Smoothness of Lipschitz minimal intrinsic graphs in Heisenberg groups $\mathbb{H}^n$, $n > 1$

By Luca Capogna at Fayetteville, and Giovanna Citti and Maria Manfredini at Bologna

Abstract. We prove that Lipschitz intrinsic graphs in the Heisenberg groups $\mathbb{H}^n$, with $n > 1$, which are vanishing viscosity solutions of the minimal surface equation, are smooth and satisfy the PDE in a strong sense.

1. Introduction

The Heisenberg group $\mathbb{H}^n$, $n > 1$ is a Lie group with a $2n + 1$ dimensional Lie algebra $\mathfrak{h}^n$ endowed with a stratification $V_1 \oplus V_2$, where $V_1$ has dimension $2n$, and $V_2 = [V_1, V_1]$ has dimension 1. We will call $V_1$ the horizontal layer and the corresponding left-invariant bundle $H_{p \in \mathbb{H}^n}$ in $\mathbb{H}^n$ the horizontal bundle. Moreover, given the stratification one must be able to choose $\{X_s, X_1, \ldots, X_{2n-1}\}$ a basis of $V_1$ and a non-zero vector $X_{2n} \in V_2$ such that $[X_s, X_{2n-1}] = [X_i, X_{i+n-1}] = X_{2n}$, for $i = 1, \ldots, n - 1$ while all other commutators vanish. To denote elements of the group we use canonical coordinates of the second kind (the so called polarized coordinates [7]) and denote by $(s, x)$ the elements of the group,

$$\exp(sX_s)\exp(x_1X_1)\ldots\exp(x_{2n}X_{2n})$$

where $x = (x_1, \ldots, x_{2n})$. Using the Baker-Campbell-Hausdorff formula we can represent the basis of the Horizontal tangent space $V_1$ as follows:

$$X_s = \partial_s, \quad X_i = \partial_i \quad \text{for} \quad i = 1, \ldots, n - 1,$$

$$X_i = \partial_i - x_{i-n+1}\partial_{2n} \quad \text{for} \quad i = n, \ldots, 2n - 2,$$

$$X_{2n-1} = \partial_{2n-1} + s\partial_{2n}$$

while $X_{2n} = \partial_{2n} \in V_2$. We introduce a sub-Riemannian metric $g_0$ in $\mathbb{H}^n$ by requiring that $\{X_s, \ldots, X_{2n-1}\}$ form a $g_0$-orthonormal basis of $V_1$. For each $\varepsilon > 0$ we extend $g_0$ to a Riemannian metric $g_\varepsilon$ by asking that $\{X_s, \ldots, X_{2n-1}, \varepsilon X_{2n}\}$ form a $g_1$-orthonormal basis of $\mathfrak{h}^n$.

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Let $M \subset \mathbb{H}^n$ be a $C^1$ smooth hypersurface. A point $p \in M$ is characteristic if $T_pM$ is entirely horizontal. We will call intrinsic regular surfaces or graphs hypersurfaces of the form

$$M = \{(s, x) : s = u(x)\}$$

where $u : \mathbb{R}^{2n} \to \mathbb{R}$. Such graphs were studied in [24] and [15], and (if $u$ is at least $C^1$) are always non-characteristic.

Note that intrinsic graphs represent, in a sense, generic non-characteristic hypersurfaces. In fact, according to a version of the implicit function theorem ([24] and [15]), any level surface $\{f(s, x) = c\} \subset \mathbb{H}^n$ of functions $f : \mathbb{H}^n \to \mathbb{R}$ with continuous derivatives along the directions (1.1), can locally (near non-characteristic points) be expressed as an intrinsic graph of a function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^{2n}$. Moreover the smoothness of $f$ implies that the function $u$ is regular with respect to the projection on its domain of the vector fields in (1.1) (see [15] and [1]). Since $X_s$ has null projection on the domain of $u$, the regularity of this function will be described in terms of the vector fields:

$$(1.2) \quad X_{i,u} = X_i \quad \text{for} \quad i \leq 2n - 2, \quad X_{2n-1,u} = \partial_{2n-1} + u(x)\partial_{2n}.$$ 

In particular $X_{2n-1,u}$ is a non-linear vector field, since it depends on $u$. Note that the vector fields $X_{1,u}, \ldots, X_{2n-1,u}$ satisfy Hörmander’s finite rank condition in $\mathbb{R}^{2n}$. Consequently they give rise to a control distance $d_u$, whose metric balls $B_u(x, r)$ have volume comparable to $r^Q$, with $Q = 2n + 1$ the homogeneous dimension of the space ($\mathbb{R}^{2n}, d_u$).

**Minimal graphs.** Several equivalent notions of horizontal mean curvature $H_0$ for a $C^2$ surface $M \subset \mathbb{H}^n$ (outside characteristic points) have been given in the literature (see for instance [20], [8], [10], [5], [36], [37], [29], [30], [39], [40]). In particular, $H_0$ arises in the first variation formulae\(^{(1)}\) of the area functional. For a $C^2$ intrinsic graph of the form $M = \{(s, x) : s = u(x)\}$ one has

$$H_0 = \frac{1}{|V_u|} \sum_{i=1}^{2n-1} X_{i,u} \left( \frac{X_{i,u} u}{\sqrt{1 + |V_u u|^2}} \right) \text{ with } V_u = (X_{1,u}, \ldots, X_{2n-1,u}).$$

This formula can be obtained by taking the limit of the mean curvatures $H_\varepsilon$ in the Riemannian metrics $g_\varepsilon$ as $\varepsilon \to 0$

$$(1.3) \quad H_\varepsilon = \frac{1}{|V_u\varepsilon|} \sum_{i=1}^{2n-1} X_{i,u}^{\varepsilon} \left( \frac{X_{i,u}^{\varepsilon} u}{\sqrt{1 + |V_u^{\varepsilon} u|^2}} \right),$$

where

$$(1.4) \quad X_{i,u}^{\varepsilon} = X_{i,u} \quad \text{for} \quad i \leq 2n - 1, \quad X_{2n-1,u}^{\varepsilon} = \varepsilon \partial_{2n} \quad \text{and} \quad V_u^{\varepsilon} = (X_{1,u}^{\varepsilon}, \ldots, X_{2n-1,u}^{\varepsilon}).$$

A $C^2$ non-characteristic surface $M \subset \mathbb{H}^n$ is called minimal if it satisfies $H_0 = 0$ identically. In particular for a $C^2$ intrinsic graph (over $\Omega \subset \mathbb{R}^{2n}$) the PDE becomes

$$(1.5) \quad Lu = \sum_{i=1}^{2n-1} X_{i,u} \left( \frac{X_{i,u} u}{\sqrt{1 + |V_u u|^2}} \right) = 0 \quad \text{for} \quad x \in \Omega \subset \mathbb{R}^{2n}.$$

\(^{(1)}\) For variations which do not move the characteristic set.
If \( u \in C^2(\Omega) \) is a solution of (1.5) then its graph is a critical point of the perimeter and consequently it is called a minimal intrinsic graph.

Properties of regular minimal surfaces have been investigated in [26], [33], [10], [8], [25], [21], [2] and [32].

Since minimal surfaces arise as critical points of the perimeter functional, the variational formulation naturally provides several notions of non-regular solutions (see for instance [26], [33] and [8]). Indeed existence of BV and Sobolev minimizers of the perimeter is proved respectively in [26], [33] using direct methods of the calculus of variations. More recently, existence of Lipschitz continuous vanishing viscosity solutions has been studied in [8]. Such solutions are defined as uniform limits of smooth minimal surfaces in the \( g_\varepsilon \) approximating Riemannian metrics (see [33], [7] and [8] for the relation between Riemannian and sub-Riemannian curvature). The Riemannian approximation of (1.5) is

\[
\sum_{i=1}^{2n} X_{i,u} \frac{X_{i,u}^\varepsilon}{\sqrt{1 + |\nabla_x u|^2}} = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^{2n}.
\]

The existence results in [8] motivate the following

**Definition 1.1.** Letting \( C^1_E \) denote the standard Euclidean \( C^1 \) norm, we will say that a Euclidean Lipschitz continuous function \( u \) is a vanishing viscosity solution of (1.5) in an open set \( \Omega \), if there exists a sequence \( u_\varepsilon \) of smooth solutions of (1.6) in \( \Omega \) such that for every compact set \( K \subset \Omega \)

- \( \|u \|_{C^1_E(K)} \leq C \) for every \( \varepsilon \),
- \( u_\varepsilon \to u \) as \( \varepsilon \to 0 \) pointwise a.e. in \( \Omega \).

As mentioned above, existence of this type of viscosity solutions in the case of \( t \)-graphs, i.e. graphs of the form \( x_{2n} = g(x, x_1, \ldots, x_{2n-1}) \), has been proved in [8], Theorem A and Theorem 4.5. For such a graph the corresponding PDE is more degenerate than (1.6) as characteristic points are allowed (indeed, much of the analysis in [8] and [10] is focused on the study of solutions near such points). In [8] the authors prove that such solutions are minimizers of the perimeter and address questions of uniqueness and comparison theorems as well. The problem of regularity of minimal surfaces is still largely open. In this paper we address the issue of regularity away from characteristic points. Our goal is to prove the following

**Theorem 1.2.** The Lipschitz continuous vanishing viscosity solutions of (1.5) are smooth functions. Moreover they satisfy pointwise the minimal graph PDE (1.5).

**Comparison with other regularity results.** We describe the relation between our results in Theorem 1.2 and the regularity results in [9]. In [9] Cheng, Hwang and Yang prove that any \( C^1 \) weak solution of the prescribed (continuous) horizontal mean curvature PDE, has \( C^2 \) smooth Legendrian foliation outside of the characteristic set. On the one hand, in the present paper we require only Lipschitz continuity of \( u \) and prove Euclidean smoothness, thus implying the result in [9]. On the other hand, we only deal with the case \( H_0 = 0 \).
and with those solutions which are limits of Riemannian minimal graphs. In this sense our results are more specialized and less general than the ones in [9]. A comparison with other regularity results for the case $n = 1$ due to Bigolin and Serra-Cassano [3] is discussed in our paper [6].

**Applications of Theorem 1.2.** Invoking the implicit function theorem, we want to apply Theorem 1.2 to the study of the regularity away from the characteristic locus of the Lipschitz perimeter minimizers found in [8] for the case $\mathbb{H}^n$, $n > 1$. Here and in the following, $\nabla_E$ denotes the Euclidean gradient in $\mathbb{R}^n$. We also define exponential coordinates $(z_1, \ldots, z_{2n+1})$ in $\mathbb{H}^n$ as

$$\exp(z_1X_s + \cdots + z_{2n+1}X_{2n}) = \exp(sX_s) \cdots \exp(x_{2n}X_{2n}).$$

**Corollary 1.3.** Let $O \subset \mathbb{R}^n$ be a strictly convex, smooth open set, $\varphi \in C^{2,\alpha}(\overline{O})$ and for each $(z_1, \ldots, z_{2n}) \in O$ denote $(z_1, \ldots, z_{2n})^* = (z_2, -z_1, z_4, -z_3, \ldots)$. Consider the family

$$\{g_\varepsilon(z_1, \ldots, z_{2n})\} \sup_{O} |g_\varepsilon| + \sup_{O} |\nabla_E g_\varepsilon| \leq C \quad \text{(uniformly in $\varepsilon$)},$$

of smooth solutions of the approximating minimal surface PDE

$$\text{div} \left( \frac{\nabla_E g_\varepsilon + (z_1, \ldots, z_{2n})^*}{\sqrt{\varepsilon^2 + |\nabla_E g_\varepsilon + (z_1, \ldots, z_{2n})^*|^2}} \right) = 0 \quad \text{in } O \quad \text{and} \quad g_\varepsilon = \varphi \quad \text{in } \partial O$$

found in [8], Theorem 4.5. If for $p_0 = (p_0^1, p_0^2, \ldots, p_0^{2n}) \in O$, $a > 0$ and for every $\varepsilon > 0$ we have $|c_{\partial_0}g_\varepsilon(p_0)| > a > 0$ (or any other partial derivative is non-vanishing at $p_0$ uniformly in $\varepsilon$) then there is a sequence $\varepsilon_k \to 0$ such that the Lipschitz perimeter minimizer $g = \lim_{\varepsilon_k \to 0} g_{\varepsilon_k}$ is smooth in a neighborhood of the point $p_0$.

**Proof.** The implicit function theorem implies that the level set of

$$z_{2n+1} = g_\varepsilon(z_1, \ldots, z_{2n})$$

can be written as smooth intrinsic graphs $s = u_\varepsilon(x)$ in a neighborhood $\Omega$ of $(p_0^1, \ldots, p_0^{2n}, g(p_0))$. The Lipschitz bounds on $g_\varepsilon$ (proved in [8], Propositions 4.2–4) yield uniform Lipschitz bounds on $u_\varepsilon$, thus allowing to apply Theorem 1.2 and conclude the proof. \hfill \Box

From this result one may conclude immediately the following

**Corollary 1.4.** Let $z_{2n+1} = g(z_1, \ldots, z_{2n})$, $(z_1, \ldots, z_{2n}) \in \Omega \subset \mathbb{R}^{2n}$ be a $C^1$ minimal $t$-graph which is the $C^1$ limit of Riemannian minimal graphs as in [8], Theorem 4.5. In the neighborhood of any non-characteristic point $g$ is smooth and a strong solution of the minimal surface PDE.

**Remark 1.5.** Given our regularity result and the wealth of examples of area minimizers with low regularity constructed recently by Cheng, Hwang and Yang in [8], by Pauls in [34] and by Ritoré in [35], it is clear that not all minimal surfaces, not even those that are area minimizers arise as limits of Riemannian minimal surfaces or area minimizers in the Riemannian approximation scheme.
The one dimensional case. We remark that in the case \( n = 1 \) of the first Heisenberg group the regularity of vanishing viscosity minimal intrinsic graphs is quite different. In the forthcoming paper [6] we study this problem and prove a form of intrinsic regularity, with differentiability only along the Legendrean foliation of the minimal graph.

Sketch of the proof. Equation (1.5) is a uniformly elliptic approximation of a sub-elliptic equation. The defining vector fields have Lipschitz coefficients and satisfy a weak Hörmander condition, since together with their first order vector fields they span the space at every point. The main difficulty of the proof is to handle the vector field

\[
X_{2n-1,u} = \partial_{2n-1} + u(x)\partial_{2n}
\]

and the dependence on \( \varepsilon \). A similar difficulty arises in problems of mathematical finance. For example in [18], [19], it was proved that the viscosity solutions of the following equation are \( C^\infty \):

\[
X_1^2 u + X_2 u = 0
\]

where \( X_1 = \partial_{xx}, X_2 = \partial_y u + u\partial_z u \) satisfy a weak Hörmander condition analogous to the one in the present paper. The techniques in [18], [19] provide the main inspiration for the proof of Theorem 1.2.

The regularity of solutions will be measured in terms of the natural norm of the intrinsic Hölder class \( C^{1,\alpha} \), i.e. functions \( f \) such that \( \nabla^\varepsilon f \) is Hölder continuous, with respect to the control distance \( d_u \). The proof will be accomplished in two steps:

Step 1. First prove that the Lipschitz continuous solutions are of class \( C^{1,\alpha} \). Since the operator \( L_\varepsilon \) in (1.6) is represented in divergence form, by differentiating the PDE and combining several horizontal and “vertical” energy estimates, it is possible to prove a Euclidean Cacciopoli-type inequality for the intrinsic gradient \( \nabla^\varepsilon u \) of the solution. The Moser iteration technique will then lead to Hölder continuous estimates uniform in \( \varepsilon \) for the gradient. This step holds also for \( n = 1 \).

