Thermal fluctuations of granular gas driven by a Gaussian thermostat based on two-point kinetic theory

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Abstract

This paper investigates thermal fluctuations of the granular gas, which is driven by a Gaussian thermostat, on the basis of the two-point kinetic theory. In particular, we consider thermal fluctuations of the inelastic variable sphere, which was proposed by Yano [1]. Time correlations of thermal fluctuations of the pressure deviator and two times of the heat flux are calculated on the basis of the two-point kinetic theory and compared with their numerical results, which are calculated using the direct simulation Monte Carlo (DSMC) method. Finally, we compare transport coefficients, which are calculated on the basis of the kinetic theory, with those calculated using the DSMC method.

Keywords: Green–Kubo expression, two-point kinetic theory, granular gases

(Some figures may appear in colour only in the online journal)

1. Introduction

The first study of thermal fluctuations of the granular gas was performed by Goldhirsch and van Noije [2]. In their study, the Green–Kubo (GK) expression was considered for the viscosity coefficient ($\mu$) of the granular gas on the basis of the scaled Sonine polynomials in the Chapman–Enskog (CE) method. Afterwards, Dufty and Brey [3] formulated GK expressions for the thermal conductivity ($\kappa$) and diffusive thermal conductivity ($\eta$) together with $\mu$. Brey et al [4] proved that their GK expressions for the transport coefficients of the granular gas reproduce transport coefficients, which are calculated by the CE method, with some good accuracies under the homogeneous cooling state (HCS), using the direct simulation Monte Carlo (DSMC) method [5], whereas their numerical results also indicated that $\kappa$ and $\eta$, which are calculated on the basis of GK expressions using the DSMC method, deviate from $\kappa$ and $\eta$, which are calculated by the first order approximations in the CE method, in the range of $0 \leq \alpha \leq 0.7$ (\(\alpha\): restitution coefficient) and $\mu$, which is calculated using the DSMC method, is similar to $\mu$, which is calculated by the first order approximation in the CE method, all in the range of $\alpha$ ($0 \leq \alpha \leq 1$). Meanwhile, Garzo, Santos and Montanero [6] proposed the modified Sonine polynomials in the CE method by expanding the velocity distribution function (VDF) not around the Maxwell–Boltzmann (MB) distribution but the zeroth order approximation of the VDF, which corresponds to the VDF under the HCS. On the basis of such a modified CE method, Garzo, Santos and Montanero [6] obtained $\kappa$ and $\eta$, which are much more similar to $\kappa$ and $\eta$, which were calculated on the basis of GK expressions using the DSMC method by Brey et al [4], than $\kappa$ and $\eta$, which were calculated using the first order approximation in the conventional CE method [4]. In a recent study related to thermal fluctuations of the granular gas, the thermally fluctuating hydrodynamics equation, namely, the Landau–Lifshitz–Navier–Stokes-Fourier (LLNSF) equation [7], for the granular gas was formulated using the inelastic Boltzmann equation with the noise term by Brey et al [8].

In this paper, we aim to calculate general solutions for time correlations of thermal fluctuations of the pressure deviator and two times of the heat flux for the granular gas driven by a Gaussian thermostat (GT), which was proposed by Montanero and Santos [9]. GT is also applied to the kinetic modeling of the collective motion of biological agents (see [10–13]). In the previous study on GK expressions for $\mu$, $\kappa$ and $\eta$ by Dufty and Brey [3], $\mu$, $\kappa$ and $\eta$ were calculated using ensemble averages of moments, which were obtained using...
two different fluxes at the same or different time. Therefore, GK expressions for $\mu$ and $\kappa$ under the elastic limit were obtained as [3]

$$
\mu = \mathcal{P} \int_0^\infty \langle (H^{(2)}_0(\hat{\xi}), H^{(2)}_0(\hat{\xi} + \hat{\xi}')) \rangle d\hat{\xi}', \quad \mathcal{P} \in \mathbb{R}^+,
$$

$$
\kappa = \mathcal{Q} \int_0^\infty \langle (H^{(3)}_0(\hat{\xi}), H^{(3)}_0(\hat{\xi} + \hat{\xi}')) \rangle d\hat{\xi}', \quad \mathcal{Q} \in \mathbb{R}^+,
$$

where $\langle (X, Y) \rangle := \int_{\mathbb{V}} f_{\text{MB}}(\hat{\xi})XYde = (c \in \mathbb{V}^1 \subseteq \mathbb{R}^3; \text{velocity of a particle}), H^{(2)}_0$ and $H^{(3)}_0$ are Hermite polynomials, which will be defined in section 2, and $\hat{\xi}$ and $\hat{\xi}'$ are non-dimensionalized time [3]. GK expressions for $\mu$, $\kappa$ and $\eta$ by Dufty and Brey [3] are indeed accurate in themselves, as confirmed by the numerical analysis by Brey et al. [4], whereas they are basically different from the conventional GK expressions for $\mu$ and $\kappa$ such as $\mu \propto \langle p_i(t), p_i(0) \rangle$ ($p_i$: pressure deviator) or $\kappa \propto \langle q_i(t), q_i(0) \rangle$ ($q_i$: heat flux) ($t \in \mathbb{R}^+$: time) by Zwanzig [14] under the perfectly elastic case, because Dufty and Brey [3] calculated the transport coefficients on the basis of one particle distribution function on the basis of the CE method. Consequently, we must consider two particle distribution function to obtain the conventional GK expressions for the transport coefficients by Zwanzig such as $\mu \propto \langle p_i(t), p_i(0) \rangle$ or $\kappa \propto \langle q_i(t), q_i(0) \rangle$. Readers must remember that GK expressions for the transport coefficients by Dufty and Brey approach those by Zwanzig, when time interval $(\Delta t)$ is small enough to neglect the change in $f_{\text{MB}}(\hat{\xi})$ during $\hat{\xi}' \in [0, \hat{\xi}]$. Of course, $\Delta t \ll 1$ in the DSMC calculation satisfies such a condition.

To attain our aim, we extend the two-point kinetic theory for the elastic gas by Tsuge and Sagara [15] to the granular gas driven by GT. In particular, we discuss the inelastic variable hard sphere (IVHS), which was proposed by Yano [1], as the component of the granular gas. The IVHS model is useful for understanding the characteristics of the thermal fluctuations of the granular gas driven by GT, because we can estimate how the characteristics of thermal fluctuations change in accordance with the change of the dependency of the collision frequency on the relative velocity between two colliding granular particles. Additionally, the investigation of thermal fluctuations of the granular gas driven by GT is interesting, because the investigation of thermal fluctuations under the thermally nonequilibrium steady state is a universal problem in the nonequilibrium statistical mechanics. Time correlations of thermal fluctuations of the pressure deviator and two times of the heat flux for the granular gas driven by GT are calculated on the basis of the two-point kinetic theory to obtain GK expressions for the transport coefficients of the IVHS driven by GT. These analytical solutions of time correlations of thermal fluctuations of the pressure deviator and two times of the heat flux. Here, we note that three cases among the IVHS, namely, inelastic hard sphere (IHS), IVHS with $\Omega = 0.6$ and inelastic Maxwell sphere (IMS) [1], are calculated, where $\Omega$ will be defined in section II. Finally, $\mu$, $\kappa$ and $\eta$, which are calculated on the basis of the kinetic theory, are compared with $\mu$, $\kappa$ and $\eta$, which are obtained by our GK expressions using the DSMC method. In particular, the numerical results of the IMS are interesting, because effects of the nonlinear collisional moments on time correlations of thermal fluctuations of the pressure deviator and two times of the heat flux or cooling rate can be removed in the case of the IMS.

This paper is organized as follows. The fourth and sixth order spherically symmetric moments are calculated for the IVHS in order to define the zeroth order approximation of the VDF in section 2. On the basis of the zeroth order approximation of the VDF, we discuss the two-point kinetic theory for the granular gas driven by GT to obtain GK expressions for the transport coefficients, in section 3. Analytical solutions of time correlations of thermal fluctuations of the pressure deviator and two times of the heat flux are compared with DSMC results in section 4, when the granular gas driven by GT is composed of the IHS. IVHS with $\Omega = 0.6$ or IMS. Additionally, $\mu$, $\kappa$ and $\eta$, which are calculated on the basis of the kinetic theory, are compared with $\mu$, $\kappa$ and $\eta$, which are obtained by our GK expressions using the DSMC method. Finally, we make the concluding remarks in section 5.

2. The fourth and sixth order spherically symmetric moments for granular gas driven by GT

Firstly, we calculate the fourth and sixth order spherically symmetric moments, when the granular gas driven by GT is composed of the IVHS. The inelastic Boltzmann equation for the IVHS is written under the spatially homogeneous state as

$$
\frac{df(e, t)}{dt} + \frac{\zeta}{2} \frac{\partial C_i f(e, t)}{\partial v_i} = \mathcal{A} \int_{V^1} \int_0^{2\pi} \int_0^{\pi} g^{1-\xi} \frac{1}{\alpha^2} j(f(e')) f(e') \sin \chi d\chi de d\xi, \quad (1)
$$

where $f(e, t)$ is the VDF, $e \in \mathbb{V}^3 \subseteq \mathbb{R}^3$ and $c_i \in \mathbb{V}^1 \subseteq \mathbb{R}^3$ are velocities of two colliding IVHSs. $\mathcal{A} \in \mathbb{R}^+$ is a constant cross section. $\xi$ is the cooling rate and $C_i := c_i - u_i$ ($u_i$: flow velocity) is the peculiar velocity, $g = |e - c_i|$ is the magnitude of the relative velocity of two colliding IVHS, $\chi \in [0, \pi]$ is the deflection angle and $e \in [0, 2\pi]$ is the scattering angle. We consider $\xi \in [0, 1]$, where $\xi = 0$ corresponds to the IHS and $\xi = 1$ corresponds to the IMS. Of course, $(e^a, e^b) \rightarrow (e, c_i)$ is obtained after a binary collision. Readers are reminded that the inelastic Maxwell model (IMM) proposed by Bobylev and Cercignani [16] is different from the IMS, because the collisional term of the IMM is obtained by $\sqrt{T_B} \int_{V^1} \int_0^{2\pi} \int_0^{\pi} [(1/\alpha)f(e') f(e') - f(e) f(c_i)] d\chi de d\xi (T: temperature, B \in \mathbb{R}^+).$ As a result, the dependency of the collision frequency on the temperature by the IMM is the same as that by the IHS and $\sin \chi$ in the right hand side of equation (1) is set to unity in the IMM. Finally, the IVHS is a straightforward extension of the VHS, which was proposed by Bird [5] to calculate molecules with the inverse power low potential without facing to the angular cut-off problem in the
DSMC method, to the inelastic collision, as described in the author’s previous study [1]. Of course, our assumption of the inverse power law potential for the granular gas is physically unrealistic, except for the charged granular particles [17], which interact with each other through Coulomb force. Thus, the IVHS is a mathematical model to investigate the characteristics of the IHS, furthermore, as well as the IMM.

