MERTENS’ THIRD THEOREM FOR NUMBER FIELDS: A NEW PROOF, CRAMÈR’S INEQUALITY, OSCILLATIONS, AND BIAS

SHEHZAD HATHI AND ETHAN S. LEE

Abstract. The first result of our article is another proof of Mertens’ third theorem in the number field setting, which generalises a method of Hardy. The second result concerns the sign of the error term in Mertens’ third theorem. Diamond and Pintz showed that the error term in the classical case changes sign infinitely often and in our article, we establish this result for number fields assuming a reasonable technical condition. In order to do so, we needed to prove Cramér’s inequality for number fields, which is interesting in its own right. Lamzouri built upon Diamond and Pintz’s work to prove the existence of the logarithmic density of the set of real numbers \( x \geq 2 \) such that the error term in Mertens’ third theorem is positive, so the third result of our article generalises Lamzouri’s results for number fields. We also include numerical investigations for the number fields \( \mathbb{Q}(\sqrt{5}) \) and \( \mathbb{Q}(\sqrt{13}) \), building upon similar work done by Rubinstein and Sarnak in the classical case.

1. Introduction

Suppose that a number field \( K \) has degree \( n_K \), discriminant \( \Delta_K \), and ring of integers \( \mathcal{O}_K \). The Dedekind zeta-function associated to \( K \), denoted \( \zeta_K(s) \), is regular throughout \( \mathbb{C} \) aside from one pole at \( s = 1 \) which is simple and has residue \( \kappa_K \). Recall that the Generalised Riemann Hypothesis (GRH) postulates there is no exceptional (or Landau–Siegel) zero, and every non-trivial zero of \( \zeta_K(s) \) lies on the line \( \text{Re}(s) = 1/2 \). Moreover, the assumption that these zeros are linearly independent over \( \mathbb{Q} \) will be referred to as the Generalised Linear Independence Hypothesis (GLI). Throughout this paper, we use the notations \( \ll \) and \( O_K \), in which the implied constant may depend on the invariants of \( K \), although this should be clear from the context.

Background. In 1874, Mertens \cite{mertens1874} established the product formula

\[
\prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} = e^\gamma \log x + O(1),
\]

(1)

where \( \gamma \) is the Euler–Mascheroni constant. Without an explicit description of the error term, Lebacque \cite{lebacque1991} and Rosen \cite{rosen2002} generalised (1) for number fields \( K \) with \( n_K \geq 2 \):

\[
\prod_{N(p) \leq x} \left( 1 - \frac{1}{N(p)} \right)^{-1} = e^\gamma \kappa_K \log x + O_K(1).
\]

(2)

The product in (2) runs over the prime ideals \( p \) of \( \mathcal{O}_K \), where \( N(p) \) denotes the norm of \( p \). Garcia and the second author \cite{garcia2018} have unconditionally established (2) with an explicit description of the error term for \( x \geq 2 \).
In the setting $\mathbb{K} = \mathbb{Q}$, Rosser and Schoenfeld [29] observed that
\[
\prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} > e^\gamma \log x, \quad \text{for} \; 2 \leq x \leq 10^8.
\]
This is an inequality between the product and main term in (1). Building upon this observation, Diamond and Pintz [10] have shown that these quantities actually take turns exceeding one another. More precisely, they showed that
\[
\sqrt{x} \left( \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} - e^\gamma \log x \right)
\]
attains arbitrarily large positive and negative values as $x \to \infty$. Finally, suppose that
\[
\mathcal{M}_\mathbb{K} = \left\{ x \geq 2 : \prod_{N(p) \leq x} \left( 1 - \frac{1}{N(p)} \right)^{-1} > e^\gamma \kappa_\mathbb{K} \log x \right\}.
\]
Lamzouri [19] showed $\mathcal{M}_\mathbb{Q}$ and its complement have positive lower logarithmic densities under the Riemann Hypothesis (RH). Moreover, assuming the RH, the set $\mathcal{M}_\mathbb{Q}$ has a logarithmic density, and additionally assuming the Linear Independence Hypothesis (LI), this logarithmic density is known to be $\delta(\mathcal{M}_\mathbb{Q}) = 0.99999973 \ldots$. Therefore, the error term in Mertens’ product formula has a strong bias towards the positive sign. Further, it turns out that the logarithmic density of $\mathcal{M}_\mathbb{Q}$ is equal to the logarithmic density of the set of real numbers $x \geq 2$ such that $\pi(x) < Li(x)$. The latter was calculated by Rubinstein and Sarnak in [30].

Results. In Section 2, we introduce several results which will be important for proving the main results of this paper. One preliminary result we needed to generalise to number fields is a well-known inequality of Cramér; see [9] or [27, Thm. 13.5]. This generalisation of Cramér’s bound is presented in Theorem 5, and may be of independent interest.

In Section 3, we provide another proof of (2), using a different technique to Lebacque [22] or Rosen [28]. Our motivation for sharing this new proof of (2) is that it is not well known and it generalises a method due to Hardy [14].

In Section 4, we address [11, Qn. 16], in which the second author and Garcia raised the question whether the difference,
\[
\Delta = \prod_{N(p) \leq x} \left( 1 - \frac{1}{N(p)} \right)^{-1} - e^\gamma \kappa_\mathbb{K} \log x,
\]
changes sign infinitely often when $\mathbb{K} \neq \mathbb{Q}$. To this end, we prove Theorem 4 which is a number field analogue of [10, Thm. 1.1], that demonstrates $\Delta$ does change sign infinitely often when $\mathbb{K} \neq \mathbb{Q}$, assuming a reasonable, technical condition.

\footnote{This method of proof was also suggested to the second author and S. R. Garcia during another project by T. Freiberg, who used a similar technique for a different setting in [5].}
Theorem 1. If there exists a non-real zero $\sigma_K$ of $\zeta_K(s)$ such that $1/2 \leq \text{Re}(\sigma_K) < 1$ and there is no zero in the right-half plane $\text{Re}(s) > \text{Re}(\sigma_K)$, then the quantity

$$E_1(x) := \sqrt{x} \left\{ \prod_{N(p) \leq x} \left(1 - \frac{1}{N(p)}\right)^{-1} - e^{\gamma} K \log x \right\}$$

attains arbitrarily large positive and negative values as $x \to \infty$.

The technical condition in Theorem 1 has been carefully worded, so that one can apply Landau’s oscillation theorem at the opportune moment; it ensures that the right-most zero of $\zeta_K(s)$ is not a Landau–Siegel (or exceptional) zero.

Remark 2. In the case where this technical condition in Theorem 1 is not satisfied, i.e. an exceptional zero exists, we expect an additional term in $E_1(x)$ (corresponding to the exceptional zero) to recover a similar type of oscillatory behaviour. However, we have not pursued that line of investigation in this paper.

In Section 5, we consider the set $M_K$. Generalising Lamzouri’s work in [19], we show conditionally that $M_K$ and its complement are unbounded.

Theorem 3. Assume GRH. Then, for any number field $K$, $\delta(M_K) > 0$ and $\delta(M_K) < 1$.

Moreover, we show that $M_K$ has a logarithmic density (see Theorem 24). We calculate this logarithmic density (see Table 1) for two quadratic fields, $K = \mathbb{Q}(\sqrt{5})$ and $K = \mathbb{Q}(\sqrt{13})$, adapting the numerical work done by Rubinstein and Sarnak in [30] concerning Chebyshev’s bias. These computations are useful beyond Mertens’ third theorem for number fields because the logarithmic density of $M_K$ is equal to $\delta(P_K)$, the logarithmic density of the set of reals $x \geq 2$ such that the error term in the prime ideal theorem is negative, i.e. $\pi_K(x) - \text{Li}(x) < 0$. We also show that the logarithmic density of $M_K$ (and consequently $P_K$) goes to $1/2$ as the discriminant of the quadratic field grows.

Acknowledgements. We thank Tim Trudgian for helpful feedback on this project. We also thank Tristan Freiberg for bringing Hardy’s approach in [14] to the second author’s attention, and Stephan Garcia for helpful comments and discussions on its implementation. We also thank Greg Martin and Peter Humphries for the helpful correspondence, especially concerning Remark 2.

2. Preliminary Results

2.1. The Dedekind zeta-function. Suppose that the degree $n_K = r_1 + 2r_2$, in which $r_1$ is the number of real places and $r_2$ is the number of complex places of $K$. Further, suppose $r = r_1 + r_2 - 1$, $R_K$ is the regulator of $K$, and $h_K$ is the class number of $K$. Landau established all of the knowledge we state here in [21].

The Dedekind zeta-function is denoted and defined for $\sigma > 1$ by

$$\zeta_K(s) = \sum_{a} N(a)^{-s} = \prod_{p} \left(1 - N(p)^{-s}\right)^{-1},$$

which converges absolutely. Now, $\zeta_K(s)$ may be continued to the entire plane $\mathbb{C}$, apart from a simple pole at $s = 1$ using a functional equation. That is, $\zeta_K(s)$ is regular for all $s \in \mathbb{C}$,
aside from one simple pole at \( s = 1 \) whose residue is
\[
\kappa_K = \frac{2^{r_1 + r_2} \pi^{r_2} h_K R_K}{w_K|\Delta_K|^{3/2}};
\]
this is called the class analytic formula.

