MORSE-NOVIKOV COHOMOLOGY ON FOLIATED MANIFOLDS

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Abstract. The idea of Lichnerowicz or Morse-Novikov cohomology groups of a manifold has been utilized by many researchers to study important properties and invariants of a manifold. Morse-Novikov cohomology is defined using the differential \( d_\omega = d + \omega \wedge \), where \( \omega \) is a closed 1-form. We study Morse-Novikov cohomology relative to a foliation on a manifold and its homotopy invariance and then extend it to more general type of forms on a Riemannian foliation. We study the Laplacian and Hodge decompositions for the corresponding differential operators on reduced leafwise Morse-Novikov complexes. In the case of Riemannian foliations, we prove that the reduced leafwise Morse-Novikov cohomology groups satisfy the Hodge theorem and Poincaré duality. The resulting isomorphisms yield a Hodge diamond structure for leafwise Morse-Novikov cohomology.

1. Introduction

Consider an \( n \)-dimensional smooth manifold \( M \); denote by \( \Omega^k(M) \) the collection of all degree \( k \) differential forms on \( M \) and by \( H^k(M) \) the corresponding de Rham cohomology group. Let \( \omega \) be a closed 1-form that is not necessarily exact. We consider the twisted operator \( d_\omega : \Omega^k(M) \to \Omega^{k+1}(M) \) defined by \( d_\omega = d + \omega \wedge \), where \( d \) is the usual exterior derivative, so that \( (d_\omega)^2 = 0 \). The differential cochain complex \( (\Omega^\ast(M), d_\omega) \) is called the Morse-Novikov complex of the manifold \( M \). The cohomology groups \( H_\omega^k(M) \) of this cochain complex are called the Morse-Novikov or Lichnerowicz cohomology groups of \( M \) and have been utilized by many researchers. Morse-Novikov cohomology was first studied by A. Lichnerowicz in [13], and used in the context of Poisson geometry. The idea of Lichnerowicz has been exploited to study many properties of manifolds. In [15] and [16] S.P. Novikov proved a generalization of the Morse inequalities by comparing the ranks of these cohomology groups with combinatorial invariants derived from the zeros of the form \( \omega \). Pazhintov [21] gave an analytic proof of the real part of Novikov’s inequalities. E. Witten used the Morse-Novikov cohomology for exact \( \omega \) in his famous discovery [27] of what is now known as Witten deformation. In this case Morse-Novikov cohomology is isomorphic to de Rham cohomology. M. Shubin and S. P. Novikov applied the deformation method to a rigorous treatment of eigenvalue limits of Witten Laplacians for more general 1-forms and vector fields in [17] and [23]. Many other researchers have extended and generalized this work, such as Braverman and Farber [4] in cases of nonisolated zeros of 1-forms and vector fields. See [8] for a good reference on these related topics. Alexandra Otiman studied Morse-Novikov cohomology for particular classes of closed 1-forms in [20]. I. Vaisman studied locally conformal symplectic manifolds in [24], and L. Ornea, and M. Verbitsky studied Morse-Novikov cohomology of locally conformally Kähler manifolds in [19]. In [5], X. Chen showed that if a Riemannian manifold \( M \) has almost non-negative sectional curvature and nonzero first de Rham cohomology group, then
all the Morse-Novikov cohomology groups of $M$ vanish irrespective of the choice of the closed non-exact 1-form $\omega$. In [14], L. Meng established an analogue of the Leray-Hirsch Theorem for de Rham cohomology and a blowup formula for Dolbeault-Morse-Novikov cohomology on complex manifolds. Morse-Novikov cohomology theory has also been used to study locally conformal symplectic manifolds (see [25], [24], and [26]).

Two types of Morse-Novikov cohomology associated to foliations are basic (see [22]) and leafwise. Liviu Ornea and Vladimir Slesar studied basic Morse-Novikov cohomology in [18]. K. Richardson and G. Habib used basic Morse-Novikov cohomology to prove that the basic signature and the Álvarez class of a Riemannian foliation are homotopy invariants [10]. They have also used a modified differential as in Morse-Novikov cohomology to define a twisted basic cohomology for Riemannian foliations that satisfies Poincaré duality [9]. J. A. Álvarez Lopez, Y. Kordyukov, and E. Leichtnam studied leafwise Hodge decomposition on Riemannian foliations with bounded geometry and extended the Morse-Novikov differential complex [2].

This paper is constructed as follows. Using $d + \omega \wedge$ as the differential for a leafwise closed 1-form $\omega$, we study leafwise Morse-Novikov cohomology groups whose isomorphism classes turn out to be smooth invariants of the foliation. In the cases where $\omega$ is truly a closed 1-form on the manifold, one further obtains Morse-Novikov cohomology groups from the foliation. In Section 3 we study the basic properties of leafwise Morse-Novikov cohomology groups, including the homotopy axiom in Proposition [3.7] and foliated homotopy invariance in Corollary [3.8]. With the additional assumption that the foliation is Riemannian, we give a proof of the Hodge decomposition in Corollary [4.6] and Poincaré duality in Corollary [4.7]. We extend these results to more general settings of forms of $p, q$ type: homotopy axioms for general leafwise Morse-Novikov cohomology in Proposition [5.3], Hodge decomposition for general leafwise Morse-Novikov cohomology in Theorem [6.1], Poincaré duality for general leafwise Morse-Novikov cohomology in Proposition [6.3]. The assumption that the foliation is Riemannian is required to obtain Hodge theory and Poincaré duality; for general smooth foliations, those results are false, even for the case when $\omega = 0$. In the remainder of Section [9] the isomorphisms between the Morse-Novikov cohomology groups are discussed, leading to Corollary [6.8] that shows the Hodge diamond structure.

Much but not all of the results in this paper were part of the authors Ph.D thesis [12].

2. Leafwise de Rham cohomology

In this section we review notations and known results. Let $M$ be a closed compact oriented Riemannian manifold. Suppose we are given a smooth foliation $(M, F)$ on $M$. Let $TF$ denote the leafwise tangent bundle and $T^*F$ denotes its dual bundle. Let the conormal bundle $N^*_xF$ be defined at each point $x \in M$ as the set of all linear functionals that map each vector in $T_xF$ to zero, and this bundle can be canonically identified with a subbundle of $T^*_xM$ independent of the metric. Using the metric, the normal bundle $Q = TM/TF$ may be uniquely identified with a subbundle $N^*_F = TF^\perp$ of $TM$. Also, we can identify the dual bundle $T^*F$ with the set of covectors that kill $N_F$. Now we can decompose all differential forms using

$$(2.1) \quad \Lambda^{u,v}(M, F) = \Lambda^u N^*F \wedge \Lambda^v T^*F$$

Let $\Omega^{u,v}(M, F) = \Gamma \Lambda^{u,v}(M, F)$. The exterior derivative can then be decomposed as $d = d_{0,1} + d_{1,0} + d_{2,-1}$ with

$$d_{i,j} \omega \in \Omega^{u+i,v+j}(M, F)$$
for all $\omega \in \Omega^{n,v}(M, F)$. Then it is easy to see that since $d^2 = 0$, we also have $d_{0,1}^2 = 0$.

