Unitarily-invariant integrable systems and geometric curve flows in \( SU(n+1)/U(n) \) and \( SO(2n)/U(n) \)

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Abstract

Bi-Hamiltonian hierarchies of soliton equations are derived from geometric non-stretching (inelastic) curve flows in the Hermitian symmetric spaces \( SU(n+1)/U(n) \) and \( SO(2n)/U(n) \). The derivation uses Hasimoto variables defined by a moving parallel frame along the curves. As main results, new integrable multi-component versions of the Sine–Gordon (SG) equation and the modified Korteweg–de Vries (mKdV) equation, as well as a novel nonlocal multi-component version of the nonlinear Schrödinger (NLS) equation are obtained, along with their bi-Hamiltonian structures and recursion operators. These integrable systems are unitarily invariant and correspond to geometric curve flows given by a non-stretching wave map and a mKdV analog of a non-stretching Schr"{o}dinger map in the case of the SG and mKdV systems, and a generalization of the vortex filament bi-normal equation in the case of the NLS systems.

Keywords: integrable system, curve flow, symmetric space

1. Introduction and summary

The nonlinear Schrödinger (NLS) equation \( u_t = i(u_{xx} + |u|^2u) \) is one of the most prominent examples of an integrable system. Its possesses an infinite hierarchy of symmetries and a corresponding infinite hierarchy of conservation laws, which are generated by a recursion...
operator. It also possesses two different but compatible Hamiltonian structures and corresponding Poisson brackets. As well, it possesses a Lax pair from which an inverse scattering transform arises that can be used for solving the initial-value problem.

All of these integrability aspects of the NLS equation have a remarkable geometrical origin when non-stretching (inelastic) curve flows are considered in Euclidean space [1–4]. A non-stretching curve flow is an equation $\gamma_t = aT + bN + cB$ formulated in a Frenet frame $(T, N, B)$ along a curve $\gamma$ with an arclength-parameterization, so that $\gamma_s = T$ is the unit tangent vector. The normal coefficients $b, c$ in the flow equation are functions of the curvature $\kappa$ and the torsion $\tau$ of the curve, given by the Frenet equations $T_\tau = \kappa N, \ N_\tau = -\kappa T + \tau B, \ B_\tau = -\tau N$, while the tangential coefficient $a$ is determined by $a_\tau = \kappa b$ due to the non-stretching property of the curve. When $b = a = 0$ and $c = \kappa$, the curve $\gamma$ undergoes a bi-normal flow $\gamma_t = \kappa B$. This flow equation physically describes the motion of a vortex filament in incompressible fluids [5]. The induced flow on $(\kappa, \tau)$ turns out to be equivalent to the NLS equation for the Hasimoto variable $u = \kappa \exp(i \int \tau \, dx)$ [5]. Moreover, the Lax pair and bi-Hamiltonian operators for the NLS equation turn out to be encoded in a simple way in the structure equations of a moving frame formulation of the curve flow [3, 4], where the Hasimoto transformation from $(\kappa, \tau)$ to $u$ corresponds geometrically to a gauge transformation from a Frenet frame to a parallel frame given by rotating the vectors $(N, B)$ in the normal plane by an angle $\theta(x) = -\int \tau \, dx$ along the curve [6]. Unlike a Frenet frame, a parallel frame has a rigid gauge freedom consisting of a constant rotation $\phi$ applied to the vectors $(N, B)$ in the normal plane. Under this rigid gauge transformation, $u$ transforms to $e^{i\phi}u$ by a constant phase rotation, and so $u$ is not an invariant of the curve like $(\kappa, \tau)$ but instead has the geometrical meaning of a $U(1)$-covariant of the curve [4].

Similarly, all of the symmetries of the NLS equation themselves correspond to geometrical curve flows in Euclidean space, and in particular the first higher symmetry in the NLS hierarchy is given by $\gamma_t = \kappa \tau B + \kappa^2 N + \frac{1}{2}T$ with the Hasimoto variable $u$ satisfying the complex modified Korteweg–de Vries (mKdV) equation $u_t = u_{xxx} + \frac{1}{2}[u]^2u$. This flow equation physically describes axial motion of a vortex filament [7]. It is also an integrable system, sharing the same integrability properties as the NLS equation.

A broad generalization of parallel frames and Hasimoto variables has been obtained in work over the past few decades on non-stretching curve flows in more general geometric spaces, starting with constant-curvature Riemannian manifolds [2, 8–10] $S^n$ (spheres) and $H^n$ (hyperbolic spaces), and continuing with various other Riemannian geometries [11–17] given by symmetric spaces $M = G/H$ (also called reductive Klein geometries). As a culmination of this line of work, for general Riemannian symmetric spaces $M = G/H$, a complete theory of parallel frames, Hasimoto variables, and integrable systems of mKdV type as well as Siné–Gordon (SG) type arising from geometrical non-stretching curve flows was presented in [18] by the second author. In particular, a pair of compatible Hamiltonian operators (Poisson brackets) was shown to be encoded in the structure equations of a $H$-parallel frame for non-stretching curve flows in a general Riemannian symmetric space $M = G/H$. This integrability structure provides a recursion operator that yields a hierarchy of mKdV-type integrable systems, including a SG-type integrable system.

Riemannian symmetric spaces are curved generalizations of Euclidean space in which the Euclidean isometry group $SO(3) \ltimes \mathbb{R}^3$ is replaced by a (real) simple Lie group $G$, and the $SO(3)$ gauge group of the frame bundle is replaced by a subgroup $H \subset G$ that is invariant under an involutive automorphism of $G$. There is a well-known classification of these spaces (see e.g. [19]), based on the classification of real simple Lie groups. These respective classifications each have a division into classical types, known as the A, B, C, D series, and exceptional types, known as the E, F, G series.
To-date, mKdV and SG integrable systems, along with their integrability structure and their corresponding geometrical realizations as non-stretching curve flows, have been derived in the following classical types:

- $SO(n + 1)/SO(n)$, BD I [12]
- $SU(n)/SO(n)$, A I [12]
- $Sp(n + 1)/Sp(1)Sp(n)$, C II [14–16]
- $SU(2n)/Sp(n)$, A II [16]
- $Sp(n)/U(n)$, C I [17]

The purpose of the present paper is to consider the two remaining simplest classical types:

- $SU(n + 1)/U(n)$, A III; and $SO(2n)/U(n)$, D III

In section 2, the general theory from [18] will be applied to these two Riemannian symmetric spaces. As one new development, the theory will be extended to derive an NLS-type integrable system by exploiting a $U(1)$ subgroup given by the center of the equivalence group of the $U(n)$-parallel frame. In sections 3 and 4, the specific features of the resulting integrable NLS systems in $SU(n + 1)/U(n)$ and $SO(2n)/U(n)$ will be worked out, as well as the integrable mKdV and SG systems, along with their integrability structure. In the case of $SU(n + 1)/U(n)$, these integrable systems involve a real scalar variable and a complex vector variable, whereas in the case of $SO(2n)/U(n)$, they involve a real scalar variable and a pair of complex vector variables. In section 5, the corresponding geometrical non-stretching curve flows will be derived for each of these integrable systems. Most interestingly, the integrable NLS systems are found to correspond to a generalized bi-normal curve flow in $SU(n + 1)/U(n)$ and $SO(2n)/U(n)$. Finally, some concluding remarks will be made in section 6.

We mention that all of the integrable systems derived here also possess a Lax pair that comes directly from the frame structure equations in a $U(n)$-parallel frame, since these equations have the form of a zero-curvature matrix equation in the Lie algebra of the isometry groups $SU(n + 1)$ and $SO(2n)$. The details of this will be given in a subsequent paper [20] in which we will derive the Lax pair associated to the structure equations of a $H$-parallel frame in a general Riemannian symmetric space $M = G/H$.

We also point out that the general theory in [18], as well as its application in the present paper, is different than the work of Fordy and Kullish [21] in which integrable NLS systems are derived by writing down a linear isospectral problem in Hermitian symmetric spaces. In particular, those systems essentially utilize the Hermitian structure of the space, and as a consequence they have different form and a different number of components than the ones we derive. Our systems are constructed without using the Hermitian structure of $SU(n + 1)/U(n)$ and $SO(2n)/U(n)$, since this structure is not part of the equivalence group of the $U(n)$-parallel frame that is used in the construction. Instead we use an almost complex structure that exists in the tangent space of the symmetric space along the curve flow and comes from the center of the frame equivalence group. Moreover, the geometrical realization of the Fordy–Kullish systems involves stretching (elastic) curve flows [22] rather than the non-stretching (inelastic) curve flows that we consider here.

2. $U(n)$-parallel frames and curve flow equations

The symmetric spaces $M = G/U(n)$ with $G = SU(n + 1)$, $SO(2n)$ have a natural Riemannian structure which comes from a soldering identification between the tangent space $T_x M$ at points $x$ and the vector space $m = g/u(n)$. This soldering $T_x M \cong m$ relies on the algebraic properties of $g \supset u(n)$ as a symmetric Lie algebra. In particular,
\[ g = u(n) \oplus \mathfrak{m}, \quad K(u(n), \mathfrak{m}) = 0 \]  
(2.1) is an orthogonal direct sum decomposition relative to the Cartan–Killing form \( K \) on \( g \), with the Lie bracket relations
\[ [u(n), u(n)] \subset u(n), \quad [u(n), \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset u(n) \]  
(2.2) induced from the Lie bracket on \( g \).

A simple formulation of the soldering identification is provided by [23] a \( \mathfrak{m} \)-valued linear coframe \( e \) and a \( u(n) \)-valued linear connection \( \omega \) whose torsion and curvature
\[ \mathcal{T} := de + [\omega, e], \quad \mathcal{R} := d\omega + \frac{1}{2} [\omega, \omega] \]  
(2.3) are 2-forms with respective values in \( \mathfrak{m} \) and \( u(n) \), given by the Cartan structure equations
\[ \mathcal{T} = 0, \quad \mathcal{R} = -\frac{1}{2} [e, e]. \]  
(2.4)

Here \([\cdot, \cdot]\) denotes the Lie bracket on \( g \) composed with the wedge product on \( T^*_x M \). This structure together with the Cartan–Killing form determines a Riemannian metric \( g \) and a Riemannian connection (i.e. covariant derivative) \( \nabla \) on the space \( M = G/U(n) \) from the following soldering relations:
\[ g(X, Y) := -K(e_X, e_Y), \]  
(2.5)
\[ e \mid \nabla_X Y := \partial_X e_Y + [\omega_X, e_Y], \]  
(2.6) for all \( X, Y \) in \( T_x M \), where \( e[X = e_X, e] Y = e_Y \in \mathfrak{m} \). In addition, the 2-forms \( \mathcal{T} \) and \( \mathcal{R} \) determine the torsion tensor \( T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] \) and the curvature tensor \( R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \) through the soldering relations
\[ e \mid T(X, Y) = \mathcal{T}(X \wedge Y) = 0, \]  
(2.7)
\[ e \mid R(X, Y)Z = [\mathcal{R}](X \wedge Y), e_Z] = -[[e_X, e_Y], e_Z]. \]  
(2.8)

These expressions (2.5)–(2.8) show that \( \nabla \) is metric compatible, \( \nabla g = 0 \), and torsion-free, \( T = 0 \), while its curvature is covariantly constant, \( \nabla R = 0 \).

This formulation of the Riemannian structure of \( M = G/U(n) \) has an intrinsic gauge freedom consisting of the transformations
\[ e \longrightarrow \text{Ad}(f^{-1}) e, \quad \omega \longrightarrow \text{Ad}(f^{-1}) \omega + f^{-1} df \]  
(2.9) as defined in terms of an arbitrary function \( f : M \to U(n) \subset G \). The gauge transformations (2.9) comprise a local (\( x \)-dependent) representation of the linear transformation group \( \text{Ad}(U(n)) \) which defines the gauge group [24] of the frame bundle of \( M \). Both the metric tensor \( g \) and curvature tensor \( R \) on \( M \) are gauge invariant.

Geometrically, \( G \) represents the isometry group of \( M \), while the subgroup \( U(n) \subset G \) represents the isotropy subgroup of the origin \( o \) in \( M \). In terms of the symmetric Lie algebra structure (2.2), the Lie subalgebra \( u(n) \) is identified with the generators of isometries that leave fixed the origin \( o \) in \( M \), and the vector space \( \mathfrak{m} \) is identified with the generators of isometries that carry the origin \( o \) to any point \( x \neq o \) in \( M \).

The symmetric space \( M = G/U(n) \) also has a Hermitian structure, consisting of a complex structure tensor \( J \) that satisfies the properties \( J^2 = -\text{id}, \nabla J = 0 \), and \( g(JX, JY) = g(X, Y) \) for all \( X, Y \) in \( T_x M \). In particular, \( J \) can be identified with the linear map \( \text{Ad}(U(1)_c) \) on \( T_x M \) where \( U(1)_c \) is the center of \( U(n) \).
Let $\gamma(x)$ be any smooth curve in $M = G/U(n)$. A moving frame consists of a set of orthonormal vectors that span the tangent space $T_xM$ at each point $x$ on the curve $\gamma$. The Frenet equations of a moving frame yield a connection matrix consisting of the set of frame components of the covariant $x$-derivative of each frame vector along the curve [23]. A moving coframe consists of a set of orthonormal covectors that are dual to the frame vectors relative to the Riemannian metric $g$. Such a framing for $\gamma(x)$ is determined by the Lie-algebra components of $e$ and $\omega|\gamma_x$ when an orthonormal basis is introduced for $\mathfrak{g} = g/\mathfrak{u}(n)$ with respect to the Cartan–Killing form, where the Frenet equations are defined by the frame components of the transport equation

$$\nabla_x e = -\text{ad}(\omega|\gamma_x) e$$

(2.10)

along the curve. In particular, if $\{m_l\}$, $l = 1, \ldots, \dim \mathfrak{m}$, is any fixed orthonormal basis for $\mathfrak{m}$, then a frame at each point $x$ along the curve is given by the set of vectors $X_l := -\langle e^* \cdot m_l \rangle$, $l = 1, \ldots, \dim \mathfrak{m}$. Here $e^*$ is a $\mathfrak{m}$-valued linear frame defined as the dual to the linear coframe $e$ by the condition that $-\langle e^*, e \rangle = \text{id}$ is the identity map on each tangent space $T_xM$ (see [18, 23]).

