On the WDVV-equation in quantum K-theory

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0. Introduction. Quantum cohomology theory can be described in general words as intersection theory in spaces of holomorphic curves in a given Kähler or almost Kähler manifold $X$. By quantum K-theory we may similarly understand a study of complex vector bundles over the spaces of holomorphic curves in $X$. In these notes, we will introduce a $K$-theoretic version of the Witten-Dijkgraaf-Verlinde-Verlinde equation which expresses the associativity constraint of the “quantum multiplication” operation on $K^*(X)$.

Intersection indices of cohomology theory,

$$\int_{\text{space of curves}} \omega_1 \wedge ... \wedge \omega_k$$

obtained by evaluation on the fundamental cycle of cup-products of cohomology classes, are to be replaced in $K$-theory by Euler characteristics

$$\chi(\text{space of curves} ; V_1 \otimes ... \otimes V_k)$$

of tensor products of vector bundles. The hypotheses needed in the definitions of the intersection indices and Euler characteristics — that the spaces of curves are compact and non-singular, or that the bundles are holomorphic — are rarely satisfied. We handle this foundational problem by restricting ourselves throughout the notes to the setting where the problem disappears. Namely, we will deal with the so called moduli spaces $X_{n,d}$ of degree $d$ genus 0 stable maps to $X$ with $n$ marked points assuming that $X$ is a homogeneous Kähler space. Under the hypothesis, the moduli spaces $X_{n,d}$ (we will review their definition and properties when needed) are known to be compact complex orbifolds (see [3, 4]). We use their fundamental cycle $[X_{n,d}]$,
well-defined over $\mathbb{Q}$, in the definition of intersection indices, and we use sheaf cohomology in the definition of the Euler characteristic of a holomorphic orbi-bundle $V$:

$$\chi(X_{n,d}; V) := \sum (-1)^k \dim H^k(X_{n,d}; \Gamma(V)).$$

1. Correlators. The WDVV-equation is usually formulated in terms of the following generating function for correlators:

$$F(t, Q) = \sum_d \sum_{n=0}^\infty \frac{Q^d}{n!} (t, ..., t)_{n,d}.$$ 

Here $d \in H_2(X, \mathbb{Z})$ runs the Mori cone of degrees, that is homology classes represented by fundamental cycles of rational holomorphic curves in $X$, and the correlators $(\phi_1, ..., \phi_n)_{n,d}$ are defined using the evaluation maps at the marked points:

$$\text{ev}_1 \times ... \times \text{ev}_n : X_{n,d} \to X \times ... \times X.$$ 

In cohomology theory, we pull-back to the moduli space $X_{n,d}$ the $n$ cohomology classes $\phi_1, ..., \phi_n \in H^*(X, \mathbb{Q})$ of $X$ and define the correlator among them by

$$(\phi_1, ..., \phi_n)_{n,d} := \int_{[X_{n,d}]} \text{ev}_1^*(\phi_1) \wedge ... \wedge \text{ev}_n^*(\phi_n).$$ 

In K-theory, we pull-back $n$ elements $\phi_1, ..., \phi_n \in K^*(X)$ (representable under our restriction on $X$ by holomorphic vector bundles or their formal differences) and put

$$(\phi_1, ..., \phi_n)_{n,d} := \chi(X_{n,d}; \text{ev}_1^*(\phi_1) \otimes ... \otimes \text{ev}_n^*(\phi_n)).$$ 

We will treat the series $F$ as a formal function of $t \in H$ depending on formal parameters $Q = (Q_1, ..., Q_{\text{Betti}(X)})$, where $H = H^*(X, \mathbb{Q})$ or $H = K^*(X)$.