The elements of $\Omega^{0,k}(M, F)$ are called leafwise $k$-forms. Let $\Gamma(TF)$ be the set of smooth sections of $TF$. If $X, Y \in \Gamma(TF)$, then by the Frobenius theorem $[X, Y] \in \Gamma(TF)$. The leafwise exterior differential operator $d^k_{0,1} = d^k_d : \Omega^{0,k}(M, F) \to \Omega^{0,k+1}(M, F)$ may also be defined by

$$d^k_d \omega (X_0, \cdots, X_{k+1}) = \sum_{0 \leq i \leq k} (-1)^i [X_i \omega (X_0, \cdots, \hat{X}_i, \cdots, X_k)]$$

$$+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega ([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k)$$

for $X_0, \cdots, X_{k+1} \in \Gamma(TF)$. The differential operator $d_d$ is the restriction of the usual differential on differential forms on the leaves of $F$. Similar to the usual exterior differential, the leafwise differential satisfies $d^k_d \circ d^k_d = 0$. For $k \geq 0$, the $k$th leafwise cohomology group $\bar{H}^k (M, d_d)$ is the $k$th cohomology group

$$\bar{H}^k (M, d_d) = \frac{\ker d^k_d}{\text{im} d^k_{d-1}}$$

of the cochain complex $(\Omega^{0,*}(M, F), d_d)$.

For the purpose of having Laplacian and Hodge decompositions, we need to consider reduced leafwise cohomology

$$\bar{H}^k (M, d_d) = \frac{\ker d^k_d}{\text{im} d^k_{d-1}}.$$  

Here the closure $\text{im} d^k_{d-1}$ is taken with respect to the Frechét topology on $\Omega^{0,k}(M, F)$. The cup product induced from exterior product of forms makes $\bar{H}^*(M, d_d)$ into a graded commutative algebra over $C^\infty(M)$.

Let $f : M \to N$ be a smooth map of the foliated manifold which maps leaves into leaves. Then the pullback maps

$$f^* : \Gamma(\Lambda^k T^* F_N) \to \Gamma(\Lambda^k T^* F_M)$$

are defined for all $k$. They commute with $d_d$ and respect the exterior product; therefore they induce a continuous map of the reduced cohomology ring.

$$f^* : \bar{H}^k (N, d_{F_0}) \to \bar{H}^k (M, d_{F_1}).$$

Such maps are called foliated maps. Two smooth foliated maps $f, g : (N, F_0) \to (M, F_1)$ between two foliated manifolds are leafwise homotopic if there is a map $F : N \times [0, 1] \to M$ such that if $F_t$ denotes the restriction $F_t = F|_{N \times \{t\}} : N \to M$ for $t \in [0, 1]$, then $F|_0 = f, F|_1 = g$, $F_t (F_0) \in F_1$ for all $t \in [0, 1]$, and for every $x \in N$ the points $F_{t_1} (x), F_{t_2} (x)$ lie in the same leaf of $F_1$ for all $t_1, t_2 \in [0, 1]$. Thus, a leafwise homotopy consists of leaf-preserving maps. Denote the identity maps of $(N, F_0), (M, F_1)$ by $l_{F_0}, l_{F_1}$ respectively. A leafwise map $f : (N, F_0) \to (M, F_1)$ is a leafwise homotopy equivalence if there exists a leafwise map $g : (M, F_1) \to (N, F_0)$ with $f \circ g$ is leafwise homotopic to $l_{F_0}$ and $g \circ f$ is leafwise homotopic to $l_{F_1}$.

**Proposition 2.1.** [11 Theorem I, 3.2] If the maps $f, g : (N, F_0) \to (M, F_1)$ between two foliated manifolds are leafwise homotopic, then $f^* = g^* : H^k (M, d_{F_1}) \to H^k (N, d_{F_0})$. That is, leafwise homotopic maps induce the same map on leafwise cohomology groups.
Corollary 2.2. If a map \( f : (N, \mathcal{F}_0) \to (M, \mathcal{F}_1) \) is a smooth foliated homotopy equivalence, then \( f^* \) induces an isomorphism between \( H^k(N, d\mathcal{F}_0) \) and \( H^k(M, d\mathcal{F}_1) \).

3. Leafwise Morse-Novikov Cohomology and Hodge Theory

From now we assume that our foliation is Riemannian, characterized by the existence of a bundle-like metric \( g \) such that a geodesic of the metric \( g \) is orthogonal to all leaves that it meets whenever it is orthogonal to one of them. We assume that a bundle like metric has been chosen.

Let \( M \) be an oriented manifold endowed with a foliation \( \mathcal{F} \) of dimension \( p \). The graded Frechét space \( \Omega^{0,*}(M, \mathcal{F}) \) can be endowed with the natural metric
\[
(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle_{\mathcal{F}} \vol.
\]
In this formula \( \langle \cdot, \cdot \rangle_{\mathcal{F}} \) is the Riemannian metric on \( \Lambda T^* \mathcal{F} \) induced from the Riemannian metric \( g \) on \( M \), and \( \vol \) is the volume form associated to the metric \( g \). We denote the formal adjoint of the leafwise differential \( d_{\mathcal{F}} \) with respect to this inner product by \( \delta_{\mathcal{F}} \); then the corresponding Laplacian is
\[
\Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}.
\]
Since \( \mathcal{F} \) is Riemannian, the restriction of \( \delta_{\mathcal{F}} \) to any leaf is the codifferential of the leaf with respect to the induced metric \footnote[3]{Lemma 3.2}, i.e.
\[
(\delta_{\mathcal{F}} \alpha) |_{F} = \delta_{F} (\alpha) |_{F} \text{ for all } \alpha \in \Omega^{0,k}(M, \mathcal{F}),
\]
where \( F \) denotes a leaf of the foliation. Now we assume that the tangent bundle \( T\mathcal{F} \) is orientable. The choice of an orientation determines a volume form \( \chi_{\mathcal{F}} \in \Omega^{0,p}(M, \mathcal{F}) \). Now we can define leafwise Hodge star-operator
\[
\ast_{\mathcal{F}} : \Lambda^{0,k} T^* \mathcal{F} \to \Lambda^{0,p-k} T^* \mathcal{F} \text{ for each } k \text{ and } x \in M,
\]
and it is determined by the relation
\[
\alpha \wedge \ast_{\mathcal{F}} \beta = \langle \alpha, \beta \rangle \chi_{\mathcal{F}}, \text{ for } \alpha, \beta \in \Lambda^{0,k} T^*_x \mathcal{F}.
\]
This fibrewise star-operator determines the leafwise star-operator
\[
\ast_{\mathcal{F}} : \Omega^{0,k}(M, \mathcal{F}) \to \Omega^{0,p-k}(M, \mathcal{F}) \text{ for each } k.
\]

Now we state some important properties of leafwise cohomology. Suppose \( M \) is compact, and \( \mathcal{F} \) is a \( p \)-dimensional oriented Riemannian foliation of \( M \) with a bundle-like metric \( g \).

