MAXIMAL ESTIMATES FOR WEYL SUMS ON $\mathbb{T}^d$

CHANGXING MIAO, JIYE YUAN, AND TENGFEI ZHAO

(With an Appendix by Alex Barron)

Abstract. In this paper, we obtain the maximal estimate for the Weyl sums on the torus $\mathbb{T}^d$ with $d \geq 2$, which is sharp up to the endpoint. We also consider two variants of this problem which include the maximal estimate along the rational lines and on the generic torus. Applications, which include some new upper bound on the Hausdorff dimension of the sets associated to the large value of the Weyl sums, reflect the compound phenomenon between the square root cancellation and the constructive interference. In the Appendix, an alternate proof of Theorem 1.1 inspired by Baker’s argument in [1] is given by Barron, which also improves the $N^s$ loss in Theorem 1.1, and the Strichartz-type estimates for the Weyl sums with logarithmic losses are obtained by the same argument.

Key Words: Periodic Schrödinger equation; Weyl sums; Maximal estimate; Hausdorff dimension; Strichartz-type estimate.

AMS Classification: 42B25, 42B37, 35Q41.

1. Introduction

In this paper, we study the maximal estimate

$$\left\| \sup_{0 < t < 1} |u(x, t)| \right\|_{L^p(\mathbb{T}^d)} \lesssim N^s \left( \sum_{n \in \mathbb{Z}^d} |a_n|^2 \right)^{\frac{1}{2}}$$

(1.1)

of the following function

$$u(x, t) = \sum_{n \in \mathbb{Z}^d} a_n e^{2\pi i (n \cdot x + |n|^2 t)}$$

(1.2)

for $1 \leq p < \infty$ and $s > 0$, which is the solution of the Schrödinger equation

$$i \partial_t u - \Delta u = 0$$

(1.3)

with the periodic boundary condition

$$u(x, 0) = \sum_{n \in \mathbb{Z}^d} a_n e^{2\pi i n \cdot x}.$$

(1.4)

One of the prominent applications of the maximal estimates is the pointwise convergence of the solution to the corresponding equations. In the Euclidean setting, in 1980 Carleson in [11] first put out a question about finding the minimal $s$ such that for all $f \in H^s(\mathbb{R})$ there holds

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x), \quad \text{for a.e. } x \in \mathbb{R},$$

(1.5)

and he proved (1.5) holds for $s \geq \frac{1}{4}$ in $\mathbb{R}$. Later Dahlberg and Kenig in [18] proved that the result obtained by Carleson is sharp.
For the higher dimension cases, the geometric structure is quite different and seems more complicated. In dimensions $d \geq 2$, Sjölin in [42] and Vega in [46] established the pointwise convergence for $s > \frac{1}{2}$ independently. With the progress of Fourier restriction theory for the paraboloid or the sphere, many harmonic analysis tools are also applied to the pointwise convergence problem. By employing the localization lemmas and a crucial relation (defined in [43]) between tubes and cubes corresponding to the wave packet decomposition, Lee in [34] proved the pointwise convergence holds for $s > \frac{3}{8}$ in $\mathbb{R}^2$. On the other hand, for $d \geq 2$, by making use of the multilinear estimates for the Fourier extension operator, Bourgain in [7] proved $s > \frac{1}{2} - \frac{1}{4d}$ for $p \geq 2$. For negative results, Bourgain [8] proved the necessity of $s \geq d^2(d+1)$ with $d \geq 2$ for the pointwise convergence and Lucà and Rogers in [35] gave an alternative proof.

Recently, by establishing the refined Strichartz estimates, Du-Guth-Li in [21] proved the result of $s > \frac{1}{3}$ in dimension $d = 2$ via polynomial partition and $\ell^2$ decoupling estimates, see [9]. Later, Du-Zhang in [24] obtained that pointwise convergence holds for $s > \frac{d}{d+2}$ with $d \geq 3$, which is sharp up to the endpoint.

The refined Strichartz estimates are closely related to the refined decoupling estimates, see Du-Guth-Li-Zhang [22], Demeter [19] and Guth-Iosevich-Ou-Wang [28].

As in $\mathbb{R}^d$, considering the counterexample in Subsection 4.2 as well as in [23] and [24], one may conjecture that the maximal estimate (1.6) holds for $1 \leq p \leq 2$ and $s_p > \frac{d}{2(d+1)}$.

From preliminary analysis, compared to the Euclidean setting, one may find that the resonances in the periodic setting are more intense, so that maximal estimates (1.6) seem harder to justify. More precisely, the time localization arguments play an important role in the proof for the maximal estimate of the Schrödinger equation in $\mathbb{R}^d$, however, is invalid in $\mathbb{T}^d$, see [7, 21, 24, 34, 37]. From the Strichartz estimates in $\mathbb{T}^d$, Moyua and Vega in [34], Compaan, Lucà, and Staffilani in [17] proved that (1.6) holds for $1 \leq p \leq \frac{d+2}{d}$ and $s_p > \frac{d}{d+2}$, which seems the best result for the general initial value of the periodic Schrödinger equation up to now.

On the other hand, Barron [4] studied the maximal estimate (1.6) for a special case ($a_n = 1$) on $\mathbb{T}$ via induction on scales based on the classical Weyl estimates and the Hardy-Littlewood circle method. The key ingredients are exploring and handling the large values of the Weyl sums by better Diophantine approximation properties. Later, Baker in [11] gave another proof based on the Dirichlet approximation lemma for time $t$ and refined upper bound of Weyl sums, which exposes some direct information of the spatial variable $x$. Moreover, Baker, Chen and Shparlinski [2, 3] consider generalized maximal estimates of Weyl sums of the degree $d$.

In this paper, we first consider the maximal estimate for the Weyl sums in higher dimensions for $d \geq 2$.

**Theorem 1.1.**

\[ \left\| \sup_{0 < t < 1} \left\| \prod_{1 \leq j \leq d} \sum_{1 \leq n_j \leq N} a_n e^{2\pi i (n_j x_j + |n_j|^2 t)} \right\|_{L^p(T^d)} \right\| \lesssim N \frac{d}{2} + s_p, \] (1.7)
where \( s_p > \frac{d}{2(d+1)} \) when \( 1 \leq p \leq \frac{2(d+1)}{d} \) and \( s_p > \frac{d}{2} - \frac{d}{p} \) when \( p \geq \frac{2(d+1)}{d} \).

We prove this theorem by analysing the set where large values of the Weyl sums are obtained, inspired by the methods of \cite{1,4}. More precisely, by the Dirichlet theorem and the classical Weyl estimates, we find that the Weyl sums become large for certain time \( t \) which is in a neighborhood of \( a/q \) with coprime \( q \) satisfying \( 1 \leq a < q \). On the one hand, if there are some separating Weyl sums with respect to one direction \( x_j \) of \( x \) which has the upper bound \( N^{\frac{1}{2} + \frac{1}{d(d+1)}} \), then applying induction on the dimension \( d \), we would prove the maximal estimate \( 1.7 \). On the other hand, if all separating Weyl sums in every direction have the lower bound \( N^{\frac{1}{2} + \frac{1}{d(d+1)}} \), then from a Weyl-type lemma in \cite{9} and the refined Diophantine approximation in \cite{1,4}, one can utilize the Hardy-Littlewood circle method to deduce some additional restrictions on the parameter \( q \). Then, by utilizing the refined Weyl sums estimates of \cite{40}, we can obtain the restricted region of the spatial variable \( x \) in each dimension, and conclude a refined upper bound on the Weyl sums with respect to \( x \).

In the Appendix, based on the Baker’s argument in \cite{1}, another proof of Theorem 1.1 is given by Barron, which improves the \( N^\epsilon \) loss. The similar argument also implies the Strichartz-type estimates for the Weyl sums.

**Remark 1.2.** In view of a simple counterexample, we see that for \( p \geq \frac{2(d+1)}{d} \), the maximal estimate \( 1.7 \) is almost sharp up to the endpoint. Besides, for \( 1 \leq p \leq \frac{2(d+1)}{d} \), the loss in \( N \) for \( 1.7 \) cannot be improved even if \( p \) is decreased. In fact, one can construct a set \( E \subset \mathbb{T}^d \) with \( |E| \geq 1 \) such that for \( x \in E \),

\[
\sup_{0 < t < 1} \left| \prod_{1 \leq j \leq d} \sum_{n_j = 1}^N e^{2\pi i (n_j x_j + |n_j|^2 t)} \right| \gtrsim N^{\frac{d}{2} + \frac{1}{d(d+1)}}.
\]

See Section 4 for detail.

Besides the maximal estimates \( 1.7 \), we consider two invariant versions of the maximal estimates for the Weyl sums related to the pointwise convergence along the line \( (x - rt, t) \) for \( r \in \mathbb{Q}^d \) with \( 0 < t < 1 \) and the maximal estimates on the irrational tori defined below.

We first introduce the following maximal estimates.

**Theorem 1.3.** Let \( r = (r_1, r_2, \cdots, r_d) \in \mathbb{Q}^d \), then we have

\[
\left\| \sup_{0 < t < 1} \left| \prod_{1 \leq j \leq d} \sum_{1 \leq n_j \leq N} e^{2\pi i (n_j (x_j - rt) + |n_j|^2 t)} \right| \right\|_{L^p(\mathbb{T}^d)} \lesssim N^{\frac{d}{2} + s_p},
\]

where \( s_p > \frac{d}{2(d+1)} \) when \( 1 \leq p \leq \frac{2(d+1)}{d} \) and \( s_p > \frac{d}{2} - \frac{d}{p} \) when \( p \geq \frac{2(d+1)}{d} \).

**Remark 1.4.** (i) As a consequence of the maximal estimates \( 1.8 \), the Weyl sums converge to the initial datum along the lines \( (x - rt, t) \) for \( r \in \mathbb{Q}^d \) pointwise. In particular, the maximal estimates \( 1.7 \) is the special case of \( 1.8 \) for \( r = 0 \) which shows the pointwise convergence along the vertical line.

(ii) In \( \mathbb{R} \), using stationary phase analysis and the \( TT^* \) argument, Shiraki in \cite{41} and together with Cho in \cite{16} obtained the pointwise convergence for the Schrödinger equation along the non-tangential lines and the tangential lines respectively. The authors in \cite{47} and together with Zheng in \cite{47,38} studied these several variants of the pointwise convergence problem for the fractional Schrödinger operator with complex
On $\mathbb{T}$, Baker, Chen and Shparlinski in [2] dealt with the maximal estimates for the Weyl sums along rational and irrational non-tangential lines. However, because of the different properties between the space $\mathbb{T}^d$ and $\mathbb{R}^d$ such as Dirichlet lemma and Hardy-Littelwood circle method, here we only obtain the pointwise convergence along rational non-tangential lines on $\mathbb{T}^d$ for $d \geq 2$. One may also consider the pointwise convergence of the solutions to the Schrödinger equation on $\mathbb{T}^d$ along general curves via analysing the maximal estimates for Weyl sums.

Next, we consider the maximal estimates for the Weyl sums on the generic torus. Without loss of generality, we assume that each component parameter $\beta_i$ of $\beta$ satisfies

$$\beta_i \in [1, 2], \; \forall i \in \{1, \cdots, d\}.$$  

We recall the definition of genericity which one can find in [20].

**Definition 1.5.** We will call a property generic in $(\beta_1, \cdots, \beta_d)$ if it is true for all $(\beta_1, \cdots, \beta_d)$ outside of a null set (set with measure zero) of $[1, 2]^d$.

Here, the definition of genericity of $\beta$ is based on the classical result on Diophantine approximation. We call $\beta = (\beta_2, \cdots, \beta_d)$ is generic if there exists some constant $C$ such that

$$|k_1 + \beta_2k_2 + \cdots + \beta dk_d| \geq C \quad (1.9)$$

for any $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$, see [12].

We obtain the maximal estimates of Weyl sums on generic tori as follows.

**Theorem 1.6.** For generic $\beta_2, \cdots, \beta_d$ in $[1, 2]$ and polynomial $Q(n) = |n_1|^2 + |n_2|^2 + \cdots + |n_d|^2$, we have

$$\left\| \sup_{0 < t < 1} \left| \sum_{n \in [-N,N]^d \cap \mathbb{Z}^d} e^{2\pi i(x \cdot n + tQ(n))} \right| \right\|_{L^2(\mathbb{T}^d)} \lesssim N^{\frac{d+1}{2}} + \varepsilon. \quad (1.10)$$

**Remark 1.7.** The maximal estimate (1.10) extends the maximal estimates for the Laplacian operator $\Delta_{\mathbb{T}^d}$ to the generic case $Q(D)$ associated with the following Schrödinger equation,

$$\begin{cases}
  i\partial_t u + Q(D)u = 0, \\
  u(x, 0) = \sum_{n \in [-N,N]^d \cap \mathbb{Z}^d} e^{2\pi i x \cdot n}.
\end{cases} \quad (1.11)$$

The sharp maximal estimate for the solution to the equation (1.11) with the general initial value also remains open.

The main ingredients of proof for Theorem 1.6 are the maximal estimate for $\frac{2}{N} < t < 1$ which is related to the Strichartz estimates in [20] and local-in-time maximal estimate (3.3) for $0 < t < \frac{2}{N}$ in Section 3, see Lemma 3.2.
1.1. Applications: large values of the Weyl sums. Let
\[
\omega_N(x, t) := \sum_{\substack{1 \leq n_j \leq N, \\ 1 \leq j \leq d}} e^{2\pi i (n \cdot x + |n|^2 t)}
\]
\[
= \prod_{j=1}^{d} \left( \sum_{1 \leq n_j \leq N} e^{2\pi i (n_j x_j + n_j^2 t)} \right)
\]
\[
= \prod_{j=1}^{d} \omega_{N,j}(x_j, t),
\]
where \( n = (n_1, n_2, \ldots, n_d) \).

