Geometry and symmetries of null G-structures

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Received 4 December 2018, revised 25 April 2019
Accepted for publication 2 May 2019
Published 28 May 2019

Abstract
We present a definition of null G-structures on Lorentzian manifolds and investigate their geometric properties. This definition includes the Robinson structure on 4-dimensional black holes as well as the null structures that appear in all supersymmetric solutions of supergravity theories. We also identify the induced geometry on some null hypersurfaces and that on the orbit spaces of null geodesic congruences in such Lorentzian manifolds. We give the algebra of diffeomorphisms that preserves a null G-structure and demonstrate that in some cases it interpolates between the BMS algebra of an asymptotically flat spacetime and the Lorentz symmetry algebra of a Killing horizon.

Keywords: general relativity, special Lorentzian structures, G-structures

1. Introduction

It has been known for sometime that many solutions of 4-dimensional gravity theories admit a Lorentzian holomorphic structure. Such solutions include black hole solutions, like Schwarzschild, Reissner–Nordström and Kerr, and other solutions like Gödel. Such a structure has originally been introduced by Robinson and has been used to find new solutions like the Robinson–Trautman solution. Apart from the holomorphic structure another characteristic of Robinson manifolds is the existence of a nowhere vanishing null vector field $X$, which may not be Killing, whose integral curves are null geodesics. It can also be shown that the orbit space of the null geodesic congruence generated by $X$ admits a Cauchy–Riemann (CR) structure, for a review and a proof of some of the above statements see [1]. CR is the structure inherited on hypersurfaces in complex manifolds from the ambient complex structure of the whole space and has extensively been investigated in the literature, see e.g. [2–7].

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Many of the features of Robinson manifolds like the holomorphic structure and the existence of a null vector field are reminiscent of the properties of some supersymmetric backgrounds, see \cite{8} for a review and references within. Moreover the systematic exploration of supersymmetric backgrounds has revealed many new Lorentzian geometries. As we shall demonstrate some of them generalize that of Robinson manifolds. Nevertheless both the Robinson structure and those that appear on supersymmetric solutions, particularly those for which one of bilinears of a Killing spinor is a null vector field, belong to the same family. This is because of the existence of a null nowhere vanishing vector field $X$ on the spacetime. We refer to all of these structures as null G-structures\textsuperscript{2}.

The investigation of geometry of supersymmetric backgrounds has given a new perspective in the description of geometry of null structures. This is because it is based on the properties of \textit{globally defined} fundamental forms on the spacetime. As an approach it is similar to that taken for the description of geometry of $n$-dimensional Riemannian manifolds with a G-structure for $G \subseteq O(n)$. However there are some key differences. One is that for null G-structures all the fundamental forms are null. Another is that the null structure is a conformally invariant concept and so any geometric description of the structure is required to accommodate this in its formulation. These have consequences in the description of the spacetime geometry and in particular the patching conditions of various tensor like objects that appear on spacetime. Despite pointing to a new way of investigating null G-structures, the null structures that appear in supersymmetric backgrounds are special and they must be further generalized to apply to non-supersymmetric solutions.

One of the objectives of this paper is to give the definition of a null G-structure and develop the tools to investigate the geometry of associated Lorentzian manifolds. In particular, the structure of the tangent and co-tangent bundles of manifolds with a null G-structure will be examined in detail. The emphasis will be on the construction of the fundamental forms of a null G-structure and the exploration of their properties. This will generalize the null structures beyond the Robinson structure and those that appear in the context of supersymmetric solutions. One of the expectations is that the null G-structures developed here will be sufficiently general to describe the geometry of most of the solutions in four and higher dimensions that have appeared in recent years in the context of string and M-theories.

Under certain conditions, a spacetime with a null G-structure admits a null geodesic congruence generated by a vector field $X$. This is always the case for spacetimes with a Robinson structure. This will be explored further in the context of null G-structures. The geometry induced from the spacetime with a null G-structure on a null hypersurface $\mathcal{H}$ transversal to this null geodesic congruence as well as that on the space of orbits $\mathcal{M}$ of the null geodesics will be determined. These have applications in the context of black holes and asymptotically flat spacetimes as such hypersurfaces are identified with horizons and the asymptotic null infinity, respectively. We shall also give the conditions how to construct a Lorentzian manifold with null G-structure from data on $\mathcal{M}$.

Furthermore, we shall examine the local symmetries of null G-structures. In particular, the symmetry Lie algebras of a variety of null G-structures will be computed. These are typically infinite dimensional and are closely related to the Lie algebras of path groups. Moreover, we shall demonstrate for a certain class of spacetimes the symmetry Lie algebras of some null G-structures interpolate between the Lorentz symmetry of a Killing horizon hypersurface and the Bondi–Metzner–Sachs (BMS) algebra \cite{9, 10} of asymptotic null infinity.

\textsuperscript{2}Typically null structures have been investigated on hypersurfaces of a spacetime with a null normal vector field. The null structures investigated here will be on the whole spacetime.
In addition to developing the general theory of the null G-structures, we shall investigate in detail the null G-structures associated with the groups $SO(n-2)$, $U(k)$ and $SU(k)$, $n = 2k + 2$, on a $n$-dimensional spacetime $M$. We shall demonstrate that the null structure associated to $U(k)$ can be identified with that of almost Robinson manifolds. We also give some examples of Lorentzian spacetimes with a null G-structure which include black holes in all dimensions and brane solutions and their intersections.

This paper is organized as follows. In section 2, we give the definition of a null G-structure and investigate the properties of the tangent and contangent bundles of the associated spacetime. In section 3, we explore the properties of the fundamental forms of general null G-structures. We also examine the induced geometry on null hypersurfaces transversal to null geodesic congruences generated by the structure as well as that on orbit spaces of null geodesics. In addition, we give the definition of invariance of a null G-structure and present examples of spacetimes with a null G-structure. In section 4, we investigate in detail the geometry of null G-structures based on the groups $U(k)$ and $SU(k)$ and clarify the relation of the former to the Robinson structure. In section 5, we investigate the symmetries of null G-structures with emphasis on those related to the groups $SO(n-2)$ and $U(k)$. In particular, we explain how on certain spacetimes the Lie algebra of spacetime diffeomorphisms which preserves a null G-structure interpolates between the Lorentz algebra of a Killing horizon and the BMS algebra of asymptotic null infinity, and in section 6, we present our conclusions.

2. Null G-structures

2.1. Definition of null G-structures

As it has been mentioned in the introduction, null G-structures are characterised by the existence of a nowhere vanishing vector field, considered up to a conformal rescaling, which defines a null direction on the spacetime. Because of this let us consider the isotropy group $H_L \subset SO^+(n-1,1)$ of a null line $L$ passing through the origin in $\mathbb{R}^{n-1,1}$. This is spanned by the matrices

$$H_L = \left\{ \begin{pmatrix} \ell & -\frac{1}{2}\ell^{-1}v^2 & v \\ 0 & \ell^{-1} & 0 \\ 0 & -\frac{1}{2}Av & A \end{pmatrix} | v \in \mathbb{R}^{n-2}, A \in SO(n-2), \ell \in \mathbb{R} - \{0\} \right\},$$

(1)

while the isotropy group $H_X$ of a non-vanishing null vector $X$ is

$$H_X = \{(\ell, v, A) \in H_L | \ell = 1 \}.$$

(2)

Let us denote with $H_L^+$ the subgroup of $H_L$ for which $\ell > 0$. $H_L^+$ is the isotropy group of an oriented null line and it is more suitable to model a null direction on the spacetime than $H_L$. Because of this, we shall focus on the use of $H_L^+$ but most of the analysis that follows with some modifications also applies to $H_L$. Clearly $H_X \subset H_L^+$.

**Definition 2.1.** A Lorentzian manifold $(M, g)$ admits a (time oriented) null structure iff the structure group of $M$ reduces to a subgroup of $H_L^+$.

Clearly a special case of a null structure arises whenever the structure group of $(M, g)$ reduces to a subgroup of $H_X$ instead. In such a case, the null structure is associated to a nowhere vanishing null vector field on the spacetime instead of a null direction. As the $H_X \subset H_L^+$,
manifolds with structure group\(^3\) \(H^+_L\) are more general than those with structure group \(H^+_X\) so we shall investigate the properties of the former and specialize where necessary on the latter. Another reason to focus on spacetimes with a \(H^+_L\)-structure is that \(H^+_L\) accommodates better the conformal properties of the null structure than \(H^+_X\).

Note that \(SO(n - 2) = \{(\ell, v, A) \in H^+_L | \ell = 1, v = 0\}\) is a subgroup of \(H^+_L\). Rewriting \(H^+_L\) as \(H^+_L(SO(n - 2))\), a distinguished class of subgroups of \(H^+_L(SO(n - 2))\) are those for which the matrices \(A\) are restricted to lie into a subgroup \(K\) of \(SO(n - 2)\). Denoting these subgroups with \(H^+_L(K)\), the investigation that follows will focus on Lorentzian manifolds whose structure group is \(H^+_L(K)\) characterized by the existence of fundamental forms. The examples that will be explored in more detail are those null G-structures for which \(K = SO(n - 2), U(k)\) and \(SU(k), n = 2k + 2\). Though those with \(K = \text{Spin}(7)\), \(G_2\) and others are known to occur in certain supersymmetric backgrounds, see [16] for a list of isotropy groups of spinors in \(\text{Spin}(n - 1, 1)\) for \(n = 10, 11\). Another class of subgroups of \(H^+_L(SO(n - 2))\) that are known to occur as structure groups of Lorentzian manifolds, which again will not be investigated here, are those which are products \(H^+_L(K_1) \times K_2\), where \(K_1 \times K_2 \subseteq SO(n - 2)\). For example compactification backgrounds have such a structure group.

Null G-structures in four dimensions \((n = 4)\) are special and can be described in terms spinors. The transformations on the spinors that give rise to \(H^+_L\)-structure. A conformation by strictly positive\(^4\) functions on \(M\). This is because such transformations retain \(X\) and \(\kappa\) as sections \(N\) and \(\tilde{N}\), respectively. In addition the geometric properties under investigation here depend only of the null direction of \(X\) and \(\kappa\) instead of \(X\) and \(\kappa\) themselves. These

\(^3\)For all manifolds with a \(H^+_L\)-structure, the topological structure group reduces to a subgroup of \(SO(n - 2)\).

\(^4\)The functions are taken to be positive to preserve the orientation defined by \(X\).
null directions are not affected by such conformal rescaling. Although \( \kappa \) and \( X \) are related via a metric, we shall take the conformal rescalings of \( X \) and \( \kappa \) to be independent unless otherwise is explicitly stated.

