Resonant Excitation of Disk Oscillations in Deformed Disks IV: A New Formulation Studying Stability

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Abstract

The possibility has been suggested that high-frequency quasi-periodic oscillations observed in low-mass X-ray binaries are resonantly excited disk oscillations in deformed (warped or eccentric) relativistic disks (Kato 2004). In this paper we examine this wave excitation process from a viewpoint somewhat different from that of previous studies. We consider how amplitudes of a set of normal mode oscillations change secularly with time by their mutual couplings through disk deformation. As a first step, we consider the case where the number of oscillation modes contributing to the resonant process is not always limited to two. In order to understand the essence of this instability, however, we assume in this paper that only two modes of oscillations contribute to this resonant process. More general cases will be a subject in the future.

1. Introduction

The origin of high-frequency quasi-periodic oscillations (HF QPOs) observed in low-mass X-ray binaries (LMXBs) is one of challenging subjects to be examined, since its examination will clarify the structure of the innermost part of relativistic accretion disks and the spin of central sources. One promising possibility is that the QPOs are disk oscillations excited in the innermost region of relativistic disks. Particularly, the idea that HF QPOs are disk oscillations resonantly excited in deformed (warped or eccentric) disks has been suggested by Kato (2004, 2008a, 2008b) by analytical considerations, and studied by Ferreria and Ogilvie (2008) as well as Oktariani et al. (2010) by numerical calculations. In this model, a disk oscillation (hereafter, original oscillation) interacts non-linearly with the disk deformation to produce an oscillation (hereafter, intermediate oscillation). The intermediate oscillation is a forced oscillation due to coupling between the original oscillation and the deformation. The intermediate oscillation then resonantly responds to the forcing term at a certain radius (Lindblad resonance). After having resonance, the intermediate oscillation interacts non-linearly again with the disk deformation to feed back to the original oscillation. Through this feedback process the original and intermediate oscillations are excited, if certain conditions are satisfied.

Some important consequences have been obtained so far in relation to the origin of the instability: (i) The wave energies of the original and intermediate oscillations must have opposite signs (Kato 2004, 2008a). That is, the instability is a result of wave-energy exchange between two oscillations with opposite signs of energy through a disk deformation. (ii) Since the intermediate oscillation resonantly responds to forcing terms resulting from the coupling between the disk deformation and the original oscillation, its amplitude is not necessarily small compared with that of the original oscillation [see figure 3 of Oktariani et al. (2010)]. This means that the terminology of the “original” and “intermediate” oscillations has no particular meaning. The two oscillations are equal partners, i.e., the role of the original oscillation mentioned in the previous paragraph can also be performed by the intermediate oscillation. Hence, the interaction between two oscillations can be schematically sketched, as in figure 1. This has been correctly acknowledged by Ferreira and Ogilvie (2008) (see figure 3 of their paper).

Based on the above considerations, we develop here a perspective analytical method to study the instability. In previous analytical studies, we considered only those cases where the disk deformation is time-independent (i.e., its frequency of the disk deformation, \(\omega_D\), is zero) and the frequencies of the two oscillations, \(\omega_1\) and \(\omega_2\), coupling through disk deformation are the same, i.e., \(\omega_1 = \omega_2\). In the present analyses, the resonant condition is extended to \(\omega_1 = \omega_2 \pm \omega_D\) with non-zero \(\omega_D\), and the effects of a weak deviation from the resonant condition, \(\omega_1 = \omega_2 \pm \omega_D\), on the growth rate of oscillations are also examined.

It is noted that the number of oscillations contributing to this resonant process is not always limited to two. In order to understand the essence of this instability, however, we assume in this paper that only two modes of oscillations contribute to this resonant process. More general cases will be a subject in the future.
2. Basic Hydrodynamical Equations and Some Other Relations

We summarize here basic equations and relations [see Kato (2008a, 2008b) for details] to be used to discuss wave couplings through a disk deformation.

2.1. Non-linear Hydrodynamical Equations

When we try to apply the present excitation process quantitatively to HF QPOs observed in GBHCs (galactic black-hole candidates) and LMXBs, the effects of general relativity should be taken into account. General relativity, however, is not essentially to understand the essence of the instability mechanism. Hence, in this paper, for simplicity, we formulated the system using a pseudo-Newtonian potential, which was introduced by Paczyński and Wiita (1980). We adopt a Lagrangian formulation by Lynden-Bell and Ostriker (1967).

The unperturbed disk is in a steady equilibrium state. Over the equilibrium state, weakly non-linear perturbations are superposed. By using a displacement vector, \( \mathbf{x} \), the weakly non-linear hydrodynamical equation describing adiabatic, non self-gravitating perturbations is written as, after lengthy manipulation (Lynden-Bell & Ostriker 1967),

\[
\frac{\partial^2 \mathbf{x}}{\partial t^2} + 2 \rho_0 \left( \mathbf{u}_0 \cdot \nabla \right) \frac{\partial \mathbf{x}}{\partial t} + L(\mathbf{x}) = \rho_0 C(\mathbf{x}, \mathbf{x}),
\]

where \( L(\mathbf{x}) \) is a linear Hermitian operator with respect to \( \mathbf{x} \), and is

\[
L(\mathbf{x}) = \rho_0 (\mathbf{u}_0 \cdot \nabla)(\mathbf{u}_0 \cdot \nabla)\mathbf{x} + \rho_0 (\mathbf{x} \cdot \nabla)(\nabla \psi_0)
+ \nabla \left( 1 - \Gamma_1 \right) \rho_0 \text{div} \mathbf{x} - \rho_0 \nabla \left( \text{div} \mathbf{x} \right)
- \nabla \left( \mathbf{x} \cdot \nabla \right) \rho_0 + \left( \mathbf{x} \cdot \nabla \right)(\nabla \rho_0);
\]

