FINITE DIFFERENCE SCHEME FOR TWO-DIMENSIONAL PERIODIC NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. A nonlinear Schrödinger equation (NLS) on a periodic box can be discretized as a discrete nonlinear Schrödinger equation (DNLS) on a periodic cubic lattice, which is a system of finitely many ordinary differential equations. We show that in two spatial dimensions, solutions to the DNLS converge strongly in $L^2$ to those of the NLS as the grid size $h > 0$ approaches zero. As a result, the effectiveness of the finite difference method (FDM) is justified for the two-dimensional periodic NLS.

1. Introduction

We consider the nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u - \lambda |u|^{p-1}u = 0$$

(1.1)

on the periodic box $\mathbb{T}^d = \mathbb{R}^d / 2\pi \mathbb{Z}^d$, where $p > 1$, $\lambda = \pm 1$, and

$$u = u(t, x) : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}.$$ 

The NLS is a canonical model that describes the propagation of nonlinear waves. When the nonlinearity is either cubic or quintic, or a combination of these two types, the equation (1.1) arises in various physical contexts including nonlinear optics and low-temperature physics. In particular, if a huge number of boson particles are trapped in a box with the periodic boundary condition and they are cooled to a temperature approaching absolute zero, they form a Bose–Einstein condensate and their mean-field dynamics is determined by the periodic NLS. We refer to [19, 8, 21, 7] for a rigorous proof for this.

The periodic NLS (1.1) may be studied numerically by employing the following standard semi-discrete finite difference method (FDM). For a mesh size $h = \frac{\pi}{M} > 0$ with a large integer $M > 0$, we denote the dense periodic lattice by

$$\mathbb{T}_h^d := h\mathbb{Z}^d / 2\pi \mathbb{Z}^d,$$

(1.2)

that is, the additive group

$$\left\{ x = h(m_1, \ldots, m_d) : m_j = -M, \ldots, -2, -1, 0, 1, 2, \ldots, M - 1 \right\}$$

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of \((2M)^d\) points (see Figure 1.1), and define the discrete Laplacian \(\Delta_h\) by
\[
(\Delta_h u_h)(x) := \sum_{j=1}^{d} \frac{u_h(x + he_j) + u_h(x - he_j) - 2u_h(x)}{h^2}, \quad \forall x \in \mathbb{T}_h^d,
\]
(1.3)
which acts on complex-valued functions on the periodic lattice. Then, we formulate the discrete nonlinear Schrödinger equation (DNLS) as
\[
i\partial_t u_h + \Delta_h u_h - \lambda |u_h|^{p-1} u_h = 0,
\]
(1.4)
where \(p > 1\), \(\lambda = \pm 1\) and 
\[
u_h = u_h(t, x) : \mathbb{R} \times \mathbb{T}_h^d \rightarrow \mathbb{C}.
\]
In this way, the partial differential equation is translated into the system of \((2M)^d\)-many first-order ordinary differential equations.

The main purpose of the work presented in this article is to justify the effectiveness of the above numerical scheme. We introduce the following operators to formulate the problem precisely.

**Definition 1.1** (Discretization and linear interpolation). (i) For a function \(f : \mathbb{T}^d \rightarrow \mathbb{C}\), its discretization is defined by
\[
(d_h f)(x) := \frac{1}{h^d} \int_{x+[0,h)^d} f(y)dy, \quad \forall x \in \mathbb{T}_h^d.
\]
(1.5)
(ii) Given a function \(f_h : \mathbb{T}_h^d \rightarrow \mathbb{C}\), its linear interpolation is defined by
\[
(p_h f_h)(x) := f_h(x) + \sum_{j=1}^{d} D_{h,j}^+ (x)(x_j - x_j)
\]
(1.6)

**Figure 1.1.** Representation of the two-dimensional periodic lattice \(\mathbb{T}_h^2\). All points marked by \(\bullet\) are contained in \(\mathbb{T}_h^2\), whereas \((4M + 1)\)-points marked by \(\circ\) are excluded.
for \( x \in \mathbb{R} + (0,h)^d \) with \( x \in \mathbb{T}_h^d \), where \( e_j \) is the \( j \)-th standard unit vector, \( x_j \) denotes the \( j \)-th component of \( x \in \mathbb{T}^d \), and \( D^+_{h,j} \) is the discrete right-hand side derivative on \( \mathbb{T}_h^d \), i.e.,

\[
D^+_{h,j}f_h(x) := \frac{f_h(x + he_j) - f_h(x)}{h}.
\]

**Definition 1.2 (Nonlinear propagators).** We denote the nonlinear propagator for NLS (1.1) by \( U(t) \). In other words, \( U(t)u_0 \) is the solution to the NLS (1.1) with initial data \( u_0 \). Similarly, we denote the nonlinear propagator for DNLS (1.4) by \( U_h(t) \).

**Remark 1.3.** (i) The discretization operator \( d_h \) sends functions on the periodic box to functions on a periodic lattice. Conversely, the linear interpolation operator \( p_h \) maps discrete functions to continuous functions.

(ii) The nonlinear propagators \( U(t) \) and \( U_h(t) \) are well defined because the equations are locally well posed under suitable assumptions (see Propositions B.1 and 4.1).

Our goal is then to show that

\[
(p_h \circ U_h(t) \circ d_h)u_0 - U(t)u_0 \to 0
\]

as \( h \to 0 \) in a proper sense (see Figure 1.2). The convergence (1.8) is referred to as the **continuum limit** for DNLS. Obviously, proving the continuum limit implies the effectiveness of the numerical scheme.

Despite its physical importance, to the best of the authors’ knowledge, the continuum limit for a nonlinear dispersive equation on a compact manifold has not previously been studied. However, the continuum limit from DNLS on \( h\mathbb{Z}^d \) to NLS on \( \mathbb{R}^d \) has been investigated by Ignat–Zuazua [13, 14, 15, 16], Kirkpatrick–Lenzmann–Staffilani [18], and the first and fourth authors of this work [10, 11]. An important remark is that the linear discrete

\[
\begin{align*}
&\text{Figure 1.2. Schematic representation of the } \textit{continuum limit} \text{ for DNLS.} \\
&\text{The nonlinear propagator } U(t) \text{ maps initial data } u_0 \text{ to the solution } u(t) \text{ to NLS (1.1). On the other hand, discretizing initial data } u_0 \text{ to } u_{h,0} \text{ and then } \\
&\text{acting the nonlinear propagator } U_h(t) \text{ on } u_{h,0} \text{ enables generating a solution } u_h(t) \text{ to DNLS (1.4). Theorem 1.6 asserts that its linear interpolation } \\
&(p_h u_h)(t) \text{ converges to } u(t) \text{ as } h \to 0. \\
&\begin{array}{ccc}
\text{u_0} & \xrightarrow{d_h} & u_{h,0} \\
\downarrow U(t) & & \downarrow U_h(t) \\
\text{u(t)} & \xleftarrow{p_h \text{ with } h \to 0} & u_h(t)
\end{array}
\end{align*}
\]
model
\[ i \partial_t u_h + \Delta_h u_h = 0, \tag{1.9} \]
where \( u_h = u_h(t, x) : \mathbb{R} \times h\mathbb{Z}^d \to \mathbb{C} \), enjoys weaker dispersion than the continuous model \[13, 22\], and this causes difficulties in proving the continuum limit. In \[14, 16\], the authors circumvented the weak dispersion phenomena by introducing a new numerical scheme, that is, the two-grid algorithm, to exclude bad frequencies generating weak dispersions. Subsequently, in \[10, 11\], the authors discovered that the space–time norm bounds, namely Strichartz estimates, for (1.9) hold uniformly in \( h \in (0, 1] \) with some derivative on the right-hand side. As an application, convergence of the discrete NLS on \( h\mathbb{Z}^d \) is established without modifying the numerical scheme.

Returning to the problem discussed here, one would attempt to adopt the approach in \[10, 11\] to the periodic setting. However, several new issues are raised, in particular, for the desired uniform-in-\( h \) Strichartz estimates.

**Remark 1.4.** In the celebrated work of Bourgain \[3\], Strichartz estimates are established for the linear Schrödinger equation \( i \partial_t u + \Delta u = 0 \) on a periodic box, and they are applied to prove the local well-posedness of the periodic NLS \[11\] in a low regularity Sobolev space. Importantly, these Strichartz estimates can be captured from the gain of regularity in the multi-interaction of linear solutions localized in same frequencies but different modulations. This phenomenon is known as the dispersive smoothing effect, and its proof requires an understanding of the geometry of the support of the spacetime Fourier transform of linear solutions, that is, the hypersurface \( \{(\tau, k) \in \mathbb{R} \times \mathbb{Z}^d : \tau + |k|^2 = 0\} \). We also refer to the work of Guo, Oh and Wang \[9\] for a further context of NLS on irrational torus.

Unfortunately, we are currently unable to capture dispersive smoothing in the discrete setting. Indeed, the hypersurface for the linear equation (1.9) is given by \( \{(\tau, k) \in \mathbb{R} \times \mathbb{Z}^d : \tau + \sum_{j=1}^d \frac{2}{h^2}(1 - \cos hk_j) = 0\} \). Following Bourgain’s approach, it is necessary to count the maximal number of points in the intersection of twisted annuli \( \{\tilde{k} \in \mathbb{Z}^d : M \leq |\tilde{\tau} + \sum_{j=1}^d \frac{2}{h^2}(1 - \cos h\tilde{k}_j)| \leq 2M\} \) and \( \{\tilde{k}' \in \mathbb{Z}^d : N \leq |\tilde{\tau}' + \sum_{j=1}^d \frac{2}{h^2}(1 - \cos h\tilde{k}'_j)| \leq 2N\} \) restricted to the hyperplane \( \tilde{k} + \tilde{k}' = k \) with \( \tilde{\tau} + \tilde{\tau}' = \tau \). Compared to the continuous case, the situation is much more complicated because of the complexity of the geometry. Moreover, because local smoothing is known to fail on the noncompact lattice \( h\mathbb{Z}^d \) \[13\], this may not simply be a matter of technicality but may indicate that a new idea is needed. It is also worth to mention that Strichartz estimates on \( \mathbb{T}^d \) for higher dimension were established by Bourgain and Demeter \[5\] as a corollary of their main theorem on the decoupling inequality (Wolff’s inequality). It may be one of possible ways to follow the decoupling approach to our problem. We leave this question for future study.