Now consider any smooth flow $\gamma(x,t)$ of a curve in $M = G/U(n)$. We write $X = \gamma_x$ for the tangent vector and $Y = \gamma_t$ for the evolution vector at each point $x$ along the curve. The flow is non-stretching (inelastic) provided that it preserves the $G$-invariant arclength $ds = |\gamma| dx$, or equivalently

$$\nabla_x |\gamma_x| = 0$$

(2.11)

in which case we have $g(\gamma_x, \gamma_x) = |\gamma_x|^2 = 1$ without loss of generality. For flows that are transverse to the curve (such that $X$ and $Y$ are linearly independent), $\gamma(x,t)$ will describe a smooth two-dimensional surface in $M$. The pullback of the torsion and curvature equation (2.4) to this surface yields

$$D_x h - D_t e + [u, h] - [\varpi, e] = 0,$$

(2.12)

$$D_x \varpi - D_t u + [u, \varpi] = -[e, h],$$

(2.13)

with

$$e := e|X = e|\gamma_x, \quad u := \omega|X = \omega|\gamma_x,$$

(2.14)

$$h := e|Y = e|\gamma_t, \quad \varpi := \omega|Y = \omega|\gamma_t,$$

(2.15)

where $D_x, D_t$ denote derivative operators with respect to $x,t$. In terms of these variables, the curve flow is given by

$$\gamma_t = -K(e^*, h)$$

(2.16)

where the linear frame $e^*$ is determined in terms of $u$ from the dual of the Frenet equation (2.10),

$$\nabla_x e^* = -\text{ad}(u) e^*.$$

(2.17)

For any non-stretching curve flow $\gamma(x,t)$, these structure equations (2.12)–(2.15) turn out to encode a pair of compatible Hamiltonian operators, as shown in theorem 1 later. The encoding looks simplest when we utilize an $U(n)$-parallel framing for $\gamma(x,t)$ as follows.

A $U(n)$-parallel frame along a curve in $M = G/U(n)$ is a direct algebraic generalization of a parallel moving frame in Euclidean geometry [6], as defined by the properties [18]:
(i) e is a constant unit-norm element lying in a Cartan subspace a ⊂ m that is contained in the centralizer subspace m⊥ of e, i.e. \( D_e e = D_e = 0 \), \( K(e, e) = -1 \), and \( a(m_\|) e = 0 \) where \( m_\| \oplus m_\perp = m \) and \( K(m_\|, m_\perp) = 0 \).

(ii) \( u \) lies in the perp space \( u(n)_\perp \) of the Lie subalgebra \( u(n)_\| \subset u(n) \) of the linear isotropy group \( \text{Ad}(U(n))_\| \subset \text{Ad}(U(n)) \) that preserves \( e \), i.e. \( \text{ad}(u(n)_\|) e = 0 \) and \( K(u, u(n)_\|) = 0 \) where \( u(n)_\| \oplus u(n)_\perp = u(n) \) and \( K(u(n)_\|, u(n)_\perp) = 0 \).

Existence of such moving frames can be established by applying a suitable gauge transformation (2.9) to an arbitrary linear frame at each point \( x \) along the curve [18]. The necessary transformation is unique up to a residual gauge freedom given by rigid transformations in \( \text{Ad}(U(n))_\| \subset \text{Ad}(U(n)) \) preserving the tangent vector \( X \), where the subgroup \( U(n)_\| \subset U(n) \) is generated by the Lie subalgebra \( u(n)_\| \subset u(n) \). This residual gauge freedom is called the equivalence group of the frame.

The resulting \( U_\| \)-parallel coframe \( e \) provides an isomorphism between \( T_x M \) and \( m = g / u(n) \), which yields a correspondence between the vectors \( \{ X_i \} \) in a moving frame for \( T_x M \) and the vectors \( \{ m_i \} \) in a basis for \( m \). Under this isomorphism, the tangent vector \( X = \gamma_s \) corresponds to the Cartan element \( e = e[X] \in a \).

**Remark 1.** The set of inequivalent \( U(n)_\| \)-parallel frames admitted by a smooth arclength-parameterized curve \( \gamma(x) \) in \( M = G / U(n) \) can be characterized by the set of orbits of all unit-norm elements \( e[X] \gamma_s = e \) in the Cartan subspace \( a \subset m \) under the action of the subgroup in \( U(n) \) that preserves this subspace [18]. The dimension of \( a \) is equal to the rank of the symmetric space \( G / U(n) \), which is 1 in the case \( G = SU(n+1) \) and \([n/2] \) (integer part) in the case \( G = SO(2n) \). Thus, up to equivalence, a \( U(n)_\| \)-parallel frame is unique for the symmetric space \( M = SU(n+1) / U(n) \), but is not non-unique for the symmetric space \( M = SO(2n) / U(n) \). In both spaces \( M \), the equivalence group \( U(n)_\| \) of a \( U(n)_\| \)-parallel frame does not contain the circle group \( \exp(J\phi) \subset U(n) \) \( \phi \in \mathbb{R} \) generated by the complex structure \( J \), since no vector in \( T_x M \) is invariant under this group.

The properties and construction of \( U(n)_\| \)-parallel frames rely on the Lie bracket relations for the subspaces \( m_\|, m_\perp, u(n)_\|, u(n)_\perp \) coming from the structure of \( g \) as a symmetric Lie algebra (2.2). These relations consist of

\[
[m_\|, m_\|] \subseteq u(n)_\|, \quad [m_\|, u(n)_\|] \subseteq m_\|, \quad [u(n)_\|, u(n)_\|] \subseteq u(n)_\|, \quad (2.18)
\]

\[
[u(n)_\|, m_\perp] \subseteq u(n)_\perp, \quad [u(n)_\|, u(n)_\perp] \subseteq u(n)_\perp, \quad (2.19)
\]

\[
[m_\|, m_\perp] \subseteq u(n)_\perp, \quad [m_\|, u(n)_\perp] \subseteq u(n)_\perp, \quad (2.20)
\]

while the remaining Lie brackets obey the general relations

\[
[m_\perp, m_\perp] \subset u(n), \quad [u(n)_\perp, u(n)_\perp] \subset u(n), \quad [u(n)_\perp, m_\perp] \subset m. \quad (2.21)
\]

2.1. Non-stretching curve flow equations

We will consider non-stretching curve flows \( \gamma(x, t) \) in \( M = G / U(n) \) having a \( U(n)_\| \)-parallel framing. By projecting the Cartan structure equations (2.12)–(2.13) into the subspaces \( m_\|, m_\perp, u(n)_\|, u(n)_\perp \), we obtain the system

\[
0 = D_t h_\| + [u, h_\|] \| = m_\|, \quad (2.22)
\]

\[
0 = D_t h_\perp + [u, h_\|] + [u, h_\perp] + [e, \pi^+] \in m_\perp, \quad (2.23)
\]
0 = D_x ω^|| + [u, ω^⊥] || ∈ u(n)_||, \quad (2.24)

0 = D_x ω^⊥ - D_x u + [u, ω^||] + [u, ω^⊥] _⊥ + h^⊥ ∈ u(n)_⊥, \quad (2.25)

where

\[ u = u_⊥ ∈ u(n)_⊥, \quad h = h_|| + h_⊥ ∈ m_|| ⊕ m_⊥, \quad ω = ω^|| + ω^⊥ ∈ u(n)_|| ⊕ u(n)_⊥, \quad (2.26) \]

and

\[ h^⊥ = \text{ad}(e)h_⊥ ∈ u(n)_⊥. \quad (2.27) \]

Through equations (2.22) and (2.24), the variables \( h_|| \) and \( ω^|| \) can be eliminated, while from equations (2.25) and (2.27), \( ω^⊥ \) can be expressed in terms of \( h^⊥ \). General results in [18] show that the resulting equations encode a pair of compatible Hamiltonian operators [25, 26] as follows.

**Theorem 1.** In a \( U(n)\)-parallel framing for non-stretching curve flows in \( G/U(n) \), with \( G = SU(n+1), SO(2n) \), the frame structure equations take the form

\[ u_t = H(ω^⊥) + h^⊥, \quad ω^⊥ = J(h^⊥) \quad (2.28) \]

where \( H = K|_{u(n)_||} \) is a Hamiltonian operator, and \( J = \text{ad}(e)^{-1}K|_{m_⊥} \text{ad}(e)^{-1} \) is a compatible symplectic operator, given by

\[ K = D_x + [u, .] _⊥ + [u, D_x^{-1}[u, .]]_||. \quad (2.29) \]

These operators and the flow equation on \( u \) are invariant under the unitary equivalence group \( U(n)_|| ⊂ U(n) \) of the \( U(n)\)-parallel frame.

We emphasize that the Hamiltonian operator structure of the flow equation (2.28) in theorem 1 is universal for all non-stretching curve flows in \( M = G/U(n) \). This has an important application when the flow equation is written in the form

\[ u_t = R(h^⊥) + h^⊥, \quad R = HJ \quad (2.30) \]

with \( R \) being a hereditary recursion operator. By Magri’s theorem [27], if the flow component \( h^⊥ \) is chosen to be the generator of a symmetry \( X = h^⊥ ∂_u \) of the operators \( H \) and \( J \), then the resulting flow equation on \( u \) will be an integrable system possessing a bi-Hamiltonian structure and a hierarchy of higher symmetries and associated conservation laws. Moreover, the corresponding curve flow (2.16) will be a geometrical flow that will possess a similar integrability structure.

One obvious symmetry of \( H \) and \( J \) is \( x \)-translations, as generated by \( X = u_⊥ ∂_u \). Hence

\[ u_t - u_x = R(u_x) \quad (2.31) \]

is an integrable system. It possesses \( R \) as a recursion operator, which generates the hierarchy of higher symmetries

\[ X^{(k)} = h^{(k)}_|| ∂_u, \quad h^{(k)}_|| = R^k(u_x), \quad k = 0, 1, 2, \ldots \quad (2.32) \]

starting from the \( x \)-translation symmetry \( X^{(0)} = u_x ∂_u \). Note that the convective term \(- u_x \) can be removed by a Galilean transformation. Then the resulting system \( u_t = R(u_x) \) is of complex mKdV type.
Another symmetry of $\mathcal{H}$ and $\mathcal{J}$ consists of unitary transformations on $u$, which come from the equivalence group $U(n)_{1} \subset U(n)$ of the $U(n)$-parallel frame. The center of $U(n)_{1}$ consists of a $U(1)$ subgroup whose action on $m_{1}$ is isomorphic to that of the circle group $\exp(\phi) \subset U(n)$ ($\phi \in \mathbb{R}$). This provides a circle-group symmetry which is generated by $X = (\text{ad}(j)u) \partial_{u}$ where $j \in u(1) \subset u(n)_{1} \subset u(n)$ satisfies $\text{ad}(j)^{2} = -\text{id}$ on $m_{1}$. Then

$$u_{t} = \text{ad}(j)u = \mathcal{R}(\text{ad}(j)u) \tag{2.33}$$

is an integrable system possessing the hierarchy of higher symmetries

$$X^{(k)} = \frac{h_{(k)}}{h_{(0)}} \partial_{u}, \quad h_{(k)} = \mathcal{R}^{k}(\text{ad}(j)u), \quad k = 0, 1, 2, \ldots \tag{2.34}$$

starting from the $U(1)$ symmetry $X^{(0)} = (\text{ad}(j)u) \partial_{u}$. The term $\text{ad}(j)u$ can be removed by a phase transformation $u \rightarrow \exp(\text{rad}(j))u$, and this yields a system $u_{t} = \mathcal{R}(\text{ad}(j)u)$ which is of NLS type.

In addition to the two integrable systems (2.31) and (2.33), an integrable system of Sine–Gordon type can be obtained by considering the kernel of the symplectic operator,

$$u_{t} = h_{1}, \quad \mathcal{J}(h_{1}) = 0. \tag{2.35}$$

This system shares the hierarchy of higher symmetries admitted by the mKdV-type integrable system (2.32).

The specific structure of all three integrable systems (2.31), (2.33) and (2.35) for the two symmetric spaces $G/U(n)$ will be discussed in the next two sections.

### 3. Integrable systems in $SU(n + 1)/U(n)$

Since the symmetric space $SU(n + 1)/U(n)$ has rank 1, there is a unique choice of a $U(n)$-parallel frame, up to equivalence, for non-stretching curve flows $\gamma(x, t)$ (see remark 1). The equivalence group of this frame is $\text{Ad}(U(n - 1)) \subset \text{Ad}(U(n))$.

To construct the $U(n)$-parallel frame, we will need the following matrix decomposition of the symmetric Lie algebra $su(n + 1) = u(n) \oplus m$:

$$\begin{pmatrix} -\text{tr}(B) & a \\ -a^{t} & B \end{pmatrix} \in su(n + 1), \quad B \in u(n), a \in \mathbb{C}^{n}, \text{tr}(B) \in \mathbb{R}, \tag{3.1}$$

where we will write

$$(B) := \begin{pmatrix} -\text{tr}(B) & 0 \\ 0 & B \end{pmatrix} \in u(n) \subset su(n + 1), \quad B \in u(n), \text{tr}(B) \in \mathbb{R}, \tag{3.2}$$

$$(a) := \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix} \in m = su(n + 1)/u(n) \simeq \mathbb{C}^{n}, \quad a \in \mathbb{C}^{n}. \tag{3.3}$$

The Cartan subspace $a \subset m \simeq \mathbb{C}^{n}$ is the real span of the element $(\vec{e})$ given by $\vec{e} = (1, 0) \in \mathbb{C}^{n} = \mathbb{C} \oplus \mathbb{C}^{n-1} \simeq m$, where $0 \in \mathbb{C}^{n-1}$. Using this Cartan element $(\vec{e})$, we obtain a decomposition of $m = m_{\parallel} \oplus m_{\perp}$ and $u(n) = u(n)_{\parallel} \oplus u(n)_{\perp}$ into centralizer subspaces and perp subspaces:

$$a = (a_{\parallel} + ia_{\perp}, a_{\perp}) \in \mathbb{C}^{n}, \quad a_{\perp} \in \mathbb{C}^{n-1}, a_{\parallel}, a_{\perp} \in \mathbb{R}, \tag{3.4}$$

$$B = \begin{pmatrix} i b_{\perp} - \frac{i}{2} \text{tr}(B_{\parallel}) & b_{\perp} \\ -b_{\perp}^{t} & B_{\parallel} \end{pmatrix} \in u(n), \quad B_{\parallel} \in u(n - 1), b_{\perp} \in \mathbb{C}^{n-1}, b_{\parallel} \in \mathbb{R}, \text{tr}(B_{\parallel}) \in \mathbb{R}, \tag{3.5}$$
with the notation

$$\mathbf{B}_\parallel := \begin{pmatrix} \begin{pmatrix} \frac{1}{2} \text{tr} (\mathbf{B}_\parallel) - \frac{1}{2} \text{tr} (\mathbf{B}_\parallel) & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in u(n), \quad (3.6)$$

$$((\mathbf{b}_\perp, \mathbf{b}_\perp)) := \begin{pmatrix} -\mathbf{b}_\perp & (0, 0) \\ (0, 0)^t & -\mathbf{b}_\perp \end{pmatrix} \in u(n), \quad (3.7)$$

$$(a_i) := \begin{pmatrix} 0 \\ (a_i, 0)^t \end{pmatrix} \in m, \quad (3.8)$$

$$((\mathbf{i}a_i, \mathbf{a}_\perp)) := \begin{pmatrix} 0 \\ (\mathbf{i}a_i, \mathbf{a}_\perp)^t \end{pmatrix} \in m_\perp. \quad (3.9)$$

Details of this decomposition and its algebraic properties, including the Lie bracket structure, are summarized in appendix B.

We now write out the frame structure equations (2.22)–(2.25) by using these matrix representations (3.6)–(3.9).

