Affine quantisation of the Brans-Dicke theory: Smooth bouncing and the equivalence between the Einstein and Jordan frames

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October 2, 2018

Abstract

In this work, we present a complete analysis of the quantisation of the classical Brans-Dicke Theory using the method of affine quantisation in the Hamiltonian description of the theory. The affine quantisation method is based on the symmetry of the phase-space of the system, in this case the (positive) half-plane, that is identified with the affine group. We consider a FLRW-type spacetime, and since the scale factor is always positive, the affine method seems to be more suited than the canonical quantisation for our Quantum Cosmology. We find the wave function of the Brans-Dicke universe, and its energy spectrum. A smooth bounce is expected at the semi-classical level in the quantum phase-space portrait. We also address the problem of equivalence between the Jordan and Einstein frames.

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1 Introduction

After the formulation of General Relativity (GR), some modified theories arose in an attempt to explain open problems in Cosmology, such as inflation and the observed accelerated expansion. One of the oldest modifications of GR is the Brans-Dicke theory (BDT), proposed in the early 1960’s by Carl H. Brans and Robert H. Dicke [1], in which there is a non-minimal time-dependent coupling of the long-range scalar field with geometry, that is, with gravity. The BDT also introduces an adimensional constant $\omega$ such that, for a constant gravitational coupling, GR is recovered at the limit $\omega \to \infty$, if the trace of the energy-momentum tensor is not null [2, 3, 4]. Today it is well known that, classically, the BDT is practically indistinguishable from GR, with the constant $\omega$ estimated to be over 40,000 [5, 6]. Nevertheless, the Brans-Dicke scalar field arises naturally in Superstring Cosmology, associated with the dilaton, which couples directly with the matter field [7]. The dilaton is equivalent to the graviton for a theory with dynamical gravitational coupling. In spite of the fact that the BDT is classically no different from GR, quantumly it can reveal new dynamics for the primordial universe. There are also claims that the BDT can not reproduce GR for a scale-invariant matter content. In fact, in this case, it has been shown that $\omega$ can display various effects depending on its value, such as a symmetry breaking resulting in a binary phase structure. However, for a strong coupling $\omega \to \infty$, the BDT reproduces GR only in the quantised version [8].

With the assumption that quantum effects cannot be ignored at early stages of the universe, the quantisation of the classical Brans-Dicke theory in its Hamiltonian description is relevant to better understand it. We will assume a mini-superspace, a configuration with reduced degrees of freedom for homogeneous cosmologies being an approximation of the whole superspace, containing only the largest wavelength modes of the size of the universe [9]. With this, we obtain a fairly good approximation of the superspace we live in, with infinite degrees of freedom, but targeting specific behaviors such as the dynamics of the volume of the universe, investigating the nature of the initial singularity and the inflationary phase, for example. We choose to explore the quantisation of the BDT with affine quantisation instead of the canonical one, since the domain of the variables involved (scale factor and scalar field) is the real half-line, and its phase space can be identified with the affine group. With this, we also avoid the operator-ordering problem that arises in the case of canonical quantisation (see, for example, discussions in [10, 11]). The affine quantisation is also equipped with an inverse quantisation map that allows us to obtain classical expressions for quantum operators. In the canonical quantisation, the classical measurements are obtained by the expected values of the classical observables, but in the affine quantisation the classical system is recovered with possible corrections through this inverse map, called quantum correction or lower symbols.

This work, in which we will investigate the quantization of the BDT applying the affine method, is a continuation of the analysis initiated in [12]. We will find the wave function for a Brans-Dicke universe, and draw a quantum phase space out of it. We will then show that a bounce is expected, avoiding the initial...
singularity. We also raise the question about the equivalence between the Jordan and Einstein frames. This paper is organized as follows: in Section 2, we review the classical derivation of the Brans-Dicke Theory with a perfect fluid introduced via the Schutz formalism. In Section 3, we introduce the affine quantization method as well as a more direct way to obtain classical estimates: the quantum phase-space portraits. In Section 4, we apply the affine quantization to the BDT to obtain the Wheeler-DeWitt equation in the Jordan frame and in the Einstein frame. Finally, we derive the semi-classical version for the Hamiltonian constraint in both frames. In the last Section, we present our results and discuss the dependence of the parameters on the solutions.

2 The Brans-Dicke theory with a perfect fluid

The Brans-Dicke Theory is characterized by the introduction of a scalar field non-minimally coupled to gravity, and it is described by the gravitational Lagrangian

\[ L_G = \sqrt{-g} \left\{ \varphi R - \omega \frac{\varphi \varphi' p}{\varphi} \right\}. \]  

(1)

The Brans-Dicke coupling parameter \( \omega \) is chosen to be a constant in this work. Let us consider a homogeneous and isotropic universe,

\[ ds^2 = N^2(t) dt^2 - a^2(t) \left[ dx^2 + dy^2 + dz^2 \right], \]  

(2)

where \( N \) and \( a \) are respectively the lapse function and the scale factor. Then, the Lagrangian (1) becomes

\[ L_G = \frac{1}{N} \left\{ 6 \left[ \varphi a \dot{a}^2 + a^2 \dot{\varphi} \right] - \omega a^3 \dot{\varphi}^2 \right\}, \]  

(3)

where we have already discarded the surface terms. The Lagrangian of the system is completed with a matter component, which we will consider to be a radiative perfect fluid, defined by the equation of state \( p = \rho/3 \).

Let us use the Schutz formalism to introduce the perfect fluid \[13\], in which the four-velocity of a baryonic perfect fluid is described by four potentials, the specific enthalpy \( \mu \) and the entropy \( s \) of the fluid and another two with no clear physical meaning, let us call them \( \epsilon \) and \( \theta \). After some considerations \[13, 14, 15\], the matter Lagrangian becomes

\[ L_M = -\frac{1}{3} \left( \frac{3}{4} \right) a^3 \left( \dot{\epsilon} + \theta \dot{s} \right) e^{-\frac{s}{3}}. \]  

(4)

Since we are interested in the quantum corrections of this system, we must describe the theory with the Hamiltonian formalism. To do so, let us write the
Lagrangians above as functions of the conjugate momenta, defined by
\[ \pi_q = \frac{\partial L}{\partial \dot{q}}. \] (5)

With this, from (4) we obtain the matter super-Hamiltonian
\[ H_M = -\pi^a e^a, \] (6)
where \( \pi_e = -N \rho_0 U^0 a^3 \), with \( \rho_0 \) the rest mass density of the fluid and \( U \) the four-velocity. Let us introduce the following canonical transformation \( T \)
\[ T = -\pi_s e^{-s} \pi^s e^s; \quad \pi_T = \pi^s e^s; \quad \epsilon = \epsilon - \frac{4}{3} \pi_s \pi^s; \quad \pi_e = \pi_e. \] (7)

Then, the super-Hamiltonian for the matter component becomes
\[ H_M = -\frac{N}{a} \pi_T. \] (8)

The Hamiltonian for the gravitational part is given by the Legendre transformation of \( L_G \):
\[ H_G = \dot{a} \pi_a + \dot{\varphi} \pi_\varphi - L_G, \] (9)
where the conjugate momenta are
\[ \pi_a = \frac{6}{N} (2 \varphi a \dot{a} + a^2 \dot{\varphi}) \quad ; \quad \pi_\varphi = \frac{6}{N} a^2 \dot{a} - 2 \frac{\omega}{N} a^2 \frac{\dot{\phi}}{\varphi}, \] (10)
thus leading to
\[ H_G = \frac{1}{N} \left( 6a \dot{a} \dot{\varphi} + 6a^2 \dot{a} \dot{\varphi} - \omega a^3 (\dot{\phi}^2 / \varphi) \right). \] (11)

