The Sub-Riemannian Limit of Curvatures for Curves and Surfaces and a Gauss-Bonnet Theorem in the Group of Rigid Motions of Minkowski Plane with General Left-Invariant Metric

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The group of rigid motions of the Minkowski plane with a general left-invariant metric is denoted by \((E(1,1), g(\lambda_1, \lambda_2))\), where \(\lambda_1, \lambda_2 > 0\). It provides a natural 2-parametric deformation family of the Riemannian homogeneous manifold \(\text{Sol}_1 = (E(1,1), g(1,1))\) which is the model space to solve geometry in the eight model geometries of Thurston. In this paper, we compute the sub-Riemannian limits of the Gaussian curvature for a Euclidean \(C^2\)-smooth surface in \((E(1,1), g_L(\lambda_1, \lambda_2))\) away from characteristic points and signed geodesic curvature for the Euclidean \(C^2\)-smooth curves on surfaces. Based on these results, we get a Gauss-Bonnet theorem in the group of rigid motions of the Minkowski plane with a general left-invariant metric.

1. Introduction

In [1], Proposition 2.6 stated that any left-invariant metric on the group of rigid motions of the Minkowski plane \(E(1,1)\) is isometric to one of the metric \(g(\lambda_1, \lambda_2, \lambda_3)\) with \(\lambda_1, \lambda_2 > 0\) and \(\lambda_3 = 1/\lambda_1 \lambda_2\). In [2], the metric \(g(\lambda_1, \lambda_2, \lambda_3)\) was denoted by \(g(\lambda_1, \lambda_2) = g(\lambda_1, \lambda_2, 1/\lambda_1 \lambda_2)\) as we take in this paper, and the authors classified parallel surfaces in the groups of rigid motions of the Euclidean plane and the Minkowski plane. In [3], they completed the classification of parallel and totally geodesic surfaces in all three-dimensional homogeneous spaces by solving the problem in three-dimensional Lie groups with a left-invariant metric yielding a three-dimensional isometry group. In this paper, we consider \((E(1,1), g(\lambda_1, \lambda_2))\) which is the group of rigid motions of the Minkowski plane with the general left-invariant metric \(g(\lambda_1, \lambda_2)\). This group is very interesting and important for the reason that it provides a natural 2-parametric deformation family of one of the Riemannian homogeneous manifold \(\text{Sol}_1 = (E(1,1), g(1,1))\) which is the model space to solve geometry in the eight model geometries of Thurston.

In [4, 5], Balogh et al. used a Riemannian approximation scheme to define a notion of intrinsic Gaussian curvature for a Euclidean \(C^2\)-smooth surface in the Heisenberg group \(H^1\) away from characteristic points, and a notion of intrinsic signed geodesic curvature for the Euclidean \(C^2\)-smooth curves on surfaces. These results were then used to prove a Heisenberg version of the Gauss-Bonnet theorem. They proposed an interesting question to understand to what extent similar phenomena hold in other sub-Riemannian geometric structures. In [6, 7], Wang and Wei solved this problem for the affine group, the group of rigid motions of the Minkowski plane \((E(1,1), g(1,1))\), the BCV spaces, and the twisted Heisenberg group. Recently, we got the Gauss-Bonnet theorems in the rototranslation group and the Lorentzian Sasakian space forms [8, 9]. In this paper, we try to solve this problem for the group of rigid motions of the Minkowski plane with the general left-invariant metric \(g(\lambda_1, \lambda_2)\). We compute the sub-Riemannian limits of the Gaussian curvature for a Euclidean \(C^2\)-smooth surface in \((E(1,1), g_L(\lambda_1, \lambda_2))\) away from characteristic points and signed geodesic curvature for the Euclidean \(C^2\)-smooth curves on surfaces. We get a generalized Gauss-Bonnet theorem in \((E(1,1), g_L(\lambda_1, \lambda_2))\).
In Section 2, we provide a short introduction to \((E(1, 1), g_t(\lambda_1, \lambda_2))\) and the notions which we will use throughout the paper, such as the Levi-Civita connection in the Riemannian approximants of \((E(1, 1), g(\lambda_1, \lambda_2))\). Furthermore, we compute the sub-Riemannian limit of the curvature of curves in \((E(1, 1), g_t(\lambda_1, \lambda_2))\). In Sections 3 and 4, we compute the sub-Riemannian limits of the geodesic curvature of curves on surfaces and the Riemannian Gaussian curvature of surfaces in \((E(1, 1), g_t(\lambda_1, \lambda_2))\). In Section 5, we get the Gauss-Bonnet theorem in the group of rigid motions of the Minkowski plane with the general left-invariant metric.

2. The Sub-Riemannian Limit of Curvature of Curves in \((E(1, 1), g_t(\lambda_1, \lambda_2))\)

In this section, some basic notions in the motion group of the Minkowski plane will be introduced. Let \(E(1, 1)\) be the motion group of the Minkowski plane:

\[
E(1, 1) = \left\{ \begin{pmatrix} e^x & 0 & x \\ 0 & e^y & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}. \tag{1}
\]

The Lie algebra \(\mathfrak{e}(1, 1)\) is given explicitly by

\[
\mathfrak{e}(1, 1) = \left\{ \begin{pmatrix} \omega & 0 & u \\ 0 & -\omega & v \\ 0 & 0 & 0 \end{pmatrix} \middle| u, v, \omega \in \mathbb{R} \right\}. \tag{2}
\]

We consider the group of rigid motions of the Minkowski plane with a general left-invariant metric, \(\left( E(1, 1), g(\lambda_1, \lambda_2) \right) \). As a model of \(\left( E(1, 1), g(\lambda_1, \lambda_2) \right)\), we choose the underlying manifold \(\mathbb{R}^3\). On \(\mathbb{R}^3\), we let

\[
X_1 = \lambda_1 \lambda_2 \frac{\partial}{\partial z},
\]

\[
X_2 = \frac{1}{\lambda_1 \sqrt{2}} \left( -e^y \frac{\partial}{\partial x} + e^{-z} \frac{\partial}{\partial y} \right),
\]

\[
X_3 = -\frac{1}{\lambda_2 \sqrt{2}} \left( e^y \frac{\partial}{\partial x} + e^{-z} \frac{\partial}{\partial y} \right). \tag{3}
\]

Then, we have

\[
\frac{\partial}{\partial x} = -\frac{\sqrt{2}}{2} e^{-y}(\lambda_1 X_2 + \lambda_2 X_3),
\]

\[
\frac{\partial}{\partial y} = \frac{\sqrt{2}}{2} e^y(\lambda_1 X_2 - \lambda_2 X_3), \tag{4}
\]

\[
\frac{\partial}{\partial z} = \frac{1}{\lambda_1 \lambda_2} X_1,
\]

\[
\text{span}\{X_1, X_2, X_3\} = T((E(1, 1), g(\lambda_1, \lambda_2))). \tag{5}
\]

Let \(H = \text{span}\{X_1, X_2\}\) be the horizontal distribution on \((E(1, 1), g(\lambda_1, \lambda_2))\). Let

\[
\omega_1 = \frac{1}{\lambda_1 \lambda_2} dz,
\]

\[
\omega_2 = \frac{\lambda_1}{\sqrt{2}} (e^{-y} dx + e^y dy), \tag{6}
\]

\[
\omega = -\frac{\lambda_2}{\sqrt{2}} (e^y dx + e^{-y} dy),
\]

be the dual coframe field. Then, \(H = \ker \omega\). The Riemannian approximation scheme used in [4] can in general depend on the choice of the complement to the horizontal distribution. In the context of \((E(1, 1), g(\lambda_1, \lambda_2))\), the choice is similar to \((E(1, 1), g(1, 1))\) in [6]. Let \(L > 0\) and define a metric \(g_L(\lambda_1, \lambda_2) = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L \omega \otimes \omega\), so that \(X_1, X_2, X_3 = L^{-1/2} X_\lambda\), where \(X_\lambda\) are orthonormal basis on \(T(E(1, 1), g_L(\lambda_1, \lambda_2))\) with respect to \(g_L(\lambda_1, \lambda_2)\).

Notice that

\[
g_L(\lambda_1, \lambda_2) = \frac{1}{\lambda_1^2 \lambda_2^2} dz^2 + \frac{\lambda_1^2}{2} (e^{-y} dx + e^y dy)^2 + \frac{\lambda_2^2}{2} (e^y dx + e^{-y} dy)^2, \tag{7}
\]

and \(g(\lambda_1, \lambda_2) = g_L(\lambda_1, \lambda_2)\) be the Riemannian metric on \((E(1, 1), g(\lambda_1, \lambda_2))\).

