CRITICAL WELL-POSEDNESS AND SCATTERING RESULTS
FOR FRACTIONAL HARTREE-TYPE EQUATIONS

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Abstract. Scattering for the mass-critical fractional Schrödinger equation with a cubic Hartree-type nonlinearity for initial data in a small ball in the scale-invariant space of three-dimensional radial and square-integrable initial data is established. For this, we prove a bilinear estimate for free solutions and extend it to perturbations of bounded quadratic variation. This result is shown to be sharp by proving the discontinuity of the flow map in the super-critical range.

1. Introduction

Let $n \in \mathbb{N}$, $1 \leq \alpha \leq 2$, and $\sigma \in \mathbb{R}$. We consider the following initial value problem for a fractional Schrödinger equation with a cubic Hartree-type nonlinearity:

$$
- i \partial_t u + (-\Delta)^{\frac{\alpha}{2}} u = \sigma(|\cdot|^{-\alpha} * |u|^2)u
$$

(1.1)

$$
u(0, \cdot) = \varphi
$$

Here, the unknown is a function $u : (-T, T) \times \mathbb{R}^n \to \mathbb{C}$, the initial datum is $\varphi : \mathbb{R}^n \to \mathbb{C}$ and $(-\Delta)^{\frac{\alpha}{2}}$ is defined as the spatial Fourier multiplier with symbol $|\cdot|^{\alpha}$ on $\mathbb{R}^n$, and $*$ denotes spatial convolution. We will consider initial data $\varphi \in H^s(\mathbb{R}^n)$ and solutions will be continuous curves in $H^s(\mathbb{R}^n)$.

We may rescale solutions according to

$$
u(t, x) \to \nu_\lambda(t, x) := \lambda^{\frac{2}{\alpha}} \nu(\lambda^{\frac{1}{\alpha}} t, \lambda x),
$$

(1.2)

for fixed $\lambda > 0$. The mass

$$
M(\nu(t)) := \|\nu(t)\|_{L^2(\mathbb{R}^n)}^2
$$

of sufficiently smooth and decaying solutions $\nu$ of (1.1) is conserved and invariant under this rescaling, i.e. $M(\nu_\lambda(t)) = M(\nu(t)) = M(\varphi)$ for any $t \in \mathbb{R}$. For this reason the equation (1.1) is referred to as being mass-critical.

In addition, for sufficiently smooth and decaying solutions $\nu$ of (1.1), the energy

$$
E(\nu(t)) := \frac{1}{2} \langle (-\Delta)^{\frac{\alpha}{2}} \nu(t), \nu(t) \rangle + \frac{\sigma}{4} \langle |\cdot|^{-\alpha} * |\nu(t)|^2 \nu(t), \nu(t) \rangle,
$$

is conserved and the Sobolev space $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ serves as the energy space for equation (1.1). Here, $\langle \cdot, \cdot \rangle$ is the complex inner product in $L^2(\mathbb{R}^n)$.

The Cauchy problem with Hartree-type nonlinearities has been studied intensively. If $\alpha = 1$ and $n = 3$, in which case (1.1) arises as a model system for the dynamics of boson stars, Lenzmann and the first named author proved local...
well-posedness for radial initial data in the full subcritical range \( s > 0 \) using \( X^{s,b} \) spaces \([9]\). In the present paper the case of generalized dispersion, i.e. \( \alpha > 1 \), will be addressed. In \([11]\), Kirkpatrick, Lenzmann and Staffilani rigorously derived a fractional Schrödinger equation with a cubic power-type nonlinearity if \( n = 1 \) as the continuum limit of certain discrete physical systems with long-range lattice interactions. As an open problem they suggested that their argument might be generalized to the fractional Schrödinger equation with other nonlinear terms in higher dimension. Equation \( 1.1 \) with \( \alpha = 2 \) with Hartree-linearity, i.e. convolution with \(|\cdot|^{-1}\), in \( n = 3 \) was derived from the quantum theory of large systems of bosons. With regard to well-posedness of \( 1.1 \), Cho, Hajaiej, Hwang, and Ozawa showed global well-posedness in the critical space for sufficiently small radial initial data if \( \frac{2n}{2n-1} < \alpha < 2 \) by using radial Strichartz estimates \([2]\). Our main result fills the gap in the range \( 1 < \alpha \leq \frac{2n}{2n-1} \).

Concerning the scattering problem associated with \( 1.1 \) in the case \( \alpha = 1 \), the first named author and Tesfahun \([10]\) proved scattering of solutions for small radial initial data with \( s > 0 \) in the case of Yukawa potentials in \( n = 3 \), while in case of the Coulomb potential a modified scattering result has been established by Pusateri \([13]\). The classical case \( \alpha = 2 \) has been treated in \([8]\). In the fractional case \( 1 < \alpha < 2 \), Cho, Hwang and Ozawa \([3]\) proved scattering for small initial data when \( s > \frac{2-\alpha}{2} \) in the case of generalized potentials including the Yukawa potentials in \( n \geq 3 \).

For a more complete account on previous and related results, we refer to \([2, 3]\). Concerning more references to the physics literature, we refer to \([11]\).

We address the question of well-posedness and scattering of \( 1.1 \) in the critical space in the range \( 1 < \alpha \leq 2 \). To obtain a result in the full range, we apply a contraction argument in a function space whose construction is based on the space of bounded quadratic variation \( V^2 \) and we extend a bilinear estimate for free solutions to this space. In part, the strategy of proof is similar to \([10]\), but here we work in the critical regime.

In the super-critical range, i.e. \( s < 0 \), we provide a counterexample which implies the discontinuity of the flow map.

Our aim is to prove the existence and scattering of solutions to the IVP \( 1.1 \) in the critical space \( L^2(\mathbb{R}^3) \), i.e. we will focus on \( n = 3 \). We will consider the subspace of radial functions. For \( s \in \mathbb{R} \) define

\[
H^s_{\text{rad}}(\mathbb{R}^3) := \{ \varphi \in H^s(\mathbb{R}^3) : \exists \varphi_0 : [0, \infty) \to \mathbb{R} \ s.t. \varphi(x) = \varphi_0(|x|) \ a.e. \},
\]

with norm \( \| \cdot \|_{H^s} \). We write \( L^2\text{rad}(\mathbb{R}^3) := H^0_{\text{rad}}(\mathbb{R}^3) \). Define \( (S\varphi)(t, x) = (S(t)\varphi)(x) \), for \( S(t)\varphi(\xi) = e^{-it|\xi|^2} \hat{\varphi}(\xi) \). Let us state our first main result on global well-posedness and scattering for radial initial data which is small in the critical space.

