Magnetic and Ising quantum phase transitions in a model for isoelectronically tuned iron pnictides

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Considerations of the bad-metal behavior led to an early proposal for a quantum critical point under a P for As doping in the iron pnictides, which has since been experimentally observed. We study here an effective model for the isoelectronically tuned pnictides using a large-$N$ approach. The model contains antiferromagnetic and Ising-nematic order parameters appropriate for $J_1$-$J_2$ exchange-coupled local moments on an Fe square lattice, and a damping caused by coherent itinerant electrons. The zero-temperature magnetic and Ising transitions are concurrent and essentially continuous. The order-parameter jumps are very small, and are further reduced by the inter-plane coupling; quantum criticality hence occurs over a wide dynamical range. Our results provide the basis for further studies on the quantum critical properties in the P-doped iron arsenides.

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Introduction.— Iron based materials not only show high temperature superconductivity [1], but also feature a rich phase diagram. For the parent iron arsenides, the ground state has a collinear $(\pi,0)$ magnetic order [2]. Because superconductivity occurs at the border of such an antiferromagnetic (AF) order, a natural question is whether quantum criticality plays a role in the phase diagram. It was theoretically proposed early on that tuning the parent iron pnictides by an isoelectronic P-for-As doping induces a quantum critical point (QCP), where both the $(\pi,0)$ AF order and an Ising-nematic spin order are suppressed [3]. This proposal was made within a strong-coupling approach, which attributes the bad-metal behavior of iron arsenides [4–7] to correlation effects that are on the verge of localizing electrons [8–11]. The P doping increases the in-plane electronic kinetic energy and thus the coherent electronic spectral weight while leaving other model parameters little changed [12, 13], thereby weakening the magnetic order and the associated Ising-nematic spin order [3, 14].

Such a QCP has since been observed by measurements in the P-doped CeFeAsO [15, 16] and BaFe$_2$As$_2$ [17–20]. Neutron scattering in the former has shown that the tetragonal-to-orthorhombic structural distortion vanishes at the same doping ($x_c \approx 0.4$) where the AF order goes away [16], providing evidence for the simultaneous suppression of the AF and Ising-nematic spin orders. In the P-doped BaFe$_2$As$_2$, a large non-Fermi liquid regime has been shown in the phase diagram [17–20]. Static structural order is also suppressed around the same P-doping concentration ($x_c \approx 0.33$) where the AF order goes away, although there may be an additional channel of electronic anisotropy [21]; While there is evidence for a QCP “hidden” inside the superconducting dome [19] quantum criticality has now been observed and studied in the normal state where superconductivity is suppressed by a high field [20]. The accumulated experimental evidences for a QCP in the P-doped parent iron arsenides motivate further theoretical analyses on the underlying quantum phase transitions.

In this letter, we study the zero-temperature phase transitions in a continuum model introduced earlier [3, 14]. This effective field theory contains antiferromagnetic (vector) and Ising-nematic (scalar) order parameters appropriate for a $J_1$-$J_2$ model of local moments on a square lattice [8, 22–24]. It also incorporates a damping term caused by coupling of the local moments to the coherent itinerant electrons. Since it is important to establish the nature of quantum criticality in the absence of superconductivity [15, 20], we will focus on the transitions in the normal state and will in particularly not consider the effect of superconductivity [25]. Using a large-$N$ approach [26, 27], we demonstrate that the magnetic and Ising transitions are concurrent at zero temperature both for the case of a square lattice and in the presence of interlayer coupling. Moreover, the transitions in the presence of damping are essentially continuous, leading to quantum criticality over a wide dynamical range.

The model.— The proximity of a bad metal to a Mott transition can be measured by the parameter $w$, the percentage of the single-electron spectral weight in the coherent itinerant part [8, 9, 28]. To the zeroth order in $w$, all the single-electron excitations are incoherent; integrating them out leads to an effective model of local moments with couplings $J_1$ and $J_2$:

$$H = \sum_{\langle i,j \rangle} J_1 \vec{S}_i \cdot \vec{S}_j + \sum_{\langle\langle i,j \rangle\rangle} J_2 \vec{S}_i \cdot \vec{S}_j$$

where $\langle \cdot \cdot \cdot \rangle$ and $\langle\langle \cdot \cdot \cdot \cdot \rangle\rangle$ respectively denote the nearest neighbor and next nearest neighbor sites; see Fig. 1(a). Both general considerations [8] and the first-principal calculations [29, 30] suggest $J_2 > J_1/2$. In this regime, we consider two interpenetrating sublattices [the dotted squares in Fig. 1(a)] with independent staggered magnetizations ($\vec{n}$ vectors) $\vec{m}_A$ and $\vec{m}_B$. While the mean-
field energy is independent of the angle $\phi$ between $\vec{m}_A$ and $\vec{m}_B$, this degeneracy is not protected by symmetry. Quantum or thermal fluctuations can break the degeneracy, leading to the collinear order with $\phi = 0$ or $\pi$ \cite{22, 31}. Thus $\vec{m}_A \cdot \vec{m}_B = \pm 1$ becomes an Ising variable.

To non-vanishing orders in $w$, the coherent itinerant electrons provide Landau damping. This leads to the following Ginzburg-Landau action \cite{3, 14}:

$$S = S_2 + S_4$$

with

$$S_2 = \sum_{\vec{q}, i \omega} \left\{ \chi_0^{-1}(\vec{q}, i \omega) \left[ |\vec{m}_A(\vec{q}, i \omega)|^2 + |\vec{m}_B(\vec{q}, i \omega)|^2 \right] + 2v (q_x^2 - q_y^2) \vec{m}_A(\vec{q}, i \omega) \cdot \vec{m}_B(-\vec{q}, -i \omega) \right\}$$

$$S_4 = \int_0^\beta d\tau \int d\vec{r} \left\{ u_1 \left( |\vec{m}_A|^4 + |\vec{m}_B|^4 \right) + u_2 \left| \vec{m}_A \right|^2 \left| \vec{m}_B \right|^2 - u_I \left( \vec{m}_A \cdot \vec{m}_B \right)^2 \right\}$$

where $\vec{m}_{A/B}(\vec{r}, \tau) = (m^1_{A/B}, m^2_{A/B}, m^3_{A/B})$ are the $O(3)$-vector fields of sublattices A and B, and

$$\chi_0^{-1}(\vec{q}, i \omega) = r + \omega^2 + c |\vec{q}|^2 + \gamma |\omega|$$

(5)

where $c$ is the square of the spin-wave velocity. The parameter $v$ leads to an anisotropic distribution of the spin spectral weight in the momentum space, which is described by the ellipticity

$$\epsilon \equiv \sqrt{(e - v)/(e + v)},$$

(6)

which goes from full isotropy $\epsilon = 1$ ($v = 0$) to extreme anisotropy $\epsilon = 0$ ($v = c$). For latter convenience, we also introduce the parameter

$$a_c \equiv c/\sqrt{c^2 - v^2} = (\epsilon + 1/\epsilon)/2.$$ (7)