Next consider \( N^\perp \), the orthogonal subbundle of \( N \) in \( TM \), whose fibres are

\[ N^\perp_p = \{ v \in T_p M | \kappa(v) = 0 \}. \tag{6} \]

Note that two metrics in the same conformal class give rise to the same \( N^\perp \). Clearly \( N \) is a subbundle in \( N^\perp \) and so

\[ 0 \to N \to N^\perp \to N^\perp/N \to 0. \tag{7} \]

The spacetime metric \( g \) restricts well on the fibres of \( N^\perp/N \). In fact it restricts to a non-degenerate Euclidean signature metric. This is significant in many general relativity computations.

For reasons that will become more apparent later, let us develop a tensor calculus based on \( TM/N \) and \( T^* M/\tilde{N} \). Let us denote the smooth sections of a vector bundle \( E \) over \( M \) with \( \Gamma(E) \). First observe that \( W \in \Gamma(TM/N) \) and \( \alpha \in \Gamma(T^*M/\tilde{N}) \) can be represented by a vector field \( W \) and a 1-forms \( \alpha \) on the spacetime \( M \), respectively, up to the equivalence relation \( \sim \), where \( W \sim W' \) and \( \alpha \sim \alpha' \), iff \( W = W' + aX \) and \( \alpha = \alpha' + b\kappa \), for some spacetime functions \( a \) and \( b \). This can be extended to sections of the tensor bundle \( \otimes^k (TM/N) \otimes^\ell (T^*M/\tilde{N}) \).

In particular \( I \in \Gamma \left( (TM/N) \otimes (T^*M/\tilde{N}) \right) \) can be represented by an \((1,1)\) tensor \( I \) up to the equivalence \( \sim \), where \( I \sim I' \) iff \( I = I' + X \otimes \alpha + W \otimes \kappa \) with \( \alpha \) and \( W \) be a 1-form and a vector field on the spacetime, respectively. It is often convenient to use instead of \( I \sim I' \),

\[ I = I' \mod (X, \kappa). \]

Similarly sections of \( N^\perp/N \) and \( \tilde{N}^\perp/\tilde{N} \) can be viewed as sections of \( N^\perp \) and \( \tilde{N}^\perp \) up to the same identifications as those for the sections of \( TM/N \) and \( T^* M/\tilde{N} \) described in the previous paragraph. However, the identification for the sections of \( \otimes^k (N^\perp/N) \otimes^\ell (\tilde{N}^\perp/\tilde{N}) \) is somewhat different from that in \( \otimes^k (TM/N) \otimes^\ell (T^*M/\tilde{N}) \). In particular \( I \in \Gamma \left( (N^\perp/N) \otimes (\tilde{N}^\perp/\tilde{N}) \right) \) can be viewed as a section \( I \) of \( N^\perp \otimes \tilde{N}^\perp \) up to the identification \( I = I' + X \otimes \alpha + W \otimes \kappa \), where now \( \alpha \in \Gamma(\tilde{N}^\perp) \) and \( W \in \Gamma(N^\perp) \) instead of sections of \( T^* M \) and \( TM \), respectively.

Next define a Lie derivative type of operation, \( \hat{L}_W \), with respect to a vector field \( W \) on \( M \) on \( \Gamma(TM/N) \) as follows

\[ \hat{L}_W \hat{V} := [W, V] \mod X, \quad W \in \Gamma(TM), \quad \hat{V} \in \Gamma(TM/N). \tag{8} \]

This operation is not always well defined unless \( W \) is appropriately restricted. In particular, one has the following.

**Proposition 2.1.** \( \hat{L} \) is well defined provided that \( W \) preserve the null structure associated to \( X \), i.e. \( [W, X] \mod X = 0 \)

**Proof.** Indeed consider another representative, \( V + fX \), of the section \( \hat{V} \) of \( TM/N \). Then

\[ \hat{L}_W (V + fX) = [W, V + fX] \mod X = \hat{L}_W \hat{V} + f[W, X] \mod X = \hat{L}_W \hat{V}. \tag{9} \]

where in the last equality we have used the assumption that \( W \) preserves the null structure. \( \square \)

Observe if \( W \) and \( W' \) preserve the null structure, \([W, W']\) also preserves the null structure. Using this, one can prove that

\[ \hat{L}_{[W, W']} \hat{V} = \hat{L}_W \hat{L}_W \hat{V} - \hat{L}_W \hat{L}_W \hat{V}. \tag{10} \]
Similarly a Lie derivative can be defined on $\Gamma(T^*M/\tilde{N})$ as

$$\dot{\mathcal{L}}_W \alpha := \mathcal{L}_W \alpha \mod \kappa.$$  

(11)

This is well defined provided that again $W$ preserves the null structure associated to $\kappa$, i.e. $\mathcal{L}_W \kappa \mod \kappa = 0$. This can be generalized to sections of $\otimes^k(TM/N) \otimes^\ell (T^*M/\tilde{N})$ by applying appropriately the mod $(X, \kappa)$ operation. In such a case $\dot{\mathcal{L}}_W$ is well-defined provided that $W$ preserve both $X$ and $\kappa$ null structures. Observe that the condition $\mathcal{L}_W X \mod X = 0$ is not implied from the $\mathcal{L}_W X \mod X = 0$ unless $W$ is further restricted to satisfy $i_{\dot{\mathcal{L}}_W g} = 0 \mod \kappa$.

Furthermore an exterior derivative can be defined on the sections of exterior bundle $\Lambda^k(TM/N)$ of $TM/N$ as $d\alpha = d\alpha \mod \kappa$. This is well defined provided that $d\kappa = 0 \mod \kappa$ (or equivalently $\kappa \wedge d\kappa = 0$).

A Lie derivative operation $\mathcal{L}$ can also be defined on $\Gamma(\otimes^k(N^\perp/N) \otimes^\ell (\tilde{N}^\perp/\tilde{N}))$. However now the coefficient tensors that appear in the modulus operation are sections of either $\otimes^{k-1}N^\perp \otimes^\ell \tilde{N}^\perp$ or $\otimes^kN^\perp \otimes^{\ell-1} \tilde{N}^\perp$. For example let $\dot{\alpha} \in \Gamma(\Lambda^2(\tilde{N}^\perp/\tilde{N}))$, then one has $\dot{\mathcal{L}}_{g^T \alpha} = \mathcal{L}_{g^T \alpha} \mod \kappa$, where the modulus operation is up to sections $\beta \wedge \kappa$ with $\beta \in \Gamma(N^\perp)$.

The operation of $\dot{\mathcal{L}}_W$ is well defined provided that $W$ preserve the null structure associated to both $X$ and $\kappa$ in all cases.

Although this will not be used later for completeness consider a connection $D$ in $TM$ and define a connection $\tilde{D}$ in $TM/N$ as

$$\tilde{D}_W \tilde{V} := D_W V \mod X, \quad W \in \Gamma(TM).$$  

(12)

It can be seen that $\tilde{D}$ is well defined provided that $D$ satisfies

$$D_W X = \eta(W)X,$$  

(13)

where $\eta$ is a spacetime 1-form. Similarly given a connection $D$ in $T^*M$, one can define a connection $\tilde{D}$ in $T^*M/\tilde{N}$ as $\tilde{D}_W \tilde{\alpha} := D_W \tilde{\alpha} \mod \kappa$ provided that $D_W \kappa = \theta(W)\kappa$ for some spacetime form $\theta$. One can then extend $\tilde{D}_W$ to the sections of $\otimes^k(TM/N) \otimes^\ell (T^*M/\tilde{N})$.

2.2.2. Splitting of the tangent and cotangent bundles. There is not a natural way to identify $TM/N$ as a subbundle of $TM$. However, this will be the case if one splits of the sequence (4). One way to do this is to choose $\lambda$ a null 1-form on $M$, $g^{-1}(\lambda, \lambda) = 0$, such that $\lambda(X) = 1$. Then $TM = N \oplus Z$, where $Z$ is the bundle whose fibres, $Z_p$, are $Z_p = \{v \in T_pM|\lambda(v) = 0\}$. Indeed notice that the splitting map $\tilde{\lambda} : TM/N \rightarrow TM$ defined as $\tilde{\lambda}(W) := W - \lambda(W)X$ is independent from the representative $W$ of $W$, is an injection and the image of $\tilde{\lambda}$ is $Z$.

Similarly a splitting can be chosen for the sequence (7) via the use of the 1-form $\lambda$ as above now restricted on the sections of $N^\perp$. As a result $N^\perp = N \oplus T$, where the fibres of $T$, $T_p$, are $T_p = \{u \in N^\perp|\lambda(u) = 0\}$. $T$ is the image of the splitting map $\lambda$ now restricted on $N^\perp/N$. The metric of $M$ can be decomposed as

$$g = \kappa \otimes \lambda + \lambda \otimes \kappa + g_T,$$  

(14)

where $g_T$ is the restriction of $g$ on the fibres of $T$. $g_T$ is a metric with Euclidean signature on the fibres of $T$. $T$ is also called a screening space.

One can also define a splitting of the sequence (5) for the cotangent bundle. In particular using $Y(\alpha) := g^{-1}(\alpha, \lambda)$, one has that $T^*M = N \oplus \tilde{Z}$, where $\tilde{Z}_p = \{\alpha \in T^*_pM|Y(\alpha) = 0\}$. Similarly,

$^5$Note that $\lambda$ could also be chosen as $\lambda(X) = f$, where $f$ is a no-where vanishing function of $M$. However, in such a case one could consider $f^{-1}\lambda$ instead.
one defines the orthogonal bundle to $\tilde{N}$, $\tilde{N}^\perp$, in $TM$ and the decomposition $\tilde{N}^\perp = \tilde{N} \oplus \tilde{T}$, where $\tilde{T}_p = \{ \alpha \in N^\perp_p | Y(\alpha) = 0 \}$.

In four dimensions $2 + 2$ splitting of the spacetime have a long history, see e.g. the GHP formalism and [12]. The splitting investigated here is that of $TM$. In particular, it is not assumed that there is a submanifold $S$ in $M$ such that the restriction of the subbundle $T$ on $S$ can be identified with $TS$.

**Definition 2.2.** Let $(M, g)$ be a Lorentzian manifold with a null structure contained in $H^+_2(SO(n - 2))$. A $k$-form $\alpha$ on $(M, g)$ is null iff $\kappa \wedge \alpha = 0$.

**Proposition 2.2.** Let $\alpha$ be a null $k$-form, then $\alpha = \kappa \wedge \beta$, where $\beta$ is a $(k - 1)$-form.

