Almost automorphic functions on time scales and almost automorphic solutions to shunting inhibitory cellular neural networks on time scales*

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Abstract

In this paper, we first propose a new concept of almost periodic time scales, a new definition of almost automorphic functions on almost periodic time scales, and study some of their basic properties. Then we prove a result ensuring the existence of an almost automorphic solution for both the linear nonhomogeneous dynamic equation on time scales and its associated homogeneous equation, assuming that the associated homogeneous equation admits an exponential dichotomy. Finally, as an application of our results, we establish the existence and global exponential stability of almost automorphic solutions to a class of shunting inhibitory cellular neural networks with time-varying delays on time scales. Our results about the shunting inhibitory cellular neural network with time-varying delays on time scales are new even for the both cases of differential equations (the time scale $\mathbb{T} = \mathbb{R}$) and difference equations (the time scale $\mathbb{T} = \mathbb{Z}$).

Key words: Time scales; Almost automorphic functions; Dynamic equations; Exponential dichotomy.

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1 Introduction

The theory of time scales was initiated by Hilger [1] in his Ph.D. thesis in 1988, which can unify the continuous and discrete cases. The theory of dynamic equations on time scale contains, links and extends the classical theory of differential and difference equations. In

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the recent years, there has been an increasing interest in studying the existence of periodic solutions and almost periodic solutions of various dynamic equations on time scales; we refer the reader to the papers [2-23].

The concept of almost automorphy was introduced in the literature by S. Bochner in 1955 in the context of differential geometry [24] (see also Bochner [25, 26]). Since then, this concept has been extended in various directions.

**Remark 1.1.** Although every almost periodic function is almost automorphic, the converse, however, is not true.

In order to study almost periodic, pseudo almost periodic and almost automorphic dynamic equations on time scales, the following concept of almost periodic time scales was proposed in [27].

**Definition 1.1.** [27] A time scale \( T \) is called an almost periodic time scale if
\[
\Pi = \{ \tau \in \mathbb{R} : t \pm \tau \in T, \forall t \in T \} \neq \{0\}.
\]

Based on Definition 1.1, almost periodic functions [27], pseudo almost periodic functions [28], almost automorphic functions [29, 30] and weighted piecewise pseudo almost automorphic functions [31] on time scales were defined successfully. For example, the authors of [30] proposed the following concept of almost automorphic functions on time scales.

**Definition 1.2.** [30] Let \( T \) be an almost periodic time scale and \( X \) be a Banach space. A bounded continuous function \( f : T \to X \) is said to be almost automorphic, if for every sequence \((s_n') \subset \Pi\), there is a subsequence \((s_n) \subset (s_n')\) such that
\[
\lim_{n \to \infty} f(t + s_n) = \bar{f}(t)
\]
is well defined for each \( t \in T \), and
\[
\lim_{n \to \infty} \bar{f}(t - s_n) = f(t)
\]
for each \( t \in T \).

A continuous function \( f : T \times X \to X \) is said to be almost automorphic if \( f(t, x) \) is almost automorphic in \( t \in T \) uniformly in \( x \in B \), where \( B \) is any bounded subset of \( X \).

Since the concept of almost periodic time scales in sense of Definition 1.1 is very restrictive in the sense that it is a kind of periodic time scale (see [2]). This excludes many interesting time scales. Therefore, it is a challenging and important problem in theories and applications to find a new concept of almost periodic time scales.

Motivated by the above discussions, our main aim of this paper is to propose a new concept of almost periodic time scales and a new definition of almost automorphic functions on almost periodic time scales, and to study the existence of an almost automorphic solution for both the linear nonhomogeneous dynamic equation on time scales and its associated homogeneous equation. As an application of our results, the existence and global exponential stability of
almost automorphic solutions to a class of shunting inhibitory cellular neural networks with time-varying delays on time scales is established.

The organization of this paper is as follows: In Section 2, we introduce some notations and definitions and state some preliminary results which are needed in later sections. In Section 3, we first introduce a new concept of almost time scales and a new definition of almost automorphic functions on time scales, then we discuss some of their properties. Finally, we propose two open problems. In Section 4, we prove a result ensuring the existence of an almost automorphic solution for both the linear nonhomogeneous dynamic equation on time scales and its associated homogeneous equation, assuming that the associated homogeneous equation admits an exponential dichotomy. In Section 5, as an application of our results, we establish the existence and global exponential stability of almost automorphic solutions to a class of shunting inhibitory cellular neural networks with time-varying delays on time scales. Finally, we draw a conclusion in Section 6.

2 Preliminaries

In this section, we shall first recall some fundamental definitions, lemmas which are used in what follows.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\sigma(t) = \inf \{s \in \mathbb{T}, s > t\}$ for all $t \in \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) = \sup \{s \in \mathbb{T}, s < t\}$ for all $t \in \mathbb{T}$. Finally, the graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$.

If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. A function $r : \mathbb{T} \rightarrow \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}_k$. The set of all regressive and right-dense continuous functions $r : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{r \in \mathcal{R} : 1 + \mu(t)r(t) > 0, \forall t \in \mathbb{T}\}$. Let $A$ be an $m \times n$-matrix-valued function on $\mathbb{T}$. We say that $A$ is rd-continuous on $\mathbb{T}$ if each entry of $A$ is rd-continuous on $\mathbb{T}$. We denote the class of all rd-continuous $m \times n$ matrix-valued functions on $\mathbb{T}$ by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}^{m \times n})$. An $n \times n$-matrix-valued function $A$ on a time scale $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided $(1 + \mu(t)A(t))$ is invertible for all $t \in \mathbb{T}_k$. The set of all regressive and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$. For more knowledge of time scales, one can refer to [32, 33].

**Definition 2.1.** ([34]) Let $x \in \mathbb{R}^n$ and $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$, the linear system

$$x^\Delta(t) = A(t)x(t), \ t \in \mathbb{T} \quad (2.1)$$

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is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants $k, \alpha$, a projection $P$ and a fundamental solution matrix $X(t)$ of (2.1), satisfying

$$|X(t)PX^{-1}(\sigma(s))| \leq k e^{\alpha(t, \sigma(s))}, \quad s, t \in \mathbb{T}, t \geq \sigma(s),$$

$$|X(t)(I - P)X^{-1}(\sigma(s))| \leq k e^{\alpha(\sigma(s), t)}, \quad s, t \in \mathbb{T}, t \leq \sigma(s),$$

where $|\cdot|$ is a matrix norm on $\mathbb{T}$, that is $A = (a_{ij})_{n \times n}$, then we can take $|A| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$.

**Definition 2.2.** [35] Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales, we define

$$\text{dist}(\mathbb{T}_1, \mathbb{T}_2) = \max\{\text{sup}_{t \in \mathbb{T}_1}\{\text{dist}(t, \mathbb{T}_2)\}, \text{sup}_{t \in \mathbb{T}_2}\{\text{dist}(t, \mathbb{T}_1)\}\}$$

where, $\text{dist}(t, \mathbb{T}_2) = \inf\{|t - s|\}, \text{dist}(t, \mathbb{T}_1) = \inf\{|t - s|\}$. Let $\tau \in \mathbb{R}$ and $\mathbb{T}$ be a time scale, we define

$$\text{dist}(\mathbb{T}, \mathbb{T}_\tau) = \max\{\text{sup}_{t \in \mathbb{T}}\{\text{dist}(t, \mathbb{T}_\tau)\}, \text{sup}_{t \in \mathbb{T}_\tau}\{\text{dist}(t, \mathbb{T})\}\},$$

where $\mathbb{T}_\tau := \mathbb{T} \cap \{t - \tau \mid \forall t \in \mathbb{T}\}, \text{dist}(t, \mathbb{T}_\tau) = \inf\{|t - s|\}, \text{dist}(t, \mathbb{T}) = \inf\{|t - s|\}.$

**Definition 2.3.** [35] A time scale $\mathbb{T}$ is called an almost periodic time scale if for every $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon)$ such that $\mathbb{T}_\tau \neq \emptyset$ and $\text{dist}(\mathbb{T}, \mathbb{T}_\tau) < \varepsilon$, that is, for any $\varepsilon > 0$, the set $\Pi(\mathbb{T}, \varepsilon) = \{\tau \in \mathbb{R}, \text{dist}(\mathbb{T}, \mathbb{T}_\tau) < \varepsilon\}$ is relatively dense. $\tau$ is called the $\varepsilon$-translation number of $\mathbb{T}$.

Obviously, if $\mathbb{T}$ is an almost periodic time scale, then $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$, if $\mathbb{T}$ is a periodic time scale (see [35]), then $\text{dist}(\mathbb{T}, \mathbb{T}_\tau) = 0$, that is $\mathbb{T} = \mathbb{T}_\tau$.

**Remark 2.1.** One can easily see that if a time scale is an almost time scale under Definition 1.1, then it is also an almost time scale under Definition 2.3.

**Lemma 2.1.** [35] Let $\mathbb{T}$ be an almost periodic time scale under Definition 2.3, then

(i) if $\tau \in \Pi(\mathbb{T}, \varepsilon)$, then $t + \tau \in \mathbb{T}$ for all $t \in \mathbb{T}_\tau$;

(ii) if $\varepsilon_1 < \varepsilon_2$, then $\Pi(\mathbb{T}, \varepsilon_1) \subset \Pi(\mathbb{T}, \varepsilon_2)$;

(iii) if $\tau \in \Pi(\mathbb{T}, \varepsilon)$, then $-\tau \in \Pi(\mathbb{T}, \varepsilon)$ and $\text{dist}(\mathbb{T}_\tau, \mathbb{T}) = \text{dist}(\mathbb{T}_{-\tau}, \mathbb{T})$;

(iv) if $\tau_1, \tau_2 \in \Pi(\mathbb{T}, \varepsilon)$, then $\tau_1 + \tau_2 \in \Pi(\mathbb{T}, 2\varepsilon)$.
3 Almost time scales and almost automorphic functions

In this section, we first introduce a new concept of almost time scales and a new definition of almost automorphic functions on time scales, then we discuss some of their properties.

