ON MATRIX ELEMENTS FOR THE QUANTIZED CAT MAP MODULO PRIME POWERS

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ABSTRACT. The quantum cat map is a model for a quantum system with underlying chaotic dynamics. In this paper we study the matrix elements of smooth observables in this model, when taking arithmetic symmetries into account. We give explicit formulas for the matrix elements as certain exponential sums. With these formulas we can show that there are sequences of eigenfunctions for which the matrix elements decay significantly slower than was previously expected. We also prove a limiting distribution for the fluctuation of the normalized matrix elements around their average.

1. Introduction

The quantum cat map is a model for a quantum system with underlying chaotic dynamics that was originally introduced by the physicists Hannay and Berry [10]. This model can be used to study the semiclassical properties of such systems [2, 6, 15, 16]. The classical dynamics underlying this model is the discrete time iteration of a hyperbolic map, \( A \in \text{SL}(2, \mathbb{Z}) \), on the torus, \( \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \). In order to quantize the cat map, for every integer \( N \) (playing the role of the inverse of Planck’s constant) the Hilbert space of states is \( \mathcal{H}_N = L^2(\mathbb{Z}/N\mathbb{Z}) \). For every smooth real valued function \( f \) there is a quantum observable, i.e., a Hermitian operator \( \text{Op}_N(f) : \mathcal{H}_N \to \mathcal{H}_N \). The quantum evolution is given by a unitary operator \( U_N(A) \) on \( \mathcal{H}_N \).

For generic quantum systems with underlying chaotic dynamics, it is believed that matrix elements of smooth observables tend to the phase space average of the observable in the semiclassical limit. In order to test this phenomenon in the quantum cat map model, Kurlberg and
Rudnick introduced hidden symmetries of this model, a group of commuting operators that commute with $U_N(A)$, they called Hecke operators [15]. They showed that for any sequence of Hecke eigenfunctions (i.e., joint eigenfunctions of all Hecke operators), the corresponding matrix elements converge to the phase space average as $N \to \infty$. To be more precise they showed [15, Theorem 1] that for any $f \in C^\infty(T^2)$ and $\psi \in \mathcal{H}_N$ a Hecke eigenfunction the matrix elements satisfy

$$|\langle \text{Op}_N(f)\psi, \psi \rangle - \int_{T^2} f| \ll_{f, \epsilon} N^{-\frac{1}{4}+\epsilon}. $$

They remarked [15, Remark 1.2] that the exponent of $\frac{1}{4}$ is not optimal and that the correct bound should be $O(N^{-\frac{1}{2}+\epsilon})$, in accordance to the second and fourth moments. For $N$ prime (and consequently also for $N$ square free) this is indeed the correct bound [4, 9].

**Remark 1.1.** We note that without the arithmetic symmetries these bounds hold only if the spectral degeneracies are sufficiently small. In fact, there are sequences of eigenfunctions (where the degeneracies are exceptionally large) that don’t converge to the phase space average at all. For these eigenfunctions the matrix elements localize around short periodic orbits in the sense that the corresponding limiting measure contains a component that is supported on the periodic orbit [6].

In [16] Kurlberg and Rudnick went on to investigate the fluctuation of the normalized matrix elements,

$$F_j^{(N)} = \sqrt{N} \left( \langle \text{Op}_N(f)\psi_j, \psi_j \rangle - \int_{T^2} f dx \right),$$

where $\psi_j$ are Hecke eigenfunctions and $N \to \infty$ through primes. For this purpose they introduced the quadratic form $Q(n) = \omega(nA,n)$ (with $\omega(n,m) = n_1 m_2 - n_2 m_1$ the standard symplectic form) and used it to define twisted Fourier coefficients. For a smooth function $f \in C^\infty(T^2)$ with Fourier coefficients $\hat{f}(n)$ for $n \in \mathbb{Z}^2$, the twisted coefficients are given by

$$f^\#(\nu) = \sum_{Q(n) = \nu} (-1)^{n_1 n_2} \hat{f}(n).$$

**Conjecture** (Kurlberg-Rudnick [16]). As $N \to \infty$ through primes, the limiting distribution of the normalized matrix elements $F_j^{(N)}$ is that of the random variable

$$X_f = \sum_{\nu \neq 0} f^\#(\nu) \text{Tr}(U_\nu)$$
where $U_{\nu}$ are independently chosen random matrices in $SU(2)$ endowed with Haar probability measure.

As evidence, the second and fourth moment were computed to show agreement with this conjecture. In particular, the moment calculation implies that the limiting distribution is not Gaussian, in contrast to generic chaotic systems where the fluctuations are believed to be Gaussian [5, 7].

In this paper we further study the matrix elements for the cat map for composite $N$. In fact, it is sufficient to understand the case of prime powers (see [15, Section 4.1]), and so we restrict ourselves to this case. For $N$ a prime power, we give an explicit formula for the matrix elements as a weighted sum of certain exponential sums. We then use this formula to show that there are sequences of eigenfunctions such that the matrix elements decay like $N^{-1/3}$ rather than the expected rate of $N^{-1/2+\epsilon}$. We further show that when $N = p^k$ with $k > 1$, the matrix elements have a limiting distribution as $p \to \infty$. This distribution is not Gaussian and it is also different from the (conjectured) distribution for $k = 1$. Instead of behaving like traces of random elements from $SU(2)$, here the normalized matrix elements vanish for half of the eigenfunctions and for the rest they behave like $2\cos(\theta)$ where the angle is chosen at random.

1.1. Results. For every $N = p^k$ denote by

$$C(p^k) = \{B \in SL(2, \mathbb{Z}/p^k\mathbb{Z}) | AB = BA \pmod{p^k}\},$$

the group of Hecke operators. For $\nu \in \mathbb{Z}$ and $\chi$ a character of $C(p^k)$ define the exponential sum

$$E_{p^k}(\nu, \chi) = \sum_{x \in X(p^k)} c_{p^k}(\nu x)\chi(\beta(x)),$$

where

$$X(p^k) = \{x \in \mathbb{Z}/p^k\mathbb{Z}|(\text{Tr}(A)^2 - 4)x^2 \neq 1 \pmod{p}\}$$

and $\beta: X(p^k) \hookrightarrow C(p^k)$ is an injection of $X(p^k)$ into $C(p^k)$ given by a rational function (defined by (3.11)).

**Theorem 1.** For each prime power $p^k$, there is a subset $\hat{C}_0(p^k) \subset \hat{C}(p^k)$ of characters, with $\lim_{p \to \infty} \frac{|\hat{C}_0(p^k)|}{p^k} = 1$ such that

1. For any $\chi \in \hat{C}_0(p^k)$ there is a unique Hecke eigenfunction $\psi$, s.t., $\chi$ is a joint eigenvalue.
For this eigenfunction, and any elementary observable \( f_n(x) = \exp(2\pi i n \cdot x) \) with \( Q(n) \not\equiv 0 \pmod{p} \)
\[
\langle \text{Op}\_p(f_n)\psi, \psi \rangle = \pm \frac{(-1)^{n_1n_2}}{\#C(p^k)} E_{p^k}(\frac{Q(n)}{2}, \chi \chi_0),
\]
where \( \chi_0 \) is a fixed character of \( C(p^k) \) and the sign \( \pm \) depends on \( p, k \) but not on \( \psi \).

If we consider nontrivial prime powers (i.e., \( k > 1 \)) we can use elementary methods to evaluate these sums. In particular we find that there are matrix elements that decay much slower then the expected rate of \( N^{-\frac{1}{2} + \epsilon} \).

**Theorem 2.** There are smooth observables \( f \in C^\infty(T^2) \), and sequences of Hecke eigenfunctions satisfying
\[
|\langle \text{Op}_N(f)\psi_j, \psi_j \rangle - \int_{T^2} f| \gg N^{-\frac{1}{4}}.
\]

We note, however, that these exceptional matrix elements are quite rare, in the sense that for a fixed observable the number of matrix elements decaying slower then \( N^{-\frac{1}{2} + \epsilon} \) is bounded by \( O(p^{k-1}) \) (see Corollary 1).

**Remark 1.2.** In [17] Olofsson studied the supremum norm of Hecke eigenfunctions for the quantized cat map. He showed that for composite \( N \) the supremum norm can be of order \( N^{\frac{1}{4}} \), which is much larger then the case of \( N \) prime (or square free) where all Hecke eigenfunctions satisfy \( ||\psi||_\infty \ll N^\epsilon \) [8, 14]. Although the two phenomena look similar, there does not seem to be any apparent connection between them. At least in the sense that the eigenfunctions with large matrix elements are usually not the eigenfunctions with large supremum norm.

For nontrivial prime powers, we can also show that the exponential sums \( E_{p^k}(\nu, \chi) \) (and hence also the matrix elements) have a limiting distribution as \( p \to \infty \). (See [13] for similar results on twisted Kloosterman sums). To simplify the discussion we will assume from here on that the observable \( f \) is a trigonometric polynomial and let \( E^{(N)}_j \) be the normalized matrix element as in (1.1). Let \( \mu \) denote the measure on \([0, \pi]\) defined by
\[
\mu(f) = \frac{1}{2} f\left(\frac{\pi}{2}\right) + \frac{1}{2\pi} \int_0^\pi f(\theta) d\theta.
\]

**Theorem 3.** Let \( f \) be a trigonometric polynomial. For any \( k > 1 \), as \( p \to \infty \) through primes, the limiting distribution of the normalized
matrix elements $F_j^{(p^k)}$ is that of the random variable

$$Y_f = 2 \sum_{\nu \neq 0} f^\#(\nu) \cos(\theta_\nu)$$

where $\theta_\nu$ are independently chosen from $[0, \pi)$ with respect to the measure $\mu$.

