r B_1) y^2 e^{-\tau y} F - (A_0 + r A_1) y e^{-\tau y} F$$

$$+ (Q^T - r I) e^{-\tau y} F$$

$$= e^{-\tau y} \left( \frac{dF}{dt} + \frac{1}{2} y^2 B_0 F - y A_0 F + Q^T F \right)$$

$$+ r e^{-\tau y} \left( -\frac{dy}{dt} I + \frac{1}{2} y^2 B_1 - y A_1 - I \right) F.$$ 

But recalling (3.1), (3.2), and (3.10), we obtain

$$R(t, r) = r e^{-\tau y(t)} G(t) F(t). \quad (3.11)$$

Next, let us take closer look at (3.10). Defining

$$D(t) = -\frac{1}{2} y(t)^2 B_0(t) + y(t) A_0(t),$$

we see that $D(t)$ is a diagonal matrix since both $A_0(t)$ and $B_0(t)$ are. In fact, $D(t) = \text{diag}(d_1(t), d_2(t), \ldots, d_n(t))$, where $d_j(t)$ for $j = 1, 2, \ldots, n$ are defined in (3.4). Then (3.10) can be rewritten as

$$\frac{dF}{dt} = [D(t) - Q^T] F, \quad F(T) = I. \quad (3.12)$$

Decompose $F(t)$ into $F(t) = F_1(t) F_2(t)$, where $F_1$ and $F_2$ are as given in (3.3) and (3.5), respectively. We see that $F(T) = F_1(T) F_2(T) = I$. Furthermore,

$$\frac{dF}{dt} - [D(t) - Q^T] F = \frac{dF_1}{dt} F_2 + F_1 \frac{dF_2}{dt} - D(t) F_1 F_2 + Q^T F_1 F_2$$

$$= \left[ \frac{dF_1}{dt} - D(t) F_1 \right] F_2 + F_1 \frac{dF_2}{dt} + Q^T F_1 F_2.$$ 

Suppose that $F_1$ is the solution of the final value problem

$$\frac{dF_1}{dt} = D(t) F_1, \quad F_1(T) = I.$$

As $D(t)$ is a diagonal matrix, $F_1(t)$ will also be a diagonal matrix whose diagonal elements are obtained by solving the corresponding uncoupled first-order linear ODEs, thus yielding (3.3). Since $\det F_1(t) \neq 0$, we know that $F_1(t)^{-1}$ exists. In fact,

$$F_1(t)^{-1} = \text{diag}\left(e^{T_1} d_1(s) ds, e^{T_2} d_2(s) ds, \ldots, e^{T_n} d_n(s) ds\right).$$

If $F_2$ solves (3.5), then

$$F_1 \frac{dF_2}{dt} + Q^T F_1 F_2 = O.$$

Hence

$$\frac{dF}{dt} - [D(t) - Q^T] F = O$$

and therefore $F(t) = F_1(t) F_2(t)$ solves (3.12).
From (3.11), we see that
\[\|R(t,r)\|^2 = r^2 e^{-2y(t)} \|G(t)F_1(t)F_2(t)\|^2.\]
The function \(r \mapsto r^2 e^{-\alpha r}\), where \(\alpha > 0\), is maximized when \(r = \frac{2}{\alpha}\) and its maximum value is \(\frac{4}{e^{2\alpha}}\). Taking \(\alpha = 2y(t)\), we deduce that
\[\|R(t,r)\|^2 \leq \frac{1}{e^{2y(t)}} \|G(t)F_1(t)F_2(t)\|^2.\]
For every \(r > 0\), we integrate both sides to get
\[\int_0^T \|R(t,r)\|^2 dt \leq \int_0^T \frac{1}{e^{2y(t)}} \|G(t)F_1(t)F_2(t)\|^2 dt.\]
We deduce that
\[0 \leq \int_0^T \|R(t,r)\|^2 dt \leq J(y) \quad \text{for every } y \in Y. \quad (3.13)\]
If there exists \(y^* \in Y\) such that \(J(y^*) = 0\), then
\[R(t,r; y^*) = re^{-y^*(t)}G(t; y^*)F_1(t; y^*)F_2(t; y^*) = O\]
for every \(r > 0\) and for every \(t \in [0,T]\). This implies that
\[\Phi(t,r; y^*) = e^{-y^*(t)}F_1(t; y^*)F_2(t; y^*)\]
satisfies (2.9). Moreover, \(\Phi(T,r; y^*) = I\). Thus \(\Phi\) is the solution of the final value problem (2.9), (2.10). Equivalently,
\[v(t,r; y^*) = e^{-y^*(t)}F_1(t; y^*)F_2(t; y^*)v_T\]
is the solution of the final value problem (2.7), (2.8). This proves the claim.