Expressing the generalized velocities in terms of the momenta, we obtain
\[ \dot{a} = \frac{\omega N}{(3 + 2\omega) \varphi a} \left( \frac{\pi_a}{6} + \frac{\varphi \pi_\varphi}{2 \omega a} \right) \quad ; \quad \dot{\varphi} = \frac{N \varphi}{2(3 + 2\omega) a^3} \left( \frac{a \pi_a}{\varphi} - 2 \pi_\varphi \right), \] (12)
which gives us, after some algebra:
\[ H_G = \frac{N}{3 + 2\omega} \left[ \frac{\omega}{12 \varphi a} \pi_a^2 + \frac{1}{2a^2 \varphi^2} \pi_a \pi_\varphi - \frac{\varphi}{2a^2 \varphi^2} \right]. \] (13)

Therefore, the Hamiltonian of the BDT is given by
\[ \mathcal{H} = N \left\{ \frac{1}{(3 + 2\omega)} \left[ \frac{\omega}{12 \varphi a} \pi_a^2 + \frac{1}{2a^2 \varphi^2} \pi_a \pi_\varphi - \frac{\varphi}{2a^2 \varphi^2} \right] - \frac{1}{a} \pi_T \right\}, \] (14)
where \( \pi_T, \pi_a \) and \( \pi_\varphi \) are the conjugate momenta associated with the matter component, the scale factor \( a \) and the field \( \varphi \), respectively.

The classical Hamiltonian constraint \( \mathcal{H} \approx 0 \) still holds. Notice that here \( \approx \) means "weakly equal", so that \( \mathcal{H} \) is a second-class constraint, i.e. its Poisson's
brackets are non-necessarily vanishing for the BDT with a perfect fluid \[17, 18\]. Thus, we have

\[
\frac{\omega}{12\varphi^2} \pi_a^2 + \frac{1}{2a} \pi_a \pi_{\varphi} - \frac{\varphi^2}{2a^2} \pi_\varphi^2 = (3 + 2\omega)T.
\]

The quantisation of this constraint results in the Wheeler-DeWitt equation, and we can interpret it as a Schrödinger-like equation and, from it, obtain the cosmological scenarios at a quantum level \[19\]. However, we will introduce another quantisation method, based on the symmetry group of the system’s phase space. This kind of quantisation is completed with a quantum phase space portrait, which accounts for a quantum correction for the classical trajectories of the theory, that we will use to analyse the BDT at early cosmological times.

3 The affine quantisation

3.1 Mathematical background

First, let us introduce the affine quantisation method mentioned earlier. The model requires the scale factor and the scalar field, our two dynamical variables, to be positive, with the zero value being a geometrical singularity. Thus, the phase space is a four-dimensional space which is the cartesian product of two half-planes,

\[
\Pi^+_2 := \{(a, \pi_a) \times (\varphi, \pi_\varphi) \mid a > 0, \varphi > 0, \pi_a, \pi_\varphi \in \mathbb{R}\}.
\]

Since it is a cartesian product, we can analyse each half-plane separately. Thus, we will present the theory behind this method of quantisation for a generical phase space, and then apply it to our specific case.

The half-plane \(\Pi_+ := \{(q, p) \mid q > 0, p \in \mathbb{R}\}\) with a multiplication operation defined by

\[
(q, p) (q_0, p_0) = (qq_0, \frac{p_0}{q} + p);
\]

\(q \in \mathbb{R}^+, \quad p \in \mathbb{R},
\]

is identified with the affine group \(\text{Aff}_{+}(\mathbb{R})\) of the real line. The group acts on \(\mathbb{R}\) as follows

\[
(q, p) \cdot x = \frac{x}{q} + p, \quad \forall x \in \mathbb{R}.
\]

On a physical level, one can interpret \[18\] as a contraction/dilation (depending if \(q > 1\) or \(q < 1\)) of space plus a translation. We shall equip the half-plane with the measure \(dq dp\), which is invariant under the left action of the affine group on itself \[20\].

Rigorously, the affine quantisation is a covariant integral method, that combines the properties of symmetry from the affine group with all the resources

\[2\]In the case of radiative matter, at least \[19\].
of integral calculus. This method makes use of coherent states \(^{21}\) to construct the quantisation map, which definition is connected with the symmetry of the phase-space, as we will see. First, let us explain the integral quantisation method. Given a group \(G\) and an unitary irreducible representation (UIR) of it, the quantisation map transforms a classical function (or distribution) into an operator using the assistance of a limited square-integrable operator \(M\), such as

\[
\int_G M(g) \, d\nu(g) = I, \tag{19}
\]

where \(g \in G\) and \(M(g) = U(g)M\gamma\nu^{-1}(g)\). This is the identity resolution of the operator \(M\). With this, from a classical observable \(f(g)\), we obtain the correspondent operator

\[
A_f = \int_G M(g) \, f(g) \, d\nu(g). \tag{20}
\]

For the affine group, that is \(G = \text{Aff}_+(\mathbb{R})\), we have two non-equivalent unitary irreducible representations (UIR) \(U\), plus a trivial one \(U_0\). Let us choose \(U = U_\pm\), which acts on the Hilbert space \(L^2(R^*_+, dx)\) as

\[
(U(q,p)\psi)(x) = e^{ipx}\sqrt{q} \, \psi\left(\frac{x}{q}\right). \tag{21}
\]

We choose the operator \(M\) such as

\[
M = |\psi\rangle \langle \psi|; \quad \psi \in L^2(R^*_+, dx) \cap L^2(R^*_+, dx/x). \tag{22}
\]

The normalised vectors \(\psi\) are arbitrarily chosen providing the square-integrability condition \(^{22}\), and they are known as fiducial vectors. For simplicity, we will consider only real fiducial vectors. The action \(^{21}\) of the UIR \(U\) over them produces the quantum states

\[
|q,p\rangle := U(q,p)|\psi\rangle. \tag{23}
\]

They are called affine coherent states (ACS) or wavelets. It is easy to show that

\[
\int_{\Pi_+} |q,p\rangle \langle q,p| \frac{dq dp}{2\pi c_{\gamma}} = I, \tag{24}
\]

where the constant \(c_{\gamma}\) depends on the choice of \(\psi\) as defined in the following:

\[
c_\gamma = c_\gamma(\psi) := \int_0^\infty |\psi(x)|^2 \frac{dx}{x^{2+\gamma}}. \tag{25}
\]

Hence, the quantisation maps \(^{20}\) becomes

\[
f(q,p) \mapsto A_f = \int_{\Pi_+} f(q,p) \, |q,p\rangle \langle q,p| \frac{dq dp}{2\pi c_{\gamma}}. \tag{26}
\]
With this, one can easily verify that the quantisation of the elementary functions position \( q^\beta \) (for any \( \beta \)), momentum \( p \) and kinetic energy \( p^2 \) yields:

\[
A q^\beta = \frac{c^{(1)} - 1}{c_{-1}} \hat{Q}^\beta \quad ; \quad A p = -i \frac{\partial}{\partial x} = \hat{P} \quad ; \quad A p^2 = \hat{P}^2 + \frac{c^{(1)}}{c_{-1}} \hat{Q}^{-2},
\]

