A New Set of Maxwell-Lorentz Equations and Rediscovery of Heaviside-Maxwellian (Vector) Gravity from Quantum Field Theory

Harihar Behera and N. Barik

1 Physics Department, BIET Higher Secondary School, Govindpur, Dhenkanal-759001, Odisha, India
2 Department of Physics, Utkal University, Vani Vihar, Bhubaneswar-751004, Odisha, India

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Abstract We show that if we start with the free Dirac Lagrangian, and demand local phase invariance, considering the total phase coming from two independent contributions associated with the charge and mass degrees of freedom of charged Dirac particles, then we are forced to introduce two massless independent vector fields for charged Dirac particles that generate all of electrodynamics and gravitodynamics of Heaviside’s Gravity of 1893 or Maxwellian Gravity and specify the charge and mass currents produced by charged Dirac particles. From this approach we found: (i) a new mathematical representation of Lorentz-Maxwell’s equations of electrodynamics physically equivalent to the standard equations and (ii) two equivalent sets of gravitolo-Lorentz-Maxwell’s equations of vector gravity by correcting Heaviside’s speculative gravito-Lorentz force. The gravitolo-Lorentz-Maxwell equations obtained here match with several classical (Galilean or Special or General Relativistic) versions of gravitoelectromagnetism at appropriate limits reported recently [Behera, H. Eur. Phys. J. C (2017) 77: 822]. Our approach naturally renders a gravitoelectromagnetic or gravitodynamics correction to the standard Lagrangian of quantum electrodynamics, which, for a neutral massive Dirac particle, reduces to the Lagrangian of quantum gravitodynamics. The resulting spin-1 vector gravity is shown to produce attractive interaction between two static like masses, contrary to the prevalent view.

1 Introduction

Many field theorists, like Gupta [1], Feynman [2], Zee [3] and Gasperini [4], to name a few, have rejected spin-1 vector theory of gravity on the ground that if gravitation is described by a vector field theory like Maxwell’s electromagnetic theory, then vector-like interactions will produce repulsive static interactions between sources of the same sign, while - according to Newton’s gravitational theory - the static gravitational interaction between masses of the same sign is attractive. However, here we show that this not true, if one considers appropriate field equations for vector gravity derived using the well established principle of local phase (or gauge) invariance of quantum field theory (QFT). Inspired by Feynman’s view [2] that “space-time curvature is not essential to physics”, and following the usual procedure of quantum electrodynamics (in flat space-time), here we show that if we start with the Dirac Lagrangian, and demand local phase invariance, considering the total phase coming from two independent contributions associated with the charge and mass degrees of freedom of charged Dirac particles, then we are forced to introduce two massless independent vector fields for charged Dirac particles that generate all of electromagnetism and gravitoelectromagnetism of Heaviside’s Gravity (HG)[5,6,7,8,9,10,11,12] of 1893 or Maxwellian Gravity (MG)[13] and specify the charge and mass currents produced by charged Dirac particles. Our approach naturally renders a gravitoelectromagnetic or gravitodynamics correction to the standard Lagrangian of quantum electrodynamics, which, for a neutral massive Dirac particle, reduces to the Lagrangian of quantum gravitodynamics. The resulting spin-1 vector gravity is shown to produce attractive interaction between two static like masses, contrary to the prevalent view.

1 Heaviside had speculated a gravitational analogue of Lorentz force law with a sign error that is corrected in this work.
2 Which looks mathematically different from Heaviside’s Gravity due to some differences in the sign of certain terms. But HG and MG are shown here to represent a single physical theory called Heaviside-Maxwellian Gravity (HMG) by correct representations of their respective field and force equations.
2 Consequences of Local Phase Invariance for Charge and Mass Degrees of Freedom

It is well known that the free Dirac Lagrangian density (in SI units)
\[
\mathcal{L} = i \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi - m_0 \bar{\psi} \psi
\]  
(1)
is invariant under the transformation\footnote{Here we adopt SI units for clarity to the general readers.}
\[
\psi \rightarrow e^{i \theta} \psi \quad \text{(global phase transformation)}
\]  
(2)
where \( \theta \) is any real number. This is because under global phase transformation \( \bar{\psi} \rightarrow e^{-i \theta} \bar{\psi} \) which leaves \( \mathcal{L} \) unchanged as the exponential factors cancel out. But the Lagrangian density \( \mathcal{L} \) is not invariant under the following transformation
\[
\psi \rightarrow e^{i \theta(x)} \psi \quad \text{(local phase transformation)}
\]  
(3)
where \( \theta \) is now a function of space-time \( x = x^\mu = (ct, \mathbf{x}) \), because the factor \( \partial_\mu \psi \) in \( \mathcal{L} \) now picks up an extra term from the derivative of \( \theta(x) \):
\[
\partial_\mu \psi \rightarrow \partial_\mu \left( e^{i \theta(x)} \psi \right) = i (\partial_\mu \theta) e^{i \theta} \psi + e^{i \theta} \partial_\mu \psi
\]  
(4)
so that under local phase transformation,
\[
\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} - \hbar c (\partial_\mu \theta) \bar{\psi} \gamma^\mu \psi.
\]  
(5)
Now suppose that the phase \( \theta(x) \) is made up of two parts:
\[
\theta(x) = \theta_1(x) + \theta_2(x),
\]  
(6)
which come from two independent contributions. Then \( \mathcal{L} \) becomes
\[
\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} - \hbar c (\partial_\mu \theta_1) \bar{\psi} \gamma^\mu \psi - \hbar c (\partial_\mu \theta_2) \bar{\psi} \gamma^\mu \psi.
\]  
(7)
For an electrically charged Dirac particle of charge \( q \) and mass \( m_0 \), we can re-write the transformed Lagrangian density \( \mathcal{L}' \) in equation (7) as
\[
\mathcal{L}' = \mathcal{L} - \frac{\hbar}{q} \partial_\mu \theta_1 q + \frac{\hbar}{m_0} \partial_\mu \theta_2 m_0 \bar{\psi} \gamma^\mu \psi
\]  
(8)
\[
= \mathcal{L} + j^\mu_1 \partial_\mu \lambda_1(x) + j^\mu_2 \partial_\mu \lambda_2(x),
\]  
\[
\text{where}
\]
\[
j^\mu_1 = \frac{qc}{\hbar} \bar{\psi} \gamma^\mu \psi = 4\text{-charge-current density},
\]  
(9)
\[
j^\mu_2 = \frac{mqc}{\hbar} \bar{\psi} \gamma^\mu \psi = 4\text{-mass-current density},
\]  
(10)
\[\text{and } \lambda_1(x) \text{ and } \lambda_2(x) \text{ stand for}
\]
\[
\lambda_1(x) = -\frac{\hbar}{q} \theta_1(x),
\]  
(11)
\[
\lambda_2(x) = -\frac{\hbar}{m_0} \theta_2(x).
\]  
(12)
In terms of \( \lambda_1 \) and \( \lambda_2 \) then,
\[
\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + j^\mu_1 \partial_\mu \lambda_1 + j^\mu_2 \partial_\mu \lambda_2,
\]  
(13)
under the local transformation
\[
\psi \rightarrow e^{i \theta(x) \lambda_1(x) + \theta_2(x)} \psi.
\]  
(14)
Now, we demand that the complete Lagrangian be invariant under local phase transformations. Since, the free Dirac Lagrangian density \( \mathcal{L} \) is not locally phase invariant, we are forced to add something to swallow up or nullify the extra term in eq. (13). Specifically, we suppose
\[
\mathcal{L} = [i \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi - mc \bar{\psi} \psi] - j^\mu_1 A_{\mu \nu} - j^\mu_2 A_{\mu \nu}
\]  
(15)
where \( A_{\mu \nu} \) and \( A_{2\mu} \) are some new fields, which change in coordination with the local phase transformation of \( \psi \) according to the rule
\[
A_{\mu \nu} \rightarrow A_{\mu \nu} + \partial_\mu \lambda_1,
\]  
(16)
\[
A_{2\mu} \rightarrow A_{2\mu} + \partial_\mu \lambda_2,
\]  
(17)
The ‘new, improved’ Lagrangian \( \mathcal{L} \) is now locally invariant. But this was ensured at the cost of introducing two new vector fields that couples to \( \psi \) through the last terms in eq. (15). But the eq. (15) is devoid of ‘free’ terms for the fields \( A_{\mu \nu} \) and \( A_{2\mu} \) (having the dimensions of velocity: \( [L][T]^{-1} \)). Since these are independent vectors, we look to the Proca-type Lagrangians for these fields \footnote{These are the dimensional constants.}:
\[
\mathcal{L}_{\text{free}}^\mu = \frac{k_1}{4} F^\mu \nu F_{\mu \nu} + \kappa_{01} \left( \frac{m_1}{c} \right)^2 A_\mu^\nu A_{\mu \nu}
\]  
(18)
\[
\mathcal{L}_{\text{free}}^2 = \frac{k_2}{4} F^\mu \nu F_{\mu \nu} + \kappa_{02} \left( \frac{m_2}{c} \right)^2 A_\mu^\nu A_{\mu \nu}
\]  
(19)
where \( k_1, k_2, \kappa_{01}, \text{and } \kappa_{02} \) are some dimensional constants and \( m_1 \) is the mass of the free field \( A_{\mu \nu} \) while \( m_2 \) is the mass of the free field \( A_{2\mu} \). But there is a problem here, for whereas
\[
F^\mu = \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) \text{ or } F_{\mu \nu} = \left( \partial_\mu A_{\nu \rho} - \partial_\nu A_{\mu \rho} \right)
\]  
(20)
is invariant under \(11\), \(A^\mu_0 A_{\mu 0}\) is not. Similarly

