A Turán-type problem on distance two

Xueliang Li, Jing Ma, Yongtang Shi, Jun Yue
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China
Email: lxl@nankai.edu.cn, majingnk@gmail.com, shi@nankai.edu.cn, yuejun06@126.com

Abstract

A new Turán-type problem on distances on graphs was introduced by Tyomkyn and Uzzell. In this paper, we focus on the case that the distance is two. We primely show that for any value of \( n \), a graph on \( n \) vertices without three vertices pairwise at distance 2, if it has a vertex \( v \in V(G) \), whose neighbours are covered by at most two cliques, then it has at most \( \left( \frac{n^2}{4} - 1 \right) + 1 \) pairs of vertices at distance 2. This partially answers a guess of Tyomkyn and Uzzell [Tyomkyn, M., Uzzell, A.J.: A new Turán-Type problem on distances of graphs. Graphs Combin. 29(6), 1927–1942 (2012)].

Keywords: distance, Turán-type problem, forbidden subgraph

1 Introduction

In [10], Tyomkyn and Uzzell introduced a new Turán-type problem on distances in graphs, which is an extension of the problem studied by Bollobás and Tyomkyn in [6], namely, determining the maximum number of paths with length \( k \) in a tree \( T \) on \( n \) vertices.

The problem on counting paths of a given length in a graph \( G \) has been studied since 1971, see, e.g., [1 2 3 1 5 8 0] and the references therein. On the other hand, counting paths of length \( k \) in trees can be interpreted as counting pairs of vertices at distance \( k \). Tyomkyn and Uzzell asked a natural question as follows.

Question. For a graph \( G \) on \( n \) vertices, what is the maximum possible number of pairs of vertices at distance \( k \)?

*Supported by NSFC.
Let $G = (V, E)$ be a connected simple graph. The distance between two vertices $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length of a shortest path between $u$ and $v$ in $G$. Let $N_G(v)$ be the neighborhood of $v$, and $d_G(v) = |N_G(v)|$ denote the degree of vertex $v$. The greatest distance between any two vertices in $G$ is the diameter of $G$, denoted by $\text{diam}(G)$. The set of neighbors of a vertex $v$ in $G$ is denoted by $N(v)$ or $N_1(v)$, and the set of vertices, whose distance is $i$ from $v$, is denoted by $N^i(v)$, where $i \in \{1, 2, 3, \ldots, \text{diam}(G)\}$. Suppose that $V'$ is a nonempty subset of $V$. The subgraph of $G$ whose vertex set is $V'$ and whose edge set is the set of those edges of $G$ that have both ends in $V'$ is called the subgraph of $G$ induced by $V'$ and is denoted by $G[V']$; we say that $G[V']$ is an induced subgraph of $G$. A clique in a graph $G$ is a subset of its vertices such that every two vertices in the subset are connected by an edge.

If $H$ is a graph, then we say that $G$ is $H$-free if $G$ does not contain a copy of $H$ as an induced subgraph. The claw, denoted by $C$, is the complete graph $K_{1,3}$ (see Figure 1). Thus, $G$ is said to be claw-free if it does not contain an induced subgraph that is isomorphic to $C$.

![Figure 1: A claw](image)

A graph $G$ is a quasi-line graph if for every vertex $v \in V(G)$, the neighborhood of $v$ can be partitioned into two sets $A, B$ in such a way that $A$ and $B$ are both cliques. (Note that there may be edges between $A$ and $B$.) Thus all line graphs are quasi-line graphs, and all quasi-line graphs are claw-free, but if we converse either of the statements, it is not true. For the other notations and terminology we refer to \cite{7}.

For a graph $G$, a new graph $G_k$ is defined to be the graph with vertex set $V(G)$ and $\{x, y\} \in E(G_k)$ if and only if $x$ and $y$ are at distance $k$ in $G$. We call $G_k$ the distance-$k$ graph. Such vertices $x$ and $y$ are called $k$-neighbors. We call $d_{G_k}(x)$ the $k$-degree of $x$. Let $\omega(G)$ denote the clique number of graph $G$, which is the maximal number of vertices of a clique in $G$.

It is interesting to maximize the number of edges in $G_k$ over all graphs $G$ on $n$ vertices. In \cite{6}, Bollobás and Tyomkyn proved that if $G$ is a tree, then $e(G_k)$ is maximal when $G$ is a $t$-broom for some $t$.