Proposition 3.1. \footnote[5]{Theorem 0.2} The map \( \phi : \ker \Delta^k_{\mathcal{F}} \to \check{H}^k(M, d_{\mathcal{F}}) \) defined by \( \phi(\omega) = \omega \mod \im d^k_{\mathcal{F}} \) is a topological isomorphism of Frechét spaces. This isomorphism, in general, does not hold for non-Riemannian foliations.

Under the same assumptions, the next deep result is due to Álvarez López and Kordyukov.

Theorem 3.2. \footnote[4]{Corollary C} The Hodge star-operator induces an isomorphism
\[
\ast_{\mathcal{F}} : \ker \Delta^k_{\mathcal{F}} \to \ker \Delta^{p-k}_{\mathcal{F}}.
\]
Moreover \( \ast_{\mathcal{F}} \) commutes with \( \Delta^k_{\mathcal{F}} \) up to a sign. From the previous proposition we have the following isomorphism
\[
\ast_{\mathcal{F}} : \check{H}^k(M, d_{\mathcal{F}}) \to \check{H}^{p-k}(M, d_{\mathcal{F}}).
\]
Let \( \omega \) be a leafwise closed 1-form, which is not necessarily leafwise exact. We consider the twisted operator \( \tilde{d}_F^k : \Omega^{0,k}(M, \mathcal{F}) \to \Omega^{0,k+1}(M, \mathcal{F}) \) defined by \( \tilde{d}_F^k = d_F + \omega \wedge \), where \( d_F \) is the exterior derivative along the leaf. Since \( d_F \circ d_F = d_F^2 = 0 \), \( \omega \wedge \omega = 0 \), and \( d_F(\omega \wedge \alpha) = -\omega \wedge d_F \alpha \) for any \( k \)-form \( \alpha \), it follows that \( (d_F^\omega)^2 = 0 \). The differential cochain complex \( (\Omega^{0,*}(M, \mathcal{F}), d_F^\omega) \) is called the leafwise Morse-Novikov complex of the foliated manifold \((M, \mathcal{F})\). Let \( d_F^{-k} \) be the restriction of \( d_F^k \) to \( \Omega^{0,k}(M, \mathcal{F}) \). The cohomology groups

\[
H^k_\omega(M, d_F) = \frac{\ker \left( d_F^{-k} \right)}{\text{im} \left( d_F^{k-1} \right)}
\]

of this cochain complex are called the leafwise Morse-Novikov cohomology groups of \((M, \mathcal{F})\). For the purpose of obtaining Hodge decomposition, we need to consider the reduced leafwise Morse-Novikov cohomology

\[
\tilde{H}^k_\omega(M, d_F) = \frac{\ker \left( d_F^{-k} \right)}{\text{im} \left( d_F^{k-1} \right)}.
\]

Here the closure \( \overline{\text{im}(d_F^{-k-1})} \) is taken with respect to the Frechet topology on \( \Omega^{0,k}(M, \mathcal{F}) \). The cup product induced from exterior product of forms makes \( \tilde{H}^*_\omega(M, d_F) \) into a graded commutative algebra over \( C^\infty(M) \).

**Proposition 3.3.** If \( \omega \) and \( \theta = \omega + d_F g \) are cohomologous in \( H^1(M, d_F) \), then for each \( k \), the leafwise Morse-Novikov cohomology groups \( H^k_\omega(M, d_F) \) and \( H^k_\theta(M, d_F) \) are isomorphic. That is, the map \( \Phi : H^k_\omega(M, d_F) \to H^k_\theta(M, d_F) \) given by \( \Phi([\alpha]) = [e^{-g}\alpha] \) is an isomorphism.

**Proof.** If \( \omega \) and \( \theta \) are cohomologous, then there exists \( g \in \Omega^0(M) \), such that \( \theta - \omega = d_F g \). Define the mapping \( \phi : H^k_\omega(M, d_F) \to H^k_\theta(M, d_F) \) by \( \phi([\alpha]) = [e^{-g}\alpha] \). One can check that \( \phi \) is well-defined and is a group homomorphism, since

\[
(d_F + \theta \wedge) (e^{-g}\alpha) = (d_F + \omega \wedge + d_F g \wedge) (e^{-g}\alpha),
\]

\[
= e^{-g} (d_F + \omega \wedge) (\alpha),
\]

for all \( \alpha \in \Omega^k(M, d_F) \).

Suppose \( \alpha, \beta \in \Omega^k(M, d_F) \) are cohomologous, then there exists \( \nu \in \Omega^{k-1}(M, d_F) \) such that \( \alpha - \beta = (d_F + \omega \wedge) \nu \). We have

\[
\phi([\alpha - \beta]) = [e^{-g} (d_F + \omega \wedge) \nu],
\]

\[
\Rightarrow [(d_F + \theta \wedge) (e^{-g}\nu)] = [0].
\]

Similarly if, \( \phi([\alpha]) = 0 \), Then \( [\alpha] = 0 \in H^k_\omega(M, d_F) \), and \( \phi \) is injective. If \( [\alpha] \in H^k_\theta(M, d_F) \) then we find similarly that \( [\alpha] = 0 \) in \( H^k_\omega(M, d_F) \), so that \( \phi \) is surjective. \( \square \)

**Corollary 3.4.** If \( \omega \) is a \( d_F \) exact 1-form, then for each \( k \) the leafwise Morse-Novikov cohomology group \( H^k_\omega(M, d_F) \) and the leafwise de Rham cohomology group \( H^k(M, d_F) \) are isomorphic.

\[
H^k_\omega(M, d_F) \cong H^k(M, d_F).
\]

**Corollary 3.5.** If the first leafwise de Rham cohomology group \( H^1(M, d_F) \) equals 0, then for every \( d_F \) closed 1-form \( \omega \) and for each \( k \) the leafwise Morse-Novikov cohomology groups satisfy \( H^k_\omega(M, d_F) = H^k(M, d_F) \).
Lemma 3.6. For any smooth foliation \((M, \mathcal{F})\) the leafwise Morse-Novikov cohomology \(H^0_\omega(M, d_{\mathcal{F}}) = \{0\}\) if and only if \(\omega\) is not \(d_{\mathcal{F}}\) exact.

