Algebraic independence of certain Mahler numbers

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Abstract: In this note we prove algebraic independence results for the values of a special class Mahler functions. In particular, the generating functions of Thue-Morse, regular paperfolding and Cantor sequences belong to this class, and we obtain the algebraic independence of the values of these functions at every non-zero algebraic point in the open unit disk. The proof uses results on Mahler’s method.

Keywords: Algebraic independence of numbers, Mahler’s method, Thue-Morse-Mahler numbers, regular paperfolding numbers.

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1. Introduction and results

In the present paper we are interested in the values of special degree 1 Mahler functions \( F(z) \) satisfying a functional equation of the form

\[
p(z) + p_0(z)F(z) + p_1(z)F(z^d) = 0,
\]

where \( d \geq 2 \) is an integer and \( p(z), p_0(z), p_1(z) \) are polynomials satisfying \( p_0(z)p_1(z) \neq 0 \). The values \( F(1/b) \) with integers \( b \geq 2 \) are called Mahler numbers. The arithmetic properties of such numbers has been an active research area in last years. In a remarkable work Bugeaud [1] proved that the irrationality exponent of the Thue-Morse-Mahler numbers \( f_{TMM}(1/b) \) equals 2, here

\[
f_{TMM}(z) = \sum_{n=0}^{\infty} t_n z^n,
\]

and \( (t_n) \) is the famous Thue-Morse sequence defined recursively by \( t_0 = 0, t_{2n} = t_n, t_{2n+1} = 1 - t_n \) \((n \geq 0)\). Then similar results were proved by Coons [3] for the values of the functions

\[
G(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 - z^{2^n}},
\]

\[
F(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 + z^{2^n}},
\]

and by Guo, Wen and Wu [5] and Wen and Wu [10] for the values of

\[
f_{RPF}(z) = \sum_{n=0}^{\infty} u_n z^n, \quad f_C(z) = \sum_{n=0}^{\infty} v_n z^n,
\]

respectively, where \( (u_n) \) is the regular paperfolding sequence defined by \( u_{4n} = 1, u_{4n+2} = 0, u_{2n+1} = u_n \) \((n \geq 0)\) and \( (v_n) \) is the Cantor sequence on \( \{0, 1\} \) such that \( v_n = 1 \) \((n \geq 0)\) if and only if the ternary expansion of \( n \) does not contain the digit 1. For a unified expression of these (and other) results we refer to [2]. It is also well-known that all these functions obtain
transcendental values at every non-zero algebraic point in the open unit disk \( \mathbb{D} \). Here our aim is to consider the algebraic independence of these special Mahler numbers.

**Theorem 1.** For every non-zero algebraic \( \alpha \in \mathbb{D} \) the numbers \( f_{TMM}(\alpha) \), \( f_{RPF}(\alpha) \) and \( G(\alpha) \) are algebraically independent over \( \mathbb{Q} \). The same holds if we replace \( G(\alpha) \) by \( F(\alpha) \).

**Corollary 1.** For every integer \( b \geq 2 \), the three numbers

\[
f_{TMM}(\frac{1}{b}), \ f_{RPF}(\frac{1}{b}), \sum_{n=0}^{\infty} \frac{1}{b^{2n} + 1}
\]

are algebraically independent over \( \mathbb{Q} \).

Note that if \( b = 2 \), then the latest number above is the reciprocal sum of Fermat numbers.

Theorem 1 is obtained from the following more general result. To introduce this, let \( d \geq 2 \) be a fixed integer, and denote

\[
T_d(z) = \prod_{n=0}^{\infty} (1 - z^{dn}), \quad U_d(z) = \prod_{n=0}^{\infty} (1 + z^{2dn}),
\]

\[
G_{d,j}(z) = \sum_{n=0}^{\infty} \frac{z^{dn}}{1 - z^{dn+j}}, \quad j \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}.
\]

Then we have

**Theorem 2.** Let \( \alpha \in \mathbb{D} \setminus \{0\} \) be an algebraic number. Then the numbers \( T_2(\alpha) \) and \( G_{2,j}(\alpha), j \in \mathbb{N}_0 \setminus \{1\}, \) are algebraically independent over \( \mathbb{Q} \).

