IMPULSIVE MOTION ON SYNCHRONIZED SPATIAL TEMPORAL GRIDS

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Abstract. We introduce a family of kinetic vector fields on countable space-time grids and study related impulsive second order initial value Cauchy problems. We then construct special examples for which orbits and attractors display unusual analytic and geometric properties.

1. Introduction. The paper is a contribution to the research in two related though distinct fields. In loose terms, the first field is the study of evolution processes involving an increasing number of particles. The second field is the study of the motion of boundaries and interfaces of planar open domains with increasing lengths.

More specifically, we introduce a new class of second order ODEs and related initial value problems, which describe the non autonomous evolution in continuous time of an increasing number of particles subject to a forcing vector field obtained as the superposition of a smooth force field and of a concentrated force field on a countable grid in the space-time cylinder. As the number of particles increases to infinity, a limit system is obtained which is continuous in space. However, in our applications, such a limit system cannot be described by a differential equation. In other words, the discrete objects underlying our ODEs are not automata in the sense of physics, because the number of particles becomes infinite in the limit, and at the same time they are not numerical approximations of differential equations, because the limit system is not differential. In this collocation between discrete and continuous structures lies the main novelty of our study, as part of the first field of research mentioned before.

In the second field of research mentioned before, the main contribution of this paper consists in the construction of open simply-connected planar domains of finite area, topologically bounded by oriented Jordan curves of any given Hausdorff dimension between 1 and 2. Such Jordan curves are the common boundary of two adjacent open domains, hence, in physical terms, the are interfaces. Such a family of curves is constructed by means of an ODEs evolution problem of the kind mentioned before, with the forcing vector field being interpreted now as a highly irregular concentrated vector curvature field. The construction of such curvature field is obtained by combining the action of a family of similarities with a suitable

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finite group of rotations in the plane. Similarities and rotations can be given the role of control variables in a control problem aimed at designing optimal interfaces of the kind explained before. In this perspective, our study is a contribution to the mathematical modeling of small cells with highly rippled boundaries.

All complex nonlinear systems loosely described before combine short-range spatial interactions with fast time observations, with equations that take place on a sequence of increasing synchronized finite spatial-temporal grids of decreasing spatial and time sizes. In this perspective, it is also the purpose of this paper to open new lines of study for various synchronized systems that occur in the applications. A few examples will be discussed in Section 5.

We now describe the structure of the paper in some more detail. We introduce a family of impulsive initial value problem of second order:

\[
\begin{cases}
\ddot{y}(t) = g(y(t), t) & \text{if } (y(t), t) \in \mathbb{R}^2 \times [0, \infty) \\
y(0) = x \\
\dot{y}_+(0) = \mathbf{v}_0
\end{cases}
\]

with \( x \in \mathbb{R}^2 \) and \( \mathbf{v}_0 \in \mathbb{R}^2 \) given initial data and \( g(y, t) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) \) a highly irregular non-autonomous vector field on the cylinder \( (y, t) \in \mathbb{R}^2 \times [0, +\infty) \).

The vector field \( g : \mathbb{R}^2 \times [0, +\infty) \mapsto \mathbb{C} \) is supported on a subset \( S \subset G^\infty \times \mathcal{T}^\infty \) where \( G^\infty \times \mathcal{T}^\infty \) is a discrete, countable subset of the space-time cylinder \( \mathbb{R}^2 \times [0, +\infty) \). More precisely, \( g \) is of the type

\[
g : \mathbb{R}^2 \times [0, +\infty) \mapsto \mathbb{C}
\]

with

\[
\begin{cases}
g(P, t) = \gamma(P, t), & \text{if } (P, t) \in S \\
g(P, t) = (0, 0), & \text{if } (P, t) \in \mathbb{R}^2 \times [0, +\infty) \setminus S
\end{cases}
\]

and \( \gamma \) a vector field \( \gamma : S \mapsto \mathbb{C} \) with domain \( S \). The field \( g : \mathbb{R}^2 \times [0, +\infty) \mapsto \mathbb{C} \) in (1.1) is thus very irregular, both in space and time. Problem 1.1 describes an impulsive motion in \( \mathbb{R}^2 \), driven by a non autonomous force field \( \gamma \) of purely discrete nature, supported on the set \( S \).

Our main motivation for studying such a kind of highly singular, impulsive evolution equations comes from fractal theory, in particular from the classical constructions of fractals based on countable iterations of a finite family of similarities in the plane, e.g. see [6]. The description of these classical constructions as an evolution ODE in continuous time is new in the fractal literature to date.

We give various examples of this kind of problems. The force fields \( \gamma \) are constructed by exploiting suitable symmetry and similarities maps in the space \( \mathbb{R}^2 \). The actions of these maps in space is synchronized with the ticking of time, short steps in space being accompanied by fast ticks in time. Space-time synchronization is regulated by a set of multi-indices (words)

\[
\mathcal{W}^\infty = \{n\kappa i/n : n \in \{0, N\}, \kappa \in \{0, 1, \ldots, K - 1\}, i/n \in \{0, 1, \ldots, N - 1\}^n\}
\]

where \( i/n = i_1 \ldots i_n \) and \( K \geq 2, N \geq 2 \) are two integers, ordered lexicographically. Both the discrete grids \( G^\infty \times \mathcal{T}^\infty \) as well as the set \( S \) and the field \( \gamma \) inherit synchronization from the set \( \mathcal{W}^\infty \). In these examples, the trajectories traced in \( \mathbb{R}^2 \) by the solutions as time runs in \( [0, +\infty) \) have interesting geometric properties. They produce orbits and attractors with fractal features.
Section 2 is dedicated to introduce our general impulsive initial value problems and provide a definition of weak solution. Theorem 2.1 gives the existence and uniqueness of the weak solution for general impulsive problems. Theorem 2.2 and Theorem 2.3 deal with the special case of purely impulsive problems. Theorem 2.2 gives the special expression taken by the weak solution in the purely impulsive case. Theorem 2.3 shows that the problem in continuous time considered in Theorem 2.2 can be equivalently formulated as a countable set of vector inequalities, solvable iteratively.

Section 3 is dedicated to the study of fully discrete impulsive problems on synchronized space-time grids. By this we mean that not only the time variable is discretized, as done in the theory developed in Section 2, but now also the space variable is discretized and required to belong to a discrete space grid. The non-autonomous problems in this section are thus formulated on synchronized discrete space-time grids. Such a fully discrete theory allows for the applications to particle systems, as mentioned before. Theorem 3.1 establishes the existence and uniqueness of the solutions in this case.

The rest of the paper is dedicated to supply special examples of the general theory developed in Section 2 and Section 3. Our main application is given in Section 4. It consists in providing an explicit construction of a family of vector force fields to which the impulsive theory applies. The construction is based on special families of similarities and rotations of the plane. Our main result is given in Theorem 4.1. This result applies to the construction of the boundaries of open set with large Hausdorff dimensions, as illustrated before.

Analytic and geometric features of the problems treated in 4.1, as well as further applications and developments are described in the final Section 5.

As the proof of Theorem 4.1 involves rather complicated, though elementary, computations, we put these in the Appendix.

2. Impulsive initial value problems. In this section we prove three main results. The first result, Theorem 2.1, is about the existence and uniqueness of the weak solution of a second order Cauchy initial value problem for a vector force-field in the plane resulting from the superposition of a continuous (smooth) component and of a (discontinuous) impulsive component. The second result, Theorem 2.2, refers to the special case of a purely impulsive force-field, for which the continuous component vanishes. The third result, Theorem 2.3, shows that the purely impulsive Cauchy problem can be equivalently formulated as a countable family of vector equations for the force vectors, which can be solved iteratively.

The problems in this section, formally stated, are of the following kind

\[
\begin{align*}
\dot{y}(t) &= g(y(t), t) \quad (y(t), t) \in \mathbb{R}^2 \times [0, \infty) \\
y(0) &= x \\
\dot{y}(0) &= \vec{v}_0
\end{align*}
\]

where

\[
g : \mathbb{R}^2 \times [0, +\infty) \mapsto \mathbb{R}^2
\]

is a non autonomous vector field, possibly very irregular in both space and time variables, and where

\[
x \in \mathbb{R}^2
\]

and

\[
\vec{v}_0 \in \mathbb{R}^2
\]
are assigned initial value conditions.

The impulsive character of this problems is due to a suitable discretization of the time variable. We fix two integers

\[ K \geq 2, \quad N \geq 2 \]

and we define \( T^\infty \) to be the set of all \( \text{mod-}K \) rationals:

\[ \tau_{nk_i/n} := Kn + \kappa + \sum_{m=1}^{n} i_m N^{-m} \subset [0, +\infty) \]

with \( n \in \{0, N\}, \kappa \in \{0, 1, \ldots, K - 1\}, i/n = (i_1, \ldots, i_n) \in \{0, 1, \ldots, N - 1\}^n \). We write

\[ T^\infty := \bigcup_{n=0}^{\infty} T^n \]

where for every \( n \in \{0, N\} \)

\[ T^n := \{\tau_{nk_i/n} : \kappa \in \{0, 1, \ldots, K - 1\}, i/n \in \{0, 1, \ldots, N - 1\}^n\} \]

To simplify notation, we also write

\[ (i_1, \ldots, i_n) = i/n = i_1i_2\ldots i_n \quad (n \geq 1), \quad i/n = i/0 = \emptyset \quad (n = 0) \]

and, occasionally, when \( i_1 = \cdots = i_n = 0 \) we write \( i/n = i_1 \ldots i_n = 0^n \). The set \( T^\infty \) is an ordered set with the order relation induced on \( T^\infty \) by the lexicographic order \( \preceq \) \( (\prec, \succeq, \succ) \) of the set of multi-indices

\[ W^\infty = \{nk_i/n : n \in \{0, N\}, \kappa \in \{0, 1, \ldots, K - 1\}, i/n \in \{0, 1, \ldots, N - 1\}^n\}. \]

The multi-index following \( nk_i/n \) in the lexicographic order of \( W \) is denoted by \( nk_i/n+ \), the one preceding \( nk_i/n \neq 000^n \) is denoted by \( nk_i/n- \). The time that follows \( \tau_{nk_i/n} \) in \( T^\infty \) is \( \tau_{nk_i/n+} \), and the time that precedes \( \tau_{nk_i/n} > 0 \) is \( \tau_{nk_i/n-} \). We have

\[ \Delta \tau = \tau_{nk_i/n+} - \tau_{nk_i/n} = N^{-n} \]

for all \( nk_i/n \in W^\infty \).

With the set \( T^\infty \) we associate the space \( Y_{T^\infty} (0, +\infty) \) of vector functions \( y : [0, +\infty) \rightarrow \mathbb{R}^2 \), defined according to the

**Definition 2.1.** \( Y_{T^\infty} (0, +\infty) \) is the space of all vector functions \( y : [0, +\infty) \rightarrow \mathbb{R}^2 \) which have the following properties:

: (i) \( y(t) \), for \( t \in [0, +\infty) \), is continuous on \( (0, +\infty) \) and right-continuous at \( t = 0 \) with value denoted by \( y_+(0) \), \( y_+(0) = y(0) \);

: (ii) \( y(t) \) has a continuous derivative \( \dot{y} \) in each open interval \( (\tau_{nk_i/n}, \tau_{nk_i/n+}) \) with \( \tau_{nk_i/n} \in T^\infty \); \( \dot{y} \) is right-continuous at each \( \tau_{nk_i/n} \in T^\infty \), that is, \( \dot{y}(\tau_{nk_i/n}) = \dot{y}_+(\tau_{nk_i/n}) \) where

\[ \dot{y}_+(\tau_{nk_i/n}) := \lim_{0<\epsilon \rightarrow 0} \frac{y(\tau_{nk_i/n} + \epsilon) - y(\tau_{nk_i/n})}{\epsilon}; \]

\( \dot{y} \) possesses the left-limit

\[ \dot{y}_-(\tau_{nk_i/n}) := \lim_{0<\epsilon \rightarrow 0} \frac{y(\tau_{nk_i/n}) - y(\tau_{nk_i/n} - \epsilon)}{\epsilon}; \]

at each \( \tau_{nk_i/n} \in T^\infty \), \( \tau_{nk_i/n} > 0 \), the limits being taken in \( \mathbb{R}^2 \);
(iii) $\dot{y}$ has a bounded continuous (classical) derivative $\ddot{y}$ in each open interval $(\tau_{n\kappa i/n}, \tau_{n\kappa i/n+})$, $\tau_{n\kappa i/n} \in T^\infty$, while globally on $(0, +\infty)$ the derivative of $\dot{y}$ is a measure $\dot{y} := \frac{d\dot{y}}{dt} \in M(0, +\infty)$, where $M(0, +\infty)$ denotes the space of regular $\mathbb{R}^2$-vector-valued Borel measures on $(0, +\infty)$.

Property (iii) for $\dot{y}$ can be stated more explicitly as follows. In each open interval $(\tau_{n\kappa i/n}, \tau_{n\kappa i/n+})$, the $\mathbb{R}^2$-valued function $\dot{y}$ of (ii) has a $\mathbb{R}^2$-valued bounded continuous derivative $\ddot{y}$ and the identity

$$\langle \ddot{y}, \phi \rangle := -\int_0^{+\infty} \frac{d\dot{y}}{dt}(t) \dot{\phi}(t) dt = \sum_{\tau_{n\kappa i/n} \in T^\infty} \chi(\tau_{n\kappa i/n}, \tau_{n\kappa i/n+}) \dot{y}(t) \dot{\phi}(t) + \sum_{0 < t_{n\kappa i/n} \in T^\infty} \left[ \dot{y}(t_{n\kappa i/n}) - \dot{y}-(t_{n\kappa i/n}) \right] \phi(t_{n\kappa i/n})$$

is satisfied for every real-valued differentiable function $\phi$ with support in $(0, +\infty)$.

We assume that a vector field $g = g_0 + \gamma : \mathbb{R}^2 \times [0, +\infty) \mapsto \mathbb{R}^2$ (2.14) is given, which is the superposition of a bounded, Lipschitz vector field $g_0 : \mathbb{R}^2 \times (0, +\infty) \mapsto \mathbb{C}$ (2.15) and of a vector field $\gamma : \mathbb{R}^2 \times T^\infty \mapsto \mathbb{C}$ (2.16) supported in time on the discrete set $T^\infty$. We point out that, because of the discrete nature of the time-set $T^\infty$ and of the absence of any regularity assumption of $\gamma = \gamma(y, t)$ in the space-variable $y \in \mathbb{R}^2$, the vector field $g$ allows for sharp variations of the field at concentrated instants of time, as well as for very sharp discontinuities in space. Special examples of such irregular vector fields will be given later on.

