Supplementary Information: Evolutionary dynamics of networked multi-person games: Mixing opponent-aware and opponent-independent strategy decisions

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In the supplementary information, here we extend our model in the main text to the mutation-selection process, and derive the criterion under which strategy A is more abundant than strategy B. As a result, we find that such a criterion is akin to the condition (11) shown in the main text. In order to elucidate the effect of the effective payoff function (a concept similar to the decision-making function in the main text) on the fate of strategy evolution, we particularly display some results about the evolution of cooperation in the context of several prototypical games, such as linear public goods games, threshold public goods games, d-person snowdrift games, and the volunteer’s dilemma.

1 An extension in the mutation-selection process

The mutation-selection process is a canonical framework to model the evolution of different traits across various disciplines [1, 2, 3]. Compared with the opponent-aware and opponent-independent decision-making processes, it more lays emphasis on the properties in biological inheritance or social learning. As an extension, nevertheless, we find that an analogous result to the Corollary 1 in our main text holds in this context. In a graph-structured population of size N, we consider a mutation-selection process where individuals’ reproduction is subject to the tradeoff between mutation and selection. The reproductive ability of each individual is proportional to its effective payoff (i.e., fitness), which is a mapping that translates the payoff derived from game interactions into reproductive success [4, 5]. Instead of a specific function adopted by previous studies [2, 6, 7], here we adopt a generic mapping \( \psi(\beta, \pi_i) \) [3], where \( \beta \) and \( \pi_i \) denote the intensity of selection and the payoff of individual i, respectively. With probability \( \varepsilon \), the offspring adopts a random strategy, A or B. Otherwise, with probability \( 1 - \varepsilon \) the offspring inherits its parent’s strategy. Using a similar method adopted in the main text, we model the evolution of strategy with a Markov chain. In this case, the transition probability \( p_{u,v} \) is a function of the effective payoff, \( p_{u,v}[\psi(\beta, \pi_i)] \). Following a similar process of proving Theorem 1 in the main text, under weak selection, we find that if condition \( [\partial \psi(\beta, \pi_i)/\partial \beta]_{\beta=0} = k_0 \cdot \pi_i + c_0 \) is satisfied, the criterion under which strategy A
is more abundant than strategy B in the stationary distribution is \( k_0 \sum_{j=0}^{d-1} \sigma_j (a_j - b_{d-1-j}) > 0 \). Different from the Corollary 1 in the main text, here \( \sigma_j \)s are structure coefficients \([2, 6, 7]\), which depend on the model, population structure, update rule, and mutation rate. Accordingly, we have the following corollary.

**Corollary S1.** Consider a mutation-selection process in the population where individuals adopt a homogeneous effective payoff function \( \psi(\beta, \pi_i) \) with \( \psi(0, \pi_i) \equiv \text{Constant} \), which satisfies the following conditions: i) the effective payoff function is differentiable at \( \beta = 0 \); ii) the update rule is symmetric for the strategies, A and B. Then, in the limit of weak selection, when the effective payoff function \( \psi(\beta, \pi_i) \) satisfies \[ \frac{\partial \psi(\beta, \pi_i)}{\partial \beta}_{\beta=0} = k_0 \cdot \pi_i + c_0, \] strategy A is favoured over strategy B if

\[
k_0 \sum_{j=0}^{d-1} \sigma_j (a_j - b_{d-1-j}) > 0, \tag{S1}
\]

where \( \sigma_j \)s are structure coefficients that depend on the model and the dynamics (population structure, update rule, and mutation rate) but not on the effective payoff function and the entries of the payoff table, \( a_j \) and \( b_j \).

According to this corollary, we can find that the effective payoff function plays an analogous role to the decision-making function. The constant \( k_0 \) determined by the effective payoff function curbs the direction of strategy evolution, which is in line with the results shown in the Corollary 1 and meanwhile extends a recent finding obtained in pairwise games \([3]\) to the case of multi-player. When an effective payoff function is chosen such that \( k_0 \) is positive, condition (S1) reduces directly to the multi-person version of \( \sigma \)-rule \([7]\). It predicts that strategy A is favoured over strategy B. On the contrary, when \( k_0 \) is negative, it changes to the opposite of \( \sigma \)-rule and predicts that selection favours B to dominate A. To elaborate this effect clearly, in the following, we study several specific microscopic processes in the limit of small mutation (\( \varepsilon \rightarrow 0 \)), where a canonical multi-person game is played in well-mixed or structured populations. Particularly, for small mutations, we notice that strategy A is more abundant than strategy B if and only if \( \varepsilon \rho_B / \varepsilon \rho_A < 1 \) \([8, 9]\), where \( \rho_A \) (resp. \( \rho_B \)) denotes the fixation probability which quantifies how likely a mutant A (resp. B) invades a resident B (resp. A) population and takes it over \([4, 10]\). That is, \( \rho_A > \rho_B \) is an equivalent condition of (S1) in the limit of small mutation. Therefore, in order to determine whether strategy A wins over strategy B, we merely compare \( \rho_A \) with \( \rho_B \) in the following.

