The standard deviation effect
(or why one should sit first base playing blackjack)

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Abstract. For a balanced cardcounting system we study the random variable of the true count after a number of cards are removed from the remaining deck and we prove a close formula for its standard deviation. As expected, the formula shows that the standard deviation increases with the number of cards removed. This creates a "standard deviation effect" with a two fold consequence: longer long run and presumably larger fluctuations of the bankroll, but a small gain in playing accuracy for the player sitting third base. The opposite happens for the player sitting first base. Thus the optimal position in casino blackjack in terms of shorter long run is first base.

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Disclaimer.

Cardcounters are asked to forgive us the simplistic presentation of their activity aimed to (non-cardcounter) mathematicians, and mathematicians those elementary computations in the last section aimed to (non-mathematician) cardcounters.

Not being gamblers or probability specialists this article is written for fun and with no pretention.

1) Introduction.

This article is about casino blackjack and not about tournament blackjack. Cardcounters in the game of blackjack use different count systems to keep track of the ratio of favorable and unfavorable cards that remain on the deck. It is well known now and documented ([Th]) that one can get an edge (of the order of 2% ) over the house by increasing the bets in favorable counts and keep them to a minimum in unfavorable ones. The first person who published and analyzed this finding, mathematically and with computer simulations, was E. Thorp (who at the time was a graduate student in the mathematics department in UCLA).

One of the main questions that a professional cardcounter player faces is the money management. He plays with a given bankroll. He has to decide the amount to bet at each moment taking into account the limit of the table and the upper bound fixed by his own bankroll. Obviously the maximum bet planned in his strategy should be inferior to the maximum of the table. But how much to bet depending on how favorable is the count ? The goal is to follow a betting pattern that minimizes the risk (that is losing the whole bankroll) but maximizes the growth rate. The well known sharp strategy is Kelly’s criterion : The bet is a fraction of the total bankroll equal to the advantage you have. This maximizes the expected exponential rate of growth of the bankroll. Kelly’s criterion can be proven to be optimal in a very strong sense: Any other strategy will have a longer expected time to achieve a given amount of the bankroll (L. Breiman [Br] *). For establishing this type of results one makes the assumption that there is no minimal unit bet. In section 2 we give a simple derivation of Kelly criterion. For a basic bibliography we refer to [RT]. When the advantage is not know precisely but it is a random variable, Kelly criterion also applies (see section 3.c): One should bet according to the expected value of the advantage. The standard deviation of the exponential rate of growth of the bankroll turns out to increase in a significant way with the standard deviation of the advantage. This makes longer the ”long run” (see section 3) and presumably induces larger fluctuations. Thus one should prefer playing conditions that minimize the standard deviation of the advantage.

In practice, count systems associate a value to each card. We restrict the discussion here to balanced counts, that is, those for which the total sum of values of a cards in a

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complete deck is zero. In blackjack only the numerical value of the card matters. A very simple balanced count system called Hi-Lo gives the weight $-1$ to high cards (10,J,Q,K,A), the weight $+1$ to low cards (2,3,4,5,6) and the weight 0 to medium cards (7,8,9). The cardcounter player adds up the weights of the cards as they are revealed. The numerical value obtained is the running count (RC). It tells the player how favorable is the remaining deck (from the Hi-Lo values you can tell that high cards are in favor of the player, low cards are in favor of the dealer). Of course how favorable is the deck depends not directly on the RC but on the true count (TC) that is the ratio of RC by the number of cards remaining (one can also divide by the number of decks remaining, this is just a change of units, and it is what is done in practice). The quantity $52 \cdot TC$ (i.e. the running count in deck units, 52 is the total number of cards in a deck) gives approximately the percentage in edge gained by the player due to the unbalanced composition of the deck.

The advantage of the house with a game with favorable rules facing a perfect play with no counting from the player (this is called basic strategy) is of the order of 0.5% (see for example, [Gr2] p.139). Thus it is recommended to beat the minimum when $52 \cdot TC < 2$ and to bet $52 \cdot TC$ units when $52 \cdot TC > 1$.

The blackjack tables are semi-circular. The dealer stands on the flat side and at most seven players (in some tables five players) sit around the circular border. Cards are dealt from the left side to the right side of the dealer. The first position at the left of the dealer is the first base and the last position at the right of the dealer is the third base. The dealer deals the cards from left to right.

It seems to be a common belief in the blackjack literature that there is no preferred position for the card counter and that the company of other players at the table has no influence (see [HC] p.68, [Wo] p. 220, for example). In this article we prove that the positions in the blackjack table are not all equivalent. There is an advantage in terms of shorter long run (and presumably of smaller fluctuations of the bankroll) for the player to be as close as possible to first base. More precisely, a player betting according to Kelly’s criterium will have a larger fluctuations of the exponential rate of growth of his bankroll if he sits on first base than if he sits in third base. The main noticeable effect of this is in the ”long run”. A player sitting third base will need to play about 2% more favorable hands than a player sitting first base in order to achieve the same standard deviation for the exponential rate of growth of his bankroll. On the other hand, from the edge point of view, a player sitting closer to third base has a slight playing advantage over the players at his right.

The main goal of this article is to explain theoretically these differences in the position of the players. They both have the same origin: the standard deviation effect. In a few words and in a simplistic form, the standard deviation effect appears when one has to take an irreversible decision with respect to a future situation. More the lag is important, more the decision has chances to be incorrect.

The game of blackjack is played as follows. First the players decide their bets. A card counter will do this according to the value of the true count, and he will bet following Kelly’s criterium. Then the dealer will deal two cards to each player and two for him.
The cards are dealt face up or down, this is not important for our purposes (in all cases there is always a hidden card, the so-called hole card of the dealer). Then the players will make their play decisions (to split, double, hit or stand) in turn from first to third base, and play the hand. When the moment of the play decision comes the true count of the player has changed because he has seen some of the new cards (at least the two he has received). Thus he may not be betting in a favorable situation. The so-called “True count theorem” justifies the action. The expected value of the true count after some cards have been removed from the remaining deck is the same as the true count before removing the cards. As far as we know, the proof of this theorem in the blackjack context was published for the first time by Abdul Jalib M’Hall ([JM]) in a message posted to the rec.gambling.blackjack newsgroup.

**Theorem (True count theorem, A. J. M’Hall).** For any balanced count, the expected value of the true count after several cards (but not all) are removed from the remaining deck is the value of the true count before removing the cards.

This type of result goes back to the origins of probability theory. There is a well known problem: Each person in a room is asked to take a ticket from a box. One indicates the winner. Should you try to take your ticket as soon as possible or as late as possible? We assume that everyone waits that all tickets have been chosen to look at their ticket (otherwise if they look and reveal the result the problem is not the same). The answer is that it doesn’t matter of course. The expected value of the probability of being a winner (running count) stays the same at all moments.

Coming back to the card counting problem, even if the expected value of the true count is independent of the number of cards removed, the standard deviation of the true count will increase as we remove cards from the deck. This results in particular from the exact formula proved in the theorem below. Thus the precise knowledge of the true count ”dilutes” as more cards are played. This makes that often the advantage the player has when his turn of play comes (and also when the dealer’s turn comes) differs from the expected value at the moment of betting. This induces a longer long run. Since the standard deviation increases with the number of cards played, the player sitting in third base will be systematically hurt by a larger standard deviation, thus will experience a longer long run. This is one of the consequences of the standard deviation effect. On the other hand, when the player in third base has to play, he is closer to the dealer’s play. Thus, compared to the player in first base, he knows more accurately the true count at the moment the dealer will play. Thus he can adjust more effectively his play. He has a supplementary bet advantage than the player in first base because he can deviate in a more efficient way from basic strategy. This is the second consequence of the standard deviation effect. The gain from deviating from basic strategy is small.
Now we present the "True count standard deviation formula" which proves the increasing of the standard deviation with the number of cards removed from the remaining deck. Let \( \sigma_n \) be the standard deviation of the true count after having removed \( n \geq 1 \) cards.