3.1. Exact analytical bond pricing formula. A special case is if the entries of each of the diagonal matrices \(A_1(t)\) and \(B_1(t)\) are the same. This was the assumption made in [12] for the regime-switching Vasicek and CIR models. Note, however, that the diagonal matrices \(A_0(t)\) and \(B_0(t)\) are still assumed to have arbitrary (possibly distinct) entries. Then choose \(\tilde{a}\) and \(\tilde{b}\) such that
\[\tilde{a}(t) = a_{1,1}(t) = \cdots = a_{1,n}(t), \quad \tilde{b}(t) = b_{1,1}(t) = \cdots = b_{1,n}(t), \quad (3.14)\]
so that \(y = y^*\) with these coefficient functions \(\tilde{a}\) and \(\tilde{b}\) solves (3.1). Then from (3.2) we see that \(G(t; y^*) = O\) and therefore \(J(y^*) = 0\). Hence (3.7) is the exact solution of (2.7), (2.8). Of course, even though (3.7) is exact, we still need to know the solution of the linear first-order system (3.5).

3.2. Approximate analytical bond pricing formula. In general, the diagonal matrices \(A_1(t)\) and \(B_1(t)\) are not time-varying multiples of the identity matrix. Then \(G(t; y)\) in (3.2) is not necessarily zero, and neither is \(J(y)\) in (3.6). The problem now becomes how to choose \(y\) (equivalently, the functions \(\tilde{a}\) and \(\tilde{b}\)) such that \(J(y)\) is minimized; see (3.13). One way of proceeding is to choose \(\tilde{a}(t)\) and \(\tilde{b}(t)\) as weighted combinations of the diagonal entries of \(A_1(t)\) and \(B_1(t)\), respectively. That is, let
\[\tilde{a}(t) = \sum_{j=1}^n \omega_{a,j} a_{1,j}(t), \quad \tilde{b}(t) = \sum_{j=1}^n \omega_{b,j} b_{1,j}(t),\]
where $\omega_{a,j}$ and $\omega_{b,j}$ for $j = 1, 2, \ldots, n$ are nonegative constants such that
\[
\sum_{j=1}^{n} \omega_{a,j} = 1 = \sum_{j=1}^{n} \omega_{b,j}.
\]

Using a standard variational analysis, we can choose the weights so as minimize the error functional $J(y)$ to within a desired tolerance. In particular, if we assume that all the entries of the diagonal matrices $A_1(t)$ and $B_1(t)$ are the same, then we may choose
\[
\omega_{a,j} = \frac{1}{2} = \omega_{b,j}, \quad j = 1, 2, \ldots, n,
\]
so that (3.14) is recovered in this special case.

4. **Numerical and analytical approximations for the bond price.** The Riccati equation (3.1) is not analytically solvable in general except for a few short rate models, e.g., those due to Vasicek, Hull-White [24], Cox-Ingersoll-Ross, etc. (although see Rodrigo and Mamon [28] for a solution in series form when the coefficient functions $\bar{a}$ and $\bar{b}$ are arbitrary analytic functions of $t$). The linear system (3.5) is also not analytically tractable in general since the associated coefficient matrix is not constant. But both (3.1) and (3.5) can easily be solved numerically. In this section, we explore approximate numerical and analytical methods for solving the linear system (3.5).

