Cosmological Evolution of a Statistical System of Degenerate Scalarly Charged Fermions with an Asymmetric Scalar Doublet.

I. Two-Component System of Assorted Charges

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Abstract—Based on the mathematical model of a statistical system with scalar interaction of fermions, formulated earlier, a cosmological model based on a two-component statistical system of scalarly charged degenerate fermions interacting with an asymmetric scalar doublet of canonical and phantom scalar fields, is investigated. The asymptotic and limiting properties of the cosmological model are investigated, and it is shown that among all models there is a class of those with finite lifetime. The asymptotic behavior of the models near the corresponding singularities is investigated, and numerical implementations of such models are constructed.

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1. INTRODUCTION

Two-field models with intersection of the barrier of the cosmological constant $w_{DE} = -1$ are known as quintom models and include one phantom scalar field and one standard (canonical) scalar field (see [1–4]). These models have been intensively studied since 2005 due to the need to explain the observed acceleration of the Universe expansion [5, 6]

$$\Omega = \frac{\ddot{a} a}{a^2} \equiv 1 + \frac{H}{H^2} \equiv -\frac{1}{2}(1 + 3w),$$

where $H$ is the Hubble parameter, $w$ is the barotropic coefficient,

$$w = \frac{p}{\epsilon};$$

$p$ and $\epsilon$ being the total pressure and energy density of cosmological matter.

From a theoretical point of view, such models are very interesting and insufficiently studied. In this regard, we should note the paper [7], in which, based on a study of the dynamical system of a quintom, a counterexample for the typical behavior of the quintom, including attractors with $w \leq 0$, was presented. In [8], a qualitative and numerical analysis of quintom models was carried out under the assumption of nonnegativity of the Hubble parameter ($H \geq 0$). However, it was shown in [9] that this assumption may contradict the complete set of Einstein equations and the scalar field at $H \to 0$. In the same paper, the concept of an Einstein-Higgs hypersurface was introduced, the topology of which largely determines the global properties of cosmological models based on an asymmetric scalar doublet. Thereby, a systematic study of the complete model of the cosmological evolution of an asymmetric scalar doublet (quintom) was carried out in [10] using the methods of qualitative theory of dynamical systems and numerical integration. A wide variety of behaviors of cosmological models depending on the topology of the Einstein-Higgs hypersurface was revealed, in particular, oscillatory modes characterized by alternating phases of expansion and compression were found. Note that in the recent independent papers [11] and [12], a qualitative analysis of cosmological models based on an asymmetric doublet with exponential potentials and chiral fields, respectively, was carried out.

Note that we have no arguments in favour of the vacuum nature of cosmological scalar fields. These fields, like other physical fields, can be associated with the corresponding charges, which, as usual, can correspond to some hypothetical fermions, $\zeta_a$, having scalar charges $q_a$. The interactions between these

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¹¹We did not set the task to write any detailed review of papers on two-field cosmological models based on canonical and phantom vacuum scalar fields. Such a review is contained, for example, in the paper [10] cited above.
fermions and quanta of the scalar field $\phi$ can be carried out in reactions of the type

$$\phi + \overline{\phi} \leftrightarrow \zeta + \overline{\zeta}; \quad \phi \leftrightarrow \zeta + \nu; \ldots \quad (3)$$

This circumstance leads, firstly, to the necessity of constructing a self-consistent mathematical model of the statistical system of scalarly charged fermions interacting with an asymmetric scalar doublet, secondly, to a comprehensive study of the cosmological evolution of such a statistical system, and thirdly, to a physical identification of $\zeta$-fermions, included in the composition of cosmic matter. In this sense, we believe that it is promising to study statistical systems of completely degenerate $\zeta$-fermions, which, using the well-known processes of formation of Cooper pairs, can be transformed into a cold Bose condensate and create a basis for dark matter.

Scalar fields in general-relativistic statistics and kinetics were introduced in the early 80s in the paper [13], and further, a nonminimal theory of scalar interaction was consistently developed on the basis of the concept of a fundamental scalar charge, for both canonical and phantom scalar fields [14–16]. In particular, in these papers, some features of phantom fields were revealed, for example, features of particles’ interaction. Further, in [17–19], a mathematical model of the statistical system of scalarly charged particles was formulated, based on a microscopic description and the subsequent procedure for switching to kinetic and hydrodynamic models. Later, these studies were deepened to extend the theory of scalar fields, including phantom fields, to the sector of negative particle masses, degenerate Fermi systems, conformally invariant interactions, etc. [17–20].

The mathematical models of scalar fields constructed in this way were applied to a study of the cosmological evolution of systems of interacting particles and scalar fields, both of the canonical and phantom types [21–23]. These studies revealed unique features of the cosmological evolution of a plasma with phantom scalar particles’ interaction, such as the existence of giant bursts of cosmological acceleration, the presence of a plateau with constant acceleration, and other anomalies that sharply distinguish the behavior of cosmological models with a phantom scalar field from models with a canonical scalar field.

