Ground state instability in Nonrelativistic QFT and Euler-Heisenberg Lagrangian via holography

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Abstract: We study the ground state instability of mesons in the presence of the electric field on Schrödinger geometry which is a solution of type IIB string theory. The instability takes place as a result of Schwinger like effect. We calculate decay rate of instability and pair production probability by using gauge/gravity duality. At zero temperature for massive mesons, the critical electric field would be the same as AdS result if we replace the mass of mesons with the effective mass. This effective mass depends on compactification parameter \( \mu \), which it would be dual to the chemical potential, and also \( \lambda \) coupling. We also find Euler–Heisenberg Lagrangian from holography in Schrödinger space-time which behaves similarly to the AdS one for the gapless system concerning the electric field. For the massive one the functionality of effective action to the external electric field would be the same as AdS geometry if we call \( E \sqrt{1 - \frac{\pi^2 m^2}{|\mu| x}} \) instead of the electric field.

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1 Introduction

The decay of false vacuum to true vacuum might consider as pair production mechanism. Schwinger pair production came from an instability of the vacuum in QED [2, 1]. Pair production probability or decay rate of instability of QED vacuum known as Euler – Heisenberg effective Lagrangian.

Ground state as a condensed matter counterpart of QFT vacuum also could feel instability. In the presence of the external electric field, the decay of ground state to ground state in condensed matter physics has been considering as Zener breakdown of Mott(or band) insulator [6]. Revisited version of Euler-Heisenberg effective Lagrangian in condensed matter physics is ground state to ground state transition amplitude or Zener tunneling rate [7].

Generally, the Schwinger effect as a vacuum instability is a non-perturbative phenomenon. In strongly correlated systems calculating effective action is demanding great exertion. The well-known toolbox for studying strong interaction in quantum theory is AdS/CFT correspondence or more generally the gauge/gravity duality.

The rate of vacuum to vacuum transition of supersymmetric QED (QCD) has been studied through AdS/CFT correspondence [8]. Results are in agreement with Schwinger pair production in QED. So the question will arise: What is a rate of false vacuum decay to the true vacuum, generalizing this idea to other theory with different kind of symmetries? Such as deforming conformal field theory to anisotropic scaling theory such as theories with Lifshitz or Schrödinger symmetries.

The non-relativistic limit of conformal symmetry reduces to the Schrödinger symmetry [9]. Null Melvin Twist transformations of $AdS_5 \times S^5$ will bring up geometry with Schrödinger symmetry as its isotropy.

It’s more than a decade that physicists make more attention to $AdS/CFT$ as a well-known example of holography or $Gauge/Gravity$ duality. The holography picture for other field theories also have some successes [12, 13, 14, 15]. Extending that idea to many-body physics and condensed matter systems with different symmetries, which are believed to have gravity duals, have been attracted a lot of attention to extract and justify the experimental result.

The $AdS_5 \times S^5$ as a near extremal solution of $N_c$ coincident $D3$ branes is dual to $3 + 1$ dimensional conformal field theory with $\mathcal{N} = 4$ super-symmetries with $SU(N_c)$ gauge degrees of freedom. For adding
other degrees of freedom besides of gauge fields, we add $N_f$ $D7$ branes [16] and for simplicity take them as a probe ($N_f \ll N_c$). This configuration is describing a QCD-like system with $\mathcal{N} = 2$ supersymmetry. Dynamics of the $D3/D7$ system is given by $DBI$ action of probe $D7$ branes. From gauge/gravity dictionary this action would be an effective action of fermionic degrees of freedom addressed by the strings with one end on $D7$ branes and the other on $D3$ branes.

Generalizing this QCD like framework to other theories is more interesting. By $TsT$ [18] or null Melvin twist ($NMT$) [19] transformations the type $IIB$ superstring theory or its low energy supergravity solution with $AdS$ geometry which has conformal symmetry transforms to geometry with Schrödinger symmetry. Studying the $DBI$ action of probe $D7$ branes in this background gives us some information about fermions in a field theory with Schrödinger symmetry. The cold atoms system known as fermions at unitarity is a famous example of a system with Schrödinger symmetry [28]. As a framework, we study the $D7$ brane as a probe in 10-dimensional Schrödinger space-time.

By turning on the electric field on the $D7$ branes and calculating the effective action, which is nothing but $DBI$ action, we study the instability that could potentially happen and produce fermion and anti-fermion pairs through the Schwinger effect which could occur in the equilibrium.

## 2 Probe branes and Instability

Introducing the $N_f$ number of flavor $D7$ branes, we have matter content besides of gauge degrees of freedom [16]. In the case of $AdS_5 \times S^5$, i.e., the $N_c$ stack of $D3$ branes, adding the $N_f$ copy of $D7$ branes at probe limit represent $\mathcal{N} = 2$ hypermultiplet of fundamental degrees of freedom besides of $SU(N_c)$ degrees of freedom in the dual field theory. At probe limit i.e.,

$$N_f \ll N_c$$

dynamics of this configuration would be described by $DBI$ action of $D7$ branes [4]. The DBI action is

$$S_{D7} \equiv -N_f T_{D7} \int d\xi^8 e^{-\Phi} \sqrt{\det ([g + B]_{ab} + (2\pi \alpha') F_{ab})}. \tag{1}$$