**Theorem 1.1.** Let $n \geq k$. If $G$ is a tree on $n$ vertices, then $e(G_k)$ is maximal when $G$ is
a t-broom. If $k$ is odd, then $t = 2$. If $k$ is even, then $t$ is within 1 of

$$\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{n-1}{k-2}}.$$  

For a general graph $G$, Tyomkyn and Uzzell [10] gave a conjecture and proved a part of it in the following theorem.

**Conjecture 1.1.** Let $k \geq 3$ and $t \geq 2$. There is a function $h_2 : N \times N \rightarrow N$ such that if $n \geq h_2(k,t)$, then $e(G_k)$ is maximised over all $G$ with $|G| = n$ and $\omega(G_k) \leq t$ when $G$ is $k$-isomorphic to a $t$-broom for some $t$.

**Theorem 1.2.** There is a constant $k_0$ and a function $n_0 : N \rightarrow N$ such that for all $k \geq k_0$, all $n \geq n_0(k)$ and all graphs $G$ of order $n$ with no three vertices pairwise at distance $k$,

$$e(G_k) \leq (n - k + 1)^2/4.$$  

Moreover, if the equality holds, then $G$ is $k$-isomorphic to the double broom.

For the detailed proof and some terminology, we refer to [10]. Actually, Tyomkyn and Uzzell try to do better about the bound, but there is no good way. For the case of $k = 2$, they believe that for $n \geq 6$, a triangle-free $G_2$ can have no more than $(n - 1)^2/4 + 1$ edges. They mentioned that this is clearly true for $n = 5$ and a computer search verifies that it also holds for $6 \leq n \leq 11$. However, they cannot prove it in general. In this paper, we will give an partially answer to it. Our main results are as follows:

**Theorem 1.3.** Let $G$ be a graph on $n$ vertices, which has no three vertices pairwise at distance 2. If there exists an vertex $v \in V(G)$, whose neighbors are covered by at most two cliques, then it has at most $(n^2 - 1)/4 + 1$ pairs of vertices at distance 2.

From theorem 1.3, we can get the following corollary.

**Corollary 1.1.** Let $G$ be a quasi-line graph on $n$ vertices, which has no three vertices pairwise at distance 2, then it has at most $(n^2 - 1)/4 + 1$ pairs of vertices at distance 2.

## 2 Preliminaries

In this section, we will discuss some restrictions on the structure of graph $G$ on condition that $G_2$ is triangle-free.

Let $C_k$ be the cycle with $k$ vertices. First of all, we define two graphs $C'_6$ and $C''_6$ which can be obtained from $C_6$ as follows: $C'_6 = C_6 + v_1v_3$, $C''_6 = C_6 + v_1v_3 + v_3v_5$.  

3
Lemma 2.1. If $G_2$ is triangle-free, then $G$ is claw-free and $C_6$-free, $C'_6$-free, $C''_6$-free.

Proof. By contradiction. Suppose $G$ has a claw $C$ as an induced subgraph. Let $V(C) = \{v, u_1, u_2, u_3\}$ and $v$ is adjacent to $u_i$ ($i = 1, 2, 3$). Then the three vertices $u_1$, $u_2$ and $u_3$ form a triangle in $G_2$, a contradiction.

Suppose that $G$ is not $C_6$-free. Let $C_6 = v_1v_2 \ldots v_5v_1$, then $v_1v_2$, $v_3v_5$, $v_1v_5 \in e(G_2)$, which implies that the three vertices $v_1$, $v_3$ and $v_5$ form a triangle in $G_2$, a contradiction.

Similarly, $G$ is $C'_6$-free and $C''_6$-free. □

The stability number $\alpha(G)$ of a graph $G$ is the cardinality of the largest stable set. Recall that a stable set of $G$ is a subset of vertices such that no two of them are connected by an edge. For a claw-free graph, there is a well-known result [9] as follows.

Lemma 2.2. Let $G$ be a claw-free graph with stability number at least three, then every vertex $v$ satisfies exactly one of the following:

1. $N_G(v)$ is covered by two cliques;
2. $N_G(v)$ contains an induced $C_5$.

From Lemma 2.2, we know that for a claw-free graph $G$, the subgraph induced by the neighborhood of a vertex $v \in G$ is covered by at most two cliques, or contains an included $C_5$. In the following, we give an observation about the subgraph induced by $N^2_G(v)$.

Observation 1. Let $G$ be a graph with diameter two and $v \in V(G)$. If $G_2$ is triangle-free, then $G[N^2_G(v)]$ is a clique.