One way to circumvent the aforementioned difficulties would be to approximate the linear propagator on a periodic box by that on an entire space. Ultimately it would seem that, by suitably adjusting the argument of Vega \[25\] to the discrete setting, the time-localized
uniform-in-$h$ Strichartz estimates can be obtained on a periodic lattice. For the statement, we define the finite-dimensional vector space $L^r_h = L^r_h(\mathbb{T}^d_h)$ equipped with the norm

$$
\|f\|_{L^r_h} := \begin{cases} 
\left\{ h^d \sum_{x \in \mathbb{T}^d_h} \left| f(x) \right|^r \right\}^{1/r} & \text{if } 1 \leq r < \infty, \\
\sup_{x \in \mathbb{T}^d_h} |f(x)| & \text{if } r = \infty,
\end{cases}
$$

and define the fractional derivative $\langle \nabla_h \rangle^s$ as the Fourier multiplier of symbol $\langle k \rangle^s$ via the discrete Fourier transform, where $\langle k \rangle = \sqrt{1 + |k|^2}$ (see Section 2). We say that $(q, r)$ is lattice-admissible if $2 \leq q, r \leq \infty$, $3 \frac{q}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q, r, d) \neq (2, \infty, 3)$.

**Theorem 1.5** (Strichartz estimates on a periodic lattice). Let $h \in (0, 1]$. For a lattice-admissible pair $(q, r)$, there exists $C > 0$, independent of $h$, such that

$$
\|e^{it\Delta_h} u_{h,0}\|_{L^q_t([0,1];L^r_h)} \leq C\|(\nabla_h)^\frac{2}{q} + \epsilon u_{h,0}\|_{L^2_h}
$$

(1.12)

for any $\epsilon > 0$.

Strichartz estimates are one of the fundamental tools to study dispersive equations because they quantify the smoothing and/or decay properties of solutions. On the unbounded lattice $\mathbb{Z}^d$, the smoothing and decay properties have been investigated for various models: we refer to [22, 13, 12] for the Schrödinger equation, [20] for the wave equation, and [1] for the Klein–Gordon equation. Theorem 1.5 is the first result on a compact discrete domain as far as the authors know. It should be noted that the inequality (1.12) holds uniformly in $h \in (0, 1]$. Indeed, it is easy to show the inequality $\|e^{it\Delta_h} u_{h,0}\|_{L^q_t([0,1];L^r_h)} \leq C_h\|u_{h,0}\|_{L^2_h}$ for all $1 \leq q, r \leq \infty$, since $\mathbb{T}^d_h$ is finite-dimensional. However, this inequality is not useful at all for our purpose because the constant $C_h$ blows up as $h \to 0$. As in [10], for which uniform Strichartz estimates are proven on $h\mathbb{Z}^d$, we could obtain an appropriate (uniform-in-$h$) Strichartz estimates by placing some derivatives on the norm on the right-hand side (Theorem 1.5). We also note that we do not claim optimality of the Strichartz estimates (1.12). In fact, the order of the derivative could be reduced by solving the counting problem mentioned in Remark 1.4.

Although there is still room for improvement, Theorem 1.5 is sufficient to establish the global-in-time continuum limit for the two-dimensional periodic NLS, which is the main result of this work.

**Theorem 1.6** (Continuum limits). Let $d = 2$. We assume

$$
\begin{cases} 
1 < p < \infty & \text{when } \lambda = 1 \quad \text{(defocusing)}, \\
1 < p < 3 & \text{when } \lambda = -1 \quad \text{(focusing)},
\end{cases}
$$

(1.13)
There exist constants $A, B > 0$, independent of $h \in (0, 1]$, such that for all $t \in \mathbb{R}$,
\[
\|(p_h \circ U_h(t) \circ d_h) u_0 - U(t)u_0\|_{L^2(T^2)} \leq A\sqrt{he^{B|t|}} \left(1 + \|u_0\|_{H^1(T^2)}\right)^p.
\]

The proof of Theorem 1.6 follows the argument outlined in [11]. Precisely, we consider two solutions in Duhamel’s formulas,
\[
u_h(t) = e^{-it(-\Delta_h)}(d_h u_0)_h - i\lambda \int_0^t e^{-i(t-s)(-\Delta_h)}(|u_h|^{p-1}u_h)(s)ds
\]
and
\[
u(t) = e^{-it(-\Delta)}u_0 - i\lambda \int_0^t e^{-i(t-s)(-\Delta)}(|u|^{p-1}u)(s)ds,
\]
where $u_h(t) = U_h(t)(d_h u_0)$ and $u(t) = U(t)u_0$. We aim to estimate the difference $p_hu_h(t) - u(t)$ directly by the standard Gr"onwall’s inequality. We accomplish this by making use of a “time-averaged” uniform-in-$h$ $L^\infty_h$-bound for nonlinear solutions $\{u_h(t)\}_{h \in (0,1]}$. Such a uniform bound can be obtained by applying uniform-in-$h$ Strichartz estimates for the discrete linear equation to the nonlinear problem.

Remark 1.7. (i) The essential part of our analysis lies in proving the uniform Strichartz estimates for the linear equation. For this proof, we employ the Fourier analysis on a periodic lattice, and we develop harmonic analysis tools on the lattice, including the Littlewood–Paley theory. Indeed, a periodic lattice is a finite abelian group; thus, the Fourier and its inverse transforms are properly defined (see Section 2.2).
(ii) As mentioned in Remark 1.4, if the classical Bourgain’s argument is adopted, the proof of the Strichartz estimates is transferred to a certain counting problem, but this is ultimately quite challenging. Instead, we employ an alternative approach of Vega [25]. This approach is simpler and can also be applied to more general settings [6], but optimality is far from guaranteed.
(iii) In higher dimensions $d \geq 3$, only local-in-time convergence can be derived from Theorem 1.5, because uniform Strichartz estimates hold for more regular initial data than those in the energy space. Indeed, if $d \geq 3$, the regularity $\frac{2}{q} + \epsilon$ of the Sobolev norm on the right-hand side in (1.12) is always strictly greater than one (when $r = \infty$).

The remainder of the paper is organized as follows: In Section 2 we provide the collection of basic analysis tools. In particular, Fourier analysis on a periodic lattice is briefly presented, but some important inequalities, such as the Sobolev and the Gagliardo–Nirenberg inequalities, are also introduced. In Section 3 we prove the key uniform Strichartz estimates (Theorem 1.5). In Section 4 we establish a well-posedness theory for DNLS (1.4) as well as uniform bounds for the nonlinear solutions. Finally, in Section 5 we prove the main theorem (Theorem 1.6).
2. Preliminaries

2.1. Basic inequalities on a periodic lattice. Recall the definition of the Lebesgue spaces on a periodic lattice (see (1.10)). On a lattice, we often have a larger class of inequalities, compared to those in the continuum domain $T^d$. For instance, by the definition, one can easily show the inequality

$$\|u\|_{L^q_h} \lesssim h^{-\left(\frac{1}{p} - \frac{1}{q}\right)}\|u\|_{L^p_h} \quad \text{for all } q > p,$$

(2.1)

while the embedding $L^p \hookrightarrow L^q$ fails on $T^d$. However, these inequalities become meaningless in the continuum limit $h \to 0$. Therefore, we would have to use inequalities wherein the implicit constants are independent of $h \in (0, 1]$.

We state the following inequalities, which hold uniformly in $h \in (0, 1]$.

**Lemma 2.1.** (i) (Hölder’s inequality). If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $1 \leq p, q, r \leq \infty$, then

$$\|uv\|_{L^r_h} \leq \|u\|_{L^p_h}\|v\|_{L^q_h}.$$  

(2.2)

(ii) (Young’s inequality) If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} \geq 1$, then

$$\|u \ast v\|_{L^r_h} \leq \|u\|_{L^p_h}\|v\|_{L^q_h},$$

(2.3)

where $\ast$ denotes the convolution operator defined by

$$(u \ast v)(x) = h^d \sum_{y \in T^d_h} u(x - y)v(y).$$

(2.4)

**Proof.** Based on Hölder’s and Young’s inequalities for sequences, we prove that

$$\|uv\|_{L^r_h} = h^d\|uv\|_{L^r_{\mathbb{Z}^d}} \leq h^d\|u\|_{L^p_{\mathbb{Z}^d}}\|v\|_{L^q_{\mathbb{Z}^d}} = \|u\|_{L^p_h}\|v\|_{L^q_h}$$

and

$$\|u \ast v\|_{L^r_h} = h^{d\left(1 + \frac{1}{r}\right)}\left|\sum_{y \in T^d_h} u(x - y)v(y)\right|_{L^r_{\mathbb{Z}^d}} \leq h^{d\left(\frac{1}{p} + \frac{1}{q}\right)}\|u\|_{L^p_h}\|v\|_{L^q_h} = \|u\|_{L^p_h}\|v\|_{L^q_h}.$$ 

$\square$
2.2. Fourier transform on a periodic lattice. Fix a large integer $M > 0$. For the periodic lattice $T^d_h$ with $h = \frac{\pi}{M}$ (see (1.2)), we denote its Fourier dual space, that is, the sparse periodic lattice, by \[ (T^d_h)^* = \mathbb{Z}^d / \frac{2\pi}{h} \mathbb{Z}^d = (\mathbb{Z} / \frac{2\pi}{h} \mathbb{Z})^d \]

\[ = \left\{ -\frac{2\pi}{h}, ..., -2, -1, 0, 1, 2, ..., \frac{2\pi}{h} - 1 \right\}^d \]

\[ = \left\{ -M, ..., -2, -1, 0, 1, 2, ..., M - 1 \right\}^d. \tag{2.5} \]

For a function $u : T^d_h \to \mathbb{C}$, its Fourier transform $F_h u : (T^d_h)^* \to \mathbb{C}$ is defined by

\[ (F_h u)(k) := h^d \sum_{x \in T^d_h} u(x)e^{-ik \cdot x}. \]

The inverse Fourier transform of a function $u : (T^d_h)^* \to \mathbb{C}$ is given by

\[ (F_h^{-1} u)(x) := \frac{1}{(2\pi)^d} \sum_{k \in (T^d_h)^*} u(k)e^{ik \cdot x}. \]

With abuse of notation, we write $\sum_{x \in T^d_h} = \sum_x$ and $\sum_{k \in (T^d_h)^*} = \sum_k$ unless there is confusion.

