Using the WOWA operator in robust discrete optimization problems

Adam Kasperski*
Department of Operations Research, Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland, adam.kasperski@pwr.edu.pl

Paweł Zieliński
Department of Computer Science (W11/K2), Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland, pawel.zielinski@pwr.edu.pl

Abstract

In this paper a class of discrete optimization problems with uncertain costs is discussed. The uncertainty is modeled by introducing a scenario set containing a finite number of cost scenarios. A probability distribution in the scenario set is available. In order to choose a solution the weighted OWA criterion (WOWA) is applied. This criterion allows decision makers to take into account both probabilities for scenarios and the degree of pessimism/optimism. In this paper the complexity of the considered class of discrete optimization problems is described and some exact and approximation algorithms for solving it are proposed. An application to a selection problem, together with results of computational tests are shown.

Keywords: robust optimization, weighted OWA, computational complexity, approximation algorithms

1 Introduction

Most practical decision making problems arise in a risky or uncertain environment, which means that an outcome of each decision is unknown and depends on a state of the world, which may occur with a positive probability. If probabilities for the states of the world are available, then each decision leads to a lottery, i.e. a probability distribution in the set of all possible outcomes. A decision problem can be then reduced to establishing an order in the set of lotteries. According to the classic expected utility theory by von Neumann and Morgenstern [25, 16], if decision maker accepts some simple and appealing axioms, then he can assign an utility to each outcome. He can then compute an expected utility of each lottery and choose a decision which leads to a lottery with the largest expected utility.

The expected utility can be seen as a weighted average of outcomes, where the weight of each outcome is just the probability of obtaining it. Thus, in the von Neumann and Morgenstern theory, the weights are independent of the outcomes and other probabilities of the lottery. However, it has been observed in human behavior that this assumption is

*Corresponding author
often violated (see \cite{5} for a deeper discussion on this topic). Many decision makers pay more attention to unfavorable outcomes and would assign larger weights to such outcomes. In such a situation the weight of each outcome depends not only on its probability, but also on its rank in the lottery. Such weights may better reflect the pessimism/optimism of decision makers. A theory of such rank dependent, transformed probabilities was introduced by Quiggin \cite{21} (see also \cite{23}).

In many practical situations the probabilities for scenarios are not available. We then obtain a decision problem under uncertainty. In this case, decision makers may assign subjective probabilities to scenarios \cite{22} and compute the expected utility with respect to these subjective probabilities. However, determining the subjective probabilities may be not an easy task. An alternative approach is to apply some decision criteria such as the min-max, min-max regret, Hurwicz, or Laplace ones. In particular, in the Laplace criterion we apply the principle of insufficient reason and assign equal probability to each scenario. Each decision is then evaluated as the average utility of all possible outcomes. For a deeper discussion on decision making under uncertainty and description of the criteria we refer the reader to \cite{16}.

In this paper we discuss a class of discrete optimization problems, in which a finite set of feasible solutions is specified. In the deterministic case a cost of each solution is known and a decision problem consists in choosing a solution of the minimum cost. Discrete optimization problems are often represented as integer programming ones, in which the set of feasible solutions is described in compact form by a system of constraints. A class of deterministic discrete optimization problems was described, for example, in \cite{19}. In many practical situations, the cost of each solution is unknown and depends on a state of the world which may occur with a positive probability. Each state of the world induces a cost scenario, and a \textit{scenario set}, containing all possible cost scenarios is a part of the input. In this paper we assume that this scenario set contains a finite number of explicitly listed scenarios. We also assume that probabilities for the scenarios are available. Notice, that under uncertainty, the principle of insufficient reason can be applied, which assigns equal probabilities to scenarios \cite{16}. In order to choose a solution, we will apply the \textit{Weighted Ordered Weighted Averaging} (WOWA for short) operator, proposed by Torra \cite{24}. Given a solution, this operator allows us to define a rank-dependent weight for this solution under each scenario. This weight can be seen as a distorted scenario probability and the WOWA criterion is then a special case of Choquet integral with respected to distorted probabilities \cite{10}. We can evaluate each solution as a weighted average of its costs over all scenarios. The WOWA criterion contains basic criteria used in decision making under risk and uncertainty, such as the expectation (weighted mean), maximum, minimum, Hurwicz, and Laplace ones. Furthermore, if the principle of insufficient reason is applied, then WOWA becomes the OWA criterion proposed by Yager \cite{26}.

Under the discrete scenario uncertainty representation, the robust approach to compute a solution is typically used \cite{15}. In this approach we assume that decision makers are risk averse and we seek a solution which minimize the cost in a worst case. This leads to applying the min-max or min-max regret criteria to choose a solution. The traditional robust approach has, however, several drawbacks. The min-max criterion is extremely conservative and it is not difficult to show examples in which it gives unreasonable solutions \cite{16}. In particular, applying this criterion we may get a solution which is not Pareto optimal. Furthermore, the so-called drowning effect may also appear \cite{6}. If the costs under some scenario are large in comparison with the costs under the remaining scenarios, then only this bad scenario is taken into account in the process of computing a solution (information connected with the remaining scenarios is ignored). Hence, in many applications a criterion which takes into account all (or at least
a subset) of scenarios into account is required. The traditional robust approach assumes also that probabilities for scenarios are not available, which is not always true. By using the WOWA criterion we can overcome this drawback. We can use the information connected with scenario probabilities and soften the very conservative min-max criterion. Furthermore, the WOWA criterion is consistent with the theory of rank-dependent probabilities and, in consequence, can better reflect the real attitude of decision makers towards the risk. This is particularly important when decisions are not repetitious, i.e. they are implemented only once. The WOWA operator allows us to establish a link between the stochastic and robust optimization frameworks. The distorted (rank-dependent) probabilities allows us to establish a trade-off between the expected and the maximum solution costs.

In this paper we focus on the computational properties of the considered problem. Since the maximum criterion is a special case of WOWA, all the negative results known for the robust min-max problems remain valid if the WOWA criterion is used. Unfortunately, the min-max versions of all basic discrete optimization problems become NP-hard even for two scenarios. This is the case for the shortest path, minimum spanning tree, minimum assignment, minimum cut, or minimum selecting items problems [15, 2, 4]. All these aforementioned problems become strongly NP-hard and also hard to approximate when the number of scenarios is a part of the input [12, 13, 11]. Furthermore, when the OWA operator is used to choose a solution, then network problems (the shortest path, minimum spanning tree, minimum assignment, minimum cut) are not at all approximable [14]. However, for an important case of nondecreasing weights in the OWA operator, there exists an approximation algorithm with some guaranteed worst case ratio and the aim of this paper is to generalize this algorithm to more general WOWA criterion. In the existing literature, the OWA operator and more general Choquet integral have been recently applied to some multiobjective optimization problems in [9, 8, 7]. In these papers the authors propose some exact methods of solving the problems, which are based on a MIP formulation and branch and bound method.

This paper is organized as follows. In Section 2 we present the problem formulation and show a motivation for using WOWA as a criterion for choosing a solution under risk and uncertainty. In Section 3 we recall some known complexity results for the considered problem. In Section 4 we propose an approximation algorithm for solving the problem, which can be applied to a large class of discrete optimization problems. Section 5 describes a method of constructing a mixed integer programming formulation for the problem, which can be used to solve the considered problem exactly. This method will be adopted from [18]. Finally, in Section 6 we show an application of the proposed model to a selection problem. This section also contains results of computational tests, which describe the efficiency of the MIP formulation and the quality of solutions returned by the approximation algorithm designed in Section 4.