**Definition 3.1.** A time scale $\mathbb{T}$ is called an almost periodic time scale if

(i) $\Pi := \{ \tau \in \mathbb{R} : \mathbb{T}_\tau \neq \emptyset \} \neq \{0\}$ and $\mathbb{T} \neq \emptyset$,

(ii) if $\tau_1, \tau_2 \in \Pi$, then $\tau_1 \pm \tau_2 \in \Pi$,

where $\mathbb{T}_\tau = \mathbb{T} \cap \{ T - \tau \} = \mathbb{T} \cap \{ t - \tau : \forall t \in \mathbb{T} \}$ and $\mathbb{T} = \bigcap_{\tau \in \Pi} \mathbb{T}_\tau$.

It is obvious that if $\tau \in \Pi$, then $\pm \tau \in \Pi$ and $t \pm \tau \in \mathbb{T}$ for all $t \in \mathbb{T}$.

**Remark 3.1.** Noticing the fact that $\Pi = \{ \tau \in \mathbb{R} : \mathbb{T}_\tau \neq \emptyset \} \supset \Pi(\mathbb{T}, \varepsilon)$, from Lemma 2.1, one can easily see that if a time scale is an almost time scale under Definition 2.3, then it is also an almost time scale under Definition 3.1.

**Definition 3.2.** Let $\mathbb{T}$ be an almost periodic time scale and $X$ be a Banach space. A bounded rd-continuous function $f : \mathbb{T} \rightarrow X$ is said to be almost automorphic, if for every sequence $(s'_n) \subset \Pi$, there is a subsequence $(s_n) \subset (s'_n)$ such that

$$\lim_{n \rightarrow \infty} f(t + s_n) = \bar{f}(t)$$

is well defined for each $t \in \mathbb{T}$, and

$$\lim_{n \rightarrow \infty} \bar{f}(t - s_n) = f(t)$$

for each $t \in \mathbb{T}$.

Denote by $AA(\mathbb{T}, X)$ the set of all almost automorphic functions on time scale $\mathbb{T}$.

**Remark 3.2.** From Definition 3.2, one can easily see that if a function $f : \mathbb{T} \rightarrow X$ is an almost automorphic function under Definition 1.2, then it is also an almost automorphic function under Definition 3.2.

**Lemma 3.1.** Let $\mathbb{T}$ be an almost periodic time scale and suppose $f, f_1, f_2 \in AA(\mathbb{T}, X)$. Then we have the following

(i) $f_1 + f_2 \in AA(\mathbb{T}, X)$;

(ii) $\alpha f \in AA(\mathbb{T}, X)$ for any constant $\alpha \in \mathbb{R}$;

(iii) $f_c(t) \equiv f(c + t) \in AA(\mathbb{T}, X)$ for each fixed $c \in \mathbb{T}$. 

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Proof. (i) Let $f_1, f_2 \in AA(T, X)$. Then, for every sequence $(s'_n) \subset \Pi$, there is a subsequence $(s_n) \subset (s'_n)$ such that
\[
\lim_{n \to \infty} f_1(t + s_n) = \tilde{f}_1(t) \quad \text{and} \quad \lim_{n \to \infty} f_2(t + s_n) = \tilde{f}_2(t)
\]
are well defined for each $t \in \tilde{T}$, and
\[
\lim_{n \to \infty} \tilde{f}_1(t - s_n) = f_1(t) \quad \text{and} \quad \lim_{n \to \infty} \tilde{f}_2(t - s_n) = f_2(t)
\]
for each $t \in \tilde{T}$. Therefore, we obtain
\[
\lim_{n \to \infty} (f_1 + f_2)(t + s_n) = \lim_{n \to \infty} (f_1(t + s_n) + f_2(t + s_n)) = \tilde{f}_1(t) + \tilde{f}_2(t)
\]
is well defined for each $t \in \tilde{T}$, and
\[
\lim_{n \to \infty} (\tilde{f}_1 + \tilde{f}_2)(t - s_n) = \lim_{n \to \infty} (\tilde{f}_1(t - s_n) + \tilde{f}_2(t - s_n)) = f_1(t) + f_2(t)
\]
for each $t \in \tilde{T}$.

(ii) Since $f \in AA(T, X)$, then for every sequence $(s'_n) \subset \Pi$, there is a subsequence $(s_n) \subset (s'_n)$ such that
\[
\lim_{n \to \infty} (\alpha f)(t + s_n) = \lim_{n \to \infty} \alpha f(t + s_n) = \alpha \tilde{f}(t) = (\alpha \tilde{f})(t)
\]
is well defined for each $t \in \tilde{T}$, and
\[
\lim_{n \to \infty} (\alpha \tilde{f})(t - s_n) = \lim_{n \to \infty} \alpha \tilde{f}(t - s_n) = \alpha f(t) = (\alpha f)(t)
\]
for each $t \in \tilde{T}$.

(iii) It follows from $f \in AA(T, X)$, $c \in \tilde{T}$ that for every sequence $(s'_n) \subset \Pi$, there is a subsequence $(s_n) \subset (s'_n)$ such that
\[
\lim_{n \to \infty} f_c(t + s_n) = \lim_{n \to \infty} f((c \cdot t) + s_n) = \tilde{f}(c \cdot t) = \tilde{f}_c(t)
\]
is well defined for each $t \in \tilde{T}$, and
\[
\lim_{n \to \infty} \tilde{f}_c(t - s_n) = \lim_{n \to \infty} \tilde{f}((c \cdot t) - s_n) = f(c \cdot t) = f_c(t)
\]
for each $t \in \tilde{T}$. The proof is completed.  

Lemma 3.2. Let $T$ be an almost periodic time scale and functions $f, \phi : T \to X$ be almost automorphic, then the function $\phi f : T \to X$ defined by $(\phi f)(t) = \phi(t)f(t)$ is also almost automorphic.
Proof. Both functions $\phi$ and $f$ are bounded since they are almost automorphic. So we let $K_1 = \sup_{t \in \mathbb{T}} \| \phi(t) \|$.

Let the sequence $(s'_n) \subset \Pi$, then there exists a subsequence $(s''_n) \subset (s'_n)$ such that $\lim_{n \to \infty} \phi(t + s''_n) = \bar{\phi}(t)$ is well defined for each $t \in \mathbb{\bar{T}}$ and $\lim_{n \to \infty} \bar{\phi}(t - s''_n) = \phi(t)$ for each $t \in \mathbb{T}$. Since $f$ is almost automorphic, there exists a subsequence $(s_n) \subset (s''_n)$ such that $\lim_{n \to \infty} f(t + s_n) = \bar{f}(t)$ is well defined for each $t \in \mathbb{T}$ and $\lim_{n \to \infty} \bar{f}(t - s_n) = f(t)$ for each $t \in \mathbb{T}$.

Now, we have

\[
\| \phi(t + s_n)f(t + s_n) - \bar{\phi}(t)\bar{f}(t) \|
\leq \| \phi(t + s_n)f(t + s_n) - \phi(t + s_n)f(t) \| + \| \phi(t + s_n)f(t) - \bar{\phi}(t)f(t) \|
\leq K_1 \| f(t + s_n) - \bar{f}(t) \| + K_2 \| \phi(t + s_n) - \bar{\phi}(t) \|
\leq (K_1 + K_2)\varepsilon
\]

for $n$ sufficiently large, where $K_2 = \sup_{t \in \mathbb{T}} \| \bar{f}(t) \| < \infty$. Thus, we obtain

\[
\lim_{n \to \infty} \phi(t + s_n)f(t + s_n) = \bar{\phi}(t)\bar{f}(t)
\]

for each $t \in \mathbb{T}$.

It is also easy to check that

\[
\lim_{n \to \infty} \bar{\phi}(t - s_n)\bar{f}(t - s_n) = \phi(t)f(t)
\]

for each $t \in \mathbb{T}$. The proof is now complete. \qed

Lemma 3.3. Let $\mathbb{T}$ be an almost periodic time scale and $(f_n)$ be a sequence of almost automorphic functions such that $\lim_{n \to \infty} f_n(t) = f(t)$ converges uniformly for each $t \in \mathbb{T}$. Then, $f$ is an almost automorphic function.

Proof. Let the sequence $(s'_n) \subset \Pi$. As in the standard case of the almost automorphic functions the approach follows across the diagonal procedure. Since $f_1 \in AA(\mathbb{T}, \mathbb{X})$, then there exists a subsequence $(s^{(1)}_n) \subset (s'_n)$ such that

\[
\lim_{n \to \infty} f_1(t + s^{(1)}_n) = \bar{f}_1(t)
\]

is well defined for each $t \in \mathbb{T}$, and

\[
\lim_{n \to \infty} \bar{f}_1(t - s^{(1)}_n) = f_1(t)
\]

for each $t \in \mathbb{T}$. Since $f_2 \in AA(\mathbb{T}, \mathbb{X})$, then there exists a subsequence $(s^{(2)}_n) \subset (s^{(1)}_n)$ such that

\[
\lim_{n \to \infty} f_2(t + s^{(2)}_n) = \bar{f}_2(t)
\]
is well defined for each $t \in \mathbb{T}$, and
\[
\lim_{n \to \infty} \tilde{f}_2(t - s^{(2)}_n) = f_2(t)
\]
for each $t \in \mathbb{T}$. Following, by this procedure, we can construct a subsequence $(s^{(n)}_n) \subset (s'_n)$ such that
\[
\lim_{n \to \infty} f_i(t + s^{(n)}_n) = \tilde{f}_i(t)
\]  
(3.1)
for each $t \in \mathbb{T}$ and for all $i = 1, 2, \ldots$. Let us consider
\[
\|f_i(t + s^{(n)}_n) - f_j(t + s^{(n)}_n)\| < \varepsilon
\]  
(3.3)
for all $t \in \mathbb{T}$ and all $n = 1, 2, \ldots$.

