New 2d $\mathcal{N} = (0, 2)$ dualities from four dimensions

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Abstract: We propose some new infra-red dualities for 2d $\mathcal{N} = (0, 2)$ theories. The first one relates a $USp(2N)$ gauge theory with one antisymmetric chiral, four fundamental chirals and $N$ Fermi singlets to a Landau–Ginzburg model of $N$ Fermi and $6N$ chiral fields with cubic interactions. The second one relates $SU(2)$ linear quiver gauge theories of arbitrary length $N - 1$ with the addition of $N$ Fermi singlets for any non-negative integer $N$. They can be understood as a generalization of the duality between an $SU(2)$ gauge theory with four fundamental chirals and a Landau–Ginzburg model of one Fermi and six chirals with a cubic interaction. We derive these dualities from already known 4d $\mathcal{N} = 1$ dualities by compactifications on $S^2$ with suitable topological twists and we further test them by matching anomalies and elliptic genera. We also show how to derive them by iterative applications of some more fundamental dualities, in analogy with similar derivations for parent dualities in three and four dimensions.
1 Introduction

Among the most fascinating phenomena that may characterize the low energy dynamics of a quantum field theory are infra-red dualities. This occurs when two different microscopic theories become equivalent in the infra-red (IR). A vast amount of such dualities have been discovered over the years since the first example of Seiberg duality in four dimensions [1], especially for theories with supersymmetry. These theories possess indeed several protected quantities that can be used to test dualities, such as partition functions on different backgrounds which have been computed exactly using localization techniques (for a review see [4], especially contribution [5] for localization in two dimensions).

1See for example [2, 3] for two-dimensional versions of Seiberg duality with $\mathcal{N} = (2, 2)$ supersymmetry.
Nevertheless, a complete understanding of an organizing principle behind the currently known IR dualities in two, three and four dimensions hasn’t been achieved yet. One possible approach towards this goal resides in the study of dimensional reductions of dualities. One can indeed hope that all of the dualities in lower dimensions can be derived from a small, restricted set of dualities in higher dimensions. Remarkable results have been obtained over the last few years in this direction.

A different point of view on the understanding of the dimensional reduction limit of IR dualities is that this can be used to derive new dualities from already known ones. More precisely, we can start from a known duality in $d$ dimensions, compactify both of the dual theories on a $(d - d')$-dimensional manifold and flow to energies much smaller than the compactification scale. In this way we obtain two theories in $d'$ dimensions and we can ask ourselves if they are still dual or not. In some cases, when enough insight is gained about the dimensional reduction limit, one can even push this approach further and try to reverse the logic. Namely, we can start from a duality in lower dimensions and use it to guess a still unknown parent duality in higher dimensions. This should not be intended as an attempt of reversing the RG flow, but just as a hint for the existence of the higher dimensional duality.

There are several subtleties that one has to take into account when studying the dimensional reduction of a duality and in general it may just happen that the two lower dimensional theories are not dual. This is due to the fact that two different limits are involved in the dimensional reduction and issues of order of limits are typically involved. The first limit consists of flowing to low energies while keeping the compactification radius $r$ fixed. Here is where the duality holds and we expect the two $d$-dimensional theories to flow to the same fixed point. The second limit is the strict dimensional reduction limit $r \to 0$ while keeping the energy scale fixed. Taking the two limits in this order would give us the lower dimensional version of the fixed point theory of the original $d$-dimensional dual theories. If we instead take the limit $r \to 0$ first, we obtain two lower dimensional theories that we can conjecture being two different UV description of the aforementioned $d'$-dimensional fixed point theory. Thus, we understand that this conjecture is true and that the duality survives the dimensional reduction only if the two limits commute, but this is not always true.

This problem has so far been understood at a different level depending on the setup considered. The most understood case is the dimensional reduction of $4d \mathcal{N} = 1$ dualities on $S^1$, giving dualities between $3d \mathcal{N} = 2$ theories [6, 7]. A crucial role is played here by monopole operators, in the sense that the two UV theories in three dimensions are dual to each other provided that they are supplemented by some monopole superpotential. These additional superpotential terms also explain a mismatch of symmetries that we can have between the $4d$ and the $3d$ theories. Indeed, four-dimensional theories typically possess anomalous $U(1)$ axial symmetries, which are not anomalous in three dimensions. The monopole superpotential has precisely the effect of breaking the symmetries of the three-dimensional theory that were anomalous in $4d$.

Another set-up that was analyzed in details is the dimensional reduction of $3d \mathcal{N} = 2$ theories on $S^1$, giving dualities between $2d \mathcal{N} = (2, 2)$ theories [8, 9]. In this case there are subtleties related to the fact that the resulting two-dimensional theories may have a non-
compact target space [10]. Indeed, in 2d the ground state can explore the entire moduli space of the theory because of quantum fluctuations, so that we can’t just focus on a single region of it. Moreover, the metric on the target space, which is not protected by supersymmetry, is classically marginal in two dimensions. Consequently, in order to claim for a duality we need a complete knowledge of the target space of the theories at the quantum level, which is in general extremely difficult to achieve. This problem doesn’t appear when the theories have a compact target space or when massive deformations are turned on, since these have the effect of lifting the vacua of the theory, leaving just discrete isolated vacua, but in the non-compact case it is not guaranteed that the duality survives when massive deformations are switched off.

In this paper we are interested in the dimensional reduction of 4d $\mathcal{N} = 1$ dualities on $S^2$ with a topological twist to give dualities between 2d $\mathcal{N} = (0, 2)$ theories [11]. There has been renewed interest in 2d $\mathcal{N} = (0, 2)$ dualities after the discovery of the trialities of [12, 13] (see also [14, 15] for examples with a different amount of chiral supersymmetry) and in [11] it was shown how to derive them by dimensional reduction of Seiberg duality in four dimensions. This type of dimensional reduction is characterized by many more subtleties than the aforementioned cases, some of them being the followings:

- Truncation to the zero modes. A single 4d theory compactified on $S^2$ may lead in general to an infinite direct sum of 2d theories describing the various KK modes of the 4d fields. A prescription was given in [11] for how to obtain a single 2d theory describing the zero modes only, which we are going to review.

- Non-compact target space. Also in this set-up there could be problems related to the non-compactness of the target spaces of the resulting 2d theories. For this reason, all the dualities discussed in this paper should be more appropriately considered as dualities between mass deformed theories.

- Anomalous vs. non-anomalous global symmetries. Because of the different nature of anomalies in two and four dimensions, it may happen that a $U(1)$ global symmetry which was anomalous in 4d is not anomalous in 2d. Typically the two-dimensional dualities obtained from four dimensions only hold provided that this symmetry is broken also in the 2d theory. This situation is reminiscent of the monopole superpotential that is non-perturbatively generated when going from 4d to 3d, but it is still not clear what should cause such a symmetry breaking in 2d. The majority of the examples we will discuss are not affected by such a problem.

On top of these issues, it may still happen that the resulting two-dimensional theories are not dual to each other and that standard tests of the duality, such as matching global symmetries, anomalies or elliptic genera, don’t work. In this paper we are going to present some cases where the dimensional reduction seems to work, leading to new dualities for 2d $\mathcal{N} = (0, 2)$ theories starting from known 4d $\mathcal{N} = 1$ dualities. We will perform several tests of the proposed dualities and show how to derive them by iterative applications of some already known dualities.
The paper is organized as follows. In Section 2 we review some known 2$d$ $N = (0, 2)$ dualities that will play a role in our discussion and the prescription of [11] for how to derive them from four dimensions. In Section 3 we propose a new 2$d$ duality for a $USp(2N)$ gauge theory with antisymmetric matter, which is obtained from a parent four-dimensional duality discussed in [16]. In Section 4 we propose an infinite dimensional family of dual frames made of $SU(2)$ linear quiver gauge theories of arbitrary length $N-1$ and with $N$ Fermi singlets, where $N$ is a non-negative integer number, which is derived from one of the 4$d$ mirror-like dualities of [17]. In Appendix A we comment on the possibility of another duality for a $USp(2N)$ gauge theory with antisymmetric matter, but with a different number of fundamental matter fields than the one of Section 3. Finally, in Appendix B we summarize our conventions for the elliptic genus.

2 Review material

In this section we review some known results that will be important in our next discussion. We first describe some aspects of the 2$d$ $N = (0, 2)$ duality between the $SU(2)$ gauge theory with 4 fundamental chiral fields and the Landau–Ginzburg (LG) model of one Fermi and 6 chiral fields with a cubic interaction proposed in [11] and further analyzed in [18]. Then, we briefly explain the prescription of [11] for the reduction of 4$d$ $N = 1$ theories to 2$d$ $N = (0, 2)$ on $S^2$ with a topological twist. We conclude the section revisiting the dimensional reduction of Intriligator–Pouliot duality studied in [11], focusing in particular on the confining case.

2.1 Duality for the $SU(2)$ gauge theory with 4 chiral fields

We consider the following 2$d$ $N = (0, 2)$ duality [11, 18]:

Theory $\mathcal{T}_A$: $SU(2)$ gauge theory with four fundamental chiral fields $Q_a$ and no interaction\(^2\)

$$W_{\mathcal{T}_A} = 0.$$ (2.2)

Theory $\mathcal{T}_B$: LG model of one Fermi field $\Psi$ and six chiral fields $\Phi_{ab}$ for $a < b = 1, \cdots, 4$ with cubic interaction

$$W_{\mathcal{T}_B} = \Psi \text{Pf} \Phi.$$ (2.3)

The global symmetry of both of the theories is $SU(4)_u \times U(1)_s^3$, under which the fields transform according to

\(^2\)Recall that in a 2$d$ $N = (0, 2)$ theory we can have $E$ and $J$ interactions between Fermi multiplets $\Psi^i$ and chiral multiplets $\Phi^a$ (see [19] for a review). In all the examples we will consider in this paper, only $J$-interactions will be involved, which take the form

$$\int d\bar{\theta}^+ W(\Psi^i, \Phi^a) = \int d\bar{\theta}^+ \Psi^i J_i(\Phi^a),$$ (2.1)

where $J_i$ are generic holomorphic functions of the chiral multiplets $\Phi^a$ only. Hence, $W$ has to be a Fermi operator of 2$d$ $N = (0, 2)$ R-charge one. By abuse of notation, we will call $W$ the “superpotential” of the 2$d$ theory.

\(^3\)Throughout the paper we will label the factors in the global symmetry groups with the names we will use for the corresponding fugacities in the elliptic genus.
where we also introduced a possible choice of UV trial right-moving R-symmetry $U(1)_{R_0}$. When we flow to low energies this can mix with all the other abelian global symmetries of the theory and the exact superconformal one of the IR theory will take the form

$$R = R_0 + q_s R_s,$$

(2.4)

where $q_s$ is the charge under $U(1)_s$ and $R_s$ is the mixing coefficient, which can be fixed with $c$-extremization [20].

On top of matching global symmetries, there are several tests that we can perform for this duality. One consists of matching anomalies for the global symmetries. For both of the theories we find

$$\text{Tr} \gamma^3 U(1)^2_s = 8, \quad \text{Tr} \gamma^3 SU(4)^2_u = 1.$$

(2.8)

Using the generic parametrization of the R-symmetry (2.4), we can compute the trial central charges of the dual theories and verify that they match

$$c_R = 3 \text{Tr} \gamma^3 U(1)^2_R = 15 - 48 R_s + 24 R_s^2, \quad c_R - c_L = \text{Tr} \gamma^3 = 5.$$

(2.9)

Here we encounter a curious feature of this theory. If we try to extremize $c_R$ to find the value of $R_s$ corresponding to the superconformal R-charge we get $R_s = 1$. Plugging this back into the trial central charges we obtain $c_R = -9$ and $c_L = -14$, which violate the unitarity bound. This signals that $c$-extremization fails in this case. There are two possible explanation for such a phenomenon: either the theory is SUSY breaking in the IR or it has a non-compact target space. For this particular duality, the failure of $c$-extremization was interpreted in [18] as due to the non-compactness of the target space.

We use conventions where the chirality matrix $\gamma^3$ takes value $+1$ on right-handed fermions and $-1$ on left-handed fermions. For example

$$\text{Tr} \gamma^3 = n_{\text{chir}} - n_{\text{term}} - d_G,$$

(2.5)

where $n_{\text{chir}}$ is the number of chiral multiplets in the theory, $n_{\text{term}}$ is the number of fermi multiplets and the last term is the contribution of the vector multiplet, with $d_G$ being the dimension of the adjoint representation of the gauge group $G$.