Remark 1.3. As mentioned above, there can be exceptionally large matrix elements for which $F_j^{(N)} \gg N^{1/6}$ are not bounded. Such matrix elements would cause the moments (above the 6'th moment) to blow up as $N \to \infty$. Nevertheless, since the number of exceptional matrix elements is of limiting density zero, they do not influence the limiting distribution (see section 2.5 for more details).

1.2. Outline. The outline of the paper is as follows: In section 2 we provide some background on the cat map and its quantization and on the notion of a limit distributions. In section 3 we compute the formulas for the matrix elements proving Theorem 1. In section 4 we compute the exponential sums appearing in these formulas for non trivial prime powers, and establish the limiting distribution as $p \to \infty$. Then in section 5 we deduce both of the results on the matrix elements (Theorems 2 and 3) from the analysis of the exponential sums.

2. Background

The full details for the cat map and it’s quantization can be found in [15]. We briefly review the setup and go over our notation.

2.1. Classical dynamics. The classical dynamics are given by the iteration of a hyperbolic linear map $A \in \text{SL}(2, \mathbb{Z})$.

$$x = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{T}^2 \mapsto Ax \pmod{1}.$$  

Given an observable $f \in C^\infty(\mathbb{T}^2)$, the classical evolution defined by $A$ is $f \mapsto f \circ A$.

2.2. Quantum kinematics. For doing quantum mechanics on the torus, one takes Planck’s constant to be $1/N$, as the Hilbert space of states one takes $\mathcal{H}_N = L^2(\mathbb{Z}/N\mathbb{Z})$, where the inner product is given by:

$$\langle \phi, \psi \rangle = \frac{1}{N} \sum_{y \in \mathbb{Z}/N\mathbb{Z}} \overline{\phi(y)} \psi(y).$$
For $n = (n_1, n_2) \in \mathbb{Z}^2$ define elementary operators $T_N(n)$ acting on $\psi \in \mathcal{H}_N$ via:

$$T_N(n)\psi(y) = e_{2N}(n_1n_2)e_N(n_2y)\psi(y + n_1),$$

where $e_N(x) = e^{2\pi i x/N}$. For any smooth classical observable $f \in C^\infty(\mathbb{T}^2)$ with Fourier expansion $f(x) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)e^{2\pi in \cdot x}$, its quantization is given by

$$\text{Op}_N(f) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)T_N(n).$$

2.3. Quantum dynamics: For any $A \in \text{SL}(2, \mathbb{Z})$, we assign unitary operators $U_N(A)$, acting on $L^2(\mathbb{Z}/N\mathbb{Z})$ having the following important properties:

- "Exact Egorov": For $A \equiv I \pmod{2}$, and any $f \in C^\infty(\mathbb{T}^2)$
  $$U_N(A)^{-1}\text{Op}_N(f)U_N(A) = \text{Op}_N(f \circ A).$$

- The map $A \mapsto U_N(A)$ is a representation of $\text{SL}(2, \mathbb{Z}/N\mathbb{Z})$: If $C \equiv AB \pmod{N}$ then $U_N(A)U_N(B) = U_N(C)$.

We will make use of the following formula for $U_N(A)$, (valid for odd $N$ and any $A \in \text{SL}(2, \mathbb{Z})$) [12 Proposition 1.4].

$$U_N(A) = \frac{\sigma_N(A)}{|\ker_N(A-I)|N} \sum_{m \in (\mathbb{Z}/N\mathbb{Z})^2} \tilde{T}_N(m)\tilde{T}_N(-mA)$$

where $\sigma_N(A) = \text{Tr}(U_N(A))$ is the character of the representation,

$$|\ker_N(A-I)| = \# \{ n \in (\mathbb{Z}/N\mathbb{Z})^2 | n(A-I) \equiv 0 \pmod{N} \},$$

and $\tilde{T}_N(n) = (-1)^{n_1,n_2}T_N(n)$ are twisted elementary operators.

Remark 2.1. The twisted operators $\tilde{T}_N(n)$ have the convenient feature that

$$U_N(A)^*\tilde{T}_N(n)U_N(A) = \tilde{T}_N(nA)$$

for any $A \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ (without the parity condition).

2.4. Hecke eigenfunctions. Let $\alpha > \alpha^{-1}$ be the eigenvalues of $A$ in a (real) quadratic extension $K/\mathbb{Q}$. Then the vectors $\vec{v}_\pm = (c, \alpha^{\pm 1}a)$ are corresponding eigenvectors $\vec{v}_\pm A = \alpha^{\pm 1}\vec{v}_\pm$. Denote by $D = \text{Tr}(A)^2 - 4 \in \mathbb{Z}^+$ so that $\sqrt{D} = \alpha - \alpha^{-1}$. Consider the ring $\mathcal{O} = \mathbb{Z}[\alpha]$ and denote by $\iota: \mathcal{O} \to \text{Mat}(2, \mathbb{Z})$ the map sending $\beta = n + ma \mapsto B = n + mA$ (this map is a ring homomorphism as $\alpha$ and $A$ have the same minimal polynomial).

For any integer $N$ the norm map, $\mathcal{N}_{K/\mathbb{Q}} : K^* \to \mathbb{Q}^*$, induce a well defined map $\mathcal{N}_N : (\mathcal{O}/N\mathcal{O})^* \to (\mathbb{Z}/N\mathbb{Z})^*$. Let $C(N) = \ker \mathcal{N}_N$ be its kernel, then its image $\iota(C(N)) \subset \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ is a commutative
subgroup of $\text{SL}(2,\mathbb{Z}/N\mathbb{Z})$ that commutes with $A \pmod{N}$. The Hecke operators are then $\{U_N(B) | B \in \iota(C(N))\}$, and Hecke eigenfunctions are joint eigenfunctions of $U_N(A)$ and all the Hecke operators.

The eigenvalues corresponding to each Hecke eigenfunction define a character $\chi$ of $C(N)$ i.e., $U_N(\iota(\beta))\psi = \chi(\beta)\psi$. We can thus decompose our Hilbert space into a direct sum of joint eigenspaces $H_N = \bigoplus \mathcal{H}_\chi$, parameterized by the characters of $C(N)$. We say that a character $\chi$ appears with multiplicity one in the decomposition when the corresponding eigenspace is one dimensional.

2.5. Limit distribution. We recall the notion of a limiting distribution for a sequence of points on the line. For each $N$ let $\{F_j^{(N)}\}_{j=1}^N$ be a set of points on the line. We say that these points have a limiting distribution $Y$ (where $Y : \Omega \rightarrow \mathbb{R}$ is some random variable on a probability space $\Omega$) if for any segment $[a, b] \subset \mathbb{R}$ the limit

$$\lim_{N \rightarrow \infty} \frac{\# \left\{ j | a \leq F_j^{(N)} \leq b \right\}}{N} = \text{Prob}(Y \in [a, b]).$$

From this definition it is strait forward that making an arbitrary change in a density zero set of points (i.e., changing $S_N$ points for each $N$ with $\frac{S_N}{N} \rightarrow 0$), does not affect the limiting distribution.

An equivalent condition for having a limiting distribution $Y$, is that for any continues bounded function $g$ the average $\frac{1}{N} \sum_j g(F_j^{(N)})$ converges as $N \rightarrow \infty$ to $\int_\Omega g(Y(\omega))d\omega$. Note that the condition that the test function $g$ is bounded is necessary unless both the variable $Y$ and the points $F_j^{(N)}$ are uniformly bounded. In particular, if the points $F_j^{(N)}$ are not uniformly bounded then their moments don’t necessarily converge to the moments of $Y$.

2.6. Notation. We use the notation $e(x) = e^{2\pi ix}$. For any $N \in \mathbb{N}$ we denote by $e_N(\cdot)$ the character of $\mathbb{Z}/N\mathbb{Z}$ given by $e_N(x) = e(\frac{x}{N})$. When there is no risk of confusion we will slightly abuse notation and write $e_N(\frac{a}{b})$ for $e_N(ab^{-1})$ (where $b^{-1}$ denotes the inverse of $b$ modulo $N$). For example, for $N$ odd and $a \in \mathbb{Z}$ we may write $e_{2N}(a) = (-1)^a e_N(\frac{a}{2})$.