2 Problem formulation

Let \( E = \{e_1, \ldots, e_n\} \) be a finite set of elements and let \( \Phi \subseteq 2^E \) be a set of feasible solutions. In a deterministic case, each element \( e_i \in E \) has a nonnegative cost \( c_i \) and we seek a feasible solution \( X \in \Phi \), which minimizes the total cost \( F(X) = \sum_{e_i \in X} c_i \). We denote such a deterministic discrete optimization problem by \( \mathcal{P} \). This formulation encompasses a wide class of problems (see, e.g. [19, 11]). We obtain, for example, a class of network problems by identifying \( E \) with edges of a graph \( G \) and \( \Phi \) with some objects in \( G \) such as paths, spanning
trees, matchings, or cuts. Usually, $P$ is represented as an integer 0-1 programming problem whose constraints describe $\Phi$ in compact form.

Assume that the element costs are uncertain and their values depend on a state of the world which may occur with a positive probability. Each such a state of the world induces an element cost scenario (scenario for short) $c_j = (c_{j1}, \ldots, c_{jn})$. Let scenario set $\Gamma = \{c_1, \ldots, c_K\}$ contain $K$ explicitly listed scenarios. Let $p = (p_1, \ldots, p_K)$ be a vector of probabilities for scenarios, i.e. $p_j$ is a probability of the event that scenario $c_j$ will occur. The cost of a solution $X$ depends on scenario $c_j \in \Gamma$ and we will denote it by $F(X, c_j) = \sum_{e_i \in X} c_{ji}$. Choosing a solution $X$ leads to a lottery, i.e. a probability distribution $(p_1 F(X, c_1), \ldots, p_K F(X, c_K))$ over the costs of $X$ under scenarios in $\Gamma$. In order to choose the best solution, we need to evaluate each lottery. To do this, we should assign a weight to each scenario and compute the weighted average solution cost. Under the assumption that the weight of $j$th scenario depends only on $p_j$, we obtain that the weighted average is just the expected solution cost, i.e. $f(X) = E[X] = \sum_{j \in [K]} p_j F(X, c_j)$.

Figure 1: A sample Shortest Path problem with four scenarios $c_1 = (5, 6, 0, 5, 0)$, $c_2 = (1, 6, 4, 0, 0)$, $c_3 = (1, 6, 6, 0, 0)$, $c_4 = (2, 6, 6, 0, 0)$. The costs of all three paths under all scenarios are shown in the table.

| Path   | $c_1$ | $c_2$ | $c_3$ | $c_4$ | Exp. | Max |
|--------|-------|-------|-------|-------|------|-----|
| $X_1$  | 10    | 1     | 1     | 2     | 5.6  | 10  |
| $X_2$  | 5     | 5     | 7     | 8     | 5.7  | 8   |
| $X_3$  | 6     | 6     | 6     | 6     | 6     | 6   |

Consider a sample Shortest path problem depicted in Figure 1. The set of elements $E = \{e_1, \ldots, e_5\}$, contains 5 arcs of network $G$ and the set of feasible solutions consists of three paths $X_1$, $X_2$ and $X_3$ from $s$ to $t$ in $G$. There are 4 costs scenarios with the probabilities 0.5, 0.2, 0.2 and 0.1, respectively, and the costs of each path under these scenarios are shown in Figure 1. Path $X_1 = \{e_1, e_4\}$ has the smallest expected cost and thus should be chosen when the expected value is used. However, this choice may be unreasonable for some risk averse or pessimistic decision makers. Observe that the probability that the path $X_1$ will have a large cost equal to 10 equals 0.5 which may be too large and cause some decision makers to reject $X_1$. On the other hand, the path $X_3 = \{e_2, e_5\}$ has the smallest maximum cost and should be chosen when the min-max criterion is used and the probabilities of scenarios are ignored. Notice that the path $X_3$ has a deterministic cost equal to 6. However, some decision makers may feel that path $X_2 = \{e_1, e_3, e_5\}$ is better, since the probability that the cost of $X_2$ will be less than 6 equals 0.7 and the probability that $X_2$ will have a large cost, equal to 8, is only 0.1. The sample problem illustrates that there is a need of defining aggregation weights, which would depend not only on the scenario probabilities, but also on the rank positions of the costs of a solution under scenarios. For example, risk averse decision makers would assign a weight larger that 0.5 to scenario $c_1$, when solution $X_1$ is considered. In the following, we will present a criterion which allows us to define such rank-dependent weights.

In [24] Torra proposed an aggregation criterion, called the Weighted Ordered Weighted Averaging operator (WOWA for short), defined as follows. Let $v = (v_1, \ldots, v_K)$ and $p = \{
(\(p_1, \ldots, p_K\)) be two weight vectors such that \(v_j, p_j \in [0, 1]\) for each \(j \in [K]\), \(\sum_{i \in [K]} v_j = 1\), and \(\sum_{j \in [K]} p_j = 1\) (we use the notation \([K] = \{1, \ldots, K\}\)). Given a vector of reals \(a = (a_1, \ldots, a_K)\), let \(\sigma\) be a sequence of \([K]\) such that \(a_{\sigma(1)} \geq \cdots \geq a_{\sigma(K)}\). Then
\[
wowa_{v, p}(\alpha) = \sum_{j \in [K]} \omega_j a_{\sigma(j)},
\]
where
\[
\omega_j = w^*(\sum_{i \leq j} p_{\sigma(i)}) - w^*(\sum_{i < j} p_{\sigma(i)}),
\]
and \(w^*\) is a nondecreasing function that interpolates the points \((0, 0)\) and \((j/K, \sum_{i \leq j} v_i)\) for \(j \in [K]\). The function \(w^*\) is required to be a straight line when the points can be interpolated in this way. In this paper, we will assume that \(v_1 \geq v_2 \geq \cdots \geq v_K\) and the function \(w^*\) is linear between the points \((0, 0)\), \((j/K, \sum_{i \leq j} v_i)\), \(j \in [K]\). Under these assumptions, \(w^*\) is a concave and piecewise linear function. The value of \(\omega_j\) is a weight assigned to number \(a_{\sigma(j)}\).

It is not difficult to show that \(\omega_j \in [0, 1]\) for each \(j \in [K]\), and \(\sum_{j \in [K]} \omega_j = 1\).

Figure 2: Three sample functions \(w^*\) for \(K = 5\)

Figure 2 presents three sample functions \(w^*\) for \(K = 5\), with two boundary cases, where \(v^1 = (1, 0, \ldots, 0)\) and \(v^2 = (1/K, \ldots, 1/K)\). The vector \(v^2\) models the weighted mean, i.e. in this case we get \(wowa_{v^1, p}(\alpha) = \sum_{j \in [K]} p_j a_j\). The vector \(v^1\) models the weighted maximum, which in the case of uniform \(p = (1/K, \ldots, 1/K)\) is the usual maximum operator. In general, for arbitrary \(v\) and uniform \(p = (1/K, \ldots, 1/K)\), WOWA becomes the OWA operator proposed by Yager in [26], which contains the maximum, minimum, median, average and Hurwicz criteria as special cases (see, e.g. [14]). It is easy to verify that the WOWA operator is monotone, i.e. when \(a\) and \(a'\) are such that \(a_j \geq a'_j\) for all \(j \in [K]\), then \(wowa_{v, p}(\alpha) \geq wowa_{v, p}(\alpha')\). Also, since it is a convex combination of the components of \(a\), it holds \(\min_{j \in [K]} a_j \leq wowa_{v, p}(\alpha) \leq \max_{j \in [K]} a_j\). Additionally, when \(v_1 \geq \cdots \geq v_K\), then it holds \(\sum_{j \in [K]} p_j a_j \leq wowa_{v, p}(\alpha) \leq \max_{j \in [K]} a_j\).