**Theorem 1.1.** Let \( 1 < \alpha \leq 2 \). There exists \( \delta > 0 \), such that for all \( \varphi \in L^2_{\text{rad}}(\mathbb{R}^3) \) satisfying \( \|\varphi\|_{L^2} \leq \delta \), there exist a global solution \( u \in C_0(\mathbb{R}, L^2_{\text{rad}}(\mathbb{R}^3)) \) of \( 1.1 \). \( u \) is unique in a certain subspace and the flow map \( \varphi \to u \) is smooth.

Moreover, the solution scatters as \( t \to \pm \infty \), i.e. there exist \( \varphi_{\pm} \in L^2_{\text{rad}}(\mathbb{R}^3) \), such that

\[
\| u(t) - S(t)\varphi_{\pm} \|_{L^2(\mathbb{R}^3)} \to 0 \quad (t \to \pm \infty).
\]
In order to state the second result, let \( P_{>\Lambda} \) be the projector onto frequencies greater than \( \Lambda \), see Subsection 1.1 and let us define
\[
B_{r,\Lambda} := \{ \varphi \in L^2_{\text{rad}}(\mathbb{R}^3) : \| \varphi \|_{L^2} \leq r, \| P_{>\Lambda} \varphi \|_{L^2} \leq \eta r^{-1} \},
\]
for \( r, \Lambda \geq 1 \) and some parameter \( 0 < \eta \ll 1 \), which will be fixed in Subsection 3.2 independently of \( r, \Lambda \). Notice that for each \( \varphi \in L^2_{\text{rad}}(\mathbb{R}^3) \) and \( \eta > 0 \) there exist \( \Lambda > 0 \) such that \( \| P_{>\Lambda} \varphi \|_{L^2} \leq \eta r^{-1} \). Therefore, for any \( \varphi \in L^2_{\text{rad}}(\mathbb{R}^3) \) and any \( r \geq \| \varphi \|_{L^2} \) there exists \( \Lambda \geq 1 \), such that \( \varphi \in B_{r,\Lambda} \). For large radial initial data in the critical space, we have local well-posedness.

**Theorem 1.2.** Let \( 1 < \alpha \leq 2 \). For all \( r, \Lambda \geq 1 \) and all \( \varphi \in B_{r,\Lambda} \subset L^2_{\text{rad}}(\mathbb{R}^3) \), there exists \( T = T(r, \Lambda) \) and a solution \( u \in C([0, T], L^2_{\text{rad}}(\mathbb{R}^3)) \) of (1.1). \( u \) is unique in a certain subspace and the flow map \( \varphi \to u \) is smooth.

Our next result shows that Theorem 1.2 is optimal.

**Theorem 1.3.** Let \( s < 0 \) and \( T > 0 \). The flow map is is discontinuous in \( H^s \) at the origin, i.e. there exists a sequence \( (\varphi_n) \) of initial in \( L^2_{\text{rad}}(\mathbb{R}^3) \) with solutions \( (u_n) \), such that \( \| \varphi_n \|_{H^s(\mathbb{R}^3)} \to 0 \) while \( \sup_{0 \leq t \leq T} \| u_n(t) \|_{H^s(\mathbb{R}^3)} \neq 0 \).

The proof is based on an adaptation of the counterexample for the case \( \alpha = 1 \) from [9] and the abstract ill-posedness result from [11].

1.1. **Notation.** Dyadic numbers \( \lambda \in 2^\mathbb{Z} \) will always be denoted by greek letters, e.g. \( \mu, \lambda, \lambda_1, \lambda_2, \) and sums with respect to greek letters are implicitly assumed to range over (subsets of) \( 2^\mathbb{Z} \).

Let \( \rho \in C^\infty_c(-2, 2) \) be even and satisfy \( \rho(s) = 1 \) for \( |s| \leq 1 \). For \( \chi(\xi) := \rho(|\xi|) \rho(2|\xi|) \) define \( \chi_\lambda(\xi) = \chi(\lambda^{-1}\xi) \). Then, \( \sum_{\lambda \in 2^\mathbb{Z}} \chi_\lambda = 1 \) on \( \mathbb{R}^n \setminus \{0\} \) at it is locally finite. We define the (spatial) Fourier localization operator \( P_\lambda f = F^{-1}(\chi_\lambda F f) \). Further, we define \( \chi_{\leq \lambda} = \sum_{\mu \in 2^\mathbb{Z}, \mu \leq \lambda} \chi_\mu \) and \( P_{\leq \lambda} f = F^{-1}(\chi_{\leq \lambda} F f) \), \( P_{> \lambda} f = f - P_{\leq \lambda} f \). Let \( \tilde{\chi}_\lambda = \chi_{\lambda/2} + \chi_\lambda + \chi_{2\lambda} \) and \( \tilde{P}_\lambda f = F^{-1}(\tilde{\chi}_\lambda F f) \). Then \( \tilde{P}_\lambda P_\lambda = P_\lambda \tilde{P}_\lambda = P_\lambda \).

2. **Bilinear estimates for radial functions**

2.1. **Free solutions.** Since the characteristic hypersurface in \( \mathbb{R}^{1+n} \) defined by the phase function \( |\xi|^\alpha \) has \( n \) nonvanishing principal curvatures in the case \( \alpha > 1 \), there are similar Strichartz estimates as for the Schrödinger equation, up to a loss of derivatives dictated by scaling. For the following Lemma, its proof and more information we refer to [4].

**Lemma 2.1.** Let \( 1 < \alpha \leq 2, q > 2, r \geq 2, \frac{a}{q} + \frac{b}{2} = \frac{1}{q}, \theta = \frac{a}{q}(2-\alpha)(\frac{1}{q} - \frac{1}{2}). \) Then,
\[
|S\varphi|_{L_q^\infty(\mathbb{R}^n)} \lesssim \| \varphi \|_{H^\alpha(\mathbb{R}^n)}.
\]

Next, we adapt [10] Lemma 3.2 to the case of generalized dispersion.