Let $\{\phi_\alpha\}$ be a graded basis in $H^*(X, \mathbb{Q})$, and

$$g_{\alpha\beta} := \langle \phi_\alpha, \phi_\beta \rangle = \int_{[X]} \phi_\alpha \wedge \phi_\beta$$ 

denote the intersection matrix. Let $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ be the inverse matrix (so that $\sum (\phi_\alpha \otimes 1)g^{\alpha\beta}(1 \otimes \phi_\beta)$ is Poincare-dual to the diagonal in $X \times X$).
In quantum cohomology theory, one defines the *quantum cup-product* $\bullet$ on the tangent space $T_t H$ by

$$\langle \phi_\alpha \bullet \phi_\beta, \phi_\gamma \rangle := F_{\alpha\beta\gamma}(t)$$

(where the subscripts on the RHS mean partial derivatives in the basis $\{\phi_\alpha\}$). In the above notation the associativity of the quantum cup-product is equivalent to the WDVV-identity:

$$\sum_{\varepsilon, \varepsilon'} F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta} \text{ is totally symmetric in } \alpha, \beta, \gamma, \delta.$$  

### 2. Stable maps, gluing and contraction.

In order to explain the proof of the WDVV-identity we have to discuss some properties of the moduli spaces $X_{n,d}$ (see [9, 1, 4] for more details).

We consider prestable marked curves $(C, z)$, that is compact connected complex curves $C$ with at most double singular points and with $n$ marked points $z = (z_1, ..., z_n)$ which are non-singular and distinct. Two holomorphic maps, $f : (C, z) \to X$ and $f' : (C', z') \to X$, are called *equivalent* if they are identified by an isomorphism $(C, z) \to (C', z')$ of the curves. This definition induces the concept of *automorphism* of a map $f : (C, z) \to X$, and one calls $f$ *stable* if it has no non-trivial infinitesimal automorphisms. The moduli spaces $X_{n,d}$ consist of equivalence classes of stable maps with fixed number $n$ of marked points, degree $d$ and arithmetical genus $0$ (it is defined as $g = \dim H^1(C, \mathcal{O}_C)$).

In plain words, the space of degree $d$ holomorphic spheres in $X$ with $n$ marked points is compactified by prestable curves which are trees of $\mathbb{C}P^1$’s and satisfy the stability condition: each irreducible component $\mathbb{C}P^1$ mapped to a point in $X$ must carry at least 3 marked or singular points. Under the hypothesis that $X$ is a homogeneous Kähler space, the moduli space $X_{n,d}$ has the structure of a compact complex orbifold of dimension $\dim_{\mathbb{C}} X + \int d c_1(T_X) + n - 3$.

In the case when $X$ is a point the moduli spaces coincide with the Deligne-Mumford compactifications $\mathcal{M}_{0,n}$ of moduli spaces of configurations of marked points on $\mathbb{C}P^1$. For instance, $\mathcal{M}_{0,4}$ is the set $\mathbb{C}P^1 - \{0, 1, \infty\}$ of legitimate values of the cross-ratio of 4 marked points on $\mathbb{C}P^1$. The compactification $\mathcal{M}_{0,4} = \mathbb{C}P^1$ fills-in the forbidden values of the cross-ratio by equivalence classes of the reducible curves $\mathbb{C}P^1 \cup \mathbb{C}P^1$ with one double point and two marked point on each irreducible component.
For \( n \geq 3 \), there is a natural contraction map \( X_{n,d} \rightarrow \overline{\mathcal{M}}_{0,n} \) defined by composing the map \( f : (C, z) \rightarrow X \) with \( X \rightarrow \text{pt} \) (so that the components of \( C \) carrying \(< 3 \) special points become unstable) and contracting the unstable components. Similarly, one can define the forgetting maps \( f_t : X_{n+1,d} \rightarrow X_{n,d} \) by disregarding the \( i \)-th marked point and contracting the component if it has become unstable.

