INEQUALITIES FOR \( \log -\)CONVEX FUNCTIONS VIA THREE TIMES DIFFERENTIABILITY

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Abstract. In this paper, we obtain some new integral inequalities like Hermite-Hadamard type for third derivatives absolute value are \( \log -\)convex. We give some applications to quadrature formula for midpoint error estimate.

1. INTRODUCTION

We shall recall the definitions of convex functions and \( \log -\)convex functions:

Let \( I \) be an interval in \( \mathbb{R} \). Then \( f : I \to \mathbb{R} \) is said to be convex if for all on \( x, y \in I \) and all \( \alpha \in [0, 1] \),

\[
f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)
\]

(1.1)

holds. If (1.1) is strict for all \( x \neq y \) and \( \alpha \in (0, 1) \), then \( f \) is said to be strictly convex. If the inequality in (1.1) is reversed, then \( f \) is said to be concave. If it is strict for all \( x \neq y \) and \( \alpha \in (0, 1) \), then \( f \) is said to be strictly concave.

A function is called \( \log -\)convex or multiplicatively convex on a real interval \( I = [a, b] \), if \( \log f \) is convex, or, equivalently if for all \( x, y \in I \) and all \( \alpha \in [0, 1] \),

\[
f(\alpha x + (1 - \alpha)y) \leq f(x)^\alpha + f(y)^{(1-\alpha)}
\]

(1.2)

It is said to be \( \log -\)concave if the inequality in (1.2) is reversed.

For some results for \( \log -\)convex functions see [1]-[4].

The following inequality is called Hermite-Hadamard inequality for convex functions:

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). Then double inequality

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

holds.

The main purpose of this paper is to obtain some new integral inequalities like Hermite-Hadamard type for third derivatives absolute value are \( \log -\)convex.

In order to prove our main results for \( \log -\)convex functions we need the following Lemma from [5]:

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Lemma 1. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a three times differentiable mapping on \( I^o \) and \( a, b \in I^o \) with \( a < b \). If \( f^{(3)} \in L_1([a, b]) \), then

\[
\frac{1}{b-a} \int_a^b f(x)dx - f \left( \frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f'' \left( \frac{a+b}{2} \right) = \frac{(b-a)^3}{96} \left[ \int_0^1 t^3 f^{(3)} \left( \frac{t}{2} a + \frac{2-t}{2} b \right) dt - \int_0^1 t^3 f^{(3)} \left( \frac{2-t}{2} a + \frac{t}{2} b \right) dt \right].
\]

2. INEQUALITIES FOR log–CONVEX FUNCTIONS

We shall start the following result:

Theorem 1. Let \( f : I \to [0, \infty) \), be a three times differentiable mapping on \( I^o \) such that \( f''' \in L[a, b] \) where \( a, b \in I^o \) with \( a < b \). If \( |f'''| \) is log–convex on \([a, b] \), then the following inequality holds:

\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f \left( \frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f'' \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left( |f'''(b)| \mu_K + |f'''(a)| \mu_M \right)
\]

where

\[
\mu_K = \frac{2K}{(ln K)^2} + \frac{48K}{(ln K)^4} + \frac{96}{(ln K)^6},
\]

\[
\mu_M = \frac{2M}{(ln M)^2} + \frac{48M}{(ln M)^4} + \frac{96}{(ln M)^6}
\]

and

\[
K = \frac{|f'''(a)|}{|f''(a)|}, \quad M = \frac{|f'''(b)|}{|f''(b)|}.
\]

Proof. From Lemma[1] property of the modulus and log–convexity of \( |f'''| \) we have

\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f \left( \frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f'' \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \int_0^1 t^3 \left| f''' \left( \frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt + \int_0^1 t^3 \left| f''' \left( \frac{2-t}{2} a + \frac{t}{2} b \right) \right| dt \right\}
\]

\[
\leq \frac{(b-a)^3}{96} \left\{ \int_0^1 t^3 \left| f'''(a) \right|^{\frac{1}{2}} \left| f''(b) \right|^{\frac{1}{2}} dt + \int_0^1 t^3 \left| f'''(b) \right|^{\frac{1}{2}} \left| f''(a) \right|^{\frac{1}{2}} dt \right\}
\]

\[
= \frac{(b-a)^3}{96} \left\{ |f'''(b)| \int_0^1 t^3 \left[ \frac{|f''(a)|}{|f''(b)|} \right]^{\frac{1}{2}} dt + |f'''(a)| \int_0^1 t^3 \left[ \frac{|f''(b)|}{|f''(a)|} \right]^{\frac{1}{2}} dt \right\}.
\]

The proof is completed by making use of the necessary computation. \( \square \)