We first consider a numerical approximation. Let $M(t) = F_1(t)^{-1}Q^t F_1(t)$. Then (3.5) becomes
\[
\frac{dF_2}{dt} = -M(t) F_2, \quad F_2(T) = I. \tag{4.1}
\]
Partition $[0, T]$ into $\{t_1, t_2, \ldots, t_K\}$ in such a way that
\[
t_k = (k - 1)\Delta t, \quad k = 1, 2, \ldots, K, \quad \Delta t = \frac{T}{K - 1},
\]
where $K$ is a sufficiently large positive integer. Note that $t_1 = 0$ and $t_K = T$. We denote an approximation of $F_2(t_k)$ at any time $t_k$ by $F_{2,k}$, i.e., $F_2(t_k) \simeq F_{2,k}$, so that $F_{2,k} = I$. An Euler approximation of the system of ODEs in (4.1) is given by
\[
\frac{1}{\Delta t} (F_{2,k+1} - F_{2,k}) = -M(t_k) F_{2,k} \quad \text{or} \quad F_{2,k+1} = [I - \Delta t M(t_k)] F_{2,k}.
\]
Therefore $F_{2,k}$ is recursively determined by the algorithm
\[
F_{2,K} = I, \quad F_{2,k} = [I - \Delta t M(t_k)]^{-1} F_{2,k+1}, \quad k = K - 1, K - 2, \ldots, 1.
\]
As (4.1) is a final value problem, we march backward in time to calculate $F_{2,k}$ using $F_{2,k+1}$. Observe that
\[
F_{2,K-1} = [I - \Delta t M(t_{K-1})]^{-1},
\]
\[
F_{2,K-2} = [I - \Delta t M(t_{K-2})]^{-1} F_{2,K-1} = [I - \Delta t M(t_{K-2})]^{-1} [I - \Delta t M(t_{K-1})]^{-1}
\]
\[
= \{[I - \Delta t M(t_{K-1})][I - \Delta t M(t_{K-2})]\}^{-1},
\]
and so on. By induction on $k$, we see that
\[
F_2(t_k) \simeq F_{2,k} = \left\{ \prod_{m=1}^{K-k} [I - \Delta t M(t_{K-m})] \right\}^{-1}, \quad k = K - 1, K - 2, \ldots, 1.
\]
Hence the bond price formula (3.7) is approximated by
\[
v(t_k, r; y^*) = e^{-rv^*(t_k)}F_1(t_k; y^*)F_2(t_k; y^*)v_T
\]
\[
\simeq e^{-rv^*(t_k)}\text{diag}(e^{-\int_{t_k}^{T} d_1(s; y^*) \, ds}, e^{-\int_{t_k}^{T} d_2(s; y^*) \, ds}, \ldots, e^{-\int_{t_k}^{T} d_n(s; y^*) \, ds}) \\
\times \left\{ \prod_{m=1}^{K-k} \left[ I - \Delta t M(t_{K-m}; y^*) \right] \right\}^{-1} v_T
\]
(4.2)
at each \( t_k \in [0, T] \), where \( k = 1, 2, \ldots, K - 1 \). We remark that the number \( n \) of regimes is typically small, say \( n = 2 \) or \( n = 3 \). Therefore the matrix inversion in (4.2) is not costly since it is performed only once after taking the matrix product. Nevertheless, since \( \Delta t \) is small, we can use the Neumann series
\[
(I - A)^{-1} = I + A + A^2 + \cdots, \quad \|A\| < 1
\]
to approximate
\[
[I - \Delta t M(t_{K-m})]^{-1} \simeq I + \Delta t M(t_{K-m}),
\]
thus yielding
\[
F_2(t_k) \simeq F_{2,k} = \prod_{m=K-k}^{1} [I - \Delta t M(t_{K-m})]^{-1}
\]
\[
\simeq \prod_{m=K-k}^{1} [I + \Delta t M(t_{K-m})], \quad k = K - 1, K - 2, \ldots, 1.
\]

Therefore an alternative numerical approximation of the bond price formula (3.7) that does not involve matrix inversion is
\[
v(t_k, r; y^*) = e^{-rv^*(t_k)}F_1(t_k; y^*)F_2(t_k; y^*)v_T
\]
\[
\simeq e^{-rv^*(t_k)}\text{diag}(e^{-\int_{t_k}^{T} d_1(s; y^*) \, ds}, e^{-\int_{t_k}^{T} d_2(s; y^*) \, ds}, \ldots, e^{-\int_{t_k}^{T} d_n(s; y^*) \, ds}) \\
\times \prod_{m=K-k}^{1} [I + \Delta t M(t_{K-m})]v_T
\]
(4.3)
at each \( t_k \in [0, T] \), where \( k = 1, 2, \ldots, K - 1 \).

