Effective Field Theory

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ABSTRACT: I give a brief review of effective field theory, discussing the contribution of Feza Gürsey in particular and focussing on the literature I am most familiar with.

I would like to begin by saying a few words about Feza Gürsey (1921-1992), whom I regard as one of the last “gentleman-physicists.” Unlike many of the other speakers, I never had the pleasure of any direct physics interaction with him. But I have met him on a number of occasions and Feza and Suha have always been exceptionally nice to me, ever since the beginning of my career. The last time I saw them was a few years ago in a small town in eastern Hungary. Every time Feza saw me, he told me to come to Istanbul. Well, here I am finally, but unfortunately in his absence. The community has lost a true gentleman scholar.

My subject today is the low energy or long distance effective field theory, a concept that has pervaded throughout much of modern physics. In a sense, all of physics involves the concept of effective field theory. Hydrodynamics, for example, studies the behavior of a collection of particles on distance scales large compared to the separation between the particles. One can even say that all of known physics may be described by the effective low effective Lagrangian of string theory. In recent years, the concept of effective field theory has played an increasingly important role in condensed matter physics as well as in particle physics.

Perhaps the two most studied effective Lagrangians are the Landau- Ginzburg theory of superconductivity and the sigma model of the interaction between pions and nucleons.

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The Landau-Ginzburg theory went on to great glory as the prototype of a spontaneously broken gauge theory which underlies electroweak unification and grand unification. For its part, the non-linear sigma model has been studied intensively in recent years in connection with quantum spin systems, both ferromagnetic and anti-ferromagnetic. The discovery of high temperature superconductivity have thrust these studies into prominence since the relevant materials are known to be anti-ferromagnetic at low doping concentrations. In these applications, it is the non-linear sigma model, rather than the linear sigma model, that enters.

The ubiquitous non-linear sigma model first appeared in the work of Feza Gürsey.\textsuperscript{1,2,3,4} Indeed, even the notation and the philosophical underpinning in Gürsey’s first paper\textsuperscript{1} were already remarkably close to what is used in modern times. Starting with the chiral transformation of the nucleon field

$$\psi(x) \rightarrow e^{2i\gamma_5 \vec{\tau} \cdot \vec{\theta}} \psi(x)$$

Gürsey jumped to the non-linear transformation

$$\psi(x) \rightarrow e^{2i\gamma_5 \vec{\tau} \cdot \vec{\phi}(x)} \psi(x)$$

where $\phi$ denoted the pion field. Incidentally, Gürsey cited Nishijima\textsuperscript{5} for this crucial step of replacing the parameter of a symmetry transformation, $\vec{\theta}$, by a field, $\vec{\phi}(x)$. He then identified the unitary matrix field $\Phi = e^{2i\vec{r} \cdot \vec{\phi}}$, its kinetic energy term $tr\partial\Phi \partial\Phi^\dagger$ and its interaction with nucleons. As another historical note, we mention here that Gürsey cited Glauber\textsuperscript{6} as having written down, in 1951, a non-linear interaction of pions with nucleons in order to account for multiple pion production in nucleon nucleon collisions. Of course, back in 1951, chiral invariance was not yet appreciated.

Shortly afterwards, indeed, in the same volume of Il Nuovo Cimento, Gell-Mann and Lévy\textsuperscript{7} wrote down the linear sigma model. They were able to make the theory renormalizable, at the cost of introducing another field, the sigma field. The meson described by the sigma field, while its existence was, and remains controversial, has managed to give its name to this class of field theories. To put all this into perspective, we must remember that in the early sixties, renormalizability was considered “sacred,” and non-renormalizable field theories, such as the non-linear sigma model, were regarded with distaste. More on this later.

Indeed, at that time even the relevance of field theory for strong interaction physics was much in doubt, and the emphasis was decisely on the S-matrix and the dispersion
theoretic approach. The notion of broken chiral invariance was established only with the successes of the current algebra approach \(^8\) championed by Gell-Mann and others in the mid 1960’s. Using current algebra Adler and Weisberger were able to calculate what amounts to the low-energy interaction between pions and nucleons, an approach developed further by a number of authors. In particular, Weinberg \(^9\) showed how to calculate pion pion scattering at low energies, and in effect re-discovered the non-linear sigma model. In a series of influential papers, Weinberg not only brought respectability back to the non-linear sigma model, but to the entire Lagrangian field theory approach, thus sweeping away the S-matrix worship of the late fifties and early sixties and paving the way for electroweak unification.