Let $G^{(d)}$ denote a graph $G$ with diameter $d$. Let $v_0v_1 \ldots v_d$ be a spindle of $G^{(d)}$ and $V_d = N_{G^{(d)}}(v_{d-1}) \setminus v_{d-2}$. We define an operation as follows: delete all the edges between $v_{d-1}$ and $V_d$, and then join $v_{d-2}$ with all the vertices of $V_d$. We call this operation “move $V_d$ to $v_{d-2}$”.

Lemma 2.3. If $G_2$ is triangle-free, then $e(G^{(d)}_2) \leq e(G^{(d-1)}_2)$, where $G^{(d-1)}$ is obtained by applying the above operation on $G^{(d)}$.

Proof. Let $v_0v_1 \ldots v_d$ be a spindle of $G^{(d)}$. After applying the above operation on $G^{(d)}$, the only change on the number of vertex pairs $\{u, v\}$ such that $d_{G^{(d)}}(u, v) = 2$ is brought by the movement of $V_d$. Thus to prove $e(G^{(d)}_2) \leq e(G^{(d-1)}_2)$, it suffices to show that for every spindle of $G^{(d)}$, the number of vertex pairs such that the distance between them is two is not decreasing after using the operation “move $V_d$ to $v_{d-2}$”. So we only need to show that for some spindle, after using the above operation, $|\{u|d_{G^{(d)}}(u, v_d) = 2\}| \leq |\{u'|d_{G'}(u', v_d) = 2\}|$, where $G'$ is obtained by moving $V_d$ to $v_{d-2}$ in $G^{(d)}$. Since $|\{u|d_{G^{(d)}}(u, v_d) = 2\}| = \ldots$
Theorem 3.1. can be stated as the following theorem.

Since

\[ |N_{G^{(d)}}(v_{d-1}) \setminus v_d| \text{ and } |\{u'|d_{G'}(u', v_d) = 2\}| = |N_{G^{(d)}}(v_{d-2}) \setminus N_{G^{(d)}}(v_{d-1}) \setminus v_d| \text{, we can get that } |\{u|d_{G'}(u, v_d) = 2\}| \leq |\{u'|d_{G'}(u', v_d) = 2\}|. \]

Therefore, we have \( e(G_2^{(d)}) \leq e(G_2^{(d-1)}) \).

\[ \square \]

Corollary 2.1. If \( G_2 \) is triangle-free and \( d \geq 2 \) is the diameter of \( G \), then \( e(G_2^{(d)}) \leq e(G_2^{(2)}) \).

Proof. By Lemma 2.3 we know that \( e(G_2^{(d)}) \leq e(G_2^{(d-1)}) \leq \cdots \leq e(G_2^{(2)}) \).

By Corollary 2.1, we know that to maximize \( |e(G_2^{(d)})| \) \( d \geq 2 \), it suffices to get the maximum value of \( |e(G_2^{(2)})| \). Thus, in the following part of the paper, we focus on the graph \( G^{(2)} \). For convenience, we write \( G \) instead of \( G^{(2)} \).

3 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. For convenience, we use \( \{x, y\} \) or vertex-pair to stand for the vertex-pair such that \( d_{G}(x, y) = 2 \). Actually, Theorem 1.3 can be stated as the following theorem.

Theorem 3.1. Let \( G \) be a graph with \( |V(G)| \geq 5 \). If there is a vertex \( v \in V(G) \) whose neighbourhood is covered by at most two cliques, then a triangle-free \( G_2 \) can have no more than \( (n-1)^2/4 + 1 \) edges.

Proof. Since \( G_2 \) is triangle-free, Lemma 2.1 implies that \( G \) is claw-free. Let \( v \in V(G) \), whose neighbours is covered by at most two cliques. Then the proof will be given by the following cases.

Case 1. \( N_G(v) \) is covered by only one clique.

Let \( V_1 = N_G(v) \) and \( V_2 = N_G^2(v) \). By Observation 1, \( G[N_G^2(v)] \) is a clique. Suppose \( V_1 = V_{11} \cup V_{12} \) and \( V_2 = V_{21} \cup B \cup V_{22} \), where \( V_{21} \) is only adjacent to \( V_{11} \), \( V_{22} \) is only adjacent to \( V_{12} \), \( B \) is adjacent to both \( V_{11} \) and \( V_{12} \). Without loss of generality, we suppose that \( |V_{11}| \geq |V_{12}| \) and \( |V_{21}| \geq |V_{22}| \).