Step 2. We prove the smoothness of the solution. In order to do so we first note that the operator \( L_\varepsilon \) can also be represented in a divergence form:

\[
L_\varepsilon u = \sum_{i,j=1}^{2n} a^\varepsilon_{ij}(\nabla^\varepsilon u) X^\varepsilon_{i,u} X^\varepsilon_{j,u} u,
\]

where

\[
a^\varepsilon_{ij} : \mathbb{R}^{2n} \to \mathbb{R}, \quad a^\varepsilon_{ij}(p) = \delta_{ij} - \frac{p_ip_j}{1 + |p|^2}.
\]

For every fixed point \( x_0 \) we will approximate the assigned operator with a linear, uniformly subelliptic operator in divergence form \( L_{\alpha,x_0} \), with \( C^\infty \) coefficients. The approximation is carried out through a ad-hoc freezing technique, where the function \( u \) in the coefficients of the vector field is substituted with polynomials, in a technique reminiscent of the work of
Rothschild and Stein [38]. The novel difficulty arises from the non-smoothness of \( u \), and has to be dealt with through a delicate bootstrap argument. The existence of a fundamental solution \( \Gamma_{x_0}^e \) for such operator as well as its estimates, uniform in \( e \), have previously been proved in the papers [4] and [14]. Eventually \( \Gamma_{x_0}^e \) will be used to define a parametrix for the fundamental solution of \( L_e \) and to obtain estimates independent of \( e \), of the derivatives of any order of the solution.

2. Notations and known results

2.1. Hölder classes. In the sequel we will always keep a function \( \bar{u} \in C^\infty = C^\infty (\Omega) \) with \( \Omega \subset \mathbb{R}^{2n} \) fixed and consider the vector fields \( X_{i,\bar{u}}^e \) in (1.4) with coefficients depending on the fixed function \( \bar{u} \). Let us define a new vector field

\[
X_{2n+1,\bar{u}}^e = \partial_2 n = [X_{1,\bar{u}}^e, X_{n,\bar{u}}^e],
\]

which act as a second order derivative, in the sense that it is a homogenous differential operator of order two with respect to the group dilations. We set the degree of \( \sigma_i \) to be the integer \( \deg(\sigma_i) = 1 \) for \( \sigma_i \leq 2n \), and \( \deg(2n+1) = 2 \). Correspondingly the degree of any multi-index \( \sigma = (\sigma_1, \ldots, \sigma_m) \), \( \sigma_r \in \{1, \ldots, 2n+1\} \), \( 1 \leq r \leq m \in \mathbb{N} \), will be

\[
\deg(\sigma) = \sum_{i=1}^{m} \deg(\sigma_i).
\]

We will also denote the cardinality of \( \sigma = (\sigma_1, \ldots, \sigma_m) \) the number of its elements:

\[
\#(\sigma) = m.
\]

We define the intrinsic derivative

\[

(2.1)
\]

\[
\nabla_{\sigma,\bar{u}} = X_{\sigma_1,\bar{u}}^e \cdots X_{\sigma_m,\bar{u}}^e,
\]

and \( \nabla_{\bar{u}}^{ek} \) the vector field with components \( \nabla_{\sigma,\bar{u}}^{\deg(\sigma)=k} \).

Since the vector fields \( X_{1,\bar{u}}^e, \ldots, X_{2n,\bar{u}}^e \) are the Riemannian completion of a Hörmander type set of vectors, they give rise to a control distance \( d_{e,\bar{u}} \). The corresponding metric balls are denoted \( B_{e,\bar{u}}(x, r) \). As \( e \to 0 \) the metric space \( (\Omega, d_{e,\bar{u}}) \) converges in the Gromov-Hausdorff sense to \( (\Omega, d_{\bar{u}}) \) (see [7]).

We next define the spaces of Hölder continuous functions related to the fixed function \( \bar{u} \).

Definition 2.1. Let \( x_0 \in \Omega \), \( 0 < \alpha < 1 \), assume that \( \bar{u} \) is a fixed Lipschitz continuous function, and that \( u \) is defined on \( \Omega \). We say that \( u \in C^{\alpha}_{\bar{u}}(\Omega) \) if for every compact set \( K \) there exists a positive constant \( M \) such that for every \( x, x_0 \in K \) and \( \varepsilon > 0 \)

\[

(2.2)
\]

\[
|u(x) - u(x_0)| \leq Md_{e,\bar{u}}^{\alpha}(x, x_0).
\]
Iterating this definition, if \( k \geq 1 \) and \( \vec{u} \in C^{k-1,2}_{\vec{a}}(\Omega) \) we say that \( u \in C^{k,2}_{\vec{a}}(\Omega) \), if \( \nabla^2_{\vec{a}}u \in C^{k-1,2}_{\vec{a}}(\Omega) \). Here we have set \( C^{0,2}_{\vec{a}}(\Omega) = C^{2}_{\vec{a}}(\Omega) \).

### 2.2. Taylor approximation.

The following result is well known for vector fields with \( C^\infty \) coefficients (see [32]) and also holds for vector fields of the form \( \partial_1 + \vec{u} \partial_{2n} \), with \( \vec{u} \) Lipschitz continuous with respect to the Euclidean distance. Let us first denote by \( e_1, \ldots, e_{2n} \) the canonical coordinates of a point \( x \) around \( x_0 \),

\[
    x = \exp\left( \sum_{i=1}^{2n-1} e_i X^e_{i,a} + e_{2n} X^e_{2n+1,a} \right)(x_0)
\]

and, for a multi-index \( \sigma = (\sigma_1, \ldots, \sigma_{2n}) \) we will denote \( e_\sigma = (e_{\sigma_1}, \ldots, e_{\sigma_{2n}}) \).

We explicitly note that, since \( X^e_{2n,u} = e \partial_{2n} \) and \( X^e_{2n+1,u} = \partial_{2n} \) are parallel, only one of them can appear in the definition of canonical coordinates, otherwise the values of \( e_i \) would not be uniquely determined. Although both vector fields would work equally well to define canonical coordinates, we use \( X^e_{2n+1,u} = \partial_{2n} \) rather than \( X^e_{2n,u} \) as this choice will bring a slight simplification of notations as we proceed with our estimates.

For every multi-index \( \sigma = (\sigma_1, \ldots, \sigma_m) \), with components in \( \{1, \ldots, 2n\} \) we will denote \( I(\sigma) = (\varrho_1, \ldots, \varrho_m) \), where \( \varrho_i = \sigma_i \) if \( \sigma_i \neq 2n \) and \( \varrho_i = 2n + 1 \) if \( \sigma_i = 2n \).

**Theorem 2.2.** Let \( x_0 \in \Omega \), \( 0 < \alpha < 1 \), \( k \in \mathbb{N} \cup \{0\} \) and assume that \( u \in C^{k,2}_{\vec{a}}(\Omega) \). Then we can define as Taylor polynomial of order \( k \) the function

\[
    P^k_{x_0} u(x) = \sum_{h=1}^{k} \sum_{\deg(\sigma) = n \atop \sigma_i \neq 2n+1} \frac{1}{\#(\sigma)!} e_\sigma \nabla^2_{\vec{a}} u(x_0)
\]

and we have

\[
    (2.3) \quad u(x) = P^k_{x_0} u(x) + O(d_{\vec{a}}(x_0, x)^{k+2}) \quad \text{as } x \to x_0.
\]

We will also set \( P^k_{x_0} u = 0 \) for any negative integer \( k \).

Note that

\[
    P^1_{x_0} u(x) = \sum_{i=1}^{2n-1} e_i(x) X^e_{i,a} u(x_0) + e_{2n}(x) X^e_{2n+1,a} u(x_0).
\]

From the explicit expression of the Taylor polynomials of order less than 4 it is possible to directly deduce the following result.

**Remark 2.3.** If \( u \in C^{k,2}_{\vec{a}}(\Omega) \), \( 0 \leq k \leq 4 \), \( K \) is a compact subset of \( \Omega \) and \( \sigma \) is a multi-index, then there exists \( C > 0 \) such that

\[
    (2.4) \quad |P^k_{x_0} u(\xi) - P^k_{x} u(\xi)| \leq Cd^{2}_{\vec{a}}(x_0, x) d^{2}_{\vec{a}}(x_0, \xi)
\]

for every \( x, x_0, \xi \in K \), see [13], Lemma 3.6, and [18], Remarks 2.24 and 2.25.
2.3. Derivatives and frozen derivatives. We will introduce here first order operators with polynomial coefficients which locally approximate the vector fields $X_{i,a}^e$. These new vector fields are defined in terms of the Taylor expansions of the coefficients of $X_{i,a}^e$. Precisely, for any fixed point $x_0$ we will call an operator frozen at the point $x_0$ 

$$X_{i, x_0}^e = X_{i, u}^e \quad \text{if } i \neq 2n - 1,$$

$$X_{2n-1, x_0}^e = \partial_{2n-1} + P_{x_0}^1 \tilde{u}(x) \partial_{2n}$$

and for every multi-index $\sigma$,

$$\nabla_{\sigma, x_0}^e = X_{\sigma_1, x_0}^e \cdots X_{\sigma_m, x_0}^e,$$

and $\nabla_{\sigma, x_0}^e$ will be the vector field with components $(\nabla_{\sigma, x_0}^e)_{\deg(\sigma) = k}$.

These frozen derivatives have been defined as approximation of the intrinsic derivatives, depending on $\tilde{u}$. In order to clarify this point, we recall the following definition, given in [23] and [32]. If $\alpha \in \mathbb{R}$ and $f(x, x_0) = O(d_{x, x_0}^\alpha(x))$ as $x \to x_0$, we will say that the differential operator $f(x, x_0)\nabla_{\alpha, x_0}^e$ has degree $\deg(\sigma) - \alpha$. We have

$$X_{i, x_0}^e = X_{i, u}^e \quad \text{if } i \neq 2n - 1,$$

$$X_{2n-1, x_0}^e = \partial_{2n-1} + (P_{x_0}^1 \tilde{u}(x)) \partial_{2n} \quad \text{if } i = 2n - 1.$$ 

Hence, if $\tilde{u}$ is of class $C^{k-1, \alpha}_{\tilde{u}}$, then $(\tilde{u} - P_{x_0}^1 \tilde{u}(x)) \partial_{2n}$ is a differential operator of degree $1 - \alpha$, while $X_{i, a}^e$ and $X_{i, x_0}^e$ have degree 1. This means that the intrinsic derivative is expressed as the frozen derivative, plus a lower order term.

More generally the following approximation result holds:

**Lemma 2.4.** If $\tilde{u} \in C^{k-1, \alpha}_{\tilde{u}}(\Omega)$, and $\sigma$ is a multi-index such that $\deg(\sigma) \leq k$, then for every function $\varphi \in C^0_{\alpha}(\Omega)$ the derivative $\nabla_{\sigma, \alpha}^e \varphi$ can be represented as

$$\nabla_{\sigma, \alpha}^e \varphi = \nabla_{\sigma, x_0}^e \varphi + \sum_{\deg(\mu) \leq \deg(\sigma)} (\tilde{u} - P_{x_0}^1 \tilde{u})^h$$

$$\times \sum_{\deg(\mu) \leq \deg(\sigma)} C_{\rho, \mu, \alpha, h} \prod_{1 \leq \deg(\mu)} \nabla_{\mu, \alpha}^e (\tilde{u} - P_{x_0}^1 \tilde{u}) \nabla_{\rho, x_0}^e \varphi,$$

where $C_{\rho, \mu, \alpha, h}$ are suitable constants. In particular the operator $\nabla_{\sigma, \alpha}^e \varphi(x)$ can be identified as a differential operator of degree $\deg(\sigma)$ and represented in terms of frozen derivatives.

**Proof.** Since the function $\varphi$ is of class $C^0_{\alpha}(\Omega)$, its Lie derivatives can be simply computed as directional derivatives. By definition

$$X_{i, \alpha}^e \varphi = X_{i, x_0}^e \varphi + \partial_{i, 2n-1} (\tilde{u} - P_{x_0}^1 \tilde{u}) \partial_{2n} \varphi.$$ 

Hence the assertion is true if $\deg(\sigma) = 1$. If the assertion is true for any $\sigma$ such that $\deg(\sigma) = k$, then we consider a multiindex $\sigma$ such that $\deg(\sigma) = k + 1$. In this case

$$\sigma = (\sigma_1, \sigma),$$
where
\[
\begin{aligned}
\text{deg}(\tilde{\sigma}) &= k & \text{if} \; \sigma_1 \neq 2n + 1, \\
\text{deg}(\tilde{\sigma}) &= k - 1 & \text{if} \; k \geq 3 \text{ and } \sigma_1 = 2n + 1.
\end{aligned}
\]

We have
\[
V^{E}_{\tilde{\sigma},\tilde{\mu}} \phi(x) = X^{E}_{\sigma_1,\tilde{\mu}} V^{E}_{\sigma_1,\mu} \phi(x)
\]
(by inductive assumption)
\[
= X^{E}_{\sigma_1,\tilde{\mu}}(V^{E}_{\tilde{\sigma},\tilde{\nu}} \phi) + \sum_{\text{deg}(\rho) - h \leq \text{deg}(\tilde{\sigma})} X^{E}_{\sigma_1,\tilde{\mu}} X^{E}_{\sigma_1,\tilde{\nu}}
\times \left( \left( \tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u} \right)^h \sum_{\text{deg}(\mu_1) + \cdots + \text{deg}(\mu_k) \leq k - 1} C_{\rho,\mu_1,\tilde{\sigma},h} \prod_{1 \leq \text{deg}(\mu_i) \atop \text{deg}(\mu_i) \geq 0} V^{E}_{\mu_1,\tilde{u}}(\tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u}) V^{E}_{\rho,\chi_{x_0}} \phi \right)
\]
(also using (2.6))
\[
= X^{E}_{\sigma_1,\chi_{x_0}} V^{E}_{\chi_{x_0}} \phi + \delta_{\sigma_1,2n-1} \left( \tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u} \right) \tilde{c}_{2n} V^{E}_{\chi_{x_0}} \phi
\]
\[
+ \sum_{\text{deg}(\rho) - h \leq \text{deg}(\tilde{\sigma})} \left( \tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u} \right)^{h-1} \prod_{1 \leq \text{deg}(\mu_i) \atop \text{deg}(\mu_i) \geq 0} V^{E}_{\mu_1,\tilde{u}}(\tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u}) X^{E}_{\sigma_1,\tilde{\nu}}
\times \left( \prod_{1 \leq \text{deg}(\mu_i)} V^{E}_{\mu_1,\tilde{u}}(\tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u}) \right) V^{E}_{\rho,\chi_{x_0}} \phi
\]
\[
+ \sum_{\text{deg}(\rho) - h \leq \text{deg}(\tilde{\sigma})} \left( \tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u} \right)^{h} \prod_{1 \leq \text{deg}(\mu_i) \atop \text{deg}(\mu_i) \geq 0} \prod_{1 \leq \text{deg}(\mu_i)} V^{E}_{\mu_1,\tilde{u}}(\tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u}) X^{E}_{\sigma_1,\tilde{\nu}}(V^{E}_{\rho,\chi_{x_0}} \phi)
\]
\[
= \delta_{\sigma_1,2n-1} \sum_{\text{deg}(\rho) - h \leq \text{deg}(\tilde{\sigma})} \left( \tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u} \right)^{h+1} \prod_{1 \leq \text{deg}(\mu_i) \atop \text{deg}(\mu_i) \geq 0} C_{\rho,\mu_1,\tilde{\sigma},h}
\times \prod_{1 \leq \text{deg}(\mu_i)} V^{E}_{\mu_1,\tilde{u}}(\tilde{u} - P^{1}_{\chi_{x_0}} \tilde{u}) \tilde{c}_{2n} V^{E}_{\rho,\chi_{x_0}} \phi.
\]

Note that the first term satisfies
\[
X^{E}_{\sigma_1,\chi_{x_0}} V^{E}_{\chi_{x_0}} \phi = V^{E}_{\chi_{x_0}} \phi.
\]
The second term (when non-zero) is \((\bar{u} - P^1_{x_0} \bar{u}) \partial_{2n} \nabla_{\nabla_{x_0}} \varphi\). It can be considered as one of the terms listed in the thesis, with \(h = 1\), while \(\partial_{2n} \nabla_{\nabla_{x_0}} = \nabla_{\nabla_{x_0}}\), for a suitable \(\varphi\), of degree \(\deg(\varphi) = k + 2\). Hence \(\deg(\varphi) - h = k + 1\). Similarly, all the other terms are in the form indicated in the thesis (in both case, \(\sigma_1 = 2n + 1\) or \(k \geq 3\) and \(\sigma_1 = 2n + 1\).

