On a relation of bicomplex pseudoanalytic function theory to the complexified stationary Schrödinger equation

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Abstract

Using three different representations of the bicomplex numbers $\mathbb{T} \cong \mathbb{Cl}_c(1,0) \cong \mathbb{Cl}_c(0,1)$, which is a commutative ring with zero divisors defined by $\mathbb{T} = \{w_0 + w_1i_1 + w_2i_2 + w_3j \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}$ where $i_1^2 = -1$, $i_2^2 = -1$, $j^2 = 1$ and $i_1i_2 = j = i_2i_1$, we construct three classes of bicomplex pseudoanalytic functions. In particular, we obtain some specific systems of Vekua equations of two complex variables and we established some connections between one of these systems and the classical Vekua equations. We consider also the complexification of the real stationary two-dimensional Schrödinger equation. With the aid of any of its particular solutions, we construct a specific bicomplex Vekua equation possessing the following special property. The scalar parts of its solutions are solutions of the original complexified Schrödinger equation and the vectorial parts are solutions of another complexified Schrödinger equation.

Keywords: Bicomplex Numbers, Hyperbolic Numbers, Complex Clifford Algebras, Pseudoanalytic Functions, Second Order Elliptic Operator, Two-dimensional Stationary Schrödinger Equation.

∗Research supported by CRSNG(Canada) and FQRNT(Québec)
1 Introduction

The pseudoanalytic function theory was independently developed by two prominent mathematicians, I.N. Vekua (see [27]) and L. Bers (see [1, 4, 5]). Historically, the theory became one of the important impulses for developing the general theory of elliptic systems. More recently, it has been established by V.V. Kravchenko (see [12, 14]) that with the aid of any particular solutions of the real stationary two-dimensional Schrödinger equation we can construct a Vekua equation possessing the following special property. The real parts of its solutions are solutions of the original Schrödinger equation and the imaginary parts are solution of an associated Schrödinger equation with a potential having the form of a potential obtained after a Darboux transform. Using Bers’s theory of Taylor series for pseudoanalytic function, the author obtain a locally complete system of solutions of the original Schrödinger equation which can be constructed explicitly for an ample class of Schrödinger equation. Subsequently, V.V. Kravchenko (see [24]) gives a generalization of the factorization technics developed in [12] for the more general two-dimensional elliptic operator $(\text{div} \, p \, \text{grad} + q)u = 0,$ and consider the case where $p$ and $q$ are complex functions. In particular, the author had to consider a bicomplex Vekua equation of a special form to be able to obtain the following more general property. The scalar parts of the bicomplex Vekua equation considered are solutions of the original Schrödinger equation with a complex-valued potential. However, the case using the complex functions is more complicated and the author let as an open question the proof of expansion and convergence theorems for the bicomplex Vekua equation considered.

In this article, using three different representations of the bicomplex numbers (see, e.g. [7, 18, 19, 20, 22, 23, 24]), which is a commutative ring with zero divisors defined by\[ T = \{w_0 + w_1 i_1 + w_2 i_2 + w_3 j \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\} \]where $i_1^2 = -1, i_2^2 = -1, j^2 = 1$ and $i_1 i_2 = j = i_2 i_1,$ we construct three classes of bicomplex pseudoanalytic functions. For every class we obtain a special type of bicomplex Vekua equation of two complex variables, of which the one considered by A. Castaneda and V.V. Kravchenko (see [6, 13]) when the domain is restricted to the complex (in $i_2$) plane. Moreover, we established some connections between one of these systems of bicomplex Vekua equation and the classical Vekua equations. We consider also the complexification of the real stationary two-dimensional Schrödinger equation:\[ (\Delta_C - \nu(z_1, z_2)) f = 0 \]where $\omega = z_1 + z_2 i_2 \in T$ with $z_1, z_2 \in \mathbb{C}(i_1)$ and $\Delta_C$ is the complexified Laplacian operator i.e. $\Delta_C = \partial^2_{z_1} + \partial^2_{z_2}.$ With the aid of any of its particular solutions $f_0$ and the bicomplex operators:\[ \partial_{\omega} = \frac{1}{2} (\partial_{z_1} + i_2 \partial_{z_2}) \quad \text{and} \quad \partial_{\omega} = \frac{1}{2} (\partial_{z_1} - i_2 \partial_{z_2}), \]
we obtain the following factorization of the complexified Schrödinger equation

$$(\Delta C - \nu)\varphi = 4 \left( \partial_{\omega^1} + \frac{\partial_{\omega} f_0}{f_0} C \right) \left( \partial_{\omega} - \frac{\partial_{\omega} f_0}{f_0} C \right) \varphi$$

where $C$ denote the $\dagger_2$-bicomplex conjugation operator, and we consider a specific bicomplex Vekua equation

$$\left( \partial_{\omega^1} - \frac{\partial_{\omega^1} f_0}{f_0} C \right) W = 0$$

possessing the following special property. The scalar parts of its solutions are solutions of the original complexified Schrödinger equation and the vectorial parts are solutions of another complexified Schrödinger equation with the following potential $-\nu(z_1, z_2) + \frac{2|\nabla C f_0|^2}{f_0^2}$ where $|\omega|_{i_1}^2 = \omega \omega^1 \forall \omega \in \mathbb{T}$ and $\nabla C$ is the complexified gradient operator i.e. $\nabla C = \partial_{z_1} + i_2 \partial_{z_2}$.

Finally, from the fact that the complexified Schrödinger equation contains the stationary two-dimensional Schrödinger equation

$$(\Delta - \nu(x,p)) f = 0$$

and the Klein-Gordon equation

$$(\Box - \nu(x,q)) f = 0,$$

we show that our factorization of the complexified Schrödinger equation is a generalization of the factorization obtained in [12] for the stationary two-dimensional Schrödinger equation and for the factorization obtained in [15] for the Klein-Gordon equation.

2 Preliminaries

2.1 Bicomplex Numbers

Bicomplex numbers are defined as

$$\mathbb{T} := \{ z_1 + z_2 i_2 \mid z_1, z_2 \in \mathbb{C}(i_1) \}$$

(2.1)

where the imaginary units $i_1, i_2$ and $j$ are governed by the rules: $i_1^2 = i_2^2 = -1, \ j^2 = 1$ and

$$
i_1 i_2 = i_2 i_1 = j, \quad i_1 j = ji_1 = -i_2, \quad i_2 j = ji_2 = -i_1. \tag{2.2}$$

Note that we define $\mathbb{C}(i_k) := \{ x + y i_k \mid i_k^2 = -1 \ \text{and} \ x, y \in \mathbb{R} \}$ for $k = 1, 2$. Hence, it is easy to see that the multiplication of two bicomplex numbers is commutative. In fact, the bicomplex numbers

$$\mathbb{T} \cong \mathbb{C}(1,0) \cong \mathbb{C}(0,1)$$

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are *unique* among the complex Clifford algebras in that they are commutative but not division algebras. It is also convenient to write the set of bicomplex numbers as

\[ T := \{ w_0 + w_1 i_1 + w_2 i_2 + w_3 j | w_0, w_1, w_2, w_3 \in \mathbb{R} \}. \quad (2.3) \]

In particular, in equation (2.1), if we put \( z_1 = x \) and \( z_2 = y i_1 \) with \( x, y \in \mathbb{R} \), then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers (see, e.g. [22, 26]):

\[ D := \{ x + y j | j^2 = 1, \ x, y \in \mathbb{R} \} \cong \text{Cl}_{\mathbb{R}}(0, 1). \]

Complex conjugation plays an important role both for algebraic and geometric properties of \( \mathbb{C} \). For bicomplex numbers, there are three possible conjugations. Let \( w \in T \) and \( z_1, z_2 \in \mathbb{C}(i_1) \) such that \( w = z_1 + z_2 i_2 \). Then we define the three conjugations as:

\[ w^{\dagger_1} = (z_1 + z_2 i_2)^{\dagger_1} := \overline{z}_1 + \overline{z}_2 i_2, \quad (2.4a) \]
\[ w^{\dagger_2} = (z_1 + z_2 i_2)^{\dagger_2} := z_1 - z_2 i_2, \quad (2.4b) \]
\[ w^{\dagger_3} = (z_1 + z_2 i_2)^{\dagger_3} := \overline{z}_1 - \overline{z}_2 i_2, \quad (2.4c) \]

where \( \overline{z}_k \) is the standard complex conjugate of complex numbers \( z_k \in \mathbb{C}(i_1) \). If we say that the bicomplex number \( w = z_1 + z_2 i_2 = w_0 + w_1 i_1 + w_2 i_2 + w_3 j \) has the “signature” \((+ + ++)\), then the conjugations of type 1, 2 or 3 of \( w \) have, respectively, the signatures \((+ − + −)\), \((+ + − −)\) and \((+ − − +)\). We can verify easily that the composition of the conjugates gives the four-dimensional abelian Klein group:

\[
\begin{array}{c|cccc}
  & 0 & \dagger_0 & \dagger_1 & \dagger_2 & \dagger_3 \\
\hline
0 & \dagger_0 & \dagger_0 & \dagger_1 & \dagger_2 & \dagger_3 \\
\dagger_0 & \dagger_0 & \dagger_1 & \dagger_2 & \dagger_3 \\
\dagger_1 & \dagger_1 & \dagger_0 & \dagger_3 & \dagger_2 \\
\dagger_2 & \dagger_2 & \dagger_3 & \dagger_0 & \dagger_1 \\
\dagger_3 & \dagger_3 & \dagger_2 & \dagger_1 & \dagger_0 \\
\end{array}
\quad (2.5)
\]

where \( w^{\dagger_0} := w \ \forall w \in T \).