Corollary 1. Let \( \mu_K, \mu_M, K, M \) be defined as in Theorem[1]. If we choose \( f'' \left( \frac{a+b}{2} \right) = 0 \) in Theorem[1] we obtain the following inequality

\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left( |f'''(b)| \mu_K + |f'''(a)| \mu_M \right).
\]
Theorem 2. Let \( f : I \to [0, \infty) \), be a three times differentiable mapping on \( I \) such that \( f''' \in L[a, b] \) where \( a, b \in I \) with \( a < b \). If \( |f'''|^q \) is log–convex on \([a, b]\), then the following inequality holds for some fixed \( K \): 

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f'' \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \left( \int_0^1 t^{3p} \, dt \right)^{\frac{q}{p}} \left( \int_0^1 |f'''(t)|^q \, dt \right)^{\frac{1}{q}} \right\}
\]

where \( K = \frac{|f'''(a)|}{|f'''(b)|} \), \( M = \frac{|f'''(b)|}{|f'''(a)|} \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. From Lemma 1 and using the Hölder’s integral inequality, we obtain

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f'' \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \left( \int_0^1 t^{3p} \, dt \right)^{\frac{q}{p}} \left( \int_0^1 |f'''(a)|^q \, dt \right)^{\frac{1}{q}} \right\}
\]

where \( K = \frac{|f'''(a)|}{|f'''(b)|} \), \( M = \frac{|f'''(b)|}{|f'''(a)|} \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

If we use the log-convexity of \( |f'''|^q \) above, we can write

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f'' \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^3}{96} \left\{ \left( \int_0^1 t^{3p} \, dt \right)^{\frac{q}{p}} \left( \int_0^1 |f'''(a)|^q \, dt \right)^{\frac{1}{q}} \right\}
\]

where \( K = \frac{|f'''(a)|}{|f'''(b)|} \), \( M = \frac{|f'''(b)|}{|f'''(a)|} \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

The proof is completed. \( \square \)
Corollary 2. Let $K$ and $M$ be defined as in Theorem $\text{2}$. If we choose $f'' \left( \frac{a+b}{2} \right) = 0$ in Theorem $\text{2}$ we obtain the following inequality

$$
\left| \frac{1}{b-a} \int_{a}^{b} f(x) dx - f \left( \frac{a+b}{2} \right) \right| \\
\leq \frac{(b-a)^{3}}{96} \left( \frac{1}{3p+1} \right)^{\frac{1}{2}} \left\{ |f'''(b)| \left( \frac{2}{q \ln K} \left[ K^{\frac{q}{q}} - 1 \right] \right)^{\frac{1}{q}} + |f'''(a)| \left( \frac{2}{q \ln M} \left[ M^{\frac{q}{q}} - 1 \right] \right)^{\frac{1}{q}} \right\}
$$

where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 3.** Let $f : I \to [0, \infty)$, be a three times differentiable mapping on $I^{\circ}$ such that $f''' \in L[a,b]$ where $a, b \in I^{\circ}$ with $a < b$. If $|f'''|^{q}$ is log-convex on $[a, b]$, then the following inequality holds for some fixed $q \geq 1$, then the following inequality holds:

$$
\left| \frac{1}{b-a} \int_{a}^{b} f(x) dx - f \left( \frac{a+b}{2} \right) \right| \\
\leq \frac{(b-a)^{3}}{96} \left( \frac{1}{4} \right)^{1-\frac{1}{q}} \left\{ |f'''(b)| \left( \mu_{K,q} \right)^{\frac{1}{q}} + |f'''(a)| \left( \mu_{M,q} \right)^{\frac{1}{q}} \right\}
$$

where

$$
\mu_{K,q} = \frac{2K^{\frac{q}{q}} + 48K^{\frac{q}{q}} (q \ln K - 6)}{(q \ln K)^{2} + \frac{96}{(q \ln K)^{2}}},
$$
$$
\mu_{M,q} = \frac{2M^{\frac{q}{q}} + 48M^{\frac{q}{q}} (q \ln M - 6)}{(q \ln M)^{2} + \frac{96}{(q \ln M)^{2}}}
$$

and

$$
K = \frac{|f'''(a)|}{|f'''(b)|} \quad M = \frac{|f'''(b)|}{|f'''(a)|}
$$

**Proof.** From Lemma $\text{2}$ using the well-known power-mean integral inequality and log-convexity of $|f'''|^{q}$ we have

