COMBINATORICS OF THE SPRINGER CORRESPONDENCE FOR CLASSICAL LIE ALGEBRAS AND THEIR DUALS IN CHARACTERISTIC 2

TING XUE

Abstract. We give a combinatorial description of the Springer correspondence for classical Lie algebras of type $B, C$ or $D$ and their duals in characteristic 2. The combinatorics used here is of the same kind as those appearing in the description of (generalized) Springer correspondence for unipotent case of classical groups by Lusztig in odd characteristic and by Lusztig and Spaltenstein in characteristic 2.

1. Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed field of characteristic $p$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. When $p$ is large enough, Springer [13] constructs a correspondence which associates to an irreducible character of the Weyl group of $G$ a unique pair $(x, \phi)$ with $x \in \mathfrak{g}$ nilpotent and $\phi$ an irreducible character of the component group $A_G(x) = Z_G(x)/Z_G^0(x)$. For arbitrary $p$, Lusztig [5] constructs the generalized Springer correspondence which is related to unipotent conjugacy classes in $G$. Assume $p = 2$ and $G$ is of type $B, C$ or $D$, a Springer correspondence for $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) is constructed in [14] (resp. [15]) using a similar construction as in [5, 7].

Assume $G$ is classical. When $p$ is large, Shoji [10] describes an algorithm to compute the Springer correspondence which does not provide a close formula. A combinatorial description of the generalized correspondence for $G$ is given by Lusztig [5] for $p \neq 2$ and by Lusztig, Spaltenstein [9] for $p = 2$. Spaltenstein [11] describes a part of the Springer correspondence for $\mathfrak{g}$ under the assumption that the theory of Springer representation is valid for $\mathfrak{g}$ when $p = 2$. We describe the Springer correspondence for $\mathfrak{g}$ and $\mathfrak{g}^*$ using similar combinatorics that appears in [5, 9]. It is very nice that this combinatorics gives a unified description for (generalized) Springer correspondences of classical groups in all cases, namely, in $G$, $\mathfrak{g}$ and $\mathfrak{g}^*$ in all characteristics. Moreover, it gives rise to close formulas for computing the correspondences.

2. Recollections and outline

2.1. Throughout this paper, $K$ denotes an algebraically closed field of characteristic 2 unless otherwise stated, $G$ denotes a connected algebraic group of type $B, C$ or $D$ over $K$, $\mathfrak{g}$ the Lie algebra of $G$ and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. There is a natural coadjoint action of $G$ on $\mathfrak{g}^*$, $g.\xi(x) = \xi(Ad(g^{-1})x)$ for $g \in G, \xi \in \mathfrak{g}^*, x \in \mathfrak{g}$, where $Ad$ is the adjoint action of $G$ on $\mathfrak{g}$.

2.2. For a finite group $H$, we denote $H^\wedge$ the set of irreducible characters of $H$.

Let $W_G$ be the Weyl group of $G$. Denote $A_\mathfrak{g}$ (resp. $A_{\mathfrak{g}^*}$) the set of all pairs $(c, \mathcal{F})$ with $c$ a nilpotent $G$-orbit in $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) and $\mathcal{F}$ an irreducible $G$-equivariant local system on $c$ (up to isomorphism), which is the same as the set of all pairs $(x, \phi)$ (resp. $(\xi, \phi)$) with...
moreover, corresponding component groups are isomorphic. Hence the sets two Lie algebras (resp. duals of the Lie algebras) of groups in the same isogeny class, and 2.4. For a Borel subgroup

\[ A \] is adjoint (resp. simply connected). The correspondence is a bijective map from \( \mathfrak{A}_g \) (resp. \( \mathfrak{A}_{g^*} \)) to \( W^G_{\mathfrak{L}} \). This induces a Springer correspondence for any \( g \) (resp. \( g^* \)) as in [2,1] (char(\( k \)) = 2). In fact, there are natural bijections between the sets of nilpotent orbits in two Lie algebras (resp. duals of the Lie algebras) of groups in the same isogeny class, and moreover, corresponding component groups are isomorphic. Hence the sets \( \mathfrak{A}_g \) (resp. \( \mathfrak{A}_{g^*} \)) are naturally identified in each isogeny class.

2.4. For a Borel subgroup \( B \) of \( G \), we write \( B = TU \) a Levi decomposition of \( B \) and denote \( b, l \) and \( n \) the Lie algebra of \( B, T \) and \( U \) respectively. Define \( n^* = \{ \xi \in g^*|\xi(b) = 0 \} \) and \( b^* = \{ \xi \in g^*|\xi(n) = 0 \} \).

For a parabolic subgroup \( P \) of \( G \), we denote \( U_P \) the unipotent radical of \( P \), \( p \) and \( n_P \) the Lie algebra of \( P \) and \( U_P \) respectively. For a Levi subgroup \( L \) of \( P \), denote \( l \) the Lie algebra of \( L \). Define \( p^* = \{ \xi \in g^*|\xi(n_P) = 0 \} \), \( n^*_P = \{ \xi \in g^*|\xi(l \oplus n_P) = 0 \} \) and \( l^* = \{ \xi \in g^*|\xi(n_P \oplus n_P) = 0 \} \) where \( g = l \oplus n_P \oplus n^*_P \). We have \( p^* = l^* \oplus n^*_P \). Let \( \pi_{p^*} : p^* \rightarrow l^* \) be the natural projection.

2.5. Let \( P \) be a parabolic subgroup of \( G \) with a Levi decomposition \( P = LU_P \), where \( \text{rank}(L) = \text{rank}(G) - 1 \). We identify \( L \) with \( P/U_P \) and \( l \) with \( p/n_P \). Let \( x \in g \) and \( x' \in l \) be nilpotent elements. Consider the variety

\[ Y_{x,x'} = \{ g \in G|\text{Ad}(g^{-1}) (x) \in x' + n_P \}. \]

The group \( Z_G(x) \times Z_L(x')U_P \) acts on \( Y_{x,x'} \) by \( (g_0, g_1).g = g_0 g_1^{-1} \).

Let \( d_{x,x'} = (\text{dim} Z_G(x) + \text{dim} Z_L(x'))/2 + \text{dim} n_P \). We have \( \text{dim} Y_{x,x'} \leq d_{x,x'} \) (see Proposition 3.1 (ii)).

Let \( S_{x,x'} \) be the set of all irreducible components of \( Y_{x,x'} \) of dimension \( d_{x,x'} \). Then the group \( A_G(x) \times A_L(x') \) acts on \( S_{x,x'} \). Denote \( \xi_{x,x'} \) the corresponding representation. We prove in section 3 the following restriction formula

\[ (R) \quad \langle \phi \otimes \phi', \xi_{x,x'} \rangle = \langle \text{Res}_{W_L}^G \rho^G_{x,\phi}, \rho^L_{x',\phi'} \rangle w_L, \]

where \( \phi \in A_G(x)^\wedge, \phi' \in A_L(x')^\wedge \) and \( \rho^G_{x,\phi} \in W^G_{\mathfrak{L}}, \rho^L_{x',\phi'} \in W^L_{\mathfrak{L}} \) correspond to the pairs \( (x, \phi) \in \mathfrak{A}_g, (x', \phi') \in \mathfrak{A}_l \) respectively under the Springer correspondence.

It suffices to consider the case where \( G \) is adjoint (see 2.3). The proof is essentially the same as that of the restriction formula in unipotent case [4].

2.6. We preserve the notations from 2.5. Let \( \xi \in g^* \) and \( \xi' \in l^* \) be nilpotent elements. We define \( Y_{\xi,\xi'}, S_{\xi,\xi'}, \xi_{x,\xi'} \) as \( Y_{x,x'}, S_{x,x'}, \xi_{x,x'} \) replacing \( x, x', p, n_P \) by \( \xi, \xi', p^*, n^*_P \) respectively and adjoint \( G \)-action on \( g \) by coadjoint \( G \)-action on \( g^* \). We identify \( l^* \) with \( p^*/n^*_P \). We have the following restriction formula

\[ (R') \quad \langle \phi \otimes \phi', \xi_{\xi,\xi'} \rangle = \langle \text{Res}_{W_L}^G \rho^G_{\xi,\phi}, \rho^L_{\xi',\phi'} \rangle w_L, \]

where \( \phi \in A_G(\xi)^\wedge, \phi' \in A_L(\xi')^\wedge \) and \( \rho^G_{\xi,\phi} \in W^G_{\mathfrak{L}}, \rho^L_{\xi',\phi'} \in W^L_{\mathfrak{L}} \) correspond to the pairs \( (\xi, \phi) \in \mathfrak{A}_{g^*}, (\xi', \phi') \in \mathfrak{A}_{l^*} \) respectively under the Springer correspondence. The proof of \( (R') \) is entirely similar to that of \( (R) \) and is omitted.
2.7. Let $V$ be a vector space of dimension $2n$ over $k$ equipped with a non-degenerate symplectic form $\beta : V \times V \to k$. The symplectic group is defined as $Sp(2n) = Sp(V) = \{g \in GL(V) \mid \beta(gv, gw) = \beta(v, w), \forall v, w \in V\}$ and its Lie algebra is $\mathfrak{sp}(2n) = \mathfrak{sp}(V) = \{x \in \mathfrak{gl}(V) \mid \beta(xv, w) + \beta(v, xw) = 0, \forall v, w \in V\}$.

Recall (see [15]) that for a nilpotent element $\xi \in \mathfrak{sp}(2n)^*$, we associate a well-defined quadratic form $\alpha_\xi : V \to k$, $\alpha_\xi(v) = \beta(v, Xv)$ and a nilpotent endomorphism $T_\xi : V \to V$, $\beta(T_\xi(v), w) = \beta_\xi(v, w)$, where $X \in \text{End}(V)$ is such that $\xi(-) = \text{tr}(X-)$ and $\beta_\xi$ is the bilinear form associated to $\alpha_\xi$. The function $\chi_\xi : \mathbb{N} \to \mathbb{N}$ is defined as $\chi_\xi(m) = \min\{k \geq 0 | T_\xi^m v = 0 \Rightarrow \alpha_\xi(T_\xi^k v) = 0\}$.

2.8. Let $V$ be a vector space of dimension $N$ over $k$ equipped with a non-degenerate quadratic form $\alpha : V \to k$. Let $\beta : V \times V \to k$ be the bilinear form associated to $\alpha$. The orthogonal group is defined as $O(N) = O(V) = \{g \in GL(V) \mid \alpha(gv, gw) = \alpha(v, w), \forall v, w \in V\}$ and its Lie algebra is $\mathfrak{o}(N) = \mathfrak{o}(V) = \{x \in \mathfrak{gl}(V) \mid \beta(xv, w) + \beta(v, xw) = 0, \forall v, w \in V \text{ and } \text{tr}(x) = 0\}$. When $N$ is even, we define $SO(N)$ to be the identity component of $O(N)$.

Recall (see [15]) that for a nilpotent element $\xi \in \mathfrak{o}(2n + 1)^*$, we associate a well-defined bilinear form $\beta_\xi : V \times V \to k$, $\beta_\xi(v, w) = \beta(Xv, w) + \beta(v, Xw)$ where $X \in \text{End}(V)$ is such that $\xi(-) = \text{tr}(X-)$.

Assume $m \geq 1$. Let $u_i, i = 0, \ldots, m-1$ be a set of vectors as in [15] Lemma 3.6, $V_{2m+1}$ the vector space spanned by $u_i, i = 0, \ldots, m-1, v_i, i = 0, \ldots, m$, and $W = \{v \in V \mid \beta(v, V_{2m+1}) = \beta(v, V_{2m+1}) = 0\}$. Then $V = V_{2m+1} \oplus W$. Define $T_\xi : W \to W$ by $\beta(T_\xi(w), w') = \beta_\xi(v, w')$ and a function $\chi_W : \mathbb{N} \to \mathbb{N}$ by $\chi_W(s) = \min\{k \geq 0 | T_\xi^k w = 0 \Rightarrow \alpha(T_\xi^k w) = 0\}$. Note that the set $\{u_i\} \text{ and } W$ depends on the choice of $u_0$ and is uniquely determined by $u_0$. Suppose we take $\bar{u}_0 = u_0 + w_0$, where $w_0 \in W$ and $\alpha(w_0) = 0$, then $\bar{u}_i = u_i + T_\xi^{s_i} w_0$ defines another set of vectors as in [15] Lemma 3.6. Let $\bar{V}_{2m+1}, \bar{W}, \bar{T}_\xi$ be defined as $V_{2m+1}, W, T_\xi$ replacing $u_i$ by $\bar{u}_i$. Then $V = V_{2m+1} \oplus \bar{W}$ and $\bar{W} = \{\sum_{i=0}^m \beta(w, T_\xi^{s_i} w_0) v_i + w \mid w \in W\}$. On $\bar{W}$, we have $T_\xi(\sum_{i=0}^m \beta(w, T_\xi^{s_i} w_0) v_i + w) = \sum_{i=0}^m \beta(w, T_\xi^{s_i+1} w_0) v_i + T_\xi w$.

From now on, we always assume the decomposition $V = V_{2m+1} \oplus \bar{W}$ is a norm form of $\xi$, namely, if $W = W_{\chi_\lambda_1}(\lambda_1) \oplus \cdots \oplus W_{\chi_\lambda_s}(\lambda_s)$ with $\lambda_1 \geq \cdots \geq \lambda_s$ (notation as in [15]), then $m \geq \lambda_1 - \chi_\lambda(\lambda_1)$. Note if $\bar{V}_{2m+1}$ and $\bar{W}$ are obtained from $V_{2m+1}$ and $W$ as above, then $V = V_{2m+1} \oplus \bar{W}$ is also a norm form, in particular, $\bar{W} \cong W_{\chi_\lambda} \lambda_1 \oplus \cdots \oplus W_{\chi_\lambda}(\lambda_s)$.

2.9. For $n \geq 1$, let $W_n$ be a Weyl group of type $B_n$ (or $C_n$). The set $W_n^\wedge$ is parametrized by ordered pairs of partitions $(\mu, \nu)$ with $\sum \mu_i + \sum \nu_i = n$. We use the convention that the trivial representation corresponds to $(\mu, \nu)$ with $\mu = (n)$ and the sign representation corresponds to $(\mu, \nu)$ with $\nu = (1^n)$. Moreover, $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0)$, $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq 0)$.

For $n \geq 2$, let $W_n' \subset W_n$ be a Weyl group of type $D_n$. Let $W_0' = W_1' = \{1\}$. Let $W_n'^\wedge$ be the quotient of $(W_n')^\wedge$ by the natural action of $W_n/W_n'$. The parametrization of $W_n$ by ordered pairs of partitions induces a parametrization of $W_n'^\wedge$ by unordered pairs of partitions $\{\mu, \nu\}$. Moreover, $\{\mu, \nu\}$ corresponds to one (resp. two) element(s) of $(W_n')^\wedge$ if and only if $\mu \neq \nu$ (resp. $\mu = \nu$). We say that $\{\mu, \nu\}$ and the corresponding elements of $W_n'^\wedge$ and $(W_n')^\wedge$ are non-degenerate (resp. degenerate).
2.10. Assume $G = Sp(2n)$ or $G = O(2n + 1)$. The Springer correspondence for $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) is a bijective map

$$\gamma : \mathfrak{g}_x \rightarrow \mathfrak{w}_n^\wedge \text{ (resp. } \gamma' : \mathfrak{g}_x^* \rightarrow \mathfrak{w}_n^\wedge).$$

Assume $G = SO(2n)$. The Springer correspondence for $\mathfrak{g}$ (or $\mathfrak{g}^*$) is a bijective map

$$(2.1) \quad \gamma : \mathfrak{g}_x \cong \mathfrak{g}_x^* \rightarrow (\mathfrak{w}_n^\wedge).$$

Let $\tilde{G} = O(2n)$. The group $\tilde{G}/G$ acts on $\mathfrak{g}_x$ and on the set of all nilpotent $G$-orbits in $\mathfrak{g}$. An element in $\mathfrak{g}_x$ or a nilpotent orbit in $\mathfrak{g}$ is called non-degenerate (resp. degenerate) if it is fixed (resp. not fixed) by this action. Then $(x, \phi) \in \mathfrak{g}_x$ is degenerate if and only if $x$ is degenerate, in this case $A_G(x) = 1$ and thus $\phi = 1$. Let $\mathfrak{g}_x$ be the quotient of $\mathfrak{g}_x$ by $\tilde{G}/G$. Then (2.1) induces a bijection

$$(2.2) \quad \tilde{\gamma} : \mathfrak{g}_x \rightarrow \mathfrak{w}_n^\wedge.$$

2.11. Assume $G = SO(V)$. Let $\tilde{G} = O(V)$. Note that $\tilde{G} \neq G$ if and only if dim($V$) is even. Let $\Sigma \subseteq V$ be a line such that $\alpha|_\Sigma = 0$. Let $\tilde{P}$ be the stabilizer of $\Sigma$ in $\tilde{G}$ and $P$ the identity component of $\tilde{P}$. Then $P$ is a parabolic subgroup of $\tilde{G}$. Let $L$ be a Levi subgroup of $P$ and $L = N_P(L)$. Let $U_P$, $\mathfrak{p}$, $\mathfrak{n}_P$ be as in 2.4. Then $\tilde{P} = LU_P$ and $L = \tilde{L}$. Fix a Borel subgroup $B \subset P$ and let $\tilde{B} = N_G(B)$. Denote $\tilde{B} = \{ g\tilde{B}g^{-1} | g \in \tilde{G} \}$, $\tilde{P} = \{ g\tilde{P}g^{-1} | g \in \tilde{G} \}$.

Let $x \in \mathfrak{g}$ be nilpotent. Define $\tilde{B}_x = \{ g\tilde{B}g^{-1} \in \tilde{B} | \text{Ad}(g^{-1})(x) \in \mathfrak{b} \}$ and $\tilde{P}_x = \{ g\tilde{P}g^{-1} \in \tilde{P} | \text{Ad}(g^{-1})(x) \in \mathfrak{p} \}$. The natural morphism $\varphi_x : \tilde{B}_x \rightarrow \tilde{P}_x$, $g\tilde{B}g^{-1} \rightarrow g\tilde{P}g^{-1}$ is $Z_{\tilde{G}}(x)$ equivariant. We have a well defined map

$$f_x : \tilde{P}_x \rightarrow \mathcal{CN}(\mathfrak{p}/\mathfrak{n}_P), \quad g\tilde{P}g^{-1} \mapsto \text{orbit of Ad}(g^{-1})x + \mathfrak{n}_P,$$

where $\mathcal{CN}(\mathfrak{p}/\mathfrak{n}_P)$ is the set of nilpotent $\tilde{P}/U_P$-orbits in $\mathfrak{p}/\mathfrak{n}_P$. Let $c' \in f_x(\tilde{P}_x)$ be a nilpotent orbit. Define $Y = f_x^{-1}(c')$ and $X = \varphi_x^{-1}(Y)$.

We can assume $P \subseteq Y$. We identify $L$ with $\tilde{P}/U_P$, $L$ with $\mathfrak{p}/\mathfrak{n}_P$. Let $x'$ be the image of $x$ in $L$ and $\tilde{A}'(x') = A_L(x') = Z_L(x')/Z_L^0(x')$, $H = Z_{\tilde{G}}(x) \cap \tilde{P} = Z_{\tilde{P}}(x)$, $K = Z_{\tilde{G}}^0(x) \cap \tilde{P}$. The natural morphisms $H \rightarrow Z_{\tilde{G}}(x)$, $H \rightarrow Z_L(x')$ and $K \rightarrow Z_L(x')$ induce morphisms $H \rightarrow A_{\tilde{G}}(x)$, $H \rightarrow A_L(x')$ and $K \rightarrow A_L(x')$. Let $\tilde{A}_P$ be the image of $H$ in $A_{\tilde{G}}(x)$ and $\tilde{A}_P$ be the image of $K$ in $A_L(x')$. Then we have a natural morphism $A_P \rightarrow \tilde{A}'(x')/\tilde{A}'_P$.

If $G = \tilde{G}$, then we omit the tildes from the notations, for example, $A_P = \tilde{A}_P$ and etc.

2.12. We preserve the notations in 2.11. Let $\tilde{Y}_{x,x'}$ and $\tilde{S}_{x,x'}$ be defined as in 2.5 replacing $G$ by $\tilde{G}$ and $L$ by $\tilde{L}$. Note that $\tilde{S}_{x,x'} \neq \emptyset$ if and only if dim $X = \dim \tilde{B}_x$, where $X$ is defined as in 2.11 with $c'$ the orbit of $x'$. If $\tilde{S}_{x,x'} \neq \emptyset$, then $\tilde{Y}_{x,x'}$ is a single orbit under the action of $Z_{\tilde{G}}(x) \times Z_L(x')U_P$ (see Proposition 1.3). It follows that $\tilde{S}_{x,x'}$ is a single $A_{\tilde{G}}(x) \times A_L(x')$-orbit. Hence $\tilde{S}_{x,x'} = A_{\tilde{G}}(x) \times A_L(x')/\tilde{H}_{x,x'}$ for some subgroup $\tilde{H}_{x,x'} \subseteq A_{\tilde{G}}(x) \times A_L(x')$. The subgroup $\tilde{H}_{x,x'}$ is described as follows.