The paper [23] analyzes the types of behavior of cosmological models based on a scalarly charged system of degenerate fermions with a quadratic interaction potential and identifies 4 main types of behavior that allow, among other things, for intermediate nonrelativistic and ultrarelativistic expansion stages. Let us note that intermediate nonrelativistic stages of cosmological expansion are a necessary condition for the emergence of a large-scale structure of the Universe as a result of the development of gravitational instability. Also let us note that in models based on both vacuum scalar fields and vacuum fields with a perfect fluid, such steps do not occur spontaneously.

In [24], a model of the cosmological evolution of a statistical system of degenerate scalarly charged fermions interacting by means of a single Higgs scalar, canonical or phantom, field is formulated. This paper also provides examples of numerical models of such systems that radically differ in their behavior from the behavior of models based on vacuum scalar fields. On the basis of this model, in the papers by one of the authors [25], the theory of gravitational perturbations of a cosmological system of scalarly charged fermions in the case of a scalar Higgs singlet (canonical or phantom) is developed, and the stability of this cosmological model with respect to short-wave perturbations of the gravitational and scalar fields is investigated. In these papers, it is shown that the fermionic system is unstable at early stages of the cosmological expansion in the case of canonical Higgs interaction and is stable in the case of phantom Higgs interaction.

Finally, in [28], a macroscopic theory of statistical systems of scalarly charged particles was justified at the microscopic level. In this paper, in particular, it was proved that the correct formula for the dynamic mass of scalarly charged particles is [19]

$$m = m_0 + \sum_r q_r \Phi_r, \quad (4)$$

where $m_0$ is some seed (bare) mass of the particle, and $q_r$ is the scalar charge of the particle with respect to the $r$-th scalar field $\Phi_r$. In this case, the total mass of the particle is determined by an absolute value of the dynamic mass

$$m_* = \sqrt{m^2 + p^2},$$

where $p$ is the particle’s momentum. In this paper, as in [19], it is shown that despite the fact that the dynamic mass (4) can take negative values, the macroscopic quantities determined by this mass give physically correct values.

Note that the question of the need for the seed mass $m_0$ in Eq. (4) is quite subtle because it is determined by the fundamental factor of the scalar fields $\Phi_r$ included in this relation. In the event that these fields are fundamental on a pair with the gravitational field and thus determine the masses of all particles, we must put $m_0 = 0$, since otherwise the fundamental property of equality of the masses of particles and antiparticles will be violated. In the case where the scalar fields $\Phi_r$ are secondary and play an intermediate role in the structure and evolution of the Universe, we can store the seed mass in Eq. (4), assuming that it will be removed at a deeper level of interactions.
In this connection, we note the following important property of statistical systems with scalar interaction [28]: in the case of a zero seed mass, \( m_0 \equiv 0 \), such a system, unlike that with electromagnetic interaction, admits a state with zero scalar fields. The interaction with scalar fields, including the appearance of scalar charge density, is purely nonlinear. Thus, the vacuum state of such a system corresponds to zero potentials of scalar fields. Note that the complete Lagrangian of a cosmological system can, in principle, consist of a sum of the form

\[
L = \sum_s L^{(s)} + L_G
\]

\[
\equiv \sum_{r_1} L^{(1)}_{r_1} + \sum_{r_2} L^{(2)}_{r_2} + \ldots + \sum_{r_n} L^{(n)}_{r_n} + L_G,
\]

where \( L_G \) is the Lagrangian of the gravitational field, and \( L^{(s)} \) are Lagrangians of non-interacting physical subsystems. Thus, the real Universe can be an object in the same Riemannian space with minimally interacting physical worlds (physical subsystems). Indirectly, we can estimate the number of such parallel worlds, given the fact that dark matter contributes 22% of the total gravitational density of the Universe, while visible matter is only 4%—this number, taking into account our physical world, can be on the order of \( 5 \div 6 \) (22/4 = 5.5). At the same time, the question of the interaction of fundamental scalar fields, both among themselves and with particles of physical subsystems, remains open.

The present paper is devoted to a study of cosmological models based on the statistical system of scalarly charged fully degenerate fermions with an asymmetric scalar doublet. Thus, our goal is, first, to extend the results of previous works to arbitrary values of the Hubble parameter based on the complete set of Einstein equations; second, to extend the results of previous works to the negative segment of the dynamic mass of fermions, and third, to extend the theoretical model of interaction of fermions with scalar fields to the model of the asymmetric Higgs doublet.