The approach in this paper is to define sub-Riemannian objects as limits of horizontal objects in \((E(1, 1), g_L(\lambda_1, \lambda_2))\), where a family of metrics \(g_L(\lambda_1, \lambda_2)\) is essentially obtained as an anisotropic blow-up of the Riemannian metric \(g(\lambda_1, \lambda_2)\).

At the heart of this approach is the fact that the intrinsic horizontal geometry does not change with \(L\). In general, the metric \(g_L(\lambda_1, \lambda_2)\) does not fit in the family \(g(\lambda_1, \lambda_2)\) for the case of \(L \neq 1\). We have

\[
[X_1, X_2] = \lambda_1^2 X_3,
\]

\[
[X_2, X_3] = 0,
\]

\[
[X_1, X_3] = \lambda_1^{-2} X_2. \tag{8}
\]

To compute the curvatures of curves and surfaces in the motion group of the Minkowski plane with respect to \(g_L(\lambda_1, \lambda_2)\), we use the Levi-Civita connection \(\nabla^{g_L}\) on \((E(1, 1), g_L(\lambda_1, \lambda_2))\). A straightforward calculation shows the following proposition.

**Proposition 1.** Let \((E(1, 1), g_L(\lambda_1, \lambda_2))\) be the motion group of the Minkowski plane with a general left-invariant metric, relative to the coordinate frame \(X_1, X_2, X_3\); then, the Levi-Civita connection on \((E(1, 1), g_L(\lambda_1, \lambda_2))\) is given by...
\[ \nabla^2_j X_j = 0, \quad 1 \leq j \leq 3, \]
\[ \nabla^L_j X_j = \frac{\lambda_j^2 L - \lambda_j^2}{2L} X_j, \]
\[ \nabla^L_j X_j = \frac{-\lambda_j^2 L - \lambda_j^2}{2L} X_j, \]
\[ \nabla^L_j X_j = \frac{\lambda_j^2 - \lambda_j^2 L}{2} X_j, \]
\[ \nabla^L_j X_j = \frac{-\lambda_j^2 - \lambda_j^2 L}{2} X_j, \]
\[ \nabla^L_j X_j = \nabla^L_j X_j = \frac{\lambda_j^2 + \lambda_j^2 L}{2} X_j. \]

Proof. It follows from a direct application of the Koszul identity, which here simplifies

\[ 2 \left( \nabla^L_j X_j \right) = \left( [X_i, X_j] \right) - \left( [X_i, X_j] \right) + \left( [X_i, X_j] \right), \]

where \( i, j, k = 1, 2, 3 \). By (8) and (9), we have

\[ 2 \left( \nabla^L_j X_j \right) = \left( [X_i, X_j] \right) + \left( [X_i, X_j] \right) + \left( [X_i, X_j] \right) - 2 \left( [X_i, X_j] \right). \]

When \( j = 1 \), we compute \( \nabla^L_j X_j = \left( [X_1, X_1] \right) + \left( [X_1, X_1] \right) + \left( [X_1, X_1] \right) = 0 \). Similarly, \( \nabla^L_j X_j = 0 \) and \( \nabla^L_j X_j = 0 \). By the following equation,

\[ 2 \left( \nabla^L_j X_j \right) = \left( [X_1, X_1] \right) + \left( [X_1, X_1] \right) + \left( [X_1, X_1] \right) - 2 \left( [X_1, X_1] \right), \]

we get \( \nabla^L_j X_j = (\lambda_j^2 L - \lambda_j^2 L/2) L \). Other cases follow by similar computations. \( \Box \)

Definition 3. Let \( \gamma : [a, b] \rightarrow (E(1, 1), g_{L_1}(\lambda_1, \lambda_2)) \) be a Euclidean \( C^2 \)-smooth regular curve in the Riemannian manifold \((E(1, 1), g_{L_1}(\lambda_1, \lambda_2))\). The curvature \( k^L_\gamma \) of \( \gamma \) at \( \gamma(t) \) is defined as follows:

\[ k^L_\gamma = \left\{ \left[ \left[ \frac{1}{\lambda_1} \gamma_\gamma(t) + \frac{\lambda_2}{\lambda_1^2} \gamma_\gamma^2 \right] - (e^{\gamma_\gamma(t)} \gamma^3 + e^{\gamma_\gamma(t)} \gamma^3) \omega(\gamma(t)) \right]^2 \right\}^{1/2}. \]

Proposition 4. Let \( \gamma : [a, b] \rightarrow (E(1, 1), g_{L_1}(\lambda_1, \lambda_2)) \) be a Euclidean \( C^2 \)-smooth regular curve in the Riemannian manifold \((E(1, 1), g_{L_1}(\lambda_1, \lambda_2))\). Then, we have

\[ k^L_\gamma = \left\{ \left[ \frac{1}{\lambda_1} \gamma_\gamma(t) + \frac{\lambda_2}{\lambda_1} \gamma_\gamma(t) + \frac{\lambda_2}{\lambda_1^2} \gamma_\gamma(t) \right] - (e^{\gamma_\gamma(t)} \gamma^3 + e^{\gamma_\gamma(t)} \gamma^3) \omega(\gamma(t)) \right]^2 \right\}^{1/2}. \]

In particular, if \( \gamma(t) \) is a horizontal point of \( \gamma \),

\[ k^L_\gamma = \left\{ \left[ \frac{1}{\lambda_1} \gamma_\gamma(t) + \frac{\lambda_1}{\lambda_1^2} \gamma_\gamma(t) + \frac{\lambda_2}{\lambda_1^2} \gamma_\gamma(t) \right] - \left[ \frac{1}{\lambda_1} \gamma_\gamma(t) + \frac{\lambda_1}{\lambda_1^2} \gamma_\gamma(t) \right] \right\}^{1/2}. \]
Proof. By (4), we have
\[
\dot{y}(t) = \dot{y}_1(t) + \frac{\partial}{\partial s} \dot{y}_2(t) + \frac{\partial}{\partial t} \dot{y}_3(t) = \begin{bmatrix}
\frac{\lambda_1}{1 + \lambda_1^2} y_1(t) + \frac{\sqrt{2}}{2} (\gamma(t) - x) y_2(t) + \frac{\lambda_1}{1 + \lambda_1^2} y_3(t) \\
\frac{\lambda_1}{1 + \lambda_1^2} y_1(t) + \frac{\lambda_1}{1 + \lambda_1^2} y_3(t) \\
\frac{\lambda_1}{1 + \lambda_1^2} y_1(t) + \frac{\lambda_1}{1 + \lambda_1^2} y_3(t)
\end{bmatrix} X_1 + \frac{\lambda_1}{1 + \lambda_1^2} (-e^{\theta(t)} \dot{y}_2(t) + e^{\theta(t)} \dot{y}_2(t)) X_2 + \omega(\dot{y}(t)) X_3.
\]
(17)

By Proposition 1 and (17), we have
\[
\nabla_x^2 y_1(t) = \begin{bmatrix}
\frac{1}{1 + \lambda_1^2} y_1(t) y_1(t) + \frac{\lambda_1}{1 + \lambda_1^2} (-e^{\theta(t)} \dot{y}_2(t) + e^{\theta(t)} \dot{y}_2(t)) y_1(t) \\
\frac{\lambda_1}{1 + \lambda_1^2} y_3(t) + \omega(\dot{y}(t)) y_1(t) \\
\frac{\lambda_1}{1 + \lambda_1^2} y_3(t) + \omega(\dot{y}(t)) y_1(t)
\end{bmatrix} X_1 + \frac{\lambda_1}{1 + \lambda_1^2} (-e^{\theta(t)} \dot{y}_2(t) + e^{\theta(t)} \dot{y}_2(t)) y_1(t) X_2 + \omega(\dot{y}(t)) y_1(t) X_3.
\]
(18)

\[
\nabla_x^2 y_2(t) = \begin{bmatrix}
\frac{1}{1 + \lambda_1^2} y_1(t) y_1(t) + \frac{\lambda_1}{1 + \lambda_1^2} (-e^{\theta(t)} \dot{y}_2(t) + e^{\theta(t)} \dot{y}_2(t)) y_1(t) \\
\frac{\lambda_1}{1 + \lambda_1^2} y_3(t) + \omega(\dot{y}(t)) y_1(t) \\
\frac{\lambda_1}{1 + \lambda_1^2} y_3(t) + \omega(\dot{y}(t)) y_1(t)
\end{bmatrix} X_1 + \frac{\lambda_1}{1 + \lambda_1^2} (-e^{\theta(t)} \dot{y}_2(t) + e^{\theta(t)} \dot{y}_2(t)) y_1(t) X_2 + \omega(\dot{y}(t)) y_1(t) X_3.
\]
(19)