**Theorem (True count standard deviation formula).** Let \( N \) be the number of cards remaining in the deck. After removing \( 1 \leq n < N \) cards, the standard deviation of the true count is

\[
\sigma_n = \sqrt{\frac{N - 1}{N - n}} \sqrt{n} \sigma_1.
\]

The combinatorics in the proof of the true count standard deviation formula reveal a beautiful set of identities that we call the *true count algebra*. Each one of these identities has a probabilistic interpretation. We only develop in this article the minimum set of identities necessary in the proof of the main theorem. The distribution of the true count does depend on the composition of the deck and the weights of the counting system. This family of distributions contains the hypergeometric distribution. We don’t know a reference for the true count distribution in the classical literature. The authors will be grateful if any reader can provide one.

We have the following corollaries (using also the results in section 2).

**Corollary 1.** The standard deviation \( \sigma_n \) is strictly increasing with \( n \), even faster when the deck contains fewer cards.

**Corollary 2.** For a given player, at the moment of taking his betting decision, the standard deviation of the true count at the moment of his play decision will be larger if he sits further away from the first base.

**Corollary 3.** The theoretical long run of players sitting further away from the first base are larger.

**Corollary 4.** In terms of shorter theoretical long run, the optimal seat is first base.

**Corollary 5.** For a given player, at the moment of taking his playing decision, the standard deviation of the true count at the moment of the play of the dealer will be smaller if he sits further away from first base.

**Corollary 6.** The playing advantage of players sitting further away from the first base is increased.

**Corollary 7.** In terms of playing advantage the optimal seat is third base.

We mainly discuss the effects on long run that is the most relevant one. There is also an effect on larger fluctuations of the bankroll and risk of ruin that is more difficult to analyze with precision. Classically the trade-off between betting efficiency and playing efficiency is something the card counter has to consider carefully when choosing his system (these quantities differ for different systems). Empirical "proper balances" have been proposed (see [Gr1] p.40-49), but of course the proper balance between this two different quantities is up to each player and his goal.
Practical gambling.

In the last section we present a closer study of how the playing conditions do influence the standard deviation of the true count, thus the long run. The true count standard deviation formula only gives the relation of $\sigma_n$ with $\sigma_1$. The actual value of $\sigma_1$ does depend on the count system and on the remaining distribution of weights in the remaining cards. One has a quite accurate approximate formula only depending on the count system and the number of cards on the deck

$$\sigma_1 \approx \frac{\Sigma_0}{N}$$

where $\Sigma_0$ is the standard deviation of weights used in the count system. We discuss some relevant consequences for practical play. Some amusing consequence of the formulas is that for a continuous model of the deck the standard deviation goes to $+\infty$ when $N \to 0$. Fortunately (!) the casinos do not use to practice a 100% penetration on the decks. Of course, in normal conditions an important penetration will increase in a substantial amount the playing advantage that will become the predominant effect. Nevertheless it is not excluded that casinos could device a set of rules that look advantageous to the players (according to the classical literature) but induce large fluctuations that will wipe out the bankrolls of card counters in the long run.

The methods exposed in this article provide also information about higher moments of the true count. The combinatorics one faces is more involved. We plan to study this in the future.

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2) Standard deviation of the true count.

a) Proof of the true count theorem.

To set up the notations we recall first Abdul Jalib M’Hall’s proof [JM]. It is enough to prove the result when we remove one card.

We denote by $N$ the number of cards remaining in the deck. We denote by $w$ the different possible weights of the cards given by the count system. Let $s_w$ be the total number of cards in the full deck with weight $w$. Since the system count is balanced, we have

$$\sum_w ws_w = 0.$$ 

Let $l_w$ be the total number of cards in the remaining deck with weight $w$ (so $1 = \sum_w l_w/N$). Then if we denote by $k_w = s_w - l_w$ we have that

$$R = \sum_w wk_w = -\sum_w wl_w$$

is the running count. The true count is $T = R/N$.

We compute the expected value $T_1$ of the true count after removing 1 card. It is given by

$$T_1 = \sum_w \frac{R + w l_w}{N - 1} \frac{1}{N}$$

$$= \sum_w \frac{R}{N - 1} \frac{l_w}{N} + \sum_w \frac{w}{N - 1} \frac{l_w}{N}$$

$$= \frac{R}{N - 1} - \frac{R}{(N - 1)N}$$

$$= \frac{R}{N}$$

$$= T$$

and the result follows.

b) The true count algebra.

For $1 \leq p \leq N$ we define

$$l_{w_1 \ldots w_p}^w = l_w - |\{1 \leq i \leq p; w_i = w\}|,$$

that is the number of cards of weight $w$ remaining once $p$ cards of weight $w_1, \ldots, w_p$ have been removed. Observe that we have

$$\sum_w l_{w_1 \ldots w_p}^w \frac{1}{N - p} = 1.$$ 

These coefficients have a beautiful and rich combinatorics. We call this system of identities the true count algebra. The name is chosen because most relevant formulas have a “true
count” probabilistic interpretation. We concentrate here into the relevant identities in order to compute the standard deviation. The full algebra will be studied in a forthcoming article.

Lemma 1. For $0 \leq p \leq N - 2$ we have

$$S_0 = \sum_w l_{w_1 \ldots w_p}^{w_1 \ldots w_p} \frac{v_0}{N-p} \frac{l_{v_0}^{w_1 \ldots w_p v_0}}{N-p-1} = \frac{l_{v_0}^{w_1 \ldots w_p}}{N-p}.$$ 

One can translate this relation in a probabilistic statement. Picking at random one card on the remaining deck, the probability that this card has a weight $v_0$ is the same than the probability of this same occurrence using the same deck depleted from one card at random. Or, in card counters terms, the running count for weight $v_0$ does not change when we remove one card at random.

Proof. We split the sum into a sum over all $w \neq v_0$ and the term for $w = v_0$. Observe that for $w \neq v_0$ we have

$$l_{v_0}^{w_1 \ldots w_p} = l_{v_0}^{w_1 \ldots w_p}.$$ 

Also for $w = v_0$ we have

$$l_{v_0}^{w_1 \ldots w_p v_0} = l_{v_0}^{w_1 \ldots w_p} - 1.$$ 

Thus we have

$$S_0 = \left( \sum_{w \neq v_0} l_{w_1 \ldots w_p}^{w_1 \ldots w_p} \frac{l_{v_0}^{w_1 \ldots w_p}}{N-p} \frac{l_{v_0}^{w_1 \ldots w_p v_0}}{N-p-1} \right) + l_{v_0}^{w_1 \ldots w_p} \frac{l_{v_0}^{w_1 \ldots w_p v_0}}{N-p} \frac{l_{v_0}^{w_1 \ldots w_p v_0}}{N-p-1}$$

$$= l_{v_0}^{w_1 \ldots w_p} \left( \sum_{w \neq v_0} l_{w_1 \ldots w_p}^{w_1 \ldots w_p} \frac{l_{v_0}^{w_1 \ldots w_p}}{N-p} \frac{l_{v_0}^{w_1 \ldots w_p v_0}}{N-p-1} \right) + l_{v_0}^{w_1 \ldots w_p} \frac{l_{v_0}^{w_1 \ldots w_p v_0}}{N-p} \frac{l_{v_0}^{w_1 \ldots w_p v_0}}{N-p-1}$$

$$= l_{v_0}^{w_1 \ldots w_p} \left( \frac{1}{N-p-1} - \frac{1}{(N-p)(N-p-1)} \right)$$

$$= l_{v_0}^{w_1 \ldots w_p} \frac{N-p}{N-p-1}$$

q.e.d.