### 1.1 Well-mixed populations

When the population is well-mixed (i.e., a complete graph), we take into account two prevalent mechanisms of strategy evolution, Moran-like process \([4]\) and pairwise comparison process \([11]\). For the former, with a probability proportional to the fitness \( \psi(\beta, \pi_i) \), an individual is selected randomly for reproduction, and then one identical offspring replaces another randomly chosen individual. While for the latter, two individuals are sampled randomly and then a focal player imitates the strategy of the role model with a probability depending on the payoff comparison (termed imitation probability), \( \psi(\beta, \Delta \pi_i) \), where \( \Delta \pi_i := \pi_{-i} - \pi_i \). For these two microscopic processes, it has been proved that the structure coefficients are given by \([12, 13]\)

\[
\sigma_j = \begin{cases} 
N, & \text{if } 0 \leq j \leq d-2; \\
N-d, & \text{if } j = d-1.
\end{cases} \tag{S2}
\]

Inserting them into inequality (S1), then we can determine whether strategy A is favoured by natural selection over strategy B.
In Fig. S1, we show the fixation probabilities of strategies $A$ and $B$ to quantify their competitive advantages in a Moran process and a pairwise comparison process, respectively. Therein, individuals play a public goods game. Each $A$ player (who plays a role of cooperators) contributes a fixed amount $c$ to the common pool whereas all $B$ players (who play a role of defectors) do nothing. Then, the sum of all contributions in the group is multiplied by a synergy factor $r$ and subsequently allotted equally among all group members irrespective of their contributions. Thus, the entries of the payoff table are given by $a_j = (j + 1)rc/d - c$ and $b_j = jrc/d$, respectively. By theoretical calculations and computer simulations, we obtain the fixation probabilities as illustrated in Fig. S1. Analytical results are in good agreement with the simulations, and consistent with the theoretical prediction of condition (S1). When a fitness or an imitation probability function is chosen such that constant $k_0$ is positive, condition (S1) predicts the occurrence of the tragedy of the commons [14] where all players are tempted to defect (red lines in Fig. S1(a) and (c)). On the contrary, if $k_0$ is negative, the direction of strategy evolution is reversed, which leads strategy $A$ to dominating strategy $B$ (blue lines in Fig. S1(a) and (c)). This effect is also confirmed by the results in threshold public goods games and $d$-person snowdrift games, as shown in Figs. S2 and S3.

1.2 Structured populations

In structured populations, we investigate the evolutionary dynamics on a regular graph with degree $d - 1$ where each individual occupies a vertex and interactive ties are indicated by edges. The mechanism of strategy propagation is modeled by a Moran death-birth process on graphs [15, 16]. At each time step, a random individual is chosen to die, and then its neighbors compete for the empty spot proportional to their effective payoffs (i.e., fitness), $\beta, \pi$. By theoretical calculations and computer simulations, we obtain the fixation probabilities as illustrated in Fig. S1. Analytical results are in good agreement with the simulations, and consistent with the theoretical prediction of condition (S1). When a fitness or an imitation probability function is chosen such that constant $k_0$ is positive, condition (S1) predicts the occurrence of the tragedy of the commons [14] where all players are tempted to defect (red lines in Fig. S1(a) and (c)). On the contrary, if $k_0$ is negative, the direction of strategy evolution is reversed, which leads strategy $A$ to dominating strategy $B$ (blue lines in Fig. S1(a) and (c)). This effect is also confirmed by the results in threshold public goods games and $d$-person snowdrift games, as shown in Figs. S2 and S3.

As illustrated in Fig. S4, we have displayed the evolution of cooperation in a simple nonlinear multi-person game—volunteer’s dilemma [16, 17]. In such a game, each player can obtain a payoff $B$ from the public good if there is at least one $A$ player, who is assigned as a cooperator and contributes a fixed amount $C$ to the common pool, within the group. In contrast, all $B$ players (defectors) contribute nothing and produce no any benefit. In this way, the entries of the payoff table are given by $a_j = B - C$ for all $j$, $b_0 = 0$, and $b_j = B$ for $j > 0$. Substituting these payoff entries and the structure coefficients given above into inequality (S1), then, we can determine whether strategy $A$ is more abundant than strategy $B$ in structured populations.

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benefit-to-cost ratio is the watershed that $A$ dominates $B$ (middle row in Fig. S4). On the contrary, when $k_0$ is negative, it changes to the one that $B$ dominates $A$ (bottom row in Fig. S4). The direction of strategy evolution is completely reversed. This effect is also confirmed when the cost of producing the public good is shared among cooperators [16, 18] (in this case the entries of the payoff table are $a_j = B - C/(j + 1)$ for all $j$, $b_0 = 0$, and $b_j = B$ for $j > 0$), as shown in Fig. S5.

