The Interaction of Gravity with Other Fields

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Abstract. We consider the interaction of gravity, as expressed by Einstein’s Equations of General Relativity, to other force fields. We describe some recent results, discussing both the mathematics, and the physical interpretations. These results concern both elementary particles, as well as cosmological models. (This paper describes joint work variously done with with F. Finster, N. Kamran, B. Temple, and S.-T. Yau.)

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1 Introduction

We wish to describe how General Relativity modifies classical physics on two different scales: the scale of elementary particles, and on cosmological scales, the large scale structure of the Universe. We shall discuss a few different scenarios, describing the main ideas. The more detailed mathematical discussions can be found in our original papers.

The mathematics of General Relativity (GR) has several features which are extremely interesting, and are quite different from the classical Newtonian theory of gravity. We will show how rich a subject GR is, both mathematically, and in its physical applications and interpretations. Indeed, GR is most interesting from a physical viewpoint because it gives a different understanding of physical phenomena.

The main feature of Einstein’s Theory of GR is that the gravitational field is the metric in 4-dimensional spacetime. Thus GR is actually a theory of spacetime in the sense that the underlying spacetime continuum is not fixed, but is allowed to vary; indeed, massive bodies modify the curvature of their surrounding spacetime geometry, and free particles traverse along geodesics determined by the metric. The second important feature of GR is the existence of black holes; this notion has no classical analogue. We shall see the role played by black holes in various aspects of this paper. Finally, we remark that GR in and of itself, is a very beautiful subject. This was already recognized by Einstein himself who in 1915, in his presentation of the theory of GR to the Prussian Academy in Berlin stated, “Hardly anyone who has grasped the theory, will be able to escape from its magic.” This is undoubtedly still true today.

2 Background Material

In this section we shall review some basic ideas in GR, Yang-Mills (YM) equations and the Dirac (D) equation in four-dimensional Lorentzian spacetime.
A. General Relativity

General Relativity is Einstein’s Theory of Gravity, and is based on the following 3 hypotheses.

\((E_1)\) The gravitational field is the metric \(g_{ij}\) in 4-dimensional spacetime. The metric is assumed to be symmetric: \(g_{ij} = g_{ji}, \ i, j = 0 \rightarrow 3.\)

\((E_2)\) At each point, \(g_{ij}\) can be diagonalized as

\[ g_{ij} = \text{diag}(-1, 1, 1, 1). \]

\((E_3)\) The equations should be independent of the particular coordinates involved, and thus the equations should be tensor equations.

\((E_1)\) is Einstein’s brilliant insight, whereby he “geometrizes” the gravitational field, replacing one Newtonian potential for the gravitational field by the ten metric potential functions, \(g_{ij}, \ i \leq j.\) \((E_2)\) means that Special Relativity is included in GR, (and also that the metric tensor is everywhere non-singular), while \((E_3)\) asserts that coordinates are merely an artifact, and physics shouldn’t depend on the choice of coordinates.

The metric \(g_{ij} = g_{ij}(x), \ i, j = 0 \rightarrow 3, \ x = (x^0, x^1, x^2, x^3), \ x^0 = ct (c = \text{speed of light}),\) is a tensor defined on 4-d spacetime. Einstein’s equations are ten (tensor) equations for the unknown metric \(g_{ij}\) (gravitational field), and take the form

\[ R_{ij} - \frac{1}{2} R g_{ij} = \sigma T_{ij}. \tag{2.1} \]

The left-hand side \(G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}\) is the Einstein tensor and is a geometric quantity, depending only on \(g_{ij}\) and its derivatives, while \(T_{ij}\), the energy-momentum tensor, represents the source of the gravitational field, and encodes the distribution of matter. The word “matter” in GR refers to everything which can produce a gravitational field, including elementary particles and electromagnetic fields. Since the Einstein tensor identically satisfies the equation \(G_{ji}^{\ j} = 0\) (the covariant divergence vanishes) it follows that on solutions of (1), \(T_{ji}^{\ j} = 0,\) and this in turn expresses the laws of conservation and
momentum, [23]. The quantities which make up the Einstein tensor $G_{ij}$ are given as follows. First from the metric tensor $g_{ij}$, we form the Levi-Civita connection (Christoffel symbols) $\Gamma^k_{ij}$:

\[
\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left( \frac{\partial g_{ij}}{\partial x^\ell} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right), \quad i, j, k = 0 - 3,
\]

where $[g^{k\ell}] = [g_{k\ell}]^{-1}$, and summation convention is employed; namely every up-down index is to be summed from 0 - 3. Using these $\Gamma^k_{ij}$, we construct the Riemann curvature tensor $R^i_{qk\ell}$:

\[
R^i_{qk\ell} = \frac{\partial \Gamma^i_{q\ell}}{\partial x^k} - \frac{\partial \Gamma^i_{qk}}{\partial x^\ell} + \Gamma^p_{pk} \Gamma^i_{q\ell} - \Gamma^p_{q\ell} \Gamma^i_{pk}.
\]

Finally, we can explain the terms $R_{ij}$ and $R$ in $G_{ij}$; namely $R_{ij} = R^{s}_{isj}$ is the Ricci tensor and $R = g^{ij}R_{ij}$ is the scalar curvature.