The vector fields \(X_i^{e_{x_0}}\) satisfy a Hörmander type condition, hence they define a control distance \(d_{e, x_0}(x, \xi)\). The corresponding metric balls \(B_{e, x_0}(x, r)\) have volume comparable to \(r^{2n+1}\), and we will call

\[
Q = 2n + 1
\]

the homogeneous dimension of the space \((\mathbb{R}^{2n}, d_{e, x_0})\). Note that the homogeneous dimension is the same as the Hausdorff dimension of \((\mathbb{R}^{2n}, d_\varphi)\), defined in the introduction. A simple modification of [18], Proposition 2.4 yields the following relation between \(d_{e, x_0}\) and the control distance \(d_{e, \varphi}\) associated to the vector fields \(X_i^{e_{x_0}}\):

**Proposition 2.5.** For every compact subset \(K\) of \(\Omega\), there exists a positive constant \(C = C(K)\) such that for every \(x, x_0 \in K\):

\[
C^{-1} d_{e, x_0}(x_0, \xi) \leq d_{e, \varphi}(x_0, \xi) \leq C d_{e, x_0}(x_0, \xi),
\]

\[
d_{e, x_0}(x_0, x) \leq C(d_{e, x_0}(x_0, \xi) + d_{e, \varphi}(\xi, x)),
\]

\[
d_{e, \varphi}(x_0, x) \leq C(d_{e, \varphi}(x_0, \xi) + d_{e, \varphi}(\xi, x)).
\]

2.4. Linearized and frozen operator. In analogy with the definition of linear vector fields, in terms of a fixed function \(u\), we can also define a linearization \(L_{e, \varphi}\) of the operator \(L_e\), written in terms of the linearized vector fields \(X_i^{e_{x_0}}\):

\[
L_{e, \varphi} u = \sum_{i, j=1}^{2n} a_{ij}^e(\nabla_{\varphi} u) X_i^{e_{x_0}} X_j^{e_{x_0}} u,
\]

where \(a_{ij}^e\) are defined in (1.7). Since the function \(u\) is fixed, the operator is a linear non-divergence type operator, whose coefficients have the regularity of the function \(u\). In case \(u\) is not smooth, it is natural to approximate it with a frozen operator, defined in terms of the vector fields \(X_i^{e_{x_0}}\):

\[
L_{e, x_0} u = \sum_{i, j=1}^{2n} a_{ij}^e(\nabla_{x_0} u(x_0)) X_i^{e_{x_0}} X_j^{e_{x_0}} u,
\]

where \(a_{ij}^e\) are defined in (1.7). This is a divergence form uniformly subelliptic operator with \(C^\infty\) coefficients, which depends on \(e\). Hence it has a fundamental solution \(\Gamma_{x_0}^e\) (see [4]), and its dependence on \(e\) can be handled as in [14]. Since \(\Gamma_{x_0}^e\) depends on many variables, the notation

\[
X_i^{e_{x_0}}(x) \Gamma_{x_0}^e(s, \xi)
\]

shall denote the \(X_i^{e_{x_0}}\)-derivative of \(\Gamma_{x_0}^e(s, \xi)\) with respect to the variable \(s\), evaluated at the point \(x\).
**Theorem 2.6** ([14], Theorem 1.1). Let \( x_0 \in \Omega \) For every compact set \( K \subset \Omega \) and for every \( p \in \mathbb{N} \) there exist two positive constants \( C, C_p \) independent of \( \varepsilon \), such that

\[
|\nabla_{\sigma, x_0}^\varepsilon (x) \Gamma_{x_0}^\varepsilon (\cdot, \xi)| \leq C_p \frac{d_{e,x_0}^{2-p}(x, \xi)}{B_{e,x_0}(x, d_{e,x_0}(x,\xi))}, \quad \deg(\sigma) = p,
\]

for every \( x, \xi \in K \) with \( x \neq \xi \), where \( B_{e,x_0}(x,r) \) denotes the ball with center \( x \) and radius \( r \) of the distance \( d_{e,x_0} \). If \( p = 0 \) we mean that no derivative is applied on \( \Gamma_{x_0}^\varepsilon \).

**Remark 2.7.** With the same notation as in the preceding theorem, from Lemma 2.4, and inequality (2.10) it follows that

\[
|\nabla_{\sigma, x_0}^\varepsilon (x) \Gamma_{x_0}^\varepsilon (\cdot, \xi)| \leq C_p \frac{d_{e,x_0}^{2-p}(x, \xi)}{B_{e,x_0}(x, d_{e,x_0}(x,\xi))}, \quad \deg(\sigma) = p,
\]

for every \( x, \xi \in K \) with \( x \neq \xi \).

The first step in our bootstrap argument for regularity consists in establishing representation formulae for the solution \( u \) and for its derivatives up to order four. Such representation formulae involve derivatives of the fundamental solution \( \Gamma_{x_0}^\varepsilon \) evaluated at points different from \( x_0 \). In order to provide estimates for such terms we invoke [18], Propositions 2.20 and 2.21, to obtain the following

**Proposition 2.8.** Let \( k \in \mathbb{N}, 2 \leq k \leq 4 \). Let \( \tilde{u} \in C^{k-1,2}_{\tilde{a}}(\Omega) \) and \( K \) be a compact subset of \( \Omega \). For every multi-index \( \sigma, \deg(\sigma) = k \), there are positive constants \( C \) and \( M \), independent of \( \varepsilon \), such that

\[
|\nabla_{\sigma, x_0}^\varepsilon \tilde{u}(x) - \nabla_{\sigma, x_0}^\varepsilon (x_0)| \Gamma_{x_0}^\varepsilon (\cdot, \xi)\]

\[
\leq C(d_{e,x_0}(x_0,x))d_{e,x_0}(x_0,\xi)^{-Q-\deg(\sigma)+1} + d_{e,x_0}(x_0,x)^2d_{e,x_0}(x_0,\xi)^{-Q-\deg(\sigma)+2},
\]

and

\[
|\nabla_{\sigma, x_0}^\varepsilon \tilde{u}(x) \Gamma_{x_0}^\varepsilon (\cdot, \xi) - \nabla_{\sigma, x_0}^\varepsilon (x_0) \Gamma_{x_0}^\varepsilon (\cdot, \xi)|
\leq C(d_{e,x_0}(x_0,x))d_{e,x_0}(x_0,\xi)^{-Q-\deg(\sigma)+1} + d_{e,x_0}(x_0,x)^2d_{e,x_0}(x_0,\xi)^{-Q-\deg(\sigma)+2},
\]

for every \( x, x_0 \in K \) and \( \xi \) such that \( d_{e,x_0}(x_0,\xi) \geq Md_{e,x_0}(x_0,x) \). The constant \( Q \) is the homogeneous dimension of the space, defined in (2.7).

Estimates of this type for the fundamental solution are the key elements used in [18], Proposition 3.9, to prove the following result:
Proposition 2.9. Let $k \in \mathbb{N}$, $2 \leq k \leq 4$. Assume that $u$ is a function of class $C^{k-1}_{\partial}(\Omega)$ and that there are open sets $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \Omega$ such that for every $x \in \Omega_1$ the function $u$ admits the following representation:

$$
u(x) = \int_{\Omega} \Gamma^{e}_{x_0}(x, \xi) N_{1}(\xi, x_0) \, d\xi + \int_{\Omega} \Gamma^{e}_{0}(x, \xi) N_{2,k}(\xi, x_0) \, d\xi$$

$$+ \sum_{i=1}^{2n} \int_{\Omega} X^{e}_{i, x_0}(x, \xi) N_{2,kl}(\xi, x_0) \, d\xi + \int_{\Omega} \Gamma^{e}_{x_0}(x, \xi) N_{3,k}(\xi, x_0) \, d\xi.$$

Also assume that for every $x_0$ fixed $\in \Omega_1$, the kernels $N_{1}(\cdot, x_0)$ are supported in $\overline{\Omega_3} \setminus \overline{\Omega_2}$, as functions of their first variable and there exists a constant $C_1$ such that the kernels satisfy the following conditions:

(i) If $x_0$ is fixed $\in \Omega_1$ the $N_{1}(\cdot, x_0)$ is supported in $\overline{\Omega_3} \setminus \overline{\Omega_2}$,

$$|N_{1}(\xi, x_0) - N_{1}(\xi, x)| \leq C_1 d^{x}_{e, x_0}(x_0, x).$$

(ii) $N_{2,k}(\cdot, x_0)$ and $N_{2,kl}(\cdot, x_0)$ are smooth functions and all derivatives are uniformly Hölder continuous in the variable $x_0$, satisfying condition (2.15) with the same constant $C_1$ as $N_{1}$.

(iii) For every $\xi \in \Omega_3$ and $x, x_0 \in \Omega_1$,

$$|N_{3,k}(\xi, x_0)| \leq C_1 d^{x}_{e, x_0}(x_0, \xi),$$

and

$$|N_{3,k}(\xi, x_0) - N_{3,k}(\xi, x)| \leq C_1 d^{x}_{e, x_0}(x_0, x) d^{x}_{e, x_0}(x_0, \xi).$$

Then $u \in C^{k}_{\partial}$ and for every $\sigma$ such that $\deg(\sigma) = k$,

$$\nabla^{e}_{\sigma, \partial}(u)(x_0) = \int_{\Omega} \nabla^{e}_{\sigma, \partial}(x_0) \Gamma^{e}_{x_0}(\cdot, \xi) N_{1}(\xi, x_0) \, d\xi$$

$$+ \int_{\Omega} \Gamma^{e}_{x_0}(\xi, 0) \nabla^{e}_{\sigma, \partial}(x_0) (N_{2,k}(x_0 \circ \xi^{-1}, x_0)) \, d\xi$$

$$+ \sum_{i=1}^{2n} \int_{\Omega} X^{e}_{i, x_0}(x_0) \Gamma^{e}_{x_0}(\xi, 0) \nabla^{e}_{\sigma, \partial}(x_0) (N_{2,kl}(x_0 \circ \xi^{-1}, x_0)) \, d\xi$$

$$+ \int \nabla^{e}_{\sigma, \partial}(x_0) \Gamma^{e}_{x_0}(x, \xi) N_{3,k}(\xi, x_0) \, d\xi.$$

Besides, for any $\alpha' < \alpha$, there exists a constant $C$ only dependent on $C_1$ and on $C_{\alpha}$ in (2.10) such that

$$\|u\|_{C^{\alpha', \alpha}_{\partial}} \leq C.$$

Remark 2.10. The derivatives $\nabla^{e}_{\sigma, \partial}(x_0)$ in (2.18) can be computed by Lemma 2.4 in terms of the frozen derivatives $\nabla^{e}_{\sigma, x_0}(x_0)$. In particular the frozen derivatives $\nabla^{e}_{\sigma, x_0}(x_0)(h(\cdot \circ \xi^{-1}))$ can be calculated by the formula in [18], Proposition 2.23.
Remark 2.11. It is not difficult to prove that the same result is still true, if \( u \) has a more general representation

\[
(2.19) \quad u(x) = \int_{\Omega} \Gamma_{x_0}^e(x, \xi) N_1(\xi, x_0) \, d\xi + \int_{\Omega} \Gamma_{x_0}^e(x, \xi) N_{2,k}(\xi, x_0) \, d\xi \\
+ \sum_{i=1}^{2n} \int X_{i,x_0}^e(x, \xi) N_{2,ki}(\xi, x_0) \, d\xi + \int \Gamma_{x_0}^e(x, \xi) N_{3,k}(\xi, x_0) \, d\xi \\
+ \sum_{i=1}^{2n} \int X_{i,x_0}^e(x, \xi) N_{4,ki}(\xi, x_0) \, d\xi,
\]

where the kernels \( N_{4,ki}(\xi, x_0) \) satisfy assumptions similar to \( N_{3,k} \):

For every \( \xi \in \Omega_3 \) and \( x, x_0 \in \Omega_1 \),

\[
(2.20) \quad |N_{4,ki}(\xi, x_0)| \leq C_1 d_{x_0}^{k-1+\theta}(x_0, \xi),
\]

and

\[
(2.21) \quad |N_{4,ki}(\xi, x_0) - N_{4,ki}(\xi, x)| \leq C_1 d_{x_0}^{k-1}(x_0, \xi).
\]

3. From Lipschitz to \( C^1 \) regularity

Let us now start the first step in the proof of the regularity result. Using in full strength the nonlinearity of the operator \( L_e \), we prove here some Cacciopoli-type inequalities for the intrinsic gradient of \( u \), and for the derivative \( \partial_{2n}u \). The main novelty of the proof is that putting together two intrinsic subelliptic Cacciopoli inequalities we will end up with a Euclidean Cacciopoli inequality. In this way we can obtain the Hölder-regularity of the gradient via a standard Moser iteration.

We first observe that

\[
(3.1) \quad (X_{i,u}^e)^* = -X_{i,u}^e \quad \text{if} \quad i = 1, \ldots, 2n - 2, 2n, \quad \text{and} \\
(X_{2n-1,u}^e)^* = -X_{2n-1,u}^e - \partial_{2n}u,
\]

where \( (X_{i,u}^e)^* \) is the \( L^2 \) adjoint of the differential operator \( X_{i,u}^e \), and

\[
\partial_{2n}X_{i,u}^e u = -(X_{i,u}^e)^* \partial_{2n}u.
\]

We now prove that if \( u \) is a smooth solution of \( L_e u = 0 \) in \( \Omega \subset \mathbb{R}^{2n} \) then its derivatives \( \partial_{2n}u \) and \( X_{k,u}^e u \) are solutions of a similar mean curvature type equation with different right.

More precisely, we consider \( \omega = \partial_{2n}u + 2\|u\|_{\text{Lip}} \) and \( z = X_{k,u}^e u + 2\|u\|_{\text{Lip}} \). Note that positive constants have been added so that \( z \) and \( \omega \) are always strictly positive. This is a standard and necessary device which has to be used because negative powers of \( \omega \) and \( z \) have to be considered later in the course of the Moser iteration scheme.
Lemma 3.1. If \( u \) is a smooth solution of \( L u = 0 \) then \( \omega = \partial_{2n} u + 2\|u\|_{\text{Lip}} \) is a solution of the equation

\[
(3.2) \quad \sum_{i,j} (X_{i,u}^e)^* \left( \frac{a_{ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla_u^e u|^2}} (X_{j,u}^e)^* \omega \right) = 0,
\]

where \( a_{ij} \) are defined in (1.7).

