CANCELLATION CONJECTURE FOR FREE ASSOCIATIVE ALGEBRAS

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Abstract. We develop a new method to deal with the Cancellation Conjecture of Zariski in different environments. We prove the conjecture for free associative algebras of rank two. We also produce a new proof of the conjecture for polynomial algebras of rank two over fields of zero characteristic.

1. Introduction and main results

There is a famous

Conjecture 1.1. (Cancellation Conjecture of Zariski) Let $R$ be an algebra over a field $K$. If $R[z]$ is $K$-isomorphic to $K[x_1, \ldots, x_n]$, then $R$ is isomorphic to $K[x_1, \ldots, x_{n-1}]$.

Conjecture 1.1 is proved for $n = 2$ by Abhyankar, Eakin and Heizer [1], and Miyanishi [10]. For $n = 3$, the Conjecture is proved by Fujita [5], and Miyanishi and Sugie [11] for zero characteristic, and by Russell [12] for arbitrary fields $K$. For $n \geq 4$, the Conjecture remains open to the best of our knowledge. See [4, 6, 7, 8, 9, 14] for Zariski’s conjecture and related topics.

In view of Conjecture 1.1, it is natural and interesting to raise

Conjecture 1.2. (Cancellation Conjecture for Free Associative Algebras) Let $R$ be an algebra over a field $K$. If the free product $R \ast K[z]$ is $K$-isomorphic to $K\langle x_1, \ldots, x_n \rangle$, then $R$ is $K$-isomorphic to $K\langle x_1, \ldots, x_{n-1} \rangle$.

In this paper we develop a new method based on conditions of algebraic dependence, which can be used in different environments. In particular, by this method we prove Conjecture 1.2 for $n = 2$.

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**Theorem 1.3.** Let $R$ be an algebra over an arbitrary field $K$. If $R \ast K[z]$ is $K$-isomorphic to $K \langle x, y \rangle$, then $R$ is $K$-isomorphic to $K[x]$.

We also produce a new and simple proof for Conjecture 1.1 for $n = 2$ in the zero characteristic case [1, 10]:

**Proposition 1.4.** Let $R$ be an algebra over a field $K$ of zero characteristic. If $R[z]$ is $K$-isomorphic to $K[x, y]$, then $R$ is isomorphic to $K[x]$.

2. Preliminaries

First let us recall the structure of the free product $R \ast K[z]$. If as a vector space the (not necessarily commutative) unitary algebra $R$ has a basis $\{r_i \mid i \in I\}$ and its multiplication is defined by

$$r_ir_j = \sum_{k \in I} \alpha_{ij}^k r_k,$$

then the basis of $R \ast K[z]$ consists of all products $r_{i_0}zr_{i_1}z\cdots r_{i_a-1}zr_{i_a}$ and the multiplication in $R \ast K[z]$ is defined by

$$(r_{i_0}z\cdots r_{i_{a-1}}zr_{i_a})(r_{j_0}zr_{j_1}z\cdots r_{j_b}) = \sum_{k \in I} \alpha_{i_0, j_0}^k r_{i_0}z\cdots r_{i_{a-1}}zr_kzr_{j_1}z\cdots r_{j_b}.$$

The free associative algebra of rank $n$ can be defined as

$$K\langle x_1, \ldots, x_n \rangle \cong K[x_1] \ast \cdots \ast K[x_n].$$

To prove the main results, we need the well-known necessary and sufficient conditions for algebraic dependence.

**Lemma 2.1.** Let $K$ be an arbitrary field, $f, g \in K\langle x_1, \ldots, x_n \rangle$. Then $f$ and $g$ are algebraically dependent over $K$ if and only if $[f, g] = 0$, where $[f, g] = fg - gf$ is the commutator of $f$ and $g$.

See Bergman [2] (or Cohn [3]), for a proof.

**Lemma 2.2.** Let $K$ be a field of zero characteristic, $f, g \in K[x_1, \ldots, x_n]$. Then $f$ and $g$ are algebraically dependent over $K$ if and only if $J_{x_i, x_j}(f, g) = 0$ for all $1 \leq i < j \leq n$, where $J_{x_i, x_j}(f, g)$ is the Jacobian determinant of $f$ and $g$ with respect to $x_i$ and $x_j$.

See, for instance, Jie-Tai Yu [15], for a proof.

We also need a description of the subset of all elements of a polynomial or free associative algebra which are algebraically dependent to a fixed element. The following result is due to Bergman [2], see also Cohn [3].

**Lemma 2.3.** Let $K$ be an arbitrary field, $f \in K\langle x_1, \ldots, x_n \rangle \backslash K$, and let $\mathcal{C}(f)$ be the subset of $K\langle x_1, \ldots, x_n \rangle$ consisting of all $g$ such that $[f, g] = 0$. Then $\mathcal{C}(f) = K[u]$ for some $u \in K\langle x_1, \ldots, x_n \rangle$. 

Lemma 2.4. Let $K$ be a field of zero characteristic, $f \in K[x_1, \ldots, x_n] \setminus K$, and let $C(f)$ be the subset of $K[x_1, \ldots, x_n]$ consisting of all $g$ such that $J_{x_i,x_j}(f,g) = 0$ for all $1 \leq i < j \leq n$. Then $C(f) = K[u]$ for some $u \in K[x_1, \ldots, x_n]$.

3. Proofs of the main results

Proof of Theorem 1.3. Let $R \ast K[z] \cong K\langle x, y \rangle$ and let $(z)$ be the ideal of $R \ast K[z]$ generated by $z$. Clearly, $(R \ast K[z])/(z) \cong R$. Since the algebra $R \ast K[z]$ is isomorphic to the free algebra of rank 2, it is two-generated and the same holds for its homomorphic image $(R + K[z])/(z) \cong R$. Hence $R$ is generated by $v, w \in R$. Now we use that $R$ is a subalgebra of the free associative algebra $R \ast K[z] \cong K\langle x, y \rangle$. If $v$ and $w$ are algebraically independent over $K$, then $R$ is isomorphic to the free algebra $K\langle t_1, t_2 \rangle$ and $R \ast K[z] \cong K\langle t_1, t_2, z \rangle$ is the free algebra of rank 3, which is impossible. Hence $v$ and $w$ are algebraically dependent. It follows that any element $f \in R$ and $v$ are algebraically dependent over $K$. By Lemmas 2.1 and 2.3, $R \subset K[u]$ for some $u \in R \ast K[z]$. Write $u = u_0 + u_1$, where $u_0 \in R$ and $u_1$ contains all monomials of $u$ with $z$-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, $h$ is a polynomial over $K$ in one variable. Substituting $z = 0$, we obtain $f = h(u_0)$. Therefore $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. It forces $R = K[u_0]$. Hence $R$ is $K$-isomorphic to $K[x]$. □

Proof of Proposition 1.4. As $R[z]$ is $K$-isomorphic to $K[x, y]$, it is easy to know that $R$ has transcendental degree 1 over $K$. Therefore there exists a $g \in R \setminus K$ such that for all $f \in R$, $f$ and $g$ are algebraically dependent over $K$. By Lemmas 2.2 and 2.3, $R \subset K[u]$ for some $u \in R[z]$. Write $u = u_0 + u_1$, where $u_0 \in R$ and $u_1$ contains all monomials of $u$ with $z$-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, $h$ is a polynomial over $K$ in one variable. Substituting $z = 0$, we obtain $f = h(u_0)$. Therefore $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. It forces $R = K[u_0]$. Hence $R$ is $K$-isomorphic to $K[x]$. □

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