The zeroth order approximation of \( f(c, t) \), namely, \( f^{(0)}(c, t) \) can be approximated using the fourth and sixth order spherically symmetric moments \( a_4 \) and \( a_6 \) as follows:

\[
f^{(0)}(c, t) = f_{\text{MB}}(c, t) \left( 1 + \frac{1}{120} a_4 H^{(4)} + \frac{1}{5400} a_6 H^{(6)} \right), \tag{2}
\]

where \( f_{\text{MB}}(c, t) \) is the MB distribution, and \( H^{(n)} = v^n - 10v^{n-2} + 15 \) and \( H^{(6)} = v^6 - 21v^4 + 105v^2 - 105 \), in which \( v \) is the MB distribution, and \( H^{(n)} = \int f^{(n)}(c, t) \, dc \) and \( a_4 = (1/\rho) \int f^{(4)}(c, t) \, dc \) and \( a_6 = (1/\rho) \int f^{(6)}(c, t) \, dc \). (\( \rho \): density).

Substituting \( f(c, t) = f^{(0)}(c, t) \) into both sides of equation (1), multiplying both sides of equation (1) by \( C^2/3 \) and integrating over \( V^3 \), we obtain

\[
\rho \frac{dT}{dt} + \rho \tau RT = -\frac{5}{2(5+\Omega)}(1-\alpha^2)\rho RT, \tag{3}
\]

where \( \Omega = 1 - \xi \) and \( \tau = p/\mu_{\text{e}}(\Omega) \) (\( \mu_{\text{e}}(\Omega) \): viscosity coefficient of the elastic VHS, \( p \): static pressure).

GT postulates that the total energy is always conserved. As a result, we obtain the cooling rate \( \zeta \) from \( dT/dt = 0 \) in equation (3), when the flow velocity is always equal to zero, namely, \( |u| = 0 \), as

\[
\zeta = \frac{5}{2(5+\Omega)}(1-\alpha^2), \tag{4}
\]

where \( \zeta \) in equation (4) is used throughout the analytical discussion, because contributions of \( a_4 \) and \( a_6 \) on the cooling rate are markedly small in cases of the IHS, IVHS with \( \Omega = 0.6 \) and IMS, as described in appendix C.

In later numerical analysis, the definition of \( \zeta \) in equation (4) is not used, because \( |u| = 0 \) is not satisfied, owing to velocity fluctuations in the DSMC calculation. Therefore, \( \zeta \) is calculated from the energy-loss via inelastic collisions by each time step in the DSMC calculation.

Substituting \( f(c, t) = f^{(0)}(c, t) \) into both sides of equation (1), multiplying both sides of equation (1) by \( \rho^{-1}H^{(4)} \) and \( \rho^{-1}H^{(6)} \), respectively, and integrating over \( V^3 \), we obtain

\[
\frac{da_4}{dt} = A_0 + A_1 a_4 + A_2 a_6 \\
\frac{da_6}{dt} = B_0 + B_1 a_4 + B_2 a_6, \tag{5}
\]

where \( A_i \) and \( B_i \) (\( i = 0, 1, 2 \)) are functions of \( \alpha \) and \( \Omega \). In equation (5), we neglected nonlinear terms \( a_4^2, a_4 a_6 \) and \( a_6^2 \) to simplify our discussions. Readers are recommended to read the paper by Santos and Montanero [18] to confirm effects of those nonlinear terms. Concrete forms of \( A_i \) and \( B_i \) (\( i = 0, 1, 2 \)) in equation (5) are defined by equations (A1)–(A6) in appendix A. Stationary solutions of \( a_4 \) and \( a_6 \) in equation (5) are obtained by \( d_4 a_4 = d_6 a_6 = 0 \). Finally, stationary solutions of \( a_4 \) and \( a_6 \) are obtained as

\[
a_4 = \frac{\sum_{i=0}^{11} \beta_{1i} \alpha^i}{\sum_{i=0}^{11} \beta'_{1i} \alpha^i}, \tag{6}
\]

\[
a_6 = \frac{\sum_{i=0}^{6} \beta_{2i} \alpha^i}{\sum_{i=0}^{6} \beta'_{2i} \alpha^i}. \tag{7}
\]

\( \beta, \gamma, \beta', \gamma' \) in equation (6) and \( \beta', \gamma' \) in equation (7) are defined by equations (A7)–(A10) in appendix A. \( a_4 \) and \( a_6 \) in equations (6) and (7) are equal to those under the HCS. On the basis of the CE method, Brilliantov and Pöschel [19] obtained \( a_4 \) and \( a_6 \) for the IHS under the HCS in a different form from equations (6) and (7).

Figures 1, 2 and 3 show \( a_4 \) (left frame) and \( a_6 \) (right frame) versus \( \alpha \) for the IHS, IVHS with \( \Omega = 0.6 \) and IMS, respectively. DSMC results of \( a_4 \) and \( a_6 \) for the IHS, IVHS with \( \Omega = 0.6 \) and IMS are obtained using 5000 sample particles per a cell in \( 5 \times 5 \) grids in the square domain \( x \in [0, 1], y \in [0, 1] \), in which the periodic boundary condition is used.
Firstly, we investigate \( a_4 \) and \( a_6 \) for the IHS. Of course, \( a_4 \) and \( a_6 \) are different from the second and third order cumulants, which are calculated using the Sonine polynomials in the CE method, because we use Grad’s method in this paper. Then, \( a_4 \) is 15 times of the second order cumulant \([9, 10]\) and \( a_6 \) is \(-900/8\) times of the third order cumulant \([9, 10]\). \( a_4 \) and \( a_6 \) for the IHS in equations (6) and (7) are, however, different from 15 times of the second order cumulant and \(-900/8\) times of the third order cumulant, which were obtained by Brilliantov and Pöschel \([9, 10]\). Indeed, calculations of \( a_4 \) and \( a_6 \) for the IHS, IVHS with \( \Omega = 0.6 \) and IMS are significant for understanding the changes of \( a_4 \) and \( a_6 \) in accordance with the change of \( \Omega \). The left frame of figure 1 shows that \( a_4 \) (symbols), which is calculated using the DSMC method, namely, \( (a_4)_{\text{DSMC}} \) is similar to \( a_4 \) (dashed-dot line), which is obtained as a stationary solution by setting \( A_2 = 0 \) in equation (5) and equal to \( a_4 \), which is 15 times of the second order cumulant obtained by Noije-Ernst \([11, 12]\). \( a_4 \) in equation (6) (solid line) is similar to \( (a_4)_{\text{DSMC}} \) in the range of \( 0.6 \leq \alpha \leq 1 \), whereas the difference between \( a_4 \) in equation (6) and \( (a_4)_{\text{DSMC}} \) increases, as \( \alpha \) decreases in the range of \( 0 \leq \alpha \leq 0.6 \). Additionally, \( a_4 \) in equation (6) is similar to \( a_4 \) (dashed line), which was obtained by Brilliantov and Pöschel \([9, 10]\), in the range of \( 0.4 \leq \alpha \leq 1 \). The right frame of figure 1 shows that \( a_6 \) (symbols), which is calculated using the DSMC method, namely, \( (a_6)_{\text{DSMC}} \) is similar to \( a_6 \) (solid line) in equation (7) in the range of \( 0.3 \leq \alpha \leq 1 \), whereas \( (a_6)_{\text{DSMC}} \) is similar to \( a_6 \) (dashed line), which was obtained by Brilliantov and Pöschel \([9, 10]\), in the range of \( 0.5 \leq \alpha \leq 1 \). Therefore, \( a_6 \) in equation (7) is more similar to \( (a_6)_{\text{DSMC}} \) than \( a_6 \), which was obtained by Brilliantov and Pöschel \([9, 10]\).

Next, we investigate \( a_4 \) and \( a_6 \) for the IVHS with \( \Omega = 0.6 \). The left frame of figure 2 shows that \( (a_4)_{\text{DSMC}} \) is similar to \( a_4 \) (dashed line), which is obtained as a stationary solution by setting \( A_2 = 0 \) in equation (5), in the range of \( 0.4 \leq \alpha \leq 1 \). \( a_4 \) in equation (6) (solid line) is similar to \( (a_4)_{\text{DSMC}} \) in the range of \( 0.7 \leq \alpha \leq 1 \), whereas the difference between \( a_4 \) in equation (6) and \( (a_4)_{\text{DSMC}} \) increases, as \( \alpha \) decreases in the range of \( 0 \leq \alpha \leq 0.7 \). The right frame of figure 2 shows that \( (a_6)_{\text{DSMC}} \) is similar to \( a_6 \) (solid line) in equation (7) in the range of \( 0.2 \leq \alpha \leq 1 \).