\[
f^{\mu \nu} = (\partial^\mu A^\nu_0 - \partial^\nu A^\mu_0) \quad \text{or} \quad f_{\mu \nu} = (\partial_\mu A^\nu_0 - \partial_\nu A^\mu_0)
\]

(21)
is invariant under \(17\), \(A^\mu_0 A_{\mu 0}\) is not; Evidently, the new fields \(A^\mu_0\) and \(A^\mu_0\) must be mass-less \((m_1 = 0 = m_2)\), otherwise the invariance will be lost for these two independent fields. The complete Lagrangian density then becomes

\[
\mathcal{L} = [i\hbar c \gamma^\mu \partial_\mu \psi - m_0 c^2 \overline{\psi} \psi] + \mathcal{L}_c + \mathcal{L}_g
\]

(22)
where

\[
\mathcal{L}_c = \frac{k_1}{4} F^{\mu \nu} F_{\mu \nu} - j^\mu A_{\mu 0}.
\]

(23)

\[
\mathcal{L}_g = \frac{k_2}{4} f^{\mu \nu} f_{\mu \nu} - j^\mu A_{\mu 0}.
\]

(24)
The equations of motion of these new fields can be obtained using the Euler-Lagrange equations of motion:

\[
\partial^\beta \frac{\partial \mathcal{L}_c}{\partial (\partial^3 A^\alpha)} = \frac{\partial \mathcal{L}_c}{\partial A^\alpha} \quad \text{&} \quad \partial^\beta \frac{\partial \mathcal{L}_g}{\partial (\partial^3 A^\alpha)} = \frac{\partial \mathcal{L}_g}{\partial A^\alpha}
\]

(25)

A bit calculation (see for example, Jackson \(14\)) yields

\[
\frac{\partial \mathcal{L}_c}{\partial A^\alpha} = - j^\alpha \quad \text{&} \quad \frac{\partial \mathcal{L}_g}{\partial A^\alpha} = - j^\alpha.
\]

(26)

and

\[
\frac{\partial \mathcal{L}_c}{\partial (\partial^3 A^\alpha)} = - k_1 F_{\alpha \beta} \quad \text{&} \quad \frac{\partial \mathcal{L}_g}{\partial (\partial^3 A^\alpha)} = - k_2 f_{\alpha \beta}.
\]

(27)

Using the calculated results \(23\) - \(27\) in Euler-Lagrange eqs. \(25\), we get the equations of motion of the new fields as

\[
\partial^\beta F_{\alpha \beta} = \frac{1}{k_1} j^\alpha.
\]

(28)

\[
\partial^\beta f_{\alpha \beta} = \frac{1}{k_2} j^\alpha.
\]

(29)
Eqs. \(28\) express the generation of \(F_{\alpha \beta}\) fields by the 4-charge-current density \(j^\alpha\), associated with the electric charge of the Dirac particles, while eqs. \(22\) express the generation of \(f_{\alpha \beta}\) fields by the 4-mass-current density associated with the proper (or rest) mass of massive (charged or neutral) Dirac particles as given in \(10\).

2.1 Maxwell’s Fields from Charge Degree of Freedom

For classical fields, the 4-current charge density is represented by

\[
j_e^\alpha = (c e, j_x), \quad j_e^\alpha = (c e, -j_x)
\]

(30)
where \(j_x = \rho_x v\), with \(\rho_x\) = electric charge density. For static charge distributions, the current density \(j_{\alpha 0} = j_{\alpha 0} = c e\). It produces a time-independent - static-field, given by \(28\):

\[
\frac{1}{\epsilon_0 c} \frac{\partial F_{00}}{\partial t} + \frac{\partial F_{01}}{\partial x} - \frac{\partial F_{02}}{\partial y} - \frac{\partial F_{03}}{\partial z} = \frac{\rho_e}{\kappa_1}
\]

(31)
where we use \([\partial_\alpha \equiv (\partial/c \partial t, \nabla) & \partial^\alpha \equiv (\partial/c \partial t, -\nabla)].\) Multiplying eq. \(31\) by \(c\) we get

\[
\frac{\partial (c F_{01})}{\partial x} + \frac{\partial (c F_{02})}{\partial y} + \frac{\partial (c F_{03})}{\partial z} = - \frac{\rho_e c^2}{\kappa_1}.
\]

(32)
Eq. \(32\) gives us Coulomb field \(\mathbf{E}\) as expressed in the Gauss’s law of electrostatics, viz.,

\[
\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho_e}{\epsilon_0}
\]

(33)
(\(\epsilon_0\) = permitivity of vacuum), if we make the following identifications:

\[
F_{01} = E_x/c, \quad F_{02} = E_y/c, \quad F_{03} = E_z/c \quad \text{&} \quad \kappa_1 = -\epsilon_0 c^2.
\]

(34)
With these findings, we write the Eqs. \(23\) and \(28\) as

\[
\mathcal{L}_c = - \frac{\epsilon_0 c^2}{4} F^{\mu \nu} F_{\mu \nu} - j^\alpha A_{\alpha 0},
\]

(35)
Eq. \(35\) is applicable for Dirac current density \(41\) as well as classical current density \(40\). From the anti-symmetry property of \(F_{\alpha \beta}\) \((F_{\alpha \beta} = - F_{\beta \alpha})\) it follows form the results \(34\) that

\[
F_{10} = -E_x/c, \quad F_{20} = -E_y/c, \quad F_{30} = -E_z/c \quad \text{&} \quad F_{\alpha \alpha} = 0.
\]

(37)
The other elements of \(F_{\alpha \beta}\) can be obtained as follows. For \(\alpha = 1\), i.e. \(j_{x 1} = -j_{x 0}\), Eq. \(36\) gives us

\[
-\mu_0 j_{e 1} = \mu_0 j_{e 0}.
\]

(38)

\[
\frac{\partial F_{10}}{\partial t} + \frac{\partial F_{11}}{\partial x} + \phi \partial F_{12} + \phi \partial F_{13} = 0
\]

(38)

For SME

\[
\frac{1}{c^2} \partial E_x / \partial t - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} \quad \text{(For SME)}
\]

For NME

\[
\frac{1}{c^2} \partial E_x / \partial t - \nabla \times \mathbf{B}_x \quad \text{(For NME)}
\]
where $F_{12} = -B_z$ and $F_{13} = B_y$ for the standard Maxwell’s Equations (SME); $F_{12} = B_z$ and $F_{13} = -B_y$ for a possible form of New Maxwell’s Equations (NME).

This way, we determined all the elements of the anti-symmetric ‘field strength tensor’ $F_{\alpha \beta}$:

$$F_{\alpha \beta} = \begin{cases} 
0 & E_x/c \ E_y/c \ E_z/c \\
-E_x/c & 0 & -B_z \\
-E_y/c & B_z & 0 \\
-E_z/c & -B_y & B_x \\
\end{cases} 
$$

For SME

$$F_{\alpha \beta} = \begin{cases} 
0 & E_x/c \ E_y/c \ E_z/c \\
-E_x/c & 0 & -B_y \\
-E_y/c & B_z & 0 \\
-E_z/c & -B_x & B_y \\
\end{cases} 
$$

For NME

and the Ampère-Maxwell law of SME and NME:

$$\nabla \times \mathbf{B} = \begin{cases} 
+\mu_0 \mathbf{j} & + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} & \text{ (For SME)} \\
-\mu_0 \mathbf{j} & - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} & \text{ (For NME)} 
\end{cases}$$

where $\mathbf{B}$ is magnetic field, which is generated by electric charge current and time-varying electric field $\mathbf{E}$.