**Theorem 3.** Let \( \alpha \in \mathbb{D} \setminus \{0\} \) be an algebraic number. If \( d \geq 3 \), then the numbers \( T_d(\alpha), U_d(\alpha) \) and \( G_{d,j}(\alpha), j \in \mathbb{N}_0, \) are algebraically independent over \( \mathbb{Q} \).

Since

\[
T_2(z) = \frac{1}{1 - z} - 2f_{TMM}(z)
\]

and

\[
G_{2,2}(z) = zf_{RPF}(z),
\]

see [2], Theorem 2 implies Theorem 1.

**Theorem 4.** Let \( \alpha \in \mathbb{D} \setminus \{0\} \) be an algebraic number. Then the numbers

\[
T_d(\alpha), U_3(\alpha), G_{3,1}(\alpha), G_{d,j}(\alpha), \quad d = 2, 3; j \in \mathbb{N}_0 \setminus \{1\};
\]

are algebraically independent over \( \mathbb{Q} \).

Since \( U_3(z) = f_C(z) \), we immediately obtain the following
Corollary 2. Let \( \alpha \in \mathbb{D} \setminus \{0\} \) be an algebraic number. Then the numbers \( f_{TMM}^{\mathcal{T}}(\alpha), f_{RPF}^{\mathcal{R}}(\alpha), F(\alpha) \) and \( f_C(\alpha) \) are algebraically independent over \( \mathbb{Q} \). In particular, for every integer \( b \geq 2 \), the four numbers
\[
f_{TMM}^{\mathcal{T}}\left(\frac{1}{b}\right), f_{RPF}^{\mathcal{R}}\left(\frac{1}{b}\right), \sum_{n=0}^{\infty} \frac{1}{b^{2n} + 1}, f_C\left(\frac{1}{b}\right)
\]
are algebraically independent over \( \mathbb{Q} \).

To prove Theorems 2, 3 and 4 we consider in Chapter 2 algebraic independence over \( \mathbb{C}(z) \) of the functions \( \mathcal{I} \). Then Mahler’s method can be used to prove our theorems in Chapter 3.

2. Algebraic independence of functions

To study algebraic independence of the functions \( \mathcal{I} \) we use the following special case of a result of Kubota \([6]\) to be found also in Nishioka \([8, \text{Theorem 3.5}]\).

**Theorem K.** Let us assume that \( f_j(z) \in \mathbb{C}[[z]] \setminus \{0\} \) (\( j = 0, 1, \ldots, m + h \)) converge on \( \mathbb{D} \) and satisfy the functional equations
\[
f_j(z^d) = a(z)f_j(z) + a_j(z), \; j = 0, 1, \ldots, m, \; f_m(z^d) = b_i(z)f_{m+i}(z), \; i = 1, \ldots, h,
\]
with \( a(z), a_j(z), b_i(z) \in \mathbb{C}(z) \setminus \{0\} \). Then the functions \( f_j(z) \) (\( j = 0, 1, \ldots, m + h \)) are algebraically independent over \( \mathbb{C}(z) \), if \( a(z), a_j(z) \) and \( b_i(z) \) satisfy the following conditions.

(i) If \( c_0, \ldots, c_m \in \mathbb{C} \) are not all zero, then the functional equation
\[
g(z^d) = a(z)g(z) - \sum_{j=0}^{m} c_j a_j(z)
\]
does not have a solution \( g(z) \in \mathbb{C}(z) \).

(ii) For any \( (n_1, \ldots, n_h) \in \mathbb{Z}^h \setminus \{0\} \) the functional equation
\[
r(z^d) = \prod_{i=1}^{h} b_i(z)^{n_i} r(z)
\]
does not have a solution \( r(z) \in \mathbb{C}(z) \setminus \{0\} \).

