Examples of centralizers in the Artin braid groups

Nikolai V. Ivanov

1 Introduction

In the first version of the preprint [F-GM], N. Franco and J. González-Meneses made the following conjecture (discussed also in the second version [F-GM]):

The normalizer of an element of the Artin braid group \( B_n \) on \( n \) strings is generated by no more than \( n - 1 \) elements.

The normalizer of an element of a group is defined as the subgroup of all elements commuting with it; this subgroup is often also called the centralizer of the given element. In the second version [F-GM] the authors switched to the latter terminology, and we will also use the term “centralizer”.

The conjecture was supported by a new algorithm for finding generators of a centralizer, suggested in [F-GM], and by extensive computations based on this algorithm.

The goal of this paper is to present simple examples of elements of \( B_n \) for which the centralizer cannot be generated by a less than quadratic in \( n \) number of generators. In particular, the above conjecture is disproved. The key insight is that the centralizer may be closer to a pure braid group than to a braid group (as one may think initially), and that a pure braid group requires the number of generators quadratic in the number of strings.

Our methods are based on Thurston’s theory of the surface diffeomorphisms and on the well known relation between the Artin braid groups and mapping class groups, in contrast with the purely algebraic methods of [F-GM]. In fact, the effectiveness of Thurston’s theory for studying the centralizers in the context of the mapping class groups was established long
ago by J. McCarthy [M] and the author [I1, I2]. Centralizers, considered from the point of view of Thurston’s theory, also play a key role in the recent work [IM] about braid groups of surfaces of higher genus of E. Irmak, J. McCarthy and the author.

One should also mention a nearly forgotten paper of G. Burde [B] in which he gave, in particular, a description of the centralizers of the elements of the pure braid group in the whole Artin braid group. The methods of Burde are fairly geometrical; Thurston’s theory clarifies the picture further. It would be interesting to compare in detail the methods of Burde and the methods based on Thurston’s theory.

The present work was reported in the author’s talk [I3]. After learning about these results, J. González-Meneses jointly with B. Wiest [GM-W] applied Thurston’s theory to give a rather complete description of the centralizers in the Artin braid group and found a conjecturally exact upper bound for the minimal number of generators of a centralizer. This bound is exact for the examples discussed in the present paper.

J. González-Meneses and B. Wiest [GM-W], while using Thurston’s theory, present many of their arguments in the framework of braids. In contrast, in the present note we work entirely in the framework of the mapping class groups. One may hope that this point of view may clarify some of the more obscure parts of [GM-W] (the authors themselves point out that some proofs seem to be more complicated than necessary).

2 Preliminaries

Let $B_n$ be the Artin braid group on $n$ strings. The group $B_n$ can be realized as the group of isotopy classes of diffeomorphisms of the disc $D^2$ pointwise fixed on the boundary $\partial D^2$ and preserving a set of $n$ distinguished points in $\text{int} D^2$, called punctures. This result is, essentially, due to Artin himself [Art]. It is usually assumed that $D^2$ is the standard unit disc in $\mathbb{R}^2$ and the punctures are lying on the real line; this allows a canonical identification of the group $B_n$ defined in terms of Artin generators and relations with the above group of isotopy classes.

If we remove the condition that the diffeomorphisms of $D^2$ are pointwise fixed on the boundary, the resulting group of isotopy classes is known as the mapping class group of the disc with $n$ punctures and will be denoted by
$M_n$. Obviously, there is a canonical forgetting homomorphism

$$B_n \to M_n,$$

where $B_n$ is realized, as above, as the group of isotopy classes. The following facts are well known. First, this homomorphism is surjective. Its kernel is equal to the center of $B_n$ and is an infinite cyclic group generated by the Dehn twist about the boundary $\partial D^2$.

This relation between $B_n$ and $M_n$ shows that in order to find elements with the centralizers generated by, say, no less than a quadratic in $n$ number of elements it is sufficient to find elements with the same property in $M_n$.

One may also consider the pure braid group $PB_n$ and the corresponding mapping class group $PM_n$; in order to define them in terms of the isotopy classes, one needs only to add the condition that the considered diffeomorphisms preserve each puncture individually. Similarly to the above, there is a surjective homomorphism $PB_n \to PM_n$ having central kernel isomorphic to $\mathbb{Z}$. A crucial fact about pure braid groups is the following theorem of V. I. Arnold [Arn].

*The first homology group $H_1(PB_n, \mathbb{R})$ is isomorphic to $\mathbb{R}^{n(n-1)/2}$.*

This theorem immediately implies that the group $PB_n$ cannot be generated by less than $n(n - 1)/2$ elements, and its quotient group $PM_n$ by less than $\frac{n(n-1)}{2} - 1$ elements.