First, the tangential components (2.14) of the linear coframe $\mathbf{e}$ and the linear connection $\mathbf{\omega}$ along the curve are given by the variables

$$\mathbf{e} = \frac{1}{\sqrt{\chi}} ((1, 0)) \in m_\parallel \subset m, \quad (3.10)$$

$$\mathbf{u} = ((\mathbf{i}u, \mathbf{u})) \in u(n)_\perp, \quad (3.11)$$

where $u \in \mathbb{R}$ is a real scalar variable, and $\mathbf{u} \in \mathbb{C}^{n-1}$ is a complex vector variable. These variables geometrically represent the components of the Cartan matrix of the $U(n)$-parallel frame. Here $\chi = 2\sqrt{n+1}$ is a normalization constant determined by the Cartan–Killing metric (see proposition B.1 in appendix B).

Next, the flow components (2.15) of the linear coframe and the linear connection are expressed in terms of the variables

$$h_\parallel = \lambda (h_\parallel) \in m_\parallel, \quad (3.12)$$

$$h_\perp = ((\mathbf{i}h_\perp, \mathbf{h}_\perp)) \in m_\perp, \quad (3.13)$$

$$\mathbf{\omega}_\parallel = (\Theta) \in h_\parallel, \quad (3.14)$$

$$\mathbf{\omega}_\perp = ((\mathbf{i}w, \mathbf{w})) \in h_\perp, \quad (3.15)$$

as well as

$$h_\perp = \lambda \frac{1}{\sqrt{\chi}} (h_\perp, h_\perp) \in h_\perp, \quad h_\perp = -2\lambda^{-1} h_\perp, \quad h_\perp = -\lambda^{-1} h_\perp \quad (3.16)$$

from relation (2.27). Here $h_\parallel, h_\perp, w \in \mathbb{R}$ are real scalar variables; $h_\perp, h_\perp, w \in \mathbb{C}^{n-1}$ are complex vector variables; $\Theta \in u(n-1)$ is an anti-hermitian matrix variable. Then, the frame structure equations (2.22)–(2.25) are given by
\[ D_{x}h_{\parallel} = h_{\perp} + \langle h_{\perp}, u \rangle, \]  
(3.17)

\[ D_{x}\Theta = u \wedge w, \]  
(3.18)

\[ \frac{1}{\sqrt{\lambda}} \lambda^{-1} w = \frac{1}{4} D_{x}h_{\perp} + h_{\parallel}u + \frac{1}{2} \text{Im} \left( \bar{u} \cdot h_{\perp} \right), \]  
(3.19)

\[ \frac{1}{\sqrt{\lambda}} \lambda^{-1} w = D_{x}h_{\perp} + h_{\parallel}u - i(\bar{u}h_{\perp} + \frac{1}{2} h_{\perp}u), \]  
(3.20)

\[ u_{t} = D_{x}w + \lambda \frac{1}{\sqrt{\lambda}} h_{\perp}, \]  
(3.21)

\[ \frac{1}{\sqrt{\chi}} \sqrt{\lambda} w = D_{x}h_{\perp} + h_{\parallel}u - i \left( \bar{u} + \frac{1}{2} h_{\perp}u \right), \]  
(3.22)

with the outer product notation (A.1)–(A.4) shown in appendix A.

Applying theorem 1 to this system (3.17)–(3.22), we obtain a Hamiltonian operator

\[ H = \begin{pmatrix} D_{x} & \text{Im} \bar{u} \\ -i\bar{u} & D_{x} + iu + iuD_{x}^{-1} \text{Im} \bar{u} \cdot + u |D_{x}^{-1}| u \wedge \end{pmatrix}, \]  
(3.23)

and a compatible symplectic operator

\[ J = \begin{pmatrix} \frac{1}{4} D_{x} + uD_{x}^{-1}u & \frac{1}{2} \text{Im} \bar{u} \cdot + uD_{x}^{-1} \text{Re} \bar{u} \\ -i\frac{1}{2} u + uD_{x}^{-1}u & D_{x} - iu + uD_{x}^{-1} \text{Re} \bar{u} \end{pmatrix}, \]  
(3.24)

with the system being given by

\[ \begin{pmatrix} u' \\ w \end{pmatrix} = H \begin{pmatrix} w \\ u \end{pmatrix} + \lambda \frac{1}{\sqrt{\lambda}} \begin{pmatrix} h_{\perp} \\ h_{\perp} \end{pmatrix}, \quad \frac{1}{\sqrt{\chi}} \lambda^{-1} \begin{pmatrix} w \\ u \end{pmatrix} = J \begin{pmatrix} h_{\perp} \\ h_{\perp} \end{pmatrix}. \]  
(3.25)

Composition of these two operators yields

\[ \lambda \frac{1}{\sqrt{\lambda}} \begin{pmatrix} u' \\ w \end{pmatrix} = R \begin{pmatrix} h_{\perp} \\ h_{\perp} \end{pmatrix} + \chi^{-1} \begin{pmatrix} h_{\perp} \\ h_{\perp} \end{pmatrix}, \]  
(3.26)

where

\[ R = HJ = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \]  
(3.27)

is a hereditary recursion operator given by

\[ R_{11} = \frac{1}{4} D_{x}^{2} - \frac{1}{2} |u|^{2} + D_{x}uD_{x}^{-1}u, \]  
(3.28a)

\[ R_{12} = \frac{1}{2} D_{x} \text{Im} u \cdot + \text{Im} \bar{u} \cdot D_{x} - u \text{Re} \bar{u} \cdot + D_{x}uD_{x}^{-1} \text{Re} \bar{u}. \]  
(3.28b)
\[ R_{22} = D_x^2 + u^2 - \frac{1}{2} i u \text{Im} \cdot D_x i u + D_x u D_x^{-1} \text{Re} \cdot + i u D_x^{-1} \text{Im} \cdot D_x - i u D_x^{-1} uu \odot. \] (3.28c)

Note that all operations \( \text{Re}, \text{Im}, \cdot, \wedge, \bar{\cdot}, \odot \) are meant to act in rightmost to leftmost order.

**Proposition 1.** All non-stretching curve flows \( \gamma(x, t) \) in \( SU(n+1)/U(n) \) are described by the system (3.25) which is formulated by using a \( U(n) \)-parallel frame, with \( (u, u) \) being the components of the Cartan matrix \( u = ((u, u)) \). This system encodes a Hamiltonian operator (3.23) and a compatible symplectic operator (3.24), which yields a hereditary recursion operator (3.27) and (3.28). The action of the equivalence group \( U(n-1) \subset U(n) \) of the frame is given by \( (u, u) \to (u, \det(X^{-1})^{1/2} u X^{-1} u) \), where \( X^{-1} \) is an arbitrary \( x \)-independent \( (n-1) \times (n-1) \) unitary matrix.

Both operators (3.23) and (3.24) are invariant under \( x \)-translation symmetries and \( U(1) \) symmetries, which are respectively generated by

\[ X = u_x |\partial_u + u_x |\partial_u \] (3.29)

and

\[ X = i u |\partial_u. \] (3.30)

### 3.1. mKdV flow

Using the \( x \)-translation symmetry generator (3.29), we define a flow on \( (u, u) \) by taking

\[ (h^+, h^-) = (u_x, u_x). \] (3.31)

This yields, from the frame structure equations (3.17)–(3.20),

\[ h_x = \frac{1}{2} u^2 + \frac{1}{2} |u|^2, \] (3.32)

\[ \lambda^{-1} \frac{1}{\sqrt{\chi}} \Theta = u \wedge u_x - \frac{1}{2} i u u \odot u, \] (3.33)

\[ \lambda^{-1} \frac{1}{\sqrt{\chi}} w = \frac{1}{4} u_{xx} + \frac{1}{2} (u^2 + |u|^2) u + \frac{1}{2} \text{Im} (u \cdot u_x), \] (3.34)

\[ \lambda^{-1} \frac{1}{\sqrt{\chi}} w = u_{xx} + \frac{1}{2} (u^2 + |u|^2) u - i \left( \frac{1}{2} u_x u + uu_x \right). \] (3.35)

(See appendix A for notation.) Hence, the resulting flow equations (3.21)–(3.22) are given by

\[ \lambda^{-1} \frac{1}{\sqrt{\chi}} u_x - \lambda^{-1} u_x = \frac{1}{4} u_{xxx} + \frac{3}{2} u^2 u_x + \frac{3}{2} \text{Im} (u \cdot u_x), \] (3.36)

\[ \lambda^{-1} \frac{1}{\sqrt{\chi}} u_x - \lambda^{-1} u_x = u_{xxx} - \frac{3}{4} (2 i u |u|^2 + 6 i \text{Im} (u_i \cdot u) + i u_x - (u^2) u_x \]

\[ + \frac{3}{2} (u^2 + |u|^2 - i u_x) u_x, \] (3.37)
This is the integrable mKdV-type system (2.31). It has the bi-Hamiltonian structure
\[
\left( \frac{du}{u} \right)_t = \mathcal{H} \left( \frac{\delta \mathcal{H}}{\delta u} \right) = \mathcal{E} \left( \frac{\delta \mathcal{E}}{\delta u} \right),
\]
where the Hamiltonians are given by
\[
\mathcal{E} = \int \frac{1}{2} (u^2 + |u|^2) \, dx,
\]
\[
\mathcal{H} = \int \left( - \frac{1}{8} (u_t)^2 - \frac{1}{2} |u_x|^2 + \frac{1}{2} (u^2 + |u|^2)^2 + \frac{1}{2} u \, \text{Im} (\bar{u} \cdot u_x) \right) \, dx,
\]
and where the second Hamiltonian operator is given by
\[
\mathcal{E} = \mathcal{R} \mathcal{H} = \mathcal{H} \mathcal{J} \mathcal{H}
\]
in terms of the recursion operator (3.27).

3.2. NLS flow

Using the \( U(1) \) symmetry generator (3.30), we define another flow on \((u, u)\) by taking
\[
(h^+, h^-) = (0, iu).
\]
This yields, from the frame structure equations (3.17)–(3.20),
\[
h_\parallel = 0,
\]
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} w = \frac{1}{2} |u|^2,
\]
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} w = iu_x + uu,
\]
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} \Theta = iu^t u = \frac{1}{2} iu \otimes u, \quad \lambda^{-1} \frac{1}{\sqrt{\chi}} \text{tr} (\Theta) = i|u|^2.
\]
(See appendix A for notation.) Then the resulting flow equations (3.21) and (3.22) are given by
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} u_t = (|u|^2)_x,
\]
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} u_t - \chi^{-1} u_x = iu_{xx} + i(u^2 + |u|^2)u + u_x u,
\]
which is the integrable NLS-type system (2.33). It has the bi-Hamiltonian structure
\[
\left( \frac{du}{u} \right)_t = \mathcal{H} \left( \frac{\delta \mathcal{H}}{\delta u} \right) = \mathcal{E} \left( \frac{\delta \mathcal{E}}{\delta u} \right),
\]
where the first Hamiltonian is given by
\[
\mathcal{E} = \int \left( \frac{1}{2} |u|^2 - \text{Im} (\bar{u} \cdot u_x) \right) \, dx.
\]
To explain the second Hamiltonian, we observe that the Hamiltonian operator $\mathcal{E} = \mathcal{R} \mathcal{H}$ will give the NLS system (3.47) and (3.48) if
\[
\mathcal{H} \left( \frac{\delta \mathcal{E}}{\delta u} \right) = \begin{pmatrix} 0 \\ iu \end{pmatrix}.
\]  
(3.51)

This relation can be made to hold if we take $\mathcal{E} = 0$ and allow $D_x^{-1}(0) = c = \text{const.}$ and $D_x^{-1}(0) = C = \text{const.}$, since then we have
\[
\mathcal{H} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ cu + u \end{pmatrix} C
\]  
(3.52)

which reduces to the desired flow (3.51) when $c = 1$ and $C = 0$. This type of Hamiltonian structure has been considered previously in other contexts [28, 29].

We remark that this NLS system (3.47) and (3.48) and its Lax pair has been derived previously [30] by applying a Miura transformation to a multi-component version of the Yajima–Oikawa system [31].

3.3. SG flow

We obtain a SG-type system (2.35) by putting $w = 0$ and $\tilde{w} = 0$ in the frame structure equations (3.17) and (3.22), which yields
\[
\begin{align*}
\dot{u}_t &= \lambda \frac{1}{\sqrt{\chi}} h^\perp, \\
\dot{u}_x &= \lambda \frac{1}{\sqrt{\chi}} h^\perp
\end{align*}
\]  
(3.53)

with
\[
\begin{align*}
D_x h^\parallel - h^\perp u - (h^\perp, u) &= 0, \\
D_x h^\perp + 4h^\parallel u + 2 \text{Im} (\bar{u} \cdot h^\perp) &= 0, \\
D_x h^\perp + h_x u - i(uh^\perp + \frac{1}{2} h^\perp u) &= 0.
\end{align*}
\]  
(3.54)\quad(3.55)\quad(3.56)

These equations possess a conservation law
\[
D_x \left( h_x^2 + \frac{1}{4} (h^\perp)^2 + |h^\perp|^2 \right) = 0
\]  
(3.57)

from which we obtain
\[
h_x^2 + \frac{1}{4} (h^\perp)^2 + |h^\perp|^2 = 1
\]  
(3.58)

after $t$ is conformally rescaled. Hence we have
\[
h_x^2 = \pm \sqrt{1 - \frac{1}{4} (h^\perp)^2 + |h^\perp|^2}.
\]  
(3.59)

Substituting this conservation law along with the flow equations (3.53) into (3.55) and (3.56), we get
\[
\begin{align*}
\dot{u}_x &= \mp 4 \sqrt{\chi^{-1} \lambda^2 - \frac{1}{4} u_t^2 + |u_t|^2} u + \text{Im} (\bar{u}_t \cdot u).
\end{align*}
\]  
(3.60)
\[ u_{rt} = \mp \frac{1}{\sqrt{\lambda}} \left( \lambda^2 - \left( \frac{1}{4} u_t^2 + |u_t|^2 \right) u + i(u u_t + \frac{1}{2} u_t u) \right). \]  
(3.61)

3.4. Hierarchies of integrable systems

The mKdV system (3.36) and (3.37) and the NLS system (3.47) and (3.48) are each a root system in a hierarchy of integrable systems generated by the recursion operator (3.27), (3.28).

**Theorem 2.** There is a mKdV hierarchy of integrable systems

\[ \begin{pmatrix} u_t \\ u_r \end{pmatrix} - \lambda \frac{1}{\sqrt{\lambda}} \begin{pmatrix} u_r \\ u_t \end{pmatrix} = R^k \begin{pmatrix} u_t \\ u_r \end{pmatrix}, \quad k = 0, 1, 2, \ldots \]
(3.62)

as well as a NLS hierarchy of integrable systems

\[ \begin{pmatrix} u_t \\ u_r \end{pmatrix} - \lambda \frac{1}{\sqrt{\lambda}} \begin{pmatrix} 0 \\ iu \end{pmatrix} = R^k \begin{pmatrix} 0 \\ iu \end{pmatrix}, \quad k = 0, 1, 2, \ldots \]
(3.63)

arising from the structure equations (3.17)–(3.22) of a \( U(n) \)-parallel frame for non-stretching curve flows in \( SU(n + 1)/U(n) \). Associated to the mKdV hierarchy is an integrable SG system (3.60), (3.61). All of these integrable systems are invariant under the unitary symmetry group \( U(n-1) \) which acts as \( (u, u_1, u_2) \to (u, u_1 X_{n-2}^{-1}, u_2 X_{n-2}^{-1}) \) where \( X_{n-2} \) is an arbitrary \( x \)-independent \( (n-2) x (n-2) \) unitary matrix.