Hence, taking $i, j$ sufficiently large in (3.2) and using (3.3) and the limit (3.1), we can conclude that $(\tilde{f}_i(t))$ is a Cauchy sequence. Since $\mathbb{X}$ is a Banach space, then $(\tilde{f}_i(t))$ is a sequence which converges pointwisely on $\mathbb{X}$. Let $\bar{f}(t)$ be the limit of $(\tilde{f}_i(t))$, then for each $i = 1, 2, \ldots$, we have
\[
\|f_i(t + s^{(n)}_n) - \bar{f}_i(t + s^{(n)}_n)\| < \varepsilon
\]  
(3.4)
for $i$ sufficiently large, by (3.4) and using the almost automorphicity of $f_i$ and the convergence of the functions $f_i$ and $\bar{f}_i$, we obtain
\[
\lim_{n \to \infty} f(t + s^{(n)}_n) = \bar{f}(t)
\]
for each $t \in \mathbb{T}$. Similarly, we can get
\[
\lim_{n \to \infty} \bar{f}(t - s^{(n)}_n) = f(t)
\]
for each $t \in \mathbb{T}$. This completes the proof.

\textbf{Lemma 3.4.} Let $\mathbb{T}$ be an almost periodic time scale and $\mathbb{X}, \mathbb{Y}$ be Banach spaces. If $f : \mathbb{T} \to \mathbb{X}$ is an almost automorphic function and $\varphi : \mathbb{X} \to \mathbb{Y}$ is a continuous function, then the composite function $\varphi \circ f : \mathbb{T} \to \mathbb{Y}$ is an almost automorphic function.
Proof. Since \( f \in AA(T, X) \), then for every sequence \((s'_n) \subset \Pi\), there exists a subsequence \((s_n) \subset (s'_n)\) such that \( \lim_{n \to \infty} f(t + s_n) = \bar{f}(t) \) is well defined for each \( t \in \tilde{T} \) and \( \lim_{n \to \infty} \bar{f}(t - s_n) = f(t) \) for each \( t \in \tilde{T} \).

By the continuity of function \( \varphi \), it follows that
\[
\lim_{n \to \infty} \varphi(f(t + s_n)) = \varphi\left( \lim_{n \to \infty} f(t + s_n) \right) = (\varphi \circ \bar{f})(t).
\]

On the other hand, we can get
\[
\lim_{n \to \infty} \varphi(\bar{f}(t - s_n)) = \varphi\left( \lim_{n \to \infty} \bar{f}(t - s_n) \right) = (\varphi \circ f)(t)
\]
for each \( t \in \tilde{T} \). Thus, \( \varphi \circ f \in AA(T, Y) \). The proof is complete.

Definition 3.3. Let \( T \) be an almost periodic time scale and \( X \) be a Banach space. A bounded rd-continuous function \( f : T \times X \to X \) is said to be almost automorphic, if for any sequence \((s'_n) \subset \Pi\), there is a subsequence \((s_n) \subset (s'_n)\) such that
\[
\lim_{n \to \infty} f(t + s_n, x) = \bar{f}(t, x)
\]
is well defined for each \( t \in \tilde{T} \), \( t + s_n \in T \), \( x \in X \), and
\[
\lim_{n \to \infty} \bar{f}(t - s_n, x) = f(t, x)
\]
for each \( t \in \tilde{T} \), \( x \in X \).

Denote by \( AA(T \times X, X) \) the set of all such functions.

Remark 3.3. From Definition 3.3, one can easily see that if a function \( f : T \times X \to X \) is an almost automorphic function under Definition 1.2, then it is also an almost automorphic function under Definition 3.3.

Lemma 3.5. Let \( f \in AA(T \times X, X) \) and \( f \) satisfy the Lipschitz condition in \( x \in X \) uniformly in \( t \in \tilde{T} \). If \( \varphi \in AA(T, X) \), then \( f(t, \varphi(t)) \) is almost automorphic.

Proof. For any sequence \((s'_n) \subset \Pi\), there is a subsequence \((s_n) \subset (s'_n)\) such that
\[
\lim_{n \to \infty} f(t + s_n, x) = \bar{f}(t, x)
\]
is well defined for each \( t \in \tilde{T} \), \( x \in X \), and
\[
\lim_{n \to \infty} \bar{f}(t - s_n, x) = f(t, x)
\]
for each \( t \in \tilde{T} \), \( x \in X \). Since \( \varphi \in AA(T, X) \), there exists a subsequence \((\tau_n) \subset (s_n)\) such that
\[
\lim_{n \to \infty} \varphi(t + \tau_n) = \bar{\varphi}(t)
\]
is well defined for each \( t \in \mathbb{T} \), and

\[
\lim_{n \to \infty} \varphi(t - \tau_n) = \varphi(t)
\]

for each \( t \in \mathbb{T} \). Since \( f \) satisfies the Lipschitz condition in \( x \in \mathbb{X} \) uniformly in \( t \in \mathbb{T} \), there exists a positive constant \( L \) such that

\[
\|f(t + \tau_n, \varphi(t + \tau_n)) - \tilde{f}(t, \tilde{\varphi}(t))\| \\
\leq \|f(t + \tau_n, \varphi(t)) - f(t + \tau_n, \tilde{\varphi}(t))\| + \|f(t + \tau_n, \tilde{\varphi}(t)) - \tilde{f}(t, \tilde{\varphi}(t))\| \\
\leq L\|\varphi(t + \tau_n) - \tilde{\varphi}(t)\| + \|f(t + \tau_n, \tilde{\varphi}(t)) - \tilde{f}(t, \tilde{\varphi}(t))\| 
\to 0, \; n \to \infty
\]

and

\[
\|\tilde{f}(t - \tau_n, \varphi(t - \tau_n)) - f(t, \varphi(t))\| \\
\leq \|\tilde{f}(t - \tau_n, \varphi(t)) - \tilde{f}(t - \tau_n, \tilde{\varphi}(t))\| + \|\tilde{f}(t - \tau_n, \tilde{\varphi}(t)) - f(t, \varphi(t))\| \\
\leq L\|\varphi(t - \tau_n) - \tilde{\varphi}(t)\| + \|\tilde{f}(t - \tau_n, \tilde{\varphi}(t)) - f(t, \varphi(t))\| 
\to 0, \; n \to \infty.
\]

Hence, for any sequence \( (s_n') \subset \Pi \), there is a subsequence \( (\tau_n) \subset (s_n') \) such that

\[
\lim_{n \to \infty} f(t + \tau_n, \varphi(t + \tau_n)) = \tilde{f}(t, \tilde{\varphi}(t))
\]

is well defined for each \( t \in \mathbb{T} \), and

\[
\lim_{n \to \infty} \tilde{f}(t - \tau_n, \varphi(t - \tau_n)) = f(t, \varphi(t))
\]

for each \( t \in \mathbb{T} \). That is, \( f(t, \varphi(t)) \) is almost automorphic. This completes the proof. \( \blacksquare \)

**Definition 3.4.** Let \( \mathbb{T} \) be an almost periodic time scale. The graininess function \( \mu : \mathbb{T} \to \mathbb{R}_+ \) is said to be almost automorphic, if for every sequence \( (s_n') \subset \Pi \), there is a subsequence \( (s_n) \subset (s_n') \) such that

\[
\lim_{n \to \infty} \mu(t + s_n) = \bar{\mu}(t)
\]

is well defined for each \( t \in \mathbb{T} \), and

\[
\lim_{n \to \infty} \bar{\mu}(t - s_n) = \mu(t)
\]

for each \( t \in \mathbb{T} \).

In the following, in order to make the graininess function \( \mu \) have a better property, we will use Definition 2.3 as the definition of almost periodic time scales.

From Corollary 15, Theorem 19 in [35] and Definition 3.4, we can obtain

**Lemma 3.6.** Let \( \mathbb{T} \) be an almost periodic time scale under Definition 2.3. Then the graininess function \( \mu \) is almost automorphic.

**Open problem 1:** Let \( \mathbb{T} \) be an almost periodic time scale under Definition 3.4. Whether or not the graininess function \( \mu \) is almost automorphic?

**Open problem 2:** Let the graininess function \( \mu \) of \( \mathbb{T} \) be almost automorphic. Whether or not \( \mathbb{T} \) is an almost periodic time scale under Definition 3.4?
4 Automorphic solutions to linear dynamic equations

Consider the linear nonhomogeneous dynamic equation on time scales

\[ x^{\Delta}(t) = A(t)x(t) + f(t), \quad t \in \mathbb{T}, \]

(4.1)

where \( A : \mathbb{T} \to \mathbb{R}^{n \times n}, f : \mathbb{T} \to \mathbb{R}^n \) and its associated homogeneous equation

\[ x^{\Delta}(t) = A(t)x(t), \quad t \in \mathbb{T}. \]

(4.2)

Throughout this section, we restrict our discussions on almost periodic time scales under Definition 2.3 and we assume that \( A(t) \) is almost automorphic on \( \mathbb{T} \), which means that each entry of the matrix \( A(t) \) is almost automorphic.

Lemma 4.1. Let \( \mathbb{T} \) be an almost periodic time scale, \( A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}) \) be almost automorphic and nonsingular on \( \mathbb{T} \) and the sets \( \{A^{-1}(t)\}_{t \in \mathbb{T}} \) and \( \{(I + \mu(t)A(t))^{-1}\}_{t \in \mathbb{T}} \) be bounded on \( \mathbb{T} \). Then, \( A^{-1}(t) \) and \( (I + \mu(t)A(t))^{-1} \) are almost automorphic on \( \mathbb{T} \). Moreover, suppose that \( f \in \mathcal{AA}(\mathbb{T}, \mathbb{R}^n) \) and (4.2) admits an exponential dichotomy, then (4.1) has a solution \( x(t) \in \mathcal{AA}(\mathbb{T}, \mathbb{R}^n) \) and \( x(t) \) is expressed as follows

\[ x(t) = \int_{-\infty}^{t} X(t)P X^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I - P) X^{-1}(\sigma(s))f(s)\Delta s, \]

(4.3)

where \( X(t) \) is the fundamental solution matrix of (4.2).