We recall the Dynkin indices for fundamental, adjoint and antisymmetric representations of $SU(N)$

$$T_{SU(N)}(N) = \frac{1}{2}, \quad T_{SU(N)}(N^2 - 1) = N, \quad T_{SU(N)}(N(N - 1)/2) = \frac{N - 2}{2},$$

(2.6)

and of $USp(2N)$

$$T_{USp(2N)}(N) = \frac{1}{2}, \quad T_{USp(2N)}(N(2N + 1)) = N + 1, \quad T_{USp(2N)}(N(2N - 1) - 1) = N - 1.$$

(2.7)
Finally, we can match the elliptic genera of the two theories [21–24] (see also Appendix B for our conventions)

\[ \mathcal{I}_{TA} = \oint \prod_{a=1}^{4} \frac{dz_1}{\theta(s u_a z^\pm_1; q)} = \prod_{a<b} \theta(s^2 u_a u_b; q) = \mathcal{I}_{TB}, \]  

(2.10)

where we defined the following integration measure over USp(2N) gauge fugacities

\[ dz_N = (\frac{q^2}{q; q})^{2N} \prod_{i=1}^{N} \frac{dz_i}{2 \pi i z_i} \theta(z_i^\pm 2; q) \prod_{i<j} \theta(z_i^\pm 1 z_j^\pm 1; q) \]  

(2.11)

and we introduced fugacities \( u_a, s \) in the Cartan of the global symmetry group \( SU(4)_u \times U(1)_s \), with the constraint \( \prod_{a=1}^{4} u_a = 1 \). The integral on the l.h.s. of (2.10) is defined with the prescription of taking the residues only at poles coming from fields with the same charge under the \( U(1) \) in the Cartan of the \( SU(2) \) gauge symmetry. This integral identity first appeared in [14] and one way to test it consists of expanding perturbatively in \( q \) both sides and matching them order by order.

In Section 4.3 we will use a slightly different version of the duality. This is obtained adding on both side of the duality two Fermi fields, which are singlets under the gauge symmetry of theory \( TA \), and coupling them to some of the gauge invariant operators. Specifically, on the side of theory \( TA \) we split the 4 chirals \( Q_a \) into two pairs of chirals \( L_i, R_i \) with \( i = 1, 2 \) and we add two Fermi singlets \( \Psi_L, \Psi_R \) that flip the mesonic operators

\[ WT_A = \Psi_L L L + \Psi_R R R. \]  

(2.12)

This has the effect of explicitly breaking the original \( SU(4)_u \) global symmetry to the subgroup \( SU(2)_l \times SU(2)_r \times U(1)_d \). Schematically, we can represent the new version of theory \( TA \) with the following quiver diagram:

\[ \text{2} \quad \times \quad \text{2} \quad \text{2} \]

where the circle node with the label 2 denotes the \( SU(2) \) gauge symmetry, the two square nodes with the label 2 denote the \( SU(2)_l \times SU(2)_r \) global symmetries, the straight lines represent the chiral multiplets \( L, R \) and the two dashed crosses represent the Fermi singlets \( \Psi_L, \Psi_R \).

On the side of theory \( TB \) the deformation has the effect of making both the new Fermi fields \( \Psi_L, Psit_R \) and two of the original six chirals \( \Phi_{ab} \), specifically those uncharged under the \( SU(2)_l \times SU(2)_r \) subgroup of \( SU(4)_u \), massive. Hence, we end up with an \( SU(2)_l \times SU(2)_r \) bifundamental chiral field \( Q_{ij} \) and a Fermi field \( \Psi \) interacting with

\[ WT_B = \Psi Q Q. \]  

(2.13)

\[ ^6 \text{We will often omit contractions of gauge and flavor indices, which should be understood from the context. For example in this case } \Psi Q Q = c^{ij} c^{kl} \Psi Q_{ik} Q_{jl}. \]
Schematically, we can represent the new version of theory $\mathcal{T}_B$ with the following quiver diagram:

![Quiver Diagram](image)

The equality of the elliptic genera associated to this duality, which we will use intensively in Subsubsection 4.3.2, is obtained from (2.10) by simply re-expressing it in terms of the fugacities for the subgroup $SU(2)_l \times SU(2)_r \times U(1)_d \subset SU(4)_u$ and moving the contributions of two of the chiral fields from the r.h.s. to the l.h.s.\(^7\)

\[
\mathcal{I}_{\mathcal{T}_A} = \theta \left( q s^{-2} d^{\pm 2}; q \right) \oint \frac{dz_1}{\theta \left( s d^{l\pm 1} z^{\pm 1}; q \right) \theta \left( s d^{-r\pm 1} z^{\pm 1}; q \right)} = \theta \left( q s^{-4}; q \right) \frac{\theta \left( s^2 d^{l\pm 1} z^{\pm 1}; q \right)}{\theta \left( s^2 d^{-r\pm 1} z^{\pm 1}; q \right)} = \mathcal{I}_{\mathcal{T}_B}.
\]

(2.14)

Observe that the fugacity $d$ completely disappeared from the expression for $\mathcal{I}_{\mathcal{T}_B}$. Indeed, $U(1)_d$ is not a symmetry of this alternative version of theory $\mathcal{T}_B$, since no fields charged under it remained. The equality (2.14) then implies that even if we refine the elliptic genus of theory $\mathcal{T}_A$ with the fugacity for $U(1)_d$, in the end it is actually independent of $d$. This fact can be checked by computing perturbatively in $q$ the elliptic genus $\mathcal{I}_{\mathcal{T}_A}$. This means that $U(1)_d$ is not actually a symmetry of the low energy theory to which $\mathcal{T}_A$ flows. We will come back to this point in Subsection 4.3.

### 2.2 Dimensional reduction of 4d $\mathcal{N} = 1$ dualities on $S^2$

The dimensional reduction of 4d $\mathcal{N} = 1$ theories to 2d $\mathcal{N} = (0, 2)$ theories has been discussed in details in [11]. The first step consists of defining the four-dimensional theory on $\mathbb{R}^2 \times S^2$ preserving half of its supersymmetry. As discussed also in [25–27], this can be done by introducing a background vector multiplet for a $U(1)^{4d}_R$ R-symmetry with a quantized flux through $S^2$ so to cancel the contribution of the spin connection of $S^2$ in the supersymmetry variation of the fermionic fields. In this way we preserve two supercharges with same chirality on $\mathbb{R}^2$, giving a 2d theory with $\mathcal{N} = (0, 2)$ supersymmetry. The quantization of the flux translates into the requirement that the $U(1)^{4d}_R$ symmetry should be such that all the chiral fields of the theory have integer R-charge. This $U(1)^{4d}_R$ doesn’t have to be the superconformal one of the theory to which our 4d $\mathcal{N} = 1$ theory flows in the IR, but it has to be non-anomalous. Indeed, it is usually chosen taking specific mixing coefficients with any $U(1)$ in the Cartan of the global symmetries of the theory, possibly also non-abelian ones, which don’t necessarily correspond to the superconformal R-symmetry. This procedure is also known as topological twist [28, 29].

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\(^7\)We also used the property of the theta-function $\theta \left( x; q \right) = \theta \left( q x^{-1}; q \right)$ which trivially follows from its definition $\theta \left( x; q \right) = \left( x; q \right)_{\infty} \left( q x^{-1}; q \right)_{\infty}$, where $\left( x; q \right)_{\infty} = \prod_{k=0}^{\infty} (1 - x q^k)$. This is the translation at the level of the elliptic genus of the fact that a Fermi multiplet in a representation $\mathcal{R}$ is equivalent to a Fermi multiplet in a representation $\bar{\mathcal{R}}$ with $E$ and $J$ interactions swapped.
When defined on $\mathbb{R}^2 \times S^2$, each multiplet of the $4d \mathcal{N} = 1$ theory decomposes in an infinite tower of KK modes which can be re-arranged into different $2d \mathcal{N} = (0, 2)$ multiplets \cite{30}. This can also be understood from the $T^2 \times S^2$ partition function of the $4d \mathcal{N} = 1$ theory, which was computed using localization methods in \cite{26} provided that we choose an integrally quantized R-symmetry $U(1)^{4d}_R$ as we just discussed. Such a partition function takes indeed the form of an infinite sum of contour integrals that are precisely of the form of elliptic genera of $2d \mathcal{N} = (0, 2)$ theories. In \cite{11} it was shown that this sum actually truncates and that it reduces to a single term corresponding to the zero modes of the $4d$ fields provided that we also choose $U(1)^{4d}_R$ to be non-negative\footnote{The dimensional reduction from $4d \mathcal{N} = 1$ to $2d \mathcal{N} = (0, 2)$ without this constraint on the non-negativity of the R-charges was first discussed in \cite{31}.}. Hence, with this choice we get a single $2d \mathcal{N} = (0, 2)$ theory, which is the one describing the zero modes of the $4d$ fields.

The zero modes of the $4d$ fields can be re-organized into $2d \mathcal{N} = (0, 2)$ multiplets according to the following rules \cite{11, 30}. A $4d \mathcal{N} = 1$ chiral multiplet of R-charge $r$ gives

- $r - 1$ Fermi multiplets when $r > 1$;
- $1 - r$ chiral multiplets when $r < 1$;
- no multiplets when $r = 1$.

Instead, the zero modes of a $4d \mathcal{N} = 1$ vector multiplet consist only of a $2d \mathcal{N} = (0, 2)$ vector multiplet. Hence, a four-dimensional gauge theory will reduce to a two-dimensional gauge theory with the same gauge group, but with different matter content. These matter fields will have interactions that can be understood from the original superpotential of the $4d$ theory.

If we apply this procedure to two dual $4d \mathcal{N} = 1$ theories, we obtain a putative $2d \mathcal{N} = (0, 2)$ duality. As we mentioned in the introduction, it is not always true that the resulting duality is valid and one should perform all the standard tests to understand if this is the case or not. Moreover, there are in principle many choices of the R-symmetry $U(1)^{4d}_R$ that we can make, which lead to different possible $2d$ dualities starting from a single duality in $4d$. Nevertheless, these possibilities are strictly constrained from the requirement that the $4d$ R-charges should be non-negative integers and from the cancellation of gauge anomalies in the resulting $2d$ theories.

As a particular example of the application of the prescription of \cite{11} we will review how to obtain the duality discussed in the previous subsection from Seiberg duality in four dimensions \cite{1}. More precisely, the $4d \mathcal{N} = 1$ duality we should start with is the one relating the $SU(2)$ gauge theory with 6 fundamental chirals and the Wess–Zumino (WZ) model of 15 chirals with a cubic superpotential, which corresponds to the case $N_c = 2$ and $N_f = 3$ of Seiberg duality.

The global symmetry of the $4d$ gauge theory is $SU(6)_v$, since the $U(1)$ part is anomalous. In particular, requiring the existence of a non-anomalous R-symmetry we can uniquely fix the superconformal R-charge of the chirals, which has to be $1\over 3$. Nevertheless, we know that in the dimensional reduction we can choose another R-symmetry $U(1)^{4d}_R$ that differs for a mixing with any $U(1)$ in the Cartan of the $SU(6)_v$ flavor symmetry. In order to choose the correct
$U(1)^{4d}_R$ symmetry that reproduces the $2d$ duality, we consider the subgroup $SU(4)_u \times SU(2) \times U(1)_s \subset SU(6)_v$ of the global symmetry$^9$. We then allow for a mixing of the non-anomalous R-symmetry with $U(1)_s$. The chirals which transform in the fundamental representation of $SU(6)_v$ accordingly decompose as

$$6 \rightarrow (4, 1)^1 \oplus (1, 2)^{-2}. \quad (2.15)$$

Choosing the mixing coefficient with $U(1)_s$ to be $R_s = -\frac{1}{3}$ we see that the four chirals in the representation $(4, 1)^1$ have R-charge $\frac{1}{3} + R_s = 0$ and become four chirals in the $2d$ theory, while the two chirals in the representation $(1, 2)^{-2}$ have R-charge $\frac{1}{3} - 2R_s = 1$ and don’t survive the dimensional reduction. In conclusion, on this side of the duality we get a $2d$ $\mathcal{N} = (0, 2)$ $SU(2)$ gauge theory with 4 fundamental chirals.

