Towards Resolution of the Scalar Meson Nonet Enigma II. Gell-Mann–Okubo Revisited

L. Burakovsky∗ and T. Goldman†

Theoretical Division, MS B285
Los Alamos National Laboratory
Los Alamos, NM 87545, USA

Abstract

The new SU(3) nonet mass formula $2M^2(s\bar{s}) + 3M^2(n\bar{n}, I = 1) = 4M^2(s\bar{n}) + M^2(n\bar{n}, I = 0)$ ($n = u, d$), obtained in our previous paper by using Regge phenomenology, is rederived for the pseudoscalar and scalar mesons in the Nambu–Jona-Lasinio model with instanton-induced interaction and applied to the problem of the correct $q\bar{q}$ assignment for the scalar meson nonet. The results strongly favor the masses of the scalar isoscalar mostly octet and mostly singlet states in the vicinity of 1.45 GeV and 1.1 GeV, respectively.

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The spectrum of the scalar meson nonet is a long-standing problem of light meson spectroscopy. The number of resonances found in the region of 1–2 GeV exceeds the number of states that conventional quark models can accommodate. Extra states are interpreted alternatively as $K\bar{K}$ molecules, glueballs, multi-quark states or hybrids. In particular, except for a well established scalar isodoublet state, the $K_0^*(1430)$, the Particle Data Group (PDG) lists two isovector states, the $a_0(980)$ and $a_0^*(1450)$. The latter, having mass and width $1450\pm 40$ MeV, $270\pm 40$ MeV, respectively, was discovered recently by the Crystal Barrel collaboration. A third isovector state (not included in the PDG), $a_0(1320)$, having mass and width $1322\pm 30$ MeV...
and $130 \pm 30$ MeV, was seen by GAMS \[3\] and LASS \[4\] in the partial wave analyses of the $\eta \pi$ and $K_sK_s$ data, respectively.

There are four isoscalar states in \[1\], the $f_0(400-1200)$ (or $\sigma$), the interpretation of which as a particle is controversial due to a huge width of 600–1000 MeV, $f_0(980)$, $f_0(1370)$ (which stands for two separate states, $f_0(1300)$ and $f_0(1370)$, of a previous edition of PDG \[5\]), and $f_0(1500)$ (which also stands for two separate states, $f_0(1525)$ and $f_0(1590)$, of a previous edition of PDG), and two more possibly scalar states, the $f_J(1710)$, $J = 0$ or 2, seen in radiative $J/\Psi$ decays, and an $\eta\eta$ resonance $X(1740)$ with uncertain spin, produced in $p\bar{p}$ annihilation in flight and in charge-exchange. Recently several groups claimed different scalar isoscalar structures close to 1500 MeV, including: 1) a narrow state with mass 1445 ± 5 MeV and width 65 ± 10 MeV seen by the WA91 collaboration at CERN in central production of $4\pi$ in high-energy $pp$ collisions \[3, 2\], the lightest of the three states with masses 1505 MeV, 1750 MeV and 2104 MeV revealed upon reanalyzing of data on $J/\Psi \rightarrow \gamma 2\pi^+2\pi^-$ \[7\], and 3) the $f_0(1400)$, $f_0(1500)$, $f_0(1520)$. The masses, widths and decay branching ratios of these states are incompatible within the errors quoted by the groups. We do not consider it as plausible that so many scalar isoscalar states exist in such a narrow mass interval. Instead, we take the various states as manifestation of one object which we identify tentatively with the $f_0(1450)$.

It has been convincingly argued that the narrow $a_0(980)$, which has also been seen as a narrow structure in $\eta\pi$ scattering, can be generated by meson-meson dynamics alone \[3, 2\]. This interpretation of the $a_0(980)$ leaves the $a_0(1320)$ or $a_0(1450)$ (which may be manifestations of one state having a mass in the interval 1350-1400 MeV) as the $1^3P_0$ $q\bar{q}$ state. Similarly, it is usually assumed that the $f_0(980)$ is a $K\bar{K}$ molecule, as suggested originally by Weinstein and Isgur \[8\]. The mass degeneracy with the $a_0(980)$ and their proximity to the $K\bar{K}$ threshold seem to require that the nature of both states should be the same. On the other hand, the $K\bar{K}$ interaction in the $I = 1$ and $I = 0$ channels is very different: the extremely attractive $I = 0$ interaction may not support a loosely bound state. Instead, it may just define the pole position of the $f_0(980)$ $q\bar{q}$ resonance. Indeed, Morgan and Pennington \[10\] find the $f_0(980)$ pole structure characteristic for a genuine resonance of the constituents and not of a weakly bound system. The $I = 1$ $K\bar{K}$ interaction is weak and may generate a $K\bar{K}$ molecule. Alternatively, Tönnqvist \[11\] interprets both the $f_0(980)$ and $a_0(980)$ as the members of the $q\bar{q}$ nonet with strong coupling to the decay channels. However, this does not account for the recently discovered $a_0(1320)$ and $a_0(1450)$.