By the maximal estimates (1.7), we give some applications associated with the large values of the Weyl sums. One application is to estimate the Lebesgue measure of the set where Weyl sums attain large value and are bounded from below by \( N^\alpha \) in Section 5 as follows,
\[
S_\alpha(N) := \{ x \in \mathbb{T}^d : \sup_{0 < t < 1} |\omega_N(x, t)| \geq N^\alpha \},
\]
where \( 0 < \alpha \leq d \). We extend the results of Chen and Shparlinski in [13] and Barron in [4] in \( \mathbb{T} \) to the higher dimension \( \mathbb{T}^d \) for \( d \geq 2 \), and obtain
\[
|S_\alpha(N)| \lesssim \varepsilon N^{(d+2) - \frac{2(d+1)}{d+1} + \varepsilon}
\]
for \( d \geq 2 \) and \( \frac{d}{2} + \frac{d}{2(d+1)} < \alpha \leq d \).

The estimate (1.14) for \( S_\alpha(N) \) characterizes the extent of the occurrence for the square root cancellation for the Weyl sums \( \omega_N(x, t) \). It is easy to see that when \( (x, t) \) equals \((0, 0) \in \mathbb{T}^{d+1} \) or approximates \((0, 0) \), constructive interference for \( \omega_N(x, t) \) occurs, that is
\[
|\omega_N(x, t)| \sim N^d \text{ for } (x, t) \in \mathbb{T}^d \setminus \mathbb{N}^{-2}(0, 0).
\]

In the pointwise sense, the square root cancellation which corresponds to \( |\omega_N(x, t)| \sim N^{\frac{d}{2}} \) hardly happens. However, our result about the estimate for \( S_\alpha(N) \) falling in between these two cases can be obtained, which exhibits the compound effect between the constructive interference and the square root cancellation for Weyl sums.

In Section 6, as the second application corresponding to the large values of the Weyl sums, we consider the Hausdorff dimension of the set on which the Weyl sums are large and have the lower bound \( N^\alpha \) for many \( N \), that is
\[
L_\alpha := \{ x \in \mathbb{T}^d : \sup_{0 < t < 1} |\omega_N(x, t)| \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N} \}.
\]

In one dimension, utilizing the completion method, Chen and Shparlinski in [14, 15] and Barron in [4] studied the corresponding set. Using the higher dimension version of the completion method and the fractal version of the maximal estimates (1.7), we have the following estimate for the set \( L_\alpha \).

**Proposition 1.8.** If \( d \frac{2}{2} + \frac{d}{2(d+1)} \leq \alpha \leq d \), then
\[
\dim_H(L_\alpha) = \frac{2(d+1)}{d}(d - \alpha).
\]

**Remark 1.9.** For \( \alpha = d \), \( \dim_H(L_\alpha) = 0 \) indicates that the set where the constructive interference for the Weyl sums occurs is a small part in \( \mathbb{T}^d \). While
\[
\dim_H(L_\alpha) = d \quad \text{for} \quad \alpha = \frac{d}{2} + \frac{d}{2(d+1)} \quad \text{shows that at almost every point, the square root cancellation happens.}
\]

1.2. Notation. We write \( A \lesssim B \) to mean that there exists a constant \( C > 0 \) such that \( A \leq CB \). If there exists a constant \( C(\beta) \) that depends on some parameter \( \beta \) such that \( A \leq C(\beta)B \), then we write \( A \lesssim_\beta B \). We write \( A \ll B \) if \( A \leq C_0B \) and \( A \gtrsim_\beta B \) if \( A \geq C_0B \). We use \( A \ll B \to 0 \) to denote the statement \( A \leq cB \) where \( c \) is a very small constant. Throughout the paper, we often omit the constant \( C, \beta \) and \( c \) which may differ from line to line according to the context.

Let \( E \) be a Borel set in \( \mathbb{R}^d \). For \( 0 \leq s \leq d \), the \( s \)-dimensional Hausdorff measure of \( E \) is defined as follows
\[
H^s(E) := \lim_{\delta \to 0} H^s_\delta(E),
\]
where for \( 0 < \delta \leq \infty \),
\[
H^s_\delta(E) := \inf \{ \sum_j d(B_j)^s : E \subset \bigcup_j B_j, d(B_j) < \delta \}. \quad (1.18)
\]
We define the Hausdorff dimension of the Borel set \( E \) by
\[
\dim_H(E) := \inf \{ 0 \leq s \leq d : H^s(E) = 0 \} = \sup \{ 0 \leq s \leq d : H^s(E) = \infty \}. \quad (1.20)
\]

In Section 2 we give the proof of Theorem 1.1 and Theorem 1.3. In Section 3 we give the proof of Theorem 1.6.

2. Proof of Theorem 1.1 and Theorem 1.3

In this section, we prove Theorem 1.1 and Theorem 1.3. Enlightened by the method developed by Baker in [1] who dealt with the case \( d = 1 \), we give the proof in higher dimension. Nonetheless, the method in [1] cannot be applied to deal with the case in higher dimension directly, and we need to investigate a better upper bound for the Weyl sums to give a better control. Also we can see that the method for the proof of Theorem 1.1 and Theorem 1.3 in higher dimension for \( d \geq 2 \) in this section can be applied to the case for \( d = 1 \).

Before the proof, we supply some preliminaries. We write \( u(x, t) = \omega_N(x, t) \) as in Section 1.1 for the following sections.

2.1. Preliminaries. Let \( k \geq 2 \) and \( K = 2^{k-1} \). Let \( \| \alpha \| \) be the distance between \( \alpha \in \mathbb{R} \) and \( \mathbb{Z} \). Denote
\[
S(f) := \sum_{1 \leq n \leq N} e(f(n)) = \sum_{1 \leq n \leq N} e^{2\pi if(n)},
\]

Lemma 2.1 (Bourgain [6], Lemma 2.2 of Barron [4]). Let \( q, a \) be coprime, with \( 1 \leq a < q \leq N \). Suppose that \( |t - \frac{a}{q}| < \frac{1}{qN} \), then we have
\[
|S(xn + tn^2)| \ll N^{1+\varepsilon} \min(q^{-\frac{1}{2}}, N^{-1}|qt - a|^{-\frac{1}{2}}).
\]

The constant implied by \( \ll \) depends only on \( \varepsilon \) with \( 0 < \varepsilon < \frac{1}{1000} \).

By this lemma, we can obtain a better result for the approximation property for time \( t \) than that Dirichlet lemma shows, if the Weyl sums have a nontrivial lower bound. Indeed, utilizing this lemma, we have if \( P \geq N^{\frac{1}{2}+\delta} \) and
\[
P \leq |S(xn + tn^2)|, \quad (2.2)
\]
for some $\delta > 0$, then
\begin{equation}
1 \leq q \leq (NP^{-1})^2N^{2\varepsilon} \leq N^{1-2(\delta - \varepsilon)} \quad \text{and} \quad |qt - a| \leq N^{2\varepsilon}P^{-2} \leq N^{-1-2(\delta - \varepsilon)}, \tag{2.3}
\end{equation}
where $0 \ll \varepsilon \ll \delta < 1$.

Furthermore, the following lemmas would supply additional constraints on the other variable $x$, which satisfies the large condition (2.2).

**Lemma 2.2** (Lemma 10D in Schmidt [40]). Suppose $\varepsilon > 0, k \geq 2$ and $L < N$. Recall $K = 2^{k-1}$. Suppose $f(x) = \alpha x^k + \beta x^{k-1} + \gamma x^{k-2} + \cdots + c_0$. Then if we write $S_m(f) = S(mf)$ for integers $m$, we have

\begin{equation}
\sum_{m=1}^{L} |S_m(f)|^K \ll N^{K-k+\varepsilon} \sum_{z=1}^{kN^{k-2}L} \sum_{u=1}^{2kN} \min(N, \|z(\alpha u + 2\beta)\|^{-1}). \tag{2.4}
\end{equation}
The constant implied by $\ll$ depends only on $k, \varepsilon$.

With Lemma 2.2 in mind, if we take $k = K = 2$ and $L = 1$, then

\begin{equation}
|S(xn + tn^2)|^2 \ll_\varepsilon N^{\varepsilon} \sum_{z=1}^{2N} \sum_{u=1}^{4N} \min(N, \|z(ut + 2x)\|^{-1})
\end{equation}

\begin{equation}
\leq N^\varepsilon \left( \sum_{u=1}^{8N} \min(N, \|ut + 2x\|^{-1}) + \sum_{u=1}^{8N} \min(N, \|ut + 4x\|^{-1}) \right).
\end{equation}

The next lemma provides further marvellous bounds of these sums.

**Lemma 2.3** (Lemma 9D in [40]). Let $q, a$ be coprime, with $1 \leq q \leq N$. Suppose that

\begin{equation}
|\alpha q - a| = \|\alpha q\| < \frac{1}{(1 + c)N},
\end{equation}

where $c > 1$ is a given constant. Then

\begin{equation}
\sum_{n=1}^{cN} \min(N, |\alpha n + \beta|^{-1}) \ll (\log N) \min(N, \|\beta q\| \frac{1}{\|\alpha q\|}). \tag{2.6}
\end{equation}

**Lemma 2.2** and 2.3 yield that

\begin{equation}
P \leq |S(xn + tn^2)| \ll CN^{2\varepsilon} \min\left( \frac{N}{q^{\frac{1}{2}}}, \frac{N^{\frac{1}{2}}}{\|2xq\|^{\frac{1}{2}}}, \frac{N^{\frac{1}{2}}}{\|4xq\|^{\frac{1}{2}}}, \frac{1}{\|tq\|^{\frac{1}{2}}} \right). \tag{2.7}
\end{equation}

Thus, there exists $j = 2$ or $4$ such that

\begin{equation}
\|jxq\| \leq N^{1+4\varepsilon}P^{-2}. \tag{2.8}
\end{equation}

### 2.2. Proof of Theorem 1.1

For $d \geq 2$, let $\alpha \in [\frac{d}{2} + \frac{d}{2(\alpha_1 + 1)}, d]$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_d = \alpha$. For any measurable function $t(x)$ on $\mathbb{T}^d$, we consider $x$ such that

$|S(x_1n_1 + t(x)n_1^2)| = N^{\alpha_1}, |S(x_2n_2 + t(x)n_2^2)| = N^{\alpha_2}, \cdots, |S(x_dn_d + t(x)n_d^2)| = N^{\alpha_d}$.

We pigeonhole the regions of $x$ in $\mathbb{T}^d$ into two parts corresponding to two cases. For the first case in which $x$ satisfies that there exists at least one $\alpha_{i_0}$ such that $\alpha_{i_0} \leq \frac{d}{2(\alpha_1 + 1)} + \frac{d}{2} - \frac{d}{2(\alpha_1 + 1)} = \frac{d}{2} + \frac{d}{2(\alpha_1 + 1)}$, exerting induction on the dimension $d$, by corresponding results on $\mathbb{T}^{d-1}$, we have

\begin{equation}
\|u(x, t(x))\|_L^{\frac{2d+1}{d}}(\mathbb{T}^d) \leq N^{\alpha_0}N^{\frac{d}{2(\alpha_1 + 1)} + \frac{d}{2}} \leq N^{\frac{d}{2(\alpha_1 + 1)} + \frac{d}{2}}. \tag{2.9}
\end{equation}
For the second case, it suffices to consider $x$ such that $\frac{1}{2} + \frac{1}{2a(d+1)} < \alpha_1, \ldots, \alpha_d \leq 1$. According to the analysis above and (2.3), there exist integers $q, a$ with $(q, a) = 1$ such that

$$0 \leq a < q \leq N^{2 - 2\alpha_1 + 2\varepsilon}$$

for $1 \leq l \leq d$. Moreover, as per the restriction of $x$ in (2.3), there exist $j_1, \ldots, j_d \in \{2, 4\}$, such that for $l = 1, 2, \ldots, d$, we have

$$\|x_l q\| \leq 2N^{1 - 2\alpha_1 + 4\varepsilon}.$$  

(2.11)

This of course implies that for $l = 1, 2, \ldots, d$, there exist $b_l \in \frac{1}{2} \mathbb{Z} \cap [1, 4q]$, such that

$$|\beta_l| = |x_l - b_l| \leq q^{-1} N^{1 - 2\alpha_1 + 4\varepsilon}.$$  

On the other hand, by the estimate (2.7), we have for $l = 1, 2, \ldots, d$,

$$|S(x \cdot n_l + t(x) n_l^2)| \leq C N^{\frac{1}{2} + d\varepsilon} q^{-\frac{1}{2}} \prod_{l=1}^d \min(N^{\frac{1}{2}}, |\beta_l|^{-\frac{1}{2}}).$$  

(2.12)

This estimate implies

$$|u(x, t(x))| = \prod_{l=1}^d |S(x \cdot n_l + t(x) n_l^2)| \leq C N^{\frac{1}{2} + d\varepsilon} q^{-\frac{1}{2}} \prod_{l=1}^d \min(N^{\frac{1}{2}}, |\beta_l|^{-\frac{1}{2}}).$$  

(2.13)

Using this bound and defining $\tilde{\alpha} = \frac{1}{2} + \frac{1}{2a(d+1)}$, we obtain

$$\int_{x \in \mathbb{T}^{d}((\alpha_1, \ldots, \alpha_d)} |u(x, t(x))|^{\frac{2(d+1)}{d}} \, dx$$

$$\leq C \sum_{1 \leq q \leq N^{2 - 2\tilde{\alpha} + 2\varepsilon}} \sum_{b_l \in \frac{1}{2} \mathbb{Z} \cap [1, 4q]} \prod_{l=1}^d N^{d+1 + C\varepsilon} q^{- (d+1)} \int_{|\beta_l| \leq q^{-1} N^{1 - 2\alpha_1 + 4\varepsilon}} \min(N^{\frac{1}{2}}, |\beta_l|^{-\frac{1}{2}})^{\frac{2(d+1)}{d}} d\beta_l$$

$$\leq C \sum_{1 \leq q \leq N^{2 - 2\tilde{\alpha} + 2\varepsilon}} N^{d+1 + C\varepsilon} q^{-1} \prod_{l=1}^d \int_{|\beta_l| \leq q^{-1} N^{1 - 2\alpha_1 + 4\varepsilon}} \min(N^{\frac{1}{2}}, |\beta_l|^{-\frac{1}{2}})^{\frac{2(d+1)}{d}} d\beta_l$$

$$\leq C \sum_{1 \leq q \leq N^{2 - 2\tilde{\alpha} + 2\varepsilon}} N^{d+2 + C\varepsilon} q^{-1}.$$  

(2.14)

Thus, we have

$$\|u(x, t(x))\|_{L^{\frac{2(d+1)}{d}}(\mathbb{T}^d)} \leq C N^{\frac{d(d+2)}{d} + \varepsilon} = C N^{\frac{d}{2} + \frac{d}{2a(d+1)} + \varepsilon},$$  

(2.15)

for any $\varepsilon > 0$. For the case $1 \leq p \leq \frac{2(d+1)}{d}$, by Hölder’s inequality, we have

$$\left\| \sup_{0 < t < 1} |u(x, t)| \right\|_{L^p(\mathbb{T}^d)} \leq \left\| \sup_{0 < t < 1} |u(x, t)| \right\|_{L^\infty(\mathbb{T}^d)}^{\frac{2(d+1)}{d}} \left\| \sup_{0 < t < 1} |u(x, t)| \right\|_{L^p(\mathbb{T}^d)} \leq N^{\frac{d}{2} + \frac{d}{2a(d+1)} + \varepsilon}.$$  

(2.16)

On the other hand, for general $p \geq \frac{2(d+1)}{d}$ we have, akin to the estimate (2.12),

$$\left\| \sup_{0 < t < 1} |u(x, t)| \right\|_{L^p(\mathbb{T}^d)} \lesssim N^{d - \frac{d}{p} + \varepsilon},$$  

(2.17)

for $p \geq \frac{2(d+1)}{d}$. 


