On the Classification of Stable Solutions of some elliptic equations in half-space.

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Abstract

In this paper, we are concerned with stable solutions, possibly unbounded and sign-changing, of some semi-linear elliptic problem with mixed nonlinear boundary conditions. We establish the nonexistence of stable solutions, the main methods used are the Pohozaev identity, monotonicity formula of solutions together with a blowing down sequence.

Keywords: Liouville type theorems, stable solutions, nonlinear boundary conditions, monotonicity.

1. Introduction and main results

This paper is devoted to the study of the following semi-linear elliptic problem

\[(P)\quad -\Delta u + lu = |u|^{p-1}u \quad \text{in } \mathbb{R}^N_+,
\]

where \(\mathbb{R}^N_+ = \{x = (x', x_N), x' \in \mathbb{R}^{N-1}, x_N > 0\}, N \geq 2, p > 1\) and \(l\) is a positive real parameter. The motivation of studying such an equation is originated from the classical Lane-Emden equation

\[-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N\]

(1.1)

A fundamental result on equation (1.1) is the celebrated Liouville-type theorem due to Gidas and Spruck \cite{9}. The equation (1.1) has no positive classical solution if \(1 < p < \frac{N+2}{N-2}\). Moreover, it was also proved that the exponent \(\frac{N+2}{N-2}\) is optimal, in the sense that problem (1.1) admits a positive solution for \(p \geq \frac{N+2}{N-2}\) and \(N \geq 3\). Soon afterward, similar results were established in \cite{10} for positive solutions of the subcritical problem (1.1) in the upper half-space \(\mathbb{R}^N_+\). These results received wide attention as regards the theory itself and its applications. Particularly, when variational methods cannot be employed, one use them to establish a prior bound of solutions for general operator, and therefore existence of solutions may be dealt with via topological methods; see for instance \cite{9, 10}.

On the other hand, the idea of using the Morse index of a solution for a semilinear elliptic equation was first explored by Bahri and Lions \cite{1} to get further proved that when \(1 < p < \frac{N+2}{N-2}\), no sign-changing solution exists for (1.1). To prove this result, they first deduced some integrable conditions on the solution based on finite Morse index; then they used the Pohozaev identity to prove the nonexistence result. So, motivated by \cite{10}, they used blow-up argument to obtain a relevant \(L^\infty\)-bound for solutions of semilinear boundary value problems in bounded domain from the boundedness of Morse index (see also \cite{1, 2, 3, 10, 11}). We mention also that when the Palais-Smale; or the Cerami compactness conditions for the energy functional do not seem to follow readily, the proof of existence of solutions is essentially reduced to deriving \(L^\infty\)-estimate from Liouville-type theorems via Morse index (see for instance \cite{3, 22, 23}). After these works, many authors investigated various Liouville type theorems for solutions with finite Morse

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indices in subcritical case such as problems with Neumann boundary condition, Dirichlet-Neumann mixed boundary and nonlinear boundary conditions (see [2, 11, 14, 15, 24, 25, 26]). In the supercritical case, the finite Morse index solutions to the corresponding nonlinear problem (1.1) have been completely classified by Farina [8].

A natural question is to understand more about finite Morse index solutions of the problem (P) when \( l > 0 \). The case of Dirichlet boundary condition was studied by A. Selmi, A. Harrabi and C. Zaidi in [18]. They prove various Liouville type theorems for stable solutions possibly unbounded and sign-changing. While when the half space is replaced by strip domain similar result was obtained in [19].

In this paper we study solutions, possibly unbounded and sign-changing of the following mixed problems with mixed boundary value conditions

\[
\begin{cases}
-\Delta u + lu = |u|^{p-1}u, & \text{in } \mathbb{R}^N_+ \\
\frac{\partial u}{\partial \nu} = |u|^{q-1}u, & \text{on } \Sigma_1, \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \Sigma_2,
\end{cases}
\]

(1.2) or

\[
\begin{cases}
-\Delta u + lu = |u|^{p-1}u, & \text{in } \mathbb{R}^N_+ \\
\frac{\partial u}{\partial \nu} = |u|^{q-1}u, & \text{on } \Sigma_1, \\
u = 0 & \text{on } \Sigma_2,
\end{cases}
\]

(1.3)

where

\[
\begin{align*}
\mathbb{R}^N_+ &= \{x = (x', x_N), x' \in \mathbb{R}^{N-1}, x_N > 0\}, \\
\Sigma_1 &= \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N; x_N = 0, x_1 > 0\}, \\
\Sigma_2 &= \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N; x_N = 0, x_1 < 0\}.
\end{align*}
\]

In the case \( l = 0 \), Liouville-type theorems and properties for problem (1,2) or (1,3) has been studied by [24, 26]. In [24] it was shown that when \( l = 0 \) there is no nontrivial bounded solution of problem (1.2) or (1.3) with finite morse index, provided that

\[
1 < p \leq \frac{N+2}{N-2}, 1 < q \leq \frac{N}{N-2} \text{ and } (p, q) \neq \left( \frac{N+2}{N-2}, \frac{N}{N-2} \right).
\]

Recently in [12] Harrabi and Rahal have improved the results in [26] for a large class of the exponents \( p \) and \( q \).

Our aim in this work is to analyze the influence of the linear term \( lu \) when \( l > 0 \) in order to classify regular stable solutions possibly unbounded and sign-changing. We will prove various Liouville type theorems for \( C^2 \) solutions which are stable or stable outside a compact set. Our analysis reveals the nonexistence of nontrivial finite Morse index solutions for all \( p > 1 \). Following [6, 7, 13, 18, 20], we establish a variant of the monotonicity formula to provide nonexistence results in the supercritical range. However, in the subcritical case, we require a restrictive condition on \( u \) namely

\[
|u|_{L^\infty(\mathbb{R}^N_+)}^{p-1} < l \left( \frac{N}{2} - \frac{N-1}{q+1} \right) \left( \frac{N}{p+1} - \frac{N-1}{q+1} \right)^{-1},
\]

which allows us to provide nonexistence result from the Pohozaev identity.

We mention that Liouville type theorems in unbounded domains play a crucial role to obtain a priori \( L^\infty \)-bounds for solutions of semilinear boundary value problems in bounded domain, see [9] for the case of positive solutions and [1, 3, 8] and [14] for sign changing solutions and having finite Morse index. Also, nonlinear Liouville type theorems combined with the degree type arguments, are useful to obtain the existence of solutions (see for instance [3]).
In order to state our results we need to recall the following.

