Directional Short-Time Fourier Transform of Ultradistributions

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Abstract
We define and analyze the \(k\)-directional short-time Fourier transform and its synthesis operator over Gelfand–Shilov spaces \(S^\alpha_\beta (\mathbb{R}^n)\) and \(S^\alpha_\beta (\mathbb{R}^{k+n})\), respectively, and their duals. Also, we investigate directional regular sets and their complements—directional wave fronts, for elements of \(S^\alpha_\alpha (\mathbb{R}^n)\).

Keywords \(k\)-Directional short-time Fourier transform · Ultradistributions · Wave front sets

Mathematics Subject Classification 42A38 · 46F05 · 35A18

1 Introduction

Grafakos and Sansing [13] developed the concept of directional sensitive variant of the short-time Fourier transform (STFT) by introducing the Gabor ridge functions that can be viewed as a time-frequency analysis in a certain domain. A slightly different
version of their concept was considered by Giv in [11], where he defined the directional short-time Fourier transform (DSTFT). In [27], the authors analyzed the DSTFT on Schwartz test function spaces, proving the continuity theorems and extending the DSTFT to the spaces of tempered distributions. Moreover, in [1], the results of Giv are extended through the investigations of the STFT in the direction of \( u \in \mathbb{S}^{n-1} \) on the distributions of exponential type and through the analysis of the (multi)directional wave front sets for tempered distributions.

In the past several decades, there has been a trend in investigating integral transforms on the spaces of ultradistribution, such as the wavelet transform, STFT, Laplace and Hilbert transform, [2,6,17,24,28], as natural generalization of these transforms over the spaces of distributions. We will develop our theory on Gelfand–Shilov spaces (of Roumieu and Beurling type) [12]. Gelfand–Shilov type spaces and their duals have been successfully used in the theory of differential operators, spectral analysis, and more recently in the theory of pseudo-differential operators [24], quantum mechanics and traveling waves as well as in various hyperbolic and weak hyperbolic problems [10,26]. Such spaces have become a frame for the time-frequency analysis. They are especially related to modulation spaces [15], localization operators [3] and the corresponding pseudodifferential calculus [29,30]. In this paper, we define the multi-dimensional STFT in the direction of \( u^k = (u_1, \ldots, u_k) \), where \( u_i, i = 1, \ldots, k \) are independent vectors of \( \mathbb{S}^{n-1} \). Moreover, we analyze the corresponding synthesis operator. By a linear transformation of coordinates, we simplify our exposition considering direction \( e^k = (e_1, \ldots, e_k) \). By the continuity results presented in Theorem 2.3 and 2.5, we show in Sect. 2.3 that both transforms defined as a transposed mappings and as actions on appropriate window functions coincide on Gelfand–Shilov (GS) spaces of ultradistributions.

For our results, Hörmander’s notion of the wave front set [16] will be of interest. In many situations, it is essential to identify the location and the orientation of a distributed singularity of a signal, in the sense of resolving the wave front set. This is useful in the study of certain phenomena in local solvability of PDEs. In the recent years, results on characterizing wave front set of a (ultra)distributions in terms of different continuous integral transforms have been presented [7,19,22,23]. In the second part, in Section 3, we analyze the regularity properties of a GS ultradistributions \( f \in S^\omega(\mathbb{R}^n) = S^\omega_\alpha(\mathbb{R}^n) \) by introducing the \( k \)-directional regular sets and the wave front sets using the \( k \)-DSTFT of \( f \). The main result is presented in Theorem 3.4, where we show that the \( k \)-directional wave front does not depend on the window function. We also consider partial wave front in the sense of [16] and show that this notion is equivalent with the \( k \)-directional wave front. Actually, the partial wave front was not considered neither for distribution spaces nor for GS type spaces. More on, the micro-local analysis in spaces of ultradistributions can be found in [8,9], as well as in [4,5] and the references therein.

Let us note that we follow [1] (related to distributions) and give here new details which are important when developing the analysis on GS spaces. Similarly, in the last subsection when we compare directional wave fronts, we follow [22] but again with details related to ultradistributions which makes the proof more complex. The most important remark is that we corrected mistakes appeared in [1] related to the transfer from one to another window function of the \( k \)-DSTFT (see Theorem 2.11).
1.1 Notation

We employ the notation \( \mathbb{N}_0, \mathbb{R} \) and \( \mathbb{C} \) for the sets of natural (including zero), real and complex numbers, respectively; \( S^{n-1} \) stands for the unit sphere of \( \mathbb{R}^n \). For given multi-indexes \( p = (p_1, \ldots, p_n) \), \( v = (v_1, \ldots, v_n) \in \mathbb{N}_0^n \) and \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n \), we write

\[
t^p = t_1^{p_1} \cdots t_n^{p_n}, \quad (-i)^{|p|} D^p = \partial_1^{p_1} \cdots \partial_n^{p_n}, \quad |p| = p_1 + \cdots + p_n,
\]

and for \( v \leq p \), \( (^p v) = \prod_{i=1}^n (v_i)! \), where \( v \leq p \) means \( v_i \leq p_i, \ i = 1, \ldots, n \). Points in \( \mathbb{R}^k \) are denoted by \( \tilde{x}, \tilde{y}, \ldots \), while points in \( \mathbb{R}^{n-k} \) are denoted by \( \tilde{x}, \tilde{y}, \ldots \). So

\[
x = (x_1, \ldots, x_n) = (\tilde{x}, \tilde{x}), \quad \text{where} \quad \tilde{x} = (x_1, \ldots, x_k), \quad \tilde{x} = (x_{k+1}, \ldots, x_n).
\]

For an open set \( \Omega \subseteq \mathbb{R}^n \), the symbol \( K \subset \subset \Omega \) means that \( K \) is a compact set contained in \( \Omega \). The support of a given function (or ultradistribution) \( f \) is denoted by \( \text{supp} \ f \).

We say that a function \( f \) is compactly supported if there exists a \( K \subset \subset \mathbb{R}^n \) such that \( \text{supp} \ f \subset K \). The Fourier transform of a function \( f \) is defined as \( \hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx, \ \xi \in \mathbb{R}^n \). For the dual pairing, we use \( \langle f, g \rangle \); for the \( L^2 \) inner product of \( f \) and \( g \) we use the notation \( (f, g) \).

1.2 Ultradistribution Spaces and Ultradifferential Operators

Following the approach of \cite{ref18}, we introduce test spaces defined by the use of Gevrey sequences \( M_\rho = p!^\alpha, \alpha > 1 \). All the results can be given for general sequences \( (M_\rho)_{\rho \in \mathbb{N}} \) satisfying appropriate conditions. We only consider spaces of Roumieu type \cite{ref18}. Moreover, the results of this paper also hold for the Beurling type spaces. The topology for them is the same as in the case of Schwartz space of rapidly decreasing functions. Because of that, we omit this part of analysis.

Let \( K \subset \subset \Omega \) and \( h > 0 \). We recall the definitions of some spaces of test functions \cite{ref18} (of the Roumieu type):

\[
E_h^\alpha (K) = \left\{ \varphi \in C^\infty (\Omega) : \sup_{t \in K, p \in \mathbb{N}_0^n} \frac{h^{|p|}}{p!^\alpha} |D^p \varphi (t)| < \infty \right\};
\]

\[
D_h^\alpha (K) = E_h^\alpha (K) \cap \left\{ \varphi \in C^\infty (\Omega) : \text{supp} \varphi \subset K \right\};
\]

\[
E^\alpha (K) = \lim_{h \to 0} E_h^\alpha (K) ; \quad E^\alpha (\Omega) = \lim_{K \subset \subset \Omega} E^\alpha (K) ;
\]

\[
D^\alpha (K) = \lim_{h \to 0} D_h^\alpha (K) ; \quad D^\alpha (\Omega) = \lim_{K \subset \subset \Omega} D^\alpha (K) ;
\]

\[
D_{L^\infty}^\alpha (\mathbb{R}^n) = \lim_{h \to 0} \{ \varphi \in C^\infty (\mathbb{R}^n) : \sup_{t \in \mathbb{R}^n, p \in \mathbb{N}_0^n} \frac{h^{|p|}}{p!^\alpha} |D^p \varphi (t)| < \infty \}.
\]
The spaces of ultradistributions $D'_{\alpha}(\Omega)$ are a strong dual of $D_{\alpha}(\Omega)$; its subspace $E'_{\alpha}(\Omega)$ consists of compactly supported ultradistributions. All these spaces are complete, bornological, Montel and Schwartz [18].