Finally, we investigate \( a_4 \) and \( a_6 \) for the IMS. The left frame of figure 3 shows that \( (a_4)_{\text{DSMC}} \) (symbols) is similar to \( a_4 \) (solid line) in equation (6) in the range of \( 0.1 \leq \alpha \leq 1 \). Here, it is important to note that \( A_2 = 0 \) in equation (5) is always obtained for the IMS. On the other hand, \( a_6 \) in equation (7) diverges at \( \alpha \approx 0.366 \) and becomes negative in
the range of $0 \leq \alpha < 0.366$, as shown in the right frame of figure 3. $a_6$ in equation (7) approximates $(a_6)$DSCM with good accuracies in the range of $0.6 \leq \alpha \leq 1$. To avoid the divergence of $a_6$ in equation (7), we should have considered nonlinear collisional moments for the IMS such as $a_1^2$ or $a_1a_6$ in the time evolution of $a_6$ in equation (5). However, such an inclusion of nonlinear collisional moments such as $a_1^2$ or $a_1a_6$ is beyond the scope of this paper. We conjecture that nonlinear terms $a_6^2$ and $a_2a_6$ ($4 \leq n$) never appear in the time evolution of $a_6$ in the case of the IMS, as $a_1^2$ and $a_6$ never appear in the time evolution of $a_4$ in the case of the IMS, whereas $a_2^2$ and $a_2a_6$ are always equal to zero in the time evolution of $a_4$ owing to $a_2 = 0$. The tendency of $a_3$ versus $\alpha$ in the case of the 1D IMM is investigated by Ernst and Brito [21] in detail. The 2nth order moment diverges at $n \geq 3$ in the 1D IMM. Thus, the sixth order moment $(a_6)$ cannot be defined, mathematically, in the case of 1D IMM. Such a lack in mathematical definition of the moment in the IMM is dismissed by DSMC calculation, because the finite moments are always guaranteed owing to a finite number of sample particles. At any rate, the question that $a_6$ always diverges will be answered by checking the effects of nonlinear collisional terms in the right hand side of $d_t a_6$ in equation (5). From numerical results of $a_4$ and $a_6$ for the IHS, IVHS with $\Omega = 0.6$ and IMS, we conclude that both $a_4$ and $a_6$ at $\alpha = 0$ increase in accordance with the decrease of $\Omega$.

On the basis of $a_4$ and $a_6$, we define the zeroth order approximation of the VDF, namely, $f^{(0)}(e)$, in the next section.

3. Two-point kinetic theory for granular gas driven by GT

The two-point kinetic theory was firstly discussed for the elastic gas by Tsuge and Sagara [15]. In this section, we consider the two-point kinetic theory for the granular gas driven by GT. The stochastic inelastic Boltzmann equation is the most fundamental equation in the two-point kinetic theory. In particular, the stochastic inelastic Boltzmann equation including the GT heating term is written as

$$B \psi(t, l) = \left( \frac{\partial}{\partial t} + e \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial x} \cdot C \right) \psi(t, l) - \mathcal{J}(\partial)[\psi(t, l) \psi(t, \hat{l})] = 0,$$ (8)

where $\psi$ is the microscropic distribution function defined by

$$\psi(t, l) := \sum_{s=1}^{N} \delta[t - \ell^{(s)}(t)],$$ (10)

where $s$ is the index of the particles, $N$ is the total number of particles, and $\ell := (e, x) \ (x \in \mathbb{R}^3)$. Of course, $(\ell^{(s)}, t^s) \to (\ell, t)$ is obtained after the binary inelastic collision. From equation (8), we assume $\psi(t, l) \in C^1(\mathbb{R}^+ \times C(\mathbb{R}^3) \times C^0(\mathbb{R}^3))$.

For convenience, we set $\xi := (t, l)$. In $\xi$-space, we consider the correlation between two points $\hat{\alpha}$ and $\hat{\beta}$. As a result, definitions $\varphi(\hat{\alpha}) := \varphi(\xi(\hat{\alpha}))$ and $\varphi(\hat{\beta}) := \varphi(\xi(\hat{\beta}))$ are used.

At a point $\alpha$, the stochastic inelastic Boltzmann equation including the GT heating term in equation (8) is rewritten as

$$B(\alpha) \varphi(\hat{\alpha}) = \left[ \frac{\partial}{\partial \alpha(\hat{\alpha})} + e(\alpha) \frac{\partial}{\partial x(\hat{\alpha})} + \frac{1}{2} \frac{\partial}{\partial x(\hat{\alpha})} \cdot C(\hat{\alpha}) \right] \varphi(\hat{\alpha}) + \mathcal{J}(\hat{\alpha})[\varphi(\hat{\alpha}) \varphi(\hat{\beta}) - \varphi(\hat{\alpha}) \varphi(\hat{\beta})] = 0,$$ (11)

Next, we define the averaged quantity in $\ell$-space using finite volumes $(x \in \Delta V(\subseteq \mathbb{R}^3)$ and $(\ell \in \Delta V$) \subseteq \mathbb{R}^3$) as follows

$$\bar{\psi} := \frac{1}{\Delta V \Delta V} \int_{\Delta V} dx \int_{\Delta V} d\psi, \quad \bar{\psi} := \psi - \bar{\psi},$$ (13)

where $\psi$ is the arbitrary function defined in $\ell$-space.

On the basis of equations (12) and (13), we obtain the following equation by multiplying $\Delta \varphi(\hat{\beta})$ by both sides of equation (11) and taking the average in $\ell$-space,

$$\left[ \frac{\partial}{\partial \alpha(\hat{\alpha})} + e(\alpha) \frac{\partial}{\partial x(\hat{\alpha})} + \frac{1}{2} \frac{\partial}{\partial x(\hat{\alpha})} \cdot C(\hat{\alpha}) \right] \varphi(\hat{\alpha}) \varphi(\hat{\beta}) = \left[ \varphi(\hat{\alpha}) \right] \left( f(\hat{\beta}) \varphi(\hat{\alpha}) - \varphi(\hat{\alpha}) \varphi(\hat{\beta}) \right),$$ (14)

where $f(\hat{\beta}) = \varphi(\hat{\beta})$ \& $(\hat{\beta}) = \varphi(\hat{\beta})$.

In equation (14), $\varphi(\hat{\alpha}) \varphi(\hat{\beta})$ is decomposed as

$${\phi}^{(\alpha)}(\hat{\alpha}) f(\hat{\beta}) = f(\hat{\alpha}) \varphi(\hat{\beta}) + g(\hat{\alpha}) \varphi(\hat{\beta}),$$ (15)

where $f(\hat{\beta}) \varphi(\hat{\beta})$ is the two-point phase density of different particles and $g(\hat{\alpha}) \varphi(\hat{\beta})$ is the probability of finding same particles at $\ell = \ell(\hat{\alpha})$ \& $t = \ell(\hat{\alpha})$ and $\ell = \ell(\hat{\beta}) \& t = \ell(\hat{\beta})$, $f(\hat{\alpha}) \varphi(\hat{\beta})$ is related to the hydrodynamic fluctuation term $\phi(\hat{\alpha}) \varphi(\hat{\beta})$ as follows

$$f(\hat{\alpha}) \varphi(\hat{\beta}) = \phi(\hat{\alpha}) \varphi(\hat{\beta}) + \left( 1 + \frac{1}{N} \right) f(\hat{\alpha}) f(\hat{\beta}).$$ (16)
\[ g(\alpha; \beta) = \omega(\alpha)\omega(\beta) \sum_{m,n} Q_{ij,mn}^{(j,k)} c(\alpha)^j c(\beta)^k J K! H_{lm}^{(j)}(\alpha) H_{lm}^{(k)}(\beta) \times \int H_{y}^{(j)}g(\alpha, \beta) d\alpha d\beta. \] (22)
Here, we note that \( \mu(\alpha, \Omega), \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \) for the granular gas driven by GT are written as [23]

\[
\mu(\alpha, \Omega) = \frac{p(\bar{\alpha})}{\nu_0(\bar{\alpha}) - \zeta(\bar{\alpha})}, \tag{38}
\]

\[
\kappa(\alpha, \Omega) = \left(\frac{5}{2} + \frac{a_4}{3}\right) \frac{p(\bar{\alpha}) R}{\nu_0(\bar{\alpha}) - \frac{3}{2} \zeta(\bar{\alpha})}, \tag{39}
\]

\[
\eta(\alpha, \Omega) = \frac{a_4 p(\bar{\alpha}) R T(\bar{\alpha})}{6 \rho(\bar{\alpha})} \frac{1}{\nu_0(\bar{\alpha}) - \frac{3}{2} \zeta(\bar{\alpha})}. \tag{40}
\]

where \( \mu(\alpha, 1), \kappa(\alpha, 1) \) and \( \eta(\alpha, 1) \) for the IHS driven by GT are quite similar to those obtained by Santos [23].

From equations (38)–(40), we obtain

\[
\mu(\alpha, \Omega) = p(\bar{\alpha}) \int_0^\infty Q_{ij,lm}^{(2,2)}(\epsilon) \, d\epsilon \, [Q_{ij,lm}^{(2,2)}]_{k=0}^{-1}, \tag{41}
\]

\[
\kappa(\alpha, \Omega) = \left(\frac{5}{2} + \frac{a_4}{3}\right) p(\bar{\alpha}) R \int_0^\infty Q_{ij,lm}^{(3,3)}(\epsilon) \, d\epsilon \, [Q_{ij,lm}^{(3,3)}]_{k=0}^{-1}, \tag{42}
\]

\[
\eta(\alpha, \Omega) = \frac{a_4 p(\bar{\alpha}) R T(\bar{\alpha})}{6 \rho(\bar{\alpha})} \int_0^\infty Q_{ij,lm}^{(3,3)}(\epsilon) \, d\epsilon \, [Q_{ij,lm}^{(3,3)}]_{k=0}^{-1}. \tag{43}
\]

Equations (41)–(43) are definitions of the transport coefficients of the granular gas driven GT. Substituting \( \alpha = 1 \) into equations (41)–(43), we can readily obtain GK expressions for the transport coefficients in the elastic gas by Zwanzig [14].