By considering whether \( V_{12} = \emptyset \) or not, we give the discussion as follows.

Subcase 1.1. \( V_{12} = \emptyset \).

Since \( V_{22} \) is only adjacent to \( V_{12} \), we have \( V_{22} = \emptyset \). That is, \( G[V_{1}, V_{2}] \) is a complete bipartite graph. Hence, \( e(G_2) = |V_2| \leq n - 2 \leq (n-1)^2/4 + 1 \).

Subcase 1.2. \( V_{12} \neq \emptyset \).

In this subcase, we will give the proof in detail as follows.

Subsubcase 1.2.1. \( V_{22} \neq \emptyset \).
Let \( d = |V_{22}| \). We can define a new graph \( G' \) (see Figure 2) as follows. \( V(G') = V(G) \), \( V'_1 = N_{G'}(v) = V'_1 \cup V'_{12} \) and \( V'_2 = N_{G'}(v) = V'_2 \cup V'_{22} \), where \( G'[V'_1] \) and \( G'[V'_2] \) are both cliques, \( G'[V'_{11}, V'_{21}] \) and \( G'[V'_{12}, V'_{22}] \) are both complete bipartite graphs, and \( |V'_{11}| = 1, |V'_{12}| = |V_{11}| + |V_{12}| - 1, |V'_{21}| = |V_{21}| + |B|, |V'_{22}| = |V_{22}| \).

![Figure 2: A new graph \( G' \)](image_url)

Suppose that \( d = 1 \). In the following, we show that \( G' \) can be obtained from \( G \) by applying the corresponding operations mentioned in the following paper. What is more, we show that such operations ensure that the number of vertices-pairs remains the same or increases.

Firstly, delete the edges between \( V_{12} \) and \( B \), and it is obvious that the number of \( \{u_1, u_2\} \) is not decreasing, where \( u_1, u_2 \in V(G) \). Then \( V'_{21} = V_{21} \cup B \) and \( V'_{22} = V_{22} \).

Secondly, suppose \( u \in V_{11} \) such that \( u \) is adjacent to all the vertices in \( V'_{21} \). Delete the edges between \( V_{11} \setminus \{u\} \) and \( V'_{21} \), meanwhile, connect all the vertices in \( V_{11} \setminus u \) and all the vertices in \( V'_{22} \). Then \( V'_{11} = \{u\} \) and \( V'_{12} = \{V_{11} \setminus \{u\}\} \cup V_{12} \). Therefore, we get the graph \( G' \) (see Figure 2). Let \( a = |V_{11}|, b = |V_{12}|, c = |V_{21}| + |B| \). Since \( e(G_2) = a + b + c + 1 \) and \( e(G'_2) = (a + c - 1)b + (a + c + 1) \), then \( e(G'_2) - e(G_2) = (a - 1)b \geq 0 \), that is, after applying the above operation, we ensure that the number of \( \{u, v\} \) is not decreasing.

For the graph \( G' \), we have \( e(G'_2) = xy + x + 2 \) where \( x = |V'_{12}|, y = |V'_{21}| \), such that \( x + y = n - 3 \) and \( x \geq 1, y \geq 1 \). Thus, \( e(G'_2) \leq (n - 2)^2/4 + 2 < (n - 1)^2/4 + 1 \), where \( n \geq 5 \).

Now suppose that \( d \geq 2 \). Let \( u_1 \in V_{21} \) and \( u_2 \in V_{12} \). Delete the edges between \( V_{21} \setminus \{u_1\} \cup B \) and \( V_{22} \) and the edges between \( V_{12} \setminus \{u_2\} \) and \( V_{11} \); meanwhile move the vertices of \( V_{21} \setminus \{u_1\} \cup B \) to \( V_{11} \) (the new vertex set obtained is denoted by \( V'_{12} \)), and the vertices in \( V_{12} \setminus \{u_2\} \) with \( V_{22} \) (the new vertex set obtained is denoted by \( V'_{21} \)), therefore, we get the graph \( G' \) (see Figure 3), where \( V'_{11} = \{u_1\} \) and \( V'_{22} = \{u_2\} \). Let \( a = |V_{11}|, b = |V_{12}|, c = |V_{21}| + |B| \). Since \( e(G_2) = ad + bc + c + d \) and \( e(G'_2) = (a + c - 1)(b - d + 1) + (a + c + 1) + 1 \), then \( e(G'_2) - e(G_2) = a(b - 1) + (c - 1)(d - 2) \geq 0 \), that is, using the above operation we ensure that the number of \( \{w_1, w_2\} \) is not decreasing, where \( w_1, w_2 \in V(G') \).
For the graph $G'$, we have $e(G'_2) = xy + x + 2$, where $x = |V_{12}'|$, $y = |V_{21}'|$ such that $x + y = n - 3$ and $x \geq 1$, $y \geq 1$. Thus, $e(G'_2) \leq (n - 2)^2/4 + 2 < (n - 1)^2/4 + 1$, where $n \geq 5$.