By (17) and (18), we have
\[
\nabla_x^2 \dot{y}(t) = \nabla_x^2 \dot{y}_1(t) X_1 + \frac{\lambda_1}{1 + \lambda_1^2} (-e^{\theta(t)} \dot{y}_2(t) + e^{\theta(t)} \dot{y}_2(t)) X_2 + \omega(\dot{y}(t)) X_3.
\]
(20)

By (14), (17), and (21), we get
\[
\|\nabla_\chi^2 \dot{y}(t)\|^2 = \begin{bmatrix}
\frac{1}{1 + \lambda_1^2} \dot{y}_1(t) + \frac{\lambda_1^2}{2} \dot{y}_2(t) + \frac{\lambda_1^2}{2} \dot{y}_3(t) \omega(\dot{y}(t)) \\
\frac{\lambda_1^2}{2} (\dot{y}_2(t) + \dot{y}_3(t) - \dot{y}_1(t) \omega(\dot{y}(t))) \\
\frac{\lambda_1^2}{2} (\dot{y}_2(t) + \dot{y}_3(t) - \dot{y}_1(t) \omega(\dot{y}(t)))
\end{bmatrix}^2 + \left[ \frac{d}{dt} \omega(\dot{y}(t)) - \frac{\lambda_1^2}{2} \omega(\dot{y}(t)) \right]^2.
\]
(22)

By the definition of \(k_1^\omega\), we get Proposition 4.

Definition 5. Let \(\gamma : [a, b] \rightarrow (E(1, 1), g_\chi(\lambda_1, \lambda_2))\) be a Euclidean \(C^2\)-smooth regular curve in the Riemann manifold \((E(1, 1), g_\chi(\lambda_1, \lambda_2))\), we define the intrinsic curvature \(k_1^\omega\) of \(\gamma\) at \(y(t)\) to be
\[
k_1^\omega = \lim_{L \rightarrow \infty} k_1^\omega,
\]
(23)
if the limit exists.

We introduce the following notation: for continuous functions \(f_1, f_2 : [0, +\infty) \rightarrow \mathbb{R}\),
\[
f_1(L) \sim f_2(L), \quad \text{as } L \rightarrow +\infty \iff \lim_{L \rightarrow \infty} \frac{f_1(L)}{f_2(L)} = 1.
\]
(24)

Proposition 6. Let \(\gamma : [a, b] \rightarrow (E(1, 1), g_\chi(\lambda_1, \lambda_2))\) be a Euclidean \(C^2\)-smooth regular curve in the Riemann manifold \((E(1, 1), g_\chi(\lambda_1, \lambda_2))\). Then, we have
\[
k_1^\omega = \sqrt{\frac{\lambda_1^2}{2 \lambda_2}} \frac{(-e^{\theta(t)} \dot{y}_2(t) + e^{\theta(t)} \dot{y}_2(t))^2 + \dot{y}_2(t)^2}{\omega(\dot{y}(t))}, \quad \text{if } \omega(\dot{y}(t)) \neq 0.
\]
\[ \kappa_\gamma^\alpha = \left\{ \left[ \frac{1}{\lambda_1 A_2} \tilde{y}_1(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (\tilde{y}_2 e^{\alpha t} + \tilde{y}_2 \tilde{y}_2 e^{\alpha t} - \tilde{y}_2 e^{\gamma t} + \tilde{y}_2 \tilde{y}_2 e^{\gamma t}) \right] \right\} \cdot \left\{ \left[ \frac{1}{\lambda_1 A_2} \tilde{y}_2(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (e^{-\gamma t} \tilde{y}_2(t) + \tilde{y}_2 e^{\gamma t}) \right] \right\}^{-2} \left\{ \left[ \frac{1}{\lambda_1 A_2} \tilde{y}_2(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (e^{-\gamma t} \tilde{y}_2(t) + \tilde{y}_2 e^{\gamma t}) \right] \right\}^{-3/2}, \]

if \( \omega(\tilde{y}(t)) = 0, \frac{d}{dt}(\omega(\tilde{y}(t))) = 0. \)

\[ \lim_{L \to \infty} \frac{\kappa_\gamma^\alpha}{\sqrt{L}} = \frac{\left\{ (d/dt)(\omega(\tilde{y}(t))) \right\}^2}{\left\{ (1/\lambda_1 A_2) \tilde{y}_2(t) \right\}^2 + \left\{ (\lambda_1 \sqrt{2}) (e^{-\gamma t} \tilde{y}_2(t) + \tilde{y}_2 e^{\gamma t}) \right\}^2}, \]

if \( \omega(\tilde{y}(t)) = 0, \frac{d}{dt}(\omega(\tilde{y}(t))) \neq 0. \)

(25)

Proof. When \( \omega(\tilde{y}(t)) \neq 0, \) we have

\[ \left( \nabla_{\tilde{y}}^L \tilde{y}, \nabla_{\nabla_{\tilde{y}}^L \tilde{y}} ^L \right) = \frac{\lambda_1^2 A_2}{2} (e^{-\gamma t} \tilde{y}_1(t) + \tilde{y}_2 e^{\gamma t})^2 + \left( \tilde{y}_2(t) \right)^2 \]

\[ \left[ \omega(\tilde{y}(t)) \right]^2 \sqrt{L} \]

as \( L \to +\infty, \)

\[ \left\| \tilde{y} \right\|^2 \sim L(\omega(\tilde{y}(t)))^2, \]

\[ \left\| \nabla_{\nabla_{\tilde{y}}^L \tilde{y}} ^L \tilde{y} \right\|^2 \sim O(L^2) \]

as \( L \to +\infty. \)

(26)

Therefore, we have

\[ \left\| \nabla_{\nabla_{\tilde{y}}^L \tilde{y}} ^L \tilde{y} \right\| \left\| \tilde{y} \right\|^2 \left\| \omega(\tilde{y}(t)) \right\|^2 \]

\[ \sim \frac{\lambda_1^2 A_2}{2} (e^{-\gamma t} \tilde{y}_1(t) + \tilde{y}_2 e^{\gamma t})^2 + \left( \tilde{y}_2(t) \right)^2, \]

as \( L \to +\infty, \)

\[ \left\| \nabla_{\nabla_{\tilde{y}}^L \tilde{y}} ^L \tilde{y} \right\|^2 \left\| \tilde{y} \right\|^2 \left\| \omega(\tilde{y}(t)) \right\|^2 \]

\[ \sim 0, \]

as \( L \to +\infty. \)

(27)

If \( \omega(\tilde{y}(t)) \neq 0, \) by (14), we have

\[ \kappa_\gamma^\alpha = \sqrt{\frac{\lambda_1^2 A_2}{2} (e^{-\gamma t} \tilde{y}_1(t) + \tilde{y}_2 e^{\gamma t})^2 + \left( \tilde{y}_2(t) \right)^2}{\left\| \omega(\tilde{y}(t)) \right\|^2}. \]

(28)

By (16) and \((d/dt)(\omega(\tilde{y}(t))) = 0, \) we have

\[ \kappa_\gamma^\alpha = \left\{ \left[ \frac{1}{\lambda_1 A_2} \tilde{y}_1(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (\tilde{y}_2 e^{\alpha t} + \tilde{y}_2 \tilde{y}_2 e^{\alpha t} - \tilde{y}_2 e^{\gamma t} + \tilde{y}_2 \tilde{y}_2 e^{\gamma t}) \right] \right\} \cdot \left\{ \left[ \frac{1}{\lambda_1 A_2} \tilde{y}_2(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (e^{-\gamma t} \tilde{y}_2(t) + \tilde{y}_2 e^{\gamma t}) \right] \right\}^{-2} \left\{ \left[ \frac{1}{\lambda_1 A_2} \tilde{y}_2(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (e^{-\gamma t} \tilde{y}_2(t) + \tilde{y}_2 e^{\gamma t}) \right] \right\}^{-3/2}. \]

(29)

When \( \omega(\tilde{y}(t)) = 0 \) and \((d/dt)(\omega(\tilde{y}(t))) \neq 0, \) we have

\[ \left\| \nabla_{\nabla_{\tilde{y}}^L \tilde{y}} ^L \tilde{y} \right\|^2 \sim L \left[ \frac{d}{dt}(\omega(\tilde{y}(t))) \right]^2, \]

as \( L \to +\infty. \)

