THE EXCESS DEGREE OF A POLYTOPE

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Abstract. We define the excess degree $\xi(P)$ of a $d$-polytope $P$ as $2f_1 - df_0$, where $f_0$ and $f_1$ denote the number of vertices and edges, respectively.

We first prove that the excess degree of a $d$-polytope does not take every natural number: the smallest possible values are 0 and $d-2$, and the value $d-1$ only occurs when $d = 3$ or 5. On the other hand, for fixed $d$, the number of values not taken by the excess degree is finite (subject to the restraint that if $d$ is even, then the excess degree must be even).

Our study of the excess degree is then applied in three different settings. We show that polytopes with small excess (i.e., $\xi(P) < d$) behave in a similar manner to simple polytopes in terms of Minkowski decomposability: each such polytope is either decomposable or a pyramid, and their duals are always indecomposable. (This is no longer true when $\xi(P) \geq d$.) Secondly, we characterise the decomposable $d$-polytopes with $2d + 1$ vertices. And thirdly, we characterise all pairs $(f_0, f_1)$ for which there exists a 5-polytope with $f_0$ vertices and $f_1$ edges.

1. Introduction

This paper revolves around the excess degree of a $d$-dimensional polytope $P$, or simply $d$-polytope, and some of its applications. We define the excess degree $\xi(P)$, or simply excess, of a $d$-polytope $P$ as the sum of the excess degrees of its vertices, i.e.

$$
\xi(P) = 2f_1 - df_0 = \sum_{u \in \text{vert} P} (\text{deg } u - d).
$$

Here as usual $\text{deg } u$ denotes the degree of a vertex $u$, i.e the number of edges of $P$ incident with the vertex; $\text{vert } P$ denotes the set of vertices of $P$; and $f_0$ and $f_1$ denote the number of vertices and edges of the polytope.

Our first substantial result, in §3, is the excess theorem: the smallest values of the excess degree of $d$-polytopes are 0 and $d-2$; clearly a polytope is simple if, and only if, its excess degree is 0. Note that for fixed $d$ and $f_0$, the possible values of the excess are either all even or all odd. We further show that if $d$ is even, the excess degree takes every even natural number from $d\sqrt{d}$ onwards; while, if $d$ is odd, the excess degree takes every natural number from $d\sqrt{2d}$ onwards. So, for a fixed $d$, only a finite number of gaps in the values of the excess of a $d$-polytope are possible.

In §4, we study $d$-polytopes with excess strictly less than $d$, and establish some similarities with simple polytopes. In particular, they are either pyramids or decomposable. (A polytope $P$ is (Minkowski) decomposable if it can be written as the Minkowski sum of two polytopes, neither of which are homothetic to $P$; otherwise it is indecomposable.) A polytope can have excess $d-1$ only if $d = 3$ or 5, which shows that there are infinitely many gaps in the possible values of the excess degree. Polytopes with excess $d-2$ exist in all dimensions, but their structure is very restricted: either there is a unique nonsimple vertex, or there is a $(d-3)$-face containing only vertices with excess degree 1. A similar, but slightly more complicated, situation arises for excess $d-1$: a 5-polytope with excess 4 may also contain 2 vertices with excess degree 2.
In §5, we characterise all the decomposable $d$-polytopes with $2d + 1$ or fewer vertices; this incidentally proves that a conditionally decomposable $d$-polytope must have at least $2d + 2$ vertices.

The final application, in §7, is the completion of the $(f_0, f_1)$ table for $d \leq 5$; that is, we give all the possible values of $(f_0, f_1)$ for which there exists a $d$-polytope with $d = 5$, $f_0$ vertices and $f_1$ edges. The solution of this problem for $d \leq 4$ was already well known [5, Chap. 10]. The same result has recently been independently obtained by Kusunoki and Murai [13]. Our proof requires some results of independent interest; in particular, a characterisation of the 4-polytopes with 10 vertices and minimum number of edges (namely, 21); this is completed in §6. We have more comprehensive results characterising polytopes with a given number of vertices and minimum possible number of edges, details of which will appear elsewhere [17].

Most of our results tacitly assume that the dimension $d$ is at least 3. When $d = 2$, all polytopes are both simple and simplicial, and the reader can easily see which theorems remain valid in this case.

2. Background: some special polytopes and previous results

2.1. Some basic results on polytopes. In this subsection we group a number of basic results on polytopes which we will use throughout the paper. Recall that a facet of a $d$-polytope is a face of dimension $d - 1$; a ridge is a face of dimension $d - 2$; and a subridge is a face of dimension $d - 3$. A fundamental property of polytopes is that every ridge is contained in precisely two facets.

Recall that a vertex $u$ in a $d$-polytope $P$ is called simple if its degree in $P$ is precisely $d$, equivalently if it is contained in exactly $d$ facets. Otherwise it is nonsimple. Note that a nonsimple vertex in a polytope $P$ may be simple in a proper face of $P$; we often need to make this distinction. A polytope is simple if every vertex is simple; otherwise it is nonsimple. We will call $H \cap P$ the underfacet corresponding to $F$. Klee [11] called this the face figure, but that term is sometimes used for a different concept [23, p. 71]. When $F$ has dimension 0, the underfacet is simply the vertex figure.

Let $H$ be a hyperplane intersecting the interior of $P$ and containing no vertex of $P$, and let $H^+$ be one of the two closed half-spaces bounded by $H$. Set $P' := H^+ \cap P$. If the vertices of $P$ not in $H^+$ are the vertices of a face $F$, then the polytope $P'$ is said to be obtained by truncation of the face $F$ by $H$. We often say that $P$ has been sliced or cut at $F$.

**Lemma 2.1.** Let $P$ be a $d$-polytope and let $F$ be a face of $P$. Suppose the neighbours outside $F$ of every vertex of $F$ are all simple in $P$. Let $H$ be a hyperplane as above, and let $P' := H^+ \cap P$ be obtained by truncation of $F$ by $H$. Then every vertex in the facet $H \cap P$ is simple in $P'$, and thus the facet $H \cap P$ is a simple $(d - 1)$-polytope.

**Proof.** Every vertex $u_e$ in the facet $H \cap P$ is the intersection of $H$ and an edge $e$ of $P$ outside $F$ but incident to a vertex of $F$. Consequently, the facets of $P'$ containing the vertex $u_e$ in $H \cap P$ are precisely the facets of $P$ containing the edge $e$ plus the facet $H \cap P$. Since the edge $e$ is contained in exactly $d - 1$ facets of $P$, the vertex $u_e$ in $P'$ is contained in exactly $d$ facets of $P'$, thus is simple in $P'$. \hfill $\square$

An interesting corollary of Lemma 2.1 arises when $F$ is a vertex and reads as follows.

**Lemma 2.2.** Let $P$ be a $d$-polytope and let $v$ be a vertex whose neighbours are all simple in $P$. Then the vertex figure of $P$ at $v$ is a simple $(d - 1)$-polytope.
The next few remarks are simple but useful.

**Remark 2.3.** For any edge of a polytope, and any vertex outside that edge, there is a facet containing the edge but not the vertex. Hence for any two vertices of a polytope, there is a facet containing one of the vertices but not the other.

**Remark 2.4.** Let $P$ be a $d$-polytope and let $v$ be a vertex simple in a facet $F$. Suppose that every facet containing $v$ intersects $F$ at a ridge. Then $v$ is a simple vertex in $P$.

**Proof.** The other facets containing $v$ are given by the ridges of $F$ containing $v$, whose number is $d - 1$ since $v$ is simple in $F$. Thus, in total there are $d$ facets containing $v$. □

**Remark 2.5.** Let $P$ be a polytope, $F$ a facet of $P$ and $u$ a nonsimple vertex of $P$ which is contained in $F$. If $u$ is adjacent to a simple vertex $x$ of $P$ in $P \setminus F$, then $u$ must be adjacent to another vertex in $P \setminus F$, other than $x$.

**Proof.** Since $u$ is nonsimple, it is contained in at least $d + 1$ facets of $P$. The edge $ux$ is contained in exactly $d - 1$ facets of $P$, since $x$ is simple. Hence there are at least two edges of $u$ outside $F$, as desired. □

**Lemma 2.6.** Let $F$ and $G$ be any two distinct nondisjoint facets of a $d$-polytope $P$ and let $j := \dim F \cap G$.

(i) Every vertex in $F \cap G$ has excess degree at least $(d - 2 - j)$.

(ii) The total excess degree of $P$ is at least $\xi(F \cap G) + (d - 2 - j)(j + 1)$.

(iii) If $F \cap G$ is not a ridge, then $P$ is not simple.

**Proof.** Set $R := F \cap G$. Then, for any vertex $u$ in $R$, the degrees of $u$ in $R$, $F \setminus R$ and $G \setminus R$ are at least $j$, $d - 1 - j$ and $d - 1 - j$, respectively. So the total degree of $u$ in $F_1 \cup F_2$ is at least $2d - 2 - j$. This implies that the excess degree of each vertex in $R$ is at least $d - 2 - j$. Since there are at least $j + 1$ vertices in $R$, the total excess degree of $P$ is at least $(d - 2 - j)(j + 1)$. □

In particular, if two facets of a $d$-polytope intersect in a face $K$ with $\dim K < d - 2$, then every vertex in $K$ is nonsimple in the polytope. We will call a polytope **semisimple** if every pair of facets is either disjoint, or intersects in a ridge. Clearly every simple polytope is semisimple.

**Remark 2.7.** Let $P$ be a semisimple $d$-polytope. Then every facet containing a nonsimple vertex of $P$ is also nonsimple. Furthermore, each nonsimple vertex of $P$ is nonsimple in each of the facets containing it.

**Proof.** If $d = 3$, then Remark 2.4 gives at once that the polytope is simple. So we let $d > 3$. Let $u$ be a nonsimple vertex of $P$ and let $F$ be some facet containing it. Since every other facet containing $u$ intersects $F$ at a ridge, there are at least $d$ ridges of $F$ containing $u$, which implies that $u$ is nonsimple in $F$, and $F$ is therefore nonsimple. □

2.2. **Taxonomy of polytopes.** In this subsection we introduce or recall families of polytopes which are important for this work.

There is a 3-polytope with six vertices, ten edges and six facets (four quadrilaterals and two triangles), which can fairly be described as the simplest polyhedron with no name. Accordingly, we will call it **NN**; see Fig. 1(d). This humble example will naturally appear several times in this paper. (Kirkman [9, p. 345] called it a 2-ple zoneless monaxine heteroid, having an amphigrammic axis, but this terminology never caught on.)
Call simplicial $d$-prism any prism whose base is a $(d − 1)$-simplex. We will often refer to these simply as prisms. They each have $2d$ vertices, $d^2$ edges, and $d + 2$ facets. Being simple, they have excess degree 0.

Define a $(k, d − k)$-triplex as a $(d − k)$-fold pyramid over the simplicial $k$-prism, $1 ≤ k ≤ d$, and denote it by $M_{k,d−k}$. Any triplex is of this form for some values of $d$ and $k$, and clearly has $d + k$ vertices. Clearly $M_{1,d−1}$ is a simplex, and $M_{d,0}$ is a simplicial $d$- prism. It is well known (see [15] and [5, Ch. 10]) and easily checked, that triplexes (other than the simplex) have $d + 2$ facets. They were studied in some detail in [16]. It is routine to check that $M_{k,d−k}$ has excess degree $(k − 1)(d − k)$.

More generally, denote by $\Delta_{m,n}$ the sum of an $m$-dimensional simplex and an $n$-dimensional simplex lying in complementary subspaces. The polytope $\Delta_{m,n}$ is a simple $(m + n)$-dimensional polytope with $(m + 1)(n + 1)$ vertices, $\frac{1}{2}(m + n)(m + 1)(n + 1)$ edges and $m + n + 2$ facets. Likewise $\Delta_{m,n,p}$ will denote the sum of three simplices, $m$-dimensional, $n$-dimensional and an $p$-dimensional respectively, lying in complementary subspaces.

Remark 2.8. Any polytope with $d + 2$ facets is combinatorially isomorphic to an $r$-fold pyramid over $\Delta_{m,n}$ for some values of $m, n, r$; see [5, Sec 6.1] and [15, p. 352]. In particular, a simple polytope with $d + 2$ facets must be $\Delta_{m,n}$ for some $m, n$ with $m + n = d$.

The $d$-dimensional pentasm, or just $d$-pentasm, was defined in [16, §4] as the Minkowski sum of a simplex and a line segment which is parallel to one triangular face, but not parallel to any edge, of the simplex; or any polytope combinatorially equivalent to it. The same polytope is obtained by truncating a simple vertex of the triplex $M_{2,d−2}$. Pentasms have $2d + 1$ vertices and excess degree $d − 2$; see Fig. 1(a)-(b) for drawings of them.

Remark 2.9 (Facets of a $d$-pentasm). The facets of $d$-pentasm, $d + 3$ in total, are as follows.

- $d − 2$ copies of a $(d − 1)$-pentasm,
- 2 simplicial prisms, and
- 3 simplices.

We can consider the pentagon as a 2-dimensional pentasm.

Take any polytope $P$ with dimension $d ≥ 4$ and stack a vertex $v_0$ beyond one facet and move it slightly so that $v$ is contained in the affine hull of $\ell$ other facets ($0 ≤ \ell ≤ d − 3$). By stacking we mean as usual adding a vertex beyond a facet of $P$ and taking the convex hull. A point $w$ is beyond a face $R$ of $P$ if the facets of $P$ containing
$R$ are precisely those that are visible from $w$, where a facet $F$ of $P$ is said to be visible from the point $w$ if $w$ belongs to the open half-space determined by the affine hull of $F$ which does not meet $P$. All the $d-2$ polytopes constructed in this manner have the same graph, some have even the same $(d-3)$-skeleton. If $P$ is taken to be a simplicial $d$-prism, we call the resulting polytopes capped $d$-prisms.

Consider a capped $d$-prism. Denote by $k$ the minimum dimension of any face of the simplicial $d$-prism whose affine hull contains the extra vertex $v_0$. The combinatorial type of such a polytope depends on the value of $k$; let us denote it by $CP_{k,d}$. Note that $k \geq 1$, otherwise $v_0$ would be a vertex of the prism. For $k = 1$, the capped prism $CP_{1,d}$ will be (combinatorially) just another prism. For $k = 2$, $CP_{2,d}$ is a pentasm, with $d^2 + d - 1$ edges. For $3 \leq k \leq d$ and fixed $d \geq 4$, the $d-2$ polytopes $CP_{k,d}$ are combinatorially distinct, although their graphs are all isomorphic, with $2d + 1$ vertices and $d^2 + d$ edges. In particular, they all have excess degree $d$. However $CP_{k,d}$ has $d + k + 1$ facets, so their f-vectors are all distinct.

Denote by $A_d$ a polytope obtained by slicing a one nonsimple vertex of a $(2,d-2)$-triplex. This polytope can be also realised as a prism over a $(d-3)$-fold pyramid over a quadrilateral. These polytopes have $2d + 2$ vertices and excess degree $2d - 6$.

**Remark 2.10 (Facets of $A_d$).** The facets of the $d$-polytope $A_d$, $d + 3$ in total, are as follows.

- $d - 3$ copies of $A_{d-1}$,
- 4 simplicial prisms, and
- 2 copies of $M_{2,d-3}$.

Denote by $B_d$ a polytope obtained by truncating a simple vertex of a $(3,d-3)$-triplex. The polytope $B_3$ is the well known 5-wedge; it can be obtained as the wedge [12, pp. 57–58] at an edge over a pentagon. The polytope $B_d$ can also be realised as follows: the convex hull of $B_3$ and a simplicial $(d-3)$- prism $R$, where each vertex of one copy of the $(d-4)$-simplex in $R$ is connected to each of the three vertices in a triangle of $B_3$, and each vertex of the other copy of the $(d-4)$-simplex in $R$ is connected to each of the remaining five vertices of $B_3$. These polytopes also have $2d + 2$ vertices and excess degree $2d - 6$.

**Remark 2.11 (Facets of $B_d$).** The facets of the $d$-polytope $B_d$, $d + 3$ in total, are as follows.

- $d - 3$ copies of $B_{d-1}$,
- 2 simplices,
- 1 simplicial prism,
- 1 copy of $M_{2,d-3}$, and
- 2 pentasms.

