Entropy Numbers of Spheres in Banach and quasi-Banach spaces

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Abstract
We prove sharp upper bounds on the entropy numbers $e_k(S^d_{p}, \ell_q^d)$ of the $p$-sphere in $\ell_q^d$ in the case $k \geq d$ and $0 < p \leq q \leq \infty$. In particular, we close a gap left open in recent work of the second author, T. Ullrich and J. Vybiral. We also investigate generalizations to spheres of general finite-dimensional quasi-Banach spaces.

1 Introduction
Entropy numbers are a central concept in approximation theory. They quantify the compactness of a given set with respect to some reference space. For $K$ being a subset of a finite-dimensional (quasi-)Banach space $Y$, the $k$-th (dyadic) entropy number $e_k(K, Y)$ is defined as

$$e_k(K, Y) = \min \left\{ \varepsilon > 0 : K \subset \bigcup_{j=1}^{2^{k-1}} (x_j + \varepsilon B_Y) \text{ for some } x_1, \ldots, x_{2^{k-1}} \in Y \right\}. \quad (1)$$

A detailed discussion and historical remarks can be found in the monographs [1, 2].

Recall the (quasi-)norms $\|x\|_p := \left( \sum_{i=1}^d |x_i|^p \right)^{1/p}$ for $0 < p < \infty$ and $\|x\|_{\infty} := \max_{i=1}^d |x_i|$. The entropy numbers $e_k(B_p^d, \ell_q^d)$ of the unit balls $B_p^d = \{ x \in \mathbb{R}^d : \|x\| \leq 1 \}$
in $\ell_q^d = (\mathbb{R}^d, \| \cdot \|_q)$ for $0 < p, q \leq \infty$ are well-understood for more than a decade now \cite{5, 7, 8}. Regarding $p$-spheres $S_p^{d-1} = \{ x \in \mathbb{R}^d : \| x \|_p = 1 \}$, a rigorous proof has been provided just recently \cite{6}. For the reader’s convenience, let us restate the result.

**Lemma 1.** Let $d \in \mathbb{N}$, $d \geq 2$, $0 < p \leq q \leq \infty$ and $\bar{p} = \min\{1, p\}$. Then,

(i) $$2^{-k/(d-1)}d^{1/q-1/p} \lesssim e_k(S_p^{d-1}, \ell_q^d) \lesssim 2^{-k/(d-\bar{p})}d^{1/q-1/p} \quad \text{for } k \geq d.$$  

(ii) $$e_k(S_p^{d-1}, \ell_q^d) \asymp \begin{cases} 1 & \text{for } 1 \leq k \leq \log d, \\ \left(\frac{\log(1+d/k)}{k}\right)^{1/p-1/q} & \text{for } \log d \leq k \leq d. \end{cases}$$

To a large extent, the proof just mimics the corresponding proof for $p$-unit balls, including the volume arguments which are utilized in the case $k \geq d$. Once $p < 1$ however, the volume arguments get too coarse for the upper bound. They do no longer lead to the correct order of decay; instead of $2^{-k/(d-1)}$, we only see $2^{-k/(d-\bar{p})}$.

In this work, we present an alternative proof for the case $k \geq d$ which gives the correct decay behaviour in $k$. In essence, the proof relies on a scheme to construct a covering of $S_p^{d-1}$ from a covering of $B_p^{d-1}$. In Section 2 we prove

**Theorem 2.** Let $d \in \mathbb{N}$, $d \geq 2$, $0 < p \leq q \leq \infty$. Then,

$$e_k(S_p^{d-1}, \ell_q^d) \asymp 2^{-k/(d-1)}d^{1/q-1/p}, \quad \text{for } k \geq 3d.$$  

Finally in Section 3 we generalize the methods presented in Section 2 to study entropy numbers $e_k(S_X, Y)$ for general finite-dimensional quasi-Banach spaces $X$ and $Y$.