Remark 2.2. The above definitions are consistent with those on the periodic box $T^d$. Indeed, formally, we have

\[ T^d_h \to T^d, \quad (T^d_h)^* \to \mathbb{Z}^d, \quad F_h \to F, \quad F_h^{-1} \to F^{-1} \]

as $h \to 0$, where $F$ and $F^{-1}$ are the Fourier and the inverse transforms on $T^d$, respectively,

\[ (Fu)(k) := \int_{T^d} u(x)e^{-ik \cdot x} \, dx, \quad (F^{-1} u)(x) := \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} u(k)e^{ik \cdot x}. \]

We collect the properties of the Fourier and inverse Fourier transforms.

Lemma 2.3 (Properties of the Fourier transform on a periodic lattice).

1. (Inversion)
\[ F_h^{-1} \circ F_h = \text{Id} \text{ on } L^2(T^d_h), \quad F_h \circ F_h^{-1} = \text{Id} \text{ on } L^2((T^d_h)^*). \]

2. (Plancherel’s theorem)
\[ \frac{1}{(2\pi)^d} \sum_k (F_h u)(k)(F_h v)(k) = h^d \sum_x u(x)v(x). \]

3. (Fourier transform of a product)
\[ F_h(uv)(k) = \frac{1}{(2\pi)^d} \sum_{k'} (F_h u)(k')(F_h v)(k - k'). \]

To prove Lemma 2.3 we need the following identities.
Lemma 2.4.

\[
\frac{h^d}{(2\pi)^d} \sum_k e^{ik \cdot x} = \delta(x) := \begin{cases} 
1 & \text{if } x = 0, \\
0 & \text{if } x \neq 0 
\end{cases}
\]  
(2.6)

and

\[
\frac{h^d}{(2\pi)^d} \sum_x e^{ik \cdot x} = \delta(k). 
\]  
(2.7)

Proof. We only prove (2.6), because the proof of (2.7) is similar. Recalling that \( h = \frac{2\pi}{M} \) and \( x = (x_1, \ldots, x_d) = (hm_1, \ldots, h m_d) \in \mathbb{T}_h^d \) where \( m_j \in \{-M, -M+1, \ldots, M-1, M-1\} \), we evaluate the geometric sum

\[
\sum_{k_j=-M}^{M-1} e^{ik_j x_j} = \begin{cases} 
2M = \frac{2\pi}{h} & \text{if } x_j = 0, \\
\frac{e^{-iMx_j}(e^{2iMx_j} - 1)}{e^{ix_j} - 1} = \frac{e^{-iMx_j}(e^{2\pi im_j} - 1)}{e^{ix_j} - 1} = 0 & \text{if } x_j \in \mathbb{T}_h \setminus \{0\} 
\end{cases}
\]

Thus, we conclude that

\[
\sum_k e^{ik \cdot x} = \prod_{j=1}^d \sum_{k_j=-M}^{M-1} e^{ik_j x_j} = \prod_{j=1}^d \frac{2\pi}{h} \delta(x_j) = \frac{(2\pi)^d}{h^d} \delta(x),
\]

where we use the fact that \((\mathbb{T}_h^d)^* = \{-M, -M+1, \ldots, M-1\}^d\) \(\square\)

Proof of Lemma 2.3. (1) A direct calculation in addition to (2.6) yields

\[
(F_h^{-1}(F_h u))(x) = \frac{1}{(2\pi)^d} \sum_k \left\{ h^d \sum_{x'} u(x') e^{-ik \cdot x'} \right\} e^{ik \cdot x} = \sum_{x'} \left\{ \frac{h^d}{(2\pi)^d} \sum_k e^{ik \cdot (x-x')} \right\} u(x') = \sum_{x'} \delta(x - x') u(x') = u(x).
\]

Analogously, one can show that \((F_h(F_h^{-1} u))(x) = u(x)\).

(2) Similarly, using (2.6), we prove that

\[
\frac{1}{(2\pi)^d} \sum_k (F_h u)(k)(F_h v)(k) = \frac{1}{(2\pi)^d} \sum_k \left\{ h^d \sum_{x} u(x) e^{-ik \cdot x} \right\} \left\{ h^d \sum_{x'} v(x') e^{ik \cdot x'} \right\} = h^d \sum_x \sum_{x'} u(x) v(x') \left\{ \frac{h^d}{(2\pi)^d} \sum_k e^{-ik \cdot (x-x')} \right\} = h^d \sum_x u(x) v(x).
\]
(3) We write
\[ F_h(uv)(k) = h^d \sum_x u(x)v(x)e^{-ik \cdot x} \]
\[ = h^d \sum_x \left\{ \frac{1}{(2\pi)^d} \sum_\ell (F_h u)(\ell)e^{i\ell \cdot x} \right\} \left\{ \frac{1}{(2\pi)^d} \sum_{\ell'} (F_h v)(\ell')e^{i\ell' \cdot x} \right\} e^{-ik \cdot x} \]
\[ = \frac{1}{(2\pi)^d} \sum_\ell \sum_{\ell'} (F_h u)(\ell)(F_h v)(\ell') \left\{ \frac{h^d}{(2\pi)^d} \sum_x e^{i(\ell+\ell'-k) \cdot x} \right\}. \]

Then, applying (2.7) and summing out \( \ell' \), we prove the desired identity. \( \square \)

By the Fourier transform, we see that the discrete Laplacian is a Fourier multiplier operator.

**Lemma 2.5** (Discrete Laplacian as a Fourier multiplier operator). The discrete Laplacian
\[-\Delta_h\]
is the Fourier multiplier of the symbol \( \sum_{j=1}^d \frac{4}{h^2} \sin^2(\frac{hk_j}{2}) = \sum_{j=1}^d \frac{2}{h^2} (1 - \cos hk_j) \).

**Proof.** By the definition (1.3),
\[ F_h((-\Delta_h)u)(k) = \sum_{j=1}^d \frac{2 - e^{ikh_j} - e^{-ikh_j}}{h^2} (F_h u)(k) = \sum_{j=1}^d \frac{2(1 - \cos hk_j)}{h^2} (F_h u)(k). \]

\( \square \)

**Remark 2.6.** The discrete Laplacian formally converges to the Laplacian on \( \mathbb{T}^d \) as \( h \to 0 \), because given \( k \in (\mathbb{T}^d_h)^* \), the multiplier for the discrete Laplacian converges to that for the Laplacian on \( \mathbb{T}^d \), i.e., \( \sum_{j=1}^d \frac{2}{h^2} (1 - \cos hk_j) \to |k|^2 \) as \( h \to 0 \).

2.3. Dyadic decompositions and Sobolev spaces. Let
\[ N_* = 2^{\ell_*} \quad \text{with} \quad \ell_* = \lceil \log_2(\frac{h}{\pi}) \rceil - 1, \]
where \( \lceil a \rceil \) denotes the smallest integer greater than or equal to \( a \). For a dyadic number \( N = 2^\ell \) with \( \ell \in \mathbb{Z} \) such that \( N_* \leq N \leq 1 \), we define the frequency projection operator \( P_N = P_N^h \) by
\[ (P_N u)(x) := \begin{cases} \frac{1}{(2\pi)^d} \sum_{\frac{2\pi^*}{h} < \max|k_j| \leq \frac{2\pi}{h}} (F_h u)(k)e^{ik \cdot x} & \text{if} \quad 2N_* \leq N \leq 1, \\
\frac{1}{(2\pi)^d} (F_h u)(0) & \text{if} \quad N = N_* \end{cases} \] (2.8)

For \( s \in \mathbb{R} \), we define the Sobolev space \( H^s_h \) by the Hilbert space equipped with the norm
\[ \|u\|_{H^s_h} := \left\{ \frac{1}{(2\pi)^d} \sum_k \langle k \rangle^{2s} |(F_h u)(k)|^2 \right\}^{1/2}. \] (2.9)
We observe that
\[ \|u\|_{H^s_h}^2 \sim \sum_{N_* \leq N \leq 1} \left( \frac{N}{h} \right)^{2s} \|P_N f\|_{L^2_h}^2. \]

The following Sobolev and Gagliardo–Nirenberg inequalities are used in our analysis.

**Lemma 2.7** (Sobolev embedding). Suppose that \( 0 < s \leq \frac{d}{4} \), \( q \geq 2 \) and \( \frac{1}{q} = \frac{1}{2} - \frac{s}{d} \). Then, for any \( \epsilon > 0 \), we have
\[ \|u\|_{L^q_h} \lesssim \|u\|_{H^{s+\epsilon}_h}. \] (2.10)

**Lemma 2.8** (Gagliardo–Nirenberg inequality). Suppose \( \frac{1}{p} = \frac{1}{2} - \theta \frac{d}{2} \), \( 1 < p \leq \infty \) and \( 0 < \theta < 1 \). Then we have
\[ \|f\|_{L^p_h} \lesssim \|f\|_{L^2_h}^{1-\theta} \|f\|_{H^1_h}^{\theta}. \]

Proofs of Lemmas 2.7 and 2.8 are given in Appendix A. We expect the inequality (2.10) to be improved to the sharp version (\( \epsilon = 0 \)) by adopting the argument in [2] for instance. Nevertheless, in this study, we employ a nonsharp version, because its proof is simpler but also sufficient for our analysis.

### 2.4. Norm equivalence.

There are several ways to define Sobolev spaces on a periodic lattice. The following lemma shows that the Sobolev norm defined by (2.9) is equivalent to that by the discrete derivatives (1.7) as well as that by \( \sqrt{1 - \Delta_h} \), that is, the Fourier multiplier of the symbol \( (1 + \sum_{j=1}^d \frac{4}{h^2} \sin^2(\frac{hk}{2}))^{1/2} \).

**Lemma 2.9** (Norm equivalence).
\[ \|u\|_{H^1_h} \sim \|\sqrt{1 - \Delta_h} u\|_{L^2_h} = \left\{ \|u\|_{L^2_h}^2 + \|D_h^+ u\|_{L^2_h}^2 \right\}^{1/2}, \]
where \( D_h^+ = (D_{h,1}^+, ..., D_{h,d}^+) \).