Denote by $C_d$ a polytope obtained by slicing one simple edge, i.e. an edge whose endvertices are both simple, of a $(2,d-2)$-triplex. It has $3d - 2$ vertices and excess degree $d - 2$.

**Remark 2.12 (Facets of $C_d$).** The facets of the $d$-polytope $C_d$, $d + 3$ in total, are as follows.

- $d - 2$ copies of $C_{d-1}$,
- 3 simplicial prisms,
- 1 copy of $\Delta_{2,d-3}$, and
- 1 simplex.
For consistency here, we can define $C_2$ as a quadrilateral.

Denote by $\Sigma_d$ a certain polytope which can be expressed as the sum of two simplices. One concrete realisation of it is given by the convex hull of

$$\{0, e_1, e_1 + e_k, e_2 + e_k, e_2, e_1 + e_2, e_1 + 2 e_k : 3 \leq k \leq d\},$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^d$. It is easily shown to have $3d - 2$ vertices; of these, one has excess degree $d - 2$, and the rest are simple.

**Remark 2.13 (Facets of $\Sigma_d$).** The facets of the $d$-polytope $\Sigma_d$, $d + 3$ in total, are as follows.

- $d - 1$ copies of $\Sigma_{d-1}$,
- 2 simplicial prisms,
- 2 simplices.

For consistency, we can also define $\Sigma_2$ as a quadrilateral.

Diagrams of $A_4, B_4, C_4$ and $\Sigma_4$ appear in Fig. 2(b) in §6.

Let us denote by $\Gamma_{m,n}$ the result of truncating one vertex from $\Delta_{m,n}$. This is a simple $(m + n)$-dimensional polytope with $mn + 2m + 2n$ vertices and $m + n + 3$ facets.

**Remark 2.14 (Facets of $\Gamma_{m,n}$).** The facets of the $d = m + n$-polytope $\Gamma_{m,n}$, $d + 3$ in total, are as follows.

- $m$ copies of $\Gamma_{m-1,n}$,
- $n$ copies of $\Gamma_{m,n-1}$,
- 1 copy of $\Delta_{m-1,n}$,
- 1 copy of $\Delta_{m,n-1}$,
- 1 simplex.

Denote by $J_d$ the special case $\Gamma_{d-1,1}$, i.e. the polytope obtained by slicing one vertex of a simplicial $d$-prism; it clearly has $3d - 1$ vertices. Observe that $J_2$ is a pentagon, and that $B_3$ coincides with $J_3$; so $J_d$ can be considered a generalisation of the 5-wedge.

**Remark 2.15 (Facets of $J_d$).** The facets of the $d$-polytope $J_d$, $d + 3$ in total, are as follows.

- $d - 1$ copies of $J_{d-1}$,
- 2 simplicial prisms,
- 2 simplices.

It can be shown that, for $d \neq 3, 7$, $J_d$ is the unique simple polytope with $3d - 1$ vertices; see Lemma 2.17 below.

Some of the examples we have just defined coincide in low dimensions. In particular, $A_3 = \Delta_{1,1,1}$, $B_3 = J_3$ and $C_3 = \Sigma_3$. By definition $\Delta_{1,d-1} = M_{d,0}$ and $J_d = \Gamma_{d-1,1}$.

With the fundamental examples now elucidated, we can characterise all the simple $d$-polytopes with up to $3d$ vertices, which will be useful later in the paper. First we need the following result, of independent interest. We omit the case $d = 2$ in the statement of the next two results, since all polygons are simple.
Proposition 2.16. A simple $d$-polytope $P$ in which every facet is a prism or a simplex is either a $d$-simplex, a simplicial $d$-prism, $\Delta_{1,1}$ or $\Delta_{2,2}$.

Proof. If $P$ is not a simplex, then it is not simplicial, so assume that some facet $F$ is a prism. Choose a ridge $R$ in $F$ which is also a prism, and let $G$ be the other facet containing $R$. Then $G$ is also a prism and every vertex in $R$ has degree $d$ in the subgraph $F \cup G$. So no vertex in $R$ is adjacent to any vertex outside $F \cup G$. If we remove the four vertices in $(F \cup G) \setminus R$ from the graph of $P$, the resulting graph will be disconnected. By Balinski’s Theorem [23, Thm. 3.14], we have $d \leq 4$. Denote by $t$ the number of triangular 2-faces of $P$, and by $q$ the number of quadrilateral 2-faces.

If $d = 3$, simplicity implies $2f_1 = 3f_0$. Then $f_0$ is even and

$$0 \leq t = 4(t + q) - (3t + 4q) = 4f_2 - 2f_1 = 4(f_1 - f_0 + 2) - 2f_1 = 2f_1 - 4f_0 + 8 = 3f_0 - 4f_0 + 8 = 8 - f_0.$$

Thus either $f_0 = 4$ and $P$ is a simplex, or $f_0 = 6$ and $P$ is a prism, or $f_0 = 8$ and $P$ is a cube.

If $d = 4$, simplicity implies $2f_1 = 4f_0$, whence $f_2 = f_1 + f_3 - f_0 = f_0 + f_3$. Simplicity also ensures that every edge belongs to precisely three 2-faces. By hypothesis, every facet contains at least 2 triangular faces, and every triangular face, being a ridge, belongs to precisely 2 facets. Thus $t \geq f_3$. The Lower Bound Theorem [1] tells us $f_0 \geq 3f_3 - 10$. Then

$$\frac{1}{2}f_0 \leq \frac{1}{2}t = 2(t + q) - \frac{1}{2}(3t + 4q) = 2f_2 - 1.5f_1 = 2(f_0 + f_3) - 3f_0 = 2f_3 - f_0 \leq 2f_3 - (3f_3 - 10) = 10 - f_3.$$

Thus $1.5f_3 \leq 10$. So either $f_3 = 5$ and $P$ is a simplex, or $f_3 = 6 = d + 2$, and Remark 2.8 tells us that $P$ has the form $\Delta_{m,n}$ for some $m, n$ with $m + n = 4$. \hfill $\Box$

Lemma 2.17. Up to combinatorial equivalence,

(i) the simplex $\Delta_{0,d}$ is the only simple $d$-polytope with strictly less than $2d$ vertices;
(ii) the simplicial prism $\Delta_{1,d-1}$ is the only simple $d$-polytope with between $2d$ and $3d - 4$ vertices;
(iii) $\Delta_{2,d-2}$ is the only simple $d$-polytope with $3d - 3$ vertices;
(iv) the only simple $d$-polytope with $3d - 2$ vertices is the 6-dimensional polytope $\Delta_{3,3}$;
(v) the only simple $d$-polytopes with $3d - 1$ vertices are the polytope $J_d$, the 3-dimensional cube $\Delta_{1,1,1}$ and the 7-dimensional polytope $\Delta_{3,4}$;
(vi) there is a simple polytope with $3d$ vertices if and only if $d = 4$ or 8; the only possible examples are $\Delta_{1,1,2}$, $\Gamma_{2,2}$ and $\Delta_{3,5}$.

Proof. Assertions (i) to (v) are simply a rewording of [19, Lem. 10(ii)-(iii)]. We prove (vi). Recall that two simple polytopes with the same graph must be combinatorially equivalent [23, §3.4].

For $d = 4$ or 8, the validity of the three examples given is easy to verify. Conversely, suppose $P$ is a simple polytope with $3d$ vertices. Then $d$ must be even.
Every facet of $P$ is simple, and so has an even number of vertices (because $d - 1$ is odd).

If there were a facet with $3d - 2$ vertices, there would be $3d - 2$ edges running out of it and $2(d - 1)$ edges running out of the two external vertices. But $2d - 2 \neq 3d - 2$.

So any facet has at most $3d - 4 = 3(d - 1) - 1$ vertices. We know from (i) to (v) that any facet must be a simplex, a prism, or have $3d - 6$ or $3d - 4$ vertices.

First suppose some $F$ has $3d - 4$ vertices. The four vertices outside all have degree $d$, so the number of edges between them and $F$ is at least $4(d - 3)$. On the other hand, there are exactly $3d - 4$ edges running out of $F$. So $4(d - 3) \leq 3d - 4$ whence $d \leq 8$.

Next suppose some facet $F$ has $3(d - 1) - 3$ vertices. Each of the six vertices outside each has degree $d$, so the number of edges between them and $F$ is at least $6(d - 5)$. On the other hand, there are exactly $3d - 6$ edges running out of $F$. Again $d \leq 8$.

Now we show that the 3 examples in the first paragraph are the only possibilities when $d = 4$ or 8. If $d = 8$, every facet must be $\Delta_{2,5}$ or $\Delta_{3,4}$; a short calculation shows that $P$ must be $\Delta_{3,5}$.

If $d = 4$, simplicity ensures that no facet has 10 vertices. Thus every facet is either a simplex, a prism, a cube or a 5-wedge. Proposition 2.16 ensures that at least one facet is neither a simplex nor a prism. If some facet is a cube, it is not hard to verify that four of the other facets must be prisms, and two of them must be cubes; this is the only way the 4 vertices outside can be connected to give us the graph of a simple polytope, and the graph is that of $\Delta_{1,1,2}$. Otherwise some facet is a 5-wedge, and there are a number of cases to consider. Since each such facet contains two pentagonal faces, and each pentagonal ridge belongs to two such facets, we can find a collection of facets $W_1, W_2, \ldots, W_6 = W_6$, with each $W_i$ being a 5-wedge, and $W_i \cap W_{i+1} = P_i$ being a pentagon for each $i$. Clearly there must be at least 3 pentagonal faces, so $k \geq 3$. We cannot have $k \geq 5$, or there would be too many vertices. If $k = 3$, then $P_1 \cup P_2$ would contain at least 3 of the vertices in $P_3$. Then the affine hyperplane containing $W_1$ would also contain $P_3$. This means that all 3 facets lie in the same 3-dimensional affine subspace, which is absurd. So we must have $k = 4$. The graphs of these four facets actually determine the entire graph of $P$; it is the graph of $\Gamma_{2,2}$.

Finally we need to show that the case $d = 6$ does not arise. This was first proved by Lee [14, Example 4.4.17], using the $g$-theorem, but we give an independent argument. Suppose $P$ is such a 6-polytope; then every facet must be either a simplex, a prism, $J_5$ or $\Delta_{2,3}$. Consider a ridge $R$ which is a 4-prism; clearly such ridges exist, because $P$ is not simplicial. Denote by $F$ and $G$ the two facets containing $R$; since $d > 4$, the previous result ensures that one of them is not a prism. They cannot both be $J_5$, because then $F \cup G$ would contain more than 18 vertices. If $F$ is $J_5$ and $G$ is $\Delta_{2,3}$, then $F \cup G$ would contain 18 vertices and 49 edges, with the 6 vertices in $F \setminus R$ guaranteeing another 6 edges; but $49 + 6 > 54$, contradicting simplicity. If $F$ is $J_5$ and $G$ is a prism, then $F \cup G$ would contain 16 vertices and 44 edges, with the 2 vertices outside $F \cup G$ guaranteeing another 11 edges; but $44 + 11 > 54$, again contradicting simplicity. Likewise if $F$ and $G$ are both $\Delta_{2,3}$, then $F \cup G$ would contain 16 vertices and 44 edges, with the 2 vertices outside $F \cup G$ guaranteeing another 11 edges. Finally if $F$ is $\Delta_{2,3}$ and $G$ is a prism, then $F \cup G$ would contain 14 vertices and 39 edges, with the 2 vertices outside $F \cup G$ guaranteeing another 18 edges; but $39 + 18 > 54$. This exhausts all the possibilities, so there is no simple 6-polytope with 18 vertices.

2.3. Decomposability of polytopes. A polytope $P$ is (Minkowski) decomposable if it can be written as the Minkowski sum of two polytopes, neither of which are homothetic to $P$; otherwise it is indecomposable. See [5,
Chap. 15] for a more detailed account and historical references. As usual, the (Minkowski) sum of two polytopes $Q + R$ is defined to be $\{x + y : x \in Q, y \in R\}$, and a polytope $P$ is homothetic to a polytope $Q$ if $Q = \lambda P + t$ for $\lambda > 0$ and $t \in \mathbb{R}^d$. All polygons other than triangles are decomposable; this topic becomes more serious when $d \geq 3$. A polytope can have a (geometric) realisation that is decomposable and another realisation that is indecomposable. We will generally not distinguish between polytopes which are combinatorially equivalent, i.e. have isomorphic face lattices. Smilansky [21, p. 43] calls a polytope $P$ combinatorially decomposable (resp. indecomposable) if whenever $Q$ is combinatorially equivalent to $P$, then $Q$ is also decomposable (resp. indecomposable); otherwise $P$ is called conditionally decomposable. In this paper we only come across polytopes which are combinatorially decomposable, just called decomposable henceforth, or combinatorially indecomposable polytopes, just called indecomposable henceforth. But it is important to be aware that conditionally decomposable polytopes do exist; see [8, §5], [21, Fig. 1] or [18, Example 11] for further discussion.

The main tool for establishing decomposability is the following concept. We will say that a facet $F$ in a polytope $P$ has Shephard's property if every vertex in $F$ has a unique neighbour outside $F$. This is not an intrinsic property of $F$, but rather of the way that $F$ sits inside $P$. This concept appears implicitly in the proof of [20, (15)], and leads to the following result.

**Theorem 2.18.** A polytope $P$ is decomposable whenever there is a facet $F$ with Shephard's property, and $P$ has at least two vertices outside $F$. In other words, if a polytope $P$ has a facet $F$ with Shephard's property, then it is either decomposable or a pyramid with base $F$.

This result was essentially proved by Shephard [20, Result (15)]. He made the stronger assumption that every vertex in $F$ is simple in $P$, but the general statement does follow from his proof. Another proof appears in [19, Prop. 5]. We will say that a polytope is a Shephard polytope if it has at least one facet with Shephard’s property. It is not hard to see that a polytope is simple if and only if every facet has Shephard’s property.

Theorem 2.18 implies that any simple polytope is either decomposable or a simplex. Later we will generalise this, to show that any polytope with small excess is either decomposable, or a pyramid.

The following related concept will be useful in §4. Let us say that a facet $F$ in a polytope $P$ has Kirkman’s property if every other facet $G$ intersects $F$ at a ridge. Again this is not an intrinsic property of $F$ alone. Lee [14, p. 110] calls a Kirkman polytope any polytope in which at least one facet has Kirkman’s property; Klee [10, p. 2] had earlier made an equivalent definition for simple polytopes. The generalisation to non-simple polytopes in [14] is quite natural. Following Klee, we call a super-Kirkman polytope any polytope in which every facet has Kirkman’s property. This is equivalent to the dual polytope being 2-neighborly.

Numerous examples (e.g. the cube) show that Shephard’s property does not imply Kirkman’s property. In the other direction, if $P$ is a pyramid over $\Delta_{m,n}$, where $m, n \geq 2$, then $P$ is a super-Kirkman polytope, but only one facet has Shephard’s property.

As a common weakening of both properties, we could say that a facet $F$ in a polytope $P$ has the weak Kirkman-Shephard property if every other facet $G$ is either disjoint from $F$, or intersects $F$ at a ridge. Although this defines a natural property, we have no applications for it so far. Obviously a polytope is semisimple if every facet has the weak Kirkman-Shephard property.

Turning now to indecomposability, a much bigger toolkit of sufficient conditions is available. Rather than giving a detailed survey of this topic, we will just summarize the results we need. The main concept we need is due to
Kallay [8]; he defined a geometric graph as any graph $G$ whose vertex set $V$ is a subset of a finite-dimensional real vector space $X$, and whose edge set $E$ is a subset of the line segments joining members of $V$.

Then a function $f : V \to X$ is called a decomposing function for the graph $(V, E)$ if it has the property that $f(v) - f(w)$ is a scalar multiple of $v - w$ for each edge $[v, w] \in E$. A geometric graph $G = (V, E)$ is then called decomposable if there is a decomposing function which is not the restriction of a homothety on $X$. If the only decomposing functions are homotheties then $G$ is called indecomposable. Kallay showed [8, Thm. 1a] that a polytope is decomposable if and only if its edge graph is decomposable in this sense. The study of geometric graphs is actually a useful tool for establishing indecomposability. For a detailed proof of the next result, see [18] and the references therein.

**Theorem 2.19.** (i) If the vertices of a geometric graph are affinely independent, and its edges form a cycle, then the graph is indecomposable.