Proof. We divide the proof into several steps.

Step 1. \( A^{-1}(t) \) is almost automorphic on \( \mathbb{T} \).

Let the sequence \( (s_n') \subset \Pi \). Since \( A(t) \) is almost automorphic on time scales, there exists a subsequence \( (s_n) \subset (s_n') \) such that

\[ \lim_{n \to \infty} A(t + s_n) = \bar{A}(t) \]

is well defined for each \( t \in \tilde{T} \), and

\[ \lim_{n \to \infty} \bar{A}(t - s_n) = A(t) \]

for each \( t \in \tilde{T} \).

Given \( t \in \tilde{T} \) and define \( A_n = A(t + s_n), \ n \in \mathbb{N} \). By hypothesis, the set \( \{A_n^{-1}\}_{n \in \mathbb{N}} \) is bounded, that is, there exists a positive constant \( M \) such that \( |A_n^{-1}| < M \). From the identity \( A_n^{-1} - A_m^{-1} = A_n^{-1}(A_m - A_n)A_m^{-1} \), it implies that \( \{A_n\} \) is a cauchy sequence, in addition, it follows that \( \{A_n^{-1}\} \) is cauchy sequence. Hence, for each \( t \in \tilde{T} \) fixed, there exists a matrix \( S \) such that

\[ A_n^{-1}(t) \to S(t). \]

Then we have

\[ \lim_{n \to \infty} A_nA_n^{-1} = \bar{A}\bar{A}^{-1} = I, \]
where $I$ denotes the identity matrix, we obtain that $\tilde{A}(t)$ is invertible and $\tilde{A}^{-1}(t) = S(t)$ for each $t \in \mathbb{T}$. Since the map $A \rightarrow A^{-1}$ is continuous on the set of nonsingular matrices, we can obtain
\[
\lim_{n \rightarrow \infty} A^{-1}(t + s_n) = \tilde{A}^{-1}(t)
\]
is well defined for each $t \in \mathbb{T}$. Similarly, we can get
\[
\lim_{n \rightarrow \infty} \tilde{A}^{-1}(t - s_n) = A^{-1}(t)
\]
for each $t \in \mathbb{T}$.

Step 2. $(I + \mu(t)A(t))^{-1}$ is almost automorphic on $\mathbb{T}$.

Since $A(t)$ and $\mu(t)$ are almost automorphic functions, then for every sequence $(s'_n) \subset \Pi$, there exists a subsequence $(s_n) \subset (s'_n)$ such that
\[
\lim_{n \rightarrow \infty} A(t + s_n) = \tilde{A}(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(t + s_n) = \tilde{\mu}(t)
\]
are well defined for each $t \in \mathbb{T}$, and
\[
\lim_{n \rightarrow \infty} \tilde{A}(t - s_n) = A(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \tilde{\mu}(t - s_n) = \mu(t)
\]
for each $t \in \mathbb{T}$. Therefore, we have
\[
\lim_{n \rightarrow \infty} (I + A(t + s_n)\mu(t + s_n)) = I + \tilde{A}(t)\tilde{\mu}(t)
\]
for each $t \in \mathbb{T}$, and
\[
\lim_{n \rightarrow \infty} (I + \tilde{A}(t - s_n)\tilde{\mu}(t - s_n)) = I + A(t)\mu(t)
\]
for each $t \in \mathbb{T}$. Hence, $(I + A(t)\mu(t))$ is almost automorphic on $\mathbb{T}$. In addition, since $A(t)$ is a regressive matrix, $(I + A(t)\mu(t))$ is nonsingular on $\mathbb{T}$. By hypothesis, the set $\{(I + A(t)\mu(t))^{-1}\}_{t \in \mathbb{T}}$ is bounded. The proof is similar to the proof of Step 1, we obtain that $(I + A(t)\mu(t))^{-1}$ is almost automorphic on $\mathbb{T}$.

Step 3. The system (4.1) has an automorphic solution.

Since the linear system (4.2) admits an exponential dichotomy, system (4.1) has a bounded solution $x(t)$ given by
\[
x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s.
\]
Since $A(t)$, $\mu(t)$ and $f(t)$ are almost automorphic functions, then for every sequence $(s'_n) \subset \Pi$, there exists a subsequence $(s_n) \subset (s'_n)$ such that
\[
\lim_{n \rightarrow \infty} A(t + s_n) = \tilde{A}(t), \quad \lim_{n \rightarrow \infty} \mu(t + s_n) = \tilde{\mu}(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} f(t + s_n) = \tilde{f}(t)
\]
for each $t \in \mathbb{T}$. Similarly, we can get
\[
\lim_{n \rightarrow \infty} \tilde{A}(t - s_n) = A(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \tilde{\mu}(t - s_n) = \mu(t)
\]
for each $t \in \mathbb{T}$.
are well defined for each \( t \in \tilde{T} \), and
\[
\lim_{n \to \infty} \bar{A}(t - s_n) = A(t), \quad \lim_{n \to \infty} \bar{\mu}(t - s_n) = \mu(t) \quad \text{and} \quad \lim_{n \to \infty} \bar{f}(t - s_n) = f(t)
\]
for each \( t \in \tilde{T} \).

We let
\[
B(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s
\]
and
\[
\bar{B}(t) = \int_{-\infty}^{t} M(t, s)\bar{f}(s)\Delta s,
\]
where \( M(t, s) = \lim_{n \to \infty} X(t + s_n)PX^{-1}(\sigma(s + s_n)) \).

Then, we obtain

\[
\|B(t + s_n) - \bar{B}(t)\| = \left\| \int_{-\infty}^{t + s_n} X(t + s_n)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{-\infty}^{t} M(t, s)\bar{f}(s)\Delta s \right\|
\]
\[
= \left\| \int_{-\infty}^{t} X(t + s_n)PX^{-1}(\sigma(s + s_n))f(s + s_n)\Delta s - \int_{-\infty}^{t} M(t, s)\bar{f}(s)\Delta s \right\|
\]
\[
\leq \left\| \int_{-\infty}^{t} X(t + s_n)PX^{-1}(\sigma(s + s_n))f(s + s_n)\Delta s - \int_{-\infty}^{t} X(t + s_n)PX^{-1}(\sigma(s + s_n))\bar{f}(s)\Delta s \right\|
\]
\[
+ \left\| \int_{-\infty}^{t} X(t + s_n)PX^{-1}(\sigma(s + s_n))\bar{f}(s)\Delta s - \int_{-\infty}^{t} M(t, s)\bar{f}(s)\Delta s \right\|
\]
\[
= \left\| \int_{-\infty}^{t} X(t + s_n)PX^{-1}(\sigma(s + s_n))(f(s + s_n) - \bar{f}(s))\Delta s \right\|
\]
\[
+ \left\| \int_{-\infty}^{t} (X(t + s_n)PX^{-1}(\sigma(s + s_n)) - M(t, s))\bar{f}(s)\Delta s \right\|. \quad (4.4)
\]

In addition, since the function \( f \) is almost automorphic, we have \( \bar{f} \) is a bounded function, limit both sides with \( (1.4) \), then we can get
\[
\lim_{n \to \infty} B(t + s_n) = \bar{B}(t) \quad (4.5)
\]
for each \( t \in \tilde{T} \). Analogously, one can prove that
\[
\lim_{n \to \infty} \bar{B}(t - s_n) = B(t) \quad (4.6)
\]
for each \( t \in \tilde{T} \).

On the other hand, let
\[
C(t) = \int_{t}^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s
\]
\[
\bar{C}(t) = \int_{-\infty}^{t} N(t, s) \bar{f}(s) \Delta s,
\]
where \(N(t, s) = \lim_{n \to \infty} X(t+s_n)(I-P)X^{-1}(\sigma(s+s_n))\). Then, similar to the proofs of (4.5) and (4.6), we can prove that given a sequence \((s'_n) \subset \Pi\), there exists a subsequence \((s_n) \subset (s'_n)\) such that
\[
\lim_{n \to \infty} C(t+s_n) = \bar{C}(t)
\]
for each \(t \in \tilde{T}\), and
\[
\lim_{n \to \infty} \bar{C}(t-s_n) = C(t)
\]
for each \(t \in \tilde{T}\).

Finally, we define \(\bar{x}(t) = \bar{B}(t) - \bar{C}(t)\), then by the definition of \(x\) from (4.3), one can prove that given a sequence \((s'_n) \subset \Pi\), there exists a subsequence \((s_n) \subset (s'_n)\) such that
\[
\lim_{n \to \infty} x(t+s_n) = \bar{x}(t)
\]
is well defined for each \(t \in \tilde{T}\), and
\[
\lim_{n \to \infty} \bar{x}(t-s_n) = x(t)
\]
for each \(t \in \tilde{T}\). Hence, \(x(t)\) is an almost automorphic solution to system (4.1). This completes the proof.

Similar to the proof of Lemma 2.15 in [36], one can easily show that

\textbf{Lemma 4.2.} Let \(c_i(t)\) be almost automorphic on \(\mathbb{T}\), where \(c_i(t) > 0, -c_i(t) \in \mathcal{R}^+, t \in \tilde{T}, i = 1, 2, \ldots, n\) and \(\min_{1 \leq i \leq n} \{\inf_{t \in \tilde{T}} c_i(t)\} = \bar{m} > 0\), then the linear system
\[
x^\Delta(t) = \text{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t)
\]
admits an exponential dichotomy on \(\mathbb{T}\).

