A technical critique of the free energy principle as presented in “Life as we know it” and related works

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Abstract

We summarize the argument in \textcite{Friston:2013} and highlight some technical errors. We also discuss how these errors affect the very similar \textcite{Friston:2014} and, where appropriate, mention consequences for the newer proposals in \textcite{Friston:2019, Parr:2019}. The errors call into question the purported interpretation that the internal coordinates of every system with a Markov blanket will appear to engage in Bayesian inference. In particular, in addition to highlighting the implicit restriction to linear models, we identify three formal errors in the main argument of \textcite{Friston:2013}: The first concerns the rewriting of the equations of motion of systems with Markov blankets which turns out not to be generally correct. We prove the non-equivalence with a counterexample that exhibits a Markov blanket but does not satisfy the rewritten equations. Our counterexample also invalidates the corresponding (but more general) rewritten equations in the more recent \textcite{Friston:2019}. The second error concerns the Free Energy Lemma itself, which we prove, by counterexample, to be wrong in general. The third is the claim that the Free Energy Lemma, when it does hold, implies equality of variational density and ergodic conditional density. The interpretation in terms of Bayesian inference hinges on this point, and we hence conclude that it is unjustified. Additionally, we highlight that the definitions of the Markov blanket in \textcite{Friston:2013, Parr:2019} are not equivalent and that the assumptions in \textcite{Parr:2019} may be too strong to allow for meaningful interpretation.

Overview

In \textcite{Friston:2013} it is argued that the internal coordinates of an ergodic random dynamical system with a Markov blanket necessarily appear to engage in active

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Figure 1: Argument visualization. Numbers labelling edges indicate corresponding steps in this paper. Struck out edges indicate implications that we prove incorrect. The main argument in Friston (2013) takes the left path. The box in the top right indicates the relations between Conditions 1 to 3 and their role in Parr et al. (2019). Merged edges indicate a logical AND combination of the parent nodes.
Bayesian inference. Here, we reproduce the argument supporting this interpretation in detail and highlight at which points it fails on technical grounds. In the course of our critique, we rule out some closely related alternative arguments. In cases where our results have clear consequences for the more recent related publications Friston (2019); Parr et al. (2019) we also mention those. However, our analysis is confined to only a few of the significant differences contained in the latter publications. In an additional section we discuss the effect of our argument on Friston et al. (2014). The logical structure of the present paper is depicted in Fig. 1. We note that the technical issues presented here do not affect the validity of approaches where an (expected) free energy minimizing agent is assumed a priori, as presented in, e.g. Friston et al. (2015). None of Friston (2013); Friston et al. (2014); Friston (2019); Parr et al. (2019) make this assumption, they instead aim to identify the conditions under which such agents will emerge within a given stochastic process. We now briefly introduce the setting of Friston (2013) and then sketch the content of this paper.

The starting point is a random dynamical system whose evolution is governed by the Langevin equation

$$\dot{x} = f(x) + \omega,$$

where the system state $x$ and vector field $f(x)$ are multi-dimensional, and $\omega$ is a Gaussian noise term. There is an additional assumption that the system is ergodic, such that the steady state probability density $p^*(x)$ is well defined.\footnote{In the original paper, the ergodic density is simply denoted $p(x)$. We here add a star to highlight that it is a time independent probability density.}

It is then assumed that there is a coordinate system $x = (\psi, s, a, \lambda)$ with $\psi = (\psi_1, ..., \psi_n)$, $s = (s_1, ..., s_n)$, $a = (a_1, ..., a_n)$, and $\lambda = (\lambda_1, ..., \lambda_n)$, referred to as external, sensory, active, and internal coordinates\footnote{These are called “states” in Friston (2013).} respectively, such that the following condition holds:

**Condition 1.** The function $f(x)$ can be written as

$$f(x) = \begin{pmatrix} f_\psi(\psi, s, a) \\ f_s(\psi, s, a) \\ f_a(s, a, \lambda) \\ f_\lambda(s, a, \lambda) \end{pmatrix}.$$