On the other side of the duality we have 15 chirals that can be collected in a matrix $M$ transforming in the antisymmetric representation of $SU(6)_v$ and which interact with cubic superpotential

$$W = \text{Pf} M. \quad (2.16)$$

This is a cubic interaction that fixes the R-charges of all the chirals to $\frac{2}{3}$. In this case we use the branching rule

$$15 \rightarrow (4, 2)^{-1} \oplus (6, 1)^2 \oplus (1, 1)^{-4}. \quad (2.17)$$

and choosing $R_s = -\frac{1}{3}$ we see that only the representations $(6, 1)^2$ and $(1, 1)^{-4}$ survive the dimensional reduction, becoming the chirals $\Phi_{ab}$ and the Fermi $\Psi$ respectively. Of the original $4d$ superpotential, we are left with the $J$-interaction

$$W_{T_4} = \Psi \text{Pf} \Phi. \quad (2.18)$$

We thus recovered the duality discussed in the previous subsection. Notice also that of the original $4d$ global symmetry $SU(4)_u \times SU(2) \times U(1)_s$ we are left only with $SU(4)_u \times U(1)_s$, since all the fields charged under the $SU(2)$ factor didn’t survive the dimensional reduction.

### 2.3 $2d$ $\mathcal{N} = (0, 2)$ version of the confining Intriligator–Pouliot duality

We conclude this review part with the dimensional reduction of the four-dimensional Intriligator–Pouliot duality [32], which has been discussed in [11]. We will focus on the confining case $N_c = N$, $N_f = N + 2$ since for different values of $N_f$ we don’t get new $2d$ dualities. This is the following IR duality between $4d$ $\mathcal{N} = 1$ theories:

**Theory $T^{4d}_A$:** $USp(2N)$ gauge theory with $2N + 4$ fundamental chirals and no superpotential

$$W_{T^{4d}_A} = 0. \quad (2.19)$$

$^9$This is not the only choice leading to a consistent two-dimensional duality. For other choices of $U(1)^{4d}_R$ see [11].
Theory $\mathcal{T}_A^{4d}$: WZ model of $(N+2)(2N+3)$ chirals collected in an antisymmetric $(2N+4) \times (2N+4)$ matrix $M_{ab}$ for $a < b = 1, \cdots, 2N+4$ interacting with a superpotential of degree $N+2$

$$W_{\mathcal{T}_A^{4d}} = \text{Pf} \, M.$$ \quad (2.20)

Notice that for $N = 1$ this reduces to the Seiberg duality we considered in the last subsection. The non-anomalous global symmetry of the dual theories is $SU(2N+4)_v$. Again the requirement of the existence of a non-anomalous R-symmetry fixes the R-charges of the chirals in the gauge theory to be $\frac{1}{N+2}$, while on the WZ dual side the R-charges of the chirals $M$ are fixed by the superpotential to the value $\frac{2}{N+2}$. The R-symmetry $U(1)^{4d}_R$ we want to use in the reduction is obtained decomposing $SU(2N+2)_u \times SU(2) \times U(1)_s \subset SU(2N+4)_v$ and considering a mixing $R_s$ of the non-anomalous R-symmetry we just found with $U(1)_s$. Under this subgroup, the fundamental and the antisymmetric representations of $SU(2N+4)_v$ decompose according to

$$2N+4 \rightarrow (2N+2, 1)^1 \oplus (1, 2)^{-(N+1)}$$

$$(N+2)(2N+3) \rightarrow (2N+2, 2)^{-N} \oplus ((N+1)(2N+1), 1)^2 \oplus (1, 2)^{-2(N+1)}.$$ \quad (2.21)

Choosing $R_s = -\frac{1}{N+2}$ we get the following $2d$ $\mathcal{N} = (0, 2)$ multiplets:

- on the side of theory $\mathcal{T}_A^{4d}$ we have $2N+2$ chiral multiplets forming the fundamental representation of $SU(2N+2)_u$ which have charge 1 under $U(1)_s$;

- on the side of theory $\mathcal{T}_B^{4d}$ we have $(N+1)(2N+1)$ chiral multiplets forming the antisymmetric representation of $SU(2N+2)_u$ which have charge 2 under $U(1)_s$ and one Fermi multiplet which is a singlet of $SU(2N+2)_u$ and has charge $-2(N+1)$ under $U(1)_s$.

The $2d$ $\mathcal{N} = (0, 2)$ duality we get from the dimensional reduction of the $4d$ $\mathcal{N} = 1$ Intriligator–Pouliot duality in the confining case is

Theory $\mathcal{T}_A$: $USp(2N)$ gauge theory with $2N+2$ fundamental chirals and no superpotential

$$W_{\mathcal{T}_A} = 0.$$ \quad (2.22)

Theory $\mathcal{T}_B$: LG model of one Fermi $\Psi$ and $(N+1)(2N+1)$ chirals collected in an antisymmetric $(2N+2) \times (2N+2)$ matrix $\Phi_{ab}$ for $a < b = 1, \cdots, 2N+2$ interacting with a superpotential of degree $N+2$

$$W_{\mathcal{T}_B} = \Psi \text{Pf} \, \Phi.$$ \quad (2.23)

Notice that for $N = 1$ this reduces to the duality we reviewed in Subsection 2.1. The transformation rules of the fields of the two theories under the $SU(2N+2)_u \times U(1)_s$ global symmetry, which can either be obtained from the $4d$ ones or from the superpotential constraints, are summarized in the following table:
One simple test we can perform for this duality consists of matching anomalies

\[ \text{Tr} \gamma^3 U(1)^2_s = 4N(N+1), \quad \text{Tr} \gamma^3 SU(2N+2)^2_{s_0} = N. \]  

(2.24)

We can also compute the trial central charges

\[ c_R = 3N \left( 4(N+1)(R_s - 1)^2 - 2N - 1 \right), \quad c_R - c_L = N(2N + 3). \]  

(2.25)

where again \( U(1)_R \) is defined taking into account a generic mixing with \( U(1)_s \) as in (2.4).

Performing \( c \)-extremization we get, as in the \( N = 1 \) case, \( R_s = 1 \) and a corresponding value of the central charges

\[ c_R = -3N(2N+1), \quad c_L = -2N(4N + 3). \]  

(2.26)

These central charges are negative for any \( N \). Again we interpret this as due to the fact that the theories have a non-compact target space, since the number of chirals on the gauge theory side is large enough to expect no SUSY breaking in the IR\(^{10}\).

Another test is matching the elliptic genera of the two theories. In particular, the duality translates into the following integral identity

\[ \mathcal{I}_{T_A} = \oint \frac{d\vec{z}_N}{\prod_{i=1}^N \prod_{a=1}^{2N+2} \theta(su_a s_i^{\pm 1}; q)} = \frac{\theta(q s^{-2(N+1)}; q)}{\prod_{a<b}^{2N+2} \theta(s^2 u_a u_b; q)} = \mathcal{I}_{T_B}, \]  

(2.27)

This equality can also be understood as the matching of the \( \mathbb{T}^2 \times S^2 \) partition functions of the original 4d dual theories. We couldn’t find an analytical proof of this result in the mathematical literature, as instead can be done for the matching of the \( S^3 \times S^1 \) partition functions of the 4d theories [33, 34]. Nevertheless, this identity can be tested perturbatively in \( q \) for low values of the rank \( N \). As we will show in Subsubsection 3.3.2, equation (2.27) will play a key role in the derivation of the identity for one of the dualities we are going to propose.

### 3 Duality for \( USp(2N) \) gauge theory with antisymmetric matter

In this section we discuss an higher rank generalization of the duality for the \( SU(2) \) theory with 4 fundamental chirals, where instead of increasing the number of fundamental flavors as

---

\(^{10}\)This can be understood for example from the elliptic genus. Since it is a refined version of the Witten index, we can figure out if the theory is SUSY breaking or not by computing it and verifying that it is non-zero. This may happen when the theory has too few fundamental chirals, since we don’t have enough poles to make the integral (B.2) in terms of which we can express the elliptic genus non-vanishing \([12, 13]\).
in the Intriligator–Pouliot duality we add a chiral field in the antisymmetric representation of the gauge group.

If we consider a 2d $\mathcal{N} = (0,2)$ theory with $USp(2N)$ gauge group, one antisymmetric chiral, $N_b$ fundamental chirals and $N_f$ fundamental Fermis, cancellation of gauge anomalies requires that $N_b - N_f - 4 = 0$. We would like to find a dual for this class of theories. We will do so starting from a 4d duality for a theory with 6 fundamental chirals only. This means that from it we can only derive a 2d duality for a theory with $N_b + N_f \leq 6$. Combining this constraint with the one coming from anomaly cancellation we see that we can have only two possibilities: $N_b = 4$, $N_f = 0$ or $N_b = 5$, $N_f = 1$. In this section we will discuss in detail the former case, while we will comment on the latter in Appendix A.

We start reviewing the 4d ancestor duality, then we discuss its dimensional reduction and finally we perform some tests for the resulting 2d duality. In particular, we show how to derive the proposed 2d duality by iterative applications of the duality we reviewed in Subsection 2.3 corresponding to the dimensional reduction of Intriligator–Pouliot duality in the confining case, in complete analogy to a similar derivation that can be done for the original 4d duality.

### 3.1 The 4d duality

The 4d duality we are interested in was first proposed by Csaki, Skiba and Schmaltz in [16]:

**Theory $\mathcal{T}_A^{4d}$**: $USp(2N)$ gauge theory with one antisymmetric chiral $A$, six fundamental chirals $Q_a$ and $N$ chiral singlets $\beta_i$ with superpotential

$$W_{\mathcal{T}_A^{4d}} = \sum_{i=1}^{N} \beta_i \text{Tr}_N A^i. \quad (3.3)$$

**Theory $\mathcal{T}_B^{4d}$**: WZ model with $15N$ chiral singlets $\mu_{abc}$ for $i = 1, \cdots, N$, $a < b = 1, \cdots, 6$ interacting with the cubic superpotential

$$W_{\mathcal{T}_B^{4d}} = \sum_{i,j,k=1}^{N} \sum_{a,b,c,d,e,f=1}^{6} \epsilon^{abcdef \mu_{abc} i \mu_{cdj} j \mu_{efj} k} \delta_{i+j+k,2N+1}. \quad (3.4)$$

Notice that for $N = 1$ this duality reduces to the Seiberg duality between $SU(2)$ with 6 chirals and the WZ model of 15 chirals whose dimensional reduction we studied in Subsection 2.2. Indeed, the antisymmetric of $SU(2)$ is just a singlet and the superpotential $W = \beta_1 A$ is a mass term for both the singlets $\beta_1$ and $A$. Integrating them out we recover the aforementioned duality.

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11 Throughout the paper, all the $USp(2n)$ indices are contracted using the totally antisymmetric tensor

$$J^{(n)} = \mathbb{I}_n \otimes i \sigma_2 . \quad (3.1)$$

For example, the trace of the $USp(2n)$ antisymmetric operator $A$ is

$$\text{Tr}_n A = J^{(n)}_{ij} A^{ij}. \quad (3.2)$$
The non-anomalous global symmetry of the dual theories is
\[ SU(6)_v \times U(1)_x, \] (3.5)
under which the chiral fields transform according to

| \( \beta_i \) | \( SU(6)_v \) | \( U(1)_x \) | \( U(1)_{R_0} \) |
|---|---|---|---|
| \( 6 \) | \( -i \) | \( 1 - N/3 \) | \( 1/3 \) |
| \( 1 \) | \( 1 \) | \( 1 \) | \( 0 \) |

where \( U(1)_{R_0} \) is a possible choice of UV trial R-symmetry. The charges under \( U(1)_x \) are determined on the gauge theory side requiring that \( U(1)_R \) is not anomalous, where
\[ R = R_0 + q_x R_x, \] (3.6)
with \( q_x \) being the charge under \( U(1)_x \) and \( R_x \) the mixing coefficient of \( U(1)_x \) with the R-symmetry, while on the WZ side they are fixed by the superpotential.

This duality can be derived by iterative applications of the Intriligator–Pouliot duality in the confining case. This strategy was used in [34] to prove the equality of the \( S^3 \times S^1 \) partition functions of the dual theories. In Subsubsection 3.3.2 we will present a similar derivation for the 2d version of the duality that we are going to discuss.