At \( s = 0 \), \( \zeta_K(s) = 0 \) as long as \( r = r_1 + r_2 - 1 > 0 \) and this zero at \( s = 0 \) has order \( r \). If \( r = 0 \), then \( K \) is \( \mathbb{Q} \) satisfying \( (r_1, r_2) = (1, 0) \) or \( K \) is an imaginary quadratic field satisfying \( (r_1, r_2) = (0, 1) \). Moreover, \( \zeta_K(s) = 0 \) whenever \( s \) is a negative, even integer (these zeros have order \( r_1 + r_2 \)). If \( r_2 = 0 \), then \( K \) is \( \mathbb{Q} \) satisfying \( (r_1, r_2) = (1, 0) \) or \( K \) is an imaginary quadratic field satisfying \( (r_1, r_2) = (0, 1) \). Moreover, \( \zeta_K(s) = 0 \) whenever \( s \) is a negative, odd integer (these zeros only occur when \( r_2 > 0 \) and they have order \( r_2 \)). Alongside the zero at \( s = 0 \) (whenever \( r > 0 \)), these zeros are called trivial.

The non-trivial zeros of \( \zeta_K(s) \) satisfy \( 0 < \text{Re}(s) < 1 \), and we note that there might exist a single, simple, real zero \( 0 < \beta_0 < 1 \), which is called an exceptional zero. Explicit bounds for \( \beta_0 \) may be found in [1, 16, 23].

2.2. The prime ideal theorem. Let \( s = \sigma + it, \ a \) denote an integral ideal of \( K \), and \( \mathfrak{p} \) denote a prime ideal of \( K \). Suppose that
\[
\psi_K(x) = \sum_{\mathcal{N}(a) \leq x} \Lambda_K(a) \quad \text{where} \quad \Lambda_K(a) = \begin{cases} \log N(\mathfrak{p}) & \text{if } a = \mathfrak{p}^m, \\ 0 & \text{otherwise}. \end{cases}
\]

The prime ideal theorem was initially proved by Landau in [20]. Explicit conditional versions of the same have been established in [12], and an explicit, unconditional generalisation has been established by Lagarias and Odlyzko in [18]. Corollary 4 is a special case of [18, Thm. 7.1], and can be obtained using Kadiri and Ng’s zero-density estimate from [17] with \( L = K = \mathbb{K} \) in Lagarias and Odlyzko’s notation.\(^2\)

**Corollary 4.** Suppose \( K \) is a number field such that \( n_K \geq 2 \) and \( 2 \leq T \leq x \). Then
\[
\psi_K(x) = x - \sum_{|\gamma| \leq T} \frac{x^\theta}{\theta} + R_K(x, T),
\]
where \( R_K(x, T) \ll \frac{x \log x}{T} (n_K \log x + \log |\Delta_K|) \ll \frac{x \log^2 x}{T} \) and \( \gamma \) are the ordinates of the non-trivial zeros \( \gamma \) of \( \zeta_K(s) \).

2.3. Cramér’s inequality for number fields. The next result we require is a generalisation of Cramér’s inequality for number fields, which we present in Theorem 5. A consequence of Theorem 5 is that \( \psi_K(x) = x + O(x^{1/2}) \) on average.

**Theorem 5.** Assume GRH. For \( x \geq 2 \), we have
\[
\int_0^{2x} (\psi_K(t) - t)^2 \, dt \ll x^2.
\]
Once Theorem 5 is established, the Cauchy–Schwarz inequality ensures that
\[
\left( \int_1^x |\psi_K(t) - t| \, dt \right)^2 \leq x \int_1^x (\psi_K(t) - t)^2 \, dt.
\]
\(^2\)Grenié et al. make the same claim in [13, Eqn. (6)].
Therefore,
\[
\int_x^{2x} |\psi_K(t) - t| \, dt \ll \int_1^{2x} |\psi_K(t) - t| \, dt \ll x^{3/2},
\]
which is the form of Cramér’s inequality used in [10].

**Remark 6.** The implied constant in Theorem 5 is of independent interest. In the classical setting, Brent et al. [8] estimate this to be \( \leq 0.8603 \).

Before we can prove Theorem 5, we need two lemmas; the first is given in Lemma 7.

**Lemma 7.** Suppose that \( N_K(T) \) is the number of non-trivial zeros (counted with multiplicity) of \( \zeta_K(s) \) up to height \( T \geq 1 \). We have

\[
N_K(T + 1) - N_K(T) \ll \log(T + 1)
\]

*Proof.* Kadiri and Ng [17] showed that there exist constants \( C_1, C_2, C_3 \) such that

\[
\left| N_K(T) - \frac{T}{\pi} \log \left( |\Delta_K| \left( \frac{T}{2\pi e} \right)^n_K \right) \right| \leq C_1 (\log |\Delta_K| + n_K \log T) + C_2 n_K + C_3. \tag{5}
\]

Trudgian [31], and more recently Hasanalizade et al. [15], provided explicit constants for \( C_i \), but these are not necessary for our purposes. It follows that

\[
N_K(T + 1) - N_K(T) < T \pi \log \left( \left( \frac{T + 1}{T} \right)^n_K \right) + \frac{1}{\pi} \log(T + 1) + 2 r_K(T + 1)
\]

\[
\ll \log(T + 1)
\]

for all \( T \geq 1 \). \[Q.E.D.\]

Using Lemma 7, we establish Lemma 8, which is important in the proof of Theorem 5.

**Lemma 8.** Suppose \( \gamma_1, \gamma_2 \) are ordinates of zeros \( \varrho_1, \varrho_2 \) of \( \zeta_K(s) \). Then the sum

\[
\sum_{\gamma_1, \gamma_2} \frac{1}{|\gamma_1 \gamma_2|((1 + |\gamma_1 - \gamma_2|)} < \infty.
\]

*Proof.* Fix the ordinate \( \gamma_1 \). It suffices to determine that the sum

\[
\sum_{\gamma_2} \frac{1}{|\gamma_2|((1 + |\gamma_1 - \gamma_2|)}
\]

is finite for each \( \gamma_1 > 0 \), using the symmetry of zeros. To do this, split the sum into five parts. Using Lemma 7 and following an analogous method to [27 Thm. 13.5], the result follows. \[Q.E.D.\]

**Proof of Theorem 5.** Our proof is modelled on [27 Thm. 13.5]. For \( x \geq 2 \), using Corollary 4 with \( T = t \), we have

\[
\int_x^{2x} |\psi_K(t) - t|^2 \, dt \leq \int_x^{2x} \left( \sum_{|\gamma| \leq t} \frac{t^\theta}{\theta} \right)^2 \, dt + 2 \int_x^{2x} \left( \sum_{|\gamma| \leq t} \frac{t^\theta}{\theta} \right) |R_{\zeta_K}(t)| \, dt + \int_x^{2x} |R_{\zeta_K}(t)|^2 \, dt.
\]
Now, using Lemma 8, the main term satisfies
\[ \int_x^{2x} \left| \sum_{\gamma \leq t} \frac{t^\gamma}{\gamma} \right|^2 dt = \sum_{\gamma_1, \gamma_2 \leq x} \frac{1}{\varrho_{1,2}} \left| \frac{t^{2+i(\gamma_1+\gamma_2)}}{2+i(\gamma_1+\gamma_2)} \right|^2 \ll x^2 \sum_{\gamma_1, \gamma_2 \leq x} \frac{1}{\varrho_{1,2}} |2+i(\gamma_1+\gamma_2)| \]
\[ \ll x^2. \]

Next, using GRH, the middle term satisfies
\[ \int_x^{2x} \left| \sum_{\gamma \leq t} |R(\gamma)| \right|^2 dt \ll \int_x^{2x} \frac{1}{t^2} \log^2 t \left| \sum_{|\gamma| \leq t} \frac{1}{\varrho} \right| dt \ll \int_x^{2x} \frac{1}{t^2} \log^4 t dt \ll x^2. \]

Therein, one can show that the sum over zeros in the second inequality is \( \ll \log^2 t \) using partial summation and applying (5). Finally, the remainder term satisfies
\[ \int_x^{2x} |R(\gamma)|^2 dt \ll \int_x^{2x} \log^4 t dt \ll x \log^4 x. \]

The result follows naturally. \( \blacksquare \)

**Remark 9.** By altering the weight in the integral considered in Theorem 5, we can obtain an exact formula for the integral. That is, alter the weight to its natural weighting to consider
\[ \int_x^{2x} \left( \frac{\psi(t) - t}{\sqrt{t}} \right)^2 \frac{dt}{t}. \]

Apply Corollary 4 with \( T = t \) and Lemma 7 to yield
\[ \int_x^{2x} \left( \frac{\psi(t) - t}{\sqrt{t}} \right)^2 \frac{dt}{t} = \int_x^{2x} \left( \sum_{|\gamma| \leq t} \frac{t^\gamma}{\varrho} \right)^2 \frac{dt}{t} + O \left( \frac{\log^3 x}{\sqrt{x}} \right). \]

If the interested reader wanted to compute the integral precisely, then careful examination of this main term is a good starting point. However, to explore this further would transcend the scope of this paper, so the authors propose this as an open problem for the future.

### 2.4. Littlewood’s result for number fields.

Suppose that
\[ \Pi(x) = \sum_{\ell \geq 1} \frac{\pi(x/\ell)}{\ell}, \quad \Pi_{\mathbb{K}}(x) = \sum_{\ell \geq 1} \frac{\pi_{\mathbb{K}}(x/\ell)}{\ell}, \quad \text{li}^*(x) = \int_1^x \frac{1-t^{-1}}{\log t} dt, \]
\[ \Delta^*(x) = \Pi(x) - \text{li}^*(x) \text{ and } \Upsilon^*(x) = \Pi_{\mathbb{K}}(x) - \text{li}^*(x). \]

Littlewood [25] proved that
\[ \frac{\Delta^*(x)}{x} = \Omega_\pm \left( \frac{\log \log \log x}{\sqrt{x} \log x} \right). \]  

(6)

We extend Littlewood’s result, and our result is presented in Lemma 10.

