WHY THE CIGAR CANNOT BE ISOMETRICALLY IMMERSED INTO THE 3-SPACE

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Abstract. In this paper, we study the question if there is an isometric immersion of the cigar soliton into $R^3$. We show that the answer is negative. This gives a counterexample to the classical Weyl problem on $R^2$. A similar result in higher dimensions is also true for steady Bryant solitons.

1. Introduction

The classical Weyl problem asks if there exists a global $C^2$ isometric embedding $X : (S^2, g) \to (R^3, \sigma)$, where $\sigma$ is the standard flat metric in $R^3$, for a two-sphere $S^2$ with $g$ being a Riemannian metric on $S^2$ whose Gauss curvature is everywhere positive. This problem is solved affirmatively by L. Nirenberg in [4]. Recall here that such an embedding $X : (S^2, g) \to R^3$ is isometric if in the local coordinates $(u^i)$, $g = g_{ij}du^i du^j$, we have the system $\partial_{u^i}X \cdot \partial_{u^j}X = g_{ij}$. For more recent progress related to the Weyl problem one may see the paper [2]. One may ask a similar question of problem on the plane $R^2$ with a Riemannian metric $g$ whose Gauss curvature is everywhere positive. We give a counterexample in this short note by considering the cigar soliton in the study of Ricci flow [1]. Our example also shows that a similar Minkowski problem is not true on complete noncompact surfaces in $R^3$, namely, for a given strictly positive real function $f$ defined on $R^2$, one cannot find a complete noncompact convex surface $\Sigma \subset R^3$ such that the Gauss curvature of $\Sigma$ at the point $x$ equals $f(n(x))$, where $n(x)$ denotes the normal to $\Sigma$ at $x$.

The problem we consider in this short note is if there is a nontrivial $n$-dimensional steady Ricci soliton which can be embedded as a hypersurface in $R^{n+1}$. Recall that steady Ricci solitons are special solutions to Ricci flow introduced by R. Hamilton [3], [1]. In dimension two, the only nontrivial complete Ricci soliton is the cigar. We show that it is impossible to realize it as a surface in $R^3$. The key step in proving it is to use the deep result of H. Wu [5] about the convex surfaces. A similar result is also true for steady Bryant solitons in higher dimensions. We believe that a similar result is also true for radially symmetric expanding Ricci solitons. We shall use the notation $u = 0(r)$ for large $r > 0$ to denote by $C^{-1}r \leq u \leq Cr$ for some uniform constant $C > 0$ and the uniform constant $C$ may vary from line to line.

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2. The cigar cannot be isometrically immersed into $\mathbb{R}^3$

Recall that the cigar soliton is a two-dimensional Riemannian manifold $(\mathbb{R}^2, g_\Sigma)$ with the Riemannian metric (\cite{1}, \cite{3})

$$g_\Sigma = \frac{dx^2 + dy^2}{1 + x^2 + y^2} = \frac{dr^2 + r^2d\theta^2}{1 + r^2}.$$  

It has positive Gauss curvature

$$K = \frac{2}{1 + r^2}.$$  

If the cigar can be isometrically immersed into $\mathbb{R}^3$ (according to a theorem of Sacksteder-Heijenoort and the main theorem ($\delta$) of H. Wu in \cite{5}), it is the graph of a nonnegative strictly convex function $u$ defined in the plane $\{x_3 = 0\}$.

Recall that the induced metric of the graph of the function $z = u(x_1, x_2)$ is given by

$$g = (\delta_{ij} + u_i u_j)dx^i dx^j = g_{ij} dx^i dx^j$$

with its second fundamental form

$$II = h_{ij} dx^i dx^j,$$

where $(x^i) = (x_i)$, $u_i = \frac{\partial u}{\partial x_i}$, $F(x) = (x, u(x))$, $F_j(x) = e_j + u_j(x)e_{n+1}$,