We give a precise meaning to problem (2.2) by the following definition

**Definition 2.2.** Given a vector field $g$ as in (2.14), a weak solution of the Cauchy problem (2.2) is a function $y \in Y_{T^\infty} \cap C(0, +\infty)$ that satisfies the initial conditions $y(0) = x$, $\dot{y}+(0) = \vec{v}_0$ and the equation $\ddot{y}(t) = g(y(t), t)$ on $(0, +\infty)$ in the measure sense of $\mathcal{M}(0, +\infty)$.

This definition is validated by the following result, which show that a weak solution of problem (2.2) does in fact exists and is unique.

**Theorem 2.1.** For every given $g$ as in (2.14) and for every given $x$ and $\vec{v}_0$ as in (2.3), (2.4), there exists a unique weak solution of problem (2.2) according to Definition 2.2. Such a solution is given by

$$y(t) = \sum_{\tau_{n\kappa i/n} \in T^\infty} \chi(\tau_{n\kappa i/n}, \tau_{n\kappa i/n+}) \left(y_{g_0, \tau_{n\kappa i/n}}(t) + \gamma(y_{g_0, \tau_{n\kappa i/n}}(t), \tau_{n\kappa i/n+}) \right)$$

(2.17)
where for every \( \tau_{n\kappa}/n \in T^\infty \), \( y_{g_0,\tau_{n\kappa}/n} \) is the solution on \([\tau_{n\kappa}/n, \tau_{n\kappa}/n+1] \) of the problem

\[
\begin{align*}
\dot{y}(t) &= g_0(y(t), t), \quad (y(t), t) \in \mathbb{R}^2 \times [\tau_{n\kappa}/n, \tau_{n\kappa}/n+1) \\
y(0) &= x, \quad y(\tau_{n\kappa}/n) = y_{g_0, \tau_{n\kappa}/n-} \quad (\tau_{n\kappa}/n-), \quad \tau_{n\kappa}/n \geq 0 \\
\dot{y}(0) &= \dot{\nu}_0 \\
\dot{y}(\tau_{n\kappa}/n) &= y_{g_0, \tau_{n\kappa}/n} (\tau_{n\kappa}/n- + \gamma(y(\tau_{n\kappa}/n), \tau_{n\kappa}/n)) \quad \text{for } \tau_{n\kappa}/n \geq 0.
\end{align*}
\]  

(2.18)

**Proof.** We first prove the uniqueness. Let \( y_1, y_2 \) be two solutions of (2.2) and let \( y := y_1 - y_2 \). Then \( y \) belongs to \( \mathcal{V}_{T^\infty}(0, +\infty) \) and is a solution of (2.2) with \( x = 0 \in \mathbb{R}^2 \), \( \dot{\nu}_0 = 0 \in \mathbb{R}^2 \), \( \dot{g} = 0 \in \mathcal{M}(0, +\infty) \). In particular, \( y \) has first order derivative \( \dot{y} \equiv 0 \) and second order derivative \( \ddot{y} \equiv 0 \) in each open interval \((t_{n\kappa}/n, t_{n\kappa}/n+1)\) with \( t_{n\kappa}/n \in T^\infty \), satisfying the identity

\[
\langle \dot{y}, \phi \rangle = \left( \frac{d}{dt} \right)_{t_n\kappa/n} \phi = \sum_{0 < t_n\kappa/n \in T^\infty} [\dot{y}(t_{n\kappa}/n) - \dot{y}(t_{n\kappa}/n)] \phi(t_{n\kappa}) = 0
\]  

(2.19)

for every real-valued continuous function \( \phi \) with support in \((0, +\infty)\). By choosing \( \phi \) such that \( \phi \equiv 1 \) on \([t_{n\kappa}/n - \epsilon, t_{n\kappa}/n + \epsilon]\) and \( \phi \equiv 0 \) outside \([t_{n\kappa}/n - 2\epsilon, t_{n\kappa}/n + 2\epsilon]\) for a small \( \epsilon \), we get

\[
\dot{y}(t_{n\kappa}/n) = \dot{y}(t_{n\kappa}/n).
\]

Since \( \dot{y} \) is right-continuous at \( t_{n\kappa}/n \), \( y \) is differentiable on \((0, +\infty)\), hence \( \dot{y}(t) = 0 \) for all \( t \in (0, +\infty) \) and \( \dot{y}(0) = 0 \). Thus, \( y(t) \) is constant over \([0, +\infty)\). Since \( \dot{y}(0) = 0 \), we conclude that \( y(t) \equiv 0 \) on \([0, +\infty)\). This proves that \( y_1 = y_2 \).

We now prove the existence. We start with a sample function \( y \in \mathcal{V}_{T^\infty}(0, +\infty) \) and make it a weak solution of (2.2) by proceeding iteratively from one time period \((\tau_{0\kappa}/n, \tau_{0\kappa}/n+1)\) to the next, beginning with the first interval \([\tau_{0\kappa}/n, \tau_{0\kappa}/n+1) = [0, 1)\). The initial conditions determine the value \( y(t_{0\kappa}/n) = y(0) = x \), as well as the value \( \dot{y}(t_{0\kappa}/n) = \dot{y}(0) = \dot{\nu}_0 \). By the properties of functions in \( \mathcal{V}_{T^\infty}(0, +\infty) \), \( y \) has a continuous derivative \( \dot{y} \) and a bounded continuous second order derivative \( \ddot{y} \) in the open interval \((0, 1)\). In this interval, the vector field \( g \) is reduced to its smooth component \( g_0 \) over \( \mathbb{R}^2 \times (0, 1) \). Therefore, a smooth solution \( y_{g_0, 0} \) of (2.2) exists in \( C^1((0, 1), \mathbb{R}^2) \), bounded with bounded derivative over \([0, 1]\). Moreover, \( y_{g_0, 0} \) has a finite left-limits \( y_{g_0, 0}(1-) \) at \( t = 1 \) over \([0, 1]\). We define \( y(t) := y_{g_0, 0}(t) \) over \([0, 1]\). Since \( y \in \mathcal{V}_{T^\infty}(0, +\infty) \) is continuous over \([0, +\infty)\), in particular on \([0, 1]\), we extend continuously \( y = y_{g_0, 0} \) from \([0, 1]\) to \([0, 1]\) by defining \( y(1) := y_{g_0, 0}(1-) \). We move now to the point \((P, 1), \) where \( P := y(1), \) and to the next contiguous interval to \([\tau_{0\kappa}/n, \tau_{0\kappa}/n+1) = [0, 1) \) in \( T^\infty \), which is the interval \([\tau_{0\kappa}/n, \tau_{0\kappa}/n+1) = [1, 2]\). We want to extend the solution \( y \) to the interval \([1, 2]\). The initial condition \( y(1) \) has already been determined. We must prescribe the initial condition for the derivative, that is the value \( \dot{y}(1) \). At the time \( t = 1 \), by property (ii) of the function \( y \in \mathcal{V}_{T^\infty}(0, +\infty) \), the left-limit value \( \dot{y}_{g_0, 0}(1-) \) is determined. On the other hand, since \( 1 \in T^\infty \), the vector field \( \gamma \) is defined at the point \((P, 1) \) in \( \mathbb{R}^2 \times T^\infty \), where it has the value \( \gamma(1) = \gamma(y(1), 1) \). We then define the initial condition at \( t = 1 \) to be \( \gamma(1) := \dot{y}_{g_0, 0}(1-) + \gamma(y(1), 1) \). We can now extend \( y \) to the interval \([1, 2]\), by proceeding similarly as done in the preceding interval \([0, 1]\). In fact, as a function in \( \mathcal{V}_{T^\infty}(0, +\infty) \), \( y \) has a continuous derivative \( \dot{y} \) and a bounded continuous second order derivative \( \ddot{y} \) also in the open interval \([1, 2]\), where \( g = g_0 \). Therefore, a smooth solution \( y_{g_0, 1} \) of (2.2) exists in \( C^1([1, 2], \mathbb{R}^2) \), bounded with bounded derivative
over $[1, 2]$ and $y_{g_0, 1}$ has a finite left-limit $y_{g_0, 1}(2−)$ at $t = 2$ over $[1, 2)$. We thus define $y(t) = y_{g_0, 1}(t)$ over $[1, 2)$, and since $y \in \mathcal{Y}_{\tau} = (0, +\infty)$ is continuous over $[0, +\infty)$ in particular on $[1, 2]$, we extend $y = y_{g_0, 1}$ continuously from $[1, 2)$ to $[1, 2]$ by defining the value of $y$ at $t = 2$ by setting $y(2) = y_{g_0, 1}(2−)$. It is clear how to proceed in a similar way to get for every interval $(\tau_{n\kappa/n}, \tau_{n\kappa/n+})$ a function $y := y_{g_0, \tau_{n\kappa/n}}$ defined on $(\tau_{n\kappa/n}, \tau_{n\kappa/n+})$ which has the following properties: $y = y_{g_0, \tau_{n\kappa/n}}$ satisfies at $t = \tau_{n\kappa/n}$ the initial conditions

$$
y(\tau_{n\kappa/n}) = y_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n})$$

$$
y'(\tau_{n\kappa/n}) = y'_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n}) + \gamma(y(\tau_{n\kappa/n}), \tau_{n\kappa/n})$$

where the values $y_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n})$ and $y'_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n})$ are determined as before. Moreover, $y_{g_0, \tau_{n\kappa/n}}$ is a smooth solution of (2.2) in the open interval $(\tau_{n\kappa/n}, \tau_{n\kappa/n+})$ that can be extended to a function in $C^1((\tau_{n\kappa/n}, \tau_{n\kappa/n+}))$ – that is, to a function which is bounded with a bounded derivative over $(\tau_{n\kappa/n}, \tau_{n\kappa/n+})$ – such extension having a finite left-limit $y_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n+})$ at $t = \tau_{n\kappa/n+}$. We can thus consider the following function of $t \in [0, +\infty)$:

$$y(t) = \sum_{\tau_{n\kappa/n} \in \mathcal{T}} x(\tau_{n\kappa/n}, \tau_{n\kappa/n+})(t) y_{g_0, \tau_{n\kappa/n}}(t) + \gamma(y_{g_0, \tau_{n\kappa/n+}}(\tau_{n\kappa/n+}), \tau_{n\kappa/n+}).$$

What remains to prove is that this function is in the space $y \in \mathcal{Y}_{\tau} = (0, +\infty)$ and is a weak solution of problem (2.2) in the sense of Definition 2.2. Let us first check that $y \in \mathcal{Y}_{\tau} = (0, +\infty)$. Properties (i) and (ii) are satisfied by the very construction of $y$. The first part of property (iii) is also satisfied, because $\dot{y}$ has a bounded continuous classical derivative $\ddot{y}$ in each open interval $(\tau_{n\kappa/n}, \tau_{n\kappa/n+})$, $\tau_{n\kappa/n} \in \mathcal{T}$. It only remains to verify that, globally on $(0, +\infty)$, the derivative of $y$ is a measure $\dot{y} := dy/dt \in \mathcal{M}(0, \infty)$. In view of (2.1) and (2.19), by a simple partition of unit argument it suffices to compute the derivative in weak form at each point $\tau_{n\kappa/n}$. For every small $\epsilon > 0$ and every real-valued continuous function $\phi$ such that $\phi \equiv 1$ on $(\tau_{n\kappa/n} - \epsilon, \tau_{n\kappa/n})$ and $\phi \equiv 0$ outside $(\tau_{n\kappa/n} - 2\epsilon, \tau_{n\kappa/n} + 2\epsilon)$, by identifying $\dot{y}$ separately to the left and to the right of $\tau_{n\kappa/n}$, we get by integration by parts for every $\epsilon > 0$ small enough

$$\langle \dot{y}, \phi \rangle = -\langle y, \phi' \rangle = -\int_{\tau_{n\kappa/n} - 2\epsilon}^{\tau_{n\kappa/n} + 2\epsilon} \dot{y}(t) \phi(t) dt =$$

$$= \int_{\tau_{n\kappa/n} - 2\epsilon}^{\tau_{n\kappa/n}} \dot{y}_{g_0, \tau_{n\kappa/n}}(t) \phi(t) dt - \int_{\tau_{n\kappa/n}}^{\tau_{n\kappa/n} + 2\epsilon} \dot{y}_{g_0, \tau_{n\kappa/n}}(t) \phi(t) dt =$$

$$= (\ddot{y}_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n}) \phi(\tau_{n\kappa/n}) + \int_{\tau_{n\kappa/n} - 2\epsilon}^{\tau_{n\kappa/n}} \ddot{y}_{g_0, \tau_{n\kappa/n}}(t) \phi(t) dt) +$$

$$+ (\ddot{y}_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n}) \phi(\tau_{n\kappa/n}) + \int_{\tau_{n\kappa/n}}^{\tau_{n\kappa/n} + 2\epsilon} \ddot{y}_{g_0, \tau_{n\kappa/n}}(t) \phi(t) dt).$$

In the limit as $\epsilon \to 0$, since $\phi(t_{n\kappa/n}) = 1$, this gives

$$\langle \dot{y}, \phi \rangle = [\ddot{y}_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n}) - \ddot{y}_{g_0, \tau_{n\kappa/n} -}(\tau_{n\kappa/n} -)]$$

that shows that $\dot{y}$ at $\tau_{n\kappa/n} -$ is a Dirac measure of mass $[\ddot{y}_{g_0, \tau_{n\kappa/n}}(\tau_{n\kappa/n}) - \ddot{y}_{g_0, \tau_{n\kappa/n} -}(\tau_{n\kappa/n} -)]$. Hence $\dot{y}$ at $\tau_{n\kappa/n} - \in \mathcal{M}(0, +\infty)$. Thus the function $y$ belongs to the space $\mathcal{Y}_{\tau} = (0, +\infty)$. We now show that $y$ is a weak solution of (2.2)
over \([0, +\infty)\). In fact, by (2.18), we have
\[
\dot{y}(t_{n+1}/n) - y(t_{n+1}/n) = \gamma(y(t_{n+1}/n), \tau_{n+1}/n)
\]
and the function \(y\) can be written over the whole time range \([0, +\infty)\) as
\[
y(t) = \sum_{\tau_{n+1}/n \in \mathcal{T}} \chi_{[\tau_{n+1}/n, \tau_{n+1}/n+)}(t) y_{g_0, \tau_{n+1}/n}(t) + \gamma(y(t_{n+1}/n+), \tau_{n+1}/n+). \tag{2.20}
\]

In this expression, by our construction, each function \(y_{g_0, \tau_{n+1}/n}\) is the solution of problem (2.18). This proves that \(y\) is a solution of problem (2.2) according to Definition 2.2, hence \(y\) is the weak solution of this problem and is of the form stated in Theorem 2.1. In particular, we have proved that the weak solution of problem (2.2) does in fact exist. The proof of Theorem 2.1 is now complete. □

In the rest of this paper we shall focus on the special case of purely impulsive problems, that is, on the case where the field \(g\) in (2.14) consists only of an impulsive component, supported on a subset of \(\mathbb{R}^2 \times [0, +\infty)\), and \(g_0 = 0\). Such special, synchronized vector fields are called kinetic vector field and will be defined precisely later. Clearly, in a purely impulsive problem, the intermediate solutions \(y_{g_0, \tau_{n+1}/n}\), evolving between two consecutive times of \(\mathcal{T}\), vanish, thus the solution over the whole time range \([0, +\infty)\) takes a simplified form, as we now describe in more detail.