In the above, $\xi^a$ are $D7$ worldvolume coordinates and

$$T_{D7} = \frac{1}{(2\pi)^7 g_s \alpha'^4}. \tag{2}$$

The $g_{ab}$ and $B_{ab}$, which are induced metric and induced $B$ field from the background on the branes respectively, given by

$$g_{ab} = G_{MN} \partial_a X^M \partial_b X^N$$
$$B_{ab} = B_{MN} \partial_a X^M \partial_b X^N. \tag{3}$$

where $\partial_a = \frac{\partial}{\partial \xi^a}$ and capital letters i.e. $M, N, ...$ denote background space time coordinates. Let embed the $D7$ branes in 10 dimension space-time as follow :

$$x^+ \ x^- \ \vec{x} \ r \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \theta \ \chi$$
$$D3 \ \times \ \times \ \times$$
$$D7 \ \times \ \times \ \times \ \times \ \times \ \times \ \times$$

---

[2] There is a lot of interest in the Schwinger effect in the case of ultracold atoms, see [17] and references therein.

[3] vacuum

[4] Full dynamics also has a $Wess - Zumino$ term. With proper embedding as we considered the $WZ$ term has no contribution and we only concern about $DBI$ term.
The dual theory would live on intersection of $D3$ and $D7$ branes at $r = 0$ which is $(x^+, x^-, x)$. There is an $O(2)$ symmetry in $(\chi, \theta)$ direction. Assume without loss of generality that $\chi = 0$ and $\theta = \theta(r)$. At zero temperature background and zero $F_{ab}$ it could be shown that DBI action will not be affected by $NMT$ transformation\(^5\). So as \([32, 33]\) the shape of the branes which is determined by $\theta(r)$ would be

$$\theta(r) = \arcsin(c \, r).$$

(5)

From holography dictionary $c$ is related to the mass of flavor fermions (quarks) in dual theory, $m = \frac{c}{2\pi\alpha'}$ \([32, 33, 31]\). With zero gauge field on $D7$ branes, there are different kinds of embedding in AdS background which ratio of mass and temperature ($\frac{m}{T}$) would tell us which embedding is thermodynamically favorable \([32, 33, 34]\). For zero temperature background, the Mankowski embedding (ME) is the preferred solution of the $D7$ branes shape. With schrödinger geometry, this embedding also will exist (same as AdS). For small $\frac{m}{T}$ the black hole embedding (BE) is favorable \(^6\). To compare instability of mesons which live on $D7$-branes in Schrödinger background with $AdS$ result \([8]\), we turn on one form $A = E x \, dt + A_x(r) \, dx$ as a gauge field on the $D7$ branes. Using (A.7) we have $A = A_+ dx^+ + A_- dx^- + A_x dx$ \(^7\) which

$$A_+(x, r) = \frac{E_\beta}{L} x, \quad A_-(x, r) = -\beta^2 LE_\beta x, \quad A_x = A_x(r).$$

(6)

we have relabeled the electric field to be $E_\beta = E/(2\beta)$. Scaling dimension of $E_\beta$ is $[E_\beta] = [L]^3$ which tell us $E_\beta$ is non relativistic quantity. From now we go into the units of AdS radius, $L = 1$. With some calculation with help from (A.5), (A.8) and (A.9) we get $D7$-brane action \([21]\)

$$S_{D7} = -N_f T_{D7} \int dr \, d^3x \, dx^- d^2\vec{x} \, e^{-\Phi} \sqrt{-\det \left[ g + B \right]} + (2\pi\alpha') F_{ab}$$

\(= -NVol_{(+\, \vec{x})} \int dr \sqrt{-K(r)} \det M_{ab},\)

(7)

with

$$N = N_f T_{D7} \int \sin \alpha_1 \alpha_2 \alpha_3 = N_f T_{D7} 4\pi^2.$$

(8)

and

$$\det M_{ab} \equiv g_{xx} g_{\alpha_1 \alpha_2} \left\{ g_{\alpha_1 \alpha_2} \left[ (2\pi\alpha')^2 E_\beta^2 H_1 + g_{xx} H_2 \right] + (2\pi\alpha')^2 H_2 A_x^2 \right\},$$

(9)

where we have defined

$$H_1 = \frac{\beta^2 r^2 - \beta^2 f(r) \sin^2 \theta(r)}{4r^4 K(r)} \cos^4 \theta(r), \quad H_2 = -\frac{f(r) \cos^4 \theta(r)}{16r^4 K(r)}.$$

(10)

From (7) we deal with one dimensional Lagrangian. We hence find that equation of motion :

$$D = \frac{N K g_{xx} \alpha_1 \alpha_2}{\sqrt{-K \det M_{ab}}} (2\pi\alpha')^2 H_2 A_x^2.$$

(11)

$D$ is a constant of motion. Near the $r = 0$ boundary, the gauge field $A_x(r)$ asymptotically behaves\(^8\)

$$A_x(r) = \frac{D}{2(2\pi\alpha')^2 N} r^2 + ...$$

(12)

\(^5\)At zero temperature $g_{--}$ is zero and also $B_{a_2 a_3}$ so the determinant would be the same as $AdS$ (in light-like coordinate).

\(^6\)Analysis of embedding for anisotropic background from type IIB supergravity have been done in \([39]\).

\(^7\)For more general case see \([21]\) and also \([22]\).