For reference, we note the field strength tensor with two contravariant indices:

$$\mathcal{F}_{\alpha \beta} = \eta^{\alpha \gamma} F_{\gamma \delta} \eta^{\delta \beta} = \begin{cases} 
0 & -E_x/c \ -E_y/c \ -E_z/c \\
E_x/c & 0 & -B_y \\
E_y/c & B_z & 0 \\
E_z/c & -B_x & B_y \\
\end{cases} 
$$

For SME

$$\mathcal{F}_{\alpha \beta} = \begin{cases} 
0 & -E_x/c \ -E_y/c \ -E_z/c \\
E_x/c & 0 & -B_y \\
E_y/c & B_z & 0 \\
E_z/c & -B_x & B_y \\
\end{cases} 
$$

For NME

where, the usual symmetric metric tensor $\eta_{\alpha \beta} = \eta^{\alpha \beta}$ is a diagonal matrix with diagonal elements

$$\eta_{00} = 1, \quad \eta_{11} = \eta_{22} = \eta_{33} = -1.$$  

From Eq. (36) and the anti-symmetry property of $F^{\alpha \beta}$, it follows that $j^\alpha_c$ is divergence-less:

$$\partial_\alpha j^\alpha_c = 0 = \frac{1}{c} \frac{\partial (\rho_c c)}{\partial t} + \nabla \cdot j_c = \nabla \cdot j_c + \frac{\partial \rho_c}{\partial t}$$  

This is the \textit{continuity equation} expressing the local conservation electric charge.

Equation (36) gives us two in-homogeneous equations of SME and NME. The very definition of $F_{\alpha \beta}$ in eq. (20), automatically guarantees us the Bianchi identity:

$$\partial_\alpha F_{\beta \gamma} + \partial_\beta F_{\gamma \delta} + \partial_\gamma F_{\delta \alpha} = 0,$$

(44)

(where $\alpha, \beta, \gamma$ are any three of the integers 0, 1, 2, 3), from which two homogeneous equations emerge naturally:

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(For both SME and NME)}$$

(45)

$$\nabla \times \mathbf{E} = \begin{cases} 
-\frac{\partial B_z}{\partial t} & \text{ (For SME)} \\
+\frac{\partial B_y}{\partial t} & \text{ (For NME)} 
\end{cases}$$

(46)

The Bianchi identity (41) may concisely be expressed by the zero divergence of a dual field-strength tensor $\mathcal{F}^{\alpha \beta}$, viz.,

$$\nabla_\alpha \mathcal{F}^{\alpha \beta} = 0,$$

(47)

where $\mathcal{F}^{\alpha \beta}$ is defined by

$$\mathcal{F}^{\alpha \beta} = \frac{1}{2} \epsilon^{\alpha \beta \gamma \delta} F_{\gamma \delta} = \begin{cases} 
0 & -B_x & -B_y & -B_z \\
B_x & 0 & -E_z/c & E_y/c \\
B_y & E_z/c & 0 & -E_x/c \\
B_z & -E_y/c & E_x/c & 0 
\end{cases} 
$$

(48)

for SME

and the totally anti-symmetric fourth rank tensor $\epsilon^{\alpha \beta \gamma \delta}$ (known as Levi-Civita Tensor) is defined by

$$\epsilon^{\alpha \beta \gamma \delta} = \begin{cases} 
+1 & \text{ for } \alpha = 0, \beta = 1, \gamma = 2, \delta = 3, \text{and any even permutation} \\
-1 & \text{ for any odd permutation} \\
0 & \text{ if any two indices are equal.} 
\end{cases}$$

(49)

The dual field-strength tensor $\mathcal{F}^{\alpha \beta}$ for the NME can be obtained from eq. (15) by substitution $\mathbf{B} \rightarrow -\mathbf{B}$, with $\mathbf{E}$ remaining the same.

Eq. (15) suggests that $\mathbf{B}$ can be defined as the curl of a vector function $\mathbf{A}_e$ (say). If we define

$$\mathbf{B} = \begin{cases} 
+\nabla \times \mathbf{A}_e & \text{ (For SME)} \\
-\nabla \times \mathbf{A}_e & \text{ (For NME)} 
\end{cases}$$

(50)

then using these definitions in (49), we find

$$\nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}_e}{\partial t} \right) = 0 \quad \text{(For SME and NME)},$$

(51)
which is equivalent to say that the vector quantity inside the parentheses of eq. (51) can be written as the gradient of a scalar potential, $A_0$:

$$E = -\nabla A_0 - \frac{\partial A_\omega}{\partial t} \quad \text{ (For SME and NME).}$$

(52)

In relativistic notation, eqs. (51) and (52) become

$$F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha,$$

(53)

(as they must, because of their common origin) where

$$A_\alpha = (A_0/c, A_\omega) = (\phi/c, \mathbf{A}_\mathbf{e}).$$

(54)

In terms of this 4-potential, the in-homogeneous eqs. (36) of SME and NME read:

$$\partial_\beta \partial^\beta A_\alpha - \partial_\alpha (\partial_\beta A^\beta) = \mu_0 j^\alpha_c.$$  

(55)

Under the Lorenz condition,

$$\partial_\beta A^\beta = 0,$$  

(56)

the in-homogeneous equations (55) simplify to the following equations:

$$\partial_\beta \partial^\beta A_\alpha = \Box A_\alpha = \mu_0 j^\alpha_c \quad \text{ (For SME & NME),}$$  

(57)

where

$$\Box = \partial_\alpha \partial^\alpha = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

(58)

is the D’Alembertian.

The relativistic Lagrangian (not Lagrangian density) for a single particle of proper mass $m_0$ and electric charge $q$ moving in the external field of SME and NME, is written as

$$L_c = -\left[ m_0 c \sqrt{\eta_{\alpha \beta} \frac{dx_\alpha}{dx} \frac{dx_\beta}{dx}} + q c \frac{dx_\alpha}{dx} A^\alpha_c (x) \right]$$

(59)

where $ds = c d\tau$. $\eta_{\alpha \beta} = \sqrt{1 - v^2}$ is the proper time along the particle’s world-line. From the Lagrangian (59), one obtains the co-variant equation of motion of classical electrodynamics:

$$\frac{d^2 x^\alpha}{d\tau^2} = \frac{q}{m_0} F^{\alpha \beta} \frac{dx_\beta}{d\tau}. \quad \text{ (60)}$$

Now, if we introduce the energy momentum four vector:

$$p^\alpha = (p_0, \mathbf{p}) = m_0 (U_0, \mathbf{U}) \quad \text{ (61)}$$

where $p_0 = E/c$ and $U^\alpha = (U_0, \mathbf{U}) = (c \gamma u, \mathbf{u} \gamma u)$ is the 4-velocity with $\gamma = (1 - u^2/c^2)^{-1/2}$, then we can re-write eq. (60) in terms of $p^\alpha$ as

$$\frac{dp^\alpha}{d\tau} = \frac{q}{m_0} F^{\alpha \beta} p_\beta.$$  

(62)

In three dimensional form the equations of motion (62), take the following forms:

$$\frac{dp_0}{dt} = \begin{cases} \frac{q}{m_0} [E + \mathbf{u} \times \mathbf{B}] & \text{ (For SME)} \\ \frac{q}{m_0} [E - \mathbf{u} \times \mathbf{B}] & \text{ (For NME)} \end{cases} \quad \text{ (63)}$$

$$\frac{dE}{dt} = q \mathbf{u} \cdot \mathbf{E} \quad \text{ (For both SME and NME)} \quad \text{ (64)}$$

Two basic sets of Maxwell’s Equations (ME) producing the same physical effects are tabulated in Table 1.

| Standard ME (SME) | New ME (NME) |
|-------------------|-------------|
| $\nabla \cdot \mathbf{E} = \frac{\partial \phi}{\partial t}$ | $\nabla \cdot \mathbf{E} = \frac{\partial \phi}{\partial t}$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \mathbf{B} = 0$ |
| $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ | $\nabla \times \mathbf{B} = -\frac{\partial \mathbf{E}}{\partial t}$ |
| $\nabla \times \mathbf{E} = \mu_0 \mathbf{J}$ | $\nabla \times \mathbf{E} = \mu_0 \mathbf{J}$ |
| $\frac{\partial \mathbf{E}}{\partial t} = q \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}$ | $\frac{\partial \mathbf{E}}{\partial t} = q \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}$ |
| $\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$ | $\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$ |