Lemma 2. Let $p \geq 0$, $q \geq 0$ such that $p + q \leq N - 2$. We have the formula

$$S_q = \sum_w l_{w_1 \ldots w_p}^{w_1 \ldots w_p} l_{v_0}^{w_1 \ldots w_p v_0} l_{v_1}^{w_1 \ldots w_p v_0 v_1} \cdots l_{v_q}^{w_1 \ldots w_p v_0 v_1 \ldots v_q-1} \frac{l_{v_q}^{w_1 \ldots w_p v_0 v_1 \ldots v_q-1}}{N-p} \frac{l_{v_0}^{w_1 \ldots w_p v_0 v_1 \ldots v_q-1}}{N-p-1} \cdots \frac{l_{v_q}^{w_1 \ldots w_p v_0 v_1 \ldots v_q-1}}{N-p-q}$$

$$= l_{v_0}^{w_1 \ldots w_p} \frac{l_{v_1}^{w_1 \ldots w_p v_0}}{N-p} \frac{l_{v_1}^{w_1 \ldots w_p v_0 v_1}}{N-p-1} \cdots \frac{l_{v_q}^{w_1 \ldots w_p v_0 v_1 \ldots v_q-1}}{N-p-q}$$

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Clearing denominators one can also write the less cumbersome formula

$$\sum_w l_{w_1 \cdots w_p} l_{w_1 \cdots w_p v_0} l_{w_1 \cdots w_p v_0 v_1} \cdots l_{v_q}$$

$$= (N - p - q - 1) l_{v_0} l_{v_1} \cdots l_{v_q}$$

One can translate the identity of the lemma in a probabilistic statement. Picking at random $q$ cards on the remaining deck one after the other, the probability that the cards have weights $v_0, v_1 \ldots$ and $v_q$ in this order is the same than the probability of this same occurrence using the same deck depleted from one card at random. Or, in card counters terms, the running count for an ordered clump of $q + 1$ cards to have respective weights $v_0, v_1, \ldots$ and $v_q$ does not change when we remove one card at random.

**Proof.** The proof is done by induction on $q \geq 0$. The result holds for $q = 0$ because of lemma 1. We assume that the result holds for $q \geq 0$ and we prove it for $q + 1$. We split the sum

$$\sum_w l_{w_1 \cdots w_p} l_{w_1 \cdots w_p v_0} l_{w_1 \cdots w_p v_0 v_1} \cdots l_{v_{q+1}}$$

into the sum for $w \neq v_0$,

$$A = \sum_{w \neq v_0} l_{w_1 \cdots w_p} l_{w_1 \cdots w_p v_0} l_{w_1 \cdots w_p v_0 v_1} \cdots l_{v_{q+1}}$$

and the term of the sum for $w = v_0$

$$B = l_{v_0} l_{v_1} \cdots l_{v_{q+1}}$$

For $w \neq v_0$ we have $l_{v_0} l_{w_1 \cdots w_p v_0} = l_{v_0} l_{v_1} \cdots l_{v_{q+1}}$, so

$$A = l_{v_0} l_{w_1 \cdots w_p} \sum_{w \neq v_0} l_{w_1 \cdots w_p} l_{w_1 \cdots w_p v_0 v_1} \cdots l_{v_{q+1}}$$

Since for $w \neq v_0$ we have $l_{w_1 \cdots w_p} = l_{w_1 \cdots w_p v_0}$, we have

$$A = l_{v_0} \sum_{w \neq v_0} l_{w_1 \cdots w_p v_0} l_{w_1 \cdots w_p v_0 v_1} \cdots l_{v_{q+1}}$$

Using the induction hypothesis for $q$ we conclude that

$$A = l_{v_0} (N - (p + 1) - q) l_{v_1} \cdots l_{v_{q+1}}$$

$$- l_{v_0} l_{v_1} \cdots l_{v_{q+1}}$$

$$= (N - p - (q + 1)) l_{v_0} l_{v_1} \cdots l_{v_{q+1}} - B$$

And the result follows. \(\Box\)
From the probabilistic interpretation we obtain that there are corresponding formulas for the removal of $k$ cards instead of only one. We prove these formulas by induction on $k$ using lemma 1 and lemma 2.

**Lemma 3.** Let $k \geq 1$ and $0 \leq p \leq N - k - 1$. We have

$$
\sum_{i_1, \ldots, i_k} \frac{w_{i_1} \cdots w_{p} i_1}{N - p} \frac{w_{i_2} \cdots w_{p} i_1 i_2}{N - p - 1} \cdots \frac{w_{i_k} \cdots w_{p} i_1 \cdots i_{k - 1}}{N - p - k + 1} \frac{w_{i_0} \cdots w_{p} i_1 \cdots i_k}{N - p - k} = \frac{w_{i_0} \cdots w_{p}}{N - p}
$$

**Lemma 4.** Let $p \geq 0$, $q \geq 0$ and $k \geq 1$ such that $p + k + q \leq N - 1$. We have

$$
\sum_{i_1, \ldots, i_k} \frac{w_{i_1} \cdots w_{p} i_1}{N - p} \frac{w_{i_2} \cdots w_{p} i_1}{N - p - 1} \cdots \frac{w_{i_k} \cdots w_{p} i_1 \cdots i_{k - 1}}{N - p - k + 1} \frac{w_{i_0} \cdots w_{p} i_1 \cdots i_k}{N - p - k - 1} = \frac{w_{i_1} \cdots w_{p} w_0}{N - p - k - 1} \cdots \frac{w_{i_q} \cdots w_{p} w_0 \cdots \cdots w_{q - 1} i_1 \cdots i_k}{N - p - k - q}
$$

There are more general formulas, but we don’t need them for the purpose of this article. Next lemma is an application of the formulas.

**Lemma 5.** Let $W = \{w\}$ be the set of weights and $f : W \to \mathbb{R}$ a real valued function. The expected value of $f(w)$, for any $1 \leq n \leq N$ and $1 \leq j \leq n$, is given by

$$
\bar{f} = \sum_w f(w) \frac{w}{N} = \sum_{w_1 \ldots w_n} f(w_j) \frac{w_1}{N} \frac{w_2}{N - 1} \cdots \frac{w_{n - 1}}{N - n + 1}.
$$

**Proof.** We have

$$
\sum_{w_1 \ldots w_n} f(w_j) \frac{w_1}{N} \frac{w_1}{N - 1} \cdots \frac{w_{n - 1}}{N - n + 1} = \sum_{w_1 \ldots w_j} f(w_j) \frac{w_1}{N} \frac{w_1}{N - 1} \cdots \frac{w_{j - 1}}{N - j + 1} \sum_{w_{j + 1} \ldots w_n} \frac{w_{j + 1}}{N - j} \cdots \frac{w_{n - 1}}{N - n + 1} = \sum_{w_1 \ldots w_j} f(w_j) \frac{w_1}{N} \frac{w_1}{N - 1} \cdots \frac{w_{j - 1}}{N - j + 1} = \sum_{w_j} f(w_j) \frac{w_1}{N} \frac{w_2}{N - 1} \cdots \frac{w_{j - 1}}{N - j + 1} = \sum_{w_j} f(w_j) \frac{w_j}{N} = \bar{f}
$$

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where we used lemma 4 that gives
\[
\sum_{w_j+1,\ldots,w_n} \frac{l_{w_j+1\ldots w_j}}{N-j} \ldots \frac{l_{w_n}}{N-n+1} = \sum_{w_n} \frac{l_{w_n}}{N-j} = 1
\]
and
\[
\sum_{w_1,\ldots,w_{j-1}} \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1} \ldots \frac{l_{w_j-1}}{N-j+1} = \frac{l_{w_j}}{N}
\]

\[\Diamond\]

c) The true count theorem without induction.