\( \rho_0(\mathbf{r}) \) and \( \rho_0(\mathbf{r}) \) are, respectively, the density and pressure in the unperturbed state, and \( \Gamma_1 \) is the barotropic index specifying the linear part of the relation between the Lagrangian variations, \( \delta \rho \) and \( \delta \rho \), i.e., \( \langle \delta \rho / \rho_0 \rangle_{\text{linear}} = \Gamma_1 \langle \delta \rho / \rho_0 \rangle_{\text{linear}} \).

Since the self-gravity of the disk gas has been neglected, the gravitational potential, \( \psi_0(\mathbf{r}) \), is a given function, and has no Eulerian perturbation. In the above hydrodynamical equations (1) and (2), there is no restriction on the form of the unperturbed flow, \( \mathbf{u}_0 \). However, in the following we assume that the unperturbed flow is a cylindrical rotation alone, i.e., \( \mathbf{u}_0 = (0, r \Omega, 0) \), in cylindrical coordinates \( (r, \varphi, z) \), where the origin is at the disk center and the \( z \)-axis is in the direction perpendicular to the unperturbed disk plane with \( \Omega(\mathbf{r}) \) being the angular velocity of disk rotation.

The right-hand side of wave equation (1) represents weakly non-linear terms. No detailed expression for \( C \) is given here [for detailed expressions, see Kato (2004, 2008a)], but an important characteristics of \( C \) is that we have commutative relations (Kato 2008a) for an arbitrary set of \( \eta_1, \eta_2, \) and \( \eta_3, \) e.g.,

\[
\int \rho_0 \eta_1 \cdot C(\eta_2, \eta_3) dV = \int \rho_0 \eta_1 \cdot C(\eta_3, \eta_2) dV = \int \rho_0 \eta_3 \cdot C(\eta_1, \eta_2) dV.
\]

As shown later, the presence of these commutative relations leads to a simple expression of the instability criterion. We suppose that the presence of these commutative relations is a general property of conservative systems beyond the assumption of weak non-linearity.

2.2. Orthogonality of Normal Modes

In preparation for subsequent studies, some orthogonality relations are summarized here. Eigen-functions describing linear oscillations in non-deformed disks are denoted by \( \xi_\alpha(\mathbf{r}, t) \). Here, the subscript \( \alpha \) is used to distinguish all eigen-functions. The time-dependent part of \( \xi_\alpha(\mathbf{r}, t) \) is expressed as \( \exp(i \omega_\alpha t) \), where \( \omega_\alpha \) is real. Then, \( \xi_\alpha(\mathbf{r}, t) \) satisfies

\[
-\omega_\alpha^2 \rho_0 \xi_\alpha + 2i \omega_\alpha \rho_0 (\mathbf{u}_0 \cdot \nabla) \xi_\alpha + L(\xi_\alpha) = 0.
\]

Now, this equation is multiplied by \( \xi_\beta^*(\mathbf{r}, t) \), and integrated over the whole volume, where the superscript * denotes the complex conjugate and \( \beta \neq \alpha \). The volume integral of \( \rho_0 \xi_\beta^*(\mathbf{r}, t) \xi_\alpha(\mathbf{r}, t) \) over the whole volume is hereafter written as \( \langle \rho_0 \xi_\beta^* \xi_\alpha \rangle \). Then, we have

\[
-\omega_\alpha^2 \langle \rho_0 \xi_\beta^* \xi_\alpha \rangle + 2i \omega_\alpha \langle \rho_0 \xi_\beta^*(\mathbf{u}_0 \cdot \nabla) \xi_\alpha \rangle + \langle \xi_\beta^* \cdot L(\xi_\alpha) \rangle = 0.
\]

Similarly, after integrating the linear wave equation of \( \xi_\beta^* \) over the whole volume after the equation being multiplied by \( \xi_\alpha \), we obtain

\[
-\omega_\beta^2 \langle \rho_0 \xi_\beta \xi_\alpha^* \rangle - 2i \omega_\beta \langle \rho_0 \xi_\beta(\mathbf{u}_0 \cdot \nabla) \xi_\alpha^* \rangle + \langle \xi_\beta^* \cdot L(\xi_\alpha^*) \rangle = 0.
\]

Since the operator \( L \) is a Hermitian (Lynden-Bell & Ostriker 1967), we have the relation

\[
\langle \xi_\beta^* \cdot L(\xi_\alpha^*) \rangle = \langle [L(\xi_\alpha^*)]^* \cdot \xi_\beta \rangle = \langle L(\xi_\alpha^*) \cdot \xi_\beta \rangle.
\]

Hence, the difference in the above two equations [equations (5) and (6)] gives, when \( \omega_\beta \neq \omega_\alpha \),

\[
\langle \omega_\alpha + \omega_\beta \rangle \langle \rho_0 \xi_\beta \xi_\alpha^* \rangle = 2i \langle \rho_0 \xi_\beta^*(\mathbf{u}_0 \cdot \nabla) \xi_\alpha \rangle = -2i \langle \rho_0 \xi_\beta(\mathbf{u}_0 \cdot \nabla) \xi_\alpha^* \rangle.
\]
assuming that $\rho_0$ vanishes on the disk surface.