(30)

If \( \omega(\tilde{y}(t)) = 0 \) and \((d/dt)(\omega(\tilde{y}(t))) \neq 0, \) by (14), we get

\[ \lim_{L \to \infty} \frac{\kappa_\gamma^\alpha}{\sqrt{L}} = \frac{\left\{ (d/dt)(\omega(\tilde{y}(t))) \right\}^2}{\left\{ (1/\lambda_1 A_2) \tilde{y}_2(t) \right\}^2 + \left\{ \lambda_1 \sqrt{2} (e^{-\gamma t} \tilde{y}_2(t) + \tilde{y}_2 e^{\gamma t}) \right\}^2}. \]

(31)

3. The Sub-Riemannian Limit of Geodesic Curvature of Curves on Surfaces in \((E(1, 1), g_L(\lambda_1, \lambda_2))\)

In this section, we will compute the sub-Riemannian limit of the geodesic curvature of curves on surfaces in \((E(1, 1), g_L(\lambda_1, \lambda_2))\). We will say that a surface \( \Sigma \subset (E(1, 1), g_L(\lambda_1, \lambda_2)) \) is regular if \( \Sigma \) is a Euclidean \( C^2 \)-smooth compact and oriented surface. In particular, we will assume that there exists a Euclidean \( C^2 \)-smooth function \( u : E(1, 1) \to \mathbb{R} \) such that

\[ \Sigma = \{ (x, y, z) \in E(1, 1) : u(x, y, z) = 0 \}, \]

and \( u_\alpha \partial_\alpha + u_\beta \partial_\beta + u_\gamma \partial_\gamma \neq 0. \) Let \( V_{H^L} = X_1(u)X_1 + X_3(u)X_2. \) A point \( (x, y, z) \in \Sigma \) is called characteristic if \( V_{H^L}u(x, y, z) = 0. \) We define the characteristic set as follows:

\[ C(\Sigma) = \{ (x, y, z) \in \Sigma \mid V_{H^L}u(x, y, z) = 0 \}. \]

(32)

(33)

Note that the computations in the present paper will be local and away from characteristic points of \( \Sigma. \) We define \( p \)
\[ \begin{align*}
\rho &= \sqrt{p^2 + q^2}, \\
I_k &= \sqrt{p^2 + q^2 + r^2}, \\
\rho_1 &= \frac{\rho}{I_k}, \\
Q &= \frac{q}{I_k}, \\
\bar{r}_k &= \frac{r}{I_k}.
\end{align*} \]

In particular, \( \rho^2 + Q^2 = 1 \). We remark that these functions are well defined at every noncharacteristic point. Let

\[ \begin{align*}
\nu_k &= \rho_1 X_1 + Q X_2 + \bar{r}_k \tilde{X}_3, \\
e_1 &= Q X_1 - \rho_1 X_2, \\
e_2 &= \bar{r}_k \rho X_1 + \rho_1 Q X_2 - \frac{1}{I_k} \tilde{X}_3,
\end{align*} \]

then \( \nu_k \) is the Riemannian unit normal vector to \( \Sigma \), and \( e_1, e_2 \) are the orthonormal basis of \( \Sigma \). On \( T \Sigma \), we define a linear transformation \( J_L : T \Sigma \rightarrow T \Sigma \) such that

\[ \begin{align*}
J_L(e_1) &= e_2, \\
J_L(e_2) &= -e_1.
\end{align*} \]

For every \( U, V \in T \Sigma \), we define \( \nabla_{U, L} V = \pi \nabla_U V \), where \( \pi : TE(1,1) \rightarrow T \Sigma \) is the projection. Then, \( \nabla_{U, L} \) is the Levi-Civita connection on \( \Sigma \) with respect to the metric \( g_L(\lambda_1, \lambda_2) \). By (21), (34), and

\[ \nabla_{U, L} \nu = \left< \nabla_{U, L} \nu_1, e_1 \right>_L + \left< \nabla_{U, L} \nu_2, e_2 \right>_L, \]

we have

\[ \begin{align*}
\nabla_{U, L} \nu &= \left\{ \begin{array}{l}
\left[ \frac{1}{\lambda_1 \lambda_2} \gamma(t) + \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( -e^{\gamma(t)} \gamma_1(t) + e^{\gamma(t)} \gamma_2(t) \gamma(t) \right) \right] e_1 \\
- \rho_1 \left[ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( \gamma_1(t) + \gamma_2(t) \gamma(t) \right) \right] e_2 \\
+ \frac{1}{\lambda_1 \lambda_2} r L \left[ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( -e^{\gamma(t)} \gamma_1(t) + e^{\gamma(t)} \gamma_2(t) \gamma(t) \right) \right] e_1 \\
+ \rho_1 \left[ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( \gamma_1(t) + \gamma_2(t) \gamma(t) \right) \right] e_2 \\
- \frac{1}{I_k} L \left[ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( -e^{\gamma(t)} \gamma_1(t) + e^{\gamma(t)} \gamma_2(t) \gamma(t) \right) \right] e_1 \\
+ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( \gamma_1(t) + \gamma_2(t) \gamma(t) \right) e_2.
\end{array} \right. 
\end{align*} \]

Moreover, if \( \omega(\nu(t)) = 0 \), then

\[ \begin{align*}
\nabla_{U, L} \omega &= \left\{ \begin{array}{l}
\left[ \frac{1}{\lambda_1 \lambda_2} \gamma(t) + \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( -e^{\gamma(t)} \gamma_1(t) + e^{\gamma(t)} \gamma_2(t) \gamma(t) \right) \right] e_1 \\
- \rho_1 \left[ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( \gamma_1(t) + \gamma_2(t) \gamma(t) \right) \right] e_2 \\
+ \frac{1}{\lambda_1 \lambda_2} r L \left[ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( -e^{\gamma(t)} \gamma_1(t) + e^{\gamma(t)} \gamma_2(t) \gamma(t) \right) \right] e_1 \\
+ \rho_1 \left[ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( \gamma_1(t) + \gamma_2(t) \gamma(t) \right) \right] e_2 \\
- \frac{1}{I_k} L \left[ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( -e^{\gamma(t)} \gamma_1(t) + e^{\gamma(t)} \gamma_2(t) \gamma(t) \right) \right] e_1 \\
+ \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( \gamma_1(t) + \gamma_2(t) \gamma(t) \right) e_2.
\end{array} \right. 
\end{align*} \]

Definition 7. Let \( \Sigma \subset (E(1,1), g_L(\lambda_1, \lambda_2)) \) be a regular surface, \( \gamma : [a, b] \rightarrow \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. The geodesic curvature \( \kappa_{\gamma, \Sigma}^L \) of \( \gamma \) at \( \gamma(t) \) is defined as follows:

\[ \kappa_{\gamma, \Sigma}^L = \sqrt{\frac{\left\| \nabla_{U, L} \gamma \right\|^2_{\Sigma, L}}{\left\| \gamma \right\|^2_{\Sigma, L}}} - \frac{\left\langle \nabla_{U, L} \gamma, \gamma \right\rangle^2_{\Sigma, L}}{\left\| \gamma \right\|^4_{\Sigma, L}}. \]

Definition 8. Let \( \Sigma \subset (E(1,1), g_L(\lambda_1, \lambda_2)) \) be a regular surface, \( \gamma : [a, b] \rightarrow \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. We define the intrinsic geodesic curvature \( \kappa_{\gamma, \Sigma}^{\infty} \) of \( \gamma \) at \( \gamma(t) \) to be

\[ \kappa_{\gamma, \Sigma}^{\infty} = \lim_{L \to \infty} \kappa_{\gamma, \Sigma}^L, \]

if the limit exists.

