Harmonicity of a function via harmonicity of its spherical means

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Abstract
It is proved that harmonic functions are characterized by harmonicity of their spherical means, for which purpose the iterated spherical means are used. The similar characterization of solutions to the modified Helmholtz equation (panharmonic functions) is given. Another description of harmonic functions is the pointwise equality of a function and its iterated mean over an admissible pair of spheres.

1 Introduction and the main result

A function $u \in C^2(D)$ is called harmonic (see [1], p. 25, for the origin of this term), if it satisfies the equation $\nabla^2 u(x) = 0$ in a domain $D \subset \mathbb{R}^m$, $m \geq 2$; $\nabla = (\partial_1, \ldots, \partial_m)$ denotes the gradient operator, $\partial_i = \partial / \partial x_i$ and $x = (x_1, \ldots, x_m)$ is a point of $\mathbb{R}^m$. Studies of mean value properties of harmonic functions date back to the Gauss theorem of the arithmetic mean over a sphere; see [4], Article 20. Nowadays, its standard formulation is as follows.

Theorem 1. Let $u \in C^2(D)$ be harmonic in a domain $D \subset \mathbb{R}^m$, $m \geq 2$. Then for every $x \in D$ the equality $M(x, r, u) = u(x)$ holds for each admissible sphere $S_r(x)$.

Here and below the following notation and terminology are used. The open ball of radius $r$ centred at $x$ is denoted by $B_r(x) = \{y : |y - x| < r\}$; the latter is called admissible with respect to a domain $D$ provided $B_r(x) \subset D$, and $S_r(x) = \partial B_r(x)$ is the corresponding admissible sphere. If $u \in C^0(D)$, then its spherical mean value over $S_r(x) \subset D$ is

$$M(x, r, u) = \frac{1}{|S_r|} \int_{S_r(x)} u(y) \, dS_y = \frac{1}{\omega_m} \int_{S_1(0)} u(x + ry) \, dS_y,$$

where $|S_r| = \omega_m r^{m-1}$ and $\omega_m = 2 \pi^{m/2}/\Gamma(m/2)$ is the total area of the unit sphere (as usual $\Gamma$ stands for the Gamma function), and $dS$ is the surface area measure.

An immediate consequence of Theorem 1 involves the domain $D_r \subset D$ with boundary ‘parallel’ to $\partial D$ at the distance $r > 0$; namely, $D_r = \{x \in D : B_r(x) \subset D\}$. Thus, $D_r$ is nonempty only when $r$ is less than the radius of the open ball inscribed into $D$.

Corollary 1. Let $u \in C^2(D)$ be harmonic in a domain $D \subset \mathbb{R}^m$, $m \geq 2$. If $D_r$ is nonempty for $r > 0$, then the function $M(\cdot, r, u)$ is harmonic in this domain.

The proof apparently follows by applying $\nabla$ twice to the right-hand side in (1). In view of the latter assertion, it is natural to investigate whether $u$ is harmonic in $D$ provided each $M(\cdot, r, u)$ with sufficiently small $r$ is harmonic in $D_r$. The following positive answer to this question is the main result of this note.
Theorem 2. Let $D$ be a bounded domain in $\mathbb{R}^m$, $m \geq 2$, and let $u \in C^0(\overline{D}) \cap C^2(D)$ be real-valued. If $M(\cdot, r, u)$ is harmonic in $D_r$ for all $r \in (0, r_*)$, where $r_*$ is a positive number such that $D_{r_*} \neq \emptyset$, then $u$ is harmonic in $D$.

The author failed to find a result of this kind in the literature; in particular, there is no mention of anything similar in the extensive survey [9].