Remark 2.4. One may also expect to obtain the results of Theorem 1.1 via the argument of [4] for \( T \). However, when one decomposes the Weyl sums as in \[4\] directly, there would be more cross terms for the upper bound of Weyl sums with the increment of the dimension. These cross terms seem to be more difficult to control, which is the main difference from the situation in \( T \). Due to this difference, if one applies the method in \[4\], the corresponding result in \( T^d \) would seem unable to reach the sharpness of \( s \) for Theorem 1.1 to hold.

2.3. Proof of Theorem 1.3. For \( d \geq 2 \), let \( r = (r_1, r_2, \cdots, r_d) \in \mathbb{Q}^d \). Similarly, we only consider \( \alpha \in \left(\frac{d}{2} + \frac{d}{4(d^2+1)}, d\right] \) and \( \alpha_1 + \alpha_2 + \cdots + \alpha_d = \alpha \). For any measurable function \( t(x) \) on \( T^d \), we consider \( x \) such that

\[
|S((x_1 - r_1 t(x)) n_1 + t(x) n_1^2)| = N^{\alpha_1}, \cdots, |S((x_d - r_d t(x)) n_d + t(x) n_d^2)| = N^{\alpha_d}.
\]

Similar to the proof of Theorem 1.1 in Subsection 2.2, we also consider two cases for \( x \) in \( T^d \). For the first case in which \( x \) satisfies that there exists at least one \( \alpha_{l_0} \) such that \( \alpha_1 \leq \frac{2q}{2q(d^2+1)} + \frac{1}{2q} - \frac{d}{2qd} = \frac{1}{2} + \frac{1}{2q(d^2+1)} \), using induction on the dimension \( d \), by corresponding results on \( T^{d-1} \), we have

\[
\|u(x - rt(x), t(x))\|_{L^{\frac{2(d+1)}{d}}(T^d)} \leq N^{\alpha_0} N^{\frac{1}{2} + \frac{1}{2q(d^2+1)}} \leq N^{\frac{1}{2} + \frac{1}{2d^2+1} + \frac{1}{2}}.
\]

For the second case, it suffices to consider \( x \) such that \( \frac{1}{2} + \frac{1}{2d^2+1} < \alpha_1, \cdots, \alpha_d \leq 1 \). According to the analysis in Section 2.1, we have for \( l = 1, 2, \cdots, d \), such that for \( l = 1, 2, \cdots, d \),

\[
\|j_l(x_l - r_l t(x)) q\| \leq 2N^{1-2\alpha_l+4\epsilon}.
\]

Let \( r_l = \frac{r_l}{q} \) with \( r_{l,1}, r_{l,2} = 1 \). From the estimates (2.18) and (2.19), we have

\[
\min(|x_l - \frac{a}{q} \frac{r_{l,1}}{q} x_{l,1} - \frac{k_l}{2q}|, |x_l - \frac{a}{q} \frac{r_{l,1}}{q} x_{l,2} - \frac{k_l}{4q}|) < \frac{2}{q} N^{1-2\alpha_l+4\epsilon},
\]

for some \( k_l \in \mathbb{Z} \). This estimate implies that for \( l = 1, 2, \cdots, d \), there exist \( b_l \in \mathbb{Z} \cap [0, q] \), such that

\[
|\beta_l| = |x_l - \frac{k_l}{q}| = \min(|x_l - \frac{a}{q} \frac{r_{l,1}}{q} x_{l,1} - \frac{k_l}{2q}|, |x_l - \frac{a}{q} \frac{r_{l,1}}{q} x_{l,2} - \frac{k_l}{4q}|) \leq C q^{-1} N^{1-2\alpha_l+4\epsilon}.
\]

On the other hand, by the estimate (2.21), we have for \( l = 1, 2, \cdots, d \),

\[
|S((x_l - r_l t(x)) n_l + t(x) n_l^2)| \leq CN^{\frac{1}{3} + \epsilon} q^{-\frac{4}{3}} \min(N^{\frac{1}{3}}, |\beta_l|^{-\frac{4}{3}}),
\]

which yields

\[
|u(x - rt(x), t(x))| = \prod_{l=1}^d |S((x_l - r_l t(x)) n_l + t(x) n_l^2)| \leq CN^{\frac{1}{3} + \epsilon} q^{-\frac{4}{3}} \prod_{l=1}^d \min(N^{\frac{1}{3}}, |\beta_l|^{-\frac{4}{3}}).
\]
Proposition 3.1

Define \( \tilde{\alpha} = \frac{1}{2} + \frac{1}{2(d+1)} \) and we have
\[
\int_{x \in \mathbb{T}^d(\alpha_1, \ldots, \alpha_d)} |u(x - rt(x), t(x))|^{2(d+1)} \, dx
\leq C \sum_{1 \leq q \leq N^{2-2\delta+2\varepsilon}} \sum_{b_1 \in \mathbb{Z} \cap [0,q]} \cdots \sum_{b_d \in \mathbb{Z} \cap [0,q]} N^{d+1+C\varepsilon} q^{-(d+1)} \int_{|\beta| \leq q^{-1} N^{1-2\alpha_1+4\varepsilon}} \cdots \int_{|\beta| \leq q^{-1} N^{1-2\alpha_d+4\varepsilon}} \prod_{l=1}^d \min(N^{\frac{1}{2}}, |\beta_l|^{-\frac{1}{2}}) \frac{2(d+1)}{d} \beta_1 \beta_2 \cdots \beta_d
\leq C_r \sum_{1 \leq q \leq N^{2-2\delta+2\varepsilon}} N^{d+1+C\varepsilon} q^{-1} \prod_{l=1}^d \int_{|\beta| \leq q^{-1} N^{1-2\alpha_l+4\varepsilon}} \min(N^{\frac{1}{2}}, |\beta_l|^{-\frac{1}{2}}) \frac{2(d+1)}{d} \beta_l
\leq C_r \sum_{1 \leq q \leq N^{2-2\delta+2\varepsilon}} N^{d+2+C\varepsilon} q^{-1}.
\]

This estimate implies that
\[
\|u(x - rt(x), t(x))\|_{L^{2(d+1)}(\mathbb{T}^d)} \leq CN^{\frac{d(d+2)}{d+1}+\varepsilon} = CN^{\frac{d}{2}+\frac{d+1}{2}+\varepsilon},
\]
for any \( \varepsilon > 0 \), where \( C \) only depends on \( r \) and \( \varepsilon \). If \( 1 \leq p < \frac{2(d+1)}{d} \), by Hölder’s inequality, we have
\[
\|u(x - rt(x), t(x))\|_{L^p(\mathbb{T}^d)} \leq \|u(x - rt(x), t(x))\|_{L^{\frac{2(d+1)}{d+1}}(\mathbb{T}^d)} ^{\frac{d}{2} \cdot \frac{2(d+1)}{d}} \leq N^{\frac{d}{2}+\frac{d+1}{2}+\varepsilon},
\]
for any \( \varepsilon > 0 \), where \( C \) only depends on \( r \) and \( \varepsilon \).

For general \( p \geq \frac{2(d+1)}{d} \), similar to (2.22), we have
\[
\|u(x - rt(x), t(x))\|_{L^p(\mathbb{T}^d)} \leq CN^{\frac{d}{2}+\frac{d+1}{2}+\varepsilon},
\]
for any \( \varepsilon > 0 \), where \( C \) only depends on \( r \) and \( \varepsilon \).

3. Proof of Theorem 1.6

For the proof of Theorem 1.6 we divide the maximal estimate (1.10) on \( 0 < t < 1 \) into two parts on \( 0 < t < \frac{1}{2} \) and \( \frac{1}{2} < t < 1 \), for which the maximal estimate on \( \mathbb{R}^d \) and the following Proposition 3.1 related to the Strichartz estimate are exploited respectively.

Recall the Definition 1.5 for the genericity of \( \beta \in [1, 2]^{d-1} \) in Section 1 and \( Q(n) = |n_1|^2 + |n_2|^2 + \cdots + |n_d|^2 \). Deng, Germain and Guth in [20] established pointwise bound of the Weyl sums on irrational tori as follows.

**Proposition 3.1** (Proposition 4.6 in [20]). Assume that \( \beta = (\beta_2, \cdots, \beta_d) \) is chosen generically. Then for \( \frac{2}{N} < t < N^K \) for a positive constant \( K \), we have
\[
|K_N(t, x)| \leq N^{\frac{d}{2}+\varepsilon} t^{\frac{d}{2}},
\]
where
\[
K_N(t, x) := \sum_{n \in \mathbb{Z}^d} e^{2\pi i (x \cdot n - t Q(n))} \frac{n_1}{N} \cdots \frac{n_d}{N} \chi \left( \frac{n_1}{N} \right) \cdots \chi \left( \frac{n_d}{N} \right)
\]
and \( \chi \) is a smooth, nonnegative function, supported on \( B(0, 2) \), and equal to 1 on \( B(0, 1) \).
The proof for the maximal estimate \( (1.10) \) on \( \frac{2}{N} < t < 1 \) is furnished with this proposition. It suffices to consider the remaining part for the maximal estimate on \( 0 < t < \frac{2}{N} \), which can be solved by the corresponding local-in-time maximal estimate below.

Moyua and Vega in \cite{[37]} proved a local-in-time maximal estimate for the periodic Schrödinger equation in \( T \). Here, by similar localization arguments of Bourgain-Demeter \cite{[9]} (or Hickman \cite{[30]}), we extend their result to that in the higher dimensions for \( d \geq 2 \) and generic Laplacian operator \( Q(D) \) based on the results of Du-Guth-Li in \cite{[21]} and Du-Zhang in \cite{[24]} in \( \mathbb{R}^d \).

For simplicity, we first consider the two-dimensional case.

**Lemma 3.2.** (1) Suppose \( u \) is the solution to the Schrödinger equation on \( T^2 \), and \( u(x, 0) = f(x) \) with \( \text{supp } f \subset [-N, N]^2 \). Then for \( 2 \leq p \leq 3 \), we have
\[
\left\| \sup_{0 < t < N^{-1}} |u(x, t)| \right\|_{L_p(T^2)} \lesssim N^{\frac{2}{3} + \varepsilon} \|f\|_{L^2(T^2)},
\]
where \( u(x, t) = e^{it\Delta} f(x) \).

(2) Suppose \( v \) is the solution to the Schrödinger equation on the irrational torus of dimension 2, and \( v(x, 0) = g(x) \) with \( \text{supp } g \subset [-N, N]^2 \). Then for \( 2 \leq p \leq 3 \), we have
\[
\left\| \sup_{0 < t < N^{-1}} |v(x, t)| \right\|_{L_p(T^2)} \lesssim N^{\frac{2}{3} + \varepsilon} \|g\|_{L^2(T^2)},
\]
where \( v(x, t) = e^{itQ(D)} g(x) \) and \( Q(u) = |n_1|^2 + |\beta_2| |n_2|^2 \) with \( \beta_2 \in [1, 2] \).

**Proof.** We divide the proof into two steps as follows.

Through the usual localization theory (see for example \cite{[31]}), we establish some equivalent estimates in the former step. Then, in the latter step, following the arguments of \cite{[30]}, we deduce the local-in-time maximal estimates of this lemma as a consequence of the first step.

**Step 1.** We first prove the equivalence of the following three estimates:
\[
(a) \quad \left\| \sup_{0 < t < N} |\hat{f}(x)| \right\|_{L^p(B_N)} \lesssim N^\frac{\delta}{2} - \frac{\delta}{2} + \varepsilon \|f\|_{L^2}
\]
holds for \( \text{supp } \hat{f} \subset B_1 \subset \mathbb{R}^2 \);
\[
(b) \quad \left\| \sup_{0 < t < N} |F(x, t)| \right\|_{L^p(B_N)} \lesssim N^\frac{\delta}{2} - \frac{\delta}{2} + \varepsilon \|\hat{F}\|_{L^2(\mathcal{N}_{N^{-1}}(\mathbb{P}^2))}
\]
holds for \( \text{supp } \hat{F} \subset \mathcal{N}_{N^{-1}}(\mathbb{P}^2) \), where \( \mathbb{P}^2 = \{(x, |x|^2) : x \in [-1, 1]\} \);
\[
(c) \quad \left\| \sup_{0 < t < N} |R(x, t)| \right\|_{L^p(B_N)} \lesssim N^\frac{\delta}{2} - \frac{\delta}{2} + \varepsilon \|\hat{R}\|_{L^2(\mathcal{N}_{N^{-1}}(\mathbb{E}^2))}
\]
holds for \( \text{supp } \hat{R} \subset \mathcal{N}_{N^{-1}}(\mathbb{E}^2) \), where \( \mathbb{E}^2 = \{(x, Q(x)) : x \in B(0, \frac{1}{2})\} \).

