Some aspects of the Standard Model in gravitational backgrounds with torsion

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Abstract

Torsion appears in a natural way in modern formulations of the gravitational theories. In this work we study several aspects of the interplay between the Standard Model and a classical gravitational background with torsion. In particular we consider the problem of the gauge and gravitational anomalies, $B$ and $L$ anomalies, the effective action for the torsion and the propagation of electromagnetic radiation in the presence of torsion.
1 Introduction

As is well known, all of our positive knowledge about natural forces can be summarized in the Standard Model (SM) and General Relativity. The Standard Model is a $SU(3)_c \times SU(2)_L \times U(1)_Y$ quantum gauge theory which successfully describes the strong and electroweak interactions, even at the high level of precision reached at LEP. The current versions and generalizations of General Relativity can also in some sense be considered as gauge theories of the Lorentz group $SO(1, 3)$. However a consistent and generally accepted formulation of the quantum theory of gravitation is still lacking. Thus so far our description of the gravitational phenomena is classical.

On the other hand, one important point concerning the modern gravitational theories, such as supergravity [1] or superstrings [2] is that they consider the metric (or the vierbein) and the affine connection as different structures. This is much more natural from the mathematical point of view and leads to the appearance of torsion. Therefore, while waiting for a completely consistent theory of gravitation, it seems to be interesting to study the new effects that could appear when the Standard Model is formulated in a classical gravitational background with torsion. We understand that any future theory should contain this approach as some kind of low-energy limit valid in a regime of weak gravitational fields. In this work we will discuss some important aspects of this formulation and we will make some remarks on possible new observable effects. The work is organized as follows. In Section 2 we do a brief review of the classical formulation of the Standard Model in the presence of gravitational fields with torsion. In order to define a proper quantum theory one important requirement is the absence of gravitational and gauge anomalies. Anomalies arise when some symmetry of the classical lagrangian is spoiled by the regularization procedure in the corresponding quantum theory, we deal with this issue in Sections 3 and 4. In Section 5 we reconsider the Standard Model hypercharge assignments and in Section 6 we study the effects of torsion on the leptonic and baryonic charges. In section 7 we give the effective action obtained for the torsion and electromagnetic fields when matter is integrated out and study its main properties. It is then shown that quantum effects give rise to an interaction term between these two fields which would produce observable effects. Finally in Section 8 we end with the conclusions.
2 The Standard Model lagrangian in presence of curvature and torsion

Let us start by introducing the notion of torsion from a geometrical point of view. Consider a pseudo-Riemannian space-time manifold with metric tensor $g_{\mu\nu}$. As usual, in order to define the parallel transport of vectors, we should introduce a new object, an affine connection, whose components we denote by $\hat{\Gamma}^{\lambda}_{\mu\nu}$. Such arbitrary connection is in principle independent of the metric. However if we want the lengths and angles of vectors to be invariant under parallel transport, it is needed that the connection is metric, that is:

$$(\hat{\nabla}_\lambda g)_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \hat{\Gamma}^\kappa_{\lambda\mu}g_{\kappa\nu} - \hat{\Gamma}^\kappa_{\lambda\nu}g_{\kappa\mu} = 0 \tag{1}$$

where $\hat{\nabla}$ is the corresponding covariant derivative. This condition allows us to find the following general form for this kind of connections:

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2} \left( T^\lambda_{\nu\mu} + T^\lambda_{\mu\nu} + T^\lambda_{\mu\nu} \right) \tag{2}$$

The antisymmetric part, $T^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\nu\mu}$ is known as the torsion tensor and $\Gamma^\lambda_{\mu\nu}$ are the usual Christoffel symbols that can be obtained from the metric. Thus we see that only for the metric and torsion free connection, both objects (metric and connection) are not independent.

The formulation of the SM on curved spaces with torsion, can be obtained as usual by means of the Strong Equivalence Principle (SEP). This principle states that in the free falling reference frame, where the gravitational interaction is switched off, the physical laws are locally the same as in absence of gravitational fields. The SEP is related to the minimality of the SM lagrangian coupled to gravity and yields a simple procedure to couple gravity to any field theory built in a flat space-time. In the following, we will first apply this principle to work out the gravitational interaction of Dirac spinors. At the end of this section we will also obtain the lagrangians for scalar and gauge fields interacting with gravity.