1.2.1 Gelfand–Shilov Type Spaces

Let $\alpha, \beta, \alpha > 0$. By $(S_a^\alpha_{\beta}(\mathbb{R}^n))$ is denoted the Banach space of all smooth functions $\varphi$ on $\mathbb{R}^n$ such that the norm

$$
\sigma_{\alpha, \beta}^a(\varphi) = \sup_{t \in \mathbb{R}^n, p, q \in \mathbb{N}^n_0} a^{p |q| + |p|} p! q!^{\alpha} |t^p \varphi(q)(t)|
$$

is finite. The space $S_{\alpha, \beta}^\alpha(\mathbb{R}^n)$ is defined as an inductive limit of the space $(S_a^\alpha_{\beta}(\mathbb{R}^n))$:

$$
S_{\alpha, \beta}^\alpha(\mathbb{R}^n) = \lim_{a \to 0} (S_a^\alpha_{\beta}(\mathbb{R}^n)).
$$

This is a DFS (dual of Fréchet Schwartz) nuclear space. Its strong dual is called Gelfand–Shilov space of ultradistributions. The space $S_{\alpha, \beta}^\alpha(\mathbb{R}^n)$ is nontrivial if and only if $\alpha + \beta \geq 1$. The Fourier transform is a topological isomorphism between $S_{\alpha, \beta}^\alpha(\mathbb{R}^n)$ and $S_{\beta, \alpha}^\beta(\mathbb{R}^n)$, which extends to a continuous linear transform from $S_{\alpha, \beta}^\alpha(\mathbb{R}^n)$ onto $S_{\beta, \alpha}^\beta(\mathbb{R}^n)$. If $\alpha = \beta$, the space $S_{\alpha, \beta}^\alpha(\mathbb{R}^n)$ is denoted by $S_{\alpha}^\alpha(\mathbb{R}^n)$.

Remark 1.1 We will often use equivalent families of norms in which in (1.1) $t^p \varphi(q)(t)$ is replaced by $(t^p \varphi)^{(q)}$ or $(t^q \varphi)^{(p)}$, because they give rise to an equivalent topology on $(S_a^\alpha_{\beta}(\mathbb{R}^n))$ (see [2,20,24]). Also we will use the fact that the sup norm in (1.1), substituted by any $L^p, p \geq 1$, norm, gives equivalent system of norms.

1.2.2 Ultradifferential Operators

A formal expression $P(D) = \sum_{p \in \mathbb{N}^n_0} a_p D^p (a_p \in \mathbb{R})$ corresponds to the ultrapolynomial $P(\xi) = \sum_{p \in \mathbb{N}^n_0} a_p \xi^p (\xi \in \mathbb{R}^n)$, [18]. It is called an ultradifferential operator of the Roumieu type $\alpha$ if for every $a > 0$, there exists a constant $C = C(a) > 0$ such that the coefficients $a_p$ satisfy the estimate

$$
|a_p| \leq \frac{C a^{|p|}}{p!^{\alpha}} , \quad \forall p \in \mathbb{N}^n_0,
$$

(1.2)

It is a $C^\infty$ function on $\mathbb{R}^n$. We will use the following representation theorem for an $f \in S_{\alpha, \beta}^\alpha(\mathbb{R}^n)$:

For any $f \in S_{\alpha, \beta}^\alpha(\mathbb{R}^n)$, there exist a $P_1(D)$-ultradifferential operator of the Roumieu type $\alpha$, an ultrapolynomial $P_2(x)$ of Roumieu type $\beta$ and an $F \in L^2(\mathbb{R}^n)$ so that

$$
f(x) = P_1(D)(P_2(x) F(x)).$$

(1.3)
We note that $\phi \mapsto P_1(D)\phi$ and $\phi \mapsto P_2(x)\phi$ are continuous mappings of $S^\alpha_\beta(\mathbb{R}^n)$ into itself.

We will consider elliptic operators of this type. For them, one has that the function $P(\xi)$ satisfies [18, Proposition 4.5]

$$\forall a > 0 \exists C > 0 \forall \xi \in \mathbb{R}^n \quad C^{-1}e^{a|\xi|^{1/\alpha}} \leq |P(\xi)| \leq Ce^{a|\xi|^{1/\alpha}}.$$  \hspace{1cm} (1.4)

We point out that $P(D)$ defines the continuous mappings on $S^\alpha_\beta(\mathbb{R}^n)$. Moreover,

$$P(D)\phi = \lim_{n \to \infty} \sum_{|p| < n} a_p D^p \phi \quad \text{in} \quad S^\alpha_\beta(\mathbb{R}^n) \quad \text{for every} \quad \phi' \in S^\alpha_\beta(\mathbb{R}^n).$$

We recall the following estimates given in Lemma 2.1 and (2.3) of [25]: For any $r \geq 1$, there exists $C > 0$ such that for all $\xi \in \mathbb{R}^n$, $j \in \mathbb{N}_0$

$$\left| D^j_{\xi} \frac{1}{P(\xi)} \right| \leq C \frac{j!}{r^{|j|} |P(\xi)|}.$$  \hspace{1cm} (1.5)

### 1.3 STFT and the Synthesis Operator on $L^2$

Let a window function $g \in L^2(\mathbb{R}^n) \setminus \{0\}$. Then, STFT is defined by

$$V_g f(y, \xi) = \int_{\mathbb{R}^n} f(t)g(t-y)e^{-2\pi i \xi \cdot t} dt, \quad y, \xi \in \mathbb{R}^n, \ f \in L^2(\mathbb{R}^n).$$

The synthesis operator $V^*_g$ is defined on $L^2(\mathbb{R}^{2n})$ by

$$V^*_g F(t) = \int_{\mathbb{R}^{2n}} F(y, \xi)g_{y,\xi}(t)dyd\xi, \quad t \in \mathbb{R}^n$$

where $g_{y,\xi}(t) = g(t-y)e^{2\pi i \xi \cdot t}$. Let $\phi \in L^2(\mathbb{R}^n)$ be a synthesis window for $g ((g, \phi) \neq 0)$. Then, for any $f \in L^2(\mathbb{R}^n)$

$$f(t) = \frac{1}{(g, \phi)} \int_{\mathbb{R}^{2n}} V_g f(y, \xi)\phi_{y,\xi}(t)dyd\xi, \hspace{1cm} (1.6)$$

where $\phi_{y,\xi}(t) = \phi(t-y)e^{2\pi i \xi \cdot t}$. It is well known that if $g \in S^\alpha_\beta(\mathbb{R}^n) \setminus \{0\}$ is a fixed window, then $V_g : S^\alpha_\beta(\mathbb{R}^n) \to S^\alpha_\beta(\mathbb{R}^{2n})$ is a continuous mapping. Moreover, for $f \in S^\alpha_\beta(\mathbb{R}^n)$, equation (1.6) holds pointwise (see [28]).

### 2 $k$-Directional STFT and the $k$-Directional Synthesis Operator

We define the $k$-DSTFT and the $k$-directional synthesis operator (DSO) for a fixed direction, over $S^\alpha_\beta(\mathbb{R}^n)$ and its dual.
Definition 2.1 Let \( u^k = (u_1, \ldots, u_k) \), where \( u_i, i = 1, \ldots, k \) are independent vectors of \( \mathbb{S}^{n-1} \). Let \( \tilde{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k \) and \( g \in \mathcal{S}_\phi^a(\mathbb{R}^k) \setminus \{0\} \). The k-DSTFT of \( f \in L^2(\mathbb{R}^n) \) is defined by
\[
DS_{g, u^k} f(\tilde{y}, \xi) := \int_{\mathbb{R}^n} f(t)g((u_1 \cdot t, \ldots, u_k \cdot t) - (y_1, \ldots, y_k))e^{-2\pi it \cdot \xi} dt \quad (2.1)
\]
and the k-DSO of \( F \in L^2(\mathbb{R}^k \times \mathbb{R}^n) \) is defined by
\[
DS_{g, u^k}^* F(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(\tilde{y}, \xi)g(u^k, \tilde{y}, \xi)(t)d\tilde{y}d\xi, \quad t \in \mathbb{R}^n, \quad (2.2)
\]
where \( g(u^k, \tilde{y}, \xi)(t) = g((u_1 \cdot t, \ldots, u_k \cdot t) - (y_1, \ldots, y_k))e^{2\pi i t \cdot \xi}, \quad t \in \mathbb{R}^n. \)