$$\nu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}},$$

eq, and

$$h_{ij} = (D_{F_i(x)} \nu, F_j(x)) = \frac{-u_{ij}}{\sqrt{1 + |Du|^2}}.$$  

Hence,

$$II = h_{ij} dx^i dx^j = \frac{D^2 u}{\sqrt{1 + |Du|^2}}$$

and for $u = u(r)$,

$$II = \frac{u_{rr} dr^2 + ru_{r} d\theta^2}{\sqrt{1 + u_r^2}}.$$  

Then the Gauss curvature of the immersed surface can be computed by

$$K = \det(h_{ij})/\det(g_{ij}) = \det(u_{ij})/(1 + |Du|^2)^2.$$  

**Theorem 1.** The cigar cannot be isometrically immersed into $\mathbb{R}^3$.

**Proof.** Assume that we can have such an immersion into $\mathbb{R}^3$.

Since $g = g_\Sigma$ is radially symmetric, we have $z = u(\rho)$ and

$$g = (1 + u_\rho^2) d\rho^2 + \rho^2 d\theta^2$$

where $u_\rho = \frac{\partial u}{\partial \rho}$, etc. Hence, we have

(1) \hspace{1cm} (1 + u_\rho^2) d\rho^2 = \frac{dr^2}{1 + r^2}  

and

(2) \hspace{1cm} \rho^2 = \frac{r^2}{1 + r^2}.$$
By (2) we have
\[ \rho = \frac{r}{\sqrt{1 + r^2}}, \quad \frac{d\rho}{dr} = \frac{1}{(\sqrt{1 + r^2})^3}. \]
By (1) we have
\[ \sqrt{1 + u^2} d\rho = \frac{dr}{\sqrt{1 + r^2}}, \]
which implies that
\[ \sqrt{1 + u^2} \cdot \frac{1}{(\sqrt{1 + r^2})^3} = \frac{1}{\sqrt{1 + r^2}}. \]
Hence we have
\[ u^2 = r^2 \]
and then
\[ u_\rho u_{\rho\rho} = r \frac{dr}{\rho} = r(1 + r^2)^{3/2}. \]
By direct computation we know that the second fundamental form can be written as
\[ II = \frac{1}{\sqrt{1 + u^2}} [u_{\rho\rho} d\rho^2 + \rho u_\rho d\theta^2]. \]
This would imply that the Gauss curvature \( K \) is
\[ K = \frac{u_{\rho\rho} u_\rho \rho}{(1 + u^2)^2 \rho^2} = 1, \]
which is absurd. This completes the proof of Theorem 1.

3. Higher dimensional generalization

It is quite possible to show a higher dimensional analog of the result above. Namely, we may have

**Theorem 2.** The \( n \)-dimensional radially symmetric Bryant soliton cannot be isometrically immersed into \( \mathbb{R}^{n+1} \).

Recall that the \( n \)-dimensional Bryant soliton is \((\mathbb{R}^n, g), (n \geq 2)\) with its Riemannian metric (11)
\[ g = dr^2 + w(r)^2 d\theta^2, \]
where \( w(r) \) is a smooth function with \( w(0) = 0, w(r) = 0(r^{1/2}) \), and \( d\theta^2 \) is the metric on \( S^{n-1} \). It is well known that it has its positive sectional curvatures
\[ k_1 = -\frac{w''}{w} = 0(r^{-2}), \quad k_2 = \frac{1 - (w')^2}{w^2} = 0(r^{-1}), \]
where \( k_1 \) is the curvature for the planes tangent to the radial direction \( e_1 = \partial_r \) and \( k_2 \) is the curvature for the planes tangent to the sphere.

If the \( n \)-dimensional Bryant soliton can be imbedded into \( \mathbb{R}^{n+1} \), then we can use H. Wu’s result [5] as above to have it as the graph of a strictly convex radially symmetric function \( u = u(x) = u(\rho), x \in \mathbb{R}^3, \rho = |x| \). Then we have
\[ g = (1 + u_\rho^2) d\rho^2 + \rho^2 d\theta^2 \]
and
\[ dr = \sqrt{1 + u_\rho^2} d\rho, \quad w(r) = \rho = 0(r^{1/2}). \]
Hence, \( r = 0(\rho^2) \) and \( u_\rho = 0(\rho) \). We then have some uniform constant \( C > 0 \) such that \( u_{\rho\rho} = 0(1) \) for all large \( \rho > 0 \). Recall that using the components of the second fundamental form \( (h_{ij}) \) and \( |g| = \rho^{2(n-1)}(1 + u_\rho^2) \), the radial Riemannian curvature \( k_1 \) with large \( \rho \) can also be written as

\[
 k_1 = \frac{R_{1212}}{|g|} = \frac{u_{\rho\rho} u_\rho}{\rho^{2n-2}(1 + u_\rho^2)^2} = 0(\rho^{-2n}) = 0(r^{-n}).
\]

We may use this to find a contradiction with (3) as above.

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