(27)

with \( \hat{Q} \) being the position operator defined by \( \hat{Q} f(x) = x f(x) \) and the constant \( c_{-3}^{(1)} \) is defined as in

\[
c^{(\beta)}_{\gamma} (\psi) := \int_0^\infty |\psi^{(\beta)}(x)|^2 \frac{dx}{x^2 + \gamma}. \quad (28)
\]

Notice that, in this affine quantisation method, the only dependance of the fiducial vector \( \psi \) are in the constants obtained as coefficients for the quantum operators. Thus, the arbitrariness of such \( \psi \) does not play a fundamental role in the quantisation. It is, in fact, an advantage to be explored, once we can adjust the fiducial vectors to regain a certain property \[20\], as for example the self-adjoint character of the operator \( p^2 \). Choosing \( \psi \) such as \( 4 c_{-3}^{(1)} \geq 3 c_{-1} \), the kinetic operator becomes self-adjoint \[24\]. Self-adjointness is a well-known problem in the canonical quantisation of this theory, and it has been studied extensively in \[19\]. Besides, with the affine quantisation we recover naturally the quantum symmetrisation of the classical product momentum position:

\[
qp \mapsto \ A_{qp} = \frac{c_0}{c_{-1}} \hat{Q} \hat{P} + \hat{P} \hat{Q},
\]

(29)

up to a constant that once again depends on the choice of the fiducial vector.

### 3.2 Quantum phase-space portraits

The construction of the affine quantisation method presented in the previous section using coherent states allows us to invert the quantisation map in a very obvious way: by calculating the expected value of an operator with respect to the coherent states. That is, given a quantum operator \( A_f \), we obtain a classical function \( \hat{f} \) such that

\[
\hat{f}(q,p) = \langle q,p | A_f | q,p \rangle.
\]

(30)

If the operator is obtained from a classical function \( f \), as suggested in the notation, then \( \hat{f} \) is a quantum correction or lower symbol of the original \( f \) \[25\]. It corresponds to the average value of \( f(q,p) \) with respect to the probability density distribution

\[
\rho_\phi(q,p) = \frac{1}{2\pi c_{-1}} |\langle q,p | \phi \rangle|^2,
\]

(31)

with \( |\phi \rangle = |q',p'\rangle \). We can also define the time evolution of the distribution \[31\] with respect to time through a Hamiltonian operator \( \hat{H} = A_H \), using the time

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\( ^3 \) up to a factor.
evolution operator $e^{-i\hat{H}t}$. Then,

$$\rho_\phi(q, p, t) := \frac{1}{2\pi c_{c-1}} |\langle q, p|e^{-i\hat{H}t}|\phi\rangle|^2. \quad (32)$$

From equation (30), using (26), the quantum correction $\hat{f}$ of a classical function $f$ is given by

$$\hat{f}(q, p) = \frac{1}{2\pi c_{c-1}} \int_{-\infty}^{\infty} \int_{0}^{\infty} dq' dp' \int_{0}^{\infty} \int_{0}^{\infty} dx dx' f(q', p') \left[ e^{ip(x' - x)} \times e^{-ip'(x' - x)} \psi\left(\frac{x}{q}\right) \psi\left(\frac{x'}{q}\right) \psi\left(\frac{x}{q'}\right) \psi\left(\frac{x'}{q'}\right) \right]. \quad (33)$$

Thus, it is not necessary to find the operator $A_f$ of a classical function $f$ to obtain its lower symbol. One can use the above formulae (33) to do so. For example, the quantum correction of the classical functions $q^\beta$, $p$ and $p^2$ are given by

$$\hat{q}^\beta = \frac{c_{\beta-1}c_{\beta-2}}{c_{c-1}} q^\beta; \quad \hat{p} = p; \quad \hat{p}^2 = p^2 + \left( \frac{c^{(1)}_{\alpha-2}}{c_{c-1}} + \frac{c_0 c^{(1)}_{\alpha-3}}{c_{c-1}} \right) \frac{1}{q^2}. \quad (34)$$

with the constants $c_\gamma$ and $c^{(\beta)}_\gamma$ defined in (25) and (28), respectively. Notice that the corrections also depend on the choice of specific fiducial vectors to determine these constants.

### 4 The affine quantisation of the BDT

#### 4.1 Quantisation in Jordan Frame

Now that we have introduced the affine quantisation method and the quantum phase-space portrait coming from it, we can apply the method to the BDT presented in Section 2 since the variables $a$ and $\varphi$ are both positively defined. However, the Schutz variable associated to the fluid has the whole real line as its spectrum, therefore we can not apply the affine method in it. Nevertheless, we can use another integral quantisation method to it based on the Weyl-Heisenberg group, which acts on the real line [26]. Or we can use the canonical quantisation for this variable, since it works just fine for parameters in the whole line, a domain that does not have any singularity and, therefore, no problems of self-adjointness. In both cases, we have

$$p_T \mapsto \hat{p}_T = -i \frac{\partial}{\partial T}; \quad p_T \mapsto \hat{p}_T = p_T = E. \quad (35)$$
To build the coherent states of the variables $a$ and $\varphi$, let us name the respective fiducial vectors as $\psi_a$ and $\psi_\varphi$, which are not the same at first. Then, the coherent states are given by

$$|a, p_a = U_a |\psi_a\rangle \quad \Rightarrow \quad \langle x |a, p_a\rangle = \frac{e^{ip_a x}}{\sqrt{a}} \psi_a \left( \frac{x}{a} \right) \quad (36)$$

$$|\varphi, p_\varphi = U_\varphi |\psi_\varphi\rangle \quad \Rightarrow \quad \langle y |\varphi, p_\varphi\rangle = \frac{e^{ip_\varphi y}}{\sqrt{\varphi}} \psi_\varphi \left( \frac{y}{\varphi} \right). \quad (37)$$

With this, the quantisation of equation (15) results in the following Wheeler-DeWitt equation:

$$\left\{ -\frac{\omega}{12c_{-1}(\varphi)} \varphi \partial_a^2 + \left( \omega \lambda_1 - \lambda_2 \right) \frac{1}{4a^2} - \frac{1}{2c_{-1}(a)} \frac{1}{a} \partial_a \partial_\varphi + \lambda_3 \frac{\varphi}{a^2} \partial_\varphi^2 + \lambda_4 \frac{1}{a^2} \partial_\varphi \right\} \Psi (a, \varphi, T) = -i \left( 3 + 2\omega \right) \partial_T \Psi (a, \varphi, T), \quad (38)$$

where $\Psi (a, \varphi, T)$ is the wave function, and the constants $\lambda_i$ are given by

$$\lambda_1 = \frac{1}{12c_{-1}(\varphi)} c_{-1}^{(1)} (a); \quad \lambda_2 = \frac{1}{2} c_{-1} (a) c_{-1}^{(1)} (\varphi);$$

$$\lambda_3 = \frac{1}{2} c_{-3} (a) c_{-1} (\varphi); \quad \lambda_4 = \frac{1}{2} c_{-1} (a) c_{-1} (\varphi) + \frac{1}{4} c_{-1} (a) c_{-1} (\varphi).$$

and we defined

$$c_j^{(1)} (a) = \int_0^\infty \left[ \psi_j^{(1)} (x) \right]^2 \frac{dx}{x^{2+\gamma}}; \quad c_j^{(1)} (\varphi) = \int_0^\infty \left[ \psi_j^{(1)} (\varphi) \right]^2 \frac{dx}{x^{2+\gamma}}. \quad (39)$$