This particular structure is described as “[formalizing] the dependencies implied by the Markov blanket” (Friston, 2013). In contrast, more recent works Friston (2019); Parr et al. (2019) formulate the Markov blanket in terms of statistical dependencies of the ergodic density $p^*(x) = p^*(\psi, s, a, \lambda)$. Specifically, the following condition is presented:

**Condition 2.** The ergodic density factorises as

$$p^*(\psi, s, a, \lambda) = p^*(\psi|s, a)p^*(\lambda|s, a)p^*(s, a).$$

In other words, the internal and external coordinates are independently distributed when conditioned on the sensory and active coordinates. This means we have two different formal expressions of what constitutes a Markov blanket in these publications, and their relationship has not previously been established.

Taking Condition 1 to hold, the argument of Friston (2013) then proceeds along the following steps:
Step 1 Rewrite the vector field \( f(\psi, s, a, \lambda) \) describing the dynamics of the system in terms of the gradient of negative logarithm of the ergodic density \( p^*(\psi, s, a, \lambda) \) of that system.

Step 2 Rewrite the components \( f_\lambda(s, a, \lambda) \) and \( f_a(s, a, \lambda) \) of the vector field \( f(\psi, s, a, \lambda) \) in terms of only partial gradients of the negative logarithm of \( p^*(\psi, s, a, \lambda) \).

Step 3 Assert (in the Free Energy Lemma) the existence of a density \( q(\psi|\lambda) \) over the external coordinates \( \psi \) parameterized by the internal coordinates \( \lambda \), and that \( f(\psi, s, a, \lambda) \) can again be rewritten, this time in terms of a free energy depending on \( q(\Psi|\lambda) \).

Step 4 Claim that equivalence of the equations of motion in Step 2 and Step 3 implies that certain partial gradients of the KL divergence between \( q(\Psi|\lambda) \) and the conditional ergodic density \( p^*(\Psi|s, a, \lambda) \) must vanish.

Step 5 Claim that it follows from Step 4 that \( q(\Psi|\lambda) \) and \( p^*(\Psi|s, a, \lambda) \) are “rendered” equal.

Step 6 Interpret

- \( p^*(\Psi|s, a, \lambda) \) as a posterior over external coordinates given particular values of sensor, active, and internal coordinates,
- \( q(\Psi|\lambda) \) as encoding Bayesian beliefs about the external coordinates by the internal coordinates, and
- their equality as the internal coordinates appearing to “solve the problem of Bayesian inference”.

In the present paper, we make the following main observations

- The re-expression of Eq. (1) in the form chosen in Step 1 is derived under the assumption that the system is linear and subject to Gaussian and Markov noise.

- Condition 1 and Condition 2 are independent from each other.

- Condition 1 and Condition 3 together lead to a system where the interpretation of \( s \) and \( a \) as sensory and active coordinates is questionable.

- Under both Conditions 1 and 2, the expressions of \( f_\lambda(s, a, \lambda) \) and \( f_a(s, a, \lambda) \) resulting from Step 2 are not as general as those contained in the result of Step 1. The more general alternative expression derived in Friston (2019) remains insufficiently general.

- Under both Conditions 1 and 2, the Free Energy Lemma is wrong and cannot be salvaged by using alternatives in Step 2.

- Under both Conditions 1 and 2, contrary to Step 5 the vanishing of the gradient of the KL divergence does not imply equality of \( q(\Psi|\lambda) \) and \( p^*(\Psi|s, a, \lambda) \).

\(^3\)Here, and whenever it would otherwise be ambiguous, we use a capitalized \( \Psi \) to indicate full distributions, rather than the probability density for specific value of \( \psi \).
• As a consequence, the basic preconditions for the interpretations in Step 6 are not implied by either of the two proposed Markov blanket Conditions 1 and 2.