3. Formulas for Matrix Elements

For $N$ a prime power we give formulas for the matrix elements of elementary observables explicitly as exponential sums. When $N$ is prime these formulas appeared in [16] (for primes that split in $\mathcal{O}$) and in [12] (for inert primes).
We will make use of the following parametrization of the Hecke operators. For any integer \(1 \leq l \leq k\) we define subgroups \(C_p(k, l) \subset C(p^k)\) by
\[
C_p(k, l) = \{ \beta \in C(p^k) | \beta \equiv 1 \pmod{p^l} \}.
\]
For notational convenience we will also define \(C_p(k, k + 1) = \{1\}\). Let
\[
X(p^k) = \{ x \in \mathbb{Z}/p^k\mathbb{Z} | Dx^2 \neq 1 \pmod{p} \}
\]
then the map
\[
\beta(x) = \frac{\sqrt{D}x + 1}{\sqrt{D}x - 1}
\]
is a bijection between \(X(p^k)\) and \(C(p^k) \setminus C_p(k, 1)\) with inverse map given by \(x = \frac{1 + \beta(x)}{\sqrt{D}(1 - \beta(x))} \pmod{p^k}\) (note that for \(\beta \neq 1 \pmod{p}\) the inverse map is indeed well defined). For every character \(\chi\) of \(C(p^k)\) and any \(\nu \in (\mathbb{Z}/p^k\mathbb{Z})^*\) we have the exponential sum
\[
E_{p^k}(\nu, \chi) = \sum_{x \in X(p^k)} e_{p^k}(\nu x)\chi(\beta(x)).
\]
To prove Theorem 1 we will show that for any \(n \in \mathbb{Z}^2\) with \(Q(n) = \nu \neq 0 \pmod{p}\), and for every character \(\chi\) of \(C(p^k)\) that appears with multiplicity one, the corresponding matrix element is given by
\[
\langle T_{p^k}(n)\psi, \psi \rangle = \pm\frac{1}{\#C(p^k)} \sum_{x \in X(p^k)} e_{p^k}(\nu x)\chi_0(\beta(x)),
\]
(where \(\chi_0\) is a fixed character of \(C(p^k)\) and the sign is \(-1\) when \(p\) is inert and \(k\) is odd and \(+1\) otherwise). We can then take our set \(\hat{C}_0(p^k)\) to be the set of characters appearing with multiplicity one. This set is of order \(p^k\) if \(p\) is inert (Lemma 3.2) and of order \(p^k - p^{k-1}\) if \(p\) splits (Lemma 3.1). Hence, indeed \(\frac{\hat{C}_0(p^k)}{p^k} = 1 + O(\frac{1}{p})\). We will compute the matrix elements separately for the inert and split cases.

3.1. **Split case.** When \(p\) is split, we can give explicit formulas for the Hecke eigenfunctions and use them to compute the matrix elements. Since we assume that \(p\) splits in \(\mathfrak{O}\), there is a matrix \(M \in \text{SL}(2, \mathbb{Z}/p^k\mathbb{Z})\) satisfying that \(M^{-1}AM = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \pmod{p^k}\). Consequently, the Hecke group is given by
\[
C(p^k) = \left\{ M \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} M^{-1} | x \in (\mathbb{Z}/p^k\mathbb{Z})^* \right\},
\]
which is naturally isomorphic to \((\mathbb{Z}/p^k\mathbb{Z})^*\). We recall that

\[(3.2) \quad U_{p^k}(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix})\psi(y) = \chi_0(x)\psi(xy),\]

where \(\chi_0\) is a fixed character of \((\mathbb{Z}/p^k\mathbb{Z})^*\) \([15\text{, Section 4.3}].\)

**Lemma 3.1.** For any character \(\chi\) of \((\mathbb{Z}/p^k\mathbb{Z})^*\) (extended to a function on \(\mathbb{Z}/p^k\mathbb{Z}\) by setting \(\chi(px) = 0\)), the function \(\psi = \sqrt{\frac{p}{p^k-1}}U_{p^k}(M)\chi\) is a normalized joint eigenfunction of all Hecke eigenfunctions with eigenvalue \(\chi\). Furthermore, if \(\chi\) is not trivial on the subgroup \(C_p(k, k-1)\) then this is the only eigenfunction.

**Proof.** The first assertion is an immediate consequence of \((3.2)\). For the second part, assume that \(\psi\) is an eigenfunction with eigenvalue \(\chi\), and that \(\chi\) is not trivial on \(C_p(k, k-1)\). Then there is \(x_0 \in C_p(k, k-1)\) with \(\chi(x_0) \neq 1\). Now, let \(\phi = U_{p^k}(M)^{-1}\psi\), then for any \(x \in (\mathbb{Z}/p^k\mathbb{Z})^*\),

\[\chi\chi_0(x)\phi(y) = U_{p^k}(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix})\phi(y) = \chi_0(x)\phi(xy),\]

hence \(\phi(xy) = \chi(x)\phi(y)\). For any \(y \equiv 0 \pmod{p}\) we have that \(x_0y \equiv y \pmod{p^k}\) (as \(x_0 \equiv 1 \pmod{p^{k-1}}\)). Consequently, \(\phi(y) = \phi(x_0y) = \chi(x_0)\phi(y)\) implying that \(\phi(y) = 0\). On the other hand, for \(y \not\equiv 0 \pmod{p}\) we have \(\phi(y) = \chi(y)\phi(1)\) so \(\phi\) is uniquely determined (up to normalization).

**Remark 3.1.** In the case that the character \(\chi\) is trivial on the group \(C_p(k, l)\) (but not on \(C_p(k, l-1)\)) then the above argument implies that the corresponding eigenspace is of dimension \(k - l + 1\).

**Proof of Theorem 1 (split case).** Let \(\chi\) be a character not trivial on \(C_p(k, k-1)\). Then, \(\psi = \sqrt{\frac{p}{p^k-1}}U_{p^k}(M)\chi\) is an eigenfunction with character \(\chi\), where \(A = M\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}M^{-1} \pmod{p^k}\). Consequently, for any (twisted) elementary observable

\[\langle \hat{T}_{p^k}(n)\psi, \psi \rangle = \frac{p}{p-1}\langle U_{p^k}(M)^*\hat{T}_{p^k}(n)U_{p^k}(M)\chi, \chi \rangle = \frac{p}{p-1}\langle \hat{T}_{p^k}(m)\chi, \chi \rangle\]

with \(m = nM \pmod{p^k}\).
Now, let \(d = y - y^{-1}\) so that \(d^2 \equiv D \pmod{p^k}\) (recall \(\text{Tr}(A) \equiv y + y^{-1} \pmod{p^k}\)). Then
\[
\langle \tilde{T}_{p^k}(m)\chi, \chi \rangle = \frac{1}{p^k} \sum_{x \in \mathbb{Z}/p^k\mathbb{Z}} e_{p^k}(\frac{m_1m_2}{2})e_{p^k}(m_2x)\chi(\frac{x + m_1}{x})
\]
\[
= \frac{1}{p^k} \sum_{t \in X(p^k)} e_{p^k}(\frac{dm_1m_2}{2}t)\chi(\frac{dt + 1}{dt - 1})
\]
where we made the change of variables \(2x = m_1(dt - 1)\). Finally, notice that for \(m = nM \pmod{p^k}\) we have that
\[
Q(n) = \omega(nA, n) \equiv m_1m_2(y - y^{-1}) \equiv dm_1m_2 \pmod{p^k}.
\]
Hence indeed
\[
\langle \tilde{T}_{p^k}(n)\psi, \psi \rangle = \frac{1}{\#C(p^k)} \sum_{x \in X(p^k)} e_{p^k}(\frac{Q(n)x}{2})\chi(\beta(x))
\]
\[\square\]

3.2. **Inert case.** First we show that for \(p\) inert, any joint eigenspace is one dimensional.

**Lemma 3.2.** For \(N = p^k\) and \(p\) inert, the dimension of any joint eigenspace satisfies \(\dim \mathcal{H}_\chi \leq 1\).

**Proof.** The trace of the quantum propagators satisfy [12, Corollary 1.6]
\[
|\text{Tr}(U_{p^k}(B))|^2 = \# \left\{ n \in (\mathbb{Z}/p^k\mathbb{Z})^2 : n(B - I) \equiv 0 \pmod{p^k} \right\}.
\]
For \(p\) inert, the group \(C(p^k)\) is of order \(\#C(p^k) = p^{k-1}(p + 1)\), and the groups
\[
C_p(k, l) = \left\{ \beta \in C(p^k) | \beta \equiv 1 \pmod{p^l} \right\},
\]
are of order \(\#C_p(k, l) = \frac{\#C(p^k)}{\#C(p^l)} = p^{k-l}\). Moreover, for any \(\beta \in C_p(k, l) \setminus C_p(k, l + 1)\) we have \(|\text{Tr}(U_{p^k}(\iota(\beta)))|^2 = p^{2l}\). Consequently
\[
\sum_{\beta \in C(p^k)} |\text{Tr}(U_{p^k}(\iota(\beta))))|^2 = p^k + \sum_{l=1}^{k-1} \sum_{\beta \in C_p(k, l) \setminus C_p(k, l+1)} |\text{Tr}(U_{p^k}(\iota(\beta))))|^2 + p^{2k}
\]
\[
= p^k + \sum_{l=1}^{k-1} (p^{k-l} - p^{k-l-1})p^{2l} + p^{2k}
\]
\[
= p^k + p^{2k-1} - p^k + p^{2k} = p^k \#C(p^k)
\]
On the other hand, if we denote by $n_\chi = \dim \mathcal{H}_\chi$ then
\[
\frac{1}{\# C(p^k)} \sum_{\beta \in C(p^k)} |\Tr(U_{p^k}(\iota(\beta)))|^2 = \sum_\chi n_\chi^2.
\]
Comparing the two expressions we get
\[
\sum_\chi n_\chi^2 = \frac{1}{\# C(p^k)} \sum_{\beta \in C(p^k)} |\Tr(U_{p^k}(\iota(\beta)))|^2 = p^k = \dim \mathcal{H} = \sum_\chi n_\chi.
\]
Since $n_\chi$ are non negative integers this implies $n_\chi \leq 1$. \hfill \qedsymbol

After establishing this fact, following the idea of Gurevich and Hadani [9], we can write the matrix elements of elementary observables as
\[
\langle T_{p^k}(n)\psi_j, \psi_j \rangle = \Tr(T_{p^k}(n)\mathcal{P}_{\chi_j}),
\]
with
\[
\mathcal{P}_{\chi_j} = \frac{1}{\# C(p^k)} \sum_{\beta \in C(p^k)} U_{p^k}(\iota(\beta))\bar{\chi}_j
\]
the projection operator to the (one dimensional) eigenspace spanned by $\psi_j$. We then use formula (2.2) for $U_{p^k}(\iota(\beta))$ in order compute $\Tr(T_{p^k}(n)U_{p^k}(\iota(\beta)))$. However, in order to do this we first need to give a formula for the character of the representation $\sigma(B) = \Tr(U_{p^k}(B))$ (appearing in (2.2)), for any $B \in \iota(C(p^k))$.