**Definition 1.1.** We say that a solution $u$ of \( (1.2) \) belonging to $C^2(\mathbb{R}^N_+)$

- is stable if

\[
Q_u(\varphi) = \int_{\mathbb{R}^N_+} |\nabla \varphi|^2 - q \int_{\Sigma_1} |\varphi|^{q-2} \varphi \varphi^2 + l \int_{\mathbb{R}^N_+} \varphi^2 - p \int_{\mathbb{R}^N_+} |\varphi|^{p-1} \varphi^2 \geq 0 \quad \forall \varphi \in C^1_c(\mathbb{R}^N_+),
\]  

- has a Morse index equal to $K \geq 1$ if $K$ is the maximal dimension of a subspace $X_K$ of $C^1_c(\mathbb{R}^N_+)$ such that $Q_u(\varphi) < 0$ for any $\varphi \in X_K \setminus \{0\}$,

- is stable outside a compact set $K$ if $Q_u(\varphi) \geq 0$ for any $\varphi \in C^1_c(\mathbb{R}^N_+ \setminus K)$.

Similarly, we say that a solution $u$ of \( (1.3) \) belonging to $C^2(\mathbb{R}^N_+)$ is stable (respectively, stable outside a compact set $K$) if $Q_u(\varphi) \geq 0$ for every $\varphi \in C^1_c(\mathbb{R}^N_+ \cup \Sigma_1)$ (respectively, $\varphi \in C^1_c((\mathbb{R}^N_+ \cup \Sigma_1) \setminus K)$).

**Remark 1.1.**

(i) Clearly a solution is stable if and only if its Morse index is equal to zero.

(ii) Any finite Morse index solution $u$ is stable outside a compact set $K \subset \mathbb{R}^N_+$. Indeed there exist $K \geq 1$ and $X_K := \text{span}\{\varphi_1, \ldots, \varphi_K\} \subset C^1_c(\mathbb{R}^N_+)$ such that $Q_u(\varphi) < 0$ for any $\varphi \in X_K \setminus \{0\}$. Then, $Q_u(\varphi) \geq 0$ for every $\varphi \in C^1_c(\mathbb{R}^N_+ \setminus K)$, where $K := \cup_{j=1}^K \text{supp}(\varphi_j)$.

The first result of this paper is

**Theorem 1.1.** Let $u \in C^2(\mathbb{R}^N_+)$, be a stable solution of \( (1.2) \) or \( (1.3) \) and $l \geq 0$. Assume that

1. $p > 1$, $q > 1$ if $N = 2$,

2. $p \in (1, \frac{N+2}{N-2}]$ and $q > 1$, or $p > \frac{N+2}{N-2}$ and $2q - p - 1 \geq 0$ if $N \geq 3$.

Then $u \equiv 0$.

**Remark 1.2.** In statement (2) of Theorem 1.1 if $p > \frac{N+2}{N-2}$ then $2q - p - 1 \geq 0$ is satisfied if $p \leq q$. If $p > q$ the condition $2q - p - 1 \geq 0$ implies that necessarily $q > \frac{N}{N-2}$.

In the case of solutions of \( (1.2) \) or \( (1.3) \) which are stable outside a compact set of $\mathbb{R}^N_+$ we prove that

**Theorem 1.2.** Let $N \geq 3$, $l > 0$ and $u \in C^2(\mathbb{R}^N_+)$ be a solution of \( (1.2) \) or \( (1.3) \) which is stable outside a compact set. Assume that

\[
|u|^{p-1}_{L^\infty(\mathbb{R}^N_+)} < l \left(\frac{N-1}{2} p + 1 \right) \left(\frac{N-1}{2} q + 1 \right)^{-1}, \quad p \in \left(1, \frac{N+2}{N-2}\right) \quad \text{and} \quad q > \frac{N}{N-2},
\]

then $u \equiv 0$.

**Theorem 1.3.** Let $N \geq 3$, $l > 0$ and $u \in C^2(\mathbb{R}^N_+)$, be a solution of \( (1.2) \) or \( (1.3) \) which is stable outside a compact set of $\mathbb{R}^N_+$. Assume that $p \geq \frac{N+2}{N-2}$ with $2q - p - 1 \geq 0$, then $u \equiv 0$.

The proof of Theorem 1.2 or 1.1 uses a version of monotonicity formula of equation \( (1.2) \) or \( (1.3) \). We mention that the monotonicity formula is a powerful tool to understand supercritical elliptic equations or systems. This approach has been used successfully for the Lane-Emden equation in [21].

For $R > 0$, denote $A_R = \{ R < |x| < 2R \}$, $B_R^+ = B_R \cap \mathbb{R}^N_+$ and $\partial B_R^+ = \partial B_R \cap \mathbb{R}^N_+$ where $B_R$ is the open ball centered at the origin and with radius $R$. The key step of the proofs of Theorems 1.1, 1.2 and 1.3 is the following integral estimates which are useful in the subcritical, critical and supercritical cases.

**Proposition 1.1.** Let $u \in C^2(\mathbb{R}^N_+)$ a solution of \( (1.2) \) or \( (1.3) \) which is stable outside a compact set $K \subset B_{R_0}^+$ for some $R_0 > 0$. Then for all $R > 2R_0$, we have.

\[
\int_{B_R^+} |\nabla u|^2 + \int_{B_R^+} l u^2 + \int_{B_R^+} |u|^{p+1} + \int_{\Sigma_1 \cap B_R^+} |u|^{q+1} \leq C_0 + CR^{-2} \int_{A_R} u^2,
\]  

(1.5)
and
\[ \int_{B_R^c} |\nabla u|^2 + \int_{B_R^c} |u|^p + \int_{\Sigma_1 \cap B_R^c} |u|^{q+1} \leq C_0 + CR^N - 2 \frac{p+1}{p}. \] (1.6)

Here \( C = C(N, p) \) and \( C_0 = C_0(u, R_0, N, p) \) are positive constants independent of \( R \). Furthermore, if \( u \) is a stable solution, then (1.5) and (1.6) hold with \( C_0 = 0 \).