In the early seventies, these considerations were extended to the interaction between pions and photons. Indeed, Schwinger’s calculation of the pion decaying into two photons amounts to finding the effective Lagrangian at energies low compared to the nucleon mass, and in this sense represents an intellectual descendant of Euler and Heisenberg’s calculation of the effective four-photon interaction at energies low compared to the electron mass. Adler, B. Lee, Treiman, and I, without ever mentioning the word Lagrangian or the word field, used various consistency requirements to determine \(^10\), \(^11\), \(^12\) the amplitudes for the processes \(\gamma \rightarrow 3\pi\) and for \(2\gamma \rightarrow 3\pi\). At the same time, and completely independently, Wess and Zumino \(^13\) found the effective Lagrangian describing these processes. The Lagrangian appeared quite strange: its action can only be written as a five dimensional integral.

Again, to put things into perspective, we should recall that the emphasis at that period in the history of particle physics was on momentum-space amplitudes, on the differential cross sections \(^14\) that actually could be measured at various newly built electron accelerators. An effective Lagrangian was regarded as only a mnemonic device. Independently of Wess and Zumino, but during and shortly after their work had already appeared, Aviv and I used Schwinger’s proper time method \(^15\) to determine the effective Lagrangian describing the interaction of an arbitrary number of SU(3) octet mesons and photons. \(^16\) Lacking the insight of Wess and Zumino, we wrote out the Lagrangian as an infinite series. These days this exercise would be referred to as evaluating the fermion determinant.

Since as an intermediate step we had to work out the quark propagator in the presence of external meson and electromagnetic fields, we could close the fermion line to obtain not only the effective Lagrangian, but also other fermion bilinears as well. In particular, Aviv and I also wrote down the effective current. \(^17\) Some years later, Goldstone and Wilczek \(^18\) rediscovered that using the matrix field \(\Phi\) introduced by Gürsey (with the Pauli matrices
promoted to the $SU(3)$ Gell-Mann matrices $\lambda$) the effective baryon current could be written in a compact form

$$J^\mu = \text{constant} \, \epsilon^{\mu \nu \lambda \sigma} tr(\Phi^\dagger \partial_\nu \Phi \Phi^\dagger \partial_\lambda \Phi \Phi^\dagger \partial_\sigma \Phi)$$

Aviv and I used another representation of the non-linear sigma model, discussed by Gürsey and by Gell-Mann and Lévy, in which $\sigma$ and $\varphi$ were constrained by $\sigma^2 + \varphi^2 = f^2$, and thus failed to see the invariant group structure made apparent in the Goldstone-Wilczek form. Various strands in the development of effective Lagrangian and effective currents were all interconnected in an interesting way.\textsuperscript{19}

Years later, Witten\textsuperscript{20} developed the subject further, starting with the five-meson scattering amplitude and realizing that this could not be described by a Lagrangian in four-dimensional spacetime. Nowadays, this can be seen quite easily by noting that in analogy with the effective current written above, the effective Lagrangian would have the form $\mathcal{L} \sim \epsilon^{\mu \nu \lambda \sigma \tau} tr(\Phi^\dagger \partial_\mu \Phi \Phi^\dagger \partial_\nu \Phi \Phi^\dagger \partial_\lambda \Phi \Phi^\dagger \partial_\sigma \Phi \Phi^\dagger \partial_\tau \Phi)$ and an antisymmetric symbol with five indices is available only in five-dimensional spacetime. The five-pion amplitude is forbidden by G-parity, but in $SU(3) \times SU(3)$ five-meson scattering is allowed. Witten brought the theory into the modern form.