We assume that we are given a subset
\[
\mathcal{S} \subset \mathbb{R}^2 \times \mathcal{T}, \tag{2.21}
\]
a vector
\[
x \in \mathbb{R}^2 \quad \text{such that } (x, 0) \in \mathcal{S}. \tag{2.22}
\]
We define a kinetic vector field to be a map
\[
\gamma = \mathcal{S} \setminus \{(x, 0)\} \to \mathbb{C} \tag{2.23}
\]
that assigns a vector \(\gamma(P, \tau) \in \mathbb{C}\) to each space-time location \((P, \tau) \in \mathcal{S}\) for \((P, \tau) \neq (x, 0)\) and satisfies the property
\[
(P, \tau) \in \mathcal{S} \implies (P + \gamma(P, \tau) \Delta \tau, \tau) \in \mathcal{S} \tag{2.24}
\]
where, according to (2.7) and (2.12), \(\Delta \tau = \tau_{n+1}/n - \tau_{n}/n = N^{-n}\) for \(\tau \in [K_n, K(n+1)]\). Note that, by its very definition, a kinetic vector field should be more accurately defined as the triple \(\{\mathcal{S}, x, \gamma\}\), to make clear that the field is only defined on the domain \(\mathcal{S} \setminus \{(x, 0)\} \subset \mathbb{R}^2 \times \mathcal{T}\). A kinetic vector field \(\{\mathcal{S}, x, \gamma\}\) is canonically extended from its domain \(\mathcal{S} \setminus \{(x, 0)\}\) to the whole space-time cylinder \(\mathbb{R}^2 \times [0, +\infty)\), by defining the field
\[
g = g_\gamma : \mathbb{R}^2 \times [0, +\infty) \to \mathbb{C} \tag{2.25}
\]
to be
\[
\begin{cases}
g(P, t) = \gamma(P, t), & \text{if } (P, t) \in \mathcal{S} \\g(P, t) = (0, 0), & \text{if } (P, t) \in \mathbb{R}^2 \times [0, +\infty) \setminus \mathcal{S}.
\end{cases} \tag{2.26}
\]
Such a canonical extension, more accurately denoted by \(g = g(\mathcal{S}, x, \gamma)\), will also be named a kinetic vector field on \(\mathbb{R}^2 \times [0, +\infty)\).

We now assume that a kinetic vector field \(g = g(\mathcal{S}, x, \gamma)\) is given, and that, additionally, a second vector
\[
\overrightarrow{v}_0 \in \mathbb{R}^2 \quad \text{such that } (x + \overrightarrow{v}_0, 1) \in \mathcal{S} \tag{2.27}
\]
is also assigned. With the data \( \{S, x, \gamma, \vec{v}_0\} \), and the associated vector field \( g = g_{\gamma} \) at hand, we consider the problem formally stated as

\[
\begin{align*}
\dot{y}(t) &= g(y(t), t) \quad \text{if } (y(t), t) \in \mathbb{R}^2 \times [0, \infty) \\
y(0) &= x \\
\gamma_{\tau} = 0
\end{align*}
\]  

(2.28)

Problem (2.28) is a special impulsive initial value problem of the kind of problem (2.2) introduced before. Therefore, we can define a weak solution for Problem (2.28) by just applying the Definition 2.2 given for the problems (2.2).

Our new result for the purely impulsive problem (2.28) is the following

**Theorem 2.2.** Given a kinetic vector field \( g = g_{S, x, \gamma} \) and a vector \( \vec{v}_0 \) as in (2.26) and (2.27), problem (2.28) has a unique weak solution \( y = y(t) \) for \( t \in [0, +\infty) \) according to Definition 2.2. The solution \( y = y(t) \) has the expression

\[
y(t) = \sum_{\tau_{n\kappa}/n \in \mathcal{T}^\infty} \chi_{\{\tau_{n\kappa}/n, \tau_{n\kappa}/n+\}}(t) \cdot \left( y(\tau_{n\kappa}/n) + \frac{t - \tau_{n\kappa}/n}{\tau_{n\kappa}/n+ - \tau_{n\kappa}/n} \cdot (y(\tau_{n\kappa}/n) - y(\tau_{n\kappa}/n)) \right)
\]  

(2.29)

for all \( t \in [0, +\infty) \). Moreover, \((y(\tau_{n\kappa}/n), \tau_{n\kappa}/n)\) \( \in S \) for every \( \tau_{n\kappa}/n \in \mathcal{T}^\infty \).

**Proof.** As observed before, Theorem 2.1 applies to the special case at hand. Therefore, there exists a unique weak solution \( y \) of problem (2.28) according to Definition 2.2. The function \( y \) belongs to the space \( \mathcal{Y}_{\tau^\infty}(0, +\infty) \). Again by Theorem 2.1, \( y \) is given by (2.17), hence \( y \) is given on every time interval \([\tau_{n\kappa}/n, \tau_{n\kappa}/n+\]) by

\[
y(t) = y_{\tau_0, \tau_{n\kappa}/n}(t) + \gamma(y_{\tau_0, \tau_{n\kappa}/n}(\tau_{n\kappa}/n), \tau_{n\kappa}/n) \quad \text{for } t \in [0, +\infty),
\]

where \( y_{\tau_0, \tau_{n\kappa}/n} \) is the solution on \([\tau_{n\kappa}/n, \tau_{n\kappa}/n+\]) of the problem (2.18). In the special case at hand, in problem (2.18) we have \( g_0 = 0 \). Therefore, \( y \) is given on every time interval \([\tau_{n\kappa}/n, \tau_{n\kappa}/n+\]) by

\[
y(t) = y_{\tau_0, \tau_{n\kappa}/n}(t) + \gamma(y_{\tau_0, \tau_{n\kappa}/n}(\tau_{n\kappa}/n), \tau_{n\kappa}/n) + \gamma(y(\tau_{n\kappa}/n), \tau_{n\kappa}/n), \tau_{n\kappa}/n) > 0
\]

where \( y_{\tau_0, \tau_{n\kappa}/n} \) is the solution of the problem

\[
\begin{align*}
\dot{y}(t) &= 0, \quad (y(t), t) \in \mathbb{R}^2 \times [\tau_{n\kappa}/n, \tau_{n\kappa}/n+]
\end{align*}
\]

(2.30)

This implies in particular that for every \( \tau_{n\kappa}/n \geq 0 \) the solution \( y(t) \) follows over the time interval \([\tau_{n\kappa}/n, \tau_{n\kappa}/n+\]) a rectilinear trajectory

\[
y(\tau_{n\kappa}/n) + \frac{t - \tau_{n\kappa}/n}{\tau_{n\kappa}/n+ - \tau_{n\kappa}/n} \cdot (y(\tau_{n\kappa}/n+), \tau_{n\kappa}/n) \leq t < \tau_{n\kappa}/n+
\]

connecting the point \( y(\tau_{n\kappa}/n) \) to the point

\[
y(\tau_{n\kappa}/n+) = \lim_{t \to \tau_{n\kappa}/n+} y(t)
\]

Since \( y \in \mathcal{Y}_{\tau^\infty}(0, +\infty) \), the function \( y \) is continuous from \( t \in [0, +\infty) \) to \( \mathbb{R}^2 \) and \( y(\tau_{n\kappa}/n+) = y(\tau_{n\kappa}/n) \) for every \( \tau_{n\kappa}/n \geq 0 \). Therefore, \( y(t) \) has the expression

\[
y(\tau_{n\kappa}/n) + \frac{t - \tau_{n\kappa}/n}{\tau_{n\kappa}/n+ - \tau_{n\kappa}/n} \cdot (y(\tau_{n\kappa}/n+), \tau_{n\kappa}/n) - y(\tau_{n\kappa}/n)
\]
over \([\tau_{nki}/n,\tau_{nki}/n+]) for all \(\tau_{nki}/n \geq 0\). This proves (2.29).

In order to complete the proof of the Theorem we need, preliminarily, a family of recursive identities over the sequence of time intervals \([nki/n, nki/n+]\) defined by (2.7). We obtain these identities, stated below in (2.31), as follows.

From (2.29) it follows that for all intervals \([nki/n, nki/n+]\), \(y\) has a constant vector derivative \(\dot{y}(t)\) for every \(t \in (\tau_{nki}/n, \tau_{nki}/n+).\) Given by

\[
\dot{y}(t) = \frac{y(\tau_{nki}/n +) - y(\tau_{nki}/n)}{\tau_{nki}/n + - \tau_{nki}/n} = \dot{y}+(\tau_{nki}/n) \quad t \in [\tau_{nki}/n, \tau_{nki}/n+]
\]

where \(\dot{y}(t)\) is taken to be the right derivative \(\dot{y}+(\tau_{nki}/n)\) at \(t = \tau_{nki}/n\). Therefore, we have

\[
y(\tau_{nki}/n+) = y(\tau_{nki}/n) + \dot{y}+(\tau_{nki}/n)(\tau_{nki}/n+ - \tau_{nki}/n)\quad (2.30)
\]

The initial condition at \(t = \tau_{nki}/n\) in problem (2.28), in the present purely impulsive case, is

\[
\dot{y}+(\tau_{nki}/n) = y(\tau_{nki}/n-) + \gamma(y(\tau_{nki}/n), \tau_{nki}/n)\quad (2.29)
\]

By replacing this expression of \(\dot{y}+(\tau_{nki}/n)\) into the previous identity we get

\[
y(\tau_{nki}/n+) = y(\tau_{nki}/n-) + \gamma(y(\tau_{nki}/n), \tau_{nki}/n)[(\tau_{nki}/n+ - \tau_{nki}/n)]
\]

We recall that for every \(nki/n\) we have

\[
\tau_{nki}/n+ - \tau_{nki}/n = \tau_{nki}/n - \tau_{nki}/n- = \mathcal{N}^{-n}
\]

Therefore we can rewrite the previous identity as

\[
y(\tau_{nki}/n+) = y(\tau_{nki}/n-) + \gamma(y(\tau_{nki}/n), \tau_{nki}/n)(\tau_{nki}/n+ - \tau_{nki}/n)
\]

An expression for \(y(\tau_{nki}/n)\) quite analogue to the expression of \(y(\tau_{nki}/n+)\) found before can be obtained by replicating in the interval \([nki/n-, nki/n]\) the argument done before in the interval \([nki/n, nki/n+]\). We obtain

\[
y(\tau_{nki}/n) = y(\tau_{nki}/n-) + \dot{y}(\tau_{nki}/n-)\tau_{nki}/n- - \tau_{nki}/n-\)
\]

By replacing this expression into the preceding equality, we finally get the relation

\[
y(\tau_{nki}/n+) = y(\tau_{nki}/n) + \gamma(y(\tau_{nki}/n), \tau_{nki}/n)(\tau_{nki}/n+ - \tau_{nki}/n)
\]

for every interval \((\tau_{nki}/n, \tau_{nki}/n+)\) and all \(nki/n \in W^\infty\). We can rewrite these relations in space-time notation as

\[
(y(\tau_{nki}/n+), \tau_{nki}/n+) = (y(\tau_{nki}/n), \tau_{nki}/n)\Delta \tau, \tau_{nki}/n) \quad (2.31)
\]

with \(\Delta \tau = \tau_{nki}/n+ - \tau_{nki}/n = \mathcal{N}^{-n}\) for every \(nki/n \in W^\infty\).

As said before, we shall now make use of these relations in order to prove, iteratively, that for every \(\tau_{nki}/n \in T^\infty\) we have \((y(\tau_{nki}/n), \tau_{nki}/n) \in S\). We proceed recursively in the lexicographic order of \(nki/n \in W^\infty\).

We consider the initial interval \([\tau_{00}/0, \tau_{00}/0+] = [0, 1]\). By the initial condition on \(y\) at \(t = 0\) in problem (2.28), we have \(y(0) = x\), hence \((y(0), 0) = (x, 0)\) that belongs to \(S\) by the assumption (2.22). Always at \(t = 0\), the initial condition on \(y\) gives \(\dot{y}(0) = \tau_0\), where \(\tau_0\) is assigned in (2.27). As seen before, in the interval \([0, 1]\) the derivative of \(y\) is constant, equal to \(y(\tau_{00}/0+ - \tau_{00}/0)) / (\tau_{00}/0+ - \tau_{00}/0) = y(1) - y(0)\). Therefore, we have \(y(1) = y(0) + \tau_0 = x + \tau_0\). This vectors belongs to \(S\) by assumption (2.27). Therefore, we have proved that \((y(\tau_{00}/0), \tau_{00}/0) = (y(0), 0) \in S\) as well as \((y(\tau_{01}/0), \tau_{01}/0) = (y(1), 1) \in S\).
Theorem 2.3. Given a kinetic vector field \( g = g(S, x, \gamma) \) and a vector \( \bar{v}_0 \) as in (2.26) and (2.27), a function \( y \in \mathcal{Y}_{\mathcal{T}}(0, +\infty) \) is the unique weak solution of problem (2.28) if and only if \( y(\tau_{n\kappa_i/n}, \tau_{n\kappa_i/n}) \in S \) for every \( \tau_{n\kappa_i/n} \in \mathcal{T}^{\infty} \) and the following vector equations in the space \( \mathbb{R}^2 \times \mathcal{T}^{\infty} \) are satisfied

\[
\begin{align*}
\left\{ \begin{array}{l}
(0, 0) = (x, 0) \\
(1, 1) = (x + \bar{v}_0, 1)
\end{array} \right. \\
\gamma_{n\kappa_i/n} = \gamma(\tau_{n\kappa_i/n}, \tau_{n\kappa_i/n}) \quad \text{for all } n\kappa_i/n \in \mathcal{W}^{\infty} \setminus \{00\} \\
\end{align*}
\]

where the vectors \( \gamma_{n\kappa_i/n} \) are given by

\[
\gamma_{n\kappa_i/n} = \gamma(\tau_{n\kappa_i/n}, \tau_{n\kappa_i/n}) \quad \text{for all } n\kappa_i/n \in \mathcal{W}^{\infty} \setminus \{00\}
\]

and the vectors \( \mathbf{a}_{n\kappa_i/n} \) are given by (2.33). Such a solution \( y \) does exists. Moreover, the system of equations (2.34) can be solved iteratively in the lexicographic order of \( \mathcal{W}^{\infty} \) by setting

\[
\begin{align*}
t_{n\kappa_i/n} &= t_{n\kappa_i/n} - \gamma_{n\kappa_i/n} \quad \text{for all } n\kappa_i/n \in \mathcal{W}^{\infty} \setminus \{00\} \\
y(\tau_{n\kappa_i/n+}) &= y(\tau_{n\kappa_i/n}) + t_{n\kappa_i/n} \Delta \tau, \quad \Delta \tau = N^{\tau_{n\kappa_i/n}}
\end{align*}
\]

Note. The vectors \( \gamma(\tau_{n\kappa_i/n}, \tau_{n\kappa_i/n}) \), which are the data of the system (2.34), are associated with the function \( y \) and the given vector field \( \gamma \), and the vectors

We now move to the next interval, \([\tau_{01\theta}, \tau_{01\theta+}) = [1, 2)\). Since we have already proved that \((y(1), 1) \in S\), we must only prove that \((y(2), 2) \in S\). By the relations (2.31) we have

\[(y(2), 2) = (y(1), 1) + \gamma((y(1), 1)).\]

Since, by what seen before, \((y(1), 1) \in S\), by the assumption (3.56) on the field \( \gamma \) and its domain \( S \), applied to \((P, \tau) = (y(1), 1)\), we get \((y(1), 1) + \gamma((y(1), 1)) \in S\). This proves that \((y(2), 2) \in S\). By iterating the argument for all intervals \([\tau_{n\kappa_i/n}, \tau_{n\kappa_i/n+})\) as \(n\kappa_i/n\) runs increasingly in the lexicographic order of \( \mathcal{W}^{\infty} \), we prove that \((y(\tau_{n\kappa_i/n}, n\kappa_i/n)) \in S\) for every \( n\kappa_i/n \in \mathcal{W}^{\infty}\), that is, for every \( \tau_{n\kappa_i/n} \in \mathcal{T}^{\infty}\). This concludes the proof of Theorem 2.2.
\(a_{\kappa i/n}\) also depend on \(y\), therefore the system to be solved in \(\{y(\tau_{\kappa i/n})\}\) has an implicit dependence on \(y\).

