A BROAD CLASS OF SHELLABLE LATTICES

JAY SCHWEIG AND RUSS WOODROOFE

ABSTRACT. We introduce a new class of lattices, the modernistic lattices, and their duals, the comodernistic lattices. We show that every modernistic or comodernistic lattice has shellable order complex. We go on to exhibit a large number of examples of (co)modernistic lattices. We show comodernism for two main families of lattices that were not previously known to be shellable: the order congruence lattices of finite posets, and a weighted generalization of the k-equal partition lattices.

We also exhibit many examples of (co)modernistic lattices that were already known to be shellable. To begin with, the definition of modernistic is a common weakening of the definitions of semimodular and supersolvable. We thus obtain a unified proof that lattices in these classes are shellable.

Subgroup lattices of solvable groups form another family of comodernistic lattices that were already proven to be shellable. We show not only that subgroup lattices of solvable groups are comodernistic, but that solvability of a group is equivalent to the comodernistic property on its subgroup lattice. Indeed, the definition of comodernistic exactly requires on every interval a lattice-theoretic analogue of the composition series in a solvable group. Thus, the relation between comodernistic lattices and solvable groups resembles, in several respects, that between supersolvable lattices and supersolvable groups.

1. Introduction

Shellings are a main tool in topological combinatorics. An explicit shelling of a simplicial complex $\Delta$ simultaneously shows the sequentially Cohen-Macaulay property, computes homotopy type, and gives a cohomology basis. Frequently, a shelling also gives significant insight into the homeomorphy of $\Delta$. The downside is that shellings are often difficult to find, and generally require a deep understanding of the complex.

In this paper, we describe a large class of lattices whose order complexes admit shellings. The shellings are often straightforward to explicitly write down, and so give a large amount of information about the topology of the order complex. Included in our class of lattices are many examples which were not previously understood to be closely related.

The question that first motivated this research project involved shelling a particular family of lattices. The order congruence lattice $\mathcal{O}(P)$ of a finite poset $P$ is the subposet of the partition lattice consisting of all equivalence classes arising as the level sets of an order-preserving function. Order congruence lattices interpolate between Boolean lattices and partition lattices, as we will make precise later. Such lattices were already considered by Sturm in [39]. More recently, Körtesi, Radeleczki and Szilágyi showed the order congruence
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A lattice of any finite poset to be graded and relatively complemented \[22\], while Jenča and Sarkoci showed such lattices to be Cohen-Macaulay \[21\].

The Cohen-Macaulay result naturally suggested to us the question of whether every order congruence lattice is shellable. After proving the answer to this question to be “yes”, we noticed that our techniques apply to a much broader class of lattices. Indeed, a large number of the lattices previously shown to be shellable lie in our class. Thus, our main result (Theorem 1.2 below) unifies numerous results on shellability of order complexes of lattices, in addition to proving shellability for new examples. It is our belief that our results will be useful to other researchers. Finding a shelling of a lattice can be a difficult problem. In many cases, showing that a lattice is in our class may be simpler than constructing a shelling directly.

All lattices, posets, simplicial complexes, and groups considered in this paper will be finite.

1.1. Modernistic and comodernistic lattices. We now define the broad class of lattices described in the title and introduction. Our work relies heavily on the theory of modular elements in a lattice. Recall that an element \( m \) of a lattice \( L \) is left-modular if whenever \( x < y \) are elements of \( L \), then the expression \( x \lor m \land y \) can be written without parentheses, that is, that \( (x \lor m) \land y = x \lor (m \land y) \). An equivalent definition is that \( m \) is left-modular if \( m \) is not the nontrivial element in the short chain of any pentagonal sublattice of \( L \); see Lemma 2.10 for a more precise statement.

Our key object of study is the following class of lattices.

**Definition 1.1.** We say that a lattice \( L \) is modernistic if every interval in \( L \) has an atom that is left-modular (in the given interval). We say that \( L \) is comodernistic if the dual of \( L \) is modernistic, that is, if every interval has a left-modular coatom.

Our main theorem is as follows. (We will recall the definition of a CL-labeling in Section 2.3 below.)

**Theorem 1.2.** If \( L \) is a comodernistic lattice, then \( L \) has a CL-labeling.

**Corollary 1.3.** If \( L \) is either comodernistic or modernistic, then the order complex of \( L \) is shellable.

The CL-labeling is explicit from the left-modular coatoms, so Theorem 1.2 also gives a method for computing the Möbius function of \( L \). See Lemma 3.7 for details.

We find it somewhat surprising that Theorem 1.2 was not proved before now. We speculate that the reason may be the focus of previous authors on atoms and CL-labelings, whereas Theorem 1.2 requires dualizing exactly one of the two.

**Remark 1.4.** The name “modernistic” comes from contracting “atomically modular” to “atomically mod”. Since atomic was a common superlative from the the late 1940’s, and since the mod (or modernistic) subculture was also active at about the same time, we find the name to be somewhat appropriate, as well as short and perhaps memorable.
1.2. **Examples and applications.** Theorem 1.2 has a large number of applications, which we briefly survey now. First, we can now solve the problem that motivated the project.

**Theorem 1.5.** If $P$ is any poset, then the order congruence lattice $\mathcal{O}(P)$ is comodernistic, hence $CL$-shellable.

We also recover as examples many lattices already known to be shellable. The following theorem lists some of these, together with references to papers where they are shown to be shellable.

**Proposition 1.6.** The following lattices are comodernistic, hence $CL$-shellable:

1. Supersolvable and left-modular lattices, and their order duals [2, 23, 26].
2. Order duals of semimodular lattices [2]. (I.e., semimodular lattices are modernistic.)
3. $k$-equal partition lattices [7], and their type $B$ analogues [5].
4. Subgroup lattices of solvable groups [33, 43].

We comment that many of these lattices are shown in the provided references to have $EL$-labelings. Theorem 1.2 provides only a $CL$-labeling. Since the $CL$-labeling constructed is explicit from the left-modular elements, Theorem 1.2 provides many of the benefits given by an $EL$-labeling. We do not know if every comodernistic lattice has an $EL$-labeling, and leave this interesting question open.

Experts in lattice theory will immediately recognize items (1) and (2) from Proposition 1.6 as being comodernistic. Theorem 1.2 thus unifies the theory of these well-understood lattices with the more difficult lattices on the list. The $CL$-labeling that we construct in the proof of Theorem 1.2 can moreover be seen as a generalization of the standard $EL$-labeling for a supersolvable lattice, further connecting these classes of lattices.

We will prove that $k$-equal partition lattices and their type $B$ analogues are comodernistic in Section 6. In Section 6.3 we will show the same for a new generalization of $k$-equal partition lattices. The proofs show, broadly speaking, that coatoms of (intervals in) these subposets of the partition lattice inherit left-modularity from that in the partition lattice.

Although modernism and comodernism give a simple and unified framework for showing shellability of many lattices, not every shellable lattice is (co)modernistic. For an easy example, the face lattice of an $n$-gon has no left-modular elements when $n > 3$, so is neither modernistic nor comodernistic.

1.3. **Further remarks on subgroup lattices.** We can expand on the connection with group theory suggested by item (4) of Proposition 1.6. For a group $G$, the subgroup lattice referred to in this item consists of all the subgroups of $G$, ordered by inclusion; and is denoted by $L(G)$.

**Theorem 1.7.** If $G$ is a group, then $G$ is solvable if and only if $L(G)$ is comodernistic.

Stanley defined supersolvable lattices in [34] to abstract the interesting combinatorics of the subgroup lattices of supersolvable groups to general lattices. Theorem 1.7 says that comodernism is one possibility for a similar abstraction for soluble groups. A result of a
similar flavor was earlier proved by Schmidt [30]; our innovation with comodernism is to require a lattice-theoretic analogue of a composition series in every interval of the lattice. We further discuss possible notions of solvability for lattices in Section 5.

Shareshian in [33] showed that a group $G$ is solvable if and only if $L(G)$ is shellable. Theorems 1.2 and 1.7 give a new proof of the “only if” direction of this result. For the “if” direction, Shareshian needed a hard classification theorem from finite group theory. Our proof of Theorem 1.7 does not rely on hard classification theorems. On the other hand, it follows directly from Shareshian’s proof that $G$ is solvable if $L(G)$ is sequentially Cohen-Macaulay. Thus, Shareshian’s Theorem gives a topological characterization of solvable groups. The characterization given in Theorem 1.7 is lattice-theoretic, rather than topological. It is an interesting open problem to give a classification-free proof that if $L(G)$ is sequentially Cohen-Macaulay, then $G$ is solvable.