With respect to the $f_0(1370)$ (or two separate states, $f_0(1300)$ and $f_0(1370)$, according to a previous edition of PDG), one may follow the arguments of Morgan and Pennington \[10\] and assume that the $\pi\pi$ interaction produces both very broad, $f_0(1000)$, and narrow, $f_0(980)$, states, giving rise to a dip at 980 MeV in the squared $\pi\pi$ scattering amplitude $T_{11}$. In this picture, the $f_0(1370)$ is interpreted as the high-mass part of the $f_0(1000)$ (the low-mass part may be associated with the $\sigma$ of the most recent PDG). In experiments, the $f_0(1000)$ shows up at $\sim 1300$ MeV because of the pronounced dip in $|T_{11}|^2$ at $\sim 1$ GeV. The $f_0(1000)$ has an extremely large width; thus a resonance interpretation is questionable. It could be generated by $t$-channel
exchanges instead of inter-quark forces \[12\].

The $f_0(1500)$ resonance has been recently observed by the Crystal Barrel collaboration in $p\bar{p}$ annihilations \[13\]. It was claimed that this state has a peculiar decay pattern \[15\]

$$\pi\pi : \eta\eta : \eta\eta' : K\bar{K} = 1.45 : 0.39 \pm 0.15 : 0.28 \pm 0.12 : < 0.15. \quad (1)$$

This pattern can be reproduced by assuming the existence of an additional scalar state which is mainly $s\bar{s}$ and should have a mass of about 1700 MeV, possibly the $f_J(1710)$, and tuning the mixing of the $f_0(1500)$ with the $f_0(1370)$ $n\bar{n}$ ($n = u, d$) and the (predicted) $f_0(1700)$ $s\bar{s}$ states \[12\]. In this picture, the $f_0(1500)$ is interpreted as a glueball state with strong mixing with the close-by conventional scalar mesons.

On the other hand, Lee and Weingarten \[16\] interpret the $f_0(1500)$ as a mainly $s\bar{s}$ state which mixes strongly with the close-by mainly $n\bar{n}$ and scalar glueball states which show up as resonances at 1390 MeV and 1710 MeV, respectively. The latter is in agreement with the values for the scalar glueball mass $1740 \pm 71$ MeV and $1710 \pm 50$ MeV obtained from QCD lattice calculations by Sexton et al. \[17\] and Luo et al. \[18\], respectively. An interpretation of the $f_0(1500)$ as a conventional $q\bar{q}$ state, as well as a qualitative explanation of its reduced $K\bar{K}$ partial width, were also given by Klempt et al. \[19\] in a relativistic quark model with linear confinement and instanton-induced interaction. A quantitative explanation of the reduced $K\bar{K}$ partial width of the $f_0(1500)$ was given in a recent publication by the same authors \[20\]. The decay pattern obtained in \[20\] (with the $f_0$-$f_0'$ mixing angle $\approx 25^o$),

$$\pi\pi : \eta\eta : \eta\eta' : K\bar{K} = 1.45 : 0.32 : 0.18 : 0.03,$$

is in excellent agreement with (1).

The above arguments lead one to the following spectrum of the scalar meson nonet (in the order: isovector, isodoublet, isoscalar mostly octet, isoscalar mostly singlet),

$$a_0(1320) \text{ or } a_0(1450), \quad K_0^*(1430), \quad f_0(1500), \quad f_0(980) \text{ or } f_0(1000). \quad (2)$$

This spectrum agrees essentially with the $q\bar{q}$ assignments found by Klempt et al. \[19\], and Dmitrasinovic \[21\] who considered the Nambu–Jona-Lasinio model with a $U_A(1)$ breaking instanton-induced 't Hooft interaction. The spectrum of the meson nonet given in \[19\] is

$$a_0(1320), \quad K_0^*(1430), \quad f_0(1470), \quad f_0(980), \quad (3)$$

while that suggested by Dmitrasinovic, on the basis of the sum rule

$$m_{f_0}^2 + m_{f_0'}^2 + m_\eta^2 + m_{\eta'}^2 = 2(m_K^2 + m_{K^*_0}^2) \quad (4)$$

$^1$New preliminary results by Crystal Barrel are \[14\]

$$\pi\pi : \eta\eta : \eta\eta' : K\bar{K} = 1.45 : 0.34 : 0.48 : 0.48 \pm 0.24,$$

with normalization of $\pi\pi$ in agreement with (1).
The $q\bar{q}$ assignment obtained by one of the authors by the application of the linear mass spectrum discussed in ref. \[22\] to a composite system of the two, pseudoscalar and scalar nonets, is \[23\]

\[ a_0(1320), \ K_0^*(1430), \ f_0(1590), \ f_0(1000). \]  

in essential agreement with (3) and (5). The assignment (6) has found further justification in the constituent quark model explored by us in ref. \[24\].

In our previous paper \[25\], by using Regge phenomenology, we derived a new mass relation ($I$ stands for isospin):

\[ 2M^2(s\bar{s}) + 3M^2(n\bar{n}, I = 1) = 4M^2(s\bar{n}) + M^2(n\bar{n}, I = 0). \]  

This relation was obtained for the pseudoscalar mesons, and further generalized to every meson multiplet. It differs from the Sakurai mass formula \[26\] obtained in the case of the ideal nonet mixing,

\[ \theta_{id} = \arctan \frac{1}{\sqrt{2}} \approx 35.3^o, \]  

by only a term which depends explicitly on isospin variation, \[\sim (M^2(n\bar{n}, I = 1) - M^2(n\bar{n}, I = 0))\]. This term improves the accuracy of the Sakurai formula, which is not bad by itself, (\[\sim 2\text{-}3\%\]) by a factor of 2 \[25\].