\section{An application}

In the last forty years, shunting inhibitory cellular neural networks (SICNNs) have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Hence, they have been the object of intensive analysis by numerous authors in recent years. Many important results on the dynamical behaviors of SICNNs have been established and successfully applied to signal processing, pattern recognition, associative memories, and so on. We refer the reader to [37-44] and the references cited therein. However, to the best of our knowledge, there is no paper published on the existence
of almost automorphic solutions to SICNNs governed by differential or difference equations. So, our main purpose of this section is to study the existence of almost automorphic solutions to SICNNs governed by differential or difference equations.

\[ x_{ij}^*(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(t)f(x_{kl}(t - \tau_{kl}(t)))x_{ij}(t) - \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(t) \int_{t-\delta_{kl}(t)}^t g(x_{kl}(u))\Delta u x_{ij}(t) + L_{ij}(t), \]

where \( t \in \mathbb{T} \), \( \mathbb{T} \) is an almost periodic time scale under the sense of Definition 2.3, \( i = 1,2,\ldots,m, j = 1,2,\ldots,n \), \( C_{ij} \) denotes the cell at the \((i,j)\) position of the lattice, the \( r\)-neighborhood \( N_r(i,j) \) of \( C_{ij} \) is given by

\[ N_r(i,j) = \{ C_{kl} : \max(|k-i|,|l-j|) \leq r, 1 \leq k \leq m, 1 \leq l \leq n \}, \]

\( N_p(i,j) \) is similarly specified, \( x_{ij} \) is the activity of the cell \( C_{ij} \), \( L_{ij}(t) \) is the external input to \( C_{ij} \), \( a_{ij}(t) > 0 \) represents the passive decay rate of the cell activity, \( B_{ij}^{kl}(t) \geq 0 \) and \( C_{ij}^{kl}(t) \geq 0 \) are the connections or coupling strengths of postsynaptic activity of the cells in \( N_r(i,j) \) and \( N_p(i,j) \) transmitted to cell \( C_{ij} \) depending upon variable delays and continuously distributed delays, respectively, and the activity functions \( f(\cdot) \) and \( g(\cdot) \) are continuous functions representing the output or firing rate of the cell \( C_{kl} \), \( \tau_{kl}(t) \) and \( \delta_{kl}(t) \) are transmission delays at time \( t \) and satisfy \( t - \tau_{kl}(t) \in \mathbb{T} \) and \( t - \delta_{kl}(t) \in \mathbb{T} \) for \( t \in \mathbb{T}, k = 1,2,\ldots,m, l = 1,2,\ldots,n \).

The initial condition associated with system \((5.1)\) is of the form

\[ x_{ij}(s) = \varphi_{ij}(s), \quad s \in [-\theta,0]_\mathbb{T}, \quad i = 1,2,\ldots,m, j = 1,2,\ldots,n, \]

where \( \theta = \max \{ \max_{1 \leq k \leq m, 1 \leq l \leq n} \tau_{kl}, \max_{1 \leq k \leq m, 1 \leq l \leq n} \delta_{kl} \} \), \( \varphi_{ij}(\cdot) \) denotes a real-value bounded rd-continuous function defined on \([-\theta,0]_\mathbb{T}\).

Throughout this section, we let \([a,b]_\mathbb{T} = \{ t | t \in [a,b] \cap \mathbb{T} \}\) and we restrict our discussions on almost periodic time scales under the sense of Definition 2.3. For convenience, for an almost automorphic function \( f : \mathbb{T} \to \mathbb{R} \), denote \( \underline{f} = \inf_{t \in \mathbb{T}} f(t), \overline{f} = \sup_{t \in \mathbb{T}} f(t) \). We denote by \( \mathbb{R} \) the set of real numbers, by \( \mathbb{R}^+ \) the set of positive real numbers.

Set

\[ x = \{ x_{ij}(t) \} = (x_{11}(t), \ldots, x_{1n}(t), \ldots, x_{i1}(t), \ldots, x_{in}(t), \ldots, x_{m1}(t), \ldots, x_{mn}(t)) \in \mathbb{R}^{m \times n}. \]

For \( \forall x = \{ x_{ij}(t) \} \in \mathbb{R}^{m \times n} \), we define the norm \( \| x(t) \| = \max_{i,j} |x_{ij}(t)| \) and \( \| x \| = \sup_{t \in \mathbb{T}} \| x(t) \| \).

Let \( \mathbb{B} = \{ \varphi | \varphi = \{ \varphi_{ij}(t) \} = (\varphi_{11}(t), \ldots, \varphi_{1n}(t), \ldots, \varphi_{i1}(t), \ldots, \varphi_{in}(t), \ldots, \varphi_{m1}(t), \ldots, \varphi_{mn}(t)) \} \), where \( \varphi \) is an almost automorphic function on \( \mathbb{T} \). For \( \forall \varphi \in \mathbb{B} \), if we define the norm \( \| \varphi \|_\mathbb{B} = \sup_{t \in \mathbb{T}} \| \varphi(t) \| \), then \( \mathbb{B} \) is a Banach space.

**Definition 5.1.** The almost automorphic solution \( x^*(t) = \{ x_{ij}^*(t) \} \) of system \((5.1)\) with initial value \( \varphi^* = \{ \varphi_{ij}^*(t) \} \) is said to be globally exponentially stable. If there exist positive constants
\( \lambda \) with \( \ominus \lambda \in \mathcal{R}^+ \) and \( M > 1 \) such that every solution \( x(t) = \{x_{ij}(t)\} \) of system \((5.1)\) with initial value \( \varphi = \{\varphi_{ij}(t)\} \) satisfies

\[
\|x - x^*\|_B \leq Me^{\lambda}(t, t_0)\|\varphi - \varphi^*\|_B, \quad \forall t \in [t_0, +\infty)_T, \ t_0 \in T.
\]

**Theorem 5.1.** Suppose that

\( (H_1) \) for \( ij \in \Lambda = \{11, 12, \ldots, 1n, \ldots, m1, m2, \ldots, mn\} \), \(-a_{ij} \in \mathcal{R}^+\), where \( \mathcal{R}^+ \) denotes the set of positively regressive functions from \( T \) to \( \mathbb{R} \), \( a_{ij}, B_{ij}^{kl}, C_{ij}^{kl}, \tau_{kl}, L_{ij} \in \mathbb{B}; \)

\( (H_2) \) functions \( f, g \in C(\mathbb{R}, \mathbb{R}) \) and there exist positive constants \( L^f, L^g, M^f, M^g \) such that for all \( u, v \in \mathbb{R}, \)

\[
|f(u) - f(v)| \leq L^f|u - v|, \quad f(0) = 0, \quad |f(u)| \leq M^f,
\]

\[
|g(u) - g(v)| \leq L^g|u - v|, \quad g(0) = 0, \quad |g(u)| \leq M^g;
\]

\( (H_3) \) there exists a constant \( \rho \) such that

\[
\max_{i,j} \left\{ \frac{\sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}M^f \rho^2 + \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}M^g \delta_{kl} \rho^2 + L_{ij}}{a_{ij}} \right\} < \rho
\]

and

\[
\max_{i,j} \left\{ \frac{\sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}M^f \rho + \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}M^g \rho + L_{ij}}{a_{ij}} \right\} < 1.
\]

Then, system \((5.1)\) has a unique almost automorphic solution in \( \mathbb{E} = \{\varphi \in \mathbb{B} : \|\varphi\|_B \leq \rho\}. \)

**Proof.** For any given \( \varphi \in \mathbb{B} \), we consider the following system

\[
x_{ij}^\Delta(t) = \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}f(\varphi_{kl}(t - \tau_{kl}(t)))\varphi_{ij}(t) - \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}g(\varphi_{kl}(u))\Delta u\varphi_{ij}(t) + L_{ij}(t), \quad ij = 11, 12, \ldots, mn. \quad (5.2)
\]

Since \( \min_{1 \leq i \leq m; 1 \leq j \leq n} \{\inf_{t \in T} a_{ij}(t)\} > 0 \) and \( -a_{ij} \in \mathcal{R}^+ \), it follows from Lemma 2.12 that the linear system

\[
x_{ij}^\Delta(t) = -a_{ij}(t)x_{ij}(t), \quad ij = 11, 12, \ldots, mn
\]

admits an exponential dichotomy on \( T \). Thus, by Lemma 2.11, we obtain that system \((5.2)\) has exactly one almost automorphic solution as follows

\[
x^\varphi(t) := \{x_{ij}^\varphi(t)\} = \int_{-\infty}^{t} e_{-a_{ij}(t, \sigma(s))} \left( -\sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)f(\varphi_{kl}(s - \tau_{kl}(s)))\varphi_{ij}(s) + \ldots \right). 
\]
\[
- \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(s) \int_{t-\delta_k(s)}^{s} g(\varphi_{kl}(u)) \Delta u \varphi_{ij}(s) + L_{ij}(s) \Delta s \}.
\]

Now, we define the following operator \( T : \mathbb{B} \to \mathbb{B} \) by setting

\[
T(\varphi(t)) = x^{\varphi}(t), \ \forall \varphi \in \mathbb{B}.
\]