\(^8\)Of course we consider $A_x(r)$ has a zero expectation value. See \([31]\).
From the holographic picture, we know that $\langle J^x \rangle = D$ \cite{31} which $\langle J^x \rangle$ is the conserved electric current of charged fermions in dual field theory. Solving $A'_x$ from (11) and then inserting the solution into (7) we get the on-shell action \footnote{We redefine on-shell action or effective action as $S_{D7}/Vol(+-\vec{x})$.} 

$$ S_{D7} = -\mathcal{N}^2 \int_0^{r_H} dr \ K(r) g_{xx} g_{\alpha_1 \alpha_1} \ g_{rr} \sqrt{\frac{V(r)}{U(r)}} $$ \hspace{1cm} (13) 

which we define 

$$ U(r) = \frac{\langle J^x \rangle^2}{(2\pi\alpha')^2 H_2} + \mathcal{N}^2 K(r) g_{xx} g_{\alpha_1 \alpha_1} \quad V(r) = g_{xx}|H_2| - (2\pi\alpha')^2 E_\beta^2 H_1. $$ \hspace{1cm} (14) 

The Legendre transformation of the action,

$$ \tilde{S}_{D7} = S_{D7} - A'_x \frac{\delta S_{D7}}{\delta A'_x}, $$ \hspace{1cm} (15) 

would be 

$$ \tilde{S}_{D7} = -\int_0^{r_H} d r \ g_{rr}^{1/2} \sqrt{V(r)U(r)} $$ \hspace{1cm} (16) 

This action in a dual field theory resemble effective action \footnote{In the vacuum or equilibrium case this would be free energy, but in here we deal with non-equilibrium steady state.\cite{35, 36, 34}}. So turning on electric field the system will respond to it via current of charge carriers. This configuration is stable and makes (13) (or (15)) be real function. $U(r)$ and $V(r)$ could be negative or positive. The reality condition force $U$ and $V$ to change sign at a same point where we call it $r_*(0 < r_* < r_H)$ . This place came out from following equations:

$$ \left[ g_{xx}|H_2| - (2\pi\alpha')^2 E_\beta^2 H_1 \right] \bigg|_{r_*} = 0, $$

$$ \left[ \frac{\langle J^x \rangle^2}{(2\pi\alpha')^2 H_2} + \mathcal{N}^2 K(r) g_{xx} g_{\alpha_1 \alpha_1} \right] \bigg|_{r_*} = 0 $$ \hspace{1cm} (17) 

Geomtrization of this point is the horizon of effective metric (??). It could be assign a temperature to this effective horizon, which is different from background Hawking $T_H$ temperature. This situation will elucidate non-equilibrium condition. Dealing with (17), the nonlinear DC conductivity, $\langle J^x \rangle = \sigma(E_\beta)E_\beta$, is 

$$ \sigma = \mathcal{N} \sqrt{\frac{f(r_*)}{8E_\beta r_*^4}} \cos^3 \theta(r_*) $$ \hspace{1cm} (18) 

Where we use $\frac{1}{E_\beta} = \frac{4\beta^2 f(r_*) [r_*^2 - \beta^2 f(r_*) \sin^2 \theta(r_*)]}{J(r_*)}$. With the assumption $A_+(r) = 0$, here we have not explicit charge carriers but the conductivity is because of charges which thermally produced from charge neutral pairs. In other words, electric field would rip apart strings which bounded on D7 brane i.e., meson would break to the fermion-antifermion (or flavour quark-anti quark).

Let’s for a moment forget about non-equilibrium state and consider $\tilde{S}_{D7}$ in (16) as free energy same as equilibrium thermodynamic, $f$. The heat capacity, 

$$ C_V = -T \frac{\partial^2 f}{\partial T^2} \bigg|_{E_\beta, \beta} $$ \hspace{1cm} (19)
feel singularity exactly at (17), resembles to phase transition requirement. For zero electric field we do not have any singularity. What we can see now is that: At non zero temperature, electric field breaks bound state of neutral charge pairs, which could be mesons or Cooper pairs, to free fermionic charged systems. Also for zero background temperature this phenomena exist because of electric field. Changing from \( \langle J^x \rangle = 0 \) state to \( \langle J^x \rangle \neq 0 \), due to electric field, can be take as the metastable ground state (or vacuum) transition to other genuine state. Before ending this section let define D7-branes tension \( T_{D7} \) in term of t’Hooft coupling \( \lambda = g_{QFT}^2 N_c \). Since we take the radius of Schr"odinger spacetime (or AdS) equal to unity, from (2) and

\[
(2\pi \alpha')^2 = 2\pi^2 \lambda^{-1} \quad 2\pi g_s = g_{QFT}^2
\]

we will receive to

\[
T_{D7} = \frac{N_c \lambda}{(2\pi)^4 \pi^2}.
\]

3 Mesons Instability: In the state of \( \langle J^x \rangle = 0 \)

The vacuum (ground state) to vacuum tunneling or decay in the presence of the external electric field is related to the Euler-Heisenberg action [1] which for QED vacuum instability known as Schwinger effect, after [2]. In condensed matter proposed that the effective action is the longtime asymptotic of ground state to ground state tunneling rate [6]. From the holographic point of view, this effective action have been studied in [8]. Following [8], for a system with Schrödinger symmetry such as cold atoms, we study the decay rate of ground state to ground state transition via probe branes holography. Taking \( \langle J^x \rangle = 0 \) in (16) or (13)\(^{11}\), the moment when the electric field is turned on and the system does not respond to it with current \( \langle J^x \rangle \) yet, we got:

\[
\hat{S}_{D7} = -N \int_0^{r_H} dr K^{1/2}(r) \sqrt{g_{xx}g_{rr}g_{\alpha_1 \alpha_1}} \left[ g_{xx}|H_2| - (2\pi \alpha')^2 E_3^2 H_1 \right].
\]