The comparison of the viscosity coefficient of the granular gas driven by GT, which is calculated using the DSMC method, with that obtained by the CE method, was done using the solution of the uniform shear flow by Garzo and Montanero [24]. In section 4, we calculate the following four parameters using the DSMC method:

\[
\psi_2(\alpha) = [Q_{xx,xx}^{(2,2)}]_{k=0}^{-1}(\alpha)/[Q_{xx,xx}^{(2,2)}]_{k=0}^{-1}, \tag{44}
\]

\[
\psi_3(\alpha) = [Q_{xx,xx}^{(3,3)}]_{k=0}^{-1}(\alpha)/[Q_{xx,xx}^{(3,3)}]_{k=0}^{-1}, \tag{45}
\]

\[
\phi_2(\epsilon) = Q_{xx,xx}^{(2,2)}(\epsilon)/Q_{xx,xx}^{(2,2)}(\epsilon)_{k=0}^{-1}. \tag{46}
\]

\[
\phi_3(\epsilon) = Q_{xx,xx}^{(3,3)}(\epsilon)/Q_{xx,xx}^{(3,3)}(\epsilon)_{k=0}^{-1}. \tag{47}
\]

The DSMC results show \([Q_{xx,xx}^{(2,2)}]_{k=0}^{-1} = 1.33\) and \([Q_{xx,xx}^{(3,3)}]_{k=0}^{-1} = 10\) for the IHS, IVHS with \( \Omega = 0.6 \) and IMS, when \( \alpha = 1 \), namely, \( a_4 = a_6 = 0 \) in equations (26) and (27). Therefore, DSMC results reproduce equations (26) and (27) with good accuracies, when \( \alpha = 1 \).

Figures 4, 5 and 6 show \( \psi_2(\alpha) \) versus \( \alpha \) (left frame) and \( \psi_3(\alpha) \) versus \( \alpha \) (right frame) for the IHS, IVHS with \( \Omega = 0.6 \) and IMS, respectively. \( \psi_2(\alpha) \) in equation (44) is calculated using \( a_4 \) in equation (6) or \((a_4)_{DSMC}\), whereas \( \psi_3(\alpha) \) in equation (45) is calculated using \( a_6 \) in equation (6) and \( a_6 \) in equation (7) or \((a_6)_{DSMC}\) and \((a_6)_{DSMC}\). The difference between \([\psi_2(\alpha)]_{DSMC}\), which is calculated using the DSMC method, and \( \psi_2(\alpha) \) in equation (44), which is obtained using \( a_4 \) in equation (6) or \((a_4)_{DSMC}\), increases in the range of \( 0 \leq \alpha \leq 0.5 \), as \( \alpha \) decreases. \( \psi_2(\alpha) \) in equation (44), which is obtained using \((a_4)_{DSMC}\), is more similar to \( \psi_2(\alpha) \) in equation (44), which is obtained using \( a_4 \) in equation (6). In the right frame of figure 4, the difference
between $[\psi_2(\alpha)]_{\text{DSMC}}$, which is calculated using the DSMC method, and $\psi_3(\alpha)$ in equation (45), which is obtained using $a_4$ in equation (6) and $a_6$ in equation (7) or $(a_4)_{\text{DSMC}}$ and $(a_6)_{\text{DSMC}}$, increases in the range of $0 \leq \alpha \leq 0.5$, as $\alpha$ decreases. We, however, find that $\psi_2(\alpha)$ in equation (45), which is obtained using $(a_4)_{\text{DSMC}}$ and $(a_6)_{\text{DSMC}}$, is much more similar to $[\psi_3(\alpha)]_{\text{DSMC}}$ than $\psi_3(\alpha)$ in equation (45), which is obtained using $a_4$ in equation (6) and $a_6$ in equation (7).

In the left frame of figure 5, the difference between $[\psi_2(\alpha)]_{\text{DSMC}}$ and $\psi_2(\alpha)$ in equation (44), which is obtained using $a_4$ in equation (6) or $(a_4)_{\text{DSMC}}$, increases in the range of $0 \leq \alpha \leq 0.6$, as $\alpha$ decreases. $\psi_2(\alpha)$ in equation (44), which is obtained using $(a_4)_{\text{DSMC}}$, is more similar to $[\psi_2(\alpha)]_{\text{DSMC}}$ than $\psi_2(\alpha)$ in equation (44), which is obtained using $a_4$ in equation (6). The difference between $[\psi_3(\alpha)]_{\text{DSMC}}$ and $\psi_3(\alpha)$ in equation (45), which is obtained using $a_4$ in equation (6) and $a_6$ in equation (7) or $(a_4)_{\text{DSMC}}$ and $(a_6)_{\text{DSMC}}$, increases in the range of $0 \leq \alpha \leq 0.4$, as $\alpha$ decreases. We, however, find that $\psi_2(\alpha)$ in equation (45), which is obtained using $(a_4)_{\text{DSMC}}$ and $(a_6)_{\text{DSMC}}$, is much more similar to $[\psi_3(\alpha)]_{\text{DSMC}}$ than $\psi_3(\alpha)$ in equation (45), which is obtained using $a_4$ in equation (6) and $a_6$ in equation (7).

In the left frame of figure 6, the difference between $[\psi_2(\alpha)]_{\text{DSMC}}$ and $\psi_2(\alpha)$ in equation (44), which is obtained using $a_4$ in equation (6) or $(a_4)_{\text{DSMC}}$, is smaller than that in the case of the IHS. There is only a marked difference between $[\psi_2(\alpha)]_{\text{DSMC}}$ and $\psi_2(\alpha)$ in equation (44), which is calculated using $(a_4)_{\text{DSMC}}$ or $a_4$ in equation (6), when $\alpha = 0$. In the right frame of figure 6, there are marked differences between $[\psi_3(\alpha)]_{\text{DSMC}}$ and $\psi_3(\alpha)$ in equation (45), which is obtained
using \((a_4)_{\text{DSMC}}\) and \((a_6)_{\text{DSMC}}\), when \(\alpha = 0.1\) and 0.2. Additionally, there are marked differences between \(\psi_2(\alpha)\) in DSMC and \(\psi_3(\alpha)\) in equation (45), which is calculated using \(a_4\) in equation (6) and \(a_6\) in equation (7), when 0 \(\leq \alpha \leq 0.5\), because \(a_6\) in equation (7) diverges at \(\alpha = 0.366\) as shown in the right frame of figure 3.

In summary, the difference between \(a_4\) in equation (6) and \((a_4)_{\text{DSMC}}\) or \(a_6\) in equation (7) and \((a_6)_{\text{DSMC}}\) is one of causes of the difference between \(\psi_2(\alpha)\) in equation (44) and \([\psi_2(\alpha)]_{\text{DSMC}}\) or \(\psi_3(\alpha)\) in equation (45) and \([\psi_3(\alpha)]_{\text{DSMC}}\).

4.2. DSMC results of \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\) and their comparisons with equations (46) and (47)

We investigate \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\) using the DSMC method and equations (46) and (47). Figures 7, 8, and 9 show \(\phi_2(\tilde{\epsilon})\) versus \(\tilde{\epsilon}\) and \(\phi_3(\tilde{\epsilon})\) versus \(\tilde{\epsilon}\) for the IHS, IVHS with \(\Omega = 0.6\) and IMS, when \(\alpha = 0, 0.2, 0.4, 0.6, 0.8\) and 1, respectively. Figure 7 shows that \([\phi_2(\tilde{\epsilon})]_{\text{DSMC}}\), which is calculated using the DSMC method, is quite similar to \(\phi_2(\tilde{\epsilon})\) in equation (46). Meanwhile, \([\phi_3(\tilde{\epsilon})]_{\text{DSMC}}\), which is calculated using the DSMC method, is slightly smaller than \(\phi_3(\tilde{\epsilon})\) in equation (47), when \(\alpha = 0.366\) as shown in the right frame of figure 3.

In equation (47), when \(\alpha = 0\) and 0.2. The difference between \([\phi_3(\tilde{\epsilon})]_{\text{DSMC}}\) and \(\phi_3(\tilde{\epsilon})\) in equation (47) increases, as \(\alpha\) decreases, as shown in frames of \(\alpha = 0\) and 0.2. Additionally, we can confirm that the difference between \([\phi_3(\tilde{\epsilon})]_{\text{DSMC}}\) and \(\phi_3(\tilde{\epsilon})\) in equation (47) when \(\alpha = 0\) increases, as \(\Omega\) decreases, as shown in figures 7–9. Finally, we can conclude that the difference between \([\phi_3(\tilde{\epsilon})]_{\text{DSMC}}\) and \(\phi_3(\tilde{\epsilon})\) in equation (47), when \(\alpha = 0\) or 0.2, is not caused by nonlinear terms in the collisional moments of \(Q^{(3,3)}_{\alpha,\xi}\) [1] for the IHS and IVHS with \(\Omega = 0.6\), which are not included in equation (29), because nonlinear terms in the collisional moments of \(Q^{(3,3)}_{\alpha,\xi}\) never appear in the right hand side of equation (29) for the IMS, as discussed in appendix C.

4.3. DSMC results of transport coefficients and their comparisons with equations (38)–(40)

Finally, we compare transport coefficients, \(\mu\), \(\kappa\), and \(\eta\) in equations (41)–(43), which are calculated using the DSMC method, with those in equations (38)–(40). As shown in equations (41)–(43), we can understand that transport coefficients are related to areas of \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\) in \(\tilde{\epsilon} \in [0, +\infty]\). Therefore, accuracies of numerical integrations of \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\), which are calculated using the DSMC method, are significant. Meanwhile, \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\) fluctuate around 0. Therefore, \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\) become negative values in the ranges of \(\phi_2(\tilde{\epsilon}) \ll 1\) and \(\phi_3(\tilde{\epsilon}) \ll 1\). Such fluctuations around 0 can be reduced by increasing the number of sample particles. We, however, find that a further increase of the number of sample particles requires parallel computations.