Finally, in [28], two simplest models of the fermions’ interaction with an asymmetric scalar doublet were proposed: in the first model, such an interaction is carried out by two types of assorted fermions, one of which is the source of a canonical scalar field, and the other of a phantom field (model \( \mathcal{M}_1 \)); in the second model, there is a class of fermions with a paired charge — the canonical and phantom (model \( \mathcal{M}_2 \)). A qualitative analysis of the dynamic systems of the corresponding models was also carried out there. In the present paper, we first investigate the first model (\( \mathcal{M}_1 \)) of a two-component statistical system with asymmetric scalar interaction and compare it with the results of previous studies. In this part of the paper, due to a large number of parameters of the model (11), we will restrict ourselves to the case of zero seed mass of \( \zeta \) fermions and to the demonstration of the most typical cases of the behavior of the cosmological model.

2. MATHEMATICAL MODEL OF A COSMOLOGICAL PLASMA OF CHARGED FERMIONS WITH QUINTOM INTERACTION

Consider the space-plane model of the Friedman universe,\(^2\)

\[
ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2).
\]  

Consider a statistical system consisting of \( n \) varieties of degenerate scalar-charged fermions with scalar charges \( q^{(a)} \) with respect to \( N \) scalar fields \( \Phi_{r} \). Let further

\[
m^{(a)} = m_{0}^{(a)} + \sum_{r} q^{(a)}_{r} \Phi_{r},
\]

be the dynamic masses of these fermions; \( L_{s} \) the Lagrange function of non-interacting Higgs scalar fields \( L_{s} = \sum_{r} L_{(r)} \),

\[
L_{s} = \frac{1}{16\pi} \sum_{r} \left( e_{r} g^{ik} \Phi_{r,i} \Phi_{r,k} - 2V_{r}(\Phi_{r}) \right),
\]

\[
V_{r}(\Phi_{r}) = -\frac{\alpha_{r}}{4} \left( \Phi_{r}^{2} - \frac{m_{r}^{2}}{\alpha_{r}} \right)^{2}
\]

is the potential energy of the corresponding scalar fields, \( \alpha_{r} \) are their self-action constants, \( m_{r} \) are masses of their quanta, \( e_{r} = \pm 1 \) are indicators (the “+” sign corresponds to canonical scalar fields, the “−” sign corresponds to phantom fields).

Further, the energy-momentum tensor of scalar fields with respect to the Lagrange function (30) is

\[
T^{i}_{(s)} = \frac{1}{16\pi} \sum_{r} \left( 2e_{r} \Phi_{r}^{i} \Phi_{r,i} - e_{r} \delta^{i}_{k} \Phi_{r,j} \Phi_{r}^{j} + 2V_{r}(\Phi_{r}) \delta^{i}_{k} \right),
\]

and the energy-momentum tensor of an equilibrium statistical system is

\[
\varepsilon T^{i}_{(p)} = \left( \varepsilon_{p} + p_{p} \right) u^{i} u_{k} - \delta^{i}_{k} p,
\]

where \( u^{i} \) is the macroscopic velocity vector of the statistical system, \( \varepsilon_{p} \) and \( p_{p} \) are its energy density and
pressure. Einstein’s equations for the system “scalar fields+particles” have the form

$$R^i_k - \frac{1}{2} \delta^i_k R = 8\pi T^i_k + \delta^i_k \Lambda_0,$$  
(11)

where

$$T^i_k = T^i_{(s)k} + T^i_{(p)k},$$

$\Lambda_0$ is the seed value of the cosmological constant associated with its observed value $\Lambda$ by the relation

$$\Lambda = \Lambda_0 - \frac{1}{4} \sum_r \frac{m^4}{\alpha_r}.$$  
(12)

2.1. General Relations for Models of Degenerate Fermions with Scalar Interaction

It can be proved that a strict consequence of the general-relativistic kinetic theory of statistical systems of completely degenerate fermions is the Fermi momentum conservation law $\pi(a)$ for each component,

$$a(t)\pi(a)(t) = \text{const.}$$  
(13)

Assuming further for certainty $a(0) = 1$ and $\xi = \ln a, \xi \in (-\infty, +\infty), \xi(0) = 0,$ we introduce the dimensionless functions

$$\psi(a) = \frac{\pi^0(a) e^{-\xi}}{|m(a)|} (\pi^0(a) = \pi(a)(0));$$  
(15)

equal to the ratio of the Fermi momentum $\pi(a)$ to the total energy of the fermion, as well as the functions $F_1(\psi)$ and $F_2(\psi)$:

$$F_1(\psi) = \psi \sqrt{1 + \psi^2} - \ln(\psi + \sqrt{1 + \psi^2}),$$  
(16)

$$F_2(\psi) = \psi \sqrt{1 + \psi^2} - \ln(\psi + \sqrt{1 + \psi^2}),$$  
(17)

with the help of which we define the macroscopic scalars of the statistical system: the particle number density of sort $a$,

$$n(a) = \frac{1}{\pi^2} \pi^3;$$  
(18)

the energy density of the fermion system,

$$\varepsilon_p = \sum_a \frac{m^4}{8\pi^2} F_2(\psi(a)),$$  
(19)

the pressure

$$p_p = \sum_a \frac{m^4}{24\pi^2} (F_2(\psi(a)) - 4F_1(\psi(a))),$$  
(20)

and the density of the scalar charge of the fermion system with respect to the scalar field $\Phi_r$,

$$\sigma_r = \sum_a q^r(a) \frac{m^3}{2\pi^2} F_1(\psi(a)).$$  
(21)

Note also the useful relation

$$\varepsilon_p + p_p = \frac{1}{3\pi^2} \sum_a m^4(\psi(a)) \sqrt{1 + \psi^2}.$$  
(22)

Thus, in a cosmological model consisting of a system of degenerate scalar-charged fermions and scalar fields, all macroscopic scalars are defined by explicit algebraic functions of the scalar potentials $\Phi_1(t), \ldots, \Phi_N(t)$.