If we choose $\psi_a = \psi_\varphi$, then $c_j^{(1)} (a) = c_j^{(1)} (\varphi) = c_j^{(1)}$. So, let us choose a fiducial vector such that

$$\psi_a = \psi_\varphi = \sqrt{\frac{9}{6}} \frac{x^2 e^{-\frac{x^2}{2}}}. \quad (40)$$

With these vectors, we have $c_{-2} = c_{-1} = 1$, and $c_{-3}^{(1)} = 3/4$, which is necessary condition for the quantised kinetic energy to be a self-adjoint operator [24]. In turn, this gives us the Wheeler-DeWitt equation in the Jordan frame

$$\left\{ -\frac{\omega}{12} \varphi \partial_a^2 + \left( \frac{\omega}{16} - \frac{3}{4} \right) \frac{1}{\varphi a^2} - \frac{1}{2a} \partial_a \partial_\varphi + \frac{\varphi}{a^2} \partial_\varphi^2 + \frac{5}{4a^2} \partial_\varphi \right\} \Psi = -i \left( 3 + 2\omega \right) \partial_T \Psi. \quad (41)$$

From this equation, absorbing the constant $12 \left( 3 + 2\omega \right) \varphi^{-1}$ into the temporal parameter, that is, accounting it as energy, we obtain the Hamiltonian for the BDT in the Jordan frame:

$$H_J = \frac{1}{\varphi} \partial_a^2 + \frac{12}{\omega} \left( \frac{\omega}{16} - \frac{3}{4} \right) \frac{1}{\varphi a^2} + \frac{6}{\omega a} \partial_\varphi \partial_\varphi - \frac{12}{\omega} \frac{\varphi}{a^2} \partial_\varphi^2 - \frac{15}{\omega a^2} \partial_\varphi. \quad (42)$$

It is easy to see that Hamiltonian (42) is self-adjoint for the usual measure $da \ d\varphi$ on the Hilbert space, as expected. One can notice that equation (41) is not separable. We can work around this problem considering the Einstein frame instead.
4.2 Conformal transformation of affine operators

The Jordan and Einstein frames are related to each other by a conformal transformation given by \( g_{\mu\nu} = \phi^{-1} \tilde{g}_{\mu\nu} \), where \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \) represent the metric tensors in each frame, respectively. Thus, before analysing the equivalence between these frames, let us first comment on how affine operators change with a conformal transformation.

As opposed to what happens in the canonical quantisation (see \[12\]), the affine operators are uniquely defined by equation (26). Also, if \( A_f \) is the operator obtained from a classical function \( f(x) \), then for a general conformal scaling factor \( \Omega \), we have

\[
\Omega^2(x)A_f \neq A_{\Omega^2(x)f},
\]

Therefore, in spite of the operator being uniquely defined, one should still be careful when comparing the quantisation of two conformal models, since we quantise classical equations, that is, the energy constraint of the Hamiltonian formulation of classical models.

Classically, it is always possible to cancel non-null coefficients, that is,

\[
h(x)f(x) = h(x)g(x) \iff f(x) = g(x),
\]

if \( h(x) \neq 0 \). However, quantising the constraint in different frames can result in very different scenarios, because of (43). Then, if \( h(x)f(x) = h(x)g(x) \) holds, then \( A_{hf} = A_{hg} \) and

\[
h(x)A_f \neq A_{hf} = A_{hg} \neq h(x)A_g.
\]

Therefore \( A_f = A_g \) does not hold. In conclusion, to compare the quantisation of two different frames connected by a transformation of coordinates, it is important to remember that property [44] cannot be used in order to obtain equivalent quantised models.

4.3 Quantisation in the Einstein’s frame

Since the seminal paper of Brans and Dicke \[1\], we know that two formulations of the theory (and in fact, for every scalar-tensor theory) are possible. These formulations, related by a conformal transformation, are the target of a long debate on which of these frames is physically relevant. Some authors claim they are equivalent classically but should be different at the quantum level \[27, 28\], while others claim that both are equivalent at classical and quantum level \[29, 30, 31\]. Since theoretical predictions depend entirely on the conformal frame we are working on, a natural question arising is if there is a preferred frame or not, and which one is the most suitable to observations. We found the differential equation governing the wave function evolution (54), however this equation was performed in the Jordan frame and, as a crossed term appeared in the partial derivatives, it makes its resolution difficult. Then, let us analyse the problem in the Einstein frame instead.
The Brans-Dicke Lagrangian, with a non-minimally coupled scalar field is given by (1), but a conformal transformation, $g_{\mu\nu} = \frac{1}{\phi} \tilde{g}_{\mu\nu}$, where $g_{\mu\nu}$ is the metric in the non-minimal coupling frame, the Lagrangian reads as

$$L_G = \sqrt{-\tilde{g}} \left[ \tilde{R} - \left( \omega + \frac{3}{2} \right) \frac{\varphi \dot{\varphi}^2}{\phi^2} \right],$$

(46)

which is the Lagrangian for General Relativity with a scalar field minimally coupled. The Lagrangian (1) is written in the Jordan frame, and (46) is written in the Einstein frame. The conformal transformation is given by the change of coordinates

$$N' = \varphi^{\frac{1}{2}} N \quad ; \quad b = \varphi^{\frac{1}{2}} a \quad ; \quad \varphi' = \varphi,$$

(47)

and, applying it to (3), we obtain

$$L_G = \frac{1}{N^2} \left[ 6 b \dot{b}^2 - \left( \omega + \frac{3}{2} \right) b^3 \left( \frac{\varphi'}{\varphi} \right)^2 \right].$$

(48)

We choose to drop the prime on $N$ and $\varphi$ for simplicity. The total Hamiltonian is thus

$$H_T = N \left( \frac{\pi_b^2}{24b} - \frac{\varphi^2}{2(3 + 2\omega)b^3} \frac{\pi_T}{b} \right).$$

(49)

The constraint $H_T = 0$ gives us\footnote{We keep the $1/b$ factor in order to avoid inconsistencies in the quantisation (see the discussion in Subsection 1.2).}

$$\frac{\pi_b^2}{24b} - \frac{\varphi^2}{2(3 + 2\omega)b^3} \pi_T = \frac{\pi_T}{b}.$$  

(50)

Using (26), we obtain

$$A_{\varphi^2\pi_b^2} = -\frac{1}{c_{-1}(b)} \frac{1}{b} \partial_b^2 + \frac{1}{c_{-1}(b)} \frac{1}{b^2} \partial_b - \left( \frac{1 - c_{-1}^{(1)}(b)}{c_{-1}(b)} \right) \frac{1}{b^2};$$

$$A_{\varphi^2\pi_b^2} = -\frac{c_1(\varphi)}{c_{-1}(\varphi)} \varphi^2 \dot{\varphi}^2 - 2 \frac{c_1(\varphi)}{c_{-1}(\varphi)} \varphi \partial_{\varphi} + \frac{c_1(\varphi) - c_{-1}(\varphi)}{c_{-1}(\varphi)}.$$  

(51)

(52)

Then, the quantisation of equation (50) results in

$$-\frac{\varpi}{c_{-1}(b)} \partial_b^2 + \frac{\varpi}{c_{-1}(b)} \frac{1}{b} \partial_b + \left[ \varpi \left( \frac{c_{-1}^{(1)}(b) - 1}{c_{-1}(b)} \right) + \frac{6 c_{-4}(b)}{c_{-1}(\varphi)} \frac{c_1(\varphi) - c_{-1}(\varphi)}{c_{-1}(\varphi)} \right] \frac{1}{b^2} \right.$$

$$+ \frac{6 c_{-4}(b)}{c_{-1}(b)} \frac{c_1(\varphi)}{c_{-1}(\varphi)} \frac{1}{b^2} \left( \varphi^2 \dot{\varphi}^2 + 2 \varphi \partial_{\varphi} \right) \Psi = -24i \varpi \partial_{\varphi} \Psi,$$

(53)
with $\varpi = \omega + \frac{3}{4}$, and this is a separable equation. On the other hand, one can change variables as in (47) directly on (41). This yields

\[
-\omega \frac{\partial^2}{\partial b^2} + \frac{1}{8b} \partial_b + \left(\frac{\omega}{16} - \frac{3}{4}\right) \frac{1}{b^2} + \frac{\varphi^2}{b^2} \partial^2 \varphi + \frac{5\varphi}{4b^2} \partial \varphi \right] \Psi = -i(3 + 2\omega)\partial_T \Psi. \tag{54}
\]

Thus, considering the liberty in the choice of the fiducial vectors, we conclude that the Schrödinger equations in the Einstein and Jordan frames are equivalent at this level.