Let \( \varphi \in \mathcal{S}_\phi^a(\mathbb{R}^k) \) be the synthesis window for \( g \in \mathcal{S}_\phi^a(\mathbb{R}^k) \), which means \( (g, \varphi)_{L^2(\mathbb{R}^k)} \neq 0 \). We will show in Proposition 2.4 that for \( f \in \mathcal{S}_\phi^a(\mathbb{R}^n) \) the following reconstruction formula holds pointwise:
\[
f(t) = \frac{1}{(g, \varphi)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} DS_{g, u^k} f(\tilde{y}, \xi)\varphi_{u^k, y, \xi}(t)d\tilde{y}d\xi, \quad (2.3)
\]
where \( \varphi_{u^k, y, \xi}(t) = \varphi((u_1 \cdot t, \ldots, u_k \cdot t) - (y_1, \ldots, y_n))e^{2\pi i t \cdot \xi}, \quad t \in \mathbb{R}^n. \) Relation (2.3) takes the form
\[
(DS_{\varphi, u^k}^* \circ DS_{g, u^k})f = (g, \varphi)f.
\]

2.1 Coordinate Transformation

Let \( A_{k,n} = [u_{i,j}] \) be a \( k \times n \) matrix with rows \( u^i, i = 1, \ldots, k \) and \( I_{n-k,n-k} \) be the identity matrix. Let \( B \) be an \( n \times n \) matrix determined by \( A \) and \( I_{n-k,n-k} \) so that \( Bt = s \), where
\[
s_1 = u_{1,1}t_1 + \cdots + u_{1,n}t_n, \ldots, s_k = u_{k,1}t_1 + \cdots + u_{k,n}t_n, \quad s_{k+1} = t_{k+1},
\]
\[
\ldots, s_n = t_n.
\]
Clearly, this matrix is invertible (regular). Put \( C = B^{-1} \) and \( e^k = (e_1, \ldots, e_k) \), where \( e_1 = (1, 0, \ldots, 0), \ldots, e_k = (0, \ldots, 1) \) are unit vectors of the coordinate system of \( \mathbb{R}^k \). Then, with the change of variables \( t = Cs \), and \( \eta = C^T \xi \) (\( C^T \) is the transposed matrix for \( C \)), one obtains, for \( f \in L^2(\mathbb{R}^n) \), \( g \in \mathcal{S}_\phi^a(\mathbb{R}^k) \) that (2.1) is transformed into:
\[
DS_{g, e^k} f(\tilde{y}, \xi) = (DS_{g, e^k} h)(\tilde{y}, \eta) = \int_{\mathbb{R}^n} h(s)g(\tilde{s} - \tilde{y})e^{-2\pi i s \cdot \eta} ds, \quad (2.4)
\]
where \( h(s) = |C| f(Cs) \), \(|C|\) is the determinant of \( C \), and (2.2) is transformed, for \( F \in L^2(\mathbb{R}^k \times \mathbb{R}^n) \) and \( g \in \mathcal{S}_\beta^\alpha(\mathbb{R}^k) \), into:

\[
DS^{\alpha}_{g,e_k} F(s) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(\tilde{\eta}, \eta) g(\tilde{s} - \tilde{\eta}) e^{2\pi i s \cdot \eta} d\tilde{\eta} d\eta, \quad s \in \mathbb{R}^n. \tag{2.5}
\]

**Remark 2.2**

1. Let \( f \in \mathcal{S}^\alpha_{\beta}(\mathbb{R}^n) \). Then, \( h(s) = |C| f(Cs) \in \mathcal{S}^\alpha_{\beta}(\mathbb{R}^n) \).
2. If \( g(s_1, \ldots, s_k) = g_1(s_1) \cdots g_k(s_k) \), \( g_j \in \mathcal{S}^\alpha_{\beta}(\mathbb{R}) \), \( j = 1, 2, \ldots, k \), then

\[
DS^{\alpha}_{g,u^k} f(\tilde{\eta}, \xi) := \int_{\mathbb{R}^n} f(t) g_1(u_1 \cdot t - y_1) \cdots g_k(u_k \cdot t - y_k) e^{2\pi i t \cdot \xi} dt
\]

\[
= \int_{\mathbb{R}^n} h(s) g_1(s_1 - y_1) \cdots g_k(s_k - y_k) e^{-2\pi i s \cdot u} ds,
\]

and we call it, the partial short-time Fourier transform.

### 2.2 Continuity Properties

**Theorem 2.3**

Defining

\[
(DS^{\alpha}_{g,e_k} h)(\tilde{\eta}, \eta) = H(\tilde{\eta}, \eta) = \int_{\mathbb{R}^n} h(s) g(\tilde{s} - \tilde{\eta}) e^{-2\pi i s \cdot \eta} ds \tag{2.6}
\]

we obtain a continuous bilinear mapping

\[
\mathcal{S}^\alpha_{\beta}(\mathbb{R}^n) \times \mathcal{S}^\alpha_{\beta}(\mathbb{R}^k) \rightarrow \mathcal{S}^\alpha_{\beta}(\mathbb{R}^{k+n}),
\]

\[
(h, g) \mapsto H = DS^{\alpha}_{g,e_k} h.
\]

**Proof** Let \( v, p \in \mathbb{N}_0^n \), \( \tilde{\eta}, \tilde{\eta} \in \mathbb{N}_0^k \), \( \eta \in \mathbb{R}^n \), \( \tilde{\eta} \in \mathbb{R}^k \). Using (2.4), we have

\[
J = \eta^v \tilde{\eta}^\nu \sum_{\gamma} \partial^\nu_{\tilde{\eta}} \partial^\gamma_{\eta} DS^{\alpha}_{g,e_k} h(\tilde{\eta}, \eta)
\]

\[
= \eta^v \tilde{\eta}^\nu (\tilde{\eta} \cdot \eta) |p| \int_{\mathbb{R}^n} (\tilde{\eta} \cdot \eta) \sum_{\gamma} \partial^\nu_{\tilde{\eta}} \partial^\gamma_{\eta} h(s) g(\tilde{s} - \tilde{\eta}) e^{-2\pi i s \cdot \eta} ds
\]

\[
= \eta^v (\tilde{\eta} \cdot \eta) |p| \int_{\mathbb{R}^n} \sum_{\gamma} \partial^\nu_{\tilde{\eta}} \partial^\gamma_{\eta} h(s) \tilde{\eta}^\nu g(\tilde{s} - \tilde{\eta}) e^{-2\pi i s \cdot \eta} ds
\]

\[
= (-2\pi i)^{|p|-|v|} |\nu| \int_{\mathbb{R}^n} \sum_{\gamma} \partial^\nu_{\tilde{\eta}} \partial^\gamma_{\eta} h(s) \tilde{\eta}^\nu g(\tilde{s} - \tilde{\eta}) e^{-2\pi i s \cdot \eta} ds
\]

\[
= (-2\pi i)^{|p|-|v|} \sum_{\gamma} \partial^\nu_{\tilde{\eta}} \partial^\gamma_{\eta} h(s) \tilde{\eta}^\nu g(\tilde{s} - \tilde{\eta}) e^{-2\pi i s \cdot \eta} ds.
\]

Above, \( v - \tilde{\eta} = (v_1 - j_1, \ldots, v_k - j_k, \tilde{\eta}) \). In the sequel, we will use

\[
(v - \tilde{\eta})^{\alpha} \tilde{\eta}^{\beta} \leq v^{\alpha} \tilde{\eta}^{\beta}, \quad (\tilde{\eta} + \tilde{\eta})^{\alpha} \leq 2 \tilde{\eta}^{\alpha} \tilde{\eta}^{\alpha}.
\]

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We put \( J = J_1 + J_2 \) and \( r_p^{\nu, \tilde{\gamma}} = (-2\pi i)^{|p| - |v|} (1)^{|\tilde{\gamma}|} \). Then,

\[
J_1 = r_p^{\nu, \tilde{\gamma}} \sum_{j \leq \tilde{v}} \left( \frac{v}{j} \right) \int_{\mathbb{R}^n} \frac{\partial^{v-j}}{\partial s^{v-j}} (s^p h(s)) g^{(\gamma+j)}(\tilde{s} - \tilde{y})(\tilde{y} - \tilde{w}) e^{-2\pi is \cdot \eta} ds.
\]

\[
J_2 = r_p^{\nu, \tilde{\gamma}} \sum_{j \leq \tilde{v}} \left( \frac{v}{j} \right) \int_{\mathbb{R}^n} \frac{\partial^{v-j}}{\partial s^{v-j}} (s^p h(s)) g^{(\gamma+j)}(\tilde{s} - \tilde{y}) e^{-2\pi is \cdot \eta} ds.
\]

