A GENERAL MAXIMUM PRINCIPLE FOR STOCHASTIC
CONTROL PROBLEM WITH RANDOM JUMPS∗

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Abstract. In this paper, we obtain the maximum principle for optimal controls of stochastic systems with jumps by introducing a new method of variation. The control is allowed to enter both diffusion and jump term and the control domain need not to be convex.

Key words. Maximum Principle, random jumps, spike variation, adjoint equation

AMS subject classifications. 93E20, 60H10

1. Introduction. The stochastic optimal control problems is an important kind of problems in control theory. Maximum principle, the necessary conditions for the optimal control, is one of the central results. A lot of work has been done on this topic, Peng [2] proved the general maximum principle for forward stochastic control system without jump by using second-order variation equation to overcome the difficulty appeared along with the non-convex control domain and control entering the diffusion term. Situ [6] obtained the maximum principle for forward stochastic control system with jumps, but in his system the jump coefficient doesn’t contain the control variable. Tang and Li [7] proved the maximum principle for forward control system where the control variable is allowed into both diffusion and jump coefficients. There are many results for other stochastic control systems, we refer the reader for Peng [3], Wu [8], Shi and Wu [5] for forward-backward system.

The purpose of our paper is to overcome the deficiencies in Tang and Li [7]. In [7], the third estimate in (2.10) is not precise enough, if we set \( g_0 := 1, p = 2 \) and suppose \( \pi \) is a finite measure, then the left of the inequality is of order \( O(|I_\rho|) \), but the right hand is of order \( O(|I_\rho|)^2 \), this leads to a contradiction. On the other hand, if the inequality is true, then we can infer that the integral with \( \tilde{N} \) has a continuous modification by Kolmogorov’s lemma, this also leads to a contradiction. The reason to derive the contradiction is that the variation method does not consider the influence of jumps. Our approach to improve this is to introduce another variation method to avoid the influence of random jumps.

The rest of this paper is organized as follows. In section 2, we give some preliminaries about the stochastic integral with respect to jumps. The difference between our model with the model in [7] is that we need the integrand to be progressive in order to make our variation effective. Our main results are in section 3, 4 and 5. In these sections, we employ the new spike variation and introduce second order variation equations to get the desired maximum principle, which is the rigorous version in strict mathematical framework. In section 6, we explain the characteristic of our results and show our future research directions. Some results about stochastic differential equation (SDE) and backward stochastic differential equation (BSDE) with

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jumps are put in appendix.

2. Preliminaries. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete probability space with filtration, and on the probability space, there is a \(\mathcal{F}_t\)-Brownian motion \(\{B_t\}_{t \geq 0}\); and a Poisson random measure \(N\) on \(R_+ \times E\) adapted to \(\mathcal{F}_t\), where \(E\) is a standard measure space with a \(\sigma\)-field \(\mathcal{E}\). The mean measure of \(N\) is a measure on \((R_+ \times E, \mathcal{B}(R_+) \otimes \mathcal{E})\) which has the form \(\text{Leb} \times \lambda\), where \(\text{Leb}\) denotes the Lebesgue measure on \(R_+\) and \(\lambda\) is a finite measure on \(E\). For any \(B \in \mathcal{E}\) and \(t \in R_+\), since \(\lambda(B) < \infty\), we set \(\tilde{N}(\omega, [0, t] \times B) := N(\omega, [0, t] \times B) - t\lambda(B)\). It’s well known that \(\tilde{N}(\omega, [0, t] \times B)\) is a martingale for every \(B\). We assume that \(\{\mathcal{F}_t\}_{t \geq 0}\) is generated by \(B, N\); that is

\[
\mathcal{F}_t := \sigma(N([0, s], A), 0 \leq s \leq t, A \in \mathcal{E}) \vee \sigma(B_s, 0 \leq s \leq t) \vee \mathcal{N}
\]

where \(\mathcal{N}\) denotes the totality of \(P\)-null sets. Then \(\mathcal{F}_t\) satisfies the usual condition.

Suppose that \(H\) is a Euclid space, \(\mathcal{B}(H)\) is the Borel \(\sigma\)-field on \(H\). Given \(T > 0\), a process \(X : [0, T] \times \Omega \to H\) is called progressive (predictable) if \(X\) is \(\mathcal{G} \otimes \mathcal{B}(H)(\mathcal{P} \otimes \mathcal{B}(H))\) measurable, where \(\mathcal{G}\) is the progressive (predictable) \(\sigma\)-field on \([0, T] \times \Omega\); a process \(X : [0, T] \times \Omega \times E \to H\) is called \(E\)-progressive (\(E\)-predictable) if \(X\) is \(\mathcal{G} \otimes \mathcal{E} \otimes \mathcal{B}(H)(\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{B}(H))\) measurable. Different from [7], the stochastic integral we used is more general, that is the integrand of the stochastic integral in our paper is \(E\)-progressive rather than \(E\)-predictable.

Now we introduce some notations. Given a process \(X_t\) with càdlàg paths, \(X_{0-} := 0\) and \(\Delta X_t := X_t - X_{t-}, t \geq 0\). Let \(\mu\) denote the measure on \(\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{E}\) generated by \(N\), that is \(\mu(A) = E \int_0^T \int_E I_A N(ds, de)\), and for any \(\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{E} \otimes \mathcal{B}(R)\) measurable integrable process \(X\) we denote \(E[X] = \int Xd\mu\) and \(E[X|\mathcal{P} \otimes \mathcal{E}]\) to be the Radon-Nicotine derivatives with respect to \(\mathcal{P} \otimes \mathcal{E}\), actually \(E\) is not an expectation since \(\mu\) is not a probability measure, but we write it in the form of expectation because it has similar property to expectation. Then we introduce the definition of stochastic integral of random measure which is more general than that in [7] based on the theory of stochastic integral of process. We will use the theory of dual predictable projection and we will not give the definition here, the definition and other details of the theory can be found in [1]. These theories are mainly prepared to prove the existence and uniqueness of the solution of BSDE in our paper which we give in the appendix.

Suppose \(H = I_{A \times B}, A \in \mathcal{G}, B \in \mathcal{E}\). We define

\[
\int_0^T \int_E H \tilde{N}(dt, de) := \int_0^T I_A \tilde{N}(dt, B)
\]

Then for any \(E\)-progressive simple function with the form \(H = \sum_{i=1}^n a_i I_{A_i \times B_i}, a_i \in R, A_i \in \mathcal{G}, B_i \in \mathcal{E}\), we can define by linear extension.

For \(E\)-progressive process \(H\) that \(E\left[\int_0^T \int_E H^2 N(dt, de)\right] < \infty\), there exist a sequence of \(E\)-progressive simple functions \(H_n\) which have the form above that

\[
\lim_{n \to \infty} E\left[\int_0^T \int_E (H - H_n)^2 N(dt, de)\right] = 0
\]

We can verify that \(\{H_n.\tilde{N}\}_T\}_{n \geq 1}\) is a cauchy sequence in \(L^2\), so we define

\[
\int_0^T \int_E H \tilde{N}(dt, de) := \langle L^2 \rangle \lim_{n \to \infty} \int_0^T \int_E H_n \tilde{N}(dt, de)
\]
Proposition 2.1. If $H$ is a positive $E$-progressive process that

$$E \left[ \int_0^T \int_E HN(dt, de) \right] < \infty,$$

then

$$(2.1) \quad \left( \int_0^T \int_E HN(ds, de) \right)^p_t = \int_0^t \int_E \mathbb{E}[H|\mathcal{P} \otimes \mathcal{E}] \lambda(de) ds$$

where $X^p$ means the dual predictable projection of $X$.

Proof. If $H = I_{A \times B}, A \in \mathcal{G}, B \in \mathcal{E}$, then

$$\left( \int_0^T \int_E HN(ds, de) \right)^p_t = \left( \int_0^T I_A N(ds, B) \right)^p_t = \int_0^t \mathbb{E}_B [I_A |\mathcal{P}] \lambda(B) ds$$

where $\mathbb{E}_B$ is the measure on $\mathcal{B}([0, T]) \otimes \mathcal{F}$ generated by $N([0, t] \times B)$. Now we need a claim.