**Proof.** Let \(y \in \mathcal{Y}_T = (0, +\infty)\) be the weak solution of problem (2.28). By (2.29) of Theorem 2.2 the first derivative \(\dot{y}\) of \(y\) has the expression

\[
\dot{y}(t) = \sum_{\tau_{\kappa i/n} \in T^\infty} \chi|_{\tau_{\kappa i/n}, \tau_{\kappa i/n}^+}|(t) a_{\kappa i/n} \delta_{\tau_{\kappa i/n}}(dt) \in \mathcal{M}_{T^\infty}, \quad t \in [0, +\infty), \tag{2.37}
\]

in agreement with property (ii) of the functions of the space \(\mathcal{Y}_T = (0, +\infty)\) of Definition 2.1. By moving from the interval \([\tau_{\kappa i/n^-}, \tau_{\kappa i/n^+}]\) to the next \([\tau_{\kappa i/n}, \tau_{\kappa i/n}^-]\), \(\dot{y}\) undergoes a vector-change across \(t = \tau_{\kappa i/n} > 0\) given by the vector \(\tau_{\kappa i/n} - t_{\kappa i/n^-}\). By property (iii) of functions of \(y \in \mathcal{Y}_T = (0, +\infty)\), the second derivative \(\ddot{y}\) of \(y\) on the open time range \((0, +\infty)\) exists in the measure sense of \(\mathcal{M}_{T^\infty}\) and is given by

\[
\ddot{y} = \sum_{0 < \tau_{\kappa i/n} \in T^\infty} \chi|_{\tau_{\kappa i/n}, \tau_{\kappa i/n}^+}|(t) a_{\kappa i/n} \delta_{\tau_{\kappa i/n}}(dt) \in \mathcal{M}_{T^\infty}, \tag{2.38}
\]

where \(a_{\kappa i/n}\) are the vectors in (2.33). Since \(y\) is the weak solution of problem (2.28) for the vector field \(g\), by Definition 2.2 the equation \(\ddot{y}(t) = g(y(t), t)\) is satisfied on \((0, +\infty)\) in the measure sense of \(\mathcal{M}(0, +\infty)\), therefore

\[
\ddot{y} = \sum_{0 < \tau_{\kappa i/n} \in T^\infty} \chi|_{\tau_{\kappa i/n}, \tau_{\kappa i/n}^+}|(t) \gamma_{\tau_{\kappa i/n}} \delta_{\tau_{\kappa i/n}}(dt) \in \mathcal{M}_{T^\infty},
\]

where the vector \(\gamma_{\tau_{\kappa i/n}}\) are given by (2.35). This identifies the vectors \(a_{\kappa i/n} = \gamma_{\tau_{\kappa i/n}}\) for all \(\kappa i/n \in \mathcal{W}_\infty\) and shows that \(y\) satisfies the system of inequalities (2.34).

Conversely, let \(y\) be a generic function of the space \(\mathcal{Y}_T = (0, +\infty)\) such that \((y(\tau_{\kappa i/n}), \tau_{\kappa i/n}) \in S\) for every \(\tau_{\kappa i/n} \in T^\infty\) and such that the identities (2.34) hold. We must prove that \(y\) is a weak solution of (2.28), and for that it suffices to prove that the values \(y(\tau_{\kappa i/n})\) in (2.34) can be identified to be the values that make \(y\) the weak solution of problem (2.28) for the vector field \(g\). By Definition 2.2 this requires to prove the initial conditions \(y(0) = x\), \(\dot{y}(0) = \overrightarrow{\gamma_0}\), and that the equation \(\ddot{y}(t) = g(y(t), t)\) is satisfied on \((0, +\infty)\) in the measure sense of \(\mathcal{M}(0, +\infty)\). The conditions \(y(0) = x\) and \(\dot{y}(0) = \overrightarrow{\gamma_0}\) follow from the first two equations of (2.34). Since \(y \in \mathcal{Y}_T = (0, +\infty)\), as seen before, the second derivative \(\ddot{y}\) of \(y\) on \((0, +\infty)\) exists in the measure sense of \(\mathcal{M}_{T^\infty}\) and is given by

\[
\ddot{y} = \sum_{0 < \tau_{\kappa i/n} \in T^\infty} \chi|_{\tau_{\kappa i/n}, \tau_{\kappa i/n}^+}|(t) a_{\kappa i/n} \delta_{\tau_{\kappa i/n}}(dt) \in \mathcal{M}_{T^\infty}, \tag{2.39}
\]

where \(a_{\kappa i/n} = t_{\kappa i/n} - t_{\kappa i/n^-}\) and \(t_{\kappa i/n^-}\) are the vectors (2.32). We determine the vectors \(t_{\kappa i/n}\) for all \(\kappa i/n \in \mathcal{W}_\infty \setminus \{00\}\) iteratively in the lexicographic order of \(\mathcal{W}_\infty\), starting with \(t_{010} = y(1) - y(0)\) for \(\kappa i/n = 01\), by means of the equations

\[
t_{\kappa i/n} = t_{\kappa i/n^-} + \gamma_{\kappa i/n}\]

in (2.36). Then, we determine the values \(y(\tau_{\kappa i/n})\) for all \(\kappa i/n \in \mathcal{W}_\infty\), by starting with \(y(00) = y(0) = x\) for \(\kappa i/n = 00\) and by applying iteratively the equations \(y(\tau_{\kappa i/n}^-) = y(\tau_{\kappa i/n}) + t_{\kappa i/n}\) also from (2.36).
With all the values $y(\tau_{n\kappa i/n})$ so determined, the function

$$y(t) = \sum_{\tau_{n\kappa i/n} \in T} \chi(\tau_{n\kappa i/n}, \tau_{n\kappa i/n+1})(t),$$

for all $t \in [0, +\infty)$ is the weak solution of problem (2.28) for the vector field $g$. In fact, by the very construction of the values $y(\tau_{n\kappa i/n})$, we have

$$y(\tau_{n\kappa i/n}) = \sum_{0 < \tau_{n\kappa i/n} \in T^\infty} \chi(\tau_{n\kappa i/n}, \tau_{n\kappa i/n+1})(t) a_{\tau_{n\kappa i/n}} \delta_{\tau_{n\kappa i/n}}(dt).$$

Here, by (2.36), we have

$$a_{\tau_{n\kappa i/n}} = \gamma_{n\kappa i/n}$$

Since $(y(\tau_{n\kappa i/n}), \tau_{n\kappa i/n}) \in S$, we have

$$g((y(\tau_{n\kappa i/n}), \tau_{n\kappa i/n}) = \gamma(y(\tau_{n\kappa i/n}), \tau_{n\kappa i/n}) = \gamma_{n\kappa i/n}$$

for every $n\kappa i/n \in W^\infty$. This implies that the equation $\dot{y}(t) = g(y(t), t)$ is satisfied on $(0, +\infty)$ in the measure sense of $M(0, +\infty)$. This concludes the proof that $y$ is the weak solution of the Cauchy problem (2.28). This completes also the proof of Theorem 2.3.

**Remark 2.1.** From the expression (2.29) of the solutions $y(t)$, we see that a trajectory $y = y(t)$ changes its direction at a location $y(\tau_{n\kappa i/n})$ if and only if the three locations $y(\tau_{n\kappa i/n}, n\kappa i/n, \tau_{n\kappa i/n+1})$, $y(\tau_{n\kappa i/n})$, $y(\tau_{n\kappa i/n+1})$ do not lie on the same straight line of $\mathbb{R}^2$, in which case the two consecutive vectors $t_{n\kappa i/n}$ and $t_{n\kappa i/n}$ are not collinear and $a_{n\kappa i/n} \neq 0$. In the opposite case, that is when the three locations $y(\tau_{n\kappa i/n}, n\kappa i/n, \tau_{n\kappa i/n+1})$, $y(\tau_{n\kappa i/n})$, $y(\tau_{n\kappa i/n+1})$ do lie on the same straight line of $\mathbb{R}^2$, the two consecutive vectors $t_{n\kappa i/n}$ and $t_{n\kappa i/n}$ and the vector $a_{n\kappa i/n}$ also lie on the same line. In this case there will be no directional change of the vector derivative of $y(t)$ at the location $y(\tau_{n\kappa i/n})$. However, at such a location we may still have a change of the scalar derivative, and, since the time intervals $\tau_{n\kappa i/n+1} - \tau_{n\kappa i/n} = \tau_{n\kappa i/n} - \tau_{n\kappa i/n+1} = N^{-n}$ are all equal, that will happen if and only if $|y(\tau_{n\kappa i/n+1}) - y(\tau_{n\kappa i/n})| \neq |y(\tau_{n\kappa i/n}) - y(\tau_{n\kappa i/n-1})|$, in which case the two vectors $t_{n\kappa i/n}$ and $t_{n\kappa i/n}$, though being aligned, have different magnitudes and $|a_{n\kappa i/n}| \neq 0$. Finally, it can be easily checked that if the derivative $dy(t)/dt$ has a vector-jump at the time $t = \tau_{n\kappa i/n} > 0$, then the jump is the vector

$$a_{\tau_{n\kappa i/n}} = [y(\tau_{n\kappa i/n+1}) - 2y(\tau_{n\kappa i/n}) + y(\tau_{n\kappa i/n+1})] N^{-n}, \quad n\kappa i/n \in W^\infty \quad (2.40)$$

3. Kinetic vector fields on synchronized grids. The objective of this section is to perform discretization not only in the time variable, as done in Section 2 but in both time and space variables simultaneously, what introduces synchronization on the discrete space-time grids. We accomplish this goal by replacing the space-time cylinder $\mathbb{R}^2 \times [0, +\infty)$ with a discrete subset $G^\infty \times T^\infty \subset \mathbb{R}^2 \times [0, +\infty)$ in such a way that while the evolution takes place on $G^\infty \times T^\infty$ spatial steps become shorter and shorter and, simultaneously, time ticking becomes quicker and quicker. Synchronization of this sort plays an important role in the constructions carried out in this work.

We put ourselves in the setting and notation of Section 2 In particular, we consider the set $W^\infty$ in (2.11) and the map $W^\infty \rightarrow T^\infty$ that associate the time
\[ \tau_{n\kappa i/n} \text{ with any (multi-) index } n\kappa i/n \in \mathcal{T}^\infty. \] We recall that \( i/n = i_1 \ldots i_n \in \{0, 1, \ldots, \mathcal{N} - 1\}^n. \)

We proceed by discretizing space. In addition to the integers parameters \( K \geq 2 \) and \( \mathcal{N} \geq 2 \) fixed in Section 2 we now fix the real parameters
\[ L > 0 \] (3.41)and
\[ \alpha_i > 1, \ i \in \{0, \ldots, \mathcal{N} - 1\}. \] (3.42)
We assume that a map \( P : \mathcal{W}^\infty \mapsto \mathbb{R}^2 \) is given,
\[ P : n\kappa i/n \mapsto P_{n\kappa i/n} \in \mathbb{R}^2, \ n\kappa i/n \in \mathcal{W}^\infty, \] (3.43)
which has the following two properties:

1. The sets
\[ G^n := \bigcup_{\kappa \in \{0,1,\ldots,K\}} \bigcup_{i/n \in \{0,1,\ldots,N\}} P_{n\kappa i/n} \subset \mathbb{R}^2, \ n \in \{0,\ldots,N\} \] (3.44)
satisfy the condition
\[ G^n \subset G^{n+1}, \ n \in \{0,\ldots,N\}; \] (3.45)
2. For all \( n\kappa i/n \in \mathcal{W}^\infty, \)
\[ |P_{n\kappa i/n}+ - P_{n\kappa i/n}| = 2L\alpha_{i/n}^{-1} \] (3.46)
where \( |\cdot| \) is the Euclidean distance of \( \mathbb{R}^2 \) and
\[ \alpha_{i/n} = \alpha_{i_1} \cdot \alpha_{i_2} \cdots \cdot \alpha_{i_n}. \] (3.47)

Under these assumptions, we introduce the countable set
\[ G^\infty = \bigcup_{n=0}^{\infty} G^n \subset \mathbb{R}^2 \] (3.48)
in the Euclidean plane \( \mathbb{R}^2 \) and the countable space-time grid
\[ G^\infty \times \mathcal{T}^\infty \subset \mathbb{R}^2 \times [0, +\infty) \] (3.49)
in the space time cylinder \( \mathbb{R}^2 \times [0, +\infty). \) We see that as \( n \) increases and we move from one time period \([Kn, K(n + 1)]\) to the next, spacial steps \( |P_{n\kappa i/n+} - P_{n\kappa i/n}| \) and time intervals \( \tau_{n\kappa i/n+} - \tau_{n\kappa i/n} \) become simultaneously smaller and smaller.