1.4. Organization. This paper is organized as follows. In Section 2 we recall some of the necessary background material. We prove Theorem 1.2 (our main theorem) in Section 3.

In the remainder of the paper we show how to apply comodernism and Theorem 1.2 to various classes of lattices. The techniques may be illustrative for those who wish to prove additional classes of lattices to be comodernistic. In Section 4 we examine order congruence lattices, and prove Theorem 1.5. In Section 5 we prove Theorem 1.7 and argue for comodernism as a notion of solvable for lattices. We close in Section 6 by showing that $k$-equal partition lattices (and variations thereof) are comodernistic.

Acknowledgements

We would like to thank Vincent Pilaud and Michelle Wachs for their helpful remarks. We also thank the anonymous referee for his or her thoughtful comments. The example in Figure 5.1 arose from a question of Hugh Thomas. The second author is grateful to the University of Miami for their hospitality in the spring of 2016, during which time a part of the paper was written.

2. Preliminaries

We begin by recalling some necessary background and terminology. Many readers will be able to skip or skim this section, and refer back to it as necessary.

2.1. Posets, lattices, and order complexes. A poset $P$ is bounded if $P$ has a unique least element $\hat{0}$ and greatest element $\hat{1}$.

Associated to a bounded poset $P$ is the order complex, denoted $\Delta P$, a simplicial complex whose faces consist of all chains (totally ordered subsets) in $P \setminus \{\hat{0}, \hat{1}\}$. In particular, the vertices of $\Delta P$ are the chains of length 0 in $P \setminus \{\hat{0}, \hat{1}\}$, that is, the elements of $P \setminus \{\hat{0}, \hat{1}\}$. The importance of the order complex in poset theory arises since the Möbius function $\mu(P)$ (important for inclusion-exclusion) is given by the reduced Euler characteristic $\tilde{\chi}(\Delta P)$.

We often say that a bounded poset $P$ possesses a property from simplicial topology (such as “shellability”), by which we mean that $\Delta P$ has the same property.
We say that a poset $P$ is Hasse-connected if the Hasse diagram of $P$ is connected as a graph. That is, $P$ is Hasse-connected if and only if for any $x, y \in P$, there is a sequence $x = x_0, x_1, \ldots, x_k = y$ such that $x_i$ is comparable to $x_{i+1}$ for each $i$.

A poset $L$ is a lattice if every two elements $x, y \in L$ have a unique greatest lower bound (the meet $x \wedge y$) and unique least upper bound (the join $x \vee y$). It is obvious that every lattice is bounded, hence has an order complex $\Delta L$.

A poset is graded if all its maximal chains have the same length, where the length of a chain is one less than its cardinality. The height of a bounded poset $P$ is the length of the longest chain in $P$, and the height of an element $x$ is the height of the interval $[\hat{0}, x]$. An atom of a bounded poset $P$ is an element of height 1.

The order dual of a poset $P$ is the poset $P^*$ with reversed order relation, so that $x <^* y$ in $P^*$ exactly when $x > y$ in $P$. Poset definitions may be applied to the dual by prepending a “co”: for example, an element $x$ is a coatom if $x$ is an atom in $P^*$.

For more background on poset and lattice theory from a general perspective, we refer to [37]. For more on order complexes and poset topology, we refer to [42].

2.2. Simplicial complexes and shellings. We assume basic familiarity with homology and cohomology, as exposited in e.g. [18, 27].

A shelling of a simplicial complex $\Delta$ is an ordering of the facets of $\Delta$ that obeys certain conditions, the precise details of which will not be important to us. Not every simplicial complex has a shelling; those that do are called shellable.

We remark that in the early history of the subject, shellings were defined only for balls and spheres [29]. Later, shellings were considered only for pure complexes, that is, complexes all of whose facets have the same dimension. Nowadays, shellings are studied on arbitrary simplicial complexes [7].

Shellable complexes are useful for showing a complex to satisfy the Cohen-Macaulay (in the pure case) or sequentially Cohen-Macaulay property (more generally). These properties are important in commutative algebra as well as combinatorics.

We refer to [36] for more on shellable and Cohen-Macaulay complexes.

2.3. CL-labelings and EL-labelings. The definition of a shelling is often somewhat unwieldy to work with directly, and it is desirable to find tools through which to work. One such tool is given by a CL-labeling, which we will now define.

If $x$ and $y$ are elements in a poset $P$, we say that $y$ covers $x$ when $x < y$ but there is no $z \in P$ so that $x < z < y$. In this situation, we write $x \lessdot y$, and may also say that $x \lessdot y$ is a cover relation. Thus, a cover relation is an edge in the Hasse diagram of $P$. A rooted cover relation is a cover relation $x \lessdot y$ together with a maximal chain from $\hat{0}$ to $x$ (called the root).

A rooted interval is an interval $[x, y]$ together with a maximal chain $r$ from $\hat{0}$ to $x$. In this situation, we use the notation $[x, y]_r$. Notice that every atomic cover relation of $[x, y]_r$ can be rooted by $r$.

A chain-edge labeling of a bounded poset $P$ is a function $\lambda$ that assigns an element of an ordered set (which will always for us be $\mathbb{Z}$) to each rooted cover relation of $P$. Then $\lambda$ assigns
a word over \(\mathbb{Z}\) to each maximal chain on any rooted interval by reading the cover relation labels in order, so e.g. the word associated with \(\hat{0} \preceq x_1 \preceq x_2 \preceq x_3 \preceq \ldots\) is \(\lambda(\hat{0} \preceq x_1, \hat{0})\lambda(x_1 \preceq x_2, \hat{0} < x_1)\lambda(x_2 < x_3, \hat{0} < x_1 < x_2) \cdots\).

**Remark 2.1.** Since many researchers may be less familiar with \(CL\)-labelings and the machinery behind them, it may be helpful to think of a chain-edge labeling via the following dynamical process. Begin at \(\hat{0}\), and walk up the maximal chain \(\hat{0} = x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq \hat{1}\). At each step \(i\), assign a label to the label \(x_{i-1} \preceq x_i\). In assigning the label, you are allowed to look backwards at where you have been, but are not allowed to look forwards at where you may go. At each step, you add the assigned label to the end of a word associated with the maximal chain.

We say that a maximal chain \(c\) is **increasing** if the word associated with \(c\) is strictly increasing, and **decreasing** if the word is weakly decreasing. We order maximal chains by the lexicographic order on the associated words.

**Definition 2.2.** A \(CL\)-labeling is a chain-edge labeling that satisfies the following two conditions on each rooted interval \([x, y]_r\):

1. There is a unique increasing maximal chain \(m\) on \([x, y]_r\), and
2. the increasing chain \(m\) is strictly earlier in the lexicographic order than any other maximal chain on \([x, y]_r\).

If a \(CL\)-labeling \(\lambda\) assigns the same value to every \(x \preceq y\) irrespective of the choice of root, then we say \(\lambda\) is an **EL-labeling**.

Björner [2] and Björner and Wachs [6, 7] introduced \(CL\)-labelings, and proved the following theorem.

**Theorem 2.3.** [8, Theorem 5.8] If \(\lambda\) is a \(CL\)-labeling of the bounded poset \(P\), then the lexicographic order on the maximal chains of \(P\) is a shelling order of \(\Delta P\). In this case, a cohomology basis for \(\Delta P\) is given by the decreasing maximal chains of \(P\), and \(\Delta P\) is homotopy equivalent to a bouquet of spheres in bijective correspondence with the decreasing maximal chains.

For this reason, bounded posets with a \(CL\)- or \(EL\)-labeling are often called \(CL\)- or \(EL\)-shellable. Since the order complex of \(P\) and that of the order dual of \(P\) coincide, either a \(CL\)-labeling or a dual \(CL\)-labeling implies shellability of \(\Delta P\).

From the cohomology basis, it is straightforward to compute Euler characteristic, hence also Möbius number.