We present here a direct computation of the expected value of the true count when \(n\) cards have been withdrawn. It uses the true count theorem for \(n=1\) but proves the general theorem without induction. This is certainly useless, but it prepares the field and the notations to prove the standard deviation true count formula.

Remember that we denote by \(R\) the running count. The following lemma is immediate by direct computation.

**Lemma 6.** We have the identity
\[
\frac{R + w_1 + \ldots + w_n}{N-n} - \frac{R}{N} = \frac{N-1}{N-n} \left[ \left( \frac{R + w_1}{N-1} - \frac{R}{N} \right) + \ldots + \left( \frac{R + w_n}{N-1} - \frac{R}{N} \right) \right]
\]

Now the expected value of the true count after removing \(n\) cards of the deck is (we use lemma 5 here)
\[
T_n = \sum_{w_1,\ldots,w_n} \frac{R + w_1 + \ldots + w_n}{N-n} \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1} \ldots \frac{l_{w_n}}{N-n+1}
= \frac{R}{N} + \sum_{w_1,\ldots,w_n} \left( \frac{R + w_1 + \ldots + w_n}{N-n} - \frac{R}{N} \right) \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1} \ldots \frac{l_{w_n}}{N-n+1}
\]

Consider the function
\[
f(w) = \frac{R + w}{N-1} - \frac{R}{N},
\]
We have (using the true count theorem for \(n=1\) (!))
\[
\bar{f} = 0
\]

Thus using lemma 6 and lemma 5,
\[
T_n = \frac{R}{N} + \frac{N-1}{N-n} n \bar{f} = \frac{R}{N}
\]
d) The true count standard deviation formula.

We still denote by $f$ the function defined in the previous section. The square of the standard deviation is given by

$$
\sigma_n^2 = \sum_{w_1, \ldots, w_n} \left( \frac{R + w_1 + \ldots + w_n}{N - n} - \frac{R}{N} \right)^2 \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1} \ldots \frac{l_{w_n}}{N-n+1}
$$

Using lemma 6 and developing the square we have

$$
\sigma_n^2 = \left( \frac{N-1}{N-n} \right)^2 \sum_{1 \leq j_1, j_2 \leq n} \sum_{w_1, \ldots, w_n} f(w_{j_1}) f(w_{j_2}) \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1} \ldots \frac{l_{w_n}}{N-n+1}
$$

Immediately from lemma 5 we get

Lemma 7. For $j_1 = j_2 = j$, we have

$$
\sum_{w_1, \ldots, w_n} f(w_j)^2 \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1} \ldots \frac{l_{w_n}}{N-n+1} = \sigma_1^2
$$

Using next lemma we can finish the computation:

$$
\sigma_n^2 = \left( \frac{N-1}{N-n} \right)^2 \left( n \sigma_1^2 - n(n+1) \frac{1}{N-1} \sigma_1^2 \right)
$$

$$
= \left( \frac{N-1}{N-n} \right)^2 \left( 1 - \frac{n-1}{N-1} \right) n \sigma_1^2
$$

$$
= \left( \frac{N-1}{N-n} \right) n \sigma_1^2
$$

q.e.d.

Lemma 8. We have for $j_1 \neq j_2$,

$$
\sum_{w_1, \ldots, w_n} f(w_{j_1}) f(w_{j_2}) \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1} \ldots \frac{l_{w_n}}{N-n+1} = - \frac{1}{N-1} \sigma_1^2
$$

Proof. We assume $j_1 < j_2$ for example. The proof follows the same ideas than the
proof of lemma 5. We have using lemma 4 several times

\[
\sum_{w_1, \ldots, w_n} f(w_{j_1}) f(w_{j_2}) \frac{l_{w_1}}{N} \frac{l_{w_2}^1}{N-1} \cdots \frac{l_{w_n}^{w_1 \ldots w_{n-1}}}{N-n+1} = \\
\sum_{w_1, \ldots, w_{j_2}} f(w_{j_2}) \sum_{w_{j_1}} f(w_{j_1}) \frac{l_{w_1}}{N} \frac{l_{w_2}^{w_1}}{N-1} \cdots \frac{l_{w_{j_2}^{w_1 \ldots w_{j_2-1}}}}{N-j_2+1} \\
= \sum_{w_1, \ldots, w_{j_2}} f(w_{j_2}) \sum_{w_{j_1+1}, \ldots, w_{j_2-1}} f(w_{j_1}) \frac{l_{w_1}}{N} \frac{l_{w_2}^{w_1 \ldots w_{j_1}}}{N-1} \cdots \frac{l_{w_{j_1+1}^{w_1 \ldots w_{j_1-1}}}}{N-j_1+1} \frac{l_{w_{j_2}^{w_1 \ldots w_{j_2-1}}}}{N-j_2+1} \\
= \sum_{w_1, \ldots, w_{j_1}} f(w_{j_1}) \sum_{w_{j_2}^{w_1 \ldots w_{j_1-1}}} f(w_{j_2}) \frac{l_{w_1}}{N} \frac{l_{w_2}^{w_1 \ldots w_{j_1}}}{N-1} \cdots \frac{l_{w_{j_2}^{w_1 \ldots w_{j_1-1}}}}{N-j_1+1} \frac{l_{w_{j_2}^{w_1 \ldots w_{j_2-1}}}}{N-j_2+1} \\
= \sum_{w_{j_1}, w_{j_2}} f(w_{j_1}) f(w_{j_2}) \frac{l_{w_{j_1}}^{w_{j_1}}}{N} \frac{l_{w_{j_2}^{w_1 \ldots w_{j_1-1}}}}{N-1} \\
= -\frac{1}{N-1} \sigma_1^2
\]

In the last computation we use the following lemma applied to the function \(g(w_1, w_2) = f(w_1) f(w_2)\) (observe that \(\sum_w g(w, w)(l_{w}/N) = \sum_w f(w)^2(l_{w}/N) = \sigma_1^2\)). \(\diamondsuit\)

**Lemma 9.** Let \(g : W \times W \to \mathbb{R}\). We have

\[
\sum_{w_1, w_2} g(w_1, w_2) \frac{l_{w_1}}{N} \frac{l_{w_2}^{w_1}}{N-1} = \frac{N}{N-1} \bar{g} - \frac{1}{N-1} \sum_w g(w, w) \frac{l_{w}}{N}
\]

where \(\bar{g}\) is the expected value of \(g\)

\[
\bar{g} = \sum_{w_1, w_2} g(w_1, w_2) \frac{l_{w_1}^{w_1}}{N} \frac{l_{w_2}^{w_2}}{N}.
\]
Proof. We have

\[
\sum_{w_1, w_2} g(w_1, w_2) \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1}
\]

\[
= \sum_{w_1 \neq w_2} g(w_1, w_2) \frac{l_{w_1}}{N} \frac{l_{w_2}}{N-1} + \sum_w g(w, w) \frac{l_w}{N} \frac{l_w - 1}{N - 1}
\]

\[
= \frac{N}{N-1} \left( \tilde{g} - \sum_w g(w, w) \frac{l_w}{N} \right) + \frac{N}{N-1} \left( \sum_w g(w, w) \frac{l_w}{N} \frac{l_w - 1}{N - 1} \right)
\]

\[
= \frac{N}{N-1} \tilde{g} + \frac{N}{N-1} \left( \sum_w g(w, w) \frac{l_w}{N} \frac{N}{N - 1} \right)
\]

\[
= \frac{N}{N-1} \tilde{g} - \frac{1}{N-1} \sum_w g(w, w) \frac{l_w}{N}
\]
3) Long run.

a) Kelly for binomial games.