**Remark 1.3.** Observe that classification of bounded stable solutions of (1.2) or (1.3) follows immediately from (1.1) for all \( p > 1 \). In fact, since \( u \) is bounded, then from [22] we can find a positive constant \( C \) and a sequence \( R_n \to \infty \) as \( n \to \infty \) such that
\[ \int_{R_n^+ \cap B_{R_n}} u^2 \leq C \int_{R_n^+ \cap B_{R_n}} u^2. \] (1.7)

According to (1.6) (with \( C_0 = 0 \)), we derive
\[ l \int_{R_n^+ \cap B_{R_n}} u^2 \leq CR_n^{-2} \int_{R_n^+ \cap B_{R_n}} u^2. \]

As \( R_n \to \infty \) as \( n \to \infty \), there exists \( n_0 \in \mathbb{N} \) such that \( CR_n^{-2} < \frac{1}{2} \) for all \( n > n_0 \), and therefore we deduce that
\[ \int_{R_n^+ \cap B_{R_n}} u^2 \leq 0, \quad \forall n > n_0, \]
which implies that \( u \equiv 0 \). Point out that for \( l > 0 \), we cannot always use the doubling lemma technique as in the case \( l = 0 \) (see [22]), to reduce the classification for only bounded solutions of (1.2) or (1.3). Therefore, we shall pay special attention to the delicate case of unbounded solutions by exploiting the following variant of the monotonicity formula.

For \( \tau > 0 \), define the function \( u^\tau \) by
\[ u^\tau(x) = \tau^{-\frac{2}{p-1}} u(\tau x), \quad x \in \mathbb{R}_+^N, \]
then we have

**Proposition 1.2.** Let \( u \in C^2(\mathbb{R}_+^N) \) be a solution of (1.2) or (1.3) and \( \tau > 0 \) be a constant. Set
\[ E(u, \tau) = \int_{B_1^c \cap \mathbb{R}_+^N} \left( \frac{1}{2} |\nabla u|^2 + \frac{\lambda \tau}{2} |u|^2 - \frac{1}{p+1} |u|^p \right) + \frac{\lambda \tau}{2} |u|^2 - \frac{1}{p+1} |u|^p \] (1.8)

Then
\[ \frac{dE}{d\tau} = \tau \int_{\partial B_1^c \cap \mathbb{R}_+^N} (\frac{du^\tau}{d\tau})^2 ds + \lambda \tau \int_{B_1^c \cap \mathbb{R}_+^N} (u^\tau)^2 dx + \frac{2q - p - 1}{(p-1)(q+1)} \int_{\Sigma_1 \cap B_1^c} |u^\tau|^{q+1} \] (1.9)

for all \( p, q > 1 \). Furthermore, \( E \) is a nondecreasing function of \( \tau \) if \( 2q - p - 1 \geq 0 \).

This paper is organized as follows. In section 2 we prove Propositions 1.1 and 1.2. The proofs of Theorems 1.1 and 1.2,3 are given respectively in sections 3 and 4.

2. Proofs of Propositions 1.1 and 1.2

**Proof of Proposition 1.1** Let \( u \in C^2(\mathbb{R}_+^N) \) be a solution of (1.2) or (1.3) which is stable outside a compact set \( K \subset B_{R_0}^c \). For all \( R > 2R_0 \), define the family of test functions \( \psi = \psi_R \in C_c^0(\mathbb{R}_+^N) \)
satisfying
\[
\begin{align*}
0 & \leq \psi \leq 1, \\
\psi & = 1 \text{ if } 2R_0 < |x| < R, \quad \psi \equiv 0 \text{ if } |x| < R_0 \text{ or } |x| > 2R, \\
|\nabla \psi| & \leq CR^{-1} \text{ if } R < |x| < 2R.
\end{align*}
\] (2.1)

If \( u \) is a stable solution, we use \( \psi = \psi_R \) verifying (2.1) with \( R_0 = 0 \) (that is \( \psi = 1 \) if \( |x| < R \)). For \( m > 1 \) we have \( u\psi^m \in C^1_c(B^+_2 \setminus B^+_R) \), then it can be chosen as a test function and in view of (1.4), we have
\[
\int_{B^+_2} |\nabla (u\psi^m)|^2 + l \int_{B^+_2} u^2 \psi^{2m} dx - p \int_{B^+_2} |u|^{p+1} \psi^{2m} dx - q \int_{\Sigma_1 \cap B^+_2} |u|^{q+1} \psi^{2m} \geq 0, \quad \forall \ R > 2R_0. \] (2.2)

Multiply (1.2) or (1.3) by \( u\psi^m \) and integrate by parts and taking into account that
\[
\nabla u \nabla (u\psi^m) = |\nabla (u\psi^m)|^2 - u^2 |\nabla (\psi^m)|^2,
\]
so, we get
\[
\begin{align*}
\int_{B^+_2} |\nabla (u\psi^m)|^2 + l \int_{B^+_2} u^2 \psi^{2m} dx - p \int_{B^+_2} |u|^{p+1} \psi^{2m} dx \\
= - \int_{\Sigma_1 \cap B^+_2} |u|^{q+1} \psi^{2m} dx + \int_{B^+_2} u^2 |\nabla (\psi^m)|^2.
\end{align*}
\] (2.3)

Combine (2.2) and (2.3), then for all \( R > 2R_0 \) we derive
\[
\begin{align*}
\int_{B^+_2} |\nabla (u\psi^m)|^2 + l \int_{B^+_2} u^2 \psi^{2m} dx + \int_{B^+_2} |u|^{p+1} \psi^{2m} dx + \int_{\Sigma_1 \cap B^+_2} |u|^{q+1} \psi^{2m} dx \\
\leq C_p \int_{B^+_2} u^2 |\nabla (\psi^m)|^2,
\end{align*}
\] (2.4)

which implies that
\[
\begin{align*}
\int_{B^+_2} |\nabla (u\psi^m)|^2 + l \int_{B^+_2} u^2 \psi^{2m} dx + \int_{B^+_2} |u|^{p+1} \psi^{2m} dx + \int_{\Sigma_1 \cap B^+_2} |u|^{q+1} \psi^{2m} dx \\
\leq C_0 + C_{p,m} R^{-2} \int_{A^+_R} u^2 \psi^{2m-2},
\end{align*}
\] (2.5)

where \( C_0 = 0 \) if \( u \) is a stable solution. Hence inequality (1.5) follows from the last inequality. Using now Young’s inequality and choose \( m = \frac{p+1}{p-1} > 1 \), we deduce that
\[
R^{-2} \int_{A^+_R} u^2 \psi^{2m-2} \leq R^{-2} \int_{B^+_2} u^2 \psi^{2m-2} \leq C_{p,N} R^{N-2} \frac{p+1}{p-1} + \frac{2}{p+1} \int_{B^+_2} |u|^{p+1} \psi^{2m}.
\]

