Canonical quantization of classical systems with generalized entropies

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Abstract

We present, in the framework of the canonical quantization, a class of nonlinear Schrödinger equations with a complex nonlinearity describing, in the mean field approximation, systems of collectively interacting particles. The quantum evolution equation is obtained starting from the study of a $N$-body classical system where the underlined nonlinear kinetics is governed by a kinetic interaction principle (KIP) recently proposed [G. Kaniadakis: Physica A 296 (2001), 405–425]. The KIP, both imposes the form of the generalized entropy associated to the classical system, and determines the Fokker-Planck equation describing the kinetic evolution of the system towards equilibrium.

Keywords: Nonlinear Schrödinger equation, Nonlinear kinetics, Generalized entropy.
The Schrödinger equation is one of the most studied equations both from a mathematical and a physical point of view. A particular interest concerns the possible nonlinear extension of this equation. Just one year after its discovery, Fermi proposed the first nonlinear generalization \[1\].

Among the many nonlinear extensions of the Schrödinger equation (NSE) it is worthy to recall the effort made by Bialynicki-Birula and Mycielski (BBM) \[2\] who proposed to introduce the simple nonlinearity \(-b \ln(|\psi|^2) \psi\), selected by assuming the factorization of the wave function for composed systems. Motivated by this work, some experimental tests were carried out based on neutrons interferometry \[3\], and Fresnel diffraction with slow neutrons \[4\]. They gave the upper limits \(b < 3.4 \times 10^{-13}\) and \(b < 3.3 \times 10^{-15}\), respectively, suggesting that there is no a real ground for a such nonlinear term in the Schrödinger equation.

Weinberg \[5\], in order to save partially the superposition principle of the standard quantum mechanics, proposed a very general class of NSEs with a homogeneous nonlinearity. A further experimental test \[6\] shows again a sharper bound on the deviation from linearity.

Notwithstanding, many complex systems can be described by introducing nonlinear terms in the linear Schrödinger equation, to take into account the interactions among many particles. Applications of the BBM equation to many particles system, in particular in the nuclear physics, have been considered in \[7\] (and references therein). There, it has been suggested that, in spite of the negative tests based on neutrons, the BBM equation and more generally, NSEs, can be usefully applied in the description of extended objects.

The NSE with the cubic nonlinearity \[8, 9\] has been used to study the dynamical evolution of a boson gas with a \(\delta\)-function pair-wise repulsion or attraction \[10\], and in the description of the Bose-Einstein condensation of alcali atoms like \(^7\)Li, \(^{23}\)Na and \(^{87}\)Rb \[11\]. In \[12\] a NSE with complex nonlinearity has been introduced in order to account for a generalized Pauli exclusion-inclusion principle between the quantum particles of the system. Finally it is worthy to mention the family of the Doebner-Goldin equations \[13\], introduced from topological considerations, as the most general class of Schrödinger equations compatible with the Fokker-Planck equation, for the probability density \(\rho = |\psi|^2\).

The BBM equation is closely connected with the Boltzmann-Gibbs (BG) entropy. In fact, it can be shown that, within this theory, the binding energy released in the splitting of an arbitrary wave function, for an isolated system, in non-overlapping parts of the same
form $\psi(x) \rightarrow \sum_i p_i^{1/2} \psi(x - x_i)$, is given by $\Delta E = -b \sum_i p_i \ln p_i$, where $p_i$ are normalized according to $\sum_i p_i = 1$. In Ref. [7] it has been suggested that the BBM equation can be used to formulate the thermodynamical description of spatially extended or self-interacting quantum mechanical objects.

In the present paper we approach to the canonical quantization of a classical system, where the statistical information is supplied by a very general entropy.

We recall that, in the recent years nonlinear generalizations of the Fokker-Planck equation and their connection with generalized entropies have attracted the interest of many authors [14, 15, 16]. Pure-electron plasma [17], bremsstrahlung [18], self-gravitating systems [19] are few examples, among many others, where we can find experimental evidence for the relevance of the use of generalized entropies and nonlinear Fokker-Planck equations (NFPEs).

In Ref. [20], it has been proposed a kinetic interaction principle (KIP) which defines a special collective interaction among the $N$-identical particles of a classical system. The KIP imposes the form of the generalized entropy and governs the evolution of the system toward the equilibrium, by fixing the expression of the nonlinear current appearing in the NFPE.