First we show \( (3.4) \) implies \( (3.5) \). Since supp \( \hat{F} \subset \mathcal{N}_{N^{-1}}(\mathbb{P}^2) \), we have
\[
F(x, t) = \int_{|\xi| < 1} \int_{|\tau| < |\xi|^2 < N^{-1}} e^{2\pi i (x \xi + t \tau)} \hat{F}(\xi, \tau) \, d\xi d\tau = \int_{|\tau| < N^{-1}} \int_{|\xi| < 1} e^{2\pi i (x \xi + (\tau + |\xi|^2) t)} \hat{F}(\xi, \tau + |\xi|^2) \, d\xi d\tau.
\]
From the maximal estimate (3.4), Fubini’s theorem and Minkowski’s inequality, we have
\[
\left\| \sup_{0 < t < N} |F(x, t)| \right\|_{L^p(B_N)} \lesssim \int_{|\tau| < N^{-1}} \left\| \sup_{0 < t < N} \int_{|\xi| < 1} e^{2\pi i (x \cdot \xi + t |\xi|^2)} \hat{F}(\xi, \tau + |\xi|^2) \, d\xi \right\|_{L^p(B_N)} \, d\tau \\
\lesssim \int_{|\tau| < N^{-1}} N^{\frac{\beta}{p} - \frac{7}{p} + \epsilon} \|\hat{F}(\xi, \tau + |\xi|^2)\|_{L^2(|\xi| < 1)} \, d\tau \\
\lesssim N^{\frac{\beta}{p} - \frac{7}{p} + \epsilon} \|\hat{F}\|_{L^2(N_{N^{-1}}(F^3))}.
\]

Next, we prove (3.5) implies (3.6). Let \( A = \{ (\xi_1, \sqrt{2} \xi_2) : \xi \in B(0, \frac{1}{2}) \} \). By the fact that \( \beta_2 \in [1, 2] \), it is easy to see that \( A \subset B(0, 1) \). Since \( \text{supp} \hat{F} \subset N_{N^{-1}}(F^2) \), using change of variables, we have
\[
R(x, t) = \int_{|\xi| < \frac{1}{2}} \int_{|\tau - Q(\xi)| < N^{-1}} e^{2\pi i (x \cdot \xi + t \tau)} \hat{R}(\xi, \tau) \, d\xi \, d\tau \\
= \int_{|\tau| < N^{-1}} \int_{|\xi| < \frac{1}{2}} e^{2\pi i (x \cdot \xi + t(\tau + Q(\xi)))} \hat{R}(\xi, \tau + Q(\xi)) \, d\xi \, d\tau \\
= \frac{1}{\sqrt{\beta_2}} \int_{|\tau| < N^{-1}} \int_{|\xi| < \frac{1}{2}} e^{2\pi i (\hat{x} \cdot \xi + t(\tau + |\xi|^2))} \hat{R}(\hat{x}, \tau + |\xi|^2) \, d\xi \, d\tau, \\
\]
where \( \hat{x} = (x_1, \sqrt{2}^{-1}x_2) \) and \( \hat{\xi} = (\xi_1, \sqrt{2}^{-1} \xi_2) \). Let \( \hat{\hat{R}}(\xi, \tau + |\xi|^2) = \hat{R}(\hat{\xi}, \tau + |\hat{\xi}|^2) \). By the representation formula (3.8) of \( R(x, t) \) and the support of \( \hat{R} \), it is easy to see that \( \text{supp} \hat{\hat{R}}(\hat{\xi}, \tau) \subset N_{N^{-1}}(F^2) \). Then by the estimate (3.5), we have
\[
\left\| \sup_{0 < t < N} |R(x, t)| \right\|_{L^p(B_N)} \lesssim N^{\frac{\beta}{p} - \frac{7}{p} + \epsilon} \|\hat{\hat{R}}(\hat{\xi}, \tau)\|_{L^2(N_{N^{-1}}(F^2))} \\
\approx N^{\frac{\beta}{p} - \frac{7}{p} + \epsilon} \|\hat{\hat{R}}(\hat{\xi}, \tau)\|_{L^2(N_{N^{-1}}(F^2))} \\
\lesssim N^{\frac{\beta}{p} - \frac{7}{p} + \epsilon} \|\hat{\hat{R}}\|_{L^2(N_{N^{-1}}(F^2))}.
\]

Finally, we prove (3.4) by (3.5). We can rewrite \( e^{it\Delta} f(x) \) with \( \hat{f} \subset B(0, 1) \subset \mathbb{R}^2 \) as follows,
\[
e^{it\Delta} f(x) = \int e^{ix \cdot \xi - it|\xi|^2} \hat{f}(\xi) \, d\xi \\
= 4 \sqrt{\beta_2} \int e^{ix \cdot (2 \xi_1, 2 \sqrt{2} \xi_2)} e^{-itQ(\xi)} \hat{f}(2 \xi_1, 2 \sqrt{2} \xi_2) \, d\xi \\
= 4 \sqrt{\beta_2} \int e^{ix \cdot \hat{x} \cdot \xi} e^{-itQ(\xi)} \hat{h}(\xi) \, d\xi \\
= 4 \sqrt{\beta_2} e^{itQ(\hat{x})} \hat{h}(2 \hat{x}),
\]
where \( \hat{x} = (x_1, \sqrt{2}^{-1}x_2) \) and \( \hat{\hat{h}}(\hat{\xi}) = \hat{f}(2 \xi_1, 2 \sqrt{2} \xi_2) \) with \( \text{supp} \hat{h} \subset \{ \xi : (\xi_1, \sqrt{2}^{-1} \xi_2) \in B(0, \frac{1}{2}) \} \subset B(0, \frac{1}{2}) \). Then, by the translation invariance of the operator \( e^{it\Delta} \) and
change of variables, we have
\[ \left\| \sup_{0<t<N} |e^{it\Delta} f(x)| \right\|_{L^p(B_N)} \lesssim \left\| \sup_{0<t<N} |e^{itQ(D)} h(x)| \right\|_{L^p(B_{\tilde{N}})} \]
\[ \lesssim \left\| \sup_{0<t\leq 4N} |e^{itQ(D)} h(x)| \right\|_{L^p(B_N)} \]
\[ \lesssim \left\| \sup_{0<t\leq N} |e^{itQ(D)} h(x)| \right\|_{L^p(B_N)} \]  \hspace{1cm} (3.11)

where \( \tilde{N} = 4N \).

Choose \( \varphi(x,t) \in \mathcal{S}(\mathbb{R}^3) \) with Fourier support in \( B_1^3 \) such that \( |\varphi(x,t)| \sim 1 \) on \( B_1^3 \). Let \( \varphi_N(x,t) := \varphi(\frac{t}{\tilde{N}}) \) with \( \text{supp} \hat{\varphi}_N \subset B_{\tilde{N}}^3 \). Then it is easy to see \( \text{supp} \mathcal{F}_{x,t}(e^{itQ(D)} h(x) \cdot \varphi_N(x,t)) \subset \mathcal{N}_{\tilde{N}}(\mathbb{R}^2) \). Through a direct computation, we have
\[ \mathcal{F}_{x,t}(e^{itQ(D)} h(x) \cdot \varphi_N(x,t))(\xi, \tau) = \int_{B(\xi, \tilde{N}^{-1})} (\mathcal{F}_{x,t} \varphi_N)(\xi - y, \tau + |y|^2) \hat{h}(y) \, dy. \]  \hspace{1cm} (3.12)

By Hölder’s inequality and the properties of \( \text{supp} \mathcal{F}_{x,t}(\varphi_N) \) and \( \text{supp} \mathcal{F}_{x,t}(e^{itQ(D)} h(x) \cdot \varphi_N(x,t)) \), we obtain
\[ |\mathcal{F}_{x,t}(e^{itQ(D)} h(x) \cdot \varphi_N(x,t))(\xi, \tau)| \]
\[ \leq \int_{B(\xi, \tilde{N}^{-1})} |(\mathcal{F}_{x,t} \varphi_N)(\xi - y, \tau + |y|^2) \hat{h}(y)| \, dy \]
\[ \leq \tilde{N}^{-1} \left\| (\mathcal{F}_{x,t} \varphi_N)(\xi - y, \tau + |y|^2) \hat{h}(y) \right\|_{L^2} \]

Thus, by the estimate (3.13), \( \tilde{N} = 4N \); Fubini’s theorem and Plancherel’s theorem, we have
\[ \left\| \sup_{0<t\leq N} |e^{it\Delta} f(x)| \right\|_{L^p(B_N)} \]
\[ \lesssim \left\| \sup_{0<t\leq N} |e^{itQ(D)} h(x)| \right\|_{L^p(B_N)} \]
\[ \lesssim \tilde{N}^{\frac{3}{2} - \frac{\epsilon}{4}} \left\| \mathcal{F}_{x,t}(e^{itQ(D)} h(x) \cdot \varphi_N(x,t))(\xi, \tau) \right\|_{L^2(\mathcal{N}_{\tilde{N}})} \]
\[ \lesssim \tilde{N}^{\frac{3}{2} - \frac{\epsilon}{4}} \left\| (\mathcal{F}_{x,t} \varphi_N)(\xi - y, \tau + |y|^2) \hat{h}(y) \right\|_{L^2_{\xi,\tau,y}(\mathcal{N}_{\tilde{N}})} \]
\[ \lesssim \tilde{N}^{\frac{3}{2} - \frac{\epsilon}{4}} \left\| \varphi_N(x,t) \right\|_{L^2_{\xi,\tau}} \| h \|_{L^2} \]
\[ \lesssim \tilde{N}^{\frac{3}{2} - \frac{\epsilon}{4}} \left\| f(2\xi, 2\sqrt{\beta_2} \xi_2) \right\|_{L^2} \]
\[ \lesssim \tilde{N}^{\frac{3}{2} - \frac{\epsilon}{4}} \| f \|_{L^2}. \]

**Step 2.** Utilizing the result in step 1, we prove that if (3.13) holds, then
\[ \left\| \sup_{0<t\leq N^{-1}} \left| \sum_{1 \leq n_1, n_2 \leq N} a_n e^{2\pi i (x \cdot n + |n|^2 t)} \right| \right\|_{L^p(B_N^2)} \leq N^{\frac{3}{2} + \epsilon} \left( \sum_{1 \leq n_1, n_2 \leq N} |a_n|^2 \right)^{\frac{1}{2}}. \]  \hspace{1cm} (3.13)
Also, we can see that if \((3.6)\) holds, then
\[
\left\| \sup_{0 < t < N^{-1}} \left| \sum_{1 \leq n_1, n_2 \leq N} a_n e^{2\pi i (x \cdot n + t|n|^2)} \right| \right\|_{L^p(B_1^2)} \lesssim N^{\frac{3}{4} + \epsilon} \left( \sum_{1 \leq n_1, n_2 \leq N} |a_n|^2 \right)^{\frac{1}{2}}.
\]
(3.14)

Since the proof of \((3.14)\) by \((3.6)\) is similar to that of \((3.13)\) by \((3.5)\), we just present the proof of the former estimate \((3.13)\).

Let \(\Lambda = N^{-1} \mathbb{Z}^2 \cap B^2(0,1)\) and \(G(x,t) = \sum_{n \in \Lambda} a_n e^{2\pi i (x \cdot n + |n|^2 t)}\). Since \(\hat{G}(\xi,\tau)\) is supported in \(\mathcal{N}_{N^{-1}}(\mathbb{F})\), by the estimate \((3.6)\), we have
\[
\left| G(x,t) \phi_N \right|_{L^p(B_3^4)} \lesssim N^\frac{d}{2} N^{-\epsilon} \| \hat{G}(\phi_N) \|_{L^2(\mathcal{N}_{N^{-1}}(\mathbb{F}))}. \tag{3.15}
\]
By rescaling,
\[
\text{LHS of } (3.15) \gtrsim N^\frac{d}{2} \left\| \sup_{0 < t < N^{-1}} \left| \sum_{1 \leq n_1, n_2 \leq N} a_n e^{2\pi i (x \cdot n + |n|^2 t)} \right| \right\|_{L^p(B_3^2)} \tag{3.16}
\]
On the other hand, through a direct calculation, we have
\[
\| \hat{G}(\phi_N) \|_{L^2(\mathcal{N}_{N^{-1}}(\mathbb{F}))} \leq \left( \sum_{n \in \Lambda} |a_n|^2 \right) \left( \int_{\mathbb{R}^2} |\hat{\phi}(\xi - n, \tau - |n|^2)|^2 d\xi d\tau \right)^{\frac{1}{2}} \lesssim N^\frac{d}{2} \left( \sum_{n \in \Lambda} |a_n|^2 \right)^{\frac{1}{2}}. \tag{3.17}
\]
Combining \((3.16)\), \((3.10)\) and \((3.10)\) together, we obtain \((3.13)\). This completes the proof.

In higher dimensions \(d \geq 3\), as leading to the proof as in Lemma \(3.2\), we can also establish the local-in-time maximal estimates for the operators \(e^{it\Delta}\) and \(e^{itQ(D)}\) as follows.

**Corollary 3.3.** For \(d \geq 3\) and \(\epsilon > 0\), we have
\[
\left\| \sup_{0 < t < N^{-1}} \left| u(x,t) \right| \right\|_{L^2(\mathbb{T}^d)} \lesssim N^{\frac{d}{2} + \frac{1}{2} - \frac{1}{2d} + \epsilon} \| f \|_{L^2(\mathbb{T}^d)}, \tag{3.18}
\]
where \(u(x,t) = e^{it\Delta} f(x)\) with \(\text{supp} \hat{f} \subset [-N,N]^d\); and
\[
\left\| \sup_{0 < t < N^{-1}} \left| v(x,t) \right| \right\|_{L^2(\mathbb{T}^d)} \lesssim N^{\frac{d}{2} + \frac{1}{2} + \epsilon} \| g \|_{L^2(\mathbb{T}^d)}, \tag{3.19}
\]
where \(v(x,t) = e^{itQ(D)} g(x)\) with \(\text{supp} \hat{g}(\xi) \subset [-N,N]^d\) and \(Q(n) = |n_1|^2 + \beta_2 |n_2|^2 + \cdots + \beta_d |n_d|^2\) with \(\beta_j \in [1,2]\) for \(2 \leq j \leq d\).