The use of positive constants \( c_1, c_2, C_1, C_2 \), below, will be clear from the context. With suitable \( c_1, C_1 \), we have

\[
\frac{c_1^{\nu + |p| + |\tilde{w}| + |\tilde{y}|}}{v^{\nu} p^{1/p} \tilde{y} |\tilde{w}|^{1/\tilde{w}}} |J_1|/|r_p^{\nu, \tilde{\gamma}}| \leq \frac{c_1^{\nu + |p| + |\tilde{w}| + |\tilde{y}|}}{v^{\nu} p^{1/p} \tilde{y} |\tilde{w}|^{1/\tilde{w}}} \sum_{j \leq \tilde{v}} \left( \frac{v}{j} \right) \int_{\mathbb{R}^n} \frac{\partial^{v-j}}{\partial s^{v-j}} (s^p h(s)) g^{(\gamma+j)}(\tilde{s} - \tilde{y})(\tilde{y} - \tilde{w}) e^{-2\pi is \cdot \eta} ds \leq C_1.
\]

Next, with suitable \( c_2, C_2 \),

\[
\frac{c_2^{\nu + |p| + |\tilde{w}| + |\tilde{y}|}}{v^{\nu} p^{1/p} \tilde{y} |\tilde{w}|^{1/\tilde{w}}} |J_2|/|r_p^{\nu, \tilde{\gamma}}| \leq \frac{c_2^{\nu + |p| + |\tilde{w}| + |\tilde{y}|}}{v^{\nu} p^{1/p} \tilde{y} |\tilde{w}|^{1/\tilde{w}}} \sum_{j \leq \tilde{v}} \left( \frac{v}{j} \right) \int_{\mathbb{R}^n} \frac{\partial^{v-j}}{\partial s^{v-j}} (s^p h(s)) g^{(\gamma+j)}(\tilde{s} - \tilde{y}) e^{-2\pi is \cdot \eta} ds \leq C_2.
\]

By the use of Young’s convolution inequality, we pass from the sup norm to the \( L^1 \) norm for the part of integral related to \( h \) (see the last part of Remark 1.1) and conclude

\[
\sup_{s \in \mathbb{R}^n, l, q \in \mathbb{N}_0^k, \tilde{w} \in \mathbb{N}_0^k} \frac{c_1^{\nu + |l| + |\tilde{w}|} |\tilde{s} \tilde{w}| (s^l h(s)(q)) |}{q^{1/\tilde{w}} p^{1/p} \tilde{y} |\tilde{w}|^{1/\tilde{w}}} < \infty, \text{ for some } a > 0.
\]

So, with \( \sum_{j \leq \tilde{v}} \left( \frac{v}{j} \right) \leq 2^v \), and new constants \( c \) and \( C \) we have

\[
|J| \leq C \sigma_c^{\alpha, \beta} (h) \sigma_c^{\alpha, \beta} (g).
\]
Proposition 2.4 Let \( f \in S^\alpha_\beta(\mathbb{R}^n) \) and \( g, \varphi \in S^\alpha_\beta(\mathbb{R}^k) \). Then, the reconstruction formula (2.3) holds pointwise.

Proof Indeed, by using the Parseval identity for given \( f_1, f_2 \in L^2(\mathbb{R}^n) \) and \( g, \varphi \in S^\alpha_\beta(\mathbb{R}^k) \), and after the change of variables as in the representation (2.4), that is \( h_i(\cdot) = |C| f_i(C \cdot), i = 1, 2 \), we have

\[
(DS_{g, u^k} f_1(\tilde{y}, \xi), DS_{\varphi, u^k} f_2(\tilde{y}, \xi))_{L^2(\mathbb{R}^k \times \mathbb{R}^n)} = (DS_{g, e^k} h_1(\tilde{y}, \eta), DS_{\varphi, e^k} h_2(\tilde{y}, \eta))_{L^2(\mathbb{R}^k \times \mathbb{R}^n)} = (h_1, h_2)_{L^2(\mathbb{R}^n)}(\tilde{y}, \varphi)_{L^2(\mathbb{R}^k)}.
\] 

(2.7)

We obtain the reconstruction formula (2.3) as a consequence of (2.7) as in [14, Theorem 3.2.1. and Corollary 3.2.3]. \( \square \)

Now we will consider the continuity properties of (2.5). We fix \( g \in S^\alpha_\beta(\mathbb{R}^k) \).

Theorem 2.5 Setting \( DS^*_{g, e^k} (H(\tilde{y}, \eta))(s) = h(s), s \in \mathbb{R}^n, \) given by (2.5), we obtain a continuous linear mapping

\[
S^\alpha_\beta(\mathbb{R}^k \times \mathbb{R}^n) \rightarrow S^\alpha_\beta(\mathbb{R}^n),
\]

\[
H \mapsto h = DS^*_{g, e^k} H.
\]

Proof We will estimate

\[
|s^v \partial_s^p h(s)| = \left| s^v \sum_{j \leq \beta} \left( \begin{array}{c} p \\ \tilde{j} \end{array} \right) (2\pi i)^p-j \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(\tilde{y}, \eta) \eta^{p-j} g^{(j)}(\tilde{s} - \tilde{y}) e^{2\pi i s \cdot \eta} d\tilde{y} d\eta \right|.
\]

\[
= \left| \sum_{j \leq \beta} \left( \begin{array}{c} p \\ \tilde{j} \end{array} \right) (2\pi i)^p-j-v \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \partial^v_\eta F(\tilde{y}, \eta) \eta^{p-j} g^{(j)}(\tilde{s} - \tilde{y}) e^{2\pi i s \cdot \eta} d\tilde{y} d\eta \right|.
\]

We continue with

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \sum_{v \geq r, p-j \geq r} \left( \begin{array}{c} v \\ r \end{array} \right) \partial^v_\eta F(\tilde{y}, \eta) \left( \begin{array}{c} p-j \\ r \end{array} \right) r! \eta^{r-j-r} g^{(j)}(\tilde{s} - \tilde{y}) e^{2\pi i s \cdot \eta} d\tilde{y} d\eta.
\]

This leads to

\[
|J| \leq \frac{c_{\alpha} (v-r+p-j-r)!}{(v-r)! \beta-r(p-j-r)!} \frac{||\partial^v_\eta F(\tilde{y}, \eta) \eta^{p-j-r}||_{L^1(\mathbb{R}^k \times \mathbb{R}^n)}}{c^j_\alpha ||g^{(j)}(\tilde{s} - \tilde{y})||_{L^\infty(\mathbb{R}^k)}}.
\]
With this, we finish the proof. □

**Remark 2.6** This proof shows that one can assume less restrictive conditions on \( g \) since we just differentiate \( g \). For example, we can assume the weaker condition \( g \in \mathcal{S}_0^\alpha(\mathbb{R}^k) \):

**Corollary 2.7** \( (DS^\alpha g, e^k) (H(\tilde{\gamma}, \eta))(s) = h(s), s \in \mathbb{R}^n \) defined as in Theorem 2.5, extends to a continuous bilinear mapping

\[
\mathcal{S}^\alpha_\beta(\mathbb{R}^k \times \mathbb{R}^n) \times \mathcal{S}_0^\alpha(\mathbb{R}^k) \to \mathcal{S}^\alpha_\beta(\mathbb{R}^n),
\]

\((H, g) \mapsto h = DS^\ast g, e^k H.\)

### 2.3 k-DSTFT and k-DSO on \( \mathcal{S}^\alpha_\beta \)

Let \( u^k = (u_1, \ldots, u_k) \), where \( u_i, i = 1, \ldots, k \) are independent vectors of \( \mathbb{R}^{n-1} \), and \( g \in \mathcal{S}^\alpha_\beta(\mathbb{R}^k) \). The continuity results allow us to define k-DSTFT of \( f \in \mathcal{S}^\alpha_\beta(\mathbb{R}^n) \) as an element \( DS_{g, u^k} f \in \mathcal{S}^\alpha_\beta(\mathbb{R}^k \times \mathbb{R}^n) \) whose action on test functions is given as a transposed mapping

\[
\langle DS_{g, u^k} f, \Phi \rangle = \langle f, DS_{g, u^k}^\ast \Phi \rangle, \quad \Phi \in \mathcal{S}_\beta^\alpha(\mathbb{R}^k \times \mathbb{R}^n). \quad (2.8)
\]

We use notation \( \mathbb{R}^k \times \mathbb{R}^n = \mathbb{R}^{k+n} \) just to indicate the domain of the above mapping. Since \( g \in \mathcal{S}^\alpha_\beta(\mathbb{R}^k) \), one can define the k-DSTFT of \( f \in \mathcal{S}^\alpha_\beta(\mathbb{R}^n) \) as

\[
DS_{g, u^k} f(\tilde{\gamma}, \xi) = \langle f(t), g((t \cdot u_1, \ldots, t \cdot u_k) - \tilde{\gamma})e^{-2\pi t \cdot \xi} \rangle, \quad \tilde{\gamma} \in \mathbb{R}^k, \; \xi \in \mathbb{R}^n. \quad (2.9)
\]

This is a direct method of the definition of an integral transform. We have

**Proposition 2.8** The two definitions (2.8) and (2.9) of the k-DSTFT of an \( f \in \mathcal{S}^\alpha_\beta(\mathbb{R}^n) \) coincide.