It’s clear that function under the square root in (22) could have a minus sign. In this situation there will be a special point (location), which we call it \( r_I \), that the action (22) is imaginary below that point:

\[
\hat{S}_{D7} = i \text{Im } \hat{S}_{D7} \bigg|_{r_I} + \text{Re } \hat{S}_{D7} \bigg|_{0}
\]

This imaginary action is a well-known symbol of the instability of the system at the false vacuum or ground state. Decay to the stable vacuum from the unstable one can be interpreted as the Schwinger-like pair production\(^{12}\). The \( r_I \) (0 < \( r_I < r_H \)) came from

\[
[g_{xx}|H_2| - (2\pi \alpha')^2 E_3^2 H_1]|_{r_I} = 0
\]

In the following we study this point and imaginary part of DBI action which is imaginary part of the effective action in dual field theory [8].

\(^{11}\)If \( \langle J^x \rangle = 0 \), (16) is a same as (13)

\(^{12}\)vacuum is not an exact word in this configuration.
3.1 Gapless hot cold atoms: Ground state instability

For massless embedding (or roughly gapless system), we should consider $\theta(r) = 0$. So inserting (10) into (24) we will obtain

$$r_I = r_H \left( (2\pi \alpha')^2 4E^2_\beta \beta^2 r_H^4 + 1 \right)^{-1/4}$$

or in term of (20) and (A.13)

$$r_I = r_H \left( 16E^2_\beta \frac{|\mu|}{\pi^2 T^4 \lambda} + 1 \right)^{-1/4} = \kappa r_H$$

where we define

$$\kappa = \frac{1}{(1 + 16E^2_\beta \frac{|\mu|}{\pi^2 T^4 \lambda})^{1/4}} < 1.$$  

(26a)

So it’s clear that $0 < r_I < r_H$.  

If we redo $E_\beta = E/2\beta$, (25) is the same as the result of massless flavor in AdS background which have been done in [8].

3.2 Imaginary part of action

The imaginary part of the action (22) for massless fermions is

$$\text{Im } L = -N \int_{r_I}^{r_H} \frac{dr}{r^5} \sqrt{-f(r) + (2\pi \alpha')^2 4\beta^2 E^2_\beta r^4}$$

$$= -\frac{N}{8r_H^2} \int_{r_I}^{r_H} dr \frac{1}{r^3} \sqrt{\frac{r^4 - r_I^4}{r_H^4 - r^4}}$$

Which from now on we relabel $S_{D7}$ to $L$. Defining $x^4 = \frac{r^4}{r_H^4}$ and also recalling $r_I = \kappa r_H$ we get

$$\text{Im } L = -\frac{N}{8\kappa^2 r_H^4} \int_{\kappa}^{1} dx \frac{1}{x^3} \sqrt{\frac{x^4 - \kappa^4}{1 - x^4}} = -\frac{N}{8\kappa^2 r_H^4} \left( \frac{1 - \kappa^4}{8\kappa^2 - \pi} \right)$$

(28)

After replacing $\kappa$ with (25) we will find that

$$L = \frac{N\pi}{64} (2\pi \alpha')^2 4\beta^2 E^2_\beta$$

(29)

Remembering that $N = N_f T_{D7} 4\pi^2$ and also

$$(2\pi \alpha')^2 = 2\pi^2 \lambda^{-1} \quad 2\pi g_s = g_{QFT}^2 \quad T_{D7} = \frac{1}{(2\pi)^7 g_s \alpha'^4}$$

(30)

we get

$$\text{Im } L = \frac{N_f N_c}{64\pi} 4\beta^2 E^2_\beta = \frac{N_f N_c}{32\pi |\mu|} E^2_\beta$$

(31)

which differs from AdS result [8] by the $\frac{1}{2}$ if we replace $E_\beta = E/2\beta$.  

\[13\] Remember that $\mu$ is negative
\[14\] We use $\lambda = g_{QFT}^2 N_c$
\[15\] This is because of Hopf fibration (A.2).
3.3 Real part of action: Euler-Heisenberg action

The real part of (22) for massless embedding would be

$$\text{Re } \mathcal{L} = -\frac{N}{8\kappa^2 r_H^4} \int_0^\kappa dx \frac{1}{x^3} \sqrt{\frac{k^4 - x^4}{1 - x^4}}$$

$$= -\frac{N}{8\kappa^2 r_H^4} \left( \frac{\sqrt{\kappa^3}}{1} \, _2F_1 \left( \frac{1}{4}, \frac{1}{2}; \frac{3}{4}; k^4 \right) \right). \quad (32)$$

For $E_\beta^2 \ll \frac{\mu^4}{\lambda}$, we would find that

$$\text{Re } \mathcal{L} = N(c_1 + c_2 |\mu| E_\beta^2 \ln \left( \frac{E_\beta^2 |\mu|}{\lambda T^4} \right) + O(\left( \frac{E_\beta^2 |\mu|}{\lambda T^4} \right)^2)) \quad (33)$$

where $c_1$ and $c_2$ are numerical constant. The real part of effective action known as Euler – Heisenberg Lagrangian, see [8] and references therein.