**Table 1** Equivalent Sets of Maxwell’s Equations (ME).

2.2 Maxwell-like Fields from Mass Degree of Freedom

For classical fields, the 4-current mass density or 4-momentum density is represented by

$$j^\alpha_g = (c p_0, \mathbf{j}_g), \quad j_g = (c p_0, -\mathbf{j}_g) \quad \text{ (65)}$$

where $\mathbf{j}_g = \rho_0 \mathbf{v}$, with $\rho_0$ proper mass density. For static mass distributions, the current density $j_g = j_0 = c \rho_0$. It produces a time-independent - static - field, given by Eq. (66):

$$\frac{1}{c^2} \frac{\partial f_{00}}{\partial t} + \frac{\partial f_{01}}{\partial x} + \frac{\partial f_{02}}{\partial y} + \frac{\partial f_{03}}{\partial z} = - \frac{\rho_0 c^2}{\kappa_2}$$

(66)

Multiplying eq. (66) by c we get

$$\frac{\partial (c f_{00})}{\partial x} + \frac{\partial (c f_{02})}{\partial y} + \frac{\partial (c f_{03})}{\partial z} = - \frac{\rho_0 c^2}{\kappa_2} \quad \text{ (67)}$$

Equation (67) gives us Newton’s gravitational field ($\mathbf{g}$) as expressed in the Gauss’s law of gravitostatics, viz.,

$$\nabla \cdot \mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} = -\frac{4 \pi G \rho_0}{\kappa_2} \quad \text{ (68)}$$

$(G = Newton’s gravitational constant)$, if we make the following identifications:

$$f_{01} = \frac{g_x}{c}, \quad f_{02} = \frac{g_y}{c}, \quad f_{03} = \frac{g_z}{c} \quad \text{and} \quad \kappa_2 = \frac{c^2}{4 \pi G} \quad \text{ (69)}$$
With these findings, we write the Eqs. (24) and (28) as
\[ L_g = \frac{e^2}{16\pi G} f^{\mu\nu} f_{\mu\nu} - j^\mu_y A_{g\mu} \\ = \frac{\epsilon_0 e^2}{4} f^{\mu\nu} f_{\mu\nu} - j^\mu_y A_{g\mu}, \] (70)
\[ \partial^\beta f_{\alpha\beta} = \frac{4\pi G}{c^2} j_{g\alpha} = \mu_0 j_{g\alpha}. \] (71)
where we have introduced two new constants \( \epsilon_0 \) and \( \mu_0 \) by defining
\[ \epsilon_0 = \frac{1}{4\pi G} \quad \text{and} \quad \mu_0 = \frac{4\pi G}{c^2}, \] (72)
so that they are related by the following equation
\[ c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \] (73)
in complete analogy with the electromagnetic case where \( c = (\epsilon_0 \mu_0)^{-1/2} \).

Now following the methods adopted in the previous section we get the following results for gravitoelectromagnetic (GEM) theory. The anti-symmetric ‘field strength tensor’ \( f_{\alpha\beta} \) of what we call Maxwellian Gravity (MG) and Heaviside Gravity (HG):
\[ f_{\alpha\beta} = \begin{cases} 0 & \frac{g_x}{c} \frac{g_y}{c} \frac{g_z}{c} \\ -\frac{g_x}{c} & 0 & -b_x \\ -\frac{g_y}{c} & b_x & 0 \\ -\frac{g_z}{c} & -b_y & b_x \end{cases} \quad \text{(For MG)} \]
\[ = \begin{cases} 0 & \frac{g_x}{c} \frac{g_y}{c} \frac{g_z}{c} \\ -\frac{g_x}{c} & 0 & -b_x \\ -\frac{g_y}{c} & b_x & 0 \\ -\frac{g_z}{c} & -b_y & b_x \end{cases} \quad \text{(For HG)} \] (74)
and the Gravito-Ampère-Maxwell law of MG and HG:
\[ \nabla \times \mathbf{b} = \begin{cases} -\frac{4\pi G}{c^2} j_g + \frac{1}{c} \frac{\partial \mathbf{b}}{\partial t} & \text{(For MG)} \\ +\frac{4\pi G}{c^2} j_g - \frac{1}{c} \frac{\partial \mathbf{b}}{\partial t} & \text{(For HG)} \end{cases} \] (75)
where \( \mathbf{b} \) is named as gravitomagnetic field, which is generated by gravitational charge (or mass) current and time-varying gravitational or gravitoelectric field \( \mathbf{g} \). The field strength tensor \( f^{\alpha\beta} \) is obtained as:
\[ f^{\alpha\beta} = \eta^{\alpha\gamma} f_{\gamma\delta} \eta^{\delta\beta} = \begin{cases} \begin{pmatrix} 0 & \frac{g_x}{c} & -\frac{g_y}{c} & -\frac{g_z}{c} \\ -\frac{g_x}{c} & 0 & -b_x & b_y \\ -\frac{g_y}{c} & b_z & 0 & -b_x \\ -\frac{g_z}{c} & -b_y & b_x & 0 \end{pmatrix} & \text{For MG} \\ \begin{pmatrix} 0 & \frac{g_x}{c} & -\frac{g_y}{c} & -\frac{g_z}{c} \\ -\frac{g_x}{c} & 0 & -b_x & b_y \\ -\frac{g_y}{c} & b_z & 0 & -b_x \\ -\frac{g_z}{c} & -b_y & b_x & 0 \end{pmatrix} & \text{For HG} \end{cases} \] (76)

The Bianchi identity for GEM:
\[ \partial_{\alpha} f_{\beta\gamma} + \partial_{\beta} f_{\gamma\delta} + \partial_{\gamma} f_{\delta\alpha} = 0, \] (77)
(where \( \alpha, \beta, \gamma \) are any three of the integers 0, 1, 2, 3), yields the two homogeneous equations:
\[ \nabla \cdot \mathbf{b} = 0 \quad \text{(For both MG and HG)} \] (78)
\[ \nabla \times \mathbf{g} = \begin{cases} \frac{\partial \mathbf{b}}{\partial t} & \text{(For MG)} \\ +\frac{\partial \mathbf{b}}{\partial t} & \text{(For HG)} \end{cases} \] (79)
The Eq. (79) represents the gravito-Faraday’s law for MG and HG. The Bianchi identity (77) may concisely be expressed by the zero divergence of a gravitational dual field-strength tensor \( \mathcal{F}^{\alpha\beta} \), viz.,
\[ \partial_{\alpha} \mathcal{F}^{\alpha\beta} = 0, \] (80)
where \( \mathcal{F}^{\alpha\beta} \) is defined by
\[ \mathcal{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma} f_{\gamma\delta} = \begin{cases} \begin{pmatrix} b_y & -\frac{g_y}{c} & -\frac{g_z}{c} \\ b_z & 0 & \frac{g_z}{c} \\ b_x & -\frac{g_z}{c} & 0 \end{pmatrix} & \text{For MG} \end{cases} \] (81)
The gravitational dual field-strength tensor \( \mathcal{F}^{\alpha\beta} \) for HG can be obtained from eq. (81) by substitution \( \mathbf{b} \rightarrow -\mathbf{b} \), with \( \mathbf{g} \) remaining the same.

Equation (78) suggests that \( \mathbf{b} \) can be defined as the curl of a vector function \( \mathbf{A}_g \) (say). If we define
\[ \mathbf{b} = \begin{cases} +\nabla \times \mathbf{A}_g & \text{(For MG)} \\ -\nabla \times \mathbf{A}_g & \text{(For HG)} \end{cases} \] (82)
then using these definitions in (79), one gets
\[ \nabla \times \left( \mathbf{g} + \frac{\partial \mathbf{A}_g}{\partial t} \right) = \mathbf{0} \quad \text{(For both MG and HG)} \] (83)
which is equivalent to say that the vector quantity inside the parentheses of eq. (83) can be written as the gradient of a scalar potential, $A_{\gamma 0}$:

$$g = - \nabla A_{\gamma 0} - \frac{\partial A_{\alpha 0}}{\partial t} \quad \text{(For both MG and HG)}. \quad (84)$$

In relativistic notation, eqs. (82) and (83) become

$$f^{\alpha \beta} = \partial^\alpha A_\beta^2 - \partial^\beta A_\alpha^2,$$ \hspace{1cm} (85)

where

$$A_\alpha = (A_{\gamma 0}/c, A_\gamma) = (\phi_0/c, A_\gamma). \quad (86)$$

In terms of this 4-potential, the in-homogeneous eqs. (71) of MG and HG read:

$$\partial_\beta \partial^\beta A_\gamma + 4\pi G \rho_\gamma = - \mu_0 j_\gamma^\alpha. \quad (87)$$

Under Gravito-Lorentz condition,

$$\partial_\beta A_\gamma^\beta = 0,$$ \hspace{1cm} (88)

the in-homogeneous eqs. (87) simplify to the following equations:

$$\partial_\beta \partial^\beta A_\gamma^\beta = \Box A_\gamma^\alpha = - \mu_0 j_\gamma^\alpha \quad \text{(For MG & HG)}. \quad (89)$$

This eq. (89) when compared with the corresponding to eq. (57) of Maxwell’s theory of electromagnetism, one finds a crucial sign difference before their source terms, which will explain why two like masses attract each other under static conditions, while two like electric charges repel each other under static conditions as we shall see. Since the fundamental field equations are the same for MG and HG, they represent the same physical thing and any sign difference in some particular terms arise due to particular definitions which will not change the nature of physical interactions. In other words MG and HG are two equivalent representations of a single physical theory, that we call Heaviside Maxwellian Gravity (HMG). The same remark applies to SME and NME, which are mere two different representations of one physical theory: Maxwell’s Electromagnetic Theory. In what follows, what we call MG is to be understood as HMG.