Insert the last inequality into the right-hand-side of (2.5), so derive the main integral estimate (1.6). The proof of Proposition 1.1 is thereby completed. \( \Box \)

**Proof of Proposition 1.2** Since \( u \) is a solution of (1.2), it follows that \( u^\tau \) satisfies
\[
\begin{align*}
\begin{cases}
-\Delta u^\tau + l \tau^2 u^\tau = |u^\tau|^{p-1} u^\tau & \text{in } \mathbb{R}_+^N, \\
\frac{\partial u^\tau}{\partial \nu} = \tau^{1-\frac{2m}{p}} |u^\tau|^{q-1} u^\tau & \text{on } \Sigma_1, \\
\frac{\partial u^\tau}{\partial \nu} = 0 & \text{on } \Sigma_2.
\end{cases}
\end{align*}
\] (2.6)
Indeed, we have \( u^\tau \) variable. In the definition of \( \int_{B_1 \cap \mathbb{R}^N_+} \),

\[
\mathcal{E}(u, \tau) = \int_{B_1 \cap \mathbb{R}^N_+} \left( \frac{1}{2} \nabla u^\tau \right)^2 + \frac{l}{2} \tau^2 (u^\tau)^2 - \frac{1}{p+1} |u^\tau|^{p+1},
\]

then

\[
\frac{d}{d\tau} \mathcal{E}(u, \tau) = \int_{B_1 \cap \mathbb{R}^N_+} \nabla u^\tau \nabla \frac{du^\tau}{d\tau} + l\tau \int_{B_1} (u^\tau)^2 + l\tau^2 \int_{B_1} u^\tau \frac{du^\tau}{d\tau} - \int_{B_1} |u^\tau|^p u^\tau \frac{du^\tau}{d\tau}. \tag{2.7}
\]

Integrating by parts and using the fact that \( u^\tau \) is a solution of (2.10), we get

\[
\frac{d}{d\tau} \mathcal{E}(u, \tau) = \int_{\partial B_1 \cap \mathbb{R}^N_+} \frac{\partial u^\tau}{\partial r} \frac{d u^\tau}{d\tau} + l\tau \int_{B_1} (u^\tau)^2 + l\tau^2 \int_{\Sigma_1 \cap B_1} |u^\tau|^{q-1} u^\tau \frac{d u^\tau}{d\tau}. \tag{2.8}
\]

In what follows, we express all derivatives of \( u^\tau \) in the \( r = |x| \) variable in terms of derivatives in the \( \tau \) variable. In the definition of \( u^\tau \), directly differentiating in \( \tau \) gives

\[
\frac{d u^\tau}{d\tau} = \frac{2}{p-1} u^\tau + \tau \frac{\partial u^\tau}{\partial r}. \tag{2.10}
\]

From (2.9) and (2.10), we obtain

\[
\frac{d}{d\tau} \mathcal{E}(u, \tau) = \tau \int_{\partial B_1 \cap \mathbb{R}^N_+} \left( \frac{du^\tau}{d\tau} \right)^2 - \frac{1}{p-1} \int_{\partial B_1 \cap \mathbb{R}^N_+} \frac{d(u^\tau)^2}{d\tau} + \lambda \tau \int_{B_1} (u^\tau)^2 + \tau^{1-2 \frac{q-1}{p+1}} \int_{\Sigma_1 \cap B_1} |u^\tau|^{q-1} u^\tau \frac{d u^\tau}{d\tau}. \tag{2.11}
\]

On the other hand we have

\[
\tau^{1-2 \frac{q-1}{p+1}} \int_{\Sigma_1 \cap B_1} |u^\tau|^{q-1} u^\tau \frac{d u^\tau}{d\tau} = \frac{d}{d\tau} \left[ \tau^{1-2 \frac{q-1}{p+1}} \int_{\Sigma_1 \cap B_1} |u^\tau|^{q+1} \right] - \frac{(p+2q-2\tau^{-2 \frac{q-1}{p+1}})}{(p-1)(q+1)} \int_{\Sigma_1 \cap B_1} |u^\tau|^{q+1}. \tag{2.12}
\]

Combining (2.7), (2.11) and (2.12), we get (1.8) and (1.9).

Concerning problem (1.3) the proof can be obtained with only minor modifications. Since \( u \) is a solution of (1.3), then \( u^\tau \) satisfies

\[
\begin{cases}
-\Delta u^\tau + l\tau^2 u^\tau = |u^\tau|^{p-1} u^\tau, & \text{in } \mathbb{R}^N_+, \\
\frac{\partial u^\tau}{\partial r} = \tau^{1-2 \frac{q-1}{p+1}} |u^\tau|^{q-1} u^\tau, & \text{on } \Sigma_1, \\
u^\tau = 0, & \text{on } \Sigma_2.
\end{cases}
\tag{2.13}
\]

Integrating by parts in (2.8) and using the fact that \( u^\tau \) is a solution of (2.13), we get

\[
\frac{d}{d\tau} E(u, \tau) = \int_{\partial B_1 \cap \mathbb{R}^N_+} \frac{\partial u^\tau}{\partial r} \frac{d u^\tau}{d\tau} + l\tau \int_{B_1} (u^\tau)^2 + \int_{\Sigma_1 \cup \Sigma_2 \cap B_1} \frac{\partial u^\tau}{\partial \nu} \frac{d u^\tau}{d\tau} \tag{2.14}
\]

Indeed, we have \( u^\tau \equiv 0 \) in \( \Sigma_2 \cap B_1 \) for all \( \tau > 0 \), then \( \frac{d u^\tau}{d\tau} = 0 \) in \( \Sigma_2 \cap B_1 \). The rest of the proof is unchanged. Now, since \( \lambda \) is a positive scalar and \( 2q - p - 1 \geq 0 \) we have that \( E \) is a nondecreasing function of \( \tau \). This completes the proof of Proposition 1.2.