(ii) If $G_1$ and $G_2$ are two indecomposable geometric graphs with (at least) two vertices in common, then the graph $G_1 \cup G_2$ is also indecomposable.

(iii) If $P$ is a polytope, $G$ is a subgraph of the skeleton of $P$, $G$ contains at least one vertex from every facet of $P$, and $G$ is an indecomposable graph, then $P$ is an indecomposable polytope.

(iv) Conversely, if $P$ is an indecomposable polytope, then its skeleton $G$ is an indecomposable geometric graph.

A Hamiltonian cycle through the vertices of a simplex is an affinely independent cycle; this is one way to see that a simplex is indecomposable. Repeated application of part (ii) then shows that any simplicial polytope is indecomposable. In particular, the dual of any simple polytope must be indecomposable. The corollary below generalises that. It depends on the following result, which seems to be a basic and fundamental result on decomposability, but we could not find a reference.

**Theorem 2.20.** Let $P$ be a $d$-polytope with strictly less than $d$ decomposable facets. Then $P$ is indecomposable.

**Proof.** Let $n$ denote the number of decomposable facets of $P$, with $n < d$. Consider the corresponding collection of vertices in the dual polytope $P^*$. Thanks to Balinski’s Theorem [23, Thm. 3.14], their removal does not disconnect $P^*$, so all the other vertices (which correspond to indecomposable facets of $P$) can be ordered into a sequence, possibly with repetition, so that any successive pair defines an edge of $P^*$.

This means that the indecomposable facets of $P$ form a strongly connected family, in the sense that each successive pair intersects in a ridge. This is much more than we need; intersecting in an edge gives us a suitably connected family of indecomposable faces. The union of this family is clearly an indecomposable graph. It touches every facet because there are only $n$ other facets, and any facet intersects at least $d$ other facets. Hence Theorem 2.19 gives the conclusion.

Furthermore, this is best possible: there are decomposable $d$-polytopes with precisely $d$ decomposable facets, namely $d$-prisms and capped $d$-prisms.

**Corollary 2.21.** Let $P$ be a $d$-polytope at most $d - 1$ nonsimple vertices. Then the dual polytope is indecomposable.

**Proof.** Theorem 2.20 gives the result at once, as duals of $d$-polytopes with at most $d - 1$ nonsimple vertices have at most $d - 1$ decomposable facets.

This is also best possible. The bipyramid over a $(d - 1)$-simplex has exactly $d$ nonsimple vertices, but its dual is the prism, which is decomposable.
2.4. On the number of edges of polytopes. Following Grünbaum’s notation [5, p. 184], define
\[ \phi(v, d) = \binom{d+1}{2} + \binom{d}{2} - \binom{2d+1-v}{2} \]

It is convenient to note that for \( 1 \leq k \leq d \)
\[ \phi(d+k, d) = \frac{1}{2}d(d+k) + \frac{1}{2}(k-1)(d-k). \]
\[ = \binom{d}{2} - \binom{k}{2} + kd. \]

Theorem 2.22 ([16, Thm. 6]). Let \( P \) be a \( d \)-polytope with \( d+k \) vertices with \( 1 \leq k \leq d \). Then \( P \) has at least \( \phi(d+k, d) \) edges. Equivalently, we can say that \( \xi(P) \geq (k-1)(d-k) \). Furthermore, equality is obtained only if \( P \) is a \( (k, d-k) \)-triplex.

Theorem 2.23 ([16, Thm. 13]). The polytopes with \( 2d+1 \) vertices and \( d^2 + d - 1 \) or fewer edges are as follows.

(i) For \( d = 3 \), there are exactly two polyhedra with 7 vertices and 11 edges; the pentasm, and \( \Sigma_3 \). None have fewer edges.

(ii) For \( d = 4 \), a sum of two triangles \( \Delta_2 \Delta_2 \) is the unique polytope with 18 edges, and the pentasm is the unique polytope with 19 edges. None have fewer edges.

(iii) For \( d \geq 5 \), the pentasm is the unique polytope with \( d^2 + d - 1 \) edges. None have fewer edges.

3. Possible values of the excess degree

We will denote by \( \Xi(d) \) the set of possible values of the excess of all \( d \)-dimensional polytopes. The main result here is that the smallest two values in \( \Xi(d) \) are 0 and \( d-2 \); nothing in between is possible.

If \( j \leq d-3 \), then \( (d-2-j)(j+1) \geq d-2 \), so Lemma 2.6 establishes this for any polytope which is not semisimple. Accordingly, we restrict our attention to semisimple polytopes. First consider the case in which every facet is simple.

Proposition 3.1. Let \( P \) be a semisimple \( d \)-polytope in which every facet is simple. Then \( P \) is simple.

Proof. This is immediate from Remark 2.4. \( \square \)

In addition to the assumption that every two nondisjoint facets intersect at a ridge, thanks to Proposition 3.1, we can now assume that our nonsimple \( d \)-polytope \( P \) contains a nonsimple facet.

Lemma 3.2. The excess of a nonsimple polytope is strictly larger than the excess of any of its facets.

Proof. Suppose otherwise. Let \( F \) be a facet with equal excess to the polytope. Then, every vertex outside \( F \) is simple. Take a nonsimple vertex \( u \) in \( F \) and neighbour \( x \) of \( u \) lying in \( P \setminus F \). Since \( x \) is simple the edge \( ux \) is contained in exactly \( d-1 \) facets, but \( u \) must be contained in at least \( d+1 \) facets. Thus, \( u \) has at least two neighbours outside \( F \) (cf. Remark 2.5), implying the excess of the polytope is larger than the excess of \( F \). \( \square \)

We are now ready to prove the theorem.

Theorem 3.3 (Excess Theorem). Let \( P \) be a \( d \)-polytope. Then the smallest values in \( \Xi(d) \) are 0 and \( d-2 \).
Proof. Proceed by induction on $d$, with the base case $d = 3$ being easy. A tetrahedron is simple, and square pyramid has excess 1.

If $P$ is a simple polytope we get excess 0, so assume that $P$ is nonsimple. By virtue of Lemma 2.6 we can assume that every nondisjoint pair of facets intersect at a ridge, and by Proposition 3.1 that there exists a nonsimple facet $F$ with excess at least $d - 3$ by the inductive hypothesis.

If there is a nonsimple vertex outside $F$, we are done. If a nonsimple vertex of $P$ is simple in $F$, we are also at home. So we can further assume that the facet $F$ contains all the nonsimple vertices of $P$ and that each nonsimple vertex $u$ of $P$ is nonsimple in $F$ and has exactly one neighbour $x$ outside $F$. In this case, the facet $F$ would contain all the excess of the polytope, which is ruled out by Lemma 3.2. □

The existence, in every dimension, of $d$-polytopes with excess $d - 2$ and $d$ has been observed in §2. The existence of $d$-polytopes with excess $d + 2$, e.g. the cyclic polytope with $d + 2$ vertices, is also well known. (The characterisation of $d$-polytopes with $d + 2$ vertices [5, §6.1] gives many more examples.) In the next section, we will show that polytopes with excess $d - 1$ only exist in dimensions 3 and 5. What about higher values? It is clear that the excess degree can be any natural number if $d = 3$, and any even number if $d = 4$. We can show that the excess degree takes all possible values above $d - 2$ if $d = 5$ or 6. We suspect that there are gaps in the possible values of $\xi(P)$ for $d \geq 7$. For $d = 7$, we can show that the excess degree takes all possible values from 7 onwards, expect perhaps 11. Apart from the excess $d - 1$, we have so far been unable to prove the existence of any other such gap above $d - 2$. The next Theorem shows that for each dimension, the number of further gaps, if any, is finite.

**Lemma 3.4.** If the $d$-polytope $P$ is a pyramid with base $F$, and $F$ has $v$ vertices, then $\xi(P) = \xi(F) + v - d$.

**Theorem 3.5.** If $d$ is even, then $\Xi(d)$ contains every even integer in the interval $[d\sqrt{d}, \infty)$. If $d$ is odd, then $\Xi(d)$ contains every integer in the interval $[d\sqrt{2d}, \infty)$.

Proof. According to [5, §10.4], for any integer $v \geq 6$ and any integer $e$ in the interval $[2v + 2, (\frac{v}{2})^2]$ there is a 4-polytope with $v$ vertices and $e$ edges. (A stronger assertion can be made if we exclude the values $v = 6, 7, 8, 10$, but we do not need it here.) It follows that for any $v \geq 6$ and any even $\xi$ in the interval $[4, \sqrt{v^2 - 5v}]$ there is a 4-polytope with $v$ vertices and excess $\xi$. Let us denote by $I(v)$ the collection of all integers in the interval

$$[4 + (d - 4)(v - 5), v^2 - 5v + (d - 4)(v - 5)]$$

which have the **same parity** as the two endpoints. Applying Lemma 3.4 to $(d - 4)$-fold pyramids, we see that for any $d > 4$, $v \geq d + 2$ and any $\xi \in I(v)$, there is a $d$-polytope with $v + d - 4$ vertices and excess $\xi$. We now consider three cases, depending on the value of $d$.

First consider the case when $d \geq 5$ is odd. Let $v_0$ be the smallest integer greater than $\sqrt{2d} + 3$; clearly $v_0 \geq 6$. For any $v \geq v_0$ it is clear that $\Xi(d)$ contains the (parity based) interval $I(v)$. It is easily seen that $v^2 - 5v \geq (\sqrt{2d} + 3)(\sqrt{2d} - 2) > 2d - 6$, so $4 + 2(d - 4) < v^2 - 5v + 2$, whence $\min I(v + 2) < \max I(v) + 2$, i.e. the intervals $I(v)$ and $I(v + 2)$, which have elements of the same parity, overlap. Consider separately the intervals $I(v_0), I(v_0 + 2), I(v_0 + 4) \ldots$, whose elements all have the same parity, and the intervals $I(v_0 + 1), I(v_0 + 3), I(v_0 + 5) \ldots$ whose elements all have the other parity. Since $\min I(v_0 + 1) - 1 \in I(v_0)$, a moment’s reflection shows that $\Xi(d)$ contains every integer, both even and odd, from $\min I(v_0 + 1) - 1$ onwards. But $v_0 < \sqrt{2d} + 4$, and $4\sqrt{2d} > 12$, so

$$\min I(v_0 + 1) - 1 = 3 + (d - 4)(v_0 + 1 - 5) < 3 + (d - 4)v_0 \sqrt{2d} < d\sqrt{2d} - 9.$$
Thus \( \Xi(d) \) contains every integer greater than \( d\sqrt{2d} - 9 \).

Now consider the case when \( d \geq 16 \) is even. Let \( v_0 \) be the smallest integer greater than or equal to \( \sqrt{d} + 3 \). Then \( v_0 \geq 7 \). For any \( v \geq v_0 \) we have

\[
v(v - 5) \geq (\sqrt{d} + 3)(\sqrt{d} - 2) = d + \sqrt{d} - 6 \geq d - 2.
\]

It follows that

\[
4 + (d - 4)(v + 1 - 5) \leq \sqrt{d} - 5v + (d - 4)(v - 5) + 2,
\]

which means that \( \min l(v) \leq \max l(v + 1) \), i.e. there is no gap between the parity-based intervals \( l(v) \) and \( l(v + 1) \). Each such interval is contained in \( \Xi(d) \), which thus contains all even integers in \([4 + (d - 4)(v_0 - 5), \infty)\). On the other hand, \( v_0 < \sqrt{d} + 4 \), and \( 4\sqrt{d} + d \geq 32 \), so

\[
4 + (d - 4)(v_0 - 5) < 4 + (d - 4)(\sqrt{d} - 1) \leq d^{1.5} - 24.
\]

Thus \( \Xi(d) \) contains every even integer greater than or equal to \( d^{1.5} - 24 \).

For lower dimensions, we recall that if \( d \geq 4 \) and \( \xi \) is an even number in the interval \([2d - 6, 2d + 6]\), then there is a \( d \)-polytope \( P \) with excess \( \xi \) (and \( d + 3 \) vertices). This is a reformulation of [16, Lemma 14]; a tiny modification of the proof there shows that \( P \) can be chosen to have at least one (in fact every) facet being a simplex. Now stacking a vertex on a simplex facet will increase the excess by precisely \( d \). Thus we obtain \( d \)-polytopes whose excesses are even every number in the intervals \([3d - 6, 3d + 6], [4d - 6, 4d + 6], [5d - 6, 5d + 6] \) etc. If \( d \leq 14 \), the union of these intervals in \([2d - 6, \infty)\); clearly \( 2d \leq d^{1.5} \), so this completes the proof.

We now note that the excess degree takes all possible values above \( d - 2 \) if \( d \) is sufficiently small. In particular, if \( d \leq 6 \), the only impossible values of \( \Xi(d) \) are those excluded by Theorem 3.3 and parity considerations.

In case \( d = 6, 8 \) or 10, the proof just given shows that every even integer from \( 2d - 6 \) onwards is the excess degree of some polytope; and the existence of \( d \)-polytopes with excess \( d - 2, d \) and \( d + 2 \) has already been noted. It is clear then that every even value above \( d - 2 \) is realised as the excess of some polytope.

In case \( d = 5 \), this proof shows that every integer from 7 onwards is the excess degree of some polytope; the existence of 5-polytopes with excess 3 or 5 should be clear. The triplex \( M_{3,2} \) has excess degree 4, and a 3-fold pyramid over a pentagon has excess degree 6.

4. Structure of polytopes with small excess

Corollary 2.21 showed the indecomposability of the duals of \( d \)-polytopes with at most \( d - 1 \) nonsimple vertices. In this section, we establish stronger conclusions for polytopes with small excess, namely \( \xi = d - 2 \) or \( d - 1 \). We show that this imposes strong restrictions on the distribution of nonsimple vertices. A \( d \)-polytope with excess \( d - 2 \) either has a single vertex with excess \( d - 2 \), or \( d - 2 \) vertices each with excess 1; in the latter case, the nonsimple vertices form a \((d - 3)\)-face. A polytope can have excess \( d - 1 \) only if \( d = 3 \) or 5, and cannot have 3 nonsimple vertices. Thus there is at least one further gap in the possible values of the excess degree, from dimension 7 onwards.

We begin by examining the intersection patterns of facets; these are also severely restricted by the assumption of low excess. With the exception of ten 3-dimensional polyhedra, every \( d \)-polytope with excess \( d - 2 \) or \( d - 1 \) is a Shephard polytope. Nine of these ten turn out to be (Minkowski) decomposable. Thus every polytope with low excess is either decomposable or a pyramid or \( NN \) (see Fig. 1(d)); and we will see that in all cases their duals are indecomposable. Of course a polytope with excess 0 is either decomposable or a simplex.
Thus the excess theorem is highly applicable to questions about the decomposability of polytopes. The results just mentioned are best possible since there are $d$-polytopes with excess $d$ which are decomposable and $d$-polytopes that are indecomposable. For instance, the capped prism $CP_{d,d}$ is decomposable and self-dual, a bipyramid over a $(d-1)$-simplex is indecomposable with decomposable dual, and the capped prisms $CP_{k,d}$ for $k < d$ are decomposable with indecomposable duals. For $d = 3$, all polyhedra with 7 vertices and 7 faces have excess $d$; with the exception of the capped prism, they are all indecomposable with indecomposable duals.

For simplicity of expression, we will often simply state the conclusion that we have a Shephard polytope; bear in mind that Theorem 2.18 then guarantees that the polytope is either decomposable or a pyramid.

We begin by making explicit the following easy consequence of Lemma 2.6.

**Lemma 4.1.** Let $F$ and $G$ be distinct nondisjoint facets of a polytope $P$ and set $j = \dim(F \cap G)$. If $\xi(P) = d - 2$, then $j$ is either 0, $d - 3$ or $d - 2$. If $\xi(P) = d - 1$, then either $j = 0$, $d - 3$ or $d - 2$; or $j = 1$ and $d = 5$. In case $j \neq d - 2$, $F \cap G$ is either a simplex or a quadrilateral.

**Proof.** Let $k$ be the number of vertices in $F \cap G$; then Lemma 2.6 informs us that $(j + 1)(d - 2 - j) \leq k(d - 2 - j) \leq \xi(P) \leq d - 1$. Clearly $j = d - 2$ is one possibility. Henceforth, it is enough consider only the case $j \leq d - 3$. The inequality $(j + 1)(d - 2 - j) \leq d - 1$ is equivalent to $j(d - 3 - j) \leq 1$. The only solutions for this are $j = 0$, $j = d - 3$ and $j = 1 = d - 3$. Clearly $F \cap G$ is a simplex if $j = 0$ or 1.