In numerical integrations of \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\), we integrate \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\) in the range of \([0, \tilde{\epsilon}_c]\), in which both \(\phi_2(\tilde{\epsilon})\) and \(\phi_3(\tilde{\epsilon})\) are always positive. In particular, we are interested in \(\mu(\alpha) = \mu(0)\), \(\kappa(\alpha) = \kappa(0)\) and \(\eta(\alpha) = \eta(0)\rho_{\infty}^{-1}(RT_{\infty})^{-2}\). Additionally, we note that the transport coefficients do not depend on \([Q^{(2,3)}_{\alpha,\xi}]_{\tilde{k}=0}\) and \([Q^{(3,3)}_{\alpha,\xi}]_{\tilde{k}=0}\), as shown in equations (41)–(43). Consequently, differences between \([Q^{(2,3)}_{\alpha,\xi}]_{\tilde{k}=0}\) and \([Q^{(3,3)}_{\alpha,\xi}]_{\tilde{k}=0}\), which are calculated using the

\[\begin{align*}
\psi_2(\alpha) &\quad \text{for the IMS.} \\
\psi_3(\alpha) &\quad \text{for the IMS.}
\end{align*}\]
DSMC method, and those in equations (26) and (27) do not contribute to any differences between transport coefficients, which are calculated using the DSMC method in equations (41)–(43), and those in equations (38)–(40).

Figure 10 shows $\mu(\alpha)/\mu(0)$, $\kappa(\alpha)/\kappa(0)$ and $\eta(\alpha)$ versus $\alpha$. $a_4$ in equations (39) and (40) are equal to a stationary solution of equation (5) with $A_2 = 0$ or equation (6). As mentioned above, $a_4$, which is a stationary solution of equation (5) with $A_2 = 0$, coincides with $a_4$ in equation (6) only for the IMS. Figure 10 shows that $[\mu(\alpha)/\mu(0)]_{\text{DSMC}}$, which is calculated using the DSMC method in equation (41), is quite similar to that in equation (38) for the IHS, IVHS with $\Omega = 0.6$ and IMS in all the range of $\alpha$. Such a similarity between $\mu(\alpha)/\mu(0)$ in equation (38) and $[\mu(\alpha)/\mu(0)]_{\text{DSMC}}$ was also confirmed for the IHS by Garzo and his coworkers [6, 24].

Figure 10 shows that $\kappa(\alpha)/\kappa(0)$ in equation (39) with $a_4$ in equation (6) or $a_4$, which is a stationary solution of equation (5) with $A_2 = 0$, does not fit $[\kappa(\alpha)/\kappa(0)]_{\text{DSMC}}$, which is calculated using the DSMC method in equation (42), in the ranges of $0.4 \leq \alpha < 1$ and $0 \leq \alpha \leq 0.1$ in the case of the IHS. Meanwhile, $\kappa(\alpha)/\kappa(0)$ in equation (39) with $a_4$ in equation (6) is more similar to $[\kappa(\alpha)/\kappa(0)]_{\text{DSMC}}$ than $\kappa(\alpha)/\kappa(0)$ in equation (39) with $a_4$, which is a stationary solution of equation (5) with $A_2 = 0$, in the range of
0 \leq \alpha \leq 0.3 \text{ in the case of the IHS. } \kappa(\alpha)/\kappa(0) \text{ in equation } (39) \text{ with } a_4 \text{ in equation } (6) \text{ or } a_4, \text{ which is a stationary solution of equation } (5) \text{ with } A_2 = 0, \text{ does not fit } [\kappa(\alpha)/\kappa(0)]_{\text{DSMC}} \text{ in the range of } 0 \leq \alpha \leq 0.7 \text{ in the case of the IVHS with } \Omega = 0.6. \text{ Meanwhile, } \kappa(\alpha)/\kappa(0) \text{ in equation } (39) \text{ with } a_4 \text{ in equation } (6) \text{ is more similar to } [\kappa(\alpha)/\kappa(0)]_{\text{DSMC}} \text{ than } \kappa(\alpha)/\kappa(0) \text{ in equation } (39) \text{ with } a_4, \text{ which is a stationary solution of equation } (5) \text{ with } A_2 = 0, \text{ in the range of } 0 \leq \alpha \leq 0.7 \text{ in the case of the IVHS with } \Omega = 0.6. \text{ } \kappa(\alpha)/\kappa(0) \text{ in equation } (39) \text{ with } a_4 \text{ in equation } (6) \text{ does not fit } [\kappa(\alpha)/\kappa(0)]_{\text{DSMC}} \text{ in the range of } 0 \leq \alpha \leq 0.7 \text{ in the case of the IMS. } \text{ Meanwhile, the difference between } [\kappa(\alpha)/\kappa(0)]_{\text{DSMC}} \text{ and } \kappa(\alpha)/\kappa(0) \text{ in equation } (39) \text{ with } a_4 \text{ in equation } (6) \text{ increases markedly, as } \alpha \text{ decreases from } \alpha = 0.7 \text{ to } 0, \text{ in the case of the IMS. As a result, the difference in the range of } 0 \leq \alpha \leq 0.7 \text{ between } [\kappa(\alpha)/\kappa(0)]_{\text{DSMC}} \text{ and } \kappa(\alpha)/\kappa(0) \text{ in equation } (39) \text{ with } a_4 \text{ in equation } (6) \text{ increases, as } \Omega \text{ decreases from unity (IHS) to zero (IMS).} \text{ Figure } 10 \text{ shows that } \tilde{\eta}(\alpha) \text{ in equation } (40) \text{ with } a_4 \text{ in equation } (6) \text{ or } a_4, \text{ which is a stationary solution of equation } (5) \text{ with } A_2 = 0, \text{ does not fit } \tilde{\eta}(\alpha)_{\text{DSMC}}, \text{ which is calculated using the DSMC method in equation } (43), \text{ in the range of } 0 \leq \alpha \leq 0.6 \text{ in the case of the IHS. } \text{ Meanwhile, } \tilde{\eta}(\alpha) \text{ in equation } (40) \text{ with } a_4 \text{ in equation } (6) \text{ is more similar to}
\(\tilde{\eta}(\alpha)_{\text{DSMC}}\) than \(\tilde{\eta}(\alpha)\) in equation (40) with \(a_4\), which is a stationary solution of equation (5) with \(A_2 = 0\), in the range of \(0 \leq \alpha \leq 0.5\) in the case of the IHS. \(\tilde{\eta}(\alpha)\) in equation (40) with \(a_4\) in equation (6) or \(a_4\), which is a stationary solution of equation (5) with \(A_2 = 0\), does not fit \(\tilde{\eta}(\alpha)_{\text{DSMC}}\) in the range of \(0 \leq \alpha \leq 0.5\) in the case of the IVHS with \(\Omega = 0.6\). Meanwhile, \(\tilde{\eta}(\alpha)\) in equation (40) with \(a_4\) in equation (6) is more similar to \(\tilde{\eta}(\alpha)_{\text{DSMC}}\) than \(\tilde{\eta}(\alpha)\) in equation (40) with \(a_4\), which is a stationary solution of equation (5) with \(A_2 = 0\), in the range of \(0 \leq \alpha \leq 0.5\) in the case of the IVHS with \(\Omega = 0.6\). \(\tilde{\eta}(\alpha)\) in equation (40) with \(a_4\) in equation (6) does not fit \(\tilde{\eta}(\alpha)_{\text{DSMC}}\) in the range of \(0 \leq \alpha \leq 0.6\) in the case of the IMS.

Meanwhile, the difference between \(\tilde{\eta}(\alpha)_{\text{DSMC}}\) and \(\tilde{\eta}(\alpha)\) in equation (40) with \(a_4\) in equation (6) increases markedly, as \(\alpha\) decreases from \(\alpha = 0.5\) to 0, in the case of the IMS. As a result, the difference in the range of \(0 \leq \alpha \leq 0.5\) between \(\tilde{\eta}(\alpha)_{\text{DSMC}}\) and \(\tilde{\eta}(\alpha)\) in equation (40) with \(a_4\) in equation (6) increases, as \(\Omega\) decreases from unity (IHS) to zero (IMS).

5. Concluding remarks

In this paper, we investigated thermal fluctuations of the granular gas driven by GT on the basis of the two-point kinetic theory. In particular, we considered the IVHS as the component of the
granular gas. GK expressions for the transport coefficients, which were proposed in this paper, approximate to those for the elastic gas by Zwanzig under the elastic limit. Therefore, GK expressions for the transport coefficients by Dufty and Brey. Spherically symmetric moment \( a_6 \), which is calculated using the DSMC method, is similar to \( a_6 \), which is analytically obtained by including \( a_6 \) in the collisional term of \( a_6 \) in the cases of the IHS and IVHS with \( \Omega = 0.6 \). \( a_6 \), which is calculated using the DSMC method, is similar to \( a_6 \), which is analytically obtained, in the range of \( 0.5 \leq \alpha \leq 1 \) in the case of the IMS and in the range of \( 0.2 \leq \alpha \leq 1 \) in the case of IVHS with \( \Omega = 0.6 \), whereas \( a_6 \), which is analytically obtained for the IMS, diverges at \( \alpha \approx 0.366 \). Correlations of thermal fluctuations of the pressure deviator and two times of the heat flux at the same time were evaluated using two parameters \( \psi_2(\hat{\alpha}) \) and \( \psi_3(\hat{\alpha}) \), which is calculated using the DSMC method, is similar to \( \psi_2(\alpha) \) in equation (44) in the range of \( 0.6 \leq \alpha \leq 1 \) in the cases of the IHS and IVHS, whereas \( \psi_2(\alpha) \), which is calculated using the DSMC method, is similar to \( \psi_2(\alpha) \) in equation (44) in the range of \( 0.1 \leq \alpha \leq 1 \) in the case of the IMS. Meanwhile, \( \psi_3(\alpha) \) in equation (45) is similar to \( \psi_3(\alpha) \), which is calculated using the DSMC method, in the range of \( 0.6 \leq \alpha \leq 1 \) in the case of the IHS and in the range of \( 0.5 \leq \alpha \leq 1 \) in cases of the IVHS with \( \Omega = 0.6 \) and IMS. The use of \( (a_6)_{\text{DSMC}} \) and \( (a_6)_{\text{DSMC}} \) in equations (44) and (45) improved similarities between \( \psi_2(\alpha) \), which is calculated using the DSMC method, and \( \psi_2(\alpha) \) in equation (44) with \( a_4 \) in equation (6) or \( \psi_3(\alpha) \), which are calculated using the DSMC method, and \( \psi_3(\alpha) \) in equation (45) with \( a_4 \) and \( a_6 \) in equations (6) and (7). Time correlations of thermal fluctuations of the pressure deviator and two times of the heat flux were evaluated using two parameters, namely, \( \phi_2(\hat{\xi}) \) and \( \phi_3(\hat{\xi}) \), which is calculated using the DSMC method, is similar to \( \phi_2(\hat{\xi}) \) in equation (46) in the cases of the IHS, IVHS with \( \Omega = 0.6 \) and IMS. \( \phi_2(\hat{\xi}) \), which is calculated using the DSMC method, is quite similar to the viscosity coefficient, which is calculated using the DSMC method, in the range of \( 0 \leq \alpha \leq 0.1 \) in the case of the IHS. The thermal conductivity, which is calculated using the DSMC method, does not fit the thermal conductivity, which is obtained by the kinetic theory, in ranges of \( 0.4 \leq \alpha \leq 1 \) and \( 0 \leq \alpha \leq 0.1 \) in the case of the IHS. The thermal conductivity, which is calculated using the DSMC method, does not fit the thermal conductivity, which is obtained by the kinetic theory, in the range of \( 0 \leq \alpha \leq 0.7 \) in cases of the IVHS with \( \Omega = 0.6 \) and IMS. The diffusive thermal conductivity, which is calculated using the DSMC method, is
similar to the diffusive thermal conductivity, which is obtained by the kinetic theory, in the range of $0.7 \leq \alpha \leq 1$ in cases of the IHS, IVHS with $\Omega = 0.6$ and IMS. Finally, differences between the thermal conductivity and diffusive thermal conductivity at $\alpha = 0$, which are calculated using the DSMC method and those obtained by the kinetic theory, increases, as $\Omega$ decreases.