The quantity $\sigma$ is a universal constant defined by

\[
\sigma = \frac{8\pi \kappa}{c^4}
\]

where $\kappa$ is Newton’s gravitational constant, and $c$ is the speed of light (both in “appropriate” units; we shall often choose units in which $\kappa = 1 = c$).

From these definitions, one immediately sees the enormous complexity of the Einstein equations (2.1). For this reason, we shall often seek solutions which have a high degree of symmetry, the aim being to make the resulting equations mathematically tractable.

### B. Black Hole Solutions

Consider the gravitational field outside of a ball of mass $M$ in $\mathbb{R}^3$. Solving Einstein’s equations $G_{ij} = 0$, gives the celebrated *Schwarzschild solution* (1916):

\[\text{In a letter to a friend Einstein wrote, “I have made a great discovery in mathematics; I have suppressed the summation sign every time the summation must be made over an index that occurs twice...”}\]
where \( m = \frac{\kappa M}{c^4} \), and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the standard metric on the unit 2-sphere. Since \( 2m \) has dimensions of length, it is called the Schwarzschild radius. Observe that on the sphere \( r = 2m \), the metric becomes singular; indeed \( g_{tt} = 0 \) and \( g_{rr} = \infty \). By transforming the metric (2.2) to so-called Kruskal coordinates, [1], one sees that the Schwarzschild sphere \( r = 2m \), has the physical characteristics of a black hole: light and nearby particles can enter, nothing can escape, and there is an intrinsic (non-removable) singularity at the center \( r = 0 \).

More generally, we could consider a metric of the form
\[
ds^2 = -T(r)^2 dt^2 + A^{-1}(r) dr^2 + r^2 d\Omega^2, \tag{2.3}
\]
where
\[
A(r) = 1 - \frac{2m(r)}{r}, \quad 2m(r) = r(1 - A(r));
\]
\( m(r) \) is called the “mass” function. For this metric, we define a black hole solution to be a solution of Einstein’s equations which satisfies
\[
A(\rho) = 0, \quad \text{for some } \rho > 0,
\]
and \( A(r) > 0 \) if \( r > \rho \). \( \rho \) is the radius of the black hole, and is often referred to as the event horizon. (Defined by the condition that it be the largest zero of the \( dr^2 \) term in the metric.)

C. Yang-Mills Equations

The Yang-Mills equations are a generalization of Maxwell’s equations of electromagnetism. To see this, we first write Maxwell’s equations in an invariant way. For this, let \( A \) be the scalar-valued 1-form
\[
A = A_i dx^i, \quad A_i \in \mathbb{R},
\]
called the *electromagnetic potential* (by physicists), or a *connection* (by geometers). Then, $F$, the electromagnetic field (curvature) is the 2-form defined by

$$F = dA. \quad (2.4)$$

In coordinates, $F$ can be written as

$$F = F_{ij} dx^i \wedge dx^j, \quad F_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}. $$

In this set-up, Maxwell’s equations take the form

$$dF = 0, \quad d^* F = 0.$$ 

The first equation follows trivially from (2.4), while for the second equation, the “star” is called the Hodge star operator and in four dimensions, it maps 2-forms to 2-forms and is defined by

$$(^* F)_{k \ell} = \frac{1}{2} \sqrt{|g|} \varepsilon_{ijkl} F^{ij},$$

where $g = \det(g_{ij})$, and $\varepsilon_{ijkl}$ is the completely anti-symmetric symbol defined as $\varepsilon_{ijkl} = \text{sgn}(i, j, k, \ell)$. Here, as usual, we always use the metric to raise (or lower) indices, so that

$$F^{ij} = g^{\ell i} g^{mj} F_{\ell m}. $$

It is important to notice that $^* F$ depends on metric $g_{ij}$.

The *Yang-Mills* equations are a set of equations which generalize Maxwell’s equations. Thus, to each Yang-Mills equation is associated a Lie group $G$ (of symmetries), called the *gauge group*. For such $G$, we consider its Lie algebra $\mathfrak{g}$, defined as being the tangent space at the identity of $G$. Now suppose that $A$ is a $\mathfrak{g}$-valued 1-form; i.e.

$$A = A_i dx^i,$$

where each $A_i$ is in $\mathfrak{g}$. In this (slightly!) more general case, the curvature 2-form $F$ is defined by

$$F = dA + A \wedge A,$$
and in components,

\[ F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j], \quad A_i \in \mathfrak{g}. \]

Notice that the commutator, \([A_i, A_j] = 0\) if \(G\) is an abelian group, but is generally non-zero if \(G\) is a matrix group. The Yang-Mills equations are again given by

\[ dF = 0 \quad \text{and} \quad d^*F = 0. \tag{2.5} \]

These generalize Maxwell’s equation because for Maxwell’s equation, \(G = U(1)\), the circle group \((U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\})\), so \(\mathfrak{g}\) consists of scalars and the commutator term vanishes. Note that for Maxwell’s equations \(d^*F = 0\) is a linear equation for the \(F_{ij}\)’s, or for the unknown “connection coefficients” \(A_i\). However if \(G\) is non-abelian, say \(G = SU(2)\), then the Yang-Mills equations \(d^*F = 0\) become non-linear equations for the (unknown) matrices \(A_i\).