Next, to obtain an approximate analytical bond pricing formula, let us reexamine (3.5). As mentioned previously, the difficulty arises from the fact that the associated matrix \( M(t) = F_1(t)^{-1}Q^T F_1(t) \) depends on \( t \). Suppose that we average out \( F_1(t) \) over \([0, T]\) by replacing it with the continuous average
\[
\bar{F}_1 = \frac{1}{T} \int_{0}^{T} F_1(t) \, dt.
\]
Note that \( \bar{F}_1 \) is an \( n \times n \) constant diagonal matrix. This is equivalent to taking the continuous average of each of the diagonal entries in (3.3), i.e., the diagonal entries of \( \bar{F}_1 \) are of the form
\[
\frac{1}{T} \int_{0}^{T} e^{-\int_{0}^{t} d_j(s; y^*) \, ds} \, dt, \quad j = 1, 2, \ldots, n.
\]
Therefore $F_2$ approximately satisfies the final value problem

$$ \frac{dF_2}{dt} = -(\bar{F}_1^{-1}Q^i\bar{F}_1)F_2, \quad F_2(T) = I,$$

whose solution is the matrix exponential

$$ F_2(t) = e^{(T-t)\bar{F}_1^{-1}Q^i\bar{F}_1}. $$

Hence an approximate analytical bond pricing formula in a time-varying regime-switching ATSM framework is given by

$$ v(t, r; y^*) = e^{-ry^*(t)}F_1(t; y^*)F_2(t; y^*)v_T $$

$$ \simeq e^{-ry^*(t)} \text{diag}(e^{-\int_t^T d_1(s; y^*)\,ds}, e^{-\int_t^T d_2(s; y^*)\,ds}, \ldots, e^{-\int_t^T d_n(s; y^*)\,ds}) $$

$$ \times e^{(T-t)\bar{F}_1^{-1}Q^i\bar{F}_1}v_T $$

for all $t \in [0, T]$.

**Remark 4.1.** In the practical implementation of (4.4), it is possible to modify $\bar{F}_1$ by replacing it with a weighted average, thus allocating more weight to information deemed more reliable, e.g., market prices/data from investment contracts with short-term maturity.

**5. Example: Regime-switching Vasicek model.** Here, for simplicity, we illustrate our result in the case of a regime-switching Vasicek model as considered by Elliott and Siu [12]. We emphasize, however, that our result holds for general ATSMs as described in (3.1)–(3.7). In regime $j$, where $j = 1, 2, \ldots, n$, the short rate evolves according to

$$ dr_t = \alpha_j (\beta_j - r_t) \, dt + \sigma_j \, dW_t, $$

where $\alpha, \beta, \sigma > 0$ for all $j = 1, 2, \ldots, n$. In this case, we have

$$ a_{0,j}(t) = \alpha_j, \quad a_{1,j}(t) = -\alpha, \quad b_{0,j}(t) = \sigma_j^2, \quad b_{1,j}(t) = 0, \quad j = 1, 2, \ldots, n. $$

Eq. (2.6) becomes

$$ A_0(t) = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad B_0(t) = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2), $$

$$ A_1(t) = \text{diag}(-\alpha, -\alpha, \ldots, -\alpha) = -\alpha I, \quad B_1(t) = \text{diag}(0, 0, \ldots, 0) = O. $$

Note that these imply that (3.14) holds, hence we are able to obtain an exact analytical bond price formula. The Riccati ODE (3.1), choosing $a(t) = -\alpha$ and $b(t) = 0$, reduces to the linear ODE

$$ \frac{dy^*}{dt} = \alpha y^* - 1, \quad y^*(T) = 0, \tag{5.1} $$

whose exact solution is

$$ y^*(t) = \frac{1}{\alpha} [1 - e^{-\alpha(T-t)}]. \tag{5.2} $$

Using (3.4), we obtain

$$ d_j(t; y^*) = -\frac{\sigma_j^2}{2} y^*(t)^2 + \alpha_j y^*(t), \quad j = 1, 2, \ldots, n. $$

To find $F_1(t; y^*)$ from (3.3), we need to calculate

$$ e^{-\int_t^T d_j(s; y^*)\,ds} = e^{-\int_t^T [-\frac{\sigma_j^2}{2} y^*(s)^2 + \alpha_j y^*(s)]\,ds}, \quad j = 1, 2, \ldots, n.
Integrating (5.1) from $t$ to $T$ yields
\[ \int_t^T y^*(s) \, ds = -\frac{1}{\alpha} [y^*(t) - (T - t)]. \]

Multiplying both sides of (5.1) by $y^*$ and integrating from $t$ to $T$, we obtain
\[ \int_t^T y^*(s)^2 \, ds = -\frac{1}{2\alpha} y^*(t)^2 - \frac{1}{\alpha^2} [y^*(t) - (T - t)]. \]