### 3.2 Dimensional reduction

In order to dimensionally reduce the 4d duality, we follow a strategy similar to the one we used in Subsection 2.2. We first decompose the non-abelian part of the global symmetry into \( SU(4)_u \times SU(2) \times U(1)_s \subset SU(6)_v \)\(^{12}\). Then we introduce a possible mixing of \( U(1)_x \) and \( U(1)_s \) with the trial R-symmetry \( U(1)_{R_0} \)
\[ R = R_0 + q_x R_x + q_s R_s. \] (3.7)

The charges of the chiral fields of theory \( T^{4d}_A \) under this R-symmetry are
\[ R[Q_\alpha] = \frac{1}{3} - \frac{N}{3} R_x - 2 R_s, \quad \alpha = 1, 2 \]
\[ R[Q_a] = \frac{1}{3} + \frac{1 - N}{3} R_x + R_s, \quad a = 3, \cdots, 6 \]
\[ R[A] = R_x \]
\[ R[\beta_i] = 2 - i R_x, \quad i = 1, \cdots, N \] (3.8)

\(^{12}\)In Appendix A we will consider the only other possible decomposition that leads to a potential 2d duality, which will be for a theory with \( N_b = 5 \) fundamental chirals and \( N_f = 1 \) fundamental Fermis.
where we used the charges under $U(1)_s$ dictated by the branching rule (2.15). For theory $T_{B}^{4d}$ we have

\[
R[\mu_{12;\ i}] = \frac{2}{3} + \left( i - \frac{2N + 1}{3} R_x \right) - 4R_s
\]

\[
R[\mu_{ab;\ i}] = \frac{2}{3} + \left( i - \frac{2N + 1}{3} R_x \right) + 2R_s, \quad a < b = 3, \cdots, 6
\]

\[
R[\mu_{a\alpha;\ i}] = \frac{2}{3} + \left( i - \frac{2N + 1}{3} R_x \right) - R_s, \quad \alpha = 1, 2, \quad a = 3, \cdots, 6,
\]

where we instead used the branching rule (2.17). The R-symmetry $U(1)_R^{4d}$ we choose for the dimensional reduction corresponds to $R_x = 0$ and $R_s = -\frac{1}{3}$. With this choice we get the following 2d $\mathcal{N} = (0, 2)$ multiplets:

- on the side of theory $T_{A}^{4d}$ the fields $Q_i$ don’t survive the dimensional reduction, while $Q_a$, $A$ become chirals and $\beta_i$ become Fermi fields;
- on the side of theory $T_{B}^{4d}$ the fields $\mu_{ab;\ i}$ don’t survive the dimensional reduction, while $\mu_{a\alpha;\ i}$ become chirals and $\mu_{12;\ i}$ become Fermi fields.

### 3.3 The 2d duality

From the dimensional reduction we got the following putative 2d $\mathcal{N} = (0, 2)$ duality:

**Theory $T_{A}$**: $USp(2N)$ gauge theory with one antisymmetric chiral $A$, four fundamental chirals $Q_a$ and $N$ Fermi singlets $\beta_i$ with superpotential

\[
W_{T_{A}} = \sum_{i=1}^{N} \beta_i \text{Tr}_N A_i^i.
\]

**Theory $T_{B}$**: LG model with $N$ Fermi fields $\Psi_i$ and $6N$ chiral fields $\Phi_{ab;\ i}$ for $i = 1, \cdots, N$, $a < b = 1, \cdots, N$ interacting with cubic superpotential

\[
W_{T_{B}} = \sum_{i,j,k=1}^{N} \sum_{a,b,c,d=1}^{4} \epsilon_{abcd} \Psi_i \Phi_{ab;\ j} \Phi_{cd;\ k} \delta_{i+j+k,2N+1}.
\]

This duality can be understood as a higher rank generalization of the duality for $SU(2)$ with 4 chirals, to which it reduces in the particular case $N = 1$. The global symmetry group of the dual theories is

\[
SU(4)_a \times U(1)_s \times U(1)_x,
\]

where the $U(1)_x$ factor is present only if $N > 1$. This symmetry is indeed associated to the antisymmetric chiral $A$, which is a massive singlet together with $\beta_1$ for $N = 1$. The fields transform under this symmetry according to
The charges under the $U(1)$ symmetries can be determined by imposing superpotential constraint, but they also coincide with those of the original $4d$ fields.

In the following we perform some tests for the validity of this duality.

### 3.3.1 Anomalies

The anomalies for the abelian factors of the global symmetry group are for both of the theories

$$\text{Tr} \gamma^3 U(1)_s^2 = 8N, \quad \text{Tr} \gamma^3 U(1)_x^2 = \frac{5}{18} N(N - 1)(2N + 1), \quad \text{Tr} \gamma^3 U(1)_s U(1)_x = -\frac{8}{3} N(N - 1),$$

(3.13)

while the anomaly for the $SU(4)_u$ factor is

$$\text{Tr} \gamma^3 SU(4)_u^2 = N.$$  (3.14)

These results coincide with (2.8) for $N = 1$. In particular, the anomalies involving $U(1)_x$ are zero when $N = 1$, as expected since this symmetry disappears in this particular case.

We can also match the trial central charges

$$c_R = \frac{1}{6} N \left(-96R_s((N - 1)R_x + 3) + (N - 1)R_x(5(2N + 1)R_x + 42) + 144R_s^2 + 90\right)$$

$$c_R - c_L = 5N,$$  (3.15)

which again coincide with (2.9) for $N = 1$, with the mixing coefficient with $U(1)_x$ disappearing. From $c$-extremization we get

$$R_s = -\frac{N + 4}{2N - 7}, \quad R_x = -\frac{9}{2N - 7},$$  (3.16)

from which we find the values of the central charges

$$c_R = \frac{45N(N + 1)}{2(2N - 7)}, \quad c_L = \frac{5N(5N + 23)}{2(2N - 7)}.$$  (3.17)

Notice that for a value of the rank of the gauge group above the critical value $N > N^* = \frac{7}{2}$ the central charges jump from negative to positive. This might suggest that $c$-extremization becomes reliable in this regime. Nevertheless, for $N > N^*$ it also happens that the antisymmetric chiral field gets a negative R-charge. Hence, if we consider the gauge invariant operators

$$R[Q_a A^{i-1} Q_b] = R[\Phi_{a b; i}] = 2 - \frac{9(i - 1)}{2N - 7},$$  (3.18)
which are independent operators for \( i = 1, \cdots, N \), those with \( i = 2, \cdots, N \) will fall below the unitarity bound for \( N > N^* \).

This phenomenon is common in four dimensions, where in order to get a duality between theories flowing to an interacting CFT one should add some gauge singlets that flip the operators violating the bound, since they represent a decoupled free sector of the low energy theory [35]. The procedure of “flipping” an operator \( O \) consists of adding a gauge singlet \( S \) together with the superpotential deformation \( \delta W = S O \), so that the equation of motion of \( S \) sets \( O \) to zero. In order to preserve the duality, one should consistently add the same singlets to both of the theories. For example in our case, if \( c \)-extremization were applicable and if we wanted to get rid of the decoupled free sector represented by the operators \( Q_a A_i^{-1} Q_b \) with \( i = 2, \cdots, N \), we would need to add \( 6(N - 1) \) Fermi singlets \( \alpha_{ab};i \) and deform theory \( T_A \) with the superpotential term

\[
\delta W_{T_A} = \sum_{i=2}^{N} \sum_{a,b,c,d=1}^{4} \epsilon_{abcd} \alpha_{ab};i Q_c A_i^{-1} Q_d .
\]

(3.19)

This deformations maps under the duality into the following deformation of theory \( T_B \)

\[
\delta W_{T_B} = \sum_{i=2}^{N} \sum_{a,b,c,d=1}^{4} \epsilon_{abcd} \alpha_{ab};i \Phi_{cd};i .
\]

(3.20)

In general this is not the end of the story, since it may happen that in this alternative version of the duality there are new operators violating the unitarity bound. In our case, we see that on the side of theory \( T_B \) the deformation \( \delta W_{T_B} \) has the effect of making the chiral fields \( \Phi_{ab};i \) with \( i = 2, \cdots, N \) massive, so that the remaining chiral \( \Phi_{ab};1 \) and Fermi fields \( \Psi_i \) are now non-interacting. The fact that the deformed version of theory \( T_A \) flows to a free theory in the IR is not manifestly obvious, but it can be understood exploiting the duality.

We conclude that the faithful version of the duality is the original one of (3.10)–(3.11) and that either \( c \)-extremization is not applicable to our theories for any \( N \) because of the non-compactness of the target space or the duality is between two theories that necessarily have a decoupled free sector.

### 3.3.2 Elliptic genus and derivation

In this section we show how to derive the proposed duality by iterative applications of the 2\( d \) version of the confining Intriligator–Pouliot duality we reviewed in Subsection 2.3. We will first sketch the derivation at the level of quivers and then apply it in details to derive the corresponding equality of elliptic genera.

We remark that a completely analogous derivation exists for the original 4\( d \) duality of [16], which is based on iterative applications of the confining Intriligator–Pouliot duality [32]. This strategy was used in [34] to prove the equality of the \( S^3 \times S^1 \) partition functions of the dual theories. Moreover, it also exists a 3\( d \) \( \mathcal{N} = 2 \) version of the duality, which is obtained from the 4\( d \) one by compactification on \( S^1 \) followed by a series of real mass deformations [36]. This three-dimensional duality relates a \( U(N) \) gauge theory with
one adjoint chiral $A$, one fundamental flavor $Q$ and $N$ chiral singlets with superpotential $W = \sum_{i=1}^{N} \beta_i \text{Tr} A^i$ and a WZ model of $3N$ chirals $\alpha_i$, $T_{j}^\pm$ interacting with the cubic superpotential $\hat{W} = \sum_{i,j,l=1}^{N} \alpha_i T_j^+ T_{N-l+1}^+ \delta_{i+j+l,2N+1}$\textsuperscript{13}. Also this duality can be derived by iterative applications of some more fundamental confining dualities, as it was shown in [38] at the level of the $S^2$ partition function\textsuperscript{14}. Such more fundamental dualities correspond to the confining cases of Aharony duality [41] and a variant with monopole superpotential [42], which can also be derived as limits of the compactification of the $4d$ Intriligator–Pouliot duality. Hence, the $2d$ duality we are proposing and its derivation complete this picture: we have an analogue of the same duality in $2d$, $3d$ and $4d$, as well as an analogue of their derivation by iteration of confining dualities.

The derivation consists of two fundamental steps that are iterated $N$ times. At the first step, we start with the $USp(2N)$ gauge theory and trade the antisymmetric chiral field $A$ for an auxiliary $USp(2(N-1))$ gauge node by using the $2d$ version of Intriligator–Pouliot duality we reviewed in Subsection 2.3. In the process one of the Fermi singlets, specifically $\beta_N$, becomes massive so that we get the following dual frame:

\[
\begin{align*}
W_{T_A} &= \sum_{i=1}^{N} \beta_i \text{Tr}_N A^i \\
W_{T_{\text{aux}}^{(1)}} &= \sum_{i=1}^{N-1} \beta_i \text{Tr}_N (\text{Tr}_{N-1} P P)^i
\end{align*}
\]

In this quiver notation, we represent gauge symmetries with circle nodes and global symmetries with square nodes, with single lines standing for $USp(2n)$ groups and double lines for $SU(n)$ groups. Solid lines connecting the nodes represent $2d\ \mathcal{N} = (2,0)$ chiral multiplets charged under the corresponding symmetries.

At this point we observe that in the new dual frame $T_{\text{aux}}^{(1)}$ we found, the fields charged under the original $USp(2N)$ gauge node consist of only $2N + 2$ fundamental chiralss. This means that applying the basic duality again we can confine this node. Recall that the duality produces one Fermi and $(N+1)(2N+1)$ chiral fields. The Fermi singlet will correspond to the field $\Psi_N$ of the final theory $T_B$. Instead, of the $(N+1)(2N+1)$ chiral fields, $(N-1)(2N-3)$ of them become an antisymmetric chiral field of the remaining $USp(2(N-1))$ gauge node, other $8(N-1)$ of them become $4$ fundamental chirals of $USp(2(N-1))$ and the remaining

\textsuperscript{13}In [36, 37] it was actually proposed another 3d $\mathcal{N} = 2$ duality that is more similar to the $2d$ one we are discussing here, which relates a $USp(2N)$ gauge theory with one antisymmetric chiral and $4$ fundamental chirals to a WZ model of $7N$ chiralss interacting with a cubic superpotential. We expect that also this duality can be derived by iterative applications of some dimensional reduction to $3d$ of Intriligator–Pouliot duality.