The three kinds of conjugation all have some of the standard properties of conjugations, such as:

\[ (s + t)^{\dagger_k} = s^{\dagger_k} + t^{\dagger_k}, \quad (2.6) \]
\[ (s t)^{\dagger_k} = s^{\dagger_k}, \quad (2.7) \]
\[ (s \cdot t)^{\dagger_k} = s^{\dagger_k} \cdot t^{\dagger_k}, \quad (2.8) \]

for \( s, t \in T \) and \( k = 0, 1, 2, 3 \).
We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in \( \mathbb{R}^2 \). The analogs of this, for bicomplex numbers, are the following. Let \( z_1, z_2 \in \mathbb{C}(i_1) \) and \( w = z_1 + z_2i_2 \in \mathbb{T} \), then we have that [22]:

\[
|w|^2_{i_1} := w \cdot w^{i_1} = z_1^2 + z_2^2 \in \mathbb{C}(i_1), \quad (2.9a)
\]

\[
|w|^2_{i_2} := w \cdot w^{i_2} = (|z_1|^2 - |z_2|^2) + 2\text{Re}(z_1z_2)i_2 \in \mathbb{C}(i_2), \quad (2.9b)
\]

\[
|w|^2_{i_3} := w \cdot w^{i_3} = (|z_1|^2 + |z_2|^2) - 2\text{Im}(z_1z_2)j \in \mathbb{D}, \quad (2.9c)
\]

where the subscript of the square modulus refers to the subalgebra \( \mathbb{C}(i_1) \), \( \mathbb{C}(i_2) \) or \( \mathbb{D} \) of \( \mathbb{T} \) in which \( w \) is projected.

Note that for \( z_1, z_2 \in \mathbb{C}(i_1) \) and \( w = z_1 + z_2i_2 \in \mathbb{T} \), we can define the usual (Euclidean in \( \mathbb{R}^4 \)) norm of \( w \) as \( |w| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\text{Re}(|w|^2_{i_1})} \).

It is easy to verify that \( w \cdot \frac{w^{i_2}}{|w|^2_{i_1}} = 1 \). Hence, the inverse of \( w \) is given by

\[
w^{-1} = \frac{w^{i_2}}{|w|^2_{i_1}}. \quad (2.10)
\]

From this, we find that the set \( \mathcal{NC} \) of zero divisors of \( \mathbb{T} \), called the null-cone, is given by \( \{ z_1 + z_2i_2 \mid z_1^2 + z_2^2 = 0 \} \), which can be rewritten as

\[
\mathcal{NC} = \{ z(i_1 \pm i_2) \mid z \in \mathbb{C}(i_1) \}. \quad (2.11)
\]

It is also possible to define differentiability of a function at a point of \( \mathbb{T} \):

**Definition 1** Let \( U \) be an open set of \( \mathbb{T} \) and \( w_0 \in U \). Then, \( f : U \subseteq \mathbb{T} \rightarrow \mathbb{T} \) is said to be \( \mathbb{T} \)-differentiable at \( w_0 \) with derivative equal to \( f'(w_0) \in \mathbb{T} \) if

\[
\lim_{(w \rightarrow w_0 \text{ inv.})} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0).
\]

We also say that the function \( f \) is \( \mathbb{T} \)-holomorphic on an open set \( U \) if and only if \( f \) is \( \mathbb{T} \)-differentiable at each point of \( U \).

As we saw, a bicomplex number can be seen as an element of \( \mathbb{C}^2 \), so a function \( f(z_1 + z_2i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2 \) of \( \mathbb{T} \) can be seen as a mapping \( f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2)) \) of \( \mathbb{C}^2 \). Here we have a characterization of such mappings:

**Theorem 1** Let \( U \) be an open set and \( f : U \subseteq \mathbb{T} \rightarrow \mathbb{T} \) such that \( f \in \mathbb{C}^1(U) \). Let also \( f(z_1 + z_2i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2 \). Then \( f \) is \( \mathbb{T} \)-holomorphic on \( U \) if and only if:

- \( f_1 \) and \( f_2 \) are holomorphic in \( z_1 \) and \( z_2 \).
and,
\[ \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \text{ and } \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \text{ on } U. \]

Moreover, \( f' = \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2}i_2 \) and \( f'(w) \) is invertible if and only if \( \det f'(w) \neq 0 \).

This theorem can be obtained from results in [18] and [21]. Moreover, by the Hartogs theorem [25], it is possible to show that \( f \in C^4(U) \) can be dropped from the hypotheses. Hence, it is natural to define the corresponding class of mappings for \( \mathbb{C}^2 \):

**Definition 2** The class of \( \mathbb{T} \)-holomorphic mappings on a open set \( U \subseteq \mathbb{C}^2 \) is defined as follows:

\[ \text{TH}(U) := \{ f : U \subseteq \mathbb{C}^2 \rightarrow \mathbb{C}^2 \mid f \in H(U) \text{ and } \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \text{ on } U \}. \]

It is the subclass of holomorphic mappings of \( \mathbb{C}^2 \) satisfying the complexified Cauchy-Riemann equations.

We remark that \( f \in \text{TH}(U) \) in terms of \( \mathbb{C}^2 \) if and only if \( f \) is \( \mathbb{T} \)-differentiable on \( U \). It is also important to know that every bicomplex number \( z_1 + z_2i_2 \) has the following unique idempotent representation:

\[ z_1 + z_2i_2 = (z_1 - z_2i_1)e_1 + (z_1 + z_2i_1)e_2. \tag{2.12} \]

where \( e_1 = \frac{1+i}{2} \) and \( e_2 = \frac{1-i}{2} \).

This representation is very useful because: addition, multiplication and division can be done term-by-term. Also, an element will be non-invertible if and only if \( z_1 - z_2i_1 = 0 \) or \( z_1 + z_2i_1 = 0 \).

The notion of holomorphicity can also be seen with this kind of notation. For this we need to define the projections \( P_1, P_2 : \mathbb{T} \rightarrow \mathbb{C}(i_1) \) as \( P_1(z_1 + z_2i_2) = z_1 - z_2i_1 \) and \( P_2(z_1 + z_2i_2) = z_1 + z_2i_1 \). Also, we need the following definition:

**Definition 3** We say that \( X \subseteq \mathbb{T} \) is a \( \mathbb{T} \)-cartesian set determined by \( X_1 \) and \( X_2 \) if \( X = X_1 \times_c X_2 := \{ z_1 + z_2i_2 \in \mathbb{T} : z_1 + z_2i_2 = w_1e_1 + w_2e_2, (w_1, w_2) \in X_1 \times X_2 \} \).

In [11] it is shown that if \( X_1 \) and \( X_2 \) are domains of \( \mathbb{C}(i_1) \) then \( X_1 \times_c X_2 \) is also a domain of \( \mathbb{T} \). Now, it is possible to state the following striking theorems [18]:

**Theorem 2** If \( f_{x_1} : X_1 \rightarrow \mathbb{C}(i_1) \) and \( f_{x_2} : X_2 \rightarrow \mathbb{C}(i_1) \) are holomorphic functions of \( \mathbb{C}(i_1) \) on the domains \( X_1 \) and \( X_2 \) respectively, then the function \( f : X_1 \times_c X_2 \rightarrow \mathbb{T} \) defined as

\[ f(z_1 + z_2i_2) = f_{x_1}(z_1 - z_2i_1)e_1 + f_{x_2}(z_1 + z_2i_1)e_2 \quad \forall \ z_1 + z_2i_2 \in X_1 \times_c X_2 \]

is \( \mathbb{T} \)-holomorphic on the domain \( X_1 \times_c X_2 \) and

\[ f'(z_1 + z_2i_2) = f'_{x_1}(z_1 - z_2i_1)e_1 + f'_{x_2}(z_1 + z_2i_1)e_2 \quad \forall \ z_1 + z_2i_2 \in X_1 \times_c X_2. \]
Theorem 3 Let $X$ be a domain in $\mathbb{T}$, and let $f : X \rightarrow \mathbb{T}$ be a $\mathbb{T}$-holomorphic function on $X$. Then there exist holomorphic functions $f_{c1} : X_1 \rightarrow \mathbb{C}(i_1)$ and $f_{c2} : X_2 \rightarrow \mathbb{C}(i_1)$ with $X_1 = P_1(X)$ and $X_2 = P_2(X)$, such that:

$$f(z_1 + z_2i_2) = f_{c1}(z_1 - z_2i_1)e_1 + f_{c2}(z_1 - z_2i_1)e_2 \forall z_1 + z_2i_2 \in X.$$ 

We note here that $X_1$ and $X_2$ will also be domains of $\mathbb{C}(i_1)$.