**Corollary 2.4.** [8, Proposition 5.7] If \(\lambda\) is a \(CL\)-labeling of the bounded poset \(P\), then the Möbius number of \(P\) is given by

\[
\mu(P) = \tilde{\chi}(\Delta P) = \#\text{even length decreasing maximal chains in } P - \#\text{odd length decreasing maximal chains in } P.
\]
2.4. Order congruence lattices. If \( P \) and \( Q \) are posets, then a map \( \varphi : P \to Q \) is order-preserving if whenever \( x \leq y \), it also holds that \( \varphi(x) \leq \varphi(y) \). The level set partition of a map \( \varphi : P \to Q \) is the partition with blocks of the form \( \varphi^{-1}(q) \). If \( \pi \) is the level set partition of an order preserving map \( \varphi : P \to Q \), then \( \pi \) is an order partition of \( P \). Since every poset has a linear extension, it is easy to see that it would be equivalent to restrict the definition of order partition to the case where \( Q = \mathbb{Z} \).

As previously defined, the order congruence lattice \( \mathcal{O}(P) \) is the subposet of the partition lattice \( \Pi_P \) consisting of all order partitions of \( P \). The cover relations in \( \mathcal{O}(P) \) correspond to merging blocks in an order partition, subject to a certain compatibility condition.

Example 2.5. Consider \( P = [3] \) with the usual order. Then the function mapping 1, 2 to 1 and 3 to 2 is order-preserving, so \( 12|3 \in \mathcal{O}([3]) \). Similarly, the partition \( 1|23 \in \mathcal{O}([3]) \). It is not difficult to see, however, that there is no order-preserving map with level set partition \( 13|2 \). Thus, the lattice \( \mathcal{O}([3]) \) is isomorphic to the Boolean lattice on 2 elements.

More generally, an elementary argument shows that the order congruence lattice of a chain on \( n \) elements is isomorphic to a Boolean lattice on \( n - 1 \) elements. It is obvious that the order congruence lattice of an antichain is the usual partition lattice. Thus, order congruence lattices interpolate between Boolean lattices and partition lattices.

It is easy to confuse \( \mathcal{O}(P) \) with another closely related lattice defined on a poset \( P \). We say that a subset \( S \subseteq P \) is order convex if whenever \( a \leq b \leq c \) with \( a, c \in S \), then also \( b \in S \). The order convexity partition lattice of \( P \), denoted \( \mathcal{O}^{\text{conv}}(P) \), consists of all partitions where every block is order convex. There is some related literature on the related lattice of all order convex subsets of a poset, going back to [1].

We do not know if order convexity lattices must always be comodernistic or shellable, as intervals of the form \([\pi, \hat{1}]\) in \( \mathcal{O}^{\text{conv}} \) seem difficult to describe.

Example 2.6. Consider the bowtie poset \( B \), with elements \( a_1, a_2, b_1, b_2 \) and relations \( a_i < b_j \) (for \( i, j \in \{1, 2\} \)). As \( B \) has height 1, all subsets are order convex, so that \( \mathcal{O}^{\text{conv}}(B) \cong \Pi_4 \). However, the partitions \( a_1 b_1 | a_2 b_2 \) and \( a_1 b_2 | a_2 b_1 \) are not order congruence partitions, so are not in \( \mathcal{O}(B) \).

We additionally caution the reader that the notion of order congruence considered here is less restrictive than that considered in [28], where congruences are required to respect lower/upper bounds.

In another point of view, it is straightforward to show that order-preserving partitions are in bijective correspondence with certain quotient objects of \( P \). Thus, the order congruence lattice assigns a lattice structure to quotients of \( P \). See [21, Section 3] for more on the quotient view of \( \mathcal{O}(P) \).

For our purposes, it will be enough to understand intervals above atoms in \( \mathcal{O}(P) \). Say that elements \( x, y \) of poset \( P \) are compatible if either \( x < y \), \( y < x \), or \( x, y \) are incomparable. If \( x, y \) are compatible in \( P \), then \( P_{x \sim y} \) is the poset obtained by identifying \( x \) and \( y \). That is, \( P_{x \sim y} \) is obtained from \( P \) by replacing \( x, y \) with \( w \), subject to the relations \( z < w \) whenever
z < x or z < y, and z > w whenever z > x or z > y. We remark in passing that this identification is an easy special case of the quotienting viewpoint discussed above.

**Lemma 2.7.** Let \( P \) be a poset. A partition \( \pi \) of \( P \) is an atom of \( \mathcal{O}(P) \) if and only if \( \pi \) has exactly one non-singleton block consisting of compatible elements \( \{x, y\} \). In this situation, we have the lattice isomorphism \( [\pi, \hat{1}]_{\mathcal{O}(P)} \cong \mathcal{O}(P_{x \sim y}) \).

Repeated application of Lemma 2.7 allows us to understand any interval of the form \([\pi, \hat{1}]\) in \( \mathcal{O}(P) \).

Although we will not use need this, intervals of the form \([\hat{0}, \pi]\) are also not difficult to understand. Let \( \pi \) have blocks \( B_1, B_2, \ldots, B_k \). It is well-known (see e.g. [37, Example 3.10.4]) that an interval of this form in the full partition lattice is isomorphic to the product of smaller partition lattices \( \Pi_{B_1} \times \cdots \times \Pi_{B_k} \). It is straightforward to see via the order-preserving mapping \( P \to \mathbb{Z} \) definition that a similar result holds in the order congruence lattice. That is, \([\hat{0}, \pi]\) in \( \mathcal{O}(P) \) is lattice-isomorphic to \( \mathcal{O}(B_1) \times \cdots \times \mathcal{O}(B_k) \), where \( B_i \) refers to the induced subposet on \( B_i \subseteq P \). Combining this observation with Lemma 2.7 allows us to write any interval in \( \mathcal{O}(P) \) as a product of order congruence lattices of quotients of subposets. We find it simpler to give more direct arguments, but readers familiar with poset products may appreciate this connection.

### 2.5. Supersolvable and semimodular lattices

We previously defined an element \( m \) of a lattice \( L \) to be left-modular if \((x \lor m) \land y = x \lor (m \land y)\) for all pairs \( x < y \).

A lattice is modular if every element is left-modular. A lattice \( L \) is usually defined to be semimodular if whenever \( a \land b \preceq a \) in \( L \), then \( b \preceq a \lor b \). We prefer the following equivalent definition, which highlights the close connection between semimodularity and comodernism:

**Lemma 2.8** (see e.g. [38, essentially Theorem 1.7.2]). A lattice \( L \) is semimodular if and only for every interval \([x, y]\) of \( L \), every atom of \([x, y]\) is left-modular (as an element of \([x, y]\))

Thus, the definition of a modernistic lattice is obtained from that of a semimodular lattice by weakening a single universal quantifier to an existential quantifier.

An \( M \)-chain in a lattice is a maximal chain consisting of left-modular elements. A lattice is left-modular if it has an \( M \)-chain, and supersolvable if it is graded and left-modular.

Supersolvable lattices were originally defined by Stanley [34], in a somewhat different form. The theory of left-modular lattices was developed in a series of papers [10, 23, 24, 26], and it was only in [26] that it was noticed that Stanley’s original definition of supersolvable is equivalent to graded and left-modular.

There is an explicit cohomology basis for a supersolvable lattice, which does not seem to be as well-known as is deserved. A chain of complements to an \( M \)-chain \( m = \{\hat{0} = m_0 \preceq m_1 \preceq \cdots \preceq m_n = \hat{1}\} \) is a chain of elements \( c = \{\hat{0} = c_n \prec c_{n-1} \prec \cdots \prec c_0 = \hat{1}\} \) so that each \( c_i \) is a complement to \( m_i \), that is, so that \( c_i \lor m_i = \hat{1} \) and \( c_i \land m_i = \hat{0} \). A less explicit form of the following appears in [2, 34], and a special case in [41].
Theorem 2.9. If $L$ is a supersolvable lattice with a fixed $M$-chain $m$, then a cohomology basis for $\Delta L$ is given by the chains of complements to $m$. In particular, the Möbius number of $L$ is (up to sign) the number of such chains.

A (strong form of a) homology basis for supersolvable lattices appears in [32].

2.6. Left-modularity. We now recall some additional basic properties of left-modular elements. First, we state more carefully the equivalent “no pentagon” condition mentioned in the Introduction.

Lemma 2.10. \[24, Proposition 1.5\] An element $m$ of the lattice $L$ is left-modular if and only if for every $a < c$ in $L$, we have $a \land m \neq c \land m$ or $a \lor m \neq c \lor m$.

The pentagon lattice (usually notated as $N_5$) consists of elements $\hat{0}, \hat{1}, a, b, c$ with the only nontrivial relation being $a < c$. Lemma 2.10 says exactly that $m$ is left-modular if and only if $m$ never plays the role of $b$ in a sublattice of $L$ isomorphic to $N_5$. Thus, Lemma 2.10 is a pleasant generalization of the characterization of modular lattices as those with no pentagon sublattices.