**Proposition 3.3.** There is a character $\chi_0 \in \hat{\mathcal{C}}(p^k)$ such that for any $\beta \in C(p^k)$, we have
\[
\Tr(U_{p^k}(\iota(\beta))) = (-1)^k(-p)^l\chi_0(\beta),
\]
with $1 \leq l \leq k$ the maximal integer such that $\beta \equiv 1 \pmod{p^l}$.

**Proof.** For any $1 \leq l \leq k + 1$ consider the subgroup of characters
\[
\hat{\mathcal{C}}(l)(p^k) = \left\{ \chi \in \hat{\mathcal{C}}(p^k) | \chi(\beta) = 1, \forall \beta \in C_p(k, l) \right\}.
\]
For $1 \leq l \leq k$, the group $\hat{\mathcal{C}}(l)(p^k)$ is the kernel of the restriction map from $\hat{\mathcal{C}}(p^k)$ to $\hat{\mathcal{C}}_p(k, l)$ and hence of order $\# C(p^k) = p^{l-1}(p + 1)$ (and for $l = k + 1$ we have $\hat{\mathcal{C}}(k+1)(p^k) = \hat{\mathcal{C}}(k)(p^k) = C(p^k)$).

We will first prove the following: For each $1 \leq l \leq k + 1$ there is a character $\chi_l \in \hat{\mathcal{C}}(l)(p^k)$ and a subset $S_l \subset \hat{\mathcal{C}}(l-1)(p^k)$ of order $\# S_l = p^{l-1}$ such that for any $\beta \in C(p^k) \setminus C_p(k, l)$,
\[
\Tr(U_{p^k}(\iota(\beta))) = (-1)^{k+l+1}\chi_l\chi_{l+1}\cdots\chi_k(\beta)\sum_{\chi \in S_l} \chi(\beta).
\]
First for \( l = k + 1 \) we take the character to be the trivial character and the set \( S_{k+1} \subset \hat{C}(p^k) = \hat{C}(p^k) \) to be the set of characters that appear in the decomposition of \( H_{p^k} \) (there are \( p^k \) such characters each appearing with multiplicity one). Then indeed \( \text{Tr}(U^k(\iota(\beta))) = \sum_{\chi \in S_{k+1}} \chi(\beta) \).

If \( k = 1 \) the sum is over all but one of the characters, say \( \chi_0 \in \hat{C}(p) \), and hence \( \text{Tr}(U^k(\iota(\beta))) = -\chi_0(\beta) \) as claimed. For \( k > 1 \) we proceed by induction as follows.

We assume the assertion is true for \( 1 < l \leq k + 1 \) and show that it is true for \( l - 1 \). For any \( \beta \in C(p^k) \setminus C_p(k, l) \), by our assumption

\[
\text{Tr}(U^k(\iota(\beta))) = (-1)^{k+l+1} \chi_l \chi_{l+1} \cdots \chi_k(\beta) \sum_{\chi \in S_l} \chi(\beta)
\]

with \( S_l \subset \hat{C}^{(l-1)}(p^k) \) of order \( \#S_l = p^{l-1} \). The order \( \#\hat{C}^{(l-1)}(p^k) = p^{l-1} \) hence the complement \( S_l^c \) in \( \hat{C}^{(l-1)}(p^k) \) is of order \( p^{l-2} \). Now, if \( \beta \notin C_p(k, l - 1) \) then the sum over all characters in \( \hat{C}^{(l-1)}(p^k) \) vanish, and hence \( \sum_{\chi \in S_l} \chi(\beta) = -\sum_{\chi \in S_l^c} \chi(\beta) \). We thus have that

\[
\text{Tr}(U^k(\iota(\beta))) = (-1)^{k+l} \chi_l \chi_{l+1} \cdots \chi_k(\beta) \sum_{\chi \in S_l^c} \chi(\beta).
\]

On the other hand, for \( \beta \in C_p(k, l - 2) \setminus C_p(k, l - 1) \) we have that \( |\text{Tr}(U^k(\iota(\beta)))| = p^{l-2} \), which could happen only if \( \chi(\beta) \) takes the same value for all \( \chi \in S_l^c \). Now take \( \chi_{l-1} \) to be any character from \( S_l^c \) and let \( S_{l-1} = \chi_{l-1}^{-1} S_l^c \). Then \( S_{l-1} \subset \hat{C}^{(l-2)}(p^k) \) is of order \( p^{l-2} \) and

\[
\text{Tr}(U^k(\iota(\beta))) = (-1)^{k+l+1} \chi_l \chi_{l-1} \chi_{l+1} \cdots \chi_k(\beta) \sum_{\chi \in S_{l-1}} \chi(\beta).
\]

Now, let \( \chi_0 = \chi_1 \cdot \chi_2 \cdots \chi_k \) and let \( \beta \in C_p(k, l) \setminus C_p(k, l + 1) \). Since \( \beta \notin C_p(k, l + 1) \) we have,

\[
\text{Tr}(U^k(\iota(\beta))) = (-1)^{k+l} \chi_{l-1} \chi_{l+1} \cdots \chi_k(\beta) \sum_{\chi \in S_{l+1}} \chi(\beta).
\]

On the other hand we also assume \( \beta \in C_p(k, l) \), hence, for all \( \chi \in S_{l+1} \subset \hat{C}^{(l)}(p^k) \) we have \( \chi(\beta) = 1 \) implying that \( \sum_{\chi \in S_{l+1}} \chi(\beta) = \#S_{l+1} = p^l \). Also for any \( m \leq l \), \( \chi_m \in C^{(m)}(p^k) \subset C^{(l)}(p^k) \), so \( \chi_m(\beta) = 1 \). We thus get that indeed

\[
\text{Tr}(U^k(\iota(\beta))) = (-1)^k (-p)^l \chi_0(\beta).
\]
Proposition 3.4. Let \( n \in \mathbb{Z}^2 \) and \( B \in \iota(C(p^k)) \). For \( B \equiv I \pmod{p} \) the trace \( \text{Tr}(\tilde{T}_{p^k}(n)U_{p^k}(B)) = 0 \). Otherwise, there is \( x \in X(p^k) \) such that \( B = \iota(\beta(x)) \) and

\[
\text{Tr}(\tilde{T}_{p^k}(n)U_{p^k}(B)) = (-1)^k \chi_0(\beta(x)) e_{p^k}(\frac{Q(n)x}{2}).
\]

Proof. Use formula (2.2) for \( U_{p^k}(B) \) to get that

\[
\text{Tr}(\tilde{T}_{p^k}(n)U_{p^k}(B)) = \frac{\sigma_{p^k}(B)}{|\ker_B(B - I)|p^k} \sum_{m \in (\mathbb{Z}/p^k\mathbb{Z})^2} \text{Tr}(\tilde{T}_{p^k}(n)\tilde{T}_{p^k}(m)\tilde{T}_{p^k}(-mB))
\]

Note that up to a phase \( \tilde{T}_{p^k}(n)\tilde{T}_{p^k}(m)\tilde{T}_{p^k}(-mB) = e^{i\alpha}\tilde{T}_{p^k}(n - mB) \) and recall that \( \text{Tr}(\tilde{T}_{p^k}(n)) = 0 \) unless \( n \equiv 0 \pmod{p^k} \) (see e.g., [15, Lemma 4]). Hence, the only summand that does not vanish is the one satisfying \( n = m(B - I) \pmod{p^k} \). We can assume \( n \neq 0 \pmod{p} \), so that the trace vanishes whenever \( B \equiv I \pmod{p} \). Otherwise, \( B = \iota(\beta) \) for some \( \beta \in C(p^k) \setminus C_p(k, 1) \) and \( \sigma_{p^k}(B) = (-1)^k \chi_0(\beta) \) so that

\[
\text{Tr}(\tilde{T}_{p^k}(n)U_{p^k}(B)) = (-1)^k \chi_0(\beta)e_{p^k}(\frac{\omega(m, mB)}{2}),
\]

with \( m = n(B - I)^{-1} \pmod{p^k} \).

