An algorithm to compute a presentation of pushforward modules

M. E. Hernandes, A. J. Miranda, G. Peñafort-Sanchis

March 10, 2017

Abstract

We describe an algorithm to compute a presentation of the pushforward module $f_\ast \mathcal{O}_X$ for a finite map germ $f: \mathcal{X} \to (\mathbb{C}^{n+1}, 0)$, where $\mathcal{X}$ is Cohen-Macaulay of dimension $n$. The algorithm is an improvement of a method by Mond and Pellikaan. We give applications to problems in singularity theory, computed by means of an implementation in the software SINGULAR.

Mathematics Subject Classification. Primary 58K05; Secondary 32S10, 14Q99, 13C10.

Keywords: presentation matrices, Fitting ideals, multiple-point schemes.

1 Introduction

Let $M$ be a module over a commutative unitary ring $R$. A presentation of $M$ is an exact sequence

$$R^p \xrightarrow{\lambda} R^q \xrightarrow{\psi} M \to 0.$$  

If $M$ admits a presentation, then we say that $M$ is a finitely presented module, and any matrix $\Lambda$ associated to $\lambda$ is called a presentation matrix of $M$. It is well known that any finitely generated module over a Noetherian ring is finitely presented (see [GP]).

In this work, we introduce an algorithm to compute presentation matrices in the particular case where $M$ is the pushforward of the ring of holomorphic functions in the source of a finite map $f: \mathcal{X} \to (\mathbb{C}^{n+1}, 0)$, and $\mathcal{X}$ is a Cohen-Macaulay space of dimension $n$. The algorithm is based on a method by Mond and Pellikaan [MP], but introduces an improvement which allows to circumvent certain problems, concerning the limitation to polynomial inputs and outputs of commutative algebra systems, such as SINGULAR [DGPS]. As we will see, this improvement also makes the algorithm more efficient from a computational point of view. The reader can find in [A] a SINGULAR library containing an implementation of the algorithm.

In the last section, we illustrate some applications of presentation matrices that can be computed by means of our algorithm. We show the computation of target and source multiple-point schemes for map germs $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, discriminants and certain topological invariants of maps —leading, for example, to the answer of a question, due to Gaffney and Mond, about the topological classification for corank 2 map germs from $\mathbb{C}^2$ to $\mathbb{C}^2$. All these applications are based on Fitting ideals, which play a crucial role in the theory of singularities of map germs and in enumerative geometry (see for instance [KLU]).
If $M$ is a finitely presented module as above, then its $k$-th Fitting ideal is given by

$$F_k(M) = \begin{cases} 
0 & \text{if } k < 0; \\
\langle \text{minors of order } q - k \text{ of } \Lambda \rangle & \text{if } 0 \leq k < \min(q,p); \\
R & \text{if } \min(q,p) \leq k.
\end{cases}$$

Fitting ideals are invariant under module isomorphisms and they do not depend on the chosen presentation of $M$ (see [L]).

Let $f: X \to Y$ be a finite map germ and $O_X, O_Y$ the rings of holomorphic functions of $X$ and $Y$. The pushforward module $f_*O_X$ is just $O_X$, regarded as an $O_Y$-module via $f$. Finiteness of $f$ implies that $f_*O_X$ is finite, and hence finitely presented. For simplicity, we write the corresponding Fitting ideals as $F_k(f) = F_k(f_*O_X)$. As shown by Mond and Pellikaan [MP], the $k$th Fitting ideal of $f_*O_X$ defines the $(k - 1)$th multiple point space of $f$ in $Y$, which we write as $M_k(f) = V(F_{k-1}(f))$.

It is worth saying that the SINGULAR implementation of the algorithm has already been used in the works of other authors: In [BOT], Nuño Ballesteros, Oréfice Okamoto and Tomazella use it to compute discriminant curves of map germs from a complete intersection surface to the plane. Oset Sinha, Ruas and Wik Atique have used the algorithm to compute the image of a stable map germ of corank 2 from $C^8$ to $C^9$, which plays an important role in their work on the extra-nice dimensions [ORW]. Recently, O. N. Silva has used the algorithm to compute source double points of certain maps (as in Section 3.5) [S], obtaining the first known counter-example to a conjecture by M. A. Ruas, on the equivalence between Whitney equisingularity and Topological triviality. For convenience to the reader, we describe Mond-Pellikaan’s original method to obtain presentation matrices:

### 1.1 Mond-Pellikaan algorithm

Let $X$ be an $n$-dimensional germ of Cohen Macaulay space, and let

$$f: X \to (\mathbb{C}^{n+1}, 0)$$

satisfy the following extra condition: If we let

$$\tilde{f}: X \to (\mathbb{C}^n, 0),$$

be the germ obtained by composing $f$ with the projection $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$ which forgets the last coordinate, then $\tilde{f}$ is a finite map germ. Since our interest is to obtain a computer implementation, we are going to switch from the holomorphic setting of Mond and Pellikaan to the rational setting software like SINGULAR can handle. This is mostly a matter of language, and the results of Mond and Pellikaan apply here exactly in the same way.