**Lemma 10.** We have
\[ \frac{\Upsilon^*(x)}{x} = \Omega_\pm \left( \frac{\log \log \log x}{\sqrt{x} \log x} \right). \]

(7)

\[ \text{The authors thank Peter Humphries for raising this comment.} \]
Proof. Observe that
\[
\frac{\Upsilon^*(x)}{x} = \frac{\delta(x)}{x} + \frac{\Delta^*(x)}{x},
\]
where \(\delta(x) := \Pi_{\mathcal{K}}(x) - \Pi(x)\). Since there are at most \(n_{\mathcal{K}}\) prime ideals in \(\mathcal{K}\) which lie over each rational prime \(p\),
\[
\delta(x) \leq (n_{\mathcal{K}} - 1) \sum_{\ell \geq 1} \frac{\pi(x^{1/\ell})}{\ell} = (n_{\mathcal{K}} - 1) \pi(x) + (n_{\mathcal{K}} - 1) \sum_{\ell = 2}^{\left\lfloor \log x \log 2 \right\rfloor} \frac{\pi(x^{1/\ell})}{\ell}.
\]
Therefore, using \(\pi(x) \ll x / \log x\), we see that
\[
\frac{\delta(x)}{x} \ll \frac{n_{\mathcal{K}} - 1}{\log x} + (n_{\mathcal{K}} - 1) \sum_{\ell = 2}^{\left\lfloor \log x \log 2 \right\rfloor} \frac{x^{1/\ell - 1}}{\log x}
\ll \frac{(n_{\mathcal{K}} - 1)}{\log x} + (n_{\mathcal{K}} - 1) \sqrt{x} \log x \sum_{\ell = 2}^{\left\lfloor \log x \log 2 \right\rfloor} 1 \ll (n_{\mathcal{K}} - 1) \left( \frac{1}{\log x} + \frac{1}{\sqrt{x}} \right).
\]
Hence, for a fixed \(\mathcal{K}\), \(\delta(x)/x \to 0\) as \(x \to \infty\). Since \(\delta(x) \geq 0\),
\[
\limsup_{x \to \infty} \frac{\Delta^*(x)}{x} \leq \limsup_{x \to \infty} \left( \frac{\delta(x)}{x} + \frac{\Delta^*(x)}{x} \right) \leq \limsup_{x \to \infty} \frac{\delta(x)}{x} + \limsup_{x \to \infty} \frac{\Delta^*(x)}{x}
\]
using the subadditivity of limit superior. The limit of \(\delta(x)/x\) exists as \(x \to \infty\) and is 0, which implies \(\limsup_{n \to \infty} \Upsilon^*(x)/x = \limsup_{n \to \infty} \Delta^*(x)/x\). Similarly,
\[
\liminf_{x \to \infty} \left( \frac{\delta(x)}{x} + \frac{\Delta^*(x)}{x} \right) \geq \liminf_{x \to \infty} \frac{\delta(x)}{x} + \liminf_{x \to \infty} \frac{\Delta^*(x)}{x}
\]
using the superadditivity of limit inferior, and
\[
\liminf_{x \to \infty} \left( \frac{\delta(x)}{x} + \frac{\Delta^*(x)}{x} \right) \leq \limsup_{x \to \infty} \frac{\delta(x)}{x} + \liminf_{x \to \infty} \frac{\Delta^*(x)}{x}.
\]
Again, using the fact that \(\delta(x)/x \to 0\) as \(x \to \infty\), we obtain
\[
\liminf_{x \to \infty} \frac{\Upsilon^*(x)}{x} = \liminf_{x \to \infty} \frac{\Delta^*(x)}{x}.
\]
Using (6), the result follows.

3. Mertens’ Product Formula for Number Fields

Hardy proved (1) in [14], with an elegant proof consisting of four core ingredients. In this section, we have generalised Hardy’s arguments into the number field setting. Note that one can use this method to prove other generalisations of (1), as long as the prerequisite ingredients are available.
3.1. **Ingredients.** The simplest ingredients we require are partial summation and Chebyshev-type bounds:

$$\pi_K(x) = O \left( \frac{x}{\log x} \right).$$

To see that we have Chebyshev-type bounds for $$\pi_K(x)$$, note that

$$\pi_Q(x) \leq \pi_K(x) \leq n_K \pi_Q(x),$$

using the fact that every prime ideal $$p$$ lies over a unique rational prime $$p$$, and there are at most $$n_K$$ distinct prime ideals lying over each $$p$$.

Now, [4, Thm. 4.6] and [29, Eqn. (3.6)] establish

$$\frac{1}{6} \frac{x}{\log x} < \pi_Q(x) < 1.25506 \frac{x}{\log x}$$

for $$x > 1$$, hence

$$\pi_K(x) = O \left( \frac{x}{\log x} \right),$$

because

$$\frac{1}{6} \frac{x}{\log x} < \pi_K(x) < 1.25506 n_K \frac{x}{\log x}.$$

We also require the relationship

$$\lim_{\delta \to 0} \left( \int_a^\infty \frac{e^{-\delta t}}{t} \, dt - \log \frac{1}{\delta} \right) = -\log a - \gamma,$$

which Hardy states is "familiar in the theory of the Gamma function". Finally, we require a Tauberian theorem [14, Eqn. (D)] that states if $$f(t) = O \left( \frac{1}{t \log t} \right)$$ and

$$J(\delta) = \int_a^\infty f(t) t^{-\delta} \, dt \to \ell \text{ as } \delta \to 0,$$

then $$J(0) = \ell$$.

3.2. **Proof.** We are now in position to prove (2) following Hardy’s approach. Using partial summation, we have

$$\sum_{N(p) \leq x} \log(1 - N(p)^{-s}) = \pi_K(x) \log(1 - x^{-s}) - s \int_2^x \frac{\pi_K(t)}{t(t^s - 1)} \, dt,$$

where $$\text{Re}(s) > 1$$ and $$p$$ are prime ideals in $$K$$. Let $$x \to \infty$$ in (10) and re-write the integral therein to obtain

$$\log \zeta_K(s) = s \int_2^\infty \left\{ \pi_K(t) - \frac{t}{\log t} \right\} \frac{dt}{t^{1+s}} + s \int_2^\infty \frac{\pi_K(t)}{t^{1+s}(t^s - 1)} \, dt + s \int_2^\infty \frac{t^{-s}}{\log t} \, dt.$$

Next, let $$s \to 1^+$$, then we have

$$\log \zeta_K(s) \to -\log(s - 1) + \log \kappa_K,$$

$$K_2(s) \to K_2(1), \quad \text{by uniform convergence}$$

$$K_3(s) \to -\log(s - 1) - \log \log 2 - \gamma. \quad \text{by (8)}$$

It follows that

$$K_1(s) = \log \zeta_K(s) - K_2(s) - K_3(s) \to \log \kappa_K + \log \log 2 + \gamma - K_2(1) \quad \text{as } s \to 1.$$
Therefore, it follows from invoking the Tauberian theorem (9) (which we may do by virtue of the Chebyshev-type observation on $\pi_K(x)$), that
\[
\int_2^\infty \left\{ \pi_K(t) - \frac{t}{\log t} \right\} \frac{dt}{t^2} = \log \kappa_K + \log \log 2 + \gamma - K_2(1).
\] (11)

Now, suppose $s = 1$ in (10), then
\[
\sum_{N(p) \leq x} \log \left( 1 - \frac{1}{N(p)} \right) = \pi_K(x) \log(1 - x^{-1}) - \int_2^x \left\{ \pi_K(t) - \frac{t}{\log t} \right\} \frac{dt}{t^2} - \int_2^x \frac{\pi_K(t) dt}{t^2(t - 1)} - \int_2^x \frac{dt}{t \log t}
\]
\[
\to - \log \kappa_K - \gamma - \log \log x
\]
as $x \to \infty$, using (11). This is equivalent to Mertens’ third theorem for number fields (2), because $\exp(o(1)) = 1 + o(1)$ and
\[
\prod_{N(p) \leq x} \left( 1 - \frac{1}{N(p)} \right) = \exp \left( \sum_{N(p) \leq x} \log(1 - N(p)^{-1}) \right) = \frac{e^{-\gamma}}{\kappa_K \log x} (1 + o(1)).
\] (12)

**Remark 11.** To provide an explicit description of the error term in (12), one would need to show
\[
\int_2^x \left\{ \pi_K(t) - \frac{t}{\log t} \right\} \frac{dt}{t^2} = -\log \kappa_K - \log \log 2 - \gamma + \int_2^x \frac{\pi_K(t) dt}{t^2(t - 1)} + o(1),
\]
using explicit constants. Unfortunately, it appears as though one needs to rely on the prime ideal theorem to achieve this goal, which represents a limitation of this method.