\( \square \)
3. Proof of Theorem 1.1

Let \( u \in C^2(\mathbb{R}^N) \) be a stable solution of (1.2) or (1.3). If \( N = 2 \) from (1.6) we have

\[
\int_{B^+_R} |u|^{p+1} \leq CR^{-\frac{p+1}{2}},
\]

which yields that \( u \equiv 0 \) for all \( p > 1 \).

If \( N \geq 3 \), three cases may occur.

Case 1. If \( 1 < p < \frac{N+2}{N-2} \). From the main integral estimate (1.6) of Proposition 1.1 (with \( C_0 = 0 \)), we have

\[
\int_{B^+_R} |u|^{p+1} \leq CR^{N-2\frac{p+1}{N-2}}, \quad \forall R > 0.
\]

As \( N - \frac{2(p+1)}{p-1} < 0 \) if \( 1 < p < \frac{N+2}{N-2} \), we may readily see that \( u \equiv 0 \).

Case 2. If \( p = \frac{N+2}{2} \). Apply again (1.6), we derive

\[
\int_{B^+_R} |u|^{\frac{2N}{N-2}} < \infty \quad \text{and} \quad \int_{A_R \cap \mathbb{R}^N_+} |u|^{\frac{2N}{N-2}} \to 0 \quad \text{as} \quad R \to \infty,
\]

where \( A_R = \{ R < |x| < 2R \} \). Invoking now (1.5) of Proposition 1.1 (with \( C_0 = 0 \)) and applying Hölder’s inequality, we derive that

\[
\int_{B^+_R} |u|^{\frac{2N}{N-2}} \leq CR^{-2} \int_{A_R \cap \mathbb{R}^N_+} |u|^2 \leq C \left( \int_{A_R \cap \mathbb{R}^N_+} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}, \quad \forall R > 0.
\]

Hence, we deduce that \( \int_{\mathbb{R}^N_+} |u|^{\frac{2N}{N-2}} = 0 \) and so \( u \equiv 0 \).

Case 3. If \( p > \frac{N+2}{2} \). This case needs more involving analysis using the powerful monotonicity formula. The fact that \( \lambda > 0 \) allows-us to provide nonexistence result for all supercritical exponent \( p \).

Substitute \( R \) by \( \tau R \) in (1.6) (with \( C_0 = 0 \)), then \( \forall R > 0 \) and \( \tau > 1 \), one has

\[
\int_{B^+_{\tau R}} \left( |\nabla u|^2 + |u|^p+1 + l|u|^2 \right) + \int_{\Sigma_1 \cap B^+_{\tau R}} |u|^{q+1} \leq C\tau^{N-2\frac{p+1}{p-1}} R^{N-2\frac{p+1}{p-1}}. \tag{3.1}
\]

Scaling back\( \tau \) we obtain

\[
\int_{B^+_{R}} \left( |\nabla \tau^r|^2 + |\tau^r|^p+1 + l\tau^2|\tau^r|^2 \right) + \tau^{1-2\frac{p-1}{p-1}} \int_{\Sigma_1 \cap B^+_{R}} |\tau^r|^{q+1} \leq C\tau^{N-2\frac{p+1}{p-1}}, \quad \forall \tau > 0, \forall \tau > 1. \tag{3.2}
\]

So, \( (u^\tau)_{\tau \geq 1} \) is uniformly bounded in \( H^1 \cap L^{p+1}(B^+_R) \) and \( (\tau^{\frac{p+1}{p-1}+\frac{q}{p-1}} u^\tau)_{\tau \geq 1} \) is uniformly bounded in \( L^{q+1}(\Sigma_1 \cap B^+_R) \) for any \( R > 0 \). Consequently, from a standard diagonal subsequence argument, we can find a sequence \( (u^\tau) \) which converges strongly in \( L^2(B^+_R) \) and weakly in \( H^1 \cap L^{p+1}(B^+_R) \) to some function \( u^\infty \) for every \( R > 0 \), as \( \tau_j \to +\infty \). Multiply the following equation

\[ u^\tau = \lambda^{-1} \tau^{-2}(\Delta u^\tau + |u^\tau|^{p-1} u^\tau) \quad \text{in} \quad \mathbb{R}^N_+, \]

by \( \phi \in C^1_c(B^+_R) \), then integrate by parts and using Hölder’s inequality, we derive from (3.2) that

\[
\left| \int_{B^+_R} u^\tau \phi \right| \leq \frac{C_{R, \phi}}{\tau^2}, \quad \text{where} \quad C_{R, \phi} > 0.
\]

\(^1\)In fact, multiply (3.1) by \( \tau^{\frac{2p+1}{p-1}} \) and use the change of variable \( x = \tau y \).
Consequently, \((u^\tau)\) converges weakly to 0 in \(L^2(B^+_R)\), so \(u^\infty = 0\) and then \((u^\tau)\) converges strongly to 0 in \(L^2(B^+_R)\). Invoking now inequality (1.5) of Proposition 1.1 where we substitute \(R\) by \(\tau R\) we obtain

\[
\int_{B^+_R} (|\nabla u|^2 + |u|^{p+1} + lu^2) + \int_{\Sigma_1 \cap B^+_R} |u|^{q+1} \leq CR^{-2} \tau - 2 \int_{B^+_R} u^2.
\]

Thus, scaling back, we deduce

\[
\int_{B^+_R} (|\nabla u|^2 + |u|^p + l\tau^2|u|^2) + \tau^{1-2\frac{p}{q-1}} \int_{\Sigma_1 \cap B^+_R} |u|^q \leq CR^{-2} \int_{B^+_R} (u^\tau)^2.
\]

As a consequence, \((u^\tau)\) converges strongly to 0 in \(H^1 \cap L^{p+1}(B^+_R)\) and \((\tau u^\tau)\) converges strongly to 0 in \(L^2(B^+_R)\) as \(\tau \to \infty\). Moreover, we have

\[
\lim_{\tau \to +\infty} E_2(u, \tau) = 0,
\]

where

\[
E_2(u, \tau) = \int_{B^+_R} \left( \frac{1}{2} |\nabla u|^2 + \lambda \tau^2 |u|^2 - \frac{1}{p+1} |u|^p \right) - \frac{\tau^{1-2\frac{p}{q-1}}}{q+1} \int_{\Sigma_1 \cap B^+_R} |u|^q.
\]