As mentioned earlier, the non-linear sigma model has been extensively studied in connection with ferromagnetism and anti-ferromagnetism in the condensed matter physics literature. Naively, it seems reasonable enough that in a ferromagnetic or anti-ferromagnetic system the magnetization (or the staggered magnetization) can be represented in the continuum limit by a 3-vector field $\vec{n}(x,t)$. We are typically not interested in the fluctuation of the magnitude of $\vec{n}(x,t)$ and thus $\vec{n}(x,t)$ may be taken to be a vector of unit length. The Lagrangian is naturally taken to be a non-linear sigma model

$$\mathcal{L} = \frac{1}{2f^2} \partial_\mu \vec{n} \cdot \partial^\mu \vec{n}$$

For a long time, a number of field theorists puzzled over how quantum spin, even a single quantum spin with its non-commuting components, could be incorporated into the path integral formalism. I understand that Feynman himself was troubled over this point. In hindsight, as if often the case, the solution as now presented in textbooks\textsuperscript{21} seems straightforward enough, and it appears as if one only needs to have “one’s head screwed on straight” to be led by the formalism step-by-step towards the correct answer, namely that the Wess-Zumino-Witten term has to be included.
In fact, there is another approach which avoids having to write the action as a higher dimensional integral. To explain this, I must regrettably mention that particle physicists were traditionally confused by the difference between the ferromagnet and the anti-ferromagnet, as indicated by the discussion in the paragraph preceding the one above. A clue is provided by the fact, as shown in elementary solid state physics texts, that the spin wave disperses linearly (that is, $\omega \propto k$) in an antiferromagnet and quadratically (that is, $\omega \propto k^2$) in a ferromagnet. Thus, on the face of it, the ferromagnet cannot be represented by the (relativistic) non-linear sigma model written above. This suggests that the Lagrangian has to involve only one derivative in time, rather than two, but there is no such term involving $\vec{n}$! The term $\vec{n}.\partial_t \vec{n}$ is a total derivative.

The solution, as it turns out, was to write $\vec{n} = z^\dagger \vec{\sigma} z$ with $z$ a spinor such that $z^\dagger z = 1$. The desired Lagrangian is then

$$\mathcal{L} = -iz^\dagger \dot{z} - V(\vec{n})$$

with $V(\vec{n}) = (\nabla \vec{n})^2 + \text{possible other terms such as the coupling of } \vec{n} \text{ to an external magnetic field. Indeed, using the identity } \delta(z^\dagger \dot{z}) \propto \delta\vec{n}(\vec{n} \times \dot{\vec{n}}) \text{ we obtain}$

$$\vec{n} \times \dot{\vec{n}} = \frac{\delta V}{\delta \vec{n}}$$

Taking the scalar product of this equation with $\vec{n}$ we obtain

$$\dot{\vec{n}} = \vec{n} \times \frac{\delta V}{\delta \vec{n}}$$

the familiar equation of motion for a spin. A straightforward exercise shows that the dispersion laws around a ferromagnetic and an anti-ferromagnetic background are indeed different and as stated above.

My generation of physicists was taught the notion that quantum field theory, quantum electrodynamics, quantum chromodynamics, and so forth, were “fundamental,” that these theories hold at arbitrarily short distance scales. We used renormalization group to study the ultraviolet flow towards short distances. We would write down a Lagrangian and use “renormalizability” and symmetry to limit the number of possible terms. A “more modern” view, which has emerged from condensed matter physics, in particular the theory of critical phenomena, and which may be called “Wilsonian”, acknowledges and emphasizes that the short distance physics may be extremely complicated or unknown, or perhaps even “unknowable,” depending on your philosophical persuasion. In condensed matter physics,
the short distance physics may be described by lattice dynamics. In particle physics, the short distance physics is allegedly that of a string. Instead of the ultraviolet flow, we should study the infrared flow towards long distances and times and hope that most terms become irrelevant in that limit. We would then arrive at an effective field theory described by a small number of terms. The better among the more modern textbooks, such as the one by Polyakov, specifically emphasize this view. The relevant equations are essentially the same, but the mindset is different.