The later Friston et al. (2014) presents an argument almost identical to the one in the original Friston (2013). In Section 7 we discuss how our observations apply to this publication.

1 Expression via the gradient of the ergodic density

Here we introduce the expression of the system’s dynamics Eq. (1) in the form used for the Free Energy Lemma (Lemma 2.1. in Friston, 2013). This form expresses the dynamics of internal and active coordinates of the given ergodic random dynamical system in terms of the gradient of the ergodic density $p^*(x)$.

In accordance with the results of Kwon et al. (2005), $f(x)$ is rewritten as (see Eq.(2.5) in Friston, 2013):

$$f(x) = (\Gamma + R) \cdot \nabla \ln p^*(x),$$

where $\Gamma$ is the diffusion matrix, which we will take to be block diagonal, and $R$ is an antisymmetric matrix, defined through the relation

$$MR + RM^T = M\Gamma - \Gamma M^T,$$

with

$$M_{ij} = \nabla_j f_i(x).$$

Both $\Gamma$ and $R$ are assumed constant. We emphasise here that Eq. (4) is derived in the literature under the explicit assumption that the fluctuations $\omega$ be Gaussian and Markov, and that $f(x)$ is a linear function (Ao, 2004; Kwon et al., 2005). When $f(x)$ is nonlinear, Eq. (4) must be modified (Kwon and Ao, 2011). In other words, by requiring Eq. (4) to hold generically, one is restricted to the class of Ornstein-Uhlenbeck processes, and the ergodic density $p^*(x) = p^*(\psi, s, a, \lambda)$ is necessarily a multivariate Gaussian with zero mean.

Specifically, following Kwon et al. (2005),

$$p^*(\psi, s, a, \lambda) := \frac{1}{Z} \exp \left[ -\frac{1}{2}(\psi, s, a, \lambda)U(\psi, s, a, \lambda)^\top \right],$$

where $(\psi, s, a, \lambda)$ is a row vector and $Z$ is a suitable normalisation constant.

From Eq. (4) it can be seen that,

$$U = -(\Gamma + R)^{-1}M.$$

This concludes Step 1.

4In Friston (2013), and later work such as Friston (2019), $\Gamma$ is taken to be proportional to the identity matrix.

5There may be more general processes for which Eq. (50) of Kwon and Ao (2011) is zero (and hence Eq. (4) is valid), but these will necessarily be finely tuned, and Eq. (4) will not be robust under small perturbations to the dynamics.
Before moving on to Step 2, we note that, under the assumptions implicit in Step 1, we can express Condition 1 and Condition 2 in terms of the matrices $M$ and $U$. Firstly, since it effectively states that

$$\nabla \psi f_a(x) = \nabla \psi f_\lambda(x) = \nabla \lambda f_s(x) = \nabla \lambda f_\psi(x) = 0,$$

Condition 1 $\iff$ $M_{a\psi} = M_{\lambda \psi} = M_{s\lambda} = M_{\psi \lambda} = 0$, \hspace{1cm} (9)

with $M_{\alpha\beta}$ a block sub-matrix of $M$ in general. Secondly, because of the multivariate Gaussian nature of $p^*(\psi, s, a, \lambda)$, the dependencies of conditional distributions are encoded in the inverse $U$ of the covariance matrix; we therefore have that

$$\text{Condition 2 } \iff \text{ } U_{\psi \lambda} = U_{\lambda \psi} = 0,$$

(10)

where $U_{\alpha\beta}$ is a block sub-matrix of $U$. These implications bring us to our first observation:

**Observation 1.** Neither one of Condition 1 (the vector field dependency structure) or Condition 2 (conditional independence in the ergodic distribution) implies the other:

Condition 1 $\nRightarrow$ Condition 2 \hspace{1cm} (11)
Condition 1 $\nLeftrightarrow$ Condition 2. \hspace{1cm} (12)

**Proof.** In Appendix A, we provide direct counterexamples, using the equivalent constraints on the matrices $M$ and $U$ in Eqs. (9) and (10), to implication in either direction. That is, there exists a system obeying Condition 1 that does not obey Condition 2 (proving Eq. (11)), and there exists one obeying Condition 2 that does not obey Condition 1 (proving Eq. (12)). \hfill \Box

Henceforth, unless otherwise stated, we will assume both Condition 1 and Condition 2. Any implications that fail to hold in this special case cannot hold generally.

