The Characterizations of Anisotropic Mixed-Norm Hardy Spaces on $\mathbb{R}^n$ by Atoms and Molecules

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Abstract Let $\vec{p} \in (0, \infty)^n$, $A$ be an expansive dilation on $\mathbb{R}^n$, and $H^{\vec{p}}_A(\mathbb{R}^n)$ be the anisotropic mixed-norm Hardy space defined via the non-tangential grand maximal function studied by [13]. In this paper, the authors establish new atomic and molecular decompositions of $H^{\vec{p}}_A(\mathbb{R}^n)$. As an application, the authors obtain a boundedness criterion for a class of linear operators from $H^{\vec{p}}_A(\mathbb{R}^n)$ to $H^{\vec{p}}_A(\mathbb{R}^n)$. Part of results are still new even in the classical isotropic setting (in the case $A := 2I_{n \times n}$, $I_{n \times n}$ denotes the $n \times n$ unit matrix).

1 Introduction

The theory of Hardy spaces on the Euclidean space $\mathbb{R}^n$ has been developed and plays an important role in various fields of analysis and partial differential equations; see, for examples, [5, 8, 9, 22, 24, 26]. On the other hand, the mixed-norm Lebesgue space $L^\vec{p}(\mathbb{R}^n)$, with the exponent vector $\vec{p} \in (0, \infty)^n$, is a natural generalization of the classical Lebesgue space $L^p(\mathbb{R}^n)$, via replacing the constant exponent $p$ by an exponent vector $\vec{p}$. The study of mixed-norm Lebesgue spaces originates from Benedek and Panzone [2]. Later on, function spaces in mixed-norm setting have attracted considerable attention and have rapidly been developed; see, for instance, [3, 6, 7, 15].

Let vector $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty)^n$. Recently, Cleanthous et al. [4] introduced the anisotropic mixed-norm Hardy space $H^{\vec{p}}_A(\mathbb{R}^n)$, where $\vec{a} := (a_1, \ldots, a_n) \in [1, \infty)^n$, via the non-tangential grand maximal function, and then obtained its maximal function characterizations. Not long afterward, Huang et al. [11] further completed some real-variable characterizations of the space, such as the characterizations in terms of the atomic characterization and the Littlewood-Paley function characterization. Moreover, they obtained the boundedness of $\delta$-type Calderón-Zygmund operators from $H^{\vec{p}}_A(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$ or from $H^{\vec{p}}_A(\mathbb{R}^n)$ to itself. For more information about this space, see [10, 12, 16].

Very recently, Huang et al. [13] also introduced the new anisotropic mixed-norm Hardy space $H^{\vec{p}}_A(\mathbb{R}^n)$ associated with a general expansive matrix $A$, via the non-tangential grand maximal function, and then established its various real-variable characterizations of $H^{\vec{p}}_A$, respectively, in terms of the atomic characterization and the Littlewood-Paley function characterization. For more information about Hardy space, see [13, 14, 21, 23, 25]. Nevertheless, the molecular decompositions of $H^{\vec{p}}_A(\mathbb{R}^n)$ has not been established till now. Once
its molecular decomposition is established, it can be conveniently used to prove the boundedness of many important operators on the space $H_A^p(\mathbb{R}^n)$, for example, one of the most famous operator in harmonic analysis, Calderón-Zygmund operators. To complete the theory of the new anisotropic mixed-norm Hardy space $H_A^p(\mathbb{R}^n)$, in this article, we establish the molecular decompositions of $H_A^p(\mathbb{R}^n)$. Then, as application, we further obtain a boundedness criterion for a class of linear operators from $H_A^p(\mathbb{R}^n)$ to $H_A^p(\mathbb{R}^n)$.

Precisely, this article is organized as follows.

In Section 2, we first recall some notations and definitions concerning expansive dilations, the mixed-norm Lebesgue space $L^p(\mathbb{R}^n)$ and the anisotropic mixed-norm Hardy space $H_A^p(\mathbb{R}^n)$, via the non-tangential grand maximal function.

The aim of Section 3 is to establish a new atomic characterization of $H_A^p(\mathbb{R}^n)$.

In Section 4, motivated by Liu et al. [18, 19] and Huang et al. [13], we introduce the anisotropic mixed-norm molecular Hardy space $H_{A,\text{mol}}^{p,q,s,\varepsilon}(\mathbb{R}^n)$ and establish its equivalence with $H_A^p(\mathbb{R}^n)$ in Theorem 4.4. When it comes back to the isotropic setting, i.e., $A := 2I_{n\times n}$, $I_{n\times n}$ denotes the $n \times n$ unit matrix, this result is still new, see Remark 4.5 for more details. It is worth pointing out that some of the proof methods of the molecular characterization of $H_A^p(\mathbb{R}^n) = H^{p,p}_A(\mathbb{R}^n)$ ([19, Theorem 3.9]) don’t work anymore in the present setting. For example, we search out some estimates related to $L^\varphi(\mathbb{R}^n)$ norms for some series of functions which can be reduced into dealing with the $L^q(\mathbb{R}^n)$ norms of the corresponding functions (see Lemma 4.6). Then, by using this key lemma and the Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator $M_{\text{HL}}$ on $L^\varphi(\mathbb{R}^n)$ (see Lemma 2.9), we prove their equivalences with $H_A^p(\mathbb{R}^n)$ and $H_{A,\text{mol}}^{p,q,s,\varepsilon}(\mathbb{R}^n)$.

In Section 5, as an application of the molecular characterization of $H_A^p(\mathbb{R}^n)$, we obtain a boundedness criterion for a class of linear operators from $H_A^p(\mathbb{R}^n)$ to $H_A^p(\mathbb{R}^n)$ (see Theorem 5.1 below). Particularly, when it comes back to the isotropic setting, i.e., $A := 2I_{n\times n}$, $I_{n\times n}$ denotes the $n \times n$ unit matrix, this result is also new.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. Throughout the whole paper, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. For any $q \in [1, \infty]$, we denote by $q'$ its conjugate index, namely, $1/q + 1/q' = 1$. For any $a \in \mathbb{R}$, $|a|$ denotes the maximal integer not larger than $a$. The symbol $D \lesssim F$ means that $D \leq CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. If $E$ is a subset of $\mathbb{R}^n$, we denote by $\chi_E$ its characteristic function. If there are no special instructions, any space $X(\mathbb{R}^n)$ is denoted simply by $X$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by $L^2$. The symbol $C^\infty$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^n$. Denote by $S$ the space of all Schwartz functions and $S'$ its dual space (namely, the space of all tempered distributions).
2 Preliminary

In this section, we first recall the notion of anisotropic mixed-norm Hardy space $H^p_A$, via the non-tangential grand maximal function $M_N(f)$, and then given its molecular decom- position.

We begin with recalling the notion of expansive dilations on $\mathbb{R}^n$; see [1, p.5]. A real $n \times n$ matrix $A$ is called an expansive dilation, shortly a dilation, if $\min_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all eigenvalues of $A$. Let $\lambda_-$ and $\lambda_+$ be two positive numbers such that
\[ 1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+. \]

Then there exists a positive constant $C$, such that, for any $x \in \mathbb{R}^n$, when $j \in \mathbb{Z}_+$,
\[ C^{-1}(\lambda_-)^j |x| \leq |A^j x| \leq C(\lambda_+)^j |x| \]
and, when $j \in \mathbb{Z}\setminus\mathbb{Z}_+$,
\[ C^{-1}(\lambda_+)^j |x| \leq |A^j x| \leq C(\lambda_-)^j |x|. \]

From [1, p.5, Lemma 2.2] that, for a fixed dilation $A$, there exist a number $r \in (1, \infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$, where $P$ is some non-degenerate $n \times n$ matrix, such that
\[ \Delta \subset r\Delta \subset A\Delta, \]
and we may assume that $|\Delta| = 1$, where $|\Delta|$ denotes the $n$-dimensional Lebesgue measure of the set $\Delta$. Let $B_k := A^k \Delta$ for $k \in \mathbb{Z}$. Then $B_k$ is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$, here and hereafter, $b := |\det A|$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a dilated ball. Denote by $\mathcal{B}$ the set of all such dilated balls, namely,
\[ \mathcal{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}. \]

Throughout the whole paper, let $\sigma$ be the smallest integer such that $2B_0 \subset A^\sigma B_0$ and, for any subset $E$ of $\mathbb{R}^n$, let $E^\complement := \mathbb{R}^n \setminus E$. Then, for all $k, j \in \mathbb{Z}$ with $k \leq j$, it holds true that
\[ B_k + B_j \subset B_{j+\sigma}, \]
\[ B_k + (B_{k+\sigma})^\complement \subset (B_k)^\complement, \]
where $E + F$ denotes the algebraic sum $\{x + y : x \in E, y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

**Definition 2.1.** A quasi-norm, associated with dilation $A$, is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$, for simplicity, denoted by $\rho$, satisfying
\begin{align*}
(i) & \quad \rho(x) > 0 \text{ for all } x \in \mathbb{R}^n \setminus \{0\}, \text{ here and hereafter, } 0 \text{ denotes the origin of } \mathbb{R}^n; \\
(ii) & \quad \rho(Ax) = b \rho(x) \text{ for all } x \in \mathbb{R}^n, \text{ where, as above, } b := |\det A|; \\
(iii) & \quad \rho(x + y) \leq C_A [\rho(x) + \rho(y)] \text{ for all } x, y \in \mathbb{R}^n, \text{ where } C_A \in [1, \infty) \text{ is a constant independent of } x \text{ and } y.
\end{align*}
In the standard dyadic case \( A := 2I_{n \times n} \), \( \rho(x) := |x|^n \) for all \( x \in \mathbb{R}^n \) is an example of homogeneous quasi-norms associated with \( A \), here and hereafter, \( I_{n \times n} \) denotes the \( n \times n \) unit matrix, \( | \cdot | \) always denotes the Euclidean norm in \( \mathbb{R}^n \).

