The circular loop equation of a cosmic string with time-varying tension

Hongbo Cheng∗ Yunqi Liu
Department of Physics, East China University of Science and Technology, Shanghai 200237, China

Abstract

The equation of circular loops of cosmic string with time-dependent tension is studied in the Minkowski spacetime and Robertson-Walker universe. We find that, in the case where the tension depends on some power of the cosmic time, cosmic string loops with time-varying tension should not collapse to form a black hole if the power is lower than a critical value.

PACS number(s): 98.80.Cq

∗E-mail address: hbcheng@public4.sta.net.cn
More than twenty years ago, as a kind of topological defects, cosmic strings including their formation, evolution and observational effects, started to attract much attention [1-3]. As linear defects at a symmetry breaking phase transition in the early universe, cosmic strings can be produced at the end of inflation. They have several potentially important astrophysical features. In particular, one is providing an explanation for the origin of the primordial density perturbations leading to the existence of galaxies and clusters. Another is their spacetime metric with a deficit angle which could give rise to some observational results. However, the improved observational data like CMB anisotropy observations and the WMAP experiments rule out cosmic strings as seeds of large scale structure formation in the universe because of limits on their tension ($G\mu \leq 10^{-6}$) [4-6].

Recently there has been a resurgence of interest in astrophysical results of cosmic strings for both theoretical and observational reasons although cosmic strings can not play an important role in the formation of large scale structure in our universe. Topological defects, including cosmic strings, are inevitably formed at the end of inflation. They can also generate at the end of brane inflation [7, 8], which provides us with a potential window on M theory [9-11]. Furthermore cosmic strings still have strong influence on various astrophysics [1-3] such as gravitational lensing effects [12, 13], gravitational wave background [14, 15], early reionization [16, 17] and so on. A gravitational lens called CSL-1 invoking two imagines of comparable magnitude of the same giant elliptical galaxy were discovered. It is interesting that many similar objects were found in the vicinity [18, 19]. The cosmic string can become an important and powerful explanation for these experimental findings.

It is necessary to explore the evolution of cosmic string loops extensively and deeply. Once cosmic string formed at any epoch in the history of the universe, they are not static and would envolve under their own force of tension continuously instead. They can collide and intersect to undergo reconnections although the strings stretch under the influence of the Hubble expansion or the environment and the strings lose energy to gravitational radiation when they oscillate. The reconnections of long strings and large loops will produce small loops copiously. In general the methods of existence of cosmic strings are string network consisting of long strings and closed string loops. As a complicated time dependent gravitational source, the cosmic string loops oscillate with time rather randomly. On the experimental side, Schild et al observed and analysed the anomalous brightness fluctuation in the multiple-image lens system Q0957+561A, B which has been investigated for 25 years [20, 21]. They think that system consists of two quasar images separated by approximately 6″. The phenomena are known to be images of the same quasar not only because of the spectroscopic match, but also because the images fluctuate in brightness, and the time delay between fluctuations is always the same. The effect may be due to lensing by an oscillating loop of cosmic string between us and the lensing system, because loops of cosmic strings supply a quantitative explanations of such synchronous variations in two images. The cosmic string loops can also generate more gravitational waves and distinct signatures [14, 15, 22]. In a word once the cosmic strings appeared, certainly they evolve to produce the cosmic string loops supported by the possible examples including those above. The evolution and fate of cosmic string loops
attracted more attention. More efforts have been paid for the research in various backgrounds. In some cases like Minkowski spacetime and Robertson-Walker universe, the loops will collapse to form black holes or become a long cosmic string instead of remaining oscillating loops [23, 24]. In de Sitter universe, only loops with larger initial radii can survive [23, 25]. In the Kerr-de Sitter environment, around rotating gravitational sources with positive cosmological constant, a lot of cosmic string loops, including smaller ones, can evolve to survive when the gravitational source rotates faster [26]. According to the examples mentioned above, it is clear that most of cosmic string loops will become black holes unless the loops live around the rotating gravitational source with greater angular momentum in the de Sitter spacetime. It should be pointed out that all results mentioned above are obtained when the tension of cosmic string is chosen to be constant.