**Proposition 9.** Let \( \Sigma \subset (E(1,1), g_L(\lambda_1, \lambda_2)) \) be a regular surface and \( \gamma : [a, b] \rightarrow \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. Then, we have

\[ \kappa_{\gamma, \Sigma}^{\infty} = 0, \quad \text{if} \ \omega(\nu(t)) = 0, \quad \frac{d}{dt}(\omega(\nu(t))) = 0, \]

\[ \lim_{L \to \infty} \left( \frac{[\omega(\nu(t))]^{\infty}}{\left\| \gamma \right\|_{\Sigma, L}^2} \right) = 0, \quad \text{if} \ \omega(\nu(t)) \neq 0. \]

**Proof.** By (17) and \( \nu \in T \Sigma \), we have

\[ \gamma(t) = \frac{1}{\lambda_1 \lambda_2} \gamma_1(t) X_1 + \frac{\lambda_1^2 \lambda_2^2}{2} \gamma^2 \left( \gamma_1(t) + \gamma_2(t) \gamma(t) \right) X_2 + \omega(\nu(t)) X_3. \]
On the other hand,
\[
\dot{y}(t) = ae_1 + be_2 = a(\ddot{q}X_1 - \ddot{p}X_2) + b\left(\ddot{r}_1\dot{p}X_1 + \ddot{r}_2\dot{q}X_2 - \frac{l}{L}\dot{X}_3\right) \\
= (a\ddot{q} + b\ddot{r}_1\ddot{p})X_1 + (-a\ddot{p} + b\ddot{r}_1\ddot{q})X_2 - \frac{bl}{L}L^{-1/2}\dot{X}_3.
\]
Comparing the above equations, we get
\[
\begin{align*}
a\ddot{q} + b\ddot{r}_1\ddot{p} &= \frac{1}{\lambda_1\lambda_2}\ddot{y}_3(t), \\
a\ddot{p} - b\ddot{r}_1\ddot{q} &= \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}), \\
bl L^{-1/2} &= \omega(\dot{y}(t)),
\end{align*}
\]
from which
\[
\begin{align*}
a &= \frac{1}{\lambda_1\lambda_2}\ddot{y}_3(t) - \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}), \\
b &= -\frac{l}{T}L^{1/2}\omega(\dot{y}(t)).
\end{align*}
\]
Similarly, we get that when \(\omega(\dot{y}(t)) \neq 0\),
\[
\|\hat{y}\|_{L^2 L} = \left\{\frac{1}{\lambda_1\lambda_2} \ddot{y}_3(t)\ddot{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}\ddot{p}) + L\left(\frac{l}{T}\omega(\dot{y}(t))\right)\right\}^{1/2} - L^{1/2}\|\omega(\dot{y}(t))\|, \quad \text{as } L \to +\infty.
\]
By (37) and (46), we have
\[
\begin{align*}
\langle \nabla^L_{\gamma}, \dot{y}, \ddot{y} \rangle_{L^2 L} &\sim \left\{\frac{1}{\lambda_1\lambda_2} \ddot{y}_3(t)\ddot{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}\ddot{p}) + L\left(\frac{l}{T}\omega(\dot{y}(t))\right)\right\}^{1/2} L^{-1/2}(\omega(\dot{y}(t))) \nonumber \\
&+ \frac{d}{dt}(\omega(\dot{y}(t)))\omega(\dot{y}(t))L - N_0L, \quad \text{as } L \to +\infty,
\end{align*}
\]
where \(N_0\) does not depend on \(L\). By (39), we get
\[
k_{\omega}^{\infty} = \lim_{L \to +\infty} k_{\omega}^{L} = \frac{\sqrt{\left(\frac{\lambda_1\lambda_2\sqrt{2}}{2}\right)^2(-e^{\gamma_1(t)} + e^{\gamma_2(t)})^2 + \ddot{p}^2(\ddot{y}(t))^2}}{\omega(\dot{y}(t))},
\]
if \(\omega(\dot{y}(t)) \neq 0\).

When \(\omega(\dot{y}(t)) = 0\) and \(d\omega(\dot{y}(t))/dt = 0\), we have
\[
\|\hat{y}\|_{L^2 L} = \left\{\frac{1}{\lambda_1\lambda_2} \ddot{y}_3(t)\ddot{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}\ddot{p}) + \ddot{p}(\ddot{y}(t))^2\right\}^{1/2} - \left\{\frac{1}{\lambda_1\lambda_2} \ddot{y}_3(t)\ddot{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}\ddot{p}) + \ddot{p}(\ddot{y}(t))^2\right\}^{1/2},
\]
as \(L \to +\infty\) and
\[
\|\hat{y}\|_{L^2 L} = \left\{\frac{1}{\lambda_1\lambda_2} \ddot{y}_3(t)\ddot{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}\ddot{p}) + \ddot{p}(\ddot{y}(t))^2\right\}^{1/2},
\]
and
\[
\langle \nabla^L_{\gamma}, \dot{y}, \ddot{y} \rangle_{L^2 L} = \left\{\frac{1}{\lambda_1\lambda_2} \ddot{y}_3(t)\ddot{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}\ddot{p}) + \ddot{p}(\ddot{y}(t))^2\right\}^{1/2} - \left\{\frac{1}{\lambda_1\lambda_2} \ddot{y}_3(t)\ddot{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{\gamma_1(t)} + e^{\gamma_2(t)}\ddot{p}) + \ddot{p}(\ddot{y}(t))^2\right\}^{1/2} = AB.
\]
By (51), (52), (53), and (39), we get
\[
\kappa^\text{geo}_{x, y} = \sqrt{\frac{B^2}{A} - \frac{A^2B^2}{A^6}} = 0.
\]
(55)
When \(\omega(\dot{y}(t)) = 0\) and \((d/dt)\omega(\dot{y}(t)) \neq 0\), we have
\[
\left\| \nabla^\Sigma_L \dot{y} \right\|_{\Sigma, L}^2 \sim \left[ \frac{d}{dt} \omega(\dot{y}(t)) \right]^2,
\]
as \(L \to +\infty,
\[
\left\| \dot{y} \right\|_{\Sigma, L} \sim \left[ \frac{\sqrt{2}}{\lambda_1} \lambda_2 \frac{1}{2} (\dot{y}_1(t) + e^{\gamma} \dot{y}_2(t)) \right],
\]
\[
\left\langle \nabla^\Sigma_L \dot{y}, \dot{y} \right\rangle_{\Sigma, L} = O(1).
\]
(56)
So, we get
\[
\lim_{L \to +\infty} \frac{k^L_{x, y}}{\sqrt{L}} = \frac{|(d/dt)\omega(\dot{y}(t))|^2}{\left(1/(\lambda_1 \lambda_2) \dot{y}_2(t) \dot{y}_1(t) \dot{y}_1(t) + \dot{y}_2(t) \dot{y}_2(t) \right)^2}.
\]
(57)
\[
\begin{align*}
\left\| \nabla^\Sigma_L \dot{y} \right\|_{\Sigma, L}^2 & = \left[ \frac{\sqrt{2}}{\lambda_1} \lambda_2 \frac{1}{2} (\dot{y}_1(t) + e^{\gamma} \dot{y}_2(t)) \right]^2, \\
\left\langle \nabla^\Sigma_L \dot{y}, \dot{y} \right\rangle_{\Sigma, L} & = \left\langle \nabla^\Sigma_L \dot{y}, J_L(\dot{y}) \right\rangle_{\Sigma, L},
\end{align*}
\]
(58)
where \(J_L\) is defined by (35).

**Definition 10.** Let \(\Sigma \subset (E(1, 1), g_L(\lambda_1, \lambda_2))\) be a regular surface. Let \(\gamma : [a, b] \to \Sigma\) be a Euclidean \(C^2\)-smooth regular curve. The signed geodesic curvature \(k^L_{x, y}\) of \(\gamma\) at \(y(t)\) is defined as follows:
\[
k^L_{x, y} = \frac{\left\langle \nabla^\Sigma_L \dot{y}, J_L(\dot{y}) \right\rangle_{\Sigma, L}}{\left\| \dot{y} \right\|_{\Sigma, L}^2},
\]
(59)
if the limit exists.

