Hall magnetohydrodynamic model of the quasi-single helicity self-organization of RFP

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Abstract. In the framework of two-fluids Hall magnetohydrodynamic description of a paramagnetic z-pinch with the low safety factor (q<<1 on the pinch axis), the theoretical model of kink mode stabilization is developed. It is shown that global kink oscillations of paramagnetic pinch, whose instability had been found in the single-fluid MHD theory, generate due to the Hall effects a rotation of plasma, which, in turn, stabilizes the kinks of the plasma column. The fluctuations are considered in terms of the cylindrical boundary problem for the pinch equilibrium with arbitrary components of the magnetic field, \((0, B_\theta, B_z)\), hydrodynamic velocity, \((0, V_\theta, V_z)\), and the Hall parameter. The consideration is fulfilled in approximation of “cold” plasma when the Alfvén magnetic activity plays the key role. The hydrodynamic velocities \(V_\theta, V_z\) are determined numerically from momentum balance collision-free equations averaged over the helical fluctuations. The developed self-consistent quasi-linear magnetohydrodynamic model of the marginally stable kink spectrum reflects some physical features of the quasi-single helicity regime in reversed field pinches.

1. Introduction
The last decade has seen substantial progress in understanding the physics of toroidal discharges, qualified as a reversed field pinches (RFP) with improved confinement. The new paradigm has established, under which a current maintenance in RFP is characterized by laminar quasi-single-helical (QSH) oscillations - the kinks with numbers \(m = 1\) and \(n \gg 1\) [1-3]. Two questions are of current interest: the nature of the QSH wave spectrum and its mechanism of self-organization.

Magnetohydrodynamics (MHD) spectra are the most extensively discussed subject in the pinch physics. Linearized version of the MHD equations was first formulated as a differential boundary value problem of order 2 for linear pinches as far back as before tokamak epoch [4]. The Hain-List equation [4] allows determine the non-local frequency dispersion and radial structure of helical pinch oscillations for an arbitrary magnetic fields and parameters of plasma without flows. Many authors have rediscovered the equation in various forms. In references [5,6] the most usable view presented, and in [7] in the same terms hydrodynamic flows of the plasma, \((0, V_\theta(r), V_z(r))\), are taken into consideration. Disclaimer of immobility plasma in the spectral theory is very important in view of the extensive evidence that any equilibrium plasma is accompanied by its differential rotation, and the relatively weak low-frequency fluctuations are associated with plasma flows. However, inclusion of plasma velocities in the analysis requires the explicit knowledge of the radial distributions of \(V_\theta(r)\) and
Thus, in each case, an appropriate model matching fluctuation spectra and plasma equilibriums is in a need.

In this paper, with reference to the RFP, the model proposed, in which the boundary value problem includes Hall effects of two-fluid dynamics in addition to plasma flows. While in the standard one-fluid MHD theory of ideal plasma the zero real frequency is characteristic for unstable wave oscillations ($\text{Re}\{\omega\}=0$ whereas $\text{Im}\{\omega\}>0$), in the Hall MHD theory non-zero real frequencies arise. Therefore it turns out, the balance equation averaged over fluctuations become non-trivial and can be satisfied only taking into account plasma flows. Substituting these velocities in the computational code for the boundary value problem reveals the stabilizing effect of the induced plasma flows on each of the unstable modes. The iterative procedure minimizing increments permits us to call the amplitudes that define a marginal stable kink spectrum.

The entire treatment is performed in the cold plasma approximation, in which thermal effects are omitted. Moreover, the model considers an ideal plasma without any collisions, whereby only detects a tendency towards self-stabilization of resonant modes, while the limit of strict stabilization is unattainable in the iterative procedure because of their singularity. This significantly limits the physical relevance of the model but provides a basis for a broader description of the Hall mechanism of the RFP dynamic self-organization.