We are thus instructed to study the renormalization group flow of various operators. In many cases, simple dimensional analysis, which may be regarded as “zeroth order renormalization group,” suffices. In particle physics, this point of view was developed over a number of years. An early example is a “model independent” analysis of proton decay.\(^\text{23,24}\)

Let us say that we believe only in \(SU(3) \times SU(2) \times U(1)\) and not in grand unification. We simply say that proton decay is due to some unknown short distance physics, but whatever this mechanism might be, we can still write down a long distance effective Lagrangian to describe proton decay. The most relevant operators are those with the lowest scaling dimensions. These dimension six operators have the form

\[
\frac{1}{M^2_*}qqql
\]

where \(q\) and \(l\) represent quark and lepton fields respectively. By engineering dimensional analysis, an unknown mass \(M_*\) has to be introduced. Thus, the rate for proton decay is of course undetermined. However, by imposing \(SU(3) \times SU(2) \times U(1)\) we are able to restrict the number of possible operators enormously. In this way, we can make predictions about proton decay completely independent of what the short distance physics may be!

Incidentally, this may be construed as an argument for colored quarks. Before the invention of quarks, we were able to write down a dimension four operator of the form \(P\bar{e}\pi\), and we must arbitrarily decree that the dimensionless coefficient in front of this operator to be ridiculously small, and since we don’t understand why this coefficient is so small, we dignify this ignorance by elevating it to a principle, known as baryon conservation. It is color that forces us to go from a dimension four operator to a dimension six operator. Physics has progressed: a small dimensionless coefficient has been replaced by the ratio \((\text{proton mass})^2/M^2_*\). We can go on and study any exotic process involving the known quarks and leptons, by systematically writing down,\(^\text{25}\) in order of increasing scaling dimension, all possible operators allowed by \(SU(3) \times SU(2) \times U(1)\).
Given an effective long distance field theory, we can of course try to “induce” what the short distance physics might be. In general, of course, many possible short distance physics may give rise to the same long distance physics: this remark has been elevated to the “principle of universality. Given the long distance physics, we can arrive at the correct short distance physics only by astutely combining experimental observations and inspired guesses guided by general principles and esthetic considerations. Such is the history of physics. Currently, this enterprise is represented by string theory.

An infinitely more modest example involves a possible Majorana mass for the neutrino. The relevant long distance effective Lagrangian, as restricted by $SU(3) \times SU(2) \times U(1)$, has the form

$$L_{eff} = f \bar{\psi}_L C \psi_L \bar{\phi} \phi.$$

where $\psi_L$ and $\phi_L$ represent the left-handed lepton doublet and the Higgs doublet respectively. Here $\bar{\phi}$ denotes, as usual, $\epsilon_{ij} \phi_j$. When the neutral component of $\phi$ acquires a vacuum expectation value, the neutrino gets a Majorana mass. Can we “induce” the short distance physics responsible? In other words, can we replace effectively the dimension five operator above by operators with dimension four or less. One attempt is represented by the following

$$L = g \bar{\psi}_L C \psi_L h^+ + M \bar{\phi} \phi h^-$$

where $h$ represents a charged $SU(2)$ singlet Higgs. The neutrino Majorana mass is now calculable (in the technical sense!) in terms of the (unknown) $g$ and $M$. Short distance physics of course tells us more (again, the progress of physics!). In this particular case, we can learn something about the flavor of the neutrino Majorana mass.

In some sense, physics is possible only if the effect of a particle of mass $M$ on the low energy or long distance effective theory vanishes as $M$ tends to infinity. Thus, for Schwinger to calculate the $g$ factor of the electron, he didn’t have to know the mass of the top quark. There is however a conceptually important exception to this perfectly reasonable expectation. Suppose that after the field for the massive particle is removed, the resulting theory becomes non-renormalizable. Then the low energy theory will remember the massive particle: its absence would be missed even at arbitrarily low energies.

For example, in the standard model the left handed top quark appears in a doublet $(t \ b)_L$. Removing the top field would render the theory non-renormalizable. Now consider the Feynman diagram in which a $Z$ boson couples to a top quark loop which connects to an external up or down quark line via two gluons. This process introduces an isoscalar
contribution to the neutral current of the up and down quarks. A simple analysis shows that indeed this radiative correction goes like $\log m_t$ for large top quark mass $m_t$. Note that whether or not the theory left behind after the removal of the heavy field is renormalizable or not depends on the theory being considered. Thus, for Schwinger, the relevant theory was quantum electrodynamics in which the top quark is represented as just another charged field. Removing it leaves the theory perfectly renormalizable.