We claim that the same holds true for \(E\). To see this, simply observe that since \(E\) is nondecreasing,

\[
E(u, \tau) \leq \int_{\tau}^{2\tau} E(u, t) dt = \int_{\tau}^{2\tau} E_2(u, t) dt + \frac{\tau}{p-1} \int_{\tau}^{2\tau} \int_{\partial B^+_R} |u|^2
\]

\[
\leq \sup_{t \geq \tau} E_2(u, t) + C \int_{B^+_R} |u|^2.
\]

Thanks to this, we deduce from (3.4) that

\[
\lim_{\tau \to +\infty} E(u, \tau) = 0.
\]

In addition, since \(u \in C^2(\mathbb{R}^N_+)\), one easily verifies that \(E(u, 0) = 0\). As a consequence \(E(u, \tau) \equiv 0\), and therefore \(\frac{dE}{d\tau} = 0\). Then from (1.9)

\[
\int_{B^+_R} |u|^2 dx = 0, \forall \tau > 1.
\]

We readily deduce that \(u \equiv 0\). The proof is completed. \(\square\)

4. Proof of Theorems 1.2-1.3

In this section, we will prove Theorems 1.2-1.3. For this, we need the following well known Pohozaev identity.

Lemma 4.1. Let \(u \in C^2(\mathbb{R}^N_+)\) be a solution of (1.2) or (1.3). Then the following identity holds:

\[
\frac{N-2}{2} \int_{B^+_R} |\nabla u|^2 + \frac{N}{2} \int_{B^+_R} u^2 - \frac{N}{p+1} \int_{B^+_R} |u|^{p+1} - \frac{N-1}{q+1} \int_{B^+_R \cap \Sigma_1} |u|^{q+1}\] 

\[
= \frac{R}{2} \int_{\partial B^+_R} |\nabla u|^2 ds - R \int_{\partial B^+_R} \frac{\partial u}{\partial \nu}^2 ds + \frac{R^2}{2} \int_{\partial B^+_R} u^2 ds - \frac{R}{p+1} \int_{\partial B^+_R} |u|^{p+1} ds,
\]

\[
- \frac{R}{q+1} \int_{\partial B^+_R \cap \Sigma_1} |u|^{q+1} ds.
\] (4.1)
Proof. The proof of this lemma is standard, we give it here for completeness. We deal only with problem (1.2). The proof for problem (1.3) is almost the same except that different boundary value conditions were used. We omit the details.

Multiplying the equation (1.2) by \(\langle x, \nabla u \rangle\) and integrating on \(B_R^+\), then a direct computation shows that

\[
\int_{B_R^+} |u|^{p-1} u(x, \nabla u) dx = \frac{1}{p+1} \int_{B_R^+} \langle x, \nabla |u|^{p+1} \rangle dx
\]

\[
= -\frac{N}{p+1} \int_{B_R^+} |u|^{p+1} dx + \frac{1}{p+1} \int_{\partial B_R^+} \langle x, \nu \rangle |u|^{p+1} ds
\]

(4.2)

\[
= -\frac{N}{p+1} \int_{B_R^+} |u|^{p+1} dx + \frac{R}{p+1} \int_{\partial B_R^+} |u|^{p+1} ds.
\]

Similarly, we have

\[
\int_{B_R^+} u \langle x, \nabla u \rangle dx = \frac{N}{2} \int_{B_R^+} u^2 dx + \frac{R}{2} \int_{\partial B_R^+} u^2 ds.
\]

(4.3)

Next, we deduce

\[
\int_{B_R^+} -\Delta u \langle x, \nabla u \rangle dx = \int_{B_R^+} \nabla u \nabla \langle \langle x, \nabla u \rangle \rangle dx - \int_{\partial B_R^+} \frac{\partial u}{\partial \nu} \langle x, \nabla u \rangle ds - \int_{B_R \cap \partial B_R^+} \frac{\partial u}{\partial \nu} \langle x, \nabla u \rangle ds'
\]

\[
= \int_{B_R^+} |\nabla u|^2 dx + \frac{1}{2} \int_{\partial B_R^+} \langle x, \nabla (|\nabla u|^2) \rangle ds - R \int_{\partial B_R^+} \frac{\partial u}{\partial \nu} \langle x, \nabla u \rangle ds
\]

\[
- \frac{1}{q+1} \int_{B_R \cap \Sigma_1} \langle x', \nabla x' (|u|^{q+1}) \rangle ds'
\]

(4.4)

\[
= -\frac{N-2}{2} \int_{B_R^+} |\nabla u|^2 dx + \frac{R}{q+1} \int_{\partial B_R^+} |\nabla u|^2 ds - R \int_{\partial B_R^+} \frac{\partial u}{\partial \nu} |\nabla u|^2 |\nabla u| ds
\]

\[
+ \frac{N-1}{q+1} \int_{B_R \cap \Sigma_1} |u|^{q+1} ds'
\]

Then (4.1) follows immediately from (1.2), (4.3) and (4.4). \(\square\)

4.1. Proof of Theorem 1.2

We only prove the conclusion for problem (1.2), the proof for problem (1.3) is the same. Working by contradiction. Suppose that \(u \neq 0\) then, by virtue of the main integral estimate (1.6) of Proposition 1.1 (with \(C_0 > 0\)) we have

\[
\int_{\Sigma \cap B_R^+} |u|^{q+1}, \int_{\mathbb{R}^N_+} |u|^{p+1}, \int_{\mathbb{R}^N_+} u^2 \text{ and } \int_{\mathbb{R}^N_+} |\nabla u|^2 < \infty; \text{ for any } p \in (1, \frac{N+2}{N-2}).
\]

Consequently, we can find a sequence \(R_n \to \infty\) such that

\[
\frac{R_n}{2} \int_{\partial B_{R_n}^+} |\nabla u|^2 ds - \frac{R_n}{2} \int_{\partial B_{R_n}^+} \frac{\partial u}{\partial \nu} |\nabla u|^2 ds + \frac{R_n}{2} \int_{\partial B_{R_n}^+} u^2 ds
\]

\[
- \frac{R_n}{p+1} \int_{\partial B_{R_n}^+} |u|^{p+1} ds - \frac{R_n}{q+1} \int_{\partial B_{R_n} \cap \Sigma_1} |u|^{q+1} ds,
\]

tends to 0 as \(n\) tends to \(+\infty\). So we deduce from the Pohozaev identity (1.1) that

\[
\frac{N-2}{2} \int_{\mathbb{R}^N_+} |\nabla u|^2 + \frac{N}{p+1} \int_{\mathbb{R}^N_+} u^{p+1} = \frac{N}{q+1} \int_{\Sigma_1} |u|^{q+1} dx'.
\]