We present the Lagrangian formulation of the model. By using the minimum action principle, we obtain a class of nonlinear Schrödinger equations containing a complex nonlinearity and describing, in the mean field approximation, a quantum system of interacting particles. The form of the entropy of the ancestor classical system determines the nonlinearity in the evolution equation. Finally, by means of a gauge transformation of third kind [13, 21, 22] we show that this family of NSEs is gauge equivalent to another family of NSEs having a pure real nonlinearity depending only on the probability density field.

Some evolution equations obtained from kinetic equations known in literature within the Boltzmann-Gibbs entropy, the Tsallis entropy, the quantum-group entropy and the Kaniadakis entropy are given to illustrate the relevance of the method.

Let us consider a classical Markovian process in a $n$-dimensional space

$$\frac{\partial \rho(t, x)}{\partial t} = \int \left[ \pi(t, y \rightarrow x) - \pi(t, x \rightarrow y) \right] dy ,$$

(1)

describing the kinetics of a system of $N$ identical particles. We assume for the transition probability $\pi(t, x \rightarrow y)$ a suitable expression in terms of the populations of the initial site
According to KIP [20] we pose
\[ \pi(t, x \to y) = r(t, x, x - y) a(\rho) b(\rho') c(\rho, \rho') , \]
where \( \rho = \rho(t, x) \) and \( \rho' \equiv \rho(t, y) \) are the particle density functions in the starting site \( x \) and in the arrival site \( y \), respectively, whereas \( r(t, x, x - y) \) is the transition rate which depends only on the starting \( x \) and arrival \( y \) sites during the particle transition \( x \to y \).

The two factors \( a(x) \) and \( b(x) \) in Eq. (2) are arbitrary non-negative functions, supposed to be continuous. Their limit values for \( x \to 0 \) are subject to precise physical requirements. For instance, \( a(0) = 0 \) because if the starting site is empty the transition probability is equal to zero; \( b(0) = 1 \) because no inhibition occurs if the arrival site is empty. Finally, the last factor \( c(\rho, \rho') \) takes into account that the populations of the two sites can eventually affect the transition, collectively and symmetrically.

The expression of the functional \( b(\rho) \) plays a very important role in the particle kinetics because can stimulate or inhibit the transition \( x \to y \) allowing in this way to take into account interactions originated from collective effects.

With the assumption given in Eq. (2) for the transition probability, under the first neighbor approximation, we can expand the r.h.s of Eq. (1) up to the second order in \( y \ [20] \)
\[ \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x_i} \left[ \left( \zeta_i + \frac{\partial \zeta_{ij}}{\partial x_j} \right) \gamma(\rho) + \zeta_{ij} \gamma(\rho) \frac{\partial}{\partial x_j} \ln \kappa(\rho) \right] , \]
with \( i = 1, \cdots, n \) and summation over repeated indices is assumed.

In Eq. (3) \( \gamma(\rho) \equiv a(\rho) b(\rho) c(\rho, \rho) \) and \( \kappa(\rho) = a(\rho)/b(\rho) \), whereas the coefficients \( \zeta_i \) and \( \zeta_{ij} \) are given by
\[ \zeta_i = \int r(t, x, y) y_i \, dy , \quad \text{and} \quad \zeta_{ij} = \frac{1}{2} \int r(t, x, y) y_i y_j \, dy . \]

In the following we consider the case of linear drift by posing \( \gamma(\rho) = \rho \) and after introducing the quantity \( u_{\text{drift}} \mid _i = -\zeta_i - \partial \zeta_{ij} / \partial x_j \), the \( i \)-th component of the drift velocity and assuming \( \zeta_{ij} = D \delta_{ij} \), with \( D \) the diffusion coefficient, Eq. (3) becomes
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \left\{ \rho \left[ u_{\text{drift}} - D \nabla \ln \kappa(\rho) \right] \right\} = 0 , \]
which is a NFPE for the field \( \rho \). The total current \( j = j_{\text{drift}} + j_{\text{diff}} \) is the sum of two quantities: \( j_{\text{drift}} = \rho u_{\text{drift}} \) which is the standard linear drift current and \( j_{\text{diff}} = -D \rho \nabla \ln \kappa(\rho) \) which is a nonlinear diffusion current different from the standard Fick’s one.
Eq. 5 describes a class of nonlinear diffusive processes varying the functional $\kappa(\rho)$. In the following we assume a constant diffusion coefficient $D = \text{const.}$

The functional $\kappa(\rho)$ is related to the entropy of the classical system as can be showed in the non-equilibrium thermodynamics framework [23, 24], by relating the production of the entropy to a NFPE. In fact, we may show this link on a more general ground by starting from a constrained entropic form

$$S(\rho) = S(\rho) - \beta \int E(x) \rho \, dx - \beta' \int \rho \, dx,$$

where $\beta$ and $\beta'$ are the Lagrange multipliers associated to the constraints of the total energy $E = \int E(x) \rho \, dx$ and the normalization $\int \rho \, dx = 1$.