By [1, Lemma 2.4], we see that all homogeneous quasi-norms associated with a given dilation \( A \) are equivalent. Therefore, for a given dilation \( A \), in what follows, for simplicity, we always use the step homogeneous quasi-norm \( \rho \) defined by setting, for all \( x \in \mathbb{R}^n \),

\[
\rho(x) := \sum_{k \in \mathbb{Z}} b^k \chi_{B_{k+1}\setminus B_k}(x) \text{ if } x \neq 0, \quad \text{or else } \rho(0) := 0.
\]

By (2.4), we know that, for all \( x, y \in \mathbb{R}^n \),

\[
\rho(x + y) \leq b^\sigma \left( \max \left\{ \rho(x), \rho(y) \right\} \right) \leq b^\sigma [\rho(x) + \rho(y)].
\]

Now we recall the definition of mixed-norm Lebesgue space. Let \( \vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n \). The \textit{mixed-norm Lebesgue space} \( L^{\vec{p}} \) is defined to be the set of all measurable functions \( f \) such that

\[
\|f\|_{L^{\vec{p}}} := \left\{ \int_{\mathbb{R}} \cdots \left[ \int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1} \, dx_1 \right]^{p_2/p_1} \cdots \, dx_n \right\}^{1/p_n} < \infty
\]

with the usual modifications made when \( p_i = \infty \) for some \( i \in \{1, \ldots, n\} \).

For any \( \vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n \), let

\[
p_- := \min\{p_1, \ldots, p_n\} \quad \text{and} \quad p_+ := \max\{p_1, \ldots, p_n\}.
\]

**Lemma 2.2.** [13, Lemma 3.4] Let \( \vec{p} \in (0, \infty]^n \). Then, for any \( r \in (0, \infty) \) and \( f \in L^{\vec{p}} \),

\[
\| |f|^r \|_{L^{\vec{p}}} = \|f\|_{L^{\vec{p}}^r}^r.
\]

In addition, for any \( \mu \in \mathbb{C}, \gamma \in [0, \min\{1, p_-\}] \) and \( f, g \in L^{\vec{p}}, \|\mu f\|_{L^{\vec{p}}} = |\mu| \|f\|_{L^{\vec{p}}} \) and

\[
\|f + g\|_{L^{\vec{p}}}^\gamma \leq \|f\|_{L^{\vec{p}}}^\gamma + \|g\|_{L^{\vec{p}}}^\gamma,
\]

here and hereafter, for any \( \alpha \in \mathbb{R}, \alpha \vec{p} := (\alpha p_1, \ldots, \alpha p_n) \) and

\[
p := \min\{p_-, 1\}
\]

with \( p_- \) as in (2.6).

A \( C^\infty \) function \( \varphi \) is said to belong to the Schwartz class \( \mathcal{S} \) if, for every integer \( \ell \in \mathbb{Z}_+ \) and multi-index \( \alpha, \|\varphi\|_{\alpha, \ell} := \sup_{x \in \mathbb{R}^n} |\partial^\alpha \varphi(x)| < \infty \). The dual space of \( \mathcal{S} \), namely, the space of all tempered distributions on \( \mathbb{R}^n \) equipped with the weak*-topology, is denoted by \( \mathcal{S}' \). For any \( N \in \mathbb{Z}_+ \), let

\[
\mathcal{S}_N := \{ \varphi \in \mathcal{S} : \|\varphi\|_{\alpha, \ell} \leq 1, \ |\alpha| \leq N, \ \ell \leq N \}.
\]

In what follows, for \( \varphi \in \mathcal{S}, k \in \mathbb{Z} \) and \( x \in \mathbb{R}^n \), let \( \varphi_k(x) := b^{-k} \varphi(A^{-k}x) \).
Definition 2.3. Let $\varphi \in S$ and $f \in S'$. The non-tangential maximal function $M_\varphi(f)$ with respect to $\varphi$ is defined by setting, for any $x \in \mathbb{R}^n$,
\[
M_\varphi(f)(x) := \sup_{y \in x + B_k, k \in \mathbb{Z}} |f \ast \varphi_k(y)|.
\]
The radial maximal function $M_0^\varphi(f)$ with respect to $\varphi$ is defined by setting, for any $x \in \mathbb{R}^n$,
\[
M_0^\varphi(f)(x) := \sup_{k \in \mathbb{Z}} |f \ast \varphi_k(x)|.
\]
Moreover, for any given $N \in \mathbb{N}$, the non-tangential grand maximal function $M_N(f)$ of $f \in S'$ is defined by setting, for any $x \in \mathbb{R}^n$,
\[
M_N(f)(x) := \sup_{\varphi \in S_N} M_\varphi(f)(x).
\]
The radial grand maximal function $M_N^0(f)(x)$ of $f \in S'$ is defined by setting, for any $x \in \mathbb{R}^n$,
\[
M_N^0(f)(x) := \sup_{\varphi \in S_N} M_0^\varphi(f)(x).
\]
The following anisotropic mixed-norm Hardy space $H_{\vec{A}}^{\vec{p}}$ was introduced in [13, Definition 2.5].

Definition 2.4. Let $\vec{p} \in (0, \infty)^n$, $A$ be a dilation and $N \in \mathbb{N} \cap \lfloor (1/p - 1) \ln b / \ln \lambda_- \rfloor + 2, \infty)$, where $p$ is as in (2.7). The anisotropic mixed-norm Hardy space $H_{\vec{A}}^{\vec{p}}$ is defined as
\[
H_{\vec{A}}^{\vec{p}} := \left\{ f \in S' : M_N(f) \in L^{\vec{p}} \right\}
\]
and, for any $f \in H_{\vec{A}}^{\vec{p}}$, let $\|f\|_{H_{\vec{A}}^{\vec{p}}} := \|M_N(f)\|_{L^{\vec{p}}}$. 

Remark 2.5. Let $\vec{p} \in (0, \infty)^n$.

(i) From [13, Theorem 4.7], we know that the $H_{\vec{A}}^{\vec{p}}$ is independent of the choice of $N$, as long as $N \in \mathbb{N} \cap \lfloor (1/p - 1) \ln b / \ln \lambda_- \rfloor + 2, \infty)$.

(ii) When $\vec{p} := \{p, \ldots, p\}$, where $p \in (0, \infty)$, the space $H_{\vec{A}}^{\vec{p}}$ is reduced to the anisotropic Hardy $H_A^p$ studied in [1, Definition 3.11].

(iii) From [13, Proposition 4], we know that, when
\[
A := \begin{pmatrix}
2^{a_1} & 0 & \cdots & 0 \\
0 & 2^{a_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 2^{a_n}
\end{pmatrix}
\]
with $\vec{a} := (a_1, \ldots, a_n) \in [1, \infty)^n$, the space $H_{\vec{A}}^{\vec{p}}$ is reduced to the anisotropic Hardy $H_{\vec{a}}^{\vec{p}}$ studied in [4, 11].
We recall the following notion of anisotropic mixed-norm \((\vec{p}, q, s)\)-atoms introduced in [13, Definition 4.1].