## 2 Re-expression using only partial gradients

For Step 2 we focus on the components $f_\lambda = (f_{\lambda 1}, ..., f_{\lambda n})$ and $f_a = (f_{a 1}, ..., f_{a n})$ of $f$. Without loss of generality we can rewrite them from Eq. (4) as:

\begin{align*}
  f_a(s, a, \lambda) &= (R_{a\psi} \cdot \nabla \psi + R_{as} \cdot \nabla s + (\Gamma_{aa} + R_{aa}) \cdot \nabla a \\
                   &+ R_{a\lambda} \cdot \nabla \lambda) \ln p^*(\psi, s, a, \lambda), \hspace{1cm} (13) \\
  f_\lambda(s, a, \lambda) &= (R_{\lambda\psi} \cdot \nabla \psi + R_{\lambda s} \cdot \nabla s + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla \lambda \\
                         &+ R_{\lambda a} \cdot \nabla a) \ln p^*(\psi, s, a, \lambda), \hspace{1cm} (14)
\end{align*}

where $\Gamma_{mn}$ $(R_{nm})$ is the block of $\Gamma$ $(R)$ connecting derivatives with respect to the $m$ coordinates to the time derivatives of the $n$ coordinates. The expectation value with respect to $p^*(\psi | s, a, \lambda)$ leaves the left hand side of these equations unchanged. A few manipulations (cf. Friston, 2019, Eq.(12.14), p.129) reveal
that, on the right hand side, this leads to the ergodic density \( p^*(\psi, s, a, \lambda) \) being replaced by the marginalised ergodic density \( p^*(s, a, \lambda) \) so that we get

\[
\begin{align*}
  f_a(s, a, \lambda) &= (R_{\psi \psi} \cdot \nabla_\psi + R_{as} \cdot \nabla_s + (\Gamma_{aa} + R_{aa}) \cdot \nabla_a \\
  &\quad + R_{a\lambda} \cdot \nabla_\lambda) \ln p^*(s, a, \lambda) \\
  f_\lambda(s, a, \lambda) &= (R_{\psi \psi} \cdot \nabla_\psi + R_{\lambda\lambda} \cdot \nabla_\lambda + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla_\lambda \\
  &\quad + R_{a\lambda} \cdot \nabla_a) \ln p^*(s, a, \lambda).
\end{align*}
\]

(15)

(16)

Since \( \nabla_\psi \ln p^*(s, a, \lambda) = 0 \), the terms involving \( \nabla_\psi \) drop out:

\[
\begin{align*}
  f_a(s, a, \lambda) &= (R_{aa} \cdot \nabla_a + (\Gamma_{aa} + R_{aa}) \cdot \nabla_a + R_{a\lambda} \cdot \nabla_\lambda) \ln p^*(s, a, \lambda), \\
  f_\lambda(s, a, \lambda) &= (R_{\lambda\lambda} \cdot \nabla_\lambda + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla_\lambda) \ln p^*(s, a, \lambda).
\end{align*}
\]

(17)

(18)

We are not aware of how to further simplify this equation without additional assumptions. However, in Friston (2013, Eq. (2.5) and Eq. (2.6)) all but the diagonal terms are implicitly assumed to vanish, i.e., Eq. (4) is equated with:

\[
\begin{align*}
  f_a(s, a, \lambda) &= (\Gamma_{aa} + R_{aa}) \cdot \nabla_a \ln p^*(s, a, \lambda), \\
  f_\lambda(s, a, \lambda) &= (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla_\lambda \ln p^*(s, a, \lambda).
\end{align*}
\]

(19)

(20)

This equation is the result of Step 2.