In addition, $\gamma$ is the damping rate, with $\omega_l$ denoting Matsubara frequencies. Finally, $r = r_0 + w A_Q$, where the bare mass $r_0$ is negative, reflecting the ground-state order in the absence of damping, and $A_Q > 0$ is a quasiparticle susceptibility at $Q = (\pi, 0)$ or $(0, \pi)$ \cite{3}. When $J_1 \ll J_2$, we have $u_I, u_2 \ll u_1$. A biquadratic coupling \cite{32, 33} can also be incorporated, which primarily renormalizes $u_I$. When the damping is present, the effective dimensionality of the fluctuations with respect to the underlying $O(3)$ transition is $d + z = 4$. From a renormalization-group (RG) perspective, because “$-u_I$” is negative, it is marginally relevant w.r.t the underlying QCP of O(3) transitions at $d + z = 4$ \cite{3, 34}. So unlike the thermally-driven transitions or the case of zero-temperature transition in the absence of damping (where $u_I$ is relevant), it is expected that any splitting between the magnetic and Ising transitions would be small, leading to a qualitative phase diagram shown in Fig. 1(b) \cite{3, 14}.

![FIG. 1: (a) Illustration of the $J_1 - J_2$ model on a square lattice. The staggered magnetizations $\vec{m}_A$ and $\vec{m}_B$ are defined on two interpenetrating Néel square lattices; (b) Schematic phase diagram proposed for the P-doped iron arsenides \cite{3}. P-doping increases $w$, the spectral weight of the coherent itinerant electrons. The yellow dot denotes the tuning parameter $w_c$ for the QCP. The purple solid line and the green dashed one respectively mark the AF and structural transitions.](image-url)
limit, and we place a particular focus on the effect of damping. We note that the effect of damping on the transitions and dynamics at non-zero temperatures was studied before [35].

Large-$N$ approach. — We generalize the spin symmetry of the model from $O(3)$ to $O(N)$, with $\tilde{m}_{A/B}$ now taking $N$ components. The quartic couplings (rescaled by a $1/N$ factor) are decomposed in terms of Hubbard-Stratonovich fields $i\lambda_{A}$, $i\lambda_{B}$, and $\Delta_I$ [36]. To the leading order in $1/N$, $i\lambda_{A} = \langle m_{A}^{2} \rangle$ and $i\lambda_{B} = \langle m_{B}^{2} \rangle$ contribute to the renormalization of the mass (quadratic coefficient), and $\Delta_{I} = \langle \tilde{m}_{A} \cdot \tilde{m}_{B} \rangle$ is the Ising order parameter. We carry out our analysis from the ordered side, and set $\tilde{m}_{A/B} = (\sqrt{N} \sigma_{A/B}, \pi_{A/B})$ with $\sigma_{A/B}$ and $\pi_{A/B}$ as the static order and fluctuation fields of sublattices $A$ and $B$ respectively. To the order of $O(1/N)$ we can integrate out $\pi_{A/B}$, which yields an effective free energy density:

$$f = \frac{\Delta_{I}^{2}}{u_I} - \frac{(m^{2} - r)^{2}}{2u_{1} + u_{2}} + 2(m^{2} - |\Delta_{I}|)\sigma^{2} + \frac{\gamma^{3}a_{c}}{16c^{2}}\left\{ \left( x - \frac{1}{6} \right) \ln x + \left( y - \frac{1}{6} \right) \ln y - (x + y) \left[ \frac{1}{3} + \ln \left( 1 + \frac{4c\Lambda_{c}^{2}}{\gamma} \right) \right] \right\} (8)$$

where $\Lambda_{c}$ is a cutoff wave vector, and $m^{2} = \langle \tilde{m}_{A}^{2} \rangle = \langle \tilde{m}_{B}^{2} \rangle$. Taking $\sigma_{A} = \sigma_{B} = \sigma$ and $\sigma_{A} = -\sigma_{B} = \sigma$ correspond to $Q = (0, \pi)$ and $(\pi, 0)$ AF orders, respectively. We have introduced the notations

$$x = (m^{2} + \Delta_{I})/\gamma^{2}, \quad y = (m^{2} - \Delta_{I})/\gamma^{2} \quad (9)$$

with the physical requirement $m^{2} \geq |\Delta_{I}|$, which guarantees the free energy to be real. We note that the free energy of Eq. (8) is analytic when $\gamma \to 0$. From these, we have variational equations w.r.t $\sigma$, $m^{2}$ and $\Delta_{I}$,

$$\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial \Delta_{I}} = \frac{\partial f}{\partial m^{2}} = 0 \quad (10)$$

which correspond to [36]

$$\frac{\Delta_{I}}{u_{I}} = \frac{(m^{2} - |\Delta_{I}|)\sigma}{2u_{1} + u_{2}} = 0 \quad (11)$$

$$\frac{\Delta_{I}}{u_{I}} = \frac{-m^{2} - |\Delta_{I}|}{2u_{1} + u_{2}} + G_{1} \quad (12)$$

$$\frac{\Delta_{I}}{u_{I}} = -\frac{m^{2} - |\Delta_{I}|}{2u_{1} + u_{2}} + G_{2} \quad (13)$$

The details about $G_{1}$ and $G_{2}$ are relegated to the Supplementary Material [36]. Several limits provide a check on our approach. From Eqs. (12,13), setting $u_{I} = 0$ will lead to $\Delta_{I} = 0$; this is consistent with the Ising order being driven by the interaction $u_{I}$. In the absence of coupling to coherent itinerant fermions i.e., setting $\gamma^{2}/|\Delta_{I}| = 0$ and $w = 0$, we have a nonzero Ising order at zero temperature, which is what happens for the pure $J_{1} - J_{2}$ model [22, 31].

It follows from these equations that [36] the vanishing of the Ising order implies a vanishing magnetic order. The converse can also be shown explicitly in the limits of $\Delta_{I}/\gamma^{2}, m^{2}/\gamma^{2} \ll 1$, and is numerically confirmed for the generic cases (see below).

Nature of the magnetic and Ising transitions at zero temperature. — We are now in position to address the concurrent magnetic and Ising transitions at $T = 0$. The RG arguments we outlined earlier suggest that there will be a jump of the order parameters across the transitions, but the jump will be smaller as the damping parameter $\gamma$ increases. To see how the damping affects the transition, we first consider the parameter regime where analytical insights can be gained in our large-$N$ approach. When $\gamma$ is sufficiently large so that $x, y \ll 1$, Eq. (13) simplifies to be

$$A(\eta) = a\eta - \eta \ln \eta = \mu(w) \quad (14)$$

with $\eta = |\Delta_{I}|/\gamma^{2}$, and

$$a = -\frac{4\pi^{2}r(\alpha_{d} - \alpha_{c})}{a_{c}} - \ln 2 - 1/2, \quad (15)$$

$$\mu(w) = \frac{4\pi^{3}}{a_{c}^{2}^{(w)} r^{(w)}} \frac{\ln(1 + \frac{1}{\Gamma})}{\Gamma} \quad (16)$$

where $\Gamma = \frac{2}{c^{1/2}a_{c}}$ is the normalized damping rate, while $\alpha_{0} = \frac{r_{c}a_{c}^{3/2}}{\alpha_{c}}$ and $\alpha_{l} = \frac{r_{c}a_{c}^{3/2}}{\alpha_{c}}$ relate to the normalized interactions. As described in detail in the Supplementary Material [36], it follows from this equation that the transition is first order, with the jump of the order parameter decreasing as the damping rate $\Gamma$ is increased. The jump is exponentially suppressed when $\Gamma$ becomes large.