The relativistic Lagrangian (not Lagrangian density) for a single particle of proper mass $m_0$ moving in the external fields of HMG, is written as

$$L_g = - \left[ m_0 c^2 \sqrt{\eta^{\alpha \beta} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} + m_0 c \frac{dx_\alpha}{ds} A_\gamma^\beta (x) } \right] \quad (90)$$

where $ds = c dt$; $\tau$ is the proper time along the particle’s world-line. From the Lagrangian (90), one obtains the co-variant equation of motion of classical gravitodynamics:

$$\frac{d^2 x^\alpha}{d \tau^2} = f^{\alpha \beta} \frac{dx_\beta}{d \tau}. \quad (91)$$

Equation (91) shows us that the proper acceleration of a particle in the fields of HMG is independent of its rest mass, $m_0$. This is the relativistic generalization of Galileo’s law of Universalis of Free Fall (UFF) - known to be true both theoretically and experimentally since Galileo’s time. It states that all (non-spinning) particles of whatever rest mass, moving with same proper velocity $dx/\tau$ in a given gravitational field $f^{\alpha \beta}$, experience the same proper acceleration. It is to be noted that the equation of motion (91) holds only in an inertial frame. Appropriate modifications are necessary for its application in non-inertial frames, as is done in non-relativistic physics by introducing pseudo-forces.

In terms of the the energy momentum four vector $p^\alpha$ as defined in eq. (92), we can re-write the eq. (91) as

$$\frac{dp^\alpha}{d \tau} = f^{\alpha \beta} p_\beta. \quad (92)$$

Thus, the fields $f^{\alpha \beta}$ couples to energy-momentum 4-vector of all particles. In three dimensional form the equations of motion (72), take the following forms:

$$\frac{dp}{dt} = \begin{cases} m_0 [\mathbf{g} + \mathbf{u} \times \mathbf{b}] & \text{(For MG)} \\ m_0 [\mathbf{g} - \mathbf{u} \times \mathbf{b}] & \text{(For HG)} \end{cases} \quad (93)$$

$$\frac{dE}{dt} = m_0 \mathbf{u} \cdot \mathbf{g} \quad \text{(For both MG and HG)}. \quad (94)$$

It is to be noted that the gravito-Lorentz force law speculated by Heaviside was of MG-type in (93). The two basic sets of Lorentz-Maxwell-like Equations (ME) of gravity producing the same physical effects are given in Table 2. They represent a single vector gravitational theory, which we name as Heaviside-Maxwellian Gravity (HMG).

| Heaviside Gravity (HG) | Maxwellian Gravity (MG) |
|------------------------|-------------------------|
| $\nabla \cdot \mathbf{g} = - 4\pi G \rho_0 = - \rho_0/\epsilon_0 g$ | $\nabla \cdot \mathbf{g} = - 4\pi G \rho_0 = - \rho_0/\epsilon_0 g$ |
| $\nabla \cdot \mathbf{b} = 0$ | $\nabla \cdot \mathbf{b} = 0$ |
| $\nabla \times \mathbf{b} = + \mu_0 j_\gamma (x)$ | $\nabla \times \mathbf{b} = - \mu_0 j_\gamma (x)$ |
| $\nabla \times \mathbf{g} = + \frac{\partial \mathbf{b}}{\partial t}$ | $\nabla \times \mathbf{g} = - \frac{\partial \mathbf{b}}{\partial t}$ |
| $\frac{\partial \mathbf{b}}{\partial t} = m_0 [\mathbf{g} - \mathbf{u} \times \mathbf{b}]$ | $\frac{\partial \mathbf{b}}{\partial t} = m_0 [\mathbf{g} + \mathbf{u} \times \mathbf{b}]$ |
| $\mathbf{b} = - \nabla \times \mathbf{A}_\gamma$ | $\mathbf{b} = + \nabla \times \mathbf{A}_\gamma$ |
| $\mathbf{g} = - \nabla \phi_\gamma - \frac{\partial \mathbf{A}_\gamma}{\partial t}$ | $\mathbf{g} = - \nabla \phi_\gamma + \frac{\partial \mathbf{A}_\gamma}{\partial t}$ |

Table 2 Two equivalent representations of Heaviside-Maxwellian Gravity (HMG) with $\epsilon_0 g = \frac{1}{4\pi c^2}$ and $\mu_0 g = \frac{4\pi c^2}{\varepsilon_0}$. 
3 Discussions

The analogy between Newton’s law of gravitostatics and Coulomb’s law of electrostatics has been largely investigated since the nineteenth century, focusing on the possibility that the motion of masses could produce a magnetic-like field of gravitational origin. In the following we make a brief note of these studies before we discuss our results.

3.1 A Brief History of Heaviside-Maxwellian Gravity

By recognizing the striking structural similarity of Newton’s law of gravitational interaction between two masses and Coulomb’s law of electrostatic (or magnetostatic) interaction between two charges (or magnetic poles) and also their fundamental differences, J. C. Maxwell [15] in sect. 82 of his great 1865 paper, *A Dynamical Theory of the Electromagnetic Field*, made a note on the attraction of gravitation, where he considered whether Newtonian gravity could be extended to a form similar to the form of electromagnetic theory - a vector field theory - where the fields in a medium possess intrinsic energy. As a first step in this line of thought, Maxwell calculated the intrinsic energy $U_g$ of the static gravitational field at any place around gravitating bodies:

$$U_g = C - C' \int_{\text{All space}} g^2 d^3x$$  \hspace{1cm} (95)

where $C$ and $C'$ are two positive constants and $g$ is the gravitational field intensity at the place. If we assume that energy is essentially positive as Maxwell did, then the constant $C$ must have a value greater than $C'g^2$, where $g$ is the greatest value of the gravitational field at any place of the universe; and hence at any place where $|g| = 0$, the intrinsic energy must have an enormously great value. Being dissatisfied with this result, Maxwell, concluded his note on gravitation by stating, “As I am unable to understand in what way a medium can possess such properties, I can not go any further in this direction in searching for the cause of gravitation”.

4Which is not true if one considers gravitostatic field energy only. In fact following the electrostatic field energy calculation (see for example, Griffiths [16]) one obtains $U_g = -\frac{1}{\pi \mu_0} \int_{\text{All space}} g^2 d^3x$. Thus one can set $C = 0$ and $C' = \frac{1}{\pi \mu_0}$ in Eq. 95. The value of $U_g$ calculated by this field theoretical method by using 95 with $C = 0$ and $C' = \frac{1}{\pi \mu_0}$, for a spherical body of mass $M$, radius $R$ with uniform mass density within the body’s volume, turns out as $U_g = -\frac{G M^2}{3 R}$, which is the correct Newtonian (non-field-theoretical) result.

5By stating, “As energy is essentially positive it is impossible for any part of space to have negative intrinsic energy.”