**Proof.** Differentiating the equation \( L u = 0 \) with respect to \( \partial_{2n} \) we obtain

\[
\partial_{2n} \left( X_{i,u}^e \left( \frac{X_{i,u}^e u}{\sqrt{1 + |\nabla_u^e u|^2}} \right) \right) = 0.
\]

Using the previous remark,

\[
(X_{i,u}^e)^* \left( \partial_{2n} \left( \frac{X_{i,u}^e u}{\sqrt{1 + |\nabla_u^e u|^2}} \right) \right) = 0.
\]

Note that

\[
\partial_{2n} \left( \frac{X_{i,u}^e u}{\sqrt{1 + |\nabla_u^e u|^2}} \right) = \frac{\partial_{2n} X_{i,u}^e u}{\sqrt{1 + |\nabla_u^e u|^2}} - \frac{X_{i,u}^e u X_{i,u}^e u \partial_{2n} X_{j,u}^e u}{(1 + |\nabla_u^e u|^2)^{3/2}}
\]

\[
= - \frac{(X_{i,u}^e)^* \partial_{2n} u}{\sqrt{1 + |\nabla_u^e u|^2}} + \frac{X_{i,u}^e u X_{i,u}^e u (X_{j,u}^e)^* \partial_{2n} u}{(1 + |\nabla_u^e u|^2)^{3/2}}.
\]

The result follows immediately. \( \square \)

Differentiating the equation \( L u = 0 \) with respect to \( X_{k,u}^e \) we obtain

Lemma 3.2. If \( u \) is a smooth solution of \( L u = 0 \) then \( z = X_{k,u}^e + 2\|u\|_{\text{Lip}} \) with \( k \leq 2n - 1 \) is a solution of the equation

\[
(3.3) \quad \sum_{i,j} X_{i,u}^e \left( \frac{a_{ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla_u^e u|^2}} X_{j,u}^e z \right) = -\sum_i [X_{k,u}^e, X_{i,u}^e] \left( \frac{X_{i,u}^e u}{\sqrt{1 + |\nabla_u^e u|^2}} \right)
\]

\[
- \sum_{i,j} X_{i,u}^e \left( \frac{a_{ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla_u^e u|^2}} [X_{k,u}^e, X_{j,u}^e] u \right),
\]

where \( a_{ij} \) are defined in (1.7).
Proof. Differentiating the equation $L_u u = 0$ with respect to $X^e_{k,u}$ we obtain

$$X^e_{k,u} \left( X^e_{i,u} \left( \frac{X^e_{i,u} u}{\sqrt{1 + |\nabla^e u|^2}} \right) \right) = 0,$$

and

$$[X^e_{k,u}, X^e_{i,u}] \left( \frac{X^e_{i,u} u}{\sqrt{1 + |\nabla^e u|^2}} \right) + X^e_{i,u} \left( \frac{X^e_{i,u} u}{\sqrt{1 + |\nabla^e u|^2}} \right) = 0.$$

Note that

$$X^e_{k,u} \left( \frac{X^e_{i,u} u}{\sqrt{1 + |\nabla^e u|^2}} \right) = \frac{X^e_{k,u} X^e_{i,u} u - X^e_{i,u} u X^e_{k,u} X^e_{j,u} u}{(1 + |\nabla^e u|^2)^{3/2}}$$

$$= \frac{d_{ij}(\nabla^e u)}{(1 + |\nabla^e u|^2)^{1/2}} ([X^e_{k,u} X^e_{j,u}] u + X^e_{j,u} z),$$

concluding the proof. □

Remark 3.3. It is useful to compute explicitly the commutators that appear in the previous result.

If $k \leq n - 1$ and $i = k + n - 1$ or $i \leq n - 1$ and $k = i + n - 1$, then

$$[X^e_{k,u}, X^e_{i,u}] = \text{sign}(k - i) \partial_{2n}.$$ 

If $i = 2n - 1$ and $k < 2n - 1$, then

$$[X^e_{k,u}, X^e_{i,u}] = X^e_{k,u} u \partial_{2n}.$$ 

If $k = 2n - 1$ and $i \neq 2n - 1$ then

$$[X^e_{k,u}, X^e_{i,u}] = -X^e_{i,u} u \partial_{2n}.$$ 

If $k = 2n$ and $i = 2n - 1$ then

$$[X^e_{k,u}, X^e_{i,u}] = X^e_{k,u} u \partial_{2n}.$$ 

All other commutators vanish. As a consequence, if $u$ is a smooth solution of $L_u u = 0$ then $|[X^e_{k,u}, X^e_{i,u}]|$ is always bounded by $1 + \|u\|_{Lip}^2$.

Proposition 3.4 (First Caccioppoli type inequality for $X^e_{k,u}$). If $u$ is a smooth solution of $L_u u = 0$ in $\Omega \subset \mathbb{R}^{2n}$, and $z = X^e_{k,u} u + 2\|u\|_{Lip}^2$ with $k \leq 2n$ then for every $p \geq 2$ there exists a constant $C$, only dependent on the bounds on the gradient of $u$ and on $p$ such that for every $\varphi \in C_0^\infty(\Omega)$

$$\int |\partial_{2n} z|^p z^{p-2} \varphi^2 \leq C \left( \int |\nabla^e z|^p z^{p-2} \varphi^2 + \int z^p (\varphi^2 + |\nabla^e \varphi|^2) \right).$$
The constant $C$ is bounded if $p$ is bounded away from 2. If $p = 2$ the inequality holds in the form

$$\int |\hat{\partial}_{2n}z|^2 \varphi^2 \leq C \int z^2 (\varphi^2 + |\nabla^2_{\varphi}|^2).$$

Proof. Calling $\omega = \hat{\partial}_{2n}u + 2||u||_{\text{Lip}}$, we have

$$\int |\nabla_{\varphi}^2 \omega|^2 z^{p-2} \varphi^2 \leq C \int \frac{a_{ij}(\nabla_{\varphi}^2 u)(X^{e}_{i,u})^{*} \omega X^{e}_{j,u} \omega z^{p-2} \varphi^2}{1 + |\nabla_{\varphi}^2 u|^2}$$

$$= - C \int \frac{a_{ij}(\nabla_{\varphi}^2 u)(X^{e}_{i,u})^{*} \omega X^{e}_{j,u} \omega z^{p-2} \varphi^2}{1 + |\nabla_{\varphi}^2 u|^2}$$

$$- C \int \frac{a_{2n-1j}(\nabla_{\varphi}^2 u)(X^{e}_{j,u})^{*} \omega X^{e}_{j,u} \omega z^{p-2} \varphi^2}{1 + |\nabla_{\varphi}^2 u|^2}$$

(integrating by parts $X^{e}_{j,u}$ in the first integral)

$$= C \int (X^{e}_{j,u})^{*} \left( \frac{a_{ij}(\nabla_{\varphi}^2 u)}{1 + |\nabla_{\varphi}^2 u|^2} (X^{e}_{j,u})^{*} \omega \right) \omega z^{p-2} \varphi^2$$

$$+ C \int \frac{a_{2n-1j}(\nabla_{\varphi}^2 u)}{1 + |\nabla_{\varphi}^2 u|^2} (X^{e}_{j,u})^{*} \omega X^{e}_{j,u} \omega z^{p-2} \varphi^2$$

$$+ C(p - 2) \int \frac{a_{ij}(\nabla_{\varphi}^2 u)}{1 + |\nabla_{\varphi}^2 u|^2} (X^{e}_{j,u})^{*} \omega X^{e}_{j,u} z z^{p-3} \varphi^2$$

$$+ 2C \int \frac{a_{ij}(\nabla_{\varphi}^2 u)}{1 + |\nabla_{\varphi}^2 u|^2} (X^{e}_{j,u})^{*} \omega z^{p-2} \varphi X^{e}_{j,u} \varphi$$

The first integral vanishes by Lemma 3.1. In the other integrals we can use the fact that

$$\left| \frac{a_{ij}(\nabla_{\varphi}^2 u)}{1 + |\nabla_{\varphi}^2 u|^2} \right| \leq 1 \quad \text{and} \quad |(X^{e}_{j,u})^{*} \omega| \leq (1 + ||u||_{\text{Lip}})|\nabla_{\varphi}^2 \omega|$$

where $||u||_{\text{Lip}}$ is bounded uniformly in $\epsilon$ by assumption. Then

$$\int |\nabla_{\varphi}^2 \omega|^2 z^{p-2} \varphi^2 \leq C \left( \int |\nabla_{\varphi}^2 \omega| z^{p-2}(\varphi^2 + |\varphi \nabla_{\varphi}^2 \varphi|) + (p - 2) \int |\nabla_{\varphi}^2 \omega| |\nabla_{\varphi}^2 z| z^{p-3} \varphi^2 \right)$$

(by Hölder inequality and the fact that $z$ is bounded away from 0)

$$\leq \delta \int |\nabla_{\varphi}^2 \omega|^2 z^{p-2} \varphi^2 + C(\delta) \int |\nabla_{\varphi}^2 \omega|^2 z^{p-2} \varphi^2 + C(\delta) \int z^p (\varphi^2 + |\nabla_{\varphi}^2 \varphi|^2).$$

For $\delta$ sufficiently small this implies that

$$\int |\nabla_{\varphi}^2 \omega|^2 z^{p-2} \varphi^2 \leq C \int |\nabla_{\varphi}^2 \omega|^2 z^{p-2} \varphi^2 + C \int z^p (\varphi^2 + |\nabla_{\varphi}^2 \varphi|^2).$$
The constant $C$ above is bounded as long as $p$ is away from 2. The special case $p = 2$ follows along similar computations.

Note that, if $k \neq 2n - 1$, we have

$$|\partial_{2nz}| = |\partial_{2n}X_{k,u}^e| = |X_{k,u}^e \partial_{2n}u| \leq |\nabla u^e|.$$ 
If $k = 2n - 1$, we have

$$|\partial_{2nz}| = |\partial_{2n}X_{2n-1,u}^e| = |X_{2n-1,u}^e \partial_{2n}u| + |\partial_{2n}u|^2 \leq |\nabla u^e| + |\partial_{2n}u|^2.$$ 

Hence, using again the boundedness of $|\partial_{2n}u|$ and the fact that $z$ is bounded from below, together with inequality (3.4) we conclude the proof. □

**Proposition 3.5** (Intrinsic Caccioppoli type inequality for $X_{k,u}^e$). If $u$ is a smooth solution of $L_\epsilon u = 0$ in $\Omega \subset \mathbb{R}^{2n}$ and $z = X_{k,u}^e + 2\|u\|_{\text{Lip}}$, with $k \leq 2n$, then for every $p \geq 1$ there exists a constant $C > 0$, only dependent on the bounds on the gradient of $u$ and on $p$ such that for every $\phi \in C_0^\infty(\Omega)$

$$\int |\nabla u^e|^2 z^{p-2} \phi^2 \leq C \int z^p (\phi^2 + |\nabla u^e|^2 + |\partial_{2n} u|).$$

The constant $C$ is bounded if $p$ is bounded away from the values 1 and 2.

**Proof.** Multiplying the equation (3.3) by $z^{p-1} \phi^2$ and integrating we obtain

$$\int X_{i,u}^e \left( \frac{a_{ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla u^e|^2}} X_{j,u}^e \right) z^{p-1} \phi^2$$

$$= -\int X_{i,u}^e X_{i,u}^e \left( \frac{X_{i,u}^e u}{\sqrt{1 + |\nabla u^e|^2}} \right) z^{p-1} \phi^2 - \int X_{i,u}^e \left( \frac{a_{ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla u^e|^2}} [X_{k,u}^e, X_{j,u}^e] u \right) z^{p-1} \phi^2.$$ 

We denote by $I_1$ and $I_2$ the integrals in the right-hand side. Let us consider the left-hand side

$$\int X_{i,u}^e \left( \frac{a_{ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla u^e|^2}} X_{j,u}^e \right) z^{p-1} \phi^2$$

(since $(X_{i,u}^e)^* = -X_{i,u}^e - \delta_{i,2n-1} \partial_{2n} u$)

$$= -(p-1) \int \frac{a_{ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla u^e|^2}} X_{j,u}^e z X_{i,u}^e z z^{p-2} \phi^2 - 2 \int \frac{a_{ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla u^e|^2}} X_{j,u}^e z z z^{p-2} \phi X_{i,u}^e \phi$$

$$- \int \frac{a_{2n-1,ij}(\nabla_u^e u)}{\sqrt{1 + |\nabla u^e|^2}} X_{j,u}^e \partial_{2n} u z z z^{p-2} \phi^2.$$
Using the uniform ellipticity of $a_{ij}$ and the boundedness of $a_{ij}$ and $\partial_{2n}u$, we obtain
\[ \int |V^e_u|^2 z^{p-2} \varphi^2 \leq C(2 \int |V^e_u|^p z^{p-1} \varphi |V^e_u| + \int |V^e_u|^2 z^{p-2} \varphi^2) + |I_1| + |I_2|. \]

From here, using a Hölder inequality and the boundedness of $z$ from below one has
\[ (3.5) \quad \int |V^e_u|^2 z^{p-2} \varphi^2 \leq C \int z^{p-2} (\varphi^2 + |V^e_u|^2) + |I_1| + |I_2|. \]

Next we estimate separately the terms $I_1$ and $I_2$. We begin with the latter and observe that integrating by parts the expression of $I_2$, we have
\[
I_2 = (p - 1) \int \frac{a_{ij}(V^e_u)}{1 + |V^e_u|^2} [X^e_{k,u}, X^e_{j,u}] uX^e_{i,u} z^{p-2} \varphi^2 \\
+ 2 \int \frac{a_{ij}(V^e_u)}{1 + |V^e_u|^2} [X^e_{k,u}, X^e_{j,u}] uz^{p-1} \varphi X_i \varphi \\
+ \int \frac{a_{2n-1}(V^e_u)}{1 + |V^e_u|^2} [X^e_{k,u}, X^e_{j,u}] u\partial_{2n} uz^{p-1} \varphi^2
\]
(using the fact that both $a_{ij}$ and the brackets, computed in Remark 3.3, are bounded)
\[
\leq C \int |V^e_u|^2 z^{p-2} \varphi^2 + C \int z^{p-1} \varphi |V^e_u| + C \int z^{p-1} \varphi^2
\]
(since $z$ is bounded from below)
\[
\leq \delta \int |V^e_u|^2 z^{p-2} \varphi^2 + C(\delta) \int z^p (\varphi^2 + |V^e_u|^2),
\]
for every $\delta > 0$.

In order to estimate $I_1$ we first consider separately the cases $k \neq 2n - 1$ and $k = 2n - 1$. If $k \leq n - 1$ then by Remark 3.3
\[
I_1 = -\text{sign}(k - n)(1 - \delta_{k,2n}) \int \partial_{2n} \left( \frac{X^e_{k+n-1,u}}{1 + |V^e_u|^2} \right) z^{p-1} \varphi^2 \\
- \int X^e_{k,u} u\partial_{2n} \left( \frac{X^e_{2n-1,u}}{1 + |V^e_u|^2} \right) z^{p-1} \varphi^2
\]
(integrating by parts, using the boundness of $V^e_u$, and the fact that $\partial_{2n} X^e_{k,u} u = \partial_{2n}z$)
\[
\leq C \int |\partial_{2n}z| z^{p-2} \varphi^2 + C \int z^{p-1} |\varphi \partial_{2n} \varphi| + C \int |\partial_{2n}z| z^{p-1} \varphi^2
\]
(using Hölder inequality and the fact that $z$ is bounded away from 0)
\[
\leq \delta \int |\partial_{2n}z|^2 z^{p-2} \varphi^2 + C(\delta) \int z^p (|\varphi \partial_{2n} \varphi| + \varphi^2)
\]
(by Proposition 3.4)

\[ \leq \delta \int |\nabla^e z|^2 z^{p-2} \phi^2 + C(\delta) \int z^p (|\phi \partial_{2n} \phi| + \phi^2 + |\nabla^e \phi|^2). \]

This estimate can be proved with a similar argument in the case \( n \leq k \leq 2n - 2 \) and \( k = 2n \). Hence if \( k \geq 2n - 1 \) the conclusion follows by choosing \( \delta \) sufficiently small.