In order to work in the path integral formalism in the following sections, we will be interested in the euclidean lagrangian

$$\mathcal{L}_M = \frac{1}{2} \left( \bar{\psi}\gamma^m \partial_m \psi - \partial_m \bar{\psi}\gamma^m \psi \right) \tag{3}$$

Before coupling gravity to this lagrangian let us introduce some notation. We will use latin indices $m,n,...$ for objects referred to the locally inertial reference frame.
and Greek indices $\mu, \nu, ...$ for any other. If $\{\xi^m\}$ are the coordinates in the privileged system and $\{x^\mu\}$ the coordinates in any other then as usual:

$$g^{\mu\nu}(x) = e^\mu_m(x)e^n_\nu(x)\eta^{mn}$$

(4)

where $\eta^{mn} = (-, -, -)$ is the euclidean flat metric and $e^\mu_m(x) = \partial x^\mu/\partial \xi^m$ is the vierbein. In flat space-time, Dirac spinors change in the following way under Lorentz transformations:

$$\psi(x) \rightarrow U\psi(x) = e^{i\Sigma^{mn}\eta_{mn}}\psi(x)$$

$$\overline{\psi}(x) \rightarrow \overline{\psi}(x)U^\dagger = \overline{\psi}(x)e^{-i\Sigma^{mn}\eta_{mn}}$$

(5)

where $\Sigma_{mn} = i/4[\gamma_m, \gamma_n]$ are the hermitian generators of the $SO(4)$ group in the spinor representation. Notice that, in euclidean space $\psi$ and $\overline{\psi}$ are independent variables and the transformation rule of $\overline{\psi}$ is taken in such a way that $\psi\overline{\psi}$ is invariant. Therefore, the flat space-time Dirac lagrangian is invariant under those global transformations.

The SEP requires the invariance of the Dirac lagrangian under Lorentz transformations to be not only global but also local. With that purpose, let us introduce a covariant derivative $\partial_m \rightarrow \nabla_m$ and change from the locally inertial frame to arbitrary coordinates by means of the vierbein $\nabla_\mu = e^\mu_m\nabla_m$ so that we can write the gauged hermitian Dirac lagrangian in the following way:

$$L_M = \frac{\sqrt{g}}{2} \left( \overline{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \overline{\psi}\gamma^\mu \psi \right)$$

(6)

where we have defined the Dirac matrices in curved space-time $\gamma^\mu(x) = e^\mu_m(x)\gamma^m$. These matrices satisfy: $\{\gamma^\mu(x), \gamma^\nu(x)\} = -2g^{\mu\nu}(x)$. The covariant derivative is defined as usual by

$$\nabla_\mu = \partial_\mu + \Omega_\mu$$

(7)

where $\Omega_\mu$ is known as the spin connection. In order to keep the invariance under local Lorentz transformations, $\Omega_\mu$ should transform as follows:

$$\Omega_\mu \rightarrow \Omega'_\mu = U(x)\Omega_\mu U^{-1}(x) - (\partial_\mu U)U^{-1}(x)$$

(8)

It can be shown that the connection components in the privileged reference frame $\hat{\Gamma}_\mu^a = \eta^{bc}e^a_c(\partial_\mu e^\nu_c + e^\lambda_c\hat{\Gamma}_{\mu\nu}^\lambda)$ do have precisely the above transformation rule.
Following with the previous discussion, notice that $\overline{\Gamma}^{a b}_{\mu}$ does not have to be a Levi-Civita connection (that is, torsion free and metric), which we will denote $\{\Gamma^a_{\mu}\}$ and therefore torsion appears automatically. In fact, using the decomposition of the metric connection in eq.2, we can write the Dirac lagrangian in eq.6 in terms of the Levi-Civita connection plus an additional term depending on the torsion

$$L_M = \sqrt{g} \overline{\psi} \gamma^\mu \left( \partial_\mu - \frac{i}{2} \Gamma^a_{\mu} \Sigma_{ab} - \frac{1}{8} S_\mu \gamma_5 \right) \psi$$

(9)

where $S_\rho = \epsilon_{\mu\nu\rho\lambda} T^{\mu \nu \lambda}$. In conclusion, the lagrangian for Dirac fermions in a curved space-time with torsion is that of a fermion in a curved space-time without torsion plus a coupling between $S_\mu$ and the axial current.

Following the above arguments we can now write the corresponding expression for the SM matter sector:

$$L_M = \sqrt{g} \left( \overline{Q} D^Q Q + \overline{L} D^L L \right)$$

(10)

where:

$$D^Q = \gamma^\mu (\partial_\mu + \Omega^Q_\mu + G_\mu + W^Q_\mu P_L + ig' B^Q_\mu (Y^Q_L P_L + Y^Q_R P_R) + S^Q_\mu \gamma_5)$$

$$D^L = \gamma^\mu (\partial_\mu + \Omega^L_\mu + W^L_\mu P_L + ig' B^L_\mu (Y^L_L P_L + Y^L_R P_R) + S^L_\mu \gamma_5)$$

(11)

Here we have used the following notation: $G_\mu$, $W_\mu$ and $B_\mu$ are the gauge fields corresponding to $SU(3)$, $SU(2)_L$ and $U(1)_Y$ groups respectively. Quarks and lepton are organized in doublets $Q' = (U, D)$, $L' = (N, E)$ and $Y^{Q,L}_{L,R}$ denote the hypercharge matrices $[3]$.