A complete closed normal autonomous system of ordinary differential equations describing a cosmological model based on a statistical system of fully degenerate scalar charged fermions has the form (see [24, 28])

$$\dot{\xi} = \frac{H}{\Lambda},$$  
(23)

$$\dot{H} = -\sum_r \frac{e_r Z_r^2}{2} - \sum_a \frac{4m^2(\psi(a))}{3\pi} \sqrt{1 + \psi^2},$$  
(24)

$$\dot{\Phi}_r = Z_r \quad (r = 1, N);$$  
(25)

$$e_r \dot{Z}_r = -e_r 3HZ_r - m^2(\Phi_r + \alpha_r \Phi_r^3) - 8\pi \sigma_r(t).$$  
(26)

Remark. Let us note, first, that since the dynamical system (23)–(26) is an autonomous set of differential equations, it is invariant under time translations $t \rightarrow t_0 + t$. Therefore, when studying this system, any value can be chosen as the initial moment of time $t_0$, including $t_0 = 0$. Of course, this value has nothing to do with the “beginning of the universe” in the general case. Due to this circumstance, the time $t$ can be continued into negative values if it is not prevented by a possible singularity.

A strict consequence of the dynamic system (23)–(26) is the total energy integral

$$3H^2 - 8\pi E_{\text{eff}} = \text{const},$$

whose partial zero value corresponds to the Einstein equation (14)

$$3H^2 - \Lambda = \sum_r \left( \frac{e_r Z_r^2}{2} - \frac{m^2(\Phi_r^2)}{2} + \frac{\alpha_r \Phi_r^4}{4} \right),$$

$$-\frac{1}{\pi^2} \sum_a m^4(\psi(a)) \equiv 3H^2 - 8\pi E_{\text{eff}} = 0.$$  
(27)

Equations (27) define a certain hypersurface $\Sigma_E$ in the $2N + 2$-dimensional phase space of the dynamical system (23)–(26). This hypersurface will henceforth be called the Einstein hypersurface. The points of the phase space $\mathbb{R}_{2N+2}$ where the effective energy $E_{\text{eff}}$ is
negative are not available for the dynamical system. The inaccessible region is separated from the accessible region of the phase space by a hypersurface of zero effective energy \( S_E \subset \mathbb{R}_{2N+2} \), which is a cylinder with the \( OH \) axis:

\[
8\pi E_{\text{eff}} \equiv \Lambda + \frac{1}{\alpha} \sum_a m_a^4 F_2(\psi_a) + \sum_r \left( \frac{e_r Z^2}{2} + \frac{m_r^2 \Phi_r^2}{2} - \alpha_r \Phi_r^4 \right) = 0; \quad (28)
\]

moreover, the hypersurface of zero effective energy (28) touches the Einstein hypersurface (27) at the hyperplane \( H = 0 \):

\[
\Sigma_E \cap S_E = H = 0. \quad (29)
\]

The fact that the Einstein equation (27) is a particular first integral of the dynamical system (13)–(26) allows us to use it to determine the initial value of the Hubble parameter \( H_0 = H(0) \), which we will do in the future. Equation (27) is a square with respect to \( H(t) \), so it has two symmetric roots \( \pm H(t) \), where the positive root corresponds to expansion of the universe, and the negative one to its contraction.

2.2. The Main Relations of the Cosmological Model with Quinton Interaction of Fermions (the \( \mathcal{M}_1 \) Model)

Next, \( L_s \) is the Lagrange function of interacting canonical (\( \Phi \)) and phantom (\( \varphi \)) scalar fields,

\[
L_s = \frac{1}{16\pi} \left( g^{ik} \Phi_i \Phi_k - 2V(\Phi) \right) + \frac{1}{16\pi} \left( -g^{ik} \varphi_i \varphi_k - 2V(\varphi) \right), \quad (30)
\]

where

\[
V(\Phi) = -\frac{\alpha}{4} \left( \Phi^2 - \frac{m^2}{\alpha} \right)^2,
V(\varphi) = -\frac{\beta}{4} \left( \varphi^2 - \frac{m^2}{\beta} \right)^2
\]