Different from the case of non-rotating stars, the eigenfunctions of the normal modes of disk oscillations are not orthogonal in the sense of $\langle \rho_0 \xi_a \xi_b \rangle = 0$. In spite of this, however, the eigenfunctions of disk oscillations are orthogonal in many situations. They are classified by the azimuthal wavenumber, $m$, node number in the vertical direction, $n$, and that in the radial direction, $\ell$, in addition to the distinction of the p- and g-modes [see Kato (2001) or Kato et al. (2008) for classifying disk oscillations].

Eigenfunctions with different azimuthal wavenumbers are obviously orthogonal, i.e., $\langle \rho_0 \xi_a \xi_b \rangle = 0$ when $m_a \neq m_b$. Even if the azimuthal wavenumbers are the same, $\langle \rho_0 \xi_a \xi_b \rangle = 0$ when $n_a \neq n_b$, if the disk is geometrically thin and isothermal in the vertical direction. This comes from the fact that in such disks the $z$-dependence of eigenfunctions with $n$ node(s) in the $z$-direction ($n$ is zero or a positive integer) is described by the Hermite polynomials $H_n$ as

$$\xi_r, \xi_\varphi \propto H_n(z/H), \quad \xi_z \propto H_{n-1}(z/H),$$

(Okazaki et al. 1987), where the subscripts $r$, $\varphi$, and $z$ represent cylindrical coordinates ($r$, $\varphi$, $z$) whose origin is at the disk center, and the $z$-axis is perpendicular to the disk plane. Here, $H_n$ is the Hermite polynomial of argument $z/H$, $H$ being the half-thickness of the disk. Thus, the eigenfunctions classified by $m$ and $n$ are orthogonal. In summary, we have

$$\langle \rho_0 \xi_a \xi_b \rangle = \langle \rho_0 \xi_a \xi_b \rangle \delta_{m_a m_b} \delta_{n_a n_b},$$

(10)



where $\delta_{m,n}$ is the Kronecker delta, i.e., it is unity when $a = b$, otherwise zero.

The orthogonality of $\langle \rho_0 \xi_a \xi_b \rangle$ does not hold in the case where $m_a = m_b$ and $n_a = n_b$. Even in these cases, however, $\langle \rho_0 \xi_a \xi_b \rangle$ will be close to zero for $\xi_a \neq \xi_b$, if our interest is on short-wavelength oscillations in the radial direction, since the radial dependence of eigenfunctions is close to sinusoidal in such cases.

3. Couplings of Two Oscillations through Disk Deformation

Let us consider the case where two oscillation modes, $\xi_1(r, t)$ and $\xi_2(r, t)$, resonantly couple through a disk deformation, $\xi_D(r, t)$. Through the coupling term $C(\xi, \xi)$ [see equation (1)] many other modes than $\xi_1$ and $\xi_2$ appear, and their amplitudes as well as those of $\xi_1$ and $\xi_2$ become time dependent. Now, we assume that the normal modes of oscillations form a complete set, and expand the resulting oscillations, $\xi(r, t)$, including the disk deformation, $\xi_D(r, t)$, in the form

$$\xi(r, t) = A_1(t)\xi_1(r, t) + A_2(t)\xi_2(r, t) + A_D(t)\xi_D(r, t) + \sum_a A_a(t)\xi_a(r, t).$$

(11)

Since our main concern is on the modes 1 and 2, $\xi_1$ and $\xi_2$ are distinguished from other eigen-functions and the subscript $\alpha$ is hereafter used only to denote other eigen-functions than $\xi_1$ and $\xi_2$. Our purpose here is to derive equations describing a secular time evolution of $A_1$ and $A_2$. The disk deformation, $\xi_D(r, t)$, is assumed to have a much larger amplitude than other oscillations, and its time variation during the coupling processes is neglected, i.e., $A_D = \text{const}$.

We now express the eigen-frequencies associated with $\xi_1$, $\xi_2$, $\xi_D$, and $\xi_a$ by $\omega_1$, $\omega_2$, $\omega_D$, and $\omega_a$, respectively, i.e., $\xi_1(r, t) = \exp(\ii \omega_1 t) \xi_1(r)$ and so on. Then, substitution of equation (11) into equation (1) leads to

$$2\rho_0 \frac{d A_1}{dt}[i \omega_1 + (u_0 \cdot \nabla)]\xi_1 + 2\rho_0 \frac{d A_2}{dt}[i \omega_2 + (u_0 \cdot \nabla)]\xi_2 + \sum_a \rho_0 \frac{d A_a}{dt}[i \omega_a + (u_0 \cdot \nabla)]\xi_a$$

$$= \sum_{i=1,2} \frac{1}{2} A_i \left[ \rho C(\xi_i, \xi_D) + \rho C(\xi_D, \xi_i) \right] + \frac{1}{2} A_D \left[ \rho C(\xi_1, \xi_2) + \rho C(\xi_2, \xi_1) \right] + \sum_a \frac{1}{2} A_a \left[ \rho C(\xi_a, \xi_D) + \rho C(\xi_D, \xi_a) \right] + \frac{1}{2} A_D \left[ \rho C(\xi_1, \xi_a) + \rho C(\xi_a, \xi_1) \right],$$