To study the transition more quantitatively, we have solved the large-$N$ equations numerically. Fig. 2 shows how the Ising and magnetic order parameters change when tuning $w$, where, for comparison, we assume $r$ can still be tuned even at $\gamma = 0$. The jump of the order parameter is seen to be very small, even for the case of a very large anisotropy: $a_{c} = 2$ here corresponds to an ellipticity of $\epsilon \approx 0.27$, which is already considerably stronger than what is typically observed.
and the collinear AF order parameter $\sigma_I$.

**FIG. 2:** The evolution of the Ising order parameter $\Delta_I$ (a) and the collinear AF order parameter $\sigma$ (b) vs. the control parameter at different damping rates ($\Gamma = \gamma/(\ell^3/\Lambda)$) at a relatively strong anisotropy $a_c = 2$ (corresponding to an ellipticity $\epsilon \approx 0.27$), with fixed values of the normalized interactions $a_I$ and $a_0$. The transitions are very weakly first order, with jumps in the order parameters (insets) that are very small and decrease with damping.

in the inelastic neutron scattering experiments, which is about $\epsilon \approx 0.6 - 0.9$ [35, 37]. When the anisotropy becomes extremely large, the system effectively becomes one-dimensional, and the effective dimensionality $d + z$ becomes 3; the quartic coupling $-u_I$ will become relevant (as opposed to being marginal) w.r.t. the underlying O(3) QCP, and we expect a stronger degree of first-orderness. Indeed, as shown in Fig. S2 in the Supplementary Material [36] for an extreme value of anisotropy $a_c = 20$ (corresponding to $\epsilon = 0.025$), the order-parameter jump becomes larger.

**Effect of the third-dimensional coupling.**— Iron pnictides have a finite Néel temperature, which results from an interlayer exchange coupling. In order to understand the role of this coupling on the quantum phase transition, we have studied the effective field theory in three-dimensional space. The details of the model are described in the Supplementary Material [36], and the results for the case with the spin-wave velocity on the third dimension being equal to the in-plane velocity at $v = 0$ are shown in Figs. S3,S4. The AF and Ising transitions are still concurrent, and become genuinely continuous. Again, this is consistent with the RG considerations: given that the effective dimensionality in this case is $d + z = 5$, the quartic coupling $-u_I$ becomes irrelevant w.r.t. the underlying O(3) transition and will therefore not destabilize the continuous nature of the transitions.

In the more general case, with a varying third-dimensional coupling, it is more difficult to solve the large-$N$ equations. However, the RG considerations imply that, even in this case, the quantum transitions will be asymptotically continuous.

**Discussion.**— Our results imply that the model for the isoelectronically doped iron pnictides yield quantum phase transitions of the AF and Ising-nematic orders that are concurrent, and essentially second order. This conclusion is consistent with the experimental observations of essentially continuous quantum phase transitions in the normal states of P-doped BaFe$_2$As$_2$ [20] and Ce-FeAsO [15].

In addition, the extremely small jump of the order parameters across the quantum transitions in the 2D case is also important for understanding experimental observations. It implies that quantum criticality occurs over a wide dynamical range, with two-dimensional character.

The logarithmic divergence of the effective mass expected from such 2D quantum critical fluctuations [3, 14] has received considerable experimental support in the P-doped BaFe$_2$As$_2$. It fits well the P-doping dependence of the effective mass as extracted from the de Haas-van Alphen (dHvA) measurements [38], as well as that of the square root of the $T^2$-coefficient of the electrical resistivity [20]. In turn, the latter provides further motivation to study the coupling between the collective fluctuations and charge carriers.

Finally, our results also imply that there will be a wide dynamical range both in temperature and frequency to observe the quantum fluctuations, not only in the staggered magnetizations but also in the Ising-nematic spin channel. Initial indications for the latter have come from inelastic neutron scattering measurements in the electron-doped BaFe$_2$As$_2$ detwinned by uniaxial strain [39]. It would be very instructive to measure such effects in the P-doped BaFe$_2$As$_2$.

**Conclusion.**— We studied zero-temperature magnetic and Ising transitions in a model for isoelectronically tuned iron pnictides using a large-N approach. We demonstrated that the two transitions are concurrent at zero temperature. We also showed that the transitions in the presence of damping are essentially continuous; jumps in the order parameters are extremely small, and are further suppressed by an inter-plane coupling. Our results imply the occurrence of quantum criticality in the isoelectronically doped iron pnictides, which is consistent with the experimental observations in the P-doped iron arsenides.
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Supplemental Material – Magnetic and Ising quantum phase transitions in a model for isoelectronically tuned iron pnictides

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Effective Free Energy at the Large \( N \) Limit

The corresponding action can be written as \([3, 14]\)

\[ S = S_2 + S_4 \]  \hspace{1cm} (S1)

where

\[ S_2 = \sum_{\vec{q}, \omega_l} \left\{ \chi^{-1}_{0, \vec{q}, \omega_l} \left( |\vec{m}_{A, \vec{q}, \omega_l}|^2 + |\vec{m}_{B, \vec{q}, \omega_l}|^2 \right) + 2v (q_x^2 - q_y^2) \vec{m}_{A, \vec{q}, \omega_l} \cdot \vec{m}_{B, -\vec{q}, -\omega_l} \right\} \]  \hspace{1cm} (S2)

\[ S_4 = \int_0^\beta d\tau \int d^2\vec{r} \left\{ u_1 \left[ \left( \vec{m}_A \right)^2 + \left( \vec{m}_B \right)^2 \right] + u_2 \vec{m}_A \vec{m}_B - u_I \left( \vec{m}_A \cdot \vec{m}_B \right) \right\} \]  \hspace{1cm} (S3)

where \( \vec{m}_{A/B}(\vec{r}, \tau) \) are \( O(N) \) vector fields of the sublattices of \( A \) and \( B \), and \( \chi^{-1}_{0, \vec{q}, \omega_l} = r + \omega_l^2 + \chi q^2 + \gamma |\omega_l| \) with \( r = r_0 + wA_Q \). In addition \( u_I \sim J_1^2/J_2^2 \) \([22]\). The quartic fluctuations can be decoupled as follows (the interaction coefficients of \( u_1, u_2, u_I \) have been re-scaled as \( u_1/N, u_2/N, u_I/N \) \([40]\),