In this paper we present another derivation of the formula (7) for pseudoscalar and scalar mesons in the Nambu–Jona-Lasinio model with an instanton-induced interaction. Before doing this, let us mention that this formula can be applied to the physical states of these two nonets provided the mixing angles are known. For example, for pseudoscalar mesons, the $\eta-\eta'$ mixing angle is given by duality constraints \[27\],

\[ \tan \theta_{\eta\eta'} = -\frac{1}{2\sqrt{2}}, \quad \theta_{\eta\eta'} \approx -19.5^o, \]  

in good agreement with most of experimental data \[28\]. In view of the relations

\[ \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos \theta_{\eta\eta'} & -\sin \theta_{\eta\eta'} \\ \sin \theta_{\eta\eta'} & \cos \theta_{\eta\eta'} \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_9 \end{pmatrix}, \quad \begin{pmatrix} \eta_8 \\ \eta_9 \end{pmatrix} = \begin{pmatrix} \cos \theta_{id} & -\sin \theta_{id} \\ \sin \theta_{id} & \cos \theta_{id} \end{pmatrix} \begin{pmatrix} \eta_{\eta} \\ \eta_{\eta'} \end{pmatrix}, \]

where

\[ \eta_8 = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}, \quad \eta_9 = \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}} \]

are the isoscalar octet and singlet states, respectively, and

\[ \eta_n = \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}, \quad \eta_s = s\bar{s} \]
are the “ideal-mixture” counterparts of the physical $\eta$ and $\eta'$ states, one obtains
\[
\begin{pmatrix}
\eta_s \\
\eta_n
\end{pmatrix} = \begin{pmatrix}
\cos(\theta_{id} - \theta_{\eta\eta'}) & -\sin(\theta_{id} - \theta_{\eta\eta'}) \\
\sin(\theta_{id} - \theta_{\eta\eta'}) & \cos(\theta_{id} - \theta_{\eta\eta'})
\end{pmatrix}
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\cos \xi & -\sin \xi \\
\sin \xi & \cos \xi
\end{pmatrix}
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix},
\]
(11)
where $\xi \equiv \theta_{id} - \theta_{\eta\eta'}$ and, as follows from (8),
\[
\cos \xi = \frac{\sin \theta_{\eta\eta'} + \sqrt{2} \cos \theta_{\eta\eta'}}{\sqrt{3}}.
\]
(12)

Assuming, as usual, that the relevant matrix elements are equal to the squared masses of the corresponding states, and using the orthogonality of the $\eta$ and $\eta'$ as physical states, we obtain, using (10)-(12),
\[
m_{\eta_n}^2 = \frac{2}{3} m_{\eta}^2 + \frac{1}{3} m_{\eta'}^2,
\]
(13)
\[
m_{\eta_s}^2 = \frac{1}{3} m_{\eta}^2 + \frac{2}{3} m_{\eta'}^2,
\]
(14)
in agreement with naive expectations from the quark content of these states which is, in view of (10) \[25\],
\[
\eta = \frac{u \bar{u} + d \bar{d} - s \bar{s}}{\sqrt{3}}, \quad \eta' = \frac{u \bar{u} + d \bar{d} + 2s \bar{s}}{\sqrt{6}}.
\]
The use of the values (13),(14) in Eq. (7) leads finally to
\[
4m_K^2 = 3m_n^2 + m_s^2,
\]
(15)
which is the new Gell-Mann–Okubo mass formula for pseudoscalar mesons found in our previous publication \[25\] and which is satisfied to an accuracy of better than 1% by the measured pseudoscalar meson masses.

We now turn to the derivation of the formula (7). We shall adopt the version of the Nambu–Jona-Lasinio (NJL) model which includes the $U_A(1)$ breaking ’t Hooft interaction \[29\] in the form of a $2N_f$-point determinant nonlocal quark interaction, where $N_f$ is the number of flavors. This model has been extensively studied by Dmitrasinovic \[21, 30\]. For $N_f = 2$, and in the local interaction limit, the sum of the determinant and its Hermitian conjugate is equivalent to the following four-point interaction:
\[
L_{ij}^{(4)} = G_2 \left[ \det \left( \bar{\psi} (1 + \gamma_5) \psi \right) + \text{H.c.} \right]
\]
\[
= \frac{G_2}{2} \left[ (\bar{\psi} \psi)^2 - (\bar{\psi} \tau \psi)^2 - (\bar{\psi} i \gamma_5 \psi)^2 + (\bar{\psi} i \gamma_5 \tau \psi)^2 \right].
\]
(16)
Therefore, the resulting effective Lagrangian is
\[
L_{NJL}^{(4)} = L_{NJL} + L_{tH}^{(4)} = \bar{\psi} [i\gamma\partial - m^0] \psi \\
+ \frac{G_1}{2} \left( [\bar{\psi}\psi]^2 + (\bar{\psi}\tau\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2 \right) \\
+ \frac{G_2}{2} \left( [\bar{\psi}\psi]^2 - (\bar{\psi}\tau\psi)^2 - (\bar{\psi}i\gamma_5\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2 \right).
\]

(17)

The coupling constants \( G_1 \) and \( G_2 \) scale as
\[
G_1 = O \left( \frac{1}{N_c} \right), \quad G_2 = O \left( \frac{1}{N_c^2} \right)
\]
in the large \( N_c \) limit, \( N_c \) being the number of colors [21]. Note that in the point limit of single gluon exchange, QCD would produce the \( G_1 \) terms only (as well as their vector and axial-vector analogs) [31].