As in the flat space-time case, these operators are not hermitian due to the chiral couplings of $SU(2)_L$ and hypercharge fields. Thus the adjoint operators are

$$(D^Q)^\dagger = \gamma^\mu \overline{D}_\mu^Q = \gamma^\mu (\partial_\mu + \Omega^Q_\mu + G_\mu + W^Q_\mu P_R + ig' B^Q_\mu (Y^Q_R P_R + Y^Q_L P_L) + S^Q_\mu \gamma_5)$$

$$(D^L)^\dagger = \gamma^\mu \overline{D}_\mu^L = \gamma^\mu (\partial_\mu + \Omega^L_\mu + W^L_\mu P_R + ig' B^L_\mu (Y^L_R P_R + Y^L_L P_L) + S^L_\mu \gamma_5)$$

(12)

Notice that, since there is no right neutrino, the spin connection and torsion couplings can be written as follows for leptonic operators:

$$\Omega^L_\mu = -\frac{i}{2} \Gamma^a_{\mu} \left( P_L \Sigma_{ab} \Sigma_{ab} \right), \quad S^L_\mu \gamma_5 = -\frac{1}{8} S_\mu \left( P_L \gamma_5 \gamma_5 \right)$$

(13)
where the matrices act on the flavor space. The expressions for $\Omega$ and $S$ are obtained from eq.13 just replacing $P_L \to P_R$. For quark operators the spin connection and torsion terms are the same as for leptons but without the $P_{L,R}$ projectors.

Finally we will give the lagrangians in curved space-time for the rest of fields present in the SM.

As the scalar fields do not change under Lorentz transformations, their covariant derivative is just an ordinary derivative. Then, according to the prescription based on the SEP, we simply have to use the vierbein to perform an arbitrary coordinate transformation. Thus, the action of the minimal SM symmetry breaking sector in curved space-time reads:

$$S_{SBS} = \int d^4x \sqrt{g} \left( g^{\mu\nu} (D_\mu \phi)^\dagger (D_\nu \phi) - V(\phi) + \mathcal{L}_{YK} \right)$$

(14)

where $D_\mu = \partial_\mu + ig'^2 B_\mu + W_\mu$ and $\mathcal{L}_{YK}$ is the usual Yukawa lagrangian that is not modified by the gravitational coupling.

The Yang-Mills lagrangian in flat space-time is given by

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{a mn} F_{^a mn}$$

(15)

where $a$ is a group index. We consider the strength tensor $F_{a mn}$ as defined in a locally inertial coordinate system. $F_{a mn}$ is a Lorentz tensor and $F_{a mn} F_{^a mn}$ is invariant under global and local Lorentz transformations. Therefore we only have to transform it to an arbitrary coordinate system using the vierbein

$$F_{\mu\nu}^a = e^m_\mu e^n_\nu F_{^a mn}$$

(16)

Thus the action for the SM gauge sector reads

$$S_{YM} = \int d^4x \sqrt{g} \left( \frac{1}{2g_s^2} \text{tr} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2g^2} \text{tr} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4} \text{tr} B_{\mu\nu} B^{\mu\nu} \right)$$

(17)

3 The quantum Standard Model

Up to now we have considered the classical theory. The quantization of the SM in curved space-time with torsion can give rise to new interesting effects. In particular, some of the above minimal lagrangians are not renormalizable. In fact, the one-loop calculations require counterterms which were not present in the original
lagrangian. For instance, for the scalar sector one needs to introduce the counterterm $R \phi^2$ where $R$ is the scalar curvature. In addition, one should include in the pure gravitational sector some counterterms that absorb the vacuum divergences, which cannot be discarded by the procedures used in flat space-time (such as normal ordering). However, the total number of new counterterms that we have to add to render the theory renormalizable is finite. Furthermore, since symmetry is our only guiding principle in constructing the SM lagrangian in curved space-time, any other non-minimal term could be included, provided it respects the symmetries of the theory. All such terms are different from the minimal ones in the sense that they violate the SEP. The reason is that a term like $R \phi^2$ vanishes in flat space-time, but that is not the case in a free-falling reference frame due to the presence of the scalar curvature. In contrast, the minimal couplings are the same either in a flat space-time or in a free falling frame.

The violation of the SEP does not mean a breaking of Lorentz invariance (provided the non-minimal terms are Lorentz scalars). Nevertheless, we will see that the anomaly effects may also violate Lorentz invariance, although for consistency we will require its conservation. According to this discussion we conclude that the SEP is only a low-energy effect which will not be satisfied when higher order corrections are included in the effective lagrangian.

Concerning the quantum SM there is another issue that must be taken into account in the canonical quantization procedure. In an arbitrary curved manifold, Poincaré invariance is no longer a symmetry and $\partial/\partial t$, in general, is not a Killing field. The existence of such a Killing vector provides a natural definition of positive energy modes and therefore of creation and annihilation operators. As far as the vacuum is defined using annihilation operators, in curved space-time the vacuum is not unique. In this sense a given state which for certain observer is empty, may have some particles for a different (accelerated) observer. These processes of particle creation are typical of Quantum Field Theory (QFT) in curved space-time and they have been extensively studied [4], [5].