If $A, B$ are groups, a subgroup $C$ of $A \times B$ is characterized by the triple $(A_0, B_0, h)$ where $A_0 = \text{pr}_1(C)$, $B_0 = B \cap C$ and $h : A_0 \rightarrow N_B(B_0)/B_0$ is defined by $a \mapsto bB_0$ if $(a, b) \in C$. Then $\tilde{H}_{x,x'}$ is characterized by the triple $(A_P, \tilde{A}_P, h)$, where $h$ is the natural morphism $A_P \rightarrow A_L(x')/\tilde{A}'_P$ described in 2.11.
Assume $G = SO(2n)$. The subset $S_{x,x'}$ of $\tilde{S}_{x,x'}$ is the image in $\tilde{S}_{x,x'}$ of the subgroups of $A_G(x) \times A_L(x')$ consisting of the elements that can be written as a product of even number of generators. This is also the image of $A_G(x) \times A_L(x')$.

2.13. Assume $G = Sp(V)$ or $O(V)$ with $\dim V$ odd. The definitions in 2.11 apply to $\mathfrak{g}^*$ (if $G = Sp(V)$, there are no conditions on the line $\Sigma$). Let $g_\xi, f_\xi, A_P, A'_P$ etc. be defined in this way. Then $Y_{\xi,\xi'}$, $S_{\xi,\xi'}$ are described in the same way as $Y_{x,x'}, S_{x,x'}$ in 2.12.

2.14. The correspondence for symplectic Lie algebras is determined by Spaltenstein [11] since in this case the centralizer of a nilpotent element is connected and $\mathfrak{A}_g = \{(c,1)\}$. We rewrite his results in section 8 using different combinatorics and describe the Springer correspondence for orthogonal Lie algebras in section 9. The proof will essentially be as in [5], which is based on the restriction formula (R) and the following observation of Shoji: if $n \geq 3$, an irreducible character of $W_n$ (resp. a nondegenerate irreducible character of $W'_n$) is completely determined by its restriction to $W_{n-1}$ (resp. $W'_{n-1}$). We need to study the representations $\varepsilon_{x,x'}$, which require a description of the groups $\tilde{A}_P$ and $\tilde{A}'_P$. Using a similar method as in [12], we describe these groups for orthogonal Lie algebras, duals of symplectic Lie algebras and duals of odd orthogonal Lie algebras in section 11.5 and 6 respectively.

2.15. The Springer correspondence for the duals of symplectic Lie algebras and orthogonal Lie algebras is described in section 11. The proofs are very similar to the Lie algebra case. We omit many details.

3. Restriction Formula

Assume $G$ is adjoint. Fix a Borel subgroup $B$ of $G$ and a maximal torus $T \subset B$. Let $B$ be the variety of Borel subgroups of $G$. A proof of the restriction formula in unipotent case is given in [5]. The proof for nilpotent case is essentially the same. For completeness, we include the proof here.

3.1. We prove first a dimension formula. Let $\mathcal{P}$ be a $G$-conjugacy class of parabolic subgroups of $G$. For $P \in \mathcal{P}$, let $\tilde{P} = P/U_P$, $\tilde{p} = p/n_P$ and $\pi_p: p \rightarrow \tilde{p}$ be the natural projection. We assume given a $G$-orbit $c$ in $\mathfrak{g}$ and given for each $P \in \mathcal{P}$, a $P$-orbit $c_p \subset \tilde{p}$ with the following property: for any $P_1, P_2 \in \mathcal{P}$ and any $g \in G$ such that $P_2 = gP_1g^{-1}$, we have $\pi_{P_2}^{-1}(c_{P_2}) = \text{Ad}(g)(\pi_{P_1}^{-1}(c_{P_1}))$.

Let

$$Z' = \{(x, P_1, P_2) \in \mathfrak{g} \times \mathcal{P} \times \mathcal{P} | x \in \pi_{P_1}^{-1}(c_{P_1} \cap \pi_{P_2}^{-1}(c_{P_2}))\}.$$ 

We have a partition $Z' = \cup_{\mathcal{O}} Z'_\mathcal{O}$, according to the $G$-orbits $\mathcal{O}$ on $\mathcal{P} \times \mathcal{P}$; where $Z'_\mathcal{O} = \{(x, P_1, P_2) \in Z'| (P_1, P_2) \in \mathcal{O}\}$.

We denote $\nu_G$ the number of positive roots in $G$ and set $\tilde{\nu} = \nu_{\tilde{P}} (P \in \mathcal{P})$. Let $c = \dim c$ and $\tilde{c} = \dim c_{\tilde{p}}$ for $P \in \mathcal{P}$.

Proposition. (i) Given $P \in \mathcal{P}$ and $\bar{x} \in c_{\tilde{p}}$, we have $\dim (c \cap \pi_{\tilde{p}}^{-1}(\bar{x})) \leq \frac{1}{2}(c - \tilde{c})$.

(ii) Given $x \in c$, we have $\dim \{P \in \mathcal{P} | x \in \pi_{P}^{-1}(c_{P})\} \leq (\nu_G - \frac{c}{2}) - (\tilde{\nu} - \frac{\tilde{c}}{2})$.

(iii) If $d_0 = 2\nu_G - 2\tilde{\nu} + \tilde{c}$, then $\dim Z'_\mathcal{O} \leq d_0$ for all $\mathcal{O}$. Hence $\dim Z' \leq d_0$. 

Proof. We prove the proposition by induction on the dimension of the group. Assume \( \mathcal{P} = \{ G \} \), the proposition is clear. Thus we can assume that \( \mathcal{P} \) is a class of proper parabolic subgroups of \( G \) and that the proposition holds when \( G \) is replaced by a group of strictly smaller dimension.

Consider the map \( Z_G' \to \mathcal{O}, \,(x, P_1, P_2) \mapsto (P_1, P_2) \). We see that proving (iii) for \( Z_G' \) is the same as proving that for a fixed \( (P', P'') \in \mathcal{O} \), we have

\[
\dim \pi_{p'}^{-1}(c_{p'}) \cap \pi_{p''}^{-1}(c_{p''}) \leq 2 \nu_G - 2 \nu + \bar{c} - \dim \mathcal{O}.
\]

(3.1)

Choose Levi subgroups \( L' \) of \( P' \) and \( L'' \) of \( P'' \) such that \( L' \) and \( L'' \) contain a common maximal torus. Thus \( P' \cap L'' \) is a parabolic subgroup of \( L'' \) with unipotent radical \( U_{p''} \cap L'' \) and Levi subgroup \( L' \cap L'' \); \( P'' \cap L' \) is a parabolic subgroup of \( L' \) with unipotent radical \( U_{p'} \cap L' \) and Levi subgroup \( L' \cap L'' \). We have \( \text{Lie}(L' \cap L'') = l' \cap l'' \), \( \text{Lie}(U_{p''} \cap L'') = n_{p''} \cap l'' \) and \( \text{Lie}(U_{p'} \cap L') = n_{p'} \cap l' \). An element in \( p' \cap p'' \) can be written both in the form \( x' + n' \) (\( x' \in l', n' \in n_{p'} \)) and in the form \( x'' + n'' \) (\( x'' \in l'', n'' \in n_{p''} \)). It is easy to see that there are unique elements \( z \in l' \cap l'', u'' \in l' \cap n_{p''}, u' \in l'' \cap n_{p'} \), such that \( x' = z + u'', x'' = z + u' \). Hence (3.1) is equivalent to

\[
\dim \{(n', n'', u'', u'), (n', n'') \in n_{p'} \times n_{p''} \times (l' \cap n_{p'} \times (l'' \cap n_{p''}) \times (l' \cap l'') \mid u'' + n' = u' + n'', z + u'' \in c_{p'}, z + u' \in c_{p''} \} \leq 2 \nu_G - 2 \nu + \bar{c} - \dim \mathcal{O}.
\]

(3.2)

(We identify \( l' = p', l'' = p'' \), and thus regard \( c_{p'} \subset l', c_{p''} \subset l'' \).) When \( (u'', u') \in (l' \cap n_{p''}) \times (l'' \cap n_{p'}) \) is fixed, the variety \( \{(n', n'') \in n_{p'} \times n_{p''} \mid u'' + n' = u' + n'' \} \) is isomorphic to \( n_{p'} \cap n_{p''} \). Since \( \dim(n_{p'} \cap n_{p''}) = 2 \nu_G - 2 \nu - \dim \mathcal{O} \), we see that (3.2) is equivalent to

\[
\dim \{(u'', u'), (n', n'') \in (l' \cap n_{p''}) \times (l'' \cap n_{p'}) \times (l' \cap l'') \mid z + u'' \in c_{p'}, z + u' \in c_{p''} \} \leq \bar{c}.
\]

Note the projection \( \text{pr}_3 \) of the variety in (3.3) on the \( z \)-coordinate is a union of finitely many orbits \( \hat{c}_1 \cup \hat{c}_2 \cup \cdots \cup \hat{c}_m \) in \( l' \cap l'' \). (By the finiteness of the number of nilpotent orbits, it is enough to show that the semisimple part \( z_s \) of \( z \) can take only finitely many values up to \( L' \cap L'' \)-conjugacy; but \( z_s \) is conjugate to one of the elements in the finite set obtained by intersecting the set of semisimple parts of elements in \( c_{p'} \subset l' \) with the Lie algebra of a maximal torus in \( L' \cap L'' \).) The inverse image under \( \text{pr}_3 \) of a point \( z \in \hat{c}_i \) is a product of two varieties of the type considered in (i) for a smaller group \( \{ G \} \) replaced by \( L' \) or \( L'' \), thus by the induction hypothesis it has dimension \( \leq \frac{1}{2}(\bar{c} - \dim \hat{c}_i) + \frac{1}{2}(\bar{c} - \dim \hat{c}_i) \). Hence \( \dim \text{pr}_3^{-1}(\hat{c}_i) \leq \bar{c}, \forall 1 \leq i \leq m \). Then (3.3) holds. This proves (iii).

We show that (ii) is a consequence of (iii). Let \( Z'(c) \) be the subset of \( Z' \) defined by \( Z'(c) = \{(x, P_1, P_2) \in Z' \mid x \in c \} \). If \( Z'(c) \) is empty then the variety in (ii) is empty and (ii) follows. Hence we may assume that \( Z'(c) \) is non-empty. From (iii), we have \( \dim Z'(c) \leq d_0 \). Consider the map \( Z'(c) \to c, \,(x, P_1, P_2) \mapsto x \). Each fiber of this map is a product of two copies of the variety in (ii). It follows that the variety in (ii) has dimension equal to \( \frac{1}{2}(\dim Z'(c) - \dim c) \leq \frac{1}{2}(d_0 - c) = \nu_G - \nu + \frac{c}{2} - \frac{\nu}{2} \). Then (ii) follows.

We show that (i) is a consequence of (ii). Consider the variety \( \{(x, P) \in c \times \mathcal{P} \mid x \in \pi_p^{-1}(c_p) \} \). By projecting it to the \( x \)-coordinate and using (ii), we see that it has dimension \( \leq \nu_G - \nu + \frac{c}{2} + \frac{\nu}{2} \). If we project it to the \( P \)-coordinate, each fiber will be isomorphic to the variety \( c \cap \pi_p^{-1}(c_p), \,(P \in \mathcal{P} \text{ fixed}) \). Hence \( \dim(c \cap \pi_p^{-1}(c_p)) \leq \nu_G - \nu + \frac{c}{2} - \frac{\nu}{2} - \dim \mathcal{P} = \frac{c+\nu}{2} \). Now \( c \cap \pi_p^{-1}(c_p) \) maps onto \( c_p \) (via \( \pi_p \)) and each fiber is the variety in (i). Hence the variety in (i) has dimension \( \leq \frac{c+\nu}{2} - \bar{c} = \frac{c+\nu}{2} - \bar{c} \). The proposition is proved. \( \square \)
3.2. Let $P \supset B$ be a parabolic subgroup of $G$ with Levi subgroup $L$ such that $T \subset L$. Let $W_L = N_L(T)/T$. Then $\mathbb{Q}[W_L]$ is in a natural way a subalgebra of $\mathbb{Q}[W_G]$.

Recall that we have the map (see [14])

$$\pi : \tilde{Y} = \{(x, gT) \in Y \times G/T | \text{Ad}(g^{-1})(x) \in t_0\} \to Y, (x, gT) \mapsto x,$$

where $Y$, $t_0$ is the set of regular semisimple elements in $g$, $t$ respectively. Let

$$Y_L = \bigcup_{g \in L} \text{Ad}(g)t_0, \quad \tilde{Y}_1 = \{(x, gL) \in g \times G/L | \text{Ad}(g^{-1})(x) \in Y_L\}.$$

Then $\pi$ factors as $\tilde{Y} \xrightarrow{\pi'} \tilde{Y}_1 \xrightarrow{\pi''} Y$, where $\pi'$ is $(x, gT) \mapsto (x, gL)$ and $\pi''$ is $(x, gL) \mapsto x$. The map $\pi' : \tilde{Y} \to \tilde{Y}_1$ is a principal bundle with group $W_L$. It follows that $\text{End}(\pi_1(\tilde{Q}_{\tilde{Y}})) = Q_l[W_L]$ and that we have a canonical decomposition

$$\pi_1(\tilde{Q}_{\tilde{Y}}) = \bigoplus_{\rho' \in W_L} (\rho' \otimes (\pi_1(\tilde{Q}_{\tilde{Y}}))_{\rho'}).$$

Recall that we have

$$\pi_1(\tilde{Q}_{\tilde{Y}}) = \bigoplus_{\rho \in W_G} (\rho \otimes (\pi_1(\tilde{Y}))_{\rho}),$$

where $(\pi_1(\tilde{Q}_{\tilde{Y}}))_{\rho} = \text{Hom}_{\mathbb{Q}[W_G]}(\rho, \pi_1(\tilde{Q}_{\tilde{Y}}))$ is an irreducible local system on $Y$. We have

$$\pi_1(\tilde{Q}_{\tilde{Y}}) = \pi''_1(\tilde{Q}_{\tilde{Y}}) = \bigoplus_{\rho' \in W_L} (\rho' \otimes \pi''_1((\pi_1(\tilde{Q}_{\tilde{Y}}))_{\rho})).$$

hence

$$\pi''_1((\pi_1(\tilde{Q}_{\tilde{Y}}))_{\rho'}) = \text{Hom}_{\mathbb{Q}[W_L]}(\rho', \pi_1(\tilde{Q}_{\tilde{Y}})) = \text{Hom}_{\mathbb{Q}[W_L]}(\rho', \bigoplus_{\rho \in W_G} (\rho \otimes (\pi_1(\tilde{Q}_{\tilde{Y}}))_{\rho})).$$

We see that for any $\rho' \in W_L$,

$$\pi''_1((\pi_1(\tilde{Q}_{\tilde{Y}}))_{\rho'}) = \bigoplus_{\rho \in W_G} ((\pi_1(\tilde{Q}_{\tilde{Y}}))_{\rho} \otimes \text{Hom}_{\mathbb{Q}[W_L]}(\rho', \rho)).$$

3.3. Recall that we have the map (see [14])

$$\varphi : X = \{(x, gB) \in g \times G/B | \text{Ad}(g^{-1})(x) \in b\} \to g, (x, gB) \mapsto x.$$

Let $X_1 = \{(x, gP) \in g \times G/P | \text{Ad}(g^{-1})(x) \in p\}$. Then $\varphi$ factors as $X \xrightarrow{\varphi'} X_1 \xrightarrow{\varphi''} g$ where $\varphi'$ is $(x, gB) \mapsto (x, gP)$ and $\varphi''$ is $(x, gP) \mapsto x$. The maps $\varphi', \varphi''$ are proper and surjective. We have a commutative diagram

$$\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\pi'} & \tilde{Y}_1 & \xrightarrow{\pi''} & Y \\
\downarrow j_0 & & \downarrow j_1 & & \downarrow j_2 \\
X & \xrightarrow{\varphi'} & X_1 & \xrightarrow{\varphi''} & g
\end{array}$$

where $j_2$ is $x \mapsto x$, $j_0$ is $(x, gT) \mapsto (x, gB)$ (an isomorphism of $\tilde{Y}$ with the open subset $\varphi^{-1}(Y)$ of $X$) and $j_1$ is $(x, gL) \mapsto (x, gP)$ (an isomorphism onto the open subset $\varphi''^{-1}(Y)$ of $X_1$). Note also that $\tilde{Y}_1$ is smooth (since $\tilde{Y}$ is smooth). We identify $\tilde{Y}, \tilde{Y}_1$ with open subsets
of $X, X_1$ via the maps $j_0, j_1$ respectively. We have a commutative diagram with cartesian squares

$$
\begin{array}{ccc}
X & \xrightarrow{p_1} & X'' & \xrightarrow{p_2} & X_L \\
\varphi' \downarrow & & \phi \downarrow & & \varphi_L \downarrow \\
X_1 & \xleftarrow{p_3} & G \times n_P \times I & \xrightarrow{p_4} & I
\end{array}
$$

where

- $X_L = \{(x, g(B \cap L)) \in I \times L/(B \cap L)|\text{Ad}(g^{-1})(x) \in b \cap I\}$,
- $X'' = \{(g_1, x, pB) \in G \times p \times P/B|\text{Ad}(p^{-1})(x) \in b\}$,
- $p_1$ is $(g_1, x, pB) \mapsto (\text{Ad}(g_1)(x), g_1pB)$, a principal $P$-bundle,
- $p_2$ is $(g_1, l + n, g'B) \mapsto (l, g'(B \cap L))$ with $l \in I, n \in n_P, g' \in L$, a principal $G \times n_P$-bundle,
- $p_3$ is $(g_1, n, l) \mapsto (\text{Ad}(g_1)(l + n), g_1P)$, a principal $P$-bundle,
- $p_4$ is $(g_1, n, l) \mapsto l$, a principal $G \times n_P$-bundle,
- $\varphi_L$ is $(x, g(B \cap L)) \mapsto x$,
- $\phi$ is $(g_1, l + n, g'B) \mapsto (g_1, n, l)$, with $l \in I, n \in n_P, g' \in L$.

Since $p_3, p_4$ are principal bundles with connected groups, we have $p_3^*IC(X_1, \pi_i^i\bar{Q}_{i\bar{Y}}) = p_4^*IC(I, \pi_{iL}^i\bar{Q}_{i\bar{Y}_L})$ (both can be identified with $IC(G \times n_P \times I, p_3^*\pi_{iL}^i\bar{Q}_{i\bar{Y}_L})$). Here $\pi_L$ is $Y_L = \{(x, gT) \in I \times L/T|\text{Ad}(g^{-1})(x) \in t_0\} \to Y_L, (x, gL) \mapsto x$.