If \( k = 2n - 1 \), then Remark 3.3 yields

\[ I_1 = -\int X_{i,u}^e u \partial_{2n} \left( \frac{X_{i,u}^e u}{\sqrt{1 + |\nabla^e u|^2}} \right) z^{p-1} \phi^2 \]

(directly computing the derivative with respect to \( \partial_{2n} \))

\[ = -\int (X_{i,u}^e u)^2 \partial_{2n} \left( \frac{1}{\sqrt{1 + |\nabla^e u|^2}} \right) z^{p-1} \phi^2 - \int X_{i,u}^e u \frac{\partial_{2n} X_{i,u}^e u}{\sqrt{1 + |\nabla^e u|^2}} z^{p-1} \phi^2 \]

(since \( \sum_i (X_{i,u}^e u)^2 = |\nabla^e u|^2 \))

\[ = -\int |\nabla^e u|^2 \partial_{2n} \left( \frac{1}{\sqrt{1 + |\nabla^e u|^2}} \right) z^{p-1} \phi^2 - \frac{1}{2} \int \frac{\partial_{2n} |\nabla^e u|^2}{\sqrt{1 + |\nabla^e u|^2}} z^{p-1} \phi^2. \]

If we set \( F(s) = \frac{1}{\sqrt{1 + s}} \), then one easily computes

\[ \partial_{2n} F(|\nabla^e u|^2) = \left( |\nabla^e u|^2 \partial_{2n} \left( \frac{1}{\sqrt{1 + |\nabla^e u|^2}} \right) + \frac{1}{2} \frac{\partial_{2n} |\nabla^e u|^2}{\sqrt{1 + |\nabla^e u|^2}} \right) \]

so that the previous integral becomes

\[ I_1 = -\int \partial_{2n} F(|\nabla^e u|^2) z^{p-1} \phi^2 \]

(integrating by parts)

\[ = (p - 1) \int F(|\nabla^e u|^2) \partial_{2n} z z^{p-2} \phi^2 + 2 \int F(|\nabla^e u|^2) z^{p-1} \phi \partial_{2n} \phi \]

(using the fact that \( F \) is bounded)

\[ \leq C \int |\partial_{2n} z| z^{p-2} \phi^2 + C \int z^{p-1} |\phi \partial_{2n} \phi| \]

(by Proposition 3.4 and an Hölder inequality)

\[ \leq \delta \int |\nabla^e u|^2 z^{p-2} \phi^2 + C(\delta) \int z^p (|\phi \partial_{2n} \phi| + \phi^2 + |\nabla^e \phi|^2), \]

thus concluding the proof. \( \square \)
Next, we note that, from Propositions 3.4 and 3.5 one can derive a Euclidean Cacciopoli type inequality for \( z = X^e_{k, u} u + 2\|u\|_{\text{Lip}} \), with \( k \leq 2n \). Here and in the following \( \nabla_E \) denotes the Euclidean gradient in \( \mathbb{R}^{2n} \).

**Proposition 3.6** (Euclidean Cacciopoli inequality). If \( u \) is a Lipschitz continuous solution of \( L e u = 0 \) in \( \Omega \subset \mathbb{R}^{2n} \), and \( z = X^e_{k, u} u + 2\|u\|_{\text{Lip}} \), with \( k \leq 2n \), then for every \( p \neq 1 \) there exists a constant \( C \), only dependent on the bounds on the gradient of \( u \) and on \( p \) such that for every \( \varphi \in C_0^\infty(\Omega) \)

\[
(3.6) \quad \int |\nabla_E z|^2 z^{p-2} \varphi^2 \leq C \int z^p (\varphi^2 + |\nabla_E \varphi|^2).
\]

The constant \( C \) is bounded if \( p \) is bounded away from the values 1 and 2.

**Proof.** Observe that there exists \( C > 0 \) depending only on \( \|u\|_{\text{Lip}} \) and \( \Omega \) such that for all points in \( \Omega \),

\[
|\nabla_E z|^2 \leq C \left( \sum_{k \leq 2n} |X^e_{k, u} z|^2 + |\partial_{2n} z|^2 \right).
\]

Hence using Propositions 3.4 and 3.5 and observing that

\[
|\nabla^e_u \varphi|^2 + |\partial_{2n} \varphi|^2 \leq C|\nabla_E \varphi|^2
\]

we obtain (3.6). \( \square \)

From Proposition 3.6, using the classical Moser procedure in the Euclidean setting, we can immediately deduce the following regularity result:

**Proposition 3.7.** Let \( u \) be a solution of \( L e u = 0 \) in \( \Omega \subset \mathbb{R}^{2n} \) and set

\[ z = X^e_{k, u} u + 2\|u\|_{\text{Lip}}, \quad \text{with } k \leq 2n. \]

For every compact set \( K \subset \subset \Omega \) then there exist a real number \( \alpha \in ]0, 1[ \) and a constant \( C \), only dependent on the bounds on the gradient and on the choice of the compact set such that

\[
\|z\|_{C^{\alpha}(K)} \leq C.
\]

In particular we have the estimate

\[
\sum_{i=1}^{2n+1} \sum_{j=1}^{2n} \|X^e_{i, u} X^e_{j, u} u\|_{L^2(K)} + \|u\|_{C^1_0(z)(K)} \leq C.
\]

**Proof.** For \( p \neq 1, 2 \) and \( z \) as in the statement of the proposition define the function

\[
w = \begin{cases} 
z^e & \text{if } p \neq 0, \\
\ln z & \text{if } p = 0. \end{cases}
\]
If \( p \neq 0 \) then the Caccioppoli inequality (3.6) and the Euclidean Sobolev Embedding Theorem yield

\[
(\int |\varphi w|^{2p})^{\frac{1}{p}} \leq Cp^2 \int w^2 |\nabla \varphi|^2,
\]

for some \( \theta > 1 \) and for every \( \varphi \in C_0^\infty(\Omega) \). Let \( 0 < r_1 < r_2 \) be sufficiently small so that the Euclidean ball \( B_{r_2} \) is contained in \( \Omega \). With an appropriate choice of test function, (3.7) implies

\[
(\int |z|^{\theta p})^{\frac{1}{\theta}} \leq \left( \frac{Cp^2}{r_2 - r_1} \right)^{\frac{1}{\theta}} \left( \int |z|^p \right)^{\frac{1}{\theta}} \quad \text{if } p > 0,
\]

\[
(\int |z|^{\theta p})^{\frac{1}{\theta}} \left( \frac{Cp^2}{r_2 - r_1} \right)^{\frac{1}{\theta}} \geq \left( \int |z|^p \right)^{\frac{1}{\theta}} \quad \text{if } p < 0.
\]

If \( p = 0 \), (3.6) implies

\[
\int |\varphi \nabla w|^2 \leq C \int |\nabla \varphi|^2.
\]

Let \( r > 0 \) sufficiently small so that the Euclidean ball \( B_r \subset \Omega \). A standard choice of test function and H"older inequality yield

\[
\int_{B_r} |\nabla w| \leq CR^{2n+1}.
\]

Recalling that \( \Omega \subset \mathbb{R}^{2n} \) and using Poincaré inequality we obtain \( w \in \text{BMO}(\Omega) \).

At this point, using (3.8), the John-Nirenberg Lemma and following the standard Moser iteration process (see for instance [27], Chapter 8) we obtain the Hölder regularity of \( z \).

**4. From \( C_{u}^{1,a} \) to \( C_{u}^{\infty} \) regularity**

In this section we will conclude the proof of the regularity result. The section is organized in 3 steps. We fix a function \( \bar{u} \), and study solutions of the linearized equation

\[
L_{e, \bar{u}} u = \sum_{i,j=1}^{2n} a_{i,j}^e (\nabla_{\bar{u}}^e \bar{u}) X_{i, \bar{u}}^e X_{j, \bar{u}}^e u = 0
\]

defined in (2.8), and represented in non-divergence form. The solutions \( u \) will be represented in terms of the fundamental solution \( \Gamma_{x_0}^e \) of the approximating operator \( L_{e, x_0} \), defined in (2.9):

\[
L_{e, x_0} u = \sum_{i,j=1}^{2n} a_{i,j}^e (\nabla_{x_0}^e \bar{u}(x_0)) X_{i, x_0}^e X_{j, x_0}^e u
\]
where $a_{ij}^\varepsilon$ are defined in (1.7). Since estimates on $\Gamma_{x_0}^\varepsilon$ uniform in $\varepsilon$ are well known (and have been recalled in Section 2), from these representation formulas we will deduce a priori estimates for the solution $u$, in terms of the fixed solution $\bar{u}$. Choosing $\bar{u} = u$ we will obtain a priori estimates of the solutions of the non-linear equation $L_u = 0$. Finally, letting $\varepsilon$ go to 0, we will conclude the proof of the estimates of the vanishing viscosity solutions of $Lu = 0$.

### 4.1. Representation formulas.

**Lemma 4.1.** The difference between the operator $L_{e, \bar{u}}$ and its frozen operator can be expressed as follows:

\[
(L_{e, x_0} - L_{e, \bar{u}})u(\xi) = \sum_{ij=1}^{2n} (a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 u(x_0)) - a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(\xi))) X_i^\varepsilon X_j^\varepsilon u(\xi)
\]

\[
- \sum_{j=1}^{2n} a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 u(x_0)) (\bar{u}(\xi) - P_1 \bar{u}(\xi)) \partial_{2n} X_j^\varepsilon u(\xi)
\]

\[
- \sum_{j=1}^{2n} a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(x_0)) X_j^\varepsilon X_i^\varepsilon((\bar{u}(\xi) - P_1 \bar{u}(\xi)) \partial_{2n}) u(\xi).
\]

**Proof.** Observe that

\[
(L_{e, x_0} - L_{e, \bar{u}})u(\xi) = \sum_{ij=1}^{2n} (a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(x_0)) - a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(\xi))) X_i^\varepsilon X_j^\varepsilon u(\xi)
\]

\[
- \sum_{j=1}^{2n} a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(x_0)) (X_i^\varepsilon X_j^\varepsilon - X_i^{\varepsilon, x_0} X_j^{\varepsilon, x_0}) u(\xi)
\]

\[
= \sum_{ij=1}^{2n} (a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(x_0)) - a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(\xi))) X_i^\varepsilon X_j^\varepsilon u(\xi)
\]

\[
- \sum_{ij=1}^{2n} a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(x_0)) ((X_i^\varepsilon - X_i^{\varepsilon, x_0}) X_j^\varepsilon + X_i^{\varepsilon, x_0} (X_j^\varepsilon - X_j^{\varepsilon, x_0})) u(\xi)
\]

\[
= \sum_{ij=1}^{2n} (a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(x_0)) - a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(\xi))) X_i^\varepsilon X_j^\varepsilon u(\xi)
\]

\[
- \sum_{ij=1}^{2n} a_{ij}^\varepsilon(\nabla_{\bar{u}}^2 \bar{u}(x_0)) (\delta_{i, 2n} \bar{u}(\xi) - P_1 \bar{u}(\xi)) \partial_{2n} X_j^\varepsilon
\]

\[
+ X_i^{\varepsilon, x_0} (\delta_{i, 2n} (\bar{u}(\xi) - P_1 \bar{u}(\xi)) \partial_{2n}) u(\xi)
\]

where $\delta$ is the Kroeneker function. From this the thesis immediately follows. \(\square\)

Let us first represent the solutions of the equation $L_{e, \bar{u}} u = 0$ in terms of the fundamental solution $\Gamma_{x_0}^\varepsilon$ of the operator $L_{e, x_0}$ defined in (2.9).
Proposition 4.2. Let us assume that \( \bar{u} \) is a fixed function of class \( C^\infty(\Omega) \), and that \( u \) is a classical solution of \( L_{e,\bar{a}}u = g \in C^\infty(\Omega) \). Then for any \( \varphi \in C^0_0(\Omega) \) the function \( u\varphi \) can be represented as

\[
(4.2) \quad u\varphi(x) = \int_\Omega \Gamma^e_{x_0}(x, \xi)N_1(\xi, x_0) \, d\xi + \int_\Omega \Gamma^e_{x_0}(x, \xi)L_{e,\bar{a}}u(\xi)\varphi(\xi) \, d\xi \\
+ \sum_{j=1}^{2n} \int \Gamma^e_{x_0}(x, \xi) \left( b^{e}_{ij}(\nabla^e_{\bar{a}}\bar{u}(x_0)) - b^{e}_{ij}(\nabla^e_{\bar{a}}\bar{a}(\xi)) \right) X^e_{i,j}X^e_{j,i}u(\xi)\varphi(\xi) \, d\xi \\
+ \sum_{j=1}^{2n} \int X^e_{s,j_k} \Gamma^e_{x_0}(x, \xi) \left( \bar{a}(\xi) - P^1_{x_0} \bar{a}(\xi) \right) h_{sij}(x_0)X^e_{i,j}X^e_{j,i}u(\xi)\varphi(\xi) \, d\xi.
\]

The expressions of \( N_1, b_{ij} \) and \( h_{sij} \) are the following:

\[
(4.3) \quad N_1(\xi, x_0) = u(\xi)L_{e,x_0}\varphi(\xi) \\
+ \sum_{j=1}^{2n} a^{e}_{ij}(\nabla^e_{\bar{a}}\bar{u}(x_0)) \left( X^e_{i,j}u(\xi)X^e_{j,i}u(\xi) + X^e_{j,i}u(\xi)X^e_{i,j}u(\xi) \right) \\
- \sum_{j=1}^{2n} a^{e}_{ij}(\nabla^e_{\bar{a}}\bar{u}(x_0)) \left( \bar{u}(\xi) - P^1_{x_0} \bar{u}(\xi) \right) X^e_{i,j}u(\xi)u(\xi) \\
+ \sum_{j=1}^{2n} a^{e}_{ij}(\nabla^e_{\bar{a}}\bar{u}(x_0)) \left( \bar{a}(\xi) - P^1_{x_0} \bar{a}(\xi) \right) X^e_{i,j}X^e_{i,j}u(\xi)u(\xi) \\
- \sum_{j=1}^{2n} a^{e}_{ij}(\nabla^e_{\bar{a}}\bar{u}(x_0)) \left( \bar{u}(\xi) - P^1_{x_0} \bar{u}(\xi) \right) X^e_{i,j}X^e_{i,j}u(\xi)u(\xi),
\]

\[
b_{ij} : \mathbb{R}^{2n} \rightarrow \mathbb{R},
\]

\[
b_{i,j}(p) = -\delta_{ij}a_{k}(p) + a_{2n-1-j}(\nabla^e_{\bar{a}}\bar{a}(x_0)) p_{1} \delta_{i} \delta_{j} - p_{n} \delta_{i} a_{2n-1-j}(\nabla^e_{\bar{a}}\bar{a}(x_0)).
\]

Finally \( h_{sij} \) are real numbers, only dependent on \( x_0 \), defined as

\[
(4.5) \quad h_{sij}(x_0) = -a^{e}_{2n-1-s}(\nabla^e_{\bar{a}}\bar{u}(x_0)) \left( \delta_{i} \delta_{j} \delta_{m} - \delta_{m} \delta_{i} \delta_{j} \right) \\
+ a^{e}_{2n-1-j}(\nabla^e_{\bar{a}}\bar{u}(x_0)) \left( \delta_{i} \delta_{m} \delta_{j} - \delta_{m} \delta_{i} \delta_{j} \right).
\]

Proof. By definition of fundamental solution, we have

\[
(4.6) \quad u\varphi(x) = \int_\Omega \Gamma^e_{x_0}(x, \xi) L_{e,\bar{a}}u(\xi)\varphi(\xi) \, d\xi \\
= \int_\Omega \Gamma^e_{x_0}(x, \xi) \left( uL_{e,\bar{a}}\varphi + \sum_{j=1}^{2n} a^{e}_{ij}(\nabla^e_{\bar{a}}\bar{u}(x_0)) \\
\times \left( X^e_{i,j}uX^e_{j,i}u(\xi) + X^e_{j,i}uX^e_{i,j}u(\xi) \right) \right) \, d\xi \\
+ \int_\Omega \Gamma^e_{x_0}(x, \xi)L_{e,\bar{a}}u(\xi)\varphi(\xi) \, d\xi + \int_\Omega \Gamma^e_{x_0}(x, \xi)(L_{e,\bar{a}} - L_{e,\bar{a}})u(\xi)\varphi(\xi) \, d\xi.
\]
We can use the expression of $L_{\epsilon,x_0} - L_{\epsilon,\tilde{\theta}}$ computed in (4.1). Let us consider the second term in the right-hand side, multiplied by the fundamental solution. Since $\hat{c}_{2n} = [X^{\epsilon}_{1,x_0}, X^{\epsilon}_{n,x_0}]$, it becomes

$$
\sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) \hat{c}_{2n} X^{\epsilon}_{j,\tilde{\theta}} u \varphi \, d\xi 
$$

$$= \sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) [X^{\epsilon}_{1,x_0}, X^{\epsilon}_{n,x_0}] (X^{\epsilon}_{j,\tilde{\theta}} u) \varphi \, d\xi
$$