In addition the definition of an $S$-matrix requires a time parameter with respect to which we can define asymptotically free states in the remote past and future. In fact, in Minkowski space-time, particles can be well separated before and after the interaction. However, in curved space-time this situation does not take place in general and, as a consequence, it is not always possible to define an $S$-matrix.

Finally it is well known that chiral theories (like the Standard Model) in four dimensions are potentially inconsistent due to gauge and mixed gauge-gravitational anomalies. In flat space-time, the assignment of hypercharges for the different
fermions is done in such a way that the contributions to the anomalies of some fermion field is exactly cancelled by the contributions of the rest of fermions. However when gravity is introduced things can change since there is new contributions to the anomalies coming from curvature and torsion. In the next sections we will study these new terms and we will extract the conditions needed for their cancellation.

4 The Standard Model anomalies

Let us consider the effective action for the gauge fields, the vierbein and the spin-connection that is obtained after the functional integration of the fermionic fields in the SM:

$$e^{-W[A,\Omega,e]} = \int[d\psi d\bar{\psi}] e^{-\int d^4x L_{SM}}$$  \hspace{1cm} (18)

Here $A$ denotes the gauge fields and $\psi$ all the fermions. This effective action contains all the information about the quantum effects of the matter fields on the gravitational and gauge interactions. In particular, from eq[18] it is possible to extract, particle creation rates, gauge and gravitational anomalies, new interaction terms, etc.

In order to calculate the SM anomalies we will follow the standard Fujikawa method [6]. There exist a gauge anomaly whenever the EA is not gauge invariant. Let us consider as an example the $SU(N_c)$ gauge transformations. The effective action, being a functional in the gauge fields, may have an anomalous variation given by

$$\delta_\theta W = -\int d^4x \sqrt{g} i \theta^b (D_\mu \langle j^a_\mu \rangle)^b = -\int d^4x \sqrt{g} i \theta^a A^a(x)$$  \hspace{1cm} (19)

where $D_\mu = \nabla_\mu + [G_\mu, \cdot]$, $\theta$ is the transformation parameter and $j^{a\mu} = \overline{Q} \gamma^\mu \Lambda^a Q$ the color gauge current. This transformation comes from the change in the integration measure in eq[18] since the classical action is gauge invariant.

The anomaly $A^a(x)$ in the above equation is in general a divergent object. The corresponding renormalized expression is obtained by using the regularizing operators defined by:

$$H^{Q,\mathcal{L}}_\psi = (i D^{Q,\mathcal{L}})(i D^{Q,\mathcal{L}})^\dagger = \mathcal{D}^{Q,\mathcal{L}} \mathcal{D}^{Q,\mathcal{L}} + \mathcal{X}^{Q,\mathcal{L}}$$

$$H^{Q,\mathcal{L}}_{\overline{\psi}} = (i D^{Q,\mathcal{L}})(i D^{Q,\mathcal{L}})^\dagger = \overline{\mathcal{D}}^{Q,\mathcal{L}} \mathcal{D}^{Q,\mathcal{L}} + \overline{\mathcal{X}}^{Q,\mathcal{L}}$$  \hspace{1cm} (20)
with $X = \gamma_5 S^\mu \gamma_\mu + 2 S^\mu S^\mu - \frac{1}{4} [\gamma^\mu, \gamma^\nu] [d^\mu, d^\nu]$ and $D^\mu = D_\mu - \frac{1}{2} \gamma_5 [\gamma^\mu, \gamma^\nu] S^\nu$ whose eigenvalues are gauge and local Lorentz invariant. As before, a bar means that left and right projectors have to be exchanged.

The usual renormalization prescription for the anomaly in the Fujikawa method is given by [7]:

$$A^a_{\text{ren}}(x) = \frac{1}{(4\pi)^2} \text{Tr}(\Lambda^a(a_2(H_{\bar{\psi}}^Q, x) - a_2(H_{\psi}^Q, x)))$$  \hspace{1cm} (21)

The second coefficient in the heat-kernel expansion in curved space-time for the operators in eq.20, which is the only relevant for the anomaly calculation, reads

$$a_2(H_{\bar{\psi}}, x) = \frac{1}{12} [D_\mu, D_\nu][D^\mu, D^\nu] + \frac{1}{6} [D_\mu, [D^\mu, X]] + \frac{1}{2} X^2 - \frac{1}{6} RX - \frac{1}{30} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu}$$  \hspace{1cm} (22)

and $a_2(H_{\psi}, x)$ has similar expression but with the barred quantities.

From the previous discussion, we obtain the anomalous Ward identity

$$(D_\mu \langle j^\mu \rangle)^a = A^a_{\text{ren}}$$  \hspace{1cm} (23)

Thus we see that the presence of the anomaly represents a failure in the conservation of the gauge currents. Such non-conservation would destroy the consistency of the model and then it is necessary that the new terms depending on the curvature and torsion cancel. This could impose new constraints to the SM hypercharges. Up to now we have only considered the anomalous Ward identities in the $SU(N_c)$ case, the $SU(2)_L$ and $U(1)_Y$ results are obtained in a similar way.