4. Integrable systems in \( SO(2n)/U(n) \)

Since the symmetric space \( SO(n+1)/U(n) \) has rank \( [n/2] \geq 1 \), there is a unique choice of a \( U(n) \)-parallel frame, up to equivalence, for non-stretching curve flows \( \gamma(x,t) \) only in the cases \( n = 2, 3 \) when the rank is 1 (see remark 1). We will choose a \( U(n) \)-parallel frame that does not depend on \( n \). It is distinguished among all possible choices (when \( n \geq 4 \)) by having the maximal dimension for its equivalence group.

To construct this frame, we will need the following matrix decomposition of the symmetric Lie algebra \( so(2n) = u(n) \oplus m \):

\[
\begin{pmatrix}
\text{Re} (A + B) & \text{Im} (A + B) \\
\text{Im} (A - B) & -\text{Re} (A - B)
\end{pmatrix} \in so(2n), \quad \text{Re} B, \text{Re} A, \text{Im} A \in so(n), \text{Im} B \in g(n)
\]
(4.1)

where we will write

\[
(B) := \begin{pmatrix} \text{Re} B & \text{Im} B \\ -\text{Im} B & \text{Re} B \end{pmatrix} \in u(n), \quad B \in u(n),
\]
(4.2)

\[
(A) := \begin{pmatrix} \text{Re} A & \text{Im} A \\ \text{Im} A & -\text{Re} A \end{pmatrix} \in m = so(2n)/u(n) \simeq \mathbb{C}^{n(n-1)/2}, \quad A \in so(n, \mathbb{C}).
\]
(4.3)

In the Cartan subspace \( a \subset m \simeq \mathbb{C}^{n(n-1)/2} \), we choose the element \( (\hat{e} \wedge \check{e}) \) given by \( \hat{e} \wedge \check{e} \in so(n, \mathbb{C}) \) with \( \hat{e} = (1, 0) \in \mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1} \), where \( 0 \in \mathbb{C}^{n-1} \). This Cartan element \( (\hat{e} \wedge \check{e}) \) yields a decomposition of \( m = m_{\parallel} \oplus m_{\perp} \) and \( u(n) = u(n)_{\parallel} \oplus u(n)_{\perp} \) into centralizer subspaces and perp subspaces:
\[ A = \begin{pmatrix} 0 & a_\parallel + ia_\perp & a_\perp \\ -a_\parallel - ia_\perp & 0 & a_\parallel \\ -a_\perp & -a_\parallel & A_\parallel \end{pmatrix} \in \text{so}(n, \mathbb{C}), \]

\[ A_\parallel \in \text{so}(n - 2, \mathbb{C}), \ a_\parallel, a_\perp \in \mathbb{C}^{n-2}, \ a_\parallel, a_\perp \in \mathbb{R} \]

and

\[ B = \begin{pmatrix} ib_\parallel + ib_\perp & b_\parallel \perp \\ -b_\parallel & -ib_\parallel + ib_\perp & b_\perp \\ -b_\perp & b_\perp & B_\parallel \end{pmatrix} \in u(n), \]

\[ B_\parallel \in u(n - 2), \ b_\parallel, b_\perp \in \mathbb{C}^{n-2}, \ b_\parallel, b_\perp \in \mathbb{R}, \ b_\parallel, b_\perp \in \mathbb{C}, \]

with the notation

\[ ((a_\parallel, A_\parallel)) := \begin{pmatrix} 0 & a_\parallel & 0 \\ -a_\parallel & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in m_\parallel, \]

\[ ((a_\perp, a_\parallel, b_\perp)) := \begin{pmatrix} 0 & 0 & \text{Re} a_\perp \\ \text{Re} a_\perp & 0 & \text{Re} a_\parallel \\ -\text{Re} a_\parallel & -\text{Re} a_\perp & 0 \end{pmatrix} \begin{pmatrix} 0 & a_\parallel & a_\parallel \\ -a_\parallel & 0 & -a_\parallel \\ -a_\parallel & 0 & -a_\parallel \end{pmatrix} \in m_\perp, \]

and

\[ ((b_\parallel, b_\perp, B_\parallel)) := \begin{pmatrix} \text{Re} b_\parallel & 0 \\ -\text{Re} b_\parallel & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_\parallel & b_\parallel & 0 \\ -b_\parallel & -b_\parallel & 0 \\ 0 & 0 & 0 \end{pmatrix} \in u(n)_\parallel, \]

\[ ((b_\parallel, b_\perp, b_\perp)) := \begin{pmatrix} \text{Re} b_\parallel & 0 \\ -\text{Re} b_\parallel & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_\parallel & b_\parallel & 0 \\ -b_\parallel & -b_\parallel & 0 \\ 0 & 0 & 0 \end{pmatrix} \in u(n)_\perp. \]
Details of this decomposition and its algebraic properties, including the Lie bracket structure, are summarized in appendix C.

We now write out the frame structure equations (2.22)–(2.25) by using the matrix representations (4.6)–(4.9).

First, the tangential components (2.14) of the linear coframe \( e \) and the linear connection \( \omega \) along the curve are given by the variables

\[
e = \frac{1}{\sqrt{\chi}}((1, 0)) \in \mathfrak{m}_\parallel \subset \mathfrak{m},
\]

\[
u = ((i\mathfrak{u}, \mathfrak{u}_1, \mathfrak{u}_2)) \in \mathbb{C} \oplus \mathbb{C}^n - 2 \oplus \mathbb{C}^n - 2 \simeq \mathfrak{u}(n)_\perp,
\]

where \( \mathfrak{u} \in \mathbb{R} \) is a real scalar variable, and \( \mathfrak{u}_1, \mathfrak{u}_2 \in \mathbb{C}^{n - 1} \) are complex vector variables. These variables geometrically represent the components of the Cartan matrix of the \( U(n) \)-parallel frame. Here \( \chi = 4\sqrt{n - 1} \) is a normalization constant determined by the Cartan–Killing metric (see proposition C.1 in appendix C).

Next, the flow components (2.15) of the linear coframe \( e \) and the linear connection \( \omega \) are summarized in appendix C. Details of this decomposition and its algebraic properties, including the Lie bracket structure, are summarized in appendix C.

Then, the frame structure equations (2.22)–(2.25) are given by

\[
D_1 h_\parallel = h_\parallel^1 u + \langle \mathfrak{u}_1, h_\parallel^1 \rangle + \langle \mathfrak{u}_2, h_\parallel^2 \rangle,
\]

\[
D_1 H_\parallel = \mathfrak{u}_2 \wedge h_\parallel^1 - \mathfrak{u}_1 \wedge h_\parallel^2,
\]

\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} w_1 = \frac{1}{4} D_1 h^1 + h_{\parallel} u + \frac{1}{2} \Im(\mathfrak{u}_1 \cdot h_\parallel^1 + \mathfrak{u}_2 \cdot h_\parallel^2),
\]

\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} w_2 = D_1 h_\parallel^2 + h_{\parallel} \mathfrak{u}_1 + \mathfrak{u}_2 \mathfrak{H}_\parallel - i(\frac{1}{2} h_\parallel^1 \mathfrak{u}_1 + u h_\parallel^1),
\]

as well as

\[
h_\parallel^1 = \lambda \frac{1}{\sqrt{\chi}} ((i h_\parallel^1, h_\parallel^1 \perp)) \in \mathfrak{u}(n)_\perp.
\]
and

\[ D_\xi \Theta_1 = \text{Im} \left( w_1 \cdot u_1 - w_2 \cdot u_1 \right), \]  
(4.22)  
\[ D_\xi \Theta_2 = w_2 \cdot u_1 - u_2 \cdot w_1, \]  
(4.23)  
\[ D_\xi \Theta = u_1 \wedge w_1 + u_2 \wedge w_2, \]  
(4.24)  
\[ u_t = D_\xi w + \text{Im} \left( \bar{u}_1 \cdot w_1 + \bar{u}_2 \cdot w_2 \right) + \lambda \frac{1}{\sqrt{\chi}} h^+, \]  
(4.25)  
\[ u_{1t} = D_\xi w_1 + u_1 \mid \Theta - i\Theta_1 u_1 - \Theta_2 u_2 + i(\bar{u}w_1 - w_u_1) + \lambda \frac{1}{\sqrt{\chi}} h^{1+}, \]  
(4.26)  
\[ u_{2t} = D_\xi w_2 + u_2 \mid \Theta + i\Theta_1 u_2 + \Theta_2 u_1 + i(\bar{u}w_2 - w_u_2) + \lambda \frac{1}{\sqrt{\chi}} h^{2+}, \]  
(4.27)  

with the outer product notation (A.1)–(A.4) shown in appendix A.

We apply theorem 1 to this system (4.17)–(4.27), yielding a Hamiltonian operator

\[
\mathcal{H} = \begin{pmatrix}
-iu_1 & D_\xi + iu - u_2 D_\xi^{-1} u_2 \\
-iu_1 D_\xi^{-1} \text{Im} u_1 + u_1 \wedge u_2 \wedge & u_2 D_\xi^{-1} u_1 \cdot \mathcal{C} \\
-iu_2 & -u_1 D_\xi^{-1} u_2 \cdot \mathcal{C} \\
-iu_2 D_\xi^{-1} \text{Im} (u_1 \wedge) + u_2 \wedge D_\xi^{-1} u_1 \wedge + iu_2 D_\xi^{-1} \text{Im} (u_2 \wedge) + u_2 \wedge D_\xi^{-1} u_2 \wedge & D_\xi + iu + u_1 D_\xi^{-1} \bar{u}_1 \\

\end{pmatrix}
\]  
(4.28)  

and a compatible symplectic operator

\[
\mathcal{J} = \begin{pmatrix}
\frac{1}{2} D_\xi + u D_\xi^{-1} u & \frac{1}{2} \text{Im} \bar{u}_1 + u D_\xi^{-1} \text{Re} \bar{u}_1 \\
\frac{1}{2} \text{Im} \bar{u}_2 + u D_\xi^{-1} \text{Re} \bar{u}_2 \\

\end{pmatrix}
\]  
(4.29)  

where \( \mathcal{C} \) denotes the complex conjugation operator, with the system being given by

\[
\begin{pmatrix}
u_t \\
u_{1t} \\
u_{2t}
\end{pmatrix} = \mathcal{H} \begin{pmatrix}
w_1 \\
w_2 \\
w_1 \wedge
\end{pmatrix} + \lambda \frac{1}{\sqrt{\chi}} \begin{pmatrix}
h^+ \\
h^{1+} \\
h^{2+}
\end{pmatrix}, \]  
\frac{1}{\sqrt{\chi}} \begin{pmatrix}
h^+ \\
h^{1+} \\
h^{2+}
\end{pmatrix} = \mathcal{J} \begin{pmatrix}
w_1 \\
w_2 \\
w_1 \wedge
\end{pmatrix}. 
\]  
(4.30)  

Composition of these two operators yields

\[
\begin{pmatrix}
u_t \\
u_{1t} \\
u_{2t}
\end{pmatrix} = \mathcal{R} \begin{pmatrix}
w_1 \\
w_2 \\
w_1 \wedge
\end{pmatrix} + \lambda \frac{1}{\sqrt{\chi}} \begin{pmatrix}
h^+ \\
h^{1+} \\
h^{2+}
\end{pmatrix}, 
\]  
(4.31)
where
\[
\mathcal{R} = \mathcal{H} \mathcal{J} = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \mathcal{R}_{13} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \mathcal{R}_{23} \\ \mathcal{R}_{31} & \mathcal{R}_{32} & \mathcal{R}_{33} \end{pmatrix}
\] (4.32)

is a hereditary recursion operator given by
\[
\mathcal{R}_{11} = \frac{1}{4} D_x^2 + u^2 - \frac{1}{2} (|u_1|^2 + |u_2|^2) + D_x u D_x^{-1} u.
\] (4.33)
\[
R_{12} = \frac{1}{2} D_x \text{Im} \, u_1 \cdot + D_x u D_x^{-1} \text{Re} \, u_1 \cdot + \text{Im} \, u_1 \cdot D_x - u \text{Re} \, u_1 \cdot - 2 \text{Im} \, u_2 \cdot u_1 |D_x^{-1} u_2| \wedge.
\] (4.34)
\[
R_{13} = \frac{1}{2} D_x \text{Im} \, u_2 \cdot + D_x u D_x^{-1} \text{Re} \, u_2 \cdot + \text{Im} \, u_2 \cdot D_x - u \text{Re} \, u_2 \cdot - 2 \text{Im} \, u_1 \cdot u_2 |D_x^{-1} u_1| \wedge.
\] (4.35)
\[
R_{22} = D_x^2 + i(uD_x - D_x u) + u^2 - \frac{1}{2} i u_1 \text{Im} \, u_1 \cdot
\]
\[
+ D_x u_1 D_x^{-1} \text{Re} \, u_1 \cdot + D_x u_2 |D_x^{-1} u_2| \wedge + i u_2 D_x^{-1} u_2 \wedge
\]
\[
- u_2 D_x^{-1} u_2 \cdot D_x - i u_1 D_x^{-1} \text{Im} \, u_1 \cdot D_x + u_1 |D_x^{-1} u_1| \wedge D_x
\]
\[
+ i u_1 D_x^{-1} u_2 \cdot + i u_1 D_x^{-1} u_2 \text{Re} \, u_1 \cdot - i u_1 |D_x^{-1} u_1| \circ \bar{u}_1
\]
\[
+ u_1 D_x^{-1} u_1 \wedge u_2 |D_x^{-1} u_2| \wedge - u_1 |D_x^{-1} u_2| \wedge u_1 |D_x^{-1} u_2| \wedge.
\] (4.36)
\[
R_{23} = \frac{1}{2} i u_1 \text{Im} \, u_2 \cdot + D_x u_1 D_x^{-1} \text{Re} \, u_2 \cdot - D_x u_2 |D_x^{-1} u_1| \wedge - i u_2 D_x^{-1} u_2 \wedge
\]
\[
+ u_2 D_x^{-1} u_1 \cdot \bar{C} D_x + u_1 |D_x^{-1} u_2| \wedge D_x + i u_1 D_x^{-1} \text{Im} \, u_2 \cdot D_x
\]
\[
- i u_1 D_x^{-1} u_1 \text{Re} \, u_2 \cdot + i u_2 D_x^{-1} u_1 \cdot \bar{C} - i u_1 |D_x^{-1} u_2| \circ \bar{u}_2 \circ
\]
\[
- u_1 D_x^{-1} u_1 \wedge u_2 |D_x^{-1} u_2| \wedge + u_1 |D_x^{-1} u_2| \wedge u_1 |D_x^{-1} u_2| \wedge.
\] (4.37)
\[
R_{33} = D_x^2 + i(uD_x - D_x u) + u^2 - \frac{1}{2} i u_2 \text{Im} \, u_2 \cdot
\]
\[
+ D_x u_2 D_x^{-1} \text{Re} \, u_2 \cdot + D_x u_1 |D_x^{-1} u_1| \wedge + i u_1 |D_x^{-1} u_1| \wedge\bar{u}_1
\]
\[
+ u_1 D_x^{-1} u_1 \cdot D_x + i u_2 D_x^{-1} \text{Im} \, u_2 \cdot D_x + u_2 |D_x^{-1} u_2| \wedge D_x
\]
\[
- i u_2 D_x^{-1} u_2 \cdot - i u_2 D_x^{-1} u_2 \text{Re} \, u_2 \cdot - i u_2 |D_x^{-1} u_2| \circ \bar{u}_2 \circ
\]
\[
- u_2 D_x^{-1} u_1 \wedge u_1 |D_x^{-1} u_2| \wedge + u_2 |D_x^{-1} u_2| \wedge u_1 |D_x^{-1} u_1| \wedge.
\] (4.38)

Note that all operations \( \text{Re} \), \( \text{Im} \), , \( \wedge \), \( \circ \), \( \circ \) are meant to act in rightmost to leftmost order.