In particular, we will make use of the contraction map

\[
ct : X_{n+4,d} \rightarrow \overline{\mathcal{M}}_{0,4}
\]

defined by forgetting the map \( f : (C, z) \rightarrow X \) and all the marked points except the first four. A legitimate value \( \lambda = ct[f] \) of the cross-ratio means the following: the curve \( C \) has a component \( C_0 = \mathbb{CP}^1 \) carrying 4 special points with the cross-ratio \( \lambda \), and the first 4 marked point are situated on the branches of the tree connected to \( C_0 \) at those 4 special points. A forbidden value \( ct[f] = 0,1 \) or \( \infty \) means that \( C \) containing a chain \( C_0, \ldots, C_k \) of \( k > 0 \) of \( \mathbb{CP}^1 \)'s such that 2 of the 4 branches of the tree carrying the marked points are connected to the chain via \( C_0 \), and the other two — via \( C_k \). Such stable maps form a stratum of codimension \( k \) in the moduli space \( \mathcal{X}_{n,d} \). We will refer to them as strata (or stable maps) of depth \( k \).

A stable map of depth 1 is glued from 2 stable maps obtained by disconnecting \( C_0 \) from \( C_1 \). This gives rise to the gluing map

\[
X_{n_0+3,d_0} \times \Delta X_{n_1+3,d_1} \rightarrow X_{n_0+n_1+4,d_0+d_1}
\]
as follows. Consider the map from \( X_{n_0+3,d_0} \times X_{n_1+3,d_1} \) to \( X \times X \) defined by evaluation at the 3-rd marked points. The source of the gluing map is the preimage of the diagonal \( \Delta \subset X \times X \). It consists of pairs of stable maps which have the same image of the third marked point and which therefore can be glued at this point into a single stable map of degree \( d_0 + d_1 \) with \( n_0 + 2 + n_1 + 2 \) marked points.

Similarly, gluing stable maps of depth \( k \) from \( k + 1 \) stable maps subject to \( k \) diagonal constraints at the double points of the chain \( C_0, \ldots, C_k \) defines appropriate gluing maps parameterizing the strata of depth \( k \).

3. **Proof of the WDVV-identity.** All points in \( \overline{\mathcal{M}}_{0,4} \) represent the same (co)homology class. Thus the analytic fundamental cycles of the fibers

\[\text{Note that for a homogeneous Kähler } X, \text{ the evaluation map is conveniently transverse to the diagonal in } X \times X.\]
$\text{ct}^{-1}(\lambda)$ are homologous in $X_{n+4,d}$. The cohomological WDVV-identity follows from the fact that for $\lambda = 0, 1$ or $\infty$ the fiber $\text{ct}^{-1}(\lambda)$ consists of strata of depth $> 0$, and moreover — the corresponding gluing maps (for all splittings $d = d_0 + d_1$ of the degree and all splittings of the $n = n_0 + n_1$ marked points), being isomorphisms at generic points, identify the analytic fundamental cycle of the fiber with the sum of the fundamental cycles of $X_{n_0+3,d_1} \times \Delta X_{n_1+3,d_2}$. This allows one to equate $3$ quadratic expression of the correlators which differ by the order of the indices $\alpha, \beta, \gamma, \delta$ associated with the first 4 marked points.

We leave the reader to work out some standard combinatorial details which are needed in order to translate this argument into the WDVV-identity for the generating function $F$ and note only that the contraction with the intersection tensor $(g^{\epsilon \epsilon'})$ in the WDVV-equation takes care of the diagonal constraint $\Delta \subset X \times X$ for the evaluation maps.

In K-theory, similarly, the push-forward to $X \times X$ of the structural sheaf $O_{\Delta}$ of the diagonal is expressed as

$$
\sum (\phi_{\epsilon} \otimes 1)g^{\epsilon \epsilon'} (1 \otimes \phi_{\epsilon'})
$$

via $(g^{\epsilon \epsilon'})$ inverse to the “intersection matrix”

$$
g_{\alpha \beta} := \langle \phi_{\alpha}, \phi_{\beta} \rangle = \chi(X; \phi_{\alpha} \otimes \phi_{\beta}).
$$

The argument justifying the WDVV-equation fails, however, since the above gluing map to $\text{ct}^{-1}(\lambda)$ is one-to-one only at the points of depth 1 and does not identify the corresponding structural sheaves. Indeed, a stable map of depth $k$ can be glued from two stable maps in $k$ different ways and thus belongs to the $k$-fold self-intersection in the image of the gluing map.