The final explicit results for the different anomalous Ward identities are obtained just by taking the traces in Lorentz and internal indices in eq.21. Thus for $SU(N_c)$:

$$(D_\mu \langle j^\mu \rangle)^a = -\frac{1}{32\pi^2} g_s g' e^{\mu\nu\rho\sigma} G^a_{\mu\nu} B_{\rho\sigma} \sum_{u,d} (y_L - y_R)$$  \hspace{1cm} (24)

The $SU(2)_L$ gauge current is $j^{a\mu}_L = \overline{Q} \gamma^\mu T^a P_L Q + (Q \rightarrow L)$ and the corresponding anomaly yields:

$$(D_\mu \langle j^\mu_L \rangle)^a = -\frac{1}{32\pi^2} g g' e^{\mu\nu\rho\sigma} W^a_{\mu\nu} B_{\rho\sigma} \left( N_C \sum_{u,d} y_L + \sum_{\nu,e} y_L \right)$$  \hspace{1cm} (25)
The $U(1)_Y$ gauge current is $\mathcal{J}^\mu_Y = \mathcal{Q}\gamma^\mu(Y_L^Q P_L + Y_R^Q P_R)\mathcal{Q} + (\mathcal{Q} \rightarrow \mathcal{L})$ and the anomaly:

$$D_\mu \langle j^\mu_Y \rangle = \frac{1}{32\pi^2} \left( -\frac{1}{24} \epsilon^{\rho\sigma\gamma\delta} R_{\mu\nu\rho\sigma} R_{\gamma\delta}^{\mu\nu} - \frac{g_2}{2} \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu} G_{\rho\sigma} \right) \sum_{u,d} (y_L^u - y_R^u) + \sum_{\nu,e} (y_L^\nu - y_R^\nu)$$

$$+ \frac{g_2}{2} \epsilon^{\mu\nu\rho\sigma} W_{\mu\nu} W_{\rho\sigma} \sum_{u,d} (y_L^u - y_R^u) + \sum_{\nu,e} (y_L^\nu - y_R^\nu)$$

$$+ g^2 \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma} \sum_{u,d} (y_L^u - y_R^u) + \sum_{\nu,e} (y_L^\nu - y_R^\nu)$$

Thus we find the same result as in flat space-time for the non-abelian currents. However the abelian $U(1)_Y$ anomaly does get contributions from the curvature and the torsion, thus imposing new conditions on the hypercharges for the cancellation. The torsion contribution can be seen to be nothing but a total derivative and therefore can be removed by adding suitable counterterms to the lagrangian, for that reason we have not written it in the final result.

As we have mentioned above, the SEP states that any theory in curved space-time should be invariant under local Lorentz transformations. We consider the possible violation of this symmetry due to quantum effects when chiral fermions are present. As we will see, whenever abelian chiral gauge fields are present, as is the case of the hypercharge field, local Lorentz invariance is in principle broken.

As we saw in Sect.2, the classical Dirac lagrangian in curved space-time is invariant under the $SO(4)$ transformations given in eq.5 and eq.8 (for Euclidean signature). Therefore we can calculate the gravitational anomalies as gauge anomalies of the Lorentz group. The only difference is the appearance of an additional field, the vierbein, which also transforms under this group.

Following the same steps as in the previous section, we can obtain the corresponding anomalous Ward identities.

We can write this result more conveniently using

$$\langle j^\mu \rangle = \langle j^\mu_{\Sigma} \rangle$$

and the explicit form of the $SO(4)$ gauge covariant derivative that is nothing but: $D_\mu = \nabla_\mu + [\Omega_\mu, \cdot]$. 
Following the same steps as in the gauge case we find the anomalous Ward identity:

$$A_{ab}^{\text{ren}}(x) = -(D_{\mu}(j^{\mu}))^{ab}(x) + i(T^{ab}(x) - T^{ba}(x))$$

(28)

where $T_{ab} = e_{b\mu} \delta W/\delta e_{a\mu}$ is the expectation value of the energy-momentum tensor in the presence of the background fields.