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Appendix A. Definitions of symbols in equations (5)–(7)

$A_i$ and $B_i$ ($i = 0, 1, 2$) in equation (5) are calculated as

$$A_0 = -\frac{5(\alpha^2 - 1)(\alpha^2(\Omega + 5) + 2\Omega - 5)}{2(\Omega + 5)},$$  \hspace{1cm} (A1)

$$A_i = (-\alpha^6(\Omega^3 - 11\alpha^6\Omega^2 - 38\alpha^6\Omega - 40\alpha^2 - \alpha^2\Omega^3 + 4\alpha^2\Omega^3 - 124\alpha^2\Omega + 160\alpha^2 + 32\alpha\Omega + 160\alpha + 2\alpha^3 + 7\Omega^2 + 194\alpha + 40}{96(\Omega + 5))^{-1}}, \hspace{1cm} (A2)

$$A_2 = \{\Omega(-\alpha^6(\Omega + 2)(\Omega + 4)(\Omega + 5) - \alpha^2(\Omega + 4)(\Omega + 5) - 96\alpha(\Omega + 5) + 2\Omega^2 + 7\Omega^2 + 43\Omega + 1280)}\{8640(\Omega + 5))^{-1}, \hspace{1cm} (A3)

$$B_0 = -\frac{15(\alpha^2 - 1)(\alpha^2(\Omega + 5)(\Omega + 7) + 2\alpha^2(\Omega - 7)(\Omega + 5) + \Omega(3\Omega - 20) + 35)}{16(\Omega + 5)}.$$  \hspace{1cm} (A4)

$B_1 = \{-\alpha^6(\Omega + 4)(\Omega + 5)(\Omega + 6)(\Omega + 7) - \alpha^4(\Omega + 5)(\Omega^3 - 11\Omega^2 + 166\Omega + 1064) + 64\alpha^3(\Omega + 5)(\Omega + 7) + 12(15400 - \Omega^4 - 6\Omega^3 + 289\Omega^2 - 1318\Omega) + 64\alpha(\Omega - 7)(\Omega + 5) + 3\Omega^4 + 10\Omega^3 + 70\Omega^2 + 1854\Omega - 9240\}\{256(\Omega + 5))^{-1}, \hspace{1cm} (A5)

$$B_2 = \{-\alpha^6(\Omega + 2)(\Omega + 4)(\Omega + 5)(\Omega + 6)(\Omega + 7) + \alpha^4(\Omega + 2)(\Omega + 5)(\Omega^3 - 11\Omega^2 + 53\Omega + 2856) - 192\alpha^3(\Omega + 2)(\Omega + 5)(\Omega + 7) + \alpha^2(\Omega^3 - 4\Omega^4 + 581\Omega^3 - 2084\Omega^2 + 14796\Omega - 114000) \hspace{1cm} \nonumber \nonumber$$

$$- 192\alpha(\Omega^3 + 17\Omega + 210) - (3\Omega^5 + 16\Omega^4 + 1655\Omega^3 + 8084\Omega^2 + 56836\Omega + 30000)\{23040(\Omega + 5))^{-1}, \hspace{1cm} (A6)$$

$\beta_i$ in equation (6) are calculated as

$$\beta_0 = 2880(-6\Omega^2 - 335\Omega^4 - 2068\Omega^3 - 13967\Omega^2 + 64830\Omega - 25000),$$

$$\beta_2 = -46080(\Omega + 5)(\Omega^3 - 10\Omega^2 + 183\Omega - 420),$$

$$\beta_3 = 2880(5\Omega^5 + 191\Omega^4 - 1149\Omega^3 - 637\Omega^2 - 176890\Omega + 145000),$$

$$\beta_4 = -46080(\Omega + 5)(3\Omega^3 + 40\Omega^2 - 113\Omega + 700),$$

$$\beta_5 = 5760(2\Omega^5 + 2080\Omega^4 + 3090\Omega^3 + 13657\Omega^2 + 54950\Omega - 119400),$$

$$\beta_6 = 46080(\Omega + 5)(3\Omega^3 + 140\Omega^2 - 13\Omega + 40),$$

$$\beta_7 = -5760(\Omega + 5)(92\Omega^3 + 491\Omega^2 + 538\Omega - 14120),$$

$$\beta_8 = 46080(\Omega + 4)(\Omega + 5)^2(\Omega + 7),$$

$$\beta_9 = -2880(\Omega + 5)(2\Omega^4 + 55\Omega^3 + 581\Omega^2 + 2938\Omega + 4200),$$

$$\beta_{10} = 0,$$

$$\beta_{11} = -2880(\Omega + 2)(\Omega + 4)(\Omega + 5)^2(\Omega + 7),$$

$\gamma_i$ in equation (6) are calculated as

$$\gamma_0 = 240^2 + 2460\Omega^6 + 28360\Omega^6 + 361324\Omega^4 + 669696\Omega^3 + 25479248\Omega^2 + 16561280\Omega - 2400000,$$

$$\gamma_1 = -46080(\Omega + 5)(\Omega^3 - 10\Omega^3 + 183\Omega - 420),$$

$$\gamma_2 = 128(\Omega + 5)(\Omega^3 - 9\Omega^4 + 115\Omega^3 + 1083\Omega^2 + 75082\Omega - 9960),$$

$$\gamma_3 = -20\Omega^2 - 1540\Omega^6 + 4440\Omega^5 - 155876\Omega^4 + 1711888\Omega^3 - 34234800\Omega^2 + 20982400\Omega + 12422400,$$

$$\gamma_4 = 384(\Omega + 5)(\Omega^3 + 21\Omega^4 + 95\Omega^3 + 653\Omega^2 - 17858\Omega + 26280),$$

$$\gamma_5 = -16\Omega^2 - 2784\Omega^6 - 61392\Omega^5 - 536904\Omega^4 - 3784320\Omega^3 + 10602528\Omega^2 - 9758720\Omega + 4089600.$$
\[ \gamma_n = -384(\Omega + 5)(\Omega^5 + 11\Omega^4 + 73\Omega^3 + 477\Omega^2 + 1574\Omega + 4200), \]
\[ \gamma_i = 8(\Omega + 5)(156\Omega^2 + 198\Omega^3 + 20240\Omega^3 + 29444\Omega^2 - 258752\Omega - 459840), \]
\[ \gamma_{g} = -128(\Omega + 4)(\Omega + 5)^2(\Omega + 7)(\Omega^2 + 51\Omega + 18), \]
\[ \gamma_o = 4(\Omega + 4)(\Omega + 5)(2\Omega^5 + 103\Omega^4 + 1187\Omega^3 + 6288\Omega^2 + 18100\Omega + 21840), \]
\[ \gamma_0 = 0, \]
\[ \gamma_{11} = 4(\Omega + 2)(\Omega + 4)^2(\Omega + 5)(\Omega + 6)(\Omega + 7), \]
\[ \beta_i' \text{ in equation (7) are calculated as } \]
\[ \beta_0' = 43200(6\Omega^4 + 213\Omega^3 + 9226\Omega^2 - 8345\Omega + 12000), \]
\[ \beta_i' = -86400(6\Omega^4 + 2095\Omega^3 + 9749\Omega^2 - 8215\Omega + 10500), \]
\[ \beta_0' = 43200(13\Omega^4 + 5140\Omega^3 + 39410\Omega^2 - 46102\Omega - 11900), \]
\[ \beta_i' = -86400(\Omega + 5)(7\Omega^3 + 2582\Omega + 1473\Omega - 4060), \]
\[ \beta_0' = 43200(\Omega + 5)(11\Omega^2 + 315\Omega + 1956\Omega - 1820), \]
\[ \beta_i' = -86400(\Omega + 4)(\Omega + 5)(\Omega + 7), \]
\[ \beta_0' = 43200(\Omega + 4)(\Omega + 5)^2(\Omega + 7), \]
\[ \beta_i' \text{ in equation (7) are calculated as } \]
\[ \gamma_{0} = 6\Omega^7 + 615\Omega^6 + 7900\Omega^5 + 9331\Omega^4 + 167424\Omega^3 + 6369812\Omega^2 + 4140320\Omega - 600000, \]
\[ \gamma_{1} = -12\Omega^7 - 1198\Omega^6 - 14308\Omega^5 - 178422\Omega^4 - 281792\Omega^3 - 1016372\Omega^2 + 3413760\Omega - 393600, \]
\[ \gamma_{2} = 13\Omega^7 + 1396\Omega^6 + 21637\Omega^5 + 227544\Omega^4 + 824132\Omega^3 + 5398928\Omega^2 - 5722240\Omega + 4492800, \]
\[ \gamma_{3} = -2(\Omega + 5)(7\Omega^3 + 714\Omega^2 + 9665\Omega^2 + 8040\Omega^2 + 227052\Omega - 117728\Omega - 402240, \]
\[ \gamma_{4} = (\Omega + 5)(11\Omega^2 + 849\Omega^2 + 1711\Omega^2 + 94612\Omega^2 + 273096\Omega^2 - 43648\Omega - 462720), \]
\[ \gamma_{5} = -2(\Omega + 4)(\Omega + 5)(4\Omega^3 + 167\Omega^3 + 2005\Omega^2 + 10080\Omega^2 + 20858\Omega + 35280), \]
\[ \gamma_{6} = (\Omega + 4)(\Omega + 5)(5\Omega^3 + 175\Omega^3 + 1856\Omega^3 + 29276\Omega^2 + 24448\Omega^2 + 26880), \]
\[ \gamma_{7} = -2(\Omega + 2)(\Omega + 4)^2(\Omega + 5)(\Omega + 6)(\Omega + 7), \]
\[ \gamma_{8} = -2(\Omega + 2)(\Omega + 4)^2(\Omega + 5)^2(\Omega + 6)(\Omega + 7), \]
\[ \text{(A8)} \]