### 3 Examples

Let us assume for simplicity that $n$ is divisible by 3, say $n = 3m$. Let $D_1, \ldots, D_m$ be $m$ disjoint discs in the interior of $D^2$ such that every disc $D_i$, $1 \leq i \leq m$, contains exactly 3 punctures in its interior. Pick up for each $i$, $1 \leq i \leq m$, a pseudo-Anosov isotopy class $f_i$ of diffeomorphisms $D_i \to D_i$. We can choose such $f_i$’s that the dilatation coefficients (assumed to be $> 1$ of all these pseudo-Anosov classes are different. Represent each of these pseudo-Anosov classes by a diffeomorphism $D_i \to D_i$, pointwise fixed on the boundary $\partial D^2$, and extend these diffeomorphisms by the identity to a diffeomorphism $F$ of $D^2$. Let $f$ be the isotopy class of $F$.

The Thurston’s normal form of $f$ is obvious: $f$ is reducible; boundaries $\partial D_i$ form the canonical reduction system of $f$; the isotopy classes $f_i$ and the
identity class on the complement of the interiors of discs $D_i$ are the canonical “pieces” of $f$.

If $g \in M_n$, then the Thurston’s normal form of $gfg^{-1}$ results from applying (a representative of) $g$ to the Thurston’s normal form of $f$. In particular, if $g$ belongs to the centralizer of $f$, i.e., if $gfg^{-1} = f$, then $g$ preserves the (isotopy class of) the union of the boundaries $\partial D_i$. Moreover, since the pseudo-Anosov classes $f_i$ have different dilatation coefficients, $g$ must preserve every boundary $\partial D_i$ and every disc $D_i$ individually. The structure of the centralizer is now clear: it consists of isotopy classes of diffeomorphisms $G$ preserving discs $D_i$ and subject only to the condition that the isotopy classes of the restrictions on discs $D_i$ commute with the classes $f_i$. This condition means that these restrictions belong to the centralizers of $f_i$’s, and one can use the results of J. McCarthy [M] to describe these centralizers.

More important for us is the fact that there is no condition at all on the action of $G$ on the complement of the interiors of $D_i$’s, except that all boundary components are preserved. By restricting $G$ to this complement and collapsing boundaries $\partial D_i$ to punctures, we get a surjective homomorphism of the centralizer of $f$ to the group $PM_m$ of the isotopy classes of diffeomorphisms of a disc fixing each of $m$ punctures in the interior.

Using the above mentioned corollary of the Arnold’s theorem, we see that $PM_m$ cannot be generated by less than $\frac{m(m-1)}{2} - 1$ elements. Since there is a surjective homomorphism from the centralizer of $f$ to $PM_m$, this centralizer also cannot be generated by less than $\frac{m(m-1)}{2} - 1$ elements. In fact, an easy argument shows that for any lift of $f$ to $PB_n$, its centralizer admits a surjective homomorphism to $PB_m$, and hence cannot be generated by less than $m(m - 1)/2$ elements.

In any case, since $n = 3m$, our elements $f$ require a quadratic in $n$ number of elements to generate its centralizer. This, clearly, disproves the conjecture stated in Introduction.

4 Remarks

1. Notice that the free abelian group generated by $m$ Dehn twists about boundaries $\partial D_i$ is obviously contained in the center of the centralizers of our examples $f$. This implies that the centralizer of a lift of $f$ to $PB_n$ cannot be generated by less than $\frac{m(m-1)}{2} + m = \frac{m(m+1)}{2}$ elements.
2. The above examples admit a lot of variations. H. Hamidi-Tehrani suggested (right after the talk [I3]) a modification of these examples: instead of pseudo-Anosov classes with different dilatations one can use different powers of Dehn twists about the boundaries \( \partial D_i \). After this, it is only natural to notice that there is no need to have 3 punctures inside the discs (3 punctures were needed to have pseudo-Anosov classes on \( D_i \)'s), and therefore one can have exactly 2 punctures inside every \( D_i \). Also, one may leave (exactly) one puncture outside all these discs, if the number \( n \) of punctures is odd. This leads to examples of elements requiring at least \( \frac{m(m-1)}{2} + m = \frac{m(m+1)}{2} \) elements to generate their centralizer for \( n = 2m \) and \( \frac{m(m+1)}{2} + m = \frac{m(m+3)}{2} \) elements for \( n = 2m + 1 \). According to [GM-W], the first specific examples of this type (with 2 punctures inside the discs) were suggested by S. J. Lee. It is proved in [GM-W] that such examples are the worst possible in terms of the required number of generators of the centralizer.

3. Another illustration of the power of the geometric approach to this circle of questions is provided by some specific elements of Artin braid groups for which a computation of the centralizers was done by methods of the combinatorial group theory by G. G. Gurzo [Gu1], [Gu2]. The proofs are based on long and difficult computations. All elements considered by G. G. Gurzo have simple images in \( M_n \), and Thurston’s theory very easily leads to a description of their centralizers. We leave this as an exercise for an interested reader. Many of the specific elements treated by Gurzo can be treated also by the methods of G. Burde [B]; strangely enough, Burde’s work is cited by Gurzo, but is not used.

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Michigan State University
Department of Mathematics
Wells Hall
East Lansing, MI 48824-1027

E-mail: ivanov@math.msu.edu