Let $A = \mathbb{C}[X,Y]_{(X,Y)}$ be the localization at the maximal ideal at the origin of the ring of polynomials in the $n + 1$ variables $X = X_1, \ldots, X_n$ and $Y$. We denote $\hat{A} = \mathbb{C}[X]_{(X)}$ and $B = (\mathbb{C}[x]/I)_{(x)}$, with variables $x = x_1, \ldots, x_\ell$, and assume that $B$ is a Cohen-Macaulay ring of dimension $n$. Let

$$\phi: A \to B$$

be a morphism of local rings given by $X_i \mapsto f_i, i = 1, \ldots, n$ and $Y \mapsto f_{n+1}$, for some polynomials $f_j \in \mathbb{C}[x]$. Write

$$\tilde{\phi}: \hat{A} \to B$$
for the restricted morphism, and assume that \( B \) is minimally generated by \( g_1, \ldots, g_h \) as an \( \hat{A} \)-module. Since \( B \) is generated by \( g_1, \ldots, g_h \), there exist \( \alpha_{ij} \in \hat{A} \), \( 1 \leq i, j \leq h \), satisfying the equations
\[
Y g_i = \sum_{j=1}^{h} \alpha_{ij} g_j, \quad \text{for every} \quad 1 \leq i \leq h.
\] (1)

Let \( \lambda: A^h \to A^h \) be given by multiplication by the matrix \( \Lambda \) whose entries are
\[
\Lambda_{ij} = \alpha_{ij} - \delta_{ij} Y,
\]
where \( \delta_{ij} \) stands for the Kronecker delta function.

If \( \psi: A^h \to B \) is the epimorphism given by \( e_i \mapsto g_i \), where \( e_i \in A^h \) is the element whose only non-zero entry is 1 in the \( i \)th position, then the inclusion \( \text{Im} \lambda \subseteq \ker \psi \) follows from (1). Mond and Pellikaan show that indeed the sequence
\[
A^h \xrightarrow{\lambda} A^h \xrightarrow{\psi} B \to 0
\] (2)
is exact [MP]. Therefore, the matrix
\[
\Lambda = \begin{pmatrix}
\alpha_{11} - Y & \alpha_{12} - Y & \cdots & \alpha_{1h} - Y \\
\alpha_{21} - Y & \alpha_{22} - Y & \cdots & \alpha_{2h} - Y \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{h1} - Y & \alpha_{h2} - Y & \cdots & \alpha_{hh} - Y
\end{pmatrix}
\]
is a presentation matrix for \( B \).

**Example 1.1.** Let \( f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) be the cross-cap map (Figure 1), given by
\[
(x, y) \mapsto (x, y^2, xy).
\]
Let \( A = \mathbb{C}[X_1, X_2, Y]_{(X_1, X_2, Y)}, \hat{A} = \mathbb{C}[X_1, X_2]_{(X_1, X_2)} \) and \( B = \mathbb{C}[x, y]_{(x, y)} \) and consider the ring homomorphism \( A \to B \) given by \( X_1 \mapsto x, X_2 \mapsto y^2 \) and \( Y \mapsto xy \).

Observe that \( B \) is an \( \hat{A} \)-module minimally generated by \( g_1 = 1 \) and \( g_2 = y \), and thus we can apply Mond-Pellikaan algorithm. From the equalities
\[
Y \cdot g_1 = xy \cdot g_1 = x \cdot g_2 = X_1 \cdot g_2 \quad \text{and} \quad Y \cdot g_2 = xy \cdot g_2 = xy^2 \cdot g_1 = X_1 X_2 \cdot g_1
\]
we obtain the presentation matrix
\[
\Lambda = \begin{pmatrix}
-Y & X_1 \\
X_1 X_2 & -Y
\end{pmatrix}.
\]
The matrix \( \Lambda \) can be used to compute the Fitting ideals of \( f_* \mathcal{O}_2 \), which determined the multiple-point scheme in the target of \( f \). The image of \( f \) and the double point space are, respectively:
\[
M_1(f) = V(Y^2 - X_1^2 X_2) \quad \text{and} \quad M_2(f) = V(X_1, Y).
\]
2 Polynomial presentation matrices

Commutative algebra software such as Singular only admit polynomial inputs and outputs. In this section we deal with the problem of how to find presentation matrices whose entries are polynomial. We start with the following trivial remark:

**Remark 2.1.** The elements $\alpha_{ij} \in \tilde{A}$ in (1) are fractions $\alpha_{ij} = a_{ij}/b_{ij}$, for some polynomials $a_{ij}, b_{ij} \in \mathbb{C}[X]$ and $b_{ij}(0) \neq 0$. Multiplying the $i$th row by the least common multiple of the elements $b_{ij}$, $j = 1, \ldots, h$, we obtain another presentation matrix, whose entries are polynomial.

The previous remark guarantees, given a minimal collection of generators $g_i$, the existence of a polynomial presentation matrix of the following form:

**Definition 2.2.** Given $g_1, \ldots, g_h \in B$, an MP-matrix (for $g_i$) is a matrix

$$\Lambda = \begin{pmatrix}
\beta_{11} - u_1 X & \beta_{12} & \cdots & \beta_{1h} \\
\beta_{21} & \beta_{22} - u_2 X & \cdots & \beta_{2h} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{h1} & \beta_{h2} & \cdots & \beta_{hh} - u_h X
\end{pmatrix},$$

with $\beta_{ij}, u_j \in \mathbb{C}[X], u_j(0) = 1$, such that the equations (1) hold.