2 Proof of Theorem 2 and discussion

Prior to proving Theorem 2, let us consider some properties of the iterated spherical mean introduced by John; see [5], p. 78, but the notation used here is different:

$$I(x, r', r, u) = M(x, r', M(\cdot, r, u)) = \frac{1}{\omega_m} \int_{S_1(0)} M(x + r'y, r, u) \, dS_y.$$  \hspace{1cm} (2)

The second equality is a consequence of [1]. Since $M(\cdot, r, u)$ is defined on $D_r$, it is obvious that $I(\cdot, r', r, u)$ is defined on $D_{r'+r}$. Substituting the expression for $M$, we obtain

$$I(x, r', r, u) = \frac{1}{\omega_m^2} \int_{S_1(0)} \int_{S_1(0)} u(x + r'y + rz) \, dS_z \, dS_y,$$  \hspace{1cm} (3)

and so it is symmetric in $r'$ and $r$, that is, $I(x, r', r, u) = I(x, r, r', u)$. Moreover,

$$I(x, 0, r, u) = I(x, r, 0, u) = M(x, r, u) \quad \text{and} \quad I(x, 0, 0, u) = u(x).$$

By virtue of the iterated mean [3], Theorem 2 will be reduced to the converse of Theorem 1 due to Kellogg [9]; its modern formulation is as follows.

Theorem 3 (Kellogg). Let $D$ be a bounded domain in $\mathbb{R}^m$, $m \geq 2$, and let $u \in C^0(\overline{D})$ be real-valued. If for each $x \in D$ there exists $r(x)$ such that $\overline{B_{r(x)}(x)} \subset D$ and $M(x, r(x), u) = u(x)$, then $u$ is harmonic in $D$.

Proof of Theorem 2. It is clear that each $x \in D$ belongs to all $D_r$ with $r < \text{dist}(x, \partial D)/2$, where $\text{dist}(x, \partial D)$ is the distance from $x$ to $\partial D$. Let us fix some $r(x) \in (0, \text{dist}(x, \partial D)/2)$, and so $\overline{B_{r(x)}(x)} \subset D_r$ for all described values of $r$. Since the mean $M(\cdot, r, u)$ is harmonic in $D_r$ for each of these values, we have $M(x, r(x), M(\cdot, r, u)) = M(x, r, u)$ by Theorem 1. In view of (3) and (1), this can be written as follows:

$$\frac{1}{\omega_m} \int_{S_1(0)} \int_{S_1(0)} u(x + r(y + rz)) \, dS_z \, dS_y = \frac{1}{\omega_m} \int_{S_1(0)} u(x + ry) \, dS_y.$$ 

Letting $r \to 0$ in this equality, we obtain that $M(x, r(x), u) = u(x)$ holds for each $x \in D$ with some $r(x)$ such that $\overline{B_{r(x)}(x)} \subset D$. Now, Theorem 3 yields that $u$ is harmonic in $D$. \hfill $\blacksquare$

The proof looks simple, but it relies upon Theorem 3 which is not trivial at all. However, the strong converse of Theorem 1 is easy to prove when $D$ is the so-called Dirichlet domain; that is, a bounded domain in which the Dirichlet problem for the Laplace equation is soluble provided the function given on $\partial D$ is continuous. To illustrate the advantage of Dirichlet domains, let us prove the assertion (Theorem 4 below) similar to Theorem 3, but involving the iterated mean (2) instead of $M(\cdot, r, u)$. Theorem 4 as well as in the next proposition require admissible triples instead of admissible spheres, thus allowing us to consider $I(x, r', r, u)$ for any $x \in D$. Namely, the triple $(x, r', r)$ is admissible with respect to $D$, if $x + r'y + rz$ belongs to this domain for all $y, z \in B_1(0)$.

Proposition 1. Let $u \in C^2(D)$ be harmonic in a domain $D \subset \mathbb{R}^m$, $m \geq 2$. Then for every $x \in D$ the equality $I(x, r', r, u) = u(x)$ holds for each admissible triple $(x, r', r)$.
Proof. Since \( u \) is harmonic in \( D \), the equality \( M(x + r' y, r, u) = u(x + r' y) \) holds for every \( x \in D \) and all \( y \in B_1(0) \) provided the triple \( (x, r', r) \) is admissible. Then the result follows by using this equality in the integral on the right-hand side of (2) with subsequent application of Theorem 1 to the obtained integral.