Proof. Suppose first that \(H^0_\omega(M, d_{\mathcal{F}}) \neq \{0\}\) for a \(d_{\mathcal{F}}\) closed one form \(\omega\) on \((M, \mathcal{F})\), then there is a nonzero function \(f \in C^\infty(M)\), such that
\[
\begin{align*}
(d_{\mathcal{F}} + \omega) f &= 0 \\
d_{\mathcal{F}} f + f \omega &= 0 \\
d_{\mathcal{F}} \left( \log \left( \frac{1}{f} \right) \right) &= \omega,
\end{align*}
\]
which implies \(\omega\) is \(d_{\mathcal{F}}\) exact. Conversely, suppose that \(\omega\) is \(d_{\mathcal{F}}\) exact. There exists a function \(g \in C^\infty(M)\) such that \(d_{\mathcal{F}} g = \omega\). Then
\[
(d_{\mathcal{F}} + \omega) \left( e^{-g} \right) = -e^{-g} d_{\mathcal{F}} g + e^{-g} d_{\mathcal{F}} g = 0,
\]
which shows \(H^0_\omega(M, d_{\mathcal{F}}) \neq \{0\}\). \(\square\)

Proposition 3.7. (Homotopy axiom for the leafwise Morse-Novikov cohomology). Let \(f : (M, \mathcal{F}_0) \to (N, \mathcal{F}_1)\) and \(g : (M, \mathcal{F}_0) \to (N, \mathcal{F}_1)\) be foliated homotopic maps, and let \(\omega\) be a leafwise closed \(-1\) form on \((N, \mathcal{F}_1)\). Then there exists a positive function \(h : (M, \mathcal{F}_0) \to \mathbb{R}\) such that for all \(k\)
\[
f^* = h g^* : H^k_\omega(N, d_{\mathcal{F}_1}) \to H^k_{d_{\mathcal{F}}} (M, d_{\mathcal{F}_0})\).
\]

Proof. Since \(f\) and \(g\) are foliated homotopic maps, by the homotopy axiom of leafwise de Rham cohomology (Proposition 2.1), they induce the same map in leafwise de Rham cohomology. Therefore, for any leafwise closed \(-1\)-form \(\omega \in \Omega^1(N, \mathcal{F}_1)\), the pullback forms \(f^* \omega, g^* \omega \in H^1_\omega(M, d_{\mathcal{F}_0})\) are cohomologous. There exists a function \(\nu : (M, \mathcal{F}_0) \to \mathbb{R}\) such that \(g^* \omega - f^* \omega = d_{\mathcal{F}} \nu\). We define \(h = e^\nu\). Then from the proof of Proposition 3.3, for any \(d_{\mathcal{F}}\) closed form \(\alpha\) on \((N, \mathcal{F}_1)\), \([h g^* \alpha] = [f^* \alpha] \in H^k_{d_{\mathcal{F}}} (M, d_{\mathcal{F}_0})\). \(\square\)

Corollary 3.8. If \(f : (M, \mathcal{F}_0) \to (N, \mathcal{F}_1)\) is a foliated homotopy equivalence and \(\omega\) is a leafwise closed \(-1\)-form, then the leafwise Morse-Novikov cohomology groups \(H^*_\omega(M, d_{\mathcal{F}_0})\) and \(H^*_\omega(N, d_{\mathcal{F}_1})\) are isomorphic; i.e.
\[
H^k_\omega(M, d_{\mathcal{F}_0}) \cong H^k_{d_{\mathcal{F}}} (N, d_{\mathcal{F}_1}), \text{ for all } k.
\]

Proof. There exists a map \(g : (N, \mathcal{F}_1) \to (M, \mathcal{F}_0)\) such that \(f \circ g\) is homotopic to the identity map \(\mathbb{I}_N\) of \(N\) and \(g \circ f\) is homotopic to the identity map \(\mathbb{I}_M\) of \(M\). We have linear maps
\[
H^*_\omega(N, \mathcal{F}_1) \xrightarrow{f^*} H^*_\omega(M, \mathcal{F}_0) \xrightarrow{g^*} H^*_\omega(N, \mathcal{F}_1).
\]

By the homotopy axiom of leafwise Morse-Novikov cohomology (Proposition 3.7), there exists a positive function \(h : N \to \mathbb{R}\) such that \(g^* f^* \omega = \omega + d_{\mathcal{F}_1}(\log(h))\) then we have
\[
\mathbb{I}_N = hg^* f^* = h (f \circ g)^* : H^*_\omega(N, \mathcal{F}_1) \to H^*_\omega(N, \mathcal{F}_1).
\]
And similarly, for some positive function \(\bar{h} : M \to \mathbb{R}\) such that \(f^* g^* \omega = \omega + d_{\mathcal{F}_0}(\log(\bar{h}))\) then we have
\[
\mathbb{I}_M = \bar{h} f^* g^* = h (g \circ f)^* : H^*_\omega(M, \mathcal{F}_0) \to H^*_\omega(M, \mathcal{F}_0).
\]
Since multiplication by a positive function is an isomorphism of Leafwise Morse-Novikov cohomology, \(f^*\) and \(g^*\) are isomorphisms. \(\square\)
4. Laplacian and Hodge decomposition on leafwise Morse-Novikov cohomology

In the following, assume \( \dim(M) = n \), \( \dim(\mathcal{F}) = p \), and \( \text{codim}(\mathcal{F}) = q \). As in (Equation (2.1)), we have the bigrading
\[
\Omega^{u,v}(M, \mathcal{F}) = \Gamma(M, \Lambda^u T\mathcal{F}^* \otimes \Lambda^v T\mathcal{F}^*).
\]

We choose a tangential and a transversal orientation for \( \mathcal{F} \) on any open subset \( U \subset M \). We obtain the Hodge star operator \( \ast_{\mathcal{F}} \) on \( T\mathcal{F}^* \) and \( \ast_\perp \) on \( T\mathcal{F}^\perp \) to \( U \) such that \( \ast_\perp(1) \wedge \ast_{\mathcal{F}}(1) \) is a positive volume form on \( U \subset M \).

**Lemma 4.1.** (Lemma 3.2 in [1]) The Hodge star operator on \( \wedge (T^*M) = \wedge (T\mathcal{F}^\perp) \otimes \wedge (T\mathcal{F}^*) \) on \( U \) satisfies
\[
\ast = (-1)^{(q-u)v} \ast_\perp \otimes \ast_{\mathcal{F}} : \Lambda^u T\mathcal{F}^\perp \otimes \Lambda^v T\mathcal{F}^* \to \Lambda^{q-u} T\mathcal{F}^\perp \otimes \Lambda^{p-v} T\mathcal{F}^*.
\]

**Lemma 4.2.** \( \ast^2 = (-1)^{(u+v)(p+q+1)} \) on \( \Lambda^u T\mathcal{F}^\perp \otimes \Lambda^v T\mathcal{F}^* \).