2. Basic equations.

The Hall version of MHD theory introduces the current velocity $U$ into description as follows:

$$\frac{\partial}{\partial t} B = \text{rot}(V_e \times B - \eta \text{rot}B)$$

$$V_e = V - U, \quad U = \frac{\text{rot}B}{\sqrt{1/N}}$$

$$N(\frac{\partial}{\partial t} V + (V \nabla) V) + \nabla P = \text{rot}B \times B$$

Equations (1) are written in the dimensionless form using the Alfven unit $V = B_0/(4\pi \rho_0)^{1/2}$ for velocities, the plasma column radius $a$ and the frequency $\omega_A = V_A/a$ for all sizes and frequencies, where Alfven velocity $V_A$ is expressed through physical parameters on the pinch axis. The density $N$ and magnetic field $B$ are reduced to 1 on the axis: $N = B = 1$ at $r = 0$ in cylinder coordinates $(r, \theta, z)$. In (1), the constant Hall-Braginskii parameter $\Pi$ and resistivity $\eta$ are estimated for real RFPs.

The linear version of (1) for all fluctuating fields represented as, e.g., $B(r) + \delta B(r) \exp(i\zeta + c.c.$, where $\zeta = k_z z - m\theta - \omega t$, is suitably to express in terms of the electron displacement $\xi_e$: $\delta V_e(r) = -i(\sigma + k \cdot U)\xi_e(r)$, where $\sigma = \omega - k \cdot V(r)$. Useful variables are also $F = k \cdot B$ and $G = r[B \times k]$. With assumption $\beta_0 = 4\pi P_0/B_0^2 = 0$, i.e. neglecting thermal pressure effects, and only in this case if $1/\Pi > 0$, one can obtain a Hain-Lust-like boundary problem of order 2:

$$\frac{d}{dr} Ar \frac{d}{dr} r \xi_e + Q \xi_e = 0, \quad \xi_e(0) \neq 0, \quad \xi_e(1) = 0$$
in which $A$ and $Q$ are expressed through $\sigma$, Hall contributions and the frequency of $\theta$-rotation, $\Omega = V_\theta/r$, in explicit form:

$$
A = \frac{1}{k^2 r^2} \left[ F^2 + \sigma^2 N^2 \frac{D}{D} \bar{G}^2 \right], \quad F = k \cdot B - \frac{m B_\theta}{r}, \quad G = k \cdot r B_\theta + m B_z, \quad \bar{G} = G + \Omega^2 k^2 r^2 \sqrt{\Pi} \sigma
$$

$$
D = \sigma^2 N^2 - k^2 r^2 B^2 - \frac{1}{\sqrt{\Pi}} \left( B_\theta V_\theta - \frac{B_z}{r^3} \frac{d}{dr} r^4 \Omega \right) - \frac{1}{\Pi} \left( \sigma^2 - 2 \Omega \frac{d}{dr} r^2 \Omega \right),
$$

$$
Q = N \sigma^2 - F^2 + 4 \left( \frac{k^2 B_\theta^2}{k^2 r^2} - N \Omega^2 + m \frac{\Omega^3}{\sigma} \right) - k^2 r^2 \left( \frac{N \Omega^4}{\sigma^2} + \frac{nu^2}{D} \right) - \frac{r}{dr} \left[ \frac{B_\theta^2}{k^2 r^2} - 2 \frac{k \cdot GB_\theta}{k^2 r^2} + \frac{\sigma NGU}{D} \right]
$$

$$
U = 2 \sigma \frac{k \cdot r B_\theta}{k^2 r^2} + 2 \Omega B_z - \frac{\Omega^2 G}{\sigma} - \frac{1}{\sqrt{\Pi}} \left( \sigma^2 - 4 \Omega^2 + 2 m \frac{\Omega^3}{\sigma} \right)
$$

Equations (2) and (3) in the case $\Pi \rightarrow \infty$ and similar formulas [4] in the case $\beta_0 = 0$ (i.e. $P = 0$) are equivalent.