Let us examine the variety $\text{ct}^{-1}(\lambda)$ at a point of depth $k > 1$. One of the properties of Kontsevich’s compactifications $X_{m,d}$ is that after passing to the local non-singular covers (defined by the orbifold structure of the moduli spaces) the compactifying strata form a divisor with normal crossings [3, 4]. Moreover, analyzing (inductively in $k$) the local structure of the contraction map $\text{ct} : X_{n+4,d} \to M_{0,4}$ near a depth-$k$ point, one easily finds the local model $\lambda(x_1, ..., x_k, ...) = x_1 ... x_k$ for the map $\text{ct}$ in a suitable local coordinate system. In this model, the components $x_1 = 0, ..., x_k = 0$ of the divisor with normal crossings represent the strata of depth 1, their intersections $x_i = x_j = 0$ — the strata of depth 2, etc. Denote by $O$ the algebra of functions on our local...
chart, so that \( \mathcal{O}/(x_{i_1}, ..., x_{i_l}) \), \( i_1 < ... < i_l \), are the algebras of functions on the depth-\( l \) strata. We have the following exact sequence of \( \mathcal{O} \)-modules:

\[
0 \to \mathcal{O}/(x_1...x_k) \to \bigoplus \mathcal{O}/(x_i) \to \bigoplus \mathcal{O}/(x_{i_1}, x_{i_2}) \to \bigoplus \mathcal{O}/(x_{i_1}, x_{i_2}, x_{i_3}) \to ....
\]

Notice that the \( \bigoplus \)-terms in the sequence are the algebras of functions on the normalized strata of depth 1, depth 2, etc. Translating this local formula to a global K-theoretic statement about gluing maps, we conclude that in the Grothendieck group of orbi-sheaves on \( X_{n+4,d} \), the element represented by the structural sheaf of \( ct^{-1}(\lambda) \) for \( \lambda = 0,1 \) or \( \infty \) is identified with the structural sheaf of the corresponding alternated disjoint sum over positive depth strata:

\[
\sum X_{n_0+3,d_0} \times X_{n_1+3,d_1} - \sum X_{n_0+3,d_0} \times X_{n_1+2,d_1} \times X_{n_2+3,d_2} + ...
\]

4. Formulation and consequences. Now we can apply the above K-theoretic statement about the moduli spaces to our generating functions. Introduce

\[
G(t, Q) := \frac{1}{2} \sum g_{\alpha\beta} t^\alpha t^\beta + F(t, Q).
\]

Let \((G^{\alpha\beta})\) be the matrix inverse to \((G_{\alpha\beta}) = (\partial_\alpha \partial_\beta G)\).

**Theorem.**

\[
\sum_{\varepsilon, \varepsilon'} G^{\varepsilon\varepsilon'} G_{\varepsilon\gamma\delta} is totally symmetric in \alpha, \beta, \gamma, \delta.
\]

**Proof.** We have rewritten

\[
F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta} - F_{\alpha\beta\varepsilon} g^{\varepsilon\mu} F_{\mu\varepsilon'} g^{\mu\varepsilon'} F_{\varepsilon'\gamma\delta} + ...
\]

using the famous matrix identity \( 1 - F + F^2 - ... = (1 + F)^{-1} \).

Introduce the *quantum tensor product* on \( T_1 H \) (with \( H = \mathcal{K}^*(X) \)) by

\[
(\phi_\alpha \bullet \phi_\beta, \phi_\gamma) := G_{\alpha\beta\gamma}(t),
\]

and the metric \((,\) on \( TH \) is defined by \((\phi_\mu, \phi_\nu) := G_{\mu\nu}(t)\).
Corollary 1. The operations (,) and ◦ define on the tangent bundle the structure of a formal commutative associative Frobenius algebra with the unity 1.