First, we will check that for any \( \varphi \in \mathcal{E}, T\varphi \in \mathcal{E} \). For any \( \varphi \in \mathcal{E} \), we have

\[
\|T(\varphi)\|_{\mathbb{B}} = \sup_{t \in T} \max_{i,j} \left\{ \left| \int_{-\infty}^{t} e_{-a_{ij}}(t, \sigma(s)) \left( \sum_{C_{kl} \in N_v(i,j)} B_{ij}^{kl}(s) f(\varphi_{kl}(s - \tau_{kl}(s))) \varphi_{ij}(s) \right. \right. \right.
\]

\[
- \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(s) \int_{t-\delta_k(s)}^{s} g(\varphi_{kl}(u)) \Delta u \varphi_{ij}(s) + L_{ij}(s) \Delta s \left. \left. \right) \right\} \leq \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e_{-a_{ij}}(t, \sigma(s)) \left( \sum_{C_{kl} \in N_v(i,j)} B_{ij}^{kl}(s) f(\varphi_{kl}(s - \tau_{kl}(s))) \varphi_{ij}(s) \right. \right. \right.
\]

\[
+ \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(s) \int_{t-\delta_k(s)}^{s} g(\varphi_{kl}(u)) \Delta u \varphi_{ij}(s) \right) \Delta s \} + \max_{i,j} \frac{T_{ij}}{a_{ij}} \leq \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e_{-a_{ij}}(t, \sigma(s)) \left( \sum_{C_{kl} \in N_v(i,j)} B_{ij}^{kl}(s) f^{2} \right. \right. \right.
\]

\[
+ \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(s) \int_{t-\delta_k(s)}^{s} [\varphi_{kl}(u)] \Delta u \varphi_{ij}(s) \right) \Delta s \} + \max_{i,j} \frac{T_{ij}}{a_{ij}} \leq \max_{i,j} \left\{ \frac{\sum_{C_{kl} \in N_v(i,j)} B_{ij}^{kl}(s) f^{2} + \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(s) \int_{t-\delta_k(s)}^{s} [\varphi_{kl}(u)] \Delta u \varphi_{ij}(s) + L_{ij}}{a_{ij}} \right\} \leq \rho.
\]

Therefore, we have that \( \|T(\varphi)\|_{\mathbb{B}} \leq \rho \). This implies that the mapping \( T \) is a self-mapping from \( \mathcal{E} \) to \( \mathcal{E} \). Next, we prove that the mapping \( T \) is a contraction mapping on \( \mathcal{E} \). In fact, for any \( \varphi, \psi \in \mathcal{E} \), we can get

\[
\|T(\varphi) - T(\psi)\|_{\mathbb{B}} = \sup_{t \in T} \|T(\varphi)(t) - T(\psi)(t)\|
\]

\[
= \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e_{-a_{ij}}(t, \sigma(s)) \left( \sum_{C_{kl} \in N_v(i,j)} B_{ij}^{kl}(s) [f(\varphi_{kl}(s - \tau_{kl}(s))) \varphi_{ij}(s) \right. \right. \right.
\]

\[
- \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(s) \int_{t-\delta_k(s)}^{s} g(\varphi_{kl}(u)) \Delta u \varphi_{ij}(s) + L_{ij}(s) \Delta s \left. \left. \right) \right\} \leq \max_{i,j} \left\{ \frac{\sum_{C_{kl} \in N_v(i,j)} B_{ij}^{kl}(s) f^{2} + \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(s) \int_{t-\delta_k(s)}^{s} [\varphi_{kl}(u)] \Delta u \varphi_{ij}(s) + L_{ij}}{a_{ij}} \right\} \leq \rho.
\]
\[-f(\psi_{kl}(s - \tau_{kl}(s)))\psi_{ij}(s) + \sum_{C_{kl} \in N_p(i,j)} C_{ij}^{kl}(s) \left[ \int_{t - \delta_{kl}(s)}^{t} g(\varphi_{kl}(u)) \Delta u \varphi_{ij}(s) \right] \leq \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e^{-a_{ij}(t, \sigma(s))} \left( \sum_{C_{kl} \in N_r(i,j)} \overline{B_{ij}^{kl}} \right) f(\varphi_{kl}(s - \tau_{kl}(s))) \| \varphi_{ij}(s) - \psi_{ij}(s) \| \Delta s \right\} + \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e^{-a_{ij}(t, \sigma(s))} \left( \sum_{C_{kl} \in N_r(i,j)} \overline{B_{ij}^{kl}} \right) f(\varphi_{kl}(s - \tau_{kl}(s))) - f(\psi_{kl}(s - \tau_{kl}(s))) \right\} \times |\psi_{ij}(s)| \times \sum_{C_{kl} \in N_r(i,j)} \overline{C_{ij}^{kl}} \int_{t - \delta_{kl}(s)}^{t} (g(\varphi_{kl}(u)) - g(\psi_{kl}(u))) \Delta u \| \psi_{ij}(s) \| \Delta s \right\} \leq \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e^{-a_{ij}(t, \sigma(s))} \left( \sum_{C_{kl} \in N_r(i,j)} \overline{B_{ij}^{kl}} \right) M_f^f \| \varphi_{kl}(s - \tau_{kl}(s)) \| \| \varphi_{ij}(s) - \psi_{ij}(s) \| \Delta s \right\} + \sum_{C_{kl} \in N_r(i,j)} \overline{C_{ij}^{kl}} \int_{t - \delta_{kl}(s)}^{t} M_f \| \varphi_{kl}(u) \| \Delta u \| \varphi_{ij}(s) - \psi_{ij}(s) \| \right\} + \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e^{-a_{ij}(t, \sigma(s))} \left( \sum_{C_{kl} \in N_r(i,j)} \overline{B_{ij}^{kl}} \right) f(\varphi_{kl}(s - \tau_{kl}(s))) - f(\psi_{kl}(s - \tau_{kl}(s))) \right\} \times |\psi_{ij}(s)| \times \sum_{C_{kl} \in N_r(i,j)} \overline{C_{ij}^{kl}} \int_{t - \delta_{kl}(s)}^{t} L^g L^f \| \varphi_{kl}(u) - \psi_{kl}(u) \| \Delta u \| \psi_{ij}(s) \| \right\} \leq \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e^{-a_{ij}(t, \sigma(s))} \left( \sum_{C_{kl} \in N_r(i,j)} \overline{B_{ij}^{kl}} \right) M_f^f \rho + \sum_{C_{kl} \in N_r(i,j)} \overline{C_{ij}^{kl}} M^g \overline{\delta_{kl}(s)} \Delta s \right\} \| \varphi - \psi \|_B + \sup_{t \in T} \max_{i,j} \left\{ \int_{-\infty}^{t} e^{-a_{ij}(t, \sigma(s))} \left( \sum_{C_{kl} \in N_r(i,j)} \overline{B_{ij}^{kl}} \right) L^f \rho + \sum_{C_{kl} \in N_r(i,j)} \overline{C_{ij}^{kl}} L^g \overline{\delta_{kl}(s)} \Delta s \right\} \| \varphi - \psi \|_B \right\} \leq \max_{i,j} \left\{ \sum_{C_{kl} \in N_r(i,j)} \overline{B_{ij}^{kl}} (M_f + L^f) \rho + \sum_{C_{kl} \in N_r(i,j)} \overline{C_{ij}^{kl}} (M^g + L^g) \overline{\delta_{kl}(s)} \right\} \| \varphi - \psi \|_B.\] 

By \((H_3)\), we have that \(\|T(\varphi) - T(\psi)\|_B < \|\varphi - \psi\|_B\). Hence, \(T\) is a contraction mapping from \(E\) to \(E\). Therefore, \(T\) has a fixed point in \(E\), that is, \((5.11)\) has a unique almost automorphic solution in \(E\). This completes the proof. \(\blacksquare\)

**Theorem 5.2.** Assume that \((H_1)-(H_3)\) hold, then system \((5.1)\) has a unique almost automorphic solution which is globally exponentially stable.

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Proof. From Theorem 5.1, we see that system (5.1) has at least one almost automorphic solution \( x^*(t) = \{ x^*_ij(t) \} \) with the initial condition \( \varphi^*(t) = \{ \varphi^*_ij(t) \} \). Suppose that \( \{ x(t) \} = \{ (x^*_ij(t)) \} \) is an arbitrary solution with the initial condition \( \varphi(t) = \{ \varphi^*_ij(t) \} \). Set \( y(t) = x(t) - x^*(t) \), then it follows from system (5.1) that

\[
y^\Delta_{ij}(t) = -a_{ij}(t)y^\Delta_{ij}(t) - \sum_{Ckl \in N_r(i,j)} B^kl_{ij}(t) (f(x^*_kl(t - \tau_{kl}(t)))x^*_ij(t) - f(x^*_kl(t))x^*_ij(t)) - \sum_{Ckl \in N_p(i,j)} C^kl_{ij}(t) \left( \int_{t-\delta_{kl}(t)}^{t} g(x^*_kl(u))dx^*_ij(u) - \int_{t-\delta_{kl}(t)}^{t} g(x^*_kl(u))dx^*_ij(u) \right) \tag{5.3}
\]

for \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \), the initial condition of (5.3) is

\[
\psi_{ij}(s) = \varphi^*_ij(s) - \varphi^*_ij(s), \quad s \in [-\theta, 0], \quad ij = 11, 12, \ldots, mn.
\]

Then, it follows from (5.3) that for \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \) and \( t \geq t_0 \), we have

\[
y_{ij}(t) = y_{ij}(t_0)e^{-a_{ij}(t, t_0)} - \int_{t_0}^{t} e^{-a_{ij}(t, \sigma(s))} \left\{ \sum_{Ckl \in N_r(i,j)} B^kl_{ij}(s) (f(x^*_kl(s - \tau_{kl}(s)))x^*_ij(s) - f(x^*_kl(s))x^*_ij(s)) + \sum_{Ckl \in N_p(i,j)} C^kl_{ij}(s) \left( \int_{s-\delta_{kl}(s)}^{s} g(x^*_kl(u))dx^*_ij(u) - \int_{s-\delta_{kl}(s)}^{s} g(x^*_kl(u))dx^*_ij(u) \right) \right\} ds. \tag{5.4}
\]

Let \( S_{ij} \) be defined as follows:

\[
S_{ij}(\omega) = \frac{a_{ij}}{\omega} - \omega - \exp(\omega \sup_{s \in \mathbb{T}} \mu(s))W_{ij}(\omega), \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n,
\]

where

\[
W_{ij}(\omega) = \left( \sum_{Ckl \in N_r(i,j)} B^kl_{ij}M^f\rho + \sum_{Ckl \in N_p(i,j)} C^kl_{ij}(M^g + L^g)\delta_{kl}\rho \exp \left\{ \omega \delta_{kl} \right\} \right) + \sum_{Ckl \in N_r(i,j)} B^kl_{ij}L^f\rho \exp \left\{ \omega \delta_{kl} \right\}, \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.
\]

By (H3), we get

\[
S_{ij}(0) = a_{ij} - W_{ij}(0) > 0, \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.
\]

Since \( S_{ij} \) is continuous on \( [0, +\infty) \) and \( S_{ij}(\omega) \to -\infty \), as \( \omega \to +\infty \), so there exist \( \xi_{ij} > 0 \) such that \( S_{ij}(\xi_{ij}) = 0 \) and \( S_{ij}(\omega) > 0 \) for \( \omega \in (0, \xi_{ij}), \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n. \) Take
by (H₃) and (5.5), we have $M > 1$.