We review in this section the Kelly criterion in the case of an iterated fixed advantage game. We review also the formulas for the expected value and the variance of the exponential rate of growth and discuss the implications for Blackjack play.

We assume that we play a repetitive independent game where we have an advantage \( p > \frac{1}{2} \). We expect an exponential growth of our initial bankroll \( X_0 \) if we follow a reasonable strategy of betting. We assume that there is no minimal unit of bet. By homogeneity of the problem, the sharp strategy will consist in betting a proportion \( f(p) \) of the total bankroll. Our bankroll after \( n \) rounds of the game have been played is

\[
X_n = X_0 \prod_{i=1}^{n} (1 + \varepsilon_i f(p))
\]

where \( \varepsilon_i = +1 \) if we won the \( i \)-th hand, and \( \varepsilon_i = -1 \) if we lost the \( i \)-th hand.

The exponential rate of growth of the bankroll is

\[
G_n = \frac{1}{n} \log \frac{X_n}{X_0} = \frac{1}{n} \sum_{i=0}^{n} \log (1 + \varepsilon_i f(p))
\]

The Kelly criterion maximizes the expected value of the exponential rate of growth (the proof is elementary, see section b for a proof in the more general setting of a ”fuzzy advantage”).

**Theorem 3.1 (Kelly criterion).** The expected value of the exponential rate of growth \( G_n \) is maximized for

\[
f(p) = 2p - 1.
\]

Observe that the expected value is

\[
E(G_n) = E(G_1) = p \log(1 + f(p)) + (1 - p) \log(1 - f(p)).
\]

Also the random variables

\[
X_i = \log(1 + \varepsilon_i f(p))
\]

are independent, and

\[
\text{Var}G_n = \frac{1}{n} \text{Var}X.
\]

We compute

\[
\text{Var}X = E(X^2) - (E(X))^2
\]

\[
= p \left( \log(1 + f(p)) \right)^2 + (1 - p) \left( \log(1 - f(p)) \right)^2 - (p \log(1 + f(p)) + (1 - p) \log(1 - f(p)))^2
\]

\[
= p(1 - p) \left( \log \left( \frac{1 + f(p)}{1 - f(p)} \right) \right)^2
\]
So finally

\[ \text{Var}G_n = \frac{1}{n} p(1-p) \log \left( \frac{p}{1-p} \right)^2. \]

**Proposition 3.2.** The expected value and the standard deviation of \( G_n \) are

\[
\begin{align*}
\mathbb{E}(G_n) &= \mathbb{E}(G_1) = p \log(2p) + (1-p) \log(2-2p) \\
\sigma(G_n) &= \frac{1}{\sqrt{n}} \sqrt{p(1-p) \log \left( \frac{p}{1-p} \right)}
\end{align*}
\]

Thus for a slight advantage \( p = 1/2 + \varepsilon \) we have

\[
\begin{align*}
\mathbb{E}(G_n) &= 2\varepsilon^2 + \mathcal{O}(\varepsilon^3) \\
\sigma(G_n) &= \frac{2\varepsilon}{\sqrt{n}} (1 + \mathcal{O}(\varepsilon^2))
\end{align*}
\]

From this data in becomes clear how long is ”the long run”. The long run corresponds to play a number of rounds such that \( \sigma(G_n) \) is of the order of \( \mathbb{E}(G_n) \) (or a small fraction of it). For a smaller number of rounds played there is a good chance that \( G_n \) is negative, i.e. the bankroll has decreased. In the case of blackjack, \( \varepsilon \approx 10^{-2} \). Thus the ”long run” corresponds to have played of the order of 10000 favorable hands, thus at least 20000 hands. At 50 hands per hour this adds to 4000 hours of play. To play less hands means that we are gambling. This explains that even at the non-professional level the team play makes perfect sense in order to get into the long run quicker (divide the time by the number of members of the team).

The situation one faces when playing blackjack is different than a binomial game with a fixed advantage. First the advantage is not the same at different hands. This involves minor modifications of the above computations. Second, due to the standard deviation effect, the advantage the player has at the moment of playing is a random variable at the moment of betting. So he is betting according to a fuzzy advantage. We study this situation in the next sections. Kelly criterion is still sharp. The main effect of the fuzzy advantage is to introduce a supplementary term in the standard deviations of \( G_n \).

b) **Kelly for fuzzy advantage.**

We consider the situation one faces playing blackjack. At each round one decides the amount to bet in function of the true count.

We assume that we play multiple independent rounds of the same game. At each round the advantage we have is a random variable \( p, 0 < p < 1 \), with a known distribution \( D_x(p) \) and expected value \( p_0(x) > 1/2 \) from a family of distributions \( (D_x) \). The choice of the distribution \( D_x \) at each round is random with distribution \( \rho(x), x \in [0,1] \) (\( x \) is a mere index, we may just adjust it to have a uniform distribution \( \rho(x) = 1 \)).

We want to maximize the expected value of the exponential rate of growth of our bankroll \( X_0 \). The optimal amount to bet is a proportion \( 0 \leq f(D_x) \leq 1 \) of the bankroll.
depending only on the distribution $D_x$. For a sharp strategy we have $f(D_x) = 0$ when $p_0(x) \leq 1/2$ (that is, the game is not favorable). The quantity to maximize is
\[
\int_0^1 \int_0^1 (p \log(1 + f(D_x)) + (1 - p) \log(1 - f(D_x))) \, D_x(p) \, dp \, \rho(x) \, dx
\]
\[
= \int_0^1 (p_0(x) \log(1 + f(D_x)) + (1 - p_0(x)) \log(1 - f(D_x))) \, \rho(x) \, dx
\]

**Theorem 3.3 (Kelly criterion).** The optimal strategy is obtained for
\[
f_K(D_x) = 2p_0(x) - 1
\]
for $1/2 \leq p_0(x) \leq 1$, and $f_K(D_x) = 0$ for $0 \leq p_0(x) \leq 1/2$.