Another useful fact is:

Lemma 2.11. \[23, Proposition 2.1.5\] If $m$ is a left-modular element of the lattice $L$, and $x < y$ in $L$, then $x \lor m \land y$ is a left-modular element of the interval $[x, y]$.

Finally, we give an alternate characterization of left-modularity of coatoms, in the flavor of Lemma 2.8. This characterization will be useful for us in the proof of Theorems 1.2 and 3.5 and is also often easy to check.

Lemma 2.12 (Left-modular Coatom Criterion). Let $m$ be a coatom of the lattice $L$. Then $m$ is left-modular in $L$ if and only if for every $y$ such that $y \not\leq m$ we have $m \land y < y$.

Proof. If $m \land y < z < y$, then $z < y$ violate the condition of Lemma 2.10 with $m$, hence $m$ is not left-modular. Conversely, if $z < y$ violate the condition of Lemma 2.10 then $y \not\leq m$, and $m \land y = m \land z < z < y$.

Corollary 2.13. If $L$ is a lattice and $m$ a left-modular coatom of $L$, then for any $x < y$ in $L$ either $x \lor m \land y = y$ or else $x \lor m \land y < y$.

Proof. If $x \not\leq m$ or $y \leq m$, then $x \lor m \land y = y$. Otherwise, apply Lemmas 2.11 and 2.12.

2.7. Group theory. We recall that a group $G$ is said to be solvable if either of the following equivalent conditions is met:

(1) There is a chain $1 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = G$ of subgroups in $G$, so that each $N_i$ is normal in $G$, and so that each factor $N_i/N_{i-1}$ is abelian.

(2) There is a chain $1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = G$ of subgroups in $G$, so that each $H_i$ is normal in $H_{i+1}$ (but is not necessarily normal in $G$). Note that it follows in this case that each factor $H_i/H_{i-1}$ is cyclic of prime order.

Since every subgroup of a solvable group is solvable, an alternative form of the latter is:
(2’) For every subgroup $H \subseteq G$, there is a subgroup $K \subseteq H$ such that $K \triangleleft H$.

A subgroup $H$ of $G$ is said to be subnormal if there is a chain $H \triangleleft L_1 \triangleleft L_2 \triangleleft \cdots \triangleleft G$. Thus, Condition (2) says a group is solvable if and only if $G$ has a maximal chain consisting of subnormal subgroups.

A group is supersolvable if there is a maximal chain in $L(G)$ consisting of subnormal subgroups normal in $G$. Thus, a group is supersolvable if there is a chain which simultaneously meets the conditions in (1) and (2). One important fact about the subgroup lattice of supersolvable groups is:

**Theorem 2.14.** [20] For a group $G$, the subgroup lattice $L(G)$ is graded if and only if $G$ is supersolvable.

Subgroup lattices were one of the motivations for early lattice theorists in making the definition of (left-)modularity. It follows easily from the Dedekind Identity (see Lemma 5.3) that if $N \triangleleft G$, then $N$ is left-modular in $L(G)$. In particular, if $G$ is a supersolvable group, then $L(G)$ is a supersolvable lattice.

Moreover, a normal subgroup $N$ satisfies a stronger condition. An element $m$ of a lattice is said to be modular (or two-sided-modular) if it neither plays the role of $b$ nor of $a$ in any pentagon sublattice, where $a, b$ are as in the discussion following Lemma 2.10. A second application of the Dedekind Identity shows any normal subgroup to be modular in $L(G)$.

The following lemma, whose proof is immediate from the definitions, says that left-modularity and two-sided-modularity are essentially the same for the purpose of comodernism arguments.

**Lemma 2.15.** If $L$ is a lattice, and $m$ is a maximal left-modular element, then $m$ is modular.

We refer to e.g. [12, Chapter A] for further general background on group theory, and to [31] for the reader interested in further background on lattices of subgroups.

### 3. Proof of Theorem 1.2

#### 3.1. sub-$M$-chains.

As discussed in Section 2.5, a lattice is left-modular if it has an $M$-chain, that is, a maximal chain consisting of left-modular elements. The reader may be reminded of the maximal chain consisting of normal elements in the definition of a supersolvable group.

We extend the notion of $M$-chain to comodernistic lattices. A maximal chain $\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}$ in $L$ is a sub-$M$-chain if for every $i$, the element $m_i$ is left-modular in the interval $[\hat{0}, m_{i+1}]$. The reader may be reminded of the maximal subnormal chain in a solvable group. It is straightforward to show that a lattice is comodernistic if and only if every interval has a sub-$M$-chain.

Stanley [35] and Björner [2] showed that any supersolvable lattice has an EL-labeling, and Liu [23] extended this to any left-modular lattice. If $m^{(ss)} = \{\hat{0} = m_0^{(ss)} \lessdot m_1^{(ss)} \lessdot \cdots \lessdot m_n^{(ss)} = \hat{1}\}$
is an $M$-chain, then the $EL$-labeling is defined as follows:

$$\lambda_{ss}(x \lessdot y) = \max\{i : x \lor m^{(ss)}_{i-1} \land y = x\}$$
$$= \min\{i : x \lor m^{(ss)}_i \land y = y\}.$$  

The essential observation involved in proving Theorem 1.2 is that, if we replace the $M$-chain used for $\lambda_{ss}$ with a sub-$M$-chain, then we can still label the atomic cover relations of $L$ in the same manner as in $\lambda_{ss}$. More precisely, if $m$ is a sub-$M$-chain in a lattice $L$, then let

$$\lambda(0 \lessdot a) = 1 + \max\{i : m_i \land a = 0\}.$$  

Adding 1 is not essential, and we do so only so that the labels will be in the range 1 through $n$, rather than 0 through $n - 1$.

3.2. The $CL$-labeling. We construct the full $CL$-labeling recursively from (3.2).

We say that a chain $c$ is indexed by a subset $S = \{i_1 < \cdots < i_k\}$ of the integers if $c = \{c_{i_1} < \cdots < c_{i_k}\}$. That is, we associate an index or label with each element of the chain. Notice that we require the indices to (strictly) respect order.

We will need the following somewhat-technical lemma to handle non-graded lattices.

**Lemma 3.1.** Let $L$ be a lattice with a sub-$M$-chain $m$ of length $n$. Then no chain of $L$ has length greater than $n$.

**Proof.** We proceed by induction on $n$. The base case is trivial. Suppose that $\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_\ell = \hat{1}$ is some chain in $L$. Let $m_{n-1} < 1$ be the unique coatom in $m$, and $i$ be the greatest index such that $c_i \leq m_{n-1}$. Then $c_j \lor m_{n-1} = \hat{1}$ for any $j > i$, so by the left-modular property

$$c_{i+1} \land m_{n-1} < c_{i+2} \land m_{n-1} < \cdots < c_\ell \land m_{n-1} = m_{n-1}.$$  

Thus,

$$\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_i = c_{i+1} \land m_{n-1} < c_{i+2} \land m_{n-1} < \cdots < c_\ell \land m_{n-1} = m_{n-1}$$

is a chain of length $\ell - 1$ on $[\hat{0}, m_{n-1}]$, and by induction $\ell - 1 \leq n - 1$. \qed

**Definition 3.2.** Let $L$ be a comodernistic lattice of height $n$. Take a fixed sub-$M$-chain $m$ given as $\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}$ as the starting point for a recursive construction.

Let $x \lessdot a, r$ be a rooted cover relation. Assume by recursion that we are given a sub-$M$-chain $m^{(r)}$ on $[x, \hat{1}]$. Further assume that the elements of $m^{(r)}$ are indexed by a subset $S \subseteq [n] \cup \{0\}$, and that $\hat{1} = m^{(r)}_n$. Label $x \lessdot a$ as in (3.2) that is, as

$$\lambda(x \lessdot a) = 1 + \max\{i : m_i^{(r)} \land a = x\}.$$  

To continue the recursion, it remains to construct an indexed sub-$M$-chain $m^{(r \cup a)}$ on $[a, \hat{1}]$.

Suppose that $\lambda(x \lessdot a) = 1 + i$. It is clear that $m_i^{(r)}$ is the greatest element of $m^{(r)}$ such that $a \nleq m_i^{(r)}$. By abuse of notation, let $m_{>i}^{(r)}$ be the portion of $m$ that is greater than $m_i^{(r)}$,
and let $S_{>i}$ be the indices greater than $i$ on $m^{(r)}$. Thus, the labels of $m^{(r)}_{>i}$ are exactly $S_{>i}$. Let $S_{<i} = S \setminus (S_{>i} \cup i)$ similarly be the indices less than $i$ on $m^{(r)}$.