3 Bicomplex Pseudoanalytic Functions

3.1 Elementary Bicomplex Derivative

We will first consider the variable $z = x + yi_1$, where $x$ and $y$ are real variables and the corresponding formal differential operators

$$\partial_x = \frac{1}{2} (\partial_x + i_1 \partial_y) \quad \text{and} \quad \partial_y = \frac{1}{2} (\partial_x - i_1 \partial_y).$$

Notation $f_\xi$ or $f_\zeta$ means the application of $\partial_\xi$ or $\partial_\zeta$ respectively to a bicomplex function $f(z) = u(z) + v(z)i_1 + r(z)i_2 + s(z)j$. The derivatives $f_x$, $f_\xi$ “exist” if and only if $f_x$ and $f_y$ do. Note that

$$f_x = \frac{1}{2} \{(u_x + v_y) + (v_x - u_y)j + (r_x + s_y)i_2 + (s_x - r_y)j\}$$

and

$$f_\xi = \frac{1}{2} \{(u_x - v_y) + (v_x + u_y)i_1 + (r_x - s_y)i_2 + (s_x + r_y)j\}$$

In view of these operators,

$$f_\zeta(z) = 0 \Leftrightarrow \partial_\zeta[u(z) + v(z)i_1] = 0 \quad \text{and} \quad \partial_\zeta[r(z) + s(z)i_2] = 0. \quad (3.1)$$

i.e. $u_x = v_y$, $v_x = -u_y$ and $r_x = s_y$, $s_x = -r_y$ at $z \in \mathbb{C}(i_1)$.

We will now consider the bicomplex variable $\omega = z_1 + z_2i_2$, where $z_1 = x_1 + yi_1$, $z_2 = x_2 + y_2i_1 \in \mathbb{C}(i_1)$ and the corresponding formal differential operators

$$\partial_\omega = \partial_{\omega_{12}} = \frac{1}{2} (\partial_{z_1} + i_2 \partial_{z_2}), \quad \partial_\omega = \partial_{\omega_{10}} = \frac{1}{2} (\partial_{z_1} - i_2 \partial_{z_2})$$

$$\partial_{\omega_{13}} = \frac{1}{2} (\partial_{z_1} + i_2 \partial_{z_2}) \quad \text{and} \quad \partial_{\omega_{13}} = \frac{1}{2} (\partial_{z_1} - i_2 \partial_{z_2}).$$

Notation $f_{\omega_{1k}}$ for $k = 0, 1, 2, 3$ means the application of $f_{\omega_{1k}}$ respectively to a bicomplex function $f(\omega) = u(\omega) + v(\omega)i_1 + r(\omega)i_2 + s(\omega)j$. The derivatives $f_{\omega_{1k}}$ “exist” for $k = 0, 1, 2, 3$ if and only if $f_{x_1}$ and $f_{y_1}$ exist for $l = 1, 2$. These bicomplex operators act on sums, products, etc. just as an ordinary derivative and we have the following result in the bicomplex function theory.
Lemma 1 Let \( f(z_1 + z_2 i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2 = u(z_1, z_2) + v(z_1, z_2)i_1 + r(z_1, z_2)i_2 + s(z_1, z_2)i_3 \) be a bicomplex function. If the derivative

\[
f'(\omega_0) = \lim_{(\omega - \omega_0) \rightarrow 0} \frac{f(\omega) - f(\omega_0)}{\omega - \omega_0} \tag{3.2}
\]

exists, then \( u_x, u_y, r_x, r_y, v_x, v_y, s_x \) and \( s_y \) exist, and

1. \( f_\omega(\omega_0) = f'(\omega_0) \) \tag{3.3}
2. \( f_{\omega_11}(\omega_0) = 0 \) \tag{3.4}
3. \( f_{\omega 12}(\omega_0) = 0 \) \tag{3.5}
4. \( f_{\omega 13}(\omega_0) = 0 \). \tag{3.6}

Moreover, if \( u_x, u_y, v_x, v_y, r_x, r_y, s_x \) and \( s_y \) exist, and are continuous in a neighborhood of \( \omega_0 \), and if \( (3.4), (3.5), \) and \( (3.6) \) hold, then \( (3.3) \) exists.

Proof. First, we remark that the complexified Cauchy-Riemann equations

\[
\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \quad \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \text{at} \ \omega_0 \tag{3.7}
\]

are equivalent to \( f_{\omega 12}(\omega_0) = 0 \). Similarly, the following system of equations

\[
\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \quad \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \text{at} \ \omega_0 \tag{3.8}
\]

is equivalent to \( f_{\omega 13}(\omega_0) = 0 \) and the following one

\[
\frac{\partial f_1}{\partial \bar{z}_1} = -\frac{\partial f_2}{\partial \bar{z}_2}, \quad \frac{\partial f_2}{\partial \bar{z}_1} = \frac{\partial f_1}{\partial \bar{z}_2} \quad \text{at} \ \omega_0 \tag{3.9}
\]

is equivalent to \( f_{\omega 11}(\omega_0) = 0 \). Now, if the bicomplex derivative \( f'(\omega_0) \) exists, then from the Theorem \[1\] we obtain automatically that all partial derivatives exist at \( \omega_0 \). Moreover, the fact that \( f_1 \) and \( f_2 \) are holomorphic in \( z_1 \) and \( z_2 \) imply that

\[
\frac{\partial f_k}{\partial z_1} = \frac{\partial f_k}{\partial \bar{z}_1} = 0 \quad \text{for} \ k = 1, 2. \tag{3.10}
\]

Therefore \( f_{\omega 11}(\omega_0) = f_{\omega 13}(\omega_0) = 0 \) and the complexified Cauchy-Riemann equations imply \( f_{\omega 12}(\omega_0) = 0 \) and \( f_{\omega}(\omega_0) = f'(\omega_0) \). Conversely, if \( u_x, u_y, r_x, r_y, v_x, v_y, s_x \) and \( s_y \) exist, and are continuous in a neighborhood of \( \omega_0 \) then \( f_{\omega 11}(\omega_0) = f_{\omega 13}(\omega_0) = 0 \) imply that \( f_1 \) and \( f_2 \) are holomorphic in \( z_1 \) and \( z_2 \). Hence, from Theorem \[1\] \( f \) is \( \mathbb{T} \)-differentiable at \( \omega_0 \). \( \square \)

3.2 Bicomplex Generalization of Function Theory

Our bicomplex generalization of function theory is based on the following three different representations of bicomplex numbers. The scalar and vectorial part must be adapted to each representations.
3.2.1 Class-R1

Let \( a + bi_1 + ci_2 + dj = z_1 + z_2i_2 \) where \( z_1, z_2 \in \mathbb{C}(i_1) \). In this case, the theory will be based on assigning the part played by 1 and \( i_2 \) to two essentially arbitrary bicomplex functions \( F(\omega) \) and \( G(\omega) \). We assume that these functions are defined and twice continuously differentiable in some open domain \( D_0 \subset \mathbb{T} \).

We require that

\[
\text{Vec}\{F(\omega)^{12}G(\omega)\} \neq 0. \quad (3.11)
\]

Under this condition, \( (F, G) \) will be called a \( i_1 \)-generating pair in \( D_0 \). We remark that \( \text{Vec}\{F(\omega)^{12}G(\omega)\} = \begin{bmatrix} \text{Sc}\{F(\omega)\} & \text{Sc}\{G(\omega)\} \\ \text{Vec}\{F(\omega)\} & \text{Vec}\{G(\omega)\} \end{bmatrix} \). It follows, from Cramer’s Theorem, that for every \( \omega_0 \) in \( D_0 \) we can find unique constants \( \lambda_0, \mu_0 \in \mathbb{C}(i_1) \) such that \( w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0) \). More generally we have the following result.