The functions \( \mathcal{I} \) satisfy functional equations of the form \( \mathcal{I} \), namely
\[
T_d(z^d) = \frac{1}{1 - z} T_d(z), \; U_d(z^d) = \frac{1}{1 + z^2} U_d(z), \; G_d, j(z^d) = G_d, j(z) - \frac{z}{1 - z^d}, \; j \in \mathbb{N}_0.
\]

Applying Theorem K we prove first

**Lemma 1.** If \( d \geq 3 \), then the functions \( \mathcal{I} \) are algebraically independent over \( \mathbb{C}(z) \).

**Proof.** Assume, against Lemma 1, that there exists an integer \( m \geq 1 \) such that the functions \( T_d(z), U_d(z) \) and \( G_d, j(z) \), \( 0 \leq j \leq m \), are algebraically dependent. We shall prove that these functions satisfy conditions (i) and (ii) of Theorem K and thus obtain a contradiction with this assumption.
Let us consider (i) first. Assume that $c_0, c_1, \ldots, c_m \in \mathbb{C}$ are not all zero. If the functional equation

$$g(z^d) = g(z) - \sum_{j=0}^{m} \frac{c_j z}{1 - z^{d^j}}$$

has a rational solution $g(z)$, then by [9, Lemma 1] we have

$$g(z) = \frac{A(z)}{1 - z^{d^m}},$$

where $A(z)$ is a polynomial. By (5), $\deg A(z) \leq d^m$. Thus there exist $c \in \mathbb{C}$ and a polynomial $B(z) \neq 0$ with $\deg B(z) < d^m$ such that

$$g(z) = c + \frac{B(z)}{1 - z^{d^m}}.$$

Letting $z \to \infty$ in (5) we get $c = c + c_0, c_0 = 0$, and

$$\frac{B(z^d)}{1 - z^{d^{m+1}}} = \frac{B(z)}{1 - z^{d^m}} - \sum_{j=1}^{m} \frac{c_j z}{1 - z^{d^j}}.$$  

We use induction to prove that this is not possible.

If $m = 1$, then (6) is of the form

$$\frac{B(z^d)}{1 - z^{d^2}} = \frac{B(z)}{1 - z^d} - \frac{c_1 z}{1 - z}, \quad c_1 \neq 0,$$

and so

$$B(z^d) = (B(z) - c_1 z)(1 + z^d + z^{2d} + \cdots + z^{(d-1)d}).$$

By comparing the coefficients of $z^{kd}$ in this equation we get

$$B(z) = b_0(1 + z + \cdots + z^{d-1}) = b_0 \frac{1 - z^d}{1 - z}$$

implying

$$b_0 = b_0(1 + z + \cdots + z^{d-1}) - c_1 z.$$

Since $d \geq 3$, this leads to a contradiction $b_0 = c_1 = 0$.

Assume now that (6) is not possible, if $m$ is replaced by $m - 1 \geq 1$. If we denote

$$B(z) = \sum_{j=0}^{d^{m-1}} b_j z^j,$$

then (6) implies

$$\sum_{j=0}^{d^{m-1}} b_j z^{d^j} = (\sum_{j=0}^{d^{m-1}} b_j z^j)(1 + z^{d^m} + \cdots + z^{(d-1)d^m}) - (1 - z^{d^{m+1}}) \sum_{j=1}^{m} \frac{c_j z}{1 - z^{d^j}}.$$
We compare again the coefficients of \( z^{kd} \) in this equation to get \( b_0 = b_{dm-1} = b_{2dm-1} = \cdots = b_{(d-1)dm-1} = b_{dm-1+1} = b_{2dm-1+1} = \cdots = b_{(d-1)dm-1+1} = \cdots = b_{dm-1} \), which means that

\[
B(z) = \left( \sum_{j=0}^{dm-1-1} b_j z^j \right) (1 + z^{dm-1} + \cdots + z^{(d-1)dm-1}) =: B_1(z) \frac{1 - z^{dm}}{1 - z^{dm-1}}.
\]