Proof. As in the cohomology theory, it is a formal corollary of the Theorem, except that the statement about the unity 1 means that $G_{α,1,β} = G_{αβ}$ and follows from the simplest instance of the string equation in the K-theory: $(1, t, ..., t)_{n+1,d} = (t, ..., t)_{n,d}$. The last equality is obvious. Indeed, the push-forward of the constant sheaf 1 along the map $f_t : X_{n+1,d} \to X_{n,d}$ forgetting the first marked point is the constant sheaf 1 on $X_{n,d}$ since the fibers are curves $C$ of zero arithmetic genus, $g = \dim H^1(C, \mathcal{O}_C) = 0$, while $H^0(C, \mathcal{O}_C) = \mathbb{C}$ by Liouville’s theorem. □

We introduce on $T^*H$ the 1-parametric family of connection operators

$$\nabla_q := (1 - q)d - \sum_α (\phi_α ◦) \, dt_α \wedge .$$

Corollary 2. The connections $\nabla_q$ are flat for any $q \neq 1$.

Proof. This follows from $\phi_α ◦ \phi_β = \phi_β ◦ \phi_α$, $d^2 = 0$, and $\partial α(\phi_β ◦) = \partial β(\phi_α ◦)$:

$$\partial α(\phi_β ◦)^ν_μ = G^{μα\betaε}G^{εν} - G^{μβε}G^{εε'}G^{ε'αε''}G^{ε''ν}$$

is symmetric with respect to $α$ and $β$ due to the WDVV-identity. □

Proposition. The operator $\nabla_{-1}$ is twice the Levi-Civita connection of the metric $(G^{αβ})$ on $T^*H$.

Proof. For a metric of the form $G_{αβ} = \partial α\partial βG$ the famous explicit formulas for the Christoffel symbols yield

$$2\Gamma^γ_αβ = [G^{ε,β,α} + G^{ε,α,β} - G^{α,β,ε}]G^{εγ} = G^{αβε}G^{εγ} = (\phi_β ◦)α^γ.$$

Corollary 3. The metric (,) on $TH$ is flat.

We complete this section with a description of flat sections of the connection operator $\nabla_q$ in terms of K-theoretic “gravitational descendents”. Let us introduce the generating functions

$$S_{αβ}(t, Q) := g_{αβ} + \sum_{n,d} \frac{Q^d}{n!} (\phi_α, t, ..., t, \frac{\phi_β}{1 - qL})_{n+2,d},$$

\footnote{At $t = 0, Q = 0$ it turns into the usual multiplicative structure on $K^*(X)$.}
where the correlators are defined by
\[
(\psi_1, \ldots, \psi_\text{\textsubscript{n}} L^k)_{m,d} := \chi(X_{m,d}; \text{ev}^*_1(\psi_1) \otimes \ldots \otimes \text{ev}^*_m(\psi_m) \otimes L^\otimes k).
\]
Here \(L\) is the line orb\textsubscript{b}undle over the moduli space \(X_{m,d}\) of stable maps \((C, z) \to X\) formed by the cotangent lines to \(C\) at the last marked point (as specified by the position of the geometrical series \(1 + qL + q^2L^2 + \ldots = (1 - qL)^{-1}\) in the correlator).

**Theorem.** The matrix \(S := (S_{\mu\nu})\) is a fundamental solution to the linear PDE system:

\[
(1 - q) \partial_\alpha S = (\phi_\alpha \bullet) S.
\]

**Proof.** Taking \(\phi_\mu, \phi_\alpha, \phi_\beta\) and \(\phi_\nu/(1 - qL)\) for the content of the four distinguished marked points in the proof of the WDVV-identity, we obtain its generalization in the form:

\[
G_{\mu\alpha\varepsilon} G^{\varepsilon\varepsilon'} \partial_\beta S_{\varepsilon\nu} = G_{\mu\beta\varepsilon} G^{\varepsilon\varepsilon'} \partial_\alpha S_{\varepsilon\nu},
\]
or \((\phi_\alpha \bullet) \partial_\beta S = (\phi_\beta \bullet) \partial_\alpha S\). Now it remains to put \(\phi_\beta = 1\) and use \((1 - q) \partial_1 S = S\), which is another instance of the string equation:

\[
(1, t, \ldots, t, \phi L^k)_{n+2,d} = (t, \ldots, t, \phi(1 + L + \ldots + L^k))_{n+1,d}.
\]
The last relation is obtained by computing the push forward of \(L^\otimes k\) along \(f_1 : X_{n+2,d} \to X_{n+1,d}\).