**Theorem 4** Let \( (F, G) \) be \( i_1 \)-generating pair in some open domain \( D_0 \). If \( w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T} \), then there exist unique functions \( \phi(\omega), \psi(\omega) : D_0 \subset \mathbb{T} \to \mathbb{C}(i_1) \) such that

\[
w(\omega) = \phi(\omega)F(\omega) + \psi(\omega)G(\omega) \quad \forall \omega \in D_0.
\]

Moreover, we have the following explicit formulas for \( \phi \) and \( \psi \):

\[
\phi(\omega) = \frac{\text{Vec}\{w(\omega)^{12}G(\omega)\}}{\text{Vec}\{F(\omega)^{12}G(\omega)\}}, \quad \psi(\omega) = \frac{\text{Vec}\{w(\omega)^{12}F(\omega)\}}{\text{Vec}\{F(\omega)^{12}G(\omega)\}}.
\]

**Proof.** Let \( (F, G) \) be \( i_1 \)-generating pair in some open domain \( D_0 \). Let \( z_0 \in D_0 \) with \( w(z_0) = z_1 + z_2i_2, \ F(z_0) = z_3 + z_4i_2 \) and \( G(z_0) = z_5 + z_6i_2 \). In this case, \( w(z_0) = \phi_2(z_0)F(z_0) + \psi_2(z_0)G(z_0) \) with \( \phi_2(z_0), \psi_2(z_0) \in \mathbb{C}(i_1) \) if and only if \( z_1 = \phi_2(z_0)z_3 + \psi_2(z_0)z_5 \) and \( z_2 = \phi_2(z_0)z_4 + \psi_2(z_0)z_6 \). This is a well known

Cramer’s system of the form \( AX = B \) where \( A = \begin{pmatrix} z_3 & z_5 \\ z_4 & z_6 \end{pmatrix}, \ B = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \)

and \( X = \begin{pmatrix} \phi_2(z_0) \\ \psi_2(z_0) \end{pmatrix} \). So, the unique solution is \( X = A^{-1}B \) where \( A^{-1} = \begin{pmatrix} \frac{1}{\det A} & -z_5 \\ z_4 & \frac{1}{\det A} \end{pmatrix} \). Hence,

\[
X = \frac{1}{\text{Vec}\{F(z_0)^{12}G(z_0)\}} \begin{pmatrix} 1 \\ -z_4 \end{pmatrix} \begin{pmatrix} z_6 & -z_5 \\ -z_4 & z_3 \end{pmatrix} = \frac{1}{\text{Vec}\{F(z_0)^{12}G(z_0)\}} \begin{pmatrix} \frac{1}{\text{Vec}\{F(z_0)^{12}G(z_0)\}} & \text{Vec}\{w(z_0)^{12}G(z_0)\} \\ \text{Vec}\{w(z_0)^{12}F(z_0)\} & -\text{Vec}\{w(z_0)^{12}F(z_0)\} \end{pmatrix}.
\]

Then

\[
\phi_2(z) = \frac{\text{Vec}\{w(z)^{12}G(z)\}}{\text{Vec}\{F(z)^{12}G(z)\}}, \quad \psi_2(z) = -\frac{\text{Vec}\{w(z)^{12}F(z)\}}{\text{Vec}\{F(z)^{12}G(z)\}} \quad \forall z \in D_0. \square
\]
Now, we say that \( w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T} \) possesses at \( \omega_0 \) the \((F,G)\)-derivative \( \dot{w}(\omega_0) \) if the (finite) limit
\[
\dot{w}(\omega_0) = \lim_{\omega \to \omega_0, \omega \to \omega_0 \text{ inv.}} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \tag{3.12}
\]
exists.

In the particular case where \( w(\omega), F(\omega) \) and \( G(\omega) \) are defined on \( D_0 \subset \mathbb{C}(i_2) \to \mathbb{C}(i_2) \) then we can find unique constants \( \lambda_0, \mu_0 \in \mathbb{R} \) such that \( w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0) \) and we come back to the classical (in \( i_2 \)) pseudoanalytic theory developed by L. Bers and I.N. Vekua (see e.g. \[1, 4, 5, 27\]). In that case, using Bers’s theory of Taylor series for pseudoanalytic function, V.V. Kravchenko (see \[12\]) obtained a locally complete system of solutions of the real stationary two-dimensional Schrödinger equation. On the other hand, in the case where \( w(\omega), F(\omega) \) and \( G(\omega) \) are defined on \( D_0 \subset \mathbb{C}(j) \to \mathbb{C}(j) \) then we can also find unique constants \( \lambda_0, \mu_0 \in \mathbb{R} \) such that \( w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0) \) and we are in the hyperbolic pseudoanalytic theory developed by Guo Chun Wen in \[28\]. Moreover, we note that if we only restrict the domain \( D_0 \) to \( \mathbb{C}(i_2) \), the subclass of bicomplex pseudoanalytic functions obtained is precisely the class developed by V.V. Kravchenko and A. Castaño in \[3\] to show that in a two-dimensional situation the Dirac equation with a scalar and an electromagnetic potentials decouples into a pair of bicomplex equations. It is also the same class of functions that used V.V. Kravchenko (see \[13\]) to obtain solutions of the complex stationary two-dimensional Schrödinger equation.

The following expressions are called the \( i_k \)-characteristic coefficients of the pair \((F,G)\) for \( k = 1, 2, 3:\)
\[
\lambda_{(F,G)}^{(k)} = \frac{F^i_1 G^{i_2 k} - F^i_2 G^{i_1 k}}{F^i_1 i_2 - F^i_2 i_1 G}, \quad \beta_{(F,G)}^{(k)} = \frac{F G^{i_2 k} - F^{i_2} G^{i_1 k}}{F G^i_1 - F^{i_1} G^i_2},
\]

\[
A_{(F,G)} = -\frac{F^i_2 G_\omega - F_\omega G^i_2}{F G^i_1 - F^{i_1} G^i_2}, \quad B_{(F,G)} = \frac{F G_\omega - F_\omega G}{F G^i_1 - F^{i_1} G^i_2}.
\]

Set (for a fixed \( \omega_0 \))
\[
W(\omega) = w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega), \tag{3.13}
\]

the constants \( \lambda_0, \mu_0 \in \mathbb{C}(i_1) \) being uniquely determined by the condition
\[
W(\omega_0) = 0. \tag{3.14}
\]

Hence, \( W(\omega) \) has continuous partial derivatives if and only if \( w(\omega) \) has. Moreover, \( \dot{w}(\omega_0) \) exists if and only if \( W'(\omega_0) \) does, and if it does exist, then \( \dot{w}(\omega_0) = W'(\omega_0) \). Therefore, by Lemma \[1\] the existence of \( W_\omega(\omega_0), W_\omega(\omega_0) \) and equations
\[
W_{\omega^1} (\omega_0) = 0, W_{\omega^2} (\omega_0) = 0 \quad \text{and} \quad W_{\omega^3} (\omega_0) = 0 \tag{3.15}
\]
are necessary for the existence of (3.12), and the existence and continuity of $W_\omega(\omega_0)$, $W_\omega(\omega_0)$ in a neighborhood of $\omega_0$, together with (3.15) are sufficient. Now,

$$W(\omega) = \begin{pmatrix} w(\omega) & w(\omega_0) & w(\omega_0)^{t2} \\ F(\omega) & F(\omega_0) & F(\omega_0)^{t2} \\ G(\omega) & G(\omega_0) & G(\omega_0)^{t2} \end{pmatrix}$$

so that (3.15) may be written in the form

$$\begin{vmatrix} w_{,1}^k(\omega_0) & w(\omega_0) & w(\omega_0)^{t2} \\ F_{,1}^k(\omega_0) & F(\omega_0) & F(\omega_0)^{t2} \\ G_{,1}^k(\omega_0) & G(\omega_0) & G(\omega_0)^{t2} \end{vmatrix} = 0 \quad \text{for } k = 1, 2, 3 \quad (3.17)$$

and if (3.12) exists, then

$$\dot{w}(\omega_0) = \begin{pmatrix} w(\omega_0) & w(\omega_0) & w(\omega_0)^{t2} \\ F(\omega_0) & F(\omega_0) & F(\omega_0)^{t2} \\ G(\omega_0) & G(\omega_0) & G(\omega_0)^{t2} \end{vmatrix}.$$  

Equations (3.18) and (3.17) can be rewritten in the form

$$\dot{w} = w_\omega - A_{(F,G)}w - B_{(F,G)}w^{t2} \quad (3.19)$$

$$w_{,1}^k = a_{(F,G)}^{(k)}w + b_{(F,G)}^{(k)}w^{t2} \quad \text{for } k = 1, 2, 3. \quad (3.20)$$

Thus we have proved the following result.

**Theorem 5** Let $(F,G)$ be a $1_1$-generating pair in some open domain $D_0$. Every bicomplex function $w$ defined in $D_0$ admits the unique representation $w = \phi F + \psi G$ where $\phi, \psi : D_0 \subset \mathbb{T} \rightarrow \mathbb{C}(1_1)$. Moreover, the $(F,G)_{1_1}$-derivative $\dot{w} = \frac{d_{(F,G)_{1_1}}}{d\omega} w(\omega)$ exists at $\omega_0$ and has the form

$$\dot{w} = \phi_{,\omega}F + \psi_{,\omega}G = w_\omega - A_{(F,G)}w - B_{(F,G)}w^{t2} \quad (3.21)$$

if and only if

$$w_{,1}^1 = a_{(F,G)}^{(1)}w + b_{(F,G)}^{(1)}w^{t2}, \quad (3.22)$$

$$w_{,1}^2 = a_{(F,G)}^{(2)}w + b_{(F,G)}^{(2)}w^{t2}, \quad (3.23)$$

and

$$w_{,1}^3 = a_{(F,G)}^{(3)}w + b_{(F,G)}^{(3)}w^{t2} \quad (3.24)$$

where $w$ has continuous partial derivatives in a neighborhood of $\omega_0$.  