If $j = d - 3$, our original inequality becomes $k \leq \xi(P)$. If $\xi(P) = d - 2$, then $k \leq j + 1$ and again $F \cap G$ is a simplex. In case $\xi(P) = d - 1$, it is also possible that $k = j + 2$. But then the contribution to the excess of the edges leaving $F \cap G$ is already $d - 1$, meaning that $F \cap G$ must be simple. This is only possible if $j = 2$, and then $k = 4$ and $d = 5$. \hfill $\Box$

This result enables us to study the structure of polytopes with excess degree $d - 2$. For 3-dimensional polyhedra this is quite simple. Any such polyhedron has a unique nonsimple vertex, which has degree 4, and so the polyhedron is either a simplex or decomposable. We now proceed to higher dimensions.

**Lemma 4.2.** Let $P$ be a $d$-polytope with excess degree $d - 2$. If $P$ has a unique nonsimple vertex (necessarily with excess $d - 2$), then it has two nondisjoint facets intersecting at just this vertex. Conversely, if $F_1$ and $F_2$ be any two nondisjoint facets of $P$ such that $\dim F_1 \cap F_2 = 0$, then this vertex is the unique nonsimple vertex (with excess $d - 2$), and consequently $P$ is a Shephard polytope. Furthermore, $P$ has a facet with excess $d - 3$.

**Proof.** Suppose there is a unique nonsimple vertex $u$ with degree $2(d - 1)$. Thanks to Lemma 2.2, the vertex figure $P/u$ of $P$ at $u$ is a simple $(d - 1)$-polytope, in fact a simplicial prism. Let $R_1$ and $R_2$ be the two opposite simplices in $P/u$. Then the two facets containing $u$ arising from $R_1$ and $R_2$ intersect only at $u$.

Conversely, suppose $\{u\} = F_1 \cap F_2$. Then $u$ has degree $d - 1$ in both facets, and excess $d - 2$. Thus every other vertex in $P$ is simple. There is a facet in $P$ which does not contain $u$, which clearly has Shephard’s property.

Finally, let $F$ be the other facet arising from a ridge of $F_1$ which contains $u$. The facet $F$ must then intersect $F_2$ at a ridge; it can’t intersect $F_2$ at an edge or a $(d - 3)$-face since $u$ is the only nonsimple vertex (cf. Lemma 2.6). The degree of $u$ in $F$ is $2(d - 2) = d - 1 + d - 3$. \hfill $\Box$

**Lemma 4.3.** Let $P$ be a $d$-polytope with excess degree $d - 2$. Let $F_1$ and $F_2$ be any two facets of $P$ such that $K = F_1 \cap F_2$ has dimension $d - 3$. Then
(i) $K$ is a simplex, and each of its $d-2$ vertices has excess degree one in $P$.
(ii) $P$ is a Shephard polytope. Hence it is either decomposable, or a $(d-2)$-fold pyramid.
(iii) Every facet in $P$ intersecting $K$ but not containing it misses exactly one vertex of $K$ and every vertex of $K$ in the facet has degree $d$; thus the facet has excess $d-3$.
(iv) There are precisely four facets containing $K$, and each of them is simple.

**Proof.** The result is clearly true for $d = 3$, so assume $d \geq 4$. From Lemma 2.6 it follows that every vertex in $K$ has degree $d+1$ in $P$, that $K$ is a simplex, and that every other vertex in $P$ is simple. Moreover every vertex outside $K$ is simple in $P$, while every vertex in $K$ is simple in both $F_1$ and $F_2$, and has no neighbours outside $F_1 \cup F_2$.

Since $d \geq 4$, we can find distinct vertices $u, v \in K$. Consider a facet $F$ containing $u$ but not $v$; we claim that any such facet has Shephard’s property. Note that $F$ intersects $F_1$ and $F_2$ at a ridge, and contains every vertex in $K$ but $v$, every neighbour of $u$ in $F_1 \setminus K$, and every neighbour of $u$ in $F_2 \setminus K$. The same argument applies to every vertex in $F \cap K$. This in turn implies that every vertex in $F$ has exactly one neighbour outside $F$. In other words, the $d-3$ vertices in $F \cap K$ has degree $d$ in $F$, and hence the facet $F$ has excess $d-3$.

Thus, by Theorem 2.18 $P$ is decomposable unless $\{v\} = P \setminus F$, in which case $P$ is a pyramid over $F$ with apex $x$. As there is nothing special about $u$ or $v$, in the case of $P$ being indecomposable, we see that $P$ is a pyramid with every vertex in $K$ acting as an apex, that is, $P$ is a $(d-2)$-fold pyramid. This completes the proof of (iii).

For (iv), fix a vertex $v \in K$, let $x_1$ and $x_2$ be the neighbours of $v$ in $F_1 \setminus K$, and let $x_3$ and $x_4$ be the neighbours of $v$ in $F_2 \setminus K$. For each $i$, denote by $R_i$ the smallest face containing $K \cup \{x_i\}$. Since $K$ has codimension two in $F_1$, it must be the intersection of 2 ridges in $F_1$ (i.e. facets of $F_1$). Thus $R_1$ and $R_2$ must be ridges of $P$. Likewise $R_3$ and $R_4$ must be ridges of $P$. These four ridges are distinct, so there must be at least four facets of $P$ which contain $K$. We need to show that there are only four, not five or six.

Denote by $F_{ij}$ the smallest face of $P$ containing $K \cup \{x_i, x_j\}$ for each $i, j$. Obviously any facet containing $K$ must be of this form, and $F_1 = F_{1,2}$ and $F_2 = F_{3,4}$. With respect to the inclusion $R_1 \subset F_{1,2}$, the other facet containing $R_1$ must be either $F_{1,3}$ or $F_{1,4}$; without loss of generality suppose it is $F_{1,3}$. Since $R_1 = F_{1,2} \cap F_{1,4}$ also, the unique representation of ridges as the intersection of two facets implies that $F_{1,4}$ cannot be a facet. Continuing, we conclude that $F_{1,4} = F_{2,3} = P$ and that $F_{2,4}$ is the fourth facet containing $K$. Since $F_{1,3} \cap F_{2,4} = K$, both these facets are also simple.

We can now illuminate the structure of polytopes with excess $d-2$.

**Lemma 4.4.** Let $P$ be a 4-polytope with excess degree two. Then $P$ is not a super-Kirkman polytope.

**Proof.** If there is a unique nonsimple vertex, this follows from Lemma 4.2.

Otherwise, $P$ has two vertices, say $u$ and $v$, of degree five. They must be connected by an edge; otherwise, considering a facet $G$ containing $u$, but not $v$, would give that $u$ is nonsimple in $G$ (by Remark 2.4) with at least two neighbours outside the facet (by Remark 2.5). Consider a facet $F$ containing the edge $uv$. Then both $u$ and $v$ must be nonsimple in $F$ by Remark 2.4, which implies that the excess of $F$ is two, contradicting Lemma 3.2.

In the next result, examples for case (i) include $B_4$, $\Sigma_d$ and $M_{d-1,1}$. Examples for case (ii) include $A_d$, $C_d$, $M_{d,d-2}$ and pentasms.
Theorem 4.5. Any $d$-polytope $P$ with excess exactly $d - 2$ either

(i) has a unique nonsimple vertex, which is the intersection of two facets, or

(ii) has $d - 2$ vertices of excess degree one, which form a $(d - 3)$-simplex which is the intersection of two facets.

In either case, $P$ is a Shephard polytope and has a facet with excess $d - 3$.

Proof. We proceed by induction on the dimension. The case $d = 3$ is obvious, with (i) and (ii) coinciding. The case $d = 4$ is settled by Lemma 4.1 and Lemma 4.4.

Consequently, assume henceforth that $d \geq 5$. Lemma 4.2 and Lemma 4.3 establish the conclusion if there are two facets which intersect in a vertex or a $(d - 3)$-face. So we may assume that $P$ is a super-Kirkman polytope; we will show that this case actually does not arise.

By Proposition 3.1 and Theorem 3.3, we can assume that there exists a facet $F$ with excess at least $d - 3$, and by Lemma 3.2 that $F$ (and every nonsimple facet) has excess exactly $d - 3$.

By induction, we may suppose that $F$ satisfies either (i) or (ii).

First suppose that the facet $F$ contains a nonsimple vertex $v$, with degree $2d - 4$ therein, and two ridges $R$ and $S$ such that $R \cap S = \{v\}$. Let $G$ be the other facet corresponding to $R$. Then $v$ is not simple in $G$ because it is not simple in $P$ (cf. Remark 2.7). Consequently, $v$ must have degree $d$ in $G$, degree $2d - 4 + 2$ in $P$ and is the only nonsimple vertex of $P$. This in turn implies that the excess of $G$ is $1 = d - 3$, a contradiction.

Next suppose that $F$ contains two ridges $R$ and $S$ such that $R \cap S = K$ is a $(d - 4)$-simplex, with every vertex having excess one in $F$. If there were no other nonsimple vertex in $P$, then every vertex in $K$ would be adjacent to some simple vertex in $P \setminus F$, and hence to at least two vertices outside $F$ (by Remark 2.5), meaning the excess degree of $P$ would be at least $2(d - 3) > d - 2$. So there must another nonsimple vertex $u$ in $P \setminus F$, with excess one, and it will be the unique neighbour outside $F$ of every vertex in $K$. This implies that the other facet $G$ corresponding to $R$ must contain $u$. But then $G$ contains $d - 2$ nonsimple vertices, each having excess degree one in $G$, which is impossible by Lemma 3.2. □

We now turn our attention to the decomposability of $d$-polytopes with excess $d - 1$.

Remark 4.6. Since 3-dimensional polyhedra with excess degree $d - 1$ do not behave as neatly as those in higher dimensions, we will examine this case in detail first. Catalogues of 3-dimensional polyhedra help us to clarify the situation. Numbers in (iv) in the next statement refer to the catalogue of Federico [3]. The 3-polytope $NN$ has excess $d - 1 = 2$, yet is neither a pyramid nor decomposable; we will see shortly that it is the only such example. It has two non-simple vertices, and every facet contains at least one of them, so it also fails Shephard’s property.

Lemma 4.7. Let $P$ be 3-polytope whose excess degree is 2. Then either

(i) $P$ is a pentagonal pyramid, which is indecomposable, or

(ii) $P$ is $NN$, which is also indecomposable but not a pyramid, or

(iii) $P$ has Shephard’s property but is not a pyramid, and hence is decomposable, or

(iv) $P$ is one of $F_{27}$, $F_{31}$, $F_{37}$, $F_{105}$, $F_{109}$, $F_{140}$, $F_{159}$, $F_{163}$, $F_{172}$, which are all decomposable, but fail Shephard’s property.
Proof. Excess two means that for some $k$, $P$ has $2k$ vertices and $3k + 1$ edges. Either one vertex has degree five, or two vertices have degree four. In either case, the number of faces containing a non-simple vertex is at most eight. So if $k \geq 6$, then $P$ has $k + 3 \geq 9$ facets, one of which contains only simple vertices, and so has Shephard’s property. If a pyramid has excess 2, its base must be a pentagon.

If $k = 5$ or 4, then $2k > k + 3$, and $P$ must be decomposable by [21, Theorem 6.11(a)]. The examples which fail Shephard’s property can be verified from catalogues; they are as listed.

If $k = 3$, then $P$ is $NN$ or a pentagonal pyramid. □

We have seen that for a polytope with excess $d - 2$, the non-simple vertices form a face. We will see that this is also true for polytopes with excess $d - 1$, provided $d > 3$. For $d = 3$, it is false; in every example in case (iv) above, the two nonsimple vertices are not adjacent. Truncating simple vertices then yields many more examples which fall into case (iii).

Remark 4.8. If a $d$-polytope has excess $d - 1$, then $d$ must be odd. If also $d \geq 5$ then any two nondisjoint facets intersect, like in the case of excess $d - 2$, either at a vertex, a $(d - 3)$-face or a ridge (as a consequence of Lemma 2.6). If $d = 5$ intersections at an edge can also occur.

Lemma 4.9. Let $P$ be a $d$-polytope with excess degree $d - 1$, and two facets $F_1$ and $F_2$ intersecting at a vertex. Then $d = 3$, and the polytope is as described in Lemma 4.7.

Proof. Let $\{u\} = F_1 \cap F_2$. We consider two cases: either $u$ is the only nonsimple vertex in $P$ or there is another nonsimple vertex $v$ in $P$.

In the former case, every vertex in $P$ other than $u$ is simple and the vertex figure of $P$ at $u$ is a simple $(d - 1)$-polytope with $2d - 1$ vertices (cf. Lemma 2.2), so Lemma 2.17(ii) forces $d = 3$ or 5. Assume $d = 5$. The vertex figure $P/u$ has $2d - 1 = 3(d - 1) - 3$ vertices, and by Lemma 2.17(iii) can only be $\Delta_2,2$ (cf. Fig. 1(c)), which has the property that any two 3-faces intersect at a 2-face. Thus, every two facets in $P$ containing $u$ must intersect at a ridge, a contradiction.

In the latter case, the vertex $u$ has degree $2d - 2$ and $v$ has degree exactly $d + 1$. Lemma 2.2 ensures that $u$ and $v$ are adjacent. Without loss of generality, assume $v \in \text{vert} \ F_1$. Let $F$ be any other facet in $P$ containing $v$. If the intersection of $F$ and $F_1$ is not a ridge, then Lemma 2.6 tells us that every vertex in $F \cap F_1$ is nonsimple, meaning $v$ is the only element therein, with excess $d - 2$. This is impossible when $d > 3$. So every facet containing $v$ must intersect $F_1$ at a ridge. Remark 2.4 in turn implies that $v$ is nonsimple in $F_1$. If we choose $F$ to contain $v$ but not $u$, then Theorem 3.3 gives $1 = \xi(F) \geq d - 3$, again forcing $d = 3$.

Thus, $d = 3$ in both cases. Finally note that if a 3-polytope is nonsimple, then a nonsimple vertex must lie in the intersection of a pair of facets, so all such examples are described in Lemma 4.7. □

We now deal with the particular situation which is mentioned in Remark 4.8.

Lemma 4.10. Let $P$ be a 5-polytope with excess degree 4. Let $F_1$ and $F_2$ be any two facets of $P$ such that $\dim F_1 \cap F_2 = 1$. Then $P$ is a Shephard polytope.

Proof. Let $F_1 \cap F_2 = [u, v]$. Then $u$ and $v$ are both simple in both $F_1$ and $F_2$, and every other vertex in the polytope is simple. Consider a facet $F$ containing $u$ but not $v$. Then, in view of Lemma 4.1 and Lemma 4.9 we can assume
that $F$ intersects each $F_1$ and $F_2$ at a ridge. This implies that $u$ has exactly one neighbour outside $F$. Since every vertex in $F$ is simple, $F$ has Shephard’s property, as required.

The next two lemmas deal with the case of two nondisjoint facets intersecting at a $(d - 3)$-face or only at ridges.

**Lemma 4.11.** Let $P$ be a $d$-polytope with excess degree of $d - 1$, with $d > 3$. Let $F_1$ and $F_2$ be any two facets of $P$ whose intersection is a subridge. Then $d = 5$, the face $F_1 \cap F_2$ is a quadrilateral and $P$ is a Shephard polytope.

**Proof.** Recall that $d$ must be odd. Let $K$ denote the subridge $F_1 \cap F_2$. We consider two cases: either (1) the excess degree of $P$ comes only from the nonsimple vertices in $K$, or (2) the excess degree of $P$ comes from the nonsimple vertices in $K$ and a nonsimple vertex $v$ outside $K$. Note that, in the first case, $K$ must have $d - 2$ vertices, unless $d = 5$, in which case it could have $d - 1$ vertices. Indeed, suppose $K$ has $d - 1$ vertices. From Lemma 2.6 it follows that every vertex in $K$ has degree $d + 1$ in $P$ and that $K$ is a simple polytope, but the last assertion is only possible if $K$ is 2-dimensional, i.e. $d - 3 = 2$. So in this particular case $d = 5$ and $K$ is a quadrilateral.