We now apply the WOWA operator to the uncertain problem \(P\) and provide the interpretation of the vectors \(v\) and \(p\). For a given solution \(X \in \Phi\), let us define:

\[
WOWA(X) = wowa_{v, p}(F(X, c_1), \ldots, F(X, c_K)).
\]
We thus obtain an aggregated value for $X$, by applying the WOWA criterion to the vector of the costs of $X$ under scenarios in $\Gamma$. Given vectors $v$ and $p$, we consider the following optimization problem:

$$\text{Min-Wowa } P : \min_{X \in \Phi} \text{WOWA}(X).$$

The vector $p = (p_1, \ldots, p_K)$ denotes just the probabilities for scenarios. The vector $v$ models the level of risk aversion (or the degree of pessimism/optimism) of a decision maker. Namely, the more uniform is the weight distribution in $v$ the less risk averse a decision maker is. In particular, $v^2 = (1/K, \ldots, 1/K)$ means that decision maker is risk indifferent and minimizes the expected solution cost. On the other hand, the vector $v^1 = (1, 0, \ldots, 0)$ means that decision maker is extremely risk averse and minimizes the solution cost assuming that the worst scenario for the computed solution will occur. In general, vector $v$ allows us to model various attitudes of decision makers towards the risk. Moreover, nonincreasing weights are consistent with the concept of robustness. Given a solution $X$, let $\sigma$ be such that $F(X, c_{\sigma(1)}) \geq \cdots \geq F(X, c_{\sigma(K)})$. Then, the value of $\omega_j$ can be seen as a distorted, rank-dependent probability of scenario $c_{\sigma(j)}$, and WOWA($X$) is the expected solution cost with respect to the distorted probabilities. Notice that $\omega_j$ depends not only on the scenario probability but also on solution $X$.

Consider again the sample Shortest Path problem shown in Figure 1. Suppose that $v = (0.5, 0.2, 0.2, 0.1)$. The computation of the weights $\omega_1, \ldots, \omega_4$ for paths $X_1 = \{e_1, e_4\}$ and $X_2 = \{e_1, e_3, e_5\}$ is shown in Figure 3. For $X_1$ we get $F(X_1, c_1) \geq F(X_1, c_4) \geq F(X_1, c_2) \geq F(X_1, c_3)$ and $\omega = (0.8, 0.08, 0.12, 0)$. Hence WOWA($X_1$) = $0.8 \cdot 10 + 0.08 \cdot 2 + 0.12 \cdot 1 + 0 \cdot 1 = 8.28$. Observe that for $X_1$, the worst scenario $c_1$ has the weight equal to 0.8, which is greater than $p_1 = 0.5$ and the best scenario $c_3$ has the weight equal to 0, which is less than $p_3 = 0.2$. This example illustrates how the vector $v$ distorts the scenario probabilities for solution $X_1$, by paying more attention to worse scenarios. In a similar way we compute the weights for path $X_2$, obtaining $\omega = (0.2, 0.36, 0.44, 0)$ and WOWA($X_2$) = $0.2 \cdot 8 + 0.36 \cdot 7 + 0.44 \cdot 5 + 0 \cdot 5 = 6.32$. Observe that WOWA($X_2$) < WOWA($X_1$), so a risk averse decision maker would prefer solution $X_2$ over $X_1$, contrary to the case when the expected value is used as the criterion of choosing a solution.

![Figure 3: The weights $\omega_1, \ldots, \omega_4$ for paths a) $X_1 = \{e_1, e_4\}$ and b) $X_2 = \{e_1, e_3, e_5\}$.](image-url)
3 Complexity of the problem

In this section we discuss the complexity of MIN-WOWA \( \mathcal{P} \). Notice that MIN-WOWA \( \mathcal{P} \) becomes the MIN-OWA \( \mathcal{P} \) problem, discussed in [14], when \( p = (1/K, \ldots, 1/K) \). If additionally \( v = (1, 0, \ldots, 0) \), then MIN-WOWA \( \mathcal{P} \) is the MIN-MAX \( \mathcal{P} \) problem, widely discussed in the literature devoted to the robust discrete optimization. Hence all negative results known for MIN-OWA \( \mathcal{P} \) and MIN-MAX \( \mathcal{P} \) remain valid for MIN-WOWA \( \mathcal{P} \).

Unfortunately, MIN-MAX \( \mathcal{P} \) is typically NP-hard even when \( K = 2 \). In particular, this is the case for all basic network problems such as SHORTEST PATH, MINIMUM ASSIGNMENT, MINIMUM SPANNING TREE, or MINIMUM CUT (see, e.g. [2, 4, 15]). Furthermore, when \( K \) is a part of input, then for all these problems, MIN-MAX \( \mathcal{P} \) is strongly NP-hard and also hard to approximate within any constant factor [12, 11]. The problem complexity becomes worse when the maximum criterion is replaced with the more general OWA one. It has been shown in [14], that all the basic network problems are then not at all approximable. This negative result holds when the vector \( v \) is arbitrary. However, for nonincreasing weights in \( v \) the following positive result is known:

**Theorem 1** ([14]). When \( v_1 \geq v_2 \geq \cdots \geq v_K \) and \( \mathcal{P} \) is polynomially solvable, then MIN-OWA \( \mathcal{P} \) is approximable within \( v_1 K \).

In the next section we will generalize Theorem 1 to MIN-WOWA \( \mathcal{P} \). It has been shown in [14], then MIN-OWA \( \mathcal{P} \) can be solved in pseudopolynomial time and even admits a fully polynomial-time approximation scheme (FPTAS), when \( K \) is constant and some additional assumptions for \( \mathcal{P} \) are satisfied. We now show that the reasoning can be easily generalized to MIN-WOWA \( \mathcal{P} \). Note first that wowa\((v, p)\)(a) is nondecreasing with respect to each \( a_j \) in \( a \). This fact immediately implies, that there exists an optimal solution \( X \) to MIN-WOWA \( \mathcal{P} \), which is efficient (Pareto optimal), i.e. for which there is no solution \( Y \) such that \( F(Y, c_j) \leq F(X, c_j) \) for each \( j \in [K] \) with at least one strict inequality. Notice also that each optimal solution to MIN-WOWA \( \mathcal{P} \) must be efficient when all components of \( p \) and \( v \) are positive. For some problems, for example when \( \mathcal{P} \) is SHORTEST PATH or MINIMUM SPANNING TREE, such an optimal efficient solution can be found in pseudopolynomial time, provided that \( K \) is constant [3]. Hence in this case, for constant \( K \), MIN-WOWA \( \mathcal{P} \) can be solved in pseudopolynomial time.

In order to construct an FPTAS, we need a definition of an exact problem associated with \( \mathcal{P} \) and scenario set \( \Gamma \) (see [17]). Given a vector \((b_1, \ldots, b_K)\), we ask if there is a solution \( X \in \Phi \) such that \( F(X, c_j) = b_j \) for all \( j \in [K] \). Let us fix \( \epsilon > 0 \) and let \( P_\epsilon(\Phi) \) be the set of solutions such that for all \( X \in \Phi \), there is \( Y \in P_\epsilon(\Phi) \) such that \( F(Y, c_j) \leq (1 + \epsilon) F(X, c_j) \) for all \( j \in [K] \). Basing on the results obtained in [20], it was proven in [17] that if the exact problem associated with \( \mathcal{P} \) can be solved in pseudopolynomial time, then for any \( \epsilon > 0 \), the set \( P_\epsilon(\Phi) \) can be determined in time polynomial in the input size and \( 1/\epsilon \). This implies the following result (the reasoning is the same as in [14]):

**Theorem 2.** If the exact problem associated with \( \mathcal{P} \) can be solved in pseudopolynomial time, then MIN-WOWA \( \mathcal{P} \) admits an FPTAS.