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The equations (3.22), (3.23) and (3.24) are called the $i_1$-bicomplex Vekua equations and the solutions of these equations will be the $(F,G)_{i_1}$-pseudoanalytic functions.

**Remark 1** For $k = 1, 2, 3$, the equation

$$w_{\omega^{1k}} = a^{(k)}_{(F,G)} w + b^{(k)}_{(F,G)} w^{12}$$

(3.25)

can be rewritten in the following form

$$\phi_{\omega^{1k}} F + \psi_{\omega^{1k}} G = a^{(k)}_{(F,G)} w + \frac{F^{12}G_{\omega^{1k}} - F_{\omega^{1k}} G^{12}}{FG^{12} - F^{12}G} w.$$  (3.26)

Hence, the equation (3.25) is equivalent to

$$\phi_{\omega^{1k}} F + \psi_{\omega^{1k}} G = 0$$

if and only if

$$[G^{1k} - G^{12}]F_{\omega^{1k}} = [F^{1k} - F^{12}]G_{\omega^{1k}}$$  (3.27)

where $w$ is not identically in the null-cone on the domain.

### 3.2.2 Class-R2

Let $a + bi_1 + ci_2 + dj = z_1 + z_2 i_1$ where $z_1, z_2 \in \mathbb{C}(i_2)$. In this case, the theory will be based on assigning the part played by $1$ and $i_1$ to two essentially arbitrary bicomplex functions $F(\omega)$ and $G(\omega)$. We assume that these functions are defined and twice continuously differentiable in some open domain $D_0 \subset \mathbb{T}$. We require that

$$\text{Vec}(F(\omega)^{11}G(\omega)) \neq 0.$$  (3.28)

Under this condition, $(F,G)$ will be called a $i_2$-generating pair in $D_0$. From Cramer’s Theorem, it follows that for every $\omega_0$ in $D_0$ we can find unique constants $\lambda_0, \mu_0 \in \mathbb{C}(i_2)$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$. In fact, using the same arguments than for the Theorem 4, we have the following result.

**Theorem 6** Let $(F,G)$ be $i_2$-generating pair in some open domain $D_0$. If $w(\omega) : D_0 \subset \mathbb{T} \rightarrow \mathbb{T}$, then there exist unique functions $\phi(\omega), \psi(\omega) : D_0 \subset \mathbb{T} \rightarrow \mathbb{C}(i_2)$ such that

$$w(\omega) = \phi(\omega) F(\omega) + \psi(\omega) G(\omega) \quad \forall \omega \in D_0.$$  

Moreover, we have the following explicit formulas for $\phi$ and $\psi$:

$$\phi(\omega) = \frac{\text{Vec}(w(\omega)^{11}G(\omega))}{\text{Vec}(F(\omega)^{11}G(\omega))}, \quad \psi(\omega) = -\frac{\text{Vec}(w(\omega)^{11}F(\omega))}{\text{Vec}(F(\omega)^{11}G(\omega))}.$$  

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We say that \( w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T} \) possesses at \( \omega_0 \) the \((F,G)_{i_2}\)-derivative \( \dot{w}(\omega_0) \) if the (finite) limit

\[
\dot{w}(\omega_0) = \lim_{\omega \to \omega_0} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0}
\] (3.29)

equals. In fact, if we interchange everywhere \( i_1 \) with \( i_2 \), this case is exactly the same than \( R1 \). In particular, if we defined the function \( \pi : \mathbb{T} \to \mathbb{T} \) as

\[
\pi(a + b i_1 + c i_2 + d j) := a + c i_1 + b i_2 + d j
\] (3.30)
we obtain that \( w(\omega) \) possesses a \((F,G)_{i_1}\)-derivative at \( \omega_0 \in D_0 \) if and only if the function

\[
(\pi \circ w \circ \pi)(\omega)
\] (3.31)
possesses a \((\pi \circ F \circ \pi, \pi \circ G \circ \pi)_{i_2}\)-derivative at \( \pi(\omega_0) \in \pi(D_0) \) where

\[
(\pi \circ F \circ \pi, \pi \circ G \circ \pi)
\] (3.32)
is a \( i_2 \)-generating pair on \( \pi(D_0) \). We note that

\[
\begin{align*}
\pi(\pi(\omega)) & = \pi(\omega), \\
\pi(\omega_1 + \omega_2) & = \pi(\omega_1) + \pi(\omega_2), \\
\pi(\omega_1 \omega_2) & = \pi(\omega_1)\pi(\omega_2), \\
\pi\left(\frac{\omega_1}{\omega_2}\right) & = \frac{\pi(\omega_1)}{\pi(\omega_2)} \quad \text{if} \ \omega_2 \notin NC, \\
\pi(\omega)^{i_1} & = (\pi(\omega))^{i_1}, \\
\pi(\omega)^{i_2} & = (\pi(\omega))^{i_2}
\end{align*}
\] (3.33-3.37)

Therefore,

\[
\frac{d_{(F,G)_{i_2}}(\pi \circ w \circ \pi)}{d\omega} \bigg|_{\omega = \pi(\omega_0)}
\] (3.29)

\[
= \lim_{\omega \to \pi(\omega_0)} \frac{(\pi \circ w \circ \pi)(\omega) - \pi(\omega_0)(\pi \circ F \circ \pi)(\omega) - \pi(\mu_0)(\pi \circ G \circ \pi)(\omega)}{\omega - \pi(\omega_0)}
\]

\[
= \lim_{\omega \to \pi(\omega_0)} \frac{\pi[w(\omega) - \lambda_0 F(\pi(\omega)) - \mu_0 G(\pi(\omega))]}{\pi[\pi(\omega) - \omega_0]}
\]

\[
= \pi \left[ \lim_{\omega \to \pi(\omega_0)} \frac{w(\pi(\omega)) - \lambda_0 F(\pi(\omega)) - \mu_0 G(\pi(\omega))}{\pi(\omega) - \omega_0} \right]
\]

\[
= \pi \left[ \frac{d_{(F,G)_{i_2}} w}{d\omega} \bigg|_{\omega = \pi(\omega_0)} \right].
\]

We note that the \( i_2 \)-characteristic coefficients of the pair \((F,G)\) for \( k = 1, 2, 3 \) must be defined as:

\[
\begin{align*}
a_{(F,G)}^{(k)} & = \frac{FG^{i_1}G^{i_2} - FG^{i_1}i_2}{FG^{i_1} - F^{i_1}G}, \\
b_{(F,G)}^{(k)} & = \frac{FG^{i_1} - FG^{i_2}G}{FG^{i_1} - F^{i_1}G},
\end{align*}
\]
\[ A_{(F,G)} = \frac{F_{\xi^1}G_\omega - F_\omega G_{\xi^1}}{FG_{\xi^1} - F_{\xi^1}G}, \quad B_{(F,G)} = \frac{FG_\omega - F_\omega G}{FG_{\xi^1} - F_{\xi^1}G} \]

### 3.2.3 Class-R3

Let \( a + bi_1 + ci_2 + dj = z_1 + z_2i_1 \) (resp. \( z_1, z_2, z_3 \in \mathbb{C} \)). In this case, the theory will be based on assigning the part played by \( 1 \) and \( i_1 \) (resp. \( i_2 \)) to two essentially arbitrary bicomplex functions \( F(z) \) and \( G(z) \). We assume that these functions are defined and twice continuously differentiable in some open domain \( D_0 \subset \mathbb{T} \). We require that

\[ \text{Vec}\{F(\omega)^{\xi^3}G(\omega)\} \neq 0. \quad (3.39) \]

Under this condition, \((F,G)\) will be called a \( j \)-generating pair in \( D_0 \) for every \( \omega_0 \) in \( D_0 \). Moreover, it will be possible to find unique constants \( \lambda_0, \mu_0 \in \mathbb{C} \) such that \( w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0) \). Here also we have the following equivalence of the Theorem.

**Theorem 7** Let \((F,G)\) be \( j \)-generating pair in some open domain \( D_0 \). If \( w(\omega) : D_0 \subset \mathbb{T} \rightarrow \mathbb{T}, \) then there exist unique functions \( \phi(\omega), \psi(\omega) : D_0 \subset \mathbb{T} \rightarrow \mathbb{C} \) such that

\[ w(\omega) = \phi(\omega)F(\omega) + \psi(\omega)G(\omega) \quad \forall \omega \in D_0. \]

Moreover, we have the following explicit formulas for \( \phi \) and \( \psi \):

\[ \phi(\omega) = \frac{\text{Vec}[w(\omega)^{\xi^3}G(\omega)]}{\text{Vec}[F(\omega)^{\xi^3}G(\omega)]}, \quad \psi(\omega) = \frac{\text{Vec}[w(\omega)^{\xi^3}F(\omega)]}{\text{Vec}[F(\omega)^{\xi^3}G(\omega)]}. \]

We say that \( w(\omega) \) possesses at \( \omega_0 \) the \((F,G)\)-derivative \( \tilde{w}(\omega_0) \) if the (finite) limit

\[ \tilde{w}(\omega_0) = \lim_{\omega \to \omega_0} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \quad (3.40) \]

exists.