Moreover, we have $e_{\Theta \lambda}(t, t_0) > 1$, where $t \leq t_0$. Hence, it is obvious that

$$\|y(t)\|_{\mathcal{B}} \leq Me^{\Theta \lambda}(t, t_0)\|\psi\|_{\mathcal{B}}, \quad \forall t \in [t_0 - \theta, t_0],$$

where $\lambda \in \mathcal{R}^+$. In the following, we will show that

$$\|y(t)\|_{\mathcal{B}} \leq Me^{\Theta \lambda}(t, t_0)\|\psi\|_{\mathcal{B}}, \quad \forall t \in (t_0, +\infty). \tag{5.6}$$

To prove (5.6), we first show that for any $p > 1$, the following inequality holds:

$$\|y(t)\|_{\mathcal{B}} < pMe^{\Theta \lambda}(t, t_0)\|\psi\|_{\mathcal{B}}, \quad \forall t \in (t_0, +\infty). \tag{5.7}$$

which implies that, for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, we have

$$|y_{ij}(t)| < pMe^{\Theta \lambda}(t, t_0)\|\psi\|_{\mathcal{B}}, \quad \forall t \in (t_0, +\infty). \tag{5.8}$$

If (5.8) is not true, then there exists $t_1 \in (t_0, +\infty)$ such that

$$|y_{ij}(t)| \geq pMe^{\Theta \lambda}(t_1, t_0)\|\psi\|_{\mathcal{B}},$$

$$|y_{ij}(t)| < pMe^{\Theta \lambda}(t, t_0)\|\psi\|_{\mathcal{B}}, \quad \forall t \in (t_0, t_1).$$

Therefore, there must exist a constant $c \geq 1$ such that

$$|y_{ij}(t_1)| = cMe^{\Theta \lambda}(t_1, t_0)\|\psi\|_{\mathcal{B}},$$

$$|y_{ij}(t)| < cMe^{\Theta \lambda}(t, t_0)\|\psi\|_{\mathcal{B}}, \quad \forall t \in (t_0, t_1).$$

Note that, in view of (5.4), we have that

$$|y_{ij}(t_1)| = |y_{ij}(t_0)e^{-a_{ij}(t_1, t_0)} - \int_{t_0}^{t_1} e^{-a_{ij}(t_1, \sigma(s))}\left( \sum_{C_{kl} \in \mathcal{N},(i,j)} B_{kl}^{ij}(s)(f(x_{kl}(s - \tau_{kl}(s)))x_{ij}(s) \right)|$$
- f(x^{*}_{kl}(s - \tau_{kl}(s)))x^{*}_{ij}(s) + \sum_{C_{kl} \in N_{p}(i,j)} C^{kl}_{ij}(s) \left( \int_{s - \delta_{kl}(s)}^{s} g(x_{kl}(u))\Delta u x^{*}_{ij}(s) \right) \\
- \int_{s - \delta_{kl}(s)}^{s} g(x^{*}_{kl}(u))\Delta u x^{*}_{ij}(s) \right) \Delta s \\
\leq e_{-a_{ij}}(t_{1}, t_{0})\|\psi\|_{B} + \int_{t_{0}}^{t_{1}} e_{-a_{ij}}(t_{1}, \sigma(s)) \left\{ \sum_{C_{kl} \in N_{r}(i,j)} B^{kl}_{ij} M^{f}\left| x_{kl}(s - \tau_{kl}(s)) \right| y_{ij}(s) \right. \\
\left. + \sum_{C_{kl} \in N_{p}(i,j)} C^{kl}_{ij} \int_{s - \delta_{kl}(s)}^{s} M^{9}\left| x_{kl}(u) \right| \Delta u \left| y_{ij}(s) \right| \\
+ \sum_{C_{kl} \in N_{r}(i,j)} B^{kl}_{ij} L^{f}\left| y_{kl}(s - \tau_{kl}(s)) \right| \left| x^{*}_{ij}(s) \right| \\
+ \sum_{C_{kl} \in N_{p}(i,j)} C^{kl}_{ij} \int_{s - \delta_{kl}(s)}^{s} L^{9}\left| y_{kl}(u) \right| \Delta u \left| x^{*}_{ij}(s) \right| \right\} \Delta s \\
\leq e_{-a_{ij}}(t_{1}, t_{0})\|\psi\|_{B} + cpMe_{\omega_{\lambda}}(t_{1}, t_{0})\|\psi\|_{B} \left| \int_{t_{0}}^{t_{1}} e_{-a_{ij}}(t_{1}, \sigma(s)) \right. \\
\left. \times \left\{ \sum_{C_{kl} \in N_{r}(i,j)} B^{kl}_{ij} M^{f}\rho e_{\lambda}(s, s) + \sum_{C_{kl} \in N_{p}(i,j)} C^{kl}_{ij} M^{9}\delta_{kl}\rho e_{\lambda}(s, s - \delta_{kl}(s)) \right. \\
+ \sum_{C_{kl} \in N_{r}(i,j)} B^{kl}_{ij} L^{f}\rho e_{\lambda}(s, s - \tau_{kl}(s)) \\
+ \sum_{C_{kl} \in N_{p}(i,j)} C^{kl}_{ij} L^{9}\delta_{kl}\rho e_{\lambda}(s, s - \delta_{kl}(s)) \right\} \right| \Delta s \\
\leq e_{-a_{ij}}(t_{1}, t_{0})\|\psi\|_{B} + cpMe_{\omega_{\lambda}}(t_{1}, t_{0})\|\psi\|_{B} \left| \int_{t_{0}}^{t_{1}} e_{-a_{ij}}(t_{1}, \sigma(s)) \right. \\
\left. \times \left\{ \sum_{C_{kl} \in N_{r}(i,j)} B^{kl}_{ij} M^{f}\right. \exp \left\{ \frac{\lambda}{s_{\epsilon}} \sup_{s_{\epsilon} \in \mathbb{T}} (s_{\epsilon}) \right\} \\
+ \sum_{C_{kl} \in N_{p}(i,j)} C^{kl}_{ij} M^{9}\delta_{kl}\exp \left\{ \frac{\lambda}{s_{\epsilon}} \sup_{s_{\epsilon} \in \mathbb{T}} (s_{\epsilon}) \right\} \\
+ \sum_{C_{kl} \in N_{r}(i,j)} B^{kl}_{ij} L^{f}\exp \left\{ \frac{\lambda}{s_{\epsilon}} \sup_{s_{\epsilon} \in \mathbb{T}} (s_{\epsilon}) \right\} \right\} \right| \Delta s
Remark 5.1. Theorems 5.1 and 5.2 are new even for the both cases of differential equations ($\mathbb{T} = \mathbb{R}$) and difference equations ($\mathbb{T} = \mathbb{Z}$).
6 An example

In this section, we will give examples to illustrate the feasibility and effectiveness of our results obtained in Sections 5.

Consider the following SICNNs with time-varying delays:

\[ x^{Δ}_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{c_{kj} \in N_p(i,j)} B_{ij}^{kl}(t)f(x_{kl}(t - τ_{kl}(t)))x_{ij}(t) \]
\[ - \sum_{c_{kj} \in N_p(i,j)} C_{ij}^{kl}(t) \int_{t-τ_{kl}(t)}^{t} g(x_{kl}(u))Δux_{ij}(t) + L_{ij}(t), \]  

(6.1)

where \( i = 1, 2, 3, j = 1, 2, 3, r = p = 1, f(x) = 0.05|\cos x|, g(x) = \frac{|x|}{20}, t \in \mathbb{T}. \)