### 4. Oscillations in Mertens’ Third Theorem

In this section, we will prove Theorem 1. To ensure that the proof of Theorem 1 can be approached using analytic techniques, we observe that the problem is equivalent to showing
\[
- \sum_{N(p) \leq x} \log \left( 1 - \frac{1}{N(p)} \right) - \log \log x - \gamma - \log \kappa_K \begin{cases} > \eta/(\sqrt{x \log x}) \\ < -\eta/(\sqrt{x \log x}) \end{cases},
\] (13)
for sequences of $x$ which tend to infinity, and any large $\eta > 0$. We begin by re-interpreting the left-hand side of (13) in Lemma 12. Using this re-interpretation, we will deduce that
\[
A(x) + O \left( \frac{1}{\sqrt{x \log x}} \right) \begin{cases} > \eta/(\sqrt{x \log x}) \\ < -\eta/(\sqrt{x \log x}) \end{cases}
\]
is true for sequences of $x$ which tend to infinity, and any large $\eta > 0$ ($A(x)$ is defined in the statement of Lemma 12). We consider two separate cases to prove this assertion. First, in Section 4.2, we assume that the GRH is not true and the technical condition in the statement of Theorem 1 holds, and use Landau’s oscillation theorem [6, Thm. 6.31]. Second, in Section 4.3, we assume the GRH, and use Cramér’s inequality for number fields (4). This will automatically prove Theorem 1.
Lemma 12. The left-hand side of (13) is equivalent to
\[
\int_1^x \frac{d\Pi_K(t)}{t} - \int_1^x \frac{1 - t^{-1}}{t \log t} + O\left(\frac{1}{\sqrt{x} \log x}\right). \tag{14}
\]

4.1. Proof of Lemma 12. To prove Lemma 12 we first need to convert the sum in (13) into an integral using Lemma 13.

Lemma 13. For \(x \geq 2\),
\[
- \sum_{N(p) \leq x} \log \left(1 - \frac{1}{N(p)}\right) = \int_1^x \frac{d\Pi_K(t)}{t} + O\left(\frac{1}{\sqrt{x} \log x}\right).
\]

Proof. Clearly,
\[
\int_1^x \frac{d\Pi_K(t)}{t} = \sum_{\ell \geq 1} \frac{1}{\ell} \int_1^x \frac{d\pi_K(t^{1/\ell})}{t} = \sum_{N(p) \leq x} \frac{1}{\ell N(p)^{1/\ell}}.
\]
Moreover, there are at most \(n_K\) prime ideals which lie over any rational prime \(p\) in a number field \(K\). It follows that
\[
- \sum_{N(p) \leq x} \log \left(1 - \frac{1}{N(p)}\right) - \int_1^x \frac{d\Pi_K(t)}{t} = \sum_{N(p) \leq x} \frac{1}{\ell N(p)^{1/\ell}} \leq n_K \sum_{\substack{p \leq x \\text{prime} \\ell \mid p \\ell > x}} \frac{1}{\ell p^{1/\ell}} \ll \frac{1}{\sqrt{x} \log x},
\]
by the contents of the proof of [10, Lem. 2.1].

Second, we import [10, Lem. 2.2] in Lemma 14 which replaces \(\log \log x + \gamma\) with integrals.

Lemma 14. For \(x > 1\),
\[
\log \log x + \gamma = \int_1^x \frac{1 - t^{-1}}{t \log t} - \int_x^\infty \frac{dt}{t^2 \log t} = \int_1^x \frac{1 - t^{-1}}{t \log t} + O\left(\frac{1}{x \log x}\right).
\]

Lemma 12 follows by inserting Lemmas 13 and 14 into the left-hand side of (13).

4.2. The non-GRH case. Following a natural generalisation of the method presented in [10, Sec. 2], we can derive the Mellin formula for \(A(x)\) for \(\Re(s) > 0\),
\[
\hat{A}(s) := \int_1^\infty t^{-s-1} A(t) \, dt = \frac{1}{s} \log \frac{s \zeta_K(s + 1)}{s + 1}.
\]
Moreover, as in [10, Sec. 3], we replace the error term in (14) by
\[
B(x) := \frac{1 - x^{-1}}{\sqrt{x} \log x}
\]
which is asymptotically equivalent to the original error term. The Mellin transform of \(B(x)\) will be
\[
\hat{B}(s) = \log \frac{s + 3/2}{s + 1/2}.
\]
We want to show that for a fixed $K$, $A(x) + \eta B(x)$ changes sign infinitely often. Suppose GRH is not true, then $\zeta_K(s)$ has a zero $\sigma_K$ with $1/2 < \text{Re}(\sigma_K) < 1$. If we assume, as in Theorem 1 that there is no zero in the right-half plane $\text{Re}(s) > \text{Re}(\sigma_K)$ and $\sigma_K$ is not real, then
\[
\log \frac{s \zeta_K(s + 1)}{s + 1}
\]
has a singularity at $\sigma^* = \sigma_K - 1$,

with $\text{Re}(\sigma^*) > -1/2$ and $\text{Im}(\sigma^*) \neq 0$. As a result, $\hat{A} + \eta \hat{B}$ is holomorphic for $\text{Re}(s) > \text{Re}(\sigma^*)$ but not in any half-plane $\text{Re}(s) > \text{Re}(\sigma^*) - \epsilon$ with $\epsilon > 0$. Moreover, since $\sigma^*$ is not real, $\text{Re}(\sigma^*)$ will be a regular point and so, by Landau’s oscillation theorem [6, Thm. 6.31], $A(x) + \eta B(x)$ will change sign infinitely often for any fixed value of $\eta$.

4.3. The GRH case. Using integration by parts and similar arguments to [10, Sec. 4], we see

\[
A(x) = \frac{\Upsilon^*(x)}{x} + \int_1^x \frac{\Upsilon^*(t)}{t^2} \, dt
\]

(15)

\[
= \frac{\Upsilon^*(x)}{x} + \int_1^\infty \frac{\Upsilon^*(t)}{t^2} \, dt - \int_x^\infty \frac{\Upsilon^*(t)}{t^2} \, dt
\]

\[
= \frac{\Upsilon^*(x)}{x} - \int_x^\infty \frac{\Upsilon^*(t)}{t^2} \, dt.
\]

The integral from 1 to $\infty$ equates to zero by rearranging (15), and inserting the observations $A(x) \to 0$ and $\frac{\Upsilon^*(x)}{x} \to 0$ as $x \to \infty$.

Therein, $A(x) \to 0$ using analogous observations to Diamond and Pintz in [10, Prop. 4.1], making changes mutatis mutandis. Using (4) and dyadic interval estimates, we can also show that

\[
\int_x^\infty \frac{\Upsilon^*(t)}{t^2} \, dt \ll \frac{1}{\sqrt{x \log x}},
\]

hence, using (7), we see that

\[
A(x) = \Omega_\pm \left(\frac{\log \log \log x}{\sqrt{x \log x}}\right).
\]

5. Bias in Mertens’ Third Theorem

For a set $S \subset [0, \infty)$, upper and lower logarithmic densities of $S$ are defined by

\[
\bar{\delta}(S) = \limsup_{x \to \infty} \frac{1}{\log x} \int_{t \in S \cap [2, x]} \frac{dt}{t}, \quad \text{and} \quad \underline{\delta}(S) = \liminf_{x \to \infty} \frac{1}{\log x} \int_{t \in S \cap [2, x]} \frac{dt}{t}.
\]

Moreover, if $\bar{\delta}(S) = \underline{\delta}(S) = \delta(S)$, then $\delta(S)$ is the logarithmic density of $S$. Recall the definition of $\mathcal{M}_K$ from [3]; we know that $x \in \mathcal{M}_K$ if and only if $E_K(x) > 0$, where

\[
E_K(x) = \sqrt{x \log x} \left(\log \prod_{N(p) \leq x} \left(1 - \frac{1}{N(p)}\right)^{-1} - \log \kappa_K - \log \log x - \gamma\right).
\]

(16)

The purpose of this section is to demonstrate that $E_K(x)$ has a limiting distribution, assuming the GRH, which is equivalent to showing that the logarithmic density $\delta(\mathcal{M}_K)$
exists. To this end, always assuming the GRH, we establish a useful explicit formula for $E_K(x)$ in Corollary 16 in Section 5.1 and we prove Theorem 3 in Section 5.2. In Section 5.3, we prove Theorem 24 which uses the aforementioned explicit formula to show that the limiting distribution of $E_K(x)$ exists under the assumption of the GRH, and Theorem 3 tells us that it must be positive, but not equal to one. In Section 5.4, we confirm this by calculating the logarithmic density in two specific cases and outline a general method to perform computations in other similar cases. Finally, in Section 5.5, we address an important question concerning the keenness of the bias as the discriminant of the quadratic field grows.

5.1. Explicit formula. The following proposition generalises [19, Prop. 2.1], and is an explicit formula for $E_K(x)$.

**Proposition 15.** Suppose $K$ is a number field and the exceptional zero $\beta_0$ does not exist (see Remark 21), then, for any $x \geq 2$ and $T \geq 5$, we have

$$E_K(x) = 1 + \sum_{\mini{\rho}{\operatorname{Im} \rho \leq T}} \frac{x^{\rho - \frac{1}{2}}}{\rho - 1} + O\left(\frac{1}{\log x} \left(1 + \sum_{\mini{\rho}{\operatorname{Im} \rho \leq T}} \frac{x^{\Re(\rho)} - \frac{1}{2}}{\Im \rho^2}\right) + \frac{\sqrt{T}}{T} \left(\log x + \frac{\log^2 T}{\log x}\right)\right).$$

Here, $\rho$ are the non-trivial zeros of $\zeta_K(s)$.

Once we know Proposition 15, we can establish the following corollary, which is conditional on the GRH.