**Remark 3.4.** The proofs of Lemma \(3.2\) and Corollary \(3.3\) rely on the following two maximal estimates for the Schrödinger operator in \(\mathbb{R}^d\).

In \(\mathbb{R}^2\), Du-Guth-Li in [21] obtained that if \(2 \leq p \leq 3\), then
\[
\left\| \sup_{0 < t \leq R} \left| e^{it\Delta} f \right| \right\|_{L^p(B(0,R))} \lesssim R^{\frac{2}{p} - \frac{d}{2} + \epsilon} \| f \|_{L^2(\mathbb{R}^2)} \tag{3.20}
\]
holds for all \(R \geq 1\) and all \(f \in L^2(\mathbb{R}^2)\) with \(\text{supp} \hat{f} \subset B(0,1)\), which is exactly \((3.4)\). Later in \(\mathbb{R}^d\) with \(d \geq 3\), Du-Zhang in [24] showed that
\[
\left\| \sup_{0 < t \leq R} \left| e^{it\Delta} f \right| \right\|_{L^2(B(0,R))} \lesssim R^{\frac{2}{p} - \frac{d}{2} + \epsilon} \| f \|_{L^2(\mathbb{R}^d)} \tag{3.21}
\]
holds for all \( R \geq 1 \) and all \( f \in L^2(\mathbb{R}^d) \) with \( \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^d : |\xi| \sim 1 \} \).

From the estimates (3.20) and (3.21), we can see why the parameter \( p \) of \( L^p \) norm for the maximal estimates in Lemma 3.2 and Corollary 3.3 differ from each other.

**Proof of Theorem 1.6** Proposition 3.1 Lemma 3.2 (2) and Corollary 3.3 (\( d \)-dimensional version) would imply that for generic \( \beta = (\beta_2, \cdots, \beta_d) \) and \( Q(n) = |\sum_{k=1}^{d} \beta_k n_k|^2 \),

\[
\left\| \sup_{0 < t < 1} \left| \sum_{n \in [-N,N]^d \cap \mathbb{Z}^d} e^{2\pi i (x \cdot n + tQ(n))} \right| \right\|_{L^2(T^d)} \lesssim \varepsilon \| \frac{x}{N^{\frac{d+1}{2}}} \|_{L^2} + N^{\frac{d+1}{2}} + \varepsilon,
\]

(3.22)

for any \( \varepsilon > 0 \). In fact, according to estimates (3.1), (3.3) and (3.19), we have

\[
\left\| \sup_{0 < t < 1} \left| \sum_{n \in [-N,N]^d \cap \mathbb{Z}^d} e^{2\pi i (x \cdot n + tQ(n))} \right| \right\|_{L^2(T^d)} \lesssim \varepsilon \| \frac{x}{N^{\frac{d+1}{2}}} \|_{L^2} + N^{\frac{d+1}{2}} + \varepsilon
\]

\[
\lesssim N^{\frac{d+1}{2}} + \varepsilon.
\]

This completes the proof.

4. Lower bounds of the maximal function of Weyl sums

In this section, we consider the necessity of the exponent \( s_p \) for the maximal estimate (1.7) to hold in Theorem 1.1.

4.1. Lower bounds for Theorem 1.1. Case 1, \( p \geq \frac{2(d+1)}{d} \).

In this case, we consider a simple example in higher dimension, which is similar to that in one dimension given by Barron in [4] and indicates the occurrence of constructive interference for \( \sup_{0 < t < 1} |\omega_N(x,t)| \) on some small region of \( T^d \).

Take \( E = [0,10^{-6}N^{-1}]^d \). Then for \( x \in E \), we have

\[
\sup_{0 < t < 1} |\omega_N(x,t)| \gtrsim N^d,
\]

(4.1)

which yields that

\[
\left\| \sup_{0 < t < 1} |\omega_N(x,t)| \right\|_{L^p(E)} \gtrsim \left\| \sup_{0 < t < 1} |\omega_N(x,t)| \right\|_{L^p(E)} \gtrsim N^{d - \frac{d}{p}} = N^{\frac{d}{p} + \left( \frac{d}{p} - \frac{d}{2} \right)}. \quad (4.2)
\]

**Remark 4.1.** When \( p = \frac{2(d+2)}{d} \), the exponent \( \frac{d}{2} - \frac{d}{p} = \frac{d}{d+2} \) in the estimate (4.2) is consistent with Proposition 3.1 of [17], in which by Strichartz estimates on \( T^d \) Compaan, Luca and Staffilani obtained that the maximal estimate

\[
\left\| \sup_{0 < t < 1} |e^{it\Delta} f(x)| \right\|_{L^{\frac{2(d+2)}{d}}(T^d)} \lesssim \| f \|_{H^s(T^d)}
\]

(4.3)

holds for \( s > \frac{d}{d+2} \) and fails for \( s < \frac{d}{2(d+1)} \).
By Lemma 3.1 and Lemma 3.2 of [39] we have
\[ W \]

Theorem 1.1 via analysing the major arc of Weyl sums in \[ \equiv \].

Proof. Utilizing the partial summation Lemma 5.1 of [39] and taking

\[ \text{Proposition 4.2.} \]

by the simultaneous Dirichlet approximation in Bourgain [8] and [39], we gain the

following proposition.

**Proposition 4.2.** Fix \( N > 1 \) and suppose \( 1 \leq q \leq N^{1/4} \) is an integer with \( q \equiv 0 \pmod{4} \). Suppose \( b \) is an even integer, \( 1 \leq a, b < q \) and \( (a, q) = 1 \). For

\[ |x - \frac{b}{q}| \leq \frac{1}{100N}, \ x \in \mathbb{T} \text{ and } |t - \frac{a}{q}| \leq \frac{1}{100N}, \]

we have

\[ \left| \sum_{n=1}^{N} e^{2\pi i(nx+n^2t)} \right| \geq \frac{N}{q^2}, \quad (4.4) \]

Proof. Utilizing the partial summation Lemma 5.1 of [39] and taking \( f(n) = n\frac{b}{q} + n^2\frac{a}{q} \) and \( h(n) = (x - \frac{b}{q})n + (t - \frac{a}{q})n^2 \), we have,

\[ \left| \sum_{n=1}^{N} e^{2\pi i(nx+n^2t)} \right| \geq \sum_{n=1}^{N} e^{2\pi if(n)} \left| - \sup_{k \in [1,N]} \left| \sum_{n=1}^{k} e^{2\pi if(n)} \right| \right| \sup_{n \in [1,N]} |h'(n)|N \]

By Lemma 3.1 and Lemma 3.2 of [39] we have \( W_1 \geq \frac{N}{q^2} \). On the other hand, we have \( W_2 \leq \frac{N}{q^2} \) by our assumptions on \( (x, t) \) and Lemma 3.2 of [39]. Thus, we finish the proof. \( \square \)

Based on this proposition, we define the major arc as

\[ \mathcal{M}(q, a_1, \cdots, a_d) := \left\{ x \in \mathbb{T}^d : \left| x_j - \frac{a_j}{q} \right| \leq \frac{1}{100N}, 1 \leq j \leq N \right\}. \]

Let

\[ E = \bigcup_{1 \leq q \leq N^{\frac{d}{d+1}}, \ 1 \leq a_1, \cdots, a_d \leq q, \ \text{even}} \mathcal{M}(q, a_1, \cdots, a_d). \]

In conformity with Proposition 4.2 for \( x \in E \), there exists \( t = t(x) \) such that

\[ |u(x, t(x))| = \prod_{1 \leq j \leq d} \sum_{n_j=1}^{N} e^{2\pi i(n_jx_j+n_j^2t(x))} \gtrsim N^{d+\frac{d}{d+1}} \cdot (4.5) \]

We just need to show \( |E| \gtrsim 1 \), which one can derive from adapting the arguments of Pierce [39]. In fact, by the simultaneous Dirichlet approximation, we have

\[ E_1 := \bigcup_{1 \leq q \leq N^{\frac{d}{d+1}}, \ 1 \leq a_1, \cdots, a_d \leq q, \ \text{even}} \left\{ x \in \mathbb{T}^d : \left| x_j - \frac{a_j}{q} \right| \leq \frac{4}{qN^{\frac{d}{d+1}}}, 1 \leq j \leq N \right\} = \mathbb{T}^d. \]

We rescale \( E_1 \) by denoting

\[ E_2 := \bigcup_{1 \leq q \leq N^{\frac{d}{d+1}}, \ 1 \leq a_1, \cdots, a_d \leq q, \ \text{even}} \left\{ x \in \mathbb{T}^d : \left| x_j - \frac{a_j}{q} \right| \leq \frac{1}{qN^{\frac{d}{d+1}}}, 1 \leq j \leq N \right\}. \]

Since all the \( \frac{1}{q} \)-scaled intervals which comprise \( E_1 \) are contained in \( E_2 \), by Vitali’s covering lemma or Lemma 5.2 of [39], we have \( |E_2| \gtrsim |E_1| = 1. \)
Moreover, for $0 < c_1 \ll 1$, let
\[ E_3 := \bigcup_{e_1 N^{d/(d+1)} \leq q \leq N^{d/(d+1)}, q \equiv 0 (\text{mod } 4); \ 1 \leq a_1, \ldots, a_d \leq q, \text{ even}} \{ x \in \mathbb{T}^d : |x_j - \frac{a_j}{q}| \leq \frac{1}{q N^{d+1}}, 1 \leq j \leq N \}, \]
then we have
\[ |E_2 \setminus E_3| \leq \sum_{1 \leq q \leq c_1 N} \sum_{a_1, \ldots, a_d \leq q} \frac{1}{q^d N^{d+1}} \leq c_1. \]
Thus, by taking $c_1$ small enough, we have $|E_3| \gtrsim 1$. Analogously, by Vitali’s covering lemma again, we show that
\[ E_4 := \bigcup_{e_1 N^{d/(d+1)} \leq q \leq N^{d/(d+1)}, q \equiv 0 (\text{mod } 4); \ 1 \leq a_1, \ldots, a_d \leq q, \text{ even}} \{ x \in \mathbb{T}^d : |x_j - \frac{a_j}{q}| \leq \frac{1}{100 N}, 1 \leq j \leq N \} \]
has positive measure. Hence
\[ |E| \geq |E_4| \gtrsim 1. \quad (4.6) \]
In conclusion, from the estimates (4.5) and (4.6), we obtain
\[ \left\| \sup_{0 < t < 1} |u(x, t)| \right\|_{L^p(\mathbb{T})} \gtrsim \left\| \sup_{0 < t < 1} |u(x, t)| \right\|_{L^1(E)} \gtrsim N^{\frac{d}{2} + \frac{d}{2(2d+1)}} \quad (4.7) \]
for $1 \leq p < \frac{d}{2(d+1)}$.

4.2. Lower bounds for $L^p$ maximal estimates for certain functions. For general functions, as in [23], one may conjecture (1.7) does not hold for large $p$ and $d \geq 3$. Here we consider the $L^p$ maximal estimates for certain special functions by adapting the ideas of [17] and [23], which give the lower bounds as the Weyl sums.

Let $x = (x', x'') = \mathbb{T}^m \times \mathbb{T}^{d-m}$, for $1 \leq m \leq d - 1$. Define $D = [N^{1-\kappa}]$, for $\kappa \in (0, 1)$. We take
\[ f(x) := f_1(x')f_2(x'') := \prod_{1 \leq j \leq m} \sum_{1 \leq n_j \leq N} e^{2\pi i x_j n_j} \times \prod_{m+1 \leq j \leq d} \sum_{1 \leq n_j \leq N/D} e^{2\pi i x_j Dn_j}. \]
Define
\[ E_m := \bigcup_{e_1 N^{m/(m+1)} \leq q \leq N^{m/(m+1)}, q \equiv 0 (\text{mod } 4); \ 1 \leq a_1, \ldots, a_m \leq q, \text{ even}} \mathcal{M}(q, a_1, \ldots, a_m). \]
By the previous analysis, the set $E_m \subset \mathbb{T}^m$ has positive measure, and such that
\[ \sup_{0 < t < 1} |e^{it\Delta x_m} f_1(x')| \geq N^{\frac{m}{2(m+1)}}, \quad \text{for } x \in E_m. \]
In fact, we can choose $t = \frac{a}{q}$ with $(a, q) = 1$ for $x' \in \mathcal{M}(q, a_1, \ldots, a_m)$.

On the other hand, by letting
\[ g(x_j) := \sum_{1 \leq n_j \leq N/D} e^{2\pi i x_j Dn_j}, \]
we have
\[ |e^{it\Delta x g(x_j)}| = \sum_{1 \leq n_j \leq N/D} e^{2\pi i (x_j Dn_j + |n_j|^2 D^2t)}. \]
By Lemma 5.1 of [39] with
\[
\begin{cases}
f(n_j) = 0, \text{ and } h(n_j) = x_j D n_j + |n_j|^2 (D^2 \frac{a}{q} - k), \\
h'(n_j) = x_j D + 2n_j (D^2 \frac{a}{q} - k), \text{ for some } k \text{ such that }
|D^2 \frac{a}{q} - k| \leq \frac{1}{q},
\end{cases}
\]
we have
\[
|e^{i \frac{\pi}{2} \Delta t} g(x_j)| \geq N/D - N/D |h'(n_j)| \frac{N}{D} \\
\geq N/D \left[ 1 - \left( \frac{D}{100N} + \frac{N}{D} \right) \frac{N}{D} \right] \\
\geq \frac{N}{D}
\]
if \( D \gg \frac{N}{q^2} \sim N^{\frac{m+2}{2(m+1)}} \). This estimate implies \( \kappa \in (0, \frac{m}{2(m+1)}) \).