Claim.

$$\int_E \mathbb{E}[I_{A \times B}|\mathcal{P} \otimes \mathcal{E}] \lambda(de) = \lambda(B) \mathbb{E}_B [I_A |\mathcal{P}]$$

Proof. It’s obviously that both sides of the equation is predictable. Now for any $C \in \mathcal{P}$,

$$E \left[ \int_0^T I_C \int_E \mathbb{E}[I_{A \times B}|\mathcal{P} \otimes \mathcal{E}] \lambda(de) dt \right] = E \left[ \int_0^T \int_E \mathbb{E}[I_{C I_{A \times B}}|\mathcal{P} \otimes \mathcal{E}] N(dt, de) \right]$$

$$= E \left[ \int_0^T \int_E I_{A \cap C} I_B N(dt, de) \right] = E \left[ \int_0^T \int_E I_{A \cap C} N(dt, B) \right]$$

on the other hand,

$$E \left[ \int_0^T I_C \lambda(B) \mathbb{E}_B [I_A |\mathcal{P}] dt \right] = E \left[ \int_0^T \int_E I_B \mathbb{E}_B [I_{A \cap C} |\mathcal{P}] \lambda(de) dt \right]$$

$$= E \left[ \int_0^T \int_E I_B \mathbb{E}_B [I_{A \cap C} |\mathcal{P}] N(dt, de) \right] = E \left[ \int_0^T \int_E I_{A \cap C} N(dt, B) \right]$$

This shows the result. □

We come back to the proof of proposition. The claim above shows that (2.1) is true for functions of the form $I_{A \times B}$, now we define

$$\mathcal{C} = \{H = I_{A \times B} | A \in \mathcal{G}, B \in \mathcal{E}\}$$

$\mathcal{C}$ is a $\pi$-system that generate $\mathcal{G} \times \mathcal{E}$. And define

$$\mathcal{H} = \left\{ H \text{ is bounded and } E\text{-progressive} \left| \left( \int_0^T \int_E HN(ds, de) \right)^p_t = \int_0^t \int_E \mathbb{E}[H|\mathcal{P} \otimes \mathcal{E}] \lambda(de) ds \right. \right\}$$
then \( C \in \mathcal{H} \), and by the linear property of dual predictable projection we can verify that \( \mathcal{H} \) is a linear space. If \( H^n \uparrow H \), and \( H \) is bounded, then we have \( (\int_0^T H^n N(ds, de))^p \rightarrow (\int_0^T H N(ds, de))^p \) for each \( t \) in \( L^1 \) sense and this implies that \( H \in \mathcal{H} \). So, by monotone class theorem, we prove that all bounded \( E \)-progressive process satisfy the result.

For \( H \) \( E \)-progressive and \( E \left[ \int_0^T \int_E H N(dt, de) \right] < \infty \), we set \( H^n = H I_{\{|H| \leq n\}} \in \mathcal{H} \) and take limit to show that \( H \) satisfies (2.1).

**Proposition 2.2.** If \( H \) is \( E \)-progressive and satisfy \( E \left[ \int_0^T \int_E H^2 N(dt, de) \right] < \infty \), then we have

\[
\int_0^T \int_E H \tilde{N}(dt, de) = \int_0^T \int_E H N(dt, de) - \left( \int_0^T \int_E H N(dt, de) \right)_T^p
\]

*Proof.* If \( H = I_{A \times B}, A \in \mathcal{G}, B \in \mathcal{F} \), by the definition of stochastic integral,

\[
\int_0^T \int_E H \tilde{N}(dt, de) = \int_0^T \int_E I_A \tilde{N}(dt, B) = \int_0^T \int_E I_A N(dt, B) - \left( \int_0^T \int_E I_A N(dt, B) \right)_T^p
\]

So for any \( E \)-progressive simple process which has the form \( H = \sum_{i=1}^n a_i I_{A_i \times B_i}, a_i \in R, A_i \in \mathcal{G}, B_i \in \mathcal{F} \) the conclusion is true.

If \( H \) is positive and \( E \left[ \int_0^T \int_E H^2 N(dt, de) \right] < \infty \), then \( E \left[ \int_0^T \int_E H N(dt, de) \right] < \infty \) and there exists a sequence of positive increasing simple functions with the form above \( H_n \) that

\[
E \left[ \int_0^T \int_E (H - H_n)^2 N(dt, de) \right] \rightarrow 0
\]
as \( n \) goes to infinity, so

\[
\int_0^T \int_E H \tilde{N}(dt, de) = (L^2) \lim_{n \rightarrow \infty} \int_0^T \int_E H_n \tilde{N}(dt, de)
\]

\[
= (L^2) \lim_{n \rightarrow \infty} \left( \int_0^T \int_E H_n N(dt, de) - \left( \int_0^T \int_E H_n N(dt, de) \right)_T^p \right)
\]

\[
= \int_0^T \int_E H N(dt, de) - (L^1 \text{ or } L^2) \lim_{n \rightarrow \infty} \left( \int_0^T \int_E H_n N(dt, de) \right)_T^p
\]

If \( H \) is not positive, we decompose \( H = H^+ - H^- \) and get the result.

By the two propositions above, we have

**Proposition 2.3.** If \( H \) is \( E \)-progressive and \( E \left[ \int_0^T \int_E H^2 N(dt, de) \right] < \infty \), then

\[
\int_0^T \int_E H \tilde{N}(dt, de) = \int_0^T \int_E H N(dt, de) - \int_0^T \int_E E[H|\mathcal{P} \otimes \mathcal{I}] \lambda(de)dt
\]
Remark. Under the condition of the proposition above, we have

\[ E \left[ \int_0^T \int_E HN(dt, de) \right] = E \left[ \int_0^T \int_E [H| \mathcal{P} \otimes \mathcal{E}] \lambda(de) dt \right] \]

In particular, if \( H \) is \( E \)-predictable, we have the well-known result

\[ E \left[ \int_0^T \int_E HN(dt, de) \right] = E \left[ \int_0^T \int_E H\lambda(de) dt \right] \]

Since \( N([0, t] \times A) \) is quasi-left-continuous for each \( A \in \mathcal{E} \), we have the following two propositions.

**Proposition 2.4.** If \( H \) is \( E \)-progressive and \( E \left[ \int_0^T \int_E H^2N(dt, de) \right] < \infty \), then we have

\[ \Delta(H, N)_t = \int_E HN(\{t\}, de) \]

**Proof.** If \( H = I_{A \times B}, A \in \mathcal{G}, B \in \mathcal{E} \), then

\[ \Delta \left( \int_0^t \int_E H\tilde{N}(ds, de) \right)_t = \Delta \left( \int_0^t I_{A\tilde{N}}(ds, B) \right)_t = I_{A\tilde{N}}(\{t\}, B) = \int_E I_{A \times B} N(\{t\}, de) \]

So for any \( E \)-progressive simple functions with the form \( H = \sum_{i=1}^n a_i I_{A_i \times B_i}, a_i \in \mathbb{R}, A_i \in \mathcal{G}, B_i \in \mathcal{E} \), the conclusion is true.

Then for any positive \( H \) that \( E \left[ \int_0^T \int_E (H - H_n)^2N(dt, de) \right] < \infty \), there exists a sequence of positive increasing simple functions \( H_n \) that \( E \left[ \int_0^T \int_E (H - H_n)^2N(dt, de) \right] \to 0 \) as \( n \) goes to infinity, so

\[ \Delta \left( \int_0^t \int_E H\tilde{N}(ds, de) \right)_t = \lim_{n \to \infty} \Delta \left( \int_0^t H_n\tilde{N}(ds, B) \right)_t = \lim_{n \to \infty} \int_E H_n N(\{t\}, de) \]

\[ = \int_E HN(\{t\}, de) \]

**Proposition 2.5.** If \( H \) is \( E \)-progressive and \( E \left[ \int_0^T \int_E H^2N(dt, de) \right] < \infty \), then we have

\[ [H, \tilde{N}, H, \tilde{N}]_t = \int_0^t \int_E H^2 N(ds, de) \]