The original NJL model contains the \( U_L(2) \times U_R(2) \)-symmetric and the \( U_A(1) \) breaking terms with equal weights in the Lagrangian, i.e., with \( G_1 = G_2 \). This case corresponds to the complete vanishing of interaction in the isoscalar pseudoscalar and isovector scalar channels, and hence to the complete disappearance of these states from the spectrum of the model, as seen in Eq. (17). This situation is referred to as “maximal \( U_A(1) \) breaking” in ref. [21]. Since isoscalar pseudoscalar mesons \( \eta \) and \( \eta' \) do exist and one can construct the ideal mixture of them, i.e., a linear combination of the two that contains no \( s\bar{s} \) component (and corresponds to the isoscalar pseudoscalar state of the two-flavor version of NJL), one has to relax the severity of \( U_A(1) \) breaking in the model and thus consider the “minimally extended” [21] two-flavor NJL model, Eq. (17), with \( G_1 \neq G_2 \).

Working out this model in a standard manner, one finds the familiar NJL gap equation for the constituent quark mass \( m \) [32],
\[
m = m^0 - (G_1 + G_2) \langle \bar{\psi}(x)\psi(x) \rangle_0 \\
= m^0 + 4iN_cN_f (G_1 + G_2) \int \frac{d^4p}{(2\pi)^4} \frac{m}{p^2 - m^2},
\]
which has to be regularized either by introducing a Euclidean cutoff or following Pauli and Villars (PV) [33]. The self-consistency condition \( \Sigma_H = m \), where \( \Sigma_H \) is the quark self-energy, determines \( m \) at the one-loop level. This \( m \) is also related to the quark condensate:
\[
m = m^0 - (G_1 + G_2) \langle \bar{\psi}(x)\psi(x) \rangle_0.
\]

A nonzero value of \( m \) in the chiral limit \( m^0 = 0 \) signals the breakdown of chiral symmetry.

The meson masses are further read off from the poles of the corresponding propagators of the inhomogeneous Bethe-Salpeter equation describing the scattering of quarks and antiquarks which are
\[
- iD(k) = \frac{i(G_1 + G_2)}{1 - (G_1 + G_2)\Pi(k)},
\]

(19)
where \( k \) is the four-momentum transfer, and \( \Pi(k) \) represents the sum of all proper polarization diagrams in the relevant channel. The form of the interaction in Eq. (17) gives rise to scattering in the four channels: the isovector pseudoscalar (\( \pi \)), isoscalar pseudoscalar (\( \eta \)), isoscalar scalar (\( \sigma \)) and isovector scalar (\( \sigma \)). One finds, e.g., \( 32 \),

\[
\Pi_\pi(k) - \frac{\Sigma_H}{(G_1 + G_2)m} = -4iN_c k^2 I(k),
\]

(20)

where \( I(k) \) is a logarithmically divergent loop integral,

\[
I(k) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{[p^2 - m^2][(p + k)^2 - m^2]},
\]

(21)

which is PV-regularized. It follows from (20) that

\[
\Sigma_H = (G_1 + G_2)m \Pi_\pi(0)
\]

which, when combined with \( \Sigma_H = m \), leads to the self-consistency condition

\[
1 = (G_1 + G_2) \Pi_\pi(0).
\]

(22)

Then \( D \) of Eq. (19) for the pion mode becomes, through (20) \( 21 \)

\[
-iD_\pi(k) = \frac{i(G_1 + G_2)}{1 - (G_1 + G_2) I_\pi(k)} = \frac{1}{\frac{4N_c I(k)(k^2 + i\varepsilon)}{4N_c I(k)(k^2 + i\varepsilon)}},
\]

(23)

and similarly, for the remaining three modes \( 21 \),

\[
-iD_\sigma(k) = \frac{i(G_1 + G_2)}{1 - (G_1 + G_2) I_\sigma(k)} = \frac{-ig_{\pi qq}^2}{(k^2 + i\varepsilon)F(k) - m^2_{tH}},
\]

(24)

\[
-iD_\eta_{nn}(k) = \frac{i(G_1 - G_2)}{1 - (G_1 - G_2) I_\eta_{nn}(k)} = \frac{-ig_{\eta qq}^2}{(k^2 + i\varepsilon)F(k) - m^2_{tH}},
\]

(25)

\[
-iD_\sigma(k) = \frac{i(G_1 - G_2)}{1 - (G_1 - G_2) I_\sigma(k)} = \frac{-ig_{\sigma qq}^2}{(k^2 + i\varepsilon)F(k) - m^2_{tH}},
\]

(26)

where

\[
m^2_{tH} (N_f = 2) = \frac{2g_{\pi qq}^2 G_2}{G_1^2 - G_2^2} \approx \frac{2g_{\pi qq}^2 G_2}{G_1^2} + O\left(\frac{1}{N_c^2}\right)
\]