**Proof.** We want to maximize the functional
\[
G(f) = \int_0^1 (p_0(x) \log(1 + f(D_x)) + (1 - p_0(x)) \log(1 - f(D_x))) \, \rho(x) \, dx
\]
\[
= \int_{\{x: p_0(x) > 1/2\}} (p_0(x) \log(1 + f(D_x)) + (1 - p_0(x)) \log(1 - f(D_x))) \, \rho(x) \, dx
\]
where we assume that $f(D_x) = 0$ for $0 < p_0(x) \leq 1/2$. If $f_0$ is an extremum then for any perturbation $h$ such that $f_0 + \varepsilon h$ is an allowable strategy ($f_0 + \varepsilon h > 0$), then if we consider
\[
g(\varepsilon) = G(f_0 + \varepsilon h)
\]
we must have
\[
g'(0) = 0.
\]
A direct computation gives
\[
g'(\varepsilon) = \int_{\{x: p_0(x) > 1/2\}} \frac{h(D_x)}{1 - (f_0(D_x) + \varepsilon h(D_x))^2} [(2p_0(x) - 1) - f_0(D_x) - \varepsilon h(D_x)] \, \rho(x) \, dx > 0
\]
So
\[
g'(0) = \int_{\{x: p_0(x) > 1/2\}} \frac{h(D_x)}{1 - f_0(D_x)^2} (2p_0(x) - 1 - f_0(x)) \, \rho(x) \, dx .
\]
Thus when $p_0(x) > 1/2$ we must have $f_0(x) = 2p_0(x) - 1 = f_K(x)$. Also by direct computation we have
\[
g''(\varepsilon) = \int_{\{x: p_0(x) > 1/2\}} \frac{-h^2}{(1 - (f_0 + \varepsilon h)^2)^2} [(f_0 + \varepsilon h)^2 + 1 - 2(2p_0(x) - 1)(f_0 + \varepsilon h)] \, \rho(x) \, dx
\]
Now since $f_0 + \varepsilon h > 0$ and $2p_0(x) - 1 > 0$ in the range of integration, we have
\[
g''(\varepsilon) \geq \int_{\{x: p_0(x) > 1/2\}} \frac{h^2}{(1 - (f_0 + \varepsilon h)^2)^2} (f_0 + \varepsilon h - 1)^2 \, \rho(x) \, dx > 0
\]
Therefore $\varepsilon \mapsto g(\varepsilon)$ is concave, $g''(0) < 0$, and $f_0 = f_K$ is a maximum of the functional.\hfill\diamondsuit

c) Long run.

We assume here that the player uses a Kelly betting strategy for a repetitive game with fuzzy advantage. For a given advantage distribution $D_x$ with $p_0(x) > 1/2$, the expected exponential rate of growth after playing $n$ favorable hands with distributions $x_1, x_2, \ldots$ is

$$E(G_n) = \frac{1}{n} \sum_{i=1}^{n} \int (p \log(1 + f(p_0(x_i))) + (1 - p) \log(1 - f(p_0(x_i)))) \, D_{x_i}(p) \, dp.$$  

Thus the expected exponential rate of growth is

$$E(G_n) = \int \int (p \log(1 + f(p_0(x))) + (1 - p) \log(1 - f(p_0(x)))) \, D_x(p) \, d\rho(x) \, dx$$

$$= \int (p_0(x) \log(1 + f(p_0(x))) + (1 - p) \log(1 - f(p_0(x)))) \, \rho(x) \, dx$$

$$= \int E_x(G_1) \, \rho(x) \, dx$$

$$= E(G_1)$$

Observe also that by independence of the hands we have

$$\text{Var}(G_n) = \frac{1}{n} \text{Var}(G_1).$$

We estimate the standard deviation of $G_n$. For this we make the assumption that the distribution $D_x(p)$ is a distribution $D(p)$ with a small "noise", that is

$$D_x(p) = D(p) + Z_x \, d(p)$$

where $Z_x$ is a random variable with 0 expectation. We have

$$\int Z_x \rho(x) dx = 0$$

$$\int d(p) dp = 0$$

the second equation coming from $\int D_x(p) \, dp = 1$. According to the previous notation, the expected advantage is

$$p_0(x) = \int p D_x(p) \, dp$$

thus

$$p_0(x) = p_0 + AZ_x$$

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where \( p_0 = \int pD(p) \, dp \) and we denote

\[
A = \int pd(p) \, dp.
\]

Observe also that the quantity \( A \) is directly related to the variance of \( p_0(x) \) by

\[
\text{Var}(p_0(x)) = \int p_0(x)^2 \rho(x) \, dx - p_0^2 = A^2 \int Z_x^2 \rho(x) \, dx.
\]

Our purpose now is to compute the first order perturbation of the standard deviation of \( G_n \) introduced by the noise \( Z_x \).

We have to the second order on \( Z_x \) (we do not care about the coefficient of the first order),

\[
E_x(G_1) = (p_0 + AZ_x) \log(2p_0 + 2AZ_x) + (1 - p_0 - AZ_x) \log(2 - 2p_0 - 2AZ_x)
\]

\[
= (p_0 + AZ_x)(\log(2p_0) + \log(1 + AZ_x/p_0)) +
\]

\[
+ (1 - p_0 - AZ_x)(\log(2 - 2p_0) + \log(1 - AZ_x/(1 - p_0)))
\]

\[
= p_0 \log(2p_0) + (1 - p_0) \log(2 - 2p_0) + *AZ_x + \frac{1}{2p_0(1 - p_0)} A^2 Z_x^2 + \ldots
\]

Thus taking expected values with respect to \( x \),

\[
E(G_1) = p_0 \log(2p_0) + (1 - p_0) \log(2 - 2p_0) + \frac{1}{2} \frac{\text{Var}(p_0(x))}{p_0(1 - p_0)} + \ldots
\]

And, on the first order on \( \text{Var}(p_0(x)) \), we have

\[
E(G_1)^2 = (p_0 \log(2p_0) + (1 - p_0) \log(2 - 2p_0))^2 + (p_0 \log(2p_0) + (1 - p_0) \log(2 - 2p_0)) \frac{\text{Var}(p_0(x))}{p_0(1 - p_0)} + \ldots
\]

Now we compute the expansion of

\[
E(G_1^2) = \int E_x(G_1^2) \rho(x) \, dx.
\]

We have (after a long computation)

\[
E_x(G_1^2) = p_0(x)(\log(2p_0(x)))^2 + (1 - p_0(x))(\log(2 - 2p_0(x)))^2
\]

\[
= \ldots
\]

\[
= p_0(\log(2p_0))^2 + (1 - p_0)(\log(2 - 2p_0))^2 + *AZ_x +
\]

\[
+ \frac{1}{p_0(1 - p_0)} (1 + (1 - p_0) \log(2p_0) + p_0 \log(2 - 2p_0)) A^2 Z_x^2 + \ldots
\]

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Thus
\[
E(G_1^2) = p_0(\log(2p_0))^2 + (1 - p_0)(\log(2 - 2p_0(x)))^2 + \\
+ \frac{1}{p_0(1 - p_0)} (1 + (1 - p_0) \log(2p_0) + p_0 \log(2 - 2p_0)) \text{Var}(p_0(x)) + \ldots
\]

Finally, putting together the previous formulas, we have

**Theorem 3.4.** The variance of the exponential rate of growth in a Kelly betting strategy with fuzzy advantage is, at the first order in the noise,

\[
\text{Var}(G_1) = p_0(1 - p_0) \left( \log \left( \frac{p_0}{1 - p_0} \right) \right)^2 + \frac{1}{p_0(1 - p_0)} \left( 1 - (2p_0 - 1) \log \left( \frac{p_0}{1 - p_0} \right) \right) \text{Var}(p_0(x)) + \ldots
\]

As an application to blackjack, where we have \( p_0 = 1/2 + \varepsilon \) with \( \varepsilon \approx 10^{-2} \), and \( \text{Var}(p_0(x)) = \varepsilon \sigma_{BET} \) (where \( \sigma_{BET} \) is the standard deviation of the true count),

\[
\text{Var}(G_1) = 4(1 + \sigma_{BET}^2)\varepsilon^2 + \ldots
\]

thus

\[
\sigma(G_n) = \frac{2\sqrt{1 + \sigma_{BET}^2}}{\sqrt{n}} \varepsilon + \ldots
\]

Thus the long run is increased by a factor \( 1 + \sigma_{BET}^2 \) caused by the standard deviation effect. In order to carry out precise estimates on the "long run" we state a precise definition. Note that the factor 2 is somewhat arbitrary. We may want to choose another value in order to have a bound on the probability of losing after achieving the long run \( N \) using Tchebichev's inequality.