Now by construction, all elements of $m^{(r)}_{>i}$ are greater than $a$. By the comodernistic property, the submodular chain $m^{(r)}_{>i}$ may be completed to a sub-$M$-chain $m^{(r,ja)}$ for $[a, \hat{1}]$. Preserve the indices on $m^{(r)}_{>i}$, and index the elements of $m^{(r,ja)} \setminus m^{(r)}$ by elements of $S_{<i}$. It follows by applying Lemma 3.1 on $[\hat{0}, m_{i+1}]$ that there are enough indices available in $S_{<i}$ to perform such indexing.

The recursion can now continue, which completes the definition of the $CL$-labeling.

**Notation 3.3.** Throughout the remainder of Section 3 we fix $L$ to be a comodernistic lattice of height $n$, with a sub-$M$-chain $m = \{ \hat{0} = m_0 < m_1 < \cdots < m_n = \hat{1} \}$. Indeed, we select a sub-$M$-chain on every interval, which uniquely determines a chain-edge labeling $\lambda$ as in Definition 3.2.

**Remark 3.4.** By repeated application of Lemma 2.11 it follows that if $L$ has an $M$-chain $m = \{ \hat{0} = m_0 < m_1 < \cdots < m_n = \hat{1} \}$, then the set $\{ u \lor m_i \land v \}$ is an $M$-chain for the interval $[u, v]$. With this choice of (sub)-$M$-chain on each interval, the labeling $\lambda$ of Notation 3.3 coincides with $\lambda_{ss}$.

We are now ready to prove the following refinement of Theorem 1.2.

**Theorem 3.5.** The labeling $\lambda$ of Notation 3.3 is a $CL$-labeling.

**Proof.** It is clear from construction that $\lambda$ is a chain-edge labeling. By the recursive construction, it suffices to show that an interval of the form $[\hat{0}, y]$ has a unique increasing maximal chain, and that every lexicographically first maximal chain on $[\hat{0}, y]$ is increasing.

Let $m = \{ \hat{0} = m_0 < m_1 < \cdots < m_n = \hat{1} \}$ be the sub-$M$-chain used to define the labeling. Let $\ell = 1 + \max \{ i : m_i \land y < y \}$. It is clear from the construction that every atomic cover relation on $[\hat{0}, y]$ receives a label that is at most $\ell$. Since the elements greater than $m_{\ell-1}$ are preserved until the corresponding labels are used, no chain on $[\hat{0}, y]$ receives any label greater than $\ell$.

Similarly, the $m_{\ell-1}$ element in the sub-$M$-chain is by construction preserved until the $\ell$ label is used, and a chain receives an $\ell$ label when it leaves the interval $[\hat{0}, m_{\ell-1}]$. Since $y \notin [\hat{0}, m_{\ell-1}]$, we see that every maximal chain on $[\hat{0}, y]$ receives an $\ell$ label. Thus, an increasing chain must have the $\ell$ label on its last cover relation.

But Corollary 2.13 gives $m_{\ell-1} \land y$ to be a coatom of $[\hat{0}, y]$. It follows by the definition of the labeling that every increasing chain on $[\hat{0}, y]$ must end with $m_{\ell-1} \land y < y$. An easy induction now yields the only increasing chain to be $\hat{0} = m_0 \land y \leq m_1 \land y \leq \cdots \leq m_{\ell-1} \land y$, the “projection” of the sub-$M$-chain to $[\hat{0}, y]$. As $m_{\ell-1} \land y < y$, the projection chain is in particular maximal.

We now show that this chain is the unique lexicographically first chain. In the construction, the least label of an atomic cover relation on $[\hat{0}, y]$ corresponds with the least $m_{i+1}$ such that...
Figure 3.1. A comodernistic labeling of a lattice

$m_{i+1} \land y > \hat{0}$. But this is the (unique) first non-$\hat{0}$ element of the increasing chain. The desired follows.

3.3. More details about the CL-labeling. Any chain-edge labeling assigns a word to each maximal chain of $L$. Since when we label a cover relation with $i$ according to $\lambda$, we remove $i$ from the index set (used for available labels), we obtain a result extending one direction of [25, Theorem 1].

Lemma 3.6. The chain-edge labeling $\lambda$ assigns a word with no repeated labels to each maximal chain in $L$.

Thus, if $L$ is graded of height $n$, then $\lambda$ assigns a permutation in $S_n$ to each maximal chain.

The decreasing chains are also easy to (recursively) understand. Recall that if $x$ and $y$ are lattice elements, then $x$ is a complement to $y$ if $x \lor y = \hat{1}$ and $x \land y = \hat{0}$. The following is an extension of Theorem [2.9] for comodernistic lattices.

Lemma 3.7. If $\hat{0} \ll c_1 \ll \cdots \ll \hat{1}$ is a decreasing chain of $L$ with respect to $\lambda$, then $c_1$ is a complement to $m_{n-1}$.

Proof. As in the proof of Theorem [3.5], every maximal chain on $[\hat{0}, \hat{1}]$ contains an $n$ label. Thus $\lambda(\hat{0} \ll c_1) = n$, so $m_{n-1} \land c_1 = \hat{0}$. The result follows.

It is natural to ask whether theorems about supersolvable geometric lattices (see e.g. [34]) extend to comodernistic geometric lattices. The answer to this question is positive, but for rather uninteresting reasons:

Proposition 3.8. If $L$ is a geometric lattice, then any sub-$M$-chain of $L$ is also an $M$-chain. Thus, a lattice $L$ is geometric and comodernistic if and only if $L$ is geometric and supersolvable.
Proof. A result of Brylawski [11, Proposition 3.5] says that if $m$ is modular in a geometric lattice $L$, and $x$ is modular in $[\hat{0}, m]$, then $x$ is also modular in $L$. The result now follows from Lemma 2.15 and an inductive argument. □

In contrast to Proposition 3.8, lattices that are not geometric may have sub-$M$-chains which are not $M$-chains. We close this section by working out a small example in detail.

Example 3.9. We consider the lattice $C$ in Figure 3.1, which we obtained by removing a single cover relation from the Boolean lattice on 3 elements. It is easy to check that $m_2$ is modular in $C$, but that $m_1$ is not modular in $C$. (Indeed, $m_1$ together with $b < c$ generate a pentagon sublattice.) Since any lattice of height at most 2 is modular, the chain $\hat{0} < m_1 < m_2 < \hat{1}$ is a sub-$M$-chain, though not an $M$-chain.

With the exception of $c < \hat{1}$, the label of every cover relation in $C$ is independent of the choice of root. We have indicated these labels in the diagram. But we notice that the interval $[a, \hat{1}]$ inherits the sub-$M$-chain $a < m_2 < \hat{1}$, while the interval $[b, \hat{1}]$ has unique maximal (sub-$M$-)chain $b < c < 1$. Thus, the edge $c < \hat{1}$ receives a label of 1 with respect to root $\hat{0} < a < c$, but a label of 2 with respect to root $\hat{0} < b < c$.

The reader may have noticed that the atom $a$ is indeed left-modular. Thus, although we have shown the comodernistic labeling determined by the given sub-$M$-chain, there is also a supersolvable $EL$-labeling of $C$. We will see in Example 4.7 and Figure 4.1 a lattice that is neither geometric nor supersolvable, but that is comodernistic.

4. ORDER CONGRUENCE LATTICES

In this section, we examine the order congruence lattices of posets, as considered in the introduction and in Section 2.4. We prove Theorem 1.5 and apply Lemma 3.7 to calculate the Möbius number of $O(P)$.

4.1. Order congruence lattices are comodernistic. A useful tool for showing certain lattices to be comodernistic is given by the following lemma.

Lemma 4.1. Let $L$ be a meet subsemilattice of a lattice $L_+$. If $m \in L_+$ is a left-modular coatom in $L_+$, and $m \in L$, then $m$ is also left-modular in $L$.

Proof. Since $m$ is a coatom in $L_+$ and therefore in $L$, the join of $x$ and $m$ is either $m$ (if $x \leq m$) or $\hat{1}$ (otherwise) in both lattices. In particular, the join operations in $L$ and $L_+$ agree on $m$, and we already know the meet operations agree by the subsemilattice condition. The result now follows by Lemma 2.11. □

The following theorem follows immediately.