Subsubcase 1.2.2. $V_{22} = \emptyset$.

For this case, delete the edges between $V_{11}$ and $B$, then it returns to Subsubcase 1.2.1.

By combining all the situations in Subcase 1.1, we get that $e(G) \leq (n - 2)^2/4 + 2 < (n - 1)^2/4 + 1$, where $n \geq 5$.

Case 2. $G[N_G(v)]$ is covered by two cliques.

In this case, $G[N_G(v)]$ is covered by two cliques, denoted by $V_1$ and $U_1$. According to the condition that whether $e(U_1, V_1) = 0$ or not, we prove it by the following subcases.

Subcase 2.1 $e(U_1, V_1) = 0$.

Let $V_2 = N^2_G(v)$. $V_2$ can be divided into three parts $A$, $B$ and $C$, where $A$ is only adjacent to $V_1$, $B$ is adjacent to both $V_1$ and $U_1$, and $C$ is only adjacent to $U_1$. Now we give the following claim.

Claim 1. $G[V_1, A]$ and $G[U_1, C]$ are both complete bipartite graphs.

Proof. If $G[V_1, A]$ is not a complete bipartite graph, then there are vertices $u_1 \in A$, $u_2 \in V_1$, $w \in U_1$ such that $d_G(u_1, u_2) = d_G(u_1, w) = d_G(u_2, w) = 2$. Thus, the three vertices $u_1, u_2, w$ form a triangle in $G_2$, a contradiction.

Similarly, $G[U_1, C]$ is also a complete bipartite graph. \qed

Now we divide vertex set $B$ into three parts $B_1$, $B_2$ and $B_3$, where $B_1$ is only adjacent to $V_1$, $B_2$ is adjacent to both $V_1$ and $U_1$, and $B_3$ is only adjacent to $U_1$.

Claim 2. $G[V_1, B_1 \cup B_2]$ and $G[U_1, B_2 \cup B_3]$ are both complete bipartite graphs.

Proof. To prove Claim 2, it suffices to show that every vertex $u \in B$ is adjacent to all the vertices of $V_1$ or $U_1$, that is, at least one of $G[V_1, u]$ and $G[U_1, u]$ is a complete bipartite
Suppose that there is a vertex \( u \in B \) such that neither \( G[V_1, u] \) nor \( G[U_1, u] \) is a complete bipartite graph. Then there are vertices \( w_1 \in V_1, w_2 \in U_1 \) such that \( d_G(u, w_1) = d_G(u, w_2) = d_G(w_1, w_2) = 2 \). Thus, \( uw_1, w_1w_2, w_2u \in e(G_2) \), contradicting to the fact that \( G_2 \) is triangle-free.

In the following, we apply some operations on \( G \) to maximise \( |e(G_2)| \).

We define a new graph \( G' \) as follows. \( G' \) with \( V'_1 = V_1, U'_1 = U_1, V'_2 = A' \cup C' \) \( (A' = A \cup B_1 \cup B_2 \) and \( C' = B_3 \cup C) \), where \( G[V'_1, A'], G[V'_2, C'] \) and \( G[A', C'] \) are complete bipartite graphs, \( G[V'_1], G[V'_2], G[A'] \) and \( G[B'] \) are complete graphs.

Form \( G \) to \( G' \), we perform the following operations: delete the edges between \( B_1 \) and \( U_1 \), remove the edges between \( B_2 \) and \( U_1 \), and the edges between \( B_3 \) and \( V_1 \). It is obviously that the number of vertex-pairs at distance two is not decrease.

Now we construct a new graph \( G'' \) (see Figure 4) which is obtained from \( G' \), that is, let \( V(G'') = V(G') \), move the vertex in \( A' \setminus \{u\} \) to \( V'_1 \), and the vertices in \( C' \setminus \{w\} \) to \( U'_1 \). The new vertices sets are denoted by \( V''_1 \) and \( U''_1 \), where \( u \in A' \) and \( w \in B' \).