**Proof.** Restricted to \( \Lambda^u T\mathcal{F}^\perp \) and \( \Lambda^v T\mathcal{F}^* \), we have \( \ast^2 = (-1)^u(q+1) \) and \( \ast^2 = (-1)^v(p+1) \).
\[
\ast^2 = (-1)^{(q-u)v}(-1)^u(p-v) \ast_\perp \otimes \ast_{\mathcal{F}}
= (-1)^{(q-u)v}(-1)^u(p-v)(-1)^u(q+1)(-1)^v(p+1)id_\perp \otimes id_{\mathcal{F}}
= (-1)^{(u+v)(p+q+1)}id_\perp \otimes id_{\mathcal{F}}.
\]
\[
\square
\]

**Lemma 4.3.** (Formula 17 in [1]) The adjoint \( \delta_{\mathcal{F}} = d_{\mathcal{F}}^\ast \) of \( d_{\mathcal{F}} \) is given by
\[
\delta_{\mathcal{F}} \beta = d_{\mathcal{F}}^\ast \beta = (-1)^{pk+p+1} \ast_{\mathcal{F}} \ast_\perp \ast \beta,
\]
for any \( \beta \in \Omega^{0,k}(M, \mathcal{F}) \).

**Proof.** The standard proof that \( d^\ast = (-1)^{nk+n+1} \ast d \ast \) on \( n \)-manifolds applies on a foliated manifold in a local neighborhood.
\[
\square
\]

**Lemma 4.4.** \( \omega_\perp = (-1)^{pk+p} \ast_{\mathcal{F}} (\omega \wedge) \ast_{\mathcal{F}} \) for all \( \omega \in \Omega^{0,k}(M, \mathcal{F}) \).

**Proof.** Suppose \( \tau \) denotes the tangential volume form. For any \( \beta \in \Omega^{0,k} \), we have
\[
\omega_\perp \beta = (-1)^{nk+1} \ast (\omega \wedge) \ast \beta
\]
\[
= (-1)^{nk+1} \ast (\omega \wedge) (\ast_\perp \otimes \ast_{\mathcal{F}})(1 \otimes \beta)
\]
\[
= (-1)^{nk+1} (-1)^q (\ast \ast \wedge \ast_{\mathcal{F}} \beta)
\]
\[
= (-1)^{nk+1} (-1)^q (\ast_\perp \otimes \ast_{\mathcal{F}})(\tau \wedge \omega \wedge \ast_{\mathcal{F}} \beta)
\]
\[
= (-1)^{p+1} \ast_{\mathcal{F}} (\omega \wedge) \ast \beta.
\]
\[
\square
\]

By Lemmas above and the identity \( \ast_{\mathcal{F}}^2 = (-1)^{k(p-k)} \), we have on \( \Omega^{0,k}(M, \mathcal{F}) \)
\[
(\omega_\perp)_{\mathcal{F}} = (-1)^k (\omega \wedge)_{\mathcal{F}}
\]
\[
\ast_{\mathcal{F}} (\omega_\perp) = (-1)^{k+1} (\omega \wedge)_{\mathcal{F}}
\]
\[
\ast_{\mathcal{F}} d_{\mathcal{F}}^p = (-1)^k d_{\mathcal{F}} \ast_{\mathcal{F}}
\]
\[
d_{\mathcal{F}}^p \ast_{\mathcal{F}} = (-1)^{k+1} \ast_{\mathcal{F}} d_{\mathcal{F}}.
\]
The adjoint of the leafwise differential \((d_F + \omega \wedge)\) is \((d^*_F + \omega \wedge)\). We denote the Laplacian corresponding to the differential \(d_F + \omega \wedge\) by \(\Delta_{F,\omega}\). Then

\[
\Delta_{F,\omega} = (d_F + \omega \wedge)(d^*_F + \omega \wedge) + (d^*_F + \omega \wedge)(d_F + \omega \wedge).
\]

**Proposition 4.5.** If \(\omega \in \Omega^{0,1}(M, F)\) is a leafwise closed 1-form, then the Hodge star operator \(*_F\) satisfies

\[
*_F \Delta_{F,\omega} = \Delta_{F,-\omega} *_F.
\]

**Proof.** For all \(\beta \in \Omega^{0,k}(M, F)\), we have

\[
*_F \Delta_{F,\omega} \beta = *_F (d_F + \omega \wedge)(d^*_F + \omega \wedge) \beta + *_F (d^*_F + \omega \wedge)(d_F + \omega \wedge) \beta
\]

\[
= (-1)^k (d^*_F - \omega \wedge) *_F (d^*_F + \omega \wedge) \beta + (-1)^{k+1} (d_F - \omega \wedge) *_F (d_F + \omega \wedge) \beta
\]

\[
= (-1)^k ((-1)^k (d^*_F - \omega \wedge)(d_F - \omega \wedge) *_F \beta + (-1)^{k+1} (d_F - \omega \wedge)(d^*_F - \omega \wedge) *_F \beta
\]

\[
= \Delta_{F,-\omega} *_F \beta.
\]

Thus the operator \(*_F\) maps \(\Delta_{F,\omega}\)-harmonic forms to \(\Delta_{F,-\omega}\)-harmonic forms. \(\square\)

**Corollary 4.6.** If we restrict the Laplacian \(\Delta_{F,\omega}\) on \(\Omega^{0,v}(M, F)\), then \(\ker \Delta_{F,\omega}\) is finite dimensional, and every reduced leafwise Morse-Novikov cohomology class has a \(\Delta_{F,\omega}\) harmonic representative.

**Proof.** Notice that the operator \(\Delta_{F,\omega}\) is defined on all forms in \(\Omega^{u,v}(M, F)\), but it is elliptic when restricted on the forms \(\Omega^{0,v}(M, F)\) along the leaves of the foliation (Section 1 in [1]). Using this ellipticity and the arguments similar to [1], we can conclude that \(\mathcal{H}^{k}_{\omega}(M, d_F) = \ker \Delta_{F,\omega} \subset \Omega^{0,k}(M, F)\) is isomorphic to \(\mathcal{H}^{k}_{\omega}(M, d_F)\), and

\[
\mathcal{H}^{k}_{\omega}(M, d_F) \cong \mathcal{H}^{k}_{\omega}(M, d_F).
\]

\(\square\)

**Corollary 4.7.** \(\mathcal{H}^{k}_{\omega}(M, d_F) \cong \mathcal{H}^{k}_{-\omega}(M, d_F)\).

**Proof.** Since the operator \(*_F\) maps \(\Delta_{F,\omega}\)-harmonic forms to \(\Delta_{F,-\omega}\)-harmonic forms, it induces the isomorphism

\[
\mathcal{H}^{k}_{\omega}(M, d_F) \cong \mathcal{H}^{k}_{-\omega}(M, d_F).
\]

\(\square\)

5. Extension of leafwise Morse-Novikov cohomology to forms of general \(u, v\) type

We now extend leafwise Morse-Novikov cohomology to forms of general \(u, v\) type. Let \(\omega \in \Omega^{0,1}(M, F)\) be a leafwise closed 1-form which is not necessarily exact. We consider the twisted operator \(d^*_F : \Omega^{u,v}(M, F) \to \Omega^{u,v+1}(M, F)\) defined by \(d^*_F = d_F + \omega \wedge\), where \(d_F\) is the exterior derivative along the leaf.