The renormalized anomaly is:

$$A_{ab}^{\text{ren}}(x) = \frac{1}{(4\pi)^2} \text{tr} \left[ \Sigma^{ab} \left( a_2(H^Q, x) - a_2(H^Q_{\psi}, x) \right) \right] + (Q \to \mathcal{L})$$

(29)

After a lengthy calculation we arrive to the final expression for the Lorentz anomaly, its explicit expression is:

$$A_{mn}^{\text{ren}} = \frac{g'}{32\pi^2} \left( \frac{1}{6} \epsilon^{mnab} R_{\mu\nu} B^{\mu\nu} + \frac{1}{6} (B_\alpha^a S_{\alpha,m} - B_\alpha^m S_{\alpha,n}) 
- \frac{1}{24} \epsilon^{mnab} (B_{ac} S_b + B_{ab} S^2) - \frac{1}{6} \epsilon^{mnab} B_{ab} R - \frac{1}{2} S_{\mu}^{\mu} B^{mn} 
- \frac{1}{3} \epsilon^{mnab} \Box B_{ab} + \frac{1}{3} (S_\alpha B^{\alpha m})^{;m} - \frac{1}{3} (S_m B^{\mu m} - S^m B^{\mu m})^{;\mu} 
- \frac{1}{3} (S_\alpha B^{\alpha m})^{;m} \left( \sum_{u,d} N_c (y_L - y_R) + \sum_{\nu,e} (y_L - y_R) \right) \right)$$

(30)

Notice that pure gravity terms do not arise, in agreement with the result that there are no pure gravitational anomalies in four dimensions. Observe also that all the terms depend on the abelian $B_{ab}$ field, whereas there is no contribution from non-abelian gauge fields. Finally, the cancellation condition agrees with that of eq.[31] which ensures the vanishing of the gravity terms in the $U(1)_Y$ anomaly. This condition is satisfied with the usual hipercharge assignment in the SM.

5 Anomaly cancellation and charge quantization in the SM

From the above computation of the SM gauge and gravitational anomalies it is very easy to read the conditions that must be set on the hypercharges in order to cancel those anomalies thus making possible a proper definition of the SM as a QFT. The
three first conditions in eqs.24, 25 and 26 are the same as those obtained in flat space-time. However notice the appearance of terms depending on the curvature in the third condition, which did not occur in the case of non-abelian gauge fields. The new terms that were not present in flat space-time impose a new cancellation condition, namely, the vanishing of the sum of all hypercharges

\[ N_c \sum_{u,d} (y_L - y_R) + \sum_{\nu,e} (y_L - y_R) = 0 \quad (31) \]

The conditions for the cancellation of gauge anomalies in flat space-time, together with the gauge invariance of the SM Yukawa sector, allows us to fix all the hypercharges up to a normalization constant [8]. However, we have just seen that, in curved space-time, we have an additional constraint on the hypercharges, eq.31, coming both from the curvature terms in the \( U(1)_Y \) anomaly and from the local Lorentz anomaly. Within the minimal SM, this condition is compatible with the others. However it is possible to take a different point of view and, without assuming any specific symmetry breaking sector, try to fix the hypercharges. Then, the above conditions form a set of four equations for five unknowns

\[ y^u_L = y^e_L, \quad y^d_L = y^d_R, \quad y^u_R, \quad y^d_R. \]

Let us try to solve the system explicitly and check whether they determine all the hypercharges up to a normalization factor. First, we note that the four equations can be reduced to a single one in terms of one variable for \( y^d_R \neq 0 \) (if \( y^d_R = 0 \) all the hypercharges vanish, which is unphysical):

\[ 1 + \left( \frac{y^u_R}{y^d_R} \right)^3 + \frac{21}{6} \left( \frac{y^u_R}{y^d_R} \right) + \frac{21}{6} \frac{y^d_R}{y^d_R} = 0 \quad (32) \]

It is not difficult to see that there are three real solutions,

\[ \frac{y^u_R}{y^d_R} = -1, \quad -2, \quad -\frac{1}{2} \quad (33) \]

The rest of the hypercharges can be obtained in terms of one of them in a strightforward way. Hence, there are only three possible sets of hypercharges, up to a global normalization factor. We have listed them in Table 8.1. The first solution, whose normalization is arbitrary, together with the usual weak isospin assignment \( Q = T_3 + Y \) implies that the right component of the electron is chargeless.

The second set is the usual hypercharge assignment in the SM, normalizing as usual \( Q_e = -1 \). The third solution, keeping the same normalization, leads to different electric charges for the left and right components of the quark fields.
and therefore to chiral electromagnetism. In conclusion, gauge and local Lorentz invariance together with some physical constraints such as vector electromagnetism, allows us to fix all the hypercharges up to global normalization.

6 The leptonic and baryonic anomalies

Following the above methods we will study in this section the anomalies appearing in the leptonic and baryonic currents. Those anomalies correspond to global symmetries of the classical SM and then they do not lead to any inconsistency in the quantum theory. Moreover they could lead to new observable effects. In flat space-time neither the baryon number \( B \) nor the leptonic number \( L \) are conserved because of the axial anomaly. However, those anomalies equal so that \( B - L \) is still conserved in the presence of the anomaly.