Another striking example is given by the electric dipole moment of the neutron\textsuperscript{29} and electron.\textsuperscript{30} Consider the diagram in which a photon couples to a top quark loop which connects to an external electron line via a photon and a Higgs field. Again, we see that this contribution to the electric dipole moment of the electron grows like $\log m_t$. One instructive way of looking at this process is to consider an effective Lagrangian for the coupling of a Higgs field to two photons, which in fact consists the dimension five operator $\phi F_{\mu\nu} F^{\mu\nu}$. When we insert this effective Lagrangian into a calculation of the electric dipole moment of the electron, we would obtain a (logarithmically) divergent answer since this Lagrangian is not renormalizable. The relevant Feynman integral has to be cut off at the energy scale beyond which the low energy effective Lagrangian ceases to be valid, and this scale is set by precisely the top quark mass.

In condensed matter physics, we are not interested in ultraviolet divergences since these are always cut off by the lattice. As an example, consider the chiral spin state\textsuperscript{31,21}. Start with a single particle hopping on a square lattice. We can look at the “Feynman freshman physics” book for example and see that the energy of the particle is related to its momentum by $E = -(\cos k_x + \cos k_y)$; the spectrum, typical of band structure theory, reflects the square symmetry of the lattice. Now suppose every time the electron goes around a square plaquette on the lattice it acquires a factor of $(-1)$. Another way of saying this is that there is a magnetic flux of $\pi$ threading through each plaquette. The energy spectrum becomes\textsuperscript{32,33,34} $E = \pm \sqrt{\cos^2 k_x + \cos^2 k_y}$, still a messy looking expression. But if we want to study a half-filled system, that is, if we fill the band with fermions up to $E = 0$ in accordance with the Pauli exclusion principle, and if we are only interested in physics at long distance and time, then we expand around a point where $E$ vanishes. Writing $k_x = \frac{\pi}{2} + q_x$ and $k_y = \frac{\pi}{2} + q_y$, we find $E = \pm \sqrt{q_x^2 + q_y^2}$. This is a most remarkable result: in a system which is not even rotational invariant we obtain a Lorentz invariant dispersion!

Further, when we allow the particle to hop along the diagonal as well, such that when the particle hops around a triangle its wave function acquires a phase of $i$, a gap opens up
between the upper and lower band. The effective low energy theory describing a half-filled system, that is for energies low compared to the gap, then reads

\[ \mathcal{L} = \bar{\psi}(i\partial - m)\psi + \ldots. \]

We have discovered the Dirac Lagrangian! Finally, introducing the phase degree of freedom contained in the fermion creation and annihilation operator on the lattice, we arrive at the gauge theory

\[ \mathcal{L} = \bar{\psi}(i\partial + a - m)\psi \]

For physics below the energy scale \( m \), we can integrate out the fermion field and thus obtain

\[ \mathcal{L} = \frac{1}{4\pi^2} 2\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \ldots, \]

the famous Chern-Simon term much talked about by physicists and mathematicians.\(^{35}\)

From here we can proceed to a discussion of fractional statistics and semion superconductivity, but that is another story.\(^{36}\) One can also go on to discuss extensions to 3 + 1 dimensional spacetime and to non-abelian flux and so on.\(^{37},^{38}\)

The remarkable emergence of a relativistic Dirac Lagrangian from a lattice theory without even rotational invariance naturally prompts speculations on whether the observed quarks and leptons can emerge in this way also.\(^{39}\) In a sense, that is what lattice gauge theory\(^{40}\) is all about.

We do not want to give the impression that low energy effective theories are easy to write down. The difficulty is that the relevant degrees of freedom may be quite different from those present in the short distance theory. In quantum chromodynamics, for example, we have hadrons in the low energy theory, not quarks. In superconductivity, we have the Ginzburg-Landau field or the Cooperon. In the example just given, the relevant low energy degree of freedom turns out to be a gauge potential governed by Chern-Simons dynamics.