The cooling rate \( \zeta \) never emerges in equations (B7) and (B8), unlike equations (28) and (29), whereas \( \nu_s \) and \( \nu_s \) are a function of time, because the temperature depends on \( \epsilon \).

The solutions of equations (B7) and (B8) are obtained as
\[ Q_{ij,lm}^{(2,2)}(\epsilon) = \exp \left( -\int_0^\epsilon \nu_s(\tilde{\epsilon}) \, d\tilde{\epsilon} \right) Q_{ij,lm}^{(2,2)} \big|_{\epsilon = 0}, \]
\[ Q_{ij}^{(3,3)}(\epsilon) = \exp \left( -\int_0^\epsilon \nu_s(\tilde{\epsilon}) \, d\tilde{\epsilon} \right) Q_{ij}^{(3,3)} \big|_{\epsilon = 0}, \]
\[ \text{where } s \in t. \]
From equations (B9) and (B10), we obtain
\[
\int_0^t \tilde{Q}_{ij,lm}^{(2,2)}(\epsilon) \, d\epsilon = -\left[ \nu_\epsilon^{-1}(\alpha, \epsilon) \exp \left( - \int_0^t \nu_\epsilon(\alpha, s) \, ds \right) \right] \left[ \tilde{Q}_{ij,lm}^{(2,2)} \right]_{t=0}. \tag{B11}
\]
\[
\int_0^t \tilde{Q}_{ij}^{(3)}(\epsilon) \, d\epsilon = -\left[ \nu_\eta^{-1}(\alpha, \epsilon) \exp \left( - \int_0^t \nu_\eta(\alpha, s) \, ds \right) \right] \left[ \tilde{Q}_{ij}^{(3)} \right]_{t=0}. \tag{B12}
\]
The transport coefficients of the IVHS, which are calculated by the CE method, are written as [1]
\[
\mu(\alpha, \Omega) = \frac{p(\tilde{\alpha})}{\nu_\epsilon(\tilde{\alpha})}(1 - \chi_\rho), \tag{B13}
\]
\[
\kappa(\alpha, \Omega) = \phi_T \frac{p(\tilde{\alpha}) R}{\nu_\eta(\tilde{\alpha})}(1 - \chi_T), \tag{B14}
\]
\[
\eta(\alpha, \Omega) = \frac{1}{1 - \chi_\rho} \left( \frac{\chi_T}{2(1 - \chi_T)} \phi_T + \phi_\eta \right) \times \frac{p(\tilde{\alpha}) R T}{\nu_\eta(\tilde{\alpha})} \frac{1}{\nu_\eta(\tilde{\alpha})}, \tag{B15}
\]
where \( \chi_\rho := (\zeta/\nu_\rho) (1 - \Omega/2) \), \( \chi_T := 2 \zeta/\nu_\eta \), \( \chi_{\eta} := (2 - \Omega/2) \zeta/\nu_\eta \), \( \phi_T := a_4/6 \) and \( \phi_\rho := 5/2 + a_4/3 \). \( \mu(\alpha, \Omega) \), \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \) at \( t \) cannot be expressed with \( \int_0^t \tilde{Q}_{ij,lm}^{(2,2)}(\epsilon) \, d\epsilon \), \( \int_0^t \tilde{Q}_{ij}^{(3)}(\epsilon) \, d\epsilon \) and \( \tilde{Q}_{ij,lm}^{(2,2)} \left|_{t=0} \right. \) explicitly, because we cannot express \( \nu_\epsilon \) with \( \int_0^t \tilde{Q}_{ij,lm}^{(2,2)}(\epsilon) \, d\epsilon \) and \( \tilde{Q}_{ij,lm}^{(2,2)} \left|_{t=0} \right. \) in equation (B11) and \( \nu_\eta \) with \( \int_0^t \tilde{Q}_{ij}^{(3)}(\epsilon) \, d\epsilon \) and \( \tilde{Q}_{ij}^{(3)} \left|_{t=0} \right. \) in equation (B12). Of course, \( \mu(\alpha, 1), \kappa(\alpha, 1) \) and \( \eta(\alpha, 1) \) in equations (B13)–(B15) coincide with those calculated for the IHS by Brey et al [28]. On the other hand, we can obtain initial values of \( \nu_\epsilon \) and \( \nu_\eta \) by setting \( t = \infty \) such as
\[
\nu_\epsilon(\alpha, 0) = \left[ \tilde{Q}_{ij,lm}^{(2,2)} \right]_{t=0}, \quad \int_0^\infty \tilde{Q}_{ij,lm}^{(2,2)}(\epsilon) \, d\epsilon,
\]
\[
\nu_\eta(\alpha, 0) = \left[ \tilde{Q}_{ij}^{(3)} \right]_{t=0}, \quad \int_0^\infty \tilde{Q}_{ij}^{(3)}(\epsilon) \, d\epsilon. \tag{B16}
\]
Substituting equation (B16) into equations (B13)–(B15), we obtain the modified GK expressions for the transport coefficients using initial values of transport coefficients. Of course, such modified GK expressions obtained using initial values of transport coefficients are nonlinear responses. In the DSMC calculation, the transport coefficients at \( t \) are obtained when we calculate \( \int_0^\infty \tilde{Q}_{ij,lm}^{(2,2)}(\epsilon) \, d\epsilon \) and \( \int_0^\infty \tilde{Q}_{ij}^{(3)}(\epsilon) \, d\epsilon \) by setting \( t = \infty \).

Another approach to calculate transport coefficients for the granular gas under the HCS, we consider the map:
\[
e \to e' = \sqrt{RT_{\infty}(T/T_{\infty})^{1/2}/L_{\infty}}^{-1},
\]
in which quantities with subscript \( \infty \) correspond to representative values, in accordance with the map by Dufty and Brey [3]. This map nondimensionalizes dissipation rates \( \nu_\epsilon \) and \( \nu_\eta \), which depend on time, in time independent forms. Applying this map to equations (B7) and (B8), we obtain
\[
\int_0^\infty \tilde{Q}_{ij,lm}^{(2,2)}(e') \, de' \text{ as}
\]
\[
\int_0^\infty \tilde{Q}_{ij}^{(3)}(e') \, de' \text{ as}
\]
\[
\int_0^\infty \tilde{Q}_{ij}^{(3)}(e') \, de' = \left[ \tilde{Q}_{ij}^{(3)} \right]_{t=0}, \quad \int_0^\infty \tilde{Q}_{ij}^{(3)}(\epsilon) \, d\epsilon. \tag{B17}
\]
\[
\int_0^\infty \tilde{Q}_{ij}^{(3)}(e') \, de' = \left[ \tilde{Q}_{ij}^{(3)} \right]_{t=0}, \quad \int_0^\infty \tilde{Q}_{ij}^{(3)}(\epsilon) \, d\epsilon. \tag{B18}
\]
where \( \tilde{v}_\epsilon = e' \nu_\epsilon \) and \( \tilde{v}_\eta = e' \nu_\eta \). As mentioned above, \( \tilde{v}_\epsilon \) and \( \tilde{v}_\eta \) are time independent. Substituting \( \tilde{v}_\epsilon \) and \( \tilde{v}_\eta \) in equations (B17)–(B18) into equations (B13)–(B15), we obtain another form of the modified GK expressions for the transport coefficients. In the DSMC calculation, we use the time interval \( de' \) to integrate \( \tilde{Q}_{ij,lm}^{(2,2)}(e') \) and \( \tilde{Q}_{ij}^{(3)}(e') \) in equations (B17)–(B18), which are calculated using the temperature by each step time. The relation between the modified GK expressions for the transport coefficients, which was introduced on the basis of the two-point kinetic theory, and GK expressions by Dufty and Brey [4] must be addressed in our future work.