\[ e^{(u_I/N)} \int dx \left( \vec{m}_A \cdot \vec{m}_B \right)^2 = L_1 \int D\Delta_I e^{\int dx \left( -\frac{N\Delta_I^2}{\pi} - 2\Delta_I \vec{m}_A \cdot \vec{m}_B \right)} \]  \hspace{1cm} (S4)

and

\[ -r \sum_{\vec{q}, \omega_l} \left( |\vec{m}_{A, \vec{q}, \omega_l}|^2 + |\vec{m}_{B, \vec{q}, \omega_l}|^2 \right) - \int dx \left( \frac{u_I}{N} \left[ \left( \vec{m}_A \right)^2 + \left( \vec{m}_B \right)^2 \right] + \frac{u_2}{N} \vec{m}_A \vec{m}_B \right) \]  \hspace{1cm} (S5)

\[ = L_2 \int D\lambda_A D\lambda_B e^{-i\lambda_A \vec{m}_A^2 - i\lambda_B \vec{m}_B^2} \]  \hspace{1cm} (S6)

with the normalized factors

\[ L_1 = \prod_x \sqrt{\frac{u_I/N}{\pi}}, \quad L_2 = \prod_x \sqrt{\frac{(4u_1^2 - u_2^2)/N^2}{4\pi^2}}, \]  \hspace{1cm} (S6)

where \( x = (\tau, \vec{r}) \) with \( \int dx \equiv \int_0^\beta d\tau \int d^2\vec{r} \). Eq. \( (S5) \), for the case with positive quartic couplings, corresponds to the standard Hubbard-Stratonovich transformation. By contrast, Eq. \( (S4) \) describes the case of a negative quartic coupling and a regularization is needed in general \([40]\). The LHS of Eq. \( (S4) \) is going to diverge after functional integrations over the fields \( \vec{m}_A \) and \( \vec{m}_B \), which indicates that the functional integrals over the fields \( \vec{m}_A \) and \( \vec{m}_B \) cannot exchange with the functional integral over the field \( \Delta_I \) in the RHS of Eq. \( (S4) \). However, since in our case \( u_I \) is larger than \( u_I \), when we combine the functional integrals over the LHS’s of Eqs. \( (S4,S5) \), the total partition function is regular. (Another way of seeing this is that, the solutions to the saddle-point equations are bounded.) As a result, when we deal with the decoupling over the quartic terms simultaneously, the functional integrals over the fields \( \vec{m}_A \) and \( \vec{m}_B \) can exchange with those over the conjugate fields of \( \Delta_I, \lambda_A \) and \( \lambda_B \). Hence, after decoupling the quartic terms we can first integrate over the fluctuations in the fields \( \vec{m}_A \) and \( \vec{m}_B \), leading to the standard procedure of a large-\( N \) approach.

To the leading order in \( 1/N \), we only keep the zeroth mode \( (\omega = 0, k = 0) \) of order parameters \( i\lambda_A/B \) and \( \Delta_I \). We then integrate over the \( N - 1 \)-component fluctuation fields \( \vec{\pi}_{A/B} \) in \( \vec{m}_{A/B} = \left( \sqrt{N} \sigma_{A/B}, \vec{\pi}_{A/B} \right) \), leaving us with an effective action as a function of order parameters \( \lambda_{A/B}, \sigma_{A/B} \) and \( \Delta_I \). Because the sublattices \( A \) and \( B \) are symmetric,
we have $i\lambda_A = \langle m_A^2 \rangle = i\lambda_B = \langle m_B^2 \rangle = m^2$, and $\sigma_A = \pm \sigma_B = \sigma$. Thus to the order of $O(1/N)$ we get the effective free energy as,

$$f = \frac{\Delta_I^2}{u_I} - \frac{(m^2 - r)^2}{2u_1 + u_2} + (m^2 \pm \Delta_I)\sigma^2 + g(m^2, \Delta_I)$$ \hline (S7)

with

$$g(m^2, \Delta_I) = \frac{1}{2} \frac{1}{\beta V} \sum_{\bar{q}, i\omega_l} \ln \left[ \left( D^{-1}_{0, \bar{q}, i\omega_l} + m^2 \right)^2 - (v(q_x^2 - q_y^2) \pm \Delta_I)^2 \right]$$ \hline (S8)

where $D^{-1}_{0, \bar{q}, i\omega_l} = \chi^{-1}_{0, \bar{q}, i\omega_l} - r$, and we take + when $\sigma_A = \sigma_B = \sigma$, and − when $\sigma_A = -\sigma_B = \sigma$ in the expression of $(m^2 \pm \Delta_I)\sigma^2$.

**Saddle Point Equations and Some General Conclusions to the Order of $O(1/N)$**

From Eq. (S7) we have variational equations w.r.t $\sigma$, $m^2$ and $\Delta_I$,

$$\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial m^2} = \frac{\partial f}{\partial \Delta_I} = 0$$ \hline (S9)

After re-arranging these equations we have (for convenience here we choose the branch $\sigma_A = \sigma_B = \sigma$)

$$(m^2 \pm \Delta_I)\sigma = 0$$ \hline (S10)

$$\frac{\Delta_I}{u_I} = \frac{m^2 - r}{2u_1 + u_2} - 2\sigma^2 - \frac{1}{2\beta V} \sum_{\bar{q}, i\omega_l} \frac{1}{D^{-1}_{0, \bar{q}, i\omega_l} + v(q_x^2 - q_y^2) + m^2 + \Delta_I}$$ \hline (S11)

$$\frac{\Delta_I}{u_I} = -\frac{m^2 - r}{2u_1 + u_2} + \frac{1}{2\beta V} \sum_{\bar{q}, i\omega_l} \frac{1}{D^{-1}_{0, \bar{q}, i\omega_l} - v(q_x^2 - q_y^2) + m^2 - \Delta_I}.$$ \hline (S12)

Eqs. (S11, S12) imply that, for the branch $\sigma_A = \sigma_B = \sigma$, $\Delta_I \leq 0$. When $\Delta_I = 0$, after summing over Eq. (S11) and Eq. (S12) we immediately have $\sigma = 0$. In other words the vanishing of $\Delta_I$ can not happen before $\sigma$ vanishes.