Without loss of generality, we can assume for the entropy function the following expression

$$S(\rho) = -k \int d\mathbf{x} \int d\rho \ln \kappa(\rho).$$

The standard BG-entropy is recovered from Eq. 7 after posing $\kappa(\rho) = e^\rho$.

Quite generally, the evolution of the field $\rho$ in the configuration space is governed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

and we assume a linear relation between the current $\mathbf{J}$ and the constrained thermodynamic force $\nabla(\delta S/\delta \rho)$

$$\mathbf{J} = \frac{D}{k} \rho \nabla \left( \frac{\delta S}{\delta \rho} \right).$$

Putting Eq. 9 into Eq. 8 we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left\{ \rho \left[ -\frac{D}{k} \beta \nabla E(x) - D \nabla \ln \kappa(\rho) \right] \right\} = 0,$$

and, after introducing the drift velocity

$$u_{\text{drift}} = -\frac{D}{k} \beta \nabla E(x),$$

Eq. 10 takes the same form of the continuity equation 3.

Let us now introduce the wave function $\psi$ describing, in the mean field approximation, the quantum analogous of the classical $N$-particle interacting system depicted above. We postulate that the evolution equation of this system is given by a NSE

$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \Lambda(\psi^*, \psi) \psi + V(x) \psi,$$
where $\Lambda(\psi^*, \psi)$ is a complex nonlinearity and $V(\mathbf{x})$ is the external potential.

In the canonical picture, the motion equation (12) can be obtained after introducing the action of the system

$$\mathcal{A} = \int \mathcal{L}(\psi^*, \psi) d\mathbf{x} dt ,$$

(13)

where $\mathcal{L}$ is the Lagrangian density given by

$$\mathcal{L}(\psi^*, \psi) = i \frac{\hbar}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi \right) - \frac{\hbar^2}{2m} |\nabla \psi|^2 - U(\psi^*, \psi) - \psi^* V(\mathbf{x}) \psi ,$$

(14)

and $U(\psi^*, \psi)$ is a nonlinear potential describing the collective interaction between the particles.

The NSE (12) is obtained by applying the minimum action principle to the system described by the Lagrangian (14)

$$\frac{\delta \mathcal{A}}{\delta \psi^*} = 0 .$$

(15)

The nonlinear functional $\Lambda(\psi^*, \psi)$ in Eq. (12) is given by

$$\Lambda(\psi^*, \psi) = \frac{\delta}{\delta \psi^*} \int U(\psi^*, \psi) d\mathbf{x} dt .$$

(16)

It is convenient at this point to introduce the hydrodynamic fields $\rho$ and $\Sigma$ related to the $\psi$-function through the polar decomposition [25, 26]

$$\psi(t, \mathbf{x}) = \rho^{1/2}(t, \mathbf{x}) \exp \left( \frac{i \hbar}{\hbar} \Sigma(t, \mathbf{x}) \right) .$$

(17)

Introducing the real part $W(\psi^*, \psi)$ and the imaginary part $\mathcal{W}(\psi^*, \psi)$ of the complex nonlinearity $\Lambda(\psi^*, \psi) = W(\psi^*, \psi) + i \mathcal{W}(\psi^*, \psi)$, Eq. (12) can be separated in a coupled of real nonlinear evolution equations for the phase $\Sigma$

$$\frac{\partial \Sigma}{\partial t} + \frac{\nabla (\Sigma)^2}{2m} - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} + W(\rho, \Sigma) + V(\mathbf{x}) = 0 ,$$

(18)

and the density $\rho$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \frac{\nabla \Sigma}{m} \rho \right) - \frac{2}{\hbar} \rho \mathcal{W}(\rho, \Sigma) = 0 .$$

(19)

In Eq. (19) $j_\phi \equiv (\nabla \Sigma/m) \rho$ is the linear current of the standard quantum mechanics.