We point out that we have not required the map \( P : \mathcal{W}^\infty \mapsto \mathbb{R}^2 \) in (3.43) to be injective. As a consequence, a point \( P \in G^\infty \) can be obtained as a point \( P = P_{n\kappa i/n} \) given by the map \( P \) for possible infinitely many \( n\kappa i/n \in \mathcal{W} \). The times \( \tau_{n\kappa i/n} \in \mathcal{T}^\infty \), associated with such recurrent multi-indices \( n\kappa i/n \in \mathcal{W} \) leading to the same \( P \in G^\infty \), are the local times at \( P \) and they form the set
\[ \mathcal{T}^\infty_P := \{ \tau = \tau_{n\kappa i/n} \in \mathcal{T}^\infty : P_{n\kappa i/n} = P \}. \] (3.50)

The graph of the multi-valued map \( P \mapsto \mathcal{T}^\infty_P \) is the countable subset
\[ \mathcal{S}_P = \{ (P, \tau) : P = P_{n\kappa i/n}, \tau = \tau_{n\kappa i/n} \in \mathcal{T}^\infty_P, n\kappa i/n \in \mathcal{W}^\infty \} \] (3.51)
of \( \mathbb{R}^2 \times [0, +\infty) \) contained in \( G^\infty \times \mathcal{T}^\infty \). We point out that the projections \( P \in G^\infty \) and \( \tau \in \mathcal{T}^\infty \) of a point \( (P, \tau) \in \mathcal{S}_P \) are simultaneously determined by the same multi-index \( n\kappa i/n \) of \( \mathcal{W} \). This is how synchronization is incorporated in the set \( \mathcal{S}_P \).

Since the grids \( G^n \) are monotone increasing in \( n \), for a given \( P \in G^\infty \) there exists a smallest \( n \geq 0 \) such that
\[ P \in G^n \quad \text{for all } n \geq n. \] (3.52)
As \( P \in G^\pi \), there exists a smallest index \( \bar{i}/\bar{n} = \bar{i}_1\bar{i}_2\ldots\bar{i}_\pi \in \mathcal{W} \) in the lexicographic order of \( \mathcal{W} \), such that

\[
P = P_{\bar{i}/\bar{n}}.
\]

In turn, the index \( \bar{i}/\bar{n} \) uniquely determines the time

\[
\bar{T} = \tau_{\bar{i}/\bar{n}} \in T_\pi^\infty.
\]

The time \( \bar{T} = \tau_{\bar{i}/\bar{n}} \) is the lowest term of the sequence \( T_\pi^\infty \), it is the initial local time at \( P \). We point out that all the (finitely many) intermediate times \( \tau < \tau_{n\kappa i/n} < \tau' \) of \( T_\pi^\infty \) lying between two successive terms \( \tau < \tau' \) of the sequence \( T_\pi^\infty \) are also local times, however not at \( P \), but at different points \( Q \neq P \) of \( G^\beta,\infty \). Indeed we have

\[
T^\infty = \bigcup_{P \in G^\beta,\infty} T_P^\infty, \quad T_P^\infty \cap T_Q^\infty = \emptyset \quad \text{if} \ P \neq Q.
\]

The general setting for the synchronized problems in this section is the same as for Theorem 2.3 with the only change that the set \( \mathcal{S} \) is now specified to be the set \( \mathcal{S}_P \) associated with the map \( \mathcal{P} \), as explained before. Accordingly, the kinetic vector field \( g = g(S_{P,x,y}) \) is specified to be the vector field \( g = g(S_{P,x,y}) \), obtained as the canonical extension of a given kinetic vector field

\[
\gamma = \gamma_P = \mathcal{S}_P \setminus \{(x,0)\} \mapsto \mathbb{C}
\]

that assigns a vector \( \gamma(P,\tau) \in \mathbb{C} \) to each space-time location \( (P,\tau) \in \mathcal{S}_P \) for \( (P,\tau) \neq (x,0) \) and satisfies the property

\[
(P,\tau) \in \mathcal{S}_P \implies (P + \gamma(P,\tau) \Delta \tau,\tau) \in \mathcal{S}_P
\]

with \( \Delta \tau = \mathcal{N}^{-n} \) for \( \tau \in [Kn,Kn+1)\). Since the kinetic field \( \gamma \) depends on the map \( \mathcal{P} \), then also the field \( g \) depends on \( \mathcal{P} \). In a more precise notation, we write \( g = g(S_{P,x,y}) \). We assume that, in addition to such a field \( g \), we are given a vector

\[
\overrightarrow{0} \in \mathbb{R}^2 \quad \text{such that} \ (x + \overrightarrow{0},1) \in \mathcal{S}_P
\]

With this field \( g \) and the vectors \( x \) and \( \overrightarrow{0} \), we consider the purely impulsive Cauchy problem of the kind considered in Theorem 2.3

\[
\begin{cases}
\dot{y}(t) = g(y(t),t) & \text{if} \ (y(t),t) \in \mathbb{R}^2 \times [0,\infty) \\
y(0) = x \\
\dot{y}^+(0) = \overrightarrow{0}
\end{cases}
\]

From Theorem 2.2 and Theorem 2.3, we obtain the result

**Theorem 3.1.** Let a map \( \mathcal{P} \) of \( \mathbb{W}^\infty \) into \( \mathbb{R}^2 \) be given as in (3.33), satisfying the properties (3.44), (3.45). Let a kinetic vector field \( g = g(S_{P,x,y}) \) and a vector \( \overrightarrow{0} \) be also given, satisfying (3.54), (3.55), (3.56), and consider the problem (2.28) for such \( g \), \( x \), \( \overrightarrow{0} \). Then:

(i) there exists a unique weak solution \( y \in \mathcal{Y}_{\mathcal{P}^\infty}(0,+\infty) \) of problem (2.28), according to Definition 2.2.

(ii) a function \( y \in \mathcal{Y}_{\mathcal{P}^\infty}(0,+\infty) \) is the solution of problem (2.28) if and only if \((y(\tau_{n\kappa i/n}),n\kappa i/n) \in \mathcal{S}_P \) for every \( n\kappa i/n \in \mathbb{W}^\infty \) and the system of equation (2.34) is satisfied in the space \( \mathbb{R}^2 \times T^\infty \), the vectors \( \gamma_{n\kappa i/n} \) being given by

\[
\gamma_{n\kappa i/n} = \gamma_P(y(\tau_{n\kappa i/n}),n\kappa i/n) \quad \text{for all} \ n\kappa i/n \in \mathbb{W}^\infty \setminus \{00\}
\]

and the vectors \( a_{n\kappa i/n} \) by (2.35);
(iii) the system of equations (2.34) is solved iteratively by setting

\[ t_{n\kappa i/n} = t_{n\kappa i/n} + \gamma_{n\kappa i/n} \quad \text{for all } n\kappa i/n \in \mathcal{W}^\infty \setminus \{00\} \]  

in the lexicographic order of \( \mathcal{W}^\infty \).

Furthermore,

**Corollary 3.1.** The solution \( y \) of Theorem 3.1 has the expression

\[ y(t) = \sum_{\tau_{n\kappa i/n} \in \mathcal{T}} \chi_{[\tau_{n\kappa i/n}, \tau_{n\kappa i/n} + \Delta \tau]}(t) \cdot \left[ P(n\kappa i/n) + \frac{t - \tau_{n\kappa i/n}}{\tau_{n\kappa i/n} + \Delta \tau - \tau_{n\kappa i/n}}(P(n\kappa i/n) - P(n\kappa i/n)) \right], \]

where \( P \) is the given map (3.43).

The equivalent system of equations (2.34) is intrinsic to the discrete kinetic field \( \{S_P, x, \gamma_P\} \), as the vectors \( P(n\kappa i/n) \) stay in the domain \( S_P \) of the map \( P \). The solution \( y = y(t) \), as given in the Corollary 3.1, is a parametric equation of the geometric polygonal curve that interpolates the vertices \( P(n\kappa i/n) \) in \( \mathbb{R}^2 \). This curve, instead, is not intrinsic to \( \{S_P, x, \gamma_P\} \), because in each time interval \( (\tau_{n\kappa i/n}, \tau_{n\kappa i/n} + \Delta \tau) \) it moves into the surrounding space \( \mathbb{R}^2 \), away from the chord segment connecting the two vertices \( P(n\kappa i/n) \) and \( P(n\kappa i/n + \Delta \tau) \).

Theorem 3.1 covers a variety of interesting situations, brought to light by appropriate choices of the kinetic field \( \{P, x, \gamma_P\} \). The section that follows is dedicated to some examples of kinetic vector fields \( \{P, x, \gamma_P\} \) which by integration with Theorem 3.1 give origin to trajectories that display peculiar geometric and analytic properties. Symmetry and similarity are the basic transformations that lead to these interesting objects.

4. **Symmetry and similarity.** In this section we construct special grids and define on these grids special maps \( P \) of the kind considered in Section 3. The grids \( G^\infty \times \mathcal{T}^\infty \) are obtained by combining the action of symmetries with the action of a family of similarities in the Euclidean space \( \mathbb{R}^2 \). To keep our examples simple, we choose the similarity maps to be those occurring in the classic v.Koch fractal curves \( \mathcal{K} \). Alternative choices could be also, for example, Peano [23], Hilbert [5] and Polya [24] curves.

In this Section we fix the values

\[ K = 4, \quad N = 4 \]  

and assume that all constants \( \alpha_i \) are equal,

\[ \alpha_i = \alpha > 1, \quad i = 0, \ldots, N - 1 = 3. \]

This choice is dictated by the special symmetry and similarity maps considered here.

In the (Euclidean) plane \( \mathbb{R}^2 \) we consider the coordinate square domain \( D = D_L \) of diameter \( 2L\sqrt{2} \):

\[ D = (-L, L) \times (-L, L) \subset \mathbb{R}^2 \]  

(4.63)
with closure $\overline{D} = [-L, L] \times [-L, L]$ in $\mathbb{R}^2$. The scaling parameter $L > 0$ is left undetermined, allowing for global dilations of this domain. The boundary $\partial D$ of $D$ in $\mathbb{R}^2$ is decomposed in four half-open intervals as

$$\partial D = [Z_0, Z_1) \cup [Z_1, Z_2) \cup [Z_2, Z_3) \cup [Z_3, Z_0)$$

where

$$Z_0 = (-L, -L), \ Z_1 = (L, -L), \ Z_2 = (L, L), \ Z_3 = (-L, L)$$

(4.64)

are the vertices of $D$. The vertices $Z_\kappa$ belong to the circle centered at the origin and radius $R = L\sqrt{2}$ for $\kappa = 0, 1, 2, 3$. This choice of indices for the vertices gives a special role to the vertex $x = (-L, -L)$, which in the lexicographic order gets the index $\kappa = 0$. For $\kappa \in \{0, N\}$, we consider in complex notation the (counterclockwise) rotation maps

$$e^{i\kappa \frac{\pi}{2}} = O(\kappa \frac{\pi}{2})$$

(4.65)

where

$$O(\omega) = \left( \begin{array}{cc} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{array} \right), \quad 0 \leq \omega \leq 2\pi.$$  

(4.66)

As $\kappa$ increases in $\{0, N\}$, the product $\prod_{k=0}^{\kappa} O(\kappa \frac{\pi}{2})$ always equals the identity map $Id_C$ of $\mathbb{C}$, and $O(k \frac{\pi}{2}) = O((k+4 \pi)^2)$ for every $k \in \{0, N\}$. By identifying the vertices $Z_\kappa$ as complex numbers, we have cyclically

$$Z_{\kappa+1} = O(\frac{\pi}{2})Z_\kappa, \quad Z_\kappa = Z_{\kappa+4}, \quad \forall \kappa \in \{0, N\}.$$  

(4.72)

For $\kappa = 0, 1, 2, 3$ we define the maps $z = z_\kappa(w)$ from the plane of the variable $w = \xi + i\eta$ to the plane of the variable $z = x + iy$ by

$$z_\kappa = O(\kappa \frac{\pi}{2})2R \sin \frac{\pi}{4} w + Z_\kappa = O(\kappa \frac{\pi}{2})2Lw + Z_\kappa.$$  

(4.67)

The maps $z_\kappa$ bring the unit segment $I = \{w = \xi + i\eta : \eta = 0, 0 \leq \xi < 1\}$ onto the side $[Z_\kappa, Z_{\kappa+1})$ of $D$ in counterclockwise succession as $\kappa = 0, 1, 2, 3$. The inverse maps are

$$w = z_{\kappa}^{-1}(z) = O(-\kappa \frac{\pi}{2}) \frac{z}{2R \sin \frac{\pi}{4}} + P^*$$

(4.68)

where $P^* = -e^{-i\frac{\pi}{2} / \sqrt{2}}$, and the identities

$$w = z_{\kappa}^{-1}(Z_\kappa) = 0, \quad \kappa = 0, 1, 2, 3$$

(4.69)

hold. We proceed by defining a family of $N = 4$ similarity maps in $\mathbb{R}^2$

$$\{\psi_0^\beta, \psi_1^\beta, \psi_2^\beta, \psi_3^\beta\}$$

(4.70)

depending on a parameter

$$1 \leq \beta \leq 2.$$  

(4.71)

The parameter $1 \leq \beta \leq 2$ uniquely determines the angle

$$\theta = \theta^\beta = \arccos \left( \frac{2 - \beta}{\beta} \right), \quad 0 \leq \theta \leq \pi/2,$$

(4.72)

as an increasing function of $\beta$. We have $\sin \theta = (2/\beta) \sqrt{\beta - 1}$, $\cos \theta = (2 - \beta)/\beta$. We define the similarities as functions of $w = \xi + i\eta = (\xi, \eta) \in \mathbb{C}$ with values in $\mathbb{C}$.
as follows:

\[
\begin{align*}
\psi_0^\beta (w) & = \frac{\beta}{4} w \\
\psi_1^\beta (w) & = \frac{\beta}{4} + e^{i\theta} \frac{\beta}{4} w \\
\psi_2^\beta (w) & = \frac{1}{2} i \frac{\beta}{4} \sin \theta + e^{-i\theta} \frac{\beta}{4} w \\
\psi_3^\beta (w) & = 1 - \frac{\beta}{4} + \frac{\beta}{4} w.
\end{align*}
\] (4.73)

We note, incidentally, that only the maps \(\psi_1^\beta\) and \(\psi_2^\beta\) involve a change of direction, given by the rotation \(e^{i\theta}\) for \(\psi_1^\beta\) and by the rotation \(e^{-i\theta}\) for \(\psi_2^\beta\). All maps \(\psi_i^\beta\), \(i = 0, 1, 2, 3\), are contractive in the Euclidean metric of the plane by the factor \(1/4 \leq \beta/4 \leq 1/2\). Accordingly, we fix the parameter \(\alpha\) in (4.62) to be

\[
\alpha = \frac{4}{\beta}.
\] (4.77)

This clarifies that the choice done in (4.62), that is, to take all constants \(\alpha_i\) equal to \(\alpha\), corresponds to the simplifying choice of having all maps \(\psi_i^\beta\) to have the same contractive factor \(0 < \alpha^{-1} = \beta/4 < 1\).