The first written record of a vector gravitational theory was made by Oliver Heaviside [5,6,7,8,9,10,11,12] in 1893. Following electromagnetic analogy, Heaviside had obtained and written down a set of four field equations for gravity that had exactly same mathematical form as the field equations derived here from quantum field theory and listed under the head Heaviside Gravity (HG) in Table 2. If we replace $c$ by $c_g$ in the field equations under the head HG in Table 2, we get the original field equations of Heaviside with $c_g$ representing the speed of gravitational waves in vacuum, which might well be the speed of light $c$ in vacuum as Heaviside thought it. Here, our findings on HG testify the correctness of Heaviside’s gravitoelectromagnetic equations and his conjecture on the value of $c_g$. In Table 2, $\rho_0$ is the ordinary (rest) mass density, $j_g = \rho_0 v$ is the mass current density ($v$ is velocity) and by electro-magnetic analogy, $b$ is called the gravitational field and the Newtonian gravitational field $g$ is called the gravitoelectric field, $e_{g0}$ is called the gravitoelectric field (or gravitational) permittivity of vaccum and $\mu_{0g}$ is called the gravitomagnetic permeability of vacuum. To complete the dynamic picture, in a subsequent paper (Part II) [6,7,8,9,10,11] Heaviside speculated a gravitational analogue of Lorentz force law in the following form

$$F_{gL}^{HG} = m_0 \frac{d \mathbf{v}}{dt} = m_0 \mathbf{g} + m_0 \mathbf{v} \times \mathbf{b} \quad (\text{speculated}), \hspace{1cm} (96)$$

to calculate the effect of the $\mathbf{b}$ field (particularly due to the motion of the Sun through the cosmic aether) on Earth’s orbit around the Sun. But as shown in this paper, the correct gravito-Lorentz force law for HG has to take the following form:

$$F_{gL}^{HG} = m_0 \frac{d \mathbf{v}}{dt} = m_0 \mathbf{g} - m_0 \mathbf{v} \times \mathbf{b} \quad (\text{corrected}), \hspace{1cm} (97)$$

if the field equations are of HG-type in Table 2. This correction ensures that in both HG and MG like mass currents (parallel currents) should repel each other and unlike mass currents (anti-parallel currents) should attract each other in their gravitomagnetic interaction - opposite to the case of electromagnetism where like electric currents attract each other and unlike electric currents repel each other in their magnetic interaction. However, Heaviside calculated the precession of Earth’s orbit around the Sun by considering Eq. 96 and concluded that this effect was small enough to have gone unnoticed thus far, and therefore offered no contradiction to his hypothesis that gravitational effects propagate at the speed of light. Surprisingly, Heaviside seemed to be unaware of the long history of measurements of the precession of Mercury’s orbit as noted by McDonald [12], who reported Heaviside’s gravitational equations (in our present notation) as given in Table 2 under
the head Maxwellian Gravity (MG) - a name coined by Behera and Naik [13] in honor of J. C. Maxwell for his first attempt in this direction. Behera and Naik [13] obtained these equations demanding the Lorentz invariance of physical laws. It is to be noted that without the correction of Heaviside’s speculative gravito-Lorentz force law the effect the gravitomagnetic field of the spinning Sun on the precession of a planet’s orbit has the opposite sign to the observed effect as rightly noted by McDonald [12] and Iorio and Corda [17]. Heaviside also considered, the propagation of gravitational waves carrying energy momentum in terms of gravitational analogue of electromagnetic Heaviside-Poynting’s theorem.

Apart from Maxwell and Heaviside, prior attempts to modify Newton’s theory of gravitation were made by Lorentz in 1900 [18] and Poincaré [19] in 1905. There was a good deal of debate concerning Lorentz-covariant theory of gravitation in the years leading up to Einstein’s publication of his work in 1915 [20]. For an overview of research on gravitation from 1850 to 1915, the reader may see Roseveare [21], Renn et al. [22]. Walter [23] in ref. [22] discussed the Lorentz-covariant theories of gravitation. However, the success of Einstein’s gravitation theory, described in General Relativity (see for instance [20,21,22,23,24,25,26,27,28,29,30,31]), led to the abandonment of these old efforts. It seems, Einstein was unaware of Heaviside’s work on gravity, otherwise his remark on Newton’s theory of gravitation would have been different than what he made before the 1913 congress of natural scientists in Vienna [32], viz.,

After the un-tenability of the theory of action at distance had thus been proved in the domain of electrodynamics, confidence in the correctness of Newton’s action-at-a-distance theory of gravitation was shaken. One had to believe that Newton’s law of gravity could not embrace the phenomena of gravity in their entirety, any more than Coulomb’s law of electrostatics embraced the theory of electromagnetic processes.

However, after Sciama’s consideration [33] of MG, in 1953 to explain the origin of inertia, there have been several studies on it, see [11,13,14,15,16,17,18,19,34,35,36,37,38,39,40,41,42,43,44,45,46,47] and other references therein. The gravito-Lorentz-Maxwell (g-LM) equations of MG obtained here using the well established principles of quantum field theory in flat space-time corroborate the g-LM equations obtained by several authors using a variant of classical methods: (a) Schwinger’s Galileo-Newtonian

Relativistic approach to get the Lorentz-Maxwell equations of electromagnetism [45], (b) Special Relativistic approaches to gravity [13,14,15,16,17] modification of Newton’s law on the basis of the principle of causality [11,31], (d) some axiomatic methods [42,43] common to electromagnetism and gravitoelectromagnetism and also (e) a specific linearization scheme of General Relativity (GR) in the weak field and slow motion approximation [47]. However, in the context of GR several versions of linearized approximations exist, which are not isomorphic and predict different values of speed of gravity $c_g$ in vacuum as explicitly shown by Behera [45]. This is one of the limitations of GR. MG of GR origin will be denoted as GRMG below. Out of a number of linearized versions of GR considered in [45], here we pick out only 4 versions for our discussion on the value of $c_g$ below for explicit comparison and other purpose.

3.2 On the Speed of Gravitational Waves ($c_g$)

The speed of gravitational waves $c_g$ in vacuum provides us a new tool to test alternative theories of gravity [48,49]. While $c_g = c$ is implied by the field equations of HG or MG listed in Table 2, we discuss below the value of $c_g$ in some linearized versions of GR.

3.2.1 GRMG of Braginsky et al. and Forward (GRMG-BF)

The value of $c_g$ can be predicted from the GRMG of Braginsky et al. [50] and Forward [51], which is called as GRMG-BF [45]. This is based on certain parametrized-post-Newtonian (PPN) formalism. The Lorentz-Maxwell-type equations of GRMG-BF are noted below in our present notation and convention.

\[
\nabla \cdot \textbf{g} = - 4\pi G \rho \left[ 1 + 2 \frac{\nabla^2}{c^2} + \frac{\Pi}{c^2} + \frac{3\rho}{\rho_0 c^2} \right] + \frac{3}{c^2} \frac{\partial^2 \phi}{\partial t^2} \tag{98a}
\]

\[
\nabla \times \textbf{b} = - \frac{16\pi G}{c^2} (\rho_0 \textbf{v}) + \frac{4}{c^2} \frac{\partial \Phi}{\partial t} \tag{98b}
\]

\[
\nabla \cdot \textbf{b} = 0 \tag{98c}
\]

\[
\nabla \times \textbf{g} = - \frac{\partial \textbf{b}}{\partial t} \tag{98d}
\]

where we have put the values of PPN parameters as appropriate for GR, $\rho_0$ is the density of rest mass in the local rest frame of the matter, $\textbf{v}$ is the ordinary (co-ordinate velocity) velocity of the rest mass relative to the PPN frame, $\Pi$ is the specific internal energy

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\(^6\)Who relying on McDonald’s [12] report of HG, stated that MG is same as HG. This should not be taken for granted without a proof because a sign difference in some vector quantities or equations has different physical meaning/effect.

\(^7\)Here we use $\textbf{b}$ in place of $\textbf{H}_g$ in [50].
(energy per unit rest mass) and \(p\) is the radiation pressure and \(\phi_g\) is the electric-type scalar potential. In this formalism, the \(\phi_g\) and gravitomagnetic vector potential \(A_g\) are related to the \(g\) and \(b\) fields as

\[
g = -\nabla \phi_g - \frac{\partial A_g}{\partial t}, \quad b = \nabla \times A_g \tag{99}
\]

\[
\nabla \cdot A_g + \frac{3}{c^2} \frac{\partial \phi_g}{\partial t} = 0 \quad \text{(For Lorenz-type Gauge)} \tag{100}
\]

where the number 3 in the Lorenz-type gauge above is the GR value for some PPN parameters used in [50]. For the source and particle of velocities \(|v| < 10^5 \text{ cm/sec} << c\), Braginsky et al. [50] approximated the gravitational force (with a typographical error in eqn. (3.10) of [50], corrected in [45]) on a unit mass,

\[
\frac{F}{m_0} = \left[ 1 + \frac{1}{2}(2\gamma + 1) \frac{v^2}{c^2} \right] g + (v \times b), \tag{101}
\]

where the PPN parameter \(\gamma \simeq 1\) in GR. In vacuum \((\rho_0 = 0)\) with no radiation pressure \((p = 0)\), if we consider Coulomb–Newton gauge \((\nabla \cdot A_g = 0)\), the field eqns. (103a–103d) reduce to

\[
\begin{align*}
\nabla \cdot g &= 0, \tag{102a} \\
\nabla \times b &= +\frac{4}{c^2} \frac{\partial g}{\partial t}, \tag{102b} \\
\nabla \cdot b &= 0, \tag{102c} \\
\nabla \times g &= -\frac{\partial b}{\partial t}. \tag{102d}
\end{align*}
\]