Now recall the parametrization \( C(p^k) \setminus C_p(k, 1) = \{ \beta(x)|x \in X(p^k) \} \), with \( \beta(x) = \frac{\sqrt{Dx} + 1}{\sqrt{Dx} - 1} \). We claim that for \( B = \iota(\beta(x)) \) and \( m = nB \) we have that \( \omega(m, mB) = Q(n)x \). To show this substitute \( \beta(x) - 1)^{-1} = \frac{\sqrt{Dx} - 1}{2} \) and \( (\beta(x) - 1)^{-1} = \frac{\sqrt{Dx} + 1}{2} \). Consequently we get

\[
\omega(m, mB) = \omega \left( nu\left(\frac{\sqrt{Dx} - 1}{2}\right), nu\left(\frac{\sqrt{Dx} + 1}{2}\right)\right) = \frac{x}{2} \omega(nu(\sqrt{D}), n).
\]

Recall that \( \sqrt{D} = (\alpha - \alpha^{-1}) \) so that indeed

\[
\omega(nu(\sqrt{D}), n) = \omega(n(A - A^{-1}), n) = 2\omega(nA, n) = 2Q(n).
\]

\( \square \)

Proof of Theorem 1 (inert case). For every character \( \chi \) let

\[
P_\chi = \frac{1}{\#C(p^k)} \sum_{B \in C(p^k)} U_{p^k}(B)\bar{\chi}(B),
\]
be the projection operator to the (one dimensional) eigenspace corresponding to $\chi$. Let $\psi$ be the corresponding Hecke eigenfunction. Then

$$\langle \tilde{T}_{p^k}(n)\psi, \psi \rangle = \text{Tr}(\tilde{T}_{p^k}(n)P_\chi) = \frac{1}{\#C(p^k)} \sum_{B \in C(p^k)} \text{Tr}(\tilde{T}_{p^k}(n)U_{p^k}(B))\bar{\chi}(B).$$

Now from the Proposition 3.4

$$\text{Tr}(\tilde{T}_{p^k}(n)U_{p^k}(B)) = (-1)^k \chi_0(B)e_{p^k}(\frac{-Q(n)x}{2}),$$

implying that

$$\langle \tilde{T}_{p^k}(n)\psi, \psi \rangle = (-1)^k \frac{\#C(p^k)}{\#C(p^k)} e_{p^k}(\frac{-Q(n)x}{2}) \chi_0(\beta(x))\bar{\chi}(\beta(x)).$$

After a change of variables $x \mapsto -x$ we get

$$\langle \tilde{T}_{p^k}(n)\psi, \psi \rangle = (-1)^k \frac{\#C(p^k)}{\#C(p^k)} E_{p^k}(Q(n)/2, \chi\bar{\chi}_0)$$

$\square$

4. Analysis of the Exponential Sums

In this section we compute the exponential sums $E_{p^k}(\nu, \chi)$ for any prime power $k > 1$. This can be done using elementary methods (see, e.g., [11, section 12.3] or [1, Chapter 1.6]), however, since the setup here is slightly different we will perform this computation in full. We then evaluate all mixed moments of these exponential sums to deduce their limiting distribution.

4.1. Computation of exponential sums. For $\nu \in \mathbb{Z}/p^k\mathbb{Z}$ its “square root” (modulo $p^k$) is the set

$$\text{Sq}(\nu, p^k) = \{ x \in \mathbb{Z}/p^k\mathbb{Z} | x^2 = \nu \pmod{p^k} \}.$$

Note that for $\nu \not\equiv 0 \pmod{p}$ this set contains two or zero elements, for $\nu \equiv 0 \pmod{p^k}$ it contains $p^{k/2}$ elements (and for $\nu = p^l\tilde{\nu}$ with $\tilde{\nu}$ coprime to $p$ it contains zero or $2p^{l/2}$ elements).

Proposition 4.1. For $k = 2l$ even

$$E_{p^k}(\nu, \chi) = p^l \sum_{x \in \text{Sq}(\frac{2\chi+\nu}{p^l}, p^l) \atop Dx^2 \not\equiv 1(p)} e_{p^k}(\nu x)\chi(\beta(x)),$$

where $t_\chi \in \mathbb{Z}/p^l\mathbb{Z}$ satisfies that $\chi(1 + p^l\sqrt{D}x) = e_{p^l}(t_\chi x)$
For \( k = 2l + 1 \) odd

\[
E_{p^k}(\nu, \chi) = p^l \sum_{x \in \text{Sq}(\frac{2l + \nu}{\nu}, p^l)} e_{p^k}(\nu x) \chi(\beta(x)) \mathcal{G}(x)
\]

where \( t_\chi \in \mathbb{Z}/p^{l+1}\mathbb{Z} \) satisfies \( \chi(1 + p^l \sqrt{D}x + p^{2l} \frac{Dx^2}{2}) = e_{p^{l+1}}(t_\chi x) \), and \( \mathcal{G}(x) \) is the Gauss sum given by

\[
\mathcal{G}(x) = \sum_{y \in \mathbb{Z}/p\mathbb{Z}} e_p(f(x)y^2 + g(x)y),
\]

with \( f(x) = \frac{2t_\chi x}{Dx^2 + 1} \) and \( g(x) = p^{-l}(\nu - t_\chi \frac{2}{Dx^2 + 1}) \). (Notice that for \( x \in \text{Sq}(\frac{2l + \nu}{\nu}, p^l) \) we have \( (\nu - t_\chi \frac{2}{Dx^2 + 1}) \equiv 0 \) (mod \( p^l \)), hence \( p^{-l}(\nu - t_\chi \frac{2}{Dx^2 + 1}) \) gives a well defined residue modulo \( p \).

**Proof.** First for \( k = 2l \), write the sum as

\[
E_{p^k}(\nu, \chi) = \sum_{x \in X(p^l)} \sum_{y \in \mathbb{Z}/p\mathbb{Z}} e_{p^k}(\nu(x + p^l y)) \chi(\beta(x + p^l y)).
\]

Replace \( \beta(x + p^l y) \equiv \beta(x)(1 + \frac{\nu}{\beta}(x)p^l y) \) (mod \( p^{2l} \)) to get

\[
E_{p^k}(\nu, \chi) = \sum_{x \in X(p^l)} e_{p^k}(\nu x) \chi(\beta(x)) \sum_{y \in \mathbb{Z}/p\mathbb{Z}} e_{p^l}(\nu y) \chi(1 + p^l \frac{\nu}{\beta}(x)y).
\]

Differentiating \( \beta(x) = \frac{\sqrt{D}x + 1}{\sqrt{D}x - 1} \) we get \( \frac{\nu}{\beta}(x) = -\frac{2\sqrt{D}}{Dx^2 - 1} \), so that the inner sum takes the form

\[
\sum_{y \in \mathbb{Z}/p\mathbb{Z}} e_{p^l}(\nu y) \chi(1 - p^l \sqrt{D} \frac{2y}{Dx^2 - 1}).
\]

The map \( x \mapsto 1 + p^l \sqrt{D}x \) is an isomorphism of \( \mathbb{Z}/p^l\mathbb{Z} \) and \( C_p(2l, l) \). Hence, for any character \( \chi \) of \( C(p^{2l}) \) there is \( t_\chi \in \mathbb{Z}/p^l\mathbb{Z} \) such that \( \chi(1 + p^l \sqrt{D}x) = e_{p^l}(t_\chi x) \). We can thus write the inner sum as

\[
\sum_{y \in \mathbb{Z}/p\mathbb{Z}} e_{p^l}(\nu y) e_{p^l}(\nu y \frac{2y}{Dx^2 - 1}) = \sum_{y \in \mathbb{Z}/p\mathbb{Z}} e_{p^l}(\nu y \frac{2t_\chi}{Dx^2 - 1} y).
\]

This sum vanishes unless \( x \in \text{Sq}(\frac{2l + \nu}{\nu}, p^l) \) in which case it is equal \( p^l \).

Now for \( k = 2l + 1 \), we start again by writing

\[
E_{p^k}(\nu, \chi) = \sum_{x \in X(p^l)} \sum_{y \in \mathbb{Z}/p\mathbb{Z}} e_{p^k}(\nu(x + p^l y)) \chi(\beta(x + p^l y)),
\]
and replace

\[ \beta(x + p^l y) \equiv \beta(x)(1 + \frac{\beta'(x)p^l y + \frac{1}{2} \beta''(x)p^2 y^2}{\beta(x)}) \pmod{p^{2l+1}}. \]

It is easy to verify that the map \( x \mapsto 1 + p^l \sqrt{Dx + p^2 Dx^2} \) is an isomorphism of \( \mathbb{Z}/p^{l+1}\mathbb{Z} \) with \( C_p(2l + 1, l) \). Consequently, for every character \( \chi \) of \( C(p^k) \), there is \( t_\chi \in \mathbb{Z}/p^{l+1}\mathbb{Z} \) such that

\[ \chi(1 + p^l \sqrt{Dx + p^2 Dx^2}) = e_{p^{l+1}}(t_\chi x). \]

By differentiating \( \beta(x) = \sqrt{Dx + 1} \sqrt{Dx} - 1 \) (twice), we get that

\[ \frac{\beta'(x)y + p^l \frac{\beta''(x)y^2}{2\beta(x)}}{\beta(x)} = \sqrt{D} \left( -\frac{2(y - xp^l y^2)}{Dx^2 - 1} \right) + p^l \frac{2(y - xp^l y^2)}{Dx^2 - 1} \] (mod \( p^{l+1} \)), implying that the inner sum is of the form

\[ \sum_{y \in \mathbb{Z}/p^{l+1}\mathbb{Z}} e_{p^{l+1}}(\nu y - \frac{2t_\chi}{Dx^2 - 1} y + p^l \frac{2t_\chi x}{Dx^2 - 1} y^2). \]