**Proof.** Indeed take the inner derivation of $\kappa \wedge \alpha$ with respect to $Y$ to find

$$i_Y(\kappa \wedge \alpha) = \alpha - \kappa \wedge i_Y \alpha = 0,$$

where we have used that $i_Y(\kappa) = \kappa(Y) = \lambda(X) = 1$. This proves the statement. □

Notice that in general $\beta$ depends on the choice of splitting. However its class $\beta$ as a section of $L^{k-1}(TM/N)$ does not.

A consequence of the above proposition is that if $\kappa \wedge \alpha = \kappa \wedge \beta$ for some $k$-forms $\alpha$ and $\beta$, then $\alpha = \beta + \kappa \wedge \gamma$ for some $(k - 1)$-form $\gamma$. Note also that operation $\delta_\kappa(\alpha) := \kappa \wedge \alpha$ is a cohomology operation on the space of forms as $(\delta_\kappa)^2 = 0$. The above proposition implies that all the cohomology groups of $\delta_\kappa$ are trivial.

### 2.2.3. Dependence on the choice of splitting.

The subbundle $T$ of $TM$ depends on the choice of splitting $\lambda$ and we denote this with $T_\lambda$. To investigate the dependence of $T_\lambda$ on the choice of $\lambda$ suppose that $\lambda'$ is another choice of splitting with $\lambda'(X) = 1$. Any 1-form can be written as $\lambda' = a\lambda + bk + \gamma$, where $\gamma \in \Gamma(T_\lambda)$. Imposing $\lambda'(X) = 1$, one finds that $\lambda' = \lambda + bk + \gamma$. Furthermore imposing the condition that $\lambda'$ must be null, one finds that

$$\lambda' = \lambda - \frac{1}{2} || \gamma ||^2 \kappa + \gamma.$$  \hfill (16)

To compare $T_\lambda'$ with $T_\lambda$ consider a section $Z_\lambda'$ of $T_\lambda$. Writing $Z_\lambda = aX + bY + Z_\lambda$, where $Z_\lambda \in \Gamma(T_\lambda)$, as any vector field on $M$ can be decomposed in this way, and imposing the conditions $\kappa(Z_\lambda) = \lambda'(Z_\lambda') = 0$, one finds that

$$Z_\lambda' = Z_\lambda - \gamma(Z_\lambda)X.$$  \hfill (17)

Therefore $T_\lambda'$ is ‘shifted’ relative to $T_\lambda$ with $N$ as expected. Notice that $g_{T_\lambda'} = g_{T_\lambda}$ because as it has already been mentioned $g$ restricts well as a Euclidean signature fibre metric on $N^\perp /N$.

Next let us determine $Y_\lambda'$ in terms of $Y_\lambda$, where again the subscripts denote the dependence of the vector fields on the choice of splitting. Writing again $Y_\lambda' = aY_\lambda + bX + Z$, where $Z \in \Gamma(T_\lambda)$, and imposing that $\kappa(Y_\lambda) = 1, \lambda'(Y_\lambda') = 0$ and that $g(Y_\lambda, Z_\lambda) = 0$, where $Z_\lambda$ is any section of $T_\lambda$, one finds that

$$Y_\lambda' = Y_\lambda - \frac{1}{2} || \gamma ||^2 X + W,$$ \hfill (18)

where $W(\alpha) = g_T^{-1}(\gamma, \alpha)$ for every $\alpha \in \Gamma(T_\lambda)$. 

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It remains to find the way that the subbundle $\tilde{T}$ of $T^*M$ depends on the splitting. For this consider a section $\alpha_{\lambda'}$ of $\tilde{T}_{\lambda'}$. Then $\alpha_{\lambda'} = a\kappa + b\lambda + \alpha_\lambda$, where $\alpha_\lambda$ is a section of $\tilde{T}_\lambda$. Imposing the conditions $i_X\alpha_{\lambda'} = i_{\gamma'}\alpha_{\lambda'} = 0$ on $\alpha_{\lambda'}$, so that $\alpha_{\lambda'}$ is a section of $\tilde{T}_{\lambda'}$, one finds that
\[
\alpha_{\lambda'} = -\alpha_\lambda(W)\kappa + \alpha_\lambda. \tag{19}
\]
Therefore $\tilde{T}_{\lambda'}$ is "shifted" relative to $\tilde{T}_\lambda$ with the line bundle $\tilde{N}$.

2.3. Null geodesic congruences and null structures

Null structures are closely related to the existence of null geodesic congruences in a spacetime. For this impose the condition
\[
\kappa \wedge \mathcal{L}_X \kappa = 0, \tag{20}
\]
on $\kappa$. A key consequence of (20) is as follows [15].

**Proposition 2.3.** If $X$ and $\kappa$ satisfy (20), then integral curves of $X$ will be null geodesics.

**Proof.** Indeed
\[
\kappa \wedge \nabla_X \kappa = \kappa \wedge i_X d\kappa = \kappa \wedge \mathcal{L}_X \kappa = 0, \tag{21}
\]
where we have used that $\kappa(X) = 0$. So $\nabla_X \parallel X$. □

This is significant as the boundaries of spacetimes are orbit spaces of null geodesic congruences, e.g. the even horizons of black holes as well as the conformal boundaries at infinity of asymptotically flat spacetimes. This will be explored further below.

Clearly (20) holds whenever $X$ can be chosen to be Killing, $\mathcal{L}_X g = 0$, as $\mathcal{L}_X \kappa = 0$. So from the proposition 2.3, one has $\nabla_X X = 0$ and the integral curves of $X$ are null geodesics. All supersymmetric backgrounds whose Killing spinor bilinears include a null Killing vector field $\mathcal{L}_X\kappa = h\kappa$, where $h$ is a spacetime function. In turn one finds that $\nabla_X X = h X$.

Another special case that arises is whenever $\kappa \wedge d\kappa = 0$. In such a case, the spacetime is foliated with leaves $(n - 1)$-dimensional hypersurfaces. Taking the inner derivation of $\kappa \wedge d\kappa = 0$ with $X$, one arrives at (20). So again the integral curves of $X$ are null geodesics.

2.4. Frames and null structures

In the description of geometry of a spacetime $M$ with a null G-structure with a splitting, it is often useful to introduce a local (pseudo-)orthonormal co-frame $\{e^-, e^+; i = 1, \ldots, n - 2\}$ such that
\[
e^- = \kappa, \quad e^+ = \lambda. \tag{22}
\]
Such a notation has also extensively been used in the investigation of supersymmetric backgrounds.

The metric is written in terms of the co-frame as $g = 2e^- e^+ + \delta_i e^i$. Under local transformations in the isotropy group $H^+_L$, the co-frame transforms as

\[\text{\textsuperscript{6}The use of the term 'boundary' does not signify that the spacetime necessarily ends at those hypersurfaces. Instead it is used to denote hypersurfaces of interest like the null infinity for asymptotically flat spacetimes or the event horizon of a black hole.}\]
\[ e^- \rightarrow \ell^{-1}e^-, \quad e^+ \rightarrow \ell e^+ - \frac{1}{2} \ell^{-1}v^2 e^- + v_k e^k, \]
\[ e^i \rightarrow A^i_{j'}(e^j - \ell^{-1}v^j e^-). \]

Clearly the transformation of the co-frame under the isotropy group \( H_\chi \) is as above but now \( \ell = 1 \). Observe that a change of splitting, investigated in section 2.2.3, introduces a transformation on the frame of the type described above and so the spacetime metric does not depend on the choice of splitting as expected.

A local description of the geometry is as follows. Adapting a coordinate along \( X, X = \partial_u, \) introducing additional coordinates \( v, y^j \) and choosing a splitting, one can write for the coframe

\[ e^- = h(dv + n_j dy^j), \quad e^+ = du + \frac{1}{2} V dv + m_j dy^j, \quad e^i = e^i_j dy^j, \]

where \( h, V, n_j, m_j \) and \( e^i_j \) depend on all coordinates. The frame then is

\[ e^- = \partial_u, \quad e^+ = h^{-1}(\partial_x - \frac{1}{2} V \partial_u), \quad e^i = e^i_j (\partial_x - n_j \partial_x - m_j \partial_u + \frac{1}{2} V n_j \partial_u). \]

So one has \( X = e^+ \) and \( Y = e^- \). The above local choices of frame and co-frame are not unique. In particular the \( e^i \) frame is chosen as \( i_\chi e^i = 0 \). This choice is helpful in the local description of fundamental forms of a null G-structure as it will be explained in section 3.1. However, one can also choose \( e^i = e^i_j p^j e^- \). The form of the spacetime metric does not change as the addition of \( p e^- \) can be compensated in a redefinition of \( V \) and \( m \).

3. General null G-structures

3.1. Forms and null G-structures

Before we turn to investigate examples of individual null G-structures, it is instructive to give an overview of some of their properties. Suppose a spacetime \( M \) admits a \( H^{1+}_n(K) \)-structure characterized with fundamental forms \( \kappa \) and one or more additional forms that we collectively denote with \( \chi \). All the fundamental forms of such a null structure satisfy the conditions

\[ \kappa \wedge \chi = 0, \quad i_\chi \chi = 0. \]

These are forms on the spacetime and so do not depend on the choice of splitting \( \lambda \) of \( TM \). Moreover all the fundamental forms are specified up to a conformal rescaling by strictly positive spacetime functions. This is because the properties (26) are independent from the overall normalization of these forms. However in the presence of more than one fundamental forms \( \chi \) that satisfy certain relative normalisation conditions not all such rescaling can be independent. An example of this arises in the investigation of the null \( H^1_n(SU(k)) \)-structures below. However note that not all fundamental forms of a \( H^1_n(K) \) \( \times \) \( K_2 \)-structure are null as the fundamental forms associated to \( K_2 \) do not satisfy the first condition in (26).

The first condition in (26) implies that \( \chi \) is null and it can be solved to yield \( \chi = \kappa \wedge \alpha \) for some form \( \alpha \). Indeed taking the inner derivation of the first equation in (26) with \( Y \) and using \( \kappa(Y) = 1 \) yields the desirable result, see also proposition 2.2. Clearly \( \alpha \) is not unique as both \( \alpha \) and \( \alpha + \kappa \wedge \gamma \), for any form \( \gamma \), give rise to the same \( \chi \). Typically, \( \alpha \) depends on the choice of splitting of \( TM \) but not its class \( \hat{\alpha} \) as a section of \( \Lambda^+(TM/N) \). One of the roles of the second condition in (26) is to choose a representative for \( \alpha \) as explained in the proposition below.
Proposition 3.1. Given a splitting of \( TM \), a null fundamental \( k \)-form \( \chi \) of a null \( G \)-structure can be represented as \( \chi = \kappa \wedge \phi \), where \( \phi \) is a section of \( \Lambda^k(T) \), i.e. it satisfies \( i_Y \phi = i_Y \phi = 0 \).

