Abstract

We investigate 4d SU(2) lattice gauge theory with Regge–Einstein quantum gravity on a dynamically coupled Regge skeleton. To overview the phase diagram we perform simulations on a small 2·4³ system. Evidence for an entropy–dominated disordered, an entropy–dominated ordered and an ill–defined region is presented.
In recent years various exploratory simulations of pure 4d quantum gravity have been carried out, see ref. [1, 2] for recent overviews. In ref. [3] one of the authors reported numerical evidence for an “entropy–dominated” phase, where the partition function is well–defined, although the Regge–Einstein action is unbounded. Essentially, these results were confirmed by Beirl et al.,[4], where also some universality under changes of the measure was reported. The question of possible physical relevance of the entropy-dominated phase was first addressed in ref. [5], and it was suggested to show that the entropy–dominated phase can accommodate hadronic masses $m_h$ many order of magnitude smaller than the Planck mass $m_P$. SU(2) lattice gauge theory was suggested to represent the matter fields, because it stands out as the simplest 4d asymptotically free quantum field theory. In ref. [6], see ref. [7] for a concise summary, a research program along this line of thought was outlined, and supported by exploratory numerical results from an $2^2 \times 4^3$ lattice. For the coupled system the entropy dominated phase survives, and within it an order–disorder transition could be located. In a continuum or almost continuum limit, if it exists, the transition will have the usual interpretation of a deconfining phase transition, the ordered phase defines a quark–gluon plasma and in the disordered phase quarks are confined. To provide modest evidence for a consistent picture, one has to show that $m_h/m_P$ can be decreased. This will require a finite size scaling analysis, and simulations on $4 \cdot 8^3$ and larger systems. They are out of reach for present medium–sized computer systems. In contrast, a study of the phase diagram in the $(m_P^2, \beta)$ plane is feasible. To map out the phase diagram we performed new simulations, which are reported in this paper.

Let us recall euclidean quantum gravity on a Regge skeleton. The Regge–Einstein action is given by

$$S_{RE} = 2m_P^2 \sum_t \alpha_t A_t. \quad (1)$$

Here $m_P$ is the Planck mass, the sum is over all triangles $(d–2$ simplices) of a 4d Regge skeleton, $A_t$ denotes the area of the corresponding triangle and $\alpha_t$ its deficit angle. As in previous work, our Regge skeleton will be given by the standard tessellation of the hypercubical 4d lattice. It is well–known that the Regge–Einstein action is unbounded. Nevertheless the theory could exist due to entropy effects. For volume $V = \text{const.}$, numerical evidence was reported in ref. [3] for an entropy–dominated region, such that the partition function

$$Z = \int \prod_t \frac{dx_t}{x_t} \ e^{S_{RE}} \quad (2)$$

exists for $m_P^2 < m_{\text{MAX}}^2$. Here $m_{\text{MAX}}^2$ limits the well–defined region. As in ref. [3] we use the expectation value $< v_p >$, the average volume of a 4-simplex, to define our length unit as $l_0 = < v_p >^{1/4}$. Then $m_{\text{MAX}}^2 \geq 0.02l_0^2$ is found. Henceforth, all quantities will be expressed in units of $l_0$, which will not be explicitly written down anymore. In a fundamental length scenario $l_0$ stays finite in physical units (for instance Fermi),

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whereas \( l_0 \to 0 \) for a continuum theory. A fundamental length theory is attractive under the assumption that it will define a “natural” regularization scheme, such that no renormalization occurs when physical quantities are calculated in this scheme. If this is not the case, the continuum scenario is preferred, as it offers the possibility that (after renormalization) physical results will not depend on microscopic details of the regularization, \textit{i.e.} be universal in the usual sense of lattice gauge theory. These two scenarios are presumably indistinguishable in present day numerical simulations. Classically our condition \( V = \text{const.} \) corresponds to a cosmological constant \( \lambda = -\frac{1}{2}m_P^2V^{-1/2} \int d^4x \sqrt{g}R \). (3) This is seen by applying the variational principle to \( S = m_P^2V^{-1/2} \int d^4x \sqrt{g}R \) and comparing the result with the one obtained from \( S = m_P^2 \int d^4x \sqrt{g}R + \lambda V \). Equation (3) is consistent with the observed value \( \lambda < 10^{-122}m_P^4 \), and this would survive quantization if renormalization is not necessary.