Now taking the curl of eq. (102b) and then utilizing eq. (102c) as well as eq. (102d), we get the wave equation for the \(b\) field as

\[
\nabla^2 b - \frac{4}{c^2} \frac{\partial^2 b}{\partial t^2} = \nabla^2 b - \frac{1}{c_g^2} \frac{\partial^2 b}{\partial t^2} = 0, \tag{103}
\]

where \(c_g = c/2\). Similarly, taking the curl of (102d) and then utilizing eq. (102a) as well as eq. (102b), we get the wave equation for the \(g\) field as

\[
\nabla^2 g - \frac{4}{c^2} \frac{\partial^2 g}{\partial t^2} = \nabla^2 g - \frac{1}{c_g^2} \frac{\partial^2 g}{\partial t^2} = 0, \tag{104}
\]

where \(c_g = c/2\).

\[\text{Viz.: } \frac{F}{m_0} = [1 + \frac{1}{2}(2\gamma + 1)] \frac{v^2}{c^2} g + (v \times b) \text{ in our convention.}\]

3.2.2 GRMG of Ohanian and Ruffini (GRMG-OR)

Ohanian and Ruffini [30] (Sec. 3.4 of [30]) obtained the following equations from GR in the Non-relativistic limit and Newtonian Gravity correspondence of GR:

\[
\frac{dv}{dt} = g + v \times b \tag{105}
\]

\[
\begin{align*}
\nabla \cdot g &= -4\pi G\rho_0 \tag{106a} \\
\nabla \times g &= -\frac{1}{2} \frac{\partial b}{\partial t} \\
\nabla \cdot b &= 0 \tag{106b} \\
\nabla \times b &= -\frac{16\pi G}{c^2} j_0 \tag{106c}
\end{align*}
\]

where \(\rho_0\) is the (rest) mass density, \(j_0\) is the momentum density. The equation (106a) (representing the gravito-Ampère law) is valid for time independent field [30]. By noticing the limitation in (106d) as Maxwell noticed in the Ampère’s law of electromagnetism, recently Behera [45], has made a correction of (106d) (as Maxwell did to the Ampère law) so that the field equations are now consistent with the continuity equation for mass and mass current or momentum density:

\[
\nabla \cdot j_0 + \frac{\partial \rho_0}{\partial t} = 0. \tag{107}
\]

The corrected equation is

\[
\nabla \times b = -\frac{16\pi G}{c^2} j_0 + \frac{4}{c^2} \frac{\partial g}{\partial t}. \tag{108}
\]

Without this corrected equation there can not be gravitational waves. The corrected self-consistent field equations (106a–106c, 108) yield transverse gravitational waves; the wave equations for the \(g\) and \(b\) fields of GRMG-OR, in vacuum, take the following forms:

\[
\begin{align*}
\nabla^2 g - \frac{2}{c^2} \frac{\partial^2 g}{\partial t^2} &= \nabla^2 g - \frac{1}{c_g^2} \frac{\partial^2 g}{\partial t^2} = 0, \tag{109a} \\
\nabla^2 b - \frac{2}{c^2} \frac{\partial^2 b}{\partial t^2} &= \nabla^2 b - \frac{1}{c_g^2} \frac{\partial^2 b}{\partial t^2} = 0. \tag{109b}
\end{align*}
\]

where \(c_g = c/\sqrt{2}\) in vacuum.

3.2.3 GRMG of Pascual-Sánchez and Moore (GRMG-PS-M):

In some form of the weak field and slow motion approximation of GR, Pascual-Sánchez [52] obtained the following gravito-Lorentz-Maxwell equations which match with Moore’s findings [53]:

\[
\rho_0 \frac{dv}{dt} = m_0 (g + 4v \times b). \tag{110}
\]
\[ \nabla \cdot \mathbf{g} = -4\pi G\rho_0, \quad (111a) \]
\[ \nabla \times \mathbf{b} = -\frac{4\pi G}{c^2} \rho_0 \mathbf{v} + \frac{1}{c^2} \frac{\partial \mathbf{g}}{\partial t}, \quad (111b) \]
\[ \nabla \cdot \mathbf{b} = 0, \quad (111c) \]
\[ \nabla \times \mathbf{g} = -\frac{\partial \mathbf{b}}{\partial t}. \quad (111d) \]

The waves equations that emerge from eqs. (111) give us \( c_g = c \).

### 3.2.4 GRMG of Ummarino-Gallerati (GRMG-UG):

Recently Ummarino and Gallerati derived the following gravito-Lorentz-Maxwell equations from Einstein’s GR by linearization procedure in the weak field and slow motion approximations.

\[
\frac{d\mathbf{v}}{dt} = \mathbf{g} + \mathbf{v} \times \mathbf{b},
\]

\[ \nabla \cdot \mathbf{g} = -4\pi G\rho_0, \quad (113a) \]
\[ \nabla \cdot \mathbf{b} = 0, \quad (113b) \]
\[ \nabla \times \mathbf{b} = -\frac{4\pi G}{c^2} (\rho_0 \mathbf{v}) + \frac{1}{c^2} \frac{\partial \mathbf{g}}{\partial t}, \quad (113c) \]
\[ \nabla \times \mathbf{g} = -\frac{\partial \mathbf{b}}{\partial t}. \quad (113d) \]

The field equations (113) yield \( c_g = c \) in vacuum. Thus the reader can now realize that the predictions on the speed of gravity in the weak field and slow motion approximation of GR are not unique, but the value of \( c_g \) is uniquely and unambiguously fixed at \( c_g = c \) in the present quantum field theoretical findings of HMG or our previous findings [13,15]. It is interesting to note that the existence of gravitational waves has recently been detected [51,55,59,67], and also the existence of the gravitomagnetic field generated by mass currents has been confirmed by experiments [58,59,60,61,62,63,61,65]. These are being considered as new confirmation tests of GR [21,25,20,37,55,59,60,61,62,63,61,65]. The explanations for experimental data on gravitational waves and the gravitomagnetic field within the framework of HMG are being explored by the authors, since the explanations for the (a) perihelion advance of Mercury (b) gravitational bending of light and (c) the Shapiro time delay within the vector theory of gravity exist in the literature [38,39,40,66]. Further, very recently Hilborn following an electromagnetic analogy, calculated the waveforms of gravitational radiation emitted by orbiting binary objects that are very similar to those observed by the Laser Interferometer Gravitational-Wave Observatory (LIGO-VIRGO) gravitational wave collaboration in 2015 up to the point at which the binary merger occurs. Hilborn’s calculation produces results that have the same dependence on the masses of the orbiting objects, the orbital frequency, and the mass separation as do the results from the linear version of general relativity (GR). But the polarization, angular distributions, and overall power results of Hilborn differ from those of GR. It is to be noted that the polarization states of gravitational waves has not yet been conclusively determined from recent experimental data [51,55,59,67].

### 3.3 Does GR satisfy the correspondence principle?

By deducing Newtonian Gravity (NG) from GR, all texts books on GR teach us that GR does satisfy the correspondence principle by which a more sophisticated theory should reduce to a theory of lesser sophistication by imposing some conditions; Misner, Thorne and Wheeler [24] in a boxed item (Box 17.1, page-412) of their book “Gravitation” have put much emphasis on it by giving a host of examples. In the light of our findings on HMG here and in [45] we see that GR defies the correspondence principle (cp) in its true sense: GRMG \( \not\Rightarrow \) SRMG \( \not\Leftrightarrow \) N(R)MG \( \not\Leftrightarrow \) NG, where SRMG, N(R)MG and NG stands for Special Relativistic Maxwellian Gravity, Non-relativistic or Newtonain MG and Newtonian Gravity respectively.