**Definition 2.6.** Let \(\vec{p} \in (0, \infty)^n\), \(q \in (1, \infty]\) and \(s \in [\max\{(p_- - 1)\ln b/\ln \lambda_-, \infty]\) \cap \mathbb{Z}_+\) with \(p_-\) as in (2.6). An anisotropic mixed-norm \((\vec{p}, q, s)\)-atom is a measurable function \(a\) on \(\mathbb{R}^n\) satisfying

(i) (support) \(\text{supp } a \subset B\), where \(B \in \mathcal{B}\) and \(\mathcal{B}\) is as in (2.3);

(ii) (size) \(|a|_{L^q} \leq \|B\|^1/q \|\chi_B\|_{L^p}\);

(iii) (vanishing moment) \(\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0\) for any \(\alpha \in \mathbb{Z}_+^n\) with \(|\alpha| \leq s\).

In this paper, we call an anisotropic mixed-norm \((\vec{p}, q, s)\)-atom simply by a \((\vec{p}, q, s)\)-atom. The following anisotropic mixed-norm atomic Hardy space was introduced in [13].

**Definition 2.7.** Let \(\vec{p} \in (0, \infty)^n\), \(q \in (1, \infty]\), \(A\) be a dilation and \(s \in [\max\{(p_- - 1)\ln b/\ln \lambda_-, \infty]\) \cap \mathbb{Z}_+\) with \(p_-\) as in (2.6). The anisotropic mixed-norm atomic Hardy space \(H^{\vec{p}, q, s}_A\) is defined to be the set of all distributions \(f \in \mathcal{S}'\) satisfying that there exist \(\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}\) and a sequence of \((\vec{p}, q, s)\)-atoms, \(\{a_i\}_{i \in \mathbb{N}}\), supported, respectively, on \(\{B^{(i)}_B\}_{i \in \mathbb{N}} \subset \mathcal{B}\) such that

\[
f = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \text{in } \mathcal{S}'.
\]

Moreover, for any \(f \in H^{\vec{p}, q, s}_A\), let

\[
\|f\|_{H^{\vec{p}, q, s}_A} := \inf \left\| \left\{ \sum_{i \in \mathbb{N}} \left[ |\lambda_i| \chi_{B^{(i)}} \right] \right\}^{1/\eta} \right\|_{L^{p_\eta}},
\]

where \(\eta \in (0, \min\{1, p_-\})\) and the infimum is taken over all the decompositions of \(f\) as above.

**Lemma 2.8.** [13, Theorem 4.7] Let \(\vec{p} \in (0, \infty)^n, q \in (\max\{p_+, 1\}, \infty]\) with \(p_+\) as in (2.6), \(s \in [\max\{(p_- - 1)\ln b/\ln \lambda_-, \infty]\) \cap \mathbb{Z}_+\) with \(p_-\) as in (2.6) and \(N \in \mathbb{N} \cap [\max\{(1/2 - 1)\ln b/\ln \lambda_-, 2\}, \infty]\). Then

\[
H^{\vec{p}}_A = H^{\vec{p}, q, s}_A
\]

with equivalent quasi-norms.

We recall the definition of anisotropic Hardy-Littlewood maximal function \(M_{\text{HL}}(f)\). For any \(f \in L^1_{\text{loc}}\) and \(x \in \mathbb{R}^n\),

\[
M_{\text{HL}}(f)(x) := \sup_{x \in B \in \mathcal{B}} \frac{1}{|B|} \int_B |f(z)| dz,
\]

where \(\mathcal{B}\) is as in (2.3).
Lemma 2.9. [13, Lemma 4.4] Let $\vec{p} \in (1, \infty)^n$ and $u \in (1, \infty]$. Then there exists a positive constant $C$ such that, for any sequence $\{f_k\}_{k \in \mathbb{N}}$ of measurable functions,

$$\left\| \left\{ \sum_{k \in \mathbb{N}} [M_{HL}(f_k)]^u \right\}^{1/u} \right\|_{L^{\vec{p}}} \leq C \left\| \left( \sum_{k \in \mathbb{N}} |f_k|^u \right)^{1/u} \right\|_{L^{\vec{p}}}$$

with the usual modification made when $u = \infty$, where $M_{HL}$ denotes the Hardy-Littlewood maximal operator as in (2.8).

3 Atomic Decomposition of $H^{\vec{p}}_A$

In this section, we obtain the new atomic decomposition of $H^{\vec{p}}_A$. Let

$$\mathcal{S}_\infty := \left\{ \varphi \in \mathcal{S} : \int_{\mathbb{R}^n} x^\alpha \varphi(x) dx = 0, \forall \alpha \in \mathbb{Z}_+^n \right\}$$

equipped with the same topology as $\mathcal{S}$, and $\mathcal{S}'_\infty$ be its dual space equipped with the weak-* topology. Now we introduce the new anisotropic mixed-norm atomic Hardy space $H^{\vec{p}, q, s}_A$ in term of $\mathcal{S}'_\infty$.

Definition 3.1. Let $\vec{p} \in (0, \infty)^n$, $q \in (1, \infty]$, $A$ be a dilation and $s \in [[[1/p_- - 1)\ln b/\ln \lambda_-], \infty) \cap \mathbb{Z}_+$ with $p_-$ as in (2.6). The anisotropic mixed-norm atomic Hardy space $H^{\vec{p}, q, s}_A$ is defined to be the set of all distributions $f \in \mathcal{S}'$ satisfying that there exist $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, q, s)$-atoms, $\{a_i\}_{i \in \mathbb{N}}$, supported, respectively, on $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathcal{B}$ such that

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } \mathcal{S}'_\infty.$$ 

Moreover, for any $f \in H^{\vec{p}, q, s}_A$, let

$$\|f\|_{H^{\vec{p}, q, s}_A} := \inf \left\{ \left\| \left( \sum_{i \in \mathbb{N}} \left[ \frac{\lambda_i \chi_{B^{(i)}}}{\| \chi_{B^{(i)}} \|_{L^{\vec{p}}}} \right]^\eta \right)^{1/\eta} \right\|_{L^{\vec{p}}} \right\},$$

where $\eta \in (0, \min\{1, p_-\})$ and the infimum is taken over all the decompositions of $f$ as above.

Theorem 3.2. Let $\vec{p} \in (0, 1]^n$, $q \in (1, \infty]$, $s \in [[[1/p_- - 1)\ln b/\ln \lambda_-], \infty) \cap \mathbb{Z}_+$ with $p_-$ as in (2.6) and $N \in \mathbb{N} \cap [[[1/p - 1)\ln b/\ln \lambda_-] + 2, \infty)$. Then

$$H^{\vec{p}}_A = H^{\vec{p}, q, s}_A$$

in the following sense: if $f \in H^{\vec{p}}_A$, then $f \in H^{\vec{p}, q, s}_A$ and

$$\|f\|_{H^{\vec{p}, q, s}_A} \leq \|f\|_{H^{\vec{p}}_A}.$$
Conversely, if \( f \in H^\vec{p}_A \), then there exists a unique extension \( F \in H_A \) such that, for any \( \phi \in \mathcal{S}_\infty \), \( \langle F, \phi \rangle = \langle f, \phi \rangle \) and
\[
\|F\|_{H_A} \leq \|f\|_{H^\vec{p}_q A}.
\]

**Remark 3.3.** Let \( \vec{p} \in (0, \infty)^n \).

(i) When \( \vec{p} := \{p, \ldots, p\} \), where \( p \in (0, 1] \), this result is also new.

(ii) When
\[
A := \begin{pmatrix}
2^{a_1} & 0 & \cdots & 0 \\
0 & 2^{a_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 2^{a_n}
\end{pmatrix}
\]
with \( \vec{a} := (a_1, \ldots, a_n) \in [1, \infty)^n \), this result is reduced to the [14, Theorem 3.3].

To prove Theorem 3.2, we need some technical lemmas as following.

**Lemma 3.4.** Let \( \vec{p} \in (0, \infty)^n \), \( q \in (1, \infty) \) and \( s \in [(1/p_- - 1)\ln b/\ln \lambda_+, \infty) \cap \mathbb{Z}_+ \) with \( p_- \) as in (2.6). Then there exists a positive constant \( C \) such that, for any \((\vec{p}, q, s)\)-atom \( b \) and \( \varphi \in \mathcal{S} \),
\[
\left| \int_{\mathbb{R}^n} b(y)\varphi(y)dy \right| \leq CM_s(\varphi),
\]
where \( M_s(\varphi) := \sup\{\|\varphi\|_{\alpha, 0} : \alpha \in \mathbb{Z}_+^n, |\alpha| \leq s + 1\} \).