However, notice that equation (54) is derived directly from (41), and this can result in losing the self-adjointness character. Indeed, from the Schrödinger-like equation above, we obtain

\[
H_E = \partial_b^2 - \frac{3}{2\omega} \frac{1}{b} \partial_b - \frac{12}{\omega} \left(\frac{\omega}{16} - \frac{3}{4}\right) \frac{1}{b^2} - \frac{12\varphi^2}{\omega b^2} \partial^2 \varphi - \frac{15\varphi}{\omega b^2} \partial \varphi , \tag{55}
\]

the Hamiltonian operator of the BDT in the Einstein frame. In [19], there is a study on the self-adjointness for this type of Hamiltonian. The operator (55) is self-adjoint only with the measure

\[
(\psi|\phi) = \int_0^\infty \int_0^\infty \psi^* \phi \ b^{\frac{3}{2}\omega} \varphi^{-\frac{3}{4}} \ db \ d\varphi . \tag{56}
\]

Note that the measure $b^{\frac{3}{2}\omega} \varphi^{-\frac{3}{4}} \ db \ d\varphi$ is not equivalent to the measure $da \ d\varphi$ for which the Hamiltonian operator (42) in the Jordan frame is self-adjoint, as we already suggested. Thus, the equivalence is possible only if we consider different domains for the Hamiltonian operator in both the Jordan and the Einstein frames. On the other hand, one can consider different fiducial vectors in both frames, and then, with the right choice, the Hamiltonian operator coming from (53) will be self-adjoint without the need of adopting another measure. Thus, the equivalence is strongly connected to one of these options: considering different domains for the operators in each frame, or considering different fiducial vectors in each frame. Both are equivalent options, as shown in [19], since the measure depends on the choice of ordering parameters (for the canonical quantisation), and this reads as a choice of fiducial vectors for the affine quantisation.

Let us solve, without loss of generality, equation (54). We suppose the following separation of variables: $\Psi(b, \varphi, t) := X(b) \ Y(\varphi) \ P(T)$. We obtain, for the function of time

\[
P(T) = A \exp \left(i \frac{\omega}{12(3 + 2\omega)} ET \right) , \tag{57}
\]

where $E$ is the energy constant. Notice, however, that the actual energy of the system is then $\omega[12(3 + 2\omega)]^{-1}E$. This results in the following system of ODE’s:

\[
\begin{align*}
&\left\{-\partial_b^2 + \frac{3}{2\omega} \frac{1}{b} \partial_b + \frac{12}{\omega} \left[\frac{\omega}{16} - \frac{3}{4} - k^2\right] \frac{1}{b^2}\right\} X(b) = E \ X(b) ; \\
&\left\{\varphi^2 \partial^2 \varphi + \frac{5\varphi}{4} \partial \varphi \right\} Y(\varphi) = -k^2 Y(\varphi) , \tag{58}
\end{align*}
\]
with \( k^2 \) being a separation constant. The general solutions are given by
\[
X(b) = C_1 b^{\frac{3+2\omega}{2}} J_{\nu} \left( \sqrt{E} \, b \right) + C_2 b^{\frac{3+2\omega}{2}} Y_{\nu} \left( \sqrt{E} \, b \right) : \quad (59)
\]
\[
Y(\varphi) = D_1 \varphi^{-\frac{1}{4}} (\sqrt{1-64k^2}+1) + D_2 \varphi^{\frac{1}{4}} (\sqrt{1-64k^2}-1), \quad (60)
\]
with \( J_{\nu} \) and \( Y_{\nu} \) the Bessel functions of first and second kind, respectively, \( C_{1,2}, D_{1,2} \) are integration constants,
\[
\nu = \frac{1}{2} \sqrt{\left( 1 + \frac{3}{2\omega} \right)^2 + \frac{48}{\omega} \left( \frac{\omega}{16} - \frac{3}{4} - k^2 \right)}. \quad (61)
\]
and \( k^2 < 1/64 \).

The wave-function of the universe \( \Psi_T(b, \varphi) = X(b)Y(\varphi) \) must be square-integrable, normalizable. This is the reason for the choice of the limit set for the separation constant. Equation (58) is known as Euler equation and the solution (60) corresponds to said limit of \( k^2 \). The solution for \( k^2 = 1/64 \) gives similar results, however, \( k^2 > 1/64 \) results in a non square-integrable wave-function. This is also the reason why we choose a negative sign for the separation constant. Also, since \( Y_n \) blows up at the origin, we must consider \( C_2 = 0 \). Now, let us consider the following transformation for the variable \( \varphi \):
\[
\sigma = \ln \varphi \quad \Rightarrow \quad d\sigma = \frac{1}{\varphi} d\varphi. \quad (62)
\]
With this, the solution (60) becomes
\[
Y(\sigma) = D_1 e^{-\frac{1}{8} (\sqrt{1-64k^2}+1)} + D_2 e^{\frac{1}{8} (\sqrt{1-64k^2}-1)}. \quad (63)
\]
For the sake of simplicity, let us consider \( D_2 = 0 \). We construct the wave packet as
\[
\Psi_{T_n} = N \int_{-\frac{1}{8}}^{\frac{1}{8}} dk \, b^{\frac{3+2\omega}{2}} J_{\nu} \left( \sqrt{E_n} \, b \right) e^{-\frac{1}{8} (\sqrt{1-64k^2}+1)}, \quad (64)
\]
where \( N \) is a constant. Therefore, the product of two wave packets is
\[
\langle \Psi_{T_m} | \Psi_{T_n} \rangle = N^2 \int_0^{b_0} \int_0^{\infty} \frac{d\varphi}{\varphi} \varphi^{-\frac{1}{2}} \, db \, d\varphi \int_{-\frac{1}{8}}^{\frac{1}{8}} dk \, b^{\frac{3+2\omega}{2}} e^{-\frac{1}{8} } \times
\]
\[
\times J_{\nu} \left( \sqrt{E_n} \, b \right) J_{\nu'} \left( \sqrt{E_m} \, b \right) e^{i \left( \frac{1}{8} \sqrt{1-64k'^2} - \frac{1}{8} \sqrt{1-64k^2} \right) \sigma}, \quad (65)
\]
or, writing only in terms of \( \sigma \),
\[
\langle \Psi_{T_m} | \Psi_{T_n} \rangle = N^2 \int_0^{b_0} \int_0^{\infty} d\varphi d\sigma \int_{-\frac{1}{8}}^{\frac{1}{8}} dk \, b^{\frac{3+2\omega}{2}} J_{\nu} \left( \sqrt{E_n} \, b \right) \times
\]
\[
\times J_{\nu'} \left( \sqrt{E_m} \, b \right) e^{i \left( \frac{1}{8} \sqrt{1-64k'^2} - \frac{1}{8} \sqrt{1-64k^2} \right) \sigma}, \quad (66)
\]
\[\text{With this, it becomes more evident why it is only square-integrable for } k^2 < 1/64.\]
where the prime on the \( \nu \) indicates \( \nu(k') \) and we can take \( b_0 = 1 \) as the value of the scale factor today. The integration over \( \sigma \) and afterwards over \( k' \) gives us:

\[
\langle \Psi_{Tm}|\Psi_{Tn}\rangle = 2\pi N^2 \int_0^{b_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} db dk b^\frac{\nu + 2\omega}{2\nu} J_\nu \left( \sqrt{E_n b} \right) J_\nu \left( \sqrt{E_m b} \right).
\]