In the more recent Friston (2019, Appendix B) a more detailed discussion of Eq. (4) is presented, where it is claimed that Condition 1 implies Condition 2 (cf. our Observation 1) along with the following simplification of Eqs. (17) and (18) (Friston, 2019, Eqs. (12.8-12.11,12.15), pp.126-129):

\[
\begin{align*}
  f_a(s, a, \lambda) &= ((R_{aa} + R_{aa}) \cdot \nabla_a + R_{a\lambda} \cdot \nabla_\lambda) \ln p^*(s, a, \lambda), \\
  f_\lambda(s, a, \lambda) &= (R_{\lambda\lambda} \cdot \nabla_\lambda + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla_\lambda) \ln p^*(s, a, \lambda).
\end{align*}
\]

(21)

(22)

However, Eqs. (21) and (22) are still provably less general than Eqs. (13) and (14), even when both Condition 1 and Condition 2 are satisfied.

**Observation 2.** Given a random dynamical system obeying Eq. (1), ergodicity, and both Condition 1 and Condition 2, none of Eqs. (19) to (22) generally hold.

**Proof.** By counterexample, see Appendix B. There, we show explicitly that a model satisfying the above assumptions does not satisfy the equations in question.  

In order to arrive at Eqs. (21) and (22) from Eqs. (17) and (18) in general, one must remove the offending “solenoidal flow” terms by fiat. That is, one assumes \( R_{as} = R_{\lambda s} = 0 \). In Friston (2019, Eq. (12.4)), the following, even stronger, condition is assumed as an alternative starting point (along with Condition 2):

**Condition 3.** The blocks of the \( R \) matrix appearing in Eq. (4) coupling \( (s, a) \) coordinates to \( \lambda \) and \( \psi \) coordinates, and \( \psi \) coordinates to \( \lambda \) coordinates vanish, i.e.

\[
R_{\psi s} = R_{\psi a} = R_{\psi \lambda} = R_{a\lambda} = R_{a\lambda} = 0.
\]

(23)

This is claimed to imply \( M_{\psi \lambda} = M_{\lambda \psi} = 0 \), but not the full Condition 1. However, in Parr et al. (2019), both Condition 1 and Condition 3 are assumed (along with \( R_{as} = 0 \)). This prompts our next observation.
Observation 3. In a system satisfying both Condition 1 and Condition 3, the internal coordinates cannot be directly influenced by the sensory coordinates: $f_\lambda(s,a,\lambda) = f_\lambda(a,\lambda)$, and the external coordinates cannot be directly influenced by the active coordinates: $f_\psi(\psi,s,a) = f_\psi(\psi,s)$.

Proof. From Eq. (5), it follows that
\[ M = (\Gamma + R)M^T(\Gamma - R)^{-1}, \] (24)
with the inverse replaced by a pseudoinverse if $\Gamma - R$ is not invertible. Therefore, if $\Gamma_{\alpha\beta} = \delta_{\alpha\beta}\Gamma_{\alpha\alpha}$ and $R_{\alpha\beta} = \delta_{\alpha\beta}R_{\alpha\alpha}$ for blocks of coordinates labelled by $\alpha$ and $\beta$, then
\[ M_{\alpha\beta} = (\Gamma_{\alpha\alpha} + R_{\alpha\alpha})M^T_{\beta\alpha}(\Gamma_{\beta\beta} - R_{\beta\beta})^{-1}, \] (25)
and $M_{\beta\alpha} = 0 \Rightarrow M_{\alpha\beta} = 0$.