The baryon and lepton number currents:

\[
\begin{align*}
\mathbf{j}_B^\mu &= \frac{1}{N_c} \bar{Q} \gamma^\mu Q, \\
\mathbf{j}_L^\mu &= \bar{L} \gamma^\mu L
\end{align*}
\]

are classically conserved, i.e \( \nabla^\mu \mathbf{j}_{Q,L}^\mu = 0 \), where \( \nabla^\mu \) is the Levi-Civita covariant derivative. However these conservation laws are violated due to quantum effects. The corresponding anomalies can be obtained using Fujikawa method based on the operators defined in eq. (20). The gaussian regulators associated to these operators respect the gauge and local Lorentz symmetries of the theory. This procedure yields for the anomalies the following results (for euclidean signature):

\[
\nabla^\mu \mathbf{j}_B^\mu = \frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} \left( \frac{g^2}{2} W_{\mu\nu}^a W_{\alpha\beta}^a + g^2 B_{\mu\nu} B_{\alpha\beta} \sum_{u,d} (y_L^2 - y_R^2) \right)
\]

Table 8.1: Hypercharge assignments.
and

\[ \nabla_\mu j^\mu_L = \frac{1}{32\pi^2} \left\{ \frac{\epsilon^{\alpha\beta\gamma\delta}}{24} R_{\mu\nu\alpha\beta} R^{\mu\nu}_{\gamma\delta} + \frac{\epsilon^{\alpha\beta\gamma\delta}}{48} S_{\beta\gamma\delta\alpha} + \epsilon^{\alpha\beta\gamma\delta} \left( \frac{g^2}{2} W_{\gamma\delta}^a W_{\alpha\beta}^a \right) \right. \\
\left. + g^2 B_{\gamma\delta} B_{\alpha\beta} \sum_{\nu,e} (y_L^2 - y_R^2) - \frac{1}{6} \Box S_{\alpha\alpha}^\alpha + \frac{1}{96} (S_{\alpha\mu}^\alpha S_{\mu\alpha}^\alpha - \frac{1}{2} RS_{\mu}^\alpha S_{\alpha}^\mu) - \frac{1}{6} \left( R^{\mu\alpha} S_{\alpha}^\alpha - \frac{1}{2} RS_{\mu}^\alpha \right) \right\} (36) \]

The resulting lepton anomaly has terms depending on the curvature and the torsion that appear due to the absence of one of the chirality components of the neutrino field. Such terms are not present in the baryonic anomaly since quarks have both chirality components. Thus in contrast with flat space-time, $B - L$ is spoiled in the presence of curvature and in principle also in presence of torsion.

Therefore we have three different kinds of potential contributions to the lepton anomaly. First we have a $\epsilon^{\alpha\beta\gamma\delta} R_{\mu\nu\alpha\beta} R^{\mu\nu}_{\gamma\delta}$ term where $R^{\mu\nu}_{\gamma\delta}$ is the curvature associated to the Levi-Civita spin connection. This topological term could give rise to actual contributions to the $L$ violation through the so called gravitational instantons that would be relevant in the context of quantum gravity \[9\]. Second, there is also the well known $SU(2)_L$ and $U(1)_Y$ gauge contributions to the anomaly \[10\]. Finally, we also have the possible torsion contribution in which we are interested.

However the torsion contribution to the anomaly is a four-divergence and it can be absorbed in the redefinition of the lepton current as follows:

\[ \tilde{j}_L^\mu = j_L^\mu - \frac{1}{32\pi^2} \left( \frac{1}{6} S_{\alpha\alpha}^\alpha + \frac{1}{96} S_{\alpha}^\alpha S_{\mu}^\mu S_{\alpha}^\alpha - \frac{1}{6} \left( R^{\mu\alpha} S_{\alpha}^\alpha - \frac{1}{2} RS_{\mu}^\alpha S_{\alpha}^\mu \right) \right) \] (37)

In the absence of the other contributions to the anomaly this new current is conserved, that is, $\nabla_\mu \tilde{j}_L^\mu = 0$. Thus we observe that the possible effects of the torsion can be eliminated away by this redefinition of the lepton current. Notice that the new definition, being gauge and local Lorentz invariant, is physically meaningful and should satisfy the appropriate Ward identities for a properly defined lepton current.

An alternative way to show that torsion does not contribute to the lepton anomaly is based on a different choice of regulator. In fact let us take as regulators for the anomalies the operators $i\tilde{D}\dagger i\tilde{D}$ and $i\tilde{D}i\tilde{D}\dagger$, where $\tilde{D}$ is the $\tilde{D}$ operator in which the torsion field has been set to zero. These operators respect all the gauge and local Lorentz symmetries of the theory and therefore they are also valid as regulators. However they do not depend on torsion, this implies that the regulated anomalies cannot depend on torsion as expected.
7 The EA for torsion and electromagnetism

The effective action defined in eq.18 is a functional in the vierbein and the connection, i.e., in the curvature and the torsion. In addition it also depends on the gauge field. In this section we will concentrate in the contributions to the EA coming only from the electromagnetic and torsion fields, from them it could be possible to obtain some phenomenological effects as we will see. Therefore we will omit the curvature dependence working in a flat space-time and also ignore the effects of the rest of the gauge fields.