**Case 1.** The excess degree of $P$ comes only from the (nonsimple) vertices in $K$.

We will first rule out this case when $K$ has $d - 2$ vertices. Then it is a simplex, and there exists a vertex in $K$, say $u$, with degree $d + 2$ in $P$, while every other vertex in $K$ has degree $d + 1$ and is simple in both $F_1$ and $F_2$. If $u$ were nonsimple in $F_1$ or $F_2$, then $1 = \xi(F_1) \geq d - 3$ or $1 = \xi(F_2) \geq d - 3$ by the Excess Theorem. So $u$ is simple in $F_1$ and $F_2$. Denote by $v$ the neighbour of $u$ outside $F_1 \cup F_2$.

We consider the set $F_u$ of facets containing $u$. Since $u$ is simple in $F_1$, there are exactly $d - 1$ facets in $F_u$ intersecting $F_1$ at a ridge. Out of these $d - 1$ facets, there are $d - 3$ facets which miss one vertex in $K$; consider one such facet $F$. The facet $F$ must also intersect $F_2$ at a ridge, otherwise the excess would be much larger. Consequently, every vertex in $(F \cap K) \setminus \{u\}$ (of which there are $d - 4$) has excess 1 in $F$, i.e. has $d$ neighbours in $F$. If $v$ belonged to $F$, then $u$ would have $d + 1$ neighbours in $F$ and $\xi(F) = d - 4 + 2$. However, since $d - 1$ is even, $\xi(F)$ cannot be odd. Thus, $v \notin F$.

Suppose there is a facet $G$ in $F_u$ which intersects $F_1$ at $K$; by 4.9 the dimension of the intersection cannot be smaller than $d - 3$. Now consider any vertex in $K$ other than $u$; its two neighbours in $F_1 \setminus K$ cannot be in $G$, so $G$ contains both neighbours in $F_2$. This means that $G$ must be $F_2$. Consequently, there are exactly $d + 1$ facets in $F_u$: $F_1$, $d - 1$ facets intersecting $F_1$ at a ridge, and $F_2$. Hence there are at most 2 facets in $F_u$ containing the vertex $v$. But there must be exactly $d - 1$ facets in $F_u$ containing $v$. So $d - 1 \leq 2$.

For the particular case of $d = 5$ and $K$ having $d - 1$ vertices, every vertex in $K$ has excess degree 1, so we can choose $u$ to be any of them. Let $w \neq u$ be a vertex in $K$ (this step requires $d > 3$). Consider a facet $F$ containing $w$ but not $u$. Then from Remark 4.8 it ensues that $F$ intersects each $F_1$ and $F_2$ at a ridge. Thus $F$ contains every neighbour of $w$ in $F_1 \setminus K$ and $F_2 \setminus K$. The same argument applies to every vertex $\neq u$ in $F \cap K$. This in turn implies that $F$ has Shephard’s property.

We next rule out the second case for $d > 3$.

**Case 2.** Every vertex in $K$ has degree $d + 1$ and is simple in both $F_1$ and $F_2$, and there is a nonsimple vertex $v \notin K$ of degree $d + 1$.

If the vertex $v$ does not belong to $F_1 \cup F_2$ then all the neighbours of $v$ are simple in $P$. From Lemma 2.2 it ensues that the vertex figure of $P$ at $v$ is a simple $(d - 1)$-polytope with $d - 1 + 2$ vertices, which implies $d = 2$ by
Lemma 2.17. So \( v \) belongs to either \( F_1 \setminus K \) or \( F_2 \setminus K \), say \( F_1 \setminus K \), and must be adjacent to at least one vertex \( u \in \text{vert} \, K \).

We claim that every vertex \( w \neq u \) in \( K \) (which do exist since \( d > 3 \)) must be adjacent to \( v \). Suppose otherwise. Consider a facet \( F \) containing \( v \) but not \( u \) (cf. Remark 2.3). Then \( F \) must intersect \( F_1 \) at a ridge (note that \( u, w \notin \text{vert} \, F \)). Since \( F_1 \) cannot have excess degree precisely 1, \( v \) must be simple in \( F_1 \). Thus there is only one such facet \( F \), thereby implying that there are at least \( d \) facets containing the edge \( uv \). The edge \( uv \) is contained in exactly \( d - 2 \) facets of \( F_1 \) (which are of courses ridges in \( P \)). Thus there are at least two facets \( G_1 \) and \( G_2 \) containing the edge \( uv \) intersecting \( F_1 \) at a \((d - 3)\)-face. Since \( w \notin G_i \) \((i = 1, 2)\), each \( G_i \) misses exactly one vertex of \( K \), namely \( w \). This in turn implies that \( G_1 \cap F_1 = G_2 \cap F_1 = (K \setminus \{w\}) \cup \{v\} =: K' \). But then the intersection of each \( G_i \) with \( F_2 \) must necessarily be the unique ridge of \( F_2 \) containing \( K \setminus \{w\} \), a contradiction. Let \( R \) be the ridge of \( P \) containing \( K \cup \{v\} \). Since \( v \) is adjacent to every vertex in \( K \), \( R \) must be a \((d - 2)\)-simplex.

Consider the other facet \( F_2 \) containing the ridge \( R \). Note that every vertex in \( K \) is simple in \( F_2 \), and thus, \( v \) must be simple in \( F_2 \). Let \( e_v \) denote the unique edge incident with \( v \) not contained in \( F_1 \cup F_3 \) and let \( \mathcal{F}_e \) denote the set of facets containing \( e \). Any facet \( F \in \mathcal{F}_e \) must intersect \( F_1 \) at a \((d - 3)\)-face or at a ridge; in any case, it must miss precisely one vertex of \( K \), say \( w_F \). In this case, \( F \) intersects \( F_2 \) at a ridge, since \( d > 3 \) and \( F \) cannot intersect \( F_2 \) at a \((d - 3)\)-face. It thus follows that for each \( w_F \) there exists exactly one such facet \( F \). But there are only \( d - 2 \) vertices in \( K \) and we need \( d - 1 \) facets in \( \mathcal{F}_e \). This contradiction completes the proof of the lemma.

\[ \Box \]

Lemma 4.12. Any semisimple \( d \)-polytope, in which every vertex has degree at most \( d + 1 \), is actually simple.

Proof. We will show first that any facet \( F \) is also semisimple.

Suppose otherwise; that is there are two \((d - 2)\)-faces \( R_1 \) and \( R_2 \) in \( F \) intersecting at a \((d - 4)\)-face \( J \). (If \( J \) had lower dimension, every vertex in \( J \) would have excess at least 2 in \( F \).) Choose any \( u \in J \). Then \( u \) has excess 1 in \( F \) and so is nonsimple in \( P \); Remark 2.4 ensures that \( u \) is nonsimple in any facet containing it. Denoting by \( F_i \) the other facet containing \( R_i \) \((i = 1, 2)\), we see that \( u \) is nonsimple in \( F, F_1 \) and \( F_2 \), and it has only one neighbour outside \( F \), so \( u \) must be nonsimple in each \( R_i \). But then, the degree of \( u \) in \( F \) is the sum of the degree of \( u \) in \( R_i \) \((d - 1)\) plus the edges of \( u \in R_2 \setminus J \) \((two)\), a contradiction.

Any 3-dimensional semisimple polytope must be simple by Remark 2.4. The result now follows by induction on \( d \).

\[ \Box \]

Lemma 4.13. Let \( P \) be a semisimple \( d \)-polytope with excess \( d - 1 \). Then \( d = 5 \), there is a unique nonsimple vertex in \( P \), and its vertex figure is combinatorially equivalent to \( \Delta_{2,2} \). In particular, \( P \) is a Shephard polytope.

Proof. Recall that the excess is \( d - 1 \) only if \( d \) is odd. Note that \( d > 3 \), since Proposition 3.1 gives at once that all 3-dimensional semisimple polytopes are simple. By Proposition 3.1 and Theorem 3.3, we know that there exists a facet with excess at least \( d - 3 \), and by Lemma 3.2 that the facet (and every nonsimple facet) has excess exactly \( d - 3 \); note that since \( d - 1 \) is even, the excess of a facet cannot be \( d - 2 \). It follows that every nonsimple ridge has excess \( d - 4 \).

We begin with the important observation that when a facet \( F \) contains all the nonsimple vertices of \( P \), then \( P \) contains at most two nonsimple vertices. From Remark 2.5 it follows that every nonsimple vertex in \( F \) is adjacent to at least 2 vertices outside \( F \). The conclusion is now clear, since \( F \) has excess \( d - 3 \) and \( P \) has excess \( d - 1 \).
We consider first the case that $P$ contains a unique nonsimple vertex $u$; in fact, this is the only case which actually occurs. The vertex figure $P/u$ of $P$ at $u$ is a simple $(d - 1)$-polytope with $2d - 1$ vertices (cf. Lemma 2.2), from which Lemma 2.17(ii) forces $d \leq 5$. Since the case $d = 3$ has been excluded, we have $d = 5$, and the vertex figure $P/u$ has $2d - 1 = 3(d - 1) - 3$ vertices by virtue of Lemma 2.17. That is, $P/u$ is $\Delta_{2,2}$, which is depicted in Fig. 1(c).

Henceforth we assume that $P$ has at least two nonsimple vertices. Then any nonsimple vertex will be adjacent to at least one other nonsimple vertex. For, if a simple vertex were adjacent only to simple vertices, truncating it would give us a new simple facet with strictly less than $2(d - 1)$ vertices.

Next consider the case that $P$ contains exactly two nonsimple vertices, $u$ and $v$. They must be connected by an edge, and hence contained in some facet $F$. According to Theorem 4.5, $F$ contains exactly $d - 3$ vertices with excess one. So again $d - 3 = 2$. Consider a facet $F_u$ containing $u$ but not $v$, and a facet $F_v$ containing $v$ but not $u$ (cf. Remark 2.3). Since $F_u$ has excess $d - 3 = 2$, every vertex in $F_u$ sends exactly one edge outside $F_u$. The vertex figure $F_u/u$ of $F_u$ at $u$ is a simple $3$-polytope with six vertices; that is, it is a simplicial $3$-prism. This implies that the 3-faces $R_1$ and $R_2$ of $F_u$ arising from the opposite simplices of $F_u/u$ intersect only at $u$. Consequently, the intersection of the other two facets of $P$ containing the ridges $R_1$ and $R_2$ has dimension at most one, a contradiction.

Now consider the case that $P$ contains exactly three nonsimple vertices, say $v_1$, $v_2$ and $v_3$. Without loss of generality, we can assume that $v_2$ is adjacent to both $v_1$ and $v_3$. Thanks to Remark 2.3, we can find a facet $F_1$ containing $v_2$ and $v_3$ but not $v_1$, and a facet $F_3$ containing $v_1$ and $v_2$ but not $v_3$. Thanks to Theorem 4.5, we have $d - 3 = 2$, each $v_i$ has excess 1 in the appropriate facet, and the total excess is 4. Clearly $v_1$ and $v_3$ cannot both have excess 2, so Remark 2.5 tells us that one of them is adjacent to a nonsimple vertex outside the corresponding facet. The only possibility is that $v_1$ and $v_3$ are also adjacent. We may assume that $v_1$ and $v_3$ have excess 1, and that $v_2$ has excess 2. Let $F_2$ be a facet containing $v_1$ and $v_3$ but not $v_2$, and denote the ridge $F_i \cap F_j$ by $R_{i-1}$. Then $v_i \in R_i$ for each $i$, $v_i$ has excess 1 in $R_i$ for $i = 1, 3$, while $R_2$ is simple. Now $v_1$ has 6 neighbouring vertices, and every facet containing it is nonsimple. It follows that $v_1$ is contained in exactly 6 facets, one for each set of 5 of its neighbours. In particular, there are 4 facets containing $v_1$, $v_2$ and $v_3$, contrary to our previous observation.

If some facet contains a unique nonsimple vertex $u$, then by Theorem 4.5, the degree of $u$ in $P$ is at least $d - 3$, and there can be at most 3 nonsimple vertices in $P$. This case was dealt with in the previous paragraph, so we now assume that every nonsimple facet $F$ contains $d - 3$ nonsimple vertices, each with excess 1 in $F$. Again, there can be at most 2 nonsimple vertices outside $F$.

If $P$ contains exactly $d - 3$ nonsimple vertices, they must all belong to the same facet, a situation whose impossibility has already been demonstrated. If $P$ contains exactly $d - 1$ nonsimple vertices, they must all have excess 1, and the impossibility of this is given by 4.12. We are left with the case that $P$ contains exactly $d - 2$ nonsimple vertices; of these, one, say $v$, will have excess 2, while all the others have excess 1.

Fix a vertex $u$ with excess 1, and choose a facet $F$ containing $u$ but not $v$. Then $F$ must contain all $d - 3$ vertices with excess 1. The vertex $u$ has degree $d + 1$: its neighbours are $v$, the other $d - 4$ nonsimple vertices in $F$, and four simple vertices in $F$. In any facet which contains it, $u$ will be nonsimple and hence have $d$ neighbours. Thus every such facet contains all but one of the nonsimple vertices, and all four simple vertices in $F$. This leads to the absurdity that $u$ belongs to only $d - 3$ facets. This impossibility completes the proof. \hfill $\Box$

**Theorem 4.14.** Let $P$ be $d$-polytope with excess degree $d - 1$, where $d > 3$. Then $d = 5$ and either
(i) there is a single vertex with excess four, which is the intersection of three facets, and whose vertex figure is \( \Delta_{2,2} \); or

(ii) there are two vertices with excess two, the edge joining them is the intersection of two facets, and its underfacet is either \( \Delta_{1,1,2} \) or \( \Gamma_{2,2} \); or

(iii) there are four vertices each with excess one, which form a quadrilateral 2-face which is the intersection of two facets, whose underfacet is the tesseract \( \Delta_{1,1,1,1} \).

In all cases \( P \) is a Shephard polytope.

Proof. By Lemma 2.6 and Remark 4.8 any two nondisjoint facets in \( P \) intersect at a vertex, an edge with \( d = 5 \), a subbridge, or a ridge. Applying in order Lemmas 4.9 to 4.11 leads to cases (i), (ii) and (iii) respectively. Otherwise \( P \) is semisimple, an impossibility according to Lemma 4.13.

In case (i), the vertex figure is obviously \( \Delta_{2,2} \). In case (ii), the “edge figure” must be a simple 4-polytope with 12 vertices, so by Lemma 2.17 is either \( \Delta_{1,1,2} \) or \( \Gamma_{2,2} \).

Case (iii) is more complicated. Let \( F_1 \) and \( F_2 \) be as in Lemma 4.11. Since we only need to calculate the face figure, we can assume that \( P \) has a simple facet \( G \) containing all vertices except the four in \( K = F_1 \cap F_2 \). Each of these four is incident with two edges in \( F_1 \) and with two edges in \( F_2 \). Thus each ridge \( R_i = F_i \cap G \) has 8 vertices, and is simple, so must be either a cube or a 5-wedge. However \( F_i \) is the convex hull of \( R_i \cup K \), and is simple. If we remove an quadrilateral face from a copy of \( \Gamma_{2,2} \), the resulting graph is not the graph of a polytope. So for each \( i \), \( F_i \) is a copy of \( \Delta_{1,1,2} \), and \( R_i \) is a cube. Then \( F \), being the convex hull of \( R_1 \) and \( R_2 \), and simple, must be a tesseract. \( \square \)

Examples for cases (i), (ii) and (iii) are the pyramid over \( \Delta_{2,2} \), \( M_{3,2} \) and \( B_5 \), and \( A_5 \) respectively. All of these have simple vertices, so repeated truncation leads to more examples.

5. Characterisation of decomposable \( d \)-polytopes with \( 2d + 1 \) vertices

It is known [7, Theorem 7.1, page 39] or [19, Theorem 9] that the only decomposable \( d \)-polytope with \( 2d \) or fewer vertices is the prism. In this section, we characterise all the decomposable \( d \)-polytopes with \( 2d + 1 \) vertices. Characterising decomposable \( d \)-polytopes with \( 2d + 2 \) vertices appears to be a much harder exercise. When \( d = 3 \), there are already 11 examples, namely \([2, 245-255]\). For \( d = 4 \), a brief discussion of this problem is given in the next section.