**Proposition 5.1.** \((d_F + \omega \wedge)^2 = 0\). Therefore \(d_F + \omega \wedge\) is a differential of the sections of \(\Omega^{u,v}(M, F)\).
Proof. Observe that for any section \( \alpha \land \beta \in \Omega^{u,v} (M, \mathcal{F}) \), we have
\[
(d_\mathcal{F} + \omega \land)^2 (\alpha \land \beta) = (d_\mathcal{F} + \omega \land) ((-1)^u \alpha \land d_\mathcal{F} \beta + \omega \land (\alpha \land \beta))
\]
\[
= d_\mathcal{F} ((-1)^u \alpha \land d_\mathcal{F} \beta) + (-1)^u \omega \land (\alpha \land d_\mathcal{F} \beta) + (-1)^v (\omega \land (\alpha \land d_\mathcal{F} \beta)) + \omega \land (\omega \land (\alpha \land \beta))
\]
\[
= (-1)^{u+1} (\omega \land (\alpha \land d_\mathcal{F} \beta) - \omega \land (\alpha \land d_\mathcal{F} \beta)) = 0.
\]

We call the differential cochain complex \((\Omega^{*,*} (M, \mathcal{F}), d_\mathcal{F})\) the general leafwise Morse-Novikov complex of the foliated manifold \((M, \mathcal{F})\). The cohomology groups
\[
H^{*,*}_\omega (M, d_\mathcal{F}) = \frac{\ker (d_\mathcal{F}^*)}{\text{im} (d_\mathcal{F})}
\]
of this cochain complex are called the general leafwise Morse-Novikov cohomology groups of \((M, \mathcal{F})\). For the purpose of having Laplacian and Hodge decomposition, we need to consider reduced general leafwise Morse-Novikov cohomology
\[
\bar{H}^{*,*}_\omega (M, d_\mathcal{F}) = \frac{\ker (d_\mathcal{F}^*)}{\overline{\text{im} (d_\mathcal{F})}}.
\]
Again the closure \(\overline{\text{im} (d_\mathcal{F})}\) is taken with respect to the Frechét topology on \(\Omega^{*,*} (M, \mathcal{F})\).

**Proposition 5.2.** If \( \omega \) and \( \theta = \omega + d_\mathcal{F} g \) are cohomologous in \( H^1 (M, d_\mathcal{F}) \), then for each \( \ell, k \), the general leafwise Morse-Novikov cohomology groups \( H^{\ell,k}_\omega (M, d_\mathcal{F}) \) and \( H^{\ell,k}_\theta (M, d_\mathcal{F}) \) are isomorphic via the isomorphism \([\alpha] \mapsto [e^{-g} \alpha]\).

**Proof.** Similar to the proof of Proposition 3.3. \(\square\)

**Proposition 5.3.** (Homotopy axiom for the general leafwise Morse-Novikov cohomology). Let \( f : (M, \mathcal{F}_0) \rightarrow (N, \mathcal{F}_1) \) and \( g : (M, \mathcal{F}_0) \rightarrow (N, \mathcal{F}_1) \) be foliated homotopic maps, and \( \omega \in \Omega^{0,1} (N, \mathcal{F}_1) \) be a leafwise closed 1-form on \((N, \mathcal{F}_1)\). Then there exists a positive function \( h : (M, \mathcal{F}_0) \rightarrow \mathbb{R} \) such that, for all \( \ell, k \)
\[
f^* = h g^* : H^{\ell,k}_\omega (N, d_{\mathcal{F}_1}) \rightarrow H^{\ell,k}_{f^* \omega} (M, d_{\mathcal{F}_0}).
\]

**Proof.** Similar to the proof of Proposition 3.7. \(\square\)

6. **Leafwise Morse-Novikov Hodge Theory of Forms of General \((u, v)\) Type**

Let \( P = d_\mathcal{F} + \omega \land \), then its formal adjoint \( P^* \) is \( \delta_\mathcal{F} - \omega \land \). Let \( D = P + P^* \) be the corresponding Dirac operator. Then the corresponding Laplacian \( \Delta_{\mathcal{F}, \omega}^{u,v} = (P + P^*)^2 = PP^* + P^* P \) is a nonnegative, self-adjoint second order differential operator on the smooth sections on \( \Omega^{u,v} (M, \mathcal{F}) \). For each integer \( k \geq 0 \), let \( H_k (\Delta_{\mathcal{F}, \omega}^{u,v}) \) be the Hilbert space completion of the space \( \Omega^{u,v} (M, \mathcal{F}) \) with respect to the scalar product
\[
\langle \alpha, \beta \rangle = \sum_{j=0}^{j=k} \langle (\Delta_{\mathcal{F}, \omega}^{u,v})^j \alpha, \beta \rangle
\]
for \( \alpha, \beta \in \Omega^{u,v} (M, \mathcal{F}) \). For the corresponding norm \( \| \cdot \|_k \), we have
\[
k \leq k' \Rightarrow \| \alpha \|_k \leq \| \alpha \|_{k'} \text{ for all } \alpha \in \Omega^{u,v} (M, \mathcal{F}) .
\]
Thus we obtain the chain of continuous inclusions
\[
H = H_0 (\Delta_{\mathcal{F}, \omega}^{u,v}) \supset H_1 (\Delta_{\mathcal{F}, \omega}^{u,v}) \supset H_2 (\Delta_{\mathcal{F}, \omega}^{u,v}) \supset \cdots \supset H_{\infty} (\Delta_{\mathcal{F}, \omega}^{u,v}) .
\]
where
\[ H_\infty (\Delta_{\mathcal{F},\omega}^{u,v}) = \bigcap_{k \geq 0} H_k (\Delta_{\mathcal{F},\omega}^{u,v}) \]
equipped with the Frechét topology.