This sum vanishes unless \( x \in \text{Sq}(\frac{2t_\chi + \nu}{\nu D}, p^l) \). To see this make a change of summation variable \( y \mapsto y + p \) to get that

\[ \sum_{y \in \mathbb{Z}/p^{l+1}\mathbb{Z}} e_{p^{l+1}}(\nu y - \frac{2t_\chi}{Dx^2 - 1} y + p^l \frac{2t_\chi x}{Dx^2 - 1} y^2) = \]

\[ = e_{p^l}(\nu - \frac{2t_\chi}{Dx^2 - 1}) \sum_{y \in \mathbb{Z}/p^{l+1}\mathbb{Z}} e_{p^{l+1}}(\nu y - \frac{2t_\chi}{Dx^2 - 1} y + p^l \frac{2t_\chi x}{Dx^2 - 1} y^2) \]

Now unless \( \nu - \frac{2t_\chi}{Dx^2 - 1} \equiv 0 \pmod{p} \) we have that \( e_{p^l}(\nu - \frac{2t_\chi}{Dx^2 - 1}) \neq 1 \), implying that the sum must vanish. For \( x \in \text{Sq}(\frac{2t_\chi + \nu}{\nu D}, p^l) \) the inner sum given by \( p^l \) times the Gauss sum

\[ \mathcal{G}(x) = \sum_{y \in \mathbb{Z}/p\mathbb{Z}} e_p(f(x)y^2 + g(x)y). \]

\[ \square \]

In particular this computation implies that for most characters the exponential sum has square root cancelation.

**Corollary 1.** For any character \( \chi \) with \( 2t_\chi \not\equiv -\nu \pmod{p} \) there is \( \theta = \theta(\chi, \nu) \in [0, \pi) \) such that \( E_{p^k}(\nu, \chi) = p^{k/2} \cos(\theta(\nu, \chi)) \).
Proof. The condition $2t_\chi \not\equiv -\nu \pmod{p}$ implies $\frac{2t_\chi + \nu}{\nu D} \not\equiv 0 \pmod{p}$. Hence for any $1 \leq l \leq k$

$$\# Sq(\frac{2t_\chi + \nu}{\nu D}, p^l) = \begin{cases} 2 & \frac{2t_\chi + \nu}{\nu D} = \square \pmod{p} \\ 0 & \text{otherwise} \end{cases}$$

Now for $k = 2l$ even, recall that $E_{p^k}(\nu, \chi) = p^l \sum_{x \in Sq(\frac{2t_\chi + \nu}{\nu D}, p^l)} e^{p^k(\nu x)} \chi(\beta(x))$. If $\frac{2t_\chi + \nu}{\nu D} = \square \pmod{p}$ this sum vanishes. Otherwise it is a sum over two elements of absolute value $p^l = p^{k/2}$ hence indeed $E_{p^k}(\nu, \chi) = 2p^{k/2} \cos(\theta(\nu, \chi))$.

For $k = 2l + 1$ odd we have $E_{p^k}(\nu, \chi) = p^l \sum_{x \in Sq(\frac{2t_\chi + \nu}{\nu D}, p^l)} e^{p^k(\nu x)} \chi(\beta(x)) G(x)$. The condition $2t_\chi \not\equiv -\nu \pmod{p}$ implies that the Gauss sum $G(x)$ is not a trivial sum and hence of order $\sqrt{p}$. As before, the sum $E_{p^k}(\nu, \chi)$ either vanishes (if $\frac{2t_\chi + \nu}{\nu D} = \square \pmod{p}$) or it is a sum of two elements of absolute value $p^{l+\frac{3}{2}} = p^{k/2}$. □

On the other hand, if $2t_\chi \equiv -\nu \pmod{p^{2l'}}$ for some $l' \leq \frac{k}{2}$ then the sum contains $p^{l'}$ elements and could be much larger. Moreover, in the odd case, if $2t_\chi \equiv -\nu \pmod{p^{l'+1}}$ then $g(x) \equiv 0 \pmod{p}$. Also, in this case any $x \in Sq(\frac{2t_\chi + \nu}{\nu D}, p^l)$ satisfies $x \equiv 0 \pmod{p}$ and hence also $f(x) = \frac{2t_\chi x}{Dx^2 - 1} \equiv 0 \pmod{p}$. So that in this case the Gauss sum $|G(x)| = p$ rather then $\sqrt{p}$ and the sum is even bigger. In particular we get

**Corollary 2.** For $\nu \in \mathbb{Z}$ and $\chi \in \hat{C}(p^3)$ with $t_\chi \equiv -\nu \pmod{p^2}$, $|E_{p^l}(\nu, \chi)| = p^2$.

**4.2. Equidistribution of exponential sums.** We now show that as $p \to \infty$ the normalized exponential sums $p^{-k/2}E_{p^k}(\nu, \chi)$ become equidistributed with respect to the measure

$$\mu(f) = \frac{1}{2} f\left(\frac{\pi}{2}\right) + \frac{1}{2\pi} \int_0^\pi f(\theta) d\theta.$$  

For fixed $\nu$ and a character $\chi$, if $\frac{2t_\chi + \nu}{\nu D}$ is not a square modulo $p$ then the sum $E_{p^k}(\nu, \chi) = 0$ (or equivalently $\theta(\nu, \chi) = \frac{\pi}{2}$). The following lemma shows that this happens for roughly half the characters, and that this behavior is independent for different values of $\nu$. 

Lemma 4.2. Fix a finite set \( \bar{\nu} = \{\nu_1, \ldots, \nu_r\} \) of nonzero distinct integers. Then,
\[
\frac{1}{p} \# \left\{ t \in \mathbb{Z}/p\mathbb{Z} \mid \forall j, \frac{t - \nu_j}{D_{\nu_j}} \equiv \square \pmod{p} \right\} = \frac{1}{2^r} + O\left(\frac{1}{\sqrt{p}}\right).
\]

Proof. We can write
\[
2^r \# \left\{ t \in \mathbb{Z}/p\mathbb{Z} \mid \forall j, \frac{t - \nu_j}{D_{\nu_j}} \equiv \square \pmod{p} \right\} = \sum_{t} \prod_{j=1}^{r} \left( \chi_2 \left( \frac{t - \nu_j}{D_{\nu_j}} \right) + 1 \right),
\]
with \( \chi_2 \) the quadratic character modulo \( p \). Now expand the right hand side
\[
\sum_{J} \sum_{t} \chi_2 \left( \prod_{j \in J} \frac{t - \nu_j}{D_{\nu_j}} \right).
\]
Where the sum is over all subsets \( J \subseteq \{1, \ldots, r\} \). The contribution of the empty set is exactly \( \sum t = p \), while for nonempty \( J \) we get an exponential sum of the form \( \sum \chi_2 \left( \prod_{j \in J} \frac{t - \nu_j}{D_{\nu_j}} \right) \). Since we assumed all \( \nu_j \) are distinct, the polynomial \( g(t) = \prod_{j \in J} \frac{t - \nu_j}{D_{\nu_j}} \) is not a square and we can apply the Weil bounds \( \sum \chi_2(g(t)) = O(\sqrt{p}) \) \([18]\). Consequently, we have that indeed
\[
2^r \# \left\{ t \in \mathbb{Z}/p\mathbb{Z} \mid \forall j, \frac{t - \nu_j}{D_{\nu_j}} \equiv \square \pmod{p} \right\} = p + O(\sqrt{p}).
\]

Next we need to show that for the rest of the characters (when the exponential sum does not vanish) the angles \( \theta(\chi, \nu) \) become equidistributed (independently) in \([0, \pi]\). We will do that by computing all mixed moments. However, we recall that there are exceptional characters for which the normalized exponential sums are not bounded causing the moments to blow up. For that reason we first restrict ourself to a set of “good” characters (of limiting density one) for which the sums are bounded and only then we calculate the moments.

Fix a finite set of \( r \) nonzero distinct integers \( \bar{\nu} = \{\nu_1, \ldots, \nu_r\} \), and define the set of “good” characters to be
\[
S_{p^k}(\bar{\nu}) = \left\{ \chi \in \hat{C}(p^k) \mid \forall j, 2t_{\chi} \not\equiv -\nu_j \pmod{p} \right\},
\]
where \( t_{\chi} \) is determined by \( \chi \) as above. Then for any character \( \chi \in S_{p^k}(\bar{\nu}) \), we can write \( E_{p^k}(\nu_j, \chi) = p^{k/2}\cos(\theta(\nu_j, \chi)) \) with \( \theta(\nu_j, \chi) \in \)
[0, π). Furthermore, for any \( \nu_j \) there are precisely \( p^k-2(p\pm1) \) characters with \( 2t_\chi \equiv \nu_j \pmod{p} \) (this is the size of the kernel of the restriction map from \( \hat{\mathcal{C}}(p^k) \) to \( \hat{\mathcal{C}}_p(k,1) \)). Hence, \( \frac{|S_{\chi(p)}|}{p^k} = 1 + O(\frac{1}{p}) \) and the set \( S_{\chi(p)} \) is of (limiting) density one inside \( \hat{\mathcal{C}}(p^k) \).