Thus, by the fact that \( e^{it \Delta x} (e^{i \phi})(x) = e^{i(x-\ell)}(e^{it \Delta \phi})(x - 2t) \), we have
\[
|e^{it \Delta x} f_2(x')| \geq \frac{N^{d-m}}{D^{d-m}} \sim N^{\kappa(d-m)}
\]
on the region
\[
F_{d-m,q} := \bigcup_{1 \leq i_n \leq q} B(2a/q, \frac{1}{100N})^{d-m},
\]
which satisfies \( |F_{d-m,q}| \geq N^{q} \sim N^{\frac{m}{m+1} - (d-m)} \). Hence, we have for \( \kappa \in (0, \frac{m}{2(m+1)}) \),
\[
\frac{\|e^{it(x) \Delta} f(x)\|_{L^p}}{\|f\|_2} \gtrsim N^{\frac{m}{m+1} \kappa(d-m)} N^{\frac{m}{m+1} - \frac{d-m}{p}} N^{-\frac{d}{2}(d-m)} \\
=N^{\frac{m}{m+1} \kappa(d-m)} N^{\frac{m}{m+1} - \frac{d-m}{p}}.
\]
As a example, if we take \( m = 2, d = 3, p = \frac{8}{3}, \kappa = 1 \), the above bound behaves as \( \frac{1}{2} + \frac{1}{8} - \frac{1}{2} = \frac{3}{8} \). If we take \( m = 2, d = 3, p = \frac{10}{3}, \kappa = 1 \), the above bound behaves as \( \frac{3}{2} > \frac{3}{8} \).

5. Applications of the maximal estimate in Theorem [1.1]

In this section, we give some applications of Theorem [1.1] including the following estimate for the Lebesgue measure of the set where the maximal Weyl sums gain large values. Before that, we introduce an efficacious lemma, which extends the locally constant property of Weyl sums in one dimension obtained by Barron in [4] to the higher dimensions.

For \( 0 < \alpha < d \), we tile \( \mathbb{T}^{d+1} \) by axis-parallel rectangles of dimensions \( (N^{-(d+1)+\alpha})^d \times N^{-2(d+1)+2\alpha} \) and denote the set of these rectangles by \( \Lambda_{\alpha} \).

**Definition 5.1.** Let \( \pi_x : [0,1]^d \times [0,1] \to [0,1]^d \) be a projection in the \( x \) variables. We say \( Q \subset \Lambda_{\alpha} \) is \( d \)-dimensional at scale \( (N, \alpha) \), if for \( \forall x_0 \in [0,1]^d \), there exist at most 2 rectangles \( Q \in Q \) such that \( x \in \pi_x(Q) \).

The following lemma is the locally constant property for the higher dimensional version of the quadratic Weyl sums.
Lemma 5.2. Let $\eta > 0$ be a small constant. Assume $cN^\alpha \leq |\omega_N(x_0, t_0)| < CN^\alpha$ for some $(x_0, t_0) \in [0, 1]^d \times [0, 1]$ and $0 < \alpha < d$. Let $Q$ be a rectangle of dimensions $(N^{-(d+1)+\alpha-\eta})^d \times N^{-2(d+1)+2\alpha-\eta}$ which contains $(x_0, t_0)$. Then we have $|\omega_N(x, t)| \sim N^\alpha$ for $\forall (x, t) \in Q$.

Proof. By a simple mean value theorem, we have

$$|\omega_N(x, t) - \omega_N(x_0, t_0)| = \left| \sum_{1 \leq n \leq N, 1 \leq j \leq d} e^{2\pi i (x \cdot n + |n|^2 t)} - \sum_{1 \leq n \leq N, 1 \leq j \leq d} e^{2\pi i (n \cdot x_0 + |n|^2 t_0)} \right| \leq \sum_{1 \leq n \leq N, 1 \leq j \leq d} (|n| \cdot |x - x_0| + |n|^2 |t - t_0|) \lesssim N^{-(d+1)+\alpha-\eta} + \sum_{1 \leq n \leq N, 1 \leq j \leq d} |n|^2 N^{-2(d+1)+2\alpha-\eta} \lesssim N^\alpha - \eta.$$

This completes the proof. \qed

Using the locally constant property of the Weyl sums, first we estimate the number of the rectangles where the Weyl sums have the lower bound $N^\alpha$ for $\frac{d}{2} + \frac{d}{2(d+1)} < \alpha < d$.

Proposition 5.3. Let $\frac{d}{2} + \frac{d}{2(d+1)} < \alpha < d$, and $X$ be a union of rectangles $Q \subset [0, 1]^{d+1}$ with dimension $(N^{-(d+1)+\alpha})^d \times N^{-2(d+1)+2\alpha}$, which is also $d$-dimensional at scale $(N, \alpha)$. Let $C > 0$ be a constant independent of other parameters. Suppose that for any $Q \in X$, there holds

$$\|\omega_N(x, t)\|_{L^\infty(Q)} \geq CN^\alpha. \quad (5.1)$$

Then we have

$$\#X \lesssim N^{(d^2+2d+2)(1-\frac{d}{2})+\epsilon}. \quad (5.2)$$

Proof. According to the assumption $(5.1)$, we have

$$\left( \sum_{Q \in X} \int_Q |\omega_N(x, t)|^{\frac{2(d+1)}{d}} \, dx \, dt \right)^{\frac{d}{2(d+1)}} \geq N^\alpha (N^{-(d+1)+\alpha})^d N^{-2(d+1)+2\alpha} \frac{d^\frac{d}{2(d+1)}}{2(d+1)} \frac{d^\frac{d}{2(d+1)}}{2(d+1)} \left( \#X \right)^{\frac{d}{2(d+1)}} \geq N^{-\frac{d^2}{2} + d + \frac{d^2+4d+2\alpha}{2(d+1)}} \left( \#X \right)^{\frac{d}{2(d+1)}}.$$

Besides, we have

$$\left( \sum_{Q \in X} \int_Q |\omega_N(x, t)|^{\frac{2(d+1)}{d}} \, dx \, dt \right)^{\frac{d}{2(d+1)}} \leq (N^{-2(d+1)+2\alpha} \int_{[0, 1]^d} \sup_{0 < t < 1} |\omega_N(x, t)|^{\frac{2(d+1)}{d}} \, dx)^{\frac{d}{2(d+1)}} \leq N^{\frac{d(2\alpha-d)}{2(d+1)}+\epsilon}.$$
Combining these two estimates above, we have

\[ N^{-\left(\frac{d^2}{2} + d - \frac{d^2 + 2d - 2}{2(d+1)}\right)} \left(\#X\right)^{\frac{d}{d+1}} \lesssim N^{\frac{d(2d-d)}{d+2}} + \epsilon. \]

This inequality yields

\[ \#X \lesssim N^{(d^2 + 2d + 2)(1 - \frac{d}{d+1}) + \epsilon}. \]

For the proof of the maximal estimates (1.7), recall that the maximal estimate of the Weyl sums on the set

\[ \{ x \in \mathbb{T}^d : \sup_{0 < t < 1} |\omega_N(x, t)| \leq N^{\frac{d}{d+1}} \} \]

is trivial for the estimates (1.7), and we just need to consider the contribution of the remaining

\[ \{ x \in \mathbb{T}^d : \sup_{0 < t < 1} |\omega_N(x, t)| > N^{\frac{d}{d+1}} \}. \]

It is necessary that the Lebesgue measure of the set for \( x \in \mathbb{T}^d \) where the maximal Weyl sums cannot be too large for the estimate (1.7). Employing the maximal estimates (1.7) and Proposition 5.3, we can give a precise estimate for the Lebesgue measure of the set for \( x \in \mathbb{T}^d \) where the maximal Weyl sums gain the large value as follows.

**Corollary 5.4.** For \( \frac{d}{2} + \frac{d}{2(d+1)} < \alpha < d \), let

\[ S_\alpha(N) = \{ x \in \mathbb{T}^d : \sup_{0 < t < 1} |\omega_N(x, t)| \gtrsim N^\alpha \}. \]  

Then we have

\[ |S_\alpha(N)| \lesssim N^{(d+2) - \frac{2(d+1)\alpha}{d} + \epsilon}. \]  

**Proof.** By the locally constant property, for the Lebesgue measure of \( S_\alpha(N) \), we just need to consider the rectangles \( \prod_{j=1}^d [a_j N^{-(d+1)-\alpha}, (a_j + 1) N^{-(d+1)-\alpha}] \) for \( 0 \leq a_j \leq N^{(d+1)-\alpha} \) with \( 1 \leq j \leq d \).

By Proposition 5.3, we can obtain the number of these rectangles is at most \( N^{(d^2 + 2d + 2)(1 - \frac{d}{d+1}) + \epsilon} \), and so

\[ |S_\alpha(N)| \lesssim (N^{-(d+1)+\alpha})^d N^{(d^2 + 2d + 2)(1 - \frac{d}{d+1}) + \epsilon} = N^{(d+2) - \frac{2(d+1)\alpha}{d} + \epsilon}. \]

**Remark 5.5.** Corollary 5.4 also follows from Chebychev’s inequality and Theorem [4] with \( p = \frac{2d+1}{d} \).

6. **Hausdorff dimension of the large value set**

In this section, we consider lower and upper bounds of the Hausdorff dimension of the set \( L_\alpha \) for \( \frac{d}{2} + \frac{d}{2(d+1)} \leq \alpha \leq d \), and give the proof of Proposition 1.8. For the definition of \( L_\alpha \), see (1.10) in Section 1.
6.1. **Upper bound.** To prove the upper bound in Proposition 1.8 we utilize a completion method of higher dimension which is developed by Chen, Shparlinski in [15]. Meanwhile, we use a fractal maximal estimate of the periodic Schrödinger operator derived from Theorem 1.1 which can be obtained by the argument of Eceizabarrena and Luca in [25]. Barron in [4] studied the similar problem in one dimension.

Let
\[ L_\alpha(N) := \left\{ x \in \mathbb{T}^d : \sup_{t \in \mathbb{T}} |\omega_N(x, t)| \geq N^{\alpha} \right\}. \] (6.1)

It is easy to see that
\[ L_\alpha = \bigcap_{M \geq 1} \bigcup_{N \geq M} L_\alpha(N). \] (6.2)

Fix \( M \geq 1 \), then for \( x \in L_\alpha \), define
\[ N(x) := \min \left\{ N \geq M : \sup_{t \in \mathbb{T}} |\omega_N(x, t)| \geq N^{\alpha} \right\}, \] (6.3)

and for \( j \geq 1 \), define
\[ E_j := \{ x \in L_\alpha : N(x) \in [2^{j-1} M, 2^j M) \}. \] (6.4)

From the expression (6.2), we have
\[ L_\alpha \subset \bigcup_{j \geq 1} E_j. \] (6.5)

We recall the lower bound of \( S_{N,k}(x_k, t) \) given by Chen and Shparlinski in [13] and [15], where for \( x_k \in \mathbb{T} \) with \( 1 \leq k \leq d \), \( S_{N,k}(x_k, t) \) is defined by
\[ S_{N,k}(x_k, t) := \sum_{n=1}^{N} \frac{1}{h} e^{2\pi i n \frac{h}{N} x_k} e^{2\pi i (x_k n + tn^2)} = \sum_{n=1}^{N} \frac{1}{h} |\omega_N(x_k + \frac{h}{N}, t)|. \] (6.6)

**Lemma 6.1.** For all \((x_k, t) \in \mathbb{T}^2 \) with \( 1 \leq k \leq d \) and any \( 1 \leq N \leq M \), we have
\[ |\omega_N(x_k, t)| \lesssim |S_{N,k}(x_k, t)|. \] (6.7)

By this lemma, we can choose \( M \) to be dyadic parameter to control the Weyl sums. Indeed, for \( x \in E_j \), by Lemma 6.1 and the definition of \( E_j \), we have
\[ 1 \leq N(x)^{-\alpha} \sup_{0 < t < 1} |\omega_N(x, t)| \lesssim (2^j M)^{-\alpha} \sup_{0 < t < 1} |S_{2^j M}(x, t)|, \] (6.8)

where
\[ S_{2^j M}(x, t) := \prod_{k=1}^{d} |\omega_{2^j M,k}(x_k, t)| \]
\[ = \prod_{k=1}^{d} \left| \sum_{h_k = 1}^{2^j M} \frac{1}{h_k} \omega_{2^j M,k}(x_k + \frac{h_k}{2^j M}, t) \right| \]
\[ = \left| \sum_{1 \leq h_k \leq 2^j M, k=1,2,\ldots,d} \prod_{k=1}^{d} \frac{1}{h_k} \omega_{2^j M,k}(x_k + \frac{h_k}{2^j M}, t) \right| \]
\[ = \left| \sum_{1 \leq h_k \leq 2^j M, k=1,2,\ldots,d} \left( \prod_{k=1}^{d} \frac{1}{h_k} \right) \omega_{2^j M}(x + \frac{h}{2^j M}, t) \right|. \] (6.9)
and $S_{2^jM}(x_k,t)$ is defined by (6.6) and $h = (h_1,h_2,\cdots,h_d)$.

Recall that a Borel measure $\mu$ on $\mathbb{T}^d$ is said to be $\beta$-dimensional with $0 < \beta \leq d$ if

$$c_\beta(\mu) := \sup_{B(x,r)} \frac{\mu(B(x,r))}{r^\beta} < \infty,$$

(6.10)

where the supremum is taken over balls $B(x,r)$ in $\mathbb{T}^d$.

**Lemma 6.2** (Frostman’s Lemma, [36]). Suppose $S \subset \mathbb{T}^d$ is a Borel set and $\mathcal{H}^\gamma(S) > 0$ with $\gamma > 0$. Then there exists a nonzero $\gamma$-dimensional measure $\mu_\gamma$ supported on $S$.

Frostman’s Lemma establishes the relationship between the Hausdorff dimension of the Borel set and corresponding Borel measure supported on it.

Next we cite the following useful lemma transforming Theorem 1.1 into the fractal version, which plays an essential role in the control of the measure for $\mathbb{T}^d$ with respect to the Borel measure supported on it.

**Lemma 6.3** (Proposition 5.2 in [25]). Let $d \geq 1$, $p \geq 1$ and $s_0 \geq 0$. Fix $\epsilon > 0$ and suppose $f \in H^{s_0+\epsilon} (\mathbb{T}^d)$ is a function such that

$$\left\| \sup_{t \in \mathbb{T}} |e^{it\Delta} f| \right\|_{L^p(\mathbb{T}^d)} \lesssim \|f\|_{H^{s_0+\epsilon}(\mathbb{T}^d)},$$

(6.11)

Then for any $\gamma$-dimensional measure $\mu_{\gamma}$ on $\mathbb{T}^d$, we have

$$\left\| \sup_{t \in \mathbb{T}} |e^{it\Delta} f| \right\|_{L^p(\mathbb{T}^d, d\mu_{\gamma})} \lesssim \|f\|_{H^{s}(\mathbb{T}^d)}, \quad s > \frac{d-\gamma}{p} + s_0,$$

(6.12)

Now we show how the upper bound of the Hausdorff dimension of the set $L_\alpha$ follows after combining Theorem 1.1 with these three lemmas above.