Now, let us prove the following strong converse of Proposition 1.

**Theorem 4.** Let \( D \) be a Dirichlet domain in \( \mathbb{R}^m \), \( m \geq 2 \), and let \( u \in C^0(\overline{D}) \cap C^2(D) \) be real-valued. If for every \( x \in D \) there exist \( r'(x) \) and \( r(x) \) such that the triple \( (x, r'(x), r(x)) \) is admissible and the equality \( I(x, r'(x), r(x), u) = u(x) \) holds, then \( u \) is harmonic in \( D \).

**Proof.** First, let us demonstrate that theorem’s assumptions yield that

\[
\max_{x \in \partial D} u(x) = \max_{x \in \overline{D}} u(x).
\]

Denoting the left-hand side by \( U \), we show that the closed preimage \( u^{-1}(U) \) has a nonempty intersection with \( \partial D \). Indeed, if \( u^{-1}(U) \cap \partial D = \emptyset \), then there exists \( x_0 \in u^{-1}(U) \subset D \) that is nearest to \( \partial D \), and so for some admissible triple \((x_0, r'(x_0), r(x_0))\) we have:

\[
U = u(x_0) = \frac{1}{\omega_m^2} \int_{S_1(0)} \int_{S_1(0)} u(x_0 + r'(x_0)y + r(x_0)z) dS_x dS_y.
\]

In view of the maximality of \( U \), the equality \( u(x_0 + r'(x_0)y + r(x_0)z) = U \) holds for all \( y, z \in S_1(0) \), that is, every \( x_0 + r'(x_0)y + r(x_0)z \) belongs to \( u^{-1}(U) \). Hence the distance to \( \partial D \) from some point \( x_0 + r'(x_0)y_0 + r(x_0)z_0 \in D \) with \( y_0, z_0 \in S_1(0) \) is smaller than from \( x_0 \). The obtained contradiction yields (4).

Let \( f \) denote the trace of \( u \) on \( \partial D \); then there exists \( u_0 \in C^0(\overline{D}) \) solving the Dirichlet problem for Laplace equation in \( D \) with \( f \) as the boundary data. Therefore, Theorem 1 is valid for \( u_0 \), and so theorem’s assumptions are fulfilled for \( u - u_0 \) and \( u_0 - u \). Since both these functions vanish on \( \partial D \), equality (4) yields that \( u - u_0 \geq 0 \) and \( u_0 - u \leq 0 \) in \( D \). Thus, \( u \) is harmonic in \( D \), being equal to \( u_0 \) there.

2.1 Extension of Theorem 2 to panharmonic functions

It was Duffin [3], who introduced the convenient abbreviation ‘panharmonic functions’ for the awkward ‘solutions of the modified Helmholtz equation’ which arise in numerous applications; see [2]. Since the most important of them concerns nuclear forces, the equation

\[
\nabla^2 u - \mu^2 u = 0, \quad \mu \in \mathbb{R} \setminus \{0\},
\]

is referred to as the Yukawa equation in [3] (surprisingly, without citing the original paper [2], in which Yukawa proposed his theory of these forces). Much attention has been given to solving (5) numerically (see [2] again), but no analogue of Theorem 1 for panharmonic functions was proved until recently. The following assertion about the \( m \)-dimensional mean for spheres was obtained in [7].

**Theorem 5.** Let \( u \in C^2(D) \) be panharmonic in a domain \( D \subset \mathbb{R}^m \), \( m \geq 2 \). Then for every \( x \in D \) the equality

\[
M(x, r, u) = a(\mu r) u(x), \quad a(\mu r) = \Gamma \left( \frac{m}{2} \right) \frac{I_{(m-2)/2}(\mu r)}{I_{(m-2)/2}(\mu r)}
\]

holds for each admissible sphere \( S_r(x) \); \( I_\nu \) denotes the modified Bessel function of order \( \nu \).