Figure S 1. Fixation probabilities of strategy $A$ and the ratios between strategy $A$ and strategy $B$ in Linear Public Goods Games. In the first row, strategies update with a Moran process (where fitness functions $f_1$ and $f_2$ are adopted, respectively), whereas in the second row they are realized with a pairwise comparison process (where imitation probability functions $g_1$ and $g_2$ are adopted, respectively). In all panels, parameters are $d = 5$, $r = 3$, $c = 1$, and $N = 100$. Since $\sum_{j=1}^{d-1} \sigma_j (a_j - b_{d-1-j}) < 0$ in this case, it results in strategy $A$ (cooperation) dying out when $f_1$ is applied in the Moran process and $g_1$ in the pairwise comparison process where $k_0 > 0$ (red lines). This effect is more striking beyond weak selection. In contrast, when $f_2$ and $g_2$ are adopted such that $k_0 < 0$ (blue lines), the confusing social dilemma is solved and strategy $A$ (cooperation) becomes the dominant one. We use the formulae in [12, 13] to obtain the theoretical results and weak selection approximations.
Threshold Public Goods Games (T-PGG). T-PGG is also called the collective risk dilemma [19]. All A players (i.e., cooperators) contribute a fixed cost $c$ to the common pool whereas $B$ players (i.e., defectors) do nothing. Whenever the threshold investment $Tc$ is met, the sum of all endowments is multiplied by $r$ and then distributed among all group members evenly. Otherwise, nobody can get a benefit. Therefore, the entries of the payoff table is $a_j = (j + 1)cr/d - c$ for $j \geq T - 1$, and $a_j = 0$ otherwise; $b_j = jcr/d$ for $j \geq T$, and $b_j = 0$ otherwise. Here, we adopt parameters $d = 5$, $r = 3$, $c = 1$, $T = 3$, and $N = 100$. In this case, we have $\sum_{j=0}^{d-1} \sigma_j(a_j - b_{d-1-j}) > 0$. Accordingly, when $k_0 > 0$ (red lines), natural selection favours strategy $A$ (cooperation) to evolve and meanwhile restrains from the evolution of strategy $B$ (defection). However, when $k_0 < 0$ (blue lines), such a result is reversed.
Figure S3. Fixation probabilities of strategy A and the ratios between strategy A and strategy B in d-person Snowdrift Games (d-SG). d-SG is a multi-person version of the classical snowdrift games [20]. In a group of size d, every member can gain a benefit B if there is at least one A player (i.e., cooperator). At the same time, the cost of producing benefits C will be fully paid by the A players within the group. Then, the entries of the payoff table are given by $a_j = B - C/(j+1)$ for all $j$, $b_0 = 0$, and $b_j = B$ for $j > 0$. Here, we adopt parameters $B = 1.5$, $C = 1$, and $N = 100$. In this case, we have $\sum_{j=0}^{d-1} \sigma_j (a_j - b_{d-1-j}) < 0$. Therefore, overall we obtain an evolutionary outcome similar to Fig. S1.
Figure S 4. Analytical calculations and simulations of fixation probabilities on regular graphs with degree $d - 1$ where individuals participate in a volunteer’s dilemma. The first row shows an initial strategy distribution when the population structure is modeled by a random regular graph, ring, and lattice with degree 4, respectively. The second and third rows show the gap of fixation probabilities between strategy $A$ and strategy $B$ on these three classes of graphs, where the effective payoff functions $\psi(\beta, \pi_i) = 1 - \beta + \beta \pi_i$ (the second row) and $\psi(\beta, \pi_i) = 1 - \beta \pi_i$ (the third row) are applied, respectively. Lines are theoretical calculations whereas symbols are simulations. The fixation probability of a single $A$ or $B$ player is calculated by averaging $10^5$ runs and 100 network realizations after the population reaching the homogenous state. Parameters are $N = 100$, $\beta = 0.01$, and $C = 1.0$. 

(a) Random regular graph  
(b) Ring  
(c) Lattice
Analytical calculations and simulations of fixation probabilities on regular graphs with degree $d-1$ where individuals participate in a **volunteer’s dilemma with cost sharing** (VDCS). VDCS is a cost-sharing version of the traditional volunteer’s dilemma where the cost of producing the public good is shared among cooperators. It is also a d-SG mathematically, and thus the entries of the payoff table are given by $a_j = B - C/(j + 1)$ for all $j$, $b_0 = 0$, and $b_j = B$ for $j > 0$. Other captions keep the same as in Fig. S4. Notably, the results here are in good agreement with those in Fig. S4.
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