5. Some open questions.

(a) Definitions. It is natural to expect that the above results extend from the case of homogeneous Kähler spaces \(X\) to general compact Kähler and, even more generally, almost Kähler target manifolds.

In the Kähler case, the moduli of stable degree \(d\) genus \(g\) maps with \(n\) marked points form compact complex orbi-spaces \(X_{g,n,d}\) equipped with the 3Some details can be found in [13, 14, 15, 16, 17, 18]. Briefly, one identifies the fibers of \(f_1\) with the curves underlying the stable maps \(f : (C, z) \to X\) with \(n + 1\) marked points. It is important to realize that the pull-back \(L' := f_1^*(L)\) of the line bundle named \(L\) on \(X_{n+1,d}\) differs from the line bundle named \(L\) on \(X_{n+2,d}\). In fact, there is a holomorphic section of \(\text{Hom}(L', L)\) with the divisor \(D\) defined by the last marked point \(z_{n+1} \in C\), and the bundle \(L\) restricted to \(D\) is trivial (while \(L'|D\) is therefore conormal to \(D\)). Since \(L'\) is trivial along the fibers \(C\), we find that \(H^1(C, L^k) = 0\) and \(H^0(C, L^k) = (L')^k \otimes H^0(C, \mathcal{O}_C(kD)) \simeq (L')^k(1 + (L')^{-1} + \ldots + (L')^{-k})\).
intrinsic normal cone \[13\]. The cone gives rise \[13\] to an element in K-group of \(X_{g,n,d}\) which should be used in the definition of \(K\)-theoretic correlators in the same manner as the virtual fundamental cycle \([X_{g,n,d}]\) is used in quantum cohomology theory.

The moduli space \(X_{g,n,d}\) can be also described as the zero locus of a section of a bundle \(E \to B\) over a non-singular space. Due to the famous “deformation to the normal cone” \[3\], the virtual fundamental cycle represents the Euler class of the bundle. This description survives in the almost Kähler case and yields a topological definition and symplectic invariance of the cohomological correlators. In \(K\)-theory, there exists a topological construction of the push forward from \(B\) to the point based on Whitney embedding theorem and Thom isomorphisms. We don’t know however how to adjust the construction to our actual setting where \(B\) is non-singular only in the orbifold sense.

One (somewhat awkward) option is to define \(K\)-theoretic correlators topologically by the RHS of the Kawasaki-Riemann-Roch-Hirzebruch formula \[8\] for orbi-bundles over \(B\). This proposal deserves further study even in the Kähler case since it may lead to a ”quantum Riemann-Roch formula”.

(b) Frobenius-like structures. Our results in Section 4 show that \(K\)-theoretic Gromov-Witten invariants of genus 0 define on the space \(H = K^*(X)\) a geometrical structure very similar to the Frobenius structure \[2\] of cohomology theory, but not identical to it.

One of the lessons is that the metric tensor on \(H\), which can be in both cases described as \(F_{\alpha,1,\beta}\), is constant in cohomology theory and equal to \(g_{\alpha\beta}\) only by an ”accident”, but remains flat in \(K\)-theory even though it is not constant anymore.

The translation \(t \mapsto t + \tau l\) in the direction of \(1 \in H\) leaves the structure invariant in cohomology theory, but causes multiplication by \(e^{\tau}\) in \(K\)-theory — because of a new form of the string equation. Also, the \(\mathbb{Z}\)-grading missing in \(K\)-theory makes an important difference. It would be interesting to study the axiomatic structure that emerges here and to compare it with the structure implicitly encoded by \(K\)-theory on Deligne-Mumford spaces.