**Theorem 6.1.** Let \((M, \mathcal{F})\) be a smooth foliation of a closed Riemannian manifold with a bundle like metric and \(\omega \in \Omega^{0,1} (M, \mathcal{F})\). The Laplacian \(\Delta_{\mathcal{F},\omega}^{u,v}\) on \(\Omega^{u,v} (M, \mathcal{F})\) gives rise to an orthogonal direct sum decomposition
\[ H_\infty (\Delta_{\mathcal{F},\omega}^{u,v}) \cong \ker \Delta_{\mathcal{F},\omega}^{u,v} \oplus \text{im} \Delta_{\mathcal{F},\omega}^{u,v} \cong \ker \Delta_{\mathcal{F},\omega}^{u,v} \oplus \text{im} P_\infty \oplus \text{im} P_*^\infty, \]
where \(\Delta_{\mathcal{F},\omega}^{u,v}\), \(P_\infty\), and \(P_*^\infty\) are canonical continuous extensions of the corresponding differential operators.

**Proof.** The complexification of the Dirac operator \(D = P + P^*\) satisfies the hypothesis of Chernoff’s Lemma 2.1 in [6]. This can be verified from Corollary 1.4 of [6]. Then with the ideas explained in Section 2 of [3], we have the real Hilbert spaces \(H_k (\Delta_{\mathcal{F},\omega}^{u,v})\) and \(H_\infty (\Delta_{\mathcal{F},\omega}^{u,v})\).

We can extend the operator \(D\) to \(D_\infty\) and \(\Delta_{\mathcal{F},\omega}^{u,v}\) to \(\Delta_{\mathcal{F},\omega}^{u,v}\): \(H_\infty (\Delta_{\mathcal{F},\omega}^{u,v}) \rightarrow H_\infty (\Delta_{\mathcal{F},\omega}^{u,v})\), yielding the orthogonal decompositions
\[ H_\infty (\Delta_{\mathcal{F},\omega}^{u,v}) \cong \ker \Delta_{\mathcal{F},\omega}^{u,v} \oplus \text{im} \Delta_{\mathcal{F},\omega}^{u,v} \cong \ker \Delta_{\mathcal{F},\omega}^{u,v} \oplus \text{im} D_\infty \oplus \text{im} D_*^\infty. \]

Notice the spaces \(\Omega^{p,q} (M, \mathcal{F})\) are orthogonal to each other with respect to the inner product \(\langle , \rangle_k\) defined above, for each \(k \geq 0\). Therefore, it follows that \(D_\infty\) can be decomposed as the sum of the continuous operators
\[ P_\infty, P_*^\infty : H_\infty \rightarrow H_\infty, \]
which are extensions of \(P_\infty\) and \(P_*^\infty\) respectively. Since \(\text{im} P\) and \(\text{im} P^*\) are \(\langle , \rangle_k\)-orthogonal for each \(k \geq 0\), we obtain the following orthogonal decomposition:
\[ H_\infty (\Delta_{\mathcal{F},\omega}^{u,v}) \cong \ker \Delta_{\mathcal{F},\omega}^{u,v} \oplus \text{im} P_\infty \oplus \text{im} P_*^\infty. \]

**Corollary 6.2.** Every reduced general leafwise Morse-Novikov cohomology class has a \(\Delta_{\mathcal{F},\omega}^{u,v}\)-harmonic representative.

For any \(\alpha \wedge \beta \in \Omega^{u,v} (M, \mathcal{F})\), from formula 17 in [3] we have \(\delta_\mathcal{F} = (-1)^{n(u+v)+n+1} \ast d_\mathcal{F}^*\). By using the identities
\[ \omega \wedge = (-1)^{n(u+v)+n} \ast \omega \wedge \ast \]
and \(\ast^2 = (-1)^{(u+v)(n+1)}\),
it can be shown that
\[ (\omega \wedge) \ast = (-1)^{u+v} \ast (\omega \wedge) \ast, \]
\[ \ast(\omega \wedge) = (-1)^{u+v+1} (\omega \wedge) \ast, \]
\[ \ast \delta_\mathcal{F} = (-1)^{u+v} d_\mathcal{F}^* \ast, \]
\[ \delta_\mathcal{F}^* = (-1)^{u+v+1} \ast d_\mathcal{F}. \]
Proposition 6.3. Let \((M, \mathcal{F})\), and \(\omega\) be as in Theorem \([\text{6.1}]\). The Hodge operator \(\ast : \Omega^{u,v}(M, \mathcal{F}) \to \Omega^{q-u,p-v}(M, \mathcal{F})\) satisfies
\[
\ast \Delta_{\mathcal{F},\omega}^{u,v} = \Delta_{\mathcal{F},\omega}^{q-u,p-v} \ast .
\]

Proof. Similar to Proposition \([\text{4.3}]\) using the formulas above. Thus the operator \(\ast\) maps \(\Delta_{\mathcal{F},\omega}^{u,v}\)-harmonic forms to \(\Delta_{\mathcal{F},\omega}^{q-u,p-v}\)-harmonic forms. \(\square\)

Lemma 6.4. \(\omega \ast_\mathcal{F} = (-1)^{(p+1)} \ast_\mathcal{F} (\omega \wedge \ast_\mathcal{F}) \) for all \(\omega \in \Omega^{0,1}(M, \mathcal{F})\) on \(\Omega^{u,v}(M, \mathcal{F})\).

Proof. For any \(\alpha \wedge \beta \in \Omega^{u,v}(M, \mathcal{F})\), we have
\[
\omega \ast_\mathcal{F} (\alpha \wedge \beta) = (-1)^{n(u+v+1)} (\omega \ast_\mathcal{F}) (\alpha \wedge \beta)
\]
\[
= (-1)^{n(u+v+1)} (-1)^{(q-u)v} (-1)^{(q-u)} \ast_\mathcal{F} (\omega \wedge \ast_\mathcal{F} \beta)
\]
\[
= (-1)^{n(u+v+1)} (-1)^{(q-u)v} (-1)^{(q-u)} (-1)^{(q-u+1)(p-v+1)} \ast_\mathcal{F} (\omega \wedge \ast_\mathcal{F} \beta)
\]
\[
= (-1)^{n(u+v+1)} (-1)^{(q-u)v} (-1)^{(q-u)} (-1)^{(u)(p-v+1)} (-1)^{(u+1)} \ast_\mathcal{F}(\omega \wedge \ast_\mathcal{F} \beta)
\]
\[
= (-1)^{n(u+v+1)} (-1)^{(q-u)v} (-1)^{(q-u)} (-1)^{(u)(p-v+1)} (-1)^{(u+1)} \ast_\mathcal{F}(\omega \wedge \ast_\mathcal{F} \beta)
\]
\[
= (-1)^{p(v+1)+u} \ast_\mathcal{F} (\omega \wedge \ast_\mathcal{F} \beta)
\]
\[
= (-1)^{p(v+1)} \ast_\mathcal{F} (\omega \wedge \ast_\mathcal{F} \beta)
\]
\[
= (-1)^{p(v+1)} \ast_\mathcal{F} (\omega \wedge). \square
\]

In a Riemannian foliation, any element of \(\Omega^{u,v}(M, \mathcal{F})\), on a local foliated chart, can be expressed as a linear combination of forms of the type \(\alpha \wedge \beta\), where \(\alpha\) is a basic form in \(\Omega^{u,0}(M, \mathcal{F})\) and \(\beta\) is a form in \(\Omega^{0,v}(M, \mathcal{F})\). Then in particular \(d_\mathcal{F} \alpha = 0\). See \([1, \text{Lemma 3.4}]\).