**Proposition 2.** All non-stretching curve flows \( \gamma(x, t) \) in SO(2n)/U(n) are described by the system (4.30) which is formulated by using a \( U(n) \)-parallel frame whose equivalence group has maximal size, with \((u, u_1, u_2)\) being the components of the Cartan matrix \( u = ([u, u_1, u_2]) \). This system encodes a Hamiltonian operator (4.28) and a compatible symplectic operator (4.29), which yields a hereditary recursion operator (4.32). The action of the unitary subgroup \( U(n - 2) \) in the equivalence group \( U(n - 2) \times SU(2) \subset U(n) \) of the frame is given by
Both operators (3.23) and (3.24) are invariant under $x$-translation symmetries and $U(1)$ symmetries, which are respectively generated by
\[ X = u_x |\partial_{u_x} + u_{1x}|\partial_{u_{1x}} + u_{2x}|\partial_{u_{2x}} \]
and
\[ X = iu_x |\partial_{u_x} + iu_{1x}|\partial_{u_{1x}}. \]

4.1. mKdV flow

Using the $x$-translation symmetry generator (4.39), we define a flow on $(u, u_1, u_2)$ by taking
\[ (h^1, h^{1\perp}, h^{2\perp}) = (u_x, u_{1x}, u_{2x}). \]
The frame structure equations (4.17)–(4.24) then yield
\[ h_\parallel = \frac{1}{2}(u^2 + |u_1|^2 + |u_2|^2), \]
\[ H_\parallel = u_2 \wedge u_1, \]
\[ \lambda^{-1} \frac{1}{\sqrt{\lambda}} w_1 = \frac{1}{4} u_{1xx} + \frac{1}{2} (u^2 + |u_1|^2 + |u_2|^2)u + \frac{1}{2} \text{Im} (u_1 \cdot u_1 + u_2 \cdot u_2), \]
\[ \lambda^{-1} \frac{1}{\sqrt{\lambda}} w_2 = u_{2xx} + \frac{1}{2} (u^2 + |u_1|^2 + |u_2|^2)u_1 + |u_2|^2 u_1 - (u_2 \cdot u_1) u_2 - i \frac{1}{2} (u_2 u_1 + u_1 u_2), \]
\[ \lambda^{-1} \frac{1}{\sqrt{\lambda}} \Theta_1 = -\frac{1}{2} iu(|u_2|^2 - |u_1|^2) + \frac{1}{2} \text{tr} (u_2 \wedge u_{2x} - u_1 \wedge u_{1x}), \]
\[ \lambda^{-1} \frac{1}{\sqrt{\lambda}} \Theta_2 = u_{2x} \cdot u_1 - u_2 \cdot u_{1x} + iu u_2 \cdot u_1, \]
\[ \lambda^{-1} \frac{1}{\sqrt{\lambda}} \Theta = u_1 \wedge u_{1x} + u_2 \wedge u_{2x} - iu(u'_1 u_1 + u'_2 u_2). \]
(See appendix A for notation.) Hence, the resulting flow equations (4.25)–(4.27) are given by
\[ \frac{1}{\sqrt{\lambda}} \lambda^{-1} u_x - \lambda^{-1} u_x = \frac{1}{4} u_{1xx} + \frac{3}{2} u_x^2 u_x + \frac{3}{2} \text{Im} (u_1 \cdot u_{1xx} + u_2 \cdot u_{2x}). \]
\[
\frac{1}{\sqrt{\lambda}} \chi^{-1} \mathbf{u}_1 = \chi^{-1} \mathbf{u}_{1x} = \mathbf{u}_{1xx} + \frac{3}{2} \left( \frac{1}{2} u^2 + |u_1|^2 + |u_2|^2 \right) - \frac{3}{2} i u \mathbf{u}_2 - 3 (i u \mathbf{u}_2 + \mathbf{u}_2 \cdot \mathbf{u}_1) \mathbf{u}_2 \\
+ \frac{3}{4} (2 i u (|u_2|^2 - |u_1|^2)) + 2 u_{x} - 2 i u_{x} - 2 i \text{Im} (\bar{u}_x \cdot \mathbf{u}_1) \\
+ 3 u_{2x} \cdot \mathbf{u}_2 + \mathbf{u}_2 \cdot \mathbf{u}_{2x} \mathbf{u}_x, \\
\frac{1}{\sqrt{\lambda}} \chi^{-1} \mathbf{u}_2 = \chi^{-1} \mathbf{u}_{2x} = \mathbf{u}_{2xx} + \frac{3}{2} \left( \frac{1}{2} u^2 + |u_1|^2 + |u_2|^2 \right) - \frac{3}{2} i u \mathbf{u}_1 - 3 (i u \mathbf{u}_1 + \mathbf{u}_1 \cdot \mathbf{u}_2) \mathbf{u}_1 \\
- \frac{3}{4} (2 i u (|u_2|^2 - |u_1|^2)) - 2 u_{x} + 2 i u_{x} + 2 i \text{Im} (\bar{u}_x \cdot \mathbf{u}_2) \\
- 3 u_{1x} \cdot \mathbf{u}_1 - \bar{u}_1 \cdot \mathbf{u}_{1x} \mathbf{u}_x. 
\]

This is the integrable mKdV-type system (2.31). It has the bi-Hamiltonian structure

\[
\begin{bmatrix}
\mathbf{u} \\
\mathbf{u}_1 \\
\mathbf{u}_2
\end{bmatrix}
= \mathcal{H}
\begin{bmatrix}
\frac{\delta \mathcal{S}}{\delta u} \\
\frac{\delta \mathcal{S}}{\delta u_1} \\
\frac{\delta \mathcal{S}}{\delta u_2}
\end{bmatrix}
= \mathcal{E}
\begin{bmatrix}
\frac{\delta \mathcal{E}}{\delta u} \\
\frac{\delta \mathcal{E}}{\delta u_1} \\
\frac{\delta \mathcal{E}}{\delta u_2}
\end{bmatrix},
\]

where the first Hamiltonian is given by

\[
\mathcal{H} = \int \frac{1}{2} (u^2 + |u_1|^2 + |u_2|^2) \, dx
\]

and where the second Hamiltonian operator is given by

\[
\mathcal{E} = \mathcal{R} \mathcal{H} = \mathcal{H} \mathcal{J} \mathcal{H}
\]

in terms of the recursion operator (4.32).

### 4.2. NLS flow

Using the \( U(1) \) symmetry generator (4.40), we define another flow on \( (u, u_1, u_2) \) by taking

\[
(h^1, h^{1+}, h^{1-}) = (0, i u_1, i u_2).
\]

This yields, from the frame structure equations (4.17)–(4.24),

\[
h^1 = 0,
\]

\[
\mathbf{H}_1 = 2 i D_x^{-1} (u_2 \wedge u_1),
\]

\[
\lambda^{-1} \frac{1}{\sqrt{\lambda}} w_1 = \frac{1}{2} (|u_1|^2 + |u_2|^2),
\]

\[
\lambda^{-1} \frac{1}{\sqrt{\lambda}} w_1 = i u_{1x} + 2 i u_2 \mathcal{D}_x^{-1} (u_2 \wedge u_1) + u u_1,
\]

\[
\lambda^{-1} \frac{1}{\sqrt{\lambda}} w_2 = i u_{2x} - 2 i u_1 \mathcal{D}_x^{-1} (u_2 \wedge u_1) + u u_2
\]
and hence
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} i \Theta_1 = \frac{1}{2} i (|u_2|^2 - |u_1|^2),
\]
(4.62)
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} \Theta_2 = -i u_2 \cdot u_1,
\]
(4.63)
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} \Theta = i \frac{1}{2} (u_1 \odot u_1 + u_2 \odot u_2).
\]
(4.64)
(See appendix A for notation.) Then the resulting flow equations (4.25)–(4.27) are given by
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} u_{1t} - \lambda^{-1} u_{1x} = i u_{1xx} + u_1 u_1 + i (u^2 + |u_1|^2 + |u_2|^2) u_1 + 2 (i u_{2x} - u_{2r}) D_x^{-1} (u_2 \cdot u_1),
\]
(4.65)
\[
\lambda^{-1} \frac{1}{\sqrt{\chi}} u_{2t} - \lambda^{-1} u_{2x} = i u_{2xx} + u_2 u_2 + i (u^2 + |u_1|^2 + |u_2|^2) u_2 - 2 (i u_{1x} - u_{1r}) D_x^{-1} (u_2 \cdot u_1),
\]
(4.66)
which is the integrable NLS-type system (2.33). It has the bi-Hamiltonian structure
\[
\begin{pmatrix}
    u \\
    u_1 \\
    u_2
\end{pmatrix} = \mathcal{H} \begin{pmatrix}
    \delta \tilde{J} / \delta u \\
    \delta \tilde{J} / \delta u_1 \\
    \delta \tilde{J} / \delta u_2
\end{pmatrix} = \tilde{E} \begin{pmatrix}
    \delta \mathcal{E} / \delta u \\
    \delta \mathcal{E} / \delta u_1 \\
    \delta \mathcal{E} / \delta u_2
\end{pmatrix},
\]
(4.68)
where the first Hamiltonian is given by
\[
\tilde{J} = \int \frac{1}{2} \left( u (|u_1|^2 + |u_2|^2) - \text{Im} (u_1 \cdot u_{1x} + u_2 \cdot u_{2x}) + \text{Im} ((u_2 \cdot u_1) \cdot D_x^{-1} (u_2 \cdot u_1)) \right) dx.
\]
(4.69)
The second Hamiltonian involves the same explanation that was given for the NLS system obtained in section 3.2. We observe that the Hamiltonian operator \( \tilde{E} = \mathcal{R} \mathcal{H} \) will give the NLS system here if
\[
\mathcal{H} \begin{pmatrix}
    \delta \mathcal{E} / \delta u \\
    \delta \mathcal{E} / \delta u_1 \\
    \delta \mathcal{E} / \delta u_2
\end{pmatrix} = \begin{pmatrix}
    0 \\
    u_1 \\
    u_2
\end{pmatrix},
\]
(4.70)
To make this relation hold, we take \( \mathcal{E} = 0 \) and allow \( D_x^{-1} (0) = c = \text{const.} \) and \( D_x^{-1} (0) = \mathcal{C} = \text{const.} \). This yields
\[
\mathcal{H} \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix} = \begin{pmatrix}
    c_1 i u_1 + c_2 u_2 + u_1 \mathcal{C}_1 \\
    c_3 i u_2 + c_4 u_1 + u_2 \mathcal{C}_2
\end{pmatrix},
\]
(4.71)
which reduces to the desired flow (4.70) when \( c_1 = c_3 = 1, c_2 = c_4 = 0, \) and \( \mathcal{C}_1 = \mathcal{C}_2 = 0 \).
4.3. SG flow

We obtain a SG-type system (2.35) by putting \( w = 0 \) and \( w_1 = w_2 = 0 \) in the frame structure equations (4.17)–(4.27). This yields

\[
\begin{align*}
    u_t &= \lambda \frac{1}{\sqrt{\chi}} h^\perp, \\
    u_{1t} &= \lambda \frac{1}{\sqrt{\chi}} h_1^\perp, \\
    u_{2t} &= \lambda \frac{1}{\sqrt{\chi}} h_2^\perp \tag{4.72}
\end{align*}
\]

with

\[
\begin{align*}
    D_x h_\parallel &= h^\perp u - (\langle u_1, h_\perp \rangle + \langle u_2, h_\perp \rangle) = 0, \tag{4.73} \\
    D_x H_\parallel + u_1 \wedge h_\perp - u_2 \wedge h_\perp &= 0, \tag{4.74} \\
    D_x h_\perp^2 + 4h_\parallel u + \text{Im} (\bar{u}_1 \cdot h_\perp + \bar{u}_2 \cdot h_\perp) &= 0, \tag{4.75} \\
    D_x h_\perp^2 + (h_\parallel u_1 + \bar{u}_2 H_\parallel) - i(\frac{1}{2} h_\perp^2 u_1 + u h_\perp) &= 0, \tag{4.76} \\
    D_x h_\perp^2 + (h_\parallel u_2 - \bar{u}_1 H_\parallel) - i(\frac{1}{2} h_\perp^2 u_2 + u h_\perp) &= 0. \tag{4.77}
\end{align*}
\]

These equations possess a conservation law

\[
D_x \left( \frac{1}{4} (h_\perp)^2 + |h_\perp|^2 + |h_\parallel|^2 + h_\parallel^2 + \frac{1}{2} \text{tr} (H_\parallel H_\parallel) \right) = 0 \tag{4.78}
\]

from which we get

\[
\frac{1}{4} (h_\perp)^2 + |h_\perp|^2 + |h_\parallel|^2 + h_\parallel^2 + \frac{1}{2} |H_\parallel|^2 = 1 \tag{4.79}
\]

after \( t \) is conformally rescaled. Hence we obtain

\[
h_\parallel^2 = 1 - \frac{1}{4} (h_\perp)^2 - |h_\perp|^2 - |h_\parallel|^2 - \frac{1}{2} |H_\parallel|^2. \tag{4.80}
\]