For $\frac{d}{2} \leq \alpha \leq d$, we assume that $\mathcal{H}^\gamma(L_\alpha) > 0$ for some $\gamma > 0$. Then by Lemma 6.2, there exists a $\gamma$-dimensional Borel measure $\mu_\gamma$ supported on $L_\alpha$.

**Claim:** if $\gamma > \frac{2(d+1)}{d} (d-\alpha)$, then we have $\mu_\gamma(\mathbb{T}^d) = 0$.

We first prove this claim, then show the equivalence between this claim and the upper bound for $\dim_H L_\alpha$ in Proposition 1.8 in the end of this subsection.

By the estimates (6.3), we have

$$\mu_\gamma(\mathbb{T}^d)^{\frac{\beta}{p}} = \left( \int_{\mathbb{T}^d} d\mu_\gamma \right)^{\frac{\beta}{p}} \lesssim \left( \sum_{j \geq 1} (2^j M)^{-p\alpha} \int_{E_j} \sup_{0 < t < 1} |S_{2^jM}(x,t)|^p d\mu_\gamma \right)^{\frac{\beta}{p}}. \quad (6.13)$$
For each $j \geq 1$, by Lemma 6.3 we obtain

\[
\left( \int_{E_j} \sup_{0 < t < 1} |S_{2j}^M(x, t)|^p \, d\mu_{\gamma} \right)^{1/p} = \left( \int_{E_j} \sup_{0 < t < 1} \left| \prod_{k=1}^{d} \frac{1}{h_k} \omega_{2j}^M(x + \frac{h_k}{2j} M, t) \right|^p \, d\mu_{\gamma} \right)^{1/p} \leq \sum_{1 \leq h_k \leq 2jM} \left( \prod_{k=1}^{d} \frac{1}{h_k} \right) \left( \int_{E_j} \sup_{0 < t < 1} |\omega_{2j}^M(x + \frac{h_k}{2j} M, t)|^p \, d\mu_{\gamma} \right)^{1/p} \leq \prod_{k=1}^{d} \left( \sum_{1 \leq h_k \leq 2jM} \frac{1}{h_k} \right) \left( \int_{E_j} \sup_{0 < t < 1} |\omega_{2j}^M(x + \frac{h_k}{2j} M, t)|^p \, d\mu_{\gamma} \right)^{1/p} \leq \varepsilon (2jM)^{de} (2jM)^{\frac{d-\gamma}{2} + \frac{d-\delta}{2}}.
\]

Taking $p = \frac{2(d+1)}{d}$ and inserting (6.14) into (6.13), we have

\[
\mu_{\gamma}(T^d) \frac{d}{2(d+1)} \leq \left\{ \sum_{j \geq 1} (2jM)^{\frac{2(d+1)}{d}} (2jM)^{\frac{2(d+1)}{d}} (2jM)^{\frac{2(d+1)}{d}} \right\}^{\frac{d}{2(d+1)}} = M^{d-\alpha} \frac{d}{2(d+1)} \gamma + \frac{d}{2} \left( \sum_{j \geq 1} 2^{-j(\gamma + \frac{2(d+1)}{d}(\alpha-d) - 2(d+1)\epsilon)} \right)^{\frac{d}{2(d+1)}}.
\]

If $\gamma > \frac{2(d+1)}{d}(d-\alpha)$, say $\gamma = \frac{2(d+1)}{d}(d-\alpha) + \delta$. Then from (6.15), we have

\[
\mu_{\gamma}(T^d) \frac{d}{2(d+1)} \leq M^{-\frac{d}{2(d+1)} \delta + \frac{d}{2} \left( \sum_{j \geq 1} 2^{-j(\delta - 2(d+1)\epsilon)} \right)^{\frac{d}{2(d+1)}}} \leq M^\frac{d}{2(d+1)\delta},
\]

where we choose $\epsilon = \frac{d}{4(d+1)\delta}$. Let $M$ tend to $\infty$, we have $\mu_{\gamma}(T^d) = 0$, which proves the claim.

This claim and Lemma 6.2 imply that $\mathcal{H}^{\alpha}(L_\alpha) = 0$ for $\gamma > \frac{2(d+1)}{d}(d-\alpha)$, which yields

\[
\dim_{H}(L_\alpha) \leq \frac{2(d+1)}{d}(d-\alpha).
\]

Thus, the upper bound of the Hausdorff dimension for $L_\alpha$ in Proposition 1.8 has been proved.

6.2. Lower bound. Inspired by the results of Barron [1], one may conjecture that the lower bound of $\dim_{H} L_\alpha$ is $\frac{2(d+1)(d-\alpha)}{d}$, if $\frac{d}{2} + \frac{d}{2(d+1)} < \alpha \leq d$. Actually, this is indicated in the proof of Eceizabarrena and Lucà [25]. Indeed, they show that there exists a function $f \in H^s$ for $s < \frac{d}{2(d+1)}$ which consists of segmented Gauss sums fails to convergence on a nontrivial Hausdorff dimension set (See Theorem 1.1 of [25]). Here, we explicate the proof of the lower bound of $\dim_{H} L_\alpha$ via Proposition 4.2.

By the analysis in Eceizabarrena and Lucà [25], we have

**Lemma 6.4 (P. 16 in [25]).** Let $\tau > \frac{d+1}{d}$. Define

\[
G_\tau = \{ x \in T^d : |qx_j - p_j| \leq q^{1-\tau} \text{ holds for infinitely many odd } q \text{ and even } p_j \}. \]
Then, we have $\dim_H G_\tau = \frac{d+1}{\tau}$.

The ingredient of the proof for the estimate of the lower bound for $\dim_H L_\alpha$ is the construction of a subset included in $L_\alpha$ as in Lemma 6.4 which is followed by the result immediately.

Take $q$ odd and $N_q$ such that $q \sim (100)^2(d-\alpha)$

$$2(d+1)(d-\alpha) \leq \dim_H G_\tau \leq \dim_H L_\alpha,$$

which gives the proof for the lower bound of the Hausdorff dimension for $L_\alpha$ in Proposition 1.8 and completes the proof for Proposition 1.8 together with the estimate (6.17).

Remark 6.5. It is interesting to study the Fourier dimension of $G_\tau$ and it is reasonable to conjecture $\dim_F G_\tau = \dim_H G_\tau = \frac{d+1}{\tau}$. If this conjecture holds, $G_\tau$ is called a Salem set.

Appendix

AN ALTERNATE PROOF OF THEOREM 1.1 INSPIRED BY BAKER’S ARGUMENT

by Alex Barron

In this Appendix we give an alternate proof of Theorem 1.1 which is more directly related to the argument employed by Baker in [1]. Our argument relies on a straightforward higher-dimensional generalization of the Diophantine approximation result used by Baker in [1]. Our proof slightly improves the $N^\epsilon$ loss in Theorem 1.1 above. At the end of the Appendix we show that essentially the same argument also implies Strichartz-type estimates for the Weyl sum with logarithmic losses.

Our main tool is the following.

Proposition A.1. Let $w_N$ be the Weyl sum on $\mathbb{T}^{d+1}$ as in (1.12), given by

$$w_N(x, t) = \sum_{n \in \mathbb{Z}^d \cap [1, N]^d} e^{2\pi i (x \cdot n + t |n|^2)},$$

and suppose that $(x, t) \in \mathbb{T}^{d+1}$ with

$$|w_N(x, t)| \geq N^\alpha \geq C_d N^{\frac{d}{2}}$$

for some sufficiently large constant $C_d$ that does not depend on $N$. Then if $N$ is sufficiently large there is an integer $q$ such that

$$|t - a/q| \lesssim q^{-1} N^{-\frac{d}{2}\alpha} \quad \text{for some } a \leq q \quad \text{with } (a, q) = 1 \quad (A.1)$$

and

$$1 \leq q \lesssim N^{2-\frac{d}{2}\alpha}, \quad (A.2)$$
The proof of this proposition uses estimates that are standard if one ignores various losses on the order of $N^{\varepsilon}$. Since we are interested in applying this theorem with no logarithmic losses we will sketch the requisite background before proving Proposition A.1.

We begin by defining

$$I(\beta_1, \beta_2) = \int_0^N e^{2\pi i (\beta_1 \xi + \beta_2 \xi^2)} d\xi.$$  

We have the following classical estimate, which is a consequence of integration by parts and the Van der Corput lemma for oscillatory integrals.

**Proposition A.2.** One has

$$|I(\beta_1, \beta_2)| \leq C N \max(1, N|\beta_1|, N^2|\beta_2|)^{-1/2}.$$  

See also [44], Chapter 7, for a more general version of this estimate. We will also use the following sharp decomposition of the one-variable quadratic Weyl sum due to Vaughan [45].

**Lemma A.3** ([45], Theorem 8). Suppose there are integers $1 \leq a, q \leq N$ with $(a, q) = 1$ such that $|t - a/q| \leq q^{-1} N^{-1}$. Also suppose $b$ is chosen so that $1 \leq b \leq q$ and

$$|x - b/q| \leq 1/(2q).$$  

Writing

$$\beta_1 = x - b/q, \quad \beta_2 = t - a/q,$$

and

$$S(q) = \sum_{r=0}^{q-1} e^{2\pi i (\frac{r}{q} + \frac{q}{4} r^2)},$$

we have

$$\sum_{n=1}^N e^{2\pi i (xn + tn^2)} = q^{-1} S(q) I(\beta_1, \beta_2) + O(q^{1/2} N|\beta_2|^{1/2} q^{1/2}).$$  

The key point here is the error term in (A.4), which allows us to prove Proposition A.1 with no logarithmic loss in $N$.

**Proof of Proposition A.1.** We will use Lemma A.3 although we require some set-up. Let $w_{N,j}$ be a one-dimensional quadratic Weyl sum in the $j$-th variable. Choose $\gamma_j$ such that

$$N^{\gamma_j} \leq |w_{N,j}(x_j, t)| \leq 2N^{\gamma_j}$$

for each $j$.

Let $k$ be the number of $w_{N,j}$ such that

$$|w_{N,j}(x_j, t)| \geq C_1 N^{\frac{1}{2}},$$
where $C_1$ is a sufficiently large constant to be chosen below. Note that we must have $k \geq 1$ if $C_d$ is sufficiently large since $|w_N(x,t)| \geq C_d N^{d/2}$. After relabeling we may assume that

$$|w_{N,j}(x_j,t)| \geq N^\gamma_j \geq C_1 N^{\frac{d}{2}} \quad \text{for all } j = 1, \ldots, k \quad (A.5)$$

and

$$|w_{N,j}(x_j,t)| < C_1 N^{\frac{d}{2}} \quad \text{for } j = k + 1, \ldots, d. \quad (A.6)$$

Now by construction we have

$$N^\alpha \leq \prod_{i=j}^d |w_{N,j}(x_j,t)| \leq CN^{\gamma_1 + \ldots + \gamma_k \frac{d}{2} - (d-k)}$$

and therefore

$$N^{\gamma_1 + \ldots + \gamma_k} \gtrsim N^{\alpha - \frac{d}{2} (d-k)}. \quad (A.7)$$

We now apply Dirichlet’s theorem to find $0 \leq a < q \leq N$ with $(a,q) = 1$ such that $|t - a/q| \leq q^{-1} N^{-1}$. We also find $1 \leq b_j \leq q$ such that $|x_j - b_j/q| \leq 1/(2q)$. For $j = 1, \ldots, k$ we have $|w_{N,j}(x_j,t)| \geq C_1 N^{d/2}$. We now apply the decomposition in Lemma A.3. If $C_1$ is sufficiently large the error term is negligible, and therefore the classical Gauss sum bound $|S(q)| \lesssim q^{1/2}$ yields

$$|w_{N,j}(x_j,t)| \lesssim q^{1/2} N \max(1, N|x_j - b_j/q|, N^2|t - a/q|)^{-1/2}, \quad j = 1, \ldots, k. \quad (A.8)$$

With (A.6) this implies

$$N^\alpha \leq |w_N(x,t)| \lesssim q^{-k/2} |t - a/q|^{-k/2} N^{\frac{d-k}{2}},$$

and since $|t - a/q| \leq q^{-1} N^{-1}$ we have $N^{1/2} \leq q^{-1/2} |t - a/q|^{-1/2}$ and hence

$$N^\alpha \lesssim q^{-d/2} |t - a/q|^{-d/2}. \quad (A.9)$$

The estimate (A.1) now follows after rearranging (A.9). From (A.8) we also have

$$q \lesssim N^{2 - 2\gamma_j}, \quad j = 1, \ldots, k,$$

and therefore by (A.7)

$$q \lesssim N^{2 - \frac{\gamma_1 + \ldots + \gamma_k}{2}} \lesssim N^{2 - (\frac{\alpha}{d} - \frac{d-k}{d})}.$$

Now

$$\frac{2}{k^\alpha} - \frac{d-k}{k} \geq \frac{2}{d^\alpha}$$

since $\alpha \geq d/2$, so (A.2) follows. Finally, to prove A.3 we note that (A.8) implies

$$\|q x_j\| \lesssim N^{1 - 2\gamma_j} \quad \text{for } j = 1, \ldots, k. \quad (A.10)$$

Then from (A.10) and (A.7) we obtain

$$\prod_{j=1}^k \|q x_j\| \lesssim N^{k-2(\gamma_1 + \ldots + \gamma_k)} \lesssim N^{d - 2\alpha}.$$ 

But $\|q x_j\| < 1$ for each $j$, so (A.3) follows. \qed
We now show that Proposition A.1 implies Theorem 1.1 with $N^\epsilon$ loss replaced by $\log(N)^{2d+1}$. To begin the proof note that by pigeonholing it suffices to show that if

$$E^{d}_\alpha = \{x \in T^d : N^\alpha \leq \sup_{0 < t < 1} |w_N(x, t)| < 2N^\alpha \}$$

for some dyadic $N^\alpha$ then

$$\left( \int_{E^{d}_\alpha} \sup_{0 < t < 1} |w_N(x, t)|^{\frac{2(d+1)}{d}} \, dx \right)^{\frac{d}{2(d+1)}} \lesssim \log(N)^{(d-1)\frac{d}{d+1} + \frac{d}{2}} N^{\frac{d}{d+1} + \frac{d}{2}}.$$

Moreover, we can assume that

$$\frac{d}{2(d+1)} + \frac{d}{2} < \alpha \leq d$$

since otherwise the estimate is trivial. In particular Proposition A.1 applies.