Theorem 4.2. If $L$ is a meet subsemilattice of the partition lattice $\Pi_S$, and $m \in L$ is a partition $x \mid (S \setminus x)$ for some element $x \in S$, then $m$ is a left-modular coatom of $L$.

We now show:
Lemma 4.3. If $P$ is any poset, then the order congruence lattice $O(P)$ is a meet subsemilattice of $\Pi_P$.

Proof. It is clear from definition that $O(P)$ is a subposet of $\Pi_P$. It suffices to show that if $\pi_1, \pi_2$ are in $O(P)$, then their meet $\pi_1 \land \pi_2$ also is in $O(P)$. Let $f_1, f_2 : P \to \mathbb{Z}$ be such that $\pi_1$ and $\pi_2$ are the level sets of $f_1$ and $f_2$. But then the product map $f_1 \times f_2 : P \to \mathbb{Z} \times \mathbb{Z}$ (where $\mathbb{Z} \times \mathbb{Z}$ is taken with the product order) has the desired level set partition. □

That order congruence lattices are comodernistic now follows easily.

Proof (of Theorem 1.5). Let $P$ be a poset, and let $x$ be a maximal element of $P$. Assume by induction that the result holds for all smaller posets. It is straightforward to see $m = x \upharpoonright (P \setminus x)$ is the level set partition of an order preserving map. By Theorem 4.2 and Lemma 4.3, the element $m$ is a left-modular coatom on the interval $[\hat{0}, \hat{1}]$. Since $[0, m]$ is lattice-isomorphic to $O(P \setminus x)$, we get by induction that $[\hat{0}, \pi]$ is comodernistic when $\pi < m$. If $\pi$ is incomparable to $m$, then $\pi \land m$ is a left-modular coatom of $[0, \pi]$ by Corollary 2.13. Finally, repeated application of Lemma 2.7 and induction gives that intervals of the form $[\pi', \hat{1}]$ are comodernistic. The result follows for general intervals $[\pi', \pi]$. □

4.2. The Möbius number of an order congruence lattice. We now use the comodernism of the order congruence lattice $O(P)$ to recover the Möbius number calculation due to Jenča and Sarkoci. Denote by $\text{Compat}(x)$ the set of all $y \in P$ that are compatible with $x$. That is, $\text{Compat}(x)$ consists of all $y$ such that either $y \lessdot x$, $x \lessdot y$, or $y$ is incomparable to $x$.

Our proof is short and simple.

Theorem 4.4. [21, Theorem 3.8] For any poset $P$ with maximal element $x$, the Möbius function of the order congruence lattice satisfies the recurrence

$$\mu(O(P)) = - \sum_{y \in \text{Compat}(x)} \mu(O(P_{x \sim y})).$$

Proof. By Lemma 3.7 together with the proof of Theorem 1.5, every decreasing chain of $O(P)$ begins with a complement to the (left-modular) order partition $x \upharpoonright (P \setminus x)$. Such complements are easily seen to be atoms $a$ whose non-singleton block is $\{x, y\}$, where $y \in P$ is compatible with $x$. The result now follows by Lemma 2.7. □

Jenča and Sarkoci also show in [21] that if $P$ is a Hasse-connected poset, then the number of linear extensions of $P$ satisfies the same recurrence as $\mu(O(P))$. We give a short bijective proof of the same, which has the same flavor as the proof of the main result in [13]. Let $\mathcal{LE}(P)$ denote the set of linear extensions of $P$.

Lemma 4.5. Let $P$ be a poset and $x$ a maximal element of $P$. If $x$ is not also minimal, then there is a bijection

$$\mathcal{LE}(P) \to \bigcup_{y \in \text{Compat}(x)} \mathcal{LE}(P_{x \sim y}).$$
Proof. Since $x$ is not minimal, it cannot be the first element in any linear extension $L$ of $P$. If $y$ is the element immediately preceding $x$ in $L$, then it is clear that $x$ and $y$ are compatible. Then $L_{x\sim y}$ is a linear extension of $P_{x\sim y}$.

To show this map is a bijection, we notice that the process is reversible. If $L$ is a linear extension of $P_{x\sim y}$, replace the element corresponding to the identification of $x$ and $y$ with $x$ followed by $y$ to get a linear extension of $P$. □

Corollary 4.6. If $P$ is a Hasse-connected poset, then the decreasing chains of $O(P)$ are in bijective correspondence with the linear extensions of $P$.

In particular, $|\mu(O(P))|$ is the number of linear extensions of $P$, and $\Delta O(P)$ is homotopy equivalent to a bouquet of this number of $(|P| - 3)$-dimensional spheres.

Proof. Since $P$ is Hasse-connected, a maximal element $x$ cannot also be minimal. Moreover, if $P$ is Hasse-connected then $P_{x\sim y}$ is also Hasse-connected. The result now follows immediately by Theorem 4.4 and Lemma 4.5. □

In the case where $P$ is not Hasse-connected, a similar approach can be followed. Indeed, the same argument as in Lemma 4.5 applies, except that we must discard the linear extensions that begin with $x$ in each recursive step where they arise. This argument identifies the decreasing chains of $O(P)$ with a recursively-defined subset of the linear extensions of $P$. We do not have a non-recursive description of this subset. Jenča and Sarkoci give a somewhat different description of $|\mu(O(P))|$ for a non-Hasse-connected poset $P$ in [21, Theorem 4.5].

4.3. An order congruence lattice that is neither geometric nor supersolvable.

Example 4.7. Consider the pentagon lattice $N_5$, obtained by attaching a bottom and top element $\hat{0}$ and $\hat{1}$ to the poset with elements $a, b, c$ and relation $a < c$. In this case, $O(N_5)$ and $O^{\text{conv}}(N_5)$ coincide, and are pictured in Figure 4.1.

The reader can verify by inspection that no atom of $O(N_5)$ is left-modular, thus, the lattice $O(N_5)$ is comodernistic but neither geometric nor supersolvable. As some of the coatoms of $O(N_5)$ are not left-modular, the dual of $O(N_5)$ also fails to be geometric. We remark that order congruence lattices that fail to be geometric were examined earlier in [22].

5. Solvable subgroup lattices

In this section, we discuss applications to and connections with the subgroup lattice of a group.

5.1. Known lattice-theoretic analogues of classes of groups. Since the early days of the subject, a main motivating object for lattice theory has been the subgroup lattice of a finite group. Indeed, a (left-)modular element may be viewed as a purely lattice-theoretic analogue or extension of a normal subgroup. Focusing on the normal subgroups characterizing a class of groups then typically gives in a straightforward way an analogous class of lattices with interesting properties. For example, every subgroup of an abelian (or more
Figure 4.1. The order congruence lattice $\mathcal{O}(N_5)$ of the pentagon lattice $N_5$. Nontrivial left-modular elements are shown with rectangles.

| Group class | Lattice class | Characterizes group class? | Self-dual? |
|-------------|--------------|---------------------------|-----------|
| cyclic      | distributive | Yes                       | Yes       |
| abelian, Hamiltonian | modular   | No                        | Yes       |
| nilpotent   | lower semimodular | No              | No        |
| supersolvable | supersolvable | Yes                      | Yes       |
| solvable    | ??           | ??                        | ??        |

Table 1. Classes of groups and related classes of lattices.

generally Hamiltonian) group is normal, so a corresponding class of lattices is that of the modular lattices.

We summarize some of these analogies in Table 1.

We remark that, although every normal subgroup is modular in the subgroup lattice, not every modular subgroup is normal. Similarly, although every nilpotent group has lower
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semimodular subgroup lattice, group that are not nilpotent may also have lower semimodular subgroup lattice. For example, the subgroup lattice of the symmetric group on 3 elements \( L(S_3) \) has height 2, hence is modular (despite being neither abelian nor nilpotent). As \( L(S_3) \) is lattice isomorphic to \( L(\mathbb{Z}_3^2) \), a subgroup lattice characterization of these classes of groups is not possible.

It may then be surprising that a group \( G \) is supersolvable if and only if \( L(G) \) is a supersolvable lattice. The reason for this is more superficial than one might hope: Iwasawa proved in [20] that a group is supersolvable if and only if its subgroup lattice is graded. However, the definition of supersolvable lattice seems to capture the pleasant combinatorial properties of supersolvable groups much better than the definition of graded lattice does.