In this case, the following expressions are called the \( j \)-characteristic coefficients of the pair \((F,G)\) for \( k = 1, 2, 3 \):

\[ a^{(k)}_{(F,G)} = \frac{F_{\xi^k}G_\omega - F_\omega G_{\xi^k}}{FG_{\xi^k} - F_{\xi^k}G}, \quad b^{(k)}_{(F,G)} = \frac{FG_{\omega^{\xi^k}} - F_\omega G_{\xi^k}}{FG_{\xi^k} - F_{\xi^k}G}, \]

\[ A_{(F,G)} = \frac{F_{\xi^3}G_\omega - F_\omega G_{\xi^3}}{FG_{\xi^3} - F_{\xi^3}G}, \quad B_{(F,G)} = \frac{FG_\omega - F_\omega G}{FG_{\xi^3} - F_{\xi^3}G}. \]

Now, using the same kind of arguments than for the case \( \textbf{R2} \), we obtain the following result.
Theorem 8 Let \((F,G)\) be a \(j\)-generating pair in some open domain \(D_0\). Every bicomplex function \(w\) defined in \(D_0\) admits the unique representation \(w = \phi F + \psi G\) where \(\phi, \psi : D_0 \subset \mathbb{T} \rightarrow \mathbb{C}(j)\). Moreover, the \((F,G)_j\)-derivative \(\dot{w}\) exists at \(\omega_0\) and has the form

\[
\dot{w} = \phi \omega F + \psi \omega G = w_\omega - A(F,G)w - B(F,G)w^\dagger_3
\]

if and only if

\[
w_\omega^\dagger_1 = a^{(1)}(F,G)w + b^{(1)}(F,G)w^\dagger_3, \tag{3.42}
w_\omega^\dagger_2 = a^{(2)}(F,G)w + b^{(2)}(F,G)w^\dagger_3, \tag{3.43}
\]

and

\[
w_\omega^\dagger_3 = a^{(3)}(F,G)w + b^{(3)}(F,G)w^\dagger_3 \tag{3.44}
\]

where \(w\) has continuous partial derivatives in a neighborhood of \(\omega_0\).

The equations (3.42), (3.43) and (3.44) are called the \(j\)-bicomplex Vekua equations and the solutions of these equations will be the \((F,G)_j\)-pseudoanalytic functions.

Remark 2 For \(k = 1, 2, 3\), the equation

\[
w_\omega^\dagger_k = a^{(k)}(F,G)w + b^{(k)}(F,G)w^\dagger_3 \tag{3.45}
\]

can be rewritten in the following form

\[
\dot{\phi}_\omega^\dagger_k F + \dot{\psi}_\omega^\dagger_k G = a^{(k)}(F,G)w + \frac{F^\dagger_3 G_\omega^\dagger_k - G^\dagger_3 w_\omega^\dagger_k}{F^\dagger_3 G - F G^\dagger_3} w. \tag{3.46}
\]

Hence, the equation (3.45) is equivalent to

\[
\dot{\phi}_\omega^\dagger_k F + \dot{\psi}_\omega^\dagger_k G = 0
\]

if and only if

\[
[G^\dagger_k - G^\dagger_3]F_\omega^\dagger_k = [F^\dagger_k - F^\dagger_3]G_\omega^\dagger_k \tag{3.47}
\]

where \(w\) is not identically in the null-cone on the domain.

In this case, it is useful to consider a more specific class of generating pair.

Definition 4 Let \(D_1\) and \(D_2\) be open in \(\mathbb{C}(i_1)\). Consider that \((F_{e_1}, G_{e_1})\) and \((F_{e_2}, G_{e_2})\) are complex (in \(i_1\)), twice continuously differentiable, generating pairs in respectively \(D_1\) and \(D_2\). Under these conditions, \((F,G)\) will be called a \(j^*\)-generating pair in \(D_0 = D_1 \times_e D_2 \subset \mathbb{T}\) where

\[
F(z_1 + z_2i_2) := F_{e_1}(z_1) + F_{e_2}(z_1 + z_2i_1) \tag{3.48}
\]

and

\[
G(z_1 + z_2i_2) := G_{e_1}(z_1) + G_{e_2}(z_1 + z_2i_1) \tag{3.49}
\]
Lemma 2 Let \( F(\omega) \) and \( G(\omega) \) two arbitrary bicomplex functions defined in some domain \( D_0 \subset \mathbb{T} \). If

\[
\text{Im}\{F_e(\omega)G_e(\omega)\} \neq 0 \text{ or } \text{Im}\{F_e (\omega)G_e(\omega)\} \neq 0 \quad \forall \omega \in D_0
\]

then Vec\{\( F(\omega)^1\)G(\( \omega) \)} \( \neq 0 \) \( \forall \omega \in D_0 \).

Proof. Let \( F(\omega_0)^1\)G(\( \omega_0) \) = \( x_1 + x_2i_1 + x_3i_2 + x_4) = (x_1 + x_4) + (x_2 - x_3)i_1 = (x_1 + x_4j) + (x_3 - x_2j)i_2 \). Thus, Vec\{\( F(\omega_0)^1\)G(\( \omega_0) \)} = 0 if and only if \( x_2 = x_3 = 0 \). Moreover, Im\{\( F_e(\omega_0)G_e(\omega_0)\} = x_3 - x_2 \) and Im\{\( F_e(\omega_0)G_e(\omega_0)\} = x_2 + x_3. Hence, Vec\{\( F(\omega_0)^1\)G(\( \omega_0) \)} = 0 imply Im\{\( F_e(\omega_0)G_e(\omega_0)\} = 0 \) and Im\{\( F_e(\omega_0)G_e(\omega_0)\} = 0 \forall \omega \in D_0. \Box

Therefore, from the Lemma 2 we obtain automatically this following result.

Theorem 9 Let \( D_0 = D_1 \times D_2 \) where \( D_1 \) and \( D_2 \) are open domains in \( \mathbb{C}(i_1) \). If \( (F, G) \) is a \( j^* \)-generating pair in \( D_0 \) then \( (F, G) \) is, in particular, a \( j \)-generating pair in \( D_0 \).

Moreover, the \( j^* \)-generating pair will imply the following representation for a \( (F, G)j^* \)-pseudoanalytic function.

Theorem 10 Let \( D_0 = D_1 \times D_2 \) where \( D_1 \) and \( D_2 \) are open domains in \( \mathbb{C}(i_1) \). If \( (F, G) \) is a \( j^* \)-generating pair in \( D_0 \) then if the function \( w \) is \( (F, G)j^* \)-pseudoanalytic on \( D_0 \) then \( w \) can be decomposed in the following way

\[
w(z_1 + z_2i_2) = [w_{e_1}(z_1 - z_2i_1)]e_1 + [w_{e_2}(z_1 + z_2i_1)]e_2
\]

(3.50)

\( \forall \) \( z_1 + z_2i_2 = (z_1) - z_2i_1)e_1 + (z_1 + z_2i_1)e_2 \in D_0 \).

Proof. It is always possible to decomposed \( w(z_1 + z_2i_2) \) in term of the idempotent representation i.e.

\[
w(z_1 + z_2i_2) = [w_{e_1}(z_1 + z_2i_2)]e_1 + [w_{e_2}(z_1 + z_2i_2)]e_2.
\]

Moreover, from the definition of the derivative, the function

\[
W(z_1 + z_2i_2) = w(z_1 + z_2i_2) - \lambda_0 F(z_1 + z_2i_2) - \mu_0 G(z_1 + z_2i_2)
\]

is \( T \)-differentiable at \( z_1 + z_2i_2 \). Now, using the Theorem 2 we have that

\[
W(z_1 + z_2i_2) = [W_{e_1}(z_1 - z_2i_1)]e_1 + [W_{e_2}(z_1 + z_2i_1)]e_2
\]

and from the definition of a \( j^* \)-generating pair we obtain that

\[
w(z_1 + z_2i_2) =\ W_{e_1}(z_1 - z_2i_1)e_1 + W_{e_2}(z_1 + z_2i_1)e_2
\]

\[+ P_1(\lambda_0) F_{e_1}(z_1 - z_2i_1)e_1 + P_2(\lambda_0) F_{e_2}(z_1 + z_2i_1)e_2
\]

\[+ P_1(\mu_0) G_{e_1}(z_1 - z_2i_1)e_1 + P_2(\mu_0) G_{e_2}(z_1 + z_2i_1)e_2
\]
\[ \begin{align*}
&= [W_{e_1}(z_1 - z_2i_1) + P_1(\lambda_0)F_{e_1}(z_1 - z_2i_1) + P_1(\mu_0)G_{e_1}(z_1 - z_2i_1)]e_1 \\
&+ [W_{e_2}(z_1 + z_2i_1) + P_2(\lambda_0)F_{e_2}(z_1 + z_2i_1) + P_2(\mu_0)G_{e_2}(z_1 + z_2i_1)]e_2 \\
&= [w_{e_1}(z_1 - z_2i_1)]e_1 + [w_{e_2}(z_1 + z_2i_1)]e_2,
\end{align*} \]

where \( P_1(\lambda_0), P_2(\lambda_0), P_1(\mu_0), P_2(\mu_0) \in \mathbb{R} \).