**Proof.** The method of proof is the same as above.

3. **Statement of Problem.** Given time duration \( T > 0 \), let \( \{T_n\}_{n \geq 1} \) be the jump time of \( N([0, t] \times E) \), \( T_n := \inf \{t | N([0, t] \times E) \geq n\} \), then \( \{T_n\}_{n \geq 1} \) is a sequence of stopping times that strictly increasing. Let \( U \) be a nonempty subset of \( \mathbb{R} \). We define the admissible control set

\[ U_{ad} = \left\{ u | u \text{ is progressive, takes values in } U \text{ and } E \left[ \sup_{0 \leq t \leq T} |u_t|^p \right] < \infty \text{ for any } p > 1 \right\} \]
For any admissible control \( u \in U_{ad} \) and initial state \( x_0 \in R \), we consider the following stochastic system with jumps:

\[
X_t = x_0 + \int_0^t b(s, X_s, u_s)ds + \int_0^t \sigma(s, X_s, u_s)dB_s + \int_0^t \int_E c(s, X_{s-}, u_s, e)\tilde{N}(ds, de)
\]

along with the cost functional:

\[
J(u) = E \left[ \int_0^T f(t, X_t, u_t)dt + g(X_T) \right]
\]

where \( b : \Omega \times [0, T] \times R \times R \to R \), \( \sigma : \Omega \times [0, T] \times R \times R \to R \), \( c : \Omega \times [0, T] \times R \times R \times E \to R \), \( f : \Omega \times [0, T] \times R \times R \to R \), and \( g : \Omega \times R \to R \). The optimal control is an element \( u \) in \( U_{ad} \) that

\[
J(u) = \inf_{v \in U_{ad}} J(v).
\]

Suppose that there is an optimal control in \( U_{ad} \), our aim is to find a necessary condition for it. We need these assumptions below:

**Assumption H:**

- \( b, \sigma, f \) is \( \mathcal{F} \otimes \mathcal{B}(R) \otimes \mathcal{B}(R) / \mathcal{B}(R) \) measurable, \( c \) is \( \mathcal{F} \otimes \mathcal{C} \otimes \mathcal{B}(R) \otimes \mathcal{B}(R) / \mathcal{B}(R) \) measurable, \( g \) is \( \mathcal{F}_T \otimes \mathcal{B}(R) / \mathcal{B}(R) \) measurable.
- \( b, \sigma, c, f, g \) are twice continuously differentiable about \( x \) with bounded second-order derivatives. And \( b, \sigma, c \) are uniformly lipschitz continuous about \( u \) and have bounded first-order derivatives about \( x \).
- \( E \int_0^T |b(t, \omega, 0, 0)|^2 dt < \infty \), \( E \int_0^T |\sigma(t, \omega, 0, 0)|^2 dt < \infty \), \( E \int_0^T \int_E |c(t, \omega, e, 0, 0)|^2 \tilde{N}(ds, de) < \infty \).
- There exists a positive number \( \delta \) that \( |c_x| + 1 > \delta \)

Under these assumptions, we know that there exists a unique solution of (3.1) for any admissible control from Theorem A.1 in appendix.

**4. Variation.** Since \( U \) is not essentially convex, we need to employ spike variation. Suppose \( u \in U_{ad} \) is the optimal control, for any \( \bar{t} \in [0, T] \), the spike variation of \( u \) is defined as follow:

\[
u^\varepsilon = \begin{cases} v, & \text{if } (s, \omega) \in \mathcal{O} := [\bar{t} - \varepsilon, \bar{t} + \varepsilon] \setminus \bigcup_{n=1}^{\infty} [T_n] \\ u, & \text{otherwise.} \end{cases}
\]

where \([T_n] := \{ (\omega, t) \in \Omega \times [0, T] | T_n(\omega) = t \}\) is the graph of \( T_n \), \( v \) is a bounded \( \mathcal{F}_{\bar{t}} \) measurable function that takes values in \( U \). Since \( T_n \) is a stopping time, \([T_n]\) is a progressive set. So the spike variation \( u^\varepsilon \) is progressive, then it’s easy to show that \( u^\varepsilon \) is in \( U_{ad} \).

The method of variation is showed in Figure 1. Fix \( \omega \), we consider one path of \( u^\varepsilon \) and \( u \). The difference between the new method and the traditional method is that if there are jumps in \( (t, t + \varepsilon] \), for example, as the figure shows that \( T_1(\omega) \) is in \( (t, t + \varepsilon] \), then the value of \( u^\varepsilon \) at \( T_1(\omega) \) is equal to \( u \) rather than \( v \).

**Remark.** As we know, \( T_n \) is not a predictable time, so \([T_n]\) is not predictable which means that \( u^\varepsilon \) is not predictable, that’s the reason why we need the integrand of the stochastic integral to be progressive. Actually, \( T_n \) is a totally unpredictable time.
We use $X$ to denote the trajectory of $u$, and $X^\epsilon$ to denote the trajectory of $u^\epsilon$. By the estimate of SDE and notice that $(\text{Leb} \times P)(|\mathcal{T}_n|) = 0$, we can get that:

$$
E \left[ \sup_{0 \leq t \leq T} |X^\epsilon_t - X_t|^p \right] \leq CE \left[ \left( \int_0^T |b(t, X_t, u^\epsilon_t) - b(t, X_t, u_t)| \, dt \right)^p + \left( \int_0^T |\sigma(t, X_t, u^\epsilon_t) - \sigma(t, X_t, u_t)|^2 \, dB_t \right)^{\frac{p}{2}} + \left( \int_0^T \int_E |c(t, X_{t-}, u^\epsilon_t, e) - c(t, X_{t-}, u_t, e)|^2 \, N(dt, de) \right)^{\frac{p}{2}} \right] 
$$

$$
\leq CE \left[ \left( \frac{1}{t^\epsilon} |u - v| \right)^p + \left( \frac{1}{t^\epsilon} |u - v|^2 \right)^{\frac{p}{2}} + \left( \int_0^T |u - v|^2 \, N(dt, E) \right)^{\frac{p}{2}} \right]
$$

Since there is no jump on $\mathcal{O}$, actually we have:

$$
E \left[ \sup_{0 \leq t \leq T} |X^\epsilon_t - X_t|^p \right] = O(\epsilon^p) + O(\epsilon^{\frac{p}{2}}) \tag{4.2}
$$

That means the jump term doesn’t influence the order of variation. Actually, if we don’t subtract the jump term in variation, $E \left[ \left( \int_0^{t+\epsilon} |u - v|^2 \, N(dt, E) \right)^{\frac{p}{2}} \right]$ is always of order $O(\epsilon)$ no matter how large $p$ is. Thanks to this, we can use the method in [2] to get the desired conclusion. Then we introduce the variation equations:

$$
\hat{X}_t = \int_0^t b_x(s, X_s, u_s) \hat{X}_s + \delta bs + \int_0^t \sigma_x(s, X_s, u_s) \hat{X}_s + \delta \sigma dB_s 
+ \int_0^t \int_E c_x(s, X_{s-}, u_s, e) \hat{X}_{s-} + \delta c \hat{N}(ds, de) \tag{4.3}
$$
and
\[
\dot{Y}_t = \int_0^t b_x(s, X_s, u_s)Y_s + \frac{1}{2} b_{xx}(s, X_s, u_s)\dot{X}_s^2 ds
\]
\[
+ \int_0^t \sigma_x(s, X_s, u_s)Y_s + \frac{1}{2} \sigma_{xx}(s, X_s, u_s)\dot{X}_s^2 + \delta \dot{X}_s dB_s
\]
\[
+ \int_0^t \int_{E} c_x(s, X_{s-}, u_s, e)Y_s + \frac{1}{2} c_{xx}(s, X_{s-}, u_s, e)\dot{X}_{s-}^2 + \delta c_{x} \dot{X}_{s-} \tilde{N}(ds, de)
\]
where \( \delta \phi = \phi(s, X_s, u_s^e) - \phi(s, X_s, u_s), \delta b = b, \sigma. \delta c = c(s, X_s, u_s^e) - c(s, X_s, u_s, e), \delta \phi_x = \phi_x(s, X_s, u_s^e) - \phi(s, X_s, u_s), \delta b = b, \sigma. \delta c_x = c_x(s, X_s, u_s^e) - c_x(s, X_s, u_s, e). \)

It’s easy to show that (4.3) and (4.4) have unique solution. We have some basic estimates about \( \dot{X} \) and \( \dot{Y} \).