**Corollary 16.** Assume GRH and let $\frac{1}{2} + i\gamma_n$ represent the non-trivial zeros of $\zeta_K(s)$. Then, for any $x \geq 2$ and $T \geq 5$, we have

$$E_K(x) = 1 + 2 \Re \sum_{0 < \gamma_n < T} \frac{x^{i\gamma_n}}{-\frac{1}{2} + i\gamma_n} + O\left(\frac{\sqrt{T}}{T} \left(\log x + \frac{\log^2 T}{\log x}\right)\right). \quad (17)$$

**Proof.** Using Lemma 7 and $\gamma_n \neq 0$, one can see

$$\sum_{|n| \leq T} \frac{1}{\gamma_n^2} \ll 1. \quad (18)$$

Now, Corollary 16 follows easily from Proposition 15 and (18). \[\square\]

**Remark 17.** In the setting $K = \mathbb{Q}$, in [7, Cor. 1], Brent, Platt, and Trudgian refined Lehman’s lemma [24, Lem. 1], and used this to establish the sum in (17) over all $\gamma_n > 0$ is $0.02310499\ldots$ upto 28 decimal places. To evaluate (or bound with explicit constants) the sum in (18) one could apply similar techniques as Brent et al. [7] or Lehman [24].

The remainder of this subsection is dedicated to proving Proposition 15. To do so, we will argue along similar lines to [19], which means we will require the following lemmas.

**Lemma 18.** For $x \geq 2$, we have

$$-\sum_{N(p) \leq x} \log \left(1 - \frac{1}{N(p)}\right) = \sum_{N(a) \leq x} \frac{\Lambda_K(a)}{N(a) \log N(a)} + \frac{1}{\sqrt{x} \log x} + O\left(\frac{1}{\sqrt{x} \log^2 x}\right).$$

**Proof.** See [19, Lem. 2.3], the proof generalises naturally. \[\square\]
Lemma 19. For $\alpha > 1$, $x \geq 2$, and $T \geq 5$, we have

$$
\sum_{N(a) \leq x} \frac{\Lambda_K(a)}{N(a)^\alpha} = \frac{\zeta_K'(s)}{\zeta_K(s)}(\alpha) + \frac{x^{1-\alpha}}{1-\alpha} - \frac{x^{\beta_0-\alpha}}{\alpha-\beta_0} - \sum_{|\text{Im} \, \theta| \leq T} \frac{x^{\theta-\alpha}}{\theta-\alpha}
+ O \left( x^{-\alpha} \log x + \frac{x^{1-\alpha}}{T} \left( 4^\alpha + \log^2 x + \log^2 T \right) + \frac{1}{T} \sum_{N(a) \geq 1} \frac{\Lambda_K(a)}{N(a)^{\alpha+\frac{1}{2}}(\log x)} \right),
$$

where $\theta$ are the non-trivial zeros of $\zeta_K(s)$ and $\beta_0$ is the potential real, exceptional zero.

Proof. Almost all aspects of the proof generalise naturally from \[19\] Lem. 2.4, but there are some technical considerations one should be careful with, including the potential exceptional zero. Therefore, we present a generalised summary of Lamzouri’s method in \[19\] Lem. 2.4, with extra details whenever they are required.

Suppose $c = 1/\log x$, note that there exists a $T_0 \in [T, T+1]$ which has distance $\gg 1/\log T$ from the ordinate of the nearest zero of $\zeta_K(s)$, and consider the contour integral

$$
\mathcal{J}_K = \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} -\frac{\zeta_K'(s)}{\zeta_K(s)}(\alpha + s) \frac{x^s}{s} ds.
$$

First, evaluate $\mathcal{J}_K$ using Perron’s formula, then one obtains

$$
\mathcal{J}_K = \sum_{N(a) \leq x} \frac{\Lambda_K(a)}{N(a)^\alpha} + O \left( x^{-\alpha} \log x + \frac{1}{T} \left( x^{1-\alpha} \log^2 x + \sum_{N(a) \geq 1} \frac{\Lambda_K(a)}{N(a)^{\alpha+\frac{1}{2}}(\log x)} \right) \right).
$$

Second, evaluate $\mathcal{J}_K$ by moving the line of integration to the line $\text{Re}(s) = -U$, where $U > 0$ is large, and invoking Cauchy’s residue theorem. That is,

$$
\mathcal{J}_K = \frac{1}{2\pi i} \left( \int_{\mathcal{C}} -\frac{\zeta_K'(s)}{\zeta_K(s)}(\alpha + s) \frac{x^s}{s} ds - \sum_{i=2}^4 \int_{\mathcal{C}_i} -\frac{\zeta_K'(s)}{\zeta_K(s)}(\alpha + s) \frac{x^s}{s} ds \right),
$$

where $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ is a closed contour such that

$$
\mathcal{C}_1 = [c - iT_0, c + iT_0], \quad \mathcal{C}_2 = [c + iT_0, -U + iT_0],
\mathcal{C}_3 = [-U + iT_0, -U - iT_0], \quad \mathcal{C}_4 = [-U - iT_0, c - iT_0].
$$

It follows that $U$ should also be chosen such that $U \neq \alpha + m$ for any $m \in \mathbb{N}$, so that $-U$ does not equal a trivial zero of $\zeta_K(s)$. To estimate the integrals over $\mathcal{C}_i$, Lamzouri’s observations generalise naturally; i.e.

$$
-\frac{1}{2\pi i} \sum_{i=2}^4 \int_{\mathcal{C}_i} -\frac{\zeta_K'(s)}{\zeta_K(s)}(\alpha + s) \frac{x^s}{s} ds \ll \frac{x^{1-\alpha}}{T} \left( 4^\alpha + \log x + \frac{\log^2 T}{\log x} \right) + \frac{1}{T} \sum_{N(a) \geq 1} \frac{\Lambda_K(a)}{N(a)^{\alpha+\frac{1}{2}}(\log x)}.
$$
Next, using the properties of \( \zeta_K(s) \) we introduced in Section 2.1, invoke Cauchy’s residue theorem to evaluate the closed contour integral to obtain

\[
\frac{1}{2\pi i} \int_C \frac{-\zeta_K'(s)}{-\zeta_K(s)} \frac{x^s}{s} ds = -\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{x^{1-\alpha}}{1-\alpha} - r_\infty - \frac{x^{\beta_\infty}}{\alpha - \beta_\infty} - \sum_{|\text{Im } \varrho| \leq T} \frac{x^{\varrho - \alpha}}{\varrho - \alpha} - r_2 \sum_{m \leq \frac{U - 1}{2}} \frac{x^{-2m - \alpha}}{2m + 1 + \alpha} - (r_1 + r_2) \sum_{m \leq \frac{U - 1}{2}} \frac{x^{-2m - \alpha}}{2m + \alpha},
\]

in which \( r_\infty = x^{-\alpha}/\alpha \) if \( r > 0 \), and \( r_\infty = 0 \) otherwise. Now, we have

\[
\sum_{m \leq \frac{U - 1}{2}} \frac{x^{-2m - \alpha}}{2m + 1 + \alpha} \ll x^{-3 - \alpha}, \quad \sum_{m \leq \frac{U - 1}{2}} \frac{x^{-2m - \alpha}}{2m + \alpha} \ll x^{-2 - \alpha}, \quad \sum_{T \leq |\text{Im } \varrho| \leq T_0} \frac{x^{\varrho - \alpha}}{\varrho - \alpha} \ll \frac{x^{1 - \alpha} \log T}{T},
\]

and \( r_\infty \ll x^{-\alpha} \). Substituting these observations, the result follows naturally. ■

**Lemma 20.** For any \( x \geq 2 \),

\[
\log \zeta_K(\sigma) + \int_\sigma^\infty \frac{x^{1-t}}{1-t} dt \to \log \kappa_K + \log \log x + \gamma \quad \text{as} \quad \sigma \to 1^+.
\]

**Proof.** See [19, Lem. 2.5], the proof generalises naturally. ■

We are now in a position to prove Proposition 15.

**Proof of Proposition 15.** Suppose \( \beta_0 \) does not exist. Using Lemmas 19 and 20, we have

\[
\sum_{N(a) \leq x} \frac{\Lambda_K(a)}{N(a)^\sigma \log N(a)} = \int_\sigma^{\infty} \frac{\Lambda_K(a)}{N(a)^t} dt
\]

\[
= \log \zeta_K(\sigma) + \int_\sigma^{\infty} \frac{x^{1-t}}{1-t} dt - \sum_{|\text{Im } \varrho| \leq T} \int_\sigma^{\infty} \frac{x^{\varrho - t}}{\varrho - t} dt + \epsilon(x, T)
\]

\[
\to \log \kappa_K + \log \log x + \gamma - \sum_{|\text{Im } \varrho| \leq T} x^{\varrho} \int_1^{\infty} \frac{x^{-t}}{\varrho - t} dt + \epsilon(x, T),
\]

as \( \sigma \to 1^+ \), in which

\[
\epsilon(x, T) \ll \frac{1}{T} \left( \log x + \frac{\log^2 T}{\log^2 x} \right) + \frac{1}{x} + \frac{1}{T} \sum_{N(a) \geq 1} \frac{\Lambda_K(a)}{N(a)^{1 + \frac{1}{\log x} \log N(a)}}
\]

\[
\ll \frac{1}{T} \left( \log x + \frac{\log^2 T}{\log^2 x} \right) + \frac{1}{x}.
\]

Using a substitution, Lamzouri [19, p. 105] has shown

\[
\int_1^{\infty} \frac{x^{-t}}{\varrho - t} dt = \frac{1}{x \log x (\varrho - 1)} + O \left( \frac{1}{x \log^2 x \text{Im } \varrho^2} \right).
\]

Insert this into (19), and the result follows using Lemma 18. ■
Remark 21. In the end, we will assume the GRH, so it is not detrimental for our purposes to assume $\beta_0$ does not exist. However, as a thought experiment, suppose $\beta_0$ does exist. In this case, equation (19) in the proof of Proposition 15 would instead read

$$\rightarrow \log \kappa_K + \log \log x + \gamma - \int_1^\infty \frac{x^{\beta_0 - t}}{t - \beta_0} \, dt - \sum_{|\Im \varrho| \leq T} x^\varrho \int_1^\infty \frac{x^{-t}}{\varrho - t} \, dt + \epsilon(x, T).$$

Ahn and Kwon have shown in [2, Cor. 7.4] that there exists a constant $c$ such that

$$1 - \beta_0 \geq |\Delta_K| - c,$$

in which $c = 114.72\ldots$ is admissible. Using (20), one can bound the extra integral which arises, so we know that the extra integral would contribute $O(1)$. Therefore, (20) isn’t strong enough for our purposes.