4 Ground state decay: A gapped system

For simplicity, we consider massive fermionic degrees of freedom at zero temperature. For massive flavor fermions i.e., $\theta(r) \neq 0$ or gapped system the induced metric on the D7 branes with (4) embedding would be

$$ds^2 = \frac{M(r)}{r^2} \left( - \frac{M(r)dx^+}{r^2} + 2dx^+ dx^- + dx^2 + dy^2 \right) + \frac{1}{M(r)} \left( dr^2/r^2 + ds_\alpha^2 \right) \quad (34)$$

where $M(r) = 1 + (2\pi\alpha'm)^2 r^2$ and $ds_\alpha^2 = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$. The nonzero component of the induced B fields are

$$B_{+\alpha_2} = -\frac{1}{2r^2}, \quad B_{+\alpha_3} = -\frac{1}{2r^2} \cos \alpha_1 \quad (35)$$

with a little bit work we get also

$$H_2 = -\frac{1}{16r^4}, \quad H_1 = \frac{\beta^2(1 - (2\pi\alpha'm)^2 \beta^2)}{4r^2 M(r)} \quad . (36)$$

The instability condition (24) will reduce to

$$\left( 1 - \frac{4(2\pi\alpha')^2 \beta^2 E_\beta^2 (1 - (2\pi\alpha'm)^2 \beta^2) r^4}{M^2(r)} \right) |_{r_I} = 0 \quad (37)$$

For $(2\pi\alpha'm)^2 \beta^2 < 1$ , if we replace $4\beta^2 E_\beta^2 (1 - (2\pi\alpha'm)^2 \beta^2) = \tilde{E}^2$ we will find the $r_I$ , which looks similar to AdS result[8]:

$$r_I^2 = \sqrt{2\pi\alpha' \tilde{E} - (2\pi\alpha'm)^2}. \quad (38)$$

Clearly we have physical (real) $r_I$ if $\tilde{E} > 2\pi\alpha'm^2$. So we can see that there is a critical electric field $\tilde{E} > \tilde{E}_c$ or in term of non relativistic electric field $E_\beta > E_\beta^c$ which is

$$E_\beta^c = \frac{2\pi\alpha'm^2}{2\beta \sqrt{1 - \beta^2 (2\pi\alpha'm)^2}}. \quad (39)$$

\[16\] We should note that $E_c = 2\pi\alpha'm^2$ is relativistic critical electric field.
If we compare this critical value of electric field to the what found in AdS $E_c = (2\pi\alpha')^2 [8]$ as we considered $(2\pi\alpha'm)^2\beta^2 < 1$ or in term of $\mu$ i.e. $\frac{m^2}{\lambda} < \left|\frac{\mu}{\beta}\right|^2$,

$$E_{c}^{\text{sch}} > E_{c}^{\text{AdS}}. \quad (40)$$

This is because of compactification along $x^-$ which dual to chemical potential $\mu (\beta^2 = -\frac{1}{2\pi})$. The same bound on critical electric field could be found in highly degenerate fermionic systems as $(40)$, see for example [41]. See also [42] for similar result.

If we define an effective mass $m_{eff}$ as follows:

$$m_{eff}^2 = \frac{m^2}{\sqrt{1 - \beta^2(2\pi\alpha'm)^2}}$$

$$= \frac{m^2}{\sqrt{1 - \frac{\pi^2m^2}{\mu^2}}}$$

hence (39) will be

$$E_{c}^{\text{sch}} = 2\beta E_{c}^{\text{sch}} = 2\pi\alpha' m_{eff}^2.$$  \quad (42)

This is the AdS result which the mass of mesons (or two fermions) $m$ have been replaced by $m_{eff}$. So geometric distinction has been changed to the difference in masses of charge carriers. Hence, we could say that the potential between two flavor fermions with $l$ distances from each other would be:

$$V = \sqrt{\frac{2\pi}{\lambda}} \frac{m_{eff}^2}{l}$$

hence (43) will be

$$E_{c}^{\text{sch}} = 2\beta E_{c}^{\text{sch}} = 2\pi\alpha' m_{eff}^2.$$  \quad (42)

This result is nontrivial, but due to the compact direction $x^-$, would make sense. We could pretend that this compactification brings the same situation as a periodic potential in the study of the band structure in solid state physics. So the effective mass appears instead of mass, in here.

### 4.1 Effective action: Imaginary part of action

The imaginary part of the action for the gapped system by considering massive fermions from (22) is

$$\text{Im} \mathcal{L} = -\mathcal{N} \int_{u_j}^{u_i} du \frac{1}{8u^3} \left\{ -1 + \frac{(2\pi\alpha')^2 \tilde{E}^2 u^4}{M(u)^2} \right\}$$

$$= -\mathcal{N} \int_{u_j}^{u_i} du \frac{1}{8u^3 M(u)} \left\{ -1 + \frac{u^2}{u_0^2} \right\} \left\{ 1 + \frac{u^2}{u_0^2} \right\}$$

where we have defined

$$u_0^2 = \frac{1}{\sqrt{2\pi\alpha' \tilde{E} + (2\pi\alpha')^2 m^2}}.$$

This is the quite same effective action that found in AdS background which $E$ replaced by $\tilde{E}$ which is