With the previous notations, let $g_1, \ldots, g_h$ be a minimal system of generators of $B$ as an $\tilde{A}$-module. It follows from Remark 2.1 that $B$ admits an MP-matrix $\Lambda$ for $g_1, \ldots, g_h$ as a presentation matrix. The matrix is given by some polynomials $\beta_{ij}, u_j \in \mathbb{C}[X], u_j(0) = 1$, satisfying the conditions

$$\phi(u_j X)g_j \equiv \sum_{i=1}^{h} \phi(\beta_{ij})g_i \mod I, \text{ for all } 1 \leq j \leq h.$$  

To find the $j$th row of such $\Lambda$, one fixes polynomials up to some degree $d$:

$$\beta_{ij} = \sum_{|\alpha| \leq d} a_{i,\alpha} X^\alpha \text{ for } i = 1, \ldots, h,$$

$$u_{j,\alpha} = 1 + \sum_{1 \leq |\alpha| \leq d} b_{\alpha} X^\alpha,$$
and tries to find $a_{i,\alpha}, b_{\alpha} \in \mathbb{C}$, such that the polynomial
\[
P_d(x) = \phi(u_j Y)g_j - \sum_{i=1}^{h} \phi(\beta_{ij})g_i
\]
reduces to 0 modulo $I$. This is a linear system on $a_{i,\alpha}, b_{\alpha}$, prescribed by the vanishing of the coefficient of each $x^\alpha$ in the reduction of $P_d$ modulo $I$. If that is not possible for the degree $d$, then one increases $d$ and starts all over again.

Due to the reduction process, and to the clearing of denominators in Remark 2.1, there is no obvious way to estimate the degrees of the entries in an MP-matrix $\Lambda$ in terms of the degrees of $f_i$ and the generators of $I$. Unfortunately, the usage of reductions, and the increasing number of parameters $a_{i,\alpha}, b_{\alpha}$ involved, make the complexity of the procedure explained above grow very rapidly as $d$ increases. In order to keep the degree $d$ as low as possible, it seems a good idea to consider a class of matrices bigger than the set of MP-matrices. The following example illustrates this situation.

**Example 2.3.** Let $X = V(I) \subseteq \mathbb{C}^3$, with $I = \langle z - x^k y \rangle$, and let $f: X \rightarrow \mathbb{C}^3$ be given by
\[
(x, y, z) \mapsto (x, y^2 + xz, z).
\]
In our usual setting $A = \mathbb{C}[X_1, X_2, Y]|_{(X_1, X_2, Y)}, \bar{A} = \mathbb{C}[X_1, X_2]|_{(X_1, X_2)}, B = \left( \frac{\mathbb{C}[x, y, z]}{(y - z^k x)} \right)_{(x, y, z)}$, $\phi$ is given by $X_1 \mapsto x, X_2 \mapsto y^2 + xz$ and $Y \mapsto z$, and the pushforward module $B$ is minimally generated by $g_1 = 1$ and $g_2 = y$ as an $\bar{A}$-module. It is easy to check that the matrix
\[
\Lambda = \begin{pmatrix}
y & -X_1^k \\
-X_1^k X_2 & Y + X_1^{2k+1}
\end{pmatrix}
\]
is an MP-matrix for $g_1, g_2$. However, the matrix
\[
\Lambda' = \begin{pmatrix}
y & -X_1^k \\
-X_1^k (X_1 Y - X_2) & Y
\end{pmatrix}
\]
is also a presentation matrix (by Theorem 2.6 below). The procedure explained before will have to consider polynomials up-to degree $2k + 1$ in order to find $\Lambda$, but a more flexible version, allowing matrices such as $\Lambda'$, will stop at degree $k + 2$.

**Definition 2.4.** With the previous notations, given $g_1, \ldots, g_h \in B$, an HMP-matrix (for $g_i$) is a matrix $\Lambda$ with polynomial entries in $\mathbb{C}[X, Y]$, satisfying the following conditions:

- **C1.** $\sum_{j=1}^{h} \phi(\Lambda_{ij})g_j \equiv 0 \mod I$, for $i = 1, \ldots, h$.

- **C2.** $\Lambda_{ij}(0, Y) - \Lambda_{ij}(0, 0) = \begin{cases} Y \cdot u_i(Y); & \text{for some polynomial } u(Y) \in \mathbb{C}[Y] \text{ satisfying } u(0) \neq 0 \text{ if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$

**Remark 2.5.** Every MP-matrix is an HMP-matrix. On the other hand, the HMP-matrix $\Lambda'$ in Example 2.3 is not an MP-matrix.

**Theorem 2.6.** If $\Lambda$ is an HMP-matrix for a minimal set of generators $g_1, \ldots, g_h$ of $B$ as an $\bar{A}$-module, then $\Lambda$ is a presentation matrix for $B$ as an $A$-module.
Proof. The proof is similar to the one for Mond-Pellikaan’s algorithm. Take the sequence of $A$-modules

$$A^h \xrightarrow{\lambda} A^h \xrightarrow{\psi} B \rightarrow 0,$$

where $\psi$ is determined by $e_i \mapsto g_i$, and $e_i$ the $i$th canonical vector in $A^h$. Condition C1 implies $\text{Im } \lambda \subseteq \text{Ker } \psi$, and $\psi$ is an epimorphism, so it suffices to show that $\text{Coker } \Lambda = B$.