(27)

is the 't Hooft mass \( 21 \), and the associated zero external momentum coupling constants are \( 21 \)

\[
g_{\pi qq}^2 = g_{\sigma qq}^2 = g_{\eta_{qq}}^2 = g_{\sigma qq}^2 = \left(\frac{\partial \Pi}{\partial k^2}\right)_0^{-1} \left[ -4iN_c I(0) \right]^{-1} = \left(\frac{m}{f_\pi}\right)^2,
\]

(28)
where $f_\pi = 93$ MeV is the pion decay constant. In Eqs. (23)-(27), $F(k) = I(k)/I(0)$. This factor $F(k)$ provides the $D$’s with a more complicated analytic structure than that of the free scalar propagator; this reflects the composite nature of the bound states which they describe. However, as discussed in detail in ref. [21], one may set $F(k) = 1$ as a reliable approximation which does not affect the predictions of the model essentially. With $F(k) = 1$, reading off the poles of the corresponding $D$’s in (23)-(27), and switching to the hadron spectroscopy notations

$$f_{0n} = \sigma, \quad a_0 = \sigma,$$

one finds the following meson masses, upon introducing explicit chiral symmetry breaking in the form of nonzero current quark masses $m_0$ [21]:

$$m_{\eta_n}^2 = m_\pi^2 + m_{tH}^2 (N_f = 2),$$

$$m_{a_0}^2 = m_\pi^2 + 4m^2 + m_{tH}^2 (N_f = 2),$$

$$m_{f_{0n}}^2 = m_\pi^2 + 4m^2,$$

which lead to the mass relation

$$m_{a_0}^2 - m_{f_{0n}}^2 = m_{\eta_n}^2 - m_\pi^2 = m_{tH}^2 (N_f = 2),$$

found first by Dmitrasinovic [21], and which is a direct consequence of $U_A(1)$ breaking in this NJL model (by instanton-induced ’t Hooft interaction). Here $\eta_n$ and $f_{0n}$ are the nonstrange ideal mixtures of the $\eta, \eta'$ and $f_0, f'_0$ mesons, respectively, in the badly broken SU(3) limit.

It can now be easily understood how “maximal breaking” of $U_A(1)$ occurs in this NJL model. It is reached in the limit of equal coupling constants, $G_1 = G_2$. In this limit, in which the simplest NJL Lagrangian is recovered, as seen in Eq. (17), the ’t Hooft mass, Eq. (27), and the masses of $\eta_n$ and $a_0$, go to infinity. Hence, these heavy modes cannot propagate, so that they completely decouple from the model. That explains their absence in the original NJL model.

The three-flavor generalization of this NJL model is straightforward: The free Lagrangian and the $U(3)_L \times U(3)_R$-symmetric quartic self-interaction terms are essentially the same as in Eq. (17), the number of terms being appropriately extended to 18:

$$L_{\text{NJL}}^{(6)} = \bar{\psi} \left[ i\gamma \partial - m^0 \right] \psi + G \sum_{i=0}^8 \left[ \left( \bar{\psi} \lambda_i \psi \right)^2 + \left( \bar{\psi} i\gamma_5 \lambda_i \psi \right)^2 \right]$$

$$- K \left[ \det \left( \bar{\psi} (1 + \gamma_5) \psi \right) + \det \left( \bar{\psi} (1 - \gamma_5) \psi \right) \right],$$

where $\lambda_i$ are the SU(3) Gell-Mann structure constants, but the $U_A(1)$ breaking determinant interaction term is now of sixth order in the Fermi fields, rather than of fourth order as in the $N_f = 2$ case. Since one cannot work directly with a sixth-order operator, one has to construct an effective mean-field quartic Lagrangian using the expectation values of $\bar{\psi} \psi$ [32, 34]. In the $SU(3)$-symmetric limit, $\langle \bar{\psi} \lambda_0 \psi \rangle = \sqrt{2/3} \langle \bar{\psi} \psi \rangle \neq 0,$
where the quark condensates are defined as
\[ \langle \bar{\psi} \lambda_3 \psi \rangle = \langle \bar{\psi} \lambda_8 \psi \rangle = 0, \]
one finds the following effective Lagrangian \[33\]:

\[ L^{(4)}_{\text{eff}} = \bar{\psi} [i \gamma \partial - m^0] \psi + \left[ K_0^{(-)} (\bar{\psi} \lambda_0 \psi) \right]^2 + \sum_{i=1}^{8} K_i^{(+)} (\bar{\psi} i \gamma_5 \lambda_i \psi)^2 \]

\[ + \left[ K_0^{(+)} (\bar{\psi} i \gamma_5 \lambda_0 \psi)^2 + \sum_{i=1}^{8} K_i^{(-)} (\bar{\psi} \lambda_i \psi)^2 \right], \tag{34} \]

where
\[ K_0^{(\pm)} = G \pm K \langle \bar{q} q \rangle = \frac{1}{2} (G_1 \pm 2 G_2), \tag{35} \]
\[ K_i^{(\pm)} = G \mp \frac{1}{2} K \langle \bar{q} q \rangle = \frac{1}{2} (G_1 \pm G_2), \quad i = 1, 2, \ldots, 8, \tag{36} \]
i.e.,
\[ G_1 = 2G, \quad G_2 = -\frac{1}{3} K \langle \bar{q} q \rangle, \tag{37} \]

where the quark condensates are defined as
\[ \langle \bar{q} q \rangle = -i N_c \text{ Tr } S_F(x, x)^q = -4i N_c \int \frac{d^4 p}{(2\pi)^4} \frac{m_q}{p^2 - m_q^2 + i\varepsilon}, \quad q = u, d, s, \]
\[ \langle \bar{\psi} \psi \rangle = \langle \bar{u} u \rangle + \langle \bar{d} d \rangle + \langle \bar{s} s \rangle = -i N_c \text{ Tr } S_F(x, x). \tag{38} \]