On the other hand when $\sigma = 0$, Eq. (S11) and Eq. (S12) merge to one equation, after doing analytic continuation, then setting $T = 0$, this combined equation becomes,

$$\frac{2\Delta_I}{u_I} = \left( \frac{1}{2\pi} \right)^3 \int_{-\Lambda_f}^{\Lambda_f} d^3 q \int_0^\infty d\omega \left[ \frac{\gamma \omega}{(\omega^2 - c_0^2)^2 + \gamma^2 \omega^2} - \frac{\gamma \omega}{(\omega^2 - c_1^2)^2 + \gamma^2 \omega^2} \right]$$ \hline (S13)

where $\Lambda_f$ is the Fermi wave vector, and

$$c_0^2 = (c + v)q_x^2 + (c - v)q_y^2 + m^2 + \Delta_I$$ \hline (S14)

$$c_1^2 = (c + v)q_x^2 + (c + v)q_y^2 + m^2 - \Delta_I.$$ \hline (S15)

The integration on the right hand side (RHS) of Eq. (S13) can be done, and the Eq. (S13) becomes (see the next section for the detailed calculations)

$$\frac{2\Delta_I}{u_I} = \frac{1}{16\pi^2 \sqrt{c^2 - v^2}} \left\{ \gamma \ln \frac{m^2 - \Delta_I}{m^2 + \Delta_I} + i\sqrt{4(m^2 + \Delta_I)} - \gamma^2 \ln \frac{\gamma - i\sqrt{4(m^2 + \Delta_I)} - \gamma^2}{\gamma + i\sqrt{4(m^2 + \Delta_I)} - \gamma^2} ight. $$

$$-i\sqrt{4(m^2 - \Delta_I)} - \gamma^2 \ln \frac{\gamma - i\sqrt{4(m^2 - \Delta_I)} - \gamma^2}{\gamma + i\sqrt{4(m^2 - \Delta_I)} - \gamma^2} \right\}$$ \hline (S16)

The solution of Eq. (S16) is the easiest to see in the limit of $m^2/\gamma^2 \ll 1$, where $\Delta_I = 0$ is the only solution. This is also valid in the other limit but it is more technically involved to show it. Based on these asymptotic results, we expect that, at zero temperature and to the order $O(1/N)$, vanishing of the magnetic order implies that the Ising order vanishes too. This conclusion is also numerically confirmed.
Calculation of Summations in Eq. (S11,S12)

For the summations in Eqs. (S11,S12), we have

\[ \frac{1}{2\beta V} \sum_{\bar{q},i\omega_n} D^{-1}_{\bar{q},0} + v \left( q^2 - q^2_0 \right) + m^2 + \Delta I = \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \coth \frac{\omega}{2T} \left( \omega^2 - c^2_0 \right) + \gamma^2 \omega^2 \] (S17)

\[ \frac{1}{2\beta V} \sum_{\bar{q},i\omega_n} D^{-1}_{\bar{q},0} + v \left( q^2 - q^2_0 \right) + m^2 - \Delta I = \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \coth \frac{\omega}{2T} \left( \omega^2 - c^2_0 \right) + \gamma^2 \omega^2 \] (S18)

At \( T = 0 \), from Eq. (S17) we have

\[ \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \coth \frac{\omega}{2T} \left( \omega^2 - c^2_0 \right) + \gamma^2 \omega^2 \bigg|_{T=0} = \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \coth \frac{\omega}{2T} \left( \omega^2 - c^2_0 \right) + \gamma^2 \omega^2 \] (S19)

\[ \approx \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \sqrt{\omega^2 - c^2_0} \gamma \omega \approx \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \sqrt{\omega^2 - c^2_0} \gamma \omega \] (S20)

\[ \approx \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \frac{\gamma \omega}{\sqrt{\omega^2 - c^2_0} + \gamma^2 \omega^2} = \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \frac{\gamma \omega}{\sqrt{\omega^2 - c^2_0} + \gamma^2 \omega^2} \] (S21)

Using Eqs. (S21, S24), after some integrals, we can get an analytical expression for the free energy as a function of \( \Delta I, m^2, \sigma \). The central task here is to get a close form for Eq. (S8). We can tackle the summation as follows,

\[ \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \coth \frac{\omega}{2T} \left( \omega^2 - c^2_0 \right) + \gamma^2 \omega^2 \bigg|_{T=0} = \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \coth \frac{\omega}{2T} \left( \omega^2 - c^2_0 \right) + \gamma^2 \omega^2 \] (S22)

\[ \approx \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \frac{\gamma \omega}{\sqrt{\omega^2 - c^2_0} + \gamma^2 \omega^2} = \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \frac{\gamma \omega}{\sqrt{\omega^2 - c^2_0} + \gamma^2 \omega^2} \] (S23)

\[ = \frac{1}{(2\pi)^3} \int_{-\Lambda_f}^{\Lambda_f} d^2 q \int_0^{\Gamma_0} d\omega \frac{\gamma \omega}{\sqrt{\omega^2 - c^2_0} + \gamma^2 \omega^2} \] (S24)

Saddle Point Equations in the Ordered Regime

From Eqs. (S10,S11,S12,S21,S24), we can arrive at the following forms of the saddle-point equations in the ordered regime,

\[ - \left( \frac{1}{u_I} - \frac{1}{2u_1 + u_2} \right) |\Delta I| = \frac{r(w)}{2u_1 + u_2} + \frac{1}{16\pi^2 \sqrt{c^2 - v^2}} \left\{ \gamma \ln \frac{8 |\Delta I|}{4c A_\gamma^2 + \gamma^2} + 4 \sqrt{c A_\gamma} \tan^{-1} \left( \frac{2\sqrt{c A_\gamma}}{\gamma} \right) \right. \]

\[ \left. + \sqrt{\gamma^2 - 8 |\Delta I|} \ln \frac{\gamma + \sqrt{\gamma^2 - 8 |\Delta I|}}{\gamma - \sqrt{\gamma^2 - 8 |\Delta I|}} \right\} \] (S27)

After finishing integrations in the above equation, we can get a close form of \( g(\Delta I, m^2) \). Substituting the close form back into Eq. (S7), we arrive at the expression for the full free energy given in the main text.
\[ d = 2, z = 2 \] (a) Illustration of Eq. (14); (b) The counterpart for the generic case of \( a_{d, z} < 0 \) in 3D, illustrating Eq. (S53), with \( \eta = |\delta|/8 \).

\[ 2\sigma^2 = -\frac{2|\Delta_I|}{u_I} + \frac{1}{16\pi^2\sqrt{c^2 - v^2}} \left\{ \gamma \ln \frac{2|\Delta_I|}{\gamma^2} + \sqrt{\gamma^2 - 8|\Delta_I|} \ln \frac{\gamma + \sqrt{\gamma^2 - 8|\Delta_I|}}{\gamma - \sqrt{\gamma^2 - 8|\Delta_I|}} \right\} \] (S28)