It is interesting and significant to investigate the evolution of cosmic string loops with time-dependent tension. So far in nearly all researches, that the tensions of cosmic string are constant is just an assumption. In cosmological situations the cosmic strings with time-varying tension can often appear. M. Yamaguchi put forward the important issue [27]. The tensions of cosmic strings can depend on the cosmic time. For example, a potential containing one complex scalar field $\phi$ and one real scalar field $\chi$ can be written as $V(\phi, \chi) = \frac{\lambda}{4}(|\phi|^2 - \chi^2)^2 + \frac{1}{2}m_\chi^2\chi^2$. The backreaction to the oscillation of field $\chi$ is negligible as the coupling constant $\lambda$ is sufficiently small. The tension of string $\mu$ is associated with the root mean square of the expectation value of field $\chi$ and can be denoted as $\mu \propto a^{-3}$ where $a$ is the scale factor which is proportional to $t^{\frac{1}{2}}$ in the radiation-dominated era and $t^{\frac{3}{2}}$ in the matter-dominated era [27]. Some works on this topic are performed and the interesting and important conclusions are drawn [27, 28]. In the case where the tension depends on the power of the cosmic time assumed as $\mu \propto t^q$, such cosmic strings go into the scaling solution when $q < 1$ in the radiation domination and $q < \frac{3}{2}$ in the matter domination. They also pointed out that the CMB and matter power spectra induced by cosmic strings with time-dependent tension can be different significantly from those generated by the conventional cosmic strings with constant tension, which encourage us to consider that a lot of related topics need to be investigated. Until now little contribution is made to scrutinize the evolution of cosmic string loops which possess time-dependent tension.

The purpose of this paper is to obtain the equation of circular loops of cosmic string with time-varying tension in the Minkowski spacetime and Robertson-Walker universe. We wonder the time dependence of the tension on the evolution and fate of the cosmic string loops. First of all we derive the equations of circular loops of cosmic string in the expanding universe by means of the Nambu-Goto action with an additional factor for the time-dependent tension. We solve the equations numerically to study the evolution of loops and the time dependence of tension on the fate of cosmic string loops. Finally the conclusions and discussions are emphasized.

We start to consider the evolution of cosmic string loops whose tensions are functions of cosmic time in the expanding universe. The Robertson-Walker metric is written as,
\[ ds^2 = dt^2 - R^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2) \] (1)

where the scale factor is,

\[ R(t) = R_0 t^\beta \] (2)

here we choose \( \beta = \frac{1}{2} \) for radiation-dominated era and \( \beta = \frac{2}{3} \) for matter-dominated era. The Nambu-Goto action for a cosmic string with time-dependent tension is given by,

\[ S = -\int d^2\sigma(t)\left[\left(\frac{\partial x}{\partial \sigma^0}\right)\cdot\left(\frac{\partial x}{\partial \sigma^0}\right) - \left(\frac{\partial x}{\partial \sigma^0}\right)\left(\frac{\partial x}{\partial \sigma^1}\right)^2\right]^\frac{1}{2} \] (3)

where \( \mu(t) \) is the string tension and the function of cosmic time. \( \sigma^a = (t, \varphi) \) \((a = 0, 1)\) are timelike and spacelike string coordinates respectively. \( x^\mu(t, \varphi) \) \((\mu, \nu = 0, 1, 2, 3)\) are the coordinates of the string world sheet in the spacetime.

For simplicity let us assume that the string sheet we study lies in the hypersurface \( \theta = \frac{\pi}{2} \), then the spacetime coordinates of the world-sheet parametrized by \( \sigma^0 = t, \sigma^1 = \varphi \) can be chosen as \( x = (t, r(t, \varphi), \frac{\pi}{2}, \varphi) \).