It suffices to prove the following.

**Proposition A.4.** Suppose $\alpha > \frac{d}{2(d+1)} + \frac{d}{2}$. Then

$$|E^{d}_\alpha| \lesssim \log(N)^{d-1} N^{d+2 - \frac{2(d+1)}{d} \alpha}. \quad (A.11)$$

**Proof.** Our proof will rely on the Diophantine approximation properties of the points where $w_N$ can attain large values, as quantified by Proposition A.1. Let us momentarily fix $x \in E^{d}_\alpha$. By Proposition A.1 there must be an integer $1 \leq q \lesssim N^{\frac{2d-\frac{d}{\alpha}}{d}}$ and integers $1 \leq b_j \leq q$ such that

$$\prod_{i=1}^d |x_j - b_j/q| \lesssim q^{-d} N^{d-2\alpha}. \quad (A.12)$$

Given a vector $b = (b_1, ..., b_d) \in \mathbb{Z}^d$ with $1 \leq b_j \leq q$, let $\mathcal{A}(q; b) \subset T^d$ denote the set of $x$ for which (A.12) holds.

**Lemma A.5.** We have $|\mathcal{A}(q; b)| \lesssim \log(N)^{d-1} q^{-d} N^{d-2\alpha}$.

**Proof.** Note that $\mathcal{A}(q; b)$ is contained in a union of $O(\log(N)^{d-1})$ dyadic rectangles, each of volume proportional to $q^{-d} N^{d-2\alpha}$. Alternatively one can write $|\mathcal{A}(q; b)|$ as a $d$-dimensional integral and the result then follows from an elementary calculation. \qed

Now by the above discussion we must have

$$E^{d}_\alpha \subset \bigcup_{q=1}^{cN^{2-\frac{d}{\alpha}}} \bigcup_{b \in \mathbb{Z}^d} \mathcal{A}(q; b),$$

and therefore by Lemma A.5

$$|E^{d}_\alpha| \lesssim \log(N)^{d-1} N^{d-2\alpha} \sum_{q=1}^{cN^{2-\frac{d}{\alpha}}} \sum_{b \in \mathbb{Z}^d} q^{-d} \lesssim \log(N)^{d-1} N^{d-2\alpha} N^{2-\frac{d}{\alpha}} \lesssim \log(N)^{d-1} N^{d+2 - \frac{2(d+1)}{d} \alpha}.$$ 

This proves Proposition A.4. \qed
The desired estimate now follows immediately from Proposition A.4 as discussed before.

We end the Appendix by remarking that Proposition A.1 also implies the following mean-value or Strichartz-type estimate.

**Proposition A.6.** Let \( p_d = \frac{2(d+2)}{d} \). We have

\[
\|w_N\|_{L^{p_d}(\mathbb{T}^{d+1})} \leq C \log(N)^\sigma N^{\frac{d}{2}},
\]

with \( \sigma = \frac{d^2}{2(d+2)} \).

This was proved by Hu and Li in [32] for Weyl sums with a loss in \( N \) that is slightly worse than logarithmic (resulting from the divisor bound). It also follows with constant \( N^{\frac{d}{2}+\epsilon} \) from the much more general Strichartz estimates proved by Bourgain and Demeter in [9] using decoupling. It was shown by Bourgain in [6] that one needs \( \sigma \geq \frac{d^2}{2(d+2)} \).

**Proof.** The proof is similar to the proof of Proposition A.4. By pigeonholing we see it suffices to show that

\[
\int_{S_\alpha} |w_N(x,t)|^{p_d} \, dx \, dt \lesssim \log(N)^{d-1} N^{d+2},
\]

where

\[
F_\alpha = \{(x,t) \in \mathbb{T}^d \times \mathbb{T} : N^\alpha \leq |w_N(x,t)| < 2N^\alpha\}.
\]

Using Proposition A.1 as above, we see that

\[
F_\alpha \subset \bigcup_{q=1}^{cN^2 \Gamma^2} \left( \bigcup_{b \in \mathbb{Z}^d} \mathcal{A}(q,b) \times \bigcup_{1 \leq a \leq q} \{t : |t - a/q| \leq Cq^{-1}N^{-\frac{2\alpha}{d}}\} \right).
\]

Therefore

\[
|F_\alpha| \lesssim \log(N)^{d-1} \sum_{q=1}^{cN^2 \Gamma^2} \sum_{b \in \mathbb{Z}^d} \sum_{1 \leq a \leq q} q^{-(d+1)} N^{d-2\alpha} N^{-\frac{2\alpha}{d}}
\]

\[
\lesssim \log(N)^{d-1} N^{d+2 - \frac{2(d+1)}{d} \alpha} N^{-\frac{2\alpha}{d}}
\]

\[
\lesssim \log(N)^{d-1} N^{d+2} N^{-\frac{2(d+2)}{d} \alpha}
\]

as desired.

\[ \square \]

**Acknowledgements.** The authors are very grateful to Alex Barron for adding an appendix, which gives another new proof for Theorem 1.1 and is very precious and helpful for the readers to understand. The authors would also like to thank the associated editor and anonymous referee for their helpful comments and suggestions which helped improve the paper greatly. C. Miao was supported by the National Key Research and Development Program of China (No. 2020YFA0712900) and NSFC Grant 11831004. T. Zhao was supported by the NSFC Grant 12101040 and the Fundamental Research Funds for the Central Universities (FRF-TP-20-076A1).
REFERENCES

[1] R. Baker, $L^p$ maximal estimates for quadratic Weyl sums. Acta Math. Hungar., 2021, 165:316-325.
[2] R. Baker, C. Chen and I. E. Shparlinski, Bounds on the norms of maximal operators on Weyl sums. arXiv:2107.13974v1.
[3] R. Baker, C. Chen and I. E. Shparlinski, Large Weyl sums and Hausdorff dimension. arXiv:2108.10439v1.
[4] A. Barron, An $L^q$ maximal estimate for quadratic Weyl sums. International Math. Research Notices, 2021, 1-28.
[5] A. S. Besicovitch, Sets of Fractional Dimensions (IV): On Rational Approximation to Real Numbers. J. London Math. Soc., 1934, 9(2):126-131.
[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations part I: Schrödinger equations. Geom. and Func. Anal., 1993, 3(2):107-156.
[7] J. Bourgain, On the Schrödinger maximal function in higher dimension, Proc. Steklov Inst. Math., 2013, 286:46-60.
[8] J. Bourgain, A note on the Schrödinger maximal function. J. Anal. Math., 2016, 130: 303-396.
[9] J. Bourgain and C. Demeter, The proof of the $L^2$ decoupling conjecture. Ann. of Math., 2015, 182:351-399.
[10] J. Bourgain and L. Guth, Bounds on oscillatory integral operators based on multilinear estimates. Geom. Func. Anal., 2011, 21(6):1239-1295.
[11] L. Carleson, Some analytic problems related to statistical mechanics. Euclidean Harmonic Analysis(Proc. Sem., Univ. Maryland, College Park, Md, 1979), Lecture Notes in Math. 779, 5-45.
[12] J. W. S. Cassels, An Introduction to Diophantine Approximation. Cambr. Tracts Math. Phys., vol. 45, Cambridge University Press, New York, 1957.
[13] C. Chen and I. E Shparlinski, New bounds of Weyl sums. Int. Math. Res. Not., 2021, 11:8451-8491.
[14] C. Chen and I. E Shparlinski, On large values of Weyl sums. Adv. Math., 2020, v.370.
[15] C. Chen and I. E Shparlinski, Hausdorff dimension of the large values of Weyl sums. J. Number Theory, 2020, 214:27-37.
[16] CH. Cho and S. Shiraki, Pointwise convergence along a tangential curve for the fractional Schrödinger equations. Ann. Penn. Math., 2012, 46(2):993-1005.
[17] E. Compaan, R. Lucà, and G. Staffilani, Pointwise convergence of the Schrödinger flow. Int. Math. Res. Not., 2021, 2021(1): 596-647.
[18] B. E. J. Dahlberg and C. E. Kenig, A note on the almost everywhere behavior of solutions to the Schrödinger equation. Harmonic analysis, Minneapolis, MN, 1981, Lecture Notes in Mathematics 908(Springer, Berlin) 205-209.
[19] C. Demeter, On the refined Strichartz estimates. arXiv:2002.09552v1.
[20] Y. Deng, P. Germain, and L. Guth, Strichartz estimates for the Schrödinger equation on irrational tori. J. Funct. Anal., 2017, 273(9):2846-2869.
[21] X. Du, L. Guth and X. Li, A sharp Schrödinger maximal estimate in $R^2$. Ann. of Math., 2017, 186:607-640.
[22] X. Du, L. Guth, X. Li and R. Zhang, Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates. Forum of Mathematics, Sigma, 2018, 6:e14.
[23] X. Du, J. Kim, H. Wang, and R. Zhang, Lower bounds for estimates of the Schrödinger maximal function. Math. Res. Lett., 2020, 27(3):687-692.
[24] X. Du and R. Zhang, Sharp $L^2$ estimates of the Schrödinger maximal function in higher dimensions. Ann. of Math., 2019, 189(3):837-861.
[25] Eceizabarrena, Daniel, and Renato Lucà. Convergence over fractals for the periodic Schrödinger equation” arXiv preprint arXiv:2005.07681. To appear in Analysis & PDE.
[26] K. J. Falconer, The Geometry of Fractal Sets. Cambridge University Press, 1985.
[27] K. J. Falconer, Fractal Geometry: Mathematical Foundations and Applications, Second Edition. John Wiley & Sons, Ltd 2003.
[28] L. Guth, A. Iosevich, Y. Ou and H. Wang, On Falconer’s distance set problem in the plane. Invent. Math., 2020, 219:779-830.
[29] K. Hambrook. Explicit Salem sets in $R^2$. Adv. Math., 2017, 311:634-648.
[30] J. Hickman, Lecture 2: Introduction to $\ell^2$ decoupling and a first look at the applications, available at: https://www.math.sciences.univ-nantes.fr/~vitturi/lecture_notes/l2%20decoupling/Decoupling%20notes%202.pdf

[31] J. Hickman and M. Vitturi, Lecture 1: Classical methods in restriction theory, available at: https://www.math.sciences.univ-nantes.fr/~vitturi/lecture_notes/l2%20decoupling/Decoupling%20notes%201.pdf

[32] X. Li and Y. Hu, Discrete Fourier restriction associated with Schrödinger equations, Revista Matematica Iberoamericana 30 (2014), No. 4, 1281-1300.

[33] V. Jarník, Über die simultanen diophantischen Approximationen. Math. Z., 1931, 33(1):505-543.

[34] S. Lee, On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$, Int. Math. Res. Not., 2006, 32597, 1-21.

[35] R. Lucà and K. M. Rogers, A note on pointwise convergence for the Schrödinger equation. Math. Proc. Camb. Philos. Soc., 2019, 166(2):209-218.

[36] P. Mattila, Fourier Analysis and Hausdorff Dimension. Cambridge University Press, 2015.

[37] A. Moyua and L. Vega, Bounds for the maximal function associated to periodic solutions of one-dimensional dispersive equations. Bull. London Math. Soc., 2008, 40:117-128.

[38] T. Oh, Note on a lower bound of the Weyl sum in Bourgain’s NLS paper. Online note https://www.maths.ed.ac.uk/~toh/Files/WeylSum.pdf

[39] L. B. Pierce, On Bourgain’s Counterexample for the Schrödinger Maximal Function. Q. J. Math., 2020, 71(1): 1309-1344.

[40] W. M. Schmidt, Small fractional parts of polynomials. Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society.

[41] S. Shiraki, Pointwise convergence along restricted directions for the fractional Schrödinger equation. J. Fourier Anal. Appl., 2020, 26:1-12.

[42] P. Sjölin, Regularity of solutions to the Schrödinger equation, Duke Math. Journal, 1987, 55(d):699-715.

[43] T. Tao, A sharp bilinear restrictions estimate for paraboloids. Geom. Func. Anal., 2003, 13(6):1359-1384.

[44] R.C. Vaughan, The Hardy–Littlewood Method. Cambridge Tracts in Mathematics, vol. 125. Cambridge University Press, Cambridge (1997)

[45] R. C. Vaughan, On generating functions in additive number theory, I. In: Chen, W.W.L., Gowers, W.T., Halberstam, H., Schmidt, W.M., Vaughan, R.C. (eds.) Analytic Number Theory, Essays in Honour of Klaus Roth. Cambridge University Press, Cambridge (2009)

[46] L. Vega, Schrödinger equations: pointwise convergence to the initial data, Proc. Amer. Math. Soc., 1988, 102(4):874-878.

[47] J. Yuan, T. Zhao and J. Zheng, Pointwise convergence along non-tangential direction for the Schrödinger equation with complex time. Rev. Mat. Complut., 2021, 34:389-407.

[48] J. Yuan, T. Zhao and J. Zheng, On the dimension of divergence sets of Schrödinger equation with complex time. Nonlinear Anal., 2021, 208, 112312.

[49] J. Yuan and T. Zhao, Pointwise convergence along a tangential curve for the fractional Schrödinger equation with $0 < m < 1$. Math. Meth. Appl. Sci., 2021, 1-12.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN, URBANA, IL 61801, USA

Email address: aabarron@illinois.edu

Institute of Applied Physics and Computational Mathematics, Beijing 100088, China

Email address: miao changxing@iapcm.ac.cn

Department of Mathematics, Beijing Key Laboratory on Mathematical Characterization, Analysis, and Applications of Complex Information, Beijing Institute of Technology, Beijing 100081, China

Email address: yuan jiye@bit.edu.cn

School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China

Email address: zhao tengfei@ustb.edu.cn