Example 6.1. If \( \mathbb{T} = \mathbb{R}, \) then \( μ(t) \equiv 0. \) Take

\[
\begin{pmatrix}
    a_{11}(t) & a_{12}(t) & a_{13}(t) \\
    a_{21}(t) & a_{22}(t) & a_{23}(t) \\
    a_{31}(t) & a_{32}(t) & a_{33}(t)
\end{pmatrix} = \begin{pmatrix}
    3 + |\sin t| & 4 + |\cos \sqrt{2}t| & 2 + |\sin t| \\
    2 + |\cos(\frac{1}{3}t)| & 5 + |\sin 2t| & 1 + |\cos 2t| \\
    4 + |\cos 2t| & 3 + |\sin 2t| & 2 + |\cos t|
\end{pmatrix},
\]

\[
\begin{pmatrix}
    B_{11}(t) & B_{12}(t) & B_{13}(t) \\
    B_{21}(t) & B_{22}(t) & B_{23}(t) \\
    B_{31}(t) & B_{32}(t) & B_{33}(t)
\end{pmatrix} = \begin{pmatrix}
    C_{11}(t) & C_{12}(t) & C_{13}(t) \\
    C_{21}(t) & C_{22}(t) & C_{23}(t) \\
    C_{31}(t) & C_{32}(t) & C_{33}(t)
\end{pmatrix},
\]

\[
\begin{pmatrix}
    L_{11}(t) & L_{12}(t) & L_{13}(t) \\
    L_{21}(t) & L_{22}(t) & L_{23}(t) \\
    L_{31}(t) & L_{32}(t) & L_{33}(t)
\end{pmatrix} = \begin{pmatrix}
    0.1 |\cos t| & 0.2 |\cos 2t| & 0.1 |\sin t| \\
    0.1 |\sin 2t| & 0.1 |\cos t| & 0.2 |\cos 2t| \\
    0.1 |\cos t| & 0.2 |\cos 2t| & 0.1 |\cos t|
\end{pmatrix},
\]

\[
\begin{pmatrix}
    δ_{11}(t) & δ_{12}(t) & δ_{13}(t) \\
    δ_{21}(t) & δ_{22}(t) & δ_{23}(t) \\
    δ_{31}(t) & δ_{32}(t) & δ_{33}(t)
\end{pmatrix} = \begin{pmatrix}
    0.5 + 0.5 |\sin t| & \cos t & 0.7 + 0.2 |\sin t| \\
    0.8 + 0.2 |\sin t| & 0.2 + 0.4 |\sin t| & 0.7 + 0.2 |\sin t| \\
    0.9 + 0.1 |\sin t| & 0.6 + 0.3 |\sin t| & 0.4 + 0.4 |\cos t|
\end{pmatrix}.
\]

Obviously, (H₁) holds. Clearly, we have

\[
M^f = M^g = 0.05, \quad L^f = L^g = 0.05, \quad \sum_{c_{kj} \in N_1(1,1)} B_{11}^{kl} = \sum_{c_{kj} \in N_1(1,1)} B_{11}^{kl} = 0.5,
\]

\[
\sum_{c_{kj} \in N_1(1,2)} B_{12}^{kl} = \sum_{c_{kj} \in N_1(1,1)} C_{12}^{kl} = 1, \quad \sum_{c_{kj} \in N_1(1,3)} B_{13}^{kl} = \sum_{c_{kj} \in N_1(1,3)} C_{13}^{kl} = 0.6,
\]

\[
\sum_{c_{kj} \in N_1(2,1)} B_{21}^{kl} = \sum_{c_{kj} \in N_1(2,1)} C_{21}^{kl} = 0.8, \quad \sum_{c_{kj} \in N_1(2,2)} B_{22}^{kl} = \sum_{c_{kj} \in N_1(2,2)} C_{22}^{kl} = 1.2,
\]

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The combination of the above two inequalities means that \( H_3 \) is satisfied for \( \rho = 1 \).

Therefore, we have shown that assumptions \((H_1)-(H_3)\) are satisfied. By Theorem 3.1, system \((6.1)\) has exactly one almost automorphic solution in \( E = \{ \varphi \in B : \| \varphi \|_B \leq \rho \} \).

Moreover, by Theorem 5.2 this solution is globally exponentially stable.

**Example 6.2.** If \( T = \mathbb{Z} \), then \( \mu(t) \equiv 1 \). Take

\[
\begin{pmatrix}
  a_{11}(t) & a_{12}(t) & a_{13}(t) \\
  a_{21}(t) & a_{22}(t) & a_{23}(t) \\
  a_{31}(t) & a_{32}(t) & a_{33}(t)
\end{pmatrix}
= \begin{pmatrix}
  0.3 + 0.1 \sin t & 0.4 + 0.1 \cos 2t \\
  0.2 + 0.2 \cos (\frac{1}{2} t) & 0.5 + 0.1 \sin 2t \\
  0.4 + 0.1 \cos 2t & 0.3 + 0.2 \sin 2t
\end{pmatrix}
= \begin{pmatrix}
  0 + 1 \sin t & 0.4 + 0.1 \cos 2t \\
  0.2 + 0.2 \cos (\frac{1}{2} t) & 0.5 + 0.1 \sin 2t \\
  0.4 + 0.1 \cos 2t & 0.3 + 0.2 \sin 2t
\end{pmatrix}
\]

\[
\begin{pmatrix}
  B_{11}(t) & B_{12}(t) & B_{13}(t) \\
  B_{21}(t) & B_{22}(t) & B_{23}(t) \\
  B_{31}(t) & B_{32}(t) & B_{33}(t)
\end{pmatrix}
= \begin{pmatrix}
  C_{11}(t) & C_{12}(t) & C_{13}(t) \\
  C_{21}(t) & C_{22}(t) & C_{23}(t) \\
  C_{31}(t) & C_{32}(t) & C_{33}(t)
\end{pmatrix}
= \begin{pmatrix}
  0.02 \cos t & 0.02 \cos 2t & 0.03 \sin t \\
  0.01 \sin 2t & 0.01 \cos t & 0.02 \cos 2t \\
  0.03 \cos t & 0.02 \cos 2t & 0.01 \cos t
\end{pmatrix},
\]

\[
\begin{pmatrix}
  L_{11}(t) & L_{12}(t) & L_{13}(t) \\
  L_{21}(t) & L_{22}(t) & L_{23}(t) \\
  L_{31}(t) & L_{32}(t) & L_{33}(t)
\end{pmatrix}
= \begin{pmatrix}
  0.01 \sin t & 0.01 \cos t & 0.02 \sin t \\
  0.02 \sin t & 0.04 \sin t & 0.03 \sin t \\
  0.01 \sin t & 0.03 \sin t & 0.02 \cos t
\end{pmatrix},
\]

\[
\begin{pmatrix}
  \delta_{11}(t) & \delta_{12}(t) & \delta_{13}(t) \\
  \delta_{21}(t) & \delta_{22}(t) & \delta_{23}(t) \\
  \delta_{31}(t) & \delta_{32}(t) & \delta_{33}(t)
\end{pmatrix}
= \begin{pmatrix}
  \sin t & 0.9 \cos t & 0.2 + 0.5 \sin t \\
  0.2 + 0.8 \sin t & 0.4 + 0.6 \sin t & 0.1 + 0.9 \sin t \\
  \sin t & \sin t & 0.7 + 0.2 \cos t
\end{pmatrix}.
\]
Obviously, \((H_1)\) holds. Clearly, we have

\[ M^f = M^g = 0.05, \quad L^f = L^g = 0.05, \quad \sum_{C_{kl} \in N_1(1,1)} B_{11}^{kl} = \sum_{C_{kl} \in N_1(1,1)} C_{11}^{kl} = 0.06, \]

\[ \sum_{C_{kl} \in N_1(1,2)} B_{12}^{kl} = \sum_{C_{kl} \in N_1(1,2)} C_{12}^{kl} = 0.11, \quad \sum_{C_{kl} \in N_1(1,3)} B_{13}^{kl} = \sum_{C_{kl} \in N_1(1,3)} C_{13}^{kl} = 0.08, \]

\[ \sum_{C_{kl} \in N_1(2,1)} B_{21}^{kl} = \sum_{C_{kl} \in N_1(2,1)} C_{21}^{kl} = 0.11, \quad \sum_{C_{kl} \in N_1(2,2)} B_{22}^{kl} = \sum_{C_{kl} \in N_1(2,2)} C_{22}^{kl} = 0.17, \]

\[ \sum_{C_{kl} \in N_1(2,3)} B_{23}^{kl} = \sum_{C_{kl} \in N_1(2,3)} C_{23}^{kl} = 0.11, \quad \sum_{C_{kl} \in N_1(3,1)} B_{31}^{kl} = \sum_{C_{kl} \in N_1(3,1)} C_{31}^{kl} = 0.07, \]

\[ \sum_{C_{kl} \in N_1(3,2)} B_{32}^{kl} = \sum_{C_{kl} \in N_1(3,2)} C_{32}^{kl} = 0.1, \quad \sum_{C_{kl} \in N_1(3,3)} B_{33}^{kl} = \sum_{C_{kl} \in N_1(3,3)} C_{33}^{kl} = 0.06 \]

and we can easily check that \(\max_{i,j} \frac{T_{ij}}{a_{ij}} = 0.15\). Take \(\rho = 1\), then

\[
\max_{i,j} \left\{ \frac{\sum_{C_{ij} \in N_p(i,j)} B_{ij}^{k}(M^f \rho^2 + \sum_{C_{ij} \in N_p(i,j)} C_{ij}^{k} M^g \rho^2 \delta_{kl}}{a_{ij}} + L_{ij} \right\} \approx 0.1904 < \rho = 1
\]

and

\[
\max_{i,j} \left\{ \frac{\sum_{C_{ij} \in N_p(i,j)} B_{ij}^{k}(M^f + L^f) \rho + \sum_{C_{ij} \in N_p(i,j)} C_{ij}^{k}(M^g + L^g) \rho \delta_{kl}}{a_{ij}} \right\} \approx 0.3657 < 1.
\]

The combination of the above two inequalities means that \((H_3)\) is satisfied for \(\rho = 1\).

Therefore, we have shown that assumptions \((H_1)\)-(\(H_3)\) are satisfied. By Theorem 5.1, system \((6.1)\) has exactly one almost automorphic solution in \(E = \{ \varphi \in B : ||\varphi||_B \leq \rho \}\). Moreover, by Theorem 5.2 this solution is globally exponentially stable.

### 7 Conclusion

In this paper, first a new concept of almost periodic time scales and a new definition of almost automorphic functions on almost periodic time scales are proposed and some basic properties of them are studied. Two open problems concerning the relationship between the algebraic operation property of elements of a time scale and analytical property of the time scale are proposed. Then, based on these concepts and results, the existence of an almost automorphic solution for both the linear nonhomogeneous dynamic equation on time scales and its associated homogeneous equation is established. Finally, as an application of the results, the existence and global exponential stability of almost automorphic solutions to a class of shunting inhibitory cellular neural networks with time-varying delays on time scales are obtained.
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