**Lemma 2.2.** Let \( 1 \leq \alpha \leq 2 \). Consider the integral
\[
I(\phi, \psi)(\tau, \xi) = \int \phi(|\eta|) \psi(|\xi - \eta|) \delta(\tau - |\eta|^\alpha) d\eta
\]
for smooth \( \phi \) and \( \psi \) supported in \([-r, r]\) and \([-R, R]\), for some \( r, R > 0 \). Then for \( 0 \leq \tau \leq \alpha \max\{r, R\} \alpha^{-1} |\xi| \),
\[
I(\phi, \psi)(\tau, \xi) = \frac{2\pi}{\alpha |\xi|} \int_{a(\tau, |\xi|)}^\infty \phi(\rho) \psi(\omega(\tau, \rho)) \omega(\tau, \rho)^{2-\alpha} \rho d\rho
\]
where
\( a(\tau, |\xi|) = \frac{|\xi|^2 + \tau^{2/\alpha}}{2|\xi|} \)
and\( \omega(\tau, \rho) = (\rho^\alpha - \tau)^{1/\alpha} \).

Furthermore,
\[(2.3) \quad I(\phi, \psi)(\tau, \xi) = 0, \text{ if } \tau > \alpha \max\{|r, R|\}^{\alpha - 1}|\xi|.
\]

**Proof.** As in [10, pp. 8–9], the proof is an straightforward modification of the argument for \( \alpha = 1 \) from [5, Lemma 4.4]. First, we check that the delta function and support condition on \( \phi, \psi \) restrict the range of \( \tau \). By the mean-value theorem we have
\[ |\tau| = ||\eta|^\alpha - |\xi - \eta|^\alpha| \leq \alpha \max\{|\eta|, |\xi - \eta|\}^{\alpha - 1}|\xi|, \]
implying the second claim \((2.3)\).

Now, let \( 0 \leq \tau \leq \alpha \max\{|r, R|\}^{\alpha - 1}|\xi| \). Using \( \delta \)-calculus, we can write
\[(2.4) \quad \delta(\tau - |\eta|^\alpha + |\xi - \eta|^\alpha) = |\tau - |\eta|^\alpha - |\xi - \eta|^\alpha| \delta(|\xi - \eta|^2\alpha - (\tau - |\eta|^\alpha)^2) \]
\[ = 2(|\eta|^\alpha - \tau)\delta((|\xi|^2 - 2\xi \cdot \eta + |\eta|^2)^\alpha - (\tau - |\eta|^\alpha)^2), \]
having used \( 0 \leq |\xi - \eta|^\alpha = |\eta|^\alpha - \tau \leq \rho^\alpha - \tau \), we have \( |\xi - \eta| = \omega(\tau, \rho) \). Now, with
\[ g_{\rho^\alpha}(b) = (|\xi|^2 - 2|\xi| \rho + \rho^2)^\alpha - (|\eta|^\alpha - \tau)^2, \]
the integrand is independent of \( \theta' \) and we obtain
\[ I(\phi, \psi)(\tau, \xi) = 4\pi \int_{\tau}^{\infty} \int_{-1}^{1} \phi(\rho)\psi(\omega(\tau, \rho))(\rho^\alpha - \tau)^\alpha \rho^2 \delta(g_{\rho^\alpha}(b))dbd\rho. \]

We use the delta function to set the value of \( b \) to
\[(2.5) \quad \rho^\alpha := \frac{|\xi|^2 + \rho^2 - (\rho^\alpha - \tau)^{2/\alpha}}{2|\xi|}, \]
with the condition \( b \leq 1 \) that forces
\[ |\xi|^2 + \rho^2 \leq 2|\xi| \rho + (\rho^\alpha - \tau)^{2/\alpha} \]
and since \( (\rho^\alpha - \tau)^{2/\alpha} \leq \rho^2 - \tau^{2/\alpha} \), the domain of the \( \rho \)-integration is further restricted to
\[(2.6) \quad \left\{ \rho \geq \frac{|\xi|^2 + \tau^{2/\alpha}}{2|\xi|} \right\}. \]

We compute the integral over \( b \) as
\[(2.7) \quad \int_{-1}^{1} \delta(g_{\rho^\alpha}^\alpha(b))db = \left( \frac{d}{db}g_{\rho^\alpha}^\alpha(b) \right)^{-1} = (2\alpha|\xi| \rho(\rho^\alpha - \tau)^{2(\alpha - 1)/\alpha})^{-1}, \]
where \( b^\alpha \) is the value in \((2.3)\). With \((2.6) \) and \((2.7) \), we finally obtain
\[ I(\phi, \psi)(\tau, \xi) = 4\pi \int_{a(\tau, |\xi|)}^{\infty} \phi(\rho)\psi(\omega(\tau, \rho)) \rho(\rho^\alpha - \tau) \frac{\rho(\rho^\alpha - \tau)}{2\alpha|\xi|(\rho^\alpha - \tau)^{2\alpha - 1/\alpha}}d\rho, \]
which reduces to the desired form. \( \square \)
Proposition 2.3. Let $1 \leq \alpha \leq 2$. Consider $u^+(t) = S(t)f$ and $v^-(t) = S(-t)g$, where $f$ and $g$ are radial. Then for any $\mu > 0$ and $\lambda_1 \geq \lambda_2 > 0$, we have

\begin{equation}
\|P_{\mu}(u_{\lambda_1}^{+} v_{\lambda_2}^{-})\|_{L^{2}_{t,x}(\mathbb{R}^{1+3})} \lesssim \mu \lambda_2^{\frac{\alpha}{2}} \|f\|_{L^{2}_{x}(\mathbb{R}^{3})} \|g\|_{L^{2}_{x}(\mathbb{R}^{3})}
\end{equation}

Proof. The Fourier transform of a radial function is a radial function, and we may assume $\hat{f}, \hat{g} \geq 0$. We denote $\hat{f}\chi_{\lambda_1}, \hat{g}\chi_{\lambda_2}$ by $\psi_{\lambda_1}, \phi_{\lambda_2}$, respectively, and compute the space-time Fourier transform

\[
\mathcal{F}_{t,x}\{P_{\mu}(u_{\lambda_1}^{+} v_{\lambda_2}^{-})\}(\tau, \xi)
= \int_{\mathbb{R}} e^{-it\tau} \chi_{\mu}(\xi) \int_{\mathbb{R}^{3}} \chi_{\lambda_1}(|\xi - \eta|) e^{-it(\xi - \eta)^{\alpha}} \hat{f}(\xi - \eta) \chi_{\lambda_2}(|\eta|) e^{it|\eta|^{\alpha}} \hat{g}(\eta) d\eta dt
= \chi_{\mu}(\xi) \int_{\mathbb{R}^{3}} \psi_{\lambda_1}((\xi - \eta)) \phi_{\lambda_2}((\eta)) \delta(\tau - |\eta|^{\alpha} + |\xi - \eta|^{\alpha}) d\eta.
\]