Proof. First let us demonstrate \( \chi = \kappa \wedge \alpha \), where \( i_Y \alpha = 0 \). Indeed as \( \chi \) can be written as \( \chi = \kappa \wedge \beta \) for some form \( \beta \), we set \( \alpha = \beta - \kappa \wedge i_Y \beta \). Then clearly \( \chi = \kappa \wedge \beta = \kappa \wedge \alpha \) and \( i_Y \alpha = 0 \).

To continue decompose \( \alpha \) as \( \alpha = \lambda \wedge \zeta + \phi \), where \( \zeta \) and \( \phi \) are sections of \( \Lambda^{k-1}(T) \) and \( \Lambda^k(T) \), respectively. For this we have used that \( i_Y \alpha = 0 \). Therefore, we have that \( \chi = \kappa \wedge \lambda \wedge \zeta + \kappa \wedge \phi \). Imposing now the condition \( i_Y \chi = 0 \) in (26), we find that \( \kappa \wedge \zeta = 0 \) and so \( \chi = \kappa \wedge \phi \), where \( i_Y \phi = i_Y \phi = 0 \).

For a null \( G \)-structure \( H^+_1(K) \) the representative \( \phi \) of \( \chi \), as described in the previous proposition, is identified with a fundamental form of the compact subgroup \( K \). However other representatives have also been used in the literature to describe a null \( G \)-structure on a spacetime. Of course it is required that the geometry of spacetime to be independent of these choices.

It is clear from the comment above that the fundamental forms of a null \( H^+_1(K) \)-structure can be constructed from those of \( K \). In particular, the \( H^+_1(SO(n-2)) \)-structure apart from \( \kappa \) also admits a fundamental \((n-1)\)-form \( \chi \) which can be identified with the fibre volume form of \( N^\perp \), \( \chi = dvol(N^\perp) \). A similar construction will be made below for the fundamental forms of the \( H^+_1(U(k)) \) and \( H^+_1(SU(k)) \) structures.

In an adapted frame to a null \( G \)-structure associated to the forms \( \kappa \) and \( \chi \) as in (24), one has \( \kappa = e^{-} \) and \( \chi = e^{+} \wedge \phi \), where \( \phi = \frac{1}{|k-1|!} \phi_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \). The components of \( \phi \) in the frame are constant. Observe that \( \phi \) is not covariant under the patching conditions (23) and depends on the choice of splitting. Though of course \( \chi \) does not.

3.2. Geometry of null transversal hypersurfaces

Many of the spacetime boundaries, like Killing horizons and conformal boundaries of asymptotically flat spacetimes, are null hypersurfaces which are transversal to null geodesic congruences.

Definition 3.1. A hypersurface \( \mathcal{H} \) in \( M \) is transversal to the flow of a vector field \( X \) iff \( X \) is nowhere tangent to \( \mathcal{H} \).

Null geodesic congruences have many transversal hypersurfaces, the focus here will be on transversal hypersurfaces which in addition are null, i.e. hypersurfaces for which their tangent space, \( T_p \mathcal{H} \), at every point \( p \in \mathcal{H} \), is a null subspace of \( T_p M \). Such hypersurfaces admit normal vector field \( n \) which is null and so simultaneously tangent to the hypersurface. In such a case, \( g(X, n) \neq 0 \) as otherwise \( X \), which generates the null geodesic congruence, will be tangent to \( \mathcal{H} \). So after a possible rescaling of \( n \), one can set \( g(X, n) = 1 \).

As \( n \) is nowhere vanishing it defines a trivial line bundle \( N \) in \( T \mathcal{H} \) and so one has

\[
0 \to N \to T \mathcal{H} \to T \mathcal{H}/N \to 0. \tag{27}
\]

This sequence can be split using the pull back \( j^* \kappa \) of \( \kappa \) on \( \mathcal{H} \), where \( j \) is the inclusion of \( \mathcal{H} \) in \( M \), as \( j^* \kappa(n) = g(X, n) = 1 \). So one has \( T \mathcal{H} = N \oplus T^\mathcal{H} \), where \( T^\mathcal{H}_p = \{ v \in T_p M | j^* \kappa(v) = 0 \} \).

Observe that the spacetime metric restricted on \( \mathcal{H} \) becomes a positive definite fibre metric on \( T^\mathcal{H} \). Similarly \( T^* \mathcal{H} = \tilde{N} \oplus \tilde{T}^\mathcal{H} \), where now the fibres of \( \tilde{N} \) are spanned by \( j^* \kappa \) and \( \tilde{T}^\mathcal{H}_p = \{ \alpha \in T^*_p M | \alpha(n) = 0 \} \). Note that the above two described splittings are natural in the
sense that both \( j^*\kappa \) and \( n \) are determined in terms of the null structure and the choice of hypersurface. No other arbitrary choices are involved.

Incidentally, it is well known that \( n \) generates a null geodesic congruence in \( \mathcal{H} \). Indeed \( \nabla_n g(n, n) = 2g(\nabla_n n, n) = 0 \) implies that \( \nabla_n n \) is tangent to \( \mathcal{H} \). Then for any other tangent vector field \( Z \) to \( \mathcal{H} \), \( \nabla_n g(n, Z) = g(\nabla_n n, Z) + g(n, \nabla_n Z) = g(\nabla_n n, Z) = 0 \), where \( g(n, \nabla_n Z) \) vanishes as a consequence of the torsion free condition of \( \nabla \) and that \( [n, Z] \) is tangent to \( \mathcal{H} \). Therefore \( \nabla_n n \parallel n \).

**Theorem 3.1.** Suppose that \( M \) admits a null \( H_+^+(K) \)-structure with fundamental forms \( \kappa \) and \( \chi \), and \( \mathcal{H} \) be a null hypersurface in \( M \) transversal to a null geodesic congruence generated by \( X \). Then \( \mathcal{H} \) admits a \( \mathbb{R}^+ \times K \)-structure, where \( \mathbb{R}^+ \times K \) is the isotropy group of the conformal class of \( n \) in \( H_+^+(K) \).

**Proof.** Recall that the form \( \chi \) satisfies, \( \kappa \wedge \chi = i_\kappa \chi = 0 \). As \( j^*(\kappa \wedge \chi) = j^*\kappa \wedge j^*\chi = 0 \), \( j^*\chi = j^*\kappa \wedge \omega \), where \( j \) is the inclusion of \( \mathcal{H} \) in \( M \) and \( \omega \) is a section of \( \Lambda^{k-1}T\mathcal{H} \). Indeed acting on \( j^*\kappa \wedge j^*\chi = 0 \) with \( i_n \), one finds that \( \omega = i_n j^*\chi \) and so \( i_n \omega = 0 \). The forms \( j^*\kappa \) and \( \omega \) are the fundamental forms of \( \mathbb{R}^+ \times K \) and so \( \mathcal{H} \) admits \( \mathbb{R}^+ \times K \)-structure. Note that if \( H_+^+(K) \) admits more than one fundamental forms \( \chi \) that satisfy non-trivial normalization conditions, these are also automatically satisfied on \( \mathcal{H} \).

One of the consequences of the above proposition is as follows.

**Corollary 1.** Let \( M \) admit a \( H_+^+(SO(n-2)) \)-structure and \( \mathcal{H} \) be a null hypersurface in \( M \) defined as in the proposition above. Then \( \mathcal{H} \) is oriented.

**Proof.** As \( M \) admits a null nowhere vanishing \( (n - 1) \)-form \( \chi \). The pull-back of \( \chi \) on \( \mathcal{H} \) defines a nowhere vanishing top form on \( \mathcal{H} \) and so the hypersurface is oriented.

### 3.3. Orbit spaces of null geodesics

#### 3.3.1. Geometry of null geodesic orbit spaces

In many examples of interest, the geometry of a spacetime with a \( H_+^+(K) \)-structure can be described as a fibration. To investigate the conditions required for this, let us assume that there is an open set \( U \subseteq M \) such that the foliation on \( M \) generated by the flow of \( X \) is regular, i.e. the orbit space \( \mathcal{M} \) of the integral curves of \( X \) in \( U \) is a manifold and the projection of \( p : U \rightarrow \mathcal{M} \) is a surjection. \( U \) may be thought of as a neighbourhood of a spacetime boundary.

To find whether a null structure of a spacetime can be projected on \( \mathcal{M} \) consider the following definition whose justification is provided in the theorem that follows.

**Definition 3.2.** A fundamental form \( \chi \) of a \( H_+^+(K) \)-structure is preserved under the flow generated by \( X \) iff

\[
\mathcal{L}_X \chi = b \chi, \tag{28}
\]

for some spacetime function \( b \) which may depend on \( \chi \).

It should be noted that the definition above does not depend on the choice of \( X \) in its conformal class. Any other choice will lead to \( (28) \) up to an appropriate redefinition of the function \( b \). The same applies for the choice of \( \chi \) in its conformal class.

**Theorem 3.2.** Suppose that the spacetime \( M \) admits a \( H_+^+(K) \)-structure with fundamental forms \( \kappa \) and \( \chi \), and that both these forms are preserved under the flow generated by \( X \). In
addition assume that there is an open set $U \subseteq M$ such that the foliation generated by $X$ has orbit space $\mathcal{M}$ and $U$ admits a hypersurface $\mathcal{S}$ which is nowhere tangent to $X$. Then $\mathcal{M}$ admits a null $H(K)$-structure represented by forms in the conformal class of $\kappa$ and $\chi$.

**Proof.** First notice that the invariance on $\kappa$, $\mathcal{L}\kappa = a\kappa$, under the flow generated by $X$, is equivalent to $\kappa \wedge \mathcal{L}\kappa = 0$. So from the results of section 2.3, $X$ generates a null geodesic congruence. Consider the projection $\pi : U \rightarrow \mathcal{M}$. The necessary and sufficient conditions for a form $\alpha$ on $U$ to arise as a pull-back of a form $\beta$ on $\mathcal{M}$, $\alpha = \pi^*\beta$, are $i_X\alpha = \mathcal{L}\alpha = 0$.

Clearly $\kappa$ and $\chi$ satisfy the first condition, see (26), but not the Lie derivative condition. However both $\kappa$ and $\chi$ can be considered up to a conformal factor. So demanding invariance of $e^\ell\kappa$ under the flow of $X$, for some function $f$ on $U$, gives the differential condition

$$X(f) = a. \quad (29)$$

A similar differential equation can be derived for $\chi$.