(4.5)
On the other hand, we multiply (1.2) by \( u \) and integrate by parts, then we obtain
\[
\int_{\mathbb{R}^N_+} |\nabla u|^2 + l \int_{\mathbb{R}^N_+} u^2 = \int_{\mathbb{R}^N_+} |u|^p + \int_{\Sigma_1} |u|^{q+1} dx'.
\] (4.6)

Combining (4.5) and (4.6) we derive
\[
\left(\frac{N - 2}{2} - \frac{N - 1}{q + 1}\right) \int_{\mathbb{R}^N_+} |\nabla u|^2 + \left(\frac{N - 2}{2} - \frac{N - 1}{q + 1}\right) l \int_{\mathbb{R}^N_+} u^2 = \left(\frac{N}{p + 1} - \frac{N - 1}{q + 1}\right) \int_{\mathbb{R}^N_+} |u|^{p+1}.
\] (4.7)

Observe now since we assume that \( p < \frac{N+2}{N-2} \) and \( q > \frac{N}{N-2} \) then we have
\[
\frac{N - 2}{2} - \frac{N - 1}{q + 1} > 0, \quad \frac{N}{p + 1} - \frac{N - 1}{q + 1} > 0 \quad \text{and} \quad \frac{N}{p + 1} - \frac{N - 1}{q + 1} > 0.
\]

So we deduce
\[
\left(\frac{N - 2}{2} - \frac{N - 1}{q + 1}\right) l \int_{\mathbb{R}^N_+} u^2 \leq \left(\frac{N}{p + 1} - \frac{N - 1}{q + 1}\right) \int_{\mathbb{R}^N_+} |u|^{p+1}.
\]

On the other hand by hypothesis
\[
|u|^{p-1}_{L^{\infty}(\mathbb{R}^N_+)} < l\left(\frac{N - 2}{2} - \frac{N - 1}{q + 1}\right)\left(\frac{N}{p + 1} - \frac{N - 1}{q + 1}\right)^{-1},
\]

then the last inequality implies
\[
\left(\frac{N - 2}{2} - \frac{N - 1}{q + 1}\right) l \int_{\mathbb{R}^N_+} u^2 \leq \left(\frac{N}{p + 1} - \frac{N - 1}{q + 1}\right) |u|^{p-1}_{L^{\infty}(\mathbb{R}^N_+)} \int_{\mathbb{R}^N_+} u^2 \leq \left(\frac{N}{p + 1} - \frac{N - 1}{q + 1}\right) l \int_{\mathbb{R}^N_+} u^2.
\]

So, we reach a contradiction which completes the proof of Theorem 1.2. \( \square \)

4.2. Proof of Theorem 1.3

We divide the proof in two parts.

**Step 1.** If \( p = \frac{N+2}{N-2} \). In view of the Pohozaev identity and as in the proof of Theorem 1.2 we have
\[
\frac{N - 2}{2} \int_{\mathbb{R}^N_+} |\nabla u|^2 + \frac{N}{2} l \int_{\mathbb{R}^N_+} u^2 = \frac{N - 2}{2} \int_{\mathbb{R}^N_+} |u|^{\frac{2N}{q+2}} + \frac{N - 1}{q + 1} \int_{\Sigma_1} |u|^{q+1}.
\] (4.8)

On the other hand multiplying equation (1.2) or (1.3) by \( u \) and integrating by parts yields
\[
\int_{\mathbb{R}^N_+} |\nabla u|^2 + l \int_{\mathbb{R}^N_+} u^2 = \int_{\mathbb{R}^N_+} |u|^{\frac{2N}{q+2}} + \int_{\Sigma_1} |u|^{q+1}.
\] (4.9)

Combining (4.8) and (4.9) gives
\[
\left(\frac{N - 2}{2} - \frac{N - 2}{2}\right) l \int_{\mathbb{R}^N_+} u^2 = \left(\frac{N}{q + 1} - \frac{N - 2}{2}\right) \int_{\Sigma_1} |u|^{q+1}.
\] (4.10)

By assumption \( 2q - p - 1 \geq 0 \), then \( \frac{N - 1}{q+1} - \frac{N - 2}{2} \leq 0 \). Hence we derive
\[
l \int_{\mathbb{R}^N_+} u^2 = 0.
\]