**Nature of the Magnetic and Ising Transitions at Zero Temperature**

We consider here the concurrent magnetic and Ising transitions at \( T = 0 \). The RG arguments we outlined in the main text suggest that there will be a jump to the order parameters across the transitions, but the jump will be smaller as the damping parameter \( \gamma \) increases. To see how damping affects the transition, we consider the parameter regime where analytical insights can be gained in our large-\( N \) approach. When \( \gamma \) is sufficiently large so that \( x, y \ll 1 \) (definitions of \( x, y \) are given in the main text), it follows from the closed form of free energy [Eq. (7) in the main text] that

\[ f = -a_0 \left( \frac{r(w)}{cA^2} \right)^2 + \frac{\Gamma^3 a_c}{2\pi^2} \mu(w)m_0^2 + 2\Gamma^2 \left( m_0^2 - |\delta_0| \right) \sigma_0^2 + \cdots \] (S29)

where in “\( \cdots \)” we temporarily neglect terms at the order of \( O(|\delta_0|^2 \ln |\delta_0|) \) and \( O(m_0^4 \ln m_0^2) \), which will be resumed for getting Eq. (14). And \( m_0^2 = m^2/\gamma^2 = (x + y)/2 \), \( \delta_0 = \Delta_I/\gamma^2 = (x - y)/2 \), \( \sigma_0^2 = \sigma^2/\left( e^{-1/2}\Lambda_c \right) \). In addition \( a_c \) and \( a_0, a, \mu(w), \Gamma \) are respectively defined in Eq.(6) and Eq.(14) in the main text. The \( a_c \) here is related to the ellipticity \( e \) by \( a_c = (e + 1/e)/2 \geq 1 \); therefore, a larger \( a_c \) means a stronger anisotropy for the system. From Eq. (S29), we can observe that if \( r(w) \) is a large positive number, the minimum of the free energy only occurs at \( \sigma_0 = 0 \) and \( m_0 = 0 \), then \( \Delta_I = 0 \), corresponding to the disorder phase of the system as it should be. Eq. (S29) shows that when the system is deep inside the order phase with \( r_0 < 0 \) and \( |r_0| \gg 1 \), there is no minimum at the origin since \( \mu(w) < 0 \). This implies that when we increase \( r \) from a large negative value (deep inside the order phase) to certain critical point a phase transition must happen. And this can be made more clear when the system stays in the ordered regime (\( \sigma \neq 0 \)), where in the limit of \( \eta \equiv |\delta_0| \ll 1 \), to the order of \( (|\Delta_I|/\gamma^2)^2 \), we get Eq. (14) in the main text, from which we can see that \( a < 0 \) generally holds, which means the maximum of \( A(\eta) \) will be \( \mu_0 = e^{a-1} \) at \( \eta_0 = e^{a-1} \). The evolution of the equation is illustrated in Fig. S1(a). When the system is in the ordered regime, i.e., \( r(w) \) is a large negative number, then \( \mu < 0 \) and there is a unique global minimum (we focus on the positive branch of Ising order parameter). When \( r(w) \) increases (via increasing \( w \)) to the point that \( \mu = 0 \), there is a maximum emerging at the origin while the Ising order shrinks to \( \eta_1 = e^a \). After this, when \( r(w) \) is further increased, the maximum emerges at the origin moves away from the origin with a cusp-type local minimum generated at the origin which can not be covered by Eqs. (S12,14), meanwhile the Ising order shrinks further. And when \( r(w) \) is further increased until \( \mu = \mu_0 \), the local maximum and local minimum merge as an inflection point, and the free energy as a function of Ising order will only have a global cusp-type minimum at the origin. Therefore a first order transition happens when \( e^{a-1} < \eta < e^a \) while tuning \( w \) to \( w_c \).
FIG. S2: The evolution of Ising order (a) and antiferromagnetic order (b) as a function of the control parameter $r(w)$ at an extremely strong anisotropy $a_c = 20$ (corresponding ellipticity $\epsilon \approx 0.025$). The jump of the order parameters becomes larger compared with the case of a moderately strong anisotropy $a_c = 2$ (corresponding ellipticity $\epsilon \approx 0.27$) shown in Fig. 2 of the main text.

such that $0 < \mu(w_c) < \mu_0$. From Eq. (15) we can see larger $\Gamma$ leads to more negative $a$, since the transition happens in the regime of $e^{a-1} < \eta < e^{a}$, as a result, the first order transition would be exponentially suppressed when $\Gamma$ becomes larger, implying the transition would become essentially second order when damping becomes strong, which is consistent with RG predictions.

**The Effect of Extreme Anisotropy**

When the anisotropy becomes extremely large, the system effectively becomes 1D, and the effective dimensionality $d + z$ becomes 3; the quartic coupling $-u_I$ will become relevant (as opposed to being marginal) w.r.t. the underlying O(3) QCP, and we expect a stronger degree of first-orderness. Indeed, as shown in Fig. S2 for an extreme value of anisotropy $a_c = 20$ (corresponding to an extreme ellipticity of $\epsilon = 0.025$), the magnetic order parameter jump becomes sizable.

**The Case of Three Spatial Dimensions**

In this case, we still have the same saddle-point equation Eq. (S12), but now we take $q^2 = q_x^2 + q_y^2 + q_z^2$ in $\chi^{-1}_{0, q, \omega} = r + \omega^2 + cq^2 + \gamma |\omega|$. Then at zero temperature the summation in Eq. (S12) can be calculated as follows (using Eq. (S18) and working in the regime of $\Delta_I = -m^2 < 0$).

$$\approx \frac{1}{64\pi^3 c^{1/2} \sqrt{c^2 - v^2}} \left\{ 2i \int_0^{\sqrt{v^2 + 1 - |\delta|}} dx \sqrt{x^2 + 1 - |\delta|} \ln \frac{1 - ix}{1 + ix} + 2i \int_{\sqrt{\delta}}^{\sqrt{v^2 - (1 + |\delta|)}} dx \sqrt{x^2 + 1 - |\delta|} \ln \frac{1 - ix}{1 + ix} \right\}$$
where $\Lambda, \gamma = 4e\Delta^2/\gamma^2 = 4\Gamma^2$, $\delta = 8\Delta_I/\gamma^2$. Now let’s deal with the two integrals one by one.

\[
I = 2 \int_0^{(1-\delta)^{1/2}} dx \sqrt{1 - |\delta|} - x^2 \ln \frac{1 + x}{1 - x}
\] (S34)

\[
= 2 (1 - |\delta|)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{1}{2} \right) \left( \frac{(1 - |\delta|)^{n+1/2}}{2n + 1} \ln \frac{1 + (1 - |\delta|)^{1/2}}{1 - (1 - |\delta|)^{1/2}} - \int_0^{(1-\delta)^{1/2}} \frac{d(1-\delta)^{2n+1}}{2n + 1} \frac{2}{1 - x^2} \right) \] (S35)

\[
= 2 (1 - |\delta|)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{1}{2} \right) \left[ \frac{1 + (1 - |\delta|)^{1/2}}{2n + 1} \ln \frac{1 + (1 - |\delta|)^{1/2}}{1 - (1 - |\delta|)^{1/2}} \right] - \int_0^{(1-\delta)^{1/2}} \frac{d(1-\delta)^{2n+1}}{2n + 1} \frac{2}{1 - x^2} \] (S36)