**Proof** One has to use the representation formula (1.3), then the continuity of \( P_1(-D) \), and \( P_2(x) \) over \( \mathcal{S}^\alpha_\beta(\mathbb{R}^n) \) and, finally, the Fubini theorem. □

Next, the k-DSO \( DS^\ast_{g, u^k} : \mathcal{S}^\alpha_\beta(\mathbb{R}^k \times \mathbb{R}^n) \to \mathcal{S}^\alpha_\beta(\mathbb{R}^n) \) can be defined as

\[
\langle DS^\ast_{g, u^k} F, \varphi \rangle = \langle F, DS_{g, u^k} \varphi \rangle, \quad F \in \mathcal{S}_\beta^\alpha(\mathbb{R}^k \times \mathbb{R}^n), \varphi \in \mathcal{S}_\beta^\alpha(\mathbb{R}^n).
\]

We repeat the arguments given above. Let \( F \in \mathcal{S}_\beta^\alpha(\mathbb{R}^k \times \mathbb{R}^n) \) be of the form

\[
F(\tilde{\gamma}, \xi) = P_1(D_{\tilde{\gamma}, \xi})(P_2(\tilde{\gamma}, \xi)F_0(\tilde{\gamma}, \xi)) \quad (\text{c.f.}(1.3)),
\]

where \( P_1(D_{\tilde{\gamma}, \xi}) \) and \( P_2(\tilde{\gamma}, \xi) \) are an ultradifferential operator over \( \mathbb{R}^{k+n} \) of Roumieu class \( \alpha \) and an ultradifferential polynomial of Roumieu class \( \beta \), respectively. Again, we define \( DS^\ast_{g, u^k} F \) by a direct method

\[
DS^\ast_{g, u^k} F(t) = \langle F, g((u_1 \cdot t, \ldots, u_k \cdot t) - \tilde{\gamma})e^{2\pi i \tilde{\gamma} \cdot t} \rangle, \; t \in \mathbb{R}^n.
\]
We have

**Proposition 2.9** The two definitions of the $k$-DSO $DS_{g,u^k}f$ of $f \in S_\beta^{\alpha}(\mathbb{R}^k \times \mathbb{R}^n)$ coincide.

We immediately obtain:

**Proposition 2.10** Let $g \in S_0^{\alpha}(\mathbb{R}^k)$. The $k$-directional short-time Fourier transform, $DS_{g,u^k} : S_\beta^{\alpha}(\mathbb{R}^n) \to S_\beta^{\alpha}(\mathbb{R}^k \times \mathbb{R}^n)$ and the synthesis operator $DS^*_{g,u^k} : S_\beta^{\alpha}(\mathbb{R}^k \times \mathbb{R}^n) \to S_\beta^{\alpha}(\mathbb{R}^n)$ are continuous linear maps.

The following theorem connects the $k$-DSTFTs with respect to different windows. In this theorem, we correct our result from [1], since $\gamma$ here is defined over $\mathbb{R}^n$.

**Theorem 2.11** Let $u^k = (u_1, \ldots, u_k)$, where $u_i$, $i = 1, \ldots, k$ are independent vectors of $\mathbb{S}^{n-1}$. Let $\varphi$, $g$, $\gamma_1$ belong to $S_0^{\alpha}(\mathbb{R}^k)$ where $\gamma_1$ is the synthesis window for $g$ and $\gamma_0 \in S^{\alpha}(\mathbb{R}^{n-k})$ so that

$$\int_{\mathbb{R}^{n-k}} \gamma_0(t_{n-k+1}, \ldots, t_n)dt_{n-k+1} \ldots dt_n \neq 0. \quad (2.10)$$

Put

$$\gamma(t_1, \ldots, t_n) = \gamma_1(t_1, \ldots, t_k)\gamma_0(t_{n-k+1}, \ldots, t_n). \quad (2.11)$$

Let $f \in S_\beta^{\alpha}(\mathbb{R}^n)$, then

$$DS_{\varphi,u^k}f(\tilde{x}, \eta) = (DS_{g,u^k}f(\tilde{s}, \zeta)) \ast (DS_{\varphi,u^k}\gamma(\tilde{s}, \zeta))(\tilde{x}, \eta), \quad \tilde{x}, \tilde{s} \in \mathbb{R}^k, \ \eta, \zeta \in \mathbb{R}^n.$$

**Proof** We follow the proof in [1]. By (2.4), it is enough to prove the assertion for $e^k$. Let $F \in S_\beta^{\alpha}(\mathbb{R}^k \times \mathbb{R}^n)$. By the continuity arguments, we can assume that $F \in L^2(\mathbb{R}^k \times \mathbb{R}^n)$. Then

$$DS_{\varphi,e^k}(DS^*_{\varphi,e^k}F)(\tilde{x}, \eta)$$

$$= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^k} F(\tilde{y}, \xi)\gamma(t - y)e^{2\pi i \xi \cdot t}dyd\xi \right) \psi(t - \tilde{x})e^{-2\pi i t \cdot \eta} dt \right)$$

$$= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} (\int_{\mathbb{R}^k} \gamma(t)\psi(t - \tilde{x})e^{2\pi i t \cdot \eta} dt) F(\tilde{y}, \xi) d\tilde{y} d\xi \right)$$

$$= \int_{\mathbb{R}^n} (\int_{\mathbb{R}^k} F(\tilde{y}, \xi)DS_{\varphi,e^k}\gamma(\tilde{x} - \tilde{y}, \eta - \xi) d\tilde{y} d\xi).$$

Now, we put $F = DS_{g,e^k}f$ and obtain

$$DS_{\varphi,e^k}f(\tilde{x}, \eta) = (DS_{g,e^k}f(\tilde{s}, \zeta)) \ast (DS_{\varphi,e^k}\gamma(\tilde{s}, \zeta))(\tilde{x}, \eta). \quad (2.12)$$

\[ \square \]
3 Directional Wave Fronts

3.1 Basic Properties

In order to detect singularities determined by the hyperplanes orthogonal to the vectors \( u_1, \ldots, u_k \) using the \( k \)-DSTFT, we introduce \( k \)-directional regular sets and wave front sets for GS ultradistributions. Theorem 2.11 guarantees that the wave front set will not depend on the used window. Again, we simplify our exposition by the use of (2.4) and transfer the STFT in \( u^k \) direction to STFT in \( e^k \) direction.

Let \( k = 1 \) and \( y_0 = y_{0,1} \in \mathbb{R} \). Set \( \Pi_{e_1, y_0, \varepsilon} = \{ t \in \mathbb{R}^n : |t_1 - y_0| < \varepsilon \} \). It is a region of \( \mathbb{R}^n \) between two hyperplanes orthogonal to \( e_1 \),

\[
\Pi_{e_1, y_0, \varepsilon} = \bigcup_{y \in (y_0 - \varepsilon, y_0 + \varepsilon)} P_{e_1, y}, \quad (y_0 = (y_0, 0, \ldots, 0), \ y = (y, 0, \ldots, 0)),
\]

and \( P_{e_1, y} \) denotes the hyperplane orthogonal to \( e_1 \) passing through \( y \). With the notation of Section 2, we define

\[
\Pi_{e^k, \tilde{y}, \varepsilon} = \Pi_{e_1, y_1, \varepsilon} \cap \cdots \cap \Pi_{e_k, y_k, \varepsilon}, \quad \Pi_{e^k, \tilde{y}} = \Pi_{e_1, y_1} \cap \cdots \cap \Pi_{e_k, y_k}.
\]

The first set is a parallelepiped in \( \mathbb{R}^k \) so that in \( \mathbb{R}^n \) it is determined by \( 2k \) finite edges while the other edges are infinite. The set \( \Pi_{e^k, \tilde{y}} \) equals \( \mathbb{R}^{n-k} \) translates by vectors \( y_1, \ldots, y_k \). We will call it \((n-k)\)-dimensional element of \( \mathbb{R}^n \) and denote it as \( P_{e^k, \tilde{y}} \in \mathbb{R}^k \). If \( k = n \), then this is just the point \( y = (y_1, \ldots, y_n) \).

In the sequel, \( \alpha > 1 \). Also, with \( L_r(t) \) we denote open ball with center in \( t \in \mathbb{R} \) and radius \( r \).