In this case, we will get for the EA, considering only the electronic family in SM:

$$W[A,S] = S_{cl}[A,S] + i \text{Tr} \log \left( i \not\partial - m - \frac{1}{8} \not{\mathcal{G}} \gamma_5 \right)$$

where

$$S_{cl}[A,S] = S_{cl}[A,S] + i \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \text{Tr} \left( (i \not\partial - m)^{-1} (e A - \frac{1}{8} \not{\mathcal{G}} \gamma_5) \right)^k$$

The Dirac propagator is defined as usual:

$$(i \not\partial - m)^{-1} = \int d\bar{q} e^{-i\bar{q}(x-y)} \frac{\not{\bar{q}} + m}{q^2 - m^2 + i\epsilon}$$

where the functional traces Tr are evaluated in dimensional regularization with $D = 4 - \epsilon$ and $d\bar{q} = \mu^D d^D q/(2\pi)^D$. The result will contain divergent local pieces together with finite local and non-local terms. Let us first give the results for the divergences:

$$W_{\text{div}}[A,S] = \frac{\Delta}{(4\pi)^2} \int d^4x \left( -\frac{e^2}{3} F_{\mu\nu} F^{\mu\nu} - \frac{1}{192} S_{\mu\nu} S^{\mu\nu} + \frac{m^2}{16} S_\mu S^\mu \right)$$

where $S_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu$, $\Delta = N_\epsilon + \log(\mu^2/m^2)$ with the poles parametrized as $N_\epsilon = 2/\epsilon + \log 4\pi - \gamma$ and $\mu$ the renormalization scale. The first divergent term will be absorbed in the redefinition of the photon field. In order to absorb the divergences depending on torsion, it is necessary that the classical action contains a kinetic and a mass term for the torsion field. Thus we see that torsion behaves like a massive
abelian gauge field. Even if we had started from a theory with non-propagating torsion (such as Einstein-Cartan theory), we would generate a kinetic term due to the interaction with the fermions.

The rest of finite contributions to the EA are in general difficult to obtain, however we can study some of them in some particular limits. Thus we will consider those terms with two photon fields and one torsion field in the masless limit for fermions and for slowly varying torsion fields, i.e, \( p_A^2 >> m^2 \) and \( p_A^2 >> p_S^2 \), where \( p_A \) and \( p_S \) denote the momenta of photon and torsion respectively. To the lowest order in the number of torsion derivatives, we have:

\[
W^A_{0}[A,S] = -\frac{e^2}{4(4\pi)^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} F^{\rho\mu} A^\sigma S^\nu
\]  

(42)

This term is gauge invariant but only to the lowest order, that is, for constant torsion. It is local and finite since all the possible divergences are those in \( W_{div} \). The next order in the expansion with one torsion derivative reads:

\[
W^A_{1}[A,S] = \frac{e^2}{2(4\pi)^2} \int d^4x d^4y d\tilde{p} \frac{e^{ip(x-y)}}{p^2} \epsilon^{\alpha\beta\mu\nu} \partial_\mu A^\lambda(y) A_\nu(x) \partial_\alpha S_\beta(x)
\]  

(43)

This term is non-local and is not gauge invariant, however if we add \( W^A_{0} \) we recover gauge invariance even for arbitrary torsion fields. For constant \( S_\mu \) the term \( W^A_{1} \) vanishes and \( W^A_{0} \) has the precise form of the term needed to explain [11] the anisotropy in cosmological electromagnetic propagation recently claimed by Nodland and Ralston [12]. However two main difficulties appear in this explanation:

i) The term has been obtained assuming that \( p_A^2 >> m^2 \), i.e. it is only valid for highly energetic photons. However the Nodland-Ralston effect was found in radio galaxies and therefore one expects \( W^A_{0} \) not to be very relevant in that energy range. However the very same fact suggests to look for such an effect in gamma-rays sources.

ii) More important is the origin of the cosmological slowly varying torsion field. From eq.11 we see that torsion behaves like a massive gauge field with mass values around Planck scale. This fact would avoid the generation of long range torsion fields. However it has been suggested that the true vacuum of the theory might have a non vanishing vacuum expectation value for the torsion pseudotrace, thus generating the cosmological background [13] required in the above explanation.
8 Conclusions

We have reviewed some of the consequences of formulating the SM in a curved space-time with torsion. In particular, we have shown that the presence of torsion is not incompatible with the cancellation of gauge and gravitational anomalies and thus, the SM can be consistently formulated in a gravitational background with torsion. In addition, although the leptonic anomaly is not modified by the presence of torsion, it gets new contributions from the Levi-Civita curvature and as a consequence $B-L$ is no longer conserved in the presence of gravity. Finally we have shown that quantum effects generate a kinetic and a mass term for the torsion fields of the same form as those of massive abelian gauge fields. In addition, when we have torsion together with an electromagnetic background, the quantum effects of matter fields induce a new coupling of EM with torsion that in principle could have phenomenological consequences.

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