By substituting this conservation law along with the flow equations (4.72) into (4.75)–(4.77), we get a nonlocal SG-type system

\[
\begin{align*}
    u_{3t} + 4\tilde{h}_\parallel u + \text{Im} (\bar{u}_1 \cdot u_{1t} + \bar{u}_2 \cdot u_{2t}) &= 0, \tag{4.81} \\
    u_{13t} + (\tilde{h}_\parallel u_1 + \bar{u}_2 \tilde{H}_\parallel) - i(\frac{1}{2} u_1 u_1 + uu_{1t}) &= 0, \tag{4.82} \\
    u_{23t} + (\tilde{h}_\parallel u_2 - \bar{u}_1 \tilde{H}_\parallel) - i(\frac{1}{2} u_2 u_2 + uu_{2t}) &= 0, \tag{4.83}
\end{align*}
\]

where

\[
\begin{align*}
    \tilde{h}_\parallel &= \pm \sqrt{\chi^{-1} \lambda^2 - \frac{1}{4} u^2 - |u_{1t}|^2 - |u_{2t}|^2 - \frac{1}{2} |H_\parallel|^2}, \tag{4.84} \\
    \tilde{H}_\parallel &= D_x^{-1} (u_2 \wedge u_{1t} - u_1 \wedge u_{2t}). \tag{4.85}
\end{align*}
\]
We can obtain a local reduction of this SG system by the following ansatz. Put
\[ H_\parallel = \alpha(h_\parallel, h^\perp) h^{\perp \perp} \wedge h^{\perp \perp} \]  
(4.86)
with \( \alpha \) taken to be an unknown function of \((h_\parallel, h^\perp)\). This form for \( \alpha \) is motivated by the property that both \( h_\parallel \) and \( h^\perp \) are invariant under the frame equivalence group (see lemma B.2). We can now determine \( \alpha(h_\parallel, h^\perp) \) by substituting this ansatz (4.86) into equations (4.73)–(4.77) to get a system of linear first-order PDEs which are readily solved. Omitting the details, we find
\[ \alpha = 1/(\frac{1}{2} i h^\perp - h_\parallel) \]  
(4.87)
and hence
\[ |\alpha|^2 = \frac{1}{4} (h^\perp)^2 + (h_\parallel)^2, \quad |H|^2 = \text{tr} (HH) = 2|\alpha|^2((h^{\perp \perp})^2 |h^{\perp \perp}|^2 - |h^{\perp \perp} \cdot h^{\perp \perp}|^2). \]  
(4.88)
Then the conservation law (4.79) becomes
\[ |h^{\perp \perp}|^2 + |h^{\perp \perp}|^2 + |\alpha|^2((h^{\perp \perp})^2 |h^{\perp \perp}|^2 - |h^{\perp \perp} \cdot h^{\perp \perp}|^2) = 1, \]  
(4.89)
which is a quadratic equation for \( |\alpha|^2 \). If we write
\[ \beta = |h^{\perp \perp}|^2 |h^{\perp \perp}|^2 - |h^{\perp \perp} \cdot h^{\perp \perp}|^2, \quad \gamma = 1 - |h^{\perp \perp}|^2 - |h^{\perp \perp}|^2, \]  
(4.90)
then the solution is given by
\[ |\alpha|^2 = \frac{1}{2} (\gamma + \sqrt{\gamma^2 - 4\beta}) \]  
(4.91)
with \( \beta \geq 0 \) due to the Cauchy-Schwartz inequality, and \( \gamma > 0 \) due to the conservation law (4.79). (Note a choice of sign for the square root has been made so that \( |\alpha| \) is non-singular for all \( \beta \geq 0 \).) This determines
\[ h_\parallel = \pm \frac{1}{2} \sqrt{2(\gamma + \sqrt{\gamma^2 - 4\beta}) - (h^\perp)^2}. \]  
(4.92)
As a result, the flow equations (4.81)–(4.83) become a local SG system with
\[ \tilde{h}_\parallel = \pm \frac{1}{2} \sqrt{2(\bar{\gamma} + \sqrt{\bar{\gamma}^2 - 4\bar{\beta}}) - (\bar{h}^\perp)^2}, \quad \tilde{H}_\parallel = \frac{1}{4} (i \bar{u} + 2\bar{h}_\parallel)(\bar{\gamma} + \sqrt{\bar{\gamma}^2 - 4\bar{\beta}})u_{1\parallel} \wedge u_{2\parallel}, \]  
(4.93)
where
\[ \bar{\beta} = |u_{1\parallel}|^2 |u_{2\parallel}|^2 - |\bar{u}_{1\parallel} \cdot u_{2\parallel}|^2, \quad \bar{\gamma} = \chi^{-1} \lambda^2 - |u_{1\parallel}|^2 - |u_{2\parallel}|^2. \]  
(4.94)

4.4. Hierarchies of integrable systems

The mKdV system (4.50)–(4.52) and the NLS system (4.65)–(4.67) are each a root system in a hierarchy of integrable systems generated by the recursion operator (4.32).

**Theorem 3.** There is a mKdV hierarchy of integrable systems

\[ \begin{pmatrix} u_1 \\ u_{1\parallel} \\ u_{2\parallel} \end{pmatrix} = R^{\lambda \parallel} \begin{pmatrix} u_1 \\ u_{1\parallel} \\ u_{2\parallel} \end{pmatrix} = R^{\lambda \parallel} \]  
(4.95)
as well as a NLS hierarchy of integrable systems

\[
\begin{pmatrix}
  u_t \\
  u_x
\end{pmatrix}
- \lambda \frac{1}{\sqrt{\lambda}}
\begin{pmatrix}
  0 \\
  iu_{tx}
\end{pmatrix}
= R^2
\begin{pmatrix}
  0 \\
  iu_{xx}
\end{pmatrix}
\quad k = 0, 1, 2, \ldots
\]  

(4.96)

arising from the structure equations (4.17)–(4.27) of a \( U(n) \)-parallel frame for non-stretching curve flows in \( SO(2n)/U(n) \). Associated to the mKdV hierarchy is an integrable SG system (4.81)–(4.83). All of these integrable systems are invariant under the unitary symmetry group \( U(n-2) \times SU(2) \), with the \( U(n-2) \) subgroup acting as \( (u, u_1, u_2) \rightarrow (u, u_1X_{n-2}^{-1}, u_2X_{n-2}^{-1}) \) where \( X_{n-2} \) is an arbitrary \( x \)-independent \( (n-2) \times (n-2) \) unitary matrix, and with the \( SU(2) \) subgroup acting as \( (u_1, u_2) \rightarrow (u_1, u_2)X_2^{-1} \) where \( X_2 \) is an arbitrary \( x \)-independent \( 2 \times 2 \) unimodular unitary matrix.

5. Geometric curve flows

The hierarchies of integrable systems derived in theorems 2 and 3 correspond to geometrical non-stretching curve flows \( \gamma(x, t) \) in the respective symmetric spaces \( M = SU(n+1)/U(n) \) and \( M = SO(2n)/U(n) \). These curve flows can be constructed through the soldering relations (2.16)–(2.17) for a \( U(n) \)-parallel frame formulated in propositions 1 and 2.

Specifically, from the frame structure equations (2.22)–(2.25), the flow vector \( \gamma_t \) has the frame components \( e^*|\gamma_t = h_\perp + h_\parallel \) with \( h_\parallel = D^{-1}_e h_\perp = u \) where \( h_\perp \) is a specified function of \( x, u \), and \( x \)-derivatives of \( u \), and where \( u \) is Cartan matrix of the \( U(n) \)-parallel frame along the curve \( \gamma \). The curve flow equation is reconstructed from these frame variables by the soldering relation

\[
\gamma_t = -K(e^*, \mathcal{Y}(h_\perp)),
\]  

(5.1)

where \( e^* \) is the linear frame dual to the linear coframe \( e \), \( K \) is the Cartan–Killing inner product, and \( \mathcal{Y} \) is the linear operator

\[
\mathcal{Y} := \text{id} - D^{-1}_e \text{ad}(u)_\parallel.
\]  

(5.2)

Here \( e^* \) is understood to be determined in terms of \( u \) by the dual Frenet equation (2.17) along \( \gamma \), up to the action of the equivalence group \( U(n) \parallel \) of the \( U(n) \)-parallel frame, under which \( e^*|\gamma_t = e \) is preserved.

We will now derive the curve flow equations corresponding to the mKdV system, the NLS system, and the SG system in each symmetric space \( M = SU(n+1)/U(n) \) and \( M = SO(2n)/U(n) \). These curve flows can be expressed geometrically in a \( G \)-invariant form using the tangent vector \( X = \gamma_t \) and the principal normal vector \( N = \nabla\gamma_t \) of the curve, plus the Riemannian metric tensor \( g_{\cdot, \cdot} \) and the Riemannian curvature tensor \( R_{\cdot, \cdot, \cdot, \cdot} \) on \( M = G/U(n) \). In general, a curve flow equation (5.1) in \( M = G/U(n) \) will be \( G \)-invariant if and only if \( h_\perp \) is an equivariant function of \( x, u \), and \( x \)-derivatives of \( u \) under the frame equivalence group \( U(n) \parallel \subset U(n) \).

To proceed, we will need to use a decomposition of the tangent space \( T_\gamma M \) adapted to the tangent vector \( \gamma_t \) of the curve in the following geometrical manner.

We start with the soldering identification \( T_\gamma M \simeq m \) provided by the \( U(n) \)-parallel frame \( e \) along \( \gamma \). Recall that the vector space \( m \) has an orthogonal decomposition \( m = m_\parallel \oplus m_\perp \) where \( m_\parallel \) is the centralizer subspace of \( e = e|\gamma_t \in m \) and \( m_\perp \) is the perp subspace. We can construct a projection operator onto \( m_\perp \) in \( m \) by using the linear map \( \text{ad}(e)^2 : m \rightarrow m \) whose
null space is \( m_{\|} \). When restricted to \( m_{\perp} \), this linear map has two eigenspaces, with eigenvalues \(-4\chi^{-1}\) and \(-\chi^{-1}\), where

\[
\chi = \begin{cases} 2\sqrt{n+1}, & G = SU(n+1) \\ 4\sqrt{n-1}, & G = SO(2n) \end{cases}
\]  
(5.3)

(see lemma B.1 in appendix B and lemma C.1 in appendix C). This determines

\[
P_{\perp} = \left(-\frac{1}{4}\chi^2\right)\left[(5\chi^{-1})\text{id} + \text{ad}(e)^2\right],
\]  
(5.4)

which satisfies \( P_{\perp} m_{\perp} = m_{\perp} \) and \( P_{\perp} m_{\|} = 0 \). A complementary projection operator onto \( m_{\|} \) in \( m \) is then given by

\[
P_{\|} = \text{id} - P_{\perp},
\]  
(5.5)

which satisfies \( P_{\|} m_{\|} = m_{\|} \) and \( P_{\|} m_{\perp} = 0 \).

Along the curve \( \gamma \), the linear map on \( T_{\gamma}M \) corresponding to \( \text{ad}(e)^2 \) on \( m \) is given by

\[
\text{ad}^2(\gamma) = -R(\cdot, \gamma_{\chi})\gamma_{\chi}
\]  
(5.6)

since \( e|\text{ad}^2(\gamma)Z = -e|R(Z, \gamma_{\chi})\gamma_{\chi} = [e|Z, e|\gamma_{\chi}, e|\gamma_{\chi}] = \text{ad}(e)^2(e|Z) \) holds for all \( Z \in T_{\gamma}M \) through the soldering relation (2.8). Then we have the projection operators

\[
P_{\perp} = \left(-\frac{1}{4}\chi^2\right)\left[(5\chi^{-1})\text{id} + \text{ad}(\gamma_{\chi})^2\right],
\]  
(5.7)

\[
P_{\|} = \text{id} - P_{\perp}
\]  
(5.8)

where \( \chi \) has a geometrical meaning coming from the eigenvalues of \( R(\cdot, \gamma_{\chi})\gamma_{\chi} \). Hence, we have established the following useful result.

**Lemma 1.** For any arclength-parameterized curve \( \gamma \) in \( M = G/U(n) \) for \( G = SU(n+1), SO(2n) \), the projection operators (5.7) and (5.8) yield a geometrical decomposition

\[
T_{\gamma}M = (T_{\gamma}M)_{\perp} \oplus (T_{\gamma}M)_{\|}
\]  
(5.9)

with

\[
P_{\perp} T_{\gamma}M = (T_{\gamma}M)_{\perp}, \quad P_{\|} T_{\gamma}M = (T_{\gamma}M)_{\|}, \quad g((T_{\gamma}M)_{\perp}, (T_{\gamma}M)_{\|}) = 0
\]  
(5.10)

which depends only the tangent vector \( \gamma_{\chi} \) of the curve and the Riemannian metric and curvature of the symmetric space \( G/U(n) \).

This decomposition (5.9) is not preserved by the Hermitian structure of \( M = G/U(n) \), which is given by \( J = \text{Ad}(U(1)_{\text{c}}) \) where \( U(1)_{\text{c}} \) is the center of \( U(n) \). However, the center of the frame equivalence group \( U(n)_{\|} \subset U(n) \) is a circle group \( U(1)_{\|} \) whose action \( \text{Ad}(U(1)_{\|}) \) on \( T_{\gamma}M \) commutes with \( \text{ad}(\gamma_{\chi})^2 \). This linear map \( \text{Ad}(U(1)_{\|}) \) defines an almost complex structure \( j_{\gamma} \) in \( T_{\gamma}M \). In the \( U(n) \)-parallel frame, it is given by

\[
e_j j_{\gamma} (Z) = -\text{ad}(j)(e|Z)
\]  
(5.11)

for all \( Z \in T_{\gamma}M \). On the eigenspaces of \( \text{ad}(\gamma_{\chi})^2 \) in \( (T_{\gamma}M)_{\perp} \), with eigenvalues \(-4\chi^{-1}\) and \(-\chi^{-1}\), the linear map \( j_{\gamma}^2 \) has eigenvalues \( 0, -1 \). In particular, on the \(-1\) eigenspace, \( j_{\gamma} \) coincides with \( J \). (On \( (T_G M)_{\|} \), which is the null space of \( \text{ad}(\gamma_{\chi})^2 \), \( j_{\gamma}^2 ((T_G M)_{\|}) = 0 \).)
We now apply lemma 1 to the curve flow equation (5.1), yielding
\[
\gamma_t = (\gamma_t)_{\perp} + (\gamma_t)_{||}, \quad (\gamma_t)_{||} = -K(e^*, h_{\perp}), \quad (\gamma_t)_{\perp} = -K(e^*, Y(h_{\perp})).
\]
(5.12)
which shows that the perp component \((\gamma_t)_{\perp}\) determines the entire flow vector \(\gamma_t\). This relationship is a consequence of the non-stretching property of the flow combined with the properties of a \(U(n)\)-parallel frame. Therefore, hereafter, we will only look at the perp component \((\gamma_t)_{\perp}\).

As a final preliminary step, we recall that
\[
e e|\gamma_\epsilon = e \in m_\perp, \quad e | \nabla_x \gamma_\epsilon = [u, e] = -\text{ad}(e)u \in m_\perp
\]
from the Frenet equation (2.10) and the Cartan matrix equation (2.14).

5.1. NLS curve flows

In both symmetric spaces, the NLS-type system (2.33) is given by \(\text{ad}(e)h_{\perp} = h_{\perp} = \text{ad}(j)u\), where \(\text{ad}(j)\) corresponds to the almost complex structure \(j_\gamma\) through the soldering identification (5.11). This determines
\[
h_{\perp} = \text{ad}(e)^{-1}\text{ad}(j)u = \text{ad}(e)^{-2}\text{ad}(j)\text{ad}(e)u
\]
(5.14)
since \(\text{ad}(e)\) and \(\text{ad}(j)\) commute. Hence we obtain
\[
ed | (\gamma_t)_{\perp} = e | (-\text{ad}(\gamma_t)^{-2}j_\gamma (\nabla_x \gamma_\epsilon)),
\]
(5.15)
which gives the NLS curve flow equation
\[
(\gamma_t)_{\perp} = -\text{ad}(\gamma_t)^{-2}j_\gamma (\nabla_x \gamma_\epsilon).
\]
(5.16)
This is a geometrical non-stretching curve flow in \(M = SU(n+1)/U(n)\) and \(M = SO(2n)/U(n)\). It is a generalization of the bi-normal flow equation \(\gamma_t = \kappa B\) in Euclidean space \(\mathbb{R}^3\), since if bi-normal vector is expressed as \(B = T \times N\) then the flow equation becomes \(\gamma_t = \gamma_t \times N\) where \(N = \kappa N = \gamma_{xx}\) is the principal normal vector and \(\gamma_t = T\) is the tangent vector. In this formulation of the bi-normal equation, we see that the linear map \(\gamma_t \times\) can be viewed as an almost complex structure since \((\gamma_t \times)^2 = -\text{id}\) by the standard vector cross product identity, and hence is a counterpart of \(j_\gamma\) in the case of Euclidean space.