\textsuperscript{14}In [39] a similar strategy was used to derive a generalization of the $3d$ duality to an higher number of fundamental flavors. It would be interesting to find a $4d$ version of such $3d$ duality that generalizes the one of [16] we are considering here and study a possible $2d\ \mathcal{N} = (0,2)$ reduction [40].
6 can be identified with the fields $\Phi_{ab;1}$ of the final theory $\mathcal{T}_B$. Hence, we get to the following dual frame:

$$\mathcal{T}_{\text{aux}}^{(1)} \quad \mathcal{T}_{\text{aux}}^{(2)}$$

\[\begin{array}{c}
\begin{array}{c}
\circ \quad 2N-2 \\
\circ \quad 2N \\
\square \quad 4
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\circ \quad 2N-2 \\
\circ \quad 2N \\
\square \quad 4
\end{array}
\end{array}\]

$$\mathcal{W}_{\text{aux}}^{(1)} = \sum_{i=1}^{N-1} \beta_i \text{Tr}_N \left( \text{Tr}_{N-1} PP \right)^i$$

$$\mathcal{W}_{\text{aux}}^{(2)} = \sum_{i=1}^{N-1} \beta_i \text{Tr}_{N-1} A^i +$$

$$+ \epsilon_{abcd} \Psi_N \Phi_{ab;1} \text{Tr}_{N-1} (Q_c A^{N-2} Q_d)$$

Comparing this frame $\mathcal{T}_{\text{aux}}^{(2)}$ with the original one of theory $\mathcal{T}_A$, we can see that we recovered the same theory, but with rank of the gauge group decreased by one unit, with one of the $\beta_i$ singlets less and with the addition of one copy of the $\Psi_i$ and $\Phi_{ab;1}$ singlets.

It is now clear what we should do next: we simply iterate the previous two steps $N$ times, so to completely confine the gauge group, remove all the $\beta_i$ singlets and gain all the $\Psi_i$ and $\Phi_{ab;1}$ singlets.

The final result is precisely the LG model of $N$ Fermi and $6N$ chirals that we denoted by $\mathcal{T}_B$.

We can use the strategy we just described to prove the equality for the elliptic genera of the two theories. At the level of the elliptic genus, the duality implies the following non-trivial
We have a quiver gauge theory with gauge group $\text{USp}$. This expression can be interpreted as the elliptic genus of the auxiliary dual frame right to confine it chirals and no antisymmetric chiral anymore. Hence, we can apply again (2.27) from left to right to replace the contribution of the deformed duality. discussed in Sec. 4.4 of [23]). Hence, we can only use the matching of the elliptic genus as a test of the mass deformed duality.

where we turned on fugacities $u_a$, $s$, $x$ in the Cartan of the global symmetry group $SU(4)_u \times U(1)_s \times U(1)_x$, with $\prod_{a=1}^4 u_a = 1$.

We start from the elliptic genus of theory $\mathcal{T}_A$, which from now on we will denote by

$$\mathcal{I}_{T_A} = \mathcal{I}_N(u, s, x),$$

and use (2.27) from right to left to replace the contribution of the $USp(2N)$ antisymmetric chiral field $A$ with an auxiliary $(N-1)$-dimensional integral

$$\mathcal{I}_N(u, s, x) = \prod_{i=1}^{N-1} \theta(q x^{-i}; q) \oint \frac{d\bar{y}_N}{\prod_{i=1}^{N-1} \prod_{a=1}^4 \theta(s x^{-\frac{1}{N} i} u_a z_{i}^{\pm 1}; q) \prod_{a=1}^{N-1} \theta(x^{-i} u^{-1}_a w_{i a}; q)}.$$ (3.23)

This expression can be interpreted as the elliptic genus of the auxiliary dual frame $\mathcal{T}_A^{(1)}$, where we have a quiver gauge theory with gauge group $USp(2(N-1)) \times USp(2N)$.

Notice that in this auxiliary theory the original $USp(2N)$ node sees $2N + 2$ fundamental chirals and no antisymmetric chiral anymore. Hence, we can apply again (2.27) from left to right to confine it

$$\mathcal{I}_N(u, s, x) = \prod_{a<b} \theta(s^{2} x^{-\frac{2}{3} (N-1)} u_a u_b; q) \prod_{i=1}^{N-1} \theta(q x^{-i}; q) \times$$

$$\times \oint \frac{d\bar{y}_{N-1}}{\theta(x; q)^{N-1} \prod_{a<b} \theta(x^{-\frac{1}{N} i} w_{\alpha}^{i a} w_{\beta}^{i a}; q) \prod_{a=1}^{N-1} \prod_{a=1}^4 \theta(x^{-i} u_a z_{i}^{\pm 1} w_{\alpha}^{i a}; q)}.$$ (3.24)

We now observe that we obtained an integral of the same form of the original one, but of one dimension less, with a different prefactor and with a shift of the parameter $s$. In other words,

\[\text{We also verified this identity for } N = 2 \text{ with a perturbative computation in } q. \text{ The result explicitly shows that the elliptic genus is non-vanishing, meaning that the theory is not SUSY breaking. Moreover, in the limit in which we turn off the fugacities } u_a \to 1 \text{ the result is divergent, which we interpret has the fact that the theory has a non-compact Higgs branch (something similar happens, for example, for the } 2d \mathcal{N} = (2, 2) \text{ } SU(2) \text{ gauge theory with } N \text{ fundamental chirals which has a non-compact Coulomb branch for even } N [2], \text{ as discussed in Sec. 4.4 of [23]). Hence, we can only use the matching of the elliptic genus as a test of the mass deformed duality.}\]
we obtained the following recursive relation:

\[ \mathcal{I}_N(u, s, x) = \frac{\theta \left( q s^{-4} x^{\frac{N-1}{3}} ; q \right)}{\prod_{a < b} \theta \left( s^2 x^{\frac{2}{3}(N-1)} u_a u_b ; q \right)} \mathcal{I}_{N-1}(u, s x^{\frac{1}{3}}, x). \]  

(3.25)

This relation is extremely powerful. Indeed, as we explained before, what we want to do next is to iterate the two steps we just performed \( N \) times, so to completely confine the original \( USp(2N) \) integral and get an expression that takes the form of the elliptic genus of a LG model. Equation (3.25) allows us to do so with very little effort. Applying the two previous steps a second time, we get

\[ \mathcal{I}_N(u, s, x) = \frac{\theta \left( q s^{-4} x^{\frac{N-1}{3}} ; q \right)}{\prod_{a < b} \theta \left( s^2 x^{\frac{2}{3}(N-1)} u_a u_b ; q \right)} \mathcal{I}_{N-2}(u, s x^{\frac{1}{3}}, x) = \frac{\prod_{i=1}^{N} \theta \left( q s^{-4} x^{\frac{2N+1}{3}} ; q \right)}{\prod_{i=1}^{N} \prod_{a < b} \theta \left( s^2 x^{\frac{2}{3}(2N+1)} u_a u_b ; q \right)} \mathcal{I}_{N-2}(u, s x^{\frac{1}{3}}, x). \]  

(3.26)

We can easily get the expression we obtain after \( n \) iterations

\[ \mathcal{I}_N(u, s, x) = \frac{\prod_{i=N-n+1}^{N} \theta \left( q s^{-4} x^{\frac{2N+1}{3}} ; q \right)}{\prod_{i=1}^{n} \prod_{a < b} \theta \left( s^2 x^{\frac{2}{3}(2N+1)} u_a u_b ; q \right)} \mathcal{I}_{N-n}(u, s x^{\frac{n}{3}}, x). \]  

(3.27)

The complete confinement corresponds to the case \( n = N \), for which we find

\[ \mathcal{I}_N(u, s, x) = \prod_{i=1}^{N} \frac{\theta \left( q s^{-4} x^{\frac{2N+1}{3}} ; q \right)}{\prod_{a < b} \theta \left( s^2 x^{\frac{2}{3}(2N+1)} u_a u_b ; q \right)} \mathcal{T}_B, \]  

(3.28)

which indeed coincides with the elliptic genus of theory \( \mathcal{T}_B \), as desired.

4 Duality for \( SU(2) \) linear quiver gauge theories

In this section we discuss another duality that can be understood as a generalization of the one for the \( SU(2) \) theory with 4 fundamental chirals dual to a LG model. More precisely, we deform the \( SU(2) \) theory by introducing two Fermi singlets and propose multiple dual frames for it consisting of \( SU(2) \) linear quiver gauge theories of arbitrary length \( N - 1 \) and with \( N \) Fermi singlets, where \( N \) is any non-negative integer.

We start reviewing the 4d ancestor of this duality, then we discuss its dimensional reduction and finally we perform some tests for the resulting 2d duality, including a derivation by iterative applications of the most fundamental \( N = 1 \) duality. This derivation is reminiscent of a similar one for the parent 4d duality.
4.1 The 4d duality

The four-dimensional duality we want to consider is one of the recently proposed 4d mirror-like dualities [17]. More precisely, following the same nomenclature of [17], we are interested in the duality for the $E_6^{\sigma}[USp(2N)]$ theory with the partitions of $N \rho$ and $\sigma$ being $\rho = [N - 1, 1]$ and $\sigma = [1^N]$\(^{16}\). This duality can be considered as a four-dimensional uplift of the 3d $\mathcal{N} = 4$ abelian mirror duality that relates a $U(1)$ gauge theory with $N$ flavors of fundamental hypermultiplets and a linear abelian quiver with $N - 1$ $U(1)$ gauge nodes and one flavor attached to each end of the tail [45]. More precisely, upon dimensional reduction on $S^1$ and a series of real mass deformations, the 4d mirror duality reduces to the 3d one. Here we will consider a different dimensional reduction of the 4d duality to 2d.

Let us denote with $\mathcal{T}_{4d}^A$ and $\mathcal{T}_{4d}^B$ the two dual four-dimensional theories\(^ {17}\). The content of theory $\mathcal{T}_{4d}^A$ can be schematically represented with the following 4d $\mathcal{N} = 1$ quiver diagram:

![Quiver Diagram](image_url)

where all the nodes correspond to $USp(2n)$ groups, with circular nodes being gauge symmetries and square nodes global symmetries. Moreover, the crosses on some of the straight lines denote gauge singlet fields that are flipping the mesonic operators constructed with the corresponding chirals, while the blue line represents other gauge singlets that are charged under the non-abelian global symmetries. The full superpotential of the theory is

$$W_{\mathcal{T}_{4d}^A} = FPV + \alpha DD + \beta VV,$$

where contractions of gauge and flavor $USp(2n)$ indices are understood.

The dual theory $\mathcal{T}_{4d}^B$ is a linear quiver of $N - 1$ $SU(2)$ gauge groups with the following schematic structure:

\(^{16}\text{For } N = 2 \text{ the duality becomes a self-duality. More precisely, it corresponds to the self-dual case of Intriligator–Pouliot duality } N_c = 2, N_f = 8, \text{ but with a different configuration of singlets. It also corresponds to the self-duality of the } E[USp(4)] \text{ theory of [43] (see also [44] for other interesting IR properties of this theory).}\)

\(^{17}\text{With respect to the duality discussed in [17], the one that we consider here has actually a slightly different configuration of gauge singlets. More precisely, in [17] the singlet } \beta \text{ in theory } \mathcal{T}_{4d}^A \text{ and the singlet } A \text{ in theory } \mathcal{T}_{4d}^B \text{ are moved to the opposite side of the duality.}\)
In the quiver we are not showing some gauge singlet chiral fields charged under the non-abelian global symmetries that are analogues of the field $\Pi$ in theory $\mathcal{T}_{\mathbb{A}}^{4d}$ to avoid cluttering the drawing. These consists of $N - 1$ singlets $\pi^{(i)}$ connecting the upper-left square node with all the lower square nodes except the first one on the left and a singlet $\pi$ connecting the upper-right square node with the lower-right square node. The full superpotential of the theory, including the terms involving these singlets, is

$$W_{\mathcal{T}_{\mathbb{A}}^{4d}} = \sum_{i=1}^{N-1} A^{(i)} \left( q^{(i,i+1)} q^{(i+1,i)} - q^{(i-1,i)} q^{(i-1,i+1)} \right) + \sum_{i=1}^{N-2} v^{(i)} q^{(i,i+1)} d^{(i+1)} +$$

$$+ \sum_{i=1}^{N-1} \pi^{(i)} \left( \prod_{j=1}^{i} q^{(j-1,j)} \right) v^{(i)} + \frac{1}{N} A \sum_{i=1}^{N} q^{(i-1,i)} q^{(i,i+1)} ,$$

(4.2)

where the chiral singlet $A$ is represented with a cross in the previous quiver\textsuperscript{18}.