**Lemma 4.1.** For \( p \geq 2 \), we have the following estimate:
\[
\begin{aligned}
E \left[ \sup_{0 \leq t \leq T} |\dot{X}_t|^p \right] &\leq C e^{\frac{p}{2}} \\
E \left[ \sup_{0 \leq t \leq T} |\dot{Y}_t|^p \right] &\leq C e^p.
\end{aligned}
\]

**Proof.** By the elementary \( L^p \) estimate, for \( \dot{X} \) we have:
\[
E \left[ \sup_{0 \leq t \leq T} |\dot{X}_t|^p \right] \leq C E \left[ \left( \int_0^T |\delta b| dt \right)^p \right] + C E \left[ \left( \int_0^T |\delta \sigma|^2 dt \right)^{\frac{p}{2}} \right]
\]
\[
+ C E \left[ \left( \int_0^T \int_E |\delta c|^2 N(dt, de) \right)^{\frac{p}{2}} \right]
\]
\[
\leq C E \left[ \left( \int_0^T |u_t - u_t| dt \right)^p \right] + C E \left[ \left( \int_0^T |u_t^e - u_t| dt \right)^{\frac{p}{2}} \right]
\]
\[
+ C E \left[ \left( \int_0^T \int_E |u_t^e - u_t^e|^2 N(dt, de) \right)^{\frac{p}{2}} \right] = O(e^p) + O(e^{\frac{p}{2}})
\]

for \( \dot{Y} \), notice the boundness of \( b_{xx}, \sigma_{xx}, c_{xx} \), we have:
\[
E \left[ \sup_{0 \leq t \leq T} |\dot{Y}_t|^p \right] \leq C E \left[ \left( \int_0^T \left| \frac{1}{2} b_{xx}(s, X_s, u_s)\dot{X}_s^2 \right| dt \right)^p \right]
\]
\[
+ C E \left[ \left( \int_0^T \frac{1}{2} \sigma_{xx}(s, X_s, u_s)\dot{X}_s^2 + \delta \dot{X}_s |\dot{X}_s|^2 dt \right)^{\frac{p}{2}} \right]
\]
\[
+ C E \left[ \left( \int_0^T \int_E \frac{1}{2} c_{xx}(s, X_{s-}, u_s, e)\dot{X}_{s-}^2 + \delta c_{x} \dot{X}_{s-} |\dot{X}_{s-}|^2 N(dt, de) \right)^{\frac{p}{2}} \right]
\]
\[
\leq C E \left[ \sup_{0 \leq t \leq T} |\dot{X}_t|^{2p} \right] + C E \left[ \sup_{0 \leq t \leq T} |\dot{X}_t|^p \left( \int_0^T |\delta \sigma_x|^2 dt \right)^{\frac{p}{2}} \right]
\]
\[
+ C E \left[ \sup_{0 \leq t \leq T} |\dot{X}_t|^{2p} N([0, T] \times E) \right] + C E \left[ \left( \int_0^T \int_E |\delta c_{x}|^2 |\dot{X}_{s-}|^2 N(dt, de) \right)^{\frac{p}{2}} \right]
\]
= O(ε²)

Lemma 4.2.

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} E \left[ \sup_{0 \leq t \leq T} \left| X^*_t - X_t - \hat{Y}_t \right|^2 \right] = 0 \]

**Proof.** First find the equation that \( X_t + \hat{X}_t + \hat{Y}_t \) satisfy.

\[
X_t + \hat{X}_t + \hat{Y}_t = x_0 + \int_0^t b(s, X_s, u_s) + b_x(s, X_s, u_s) \hat{X}_s + b_x(s, X_s, u_s) \hat{Y}_s + \frac{1}{2} b_{xx}(s, X_s, u_s) |\hat{X}_s|^2 ds + \int_0^t \sigma(s, X_s, u_s) + \sigma_x(s, X_s, u_s) \hat{X}_s + \sigma_x(s, X_s, u_s) \hat{X}_s \hat{Y}_s + \int_0^t \sigma_x(s, X_s, u_s) \hat{X}_s \hat{Y}_s + \frac{1}{2} \sigma_{xx}(s, X_s, u_s) |\hat{X}_s|^2 dB_s + \int_0^t \int_E c(s, X_{s-}, u_s, e) (\hat{X}_s + \hat{Y}_s) + \frac{1}{2} c_{xx}(s, X_{s-}, u_s, e) |\hat{X}_s - \hat{Y}_s|^2 \hat{N}(ds, de)
\]

Since we have for \( \phi = b, \sigma, c \)

\[
\phi(s, X_s + \hat{X}_s, \hat{Y}_s, u_s, e) - \phi(s, X_s, u_s, e) = \phi(s, X_s + \hat{X}_s, \hat{Y}_s, u_s, e) - \phi(s, X_s, u_s, e) + \delta\phi
\]

\[
= \delta\phi + \phi_x(s, X_s, u_s, e) (\hat{X}_s + \hat{Y}_s) + \int_0^1 \int_0^s \alpha \phi_x(s, X_s + \alpha \beta(\hat{X}_s + \hat{Y}_s), u_s, e) d\alpha d\beta (\hat{X}_s + \hat{Y}_s)^2
\]

we get

\[
X_t + \hat{X}_t + \hat{Y}_t = x_0 + \int_0^t b(s, X_s + \hat{X}_s + \hat{Y}_s, u_s^*) + \Lambda ds + \int_0^t \sigma(s, X_s + \hat{X}_s + \hat{Y}_s, u_s^*) + G dB_s + \int_0^t \int_E c(s, X_{s-} + \hat{X}_{s-} + \hat{Y}_{s-}, u_s^*, e) + F \hat{N}(ds, de)
\]

where

\[
\Lambda = \frac{1}{2} b_{xx}(s, X_s, u_s) |\hat{X}_s|^2 - (b_x(s, X_s, u_s) - b_x(s, X_s, u_s)) (\hat{X}_s + \hat{Y}_s) + A_0 (\hat{X}_s + \hat{Y}_s)^2
\]

\[
G = \frac{1}{2} \sigma_{xx}(s, X_s, u_s) |\hat{X}_s|^2 - (\sigma_x(s, X_s, u_s^*) - \sigma_x(s, X_s, u_s)) \hat{Y}_s + A_0 (\hat{X}_s + \hat{Y}_s)^2
\]

\[
F = \frac{1}{2} c_{xx}(s, X_{s-}, u_s, e) |\hat{X}_{s-}|^2 - (c_x(s, X_{s-}, u_s, e) - c_x(s, X_{s-}, u_s, e)) \hat{Y}_{s-} + A_e (\hat{X}_{s-} + \hat{Y}_{s-})^2
\]

By Lemma 4.2, we have

\[
E \left[ \left( \int_0^T \Lambda ds \right)^2 + \int_0^T G^2 ds + \int_0^T \int_E F^2 \hat{N}(ds, de) \right] = o(\varepsilon^2)
\]

So by the basic estimate we have

\[
E \left[ \sup_{0 \leq t \leq T} \left| X^*_t - X_t - \hat{X}_t - \hat{Y}_t \right|^2 \right] \leq CE \left[ \left( \int_0^T \Lambda ds \right)^2 + \int_0^T G^2 ds \right] + CE \int_0^T \int_E F^2 \hat{N}(ds, de)
\]
which shows the result.

Now we get the variation equation for cost functional. We have

\[ J(u) = E \left[ \int_0^T f(t, X_t, u_t) dt + g(X_T) \right] \]

define

\[ \hat{J} = E \left[ \int_0^T f_x(t, X_t, u_t)(\hat{X}_t + \hat{Y}_t) + \frac{1}{2} f_{xx}(t, X_t, u_t)\hat{X}_t^2 + \delta f dt \right] \]

\( \text{(4.7)} \)

\[ + E \left[ g_x(X_T)(\hat{X}_T + \hat{Y}_T) + \frac{1}{2} g_{xx}(X_T)(\hat{X}_T)^2 \right] \]

Then we have the following lemma.