(integrating by part and using the fact that $X^{\epsilon}_{1,x_0} = X^{\epsilon}_{1,\tilde{\theta}}$ and $X^{\epsilon}_{n,x_0} = X^{\epsilon}_{n,\tilde{\theta}}$)

$$= -\sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} X^{\epsilon}_{x_0} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) X^{\epsilon}_{n,\tilde{\theta}} X^{\epsilon}_{j,\tilde{\theta}} u \varphi \, d\xi
$$

$$- \sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta}) X^{\epsilon}_{n,\tilde{\theta}}(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) X^{\epsilon}_{j,\tilde{\theta}} u \varphi \, d\xi
$$

$$- \sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) X^{\epsilon}_{n,\tilde{\theta}} X^{\epsilon}_{j,\tilde{\theta}} u X^{\epsilon}_{1,\tilde{\theta}} \varphi \, d\xi
$$

$$+ \sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} X^{\epsilon}_{n,\tilde{\theta}} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) X^{\epsilon}_{1,\tilde{\theta}} X^{\epsilon}_{j,\tilde{\theta}} u X^{\epsilon}_{1,\tilde{\theta}} \varphi \, d\xi
$$

$$+ \sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta}) X^{\epsilon}_{n,\tilde{\theta}}(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) X^{\epsilon}_{1,\tilde{\theta}} X^{\epsilon}_{j,\tilde{\theta}} u X^{\epsilon}_{1,\tilde{\theta}} \varphi \, d\xi
$$

The third term in (4.1) becomes

$$
\sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta}) X^{\epsilon}_{j,\tilde{\theta}} \left( (\tilde{\theta} - P^{1}_{x_0} \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) \right) \hat{c}_{2n} u(\tilde{\theta}) \varphi(\tilde{\theta}) \, d\xi
$$

(integrating by part)

$$= -\sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} X^{\epsilon}_{j,\tilde{\theta}} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) \hat{c}_{2n} u(\tilde{\theta}) \varphi(\tilde{\theta}) \, d\xi
$$

$$- \sum_{j=1}^{2n} a^{\epsilon}_{2n-1j} \left( \nabla^{\epsilon}_{\tilde{\theta}} \tilde{\theta}(x_0) \right) \int_{\Omega} \Gamma^{\epsilon}_{x_0}(x, \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta})(\tilde{\theta} - P^{1}_{x_0} \tilde{\theta}) \hat{c}_{2n} u(\tilde{\theta}) X^{\epsilon}_{j,\tilde{\theta}} \varphi(\tilde{\theta}) \, d\xi.
$$

Inserting (4.7) and (4.8) in (4.1) and using the expression of $b_{ij}$ and $h_{ij}(x_0)$, we obtain
\( \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi)(L_{x_0} - L_{e_0, \tilde{a}})u(\xi)\varphi(\xi) \, d\xi \)

\[
= \sum_{j=1}^{2n} \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \left( a_{ij}^e (\nabla_a^e \tilde{u}(x_0)) - a_{ij}^e (\nabla_{\tilde{a}}^e \tilde{u}(\xi)) \right) X_{i, \tilde{a}}^e X_{j, \tilde{a}}^e u(\xi)\varphi(\xi) \, d\xi \\
+ \sum_{j=1}^{2n} \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \left( b_{ij}^e (\nabla_a^e \tilde{a}(x_0)) - b_{ij}^e (\nabla_{\tilde{a}}^e \tilde{a}(\xi)) \right) X_{i, \tilde{a}}^e X_{j, \tilde{a}}^e u(\xi)\varphi(\xi) \, d\xi \\
+ \sum_{j=1}^{2n} \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \left( \tilde{u}(\xi) - P_{x_0}^1 \tilde{a}(\xi) \right) h_{ij}^e (x_0) X_{i, \tilde{a}}^e X_{j, \tilde{a}}^e u(\xi)\varphi(\xi) \, d\xi \\
- \sum_{j=1}^{2n} a_{ij}^e \left( \nabla_a^e \tilde{u}(x_0) \right) \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \left( \tilde{u}(\xi) - P_{x_0}^1 \tilde{a}(\xi) \right) X_{i, \tilde{a}}^e X_{j, \tilde{a}}^e u \, d\xi \\
+ \sum_{j=1}^{2n} a_{ij}^e \left( \nabla_{\tilde{a}}^e \tilde{a}(x_0) \right) \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \left( \tilde{u}(\xi) - P_{x_0}^1 \tilde{a}(\xi) \right) X_{i, \tilde{a}}^e u \, d\xi \\
- \sum_{j=1}^{2n} a_{ij}^e \left( \nabla_{\tilde{a}}^e \tilde{a}(x_0) \right) \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \left( \tilde{u}(\xi) - P_{x_0}^1 \tilde{a}(\xi) \right) \partial_\eta u(\xi) X_{j, \tilde{a}}^e \varphi(\xi) \, d\xi.
\]

From this expression, equation (4.6), and the expression of \( N_1 \) in (4.3) we obtain the asserted representation formula. \( \square \)

The following representation formula will be used to estimate higher order derivatives of the solutions.

**Proposition 4.3.** Let us assume that \( \tilde{u} \) is a fixed function of class \( C^\infty(\Omega) \), and assume that \( u \) is a classical solution of \( L_{x_0, \tilde{a}} u = g \in C^\infty(\Omega) \). Then for any \( \varphi \in C_0^\infty(\Omega) \) the function \( u\varphi \) can be represented as

\[
(4.12) \quad u\varphi(x) = \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \varphi(\xi) N_1(\xi, x_0) \, d\xi \quad + \quad \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \varphi(\xi) N_{2,k}(\xi, x_0) \, d\xi \\
+ \sum_{k=1}^{2n} \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \varphi(\xi) N_{2,ks}(\xi, x_0) \, d\xi \quad + \quad \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \varphi(\xi) N_{3,k}(\xi, x_0) \, d\xi \\
+ \sum_{k=1}^{2n} \int_{\Omega}^{\Gamma_{x_0}^e} (x, \xi) \varphi(\xi) N_{4,ki}(\xi, x_0) \, d\xi,
\]

where \( N_1(\xi, x_0) \) is defined in (4.3). If \( b_{ij} \) is the function defined in (4.4), we call \( b_{ij} = b_{ij}(\nabla_{\tilde{a}}^e \tilde{u}) \), and the other kernels are expressed:

\[
N_2(\xi, x_0) = P_{x_0}^{k-2} g(\xi) + \sum_{i=1}^{2n} \left( P_{x_0}^{k-2} b_{ij}^e(\xi) - b_{ij}^e(x_0) \right) P_{x_0}^{k-3} (X_{i, \tilde{a}}^e X_{j, \tilde{a}}^e u)(\xi),
\]

\[
N_{2,kx}(\xi, x_0) = \sum_{i=1}^{2n} \left( P_{x_0}^{k-1} \tilde{u}(\xi) - P_{x_0}^1 \tilde{a}(\xi) \right) h_{ij}^e (x_0) P_{x_0}^{k-3} (X_{i, \tilde{a}}^e X_{j, \tilde{a}}^e u)(\xi),
\]
The first two terms of this expression define
\[
N_{3,k}(\xi, x_0) = \left(g(\xi) - P_{x_0}^{k-2} g(\xi)\right) + \left(b_{ij}(\xi) - b_{ij}(x_0)\right)\left(X_{i,a}^e X_{j,a}^e u(\xi) - P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)\right)
\]
\[
+ \left(b_{ij}(\xi) - b_{ij}(x_0)\right)\left(X_{i,a}^e X_{j,a}^e u(\xi) - P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)\right) + \left(\bar{a}(\xi) - P_{x_0}^1 \bar{a}(\xi)\right)\left(h_{sij}(x_0) \left(X_{i,a}^e X_{j,a}^e u(\xi) - P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)\right) + \left(\bar{a}(\xi) - P_{x_0}^1 \bar{a}(\xi)\right)\left(h_{sij}(x_0) \left(X_{i,a}^e X_{j,a}^e u(\xi) - P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)\right).
\]

(4.13)
\[
N_{4,ks}(\xi, x_0) = \left(\bar{a}(\xi) - P_{x_0}^1 \bar{a}(\xi)\right) h_{sij}(x_0) \left(X_{i,a}^e X_{j,a}^e u(\xi) - P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)\right)
\]

Proof. We represent \( u \) as in formula (4.2), and we study each term separately. Let us start with the second term in (4.2):
\[
g(\xi) = L_{e,a} u(\xi) = P_{x_0}^{k-2} g(\xi) + \left(g(\xi) - P_{x_0}^{k-2} g(\xi)\right).
\]

The kernel in the third term of (4.2) can be expanded as follows:
\[
(4.16) \quad (b_{ij}(\xi) - b_{ij}(x_0)) X_{i,a}^e X_{j,a}^e u(\xi)
\]
\[
= \left(P_{x_0}^{k-2} b_{ij}(\xi) - b_{ij}(x_0)\right) P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)
\]
\[
+ \left(b_{ij}(\xi) - b_{ij}(x_0)\right) \left(X_{i,a}^e X_{j,a}^e u(\xi) - P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)\right)
\]
\[
+ \left(b_{ij}(\xi) - b_{ij}(x_0)\right) P_{x_0}^{k-2} b_{ij}(\xi) P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi).
\]

The first terms in (4.15) and (4.16) define \( N_{2,k} \), the sum of the other terms defines \( N_{3,k} \).

The kernel in the last term of (4.2) can be represented as
\[
(4.17) \quad (\bar{a}(\xi) - P_{x_0}^1 \bar{a}(\xi)) h_{sij}(x_0) \left(X_{i,a}^e X_{j,a}^e u(\xi) - P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)\right)
\]
\[
= \left(\bar{a}(\xi) - P_{x_0}^1 \bar{a}(\xi)\right) h_{sij}(x_0) \left(X_{i,a}^e X_{j,a}^e u(\xi) - P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)\right)
\]
\[
+ \left(\bar{a}(\xi) - P_{x_0}^1 \bar{a}(\xi)\right) P_{x_0}^{k-1} \bar{a}(\xi) h_{sij}(x_0) h_{sij}(x_0) P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi)
\]
\[
+ \left(P_{x_0}^{k-1} \bar{a}(\xi) - P_{x_0}^1 \bar{a}(\xi)\right) h_{sij}(x_0) P_{x_0}^{k-3} (X_{i,a}^e X_{j,a}^e u)(\xi).
\]

The first two terms of this expression define \( N_{4,ks} \), the third defines \( N_{2,ks} \).

4.2. A priori estimates of the solution of the linear operator.

Proposition 4.4. Assume that \( u \) and \( \bar{u} \) are of class \( C_{a}^{\infty}(\Omega) \) and that
\[
L_{e,a} u = g \in C_{a}^{\infty}(\Omega).
\]

Also assume that there exists a compact set \( K \subset \Omega \) and constant \( C_0 \) such that
\[
\|u\|_{C_{a}^{\infty}(K)} + \|g\|_{C_{a}^{1,\gamma}(K)} + \sum_{\deg(\sigma) \leq 2} \|V_{e,a} u\|_{L^{\infty}(K)} \leq C_0.
\]
Then, for every compact set $K_1 \subset K$, for every $x' < x$ there exists a constant $C > 0$ only dependent on $C_0$, $x$ and the compact sets, such that

\[
\sum_{\deg(\sigma) = 2} \|\nabla_{\sigma, a}^e u\|_{C^2(K_1)} \leq C.
\]

Moreover, for every choice of compact sets $K_2$, $K_3$ such that $K_1 \subset K_2 \subset K_3 \subset K$, for every function $\varphi \in C_0^\infty(\text{int}(K))$ such that $\varphi \equiv 1$ in $K_2$, for every multi-index $\sigma$ of length 2, we have the following representation:

\[
\nabla_{\sigma, a}^e (u \varphi) (x_0) = \int_{\Omega} \nabla_{\sigma, a}^e (x_0) \Gamma_{x_0}^e (\cdot, \xi) N_1 (\xi, x_0) d\xi
\]

\[
+ \int_{\Omega} \Gamma_{x_0}^e (\xi, 0) \nabla_{\sigma, a}^e (x_0) P^1_{x_0} g(\xi) \varphi (x_0 \circ \xi^{-1}) d\xi
\]

\[
+ \int_{\Omega} \nabla_{\sigma, a}^e (x_0) \Gamma_{x_0}^e (\cdot, \xi) (g(\xi) - P^1_{x_0} g(\xi)) \varphi (\xi) d\xi
\]

\[
+ \sum_{ij=1}^{2n} \int_{\Omega} \nabla_{\sigma, a}^e (x_0) \Gamma_{x_0}^e (\cdot, \xi) \left( h_{ij} (\nabla_{\sigma, a}^e u (x_0))
\right.
\]

\[
- h_{ij} \left( \nabla_{\sigma, a}^e (x_0) \right) X_{ij}^e \nabla_{\sigma, a}^e X_{ij}^e u (\xi) \varphi (\xi) d\xi
\]

\[
+ \sum_{ij=1}^{2n} \int_{\Omega} \nabla_{\sigma, a}^e (x_0) \Gamma_{x_0}^e (\cdot, \xi)
\]

\[
\times \left( \bar{u} (\xi) - P^1_{x_0} \bar{u} (\xi) \right) h_{ij} (x_0) X_{ij}^e \nabla_{\sigma, a}^e X_{ij}^e u (\xi) \varphi (\xi) d\xi
\]

where $N_1$ is defined in Proposition 4.2.

**Proof.** By Proposition 4.2, we know that the function $u \varphi$ admits a representation in terms of the fundamental solution of the frozen operator, and suitable kernels. Let us verify that these kernels satisfy the assumptions of Proposition 2.9 with $k = 2$.

From the expression of $N_1$ in (4.3), we see that $N_1$ is a sum of derivatives of the function $\varphi$. Since $\varphi$ is constantly equal to 1 on the set $K_2$, then $N_1$ vanishes on the same set. Hence the point (i) of Proposition 2.9 regarding the support of $N_1$ is satisfied. On the other side $N_1$ depends on $x_0$ only through the first derivatives of $\bar{u}$, while it depends on $\xi$ through the derivatives up to second order of the function $u$. Hence it is Hölder continuous in $x_0$ locally uniformly in $\xi$. Hence there exists a constant $C_1$ only depending on $C_0$ such that

\[ |N_1(\xi, x_0) - N_1(\xi, x)| \leq C_1 d_{x, x_0}^2 (x_0, x), \]

for every $x, x_0 \in K_1$ and $\xi \in K_3$ and this shows that (2.15) holds.

The second term in (4.2) is the convolution of the fundamental solution with the function $g(\xi) = L_{a, e} u (\xi) = P^1_{x_0} g(\xi) + (g(\xi) - P^1_{x_0} g(\xi))$. The function $P^1_{x_0} g(\xi) \varphi (\xi)$ will play the role of the kernel $N_2$ in Proposition 2.9. Since $g$ is of class $C^{1,2}_{\bar{u}} (K)$, its first order Taylor
polynomial is Hölder continuous in $x_0$ locally uniformly in $\zeta$ and there exists a constant $C_1$ only depending on $C_0$ such that

$$|P_{x_0}^1 g(\zeta)\varphi(\zeta) - P_{X}^1 g(\zeta)\varphi(\zeta)| \leq C_1 d_{x_0}^{1+\gamma}(x_0, \zeta),$$

for every $x, x_0 \in K_1$ and $\zeta \in K_3$. This shows that also (ii) holds.