\[
= \frac{2\pi}{4} (1 - |\delta|) \ln \frac{1 + (1 - |\delta|)^{1/2}}{1 - (1 - |\delta|)^{1/2}} + 2 (1 - |\delta|)^{1/2} \int_0^{(1-\delta)^{1/2}} \frac{x \sqrt{1 - x^2}}{1 - |\delta|} \frac{\sqrt{|\delta| - 1 \sinh^{-1}\left(\frac{x}{\sqrt{|\delta| - 1}}\right)} - 1}{1 + x^2} \] (S37)

\[
= \frac{2\pi}{4} (1 - |\delta|) \ln \frac{1 + (1 - |\delta|)^{1/2}}{1 - (1 - |\delta|)^{1/2}} + 2 \left[ (1 - |\delta|)^{1/2} + \sqrt{|\delta|} \cos^{-1}\left(\frac{1}{|\delta|}\right) \right]
\] (S38)

\[
+ 2 (1 - |\delta|)^{1/2} \int_0^{(1-\delta)^{1/2}} \frac{\sqrt{|\delta| - 1 \sinh^{-1}\left(\frac{x}{\sqrt{|\delta| - 1}}\right)}}{1 + x^2} \] (S39)

where \(\binom{n}{m} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}\) is the binomial coefficient. But

\[
2 (1 - |\delta|)^{1/2} \int_0^{(1-\delta)^{1/2}} \frac{\sqrt{|\delta| - 1 \sinh^{-1}\left(\frac{x}{\sqrt{|\delta| - 1}}\right)}}{1 + x^2} \] (S40)

\[
= 2 (1 - |\delta|)^{1/2} \left\{ \sqrt{|\delta| - 1} \left( - \tanh^{-1} x \right) \sinh^{-1}\left(\frac{x}{\sqrt{|\delta| - 1}}\right) \bigg|_{x=0}^{(1-\delta)^{1/2}} + \int_0^{(1-\delta)^{1/2}} \frac{\tan^{-1} x}{\sqrt{1 - x^2/(1 - |\delta|)}} \right\} \] (S41)

\[
= - \frac{\pi}{4} (1 - |\delta|) \ln \frac{1 + (1 - |\delta|)^{1/2}}{1 - (1 - |\delta|)^{1/2}} + 2 (1 - |\delta|)^{1/2} \left\{ \frac{\pi^2}{4} \cosh^{-1}\sqrt{|\delta|} \ln \left( -i \left( 1 - \sqrt{|\delta|} \right) / \sqrt{1 - |\delta|} \right) \right. \] (S42)

\[
+ 4 \text{Li}_2 \left( -i (1 - |\delta|)^{1/2} - \sqrt{|\delta|} \right) - 4 \text{Li}_2 \left( i (1 - |\delta|)^{1/2} + \sqrt{|\delta|} \right) \right\} \] (S43)

where \(\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}\) is the polylogarithm function. Substituting Eq. (S43) back into Eq. (S40), we have

\[
I = 2 \left\{ (1 - |\delta|)^{1/2} + \sqrt{|\delta|} \cos^{-1}\sqrt{|\delta|} \right\} + i \left( 2 (1 - |\delta|) \right) \left\{ \frac{\pi^2}{4} \cosh^{-1}\sqrt{|\delta|} \ln \left( -i \left( 1 - \sqrt{|\delta|} \right) / \sqrt{1 - |\delta|} \right) \right. \] (S44)

\[
+ 4 \text{Li}_2 \left( -i (1 - |\delta|)^{1/2} - \sqrt{|\delta|} \right) - 4 \text{Li}_2 \left( i (1 - |\delta|)^{1/2} + \sqrt{|\delta|} \right) \right\} \] (S45)

Note Eq. (S43) is an exact result for the integral \(I\) in Eq. (S33). For simplicity here we only consider the analytic limit at \(|\delta| = 8|\Delta_I|/\gamma^2 \ll 1\). Within this limit we can get an expansion series of Eq. (S44) in the order of \(|\delta|\),

\[
I = 2 \left\{ -1 + 2\alpha_0 - 2\alpha_0 |\delta| + \frac{\pi}{3} |\delta|^{3/2} - \frac{1}{4} |\delta|^2 + O \left( |\delta|^{5/2} \right) \right\} \] (S46)

where \(\alpha_0 \approx 0.91596\) is the Catalan number.

Now let’s calculate the integral \(II\) in Eq. (S33), which is straightforward in the limit of \(|\delta| = 8|\Delta_I|/\gamma^2 \ll 1\).

\[
II = 2i \int_0^{\sqrt{\Lambda}} dx \sqrt{x^2 + 1 - |\delta|} \ln \frac{1 - ix}{1 + ix} = 4 \int_0^{\sqrt{\Lambda}} dx \sqrt{x^2 + 1} \tan^{-1} x - 2 |\delta| \int_0^{\sqrt{\Lambda}} dx \frac{\tan^{-1} x}{\sqrt{1 + x^2}} + O \left( |\delta|^2 \right) \] (S47)
but
\[ 4 \int_0^{\sqrt{\Lambda}} \frac{dx}{\sqrt{x^2 + 1}} \tan^{-1} x \]
\[ = 4 \left\{ \frac{1}{2} \left[ \sqrt{1 + x^2} (x \tan^{-1} x - 1) + \tan^{-1} x \ln \frac{1 - i e^{i \tan^{-1} x}}{1 + i e^{i \tan^{-1} x}} + i \text{Li}_2 \left( -i e^{i \tan^{-1} x} \right) - i \text{Li}_2 \left( ie^{i \tan^{-1} x} \right) \right] \right\}_{x=0}^{\sqrt{\Lambda}} \]
\[ = \pi \Lambda \gamma - 4 \sqrt{\Lambda} + \frac{\pi}{2} \ln \Lambda \gamma + \frac{\pi}{2} (1 + 2 \ln 2) + 2 - 4\alpha_0 + O \left( \frac{1}{\sqrt{\Lambda \gamma}} \right) \] (S47)

and

\[ \int_0^{\sqrt{\Lambda}} \frac{dx}{\sqrt{x^2 + 1}} \tan^{-1} x = \left[ \frac{\ln x}{x} \ln \frac{1 - i e^{i \tan^{-1} x}}{1 + i e^{i \tan^{-1} x}} + i \text{Li}_2 \left( -i e^{i \tan^{-1} x} \right) - i \text{Li}_2 \left( ie^{i \tan^{-1} x} \right) \right]_{x=0}^{\sqrt{\Lambda}} \]
\[ = \frac{\pi}{2} \ln 2 - 2\alpha_0 + \frac{\pi}{4} \ln \Lambda \gamma + O \left( \frac{1}{\sqrt{\Lambda \gamma}} \right) \] (S48)