D. The Dirac Equation in Curved Spacetime

The Dirac equation brings in Quantum Mechanics, and is a generalization of Schrödinger equation to the relativistic case. It also describes the intrinsic “spin” of certain elementary particles. The Dirac equation takes the form

\[ (G - m) \Psi = 0, \tag{2.6} \]

where \(G\) is the Dirac operator and \(\Psi\) is a complex-valued 4-vector called wave the function (spinor) of a fermion\(^2\) (Dirac particle) of mass \(m\). The Dirac operator \(G\) can be written as

\[ G = iG^j(x) \frac{\partial}{\partial x^j} + B(x) \]

where the \(G^j\) are \(4 \times 4\) matrices called the Dirac matrices and \(B\) is a \(4 \times 4\) matrix. The Dirac matrices \(G^j\) and the Lorentzian metric \(g_{ij}\) (defined on 4-d spacetime) are related by

\[ g^{jk} I = \frac{1}{2} \{G^j, G^k\}, \tag{2.7} \]

\(^2\)Fermions are distinguished from bosons in that fermions have half-integral multiples of spin, while bosons have integral multiples of spin. Only fermions obey the Pauli Exclusion Principle. Examples of fermions are protons and antiprotons, neutrinos, electrons and positrons, quarks, and leptons. Examples of bosons are mesons, photons, and pions.
where $\{G^j, G^k\}$ is the anti-commutator

$$\{G^j, G^k\} = G^j G^k + G^k G^j.$$ 

The Dirac matrices thus depend on the metric.

Now let $H$ be any space-like hypersurface in $\mathbb{R}^4$, with future directed normal vector field $\nu = (\nu_i)$, and let $d\mu$ be the invariant measure on $H$ induced by the metric $g_{ij}$. Define a scalar product on solutions, $\Psi, \Phi$ of the Dirac equation by

$$\langle \Psi | \Phi \rangle = \int_H \bar{\Psi} G^j \Phi \nu_j d\mu .$$

This scalar product is positive definite, and, as a consequence of current conservation (c.f. [4])

$$\nabla_j \bar{\Psi} G^j \Phi = 0,$$

it is independent of the choice of the hypersurface $H$. By direct generalization of the expression

$$\bar{\Psi} \gamma^0 \Psi = |\Psi|^2$$

in flat Minkowski space given originally by Dirac, ([2]), where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

in our case $\bar{\Psi}$ is the adjoint spinor, defined by

$$\bar{\Psi} = \Psi^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathbf{1}$ is the $2 \times 2$ identity matrix, and $\Psi^*$ denotes complex conjugation,

$$\bar{\Psi} G^j \Phi \nu_j$$

(2.8)
is interpreted to be the probability density of the Dirac particle. We normalize solutions of the Dirac equation by requiring

\[ \langle \Psi | \bar{\Psi} \rangle = 1. \]  
(2.9)

If the Lorentzian metric admits a black-hole solution at \( r = \rho > 0 \), (2.9) is replaced by

\[ \int \{ t = \text{const.}, r > r_0 \} \bar{\Psi} G^j \Psi \nu^j d\mu < \infty \]  
(2.10)

for all \( r_0 > \rho \). For normalized solutions of the Dirac equation, the integral (2.10) gives the probability for the Dirac particle to be in the region \( r > r_0 \). We shall only consider Dirac particles which lie outside the event horizon. This is because the probability density (2.8) is not necessarily positive inside the event horizon so that a positive infinite contribution of the integral outside the event horizon can be compensated by a negative infinite contribution inside the event horizon. We thus demand that the integral (2.10) away from, and outside of the event horizon, be finite.

In the paper [4], it is shown that the Dirac matrices \( G^j \) can be chosen to be any \( 4 \times 4 \) matrices which are Hermitian with respect to the scalar product

\[ \langle \Psi | \bar{\Phi} \rangle = \int_{\mathbb{R}^4} \bar{\Phi} \Psi \sqrt{|g|} \, dx \]

and satisfy the anti-commutation relations (2.7). This gives us more flexibility in choosing the Dirac matrices. Namely, the relations (2.7) do not uniquely determine the Dirac matrices in curved spacetime. However it was proved in [4] that all choices of Dirac matrices satisfying (2.7) yield unitarily equivalent Dirac operators.