From the commutative diagram above it follows that $p_3^*\varphi_i^i\bar{Q}_{iL} = \phi p_4^i\bar{Q}_{iL} = \phi p_3^i\bar{Q}_{iL} = p_4^*\varphi_i^i\bar{Q}_{iL} = p_4^*IC(I, \pi_{iL}^i\bar{Q}_{i\bar{Y}_L})$ (the last equality comes from [14, Proposition 6.6] for $L$ instead of $G$), hence $p_3^*\varphi_i^i\bar{Q}_{iL} = p_3^*IC(X_1, \pi_i^i\bar{Q}_{i\bar{Y}})$. Since $p_3$ is a principal $P$-bundle we see that

$$
\varphi_i^i\bar{Q}_{iL} = IC(X_1, \pi_i^i\bar{Q}_{i\bar{Y}}).
$$

It follows that $\text{End}(\varphi_i^i\bar{Q}_{iL}) \cong \bar{Q}_L[W_L]$ and $\varphi_i^i\bar{Q}_{iL} = \bigoplus_{\rho' \in W_L^\wedge} (\rho' \otimes (\varphi_i^i\bar{Q}_{iL})_{\rho'})$ where

$$
(\varphi_i^i\bar{Q}_{iL})_{\rho'} = IC(X_1, (\pi_i^i\bar{Q}_{i\bar{Y}})_{\rho'}).
$$

Next we show that

$$
(\varphi_i^i\bar{Q}_{iL})_{\rho'} = IC(Y, \pi_i^i((\pi_i^i\bar{Q}_{i\bar{Y}})_{\rho'})), \text{ for any } \rho' \in W_L^\wedge.
$$

From (3.6) we see that the restriction of $\varphi_i^i((\varphi_i^i\bar{Q}_{iL})_{\rho'})$ to $Y$ is the local system $\pi_i^i((\pi_i^i\bar{Q}_{i\bar{Y}})_{\rho'})$. Since $\varphi_i^i$ is proper, (3.7) is a consequence of (3.6) and the following assertion:

$$
\text{For any } i > 0, \text{ we have } \dim \text{supp} H^i((\varphi_i^i((\varphi_i^i\bar{Q}_{iL})_{\rho'}))) < \dim \bar{Y} - i.
$$

We have $\text{supp} H^i((\varphi_i^i((\varphi_i^i\bar{Q}_{iL})_{\rho'}))) \subset \text{supp} H^i((\varphi_i^i(\varphi_i^i\bar{Q}_{iL}))) = \text{supp} H^i((\varphi_i^i\bar{Q}_{iL}))$, thus (3.8) follows from the proof of [14, Proposition 6.6]. Hence (3.7) is verified. Combining (3.7) with (3.5), we see that for any $\rho' \in W_L^\wedge$, we have

$$
\varphi_i^i((\varphi_i^i\bar{Q}_{iL})_{\rho'}) \cong \bigoplus_{\rho \in W_L^\wedge} ((\varphi_i^i\bar{Q}_{iL})_{\rho} \otimes \text{Hom}_{\bar{Q}_L[W_L]}(\rho', \rho)).
$$

(Recall that we have $\varphi_i^i\bar{Q}_{iL} = \bigoplus_{\rho \in W_L^\wedge} (\rho \otimes (\varphi_i^i\bar{Q}_{iL})_{\rho})$ and $(\varphi_i^i\bar{Q}_{iL})_{\rho} = IC(\bar{Y}, (\pi_i^i\bar{Q}_{i\bar{Y}})_{\rho})$.)
3.4. Let $(c, \mathcal{F}) \in \mathfrak{A}_{\mathfrak{g}}$ and $(c', \mathcal{F}') \in \mathfrak{A}_{\mathfrak{l}}$ correspond to \( \rho \in \mathbf{W}_{\mathfrak{g}}^\sim \) and \( \rho' \in \mathbf{W}_{\mathfrak{l}}^\sim \) under Springer correspondence. Let \( X'_1 = \{(x, gP) \in X_1|x \text{ nilpotent}\} \),

\[
R = \{(x, gP) \in \mathfrak{g} \times (G/P)|\text{Ad}(g^{-1})(x) \in \tilde{c} + \mathfrak{n}_P\} \subset X'_1.
\]

We show that

\[
\text{supp}(\varphi'(\tilde{Q}_{1X}))_{\rho'} \cap X'_1 \subset R.
\]

Let \((x, gP) \in \text{supp}(\varphi'(\tilde{Q}_{1X}))_{\rho'} \cap X'_1\). The isomorphism \( p_{3*} \varphi'(\tilde{Q}_{1X}) = p_{4*} \varphi_L \tilde{Q}_{1X} \) is compatible with the action of \( \mathbf{W}_L \). Thus \( p_{3*} \varphi'(\tilde{Q}_{1X})_{\rho'} = p_{4*} \varphi_L (\tilde{Q}_{1X}_{\rho'})_{\rho'} \) and

\[
p_{3}^{-1}(\text{supp}(\varphi'(\tilde{Q}_{1X})_{\rho'})) = p_{4}^{-1}(\text{supp}(\varphi_L (\tilde{Q}_{1X}_{\rho'})_{\rho'})).
\]

Hence there exists \((g_1, n, l) \in G \times \mathfrak{n}_P \times \mathfrak{l}\) such that \((x, gP) = (\text{Ad}(g_1)(n + l), g_1 P)\) and \(l \in \text{supp}(\varphi_L (\tilde{Q}_{1X}_{\rho'})_{\rho'})\). Since \(x\) is nilpotent, \(n + l\) is nilpotent and thus \(l\) is nilpotent. Hence \(l \in \tilde{c}^\prime\) since by [14] Proposition 6.6 (for \(L\) instead of \(G\)),

\[
(\varphi_L (\tilde{Q}_{1X}_{\rho'})_{\rho'})|_{\mathcal{N}_L} = IC(\tilde{c}^\prime, \mathcal{F}')[\dim c' - 2\nu_L]|(\text{extend by zero outside } \tilde{c}^\prime),
\]

where \(\mathcal{N}_L\) is the nilpotent variety of \(\mathfrak{l}\). We have \(g = g_1 p\) for some \(p \in \mathfrak{p}\) and \(x = \text{Ad}(g_1)(n + l)\), hence \(\text{Ad}(g^{-1})(x) = \text{Ad}(g^{-1})(n + l) \in \tilde{c}^\prime + \mathfrak{n}_P\) and \((x, gP) \in R\). This proves (3.10).

We have a partition \(R = \cup_{c} R_c^\prime\), where \(c^\prime\) runs over the nilpotent \(L\)-orbits in \(\tilde{c}^\prime\) and \(R_c^\prime = \{(x, gP) \in \mathfrak{g} \times (G/P)|\text{Ad}(g^{-1})(x) \in c^\prime + \mathfrak{n}_P\}\). Then \(R' = R_c^\prime\) is open in \(R\). It is clear that \(p_{3}^{-1}(R) = p_{4}^{-1}(c^\prime) = G \times \mathfrak{n}_P \times c^\prime\) and \(p_{3}^{-1}(R_c^\prime) = p_{4}^{-1}(c^\prime) = G \times \mathfrak{n}_P \times c^\prime\).

Let \(\tilde{\mathcal{F}}^\prime\) be the local system on \(R'\) whose inverse image under \(p_3: G \times \mathfrak{n}_P \times c^\prime \to R'\) equals the inverse image of \(\mathcal{F}'\) under \(p_4: G \times \mathfrak{n}_P \times c^\prime \to c^\prime\). Since \(p_3, p_4\) are principal bundles with connected group it follows that the inverse image of \(IC(R, \tilde{\mathcal{F}}^\prime)\) under \(p_3: G \times \mathfrak{n}_P \times c^\prime \to R\) equals the inverse image of \(IC(\tilde{c}^\prime, \mathcal{F}')\) under \(p_4: G \times \mathfrak{n}_P \times c^\prime \to c^\prime\). It follows that

\[
(\varphi'(\tilde{Q}_{1X}))_{\rho'}|_{X'_1} = IC(R, \tilde{\mathcal{F}}'|)[\dim c' - 2\nu_L]|(\text{extend by zero outside } R).
\]

(Using \(p_3^*\) this is reduced to (3.11).)

For any subvariety \(S\) of \(X_1\), we denote \(S\varphi'^\prime: S \to \hat{Y}\) the restriction of \(\varphi'^\prime: X_1 \to \hat{Y}\) to \(S\).

**Proposition.** Let \(d = \nu_G - \frac{1}{2} \dim c\), \(d' = \frac{1}{2}(\dim c - \dim c')\) and \(d'' = \nu_G - \nu_L - d'\). The following five numbers coincide:

(i) \(\dim \text{Hom}_{\mathfrak{g}_{\tilde{c}}} (\mathbf{W}_L)(\rho', \rho)\);

(ii) the multiplicity of \(\mathcal{F}\) in the local system \(\mathcal{L}_1 = \mathcal{H}^{2d}(\varphi'(\tilde{Q}_{1X}))_{\rho'}|_c\);

(iii) the multiplicity of \(\mathcal{F}\) in the local system \(\mathcal{L}_2 = \mathcal{H}^{2d'}(\text{Ad}(\rho), \mathcal{F}'|_{R^c})|_c\);

(iv) the multiplicity of \(\mathcal{F}\) in the local system \(\mathcal{L}_3 = \mathcal{H}^{2d''}(\text{Ad}(\rho), \mathcal{F}'|_{R^c})|_c\);

(v) the multiplicity of \(\mathcal{F}\) in the local system \(\mathcal{H}^{2d'} f_!(\mathcal{F})\) on \(c^\prime\), where \(f: \pi_{\mathfrak{p}}^{-1}(c^\prime) \cap c \to c^\prime\) is the restriction of \(\pi_{\mathfrak{p}}: \mathfrak{p} \to \mathfrak{l}\).

**Proof.** For \(\tilde{c} \in \mathbf{W}_G^\sim\), the multiplicity of \(\mathcal{F}\) in \(\mathcal{H}^{2d}((\varphi(\tilde{Q}_{1X}))_{\rho'})|_c\) is 1 if \(\tilde{c} = c\) and is 0 if \(\tilde{c} \neq c\). Hence it follows from (3.9) that the numbers in (i)(ii) are equal.

We show that \(\mathcal{L}_1 = \mathcal{L}_2\). By (3.12), we have \(\mathcal{L}_2 = \mathcal{H}^{2d'}(\text{Ad}(\rho), (\varphi(\tilde{Q}_{1X}))_{\rho'}|_R)|_c\). It suffices to show that \((X_1 - R)\varphi'^\prime((\varphi(\tilde{Q}_{1X}))_{\rho'}|_{X_1 - R})|_c = 0\). Assume this is not true. Then there exists \((x, gP) \in X_1 - R\) such that \(x \in c\) and \((x, gP) \in \text{supp}(\varphi(\tilde{Q}_{1X}))_{\rho'}\). Since \(x\) is nilpotent, this contradicts (3.11).
We show that \( L_2 = L_3 \). For any \( x \in c \) we consider the natural exact sequence
\[
H^{2d}(\varphi^{"-1}(x) \cap (R - R'), (\varphi^{"}_0 l_x)_{p'}) \xrightarrow{a} H^{2d}(\varphi^{"-1}(x) \cap R', (\varphi^{"}_0 l_x)_{p'})
\]
\[
\rightarrow H^{2d}(\varphi^{"-1}(x) \cap R, (\varphi^{"}_0 l_x)_{p'}) \rightarrow H^{2d}(\varphi^{"-1}(x) \cap (R - R'), (\varphi^{"}_0 l_x)_{p'})
\]
It is enough to show that \( H^{2d}(\varphi^{"-1}(x) \cap (R - R'), (\varphi^{"}_0 l_x)_{p'}) = 0 \) and that \( a = 0 \). By (3.12), we can replace \( (\varphi^{"}_0 l_x)_{p'} \) with \( IC(R, \bar{F}')[\dim c' - 2\nu_L] \). It is enough to show
\[
H^{2d\nu}(\varphi^{"-1}(x) \cap (R - R'), IC(R, \bar{F}')) = 0,
\]
\[
H^{2d\nu-1}(\varphi^{"-1}(x) \cap (R - R'), IC(R, \bar{F}')) \rightarrow H^{2d\nu}(\varphi^{"-1}(x) \cap R', IC(R, \bar{F}')) \text{ is zero.}
\]
From Proposition 3.3.1 we see that for any \( L \)-orbit \( c' \) in \( c \),
\[
\dim(\varphi^{"-1}(x) \cap R_{c'}) \leq (\nu_G - \frac{1}{2} \dim c) - (\nu_L - \frac{1}{2} \dim c').
\]
If (3.13) is not true, then using the partition
\[
\varphi^{"-1}(x) \cap (R - R') = \bigcup_{c' \neq c'} (\varphi^{"-1}(x) \cap R_{c'}),
\]
we see that \( H^{2d\nu}(\varphi^{"-1}(x) \cap R_{c'}, IC(R, \bar{F}')) \neq 0 \) for some \( c' \neq c' \). Hence there exist \( i, j \) such that \( 2d\nu = i + j \) and \( H^i_c(\varphi^{"-1}(x) \cap R_{c'}, IC(R, \bar{F}')) \neq 0 \). It follows that \( i \leq 2 \dim(\varphi^{"-1}(x) \cap R_{c'}) \leq 2\nu_G - \dim c - 2\nu_L + \dim c' \) (we use (3.15)). The local system \( \mathcal{H}^i(IC(R, \bar{F}')) \neq 0 \) so that \( R_{c'} \subset \text{supp} \mathcal{H}^i(IC(R, \bar{F}')) \) and \( \dim R_{c'} < \dim R - j \). It follows that \( j < \dim R - \dim R_{c'} = \dim c' - \dim c' \) and \( i + j < 2d\nu \) in contradiction to \( i + j = 2d\nu \). This proves (3.13).

To prove (3.14), we can assume that \( k \) is an algebraic closure of a finite field \( \mathbb{F}_q \), that \( G \) has a fixed \( \mathbb{F}_q \)-structure with Frobenius map \( F : G \rightarrow G \), that \( P, B, L, T \) (hence \( X_1, \varphi' \)) are defined over \( \mathbb{F}_q \), that any \( c' \) as above is defined over \( \mathbb{F}_q \), that \( F(x) = x \) and that we have an isomorphism \( F*: \mathcal{F}' \rightarrow \mathcal{F} \) which makes \( \mathcal{F}' \) into a local system of pure weight 0. Then we have natural (Frobenius) endomorphisms of \( H^{2d\nu-1}(\varphi^{"-1}(x) \cap (R - R'), IC(R, \bar{F}')) \) and \( H^{2d\nu}(\varphi^{"-1}(x) \cap R', IC(R, \bar{F}')) = H^{2d\nu}(\varphi^{"-1}(x) \cap R', \bar{F'}) \) compatible with \( a \). To show that \( a = 0 \), it is enough to show that
\[
H^{2d\nu}(\varphi^{"-1}(x) \cap R', IC(R, \bar{F}')) \text{ is pure of weight } 2d\nu;
\]
\[
H^{2d\nu-1}(\varphi^{"-1}(x) \cap (R - R'), IC(R, \bar{F}')) \text{ is mixed of weight } \leq 2d\nu - 1.
\]
Now (3.17) is clear since \( \dim(\varphi^{"-1}(x) \cap R') \leq d\nu \) (see (3.15)). We prove (3.18). Using the partition (3.16), we see that it is enough to prove that \( H^{2d\nu-1}(\varphi^{"-1}(x) \cap R_{c'}, IC(R, \bar{F}')) \) is mixed of weight \( \leq 2d\nu - 1 \) for any \( c' \neq c' \).

Using the hypercohomology spectral sequence we see that it is enough to prove if \( i, j \) are such that \( 2d\nu - 1 = i + j \), then \( H^i_c(\varphi^{"-1}(x) \cap R_{c'}, IC(R, \bar{F}')) \) is mixed of weight \( \leq 2d\nu - 1 \) for any \( c' \). By Gabber’s theorem [BBD, 5.3.2], the local system \( \mathcal{H}^i(IC(R, \bar{F}')) \) is mixed of weight \( \leq j \). Then by Deligne’s theorem [BBD, 5.1.14(i)], \( H^i_c(\varphi^{"-1}(x) \cap R_{c'}, \mathcal{H}^i(IC(R, \bar{F}')) \) is mixed of weight \( \leq i + j = 2d\nu - 1 \). This proves (3.18). We have shown that \( L_2 = L_3 \).

Now consider the diagram \( V \xrightarrow{f_2} V' \xrightarrow{f_1} c \), where \( V' = \varphi^{"-1}(c) \cap R' = \{(x, gP) \in c \times (G/P) | \text{Ad}(g^{-1})(x) \in c' + np \} \), \( V = P \setminus (c' \times G) \) with \( P \) acting by \( p : (x, g) \mapsto (\text{Ad}(\pi(p))(x), gp^{-1}) \), \( \pi : P \rightarrow L \) the natural projection, \( f_2(x, gP) = P\text{-orbit of } (\pi_p(\text{Ad}(g^{-1})(x)), g) \), \( f_1(x, gP) = x \).
The $G$-actions on $V$ by $g'(x,g)\mapsto (x,g'g)$, on $V'$ by $g'(x,gP)\mapsto (\text{Ad}(g')(x),g'gP)$ and on $c$ by $g'(x)\mapsto \text{Ad}(g'(x))$, are compatible with $f_1$, $f_2$ and are transitive on $V$ and $c$.

Note all fibers of $f_1$ have dimension $\leq d''$ and all fibers of $f_2$ have dimension $\leq d'$. We set $N = d'' + \dim c = d' + \dim V$. Applying [7, 8.4(a)] with $E_1 = F$ and with $E_2$ the local system on $V$ whose inverse image under the natural map $c \times G \rightarrow V$ is $F' \otimes \mathbb{Q}_l$, we see that the numbers (iv) and (v) are equal. This completes the proof of the proposition. \hfill \square

3.5. Now we are ready to prove the restriction formula (R). Let the notation be as in [2,5]. Let $c$ be the $G$-orbit of $x$ and $c'$ be the $L$-orbit of $x'$. Let $\tau : G/Z_G^0(x) \rightarrow G/Z_G(x) \simeq c$ be a covering of $c$ with group $A_G(x)$. We have the following commutative diagram

\[
\begin{array}{ccc}
Y_{x,x'} & \longrightarrow & Y_{x,x'}/Z_G^0(x) \\
\downarrow{b} & & \downarrow{i} \\
(x' + n_P) \cap c & \leftarrow & \tau^{-1}((x' + n_P) \cap c),
\end{array}
\]

where $a$ is the natural projection and $b$ is given by $g \mapsto \text{Ad}(g^{-1})(x)$. Then $a$ induces an $A_G(x)$-equivariant bijection between $S_{x,x'}$ and the set of irreducible components of $\tau^{-1}((x' + n_P) \cap c)$ of dimension $d' = \frac{1}{2}(\dim c - \dim c')$ (note that $\dim (x' + n_P) \cap c \leq d'$ by Proposition 3.1 (i)).

Assume $F$ corresponds to $\phi \in A_G(x)$ and $F'$ corresponds to $\phi' \in A_L(x')$. We have $F \simeq \text{Hom}_{A_G(x)}(\phi, \tau_* \mathbb{Q}_l)$ and thus $H^d_{\tilde{c}}((x' + n_P) \cap c, F) \simeq (H^d_{\tilde{c}}((x' + n_P) \cap c, \tau_* \mathbb{Q}_l) \otimes \phi^\Lambda)_{A_G(x)} \simeq (H^d_{\tilde{c}}(\tau^{-1}(x' + n_P) \cap c, \mathbb{Q}_l) \otimes \phi^\Lambda)^{A_L(x')}$. Then the number (v) in Proposition 3.1 is equal to

$$
\langle \phi', H^d_{\tilde{c}}(f^{-1}(x'), F) \rangle_{A_L(x')} = \langle \phi', H^d_{\tilde{c}}((x' + n_P) \cap c, F) \rangle_{A_L(x')} = \langle \phi \otimes \phi', \varepsilon_{x,x'} \rangle_{A_G(x) \times A_L(x')}.
$$

Hence the restriction formula (R) follows from Proposition 3.1 ((i)\Rightarrow(v)).

4. ORTHOGONAL LIE ALGEBRAS

In this section we assume $G = SO(N)$. Let $\tilde{G} = O(N)$.

4.1. Let $x \in g$ be nilpotent. The $G$-orbit of $c$ is characterized by the following data (2):

(d1) The sizes of the Jordan blocks of $x$ give rise to a partition $\lambda$ of $2n$, $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_s$.

(d2) For each $\lambda_i$, $i = 1, \ldots, s$, there is an integer $\chi(\lambda_i)$ satisfy $\frac{n}{2} \leq \chi(\lambda_i) \leq \lambda_i$. Moreover, $\lambda_i - \chi(\lambda_{i-1})$, $\lambda_i - \chi(\lambda_i)$ $\geq \lambda_{i-1} - \chi(\lambda_{i-1})$, $i = 2, \ldots, s$.

Let $m(\lambda_i)$ be the multiplicity of $\lambda_i$ in the partition $\lambda$. If $N$ is even, then $m(\lambda_i)$ is even for each $\lambda_i > 0$. If $N$ is odd, the the set $\{\lambda_i > 0 | m(\lambda_i) \text{ is odd} \}$ is $\{a, a - 1\}$ for some integer $a \geq 1$.

We write $x$ or $c = (\lambda, \chi) = (\lambda_s)_{\chi(\lambda_s)} \ldots (\lambda_1)_{\chi(\lambda_1)}$. The component groups $\tilde{A}(x) = Z_G(x)/Z_G^0(x)$ and $A(x) = Z_G(x)/Z_G^0(x)$ can be described as follows (see [14]). Let $\alpha_i$ correspond to $\lambda_i$, $i = 1, \ldots, s$. Then $\tilde{A}(x)$ is isomorphic to the abelian group generated by $\{a_i, 1 \leq i \leq s | \chi(\lambda_i) \neq \lambda_i/2 \}$ with relations

- (r1) $a_i^2 = 1$,
- (r2) $a_i = a_{i+1}$ if $\lambda_i = \chi(\lambda_{i+1}) > \lambda_{i+1}$,
- (r3) $a_i = 1$ if $m(\lambda_i)$ is odd.

If $N$ is even, $A(x)$ is the subgroup of $\tilde{A}(x)$ consisting of those elements that can be written as a product of even number of generators.
4.2. Let \( c' = (\lambda', \chi') \in f_x(\bar{P}_x) \), \( Y = \bar{g}_x^{-1}(c') \) and \( X = g_x^{-1}(Y) \) (see [11]). Spaltenstein [11] has described the necessary and sufficient conditions for \( \dim X = \dim B_x \) as follows.

**Proposition** ([11]). We have \( \dim X = \dim B_x \) if and only if \( (\lambda', \chi') \) satisfy (a) or (b):

(a) Assume that \( \lambda_i \neq \lambda_{i+1} \neq \lambda_{i+2} \), \( \chi(\lambda_{i+2}) = \lambda_{i+2} \), \( \lambda_j = \lambda_j \), \( j \neq i + 2, i + 1 \), \( \chi'(\lambda_j') = \chi(\lambda_j) \) if \( j > i + 2 \), \( \chi'(\lambda_j') = \lambda_j' \) if \( j \leq i + 2 \).

In this case, \( \dim Y = s - i - 2 \).

(b) Assume that \( \lambda_i = \lambda_i > \lambda_i - 1 \), \( \lambda_j = \lambda_j \), \( j \neq i + 1, i \), \( \lambda_j' = \lambda_j - 1 \), \( \chi'(\lambda_j') = \chi(\lambda_j) \) if \( j > i + 1 \) and \( \chi'(\lambda_j') = \chi'(\lambda_j') \) if \( j = i + 1 \) and \( \chi'(\lambda_j'') = \chi'(\lambda_j') \in \{ \chi(\lambda_i), \chi(\lambda_i) - 1 \} \) satisfies \( \lambda_j' \leq \chi'(\lambda_j') \leq \lambda_j' \).