**Notations** For $e \in \{-1, 1\}^d$ a diagonal vector, we denote the associated orthant by $Q_e := \{ x \in \mathbb{R}^d : 0 \leq \text{sign } e \cdot x \leq 1 \}$. Furthermore, by $H_i$ we denote the hyperplane where the $i$-th coordinate of every vector equals 0 and $C_i = \{ x \in \mathbb{R}^d : |x| \leq |x_j| \text{ for } j = 1, \ldots, d \}$. Whenever we write $|x| \leq |y|$, then this is meant coordinate-wise: $|x_j| \leq |y_j|$ for each $j = 1, \ldots, d$.

# 2 Entropy numbers of $p$-pheres

Let us start this section with a simple proof that the entropy numbers of the $p$-sphere $S_p^{d-1}$ are always bounded from below by the respective entropy numbers of the $p$-unit ball $B_p^{d-1}$. The known behaviour of the latter immediately implies all the lower bounds in Lemma 1 and Theorem 2.

**Theorem 3.** Let $0 < p, q \leq \infty$. For all natural numbers $k$, we have

$$e_k(S_p^{d-1}, \ell_q^d) \geq e_k(B_p^{d-1}, \ell_q^{d-1}).$$
Proof. Let \( \varepsilon = e_k(S_p^{d-1}, \ell_q^d) \) and choose an associated covering

\[ S_p^{d-1} \subset \bigcup_{j=1}^{2^{k-1}} (x_j + \varepsilon B_q^d). \]

Let \( P_1 \) be the orthogonal projection in \( \mathbb{R}^d \) onto the hyperplane \( H_1 \) setting the first coordinate to zero. Now, identify \( B_d^{d-1} \) and \( B_q^{d-1} \) with its natural isometric embedding into \( H_1 \). Then

\[ P_1(B_d^d) = B_q^{d-1} \quad \text{and} \quad P_1(S_p^{d-1}) = B_p^{d-1} \]

and the linearity of \( P_1 \) imply

\[ B_p^{d-1} = P_1(S_p^{d-1}) \subset \bigcup_{j=1}^{2^{k-1}} (P_1 x_j + \varepsilon P_1(B_q^d)) = \bigcup_{j=1}^{2^{k-1}} (P_1 x_j + \varepsilon B_q^{d-1}). \]

By definition of the entropy numbers this gives \( e_k(B_p^{d-1}, \ell_q^{d-1}) \leq \varepsilon \) and proves the theorem.

The upper bounds in (ii) of Lemma 4 directly follow from the corresponding upper bounds for \( B_p^d \) and the inclusion \( S_p^{d-1} \subset B_p^d \). We turn to the proof of the upper bound on the entropy numbers of \( S_p^{d-1} \) for \( k \geq d \) in Theorem 2. The covering construction is based on the bijective shifting map

\[ \Delta_e : B_p^d \cap Q_e \cap H_i \to S_p^{d-1} \cap Q_e \cap C_i, \quad x \mapsto x + s(x)e, \]

which shifts \( x \) by \( 0 \leq s(x) \leq 1 \) along the diagonal \( e \) until it hits the \( p \)-sphere. Since the vectors in the domain of \( \Psi_e^p \) all have the \( i \)-th coordinate equal to 0, the shift \( s(x) \) corresponds to the \( i \)-th coordinate of \( \Delta_e(x) \). Below we provide bounds on the difference between two shifts \( s(x), s(y) \) that depend on the \( \ell_\infty \)-distance of \( x \) and \( y \).

Lemma 4. Let \( x, y \in B_p^d \cap Q_e \cap H_i \) such that \( |x| \leq |y| \). Then

\[ s(y) \leq s(x) \leq s(y) + \|x - y\|_\infty. \]