Lemma 6.5. \(d_\mathcal{F}^* = (-1)^{p(v+1)+1} d_\mathcal{F} \ast_\mathcal{F} \) for all on \(\Omega^{u,v}(M, \mathcal{F})\).

Proof. For any \(\alpha \wedge \beta \in \Omega^{u,v}(M, \mathcal{F})\), we have
\[
d_\mathcal{F}^* (\alpha \wedge \beta) = (-1)^{n(u+v)+n+1} d_\mathcal{F}^* (\alpha \wedge \beta)
\]
\[
= (-1)^{n(u+v)+n+1+(q-u)(v+1)} (\ast_\mathcal{F} (\alpha \wedge d_\mathcal{F} \ast_\mathcal{F} \beta))
\]
\[
= (-1)^{n(u+v)+n+1+(q-u)(v+1)+u(p-v+1)+u(q+1)} (\ast_\mathcal{F} (\alpha \wedge \ast_\mathcal{F} d_\mathcal{F} \ast_\mathcal{F} \beta))
\]
\[
= (-1)^{p(v+1)+u} \ast_\mathcal{F} (\alpha \wedge \ast_\mathcal{F} d_\mathcal{F} \ast_\mathcal{F} \beta)
\]
\[
= (-1)^{p(v+1)+u+1+u(v-1)} \ast_\mathcal{F} d_\mathcal{F} \ast_\mathcal{F} (\ast_\mathcal{F} \ast_\mathcal{F} \beta) \wedge \alpha
\]
\[
= (-1)^{p(v+1)+1} \ast_\mathcal{F} d_\mathcal{F} \ast_\mathcal{F} (\alpha \wedge \beta). \square
\]

By Lemmas above and the identity \(\ast_\mathcal{F}^2 = (-1)^{v(p-v)}\), we have on \(\Omega^{u,v}(M, \mathcal{F})\)
\[
(\omega \ast_\mathcal{F}) \ast_\mathcal{F} \omega = (-1)^{p(v+1)} (\omega \ast_\mathcal{F}) \ast_\mathcal{F} \omega
\]
\[
= (-1)^{p(v+1)+v(p-v)} \ast_\mathcal{F} (\omega \wedge)
\]
\[
= (-1)^{v} \ast_\mathcal{F} (\omega \wedge),
\]
and
\[
\ast_\mathcal{F} (\omega \ast_\mathcal{F}) = (-1)^{p(v+1)} \ast_\mathcal{F}^2 (\omega \wedge) \ast_\mathcal{F} \omega
\]
\[
= (-1)^{p(v+1)} (-1)^{p(v+1)} (\omega \wedge) \ast_\mathcal{F}
\]
\[
= (-1)^{v+1} (\omega \wedge) \ast_\mathcal{F}. \]
Similarly on $\Omega^{u,v}(M, F)$ we also have

\[ *_F d_F^* = (-1)^v d_F *_F \]
\[ d_F^* *_F = (-1)^{v+1} *_F d_F. \]

**Proposition 6.6.** Let $(M, F)$, and $\omega$ be as in Theorem 6.1. The Hodge operator $*_F : \Omega^{u,v}(M, F) \to \Omega^{u-p,v}(M, F)$ satisfies

\[ *_F \Delta^{u,v}_{F,\omega} = \Delta^{u-p,v}_{F,\omega} *_F. \]

**Proof.** Similar to Proposition 4.5, using the formulas above. Thus the operator $*_F$ maps $\Delta^{u,v}_{F,\omega}$-harmonic forms to $\Delta^{u-p,v}_{F,\omega}$-harmonic forms. \[\square\]

Notice on $\Omega^{u,v}(M, F)$, we have

\[ **_F = (-1)^{(q-u)(p-v)} *_{\perp} *_F \]
\[ **_F = (-1)^{(q-u)(p-v)+v(p-v)} *_{\perp} \]
\[ *_{\perp} = (-1)^{(q-u)(p-v)+v(p-v)} *_F. \]

From the Propositions 6.3 and 6.5 we have,

**Corollary 6.7.** Let $(M, F)$, and $\omega$ be as in Theorem 6.1. The Hodge operator $*_{\perp} : \Omega^{u,v}(M, F) \to \Omega^{q-u,v}(M, F)$ satisfies

\[ *_{\perp} \Delta^{u,v}_{F,\omega} = \pm \Delta^{q-u,v}_{F,\omega} *_{\perp}. \]

**Proof.**

\[ *_{\perp} \Delta^{u,v}_{F,\omega} = (-1)^v (n-u-v) *_F \Delta^{u,v}_{F,\omega} \]
\[ = (-1)^v (n-u-v) * \Delta^{u,v}_{F,\omega} *_F \]
\[ = (-1)^v (n-u-v) \Delta^{u,v}_{F,\omega} *_F \]
\[ = (-1)^v (n-u-v) (1)^{(q-u)(p-v)+v(p-v)} *_F \Delta^{q-u,v}_{F,\omega} *_{\perp} \]
\[ = (-1)^p (q-u) \Delta^{q-u,v}_{F,\omega} *_{\perp}. \]

\[\square\]

Let $h^{u,v}_\omega$, $h^{u,v}_{-\omega}$ be the dimensions of the Cohomology groups $H^{u,v}_\omega(M, d_F)$, and $H^{u,v}_{-\omega}(M, d_F)$ respectively. We observe that, Proposition 6.3 implies, $h^{u,v}_\omega=h^{u-p,v}_{\omega}$, Proposition 6.6 implies, $h^{u,v}_\omega=h^{u-p,v}_{-\omega}$, and Corollary 6.7 implies, $h^{u,v}_\omega=h^{q-u,v}_\omega$.

**Corollary 6.8.** (Hodge Diamond Structure) For a Riemannian foliation of a manifold

\[ h^{u,v}_\omega = h^{q-u,v}_{\omega} = h^{u,v}_{-\omega} = h^{q-u,v}_{-\omega}. \]

In particular, we consider a diagram of the dimensions of the cohomology classes for a manifold of dimension $n = 5$ with $p = 2$, and $q = 3$. 
The dimensions in the same color are equal.
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