**Proposition 12.** Let \(\Sigma \subset (E(1, 1), g_L(\lambda_1, \lambda_2))\) be a regular surface. Let \(\gamma : [a, b] \to \Sigma\) be a Euclidean \(C^2\)-smooth regular curve. Then, we have
\[
k^\text{geo}_{x, y} = \frac{\dot{y}_1(t) + e^{\gamma} \dot{y}_2(t)}{|\omega(\dot{y}(t))|}, \quad \text{if } \omega(\dot{y}(t)) \neq 0,
\]
\[
k^\text{geo}_{x, y} = 0, \quad \text{if } \omega(\dot{y}(t)) = 0, \quad \frac{d}{dt} (\omega(\dot{y}(t))) = 0,
\]
if \(\omega(\dot{y}(t)) 
eq 0\),
\[
k^\text{geo}_{x, y} = 0, \quad \text{if } \omega(\dot{y}(t)) = 0, \quad \frac{d}{dt} (\omega(\dot{y}(t))) = 0,
\]
if \(\omega(\dot{y}(t)) = 0\).
When \( \omega(\gamma(t)) = 0 \) and \( (d/dt)\omega(\gamma(t)) = 0 \), we get
\[
\langle \nabla^L \gamma, J_L\gamma \rangle_{L^2} = \left[ \frac{1}{\lambda t} \gamma(t) \hat{q} - \frac{1}{\lambda t} \sqrt{2} (-e^{-\gamma t} \hat{y}_1(t) + e^\gamma \gamma(t)) \hat{p} \right]
\cdot \left\{ \hat{t} \hat{p} \left[ \frac{1}{\lambda t} \gamma(t) \right] + \hat{t} \hat{q} \left[ \frac{1}{\lambda t} \sqrt{2} (\gamma e^{\gamma t} \hat{y}_2(t) + \gamma \gamma(t)) e^{-\gamma t} + \hat{y}_2(t) \hat{e}_2(t) \right] + \frac{\hat{t}}{\lambda t} \hat{L} \left[ \frac{1}{\gamma_1} \gamma(t) + e^{\gamma t} \gamma(t) \hat{y}_2(t) \right] \right\}
- N(t) \gamma(t), \quad \text{as } L \rightarrow +\infty.
\]

where \( N(t) \) does not depend on \( L \). So, \( \kappa_L^{\omega,\gamma} = 0 \). When \( \omega(\gamma(t)) = 0 \) and \( (d/dt)\omega(\gamma(t)) \neq 0 \), we have
\[
\langle \nabla^L \gamma, J_L\gamma \rangle_{L^2} = L^{-1/2} \left[ -\frac{1}{\lambda t} \gamma(t) \hat{q} + \frac{1}{\lambda t} \sqrt{2} (-e^{-\gamma t} \hat{y}_1(t) + e^\gamma \gamma(t)) \hat{p} \right] \frac{d}{dt} (\omega(\gamma(t))), \quad \text{as } L \rightarrow +\infty.
\]

We get
\[
\kappa_L^{\omega,\gamma} = \lim_{L \rightarrow +\infty} \frac{\langle \nabla^L \gamma, J_L\gamma \rangle_{L^2}}{L^{1/2}} = \lim_{L \rightarrow +\infty} \frac{-\langle \nabla^L \gamma, J_L\gamma \rangle_{L^2}}{L^{1/2}} = \frac{\langle \nabla^L \gamma, J_L\gamma \rangle_{L^2}}{L^{1/2}}.
\]

4. The Sub-Riemannian Limit of the Riemannian Gaussian Curvature of Surfaces in \((E(1, 1), g_L(\lambda_1, \lambda_2))\)

In the following, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in \((E(1, 1), g_L(\lambda_1, \lambda_2))\). We define the second fundamental form \( II_L^L \) of the embedding of \( \Sigma \) into \((E(1, 1), g_L(\lambda_1, \lambda_2))\) by
\[
II_L^L = \begin{pmatrix}
\langle \nabla^L e_1, v_L, e_1 \rangle_L & \langle \nabla^L e_1, v_L, e_2 \rangle_L \\
\langle \nabla^L e_2, v_L, e_1 \rangle_L & \langle \nabla^L e_2, v_L, e_2 \rangle_L
\end{pmatrix}.
\]

Theorem 13. The second fundamental form \( II_L^L \) of the embedding of \( \Sigma \) into \((E(1, 1), g_L(\lambda_1, \lambda_2))\) is given by
\[
II_L^L = \begin{pmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{pmatrix},
\]

where
\[
h_{11} = \frac{l}{L} (X_1(\hat{p}) + X_2(\hat{q})) - \lambda_1^2 \hat{r} \hat{p} \hat{q} L^{-1/2},
\]
\[
h_{12} = \frac{l}{L} \left( 1 + \langle e_1, \nabla \gamma(\hat{r}) \rangle_L \right) - \sqrt{\frac{\lambda_1}{2}} \lambda_1 \left( \hat{q}^2 - \hat{p}^2 \right) + \frac{l}{L} \lambda_1^3 \left( \hat{q}^2 - \hat{p}^2 \right),
\]
\[
h_{21} = -\frac{l}{L} \left( 1 + \langle e_1, \nabla \gamma(\hat{r}) \rangle_L \right) + \lambda_1 \hat{r} \hat{p} \hat{q} \lambda_1 L^{-1/2}.
\]

Proof. Since \( \langle e_1, v_L \rangle_L = 0 \) and \( \langle e_2, v_L \rangle_L = 0 \), we have
\[
\langle \nabla^L e_1, v_L, e_1 \rangle_L = -\langle \nabla^L e_1, v_L, e_2 \rangle_L,
\]
\[
\langle \nabla^L e_1, v_L, e_2 \rangle_L = -\langle \nabla^L e_2, v_L, e_2 \rangle_L.
\]

Using the definition of the connection, the identities in (8) and grouping terms, we have
\[
\langle \nabla^L e_1, v_L, e_1 \rangle_L = \langle \nabla^L e_1, v_L, e_2 \rangle_L = -\langle \nabla^L e_1, v_L, e_2 \rangle_L
\]
\[
\langle \nabla^L e_1, v_L, e_2 \rangle_L = -\langle \nabla^L e_2, v_L, e_2 \rangle_L
\]

since \( \hat{p}^2 + \hat{q}^2 = 1 \), we have \( \hat{p} X_2 \hat{p} + \hat{q} X_2 \hat{q} = 0 \), \( i = 1, 2, 3 \). We have
\[
h_{11} = \frac{l}{L} (X_1(\hat{p}) + X_2(\hat{q})) - \lambda_1^2 \hat{r} \hat{p} \hat{q} L^{-1/2}
\]
\[
h_{12} = \frac{l}{L} (X_1(\hat{p}) + X_2(\hat{q})) - \lambda_1^2 \hat{r} \hat{p} \hat{q} L^{-1/2}
\]
To compute \( h_{12} \) and \( h_{21} \), using the definition of the connection, we obtain
The identities
\[ h_{21} = h_{12} = -\left( \langle \nabla^L e_2, v_L \rangle \right)_L = \frac{1}{L} \left( c_1 + \nabla^r \hat{H}(\xi_L) \right)_L - \frac{L}{2} \lambda_2^2 + \frac{\lambda_2^2}{2\sqrt{L}} \left( q_L^2 - p_L^2 \right) + \frac{r_L^2 \lambda_1^2}{2\sqrt{L}} \left( q^2 - p^2 \right). \] (78)

Since \( \langle \nabla^L e_1, v_L \rangle = -\langle \nabla^L e_2, v_L \rangle \), using the definition of connection, the identities in (9) and grouping terms, we have
\[ \langle \nabla^L e_1, v_L \rangle = \langle \nabla^L e_2, v_L \rangle = \frac{1}{L} \left( c_1 + \nabla^r \hat{H}(\xi_L) \right)_L - \frac{L}{2} \lambda_2^2 + \frac{\lambda_2^2}{2\sqrt{L}} \left( q_L^2 - p_L^2 \right) + \frac{r_L^2 \lambda_1^2}{2\sqrt{L}} \left( q^2 - p^2 \right). \] (79)

Taking the inner product with \( v_L \) yields
\[ \langle \nabla^L e_2, v_L \rangle = \frac{1}{L} \left( c_1 + \nabla^r \hat{H}(\xi_L) \right)_L - \frac{L}{2} \lambda_2^2 + \frac{\lambda_2^2}{2\sqrt{L}} \left( q_L^2 - p_L^2 \right) + \frac{r_L^2 \lambda_1^2}{2\sqrt{L}} \left( q^2 - p^2 \right). \] (80)

To simplify this, first use the product rule for the terms involving \( X_i(\partial^r\xi_L) \) and \( X_i(\partial^r\hat{q}) \), together with the identities \( \hat{p} X_i \hat{p} + \hat{q} X_i \hat{q} = 0 \), \( \nabla^r \hat{H}(\xi_L) \), and \( p^2 + q^2 = 1 \). Under these simplifications, terms involving \( X_i(\hat{p}) \) and \( X_i(\hat{q}) \) cancel and one is left with terms involving components
Proposition 14. Away from characteristic point, the horizontal mean curvature \( \mathcal{H}_L \) of \( \Sigma \) is defined by

\[
\mathcal{H}_L = \text{tr}(II^L) = \frac{1}{L} \langle X_1(\bar{p}) + X_2(\bar{q}) \rangle - \lambda_1^2 \bar{r}_L \bar{p}_L \bar{q}_L - \lambda_1^2 \bar{r}_L \bar{p}_L \bar{q}_L - \lambda_1^2 \bar{r}_L \bar{p}_L \bar{q}_L.
\]