**Proof.** Let us fix \( \epsilon > 0 \) and let \( Y \) be a solution of the minimum value of WOWA(Y) among all the solutions in \( P_\epsilon(\Phi) \). From the results obtained in [17, 20], it follows that we can find \( Y \) in time polynomial in the input size and \( 1/\epsilon \). Assume that \( X^* \) is an optimal solution to MIN-WOWA \( \mathcal{P} \). Define vector \( \bar{b}^* = ((1+\epsilon)F(X^*, c_1), \ldots, (1+\epsilon)F(X^*, c_K)) \). By the definition
of \( Y \) we get \( F(Y, c_j) \leq (1 + \epsilon)F(X^*, c_j) \) for all \( j \in [K] \). The monotonicity of WOWA implies \( \text{WOWA}(Y) \leq \text{wowa}_{(v, p)}(b^*) = (1 + \epsilon)\text{WOWA}(X^*) \). We have thus obtained an FPTAS for \( \text{MIN-WOWA} \mathcal{P} \). □

It turns out that the exact problem associated with \( \mathcal{P} \) can be solved in pseudopolynomial time for some particular problems \( \mathcal{P} \), provided that the number of scenarios \( K \) is constant. This is the case for shortest path, minimum spanning tree and some other problems described, for example, in [3]. However, it is worth pointing out that the running time of the FPTAS’s obtained is exponential in \( K \), so their practical applicability is limited to very small values of \( K \). In the next section we will construct an approximation algorithm, which can be applied for larger values of \( K \).

4 Approximation algorithm

In this section we construct an approximation algorithm for \( \text{MIN-WOWA} \mathcal{P} \) under the assumptions that \( v_1 \geq v_2 \geq \cdots \geq v_K \) and \( \mathcal{P} \) is polynomially solvable. We will also assume that \( p_j > 0 \) for each \( j \in [K] \). When \( p_j = 0 \) for some \( j \in [K] \), then we can remove scenario \( c_j \) from \( \Gamma \) without changing the problem. We first prove some properties of the WOWA operator. Let \( \mathbf{a} = (a_1, \ldots, a_K) \) be a vector of nonnegative numbers. Let \( \pi \) be any sequence of \([K]\). Let us define

\[
\text{f}_{\pi}(\mathbf{a}) = \sum_{j \in [K]} \omega_j a_{\pi(j)},
\]

where \( \omega_j = w^*(\sum_{i \leq j} p_{\pi(i)}) - w^*(\sum_{i < j} p_{\pi(i)}) \) and \( w^* \) is the piecewise linear function induced by the vector of weights \( v \) (as in the definition of the WWA operator). Observe that \( \text{f}_{\pi}(\mathbf{a}) = \text{wowa}_{(v, p)}(\mathbf{a}) \) when the sequence \( \pi \) is such that \( a_{\pi(1)} \geq \cdots \geq a_{\pi(K)} \). The following lemma expresses the intuitive fact that \( \text{f}_{\pi}(\mathbf{a}) \) is a lower bound on \( \text{wowa}_{(v, p)}(\mathbf{a}) \).

Lemma 1. For any vector \( \mathbf{a} = (a_1, \ldots, a_K) \) and any sequence \( \pi \) of \([K]\) it holds \( \text{wowa}_{(v, p)}(\mathbf{a}) \geq \text{f}_{\pi}(\mathbf{a}) \).

Proof. Assume w.l.o.g. that \( a_1 \geq a_2 \geq \cdots \geq a_K \). Consider two neighbor elements \( a_{\pi(i)} \) and \( a_{\pi(i+1)} \) in \( \pi \) such that \( a_{\pi(i)} \leq a_{\pi(i+1)} \). Let us interchange \( a_{\pi(i)} \) and \( a_{\pi(i+1)} \) in \( \pi \) and denote the resulting sequence by \( \pi' \). We will show that \( \text{f}_{\pi'}(\mathbf{a}) \geq \text{f}_{\pi}(\mathbf{a}) \) and the equality holds when \( a_{\pi(i)} = a_{\pi(i+1)} \). This will complete the proof since we can transform \( \pi \) into \( \sigma = (1, \ldots, K) \) by using a finite number of such element interchanges without decreasing the value of \( \text{f}_{\pi} \) and \( \text{f}_{\pi}(\mathbf{a}) = \text{wowa}_{(v, p)}(\mathbf{a}) \). It holds \( \text{f}_{\pi'}(\mathbf{a}) - \text{f}_{\pi}(\mathbf{a}) = \omega'_{i} a_{\pi(i+1)} + \omega'_{i+1} a_{\pi(i)} - \omega_i a_{\pi(i)} - \omega_{i+1} a_{\pi(i+1)} = (\omega'_{i+1} - \omega_i) a_{\pi(i)} - (\omega_{i+1} - \omega'_{i}) a_{\pi(i+1)} \). It holds \( \omega'_{i} + \omega'_{i+1} = \omega_i + \omega_{i+1} \) (see Figure 4b), so \( \omega_{i+1} - \omega_i = \omega_{i+1} - \omega'_{i} = \alpha \). Hence \( \text{f}_{\pi'}(\mathbf{a}) - \text{f}_{\pi}(\mathbf{a}) = \alpha (a_{\pi(i)} - a_{\pi(i+1)}) \). Since \( w^* \) is concave, we have \( \omega_{i+1}/p_{\pi(i+1)} \leq \omega'_{i}/p_{\pi(i+1)} \), which implies \( \alpha \leq 0 \) since \( p_{\pi(i+1)} > 0 \). Hence \( \text{f}_{\pi'}(\mathbf{a}) \geq \text{f}_{\pi}(\mathbf{a}) \). □

Lemma 2. For any vector \( \mathbf{a} = (a_1, \ldots, a_K) \) it holds \( \text{wowa}_{(v, p)}(\mathbf{a}) \leq v_1 K \sum_{j \in [K]} p_j a_j \).

Proof. Since \( w^* \) is concave and piecewise linear, it holds \( \frac{w_i}{p_{\sigma(j)}} \leq \frac{w_i}{1/K} = v_1 K \) for each \( j \in [K] \) (see Figure 4b). In consequence, \( \text{wowa}_{(v, p)}(\mathbf{a}) = \sum_{j \in [K]} \omega_j a_{\sigma(j)} \leq \sum_{j \in [K]} v_1 K p_{\sigma(j)} a_{\sigma(j)} = v_1 K \sum_{j \in [K]} p_j a_j \). □
Let $\hat{c}_i = \text{wowa}(v_i, \ldots, c_{Ki})$ be the aggregated cost of element $e_i \in E$ over all scenarios. Let $\hat{X}$ be an optimal solution for the costs $\hat{c}_i$, $i \in [n]$. The following theorem holds:

**Theorem 3.** For any $X$, it holds $\text{WOWA}(\hat{X}) \leq K v_1 \cdot \text{WOWA}(X)$.

**Proof.** Let $\sigma$ be a sequence of $[K]$ such that $F(\hat{X}, c_{\sigma(1)}) \geq \cdots \geq F(\hat{X}, c_{\sigma(K)})$ and $\omega_j = w^*(\sum_{i \leq j} p_{\sigma(i)}) - w^*(\sum_{i < j} p_{\sigma(i)})$. The definition of the WOWA operator and Lemma 1 imply the following inequality:

$$\text{WOWA}(\hat{X}) = \sum_{j \in [K]} \omega_j \sum_{e_i \in \hat{X}} c_{\sigma(j)i} \leq \sum_{e_i \in \hat{X}} \hat{c}_i.$$

Using Lemma 2, we get $\hat{c}_i \leq v_1 K \sum_{j \in [K]} p_j c_{ji}$. Hence, from the definition of $\hat{X}$, we obtain

$$\sum_{e_i \in \hat{X}} \hat{c}_i \leq \sum_{e_i \in \hat{X}} \hat{c}_i \leq K v_1 \sum_{e_i \in \hat{X}} \sum_{j \in [K]} p_j c_{ji}.$$

Since $v_1 \geq \cdots \geq v_K$, it follows that

$$\text{WOWA}(X) \geq \sum_{j \in [K]} p_j F(X, c_j) = \sum_{j \in [K]} p_j \sum_{e_i \in X} c_{ji} = \sum_{e_i \in X} \sum_{j \in [K]} p_j c_{ji}.$$

Combining (1), (2) and (3) completes the proof.