Differential equations as in (29) have been considered before in the context of manifolds [13] and they admit a unique solution. In particular given a hypersurface $\mathcal{S}$ in $U$ such that $X$ is nowhere tangent on $\mathcal{S}$, the (29) has a unique solution in a neighbourhood $W \subseteq U$ of $\mathcal{S}$ in $U$ such that $f|_S = q$, where $q$ is a smooth function of $\mathcal{S}$.

Therefore from the assumptions of the theorem, the forms $e^\ell\kappa$ and $e^p\chi$ exist and they are the pull-back of the forms $\kappa_M$ and $\chi_M$ on $\mathcal{M}$, respectively, where $X(f) = a$ and $X(h) = b$.

Furthermore $\kappa_M \wedge \chi_M = 0$ as $\pi^*$ is an inclusion. As under the projection $\pi, X = 0$, the structure group of $\mathcal{M}$ is $H(K)$, see (3). $T^*\mathcal{M}$ admits a trivial line bundle $N$ whose fibres are spanned by $\kappa_M$ but it does not naturally split as $T^*\mathcal{M} = \mathcal{N} \oplus \mathcal{T}\mathcal{M}$. Of course a splitting can always be arranged by choosing a dual vector field to $\kappa_M$.

A priori the solutions of (29) and so the choice of forms $e^\ell\kappa$ and $e^p\chi$ depends on the choice of the hypersurface $\mathcal{S}$ and the choice of the boundary condition $f|_S = q$ on $\mathcal{S}$. As two solutions $f_1$ and $f_2$ of (29) satisfy $X(f_1 - f_2) = 0$ and so $f_1 = f_2 + p$, where $p$ is a function of $\mathcal{M}$. The arbitrariness in the choice of hypersurface $\mathcal{S}$ and boundary condition can be compensated in the choice of the conformal class of $\kappa_M$ and $\chi_M$ on $\mathcal{M}$. Theorem 3.2 can be seen as an appropriate generalization to null G-structures of the result proven in [1] which states that if $M$ is a Robinson manifold, then $\mathcal{M}$ admits a CR structure, see also proposition 4.4.

It should be noted that if a $H^+_e(K)$-structure has fundamental form $\kappa$ and more than one forms that are required to obey relative normalisation conditions, $\mathcal{M}$ is induced with $H(K)$-structure provided that the flow of $X$ apart from individual forms also preserves their relative normalization conditions. An example of this arises in the description of the $H_1^+(SU(k))$ structure, see section 4.2.

**Corollary 2.** Let $M$ admit a $H_1^+(SO(n - 2))$-structure and $\kappa$ be preserved by the $X$ flow, then $d\text{Vol}(N^\perp)$ is preserved by the $X$ flow. Moreover if the remaining conditions of the theorem 3.2 are met, then $\mathcal{M}$ is oriented.

**Proof.** We have demonstrated that if $M$ admits a $H_1^+(SO(n - 2))$ structure, there is a nowhere vanishing $(n - 1)$-form $d\text{Vol}(N^\perp)$ on $M$. Now if $\mathcal{L}\kappa = a\kappa$, then $\mathcal{L}\kappa d\text{Vol}(N^\perp) = b d\text{Vol}(N^\perp)$, where $b = a + i_Xe^\ell$. Therefore, the $H_1^+(SO(n - 2))$ structure is preserved by the flow of $X$. In particular, $d\text{Vol}(N^\perp)$ induces a nowhere vanishing top form on $\mathcal{M}$ and so it is oriented. □
Next let us investigate the conditions for the component \( g_T \) of the metric \( g \) of \( M \) to arise from an appropriate tensor on \( M \). First observe that the push forward bundle \( \pi_*T \) on \( M \) is independent from the choice of the splitting \( \lambda \) used to define \( T \). Because of this, one can define \( \pi_*(N^\perp/N) := \pi_*T \). After choosing a splitting, from construction one has \( i_Xg_T = 0 \). To continue suppose there is a splitting such that the condition \( L_Xg_T = c g_T \) holds. Then using the same arguments as in theorem 3.2, there is a tensor \( g_M \) such that \( \pi_*g_M = c g_T \) for some function \( f \) of \( M \) with \( X(f) = c \). The tensor \( g_M \) is a fibre metric on the subbundle \( \pi_*(N^\perp/N) \) in \( T_M \).

3.3.2. Reconstruction of \( M \) from \( M \). One can reconstruct a Lorentzian manifolds \( M \) with a null \( G \)-structure from the geometric data on \( M \) given by \( \kappa_M \), \( \chi_M \), and \( g_M \) as follows. Consider \( M = \mathbb{R} \times \mathcal{M} \). Set \( X = \partial_u \), where \( u \) is the coordinate of \( \mathbb{R} \). The fundamental forms of the null \( G \)-structure on \( M \) are \( \kappa = a_1 \pi^*\kappa_M \) and \( \chi = a_2 \pi^*\chi_M \), where \( a_1, a_2 \) are strictly positive functions on \( M \). Note however that if there are two or more fundamental forms \( \chi \), the conformal factors \( a_2 \) should be chosen in such a way that they preserve their relative normalization. Next a compatible metric can be introduced on \( M \) as
\[
g = a_1 \pi^*\kappa_M \otimes \lambda + a_1 \lambda \otimes \pi^*\kappa_M + a_3 \pi^*g_M, \tag{30}
\]
where \( a_3 > 0 \) is a function of \( M \) and \( \lambda \) is any 1-form on \( M \) such that \( \lambda(\partial_u) = 1 \). It is straightforward to verify that by construction \( M \) admits a null \( G \)-structure. For some null \( G \)-structures, like the Robinson structure described below, there is a larger class of compatible metrics than the one described above.

3.4. Symmetries of a null structure

Let \( M \) admit a \( H^+_1(K) \)-structure with fundamental forms \( \kappa \) and \( \chi \). A diffeomorphism \( \Phi \) of \( M \) preserves the null structure iff \( \Phi^*\kappa = f_1 \kappa \) and \( \Phi^*\chi = f_2 \chi \), where \( f_1 \) and \( f_2 \) are spacetime functions which may depend on \( \Phi \). In particular, the infinitesimal diffeomorphisms generated by a vector field \( W \) preserve the null structure, iff
\[
\mathcal{L}_W\kappa = a \kappa, \quad \mathcal{L}_W\chi = b \chi, \tag{31}
\]
where \( a, b \) are again functions of \( M \).

**Proposition 3.2.** Let \( \chi = \kappa \wedge \omega \). If \( W \) preserves the \( H^+_1(K) \)-structure on \( M \) with fundamental forms \( \kappa \) and \( \chi \), then
\[
\mathcal{L}_W\omega = p \omega 
\]
for some spacetime function \( p \).

**Proof.** Indeed
\[
\mathcal{L}_W\chi = \mathcal{L}_W\kappa \wedge \omega + \kappa \wedge \mathcal{L}_W\omega = \mathcal{L}_W\kappa \wedge \omega + \kappa \wedge \mathcal{L}_W\omega = (a + p)\chi, \tag{33}
\]
and so \( p = b - a \).

Apart from preserving the forms \( \kappa \) and \( \chi \), one may also require that \( X \) is invariant under \( W \) up to a conformal rescaling. In such a case, an additional condition can be imposed as
\[
[W,X] = c X, \tag{34}
\]
for some spacetime function $c$. The condition (34) is not implied by the first condition in (31) as the additional restriction $i_X(L_w g) = q \kappa$, $q$ a spacetime function, should be imposed on the metric $g$ which may not necessarily be satisfied.

### 3.5. Examples

A large class of examples that includes many of the black holes and brane solutions in all dimensions can be described with the metric

$$g = -A^2(r)dt^2 + B^2(r)dr^2 + C^2(r)g(\Sigma),$$

where $t$ is the time coordinate, $r$ is a radial coordinate and $g(\Sigma)$ is the metric on a Riemannian manifold $\Sigma$. To bring this metric into desirable form first change coordinates to $u^* = r^*, t = v \pm r^*$, where $dr^* = \pm B(r)A^{-1}(r)dr$, to yield

$$g = \mp 2A^2(u^*)dv (du^* \pm \frac{1}{2}dv) + C^2(u^*)g(\Sigma).$$

This has the desirable form but it often convenient to introduce a coordinate $u$ such that $du = \mp A^2(u^*)du^*$. The metric then becomes

$$g = 2dv (du \pm \frac{1}{2} D(u)dv) + C^2(u)g(\Sigma).$$

Clearly $X = \partial_u, \kappa = dv$ and $\lambda = du \pm \frac{1}{2} D(u)dv$. For the $n$-dimensional Schwarzschild black hole with $n > 3$, one finds $D(u) = \mp(1 - 2Mu^{n-3})$, $C^2(u) = u^2$ and $g(\Sigma) = g(S^{n-2})$ is the round metric of $S^{n-2}$. Hence one has

$$g = 2dv \left( du - \frac{1}{2}(1 - 2Mu^{n-3})dv \right) + u^2g(S^{n-2}).$$

While for the $n$-dimensional Reissner–Nordström black hole with $n > 3$, one can show that

$$g = 2dv \left( du - \frac{1}{2} u^{n-2} - \frac{2Mu + Q^2u^{-n+4}}{u^{n-2}}dv \right) + u^2g(S^{n-2}).$$

Most metrics of brane and intersecting brane solutions in all dimensions have been brought into a similar form to (37) in [17] and the formulae will not be repeated here. This is a large class of solutions which include the M2- and M5-branes as well as their intersections. In particular, these include the intersection of two M2-branes at a 0-brane and well as that of a M2-brane and a M5-brane at a string and many others. This class of solutions also includes multi-intersections of M2- and M5-branes. As the triple and other multi-intersections of M2- and M5-branes give rise upon dimensional reduction black holes in 4- and 5-dimensions which are charged and with active scalars, the (37) ansatz includes additional black hole solutions to those of Schwarzschild and Reissner–Nordström. Other solutions are included in (37) are the type II and type I branes in 10-dimensions as well as their intersections. These also give rise upon dimensional reduction black holes in 4- and 5-dimensions which are included in the ansatz (37) and so admit a null G-structure, see [17] for more details. All the above solutions admit by construction a $H^+_q(SO(n - 2))$-structure at least in the region of validity of the described coordinates. Clearly all the examples above come with a chosen splitting of the tangent bundle.

The Kerr black hole solution admits a $H^+_q(U(1))$-structure. In fact it is associated with a Robinson structure which we shall investigate in the next section. This has been instrumental
in the discovery of the solution [18]. More examples of black hole solutions which can be shown to admit null G-structures in various dimensions can be found in [19].