Let $B' = (\mathbb{C}[x,t]/I)_{(x,t)}$ and let $\phi': A \rightarrow B'$ be given by $X_i \mapsto f_i, i = 1, \ldots, n$ and $Y \mapsto f_{n+1} + t$. From the fact that $B$ is minimally generated by $g_1, \ldots, g_h$ as an $A$-module, it follows that $B'$ is a free $A$-module minimally generated by $g_1, \ldots, g_h$. Let $\eta: A^h \rightarrow B'$ be the isomorphism given by

$$(a_1, \ldots, a_h) \mapsto \sum_{i=1}^{h} \phi'(a_i)g_i$$

and let $\varphi: B' \rightarrow B'$ be the morphism defined by extending

$$g_j \mapsto \sum_{i=1}^{h} \phi'(\Lambda_{ij}) \cdot g_i, \text{ for } j = 1, \ldots, h,$$

$A$-linearly, so that the diagram

$$
\begin{array}{ccc}
A^h & \xrightarrow{\Lambda} & A^h \\
\downarrow{\eta} & & \downarrow{\eta} \\
B' & \xrightarrow{\varphi} & B'
\end{array}
$$

commutes. We will show that $\text{Coker } \varphi = B$.

The morphism $\varphi$ extends $A$-linearly the assignations $g_j \mapsto \sum_{i=1}^{h} \phi'(\Lambda_{ij})g_i$, for $j = 1, \ldots, h$. Consider the expansion

$$
\sum_{i=1}^{h} \phi'(\Lambda_{ij}) \cdot g_i = \sum_{i=1}^{h} (\Lambda_{ij}(f_1, \ldots, f_{n+1} + t)) \cdot g_i = \sum_{i=1}^{h} (\Lambda_{ij}(f_1, \ldots, f_{n+1})) \cdot g_i + \sum_{i=1}^{h} (\sum_{k=1}^{\infty} \frac{1}{k!} \partial^k \Lambda_{ij}(f_1, \ldots, f_{n+1})t_k) \cdot g_i = t \sum_{i=1}^{h} (\sum_{k=1}^{\infty} \frac{1}{k!} \phi(\frac{\partial^k \Lambda_{ij}}{\partial Y^k})t^{k-1}) \cdot g_i.
$$

It follows that $\varphi$ splits as the composition $B' \xrightarrow{\epsilon} B' \xrightarrow{\psi} B'$, where the first morphism is multiplication by $t$ and $\psi$ is obtained by extending $g_j \mapsto g_j' = \sum_i R_{ij}g_i$, with

$$R_{ij} = \sum_{k=1}^{\infty} \frac{1}{k!} \phi(\frac{\partial^k \Lambda_{ij}}{\partial Y^k})t^{k-1}.$$ 

It suffices to show that $\psi$ is an $A$-module isomorphism, that is, that $g_1', \ldots, g_h'$ is a system of generators of $B'$. This is equivalent to show that the collection of the classes of $g_1', \ldots, g_h'$ is a $\mathbb{C}$-basis $B'/\mathfrak{m}B'$, where $\mathfrak{m}$ is the maximal ideal in $A$. If $a \in A$ is divisible by some $X_i$, then $\phi(a) \in \mathfrak{m}B' \subseteq \mathfrak{m}B'$. Therefore, condition C2 implies that the classes of the non-diagonal coefficients $R_{ij}, i \neq j$, are all zero, and the classes of $R_{ii}$ are all non-zero. This implies that the collection of classes of $g_1', \ldots, g_h'$ is a basis, as desired. □

**Remark 2.7.** It is clear that Theorem 2.6 works for holomorphic maps as well. With our usual assumptions, if a holomorphic map $f: \mathcal{X} \rightarrow \mathbb{C}^{n+1}$ has polynomial coordinate functions and $g_1, \ldots, g_h$ are polynomial minimal set of generators of $\tilde{f}_* \mathcal{O}_X$, then any HMP-matrix for $g_1, \ldots, g_h$ is a presentation matrix for $f_* \mathcal{O}_X$. 

6
3 Algorithm and applications

In this section we describe an algorithm to obtain a matrix Λ satisfying C1 and C2, and we give some applications. An implementation of this algorithm in Singular can be found in [A].

We recall previous assumptions and notations: We use variables \(X_1, \ldots, X_n\) and \(Y\), and variables \(x_1, \ldots, x_\ell\). We write \(A = C[X, Y]\) and \(\tilde{A} = C[x]\) for the base rings. I is an ideal in \(C[x]\), such that \(B = C[x]/I\) is a Cohen Macaulay \(n\)-dimensional ring. We have a morphism of rings \(\phi: A \to B\), such that \(B\) is finitely generated as \(\tilde{A}\)-module. It is well known that a minimal set of generators \(\{g_1, \ldots, g_h\}\) for \(B\) as \(\tilde{A}\)-module can be obtained as representatives of the elements of a basis of the \(C\)-vector space \(B/mB\), where \(m\) is the ideal \(\langle X_1, \ldots, X_n\rangle\) in \(\tilde{A}\). We assume that such a basis can be computed by an internal procedure of the software used for implementation. We also assume the software to be able to perform Groebner basis computations, in particular the reduction of an ideal with respect to another one (see [GP]). In Singular, this operations can be computed by using the instructions \texttt{kbase} and \texttt{reduce}, respectively.

