Effect of non-Newtonian gravity on the amplitude of neutron reflection from bulk materials

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Abstract. We have analysed the effect of non-Newtonian gravity on the amplitude of neutron reflection from bulk materials. We found that the first-order contribution of non-Newtonian gravity is significant when the neutron wavelength is nearly equal to the radius of curvature of the neutron refractive index due to non-Newtonian gravity and that it becomes negligible when the neutron wavelength is much smaller than the radius of curvature.

1. Introduction
Several speculative theories have been proposed with the aim of unifying quantum theory and general relativity. Some of these theories predict that the law of gravity will be modified at low accelerations. Several experiments have been performed in attempts to confirm the existence of this modification. However, no deviations have been observed from the predictions of Newtonian gravity [1], although the experiments do not provide strong constraints for length scales smaller than a millimetre. Attempts at verification in this range are attracting interest because several theories involving extra dimensions [2–4] or theories that postulate scalar particles originating from supersymmetry breaking [5] predict a non-Newtonian gravity-like force in this range. In experimental attempts to verify these predictions, measurements have been performed by varying the positions of several materials while measuring the forces between them [6]. When the distance between the materials is less than 100 nm, the effect of the Casimir force [7] cannot be neglected because the materials have finite dielectric constants. Therefore, to obtain experimental constraints on non-Newtonian gravity requires precisely evaluating Casimir forces.

On the other hand, it is not necessary to consider Casimir forces when performing experiments using neutrons because neutrons have a very small dielectric constant. Consequently, experiments employing neutrons can provide the strongest constraints for length scales smaller than 100 nm [8]. The neutron scattering amplitudes of materials are used when determining the constraints [9–12]. Because the neutron–nucleon scattering length, the neutron–electron scattering length, and the non-Newtonian gravity all contribute to the scattering amplitude and they have different momentum transfer dependences, three amplitudes with different momentum transfers have to be measured to estimate the non-Newtonian contribution. Further improvements could be achieved if experiments could be performed that are not affected by the neutron–electron scattering amplitude. Because the neutron charge radius is less than \(10^{-11}\) m, the neutron-electron scattering contribution to the total scattering amplitude can be ignored when very cold neutrons are used. Nesvizhevsky and coworkers
have proposed an experiment that involves measuring two neutron scattering amplitudes from rare noble gases by using very cold neutrons [13].

In this paper, we investigate the effects of non-Newtonian gravity on the reflection amplitude of very cold neutrons from bulk materials. In section 2, we introduce the notation for the Schrödinger equations inside and outside the bulk material. As the contribution of non-Newtonian gravity to the neutron potential is limited to the region near the bulk surface, we can select the two independent solutions of the Schrödinger equations that approach plane waves far from the surface. Taking the product of the reflection and transmission coefficients of these solutions, we obtain the wave functions for this setup. Using the continuities of the wave function and its logarithmic derivative on the surface of the material, we obtain the reflection amplitude from the material in section 3. In section 4, we analyse the reflection amplitude analytically by assuming that the non-Newtonian gravity potential is much smaller than the kinetic energy. In section 5, we numerically evaluate the contribution of non-Newtonian gravity to the reflection amplitude. Finally, we summarise the results obtained in this paper in section 6.

2. Schrödinger equations inside and outside a bulk material

We first parameterise the correction to Newton gravity as follows:

\[ V(r) = -\frac{GMm}{r} \left( 1 + \alpha \xi \right) \]

(1)

Here, \( G \) is the gravitational constant, \( M \) is the mass of the bulk material’s element, \( m \) is the neutron mass, \( r \) is the distance from the bulk material’s element to the neutron, \( \alpha \) is a dimensionless parameter that indicates the strength of the deviation from the Newtonian potential, and \( \xi \) is the effective range of the anomalous potential. As we are interested in the case for which \( \alpha \) is much larger than unity, we ignore the Newtonian part of the gravitational potential. We define the \( z \)-axis as the normal to the bulk surface and take the origin of the \( z \)-axis to be on the surface. Moreover, we assume the bulk extends infinitely in the \( xy \)-plane and that the material exists in the positive \( z \) range. In this case, we obtain that the potential for a neutron along the \( z \)-axis by integrating (1) over the volume of the bulk material, which gives:

\[ V(z) = \begin{cases} 
V_g e^{2k_g z} & z < 0 \\
V_f + V_g (2 - e^{-2k_g z}) & z > 0, 
\end{cases} \]

(2)

where

\[ k_g = \frac{1}{\lambda_g}, \quad V_g = \frac{\alpha}{2k_g^2}, \quad \text{and} \quad \alpha = -\frac{GM\rho}{2\xi} \]

(3)

Here, \( V_f \) and \( \rho \) are the Fermi potential and the weight density of the bulk material, respectively.