The function $(g - P_{x_0}^1 g)(\zeta)$ satisfies the assumptions (2.16) and (2.17) of the kernel $N_{3,k}$, in the same lemma with $k = 2$. Indeed, from the definition (2.2) of the Taylor polynomial we deduce that

$$|P_{x_0}^1 g(\zeta) - P_{X}^1 g(\zeta)| \leq C_1 d_{x_0}^{1+\gamma}(x_0, \zeta),$$

if $x, x_0$ are fixed in $K_1$ and $\zeta \in K_3$. Similarly, from (2.4) we deduce that

$$|P_{x_0}^1 g(\zeta) - P_{X}^1 g(\zeta)| \leq C_1 d_{x_0}^{1+\gamma}(x_0, X) d_{x_0}^{1+\gamma}(x_0, \zeta),$$

again with a constant $C_1$, depending on the $C_1^{1+\gamma}$ norm of $g$. This ensures that (2.17) is satisfied.

In the same way, using the regularity properties of $\bar{u}$, we deduce that the function

$$(b^\zeta_\bar{u}(\nabla_\bar{u}^\zeta(\bar{u}(x_0))) - b^\zeta_\bar{u}(\nabla_\bar{u}^\zeta(\bar{u}(\zeta)))) X_{i,a}^{\zeta} X_{j,a}^{\zeta} u(\zeta)$$

satisfies the assumption of the kernel $N_{3,k}$ in Proposition 2.9.

Finally, from the property (2.4) of the Taylor polynomials, we deduce that the function

$$(\bar{u}(\zeta) - P_{x_0}^1 \bar{u}(\zeta)) h_{\text{bij}}(x_0) X_{i,a}^{\zeta} X_{j,a}^{\zeta} u(\zeta) \varphi(\zeta)$$

satisfies the assumptions of the kernels $N_{4,k}$ in Remark 2.11.

In order to obtain an a-priori estimate of the second derivatives of the solution $u$ in terms of its $L^p$ norms we need to improve slightly the previous result.

**Lemma 4.5.** Assume that $u$ and $\bar{u}$ are $C^\infty$ functions, and that $u$ is a classical solution of $L_{e,\bar{a}}^u = g \in C^{\gamma}_{\bar{a}}(\Omega)$. Also assume that there exist a compact $K \subset \Omega$, and real numbers $p > 1$, $\gamma < 1$ and $C_0 > 0$ such that

$$\|\bar{u}\|_{C^{1,\gamma}_a(K)} + \|g\|_{C^{1,\gamma}_a(K)} + \|u\|_{C^{1,\gamma}_a(K)} + \sum_{\operatorname{deg}(\alpha) = 2} \|\nabla_\alpha^\gamma \bar{u}u\|_{L^p(K)} \leq C_0. \quad (4.20)$$

- If $Q - px > 0$, then, for every compact set $K_1 \subset K$ there exists a constant $C > 0$ only depending on $C_0$ such that

$$\sum_{\operatorname{deg}(\alpha) = 2} \|\nabla_\alpha^\gamma \bar{u}u\|_{L^r(K_1)} \leq C, \quad (4.21)$$

where $r = \frac{Qp}{Q - px}$, and $Q$ is the homogeneous dimension of the space, defined in (2.7).
\[ \text{If } Q - px < 0, \text{ then for every compact set } K_1 \ll K \text{ there exists a constant } C > 0 \text{ only depending on } C_0 \text{ such that} \]

\[ \sum_{\text{deg}(\sigma) = 2} \|\nabla_{\sigma, \theta}^\epsilon u\|_{L^\infty(K)} \leq C. \]

\section*{Proof}

The proof follows from the representation of the derivatives of \( u \varphi \) provided in Proposition 4.4.

Let us consider the first integral in (4.19). We first note that, by the expression (4.3) of \( N_1 \), there exist constants \( C_3 \) and \( C_4 \) only dependent on \( C_0 \) such that

\[ |N_1(\xi, x_0)| \leq C_3 + C_4 \sum_{i,j} |X_{\epsilon i, \theta}^\epsilon X_{\epsilon j, \theta}^\epsilon u| \]

for any \( \xi, x_0 \in K \). On the other hand \( N_1 \) is a sum of derivatives of the function \( \varphi \), constant in \( K_2 \), the support of the kernel \( N_1 \) is a subset of \( K_3 / C_0 K_2 \). This implies that if \( x \in K_1 \) and \( \xi \in \text{supp}(N_1) \), then \( d_{\epsilon, \theta}(x, \xi) \geq d_{\epsilon, \theta}(K_1, K_3 - K_2) \) and the function \( \nabla_{\epsilon, \theta}^\epsilon(x_0) \Gamma_{x_0}^\epsilon(x, \xi) \) is bounded uniformly in \( \epsilon \), by condition (2.11). Then, for every \( x_0 \in K_1 \)

\[ |\int \nabla_{\epsilon, \theta}^\epsilon \Gamma_{x_0}^\epsilon(x, \xi) N_1(\xi, x_0) d\xi| \leq \int_{K_3} |N_1(\xi, x_0)| d\xi \leq C(1 + \|X_{\epsilon i, \theta}^\epsilon X_{\epsilon j, \theta}^\epsilon u\|_{L^1(K)}) \leq C_1, \]

where \( C \) only depends on \( C_0 \).

The second term in (4.19) is the convolution of the fundamental solution with a regular function,

\[ \nabla_{\epsilon, \theta}^\epsilon(x_0) \left( P_{x_0}^1 g(\xi) \varphi(x_0 \circ \xi^{-1}) \right) \]

whose \( L^\infty \) norm only depends on \( \|g\|_{C^{1,1}_2} \) on the support \( K \) of the function \( \varphi \).

In the third term of (4.19), using the property (2.2) of the Taylor polynomial, and the estimate (2.10) of the fundamental solution, we obtain

\[ \int_{\Omega} \left| \nabla_{\epsilon, \theta}^\epsilon(x_0) \Gamma_{x_0}^\epsilon(\cdot, \xi) (g^{\epsilon}(\xi) - P_{x_0}^1 g(\xi)) \right| \varphi(\xi) d\xi \leq C \int_{K_3} d^2_{\epsilon, x_0} Q(x_0, \xi) d\xi \leq C_1 \]

for a suitable constant \( C_1 \) depending on \( \|g\|_{C^{1,1}_2} \), and on the compact set \( K \). Consequently, these terms belong to \( L^\infty_{\text{loc}} \).

Using again (2.2) and (2.10) the last two terms in representation formula (4.19) can be estimated by

\[ \int_{\Omega} d_{\epsilon, x_0}^{-Q+2} (x_0, \xi) |X_{\epsilon i, \theta}^\epsilon X_{\epsilon j, \theta}^\epsilon u(\xi)| d\xi. \]

Thanks to Theorem 2.6, we can then apply the standard theory of singular integrals and deduce that if \( |X_{\epsilon i, \theta}^\epsilon X_{\epsilon j, \theta}^\epsilon u(\xi)| \in L^p(K) \), with \( Q - px > 0 \) then (4.21) follows.
In order to prove (4.22) it suffices to apply the Hölder inequality to (4.23) and use the fact that $Q - px < 0$. This immediately leads to the desired $L^\infty$ bounds on the second derivatives of the solution.

**Proposition 4.6.** Let us assume that $\vec{u}$ is of class $C^\infty(\Omega)$ and that $u$ is a classical solution of $L_{\vec{a}_0}u = g \in C^\infty_{C_0}(\Omega)$. Let us also assume that there exists a compact set $K \subset \Omega$ and constant $C_0$ such that assumption (4.20) is satisfied, and such that

$$\|u\|_{C^{k-1,2}(K)} + \|\vec{u}\|_{C^{k-1,2}(K)} + \|g\|_{C^{k-2,2}(K)} \leq C_0.$$  

Then, for any $2 \leq k \leq 4$, for every compact set $K_1 \subset K$ there exists a constant $C > 0$ depending only on the choice of the compact sets, $C_0$, $k$ and $\alpha$ such that

$$\|u\|_{C^{k-2,2}(K_1)} \leq C.$$

**Proof.** A direct computation shows that the kernel in the representation formula in Proposition 4.3 satisfies the assumptions of Proposition 2.9.

### 4.3. A priori estimates of the solution of the non-linear operator.

We start with the following iteration result:

**Lemma 4.7.** Assume that $\vec{u}$ and $f_0$ are $C^\infty$ functions. If $z$ is a smooth solution of

$$\sum_{ij} a_{ij}(\nabla^2 \vec{u}) X^e_{ia} X^e_{ja} z + f_0 = 0 \quad \text{in } \Omega$$

then the function $v_h = X^e_{h,a} z$ satisfies the equation

$$\sum_{ij} a_{ij}(\nabla^2 \vec{u}) X^e_{ia} X^e_{ja} v_h + f_h = 0 \quad \text{in } \Omega.$$  

Here $f_h$ depends on $\nabla \vec{w}, \partial_{2n} \nabla \vec{u}, X^e_{h,a}, f_0, \nabla \vec{u}.$

**Proof.** Differentiating the equation (4.24) with respect to $X^e_{h,a}$ we obtain

$$0 = X^e_{h,a} \left( \sum_{ij} a_{ij}(\nabla^2 \vec{u}) X^e_{ia} X^e_{ja} z \right) + X^e_{h,a} f_0$$

$$= \sum_{ijk} \partial_{h} a_{ij} X^e_{h,a} X^e_{ka} a X^e_{ia} X^e_{ja} z + \sum_{ij} a_{ij}(\nabla^2 \vec{u}) [X^e_{h,a}, X^e_{ia}] X^e_{ja} z + \sum_{ij} a_{ij}(\nabla^2 \vec{u}) X^e_{ia} X^e_{ja} (X^e_{h,a} z) + X_{h,a} f_0.$$

Then

$$\sum_{ij} a_{ij}(\nabla^2 \vec{u}) X^e_{ia} X^e_{ja} v_h + f_h = 0,$$

where
Here the function $f_h$ clearly depends on $\nabla^2_\alpha z$, $\partial_{2n} \nabla^\epsilon_\alpha z$, $X^\epsilon_\alpha$, $f_0$, $\nabla^2_\alpha \tilde{u}$. 

**Lemma 4.8.** Assume that $\tilde{u}$ and $f_0$ are $C^\infty$ functions. If $z$ is a smooth solution of

$$
\sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) X^\epsilon_{i\alpha} X^\epsilon_{j\alpha} z + f_0 = 0 \quad \text{in } \Omega
$$

then the function $v = \partial_{2n} z$ satisfies

$$
\sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) X^\epsilon_{i\alpha} X^\epsilon_{j\alpha} v + f = 0 \quad \text{in } \Omega.
$$

Here $f$ depends on $\nabla^2_\alpha z$, $\partial_{2n} \nabla^\epsilon_\alpha z$, $\partial_{2n} f_0$.

**Proof.** Differentiating the equation (4.25) with respect to $\partial_{2n}$ we obtain

$$
0 = \partial_{2n} \left( \sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) X^\epsilon_{i\alpha} X^\epsilon_{j\alpha} z + f_0 \right)
$$

$$
= \sum_{ijk} \partial_k a_{ij} \partial_{2n} X^\epsilon_{k\alpha} \tilde{u} X^\epsilon_{i\alpha} X^\epsilon_{j\alpha} z + \sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) [\partial_{2n}, X^\epsilon_{i\alpha}]X^\epsilon_{j\alpha} z
$$

$$
+ \sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) X^\epsilon_{i\alpha} [\partial_{2n}, X^\epsilon_{j\alpha}] z + \sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) X^\epsilon_{i\alpha} X^\epsilon_{j\alpha} (\partial_{2n} z) + \partial_{2n} f_0.
$$

The latter can be rewritten as

$$
\sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) X^\epsilon_{i\alpha} X^\epsilon_{j\alpha} v + f = 0,
$$

where the function

$$
f = \sum_{ijk} \partial_k a_{ij} \partial_{2n} X^\epsilon_{k\alpha} \tilde{u} X^\epsilon_{i\alpha} X^\epsilon_{j\alpha} z + \sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) \partial_{2n-1} \partial_{2n} \tilde{u} \partial_{2n} X^\epsilon_{i\alpha} z
$$

$$
+ \sum_{ij} a_{ij} \partial_{2n-1} (\nabla^\epsilon_\alpha \tilde{u}) X^\epsilon_{i\alpha} (\partial_{2n} \tilde{u} \partial_{2n} z) + \sum_{ij} a_{ij}(\nabla^\epsilon_\alpha \tilde{u}) X^\epsilon_{i\alpha} X^\epsilon_{j\alpha} v + \partial_{2n} f_0
$$

depends on $\nabla^\epsilon_\alpha v$, $\nabla^2_\alpha z$, $\partial_{2n} \nabla^\epsilon_\alpha z$, $\nabla^2_\alpha \tilde{u}$. 

\square
In order to study the regularity of smooth solutions \( u \) of the approximating non-linear equations \( L_\epsilon u = 0 \), we apply the previous lemma substituting \( \tilde{u} \) with \( u \).

**Lemma 4.9.** Let \( u \) be a smooth classical solution of the non-linear equation \( L_\epsilon u = 0 \). Let \( \sigma \) be a multi-index with all components smaller than \( 2n \). Then the function \( v_\sigma = \nabla_{\sigma, u}^\epsilon u \) satisfies

\[
\sum_{ij} a_{ij}(\nabla_{\epsilon}^\mu u) X_{i\mu}^\epsilon X_{j\mu}^\nu v_\sigma + f_\sigma = 0,
\]

where \( f_\sigma \) depends on \( \nabla_{\epsilon}^{(k+1)} u, \partial_2 n \nabla_{\epsilon}^k u \) with \( k = \deg(\sigma) \).