**Proof.** The first equivalence follows from the Plancherel theorem and the pointwise bound \( (1 + \sum_{j=1}^d \frac{4}{h^2} \sin^2(\frac{hk}{2}))^{1/2} \sim \sqrt{1 + |k|^2} \) on \( (T_h^d)^* \). The second identity follows from \( (D_h)^* D_h = -\Delta_h \). \( \square \)

### 3. Uniform Strichartz estimates on a periodic lattice

This section is devoted to the proof of our key uniform-in-\( h \) Strichartz estimates (Theorem 1.5). First, we reduce the proof to the following dispersive estimate.

**Proposition 3.1** (Dispersive estimate). Let \( h \in (0, 1] \). For any dyadic number \( N \) with \( N_* := 2^{\lceil \log_2 \frac{1}{h} \rceil} \leq N \leq 1 \), there exists \( c > 0 \) such that if \( |t| \leq \frac{c}{N} \), then
\[ \|e^{it\Delta_h} P_{\leq N} u_{h,0}\|_{L^\infty} \lesssim \left( \frac{N}{h|t|} \right)^\frac{d}{4} \|u_{h,0}\|_{L^1_h}, \] (3.1)
where
\[ (P_{\leq N} u_{h,0})(x) := \frac{1}{(2\pi)^d} \sum_{\max |k_j| \leq \frac{N}{h}} (\mathcal{F}_h u_{h,0})(k) e^{ik \cdot x}. \]
Proof of Theorem 1.5 assuming Proposition 3.1. Applying the standard interpolation argument of Keel and Tao [17] with the dispersive estimate (3.1) but restricting this to the time interval \([0, \frac{\epsilon}{N}]\), one can prove that
\[
\|e^{it\Delta_h} P_{\leq N} u_{h,0}\|_{L^q_t([0, \frac{\epsilon}{N}]; L^r_h)} \lesssim \left( \frac{N}{h} \right)^{\frac{\delta}{q}} \|u_{h,0}\|_{L^r_h}.
\]
Hence, by changing the variables in time, \(P_N = P_{\leq N} P_N\) and the unitarity of the Schrödinger flow, we obtain
\[
\|e^{it\Delta_h} P_N u_{h,0}\|_{L^q_t([0, \frac{\epsilon}{N}]; L^r_h)} = \|e^{i\left(t + \frac{\epsilon h}{N}ight)\Delta_h} P_N u_{h,0}\|_{L^q_t([0, \frac{\epsilon}{N}]; L^r_h)}
\]
\[
= \|e^{it\Delta_h} P_{\leq N}(P_N e^{i\left(t + \frac{\epsilon h}{N} - 1\right)\Delta_h} u_{h,0})\|_{L^q_t([0, \frac{\epsilon}{N}]; L^r_h)}
\]
\[
\lesssim \left( \frac{N}{h} \right)^{\frac{\delta}{q}} \|P_N e^{i\left(t + \frac{\epsilon h}{N} - 1\right)\Delta_h} u_{h,0}\|_{L^r_h}
\]
\[
= \left( \frac{N}{h} \right)^{\frac{\delta}{q}} \|P_N u_{h,0}\|_{L^r_h}.
\]
Summing in the time interval,
\[
\|e^{it\Delta_h} P_N u_{h,0}\|_{L^q_t([0, 1]; L^r_h)}^q \leq \sum_{n=1}^{\left\lceil \frac{N}{\epsilon h} \right\rceil} \|e^{it\Delta_h} P_N u_{h,0}\|_{L^q_t([0, \frac{n}{\epsilon}; L^r_h])}^q
\]
\[
\lesssim \sum_{n=1}^{\left\lceil \frac{N}{\epsilon h} \right\rceil} \frac{N}{h} \|P_N u_{h,0}\|_{L^r_h}^q = \left( \frac{N}{h} \right)^2 \|P_N u_{h,0}\|_{L^r_h}^q.
\]
Then, summing in \(N\), we obtain
\[
\|e^{it\Delta_h} u_{h,0}\|_{L^q_t([0, 1]; L^r_h)} = \left\| \sum_{N=N_*}^{1} e^{it\Delta_h} P_N u_{h,0} \right\|_{L^q_t([0, 1]; L^r_h)}
\]
\[
\leq \sum_{N=N_*}^{1} \|e^{it\Delta_h} P_N u_{h,0}\|_{L^q_t([0, 1]; L^r_h)}
\]
\[
\lesssim \sum_{N=N_*}^{1} \left( \frac{N}{h} \right)^{\frac{\delta}{q}} \|P_N u_{h,0}\|_{L^r_h}.
\]
Because \(\mathcal{F}_h(P_N u_{h,0})\) is localized in \(|k| \sim \frac{N}{h}\), we conclude that
\[
\|e^{it\Delta_h} u_{h,0}\|_{L^q_t([0, 1]; L^r_h)} \lesssim \sum_{N=N_*}^{1} \left( \frac{N}{h} \right)^{-\epsilon} \|u_{h,0}\|_{H^{\frac{2}{r} + \epsilon}_h} \lesssim \left( \frac{N_*}{h} \right)^{-\epsilon} \|u_{h,0}\|_{H^{\frac{2}{r} + \epsilon}_h} \sim \|u_{h,0}\|_{H^{\frac{2}{r} + \epsilon}_h},
\]
where in the last step, we used that \(N_* \sim 2^{\log_2(\frac{h}{N})} = \frac{h}{N}\). □

Proposition 3.1 remains to be proved, for which we need to estimate the sums of the oscillating functions. Following Vega’s argument [25], we use Lemma 3.2 to approximate
the sums by the oscillatory integrals. Then, we employ the estimate (Lemma 3.3) of the oscillatory integral.

**Lemma 3.2** (Zygmund [27 Chapter V, Lemma 4.4]). Let $\varphi$ be a real-valued function, and let $a, b \in \mathbb{R}$ with $a < b$. If $\varphi'$ is monotonic and $|\varphi'| < 2\pi$ on $(a, b)$, then

$$\left| \int_a^b e^{i\varphi(x)} \, dx - \sum_{a < n \leq b} e^{i\varphi(n)} \right| \leq A,$$

where the constant $A$ is independent of $\varphi$, $a$, and $b$.

**Lemma 3.3.** Let $h \in (0, 1]$ and a dyadic number $N$ with $N_* \leq N \leq 1$ be given. We define

$$I_{N,h,t,x} := \int_{-\frac{\pi N}{h}}^{\frac{\pi N}{h}} e^{i(x\xi - \frac{2\pi}{h^2}(1 - \cos h\xi))} \, d\xi.$$

Then, there exists $B > 0$, independent of $h$ and $N$, such that

$$|I_{N,h,t,x}| \leq B \left( \frac{N}{h|t|} \right)^\frac{1}{4}.$$

**Proof.** If $N_* \leq N \leq \frac{1}{4}$, by the van der Corput lemma with $|\xi - \frac{2\pi}{h^2}(1 - \cos h\xi)| = 2|t||h\xi| \geq |t|$ for $|\xi| \leq \frac{\pi N}{h}$, we have $|I_{N,h,t,x}| \lesssim |t|^{-1/2}$. Hence, interpolating with the trivial bound $|I_{N,h,t,x}| \lesssim \frac{N}{h}$, we obtain the desired bound.

If $N = \frac{1}{2}$ or 1, then we decompose

$$I_{N,h,t,x} = I_{\frac{N}{2},h,t,x} + \int_{-\frac{\pi N}{h}}^{\frac{\pi N}{h}} e^{i(x\xi - \frac{2\pi}{h^2}(1 - \cos h\xi))} \, d\xi.$$

It has already been shown that $|I_{\frac{N}{2},h,t,x}| \lesssim (h|t|)^{-1/3}$. For the integral on the right-hand side, we change the variables,

$$\int_{-\frac{\pi N}{h}}^{\frac{\pi N}{h}} e^{i(x\xi - \frac{2\pi}{h^2}(1 - \cos h\xi))} \, d\xi = h^{-1} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i\left(\frac{x}{h} - \frac{2\pi}{h^2}(1 - \cos \xi)\right)} \, d\xi.$$

We observe that $|(\frac{x}{h} - \frac{2\pi}{h^2}(1 - \cos \xi))'| = \frac{2|t|}{h^3} |\cos \xi|$ and $|(\frac{x}{h} - \frac{2\pi}{h^2}(1 - \cos \xi))''| = \frac{2|t|}{h^5} |\sin \xi|$. Thus, applying the van der Corput lemma again, we prove that

$$\left| \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i(x\xi - \frac{2\pi}{h^2}(1 - \cos h\xi))} \, d\xi \right| \lesssim h^{-1} \frac{1}{(|t|/h^2)^{1/3}} = \frac{1}{(h|t|)^{1/3}}.$$

Therefore, we complete the proof of the lemma.

**Proof of Proposition 3.4.** We consider the case $N = N_*$. By the definition of $P_{N_*}$ (see (2.8)) and the Plancherel theorem, we have

$$e^{it\Delta_h} P_{N_*} u_{h,0}(x) = \frac{1}{(2\pi)^d} (\mathcal{F}_h u_{h,0})(0) \lesssim \left\{ \frac{1}{(2\pi)^d} \sum_k |\mathcal{F}_h u_{h,0}(k)|^2 \right\}^{\frac{1}{2}} = \|u_{h,0}\|_{L^2_h},$$
which implies $\|e^{it\Delta_h}P_N u_{h,0}\|_{L^\infty_h} \lesssim \|u_{h,0}\|_{L^2_h}$. Hence, interpolating it with a trivial estimate $\|e^{it\Delta_h}P_N u_{h,0}\|_{L^2_h} \lesssim \|u_{h,0}\|_{L^2_h}$, we get the bound $\|e^{it\Delta_h}P_N u_{h,0}\|_{L^q_h} \lesssim \|u_{h,0}\|_{L^q_h}$ for all $r \geq 2$. As a consequence, we obtain $\|e^{it\Delta_h}P_N u_{h,0}\|_{L^q([0,1];L^q_h)} \lesssim \|e^{it\Delta_h}P_N u_{h,0}\|_{L^{2\infty}([0,1];L^q_h)} \lesssim \|P_N u_{h,0}\|_{L^2_h}$ for any $1 \leq q \leq \infty$ and $2 \leq r \leq \infty$.