(83)

Define the curvature of a connection \( \nabla \) by

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [X, Y]Z.
\]

(84)

Let

\[
\mathcal{H}^{\Sigma L}(e_1, e_2) = -\langle R^{\Sigma L}(e_1, e_2) e_1, e_2 \rangle_{\Sigma L},
\]

(85)

\[
\mathcal{H}[L](e_1, e_2) = -\langle R^L(e_1, e_2) e_1, e_2 \rangle_{L}.
\]

By the Gauss equation, we have

\[
\mathcal{H}^{\Sigma L}(e_1, e_2) = \mathcal{H}[L](e_1, e_2) + \det (II^L).
\]

(86)

Proposition 15. Let \((E(1, 1), g_L(\lambda_1, \lambda_2))\) be the group of rigid motions of the Minkowski plane with a general left-invariant metric. Then, we have

\[
R^L(X_1, X_2)X_1 = -\frac{\lambda_1^4 + 2\lambda_1^2\lambda_2^2L + 3\lambda_2^4L^2}{4L} X_2,
\]

(89)

\[
R^L(X_1, X_2)X_2 = \frac{\lambda_1^4 - 2\lambda_1^2\lambda_2^2L - 3\lambda_2^4L^2}{4L} X_1,
\]

\[
R^L(X_1, X_2)X_3 = 0,
\]

\[
R^L(X_1, X_3)X_1 = \frac{3\lambda_1^4 + 2\lambda_1^2\lambda_2^2L - \lambda_2^4L^2}{4L} X_3,
\]

\[
R^L(X_1, X_3)X_3 = 0,
\]

\[
R^L(X_2, X_3)X_1 = \frac{-3\lambda_1^4 - 2\lambda_1^2\lambda_2^2L + \lambda_2^4L^2}{4L} X_1,
\]

\[
R^L(X_2, X_3)X_3 = 0,
\]

\[
R^L(X_2, X_3)X_2 = \frac{-\lambda_1^4 - 2\lambda_1^2\lambda_2^2L - \lambda_2^4L^2}{4L} X_3,
\]

Proposition 16. Away from characteristic points, we have

\[
\mathcal{H}[\Sigma L](e_1, e_2) = -\langle e_1, \nabla_H \left( X_1(\bar{p}) \right) \rangle_{L} - \lambda_1^2 \frac{(X_1(\bar{p}))^2}{L}, \text{ as } L \to \infty.
\]

(90)
Proof. We compute
\[
R^1(e_1, e_2)e_1 = R^1 \left( \tilde{q} \tilde{X}_1 - \tilde{p} \tilde{X}_1, r_1 \tilde{p} \tilde{X}_1 + r_1 \tilde{q} \tilde{X}_1 - \frac{L}{1_{\tilde{L}}} \tilde{X}_1 \right) \tilde{q} \tilde{X}_1 - \tilde{p} \tilde{X}_1 \\
= r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 + r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 - \frac{L}{1_{\tilde{L}}} R^0(X_1, X_1) X_1 \\
- r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 - r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 \\
+ \frac{L}{1_{\tilde{L}}} R^0(X_1, X_1) X_1 - r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 \\
- r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 + \frac{L}{1_{\tilde{L}}} R^0(X_1, X_1) X_1 \\
- r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 + r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 \\
+ \frac{L}{1_{\tilde{L}}} R^0(X_1, X_1) X_1 \\
= r_1 \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 - \tilde{p} \tilde{q} R^0(X_1, X_1) X_1 - \frac{L}{1_{\tilde{L}}} R^0(X_1, X_1) X_1 \\
- \frac{L}{1_{\tilde{L}}} R^0(X_1, X_1) X_1 \\
= - \frac{L}{1_{\tilde{L}}} R^0(X_1, X_1) X_1 \\
(91)
\]

\[
X^2(e_1, e_2) = - \langle R^2(e_1, e_2), e_1 \rangle L = - \tilde{p} \tilde{q} \left( \frac{L \tilde{X}_1^2}{4} \right) \\
- \left( \frac{L}{1_{\tilde{L}}} \right)^2 \left( \frac{L \tilde{X}_1^2}{4} \right) - \frac{L \tilde{X}_1^2}{4} + \frac{3 \tilde{X}_1^2}{4L} \\
+ \left( \frac{L}{1_{\tilde{L}}} \right)^2 \left( \frac{L \tilde{X}_1^2}{4} \right) + \frac{L \tilde{X}_1^2}{4} + \frac{3 \tilde{X}_1^2}{4L} \\
- \frac{3 \tilde{X}_1^2}{4L} + \frac{3 \tilde{X}_1^2}{4L} \\
= - \frac{L}{1_{\tilde{L}}} \tilde{X}_1^2 - \frac{L \tilde{X}_1^2}{4} + \frac{3 \tilde{X}_1^2}{4L} \\
(92)
\]

By Theorem 13 and \( V_H(\tilde{r}_L) = L^{-1/2} V_H(X_1 u/\sqrt{V_H u}) + O(L^{-1}) \) as \( L \to \infty \), we get
\[
\det (II^L) = h_{11} h_{22} - h_{12}^2 = - \frac{L \tilde{X}_1^2}{4} - \frac{L \tilde{X}_1^2}{4} + \frac{3 \tilde{X}_1^2}{4L} + O(L^{-1/2}) \\
\to \frac{L \tilde{X}_1^2}{4} - \frac{L \tilde{X}_1^2}{4} + \frac{3 \tilde{X}_1^2}{4L} + O(L^{-1/2}) \\
as L \to \infty.
\]

(93)

5. A Gauss-Bonnet

**Theorem in** \((E(1, 1), g_L(\lambda_1, \lambda_2))\)

In this section, we will prove the Gauss-Bonnet theorem in the group of rigid motions of the Minkowski plane \((E(1, 1), g_L(\lambda_1, \lambda_2))\). Let us first consider the case of a regular curve \( \gamma: [a, b] \to (E(1, 1), g_L(\lambda_1, \lambda_2)) \), and we define the Riemannian length measure as follows:
\[
ds_L = ||\gamma|| dt.
\]

(94)

**Lemma 17.** Let \( \gamma: [a, b] \to (E(1, 1), g_L(\lambda_1, \lambda_2)) \) be a Euclidean \( C^2 \)-smooth curve. Let
\[
ds = |\omega(\dot{\gamma}(t))| dt,
\]
\[
ds = \frac{1}{2} \frac{1}{|\omega(\dot{\gamma}(t))|} \left( \frac{1}{|\lambda_1 \lambda_2 \dot{\gamma}(t)|^2} + \frac{\lambda_1 \lambda_2 \dot{\gamma}(t)^2}{2} \left( -e^{\theta \dot{\gamma}(t)} + e^{\theta \dot{\gamma}(t)} \right) \right)^2 dt.
\]

(95)

Then, we have
\[
\lim_{L \to \infty} \frac{1}{\sqrt{L}} \int_a^b ds_L = \int_a^b ds.
\]

(96)

When \( \omega(\dot{\gamma}(t)) \neq 0 \), we have
\[
\frac{1}{\sqrt{L}} ds_L = ds + dsL^{-1} + O(L^{-2}), \quad as L \to +\infty.
\]

(97)

When \( \omega(\dot{\gamma}(t)) = 0 \), we have
\[
\frac{1}{\sqrt{L}} ds_L = \frac{1}{\sqrt{L}} \left( \frac{1}{|\lambda_1 \lambda_2 \dot{\gamma}(t)|^2} + \frac{\lambda_1 \lambda_2 \dot{\gamma}(t)^2}{2} \left( -e^{\theta \dot{\gamma}(t)} + e^{\theta \dot{\gamma}(t)} \right) \right)^2 dt.
\]

(98)

**Proof.** We know that
\[
||\gamma(t)||_L = \frac{1}{|\lambda_1 \lambda_2 \dot{\gamma}(t)|^2} + \frac{\lambda_1 \lambda_2 \dot{\gamma}(t)^2}{2} \left( -e^{\theta \dot{\gamma}(t)} + e^{\theta \dot{\gamma}(t)} \right) + L(\omega(\dot{\gamma}(t)))^2.
\]

(99)