**Corollary 1.** If $v_1 \geq \cdots \geq v_K$ and $\mathcal{P}$ is polynomially solvable, then WOWA $\mathcal{P}$ is approximable within $v_1 K$.

The bound obtained in Corollary 1 is tight and the worst case instance for the approximation algorithm is the same as the one shown in [14]. Observe that the approximation ratio depends on the weight distribution in $v$. The more uniform is the weight distribution the smaller is the approximation ratio. We get the largest approximation ratio equal to $K$, when WOWA is the weighted maximum. On the other hand, when $v_1 = 1/K$, i.e. when WOWA is the expected value, then we get an exact polynomial time algorithm for the problem.

In many cases the deterministic problem $\mathcal{P}$ is NP-hard, but is approximable within a factor of $\gamma$. In this case the following result can be established.
Theorem 4. If $v_1 \geq \cdots \geq v_K$ and $\mathcal{P}$ is approximable within $\gamma$, then Min-Wowa $\mathcal{P}$ is approximable within $\gamma v_1 K$.

Proof. The proof is similar to the proof of Theorem 3. In order to get a solution for costs $\hat{c}_i$ a $\gamma$-approximation algorithm is applied. It is then enough to modify inequality (2), so that
$$\sum_{e_i \in \hat{X}} \hat{c}_i \leq \gamma \sum_{e_i \in X} \hat{c}_i \leq \gamma K v_1 \sum_{e_i \in X} \sum_{j \in [K]} p_j c_{ji}.$$ The rest of the proof is the same. $\square$

5 Mixed integer programming formulation

In this section we design a mixed integer programming (MIP) formulation for Min-Wowa $\mathcal{P}$. We will use the idea proposed in [18]. Let us associate a binary variable $x_i \in \{0,1\}$ with each element $e_i \in E$. Let $\chi(\Phi) \subseteq \{0,1\}^n$ be the set of all characteristic vectors of $\Phi$. Each vector $\mathbf{x} = (x_1, \ldots, x_n) \in \chi(\Phi)$ defines a feasible solution $X$ such that $e_i \in X$ if and only if $x_i = 1$. We will assume that $\chi(\Phi)$ can be described by a set of linear constraints involving variables $x_1, \ldots, x_n$. From now on we will identify a feasible solution $X \in \Phi$ with the corresponding characteristic vector $\mathbf{x} \in \chi(\Phi)$. Let us fix a feasible solution $\mathbf{x} \in \chi(\Phi)$. Let $\sigma$ be such that $F(\mathbf{x}, \mathbf{c}_{\sigma(1)}) \geq \cdots \geq F(\mathbf{x}, \mathbf{c}_{\sigma(K)})$. Define vector $\mathbf{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_K)$ such that $\alpha_i = \sum_{j \leq i} p_{\sigma(j)}$, and $\alpha_0 = 0$. Let us define $h_{\mathbf{x}}(\theta) = F(\mathbf{x}, \mathbf{c}_{\sigma(i)})$ for $\alpha_{i-1} < \theta \leq \alpha_i$, $i \in [K]$, $\theta \in (0,1]$. The following equality holds [18]:

$$\text{WOWA}(\mathbf{x}) = K \sum_{j \in [K]} v_j \int_{\alpha_{j-1}}^{\alpha_j} h_{\mathbf{x}}(\theta) d\theta. \quad (4)$$

Figure 5: Illustration of formula (4) for the cost vector $(8, 3, 2, 6)$ with $\mathbf{p} = (0.5, 0.2, 0.2, 0.1)$.

Equality (4) has the following interpretation (see also [18]). The value of $K \int_{\alpha_{j-1}}^{\alpha_j} h_{\mathbf{x}}(\theta) d\theta$ is the average within the $j$th portion of $1/K$ largest solution costs. Then $\text{WOWA}(\mathbf{x})$ can be seen as the value of the OWA operator applied to these averages. Formula (4) is illustrated in Figure 5. The quantity $\int_{\alpha_{j-1}}^{\alpha_j} h_{\mathbf{x}}(\theta) d\theta$ is the area of the $j$th polygon weighted by $v_j$. When $p_j = 1/K$, $j \in [K]$, then $K \int_{\alpha_{j-1}}^{\alpha_j} h_{\mathbf{x}}(\theta) d\theta = F(\mathbf{x}, \mathbf{c}_{\sigma(j)})$ and $\text{WOWA}(\mathbf{x})$ becomes the OWA.
aggregation operator. Let us rewrite (4) as follows:

\[
\text{WOWA}(\mathbf{x}) = K \sum_{j \in [K]} v_j \left( \int_0^j h_\mathbf{x}(\theta) d\theta - \int_0^1 h_\mathbf{x}(\theta) d\theta \right) = K \left( \sum_{j=1}^K v_j \int_0^j h_\mathbf{x}(\theta) d\theta - \sum_{j=0}^{K-1} v_{j+1} \int_0^1 h_\mathbf{x}(\theta) d\theta \right).
\]

Define \( v_{K+1} = 0 \). Since \( \int_0^1 h_\mathbf{x}(\theta) d\theta = 0 \), it holds

\[
\text{WOWA}(\mathbf{x}) = K \left( \sum_{j=1}^K v_j \int_0^j h_\mathbf{x}(\theta) d\theta - \sum_{j=1}^K v_{j+1} \int_0^1 h_\mathbf{x}(\theta) d\theta \right),
\]

and we get the following equality:

\[
\text{WOWA}(\mathbf{x}) = K \sum_{j \in [K]} (v_j - v_{j+1}) \int_0^j h_\mathbf{x}(\theta) d\theta. \tag{5}
\]

Let us denote \( L_j(\mathbf{x}) = \int_0^j h_\mathbf{x}(\theta) d\theta \) and \( v_j' = v_j - v_{j+1}, j \in [K] \). Observe that \( v_j' \geq 0 \) for all \( j \in [K+1] \), by the assumption that \( v_1 \geq v_2 \geq \cdots \geq v_K \). We are now ready to design a MIP formulation. In order to do this we adopt the idea from [18]. Observe first that the value of \( L_j(\mathbf{x}) \) for a fixed \( \mathbf{x} \) can be computed by solving the following linear programming problem:

\[
\max \sum_{k \in [K]} z_k F(\mathbf{x}, \mathbf{c}_k) \quad \text{subject to} \quad \sum_{k \in [K]} z_k = \frac{j}{K}, \quad 0 \leq z_k \leq p_k \quad k \in [K] \tag{6}
\]

Indeed, \( L_j(\mathbf{x}) \) can be computed in a greedy way. Let \( \sigma \) be such that \( F(\mathbf{x}, \mathbf{c}_{\sigma(1)}) \geq \cdots \geq F(\mathbf{x}, \mathbf{c}_{\sigma(K)}) \).