**Lemma 4.3.**

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} (J(u^\epsilon) - J(u) - \hat{J}) = 0 \]

**Proof.**

\[ J(u) + \hat{J} = E \left[ \int_0^T f(t, X_t, u_t) + f_x(t, X_t, u_t)(\hat{X}_t + \hat{Y}_t) + \frac{1}{2} f_{xx}(t, X_t, u_t)\hat{X}_t^2 + \delta f dt \right] \]

\[ + E \left[ g(X_T) + g_x(X_T)(\hat{X}_T + \hat{Y}_T) + \frac{1}{2} g_{xx}(X_T)(\hat{X}_T)^2 \right] \]

\[ = E \left[ \int_0^T f(t, X_t + \hat{X}_t + \hat{Y}_t, u_t) + H dt \right] + E \left[ g(X_T + \hat{X}_T + \hat{Y}_T) + I \right] \]

where

\[ H = \frac{1}{2} f_{xx}(s, X_s, u_s)\hat{X}_s^2 - \delta f_x(\hat{X}_s + \hat{Y}_s) - A_f(\hat{X}_s + \hat{Y}_s)^2 \]

\[ I = - \int_0^1 \int_0^1 \alpha g(X_T + \alpha \beta(\hat{X}_T + \hat{Y}_T))d\alpha d\beta(\hat{X}_T + \hat{Y}_T)^2 + \frac{1}{2} g_{xx}(X_T)(\hat{X}_T)^2 \]

Then

\[ |J(u^\epsilon) - J(u) - \hat{J}|^2 \leq CE \left[ \int_0^T |f(t, X_t + \hat{X}_t + \hat{Y}_t, u_t) - f(t, X_t^\epsilon, u_t^\epsilon)|^2 dt + \left( \int_0^T H dt \right)^2 \right] \]

\[ + E \left[ \left| g(X_T + X_T + \hat{Y}_T) - g(X_T) \right|^2 + I^2 \right] \]

\[ \leq CE \left[ \sup_{0 \leq t \leq T} |X_t - X_t| \right] + E \left[ \left( \int_0^T H dt \right)^2 + I^2 \right] \]

\[ = o(\epsilon^2) \]

By the same method we can show that 
\[ E \left[ \left( \int_0^T H dt \right)^2 + I^2 \right] = o(\epsilon^2), \] which proves the result.
5. Adjoint Equations and Maximum Principle. We introduce the first order and second order adjoint equation.

First order:

\[
pt = g_t(X_t) + \int_t^T b_s p_s + \sigma_s q_s + f_s ds + \int_t^T \left( \int_E \frac{c_{s+}}{1 + c_s} k_s N(ds, dc) \right)dt - \int_t^T q_s dB_s - \int_t^T k_s \tilde{N}(ds, dc)
\]

(5.1)

And the second order:

\[
P_t = g_{xx}(X_t) + \int_t^T 2b_s P_s + 2\sigma_s Q_s + f_{xx} + b_{xx} p_s + \sigma_{xx} q_s + P_s \sigma_s^2 ds
\]

(5.2)

\[+ \int_t^T \left( \frac{c_{s+}}{1 + c_s} k_s + (P_{s+} + K_s) c_s^2 + 2K_s c_s \right) N(ds, dc)
\]

\[- \int_t^T Q_s dB_s - \int_t^T K_s \tilde{N}(ds, dc)
\]

where \( \phi_x = \phi_x(t, X_t, u_t), \phi_{xx} = \phi_{xx}(t, X_t, u_t) \). The main difference between (5.1) (5.2) and the adjoint equations in [7] is the jump term caused by \( N \).

By assumption, it is easy to show that \( \frac{c_{s+}}{1 + c_s}, \frac{c_{s+}^2 + 2c_s}{1 + c_s}, \frac{c_{s+}^2}{1 + c_s} \) and \( \frac{c_{s+}^3}{1 + c_s} \) are all bounded, \( \frac{1}{1 + c_s}, \frac{1}{1 + c_s} \) are invertible. So the existence and uniqueness of solution is ensured by Theorem B.2 in appendix.

Apply Itô’s formula for \( pt X_t, pt Y_t \) and \( P_t | \hat{X}_t |^2 \), we get

\[
dp_t \hat{X}_t = p_{t-} d\hat{X}_t + \hat{X}_t dpt + d[p, \hat{X}]_t = (p_{t-} \delta b + q_{t-} \delta \sigma - \hat{X}_t f_{t-}) dt + dM_t^1
\]

(5.3)

\[
dp_t \hat{Y}_t = p_{t-} d\hat{Y}_t + \hat{Y}_t dpt + d[p, \hat{Y}]_t = \left( \frac{1}{2} b_{xx} p_{t-} |\hat{X}_t|^2 + \frac{1}{2} \sigma_{xx} q_{t-} |\hat{X}_t|^2 - \hat{Y}_t f_{t-} + \delta \sigma_{x} \hat{X}_t q_{t-} \right) dt
\]

\[+ \int_E \frac{1}{2} \frac{c_{s+}}{1 + c_s} k_s |\hat{X}_t|^2 N(dt, dc) + dM_t^2
\]

(5.4)

\[
dP_t | \hat{X}_t |^2 = |\hat{X}_{t-}|^2 dP_{t-} + 2P_{t-} \hat{X}_{t-} d\hat{X}_{t-} + P_{t-} d[\hat{X}, \hat{X}]_{t-} + 2 \hat{X}_{t-} d[\hat{X}, P]_{t-} + \Delta P_t (|\hat{X}_t|^2)
\]

\[= \left( 2P_{t-} \sigma_{x} \hat{X}_{t-} \delta \sigma + 2P_{t-} \hat{X}_{t-} db + 2 \hat{X}_{t-} \delta \sigma Q_{t-} \right) dt + \left( P_{t-} (\delta \sigma)^2 - |\hat{X}_{t-}|^2 (f_{t-} + b_{xx} p_{t-} + \sigma_{xx} q_{t-}) \right) dt
\]

\[+ \int_E \left( P_{t-} c_{s+}^2 - \hat{H}_t + 2K_t c_{s+} - 2c_{s+} \hat{H}_t \right) |\hat{X}_{t-}|^2 N(dt, dc) + \Delta P_t (|\hat{X}_t|^2) + dM_t^3
\]

(5.5)

where \( M_t^1, M_t^2, M_t^3 \) are martingales, \( \hat{H}_t := (c_{t+} + 1)^{1/2} \left( \frac{c_{s+}}{1 + c_s} k_s + (P_{t-} + K_t) c_s^2 + 2K_t c_s \right) \).

Besides, we should write \( \sum_{s \leq t} \Delta P_s (|\hat{X}_s|^2) \) into a integral. For any \( t \),

\[
\Delta P_t = \int_E -\hat{H}_t \tilde{N}(\{t\}, dc) + \int_E K_t \tilde{N}(\{t\}, dc)
\]

and

\[
\Delta \hat{X}_t = \int_E c_{s+} \hat{X}_{t-} + \delta c \tilde{N}(\{t\}, dc)
\]
Notice that for any $A \in \mathcal{F}, N(\{t\}, A) = \tilde{N}(\{t\}, A) = 1$ or $0$, we have
\[
\Delta P_t(\Delta \hat{X}_t)^2 = \int_E -\hat{H}_t(c_x \hat{X}_t + \delta c)^2 + K_t(c_x \hat{X}_t + \delta c)^2 N(\{t\}, de)
\]
So
\[
\sum_{s \leq t} \Delta P_s(\Delta \hat{X}_s)^2 = \int_0^t \int_E -\hat{H}_t(c_x \hat{X}_t + \delta c)^2 + K_t(c_x \hat{X}_t + \delta c)^2 N(ds, de)
\]
By (5.3)–(5.5), we can get the form of $g_x(X_T)(X_T + Y_T)$ and $g_{xx}(X_T)X_T^2$. Then we have
\[
\hat{j} = E \left[ \int_0^T \left( p_t \delta b + q_t \delta \sigma + \delta f + \frac{1}{2} P_t(\delta \sigma)^2 \right) dt \right] + o(\epsilon)
\]
o($\epsilon$) represent $E \left[ \int_0^T \left( \delta \sigma_X \hat{X}_t q_t + P_t \sigma_X \hat{X}_t \delta \sigma + P_t \hat{X}_t \delta b + \hat{X}_t \delta Q_t \right) dt \right]$. We define $H(t, x, u, p, q) = pb(t, x, u) + q\sigma(t, x, u) + f(t, x, u)$, then we have such form of maximum principle:

**Theorem 5.1.** Under Assumption $H$, suppose that $u$ is the optimal control, $X$ is the trajectory of $u$, and $p, q$ satisfy (5.1), $P$ satisfies (5.2), then for any $v \in U$, we have a.e a.s:

\[
\hat{j} = E \left[ \int_0^T \left( p_t \delta b + q_t \delta \sigma + \delta f + \frac{1}{2} P_t(\delta \sigma)^2 \right) dt \right] + o(\epsilon)
\]
\[
H(t, X_t, v, p_t, q_t) - H(t, X_t, u, p_t, q_t) + \frac{1}{2} P_t(\sigma(t, X_t, v) - \sigma(t, X_t, u))^2 \geq 0
\]

**Proof.** Notice that $\bigcup_{n=1}^\infty [T_n]$ is negligible under $P \times \text{Leb}$, so by (5.6) we have
\[
\hat{j} = E \left[ \int_0^T I_{t, t+\epsilon} \left( p_t(b(t, X_t, v) - b(t, X_t, u)) + q_t(\sigma(t, X_t, v) - \sigma(t, X_t, u)) \right) \right.
\]
\[
+ (f(t, X_t, v) - f(t, X_t, u)) + \left. \frac{1}{2} P_t(\sigma(t, X_t, v) - \sigma(t, X_t, u))^2 \right) dt + o(\epsilon)
\]
then both sides are divided by $\epsilon$ and let $\epsilon \to 0$, we have for a.e $\tilde{i}$
\[
E \left[ \left( H(\tilde{i}, X_{\tilde{i}}, v, p_{\tilde{i}}, q_{\tilde{i}}) - H(\tilde{i}, X_{\tilde{i}}, u, p_{\tilde{i}}, q_{\tilde{i}}) + \frac{1}{2} P_t(\sigma(\tilde{i}, X_{\tilde{i}}, v) - \sigma(\tilde{i}, X_{\tilde{i}}, u))^2 \right) \right] \geq 0
\]
Then for any $A \in \mathcal{F}_i$ and $w \in U$, let $v = wI_A + uI_{A^c}$, we have
\[
E \left[ I_A \left( H(\tilde{i}, X_{\tilde{i}}, w, p_{\tilde{i}}, q_{\tilde{i}}) - H(\tilde{i}, X_{\tilde{i}}, w, p_{\tilde{i}}, q_{\tilde{i}}) + \frac{1}{2} P_t(\sigma(\tilde{i}, X_{\tilde{i}}, w) - \sigma(\tilde{i}, X_{\tilde{i}}, u))^2 \right) \right] \geq 0
\]
which means a.e a.s
\[
H(\tilde{i}, X_{\tilde{i}}, w, p_{\tilde{i}}, q_{\tilde{i}}) - H(\tilde{i}, X_{\tilde{i}}, w, p_{\tilde{i}}, q_{\tilde{i}}) + \frac{1}{2} P_t(\sigma(\tilde{i}, X_{\tilde{i}}, w) - \sigma(\tilde{i}, X_{\tilde{i}}, u))^2 \geq 0
\]
6. Conclusions. In order to overcome the deficiencies of [7], we introduce a new method of variation. To our best knowledge, there is no such method of variation before. With the help of our new variation, we overcome the difficulty that the jumps caused in $L^p$ estimate, in other words, (4.2) holds, and the order of this estimate grows with the growth of $p$, this feature is important to make the variation equations effective.

The form of our maximum principle with jumps is the same as the form of maximum principle in [2] without jumps. The reason is that both maximum principles are hold a.e a.s. In our case with jumps, since the measure of all jumps’ graphs is a negligible set under $P \times Leb$, jumps does not influence our result. In other words, our maximum principle only describe the optimal control on the area that $N$ is continuous, it has no information about the optimal control on the time $N$ jumps. However, this is a rigorous maximum principle obtained in a clear and concise mathematical framework and laid a solid foundation for further related theoretical and application research. Our future research is to find a way to characterize optimal control on the time $N$ jumps and explore wide applications in practice.

Appendix A. Existence and Uniqueness of SDE and $L^p$ estimate. Given a SDE with jump:

\[(A.1) \quad X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s + \int_E \int c(s, X_s, \omega)\tilde{N}(ds, d\omega)\]

where $x_0 \in R^n$, $b : \Omega \times [0, T] \times R^n \rightarrow R^n$, $\sigma : \Omega \times [0, T] \times R^n \rightarrow R^{n \times d}$, $c : \Omega \times [0, T] \times R^n \times E \rightarrow R^n$, $d$ is the dimension of Brownian Motion and $n$ is the dimension of $X$. We introduce a Banach space

\[S^2[0, T] := \left\{X \mid X \text{ has càdlàg paths and adapted and } E \left[ \sup_{0 \leq t \leq T} |X_t|^2 \right] < \infty \right\}\]

with norm $\|X\|^2 = E \left[ \sup_{0 \leq t \leq T} |X_t|^2 \right]$. We have the following assumptions:

**Assumption H1:**

- $b$ is $\mathcal{G} \otimes \mathcal{B}(R^n)/\mathcal{B}(R^n)$ measurable, $\sigma$ is $\mathcal{G} \otimes \mathcal{B}(R^n)/\mathcal{B}(R^{n \times d})$ measurable,
- $c$ is $\mathcal{G} \otimes \mathcal{F} \otimes \mathcal{B}(R^n)/\mathcal{B}(R^n)$ measurable.
- $b, \sigma, c$ are uniform lipschitz continuous about $x$.
- $E \int_0^T |b(t, \omega, 0)|^2dt < \infty$, $E \int_0^T |\sigma(t, \omega, 0)|^2dt < \infty$,
- $E \int_0^T \int_E |c(t, \omega, e)|^2N(ds, de) < \infty$.

**Theorem A.1.** Under Assumption H1, (A.1) has a unique solution in $S^2[0, T]$.

**Proof.** First we show that for each $X$ in $S^2[0, T]$, $\int_0^T \int_E c(s, X_s, e)\tilde{N}(ds, de)$ is well defined. Since $X_{s-}$ is left continuous, it is progressive, and $c(s, x, e)$ is $E$-progressive by assumption, this implies that $c(s, X_s, \omega)$ is $E$-progressive. And for any $t \in [0, T]$

\[E \left[ \int_0^t \int_E |c(s, X_{s-}, \omega)|^2N(ds, de) \right] \leq CE \left[ \int_0^t \int_E |c(s, \omega, 0)|^2 + |X_{s-}|^2N(ds, de) \right] \leq C + Ct\lambda(E)E \left[ \sup_{0 \leq s \leq t} |X_s|^2 \right] < \infty\]

That means that the stochastic integral is well defined.
Next we show that there is a unique solution in small time duration. We construct a map from $S^2[0,T]$ to $S^2[0,T]$: 

$$\mathcal{F}(X)_t = x_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s + \int_0^t \int_E c(s,X_{s-},e)\tilde{N}(ds,de)$$

It is easy to show that the image of $\mathcal{F}$ is actually in $S^2[0,T]$, then we show it is a contraction. For any $X,Y \in S^2[0,T]$, 

$$\|\mathcal{F}(X) - \mathcal{F}(Y)\|^2 \leq CE \left[ \left( \int_0^T |b(t,X_t) - b(t,Y_t)|dt \right)^2 \right]$$

$$+ CE \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(t,X_t) - \sigma(t,Y_t)dB_t \right|^2 \right]$$

$$+ CE \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \int_E c(t,X_{t-},e) - c(t,Y_{t-},e)\tilde{N}(dt,de) \right|^2 \right]$$

$$\leq C\|(X - Y)\|_2^2(T + T^2) + CE \left[ \int_0^T \int_E |X_{t-} - Y_{t-}|^2\lambda(de)dt \right]$$

$$\leq C\|(X - Y)\|_2^2(T + T^2)$$

$C$ is a constant not related to $T$ but changed every step. So we can choose $T$ small enough that $C(T + T^2) < 1$, then $\mathcal{F}$ is a contraction.

For arbitrary $T$, we can split $T$ into finite small pieces, then we get a unique solution on each piece and connect them together.

**Remark.** The difference between our results and the results in [4] is that in our case $c$ is $E$-progressive and in [4]'s case $c$ is $E$-predictable. In fact from the proof above, the difference is slight.