$$
\left| \frac{1}{b-a} \int_{a}^{b} f(x) dx - f \left( \frac{a+b}{2} \right) \right| \\
\leq \frac{(b-a)^{3}}{96} \left\{ \left( \int_{0}^{1} t^{3} dt \right)^{1-\frac{1}{q}} \left( \int_{0}^{1} t^{3} \left| f''' \left( \frac{t}{2}a + \frac{2-t}{2}b \right) \right|^{q} dt \right)^{\frac{1}{q}} + \left( \int_{0}^{1} t^{3} dt \right)^{1-\frac{1}{q}} \left( \int_{0}^{1} t^{3} \left| f''' \left( \frac{t}{2}b + \frac{2-t}{2}a \right) \right|^{q} dt \right)^{\frac{1}{q}} \right\}
$$

$$
\leq \frac{(b-a)^{3}}{96} \left\{ \left( \int_{0}^{1} t^{3} dt \right)^{1-\frac{1}{q}} \left( \int_{0}^{1} t^{3} \left| f'''(a) \right|^{\frac{q}{2}} \left| f'''(b) \right|^{q-\frac{q}{2}} dt \right)^{\frac{1}{q}} + \left( \int_{0}^{1} t^{3} dt \right)^{1-\frac{1}{q}} \left( \int_{0}^{1} t^{3} \left| f'''(b) \right|^{\frac{q}{2}} \left| f'''(a) \right|^{q-\frac{q}{2}} dt \right)^{\frac{1}{q}} \right\}.
$$

The proof is completed by making use of the necessary computation. $\square$
Corollary 3. Let $\mu_{K,q}, \mu_{M,q}$, $K$ and $M$ be defined as in Theorem 3. If we choose $f''(\frac{a+b}{2}) = 0$ in Theorem 3, we obtain the following inequality
\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^3}{96} \left( \frac{1}{4} \right)^{1-\frac{1}{q}} \left\{ |f'''(b)| \left( \mu_{K,q} \right)^{\frac{1}{q}} + |f'''(a)| \left( \mu_{M,q} \right)^{\frac{1}{q}} \right\}.
\]

Corollary 4. From Corollaries 3, 4, we have
\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \min \{ \chi_1, \chi_2, \chi_3 \}
\]
where
\[
\chi_1 = \frac{(b-a)^3}{96} \left\{ |f'''(b)| \frac{2K^{\frac{1}{q}}(\ln K - 6)}{(\ln K)^2} + \frac{48K^{\frac{1}{q}}(\ln K - 2)}{(\ln K)^4} + \frac{96}{(\ln K)^4} \right\},
\]
\[
\chi_2 = \frac{(b-a)^3}{96} \left( \frac{1}{3p+1} \right)^{\frac{1}{q}} \times \left\{ |f'''(b)| \left( \frac{2}{q \ln K} \left[ K^{\frac{1}{q}} - 1 \right] \right)^{\frac{1}{q}} + |f'''(a)| \left( \frac{2}{q \ln M} \left[ M^{\frac{1}{q}} - 1 \right] \right)^{\frac{1}{q}} \right\},
\]
\[
\chi_3 = \frac{(b-a)^3}{96} \left( \frac{1}{4} \right)^{1-\frac{1}{q}} \left\{ |f'''(b)| \frac{2K^{\frac{1}{q}}(q \ln K - 6)}{(q \ln K)^2} + \frac{48K^{\frac{1}{q}}(q \ln K - 2)}{(q \ln K)^4} + \frac{96}{(q \ln K)^4} \right\}^{\frac{1}{q}}
\]
\[
+ |f'''(a)| \left( \frac{2M^{\frac{1}{q}}(\ln M - 6)}{(\ln M)^2} + \frac{48M^{\frac{1}{q}}(\ln M - 2)}{(\ln M)^4} + \frac{96}{(\ln M)^4} \right)^{\frac{1}{q}} \}
\]
and $K = \frac{|f'''(a)|}{|f'''(b)|}$, $M = \frac{|f'''(b)|}{|f'''(a)|}$.

Remark 1. In Theorem 3 and Corollary 3, if we choose $q = 1$, we obtain Theorem 4 and Corollary 4 respectively.

3. APPLICATIONS TO MIDPOINT FORMULA

We give some error estimates to midpoint formula by using the results of Section 2.

Let $d$ be a division $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of the interval $[a,b]$ and consider the formula
\[
\int_a^b f(x)dx = M(f,d) + E(f,d)
\]
where $M(f,d) = \sum_{i=0}^{n-1} f\left( \frac{x_i + x_{i+1}}{2} \right) (x_{i+1} - x_i)$ for the midpoint version and $E(f,d)$ denotes the associated approximation error.
Proposition 1. Let $f : I \to [0, \infty)$ be a three times differentiable mapping on $I^\circ$ with $a, b \in I^\circ$ such that $a < b$. If $|f'''| \geq K_1 > 0$, then for every division $d$ of $[a, b]$, the midpoint error estimate satisfies

$$|E(f, d)| \leq \left| \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^4}{96} \right| \left( |f'''(x_{i+1})| \mu_1 + |f'''(x_i)| \mu_2 \right)$$