We first allocate to the interval \([0, j/K]\) the largest possible portion of \( p_{\sigma(1)} \), then the largest possible portion of \( p_{\sigma(2)} \) etc., until \([0, j/K]\) is completely filled. This is equivalent to solving (6). The dual to (6) for a fixed \( \mathbf{x} \) and \( j \) takes the following form:

\[
\min \frac{j}{K} \beta + \sum_{i \in [K]} p_i \alpha_{ij} \quad \beta_j + \alpha_{ij} \geq F(\mathbf{x}, \mathbf{c}_i) \quad i \in [K] \quad \alpha_{ij} \geq 0 \quad i \in [K] \tag{7}
\]

The strong duality theorem implies that \( L_j(\mathbf{x}) \) equals the optimal objective value of (7). Using (5) and (7) we get that Min-WOWA \( \mathcal{P} \) is equivalent to the following problem:

\[
\min \quad K \cdot \sum_{j \in [K]} v_j' \left( \frac{j}{K} \beta_j + \sum_{i \in [K]} p_i \alpha_{ij} \right) \quad \beta_j + \alpha_{ij} \geq F(\mathbf{x}, \mathbf{c}_i) \quad i \in [K], j \in [K] \quad \alpha_{ij} \geq 0 \quad i \in [K], j \in [K] \quad (x_1, \ldots, x_n) \in \chi(\Phi)
\]

We obtain a MIP formulation by substituting \( F(\mathbf{x}, \mathbf{c}_i) = \sum_{k \in [n]} x_i c_{ik} \) and replacing the expression \((x_1, \ldots, x_n) \in \chi(\Phi)\) with a system of linear constraints involving \( x_1, \ldots, x_n \). In the next section we will apply the MIP formulation to a sample problem.
6 Application to selection problem

In this section we apply the MIP formulation and the approximation algorithm designed in Section 4 to the following SELECTION problem. Assume that $E$ is a set of items and we wish to choose exactly $p$ of them to minimize the total cost. Hence $\Phi = \{X \subseteq E : |X| = p\}$. The set of characteristic vectors $\chi(\Phi)$ can be described by one constraint of the form $x_1 + \cdots + x_n = p$, where $x_1, \ldots, x_n$ are binary variables associated with the items in $E$. The SELECTION problem has been recently discussed in a number of papers. Its min-max version has been proven to be NP-hard for two scenarios [3], strongly NP-hard and hard to approximate within any constant factor when the number of scenarios is a part of the input [11]. Hence the same negative results hold for MIN-WOWA $P$. The aim of this section is to verify the following questions:

1. How efficient is the MIP formulation, i.e. how the computation time required to solve the MIP model depends on $n$, $K$ and the weight distribution in $v$?

2. What is the quality of the approximation algorithm designed in Section 4?

In order to answer these questions we have generated problem instances in the following way. The number of items $n \in \{120, 160, 200, 240, 280\}$, the number of scenarios $K \in \{2, \ldots, 20\}$, $p = 0.25n$, i.e. we assume that exactly 25% of the items must be chosen. Under each scenario the cost of item $e_i$ is an integer chosen randomly with uniform distribution in the set $\{0, \ldots, 100\}$. In order to fix the weights $v_1, \ldots, v_K$, we have used the generating function $g_\alpha(z) = \frac{1}{1-\alpha}(1-\alpha^z)$ for $\alpha \in \{0.1, 0.01, 0.001, 0.0001\}$. Notice that $g_\alpha(z)$ is concave and is such that $g_\alpha(0) = 0$, $g_\alpha(1) = 1$. Given $\alpha$ and $K$, we fix $v_j = g_\alpha(j/K) - g_\alpha((j-1)/K)$ for $j \in [K]$ (see Figure 6).

![Figure 6: The generating function $g_\alpha(z)$ for $\alpha \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ and the weights $v_1, \ldots, v_4$ for $K = 4$ and $\alpha = 10^{-1}$.](image)

The value of $\alpha$ expresses an attitude of decision maker towards a risk. The smaller is the value of $\alpha$ the more risk averse decision maker is (the less uniform is the weight.
Table 1: The minimum, average and maximum computation times in seconds, reported for \( n = 120 \).

| \( K \) | \( \alpha \) | \( \text{min} \) | \( \text{av.} \) | \( \text{max} \) | \( \text{min} \) | \( \text{av.} \) | \( \text{max} \) | \( \text{min} \) | \( \text{av.} \) | \( \text{max} \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 2  | 0.1 | 0 | 0.1 | 1 | 0 | 0 | 0 | 0 | 0.1 | 1 |
| 4  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6  | 0.1 | 1 | 0 | 0.1 | 1 | 0 | 0.2 | 1 | 0 | 0.3 |
| 8  | 0 | 0 | 0 | 0 | 0 | 0 | 1.2 | 4 | 0 | 1.2 |
| 10 | 0.1 | 1 | 0 | 0.6 | 2 | 0 | 1.3 | 4 | 0 | 1.8 |
| 12 | 0.1 | 1 | 1 | 1.1 | 2 | 1 | 4.1 | 12 | 3 | 7.7 |
| 14 | 0.2 | 1 | 0 | 5.5 | 18 | 1 | 26.7 | 157 | 1 | 35.4 |
| 16 | 0.4 | 1 | 2 | 12.7 | 43 | 5 | 38.6 | 128 | 4 | 61.2 |
| 18 | 0.4 | 1 | 1 | 15.9 | 69 | 14 | 57.5 | 192 | 20 | 95.8 |
| 20 | 0.5 | 1 | 7 | 26.8 | 46 | 52 | 99.5 | 209 | 59 | 159.8 |

Table 2: The minimum, average and maximum percent deviation from optimum, reported for \( n = 120 \).

| \( K \) | \( \alpha \) | \( \text{min} \) | \( \text{av.} \) | \( \text{max} \) | \( \text{min} \) | \( \text{av.} \) | \( \text{max} \) | \( \text{min} \) | \( \text{av.} \) | \( \text{max} \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 2  | 0.1 | 3.2 | 9.5 | 0.5 | 5.4 | 16.2 | 0 | 6.0 | 17.7 | 0 | 5.2 |
| 4  | 3.3 | 4.9 | 6.5 | 7.2 | 9.2 | 12.7 | 7.4 | 11.0 | 15.1 | 6.9 | 12.1 |
| 6  | 2.2 | 4.6 | 8.8 | 6.4 | 11.0 | 15.6 | 7.6 | 14.8 | 21.9 | 12.1 | 17.2 |
| 8  | 3.5 | 5.3 | 8.6 | 10.0 | 12.9 | 18.1 | 13.7 | 16.9 | 23.8 | 14.5 | 18.7 |
| 10 | 3.7 | 5.6 | 9.0 | 8.4 | 10.2 | 12.8 | 10.7 | 13.8 | 17.2 | 12.5 | 17.5 |
| 12 | 3.2 | 4.5 | 7.5 | 8.8 | 10.3 | 13.4 | 12.0 | 14.1 | 17.7 | 14.1 | 16.9 |
| 14 | 3.2 | 4.4 | 5.8 | 6.3 | 9.1 | 13.5 | 9.1 | 12.9 | 17.1 | 12.6 | 15.9 |
| 16 | 1.3 | 3.6 | 4.7 | 4.8 | 9.5 | 12.3 | 7.0 | 12.9 | 16.7 | 9.5 | 15.7 |
| 18 | 2.3 | 3.9 | 4.9 | 5.0 | 9.1 | 11.8 | 7.8 | 11.5 | 14.6 | 11.0 | 12.2 |
| 20 | 2.6 | 3.7 | 4.8 | 6.3 | 8.9 | 11.4 | 9.7 | 12.6 | 17.7 | 11.1 | 15.2 |

distribution \( v_1, \ldots, v_K \)). For each combination of \( n, K \) and \( \alpha \) we have generated 10 random instances. For each instance the CPLEX 12.5 solver with standard settings was used to solve the corresponding MIP formulation. We have fixed the time limit to 3600 seconds. The solver was executed on a computer equipped with 2.5 GHz processor with 8 GB RAM. For each instance the approximation algorithm, designed in Section \[ \text{Section } 4 \] was also applied. For each combination of \( n, K \), and \( \alpha \), we have computed the minimum, average and maximum computation time required to solve the MIP formulation. We also computed the minimum, average and maximum deviation from optimum, reported for the approximation algorithm. The results obtained are shown in Tables 1-10.