Semimodular and supersolvable lattices have been of great importance in algebraic and topological combinatorics. In particular, both classes of lattices are \( EL \)-shellable, and the \( EL \)-labeling gives an efficient method of computing homotopy type, Möbius invariants, etc.

5.2. Towards a definition of solvable lattice. After the previous subsection, it may come as some surprise that there is no widely-accepted definition of solvable lattice. It is the purpose of this subsection to make the case for the definition of comodernistic lattices as one good candidate.

It was independently proved by Suzuki in [40] and Zappa in [45] that solvable groups are characterized by their subgroup lattices. Later Schmidt gave an explicit characterization:

**Proposition 5.1** (Schmidt [30]; see also [31, Chapter 5.3]). For a group \( G \), the following are equivalent:

1. \( G \) is solvable.
2. \( L(G) \) has a chain of subgroups \( 1 = G_0 \subset G_1 \subset \cdots \subset G_k = G \) such that each \( G_i \) is modular in \( L(G) \), and such that each interval \( [G_i, G_{i+1}] \) is a modular lattice.
3. \( L(G) \) has a chain of subgroups \( 1 = G_0 \subset G_1 \subset \cdots \subset G_n = G \) such that each \( G_i \) is modular in the interval \( [1, G_{i+1}] \).

The reader will recognize the conditions in Proposition 5.1 as direct analogues of the conditions from Section 2.7. Despite this close correspondence, proving that Conditions (2) and (3) of the proposition imply solvability is not at all trivial.

Although Proposition 5.1 combinatorially characterizes solvable subgroup lattices, we find it somewhat unsatisfactory. We don’t know how to use the implicit lattice conditions to calculate Möbius numbers. And, as the following example will show, lattices satisfying the implicit conditions need not be shellable.

**Example 5.2.** Consider the lattice whose Hasse diagram is pictured in Figure 5.1. The element \( M \) is easily verified to be modular in this lattice, and the interval \([0, M]\) is a Boolean lattice, hence modular. But since the interval \([C, \hat{1}]\) is disconnected, the lattice is not shellable or Cohen-Macaulay.

It is our opinion that a good definition of “solvable lattice” should be equivalent to solvability on subgroup lattices, and obey as many of the useful properties possessed by supersolvable
lattices as possible. Among these are: $EL$-shellability, efficient computation of homotopy type and/or homology bases and/or Möbius numbers, and self-duality of the property. Perhaps even more importantly, such a definition should have many combinatorial examples.

We believe comodernistic lattices to be an important step towards understanding a definition or definitions of “solvable lattice”. As Theorem 1.7 states, comodernism is equivalent to solvability on subgroup lattices. While we do not know if a comodernistic lattice is always $EL$-shellable, we have shown such a lattice to be $CL$-shellable. The $CL$-labeling allows efficient calculation of homotopy type, and consequences thereof. Perhaps most importantly, there are many natural examples of comodernistic lattices.

Unfortunately, comodernism is not a self-dual property, as is made clear by Example 4.7.

5.3. **Proof of Theorem 1.7.** We begin the proof by reviewing a few elementary facts about groups and subgroup lattices, all of which can be found in [31], or easily verified by the reader. We say that $H$ permutes with $K$ if $HK = KH$.

**Lemma 5.3.** Let $H$, $K$, and $L$ be subgroups of a group $G$.

1. If $H$ permutes with $K$, then $HK = KH$ is the join in $L(G)$ of $H$ and $K$.
2. If $N \triangleleft G$, then $N$ permutes with every subgroup of $G$.
3. (Dedekind Identity) If $H \subseteq K$, then $H(K \cap L) = K \cap HL$ and $(K \cap L)H = K \cap LH$.

**Corollary 5.4.** If $K \supset H$ permutes with every subgroup on the interval $[H, G]$, then $K$ is a modular element of this interval.

By Lemma 2.15, the elements of a sub-$M$-chain of a comodernistic lattice satisfy the modularity condition as in part (3) of Proposition 5.1. It follows immediately by the same proposition that $G$ is solvable if $L(G)$ is comodernistic.

For the other direction, every subgroup of a solvable group is solvable. Thus, it suffices to find a modular coatom in the interval $[H, G]$ over any subgroup $H$. Let $1 = N_0 \subseteq N_1 \subseteq
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\[
\cdots \subseteq N_k = G \text{ be a chief series. It follows that each } HN_i \text{ is a subgroup for each } i. \text{ Let } \ell \text{ be the maximal index such that } HN_\ell < G, \text{ and let } K \text{ be any coatom of the interval } [HN_\ell, G].
\]

We will show that \( K \) permutes with every subgroup \( L \) on \([H, G]\), hence is modular on the same interval. For any such \( L \), we have

\[
H(L \cap N_{\ell+1}) = L \cap HN_{\ell+1} = L \cap G = L, \text{ and similarly }
\]

\[
(L \cap N_{\ell+1})H = L \cap N_{\ell+1}H = L \cap G = L,
\]

so that \( H \) permutes with \( L \cap N_{\ell+1} \). Moreover, \( N_\ell \subseteq K \cap N_{\ell+1} \subseteq N_{\ell+1} \), and since \( N_{\ell+1}/N_\ell \) is abelian, the Correspondence Theorem gives that \( K \cap N_{\ell+1} \triangleleft N_{\ell+1} \). Thus, \( K \cap N_{\ell+1} \) permutes with \( L \cap N_{\ell+1} \). Now

\[
KL = H(K \cap N_{\ell+1})H(L \cap N_{\ell+1}) = H(L \cap N_{\ell+1})H(K \cap N_{\ell+1}) = LK,
\]

as desired.

5.4. Homotopy type of the subgroup lattice of a solvable group. It is immediate from Lemma 5.3 that if \( N \triangleleft G \) then \( HN \) permutes with all subgroups on the interval \([H, G]\). Thus, a chief series \( \{N_i\} \) lifts to a chain of left-modular elements \( \{HN_i\} \) on the interval \([H, G]\). In a solvable group we can (by the proof of Theorem 1.7) complete the chain \( \{HN_i\} \) to a sub-M-chain. Let \( \lambda \) be constructed according to this choice of sub-M-chain in all applicable intervals, and consider the decreasing chains of \( \lambda \).

It is immediate by basic facts about left-modular elements that a decreasing maximal chain on \( L(G) \) contains as a subset a chain of complements to the chief series \( \{N_i\} \). Since a chain of complements to \( \{N_i\} \) in a solvable group is a maximal chain \([12, \text{Lemma 9.10}]\), such chains are exactly the decreasing maximal chains.

The order complex of a \( CL \)-shellable poset is a bouquet of spheres, where the spheres are in bijective correspondence with the decreasing chains of the poset. Thus, we recover the homotopy-type calculation of \([41]\) (see also \([43, 44]\)).

6. \( k \)-equal partition and related lattices

In this section, we will show that the \( k \)-equal partition lattices are comodernistic. We’ll also show two related families of lattices to be comodernistic.

6.1. \( k \)-equal partition lattices. Recall that the \( k \)-equal partition lattice \( \Pi_{n,k} \) is the sub-poset of the partition lattice \( \Pi_n \) consisting of all partitions whose non-singleton blocks have size at least \( k \).

**Theorem 6.1.** For any \( 1 \leq k \leq n \), the \( k \)-equal partition lattice \( \Pi_{n,k} \) is comodernistic.

**Proof.** Consider an interval \([\pi', \pi]\) in \( \Pi_{n,k} \). Let \( \pi \) be \( C_1 \mid C_2 \mid \ldots \mid C_m \), and assume without loss of generality that \( C_1 \) is not a block in \( \pi' \). Then \( C_1 \) is formed by merging blocks \( B_1, \ldots, B_\ell \) of \( \pi' \). Suppose that the \( B_i \)'s are ordered by increasing size, so that \( |B_1| \leq |B_2| \leq \cdots \). Consider the element

\[
m = B_1 \mid B_2 \cup \cdots \cup B_\ell \mid C_2 \ldots \mid C_m.
\]
Suppose that \( \sigma \) is some partition on \([\pi', \pi]\), and \( D \) is the block of \( \sigma \) containing \( B_1 \). If \( D = B_1 \), then \( \sigma \land m = \sigma \). Otherwise, there are two cases:

**Case 1:** \(|B_1| > 1\). Then \( \sigma \land m \) is formed by splitting \( D \) into blocks \( B_1, D \setminus B_1 \). Notice that \(|D \setminus B_1| \geq |B_1| \geq k\) by the ordering of the \( B_i \)’s.