With this introduction to the Einstein equations, the YM equations and Dirac’s equation, we turn to our first topic, the coupled Einstein-Dirac-Yang/Mills (EDYM) equations.
3 Einstein-Dirac-Yang/Mills Equations

(joint work with F. Finster and S.T.-Yau)

The EDYM equations are obtained by varying the action

$$S = \int \left[ \frac{1}{16\pi\kappa} R + \bar{\Psi}(G-m)\Psi - \frac{1}{16\pi e^2} \text{Tr}(F_{ij}F^{ij}) \right] \sqrt{|g|} \, dx$$

over Lorentzian metrics $g_{ij}$, g-valued Yang/Mills connections $A_i dx^i$ (where g is the Lie algebra of the given gauge group), and 4-spinors $\Psi$. Here $e$ is the so-called Yang/Mills coupling constant, and $\kappa$ is Newton’s gravitational constant. In general, the EDYM equations are extremely complicated. We specialize here to spherically symmetric solutions depending only on the radius $r = |x|$. Taking the gauge group as $SU(2)$ together with a special ansatz for the spinors, we can simplify the Dirac equations from complex 4-spinors to the case of an equation for real 2-spinors $(\alpha, \beta)^t$, [8]. In this set-up the (SU(2)) EDYM equations take the following form:

$$\sqrt{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} ' = \begin{pmatrix} \frac{W}{r} & -(m+\omega)T \\ -m+\omega T & -W/r \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.1}$$

$$r A' = 1 - A - \frac{\kappa}{e^2} \frac{(1 - W^2)^2}{r^2} - 2\kappa\omega T^2(\alpha^2 + \beta^2) - \frac{2\kappa}{e^2} A(W')^2 \tag{3.2}$$

$$2r A \frac{T'}{T} = -1 + A + \frac{\kappa}{e^2} \frac{(1 - W^2)^2}{r^2} + 2\kappa m T(\alpha^2 - \beta^2) - 2\kappa\omega T^2(\alpha^2 + \beta^2) + 4\kappa \frac{T}{r} W \alpha \beta - \frac{2\kappa}{e^2} A(W')^2 \tag{3.3}$$

$$r^2 A W'' = -(1 - W^2)W + e^2 r T \alpha \beta - r^2 \frac{A'T - 2AT'}{2T} W'. \tag{3.4}$$

Here $m$ is the rest mass of the Dirac particle, $\omega$ is its energy, and $W$ is the unknown connection coefficient. Equation (3.1) is the Dirac equation, (3.2) and (3.3) are the Einstein

\footnote{The gauge group $SU(2)$ corresponds to the weak nuclear force, while $SU(3)$ corresponds to the strong nuclear force.}
equations (for the Lorentzian metric of the form (2.3)), while (3.4) is the Yang/Mills equation. Notice that equations (3.1), (3.3) and (3.5) all become singular when \( A = 0 \); i.e., at a black-hole. (This is special to Einstein’s equations, and is the first significant difference that we see from classical equations.)

Smooth solutions of (3.1) - (3.4) are those which are defined for all \( r \geq 0 \), and are called particle-like solutions. Black hole (BH) solutions correspond to solutions having \( A(\rho) = 0 \) for some \( \rho > 0 \), and are defined for all \( r > \rho \), with \( A(r) > 0 \), if \( r > \rho \). In addition solutions of (3.1)-(3.4) are required to satisfy certain initial conditions, (see [8, 9]), together with the following global conditions:

\[
\int_0^\infty (\alpha^2 + \beta^2) \frac{T}{\sqrt{A}} \, dr = 1 \quad \text{(for particle-like solutions)}, \tag{3.5}
\]

\[
0 < \int_{r_0}^\infty (\alpha^2 + \beta^2) \frac{T}{\sqrt{A}} \, dr < \infty \quad \text{for each} \quad r_0 > \rho, \tag{3.6}
\]

(for BH solutions),

\[
\lim_{r \to \infty} r(1 - A(r)) < \infty, \tag{3.7}
\]

\[
\lim_{r \to \infty} T(r) = 1 \tag{3.8}
\]

\[
\lim_{r \to \infty} (W(r), W'(r)), \quad \text{is finite.} \tag{3.9}
\]

Conditions (3.5) and (3.6) state that the spinors (wave functions) are normalizable, (3.7) means that the fermions have finite (ADM) mass, (3.8) (together with (3.7)) means that the gravitational field is asymptotically Minkowskian, while (3.9) asserts that the YM field is well-behaved. We now state our main results, and discuss them afterward.

**Theorem 3.1** ([8]): There exist stable particle-like solutions of the EDYM equations for arbitrarily weak gravitational coupling constant: \( m^2 k/e^2 \to 0 \).

**Theorem 3.2** ([9]): Every black hole solution of the EDYM equations cannot be normalized; namely, the spinors must vanish identically outside of the black hole.
Both of these results are unexpected and rather surprising.

Theorem 3.1 is obtained using numerical methods. This (stability) result has been shown to be false for the (SU(2)), Einstein-Yang/Mills (EYM) equations, [21, 22], and shows that Dirac particles in a gravitational field form bound states if an additional strong coupling to a non-abelian Yang-Mills field is taken into account. In [8], we employ some new numerical techniques, including a multi-parameter “shooting method”, and a scaling technique introduced in [5]. Our result shows that weak as gravity is (e.g., it is $10^{21}$ times weaker than electromagnetism), it still has a regularizing effect. Although gravity is not renormalizable (which means that the problem cannot be treated in a perturbation expansion), our solutions of the EDYM equations are regular and well-behaved for arbitrarily weak gravitational coupling. Our stability result is obtained by using Conley Index techniques, [17].