$$\tilde{E}^2 = 4\beta^2 E_{\beta}^2 (1 - (2\pi\alpha'm)^2\beta^2).$$

\[\text{We must notice that electric field in AdS has a different scale dimension in comparison to Schrödinger geometry, so we compare } 2\beta E_{\beta}^{\text{sch}} = E_{c}^{\text{sch}} \text{ and } E_{c}^{\text{AdS}}.\]
With following changes
\[
\frac{u^2}{u_f^2} = 1 + x \quad \quad \epsilon = \frac{\tilde{E}_c}{E}
\]
where \(\tilde{E}_c = 2\pi\alpha'm^2\), we will find
\[
\mathcal{L} = N\left(1 - \epsilon\right)^{5/2}(2\pi\alpha')^2\tilde{E}_c^2 \int_0^\infty dx \frac{\sqrt{x(2 + x + \epsilon x)}}{(1 + x)^3(1 + \epsilon x)}
\]
for large electric field respect to \(\tilde{E}_c\) we could use \(\epsilon\) expansion and collect the following result
\[
(1 - \epsilon)^{5/2} \int_0^\infty dx \frac{\sqrt{x(2 + x + \epsilon x)}}{(1 + x)^3(1 + \epsilon x)} = \frac{\pi}{4}(1 + \frac{4}{\pi}Log \frac{\epsilon}{2} - \frac{1}{3\pi}\epsilon^3 + O(\epsilon^4))
\]
so we have
\[
\text{Im} \mathcal{L} = N\frac{\pi}{64}(2\pi\alpha')^2\tilde{E}^2(1 + \frac{\pi}{4}\tilde{E}_cLog \frac{\tilde{E}_c}{2E} - \frac{1}{3\pi}(\frac{\tilde{E}_c}{2E})^3 + O((\frac{\tilde{E}_c}{2E})^4))
\]
The functionality is the same as AdS with the change of
\[
E^2 \rightarrow \tilde{E}^2 = 4\beta^2E^2(1 - (2\pi\alpha'm)^2\beta^2)
\]
or
\[
\tilde{E} = \frac{E_\beta}{\sqrt{2|\mu|}}\sqrt{1 - \frac{\pi^2m^2}{|\mu|\lambda}}.
\]
Expanding through \(\frac{\pi^2m^2}{|\mu|\lambda}\) and replacing \(E = 2\beta E_\beta\)
\[
\text{Im} \mathcal{L}_{Sch} = -N(2\pi\alpha')^2\frac{\pi^2m^2}{\lambda|\mu|}E^2(1 - \frac{\pi}{8E}\tilde{E}_c) + \frac{1}{2}(1 + \frac{m^2\pi^2}{\lambda|\mu|})\mathcal{L}_{AdS} + ...
\]
\[
= -NfNc\frac{m^2}{2\lambda|\mu|}E^2(1 - \frac{\tilde{E}_c}{8E}) + \frac{1}{2}(1 + \frac{m^2\pi^2}{\lambda|\mu|})\mathcal{L}_{AdS} + ..
\]
At zero temperature and zero mass we know that the DBI action of D7 brane does not change under NMT transformations [21]. But the presence of mass will break the scaling in the induced metric so the DBI action will change. It is clear that if mass goes to zero we will back to AdS result. Effective action dependence to t’Hooft coupling also is shown in [40] for chiral mesons at the presence of external electromagnetic fields. Other terms ,which differ from AdS, might be consider as a dipole interaction that inherently exist in the dual theory of this Schrödinger spacetime[20].

### 4.2 Real part of effective action:Euler-Heisenberg

Real part of effective action for massive fermions- antifermions at zero temperature following above is

\[
\text{Re} \mathcal{L} = -N\int_0^u du \frac{1}{8u^5} \sqrt{1 - \frac{(2\pi\alpha')^2\tilde{E}^2u^4}{M(u)^2}}
\]
\[
= N\frac{(1 - \epsilon)^{5/2}(2\pi\alpha')^2\tilde{E}_c^2}{16} \int_0^1 dy \frac{\sqrt{y(2 - y - \epsilon y)}}{(1 - y)^3(1 - \epsilon y)}
\]
where we have defined
\[
y = 1 - \frac{u^2}{u_f^2}.
\]
Again, this is similar to real part of effective action in $AdS$ background [8]. For small $\epsilon$ i.e., Strong electric field $\tilde{E}$ relative to $\tilde{E}_c$ the finite part of the (50) would be

$$\text{Re } \mathcal{L} = \frac{N_c N_f}{32\pi^2} \tilde{E}^2 \left( 3 + \ln 2 + \ln \frac{\tilde{E}^2}{\tilde{E}_c^2} + \frac{\pi \tilde{E}_c}{\tilde{E}} \right).$$

(51)

Clearly if we inserting $\tilde{E} = \frac{E_0}{\sqrt{2|\mu|}}\sqrt{1 - \frac{\pi^2 m^2}{|\mu|\lambda}}$ in (51) there would be correction term, compare to $AdS$ one, in term of $\frac{\pi^2 m^2}{|\mu|\lambda}$.

There are also two other different version of real effective action, one of them would happen if we consider electric field below the critical electric field i.e., $\tilde{E} \leq \tilde{E}_c$. The other one would happen, see (37), if we respect

$$\frac{m^2}{\lambda} > \frac{|\mu|}{\pi^2}.$$  

(52)

This condition has not $AdS$ twins, but it resembles t constant magnetic field effect on the critical electric field in the study of the effective action from holography, [45].