The mean fields $B_\theta(r)$, $B_\theta(r)$ can be substituted into (3) as a solution of equations that define in a sufficiently general form the decomposition of the rot$B$ along the directions $b=B/B$ and $b^\wedge=b\times e_r$ with two coefficients $\alpha(r)$ and $\beta(r)$:

$$
\frac{1}{r} \frac{d}{dr} r B_\theta = \alpha \frac{B_z - \beta B_\theta}{B^2}, \quad \frac{d}{dr} B_z = - \frac{\alpha B_\theta + \beta B_z}{B^2}
$$

If $\alpha = \alpha_0 B_z$, $\beta = 0$, we have the force-free paramagnetic model [8]. A more general model that takes into account the fluctuations follows from the non-inertia equation of electron motion which takes role of the Ohm’s law in the Hall MHD theory

$$
E + \mathbf{V}_e \times \mathbf{B} = \eta \text{rot} \mathbf{B}
$$

Equation (5) does not contain thermal terms in our case. In steady state, $\mathbf{E} = E_0 e_r - \Phi'(r) e_r$, so the $\mathbf{b}$-projection of (5) gives in our dimensionless description: $E_0 = \alpha_0 \eta$, where $\alpha_0 = 4\pi \sigma a E/cB$ is a constant for the paramagnetic model expressed through driving electrical field $E$ and magnetic field $B$ at the pinch axis in physical units. In our calculations $\alpha_0$ is assumed equal 4 in accordance with data [1-3].

To study the effect of fluctuations on the pinch equilibrium, should be averaged the Ohm’s law components along directions $\mathbf{b}$ and $\mathbf{b}^\wedge$:

$$
\alpha = \alpha_0 B_z + \frac{B}{\eta} \left[ \left\langle B_z (\mathbf{b}^\wedge, \mathbf{V}_e) \right\rangle - \left\langle V_{re} (\mathbf{b}^\wedge, \mathbf{B}) \right\rangle \right]
$$

$$
\frac{V_{re}}{\eta} = - \frac{\alpha_0 B_\theta + \beta}{B^2} + \frac{1}{\eta B} \left[ \left\langle B_\theta (\mathbf{b}, \mathbf{V}_e) \right\rangle - \left\langle V_{re} (\mathbf{b}, \mathbf{B}) \right\rangle \right]
$$

The brackets $\langle \ldots \rangle$ denotes the averaging over the wave phase $\zeta$ in the single-mode approximation (recall that phase $\zeta$ is determined above by the complex presentation of perturbations, e.g., $\delta N(r)e^{i\zeta} + \text{c.c}$).

Smallness of parameter $\eta$ allows estimating the amplitude of fluctuations with the low scale of $\eta^{1/2}$ value to give sufficient contribution into parameters of configuration. Indeed, the non-radial flow
perturbations and the mean radial transport are very slow compared to the Alfvén velocity in reality. Transition to rescaled values, \( \xi_{re} = \eta^{1/2} \xi, V_{re} = \eta \tilde{V}_{re} \), excludes the parameter \( \eta \) in the formulas (6) and below for values with tilde. The \( \eta \)-analysis defines the parameter \( \beta \) from the radial momentum balance (1) as a zero-order value, \( \beta = -rN\Omega^2 \) under condition \( \beta_0 = 0 \), giving the diamagnetic rotational correction to the paramagnetic model.

To calculate the flows, it is enough to use the procedure \( \ldots \) for non-radial components of the momentum balance (1) where terms with time derivatives can be omitted if the quasi-steady states with rotation exist. It is easy to see that under the single-mode consideration following equations take place:

\[
\frac{1}{r^2} \frac{d}{dr} r^2 \langle NV, V_\theta \rangle = \left\langle B_r \frac{1}{r} \frac{\partial}{\partial r} r B_\theta + k \cdot B \frac{\partial}{\partial \zeta} B_\theta \right\rangle = \frac{1}{r^2} \frac{d}{dr} r^2 \langle B_r B_\theta \rangle
\]

\[
\frac{1}{r} \frac{d}{dr} r \langle NV, V_\zeta \rangle = \left\langle B_r \frac{\partial}{\partial r} B_z + k \cdot B \frac{\partial}{\partial \zeta} B_z \right\rangle = \frac{1}{r} \frac{d}{dr} r \langle B_r B_z \rangle
\]