Recall that a \( d \)-dimensional capped prism is the convex hull of a simplicial prism and a single extra vertex, say \( v_0 \), which lies beyond one of the simplex facets (and beneath all the other facets). Also recall that \( CP_{k,d} \) denotes the capped \( d \)-prism where \( k \) is the minimum dimension of any face of the simplicial prism whose affine hull contains the extra vertex. If \( k = 1 \), the capped prism will be (combinatorially) just another prism. If \( k = 2 \), then \( P \) is a pentasm, with \( d^2 + d - 1 \) edges. For \( k \geq 3 \), we can label the vertices as \( u_1, \ldots, u_d, v_0, \ldots, v_d \) in such a way that the edges, \( d^2 + d \) in total, are \([u_i, u_j] \) for all \( i, j \), \([v_i, v_j] \) for all \( i, j \), and \([u_i, v_i] \) for all \( 1 \leq i \leq d \).

Remark 5.1. In the case of a pentasm, one edge, say \([v_1, v_2] \) will be absent, and then \( u_1, u_2, v_0, v_1, v_2 \) will form a pentagonal face. In the case of a prism, the vertex \( v_0 \) will be absent.

Lemma 5.2. Let \( P \) be a decomposable \( d \)-polytope, \( F \) a facet of \( P \), and suppose that there are only two vertices of \( P \) outside \( F \). Then \( F \) is decomposable, and has Shephard’s property.
Proof. The two vertices outside $P$ are not enough to form a facet. Thus $F$ touches every facet, and so must be decomposable by Theorem 2.19. Moreover if some vertex $v \in F$ were adjacent to both vertices $x, y$ outside $F$, then the triangle $vxy$ would be an indecomposable face touching every facet, contrary to the decomposability of $P$. \qed

Lemma 5.3. (i) $\Sigma_3$ is not a facet of any decomposable 4-polytope with 9 vertices.
(ii) $\Delta_{2,2}$ is not a facet of any decomposable 5-polytope with 11 vertices.

Proof. (i) Let us entertain the possibility that $P$ is such a polytope, i.e. it has a facet $F$ of the type $\Sigma_3$. By Lemma 5.2, every vertex in $F$ belongs to only one edge not in $F$, and thus $P$ has $19=11+7+1$ edges. According to [16, Lemma 10(i)], only the pentasm has 9 vertices and 19 edges, and it does not have $\Sigma_3$ as a facet (cf. Remark 2.9).

(ii) Consider the possibility that $P$ is such a polytope, with $\Delta_{2,2}$ as a facet, say $F$. Again, by Lemma 5.2, every vertex in $F$ belongs to only one edge not in $F$, and so $P$ has $28=18+9+1$ edges. But then $P$ has excess degree only one, which is impossible (cf. Theorem 3.3). \qed

Remark 5.4. A similar argument proves that $\Delta_{2,3}$ is not a facet of any decomposable 6-polytope with 14 vertices; this will be useful to us in another context.

Lemma 5.5. Let $P$ be a decomposable $d$-polytope with $2d+1$ vertices.

(i) Every facet of $P$ with fewer than $2d-2$ vertices is indecomposable. Every facet of $P$ with exactly $2d-2$ vertices is a prism. Any facet of $P$ with $2d-1$ vertices is decomposable, and moreover every vertex therein belongs to only one edge outside the facet. No facet of $P$ has $2d$ vertices.
(ii) If some decomposable facet of $P$ is a capped prism, then $P$ is a capped prism.
(iii) If some decomposable facet of $P$ is a pentasm, then $P$ is a pentasm.
(iv) If every decomposable facet of $P$ is a prism, then $d=4$ and $P$ is $\Delta_{2,2}$.

Proof. (i) The first claim follows from [22, Prop. 6].

If a facet $F$ has $2d-2$ vertices then there are only three other vertices in $P$, which is not enough to form a facet. Thus $F$ touches every facet, and must be decomposable by Theorem 2.19, hence a prism by [19, Thm. 9].

For facets with $2d-1$ vertices, this is just Lemma 5.2.

The last assertion follows from the indecomposability of pyramids.

(ii) and (iii). Some facet $F$ is either a pentasm or a capped prism. We can write the vertex set of $F$ as $U \cup V$, where $U = \{u_i : 1 \leq i \leq d-1\}$, $V = \{v_i : 0 \leq i \leq d-1\}$ and the edges are as described above. Note that the geometric subgraphs determined by $U$ and $V$ are indecomposable.

Let $x$ and $y$ be the vertices outside $F$. We claim that one of them is adjacent only to vertices in $U$, while the other is adjacent only to vertices in $V$.

Suppose not. Then one of them, say $x$, is adjacent to both $u_i$ and $v_j$ for some $i, j$. Since $d-1 \geq 3$, $x$ is adjacent to at least 3 vertices in $F$. Thus we may assume that $i \neq j$. If $F$ is a capped prism, or if $\{i, j\} \neq \{1, 2\}$, then $x, y, v, u$ is a nonplanar 4-cycle sharing 2 vertices with $V$. If $\{i, j\} = \{1, 2\}$, then $x, v_j, v_3, u_3, u_i$ is an affinely independent 5-cycle sharing 2 vertices with $V$. Then either $V \cup \{x, u_i\}$ or $V \cup \{x, u_i, u_3\}$ is an indecomposable
subgraph containing all but \(d - 1\) or \(d - 2\) vertices of \(P\), hence touching every facet. This is inconsistent with the decomposability of \(P\) by Theorem 2.19.

So we have \(y\) (say) adjacent to every \(v_j\), and \(x\) adjacent to every \(u_i\). This describes the graph of \(P\) completely; it is the graph of either a pentasm or a capped prism.

We know already that a pentasm is determined uniquely by its graph, since it is the unique \(d\)-polytope with \(2d + 1\) and \(d^2 + d - 1\) edges, except for \(d = 3\) where \(\Sigma_3\) also share these properties; see Theorem 2.23.

In case \(F\) is a capped prism, a little more care is needed to determine the face lattice of \(P\), which we claim is also a capped prism; we may also assume that no other facet is a pentasm. Again, \(x\) is adjacent to everything in \(U\), and \(y\) to everything in \(V\), so it is reasonable to relabel \(x = u_d\) and \(y = v_d\). Thus \(P\) has the same graph as a capped prism, but we need to reconstruct the whole face lattice. Denote by \(E_i\) the edge \([u_i, v_i]\) and by \(k\) the minimum dimension of any face of the prism determined by \(E_1, \ldots, E_{d-1}\) whose affine hull contains \(v_0\). Without loss of generality, we may suppose that \(v_0\) lies in the affine hull of \(\bigcup_{j=1}^k E_i\), and not to the affine hull of any smaller collection of these edges. For \(j \leq k\), let \(F_j\) be the convex hull of \(\bigcup_{i \neq j} E_i\), and let \(S_j\) be the convex hull of \(\{v_i : i \neq 0, i \neq j\}\). For \(j > k\), let \(F_j\) be the convex hull of \(\bigcup_{i \neq j} E_i\cup \{v_0\}\). It is not hard to see that each of these is a facet of \(P\), as is \(S\), the convex hull of \(U \cup \{u_d\}\). In particular, \(F = F_d\). We claim there are no other facets.

Since each \(u_i\) is a simple vertex in \(P\), it must belong to exactly \(d\) facets of \(P\). The list just given contains \(d\) such facets for each \(u_i\). So any other facet must be contained in \(V \cup \{v_d\}\). All possible subsets are also accounted for by the list just given. We have now described the vertex-facet incidences of a capped prism.

(iv) Every decomposable facet is a prism; fix one such facet \(F\). Now there are three vertices outside \(F\); call them \(a, b, c\). Following the previous notation, \(F\) is the convex hull of two simplices \(U\) and \(V\) (both ridges in \(P\)). Note that if \(i \neq j\), then no vertex outside \(F\) can be adjacent to both \(u_i\) and \(v_j\). For if \(a\) were, then both \(a, u_i, u_j, v_j\) and \(a, u_i, v_j, v_j\) would be nonplanar 4-cycles with 3 vertices in common. Their union with \(U\) and \(V\) would then be an indecomposable graph containing all but two vertices of \(P\).

Consider the possibility that one of the external vertices, say \(a\), is adjacent to both \(U\) and \(V\). By the previous paragraph, there is a unique \(i\) with \(a\) adjacent to both \(u_i\) and \(v_i\). Now the degree of \(a\) is at most 4, so \(d = 4\), \(F\) is 3-dimensional and contains 3 quadrilateral and 2 triangular ridges. Suppose \(G\) is the other facet corresponding to one of the quadrilateral ridges. If \(G\) were indecomposable, its union with the 2 triangular faces of \(F\) would constitute an indecomposable graph containing at least 7 of the 9 vertices of \(P\). It follows that each of these 3 other facets must be a prism. This means that each of them contains only 2 of the vertices outside \(F\), each such vertex being adjacent to 2 vertices of the quadrilateral. This is only possible if \(a, b, c\) can be renamed \(w_1, w_2, w_3\) in such a way that they are adjacent to one another, and each \(w_i\) is adjacent to both \(u_i\) and \(v_i\). Thus \(P\) is simple and has the same graph as \(\Delta_{2,2}\).

The remaining case (which turns out to be impossible) is that each vertex outside \(F\) is adjacent only to \(U\) or \(V\). Without loss of generality, \(a\) and \(b\) are adjacent only to vertices in \(U\), and \(c\) is adjacent to every vertex in \(V\). Since \(d \geq 4\), we can choose an index \(i\) for which \(u_i\) is adjacent to both \(a\) and \(b\). We claim that \(c\) cannot be adjacent to both \(a\) and \(b\). If it were, then \(a, u_i, v_i, c\) and \(b, u_i, v_i, c\) would both be 4-cycles in the graph of \(P\); they cannot both be coplanar. So (at least) one of them, say the first, is an indecomposable geometric graph, which shares two vertices with the graph determined by \(U \cup \{a\}\), and shares two vertices with the graph determined by \(V \cup \{c\}\). Then union of these three graphs is indecomposable, and contains every vertex except \(b\). This would imply indecomposability of \(P\).
Without loss of generality, \([a, b] \) and \([b, c]\) are edges of \(P\), but \([a, c]\) is not. Having degree (at least) \(d\), \(a\) must then be adjacent to every vertex in \(U\). If \(b\) is adjacent to only \(d - 2\) vertices in \(U\), we will have only \(d^2 + d - 1\) edges altogether, and then \(P\) would be a pentasm. Otherwise, \(b\) is also adjacent to every vertex in \(U\), and we have the same graph as a capped prism. Either way, there are decomposable facets which are not prisms. □

**Theorem 5.6.** Let \(P\) be a decomposable \(d\)-polytope with \(2d + 1\) vertices. Then \(P\) is either a pentasm, a capped prism, \(\Sigma_3\) or \(\Delta_{2,2}\).

**Proof.** The case \(d = 3\) is easily checked via catalogues such as [2, Fig. 4]; see [18] for some examples of the reasoning. Henceforth assume \(d \geq 4\).

For \(d = 4\), suppose \(P\) has 9 vertices and is decomposable. Then it has a decomposable facet \(F\) with at most 7 vertices, according to Theorem 2.20. By Lemma 5.3(i), \(F\) cannot be \(\Sigma_3\), so it must be a prism, capped prism or pentasm. Every possibility is covered by the various cases in Lemma 5.5, so \(P\) is either \(\Delta_{2,2}\), a capped prism or pentasm.

Likewise if \(d = 5\). By Lemma 5.3(ii), no decomposable facet \(F\) can be \(\Delta_{2,2}\), so the only options are prisms, capped prisms and pentasms. All these cases have been dealt with in Lemma 5.5.

Finally, we can proceed with the induction. Suppose \(d \geq 6\), and that it has been established for every smaller dimension, and fix a \(d\)-dimensional decomposable polytope \(P\) with \(2d + 1\) vertices. Then \(P\) has decomposable facet \(F\), all with at most \(2d - 1\) vertices. By induction, each such facet must be a prism, capped prism or pentasm, and Lemma 5.5 then completes the proof. □

The last proof incidentally proves that a conditionally decomposable \(d\)-polytope must have at least \(2d + 2\) vertices. It also leads to the following.

**Corollary 5.7.** A polytope, whose graph is that of a capped prism, is a capped prism.

**Proof.** Let \(P\) be such a \(d\)-polytope; then \(P\) has \(d^2 + d\) edges. We can label the vertices as \(u_1, \ldots, u_d, v_0, \ldots, v_d\) as before. We know that \(u_1\) is simple, and that its neighbors are \(u_2, \ldots, u_d, v_0\). It follows that one facet containing \(u_1\) must contain \(u_2, \ldots, u_d\) but not \(v_0\). It is not hard to see that \(u_1, \ldots, u_d\) then form a facet, with Shephard’s property. Since pentasms, \(\Sigma_3\) and \(\Delta_{2,2}\) have strictly less than \(d^2 + d\) edges, \(P\) must be a capped prism. □

**6. Characterisations of some 4-polytopes with minimum number of edges**

Recall that a 4-polytope with ten vertices must have at least 21 edges. We give the complete characterisation of 4-polytopes with ten vertices and precisely 21 or fewer edges; we will need this in §7. See Fig. 2 for drawings of these polytopes. They are all all decomposable.

We can also characterise the decomposable 4-polytopes with ten vertices and 22 edges. There are 6 such examples, as well as (at least) two indecomposable 4-polytopes with ten vertices and 22 edges; we do not need the details of this characterisation here. More generally, we can characterise the polytopes with \(2d + 2\) vertices and minimal number of edges in all dimensions, but details will appear elsewhere [17].
Theorem 6.1. There are exactly four 4-polytopes with ten vertices and 21 or fewer edges: the polytopes $A_4$, $B_4$, $C_4$ and $\Sigma_4$, all of which have the same $f$-vector $(10, 21, 18, 7)$.

Proof. Let $P$ be a 4-polytope with ten vertices and a minimum number of edges. According to [5, Ch. 10] or Lemma 2.17, $P$ is not simple, so we can assume it has exactly 21 edges. The excess degree of $P$ is exactly two, and consequently, any facet must have excess degree at most one thanks to Lemma 3.2. This implies that any facet with an even number of vertices must be simple, and with an odd number of vertices must have excess one. In particular, any facet with eight or fewer vertices can only be a cube, a 5-wedge, a pentasym, $\Sigma_3$, a prism $\Delta_2$, a quadrilateral pyramid $M_2$, or a simplex. Note that no facet can have nine vertices, because it would have at least 14 edges, and $P$, being a pyramid thereover would have at least 23 edges, contrary to hypothesis. We now distinguish several cases.

Case 1. Some facet $F$ is the cube.

Denote by $u$ and $v$ the two adjacent vertices outside $F$.

Consider a quadrilateral $R_1$ of $F$ and the quadrilateral $R_2$ of $F$ opposite to $R_1$. Let $F_1$ and $F_2$ be the other facets containing $R_1$ and $R_2$, respectively. If $F_1$ is a pyramid with apex $u$, $F_2$ is another pyramid with apex $v$. The face lattice of the resulting polytope is easily reconstructed from the graph; it is $A_4$ (cf. Remark 2.10). If no quadrilateral of $F$ is contained in a pyramid, then each quadrilateral must be contained in a simplicial 3-prism which contains both $u$ and $v$. A moment’s reflection shows that this is impossible.

Case 2. Some facet $F$ is a 5-wedge.

First note that every vertex in $F$ has exactly one neighbour outside $F$. Denote by $u$ and $v$ the two adjacent vertices outside $F$.

Consider the other facet $F'$ containing one of the triangles. Suppose it has five vertices. There are two 3-polytopes with five vertices but only one has excess degree one, the pyramid over a quadrilateral. But in this case a vertex in the triangle would be adjacent to both $u$ and $v$, a contradiction. Thus, we can assume $F'$ is tetrahedron with apex $u$. By analogy the other facet containing the other triangle in $F$ is also a tetrahedron, with apex $v$.

As in the previous case, the other facet corresponding to either quadrilateral face of $F$ must be a pyramid or a prism. It follows that the two remaining vertices on $F$ must be adjacent to the same vertex outside $F$, say $u$. Thus, one quadrilateral is contained in a pyramid and the other is contained in a 3-prism. The face lattice can be reconstructed with no difficulty from the graph obtained; the polytope is $B_4$ (cf. Remark 2.11).
Case 3. Some facet \( F \) is the 3-pentasm.