For every multi-index \(n\kappa_i \ldots i_n \in \mathcal{W}\) we define the iterated map

\[
\psi_{i/n}^\beta = \psi_{i_1}^\beta \circ \psi_{i_2}^\beta \circ \cdots \circ \psi_{i_n}^\beta \quad (n \geq 1), \quad \psi_{i/n}^\beta = \psi_{i/0}^\beta = \text{Id}_{\mathbb{R}^2} \quad (n = 0).
\] (4.78)

The map \(\psi_{i/n}^\beta\) is contractive with factor \((\beta/4)^n\) for \(n \geq 1\) in the Euclidean distance of \(\mathbb{R}^2\). We now define the map

\[
\mathcal{P}^\beta : \mathcal{W}^\infty \mapsto \mathbb{R}^2
\] (4.79)

by setting

\[
\mathcal{P}^\beta : n\kappa i/n \mapsto \mathcal{P}^\beta_{n\kappa i/n}, \quad n\kappa i/n \in \mathcal{W}^\infty,
\] (4.80)

where

\[
\mathcal{P}^\beta_{n\kappa i/n} = \mathcal{P}(n\kappa i/n) = z_{\kappa} \circ \psi_{i/n}^\beta(0) \in \mathbb{R}^2, \quad n\kappa i/n \in \mathcal{W}^\infty.
\] (4.81)

We are using here a simplified notation, as the map \(\mathcal{P}^\beta\) depends not only on the parameter \(\beta\) but also on the symmetry maps \(z_0, \ldots, z_3\) and the similarity maps \(\psi_0^\beta, \ldots, \psi_3^\beta\) and should be denoted more precisely by \(\mathcal{P}_{\{z_0, \ldots, z_3; \psi_0^\beta, \ldots, \psi_3^\beta\}}\).

The map \(\mathcal{P}^\beta = \mathcal{P}_{\{z_0, \ldots, z_3; \psi_0^\beta, \ldots, \psi_3^\beta\}}\) has the following properties:

1. The sets

\[
G^{\beta,n} = \bigcup_{n} z_{\kappa} \circ V^{\beta,n} \quad n \in \{0, \mathbb{N}\}
\] (4.82)

where

\[
V^{\beta,n} = \bigcup_{i/n \in \{0,1,2,3\}^n} \psi_{i/n}^\beta(0)
\] (4.83)

satisfy the condition

\[
G^{\beta,n} \subset G^{\beta,n+1} \quad \text{for all } n \geq 0;
\] (4.84)

2. We have

\[
|\mathcal{P}^\beta_{n\kappa i/n} - \mathcal{P}^\beta_{n\kappa i/n}| = 2L\alpha^{n} = 2L(\frac{\beta}{4})^n \quad \text{for all } n\kappa i/n \in \mathcal{W}^\infty.
\] (4.85)

where \(|\cdot|\) is the Euclidean distance of \(\mathbb{R}^2\).
This monotonicity property follows from the relation
\[ V_{\beta,n} \subset V_{\beta,n+1} \quad \text{for all } n \geq 0 \] (4.86)
which is easily proved by remarking that for \( 0 \leq n < n+1 \) we have \( i/(n+1) = i_1i_2 \ldots i_n i_{n+1} \), hence
\[
V_{\beta,n+1} = \bigcup_{i/(n+1)} \psi_{i/(n+1)}(0) = \bigcup_{i/n} (\bigcup_{i_{n+1}=0}^{3} \psi_{i_{n+1}}(0)) \\
\supset \bigcup_{i/n} \psi_{i/n}(\psi_{0}(0)) = \bigcup_{i/n} \psi_{i/n}(0) = V_{\beta,n}.
\]
The second property is a consequence of \( \psi_{i/n} \) being contractive of a factor \( (\beta/4)^n \) for \( n \geq 1 \) in the Euclidean distance of \( \mathbb{R}^2 \), as already remarked.

We then define the countable sets
\[
V_{\beta,\infty} = \bigcup_{n=0}^{\infty} V_{\beta,n}, \quad (4.87)
\]
\[
G_{\beta,\infty} = \bigcup_{n=0}^{\infty} G_{\beta,n}, \quad (4.88)
\]
and introduce the grid
\[
G_{\beta,\infty} \times \mathcal{T}^\infty \quad (4.89)
\]
in the space-time cylinder \( \mathbb{R}^2 \times 0, +\infty \).

We now construct a family of kinetic vector fields. Attached with the map \( P_{\beta} = P_{\{z_0, \ldots, z_3; \psi_0, \ldots, \psi_3\}} : \mathcal{W}_\infty \mapsto \mathbb{R}^2 \) of (4.79), we have the set \( S_{P,\beta} \) defined by (3.51) in the space-time cylinder \( \mathbb{R}^2 \times 0, +\infty \). We single out the point \( x = Z_0 = (L, -L) \) (4.91)
and the vector \( \vec{v}_0 = Z_1 - Z_0 = (L, -L) - (L, -L) = (2L, 0) \) (4.92).
We have \( (x, 0) = (P_{\beta}(00), \tau_{00}) \in S_{P,\beta} \) and
\[
(x + \vec{v}_0, 1) = (Z_1, 1) = (P_{\beta}(01), \tau_{01}) \in S_{P,\beta}
\]
We now define the kinetic field
\[
\gamma_{\beta} = S_{P,\beta} \setminus \{(x, 0)\} \rightarrow \mathbb{C}
\]
by assigning a vector \( \gamma_{\beta}(P, \tau) \in \mathbb{C} \) to each space-time location \( (P, \tau) \in S_{P,\beta} \), \( (P, \tau) \neq (x, 0) \). The construction of \( \gamma_{\beta} \) is fully accomplished once we define the vector \( \gamma_{\beta}(P_{\beta_{n\kappa i/n}}, \tau_{n\kappa i/n}) \in \mathbb{C} \) for all indices \( n \kappa i/n \in W \) different from \( 0 \). To simplify
notation, we introduce the following maps of $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$, given in the form of complex vectors acting multiplicatively in $C$:

$$\overrightarrow{v}_\kappa := e^{i\kappa \frac{\pi}{2}} \quad (\kappa = 0, 1, 2, 3)$$

(4.94)

taken to be the identity map $Id_2$ of $\mathbb{R}^2$ when $\kappa = 0$:

$$\overrightarrow{v}_i := e^{i\theta (\delta_{i1} - \delta_{i2})} \quad (i = 0, 1, 2, 3)$$

(4.95)

where $\delta_{ij} := 1$ for $i = j$, $\delta_{ij} := 0$ for $i \neq j$ is the usual Kronecker symbol, taken to be the identity map $Id_2$ when $i = 1, 3$;

$$\overrightarrow{v}_{i_1 \cdots i_{n-1}} := \overrightarrow{v}_{i_1} \cdots \overrightarrow{v}_{i_{n-1}} = e^{i\theta \sum_{k=1}^{n-1}(\delta_{i_1k} - \delta_{i_2k})}$$

(4.96)

where for $n = 1$ the right hand side is defined to be the identity map $Id_2$, and where for $n \geq 2$ and $i/n = i_1 \cdots i_\kappa \in \{0, 1, 2, 3\}^n$, the factors $e^{i\theta (\delta_{i_1n} - \delta_{i_2n})}$ in the expansion of the sum in the exponent at the right hand side are also taken to be the identity map $Id_2$ whenever $i_n = i_1 = 1, 3$;

$$\overrightarrow{w}_n := e^{i\arctan(1/\beta)} \sqrt{1 + 1/\beta^2}$$

(4.97)

$$\overrightarrow{w}_j := (\delta_{1j} 2\sin(\theta/2) + \delta_{2j} 2\sin\theta + \delta_{3j} 2\sin(\theta/2)) \overrightarrow{w}_j \quad (j = 0, 1, 2, 3)$$

(4.98)

taken to be the identity map when $j = 0$. Moreover, for $n \geq 2$ we introduce the reduced index $\pi \kappa \overline{t}/\pi \in W$ of an index $n\kappa i/n$ at the point $P = P_{n\kappa i_1 i_2 \cdots i_n}^\beta$ of $G^{d, \infty}$, by defining $\pi \kappa \overline{t}/\pi$ to be the smallest index in the lexicographic order of $W$ such that $P_{n\kappa \pi \overline{t}/\pi}^\beta = P$. Reduced indices simplify notation.

The vector $\gamma^\beta(P_{n\kappa i/n}^\beta, \tau_{n\kappa i/n}) \in C$ is defined by setting

$$\gamma^\beta(P_{n\kappa i/n}^\beta, \tau_{n\kappa i/n}) = \gamma_{n\kappa i/n}^\beta \quad \text{for every } n\kappa i/n \in W, n\kappa i/n0 \neq 0 \odot ,$$

(4.100)

where the complex numbers $\gamma_{n\kappa i/n}^\beta$ are obtained for the various indices $n\kappa i/n$ as follows:

- $n \geq 1, \kappa = 0, i/n = 0^n$ :
  $$\gamma_{n\kappa (0^n)}^\beta := 2L\beta^n e^{i\arctan(\frac{1}{\beta})} \sqrt{1 + (1/\beta)^2}$$

- $n \geq 0, \kappa = 1, 2, 3, i/n = 0^n$ :
  $$\gamma_{n\kappa (0^n)}^\beta := 2L\beta^n e^{i(\kappa \frac{\pi}{2} + \frac{\pi}{2})} \sqrt{2}$$

- $n = 1, \kappa = 0, 1, 2, 3, i_1 = 1, 2, 3$ :
  $$\gamma_{n\kappa i_1}^\beta := 2L\beta \overrightarrow{u}_\kappa \cdot (\delta_{1i_1} 2\sin(\theta/2) + \delta_{2i_1} 2\sin\theta + \delta_{3i_1} 2\sin(\theta/2)) \overrightarrow{w}_{i_1}$$

- $n \geq 2, \kappa = 0, 1, 2, 3, (i_1 \cdots i_n) \in \{0, 1, 2, 3\}^n$, $(i_1 \cdots i_n) \neq (0^n)$ :
  $$\gamma_{n\kappa i_1 i_2 \cdots i_n}^\beta := 2L\beta^n \overrightarrow{u}_\kappa \overrightarrow{v}_{i_1} \cdots \overrightarrow{v}_{i_n} \cdot (\delta_{i_1 \tau_{i_1}} 2\sin(\frac{\theta}{2}) + \delta_{i_2 \tau_{i_2}} 2\sin(\theta) + \delta_{i_3 \tau_{i_3}} 2\sin(\frac{\theta}{2})) \overrightarrow{w}_{\tau_{i_n}}$$

where $\pi \kappa \overline{t}/\pi \in W$ is the reduced index of $n\kappa i/n$ at the point $P = P_{n\kappa i_1 i_2 \cdots i_n}^\beta$.

This completes the construction of $\gamma^\beta$ and also gives the extended kinetic field

$$g^\beta = g(s^\beta, x; \gamma^\beta) : \mathbb{R}^2 \times [0, +\infty) \rightarrow C$$

(4.101)

defined by

$$\begin{cases}
  g^\beta(P, t) := \gamma^\beta(P, t), & \text{if } (P, t) \in S^\beta \\
  g^\beta(P, t) := (0, 0), & \text{if } (P, t) \in \mathbb{R}^2 \times [0, +\infty) \setminus S^\beta.
\end{cases}$$

(4.102)
As noted previously, here the complex vector $\gamma^\beta(P, \tau)$ act multiplicatively in $C$ and can be identified with a linear map in $L(\mathbb{R}^2, \mathbb{R}^2)$, more specifically, as a unitary rotation modified by a scalar factor. We also note that at each given location $P \in G^\beta,\infty$ the vector $\gamma^\beta(P, \tau)$ changes at each turn of the time $\tau \geq \tau$ in the local time-clock $T^\infty$ at $P$. In particular at the point $x$ the set of local times $T^\infty_x$ is

$$T^\infty_x = \{ \tau = \tau_{n00}^n : n \in \{0, N\} \}$$

where $n00^n = 000$ for $n = 0$ and $\tau_{000} = 0$. We have $(x, n00^n) \in S^\beta$ for every $n \geq 0$. Therefore, for all $n \geq 1$ we have $g^\beta(x, n00^n) = \gamma^\beta_n(x, n00^n)$, where the vectors $\gamma^\beta(x, n00^n) = \gamma^\beta_n(0^n)$, by the definition before, are given by

$$\gamma^\beta_n(0^n) = 2L\beta^n e^{i\arctan(\frac{1}{2})} \sqrt{1 + (1/\beta)^2}, \quad n \geq 1 \quad (4.103)$$

and have same unit direction $e^{i\arctan(\frac{1}{2})}$ but lengths $2L\beta^n \sqrt{1 + (1/\beta)^2}$ changing with $n$.

The Cauchy initial value problem for the kinetic field $g^\beta$ with initial condition $x$ and $\vec{v}_0$ is now

$$\begin{cases}
\ddot{y}(t) = g^\beta(y(t), t) \quad \text{if } (y(t), t) \in \mathbb{R}^2 \times [0, \infty) \\
y(0) = x \\
y^+(0) = \vec{v}_0
\end{cases} \quad (4.104)$$

We have the following result.

**Theorem 4.1.** For every fixed $1 \leq \beta \leq 2$, problem $(4.104)$ with initial conditions $x = Z_0$ and $\vec{v}_0 = Z_1 - Z_0$ has a unique solution $y^\beta \in Y_{T^\infty}$ in the sense of Definition 2.2. The solution $y^\beta$ is given by

$$y^\beta(t) = \sum_{\tau_{n\kappa}/n \in T^\infty} X(\tau_{n\kappa}/n, \tau_{n\kappa}/n+)(t) \left( P^\beta_{n\kappa}/n + \frac{t - \tau_{n\kappa}/n}{\tau_{n\kappa}/n+ - \tau_{n\kappa}/n} (P^\beta_{n\kappa}/n+ - P^\beta_{n\kappa}/n) \right) \quad (4.105)$$

with

$$P^\beta_{n\kappa}/n = z_n \circ \psi^\beta_n(0), \quad (4.106)$$

$n\kappa/n \prec n\kappa/n+$ denoting two consecutive indices in the lexicographic order of $W^\infty$. Moreover, (i) $y^\beta(\tau_{n\kappa}/n) = P^\beta_{n\kappa}/n$ for every $n\kappa/n \in W^\infty$; (ii) $y^\beta(\tau_{n\kappa}/n) \in G^\beta,\infty$ for every $n\kappa/n \in W^\infty$; (iii) the closure $\Gamma^\beta = \overline{G^\beta,\infty}$ of $G^\beta,\infty$ in $\mathbb{R}^2$ is a subset of $[-L, L] \times [-L, L] = \mathcal{D}$; (iv) $y^\beta(t) \in [-L, L] \times [-L, L] = \mathcal{D}$ for all $t \in [0, +\infty)$; (v) the grid $G^\beta,\infty$ enjoys symmetry and self-similar invariance.