Both of the theories possess two non-anomalous independent $U(1)$ symmetries, which we denote by $U(1)_c$ and $U(1)_t$. They can be determined by solving the constraints coming from the superpotential and from the requirement that the NSVZ beta-functions vanish at each gauge node. A possible parametrization of these symmetries and of UV trial R-symmetry is summarized in the following table:

\textsuperscript{18}Notice that in the quiver the cross corresponding to the singlet $A$ appears only on the $q^{(0,1)}$ chiral. Nevertheless, the equations of motion of the singlets $A^{(i)}$ have the effect of identifying all the operators $q^{(i-1,i)} q^{(i,i+1)}$ for $i = 1, \cdots, N$, so the singlet $A$ is actually flipping all these operators, as we write in the superpotential (4.2).
where $R_c$ and $R_t$ are the mixing coefficients of the R-symmetry with $U(1)_c$ and $U(1)_t$ respectively.

The global symmetry group of the two theories contains also some non-abelian factors. The full IR global symmetry is

$$USp(2N) \times SU(2) \times SU(2) \times U(1)_c \times U(1)_t,$$

which is completely manifest in the UV Lagrangian description of theory $T^{4d}_A$, while in the frame of theory $T^{4d}_B$ the $USp(2N)$ factor is enhanced in the IR from the $SU(2)^N$ symmetry of the saw.

### 4.2 Dimensional reduction

In order to dimensionally reduce this duality, we choose a four-dimensional R-symmetry corresponding to the following values of the mixing coefficients $R_c$ and $R_t$:

$$R_c = 1, \quad R_t = 0.$$ (4.4)

Let us reconstruct the resulting two-dimensional theories separately using the prescription we reviewed in Subsection 2.2.

For theory $T^{4d}_A$ we can see that the fields $F$ and $\Pi$ have R-charge 1 and don’t survive the dimensional reduction. The fields $D$ and $V$ have instead R-charge 0 and reduce to two chiral fields in 2d which we denote by $L$ and $R$. Finally, the gauge singlets $\alpha$ and $\beta$ have R-charge 2 and become Fermi singlets $\Psi_L$ and $\Psi_R$. The resulting theory $T^{4d}_A$ can be summarized with the following quiver diagram:

```
  [2]  L  [2]  R  [2]
```

Of the original 4d superpotential only the flipping terms survived
\[ W_{T_A} = \Psi_L L L + \Psi_R R R. \] (4.5)

We thus see that all the theories \( T_A^{4d} \) reduce for any \( N \) to the same 2d \( \mathcal{N} = (0, 2) \) theory, namely the \( SU(2) \) gauge theory with 4 fundamental chirals and two Fermi singlets. Indeed, since the fields \( F \) and \( \Pi \) didn’t survive the dimensional reduction, the 2d theory doesn’t possess the \( USp(2N) \) factor of the original 4d global symmetry (4.3), so that the global symmetry of \( T_A \) is \( SU(2)_L \times SU(2)_R \times U(1)_c \times U(1)_t \). The charges of the 2d fields under these symmetries inherited from four dimensions are

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & SU(2)_L & SU(2)_R & U(1)_c & U(1)_t \\
\hline
L & \Box & \bullet & 1 & \frac{1}{2} \\
R & \bullet & \Box & -1 & \frac{N-1}{2} \\
\Psi_L & \bullet & \bullet & -2 & -1 \\
\Psi_R & \bullet & \bullet & 2 & 1 - N \\
\hline
\end{array}
\]

where \( U(1)_{R_0} \) denotes the UV trial R-symmetry. It is useful to redefine the abelian symmetries so to completely remove any remnant of the dependence on \( N \)
\[ d = c t^{-\frac{N-2}{4}} \quad s = t^{\frac{N}{4}}. \] (4.6)

With this choice, the charges of the fields of theory \( T_A \) inherited from four dimensions are

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & SU(2)_L & SU(2)_R & U(1)_d & U(1)_s \\
\hline
L & \Box & \bullet & 1 & 1 \\
R & \bullet & \Box & -1 & 1 \\
\Psi_L & \bullet & \bullet & -2 & -2 \\
\Psi_R & \bullet & \bullet & 2 & -2 \\
\hline
\end{array}
\]

For theory \( T_B^{4d} \) we can see that all the fields of the saw \( d^{(i)} \) and \( v^{(i)} \) as well as the singlets \( \pi^{(i)} \) and \( \pi \) have R-charge 1 and don’t survive the dimensional reduction. The bifundamental chiral fields \( q^{(i,j+1)} \) have all R-charge 0 and become chiral multiplets \( Q^{(i,j+1)} \), while the singlets \( A^{(i)} \) and \( A \) have R-charge 2 and become Fermi gauge singlets \( \Psi^{(i)} \) and \( \Psi \). The resulting theory can be summarized with the following quiver diagram:

![Quiver Diagram]

where the dashed lines represent Fermi fields. On this side of the duality we have a non-trivial dependence on \( N \) encoded in the length of the quiver and in the number of Fermi singlets. For this reason, we label these theories by \( T_B^{(N)} \).
Also in this dual frame there is no remnant of the four-dimensional \( USp(2N) \) symmetry. Moreover, all the 4d fields charged under \( U(1)_c \) didn’t survive the dimensional reduction, so that the full global symmetry of the 2d theory is only \( SU(2)_l \times SU(2)_r \times U(1)_s \). The charges of all the fields under these symmetries are

| Field | \( SU(2)_l \) | \( SU(2)_r \) | \( U(1)_s \) | \( U(1)_{R_0} \) |
|-------|--------------|--------------|-------------|----------------|
| \( Q^{(0,1)} \) | \( \square \) | \( \bullet \) | \( \frac{2}{N} \) | 0 |
| \( Q^{(i,i+1)} \) | \( \bullet \) | \( \bullet \) | \( \frac{2}{N} \) | 0 |
| \( Q^{(N-1,N)} \) | \( \bullet \) | \( \square \) | \( \frac{2}{N} \) | 0 |
| \( \Psi(i), \Psi \) | \( \bullet \) | \( \bullet \) | \( -\frac{4}{N} \) | 1 |

4.3 The 2d duality

From the dimensional reduction we got the following putative 2d \( \mathcal{N} = (0,2) \) duality:

**Theory \( \mathcal{T}_A \):** \( SU(2) \) gauge theory with four fundamental chiral fields \( L_a, R_a \) for \( a = 1, 2 \) and two Fermi singlets \( \Psi_L, \Psi_R \) interacting with

\[
W_{\mathcal{T}_A} = \Psi_L L L + \Psi_R R R.
\] (4.7)

**Theory \( \mathcal{T}_B(\mathcal{N}) \):** Linear quiver with \( N-1 \) \( SU(2) \) gauge groups connected by bifundamental chiral fields \( Q^{(i,i+1)} \) for \( i = 1, \ldots, N-2 \), with one fundamental chiral at each end of the tail \( Q^{(0,1)}, Q^{(N-1,N)} \) and \( N \) gauge singlet Fermi fields \( \Psi, \Psi^{(i)} \) for \( i = 1, \ldots, N-1 \) interacting through

\[
W_{\mathcal{T}_B} = \sum_{i=1}^{N-1} \Psi^{(i)} \left( Q^{(i,i+1)} Q^{(i,i+1)} - Q^{(i-1,i)} Q^{(i-1,i)} \right) + \frac{1}{N} \Psi \sum_{i=1}^{N} Q^{(i-1,i)} Q^{(i-1,i)}.
\] (4.8)

Notice that theory \( \mathcal{T}_A \) is always the same for any \( N \), while theory \( \mathcal{T}_B(\mathcal{N}) \) changes. This means that from infinitely many dualities in 4d we obtained a single duality, but between infinitely many dual frames. We will show that of all of the dualities relating these multiple frames only one is independent, in the sense that all the others can be derived by iterating the duality for \( N = 1 \).

Another peculiarity of this duality is related to the global symmetries of the two theories. We recall that the charges of all the fields that we predict from 4d and that we can equivalently determine directly in 2d by solving the superpotential constraints are
Notice that \( U(1)_d \) is not a symmetry of all the \( \mathcal{T}^{(N)}_B \) theories, since no 4d field charged under it survived the dimensional reduction. We expect this symmetry to decouple in the IR also on the side of theory \( \mathcal{T}_A \), in the sense that it is not a symmetry of the low energy theory. We will argue this by showing that the dependence on this \( U(1)_d \) symmetry disappears both from anomalies and from the elliptic genus.

**The case \( N = 1 \)**

For \( N = 1 \) the dual theory \( \mathcal{T}^{(1)}_B \) consists only of an \( SU(2)_L \times SU(2)_R \) bifundamental chiral field \( Q^{(0,1)} \) and a Fermi singlet field \( \Psi \), without any gauge group

\[
\begin{align*}
\mathcal{T}_A & & \mathcal{T}^{(1)}_B \\
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{fig1.png}
\end{array}
\end{array}
\end{align*}
\]

The interactions on the side of theory \( \mathcal{T}_A \) are

\[
W_{\mathcal{T}_A} = \Psi_L L L + \Psi_R R R ,
\]

while on the side of theory \( \mathcal{T}^{(1)}_B \)

\[
W_{\mathcal{T}_B} = \Psi Q^{(0,1)} Q^{(0,1)} .
\]

This precisely coincides with the alternative version of the duality between \( SU(2) \) with 4 chirals and the LG model of 6 chirals and one Fermi fields with a cubic interaction of [11, 18] we presented at the end of Subsection 2.1, where we already noticed the decoupling of the \( (1)_d \) symmetry in the IR on the side of theory \( \mathcal{T}_A \).

**The case \( N = 2 \)**

For \( N = 2 \) one may expect that the duality between \( \mathcal{T}_A \) and \( \mathcal{T}^{(2)}_B \) is trivial, since both of the theories have the same gauge group and matter content, namely they are \( SU(2) \) gauge
theories with 4 fundamental chirals and 2 Fermi singlets. Nevertheless, the gauge singlets interact in completely different ways in the two theories. On the side of theory $T_A$ we have
\[ W_{T_A} = \Psi_L L L + \Psi_R R R, \] (4.11)
while on the side of theory $T_B^{(2)}$ we have
\[ W_{T_B^{(2)}} = \Psi^{(1)} \left( Q^{(1,2)} Q^{(1,2)} - Q^{(0,1)} Q^{(0,1)} \right) + \frac{1}{2} \Psi \left( Q^{(1,2)} Q^{(1,2)} + Q^{(0,1)} Q^{(0,1)} \right). \] (4.12)

This has the crucial effect of breaking the $U(1)_d$ symmetry on the side of theory $T_B^{(2)}$. Hence, this apparently trivial duality implies the non-trivial decoupling of the $U(1)_d$ symmetry of theory $T_A$. We schematically represent this difference in the interactions by drawing differently the singlets in the quiver diagrams

As we mentioned before, this duality can be derived by applying twice the $N = 1$ duality, as we will show momentarily.

4.3.1 Anomalies

A first test of the duality consists of matching the anomalies of $T_A$ and $T_B^{(N)}$. On the side of theory $T_A$ we find that all the abelian anomalies vanish, including the mixed ones. In particular, the vanishing of the anomalies involving $U(1)_d$ is compatible with the decoupling of this symmetry in the IR. On the side of theory $T_B^{(N)}$ we consistently find
\[ \text{Tr} \gamma^3 U(1)^2_s = 0. \] (4.13)

Computing the anomalies for the non-abelian symmetries we find a perfect agreement between theory $T_A$ and theory $T_B^{(N)}$, since both of the theories have only two chiral fields transforming in the fundamental representation of each $SU(2)_l$ and $SU(2)_r$ symmetry
\[ \text{Tr} \gamma^3 SU(2)_l^2 = 2T_{SU(2)_l}(\Box) = 1 \]
\[ \text{Tr} \gamma^3 SU(2)_r^2 = 2T_{SU(2)_r}(\Box) = 1. \] (4.14)

Finally, we compute the trial central charges
\[ c_R = 9 - 24R_s, \quad c_R - c_L = 3, \] (4.15)
where $R_s$ is the mixing coefficients of $U(1)_s$ with the trial R-symmetry $U(1)_{R_0}$. Notice that the result doesn’t depend on any mixing coefficient with $U(1)_d$, which is again compatible with the decoupling of this symmetry. Moreover, the trial central charge is linear in $R_s$, which doesn’t allow us to use $c$-extremization to determine the superconformal R-charge.
4.3.2 Elliptic genus and derivation

We conclude this section showing how to derive the duality for $N > 1$ by iterating the fundamental duality for $N = 1$, which is nothing but a deformation of the duality for $SU(2)$ with 4 chiral of [11, 18].