As in the \( N_f = 2 \) case, the meson masses are read off from the poles of the corresponding propagators which are in turn constrained by the system of the gap equations \[21\]:
\[ m_u = m_u^0 - 4G \langle \bar{u} u \rangle + 2K \langle \bar{d} d \rangle \langle \bar{s} s \rangle, \]
\[ m_d = m_d^0 - 4G \langle \bar{d} d \rangle + 2K \langle \bar{s} s \rangle \langle \bar{u} u \rangle, \]
\[ m_s = m_s^0 - 4G \langle \bar{s} s \rangle + 2K \langle \bar{u} u \rangle \langle \bar{d} d \rangle. \tag{39} \]

In the chiral limit when the bare (current) masses \( m_q, q = u, d, s \) vanish, and in the \( SU(3) \)-symmetric limit where \( m_u = m_d = m_s \), one has \( \langle \bar{u} u \rangle = \langle \bar{d} d \rangle = \langle \bar{s} s \rangle = \langle \bar{q} q \rangle \), and hence the system (39) decouples:
\[ m_q = m_q^0 - 4G \langle \bar{q} q \rangle + 2K \langle \bar{q} q \rangle^2 = m_q^0 - 4K_1^{(+)} \langle \bar{q} q \rangle. \tag{40} \]

Upon introducing explicit chiral breaking in the form of nonzero current quark masses \( m_i^0 \), we find the following results from Eq. (40) and the effective Lagrangian (34) \[21\]:
\[ m_{u_0}^2 = m_u^0 + 4m_u^2 + \frac{2}{3} m_{tH}^2 (N_f = 3), \tag{41} \]
\[ m_{K_0}^2 = m_K^2 + 4m_K^2 + \frac{2}{3} m_{tH}^2 (N_f = 3), \tag{42} \]
\[ m_{\eta'}^2 = m_{\eta'}^2 + 2m_K^2 + m_{tH}^2 (N_f = 3), \tag{43} \]
\[ m_{f_0}^2 + m_{\eta'}^2 = 2m_{K_0}^2 - m_{tH}^2 (N_f = 3), \tag{44} \]
\[ m_{\eta'}^2 + m_{f_0}^2 = m_{\eta'}^2 + m_{f_0}^2 + 8m^2 - \frac{2}{3} m_{tH}^2 (N_f = 3). \tag{45} \]
where
\[ m_{tH}^2 (N_f = 3) \equiv \frac{3g_{\pi qq}^2 G_2}{G_1^2 - G_2^2} = \frac{3}{2} m_{tH}^2 (N_f = 2) \simeq \frac{3g_{\pi qq}^2 G_2}{G_1^2} + O \left( \frac{1}{N_f} \right) , \] (46)
in view of (27).

Summing up Eqs. (44) and (45), one immediately obtains the Dmitrasinovic sum rule (4). This sum rule is not, however, the only consequence of this model. Indeed, Eqs. (32),(43),(46) and \( m_\eta^2 + m_\pi^2 = m_{\eta_1}^2 + m_{\eta_2}^2 \) which follows in general from (11), and in particular, with the mixing angle (10), from (13),(14), lead to
\[ m_\eta^2 - m_\pi^2 = \frac{2}{3} m_{tH}^2 (N_f = 3) , \] (47)
\[ m_{\eta_1}^2 + m_{\eta_2}^2 = 2m_K^2 + m_{tH}^2 (N_f = 3) , \] (48)
which, upon eliminating \( m_{tH}^2 (N_f = 3) \), yields
\[ 2m_{\eta_1}^2 + 3m_\pi^2 = 4m_K^2 + m_{\eta_2}^2 . \] (49)

Similarly, Eqs. (32),(44),(46) and
\[ m_{f_0}^2 + m_{f_0'}^2 = m_{f_0}^2 + m_{f_0'}^2 , \] (50)
which follows from Eqs. (55),(56) below, lead to
\[ m_{a_0}^2 - m_{f_{0n}}^2 = \frac{2}{3} m_{tH}^2 (N_f = 3) , \]
\[ m_{f_{0n}}^2 + m_{f_{0s}}^2 = 2m_{K^*_{0s}}^2 - m_{tH}^2 (N_f = 3) , \]
which again, upon eliminating \( m_{tH}^2 (N_f = 3) \), yields
\[ 2m_{f_{0s}}^2 + 3m_{a_0}^2 = 4m_{K^*_{0s}}^2 + m_{f_{0n}}^2 . \] (51)

Eqs. (49) and (51) represent the formula (7) written for the pseudoscalar and scalar mesons, respectively.