Integrating equations (7) and using the \( \eta \)-analysis one can obtain the simple local presentation for non-radial velocities:

\[
V_{\theta,\zeta} = \frac{1}{\langle \Gamma_r \rangle} 2 \text{Re}\{\delta \tilde{B}_r, \delta \tilde{V}_{\theta,\zeta} - N \delta \tilde{V}, \delta \tilde{V}_{\theta,\zeta}^* \}, \quad \langle \Gamma_r \rangle = NV_{\theta,\zeta} + 2 \text{Re}\{\delta \tilde{N}, \delta \tilde{V}_{\theta,\zeta}^* \}
\] (8)

3. Main results

The boundary eigen-value problem (2) for modes \( m = \pm 1 \) was solved by the shooting method initially in the framework of the paramagnetic model of motionless cold plasma. It is found that \( m = 1 \) is instable within an interval of \( k_z \) in accordance with known results [6]. Figure 1 shows the growth rates (increments) within interval \( 1.66 < k_z < 2.37 \) (upper curve) and the occurrence of the oscillation frequencies, \( \text{Re}\{\omega\} \), to be low in the Alfvén scale (lower curves - note the factor \( 10^2 \)). Low frequencies are result of the weak influence of the Hall effects on magnetosonic wave dispersion in the case of \( \Pi \gg 1 \), as can be clearly argued in terms of oscillations in a uniform magnetic field. They noticeably depend on the Hall parameter within the interval \( 10^2 < \Pi < 50 \) of practical interest. The increments are saturated for these \( \Pi \) because influence of the Hall parameter on \( \text{Im}\{\omega\} \) manifests itself if \( \Pi \ll 1 \). In our case, the subinterval \( k_z > 2 \) corresponds to resonant "internal" modes, for which the condition \( F = k \cdot B = 0 \) introduces a singularity of solution (2) for stable modes. However, equation (2) is valid for the analysis and numerical solution while \( \text{Im}\{\omega\} > 0 \) for all \( k_z \).

![Figure 1. Dependence of the kink increments and frequencies on \( k_z \) and Hall parameter \( \Pi \) in cold plasma of the paramagnetic pinch without flows.](image-url)
The appearance of the real valued frequencies of unstable kinks if \( V_0 = V_z = 0 \) but \( \Pi \) is finite makes their amplitudes to be complex values. It is important under calculation of the velocities (8) because an additional contributions into mean values \( \langle \ldots \rangle \) appear due to \( \Pi \). The diagram on figure 2 illustrates the process of stabilization of the kink resonant modes under the influence of plasma rotations generated by theirs own field oscillations calculated in accordance with the expressions (8). The growth rates are plotted against the rescaled amplitude \( \xi_{re}(0) \) rising from zero when they start at corresponding values in figure 1. The set of internal kinks \( m=1, 2.1 < k_z < 2.3 \) are chosen for which the greatest amplitudes are required and a “dominant mode” can be called. The data are obtained in iterative procedure substituting all field amplitudes into equations (8) and (6), and then into (4) and (3), step by step increasing amplitude. As a result, the limiting values of amplitudes can be found although the singularity of stable modes breaks off the iterations when increments become very close to zero.

![Figure 2. Stabilization of different kinks under influence of flows with increasing amplitude \( \xi_{re}(0) \) (\( \Pi=40 \)).](image)

4. Conclusions

It is shown that the non-local kink-modes are unstable in the cold plasma of paramagnetic pinches and can be stabilized by the flows generated by own oscillations under influence of the Hall effects. The one mode may be dominant in the spectrum of wave numbers \( k_z \) that may be compared with some toroidal number \( n \). Theoretical quasi-linear model determining the rotational velocities and the algorithm taking into account the plasma rotation are proposed.

However the presented model don’t able to describe a static maintenance of saturated amplitudes as well to find the values of real frequencies of resonant modes in the strict limit \( \text{Im}\{\omega\} = 0 \) due to their singularity. More dissipative processes as well as all \( \beta \)-effects of non-scalar pressure must be involved into the model. Some convergence with the theory of tearing modes taking into consideration real low-frequency fluctuations of the pinch is to be achieved.

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