Define
\[ \phi_j(t; y^*) = \int_t^T d_j(s; y^*) \, ds \]
\[ = \frac{\sigma_j^2}{4\alpha} y^*(t)^2 + \left( \frac{\sigma_j^2}{2\alpha^2} - \beta_j \right) [y^*(t) - (T - t)], \quad j = 1, 2, \ldots, n. \]

Therefore (3.3) gives
\[ F_1(t; y^*) = \text{diag}(e^{-\phi_1(t; y^*)}, e^{-\phi_2(t; y^*)}, \ldots, e^{-\phi_n(t; y^*)}), \]
so that
\[ F_1(t; y^*)^{-1} = \text{diag}(e^{\phi_1(t; y^*)}, e^{\phi_2(t; y^*)}, \ldots, e^{\phi_n(t; y^*)}), \]
while $F_2 = F_2(t; y^*)$ is the solution of
\[ \frac{dF_2}{dt} = -M(t; y^*) F_2 = -[F_1(t; y^*)^{-1} Q F_1(t; y^*)] F_2, \quad F_2(T; y^*) = I. \]

Hence the bond pricing formula (3.7) under a regime-switching Vasicek dynamics for the short rate is given by
\[ v(t, r; y^*) = e^{-\int_t^T r(s) \, ds} \text{diag}(e^{-\phi_1(t; y^*)}, e^{-\phi_2(t; y^*)}, \ldots, e^{-\phi_n(t; y^*)}) F_2(t; y^*) v_T. \]

This is an exact representation as $G(t; y^*) = O$ from (3.2); hence $J(y^*) = 0$. More precisely, the time-zero bond price
\[ P(0, T) = v^T(0, r_0, X_0) \]
is the $j$th component of the $n \times 1$ matrix
\[ e^{-\int_0^T r(s) \, ds} \text{diag}(e^{-\phi_1(0; y^*)}, e^{-\phi_2(0; y^*)}, \ldots, e^{-\phi_n(0; y^*)}) F_2(0; y^*) v_T \]
if $X_0 = u_j$.

For a specific case, consider a two-state Markovian regime-switching Vasicek model (i.e., $n = 2$), and let the rate matrix be
\[ Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix} \]
for some given $q > 0$. State 1 (respectively, State 2) of the Markov chain represents a “good” (respectively, “bad”) economy. When the economy is good, the short rate reverts to a higher level and is less volatile. This leads to the assumption that $\beta_1 > \beta_2$ and $\sigma_1 < \sigma_2$. A two-state Markovian regime-switching Vasicek model is therefore
\[ dr_t = \begin{cases} \alpha(\beta_1 - r_t) \, dt + \sigma_1 \, dW_t & \text{if the economy is good,} \\ \alpha(\beta_2 - r_t) \, dt + \sigma_2 \, dW_t & \text{if the economy is bad.} \end{cases} \]
Let us choose the parameter values $\alpha = 0.5$, $\beta_1 = 0.3$, $\beta_2 = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $q = 0.2$, $r = 0.1$, and $T = 2$. We take $K = 200$ for the number of partition points of $[0, T]$, which gives $\Delta t = 0.01$. To three decimal places, the first numerical approximation formula (4.2) gives $v(0, r; y) = (0.757, 1.007)$, while the second numerical approximation formula (4.3) yields $v(0, r; y) = (0.757, 1.007)$. Comparing these values with those obtained from the analytical approximation formula (4.4), we obtain $v(0, r; y) = (0.750, 1.012)$. Thus we see a very good agreement among the three formulas. Further numerical simulations reveal that the bond prices using the analytical approximation formula (4.4) start to deviate more when the time to maturity $T$ becomes longer, e.g., $T = 5$ or $T = 10$. This is to be expected as we replaced $F_1(t)$ for all $t \in [0, T]$ by its average value $F_1$. However, formulas (4.2) and (4.3) remain accurate for any $T$.