5.2. mKdV curve flows

In both symmetric spaces, the mKdV-type system (2.31) is given by \(\text{ad}(e)h_{\perp} = h_{\perp} = u_\epsilon\). This determines
\[
h_{\perp} = \text{ad}(e)^{-1}u_\epsilon = \text{ad}(e)^{-2}\text{ad}(e)u_\epsilon.
\]
(5.17)
We can relate \(\text{ad}(e)u_\epsilon\) to the derivative of the principal normal vector \(N = \nabla_x \gamma_\epsilon\) through the soldering relation (2.6), which gives
\[
ed | \nabla_x N = \text{ad}(e)u_\epsilon + [u, \text{ad}(e)u].
\]
(5.18)
The commutator term \([u, \text{ad}(e)u]\) can be simplified by using the Lie brackets (2.20) and (2.21) in the symmetric spaces \(M = SU(n+1)/U(n)\) and \(M = SO(2n)/U(n)\) (see lemma B.1 in appendix B and lemma C.1 in appendix C). This yields
\[
[u, \text{ad}(e)u] = -K(\text{ad}(e)u, \text{ad}(e)u)e = g(N, N)e | \gamma_\epsilon
\]
(5.19)
for both symmetric spaces. Hence, we have
\[ \text{ad}(e)u_x = e](\nabla_s N - g(N, N)\gamma) \tag{5.20} \]

and consequently
\[ e][\gamma] = [\text{ad}(\gamma)]^{-2}(\nabla_s N - g(N, N)\gamma). \tag{5.21} \]

Thus we obtain the mKdV curve flow equation
\[ (\gamma) = \text{ad}(\gamma)^{-2}(\nabla_s \gamma - g(\gamma, \gamma)\gamma). \tag{5.22} \]

This is a geometrical non-stretching curve flow in \( M = SU(n+1)/U(n) \) and \( M = SO(2n)/U(n) \).

5.3. SG curve flows

The SG-type system (2.35) in both symmetric spaces is given by \( \varpi^\perp = 0 \), which implies \( \varpi^\parallel = 0 \) from the frame structure equation (2.24). Hence we have \( \varpi = 0 \), and so the connection matrix in the flow direction vanishes,
\[ \omega|\gamma = 0. \tag{5.23} \]

This can be expressed geometrically by observing
\[ 0 = \text{ad}(e)\omega|\gamma = [\omega|\gamma, e] = e]\nabla_{r}\gamma \tag{5.24} \]

through the soldering relation (2.6), since \( D_t e = 0 \). As a result, we obtain the SG curve flow equation
\[ \nabla_{r}\gamma = 0. \tag{5.25} \]

This can be recognized as being a non-stretching wave map equation in \( M = SU(n+1)/U(n) \) and \( M = SO(2n)/U(n) \).

In the two symmetric spaces, the SG system possesses the respective conservation laws (3.57) and (4.79). These conservation laws take the form \( D_t K(h,h) = 0 \). Using \( K(h, h) = -g(\gamma, \gamma) \), we obtain the corresponding conservation law
\[ \nabla_x|\gamma = 0 \tag{5.26} \]

for the SG curve flow. Thus, up to a conformal scaling of \( t \), the SG curve flow equation (5.25) describes a flow with unit speed, \( |\gamma| = 1 \).

6. Concluding remarks

The present paper completes a series of work in which the general theory developed in [18] for parallel frames, Hasimoto variables, and integrable systems of mKdV type as well as Sine–Gordon (SG) type arising from geometrical non-stretching curve flows has been applied to all of the simplest types of Riemannian symmetric spaces.

One new development that we have introduced here is an extension the theory to obtain integrable systems of NLS-type by exploiting a \( U(1) \) subgroup given by the center of the unitary equivalence group of the \( U(n) \)-parallel frame in the symmetric spaces \( M = SU(n+1)/U(n) \) and \( M = SO(2n)/U(n) \). In the case of \( M = SU(n+1)/U(n) \), this leads to a scalar-vector version of the Yajima–Oikawa system [30, 31], whereas in the case of \( M = SO(2n)/U(n) \), we obtain a novel nonlocal NLS system.

For future work, we plan to extend this development as far as possible to general Riemannian symmetric spaces.

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Appendix A. Notation

For vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{C}^n \), we denote the standard vector dot product as \( \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^t \mathbf{b} \), and the Hermitian inner product as \( \langle \mathbf{a}, \mathbf{b} \rangle = \text{Re} (\mathbf{a} \cdot \mathbf{b}) \), where ‘\( \cdot \)’ denotes the transpose. The tensor product of two vectors is given by \( \mathbf{a} \otimes \mathbf{b} = \mathbf{a}^t \mathbf{b} \), which is related to the dot product by \( \text{tr} (\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \), where ‘\( \text{tr} \)’ denotes the trace.

We define the following useful outer products:

\[
\begin{align*}
\mathbf{a} \wedge \mathbf{b} &= \mathbf{a}^t \mathbf{b} - \mathbf{b}^t \mathbf{a} \in \mathfrak{so}(n, \mathbb{C}), \\
\mathbf{a} \odot \mathbf{b} &= \mathbf{a}^t \mathbf{b} + \mathbf{b}^t \mathbf{a} \in \mathfrak{s}(n, \mathbb{C}), \\
\mathbf{a} \bar{\wedge} \mathbf{b} &= \mathbf{a}^t \mathbf{b} - \bar{\mathbf{b}}^t \mathbf{a} \in \mathfrak{u}(n), \\
\mathbf{a} \bar{\odot} \mathbf{b} &= \mathbf{a}^t \mathbf{b} + \bar{\mathbf{b}}^t \mathbf{a} \in \text{iu}(n),
\end{align*}
\]

where \( \text{iu}(n) \) denotes the vector space of \( n \times n \) hermitian matrices, and \( \mathfrak{s}(n) \) denotes the vector space of \( n \times n \) symmetric matrices. These outer products have the properties

\[
\begin{align*}
\mathbf{a} \wedge \mathbf{b} &= -\mathbf{b} \wedge \mathbf{a}, & (\mathbf{a} \wedge \mathbf{b})^t &= \mathbf{b} \wedge \mathbf{a}, \\
\mathbf{a} \odot \mathbf{b} &= \mathbf{b} \odot \mathbf{a}, & (\mathbf{a} \odot \mathbf{b})^t &= \mathbf{b} \odot \mathbf{a}, \\
\mathbf{a} \bar{\wedge} \mathbf{b} &= -\mathbf{b} \bar{\wedge} \mathbf{a}, & (\mathbf{a} \bar{\wedge} \mathbf{b})^t &= \mathbf{b} \bar{\wedge} \mathbf{a}, \\
\mathbf{a} \bar{\odot} \mathbf{b} &= \mathbf{b} \bar{\odot} \mathbf{a}, & (\mathbf{a} \bar{\odot} \mathbf{b})^t &= \mathbf{b} \bar{\odot} \mathbf{a}
\end{align*}
\]

and

\[
\begin{align*}
\text{tr} (\mathbf{a} \bar{\odot} \mathbf{b}) &= 2 \langle \mathbf{a}, \mathbf{b} \rangle, \\
i \text{tr} (\mathbf{a} \bar{\wedge} \mathbf{b}) &= 2 \text{Im} (\mathbf{b} \cdot \mathbf{a}) = -2 \text{Im} (\bar{\mathbf{a}} \cdot \mathbf{b}).
\end{align*}
\]

We likewise define the contraction between a vector and a matrix by

\[
\begin{align*}
c \downarrow (\mathbf{a} \odot \mathbf{b}) &= (\mathbf{a} \cdot \mathbf{c}) \mathbf{b}, & (\mathbf{a} \odot \mathbf{b}) \downarrow \mathbf{c} &= (\mathbf{b} \cdot \mathbf{c}) \mathbf{a},
\end{align*}
\]

and hence

\[
\begin{align*}
c \downarrow (\mathbf{a} \wedge \mathbf{b}) &= (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} - (\mathbf{c} \cdot \mathbf{b}) \mathbf{a}, & c \downarrow (\mathbf{a} \bar{\odot} \mathbf{b}) &= (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} + (\mathbf{c} \cdot \mathbf{b}) \mathbf{a}, \\
c \downarrow (\mathbf{a} \bar{\wedge} \mathbf{b}) &= (\mathbf{c} \cdot \bar{\mathbf{a}}) \mathbf{b} - (\mathbf{c} \cdot \bar{\mathbf{b}}) \mathbf{a}, & c \downarrow (\mathbf{a} \bar{\odot} \mathbf{b}) &= (\mathbf{c} \cdot \bar{\mathbf{a}}) \mathbf{b} + (\mathbf{c} \cdot \bar{\mathbf{b}}) \mathbf{a}.
\end{align*}
\]

We also define the full contraction between two matrices by

\[
\mathbf{A} \cdot \mathbf{B} = \text{tr} (\mathbf{A} \mathbf{B}).
\]

Finally, we extend the Hermitian inner product to matrices by defining
\[ |A|^2 = \bar{A} \cdot A = \text{tr} (\bar{A} A). \]  

(A.15)

Some useful identities:

\[ (a \otimes b) \cdot A = a \cdot (A|b) = b \cdot (a|A), \]  

(A.16)

\[ B|a = -a|B, \]  

(A.17)

\[ C|a = a|C, \]  

(A.18)

for \( A \in \mathfrak{gl}(n, \mathbb{C}), B \in \mathfrak{so}(n, \mathbb{C}), C \in \mathfrak{s}(n, \mathbb{C}). \)

Appendix B. Symmetric Lie algebra \( \mathfrak{su}(n+1)/\mathfrak{u}(n) \)

In the matrix representation (3.1)–(3.3) of the Lie algebra \( \mathfrak{su}(n+1) = \mathfrak{u}(n) \oplus \mathfrak{m} \), the Lie brackets are given by

\[ [u(n), m] = [(B_1), (a_2)] = (a_3) \in m, \quad a_3 = -\text{tr} (B_1) a_2 - a_2 B_1, \]  

(B.1)

\[ [m, m] = [(a_1), (a_3)] = (B_3) \in u(n), \quad B_3 = a_2 \wedge a_1, \]  

(B.2)

\[ [u(n), u(n)] = [(B_1), (B_2)] = ([B_1, B_2]) \in u(n). \]  

(B.3)

The vector space \( m = \mathfrak{su}(n+1)/\mathfrak{u}(n) \simeq \mathbb{C}^n \) has the following properties.

Proposition B.1.

1. The restriction of the Cartan–Killing form on \( \mathfrak{su}(n+1) \) to \( m \) yields a negative-definite inner product

\[ K((a), (b)) = 2(n+1) \text{tr} \begin{pmatrix} 0 & a \n \-\bar{a} & 0 \end{pmatrix} \begin{pmatrix} 0 & b \n -b^* & 0 \end{pmatrix} = -4(n+1)(a, b). \]  

(B.4)

2. The rank of \( m \) is 1, and there is a Hermitian structure \( J := -\frac{1}{n+1} (i I_n) \in \mathfrak{u}(n) \) which acts by \( \text{ad}_J(a) = [J, (a)] = (ia) \).

3. Up to isomorphism, the Cartan subspace \( \mathfrak{a} \subset \mathfrak{m} \) is given by the real span of any vector \( e \in \mathfrak{m} \) such that \( -K(e, e) = 1 \), with \( a = \text{span}(e) \). This vector can be chosen as

\[ \chi e = (\hat{e}), \quad \hat{e} = (1, 0) \in \mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1} \simeq m, \quad 0 \in \mathbb{C}^{n-1}, \]  

(B.5)

where \( \chi^2 = -K((\hat{e}), (\hat{e})) = 4(n+1)(\hat{e}, \hat{e}) = 4(n+1) \) determines the normalization constant

\[ \chi = 2\sqrt{n+1}. \]  

(B.6)

4. The Cartan subspace is not invariant under the Hermitian structure, since \( \text{ad}_J(e) = ie \notin \text{span}(e) \). 

The Cartan element \( \hat{e} \) produces a decomposition of \( m = m_\parallel \oplus m_\perp \) and \( \mathfrak{u}(n) = \mathfrak{u}(n)_\parallel \oplus \mathfrak{u}(n)_\perp \) into respective centralizer subspaces and perp subspaces defined by the matrix representations (3.4) and (3.5). These subspaces have the following main properties.

Lemma B.1.

1. \( \mathfrak{u}(n)_\parallel \simeq \mathfrak{u}(n-1) \) is the centralizer subalgebra of \( e \), and \( \mathfrak{u}(n)_\perp \simeq i \mathbb{R} \oplus \mathbb{C}^{n-1} \) is its perp space; \( m_\parallel \simeq \mathbb{R} \) is the centralizer subspace of \( e \), and \( m_\perp \simeq i \mathbb{R} \oplus \mathbb{C}^{n-1} \) is its perp space.
2. The Lie bracket $[m, m]$ has the decomposition
\[ ([a_\parallel], (c_\parallel)] = 0 \in h_\parallel, \]  \tag{B.7} \]
\[ ([a_\parallel], ((ia_\perp, a_\perp)] = ((-2ia_\parallel a_\perp, -a_\parallel a_\perp)) \in h_\perp, \]  \tag{B.8} \]
\[ [((ia_\perp, a_\perp)], (ic_\perp, c_\perp))] = (c_\perp \mathbf{A} a_\perp) \in h_\parallel, \]  \tag{B.9} \]
\[ [((ia_\perp, a_\perp)], ((ic_\perp, c_\perp))]_\perp = ((\frac{1}{2} \text{tr} (c_\perp \mathbf{A} a_\perp), ia_\perp c_\perp - ic_\perp a_\perp)) \in h_\perp. \]  \tag{B.10} \]

3. The Lie bracket $[m, u(n)]$ has the decomposition
\[ [a_\parallel], (B_\parallel))] = 0 \in h_\parallel, \]  \tag{B.11} \]
\[ ([a_\parallel], ((ib_\parallel, b_\parallel))] = ((2ia_\parallel b_\parallel, a_\parallel b_\parallel)) \in m_\perp, \]  \tag{B.12} \]
\[ [((ia_\perp, a_\perp)], (B_\parallel))] = ((0, \frac{1}{2} \text{tr} (B_\parallel b_\parallel + a_\parallel B_\parallel))) \in m_\perp, \]  \tag{B.13} \]
\[ [ia_\perp, a_\perp)], (ib_\perp, b_\perp))]_\parallel = ((-2a_\parallel b_\perp - (a_\parallel, b_\perp))) \in m_\parallel, \]  \tag{B.14} \]
\[ [((ia_\perp, a_\perp)], (ib_\perp, b_\perp))]_\perp = ((-\frac{1}{2} \text{tr} (b_\perp \mathbf{A} a_\perp), ib_\perp a_\perp + ia_\perp b_\perp)) \in m_\perp. \]  \tag{B.15} \]