Analyzing the results we can see that the computation time required to solve the model strongly depends on \( \alpha \). For \( \alpha = 0.1 \), i.e. when the WOWA operator is close to the expected value, the solver was able to solve all instances within the assumed time 3600 seconds, while for \( \alpha = 0.0001 \), it was unable to solve some instances even for \( n = 160 \). The computation time also quickly grows with the number of scenarios. All the tested instances can be solved within 3600 seconds for \( K \leq 10 \). For \( K = 12 \) the solver failed to solve one instance for \( n = 240 \) and...
Table 3: The minimum, average and maximum computation times in seconds, reported for \( n = 160 \). The number in brackets denotes the number of instances solved within 3600 seconds. The symbol "-" means that the computation time was greater than 3600 sec.

| \( \alpha \) | 0.1 | 0.01 | 0.001 | 0.0001 |
|-------------|-----|-----|------|--------|
| \( K \) | min av. max | min av. max | min av. max | min av. max |
| 2 | 0 0 0 | 0 0 0 | 0 0.1 1 | 0 0.2 1 |
| 4 | 0 0 0 | 0 0.2 1 | 0 0.1 1 | 0 0 0 |
| 6 | 0 0 0 | 0 0.2 1 | 0 0.3 1 | 0 0.6 3 |
| 8 | 0 0 0 | 0 0.8 2 | 0 2 5 | 1 3.1 9 |
| 10 | 0 0.2 2 | 0 2.1 8 | 0 5.8 19 | 1 8.2 24 |
| 12 | 0 0.5 3 | 2 15.9 96 | 4 56.2 334 | 6 98.5 552 |
| 14 | 0 0.9 4 | 2 20.5 104 | 16 42.8 102 | 26 79.8 163 |
| 16 | 0 1.3 5 | 4 48.6 193 | 30 152.2 409 | 77 291.6 631 |
| 18 | 0 1.3 3 | 3 121.6 459 | 31 220 491 | 20 213.8 424 |
| 20 | 0 2.1 9 | 11 340.9 2156 | 73 (9) - | 79 (7) - |

Table 4: The minimum, average and maximum percent deviation from optimum, reported for \( n = 160 \). The symbol "-" means that the optima for some instances were not obtained.

| \( \alpha \) | 0.1 | 0.01 | 0.001 | 0.0001 |
|-------------|-----|-----|------|--------|
| \( K \) | min av. max | min av. max | min av. max | min av. max |
| 2 | 0.1 2.89 6.2 | 1.1 5.71 11.9 | 1.1 7.69 13.4 | 1.3 7.82 14.0 |
| 4 | 3.5 5.63 7.2 | 9.6 12.43 17.0 | 11.6 15.23 21.4 | 12.3 16.12 21.3 |
| 6 | 4.2 6.21 8.3 | 7.9 13.31 17.1 | 9.3 16.94 22.5 | 11.5 19.66 25.4 |
| 8 | 4.1 6.10 7.3 | 9.6 11.81 16.6 | 12.7 16.38 21.3 | 11.7 17.46 25.7 |
| 10 | 3.6 5.05 7.5 | 7.8 10.70 16.3 | 11.7 14.37 21.7 | 13.3 16.20 23.3 |
| 12 | 4.3 5.05 6.0 | 8.0 10.85 13.2 | 11.0 14.02 17.1 | 13.6 15.95 19.5 |
| 14 | 2.7 4.23 6.4 | 5.5 9.36 15.9 | 9.4 13.49 17.5 | 10.0 15.91 23.3 |
| 16 | 2.0 3.79 5.4 | 6.0 9.18 13.7 | 8.4 12.69 17.6 | 10.4 14.82 17.7 |
| 18 | 2.1 3.33 4.4 | 6.8 8.24 9.5 | 7.7 12.32 14.9 | 10.4 14.65 18.9 |
| 20 | 2.1 3.31 4.5 | 6.0 8.21 10.2 | 8.5 - 14.0 | 11.9 - 15.9 |
Table 5: The minimum, average and maximum computation times in seconds, reported for \( n = 200 \). The number in brackets denotes the number of instances solved within 3600 seconds. The symbol "-" means that the computation time was greater than 3600 sec.

| \( K \) | \( 0.1 \) | \( 0.01 \) | \( 0.001 \) | \( 0.0001 \) |
|--------|----------|-----------|-----------|------------|
|        | min | av. | max | min | av. | max | min | av. | max |
| 2      | 0   | 0.1 | 0   | 0   | 0   | 0   | 0   | 0.2 | 1   |
| 4      | 0   | 0.1 | 1   | 0   | 0.1 | 1   | 0   | 0.2 | 1   |
| 6      | 0   | 0.1 | 1   | 0   | 0.2 | 1   | 0   | 0.5 | 2   |
| 8      | 0   | 0.2 | 1   | 0   | 1.7 | 5   | 0   | 9.3 | 57  |
| 10     | 0   | 0.3 | 1   | 0   | 8.7 | 23  | 6   | 23.4| 43  |
| 12     | 0   | 2   | 10  | 3   | 22.5| 38  | 0   | 113 | 227 |
| 14     | 0   | 1.4 | 7   | 1   | 67.3| 293 | 7   | 471.8| 2720 |
| 16     | 0   | 4.1 | 22  | 36  | 254.4| 692 | 305 | (9) | -   |
| 18     | 2   | 7.5 | 31  | 13  | (9) | -   | 799 | (6) | -   |
| 20     | 1   | 11.1| 59  | 76  | (8) | -   | 594 | (1) | -   |

Table 6: The minimum, average and maximum percent deviation from optimum, reported for \( n = 200 \). The symbol "-" means that the optima for some instances were not obtained.

| \( K \) | \( 0.1 \) | \( 0.01 \) | \( 0.001 \) | \( 0.0001 \) |
|--------|-----------|-----------|-----------|------------|
|        | min | av. | max | min | av. | max | min | av. | max |
| 2      | 0   | 2.75| 5   | 0   | 5.01| 10.6| 0   | 5.78| 12.2| 0 | 6.17| 12.5|
| 4      | 0.9 | 6.55| 10.1| 1.6 | 13.11| 22.1| 4.3 | 16.99| 25.4| 7 | 18.19| 26.4|
| 6      | 2.4 | 5.02| 7.2 | 6   | 11.21| 16.9| 7.7 | 15.38| 22.7| 11 | 16.96| 26.9|
| 8      | 2.7 | 4.98| 6.7 | 7.6 | 12.05| 16.3| 10.1| 16.01| 25.1| 12.4| 19.22| 28.2|
| 10     | 3.3 | 5.01| 6.8 | 7.6 | 9.97 | 12.8| 7.7 | 13.07| 16.2| 10.8| 15.53| 19.5|
| 12     | 3.6 | 5.04| 6.5 | 8.4 | 10.96| 14.3| 9.6 | 14.18| 17.9| 10.4| 16.44| 20.3|
| 14     | 2.7 | 3.76| 5.1 | 6.6 | 8.86 | 11.4| 8.5 | 12.51| 15.3| 10.1 | - | 21.3 |
| 16     | 2.5 | 3.67| 5.3 | 6.4 | 9.24 | 12.6| 11.8 | - | 17.7 | 11.6 | - | 18.6 |
| 18     | 2.3 | 3.87| 5.1 | 6   | -    | 10.9| 9.4 | - | 13.3 | 12.2 | - | 16   |
| 20     | 2.4 | 3.55| 4.5 | 5.7 | -    | 11  | 12.3 | - | 12.3 | - | - | -   |
Table 7: The minimum, average and maximum computation times in seconds, reported for \( n = 240 \). The number in brackets denotes the number of instances solved within 3600 seconds. The symbol ", - " means that the computation time was greater than 3600 sec.