In this case, \( \dim Y = s - i - 1 \) if \( \chi'(\lambda_j') = \chi(\lambda_i) \) and \( \dim Y = s - i - 1 \) if \( \chi'(\lambda_j') = \chi(\lambda_i) - 1 \).

4.3. From now on let \( c' \) be as in Proposition [4,2]. Let \( \tilde{A}_P \) and \( \tilde{A}'_P \) be defined as in [2,11].

**Proposition.** The group \( \mathbb{Z}/(x) \) acts transitively on \( Y \). The group \( \tilde{A}'_P \) is the subgroup of \( \tilde{A}(x) \) generated by the elements \( a_i \) which appear both in the generators of \( \tilde{A}(x) \) and of \( \tilde{A}'(x') \). The group \( \tilde{A}'_P \) is the smallest subgroup of \( \tilde{A}'(x') \) such that the map \( \tilde{A}_P \to \tilde{A}'(x')/\tilde{A}'_P \) given by \( a_i \mapsto a_i' \) is a morphism.

**Corollary.** \( Y \) has two irreducible components (and \( |\tilde{A}(x)/\tilde{A}_P| = 2 \) if \( c' \) is as in Proposition [4,2] (b) with \( \chi(\lambda_i) = 2 \), \( \lambda_i - 2 \), \( \chi(\lambda_i) - 1 \) and \( \chi'(\lambda_i') = 1 \).

In this case, suppose \( D = \{ 1, a_i \} = \{ 1, a_i + 1 \} \subset \tilde{A}(x) \), then \( \tilde{A}(x) = D \times \tilde{A}_P \). In the other cases, \( Y \) is irreducible and \( \tilde{A}_P \) is \( \tilde{A}(x) \).

**Corollary.** The group \( \tilde{A}'_P \) is trivial, except in the following cases where it has order 2:

(a) \( \tilde{A}_P = \{ 1, a_i + 1 \} \subset \tilde{A}(x) \) if \( c' \) is as in Proposition [4,2] (a) with \( \lambda_i + 2 + \chi(\lambda_i + 3) = \lambda_i + 1 \).

(b) \( \tilde{A}'_P = \{ 1, a_i + 1 \} \subset \tilde{A}(x) \) if \( c' \) is as in Proposition [4,2] (b) with \( \chi(\lambda_i) \neq \lambda_i + 1 \), \( \chi(\lambda_i + 2) + \chi(\lambda_i) - 1 \).

4.4. Assume \( G = O(2n + 1) \) and \( x \) correspond to the form module

\[
V = W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_k}(\lambda_k) \oplus D(\lambda_{k+1}) \oplus W_{l_{k+2}}(\lambda_{k+2}) \cdots \oplus W_{l_s}(\lambda_s),
\]

where \( l_i = \chi(\lambda_i), i = 1, \ldots, k \). (Note \( \lambda_i \) are different from those in [4,1].) We use here notations from [11]. We describe the orbits \( c' \) and the corresponding set \( Y \).

We view \( V \) as an \( A = k[t]-\text{module} \) by \( \sum a_i t^i v = \sum a_i(x^i v) \). For all \( i \geq 1 \), let

\[ W_i = \ker t \cap \text{Im } (t^{-1}). \]

We identify \( P_x \) with \( \mathbb{P}(\ker x \cap \alpha^{-1}(0)) \). Let \( Y \) be as in [4,3]. There exists a unique \( i_0 \) such that \( Y \subset \mathbb{P}(W_{i_0}) - \mathbb{P}(W_{i_0+1}) \). Then \( i_0 = \lambda_j \) for some \( j = 1, \ldots, s \) or \( i_0 = \lambda_{k+1} - 1 \). Write \( V' = \sum_1^{1}/\Sigma \). We have the following cases.

(i) Assume \( i_0 = \lambda_j, 1 \leq j \leq k \), \( \lambda_j - 1 \leq \lambda_{j+1} \) and \( \lambda_j - l_j - 1 \leq \lambda_{j+1} - l_{j+1} \).

\[
c' = W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_j}(\lambda_j - 1) \oplus \cdots \oplus D(\lambda_{k+1}) \oplus \cdots \oplus W_{l_s}(\lambda_s),
\]

\[
Y = \{ k^{t^{l_j-1}} w | t^{l_j} w = 0, w \notin \text{Im } t, \alpha(t^{l_j-1} w) \neq 0 \}, \dim Y = 2j - 1.
\]

(ii) Assume \( i_0 = \lambda_j, 1 \leq j \leq k \), \( \lambda_j - 1 \geq \lambda_{j+1} \), \( l_j - 1 \geq l_{j+1} \) and \( l_j - 1 \geq [\lambda_j/2] \). Let

\[
Y' = \{ k^{t^{l_j-1}} w | t^{l_j} w = 0, w \notin \text{Im } t, \alpha(t^{l_j-1} w) = 0 \}.
\]


\[ c' = W_{t_1}(\lambda_1) \oplus \cdots \oplus W_{t_j-1}(\lambda_j - 1) \oplus \cdots \oplus D(\lambda_{k+1}) \oplus \cdots \oplus W_{\lambda_n}(\lambda_n), \]

\[ Y = Y' \text{ except if } \lambda_a - l_a > \lambda_{j-1} - l_{j-1}, \text{ for all } \lambda_a > \lambda_{j-1}, \text{ and } l_{j-1} = l_j > \frac{\lambda_{j-1} + 1}{2}, \]

then \( Y = Y' - \{ \Sigma \in Y'| \chi_{V'}(\lambda_{j-1}) = l_{j-1} - 1 \} \) (an open dense subset in \( Y' \)),

\[ \dim Y = 2j - 2. \]

(iii) Assume \( i_0 = \lambda_j, \ j \geq k + 2 \) and \( \lambda_j \geq \lambda_{j+1} + 1 \).

\[ c' = W_{t_1}(\lambda_1) \oplus \cdots \oplus D(\lambda_{k+1}) \oplus \cdots \oplus W_{\lambda_j-1}(\lambda_j - 1) \oplus \cdots \oplus W_{\lambda_n}(\lambda_n), \]

\[ Y = \{ kt^{\lambda_j-1}w | t^{\lambda_j}w = 0, w \notin \text{Im } t, \alpha(t^{\lambda_j-1}w) = 0 \}, \dim Y = 2j - 2. \]

(iv) Assume \( i_0 = \lambda_{k+1} \) and \( l_k = \lambda_k \).

\[ c' = W_{t_1}(\lambda_1) \oplus \cdots \oplus W_{t_{k-1}}(\lambda_{k-1}) \oplus D(\lambda_{k+1} - 1) \oplus W_{\lambda_{k+1}-1}(\lambda_{k+1} - 1) \oplus W_{\lambda_{k+2}}(\lambda_{k+2}) \oplus \cdots \oplus W_{\lambda_n}(\lambda_n), \]

\[ Y = \{ k(t^{\lambda_{k+1}-1}w + t^{\lambda_k-1}w') | t^{\lambda_{k+1}-1}w \text{ spans } V', t^{\lambda_k}w' = 0, w' \notin \text{Im } t, \alpha(t^{\lambda_k-1}w') = \alpha(t^{\lambda_{k+1}-1}w) \}, \dim Y = 2k - 1. \]

(v) Assume \( i_0 = \lambda_{k+1} - 1 \) and \( \lambda_{k+1} - 2 \geq \lambda_{k+2} \). Let

\[ Y' = \{ kt^{\lambda_{k+1}-2}w | t^{\lambda_{k+1}-1}w = 0, w \notin \text{Im } t, \alpha(t^{\lambda_{k+1}-2}w) = 0 \}. \]

\[ c' = W_{t_1}(\lambda_1) \oplus \cdots \oplus W_{t_k}(\lambda_k) \oplus D(\lambda_{k+1} - 1) \oplus W_{\lambda_{k+2}}(\lambda_{k+2}) \oplus \cdots \oplus W_{\lambda_n}(\lambda_n), \]

\[ Y = Y' \text{ except if } \lambda_a - l_a > \lambda_k - l_k, \text{ for all } \lambda_a > \lambda_k, \text{ and } l_k = \lambda_{k+1} > \frac{\lambda_k + 1}{2}, \]

then \( Y = Y' - \{ \Sigma \in Y'| \chi_{V'}(\lambda_k) = l_k - 1 \} \) (an open dense subset in \( Y' \)),

\[ \dim Y = 2k. \]

4.5. Let \( x, c', Y, X \) be as in 4.4. Assume \( \Sigma \in Y \subset \mathbb{P}(W_{i_0}) - \mathbb{P}(W_{i_0+1}) \). Let \( X(\Sigma) \) be the set of nondegenerate submodules \( M \) of \( V \) satisfying the following conditions:

(1) \( \Sigma \subset M \) and \( M \) has no proper submodule containing \( \Sigma \).

(2) \( \chi_M(i_0) = \chi_V(i_0) \). Moreover, in case (v) of 4.4 \( \chi_M(\lambda_k) = \chi_V(\lambda_k) \).

We describe the set \( X(\Sigma) \) in the cases (i)-(v) of 4.4 in the following.

(i) Let \( \Sigma = k\nu \in Y \), where \( v = t^{\lambda_j-1}w \). There exists \( v' \in W_{\lambda_j} - W_{\lambda_{j+1}} \), such that \( \beta(v', w) \neq 0 \). Take \( w' \) such that \( v' = t^{\lambda_j-1}w' \). Then \( M = Aw \oplus Aw' \in X(\Sigma) \) and every module in \( X(\Sigma) \) is obtained in this way. It is easily seen that \( M = W_{t_j}(\lambda_j) \).

(ii) Let \( \Sigma = k\nu \in Y \), where \( v = t^{\lambda_j-1}w \). There exist \( v' = t^{\lambda_j-1}w' \in W_{\lambda_j} - W_{\lambda_{j+1}} \), such that \( \beta(v', w) \neq 0 \) and \( \alpha(t^{\lambda_j-1}w') \neq 0 \). Then \( M = Aw \oplus Aw' \in X(\Sigma) \) and every module in \( X(\Sigma) \) is obtained in this way. It is easily seen that \( M = W_{t_j}(\lambda_j) \).

(iii) Let \( \Sigma = k\nu \in Y \), where \( v = t^{\lambda_j-1}w \). There exists \( v' = t^{\lambda_j-1}w' \in W_{\lambda_j} - W_{\lambda_{j+1}} \), such that \( \beta(v', w) \neq 0 \) and \( \alpha(t^{\lambda_j-1}w') \neq 0 \). Then \( M = Aw \oplus Aw' \in X(\Sigma) \) and every module in \( X(\Sigma) \) is obtained in this way. It is easily seen that \( M = W_{t_j}(\lambda_j) \).

(iv) Let \( \Sigma = k\nu \in Y \), where \( v = t^{\lambda_{k+1}-1}w + t^{\lambda_k-1}w' \). There exists \( v_1 = t^{\lambda_{k+1}-2}w_1 \in W_{\lambda_{k+1}-1} - W_{\lambda_{k+1}} \) such that \( \beta(w, v_1) \neq 0 \) and \( v'_1 = t^{\lambda_k-1}w'_1 \in W_{\lambda_k} - W_{\lambda_{k+1}} \) such that \( \beta(v'_1, t^{\lambda_k-1}w') \neq 0 \). Then \( M = Aw \oplus Aw_1 \oplus Aw' \oplus Aw'_1 \in X(\Sigma) \) and every module in \( X(\Sigma) \) is obtained in this way. It is easy to see that \( M = W_{\lambda_k}(\lambda_k) \oplus D(\lambda_{k+1}) \).
Let \( \Sigma = kv \in Y \), where \( v = t^{k+1-2}w \). There exists \( v' = t^{k+1-1}w' \in W_{k+1} - W_{k+1+1} \) such that \( \beta(v', v) \neq 0 \). Then \( M = Av \oplus Aw' \in X(\Sigma) \) and every module in \( X(\Sigma) \) is obtained in this way. It is easy to see that \( M = D(\lambda_{k+1}) \).

4.6. Let \( M \in \Sigma(\Sigma) \) and \( M' \in \{ v \in V | \beta(v, M) = 0 \} \). Then \( M' \) is a non-degenerate submodule of \( V \). In cases (i)-(iii), we have that \( M = M + V' \). In cases (iv)-(v), we have \( V = M + M' \) and \( M \cap M' = V' \). The nondegenerate submodule \( M' \) has orthogonal decomposition \( M' = M' \oplus D(1) \), where \( M' \) is a non-defective submodule. Hence \( V = M' \oplus M \) (direct sum of orthogonal submodules). Now the map \( t : \Sigma(\Sigma) / \Sigma \to \Sigma(\Sigma) / \Sigma \) induced by \( t \) is given by the form module \( \Sigma(\Sigma) / \Sigma \) \( \oplus M' \), where \( M' \) is defined as above in cases (iv)-(v) and \( M' = M' \) in cases (i)-(iii). We write \( M' = \Sigma(\Sigma) / \Sigma \).

We explain case (ii) in some detail and the other cases are similar. In this case \( M = W_{l}(\lambda_{j}), M = W_{l-1}(\lambda_{j} - 1) \). Recall that \( \chi_{W_{l}}(\lambda_{j}) = \lambda_{j} : l_{j} \), where \( m : l : N \to N \) is defined by \( [m : l] = \min\{k-m+l \} \}. We have that \( \chi(\lambda_{i}) = \chi_{W_{l}}(\lambda_{i}) = \chi_{W_{l}}(\lambda_{i}) \) and \( \chi_{W_{l}}(\lambda_{i}) = \chi_{W_{l}}(\lambda_{i}) \). One easily check that \( \chi_{W_{l}}(\lambda_{i}) = \chi_{W_{l}}(\lambda_{i}) \) for \( i \leq j + 1 \). Hence \( c' \) is of the form as stated.

4.7. The form modules \( (\Sigma(\Sigma) / M) / \Sigma \) are described in the following.

(1) Assume \( x = W_{m}(2m), m \geq 1 \). Then \( \mathcal{P}_{x} = \mathbb{P}(\ker x) \) and \( f_{x}(\mathcal{P}_{x}) = W_{m}(2m-1) \).

(2) Assume \( x = W_{m+1}(2m+1), m \geq 1 \). Then \( \mathcal{P}_{x} = \mathbb{P}(\ker x), Y_{1} = f_{x}^{-1}(W_{m+1}(2m)) \) consists of two points and \( Y_{2} = f_{x}^{-1}(W_{m+1}(2m)) = \mathcal{P}_{x} - Y_{1} \).

(3) Assume \( x = W_{l}(m), (m+1)/2 < l < m \). Then \( \mathcal{P}_{x} = \mathbb{P}(\ker x), Y_{1} = f_{x}^{-1}(W_{l-1}(m-1)) \) consists of one point and \( Y_{2} = f_{x}^{-1}(W_{l-1}(m-1)) = \mathcal{P}_{x} - Y_{1} \).

(4) Assume \( x = W_{m}(m), m \geq 2 \). Then \( \mathcal{P}_{x} \) consists one point and \( f_{x}(\mathcal{P}_{x}) = W_{m-1}(m-1) \).

(5) Assume \( x = W_{l}(1) \). Then \( \mathcal{P}_{x} \) consists of two points and \( f_{x}(\mathcal{P}_{x}) = \{0\} \).

(6) Assume \( x = W_{m+1}(m) \oplus D(k), m \geq k \geq 1 \). Then \( \mathcal{P}_{x} = \mathbb{P}(\ker x \cap \alpha^{-1}(0)) \) and \( f_{x}^{-1}(D(m) \oplus W_{k-1}(k-1)) = \mathbb{P}(W_{k} - W_{k+1} \cap \alpha^{-1}(0)) \).

(7) Assume \( x = D(m), m \geq 2 \). Then \( \mathcal{P}_{x} \) consists of one point and \( f_{x}(\mathcal{P}_{x}) = D(m-1) \).

4.8. We prove Proposition 4.3 for \( O(2n+1) \). The proof for \( O(2n) \) is entirely similar and simpler. We use similar ideas as in [12]. We first show that \( Z_{G}(x) \) acts transitively on \( Y \). Consider \( Y^{*} = \{ (\Sigma, M) | \Sigma \in Y, M \in X(\Sigma) \} \), where \( X(\Sigma) = \{ M \in X(\Sigma) | C_{M}(\lambda_{a}) = \chi_{V}(\lambda_{a}), \forall a \neq j \) in cases (i)-(iii), \( a \neq k, k+1 \) in case (iv) and \( a \neq k+1 \) in case (v) \} is a nonempty subset in \( X(\Sigma) \). For \( M \in pr_{2}(Y^{*}) \), the equivalence classes of \( M, M' \) do not depend on the choice of \( \Sigma \in Y \) such that \( (\Sigma, M) \in Y^{*} \). It follows that \( Z_{G}(x) \) acts transitively on \( pr_{2}(Y^{*}) \).

Fix \( \Sigma \in Y \) and \( M \in X(\Sigma) \). Let \( Z_{M} \) be the stabilizer of \( M \) in \( Z_{G}(x) \). The quadratic form \( \alpha \) on \( V \) restricts to nondegenerate quadratic forms on \( M, M' \) (or \( M' \neq M' \)). Let
G(M), G(M^\perp) (or G(M')) be the groups preserving the respective quadratic forms and g(M), g(M^\perp) (or g(M')) the Lie algebras. Let x_M, x_{M^\perp} (or x_{M'}) be the restriction of x on M, M^\perp (or M') respectively. Then x_M \in g(M), x_{M^\perp} \in g(M^\perp) (or x_{M'} \in g(M')). We have that Z_M is isomorphic to Z_{G(M)}(x_M) \times Z_{G(M^\perp)}(x_{M^\perp}). Set \tilde{Y}^*_M = pr^{-1}_1(M) = \{\Sigma \in Y | M \in X(\Sigma)^*\}.

By examining the cases (1)-(7) from 4.7 we see that Z_M acts transitively on \tilde{Y}^*_M. Thus Z_G(x) acts transitively on \tilde{Y}^*_M and hence acts transitively on Y = pr_1(\tilde{Y}^*_M).

Let Z_\Sigma be the stabilizer of \Sigma in Z_G(x). The morphism A_P \to A'(x')/A'P is induced by the natural morphism Z_\Sigma/Z_\Sigma^0 \to A'(x'). Since X(\Sigma)^* is irreducible, Z_{\Sigma,M} = Z_\Sigma \cap Z_M meets all the irreducible components of Z_\Sigma. Thus to study the morphism A_P \to A'(x')/A'P, it suffices to study the natural morphism Z_{\Sigma,M}/Z_{\Sigma,M}^0 \to A'(x').

Let x_{\tilde{M}} be the endomorphism of \tilde{M} = (\Sigma^\perp \cap M)/\Sigma induced by x_M. Then x_{\tilde{M}} \in g(\tilde{M}). Let A'(x_{\tilde{M}}) = Z_{G(\tilde{M})}(x_{\tilde{M}})/Z_{G(\tilde{M})}^0(x_{\tilde{M}}). Let Z = \{z \in Z_{G(M)}(x_M)|z\Sigma = \Sigma\}. We have a natural isomorphism Z_{\Sigma,M} \cong Z \times Z_{G(M^\perp)}(x_{M^\perp}) and Z_{\Sigma,M}/Z_{\Sigma,M}^0 \cong Z/Z^0 \times Z(x_{M^\perp}). The morphism A(x_{M^\perp}) \to A'(x') is the one obtained as follows. Note that A(x_{M^\perp}) is naturally isomorphic to A(x_{M'}). The system of generators of A(x) is the union of the generators of A(x_M) and A(x_{M^\perp}) and the morphism A(x_M) \times A(x_{M^\perp}) \to A(x) is equal to the one induced by Z_{G(M)}(x_M) \times Z_{G(M^\perp)}(x_{M^\perp}) \cong Z_M \subset Z_G(x). On the other hand, we have a morphism A'(x_{\tilde{M}}) \times A(x_{M'}) \to A'(x') which comes from the isomorphism \Sigma^\perp/\Sigma \cong \tilde{M} \oplus M' and it is given by the system of generators. Hence the map A(x_{M^\perp}) \hookrightarrow Z_{\Sigma,M}/Z_{\Sigma,M}^0 \to A'(x') is given by generators. It remains to identify the morphism Z/Z^0 \to A'(x').

We can show by explicit calculation on the cases (1)-(7) in 4.7 that the natural morphism Z/Z^0 \to A(x_M) is injective and the image is generated by \{a_j|\lambda_j \neq \lambda_j, a_j \text{ belongs to the system of generators of } A(x_M) \text{ and } A'(x_{\tilde{M}})\}. Using this description of Z/Z^0 and the above description of the morphism A'(x_{\tilde{M}}) \times A(x_{M'}) \to A'(x') we see that the morphism Z/Z^0 \to A'(x') is given by the system of generators. So we have obtained a complete description of the morphism Z_{\Sigma,M}/Z_{\Sigma,M}^0 \to A'(x') and we deduce easily that A_P and the homomorphism A_P \to A'(x')/A'P are as in Proposition 4.3.

5. DUAL OF SYMPLECTIC LIE ALGEBRAS

Assume G = Sp(V).

5.1. Let \xi \in g^* be nilpotent and \alpha_\xi , T_\xi defined for \xi as in 2.7. The G-orbit c of \xi is characterized by the following data (15):
   (d1) The sizes of the Jordan blocks of T_\xi give rise to a partition of 2n. We write it as \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}, where \lambda_1 = 0.
   (d2) For each \lambda_i, there is an integer \chi(\lambda_i) satisfy \frac{\lambda_i - 1}{2} \leq \chi(\lambda_i) \leq \lambda_i. Moreover, \chi(\lambda_i) \geq \chi(\lambda_{i-1}), \lambda_i - \chi(\lambda_i) \geq \lambda_{i-1} - \chi(\lambda_{i-1}), i = 2, \ldots , 2m + 1.