Suppose that $2N_s \leq N \leq 1$. A direct calculation with Lemma 2.5 yields

$$e^{it\Delta_h}P_{\leq N}u_{h,0}(x) = \frac{1}{(2\pi)^d} \sum_{\max |k_j| \leq \frac{\pi N}{h}} e^{i(kx - \sum_{j=1}^d \frac{2\pi}{h^2}(1 - \cos hk_j))} \mathcal{F}_h(P_N u_{h,0})(k)$$

$$= \frac{h^d}{(2\pi)^d} \sum_{x'} P_N u_{h,0}(x') \prod_{j=1}^d \sum_{|k_j| \leq \frac{\pi N}{h}} e^{i((x_j - x'_j)k_j - \frac{2\pi}{h^2}(1 - \cos hk_j))}$$

$$= (K_{N,t} \ast P_N u_{h,0})(x),$$

where

$$K_{N,t}(x) = \frac{1}{(2\pi)^d} \prod_{j=1}^d \sum_{|k_j| \leq \frac{\pi N}{h}} e^{i(x_j k_j - \frac{2\pi}{h^2}(1 - \cos hk_j))}$$

and $\ast$ is the convolution on the lattice defined in (2.4). Hence, Young’s inequality ensures (3.1) provided the following one-dimensional inequality holds true:

$$\sup_{x \in \mathbb{T}_h} \left| \sum_{|k| \leq \frac{\pi N}{h}} e^{i(kx - \frac{2\pi}{h^2}(1 - \cos hk))} \right| \lesssim \left( \frac{N}{h|t|} \right)^\frac{1}{3}. \quad (3.2)$$

It remains to prove (3.2). For notational convenience, we write

$$\varphi(\xi) := x\xi - \frac{2t}{h^2} (1 - \cos h\xi)$$

for $x \in \mathbb{T}_h$ and $\xi \in \mathbb{R}$ with $|\xi| \leq \frac{\pi N}{h}$. A direct calculation yields

$$|\varphi'(\xi)| = \left| x - \frac{2t}{h} \sin h\xi \right| < 2\pi$$

under the restriction $|t| \leq \frac{ch}{N}$.

First, we consider the case $N \leq \frac{1}{2}$. Then $\varphi'$ is decreasing on $[-\frac{\pi N}{h}, \frac{\pi N}{h}]$. Hence, applying Lemma 3.2 and 3.3 we obtain

$$\left| \sum_{|k| \leq \frac{\pi N}{h}} e^{i\varphi(k)} \right| \leq \left| \sum_{|k| \leq \frac{\pi N}{h}} e^{i\varphi'(k)} - \int_{-\frac{\pi N}{h}}^{\frac{\pi N}{h}} e^{i\varphi(\xi)} \, d\xi \right| + \left| \int_{-\frac{\pi N}{h}}^{\frac{\pi N}{h}} e^{i\varphi(\xi)} \, d\xi \right|$$

$$\leq A + B \left( \frac{N}{h|t|} \right)^\frac{1}{2} \lesssim \left( \frac{N}{h|t|} \right)^\frac{1}{3}. \quad (3.3)$$

In the last inequality, we used $|t| \leq \frac{ch}{N}$ and $N \geq N_s \sim \frac{h}{c}$, implying $1 \lesssim (\frac{h}{|t|})^{1/3} \lesssim (\frac{N}{h|t|})^{1/3}$. Next, we consider the case $N = 1$. We divide the interval into three parts:

$$[-\frac{\pi}{h}, \frac{\pi}{h}] = [-\frac{\pi}{h}, -\frac{\pi}{2h}] \cup [-\frac{\pi}{2h}, \frac{\pi}{2h}] \cup [\frac{\pi}{2h}, \frac{\pi}{h}] =: I_1 \cup I_2 \cup I_3,$$
where $\varphi'$ is monotonic on each $I_j$. Then, we decompose
\[
\sum_{|k| \leq \frac{\pi}{h}} e^{i\varphi(k)} = \sum_{k \in I_1} e^{i\varphi(k)} + \sum_{k \in I_2} e^{i\varphi(k)} + \sum_{k \in I_3} e^{i\varphi(k)} =: S_1 + S_2 + S_3,
\]
Each $S_j$ can be estimated with the same method as above. Summing these, we complete the proof.

As an application of Theorem 1.5, we obtain the time-average uniform $L^\infty_h$ estimates.

**Corollary 3.4** (Uniform time-averaged $L^\infty_h$-bounds for the discrete linear Schrödinger flow; 2D case). Suppose that $d = 2$ and $1 \leq q < \infty$. Then,
\[
\|e^{it\Delta_h} u_{h,0}\|_{L^q_{t,\infty}(\mathbb{R}; L^\infty_h)} \lesssim \|u_{h,0}\|_{H^2_h}.
\]

**Proof.** Let $\epsilon > 0$ be a sufficiently small number such that the following inequalities hold.

For $1 \leq q \leq 3$, Hölder’s inequality in time and Theorem 1.5 yield
\[
\|e^{it\Delta_h} u_{h,0}\|_{L^q_{t,\infty}(\mathbb{R}; L^\infty_h)} \lesssim \|e^{it\Delta_h} u_{h,0}\|_{L^3_{t,\infty}(\mathbb{R}; H^{1+\epsilon}_h)} \lesssim \|u_{h,0}\|_{H^\frac{3}{2+\epsilon}_h}.
\]

Suppose that $q > 3$. By the Sobolev inequality (Lemma 2.7) and the unitarity of the Schrödinger flow, we get
\[
\|e^{it\Delta_h} u_{h,0}\|_{L^q_{t,\infty}(\mathbb{R}; L^\infty_h)} \lesssim \|e^{it\Delta_h} u_{h,0}\|_{L^\infty_{t,\infty}(\mathbb{R}; H^{1+\epsilon}_h)} = \|u_{h,0}\|_{H^1_{H^1}}.
\]

for a small $\epsilon = \epsilon(q) > 0$ appeared in Theorem 1.5. Thus, interpolating this inequality and Theorem 1.5 with $(q, r, d) = (3, \infty, 2)$ and choosing $\epsilon < \frac{1}{q}$, we obtain
\[
\|e^{it\Delta_h} u_{h,0}\|_{L^q_{t,\infty}(\mathbb{R}; L^\infty_h)} \lesssim \|u_{h,0}\|_{H^{\frac{1}{4}+\epsilon}_h} \leq \|u_{h,0}\|_{H^1_{H^1}},
\]
which completes the proof.

### 4. Uniform bound for discrete NLS

In this section, we provide a simple well-posedness theorem for DNLS (1.4). Then, as an application of the uniform-in-$h$ Strichartz estimates, we deduce a uniform time-averaged $L^\infty_h$-bound for nonlinear solutions.

4.1. **Global well-posedness.** By Duhamel’s principle, DNLS (1.4) is equivalent to the integral equation
\[
u_{h}(t) = e^{it\Delta_h} u_{h,0} - i\lambda \int_0^t e^{i(t-s)\Delta_h} (|u_h|^{p-1} u_h)(s) \, ds.
\]
We next show its global well-posedness.

**Proposition 4.1** (Global well-posedness). Let $d \geq 1$, $h > 0$ and $p > 1$. Then, for any initial data $u_{h,0} \in L^2_h$, there exists a unique global solution $u_{h}(t) \in C(\mathbb{R}; L^2_h)$ to DNLS (1.4). Moreover, it conserves the mass
\[
M_h(u_h) := \|u_h\|_{L^2_h}^2
\]

and the energy
\[ E_h(u_h) := \frac{1}{2} \| \sqrt{-\Delta_h} u_h \|_{L^2_h}^2 + \frac{\lambda}{p+1} \| u_h \|_{L^{p+1}_h}^{p+1} \]  

(4.3)

\textbf{Proof.} The proof is identical to the analogous theorem for the discrete NLS on \( h\mathbb{Z}^d \) (see [10, Proposition 6.1]). Fix \( h > 0 \). For a small \( T > 0 \) to be chosen later, let \( X_T := C_t([-T, T]; L^\infty_h) \). We denote by \( \Gamma(u_h) \) the right-hand side of (4.1). Then, by the unitarity of the linear propagator and the trivial inequality \( \| u_h \|_{L^\infty_h} \leq h^{-d/2} \| u_h \|_{L^2_h} \), one can show that

\[ \| \Gamma(u_h) \|_{X_T} \leq \| u_{h,0} \|_{L^2} + \| u_h \|_{L^2([-T,T];L^2_h)}^{p-1} \| u_h \|_{L^2([-T,T];L^\infty_h)} \| u_h \|_{C_t([-T,T];L^\infty_h)} \]

\[ \leq \| u_{h,0} \|_{L^2} + T \| u_h \|_{C_t([-T,T];L^\infty_h)} \| u_h \|_{C_t([-T,T];L^\infty_h)} \]

\[ \leq \| u_{h,0} \|_{L^2} + \frac{T}{h} \| u_h \|_{X_T}^{p-1} \| u_h \|_{X_T}^p \]

and in the same way,

\[ \| \Gamma(u_h) - \Gamma(v_h) \|_{X_T} \leq \frac{T}{h^{d/2}} \left( \| u_h \|_{X_T}^{p-1} + \| v_h \|_{X_T}^{p-1} \right) \| u_h - v_h \|_{X_T}. \]

Therefore, if \( T > 0 \) is sufficiently small depending on \( h > 0 \), \( \Gamma \) is contractive on the set \( \{ u_h \in X_T : \| u_h \|_{X_T} \leq 2 \| u_{h,0} \|_{L^2_h} \} \). Thus, DNLS (1.4) is locally well-posed in \( L^2_h \). The conservation laws (4.2) and (4.3) can be proven by direct calculations. The lifespan of local solutions is then extended by the mass conservation law (4.2). \( \square \)

4.2. Uniform bound for the 2D DNLS. Next, we show that not only linear solutions (Corollary 3.4) but also nonlinear solutions obey a time-averaged uniform \( L^\infty_h \)-bound.