(12)

where terms of $d^2 A_1/dt^2$, $d^2 A_2/dt^2$, and $d^2 A_D/dt^2$ have been neglected, since we are interested in slow secular evolutions of $A_1$. On the right-hand side of equation (12), the coupling terms that are not related to the disk deformation are neglected.\footnote{In writing down the right-hand side of equation (12), we have used the following relation:
$$\Re(A)\Re(B) = \frac{1}{2} \Re(AB + AB^*),$$
where $A$ and $B$ are complex variables.}

Now, we define the wave energy, $E_1$, of normal mode of oscillation $\xi_1$ by

$$E_1 = \frac{1}{2} \frac{\omega_1}{\omega_1} [\xi_1(\rho_0 \xi_1^* \xi_1) - i \rho_0 \xi_1^*(u_0 \cdot \nabla) \xi_1]$$

(Kato 2001, 2008a). The wave energy, $E_2$, of the normal mode, $\xi_2$, is also defined by

$$E_2 = \frac{1}{2} \frac{\omega_2}{\omega_2} [\xi_2(\rho_0 \xi_2^* \xi_2) - i \rho_0 \xi_2^*(u_0 \cdot \nabla) \xi_2].$$

(14)

Furthermore, we introduce the following quantities:

$$W_{11} = \frac{1}{2} \left( \rho_0 \xi_1^* C(\xi_1, \xi_D) + \rho_0 \xi_1 C(\xi_D, \xi_1) \right),$$

(15)

$$W_{11a} = \frac{1}{2} \left( \rho_0 \xi_1^* C(\xi_1, \xi_a) + \rho_0 \xi_1 C(\xi_a, \xi_1) \right),$$

(16)

$$W_{12} = \frac{1}{2} \left( \rho_0 \xi_1^* C(\xi_2, \xi_D) + \rho_0 \xi_1 C(\xi_D, \xi_2) \right),$$

(17)

$$W_{12a} = \frac{1}{2} \left( \rho_0 \xi_1^* C(\xi_2, \xi_a) + \rho_0 \xi_1 C(\xi_a, \xi_2) \right).$$

(18)
\[ W_{1a} = \frac{1}{2} \left( \left( \rho_0 \xi_1^* \cdot C(\xi_a, \xi_D) \right) + \left( \rho_0 \xi_1^* \cdot C(\xi_D, \xi_a) \right) \right), \quad (19) \]
\[ W_{1ar} = \frac{1}{2} \left( \left( \rho_0 \xi_1^* \cdot C(\xi_a, \xi_D) \right) + \left( \rho_0 \xi_1^* \cdot C(\xi_D, \xi_a) \right) \right). \quad (20) \]

To proceed further, the time and azimuthal dependences of the normal-mode oscillations are written explicitly as
\[ \xi_k(r, t) = \exp[i(\omega_k t - m_k \varphi)] \hat{\xi}_k \quad (k = 1, 2, \alpha, \beta). \quad (21) \]

To avoid writing down similar relations repeatedly, we have introduced the subscript \( k \), which represents all of the oscillation modes, i.e., \( k \) denotes 1, 2, \( \alpha \), and \( \beta \). Here, we take all \( m_k \) to be zero or positive integers, while \( \omega_k \) is not always positive. If \( \omega_k < 0 \), the oscillation is retrograde.

It is noted here that by using equation (21), we can express the wave energy of the normal-mode oscillation in an instructive form. Since the \( r \)- and \( \varphi \)-components of \( \xi_1 \), say \( \xi_{1r} \) and \( \xi_{1\varphi} \), are related in a geometrically thin disks by (e.g., Kato 2004),
\[ i(\omega_1 - m_1 \Omega) \xi_{1\varphi} + 2\Omega \xi_{1r} \sim 0, \]
we have
\[ E_1 \sim \frac{\omega_1}{2} \left( (\omega_1 - m_1 \Omega) \rho_0 (\xi_{1r}^* \xi_{1r} + \xi_{1\varphi}^* \xi_{1\varphi}) \right). \quad (22) \]

This shows that the sign of wave energy is determined by the sign of \( \omega_1 - m_1 \Omega \) in the region where the wave exists predominantly (e.g., Kato 2001). For example, a prograde \( (\omega_1 > 0) \) wave inside the corotation resonance has a negative energy, while a prograde wave outside it has a positive energy.

In previous papers we mainly considered the case of \( \omega_1 = \omega_2 \) with \( \omega_D \neq 0 \). In this paper, we extend our analyses to more general cases of resonance:
\[ \omega_1 \sim \omega_2 \pm \omega_D, \quad (24) \]
where \( \omega_D \) is not necessary to be small. We introduce \( \Delta_+ \) and \( \Delta_- \), defined by
\[ \Delta_+ = \omega_1 - \omega_2 - \omega_D \quad \text{and} \quad \Delta_- = \omega_1 - \omega_2 + \omega_D. \quad (25) \]
In the resonance of \( \omega_1 \sim \omega_2 + \omega_D, \Delta_+ \) is small, but \( \Delta_- \) is not small unless \( \omega_D \) is small. In the resonance of \( \omega_1 \sim \omega_2 - \omega_D \), on the other hand, \( \Delta_- \) is small, but \( \Delta_+ \) is not always so unless \( \omega_D \) is small. It is noted that the resonant condition concerning the azimuthal wavenumber is
\[ m_1 = m_2 \pm m_D, \quad (26) \]
where \( m_1 \) and \( m_2 \) are zero or positive integers, while \( m_D \) is a positive integer, since we focus our attention only on the case where the disk deformation is non-axisymmetric, i.e., \( m_D \neq 0 \).