First note that the other facet containing the pentagon can’t be a pyramid; otherwise the excess of \( P \) would be at least three. If it has eight vertices (i.e. is a 5-wedge), we can refer to Case 2. The remaining option is that it has seven vertices and must be another pentasm. Between the two pentasms we have 17 edges and the remaining vertex outside \( F \), say \( u \), must be simple. Consider the number \( f_2 \) of 2-faces in \( P \). The union of the pentasms gives 11 2-faces (cf. Remark 2.9) and the vertex \( u \) is contained in \((\binom{3}{2})\) 2-faces. Thus, \( f_2 \leq 17 \). Then Euler’s relation yields that the number of facets in \( P \) is at most \( 10 + 17 - 21 = 6 \). But every \( d \)-polytope with \( d + 2 \) facets is either a pyramid or simple [5, Sec. 6.1], and \( P \) is neither.

Case 4. Some facet \( F \) is \( \Sigma_3 \).

We may assume that no facet is a cube, 5-wedge or pentasm. Denote by \( w \) the vertex of excess degree one in \( F \).

From Theorem 4.5 it ensues that \( P \) is decomposable. Let us fix a triangular ridge \( T \) in \( F \) and consider the other facet \( F' \) containing it. If \( F' \) were a quadrilateral pyramid, then every vertex outside \( F \) would be simple in \( P \), and thus, every facet of \( P \) would touch \( F' \), implying by Theorem 2.19 that \( P \) is indecomposable, a contradiction.

Thus the other facets corresponding to triangular ridges in \( F \) can only be simplices or prisms. If both are prisms, they contain a common triangle (the three vertices outside \( F \)) and a common edge (the one from \( w \) running outside out of \( F \)), meaning they must lie on the same 3-dimensional affine space. This is also impossible.

Suppose both are simplices. Then one of the vertices outside \( F \) will be adjacent to all three vertices in one the two triangles, another will be adjacent to all three vertices in the other triangle, and the third will be adjacent to both of the other vertices in \( F \). It is routine to verify that \( P \) has the same face lattice as \( \Sigma_4 \).

In the remaining case, there must be exactly seven edges leaving \( F \) and the three vertices outside \( F \) are pairwise adjacent. We can denote one of them by \( a \), which sends three edges to \( F \), and the remaining two, by \( b \) and \( c \), which each send two edges to \( F \). Note that the vertex \( a \) must be adjacent to the vertex \( w \) by Remark 2.5. The only possible pattern of connections between \( F \) and the external vertices gives us the graph and face lattice of \( C_4 \) (cf. Remark 2.12).

Case 5. Every facet is a prism, a quadrilateral pyramid or a simplex.

By Theorem 4.5 no every facet can be simple. So it suffices to show that no facet can be a pyramid in this case. Suppose some facet \( F \) is a pyramid based on a quadrilateral ridge \( Q \). Denote by \( u \) the apex of this pyramid and let \( G \) be the other facet containing \( Q \). Then \( G \setminus Q \) contains at most two vertices. If \( u \) has degree six in \( P \), then every other vertex in \( P \) is simple. If \( u \) has degree five in \( P \), then the unique vertex adjacent to \( u \) outside \( F \) must be nonsimple (cf. Remark 2.5). In either case, every vertex in \( Q \) is simple in \( P \). Thus all the neighbours of vertices of \( Q \) are in \( F \cup G \). If we remove the vertices in \((G \setminus Q) \cup \{u\}\) from the graph of \( P \), the resulting graph will be disconnected, contrary to Balinski’s Theorem [23, Thm. 3.14].

7. \((f_0, f_1)\)-projections of 5-polytopes

We characterise all pairs \((f_0, f_1)\) for which there exists a 5-polytope with \( f_0 \) vertices and \( f_1 \) edges. In particular, we show that \( \min E(f_0, 5) = \frac{1}{2}(5f_0 + 3) \) if \( f_0 \) is odd, and \( \min E(f_0, 5) = 2.5f_0 \) if \( f_0 \) is even and not eight. It is well known that \( \min E(8, 5) = 22 \).

Lemma 7.1. Besides the simplex, there is a simple 5-polytope with \( f_0 \) vertices if, and only if, \( f_0 \) is even and \( f_0 \geq 10 \).
Theorem 7.2. There is a nonsimple 5-polytope with \( f_0 \) vertices and \( f_1 \) edges, if, and only if,

\[
\frac{1}{2} (5f_0 + 3) \leq f_1 \leq \binom{f_0}{2}
\]

and \((f_0, f_1) \neq (9, 25), (13, 35)\).

Proof. Clearly we must have \( 2.5f_0 \leq f_1 \leq \binom{f_0}{2} \). Any pair with \( 2f_1 - 5f_0 = 1 \) or 2 is impossible by the excess theorem (Theorem 3.3). The case \((9, 25)\) is also impossible by [16, Theorem 19]; the proof of that result becomes somewhat shorter if we set \( d = 5 \).

All other pairs except \((13, 35)\) are possible. How to construct them? The first step is to look at all 4-polytopes and construct pyramids over them. If a \( d \)-polytope has \( f_0 \) vertices and \( f_1 \) edges, then a pyramid thereover is a \((d + 1)\)-polytope with \( f_0 + 1 \) vertices and \( f_0 + f_1 \) edges. Grünbaum showed that \( E(6, 4) = [13, 15], E(7, 4) = [15, 21], E(8, 4) = \{16\} \cup [18, 28], E(9, 4) = [18, 36], E(10, 4) = [21, 45], \) and \( E(10, 4) = [2f_0, \binom{f_0}{2}] \) for all \( f_0 \geq 11 \).

For \( f_0 \leq 11 \), building pyramids on 4-polytopes with \( f_0 - 1 \) vertices shows that \( E(7, 5) = [19, 21], E(8, 5) \supseteq [22, 28], E(9, 5) \supseteq \{24\} \cup [26, 36], E(10, 5) \supseteq [27, 45], \) and \( E(11, 5) \supseteq [31, 55] \).

Thus we have all the alleged examples for \( f_0 \leq 11 \) except \((10, 25), (11, 29) \) and \((11, 30)\). But these are exemplified by the prism, the pentasem and the capped prisms respectively. Note that these three examples all come from slicing a simple vertex off something else: a simplex, a triplex \( M_{2,3} \) or a bipyramid over a 4-simplex, respectively.

For \( f_0 \geq 12 \), pyramids give all examples with \( f_1 \geq 3f_0 - 3 \). More precisely, for each such \( f_0 \), we know that \( E(f_0 - 1, 4) = [2f_0 - 2, \binom{f_0 - 1}{2}] \) and hence \( E(f_0, 5) \supseteq [3f_0 - 3, \binom{f_0}{2}] \). The cases \( \frac{1}{2} (5f_0 + 3) \leq f_1 < 3f_0 - 3 \) require a little explanation.

Note that if \( f_1 < 3f_0 \), then a 5-polytope with \( f_0 \) vertices and \( f_1 \) edges must have at least one simple vertex.

So suppose \( f_0 \geq 12 \) and \( \frac{1}{2} (5f_0 + 3) \leq f_1 < 3f_0 - 3 \). Let \( k \) be the smallest integer \( \geq \frac{1}{2} (3f_0 - 3 - f_1) \). Then \( f_1 - 3f_0 + 2k \) is either \(-3\) or \(-2\). Now put \( f'_1 = f_1 - 10k \) and \( f'_0 = f_0 - 4k \). Clearly \( 3f'_0 - 3 \leq f'_1 < 3f'_0 \), so there is a polytope with \( f'_1 \) edges and \( f'_0 \) vertices, at least one of which is simple, unless \( f'_1 = 25 \) and \( f'_0 = 9 \). Truncating a simple vertex \( k \) times then gives a polytope with \( f_1 \) edges and \( f_0 \) vertices.

In the case \( f'_1 = 25 \) and \( f'_0 = 9 \), we have \( 2f_1 - 5f_0 = 5 \). The case \((17, 45)\) comes from slicing a simple edge off a capped prism. All remaining cases with more than 17 vertices then come from repeated truncation of simple vertices.

It only remains to prove the unfeasibility of \((f_0, f_1) = (13, 35)\). This has a lengthy proof, to be given separately.

\( \square \)

We state this last impossibility result here, but postpone the proof of this theorem to Appendix A.

Theorem 7.3. There is no 5-polytope with 13 vertices and 35 edges.
Appendix A. Unfeasibility of $(f_0, f_1) = (13, 35)$

This appendix completes the proof of Theorem A.5. We proceed by examining the numbers of vertices in the facets of such a hypothetical polytope. The proof of Kusunoki and Murai [13], is quite different, considering instead the possible degrees of the vertices of the polytope. We first require some lemmas.
Lemma A.1. A 5-polytope \( P \) with 13 vertices and 35 edges does not contain a facet with ten vertices.

Proof. Note that the excess of \( P \) is five. Suppose \( F \) is a facet of \( P \) with ten vertices; then \( F \) has 22 or 21 edges. Let \( x, y, z \) denote the vertices in \( P \cap F \). We can assume that \( x \) and \( y \), and \( x \) and \( z \) are adjacent. Furthermore, every other facet of \( P \) intersects \( F \) at a ridge; otherwise there would be at least thirteen edges between \( F \) and \( x, y, z \), giving a minimum of 36 edges of in \( P \). This property implies that the number of facets other than \( F \) in \( P \) coincides with the number of 3-faces of \( F \), and that every vertex simple in \( F \) is simple in \( P \).

Case 1. The facet \( F \) has 22 edges.

There are 13 edges and three vertices outside \( F \).

First suppose that \( y \) and \( z \) are not adjacent. Then there is a unique vertex \( u \) in \( F \) sending two edges outside \( F \). The vertices \( x, y, z \) are all simple. If there were another nonsimple vertex in \( F \), then, by Remark 2.5, the vertex would have two neighbours outside \( F \). Consequently, the vertex \( u \) is the unique nonsimple vertex of \( F \) (and of \( P \), with degree ten in \( P \). But then, Lemma 2.2 would imply that the vertex figure of \( P \) at \( u \) is a simple 4-polytope with ten vertices, which is ruled out by Lemma 2.17.

Now suppose that \( y \) and \( z \) are adjacent, with \( y \) having degree six in \( P \). In this case, there are ten edges leaving \( F \). Since the vertices \( x \) and \( z \) are simple, there is a facet \( G \) in \( P \) containing \( x \) and \( z \) but not \( y \). By Remark 2.5 the six vertices in \( R := G \cap F \) are simple in \( P \). The ridge \( R \) must therefore be a simplicial 3-prism, according to Lemma 2.17. Looking at the structure of \( F \), we see that nine edges of \( F \) come from \( R \) and that there are six edges between \( R \) and the four vertices in \( F \setminus R \). But this counting leaves \( 7 > \binom{4}{2} \) edges to be distributed among the four vertices in \( F \setminus R \), a contradiction.

Case 2. The facet \( F \) has 21 edges.

There are 14 edges and three vertices \( x, y, z \) outside \( F \). We can assume that \( x \) and \( y \), and \( x \) and \( z \) are adjacent. Furthermore, \( F \) is one of the four polytopes presented in Theorem 6.1, all of which has \( f \)-vector \((10,21,18,7)\).

Suppose that \( y \) and \( z \) are not adjacent. Without loss of generality, there are two possible configurations: either the vertex \( x \) or the vertex \( y \) has degree six in \( P \). We estimate the number of 3-faces of \( F \). In the first configuration, there are at least six facets containing \( x \), since \( x \) is nonsimple. And there must be two further facets in \( P \), one containing \( y \) but not \( x \) or \( z \), and another one containing \( z \) but not \( x \) or \( y \), which contradicts the seven 3-faces of \( F \). Analogously, in the second configuration, there are at least six facets containing \( y \), since \( y \) is nonsimple. And there must be two further facets in \( P \), one containing \( x \) and \( z \) but not \( y \), and another one containing \( z \) but not \( x \) or \( y \), which contradicts the seven 3-faces of \( F \).

Suppose that \( y \) and \( z \) are adjacent. Then there are exactly eleven edges leaving \( F \). Since every vertex simple in \( F \) is simple in \( P \), there must be a unique nonsimple vertex, say \( u \), in \( F \) sending two edges outside \( F \). Without loss of generality, there are two possible configurations: either the vertices \( x \) and \( y \) have both degree six in \( P \) or the vertex \( y \) has degree seven in \( P \). We first prove the following claim.

Claim 1. The facet \( F \) cannot be \( A_4 \).

Proof. Refer to Fig. 2(a) and Remark 2.10. Consider both configurations mentioned above. The other nonsimple vertex \( v \) in \( F \) must be adjacent to a nonsimple vertex outside \( F \), say \( y \), by Remark 2.5. This in turn implies that \( y \) is contained in each of the five facets obtained from the five 3-faces in \( F \) containing \( v \). The sixth facet containing \( y \)
must then be the one obtained from the 3-cube. Consequently, the other facet containing the quadrilateral pyramid with $u$ as apex must contain $x$ and $z$ but not $y$, and it must be a $(3,1)$-triplex (cf. Theorem 2.22), with both $x$ and $z$ adjacent to $u$. Without loss of generality, assume the vertex $z$ is adjacent to $a_1$ and $a_2$, and $x$ is adjacent to $a_3$ and $a_4$. This is in turn implies that $y$ is adjacent to at least three vertices in \( \{b_1, b_2, b_3, b_4\} \) and $x$ is adjacent to at most one vertex in \( \{b_1, b_2, b_3, b_4\} \). In the first configuration, the other facet $G$ containing the quadrilateral pyramid $v b_2 b_3 b_4$ contains $x$ and $y$ but not $z$, with $x$ only having degree two in $G$, a contradiction. In either configuration, the edge $uz$ must be contained in exactly four facets containing $u$; three of such facets must come from simplicial 3-prisms in $F$, with at least one of them, say $Q$, also containing the vertices $x$ and $y$. Observe that the fifth facet containing $z$ must be the one obtained from the 3-cube. For the second configuration, the facet $G$ is a pyramid over $v b_2 b_3 b_4$, and without loss of generality, we can assume that the facet $Q$ contains the simplicial 3-prism $v b_2 b_3 u a_2 a_3$. In this case, $Q$ would have nine vertices, 19 edges, and two nonsimple vertices $u$ and $y$ which are not adjacent. From Theorem 2.23 it ensues that $Q$ would be the pentasm, but the two nonsimple vertices of the 4-pentasm are adjacent. This contradiction concludes the proof of the claim. □

We now estimate the number of 3-faces of $F$.

In the first configuration, since $z$ is simple, there are exactly five facets containing $z$, each containing $x$ or $y$. Since $x$ and $y$ are nonsimple and $F$ has exactly seven 3-faces, every facet containing $x$ but not $z$ must contain $y$, and the same for $y$; otherwise, to guarantee six facets for each $x$ and $y$, apart from the facets containing $z$, there would be at least three more facets: one containing $x$ but not $z$ or $y$, one containing $x$ and $y$ but not $z$, and one containing $y$ but not $x$ or $z$. Thus $F$ must be $A_4$, as it doesn’t contain simplices as 3-faces, which in turn contradicts Claim 1.

In the second configuration, there are exactly five facets containing $x$, since $x$ is simple. Each of the facets containing $x$ contains a 3-face of $F$ with at least five vertices. Since $z$ is simple, there is exactly one facet containing $z$ but not $x$; this facet contains $y$ and a 3-face of $F$ with at least five vertices. Thus we have already counted six 3-faces of $F$ with five or more vertices, so $F$ cannot be $B_4$ or $\Sigma_4$ as they both have at most five 3-faces with five or more vertices; see Remarks 2.11 and 2.13. Moreover, from Claim 1 it follows that $F$ is $C_4$. Refer to Fig. 2(c) and Remark 2.12. The other nonsimple vertex $v$ in $F$ must be adjacent to the nonsimple vertex $y$ outside $F$, by Remark 2.5. This in turn implies that $y$ is contained in each of the five facets obtained from the five 3-faces in $F$ containing $v$. The sixth facet containing $y$ must then be the one obtained from the copy of $\Sigma_3$ which does not contain $v$: $u b_1 b_2 a_1 a_2 a_3$. Let $G$ be the other facet containing the simplicial 3-prism which does not contain $u$ or $v$: $a_1 a_2 a_3 a_4 a_5 a_6$. Then the vertices $x$ and $z$ send each three edges to $G$, and $y$ sends its five edges into the four vertices of the simplex of $C_4$: $uv b_1 b_2$. This is a contradiction because the other facet containing the simplex must be another simplex. This final contradiction rules out this configuration, and completes the proof of the lemma. □

**Lemma A.2.** Any 4-polytope $P$ with 11 vertices and 23 edges must contain a facet with 8 vertices.

**Proof.** Suppose otherwise. Since the excess of $P$ is two, Theorem 4.5 tells us that it is decomposable and has a facet with excess one. Moreover, Lemma 3.2 tells us that no facet has excess two, so every facet of $P$ with an odd number of vertices has excess one.