Another class of examples with a null G-structure are all supersymmetric solutions for which the isotropy group of some of their Killing spinors, $\epsilon$, in the connected component of Spin$(n - 1, 1)$ is non-compact. In such a case, the group $G$ of the null G-structure is identified with the non-compact isotropy group $G$ of these Killing spinors. An extensive list of isotropy groups of spinors in various dimensions can be found in [16], see also [8]. An inspection of the isotropy groups of spinors in [16] reveals that supersymmetric backgrounds admit a variety of null G-structures which include that of the Robinson manifolds associated with $H^+_{N}(U(n))$, see section 4.1. All such solution are characterized by the existence of a nowhere vanishing null Killing vector field $X$ such that $X\epsilon = 0$, where $X$ is the Clifford algebra element associated to $X$. Moreover the 1-form $\kappa$ is identified with the Dirac current of $\epsilon$, $\kappa(X) = (\epsilon, X\epsilon)_D = 0$, where $(\cdot, \cdot)_D$ is the Dirac inner product. For all such backgrounds $\mathcal{L}_X\kappa = 0$. Furthermore, all the fundamental forms $\chi$ which are constructed as spinor bilinears of $\epsilon$ satisfy $\kappa \wedge \chi = 0$, $i_X\chi = 0$ and $\mathcal{L}_\chi\chi = 0$. Clearly $X$ generates a null geodesic congruence in $M$. In addition all such fundamental forms $\chi$ are invariant under $X$ and so they are the pull back of forms on the space of orbits $\mathcal{M}$ of the null geodesics. No further conformal normalization of $\chi$ is necessary for this.

4. Null $H^+_{N}(U(k))$- and $H^+_{N}(SU(k))$-structures

4.1. Null $H^+_{N}(U(k))$-structures and Robinson manifolds

4.1.1. Robinson manifolds. Lorentzian manifolds with a $H^+_{N}(U(k))$-structure are characterized by the existence of two nowhere vanishing forms $\kappa$ and $\chi$, where $\chi$ after choosing a splitting and using proposition 3.1, can be represented as $\kappa \wedge \omega$ with $\omega$ an almost Hermitian form on $T$. Furthermore $\omega$ and $g_T$ induce an almost complex structure on $T$.

On the other hand it is known for sometime that (almost) Robinson manifolds are closely associated to the existence of an (almost) complex structure on a Lorentzian manifold. To give the definition of these manifolds [20], see also [1, 21], recall that a subspace $W$ of a Lorentzian vector space $V$ is null or totally null, iff $W \cap W^\perp \neq \emptyset$ or $W \subset W^\perp$, respectively. Furthermore $W$ is maximally totally null (MTN), iff $W = W^\perp$.

**Definition 4.1.** Let $(M, g)$ be an even dimensional Lorentzian manifold. $(M, g)$ is an almost Robinson manifold, iff the complexified tangent bundle $TM \otimes \mathbb{C}$ admits a subbundle $W$ whose fibre $W_p$ is MTN subspace of $T_pM \otimes \mathbb{C}$, $p \in M$.

The definition of almost Robinson manifolds makes some use of the spacetime metric which is used to define $W^\perp$. This is unlike the definition of almost complex manifolds. Of course almost complex manifolds are always almost Hermitian. This is because given an almost complex structure $I$, one can always find a Hermitian metric $h$ with respect to $I$, $h(U, V) = h(IU, IV)$, by setting $h(U, V) = g(U, V) + g(IU, IV)$ for any metric $g$. Nevertheless the definition of almost complex manifolds does not involve the metric.

**Proposition 4.1.** Oriented and time oriented almost Robinson manifolds $(M, g)$ are Lorentzian manifolds with an $H^+_{N}(U(k))$ structure.

**Proof.** A consequence of the definition of an almost Robinson manifold is that there is a real line bundle $N$ such that $W \cap W = N \otimes \mathbb{C}$ and $W + W = N^\perp \otimes \mathbb{C}$. To see this as $W_p \cap W_p$ is invariant under complex conjugation, it is the complexification of a real vector space $N_p$,
i.e. \( W_p \cap W_p = N_p \otimes \mathbb{C} \). Next observe that \( g(W_p \cap W_p, W_p) = 0 \) and \( g(W_p \cap W_p, W_p) = 0 \) because \( W_p \cap W_p \subset W_p \) and \( W_p \cap W_p \subset W_p \). Therefore \( g|_{N_p} = 0 \). The dimension of \( N_p \) must be one as otherwise the metric would have been degenerate—there are no subspaces in Minkowski space spanned by two or more (pseudo-)orthogonal null vectors. The line bundle \( N \) is topologically trivial as the spacetime is oriented and time oriented. Furthermore, we have that

\[
0 \to N \otimes \mathbb{C} \to W \to W/N \otimes \mathbb{C} \to 0, \tag{40}
\]

and

\[
0 \to N \to N^\perp \to N^\perp/N \to 0. \tag{41}
\]

Suppose now that the two sequences above split and so in particular \( W = N \otimes \mathbb{C} \oplus R \) and \( N^\perp = N \oplus T \). Moreover \( R \) is an almost holomorphic subbundle of \( T \otimes \mathbb{C} \) and \( T \otimes \mathbb{C} = R \oplus \bar{R} \).

An almost complex structure is defined on \( T \|\mathbb{C} \) as

\[
\alpha \to \bar{\alpha} \text{ is no-where vanishing section of } \bar{N},
\]

for complex manifolds. It is often convenient to express it in terms of forms. For this define \( \alpha \) or \( e^\alpha \) and \( \partial \gamma \) or \( \bar{\partial} \gamma \),

\[
\alpha \gamma = \bar{\alpha} \gamma = \bar{\bar{\alpha}} \gamma = \bar{\bar{\bar{\alpha}}} \gamma = 0 \text{ } \forall \gamma \text{ in } \mathbb{C} W_p \}, \text{ } p \in M. \text{ Then (42) can also be expressed as}
\]

\[
d\alpha = \beta \wedge \gamma, \tag{43}
\]

where \( \alpha, \beta \in \Gamma(W^0) \) while \( \gamma \in \Gamma(\Lambda^1(M)) \otimes \mathbb{C} \). This integrability condition in terms of a local coframe \( \{ e^-, e^+, e^\alpha, e^\bar{\alpha} \} \) can be written as

\[
d e^- = e^- \wedge \rho + i h_{\alpha \bar{\beta}} e^\alpha \wedge e^\bar{\beta}, \quad d e^\alpha = e^- \wedge \mu^\alpha + e^\beta \wedge \tau^\alpha_{\beta}, \tag{44}
\]

where \( \rho, \mu, \tau \) are spacetime 1-forms.

**Proposition 4.2.** On Robinson manifolds the null vector field \( X \) associated to \( \kappa = e^- \) generates a null geodesic congruence.

**Proof.** This has been demonstrated, see e.g. [1] and references within, and the proof will be repeated here for completeness. It suffices to show that \( (20) \) holds. Indeed using the first condition in \( (44) \), one finds that

\[
\mathcal{L}_X \kappa = ix d \kappa = -ix \rho \kappa, \tag{45}
\]
and so $\kappa \wedge \mathcal{L}_\chi \kappa = 0$. Then the result follows from proposition 2.3.

It is well-known that many 4-dimensional solutions admit a Robinson structure. In particular for the black hole solutions (38) and (39), one has

$$e^{-} = dv, \ e^{1} = \sqrt{2} u \ \frac{dz}{1 + z^{2}},$$

(46)

where $z$ is the inhomogeneous complex coordinate on $S^{2} = CP^{1}$ and $\sqrt{2}$ appears for $S^{2}$ to have radius 1. Observe that the integrability condition (44) is satisfied.

4.1.2. Geometry of Robinson manifold null hypersurfaces. Before we proceed to investigate the geometry of null hypersurfaces in Robinson spacetimes, it is helpful to state the definition of Cauchy–Riemann (CR) structures.

**Definition 4.3.** A 2k + 1-manifold $\mathcal{M}$ admits an almost Cauchy–Riemann (CR) structure, if $T\mathcal{M} \otimes \mathbb{C}$ admits a rank $k$ subbundle $\mathcal{W}$ such that at every point $p \in \mathcal{M}$, $\mathcal{W}_{p} \cap \mathcal{W}_{p} = \{0\}$. Moreover $\mathcal{M}$ admits a CR structure, iff in addition

$$[\Gamma(\mathcal{W}), \Gamma(\mathcal{W})] \subset \Gamma(\mathcal{W}).$$

(47)

Any hypersurface in a complex manifold admits a CR structure.

**Proposition 4.3.** Any null hypersurface $\mathcal{H}$ in a Robinson manifold transversal to the geodesic congruence generated by $X$ admits a CR structure.

**Proof.** Let $n$ be the normal to the hypersurface. The spacetime metric $g$ restricted on $\mathcal{H}$ and $i_{n}\chi$ define a fibre almost Hermitian structure on $T^{\mathcal{H}}$, $T\mathcal{H} = \mathcal{N} \oplus T^{\mathcal{H}}$. This can be used to decompose $T^{\mathcal{H}} \otimes \mathbb{C} = \mathcal{W} \oplus \mathcal{W}$ where the fibres of $\mathcal{W}$ are the eigenvectors of the complex structure associated with the $i$ eigenvalue. The integrability condition (47) also follows from that of the Robinson manifolds. This is just the restriction of (44) on $\mathcal{H}$. 

4.1.3. Geometry of orbit spaces. Let $\mathcal{M}$ be the orbit space of null geodesics generated by $X$ in open subset $U$ of a Robinson manifold $M$ and $\pi$ the projection $\pi: U \to \mathcal{M}$.

**Proposition 4.4.** $\mathcal{M}$ admits a CR-structure

**Proof.** This has already been demonstrated in [1]. To show the statement define the $\mathcal{W} = \pi_{*}W$ and $\mathcal{W} = \pi_{*}W$. These are subbundles of $T\mathcal{M} \otimes \mathbb{C}$. Moreover as $W \cap W = N \otimes \mathbb{C}$, $\pi_{*}X = 0$ and the rank of $\pi_{*}$ is $2n - 1$, we have that $\mathcal{W}_{p} \cap \mathcal{W}_{p} = \{0\}$, $p \in \mathcal{M}$. Furthermore the integrability of the sections of $\mathcal{W}$ follows from that of $W$ as $\pi_{*}[A, B] = [\pi_{*}A, \pi_{*}B]$. 