For \( m = 3 \) formula (6) has particularly simple form because \( a(\mu r) = \sinh \mu r / (\mu r) \), and this was proved by C. Neumann [9] as early as 1896. Duffin independently rediscovered his proof (see [3], pp. 111-112), but for the two-dimensional case when \( a(\mu r) = I_0(\mu r) \).
Corollary 2. Let $u \in C^2(D)$ be panharmonic in a domain $D \subset \mathbb{R}^m$, $m \geq 2$. If for $r > 0$ the domain $D_r$ is nonempty, then the function $M(\cdot, r, u)$ is panharmonic in this domain; the coefficient in equation (5) for $M(\cdot, r, u)$ and $u$ is the same.

The proof of this corollary is similar to that of Corollary 1, but along with equation (5) formula (6) must be used. Now, let us turn to the version of strong converse of Theorem 5 valid for Dirichlet domains which are the same for harmonic and panharmonic functions; this is a well-known consequence of results obtained in [10] and [11]. In the paper [7], the following assertion was established.

Theorem 6. Let $D \subset \mathbb{R}^m$ be a Dirichlet domain, and let $u \in C^0(\overline{D}) \cap C^2(D)$ be real-valued. If for every $x \in D$ there exists $r(x)$ such that $S_{r(x)}(x)$ is admissible and equality (6) holds with $r = r(x)$ and fixed $\mu > 0$, then $u$ is panharmonic in $D$ and the coefficient in equation (5) is $\mu^2$.

This allows us to obtain the following analogue of Theorem 2 for panharmonic functions.

Theorem 7. Let $D$ be a Dirichlet domain in $\mathbb{R}^m$, $m \geq 2$, and let $u \in C^0(\overline{D}) \cap C^2(D)$ be real-valued. If for all $r \in (0, r_*)$, where $r_*$ is a positive number such that $D_{r_*} \neq \emptyset$, the mean $M(\cdot, r, u)$ satisfies equation (5) in $D_r$ and the coefficient is $\mu^2$ for all $r$, then $u$ is panharmonic in $D$ with the same coefficient in (5).

The proof is literally the same as that of Theorem 2, but the reference to Theorem 6 must be made instead of Theorem 3.

References

[1] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, 2nd ed., Springer-Verlag, New York, 2001.
[2] H. W. Cheng, J. F. Huang, T. J. Leiterman, “An adaptive fast solver for the modified Helmholtz equation in two dimensions”, *J. Comput. Phys.* 211 (2006), 616–637.
[3] R. J. Duffin, “Yukawan potential theory”, *J. Math. Anal. Appl.* 35 (1971), 105–130.
[4] C. F. Gauss, *Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstoßungskräfte*, Wiedmannschen Buchhandlung, Leipzig, 1840.
[5] F. John, *Plane Waves and Spherical Means*, Interscience, New York, 1955.
[6] O. D. Kellogg, “Converses of Gauss’ theorem on the arithmetic mean”, *Trans. Am. Math. Soc.* 36 (1934), 227–242.
[7] N. Kuznetsov, “Mean value properties of solutions to the Helmholtz and modified Helmholtz equations”, *J. Math. Sciences* 257 (2021), 673–683.
[8] I. Netuka, J. Veselý, “Mean value property and harmonic functions”, in *Classical and Modern Potential Theory and Applications*, Kluwer, Dordrecht, 1994, pp. 359–398.
[9] C. Neumann, *Allgemeine Untersuchungen über das Newtonsche Prinzip der Fernwirkungen*, Teubner, Leipzig, 1896.
[10] O. A. Oleinik, “On the Dirichlet problem for equations of elliptic type”, *Mat. Sbornik*, 24 (1949), 3–14 (in Russian).
[11] G. Tautz, “Zur Theorie der ersten Randwertaufgaben,” *Math. Nachr.*, 2 (1949), 279–303.
[12] H. Yukawa, “On the interaction of elementary particles”, *Proc. Phys.-Math. Soc. Japan* 17 (1935), 48–57.