We consider the fact that the formula (7) can be derived in (at least) two completely independent ways, viz., from Regge phenomenology, as done in ref. [25], and in the NJL model, as done in the present paper, as sufficient to treat this formula as (almost) model-independent.\(^2\) As discussed above, in the pseudoscalar meson case, this formula reduces to Eq. (15) which is satisfied to an accuracy of better than 1%; it therefore finds its direct experimental confirmation in terms of the pseudoscalar meson mass spectrum. Since this formula should describe the physical scalar meson mass spectrum as well, we shall apply it to the problem of the correct \( q\bar{q} \) assignment for the scalar meson nonet.

\(^2\)We know of at least four other distinct approaches which also yield these same results.
We note first that the masses of the ideally mixed counterparts of the physical $f_0$ and $f'_0$ mesons are easily obtained from Eqs. (44),(49),(51):

\[ m_{f_{0n}}^2 = m_{a_0}^2 - \frac{2}{3} m_{tH}^2 \left( N_f = 3 \right), \quad (52) \]
\[ m_{f_{0s}}^2 = 2m_{K_0^*}^2 - m_{a_0}^2 - \frac{1}{3} m_{tH}^2 \left( N_f = 3 \right), \quad (53) \]

where the numerical value of $m_{tH}^2 \left( N_f = 3 \right)$ is

\[ m_{tH}^2 \left( N_f = 3 \right) = m_{\eta}^2 + m_{\eta'}^2 - 2m_{K}^2 \simeq 0.72 \text{ GeV}^2. \quad (54) \]

To determine the masses of the physical $f_0$ and $f'_0$ states, one has to know the $f_0$-$f'_0$ mixing angle.

In the pseudoscalar meson case, the $\eta$-$\eta'$ mixing angle is given by the mass-matrix diagonalization as \[ ^{25} \]

\[ \tan 2\theta_{\eta\eta'} = 2\sqrt{2} \left( 1 - \frac{3m_{tH}^2 \left( N_f = 3 \right)}{2(m_{K}^2 - m_{\pi}^2)} \right)^{-1} \approx -19^\circ, \quad (55) \]

in agreement with (10). In the scalar meson case, one has, respectively, \[ ^{26} \]

\[ \tan 2\theta_{f_0f'_0} = 2\sqrt{2} \left( 1 + \frac{3m_{tH}^2 \left( N_f = 3 \right)}{2(m_{K_0^*}^2 - m_{a_0}^2)} \right)^{-1}, \quad (56) \]

since the sign of the instanton-induced 't Hooft interaction (the sign of $m_{tH}^2$) for the scalar mesons is opposite to that for the pseudoscalar mesons, as is clear from the above consideration of the NJL model.

As seen in Eqs. (47),(48),(52),(53),(55),(56), when the 't Hooft interaction is switched off, $m_{tH}^2 \left( N_f = 3 \right) = 0$, one recovers ideal mixing, $\theta_{\eta\eta'} = \theta_{f_0f'_0} = \arctan 1/\sqrt{2}$, and the corresponding ideal structures for both multiplets,

\[ m_{\eta}^2 = m_{\eta_0}^2 = 2m_{K}^2 - m_{\pi}^2, \quad m_{\eta'}^2 = m_{\eta_0}^2 = m_{\pi}^2, \]
\[ m_{f_0}^2 = m_{f_{0n}}^2 = 2m_{K_0^*}^2 - m_{a_0}^2, \quad m_{f_0}^2 = m_{f_{0s}}^2 = m_{a_0}^2, \]

which are solutions to Eqs. (47) and (49), respectively, as they should be.

Using Eqs. (52),(53),(56) and

\[ m_{f_{0n}}^2 = \sin^2 \xi m_{f_0}^2 + \cos^2 \xi m_{f'_0}^2, \quad (57) \]
\[ m_{f_{0s}}^2 = \cos^2 \xi m_{f_0}^2 + \sin^2 \xi m_{f'_0}^2, \quad (58) \]

\[ \cos \xi = \frac{\sin \theta_{f_0f'_0} + \sqrt{2} \cos \theta_{f_0f'_0}}{\sqrt{3}}, \quad (59) \]

\[ ^{3} \text{Similar relations for the pseudoscalar mesons follow from Eqs. (47),(48) above.} \]
\[ ^{4} \text{The formula (56) coincides with Eq. (51) of ref. \[ ^{27} \], and Eq. (41) of ref. \[ ^{30} \].} \]
which follow from relations similar to Eqs. (11),(12) in the scalar meson case, one can easily calculate the masses of the $f_0$ and $f'_0$ states and their mixing angle.

The results of our calculation are presented in Table I where we have considered a number of different possibilities for the mass of the isovector scalar state $a_0$ consistent with the masses of the experimental candidates $a_0(980)$, $a_0(1320)$ and $a_0(1450)$.

| $m_{a_0}$, MeV | $m_{f_0}$, MeV | $m_{f'_0}$, MeV | $\theta_{f_0f'_0}$, degrees |
|----------------|---------------|----------------|---------------------------|
| 983.5          | 1710          | 663            | 27.3                      |
| 1290           | 1511          | 1039           | 18.1                      |
| 1320           | 1490          | 1069           | 15.8                      |
| 1350           | 1471          | 1096           | 12.8                      |
| 1410           | 1437          | 1140           | 3.8                       |
| 1450           | 1424          | 1156           | -4.8                      |
| 1490           | 1423          | 1157           | -14.6                     |

Table I. Predictions for the masses and mixing angle of the scalar mesons for different values of the $a_0$ meson mass, according to Eqs. (52),(53),(56)-(59). The value of the $K_0^*$ meson mass used in the calculation is $m_{K_0^*} = 1429$ MeV.