\textbf{Proposition 4.2} (Uniform \( L^\infty_h \)-bound for the 2D DNLS). Suppose that \( p \) satisfies (1.3). Then, the solution \( u_h(t) \) to DNLS (1.4) with initial data \( u_{h,0} \in H^1_h \), constructed in Proposition 4.1, satisfies

\[ \| u_h \|_{L^\infty_t([-T,T];L^\infty_h)} \lesssim \langle T \rangle^{1/q_\ast} \| u_{h,0} \|_{H^1_h}, \quad \forall T > 0, \]  

(4.4)

where

\[ \begin{cases} 
q_\ast > p - 1 & \text{if } p \geq 3, \\
q_\ast = 2 & \text{if } 1 < p < 3.
\end{cases} \]

\textbf{Proof.} Let \( u_h \) be the solution to DNLS (1.4) constructed in Proposition 4.1 and let \( \tau > 0 \) be a sufficiently small number such that

\[ \| u_h \|_{C_t(I;H^1_h)} + \| u_h \|_{L^\infty_t(I;L^\infty_h)} \leq 4c_0 \| u_{h,0} \|_{H^1_h}, \]

where \( I = [-\tau, \tau] \), \( c_0 = \max(c_{q_\ast}, 1) \) and \( c_{q_\ast} \) is the implicit constant in (3.3) (when \( q = q_\ast \)). Such \( \tau \) is initially chosen depending on \( h > 0 \), but later it can be extended independently of \( h > 0 \).

From the integral representation of the solution (4.1), the unitarity of the linear flow yields

\[ \| u_h \|_{C_t(I;H^1_h)} \leq \| u_{h,0} \|_{H^1_h} + \| u_h \|_{L^\infty_t(I;H^1_h)}, \]

(4.5)
and by Corollary 3.3, we obtain
\[
\|u_h\|_{L_t^q(I; L_h^\infty)} \leq c_0 \|u_{h,0}\|_{H^1_h} + c_0 \|u_h^{p-1} u_h\|_{L_t^1(I; H^1_h)}.
\] (4.6)

Applying the fundamental theorem of calculus of the form
\[
|\alpha|^{p-1} \alpha - |\beta|^{p-1} \beta = \int_0^1 \frac{d}{ds} \left( |\alpha + s(\beta - \alpha)|^{p-1} (\alpha + s(\beta - \alpha)) \right) ds
\]
\[
= \frac{p+1}{2} \int_0^1 |\alpha + s(\beta - \alpha)|^{p-1} ds \cdot (\beta - \alpha)
\]
\[
+ \frac{p-1}{2} \int_0^1 |\alpha + s(\beta - \alpha)|^{p-3} (\alpha + s(\beta - \alpha))^2 ds \cdot \beta - \alpha
\] (4.7)

with \(\alpha = u_h(x + he)\) and \(\beta = u_h(x)\), we obtain
\[
\|D_{h,j}^\pm (|u_h|^{p-1} u_h)\|_{L_h^2} = \frac{1}{h} \|(u_h|^{p-1} u_h)(x + he) - |u_h|^{p-1} u_h(x)\|_{L_h^2}
\]
\[
\lesssim \frac{1}{h} \|u_h|^{p-1} u_h(x + he) - u_h(x)\|_{L_h^2} = \|u_h|^{p-1} u_h\|_{L_h^1}
\]

Hence, by the norm equivalence (Lemma 2.9), it follows that
\[
\|u_h|^{p-1} u_h\|_{H^1_h} \sim \|u_h|^{p-1} u_h\|_{L_h^2} + \sum_{j=1}^d \|D_{h,j}^\pm (|u_h|^{p-1} u_h)\|_{L_h^2}
\]
\[
\lesssim \|u_h|^{p-1} u_h\|_{L_h^2} + \sum_{j=1}^d \|u_h|^{p-1} u_h\|_{L_h^2}
\]
\[
\sim \|u_h|^{p-1} u_h\|_{H^1_h}.
\]

Inserting this bound in (4.5) and (4.6), we obtain
\[
\|u_h\|_{C_t(I; H^1_h)} + \|u_h\|_{L_t^q(I; L_h^\infty)}
\]
\[
\leq 2c_0 \|u_{h,0}\|_{H^1_h} + C(2\tau)^{1 - \frac{p-1}{p}} \|u_h|^{p-1} \|_{L_t^1(I; L_h^\infty)} \|u_h\|_{C_t(I; H^1_h)}
\]
\[
\leq 2c_0 \|u_{h,0}\|_{H^1_h} + C(2\tau)^{1 - \frac{p-1}{p}} \left(4c_0 \|u_{h,0}\|_{H^1_h} \right)^P.
\]

Thus, it follows that
\[
\|u_h\|_{L_t^q(I; L_h^\infty)} \leq 4c_0 \|u_{h,0}\|_{H^1_h}
\] (4.8)
as long as \(C(2\tau)^{1 - \frac{p-1}{p}} (4c_0 \|u_{h,0}\|_{H^1_h})^P \leq 2c_0 \|u_{h,0}\|_{H^1_h}\) is satisfied. Therefore, the time interval \(I\) can be extended to a short time interval of which the length depends on \(\|u_{h,0}\|_{H^1_h}\) but is independent of \(h > 0\).

To extend the time interval arbitrarily, we show that \(\|u_h(t)\|_{H^1_h}\) is bounded uniformly in time. Indeed, by the mass conservation law, it is sufficient to show that \(\|(-\Delta_h)^{1/2} u_h(t)\|_{L_h^2}\) is bounded globally in time. When \(\lambda > 0\), the energy conservation law immediately implies that
\[
\|(-\Delta_h)^{1/2} u_h(t)\|_{L_h^2}^2 \leq 2E_h(u_h(t)) = 2E_h(u_{h,0}) \text{ for all } t.
\]
When \(\lambda < 0\), we apply both
the mass and the energy conservation laws as well as the 2D uniform Gagliardo–Nirenberg inequality (Lemma 5.8) to obtain
\[
\frac{1}{2} \|(-\Delta_h)^{\frac{1}{2}} u_h(t)\|_{L^2_h}^2 = E_h(u_h(t)) + \frac{\lambda}{p+1} \|u_h(t)\|_{L^{p+1}_h}^{p+1}
\leq E_h(u_h(t)) + C \|u_h(t)\|_{L^2_h}^2 \|(-\Delta_h)^{\frac{1}{2}} u_h(t)\|_{L^p_h}^{p-1}
\leq E_h(u_{h,0}) + CM_h(u_{h,0}) \|(-\Delta_h)^{\frac{1}{2}} u_h(t)\|_{L^2_h}^{p-1}.
\]
By the assumption (1.13), we have \( p - 1 < 2 \). Thus, we can use Young’s inequality to bound \( \|(-\Delta_h)^{\frac{1}{2}} u_h(t)\|_{L^2_h}^2 \) only in terms of the mass \( M_h(u_{h,0}) \) and the energy \( E_h(u_{h,0}) \).
Because \( \|u_h(t)\|_{H^1_h} \) is bounded uniformly in time, (1.18) can be iterated with the new initial data \( u(\tau), u(2\tau), \ldots \) and with the bounds (4.8) on the intervals \([\tau, 2\tau], [2\tau, 3\tau], \ldots\) to cover an arbitrarily long time interval \([-T, T]\). Therefore, summing up, we obtain the desired bound (4.4).

5. PROOF OF THE CONTINUUM LIMIT

In this section, we prove the main theorem of this article (Theorem 1.6).

5.1. Preliminaries. We first provide lemmas concerning the discretization and linear interpolation (see (1.5) and (1.6)). Analogous lemmas on the lattice \( h\mathbb{Z}^d \) have been stated and proven in [11]. Thus, we omit some details. Indeed, differentiation (resp., discrete differentiation) is a local operation, thus the argument used in the non-compact domain \( \mathbb{R}^d \) (resp, \( h\mathbb{Z}^d \)) can easily be adopted to the compact domain \( T^d \) (resp, \( T_h^d \)).

Lemma 5.1 (Boundedness of discretization and linear interpolation).
\[
\|d_h(f)\|_{H^1_h(T_h^d)} \lesssim \|f\|_{H^1(T^d)} \quad \text{and} \quad \|p_h(f_h)\|_{H^1_h(T_h^d)} \lesssim \|f_h\|_{H^1_h(T_h^d)}.
\]

Proof. We compute the discrete Sobolev norm using Lemma 2.9 Then the proof follows from the same method as [11, Lemmas 5.1 and 5.2].

Lemma 5.2. Let \( h \in (0, 1] \). Then, for \( f \in H^1(T^d) \), we have
\[
\|(p_h \circ d_h)f - f\|_{L^2(T^d)} \lesssim h \|f\|_{H^1(T^d)}.
\]

Proof. The proof closely follows from the proof of [11, Proposition 5.3].

Lemma 5.3. Let \( h \in (0, 1] \). If \( f \in H^1(T^d) \) and \( g_h \in H^1_h(T_h^d) \), then
\[
\|p_h e^{it\Delta_h} g_h - e^{it\Delta} f\|_{L^2(T^d)} \lesssim \sqrt{h}|t| \|\|g_h\|_{H^1_h(T_h^d)} + \|f\|_{H^1(T^d)}\| + \|p_h g_h - f\|_{L^2(T^d)}.
\]
In particular,
\[
\|p_h e^{it\Delta_h} d_h(f) - e^{it\Delta} f\|_{L^2(T^d)} \lesssim \sqrt{h}|t| \|f\|_{H^1(T^d)}.
\]
Proof. The proof closely follows from the proof of \[11\] Proposition 5.4. First, using direct calculations, we observe that the Fourier transform of the linear interpolation of a discrete function is given by

\[
\mathcal{F}_h(p_h f_h)(k) = \mathcal{P}_h(k) (\widetilde{\mathcal{F}_h} f_h)(k), \quad \forall k \in \mathbb{Z}^d
\]

where

\[
\mathcal{P}_h(k) = \frac{1}{h^d} \int_{[0,h)^d} e^{-ix \cdot k} dx + \sum_{j=1}^d \frac{e^{ihx_j} - 1}{h} \int_{[0,h)^d} x_j e^{-ix \cdot k} dx
\]

and \(\widetilde{\mathcal{F}_h}\) denotes the \([-\frac{\pi}{h}, \frac{\pi}{h}]^d\)-periodic extension of the discrete Fourier transform \(\mathcal{F}_h\), precisely, \((\widetilde{\mathcal{F}_h} f_h)(k) = (\mathcal{F}_h f_h)(k')\) for all \(k' \in \mathbb{Z}^d + \frac{2 \pi}{h} \mathbb{Z}^d\). We also observe that

\[
|e^{-it \cdot \sum_{j=1}^d \sin(\frac{h x_j}{2})} - e^{i t |k|^2}| \lesssim |t| |h^2| |k|^4, \quad k \in (\mathbb{T}_h^d)^*.
\]

By these observations and Lemma 5.3 and 5.2, one can proceed as in the proof of \[11\] Proposition 5.4. Here, an \(O(\sqrt{h})\)-bound is obtained from the regularity gap between the norms on the left- and right-hand sides. \(\square\)

As a corollary of Lemma 5.3, we have the following.