**Proof.** For any \((\vec{p}, q, s)\)-atom \( b \), we assume \( \text{supp } b \subset B_k \). Now we claim that, for any \( k \in \mathbb{Z} \),
\[
\|\chi_{B_k}\|_{L^{\vec{p}}}^{-1} \lesssim \max\{b^{-k/p_-}, b^{-k/p_+}\}.
\]
In fact, there exists a \( K \in \mathbb{Z} \) large enough such that, when \( k \in (K, \infty) \cap \mathbb{Z} \), the
\[
\|\chi_{B_k}\|_{L^{\vec{p}}} = \left\{ \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |\chi_{B_k}(x_1, \ldots, x_n)|^{p_1} dx_1 \cdots dx_n \right\}^{1/p_1} \cdot \cdots \cdot \left\{ \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |\chi_{B_k}(x_1, \ldots, x_n)|^{p_+} dx_1 \cdots dx_n \right\}^{1/p_+}
\]
\[
\geq \left\{ \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |\chi_{B_k}(x_1, \ldots, x_n)|^{p_+} dx_1 \cdots dx_n \right\}^{1/p_+} = |B_k|^{1/p_+}.
\]
When \( k \in (-\infty, K] \), by [13, Lemma 6.8], we conclude that, for any \( \nu \in (0, 1) \),
\[
\|\chi_{B_k}\|_{L^{\vec{p}}} \lesssim b^{(1+\nu)(K-k)/p_-}.
\]
Let \( \nu \to 0 \). Then
\[
\frac{1}{\|\chi_{B_k}\|_{L^{\vec{p}}}} \lesssim \frac{b^{K/p_-}}{\|\chi_{B_k}\|_{L^{\vec{p}}}} b^{-k/p_-} \sim |B_k|^{-1/p_-}.
\]
If \( k > 0 \), we have
\[
\|b\|_{L^q} \leq \frac{|B_k|^{1/q}}{\|\chi_{B_k}\|_{L^p}} \lesssim |B_k|^{1/q-1/p_+}.
\]
If \( k \leq 0 \), we have
\[
\|b\|_{L^q} \leq \frac{|B_k|^{1/q}}{\|\chi_{B_k}\|_{L^p}} \lesssim |B_k|^{1/q-1/p_-}.
\]
Therefore, when \( k > 0 \), we have
\[
\left| \int_{\mathbb{R}^n} b(y) \varphi(y) dy \right| \leq \|\varphi\|_{L^\infty} \|b\|_{L^q} |B_k|^{1/q'} \lesssim \|\varphi\|_0 \|b\|_{L^q} |B_k|^{1/p_+ - 1/q} \lesssim M_s(\varphi).
\]
When \( k \leq 0 \), by the vanishing moment of \( b \) and the Taylor reminder theorem, we obtain
\[
\left| \int_{\mathbb{R}^n} b(y) \varphi(y) dy \right| \leq \left| \int_{\mathbb{R}^n} b(y) \left[ \varphi(y) - \sum_{|\alpha| \leq s} \frac{\partial^\alpha \varphi(0)}{\alpha!} y^\alpha \right] dy \right| \lesssim M_s(\varphi) \left| \int_{B_k} b(y)|y|^{s+1} dy \right| \lesssim M_s(\varphi) \|b\|_{L^q} |B_k|^{1/q'} \lesssim M_s(\varphi) |B_k|^{1/q-1/p_-} |B_k|^{s+1-1/q} \lesssim M_s(\varphi).
\]
This finishes the proof of Lemma 3.4.

**Lemma 3.5.** Let \( \vec{p} \in (0, 1]^n \). Then, for any \( \{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C} \) and \( \{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathcal{B} \),
\[
\sum_{i \in \mathbb{N}} |\lambda_i| \leq \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^\vec{p}}} \right]^\eta \right\}^{1/\eta}_{L^\vec{p}}
\]
where \( \eta \in (0, \vec{p}) \) with \( \vec{p} \) as in (2.7).

**Proof.** Since \( \vec{p} \in (0, 1]^n \) and the well-known inequality that, for all \( \{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C} \) and \( \vartheta \in (0, 1] \),
\[
\left( \sum_{i \in \mathbb{N}} |\lambda_i| \right)^\vartheta \leq \sum_{i \in \mathbb{N}} |\lambda_i|^\vartheta.
\]
Then we have
\[
\sum_{i \in \mathbb{N}} |\lambda_i| = \sum_{i \in \mathbb{N}} |\lambda_i| \left\| \frac{\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^\vec{p}}} \right\|_{L^\vec{p}}
\]
\[
\lambda \left\| \sum_{i \in \mathbb{N}} \frac{\lambda_i |\chi_{B(i)}|}{\|\chi_{B(i)}\|_L^p} \right\|_{L^q}
\]
\[
\lambda \left\| \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{\lambda_i |\chi_{B(i)}|}{\|\chi_{B(i)}\|_L^p} \right] \right\}^{1/\eta} \right\|_{L^q}.
\]

This finishes the proof of Lemma 3.5. \( \square \)

**Lemma 3.6.** [17] Let \( \pi : S' \to S'_\infty \) satisfy that, for any \( f \in S' \) and \( \phi \in S'_\infty \),
\[
\langle \pi(f), \phi \rangle = \langle f, \phi \rangle.
\]
Then
\[
P = \{ f \in S' : \pi(f) = 0 \}.
\]
where \( P \) denote all polynomials on \( \mathbb{R}^n \). Moreover, \( P \) is closed in \( S' \).

**Proof of Theorem 3.2.** By the definitions of \( S \) and \( S_\infty \), we find that
\[
S_\infty \subset S,
\]
which implies that
\[
S' \subset S'_\infty.
\]
From this, Lemma 2.8, we conclude that
\[
H_\vec{p} = H^p_A \subset \mathbb{H}^p_A.
\]
and for any \( f \in H_\vec{p} \),
\[
\| f \|_{H^p_A} \leq \| f \|_{\mathbb{H}^p_A}.
\]
Therefore, to prove Theorem 3.2, it suffices to show that, for any \( \phi \in \mathbb{H}^p_A \), there exists a unique extension \( F \in H^p_A \) such that, for any \( \phi \in S_\infty \),
\[
\langle F, \phi \rangle = \langle f, \phi \rangle
\]
and
\[
\| F \|_{H^p_A} \leq \| f \|_{\mathbb{H}^p_A}.
\]
To show this, for any \( f \in \mathbb{H}^p_A \), by Definition 3.1, we deduce that there exists a sequence of \( (\vec{p}, q, s) \)-atom \( \{a_i\}_{i \in \mathbb{N}} \), supported, respectively, on \( \{B(i)\}_{i \in \mathbb{N}} \subset \mathcal{B} \), such that
\[
f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } S'_\infty
\]
and
\[
\| f \|_{\mathbb{H}^p_A} \sim \left\| \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{\lambda_i |\chi_{B(i)}|}{\|\chi_{B(i)}\|_L^p} \right] \right\}^{1/\eta} \right\|_{L^q} < \infty
\]
with \( \eta \in (0, \min\{1, p_\pm\}) \).

For any \( \varphi \in \mathcal{S} \), set

\[
\langle F, \varphi \rangle := \sum_{i \in \mathbb{N}} \lambda_i \int_{\mathbb{R}^n} a_i(x) \varphi(x) \, dx.
\]

Then, using Lemmas 3.4 and 3.5, we have

\[
|\langle F, \varphi \rangle| \lesssim M^{s}(\varphi) \sum_{i \in \mathbb{N}} |\lambda_i| \lesssim M^{s}(\varphi) \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i| \chi_{B(i)}(\mathbb{R}^n)}{\| \chi_{B(i)} \|_{L^{\bar{p}}}} \right]^{\eta} \right\}^{1/\eta}, \quad \eta \in (0, \min\{1, p_\pm\}),
\]

which implies that \( F \in \mathcal{S}' \). From this, we see that, \( F \in H^{\vec{p}, q, s}_{A} \) and for all \( \phi \in \mathcal{S}_{\infty} \),

\[
\langle F, \phi \rangle = \langle f, \phi \rangle.
\]

By the Definition 3.1, we know that

\[
\| F \|_{H^{\vec{p}, q, s}_{A}} \leq \| f \|_{H^{\vec{p}, q, s}_{A}}.
\]

Finally, we only need to show that the distribution \( F \) is unique. In fact, if there exists distribution \( \tilde{F} \in H^{\vec{p}, q, s}_{A} \) such that, for any \( \psi \in \mathcal{S}_{\infty} \), \( \langle \tilde{F}, \psi \rangle = \langle f, \psi \rangle \). Then, by Lemma 3.6, we obtain

\[
F - \tilde{F} \in \mathcal{P}.
\]

By the fact that any element of \( H^{\vec{p}, q, s}_{A} \) vanishes weakly in infinity and \( F - \tilde{F} \in H^{\vec{p}, q, s}_{A} \), we deduce that \( F = \tilde{F} \) in \( \mathcal{S}'_{\infty} \). This finishes the proof of Theorem 3.2.

\[\square\]

4 Molecular Decomposition of \( H^{\vec{p}}_{A} \)

In this section, we introduce the definition of anisotropic mixed-norm molecules as follows.