Proof. We give the argument for \( p < \infty \), the proof is easily adapted for \( p = \infty \). By symmetry, it is enough to check the claim for the positive orthant associated with \( e = (1, \ldots, 1) \) and \( i = d \). Then \( x_d = y_d = 0 \) and the assumption \( |x| \leq |y| \) translates into \( 0 \leq x_j \leq y_j \) for \( j = 1, \ldots, d - 1 \). Moreover, \( \sigma = s(x) \geq 0 \) and \( \tau = s(y) \geq 0 \) are given as the unique nonnegative solutions of the equations

\[ \sum_{j=1}^{d-1} (x_j + \sigma)^p + \sigma^p = 1 \quad \text{and} \quad \sum_{j=1}^{d-1} (y_j + \tau)^p + \tau^p = 1. \]
Now $\tau \leq \sigma$ directly follows from $0 \leq x_j \leq y_j$ for $j = 1, \ldots, d - 1$. The remaining inequality $\sigma \leq \|x - y\|_\infty + \tau$ is a consequence of

$$
\sum_{j=1}^{d-1} (x_j + \sigma)^p + \sigma^p = \sum_{j=1}^{d-1} (y_j + \tau)^p + \tau^p \\
\leq \sum_{j=1}^{d-1} (x_j + (y_j - x_j) + \tau)^p + \tau^p \\
\leq \sum_{j=1}^{d-1} (x_j + \|x - y\|_\infty + \tau)^p + (\|x - y\|_\infty + \tau)^p.
$$

Now it is easy to establish the following for $\Delta^p_z$:

**Lemma 5.** For any $x, y \in B_p^d \cap Q_e \cap H_i$, we have

$$
\|\Delta^p_z(x) - \Delta^p_z(y)\|_\infty \leq 2\|x - y\|_\infty.
$$

**Proof.** For the moment, assume that $|x| \leq |y|$. The previous lemma and $|x| \leq |y|$ immediately give

$$
-\|x - y\|_\infty \leq |x_j| - |y_j| + s(x) - s(y) \leq \|x - y\|_\infty
$$

for every $j = 1, \ldots, d$. Hence, we have $\|\Delta^p_z(x) - \Delta^p_z(y)\|_\infty \leq \|x - y\|_\infty$.

Now for arbitrary $x, y \in B_p^d \cap Q_e \cap H_i$, let $z \in B_p^d \cap Q_e \cap H_i$ be the vector given by $z_j = \text{sign}(x_j, y_j)$ for $j = 1, \ldots, d$. But for this vector, both $|z| \leq |x|$ and $|z| \leq |y|$ hold true. Hence, a simple application of the triangle inequality yields $\|\Delta^p_z(x) - \Delta^p_z(y)\|_\infty \leq 2\|x - y\|_\infty$. \qed

**Theorem 6.** Let $0 < p, q \leq \infty$ and $d \geq 2$. Assume $k \geq 3d$. Then

$$
e_k(S_p^{d-1}, L_p^d) \lesssim 2^{-\frac{d}{d - 1}} d^{1/q - 1/p}.
$$

**Proof.** We first treat the case $q = \infty$. Let $\mathcal{N}_{e,i}(\varepsilon)$ be a minimal $\varepsilon$-covering of $B_p^d \cap Q_e \cap H_i$ in $\ell_\infty$. Since $B_p^d \cap Q_e \cap H_i$ is a subset of a $(d - 1)$-dimensional $\ell_p$-ball, we have

$$
|\mathcal{N}_{e,i}(\varepsilon)| \leq N_e(B_p^{d-1}, \ell_\infty^{d-1}).
$$

Lemma$\text{[\text{[}}$implies that the set $\Delta^p_z(\mathcal{N}_{e,i}(\varepsilon))$ is a $2\varepsilon$-covering of $S_p^{d-1} \cap Q_e \cap C_i$. Consequently,

$$
\tilde{\mathcal{N}}(2\varepsilon) := \bigcup_{e \in \{-1, 1\}^d} \bigcup_{1 \leq i \leq d} \Delta^p_z(\mathcal{N}_{e,i}(\varepsilon))
$$

is a $2\varepsilon$-covering of $S_p^{d-1}$. Moreover

$$
|\tilde{\mathcal{N}}(2\varepsilon)| \leq \sum_{e \in \{-1, 1\}^d} \sum_{1 \leq i \leq d} |\mathcal{N}_{e,i}(\varepsilon)| \leq 2^d d N_e(B_p^{d-1}, \ell_\infty^{d-1}).
$$
By definition of entropy numbers,
\[ e_k(S_d^{-1}, \ell_p^d) \leq 2e_k - d - \log d(B_p^{d-1}, \ell_\infty^{d-1}). \]
Hence the assertion follows from the known upper bound for the entropy numbers of
\( B_p^{d-1} \) in \( \ell_\infty^{d-1} \).