Now, we shall consider an approximation for the limit \( \omega \gg k^2 \). This approximation is relevant due to our understanding of today’s estimative of the Brans-Dicke constant. Notice that in this limit the energy of the system is only \( E_n \), the Bessel index \((61)\) becomes \( \nu = 1 \) and the \((67)\) becomes

\[
\langle \Psi_{Tm}|\Psi_{Tn}\rangle = 2\pi N^2 \int_0^{b_0} db J_1 \left( \sqrt{E_n b} \right) J_1 \left( \sqrt{E_m b} \right).
\]

This integral is null unless \( m = n \) and \( E_n \) are the zeros of the Bessel function \( J_1 \). This tells us that the energy spectrum of the universe is discrete and the energy states are orthogonal. Also, each stationary state obeys

\[
|\psi_{Tn}|^2 = \pi N^2 \left[ J_2 \left( \sqrt{E_n} \right) \right]^2.
\]

### 4.4 Quantum phase-space portrait of the BDT

Let us consider the formalism introduced in Subsection 3.2. The constraint \((15)\), \( \mathcal{H}_T = 0 \), can be rewritten in its semi-classical version, using \((33)\) to calculate each term. For the sake of simplicity, we will keep the same letter for the energy constant, then \( \tilde{p}_T = E \) and

\[
\omega \frac{1}{12} \varphi_a^2 + (\omega \kappa_1 - \kappa_2) \frac{1}{a^2} \varphi + \frac{1}{2a} p_a p_\varphi - \kappa_3 \frac{\varphi^2}{a^2} p_\varphi^2 = (3 + 2\omega)E,
\]

with the constants \( \kappa_i \) being

\[
\kappa_1 = \frac{1}{12} \left( \frac{c_0(a) c_{-3}(a)}{c_{-1}(a)} + c_{-2}(a) \right);
\]

\[
\kappa_2 = \frac{1}{2} \frac{c_0(a) c_{-3}(a)}{c_{-1}(a)} \left( \frac{c_0(\varphi) c_{-1}(\varphi)}{c_{-1}(\varphi)} + c_{-2}(\varphi) \right);
\]

\[
\kappa_3 = \frac{1}{2} \frac{c_0(a) c_{-3}(a)}{c_{-1}(a)} \frac{c_0(\varphi) c_{-3}(\varphi)}{c_{-1}(\varphi)};
\]

where \( c_i^{(j)}(a) \) and \( c_i^{(j)}(\varphi) \) are

\[
c_i^{(j)}(a) = \int_0^\infty [\psi_i^{(j)}(x)]^2 \frac{dx}{x^{2+\gamma}} \quad ; \quad c_i^{(j)}(\varphi) = \int_0^\infty [\psi_i^{(j)}(x)]^2 \frac{dx}{x^{2+\gamma}}.
\]

If we choose \( \psi_a = \psi_\varphi \), then \( c_i^{(j)}(a) = c_i^{(j)}(\varphi) = c_i^{(j)} \). With this in mind, let us choose a fiducial vector such that

\[
\psi_a = \psi_\varphi = \frac{9}{\sqrt{6}} x^\frac{3}{2} e^{-\frac{x}{2}}.
\]
With these vectors, we have \( c_{-2} = c_{-1} = 1 \), and \( c_{-3} = 3/4 \), the latter being a necessary condition for the quantised Hamiltonian to be a self-adjoint operator \([24]\). We want this condition to hold even if we are not doing the quantisation explicitly, since the semi-classical trajectories are probabilistic along the path a quantum state evolves. Then, (70) becomes

\[
\frac{\omega}{12} \frac{1}{\varphi} p_{\varphi}^2 + \frac{9}{8} (\omega - 2) \frac{1}{a^2 \varphi} + \frac{1}{2a} p_{\varphi}^2 \varphi - 2 \frac{\varphi^2}{a^2} p_{\varphi}^2 = (3 + 2\omega)E. \tag{73}
\]

The expression (73) allows us to analyse the expected behaviour of the scale factor \( a \) for the early universe, for a given initial value of the scalar field \( \varphi(t_0) = \varphi_0 \) and its momentum at this instant \( p_{\varphi}(t_0) = p_{\varphi_0} \).

Notice that equation (15) is the classical Hamiltonian constraint in the Jordan frame. To compare the expected behaviour of the scale factor in the Jordan frame with the one in the Einstein frame, let us calculate the quantum portrait of equation (50), the Hamiltonian constraint in the Einstein frame. To do so, we must use (33) to calculate the quantum correction of the classical functions \(\kappa - 1 \varphi_2 \frac{1}{\varphi} \) and \(\varphi^2 \frac{1}{\varphi^2} \). With this, the quantum correction of (50) becomes:

\[
3 + 2\omega \frac{24}{24} \left[ p_{\varphi}^2 \kappa_4 \frac{1}{b^2} - \kappa_5 \frac{1}{b^2} \left( \kappa_6 \varphi^2 p_{\varphi}^2 + \kappa_7 \right) \right] = (3 + 2\omega)E', \tag{74}
\]

with \( E' \) the energy, and the constants:

\[
\kappa_4 = \frac{c_{-1}^{(1)}(\varphi) + c_1(\varphi) c_{-1}^{(1)}(b) - c_1(b)}{c_{-1}(b)} \quad ; \quad \kappa_5 = \frac{1}{2} \frac{c_{-4}(b) c_1(b)}{c_{-1}(b)} ; \\
\kappa_6 = -\frac{c_1(\varphi) c_{-4}(\varphi)}{c_{-1}(\varphi)} \quad ; \quad \kappa_7 = \frac{c_{-1}^{(1)}(\varphi) + c_1(\varphi) c_{-1}^{(1)}(\varphi) - c_1(\varphi)}{c_{-1}(\varphi)}.
\]

By choosing the fiducial vectors such as before, we find:

\[
3 + 2\omega \frac{24}{24} p_{\varphi}^2 + \left[ -\frac{-189 + 54\omega}{32} \right] \frac{1}{b^2} - \frac{50}{b^2} \varphi^2 p_{\varphi}^2 = (3 + 2\omega)E'. \tag{75}
\]

Equations (73) and (75) are the quantum correction of the classical Brans-Dicke Theory described in both Jordan and Einstein frames, respectively. To understand what are the consequences of these corrections, let us build the quantum phase-space of the BDT in both those frames.