The proof of Theorem 3.2 is given in [9]. The result shows that the EDYM equations do not exhibit normalizable BH solutions. Thus, in the presence of quantum mechanical Dirac particles, static, spherically symmetric BH’s do not exist. Another interpretation of our result is that it indicates that Dirac particles can only either disappear into the BH or escape to infinity. We shall discuss this further in the next section.

4 Decay Rates and Probability Estimates for Massive Dirac Particles in a Charged, Rotating Black Hole Geometry

(joint work with F. Finster, N. Kamran, and S.-T. Yau)

Consider the physical situation of a Dirac particle (fermion, electron, proton, etc.) of mass $m > 0$ outside of a charged, rotating black hole. Question: What is its long time behavior, as $t \to \infty$?

In the classical case (no Dirac equation, no quantum mechanical considerations), the particle can remain in a time-periodic orbit around the black hole (see [3]). In this section we shall show that our previous “indication”, (mentioned at the end of the last
section) is valid. That is, if quantum mechanical effects are taken into account, the picture completely changes, and the Dirac particle either enters the black or hole, or tends to infinity; no other possibilities can occur.

In the paper [7], we proved that there are no normalizable, time periodic solutions of the Dirac equation in a Reissner-Nordström black-hole background; in particular there are no static solutions of the Dirac equation in such a background metric. This result was extended in [10] to the case in which the background geometry is that of a charged, rotating black hole. In this section we shall discuss such black holes and we shall show that our above “indications” are correct; i.e., in such a background geometry, a Dirac particle either tends to infinity, or enters the black hole – there are no other possibilities. In addition we shall obtain probability estimates on which of these two possibilities occurs. We begin with a brief discussion of the geometry of a charged rotating black hole.

In 1963, R. Kerr found a solution of Einstein’s equations corresponding to a rotating black hole [15]. This result was generalized by Kerr and Newman; (see [11]), to a charged rotating black hole (called the Newman-Penrose solution of Einstein’s equations):

\[
ds^2 = \frac{\Delta}{U}(dt - a\sin^2 \phi d\phi)^2 - U \left(\frac{dr^2}{\Delta} + d\theta^2\right) - \frac{\sin^2 \theta}{U} \left[adt - (r^2 + a^2)d\phi \right]^2,
\]

where

\[
U(r, \theta) = r^2 + a^2 \cos^2 \theta
\]
\[
\Delta(r) = r^2 - 2Mr + a^2 + Q^2,
\]

and \(M\), \(aM\), and \(Q\) denote the mass, angular momentum, and charge of the black hole, respectively; \(a\) is the angular velocity of the black hole. From our earlier remarks, the largest root of \(\Delta\) corresponds to the event horizon of the black hole. We assume\(^4\) \(M^2 > a^2 + Q^2\), \((4.2)\) then

\[
r = \rho \equiv M + \sqrt{M^2 - a^2 - Q^2}
\]

\(^4\)The Reissner-Nordström solution is an extension of the Schwazschild solution, and is a static solution of the coupled Einstein-Maxwell equations; see [1].

\(^5\)This is the so-called “non-extreme” black hole.
defines the event horizon, the boundary of the charged, rotating, black hole (CRBH).

Now consider the Cauchy Problem for a Dirac particle of mass \(m > 0\), and charge \(q\), in this CRBH background geometry, (see [11], where we write the Dirac equation in this different way, taking the spin connection into account) having initial data outside of the event horizon \(r = \rho\):

\[
(i \gamma^j D_j - m)\Psi(t, x) = 0
\]

\[
\Psi(0, x) = \Psi_0(x), \quad |x| > \rho.
\]

Here is our first result.

**Theorem 4.1**: Let \(\delta > 0\) be given, and let \(R > \rho + \delta\). Consider the (annular) region in \(\mathbb{R}^3\)

\[
K_{\delta, R} = \{\rho + \delta \leq r \leq R\}.
\]

Then the probability for the Dirac particle to be inside \(K_{\delta, R}\) tends to zero as \(t \to \infty\); namely,

\[
\lim_{t \to \infty} \int_{K_{\delta, R}} (\bar{\Psi} \gamma^j \Psi)(t, x) \nu_j d\mu = 0.
\]

Thus with probability one, the Dirac particle leaves every bounded set in \(\mathbb{R}^3\). The proof of this theorem is based on a result of Chandrasekhar, [3], who showed that the Dirac equation in the CRBH geometry can be separated into ODE’s. Moreover, using this, we can write the Dirac propagator as (see [11, 12])

\[
\Psi(t, x) = \frac{1}{\pi} \sum_{k, n \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-i\omega t} \sum_{a, b = 1}^{2} t_{ab}^{k,n} \Psi_a^{k,n}(x) \langle \Psi_b^{k,n} | \Psi_0 \rangle d\omega,
\]

where the (positive) scalar product is defined in [11]. Here \(k\) and \(n\) are generalized angular momentum numbers (arising from Chandrasekhar’s separation theorem), \(\Psi_a^{k,n}\) are solutions of the Dirac equation, and the \(t_{ab}^{k,n}\) are generalized transmission coefficients. These functions can be written in terms of the fundamental solutions of the resulting
ODE’s. For $|\omega| > m$, near the event horizon, the $\Psi_{a\omega n}^{k}$ go over to spherical waves. $\Psi_{1\omega n}^{k}$ corresponds to incoming waves (moving towards the black hole), and $\Psi_{2\omega n}^{k}$ corresponds to outgoing waves (moving away from the black hole). Near infinity, the $\Psi_{a\omega n}^{k}$ again go over to spherical waves. For $|\omega| < m$, the fundamental solutions (for $a = 1, 2$) near the event horizon are linear combinations of both incoming and outgoing waves. $\Psi_{1\omega n}^{k}$ (resp. $\Psi_{2\omega n}^{k}$) decays (resp. grows) exponentially at infinity. The theorem is proved using the representation (4.6).