Generically for $n > 4$, $\mathcal{M}$ does not inherit the fundamental forms associated with the Robinson structure on $U$ unless they are preserved by the flow of $X$. As $\kappa$ is preserved by the flow as a consequence of the integrability condition of the Robinson structure, there is a 1-form $\kappa_{\mathcal{M}}$ such that $a^{-1} \kappa = \pi^{*} \kappa_{\mathcal{M}}$ for some nowhere vanishing function $a$ on $M$. Similarly if $\chi = \kappa \wedge \omega$ is preserved by the flow, there is a 3-form $\chi_{\mathcal{M}}$ such that $b^{-1} \chi = \pi^{*} \chi_{\mathcal{M}}$, where $b$ is a strictly positive function on $U$. As $\pi^{*}$ is an inclusion $\kappa_{\mathcal{M}} \wedge \chi_{\mathcal{M}} = 0$, and so $\chi_{\mathcal{M}} = \kappa_{\mathcal{M}} \wedge \omega_{\mathcal{M}}$.

It is not always the case that a manifold with a CR-structure can be realized as a hypersurface of a complex manifold [3]. In modern terminology, the non-realizable CR manifolds are not ‘holomorphic’ in the context of complex geometry. However those that are solutions of the vacuum Einstein equations are [14].
Furthermore if $g_T$ is preserved by the flow, one can define a fibre metric $g_M$ on real part of $\mathcal{W} \oplus \mathcal{W}$. Then up to an appropriate conformal rescaling of either $g_M$ or $\omega_M$ one can define a fibre hermitian structure such that the holomorphic fibres are $\mathcal{W}_p$, see also section 3.3.

The $n = 4$ case is special and both the fundamental forms $\kappa$ and $\chi$ are preserved by the flow generated by $X$ as a consequence of the integrability conditions (44). As a result $\mathcal{M}$ apart from the CR structure also admits a globally defined 1- and 3-forms.

The Robinson manifolds $M$ for which all the fundamental forms and $g_T$ can be recovered from data on $\mathcal{M}$ up to appropriate conformal rescaling locally can be written as $\mathbb{R} \times \mathcal{M}$, where $X = \partial_a$ and $u$ a coordinate of $\mathbb{R}$. In particular, the spacetime metric is

$$g = a \pi^* \kappa_M \otimes \lambda + a \lambda \otimes \pi^* \kappa_M + p \pi^* g_M$$

(48)

where $a, p > 0$ are nowhere vanishing functions on the spacetime and $\lambda$ is any 1-form on the spacetime with $\lambda(\partial_u) = 1$. A more general construction of Robinson manifolds from the CR-structure on $\mathcal{M}$, $n > 4$, is given in [1]. Indeed if the integrability condition of the CR structure on $\mathcal{M}$ is expressed in terms of a local frame as in (44), then the most general metric compatible with the induced Robinson structure on $\mathcal{M} = \mathbb{R} \times \mathcal{M}$ is

$$ds^2 = a \pi^* \kappa_M \otimes \lambda + a \lambda \otimes \pi^* \kappa_M + q_{\alpha \beta} \left( \pi^* e^\alpha \otimes \pi^* e^\beta + \pi^* e^\beta \otimes \pi^* e^\alpha \right).$$

(49)

where $(q_{\alpha \beta}) = q_{\beta \alpha}$, and both $a$ and $(q_{\alpha \beta})$ depend on all coordinates. However, the $\chi$ fundamental form on $M$ will not always be the pull back of a form on $\mathcal{M}$.

We conclude this section on the Robinson structure with the remark that there are many refinements of the almost Robinson structure on $M$ as well as those induced on the hypersurfaces $\mathcal{H}$ and the orbit spaces $\mathcal{M}$. These can be investigated with similar techniques to those developed by Gray and Hervella [27] for exploring the different classes of almost Hermitian manifolds. In particular, the covariant derivatives of the fundamental forms

$$\nabla \kappa, \quad \nabla \chi,$$

(50)

where $\chi = \kappa \wedge^\mathcal{H} \omega$, can be decomposed in irreducible representations of $H^+_2(U(k))$ or even $U(k)$, $n = 2k + 2$. The different classes are then identified depending on which components of (50) are non-vanishing in such a decomposition. Some of these calculations have already been done in a similar Lorentzian context in [22, 23], see also [24, 25]. Also several special structures can arise like contact and Sasakian structures on $\mathcal{H}$ and $\mathcal{M}$.

### 4.2. Null $H^+_2(SU(k))$-structure

The $H^+_2(SU(k))$ structure apart from the two fundamental forms $\kappa$ and $\kappa \wedge \omega$ of the $H^+_2(U(k))$ structure also exhibits a third fundamental form $\kappa \wedge \epsilon$, where $\epsilon$ is the holomorphic volume fundamental $(k,0)$-form of $SU(k)$, $n = 2k + 2$. Furthermore, there are two relative normalization conditions that these forms satisfy

$$\kappa \wedge \omega \wedge \epsilon = 0, \quad \kappa \wedge \omega^k = m(k) \kappa \wedge \epsilon \wedge \epsilon,$$

(51)

where $m(k) \neq 0$ is a numerical normalization factor that depends on conventions whose precise value is not essential for the analysis that follows. As the implications of the existence of $\kappa$ and $\kappa \wedge \omega$ on the geometry of the spacetime $M$ have already been investigated in the context of almost Robinson manifolds, the focus is on the properties of $\kappa \wedge \epsilon$. It is clear that $\kappa \wedge \epsilon$ can be restricted on any null transversal hypersurface $\mathcal{H}$. This together with the two other fundamental forms induce a $\mathbb{R}^+ \times SU(k)$ structure on $\mathcal{H}$. 

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Furthermore if $\kappa$, $\kappa \wedge \omega$ and $\kappa \wedge \epsilon$ are preserved by the flow of $X$, they all, up to an appropriate conformal rescaling, are the pull-back of appropriate forms on $\mathcal{M}$. However this is not sufficient for $\mathcal{M}$ to admit a $H^+_n(SU(k))$-structure. For this the two normalization conditions (51) must also be preserved by the flow. In particular if $\mathcal{L}_X(\kappa \wedge \omega) = b \kappa \wedge \omega$ and $\mathcal{L}_X(\kappa \wedge \epsilon) = c \kappa \wedge \epsilon$, then for $\mathcal{M}$ to admit a $H^+_n(SU(k))$-structure, it is required that $kb = c + \hat{c}$.

5. Symmetries of null structures

To investigate the symmetries of the $H^+_n(K)$-structures, we shall first adapt a coordinate to $X$, $X = \partial_u$, and assume throughout that $\kappa \wedge d\kappa = 0$. The latter condition is a restriction on the type of null $G$-structure that will be investigated here. However many solutions satisfy this that include the Schwarzschild and Reissner–Nordström black holes as well as many of the black holes in 4- and 5-dimensions associated with intersecting branes, see section 3.5. The condition $\kappa \wedge d\kappa = 0$ implies that there is a coordinate $v$ such that $\kappa = hdv$, where $h$ depends on all spacetime coordinates. In such a case, the most general spacetime vector field can be written as $W = W^a \partial_a + W^v \partial_v + W^y \partial_y$, where $y^i$ are the remaining spacetime coordinates. In what follows, we shall identify the Lie algebra of $W$’s that preserve a $H^+_n(K)$-structure for $K = SO(n - 2)$ and $K = U(k)$, $n = 2k + 2$.

5.1. Symmetries of the $H^+_n(SO(n - 2))$ structure

To identify the Lie algebra of $W$’s that preserve the $H^+_n(SO(n - 2))$ structure, it seems reasonable to assume that $W$ preserve $X$, i.e. $[W, X] = aX$. Then it is straightforward to deduce that

$$W = W^a(u,v,y)\partial_a + W^v(v,y)\partial_v + W^y(v,y)\partial_y,$$

(52)

where the dependence of vector fields $W$ on the various coordinates is implicitly given. From now on, we shall always take that the vector fields that generate the symmetries of the null structure are of the form (52). Of course if the function $a$ vanishes, then $W^a = W^a(v,y)$. However, we allow in what follows the component $W^a$ to depend on $u$.

Suppose that in addition $W$ preserves also $\kappa$, $\mathcal{L}_W \kappa = b \kappa$. In such a case, one finds that

$$W = W^a(u,v,y)\partial_a + W^v(v)\partial_v + W^y(v,y)\partial_y.$$

(53)

Furthermore the invariance of the fundamental $(n - 1)$-form $\chi$ of $H^+_n(SO(n - 2))$, $\mathcal{L}_W \chi = c \chi$, for a function $c$, does not impose any additional conditions on $W$. The component $W^v \partial_v$ generates the diffeomorphisms of a real line with coordinate $v$. Let as denote the Lie algebra with $\text{diff}(\mathbb{R})$. Similarly the component $W^y \partial_y$ generates the infinitesimal diffeomorphisms in the $y$ coordinates, which can be thought of as the coordinates of a manifold $\mathcal{S}$, but now depend on $v$. This can be thought as the Lie algebra of a path group of the diffeomorphism group of $y$’s which we denote with $\text{pdiff}(\mathcal{S})$. Similarly the component $W^a \partial_a$ can be thought as generating diffeomorphisms of the coordinate $u$ depending on both the $v$ and $y$ coordinates. Denoting the Lie algebra of these with $\text{conf}(X)$, as they are a conformal rescaling of $X$, we have that the Lie algebra of symmetries is $\text{diff}(\mathbb{R}) \oplus \text{pdiff}(\mathcal{S}) \oplus \text{conf}(X)$, where $\oplus$ denotes semi-direct sum.

To get a bit more insight into these symmetries, suppose that the spacetime is asymptotically flat with a Killing horizon at $u = u_0$, $g(\partial_u, \partial_u)|_{u=u_0} = 0$. These conditions are met by the black hole solutions (38) and (39). At the horizon hypersurface $\mathcal{H}$ the vector field (53) restricts to
\[ W^N = W^r(v)\partial_r + W^I(v,y)\partial_I. \] (54)

As \( \partial_r \) is normal to \( \mathcal{H} \), \( \partial_r \) generates a null geodesic congruence in \( \mathcal{H} \) whose space of orbits is \( \mathcal{S} \). \( \mathcal{S} \) is the spatial horizon section of the horizon. The Lie algebra of \( W^N \)'s is \( \mathfrak{diff}(\mathbb{R}) \oplus \mathfrak{pdiff}(\mathcal{S}) \).

If \( \mathcal{S} \) is a surface, \( \mathcal{S} = S^{n-2} \), this algebra includes the Lorentz Lie algebra \( \mathfrak{so}(n - 1, 1) \) as the conformal transformations of \( S^{n-2} \).