For $0 \leq \tau \leq C_{\alpha} \lambda_1^{\alpha - 1} \mu$, Lemma 2.2 implies

\[
\mathcal{F}_{t,x}\{P_{\mu}(u_{\lambda_1}^{+} v_{\lambda_2}^{-})\}(\tau, \xi) \approx \chi_{\mu}(\xi) \frac{1}{|\xi|} \int_{a(\tau, |\xi|)}^{\infty} \phi_{\lambda_2}(\rho) \psi_{\lambda_1}(\omega(\tau, \rho)) \omega(\tau, \rho)^{2-\alpha} \rho d\rho,
\]

while for $\tau > C_{\alpha} \lambda_1^{\alpha - 1} \mu$ there is no contribution. If $-C_{\alpha} \lambda_1^{\alpha - 1} \mu \leq \tau \leq 0$, we similarly obtain

\[
\mathcal{F}_{t,x}\{P_{\mu}(u_{\lambda_1}^{+} v_{\lambda_2}^{-})\}(\tau, \xi) \approx \chi_{\mu}(\xi) \frac{1}{|\xi|} \int_{a(-\tau, |\xi|)}^{\infty} \phi_{\lambda_2}(\omega(-\tau, \rho)) \psi_{\lambda_1}(\rho) \omega(-\tau, \rho)^{2-\alpha} \rho d\rho,
\]

while for $\tau < -C_{\alpha} \lambda_1^{\alpha - 1} \mu$ there is no contribution. Hence,

\[
\|P_{\mu}(u_{\lambda_1}^{+} v_{\lambda_2}^{-})\|_{L^{2}_{t,x}(\mathbb{R}^{1+3})} \lesssim I_1 + I_2,
\]

where, by Plancherel’s theorem,

\[
I_1 \approx \int_{\mathbb{R}} \int_{0}^{C_{\alpha} \lambda_1^{\alpha - 1} \mu} \frac{\chi_{\mu}^{2}(\xi)}{|\xi|^{2}} \int_{a(\tau, |\xi|)}^{\infty} [\rho \phi_{\lambda_2}(\rho)] |\omega(\tau, \rho)^{2-\alpha} \psi_{\lambda_1}(\omega(\tau, \rho))| d\rho\ d\tau d\xi,
\]

\[
I_2 \approx \int_{\mathbb{R}} \int_{0}^{C_{\alpha} \lambda_1^{\alpha - 1} \mu} \frac{\chi_{\mu}^{2}(\xi)}{|\xi|^{2}} \int_{a(-\tau, |\xi|)}^{\infty} [\rho \psi_{\lambda_1}(\rho)] |\omega(\tau, \rho)^{2-\alpha} \phi_{\lambda_2}(\omega(\tau, \rho))| d\rho\ d\tau d\xi.
\]

In polar coordinates $\xi \rightarrow (r, \theta) \in [0, \infty) \times S^{2}$, the first term is

\[
I_1 \approx \int_{0}^{\infty} \chi_{\mu}^{2}(r) \int_{0}^{C_{\alpha} \lambda_1^{\alpha - 1} \mu} \int_{a(\tau, r)}^{\infty} [\rho \phi_{\lambda_2}(\rho)] |\omega(\tau, \rho)^{2-\alpha} \psi_{\lambda_1}(\omega(\tau, \rho))| d\rho\ d\tau dr.
\]

The Cauchy-Schwarz inequality implies

\[
\left| \int_{a(\tau, r)}^{\infty} [\rho \phi_{\lambda_2}(\rho)] |\omega(\tau, \rho)^{2-\alpha} \psi_{\lambda_1}(\omega(\tau, \rho))| d\rho \right|^{2}
\lesssim \left( \int_{\mathbb{R}} |\phi_{\lambda_2}(\rho)|^{2} d\rho \right) \left( \int_{\mathbb{R}} |\omega(\tau, \rho)^{2-\alpha} \psi_{\lambda_1}(\omega(\tau, \rho))\chi_{\lambda_2}(\rho)|^{2} d\rho \right)
\lesssim \|g\|_{L^{2}(\mathbb{R}^{3})}^{2} \left( \int_{\mathbb{R}} \sigma^{2-\alpha} \psi_{\lambda_1}(\sigma) |\chi_{\lambda_2}(\sigma)|^{2} d\sigma \right)^{2} \left( \frac{\lambda_1^{\alpha - 1}}{\lambda_2^{\alpha}} \right)^{2}.
\]
Here, we used the change of variables $\sigma = \omega(\tau, \rho)$, hence $\sigma^\alpha = \rho^\alpha - \tau$ and $d\rho = (\sigma^\alpha)^{-1}d\sigma$, and in the domain of integration we have $|\sigma^{-\alpha-1}| \lesssim (\lambda_2)^{-\alpha-1}$. We obtain

$$I_1 \lesssim \mu \|g_{12}\|^2_{L^2_{\pi}(\mathbb{R}^3)}\lambda_1^{\alpha-1}\mu \left( \frac{1}{\lambda_1\lambda_2} \right)^{\alpha-1} \|f_{\lambda_1}\|^2_{L^2_{\pi}(\mathbb{R}^3)}$$

$$\lesssim \mu^2 \lambda_2^{1-\alpha} \|g_{12}\|_{L^2_{\pi}(\mathbb{R}^3)}^2 \|f_{\lambda_1}\|^2_{L^2_{\pi}(\mathbb{R}^3)}.$$  

Concerning $I_2$, we obtain

$$I_2 \lesssim \mu \|f_{\lambda_1}\|^2_{L^2_{\pi}(\mathbb{R}^3)} \int_0^{C_{\alpha} \lambda_1^{1-\alpha}\mu} \int_{\mathbb{R}} |\omega(\tau, \rho)^{2-\alpha} \phi_{\lambda_2}(\omega(\tau, \rho))|^2 d\rho d\tau$$

along the same lines. Again,

$$I_2 \lesssim \mu^2 \lambda_2^{1-\alpha} \|g_{12}\|_{L^2_{\pi}(\mathbb{R}^3)}^2 \|f_{\lambda_1}\|^2_{L^2_{\pi}(\mathbb{R}^3)},$$

by the same change of variables as above. 