   Then m(\lambda_i) is even for each \lambda_i > 0. We write \xi (or c) = (\lambda, \chi) = (\lambda_{2m+1})_0(\lambda_{2m+1})_1 \ldots (\lambda_1)_0(\lambda_1)_1.

   The component group A(\xi) = Z_G(\xi)/Z_G^0(\xi) can be described as follows (15). Let a_i corresponding to \lambda_i. Then A(\xi) is isomorphic to the abelian group generated by \{a_i|\chi(\lambda_i) \neq (\lambda_i - 1)/2\} with relations
   (r1) a_i^2 = 1,
   (r2) a_i = a_{i+1} if \chi(\lambda_i) + \chi(\lambda_{i+1}) \geq \lambda_{i+1},
   (r3) a_i = 1, if \lambda_i = 0.
5.2. Let $P$ be the stabilizer of a line $\Delta = \{kv\} \subset V$ in $G$.

**Lemma.** $\xi \in p'$ if and only if $\alpha_\xi(v) = 0$ and $T_\xi(v) = 0$.

**Proof.** $P$ is the stabilizer of the flag $\{0 \subset \{v\} \subset \{v\perp \subset V\}$. Write $v_1 = v$. There exists vectors $v_i, i = 2, \ldots, 2n$ such that $v_i, i = 1, \ldots, 2n$ span $V$ and $\beta(v, v_j) = \delta_{i+j, n}$, $i \leq j$. Let $x \in n_P$. We have $xv_1 = 0, xv_i = a_i v_i, i \neq 1, n + 1$ and $xv_n = b v_1 + \sum_{i=2}^n a_{i+n}v_i + \sum_{i=2}^n a_iv_{n+i}$. Assume $\xi(x') = \text{tr}(Xx')$ for any $x' \in g$. A easy calculation shows that $\text{tr}(Xx') = \sum_{i=2}^n a_{i+n} \beta_i(v_i, v_{n+i}) + \sum_{i=2}^n a_i \beta_i(v_i, v_i) + \alpha_\xi(v_1)$. Moreover, $T_\xi(v_1) = \sum_{j=1}^n \beta_\xi(v_1, v_{n+j})v_j + \sum_{j=1}^n \beta_\xi(v_1, v_j)v_{n+j}$.

We have $\xi \in p'$ if and only if $\xi(x) = 0$ for any $x \in n_P$ if and only if $\beta_\xi(v_1, v_i) = \beta_\xi(v_1, v_{n+i}) = 0, i = 2, \ldots, n$ and $\alpha_\xi(v_1) = 0$. Thus $\xi \in p'$ if and only if $\alpha_\xi(v_1) = 0$ and $T_\xi(v_1) = av_1$ for some $a \in k$. Since $T_\xi$ is nilpotent, $T_\xi(v_1) = av_1$ if and only if $a = 0$. The lemma is proved. \(\square\)

5.3. Assume $c' = (\lambda', \chi') \in f_\xi(P_\xi), Y = f^{-1}_\xi(c')$ and $X = \hat{\phi^{-1}}(Y)$ (see \[2.13\]).

**Proposition.** We have $\dim X = \dim B_\xi$ if and only if $(\lambda', \chi')$ satisfies:

Assume $\lambda_{i+1} = \lambda_i > \lambda_{i-1}, \lambda_i' = \lambda_j, j \neq i + 1, i, \lambda_{i+1}' = \lambda_i - 1, \lambda_i' = \lambda_j, j \neq i + 1$ and $\chi(\lambda_i') = \chi(\lambda_i + 1) \in \{\chi(\lambda_i), \chi(\lambda_i - 1)\}$ satisfies $[\lambda_i/2] \leq \chi(\lambda_i') \leq \lambda_i' - \lambda_i - 1$.

We have $\dim Y = 2m - i + 1$ if $\chi(\lambda_i') = \chi(\lambda_i)$ and $\dim Y = 2m - i$ if $\chi(\lambda_i') = \chi(\lambda_i) - 1$.

5.4. From now on let $c'$ be as in Proposition 5.3. Let $A_P$ and $A''_P$ be as in \[2.13\].

**Proposition.** The group $Z_G(\xi)$ acts transitively on $Y$. The group $A_P$ is the subgroup of $A(\xi)$ generated by the elements $a_i$ which appear both in the generators of $A(\xi)$ and of $A'(\xi)$.

The group $A''_P$ is the smallest subgroup of $A'(\xi')$ such that the map $A_P \to A'(\xi')/A''_P$ given by $a_i \mapsto a_i$ is a morphism.

**Corollary.** $Y$ has two irreducible components (and $|A(\xi)/A_P| = 2$) if $c'$ is as in Proposition with $\chi(\lambda_i) = \frac{\lambda_i}{2}, \lambda_i + 2 - \chi(\lambda_i+2) > \lambda_i/2$ and $\chi'(\lambda_i') = \chi(\lambda_i) - 1$.

In this case, suppose $D = \{a_i\} = \{a_i, a_{i+1}\} \subset A(\xi)$, then $A(\xi) = D \times A_P$. In the other cases, $Y$ is irreducible and $A_P = A(\xi)$.

**Corollary.** The group $A''_P$ is trivial, except if $c'$ is as in Proposition with $\chi(\lambda_i) = \frac{\lambda_i}{2}, \chi(\lambda_i+2) + \chi(\lambda_i) = \lambda_i + 2$ and $\chi'(\lambda_i') = \chi(\lambda_i) - 1$.

In this case, we have $A''_P = \{1, a_{i+1}'a_{i+2}'\} \subset A'(\xi')$.

5.5. Propositions 5.3 and 5.4 are proved entirely similarly as in the orthogonal Lie algebra case. We describe the orbits $c'$ and the varieties $Y$. The details are omitted. Assume $\xi$ corresponds to the form module $V = *W_1(\lambda_1) \oplus \cdots \oplus *W_s(\lambda_s)$, where $l_i = \chi(\lambda_i)$. (Notations are as in \[15\].)

We regard $V$ as an $A = k[t]$-module by $\sum a_i t^i v = \sum a_i T_\xi^i v$. By Lemma 5.2 we can identify $P_\xi$ with $P(W)$, where $W = \{v \in \ker t|\alpha_\xi(v) = 0\}$. Let $\Sigma = kv \subset Y$ and $\Sigma^\perp = \{v' \in V|\beta(v', \Sigma) = 0\}$. The quadratic form $\alpha_\xi$ induces a quadratic form $\alpha_\xi : \Sigma^\perp/\Sigma \to \Sigma^\perp/\Sigma$ and $T_\xi$ induces a linear map $T_\xi : \Sigma^\perp/\Sigma \to \Sigma^\perp/\Sigma$. Then $\alpha_\xi$ defines an element $\xi' \in \frak{sp}(\Sigma^\perp/\Sigma)^* = t^*$. Moreover, $\xi' \in c', \alpha_\xi = \alpha_\xi$ and $T_\xi = T_\xi$. We have the following cases.

(i) Assume $1 \leq j \leq s, \lambda_j - 1 \geq \lambda_j + 1$ and $\lambda_j - \lambda_j - 1 \geq \lambda_j + 1 - \lambda_j + 1$.

$$c' = *W_1(\lambda_1) \oplus \cdots \oplus *W_s(\lambda_j - 1) \oplus \cdots \oplus *W_{s}(\lambda_s),$$

$$Y = \{kt^{j-1}w|t^jw = 0, w \notin \text{Im } t, \alpha_\xi(t^{j-1}w) \neq 0\}, \text{ dim } Y = 2j - 1.$$
similar argument as in [15, Lemma 3.11] shows that 
\[ T_n \leq \text{calculation shows that } \text{tr}(\xi_n) = 0, w \notin \text{Im } t, \alpha_\xi(t^jw) = 0. \]

\[ c' = ^*W_{t_1}(\lambda_1) \oplus \cdots \oplus ^*W_{t_s}(\lambda_s), \]

\[ Y = Y' \text{ except if } \lambda_a - l_a > \lambda_{j-1} - l_{j-1}, \text{ for all } \lambda_a > \lambda_{j-1}, \text{ and } l_{j-1} = l_j > \frac{\lambda_{j-1}}{2}, \]

then \[ Y = Y' - \{ \Sigma \in Y' | \chi_{V'}(\lambda_{j-1}) = l_{j-1} - 1 \} \text{ (an open dense subset in } Y'), \]

\[ \dim Y = 2j - 2. \]

6. DUAL OF ODD ORTHOGONAL LIE ALGEBRAS

Let \( G = O(2n + 1) \).

6.1. Let \( \xi \in \mathfrak{g}^* \) be nilpotent. Let \( V = V_{2m+1} \oplus W \) be a normal form of \( \xi, \beta_\xi \) and \( T_\xi : W \to W \) defined for \( \xi \) as in [23, §]. The orbit \( c \) of \( \xi \) is characterized by the following data (15):

(d1) An integer \( 0 \leq m \leq n \).

(d2) The sizes of the Jordan blocks of \( T_\xi \) give rise to a partition of \( 2n - 2m \). We write it as \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2s} \).

(d3) For each \( \lambda_i \), there is an integer \( \chi(\lambda_i) \) satisfy \( \frac{\lambda_i}{2} \leq \chi(\lambda_i) \leq \lambda_i \). Moreover, \( \chi(\lambda_i) \geq \chi(\lambda_{i-1}), \lambda_i - \chi(\lambda_i) \geq \lambda_{i-1} - \chi(\lambda_{i-1}), i = 2, \ldots, 2s \).

(d4) \( m \geq \lambda_{2s} - \chi(\lambda_{2s}) \).

Then \( m(\lambda_i) \) is even for each \( \lambda_i > 0 \). We write \( \xi \) (or \( c \)) = \( (m; \lambda, \chi) = (m; \lambda_{2s}, \chi) \cdots (\lambda_1, \lambda_1) \).

The component group \( A(\xi) = Z_G(\xi)/Z^0_G(\xi) \) can be described as follows (15). Let \( a_i \) correspond to \( \lambda_i, i = 1, \ldots, 2s \). Then \( A(\xi) \) is isomorphic to the abelian group generated by \( \{ a_i | \chi(\lambda_i) \neq \lambda_i / 2 \} \) with relations

(r1) \( a_i^2 = 1 \),

(r2) \( a_i = a_{i+1} \) if \( \chi(\lambda_i) + \chi(\lambda_{i+1}) > \lambda_{i+1} \),

(r3) \( a_{2s} = 1 \) if \( \chi(\lambda_{2s}) \geq m \).

6.2. Let \( P \) be the stabilizer of a line \( \Sigma = \{ kv \} \subset V \) in \( G \), where \( \alpha(v) = 0 \).

**Lemma.** \( \xi \in \mathfrak{p}' \) if and only if \( \beta_\xi(v, v') = 0 \) for any \( v' \in V \).

**Proof.** \( P \) is the stabilizer of the flag \( \{ 0 \subset \{ v \} \subset \{ v \}^\perp \subset V \} \). Write \( v_1 = v \). There exist vectors \( v_i, i = 2, \ldots, 2n + 1 \) such that \( v_i, i = 1, \ldots, 2n + 1 \) span \( V \) and \( \beta(v_i, v_j) = \delta_{j,i+n}, 1 \leq i < j \leq 2n, \beta(v_i, v_{2n+1}) = 0, i = 1, \ldots, 2n + 1, \alpha(v_i) = 0, i = 1, \ldots, 2n, \alpha(v_{2n+1}) = 1 \).

Let \( x \in \mathfrak{n}_P \). We have \( xv_1 = 0, xv_i = a_iv_1, i \neq 1, n + 1, 2n + 1 \) and \( xv_{n+1} = \sum_{i=2}^{n} a_{n+i}v_i + \sum_{i=2}^{n} a_{n+i}v_{n+i} + bv_{n+1}, xv_{n+1} = 0 \). Assume \( \xi(x') = \text{tr}(Xx') \) for any \( x' \in \mathfrak{g} \). A straightforward calculation shows that \( \text{tr}(Xx) = \sum_{i=2}^{n} a_{n+i}\beta(v_1, v_{n+i}) + \sum_{i=2}^{n} a_{n+i}\beta(v_1, v_{n+i}) + b\beta_x(v_1, v_{n+1}) \).

Thus if \( \xi \in \mathfrak{p}' \) then \( \beta_\xi(v_1, v_i) = 0, i \neq n + 1 \).

Now let \( W \) be the subspace of \( V \) spanned by \( v_i, i = 1, \ldots, 2n \). Then \( \beta \) is nondegenerate on \( W \). We define a map \( T : W \to W \) by \( \beta(Tw, w) = \beta_\xi(w, w') \), \( w, w' \in W \). Then similar argument as in [15, Lemma 3.11] shows that \( T \) is nilpotent. One easily shows that \( Tv_1 = \sum_{j=1}^{n} \beta_\xi(v_1, v_{n+j})v_j + \sum_{j=1}^{n} \beta_\xi(v_1, v_j)v_{n+j} \). It follows that \( Tv_1 = \beta_\xi(v_1, v_{n+1})v_1 \) and thus \( \beta_\xi(v_1, v_{n+1}) = 0 \). The lemma follows. \( \square \)
6.3. Let \( c' = (m'; \chi, \chi') \in f_\xi(P_\xi) \), \( Y = f_\xi^{-1}(c') \) and \( X = \rho_\xi^{-1}(Y) \) (see 2.13).

**Proposition.** We have \( \dim X = \dim B_\xi \) if and only if \((\lambda', \chi')\) and \(m'\) satisfy (a) or (b):

(a) Assume \( m - 1 \geq \lambda_2 - \chi(\lambda_2) \). \( m' = m - 1, \lambda'_i = \lambda_i \) and \( \chi'(\lambda'_i) = \chi(\lambda_i) \), \( i = 1, \ldots, 2s \).

We have \( \dim Y = 0 \);

(b) Assume that \( \lambda_{i+1} = \lambda_i < \lambda_{i-1} \). \( m' = m, \lambda'_j = \lambda_j, j \neq i + 1, i, \lambda'_{i+1} = \lambda_{i+1} - 1, \lambda'_i = \lambda_i - 1, \chi'(\lambda'_i) = \chi(\lambda_i), j \neq i, i + 1 \) and \( \chi'(\lambda'_i) = \chi(\lambda_i) - 1 \) satisfies \( \lambda'_i/2 \leq \chi'(\lambda'_i) \leq \lambda'_i \), \( \lambda_{i-1} \leq \chi'(\lambda'_i) \leq \chi(\lambda_{i-1}) + \lambda_i - \lambda_{i-1} - 1 \). We have \( \dim Y = 2s - i + 1 \) if \( \chi'(\lambda'_i) = \chi(\lambda_i) \) and \( \dim Y = 2s - i \) if \( \chi'(\lambda'_i) = \chi(\lambda_i) - 1 \).

6.4. From now on let \( c' \) be as in Proposition 6.3. Let \( A_P \) and \( A'_P \) be defined as in 2.13.

**Proposition.** The group \( Z_G(\xi) \) acts transitively on \( Y \). The group \( A_P \) is the subgroup of \( A(\xi) \) generated by the elements \( a_i \) which appear both in the generators of \( A(\xi) \) and of \( A'(\xi') \). The group \( A'_P \) is the smallest subgroup of \( A'(\xi') \) such that the map \( A_P \to A'(\xi')/A'_P \) given by \( a_i \mapsto a'_i \) is a morphism.

**Corollary.** The variety \( Y \) has two irreducible components (and \(|A(\xi)/A_P| = 2\)) if \( c' \) is as in Proposition 6.3 (b) with \( \chi(\lambda_i) = \frac{\lambda_i + 1}{2}, \chi'(\lambda'_i) = \chi(\lambda_i) - 1, \) and \( \lambda_{i+2} - \chi(\lambda_{i+2}) \geq (\lambda_i + 1)/2 \) if \( i < 2s - 1, m \geq (\lambda_i + 1)/2 \) if \( i = 2s - 1 \).

In this case, suppose \( D = \{1, a_i\} \subseteq A(\xi), \) then \( A(\xi) = D \times A_P \). In the other cases, \( Y \) is irreducible and \( A' = A(\xi) \).

**Corollary.** The group \( A'_P \) is trivial, except if \( c' \) is as in Proposition 6.3 (b) with \( \chi(\lambda_i) \neq \frac{\lambda_i + 1}{2}, \chi'(\lambda'_i) = \chi(\lambda_i) - 1, \) and \( \lambda_{i+2} + \chi(\lambda_{i+2}) = \lambda_{i+2} + 1 \) if \( i < 2s - 1, \chi(\lambda_i) = m + 1 \) if \( i = 2s - 1 \). We have \( A'_P = \{1, a'_1, a'_2\} \subseteq A'(\xi') \) if \( i < 2s - 1 \) and \( A'_P = \{1, a'_2\} \subseteq A'(\xi') \) if \( i = 2s - 1 \).

6.5. Write \( \xi = V_{2m+1} \oplus W, \) where \( W = W_{t_1}(\lambda_1) \oplus \cdots \oplus W_{t_m}(\lambda_m) \), \( l_i = \chi(\lambda_i), \lambda_i \leq \lambda_{i+1} \) (notation as in [13]). Let \( v_i, i = 0, \ldots, m \) be as in 2.8. We view \( W \) as a \( k[t] \) module by \( \sum a_i t^{w} = \sum a_i T^{w} \). It follows from Lemma 0.2 that \( P_\xi \) is identified with \( \mathbb{P}((k[t_0] \oplus \ker t) \cap \alpha^{-1}(0)) \). Let \( \Sigma \subseteq Y \) and \( \Sigma^\perp = \{v' \in V | \beta'(v', \Sigma) = 0\} \). The bilinear form \( \beta_\xi \) induces a bilinear form \( \tilde{\beta}_\xi \) on \( \Sigma^\perp / \Sigma \). Then \( \tilde{\beta}_\xi \) defines an element \( \xi' \in \mathfrak{o}(\Sigma^\perp / \Sigma)^* \cong t^* \). We have that \( \xi' \in c' \) and \( \beta_{\xi'} = \beta_\xi \). The variety \( Y \) is described in the following.

(i) Assume \( m \geq 1 \) and \( m - 1 \geq \lambda_1 - l_1 \).

\[
\xi' = V_{2m+1} \oplus W_{t_1}(\lambda_1) \oplus \cdots \oplus W_{t_m}(\lambda_m), \quad Y = \{kv | v = aw_0 + t^{\lambda_j - 1}w, w \in W, t^{\lambda_j}w = 0, w \notin tW, \alpha(t^{\lambda_j-1}w) \neq t^{\lambda_j-1}w \}, \quad \dim Y = 2j.
\]

(ii) Assume \( \lambda_j - l_j - 1 \geq j \geq j_1 + 1, \lambda_j \geq j_1 + 1 \). Then \( m \geq 1 \).

\[
\xi' = V_{2m+1} \oplus W_{t_1}(\lambda_1) \oplus \cdots \oplus W_{t_m}(\lambda_m), \quad Y = \{kv | v = aw_0 + t^{\lambda_j - 1}w, w \in W, t^{\lambda_j}w = 0, w \notin tW, \alpha(t^{\lambda_j-1}w) \neq t^{\lambda_j-1}w \}, \quad \dim Y = 2j.
\]

(iii) Assume \( l_j - 1 \geq l_j + 1, l_j \geq \lfloor \lambda_j/2 \rfloor + 1, \lambda_j \geq j_1 + 1 \).

\[
\xi' = V_{2m+1} \oplus W_{t_1}(\lambda_1) \oplus \cdots \oplus W_{t_m}(\lambda_m), \quad Y \subseteq \tilde{Y}' = \{kv | v = aw_0 + t^{\lambda_j - 1}w, w \in W, t^{\lambda_j}w = 0, w \notin tW, \alpha(t^{\lambda_j-1}w) \neq t^{\lambda_j-1}w \}, \quad \dim Y = 2j-1.
\]
6.6. Case (i) is clear. We explain case (iii) in details. Case (ii) is similar.

Let $\Sigma = kv \in Y$, where $v = av_0 + t^{\lambda_1-1}w$. Let $u_i \in V_{2m+1}$, $i = 0, \ldots, m - 1$ be as in 2.8. Assume $a \neq 0$. There exists $w_0 \in W$ such that $\beta(w_0, t^{\lambda_1-1}w) = 1$ and $\alpha(w_0) = 0$. Let $u_0 = aw_0 + u_0$ and we define $\tilde{V}_{2m+1}, \tilde{W}$ as in 2.8. Then $V = \tilde{V}_{2m+1} \oplus \tilde{W}$ (see 6.6). Let $(\alpha, \lambda_i)$ be the Lie algebras. The bilinear form $\beta(t^{\lambda_1-1}(\tilde{w}), w) = \sum_{i=0}^{m} \beta(aw_0, t^{\lambda_1}w_i) \in \tilde{W}$ and $\alpha(t^{\lambda_1-1}(\tilde{w})) = \alpha(t^{\lambda_1-1}(\tilde{w})) + a^2 \delta_{\lambda_1, \lambda_j - l_j}$.