The theorem below is the $L^p$ estimate:

**Theorem A.2.** For $p \geq 2$, suppose that $X^i$, $i = 1, 2$, is the solution of the follow equations

$$X^i_t = x^i_0 + \int_0^t b^i(s,X^i_s)ds + \int_0^t \sigma^i(s,X^i_s)dB_s + \int_0^t \int_E c^i(s,X^i_{s-},e)\tilde{N}(ds,de)$$

which satisfy assumption H1, then we have

$$E \left[ \sup_{0 \leq t \leq T} |X^1_t - X^2_t|^p \right] \leq C|x^1_0 - x^2_0|^p + CE \left[ \left( \int_0^T |b^1(t,X^1_t) - b^2(t,X^1_t)|dt \right)^p \right]$$

$$+ CE \left[ \left( \int_0^T |\sigma^1(t,X^1_t) - \sigma^2(t,X^1_t)|^2dt \right)^{\frac{p}{2}} \right]$$

$$+ CE \left[ \left( \int_0^T \int_E |c^1(t,X^1_{t-},e) - c^2(t,X^1_{t-},e)|^2\lambda(de)dt \right)^{\frac{p}{2}} \right]$$

(A.2)

$C$ is a positive real number related to $p,T$ and the Lipschitz constant.
Proof. By simple calculation we have
\[
E \left[ \sup_{0 \leq t \leq T} |X_t^1 - X_t^2|^p \right] \leq C|x_0^1 - x_0^2|^p + CT^p E \left[ \sup_{0 \leq t \leq T} |X_t^1 - X_t^2|^p \right] + CT^p E \left[ \sup_{0 \leq t \leq T} |X_t^1 - X_t^2|^p \right] + CE \left( \int_0^T |X_t^1 - X_t^2|^2 N(dt, E) \right)^{\frac{p}{2}} + CE \left( \int_0^T |\sigma^1(t, X_t^1) - \sigma^2(t, X_t^1)|^2 dt \right)^{\frac{p}{2}} + CE \left( \int_0^T \int_E |I^1(t, X_{t-}, e) - c^2(t, X_{t-}, e)|^2 N(dt, de) \right)^{\frac{p}{2}}
\]

Now we set $H_t := |X_t^1 - X_t^2|^2$, $A_t := f_t^t H_s N(ds, E)$, then $A_t$ is a pure jump process
and so is $A_t^\#$. Notice that the jump time of $A_t^\#$ is also a jump time of $N$ and the
jump size of $N$ is always equal to 1. So we have
\[
A_t^\# = \sum_{s \leq T} A_s^\# - A_{s-}^\# = \sum_{s \leq T} \left( A_s^\# - A_{s-}^\# \right) I_{\{|N(t, E)| \neq 0\}}
\]
\[
= \sum_{s \leq T} \left( |A_{s-} + H_s|^2 \right) N(\{s\}, E)
\]
\[
= \int_0^T |A_{s-} + H_s|^2 - A_{s-}^\# N(ds, E)
\]
\[
\leq C \int_0^T A_{s-}^\# + H_s^\# N(ds, E)
\]

Since $A_{s-}$ and $H$ are predictable, we have
\[
E \left[ A_t^\# \right] \leq CE \left[ \int_0^T A_s^\# + H_s^\# ds \right] \leq CTE \left[ A_T^\# \right] + CTE \left[ \sup_{0 \leq t \leq T} |X_t^1 - X_t^2|^p \right]
\]

So if we choose $T$ small enough that $CT < 1$, then we have
\[
E \left[ A_T^\# \right] \leq \frac{CT}{1 - CT} E \left[ \sup_{0 \leq t \leq T} |X_t^1 - X_t^2|^p \right]
\]
By the calculation of \((A.3)\), choose $T$ smaller if necessary, we have the estimate
in small time duration by subtract \((T^p + T^\# + \frac{CT}{1 - CT}) E \left[ \sup_{0 \leq t \leq T} |X_t^1 - X_t^2|^p \right]\) on
both sides of \((A.3)\). For any $T$, we can split $T$ into small pieces and get the desired
conclusion.

Remark. Without loss of generality, we can assume
\[
E \left[ \sup_{0 \leq t \leq T} |X_t^1 - X_t^2|^p \right] < \infty
\]
in the proof above. If not, we can introduce a sequence of stopping times that make
\((A.4)\) true, then get the $L^p$ estimate with stopping time and take limits. So actually
we can subtract that term on both sides of \((A.3)\).

Appendix B. Existence and Uniqueness of BSDE with Jumps. We
consider a BSDE with jumps of the following form:

\begin{equation}
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T \int_E g(s, Y_s, Z_s, K_s, e) \mathbb{N}(ds, de) - \int_t^T Z_s dB_s - \int_t^T K_s \tilde{N}(ds, de)
\end{equation}

where $f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^n$, $g : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^n \times \mathbb{E} \to \mathbb{R}^n$. $\xi$ is a random variable that is $\mathcal{F}_T$ measurable and square integrable. We introduce another two Banach spaces:

$$M^2[0, T] = \left\{ Z \mid Z \text{ is predictable and } E \left[ \int_0^T |Z_s|^2 ds \right] < \infty \right\}$$

with norm $\|Z\|^2 = E \left[ \int_0^T |Z_s|^2 ds \right]$, and

$$F^2[0, T] = \left\{ K \mid K \text{ is } E\text{-progressive and } E \left[ \int_0^T \int_E |K_s|^2 N(dt, de) \right] < \infty \right\}$$

with norm $\|K\|^2 = E \left[ \int_0^T \int_E |K|^2 N(dt, de) \right]$.

Denote $\det(M)$ as the determinant of the matrix $M$. We have the following assumptions:

**Assumption H2:**

- $f$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n \times d})/\mathcal{B}(\mathbb{R}^n)$ measurable, $g$ is $\mathcal{F} \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n \times d}) \otimes \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ measurable.
- $f$ is uniform lipschitz continuous about $(y, z, k)$. $g$ is continuously differentiable about $(y, z, k)$ with bounded derivatives.
- For any $(s, \omega, y, z, k, e)$, we have $\det(I - g_k) \neq 0$, $|(I - g_k)^{-1}| \leq M$ and for any $(s, \omega, y, z, e)$, $\{k - g(s, \omega, y, z, k, e) \mid k \in \mathbb{R}^n\} = \mathbb{R}^n$.
- $E \int_0^T |f(t, \omega, 0, 0)|^2 dt < \infty$, $E \int_0^T \int_E |g(t, \omega, e, 0, 0)|^2 N(dt, de) < \infty$.

**Lemma B.1.** Under Assumption H2, let $h(\omega, s, y, z, k, e) := k - g(\omega, s, y, z, k, e)$, then for each $(\omega, s, y, z, k, e)$, $h(\omega, s, y, z, k, e)$ is invertible, i.e. $k = h(\omega, s, y, z, k, e)$. And $h$ is continuously differentiable about $(y, z, k)$ with bounded derivatives.

**Proof.** By assumption H2, for any $(\omega, s, y, z, e)$, $\hat{k} = h(\omega, s, y, z, k, e)$ is surjective, now we show it is also injective. Suppose that there exists $k_1 \neq k_2$ that $h(\omega, s, y, z, k_1, e) = h(\omega, s, y, z, k_2, e)$, define $\phi(r) = h(\omega, s, y, z, r k_1 + (1 - r) k_2, e)$, then $\phi(0) = \phi(1)$, and $\phi$ is differentiable, so there exist some $r_0 \in (0, 1)$ such that $\phi'(r_0) = 0$. In other words, we have

$$h_k(\omega, s, y, z, r_0 k_1 + (1 - r_0) k_2, e)(k_1 - k_2) = 0$$

which contradicts the assumption that $h_k$ is invertible for any $(\omega, s, y, z, k)$. So $\hat{k} = h(\omega, s, y, z, k, e)$ is bijective and there exits a function $k = \hat{h}(\omega, s, y, z, k, e)$. By implicit function theorem, $\hat{h}$ is continuously differentiable and $\hat{h}_y = -h_y h_k^{-1}$, $\hat{h}_z = -h_z h_k^{-1}$, $\hat{h}_k = h_k^{-1}$. By assumption, $|h_k^{-1}| = |(1 - g_k)^{-1}| \leq M$ and $h_y, h_z$ are bounded, so $\hat{h}_y, \hat{h}_z, \hat{h}_k$ are all bounded. \(\square\)
Notice the important property
\[
(H, \tilde{N})_t = \int_0^t \int_E H N(ds, de) - \int_0^t \int_E E[H | \mathcal{P} \otimes \mathcal{E}] \lambda(de) ds
\]

We change the BSDE in another form:
(B.2)
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T \int_E \mathbb{E}[g(s, Y_{s-}, Z_s, K_s, e) | \mathcal{P} \otimes \mathcal{E}] \lambda(de) ds
- \int_t^T Z_s dB_s - \int_t^T \int_E K_s - g(s, Y_{s-}, Z_s, K_s, e) \tilde{N}(ds, de)
\]

We will use the notations in Lemma B.1 in the next theorem. Now we give the existence and uniqueness result:

**THEOREM B.2.** Under assumption $H2$, and assume that $\hat{h}$ is $\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{B}(R^n) \otimes \mathcal{B}(R^{n \times d}) \otimes \mathcal{B}(R^n) / \mathcal{B}(R^n)$ measurable, then (B.2) has a unique solution in $S^2[0, T] \times M^2[0, T] \times F^2[0, T]$.