Before we proceed to calculate the moments we will need to set some notations. For any \( k \) define the set

\[
Y(p^k, \tilde{\nu}) = \{ \tilde{x} \in (\mathbb{P}(p^k))^r | \nu_1(Dx_1^2 - 1) = \nu_j(Dx_j^2 - 1), \forall 2 \leq j \leq r \}
\]

For every fixed set of integers \( \tilde{n} = \{n_1, \ldots, n_r\} \) let

\[
Y_0(p^k, \tilde{\nu}, \tilde{n}) = \{ \tilde{x} \in Y(p^k) \mid \prod_j \beta(x_j)^{n_j} \equiv 1 \pmod{p} \}.
\]

For notational convenience we will sometimes use the notation \( S_{\chi(p^k)}, Y(p^k), Y_0(p^k) \) where the dependence on \( \tilde{\nu} \) and \( \tilde{n} \) is implicit. We will also denote by \( Y'(p^k) \) (respectively \( Y'_0(p^k) \)) the elements of \( Y(p^k) \) (respectively \( Y_0(p^k) \)) with all \( x_j \neq 0 \pmod{p} \).

**Lemma 4.3.** As \( p \to \infty \), the number of points in \( Y'(p^k) \) satisfy

\[
\#Y'(p^k) = p^k + O(p^{k-\frac{2}{3}})
\]

**Proof.** For any \( t \in \mathbb{Z}/p^k\mathbb{Z} \) satisfying \( \forall j, t \not\equiv \nu_j \pmod{p} \) we have that

\[
\# \{ \tilde{x} | \forall j, \nu_j(Dx_j^2 - 1) = t \} = \begin{cases} 2^r & \forall j, \frac{t - \nu_j}{D\nu_j} \equiv \square \pmod{p} \\ 0 & \text{otherwise} \end{cases}
\]

On the other hand if \( t \equiv \nu_j \pmod{p} \) for some \( j \), then

\[
\# \{ \tilde{x} | \forall j, \nu_j(Dx_j^2 - 1) = t \} \leq 2^r p^{k-1}
\]

(as there are at most two possibilities for \( x_i \) with \( i \neq j \) and at most \( 2p^{k-1} \) possibilities for \( x_j \)). We thus have

\[
\#Y(p^k) = \sum_{t \in (\mathbb{Z}/p^k\mathbb{Z})^*} \# \{ \tilde{x} | \forall j, \nu_j(Dx_j^2 - 1) = t \}
\]

\[
= 2^r p^{k-1} \# \left\{ t \in (\mathbb{Z}/p\mathbb{Z})^* | \forall j, \frac{t - \nu_j}{D\nu_j} \equiv \square \pmod{p} \right\} + O(p^{k-1}).
\]

Also note that \( \#Y'(p^k) = \#Y(p^k) + O(p^{k-1}) \). To conclude the proof we use the estimate

\[
2^r \# \left\{ t \in \mathbb{Z}/p\mathbb{Z} | \forall j, \frac{t - \nu_j}{D\nu_j} \equiv \square \pmod{p} \right\} = p + O(\sqrt{p}),
\]

from lemma 4.2. \( \square \)
Lemma 4.4. As \( p \to \infty \), the number of points in \( Y_0'(p^k) \) satisfy

\[
#Y_0'(p^k) = O(p^{k-1}).
\]

Proof. To prove this bound we will show that there is a nonzero polynomial \( F(t) \) with integer coefficients such that for any \( \bar{x} \in Y_0(p^k) \), with \( b = \nu_1(Dx_1^2 - 1) / 2 \) we have \( F(b^{-1}) \equiv 0 \pmod{p^k} \) (recall that for \( x_1 \in X(p^k) \) we have \( Dx_1^2 \not\equiv 1 \pmod{p} \) and hence \( b \not\equiv 0 \pmod{p} \) is invertible). This would imply that \( b \) can take at most \( \deg F \) values modulo \( p^k \), implying that

\[
#Y_0(p^k) \leq 2^r \deg(F)p^{k-1}.
\]

Now to define \( F \), consider the formal polynomial in the variables \( \beta_1^{\pm 1}, \ldots, \beta_r^{\pm 1} \) given by

\[
G(\beta_1, \ldots, \beta_r) = \prod_{\sigma \in \{\pm 1\}^r} \left( \prod_{j=1}^r \beta_j^{\sigma_j n_j} - 1 \right).
\]

Recall that if a polynomial in two variables \( x, y \) is symmetric under permutation then it can be written as a polynomial in the symmetric polynomials \( \sigma_1 = x + y, \sigma_2 = xy \) (see e.g., [3, Chapter 6]). The polynomial \( G \) is symmetric under any substitution \( \beta_j \mapsto \beta_j^{-1} \) and hence there is another polynomial \( \tilde{F} \) in \( r \) variables with integer coefficients, satisfying

\[
G(\beta_1, \ldots, \beta_r) = \tilde{F}(\beta_1 + \beta_1^{-1}, \ldots, \beta_r + \beta_r^{-1}).
\]

Define the polynomial \( F(t) = \tilde{F}(2 + \nu_1 t, \ldots, 2 + \nu_r t) \). For any \( x_1, \ldots, x_r \) with \( x_j^2 = \frac{2b - \nu_j}{\nu_j D} \pmod{p^k} \) we have \( \beta(x_j) + \beta(x_j)^{-1} = 2 + \nu_j b^{-1} \pmod{p^k} \) (recall \( \beta(x) = \sqrt{Dx^2 + 1} / \sqrt{Dx^2 - 1} \)). Hence,

\[
G(\beta(x_1), \ldots, \beta(x_r)) = \tilde{F}(2 + \nu_1 b^{-1}, \ldots, 2 + \nu_r b^{-1}) = F(b^{-1}).
\]

Now, if in addition \( \beta(x_1)^{n_1} \cdots \beta(x_r)^{n_r} \equiv 1 \pmod{p^k} \) then indeed \( F(b^{-1}) = G(\beta(x_1), \ldots, \beta(x_r)) \equiv 0 \pmod{p^k} \).

It remains to show that \( F(t) \) is not the zero polynomial. To do this, we think of it as a complex valued polynomial, and note that for it to be identically zero there has to be some choice of signs \( \sigma \in \{\pm 1\}^r \) so that the function

\[
G_\sigma(t) = \prod_{j=1}^r \beta(\sqrt{\frac{2t + \nu_j}{\nu_j D}})^{\sigma_j n_j}
\]
satisfies $G_\sigma(t) \equiv 1$. Assume that there is such a choice $\sigma$, so the derivative $G'_\sigma(t)$ must also vanish. But we have
\[
G'(t) = -G_\sigma(t) \sum_{j=1}^{r} \sigma_j n_j \sqrt{\frac{\nu_j}{t^2(2t + \nu_j)}}.
\]
so as $t \to -\frac{\nu_1}{2}$ the term $\sqrt{\frac{\nu_1}{t^2(2t + \nu_1)}}$ blows up while the rest of the terms remain bounded (recall that all $\nu_j$ are different). In particular $G'_\sigma(t)$ is not identically zero.  

**Remark 4.1.** The bound $\#Y'_0(p^k) = O(p^{k-1})$ is probably not optimal. Notice that if the polynomial $F(t)$ defined above is separable (i.e., if it has no multiple roots) then there are at most $\deg F$ solutions to $F(t) \equiv 0 \pmod{p^k}$ and the corresponding bound would be $\#Y'_0(p^k) = O(1)$.

We now preform the moment calculation establishing the limiting distribution of the exponential sums (when running over characters in $S_{pk}$).

**Proposition 4.5.** Let $\mu$ be as in (4.1) and let $g \in C([-1, 1]^r)$ be any continuous function then
\[
\lim_{p \to \infty} \frac{1}{p^k} \sum_{\chi \in S_{pk}} g(\cos(\theta_1), \ldots, \cos(\theta_r)) = \int_{[0, \pi]^d} g(\cos(\theta_1), \ldots, \cos(\theta_r)) d\mu(\theta_1) \cdots d\mu(\theta_r).
\]

**Proof.** We will give the proof for $k = 2l$ even, the odd case is analogous. Since we can always approximate the function $g$ by polynomials, it is sufficient to show this holds for all monomials of the form
\[
g(x) = (2x_1)^{m_1} \cdots (2x_r)^{m_r}.
\]
We thus need to show that
\[
\lim_{p \to \infty} \frac{1}{p^k} \sum_{\chi \in S_{pk}} \prod_j (2 \cos(\theta_j, \chi))^{m_j} = \prod_j \int_{[0, \pi]} (2 \cos(\theta))^{m_j} d\mu(\theta).
\]
With out loss of generality we can also assume that all the $m_j$ are nonzero (since $\mu$ is a probability measure, if $m_j = 0$ then the corresponding factor is 1 and we can consider the same problem for $r - 1$ instead of $r$). In this case the right hand side is given by
\[
\prod_j \left( \int_{0}^{\pi} (2 \cos(\theta))^{m_j} d\mu(\theta) \right) = \prod_j \left( \frac{1}{2} \int_{0}^{\pi} (2 \cos(\theta))^{m_j} d\frac{\theta}{\pi} \right).
\]
The integral in each factor is $\frac{1}{2} \binom{m_j}{n_j}$ for $m_j = 2n_j$ even and it is zero otherwise.