5 Results

As mentioned before, in this section we present the quantum phase-space portrait coming from equations (73) and (75). The aim is to understand the behaviour of the scale factor \( a \), which is connected to the volume of the universe, so the phase-spaces shown here are with reference to this variable. Notice, however, that there are still other free parameters: the scalar field \( \varphi \), the energy \( E \).
Phase-space portrait for $p_a$ and $a$ varying the range of $p_\phi$.

Figure 1: Quantum phase-space of the scalar field in the Jordan frame, using $\omega = 410.000$ and $E_0 = 10^{16}$. The left figure is for a range $1 \leq p_\phi \leq 10^3$, while for the right one the range is smaller $1 \leq p_\phi \leq 10^2$.

and the Brans-Dicke constant $\omega$. These parameters will be varied for the sake of understanding their influence on the issue. Without loss of generality, let us consider the initial state of the scalar field to be $\varphi_0 = 1$.

For the Jordan frame, let us set the energy at $E_0$ and construct the phase-space for a range of values of $p_\phi$ (Figure 1). Each curve represents a value for the velocity (momentum) of the scalar field. Notice that, up until an upper value for $p_\phi$, the curves are of a smooth bouncing for the universe, including solutions with possible inflationary phase. The second configuration includes divergent curves. If one assumes that this type of divergence does not describe a physical reality (favoring smoothness), then the scalar field must have a limit in momentum. Otherwise, this model predicts a singularity formed by an accelerated contraction of a prior universe, reaching null volume as the (module of the) momentum goes to infinity, followed by a decelerated inflation\textsuperscript{7}.

Secondly, we studied the effect of the Brans-Dicke parameter $\omega$ (Figure 2). In the left figure, we used $\omega = 41.000$ and see there are more divergent lines than in the generic case considered in Figure 1. In the right one, we increased $\omega$ to 4,100,000. Notice that it is required a much greater initial momentum for the scale factor to obtain divergent solutions. Therefore, a larger $\omega$ seems to lead to a more well-behaved theory. This is a result of interest, since the larger $\omega$ is, the greater the coupling between matter and the scalar field, that is, the smaller the effects of the scalar field are. This would correspond to the weak-field limit we observe today. Actually, for a perfect fluid (as in our case), we recover GR in this limit\textsuperscript{8}. Also, the change in the energy parameter (Figure 3) results in a change of scale in the phase-space.

\textsuperscript{7}Notice that, we are reading the graphics in the clockwise direction.
Phase-space portrait for \( p_{\omega} \) and \( a \) varying the value of \( \omega \).

Figure 2: The effect of the Brans-Dicke constant in the scalar field phase-space. Once again, we use \( E_0 = 10^{16} \) and consider the range \( 1 \leq p_{\phi} \leq 10^3 \). The left figure is for \( \omega = 41.000 \), and the right one is for \( \omega = 4.100.000 \).

Phase-space portrait for \( p_{\omega} \) and \( a \) varying the value of \( E \).

Figure 3: The change in the energy of the system results in a change of scale for the solutions. The left figure was drawn with \( E = 10^{13} \) and the right one with \( E = 10 \). Same values used as before for \( p_{\phi} \) and \( \omega \): \( 1 \leq p_{\phi} \leq 10^3 \) and \( \omega = 4.100.000 \).
Up until now, we considered the initial value of the scalar field to be $\phi_0 = 1$, but we also want to understand the effects of it on the shape of the solutions. Thus, in Figure 4, we show the direct influence of changing the value for the scalar field on the solutions. The top row shows greater values for $\phi$, from $10$ to $10^4$ (left to right). We notice that the greater $\phi$ is, the more singularities we obtain. Conversely, in the second row, we lower it from $0.1$ to $10^{-4}$. The solutions tend to bounces instead of singularities. As expected, the results are coherent with the study on $\omega$.

Phase-space portrait for $p_a$ and $a$ varying the initial value of the scalar field $\phi_0$.

Figure 4: The top row shows the solutions for high values of $\phi$: top-left $\phi = 10$, and top-right: $\phi = 10^4$. The bottom row is for low values of $\phi$: bottom-left: $\phi = 10^{-1}$, and bottom-right: $\phi = 10^{-4}$. For these, we are considering $\omega = 410.000$, $E = 10^{16}$, and $1 \leq p_\phi \leq 10^3$.

In the Einstein frame, we have symmetric bounces without any inflationary
epoch as we see in Figure 5. By varying once again $\omega$ (Figure 6) and the energy (Figure 7), we arrive to the same conclusions as in the Jordan frame, i.e. that the larger $\omega$ is, the lesser divergent curves we obtain, and varying the energy induces a scaling in the phase space. We also show the effect of the scalar field in Figure 8.

Phase-space portrait for $p_b$ and $b$ varying the range of $p_\varphi$.

Figure 5: Quantum phase-space of the scalar field in the Einstein frame, using $\omega = 410.000$ and $E_0 = 10^{16}$. The left figure is for a range $1 \leq p_\varphi \leq 10^3$, while for the right one the range is smaller $1 \leq p_\varphi \leq 10^2$.

Notice that these results are consistent with what was found in the Jordan frame, which provides further evidence that both frames are equivalent. Remember, though, that in spite of choosing specific fiducial vectors, this analysis is still qualitative, since one can always choose different wavelets and also restore the unities (we chose $c = \hbar = 1$). For our purpose, this qualitative analysis is fitting.

6 Conclusions

We presented in this work the quantisation of the Brans-Dicke Theory using the affine covariant integral method, and the cosmological scenarios arising from it. We introduced the classical Hamiltonian formalism of the BDT and the mathematical foundations of this quantisation method, in order to familiarise the reader with the concepts used later on. Our model is completed with the addition of a radiative matter component in form of a perfect fluid, introduced via the Schutz formalism, which we adopted as the clock. The affine quantisation is based on the symmetry of the phase-space of the system, and we can choose the

8Inflation may be interpreted as a "stretching" of the solutions induced by the conformal transformation by going from the Einstein frame to the Jordan frame.
Phase-space portrait for $p_b$ and $b$ varying the value of $\omega$.

Figure 6: The effect of the Brans-Dicke constant in the scalar field phase-space. Once again, we use $E_0 = 10^{16}$ and consider the range $1 \leq p_\phi \leq 10^3$. The left figure is for $\omega = 41.000$, and the right one is for $\omega = 4.100.000$.

Phase-space portrait for $p_b$ and $b$ varying the value of $E$.

Figure 7: The change in the energy of the system results in a change of scale for the solutions. The left figure was drawn with $E = 10^{13}$ and the right one with $E = 10$. Same values used as before for $p_\phi$ and $\omega$: $1 \leq p_\phi \leq 10^3$ and $\omega = 4.100.000$. 

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Phase-space portrait for $p_b$ and $b$ varying the initial value of the scalar field $\varphi_0$.

Figure 8: The top row shows the solutions for high values of $\varphi$: top-left $\varphi = 10$, and top-right: $\varphi = 10^4$. The bottom row is for low values of $\varphi$: bottom-left: $\varphi = 10^{-1}$, and bottom-right: $\varphi = 10^{-4}$. For these, we are considering $\omega = 410.000$, $E = 10^{16}$, and $1 \leq p_\varphi \leq 10^3$. 
free parameters, namely fiducial vectors, in a way to build a self-adjoint Hamiltonian operator. The quantisation of the Hamiltonian constraint results in the Wheeler-DeWitt equation, from which we obtain a Schrödinger-like equation \( (38) \), with the radiative matter providing the time parameter. One expected setback of this quantisation is that it results in a non-separable partial differential equation. We can work around this problem by changing frames, making a conformal transformation of the coordinates.