Then, by (6),

\[
\frac{B_1(z^d)}{1 - z^{dm}} = \frac{B_1(z)}{1 - z^{dm-1}} - \frac{m}{1 - z^{d^m}} \sum_{j=1}^{m} c_j z.
\]

If \( c_m = 0 \), then we have a contradiction by our induction hypothesis. Therefore we necessarily have \( c_m \neq 0 \), and

\[
B_1(z^d) = B_1(z)(1 + z^{dm-1} + \cdots + z^{(d-1)dm-1}) - (1 - z^{dm}) \sum_{j=1}^{m} c_j z.
\]

Repeating the above consideration we get

\[
B_1(z) = \left( \sum_{j=0}^{dm^2-1} b_j z^j \right) (1 + z^{dm^2-1} + \cdots + z^{(d-1)dm^2-1}) =: B_2(z) \frac{1 - z^{dm^2}}{1 - z^{dm^2-1}}.
\]

So we have

\[
\frac{B_2(z^d)}{1 - z^{dm^2}} = \frac{B_2(z)}{1 - z^{dm^2-1}} - \sum_{j=1}^{m} c_j z
\]

where \( c_m \neq 0 \). By comparing the poles on both sides of this equation we now get a contradiction.

We next consider the condition (ii). Assume that for some pair \( (n_1, n_2) \neq 0 \) the functional equation

\[
r(z^d) = (1 - z)^{-n_1} (1 + z^2)^{-n_2} r(z)
\]

has a rational solution \( r(z) \neq 0 \), and denote \( r(z) = s(z)/t(z) \) with coprime polynomials \( s(z) \) and \( t(z) \).

If \( n_1, n_2 \geq 0 \), then

\[
s(z)t(z^d) = (1 - z)^{n_1} (1 + z^2)^{n_2} s(z^d)t(z).
\]

Since \( s(z) \) and \( t(z) \) are coprime, this means that \( s(z^d) \) is a factor of \( s(z) \), and thus \( s(z) = s \in \mathbb{C} \setminus \{0\} \) and

\[
t(z^d) = (1 - z)^{n_1} (1 + z^2)^{n_2} t(z).
\]

Since the polynomials \( t(z) \) and \( t(z^d) \) have the same multiplicity of zero at \( z = 1 \), we necessarily have \( n_1 = 0 \) and \( (d - 1)D = 2n_2 \), where \( D := \deg t(z) \). If \( d \geq 4 \), then \( D < n_2 \), and so the equation

\[
(7) \quad t(z^d) = (1 + z^2)^{n_2} t(z)
\]
is not possible. If \( d = 3 \), then \( D = n_2 \). The equation \( z^3 = c \in \mathbb{C} \) may have at most one of \( i \) or \(-i\) as a root. From this it follows that (ii) is not possible, if \( d = 3 \). The case \( n_1, n_2 \leq 0 \) is similar.

If \( n_1, -n_2 \geq 0 \), we denote \( N := -n_2 \), and then
\[
 s(z^d) t(z) (1 - z)^{n_1} = (1 + z^2)^N s(z) t(z^d).
\]
Thus there exists a polynomial \( u(z) \) such that \( t(z^d) u(z) = t(z) (1 - z)^{n_1} \). Since \( t(z) \) and \( t(z^d) \) have the same multiplicity of zero at \( z = 1 \), we obtain \( u(z) = (1 - z)^{n_1} v(z) \) with some polynomial \( v(z) \neq 0 \). But then \( t(z^d) v(z) = t(z) \) giving \( v(z) = 1, t(z) = t \in \mathbb{C} \setminus \{0\} \). So \( s(z^d) (1 - z)^{n_1} = (1 + z^2)^N s(z) \), and using again the fact that \( s(z) \) and \( t(z^d) \) have the same multiplicity of zero at \( z = 1 \), we necessarily have \( n_1 = 0 \). But then \( s(z^d) = (1 + z^2)^N s(z) \), and this is analogous to (ii) and so impossible, as we saw above. The case \( -n_1, n_2 \geq 0 \) can be considered in a similar way. This proves (ii).

Theorem K gives now the truth of Lemma 1.