A striking recent example of this phenomenon is the effective field theory of quantum Hall fluid, which after all, simply consists of electrons interacting and moving in a 2 + 1 dimensional spacetime with a perpendicular magnetic field. The microscopic physics is described by a trial wave function of the \( N \sim 10^{23} \) electrons proposed by Laughlin. Here I cannot possibly give you anything more than just the flavor of the effective field theory approach.

What I will do here is to argue what the effective field theory must be from general principles.\(^{41}\) The conservation of the electromagnetic current \( \partial^\mu J_\mu = 0 \) in (2+1)-dimensional spacetime tells us that the current can be written as the curl of a vector
potential, then reads

\[ J^\mu = \frac{1}{2\pi} \varepsilon^{\mu \nu \lambda} \partial_\nu a_\lambda \]

We now note that when we transform \( a_\mu \) by \( a_\mu \rightarrow a_\mu - \partial_\mu \Lambda \), the current is unchanged. We did not go looking for a gauge potential, the gauge potential came looking for us! There is no place to hide. The existence of a gauge potential follows from completely general considerations.

Let us now try to write down a low energy effective local Lagrangian in terms of operators of lowest possible dimensions, noting that parity and time reversal are broken by the external magnetic field. Since gauge invariance forbids the dimension 2 term \( a_\mu a^\mu \) in the Lagrangian, the simplest possible term is in fact the dimension 3 Chern-Simons term \( \varepsilon^{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda \). Thus, the Lagrangian (density) is simply

\[ \mathcal{L} = \frac{k}{4\pi} \varepsilon^{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda + \ldots \]

To determine the dimensionless coefficient \( k \) we couple the system to an “external” or “additional” electromagnetic gauge potential \( A_\mu \), thus now have

\[ \mathcal{L} = \frac{k}{4\pi} \varepsilon^{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2\pi} \varepsilon^{\mu \nu \lambda} A_\mu \partial_\nu a_\lambda = \frac{k}{4\pi} \varepsilon^{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2\pi} \varepsilon^{\mu \nu \lambda} a_\mu \partial_\nu A_\lambda \]

Integrating out \( a \) we obtain

\[ \mathcal{L}_{\text{eff}} = - \frac{1}{4\pi k} \varepsilon^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda. \]

The electromagnetic current that flows in response to \( A_\mu \) is thus

\[ J^\mu \equiv - \frac{\delta \mathcal{L}_{\text{eff}}}{\delta A_\mu} = \frac{1}{2\pi k} \varepsilon^{\mu \nu \lambda} \partial_\nu A_\lambda \]

Looking at the time component of this equation, we recognize \( 1/k \) as the ratio of the density of electrons to the magnetic field. To study the elementary excitations in the system, we simply couple the current of these excitations to the gauge potential \( a \). Proceeding in this way, we can easily recover the classic Laughlin results that the elementary excitations carry fractional charge and statistics. For details and references to the original literature, I refer the reader to some pedagogical lectures I gave last winter.\(^{42}\)

Another topic I like to mention is that of tunneling in double-layered quantum Hall systems. First, in the absence of tunneling between the two layers, the current associated
with each layer is separately conserved $J_I = \frac{1}{2\pi} \epsilon \partial a_I$, $I = 1, 2$. Thus, we can generalize the effective Lagrangian above to

$$\mathcal{L} = \frac{1}{4\pi} \left( \sum_{IJ} K_{IJ} a_I \epsilon \partial a + \sum_I 2A_I \epsilon \partial a \right) + a_1 j_1 + a_2 j_2$$

with a 2-by-2 matrix $K = \begin{pmatrix} l & n \\ n & m \end{pmatrix}$ We note in passing that this $K$ matrix is in one-to-one correspondences with a class of wave functions proposed long ago by Halperin to describe double-layered Hall systems.

What happens if $K$ has a zero eigenvalue? Then some linear combination of the gauge fields becomes massless, leading to a gapless mode and some interesting physics.