**Definition 3.1** Let \( f \in S'_{\mathcal{O}}(\mathbb{R}^n) \). It is said that \( f \) is \( k \)-directionally microlocally regular at \((P_{e^k, \tilde{y}_0}, \tilde{z}_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \), that is, at every point of the form \((\tilde{y}_0, \cdot, \tilde{z}_0) \) denotes an arbitrary point of \( \mathbb{R}^{n-k} \) if there exists \( g \in \mathcal{D}'(\mathbb{R}^k), g(\tilde{0}) \neq 0 \), a product of open balls \( L_r(\tilde{y}_0) = L_r(y_0, 1) \times \cdots \times L_r(y_0, k) \subset \mathbb{R}^k \), a cone \( \Gamma_{\tilde{z}_0}, N \in \mathbb{N} \) and \( C_N > 0 \) such that

\[
\sup_{\tilde{y} \in L_r(\tilde{y}_0), \xi \in \Gamma_{\tilde{z}_0}} |DS_{G^k, e^k} f(\tilde{y}, \xi)| = \sup_{\tilde{y} \in L_r(\tilde{y}_0), \xi \in \Gamma_{\tilde{z}_0}} |\mathcal{F}(f(t) g(t - \tilde{y}))(\xi)| \leq C_N e^{-N|\xi|^{1/\alpha}}.
\]

(3.1)

As standard, the \( k \)-directional wave front is defined as the complement in \( \mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\}) \) of all \( k \)-directionally microlocally regular points \((P_{e^k, \tilde{y}_0}, \tilde{z}_0) \), and is denoted as \( WF_{e^k}(f) \).

**Remark 3.2** (a) Note that for \( k = n \) our definition is the classical Hörmander’s definition of regularity, [16, Section 8].

(b) If \( f \) is \( k \)-directionally microlocally regular at \((P_{e^k, \tilde{y}_0}, \tilde{z}_0) \), then there exist an open ball \( L_r(\tilde{y}_0) \) and an open cone \( \Gamma' \subset \Gamma_{\tilde{z}_0} \) so that \( f \) is \( k \)-directionally microlocally regular at \((P_{e^k, \tilde{z}_0}, \theta_0) \) for any \( \tilde{z}_0 \in L_r(\tilde{y}_0) \) and \( \theta_0 \in \Gamma' \). This implies that the
union of all $k$-directionally microlocally regular points $(P_{e^k, \tilde{z}_0}, \theta_0)$, $(\tilde{z}_0, \cdot), \theta_0) \in (L_r(\tilde{y}_0) \times \mathbb{R}^{n-k}) \times \Gamma$ is an open subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

(c) Denote by $P_{r}k$ the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{k}$. Then, the $k$-directionally microlocally regular point $(P_{e^k, \tilde{z}_0}, \xi_0)$, considered in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ with respect to the first $k$ variables, equals $(P_{r}^{-1}k \times I_\xi)(P_{e^k, \tilde{z}_0}, \xi_0)$ ($I_\xi$ is the identity matrix on $\mathbb{R}^n$).

### 3.2 Main Results

In this section, we give the main results concerning our wave front set $WF_{e^k}(f)$.

**Proposition 3.3** The set $WF_{e^k}(f)$ is closed in $\mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$ (and $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$).

Let $B_r(\tilde{0})$ denote the closed ball in $\mathbb{R}^k$ with center at zero and radius $r > 0$. The following theorem relates sets of $k$-directionally microlocally regular points for two $k$-DSTFT of GS ultradistributions.

**Theorem 3.4** If (3.1) holds for some $g \in \mathcal{D}^\omega(\mathbb{R}^k), g(\tilde{0}) \neq 0$, then it holds for every $h \in \mathcal{D}^\omega(\mathbb{R}^k), h(\tilde{0}) \neq 0$ supported by a ball $B_\rho(\tilde{0})$, where $\rho \leq \rho_0$ and $\rho_0$ depends on $r$ in (3.1).

**Proof** We follow the idea of our proof in [1] with the improvement which we already explained related to $\gamma$ of the form (2.11) in Theorem 2.11. Compact supports of $g$, $\gamma$ and $h$ simplify the integration. So we assume that $\varphi$, $g$, $\gamma_1$, belong to $S_0^\alpha(\mathbb{R}^k)$ where $\gamma_1$ is the synthesis window for $g$ and $\gamma_0 \in S^\alpha(\mathbb{R}^{n-k})$ satisfies (2.10). Moreover, by the structural theorem [2, Theorem 3.2.2], we know that $f = P_0(D)F$, where $F$ is a continuous function which satisfies

\[ \forall a > 0 \exists C_a > 0 \quad \forall \xi \in \mathbb{R}^n \quad |F(\xi)| \leq C_a e^{a|\xi|^{1/\alpha}} \]  

(3.2)

and $P_0(D)$ satisfies (1.2) and (1.4). So, we can use the technique of oscillatory integral, transfer the differentiation from $f$ on other factors in integral expressions and, from the beginning, assume that $f$ is a continuous function which satisfies (1.4).

Now, assume that (3.1) holds. We repeat from [1] the constructions of balls. So, $\gamma$ is chosen so that $\text{supp} \gamma \subset B_{\rho_1}(0)$ and $\rho_1 < r - r_0$. Moreover, $\text{supp} h \subset B_{\rho}(\tilde{0})$. Our aim is to find $\rho_0$ such that (3.1) holds for $DS_{h, e^k, f}(\tilde{x}, \eta)$, with $\tilde{x} \in B_{r_0}(\tilde{y}_0), \eta \in \Gamma_1 \subset \Gamma_0$, for $\rho \leq \rho_0$ ($\Gamma_1 \subset \Gamma_0$ implies that $\Gamma_1 \cap S^{n-1}$ is a compact subset of $\Gamma_0 \cap S^{n-1}$).

Note

\[ |\tilde{p}| \leq \rho_1, \quad |\tilde{x} - \tilde{y}_0| \leq r_0 \text{ and} \]

\[ |\tilde{p} - ((\tilde{x} - \tilde{y}_0) - (\tilde{y} - \tilde{y}_0))| \leq \rho \]

\[ \Rightarrow |\tilde{y} - \tilde{y}_0| \leq \rho + \rho_1 + r_0. \]  

(3.3)

We choose $\rho_0$ such that $\rho_0 + \rho_1 < r - r_0$ and

\[ \rho + \rho_1 + r_0 < r \quad \text{holds for} \quad \rho \leq \rho_0. \]  

(3.4)
Let \( \Gamma_1 \subset \subset \Gamma_{\xi_0} \). Then, there exists \( c \in (0, 1) \) such that
\[
\eta \in \Gamma_1, \ |\eta| > 1 \text{ and } |\eta - \xi| \leq c|\eta| \Rightarrow \xi \in \Gamma_{\xi_0}; \ |\eta - \xi| \leq c|\eta| \Rightarrow |\eta| \leq (1-c)^{-1}|\xi|. 
\] (3.5)

Let \( \tilde{x} \in B_{r_0}(\tilde{y}_0), \eta \in \Gamma_1 \). Then,
\[
|DS_{h,e^k} f(\tilde{x}, \eta)| = \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} DS_{g,e^k} f(\tilde{y}, \eta - \xi) DS_{h,e^k} \gamma(\tilde{x} - \tilde{y}, \xi) d\xi d\tilde{y}. 
\] (3.6)

For the later use, we put
\[
J = \int_{\mathbb{R}^n} DS_{h,e^k} \gamma(\tilde{x} - \tilde{y}, \xi) d\xi.
\]
This integral converges because
\[
J = \int_{\mathbb{R}^n} \int_{B_{r_1}(0)} \gamma(p)h(\tilde{p} - (\tilde{x} - \tilde{y})) P(D_p) (e^{-2\pi i p \cdot \xi}) dP d\xi
\]
\[
= \int_{\mathbb{R}^n} \int_{B_{r_1}(0)} P(D_p) \left( \gamma(p)h(\tilde{p} - (\tilde{x} - \tilde{y})) \right) e^{-2\pi i p \cdot \xi} dP d\xi.
\]

Rewrite
\[
|DS_{h,e^k} f(\tilde{x}, \eta)| = \int_{\mathbb{R}^k} \left( \int_{|\eta - \xi| \leq c|\eta|} \right. + \left. \int_{|\eta - \xi| \geq c|\eta|} \right) (\ldots) d\xi \quad d\tilde{y} \leq I_1 + I_2.
\]

Then,
\[
I_1 \leq \int_{\mathbb{R}^k} \left( \sup_{|\eta - \xi| \leq c|\eta|} |DS_{g,e^k} f(\tilde{y}, \eta - \xi)| \int_{|\eta - \xi| \leq c|\eta|} |DS_{h,e^k} \gamma(\tilde{x} - \tilde{y}, \xi)| d\xi \right) d\tilde{y}.
\]

Using (3.3), (3.4) and (3.5), we obtain
\[
\sup_{\tilde{x} \in B_{r_0}(\tilde{y}_0), \eta \in \Gamma_1} e^{N|\eta|^{1/\alpha}} I_1 \leq \int_{B_{r_0}(\tilde{y}_0)} \left( \sup_{\xi \in \Gamma_{\xi_0}} |DS_{g,e^k} f(\tilde{y}, \xi)| e^{N(1-c)^{-1}|\xi|^{1/\alpha}} \right.
\]
\[
\times \left. \int_{|\xi| \geq (1-c)|\eta|} |DS_{h,e^k} \gamma(\tilde{x} - \tilde{y}, \xi)| d\xi \right) d\tilde{y}. 
\] (3.7)

Now by the assumption and the finiteness of \( J \), we obtain that \( I_1 \) satisfies the necessary estimate of (3.1).