are the potential energies of the corresponding scalar fields, \( \alpha \) and \( \beta \) are their self–interaction constants, \( m \) and \( m_0 \) are masses of their quanta. As a carrier of quinton charges, we consider a two-component degenerate system of fermions, in which the fermions—carriers of the canonical charge \( q \) have a seed mass \( m_0^c \), with the canonical momentum \( \pi_0^c \), and carriers of the phantom charge \( \zeta \) have a seed mass \( m_0^f \), with the phantom charge \( \epsilon \) and the initial Fermi momentum \( \pi_0^f \). We write out a complete normal set of Einstein equations and those of scalar fields \( \Phi(t) \) and \( \varphi(t) \) for this two-component system of scalarly charged degenerate fermions [28],\(^4\)\(^5\) In an obviously nonsingular form, the normal set of ordinary differential equations of the model under study has the form

\[
\dot{\xi} = H, \quad \dot{\Phi} = Z, \quad \varphi = z, \quad (31)
\]

\[
\dot{H} = -\frac{Z^2}{2} + \frac{e^{-\xi}}{2} - \frac{e^{-3\xi}}{3\pi} \times \left( \frac{\pi_0^c}{\pi_0^f} \sqrt{\varphi^2 - 2e^3 + e^2\Phi^2} \right) + \pi_0^c \sqrt{\varphi^2 - 2e^3 + e^2\Phi^2}, \quad (32)
\]

\[
\dot{Z} = -3HZ - m^2 \Phi + \alpha \Phi^3 - 4e^2 \pi_0^c \pi_0^f e^{-\xi} \Phi \sqrt{\varphi^2 - 2e^3 + e^2\Phi^2} + \frac{4e^4}{\pi} \ln \left( \frac{\pi_0^c e^{-\xi} + \sqrt{\varphi^2 - 2e^3 + e^2\Phi^2}}{|e\varphi|} \right), \quad (33)
\]

\[
\dot{\varphi} = -3HZ + m^2 \varphi - \beta \varphi^3 + 4e^2 \pi_0^c \pi_0^f e^{-\xi} \varphi \sqrt{\varphi^2 - 2e^3 + e^2\varphi^2} - \frac{4e^4}{\pi} \varphi^3 \ln \left( \frac{\pi_0^f e^{-\xi} + \sqrt{\varphi^2 - 2e^3 + e^2\varphi^2}}{|e\varphi|} \right), \quad (34)
\]

where

\[
\psi_c = \frac{\pi_0^c}{|e\varphi|} e^{-\xi}, \quad \psi_f = \frac{\pi_0^f}{|e\varphi|} e^{-\xi}. \quad (35)
\]

The set of equations (31)–(34) has as its first integral the total energy integral [28], which can be used to determine the initial value of the function \( H(t) \):

\[
\frac{Z^2}{2} + \frac{z^2}{2} - \frac{m^2 \Phi^2}{2} + \alpha \Phi^4 - \frac{m^2 \varphi^2}{4} + \beta \varphi^4 - \frac{e^{-\xi}}{\pi} \left( \pi_0^c \varphi^2 - 2e^3 + e^2\Phi^2 \left( 2\pi_0^c e^{-2\xi} + e^2\Phi^2 \right) \right) + \pi_0^c \sqrt{\varphi^2 - 2e^3 + e^2\varphi^2} \left( 2\pi_0^c e^{-2\xi} + e^2\varphi^2 \right) \right) \right) \right) + \frac{e^4 \pi_0^c}{\pi} \ln \left( \frac{\pi_0^c e^{-\xi} + \sqrt{\varphi^2 - 2e^3 + e^2\Phi^2}}{|e\varphi|} \right) + \frac{e^4 \pi_0^f}{\pi} \ln \left( \frac{\pi_0^f e^{-\xi} + \sqrt{\varphi^2 - 2e^3 + e^2\varphi^2}}{|e\varphi|} \right)
\]

\(^4\) For a scalar singlet, this system is obtained in [24].

\(^5\) For simplicity, we have written out a system of dynamic equations for the case of the symmetric mass formula (6), where \( m_0^{(0)} \equiv 0 \), although examples with \( m_0^{(0)} > 0 \) will also be considered below.
In the future, we will call this model $\mathcal{M}_1$.

2.3. The Limit and Asymptotic Properties of the $\mathcal{M}_1$ Model

Note, first, that in the absence of fermions ($\pi_c = \pi_f = 0$), Eqs. (31)–(34), (36) continuously proceed to the set of equations for the vacuum Higgs doublet (see [10]):

$$H = -\frac{Z^2}{2} + \frac{z^2}{2}, \quad (37)$$

$$\dot{Z} = -3HZ - m^2\Phi + \alpha\Phi^3, \quad (38)$$

$$\dot{z} = -3Hz + m^2\varphi - \beta\varphi^3, \quad (39)$$

$$3H^2 - \Lambda = \frac{Z^2}{2} + \frac{z^2}{2} + \frac{m^2\Phi^2}{2} - \frac{\alpha\Phi^4}{4} - \frac{m^2\varphi^2}{2} + \frac{\beta\varphi^4}{4} = 0. \quad (40)$$