Now fix a character $\chi \in S_{p^k}$ and let $t_\chi \in \mathbb{Z}/p'\mathbb{Z}$ as above. If $\text{Sq}(\frac{2t_\chi + \nu_j}{D}, p') = \emptyset$ then $2 \cos \theta(\nu_j, \chi) = p^{-k/2} E_{p^k}(\nu_j, \chi) = 0$. Otherwise,

$$2 \cos(\theta(\nu_j, \chi)) = p^{-k/2} E_{p^k}(\nu_j, \chi) = 2 \Re(e_p^k(\frac{\nu_j x_j}{2}) \chi(\beta(x_j))),$$

with $x_j \in \text{Sq}(\frac{2t_\chi + \nu_j}{D}, p')$ (recall that for $\chi \in S_{p^k}$ we know $2t_\chi + \nu \neq 0 \pmod{p}$). Hence, the only contributions to the sum

$$\sum_{\chi \in S_{p^k}} \prod_j (2 \cos \theta(\nu_j, \chi))^{m_j},$$

comes from characters $\chi$ such that for all $j$ there is $x_j \in \text{Sq}(\frac{2t_\chi + \nu_j}{D}, p')$ (equivalently, there is $x_j \in (\mathbb{Z}/p'\mathbb{Z})^*$ satisfying $\nu_j(Dx_j^2 - 1) \equiv 2t_\chi \pmod{p'}$). Also note that if we multiply $\chi$ by any character that is trivial on $C_p(k, l)$ this does not change $t_\chi$. Let $\hat{C}(\mathbb{Z}/p')$ be the group of characters that are trivial on $C_p(k, l)$, and for any $b \in \mathbb{Z}/p'\mathbb{Z}$ let $\chi_b \in \hat{C}(\mathbb{Z}/p')$ be a representative of $\hat{C}(\mathbb{Z}/p')/\hat{C}(\mathbb{Z}/p')$ with $t_{\chi_b} = b$. We thus have that

$$\frac{1}{p^k} \sum_{\chi \in S_{p^k}} \prod_j (2 \cos \theta(\nu_j, \chi))^{m_j} = \frac{1}{2^{r} p^k} \sum_{x \in \text{Y}''(p')} \sum_{\chi \in \hat{C}(\mathbb{Z}/p')} \prod_j (2 \cos \theta(\nu_j, \chi \chi_b))^{m_j},$$

where $b = b(x) = \frac{\nu_j(Dx_j^2 - 1)}{2}$.

Now use the formula,

$$(2 \cos(\theta))^m = \sum_{n=0}^{m} \binom{m}{n} \cos((m - 2n)\theta).$$

The main contribution comes from the terms where in each factor $m_j - 2n_j = 0$. This vanishes unless all $m_j$ are even in which case it is given by

$$\frac{1}{2^{r} p^k} \sum_{x \in \text{Y}''(p')} \sum_{\chi \in \hat{C}(\mathbb{Z}/p')} \prod_j \binom{m_j}{n_j} = \prod_j \frac{1}{2} \binom{m_j}{n_j} + O\left(\frac{1}{\sqrt{p}}\right).$$

where we used Lemma 4.3 to get that $\text{#Y}''(p') \cdot \text{#}\hat{C}(\mathbb{Z}/p') = p^k + O(p^{k-\frac{1}{2}})$. 

It thus remains to bound the rest of the terms, which is reduced to
the vanishing (in the limit $p \to \infty$) of the sums

$$\frac{1}{p^k} \sum_{\bar{x} \in \tilde{Y}'(p^l)} \sum_{\chi \in \hat{C}(l)(p^k)} \prod_j \cos(n_j \theta(\nu_j, \chi \chi_b)),$$

for any nonzero integers $\{n_1, \ldots, n_r\}$.

For any $\bar{x} \in \tilde{Y}'(p^l)$ we have that

$$\cos(n_j \theta(\nu_j, \chi \chi_b)) = 2\Re(e^{p^k(n_j \nu_j / 2)} \chi \chi_b(\beta(x_j)^{n_j}))$$

with $b = \nu(D_{x^2} - 1)/2$. When expanding the product $\prod_j \cos(n_j \theta(\nu_j, \chi \chi_b))$ we get a sum over $2^r$ terms, each of the form

$$e^{p^k}\left(\sum_{j=1}^r \pm n_j \nu_j x_j / 2 \right) \chi \chi_b(\prod_{j=1}^r \beta(x_j)^{\pm n_j}).$$

We thus need to bound the exponential sum coming from each term. We will now bound the corresponding sum

$$\frac{1}{p^k} \sum_{\bar{x} \in \tilde{Y}'(p^l)} \sum_{\chi \in \hat{C}(l)(p^k)} e^{p^k}\left(\sum_{j=1}^r n_j \nu_j x_j / 2 \right) \chi \chi_b(\prod_{j=1}^r \beta(x_j)^{n_j}).$$

(the same bound obviously holds when changing any $n_j$ to $-n_j$). Now, rewrite this sum as

$$\frac{1}{p^k} \sum_{\bar{x} \in \tilde{Y}'(p^l)} e^{p^k}\left(\sum_{j=1}^r n_j \nu_j x_j / 2 \right) \chi \chi_b(\prod_{j=1}^r \beta(x_j)^{n_j}) \sum_{\chi \in \hat{C}(l)(p^k)} \chi(\prod_{j=1}^r \beta(x_j)^{n_j}),$$

and note that the inner sum vanishes unless $\prod_{j=1}^r \beta(x_j)^{n_j} \equiv 1 \pmod{p^l}$ in which case it is equal $\#\hat{C}(l)(p^k) = p^{k-l}$. We can thus rewrite this sum as

$$\frac{1}{p^k} \sum_{\bar{x} \in \tilde{Y}'(p^l)} e^{p^k}\left(\sum_{j=1}^r n_j \nu_j x_j / 2 \right) \chi \chi_b(\prod_{j=1}^r \beta(x_j)^{n_j}).$$

which is trivially bounded by $p^{-l} \#\tilde{Y}'(p^l) = O(1/p)$ (Lemma 4.4). \hfill \Box

Remark 4.2. The above proof also gives the rate at which the fluctuations of the normalized exponential sums approach their limiting distribution. If one takes the test function $g$ in Proposition 4.5 to be smooth then the rate of convergence is $O(1/\sqrt{p})$. This rate comes from the bound on the error term in Lemma 4.3 which seems to be a sharp bound.
5. Back to Matrix Elements

We can now deduce Theorems 2 and 3 from Theorem 1 and the analysis of the exponential sums.

Proof of Theorem 2. Let \( f(x) = e^{2\pi i n \cdot x} \) be any elementary observable. Take \( N = p^3 \) to be a prime cubed. Then by Corollary 2 there is a character satisfying \( |E_{p^3}(\frac{Q(n)}{2}, \chi)| = p^2 \). Let \( \psi \) be a Hecke eigenfunction corresponding to \( \chi \), then by Theorem 1 we get

\[
|\langle \text{Op}_N(f)\psi, \psi \rangle| = \frac{1}{\#C(p^3)} E_{p^3}(\frac{Q(n)}{2}, \chi) = \frac{1}{p \pm 1} \gg N^{-1/3}.
\]

Proof of Theorem 3. Let \( f \) be a trigonometric polynomial and write

\[
f = \sum_{|n| \leq R} \hat{f}(n)e(n \cdot x),
\]

for some fixed \( R > 0 \). Let \( \{\nu_1, \ldots, \nu_r\} = \{Q(n) | 0 < |n| \leq R\} \), and consider the random variable

\[
Y_f = 2 \sum_{j=1}^{r} f^\#(\nu_j) \cos(\theta_j).
\]

with \( \theta_j \) chosen independently from \([0, \pi]\) with respect to \( \mu \). We need to show that as \( p \to \infty \) the limiting distribution of \( F_j^{(p^k)} \) is that of \( Y_f \).

For any character \( \chi \) of \( C(p^k) \) consider the weighted sum of the corresponding exponential sums

\[
F_{\chi}^{(p^k)} = \sum_{j=1}^{r} f^\#(\nu_j) p^{-k/2} E_{p^k}(\frac{\nu_j}{2}, \chi_0).
\]

By Proposition 4.5 as \( p \to \infty \) the limiting distribution of \( F_{\chi}^{(p^k)} \) as \( \chi \) runs through \( S_{p^k} \) (hence, also as \( \chi \) runs through the whole group of characters) is that of \( Y_f \).

Now, for \( p \) sufficiently large (i.e., \( p > \max\{\nu_j\} \)) and \( \chi_j \in \hat{C}_0(p^k) \), we have

\[
F_j^{(p^k)} = \sum_{j=1}^{r} f^\#(\nu_j) \frac{p^{k/2}}{\#C(p^k)} E_{p^k}(\frac{\nu_j}{2}, \chi_j\chi_0).
\]

If we further assume that \( \chi_j\chi_0 \in S_{p^k}(\bar{\nu}) \) then \( |E_{p^k}(\frac{\nu_j}{2}, \chi_0)| \leq 2p^{k/2} \), and hence

\[
F_j^{(p^k)} = F_{\chi_j}^{(p^k)} + O(\frac{1}{p}).
\]
The set of characters \( \{ \chi_j \in \hat{C}_0(p^k) | \chi_j \chi_0 \in S_{p^k} \} \) is again of density one, hence, the limiting distribution of \( F_j^{(p^k)} \) is the same as of \( F_{\chi_j}^{(p^k)} \) concluding the proof. \( \square \)

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