The case of general \( q \) is a consequence of the factorization property of entropy num-
bers, see for instance \[2\], Section 1.3,\]
\[ e_k(S_p^{d-1}, \ell_q^d) \leq \| \text{id} : \ell_\infty^d \rightarrow \ell_q^d \| e_k(S_p^{d-1}, \ell_\infty^d) = d^{1/q} e_k(S_p^{d-1}, \ell_\infty^d). \]

Remark 7. For \( p = q \), the Mazur mapping provides a different technique to derive
matching bounds for \( p < 1 \) in Lemma 1. This mapping forms a homeomorphism between
\( p \)-spheres with the \( \ell_p \)-metric for different \( p \). In particular, for \( 0 < p < 2 \) the Mazur map
\( M : S_p^{d-1} \rightarrow S_p^{d-1} \) is Lipschitz with constant \( L = L(p) \) independent of the dimension, see \[9\]. Hence, any \( \varepsilon \)-covering of the 2-sphere yields an \( L\varepsilon \)-covering of the \( p \)-sphere by
mapping the centers to \( S_p^{d-1} \) with the Mazur map. This shows that
\[ e_k(S_p^{d-1}, \ell_p) \leq L e_k(S_p^{d-1}, \ell_2), \]
which allows to transfer the upper bounds for the case \( p = 2 \) in Lemma 1 to the case
\( 0 < p < 1 \).

3 Entropy numbers of spheres in general quasi-Ba-

nach spaces

In this section, we elaborate on the methods presented in the previous section, general-
izing them to arbitrary finite-dimensional quasi-Banach spaces. To this end, let \( X \) be
\( \mathbb{R}^d \) equipped with some quasi-norm \( \| \cdot \| \). Let \( S_X \) be the unit sphere in \( X \). Let \( X_i \) be
the hyperplane \( H_i \) considered as a subspace of \( X \), i.e. its unit ball is \( B_X \cap H_i \). Let \( X^i \)
be the hyperplane \( H_i \) with the quasi-norm whose unit ball is the image of \( B_X \) under the
orthogonal projection of \( \mathbb{R}^d \) onto \( H_i \). Observe that the quasi-norms of \( X_i \) and \( X^i \)
coincide for \( X = \ell_p^d, 0 < p \leq \infty \). That \( X_i \) and \( X^i \) are the same is always true if the
orthogonal projection onto \( H_i \) is a norm 1 projection, which is the case, in particular, if
\( X \) is a Banach space.

The following theorem provides a generalization of Theorem 3.

Theorem 8. Let \( X \) and \( Y \) be \( \mathbb{R}^d \) with quasi-norms as above. For all natural numbers
\( k \), we have
\[ e_k(S_X, Y) \geq \max_{1 \leq i \leq d} e_k(B_{X^i}, Y^i). \]

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Proof. The proof of Theorem 3 can be easily adapted to this situation. Let \( \varepsilon = e_k(S_X, Y) \) and choose an associated covering

\[
S_X \subseteq \bigcup_{j=1}^{2^{k-1}} (x_j + \varepsilon B_Y).
\]

Let \( P_i \) be the orthogonal projection in \( \mathbb{R}^d \) onto the hyperplane \( H_i \) setting the first coordinate to zero. Then we have

\[
P_i(B_Y) = B_{Y^i} \quad \text{and} \quad P_i(S_X) = B_{X^i}.
\]

Since \( P_i \) is linear this implies

\[
B_{X^i} = P_i(S_X) \subseteq \bigcup_{j=1}^{2^{k-1}} (P_i x_j + \varepsilon P_i(B_Y)) = \bigcup_{j=1}^{2^{k-1}} (P_i x_j + \varepsilon B_{Y^i}).
\]

By definition of the entropy numbers this gives \( e_k(B_{X^i}, Y^i) \leq \varepsilon \) and proves the theorem.