Now we can assume $V = V_{2m+1} \oplus W$ is a normal form of $\xi$, with $\Sigma = kv \subset W$ and $v = t^{\lambda_1-1}(\tilde{w})$, $w \in W$. Then $\Sigma / \Sigma = V_{2m+1} \oplus (\Sigma \cap W) / \Sigma$. We apply the results for orthogonal Lie algebras to $(\Sigma \cap W) / \Sigma$ (see 4.4). Write $W = (\Sigma \cap W) / \Sigma$. The set $Y = Y'$, except if $l_j - l_j > \frac{\lambda_{j-1} - 1}{2}$, $m > \lambda_{j-1} - l_{j-1}$ and for all $\lambda_a > \lambda_{j-1}$, $\lambda_a - l_a > \lambda_j - l_j$, then $Y$ consists of those $v$ such that $\chi_{W'}(\lambda_{j-1}) = l_{j-1}$.

6.7. We prove Proposition 6.4. In case (i), we have $L = \{kv\} \subset V_{2m+1}$. For any $g \in Z_G(\xi)$, we have that $gv_0 = v_0$. Hence $H = Z_P(\xi) = Z_G(\xi)$, $K = Z_P(\xi) \cap Z_G(\xi) = Z_G^0(\xi)$ and $A_P = A(\xi), A'_P = 1$.

In cases (ii) and (iii), we can find a normal form $V = V_{2m+1} \oplus W$ such that $\Sigma \subset W$ (see 6.6). Let $X(\Sigma)$ be the set of all such $W$. We first show that $Z_G(\xi)$ acts transitively on $Y$. Let $\tilde{Y} = \{(\Sigma, W) | \Sigma \in Y, W \in X(\Sigma)\}$. Then $Z_G(\xi)$ acts transitively on $pr_2(\tilde{Y})$. Set $\tilde{Y}_W = pr_2^{-1}(W) = \{(\Sigma) \in Y | W \in X(\Sigma)\}$. It follows from the results in the orthogonal Lie algebra case that $Z_W$ acts transitively on $\tilde{Y}_W$ (see Proposition 4.3). Then $Z_G(\xi)$ acts transitively on $\tilde{Y}$ and hence acts transitively on $Y = pr_1(\tilde{Y})$.

Fix $\Sigma \in Y$ and $W \in X(\Sigma)$. Let $Z_W$ and $Z_{\Sigma}$ be the stabilizer of $W$ and $\Sigma$ in $Z_G(\xi)$ respectively. The morphism $A_P \rightarrow A'(\xi')/A'_P$ is induced by the natural morphism $Z_{\Sigma} / Z_{\Sigma}^0 \rightarrow A'(\xi')$. Since $X(\Sigma)$ is irreducible, $Z_{\Sigma} = Z_{\Sigma} \cap Z_W$ meets all the irreducible components of $Z_{\Sigma}$. Thus to study the morphism $A_P \rightarrow A'(\xi')/A'_P$, it suffices to study the natural morphism $Z_{\Sigma} / Z_{\Sigma}^0 \rightarrow A'(\xi')$.

The quadratic form $\alpha$ on $V$ restricts to nondegenerate quadratic forms on $W$ and $W^\perp$. Let $G(W)$, $G(W^\perp)$ be the groups preserving the respective quadratic forms and $g(W), g(W^\perp)$ the Lie algebras. The bilinear form $\beta_\Sigma$ on $V$ restricts to bilinear forms on $W$ and $W^\perp$. Let $\xi_W$ and $\xi_{W^\perp}$ be the corresponding elements in $g(W)^* \times g(W^\perp)^*$ respectively. Moreover, the bilinear form $\beta_\Sigma$ induces a bilinear form on $\tilde{W} = (\Sigma \cap W) / \Sigma$. Let $\xi_W$ be the corresponding element in $g(W)$ and $A'(\xi_W) = Z_G(\xi_W) / Z_G(\xi_W^0)$.

Let $Z = \{z \in Z_G(W)(\xi_W) | z\Sigma = \Sigma\}$. Since $Z_W \cong Z_G(W)(\xi_W) \times Z_G(W^\perp)(\xi_{W^\perp})$, we have natural isomorphisms $Z_{\Sigma} \cong Z \times Z_G(W^\perp)(\xi_{W^\perp})$ and $Z_{\Sigma} / Z_{\Sigma}^0 \cong Z / Z_0 \times A(\xi_{W^\perp})$. Note that $A(\xi_{W^\perp}) = \{1\}$. On the other hand, we have a morphism $A'(\xi_W) \times A(\xi_{W^\perp}) \rightarrow A'(\xi')$ which comes from the isomorphism $\Sigma \cap W \cong \tilde{W} \oplus W^\perp$ and it is given by the system of generators. It follows form the results for orthogonal Lie algebras that the morphism $Z / Z_0 \rightarrow A'(\xi_W)$ is given by generators (see Proposition 4.3). We then deduce easily that $A'_P$ and the morphism $A_P \rightarrow A'(\xi')/A'_P$ are as in Proposition 6.4.

7. Some combinatorics

In this section we recall some combinatorics from [3, 9]. The combinatorics goes back to [3], where it is used to parametrize unipotent representations of classical groups. We will use
the same kind of combinatorial objects to describe the Springer correspondence for classical Lie algebras and their duals in characteristic 2.

7.1. Let \( r, s, n \in \mathbb{N} = \{0, 1, 2, \ldots\} \), \( d \in \mathbb{Z} \), \( e = \lfloor \frac{d}{2} \rfloor \in \mathbb{Z} \) ([—] means the integer part). Let \( X_{n,d}^{r,s} \) be the set of all ordered pairs \((A, B)\) of finite sequences of natural integers \( A = (a_1, a_2, \ldots, a_{m+d}) \) and \( B = (b_1, b_2, \ldots, b_m) \) (for some \( m \)) satisfying the following conditions:

- \( a_{i+1} - a_i \geq r + s, \; i = 1, \ldots, m + d - 1 \)
- \( b_{i+1} - b_i \geq r + s, \; i = 1, \ldots, m - 1 \)
- \( b_1 \geq s \)
- \( \sum a_i + \sum b_i = n + r(m + e)(m + d - e - 1) + s(m + e)(m + d - e) \).

The set \( X_{n,d}^{r,s} \) is equipped with a shift \( \sigma_{r,s} \). If \((A, B)\) is as above, then \( \sigma_{r,s}(A, B) = (A', B') \),

- \( A' = (0, a_1 + r + s, \ldots, a_{m+d} + r + s) \), \( B' = (s, b_1 + r + s, \ldots, b_m + r + s) \).

Let \( X_{n,d}^{r,s} \) be the quotient of \( X_{n,d}^{r,s} \) by the equivalence relation generated by the shift and

\[
X_n = \bigcup_{d \text{ odd}} X_{n,d}^{r,s}.
\]

The equivalence class of \((A, B)\) is still denoted by \((A, B)\).

Assume \( s = 0 \). Then there is an obvious bijection \( X_{n,d}^{r,0} \to X_{n,-d}^{r,0} \), \((A, B) \mapsto (B, A)\). This induces an involution on each of the following sets:

\[
X_{n,\text{even}} = \bigcup_{d \text{ even}} X_{n,d}^{r,0} \quad X_{n,\text{odd}} = \bigcup_{d \text{ odd}} X_{n,d}^{r,0}.
\]

Let \( Y_{n,\text{even}} \) (resp. \( Y_{n,\text{odd}} \)) be the quotient of \( X_{n,\text{even}} \) (resp. \( X_{n,\text{odd}} \)) by this involution. For \( d \geq 0 \), the image of \( X_{n,d}^{r,0} \) in \( Y_{n,\text{even}} \) or \( Y_{n,\text{odd}} \) is denoted \( Y_{n,d}^{r} \) and the image of \((A, B)\) is denoted \( \{A, B\} \).

7.2. When we consider simultaneously two elements \((A, B) \in X_{n,d}^{r,s} \) and \((A', B') \in X_{n',d'}^{r',s'}\) with \( d - d' \) even, with \( A = (a_1, \ldots, a_{m+d}) \), \( B = (b_1, \ldots, b_m) \) and \( A' = (a_1', \ldots, a_{m'+d'}') \), \( B' = (b_1', \ldots, b_{m'}') \), we always assume that we have chosen representatives such that \( 2m + d = 2m' + d' \). We use the same convention for \( \{A, B\} \in Y_{n,d}^{r} \) and \( \{A', B'\} \in Y_{n',d'}^{r'} \) with \( d, d' \geq 0 \) and \( d - d' \) even.

There is an obvious addition

\[
X_{n,d}^{r,s} \times X_{n',d'}^{r',s'} \to X_{n+n',d+d'}^{r+r',s+s'}, (A, B) + (A', B') = (A'', B''), a''_i = a_i + a'_i, b''_i = b_i + b'_i.
\]

The same formula defines \( Y_{n,d}^{r} \times Y_{n',d'}^{r'} \rightarrow Y_{n+n',d+d'}^{r+r'} \).

Let \( \Lambda_{0,1}^{r,s} \) (resp. \( \Lambda_{0,0}^{r,0} \)) be the element represented by \((A, B) = (0, 0)\) (resp. \((A, B) = (0, 0)\)). If \( s = 0 \), let \( \Lambda_{r,1}^{r} \) (resp. \( \Lambda_{r,0}^{r} \)) be the image of \( \Lambda_{0,1}^{r,s} \) (resp. \( \Lambda_{0,0}^{r,0} \)). We have the following bijective maps:

\[
X_{n,1}^{0} \to \Lambda^{r,s}_{n,1}, \; \Lambda \mapsto \Lambda + \Lambda_{0,1}^{r,s}, \quad Y_{n,d}^{0} \to Y_{n,d}^{r}, \; \Lambda \mapsto \Lambda + \Lambda_{0,d}^{r}.
\]

Since \( Y_{n,d}^{0}, d \geq 1, \) and \( X_{n,d}^{0} \) are obviously in bijection with the set of all pairs of partitions \((\mu, \nu)\) such that \( \sum \mu_i + \sum \nu_i = n \) and thus with \( W_n^{\Lambda} \), \( Y_{n,0}^{0} \) is in bijection with the set of all
unordered pairs of partitions \( \{\mu, \nu\} \) such that \( \sum \mu_i + \sum \nu_i = n \) and thus with \( W_n^{\land} \), we get bijections
\[
W_n^\land \to X_{r,s}^{n,1}, \quad W_n^\land \to Y_{r,n,1}^r, \quad W_n^\land \to Y_{r,n,0}^r.
\]

7.3. An element \((A, B) \in X_{r,s}^{n,d}\) is called distinguished if \( d = 0 \), \( a_1 \leq b_1 \leq a_2 \leq \cdots \leq a_m \leq b_m \) or if \( d = 1 \), \( a_1 \leq b_1 \leq a_2 \leq \cdots \leq a_m \leq b_m \leq a_{m+1} \). An element \( \{A, B\} \in Y_{r,n,d}^r \) \((d \geq 0)\) is called distinguished if \((A, B)\) or \((B, A)\) is distinguished. Let \( D_{r,s}^{n,d}, D_{r,n,even}^r, D_{r,n,odd}^r, D_{r,s}^{n,d}, D_{r,d}^r \) be the set of all distinguished elements in \( X_{r,s}^{n,d}, Y_{r,n,even}^r, Y_{r,n,odd}^r, X_{r,d}^{n,s}, Y_{r,n,d}^r \) respectively.

Assume \( r \geq 1 \). For \((A, B) \in \bar{X}_{r,s}^{n,d}\), we regard \( A, B \) as subsets of \( \mathbb{N} \). Two elements \((A, B), (C, D) \in X_{r,s}^{n,d}\) are said to be similar if \( A \cup B = C \cup D \) and \( A \cap B = C \cap D \). We define similarity in \( Y_{r,n,even}^r \) and \( Y_{r,n,odd}^r \) in the same way.

Let \( S = (A \cup B) \setminus (A \cap B) \). A nonempty subset \( I \) of \( S \) is called an interval of \((A, B)\) or \( \{A, B\}\) if it satisfies the following conditions:

(i) if \( i < j \) are consecutive elements of \( I \), then \( j - i < r + s \);
(ii) if \( i \in I, j \in S \) and \(|i - j| < r + s\), then \( j \in I \).

We call \( I \) an initial interval if there exists \( i \in I \) such that \( i < s \) and a proper interval otherwise.

Let \( S \subset X_{r,s}^n \) (resp. \( Y_{r,n,odd}^r \) or \( Y_{r,n,even}^r \)) be a similarity class and \((A, B)\) (resp. \( \{A, B\}\)) \( \in S \). Let \( E \) be the set of all proper intervals of \((A, B)\) (resp. \( \{A, B\}\)). The set \( \mathcal{A}(E) \) of all subsets of \( E \) is a vector space over \( \mathbb{F}_2 \). If \( S \subset X_{r,s}^n \), it acts simply transitively on \( S \) as follows. The image of \((A, B)\) under \( F \subset E \) is the pair \((C, D)\) such that
\[
A \cap I = D \cap I, \quad B \cap I = C \cap I \quad \text{if and only if} \quad I \in F.
\]

If \( S \subset Y_{r,n,odd}^r \) (resp. \( Y_{r,n,even}^r \)), as \( E \) transforms \((A, B)\) to \((B, A)\), the same formula defines a simply transitive action of \( \mathcal{A}(E)/\{\emptyset, E\} \) on \( S \).

For \( \Lambda \in X_{r,s}^n \) (resp. \( Y_{r,n,odd}^r \) or \( Y_{r,n,even}^r \)), let \( V_{r,s}^{\Lambda} \) (resp. \( V_{r}^{\Lambda} \)) denote the vector space \( \mathcal{A}(E) \) (resp. \( \mathcal{A}(E)/\{\emptyset, E\} \)), where \( E \) is the set of all proper intervals of \( \Lambda \). For \( F \in V_{r,s}^{\Lambda} \) (resp. \( V_{r}^{\Lambda} \)), let \( \Lambda_F \) be the image of \( \Lambda \) under the action of \( F \).

7.4. Examples. (1) \( X_{n,1}^1 \) and \( Y_{n,0}^1 \) are used in [3] to describe \( W_n^\land \) and \( W_n^{\land} \) respectively.

(2) Assume \( \text{char}(k) \neq 2 \). \( X_{n,1}^1 \), \( Y_{n,even}^2 \) and \( Y_{n,odd}^2 \) are used in [2] to describe the generalized Springer correspondence for \( Sp_{2n}, SO(2n) \) and \( SO(2n + 1) \) respectively.

(3) \( X_{n,2}^2 \) and \( Y_{n,even}^4 \) are used in [9] to describe the generalized Springer correspondence for unipotent classes of \( Sp(2n) \) (or \( SO(2n + 1) \)) and \( SO(2n) \) respectively.

(4) \( Y_{n,odd}^{4} \) and \( X_{n,even}^3 \) and \( X_{n,even}^3 = \cup_{d \text{ even}} X_{n,d}^{3,1} \) are used in [7] to describe the generalized Springer correspondence for disconnected groups \( O(2n), G_{2n+1} \) with \( G^0 \) type \( A_{2n} \), and \( G_{2n} \) with \( G^0 \) type \( A_{2n-1} \) respectively.

(5) We will use \( X_{n,1}^{n+1} \), \( X_{n,1,n+1}^{n+1} \) and \( Y_{n,1}^{n+1} \) to describe the Springer correspondence for \( \mathfrak{o}(2n+1), \mathfrak{sp}(2n) \) and \( \mathfrak{o}(2n) \) (or \( \mathfrak{o}(2n)^* \)). The set \( D_{n+1}^{2,n+1} \) (resp. \( D_{n+1,even}^{n+1}, D_{n,even}^{n+1} \)) is in bijection with the set of \( O(2n+1) \) (resp. \( Sp(2n), O(2n) \))-nilpotent orbits in \( \mathfrak{o}(2n+1) \) (resp. \( \mathfrak{sp}(2n), \mathfrak{o}(2n)^* \)).

(6) We will use \( Y_{n,odd}^{n+1} \) and \( X_{n,1,n+1}^{n+1} \) to describe the Springer correspondence for \( \mathfrak{o}(2n+1)^* \) and \( \mathfrak{sp}(2n)^* \). The set \( D_{n+1,odd}^{n+1} \) (resp. \( D_{n+1,even}^{n+1} \)) is in bijection with the set of \( O(2n+1) \) (resp. \( Sp(2n) \))-nilpotent orbits in \( \mathfrak{o}(2n+1)^* \) (resp. \( \mathfrak{sp}(2n)^* \)).
8. Springer correspondence for symplectic Lie algebras

Assume \( G = Sp(2n) \).

8.1. Let \( x \in \mathfrak{g} \) be nilpotent. The orbit \( c \) of \( x \) is characterized by the following data \([\mathfrak{2}]\):

(i) The sizes of the Jordan blocks of \( x \) give rise to a partition of \( 2n \). We write it as \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2m+1} \), where \( \lambda_1 = 0 \).

(ii) For each \( \lambda_i \), there is an integer \( \chi(\lambda_i) \) satisfy \( 0 \leq \chi(\lambda_i) \leq \frac{1}{2} n \). Moreover, \( \chi(\lambda_i) \geq \chi(\lambda_{i-1}) \), \( \lambda_i - \chi(\lambda_i) \geq \lambda_{i-1} - \chi(\lambda_{i-1}) \), \( i = 2, \ldots, 2m + 1 \).

We can partition the set \( \{1, 2, \ldots, 2m + 1\} \) in a unique way into blocks of length 1 or 2 such that the following holds:

(b1) If \( \chi(\lambda_i) = \lambda_i/2 \), then \( \{i\} \) is one block;

(b2) All other blocks consist of two consecutive integers.

Note that if \( \{i, i + 1\} \) is a block, then \( \lambda_i = \lambda_{i+1} \) and \( \chi(\lambda_i) = \chi(\lambda_{i+1}) \).

We attach to the orbit \( c \) the sequence \( c_1, \ldots, c_{2m+1} \) defined as follows:

1. If \( \{i\} \) is a block, then \( c_i = \lambda_i/2 + (n + 1)(i - 1) \);
2. If \( \{i, i + 1\} \) is a block, then \( c_i = \lambda_i - \chi(\lambda_i) + (n + 1)(i - 1) \), \( c_{i+1} = \chi(\lambda_{i+1}) + (n + 1)i \).

Taking \( a_i = c_{2i-1}, i = 1, \ldots, m + 1, b_i = c_{2i}, i = 1, \ldots, m \), we get a well defined element \( (A, B) \in X^{n+1, n+1}_{n,1} \). We denote it \( \rho_G(x) \), \( \rho(x) \) or \( \rho(c) \).

Lemma. (i) \( c \mapsto \rho(c) \) defines a bijection from the set of all nilpotent \( Sp(2n) \)-orbits in \( \mathfrak{sp}(2n) \) to \( D_n^{n+1, n+1} \).

(ii) \( A_G(x)^\wedge \) is isomorphic to \( V_{\rho(x)}^{n+1, n+1} \).

Proof. (i) It is easily checked from the definition that \( \rho(c) \in D_n^{n+1, n+1} \) and the map \( c \mapsto \rho(c) \) is injective. Note that \( X^{n+1, n+1}_{n,1} = D_n^{n+1, n+1} \) is in bijection with \( W_n^\wedge \) and the number of nilpotent orbits is equal to \( |W_n^\wedge| \) by Spaltenstein \([\mathfrak{1}]\). Hence the bijectivity of the map follows. In fact, given \( (A, B) \in D_n^{n+1, n+1} \), the corresponding nilpotent orbit can be obtained as follows.

Let \( c_1 \leq c_2 \leq \cdots \leq c_{2m+1} \) be the sequence \( a_1 \leq b_1 \leq \cdots \leq a_{m+1} \). If \( c_{i+1} < c_i + (n + 1) \), then \( \{i, i + 1\} \) is a block. We can recover \( \lambda_i = \lambda_{i+1} \) and \( \chi(\lambda_i) = \chi(\lambda_{i+1}) \) from (2) of the definition. All blocks of length 2 are obtained in this way. For the other blocks, we can recover \( \lambda_i \) and thus \( \chi(\lambda_i) = \lambda_i/2 \) from (1) of the definition.

(ii) One easily checks that \( (A, B) \) has no proper intervals. It follows that \( V_{\rho(x)}^{n+1, n+1} = \{0\} \). On the other hand, \( A(x) = 1 \) since \( Z_G(x) \) is connected by Spaltenstein \([\mathfrak{1}]\). \( \square \)

8.2. Consider a pair \( (x, \phi) \in \mathfrak{A}_\mathfrak{g} \), then \( \phi = 1 \).

Theorem. The Springer correspondence \( \gamma : \mathfrak{A}_\mathfrak{g} \rightarrow W_n^\wedge \cong X_{n}^{n+1, n+1} \) is given by \( (x, 1) \mapsto \rho(x) \).

Remark. Theorem 8.2 rewrites the description of Springer correspondence by Spaltenstein \([\mathfrak{1}]\) using pairs of partitions. Note that he works under the assumption that the theory of Springer representations is valid for \( \mathfrak{g} \) in characteristic 2.

8.3. Let \( c_{\text{reg}} \) be the nilpotent \( G \)-orbit in \( \mathfrak{g} \) which is open dense in the nilpotent variety \( \mathcal{N} \) of \( \mathfrak{g} \). Let \( c_0 \) be the 0 orbit.