**Definition 4.1.** Let \( \vec{p} \in (0, \infty)^n \), \( q \in (1, \infty] \),

\[
s \in \left[ \left( \frac{1}{p_-} - 1 \right) \frac{\ln b}{\ln \lambda_-}, \infty \right) \cap \mathbb{Z}_+.
\]

and \( \varepsilon \in (0, \infty) \). A measurable function \( \mathcal{M} \) is called an *anisotropic mixed-norm (\( \vec{p}, q, s, \varepsilon \)) molecule* associated with a dilated ball \( x_0 + B_i \in \mathcal{B} \) if

(i) for each \( j \in \mathbb{Z}_+ \), \( \| \mathcal{M} \|_{L^q(U_j(x_0 + B_i))} \leq b^{-j \varepsilon} \frac{|B_i|^{1/q}}{\| \chi_{x_0 + B_i} \|_{L^{\vec{p}}}} \), where \( U_0(x_0 + B_i) := x_0 + B_i \) and, for each \( j \in \mathbb{N} \), \( U_j(x_0 + B_i) := x_0 + (A^j B_i) \setminus (A^{j-1} B_i) \);

(ii) for all \( \alpha \in \mathbb{Z}_+^n \) with \( |\alpha| \leq s \), \( \int_{\mathbb{R}^n} \mathcal{M}(x) x^\alpha \, dx = 0 \).

**Remark 4.2.** Let \( \vec{p} \in (0, \infty)^n \).
(i) When \( \vec{p} := \{p, \ldots, p\} \), where \( p \in (0, 1] \), the definition of the molecule in Definition 4.1 is reduced to the molecule in [19, Definition 3.7].

(ii) When it comes back to the isotropic setting, i.e., \( A := 2I_{n \times n} \), and \( \rho(x) := |x|^n \) for all \( x \in \mathbb{R}^n \), the definition of the molecule in Definition 2.4 is also new.

In what follows, we call an anisotropic mixed-norm \((\vec{p}, q, s, \varepsilon)\)-molecule simply by \((\vec{p}, q, s, \varepsilon)\)-molecule. Via \((\vec{p}, q, s, \varepsilon)\)-molecules, we introduce the following anisotropic mixed-norm molecular Hardy space \( H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon} \).

**Definition 4.3.** Let \( \vec{p} \in (0, \infty)^n \), \( q \in (1, \infty] \cap (p_+, \infty] \) with \( p_+ \) as in (2.6), \( s \) be as in (4.1) and \( \varepsilon \in (0, \infty) \). The anisotropic mixed-norm molecular Hardy space \( H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon} \) is defined to be the set of all distributions \( f \in S' \) satisfying that there exist \( \{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C} \) and a sequence of \((\vec{p}, q, s, \varepsilon)\)-molecules, \( \{\mathcal{M}_i\}_{i \in \mathbb{N}} \), associated, respectively, with \( \{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathcal{B} \) such that

\[
f = \sum_{i \in \mathbb{N}} \lambda_i \mathcal{M}_i \text{ in } S'.
\]

Moreover, for any \( f \in H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon} \), let

\[
\|f\|_{H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}} := \inf \left\{ \left\| \sum_{i \in \mathbb{N}} \left[ \frac{\lambda_i |\chi_{B^{(i)}}|}{\|\chi_{B^{(i)}}\|_{L_{\vec{p}}}} \right]^{\eta} \right\|_{L_{\vec{p}}}^{1/\eta} \right\},
\]

where \( \eta \in (0, \min\{1, p_-\}) \) and the infimum is taken over all the decompositions of \( f \) as above.

The following Theorem 4.4 shows the molecular characterization of \( H_{A}^{\vec{p}} \), whose proof is given in the next section.

**Theorem 4.4.** Let \( \vec{p} \in (0, \infty)^n \) and \( q \in (1, \infty] \cap (p_+, \infty] \) with \( p_+ \) as in (2.6), \( s \) be as in (4.1), \( \varepsilon \in (\max\{1, (s + 1) \log b(\lambda_+)\}, \infty) \) and \( N \in \mathbb{N} \cap \left(\left\lfloor (1/p - 1) \log b/\log \lambda_- \right\rfloor + 2, \infty\right) \) with \( p \) as in (2.7). Then

\[
H_{A}^{\vec{p}} = H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}
\]

with equivalent quasi-norms.

**Remark 4.5.** Let \( \vec{p} \in (0, 1]^n \).

(i) Liu et al. [20] introduced the anisotropic Hardy-Lorentz space \( H_{A}^{p, q} \), where \( p \in (0, 1] \) and \( q \in (0, \infty) \). When \( \vec{p} := \{p, \ldots, p\} \) with \( p \in (0, 1] \), the molecular characterization of \( H_{A}^{\vec{p}} \) in Theorem 4.4 is reduced to the molecular characterization of anisotropic Hardy spaces \( H_{A}^{p} = H_{A}^{p, p} \) in [19, Theorem 3.9].

(ii) When it comes back to the isotropic setting, i.e., \( A := 2I_{n \times n} \), the molecular characterization of \( H_{A}^{\vec{p}} \) in Theorem 4.4 is still new.
To show Theorem 4.4, we need the following lemma. By a similar proof of [13, Lemma 4.5], we obtain the following useful conclusion; the details are omitted.

**Lemma 4.6.** Let \( \vec{p} \in (0, \infty)^n \) and \( q \in (1, \infty] \cap (p_+, \infty] \) with \( p_+ \) as in (2.6). Assume that \( \{ \lambda_i \}_{i \in \mathbb{N}} \subset \mathbb{C} \), \( \{ B^{(i)} \}_{i \in \mathbb{N}} \subset \mathcal{B} \) and \( \{ a_i \}_{i \in \mathbb{N}} \in L^q \) satisfy, for any \( i \in \mathbb{N} \), \( \text{supp} a_i \subset A^j B^{(i)} \) with some fixed \( j_0 \in \mathbb{Z} \), \( \| a \|_{L^q} \leq \| \chi_{B^{(i)}} \|_{L^q} \) and

\[
\left\| \left( \sum_{i \in \mathbb{N}} \left( \frac{|\lambda_i| \chi_{B^{(i)}}}{\| \chi_{B^{(i)}} \|_{L^q}} \right)^{\eta} \right)^{1/\eta} \right\|_{L^{\vec{p},q}} < \infty.
\]

Then

\[
\left\| \left( \sum_{i \in \mathbb{N}} |\lambda_i| a_i \right)^{\eta} \right\|_{L^{\vec{p},q}} \leq C \left\{ \sum_{i \in \mathbb{N}} \left( \frac{|\lambda_i| \chi_{B^{(i)}}}{\| \chi_{B^{(i)}} \|_{L^q}} \right)^{\eta} \right\}^{1/\eta}_{L^{\vec{p},q}},
\]

where \( \eta \in (0, \min\{1, p_-\}) \) and \( C \) is a positive constant independent of \( \{ \lambda_i \}_{i \in \mathbb{N}} \), \( \{ B^{(i)} \}_{i \in \mathbb{N}} \) and \( \{ a_i \}_{i \in \mathbb{N}} \).

**Proof of Theorem 4.4.** By the definitions of \((\vec{p}, q, s, \varepsilon)\)-atom and \((\vec{p}, q, s, \varepsilon)\)-molecule, we find that a \((\vec{p}, \infty, s)\)-atom is also a \((\vec{p}, q, s, \varepsilon)\)-molecule, which implies that

\[
H_{A^{\infty,s}}^{\vec{p},q} \subset H_{A,\text{mol}}^{\vec{p},q, s, \varepsilon}.
\]

This, combined with Lemma 2.8, further implies that, to prove Theorem 4.4, it suffices to show \( H_{A,\text{mol}}^{\vec{p},q, s, \varepsilon} \subset H_{A^{\infty}}^{\vec{p},q} \).