5.2. Lower and upper densities. Next, we prove Theorem 3, which establishes that if the logarithmic density of $M_K$ exists, then it must be positive but not equal to one. We work along similar lines to Lamzouri’s proof of [19, Thm. 1.1].

Proof of Theorem 3. We write $x = e^y$. Then, we have

$$\frac{1}{\log x} \int_{t \in M_K \cap [2, x]} \frac{dt}{t} = \frac{1}{x} \text{meas} \left\{ \log 2 \leq y \leq Y \mid e^y \in M_K \right\} = \frac{1}{x} \text{meas} \left\{ \log 2 \leq y \leq Y \mid E_K(e^y) > 0 \right\},$$

where $y = \log t$. By Corollary 16 and (18), we have for $y \geq \log 2$ and $T \geq 5$,

$$E_K(e^y) = \sum_{0 < \gamma_n < T} -\cos(\gamma_n y) + 2\gamma_n \sin(\gamma_n y) \left( \frac{1}{4} + \gamma_n^2 \right) + O\left( 1 + \frac{e^{y/2}}{T} \left( y + \log^2 \frac{T}{y} \right) \right)$$

$$= 2 \sum_{0 < \gamma_n < T} \frac{\sin(\gamma_n y)}{\gamma_n} + O\left( 1 + \frac{e^{y/2}}{T} \left( y + \log^2 \frac{T}{y} \right) \right).$$

We choose $T = e^Y$ and if $Y$ is large enough, there exists a constant $A > 0$ such that

$$2 \left( \sum_{0 < \gamma_n < e^Y} \frac{\sin(\gamma_n y)}{\gamma_n} - A \right) < E_K(e^y) < 2 \left( \sum_{0 < \gamma_n < e^Y} \frac{\sin(\gamma_n y)}{\gamma_n} + A \right)$$

for all $2 \leq y \leq Y$. Therefore, from (21),

$$\frac{1}{\log x} \int_{t \in M_K \cap [2, x]} \frac{dt}{t} \geq \frac{1}{Y} \text{meas} \left\{ 1 \leq y \leq Y \mid \sum_{0 < \gamma_n < e^Y} \frac{\sin(\gamma_n y)}{\gamma_n} > A \right\} + O\left( \frac{1}{Y} \right),$$

and

$$\frac{1}{\log x} \int_{t \in M_K \cap [2, x]} \frac{dt}{t} \leq \frac{1}{Y} \text{meas} \left\{ 1 \leq y \leq Y \mid \sum_{0 < \gamma_n < e^Y} \frac{\sin(\gamma_n y)}{\gamma_n} > -A \right\} + O\left( \frac{1}{Y} \right).$$
Now, as in [30, Sec. 2.2], using Littlewood’s approach [25], we have that
\[
\frac{1}{Y} \operatorname{meas} \left\{ 1 \leq y \leq Y \mid \sum_{0 < \gamma_n < e^y} \frac{\sin(\gamma_n y)}{\gamma_n} > \lambda \right\} \geq c_1 \exp(-\exp(-c_2 \lambda)), \tag{24}
\]
and
\[
\frac{1}{Y} \operatorname{meas} \left\{ 1 \leq y \leq Y \mid \sum_{0 < \gamma_n < e^y} \frac{\sin(\gamma_n y)}{\gamma_n} < -\lambda \right\} \geq c_1 \exp(-\exp(-c_2 \lambda)) \tag{25}
\]
for some absolute positive constants \(c_1, c_2\), if \(Y\) is large enough. Combining (22), (23), (24), and (25), we obtain
\[
\frac{c_1}{2} \exp(-\exp(-c_2 A)) \leq \frac{1}{\log x} \int_{t \in M_K \cap [2, x]} \frac{dt}{t} \leq 1 - \frac{c_1}{2} \exp(-\exp(-c_2 A)),
\]
if \(Y = \log x\) is large enough. In particular, \(\delta(M_K) > 0\) and \(\delta(M_K) < 1\). ■

5.3. Limiting distribution. Let \(\phi : [0, \infty) \to \mathbb{R}\) and let \(y_0\) be a non-negative constant such that \(\phi\) is square-integrable on \((0, y_0]\). Suppose there exists \((\lambda_n)_{n \in \mathbb{N}}\), a non-decreasing sequence of positive numbers which tends to infinity, \((r_n)_{n \in \mathbb{N}}\), a complex sequence, and \(c\) a real constant such that for \(y \geq y_0\),
\[
\phi(y) = c + \text{Re} \left( \sum_{\lambda_n \leq X} r_n e^{i \lambda_n y} \right) + \mathcal{E}(y, X) \tag{26}
\]
for any \(X \geq X_0 > 0\) and \(\mathcal{E}(y, X)\) such that
\[
\lim_{Y \to \infty} \frac{1}{Y} \int_{y_0}^{Y} |\mathcal{E}(y, e^y)|^2 \, dy = 0. \tag{27}
\]
The following result, which is a restatement of [3, Thm. 1.2], prescribes the conditions on \((\lambda_n)_{n \in \mathbb{N}}\) and \((r_n)_{n \in \mathbb{N}}\) under which \(\phi\) has a limiting distribution.

**Theorem 22.** Let \(\phi : [0, \infty) \to \mathbb{R}\) satisfy (26) and (27). Let \(\alpha, \beta > 0\) and \(\gamma \geq 0\). Assume either of the following conditions:

1. \(\beta > \frac{1}{2}\) and
\[
\sum_{T < \lambda_n \leq T + 1} |r_n| \ll \frac{(\log T)^\gamma}{T^\beta},
\]
for \(T > 0\).
2. \(\beta \leq \min\{1, \alpha\}, \alpha^2 + \alpha/2 < \beta^2 + \beta,\) and
\[
\sum_{S < \lambda_n \leq T} |r_n| \ll \frac{(T - S)^\alpha (\log T)^\gamma}{S^3},
\]
for \(T > S > 0\).

Then \(\phi(y)\) is a \(B^2\)-almost periodic function and therefore possesses a limiting distribution.

While Theorem 22 can be directly used to show the existence of a limiting distribution, the following corollary (which is a restatement of [3, Cor. 1.3]) makes the task much easier.
Corollary 23. Let \( \phi : [0, \infty) \to \mathbb{R} \) satisfy (26) and (27). Assume that \( r_n \ll \lambda_n^{-\beta} \) for \( \beta > \frac{1}{2} \), and
\[
\sum_{T < \lambda_n < T+1} 1 \ll \log T.
\]
Then \( \phi(y) \) is a \( B^2 \)-almost periodic function and therefore possesses a limiting distribution.

Using Corollary 23, we are in a position to state the final result, which establishes that the function \( E_{\mathbb{K}}(x) \), defined in (16), has a limiting distribution.

Theorem 24. Suppose GRH is true. Then \( E_{\mathbb{K}}(x) \) has a limiting distribution, that is, there exists a probability measure \( \mu_{\mathbb{K}} \) on \( \mathbb{R} \) such that
\[
\lim_{x \to \infty} \frac{1}{\log x} \int_{2}^{x} f(E_{\mathbb{K}}(t)) \frac{dt}{t} = \int_{-\infty}^{\infty} f(t) d\mu_{\mathbb{K}},
\]
for all bounded continuous functions \( f \) on \( \mathbb{R} \).

Proof. From (17), we can see that \( E_{\mathbb{K}}(x) \) can be written in the form given in (26) by setting
\[
c = 1, \quad \lambda_n = \gamma_n, \quad r_n = \frac{2}{-\frac{1}{2} + i\gamma_n}, \quad \text{and} \quad y = \log x.
\]
To apply Corollary 23 we must show that \( \mathcal{E}(y, X) \) satisfies condition (27), that is,
\[
\mathcal{E}(y, e^{Y}) = \lim_{Y \to \infty} \frac{1}{Y} \int_{0}^{Y} \left| \frac{e^{y/2}}{e^{Y}} \left( y + \frac{Y^2}{y} \right) \right|^2 dy = 0.
\]
This is straightforward using integration by parts. We also have that \( r_n \ll \gamma_n^{-1} \), and the number of zeros of \( \zeta_{\mathbb{K}}(s) \) in the interval \((T, T+1)\) is \( O(\log T) \) by Lemma 7. This enables us to apply Corollary 23, whence the result follows.