The outline of our algorithm is as follows:

\[
\begin{align*}
\text{INPUTS: } & f \text{ and } I. \\
& \text{Compute a } C\text{-basis } \{g_1, g_2, \ldots, g_h\} \text{ of } B/mB. \\
& \text{FOR } i = 1, \ldots, h \text{ do:} \\
& \quad \text{DEFINE } w := 1 \text{ and } k := 0; \\
& \quad \text{WHILE } w \neq 0 \text{ do:} \\
& \quad \quad k := k + 1; \\
& \quad \quad \text{Consider } v_{i1}, \ldots, v_{ih}, \text{ where} \\
& \quad \quad v_{ij} = \sum_{|\alpha| \leq k} a_{ij}^\alpha X^\alpha \text{ is a generic polynomial satisfying } (P) \text{ (see Remark 3.1);} \\
& \quad \quad \text{Compute the reduction } P(a_{ij}^\alpha, x) \text{ of } \sum_{j=1}^h \phi(v_{ij})g_j \text{ modulo } I; \\
& \quad \quad \text{If there exists } \tilde{a}_{ij}^\alpha \in C, \text{ such that } P(\tilde{a}_{ij}^\alpha, x) = 0 \\
& \quad \quad \quad \text{then } \lambda_{ij} := \sum_{|\alpha| \leq k} \tilde{a}_{ij}^\alpha X^\alpha \text{ and } w := 0; \\
\text{OUTPUT: } & \text{Matrix presentation } \Lambda = (\lambda_{ij}).
\end{align*}
\]

Algorithm to compute an HMP-matrix.

Remark 3.1. By a generic polynomial we mean that the coefficients \(a_{ij}^\alpha\) are parameters in the base ring. Property (P) is as follows:

- \(v_{ii}(0, \ldots, 0, Y) \equiv Y \mod (Y^2)\);
- \(v_{ij}(0, \ldots, 0, Y) \in C\), for all \(j \neq i\).

By construction, condition (P) implies that Λ satisfies condition C2, and the fact that the reduction \(P(\tilde{a}_{ij}^\alpha, X)\) vanishes ensures that C1 holds. Note that the degree of the generic polynomials \(v_{ij}\) grows with the “while” loop, and the algorithm runs over all the matrices considered in Remark 2.1. Since we assumed that \(B\) is a finitely generated \(\tilde{A}\)-module, the algorithm terminates.

In the remaining subsections we show some applications of the implementation [A] of the above algorithm. All computations were done using a computer with 2.8 Ghz Intel Core i7 processor and 8 Gb of RAM memory. We use standard singularity theory notation for which the reader can find the details in the references.
3.1 Topological invariants for \( f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \)

Let \( f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \) be the map germ given by

\[
(x, y) \mapsto (xy, x^4 + y^{37} + x^2y^{23}).
\]

The singular set is \( \Sigma(f) = V(4x^4 - 21x^2y^{23} - 37y^{37}) \) and the presentation matrix \( \Lambda \) for \( f_*\mathcal{O}_{\Sigma(f)} \) is a \( 41 \times 41 \) matrix, too big to be written here. Using our implementation, the total time to obtain such matrix was about 95 seconds.

An important topological invariant of \( f \) is the Milnor number of the discriminant \( \mu(\Delta(f)) \). Using Singular software (in the same computer), we can not obtain \( \mu(\Delta(f)) \) due to lack of memory. But, in \([GM1]\) we find that

\[
\mu(\Delta(f)) = 2(d(f) + c(f)) + \mu(\Sigma(f))
\]

where \( d(f) \) is the number of nodes and \( c(f) \) is the number of cusps of \( f \). As, \( \mu(\Sigma(f)) = 108 \) and

\[
c(f) + d(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2, 0}}{F_1(f)} = 2886,
\]

we have that \( \mu(\Delta(f)) = 5880 \). In this case, by computation, we obtain \( c(f) = 147 \) and, therefore, the number of ordinary double points is \( d(f) = 2739 \).

3.2 Topological classification in \( \mathcal{O}_2^2 \)

For corank 2 map germs from \( \mathbb{C}^2 \) to \( \mathbb{C}^2 \) Gaffney and Mond in \([GM2]\) ask the following question:

*How many different topological types are contained in a given \( \mathcal{K}(xy, x^a + y^b) \)-orbit?*

Miranda, Saia and Soares in \([MSS]\) show that for all pairs \((a, b)\), excluding \((2, 3)\) and \((2, 5)\), there exist a non finite number of distinct topological types in each \( \mathcal{K} \)-orbit, that is, for \( f(x, y) = (xy, \alpha x^a + \beta y^b) \) there is at least one family such that each element in the family is \( \mathcal{A} \)-finitely determined germ and any two of them are not \( C_0 - \mathcal{A} \)-equivalent. For \((a, b) = (2, 3)\) there is only one, and for \((a, b) = (2, 5)\) there are two distinct topological orbits.