Both Eqs (18) and (19) can be obtained from the variational problem

$$\frac{\delta \mathcal{A}}{\delta \rho} = 0 , \quad \text{and} \quad \frac{\delta \mathcal{A}}{\delta \Sigma} = 0 ,$$

(20)
where the Lagrangian density (14), after taking into account Eq. (17), assumes the expression

\[ L(\rho, \Sigma) = -\frac{\partial \Sigma}{\partial t} \rho - \frac{(\nabla \Sigma)^2}{2m} \rho - \frac{\hbar^2}{8m} \frac{\nabla^2\rho}{\rho} - U(\rho, \Sigma) - V(x) \rho . \]  

We observe that the nonlinear functionals \( W(\rho, \Sigma) \) and \( \mathcal{W}(\rho, \Sigma) \) are related to the nonlinear potential \( U(\rho, \Sigma) \) through the relations

\[ W(\rho, \Sigma) = \frac{\delta}{\delta \rho} \int U(\rho, \Sigma) dx dt, \quad \mathcal{W}(\rho, \Sigma) = \frac{\hbar}{2\rho} \frac{\delta}{\delta \Sigma} \int U(\rho, \Sigma) dx dt, \]  

and by choosing \( U(\rho, \Sigma) \) as

\[ U(\rho, \Sigma) = -D \rho \nabla \ln \kappa(\rho) \cdot \nabla \Sigma, \]  

from Eqs. (18) and (19) we obtain

\[ \frac{\partial \Sigma}{\partial t} + \frac{\nabla^2 \rho}{2m} + \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} + D \rho \frac{\partial}{\partial \rho} \ln \kappa(\rho) \Delta \Sigma + V(x) = 0 , \]  

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \left[ \frac{\nabla \Sigma}{m} \rho - D \rho \nabla \ln \kappa(\rho) \right] = 0 . \]  

By posing \( \hat{\mathbf{u}}_{\text{drift}} = \nabla \Sigma / m \), the quantum drift velocity, Eq. (25) assumes the same expression of the Fokker-Planck equation (5) describing the kinetics of the classical system obeying to KIP.

The NSE in the \( \psi \)-representation is given by

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + D \rho \frac{\partial}{\partial \rho} \ln \kappa(\rho) \Delta \Sigma \psi + i \frac{\hbar D}{2\rho} \nabla \left[ \rho \nabla \ln \kappa(\rho) \right] \psi + V(x) \psi , \]  

where the nonlinearities \( W \) and \( \mathcal{W} \) are determined through the functional \( \kappa(\rho) \) which also defines the entropy (7) of the ancestor classical system.

We introduce now the following nonlinear gauge transformation of third kind [21, 22]

\[ \psi \rightarrow \phi = \psi \exp \left( -i \frac{\hbar}{m} D \ln \kappa(\rho) \right) , \]  

which, being an unitary transformation, does not change the amplitude of the wave function, \(|\psi|^2 = |\phi|^2 = \rho\), and transforms the phase \( \Sigma \) of the old field \( \psi \) in the phases \( \sigma \) of the new field \( \phi \) according to the equation

\[ \sigma = \Sigma - m D \ln \kappa(\rho) . \]
After performing the transformation (27), Eq. (26) becomes

\[ i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \phi + mD^2 \left\{ \rho \frac{\partial}{\partial \rho} \ln \kappa(\rho) \Delta \ln \kappa(\rho) + \frac{1}{2} \left[ \nabla \ln \kappa(\rho) \right]^2 \right\} \phi + V(x) \phi , \]

(29)

where in Eq. (29) appears a purely real nonlinearity depending only on the field \( \rho \).

To illustrate the relevance of the approach described above, we derive some NSEs starting from as much entropies known in literature: BG-entropy, Tsallis-entropy, quantum-group entropy and \( \kappa \)-entropy.

(a) As a first example we consider the standard classical entropy

\[ S_{BG}(\rho) = -k \int \rho \ln \rho d\mathbf{x} , \]

(30)

which follows from Eq. (7) by posing \( a(\rho) = e\rho \), \( b(\rho) = 1 \) so that \( \kappa(\rho) = e\rho \). The corresponding continuity equation becomes

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \left( \frac{\nabla \Sigma}{m} \rho - D \nabla \rho \right) = 0 , \]

(31)

that is the linear Fokker-Planck equation \[13\].

The NSE is readily obtained from Eq. (29) and becomes

\[ i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \phi + mD^2 \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \phi + V(x) \phi , \]

(32)

which was studied previously in \[27\]. In particular, Eq. (32) is equivalent to the linear Schrödinger equation

\[ i\kappa \frac{\partial \chi}{\partial t} = -\frac{\kappa^2}{2m} \Delta \chi + V(x) \chi , \]

(33)

with \( \kappa = \hbar \sqrt{1 - (2mD/\hbar)^2} \) and the field \( \chi \) is related to the hydrodynamic fields \( \rho \) and \( \sigma \) through the relation \( \chi = \rho^{1/2} \exp(i \sigma/\kappa) \).