The BDT is described in what is known as the Jordan frame. A conformal change of coordinates transforms the BDT into GR with a scalar-field, the Einstein frame. The equivalence between both frames is still debatable (see e.g. [27, 28, 29, 30, 31]), thus our results may contribute to this debate. In the Einstein frame, the Schrödinger-like equation is separable, and becomes easier to deal with. We presented the classical GR with a scalar-field model corresponding to the BDT in the Einstein frame and quantised it using the affine method. We also performed a change of coordinates in the already quantised Schrödinger-like equation in the Jordan frame. Considering the freedom in the choice of the fiducial vectors, we found an equivalent equation. However, we conclude that the Hamiltonian operator in the Einstein frame is only self-adjoint if we consider different fiducial vectors while quantising the theory in each frame or if we change the domains (i.e. the measure) of the operators in the respective Hilbert space. In any case, one may argue that, because of it, there is no equivalence between the frames. However, the role of the fiducial vectors during the quantisation is precisely to open up opportunities for adjustment, since it is based on a statistical method (\(|\langle q, p | \phi \rangle|^2\) is interpreted as the probability density distribution of the function \( \phi \), see for example [21]). Thus, considering different fiducial vectors in different frames should not invalidate the equivalence between them. In spite of this, we choose to solve the equation obtained directly from the quantised one in the Jordan frame and change the measure, in order to do a qualitative analysis, since this equation has a simple solution. From it, we were able to obtain "static" orthogonal states for the wave function of the universe, and a discrete energy spectrum.

The affine quantisation method is completed with an inverse map, known as the quantum phase-space portrait or lower symbol, that transforms the quantised operator in a classical function, by means of their fiducial vectors expected values. This map provides a quantum correction for classical observables, from which we can analyse the behavior of these observables in semi-classical environments. Even if we cannot find the wave-function of the universe in the Jordan frame, we can use the quantum phase-space to compare the results with the ones from the quantum phase-space in Einstein frame. Thus, we find quantum corrections for the Hamiltonian constraint in both frames in Subsection 4.4 and compare the results in Section 5, drawing the phase-space portrait for the scale factor, to better understand the behaviour of the (volume of the) universe in earlier stages.

We obtained two types of solutions in both frames: bounces and singularities. For both types are predicted a prior universe. For the singular cases in the Jordan frame, there is an accelerated contraction, with a singular point where
the volume of the universe becomes null, followed by a decelerated inflationary era. However, if we limit the momentum of the scalar field, we can obtain only bouncing solutions. Thus, we may argue that the scalar field should have a limited velocity, since we can discard the singular possibilities. We also analysed the influence of other parameters in the solutions. In the limit $\omega \to \infty$, in which we would expect to reproduce GR (for our model, at least), bounces become more expected. It is interesting to see that an inflationary stage also appears for bounces in this frame. In the Einstein frame, however, we do not have any inflationary era, but similar conclusions can be drawn, with the exception that both singular and bouncing solutions are symmetric.

The use of affine quantisation in cosmology is a nascent subject, with the desirable feature of providing solutions without singularities for a natural range of parameters. This is in adequation with a very recent result on the quantum Belinski-Khalatnikov-Lifshitz scenario using the affine coherent states quantisation for a Bianchi IX universe [34], where they suggest that quantising GR should also lead to bouncing solutions. We intend to continue to explore this line of research in future works.

Acknowledgments

EF was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, and by the Institute of Cosmology and Gravitation. EF and CRA thank immensely Prof. Jean-Pierre Gazeau for his support.

References

[1] C. H. Brans and R. H. Dicke; Phys. Rev. 124, 925 (1961)
[2] V. Faraoni; Physics Letters A, 245 (1998)
[3] V. Faraoni; Phys. Rev. D 59, 084021 (1999)
[4] B. Chauvineau; Classical and Quantum Gravity, 20 (2003)
[5] C.M. Will; The Confrontation Between General Relativity and Experiment, Living Reviews in Relativity 17:4 (2014)
[6] A. Avilez, C. Skordis: Physical Review Letters, 113, 011101 (2014)
[7] J.E. Lidsey, D. Wands, E.J. Copeland; Physics Reports, 337 (2000)
[8] S. Pal, Physical Review D, 94, Issue 8, 084023 (2016)
[9] Y. Kerbrat, H. Kerbrat-Lunc,J. niatycki, Reports on Mathematical Physics, Volume 31, Issue 2, p. 205-215 (1992)
[10] C.J. Isham; *Canonical Quantum Gravity and the Problem of Time*, Integrable Systems, Quantum Groups, and Quantum Field Theories, Springer, Dordrecht (1993)

[11] B.S. DeWitt; Phys. Rev. **160**, 1113 (1967).

[12] C.R. Almeida; Doctorate thesis (2017)

[13] B. F. Schutz; Phys. Rev. **D2**, 2762 (1970).

[14] F. G. Alvarenga, J. C. Fabris, N. A. Lemos and G. A. Monerat; Gen. Rel. and Grav. **34**, 651 (2002)

[15] P. Pedram, S. Jalalzadeh, S.S. Gousheh; Phys. Lett. B **655** (2007)

[16] V. G. Lapchinskii and V. A. Rubakov; Theor. Math. Phys. **33**, 1076 (1977).

[17] P.A.M. Dirac, Paul A. M.; Canadian Journal of Mathematics, 2: 129148 (1950)

[18] A. Paliathanasis, M. Tsamparlis, S. Basilakos, J.D. Barrow; Phys. Rev. D **93**, 043528 (2016)

[19] C.R. Almeida, A.B. Batista, J.C. Fabris, P.V. Moniz; J. Math Phys. (2017)

[20] C.R. Almeida, H. Bergeron, J.-P. Gazeau, A.C. Scardua; Annals of Physics, 392, 206-228 (2018)

[21] J.-P. Gazeau; *Coherent States in Quantum Physics*, WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim (2009)

[22] E.W. Aslaksen, J.R. Klauder; J. Math. Phys. Vol. 9, No 2 (1968)

[23] C.J. Isham, A.C. Kakas. Classical and Quantum Gravity, 1(6), 621 (1984)

[24] M. Reed and B. Simon, *Methods of modern mathematical physics*, Volume 2, Academic Press, New York(1975).

[25] E.H. Lieb; *Inequalities*, Springer, Berlin (2003)

[26] S. T. Ali, J.P. Antoine, J.-P Gazeau; *Coherent States, Wavelets, and Their Generalizations*, Springer, Berlin (2014)

[27] M. Artymowski, Y. Ma, X. Zhang; Phys. Rev. D 88, 104010 (2013)

[28] N. Banerjee, B. Majumder; Phys.Lett. B754 (2016)

[29] C.R. Almeida, A.B. Batista, J.C. Fabris, N. Pinto-Neto; Grav. Cosmol. 24(3) (2018)

[30] A.Y. Kamenshchik, C.F. Steinwachs; Phys. Rev. D 91, 084033 (2015)

[31] S. Pandey, N. Banerjee; Eur. Phys. J. Plus, 132: 107 (2017)
[32] G. Arfken; Bessel Functions of the First Kind, $J_\nu(x)$ and Orthogonality., Mathematical Methods for Physicists, Academic Press, Orlando (1985)

[33] F.M. Paiva, C. Romero; Gen. Relat. Gravit. 25: 1305 (1993)

[34] A. Góźdz, W. Piechocki, G. Plewa, arXiv:1807.07434 (2018)