**Proof.** By Lemma 4.7 the assertion is true for the derivatives of order one. Assume that it is true for \( \deg(\sigma) = k \). Then, let us consider a multi-index \( \sigma \) of degree \( k + 1 \). By definition \( \sigma = (\sigma_1, \tilde{\sigma}) \), with \( \deg(\tilde{\sigma}) = k \). Then by inductive assumption the function \( z = \nabla_{\sigma_1, u}^\epsilon u \) satisfies

\[
\sum_{ij} a_{ij}(\nabla_{\epsilon}^\mu u) X_{i\mu}^\epsilon X_{j\mu}^\nu z + f_{\sigma_1} = 0,
\]

where \( f_{\sigma_1} \) depends on \( \nabla_{\epsilon}^{k+1} u, \partial_2 n \nabla_{\epsilon}^k u \). Applying Lemma 4.7 we deduce that the function \( v_{\sigma_1} = X_{\sigma_1 u}^\epsilon z \) is a solution of

\[
\sum_{ij} a_{ij}(\nabla_{\epsilon}^\mu u) X_{i\mu}^\epsilon X_{j\mu}^\nu v_{\sigma_1} + f_{\sigma_1} = 0,
\]

where \( f_{\sigma_1} \) depends on \( \nabla_{\epsilon}^{k+2} u, \partial_2 n \nabla_{\epsilon}^k u \) and \( X_{\sigma_1, u} f_0 \). Since \( f_0 \) depends on \( \nabla_{\epsilon}^{k+1} u, \partial_2 n \nabla_{\epsilon}^k u \), then \( X_{\sigma_1, u} f_0 \) depends on \( X_{\sigma_1, u} \nabla_{\epsilon}^{k+1} u \) and

\[
X_{\sigma_1, u} \partial_2 n \nabla_{\epsilon}^k u = \delta_{\sigma_1 2n-1} \partial_2 n u \partial_2 n \nabla_{\epsilon}^k u + \partial_2 n X_{\epsilon; \sigma_1, u} \nabla_{\epsilon}^k u.
\]

**Theorem 4.10.** Let \( u \) be a smooth classical solution of the non-linear equation \( L_\epsilon u = 0 \). Let us fix a compact set \( K \subset \Omega \) and assume that there exist constants \( a < 1 \), \( p > 1 \) and \( C_0 > 0 \) such that

\[
\|u\|_{C^a_{\mu}^{(K)}} + \sum_{\deg(\sigma) = 2} \|\nabla_{\sigma, u}^\epsilon u\|_{L^p(K)} \leq C_0.
\]

Then, for every \( \beta \in (0, 1) \), for every compact set \( K_1 \subset \subset K \) there exists a constant \( \tilde{C}_\beta \), which depends on \( \beta, K, K_1, C_0 \) and \( \Omega \) such that

\[
\|u\|_{C^a_{\mu}^{(K_1)}} + \|\partial_2 n u\|_{C^{a}_\mu^{(K_1)}} \leq \tilde{C}_\beta.
\]

**Proof.** We first prove that for every \( r > 1 \) for every compact set \( K_2 \) such that \( K_1 \subset \subset K_2 \subset \subset K \) there exists a constant \( C_r \), only depending on \( r, C_0 \) and on the choice of the compact sets, such that for every multi-index \( \sigma \) of degree 2, we have

\[
\|\nabla_{\sigma, u}^\epsilon u\|_{L^r(K_2)} \leq C_r.
\]

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Indeed, for every compact set $K_3$ such that $K_2 \subset K_3 \subset K$ we can apply the first assertion of Lemma 4.5, with $\bar{u} = u$ and we obtain

$$
\|\nabla_{\sigma, \bar{u}} u\|_{L^r(K_3)} \leq C_1,
$$

where $r_1 = \frac{n^2}{n - 2z} > 2$. If $n - r_1 z > 0$ we can apply again Lemma 4.5 on a new compact set $K_4$ such that $K_2 \subset K_4 \subset K_3$ and we have

$$
\|\nabla_{\sigma, \bar{u}} u\|_{L^2(K_4)} \leq C_2,
$$

with

$$
r_2 = \frac{nr_1}{n - r_1 z} = \frac{2n}{n - 4z} > r_1.
$$

For every fixed number $r$, after a finite number of iterations of the same argument, we can prove the estimate (4.27).

Consequently by (4.22) we have $\|\nabla_{\sigma, u} u\|_{L^\infty} \leq C$, then for every compact set $K_5$ such that $K_1 \subset K_2$ and for every $\beta < 1$ there exists a constant $\tilde{C}_\beta$ such that

$$
\|\nabla_{\sigma} u\|_{C^{\beta}_z(K_5)} \leq \tilde{C}_\beta.
$$

As a consequence of Proposition 4.4 we deduce that for every $\beta < 1$ for every compact set $K_6$ such that $K_1 \subset K_6 \subset K_5$ there exists a constant $\tilde{C}_\beta$ such that

$$
\|u\|_{C^{2, \beta}_z(K_6)} \leq \tilde{C}_\beta.
$$

By Proposition 4.6 we deduce that for every $\beta < 1$ and for every compact set $K_7$ such that $K_1 \subset K_7 \subset K_6$ there exists a constant $\tilde{C}_\beta$ such that

$$
\|u\|_{C^{3, \beta}_z(K_7)} \leq \tilde{C}_\beta.
$$

Applying again the same proposition with $k = 4$, we deduce that

$$
\|u\|_{C^{4, \beta}_z(K_1)} \leq \tilde{C}_\beta,
$$

which implies in particular that there exists a constant $\tilde{C}_\beta$ such that

$$
\|u\|_{C^{4, \beta}_z(K_1)} + \|\hat{\nabla} u\|_{C^{2, \beta}_z(K_1)} \leq \tilde{C}_\beta.
$$

**Theorem 4.11.** Let $u$ be a smooth classical solution of $L_x u = 0$. Let us also assume that assumption (4.26) is satisfied. Then, for any compact set $K_1 \subset K$, for every $k \in \mathbb{N}$ and $z < 1$, there exists a constant $C > 0$ depending only on $C_0$, $k$, $z$ and $K_1$ such that

$$
\|u\|_{C^{k, z}_x(K_1)} \leq C.
$$
Proof. For every $k \in \mathbb{N}$, for every $\sigma$ with $\deg(\sigma) = k$, and components in $\{1, \ldots, 2n\}$ we prove by induction that
\[
V_{\sigma, u}^e u \in C^{3, \alpha}_u(\Omega), \quad \hat{\partial}_2n V_{\sigma, u}^e u \in C^{2, \alpha}_u(\Omega),
\]
and that for every compact set $K_1$ such that $K_1 \subset K$ there exists a constant $C > 0$ depending only on $C_0$, $k$, $\alpha$ such that
\[
\|V_{\sigma, u}^e u\|_{C^{3, \alpha}(K_1)} + \|\hat{\partial}_2n V_{\sigma, u}^e u\|_{C^{2, \alpha}(K_1)} \leq C.
\]
The thesis is true for $\deg(\sigma) = 0$ by Theorem 4.10.

We assume by induction that it is true for $\deg(\sigma) = k$. Call $z = V_{\sigma, u}^e u$, then by Lemma 4.9 the function $z$ satisfies $L_{e, u} z = f_\sigma$ in $\Omega$ with
\[
f_\sigma = f_\sigma(V_u^{\ell(k+1)} u, \hat{\partial}_2n \nabla^e u) \in C^{2, \alpha}_u(\Omega),
\]
by inductive assumption. By Lemma 4.8, the function $v = \hat{\partial}_2n z$ satisfies $L_{e, u} v = f$, in $\Omega$, where the function $f = f(V_u^{\ell 2} z, \hat{\partial}_2n \nabla^e z) \in C^{1, \alpha}_u(\Omega)$.

It follows by Proposition 4.6 that $v = \hat{\partial}_2n z \in C^{3, \alpha}_u(\Omega)$, and if $K_2$ is a compact set such that $K_1 \subset K_2 \subset K$ there exists a constant $C$ depending only on $C_0$, $k$, $\alpha$ such that
\[
\|v\|_{C^{3, \alpha}(K_2)} \leq C \|f\|_{C^{1, \alpha}(K_2)} = C_2.
\]

This argument, applied to any multi-index $\sigma$ with $\deg(\sigma) = k$, implies that $\hat{\partial}_2n \nabla_u^{\ell k} u \in C^{3, \alpha}_u$. Consequently $\hat{\partial}_2n \nabla_u^{\ell k+1} u \in C^{2, \alpha}_u$, and
\[
\|\hat{\partial}_2n \nabla_u^{\ell k} u\|_{C^{2, \alpha}(K_2)} \leq C_1,
\]
for an other constant $C_1$, only dependent on $C_0$, $k$, $\alpha$.

Moreover by Lemma 4.7 the function $v_h = X_{hu}^e z$ satisfies $L v_h = f_h$, with
\[
f_h = f_h(V_u^{\ell 2} z, \hat{\partial}_2n \nabla^e z) \in C^{1, \alpha}_u(\Omega).
\]
Again Proposition 4.6 implies that $X_{hu}^e \nabla_{\sigma, u}^e u \in C^{3, \alpha}(\Omega)$, and that
\[
\|\nabla_{\sigma, u}^{\ell(k+1)} u\|_{C^{3, \alpha}(K_1)} \leq C_2,
\]
for a constant $C_2$ dependent on $C_0$, $k$, $\alpha$. This concludes the proof. □

Proof of Theorem 1.2. Let $(u_\epsilon)_\epsilon$ be smooth solutions of
\[
L_{\epsilon} u_\epsilon = 0 \quad \text{in } \Omega
\]
and converging to $u$ in Lipschitz sense. Let $K_1$ be an arbitrary compact set in $\Omega$. Then there exist compact sets $K$ and $K_2$ such that
\[
K_1 \subset K_2 \subset K.
\]
By assumption there exists a positive constant $C_0$ such that 
\[ \| \nabla E v \|_{L^\infty(K)} \leq C_0, \]
for every $\varepsilon$. By Proposition 3.7 there exists a constant $C_1$ such that 
\[ \| \nabla^1 u_\varepsilon \|_{C^1(K_2)} + \| \nabla^2 u_\varepsilon \|_{L^\infty(K_2)} + \| \nabla^2 u_\varepsilon \|_{L^2(K_2)} \leq C_1. \]
Then by Theorem 4.11 for every $k$ there is $C_k$ such that 
\[ \| u_\varepsilon \|_{C^{2k,s}(K_1)} \leq C_k. \]
In particular, 
\[ \| u_\varepsilon \|_{C^k E(K_1)} \leq C_k. \]
Since all the constants are independent of $\varepsilon$, letting $\varepsilon$ go to 0 we obtain estimates of $u$ in $C^k_E$ for every $k$. Consequently $u \in C^k_E$. □

References

[1] L. Ambrosio, F. Serra Cassano and D. Vittone, Intrinsic regular hypersurfaces in Heisenberg groups, J. Geom. Anal. 16 (2006), n. 2, 187–232.
[2] V. F. Barone Adesi, F. Serra Cassano and D. Vittone, The Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations, Calc. Var. Part. Diff. Equ. 30 (2007), no. 1, 17–49.
[3] F. Bigolin and F. Serra Cassano, Intrinsic regular graphs in the Heisenberg group vs. weak solutions of nonlinear first order PDEs, preprint 2007.
[4] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, Uniform Gaussian Estimates of the fundamental solutions for heat operators on Carnot Groups, Adv. Diff. Equ. 7, no. 10 (2002), 1153–1192.
[5] M. Bonk and L. Capogna, Mean curvature flow in the Heisenberg group, preprint 2005.
[6] L. Capogna, G. Citti and M. Manfredini, Higher regularity of Lipschitz minimal graphs in $\mathbb{H}^1$, Ind. Univ. J. Math., to appear.
[7] L. Capogna, D. Danielli, S. D. Pauls and J. Tyson, An introduction to the Heisenberg group and to the sub-Riemannian isoperimetric problem, Progr. Math. 259 (2007).
[8] J.-H. Cheng, J.-F. Hwang and P. Yang, Existence and Uniqueness for $P$-Area Minimizers in the Heisenberg Group, Math. Ann. 337 (2007), no. 2, 253–293.
[9] J.-H. Cheng, J.-F. Hwang and P. Yang, Regularity of $C^1$ smooth surfaces with prescribed $p$-mean curvature in the Heisenberg group, preprint 2007.
[10] J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang, Minimal surfaces in pseudohermitian geometry, Ann. Scu. Norm. Sup. Pisa Cl. Sci. (5) 4, no. 1 (2005), 129–177.
[11] M. Christ, A. Nagel, E. L. Stein and S. Wainger, Singular and maximal Radon transforms: Analysis and geometry, Ann. Math. (2) 150, no. 2 (1999), 489–577.
[12] G. Citti, Regularity of solutions of a quasilinear equation related to the Levi operator, Ann. Scu. Norm. Sup. Pisa Cl. Sci. (4) 23 (1996), 483–529.
[13] G. Citti and A. Montanari, Regularity properties of solutions of a class of elliptic-parabolic nonlinear Levi type equations, Trans. Amer. Math. Soc. 354 (2002), no. 7, 2819–2848.
[14] G. Citti and M. Manfredini, Uniform Estimates of the fundamental solution for a family of Hypoelliptic operators, Potent. Anal. 25 (2006), 147–164.
[15] G. Citti and M. Manfredini, Implicit function theorem in Carnot-Caratheodory spaces, Comm. Cont. Math. 8, no. 5 (2006), 657–680.
[16] G. Citti, N. Garofalo and E. Lanconelli, Harnack’s inequality for sum of squares of vector fields plus a potential, Amer. J. Math. 115 (1993), 699–734.
[17] G. Citti and G. Tomassini, Levi equation for almost complex structures, Rev. Mat. Iberoamer. 20 (2004), no. 1, 151–182.
G. Citti, A. Pascucci and S. Polidoro, On the regularity of solutions to a nonlinear ultraparabolic equation arising in mathematical finance, Diff. Integr. Equ. 14 (2001), no. 6, 701–738.

G. Citti, A. Pascucci and S. Polidoro, Regularity properties of viscosity solutions of a non-Hörmander degenerate equation, J. Math. Pures Appl. (9) 80 (2001), no. 9, 901–918.

D. Danielli, N. Garofalo and D. M. Nhieu, Sub-Riemannian calculus on hypersurfaces in Carnot groups, Adv. Math. 215 (2007), 292–378.

D. Danielli, N. Garofalo and D. M. Nhieu, A notable family of entire intrinsic minimal graphs in the Heisenberg group which are not perimeter minimizing, preprint 2006.

C. L. Fefferman and A. Sánchez-Calle, Fundamental solutions for second order subelliptic operators, Ann. Math. (2) 124 (1986), 247–272.

G. B. Folland and E. M. Stein, Estimates for the $\bar{\partial}_b$ Complex and Analysis on the Heisenberg Group, Comm. Pure Appl. Math. 20 (1974), 429–522.

B. Franchi, R. Serapioni and F. Serra Cassano, Rectifiability and Perimeter in the Heisenberg Group, Math. Ann. 321 (2001), 479–531.

N. Garofalo and S. D. Pauls, The Bernstein problem in the Heisenberg group, preprint 2003.

N. Garofalo and D.-M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, Comm. Pure Appl. Math. 49 (1996), 1081–1144.

D. Gilbarg and N. Trudinger, Second order elliptic partial differential equations, Springer Verlag, 2000.

M. Gromov, Carnot-Carathéodory spaces seen from within Bellaiche, André et al., eds., Sub-Riemannian geometry, Progr. Math. 144 (1996), 79–323.

R. K. Hadlky and S. D. Pauls, Constant mean curvature surfaces in sub-Riemannian geometry, preprint 2005.

F. Montefalcone, Alcune formule integrali nei Gruppi di Carnot, Seminario del dipartimento di matematica Università di Bologna, Tecnoprint, aprile 2005.

A. Nagel, E. M. Stein and S. Wainger, Balls and metrics defined by vector fields I: Basic properties, Acta Math. 155 (1985), 103–147.

Y. Ni, Sub-Riemannian constant mean curvature surfaces in the Heisenberg group as limits, Ann. Mat. Pura Appl. (4) 183 (2004), no. 4, 555–570.

S. D. Pauls, Minimal surfaces in the Heisenberg group, Geom. Dedic. 104 (2004), 201–231.

S. D. Pauls, $H$-minimal graphs of low regularity in the Heisenberg group, Comment. Math. Helv. 81 (2) (2006), 337–381.

M. Ritoré, Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity, Calc. Var. Part. Diff. Equ. 34 (2009), no. 2, 179–192.

M. Ritoré and C. Rosales, Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group $\mathbb{H}^n$, J. Geom. Anal. 16 (2006), no. 4, 703–720.

M. Ritoré and C. Rosales, Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$, Adv. Math. 219 (2008), no. 2, 633–671.

L. Rothschild and E. M. Stein, Hypoelliptic differential operators and nilpotent Lie groups, Acta Math. 137 (1977), 247–320.

C. Selby, Seminar at the meeting “Geometric analysis and applications” University of Illinois, Urbana-Champaign, July 12–15, 2006.

N. Sherbakova, Minimal surfaces in contact subriemannian manifolds, preprint 2006.

Department of Mathematics, University of Arkansas, Fayetteville, AR 72701
e-mail: lecapogna@uark.edu

Dipartimento di Matematica, Piazza Porta S. Donato 5, 40126 Bologna, Italy
e-mail: citti@dm.unibo.it
e-mail: manfredi@dm.unibo.it

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