On the other hand at infinity, the null hypersurface \( \mathcal{I} \) which is transversal to the geodesic congruence generated by \( X \) is given in all examples by the linear equation \( pu + qv = 0 \), where \( p, q \in \mathbb{R} - \{0\} \), e.g. for the black hole solutions (38) and (39) \(-2u + v = 0\). Adapting a coordinate \( s \) along the asymptotic null infinity hypersurface \( \mathcal{I} \), say \( s = pu - qv \), one finds that the component of (53) tangential to \( \mathcal{I} \) and restricted on \( \mathcal{I} \) reads
\[ W^\mathcal{I} = W^a(s,y)\partial_a + W^I(s,y)\partial_I. \] (55)

The Lie algebra of \( W^\mathcal{I} \)'s is the Lie algebra of the infinitesimal diffeomorphisms of \( \mathcal{I} \), \( \mathfrak{diff}(\mathcal{I}) \).

Therefore the spacetime diffeomorphisms that leave the \( H_L^+ (\mathfrak{so}(n - 2)) \)-structure invariant interpose between the symmetries of the horizon hypersurface and those of the null asymptotic infinity. Again if \( S = S^{n-2} \), \( \mathfrak{diff}(\mathcal{I}) \) includes the BMS Lie algebra. It is curious though that the symmetries that preserve the whole \( H_L^+ (\mathfrak{so}(n - 2)) \)-structure do not include the supertranslations of the BMS Lie algebra at the horizon. Though of course this will be the case if instead of the vector fields in (53) one considers the vector fields in (52) which preserve part of the \( H_L^+ (\mathfrak{so}(n - 2)) \)-structure.

We have seen that the Lie algebras which preserve the \( H_L^+ (\mathfrak{so}(n - 2)) \)-structure are much larger than either the Lorentz or the BMS algebra even when they are restricted at the asymptotic null infinity. This is not surprising as all considerations do not involve the spacetime metric even at infinity. However if one attempts to match the BMS algebra at infinity for asymptotically flat spacetimes, the vector fields in (53) will read
\[ W = W^a(pu + qv,y)\partial_a + W^r(y)\partial_r + W^I(y)\partial_I, \] (56)
where \( W^r \) is constant and \( W^I(y)\partial_I \) generate the conformal transformations on \( S^{n-2} \). The component \( W^r\partial_r \) generates translations of the affine parameter \( v \). The remaining Lie algebra is the Lie algebra of the path group of the BMS group, \( \mathfrak{pbms} \), as the supertranslations depend on the additional parameter \( pu + qv \). So the Lie algebra of vector fields (56) is \( \mathbb{R} \oplus \mathfrak{pbms} \). At the horizon \( \mathcal{H} \), the Lie algebra becomes \( \mathbb{R} \oplus \mathfrak{so}(n - 1, 1) \). The vector fields (56) generate the Lie algebra of transformations that leaves the \( H_L^+ (\mathfrak{so}(n - 2)) \)-structure of asymptotically flat spacetimes invariant.

This matching to the BMS symmetry at infinity can also be done for the vector fields (52).

In such a case, these vector fields will interpolate between the BMS algebra at the horizon and that at the asymptotic null infinity.

5.2. Symmetries of the integrable \( H_L^+ (\mathfrak{u}(k)) \) structure

As in the investigation of symmetries of the \( H_L^+ (\mathfrak{so}(n - 2)) \)-structure, we assume that \( \kappa \wedge d\kappa = 0 \) and adapt coordinates \( u, v \) such that \( X = \partial_u \) and \( \kappa = hdv \). We have already established that the transformations that preserve both \( X \) and \( \kappa \) are generated by the vector fields (53). Next instead of imposing that \( W \) also leave invariant the fundamental form \( \kappa \wedge \omega \), we impose that the condition that \( W \) leaves invariant the complex structure \( I \) in \( N^+ / N \), \( \mathcal{L}_W I = 0 \). Assuming that there are coordinates \( (u, v, z^\alpha, \bar{z}^\beta) \) on the spacetime such that \( \Gamma^\alpha_{\beta \gamma} = -\Gamma^\beta_{\alpha \gamma} = i\delta^\alpha_\beta \), a straightforward computation reveals that the condition \( \mathcal{L}_W I = 0 \) implies that
\[ W = W^\alpha \partial_\alpha + W^\alpha (u, v, z, \bar{z}) \partial_u + W^\alpha (v, z) \partial_v + W^\alpha (v, \bar{z}) \partial_{\bar{z}}. \]  \hspace{1cm} (57)

It is significant that the components of \( W^\alpha \partial_\alpha \) depend only on the holomorphic coordinates \( z \). Therefore for each \( v \) these generate the holomorphic diffeomorphisms on \( S \). The Lie algebra of the above vector fields is \( \text{diff}(\mathbb{R}) \oplus \text{phol}(S) \oplus \text{conf}(X) \), where \( \text{phol}(S) \) is the Lie algebra of the path group of the holomorphic diffeomorphisms of \( S \).

A similar argument to that we have used for \( H^\mathcal{I}_H(\text{SO}(n - 2)) \) reveals that the symmetry induced at a Killing horizon hypersurface \( \mathcal{H} \) is generated by the vector fields

\[ W^\mathcal{H} = W^\alpha (v) \partial_\alpha + W^\alpha (v, z) \partial_v + W^\alpha (v, \bar{z}) \partial_{\bar{z}}, \]  \hspace{1cm} (58)

while that at an asymptotic null hypersurface \( \mathcal{I} \) is generated by the vector fields

\[ W^\mathcal{I} = W^\alpha (pu + qv, z, \bar{z}) \partial_u + W^\alpha (z) \partial_v + W^\alpha (\bar{z}) \partial_{\bar{z}}. \]  \hspace{1cm} (59)

For \( n > 6 \)-dimensional black holes with \( S = S^{n-2} \), the above vector field do not generate either the Lorentz group at \( \mathcal{H} \) or the BMS group at \( \mathcal{I} \). This is because \( S^{2k}, k > 3 \), do not admit (almost) complex structures—\( \mathcal{H}^\mathcal{I}_H \) remains open.

However for \( 4 \)-dimensional black holes, the Lie algebra of \( W^\mathcal{H} \) vector fields is \( \text{diff}(\mathbb{R}) \oplus \text{phol}(S, C) \), where \( \text{phol}(S, C) \) is the Lie algebra of the path group of \( \text{SL}(2, \mathbb{C}) \). The latter are the holomorphic transformations of the \( 2 \)-sphere acting with M"obius transformations on the complex coordinate of \( \mathbb{C}P^1 = S^2 \). As \( \text{sl}(2, \mathbb{C}) = \text{so}(3, 1) \), the Lorentz Lie algebra is included. A similar analysis reveals that \( W^\mathcal{I} \) generate the Lie algebra \( \text{phm} \) where we have used again that \( \text{sl}(2, \mathbb{C}) = \text{so}(3, 1) \). Furthermore if the vector fields (57) are matched to those generated by the BMS group at \( \mathcal{H} \), then one finds that

\[ W = W^\alpha (pu + qv, z, \bar{z}) \partial_u + W^\alpha (z) \partial_v + W^\alpha (\bar{z}) \partial_{\bar{z}}, \]  \hspace{1cm} (60)

where \( W^\alpha \) is constant. The Lie algebra of these vector fields in \( \mathbb{R} \oplus \text{phm} \). It interpolates between the Lorentz algebra at the horizon hypersurface \( \mathcal{H} \) and the BMS algebra at \( \mathcal{I} \). Of course one can extend the BMS group by including the singular holomorphic transformations of \( S^2 \) as in [26].

6. Conclusions

We have presented a definition of null G-structures on Lorentzian manifolds which generalizes both the Robinson structure and the null structures which arise in the context of supersymmetric backgrounds in supergravity theories. The novelty of this definition is its generality. As it is inspired from the definition of G-structures on Riemannian manifolds, one can utilize the technology developed for the latter to investigate the properties of Lorentzian manifolds with a null G-structure. Then we proceed to investigate some of the geometric properties of spacetimes admitting such null G-structures by exploring the properties of fundamental forms that characterize such structures. We have also examined the relationship between null G-structures and null geodesic congruences. Then we use this to investigate the geometry of null hypersurfaces transversal to null geodesic congruences as well as that of the orbit spaces of null geodesics. We have investigated in more detail the null \( H^\mathcal{I}_H(K) \)-structures associated with the groups \( K = \text{SO}(n - 2), U(k) \) and \( SU(k) \), \( n = 2k + 2 \), of a \( n \)-dimensional spacetime. We have established that Robinson manifolds admit a \( H^\mathcal{I}_H(\mathcal{U}(k)) \) structure. Furthermore, we have examined the symmetries of some null G-structures and demonstrated that interpolate between the symmetries of Killing horizons and those of null asymptotic infinity. The symmetry algebra of a \( H^\mathcal{I}_H(K) \)-structure for \( K = \text{SO}(n - 2) \), and \( K = \mathcal{U}(1) \) and \( n = 4 \), on asymptotically flat spacetimes includes the BMS algebra.
We have established that many solutions of four and higher dimensional gravitational theories admit a null G-structure. These include the Schwarzschild and Reissner–Nordström black holes in four and higher dimensions. It also includes by construction all supersymmetric solutions which admit a Killing spinor whose isotropy group in the Lorentz group is non-compact. Such Killing spinors give rise to vector bilinears which are null and no-where vanishing and so induce a null G-structure on the spacetime. All brane solutions in 10- and 11-dimensions admit such a Killing spinor as well as many of their intersections, see section 3.5 and also [8, 16, 17]. It is expected that many more (non-supersymmetric) solutions can admit a null G-structure for G a proper subgroup of the Lorentz group. However to identify which one would require an extensive search amongst the plethora of solutions that have been constructed specially in the context of string and M-theories.

Supersymmetric solutions with a null vector bilinear admit a null G-structure everywhere on the spacetime. This is because a Killing spinor has to be defined everywhere on the spacetime, away from singularities, for a solution to be supersymmetric. In non-supersymmetric solutions though the null G-structures are typically defined on open, but not necessarily small, subsets of the spacetime. It will be of interest to explore the patching conditions that are required for a null G-structure to be extend across the whole of spacetime.

The different Gray–Hervella type of classes of a null $H^+_L(K)$-structure can be explored in the same way as those for almost Hermitian manifolds [27]. In fact it may be convenient to decompose the covariant derivatives $\nabla \kappa$ and $\nabla \chi$ of the fundamental forms $\kappa$ and $\chi$ in terms of the representations of the topological structure group $K$ instead of $H^+_L(K)$. Some such calculations in the Lorentzian case have already been done in [22, 23], see also [24, 25]. The various solutions of gravitational theories with a null $H^+_L(K)$-structure will belong to one of the classes that will arise.

Acknowledgments

I would like to thank C Bachas for an invitation to visit École Normale Superiéure, and for hospitality and a stimulating environment to complete this project. I am partially supported from the STFC rolling grant ST/P000258/1.

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