**Definition 3.5 (Long run).** The long run \( N \) is the minimal number of hands in order to have

\[
\frac{E(G_N)}{\sigma(G_N)} \geq 2.
\]

Thus from the above computation we have (disregarding integer parts),

\[
N = 2^2(1 + \sigma_{BET}^2)\varepsilon^{-2} \approx (1 + \sigma_{BET}^2).40000
\]

Typically the difference between \( \sigma_{BET}^2 \) between first and third base could be of the order of 2% which makes the long run about 2% longer, or about 800 more favorables hands to be played, thus about 2000 more hands to be played (that is at least 40 more hours of play).
4) Practical gambling.

a) Comparison of first and third base and head-on play.

a.1) Full table.

The number of cards removed from the deck between the betting decision and the play decision of the first base is exactly $8 \times 2 = 16$. The average number of cards for the third base can be computed considering that the average number of cards in a hand of players playing basic strategy is 2.6 (this number is well known). Thus in average the number of cards removed from the deck between the betting decision and the play decision is 19.6. Denoting by $\Sigma(1)$ and $\Sigma(2)$ the corresponding standard deviations of the true count, and considering the approximation $N >> n$, we have

$$\frac{\Sigma(2)}{\Sigma(1)} \approx \frac{4.42}{4} = 1.10$$

Thus $\Sigma(2)$ is in average about 10% higher than $\Sigma(1)$.

In company of ploppys.

A recurrent topic in the blackjack literature is about is the influence of inaccurate players (ploppys) in the same table. It is an easy escape gate to blame others for your losses. The systematic answer that one founds all over the blackjack literature is that this is pure superstition. We believe also that there is mostly exaggeration in these complaints, but as we explain next, there is some mathematical foundation based on the standard deviation effect.

In the first part of this section, when we were comparing the standard deviation of the first and third base, in order to estimate the number of cards played between first and third base, we made a major assumption: Other players at the table are playing basic strategy, thus the average number of cards in a hand is 2.6. Regular players report that very few players know basic strategy (less than 5% we were told) and among these maybe about 1/4 of them are more or less skilled card counters. Playing basic strategy, the casino has an edge of about 1% (0.5% with best rules), but the actual win rate of blackjack in a casino is estimated to be between 3% and 7% (depending of the sources, see [Gr2] p.137). This shows how inaccurately the average player plays blackjack. The most common mistake is to think that the game consists in approaching as accurately as possible a total sum of 21. This is totally false. Doing so one risks being busted, thus loosing the whole bet without getting the chance of letting the dealer bust. The real goal is to beat the dealer. And the best way to achieve this is to let him bust. Also ploppys like to split tens for example. This is something at which strategy players look with horror because the count has to be sky high to justify this play (nevertheless it can happen, see [Us] p.17 where J. Uston explain how he, on one occasion, he did split eight times tens in one hand...with an initial bet of $1000). This mistakes make that the average player uses more cards in his hands than the ideal basic strategy player. We do not have precise statistics on this, but one can guess that having a gang of ploppys splitting tens to your right can make that if you sit in third
base the average number of cards played between the betting and the play decision could go as high as 25. In such a table the standard deviation of third base will be 25% higher than the one for first base. Thus this will induce longer long run and larger fluctuations, and you will be right to blame the ploppys.

**a.2) Head-on play.**

The differences between the standard deviation effect when one plays head-on (that is alone with the dealer) and one plays with a full table is even more important. The exact number of cards played in between decisions for a head-on play is 4. Thus if you sit in first base at a full table your standard deviation will be twice as important (100 % more). And if you sit in third base this figure goes up to 121% more.

**b) Absolute magnitude of standard deviation.**

The standard deviation true count formula allows us to compute $\sigma_n$ from $\sigma_1$. We compute now the standard deviation $\sigma_1$. It not only depends on $N$, and $R$, and the system count but also in the actual composition of the remaining deck.

**Proposition 1.** We have

$$\sigma_1 = \frac{1}{N-1} \sqrt{\sum_w w^2 l_w \frac{1}{N} - \left( \frac{R}{N} \right)^2}.$$

**Proof.** We have

$$\sigma_1^2 = \sum_w \left( \frac{R + w}{N-1} - \frac{R}{N} \right)^2 \frac{l_w}{N}$$

$$= \frac{1}{N^2(N-1)^2} \sum_w (Nw + R)^2 \frac{l_w}{N}$$

$$= \frac{1}{N^2(N-1)^2} \left[ N \left( \sum_w w^2 l_w \right) + 2R \sum_w w l_w + R^2 \right]$$

$$= \frac{1}{N^2(N-1)^2} \left[ N \left( \sum_w w^2 l_w \right) - 2 \left( \frac{R}{N} \right) \sum_w w l_w + 2 \left( \frac{R}{N} \right)^2 \right]$$

$$= \frac{1}{(N-1)^2} \left[ N \left( \sum_w w^2 l_w \right) - \left( \frac{R}{N} \right)^2 \right]$$

$\diamond$

Let $S$ denote the balanced counting system used. A main characteristic of the counting system is the standard deviation $\Sigma_0(S)$ of the weights. Since the system is balanced we have

$$\Sigma_0 = \Sigma_0(S) = \sum_w w^2 S_0^w = \sum_w w^2 S_w \frac{52}{52}$$

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where \( S_w \) is the total number of cards of weight \( w \) in a complete deck. There follows the standard deviation of different systems round up to the third digit (for a description the different systems with their list of weights the reader can consult [Sc] p.62).

- Uston ace-five count: \( \Sigma_0 = 0.392 \)
- Hi-Opt I: \( \Sigma_0 = 0.784 \)
- C-R Point count: \( \Sigma_0 = 0.855 \)
- Canfield expert count: \( \Sigma_0 = 0.877 \)
- Hi-Lo: \( \Sigma_0 = 0.877 \)
- Uston advanced plus-minus count: \( \Sigma_0 = 0.920 \)
- Halves count: \( \Sigma_0 = 0.961 \)
- The systematic count: \( \Sigma_0 = 1.468 \)
- Canfield master count: \( \Sigma_0 = 1.569 \)
- Zen count: \( \Sigma_0 = 1.569 \)
- Uston advanced point count: \( \Sigma_0 = 1.687 \)
- The Revere point count: \( \Sigma_0 = 1.710 \)
- Uston SS count: \( \Sigma_0 = 1.797 \)
- The Revere advanced point count: \( \Sigma_0 = 2.449 \)
- Griffin seven count: \( \Sigma_0 = 3.234 \)
- Thorp ultimate count: \( \Sigma_0 = 5.798 \)

We observe, as expected, that \( \Sigma_0 \) is larger for higher level (i.e. more precise) count systems.

From the formula in proposition 1 we can get a good approximation of \( \sigma_1 \) in real gambling situations. Observe that for shoe games \( N \) is always large due to non complete penetration (typically for shoe games we always have \( N \geq 50 \)). Thus the square of the standard deviation in the remaining deck, that is

\[
\sum_w w^2 l_w \frac{t_w}{N},
\]

is well approximated by \( \Sigma_0^2 \). Thus the square of the true count \((R/N)^2\) can be neglected (remember that the units we use to compute the number of cards is card units and not deck units as is done in practice). We conclude that the quantity

\[
\sigma'_1 = \frac{\Sigma_0}{N}
\]

is a good approximation of \( \sigma_1 \) provided that the shoe is not totally depleted (note that in the approximation we replace \( N - 1 \) by \( N \) which is acceptable).