4. The Lie bracket $[u(n), u(n)]$ has the decomposition
\[ ([B_\parallel], (D_\parallel)] = ([B_\parallel], D_\parallel)) \in h_\parallel, \]  \tag{B.16} \]
\[ ([B_\parallel], ((ib_\perp, b_\perp))] = ((0, -\frac{1}{2} \text{tr} (B_\parallel b_\perp - b_\perp B_\parallel))) \in h_\perp, \]  \tag{B.17} \]
\[ [(ib_\perp, b_\perp)], (id_\perp, d_\perp))]_\parallel = (d_\perp \mathbf{A} b_\perp) \in h_\parallel, \]  \tag{B.18} \]
\[ [(ib_\perp, b_\perp)], (id_\perp, d_\perp))]_\perp = ((-\frac{1}{2} \text{tr} (d_\perp \mathbf{A} b_\perp), ib_\perp d_\perp - id_\perp b_\perp)) \in h_\perp. \]  \tag{B.19} \]

5. The center of the centralizer subalgebra $u(n)_\parallel \simeq u(n - 1)$ of $e$ is a $u(1)$ subalgebra generated by $j := \frac{2}{2n-1}(i\eta - 1) \in u(n)_\parallel$ which acts on $m_\perp$ and $m_\parallel$ by
\[ \text{ad}(j)(ia_\perp a_\perp)) = ((0, ia_\perp)), \quad \text{ad}(j)(a_\parallel) = 0. \]  \tag{B.20} \]

6. There are vector-space isomorphisms
\[ m_\perp \leftrightarrow u(n)_\perp : \text{ad}(e)((ia_\perp a_\perp)) = \frac{-1}{\sqrt{\chi}}((2ia_\perp a_\perp)), \]  \tag{B.21} \]
\[ u(n)_\perp \leftrightarrow m_\perp : \text{ad}(e)((ib_\perp b_\perp)) = \frac{1}{\sqrt{\chi}}((2ib_\perp b_\perp)), \]  \tag{B.22} \]
\[ m_\perp \leftrightarrow m_\perp : \text{ad}(e)^2((ia_\perp a_\perp)) = -\chi^{-1}(4ia_\perp a_\perp)), \]  \tag{B.23} \]
7. Under the Hermitian structure, $(\text{ad}_1((ia_{\perp},a_{\perp})))_{\perp} = ((0,ia_{\perp}))$ and $(\text{ad}_2((ia_{\perp},a_{\perp})))_{\parallel} = ((-a_{\perp},0))$.

The Lie subalgebra $u(n)_{||}$ generates a group $U(n)_{||} = U(n-1) \subset U(n)$ that leaves $e$ invariant in $m$, $\text{Ad}(U(n)_{||})e = e$.

**Lemma B.2.** The invariance group $U(n)_{||} = U(n-1) \subset U(n)$ of $e$ is given by the matrix representation

$$X = \begin{pmatrix} \det(X_{n-1})^{-1/2} & (0,0) \\ (0,0)^t & \begin{pmatrix} \det(X_{n-1})^{-1/2} & 0 \\ 0 & X_{n-1} \end{pmatrix} \end{pmatrix} \in SU(n+1), \quad X_{n-1} \in U(n-1).$$

(B.25)

This group acts on $m_{\perp}$ by

$$\text{Ad}(X)((ia_{\perp},a_{\perp})) = ((\det(X_{n-1})^{-1/2}a_{\perp}X_{n-1}^{-1})_{\parallel} \in m_{\perp}$$

and it leaves invariant $(a_{||}) \in m_{||}$.

The center of $U(n)_{||}$ is a $U(1)$ subgroup given by the matrix representation

$$Z = \exp(\phi \text{ad}_1) = \begin{pmatrix} e^{-i\phi/2} & (0,0) \\ (0,0)^t & \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi}I_{n-1} \end{pmatrix} \end{pmatrix} \in SU(n+1), \quad \phi \in \mathbb{R}.$$

(B.27)

**Appendix C. Symmetric Lie algebra $\mathfrak{so}(2n)/u(n)$**

In the matrix representation (4.1)–(4.3) of the Lie algebra $\mathfrak{so}(2n) = u(n) \oplus m$, the Lie brackets are given by

$$[u(n),m] = [(B_1),(A_2)] = (A_3) \in m, \quad A_3 = B_1A_2 - A_2B_1,$$

(C.1)

$$[m,m] = [(A_1),(A_2)] = (B_3) \in u(n), \quad B = \tilde{A}_1A_2 - \tilde{A}_2A_1,$$

(C.2)

$$[u(n),u(n)] = [(B_1),(B_2)] = ([B_1,B_2]) \in u(n).$$

(C.3)

The vector space $m = \mathfrak{so}(2n)/u(n) \simeq \mathbb{C}^n(n-1)$ has the following properties.

**Proposition C.1.**

1. The restriction of the Cartan–Killing form on $\mathfrak{so}(2n)$ to $m$ yields a negative-definite inner product

$$K((A_1),(A_2)) = -4(n-1) \text{ Re } (\text{ tr } (A_1\tilde{A}_2)).$$

(C.4)

2. The rank of $m$ is $[n]/2$ (integer part), and there is a Hermitian structure $J := -\frac{i}{2}(iI_n) \in u(n)$ which acts by $\text{ad}_J(A) = (iA)$.

3. Up to isomorphism, the Cartan subspace $a \subset m$ is given by the real span of the $[n]/2$ matrices

$$e_k := (E_{2k-1,2k} - E_{2k,2k-1}) \in m, \quad k = 1, \ldots, [n]/2,$$

(C.5)
where \( E_{ij} \in \mathfrak{gl}(n, \mathbb{C}) \) denotes the matrix such that its \((i,j)\) entry is 1 and all other entries are 0.

4. The Cartan subspace is not invariant under \( J \), due to \( \mathfrak{ad}(e_k) = ie_k \not\in \text{span}(e_1, \ldots, e_{[n/2]}) \mathbb{R} \).

The basis matrices \( e_\mathfrak{k} \) have the distinguishing property that the centralizer subspace of each one is of maximal dimension. We will now select a unit-norm element \( e \) in the Cartan subspace by choosing any one of the basis matrices

\[
\chi e = e_1 \in \mathfrak{a},
\]

where

\[
-\chi^2 = K(e_1, e_1) = 8(n - 1) \text{ Re } \text{tr} \left( (E_{12} - E_{21})^2 \right) = -16(n - 1)
\]

(determines the normalization constant \( \chi \))

\[
\chi = 4\sqrt{n - 1}.
\]

This choice of a Cartan element produces a decomposition of \( m = m_\parallel \oplus m_\perp \) and \( u(n) = u(n)_\parallel \oplus u(n)_\perp \) into respective centralizer subspaces and perp subspaces defined by the matrix representations \((4.4)\) and \((4.5)\). These subspaces have the following main properties.

**Lemma C.1.**

1. \( u(n)_\parallel \simeq \mathfrak{su}(2) \oplus u(n - 2) \) is the centralizer subalgebra of \( e \), and \( u(n)_\perp \simeq i\mathbb{R} \oplus C^{n-2} \oplus C^{n-2} \) is its perp space; \( m_\parallel \simeq \mathbb{R} \oplus \mathfrak{so}(n - 2, \mathbb{C}) \) is the centralizer subalgebra of \( e \), and \( m_\perp \simeq i\mathbb{R} \oplus C^{n-2} \oplus C^{n-2} \) is its perp space.

2. The Lie bracket \([m, m]\) has the decomposition

\[
[[\{(a_\parallel, A_\parallel)\}, \{c_\parallel, C_\parallel\}] = \{(0, 0, A_\parallel C_\parallel - C_\parallel A_\parallel)\} \in u(n)_\parallel,
\]

\[
[[\{(a_\parallel, A_\parallel)\}, \{(iA_\parallel, a_\perp, a_\parallel_\perp)\}] = \{(-2i\mathbb{A}_\parallel a_\perp, a_\parallel a_\perp) = \bar{a}_\perp A_\parallel, a_\parallel a_\perp = \bar{a}_\perp A_\parallel\} \in u(n)_\perp,
\]

\[
[[\{(iA_\parallel, a_\perp, a_\parallel_\perp)\}, \{(i\mathbb{C}_\parallel, c_\perp, c_\parallel_\perp)\}] = \{(-i\mathbb{A}_\parallel a_\perp, c_\perp a_\perp = a_\perp, c_\perp a_\perp, a_\perp = a_\perp) \in u(n)_\parallel,
\]

\[
[[\{(i\mathbb{C}_\parallel, c_\perp, c_\parallel_\perp)\}, \{(c_\perp, c_\parallel_\perp)\}] \in u(n)_\perp.
\]

3. The Lie bracket of \([m, u(n)]\) has the decomposition

\[
[[\{a_\parallel, A_\parallel\}, \{(iA_\parallel, b_\parallel, b_\parallel_\perp)\}] = \{(0, A_\parallel B_\parallel - B_\parallel A_\parallel)\} \in m_\parallel,
\]

\[
[[\{a_\parallel, A_\parallel\}, \{(ib_\parallel, b_\parallel, b_\parallel_\perp)\}] = \{(2iA_\parallel b_\parallel, a_\parallel b_\parallel_\perp - \bar{b}_\parallel A_\parallel, a_\parallel b_\parallel_\perp = \bar{b}_\parallel A_\parallel\} \in m_\perp,
\]

\[
[[\{(ib_\parallel, b_\parallel_\perp, b_\parallel_\perp)\}, \{(\mathbb{C}_\parallel, c_\perp, c_\parallel_\perp)\}] = \{(0, a_\parallel b_\parallel_\perp + ib_\parallel a_\parallel + b_\parallel a_\parallel_\perp) \in m_\perp.
\]

(32)
\[
\left[\left(ia_{1\perp}, a_{1\perp}, a_{2\perp}\right), \left(ib_{1\perp}, b_{1\perp}, b_{2\perp}\right)\right]_{\perp} = \left(-2a_{\perp}b_{\perp} + \frac{1}{2}\left(\text{tr} (b_{1\perp} \overset{\perp}{a}_{2\perp}) - \text{tr} (b_{2\perp} \overset{\perp}{a}_{1\perp})\right), b_{1\perp} \wedge a_{\perp} + b_{2\perp} \wedge a_{\perp}\right) \right) \in m_{\perp},
\]
(C.16)

\[
\left[\left(ia_{1\perp}, a_{1\perp}, a_{2\perp}\right), \left(ib_{1\perp}, b_{1\perp}, b_{2\perp}\right)\right]_{\perp} = \left(\frac{1}{2}\left(\text{tr} (b_{1\perp} \wedge a_{2\perp}) - \text{tr} (b_{2\perp} \wedge a_{1\perp})\right), i(a_{\perp}b_{2\perp} + b_{a_{\perp}}), i(-a_{\perp}b_{1\perp} + b_{a_{\perp}})\right) \right) \in m_{\perp}.
\]
(C.17)

4. The Lie bracket \([u(n), u(n)]\) has the decomposition

\[
\left[\left(\text{id}_{1\perp}, b_{2\perp}, B_{\perp}\right), \left(\text{id}_{2\perp}, d_{2\perp}, D_{\perp}\right)\right] = \left((d_{2\perp}b_{2\perp} - b_{2\perp}d_{2\perp}, 2i(b_{1\perp}d_{2\perp} - d_{1\perp}b_{2\perp}), B_{\perp}D_{\perp} - D_{\perp}B_{\perp})\right) \in u(n)_{\perp},
\]
(C.18)

5. The center of the centralizer subalgebra \(u(n)_{\perp} \simeq su(2) \oplus u(n - 2)\) of \(e\) is a \(u(1)\) subalgebra generated by \(j := -(i\theta_{n-2}) \in u(n)_{\perp}\) which acts on \(m_{\perp}\) by

\[
\text{ad}(j)(\left(ia_{1\perp}, a_{1\perp}, a_{2\perp}\right)) = \left(0, ia_{1\perp}, ia_{2\perp}\right).
\]
(C.22)

6. There are isomorphisms of vector spaces

\[
m_{\perp} \rightarrow u(n)_{\perp}: \text{ad}(e)(\left(ia_{1\perp}, a_{1\perp}, a_{2\perp}\right)) = \frac{1}{\sqrt{\lambda}}((-2ia_{1\perp}, a_{2\perp}, -a_{1\perp}),
\]
(C.23)

\[
u(n)_{\perp} \rightarrow m_{\perp}: \text{ad}(e)(\left(ib_{1\perp}, b_{1\perp}, b_{2\perp}\right)) = \frac{1}{\sqrt{\lambda}}((2ib_{1\perp}, b_{2\perp}, -b_{1\perp}),
\]
(C.24)

7. Under the Hermitian structure, \((\text{ad}_{j}(\left(ia_{1\perp}, a_{1\perp}, a_{2\perp}\right)))_{\perp} = \left(0, ia_{1\perp}, ia_{2\perp}\right)\) and \((\text{ad}_{j}(\left(ia_{1\perp}, a_{1\perp}, a_{2\perp}\right)))_{\perp} = \left(-ia_{1\perp}, 0\right)\).

The Lie subalgebra \(u(n)_{\perp}\) generates a group \(U(n)_{\perp} = SU(2) \times U(n - 2) \subset U(n)\) that leaves \(e\) invariant in \(m, \text{Ad}(U(n)_{\perp})e = e\).

**Lemma C.2.** The subgroups \(SU(2) \subset U(n)_{\perp}\) and \(U(n - 2) \subset U(n)_{\perp}\) in the invariance group \(U(n)_{\perp}\) of \(e\) are given by the respective matrix representations

\[
X_{SU(2)} = \begin{pmatrix}
\cos \theta & e^{i\phi} \sin \theta & 0 \\
-e^{i\phi} \sin \theta & \cos \theta & 0 \\
0 & 0 & 0
\end{pmatrix} \simeq SU(2) \in U(n), \quad \phi, \theta \in \mathbb{R},
\]
(C.25)
\[ X_{n-2}^{(n-2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0^t & 0^t & X_{n-2} \end{pmatrix} \approx U(n-2) \in U(n), \quad X_{n-2} \in U(n-2). \quad (C.26) \]

These subgroups act on \( m_\perp \) by
\[
\text{Ad}(X_{SU(2)}((i\alpha_\perp, 0, 0))) = ((i\alpha_\perp, 0, 0)) + (X_{SU(2)})((0, \alpha_1, \alpha_2)) \in m_\perp, \quad (C.27)
\]

\[
\text{Ad}(X_{U(n-2)}((i\alpha_\perp, \alpha_1, \alpha_2))) = ((i\alpha_\perp, \alpha_1, \alpha_2 X_{n-2}^{-1}, \alpha_2 X_{n-2}^{-1})) \in m_\perp, \quad (C.28)
\]

and they leave invariant \((\alpha_\|, 0) \in m_\parallel\). Composition of these two subgroups yields the group \( U(n)_\| = SU(2) \times U(n-2) \subset U(n) \).

The center of \( U(n)_\| \) is a \( U(1) \) subgroup given by the matrix representation
\[
Z = \exp(\phi \text{ad}_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0^t & 0^t & e^{-i\phi}I_{n-2} \end{pmatrix} \in U(n), \quad \phi \in \mathbb{R}. \quad (C.29)
\]

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