For the $a_0(980)$ taken as the isovector scalar state, our solution exhibits a number of interesting features: It predicts an $f_0$-$f'_0$ mixing angle close to the ideal mixture value (8), i.e., both isoscalar states would be almost pure $n\bar{n}$ and $s\bar{s}$; their masses are predicted to be in agreement with those of the $\sigma$ of the most recent edition of PDG and the $f_J(1710)$, respectively. The existence of the latter state, which in this case is almost pure $s\bar{s}$, would be in agreement with the prediction by Amsler and Close [15]. However, we will disregard this solution as unphysical. Indeed, as we discuss at the beginning of the paper, convincing arguments exist in the literature regarding the $a_0(980)$ not being a $q\bar{q}$ state, but rather molecular in character. The existence of the $\sigma$-meson (to be identified as the $f'_0(663)$ in this case) is controversial due to its enormous width. The spin of the $f_J(1710)$ meson is still uncertain ($J = 0$ or 2). Moreover, with this solution, the mass interval occupied by the scalar nonet turns out to be wider than 1 GeV, a typical mass scale of light hadron spectroscopy, which cannot be considered as plausible.

Thus, we restrict ourselves to the solution with $m_{a_0} = 1390 \pm 100$ MeV (to accommodate the both experimental candidates $a_0(1320)$ and $a_0(1450)$) which, according to the results presented in Table I, is

$$m_{f_0} = 1467 \pm 44 \text{ MeV}, \quad m_{f'_0} = 1098 \pm 59 \text{ MeV}, \quad \theta_{f_0f'_0} = 1.8 \pm 16.3^\circ.$$

It is interesting to note that for rather wide ranges of the $a_0$ mass and $f_0$-$f'_0$ mixing angle, the masses of two isoscalar states are accurately predicted to be in the vicinity of 1.1 GeV and 1.45 GeV. Especially if the $a_0(1450)$ is the true isovector scalar state (which we hope will be either confirmed or refuted by ongoing experiments), then for a wide range of its mass ($1450 \pm 40$ MeV) the solution picks the masses of the two isoscalars very accurately at $1430 \pm 7$ MeV and $1148 \pm 8$ MeV. Nothing can be said,
however, about their decay properties so far, since the solution, albeit specifying the
two masses, cannot choose between the mixing angles, the proper determination of
which requires a specific value of the $a_0$ mass.

The solution that we obtained provides the mass of the scalar isoscalar mostly
singlet state around 1.1 GeV. This value may be considered as the pole position
which gets shifted further down due to strong coupling to, e.g., the $KK$ threshold.
For example, recent analysis of various experimental data by Anisovich et al. [35]
reveals a lowest mass scalar resonance with the pole position at $1015 \pm 15$ MeV and
width of $43 \pm 8$ MeV. We do not address the question of choosing between the $f_0(980)$
and $f_0(1000)$ in this paper. Let us only mention that with the $f_0-f'_0$ mixing angle
$\theta_{f_0 f'_0} = -1/2 \arctan 1/(2\sqrt{2}) \approx -9.7^o$ which is achieved with $m_{a_0} \approx 1470$ MeV, the
quark content of the two isoscalars turn out to be

$$f_0 = \frac{n\bar{n} - s\bar{s}}{\sqrt{2}}, \quad f'_0 = \frac{n\bar{n} + s\bar{s}}{\sqrt{2}}, \quad n\bar{n} \equiv \frac{u\bar{u} + d\bar{d}}{\sqrt{2}},$$

i.e., the both have (50% $n\bar{n}$, 50% $s\bar{s}$) quark content. Such quark content can, e.g.,
easily explain a large branching ratio of 22% of the $f_0(980)$ decay into the $KK$ channel.

The mass of the scalar isoscalar mostly octet state predicted by our solution,
$1467 \pm 44$ MeV, is in good agreement with recent analysis by Kaminski et al. [36]
of the three coupled channel ($\pi\pi$, $KK$, $4\pi$) model which reveals a relatively narrow
($\Gamma = 135 \pm 45$ MeV) resonance with a mass of $1430 \pm 30$ MeV.

We remark also that the masses of both isoscalar scalar states predicted in this
paper are below a typical range of $1600 \pm 100$ MeV provided by QCD lattice calcu-
lations as that for the lightest scalar glueball [17, 18, 37]. They may however get
shifted from the predicted values because of possible mixture with the lightest scalar
 glueball.

The $q\bar{q}$ assignment found in this work,

$$a_0(1320) \text{ or } a_0(1450), \quad K_0^*(1430), \quad f_0(1450), \quad f_0(980) \text{ or } f_0(1000), \quad (60)$$
is consistent with the assignments (3),(5),(6) established in three different approaches.
The fact that different approaches lead to essentially the same result encourage us to
believe that we are not far from the complete resolution of the scalar meson nonet
enigma.

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