To generalize the upper bounds, we have to assume a monotonicity property of the quasi-norm of \( X \) in each orthant \( Q_e \). Let us call \( X \) monotone if for any \( e \in \{-1, 1\}^d \) and for \( x, y \in Q_e \) with \( |x| \leq |y| \) we have \( \|x\| \leq \|y\| \). Again, we define a bijective shifting map by

\[
\Delta^X_e : B_X \cap Q_e \cap H_i \to S_X \cap Q_e \cap C_i, \quad x \mapsto x + s(x)e,
\]

which shifts \( x \) by \( 0 \leq s(x) \leq 1 \) along the diagonal \( e \) until it hits the sphere of \( X \). Then analogues of Lemmas 4 and 5 hold true.

Lemma 9. Let \( x, y \in X \cap Q_e \cap H_i \) such that \( |x| \leq |y| \). Then

\[
s(y) \leq s(x) \leq s(y) + \|x - y\|_{\infty}.
\]

Lemma 10. For any \( x, y \in X \cap Q_e \cap H_i \), we have

\[
\|\Delta^X_e(x) - \Delta^X_e(y)\|_{\infty} \leq 2\|x - y\|_{\infty}.
\]

We leave the modifications of the proofs to the attentive reader. Now we obtain the following generalization of Theorem 6.

Theorem 11. Let \( X \) and \( Y \) be \( \mathbb{R}^d \) with quasi-norms as above and assume that \( X \) is monotone. Let \( k \geq d - \lceil \log d \rceil \). Then

\[
e_k(S_X, Y) \leq 2 \|Id : Y \to \ell^d_{\infty}\| \max_{1 \leq i \leq d} e_{k-d-\lceil \log d \rceil}(B_{X_i}, \ell^d_{\infty}).
\]

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Proof. At first, we treat the case \( Y = \ell^d_{\infty} \). Let \( \mathcal{N}_{E,i}(\varepsilon) \) be a minimal \( \varepsilon \)-covering of \( B_X \cap Q_e \cap H_i \) in \( \ell^d_{\infty} \). Since \( B_X \cap Q_e \cap H_i \) is a subset of \( B_X \), we have

\[
|\mathcal{N}_{E,i}(\varepsilon)| \leq N_\varepsilon(B_X, \ell^d_{\infty}).
\]

Lemma \[10\] implies that the set \( \Psi^X_e(\mathcal{N}_{E,i}(\varepsilon)) \) is a \( 2\varepsilon \)-covering of \( S_X \cap Q_e \cap C_i \). Consequently, \( \tilde{\mathcal{N}}(2\varepsilon) := \bigcup_{e \in \{-1, 1\}} \bigcup_{1 \leq i \leq d} \Delta^X_e(\mathcal{N}_{E,i}(\varepsilon)) \) is a \( 2\varepsilon \)-covering of \( S_X \). Moreover

\[
|\tilde{\mathcal{N}}(2\varepsilon)| \leq \sum_{e \in \{-1, 1\}} \sum_{1 \leq i \leq d} |\mathcal{N}_{E,i}(\varepsilon)| \leq 2^d \sum_{i=1}^d N_\varepsilon(B_X, \ell^d_{\infty}) \leq 2^d d \max_{1 \leq i \leq d} N_\varepsilon(B_X, \ell^d_{\infty}).
\]

By definition of entropy numbers,

\[
e_k(S_X, \ell^d_{\infty}) \leq 2 \max_{1 \leq i \leq d} e_{k-d-[\log d]}(B_X, \ell^d_{\infty}).
\]

The case of general \( Y \) is a consequence of the factorization property of entropy numbers:

\[
e_k(S_X, Y) \leq \|\text{id} : \ell^d_{\infty} \to Y\| e_k(S_X, \ell^d_{\infty}).
\]