**Conclusion and Summary**

We study breakdown of vacuum (ground state) of strongly correlated systems with Schrödinger symmetry in the presence of external electric field via gauge-gravity duality. By using holographic argument the decay rate of ground state to other ground states which is stable, resemble Schwinger pair production, is calculated in a large t’Hooft coupling, i.e., a theory with strong interaction. We find that false vacuum would be faded out if there is a bound for massive mesons(two fermions) in quantum field theories with Schrödinger symmetry which is due to chemical potential $\mu$. This chemical potential is conjugate to number operator. Because of $\mu$, even at zero temperature, a free particle could feel drag force [30]. The dual of this parameter is the compact $x^-$. The effective action in here look similar to relativistic one [8] if the electric field replaced by $\frac{E \sqrt{1 - \frac{\pi^2 m^2}{|\mu|\lambda}}}{\sqrt{2|\mu|}}$.

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**A Schrödinger space time**

Let us start from near horizon limit of non-extremal D3 branes solution in type IIB superstring theory which is $AdS_5$ Schwarzschild times $S^5$ (with AdS radius $L$)

$$ds^2 = \frac{L^2}{r^2} \left( \frac{dr^2}{f(r)} - f(r)dt^2 + dy^2 + dx^2 \right) + L^2 ds_{S^5}^2,$$  

(A.1)

Here $f(r) = 1 - \frac{r_H^4}{r^4}$ which tell us that black hole’s horizon located at $r_H$. The radial coordinate is $r$, and boundary located at $r = 0$ and field theory lives on $(t, y, \vec{x})$ which $y$ is singled out because we need to do Null Melvin twist (NMT) operation along with it. (A.1) has $ISO(1, 3) \times SO(6)$ isometry at extremal limit. The holographic dual of this geometry $^{18}$ is $N = 4$ superconformal $SU(N_c)$ gauge theory in large $N_c$. 

$^{18}$there is also five form RR field
limit and large 't Hooft coupling \( \lambda = N_c g_s^2 M \)\(^{19}\). We could write the \( S^5 \) metric as a Hopf fibration over \( \mathbb{CP}^2 \), with \( \chi \) the Hopf fiber direction

\[
ds_{S^5}^2 = (d\chi + A)^2 + ds_{\mathbb{CP}^2}^2
\]

where \( A \) gives the Kähler form \( J \) of \( \mathbb{CP}^2 \) via \( dA = 2J \). To write the (A.2) explicitly, we introduce \( \mathbb{CP}^2 \) coordinates \( \alpha_1, \alpha_2, \alpha_3 \), and \( \theta \) and define the \( SU(2) \) left-invariant forms

\[
\begin{align*}
\sigma_1 &= \frac{1}{2} (\cos \alpha_2 \, d\alpha_1 + \sin \alpha_1 \, \sin \alpha_2 \, d\alpha_3), \\
\sigma_2 &= \frac{1}{2} (\sin \alpha_2 \, d\alpha_1 - \sin \alpha_1 \, \cos \alpha_2 \, d\alpha_3), \\
\sigma_3 &= \frac{1}{2} (d\alpha_2 + \cos \alpha_1 \, d\alpha_3),
\end{align*}
\]

so that the metric of \( \mathbb{CP}^2 \) is

\[
ds_{\mathbb{CP}^2}^2 = d\theta^2 + \cos^2 \theta \left( \sigma_1^2 + \sigma_2^2 + \sin^2 \theta \sigma_3^2 \right),
\]

and \( A = \cos^2 \theta \, \sigma_3 \). The full solution also includes a nontrivial five-form, but it’s shown in refs. [25, 18, 24] that five-form will be unaffected by the NMT or TsT, so we will ignore it.

After Null Melvin Twist operation [20] we get the following \(^{20}\)

\[
ds^2 = \frac{L^2}{r^2} \left( \frac{dx^2}{f(r)} - f(r) \frac{1 + \beta^2 r^{-2}}{K(r)} dt^2 + \frac{1}{K(r)} \left( r^2 \frac{dy^2}{K(r)} - \frac{2}{L^2} f(r) dt \, dy + d\bar{x}^2 \right) \right)
\]

\[
= \frac{L^2}{r^2} \left( \frac{dr^2}{f(r)} - \frac{f(r)}{L^2 r^2 K(r)} dx^+ dx^- + \frac{2}{K(r)} dx^+ dx^- + \frac{1}{2K(r)} \left( \frac{dx^+}{\sqrt{2} \beta L} - \sqrt{2} \beta L dx^- \right)^2 + d\bar{x}^2 \right)
\]

\[
+ \frac{L^2}{K(r)} (d\chi + A)^2 + L^2 ds_{\mathbb{CP}^2}^2,
\]

where

\[
f(r) = 1 - \frac{r^4}{r_H^4}, \quad K(r) = 1 + \frac{\beta^2 r^2}{r_H^4},
\]

which at the end of (A.5) we introduce the light-cone like coordinates \( x^\pm \) as

\[
x^+ = \beta L (t + y), \quad x^- = \frac{1}{2 \beta L} (-t + y).
\]

The task of this \( \beta \) rescaling is special at the zero temperature limit and non zero temperature of this metric, see also [22, 23]. The solution also includes the Kalb-Ramond two-form or NS \( B \) field

\[
B = - \frac{\beta L^2}{r^2 K(r)} (d\chi + A) \wedge (f(r) \, dt + dy),
\]

\(^{19}\) at zero temperature or extremal limit we have supersymmetry. At non zero temperature, we have a thermal state in a dual field theory where supersymmetry is broken.