We remark that the derivation we are going to present is completely analogous to similar derivations for related dualities in higher dimensions. It was first proposed in [46] in the context of $3d\, \mathcal{N} = 4$ abelian mirror symmetry, which is the duality relating a $U(1)$ gauge theory with $N$ fundamental flavors to a linear abelian quiver with $N - 1$ nodes and one flavor at each end of the tail [45]. In [46] it was shown how to derive this duality for arbitrary $N$ by applying piecewise the duality for $N = 1$, which relates the $U(1)$ gauge theory with one fundamental flavor to a free hypermultiplet (see also [47] for an implementation of this procedure at the level of the three-sphere partition function and [48] for the $\mathcal{N} = 2$ case). The four-dimensional duality of which we are considering the $2d$ reduction is an higher dimensional ancestor of this mirror duality and in [17] it was shown that a similar piecewise derivation applies also in 4$d$, where the fundamental duality that should be iterated is the Seiberg duality of $SU(2)$ with 6 chirals dual to a WZ model of 15 chirals.

We will describe the piecewise derivation of the $2d$ duality at the level of the elliptic genus. In particular, by applying several times (2.14), which encodes the duality in the $d$ symmetry in the IR on the side of theory $I^\alpha$ and $B^\alpha$, we will be able to prove the following integral identity related to the duality for arbitrary $N$:

\[
I^\alpha_{\text{T}} = \theta \left( q s^{-2d \pm 2}; q \right) \int \frac{\text{d}z_1}{\theta \left( s d z \pm 1 \pm 1; q \right) \theta \left( s d^{-1} z \pm 1 \pm 1; q \right)} = \theta \left( q s^{-\frac{4}{N}}; q \right)^N \int \frac{\theta \left( s^2 \frac{2}{N} z^{(1)} \pm 1 \pm 1; q \right)}{\prod_{a=1}^{N-2} \theta \left( s^2 \frac{2}{N} z^{(a)} \pm 1 \pm 1; q \right) \theta \left( s^2 \frac{2}{N} z^{(N-1)} \pm 1 \pm 1; q \right)} = I^\alpha_{\text{T}_B^{(N)}},
\]

where $l$, $r$, $d$ and $s$ are fugacities in the Cartan of the global symmetry $SU(2)_l \times SU(2)_r \times U(1)_d \times U(1)_s$. Notice that the elliptic genus of theory $I^\alpha_{\text{T}}$ explicitly depends on the fugacity $d$, while that of theory $I^\alpha_{\text{T}_B^{(N)}}$ doesn’t. As we pointed out in Subsection 2.1, the identity (2.14), which corresponds to the case $N = 1$ of (4.16), is valid even without turning off the $d$ fugacity, implying that $I^\alpha_{\text{T}_A}$ is actually independent of $d$. Similarly, the identity (4.16) for generic $N$ works even without turning off the fugacity for the $U(1)_d$ symmetry. This is again compatible with the decoupling of the $U(1)_d$ symmetry in the IR on the side of theory $I^\alpha_{\text{T}_A}$, which is necessary in order for the duality to hold.

We start considering the elliptic genus of theory $I^\alpha_{\text{T}_B^{(N)}}$. In particular, we isolate the last $SU(2)$ integral

\[
I^{(1)} = \int \frac{\text{d}z^{(N-1)}_1}{\theta \left( s^2 \frac{2}{N} z^{(N-1)} \pm 1 \pm 1; q \right) \theta \left( s^2 \frac{2}{N} z^{(N-1)} \pm 1 \pm 1; q \right)}. \tag{4.17}
\]

Using (2.14) we can rewrite this as

\[
I^{(1)} = \frac{\theta \left( q s^{-\frac{8}{N}}; q \right)}{\theta \left( s^2 \frac{4}{N} q; q \right)^2 \theta \left( s^2 \frac{4}{N} z^{(N-2)} \pm 1 \pm 1; q \right)}. \tag{4.18}
\]
Plugging this back into the elliptic genus of theory \( T_B^{(N)} \) we get
\[
\mathcal{I}_{T_B^{(N)}} = \theta \left( q s^{-\frac{4}{N}}; q \right) N - 2 \theta \left( q s^{-\frac{4}{N}}; q \right) \times
\]
\[
\times \int \theta \left( s^\frac{2}{N} z^{(1)} \pm 1 \pm 1; q \right) \prod_{a=1}^{N} \theta \left( s^\frac{4}{N} z^{(a)} \pm 1 \pm 1; q \right) \theta \left( s^\frac{4}{N} z^{(N-2) \pm 1 \pm 1; q} \right)
\]
\[\text{(4.19)}\]

Now we consider the following integral:
\[
I^{(2)} = \int \frac{dz^{(N-2)}}{\theta \left( s^\frac{2}{N} z^{(N-2) \pm 1 \pm 1; q} \right) \theta \left( s^\frac{4}{N} z^{(N-3) \pm 1 \pm 1; q} \right)}.
\]
\[\text{(4.20)}\]

Applying the fundamental identity (2.14) again we get
\[
I^{(2)} = \frac{\theta \left( q s^{-\frac{12}{N}}; q \right)}{\theta \left( s^\frac{4}{N}; q \right) \theta \left( s^\frac{4}{N} z^{(N-3) \pm 1 \pm 1; q} \right)}
\]
\[\text{(4.21)}\]

and plugging this into the elliptic genus of \( T_B^{(N)} \) we get
\[
\mathcal{I}_{T_B^{(N)}} = \theta \left( q s^{-\frac{4}{N}}; q \right) N - 3 \theta \left( q s^{-\frac{4}{N}}; q \right) \times
\]
\[
\times \int \theta \left( s^\frac{2}{N} z^{(1)} \pm 1 \pm 1; q \right) \prod_{a=1}^{N} \theta \left( s^\frac{4}{N} z^{(a)} \pm 1 \pm 1; q \right) \theta \left( s^\frac{4}{N} z^{(N-3) \pm 1 \pm 1; q} \right)
\]
\[\text{(4.22)}\]

We want to iterate this procedure \( N - 1 \) times. At the \( n \)-th iteration we have to use the following evaluation formula, which again follows from (2.14):
\[
I^{(n)} = \int \frac{dz^{(N-n)}}{\theta \left( s^\frac{2}{N} z^{(N-n) \pm 1 \pm 1; q} \right) \theta \left( s^\frac{4}{N} z^{(N-n) \pm 1 \pm 1; q} \right)} = \frac{\theta \left( q s^{-\frac{4(n+1)}{N}}; q \right)}{\theta \left( s^\frac{4}{N}; q \right) \theta \left( s^\frac{4n}{N} z^{(N-n) \pm 1 \pm 1; q} \right)},
\]
\[\text{(4.23)}\]

where for \( n = N - 1 \) we have \( z^{(0)} = l \). Hence, after the \( (N - 1) \)-th iteration we get
\[
\mathcal{I}_{T_B^{(N)}} = \frac{\theta \left( q s^{-\frac{4}{N}}; q \right)}{\theta \left( s^{2l \pm 1 \pm 1; q} \right)} = \mathcal{I}_{T_B^{(1)}}
\]
\[\text{(4.24)}\]

which coincides with the elliptic genus of theory \( T_B^{(1)} \). Using (2.14) one last time but from right to left, we then get precisely (4.16).
5 Conclusions

In this paper we studied the dimensional reduction of some 4d $\mathcal{N} = 1$ IR dualities on $S^2$ with suitable topological twists and argued that the resulting 2d $\mathcal{N} = (0, 2)$ theories are still dual to each other, provided that massive deformations that lift any possible non-compact direction in the target space are turned on.

There are still many open questions about this kind of dimensional reduction of dualities, for which it would be interesting to find an answer. In particular, it is not clear why certain dualities survive the dimensional reduction while others don’t. From the examples of [11] and those studied in the present paper, it seems that everytime one of the two 4d dual theories is a WZ model the duality survives the dimensional reduction to 2d.

Instead, again thinking of the examples studied so far, it seems that many of the 4d self-dualities, namely dualities between theories with same gauge group and same gauge charged matter, but possibly a different number of gauge singlets interacting with different superpotentials, reduce in 2d to trivial dualities, in the sense that on both sides of the duality we get the same theory, including the same gauge singlets. This happens in the self-dual case of Intriligator–Pouliot duality [11] and it turns out to be true, for example, also for the dimensional reduction of the 72 dual frames of the $USp(2N)$ gauge theory with one antisymmetric and eight fundamental chirals [49–52], which differ for gauge singlets that flip all the possible combinations of mesons and baryons. After the dimensional reduction, all the gauge singlets become massive and the 72 different dual frames collapse to the same one in 2d.

Nevertheless, it may happen that starting from a different configuration of singlets in the 4d self-duality, one can obtain a non-trivial duality in 2d. This happens for example in the duality of Section 4. For $N = 2$ the original 4d mirror-like duality of [17] coincides with one of the self-dual cases of Intriligator–Pouliot duality, but with a different disposition of singlets which is essential in order for the two theories to enjoy a global symmetry enhancement that makes their symmetries match. This is actually a quite general phenomenon. When we have a self-duality, we can try to find an equivalent version of it where we also have the same number of singlets on both sides of the duality. In such cases, the duality acts non-trivially on the gauge invariant operators of the theory, implying that the Weyl group of the global symmetry of the theory is larger than the one manifest from the Lagrangian. This may lead to an enhancement of the global symmetry in the IR (see [44, 52–55] for some examples). It might be that the 4d self-dual theories with the different configurations of singlets needed for the symmetry enhancements reduce to non-trivial 2d dualities. It would be interesting to investigate this possibility further [56].

Another interesting possible line of future research is related to the strict analogy that there is between some of the 2d $\mathcal{N} = (0, 2)$ dualities discussed in this paper and similar dualities between 3d $\mathcal{N} = 2$ theories. For example, the duality discussed in Section 3 has a direct three-dimensional analogue, which was obtained in [36, 37] as an $S^1$ compactification of the same four-dimensional confining duality we started from [16], followed by a suitable real mass deformation. It would be interesting to understand if our duality can be derived also as a dual boundary condition of the duality of [36, 37] in the same spirit of [57] and, more
in general, if the dimensional reduction of $4d$ $\mathcal{N} = 1$ dualities on $S^2$ with a topological twist can be reinterpreted as a compactification on $S^1$ giving a $3d$ $\mathcal{N} = 2$ duality, followed by real mass deformations and by the introduction of a boundary condition preserving $2d$ $\mathcal{N} = (0, 2)$ supersymmetry.

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**A Another duality for $USp(2N)$ gauge theory with antisymmetric**

In this appendix we comment on the possibility of another duality for a $USp(2N)$ gauge theory with one antisymmetric chiral, but a different number of fundamental chiral and Fermi with respect to the one discussed in Section 3, more precisely $N_b = 5$ and $N_f = 1$.

We start again from the $4d$ duality of [16] relating a $USp(2N)$ gauge theory with one antisymmetric, 6 fundamental chirals and $N$ chiral singlets to a WZ model of $15N$ chirals with cubic superpotential, but we choose the R-symmetry $U(1)^{4d}_R$ in a different way than what we have done in Subsection 3.2. Specifically, we look at the subgroup $SU(3) \times SU(2) \times U(1)_B \times U(1)_p \times U(1)_x \subset SU(6)_v \times U(1)_x$ of the non-anomalous global symmetry and define $U(1)^{4d}_R$ allowing for a mixing with $U(1)_B$, $U(1)_p$ and $U(1)_x$.

\[ R = R_0 + q_B R_B + q_p R_p + q_x R_x . \]  

(A.1)

In order to determine the $U(1)^{4d}_R$ charge of all the chiral fields of the two dual theories, we need to use the following branching rules for the fundamental and antisymmetric representations of $SU(6)_v$ under the $SU(3) \times SU(2) \times U(1)_B \times U(1)_p$ subgroup.

\[
\begin{align*}
6 & \rightarrow (3, 1)^{(1,0)} \oplus (1, 2)^{(1,1)} \oplus (1, 1)^{(1,-2)} \\
15 & \rightarrow (3, 2)^{(0,1)} \oplus (3, 1)^{(0,-2)} \oplus (1, 3)^{(-2,0)} \oplus (2, 1)^{(2,-1)} + (1, 1)^{(2,2)} .
\end{align*}
\]  

(A.2)

We can see that we can make the 6 $4d$ chirals become 5 $2d$ chirals and one Fermi by choosing the R-symmetry $U(1)^{4d}_R$ corresponding to the values of the mixing coefficients $R_p = -\frac{2}{3}$ and $R_B = \frac{1}{3}$. We also choose $R_x = 0$ in order to make the $4d$ antisymmetric chiral field $A$ and a $2d$ antisymmetric chiral.