In the case of planar circular loops, we have \( r = r(t) \). According to the metric (1) and the spacetime coordinates mentioned above, the Nambu-Goto action with an additional factor for the time-dependent tension of cosmic string denoted as (3) is reduced to,

\[ S = -\int dt d\varphi \mu(t) r(R - R^2 \dot{r}^2)^\frac{1}{2} \] (4)

which leads to the following equation of motion for loops,

\[ r \ddot{r} + \frac{d \ln \mu}{dt} r \dot{r} (1 - R \dot{r}^2) + \frac{1}{R} - \dot{r}^2 + \frac{3}{2} \frac{\dot{R}}{R} \dot{r} - \ddot{R} r \dot{r}^3 = 0 \] (5)

In order to discuss the equation (5) carefully to explore the evolution of this kind of cosmic string loops, we represent the time-varying tension for simplicity as follow,

\[ \mu(t) = \mu_0 t^q \] (6)

then the equation of motion (5) becomes,

\[ r \ddot{r} + \frac{q}{t} r \dot{r} (1 - R \dot{r}^2) + \frac{1}{R} - \dot{r}^2 + \frac{3}{2} \frac{\dot{R}}{R} \dot{r} - \ddot{R} r \dot{r}^3 = 0 \] (7)

In the case of Minkowski spacetime, i.e. \( R(t) = constant \), equation (7) is reduced to,

\[ r \ddot{r} + \frac{q}{t} r \dot{r} (1 - R \dot{r}^2) - \dot{r}^2 + \frac{1}{R} = 0 \] (8)

If the tension of cosmic string keeps constant as \( q = 0 \), equation (8) has been solved by Vilenkin [2]. The analytical solution \( r(t) = r_0 \cos \frac{t - t_0}{R_0} \) with \( r(t_0) = r_0 \) and \( \dot{r}(t_0) = 0 \) shows the collapsing of all closed string loops. If the cosmic strings possess the time-dependent tension changing as
the power \( q \) of time like (6), equation (8) can be solved numerically. Having performed the very burden and difficult calculation, we find that there must exist a critical value \( q_f = -0.131 \). When \( q < q_f \), all of cosmic string loops will expand to evolve or contrarily will collapse to form black holes when \( q > q_f \) under \( \dot{r}(t_0) = 0 \) no matter how large the initial values of loop radius \( r(t_0) = r_0 \) is equal to. We also discover that the cosmic string loops in the case of time-dependent tension with \( q > q_f \) contract faster than ones in the case of constant tension and \( q \) is not chosen to be a negative integer only when \( \dot{r}(t_0) = 0 \). The evolution of radii of circular loops with time-varying tension in the Minkowski pacetime is depicted in Figure 1. Therefore we have now shown an argument as follow. In the Minkowski spacetime, if the expression of tension depending on some power of the cosmic time satisfies the conditions such as \( q < q_f \) for the case of \( \mu(t) \propto t^q \), the cosmic string loops expand to evolve instead of becoming black holes.

Next, we shall discuss the Robertson-Walker universe for which the scale factor \( R(t) \) denoted as (2). By using (2) and (7), we have,

\[
\ddot{r} + \frac{q}{t} \dot{r} (1 - R_0 t^\beta \dot{r}^2) - R_0 \beta t^{\beta-1} \dot{r}^3 - \dot{r}^2 + \frac{3}{2} \frac{\beta}{t} \dot{r}^2 + \frac{1}{R_0 t^\beta} = 0
\] (9)

For the case of constant tension equivalent to \( q = 0 \), equation (9) can be reduced to,

\[
\ddot{r} - R_0 \beta t^{\beta-1} \dot{r}^3 - \dot{r}^2 + \frac{3}{2} \frac{\beta}{t} \dot{r}^2 + \frac{1}{R_0 t^\beta} = 0
\] (10)