After these preparations, we integrate equation (12) over the whole volume after multiplying \( \xi_k^* (r, t) \). Then, the term with \( dA_1/dt \) becomes \( i(4E_1/\omega_1)(dA_1/dt) \), while the term with \( dA_2/dt \) vanishes, since \( m_2 \neq m_1 \) by definition. Concerning the term with \( dA_\alpha/dt \), some more consideration is necessary. If \( \omega_\alpha \neq \omega_1 \), by using equation (8) we can reduce the integration to
\[ i \sum_\alpha \frac{dA_\alpha}{dt}(\omega_\alpha - \omega_1)(\rho_0 \xi_1^* \xi_\alpha). \quad (27) \]

In the case where \( m_\alpha = m_1 \pm m_D \), \( \rho_0 \xi_1^* \xi_\alpha \) vanishes, since \( m_D \neq 0 \). On the other hand, when \( m_\alpha \neq m_1 \pm m_D \), \( A_\alpha \) does not appear in the coupling term. That is, \( A_1 \) and \( A_\alpha \) have no non-linear coupling, and thus we can take \( A_\alpha = 0 \) when we consider the time evolution of \( A_1 \). In the case of \( \omega_\alpha = \omega_1 \), the last term on the left-hand side of equation (12) does not lead to equation (27). Even in this case, by using the same arguments as above, we can neglect the term. Considering these situations we have
\[ i \frac{dA_1}{dt} \frac{4E_1}{\omega_1} = A_1(A_1D_{11} + A_1^* D_{12}) 
+ A_2(A_2D_{12} + A_2^* D_{12}) 
+ \sum_\alpha A_\alpha(A_\alpha D_{\alpha 1} + A_\alpha^* D_{\alpha 1}). \quad (28) \]

Among various coupling terms on the right-hand side of equation (28), the first two terms with \( W_{11} \) and \( W_{12} \) can be neglected if we consider non-axisymmetric disk deformations, such as warp or eccentric deformation, since the terms inside \( \{ \} \) in equations (15) and (16) are proportional to \( \exp(-im_1 \varphi) \) and \( \exp(im_1 \varphi) \), respectively, and their angular averages vanish. The last two terms with \( W_{1a} \) and \( W_{1ar} \) on the right-hand side of equation (28) are also neglected hereafter for the following reasons. The terms inside \( \{ \} \) of equations (19) and (20) are proportional to \( \exp[i(\omega_1 + \omega_a + \omega_D \varphi)] \) and \( \exp(-i(\omega_1 + \omega_a - \omega_D \varphi)) \), respectively. In general, they rapidly vary with time, since no resonant condition is assumed among \( \omega_1, \omega_a, \) and \( \omega_D \). Hence, if short timescale variations are averaged over,\(^3\) the averaged quantities are small and can be neglected.\(^4\) In summary, the remaining coupling terms are the middle two terms of equation (28) with \( W_{12} \) and \( W_{12a} \).

Based on the above preparations, we reduce equation (28) to
\[ i \frac{dA_1}{dt} \frac{4E_1}{\omega_1} = A_1(A_1D_{11} + A_1^* D_{12}) 
+ A_2(A_2D_{12} + A_2^* D_{12}) 
+ \sum_\alpha A_\alpha(A_\alpha D_{\alpha 1} + A_\alpha^* D_{\alpha 1}). \quad (29) \]

The symbol \( \delta_{a,b} \) is the Kronecker delta. Here, from \( W_{12} \) and \( W_{12a} \), the time and azimuthally dependent parts are separated as
\[ W_{12} = W_{12} \exp(-i\Delta_+ t) \delta_{m_1,m_2+\delta}, \quad (30) \]
\[ W_{12a} = W_{12a} \exp(-i\Delta_+ t) \delta_{m_1,m_2-\delta}. \quad (31) \]

The physical meaning of equation (29) is as follows. The imaginary part of \( (\omega_1/2) W_{12} \), for example, is the rate of work done on mode 1 (when mode 2 and the deformation have unit amplitudes) through the coupling of \( m_1 = m_1 + m_D \) (Kato 2008a). Hence, in a rough sense, equation (29) represents the fact that the growth rate of mode 1 is given by the energy flux
\[ F_{11} = A_2 A_D (\omega_1/2) W_{12} \] to mode 1 as

\(^3\) We are interested in solutions where all \( A \)'s vary slowly with time.