To conclude this note, let us discuss the situation that both \( X \) and \( Y \) are Banach spaces with norms being symmetric with respect to the canonical basis. Then it is possible to make the bounds in the Theorems \[8\] and \[11\] concrete by applying the results of Schütz \[7\]. We call a Banach space \( X \) with norm \( \| \cdot \| \) symmetric if the canonical basis \( \{e_1, \ldots, e_d\} \) has the following property. For all permutations \( \pi \), all sings \( \varepsilon_i \), and all \( x_i \in \mathbb{R} \), we have

\[
\left\| \sum_{i=1}^d \varepsilon_i x_{\pi(i)} e_i \right\| = \left\| \sum_{i=1}^d x_i e_i \right\|.
\]

Below we use the notation

\[
\lambda_E(k) = \left\| \sum_{i=1}^k e_i \right\|
\]

where \( E \) denotes a \( d \)-dimensional, symmetric Banach space and \( \{e_1, \ldots, e_d\} \) its canonical basis.

**Corollary 12.** Let \( X \) and \( Y \) be symmetric Banach spaces and assume the canonical basis in both spaces to be normalized. Then, we have

\[
e_k(S_X^{d-1}, Y) \geq \frac{1}{2e} \max_{\ell = k, \ldots, d - 1} \frac{\lambda_{X}(\ell)}{\lambda_{Y}(\ell)}
\]

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for \( k \leq d - 1 \) and
\[
\frac{1}{e} 2^{-k/(d-1)} \frac{\lambda_Y(d-1)}{\lambda_X(d-1)} \leq e_k(S_X^{d-1}, Y) \leq 32e 2^{-k/(d-1)} \frac{\lambda_Y(d-1)}{\lambda_X(d-1)}
\]
for \( k \geq 2d + \lceil \log d \rceil - 1 \).

**Proof.** Since \( X \) is symmetric, so is \( X_i \). Moreover, \( X_i \) and \( X_j \) are isometrically isomorphic for \( i \neq j \) by the symmetry of \( X \). The same holds true for \( Y \). Hence, \( e_k(B_{X_i}, Y_i) = e_k(B_{X_j}, Y_j) \) for \( i \neq j \) and all \( k \). By virtue of [7, Theorem 5 (1)] we know that
\[
eq \frac{1}{e} \max_{\ell = k, \ldots, d-1} \frac{\lambda_Y(\ell)}{\lambda_X(\ell)}.
\]
Combining with Theorem 8 and noting that \( \lambda_X(\ell) = \lambda_X(1) \) as well as \( \lambda_Y(\ell) = \lambda_Y(1) \) for \( \ell = 1, \ldots, d-1 \), we obtain the stated lower bound for \( k \leq d - 1 \).

Regarding the bounds for \( k \geq 2d - \lceil \log d \rceil - 1 \), we get from [7, Theorem 5 (2)] that
\[
\frac{1}{e} 2^{-k/(d-1)} \frac{\lambda_Y(d-1)}{\lambda_X(d-1)} \leq e_k(B_{X_1}, Y_1) \quad \text{for } k \geq d - 1
\]
as well as
\[
eq c 2^{-m/(d-1)} \frac{\lambda_{\ell_m^{-1}}(d-1)}{\lambda_{X_1}(d-1)} \quad \text{for } m \geq d - 1.
\]
Equation (2) and Theorem 8 immediately give the lower bound. For the upper bound, put \( m = k - d - \lfloor \log d \rfloor \). Noting that \( \| \id : Y \to \ell_{d-1}^\infty \| = \lambda_Y(d) \leq 2\lambda_Y(d-1) \) and \( \lambda_{\ell_{d-1}^{-1}}(d-1) = 1 \), Theorem 11 combined with (3) then gives
\[
\frac{1}{e} 2^d 2^{(d+\lfloor \log d \rfloor)/(d-1)} 2^{-k/(d-1)} \frac{\lambda_Y(d-1)}{\lambda_X(d-1)} \quad \text{for } k \geq 2d - \lfloor \log d \rfloor - 1.
\]
Since we have assumed that \( d \geq 2 \), we can estimate \( 2^d 2^{(d+\lfloor \log d \rfloor)/(d-1)} \leq 8 \).