|       | 0.1 |       | 0.01 |       | 0.001 |       | 0.0001 |
|-------|-----|-------|------|-------|-------|-------|--------|
|       | min | av.  | max  | min  | av.  | max  | min   | av.   | max   |
| 2     | 0   | 0.1  | 1    | 0    | 0.1  | 1    | 0     | 0.2   | 1     |
| 4     | 0   | 0.1  | 1    | 0    | 0.1  | 1    | 0     | 0.3   | 2     |
| 6     | 0   | 0.2  | 1    | 0    | 1.2  | 7    | 0     | 3.6   | 24    |
| 8     | 0   | 0.4  | 1    | 0    | 2.6  | 10   | 1.0   | 10.7  | 28.0  |
| 10    | 0   | 1.4  | 5    | 0    | 41.2 | 221  | 6     | 707.8 | 1321  |
| 12    | 0   | 1.2  | 4    | 2    | 78.1 | 286  | 11    | 423.6 | 1530  |
| 14    | 0   | 10.7 | 44   | 36   | (9)  | -    | 245   | (6)   | -     |
| 16    | 1   | 2.2  | 4    | 17   | 666.8| 2216 | 153   | (5)   | -     |
| 18    | 0   | 56.1 | 367  | 13   | (7)  | -    | -     | (0)   | -     |
| 20    | 3   | 16   | 54   | -    | (0)  | -    | -     | (0)   | -     |

Table 8: The minimum, average and maximum percent deviation from optimum reported for \( n = 240 \). The symbol ", - " means that the optima for some instances were not obtained.

|       | 0.1 |       | 0.01 |       | 0.001 |       | 0.0001 |
|-------|-----|-------|------|-------|-------|-------|--------|
|       | min | av.  | max  | min  | av.  | max  | min   | av.   | max   |
| 2     | 0   | 3.11 | 7.3  | 0    | 5.18 | 11.3 | 0     | 6.47  | 12.8  |
| 4     | 3.6 | 5.14 | 8.6  | 6.9  | 10.27| 13.5 | 7.0   | 13.94 | 21.0  |
| 6     | 4.5 | 5.76 | 6.6  | 7.2  | 11.74| 15.3 | 9.7   | 16.18 | 24.6  |
| 8     | 3.0 | 5.3  | 7.9  | 9.3  | 11.98| 17.2 | 12.3  | 15.64 | 20.5  |
| 10    | 3.8 | 5.21 | 6.7  | 9.5  | 11.22| 12.8 | 13.0  | 15.12 | 18.0  |
| 12    | 2.4 | 4.09 | 6.3  | 7.1  | 9.74 | 12.9 | 10.3  | 13.69 | 17.6  |
| 14    | 2.8 | 4.54 | 6.5  | 7.8  | -    | 12.2 | 11.8  | -    | 15.0  |
| 16    | 2.0 | 3.74 | 5.4  | 5.2  | 8.51 | 10.8 | 8.9   | -    | 14.5  |
| 18    | 2.6 | 3.66 | 4.6  | 6.7  | -    | 11.7 | -     | -    | -     |
| 20    | 2.9 | 3.89 | 4.9  | -    | -    | -    | -     | -    | -     |
Table 9: The minimum, average and maximum computation times in seconds, reported for \( n = 280 \). The number in brackets denotes the number of instances solved within 3600 seconds. The symbol "-" means that the computation time was greater than 3600 sec.

|         | 0.1 | 0.01 | 0.001 | 0.0001 |
|---------|-----|------|-------|--------|
|         | min | av.  | max  | min   | av.  | max  | min   | av.  | max  |
| 2       | 0   | 0    | 0    | 0.1   | 1    |      | 0     | 0    | 0    |
| 4       | 0   | 0    | 0    | 0.2   | 1    |      | 0.4   | 2    | 0.6  |
| 6       | 0   | 0.2  | 1    | 1.3   | 3    |      | 2     | 4    | 3.2  |
| 8       | 0   | 0.4  | 1    | 3.4   | 9    |      | 18.4  | 52   | 46.9 |
| 10      | 0   | 1.6  | 13   | 31.7  | 186  |      | 110.5 | 234  | 305.6|
| 12      | 0   | 7.3  | 30   | (9)   | -    |      | 6     | (9)  | -    |
| 14      | 0   | 14.5 | 78   | 538.6 | 1906 |      | 522   | (6)  | -    |
| 16      | 0   | 39.1 | 155  | 209   | (9)  |      | -     | (0)  | -    |
| 18      | 4   | 48.4 | 231  | -     | (0)  | -    | -     | (0)  | -    |
| 20      | 2   | 30.3 | 76   | -     | (0)  | -    | -     | (0)  | -    |

Table 10: The minimum, average and maximum percent deviation from optimum reported for \( n = 280 \). The symbol "-" means that the optima for some instances were not obtained.

|         | 0.1 | 0.01 | 0.001 | 0.0001 |
|---------|-----|------|-------|--------|
|         | min | av.  | max   | min   | av.  | max   | min   | av.  | max   |
| 2       | 0.1 | 2.72 | 4.7   | 0.1   | 5.02 | 7.8   | 0.1   | 5.80 | 8.7   |
| 4       | 4.0 | 6.37 | 9.1   | 8.5   | 12.42| 16.8  | 10.3  | 14.97| 19.2  |
| 6       | 3.8 | 5.98 | 9.1   | 7.9   | 11.52| 14.9  | 10.9  | 15.16| 21.9  |
| 8       | 3.8 | 5.25 | 7.2   | 9.6   | 11.38| 14.1  | 12.1  | 14.98| 19.0  |
| 10      | 3.4 | 4.92 | 6.1   | 6.5   | 10.48| 13.5  | 8.6   | 14.08| 17.5  |
| 12      | 2.6 | 3.93 | 5.0   | 6.9   | -    | 11.3  | 9.8   | -    | 20.1  |
| 14      | 2.3 | 3.84 | 4.9   | 6.0   | 9.04 | 12.1  | 9.6   | -    | 17.6  |
| 16      | 2.8 | 4.00 | 5.5   | 9.1   | -    | 9.1   | -     | -    | -     |
| 18      | 2.4 | 4.12 | 5.3   | -     | -    | -     | -     | -    | -     |
| 20      | 3.1 | 3.83 | 4.5   | -     | -    | -     | -     | -    | -     |
\( \alpha = 0.0001 \). For \( K \geq 14 \) and \( n \geq 240 \) more instances turned out to be difficult.

The quality of the solutions returned by the approximation algorithm seems to be good in comparison with the worst theoretical performance. The largest reported deviations from optimum are not greater than 32\%. We can thus obtain reasonable solutions as long as the distribution of the costs under scenarios is uniform. It is interesting that the deviation from the optimum depends more on \( \alpha \) than \( K \). The performance of the approximation algorithm is clearly better for larger \( \alpha \), i.e. when WOWA is close to the expected value. On the other hand for a fixed \( \alpha \), the performance is significantly better only for \( K = 2 \). For larger \( K \) it seems to be similar.

7 Conclusions

In this paper we have discussed a wide class of discrete optimization problems in which the uncertain costs are specified in the form of a discrete scenario set. A probability distribution in this scenario set is provided. We have applied the weighted OWA criterion to choose a solution. This criterion allows us to take both scenario probabilities and attitude of decision makers towards the risk into account, as the weights assigned to scenarios are distorted (rank dependent) probabilities. Our approach contains the traditional robust (min-max) and stochastic approaches as special cases. The problem of minimizing the WOWA criterion is typically NP-hard for two scenarios. It becomes strongly NP-hard and also hard to approximate when the number of scenarios is a part of input. It is thus important to provide efficient approximation algorithms for the problem. One such an algorithm has been constructed in this paper. It can be applied, if only the underlying deterministic problem is polynomially solvable. The quality of the approximation algorithm has been tested for the selection problem. The solution returned by the approximation algorithm were compared to the optimal solutions obtained by solving a MIP formulation. When the costs under scenarios are chosen randomly, then the approximation algorithm returns solutions whose deviation from the optimum do not exceed 32\%.

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