Along with GRH, if we also assume GLI, we can prove an explicit formula for the Fourier transform of \( \mu_{\mathbb{K}} \).

Proposition 25. Assume GRH and GLI. Then, for any number field \( \mathbb{K} \), the Fourier transform of \( \mu_{\mathbb{K}} \) is given by
\[
\widehat{\mu}_{\mathbb{K}}(t) = \int_{-\infty}^{\infty} e^{-it} d\mu_{\mathbb{K}} = e^{-it} \prod_{\gamma_n > 0} J_0 \left( \frac{2t}{\sqrt{\frac{1}{4} + \gamma_n^2}} \right),
\]
for all \( t \in \mathbb{R} \), where \( J_0(t) = \sum_{m=0}^{\infty} (-1)^m (t/2)^{2m}/(m!)^2 \) is the Bessel function of order 0.

The above proposition is an immediate consequence of Theorem 1.9 of [3]. Next, we will state a proposition which is analogous to Proposition 4.2 of [19].

Proposition 26. Assume GRH and GLI. Let \( X(\gamma_n) \) be a sequence of independent random variables, arranged in increasing order of the positive imaginary parts \( (\gamma_n) \) of non-trivial zeros of \( \zeta_{\mathbb{K}}(s) \), and uniformly distributed on the unit circle. Then \( \mu_{\mathbb{K}} \) is the distribution of the random variable
\[
Z = 1 + 2 \text{Re} \sum_{\gamma_n > 0} \frac{X(\gamma_n)}{\sqrt{\frac{1}{4} + \gamma_n^2}}.
\]
Assuming the Riemann Hypothesis (RH) and the Linear Independence Hypothesis (LI) for the Riemann zeta-function, Rubinstein and Sarnak \cite{30} showed that the limiting distribution of \((\pi(x) - \text{Li}(x)) / \sqrt{x}\) is the distribution of the random variable

\[ \tilde{Z} = -1 + 2 \text{Re} \sum_{\gamma > 0} \frac{X(\gamma \zeta)}{\sqrt{1 + \gamma^2}} \]

where \(\gamma \zeta\) are the imaginary parts of the zeros of the Riemann zeta-function. As shown in the proof of Theorem 1.3 of \cite{19}, \(P(\tilde{Z} > 0)\) is the logarithmic density of the set of real numbers \(x \geq 2\) for which \(\pi(x) > \text{Li}(x)\). Therefore, \(P(Z > 0) = 1 - P(\tilde{Z} > 0)\) is the logarithmic density of the set of reals \(x \geq 2\) for which \(\pi(x) < \text{Li}(x)\). Similarly, from Proposition 26, we can also deduce that the logarithmic density of the set \(\mathcal{M}_K\) (assuming GRH and GLI) is the logarithmic density of the set of real numbers \(x \geq 2\) for which \(\pi_K(x) < \text{Li}(x)\).

5.4. **Numerical investigations.** Let the logarithmic density of the set of real numbers \(x \geq 2\) for which \(\pi_K(x) < \text{Li}(x)\) be \(\delta(P_K)\), which is also equal to \(\delta(\mathcal{M}_K)\), as argued above. For the classical case, one can find the relevant computations in \cite{30, Sec. 4}. Here, we will combine the analysis done for \(\delta(P_{\text{comp}}^p)\), \(\delta(P_{5;N;R})\), and \(\delta(P_{13;N;R})\) in \cite{30} to find

\[ \delta(P_{\mathbb{Q}(\sqrt{5})}) \quad \text{and} \quad \delta(P_{\mathbb{Q}(\sqrt{13})}). \]

We can do this due to the following fact about the Dedekind zeta-function of a quadratic number field \(\mathbb{K} = \mathbb{Q}(\sqrt{q})\) when \(q \equiv 1\) mod 4 is a squarefree integer:

\[ \zeta_K(s) = \zeta(s)L(s, \chi_{1,q}), \]

where \(\chi_{1,q}\) is the real non-principal character modulo \(q\). This is a special case of the factorisation of the Dedekind zeta-function of an abelian number field into a product of Dirichlet \(L\)-functions (see \cite{32, Thm. 4.3}, for example). Note that while it is possible to do a similar analysis when \(q\) is squarefree and \(q \not\equiv 1\) mod 4, the modulus of the real non-principal character in that case would be 4\(q\) and we do not have the advantage of utilising the computational work done in \cite{30}.

Let \(f_K(x)\) be the density function of \(\mu_K\). We, instead, consider \(\omega_K(x) := f_K(x - 1)\) which is symmetric about \(x = 0\). Assuming GRH and GLI, from Proposition 25 we know that its Fourier transform is given by

\[ \hat{\omega}_K(t) = \prod_{\gamma_n > 0} J_0 \left( \frac{2t}{\sqrt{1 + \gamma_n^2}} \right), \]

where, as in Proposition 26 we write the positive ordinates of the zeros of \(\zeta_K\) as \(\gamma_n\). In fact, due to \cite{28}, the set of zeros with \(\gamma_n > 0\) is the union of the sets of zeros with \(\gamma_\zeta > 0\) and \(\gamma_{\chi_{1,q}} > 0\). Since \cite{29} is analogous to (4.1) in \cite{30} and

\[ \hat{\omega}_K(t) = \hat{\omega}_\zeta(t) \hat{\omega}_{\chi_{1,q}}(t), \]

we will follow the analysis done in \cite{30} for \(\hat{\omega}_\zeta(t)\) and \(\hat{\omega}_{\chi_{1,q}}(t)\).

Our objective is to evaluate the integral

\[ \delta(P_K) = \int_{-\infty}^{1} d\omega_K(t) \]
which, as in [30, (4.2)], can be written as

$$
\delta(P_K) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} \tilde{\omega}_K(u) du. \quad (31)
$$

We want to replace the integral in (31) by a sum that can be evaluated easily. We do this with the help of the Poisson summation formula

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} \tilde{\omega}_K(u) du = \epsilon \sum_{n \in \mathbb{Z}} \varphi(\epsilon n) - \sum_{n \in \mathbb{Z}, n \neq 0} \widehat{\varphi} \left( \frac{n}{\epsilon} \right). \quad (32)
$$

where $\epsilon$ is a small number (to be chosen later) and

$$
\varphi(u) = \frac{1}{2\pi} \frac{\sin u}{u} \tilde{\omega}_K(u),
\quad \widehat{\varphi}(x) = \frac{1}{2} \int_{x-1}^{x+1} d\omega(u).
$$

To estimate the error in replacing the integral in (31) by the first sum in (32), we need a bound on $\widehat{\varphi}(n/\epsilon)$. Following the analysis in [30], it is easy to verify here as well that the magnitude of the error is $< 10^{-20}$ with the choice of $\epsilon$ being $1/20$ for both $K = \mathbb{Q}(\sqrt{5})$ and $K = \mathbb{Q}(\sqrt{13})$. Therefore, we have

$$
\delta(P_K) = \frac{1}{2} + \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \epsilon \frac{\sin \epsilon n}{\epsilon n} \tilde{\omega}_K(\epsilon n) + \text{error}. \quad (33)
$$

Next, we need to replace the infinite sum $-\infty < n\epsilon < \infty$ with a finite sum $-C \leq n\epsilon \leq C$ and bound the error in this process. Analogous to [30, (4.9)], the magnitude of this error is bounded above by

$$
\prod_{j=1}^{M} \left( \frac{1}{4} + \gamma_j^2 \right)^{1/4} \left( \frac{2}{MC^{M/2}} + \frac{1}{20C^{M/2+1}} \right)
$$

where $\gamma_j$’s are the ordinates of the zeros of $\zeta_K$ indexed in increasing order. To generate these zeros, we used Rubinstein’s $L$-function calculator in SageMath. For $K = \mathbb{Q}(\sqrt{5})$, choosing $C = 25$ and $M = 42$, the magnitude of the error is $< 3 \times 10^{-10}$, and for $K = \mathbb{Q}(\sqrt{13})$, choosing $C = 25$ and $M = 53$, the magnitude of the error is $< 7 \times 10^{-13}$. Therefore, as in [30 (4.10)], we obtain

$$
\delta(P_K) = \frac{1}{2\pi} \sum_{-25 \leq n\epsilon \leq 25} \epsilon \frac{\sin \epsilon n}{\epsilon n} \prod_{\gamma_n > 0} J_0 \left( \frac{2\epsilon n}{\sqrt{1/4 + \gamma_n^2}} \right) + \frac{1}{2} + \text{error}. \quad (33)
$$

Finally, we would like to replace the infinite product in (33) with a finite product. In order to do so, we need to introduce a compensating polynomial, $p(t)$ that accounts for the tail of the infinite product:

$$
\tilde{\omega}_K(t) = p(t) \prod_{0 < \gamma_n \leq C} J_0 \left( \frac{2t}{\sqrt{1/4 + \gamma_n^2}} \right) + \text{error} \quad (34)
$$
for \(-C \leq t \leq C\), where \(p(t) = \sum_{m=0}^{A} b_m t^{2m}\), and
\[
\prod_{\gamma_n > X} J_0 \left( \frac{2t}{\sqrt{\frac{1}{4} + \gamma_n^2}} \right) = \sum_{m=0}^{\infty} b_m t^{2m}.
\] (35)

We choose \(A = 1\) and \(X = 9999\). From the definition of \(J_0\) and (35), we find that \(b_0 = 1\) and
\[
b_1 = -\left( \sum_{\gamma_n > 0} - \sum_{0 < \gamma_n \leq X} \right) \frac{1}{\frac{1}{4} + \gamma_n^2}.
\]