In this study, we consider the case when the kinetic energy is larger than potential energy. In this case, the Schrödinger equations along the \( z \)-axis outside and inside the material can be written as:

\[ \begin{align*}
\psi''(z) + (k_0^2 - \eta e^{2k_g z}) \psi(z) &= 0, \quad z < 0 \\
\psi''(z) + (k_1^2 + \eta e^{-2k_g z}) \psi(z) &= 0, \quad z > 0
\end{align*} \]

(4)

where

\[ k_0 = k, \quad k_1 = \sqrt{k^2 - \frac{2m}{\hbar^2} (V_f + 2V_g)}, \quad \text{and} \quad \eta = \frac{2m}{\hbar^2} V_g \]

(5)

Here, \( \hbar \) is Planck’s constant and \( k \) is the \( z \)-component of the neutron wavenumber.

3. Exact value of the reflection amplitude

The analytical solution of (4) can be obtained by using Bessel functions. Outside the material, the two independent solutions that approach \( e^{\pm ik_0 z} \) far from the boundary can be written as:
Here, $\phi_0(\xi) = \Gamma(1 + i\kappa_0) \left( \frac{\sqrt{\eta}}{2k_0} \right)^{\frac{\pm i k_0}{k_0}} I_{\pm i\kappa_0}(e^{\mp i k_0})$.

(6)

Here, $\kappa_0$ is the ratio of the neutron wavenumber to the inverse of the effective range of non-Newtonian gravity, $k_0/k_0$, $\Gamma(\mu)$ is the gamma function, and $I_{\nu}(\xi)$ is the modified Bessel function of the first kind.

Far from the boundary, the parameter of the Bessel function, $\nu$, becomes small, which allows us to employ the following approximation:

$I_{\nu}(\xi) \sim \left( \frac{\xi}{2} \right)^{\nu} \frac{1}{\Gamma(1 + \nu)}$.

(7)

Using this approximation, we can verify that (6) approaches $e^{\pm i k_0 \xi}$. Inside the material, the two independent solutions that approach $e^{\pm i k_1 \xi}$ far from the boundary are given by:

$\phi_{i,\perp}(\xi) = \Gamma(1 + i\kappa_i) \left( \frac{\sqrt{\eta}}{2k_i} \right)^{\pm i k_i} J_{\pm i\kappa_i}(e^{\mp i k_i})$.

(8)

Here, $\kappa_i$ is the ratio of neutron wavenumber inside the material to the inverse of the effective range of non-Newtonian gravity, $k_i/k_i$, and $J_{\nu}(\xi)$ is the Bessel function of the first kind. When $\xi$ is far from the boundary, the parameter of the Bessel function, $\nu$, becomes small. In this case, we can use the following approximation:

$I_{\nu}(\xi) \sim \left( \frac{\xi}{2} \right)^{\nu} \frac{1}{\Gamma(1 + \nu)}$.

(9)

Applying this approximation to (8) confirms that it approaches $e^{\pm i k_1 \xi}$.

If we normalise the absolute value of the incident wave function outside the material, $\phi_{o,+}(\xi)$ to unity, the wave functions outside and inside the material can be written as:

$\phi_{o,+}(\xi) + R \phi_{o,-}(\xi)$ and $T \phi_{o,+}(\xi)$

(10)

Here, $R$ and $T$ are determined by the continuities of wave function and its logarithmic derivative at the material surface. The result can be written as:

$R = -\frac{\phi_{o,+}(0)}{\phi_{o,-}(0)} \frac{d}{dz} \ln \phi_{o,+}(z) - \frac{d}{dz} \ln \phi_{o,-}(z) \bigg|_{z=0}$.

(11)

4. Perturbation analysis of the reflection amplitude

We now analytically analyse the reflection amplitude by assuming that the potential due to non-Newtonian gravity is much smaller than the kinetic energy. As $\eta$ is small in this case, the parameter of the Bessel functions, $\nu$, becomes small and the following approximation are valid:

$\left( \frac{j_\nu(\xi)}{l_\nu(\xi)} \right) \sim \left( \frac{\xi}{2} \right)^{\nu} \left( 1 - \left( \frac{\xi}{2} \right)^2 \frac{1}{1 + \nu} \right)$.

(12)

Using these approximations, we obtain the following approximations:

\[
\left. \left( \frac{d}{dz} \ln \phi_{o,\perp}(z) \right) \right|_{z=0} \sim \left( \frac{1 - \chi_0 \pm 1 y_{0,a}}{1 + \chi_0} \right)
\]

\[
\left. \left( \frac{d}{dz} \ln \phi_{i,\perp}(z) \right) \right|_{z=0} \sim \left( \frac{\pm i k_i \left( 1 + 2x_i \pm 1 \left( y_{i,a} - y_{0,b} \right) \right)}{1 + \chi_i \pm 1 \left( y_{i,a} - y_{i,b} \right)} \right)
\]

(13)

where

$\chi_0 = \frac{k_0}{1 + \kappa_0} \theta_{0,a} y_{0,a} = \frac{k_i^2}{1 + \kappa_i} \theta_{0,a} y_{0,b} = \frac{1}{1 + \kappa_i} \theta_{0,b}$.