(c) Deligne-Mumford spaces. When the target space \(X\) is the point, the moduli spaces \(X_{g,n,0}\) are Deligne-Mumford compactifications of the moduli spaces of genus \(g\) Riemann surfaces with \(n\) marked points. The parallel between cohomology and \(K\)-theory suggest several problems.

Holomorphic Euler characteristics of universal cotangent line bundles and
their tensor products satisfy the string and dilation equations. The K-theoretic
generalization of the rest of Witten – Kontsevich intersection theory is unclear.

The case of genus 0 and 1 has been studied in and . The formula
\[
\chi(\mathcal{M}_{0,n}; \frac{1}{(1 - q_1 L_1) \cdots (1 - q_n L_n)}) = \frac{(1 + q_1/(1 - q_1) + \cdots + q_n/(1 - q_n))^{n-3}}{(1 - q_1) \cdots (1 - q_n)}
\]
found by Y.-P. Lee is analogous to the famous intersection theory result
\[
\int_{[\mathcal{M}_{0,n}]} \frac{1}{(1 - x_1 c_1(L_1)) \cdots (1 - x_n c_1(L_n))} = (x_1 + \cdots + x_n)^{n-3}.
\]
The latter formula is a basis for fixed point computations in equivariant
cohomology of the moduli spaces \(X_{n,d}\) for toric \(X\). As it was noticed by Y.-P.
Lee, the former formula is not sufficient for similar fixed point computation in K-theory: it requires Euler characteristics accountable for invariants with respect to permutations of the marked points. Finding an \(S_n\)-equivariant version of Lee’s formula is an important open problem.

(d) Computations. The quantum K-ring is unknown even for \(X = \mathbb{C}P^1\).
It turns out that the WDVV-equation is not powerful enough in the absence of grading constraints and divisor equation (see, for instance, ).

On the other hand, for \(X = \mathbb{C}P^n\), it is not hard to compute the generating functions \(G(t, Q)\) and even \(S_{\alpha,\beta}(t, Q, q)\) at \(t = 0\) (see ). In cohomology theory, this would determine the small quantum cohomology ring due to the divisor equation which, roughly speaking, identifies the \(Q\)-deformation at \(t = 0\) with the \(t\)-deformation at \(Q = 1\) along the subspace \(H^2(X, \mathbb{Q}) \subset H\). No replacement for the divisor equation seems to be possible in K-theory.

At the same time, the heuristic study of \(S^1\)-equivariant geometry on the loop space \(LX\) suggests that the generating functions \(S = S_{1,\beta}(0, Q, q)\) should satisfy certain linear \(q\)-difference equations (instead of similar linear \(4\)The same is true not only for \(X = pt\) (see ). By the way, the push forward \(f_\ast(L)\) along \(f : X_{g,n+1,d} \to X_{g,n,d}\), described by the dilation equation, equals \(\mathcal{H} + \mathcal{H}^* - 2 + n\). Here \(\mathcal{H}\) is the \(g\)-dimensional Hodge bundle with the fiber \(H^1(C, \mathcal{O}_C)\). This answer replaces a similar factor \(2g - 2 + n\) in the cohomological dilation equation, but also shows that tensor powers of \(\mathcal{H}\) must be included to close up the list of “observables”.

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differential equations of quantum cohomology theory). This expectation is supported by the example of $X = \mathbb{CP}^n$: Y.-P. Lee \cite{Lee12} finds that the generating functions are solutions to the $q$-difference equation $D^{n+1}S = QS$ (where $(DS)(Q) := S(Q) - S(qQ)$).

In the case of the flag manifold $X$ the generating functions $S$ have been identified with the so called Whittaker functions — common eigen-functions of commuting operators of the $q$-difference Toda system. This result and its conjectural generalization \cite{GiventalLee98} to the flag manifolds $X = G/B$ of complex simple Lie algebras links quantum K-theory to representation theory and quantum groups. Originally this conjecture served as a motivation for developing the basics of quantum K-theory.

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