\(^4\) In some cases, however, some of \( \omega_a \)'s are close to \( \omega_1 \pm \omega_D \). For example, all trapped \( \pm \) mode oscillations have frequencies close to \( \kappa_{max} \) (the maximum of the epicyclic frequency), when their azimuthal wavenumber is zero. Then, \( W_{1a} \) or \( W_{1ar} \) have no rapid time variation and cannot be neglected by the above argument of time average when \( m_\alpha \) satisfies the relation of \( m_\alpha = m_1 \pm m_D \). Then, the terms with \( W_{1a} \) or \( W_{1ar} \) also contribute to resonant couplings. Such cases are outside of our present concern. See related discussions in the final section.
4. Growth Rate of Resonant Oscillations

By solving the set of equations (29) and (36), we examine how the amplitudes of \( A_1 \) and \( A_2 \) evolve with time. We consider two cases of \( m_2 = m_1 + m_D \) and \( m_2 = m_1 - m_D \), separately.

4.1. Case of \( m_2 = m_1 + m_D \)

In this case the set of equations of \( A_1 \) and \( A_2 \) are, from equations (29) and (36),

\[
4i \frac{E_1}{\omega_1} \frac{dA_1}{dt} = A_2 A_D^* \hat{W}_{12a} \exp(-i \Delta_t),
\]

\[
4i \frac{E_2}{\omega_2} \frac{dA_2}{dt} = A_1 A_D \hat{W}_{21a} \exp(i \Delta_t).
\]

By introducing a new variable, \( \tilde{A}_1 \), defined by

\[
\tilde{A}_1 = A_1 \exp(i \Delta_t),
\]

we can reduce the above set of equations to

\[
4i \frac{E_1}{\omega_1} \frac{d\tilde{A}_1}{dt} + 4i \frac{E_2}{\omega_2} \Delta \tilde{A}_1 = A_2 A_D^* \hat{W}_{12a},
\]

\[
4i \frac{E_2}{\omega_2} \frac{dA_2}{dt} = \tilde{A}_1 A_D \hat{W}_{21a}. \tag{45}
\]

Hence, by taking \( \tilde{A}_1 \) and \( A_2 \) to be proportional to \( \exp(i \sigma t) \), we obtain an equation describing \( \sigma \) as

\[
\sigma^2 - \Delta_0 \sigma - \frac{\omega_1 \omega_2}{4 E_1 E_2} |A_D|^2 |W_{12a}|^2 = 0, \tag{46}
\]

where equation (39) is used.

In the limit of an exact resonance of \( \Delta_0 = 0 \), the instability condition \( (\sigma^2 < 0) \) is found to be \( (\omega_1 / E_1)(\omega_2 / E_2) < 0 \). The meaning of this condition is discussed later. If the frequencies of two oscillations deviate from the resonant condition of \( \omega_1 = \omega_2 = \omega_0 = \omega_D \), the growth rate decreases. This can be shown from equation (46). That is, the condition of growth is

\[
\Delta_0^2 + \frac{\omega_1 \omega_2}{4 E_1 E_2} |A_D|^2 |W_{12a}|^2 < 0, \tag{47}
\]

and the growth rate tends to zero as \( \Delta_0^2 \) increases from zero. If \( \Delta_0^2 \) increases beyond a certain limit the left-hand side of inequality (47) becomes positive, and \( \sigma \) is no longer complex. That is, the amplitude of oscillations are modulated with time, but there is no secular increase in them.

4.2. Case of \( m_2 = m_1 - m_D \)

In the present case, from equations (29) and (36), we have

\[
4i \frac{E_1}{\omega_1} \frac{dA_1}{dt} = A_2 A_D \hat{W}_{12a} \exp(-i \Delta_t), \tag{48}
\]

\[
4i \frac{E_2}{\omega_2} \frac{dA_2}{dt} = A_1 A_D^* \hat{W}_{21a} \exp(i \Delta_t). \tag{49}
\]

A new variable \( \tilde{A}_1 \) is introduced here by

\[
\tilde{A}_1 = A_1 \exp(i \Delta_t). \tag{50}
\]

Then, the set of equations (48) and (49) are reduced to a set of equations of \( \tilde{A}_1 \) and \( A_2 \) as

\[
4i \frac{E_1}{\omega_1} \frac{d\tilde{A}_1}{dt} + 4i \frac{E_1}{\omega_1} \Delta_0 \tilde{A}_1 = A_2 A_D \hat{W}_{12a}, \tag{51}
\]

\[
4i \frac{E_2}{\omega_2} \frac{dA_2}{dt} = \tilde{A}_1 A_D^* \hat{W}_{21a}. \tag{52}
\]
Hence, by taking \( A_1 \) and \( A_2 \) to be proportional to \( \exp(i \sigma t) \), we have

\[
\sigma^2 - \Delta_+ \sigma - \frac{\omega_1 \omega_2}{16E_1 E_2} |A_D|^2 |\dot{W}_{12}|^2 = 0,
\]

(53)

where we have used equation (40).

Two oscillations certainly grow again at the limit of the exact resonance of \( \Delta_+ = 0 \) (i.e., \( \omega_1 = \omega_2 + \omega_D \)), if \( (\omega_1/E_1) (\omega_2/E_2) < 0 \). Even if the resonance is not exact, they grow if \( \Delta_+^* \) is small enough so that

\[
\Delta_+^* + \frac{\omega_1 \omega_2}{4E_1 E_2} |A_D|^2 |\dot{W}_{12}|^2 < 0
\]

(54)

is satisfied.