**Corollary 2.4.** Let $1 \leq \alpha \leq 2$. For all dyadic $\mu \lesssim \lambda_1 \sim \lambda_2$ and spatially radial functions $u_j = S(\cdot)\varphi_j$, we have

$$\|P_{\leq \mu}(P_{\lambda_1}u_1, P_{\lambda_2}u_2)\|_{L^2(\mathbb{R}^{2+n})} \lesssim \mu \left( \frac{\mu}{\lambda_1} \right)^{\frac{n+1}{2}} \|P_{\lambda_1}\varphi_1\|_{L^2(\mathbb{R}^3)} \|P_{\lambda_2}\varphi_2\|_{L^2(\mathbb{R}^3)}.$$  

**Proof.** This follows by dyadic summation over $\mu' \leq \mu$ from Proposition 2.3. □

### 2.2. Transference

Let $1 \leq p < \infty$. We call a finite set $\{t_0, \ldots, t_K\}$ a partition if $-\infty < t_0 < t_1 < \ldots < t_K \leq \infty$, and denote the set of all partitions by $T$. A corresponding step-function $a : \mathbb{R} \to L^2(\mathbb{R}^3)$ is called $U^p_S$-atom if

$$a(t) = \sum_{k=1}^{K} \mathbf{1}_{(t_{k-1}, t_k]}(t)S(t)\varphi_k, \quad \sum_{k=1}^{K} \|\varphi_k\|^p_{L^2(\mathbb{R}^3)} = 1, \quad \{t_0, \ldots, t_K\} \in T;$$

and $U^p_S$ is the atomic space. Further, let $V^p_S$ be the space of all right-continuous $v : \mathbb{R} \to L^2(\mathbb{R}^3)$ satisfying

$$\|v\|_{V^p_S} := \sup_{\{t_0, \ldots, t_K\} \in T} \left( \sum_{k=1}^{K} \|S(-t_k)v(t_k) - S(-t_{k-1})v(t_{k-1})\|^p_{L^2(\mathbb{R}^3)} \right)^{\frac{1}{p}}.$$  

with the convention $S(-t_K)v(t_K) = 0$ if $t_K = \infty$. For the theory of $U^p_S$ and $V^p_S$, see e.g. [6] [7] [12]. For $s \in \mathbb{R}$ let

$$\|u\|_{X^s} = \left( \sum_{\lambda \in 2^\mathbb{Z}} \lambda^{2s} \|P_{\lambda}u\|^2_{V^p_S} \right)^{\frac{1}{2}}.$$  

By the atomic structure of $U^p_S$, estimates in $L^2$ for free solutions transfer to $U^p_S$-functions, hence to $V^p_S$ for $p < 2$. However, transference to $V^p_S$ does not follow from the general theory of these spaces. Nevertheless, we prove below that in case of the bilinear estimate of the previous section it does hold true. This might also have applications in the case $\alpha = 1$, and the proof applies to certain other multilinear estimates.

**Proposition 2.5.** Let $1 \leq \alpha \leq 2$. For all dyadic $\mu \lesssim \lambda_1 \sim \lambda_2$ and spatially radial functions $u_1, u_2 \in V^p_S$, we have

$$\|P_{\leq \mu}(P_{\lambda_1}u_1, P_{\lambda_2}u_2)\|_{L^2(\mathbb{R}^{1+n})} \lesssim \mu^{\frac{2}{\alpha}} \left( \frac{\mu}{\lambda_1} \right)^{\frac{2}{\alpha}} \|P_{\lambda_1}u_1\|_{V^p_S} \|P_{\lambda_2}u_2\|_{V^p_S}.$$
Proof. 1. Step: Let \( P = \sum_{\lambda \in F} P_{\lambda} \), with a finite set \( F \) of dyadic numbers \( \lambda \) of size \( \lambda_1 \sim \lambda_2 \), such that \( PP_{\lambda_j} = P_{\lambda_j} \) for \( j = 1, 2 \). We claim that

\[
\| P_{\leq \mu} |Pw|^2 \|_{L^2} \lesssim \mu^{2^n} \left( \frac{\mu}{\lambda_1} \right)^{2^n} \| w \|_{L^2}^2
\]

for any \( \mu \ll \lambda_1 \) and spatially radial \( w \in U_S^4 \). To prove (2.11), let \( V_{\mu} = (\tilde{\chi}_{\leq 2\mu})^2 \). Then, \( V_{\mu} \geq 0 \) and we have the pointwise bound \( \chi_{\leq \mu} \lesssim \chi_{\leq 2\mu} \lesssim \chi_{\leq 4\mu} \) on the Fourier side, which implies

\[
\| P_{\leq \mu} |Pw|^2 \|_{L^2} \lesssim \| |Pw|^2 \|_{L^2} \lesssim \| P_{\leq 4\mu} |Pw|^2 \|_{L^2}.
\]

The quantity

\[
n(f) := \| \Phi(f) \|_{L^4(\mathbb{R}^3)}, \text{ for } \Phi(f) = \left( V_\mu |Pf|^2 \right)^{\frac{1}{2}},
\]

is subadditive, and \( |V_\mu |Pw|^2 \|_{L^2} = \| n(w(t)) \|_{L^4}^2 \). Indeed,

\[
\Phi^2(f_1 + f_2)(x) = \int_{\mathbb{R}^3} V_\mu(x - y)|Pf_1(y) + Pf_2(y)|^2 dy
\]

\[
\leq \int_{\mathbb{R}^3} V_\mu(x - y)|Pf_1(y) + Pf_2(y)||Pf_1(y)|dy
\]

\[
+ \int_{\mathbb{R}^3} V_\mu(x - y)|Pf_1(y) + Pf_2(y)||Pf_2(y)|dy
\]

\[
\leq \Phi(f_1 + f_2)(x)\Phi(f_1)(x) + \Phi(f_1 + f_2)(x)\Phi(f_2)(x)
\]

by Cauchy-Schwarz, so \( \Phi(f_1 + f_2) \leq \Phi(f_1) + \Phi(f_2) \) and

\[
n(f_1 + f_2) \leq \| \Phi(f_1) + \Phi(f_2) \|_{L^4} \leq n(f_1) + n(f_2)
\]

follows. Also, we obviously have \( n(cf) = |c|n(f) \) for all \( c \in \mathbb{C} \). Due to (2.9) we have

\[
n(S(t)\varphi) \|_{L^4} \leq C(\mu, \lambda_1)^{\frac{1}{2}} \| \varphi \|_{L^2}
\]

for all radial \( \varphi \in L^2(\mathbb{R}^3) \), where \( C(\mu, \lambda_1) \) denotes the constant in (2.9). Let \( w \in U_S^4 \) be radial with atomic decomposition

\[
w = \sum_j c_j a_j, \text{ s.th. } \sum_j |c_j| \leq 2 \| w \|_{U_S^4}, \text{ and radial } U_S^4\text{-atoms } a_j.
\]