Our complete action is

\[ S = S_{\text{RE}} + S_{\text{gauge}}, \]

where

\[ S_{\text{gauge}} = -\frac{\beta}{2} \sum_t W_t \text{Re}[Tr(1 - U_t)]. \]

Here \( U_t \) is the ordered product of SU(2) matrices attached to the links of triangle \( t \). Coupling of the gauge and the gravity part is entirely achieved through the dynamical weight factors \( W_t \). To obtain the correct classical limit, the weight factors \( W_t \) have to fulfill the constraint

\[ \sum_t W_t A_t^2 = \text{const.}V, \] (6a)

which should be supplemented by ref. \[ W_t \geq 0. \] (6b)

These conditions leave ample freedom for choice, and in the following we use the barycentric implementation. At \( m_P^2 = 0.005 \) this system was investigated for various \( \beta \) values. This is a data point well inside the entropy–dominated region. In the present paper we report additional results at \( m_P^2 = 0.010, 0.015, 0.020, 0.025 \) and \( 0.0375 \).

Let us first consider \( m_P^2 = 0.010, 0.015 \) and \( 0.020 \). There is evidence for the disorder–order phase transition at all of these Planck masses and Polyakov loop histograms for some relevant \( \beta \) values are shown in figures 1, 2 and 3. As in ref. \[ 20,000 \] thermalization sweeps were performed before measurement. The subsequent statistics for each histogram is \( \beta \) dependent and lies in the range 20,000–60,000 sweeps. More complete numerical results and additional data points are given in ref. \[ A \] . A three-peak structure is sometimes observed when both confined and deconfined phases coexist, for example in figure 1. This is suggestive of a first order transition in
the vicinity of the coupling. To substantiate this conjecture, would require finite size scaling investigations (see for instance ref. [11]) on $2 \cdot 6^3$, $2 \cdot 8^3$, ... systems, and has to be postponed until more powerful computational facilities are readily available. As function of $m_P^2$ the region of $\beta$ around which the transition occurs does not vary much. Tentatively, we identify it to be between $\beta = 1.46$ and $\beta = 1.57$.

The next step is to proceed towards higher Planck masses. At a high enough value of the Planck mass one does expect that the system will find itself in unstable states, similar as in the pure gravity case[3, 4]. Indeed, suggestions to that effect begin to appear at $m_P^2 = 0.025$ and $\beta = 1.42$. Figure 4 shows the time history of the gravity action $S_{RE}$. A two-peak structure, seen usually in first-order phase transitions, develops. However, it is unclear whether one can characterize the behavior as such because, as in the pure gravity case, the partition function likely ceases to exist when one is outside the entropy-dominated region. In any case the two peak structure characterizes competing effects of the entropy versus the unboundedness of the Regge-Einstein action.

At Planck mass 0.025 we seem to have reached the region where entropy effects from the measure begin to lose control over the unboundedness of the action. In figure 6 we present the time history of the gravity action at an even higher Planck mass 0.0375 and $\beta = 1.42$. It is constantly on the rise and the system is clearly unstable.

The $(m_P^2, \beta)$ phase diagram is summarized in figure 6. Regions A and C constitute the entropy-dominated region which is estimated to be $0 \leq m_P^2 \leq 0.02$, possibly up
Figure 2: Histograms of Polyakov loops at two different gauge couplings for Planck mass $m_P^2 = 0.015$.

Figure 3: Histograms of Polyakov loops at two different gauge couplings for Planck mass $m_P^2 = 0.020$. 
Figure 4: Time history of the gravity action $S_{RE}$ for $m_P^2 = 0.025$ and $\beta = 1.42$. Each point is an average over ten consecutive measurements.

Figure 5: Time history of the gravity action $S_{RE}$ for $m_P^2 = 0.0375$ and $\beta = 1.42$. Each point is an average over ten consecutive measurements.
to $m_P^2 \leq 0.03$. In region B the partition function ceases to exist and is assumed to be meaningless. Region C is the ordered phase where, in the infinite volume limit, Polyakov loops assume non-zero expectation values and region A is the disordered phase where the expectation value of the Polyakov loops are zero. In the present paper the hadronic mass (in our unit $l_0^{-1}$) is given by the “deconfining” temperature

$$T_c = \frac{1}{2 < x_l >_0},$$

(7)

where $< x_l >_0$ is the expectation value of the link length in the short ($L_0 = 2$) direction of the hypercubic lattice. Fairly independent of $m_P$ this value stays around $< x_l >_0 \approx 3$. Assuming $m_{\text{MAX}}^2 = 0.025$, one obtains then $m_P/m_h \leq 0.95$ for the $2 \cdot 4^3$ lattice. With increased computational power one may first envision a finite size scaling analysis of $2 \cdot L^3$ systems towards $L \to \infty$, to confirm and significantly improve the accuracy of the phase diagram of figure 6. Simultaneously, one should study an $4 \cdot 8^3$ lattice on the present heuristic level, to obtain evidence that the ratio $m_P/m_h$ can indeed be increased (naively by a factor of two).

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