### 3.4 Misner, Thorne and Wheeler on HMG and Experimental Tests of HMG

Misner, Thorne and Wheeler (MTW) [24], in their “Exercises on flat space-time theories of gravity”, have considered a possible vector theory of gravity within the framework of special relativity. They considered a Lagrangian density of the form \( \mathcal{L} \) and found it to be deficient in that there is no bending of light, perihelion advance of Mercury and gravitational waves carry negative energy in vector theory of gravity. As regards the classical tests of the GR, we have noted before that the explanation of these tests exist in the literature [38,39,40,66]. But the issue of energy and momentum carried by gravitational waves is far from clear yet, even within the framework of GR. In the community of general relativists, there is no unanimity of opinion on the energy carried by gravitational waves. For instance, one finds references in the literature on GR which describes (not in the gravito-electromagnetic approach) the radiation from a gravitating system as carrying away energy [24,68], bringing in energy [69], carrying no energy [70], or...
having an energy dependent on the coordinate system used \([70]\). In the gravitoelectromagnetic approach to understand gravitational waves, we can see here that except for some numerical factor of 4 or 2 in certain terms, there exist no differences between the field equations of MG and GRMG. So it seems, both HMG and GRMG suffer from this “energy ambiguity” - a resolution of which is desirable.

3.5 On the spin of graviton: spin 1 or spin 2?

Following the usual procedures of electrodynamics (see, for instance \([71]\) for obtaining the spin of photon, the spin of graviton (a quantum of gravitational wave carrying energy and momentum) in the framework of HMG can be shown to be 1 in the unit of \(\hbar\). Regarding the idea of spin-2 graviton, Wald \([25]\) (see p.76) noted that the linearized Einstein’s equations in vacuum are precisely of spin-2 graviton, Wald \([25]\) noted that the can be shown to be 1 in the unit of \(\hbar\) for obtaining the spin of photon, the

3.6 Attraction Between Static Like Masses

Let us find the static interaction between two point (positive) masses at rest, following a classical approach \([73]\) within the framework of Maxwellian Gravity as follows. For a particle having gravitational charge \(m_g = m_0\) at rest at the origin, the 4-current densities can be shown to be \([73]\):

\[
\begin{align*}
  j^0_g &= m_0 \delta^3(\mathbf{x}), \\
  j^i_g &= 0.
\end{align*}
\]

In eq. \((114)\), we can therefore set

\[
A^0_g = \phi_g/c, \quad A^i_g = 0,
\]

to get

\[
\nabla^2 \phi_g = 4\pi Gm_0 \delta^3(\mathbf{x}).
\]

This is nothing but the Poisson’s equation for gravitational potential of a point mass at rest at origin. Using Green’s Function, the potential at a distance \(r\) for a central point particle having gravitational mass \(m_0\) (i.e., the fundamental solution) is

\[
\phi_g(r) = \frac{-Gm_0}{r}, \quad (117)
\]

which is equivalent to Newton’s law of universal gravitation. The interaction between two point particles having gravitational charges \(m_0 = M_1\) and \(m_0' = M_2\) separated by a distance \(r\) is

\[
U_{12} = M_2 \phi_g = \frac{-G M_1 M_2}{r}, \quad (118)
\]

which is negative for like gravitational charges and positive for un-like gravitational charges, if they exist. With \(M_1\) at rest at the origin, the force on another stationary gravitational charge \(M_2\) at a distance \(r\) from origin is

\[
F_{21} = -M_1 \nabla \phi_g(r) = \frac{-G M_1 M_2}{r^2} \hat{r} = -F_{12}. \quad (119)
\]

This force is attractive, if \(M_1\) and \(M_2\) are of same sign and repulsive if they are of opposite sign, unlike the case of electrical interaction between two static electric charges. Besides the above classical approach, one may follow Feynman’s \([2]\) detailed quantum field theoretical approach using our eq. \((89)\) to arrive at our above conclusion. This is possible because here we have a difference in sign in the eqs. \((57)\) and \((89)\) and the interaction terms in eqs. \((65)\) and \((70)\) remaining the same in mathematical form, viz., \(-j^\mu A_\mu\) in Maxwell’s Electromagnetism and \(-j^\mu A_\mu\) in HMG. Zee’s \([3]\) path-integral approach may also be used to arrive at the same conclusion.

3.7 Gravitational Correction to the QED Lagrangian

The complete Lagrangian density \([22]\), that takes into account both the charge and mass degree of freedom and remains invariant under local phase in-variance is

\[
\mathcal{L}_{\text{QGD}} = [i \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi - m_0 c^2 \bar{\psi} \psi] + \mathcal{L}_c + \mathcal{L}_g, \quad (120)
\]

where \(\mathcal{L}_c\) is given by eq. \((89)\) and \(\mathcal{L}_g\) is given by eq. \((70)\). When \(\mathcal{L}_g\) can be neglected, the Lagrangian density in eq. \((120)\) reduces to the well known quantum electrodynamics (QED) Lagrangian density:

\[
\mathcal{L}_{\text{QED}} = [i \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi - m_0 c^2 \bar{\psi} \psi] + \mathcal{L}_c, \quad (121)
\]

For electrically neutral massive Dirac particles particles, \(\mathcal{L}_c = 0\) and eq. \((120)\) gives us what we call the Lagrangian density for quantum-gravitodynamics (QGD):

\[
\mathcal{L}_{\text{QGD}} = [i \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi - m_0 c^2 \bar{\psi} \psi] + \mathcal{L}_g. \quad (122)
\]
The above Lagrangian density $\mathcal{L}_{\text{QGD}}$ may be used to study the dynamics of electrically neutral Dirac’s spin 1/2 particles (such as neutrons and massive neutrinos) in external gravitoelectromagnetic fields of the Sun and the Earth.

It is to be noted that because of the smallness of the masses of elementary particles and the value of the Newton’s gravitational constant, our $\mathcal{L}_{\text{QGD}}$ would have ridiculously insignificant impact on QED processes occurring in small gravity environments such as the Earth or the Sun. But in strong gravity environments of very compact objects such as the neutron stars, Supernova and supper-massive galactic centers, $\mathcal{L}_{\text{QGD}}$ is expected to produce results different from the $\mathcal{L}_{\text{QED}}$ as shown by Franson [74], who considered the effects of including the Newtonian gravitational potential energy of massive particles in the Hamiltonian of QED to predict an apparent correction to the speed of light in a gravitational potential which is in reasonable agreement with experimental observations from Supernova 1987a.

4 Concluding Remarks

In this work we report

(1) a new set of Lorentz-Maxwell’s equations of electromagnetism (derived using the standard quantum field theoretical method) which is physically equivalent to the standard set of equations,

(2) our quantum field theoretical derivation of the field equations of Heaviside’s Gravity (HG) and Maxwellian Gravity (MG) as well as their respective Lorentzian force laws in which we found a correction to the Hamiltonian of QED to predict an apparent correction to the speed of light in a gravitational potential which is in reasonable agreement with experimental observations from Supernova 1987a.

(3) our findings that HG and MG are mere two different mathematical representations of a single theory which we named as Heaviside-Maxwellian Gravitoelectromagnetism or Gravity (HMG),

(4) the gravito-Lorentz-Maxwell’s equations of MG derived here using the well established methods of quantum field theory perfectly match with those obtained from a variant other established methods of study or principles of classical physics (a) Schwinger’s formalism based on Galilii-Newtonian relativity if $c_g = c$, (b) special relativistic approaches of different types, (c) principles of causality, (d) some axiomatic approaches common to electromagnetism and gravitoelectromagnetism and (e) in some specific linearization method of general relativity,

(5) Galileo’s Law of Universality of Free Fall is a consequence of HMG, not an initial assumption,

(6) our prediction of an unambiguous and unique value of speed of gravitational waves ($c_g = c$) unlike the ambiguous and non-unique value of $c_g$ obtainable from different linearized versions of GR,

(7) possible existence of spin-1 graviton, in contrast with the idea of spin-2 graviton within GR - an idea not well founded in the GR,

(8) that the spin-1 vector gravity of HMG denomination produces attractive interaction between like static masses contrary to the prevalent view of the field theorists,

(9) a gravitational correction to the QED Lagrangian in the light of HMG,

(10) a Lagrangian to study the quantum gravitodynamics of electrically neutral Dirac spin (1/2) particles,

(11) a brief discussion on the issue of negative/positive energy of gravitational waves both in HMG and GR, where our understanding is still unclear,

(12) the works of some other researchers which correctly explain some crucial test of GR viz., (a) non-Newtonian perihelion advance of planetary orbits including Mercury, (b) gravitational bending of light and (c) Shapiro time delay, in a non-GR approach but using some aspects of HMG.

Being simple, self consistent and well founded, HMG may deserve certain attention of the researchers interested in probing the classical and quantum gravitodynamics of moving bodies/particles in the presence and absence of electromagnetic or other interactions having energy-momentum 4-vector since the 4-vector potential of HMG couples to it. This line of research may shed some new light on the nature of gravity of HMG denomination and its connection to other fundamental fields of nature.

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