Now we consider \( I_2 \).
\[
I_2 \leq \int_{\mathbb{R}^k} \left( \right. \int_{|\eta - \xi| \geq c|\eta|} DS_{g,e^k} f(\tilde{y}, \eta - \xi) DS_{h,e^k} \gamma(\tilde{x} - \tilde{y}, \xi) d\xi \left. \right) d\tilde{y}.
\]
Let $K = \{ \xi : |\eta - \xi| \geq c|\eta| \}$. Denote by $\kappa_d^0$, $0 < d < 1$, the characteristic function of $K_d = \bigcup_{\xi \in K} L_d(\xi)$, where $L_d(\xi)$ is an open ball. This means that $K_d$ is an open $d$-neighborhood of $K$. Then, set

$$
\kappa_d = \kappa_d^0 * \varphi_d,
$$

where $\varphi_d = \frac{1}{d^n} \varphi(\cdot/d)$, $\varphi \in D'(\mathbb{R}^n)$ is non-negative, supported by the ball $B_1(0)$ and equals 1 on $B_{1/2}(0)$. This construction implies that $\kappa_d$ equals one on $K$, and is supported in $K_{2d}$. Moreover, all the derivatives of $\kappa_d$ are uniformly bounded, i.e., $\kappa_d \in D'_{\infty}(\mathbb{R}^n)$. We note that

$$
\int_K \ldots d\xi \leq \left| \int_{K_{2d}} \kappa_d(\xi) \ldots d\xi \right| + \left| \int_{K_{2d} \cap \{ |\eta - \xi| \leq c|\eta| \}} \kappa_d(\xi) \ldots d\xi \right|
$$

Then,

$$
I_2 \leq \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^n} \kappa_d(\xi) D_{g,e^k} f(\tilde{\eta}, \eta - \xi) D_{h,e^k} \gamma(\tilde{\xi} - \tilde{\eta}, \xi) d\tilde{\eta} \right| d\tilde{\eta}
$$

$$
+ \int_{\mathbb{R}^k} \left| \int_{K_{2d} \cap \{ |\eta - \xi| \leq c|\eta| \}} \kappa_d(\xi) D_{g,e^k} f(\tilde{\eta}, \eta - \xi) D_{h,e^k} \gamma(\tilde{\xi} - \tilde{\eta}, \xi) d\tilde{\eta} \right| d\tilde{\eta}
$$

$$
= I_{2,1} + I_{2,2}.
$$

We first estimate $I_{2,1}$.

$$
I_{2,1} \leq \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^n} \kappa_d(\eta - \xi) \int_{\mathbb{R}_p^n} f(t) g(\tilde{\eta} - \tilde{\xi}) e^{-2\pi it \cdot \tilde{\xi}} dt \right|
$$

$$
\int_{\mathbb{R}_p^n} \gamma(\tilde{p}) h(\tilde{\xi} - (\tilde{\xi} - \tilde{\eta})) e^{-2\pi i \tilde{p} \cdot (\eta - \xi)} d\tilde{p} d\tilde{\xi} d\tilde{\eta}
$$

Let $P(D)$ satisfy (1.4). We continue with $I_{2,1}$:

$$
I_{2,1} \leq \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}_p^n} P(D_p) \left( \gamma(\tilde{p}) h(\tilde{\xi} - (\tilde{\xi} - \tilde{\eta})) \right) e^{-2\pi i \tilde{p} \cdot (\eta - \xi)} d\tilde{p} \frac{\kappa_d(\eta - \xi)}{P(-2\pi (\eta - \xi))} \right|
$$

$$
\int_{\mathbb{R}_p^n} \frac{f(t) g(\tilde{\eta} - \tilde{\xi})}{P(2\pi t)} P(D_{\tilde{\xi}}) e^{-2\pi it \cdot \tilde{\xi}} dt d\tilde{\xi} d\tilde{\eta}
$$

$$
\leq \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}_p^n} P(D_p) \left( \gamma(\tilde{p}) h(\tilde{\xi} - (\tilde{\xi} - \tilde{\eta})) \right) \frac{f(t) g(\tilde{\eta} - \tilde{\xi})}{P(2\pi t)} e^{-2\pi it \cdot \tilde{\xi}} P(D_{\tilde{\xi}}) \left( \frac{\kappa_d(\eta - \xi)}{P(-2\pi (\eta - \xi))} e^{-2\pi p \cdot (\eta - \xi)} \right) d\tilde{p} d\tilde{t} d\tilde{\xi} d\tilde{\eta}
$$

Next, by (3.5), we have

$$
\sup_{\tilde{\eta} \in B_{1/2}(\tilde{\eta}_0)} e^{N|\eta|^{1/a}} I_{2,1}
$$
In order to show that this expression is finite, we have to use $P(D)$ in the form $\sum a_j D_j$, where $a_j$ satisfy (1.2), (1.4), the fact that $\kappa_d \in D'_{\mathbb{R}^\infty} (\mathbb{R}^n)$ and (1.5) as well as the equivalence of various norms discussed in Remark 1.1. With this we have that the right hand side of (3.8) is finite.

Concerning $I_{2,2}$, we note that it can be estimated in the same way as $I_1$ since integration goes over a subset of $\{\xi : |\eta - \xi| \leq c|\eta|\}$. This gives
\[
\sup_{\tilde{x} \in B_{r/2}(\tilde{y}), \eta \in \Gamma_1} e^{N|\eta|^{1/a}} I_{2,2} < \infty,
\]
which completes the proof of the claim. $\square$

The next corollary is a modification of the similar one concerning distributions (see [1]).

**Corollary 3.5** Let $g \in D^a(\mathbb{R}^k)$ with supp $g \subset B_a(\tilde{0})$, have synthesis window $\gamma_1$ with supp $\gamma_1 \subset B_{\rho_1}(\tilde{0})$ and $\rho_1 < a - \rho_0$. Then,
\[
\sup_{\tilde{y} \in L_{2r}(\tilde{y}_0), \tilde{x} \in \Gamma_{\tilde{y}_0}} |DS_{g, e^k} f (\tilde{y}, \xi)| \leq C_N e^{-N|\xi|^{1/a}}. \tag{3.9}
\]

Moreover, assume that $a < r$. Then, for any $h \in D^a(\mathbb{R}^k)$ with support in $B_\rho(\tilde{0})$, $\rho < a$, there exists $r_0$ and $\Gamma_1 \subset \subset \Gamma_{\tilde{y}_0}$ such that (3.9) holds for $DS_{h, e^k} f (\tilde{x}, \eta)$ with the supremum taken over $\tilde{x} \in B_{r_0}(\tilde{y}_0)$ and $\eta \in \Gamma_1$.

**Proof** Similarly as in (3.3),
\[
|\tilde{y} - \tilde{y}_0| \leq \rho + \rho_1 + r_0 < \rho + a - r_0 + r_0 = a + \rho < 2r.
\]
This implies $|\tilde{y} - \tilde{y}_0| < 2r$, so that the supremum in the estimate of $I_1$ holds. The proof now can be performed in the same way as in Theorem 3.4. $\square$

With the standard proof, we have

**Corollary 3.6** If $(P_{e^k, \tilde{y}, \xi})$ is a $k$-directionally microlocally regular point of $f \in {\mathcal S}^a(\mathbb{R}^n)$ for every $\xi \in \mathbb{R}^n \setminus \{0\}$, then $f \in \mathcal{E}^a(\mathbb{R}^n)$.