Substituting the results of Eqs. (S47,S48) into Eq. (S46), we have

\[ II = \pi \Lambda \gamma - 4 \sqrt{\Lambda} + \frac{\pi}{2} \ln \Lambda \gamma + \frac{\pi}{2} (1 + 2 \ln 2) + 2 - 4\alpha_0 - 2|\delta| \left( \frac{\pi}{2} \ln 2 - 2\alpha_0 + \frac{\pi}{4} \ln \Lambda \gamma \right) + O \left( \frac{1}{\sqrt{\Lambda \gamma}} \right) + O(|\delta|^2) \] (S49)

Combining the results in Eqs. (S45,S49), we finally have

\[ I + II = \pi \Lambda \gamma - 4 \sqrt{\Lambda} + \frac{\pi}{2} \ln \Lambda \gamma + \frac{\pi}{2} (1 + 2 \ln 2) - \left( \pi \ln 2 + \frac{\pi}{2} \ln \Lambda \gamma \right) |\delta| + \frac{\pi}{3} |\delta|^{3/2} + O \left( \frac{1}{\sqrt{\Lambda \gamma}} \right) + O(|\delta|^2) \] (S50)

Substituting Eq. (S50) into Eq. (S12), we have

\[ \frac{\alpha_c \Delta I}{u_I \gamma^2} = \frac{\alpha_c \Delta I}{(2u_1 + u_2) \gamma^2} + \kappa_0 - \kappa_1 |\delta| + \frac{\pi}{3} |\delta|^{3/2} + O(\delta^2) \] (S51)

with \( \alpha_c = 64\pi^3 c^{1/2} \sqrt{\nu^2 - v_2^2} \), and

\[ \kappa_0(w) = \frac{\alpha_c r(w)}{(2u_1 + u_2) \gamma^2} + \pi \Lambda \gamma - 4 \sqrt{\Lambda} + \frac{\pi}{2} \ln \Lambda \gamma + \frac{\pi}{2} (1 + 2 \ln 2) ; \kappa_1 = \pi \ln 2 + \frac{\pi}{2} \ln \Lambda \gamma = \pi \ln 2 + \frac{\pi}{2} \ln \frac{4}{1 + \nu_2^2} \] (S52)

Then we have

\[ B(|\delta|) = a_{d3z2} |\delta| - \frac{2\pi}{3} |\delta|^{3/2} = \kappa_0(w) \] (S53)

with

\[ a_{d3z2} = -\frac{\alpha_c}{4} \left( \frac{1}{u_I} - \frac{1}{2u_1 + u_2} \right) + 2\kappa_1 = -\frac{16\pi^3}{\alpha_c} \left( \frac{c^{3/2}}{u_I} - \frac{c^{3/2}}{2u_1 + u_2} \right) + 2\kappa_1 \] (S54)

From Eq. (S53) we can see that the sign of \( a_{d3z2} \) will determine the order(s) of the phase transition. If \( a_{d3z2} < 0 \), there is no first order transition since Eq. (S53) always has only one solution; the Ising order parameter will continuously go to zero as we increase the controlled parameter \( w \) in \( \kappa_0(w) \). Fig. S1(b) illustrates the process for the phase transitions at \( a_{d3z2} < 0 \).

If \( a_{d3z2} > 0 \), a first order transition can happen, since two solutions of Eq. (S53) emerge when \( \kappa_0(w) > 0 \). From the LHS of Eq. (S53), we can determine that \( a_{d3z2} > 0 \) can happen either at \( v \approx c \) (i.e., extreme anisotropy) or at extremely small damping rate. In the former case the system is effectively reduced back to the 3D problem, where we roughly recover the 2D results. For the latter case, it is equivalent to changing the effective dimension \( d + z = d + 2 \) to \( d + 1 \). Therefore in both of these two extreme situations, the effective dimension of the system becomes 4; the Ising
FIG. S3: Evolution of (a) the Ising order parameter and (b) the magnetic order parameter vs. the control parameter at a relatively strong anisotropy, for the 3D case.

FIG. S4: Evolution of (a) the Ising order parameter and (b) the magnetic order parameter vs. the control parameter at an extremely strong anisotropy, and with strong interactions, also in the 3D case.

coupling “−ui” is again marginal, and a first-order transition is to be expected from RG-based considerations. For the problem we are considering, neither case applies.

For the summation in Eq. (S11), a similar calculation can be carried out. One can easily find that it is δ-independent, which is just equal to the δ-independent part of the summation in Eq. (S12). Therefore after summing Eq. (S11) and Eq. (S12) we will get,

\[
\frac{2\Delta_I}{u_I} = -2\sigma^2 + \frac{\gamma^2}{\alpha_c} \left( -\kappa_1 |\delta| + \frac{\pi}{3} |\delta|^{3/2} + O \left( \frac{1}{\sqrt{\Lambda_\gamma}} \right) \right) + O \left( |\delta|^2 \right) \tag{S55}
\]

i.e.,

\[
\sigma_0 = \frac{\Gamma}{8\pi} \sqrt{\frac{\alpha_c}{\pi}} \left( \frac{1}{2} \left( \frac{16\pi^3 c^{3/2}}{\alpha_c u_I} - \kappa_1 \right) |\delta| + \frac{\pi}{6} |\delta|^{3/2} \right) \tag{S56}
\]

with dimensionless magnetization \(\sigma_0 = c^{1/4}\sigma/\Lambda_c\). From Eq. (S56) we can see that when \(\delta\) continues goes to zero, magnetization will also continuously go to zero, indicating a second-order magnetic phase transition, and the concurrence of Ising and magnetic phase transitions. From Eqs. (S53,S56), we can get Ising order and magnetic order vs. the control parameter \(r(w)\) in d3z2 systems in the limit of \(|\delta| = 8|\Delta_I|/\gamma^2 \ll 1\), as shown in Figs. (S3,S4), where the Ising order \(\Delta_I\) and magnetic order \(\sigma\) have been respectively re-scaled into dimensionless quantities via \(\Delta_I \rightarrow \Delta_I/(c\Lambda_\delta^2)\) and \(\sigma \rightarrow c^{1/4}\sigma/\Lambda_c = \sigma_0\) (for convenience we also introduce a group of dimensionless parameters.
\( a_I^{d+z} = c^{3/2}/u_I, a_0^{d+z} = c^{3/2}/(2u_1 + u_2), a_c = c/\sqrt{c^2 - v^2}, \Gamma = \gamma/(c^{1/2}A_c) \). At moderate strong anisotropy \( a_c = 2 \) i.e., \( \epsilon \approx 0.27 \) (Fig. S3), it shows continuous quantum phase transitions and concurrence of the Ising and magnetic orders when increasing \( w \). As in the 2D case we also study the effect of strong anisotropy at \( a_c = 20 \) i.e., \( \epsilon \approx 0.025 \) (Fig. S4), where the continuous phase transitions persist, and the two transitions are concurrent. This is consistent with the RG considerations: given that the effective dimensionality in this case is \( d + z = 5 \), the quartic coupling \(-u_I\) becomes irrelevant w.r.t. to the underlying O(3) transition and will therefore not destabilize the continuous nature of the transition.