This equation has been solved to show that all cosmic string loops will collapse to become black holes at last in the expanding universe although they will expand a little at first [2, 24]. For the loops of cosmic string whose tension changes as the power of time, equation (9) can be solved numerically by means of burden and surprisingly difficult calculation. We find that there also exist a special value for every era, \( q_r = -0.078 \) for radiation-dominated era and \( q_m = -0.062 \) for matter-dominated era. In each era when \( q < q_r \) or \( q < q_m \) respectively, all cosmic string loops will expand to evolve under \( \dot{r}(t_0) = 0 \) with any values of \( r(t_0) = r_0 \) the initial values of loop radius. Contrarily all loops will become black holes at last while they also contract faster than ones with constant tension. In the Robertson-Walker background the evolutions of radii of circular loops with changeable tension in the radiation-dominated era and matter-dominated era are depicted in Figure 2 and Figure 3 respectively. The power \( q \) is chosen to be a negative value excluding the negative integer only when \( \dot{r}(t_0) = 0 \). The results in the Robertson-Walker universe are similar to those in the Minkowski spacetime. There are only revisions of the critical values for the power \( q \) in comparison with ones in the Minkowski spacetime. It should be stressed that \( q_r > q_f \) and \( q_m > q_f \) appear because the expansion of universe leading the loops enlarge. Therefore, we have now shown an argument as follow. In the Robertson-Walker universe, when the expression of tension depending on some power of the cosmic time satisfies the conditions such as \( q < q_r \) or \( q < q_m \) for the case of \( \mu(t) \propto t^q \), the cosmic string loops expand to evolve instead of becoming black holes. It was found that the cosmic strings whose tension depends on some power of the cosmic time as mentioned above relax into the scaling solution only when \( q < 1 \) in the radiation domination and \( q < \frac{2}{3} \) in the
matter domination [27, 28]. Having compared our findings with the conclusions in [27, 28], we also find that \( q_r < 1 \) and \( q_m < \frac{2}{3} \), which means that such strings with \( q < q_r \) or \( q < q_m \) will form the expanding loops and can also go into the scaling solution. Our results are consistent with those in [27, 28].

According to the observational results, there could exist a lot of cosmic string loops in our universe [20, 21]. Here we report the evolution of planar circular loops of cosmic string with time-varying tension in the Robertson-Walker universe, investigating the fate of loops again. But in the case of cosmic string with constant tension, more efforts were paid to show that all loops will collapse to form black holes in the expanding universe and most loops also become black holes in the de Sitter spacetime except the larger ones, which means that only fewer loops can survive. In the case of cosmic string with changeable tension we let the tension is proportional to the power of time like equation (6) for simplicity and without losing generality. We discover that the loops with conditions \( q < -0.078 \) or \( q < -0.062 \) during radiation-dominated or matter-dominated era respectively will expand to evolve instead of becoming black holes. When the power is negative, the tension will be smaller and smaller and its influence leading the loops contract become weaker and weaker. It is clear that the fate of loops can change significantly if the expression of tension of cosmic string obey the conditions which is certainly different for different expressions. Our findings indicate that there may exist a considerable number of loops of cosmic string but their tension is a time function obeying the specific conditions. We can keep observing the effect of cosmic string loops.

The main results of this paper is equation (5), the circular loop equation for a cosmic string evolving in the hypersurface with \( \theta = \frac{\pi}{2} \) in the Robertson-Walker universe. A remarkable property about this equation is that a loop may never contract to one with a Schwarzschild radius, if the expression for tension of cosmic string satisfies the necessary conditions while these cosmic strings must go into the scaling solution. Therefore a lot of cosmic string loops can evolve to survive in our universe. Our research is a starting point for the case of cosmic string loops with changeable tension although we let the function of tension is some power of cosmic time here. The evolution of cosmic string loops possessing the changeable tension in other spacetime requires further research.

**Acknowledgement**

This work is supported by NSFC No. 10333020 and the Shanghai Municipal Science and Technology Commission No. 04dz05905.
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Figure 1: The solid, dot, dashed curves of \( r(t) \) the radii of circular loops as functions of cosmic time with \( q = -0.05, -0.13, -0.2 \) respectively and initial value \( r(t_0) = 0.6 \) and \( \dot{r}(t_0) = 0 \) in the Minkowski spacetime.
Figure 2: The solid, dot, dashed curves of $r(t)$ the radii of circular loops as functions of cosmic time with $q = -0.02, -0.08, -0.15$ respectively and initial value $r(t_0) = 0.6$ and $\dot{r}(t_0) = 0$ in the radiation-dominated era.
Figure 3: The solid, dot, dashed curves of $r(t)$ the radii of circular loops as functions of cosmic time with $q = -0.01, -0.08, -0.15$ respectively and initial value $r(t_0) = 0.6$ and $\dot{r}(t_0) = 0$ in the matter-dominated era.