**Corollary 5.4.** Let \(h \in (0,1]\) and \(p > 1\). Then,

\[
\left\| \left( p_h e^{i(t-s)\Delta_h} - e^{i(t-s)\Delta_h} \right) \left( |u_h|^{p-1} u_h \right) (s) \right\|_{L^2(\mathbb{T}^d)} \lesssim \sqrt{h} |t-s| \|u_h\|_{L^\infty(T_h^d)}^{p-1} \|u_h\|_{H^1_h(\mathbb{T}_h^d)}.
\]

**Proof.** An immediate application of Lemma 5.3 to the left-hand side of (5.1) yields

\[
\text{LHS of (5.1)} \lesssim h^{\frac{3}{2}} |t-s| \left( \|u_h\|^{p-1} \|u_h\|_{H^1_h(\mathbb{T}_h^d)} + \|p_h \left( |u_h|^{p-1} u_h \right)\|_{H^1_h(\mathbb{T}_h^d)} \right).
\]

Lemma 5.1 and Hölder’s inequality control the right-hand side, and we thus obtain (5.1). \(\square\)

**Lemma 5.5** (Proposition 5.7 in \[11\]). Let \(h \in (0,1]\) and \(p > 1\). Then,

\[
\|p_h \left( |u_h|^{p-1} u_h \right) - |p_h u_h|^{p-1} p_h u_h\|_{L^2(\mathbb{T}^d)} \lesssim h \|u_h\|_{L^\infty(T_h^d)}^{p-1} \|u_h\|_{H^1_h(\mathbb{T}_h^d)}.
\]

We end this section with the following lemma:

**Lemma 5.6.** Let \(h \in (0,1]\) and \(p > 1\). Then,

\[
\|p_h u_h|^{p-1} p_h u_h - |u|^{p-1} u\|_{L^2} \lesssim \left( \|u_h\|_{L^\infty_h} + \|u\|_{L^\infty} \right)^{p-1} \|p_h u_h - u\|_{L^2}.
\]

**Proof.** It follows from the calculation (4.7) with \(\alpha = p_h u_h\) and \(\beta = u_h\). \(\square\)
5.2. Proof of continuum limit. Now we are in a position to prove Theorem 1.6. Because the proof closely follows from the argument presented in [11, Section 6], we only sketch the outline.

Let \( h \in (0, 1] \) be fixed. Given initial data \( u_0 \in H^1(\mathbb{T}^2) \), let \( u(t) \in C(\mathbb{R}; H^1(\mathbb{T}^2)) \) be the global solution to NLS (1.1) (see Section B). For the discretization \( u_{h,0} = d_h u_0 \), let \( u_h(t) \) be the solution to DNLS (1.4) with the initial data \( u_{h,0} \) constructed in Section 4.

Applying the linear interpolation operator to the Duhamel formula (4.1), we write
\[
p_h u_h(t) = p_h e^{it\Delta_h} u_{h,0} - i\lambda \int_0^t p_h e^{i(t-s)\Delta_h} \left( |u_h|^{p-1} u_h \right) (s) \, ds.
\]

Then, by direct calculations, the difference of \( u \) and \( p_h u \) can be expressed as
\[
p_h u_h(t) - u(t) = p_h e^{it\Delta_h} u_{h,0} - e^{it\Delta} u_0
- i\lambda \int_0^t \left( p_h e^{i(t-s)\Delta_h} - e^{i(t-s)\Delta} \right) \left( |u_h|^{p-1} u_h \right) (s) \, ds
- i\lambda \int_0^t e^{i(t-s)\Delta} \left( p_h (|u_h|^{p-1} u_h) - (|p_h u_h|^{p-1} p_h u_h) \right) (s) \, ds
- i\lambda \int_0^t e^{i(t-s)\Delta} \left( |p_h u_h|^{p-1} p_h u_h - |u|^{p-1} u \right) (s) \, ds
=: I_1 + I_2 + I_3 + I_4.
\]

Lemma 5.3, 5.5, and 5.6 and Corollary 5.4 yield
\[
\|p_h u_h(t) - u(t)\|_{L^2} \lesssim h^{\frac{1}{2}} (t)^2 (1 + \|u_0\|_{H^1})^p + \int_0^t (\|u_h(s)\|_{L^p}_{H^1} + \|u(s)\|_{L^p}_{H^1}) \|p_h u_h(s) - u(s)\|_{H^1} \, ds,
\]
which, by applying Grönwall’s inequality in addition to Proposition 4.2 and Corollary B.5 implies
\[
\|p_h u_h(t) - u(t)\|_{L^2} \lesssim h^{\frac{1}{2}} (1 + \|u_0\|_{H^1})^p e^{B|t|}
\]
for sufficiently large \( B \gg 1 \). This completes the proof of Theorem 1.6.

Appendix A. Proof of Lemma 2.7 and 2.8

On a periodic domain, the proof of the Sobolev inequality is more involved, compared to that on the entire Euclidean space, because the explicit kernel formula for the inverse Laplacian is no longer available (see [2] for example). However, if an arbitrarily small loss of regularity is allowed, one can show the inequality in a simpler manner, as is presented in this appendix.

The key item is Bernstein’s inequality for the projection operator \( P_N \) (see [2,8]).
Lemma A.1 (Bernstein’s inequality). Suppose that $0 < s \leq \frac{d}{2}$, $q \geq 2$ and $\frac{1}{q} = \frac{1}{2} - \frac{s}{d}$. For $h \in (0, 1]$ and a dyadic number $N$ with $N_* := 2^{[\log_2(\frac{h}{\pi})]} - 1 \leq N \leq 1$, we have

$$\|P_N u\|_{L^q_h} \lesssim \left(\frac{N}{h}\right)^s \|u\|_{L^2_h}. \quad (A.1)$$

Proof. We prove the lemma by the standard $TT^*$ argument. When $q = \infty$, we have

$$\|P_N u\|_{L^\infty_h} = \left\|\frac{1}{(2\pi)^d} \sum_{N\pi < |k_j| \leq N_*} (F_h u)(k)e^{ik\cdot x}\right\|_{L^\infty_h} \lesssim \left(\frac{N}{h}\right)^d \|F_h u\|_{L^\infty} \lesssim \left(\frac{N}{h}\right)^d \|u\|_{L^1_h}.$$  

When $q = 2$, it is obvious that $\|P_N u\|_{L^2_h} \leq \|u\|_{L^2_h}$. Interpolating, we obtain

$$\|P_N u\|_{L^q_h} \lesssim \left(\frac{N}{h}\right)^{2s} \|u\|_{L^{q'}_h}$$

for $q \geq 2$. This inequality implies that

$$\|P_N u\|_{L^2_h}^2 = h^d \sum_x P_N u(x)P_N u(x) = h^d \sum_x P_N u(x)u(x) \leq \|P_N u\|_{L^2_h} \|u\|_{L^{q'}_h} \lesssim \left(\frac{N}{h}\right)^{2s} \|u\|_{L^{q'}_h}^2.$$  

Thus, (A.1) follows from the duality. □

Proof of Lemma 2.7. By the triangle inequality and Lemma A.1, we prove that

$$\|u\|_{L^q_h} \leq \sum_{N=N_*}^{1} \|P_N u\|_{L^q_h} \lesssim \sum_{N=N_*}^{1} \left(\frac{N}{h}\right)^s \|P_N u\|_{L^2_h} \lesssim \sum_{N=N_*}^{1} \left(\frac{N}{h}\right)^{-\epsilon} \|u\|_{H^{s+\epsilon}_h} \sim \left(\frac{N_*}{h}\right)^{-\epsilon} \|u\|_{H^{s+\epsilon}_h},$$

where in the last step, we used that $N_* \sim 2^{\log_2(\frac{h}{\pi})} = \frac{h}{\pi}$. □

Similarly, the Gagliardo–Nirenberg inequality can be proved.
Proof of Lemma 2.8. Replacing $f$ by $\frac{1}{\|f\|_{L_h^2}^2} f$, we may assume that $\|f\|_{L_h^2} = 1$. Suppose that $\|f\|_{H_h^1} \leq h^{-1}$. Let $R = h\|f\|_{H_h^1}$. Then, using Bernstein’s inequality, we prove that
\[
\|f\|_{L_h^q} \leq \sum_{N_s \leq N \leq 1} \|P_N f\|_{L_h^q} \leq \sum_{N_s \leq N \leq 1} \left( \frac{N}{h} \right)^\theta \|P_N f\|_{L_h^q}
\leq \sum_{N_s \leq N \leq R} \left( \frac{N}{h} \right)^\theta \|P_N f\|_{L_h^2} + \sum_{R < N \leq 1} \left( \frac{N}{h} \right)^{\theta - 1} \|P_N ((-\Delta)^{1/2} f)\|_{L_h^2}
\lesssim \left( \frac{R}{h} \right)^\theta + \left( \frac{R}{h} \right)^{\theta - 1} \|f\|_{H_h^1} \sim \|f\|_{H_h^1}.
\]
Similarly, if $\|f\|_{H_h^1} \geq h^{-1}$, then
\[
\|f\|_{L_h^q} \leq \sum_{N_s \leq N \leq 1} \|P_N f\|_{L_h^q} \leq \sum_{N_s \leq N \leq 1} \left( \frac{N}{h} \right)^\theta \|f\|_{L_h^q} \sim h^{-\theta} \leq \|f\|_{H_h^1}.
\]
\[\square\]

Appendix B. Well-posedness results for NLS on the $T^2$

We consider the (periodic) NLS (1.1)
\[
i\partial_t u + \Delta u - \lambda |u|^{p-1} u = 0,
\]
\[u(0) = u_0 \in H^s(T^d).\]
Duhamel’s principle yields that (B.1) is equivalent to the following integral equation on $[-T, T]$
\[
u(t) = \eta_T(t) e^{-i t (-\Delta)} u_0 - i \lambda \eta_T(t) \int_0^t e^{-i(t-s)(-\Delta)}(|\eta_{2T}(t) u|^{p-1} \eta_{2T}(t) u)(s) ds,
\]
where $\eta$ is a smooth (even) bump function satisfying $\eta \equiv 1$ in $[-1, 1]$ and $\eta \equiv 0$ in $(-2, 2)^c$, and $\eta_T(t) = \eta(t/T)$. Note that one may replace $\eta_T(t)$ by $\eta(t)$ in (B.2) (with a smallness assumption) when $p < 1 + \frac{4}{d}$ (in the 2D case, $p < 3$), owing to the scaling argument.