We have

\[
n(w) \|_{L^4} \leq \sum_j |c_j|n(a_j) \|_{L^4} \leq C(\mu, \lambda_1)^{\frac{1}{2}} \| w \|_{U_S^4},
\]

provided that for any \( U_S^4 \)-atom \( a \) the estimate

\[
n(a) \|_{L^4} \leq C(\mu, \lambda_1)^{\frac{1}{2}}
\]

holds true. Indeed, let \( a(t) = \sum_k 1_{I_k}(t)S(t)\varphi_k \), for some partition \((I_k)\) of \( \mathbb{R} \) and radial \( \varphi_k \in L^2(\mathbb{R}^3) \) satisfying \( \sum_k \| \varphi_k \|_{L^2}^2 \leq 1 \). Then,

\[
n(a) \|_{L^4} \leq \left( \sum_k \| 1_{I_k}(t)n(S(t)\varphi_k) \|_{L^4} \right)^{\frac{1}{2}}
\]

\[
\lesssim C(\mu, \lambda_1)^{\frac{1}{2}} \left( \sum_k \| \varphi_k \|_{L^2} \right)^{\frac{1}{2}} \lesssim C(\mu, \lambda_1)^{\frac{1}{2}}.
\]
where we used (2.12) in the third inequality, which completes the proof of (2.13).
This implies
\begin{equation}
\|P_{\leq \mu}|Pw|^2\|_{L^2(\mathbb{R}^{1+3})} \lesssim \|\nu(w)\|^2_{L^4_t} \lesssim C(\mu, \lambda_1)\|w\|^2_{U^2_G},
\end{equation}
hence the claim (2.11).

2. Step: Let \( v_j := P_\lambda u_j, j = 1, 2 \). We may assume \( \|v_j\|_{U^2_G} = 1 \). The functions \( w_{\pm} = v_1 \pm v_2 \) satisfy \( w_{\pm} = Pw_{\pm}, \|w_{\pm}\|_{U^2_G} \lesssim 1, \)
\begin{align*}
\text{Re}(v_1 \overline{v_2}) = & \frac{1}{4} \left( |w_+|^2 - |w_-|^2 \right), \quad \text{and} \quad \text{Im}(v_1 \overline{v_2}) = \text{Re}(-iv_1 \overline{v_2}).
\end{align*}
The estimate (2.11) yields
\begin{align*}
\|P_{\leq \mu}(v_1 \overline{v_2})\|_{L^2(\mathbb{R}^{1+3})} & \lesssim \|P_{\leq \mu}|Pw_+|^2\|_{L^2(\mathbb{R}^{1+3})} + \|P_{\leq \mu}|Pw_-|^2\|_{L^2(\mathbb{R}^{1+3})} \\
& \lesssim C(\mu, \lambda_1) \left( \|w_+\|^2_{U^2_G} + \|w_-\|^2_{U^2_G} \right) \\
& \lesssim C(\mu, \lambda_1),
\end{align*}
which implies
\begin{align*}
\|P_{\leq \mu}(v_1 \overline{v_2})\|_{L^2(\mathbb{R}^{1+3})} & \lesssim C(\mu, \lambda_1)\|v_1\|_{U^2_G}\|v_2\|_{U^2_G} \lesssim C(\mu, \lambda_1)\|v_1\|_{V^2_G}\|v_2\|_{V^2_G},
\end{align*}
where we used \( V^2_G \rightarrow U^2_G \), see [6].

**Proposition 2.6.** For any \( \mu, \lambda_1, \lambda_2 \in 2\mathbb{Z} \) and \( u, v \in U^2_G \),
\begin{equation}
\|P_{\mu}(P_\lambda u \overline{P_\lambda v})\|_{L^2(\mathbb{R}^{1+3})} \lesssim \min\{\mu, \lambda_1, \lambda_2\}^{\frac{2-\alpha}{4} + \frac{2-\alpha}{2}} \|P_{\lambda_1}u\|_{U^2_G}\|P_{\lambda_2}v\|_{U^2_G}.
\end{equation}

**Proof.** The Bernstein inequality implies
\begin{equation}
\|P_{\mu}(P_\lambda u \overline{P_\lambda v})\|_{L^2(\mathbb{R}^{1+3})} \lesssim \mu^{\frac{1}{4}} \|P_{\lambda_1}u \overline{P_\lambda v}\|_{L^2_{t,x}} \lesssim \mu^{\frac{1}{4}} \|P_{\lambda_1}u\|_{L^4_t L^2_x}\|P_{\lambda_2}v\|_{L^4_t L^2_x}
\end{equation}
and Lemma 2.1 gives (2.15) if \( \mu \leq \lambda_1, \lambda_2 \). Similarly,
\begin{equation}
\|P_{\mu}(P_\lambda u \overline{P_\lambda v})\|_{L^2(\mathbb{R}^{1+3})} \lesssim \|P_{\lambda_1}u\|_{L^4_t L^2_x}\|P_{\lambda_2}v\|_{L^4_t L^2_x} \lesssim \lambda_1^{\frac{1}{4}} \|P_{\lambda_1}u\|_{L^4_t L^2_x}\|P_{\lambda_2}v\|_{L^4_t L^2_x}.
\end{equation}
This concludes the proof because we can interchange the roles of \( u \) and \( v \).

**Remark 2.7.** In the case \( 1 < \alpha \leq 2 \) this gives another simple proof of a result which is slightly weaker than (2.10) but sufficient for our application: The obvious \( U^2_G \)-version of (2.10) (see [6, Prop. 2.19]) can now be interpolated with (2.15) in the case \( \mu \lesssim \lambda_1 \sim \lambda_2 \) via [6, Prop. 2.20], which gives
\begin{equation}
\|P_{\mu}(P_\lambda u \overline{P_\lambda v})\|_{L^2(\mathbb{R}^{1+3})} \lesssim \mu^{\frac{1}{4}} \left( \frac{\mu}{\lambda_1} \right)^{\frac{2-\alpha}{4} - \epsilon} \|P_{\lambda_1}u\|_{U^2_G}\|P_{\lambda_2}v\|_{V^2_G},
\end{equation}
for any fixed \( \epsilon > 0 \).