If $P$ were a pyramid, it would have at least $(10 \times 3)/2 + 10 = 25$ edges. If instead $P$ had a 3-face $F$ with nine vertices, $F$ would have more than $9 \times 3/2$ edges and $P$ would have at least $24 = 14 + 9 + 1$ edges. Thus the largest
3-face in \( P \) has at most seven vertices. Suppose there is a 3-face \( F \) with seven vertices. Then \( F \) must have 11 edges, and by Theorem 2.23 is a 3-pentasm. Denote by \( e_a \) the number of edges among the four vertices outside \( F \) and by \( e_b \) the number of edges between \( F \) and the four vertices in \( P \setminus F \). Then \( e_a + e_b = 12 \), \( 2e_a + e_b \geq 4 \times 4 \) and \( e_b \geq 7 \). This implies that \((e_a, e_b) = (4, 8), (5, 7)\). In both cases, the other 3-face of \( P \) containing the pentagon in \( R \) must contain at least three of the vertices in \( P \setminus F \), giving a facet with at least eight vertices, a contradiction.

Finally, suppose there is a facet \( F \) with at least five vertices. Then it has excess one and is a quadrilateral pyramid with apex \( u \) (cf. Theorem 2.22). Suppose there is a nonsimple vertex \( v \) in \( P \setminus F \). Then, the degrees of \( u \) and \( v \) in \( P \) are five, \( u \) and \( v \) are adjacent by Remark 2.5, and there are exactly five edges between \( F \) and \( P \setminus F \). Consider the other facet \( G \) containing the quadrilateral \( Q \) of \( F \). It follows that either \( G \) is a quadrilateral pyramid with apex \( v \) or a simplicial 3-prism. But then, removing either \( u \) and \( v \) in the former case, or \( u \) and the two vertices \( G \setminus F \) in the latter case would disconnect \( Q \), contradicting Balinski’s Theorem [23, Thm. 3.14]. If instead every vertex in \( P \setminus F \) is simple. Then, the degrees of \( u P \) are six and there are exactly six edges between \( F \) and \( P \setminus F \). Again, considering the other facet \( G \) containing the quadrilateral \( Q \) of \( F \), it follows that \( G \) is a simplicial 3-prism. But then, removing \( u \) and the two vertices \( G \setminus F \) would disconnect \( Q \), contradicting Balinski’s Theorem [23, Thm. 3.14]. □

Lemma A.3. A 5-polytope \( P \) with 13 vertices and 35 edges does not contain a facet with 11 vertices.

Proof. Let \( F \) denote a facet of \( P \) with 11 vertices. The facet cannot have more than 23 edges, since there must be at least twelve edges outside \( F \). Thus the facet \( F \) has 22 or 23 edges. In either case, any other facet \( G \) of \( P \) intersects \( F \) at a ridge; indeed, any intersection of smaller dimension would require at least thirteen edges to leave \( F \). We consider each subcase at a time.

Case 1. The facet \( F \) has 22 edges.

There is a unique vertex \( u \) in \( F \) sending two edges outside \( F \). Since \( u \) is nonsimple in \( P \), there must be five ridges in \( F \) containing \( u \), a contradiction.

Case 2. The facet \( F \) has 23 edges.

Lemma A.2 assures that there is a 3-face of \( F \) with eight vertices, in which case, the other facet in \( P \) containing it would have ten vertices, contradicting Lemma A.1. □

Lemma A.4. Suppose that \( P \) is a 5-polytope with 13 vertices and 35 edges, and that \( F \) is a facet with 8 vertices. Then \( F \) is a prism.

Proof. Clearly the excess of \( F \) can only be 0, 2 or 4. From [5, §10.4], it follows that \( F \) cannot have 17 edges, i.e. its excess cannot equal 2 (see [16, Thm. 19] for a higher dimensional version of this).

So suppose that \( F \) has excess 4, i.e. there are 18 edges in \( F \). Then \( F \) is indecomposable by [19, Thm. 9], and the five vertices \( \{x_1, x_2, x_3, x_4, x_5\} \) outside \( F \) cannot form a facet, as \( P \) would then have at least 36 edges. Hence the whole polytope is indecomposable by Theorem 2.19, and thus \( F \) cannot have Shephard’s property by Theorem 2.18. It follows that there are 9 edges running out of \( F \), 8 edges joining the \( x_i \), each \( x_i \) is simple. However every nonsimple vertex of \( P \) which is contained in \( F \) must send two edges outside \( F \), which means there would be a unique nonsimple vertex in \( P \) (actually in \( F \)), which can only have degree ten in \( P \). But then, Lemma 2.2 gives that the corresponding vertex figure of \( P \) would be a simple 4-polytope with 10 vertices, which is ruled out by Lemma 2.17.

The only remaining possibility is that \( F \) is simple, i.e. a simplicial 4-prism. □
**Theorem A.5.** There is no 5-polytope with 13 vertices and 35 edges.

**Proof.** Let $P$ denote a 5-polytope with 13 vertices and 35 edges. Throughout, $F$ will denote a fixed facet with the maximal number of vertices. We investigate various cases, based on this number.

**Case 1.** The facet $F$ has 12 vertices.

This case is impossible. The facet $F$ would have at least $(12 \times 4)/2$ edges, causing $P$ to have at least 36 edges.

**Case 2.** The facet $F$ has 10 or 11 vertices.

These cases are dealt with by Lemmas A.1 and A.3, respectively.

**Case 3.** The facet $F$ has nine vertices.

From Lemma 3.2 it ensues that the facet $F$ has excess zero (18 edges), excess two (19 edges) or excess four (20 edges). Denote by $x_1, x_2, x_3$ and $x_4$ the vertices outside $F$. We consider each subcase at a time.

**Subcase 3.1** The facet $F$ has 20 edges.

If every vertex in $P \setminus F$ were simple in $P$, then by Remark 2.5 every nonsimple vertex of $P$ which is contained in $F$ would send two edges outside $F$. In this case, there would be a unique nonsimple vertex in $F$ (and in $P$), say $u$, with degree ten in $P$. But then, Lemma 2.2 gives that the vertex figure of $P$ at $u$ would be a simple 4-polytope with ten vertices, which is ruled out by Lemma 2.17.

Consequently, there is a unique nonsimple vertex in $P \setminus F$, say $x_1$, which has degree six in $P$ and is adjacent to to every nonsimple vertex lying in $F$. Furthermore, there are exactly nine edges between $P \setminus F$ and $F$, and exactly six edges shared among the vertices $x_1, x_2, x_3$ and $x_4$. We count the number of facets involving the vertices $x_1, x_2, x_3$ and $x_4$. There are exactly five facets containing $x_3$, since $x_3$ is simple. Since $x_4$ is simple, there is exactly one facet containing $x_4$ but not $x_3$; this facet also contains $x_1$ and $x_2$. In this way, we have counted all the facets containing the vertices $x_2, x_3$ and $x_4$. Out of of the aforementioned six facets, the nonsimple vertex $x_1$ cannot be contained in all of them, since there must exist a facet containing $x_3$ but not $x_1$ (cf. Remark 2.3). But then the nonsimple vertex $x_1$ is only contained in five facets, a contradiction.

**Subcase 3.2** The facet $F$ has 19 edges.

Then $F$ is a 4-pentasm. There is 3-face $R$ of $F$ which is a 3-pentasm (cf. Remark 2.9), i.e. it has 7 vertices and 11 edges. Denote by $G$ the other facet containing $R$. We know that $G$ cannot have 10 or more vertices, and it cannot be $\Delta_{2,2}$ because it contains a pentagonal face. So if it has 9 vertices, then it has at least 19 edges. But then $F \cup G$ contains at least 27 edges, while the two vertices outside $F \cup G$ must contribute at least 9 more edges. Finally, consider the possibility that $G$ contains 6 vertices. Then $G$ is a pyramid over $R$, $F \cup G$ contains 25 edges, and the three vertices outside $F \cup G$ must contribute at least 12 more edges.

**Subcase 3.3** The facet $F$ has 18 edges.

Theorem 2.23 tells us that $F$ is $\Delta_{2,2}$, and thus, has six 3-faces, all of them being simplicial 3-prisms (cf. Fig. 1(c)).

There are 17 edges outside $F$. Denote the vertices outside by $x_1, x_2, x_3$ and $x_4$. Any subset of three of these vertices will belong to at least 12 edges; hence any one of them can be adjacent to at most five vertices in $F$. It follows that for any 3-face $R$ in $F$, the corresponding other facet is never a pyramid over $R$. This in turns implies...
that each \( x_i \) is adjacent to at least two of the others (if there were say only one edge from \( x_1 \) to the other \( x_i \), the only facet containing \( x_1 \) but not this edge would be a pyramid with apex \( x_1 \) and base in \( F \)).

For any ridge \( R \) in \( F \), the corresponding other facet must have either 8 or 9 vertices, and so must be either a 4-prism or a copy of \( \Delta_{2,2} \). So if \( T \) is any triangular face in \( F \), any one of the external vertices \( x_i \) must be adjacent to either 0, 1 or 3 of the vertices in \( T \); 2 is not possible. Recall that the 3-faces in \( \Delta_{2,2} \) can be partitioned into two groups of three; any two 3-faces within the same group intersect in a triangle, and any two 3-faces from different groups intersect in a quadrilateral. The structure of the other facets corresponding to ridges in \( F \) is then essentially determined. Without loss of generality we can assume that for \( i = 1, 2, 3 \), \( x_i \) is adjacent to \( a_i, b_i \) and \( c_i \), but not to \( a_j, b_j \) or \( c_j \) for \( j \neq i \), and that \( x_1, x_2, x_3 \) are mutually adjacent. Three of these facets are 4-prisms, three of them are copies of \( \Delta_{2,2} \), and none of them contain \( x_4 \). The last assertion will lead to an absurdity.

Suppose \( G_1 \) is any facet containing \( x_4 \). Clearly \( G_1 \) intersects \( F \) at a non-empty face of dimension < 3. Every vertex in \( F \) is adjacent to precisely one of \( x_1, x_2, x_3 \); this rules out the possibility that \( F \cap G_1 \) is a single vertex or an edge. Were \( F \cap G_1 \) a quadrilateral, each of its vertices would be adjacent to two vertices outside \( F \), and there would be at least 13 edges running out from \( F \). But then \( x_4 \) could only be adjacent to one of the other \( x_i \), contrary to our previous conclusion. Thus \( F \cap G_1 = \{x_1, x_2, x_3, x_4\} \) must be a triangle.

Now let \( G_2 \) be another facet containing \( x_4 \), but not containing \( T_1 \). Then \( F \cap G_2 = \{x_2, x_3, x_4\} \) is another triangle, each of whose vertices is also adjacent to two vertices outside \( F \). Thus there would be at least 10 edges running out from \( F \) from \( T_1 \cup T_2 \), and another 4 from the remaining vertices. But then \( x_4 \) could not be adjacent to any of the other \( x_i \).

**Case 4.** The facet \( F \) has eight vertices.

Observe that every two simplicial 3-prisms in \( F \) intersect in a quadrilateral 2-face. Consider one of the simplicial 3-prisms \( R_1 \) in \( F \). If the other facet \( F_1 \) containing \( R_1 \) is a pyramid, then \( F_1 \cup F \) contains 22 edges and there are four vertices outside \( F_1 \cup F \) incident to 13 edges; this is impossible since four vertices are incident to at least 14 edges. So \( F_1 \) (and every facet containing of one of the simplicial 3-prisms in \( F \)) must be a simplicial prism. Let \( x_1 \) and \( x_2 \) denote the vertices in \( F_1 \setminus R_1 \). Consequently, there are 12 edges outside \( F_1 \cup F \) incident to the remaining three vertices \( x_3, x_4 \) and \( x_5 \). So every pair in \( \{x_3, x_4, x_5\} \) is adjacent, each vertex in \( \{x_3, x_4, x_5\} \) is simple, and at least one vertex in \( \{x_3, x_4, x_5\} \) is adjacent to a vertex in \( \{x_1, x_2\} \). This in turn implies that every vertex \( \{x_1, x_2, x_3, x_4, x_5\} \) sends at most three edges into \( F \). The facet \( F \) has four different simplicial 3-prisms, say \( R_1, R_2, R_3 \) and \( R_4 \), and correspondingly, four different simplicial 4-prisms \( F_1, F_2, F_3 \) and \( F_4 \) containing these 3-faces. If a vertex in \( \{x_1, x_2, x_3, x_4, x_5\} \) were contained in more than one of the facets \( F_1, F_2, F_3 \) and \( F_4 \), it would send at least four edges into \( F \), a contradiction. Consequently, the pairs of vertices in \( F_1 \setminus R_1 \) must be pairwise disjoint, which is clearly a contradiction.

**Case 5.** The facet \( F \) has seven vertices.

The minimum number of edges of \( F \) is 15 (cf. Theorem 2.22), in which case \( F = M_{3,1} \), that is, a pyramid over a simplicial 3-prism \( R \). Suppose \( F \) has 15 edges and consider the other facet \( F_1 \) containing \( R \). Then \( F_1 = M_{3,1} \), and \( F \cup F_1 \) contains 21 edges. Hence there are 14 edges outside \( F \cup F_1 \) which are incident to five vertices of degree at least 5, which is impossible. Thus \( F \) has at least 16 edges. If \( F \) has exactly 16 edges, then, analysing [4, Fig. 5] we get that there are exactly two such polytopes, each being a pyramid over a 3-polytope with six vertices and ten edges; see [2, Fig. 3] for a description of the 3-polytopes. As before, let \( R \) be the 3-face with 10 edges in \( F \). The
other facet $F_1$ containing $R$ is also a pyramid, and thus, and $F \cup F_1$ contains 22 edges. Hence there are 13 edges outside $F \cup F_1$ which are incident to five vertices of degree at least five, which is again impossible.

The facet $F$ has at least 17 edges. Denote by $e_b$ the number of edges between the the six vertices outside $F$ and the facet $F$, and by $e_a$ the number of edges among the six vertices outside $F$. Then $e_b \geq 7$. On the other hand, $e_a + e_b \leq 18$ and $2e_a + e_b \geq 6 \times 5$, implying $e_a \geq 12$ and $e_b \leq 6$, a contradiction.

**Case 6.** The facet $F$ has six vertices.

The minimum number of edges of $F$ is 13 (cf. Theorem 2.22), in which case $F = M_{2,2}$, that is a two-fold pyramid over a quadrilateral. Suppose $F$ has 13 edges and consider the other facet $F_1$ containing the pyramid $R$ over a quadrilateral. Then $F_1 = M_{2,2}$, and $F \cup F_1$ contains 18 edges. Hence there are 17 edges outside $F \cup F_1$ which are incident to six vertices of degree at least 5. Let $S$ denote the set of the six vertices outside $F \cup F_1$. Denote by $e_b$ the number of edges between $S$ and $F \cup F_1$, and by $e_a$ the number of edges among the six vertices in $S$. Then $e_a + e_b = 17$ and $2e_a + e_b \geq 6 \times 5$, implying $e_a \geq 13$ and $e_b \leq 4$. However, having $e_b \leq 4$ contradicts Balinski’s Theorem [23, Thm. 3.14], since removing at most four vertices in $S$, those incident to the edges counted in $e_b$, disconnects the polytope graph.

Thus $F$ has at least 14 edges. From [5, Thm. 6.1.4] it ensues that there are exactly three 4-polytopes with 6 vertices and at least 14 edges, all of them being 2-simplicial, i.e. every 2-face is a simplex. This implies that $P$ is 2-simplicial, and therefore [6, Thm. 1.4]) gives the final contradiction as it states that $P$ would have at least 50 edges.

**Case 7.** The facet $F$ has five vertices.

The lower bound theorem for simplicial polytopes (cf. [6, Thm. 1.4]) gives the final contradiction as it states that $P$ would have at least 50 edges. \qed

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