**Proof.** Step 1: First consider the following simple BSDE:
\[
Y_t = \xi - \int_t^T Z_s dB_s - \int_t^T \int E \tilde{K}_s \tilde{N}(ds, de)
\]

We define $Y_t$ is the càdlàg version of the martingale $E[\xi | \mathcal{F}_t]$, and by the martingale representation theorem in [7] we have $\xi = \int_0^T Z_s dB_s + \int_0^T \int E \tilde{K}_t \tilde{N}(dt, de)$, where $Z$ is predictable and $\tilde{K}$ is $E$-predictable. It’s easy to verify that $(Y, Z, \tilde{K})$ is the unique solution.

Step 2: For any $(y, z, k) \in S^2[0, T] \times M^2[0, T] \times F^2[0, T]$, we consider the following BSDE:
\[
Y_t = \xi + \int_t^T f(s, y_s, z_s) ds + \int_t^T \int E [g(s, y_{s-}, z_s, k_s, e) | \mathcal{P} \otimes \mathcal{E}] \lambda(de) ds
- \int_t^T Z_s dB_s - \int_t^T \int E K_s - g(s, Y_{s-}, Z_s, K_s, e) \tilde{N}(ds, de)
\]

By step 1
\[
Y_t = E \left[ \xi + \int_0^T f(s, y_s, z_s) ds + \int_0^T \int E [g(s, y_{s-}, z_s, k_s, e) | \mathcal{P} \otimes \mathcal{E}] \lambda(de) ds | \mathcal{F}_t \right]
- \int_0^t f(s, y_s, z_s) ds - \int_0^t \int E [g(s, y_{s-}, z_s, k_s, e) | \mathcal{P} \otimes \mathcal{E}] \lambda(de) ds
\]

is the solution. And by martingale representation we can get $(Z, \tilde{K})$, by Lemma B.1, there exists a unique $K$ that $K - g(s, Y, Z, K, e) = \tilde{K}$. Notice that $\hat{h}(0, 0, h(0, 0, 0)) = 0$, so
\[
E \left[ \int_0^T \int E [\hat{h}(0, 0, 0)]^2 N(ds, de) \right] = E \left[ \int_0^T \int E [\hat{h}(0, 0, 0) - \hat{h}(0, 0, h(0, 0, 0))]^2 N(ds, de) \right]
= CE \left[ \int_0^T \int E [\hat{h}(0, 0, 0)]^2 N(ds, de) \right] < \infty
\]
Then
\[
E \left[ \int_0^T \int_E |K_s|^2 N(ds, de) \right] = E \left[ \int_0^T \int_E |h(Y_{s-}, Z_s, \hat{K}_s)|^2 N(ds, de) \right] \\
\leq E \left[ \int_0^T \int_E |h(0, 0, 0)|^2 + C \left( |Y_{s-}|^2 + |Z_s|^2 + |\hat{K}_s|^2 \right) N(ds, de) \right] \\
< \infty
\]

Obviously, $K$ is $E$-progressive. So $(Y, Z, K)$ is the unique solution.

Step 3: By step 2, we construct a map $\mathcal{S}$ from $S^2[0, T] \times M^2[0, T] \times F^2[0, T]$ to $S^2[0, T] \times M^2[0, T] \times F^2[0, T]$:

\[
\mathcal{S}(y, z, k) = (Y, Z, K)
\]

Now we prove that $\mathcal{S}$ is a contraction in small time duration. For any $(y^i, z^i, k^i) \in S^2[0, T] \times M^2[0, T] \times F^2[0, T], i = 1, 2$, let $\mathcal{S}(y^i, z^i, k^i) = (Y^i, Z^i, K^i), i = 1, 2$. $\delta \phi = \phi(y^1, z^1, k^1) - \phi(y^2, z^2, k^2)$. Then

\[
E \left[ \sup_{0 \leq t \leq T} |Y^i_t - Y^j_t|^2 \right] \leq CE \left[ \left( \int_0^T |\delta f|ds + \int_0^T \int_E |\delta g||\mathcal{P} \otimes \mathcal{E}| \lambda(de)ds \right)^2 \right] \\
\leq CTE \left[ \int_0^T |y^i_t - y^j_t|^2 + |z^i_t - z^j_t|^2 dt \right] \\
\text{+ } CTE \left[ \int_0^T \int_E |y^i_{t-} - y^j_{t-}|^2 + |z^i_t - z^j_t|^2 + \mathcal{E} \left[ |k^i_t - k^j_t|^2 |\mathcal{P} \otimes \mathcal{E}| \lambda(de)dt \right] \right] \\
\leq CTE \int_0^T |z^i - z^j|^2 dt + \int_0^T \int_E |k^i_t - k^j_t|^2 N(dt, de) + CT^2 E \left[ \sup_{0 \leq t \leq T} |y^i_t - y^j_t|^2 \right]
\]

we set $\hat{K}^i_t := h(t, Y^i_{t-}, Z^i_t, K^i_t, e), i = 1, 2$. Then

\[
E \left[ \int_0^T |Z^i_t - Z^j_t|^2 dt + \int_0^T \int_E |\hat{K}^i_t - \hat{K}^j_t|^2 N(dt, de) \right] \\
\leq CE \left[ \sup_{0 \leq t \leq T} \left( \int_0^T |\delta f|ds + \int_0^T \int_E |\delta g||\mathcal{P} \otimes \mathcal{E}| \lambda(de)ds - Y_0 \right)^2 \right] \\
\leq CE \left[ \left( \int_0^T |\delta f|ds + \int_0^T \int_E |\delta g||\mathcal{P} \otimes \mathcal{E}| \lambda(de)ds + |Y_0| \right)^2 \right] \\
\leq CTE \int_0^T |z^i - z^j|^2 dt + \int_0^T \int_E |k^i_t - k^j_t|^2 N(dt, de) \\
\text{+ } CT^2 E \left[ \sup_{0 \leq t \leq T} |y^i_t - y^j_t|^2 \right]
\]
and

\[
E \left[ \int_0^T \int_E |K_t^1 - K_t^2|^2 N(dt, de) \right] = E \left[ \int_0^T \int_E |\hat{h}(t,e,Y_{t-}^1,Z_{t-}^1,K_t^1) - \hat{h}(t,e,Y_{t-}^2,Z_{t-}^2,K_t^2)|^2 N(dt, de) \right] \\
\leq CE \left[ \int_0^T \int_E |Y_{t-}^1 - Y_{t-}^2|^2 + |Z_{t-}^1 - Z_{t-}^2|^2 + |\hat{K}_t^1 - \hat{K}_t^2|^2 N(dt, de) \right] \\
\leq CT E \left[ \int_0^T |z_t^1 - z_t^2|^2 dt + \int_0^T \int_E |k_t^1 - k_t^2|^2 N(dt, de) \right] + CT^2 E \left[ \sup_{0 \leq t \leq T} |y_t^1 - y_t^2|^2 \right]
\]

So we can choose \( T \) small enough that \( CT < 1 \), then \( \mathcal{F} \) is a contraction, we get the existence and uniqueness in small time duration. For any \( T \), we can split \( T \) into finite small pieces and connect the solution of each piece.

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