From now on we set \( X := \{ u \in X^0 \mid u \text{ spatially radial} \} \).

**Corollary 2.8.** Let \( 1 < \alpha \leq 2 \). For all \( u, v \in X \), we have
\begin{equation}
\|(-\Delta)^{\frac{\alpha-3}{4}} (uv)\|_{L^2(\mathbb{R}^{1+3})} \lesssim \|u\|_X \|v\|_X.
\end{equation}

**Proof.** We decompose
\begin{equation}
\|(-\Delta)^{\frac{\alpha-3}{4}} (uv)\|_{L^2(\mathbb{R}^{1+3})} \lesssim \sum_{\mu, \lambda_1, \lambda_2 \in 2\mathbb{Z}} \mu^{\frac{\alpha-3}{4}} \|P_{\mu}(u_\lambda \overline{v_\lambda})\|_{L^2(\mathbb{R}^{1+3})} \lesssim \Sigma_1 + \Sigma_2 + \Sigma_3,
\end{equation}
where
where $\Sigma_1$ is the contribution of $\lambda_1 \ll \lambda_2 \sim \mu$, $\Sigma_2$ is the contribution of $\lambda_2 \ll \lambda_1 \sim \mu$, and $\Sigma_3$ is the contribution of $\lambda_1 \sim \lambda_2 \gtrsim \mu$. From (2.14) and Cauchy-Schwarz we obtain

$$|\Sigma_1| \lesssim \sum_{\lambda_1 \ll \lambda_2} \left( \frac{\lambda_1}{\lambda_2} \right)^{\frac{4-\alpha}{4}} \|P_{\lambda_1}u\|_{V_2^\alpha} \|P_{\lambda_2}v\|_{V_2^\alpha} \lesssim \|u\|_X \|v\|_X.$$ 

Similarly we prove

$$|\Sigma_2| \lesssim \|u\|_X \|v\|_X.$$ 

Finally, using (2.10) we obtain

$$|\Sigma_3| \lesssim \sum_{\mu \leq \lambda_1 \sim \lambda_2} \left( \frac{\mu}{\lambda_1} \right)^{\frac{\alpha - 1}{4}} \|P_{\lambda_1}u\|_{V_2^\alpha} \|P_{\lambda_2}v\|_{V_2^\alpha} \lesssim \|u\|_X \|v\|_X,$$

where we have exploited the radiality. \hfill \square

3. Proofs of the main results

3.1. Proof of Theorem 1.1

It suffices to consider positive times. We represent the solution of (1.1) on $[0, \infty)$ using the Duhamel’s formula

$$u(t) = 1_{[0,\infty)}(t)S(t)\varphi + i\sigma J(u, u, u)(t),$$

$$J(u_1, u_2, u_3)(t) = 1_{[0,\infty)}(t) \int_0^t S(t - t') \left[ \{ | \cdot |^{-\alpha} * (u_1 \overline{u_2}) \} u_3 \right](t') dt'.$$

For all $\varphi \in H^s_{\text{rad}}$ we immediately have

$$\|1_{[0,\infty)} S\varphi\|_{X^s} \lesssim \|\varphi\|_{H^s}.$$

Next, we study the nonlinear part. For all $u_1, u_2, u_3 \in X$, we have

$$\|J(u_1, u_2, u_3)\|_X \lesssim \|u_1\|_X \|u_2\|_X \|u_3\|_X.$$ 

Indeed, we have

$$\|J(u_1, u_2, u_3)\|_X \lesssim \sup_{v \in X, \|v\|_X \leq 1} \left| \int \int (-\Delta)^{\frac{\alpha - 1}{2}} (u_1 \overline{u_2}) u_3(t) \overline{v(t)} dt dx \right|$$

by a standard duality argument, see e.g. [6]. Further, we obtain

$$\left| \int \int (-\Delta)^{\frac{\alpha - 1}{2}} (u_1 \overline{u_2}) u_3(t) \overline{v(t)} dt dx \right| \lesssim \|(-\Delta)^{\frac{\alpha - 1}{2}} (u_1 \overline{u_2})\|_{L^2} \|(-\Delta)^{\frac{\alpha - 1}{2}} (u_3 \overline{v})\|_{L^2} \lesssim \|u_1\|_X \|u_2\|_X \|u_3\|_X \|v\|_X$$

by Corollary 2.8, which implies (3.4).

Theorem 1.1 now follows from the standard approach via the contraction mapping principle. In particular, the scattering claim follows from the fact that functions in $V^2_1$ have a limit at $\infty$. We omit the details.

3.2. Proof of Theorem 1.2

Let $\Lambda, r \geq 1$ be given. Recall that

$$B_{r, \Lambda} := \{ \varphi \in L^2_{\text{rad}}(\mathbb{R}^3) : \|\varphi\|_{L^2} \leq r, \|P_{> \Lambda} \varphi\|_{L^2} \leq \eta r^{-1} \},$$

where the parameter $0 < \eta \leq 1$ will be determined below, and define

$$D_{R, \epsilon} := \{ u \in X : \|u\|_X \leq R, \|P_{> \Lambda} u\|_X \leq \epsilon \},$$
for some $0 < \epsilon \leq R$. We implicitly assume that all functions are supported in $[0, T]$. Split $J(u) = J_1(u_{\leq A}, u_{> A}) + J_2(u_{\leq A}, u_{> A})$, where $J_1$ is at least quadratic in $u_{> A}$ and $J_2$ is at least quadratic in $u_{\leq A}$. For $J_1$, we use \[3.4\] and obtain

$$
\|J_1(u_{\leq A}, u_{> A})\|_X \lesssim R\epsilon^2,
$$

for all $u \in D_{R,\epsilon}$. We turn to $J_2$. First, we have

$$
\|(-\Delta)^{\frac{3}{2}}(u_{\leq A} u_{\leq A})v\|_{L^1_t L^2_x} \lesssim \Lambda^\alpha \|u_{\leq A}\|_{L^2_t L^2_x}^2 \|v\|_{L^\infty_t L^2_x} \lesssim TA^\alpha \|u\|_X^2 \|v\|_X.
$$