**Proof** For every $\xi \in \mathbb{S}^{n-1}$, there exist $g_\xi \in D^a(\mathbb{R}^k)$, a product of open balls $L_{r_\xi}(\tilde{y}_0)$ and $\Gamma_{\xi}$ such that (3.1) holds with $C_{N_\xi}$. Since $\mathbb{S}^{n-1}$ is compact, we choose a finite set of points $\{\xi_1, \ldots, \xi_s\}$ so that union of open cones $\Gamma_{\xi_i}, i = 1, \ldots, s$ equals $\mathbb{R}^n \setminus \{0\}$. Let $r = \min\{r_{\xi_1}, \ldots, r_{\xi_s}\}$, $C_N = \max\{C_{N_{\xi_1}}, \ldots, C_{N_{\xi_s}}\}$ and $g = \prod_{i=1}^s g_{\xi_i}$. Then, by Theorem 3.4 it follows that (3.1) holds with $\xi \in \mathbb{R}^n$ instead of $\Gamma_{\xi_0}$. This implies that $g f \in \mathcal{E}^a(\mathbb{R}^n)$ and so $f \in \mathcal{E}^a(\mathbb{R}^n)$. $\square$

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3.3 Relations with the Partial Wave Front

Recall again that the partial wave front of an $f \in \mathcal{D}'(\mathbb{R}^n)$, which we give below, was not considered in the literature. So, it is clear that this was not done for ultradistribution spaces in particular, for GS spaces of ultradistributions. We will prove in this section that this notion is equivalent to the directional wave front set. In order to show that, we will consider the set of $k$-microlocally regular points distinguishing them from the $k$-directionally microlocally regular points and in the theorem which is to follow prove that such two sets coincide. Let $f \in S^{\alpha}(\mathbb{R}^n)$. Then, the point $((\tilde{y}_0, y_{0,k+1}, \ldots, y_{0,n}), \xi_0) \in (\mathbb{R}^k \times \mathbb{R}^{n-k}) \times (\mathbb{R}^n \setminus \{0\})$ is $k$-microlocally regular for $f$ if there exists $\chi \in \mathcal{D}'(\mathbb{R}^k)$ so that $\chi(\tilde{y}_0) \neq 0$ and a cone $\Gamma_{\xi_0}$ around $\xi_0$ so that there exist $N \in \mathbb{N}$ and $C_{N,\chi} > 0$ such that

$$|\mathcal{F}(\chi(\tilde{y})f(y))(\xi)| \leq C_{N,\chi}e^{-N|\xi|^{1/\alpha}}, \quad \xi \in \Gamma_{\xi_0}, \; y = (\tilde{y}, y_{k+1}, \ldots, y_n) \in \mathbb{R}^k \times \mathbb{R}^{n-k}.$$  

(3.10)

Since, in this definition, $\chi$ does not depend on $(y_{k+1}, \ldots, y_n)$, we will write in the sequel that $f$ is $k$–microlocally regular at $((\tilde{y}_0, \cdot), \xi_0)$.

**Theorem 3.7** Let $f \in S^{\alpha}(\mathbb{R}^n)$ and $((\tilde{y}_0, \cdot), \xi_0) \in (\mathbb{R}^k \times \mathbb{R}^{n-k}) \times (\mathbb{R}^n \setminus \{0\})$. The following conditions are equivalent.

(i) $((\tilde{y}_0, \cdot), \xi_0) \notin WF_{\alpha}(f)$.

(ii) There exist a compact neighborhood $\tilde{K}$ of $\tilde{y}_0$ and a cone neighborhood $\Gamma$ of $\xi_0$ such that there exist $N \in \mathbb{N}$ so that for every $\chi \in \mathcal{D}'(\tilde{K})$ there exists $C_{N,\chi} > 0$ such that (3.10) is valid.

(iii) There exist a compact neighborhood $\tilde{K}$ of $\tilde{y}_0$ and a cone neighborhood $\Gamma$ of $\xi_0$ such that there exist $N \in \mathbb{N}$, $h > 0$ and $C_{N,h} > 0$ such that

$$|DS_{\chi, e^k}f(\tilde{y}, \xi)| \leq C_{N,h} \sup_{p \in \mathbb{N}_0^k} \frac{h^{|p|}}{p!} \| D^p \chi \|_{L^\infty(\mathbb{R}^k)} e^{-N|\xi|^{1/\alpha}},$$

$$\forall \tilde{y} \in \tilde{K}, \; \forall \xi \in \Gamma, \; \forall \chi \in \mathcal{D}'(\tilde{K} - \{y_0\}),$$

where $\tilde{K} - \{y_0\} = \{\tilde{y} \in \mathbb{R}^k | \tilde{y} + \tilde{y}_0 \in \tilde{K}\}$.

(iv) There exist a compact neighborhood $\tilde{K}$ of $\tilde{y}_0$, a cone neighborhood $\Gamma$ of $\xi_0$ and $\chi \in \mathcal{D}'(\mathbb{R}^k)$, with $\chi(\tilde{0}) \neq 0$ such that there exist $N \in \mathbb{N}$ and $C_{N,\chi} > 0$ such that

$$|DS_{\chi, e^k}f(\tilde{y}, \xi)| \leq C_{N,\chi}e^{-N|\xi|^{1/\alpha}}, \; \forall \tilde{y} \in \tilde{K}, \; \forall \xi \in \Gamma.$$

**Proof** The proof follows the arguments similar to those in [22].

(i) $\Rightarrow$ (ii) The fact that $((\tilde{y}_0, \cdot), \xi_0) \notin WF_{\alpha}(f)$ implies the existence of $\chi \in \mathcal{D}'(\mathbb{R}^k)$ (\chi(\cdot) = g(\cdot - \tilde{y}_0)) with $\chi(\tilde{0}) \neq 0$ and a cone neighborhood $\Gamma_{\xi_0}$ of $\xi_0$ for which (3.10) is valid for $\xi \in \Gamma_{\xi_0}$. There exists a compact neighborhood $\tilde{K}$ of $\tilde{y}_0$ where $\chi$ never vanishes. Fix a cone neighborhood $\Gamma$ of $\xi_0$ such that $\tilde{K} \subseteq \Gamma_{\xi_0} \cup \{0\}$. Following the proof of Lemma 8.1.1 in [16], one can show that there exist $N \in \mathbb{N}$ and
\(\psi \in \mathcal{D}^\alpha(\tilde{K})\) such that \(|\mathcal{F}(\psi \chi f)(\xi)| \leq C_N |\psi \chi e^{-N|\xi|^1/\alpha}|, \forall \xi \in \Gamma\). Then, (ii) follows since \(\psi f = (\psi / \chi) \chi f\) where \(\psi / \chi \in \mathcal{D}^\alpha(\tilde{K})\).

(i) \(\Rightarrow\) (iii) By (ii), (3.10) implies that there exist \(N \in \mathbb{N}\), such that the set \(H_N = \{e^{-N|\xi|^1/\alpha} e^{-i\xi \cdot \cdot t} f |\xi \in \Gamma\}\) is weakly bounded in \(\mathcal{D}^\alpha(\tilde{B})\). So it is equicontinuous, as \(\mathcal{D}^\alpha(\tilde{B})\) is barrelled. Let \(\tilde{K} = \tilde{B}_{\gamma_0}(r/2)\). For each \(\chi \in \mathcal{D}^\alpha(\tilde{K} - \{\gamma_0\})\) and \(\tilde{y} \in \tilde{K}\), the function

\[ \tilde{t} \mapsto \chi(\tilde{t} - \tilde{y}) \]

is in \(\mathcal{D}^\beta(\tilde{B})\) and the equicontinuity of \(H_N\) implies the existence of \(C_N > 0\) and \(h > 0\) such that

\[ |\langle e^{-2\pi i t \cdot \xi} f(t), \chi(\tilde{t} - \tilde{y}) \rangle| \leq C_N e^{-N|\xi|^1/\alpha} \sup_{p \in \mathbb{N}_0} \frac{h^{|p|}}{p!\alpha} |D^p \chi(\tilde{t} - \tilde{y})| \]

\[ = C_N \sup_{p \in \mathbb{N}_0} \frac{h^{|p|}}{p!\alpha} \|D^p \chi\|_{L^\infty(\mathbb{R}^d)} e^{-N|\xi|^1/\alpha}, \forall \xi \in \Gamma, \forall \tilde{y} \in \tilde{K}, \]

which implies the validity of (iii).

(iii) \(\Rightarrow\) (iv) is simple and skipped.

Using the estimate in (iv) with \(\tilde{y} = \tilde{y}_0\), (iv) \(\Rightarrow\) (i) simply follows. The proof is complete. \(\Box\)

**Remark 3.8** (3.1) \(\Rightarrow\) (3.10) is clear since the translation of \(g\) in (3.1) given by \(\kappa(\cdot) = g(\cdot - \tilde{y}_0)\) realizes this implication. We have proved, as a part of the previous theorem ((ii) \(\Rightarrow\) (i)), that the opposite implication also holds. This means the equivalence of (3.1) and (3.10), that is, wave fronts of both definitions coincides as sets.

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