Second, note that the beginning of the universe (the cosmological singularity $a = 0$) corresponds to $\xi \to -\infty$, and the infinite future $a \to \infty$ to $\xi \to +\infty$ (if the model allows such a state). One can easily see that as $\xi \to +\infty$, the set of equations (31)–(34) also asymptotically tends to that for the vacuum Higgs doublet (37)–(40). Therefore, if the cosmological model admits the state $\xi \to +\infty$, then, to study the evolution of the cosmological model at later stages, we can apply the results of qualitative and numerical analysis of the vacuum model of the asymmetric scalar doublet [10].

Third, if $\pi_c = 0$, $\varphi = 0$ or $\pi_f = 0$, $\Phi = 0$, this model becomes a model of a single-component scalarly charged Fermi system with the corresponding scalar singlet [24].

Fourth, for zero charges of $\zeta$-fermions, Eqs. (31)–(34), (36) continuously turns to the equations for a cosmological model based on a vacuum asymmetric scalar Higgs doublet and a neutral two-component Fermi fluid.

Finally, we investigate the behavior of the model near the cosmological singularity $\xi \to -\infty$. From Eqs. (31)–(34), it follows that such a state is always possible for $\pi_c, \pi_f \neq 0$. In this case, for $\xi \to -\infty$, $H \to \pm \infty$, $\epsilon \Phi |\Phi| < 1$, $|\varphi| < \infty$, Eqs. (31)–(34) reduces to

$$\dot{\xi} = H; \quad \dot{\Phi} = Z; \quad \dot{\varphi} = z, \quad (41)$$

$$\dot{H} = -\frac{4e^{-4\xi}}{3\pi} (\pi_c^4 + \pi_f^4), \quad (42)$$

Thus near the singularity $t \to t_0$, the scale factor and the Hubble parameter have the following asymptotics:

$$a(t) \bigg|_{t \to t_0} \propto \sqrt{|t - t_0|},$$

$$H(t) \bigg|_{t \to t_0} \propto \frac{1}{|t - t_0|}. \quad (47)$$

Finally, calculating the invariant cosmological acceleration $\Omega (1)$ with (42), (45), and (46), we obtain near the singularity

$$\Omega(t) \bigg|_{t \to t_0} \propto -1, \quad (48)$$

which corresponds, as we know, to the ultrarelativistic equation of state $w = \frac{4}{3}$.

Substituting, finally, (45) and (46) into (43) and (44), we find the asymptotics for the scalar potentials near the cosmological singularity $\xi(t_0) = -\infty$ ($a(t_0) = 0$):

$$\Phi \simeq \tilde{C}_+(t-t_0)^{\frac{2}{3}(1-\theta_\pi)} + \tilde{C}_-(t-t_0)^{\frac{2}{3}(1+\theta_\pi)}$$

$$\varphi \simeq \tilde{c}_+(t-t_0)^{\frac{2}{3}(1-\theta_\pi)}$$

$$+ \tilde{c}_-(t-t_0)^{\frac{2}{3}(1+\theta_\pi)}, \quad (49)$$

6 This condition is not allowed in all cases of model parameters, as can be seen in the examples below.
where
\[ \theta_\mp = 1 \pm \frac{\rho^2}{g}; \quad \theta_\pm = 1 \pm \frac{\rho^2}{g}. \]  

(50)

3. NUMERICAL SIMULATION

3.1. Relation of the \( M_1 \) Model to the Previously Investigated \( M_{00} \) Model

We will call the cosmological model with a one-component system of scalar charged fermions with a scalar singlet with a quadratic interaction potential, a nonnegative dynamic mass, and a nonnegative Hubble parameter \( H \geq 0 \), studied in [22, 23], the \( M_{00} \) model. To move to this model, in the set of equations (31)–(34), (36), it is, first, necessary to turn to zero the potentials and their derivatives for one of the scalar fields, second, to turn to zero the Fermi momentum of the corresponding \( \zeta \)-fermions, and third, to replace the expression for the dynamic mass in the scalar charge density with its absolute value, \( m \rightarrow |m| \); finally, fourth, replace, in the dynamic equations (31)–(34), Eq. (32) with Eq. (36), from which determine the nonnegative root \( H_+ \), and finally put \( \alpha = \beta = 0 \).