Second, we have

$$
\|(-\Delta)^{\frac{3}{2}}(u_{\leq A} \tau_{> A})u_{\leq A}\|_{L^1_t L^2_x} \lesssim \Lambda^{\alpha-3} \|u_{\leq A} \tau_{> A}\|_{L^1_t L^2_x} \|u_{\leq A}\|_{L^\infty_t L^\infty_x} \lesssim TA^{\alpha-3} \|\tau_{> A}\|_{L^\infty_t L^\infty_x} \|u_{\leq A}\|_{L^2_t L^2_x}^2 \lesssim TA^\alpha \|u\|_X^2 \|v\|_X,
$$

and we obtain

$$
\|(-\Delta)^{\frac{3}{2}}(u_{\leq A} \tau)u_{\leq A}\|_{L^1_t L^2_x} \lesssim TA^\alpha \|u\|_X^2 \|v\|_X,
$$

because the contribution of $u_{\leq 4A}$ can be treated as in the first estimate above. We conclude that there exists a $C \geq 1$, such that for all $\varphi \in B_{r,A}$ and all $u \in D_{R,\epsilon}$ we have

$$
\|1_{[0,T]}S\varphi\|_X \leq Cr, \quad \|1_{[0,T]}S\varphi_{> A}\|_X \leq Cn^{-1},
$$

$$
\|J_1(u_{\leq A}, u_{> A})\|_X \leq C R\epsilon^2, \quad \|J_2(u_{\leq A}, u_{> A})\|_X \leq C TA^\alpha R^3,
$$

and similar estimates for differences. After choosing $R = 4Cr$, $\epsilon = 2^{-6}C^{-2}r^{-1}$, $T = 2^{-18}C^{-6}r^{-4}A^{-\alpha}$, and $n = 2^{-10}C^{-3}$, one checks that

$$
D_{R,\epsilon} \to D_{R,\epsilon}, \quad u \mapsto 1_{[0,T]}(S\varphi + J(u))
$$

is a strict contradiction, for given $\varphi \in B_{r,A}$. The contraction mapping principle implies Theorem \[1.2\]

### 3.3. Proof of Theorem \[1.3\]

We apply the abstract ill-posedness result established by Bejenaru and Tao in [\[1\] Proposition 1]. For this, it suffices to show that the following inequality

$$
\sup_{t \in [0,T]} \left\| \int_0^t S(t - t')(\cdot)^{-\alpha} * |S(t)\varphi|^2S(t)\varphi dt' \right\|_{H^s(\mathbb{R}^3)} \lesssim \|\varphi\|_{H^s(\mathbb{R}^3)}^3
$$

fails to hold for some radial data $\varphi \in H^s(\mathbb{R}^3)$, if $s < 0$. We modify the construction for the case $\alpha = 1$ from [\[3\] Section 3] and define the annulus $A_\lambda = \{ \xi \in \mathbb{R}^3 : |\xi| \leq 2\lambda \}$. Let $\varphi$ be the inverse Fourier transform of the characteristic function $1_{A_\lambda}$. Clearly, $\varphi$ is radial and $\|\varphi\|_{H^s(\mathbb{R}^3)} \sim \lambda^{s+\frac{3}{2}}$. With this choice of $\varphi$, let

$$
\Phi(t) := \int_0^t S(t - t')(\cdot)^{-\alpha} * |S(t)\varphi|^2S(t)\varphi dt'.
$$
We compute the spatial Fourier transform
\[
\hat{\Phi}(t, \xi) = \int_0^t \int_{\mathbb{R}^3} \mathcal{F}_x(S(t-t')\mathcal{F}_x(\varphi^2)(\eta)/|\eta|^{3-\alpha}) \mathcal{F}_x S(t) \varphi(\xi - \eta) \, d\eta \, dt'
\]
\[
= \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-i(t-t')|\xi|^\alpha} e^{-i(t-t')|\eta|^\alpha} 1_{A_\lambda}(\eta - \sigma) e^{i|\sigma|^\alpha} 1_{A_\lambda}(\sigma) e^{-i(t-t')|\xi - \eta|^\alpha} 1_{A_\lambda}(\xi - \eta) \, d\sigma \, d\eta \, dt'
\]
\[
= e^{-i|\xi|^\alpha} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{it' g_\alpha(\xi, \eta, \sigma)} dt' \frac{1_{A_\lambda}(\eta)1_{A_\lambda}(\sigma)1_{A_\lambda}(\xi - \eta - \sigma)}{|\eta + \sigma|^{3-\alpha}} \, d\sigma \, d\eta \, dt
\]
where
\[
g_\alpha(\xi, \eta, \sigma) = |\xi|^\alpha - |\eta|^\alpha + |\sigma|^\alpha - |\xi - \eta - \sigma|^\alpha.
\]
Choose \( T = \epsilon \lambda^{-\alpha} \) with \( 0 < \epsilon \ll 1 \). In the domain of integration we have
\[
|tg_\alpha(\xi, \eta, \sigma)| \lesssim |t\lambda^\alpha| \ll 1
\]
and we get
\[
(3.8) \quad \left| \int_0^t e^{it' g_\alpha(\xi, \eta, \sigma)} \, dt' \right| \gtrsim \int_0^t \cos (t' g_\alpha(\xi, \eta, \sigma)) \, dt' \gtrsim |t| \text{ for } t < T.
\]
Thus, if \( \xi \in A_\lambda \), we have
\[
|\hat{\Phi}(t, \xi)| \gtrsim |t| \int_{A_\lambda} \int_{A_\lambda} \frac{|\eta + \sigma|^{-3+\alpha} \, d\sigma \, d\eta} \gtrsim \epsilon \lambda^{-\alpha} \lambda^{3+\alpha}.
\]
From this we easily obtain \( \|\hat{\Phi}\|_{H^s(\mathbb{R}^3)} \gtrsim \epsilon \lambda^{s+\frac{3}{2}} \). In conclusion, these norm calculations and (3.7) give
\[
\epsilon \lambda^{s+\frac{3}{2}} \lesssim \|\Phi(t)\|_{H^s} \lesssim \|\varphi\|^3_{H^s} \sim \lambda^{3s+\frac{3}{2}}.
\]
So if \( s < 0 \), we make (3.7) fail to hold by choosing \( \lambda \) sufficiently large.

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