**Example 13** (Lorentz norms). For \( x \in \mathbb{R}^d \), let \( x^* = (x_1^*, \ldots, x_n^*) \) denote the non-increasing rearrangement of \( x \). We consider generalized Lorentz norms
\[
\| x \|_{w,q} := \left( \sum_{i=1}^{d} (w(i)|x_i^*|^q)^{1/q} \right)^{1/q}
\]
for \( 0 < q \leq \infty \) and a non-increasing function \( w : [1, \infty) \to (0, \infty) \) with \( w(1) = 1 \), \( \lim_{t \to \infty} w(t) = 0 \) and \( \sum_{i=1}^{\infty} w(i) = \infty \). Then
\[
\lambda_{w,q}(k) := \left\| \sum_{i=1}^{k} e_i \right\|_{w,q} = \left( \sum_{i=1}^{k} w(i)^q \right)^{1/q}.
\]
Particularly, for the choice \( w(t) = t^{1/p-1/q} \) for \( 0 < q < p < \infty \) we obtain the standard Lorentz norm
\[
\|x\|_{w,q} = \|x\|_{p,q} = \left( \sum_{i=1}^{d} t^{q/p-1}|x_i^*|^q \right)^{1/q}. 
\]
A simple calculation, where one approximates the sum by an integral, shows that \( \lambda_{p,q}(k) \asymp k^{1/p} \). So, if we consider \( X = \ell_{p,q}^d \) and \( Y = \ell_r^d \), we obtain from Corollary 12 that
\[
e_k(S_{p,q}^{d-1}, \ell_{r}^d) \asymp 2^{-k/(d-1)} \frac{d^{1/p-1/r}}{d^{1/p-1/r}} \asymp e_k(S_{p,q}^{d-1}, \ell_{r}^d)
\]
for \( k \geq 2d + \lceil \log d \rceil - 1 \).

**Example 14** (Orlicz norms). A convex function \( M : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( M(0) = 0 \) and \( M(t) > 0 \) for \( t \neq 0 \) is called an *Orlicz function*, which we associate with the norm \( \| \cdot \|_M \) given by
\[
\|x\|_M := \inf\{\rho > 0 : \sum_{i=1}^{d} M(|x_i|/\rho) \leq 1\}.
\]
It is easy to calculate that
\[
\lambda_M(k) := \lambda_{(\mathbb{R}^d,\|\cdot\|_M)}(k) = \frac{1}{M^{-1}(1/k)},
\]
where \( M^{-1} \) is the inverse function of \( M \). Let us consider some specific examples. In the following we always choose \( Y = \ell_q^d \) for some \( 0 < q < \infty \).

(i) Let \( M(t) = \exp(-1/t^2) \) for \( 0 \leq t < 1/2 \). See [3] for an application where the associated Orlicz norm appears naturally. We have \( M^{-1}(t) = \sqrt{-\ln t} \) and consequently
\[
e_k(S_{M}^{d-1}, \ell_q^d) \asymp \sqrt{\ln d} d^{1/q} 2^{-k/(d-1)}.
\]

(ii) The following Orlicz function is taken from [4]. For \( p > 1 \) and \( \alpha > 0 \), let \( M \) on \([0, t_0)\) be given by \( M(t) = t^p \ln(1/t)^\alpha \). Here, \( t_0 = 1/\exp(\beta + \sqrt{\beta^2 - \gamma}) \) with \( \beta = \alpha(2p - 1)/(p^2 - p) \) and \( \gamma = (\alpha^2 - \alpha)/(p^2 - p) \). There is a \( t_0' \leq t_0 \) such that
\[
M^{-1}(y) \asymp_{p,\alpha} \left( \frac{y}{\log(1/y)^\alpha} \right)^{1/p} \quad \text{for} \quad y < M(t_0').
\]
Hence, for \( k \) sufficiently large, we find
\[
e_k(S_X^{d-1}, \ell_q^d) \asymp \log(d)^{\alpha/p} d^{1/q-1/p} 2^{-k/(d-1)}.
\]

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