Theorem 4.1 implies that the Dirac particle must leave every bounded set in $\mathbb{R}^3$. Equivalently the Dirac wave function $\Psi$ decays to zero in $L^\infty_{loc}$, outside and away from the event horizon. It follows that the Dirac particle must eventually either disappear into the black hole, or escape to infinity; these are the only possibilities. This raises the following questions:

(A) What is the likelihood of each of these two possibilities?,

and

(B) What is the rate of decay of the Dirac wave function on a compact subset of $\mathbb{R}^3$ outside of the black hole?

We consider question (B) first. To this end, let $q$ denote the charge of the Dirac particle and assume that the Dirac particle has “small charge”, in the sense that

$$mM > |qQ|. \quad (4.7)$$

We also assume that the angular momentum of the initial data is bounded. This means that the summation in (4.6) is over a bounded set of integers $n$ and $k$; say

$$|k| \leq k_0 \quad \text{and} \quad |n| \leq n_0.$$

Footnote 6: This means that the gravitational attraction is the dominant force far from the black hole. Indeed, at large distances from the black hole the metric is asymptotically Minkowskian, and the gravitational and electromagnetic fields can be described by the Newtonian limit. In this limit the gravitational and electrical forces are $mM/r^2$ and $qQ/r^2$, the Newton and Coulomb laws, respectively.
We then have the following theorem; (see [12] for the proof).

**Theorem 4.2:** (Decay Rates): Consider the Cauchy problem (4.3), (4.4) for the Dirac equation in the non-extreme CRBH background geometry, with small charge. Assume that the Cauchy data $\Psi_0$ is smooth with compact support outside of the event horizon, and has bounded angular momentum.

(i) If for any $k$ and $n$
\[
\limsup_{\omega \searrow m} \left| \langle \Psi_2^{k \omega n} | \Psi_0 \rangle \right| \neq 0, \quad \text{or} \quad \liminf_{\omega \nearrow -m} \left| \langle \Psi_2^{k \omega n} | \Psi_0 \rangle \right| \neq 0,
\]
then
\[
|\Psi(t, x)| = ct^{-5/6} + O(t^{-5/6} - \varepsilon) \quad \text{as} \quad t \to \infty,
\]
where $c = c(x) \neq 0$ and any $\varepsilon < \frac{1}{50}$.  

(ii) If for all $k, n, \text{ and } a = 1, 2$, $\langle \Psi_a^{k \omega n} | \Psi_0 \rangle = 0$ for all $\omega$ in a neighborhood of $\pm m$, then for any fixed $x$, $\Psi(x, t)$ decays rapidly in $t$, (faster than any polynomial).

For a quantum mechanical particle, the Heisenberg Uncertainty Principle implies that its kinetic energy is not precisely known. Since the associated Hamiltonian has continuous spectrum (the operator is defined on an unbounded interval), the energy $\omega$ of the Dirac particle is a continuous parameter. Condition (4.8) means that the initial energy distribution has outgoing components near $\omega = m$ or near $\omega = -m$. Our theorem implies that in this case, the decay rate is $t^{-5/6}$. That is, the decay rate $t^{-5/6}$ quantifies the effect of the black hole’s attraction, on the long time behavior of the Dirac particle.

In flat Minkowski space solutions of the wave equation having compactly supported data, decay rapidly (Huygens principle). In the CRBH geometry, the Dirac wave function $\Psi$ behaves near infinity like solutions of the wave equation in flat Minkowski space, and near the black hole, it should behave like a massless particle, which decays like $t^{-3/2}$ (see [12]). Thus one would expect (by “interpolation”) that the Dirac particle in the CRBH geometry should decay at least as fast as $t^{-3/2}$. However, Theorem 4.2 shows that the naïve picture is incorrect, and that the gravitational field affects the behavior of the Dirac particle in a far more subtle way.
Finally, one can ask why $\omega = \pm m$ play such special roles. To answer this, note that for a particle of mass $m$ in flat space, its energy $\omega = m$, (or $-m$ since the Dirac equation has negative energy solutions). If the particle’s momentum is non-zero, then $\omega > m$. Thus in flat space, its energy is outside of the interval $(-m, m)$. But a quantum mechanical particle can have a continuous energy distribution, but only outside of $(-m, m)$ (this is called the “energy gap” or “mass gap”). However, in the presence of a gravitational field, the energy distribution can be supported on the entire real line, but the dominant term is near $\pm m$.