Next, to shorten the letter, we will specify a set of fundamental parameters of the \( M_1 \) model using the ordered list
\[ \bm{P} = [\alpha, m, e, m_c, \pi_c], [\beta, \mu, \epsilon, m_f, \pi_f], \Lambda, \]
and the initial conditions form the ordered list
\[ \bm{I} = [\Phi_0, Z_0, \varphi_0, z_0, \chi], \]
where \( \chi = \pm 1 \), and the value of \( \chi = +1 \) corresponds to a nonnegative initial value of the Hubble parameter \( H_0 = H_+ \geq 0 \), while \( \chi = -1 \) corresponds to its negative initial value, \( H_0 = H_- < 0 \). At the same time, using the autonomy of the dynamical system, we everywhere assume \( \xi(0) = 0 \). Thus the \( M_1 \) model is determined by 11 fundamental parameters and 5 initial conditions. The \( M_{00} \) model is determined by 5 fundamental parameters \( \bm{P}_{00} = [\mu, e, q, m, \pi_0, \Lambda] \) and one indicator \( e = \pm 1 \), such that \( e = +1 \) corresponds to a canonical field, and \( e = -1 \) to a phantom field. The initial conditions for this model are set by a list of two elements \( \bm{I}_{00} = [\Phi_0, Z_0] \).

3.2. Comparison of the Behavior of the \( M_1 \) and \( M_{00} \) Models

The main difference between the full \( M_1 \) model and the incomplete \( M_{00} \) model is the emergence of a vibrational nature of the metric functions \( \xi(t) \), \( a(t) \), \( H(t) \) as compared to the monotonic nature of these functions in the \( M_{00} \) model. In Figs. 1, 2, the dependence of evolution of these quantities on the mathematical model is demonstrated.

It should be noted that the monotonic increase of the metric function \( \xi(t) \) in the \( M_{00} \) model is precisely provided by the non-negativity condition for the Hubble parameter.

**Fig. 1.** Evolution of the function \( \xi(t) = \ln(a(t)) \) in different models with a single phantom field: solid line—the \( M_1 \) model, dashed—the \( M_{00} \) model. Everywhere \( \beta = 0 \), \( \mu = 5 \times 10^{-8} \).

**Fig. 2.** Evolution of the Hubble parameter \( H(t) \) in different models with a single phantom field: solid line—the \( M_1 \) model, dashed the \( M_{00} \) model for the parameters of Fig. 1.
4. EXAMPLE OF A STANDARD BEHAVIOR OF THE COSMOLOGICAL MODEL WITH $\Lambda > 0$

Let us consider an example of the “standard” behavior of the cosmological model having all main features of the corresponding model with vacuum Higgs fields:

$$P_0 = \left[[1, 1, 1, 0.1, 0.2], [1, 1, 0.5, 0.1, 0.2], 0.1\right],$$
$$I_0 = [0.1, 0.1, 0.1, 0.1, 1].$$

Figures 3–5 show the cosmological evolution of the metric functions for this model. As can be seen from these figures, this model describes a transition from the compression phase $H_- \approx -0.34, \omega = 1$ to the expansion stage $H_+ \approx +0.34, \Omega = 1$. This case is quite typical for cosmological models with a vacuum asymmetric doublet [10].

Next, in Figs. 6–7, the corresponding phase trajectories of scalar fields are presented. These trajectories are also quite typical for the model with an asymmetric vacuum Higgs doublet, when the trajectory of the canonical field is wound on the zero focus $[0, 0]$, while the trajectory of the phantom field is wound on the right focus $[1, 0]$.

5. EXAMPLE OF AN OSCILLATORY MODE WITH $\Lambda < 0$

Figure 8 shows a typical example of an oscillatory mode of the $\mathfrak{M}_1$ model for the model parameters

$$P_{11} = \left[[1, 1, 1, 0, 0.1], [1, 0.5, 1, 0, 0.1], -0.02\right]$$

and

$$P_{12} = \left[[1, 1, 1, 0, 0.2], [1, 0.5, 1, 0, 0.2], -0.02\right]$$

under the initial conditions $I_1 = [0.2, 0.1, 0.1, 0.1, 1]$.

In Fig. 8 one can see how an increase in the initial Fermi momentum by a factor of 2 for canonical and phantom $\zeta$ fermions leads to a violation of the oscillatory regime of the cosmological model and to...
finiteness of its lifetime, $\Delta t = t_2 - t_1 \approx 80 - (-50) = 130$.

Figures 9 and 10 show the phase trajectories of a dynamical system with the parameters $P_{11}$.

It should be noted that these models are very close to the models with the vacuum asymmetric scalar doublet [10].

6. EXAMPLES OF COSMOLOGICAL MODELS WITH A FINITE HISTORY

As follows from the results of Section 2.3, in systems of scalarly charged fermions it is possible to achieve the cosmological singularity $a(t_1) \to 0$ in the expansion phase of $H(t_1) \to +\infty$ and in the compression phase $a(t_2) \to 0$, $H(t_2) \to -\infty$. Such a universe seems to exist for a limited time $t_2 - t_1$. Note
Fig. 10. Phase trajectory of the dynamic system of the \( \mathcal{M}_1 \) model in the phantom subspace \( \mathbb{R}_3 = \{ \varphi, z, H \} \) for the parameters \( \mathbf{P} = \mathbf{P}_{11} \).