Now suppose that we consider the more general short rate dynamics such that in regime $j$ there holds

$$dr_t = \alpha_j(\beta_j - r_t) dt + \sigma_j dW_t,$$

where $\alpha_j, \beta_j, \sigma_j > 0$ for all $j = 1, 2, \ldots, n$. In this case, we have

$$a_{0,j}(t) = \alpha_j \beta_j, \quad a_{1,j}(t) = -\alpha_j, \quad b_{0,j}(t) = \sigma_j^2, \quad b_{1,j}(t) = 0, \quad j = 1, 2, \ldots, n.$$  

The crucial difference with the previous example is that the strength of mean reversion parameter $\alpha_j$ is now allowed to depend on the regime $j$. Eq. (2.6) becomes

$$A_0(t) = \text{diag}(\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_n \beta_n), \quad B_0(t) = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2),$$

$$A_1(t) = \text{diag}(-\alpha_1, -\alpha_2, \ldots, -\alpha_n), \quad B_1(t) = \text{diag}(0, 0, \ldots, 0) = O.$$  

Note that (3.14) does not necessarily hold anymore, so that we are not always able to obtain an exact analytical bond price formula. The Riccati ODE (3.1), choosing $\bar{a}(t) = -\bar{\alpha}$ (a constant still to be determined) and $\bar{b}(t) = 0$, reduces to the linear ODE

$$\frac{dy}{dt} = \bar{\alpha} y - 1, \quad y(T) = 0,$$

whose exact solution is

$$y(t) = \frac{1}{\bar{\alpha}} [1 - e^{-\alpha(\bar{T} - t)}].$$

Here, (3.2) simplifies to

$$G(t; y) = y(t) \text{diag}(\alpha_1 - \bar{\alpha}, \alpha_2 - \bar{\alpha}, \ldots, \alpha_n - \bar{\alpha}).$$

Following the same calculations as before, an approximate analytical bond pricing formula under a regime-switching Vasicek dynamics for the short rate is given by

$$v(t, r; y) = e^{-r y(t)} \text{diag}(e^{-\phi_1(t; y)}, e^{-\phi_2(t; y)}, \ldots, e^{-\phi_n(t; y)}) F_2(t; y) v_T,$$

where

$$\phi_j(t; y) = \frac{\sigma_j^2}{4\bar{\alpha}} y(t)^2 + \left( \frac{\sigma_j^2}{2\bar{\alpha}^2} - \frac{\alpha_j \beta_j}{\bar{\alpha}} \right) [y(t) - (T - t)], \quad j = 1, 2, \ldots, n,$$

$$F_1(t; y) = \text{diag}(e^{-\phi_1(t; y)}, e^{-\phi_2(t; y)}, \ldots, e^{-\phi_n(t; y)}),$$

$$F_1(t; y)^{-1} = \text{diag}(e^{\phi_1(t; y)}, e^{\phi_2(t; y)}, \ldots, e^{\phi_n(t; y)}),$$

and $F_2 = F_2(t; y)$ is the solution of

$$\frac{dF_2}{dt} = -M(t; y) F_2 = -[F_1(t; y)^{-1} Q^T F_1(t; y)] F_2, \quad F_2(T; y) = I.$$
We still have not assumed anything about $\bar{\alpha}$, so we suppose that it is a weighted combination of $a_{1,j} = -\alpha_j$ for $j = 1, 2, \ldots, n$, i.e.,

$$-\bar{\alpha} = \bar{a}(t) = \omega_1 a_{1,1}(t) + \omega_2 a_{1,2}(t) + \cdots + \omega_n a_{1,n}(t) = -\omega_1 \alpha_1 - \omega_2 \alpha_2 - \cdots - \omega_n \alpha_n$$

such that

$$\omega_j \geq 0, \ j = 1, 2, \ldots, n, \ \omega_1 + \omega_2 + \cdots + \omega_n = 1.$$ 

Thus

$$\bar{\alpha} = \omega_1 \alpha_1 + \omega_2 \alpha_2 + \cdots + \omega_n \alpha_n.$$ 

Using standard variational techniques, we choose the weights $\omega_1, \omega_2, \ldots, \omega_n$ such that the error functional

$$J(y) = \int_0^T \frac{1}{e^{\gamma(t)^2}} \|G(t; y)F_1(t; y)F_2(t; y)\|^2 \, dt,$$

viewed as a function of $\omega_1, \omega_2, \ldots, \omega_n$, is minimized.