**Proof.** Theorem 3.1 applies to the problem at hand, therefore the first part of the statement of theorem follows from Theorem 3.1 once we prove that the function $y^\beta \in Y_{T^\infty}$ satisfies the equations

$$a^\beta_{n\kappa}/n = \gamma^\beta_{n\kappa}/n \quad \text{for all } n\kappa/n \in W^\infty \setminus \{000\} \quad (4.107)$$

where $a^\beta_{n\kappa}/n$ are given by $(2.33)$ with $y$ replaced by $y^\beta$. In order to verify the identities $(4.107)$, we first compute the vectors $a^\beta_{n\kappa}/n$ for all indices $n\kappa/n$, then the vectors $a^\beta_{n\kappa}/n$ for all indices $n\kappa/n \neq 000$. The computations are executed in the lexicographic order for the indices $n\kappa/n \in W$. Since they are elementary, but rather lengthy, we put them in the final Appendix in the form of a sequence of six lemmas.
In order to complete the proof of Theorem 4.1 we must verify the properties listed in the second part of the statement. Property (i) follows immediately from the expression (4.105) of the solution \( y^\beta \). Property (ii): by (4.82) and (4.83) we have \( z_\kappa \circ \psi^\beta_{i/n}(0) \in G^{\beta,n} \) for every \( n \) and every \( \kappa \), hence, by (4.88) and (i), \( P^\beta_{n+i/n} = z_\kappa \circ \psi^\beta_{i/n}(0) \in G^{\beta,\infty} \) for every \( n \). This proves (ii). Property (iii): by (4.88) and (4.84), the set \( G^{\beta,\infty} \) is the set-increasing union of the sets \( G^{\beta,n} \), \( n \in \{0, N\} \), therefore, in order to prove (iii) it suffices to prove that \( G^{\beta,n} \subset [-L, L] \times [-L, L] \) for every \( n \). From the expression of the similarity maps \( \psi^\beta_i \), \( i = 0, \ldots, 3 \), in (4.73) and seq, we find the values

\[
\psi^\beta_0(0) = (0, 0), \quad \psi^\beta_1(0) = \left(\frac{\beta}{4}, 0\right), \quad \psi^\beta_2(0) = \left(\frac{1}{2}, \frac{\beta}{4} \sin \theta\right), \quad \psi^\beta_3(0) = (1 - \frac{\beta}{4}, 0)
\]

in the plane of the variable \( w \), and by iterating the maps \( \psi^\beta_i \) we find that the points \( \psi^\beta_{i/n}(0) \) remain all contained in the closed triangle of the \( w \)-plane with vertices at \((0, 0)\), \((1/2, \beta/4 \sin \theta)\), \((1, 0)\). After transformation by the symmetry maps \( z_\kappa \) for \( \kappa = 0, 1, 2, 3 \), the points \( z_\kappa \circ \psi^\beta_{i/n}(0) \) are all contained in the square \([-L, L] \times [-L, L] \). This proves (iii). Property (iv): by (4.105) we see that the set of all points \( y^\beta(t) \) for \( t \in [0, +\infty) \) is composed of intervals connecting \( P^\beta_{n+i/n} \) and \( P^\beta_{n+i/n} \), and, by (ii) (ii) (iii), these end-points belong all to \([-L, L] \times [-L, L] \), what proves (iv). Property (v) is specified and proved in the following two propositions.

**Proposition 4.1.** For every \( 1 \leq \beta \leq 2 \), we have

\[
G^{\beta,\infty} = \bigcup_{\kappa} z_\kappa \circ V^{\beta,\infty} \tag{4.108}
\]

and

\[
V^{\beta,\infty} = \bigcup_{i/n \in \{0, 1, 2, 3\}} \psi^\beta_{i/n} \circ V^{\beta,\infty} \quad \text{for every } n \in \{0, N\}. \tag{4.109}
\]

**Proposition 4.2.** For every \( 1 \leq \beta \leq 2 \), we have

\[
\Gamma^\beta = \bigcup_{\kappa} z_\kappa \circ K^\beta, \tag{4.110}
\]

where

\[
K^\beta = \overline{V^{\beta,\infty}} \tag{4.111}
\]

is the closure of \( V^{\beta,\infty} \) in \( \mathbb{R}^2 \), and

\[
K^\beta = \bigcup_{i/n \in \{0, 1, 2, 3\}} \psi^\beta_{i/n} \circ K^\beta \quad \text{for every } n \in \{0, N\}. \tag{4.112}
\]

The second proposition says that the invariance property of the discrete set \( G^{\beta,\infty} \) is inherited by its closure

\[
\Gamma^\beta = \overline{G^{\beta,\infty}} \tag{4.113}
\]

We proceed with the proofs.

**Proof of Proposition 4.1.** We start by proving that

\[
\bigcup_{i=0}^4 \psi^\beta_i \circ V^{\beta,n} = V^{\beta,n+1} \tag{4.114}
\]
for every $n \in \{0, N\}$. In fact, by applying (4.83) defining $V^\beta, n$ first with $n$ and then with $n + 1$, we get
\[
\bigcup_{i=0}^{4} \psi_i^\beta \circ \bigcup_{j/n \in \{0,1,2,3\}^n} \psi_{j/n}^\beta (0) = \bigcup_{i=0}^{4} \bigcup_{j/n \in \{0,1,2,3\}^n} \psi_i^\beta \circ \psi_{j/n}^\beta (0) = \bigcup_{i/(n+1) \in \{0,1,2,3\}^{n+1}} \psi_i^\beta (0)
\]
which is indeed (4.114). We now prove statement (4.109). From (4.114) and (4.87) we get
\[
\bigcup_{i=0}^{4} \psi_i^\beta \circ V^\beta, n \subset V^\beta, \infty
\]
for every $n \in \{0, N\}$, hence also
\[
\bigcup_{i=0}^{4} \psi_i^\beta \circ V^\beta, \infty \subset V^\beta, \infty.
\]
On the other hand (4.114) also implies
\[
V^\beta, n+1 = \bigcup_{i=0}^{4} \psi_i^\beta \circ V^\beta, n \subset \bigcup_{i=0}^{4} \psi_i^\beta \circ V^\beta, \infty
\]
for every $n \in \{0, N\}$, hence
\[
V^\beta, \infty \subset \bigcup_{i=0}^{4} \psi_i^\beta \circ V^\beta, \infty
\]
The two inclusions (4.116) and (4.117) together give (4.109) for $i/n = i_1$. To get (4.109) for arbitrary $i/n$, we apply (4.114) repeatedly. We now prove (4.108). On the one hand, by (4.82) and (4.86), we have
\[
G^\beta, n \subset G^\beta, n+1 \subset \bigcup_{\kappa} z_{\kappa} \circ V^\beta, \infty \text{ for all } n \geq 0
\]
hence by (4.88)
\[
G^\beta, \infty \subset \bigcup_{\kappa} z_{\kappa} \circ V^\beta, \infty.
\]
On the other hand, by (4.82), (4.84) and (4.86), we have
\[
\bigcup_{\kappa} z_{\kappa} \circ V^\beta, n = G^\beta, n \subset G^\beta, \infty
\]
hence, by (4.86) and (4.87),
\[
\bigcup_{\kappa} z_{\kappa} \circ V^\beta, \infty \subset G^\beta, \infty.
\]
These two inclusions together prove (4.108), concluding the proof of Proposition 4.1.

Proof of Proposition 4.2. We start by proving that for $K^\beta = \overline{V^\beta, \infty}$ we have
\[
K^\beta \subset \bigcup_{i=1}^{4} \psi_i^\beta \circ \overline{V^\beta, \infty}
\]
and
\[ \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,\infty \subseteq K^\beta. \]

We have for every \( n \)
\[ V^\beta,n+1 = \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,n \subset \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,\infty \subset \bigcup_{i=1}^{4} \psi_i^\beta \circ \overline{V^\beta,\infty} \]

hence
\[ V^\beta,\infty \subset \bigcup_{i=1}^{4} \psi_i^\beta \circ \overline{V^\beta,\infty} \]

implying that the first inclusion
\[ K^\beta = \overline{V^\beta,\infty} \subseteq \bigcup_{i=1}^{4} \psi_i^\beta \circ \overline{V^\beta,\infty}. \]

holds. On the other hand, for every \( n \) we have
\[ \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,n = V^\beta,n+1 \subset V^\beta,\infty \subset \overline{V^\beta,\infty} = K^\beta \]

that gives
\[ \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,\infty = \bigcup_{i=1}^{4} \psi_i^\beta \circ \left( \bigcup_{n=0}^{+\infty} V^\beta,n \right) = \bigcup_{n=0}^{+\infty} \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,n \subseteq K^\beta \]

hence the inclusion
\[ \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,\infty \subseteq K^\beta \]

also holds. We now prove the identity
\[ \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,\infty = \bigcup_{i=1}^{4} \psi_i^\beta \circ \overline{V^\beta,\infty}. \tag{4.119} \]

is satisfied by the similarity maps occurring in Theorem 4.1. Once this identity is proved, we conclude from the first of the two inclusions proved before that
\[ K^\beta \subseteq \bigcup_{i=1}^{4} \psi_i^\beta \circ \overline{V^\beta,\infty} = \bigcup_{i=1}^{4} \psi_i^\beta \circ K^\beta \]

and from the second that
\[ \bigcup_{i=1}^{4} \psi_i^\beta \circ K^\beta = \bigcup_{i=1}^{4} \psi_i^\beta \circ \overline{V^\beta,\infty} = \bigcup_{i=1}^{4} \psi_i^\beta \circ V^\beta,\infty \subseteq K^\beta \]

what leads to the identity
\[ K^\beta = \bigcup_{i=1}^{4} \psi_i^\beta \circ K^\beta \]

By iterating this identity over the maps \( i_1, \ldots, i_n \) we finally get the identity \( 4.111 \), concluding the proof of Proposition 4.2. The proof of Theorem 4.1 is now complete. \( \square \)
5. Remarks. In this section we collect a few examples and remarks about the applications and research perspectives open by the results of this paper.

1. Orbits and attractors.

The orbits described by the trajectory traced in the plane by the solution \( y^{\beta}(t) \) of Theorem 4.1 as \( t \) runs in \([0, +\infty)\), as mentioned in the Introduction, have a peculiar structure that depends critically on the parameter \( \beta \). We summarize the main properties for the case \( 1 \leq \beta < 2 \) and for the case \( \beta = 2 \) separately, by omitting proofs which will appear elsewhere [20].

The case \( 1 \leq \beta < 2 \). For every \( n \in \{0, N\} \), the orbit \( \Gamma^{\beta,n} := \{ y \in \mathbb{R}^2 : y = y^{\beta}(t), 4n \leq t < 4(n + 1), y^{\beta}(4n) = x, y^{\beta}(4(n + 1)) = x \} \) described by the trajectory \( y^{\beta}(\cdot) \) in the interval of time \( 4n \leq t < 4(n + 1) \) is a closed Jordan curve homeomorphic to the boundary \( \partial D \) (see next point 2.). Moreover, \( \Gamma^{\beta,n} \) is the boundary \( \Gamma^{\beta,n} = \partial D^{\beta,n} \) of an open connected domain \( D^{\beta,n} \subset \mathbb{R}^2 \), with \( \Gamma^{1,n} = \partial D \) and \( D^{1,n} = D \) for every \( n \). The attractor \( \Gamma^{\beta} = \lim_{n \to \infty} \Gamma^{\beta,n} \), \( 1 \leq \beta < 2 \), where the limit is taken in the Hausdorff metric of compact subsets of \( \mathbb{R}^2 \), is also a closed Jordan curve homeomorphic to the boundary \( \partial D \) and the boundary \( \Gamma^{\beta} = \partial D^{\beta} \) of an open connected domain \( D^{\beta} \subset D \).

The Hausdorff dimension \( d_H(\Gamma^{\beta}) \) of \( \Gamma^{\beta} \) is given by

\[
1 \leq d_H(\Gamma^{\beta}) = \frac{\log 4}{\log(4/\beta)} < 2
\]

and

\[
\lim_{\beta \to 2} \Gamma^{\beta} = \mathcal{D}
\]

in the Hausdorff metric of subsets of \( \mathbb{R}^2 \). See Figure 1.

The case \( \beta = 2 \). For every \( n \in \{0, N\} \), the orbit

\[
\Gamma^{2,n} := \{ y \in \mathbb{R}^2 : y = y^{\beta}(t), 4n \leq t < 4(n + 1), y^{\beta}(4n) = x, y^{\beta}(4(n + 1)) = x \}
\]

is a closed continuous curve with multiple points. In each time interval \([4n, 4(n + 1)]\) the trajectory visits every vertex once, any site of \( G^{2,n} \) on the boundary twice, any site of \( G^{2,n} \) in the interior at 4 different local times, moving along the segments connecting these sites twice, in opposite directions. The attractor

\[
\Gamma^{2} = \lim_{n \to \infty} \Gamma^{2,n}
\]

where the limit is taken again in the Hausdorff metric of compact subsets of \( \mathbb{R}^2 \), is the full domain \( \Gamma^{2} = \mathcal{D} \). By approximating the value 2 with smaller values \( 2 - \epsilon \), \( \epsilon > 0 \), the multiple points of the case \( \beta = 2 \) split into suitable quadruplets of the case \( 1 < \beta < 2 \), the approximation acting as a singularity resolution for curves.

2. Connections with PDEs.

As already noted in the Introduction and in the preceding Point 1, the orbits and attractors given by Theorem 4.1 for \( 1 \leq \beta < 2 \) are in fact closed Jordan curves topologically homeomorphic to the boundary \( \partial D \), that decompose the plane into an inner and an outer open domain. This is an important property, because it shows that these curves are in fact oriented interfaces. However, these interfaces have a quite unusual metric behavior, as it can inferred by noticing that for every segment \([a,b] \subset \Gamma^{\beta,m} \subset \Gamma^{\beta,n} \), \( m \leq n \), the ratio between the length of the arc connecting \( a \)
to $b$ in $\Gamma_{\beta,n}$ to the length $b - a$ of the chord $[a,b]$ connecting $a$ to $b$ in $\Gamma_{\beta,m}$ tends to $\infty$ as $n \to +\infty$. This is in sharp contrast with the case of smooth curves, for which the arc/chord ratio is finite and tends infinitesimally to 1 as the chord-length tends to zero. More details on the metric properties of the curves generated by the solutions of Theorem 4.1 are given in [20].