We now turn to question (A), and discuss the probability for our Dirac particle to escape to infinity, or to enter the black hole. Let $R > \rho$, (the event horizon), and define the probability $p$ of the Dirac particle to escape to infinity by

$$p = \lim_{t \to \infty} \int_{r > R} (\bar{\Psi} \gamma^j \Psi)(t, x) \nu_j d\mu. \quad (4.10)$$

Note that $p$ is independent of $R$; namely, if $R_2 > R_1 > \rho$, and

$$p_i = \lim_{t \to \infty} \int_{r > R_i} (\bar{\Psi} \gamma^j \Psi)(t, x) \nu_j dx, \quad i = 1, 2,$$

then

$$p_1 - p_2 = \lim_{t \to \infty} \left( \int_{r > R_1} - \int_{r > R_2} \right) = \lim_{t \to \infty} \int_{R_1}^{R_2} = 0,$$

by Theorem 4.1; thus $p_1 = p_2$ and this proves our assertion. In [12] we prove that

$$p = \frac{1}{\pi} \sum_{|k| \leq k_0} \int_{R \setminus [-m, m]} \left( \frac{1}{2} - 2|t_1^{k\omega n}|^2 \right) \left| \langle \Psi_2^{k\omega n} \mid \Psi_0 \rangle \right|^2 d\omega.$$

Accordingly, $1 - p$ gives the probability that the Dirac particle enters the black hole. The following theorem gives conditions under which $p = 0$, $p = 1$, or $0 < p < 1$, in terms of the initial energy distribution.

**Theorem 4.3 ([12]):** Consider the Cauchy problem as in Theorem 4.2, with initial data normalized by $\langle \Psi_0 \mid \Psi_0 \rangle = 1$.  

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(i) If the outgoing initial energy distribution satisfies \( \langle \Psi_2^{k\omega n} | \Psi_0 \rangle \neq 0 \) for some \( \omega \) with \( |\omega| > m \), then \( p > 0 \).

(ii) If the initial energy distribution satisfies for \( a = 1 \) or \( 2 \), \( \langle \Psi_2^{k\omega n} | \Psi_0 \rangle \neq 0 \) for some \( \omega \), \( |\omega| \leq m \), then \( p < 1 \).

(iii) If the initial energy distribution is supported in the interval \([-m, m]\), then \( p = 0 \).

(iv) If (4.8) holds, then \( 0 < p < 1 \).

(v) \( p = 1 \) if and only if for all \( k, \omega \) and \( n \), the following conditions hold:

\[
\langle \Psi_1^{k\omega n} | \Psi_0 \rangle = 0, \quad \text{if} \quad |\omega| \leq m
\]
\[
\langle \Psi_1^{k\omega n} | \Psi_0 \rangle = -2\epsilon_{12}^{k\omega n} \langle \Psi_2^{k\omega n} | \Psi_0 \rangle, \quad \text{if} \quad \omega > m.
\]

Note that in case (i), the Dirac particle has a non-zero probability of escaping to infinity; in case (ii) the particle can enter the black hole, while in case (iii) the components of the initial energy distribution, which lead to the decay rate \( t^{-5/6} \), do not have enough energy to allow the Dirac particle to tend to infinity. Indeed, \( \Psi_2^{k\omega n} \) for \( |\omega| > m \) is outgoing near the event horizon, and so the Dirac particle resists the gravitational attraction for a while, but it eventually gets turned around and is driven into the black hole. In case (iv), the particle has a positive probability of either escaping to infinity, or entering the black hole. Finally we can understand why \( p = 1 \) for special choices of initial data. To obtain such data, consider the special situation where a Dirac particle at time \( t = -\infty \) comes in from spatial infinity. Taking \( \Psi(0, x) \) as initial data and reversing the time direction, the solution of this Cauchy problem clearly escapes to infinity with probability 1.
5  Shock-Waves, Cosmology, and Black Holes

(joint work with B. Temple)

In the previous sections we have discussed the behavior of elementary particles in a gravitational field; that is, how General Relativity affects matter on small scales. We now change directions and discuss GR on large scales, that is we shall discuss some recent astrophysical implications of GR.

We again consider the Einstein equations (2.1) where now the energy-momentum tensor $T_{ij}$ is that of a perfect fluid; i.e., a fluid in which dissipation effects are neglected:

$$T_{ij} = (p + \rho)u_iu_j + pg_{ij}. \quad (5.1)$$

Here $\rho$ and $p$ are the density and pressure respectively, of the fluid, and $u = (u^0, u^1, u^2, u^3)$ is its 4-velocity, where as usual, $u_i = g_{ij}u^j$. The Einstein equations (5.1) describe the simultaneous evolution of the fluid and the gravitational field: matter is the source of spacetime curvature in Einstein’s theory.

Shock-waves are relevant here since the Relativistic Euler Equations are “embedded” within the Einstein equations $G_{ij} = \sigma T_{ij}$. This is because the Bianchi Identities in geometry, (cf. [23]), imply that the (covariant) divergence of the Einstein tensor $G_{ij}$, vanishes identically: $\text{Div} \ G_{ij} = 0$, so $\text{Div} \ T_{ij} = 0$, and this latter equation is precisely the Relativistic Euler Equations, expressing the conservation of energy and momentum.