Fig. 11. Evolution of the function \( \xi(t) = \ln(a(t)) \) for the \( \mathcal{M}_1 \) model with the parameters \( \mathbf{P}_1(0.1, 0.1) \).

Fig. 12. Evolution of the function \( \xi(t) = \ln(a(t)) \) for the \( \mathcal{M}_1 \) model with the parameters \( \mathbf{P}_1(0.2, 0.2) \).

Fig. 13. Evolution of the function \( \xi(t) = \ln(a(t)) \) for the model \( \mathcal{M}_1 \) for the parameters \( \mathbf{P}_1(0.19, 0.19) \).

that in the incomplete \( \mathcal{M}_{00} \) model, due to nonnegativity of the Hubble parameter, the universe exists for an infinite time.

We study in more detail numerical models of such a process. Let us first find out how the values of the Fermi momentum affect the lifetime of the universe \( \Delta t \). In the set of parameters of the model \( \mathbf{P}_{11} \), we fix all parameters except for the Fermi pulses,

\[
\mathbf{P}_1(\pi_c, \pi_f) \equiv \{[1, 1, 1, 0, \pi_c], [1, 0.5, 1, 0, \pi_f], -0.02\},
\]

while maintaining the initial conditions

\[
\mathbf{I}_1 = [0.2, 0.1, 0.1, 0.1, 1].
\]

Figures 11–15 show graphs of the evolution of the
scale function $\xi(t)$ depending on the value of the Fermi pulses in the range $0.1 \leq 0.2$.

At the same time, it can be noted that the longest lifetimes of the cosmological model correspond, first, to the values of the momentum $\pi_a = 0.19$, and, second, in the presence of only one of the components of the Fermi system. Third, we can find that the lifetime of the cosmological model is approximately proportional to the number $n$ of oscillations of the scale function $\xi(t)$, and the duration of each oscillation $\tau$ is approximately the same for all the presented cases, $\tau \approx 45$, except for one case (Fig. 14), in which it is equal to $\tau \approx 35$.

The lifetime of the model in the above examples varies within $\Delta t \approx 140 \cdot 10^20$. At the same time, it should be understood that in the absence of one of the charge carriers, the corresponding scalar field evolves as a minimally bound vacuum field in accordance with the corresponding initial conditions.

**CONCLUSION**

Thus, firstly, we have investigated the asymptotic and limiting properties of a cosmological model based on a two-component statistical system of degenerate scalarly charged fermions interacting with an asymmetric scalar doublet. Secondly, we have constructed a corresponding numerical model with the help of which we conducted a comparative analysis of this model with the previously studied incomplete models. Third, we have identified the possibility of oscillatory behavior of the model, as well as the possibility of the existence of models with finite lifetimes.

Summing up the results of this study, we will list its most important results.

- A closed mathematical model of a cosmological system has been studied, consisting of two varieties of degenerate scalarly charged fermions and an asymmetric pair of scalar fields, canonical $\Phi$ and phantom $\varphi$ ones, with Higgs potential energy, forming a dynamic system in 6-dimensional phase space $\mathbb{R}_6 = \{\xi, H, \Phi, Z = 0, \varphi, z = \dot{\varphi}\}$. The dynamic system is completely described by a normal autonomous set of ordinary differential equations in cosmological time $t$. Its behavior in a model with a cosmological term is determined by 11 parameters.

- The asymptotic and limiting properties of the models have been studied. It is shown that in cases that admit infinite expansion of the universe, its asymptotic properties at infinity are completely determined by vacuum scalar fields, i.e., mainly by late inflation. At the same time, the model also allows for finite histories of the universe, in these cases, an exit from a singular state and an entry into it occur according to the asymptotics of the ultrarelativistic model. The ultimate histories of the universe are made possible precisely by the factor of the scalar charge of the fermions.

- By numerical integration methods, the behavior of the cosmological model is compared both with the previously studied incomplete models of the cosmological evolution of charged degenerate fermions, and with the complete model based on a vacuum asymmetric scalar doublet. It is shown that in the first case, the behavior of the models coincides quite closely, except for the appearance of fluctuations in the Hubble parameter in the full model. In the second
case, examples of an almost vacuum behavior of the model under study are shown. Examples of an oscillatory expansion mode characteristic of a model with vacuum fields and a negative cosmological constant are found. Thus, it is shown that the properties of the model coincide with those of the previously studied models in the limiting cases of the parameters.

- The above-mentioned cosmological models with a finite history between the ultrarelativistic exit from a singularity and the ultrarelativistic entry into a new singularity are studied by numerical simulation methods (Figs. 11–15). In this case, the cosmological model in the interval between its beginning and end experiences finite oscillations, whose number is determined by the model parameters. The lifetime of the model is approximately proportional to the number of oscillations.

This preliminary analysis shows a need for a more detailed study of the formulated model and clarifying its dependence on the fundamental parameters of the system. The results of this study will be presented shortly.

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