**Case 2:** \(|B_1| = 1\). Then if \(|D| > k\), then \( \sigma \land m \) is formed by splitting \( D \) into smaller blocks \( B_1, D \setminus B_1 \). Otherwise, we have \(|D| = k\), and \( \sigma \land m \) is formed by splitting \( D \) into \( k \) singletons.

In either situation, we have \( \sigma \land m \preceq \sigma \), so Lemma 2.12 gives \( m \) to be left-modular on the desired interval.

We recover from Theorem 6.1 a weaker form of the result [7, Theorem 6.1] that \( k \)-equal partition lattices are \( EL \)-shellable. Repeated application of Lemma 3.7 recovers the same set of decreasing chains for the comodernistic labeling as in [7, Corollary 6.2]. See also [9].

**6.2. \( k, h \)-equal partition lattices in type \( B \).** By a *sign pattern* of a set \( S \), we refer to an assignment of + or − to each element of \( S \), considered up to reversing the signs of every element of \( S \). Thus, if \( S \) has an order, an equivalent notion is to assign a + to the first element of \( S \) and either + or − to each remaining element.

A *signed partition* of \( \{0, 1, \ldots, n\} \) then consists of a partition of the set, together with a sign pattern assignment for each block not containing 0. The block containing 0 is called the *zero block*, and other blocks are called *signed blocks*. The *signed partition lattice* \( \Pi_B^n \) consists of all signed partitions of \( \{0, 1, \ldots, n\} \). The cover relations in \( \Pi_B^n \) are of two types: merging two signed blocks, and selecting one of the two possible patterns of the merged set; or merging a signed block with the zero block (thereby ‘forgetting’ the sign pattern on the signed block).

The signed partition lattice is well-known to be supersolvable. Indeed, if \( \pi \) is a signed partition where every signed block is a singleton, then \( \pi \) is left-modular in \( \Pi_B^n \).

Björner and Sagan [5] considered the *signed \( k, h \)-equal partition lattice*, where \( 1 \leq h < k \leq n \). This is the subposet \( \Pi_{n,k,h}^B \) consisting of all signed partitions whose non-singleton signed blocks have size at least \( k \), and whose zero block is either a singleton or has size at least \( h + 1 \).

**Theorem 6.2.** For any \( 1 \leq h < k \leq n \), the signed \( k, h \)-equal partition lattice \( \Pi_{n,k,h}^B \) is comodernistic.

**Proof.** We proceed similarly to the proof of Theorem 6.1 Let \([\pi', \pi]\) be an interval in \( \Pi_{n,k,h}^B \), and \( \pi \) be \( C_1 \mid C_2 \mid \ldots \mid C_m \). Assume without loss of generality that \( C_1 \) is not a block in \( \pi' \). Then \( C_1 \) is formed by merging blocks \( B_1, \ldots, B_\ell \) of \( \pi' \). Let \( B_1 \) be the smallest signed block in this list, and consider the element

\[ m = B_1 \mid B_2 \cup \cdots \cup B_\ell \mid C_2 \ldots \mid C_m. \]

Now let \( \sigma \) be some partition in the interval \([\pi', \pi]\), and \( D \) be the block of \( \sigma \) containing \( B_1 \). If \( D = B_1 \), then \( \sigma \land m = \sigma \). Otherwise, there are two cases:
Case 1: $|B_1| > 1$. Since $\sigma$ is on $[\pi', \pi]$, we see that $D \subseteq C_1$. If $D$ is a signed block, then $|D \setminus B_1| \geq |B_1| \geq k$ by the ordering of the blocks. Otherwise, by choice of $B_1$, no part of $\sigma \land m$ is a signed singleton block contained in $C_1$. It follows that $|D \setminus B_1| \geq h + 1$. In either situation, it follows that $\sigma \land m$ is formed by splitting $D$ into blocks $B_1, D \setminus B_1$.

Case 2: $|B_1| = 1$. If $|D| > k$, then $\sigma \land m$ is formed by splitting $D$ into smaller blocks $B_1, D \setminus B_1$. Similarly if $0 \in D$ and $|D| > h + 1$. Otherwise, we have $|D| = k$ or $|D| = h$ (depending on whether $0 \in D$), and $\sigma \land m$ is formed by splitting $D$ into singletons.

In either case, we have $\sigma \land m \leq \sigma$, hence that $m$ is left-modular on the desired interval by Lemma 2.12.

Björner and Sagan [5, Theorem 4.4] showed $\Pi_{n,k,h}^B$ to be EL-shellable. We recover from Theorem 6.2 the weaker result of CL-shellability. However, we remark that our proof is significantly simpler, and still allows easy computation of a cohomology basis, etc.

There is also a “type D analogue” of $\Pi_{n,k}$ and $\Pi_{n,k,h}^B$. Björner and Sagan considered this lattice in [5], but left the question of shellability open. Feichtner and Kozlov gave a partial answer to the type D shellability question in [16].

Our basic technique in the proofs of Theorems 6.1 and 6.2 is to show that left-modularity of coatoms in $\Pi_n$ and $\Pi_n^B$ is sometimes inherited in the join subsemilattices $\Pi_{n,k}$ and $\Pi_{n,k,h}^B$. It is easy to verify from the (here omitted) definition that the type D analogue of $\Pi_n$ and $\Pi_n^B$ has no left-modular coatoms, see also [19]. For this reason, the straightforward translation of our techniques will not work in type $D$. We leave open the question of under what circumstances the type D analogue of $\Pi_{n,k}$ has left-modular coatoms, or is comodernistic.

6.3. Partition lattices with restricted element-block size incidences. The $k$-equal partition lattices admit generalizations in several directions. One such generalization, examined in [7], is that of the subposet of partitions where the size of every block is in some set $T$. Further generalizations in similar directions are studied in [14, 15].

We consider here a different direction. Motivated by the signed $k, h$-equal partition lattices, Gottlieb [17] examined a related sublattice of the (unsigned) partition lattice. In Gottlieb’s lattice, the size of a block is restricted to be at least $k$ or at least $h$, depending on whether or not the block contains a distinguished element.

We further generalize to allow each element to have a different block-size restriction associated to it. More formally, we consider a map $\text{aff} : [n] \rightarrow [n]$, which we consider as providing an affinity to each element $x$ of $[n]$. We consider two subposets of the partition lattice $\Pi_n$:

\[ \Pi_{\text{aff}}^\vee \triangleq \{ \pi \in \Pi_n : \text{every nonsingleton block } B \text{ has } |B| \geq \text{aff}(x) \text{ for every } x \in B \} , \quad \text{and} \]
\[ \Pi_{\text{aff}}^\exists \triangleq \{ \pi \in \Pi_n : \text{every nonsingleton block } B \text{ contains some } x \text{ such that } |B| \geq \text{aff}(x) \} . \]

It is clear that both subposets are join subsemilattices of $\Pi_n$, hence lattices.

**Theorem 6.3.** For any selection of affinity map $\text{aff}$, the lattices $\Pi_{\text{aff}}^\exists$ and $\Pi_{\text{aff}}^\vee$ are comodernistic.
Proof. As in the proof of Theorem 6.1, consider an interval $[\pi', \pi]$. Let some block of $\pi$ split nontrivially into blocks $B_1, B_2, \ldots, B_t$ of $\pi'$. As in Theorem 6.1, assume that the blocks are sorted by increasing size, and in particular that $|B_1| \leq |B_i|$ for all $i$.

Now, if there are multiple singleton blocks $B_1 = \{x_1\}, B_2 = \{x_2\}, \ldots, B_j = \{x_j\}$, then sort these by affinity. In the case of $\Pi_{\text{aff}}^3$, let $\text{aff}(x_1) \geq \text{aff}(x_2) \geq \ldots$; while for $\Pi_{\text{aff}}^\forall$, reverse to require $\text{aff}(x_1) \leq \text{aff}(x_2) \leq \ldots$.

The remainder of the proof now goes through entirely similarly to that of Theorems 6.1 and 6.2.

Theorem 6.3 has applications to lower bounds of the complexity of a certain computational problem, directly analogous to the work in [3, 4] with $\Pi_{n,k}$.

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Department of Mathematics, Oklahoma State University, Stillwater, OK, 74078
E-mail address: jay.schweig@okstate.edu
URL: https://math.okstate.edu/people/jayjs/

Department of Mathematics & Statistics, Mississippi State University, Starkville, MS 39762
E-mail address: rwoodroofe@math.msstate.edu
URL: http://rwoodroofe.math.msstate.edu/