Similar to the proof of Lemma 6.1 in [4], we can prove
\[
\lim_{L \to \infty} \frac{1}{\sqrt{L}} \int_a^b ||\gamma(t)||_L dt = \lim_{L \to \infty} \frac{1}{\sqrt{L}} \int_a^b ||\gamma(t)||_L dt
\]
\[
= \int_a^b \frac{1}{\sqrt{L}} \left( \frac{1}{|\lambda_1 \lambda_2 \dot{\gamma}(t)|^2} + \frac{\lambda_1 \lambda_2 \dot{\gamma}(t)^2}{2} \left( -e^{\theta \dot{\gamma}(t)} + e^{\theta \dot{\gamma}(t)} \right) \right)^2 + L(\omega(\dot{\gamma}(t)))^2 dt
\]
\[
= \int_a^b \omega(\dot{\gamma}(t)) dt + \int_a^b ds.
\]

(100)

is desired. When \( \omega(\dot{\gamma}(t)) \neq 0 \), we have
\[
\frac{1}{\sqrt{L}} ds_L = \sqrt{L} \left( \frac{1}{|\lambda_1 \lambda_2 \dot{\gamma}(t)|^2} + \frac{\lambda_1 \lambda_2 \dot{\gamma}(t)^2}{2} \left( -e^{\theta \dot{\gamma}(t)} + e^{\theta \dot{\gamma}(t)} \right) \right) + \omega(\dot{\gamma}(t))^2 dt.
\]

(101)

Using the Taylor expansion, we can prove
\[
\frac{1}{\sqrt{L}} ds_L = ds + dsL^{-1} + O(L^{-2}), \quad as L \to +\infty.
\]

(102)
From the definition of $ds_L$ and $\omega(\dot{y}(t)) = 0$, we get

$$
\frac{1}{\sqrt{L}} d\sigma_L = \frac{1}{\sqrt{L}} \left[ \frac{1}{\lambda_1} \dot{y}_1(t) \right]^2 + \frac{\lambda_1 \sqrt{2}}{2} \left( -e^{\eta} \dot{y}_1(t) + e^{\eta} \dot{y}_2(t) \right)^2 \, dt.
$$

(103)

\box

Proposition 18. Let $\Sigma \subset (E(1, 1), g_L(\lambda_1, \lambda_2))$ be a Euclidean $C^2$-smooth surface and $\Sigma = \{ u = 0 \}$ and $d\sigma_{\Sigma,L}$ denote the surface measure on $\Sigma$ with respect to the Riemannian metric $g_L(\lambda_1, \lambda_2)$. Let

$$
d\sigma_\Sigma = (\rho \omega_2 - \bar{q} \omega_1) \wedge \omega, \quad d\bar{\sigma}_\Sigma = X_{\bar{L}}^{-1} \omega_1 \wedge \omega_2 - \frac{(X_{\bar{L}} u)^2}{2L^2} (\rho \omega_2 - \bar{q} \omega_1) \wedge \omega.
$$

Then, we have

$$
\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = d\sigma_\Sigma + d\bar{\sigma}_\Sigma L^{-1} + O(L^{-2}), \quad \text{as } L \to +\infty.
$$

(104)

(105)

If $\Sigma = f(D)$ with $f = (u_1, u_2) = (f_1, f_2, f_3) : D \subset \mathbb{R}^2 \to (E(1, 1), g_L(\lambda_1, \lambda_2))$, then

$$
\lim_{L \to \infty} \frac{1}{\sqrt{L}} \int_D \left( \sigma_1^2 + \sigma_2^2 \right)^{1/2} du_1 du_2 = \int_D \left( \sigma_1^2 + \sigma_2^2 \right)^{1/2} du_1 du_2.
$$

(106)

where

$$
\sigma_1 = (f_{1,1}(f_2) + f_1(1, f_2), f_1(1, f_2), f_2(1, f_2) - f_2(1, f_2)),
$$

$$
\sigma_2 = \left\{ \left( f_{1,1}(f_2) + f_1(1, f_2), f_{1,1}(f_2) + f_1(1, f_2) \right) \frac{\sqrt{2}}{2} e^{z} \frac{1}{\lambda_1} + \left( f_{1,1}(f_2) + f_1(1, f_2) \right) \frac{\sqrt{2}}{2} e^{z} \frac{1}{\lambda_1} \right\}^2.
$$

(107)

Proof. It is well known that $g_l(X_{\lambda_1}, \cdot) = \omega_1$, $g_l(X_{\lambda_2}, \cdot) = \omega_2$, and $g_l(X_{\lambda_3}, \cdot) = L \omega$. We define $e_1^\ast = g_L(e_1, \cdot)$ and $e_2^\ast = g_L(e_2, \cdot)$; then, we have

$$
e_1^\ast = \bar{q} \omega_1 - \bar{\rho}_L \omega_2, \quad e_2^\ast = \bar{r}_L \bar{\rho}_L \omega_1 + \bar{r}_L \bar{q} \omega_2 - \frac{L}{L} L^{1/2} \omega.
$$

(108)

Therefore, we have

$$
\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = \frac{1}{\sqrt{L}} e_1^\ast \wedge e_2^\ast = \frac{L}{L} (\bar{q} \omega_2 - \bar{q} \omega_1) \wedge \omega + \frac{1}{\sqrt{L}} \bar{r}_L \omega_1 \wedge \omega_2.
$$

(109)

Recalling, we know that

$$
\bar{r}_L = \frac{(X_{\bar{L}} u) L^{-1/2}}{\sqrt{p^2 + q^2 + L^{-1}(X_{\bar{L}} u)^2}},
$$

(110)

and with the Taylor expansion, we have

$$
\frac{1}{\bar{L}} = 1 - \frac{1}{2L} (X_{\bar{L}} u) L^{-1} + O(L^{-2}), \quad \text{as } L \to +\infty,
$$

(111)

and we get (104). By (4), we have

$$
f_{u_1} = (f_3, u_1) \partial_x + (f_2, u_1) \partial_y + (f_1, u_1) \partial_z
$$

$$
= (f_3, u_1) \frac{1}{\lambda_1} X_1 + \left( (f_3, u_1) \left[ -\frac{\sqrt{2}}{2} e^{z} \frac{1}{\lambda_1} \right] + (f_2, u_1) \frac{\sqrt{2}}{2} e^{z} \frac{1}{\lambda_1} \right) X_2
$$

$$
+ \sqrt{L} \left( (f_3, u_1) \left[ -\frac{\sqrt{2}}{2} e^{z} \frac{1}{\lambda_1} \right] + (f_2, u_1) \frac{\sqrt{2}}{2} e^{z} \frac{1}{\lambda_1} \right) \bar{X}_3,
$$

(112)

Let

$$
\bar{v}_L = \begin{bmatrix} X_1 & X_2 & \bar{X}_3 \end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\frac{X_1}{u_1} \\
\frac{X_2}{u_1} \\
\frac{\bar{X}_3}{u_1}
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
f_{u_1} \\
f_{u_1} \\
f_{u_1}
\end{array}
\end{bmatrix}
$$

$$
= \sqrt{L} \left( (f_3, u_1) (f_2, u_1) \lambda_2 - (f_2, u_1) (f_1, u_1) \lambda_2 \right) X_1
$$

$$
+ \sqrt{L} \left[ (f_3, u_1) (f_2, u_1) \lambda_2 - (f_2, u_1) (f_1, u_1) \lambda_2 \right] X_2
$$

$$
+ \sqrt{L} \left[ (f_3, u_1) (f_2, u_1) \lambda_2 - (f_2, u_1) (f_1, u_1) \lambda_2 \right] \bar{X}_3
$$

(113)

We know that $d\sigma_{\Sigma,L} = \sqrt{\det (g_{ij})} du_1 du_2$, $g_{ij} = g_L(f_{u_1})$.
Given by Euclidean $C^2$-smooth regular and closed curves $\gamma_i : [0, 2\pi] \to (\partial \Sigma)$. Suppose that the characteristic set $C(\Sigma)$ satisfies $H^l(C(\Sigma)) = 0$, where $H^l(C(\Sigma))$ denotes the Euclidean $l$-dimensional Hausdorff measure of $C(\Sigma)$ and that $||\nabla H u||^{-1}_H$ is locally summable with respect to the Euclidean $2$-dimensional Hausdorff measure near the characteristic set $C(\Sigma)$. Then, we have

$$\int_{\Sigma} \omega \bigg( \frac{\partial}{\partial t} \bigg) \bigg|_{\gamma_i} \bigg| ds = 0,$$

(115)

where $A = -\langle e_1, \nabla H(X_3u, \nabla H u) \rangle_L - \lambda^3_i((X_3u)^2/L^2)$. Let $G$ go to the infinity, and use the dominated convergence theorem, we get the desired result. 

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interests.

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