For any \( f \in H_{A,\text{mol}}^{\vec{p},q, s, \varepsilon} \), by Definition 4.3, we have that there exists a sequence of 
\((\vec{p}, q, s, \varepsilon)\)-molecules, \( \{ \mathcal{M}_i \}_{i \in \mathbb{N}} \), associated with dilated balls \( \{ B^{(i)} \}_{i \in \mathbb{N}} \subset \mathcal{B} \), where \( B^{(i)} := x_i + B_{\ell_i} \) with \( x_i \in \mathbb{R}^n \) and \( \ell_i \in \mathbb{Z} \), such that

\[
f = \sum_{i \in \mathbb{N}} \lambda_i \mathcal{M}_i \text{ in } S'
\]

and

\[
\| f \|_{H_{A,\text{mol}}^{\vec{p},q, s, \varepsilon}} \sim \left\{ \sum_{i \in \mathbb{N}} \left( \frac{|\lambda_i| \chi_{B^{(i)}}}{\| \chi_{B^{(i)}} \|_{L^q}} \right)^{\eta} \right\}^{1/\eta}_{L^{\vec{p},q}},
\]

where \( \eta \in (0, \min\{1, p_-\}) \). To prove \( f \in H_{A}^{\vec{p},q} \), it is easy to see that, for any \( N \in \mathbb{N} \cap [(1/p - 1) \ln b / \ln \lambda_-] + 2, \infty) \),

\[
\| M_N(f) \|_{L^{\vec{p},q}}^p \leq \| M_N \left( \sum_{i \in \mathbb{N}} \lambda_i \mathcal{M}_i \right) \|_{L^{\vec{p},q}}^p \leq \sum_{i \in \mathbb{N}} |\lambda_i| \| M_N(\mathcal{M}_i) \|_{L^{\vec{p},q}}^p
\]

\[
\leq \left( \sum_{i \in \mathbb{N}} |\lambda_i| \right)^{p} \left( \sum_{i \in \mathbb{N}} |\lambda_i| \right)^{p} \chi_{A^{2^p B^{(i)}}} \| M_N(\mathcal{M}_i) \chi_{(A^{2^p B^{(i)}})} \|_{L^{\vec{p},q}}^p.
\]
By this, Lemma 4.6, q ∈ ((max{p_+}, 1), ∞) and (4.2), we obtain

\[ I_1 = \left\langle \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|\chi_{B(i)}\|_{L^\rho}} \right] \chi_{B(i)} M_N(\mathfrak{m}_i) \chi_{A^{2\sigma}B(i)} \right\}^{1/\eta} \right\rangle^{1/\eta} \leq \left\| \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|\chi_{B(i)}\|_{L^\rho}} \chi_{B(i)} M_N(\mathfrak{m}_i) \chi_{A^{2\sigma}B(i)} \right] \right\|_{L^\rho} \sim \|f\|_{H_{\eta,q,s}^p}^{1/\eta} \|x\|_{L^\rho}^{1/\eta} \sim \|f\|_{H_{\eta,q,s}^p}^{1/\eta} \|x\|_{L^\rho}^{1/\eta} \sim \|f\|_{H_{\eta,q,s}^p}^{1/\eta} \|x\|_{L^\rho}^{1/\eta}. \]

To deal with I_2, for any i ∈ \mathbb{N} and x ∈ (x_i + A^{2\sigma}B(i))^C, we need to estimate \( M_N^0(\mathfrak{m}_i)(x) \), it’s proof is similar to that of [19, (3.48)]. Suppose that P is a polynomial of degree not greater than s which is determined later. By the Hölder inequality and Definitions 4.1, we know that

\begin{align*}
|\mathfrak{m}_i \ast \varphi_k(x)| &= b^{-k} \int_{\mathbb{R}^n} \mathfrak{m}_i(y) \varphi(A^{-k}(x - y)) \, dy \\
&\leq b^{-k} \sum_{j=0}^{\infty} \left| \int_{U_j(x_i + B_{\ell_i})} \mathfrak{m}_i(y) \left[ \varphi(A^{-k}(x - y)) - P(A^{-k}(x - y)) \right] \, dy \right| \\
&\leq b^{-k} \sum_{j=0}^{\infty} \left| \frac{\mathfrak{m}_i}{\|\chi_{x_i+B_{\ell_i}}\|_{L^\rho}} \int_{U_j(x_i + B_{\ell_i})} \left[ \varphi(A^{-k}(x - y)) - P(A^{-k}(x - y)) \right] \, dy \right|^{1/q'} \\
&\leq b^{-k} \sum_{j=0}^{\infty} b^{-j \varepsilon} \left| \frac{x_i + B_{\ell_i}}{b_j^{(1/q')^{-\varepsilon}}} \sup_{z \in A^{-k}(x - x_i) + B_{\ell_i+j-k}} |\varphi(z) - P(z)| \right| \\
&= b^{\ell_i-k} \sum_{j=0}^{\infty} b^{j/1/q'} \sup_{z \in A^{-k}(x - x_i) + B_{\ell_i+j-k}} |\varphi(z) - P(z)|. 
\end{align*}

Assume that x ∈ x_i + B_{\ell_i+2\sigma+m+1}|B_{\ell_i+2\sigma+m+1} for some m ∈ Z+. Then, by (2.5), we know that, for any k ∈ Z and j ∈ Z+,

\begin{align*}
A^{-k}(x - x_i) + B_{\ell_i+j-k} &\subseteq A^{-k}(B_{\ell_i+2\sigma+m+1}|B_{\ell_i+2\sigma+m} + B_{\ell_i+j-k}) \\
&\subseteq A^{-k}(x - x_i + B_{\ell_i+j-k}).
\end{align*}
Taylor's theorem, (4.6)

\[ |φ(z)| \lesssim \sup_{z \in A^{i+j-k}} \frac{1}{(1+\rho(z))^N} \]
\[ \lesssim b^{-N(i+j+k+m)}. \]

If \( \ell_i \geq k \), let \( P = 0 \). Then we have, for any \( N \in \mathbb{N} \),

\[
\sup_{z \in A^{i-k}(x-x_i) + B_{i+j-k}} |\varphi(z)| \lesssim \sup_{z \in A^{i+j-k}(B_m)} \frac{1}{(1+\rho(z))^N} \]
\[ \lesssim b^{-N(i+j-k+m)}. \]

If \( \ell_i < k \), let \( P \) be the Taylor expansion of \( \varphi \) at the point \( A^{-k}(x-x_i) \) of order \( s \). By Taylor's theorem, (2.1), (2.2) and (4.4), we obtain, for any \( N \in \mathbb{N} \),

\[
\sup_{z \in A^{i-k}(x-x_i) + A^{-k}B_{i}} |\varphi(z) - P(z)| \]
\[ \lesssim \sup_{y \in A^{-k}B_{i}} \sup_{|\alpha| = s+1} |\partial^\alpha \varphi(A^{-k}(x-x_i) + y)| |y|^{s+1} \]
\[ \lesssim b^{j(s+1)\log \lambda^+ \lambda_{-(s+1)(\ell_i-k)}} \sup_{z \in A^{i-k}(x-x_i) + A^{-k+j}B_{i}} \frac{1}{(1+\rho(z))^N} \]
\[ \lesssim b^{j(s+1)\log \lambda^+ \lambda_{-(s+1)(\ell_i-k)}} b^{-N(\ell_i-k+j+m)}. \]

Notice that the supremum over \( k \leq \ell_i \) has the largest value when \( k = \ell_i \). Without loss of generality, we may assume that \( s := \lfloor \frac{1}{\ln b} - 1 \rfloor \) and \( N := s + 2 \). This implies \( b\lambda^{s+1} \leq b^N \) and the above supremum over \( k > \ell_i \) is attained when \( \ell_i - k + j + m = 0 \). By (4.3), (4.5), (4.6) and the fact that \( \varepsilon > (s+1)\log \lambda^+ \), we obtain

\[ M^0_N(\mathcal{M}_i)(x) = \sup_{\varphi \in \mathcal{S}_N} \sup_{k \in \mathbb{Z}} |\mathcal{M}_i \ast \varphi_k(x)| \]
\[ \lesssim \frac{b^{j(s+1)\log \lambda^+ \lambda_{-(s+1)(\ell_i-k)}} b^{-N(\ell_i-k+j+m)}}{\|\chi_{x_i + B_{\ell_i}}\|_{L^p}} \sum_{j=0}^{\infty} b^{-j\varepsilon} \max \left\{ b^{-mN(\ell_i-k+j+m)}, \right. \]
\[ \left. b^{j(s+1)\log \lambda^+ \lambda_{-(s+1)(\ell_i-k)}} b^{-N(\ell_i-k+j+m)} \right\} \]
\[ \lesssim \frac{b^{j(s+1)\log \lambda^+ \lambda_{-(s+1)(\ell_i-k)}} b^{-j\varepsilon} \max \{ b^{-mN}, (b\lambda^{s+1})^{-m} \}}{\|\chi_{x_i + B_{\ell_i}}\|_{L^p}} \]
\[ \lesssim \frac{1}{\|\chi_{x_i + B_{\ell_i}}\|_{L^p}} b^{-mN(\ln \lambda_-/\ln b)} \]
\[ \lesssim \frac{1}{\|\chi_{x_i + B_{\ell_i}}\|_{L^p}} b^{(s+1)(\ln \lambda_-/\ln b) + 1} b^{-\ell_i - \sigma + m} \]
\[ \lesssim \frac{1}{\|\chi_{x_i + B_{\ell_i}}\|_{L^p}} |x_i + B_{\ell_i}| \left| \frac{(s+1)(\ln \lambda_-/\ln b) + 1}{\rho(x - x_i)} - [(s+1)(\ln \lambda_-/\ln b) + 1] \right| \]
\[ \lesssim \frac{1}{\|\chi_{x_i + B_{\ell_i}}\|_{L^p}} |x_i + B_{\ell_i}| \left| \frac{(s+1)(\ln \lambda_-/\ln b) + 1}{\rho(x - x_i)} - [(s+1)(\ln \lambda_-/\ln b) + 1] \right| \]
This implies that