(b) As a second example we consider the Tsallis entropy \[28\]

\[ S_q(\rho) = k \int \frac{\rho^q - \rho}{1 - q} d\mathbf{x} . \]

(34)

Eq. (34) can be obtained from Eq. (7) after posing

\[ \ln \kappa(\rho) = - \ln_q \left( \frac{\alpha}{\rho} \right) , \]

(35)
where $\alpha = q^{1/(1-q)}$ is a constant and the deformed $q$-logarithm is defined by

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q},$$

(36)

which reduces to the standard logarithm for $q \to 1$ ($\ln_1(x) = \ln x$) as well as Eq. (34) reduces to the BG-entropy in the same limit.

The evolution equation for the field $\rho$ is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \frac{\nabla \Sigma_m \rho - D \nabla \rho^q}{m} \right) = 0,$$

(37)

which is the NFPE discussed in Ref. [15, 29].

The NSE associated to Eq. (37) assumes the form

$$i \hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \phi + q^2 m D^2 \rho^{2(\alpha - 1)} \left[ \frac{\Delta \rho}{\rho} - \left( \frac{3}{2} - q \right) \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \phi + V(x) \phi,$$

(38)

and reduces to Eq. (32) for $q \to 1$.

We observe that the nonlinearity in Eq. (38) coincides with the same one appearing in the NSE proposed in Ref. [30], after replacing $q \to 2 - q$, and obtained in the Tsallis statistics framework by using a different approach.

(c) Another interesting example is provided by the quantum groups entropy

$$S_q(\rho) = -k \int \frac{\rho^q - \rho^{q^{-1}}}{q - q^{-1}} d\mathbf{x},$$

(39)

obtained from Eq. (7) with the choice

$$\ln \kappa(\rho) = \frac{q \rho^{q^{-1}} - q^{-1} \rho^{q^{-1}}}{q - q^{-1}}.$$

(40)

Entropy (39) is invariant under the interchange $q \leftrightarrow q^{-1}$ and reduces to the BG-entropy in the $q \to 1$ limit.

The evolution equation for the field $\rho$ is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left[ \frac{\nabla \Sigma_m \rho - D \nabla \left( \frac{q \rho^q + q^{-1} \rho^{q^{-1}}}{q+1} \right)}{m} \right] = 0,$$

(41)

and the associated NSE is readily obtained from Eq. (29) and assumes the form

$$i \hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \phi + \frac{m D^2}{(q - q^{-1})^2} \left\{ \rho \frac{\partial f_q(\rho)}{\partial \rho} \Delta f_q(\rho) + \frac{1}{2} \left[ \nabla f_q(\rho) \right]^2 \right\} \phi + V(x) \phi,$$

(42)
where \( f_q(\rho) = q \rho^{q-1} - q^{-1} \rho^{q-1} \).

(d) As a final example let us consider the \( \kappa \)-entropy \[ S_{(\kappa)}(\rho) = -k \int \frac{\rho^{1+\kappa} - \rho^{1-\kappa}}{2\kappa} \, d\mathbf{x} , \] which can be obtained from Eq. (7) after posing \[
\ln \kappa(\rho) = \lambda \ln_{(\kappa)} \left( \frac{\rho}{\alpha} \right) .
\] The two quantities \( \lambda = \sqrt{1 - \kappa^2} \) and \( \alpha = [(1 - \kappa)/(1 + \kappa)]^{1/2} \kappa \) are constants and fulfil the relation \((1 \pm \kappa)\alpha^{\pm \kappa} = \lambda\).

The \( \kappa \)-logarithm is defined as
\[
\ln_{(\kappa)}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa} ,
\] and reduces, in the \( \kappa \to 0 \) limit, to the standard logarithm \((\ln_{(0)}(x) = \ln x)\).

From Eq. (25) we obtain the following continuity equation for the field \( \rho \)
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \left[ \frac{\nabla \Sigma}{m} \rho - \frac{D}{2} \nabla \left( \rho^{1+\kappa} + \rho^{1-\kappa} \right) \right] = 0 ,
\] which coincides with that proposed in Ref. [20].

The associated NSE is given by
\[
i \hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \phi + \frac{\lambda}{2} m D^2 \left[ (1 + \kappa) \rho^\kappa + (1 - \kappa) \rho^{-\kappa} \right] \Delta \ln_{(\kappa)} \left( \frac{\rho}{\alpha} \right) \\
+ \lambda \left[ \nabla \ln_{(\kappa)} \left( \frac{\rho}{\alpha} \right) \right]^2 \phi + V(x) \phi ,
\] and reduces to the Doebner-Golding Eq. (32) in the \( \kappa \to 0 \) limit as well as Eq. (43) reduces to the standard BG-entropy.

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