(14)
Here, $\theta_o$ is defined as

$$x_i = \frac{k_i}{1 + k_i} \theta_i, \quad y_{i,a} = \frac{k_i^2}{1 + k_i} \theta_i,$$

and $y_{i,b} = \frac{1}{1 + k_i} \theta_i$. (15)

$\theta_o$ and $\theta_i$ are equal to the phase shifts due to the potential with finite curvature outside and inside the material, respectively. Figure 1 shows the effect of $k_o$ on the ratio of the phase shift to the correction terms, $x_o/\theta_o$, $y_{o,a}/\theta_o$, and $y_{o,b}/\theta_o$. When $k_o$ is much larger than $k_*^0$, the anomalous potential only changes the phase of the wave functions at the boundary, $\phi_{o,-}(0)$. On the other hand, when $k_o = k_*^0$, the anomalous potential changes not only the phase but also the amplitude of the wave function at the boundary, $\phi_{o,-}(0)$. When $k_o$ is much smaller than $k_*^0$, the effective range of the anomalous potential becomes smaller than the interatomic distance of the bulk material. In this case, the potential shown in equation (2) becomes inappropriate. Therefore, we ignore this case in this study.

Next, we approximate the amplitude of neutron reflection from bulk material using the above approximations for the wave functions. In this study, we restrict our attention to situations in which the kinetic energy far exceeds the Fermi potential and the difference of the refractive index inside the material from unity, $\overline{\Delta n}$, is small. In this case, both $\theta_o$ and $\overline{\Delta n}$ are small. Therefore, we approximate the reflection amplitude by considering only first-order terms of both $\theta_o$ and $\overline{\Delta n}$. The resulting approximated reflection amplitude is $\theta_o - \overline{\Delta n}/2$ when $k_o = k_*^0$ and $-\overline{\Delta n}/2$ when $k_o \gg k_*^0$. When $k_o = k_*^0$, the sign of the reflection amplitude, $\theta_o - \overline{\Delta n}/2$, can be reversed if $\theta_o$ is sufficiently large or $\overline{\Delta n}$ is sufficiently small. In these cases, the argument of the reflection amplitude changes $\pi$. On the other hand, this phenomenon does not happen when $k_o \gg k_*^0$.

5. Constraints on non-Newtonian gravity

We now numerically evaluate the obtained reflection amplitudes. As we are interested in measuring the effect of non-Newtonian gravity, we have to identify the contribution from the Fermi potential. Because the absolute uncertainty is expected to be proportional to the absolute value of the Fermi potential, it seems best to measure the reflection amplitude from materials with small Fermi potentials.
On the other hand, the contribution of non-Newtonian gravity is proportional to the weight density of the material. Therefore, we numerically evaluate the reflection amplitude from materials that have small Fermi potentials and are solids at normal temperature.

As an example, we consider the precision required to achieve the strongest constraints when $\lambda = 10 \text{ nm}$ by measuring the reflection amplitude. In this case, the strongest constraint on $\alpha_\parallel$ is that $\alpha_\parallel$ is smaller than about $10^{20}$ [8,13]. Therefore, we numerically evaluated $d n$ and $\theta_0$, when $\lambda = \lambda_\parallel = 1/10 \text{ nm}$ and $\alpha_\parallel = 10^{20}$. Table 1 shows the results and parameters used for two materials that have small Fermi potentials and are solid at normal temperature. As the reflectivity is the square of the reflection amplitude, the result shows that we can obtain comparable constraints if we can measure the reflectivity due to the Fermi potential with a relative uncertainty of about 1%.

| Material   | $\rho$ (kg/m$^3$) | $V_\parallel$ (neV) | $d n$        | $\theta_0$ |
|------------|------------------|--------------------|-------------|-------------|
| Vanadium   | 6110             | -7.19              | $-1.73 \times 10^{-2}$ | $-6.46 \times 10^{-4}$ |
| Samarium   | 7353             | 6.13               | $+1.48 \times 10^{-2}$ | $-7.78 \times 10^{-4}$ |

6. Conclusions

We have analysed the effect of non-Newtonian gravity on the reflection amplitude from bulk materials and found that the first-order contribution of non-Newtonian gravity is significant when the neutron wavelength is nearly equal to the radius of curvature of the neutron refractive index due to non-Newtonian gravity and the first-order contribution is negligible when the neutron wavelength is much smaller than the radius of curvature.

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