\(^{20}\) The TsT transformation [19] also gives us similar result but a little bit different. In [20] although the SUSY is broken but 8 supercharges have been remained while in TsT [19] whole SUSY is broken.
\[
= -\frac{L^2}{2r^2K(r)}(d\chi + A) \land \left((1 + f(r)) \frac{dx^+}{L} + (1 - f(r)) 2\beta^2 Ldx^-ight)
\]

(A.8)

and also a dilaton
\[
\Phi = -\frac{1}{2}\log K(r).
\]

(A.9)

Zero temperature gain from (A.5) when \(r_H \to \infty\), that would be
\[
ds^2 = \frac{1}{r^2} \left(dr^2 - \frac{1}{r^2} dx^+ dx^- + 2dx^+ dx^- + dx^2\right) + (d\chi + A)^2 + ds_{\text{CP}^2}^2,
\]

(A.10)

Where we have set \(L = 1\), just for simplicity. As you see there is no \(\beta\) in this metric. There is also \(B\) field with no dependence on \(\beta\). If we do a compactification on \(S^{5\text{2}}\) the (A.10) will be a Schrödinger metric which introduced in [27, 28, 29] for gravity dual part of nonrelativistic CFT. In [28] it’s showed that the Schrödinger geometry i.e
\[
ds^2 = \frac{1}{r^2} \left(dr^2 - \frac{1}{r^2} dx^+ dx^- + 2dx^+ dx^- + dx^2\right)
\]

(A.11)
in which \(x^+\) would be a time in dual field theory and for compact \(x^-\), could be a dual to free fermions or fermions at unitarity and in [29] the cold atom aspect discussed. The (A.10) or (A.11) will be preserved with the following scale transformations
\[
x^+ \to \lambda^2 x^+ \quad x^- \to \lambda^0 x^- \quad r \to \lambda r \quad \vec{x} \to \lambda \vec{x}
\]

(A.12)

\(\beta\) is a dimensionful parameter which has units of length so in (A.7) \(x^+\) has the dimension of square of length i.e \([L]^2\) and \(x^-\) has no dimension. In schrödinger space time the \(x^-\) is a compact dimension so at the boundary \(r = 0\) we have 2 + 1 dimension theory. The isometry generator along \(x^-\) is a dual to number operator \(N\) in dual theory. At finite temperature i.e. (A.5) there would be a momentum along \(x^- (P_-)\). So the quantum state in the dual theory has finite number density \(N\) or chemical potential [37, 25, 26]. As mentioned in [37, 25, 26] the temperature and chemical potential of the dual quantum field theory, which is due to \(U(1)\) symmetry along \(x^-\) compact direction not charged carriers, would be
\[
T = \frac{1}{\pi r_H \beta L} \quad \mu = \frac{1}{2\beta^2 L^2}.
\]

(A.13)

One of the fascinating feature of zero temperature schrödinger space time is that a flavor quark would feel a drag force [30] and also one of the interesting comment about a Schrödinger metric (A.11) is that this geometry has a \(SL(2, R)\) asymptotic symmetry [30].

### B Effective metric

DBI action integrand is given by
\[
det(g_{ab} + A_{ab})
\]

(B.14)

where we define \(A_{ab} = B_{ab} + 2\pi\alpha'F_{ab}\). From antisymmetric tensor \(A_{ab}\) we always have:
\[
det(g_{ab} + A_{ab}) = \det(g_{ab} - A_{ab})
\]

(B.15)

So
\[
det(g_{ab} + A_{ab}) = \sqrt{\det(g_{ab} + A_{ab}) \det(g_{ab} - A_{ab})}
\]

\(21 ds_{S^5}^2 = (d\chi + A)^2 + ds_{\text{CP}^2}^2\)
\[
\sqrt{\text{det} g_{ab}} \sqrt{\text{det}(g_{ab} - A_{ac} g^{cd} A_{db})} = \sqrt{\text{det} \tilde{g}_{ab}}. \tag{B.16}
\]

Where we introduce \textit{effective metric}: \(\tilde{g}_{ab} = g_{ab} - A_{ac} g^{cd} A_{db}\).

For example, from (6) and (A.8)

\[
\tilde{g}_{++} = g_{++} + g^{\alpha_2 \alpha_2} B_{+ \alpha_2}^2 + g^{\alpha_3 \alpha_3} B_{+ \alpha_3}^2 + (2\pi \alpha')^2 F_{+x}^2 g^x \tag{B.17}
\]

\[
\tilde{g}_{--} = g_{--} + g^{\alpha_2 \alpha_2} B_{- \alpha_2}^2 + g^{\alpha_3 \alpha_3} B_{- \alpha_3}^2 + (2\pi \alpha')^2 F_{-x}^2 g^x \tag{B.18}
\]

\[
\tilde{g}_{rr} = g_{rr} + (2\pi \alpha')^2 F_{r}^2 g^r \tag{B.19}
\]

\[
\tilde{g}_{+-} = g_{+-} + B_{+ \alpha_2} B_{- \alpha_2} g^{\alpha_2 \alpha_2} + B_{+ \alpha_3} B_{- \alpha_3} g^{\alpha_3 \alpha_3} + (2\pi \alpha')^2 F_{+x} F_{-x} g^x. \tag{B.20}
\]

The horizon of \(\tilde{g}_{ab}\) will meet the reality constraint on DBI action i.e., \(r_\ast\) in (17). So we could put geometric meaning on the external electric field.

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