We pause to make the following remarks:

(A) There is no “Glimm’s Theorem” ([13, 17]) in GR, not yet anyway. However an important first step in this direction was recently made by Groah and Temple, see [14].

(B) No proof is known which demonstrates that in GR, shock-waves form from smooth data (see [16, 17]).

(C) In GR, the initial data cannot be arbitrarily prescribed on a non-characteristic surface, and must satisfy additional constraints which are imposed by geometrical
considerations (Bianchi Identities). These constraints take the form of a coupled set of 4 nonlinear elliptic equations. There are deep unresolved issues concerning these constraint equations.

In [18, 19], we constructed the first examples of shock-wave solutions in GR. The shock-waves are in the fluid variables and in the metric (gravitational field). This was done by matching two different spherically symmetric metrics Lipschitz continuously across a (spherically symmetric) surface of discontinuity for the fluid variables (shock-wave). The Inner Metric is the Friedmann-Robertson-Walker (FRW) metric of cosmology. Its line element is given by

\[ ds^2 = -dt^2 + R(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \]  

(5.2)

where by a suitable re-scaling of \( r \), we can take \( k = 0, 1, \text{or} -1 \). This metric is clearly spherically symmetric, and describes a homogeneous, isotropic spacetime (no preferred point or direction). It models an expanding universe. There is a singularity (\( R = 0 \)) in backwards time (i.e., earlier than present time) which corresponds to the Big-Bang. The function \( R(t) \) determines the red-shift factor for distant astronomical objects. (The red-shift is what astronomers see and can measure. It is actually the Doppler effect applied to astrophysics). The red shift gives information on distance, mass, and chemical composition of the objects; see [23] for a discussion of these things.

The Outer Metric is the Tolman-Oppenheimer-Volkoff (TOV) metric,

\[ ds^2 = -B(\bar{r})d\bar{t}^2 + \left( 1 - \frac{2M(\bar{r})}{\bar{r}} \right)^{-1} d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  

(5.3)

It is both time-independent and spherically symmetric, and models a static spacetime (or the interior of a star). The function \( M(\bar{r}) \) denotes the mass inside a ball of radius \( \bar{r} \), and is given by

\[ M(\bar{r}) = \int_0^{\bar{r}} 4\pi s^2 \bar{\rho}(s) ds, \]

where \( \bar{\rho} \) is the density function. Both the FRW and TOV metrics satisfy Einstein’s equations for a perfect fluid; see [23].
For cosmology, we assume that we are given an FRW metric (which models well our expanding Universe), and we solve the differential equations for an unknown TOV metric which matches the given FRW metric across an expanding (outgoing) shock wave for the fluid variables. Our idea is to match the given \((k = 0)\) FRW metric with equation of state given by

\[
p_{\text{matter}} = 0, \quad p_{\text{radiation}} = aT^4,
\]

where \(a\) is the Stefan-Boltzmann constant, and \(T\) denotes the temperature.\(^7\) In our shock-wave model for cosmology, this model is not an expansion of the entire universe, but rather a limited expansion into an outer ambient static spacetime modeled by the TOV metric.

In [20], we constructed an explicit solution of this problem, and we showed that there are unique density and pressure profiles for each initial radiation density. In our model, the initial Big-Bang is more like a classical fluid-dynamical shock-wave explosion. The mathematical theory of shock-waves, [17] implies that many solutions decay as \(t \to \infty\) to the same shock-wave. Thus in our shock-wave cosmology, we cannot recover all the information about the early details of the explosion, (due to entropy increase across shock-waves).

However, there is a problem with our shock-wave cosmology model; namely our shock-wave is not sufficiently far enough out. Indeed, in [20], we calculated the present shock position to be approximately at distance (Hubble length)

\[
\frac{c}{H_0} \approx 10^{10} \text{ light years}
\]

away, where \(H(t) = \dot{R}/R(t)\) is the Hubble “constant”, and \(H_0 = H\) (present time). But we observe quasars and distant galaxies at distances of \(10^{11} - 10^{12}\) light years away. For a long time we were puzzled as to why we cannot get our shock-wave further out. But we have recently answered this question in a most unexpected way. Indeed, we recently proved that if the shock wave is beyond the critical length scale

\[
\frac{c}{H(t)} \quad (\text{Hubble length})
\]

\(^7\)This equation of state is generally taken to be the equation of state for the Universe at present time.
for the FRW metric at any time $t > 0$, then the Universe lies “inside a black hole”, in the sense that $(1 - \frac{2M}{r})$ changes sign at the critical length scale for the FRW metric. That is, shock matching outside a black hole can only succeed for shocks lying inside one Hubble length from the center of the FRW metric at any fixed time. This is a rather surprising connection between shock-wave cosmology and black holes.

We also have proved that radiation admitted from a galaxy (or star) at the instant when it lies a distance of exactly one Hubble length from the FRW origin will be observed at the origin (at a later time) to be infinitely red shifted. Thus an observer at the FRW origin will see the shock wave “fading out of view” as $t \to \infty$.

We are in the process of modifying our shock-wave cosmology so as to allow shock waves beyond Hubble length.
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