In order to illustrate the use of the previous algorithm we fix \((a, b) = (3, 4)\). Let \( f_{u,v,w}(x, y) = (xy, y^4 + x(x + y)^2 + wxy^3 + vx^3y + wx^4) \) be a family of such maps. We use our implementation to obtain a presentation matrix of \( f_{u,v,w} \) restricted to the singular set. Now, the discriminant of this map is given by the 0-th Fitting ideal, and we have:

- if \( u \neq v - 2w + 2 \), then the Milnor number of discriminant curve is \( \mu(\Delta(f_{u,v,w})) = 54 \).
- if \( u = v - 2w + 2 \), then the family is not \( \mathcal{A} \)-finitely determined.

We set \( u = 2, v = w = 0 \) and consider the one parameter family

\[
f_s(x, y) = (xy, y^4 + x(x+y)^2 + 2xy^3 + x^4 y^{s+1})
\]

with \( s > 3 \). Now \( f_s \) is \( \mathcal{K} \)-equivalent to \((xy, x^3 + y^4)\) for all \( s \), and the set of generators of \( \mathcal{O}_2 \) as \( \mathcal{O}_{\Sigma(f_s), 0} \)-module via \( f_s^* \) is \( \{1, x, x^2, y, y^2, y^3, y^4\} \). Using our implementation we obtain the following presentation matrix for \( f_s, \mathcal{O}_{\Sigma(f_s)} \):
where \( \alpha = -\frac{1}{2} X Y - \frac{9}{4} X^3 - \frac{9}{40} X^2 - \frac{1}{40} X^8 (9 X - 10 Y) \) and \( \beta = -\frac{7}{2} X^3 - \frac{9}{40} X^2 - \frac{3}{4} X Y \).

Computing the Fitting ideals we obtain the following topological invariants:
\[
\mu(\Delta(f_s)) = 2s + 50, \quad d(f_s) = s + 14, \quad \mu(\Sigma(f_s)) = 4 \quad \text{and} \quad c(f_s) = 9.
\]
Therefore, for each \( s \) we have a distinct topological type.

### 3.3 Number of triple points

Consider the corank 2 quasi-homogeneous map germ \( f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0) \), given by
\[
(x, y, z) \mapsto (x, y z, z^{18} + y^2 + x^4 z),
\]
with weights \((17, 36, 4)\) and degrees \((17, 40, 72)\). It is easy to verify that the Jacobian ideal is \( J(f) = \langle 18 z^{18} + x^4 z - 2 y^2 \rangle \) and a set of generators for \( \mathcal{O}_3/(\bar{f} + J(f)) \) is \( \{1, y, z, ..., z^{18}\} \). The time to get the presentation matrix was about 44 seconds.

If \( I(A_{1,1}) \) is the ideal that defines the ordinary double points in the target, \( \sharp A_{(1,2)}, \sharp A_{(1,1,1)} \) and \( \sharp A_3 \) are, respectively, the number of transversal intersection between cuspidal edges and ordinary planes, the number of ordinary triple points and the number of swallowtail as in the classical Arnold notation, then by [JMS][Proposition 4.6 and 4.9] we have that
\[
\sharp A_{(1,2)} + \sharp A_{(1,1,1)} + \sharp A_3 = \text{dim}_{\mathbb{C}} \frac{O_3}{F_2(f)} = 7368 \quad \text{and} \quad \sharp A_{(1,1,1)} = \text{dim}_{\mathbb{C}} \frac{O_3}{(I(A_{1,1,1})^2 : F_0(f))} = 5120.
\]

For the quasi-homogeneous case, Ohmoto in [O] presents formulae to compute these 0-stable singularities from \( (\mathbb{C}^3, 0) \) to \( (\mathbb{C}^5, 0) \). In this case, we obtain
\[
\sharp A_3 = 136; \quad \sharp A_{1,1,1} = 5120; \quad \sharp A_{1,2} = 2112,
\]
which can be confirmed directly using the previous presentation matrix.

### 3.4 Target multiple points

A simple but important application of our algorithm is to compute the multiple spaces in the target \( M_k(f) \) of a finite map germ \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \). As mentioned in the introduction, \( M_k(f) \) is the zero set of the ideal \( F_{k-1}(f) \) in \( \mathcal{O}_{n+1} \).

**Example 3.2.** Let \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) be the corank two map germ given by
\[
(x, y) \mapsto (x^2, y^2, x^3 + y^3 + xy).
\]
Our implementation yields the following presentation matrix of $f_* \mathcal{O}_2$:

$$
\Lambda_f = \begin{bmatrix}
Y & -X_1 & -X_2 & -1 \\
-X_2 Y - X_1^2 & Y + X_1 X_2 & X_2^2 - X_1 & 0 \\
-X_2^2 - X_1 Y & -X_2 + X_1^2 & Y + X_1 X_2 & 0 \\
-X_1 X_2 & -X_2^2 & -X_1^2 & Y
\end{bmatrix}
$$

and we obtain the following Fitting ideals (see Figure 2):

- $F_0(f) = \langle X_1^2 X_2^2 - 2X_1 X_2 Y^2 + Y^4 - 2X_1^4 X_2 - 2X_1 X_2^2 - 8X_1^2 X_2^3 Y - 2X_1^3 Y^2 - 2X_2^4 + X_1^6 - 2X_1^3 X_2^3 + X_2^6 \rangle$, whose zeros define the image of $f$.