How is tunneling represented in this picture? When an electron tunnels from one layer to another, the currents $J_1^\mu = \frac{1}{2\pi} \epsilon \partial a_1$ and $J_2^\mu = \frac{1}{2\pi} \epsilon \partial a_2$ are no longer separately conserved. Since electrons are represented by flux quanta, tunneling from the first layer to the second converts flux of type $\epsilon \partial a_1$ to flux of type $\epsilon \partial a_2$. Thus tunneling corresponds to a kind of magnetic monopole into which flux of type 1 disappears and out of which flux of type 2 appears. Indeed, more formally, we have

$$\Delta(N_1 - N_2) = \pm 2 = \int dt \ d^2 x \ \partial \mu (J_1^\mu - J_2^\mu) = \frac{1}{2\pi} \int dt \ d^2 x \ \partial \mu (\epsilon^{\mu\nu\lambda} \partial_\nu a_{-,\lambda})$$

Suppose we continue from Minkowskian $(2+1)$ dimensional spacetime to Euclidean 3 space. We recognize $\partial_\mu (\epsilon^{\mu\nu\lambda} \partial_\nu a_{-,\lambda})$ as $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cdot \vec{B}$ if we identify $a_{-,\lambda}$ as a 3-vector gauge potential $\vec{A}$ in Euclidean space and $\vec{B}$ the corresponding magnetic field. This is precisely a Dirac magnetic monopole, with its flux quantized to be $\pm 4\pi$, just as Dirac said it should be.

Thus, in Euclidean 3 space we have a plasma of magnetic monopoles and anti-monopoles. At the location of each monopole and each anti-monopole there occurs a tunneling event in spacetime. Polyakov showed long ago that in the presence of a monopole plasma the photon acquires a mass with an effective sine-Gordon Lagrangian

$$\mathcal{L}_{eff} = g^2 (\partial \theta)^2 + \zeta \cos \theta$$

It is immensely pleasing that some of the most profound concepts in theoretical physics are involved here. The discretness of the electron leads to Dirac quantization of the magnetic monopoles. The quantization of monopoles leads to an angular variable as the order parameter. The appearance of an angular order parameter immediately reminds us of the
Josephson effect in superconductors. Wen and I\textsuperscript{44} were thus led to predict that there should be “Josephson-like” effects in double-layered quantum Hall systems. We were careful to use the term “Josephson-like” because the double-layered Hall system is to be sure not a superconductor, and thus the usual discussion of the Josephson effect must be taken over here with great care. A detailed discussion is beyond the scope of these lectures. We refer to the original work\textsuperscript{44,46,47} and to the recent literature.\textsuperscript{48,49,50}

An interesting probe is provided by applying a magnetic field parallel to the plane of the double-layered Hall system. When an electron tunnels from one layer to another, its wave function now acquires a phase factor

\[ e^{\pm ie \int dz A_z} \equiv e^{\pm i\xi(x)} \]  

(We denote the coordinate perpendicular to the plane by \( z \) and the two-dimensional coordinates in the plane by \( x \).) A monopole is now associated with the phase factor \( e^{+i\xi(x_a)} \), and an anti-monopole with the phase factor \( e^{-i\xi(x_a)} \). It is not difficult to verify that in the effective Lagrangian the cosine term is now modified to \( \cos(\theta + \xi) \). Wen and I\textsuperscript{46} considered a random magnetic field and showed that its effect was to reduce the tunneling parameter \( \zeta \). Yang et al\textsuperscript{48} showed that a uniform magnetic field would drive an interesting commensurate-incommensurate transition.

Again, the non-linear sigma model of Gürsey appears naturally. Here the vector field \( \vec{n} \) is an order parameter such that when it is pointing up it indicates electrons in the upper layer, and when it is pointing down, electrons in the lower layer. Thus, if we impose a voltage across the double layered system we simply add to the Lagrangian a term \( Un_z \) which says that to have an electron in the upper layer costs more energy than to have it in the lower layer. It is also easy to see that tunneling can be represented by a field driving \( \vec{n} \) to point in the \( x \)-direction (because \( S_x = S_+ + S_- \) raises and lowers electrons between the upper and lower layers.) In this way, we arrive at a more involved non-linear sigma model\textsuperscript{48}

\[ \mathcal{L} = -iz \dot{\vec{n}} \cdot \vec{n} - (\nabla \vec{n})^2 - \beta n_z^2 - \eta n_x \]

The sine-Gordon mentioned earlier can be obtained\textsuperscript{47} as an effective low energy Lagrangian from this effective Lagrangian by integrating out the fast mode, namely \( n_z \). As is in general the case, there is a hierarchy of low energy effective Lagrangians, presumably with string theory at the top of the hierarchy.