We can evaluate the first sum of \(b_1\) using (4.13)–(4.14) and Table 2 of [30]. The second sum was computed using Python. This works out to be
\[
b_1 = -0.000292143 \ldots \text{ for } K = \mathbb{Q}(\sqrt{5}) \text{ and } b_1 = -0.000307347 \ldots \text{ for } K = \mathbb{Q}(\sqrt{13}).
\]

As in [30], the magnitude of the error in (34) is bounded by
\[
\frac{1}{2\pi} \sum_{-C \leq \epsilon \leq C} |\epsilon| \frac{\sin \epsilon}{\epsilon} \prod_{0 < \gamma_n \leq X} J_0 \left( \frac{2\epsilon}{\sqrt{\frac{1}{4} + \gamma_n^2}} \right) \cdot 2 \left( T_1 \epsilon^2 \right)^{A+1} (A+1)!
\] (36)

where \(T_1 = \sum_{\gamma_n > 0} \left( \frac{1}{\frac{1}{4} + \gamma_n^2} \right)^{-1}\). For \(K = \mathbb{Q}(\sqrt{5})\), (36) evaluates to \(< 7.3 \times 10^{-7}\) and for \(K = \mathbb{Q}(\sqrt{13})\), it evaluates to \(< 2 \times 10^{-7}\). Therefore, we finally obtain
\[
\delta(P_K) = \frac{1}{2\pi} \sum_{-25 \leq \epsilon \leq 25} \epsilon \frac{\sin(\epsilon)}{\epsilon} \left( 1 + b_1(\epsilon)^2 \right) \prod_{0 < \gamma_n \leq 9999} J_0 \left( \frac{2\epsilon}{\sqrt{\frac{1}{4} + \gamma_n^2}} \right) + \frac{1}{2} + \text{error},
\]

with the magnitude of the error being \(< 10^{-6}\). We summarise the results in the table below.

| Table 1. Logarithmic density for \(K = \mathbb{Q}(\sqrt{q})\) |
|---|---|
| \(q\) | \(\delta(P_K)\) |
| 5 | 0.9876... |
| 13 | 0.9298... |

5.5. Dissipation of bias. A natural question that arises from the preceding discussion is about the keenness of the bias as \(q \to \infty\). For \(q = 5\) and \(q = 13\), the bias is quite sharp and while it need not be indicative of the overall trend at all, the bias for \(q = 13\) is less keen compared to \(q = 5\). We will show that, in fact, as \(q\) becomes large, the bias completely dissipates and \(\delta(P_K)\) tends to \(1/2\). For the following discussion, we need not limit ourselves to squarefree \(q \equiv 1 \mod 4\). Indeed, if \(q \not\equiv 1 \mod 4\) and \(q\) is squarefree, the discriminant of the quadratic field is \(4q\) and hence, the \(L\)-function in (28) would \(L(s, \chi_{1,q}, 4q)\) and so, if \(q\) is large, \(4q\) would be large as well. Theorem 1.5 of [30] implies that \(\delta(P_{q,N;R}) \to \frac{1}{2}\) as \(q \to \infty\). From [30] (4.2), we observe that the integral
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} e^{-u^2/4} du \to 0
\] (37)
as $q \to \infty$. In our case, from (30) and (31), we have
\[
\left|\delta(P_K) - \frac{1}{2}\right| = \frac{1}{2\pi} \left|\int_{-\infty}^{\infty} \frac{\sin u}{u} \hat{\omega}_{\chi_{1,q}}(u) du\right|
\leq \frac{1}{2\pi} \left|\int_{-\infty}^{\infty} \frac{\sin u}{u} \hat{\omega}_{\chi_{1,q}}(u) du\right|
\]
since $|J_0(z)| \leq 1$ for all real $z$ and so $\sup_{u} |\hat{\omega}_{\chi_{1,q}}(u)| \leq 1$. Using (37), we conclude that $\delta(P_K) = \delta(M_K) \to \frac{1}{2}$ as $q \to \infty$ ($q$ is squarefree).

REFERENCES

1. J.-H. Ahn and S.-H. Kwon, Some explicit zero-free regions for Hecke $L$-functions, J. Number Theory 145 (2014), 433–473. MR 3253314
2. ______, An explicit upper bound for the least prime ideal in the Chebotarev density theorem, Ann. Inst. Fourier (Grenoble) 69 (2019), no. 3, 1411–1458. MR 3986919
3. A. Akbary, N. Ng, and M. Shahabi, Limiting distributions of the classical error terms of prime number theory, Q. J. Math. 65 (2014), no. 3, 743–780. MR 3261965
4. T. M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York-Heidelberg, 1976, Undergraduate Texts in Mathematics. MR 0434929
5. M. Bardestani and T. Freiberg, Mertens’ theorem for splitting primes and more, (2013), https://arxiv.org/abs/1309.7482
6. P. Bateman and H. Diamond, Analytic Number Theory: An Introductory Course, Monographs in Number Theory, vol. 1, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2004. MR 2111739
7. R. P. Brent, D. J. Platt, and T. S. Trudgian, Accurate estimation of sums over zeros of the Riemann zeta-function, Math. Comp. 90 (2021), no. 332, 2923–2935. MR 4305374
8. R. P. Brent, D. J. Platt, and T. S. Trudgian, The mean square of the error term in the prime number theorem, J. Number Theory (2021).
9. H. Cramér, Ein mittelwertsatz in der primzahltheorie, Math. Z. 12 (1922), no. 1, 147–153. MR 1544509
10. H. G. Diamond and J. Pintz, Oscillation of Mertens’ product formula, J. Théor. Nombres Bordeaux 21 (2009), no. 3, 523–533. MR 2605532
11. S. R. Garcia and E. S. Lee, Unconditional explicit Mertens’ theorems for number fields and Dedekind zeta residue bounds, Ramanujan J. (accepted) (2021).
12. L. Grenié and G. Molteni, Explicit versions of the prime ideal theorem for Dedekind zeta functions under GRH, II, Funct. Approx. Comment. Math. 57 (2017), no. 1, 21–38. MR 3704223
13. L. Grenié, G. Molteni, and A. Perelli, Primes and prime ideals in short intervals, Mathematika 63 (2017), no. 2, 364–371. MR 3607233
14. G. H. Hardy, Note on a theorem of Mertens, J. London Math. Soc. 2 (1927), no. 2, 70–72. MR 1574590
15. E. Hasanalizade, Q. Shen, and P.-J. Wong, Counting zeros of Dedekind zeta functions, Math. Comp. (to appear) (2021).
16. H. Kadiri, Explicit zero-free regions for Dedekind zeta functions, Int. J. Number Theory 8 (2012), no. 1, 125–147. MR 2887886
17. H. Kadiri and N. Ng, Explicit zero density theorems for Dedekind zeta functions, J. Number Theory 132 (2012), no. 4, 748–775. MR 2887617
18. J. C. Lagarias and A. M. Odlyzko, Effective versions of the Chebotarev density theorem, Algebraic number fields: $L$-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), 1977, pp. 409–464. MR 0447191
19. Y. Lamzouri, A bias in Mertens’ product formula, Int. J. Number Theory 12 (2016), no. 1, 97–109. MR 3455269
20. E. Landau, Neuer beweis des primzahlsatzes und beweis des primidealsatzes, Math. Ann. 56 (1903), no. 4, 645–670. MR 1511191
21. ______, Einführung in die elementare und analytische theorie der algebraischen zahlen und der ideale, Chelsea Publishing Company, New York, N. Y., 1949. MR 0031002
22. P. Lebacque, Generalised Mertens and Brauer-Siegel theorems, Acta Arith. 130 (2007), no. 4, 333–350. MR 2365709
23. E. S. Lee, On an explicit zero-free region for the Dedekind zeta-function, J. Number Theory 224 (2021), 307–322. MR 4244156
24. R. S. Lehman, On the difference $\pi(x) - \text{li}(x)$, Acta Arith. 11 (1966), 397–410. MR 202686
25. J. E. Littlewood, Sur la distribution des nombres premiers, Comptes Rendus Acad. Sci. Paris 158 (1914), no. 1914, 1869–1872.
26. F. Mertens, Ein beitrag zur analytischen zahlentheorie, J. Reine Angew. Math. 78 (1874), 46–62. MR 1579612
27. H. L. Montgomery and R. C. Vaughan, Multiplicative Number Theory I: Classical Theory, reprint ed., Cambridge Studies in Advanced Mathematics (Book 97), 2012.
28. M. Rosen, A generalization of Mertens’ theorem, J. Ramanujan Math. Soc. 14 (1999), no. 1, 1–19. MR 1700882
29. J. B. Rosser and L. Schoenhfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6 (1962), 64–94. MR 0137689
30. M. Rubinstein and P. Sarnak, Chebyshev’s bias, Experiment. Math. 3 (1994), no. 3, 173–197. MR 1329368
31. T. S. Trudgian, An improved upper bound for the error in the zero-counting formulae for Dirichlet $L$-functions and Dedekind zeta-functions, Math. Comp. 84 (2015), no. 293, 1439–1450. MR 3315515
32. L. C. Washington, Introduction to cyclotomic fields, second ed., Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1997. MR 1421575

School of Science, UNSW Canberra, Northcott Drive, Australia ACT 2612
Email address: s.hathi@student.adfa.edu.au

School of Science, UNSW Canberra, Northcott Drive, Australia ACT 2612
Email address: ethan.s.lee@student.adfa.edu.au