- $F_1(f) = (X_1^2 + X_1 Y, Y + X_1 X_2, -X_2 + X_1^2) \cap \langle X_1 + X_2 - Y, X_2^2 - X_2 Y + Y^2 \rangle \cap \langle X_2 + Y, X_1 + Y \rangle \cap (-X_1 + X_2^2, Y + X_1 X_2, X_1^2 + X_2 Y)$. This ideal defines the double point space of $f$ in the target.

- $F_2(f) = (X_1, X_2, Y)$, which defines the triple point in the image of $f$. As the codimension of $F_2(f)$ is 1, this indicates that $f$ has exactly one triple point collapsed in the origin.

In the same way that the $\mathbb{C}$-codimension of $F_2(f)$ in $\mathcal{O}_3$ measures the number of triple points collapsed in the origin of a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^4, 0)$, we can use the algorithm to compute the number of $(n + 1)$-tuple points collapsed in the origin of a map germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. This invariant is just the codimension of $F_k(f)$ as a $\mathbb{C}$-vector subspace of $\mathcal{O}_{n+1}$.

### 3.5 Source double points of map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a germ of finite and generically-one-to-one map. Following Mond [Mo], the (lifted) double point space of $f$ is the space $D^2(f)$ given by the ideal

$$
I^2(f) = (f \times f)^* I_{n+1} + R(\alpha),
$$

where $I_{n+1}$ is the ideal defining the diagonal of $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$, and $R(\alpha)$ is given by the minors of any matrix $\alpha$, whose entries $\lambda_{ij} \in \mathcal{O}_{2n}$ satisfy

$$
f_j(x) - f_j(x') = \sum_{i=1}^{n} \alpha_{ij}(x, x')(x - x'),
$$

for all $1 \leq j \leq n + 1$. In [LN], it is shown that $D^2(f)$ is a Cohen-Macaulay space of dimension $n - 1$ (an extension of this result for map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $n \leq p$, can be found in [NP]). Set theoretically, $D^2(f)$ is given by the pairs $(x, x') \in \mathbb{C}^n \times \mathbb{C}^n$, such that $f(x) = f(x')$ and, if $x = x'$, then $f$ is singular at $x$. In [MM] the source double point space $D(f) \subset \mathbb{C}^n$ is defined as the image of $D^2(f)$ by the projection on the first component $\pi : (\mathbb{C}^n \times \mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, that is:

$$
D(f) = V(F_0(\pi|_{D^2(f)})�).
$$

The set $D(f)$ plays an important role. For instance, for map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$, it characterizes finite determinacy. More precisely: a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is finitely $\mathcal{A}$-determined if and only if the Milnor number $\mu(D(f))$ is finite [MM, MNP].

10
Computing \( D(f) \) for a map germ \((\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)\) can be quite involved, but in the case \( p = n + 1 \), since \( \pi: D^2(f) \to \mathbb{C}^n \) is a map from a Cohen-Macaulay space of dimension \( n - 1 \) to \( \mathbb{C}^n \), we can use our algorithm to do so.

**Example 3.3.** Consider the corank 2 map germ \( f(x, y) = (x^2, y^2, x^3 + y^3 + xy) \) of Example 3.2. The set of double points \( D^2(f) \) is given by the ideal \( I^2(f) \), generated by

\[
(x + u)(y + v), (x + u)(2y^2 + 2yv + 2v^2 + x + u),
\]

\[
(2x^2 + 2xu + 2u^2 + y + v)(y + v),
\]

\[
x^2 - u^2,
\]

\[
y^2 - v^2,
\]

\[
x^3 + y^3 + xy - u^3 - v^3 - uv.
\]

Take the projection \( \pi: (\mathbb{C}^2 \times \mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) given by \( \pi(x, y, u, v) = (x, y) \). A basis of the vector space \( \frac{\mathcal{O}_{D^2(f)}}{\pi^*m} \) is given by \( \{1, y, u, v, v^2, v^3\} \). Using our implementation, we find the following presentation matrix for \( \pi_*\mathcal{O}_{D^2(f)} \):

\[
\Lambda_\pi = \begin{bmatrix}
Y & -1 & 0 & 0 & 0 & 0 \\
0 & Y & 0 & 0 & -1 & 0 \\
2XY + X^3 & 0 & Y - X^2 & X - X^2Y & -X^2 & -2 \\
0 & X^2 & 0 & Y + X^2 & 1 & 0 \\
0 & X^2Y & 0 & X^2Y & Y & 1 \\
\frac{1}{2} X^2 & -\frac{1}{2} X^3 & \frac{1}{2} X + \frac{1}{2} X^2Y & -\frac{1}{2} XY & -\frac{1}{2} X + \frac{1}{2} X^2Y & Y + \frac{1}{2} X^2
\end{bmatrix}
\]

We rename \( X = x \) and \( Y = y \), since the target of \( \pi \) is the source of \( f \), and then we obtain the following (Figure 2):

1. The ideal \( F_0(\pi|_{D^2(f)}) = \langle (x^3 + y^3)(x + y^2)(y + x^2) \rangle \), which defines the source double point space \( D(f) \).

2. The ideal \( F_1(\pi|_{D^2(f)}) = \langle x^2, xy, y^2 \rangle \). We may regard the double points of \( \pi_{D^2(f)} \) as triple points of \( f \). The codimension of \( F_1(\pi|_{D^2(f)}) \) is 3, corresponding to the number of source points in an ordinary triple point, which here have collapsed at 0.