Let \( a = |V'_1|, b = |V'_2|, c = |A'|, d = |B'| \). Since \( e(G'_2) = ab + ad + bc + c + d \) and \( e(G''_2) = (a + c - 1)(b + d - 1) + (a + c - 1) + (b + d - 1) + 2 \), then \( e(G''_2) - e(G'_2) = (c - 1)(d - 1) \geq 0 \), that is, after using the move operation the number of vertex-pairs whose distance is two is not decreasing.

Now, for the graph \( G'' \), let \( x = |V'_1| \) and \( y = |U''_1| \). Then \( e(G''_2) = xy + x + y + 2 \), where \( x + y = n - 3 \) and \( x, y \geq 1 \). By some calculations, we get that \( e(G''_2) \leq (n - 1)^2/4 + 1 \). And the equality holds if and only if \( n \) is odd and \( x = y = (n - 1)/2 \), where \( n \geq 5 \).

![Figure 4: A new graph $G''$](image)

Combining all the situations in Case 1, \( e(G_2) \leq (n - 1)^2/4 + 1 \) follows on condition that \( N_G(v) \) is covered by at least two cliques.

**Subcase 2.2** \( e(U_1, V_1) \neq 0 \).

In this subcase, if there is not a subset \( B \) of \( V_2 \) such that the vertex in \( B \) can form the vertex-pairs with some vertices of \( V_{11} \subseteq V_1 \) and meanwhile with some vertices of \( U_{11} \subseteq U_1 \),
then we delete the edges between $U_1$ and $V_1$. Now it returns to **Subcase 2.1**.

If there is a subset $B$ of $V_2$ such that the vertices in $B$ can form the vertex-pairs both with some vertices of $V_{11} \subseteq V_1$ and $U_{11} \subseteq U_1$, then divide $B$ into two part $B_1$ and $B_2$, delete the edges between $V_1$ and $U_1$, move $B_1$ to $V_1$, and move $B_2$ to $U_1$. Now we want to prove that there always exists such $B_1$ and $B_2$ to ensure the number of the vertex-pairs not decreasing. Let $a = |V_{11}|$, $b = |U_{11}|$, $c = |B_1|$, $d = |B_2|$. By only considering the vertex-pairs of those set, the change of the number is at least $(a + c)(b + d) - (c + d)(a + b) = (a - d)(b - d)$, which is no less than 0 (by some knowledge of the inequality, no matter how much $(a + c)$ and $(c + d)$ are, we can find some number to make it right). Now this subcase returns to the **Subcase 2.1**.

Combining all the cases, we complete the proof of Theorem 3.1.

From the proof of Theorem 3.1 we can easily get Corollary 1.1. By Lemmas 2.1 and 2.2 if we can prove the following statement: all the claw-free graphs with diameter two, which has no three vertices pairwise at distance 2, then it has at most $(n^2 - 1)/4 + 1$ pairs of vertices at distance 2, then we can confirm the guess of Tyomkyn and Uzzell.

**References**

[1] Ahlswede, R., Katona, G.O.H.: Grpahs with maximal number of adjacent pairs of edges. Acta Math. Acad. Sci. Hungar 32, 97–120 (1978)

[2] Alon, N.: On the number of subgraphs of prescribed type of graphs with a given number of edges. Israel J.Math. 38, 116–130 (1981)

[3] Bollobás, B., Erdős, P.: Graphs of extremal weights. Ars. Combin. 50, 225–233 (1998)

[4] Bollobás, B., Sarkar, A.: Paths in graphs. Stud. Sci. Math. Hungar 38, 115–137 (2001)

[5] Bollobás, B., Sarkar, A.: Paths of length four. Discrete Math. 265, 357–363 (2003)

[6] Bollobás, B., Tyomkyn, M.: Walks and paths in trees. J. Graph Theory 70(1), 54–66 (2012)

[7] Bondy, J.A., Murty, U.S.R: Graph Theory. GTM 244, Springer-Verlag, New York, 2008.

[8] Byer, O.D.: Maximum number of 3-paths in a graph. Ars. Combin. 61, 73–79 (2001)

[9] Faudree, R., Flandrin, F., Ryjáček, Z.: Claw-free graphs – A survey. Discrete Math. 164(1-3), 87–147 (1997)
[10] Tyomkyn, M., Uzzell, A.J.: A new Turán-Type problem on distances of graphs. Graphs Combin. 29(6), 1927–1942 (2012)