Now, using the last theorem and the decomposition:

\[
\frac{w(z_1 + z_2i_2) - \lambda_0F(z_1 + z_2i_2) - \mu_0G(z_1 + z_2i_2)}{(z_1 + z_2i_2) - (z_0,1 + z_0,2i_1)}
\]

\[
= \frac{w_{e_1}(z_1 - z_2i_1) - P_1(\lambda_0)F_{e_1}(z_1 - z_2i_1) - P_1(\mu_0)G_{e_1}(z_1 - z_2i_1)}{(z_1 - z_2i_1) - (z_0,1 - z_0,2i_1)}e_1
\]

\[
+ \frac{w_{e_2}(z_1 + z_2i_1) - P_2(\lambda_0)F_{e_2}(z_1 + z_2i_1) - P_2(\mu_0)G_{e_2}(z_1 + z_2i_1)}{(z_1 + z_2i_1) - (z_0,1 + z_0,2i_1)}e_2
\]

we obtain the following connections with the classical theory of pseudoanalytic functions.

**Theorem 11** Let \( D_0 = D_1 \times_e D_2 \) where \( D_1 \) and \( D_2 \) are open domains in \( \mathbb{C}(i_1) \). If \((F,G)\) is a \( j \)-generating pair in \( D_0 \) with \( w : D_0 \subset \mathbb{T} \to \mathbb{T} \) a \((F,G)\)-pseudoanalytic function on \( D_0 \) then

\[
w(z_1 + z_2i_2) = [w_{e_1}(z_1 - z_2i_1)]e_1 + [w_{e_2}(z_1 + z_2i_1)]e_2
\]

(3.51)

where \( w_{e_k} \) is a \((F_{e_k},G_{e_k})\)-pseudoanalytic function on \( D_k \) for \( k = 1,2 \). Moreover,

\[
\dot{w}(z_1 + z_2i_2) = [\dot{w}_{e_1}(z_1 - z_2i_1)]e_1 + [\dot{w}_{e_2}(z_1 + z_2i_1)]e_2
\]

(3.52)

on \( D_0 \).

**Theorem 12** If \( w_{e_k} : D_k \to \mathbb{C}(i_1) \) is a \((F_{e_k},G_{e_k})\)-pseudoanalytic function on the open domain \( D_k \) for \( k = 1,2 \) then the function \( w : D_1 \times_e D_2 \to \mathbb{T} \) defined as

\[
w(z_1 + z_2i_2) = w_{e_1}(z_1 - z_2i_1)e_1 + w_{e_2}(z_1 + z_2i_1)e_2 \quad \forall \ z_1 + z_2i_2 \in D_1 \times_e D_2
\]

is a \((F,G)\)-\( j \)-pseudoanalytic function on \( D_1 \times_e D_2 \) and

\[
\dot{w}(z_1 + z_2i_2) = \dot{w}_{e_1}(z_1 - z_2i_1)e_1 + \dot{w}_{e_2}(z_1 + z_2i_1)e_2
\]

\forall \ z_1 + z_2i_2 \in D_1 \times_e D_2.

The last theorem gives another interpretation of Theorems 11 and 12 in terms of Vekua equations.

**Theorem 13** If \((F_{e_1},G_{e_1})\) and \((F_{e_2},G_{e_2})\) are complex (in \( i_1 \)) generating pairs in respectively \( D_1 \) and \( D_2 \). Then \( w \) is a solution on \( D_0 = D_1 \times_e D_2 \) of the \( j \)-bicomplex Vekua equations with the \( j^* \)-generating pair \((F,G)\) if and only if \( w(z_1 + z_2i_2) = w_{e_1}(z_1 - z_2i_1)e_1 + w_{e_2}(z_1 + z_2i_1)e_2 \) where \( w_{e_k} \) is a solution on \( D_k \) of the complex (in \( i_1 \)) Vekua equation with the generating pair \((F_{e_k},G_{e_k})\) for \( k = 1,2 \).
3.3 The Complexified Schrödinger Equation

Consider the equation

$$(\Delta_C - \nu(z_1, z_2)) f = 0 \quad (3.53)$$

in $\Omega \subset \mathbb{R}^4$, where $\Delta_C = \partial^2_{z_1} + \partial^2_{z_2}$, $\nu$ and $f$ are complex (in $i_1$) valued functions. The equation (3.53) is simply the complexification of the two-dimensional stationary Schrödinger equation where $\Delta_C$ is the complex Laplacian (see [8, 9, 10, 16]).

3.3.1 The Complex Laplacian

First of all, we will write the complex Laplacian in a more explicit way to see that it contains in the same time the classical Laplacian operator and the wave operator.

**Lemma 3** Let $\omega = z_1 + z_2 i_2$, where $z_1, z_2 \in \mathbb{C}(i_1)$ then

$$\partial_{\omega} \partial_{\bar{\omega}} = \frac{1}{4} (\partial^2_{z_1} + \partial^2_{z_2}) = \frac{1}{4} \Delta_C$$

\forall f \in C^2(\Omega)$ where $\Omega$ is an open set in $\mathbb{R}^4$.

**Proof.** Let $\partial_{\omega} = \frac{1}{2} (\partial_{z_1} - i_2 \partial_{z_2})$ and $\partial_{\bar{\omega}} = \frac{1}{2} (\partial_{z_1} + i_2 \partial_{z_2})$ then

$$4\partial_{\omega} \partial_{\bar{\omega}} = \partial_{z_1} (\partial_{z_1} + i_2 \partial_{z_2}) - i_2 \partial_{z_2} (\partial_{z_1} + i_2 \partial_{z_2})$$

$$= \partial^2_{z_1} + i_2 \partial^2_{z_1 z_2} - i_2 \partial^2_{z_2 z_1} + \partial^2_{z_2}$$

$$= \partial^2_{z_1} + \partial^2_{z_2}. \square$$

**Proposition 1** Let $\partial_{z_1} = \frac{1}{2} (\partial_x - i_1 \partial_y)$ and $\partial_{z_2} = \frac{1}{2} (\partial_p - i_1 \partial_q)$ then

$$16\partial_{\omega} \partial_{\bar{\omega}} = 4\Delta_C = (\partial^2_x - \partial^2_y + \partial^2_p - \partial^2_q) - 2i_1 (\partial^2_{xy} + \partial^2_{pq}) \quad (3.54)$$

\forall f \in C^2(\Omega)$ where $\Omega$ is an open set in $\mathbb{R}^4$.

**Proof.** Consider,

$$4\partial^2_{z_1} = \partial_x (\partial_x - i_1 \partial_y) - i_1 \partial_y (\partial_x - i_1 \partial_y)$$

$$= \partial^2_x - i_1 \partial^2_{xy} - i_1 \partial^2_{yx} - \partial^2_y$$

$$= \partial^2_x - \partial^2_y - 2i_1 \partial^2_{xy}. \square$$

Therefore,

$$4 \left( \partial^2_{z_1} + \partial^2_{z_2} \right) = \left( \partial^2_x - \partial^2_y + \partial^2_p - \partial^2_q \right) - 2i_1 \left( \partial^2_{xy} + \partial^2_{pq} \right). \square$$

**Remark 3** In the last proposition, if we let $y$ and $q$ be constant variables, then

1. $\Omega \subset \mathbb{C}(i_2)$;
2. $4\partial_z \partial_{\bar{z}} = \partial_z \partial_{\bar{z}}$ where $\partial_z = \frac{1}{2} (\partial_x + i \partial_p)$ and $\partial_{\bar{z}} = \frac{1}{2} (\partial_x - i \partial_p)$;

3. $4\triangle C = 4\partial_z \partial_{\bar{z}} = \partial_x^2 + \partial_p^2 = \Delta$, the Laplacian operator.

Similarly, if $y$ and $p$ are constant variables, then

1. $\Omega \subset \mathbb{D}$;

2. $4\partial_z \partial_{\bar{z}} = \partial_z \partial_{\bar{z}}$ where $\partial_z = \frac{1}{2} (\partial_x - j \partial_q)$ and $\partial_{\bar{z}} = \frac{1}{2} (\partial_x + j \partial_q)$;

3. $4\triangle C = 4\partial_z \partial_{\bar{z}} = \partial_x^2 - \partial_q^2 = \Box$, the wave operator.