With this choice, the $4d$ gauge theory $\mathcal{T}_A^{4d}$ became a $2d$ $USp(2N)$ gauge theory with one antisymmetric chiral field, $3 + 2$ fundamental chiral fields, one fundamental Fermi field and $N$ Fermi singlets. The $4d$ WZ theory $\mathcal{T}_B^{4d}$ instead reduced to a LG model of $(6 + 3 + 1)N$ chirals and $(3 + 2)N$ Fermis.
It is useful to redefine the abelian symmetries $U(1)_b$ and $U(1)_s$ as
\[ U(1)_d = -2U(1)_B - U(1)_p, \quad U(1)_s = -U(1)_B + 2U(1)_s. \] (A.3)

Then the transformation rules of the 2d matter fields of the two theories under the global symmetry that we expect from 4d are

|      | $SU(3)$ | $SU(2)$ | $U(1)_d$ | $U(1)_s$ | $U(1)_x$ |
|------|---------|---------|----------|----------|----------|
| $Q$  | $3$     | $\bullet$ | $2$      | $1$      | $\frac{1-N}{3}$ |
| $P$  | $\bullet$ | $2$     | $-3$     | $1$      | $\frac{1-N}{3}$ |
| $\Psi$ | $\bullet$ | $\bullet$ | $0$      | $-5$     | $\frac{1-N}{3}$ |
| $A$  | $\bullet$ | $\bullet$ | $0$      | $0$      | $1$       |
| $\beta_i$ | $\bullet$ | $\bullet$ | $0$      | $0$      | $-i$      |
| $\Phi^{(1)}_i$ | $3$     | $2$     | $-1$     | $2$      | $i - \frac{2N+1}{3}$ |
| $\Phi^{(2)}_i$ | $\bar{3}$ | $\bullet$ | $4$     | $2$      | $i - \frac{2N+1}{3}$ |
| $\Phi^{(3)}_i$ | $\bullet$ | $\bullet$ | $-6$     | $2$      | $i - \frac{2N+1}{3}$ |
| $\Psi^{(1)}_i$ | $3$     | $\bullet$ | $2$      | $-4$     | $i - \frac{2N+1}{3}$ |
| $\Psi^{(2)}_i$ | $\bullet$ | $2$     | $-3$     | $-4$     | $i - \frac{2N+1}{3}$ |

We can thus see that the true global symmetry that is manifest at the Lagrangian level is actually $SU(5)_u \times U(1)_s \times U(1)_x$, where $SU(5)_u$ is enhanced from the 4d symmetry $SU(3) \times SU(2) \times U(1)_d$ according to the branching rules
\[
5 \rightarrow (3,1)^2 \oplus (1,2)^{-3}
\]
\[
10 \rightarrow (3,2)^{-1} \oplus (\bar{3},1)^4 \oplus (1,1)^{-6}.
\] (A.4)

Summarizing, we get the following putative 2d $\mathcal{N} = (0,2)$ duality:

**Theory $\mathcal{T}_A$:** $USp(2N)$ gauge theory with one antisymmetric chiral $A$, five fundamental chirals $Q_a$, one fundamental Fermi $\Psi$ and $N$ Fermi singlets $\beta_i$ with superpotential
\[
\mathcal{W}_{\mathcal{T}_A} = \sum_{i=1}^{N} \beta_i \text{Tr} A^i.
\] (A.5)

**Theory $\mathcal{T}_B$:** LG model with $5N$ Fermi fields $\Psi_{a;i}$ and $10N$ chiral fields $\Phi_{a,b;i}$ for $i = 1, \cdots, N$, $a < b = 1, \cdots, 5$ with cubic superpotential
\[
\mathcal{W}_{\mathcal{T}_B} = \sum_{i,j,k=1}^{N} \sum_{a,b,c,d,e=1}^{5} \epsilon_{abcde} \Psi_{a;i} \Phi_{b,c;j} \Phi_{d,e;k} \delta_{i+j+k,2N+1}.
\] (A.6)

Notice that for $N = 1$ this duality corresponds to the dimensional reduction of Seiberg duality discussed in eq. (3.4) of [11] in the particular case $N_c = 2$, $N_f = 3$ and $n = 0$.

The global symmetry for the two theories and the transformation rules of the matter fields that we expect from four dimensions are
Notice that, from a purely 2d point of view, on the side of theory $\mathcal{T}_A$ we would expect an additional symmetry that instead we don’t get from 4d. Indeed, there is apparently no superpotential term that prevents the fundamental chirals $Q$ and the fundamental Fermi $\Psi$ to rotate under two independent $U(1)$ symmetries, while from 4d we only get one particular combination of these two symmetries, which we called $U(1)_s$. This is one of the problems that may affect the dimensional reduction of 4d $\mathcal{N} = 1$ dualities to 2d $\mathcal{N} = (0,2)$ that we mentioned in the introduction. It is essentially due to the different nature of anomalies in 4d and 2d, which makes it possible that a $U(1)$ symmetry that was anomalous in four dimensions is not anomalous anymore in two dimensions. As also mentioned in [11], the 2d duality typically holds only if this symmetry is broken also in 2d. This situation is reminiscent of the perturbatively generated monopole superpotential in the reduction from 4d to 3d, which explicitly breaks $U(1)$ symmetries that were anomalous in 4d but which wouldn’t be in 3d. At the moment it is not understood what the analogue of monopole operators should be in the reduction from 4d to 2d. It would be interesting to understand this fact more in details.

One test that we can perform for the validity of the duality is matching anomalies. For both of the theories we find the following abelian anomalies:

\[
\text{Tr } \gamma^3 U(1)_s^2 = -40N, \quad \text{Tr } \gamma^3 U(1)_x^2 = \frac{5}{18}N(N-1)(2N+1), \quad \text{Tr } \gamma^3 U(1)_s U(1)_x = -\frac{20}{3}N(N-1) \tag{A.7}
\]

and the following non-abelian anomaly

\[
\text{Tr } \gamma^3 SU(5)_u^2 = N. \tag{A.8}
\]

We can also match the trial central charges

\[
c_R = \frac{5}{6}N \left((N-1)R_x((2N+1)R_x+18) - 48(N-1)R_sR_x - 144R_s^2 + 18\right),
\]

\[
c_R - c_L = 5N. \tag{A.9}
\]

Performing $c$-extremization, we get

\[
R_s = \frac{N-1}{2(2N-1)}, \quad R_x = -\frac{3}{2N-1}. \tag{A.10}
\]

\[\text{Indeed, as we already pointed out before, our proposed duality reduces for } N = 1 \text{ to one of those discussed in [11], which also suffers of the same issue.}\]
which leads to the following values for the superconformal central charges:

\[ c_R = \frac{15N(N+1)}{2(2N-1)}, \quad c_L = \frac{5N(N-5)}{2(2N-1)}. \]  

(A.11)

Notice that here the situation is reversed with respect to the one of Subsubsection 3.3.1, that is the central charges are positive provided that we are below the critical value \( N \leq N^* = 5 \), while they become non-positive for \( N \geq N^* \). Nevertheless, we also see that the antisymmetric chiral field has negative R-charge for any \( N \). This has the consequence that some of the gauge invariant operators built from it violate the unitarity bound and should be flipped. Such operators are some of the dressed chiral mesons \( \Psi A^{-1} \Psi \) and some of the dressed Fermi mesons \( QA^{-1} \Psi \), which are independent operators for \( i = 1, \cdots, N \). As for the duality of Subsection 3.3, we conclude that either \( c \)-extremization is not applicable for any \( N \) or the duality is between theories with a decoupled free sector.

For the potential duality we are discussing in this appendix, differently from the one presented in Section 3, we are not able to provide a derivation using some more fundamental duality. Consequently, we are not able to analytically prove the equality of the elliptic genera of the two theories, which is

\[ I_{\mathbb{T}_2} = \prod_{i=1}^N \theta(q x^{-i}; q) \prod_{a \in \mathfrak{F}} \theta(s^{-5N} x^{\frac{1-N}{3}} z^i; q) \times \prod_{i=1}^N \prod_{a=1}^5 \theta(q s^{-4N} x^{-2N+1} u_a z^i; q) u_a \prod_{a<b} \theta(s^2 x^{i-N} u_a u_b; q) = I_{\mathbb{T}_6}. \]

(A.12)

Unfortunately, also a numerical test of this identity is extremely hard from a computational point of view for any \( N \) which is not \( N = 1 \). We only managed to test it for \( N = 2 \) to low orders in a double expansion in either \( q, s \) or \( q, x \).

B Elliptic genus conventions

In this appendix we explain our conventions for the elliptic genus of 2d \( \mathcal{N} = (0,2) \) theories. The elliptic genus was originally studied in [58–62]. More recently, it has been computed for generic 2d \( \mathcal{N} = (0,2) \) supersymmetric gauge theories in the NSNS sector in [21] (see also [22] for the case of \( \mathcal{N} = (2,2) \) supersymmetry) and in the RR sector in [23, 24], where in the last two references it was computed as a partition function on \( \mathbb{T}^2 \) with localization techniques.

We will follow the conventions of [21, 22] and define it in radial quantization as

\[ I(u; q) = \text{Tr}_{\text{NSNS}}(-1)^F q^{L_0} \prod_a u_a^{f_a}. \]

This can be understood as a refined version of the Witten index, where we also turned on fugacities \( u_a \) in the Cartan of the global symmetry group \( F \), whose corresponding generators we denoted by \( f_a \). The parameter \( q \) can also be interpreted as \( q = e^{2 \pi i \tau} \), where \( \tau \) is the complex structure of the torus.
The elliptic genus has the remarkable property of being independent of the coupling constants of the theory. This allows us to compute it in the free field limit. By doing so, we can equivalently write it in the following integral form:

\[ I(u; q) = \frac{1}{|W|} \oint \prod_{i=1}^{rk_G} \frac{dz_i}{2\pi iz_i} I_{\text{vec}}(z; q) I_{\text{chir}}(z; u; q) I_{\text{ferm}}(z; u; q), \]  

where \( G \) denotes the gauge group, \(|W|\) is the dimension of its Weyl group, \( rk_G \) is its rank and \( z_i \) are fugacities taking values in its Cartan subalgebra.

The integrand receives contributions from all the possible multiplets of the theory. Specifically, a chiral multiplet of R-charge \( R \) in the representation \( R_G \) of the gauge symmetry group \( G \) with weight vectors \( \rho \) and \( \tilde{\rho} \) of the global symmetry group \( F \) with weight vectors \( \tilde{\rho} \) contributes as

\[ I_{\text{chir}}(z; u; q) = \prod_{\rho \in R_G} \prod_{\tilde{\rho} \in R_F} \frac{1}{\theta(q^{\frac{R+1}{2}} z^\rho \tilde{u}^{\tilde{\rho}}; q)}, \]  

where \( \theta(x; q) = (x; q)_{\infty} (q x^{-1}; q)_{\infty} \), \( (x; q)_{\infty} = \prod_{k=0}^{\infty} (1 - x q^k) \) and we also introduced the short-hand notation \( z^\rho = \prod_{i=1}^{rk_G} z_i^{\rho_i} \). Instead, a Fermi multiplet contributes as

\[ I_{\text{ferm}}(z; u; q) = \prod_{\rho \in R_G} \prod_{\tilde{\rho} \in R_F} \theta(q^{\frac{R+1}{2}} z^\rho \tilde{u}^{\tilde{\rho}}; q). \]  

Finally, a vector multiplet contributes as

\[ I_{\text{vec}}(z; q) = (q; q)_{\infty}^{2rk_G} \prod_{\alpha \in \mathfrak{g}} \theta(z^\alpha; q), \]  

where \( \alpha \) are the roots in the gauge algebra \( \mathfrak{g} \).

The integrand has the important property of being an elliptic function, i.e. invariant under rescaling \( z_i \to q z_i \), provided that all the gauge anomalies of the theory vanish. This allows us compute the integral (B.2) considering poles in the fundamental domain only and neglecting all the multiple copies of poles of the theta-functions. The integration contour is defined according to the Jeffrey–Kirwan residue prescription [63] (see [24] for a detailed explanation). In the case of a gauge theory with fundamental matter only, this amounts to considering all possible \( rk_G \) simultaneous poles, one for each integration variable, coming only from positively charged chirals under the \( U(1)^{rk_G} \) Cartan of \( G \).

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