Since \( l > 0 \) we must have \( u \equiv 0 \).
Step 2. If $p > \frac{N+2}{N-2}$, substitute $R$ by $\tau R$ in (1.6) (with $C_0 > 0$) and scaling back, we obtain
\[
\int_{B^+_R} \left( |\nabla u^\tau|^2 + |u^\tau|^{p+1} + \tau l^2 |u^\tau|^2 \right) + \int_{\Sigma_1 \cap B^+_R} |u^\tau|^{q+1} \leq C_0 \tau^{2 \frac{p+1}{p-1} - N} + CR^{N-2 \frac{p+1}{p-1} - N}.
\]
As $2 \frac{p+1}{p-1} - N < 0$, then $(u^\tau)_{\tau \geq 1}$ is uniformly bounded in $H^1 \cap L^{p+1}(B^+_R)$ for any $R > 0$ and $(\tau u^\tau)_{\tau \geq 1}$ is uniformly bounded in $L^{q+1}(\Sigma_1 \cap B^+_R)$. So, we can find a sequence $(u^{\tau_j})$ which converges strongly in $L^2(B^+_R)$ and weakly in $H^1 \cap L^{p+1}(B^+_R)$ to some function $u^\infty$ for every $R > 0$, as $\tau_j \to +\infty$. Since $u^{\tau_j}$ satisfies the following PDE
\[
u = \lambda^{-1} \tau^{-2} \left( \Delta u^\tau + |u^\tau|^{p-1} u^\tau \right) \quad \text{in } \mathbb{R}^N_+,
\]
then taking limits in the sense of distributions, we get $u^\infty = 0$, and therefore $(u^{\tau_j})$ converges strongly to 0 in $L^2(B^+_R)$. Invoking now inequality (1.5) of Proposition (1.4) where we substitute $R$ by $\tau R$ we obtain
\[
\int_{B^+_R} \left( |\nabla u|^2 + |u|^{p+1} + l^2 |u|^2 \right) + \int_{\Sigma_1 \cap B^+_R} |u|^{q+1} \leq C_0 + CR^{-2} \tau^{-2} \int_{B^+_R} u^2, \forall R > R_0 \text{ and } \tau > 1.
\]
Thus, scaling back, we deduce
\[
\int_{B^+_R} \left( |\nabla u^\tau|^2 + |u^\tau|^{p+1} + \tau l^2 |u^\tau|^2 \right) + \tau^{1-2 \frac{p+1}{p-1} - N} \int_{\Sigma_1 \cap B^+_R} |u^\tau|^{q+1} \leq C_0 \tau^{2 \frac{p+1}{p-1} - N} + CR^{-2} \int_{B^+_R} (u^{\tau_j})^2.
\]
As a consequence, $(u^\tau)$ converges strongly to 0 in $H^1 \cap L^{p+1}(B^+_R)$ and $(\tau u^\tau)$ converges strongly to 0 in $L^2(B^+_R)$ as $\tau \to \infty$. Then from (4.11), we have
\[
\lim_{\tau \to +\infty} E_2(u, \tau) = 0,
\]
where
\[
E_2(u, \tau) = \int_{B^+_R} \left( \frac{1}{2} |\nabla u|^2 + \frac{\lambda \tau^2}{2} |u|^2 - \frac{1}{p+1} |u|^p \right) - \frac{\tau^{1-2 \frac{p+1}{p-1}}}{q+1} \int_{\Sigma_1 \cap B^+_R} |u|^q.
\]
At this stage the rest of the proof is similar to the Case 3 of Theorem (1.4). Then we omit it. □

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References

References

[1] A. Bahri and P. L. Lions, Solutions of superlinear elliptic equations and their Morse indices, Comm. Pure. App. Math. 45 (1992), 1205-1215.

[2] M. Ben Ayed, H. Fourti and A. Selmi, Harmonic functions with nonlinear Neumann boundary condition and their Morse indices, Nonlinear Anal. Real World Appl., 38 (2017), 96-112.

[3] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, Superlinear indefinite elliptic problems and nonlinear Liouville theorems, Topol. Methods Nonlinear Anal. 4(1) 1994, 59-78.

[4] E. N. Dancer, Finite Morse index solutions of supercritical problems, J.reine angew. Math. 620 (2008), 213-233.
[5] D.G. de Figueiredo and J. Yang, *On a semilinear elliptic problem without (PS) condition*, J. Differential Equations **187** (2003), 412-428.

[6] J. Dávila, L. Dupaigne, K. Wang and J. Wei, *A monotonicity formula and a Liouville-type theorem for a fourth order supercritical problem*, Adv. Math., **258** (2014), 240-285.

[7] L. Dupaigne and A. Harrabi, *The Lane-Emden Equation in Strips*, Proc Roy Soc Edin Sec A. **148**(1), doi:10.1017/S0308210517000142.

[8] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of \(\mathbb{R}^N\)*, J. Math.Pures Appl. **87** 2007, 537-561.

[9] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations. **6** (1981), 883-901.

[10] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525-598.

[11] A. Harrabi, M. Ahmadou, S. Rebhi, A. Selmi, *A priori estimates for superlinear and subcritical elliptic equations: the Neumann boundary condition case*, Manuscripta Math. **137** (2012) 525-544.

[12] A. Harrabi and B. Rahal, *Liouville-type theorems for elliptic equations in the half-space with mixed boundary value conditions*, J. Advances in nonlinear analysis. https://doi.org/10.1515/anona-2016-0168. (2016).

[13] A. Harrabi and B. Rahal, *On the sixth-order Joseph-Lundgren exponent*, Ann. IHP. **18** (2017), 1055-1094.

[14] A. Harrabi, S. Rebhi and A. Selmi, *Solutions of superlinear equations and their Morse indices, I*, Duke. Math. J. **94** (1998), 141-157.

[15] A. Harrabi, S. Rebhi and A. Selmi, *Solutions of superlinear equations and their Morse indices, II*, Duke. Math. J. **94** (1998), 159-179.

[16] H. Hajlaoui, A. Harrabi and F. Mtiri: *Morse indices of solutions for super-linear elliptic PDEs*, Nonlinear Analysis **116**, 180-192 (2015).

[17] A. Harrabi, F. Mtiri and D. Ye, *Explicit \(L^\infty\) -norm estimates via Morse index, the bi-harmonic and tri-harmonic semilinear problems*, Manuscripta Math Math DOI: 10.1007/s00229-018-1037-9 (2018).

[18] A. Selmi, A. Harrabi and C. Zaidi, *Nonexistence results on the space or the half space of \(-\Delta u+\lambda u=|u|^{p-1}u\) via the Morse index*, Comm. Pure Appl. Anal. doi:10.3934/cpaa.2020124.

[19] A. Selmi, A. Harrabi and C. Zaidi, *Nonexistence results in strips*, Accepted in Applicandae mathematicae.

[20] F. Pacard, *Partial regularity for weak solutions of a nonlinear elliptic equation*, Manuscripta Math. **79** (1993), no. 2, 161-172.

[21] P. Poláčik, P. Quittner and P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems*, Duke Math. J. **139** (3) (2007), 555-579.

[22] M. Ramos and P. Rodrigues *On a fourth order superlinear elliptic problem*, Electron. J. Diff. Eqns, Conf. **06** (2001), 243-255.

[23] Ramos, M, Terracini, S, Troestler, C, *Superlinear indefinite elliptic problems and Phozaev type identities*, J. Funct. Anal. **159** (1998), 596-628.

[24] X. Wang, X. Zheng *Liouville theorem for elliptic equations with mixed boundary valu conditions and finite Morse indices*, J. Inequal. Appl (2015), 860-871, DOI 10.1186/s13660-015-0867-1.

[25] X. Yu, *Solution of mixed boundary problems and their Morse indices*, Nonlinear Anal. **96** (2014), 146-153

[26] X. Yu, *Liouville theorem for elliptic equations with nonlinear boundary value conditions and finite Morse indices*, J. Math. Anal. Appl **421** (2015), 436-443.