Appendix C. Comments on other possibilities for kinetic calculation of transport coefficients

The differences between \( \kappa \) and \( \eta \), which were obtained on the basis of GK expressions by Dufty and Brey [3], and those obtained using the first order approximation in the CE method, were numerically confirmed using the DSMC method by Brey et al [4]. Then, Garzo, Santos and Montanero [6] proposed the modified CE method to calculate the transport coefficients, \( \mu \), \( \kappa \) and \( \eta \) together with the self-diffusion coefficient. In particular, \( \kappa \) and \( \eta \), which were calculated using the modified CE method [6] in a similar way to the method by Lutsko [29], are much more similar to \( \kappa \) and \( \eta \) [4], which were calculated on the basis of GK expressions by Dufty and Brey using the DSMC method, than \( \kappa \) and \( \eta \), which were calculated using the first order approximation in the conventional CE method. Such modified CE method expands \( f(e) \) not around \( f_{MB}(e) \) but \( f^{(0)}(e) \) in equation (2), where coefficients in the modified Sonine polynomial are determined to satisfy the orthogonality between two different Sonine polynomials. Provided that Grad’s method is used to expand \( f(e) \) around \( f^{(0)}(e) \), the modified CE method corresponds to the modified Grad’s method, in which \( f(e) \) is expanded around \( f^{(0)}(e) \) in equation (2) using modified Hermite polynomials, whose coefficients in Hermite polynomials and coefficients on Hermite polynomials are determined using the orthogonality between different modified Hermite polynomials or definitions of moments, respectively, such that coefficients \( \xi_j^{(2)} \) and \( \phi_j^{(3)} \) in modified Hermite polynomials, \( \hat{H}_j^{(2)} = \phi_j \psi_j - \delta_j \phi_j^{(2)} \) and \( \hat{H}_j^{(3)} = \psi_j (\psi_j^2 - 5 \phi_j^{(3)}) \) are determined by their orthogonality, namely,
\[
\int_{-\infty}^{\infty} \hat{H}_j^{(2)} f^{(0)}(e) \, de = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \hat{H}_j^{(3)} H_j^{(1)} f^{(0)}(e) \, de = 0 \quad (H_j^{(1)} = \psi_j),
\]
and coefficients on Hermite polynomials \( \xi_j^{(2)} \) and \( \phi_j^{(3)} \) are
determined by the definition of moments, \( p(\varepsilon)/p = \rho^{-1}\int_{0}^{\varepsilon} H(\varepsilon)\frac{d\varepsilon}{\varepsilon} \) and \( q_{i}/(\sqrt{\rho R T}) = 1/2\rho^{-1}\int_{0}^{\varepsilon} H(\varepsilon)\frac{d\varepsilon}{\varepsilon} \). Such modified Hermite polynomials were also applied to the quantum gas by the author [30]. We, however, note that such modified Sonine or Hermite polynomials do not always satisfy the completeness of the expansion of \( f(\varepsilon) \) in a mathematical sense. For instance, \( f(\varepsilon) \) is expanded around \( f^{(0)}(\varepsilon) \) in equation (2) using modified Hermite polynomials, namely, \( \tilde{H}_{\alpha}^{(n)} \), such as

\[
\tilde{f}(\varepsilon) = \tilde{f}_{MB}(\varepsilon) \left( 1 + \frac{\tilde{c}_{ij}^{(2)} p_{ij} \tilde{H}_{ij}^{(2)}}{2 \rho} + \frac{\varepsilon_{ij}^{(3)} q_{ij} \tilde{H}_{ij}^{(3)}}{5 \rho \sqrt{R T}} \right) \times \left( 1 + \frac{1}{120} a_{4} \tilde{H}^{(4)} + \frac{1}{5400} a_{3} \tilde{H}^{(6)} \right),
\]

where \( \tilde{c}_{ij}^{(2)} = 1 \), \( \tilde{H}_{ij}^{(2)} = H_{ij}^{(2)} \), \( \varepsilon_{ij}^{(3)} = 1 + \frac{a_{4}}{15} \). \( (C1) \)

From equation (C1), we can calculate \( \mu(\alpha, \Omega), \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \) for the IVHS, whereas such calculations of the transport coefficients will be described elsewhere.

As one possibility for the improvement of differences between \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \), which are calculated by equations (38)-(40), and \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \), which are calculated by equations (41)-(43) using the DSMC method, nonequilibrium moments are considered in the definition of the cooling rate. In this paper, we used the cooling rate \( \zeta \) in equation (4), which never depends on nonequilibrium moments. Provided that \( f(\varepsilon) = f^{(0)}(\varepsilon) \) in equation (2), the cooling rate is obtained by neglecting all the nonlinear terms as

\[
\zeta'(\alpha, \Omega) = \frac{5}{2(5 + \Omega)\tau} (1 - \alpha^{2})
\times \left[ 1 + \frac{\Omega(2 + \Omega)}{240} a_{4} - \frac{\Omega(4 - \Omega^{2})}{2160} a_{6} \right], \quad (C2)
\]

where \( \zeta'(\alpha, 1) \) for the IHS is surely obtained in cases of the IHS, IVHS with \( \Omega = 0.6 \) and IMS, which are calculated by Brilliantov and Pöschel [19].

Here, we must confirm \( \tilde{\zeta}_{\text{neq}} \ll 1 \) to validate \( \zeta \) in equation (4), which was used for the IHS, IVHS with \( \Omega = 0.6 \) and IMS in our analytical results. Figure 11 shows \( \tilde{\zeta}_{\text{neq}} \) versus \( \alpha \) in cases of the IHS, IVHS with \( \Omega = 0.6 \) and IMS. As shown in figure 11, \( \tilde{\zeta}_{\text{neq}} \ll 1 \) is surely obtained in cases of the IHS, IVHS with \( \Omega = 0.6 \) and IMS. In particular, \( \zeta \) does not depend on nonequilibrium moments in the case of the IMS. Therefore, the modification of \( \zeta \) in equation (4) with \( \zeta' \) in equation (C1) does not improve differences between \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \), which are calculated by equations (38)-(40), and \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \), which are calculated by equations (41)-(43) using the DSMC method.

As another possibility for the improvement of differences between \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \), which are calculated by equations (38)-(40), and \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \), which are calculated by equations (41)-(43) using the DSMC method, calculated by equations (41)-(43) using the DSMC method, nonlinear collisional moments are considered. For example, moment equations of \( p_{ij} \) and \( q_{ij} \) for the IVHS driven by GT are written, when we substitute Grad's 14 moment equation, namely,

\[
f(\varepsilon) = f_{MB}(\varepsilon) \left[ 1 + \frac{p_{ij}}{2(\rho R T)} + q_{ij} H^{(3)}(5(\rho R T)) + a_{4} H^{(4)}(120) \right]
\]

into equation (1), multiply \( C_{i} C_{i} - \delta_{ij} C^{2}/3 \) and \( C_{i} C^{2}/2 \) by both sides of equation (1) and integrate over \( V \), as [1]

\[
\frac{\partial p_{ij}}{\partial t} + \frac{\partial q_{ij}}{\partial x_{k}} + \frac{5}{2} \frac{\partial R T}{\partial x_{k}} + \frac{5}{4} \frac{\partial q_{ij}}{\partial x_{k}} + RT \frac{\partial p_{ij}}{\partial x_{k}} = 0
\]

\[
\frac{\partial q_{ij}}{\partial t} + \frac{\partial q_{ij}}{\partial x_{k}} + \frac{5}{2} \frac{\partial R T}{\partial x_{k}} + \frac{5}{4} \frac{\partial q_{ij}}{\partial x_{k}} + RT \frac{\partial p_{ij}}{\partial x_{k}} = 0
\]

\[
\frac{2}{5} \frac{\partial u_{k}}{\partial x_{k}} + \frac{1}{6} \frac{\partial s}{\partial x_{k}} = \left[ \frac{3}{2} C_{i} \right]
\]

\[
\nu_{q} \left( 1 + \left( \delta_{ij} - \frac{2}{2} (\Omega + \alpha (\Omega - 10) + \Omega) \right) \frac{A_{ij}}{\rho R T} \right) = 0
\]

where \( A_{ij} \) is the traceless tensor and \( s = \rho (RT)^{2} A_{ij} \).

Effects of nonlinear collisional moments are markedly small owing to \( |\beta_{\parallel}| \ll 1 \) and \( |\beta_{\perp}| \ll 1 \) in equations (C3) and (C4), as described in the author's previous study [1]. In particular, \( \beta_{\parallel} = \beta_{\perp} = 0 \) is always obtained for the IMS, because of \( \Omega = 0 \) in equations (C3) and (C4). From one to one correspondence between \( p_{ij} \) in equation (28) and \( Q_{ij,ln}^{(2,3)} \) in equation (C3) or \( q_{ij} \) in equation (29) and \( Q_{ij,ln}^{(1,3)} \) in equation (C3), inclusions of nonlinear collisional moments in equations (28) and (29) do not improve differences between \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \), which are calculated by equations (38)-(40), and \( \kappa(\alpha, \Omega) \) and \( \eta(\alpha, \Omega) \), which are calculated by equations (41)-(43) using the DSMC method.

![Figure 11. \( \tilde{\zeta}_{\text{neq}} \) versus \( \alpha \) in cases of the IHS, IVHS with \( \Omega = 0.6 \) and IMS.](image-url)
From above discussions, the only remaining way to improve differences between \(\kappa(\alpha, \Omega)\) and \(\eta(\alpha, \Omega)\), which are calculated by equations (38)–(40), and \(\kappa(\alpha, \Omega)\) and \(\eta(\alpha, \Omega)\), which are calculated by equations (41)–(43) using the DSMC method, is to expand \(f(c)\) around \(f_0(c)\) in equation (2) using the modified Sonine polynomials or modified Hermite polynomials such as equation (C1), when we restrict ourselves to the form of \(f_0(c)\) to \(f_0(c)\) in equation (2). Therefore, the expansion of \(f(c)\) around \(f_0(c)\) is worth trying to improve differences between \(\kappa(\alpha, \Omega)\) and \(\eta(\alpha, \Omega)\), which are calculated by equations (38)–(40), and \(\kappa(\alpha, \Omega)\) and \(\eta(\alpha, \Omega)\), which are calculated by equations (41)–(43) using the DSMC method. As described above, such an expansion of \(f(c)\) around \(f_0(c)\) is, however, artificial from the viewpoint of the completeness of the expansion, whereas all the modified Sonine or Hermite polynomials are a presumably insufficient basis to cover the functional space mapped by \(f(c)\) in \(\mathbb{V}\), because the expansion of \(f(c)\) around \(f_{\text{MB}}(c)\) with Sonine or Hermite polynomials is complete for the elastic gas owing to Gaussian form of \(f_{\text{MB}}(c)\), accidentally.

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