\[ \sim \| \chi_{B^{(i)}} \|_{L^p}^{-1} \frac{|B^{(i)}|^\theta}{\rho(x - x_i)^\theta} \]

\[ \lesssim \| \chi_{B^{(i)}} \|_{L^p}^{-1} [M_{HL}(\chi_{B^{(i)}})(x)]^\theta, \]

where, for any \( i \in \mathbb{N} \), \( x_i \) denotes the centre of the dilated ball \( B^{(i)} \) and

\[ \theta := \left( \frac{\ln b}{\ln \lambda_-} + s + 1 \right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{p}. \]

By this and the fact that, for any \( x \in \mathbb{R}^n \), \( M_N(f)(x) \sim M_N^0(f)(x) \) (see [1, Proposition 3.10]), we obtain

\[ M_N(M_i)(x) \lesssim \| \chi_{B^{(i)}} \|_{L^p}^{-1} \frac{|B^{(i)}|^\theta}{\rho(x - x_i)^\theta} \lesssim \| \chi_{B^{(i)}} \|_{L^p}^{-1} [M_{HL}(\chi_{B^{(i)}})(x)]^\theta, \]

where, for any \( i \in \mathbb{N} \), \( x_i \) denotes the centre of the dilated ball \( B^{(i)} \) and \( \theta \) is as in (4.7). From this, Lemmas 2.2, 2.9, (3.1) and (4.2), we deduce that

\[ I_2 \lesssim \left\| \sum_{i \in \mathbb{N}} \frac{|\lambda_i|}{\| \chi_{B^{(i)}} \|_{L^p}} [M_{HL}(\chi_{B^{(i)}})]^\theta \right\|_{L^p}^p \]

\[ \lesssim \left\{ \sum_{i \in \mathbb{N}} \frac{|\lambda_i|}{\| \chi_{B^{(i)}} \|_{L^p}} [M_{HL}(\chi_{B^{(i)}})]^\theta \right\}^{1/\theta} \| f \|_{L^p}^{\theta p} \]

\[ \lesssim \left\{ \sum_{i \in \mathbb{N}} \frac{|\lambda_i|}{\| \chi_{B^{(i)}} \|_{L^p}} \right\}^{1/\theta} \| f \|_{L^p}^{\theta p} \]

\[ \lesssim \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\| \chi_{B^{(i)}} \|_{L^p}} \right]^{n/\eta} \right\}^{1/\eta} \| f \|_{H_{A, \text{mol}}^{p,q,s,\varepsilon}}^p, \]

where \( \eta \in (0, \min\{1, p_-\}) \). This, together with \( I_1 \) and \( I_2 \), shows that

\[ \| f \|_{H_{A}^{p}} \sim \| M_N(f) \|_{L^p} \lesssim \| f \|_{H_{A, \text{mol}}^{p,q,s,\varepsilon}}. \]

This implies that \( f \in H_{A}^{p} \) and hence \( H_{A, \text{mol}}^{p,q,s,\varepsilon} \subset H_{A}^{p} \). This finishes the proof of Theorem 4.4.

5 Application

In this section, as an application, we obtain a boundedness criterion for some linear operators from \( H_{A}^{p} \) to itself. Particularly, when \( A := 2I_{n \times n} \), this result is still new.
**Theorem 5.1.** Let \( \vec{p} \in (0, \infty)^n \), \( q \in (1, \infty] \cap (p_+, \infty) \) with \( p_+ \) as in (2.6), and \( s \) be as in (4.1). Suppose that \( T \) is a bounded linear operator on \( L^r \), for any \( r \in (1, \infty] \). If for any \((\vec{p}, q, s)\)-atom \( a \), supported on dilated ball \( B^{(i_0)} \in \mathcal{B} \), as in Definition 2.6, \( T(a) \) is a harmless constant multiple of a \((\vec{p}, q, s, \varepsilon)\)-molecule, associated with dilated ball \( B^{(i_0)} \in \mathcal{B} \), where \( \varepsilon > (s + 1) \log_b^+ \), then \( T \) extends uniquely to a bounded linear operator on \( H^\vec{p}_A \). Moreover, there exists a positive constant \( C \) such that, for all \( f \in H^\vec{p}_A \),

\[
\|T(f)\|_{H^\vec{p}_A} \leq C \|f\|_{H^\vec{p}_A}.
\]

(5.1)

To prove Theorem 5.1, we need some definitions and technical lemmas.

**Definition 5.2.** [13] Let \( \vec{p} \in (0, \infty)^n \), \( q \in (1, \infty] \) and \( s \in \left[ (1/p_- - 1) \ln b / \ln \lambda_- \right], \infty \right) \cap \mathbb{Z}_+ \) with \( p_- \) as in (2.6). The anisotropic mixed-norm finite atomic Hardy space \( H^\vec{p}_A, q, s \) is defined to be the set of all distributions \( f \in S' \) satisfying that there exist \( I \in \mathbb{N} \), \( \{\lambda_i\}_{i \in [1, I]} \subset \mathbb{C} \) and a sequence of \((\vec{p}, q, s)\)-atoms, \( \{a_i\}_{i \in [1, I]} \subset \mathbb{C} \), supported, respectively, on \( \{B(i)\}_{i \in [1, I]} \subset \mathcal{B} \) such that

\[
f = \sum_{i=1}^{I} \lambda_i a_i \text{ in } S'.
\]

Moreover, for any \( f \in H^\vec{p}_A, q, s \), let

\[
\|f\|_{H^\vec{p}_A, q, s} := \inf \left\{ \left( \sum_{i=1}^{I} \left[ \frac{\|\lambda_i \chi_{B(i)}\|_{L^\vec{p}}} {\|\chi_{B(i)}\|_{L^\vec{p}}} \right]^{\eta} \right)^{1/\eta} \right\},
\]

where \( \eta \in (0, \min\{1, p_-\}) \) and the infimum is taken over all the decompositions of \( f \) as above.

**Lemma 5.3.** [13, Theorem 5.3] Let \( \vec{p} \in (0, \infty)^n \), \( A \) be a dilation and \( s \in \left[ (1/p_- - 1) \ln b / \ln \lambda_- \right], \infty \right) \cap \mathbb{Z}_+ \) with \( p_- \) as in (2.6)

(i) If \( q \in (\max\{p_+ ; 1\}, \infty) \) with \( p_+ \) as in (2.6), then \( \|\cdot\|_{H^\vec{p}_A, q, s} \) and \( \|\cdot\|_{H^\vec{p}_A} \) are equivalent quasi-norms on \( H^\vec{p}_A \).

(ii) \( \|\cdot\|_{H^\vec{p}_A, q, s} \) and \( \|\cdot\|_{H^\vec{p}_A} \) are equivalent quasi-norms on \( H^\vec{p}_A, q, s \cap C \), where \( C \) denotes the set of all continuous functions on \( \mathbb{R}^n \).

**Lemma 5.4.** [13, Theorem 4.7] Let \( \vec{p} \in (0, \infty)^n \), \( q \in (1, \infty] \cap (p_+, \infty) \) with \( p_+ \) as in (2.6), \( r \in (1, \infty] \) and \( s \in \left[ (1/p_- - 1) \ln b / \ln \lambda_- \right], \infty \right) \cap \mathbb{Z}_+ \) with \( p_- \) as in (2.6). Then, for any \( f \in H^\vec{p}_A \cap L^r \), there exist \( \{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C} \), dilated balls \( \{x_i + B_{t_i}\}_{i \in \mathbb{N}} \subset \mathcal{B} \) and \((\vec{p}, q, s)\)-atoms \( \{a_i\}_{i \in \mathbb{N}} \) such that

\[
f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } L^r,
\]

where the series also converge almost everywhere.
Lemma 5.5. [13] Let $\vec{p} \in (0, \infty)^n$ and $N \in \mathbb{N} \cap \left[\left(\frac{1}{\min\{1, p_-\}} - 1\right) \frac{\ln b}{\ln \lambda_-} + 2, \infty\right)$ with $p_-$ as in (2.6). Then $H^p_A$ is complete.