**Remark 3.4.** We can give two different analytic structures defining the \( k \)th source multiple point space of a map germ \( f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \). In the first one, we regard the \( k \)th source multiple point space as \( \pi(D^k(f)) \), where \( D^k(f) \subseteq (\mathbb{C}^n)^k \) is the lifted \( k \)th multiple point space, and \( \pi \) is the projection on the first copy of \( \mathbb{C}^n \). Thus, the defining ideal is \( F_0(\pi|_{D^k(f)}) \). The second structure is given by defining the source multiple points as the preimage by \( f \) of \( M_k(f) \). It is an open problem to decide whether or not these analytic structures coincide. In the previous example, we computed \( F_0(\pi|_{D^2(f)}) = \langle (x^3 + y^3)(x + y^2)(y + x^2) \rangle \), which is precisely the preimage by \( f \) of the ideal \( F_1(f) \) computed in Example 3.2.

**Acknowledgements:** The authors are grateful to Juan José Nuño Ballesteros, for many useful suggestions to the present work.
Figure 2: Lifted double points, double points and multiple points in the target.

References

[A] https://sites.google.com/site/aldicio/publicacoes/presentation-matrix-algorithm.

[BOT] Nuño-Ballesteros, J. J., Oréfice-Okamoto, B. and Tomazella, J. N. Equisingularity of map germs from a surface to the plane. [arXiv:1507.01483](https://arxiv.org/abs/1507.01483#v2).

[DGPS] Decker, W., Greuel, G.-M., Pfister, G. and Schönemann, H., SINGULAR 3-1-3 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2012).

[GM1] Gaffney, T. and Mond, D., Cusps and double folds of germs of analytic maps $\mathbb{C}^2 \to \mathbb{C}^2$, J. London Math. Soc. (2) 43, 1, 185–192, (1991).

[GM2] Gaffney, T. and Mond, D., Weighted homogeneous maps from the plane to the plane. Math. Proc. Camb. Phil. Soc., vol. 109, 451–470, (1991).

[GP] Greuel, G. M. and Pfister, G., A Singular Introduction to Commutative Algebra, Cambridge Studies in Advanced Mathematics, (2002).

[JMS] Jorge Perez, V. H., Miranda, A. J. and Saia, M. J., Counting Singularities via Fitting Ideals, International Journal of Mathematics, World Scientific Publishing Company, 23 6, 1–18, (2012).

[JN] Jorge Perez, V. H. and Nuño-Ballesteros, J. J., Finite determinacy and Whitney equisingularity of map germs from $\mathbb{C}^n$ to $\mathbb{C}^{2n-1}$, Manuscripta Math. 128, 389–410, (2009).

[KLU] Kleiman, S. L., Lipman, J. and Ulrich, B., The multiple-point schemes of a finite curvilinear map of codimension one, Ark. Mat. 34 (2), 285-326, (1996).

[L] Looijenga, E. J. N., Isolated singular points on complete intersections, London Mathematical Soc., Lecture Note series, 77, (1984).

[MM] Marar, W. L. and Mond, D., Multiple point schemes for corank 1 maps. J. London Math. Soc. (2) 39, no. 3, 553–567, (1989).
[MNP] Marar, W. L., Nuño-Ballesteros, J. J. and Peñafort-Sanchis, G., *Double point curves for corank 2 map germs from \( \mathbb{C}^2 \) to \( \mathbb{C}^3 \).* Topology Appl. 159, no. 2, 526-536, (2012).

[Mo] Mond, D., *Some remarks on the geometry and classification of germs of maps from surfaces to 3-space.* Topology 26, no. 3, 361–383, (1987).

[MP] Mond, D. and Pellikaan, R., *Fitting ideals and multiple points of analytic mappings.* Algebraic geometry and complex analysis (Pátzcuaro, 1987), 107–161, Lecture Notes in Math., 1414, Springer, Berlin, (1989).

[MSS] Miranda, A. J., Saia, M. J. and Soares, L. M. F., *On the number of topological orbits of complex germs in \( K \) classes (\( xy, x^a + y^b \)).* Proceedings of the Royal Society of Edinburgh, 146A, 1–20, (2016).

[NP] Nuno-Ballesteros, J. J., Penafort-Sanchis, G., *Multiple point spaces of finite holomorphic maps.* The Quarterly Journal of Mathematics, 2016. doi: 10.1093/qmath/haw042. arXiv:1509.04990

[O] Ohmoto, T., *Singularities and Characteristic Classes for Differentiable Maps.* arXiv:1309.0661

[ORW] Oset Sinha, R., Ruas, M. A. S., Wik Atique, R. *Defining the extra-nice dimensions.* Preprint.

[S] Silva, O. N. *Superfícies com singularidades não isoladas.* PhD thesis. ICMC-USP, São Carlos. (2017).

Hernandes, M. E. 
mehernandes@uem.br
DMA-UAM
Av. Colombo 5790
Maringá-PR 87020-900
Brazil

Miranda, A. J. 
aldicio@ufu.br
FAMAT-UFU
Av. João Naves de Ávila, 2121
Uberlândia - MG 38408-100
Brazil

Peñafort Sanchis, G. 
guillermo.penafort@uv.es
IMPA
Estr. Dona Castorina 110, 22460-320
Rio de Janeiro,