**Lemma 2.** The functions \( T_2(z), G_{2,j}(z), j \in \mathbb{N}_0 \setminus \{1\} \), are algebraically independent over \( \mathbb{C}(z) \).

**Proof:** The proof of the induction step in condition (i) above works also in the case \( d = 2 \), but the starting point of the induction does not hold, since \( G_{2,1}(z) = z/(1 - z) \) is a rational function, see [4, Theorem 9]. So in this case we delete \( G_{2,1}(z) \), start with \( m = 2 \) and consider the functional equation
\[
 \frac{B(z^2)}{1 - z^8} = \frac{B(z)}{1 - z^4} - \frac{c_2 z}{1 - z^4}, c_2 \neq 0.
\]
Let \( B(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 \). Then
\[
 b_0 + b_1 z^2 + b_2 z^4 + b_3 z^6 = (b_0 + b_1 z + b_2 z^2 + b_3 z^3)(1 + z^4) + c_2 z(1 + z^4),
\]
and by comparing the coefficients of even powers of \( z \) on both sides we have \( B(z) = (b_0 + b_1 z)(1 + z^2) \). Thus
\[
 \frac{b_0 + b_1 z^2}{1 - z^4} = \frac{b_0 + b_1 z}{1 - z^2} - \frac{c_2 z}{1 - z^4},
\]
and so \( b_0 + b_1 z^2 = (b_0 + b_1 z)(1 + z^2) - c_2 z \) implying \( b_0 = b_1 = c_2 = 0 \), a contradiction. So we may now start the induction from \( m = 2 \) and continue as in the proof of Lemma 1 to obtain the condition (i).

To consider the condition (ii), let us assume that for some integer \( n \neq 0 \) the functional equation
\[
 r(z^2) = (1 - z)^{-n} r(z)
\]
has a rational solution \( r(z) \neq 0 \), and denote \( r(z) = s(z)/t(z) \) with coprime polynomials \( s(z) \) and \( t(z) \). If \( n > 0 \), then
\[
 s(z^2)(1 - z)^n t(z) = s(z) t(z^2).
\]
Since \( s(z) \) and \( t(z) \) are coprime, this means that \( s(z^2) \) is a factor of \( s(z) \), and thus \( s(z) = s \in \mathbb{C} \setminus \{0\} \). Therefore \( (1 - z)^n t(z) = t(z^2) \), which leads immediately to a contradiction \( n = 0 \). The case \( n < 0 \) is similar.
Thus Lemma 2 is true.
We note that we cannot include $U_2(z)$ to the functions in Lemma 2, since $U_2(z) = 1/(1 - z^2)$.

3. Proof of Theorems 2, 3 and 4
We shall need the following basic result of Mahler’s method given in [3, Theorem 4.2.1].

**Theorem N1.** Let $K$ denote an algebraic number field. Suppose that $f_1(z), \ldots, f_m(z) \in K[[z]]$ converge in some disk $U \subset \mathbb{D}$ about the origin, where they satisfy the matrix functional equation

$$
\tau(f_1(z^d), \ldots, f_m(z^d)) = A(z) \cdot \tau(f_1(z), \ldots, f_m(z)) + \tau(b_1(z), \ldots, b_m(z))
$$

with $A(z) \in \text{Mat}_{m \times m}(K(z))$, $\tau$ indicating the matrix transpose, and $b_1(z), \ldots, b_m(z) \in K(z)$. If $\alpha \in U \setminus \{0\}$ is an algebraic number such that none of the $\alpha^{d_l} (j \in \mathbb{N}_0)$ is a pole of $b_1(z), \ldots, b_m(z)$ and the entries of $A(z)$, then the following inequality holds

$$
\text{trdeg}_K \mathbb{Q}(f_1(\alpha), \ldots, f_m(\alpha)) \geq \text{trdeg}_K K(f_1(z), \ldots, f_m(z)).
$$

This result with Lemmas 1 and 2 gives immediately the truth of Theorems 2 and 3.