In closing, I would like to mention some recent work\textsuperscript{51,52,53} on random matrix theory. First, a faster-than-lightning review of this venerable subject. In the early fifties, Wigner
posed the problem of calculating the distribution of the eigenvalues of an $N$ by $N$ (as $N \to \infty$) hermitean matrix randomly taken from a probability distribution, for example,

$$P(\phi) = \frac{1}{Z} e^{-N \text{tr} V(\phi)}$$

where $V$ is a polynomial of its argument. The operator measuring the density of eigenvalues is given by $\hat{\rho}(\mu) = \frac{1}{N} \text{tr} \delta(\mu - \phi)$. The density of eigenvalues $\rho(\mu) = <\hat{\rho}(\mu)>$ has been known for some fifteen years,\(^5^4\) and as one might expect, depends on $V$ of course. For $V$ an even polynomial, the density is non-vanishing between $-a$ and $a$ where the width of the spectrum $a$ is a complicated functional of $V$.

What about the density-density correlation function $\rho_c(\mu, \nu) = <\hat{\rho}(\mu)\hat{\rho}(\nu)> - <\hat{\rho}(\mu)> <\hat{\rho}(\nu)>$? This correlation function was determined recently for any $V$ and was found to have wild oscillations, as expected since there are $N$ eigenvalues in the spectrum. It is convenient to think of the eigenvalues as the positions of a gas of atoms living in a one-dimensional space of width $2a$. The short distance physics depends on $V$ in detail.

The surprising discovery\(^5^1,\,^5^3\) is that when $\rho_c(\mu, \nu)$ is smoothed over these short distance details, it becomes universal when expressed in terms of the obvious scaling variables $x = \mu/a$, $y = \nu/a$, that is, we found the smoothed correlation to be\(^5^1\)

$$\rho_c^{\text{smooth}}(\mu, \nu) = \frac{-1}{2N^2 \pi^2 a^2} f(x, y)$$

where the function

$$f(x, y) = \frac{1}{(x - y)^2} \frac{(1 - xy)}{[(1 - x^2)(1 - y^2)]^{1/2}}.$$

is universal in the sense that it does not depend on $V$ at all. We find this result rather remarkable since the density $\rho(\mu)$ does depend on $V$. Even after the Gaussian law of large numbers has been proved to us, it seems perhaps somewhat mysterious that random numbers would “know” about the function $e^{-x^2}$. In the same way, it appears mysterious that somehow random matrices know about the function $f(x, y)$.

Here the long distance effective theory corresponds to hydrodynamics, and not to a renormalization group flow towards low energy. Indeed, the universality can be derived using hydrodynamics.\(^5^5\) Alternate derivations have also been given.\(^5^6,\,^5^7\)

There is however also an analog\(^5^8\) of the renormalization group. In the renormalization group approach, we thin out the number of degrees of freedom. Here we can take an $N$ by $N$ matrix, integrate over its last row and column, and obtain an $N-1$ by $N-1$ matrix. In this
way, we can obtain a renormalization group flow to determine the density of eigenvalues. The calculation$^{52}$ is particularly “neat” because none of the usual complications appears.

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In reading a history of the Turks, I learned that the word “Turk” first appeared in an ancient Chinese chronicle, which describe the Turks as exceptionally hospitable “milk drinking octonion loving barbarians.” I would like to thank my Turkish colleagues for their exceptional hospitality. This work is supported in part by the National Science Foundation under Grant No. PHY89-04035.

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A note to my colleagues who are not referenced here: I feel that this may not be the appropriate place to give a comprehensive set of references to the recent literature. In my more extensive review article$^{42}$ a more complete set of references may be found.

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