Now, we show Theorem 5.1 by borrowing some ideas from the proof of [13, Theorem 8.4].

Proof of Theorem 5.1. Let $\vec{p} \in (0, \infty)^n$, $q \in (1, \infty) \cap (p_+, \infty)$ with $p_+$ as in (2.6), and $s \in \left[\left(1/p_- - 1\right) \ln b/\ln \lambda_-\right], \infty) \cap \mathbb{Z}_+$ with $p_-$ as in (2.6). Firstly, we show that (5.1) holds true for any $f \in H^p_{A, \text{fin}}$. For any $f \in H^p_{A, \text{fin}}$, by Definition 5.2, we know that $f \in H^p_A \cap L^q$. From Lemma 5.4, we know that there exist $\lambda_i \in \mathbb{C}$ and a sequence of $(\vec{p}, q, s)$-atoms, $\{a_i\}_{i \in \mathbb{N}}$, supported, respectively, on $\{B(i)\}_{i \in \mathbb{N}} \subset \mathfrak{B}$, where $B(i) := x_i + B_{\ell_i}$ with $x_i \in \mathbb{R}^n$ and $\ell_i \in \mathbb{Z}$, such that

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \text{in } L^q,$$

and

$$\left\| \left\{ \sum_{i \in \mathbb{N}} \left| \lambda_i \right| |\chi_{B(i)}| \right\|_{L^p}^{1/\eta} \right\|_{L^p}^{\eta} \lesssim \|f\|_{H^p_A}.$$  \hspace{1cm} (5.2)

where $\eta \in (0, \min\{1, p_+\})$. From this and the linear operator $T$ is bounded on $L^q$, we conclude that, for any $f \in H^p_{A, \text{fin}}$, $T(f) = \sum_{i \in \mathbb{N}} \lambda_i T(a_i)$ in $L^q$ and hence in $S'$. Therefore, for all $N \in \mathbb{N} \cap \left[\left(1/p_- - 1\right) \ln b/\ln \lambda_-\right] + 2, \infty)$,

$$\|M_N(T(f))\|_{L^p}^p \leq \left\| \sum_{i \in \mathbb{N}} \lambda_i |M_N(T(a_i))| \right\|_{L^p}^p \lesssim \left\| \sum_{i \in \mathbb{N}} \lambda_i |M_N(T(a_i))\chi_{A^{2\sigma}B(i)}| \right\|_{L^q}^p + \left\| \sum_{i \in \mathbb{N}} \lambda_i |M_N(T(a_i))\chi_{(A^{2\sigma}B(i))}\|_{L^q}^p \right\|_{L^p}^p \lesssim \left\| \sum_{i \in \mathbb{N}} |\lambda_i| |M_N(T(a_i))\chi_{A^{2\sigma}B(i)}| \right\|_{L^q}^p + \left\| \sum_{i \in \mathbb{N}} |\lambda_i| |M_N(T(a_i))\chi_{(A^{2\sigma}B(i))}\|_{L^q}^p \right\|_{L^p}^p \right\|_{L^p}^p \leq : K_1 + K_2,$$

where $A^{2\sigma}B(i)$ is the $A^{2\sigma}$ concentric expanse on $B(i)$, that is, $A^{2\sigma}B(i) := x_i + A^{2\sigma}B_{\ell_i}$, and $p$ as in (2.7).

For $K_1$, from the boundedness of $M_N$ and $T$ on $L^q$, and the size condition of $a_i$, we know that

$$\|M_N(T(a_i))\chi_{A^{2\sigma}B(i)}\|_{L^q} \lesssim \|a_i\chi_{A^{2\sigma}B(i)}\|_{L^q} \lesssim \frac{|B(i)|^{1/q}}{\|\chi_{B(i)}\|_{L^p}}.$$  

From this, Lemma 4.6 and (5.2), we further deduce that

$$K_1 \lesssim \left\| \left\{ \sum_{i \in \mathbb{N}} \left| \lambda_i \right| \chi_{B(i)} \right\|_{L^p}^{1/\eta} \right\|_{L^p}^p \lesssim \|f\|_{H^p_A}.$$
To deal with $K_2$, for any $i \in \mathbb{N}$ and $x \in (A^{2\sigma} B^{(i)})^C$, by the condition of Theorem 5.1, we see that, for any $(\vec{p}, q, s)$-atom $a_i$ supported on a ball $B^{(i)}$, $T(a_i)$ is a harmless constant multiple of a $(\vec{p}, q, s, \varepsilon)$-molecule associated with $B^{(i)}$, where $\varepsilon > (s+1) \log_2^\lambda$. From this and (4.8), we know that

$$\tag{5.4} M_N(T(a_i))(x) \lesssim \|\chi_{B^{(i)}}\|_{L^p}^{-1} [M_{HL}(\chi_{B^{(i)}})(x)]^\theta,$$

where $\theta$ is as in (4.7). By (5.4) and an argument same as that used in the proof of (4.9), we obtain

$$K_2 \lesssim \left\{ \left| \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|}{\|\chi_{B^{(i)}}\|_{L^p}} \right]^{1/\eta} \right|^{p} \right\}^{1/\eta} \lesssim \|f\|_{H_A^{\vec{p}, q, s}}.$$

Combining (5.3) and the estimates of $K_1$ and $K_2$, we further conclude that, for any $f \in H_{A, \text{fin}}^{\vec{p}, q, s}$,

$$\|T(f)\|_{H_A^{\vec{p}}} \lesssim \|f\|_{H_A^{\vec{p}}}.$$

Next, we prove that (5.1) also holds true for any $f \in H_{A, \text{fin}}^{\vec{p}}$. Let $f \in H_{A, \text{fin}}^{\vec{p}}$. By Lemma 5.3 and the obvious density of $H_{A, \text{fin}}^{\vec{p}, q, s}$ in $H_{A, \text{fin}}^{\vec{p}}$, we know that there exists a sequence $\{f_j\}_{j \in \mathbb{Z}^+} \subset H_{A, \text{fin}}^{\vec{p}, q, s}$, such that $f_j \to f$ as $j \to \infty$ in $H_{A, \text{fin}}^{\vec{p}}$. Therefore, $\{f_j\}_{j \in \mathbb{Z}^+}$ is a Cauchy sequence in $H_{A, \text{fin}}^{\vec{p}}$. By this, we see that, for any $j, k \in \mathbb{Z}^+$,

$$\|T(f_j) - T(f_k)\|_{H_{A, \text{fin}}^{\vec{p}}} = \|T(f_j - f_k)\|_{H_{A, \text{fin}}^{\vec{p}}} \lesssim \|f_j - f_k\|_{H_{A, \text{fin}}^{\vec{p}}}.$$

Notice that $\{T(f_j)\}_{j \in \mathbb{Z}^+}$ is also a Cauchy sequence in $H_{A, \text{fin}}^{\vec{p}}$. Applying Lemma 5.5, we conclude that there exists a $g \in H_{A, \text{fin}}^{\vec{p}}$ such that $T(f_j) \to g$ as $j \to \infty$ in $H_{A, \text{fin}}^{\vec{p}}$. Let $T(f) := g$. Then, $T(f)$ is well defined. In fact, for any other sequence $\{h_j\}_{j \in \mathbb{Z}^+} \subset H_{A, \text{fin}}^{\vec{p}, q, s}$ satisfying $h_j \to f$ as $j \to \infty$ in $H_{A, \text{fin}}^{\vec{p}}$, by Lemma 2.2, we have

$$\|T(h_j) - T(f)\|_{H_{A, \text{fin}}^{\vec{p}}} \leq \|T(h_j) - T(f_j)\|_{H_{A, \text{fin}}^{\vec{p}}} + \|T(f_j) - g\|_{H_{A, \text{fin}}^{\vec{p}}}.$$

which is wished.

From this, we see that, for any $f \in H_{A, \text{fin}}^{\vec{p}}$,

$$\|T(f)\|_{H_{A, \text{fin}}^{\vec{p}}} = \|g\|_{H_{A, \text{fin}}^{\vec{p}}} = \lim_{j \to \infty} \|T(f_j)\|_{H_{A, \text{fin}}^{\vec{p}}} \lesssim \lim_{j \to \infty} \|f_j\|_{H_{A, \text{fin}}^{\vec{p}}} \sim \|f\|_{H_{A, \text{fin}}^{\vec{p}}},$$

which implies that (5.1) also holds true for any $f \in H_{A, \text{fin}}^{\vec{p}}$ and hence completes the proof of Theorem 5.1. 

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