4.3. A Relation between \( A_1 \) and \( A_2 \)

Finally, it is useful to derive an instructive relation between \( A_1 \) and \( A_2 \). Let us first consider the case of \( m_2 = m_1 + m_D \). Let us multiply \( A_1^* \) to equation (41) and also \( A_1 \) to the complex conjugate of equation (41). Then, summing these two equations we have an equation describing the time evolution of \( |A_1|^2 \). Similarly, from equation (42), we can derive an equation describing the time evolution of \( |A_2|^2 \). Summing these two equations, we finally have

\[
\frac{d}{dt} \left[ \frac{E_1}{\omega_1} |A_1|^2 + \frac{E_2}{\omega_2} |A_2|^2 \right] = 0,
\]

(55)

where we have used \( \dot{W}_{21} = \dot{W}_{12}^* \). The same equation can be derived from equations (48) and (49) in the case of \( m_2 = m_1 - m_D \). To derive the equation, \( \dot{W}_{21} = \dot{W}_{12}^* \) has been used.

Equation (55) obviously shows that \( (\omega_1/E_1) (\omega_2/E_2) < 0 \) is necessary for the growth of oscillations. In the case of \( (\omega_1/E_1) (\omega_2/E_2) > 0 \), on the other hand, the amplitudes of \( A_1 \) and \( A_2 \) are limited, although the relative amplitude of both oscillations may change with time by interaction through disk deformation.

4.4. Summary of Resonant Instability Condition

The results given in the previous subsections show that when resonant conditions of \( \omega_1 = \omega_2 \pm \omega_D \) and \( m_1 = m_2 \pm m_D \) are satisfied among two oscillations characterized by \( (\omega_1, m_1) \) and \( (\omega_2, m_2) \) and disk deformation characterized by \( (\omega_D, m_D) \), the two oscillations are resonantly excited if \( (\omega_1/E_1) (\omega_2/E_2) < 0 \) is realized. A deviation from the condition of \( \omega_1 = \omega_2 \pm \omega_D \) decreases the growth rate, but oscillations grow as long as the deviation is smaller than a critical value.

In the case where both of \( \omega_1 \) and \( \omega_2 \) are positive (i.e., both oscillations are prograde), the above instability condition is \( E_1 E_2 < 0 \). This is a result suggested by Kato (2004, 2008a, 2008b) by a different approach.

In the case where \( \omega_D \) is larger than \( \omega_2 \) (>0), a resonant condition, \( \omega_1 = \omega_2 - \omega_D \), is satisfied for \( \omega_1 < 0 \) (i.e., retrograde wave). In this case, if the wave energy, \( E_1 \), is positive [see equation (23)] and the instability condition is reduced to \( E_2 / \omega_2 > 0 \), i.e., \( E_2 > 0 \). That is, when \( \omega_1 \omega_2 < 0 \), the condition of the resonant instability is \( E_1 E_2 > 0 \). It is noted that in the case where both of \( \omega_1 \) and \( \omega_2 \) are negative, both \( E_1 \) and \( E_2 \) are positive, so that the condition of the resonant instability, \( (\omega_1/E_1) (\omega_2/E_2) < 0 \), cannot be satisfied.

Among three case of (i) \( \omega_1 > 0 \) and \( \omega_2 > 0 \), (ii) \( \omega_1 < 0 \) and \( \omega_2 > 0 \), and (iii) \( \omega_1 < 0 \) and \( \omega_2 < 0 \), the interesting case in the practical sense is the first one, which is discussed in the next section.

5. Discussion

First, let us describe, in terms of the present formulation, the \( g \)- and \( p \)-modes resonant instability that was numerically studied by Ferreira and Ogilvie (2008) and Oktariani et al. (2010). They considered the resonant interaction, through a standing warp \( (\omega_D = 0) \), between \( i \) the axisymmetric \( g \)-mode oscillation whose \( \xi_r \) has one node in the vertical direction and \( ii \) the one-armed \( p \)-mode oscillation whose \( \xi_r \) has no node in the vertical direction. That is, the set of \( (\omega, m, n) \) is \((\sim \kappa_{\text{max}}, 0, 1)\) for the \( g \)-mode oscillation, and \((\sim \kappa_{\text{max}}, 1, 0)\) for the \( p \)-mode one, where \( \kappa_{\text{max}} \) is the maximum of the (radial) epicyclic frequency. The warp is taken to be \((0, 1, 1)\). In this case, the resonant conditions, \( \omega_1 - \omega_2 > 0 \) (i.e., \( \omega_2 = 0 \) and \( m_2 = m_1 + m_D \)), are satisfied. Hence, if \( E_1 E_2 < 0\), these modes are excited simultaneously. This condition of \( E_1 E_2 < 0 \) is really satisfied, since the axisymmetric \( g \)-mode oscillation has a positive energy, while the \( p \)-mode oscillation trapped between the inner edge of the disk and the barrier resulting from the boundary between the propagation and evanescent regions has a negative energy. The reason of excitation is a positive energy flow from a negative energy oscillation (\( p \)-mode) to a positive energy oscillation (\( g \)-mode). By this energy flow both oscillations grow. The disk deformation is a catalyst of this energy flow.

In the above argument the disk deformation is assumed to be a warp. Instead of a warp, we can consider a \( c \)-mode oscillation as one of other possible disk deformations. In this case the set of \( (\omega, m, n) \) of the disk deformation is \((\omega_D, 1, 1)\), where \( \omega_D \) is the frequency of the \( c \)-mode oscillation and \( \omega_D \ll \kappa_{\text{max}} \), unless the spin of the central source is high (Silbergleit et al. 2001). The resonant condition in this case is \( \omega_1 = \omega_2 \pm \omega_D \), not \( \omega_1 = \omega_2 \).