In practice one measures \( N \) and the true count in deck units, then using the previous formula one will get \( \sigma'_1 \) in deck units that is the standard deviation of the true count in deck units which is the quantity that is used to determine the advantage in percentage. We can also approximate \( \sigma_n \):

\[
\sigma_n = \sqrt{\frac{N-1}{N-n} \sqrt{n} \sigma_1} \approx \sqrt{n} \frac{\Sigma_0}{N}
\]
where we did approximate \( \sqrt{(N - 1)/(N - n)} \) by 1, which is quite accurate (except maybe if just one deck remains, then the value of \( \sigma_n \) will be larger than the approximation).

Using this approximation one can compute the following tables for standard deviation related to the bet decision (\( \sigma_{BET} \)) and the one for the play decision (\( \sigma_{PLAY} \)), for different players and different types of games.

| Position | 1   | 4   | 7   |
|----------|-----|-----|-----|
| \( \sigma_{BET} \) | 0.877 | 0.925 | 0.971 |
| \( \sigma_{PLAY} \) | 0.449 | 0.340 | 0.085 |

**Hi-Lo, Canfield expert, Uston Advanced plus-minus (\( \Sigma_0 = 0.877 \))**

8-Deck shoe, 50% shoe played

| Position | 1   | 4   | 7   |
|----------|-----|-----|-----|
| \( \sigma_{BET} \) | 1.316 | 1.388 | 1.457 |
| \( \sigma_{PLAY} \) | 0.674 | 0.510 | 0.128 |

**Hi-Lo, Canfield expert, Uston Advanced plus-minus (\( \Sigma_0 = 0.877 \))**

8-Deck shoe, 66.6% shoe played

| Position | 1   | 4   | 7   |
|----------|-----|-----|-----|
| \( \sigma_{BET} \) | 1.754 | 1.850 | 1.942 |
| \( \sigma_{PLAY} \) | 0.898 | 0.680 | 0.170 |

**Hi-Lo, Canfield expert, Uston Advanced plus-minus (\( \Sigma_0 = 0.877 \))**

8-Deck shoe, 75% shoe played

| Position | 1   | 4   | 7   |
|----------|-----|-----|-----|
| \( \sigma_{BET} \) | 5.780 | 6.096 | 6.400 |
| \( \sigma_{PLAY} \) | 2.959 | 2.241 | 0.560 |

**Thorp ultimate (\( \Sigma_0 = 5.798 \))**

8-Deck shoe, 50% shoe played

| Position | 1   | 4   | 7   |
|----------|-----|-----|-----|
| \( \sigma_{BET} \) | 8.670 | 9.144 | 9.600 |
| \( \sigma_{PLAY} \) | 4.439 | 3.362 | 0.840 |

**Thorp ultimate (\( \Sigma_0 = 5.798 \))**
8-Deck shoe, 66.6% shoe played

| Position | 1  | 4  | 7  |
|----------|----|----|----|
| $\sigma_{BET}$ | 11.56 | 12.192 | 12.8 |
| $\sigma_{PLAY}$ | 5.918 | 4.482 | 1.120 |

**Thorpe ultimate** ($\Sigma_0 = 5.798$)

8-Deck shoe, 75% shoe played

A common feature is that the fluctuations become much larger at the end of the deck. And there is where most of the action takes place because it is at this moment than the true count uses to take its largest values! This picture is scaring for the card counter and explains why blackjack is a game with high fluctuations for card counters. Let’s see what Stanford Wong and others have to say on this subject in his reference book [Wo] (p.199):

**Comments on risks.** Peter Giles said: ”The late Ken Uston was once quoted in the ”Review Journal” as saying, ”It’s really tough to make a living at blackjack. The fluctuations will really wipe out the average guy. If I had to play by myself (instead of on a team), I probably wouldn’t be in it now”. You can quote me (S. Wong) as saying the same (...) I am still trying to determine how many units one is safe with. What is recommended in books is, in my opinion, too risky. This is one area in which it is hard to trust mathematics.

Probably not all the mathematics can be found in the classical litterature. It is surprising to note that there is no complete treatment of the long run in the blackjack litterature as the one we carry out in section 3.

To limit risks some card counters use to divide their true count by the number of players at the table (as Sonia, a professional gambler, does [So]), thus limiting the action, the fluctuation, ... but also the advantage. The standard deviation tends to infinite when the deck depletes completely. We can see this formally, making $N \to 0$ in the formulas (in a continuous model $N - 1$ is replaced by $N$). Of course $N$ takes integer values but we can compute for example that if only 1/4 of a deck remains and we assume that $\Sigma_0 \approx 1$ for the count system used, then

$$\sigma_1 \approx 4$$

thus in the best case (playing head-on),

$$\Sigma(1) \approx 8.$$  

And in the case of a full table, third base placement,

$$\Sigma(2) \approx 17.7,$$

a true count of the order of 17 is certainly like $+\infty$.

When one sees these figures one wonders how it is possible that removing just one card from the deck could have such an effect in the standard deviation. Let’s compute exactly
\( \sigma_1 \) in one particular situation. We assume that in a head-on game 13 cards remain (that is 1/4 of a deck). We assume that we use Hi-Lo and the the composing of the remaining cards is 5 high cards, 5 low cards and 3 medium cards (thus \( R = 0 \) at this point). The standard deviation of the true count after removing just one card is (in deck units)

\[
\sigma_1 = 52 \sqrt{\left( \frac{1}{12} \right)^2 \frac{5}{13} + \left( \frac{1}{12} \right)^2 \frac{5}{13} + 0} = 52 \sqrt{\frac{5}{936}} \approx 3.80
\]

And if still one is not convinced, just look at what happens when we reveal one card, and this card is for example a low card. The new true count (in deck units) will be

\[
\frac{52}{N} \cdot R = \frac{52}{12} \approx 4.33
\]

That means that in such situations the true count indicator is meaningless and the standard deviation effect takes over. In an evil scenario casinos could exploit that weakness of card counters. They could offer and advertise very good penetration in order to get tables crowded with card counters. Then deal the shoes almost to bottom to exploit the huge standard deviation effect. Cardcounters will then experience too large fluctuations that will bring them into an important risk of ruin. Thus the casino will wipe out bankrolls of some unlucky card counters. It is unlikely that the casino will make any money in the operation, due to the increased benefits of the survival card counters.
Bibliography

[Br] L. BREIMAN, *Optimal gambling systems for favorable games*, Fourth Berkeley Symposium on Probability and Statistics, I, p. 65-78, 1961.

[Gr1] P.A. GRIFFIN, *The theory of blackjack*, Huntington Press, Las Vegas, 1999.

[Gr2] P.A. GRIFFIN, *Extra stuff*, Huntington Press, Las Vegas, 1991.

[HC] L. HUMBLE, C. COOPER, *The world’s greatest blackjack book*, Doubleday, New York, 1980.

[JM] A. JALIB M’HALL, *The true count theorem*, Message posted in rec.gambling.blackjack, 7/30/1996.

[RT] L.M. ROTANDO, E.O. THORP, *The Kelly criterion and the stock market*, Amer. Math. Monthly, 99, p.922-931, 1992.

[Sc] F. SCOOLETE, *Best blackjack*, Bonus Books, Chicago, 1996.

[So] SONIA, *Personal communication*, 1/2000.

[Th] E. O. THORP, *Beat the dealer*, Vintage books, New York, 1962.

[Wo] S. WONG, *Professional blackjack*, Pi Yee Press, 1994.