Lemma. The pair \( (c_{\text{reg}}, \mathcal{Q}_1) \) corresponds to the unit representation and the pair \( (c_{\text{trivial}}, \mathcal{Q}_1) \) corresponds to the sign representation.
Proof. One can show that the Weyl group action on $H^i(\mathcal{B})$ defined in [14] coincides with the classical representation. Assume $\chi \in \mathcal{W}_n^\wedge$ correspond to the pair $(c, \mathcal{F}) \in \mathfrak{A}_g$. We write $\chi = \chi_{(c, \mathcal{F})}$. Recall that we have the following decomposition
\[ \varphi|_{\mathcal{N}[\dim G - n]} = \bigoplus_{(c, \mathcal{F}) \in \mathfrak{A}_g} \chi_{(c, \mathcal{F})} \otimes IC(c, \mathcal{F})[\dim c]. \]
Thus for $x \in \mathcal{N}$,
\[ H^{2i}(\mathcal{B}_x, \mathcal{Q}_l) = \bigoplus_{(c, \mathcal{F})} \chi_{(c, \mathcal{F})} \otimes (H^{2i+\dim c-\dim G+n}IC(c, \mathcal{F}))[x]. \]
Taking $i = (\dim G - n)/2$ and $x = 0$, we get $H^{\dim G-n}(\mathcal{B}, \mathcal{Q}_l) = \chi_{(c_0, \mathcal{Q}_l)}$ since $A(c_0) = 1$. It follows that $\chi_{(c_0, \mathcal{Q}_l)}$ is the sign representation. Taking $i = 0$ and $x = 0$, we get $H^0(\mathcal{B}, \mathcal{Q}_l) = \chi_{(c_{\text{reg}}, \mathcal{Q}_l)} \otimes (H^0IC(c_{\text{reg}}, \mathcal{Q}_l))_0$ since $A(c_{\text{reg}}) = 1$. It follows that $\chi_{(c_{\text{reg}}, \mathcal{Q}_l)}$ is the unit representation. □

Proof of Theorem 8.2. By the discussion in [2.14] it is enough to show the correspondence defined is compatible with the restriction formula (R). When $n = 1$, by Lemma 8.3 the pair $(c_{\text{reg}}, 1)$ corresponds to the unit representation and the pair $(c_0, 1)$ corresponds to the sign representation. When $n = 2$, there are two representations of $\mathcal{W}_2$ restricting to unit representation and two representations of $\mathcal{W}_2$ restricting to sign representation. But again we know the pair $(c_{\text{reg}}, 1)$ corresponds to the unit representation and the pair $(c_0, 1)$ corresponds to the sign representation. When $n \geq 3$, we show that the map is compatible with the restriction formula. Let $x \in g$ and $x' \in I$ be nilpotent elements. Note that we have $A_G(x) = A_L(x') = 1$. Hence it is enough to show that
\[ (1, \varepsilon_{x, x'}) = (\text{Res}_{\mathcal{W}_n}^n \rho_G(x), \rho_L(x')) \mathcal{W}_{n-1}. \]

Note that $X_{n,d}^{n+1} = 0$ if $d \neq 1$ is odd. Thus $X_{n}^{n+1} = X_{n-1}^{n+1}$. Let $(A, B) \in X_{n}^{n+1}$ correspond to $\chi \in \mathcal{W}_n^\wedge$. The pairs $(A', B') \in X_{n-1}^{n+1}$ which correspond to the components of the restriction of $\chi$ to $\mathcal{W}_{n-1}$ are those which can be deduced from $(A, B)$ by decreasing one of the entries $c_i$ by $i$ and decreasing all other entries $c_j$ by $j - 1$. This can be done if and only if $i \geq 3$ and $c_i - c_{i-2} \geq 2n + 3$, $i = 2$, $c_i \geq n + 2$ or $i = 1$, $c_i \geq 1$. We write $(A, B) \mapsto (A', B')$ if they are related in this way.

Now (8.1) follows since $S_{x, x'} \neq 0$ if and only if $x, x'$ are as in Proposition 8.4 (see below) if and only if $\rho_G(x) \mapsto \rho_L(x')$. □

8.4. Consider a nilpotent class $c' \in f_x(\mathcal{P}_x)$ corresponding to $(\lambda_{2m+1}'(\lambda_{2m'+1}') \cdots (\lambda_i') \chi(\lambda_i')) := (\lambda', \chi')$. Suppose $Y = f_x^{-1}(c')$ and $X = g_x^{-1}(Y)$. (Notations are as in [2.11])

Proposition ([11]). The group $Z_G(x)$ acts transitively on $Y$. We have $\dim X = \dim B_x$ if and only if $(\lambda', \chi')$ satisfies a) or b):

a) Assume $\lambda_i - \lambda_{i-1} \geq 2$, $\chi(\lambda_i) = \lambda_{i}/2$ and $\chi(\lambda_{j}) \geq \lambda_{j} - \lambda_{i}/2 + 1$ for each $j < i$. $\lambda_i' = \lambda_i$, $j \neq i$, $\lambda_i' = \lambda_i - 2$, $\chi' (\lambda_i') = \chi(\lambda_j)$ for each $j \neq i$ and $\chi(\lambda_i') = \lambda_{i}/2$. In this case $\dim Y = 2m - i + 1$.

b) Assume $\lambda_i + 1 = \lambda_i > \lambda_{i-1}$. $\lambda_i = \lambda_{j-1}$, $j \neq i$, $i + 1$, $\lambda_i' = \lambda_i - 1$, $\chi' (\lambda_i') = \chi(\lambda_j)$ for each $j \neq i, i + 1$ and $\chi'(\lambda_i') = \chi'(\lambda_i' + 1) \in \{\chi(\lambda_i), \chi(\lambda_i - 1)\}$ satisfy $0 \leq \chi'(\lambda_i') \leq \lambda_{i}/2$, $\chi(\lambda_{i-1}) \leq \chi'(\lambda_i') \leq \chi(\lambda_{i-1}) + \lambda_i - \lambda_i - 1$. We have $\dim Y = 2m - i + 1$ if $\chi'(\lambda_i') = \chi(\lambda_i)$ and $\dim Y = 2m - i$ if $\chi'(\lambda_i') = \chi(\lambda_i - 1)$.
9. Springer correspondence for orthogonal Lie algebras

9.1. In this subsection we assume \( G = O(2n+1) \).

Let \( x = (\lambda_{2m+1})_{1}^{1} \cdots (\lambda_{1})_{1}^{1} \in \mathfrak{g} \) be a nilpotent element (see [4,1]). Assume \( \lambda_{1} = 0 \). There exists a unique \( 3 \leq m_{0} \leq 2m + 1 \) such that \( m_{0} \) is odd and \( \lambda_{m_{0}} > \lambda_{m_{0}-1} \). We have \( \chi(\lambda_{j}) = \lambda_{j} \) if \( j \leq m_{0} \); \( \lambda_{2j+1} = \lambda_{2j+1}^{1} \) \( \lambda_{2j} \not= \frac{m_{0}-1}{2} \) and \( \lambda_{m_{0}} = \lambda_{m_{0}-1}+1 \).

We attach to the orbit \( c \) of \( x \) the sequence \( c_{1}, \ldots, c_{2m+1} \) defined as follows:

\[
\begin{align*}
(1)\quad & c_{2j} = \begin{cases}
\lambda_{2j} - \chi(\lambda_{2j}) + n + 1 + (j-1)(n+3) & \text{if } 2j < m_{0} \\
\lambda_{2j} - \chi(\lambda_{2j}) + 1 + n + 1 + (j-1)(n+3) & \text{if } 2j \geq m_{0}
\end{cases}
\end{align*}
\]

\[
(2)\quad c_{2j-1} = \begin{cases}
\chi(\lambda_{2j-1}) + (j-1)(n+3) & \text{if } 2j - 1 < m_{0} \\
\chi(\lambda_{2j-1}) - 1 + (j-1)(n+3) & \text{if } 2j - 1 \geq m_{0}
\end{cases}
\]

Taking \( a_{i} = c_{2i-1}, i = 1, \ldots, m+1, b_{i} = c_{2i}, i = 1, \ldots, m \), we get a well-defined element \((A, B) \in X_{n,1}^{2,n+1}\). We denote it \( \rho_{G}(x), \rho(x) \) or \( \rho(c) \).

Lemma. (i) \( c \mapsto \rho(c) \) defines a bijection from the set of all nilpotent \( O(2n+1) \)-orbits in \( \mathfrak{o}(2n+1) \) to \( D_{n}^{2,n+1} \).

(ii) \( A_{G}(x)^{\wedge} \) is isomorphic to \( V_{\rho(x)}^{2,n+1} \).

Proof. (i) It is easily checked from the definition that \( \rho(c) \in D_{n}^{2,n+1} \) and the map \( c \mapsto \rho(c) \) is injective. Note that \( D_{n}^{2,n+1} \) is in bijection with the set \( \Delta \) consisting of all pairs of partitions \((\mu, \nu)\) such that \( \sum \mu_{i} + \sum \nu_{i} = n, \nu_{i} \leq \mu_{i} + 2 \). Since the number of nilpotent orbits is equal to \(|\Delta|\) by Spaltenstein [1,1], the bijectivity of the map follows. In fact, given \((A, B) \in D_{n}^{2,n+1}\), the corresponding nilpotent orbit can be obtained as follows. Let \( c_{1} \leq c_{2} \leq \cdots \leq c_{2m+1} \) be the sequence \( a_{1} \leq b_{1} \leq \cdots \leq a_{m+1} \). There exists a unique odd integer \( m_{0} \) such that \( c_{2j} > (n+1) + (j-1)(n+3) \) if and only if \( 2j > m_{0} \). If \( j < \frac{m_{0}-1}{2} \), then \( \lambda_{2j} = \lambda_{2j+1} = \chi(\lambda_{2j}) = c_{2j+1} - j(n+3) \). If \( j > \frac{m_{0}-1}{2} \), then \( \lambda_{2j} = \lambda_{2j+1} = c_{2j} + c_{2j+1} - (j-1)(n+3) - (n+1) \) and \( \chi(\lambda_{2j}) = c_{2j+1} - j(n+3) + 1 \). If \( j = \frac{m_{0}-1}{2} \), then \( \lambda_{2j} = \lambda_{2j+1} = c_{2j+1} - j(n+3) + 1 \).

(ii) The component group \( A_{G}(x) \) is described in [4,1]. Let \((A, B) = (\rho(x)) \) and \( c_{1}, \ldots, c_{2m+1} \) be as above. Let \( S = (A \cup B)\setminus(A \cap B) \). Note that \( c_{1} = 0, \ldots, c_{m_{0}} \in S \) and they belong to the same interval, which is the initial interval. For \( i > m_{0} \), \( \chi(\lambda_{i}) \not= \lambda_{i}/2 \) if and only if \( c_{i} \in S \). The relations (r2) and (r3) of [4,1] say that if \( c_{i}, c_{j} \) belong to the same interval of \((A, B)\), then \( a_{i}, a_{j} \) have the same images in \( A(x) \). Thus we get an element \( \sigma_{I} \) of \( A(x) \) for each interval \( I \) of \( A, B \) and \( \sigma_{I}^{2} = 1 \). Moreover (r3) means that \( \sigma_{I} = 1 \) if \( I \) is the initial interval.

The isomorphism \( V_{\rho(x)}^{2,n+1} \rightarrow A_{G}(x)^{\wedge} \) is given as follows. Let \( F \in V_{\rho(x)}^{2,n+1} \). We associate to \( F \) the character of \( A_{G}(x) \) which takes value \(-1\) on \( \sigma_{I} \) if and only if \( I \in \mathcal{F} \).

9.2. Let \((x, \phi) \in \mathfrak{A}_{\mathfrak{g}} \). We have defined \( \rho(x) \). Let \( \rho \) denote also the map \( A_{G}(x)^{\wedge} \rightarrow V_{\rho(x)}^{2,n+1} \).

Theorem. The Springer correspondence \( \gamma : \mathfrak{A}_{\mathfrak{g}} \rightarrow W_{n}^{\wedge} \cong X_{n}^{2,n+1} \) is given by

\[
(x, \phi) \mapsto \rho(x)_{\rho(\phi)}.
\]
Let $x$ be a nilpotent and $x' \in \mathfrak{g}$ nilpotent. Then $S_{x,x'} \neq \emptyset$ if and only if $x, x'$ are as in Proposition 12 if and only if $\Lambda = \rho_G(x) \rightarrow \Lambda = \rho_L(x')$. To verify the map is compatible with the restriction formula, it is enough to show that the set
defined as follows:

(i) $I$ is a proper interval of $\Lambda$. Then $\Lambda_F \rightarrow \Lambda_{F'}$ if and only if

(a) $F \backslash \{I\} = F' \backslash \{I', J'\}$;

(b) $F \cap \{I\} = F' \cap \{I', J'\} = \emptyset$ or $\{I\} \subset F, \{I', J'\} \subset F'$.

On the other hand, $A_G(x)$ (resp. $A_L(x')$) is an $F_2$ vector space with one basis element $\sigma_K$ (resp. $\sigma'_K$) for each proper interval $K$ of $\Lambda$ (resp. $\Lambda'$) and $S_{x,x'}$ is the quotient of $A_G(x) \times A_L(x')$ by the subgroup $H_{x,x'}$ generated by elements of the form $\sigma_I \sigma_{J'}, \sigma_I \sigma_{J'} \sigma_K \sigma_{J''}$ with $K$ a proper interval of both $\Lambda$ and $\Lambda'$. Now the compatibility between (9.1) and (9.2) is clear.

(ii) $I$ is an initial interval of $\Lambda$. Then $\Lambda_F \rightarrow \Lambda_{F'}$ if and only if $F = F'$. On the other hand, $A_G(x), A_L(x')$ and $S_{x,x'}$ are obtained by setting $\sigma_I = \sigma'_{J'} = 1$ in (i). Again the compatibility between (9.1) and (9.2) is clear.

9.3. In this subsection we assume $G = SO(2n), \tilde{G} = O(2n)$ and $\mathfrak{g} = \mathfrak{o}(2n)$. We describe $\tilde{\gamma} : \mathfrak{a}_{\mathfrak{g}} \rightarrow \mathfrak{w}_n^{\Lambda}$ instead of $\gamma : \mathfrak{a}_{\mathfrak{g}} \rightarrow (\mathfrak{w}_n)^{\Lambda}$ (see 2.10).

Let $x = (\lambda_2 \cdots (\lambda_1)$ be a nilpotent element (see 1.1). Note that we have $\lambda_{2j-1} = \lambda_{2i}$. We attach to the orbit $c$ of $x$ the sequence $c_1, \ldots, c_{2m}$ defined as follows:

(1) $c_{2j} = \lambda_{2j} + (j - 1)(n + 1)$,

(2) $c_{2j-1} = \lambda_{2j-1} - \lambda_{2j-1} + (j - 1)(n + 1)$.

Taking $a_i = c_{2i-1}, b_i = c_{2i}, i = 1, \ldots, m$, we get a well defined element $\{A, B\} \in V_n^{n+1}$. We denote it $\rho_G(x), \rho(x)$ or $\rho(c)$.

Lemma. (i) $c \mapsto \rho(c)$ defines a bijection from the set of all nilpotent $O(2n)$-orbits in $\mathfrak{o}(2n)$ to $D_n^{n+1}$.

(ii) $A_G(x)^{\Lambda}$ is isomorphic to $V_n^{n+1}$. 

25
Proof. (i) It is easily checked from the definition that $\rho(c) \in D_{n,\text{even}}^{n+1}$ and the map $c \mapsto \rho(c)$ is injective. Note that $D_{n,\text{even}}^{2n+1}$ is in bijection with the set $\Delta$ consisting of all pairs of partitions $(\mu, \nu)$ such that $\sum \mu_i + \sum \nu_i = n, \nu_i \leq \mu_i$. Since the number of nilpotent $O(2n)$-orbits in $a(2n)$ is equal to $|\Delta|$ by Spaltenstein [14], the bijectivity of the map follows. In fact, given $\{A, B\} \in D_{n,\text{even}}^{n+1}$ with preimage $(A, B) \in D_{n,0}^{n+1}$, the corresponding nilpotent orbit can be obtained as follows. Let $c_1 \leq c_2 \leq \cdots \leq c_{2m}$ be the sequence $a_1 \leq b_1 \leq \cdots \leq a_m \leq b_m$. We have $\chi(a_j) = \chi(b_j) = c_j + c_{j-1} - (2j-2)(n+1)$ and $\chi(\lambda_{2j}) = \chi(\lambda_{2j-1}) = c_{2j} - (j-1)(n+1)$.

(ii) The component group $A_G(x)$ is described in [14]. Note that in this case, the condition (r3) is void. By similar argument as in the proof of Lemma 9.1 (ii), one shows that $A_G(x)$ is a vector space over $\mathbb{F}_2$ with basis $(\sigma_i)_{i \in E}$, where $E$ is the set of all intervals of $\rho(x)$. Since $A_G(x)$ consists of the elements in $A_G(x)$ which can be written as a product of even number of generators, from the natural identification $A_G(x)^{\wedge} = A(E)$, we get the isomorphism $A_G(x)^{\wedge} \cong A(E)/\{\emptyset, E\} = V_{\rho(x)}^{n+1}$.

9.4. Let $(x, \phi) \in \mathfrak{Af}_g$. We have defined $\rho(x)$. Let $\rho$ denote also the map $A_G(x)^{\wedge} \to V_{\rho(x)}^{n+1}$.

**Theorem.** The Springer correspondence $\tilde{\gamma} : \mathfrak{Af}_g \to \mathcal{W}_{n,\text{even}}^{n+1}$ is given by $$(x, \phi) \mapsto \rho(x)_{\rho(\phi)}.$$

Proof. Again it is enough to prove the map is compatible with the restriction formula (R). Note that $Y_{n,d}^{n+1} = \emptyset$ if $d > 0$ is even. Thus $Y_{n,\text{even}}^{n+1} = Y_{n,0}^{n+1}$. Let $\{A, B\} \in Y_{n,\text{even}}^{n+1}$ correspond to $\chi \in \mathcal{W}_{n,\text{even}}^{n+1}$. The pairs $\{A', B'\} \in Y_{n-1,\text{even}}^{n+1}$ which correspond to the components of the restriction of $\chi$ to $\mathcal{W}_{n-1}$ are those which can be deduced from $\{A, B\}$ by decreasing one of the entries $a_i$ by $i$ (or $b_i$ by $i$) and decreasing all other entries $a_j$ by $j - 1, b_j$ by $j - 1$. We can decrease $a_i$ by $i$ (resp. $b_i$ by $i$) if and only if $i \geq 2, a_i - a_{i-1} \geq n + 2$ or $i = 1, a_i \geq 1$ (resp. $i \geq 2, b_i - b_{i-1} \geq n + 2$ or $i = 1, b_i \geq 1$). We write $\{A, B\} \to \{A', B'\}$ if they are related in this way. Suppose that $\{A, B\} \to \{A', B'\}$. One can easily check that if $\{A, B\}$ and $\{A', B'\}$ are similar to $\Lambda \in D_{n,\text{even}}^{n+1}$ and $\Lambda' \in D_{n-1,\text{even}}^{n+1}$ respectively, then $\Lambda \to \Lambda'$.

Let $x \in \mathfrak{g}$ nilpotent and $x' \in \mathfrak{g}$ nilpotent. Then $\mathcal{S}_{x, x'} \neq \emptyset$ if and only if $x, x'$ are as in Proposition 1.2 if and only if $\Lambda = \rho_G(x) \to \Lambda' = \rho_G(x')$. Let $c_1 \leq \cdots \leq c_{2m}$ and $c'_1 \leq \cdots \leq c'_{2m}$ correspond to $\Lambda$ and $\Lambda'$ respectively. Then $A_G(x)$ is generated by $\{a_i | c_i \neq c_j, \forall j \neq i\}$, $A_L(x')$ is generated by $\{a'_i | c'_i \neq c'_j, \forall j \neq i\}$ and $A_P$ is generated by $\{a_i | c_i \neq c_j, c'_i \neq c'_j, \forall j \neq i\}$. The discussion in [2.12] allows us to compute $\varepsilon_{x, x'}$ and the set $\{(\phi, \phi') \in A_G(x)^{\wedge} \times A_G(x')^{\wedge} | (\phi \otimes \phi', \varepsilon_{x, x'}) \neq 0\}$. One verifies the compatibility with the set $\{(F, F') \in V_{\rho_G(x)}^{n+1} 	imes V_{\rho_G(x')}^{n+1} | A_F \to A_F'\}$ under the map $\rho$ as in the proof of Theorem 9.2.

10. Springer Correspondence for Duals of Symplectic and Odd Orthogonal Lie Algebras

10.1. We assume $G = Sp(2n)$ in [10.3] and [10.7].

Let $\xi = (\lambda_2 \cdots \lambda_{2m+1})_1(\lambda_1)$ be nilpotent (see [5.4], where $\lambda_1 = 0$. We have $\lambda_2 = \lambda_2$. We attach to the orbit $c$ of $\xi$ the sequence $c_1, \ldots, c_{2m+1}$ defined as follows:

1. $c_{2j} = \lambda_{2j} + (j - 1)(n + 2)$,
2. $c_{2j-1} = \lambda_{2j-1} + (j - 1)(n + 2)$.