There are some limitations for a direct comparison of the present analytical results to the numerical ones by Ferreira and Ogilvie (2008). In the present analyses only two normal modes of oscillations are assumed to contribute to the resonance to understand the essence of the resonant instability. In the realistic case considered numerically by Ferreira and Ogilvie (2008), however, more than two normal modes of oscillations may contribute to the resonance. In their case, one of the resonant oscillations is an axisymmetric \( g \)-mode. As mentioned in footnote 4, eigen-frequencies of the axisymmetric \((m = 0)\) \( g \)-mode oscillations with different \( \ell \) (with \( n = 1 \)) are all close to \( \kappa_{\text{max}} \). Hence, the \( g \)-mode oscillations that satisfy the resonant conditions, \( \omega_1 = \omega_2 \pm \omega_D \) and \( m_1 = m_2 \pm m_D \), may not be only one, and overtones of \( g \)-mode oscillations with nodes in the radial direction may also contribute partially to the resonance. If this is the case, some coupling terms with other \( A \) than \( A_1 \) and \( A_2 \) appear in equations describing the time evolution of \( A_1 \) and \( A_2 \). In the present analytical formulation such situations are not considered. To extend our analyses to such cases, we must derive equations describing the time evolution of other \( A \)'s than \( A_1 \) and \( A_2 \), and these equations should be...
solved simultaneously with the equations describing the time evolution of $A_1$ and $A_2$. We think that the essence of the instability mechanism is already presented in the case where couplings occur only between two oscillations. However, since such extension of our formulation is formally simple, and there may be some subsidiary modifications of instability criterion, such an extension should be done in the near future.

In the present formulation, some important theoretical problems remain to be clarified. One of them is whether the set of normal modes of oscillations form a complete set. If not, it is uncertain whether the oscillations realized on the disk can be expressed in the form of equation (11).

In the case where $\omega_2$ is larger than $\omega_3$, the resonant frequency $\omega_1$, which satisfies the condition, $\omega_1 = \omega_2 - \omega_3$, is negative. In this case of $\omega_1 < 0$, $E_1$ is positive [see equation (23)]. Hence, the instability condition in this case is $E_1E_2 > 0$, as mentioned in the last section. All of the g-mode and p-mode oscillations with negative frequency are, however, not trapped in the inner region of disks. They propagate away far outside unless the disks are truncated. That is, their frequencies are continuous and will not be directly related to the QPO phenomena in disks.

It is important to note here that the innermost region of relativistic disks is a place where the present excitation mechanism works most efficiently. For the mechanism to work, two normal modes of oscillations with opposite signs of wave energy must coexist in a common region of the disks. In general, for $m \neq 0$, positive energy oscillations propagate in the region outside the radius of the corotation resonance, while negative-energy oscillations do so inside the resonance. Hence, there is a tendency that the propagation region of oscillations with opposite signs of wave energy are spatially separated, unless different types of oscillation modes are considered. If the propagation region of oscillations with opposite signs of wave energy are separated, the coupling efficiency between the two oscillations are weak, i.e., $W$’s in equations (29) and (36) are small, and practically there is no growth of oscillations. For oscillations with opposite signs of wave energy to coexist in a common region, the effects of general relativity are important. It is noted that in such cases there is a tendency that the Lindblad resonance of an oscillation (e.g., p-mode) occurs in the propagation region of the other oscillation (e.g., g-mode). This helps to increase the growth rate, since the coupling terms become large, as mentioned before.

Finally, as an application of the present resonant excitation process of oscillations, we briefly note Kato’s model of high-frequency twin QPOs (e.g., Kato & Fukue 2006). In this model the lower-frequency QPO of the twin is related to the set of g-mode and p-mode oscillations considered numerically by Ferreira and Ogilvie (2008) and Oktariani et al. (2010). The HF QPO of the twin is considered to be the set of one-armed g-mode and two-armed p-mode oscillations [see for details figures 1 to 3 of (Kato 2008a)]. Here, the g-mode oscillation has a positive energy and the p-mode one has a negative energy, and thus this set of oscillations can also grow by the resonant coupling.

In his model, the correlated time variation of twin QPOs observed in neutron-star low-mass X-ray binaries is described by assuming time variation of $\omega_3$. One of problems of this model, however, is that masses of neutron stars required to take into account observations are rather high, i.e., for example, $2.4 M_\odot$ for Sco X-1 and 4U 1636–53. See Lin et al. (2011) for detailed comparisons of various QPO models with observations. In black-hole low-mass X-ray binaries, the twin QPOs have no frequency change with a frequency ratio of 3 : 2. In Kato’s model this can be described by assuming $\omega_3 = 0$. The spin parameters $a_*$’s estimated by this model for black-hole sources with measured masses are around $a_* \sim 0.4$ (e.g., Kato & Fukue 2006). However, the spin parameters estimated by comparing the model continuum X-ray spectra with observations are generally higher than those above, say around $a_* \sim 0.8$ or more (e.g., Narayan et al. 2008).

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This can be shown by examining local dispersion relation of these oscillations. The vertical p-mode oscillations that are trapped in finite region also have $\omega_1 > 0$. 

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