As before, consider a two-state Markovian regime-switching Vasicek model:

$$\mathrm{d}r_t = \begin{cases} 
\alpha_1(\beta_1 - r_t) \, \mathrm{d}t + \sigma_1 \, \mathrm{d}W_t & \text{if the economy is good}, \\
\alpha_2(\beta_2 - r_t) \, \mathrm{d}t + \sigma_2 \, \mathrm{d}W_t & \text{if the economy is bad}.
\end{cases}$$

Now choose the parameter values $\alpha_1 = 0.5$, $\alpha_2 = 0.2$, $\beta_1 = 0.3$, $\beta_2 = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $q = 0.1$, $r = 0.2$, and $T = 2$. We take $K = 200$ for the number of partition points of $[0, T]$, which gives $\Delta t = 0.01$. Take, for instance, the matrix norm $\| \cdot \|_2$.

The following table summarizes the bond price calculations as a result of trying out different values for the weights $\omega_1$ and $\omega_2$. The corresponding values of the error $J(y)$ are also included.

| $\omega_1$ | $\omega_2$ | $v(0, r; y)$ in (4.2) | $v(0, r; y)$ in (4.3) | $v(0, r; y)$ in (4.4) |
|------------|------------|----------------------|----------------------|----------------------|
| 0.1        | 0.9        | (0.727, 1.108)       | (0.727, 1.108)       | (0.716, 1.114)       |
| $J(y)$     |            | 0.013                | 0.013                | 0.013                |
| 0.3        | 0.7        | (0.735, 1.091)       | (0.735, 1.091)       | (0.725, 1.097)       |
| $J(y)$     |            | 0.008                | 0.008                | 0.008                |
| 0.5        | 0.5        | (0.743, 1.076)       | (0.743, 1.076)       | (0.734, 1.082)       |
| $J(y)$     |            | 0.006                | 0.006                | 0.006                |
| 0.7        | 0.3        | (0.751, 1.064)       | (0.751, 1.064)       | (0.742, 1.070)       |
| $J(y)$     |            | 0.010                | 0.010                | 0.010                |
| 0.9        | 0.1        | (0.758, 1.053)       | (0.758, 1.052)       | (0.750, 1.058)       |
| $J(y)$     |            | 0.016                | 0.016                | 0.016                |

We see that the smallest error occurs when we choose $\omega_1 = 0.5$ and $\omega_2 = 0.5$, which gives $J(y) = 0.006$. Therefore the approximate regime-switching bond prices are $v(0, r; y) = (0.743, 1.076)$.

6. Concluding remarks. In [33], van Beek, et al. noted the “rather restraint body of literature” on Markov-switching affine processes and their use in computing the closed-form pricing solutions, or solutions imbedded in certain differential equations, to bonds and other fixed-income instruments. Our results in this paper, therefore, attempt to appreciably shrink this gap in a unifying way. In particular,
our results were derived under the more general time-dependent ATSM class; refer to Section 3. Thus we provided a significant extension to Elliott and Siu [12] addressing various implementability considerations for a large class of models in capturing the dynamic term structure of interest rates. Consequently, practical insights arising from our formulation and numerical work can support the conduct of future empirical studies of interest rates as they richly complement the approaches highlighted in Singleton [30].

Additionally, this paper relies heavily on lines of reasoning promoting the PDE technique, and hence it sleekly complements the probabilistic perspective of bond pricing. It is worth noting that from (3.7), our bond price representation neatly depicts how the rate matrix and other model parameter inputs will play out in the pricing calculation. Our construction of the numerical and analytic approximations underlines a commitment to financial practice and addresses the implementation concerns of industrial end users. New developments in the analysis and valuation of long-term insurance contracts with correlated risk factors described by ATSMs (e.g., [37] and [38]) as well as dynamic investment protection plan [17] could benefit further from the introduction of regime-switching features of this paper.

The natural research direction of this work is the consideration of a multi-factor model setting in which each factor may evolve as well as a regime-switching process. Exploring this extension anew may pave the way to the modeling of interest rate data with more complex features. This expected further complexity on financial data that we shall experience posits on more uncertainty brought about by a financial market that is incessantly bombarded by increased and unfamiliar pressures due to various political, economic, environmental, and sociological events unfolding around the globe. The calibration of a bond pricing model to current market prices, as explained in [29], is a cornerstone for its successful adoption. Hence another direction of this work is to consider the calibration of an exponential affine bond price model with regime switching that is being put forward in this research. The work entails probing efforts analogous to those in the pricing of options under regime switching set forth by Xi, et al. [36].

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*E-mail address: marianito.rodrigo@uow.edu.au*

*E-mail address: rmamon@stats.uwo.ca*