Interfaces occur in many applications of PDEs boundary value problems, in particular is so-called transmission problems for second order operators with transmission conditions of first order or of second order. A survey of recent results for interfaces of pre-fractal and fractal type can be found in [18]. The results of this paper open new perspectives to this kind of PDE applications.

3. Mathematical models of rippled cells.

The constructions described in Point 1 and Point 2 can be generalized to more general geometries and dynamics. The domain $D$ can be chosen to be a $K \geq 2$ sided regular polygon inscribed in a circle of radius $R = L\sqrt{2}$ and the symmetry maps can be related to the rotational symmetry of the polygon. The similarities can be chosen to be suitable $N \geq 2$ contractive maps depending on a parameter $1 \leq \beta \leq 2$, possibly with different contractive factors. An interesting example would be the Pólya curve, [24], [11].

Orbits and attractors can also be confined in a narrow ring-like neighborhood of the unit circle with arbitrarily small transversal diameter, leading to examples of mathematical models for very small cells with very rich and rippled boundaries. See Figure 2.

4. Motion of curves by curvature.

In the theory developed in this paper the motion in time of the orbits toward their attractor is regulated by a very irregular curvature field, given by the kinetic vector field $g^\beta$. This motion, however, has opposite features with respect to the classic geometric motion of curves by curvature, because in our case the length of the curves increases and becomes asymptotically infinite.
5. **Optimal design of boundaries.**

Further generalizations of the models described so far take into account alternating similarity families in the construction of the kinetic fields, subjecting the choice of the similarities, considered to be a control variable, to the minimization of a suitable objective functional along the trajectories. This opens the way to a new kind of optimal switching control problems governing the growth of curves. Optimal design of this kind can be done for the boundaries and interfaces of fractal type recently studied in [3], [10], [18], [22]. In this context, stochastic perturbations could be introduced, by adding to the deterministic evolution equations a second order term consisting of a small parameter \( \epsilon \) times the discrete Laplace operator on the similarity grids occurring in Point 1. Laplace operators on fractals go back to the early work on diffusions on fractals in [12], [2], [9] and to the analytic work in [7]. Discrete versions on the spatial grids of this paper can be constructed according to [17]. By suitably scaling the convergence of \( \epsilon \) to zero in terms of the vanishing grid size, the stochastic optimal problem can be expected to converge to the deterministic one. Such non-local stochastic perturbation for discrete deterministic control problems were studied in [13] and [4], see also [1]. The convergence tools that allow for such applications are related to the so-called M-convergence and order-\( \epsilon \) M-convergence in [14], [16]. This kind of switching deterministic and stochastic control problems is new in control theory.

6. **Sand-pile models.**

Recently, in [21] and [19] a fully-discrete self-organized-criticality model of sand-pile type has been introduced, which involve Euclidean synchronized space-time lattices of the kind described in Point 1. This work can be generalized to the more general countable grids \( G^\beta \times T^\infty \) described in Point 1, with a continuous spatial limit of Hausdorff dimension between 1 and 2. This adds universality to the self-organized-criticality paradigm.
6. Appendix. This Appendix contains the computations of the vectors $t^{\beta}_{n\kappa i/n}$ for all indices $n\kappa i/n$ and of the vectors $a^{\beta}_{n\kappa i/n}$ for all indices $n\kappa i/n \neq 00\emptyset$, which have been omitted in the proof of Theorem 4.2. The computations are executed in the lexicographic order for the indices $n\kappa i/n \in W$. They are summarized in the six lemmas that follow.

To simplify the notation we omit the superscript $\beta$ and write $t_{n\kappa i/n}$ in place of $t^{\beta}_{n\kappa i/n}$ and $a_{n\kappa i/n}$ in place of $a^{\beta}_{n\kappa i/n}$, as well as $P_{n\kappa i/n}$ in place of $P^{\beta}_{n\kappa i/n}$ for the points defined in (4.81). Similarly, we write $\theta$ for the angle $\theta^\beta$ of (4.72). The proofs of the six lemmas that follow are rather lengthy, though elementary, and for sake of brevity are omitted. The lemmas hold for every value of the parameter $1 \leq \beta \leq 2$.

The vectors $t_{n\kappa i/n} = t^{\beta}_{n\kappa i/n}$ are given in

**Lemma 6.1.** The following formulas hold:

- $n = 0$, $\kappa = 0, 1, 2, 3$:
  
  $$t_{0,\emptyset} = 2L e^{i\kappa (\pi/2 + \pi/2)}$$

- $n = 1$, $\kappa = 0, 1, 2, 3$, $i_1 = 0, 1, 2, 3$:
  
  $$t_{1,i_1} = 2L\beta e^{i\kappa (\pi/2 + \pi/2)} e^{i\theta (\delta_{1,i_1} + 2\delta_{1,i_1})}$$

- $n = 2$, $\kappa = 0, 1, 2, 3$, $i_1 = 0, 1, 2, 3$, $i_2 = 0, 1, 2, 3$:
  
  $$t_{2,i_1i_2} = 2L\beta^2 e^{i\kappa (\pi/2 + \pi/2)} e^{i\theta (\delta_{1,i_1} + \delta_{2,i_1} + \delta_{2,i_1})}$$

- $n \geq 2$, $\kappa = 0, 1, 2, 3$, $i_1 = 0, 1, 2, 3$, $i_2 = 0, 1, 2, 3$, ..., $i_n = 0, 1, 2, 3$:
  
  $$t_{n,i_1i_2...i_n} = 2L\beta^n e^{i\kappa (\pi/2 + \pi/2)} e^{i\theta (\delta_{1,i_1} + \delta_{2,i_1} + \cdots + \delta_{n,i_n})}.$$

Calculations of the vectors $a_{n\kappa i/n} = \tilde{a}_{n\kappa i/n}$ are based on the expressions of the vectors $t_{n\kappa i/n}$ obtained in Lemma 6.1. For $n = 0$ we get

**Lemma 6.2.** For all indices $n\kappa i/n \in W$, $n\kappa i/n \neq 00\emptyset$, with $n = 0$:

$$a_{0,\emptyset} = 2L e^{i(\kappa\pi/2 + \pi/2)} = 2L \tilde{u}_\kappa \sqrt{2} e^{i\pi/2}$$

$$n = 0, \kappa = 1, 2, 3).$$

For $n = 1$:

**Lemma 6.3.** For all indices $n\kappa i/n \in W$, $n\kappa i/n \neq 00\emptyset$, with $n = 1$:

$$a_{1,00} = 2L\beta e^{i\arccos(1/\sqrt{2})} \sqrt{1 + (1/\sqrt{2})} = 2L\beta \sqrt{2} \tilde{u}_\kappa (n = 1, \kappa = 0, i_1 = 0)$$

$$a_{1,\kappa 0} = \beta a_{0,\emptyset} = 2L\beta \tilde{u}_\kappa \sqrt{2} e^{i\pi/2} (n = 1, \kappa = 1, 2, 3, i_1 = 0)$$

$$a_{1,\kappa i_1} = 2L\beta \tilde{u}_\kappa \sqrt{2} i_1 (n = 1, \kappa = 0, 1, 2, 3, i_1 = 1, 2, 3).$$

For $n = 2$:

**Lemma 6.4.** For all indices $2\kappa (i_1 i_2)$, $\kappa = 0, 1, 2, 3$, $i_1, i_2 = 0, 1, 2, 3$ we have:

$$a_{2,0}(00) = 2L\beta^2 \tilde{u}_\kappa (n = 2, \kappa = 0, i_1 = 0, i_2 = 0)$$

$$a_{2,\kappa}(00) = \beta a_{1,\kappa 0} = 2L\beta^2 \tilde{u}_\kappa \sqrt{2} e^{i\pi/2} (n = 2, \kappa = 1, 2, 3, i_1 = i_2 = 0)$$

$$a_{2,\kappa}(i_1 0) = \beta a_{1,\kappa i_1} = 2L\beta^2 \tilde{u}_\kappa \tilde{u}_{i_1} i_1 (n = 2, \kappa = 0, 1, 2, 3, i_1 = 1, 2, 3, i_2 = 0)$$

$$a_{2,\kappa}(i_1 i_2) = 2L\beta^2 \tilde{u}_\kappa \tilde{u}_{i_1} \tilde{u}_{i_2} (n = 2, \kappa = 0, 1, 2, 3, i_1 = 0, 1, 2, 3, i_2 = 1, 2, 3).$$

The statement of this lemma can be simplified by using the reduced indices introduced before in our definition of the field $\gamma^\beta$. In fact, when $n \geq 2$ the last two expressions in Lemma 6.4 can be unified in a single expression, as we now show. We first observe that the cumulative range of the indices occurring in the last two formulas of Lemma 6.4 can be equivalently described as the set of all indices $n =$
2, κ = 0, 1, 2, 3, i_1, i_2 = 0, 1, 2, 3, (i_1, i_2) \neq (0, 0). We recall that the reduced index \( \overline{\kappa} (i_1, i_2) = \overline{\kappa} i_1 i_2 \in W \) of \( 2 \kappa (i_1, i_2) \in W \) at the point \( P = P_{2 \kappa (i_1, i_2)} \in G^{2, \infty} \) is the smallest index \( \overline{\kappa} i_1 i_2 \) in the lexicographic order of \( W \), with \( \overline{\kappa} \geq 1 \), such that \( P = P_{2 \kappa i_1 i_2} \). The reduced indices of \( 2 (i_1) \) at the point \( P = P_{2 \kappa (i_1, 0)} \) with \( i_1 = 0, 1, 2, 3 \) are

\[
\overline{2} (00) = 100, \overline{2} (10) = 101, \overline{2} (20) = 102, \overline{2} (30) = 102
\]

while the reduced index of any index \( 2 (i_1, i_2) \) with \( i_2 = 1, 2, 3 \) is the same as the initial index. Similar remarks apply to the case \( \kappa = 1, 2, 3 \). In conclusion, all vectors \( a_{2 \kappa (i_1, i_2)} \) with the exception of \( a_{2 \kappa (0, 0)} \), are given by

\[
a_{2 \kappa (i_1, i_2)} = 2L \beta^2 \overrightarrow{\tau}_{i_1 (\overline{\kappa} - 1)} \overrightarrow{\tau}_i \quad i_1 = 0, 1, 2, 3, \quad i_2 = 0, 1, 2, 3, \quad (i_1, i_2) \neq (0, 0) .
\]

Incidentally, note that if \( i_2 \neq 0 \), then \( \overline{\kappa} = 2 \) and \( \overline{\tau}_2 = i_2, \overline{\tau}_{(\overline{\kappa} - 1)} = 1 \), while if \( i_2 = 0 \), then \( \overline{\kappa} = 1, \overline{\tau}_i = i_1, \overline{\tau}_{(\overline{\kappa} - 1)} = i_0 \) and \( \overrightarrow{\tau}_{i (\overline{\kappa} - 1)} = \overrightarrow{\tau}_{i 0} \). Therefore, an equivalent statement of Lemma 6.4 is

**Lemma 6.5.** For every \( 1 \leq \beta \leq 2 \) and all indices \( 2 \kappa (i_1, i_2) \), \( \kappa = 0, 1, 2, 3 \), \( i_1, i_2 = 0, 1, 2, 3 \) we have:

\[
a_{2 \kappa (00)} = 2L \beta^2 \overrightarrow{\tau}_1 (1, 0) \quad (n = 2, \kappa = 0, i_1 = i_2 = 0)
\]

\[
a_{2 \kappa (00)} = \beta a_{1 \kappa 0} = 2L \beta^2 \overrightarrow{u}_{\kappa} \sqrt{2} e^{i \frac{\pi}{4}} (1, 0) \quad (n = 2, \kappa = 1, 2, 3, i_1 = i_2 = 0)
\]

\[
a_{2 \kappa (i_1, i_2)} = 2L \beta^2 \overrightarrow{u}_{\kappa} \overrightarrow{\tau}_{i_1 (\overline{\kappa} - 1)} \overrightarrow{\tau}_i \quad \text{for}
\]

\[
(n = 2, \kappa = 0, 1, 2, 3, i_1, i_2 = 0, 1, 2, 3, (i_1, i_2) \neq (0, 0)) .
\]

The proof of Lemma 6.5 hence also of Lemma 6.4 are omitted as noticed before. Finally, calculations for all indices with \( n \geq 2 \) can be carried out in a similar way and this leads to the last lemma of this series:

**Lemma 6.6.** For all indices \( n \kappa (i_1 \ldots i_n) \) with \( n \geq 2, \kappa = 0, 1, 2, 3, i_1, \ldots, i_n = 0, 1, 2, 3 \) we have:

\[
a_{n \kappa (00^n)} = 2L \beta^n \overrightarrow{\tau}_1 (1, 0) \quad (n \geq 2, \kappa = 0, i_1 = \cdots = i_n = 0)
\]

\[
a_{n \kappa (00^n)} = \beta^{n-1} a_{1 \kappa 0} = 2L \beta^n \overrightarrow{u}_{\kappa} \sqrt{2} e^{i \frac{\pi}{4}} (1, 0) \quad (n \geq 2, \kappa = 1, 2, 3, i_1 = \cdots = i_n = 0)
\]

\[
a_{n \kappa (i_1, \ldots, i_n)} = 2L \beta^n \overrightarrow{u}_{\kappa} \overrightarrow{\tau}_{i_1 \cdots i_{(\overline{n} - 1)}} \overrightarrow{\tau}_{i_n} \quad \text{for}
\]

\[
(n \geq 2, \kappa = 0, 1, 2, 3, i_1, \ldots, i_n = 0, 1, 2, 3, (i_1, \ldots, i_n) \neq (0^n))
\]

where \( \overline{n} \kappa i_1 \cdots i_{\overline{n}} \) is the reduced index of \( n \kappa (i_1 \cdots i_n) \) at the point \( P = P_{n \kappa (i_1 \cdots i_n)} \) and where the vectors \( \overrightarrow{\tau}_1, \overrightarrow{u}_{\kappa}, \overrightarrow{\tau}_{i_1 \cdots i_{(\overline{n} - 1)}}, \overrightarrow{\tau}_{i_n} \) are defined in \( \{ 4.97 \}, \{ 4.98 \}, \{ 4.99 \}, \{ 4.99 \} \), respectively.

This concludes the proof that the equations \( 4.107 \) in the proof of Theorem 4.1 are satisfied and that the properties \( 4.105 \) and \( 4.106 \) hold, which was the goal of this Appendix.

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