### 3.3.2 Factorization of the Complexified Schrödinger Operator

It is well known that if $f_0$ is a nonvanishing particular solution of the one-dimensional stationary Schrödinger equation

$$\left( -\frac{d^2}{dx^2} + \nu(x) \right)$$

then the Schrödinger operator can be factorized as follows:

$$-\frac{d^2}{dx^2} + \nu(x) = \left( \frac{d}{dx} + \frac{f_0'}{f_0} \right) \left( \frac{d}{dx} - \frac{f_0'}{f_0} \right).$$

The next result gives the analogue for the complexified Schrödinger operator. By $C$ we denote the $\tilde{t}_2$-bicomplex conjugation operator.

**Theorem 14** Let $f_0 : \Omega \subset \mathbb{R}^4 \to \mathbb{C}(i_1)$ be a nonvanishing particular solution of (3.53). Then for any $\mathbb{C}(i_1)$-valued continuously twice differentiable function $\varphi$ the following equality hold:

$$(\Delta_C - \nu) \varphi = 4 \left( \partial_z + \frac{\partial_z f_0}{f_0} C \right) \left( \partial_{\bar{z}} - \frac{\partial_{\bar{z}} f_0}{f_0} C \right) \varphi. \quad (3.55)$$

**Proof.** Let $f_0 : \Omega \subset \mathbb{R}^4 \to \mathbb{C}(i_1)$ be a nonvanishing particular solution of (3.53). Then

$$\left( \partial_z + \frac{\partial_z f_0}{f_0} C \right) \left( \partial_{\bar{z}} - \frac{\partial_{\bar{z}} f_0}{f_0} C \right) \varphi = \left( \partial_z + \frac{\partial_z f_0}{f_0} C \right) \left( \partial_{\bar{z}} \varphi - \frac{\partial_{\bar{z}} f_0}{f_0} \varphi \right)$$

$$= \partial_z \partial_{\bar{z}} \varphi - \partial_{\bar{z}} \left( \frac{\partial_z f_0}{f_0} \varphi \right) + \frac{\partial_z f_0}{f_0} \partial_{\bar{z}} \varphi - \frac{|\partial_z f_0|^2}{f_0^2} \varphi$$

$$= \frac{1}{4} \Delta_C \varphi - \partial_{\bar{z}} \left( \frac{\partial_z f_0}{f_0} \varphi \right) - \frac{|\partial_z f_0|^2}{f_0^2} \varphi$$

$$= \frac{1}{4} \Delta_C \varphi - \frac{\partial_z \partial_{\bar{z}} f_0 \cdot f_0 - |\partial_z f_0|^2}{f_0^2} \varphi - \frac{|\partial_z f_0|^2}{f_0^2} \varphi$$

$$= \frac{1}{4} \Delta_C \varphi - \frac{\Delta_C f_0}{f_0^2} \varphi.$$
However, \((\Delta_{C} - \nu)f_{0} = 0 \Rightarrow \frac{\Delta_{C} f_{0}}{f_{0}} = \nu\). Hence,
\[
(\Delta_{C} - \nu)\varphi = 4 \left( \partial_{\omega} + \frac{\partial_{\omega} f_{0}}{f_{0}} C \right) \left( \partial_{\omega} - \frac{\partial_{\omega} f_{0}}{f_{0}} C \right) \varphi. \square \tag{3.56}
\]

**Remark 4** From the Remark 3, we see that the complexified Schrödinger equation contains the stationary two-dimensional Schrödinger equation
\[ (\Delta - \nu(x, p)) f = 0 \]
and the Klein-Gordon equation
\[ (\Box - \nu(x, q)) f = 0. \]

Hence, our factorization of the complexified Schrödinger equation is a generalization of the factorization obtained in [12] for the stationary two-dimensional Schrödinger equation and for the factorization obtained in [15] for the Klein-Gordon equation.

### 3.3.3 Relationship Between Bicomplex Generalized Analytic Functions and Solutions of the Complexified Schrödinger Equation

The next Lemma has been inspired from a similar result in the complex plane (see [2], p 140).

**Lemma 4** Let \(b : \Omega \subset \mathbb{R}^{4} \rightarrow \mathbb{T}\) be a bicomplex function such that \(b_{\omega}\) is \(\mathbb{C}(i_{1})\)-valued, and let \(W = u + i_{2}v : \Omega \subset \mathbb{R}^{4} \rightarrow \mathbb{T}\) be a solution of the equation
\[ W_{\omega^{2}} = b W^{12} \text{ on } \Omega. \tag{3.57} \]

Thus, \(u : \Omega \subset \mathbb{R}^{4} \rightarrow \mathbb{C}(i_{1})\) is a solution of the equation
\[ \partial_{\omega^{2}} \partial_{\omega} u - (|b|_{i_{1}}^{2} + b_{\omega}) u = 0 \text{ on } \Omega \tag{3.58} \]
and \(v : \Omega \subset \mathbb{R}^{4} \rightarrow \mathbb{C}(i_{1})\) is a solution of the equation
\[ \partial_{\omega^{2}} \partial_{\omega} v - (|b|_{i_{1}}^{2} - b_{\omega}) v = 0 \text{ on } \Omega. \tag{3.59} \]

**Proof.** Using the \(1_{2}\) on both sides of Eq. (3.57), we have that
\[ \partial_{\omega^{2}} (u + i_{2}v) = b(u - i_{2}v) \Leftrightarrow \partial_{\omega}(u - i_{2}v) = b^{12}(u + i_{2}v) \text{ on } \Omega. \tag{3.60} \]
Therefore,
\[
\partial_{\omega} \partial_{\omega^{2}} (u + i_{2}v) = \partial_{\omega} b \cdot (u - i_{2}v) + b \partial_{\omega}(u - i_{2}v) \\
= b_{\omega}(u - i_{2}v) + bb^{12}(u + i_{2}v) \\
= b_{\omega}(u - i_{2}v) + |b|_{i_{1}}^{2}(u + i_{2}v) \text{ on } \Omega.
\]

Now, by equality of the scalar and vectorial part, we obtain the Equations (3.58) and (3.59). \square
Theorem 15 Let $W$ be a solution of the following bicomplex Vekua equation

$$
\left( \partial_{\omega^*} - \frac{\partial_{\omega^*} f_0}{f_0} C \right) W = 0 \quad (3.61)
$$

where $f_0$ is a nonvanishing solution of the complexified Schrödinger equation $(3.53)$. Then $u = \text{Sc}(W)$ is a solution of $(3.53)$ and $v = \text{Vec}(W)$ is a solution of the equation

$$
\left( \Delta_C + \nu(z_1, z_2) - 2 \left( \frac{\nabla_C f_0}{f_0} \right)^2 \right) v = 0 \quad (3.62)
$$

where $\nabla_C = \partial_{z_1} + i_2 \partial_{z_2}$.

Proof. Consider the function $b = \frac{\partial_{\omega^*} f_0}{f_0}$. Then

$$
b_{\omega} = \partial_{\omega} \left( \frac{\partial_{\omega^*} f_0}{f_0} \right) = \frac{(\partial_{\omega} \partial_{\omega^*} f_0) \cdot f_0 - (\partial_{\omega^*} f_0)(\partial_{\omega} f_0)}{f_0^2} = \frac{\Delta_C f_0}{4f_0} - \frac{|\partial_{\omega^*} f_0|_{i_1}^2}{f_0^2} = \frac{\nu(z_1, z_2)}{4} - \frac{|\partial_{\omega^*} f_0|_{i_1}^2}{f_0^2}.
$$

Therefore, $b_{\omega}$ is a $\mathbb{C}(i_1)$-valued function. Now, from Lemma [4]

$$
\frac{\Delta_C u}{4} = \left( \frac{\nu(z_1, z_2)}{4} - \frac{|\partial_{\omega^*} f_0|_{i_1}^2}{f_0^2} + \frac{|\partial_{\omega^*} f_0|_{i_1}^2}{f_0^2} \right) u
$$

i.e.

$$
(\Delta_C - \nu) u = 0,
$$

and

$$
\frac{\Delta_C v}{4} = \left( -\frac{\nu(z_1, z_2)}{4} + \frac{2|\partial_{\omega^*} f_0|_{i_1}^2}{f_0^2} \right) v
$$

i.e.

$$
(\Delta_C - \eta) v = 0
$$

where $\eta(z_1, z_2) = -\nu(z_1, z_2) + \frac{2|\nabla_C f_0|_{i_1}^2}{f_0^2}$. $\square$
Remark 5 From Theorem 5 if $W$ possesses a \((f_0, i^{12}_0)\)-derivative on an open set $\Omega \subset \mathbb{T}$ then $W$ is a solution of the bicomplex Vekua equation (3.61):

$$\left( \partial^{12} - \frac{\partial^{12}_0 f_0}{f_0} \right) W = 0 \text{ on } \Omega.$$ 

In that case, $a^{(2)}_{(F,G)} = 0$ and $b^{(2)}_{(F,G)} = \frac{\partial^{12}_0 f_0}{f_0}$ where

$$F = f_0 \quad \text{and} \quad G = \frac{i^{12}_0}{f_0}$$

is a $i_1$-generating pair for (3.61).

From the last remark, we can conclude that the bicomplex pseudoanalytic function theory open the way to find explicit solutions of the complexified Schrödinger equation.
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