In Theorem 4 we consider two different values $d = 2$ and $d = 3$. For this we recall the following special case of [1, Theorem 1], where $f_{i,1}(z), \ldots, f_{i,m_i}(z) \in K[[z]] (i = 1, 2)$ converge in $\mathbb{D}$ and satisfy

$$
f_{i,j}(z) = a_{i,j}(z)f_{i,j}(z^d) + b_{i,j}(z), \quad i = 1, 2; j = 1, \ldots, m_i,
$$

with $a_{i,j}(z), b_{i,j}(z) \in K(z)$ and $a_{i,j}(0) = 1$.

**Theorem N2.** Suppose that $\log d_1 / \log d_2 \notin \mathbb{Q}$. Let $\alpha \in \mathbb{D} \setminus \{0\}$ be an algebraic number such that none of the $\alpha^{d_k} (i = 1, 2; k \in \mathbb{N}_0)$ is a pole of $a_{i,j}(z), b_{i,j}(z)$ and $a_{i,j}(\alpha^{d_k}) \neq 0$. If, for both values $i = 1, 2$, the functions $f_{i,1}(z), \ldots, f_{i,m_i}(z)$ are algebraically independent over $K(z)$, then the values

$$
f_{i,j}(\alpha), \quad i = 1, 2; j = 1, \ldots, m_i,
$$

are algebraically independent over $\mathbb{Q}$.

We now choose $K = \mathbb{Q}, d_1 = 2, d_2 = 3$, and assume, against Theorem 4, that the numbers (2) are algebraically dependent. Then there exists an integer $m \geq 2$ such that the numbers

$$
(8) \quad T_d(\alpha), U_3(\alpha), G_{d,0}(\alpha), G_{d,1}(\alpha), G_{d,j}(\alpha), \quad d = 2, 3; j = 2, \ldots, m,
$$

are algebraically dependent. By Lemma 2 the functions $T_2(z), G_{2,0}(z), G_{2,j}(z) (j = 2, \ldots, m)$ are algebraically independent over $K(z)$, and, by Lemma 1, also the functions $T_3(z), U_3(z), G_{3,j}(z) (j = 0, 1, \ldots, m)$ are algebraically independent over $K(z).$ Thus Theorem N2 implies the algebraic independence of the numbers (8). This contradiction proves the truth of Theorem 4.
References

[1] Yann Bugeaud, On the rational approximation to the Thue-Morse-Mahler numbers, *Ann. Inst. Fourier (Grenoble)*, 61(5) (2011) 2065–2076.

[2] Yann Bugeaud, Guo-Niu Han, Zhi-Ying Wen, and Jia-Yan Yao, Hankel determinants, Padé approximations, and irrationality exponents, arXiv:1503.02797 (2015).

[3] M. Coons, On the rational approximation of the sum of the reciprocals of the Fermat numbers, *The Ramanujan Journal* 30, no. 1 (2013) 39-65.

[4] D. Duverney and Ku. Nishioka, An inductive method for proving transcendence of certain series, *Acta Arith.* 110 (2003) 305–330.

[5] Ying-Jun Guo, Zhi-Xiong Wen, and Wen Wu, On the irrationality exponent of the regular paperfolding numbers, *Linear Algebra Appl.* 446 (2014) 237 – 264.

[6] K. K. Kubota, On the algebraic independence of holomorphic solutions of certain functional equations and their values, *Math. Ann.* 227 (1977) 9–50.

[7] Ku. Nishioka, Algebraic independence by Mahler’s method and S-unit equations, *Compositio Math.*, 92 (1994) 87–110.

[8] Ku. Nishioka, *Mahler Functions and Transcendence*, LNM 1631 (Springer, Berlin et al., 1996).

[9] Ku. Nishioka, Algebraic independence of reciprocal sums of binary recurrences, *Monatsh. Math.* 123 (1997) 135–148.

[10] Zhi-Xiong Wen and Wen Wu, Hankel determinants of the Cantor sequence, *Scientia Sinica Mathematica (Chinese)*, 44 (10) (2014) 1059–1072.

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