Taking $a_i = c_{2i-1}, i = 1, \ldots, m + 1, b_i = c_{2i}, i = 1, \ldots, m$, we get a well-defined $(A, B) \in X_{n,1}^{1,n+1}$. We denote it $\rho_G(\xi), \rho(\xi)$ or $\rho(c)$. 26
Lemma. (i) $c \mapsto \rho(c)$ defines a bijection from the set of all nilpotent $Sp(2n)$-orbits in $\mathfrak{sp}(2n)^*$ to $D_n^{1,n+1}$.

(ii) $A_G(\xi)^\wedge$ is isomorphic to $V_{\rho(\xi)}^{1,n+1}$.

Proof. (i) It is easily checked from the definition that $\rho(c) \in D_n^{1,n+1}$ and the map $c \mapsto \rho(c)$ is injective. Note that $D_n^{1,n+1}$ is in bijection with the set $\Delta$ consisting of all pairs of partitions $(\mu, \nu)$ such that $\sum \mu_i + \sum \nu_i = n$, $\nu_i \leq \mu_i + 1$. Since the number of nilpotent orbits is equal to $|\Delta|$ by \[15\], the bijectivity of the map follows. In fact, given $(A, B) \in D_n^{1,n+1}$, the corresponding nilpotent orbit can be obtained as follows. Let $c_1 \leq c_2 \leq \cdots \leq c_{2m+1}$ be the sequence $a_1 \leq b_1 \leq \cdots \leq a_{m+1}$. Then $\lambda_2 j = \lambda_{2j+1} = c_{2j} + c_{2j+1} - (2j - 1)(n + 2) - (n + 1)$ and $\chi(\lambda_{2j}) = \chi(\lambda_{2j+1}) = c_{2j+1} - j(n + 2)$, $j = 1, \ldots, m$, $\lambda_1 = 0$.

(ii) The component group $A_G(\xi)$ is described in \[5.1\]. Let $(A, B) = \rho(\xi)$ and $c_1, \ldots, c_{2m+1}$ be as above. Let $S = (A \cup B) \setminus (A \cap B)$. Then $\chi(\lambda_i) \neq (\lambda_i - 1)/2$ if and only if $c_i \in S$. The relation (r2) of \[5.1\] says that if $c_i, c_j$ belong to the same interval of $(A, B)$, then $a_i, a_j$ have the same images in $A_G(\xi)$. Thus we get an element $\sigma_I$ of $A_G(\xi)$ for each interval $I$ of $(A, B)$ and $\sigma_f^2 = 1$. Moreover (r3) means that $\sigma_f = 1$ if $I$ is the initial interval.

The isomorphism $V_{\rho(\xi)}^{1,n+1} \rightarrow A_G(\xi)^\wedge$ is given as follows. We associate to $F$ the character of $A_G(\xi)$ which takes value $-1$ on $\sigma_f$ if and only if $I \in F$. 

10.2. Let $(\xi, \phi) \in \mathfrak{A}_{\mathfrak{g}^*}$. We have defined $\rho(\xi)$. Let $\rho$ denote also the map $A_G(\xi)^\wedge \rightarrow V_{\rho(\xi)}^{1,n+1}$.

Theorem. The Springer correspondence $\gamma' : \mathfrak{A}_{\mathfrak{g}^*} \rightarrow W_n \cong X_n^{1,n+1}$ is given by

$$(\xi, \phi) \mapsto \rho(\xi)_{\rho(\phi)}.$$ 

Proof. By similar argument as in the proof of Theorem \[8.2\] it is enough to prove the map is compatible with the restriction formula (R'). Note that $X_n^{1,n+1,d} = \emptyset$ if $d \neq 1$ is odd. Thus $X_n^{1,n+1} = X_n^{1,n+1,1}$. Let $(A, B) \in X_n^{1,n+1}$ correspond to $\chi \in W_n$. The pairs $(A', B') \in X_{n-1}^{1,n}$ which correspond to the components of the restriction of $\chi$ to $W_{n-1}$ are those which can be deduced from $(A, B)$ by decreasing one of the entries $a_i$ by $i$ (or $b_i$ by $i + 1$) and decreasing all other entries $a_j$ by $j - 1$, $b_j$ by $j$. We can decrease $a_i$ by $i$ (resp. $b_i$ by $i + 1$) if and only if $i \geq 2$, $a_i - a_{i-2} \geq n + 3$ or $i = 1$, $a_i \geq 1$ (resp. $i \geq 2$, $b_i - b_{i-1} \geq n + 3$ or $i = 1$, $b_i \geq n + 2$).

We write $(A, B) \rightarrow (A', B')$ if they are related in this way. Suppose that $(A, B) \rightarrow (A', B')$. One can easily check that if $(A, B)$ and $(A', B')$ are similar to $\Lambda \in D_n^{1,n+1}$ and $\Lambda' \in D_n^{1,n+1}$ respectively, then $\Lambda \rightarrow \Lambda'$.

Let $\xi \in \mathfrak{g}^*$ and $\xi' \in \mathfrak{i}^*$ be nilpotent. Then $S_{\xi, \xi'} \neq \emptyset$ if and only if $\xi, \xi'$ are as in Proposition \[5.3\] if and only if $\Lambda = \rho_G(\xi) \rightarrow \Lambda' = \rho_L(\xi')$. The verification of compatibility with restriction formula is entirely similar to the proof of Theorem \[9.2\].

10.3. We assume $G = O(2n + 1)$ in this subsection and \[10.4\].

Let $\xi = (m; (\lambda_2) \chi(\lambda_2) \cdots (\lambda_1) \chi(\lambda_1)) \in \mathfrak{g}^*$ be nilpotent (see \[6.1\]). We have $\lambda_{2j-1} = \lambda_2 j$. We attach to the orbit $c$ of $\xi$ the sequence $c_1, \ldots, c_{2s+1}$ defined as follows:

1. $c_{2j} = \chi(\lambda_{2j}) + (j - 1)(n + 1)$, $j = 1, \ldots, s$.
2. $c_{2j-1} = \lambda_{2j-1} - \chi(\lambda_{2j-1}) + (j - 1)(n + 1)$, $j = 1, \ldots, s$.
3. $c_{2s+1} = m + s(n + 1)$.

Taking $a_i = c_{2i-1}$, $i = 1, \ldots, s + 1$, $b_i = c_{2i}$, $i = 1, \ldots, s$, we get a well defined element \(\{A, B\} \in Y_n^{n+1}\). We denote it $\rho_G(\xi)$, $\rho(\xi)$ or $\rho(c)$. 

27
Lemma. (i) $c \mapsto \rho(c)$ defines a bijection from the set of all nilpotent $O(2n+1)$-orbits in $\mathfrak{a}(2n+1)^*$ to $\mathbb{D}^n_{n,odd}$.
(ii) $A_G(\xi)^\wedge$ is isomorphic to $V^\rho_{\rho(\xi)}$.

Proof. (i) It is easily checked from the definition that $\rho(c) \in \mathbb{D}^n_{n,odd}$ and the map $c \mapsto \rho(c)$ is injective. Note that $\mathbb{D}^n_{n,odd}$ is in bijection with the set $\Delta$ consisting of all pairs of partitions $(\mu, \nu)$ such that $\sum \mu_i + \sum \nu_i = n, \nu_{i+1} \leq \mu_i$. Since the number of nilpotent orbits is equal to $|\Delta|$ by [L3], the bijectivity of the map follows. In fact, given $(A, B) \in \mathbb{D}^n_{n,odd}$, assume $(A, B) \in \mathbb{D}^n_{n,odd}$, the corresponding nilpotent orbit can be obtained as follows. Let $c_1 \leq c_2 \leq \cdots \leq c_{2s+1}$ be the sequence $a_i \leq b_i \leq \cdots \leq a_{s+1}$. Then $\lambda_{2j} = \lambda_{2j+1} = c_{2j} + c_{2j-1} - (2j-2)(n+1)$, $\chi(\lambda_{2j}) = \chi(\lambda_{2j+1}) = c_{2j} - (j-1)(n+1)$, $j = 1, \ldots, s$ and $m = c_{2s+1} - s(n+1)$. The corresponding orbit is $(m; (\lambda_{2s})_\chi(\lambda_{2s}) \cdots (\lambda_1)\chi(\lambda_1))$.

(ii) The component group $A_G(\xi)$ is described in [L3]. Let $(A, B) = \rho(\xi)$, $(A, B)$ and $c_1, \ldots, c_{2s+1}$ be as above. Let $S = (A \cup B) \setminus (A \cap B)$. Note that $c_{2s} < c_{2s+1}$, thus $c_{2s+1} \in S$. The relation (r2) of [L3] says that for $1 \leq i < j \leq 2s$, if $c_i, c_j$ belong to the same interval of $(A, B)$, then $a_i, a_j$ have the same images in $A_G(\xi)$. Let $I_0$ be the interval containing $c_{2s+1}$. The relation (r3) says that $a_i = 1$ if $c_i \in I_0$. Thus we get an element $\sigma_I$ of $A_G(\xi)$ for each interval $I \neq I_0$ of $(A, B)$ and $\sigma_I^2 = 1$.

The isomorphism $V^\rho_{\rho(\xi)} \rightarrow A_G(\xi)^\wedge$ is given as follows. Let $F \in V^\rho_{\rho(\xi)} = A(E)/\{(0, E)\}$ and $\tilde{F}$ the inverse image of $F$ in $A(E)$ that does not contain $I_0$. We associate to $F$ the character of $A_G(\xi)$ which takes value $-1$ on $\sigma_I$ if and only if $I \in \tilde{F}$.

10.4. Let $(\xi, \phi) \in \mathfrak{a}_g^*$. We have defined $\rho(\xi)$. Let $\rho$ denote also the map $A_G(\xi)^\wedge \rightarrow V^\rho_{\rho(\xi)}$.

Theorem. The Springer correspondence $\gamma' : \mathfrak{a}_g^* \rightarrow W_n^\wedge \cong Y^\rho_{n,odd}$ is given by

$$(\xi, \phi) \mapsto \rho(\xi)_{\rho(\phi)}.$$  

Proof. Again it is enough to prove the map is compatible with the restriction formula (R'). Note that $Y^\rho_{n,d} = \emptyset$ if $d \neq 1$ is odd. Thus $Y^\rho_{n,odd} = Y^\rho_{n,1}$. Let $(A, B) \in Y^\rho_{n,odd}$ with inverse image $(A', B') \in X^\rho_{n,1,0}$ correspond to $\chi \in W_n^\wedge$. The pairs $(A', B') \in Y^\rho_{n,odd}$ with inverse images $(A', B') \in X^\rho_{n,1,0}$ which correspond to the components of the restriction of $\chi$ to $W_{n-1}$ are those which can be deduced from $(A, B)$ by decreasing one of the entries $a_i$ by $i$ (or $b_i$ by $i$) and decreasing all other entries $a_j$ by $j - 1$, $b_j$ by $j - 1$. We can decrease $a_j$ by $i$ (resp. $b_i$ by $i$) if and only if $i \geq 2, a_i - a_{i-1} \geq n + 2$ or $i = 1, a_i \geq 1$ (resp. $i \geq 2, b_i - b_{i-1} \geq n + 2$ or $i = 1, b_i \geq 1$). We write $(A, B) \rightarrow (A', B')$ if they are related in this way. Suppose that $(A, B) \rightarrow (A', B')$. One can easily check that if $(A, B)$ and $(A', B')$ are similar to $\Lambda \in \mathbb{D}^n_{n,odd}$ and $\Lambda' \in \mathbb{D}^n_{n-1,odd}$ respectively, then $\Lambda \sim \Lambda'$.

Let $\xi \in \mathfrak{g}^*$ and $\xi' \in \Gamma_n$ be nilpotent. Then $S_{\xi, \xi'} \neq \emptyset$ if and only if $\xi, \xi'$ are as in Proposition 6.3 if and only if $\Lambda = \rho(\xi) \rightarrow \Lambda' = \rho(\xi')$. To verify the map is compatible with the restriction formula, it is enough to show that the set

$$(10.1) \quad \{(F, F') \in V^\rho_{\rho(\xi)} \times V^\rho_{\rho(\xi')} | \Lambda_F \rightarrow \Lambda_{F'}\}$$

is the image of the set

$$(10.2) \quad \{(\phi, \phi') \in A_G(\xi)^\wedge \times A_L(\xi')^\wedge | (\phi \otimes \phi', \varepsilon_{\xi, \xi'}) \neq 0\}$$

under the map $\rho$.  

28
Let \( c_1 \leq \cdots \leq c_{2k+1} \) and \( c'_1 \leq \cdots \leq c'_{2k+1} \) correspond to the pre-image \((A, B)\) and \((A', B')\) of \( \Lambda \) and \( \Lambda' \) in \( D^{n,1,0}_{n,1} \) and \( D^{n,0,1}_{n,1} \) respectively. Then \( A_G(\xi) \) is generated by \( \{a_i|c_i \neq c_j, \forall j \neq i\} \), \( A_L(\xi') \) is generated by \( \{a'_i|c'_i \neq c'_j, \forall j \neq i\} \) and \( A_F \) is generated by \( \{a_i|c_i \neq c_j, c'_i \neq c'_j, \forall j \neq i\} \). There are various cases to consider. We describe one of the cases and the other cases are similar.

Assume \( k \geq 1, c_{2k} > c_{2k-1} + 1, c_{2k+1} = c_{2k} + n \) and \( c'_{2k} = c_{2k} - k, c'_{2i} = c_{2i} - (i-1)i \neq k, c'_{2i+1} = c_{2i+1} - i, i = 1, \ldots, s \). Let \( I \) (resp. \( I' \)) be the interval of \( \Lambda \) (resp. \( \Lambda' \)) containing \( c_{2k+1} \) (resp. \( c'_{2k+1} \)) and \( J \) the interval of \( \Lambda' \) containing \( c_{2k} \). Note that \( c_{2j} - c_{2j-1} = 2\chi(\lambda_{2j}) - \lambda_{2j} < n \) and \( c_{2j+1} - c_{2j} = n + 1 + \lambda_{2j+1} - \chi(\lambda_{2j+1}) - \chi(\lambda_{2j}) \geq 2, \) except if \( m = 0 \), \( \chi(\lambda_{2s}) = \lambda_{2s} = n \). In the latter case, \( \xi' \) correspond to \( m' = 0 \), \( \chi'(\lambda'_{2s}) = \lambda'_{2s} = n - 1 \) and \( A_G(\xi) = A_L(\xi') = 1 \). Hence all other intervals of \( \Lambda \) and \( \Lambda' \) can be identified naturally. Let \( I_0 \) (resp. \( I'_0 \)) be the interval of \( \Lambda \) (resp. \( \Lambda' \)) containing \( c_{2s+1} \) (resp. \( c'_{2s+1} \)). There are two possibilities:

(i) \( I \neq I_0 \). Let \( \tilde{F} \) (resp. \( \tilde{F}' \)) be the pre-image of \( F \) (resp. \( F' \)) in \( \mathcal{A}(E) \) (resp. \( \mathcal{A}(E') \)) that does not contain \( I_0 \) (resp. \( I'_0 \)). Then \( \Lambda_F \rightarrow \Lambda_{F'} \) if and only if

\[
\begin{align*}
(a) & \quad \tilde{F} \backslash \{I\} = \tilde{F}' \backslash \{I', J'\}; \\
(b) & \quad \tilde{F} \cap \{I\} = \tilde{F}' \cap \{I', J'\} = \emptyset \text{ or } \{I\} \subset \tilde{F}, \{I', J'\} \subset \tilde{F}'.
\end{align*}
\]

On the other hand, \( A_G(\xi) \) (resp. \( A_L(\xi') \)) is an \( \mathbb{F}_2 \) vector space with one basis element \( \sigma_K \) (resp. \( \sigma'_K \)) for each interval \( K \neq I_0 \) (resp. \( K \neq I'_0 \)) of \( \Lambda \) (resp. \( \Lambda' \)) and \( S_{\xi, \xi'} \) is the quotient of \( A_G(\xi) \times A_L(\xi') \) by the subgroup \( H_{\xi, \xi'} \) generated by elements of the form \( \sigma_I \sigma_{I'}, \sigma_I \sigma'_{I'}, \sigma_K \sigma'_K \) with \( K \neq I_0 \) an interval of both \( \Lambda \) and \( \Lambda' \). Now the compatibility between (10.1) and (10.2) is clear.

(ii) \( I = I_0 \). Then \( \Lambda_F \rightarrow \Lambda_{F'} \) if and only if \( \tilde{F} = \tilde{F}' \). On the other hand, \( A_G(\xi), A_L(\xi') \) and \( S_{\xi, \xi'} \) are obtained by setting \( \sigma_I = \sigma'_{I'} = 1 \) in (i). Again the compatibility between (10.1) and (10.2) is clear.

11. Complement

11.1. In [8], Lusztig gives an apriori description of the Weyl group representations that parametrize the pairs \((c, 1)\), where \( c \) is a unipotent class in \( G \) or a nilpotent orbit in \( g \). We list the results of [8] here.

Let \( R \) be a root system of type \( B_n, C_n \) or \( D_n \) with simple roots \( \Pi \) and \( W \) the weyl group. There exists a unique \( \alpha_0 \in R \backslash \Pi \) such that \( \alpha - \alpha_0 \notin R, \forall \alpha_i \in \Pi \). Let \( J \subset \Pi \cup \{\alpha_0\} \) be such that \( J = [\Pi] \). Let \( W_J \) be the subgroup of \( W \) generated by \( s_\alpha, \alpha \in J \).

(i) Denote \( S_W \) the set of special representations of \( W \). The set of unipotent classes when \( \text{char}(k) \neq 2 \) is in bijection with the set (see [4, 5])

\[
S_W = \{j^{\mathbb{W}}_{w_j} E, \ E \in S_{W_j}\},
\]

where \( W_j^\wedge \) is defined as \( W_J \) using the dual root system.

(ii) The set of unipotent classes when \( \text{char}(k) = 2 \) is in bijection with the set (see [6])

\[
S_W^2 = \{j^{\mathbb{W}}_{w_j} E, \ E \in S_{W_j}^1\}.
\]

(iii) The set of nilpotent classes when \( \text{char}(k) = 2 \) is in bijection with the set \( T_{W_j}^2 \) defined by induction on \( |W| \) as follows (see [11]). If \( W = \{1\}, T_{W_j}^2 = W^\wedge \). If \( W \neq \{1\} \), then \( T_{W_j}^2 \) is the set of all \( E \in W^\wedge \) such that either \( E \in S_{W_j}^1 \) or \( E = j^{\mathbb{W}}_{W_j} E_1 \) for some \( W_j \neq W \) and some \( E_1 \in T_{W_j}^2 \).
11.2. One can show that the set of nilpotent orbits in \( g^* \) when \( \text{char}(k) = 2 \) is in bijection with the set \( T_{W_2}^* \) defined by induction on \( |W| \) as follows. If \( W = \{1\} \), \( T_{W_2}^* = W^\wedge \). If \( W \neq \{1\} \), then \( T_{W_2}^* \) is the set of all \( E \in W^\wedge \) such that either \( E \in S^1_W \) or \( E = j_{W_j}^W E_1 \) for some \( W_j \neq W \) and some \( E_1 \in T_{W_j}^{2*} \).

Acknowledgement

I would like to thank Professor George Lusztig for guidance, encouragement and many helpful discussions.

References

[1] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers. Asterisque 100 (1981).
[2] W.H. Hesselink, Nilpotency in Classical Groups over a Field of Characteristic 2. Math. Z. 166 (1979), 165-181.
[3] G. Lusztig, Irreducible representations of finite classical groups. Invent. Math. 43 (1977), no. 2, 125-175.
[4] G. Lusztig, A class of irreducible representations of a Weyl group. Nederl. Akad. Wetensch. Indag. Math. 41 (1979), no. 3, 323-335.
[5] G. Lusztig, Intersection cohomology complexes on a reductive group. Invent. Math. 75 (1984), no.2, 205-272.
[6] G. Lusztig, Character sheaves II. Adv. in Math. 57 (1985), no.3, 226-265.
[7] G. Lusztig, Character sheaves on disconnected groups. II. Represent. Theory 8 (2004), 72-124(electronic).
[8] G. Lusztig, Remarks on Springer’s representations. Represent. Theory 13 (2009), 391-400
[9] G. Lusztig; N. Spaltenstein, On the generalized Springer correspondence for classical groups. Algebraic groups and related topics (Kyoto/Nagoya, 1983), 289-316, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.
[10] T. Shoji, On the Springer representations of the Weyl groups of classical algebraic groups. Comm. Algebra 7 (1979), no. 16, 1713-1745.
[11] N. Spaltenstein, Nilpotent Classes and Sheets of Lie Algebras in Bad Characteristic. Math. Z. 181 (1982), 31-48.
[12] N. Spaltenstein, Classes unipotentes et sous-groupes de Borel. Lecture Notes in Mathematics, 946. Springer-Verlag, Berlin-New York, 1982.
[13] T.A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups. Invent. Math. 36 (1976), 173-207.
[14] T. Xue, Nilpotent orbits in classical Lie algebras over finite fields of characteristic 2 and the Springer correspondence. Represent. Theory 13 (2009), 371-390 (electronic).
[15] T. Xue, Nilpotent orbits in the dual of classical Lie algebras in characteristic 2 and the Springer correspondence. Represent. Theory 13 (2009), 609-635 (electronic).

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

E-mail address: txue@math.mit.edu