For the classical well-posedness result of Bourgain [3] (see also [4]), we introduce the following function space. For $s, b \in \mathbb{R}$, we define the norm
\[
\|f\|_{X^{s,b}}^2 = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{2s} \langle \tau + |k|^2 \rangle^{2b} |\widetilde{f}(\tau, k)|^2 d\tau
\]
for $f \in \mathcal{S}(\mathbb{R} \times T^d)$, where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\widetilde{f}$ is the spacetime Fourier transform of $f$ given by
\[
\widetilde{f}(\tau, k) = \int_{\mathbb{R} \times T^d} f(t, x) e^{-ix \cdot k} e^{-i\tau t} dx dt.
\]
Then, the $X^{s,b}$ space is defined as the completion of $\mathcal{S}'(\mathbb{R} \times T^d)$ under the norm $\| \cdot \|_{X^{s,b}}$. This function space is termed the Bourgain space or the dispersive Sobolev space.
Theorem B.1 (GWP for 2D NLS [3]). Suppose that $d = 2$, and $p$ is given by (1.13). Then, NLS (B.1) is globally well-posed in $H^1(\mathbb{T}^2)$. Moreover, the solution $u$ obeys

$$\|u\|_{X^1,1/2} \lesssim \|u_0\|_{H^1}.$$  \hfill (B.3)

As a consequence, we have

$$\|u\|_{L^q_t,1(\mathbb{R}^d)} \lesssim \|u\|_{X^{s(q),b(q)}}$$  \hfill (B.4)

for $q \geq 4$, where $0 < \epsilon \ll 1$, $s(q,\epsilon) = \frac{4\epsilon}{q} + (1 + \frac{1-2\epsilon}{q-4})(1 - \frac{4}{q})$ and $b(q,\epsilon) = (\frac{1}{2} - \frac{\epsilon}{q})\frac{4}{q} + (\frac{1}{2} + \frac{1-2\epsilon}{4(q-4)})(1 - \frac{4}{q})$. In particular,

$$\|u\|_{L^4_t,1(\mathbb{R}^2)} \lesssim \|u\|_{X^{0,\frac{1}{2}}}.$$  \hfill (B.5)

Remark B.2. (i) One can immediately check $s(q,\epsilon) < 1 - \frac{2}{q}$ and $b(q,\epsilon) < \frac{1}{2}$.

(ii) In the one-dimensional case, Bourgain [3] proved the $L^4_t,1(\mathbb{T})$ estimate

$$\|u\|_{L^4_t,1(\mathbb{T})} \lesssim \|u\|_{X^{0,\frac{1}{2}}}.$$  \hfill (B.6)

This is an improvement of the $L^4$ estimate for free solutions by Zygmund [26], namely,

$$\|e^{it\partial_x}u_0\|_{L^4_t,1(\mathbb{T})} \lesssim \|u_0\|_{L^2},$$

which implies by the transference principle that

$$\|u\|_{L^4_t,1(\mathbb{R}^1)} \lesssim \|u\|_{X^{0,b}}, \quad b > \frac{1}{2}.$$  \hfill (B.7)

(iii) Bourgain employed a time-periodic function to show (B.6); however, such a restriction is not necessary (such as (B.5)), see, for instance, [23, 24].

Remark B.3. The $L^q$ estimate (B.4) follows from the interpolation between (B.5) and $\|u\|_{L^q_t,\infty(\mathbb{R}^2)} \lesssim \|u\|_{X^{1+\frac{1}{q+}}}. \frac{1}{q+}$. Together with the H"{o}lder inequality and the $L^4$ estimate (B.5), one has the (local-in-time) $L^q$ estimate for $1 \leq q < 4$, precisely,

$$\|u\|_{L^q_t,1([0,1] \times \mathbb{T}^2)} \lesssim \|u\|_{L^4_t,1(\mathbb{R}^2)} \lesssim \|u\|_{X^{0,\frac{1}{2}-\frac{1}{q}}}.$$  \hfill (B.8)

Remark B.4. The a priori bound (B.3) can be obtained by the standard iteration method in addition to the $L^q$ estimate (B.4).

As a corollary, we obtain a time-averaged bound.

Corollary B.5 (Time-averaged $L^\infty$ bound for 2D NLS). Suppose that $d = 2$, and $p$ is given by (1.13). Suppose that $u(t)$ is the global solution to periodic NLS (B.1) with initial data $u_0$, constructed in Theorem B.1. Then,

$$\|u\|_{L^{q_*}([-T,T];L^\infty(\mathbb{T}^2))} \lesssim \langle T \rangle^\frac{1}{q_*},$$

where $q_* > \max(p - 1, 2)$.

Proof. The proof follows from an analogous argument in the proof of Proposition 4.2. □
References

[1] V. Borovyk and M. Goldberg, *The Klein-Gordon equation on $\mathbb{Z}^2$ and the quantum harmonic lattice*, J. Math. Pures Appl. (9) **107** (2017), no. 6, 667–696.

[2] Á. Bényi and T. Oh, *The Sobolev inequality on the torus revisited*, Publ. Math. Debrecen **83** (2013), no. 3, 359–374.

[3] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), no. 2, 107–156.

[4] ______, *Global solutions of nonlinear Schrödinger equations*, American Mathematical Society Colloquium Publications, vol. 46, American Mathematical Society, Providence, RI, 1999.

[5] J. Bourgain and C. Demeter, *The proof of the $l^2$ Decoupling Conjecture*, Ann. of Math. **182** (2015), 351–389.

[6] N. Burq, P. Gérard, and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. **126** (2004), no. 3, 569–605.

[7] X. Chen and J. Holmer, *The derivation of the $T^3$ energy-critical NLS from quantum many-body dynamics*, to appear in Invent. Math.

[8] P. Gressman, V. Sohinger, and G. Staffilani, *On the uniqueness of solutions to the periodic 3D Gross-Pitaevskii hierarchy*, J. Funct. Anal. **266** (2014), no. 7, 4705–4764.

[9] Z. Guo, T. Oh and Y. Wang, *Strichartz estimates for Schrödinger equations on irrational tori*, Proc. London Math. Soc. **109** (2014), 975–1013.

[10] Y. Hong and C. Yang, *Uniform Strichartz estimates on the lattice*, to appear in Discrete Contin. Dyn. Syst.

[11] ______, *Strong Convergence for Discrete Nonlinear Schrödinger equations in the Continuum Limit*, to appear in SIAM J. Math. Anal.

[12] L. Ignat, *Fully discrete schemes for the Schrödinger equation. Dispersive properties*, Math. Models Methods Appl. Sci. **17** (2007), no. 4, 567–591.

[13] L. Ignat and E. Zuazua, *Dispersive properties of a viscous numerical scheme for the Schrödinger equation*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 7, 529–534.

[14] ______, *A two-grid approximation scheme for nonlinear Schrödinger equations: dispersive properties and convergence*, C. R. Math. Acad. Sci. Paris **341** (2005), no. 6, 381–386.

[15] ______, *Numerical dispersive schemes for the nonlinear Schrödinger equation*, SIAM J. Numer. Anal. **47** (2009), no. 2, 1366–1390.

[16] ______, *Convergence rates for dispersive approximation schemes to nonlinear Schrödinger equations*, J. Math. Pures Appl. (9) **98** (2012), no. 5, 479–517.

[17] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980.

[18] K. Kirkpatrick, E. Lenzmann and G. Staffilani, *On the continuum limit for discrete NLS with long-range lattice interactions*, Comm. Math. Phys. **317** (2013), no. 3, 563–591.

[19] K. Kirkpatrick, B. Schlein and G. Staffilani, *Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum dynamics*, Amer. J. Math. **133** (2011), no. 1, 91–130.

[20] P. Schultz, *The wave equation on the lattice in two and three dimensions*, Comm. Pure Appl. Math. **51** (1998), no. 6, 663–695.

[21] V. Sohinger, *A rigorous derivation of the defocusing cubic nonlinear Schrödinger equation on $\mathbb{T}^3$ from the dynamics of many-body quantum systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), no. 6, 1337–1365.

[22] A. Stefanov and P. G. Kevrekidis, *Asymptotic behaviour of small solutions for the discrete nonlinear Schrödinger and Klein-Gordon equations*, Nonlinearity **18** (2005), no. 4, 1841–1857.
[23] T. Tao, *Multilinear weighted convolution of $L^2$ functions and applications to nonlinear dispersive equations*, Amer. J. Math. 123(5) (2001) 839–908

[24] N. Tzvetkov, *Invariant measures for the nonlinear Schrödinger equation on the disc*, Dyn. Partial Differ. Equ. 3 (2006), no. 2, 111–160

[25] L. Vega, *Restriction theorems and the Schrödinger multiplier on the torus*, Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990), IMA Vol. Math. Appl., vol. 42, Springer, New York, 1992, pp. 199–211.

[26] A. Zygmund, *On Fourier coefficients and transforms of functions of two variables*, Stud. Math. 50 (1974) 189–201.

[27] A. Zygmund, *Trigonometric series. Vol. I, II*, third ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002, With a foreword by Robert A. Fefferman.

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