where

$$\mu_1 = \frac{2K_1^2}{(\ln K_1)^2} + \frac{48K_1^\frac{3}{2}}{(\ln K_1)^4} + \frac{96}{(\ln K_1)^4},$$

$$\mu_2 = \frac{2M_1^2}{(\ln M_1)^2} + \frac{48M_1^\frac{3}{2}}{(\ln M_1)^4} + \frac{96}{(\ln M_1)^4},$$

and

$$K_1 = \left( x_{i+1} - x_i \right)^3 \left( \frac{f'''(x_{i+1})}{f'''(x_i)} \right), \quad M_1 = \left( \frac{f'''(x_{i+1})}{f'''(x_i)} \right).$$

Proof. By applying Corollary 1 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, ..., n - 1)$ of the division $d$, we have

$$\left| \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x) dx - f \left( \frac{x_i + x_{i+1}}{2} \right) \right| \leq \frac{(x_{i+1} - x_i)^3}{96} \left( |f'''(x_{i+1})| \mu_1 + |f'''(x_i)| \mu_2 \right).$$

By summing over $i$ from 0 to $n - 1$, we can write

$$\left| \int_a^b f(x) dx - M(f, d) \right| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^4}{96} \left( |f'''(x_{i+1})| \mu_1 + |f'''(x_i)| \mu_2 \right).$$

which completes the proof. \hfill \Box

Proposition 2. Let $f : I \to [0, \infty)$ be a three times differentiable mapping on $I^\circ$ with $a, b \in I^\circ$ such that $a < b$. If $|f'''|^q \geq K_1 > 0$, then for every division $d$ of $[a, b]$, the midpoint error estimate satisfies

$$|E(f, d)| \leq \frac{1}{3p + 1} \left( \frac{1}{3p + 1} \right)^{\frac{1}{p}} \frac{1}{96} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^4 \left( |f'''(x_{i+1})| \left( \frac{2}{q \ln K_1} \left[ K_1^\frac{3}{2} - 1 \right] \right)^{\frac{1}{q}} + |f'''(x_i)| \left( \frac{2}{q \ln M_1} \left[ M_1^\frac{3}{2} - 1 \right] \right)^{\frac{1}{q}} \right)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $K_1, M_1$ are as defined in Proposition 1.

Proof. The proof can be maintained by using Corollary 2 like Proposition 1. \hfill \Box

Proposition 3. Let $f : I \to [0, \infty)$ be a three times differentiable mapping on $I^\circ$ with $a, b \in I^\circ$ such that $a < b$. If $|f'''|^q \geq K_1 > 0$, then for every division $d$ of $[a, b]$, the midpoint error estimate satisfies

$$|E(f, d)| \leq \frac{1}{96} \left( \frac{1}{4} \right)^{\frac{1}{4}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^4 \left( |f'''(x_{i+1})| \left( \mu_1, q \right)^{\frac{1}{4}} + |f'''(x_i)| \left( \mu_2, q \right)^{\frac{1}{4}} \right)$$
where
\[
\mu_{1,q} = \frac{2K_1^q (q \ln K_1 - 6)}{(q \ln K_1)^2} + \frac{48K_1^q (q \ln K_1 - 2)}{(q \ln K_1)^4} + \frac{96}{(q \ln K_1)^4},
\]
\[
\mu_{2,q} = \frac{2M_1^q (q \ln M_1 - 6)}{(q \ln M_1)^2} + \frac{48M_1^q (q \ln M_1 - 2)}{(q \ln M_1)^4} + \frac{96}{(q \ln M_1)^4},
\]
and \(K_1, M_1\) are as defined in Proposition \[2\].

Proof. The proof can be maintained by using Corollary ?? like Proposition \[1\].

References

[1] M. Alomari and M. Darus, On the Hadamard’s inequality for log—convex functions on the coordinates, Journal of Inequalities and Applications, Volume 2009, Article ID 283147, 13 pages.

[2] X. Zhang and W. Jiang, Some properties of log—convex function and applications for the exponential function, Computers and Mathematics with Applications 63 (2012) 1111–1116.

[3] B. G. Pachpatte, A note on integral inequalities involving two log-convex functions, Mathematical Inequalities & Applications, vol. 7, no. 4, pp. 511–515, 2004.

[4] J. Pečarić and A. U. Rehman, On logarithmic convexity for power sums and related results, Journal of Inequalities and Applications, vol. 2008, Article ID 389410, 9 pages, 2008.

[5] Y. Shuang, Y. Wang and F. Qi, Some inequalities of Hermite-Hadamard type for functions whose third derivatives are \((\alpha, m)\)—convex, J. Computational Analysis and Applications, Vol. 17, No:2, 2014.

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