Condition 3 implies that the only nonzero blocks of $R$ are $R_{\psi\psi}$, $R_{ss}$, $R_{sa}$, $R_{as}$, $R_{aa}$, and $R_{\lambda\lambda}$, and $\Gamma$ is assumed to be block diagonal. As noted in Eq. (9), Condition 1 requires that $M_{a\psi} = M_{s\psi} = M_{a\lambda} = M_{s\lambda} = 0$. Through Eq. (25), these together imply that $M_{\lambda s} = M_{s\lambda} = 0$, and hence that
\[ f(x) = \begin{pmatrix} f_\psi(\psi,s) \\ f_\lambda(\psi,s,a) \\ f_a(s,a,\lambda) \\ f_\lambda(a,\lambda) \end{pmatrix}, \] (26)
as was to be shown. \qed

In this case, the four sets of coordinates interact in a chain, and it is questionable whether the $s$ and $a$ coordinates can be meaningfully interpreted, respectively, as sensory inputs to the internal coordinates or their boundary-mediated influence on the external coordinates.

3 Free Energy Lemma

The relation of the dynamics of the internal coordinates to Bayesian beliefs is made by introducing a density (called the variational density) $q(\Psi|\lambda)$ that is then interpreted as encoding a Bayesian belief. It is parameterized by the internal coordinates $\lambda$ and claimed to be “arbitrary”. The existence of the variational density $q(\Psi|\lambda)$ is asserted by the Free Energy Lemma (see Lemma 2.1 in Friston, 2013).\footnote{Explicitly, the Free Energy Lemma asserts the existence of a free energy $F(s,a,\lambda)$ in terms of which $f(\psi,s,a,\lambda)$ can be expressed and not the existence of $q(\Psi|\lambda)$. However, since the free energy is defined as a functional of $q(\Psi|\lambda)$, it exists if and only if a suitable $q(\Psi|\lambda)$ exists.}

More precisely, the Free Energy Lemma (and Step 3) asserts that for every ergodic density\footnote{Equivalently as expressed in Friston (2013), for every Gibbs energy $G(x) := -\ln p^*(\psi,s,a,\lambda)$,} $p^*(\psi,s,a,\lambda)$ of a system obeying Eqs. (19) and (20) there is a free energy $F(s,a,\lambda)$, defined as
\[ F(s,a,\lambda) := -\ln p^*(s,a,\lambda) + \int q(\psi|\lambda) \ln \frac{q(\psi|\lambda)}{p^*(\psi|s,a,\lambda)} \, d\psi \] (27)
\[ = -\ln p^*(s,a,\lambda) + D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)], \] (28)
in terms of the “posterior density” \(p^\ast(\Psi|s,a,\lambda)\),\(^8\) such that Eqs. (19) and (20) can be rewritten as:

\[
\begin{align*}
    f_a(s,a,\lambda) &= -\left((\Gamma + R)_{aa} \cdot \nabla_a F(s,a,\lambda)\right) \\
    f_\lambda(s,a,\lambda) &= -\left((\Gamma + R)_{\lambda\lambda} \cdot \nabla_\lambda F(s,a,\lambda)\right).
\end{align*}
\]

(29) \hspace{5cm} (30)

It is worth considering what a proof of the Free Energy Lemma could look like. A proof of existence of a free energy (and therefore of the Free Energy Lemma) would need to show that, for every system satisfying the given assumptions, there always exists a \(q(\Psi|\lambda)\) such that the right hand sides of Eqs. (29) and (30) are equal to the right hand sides of Eqs. (19) and (20). Expanding Eqs. (29) and (30) using Eq. (28) leads to:

\[
\begin{align*}
    f_a(s,a,\lambda) &= \left((\Gamma + R)_{aa} \cdot \nabla_a \ln p^\ast(s,a,\lambda)\right) \\
    f_\lambda(s,a,\lambda) &= \left((\Gamma + R)_{\lambda\lambda} \cdot \nabla_\lambda \ln p^\ast(s,a,\lambda)\right).
\end{align*}
\]

(31) \hspace{5cm} (32)

For equality of the right hand sides to those of Eqs. (19) and (20) we need:

\[
\begin{align*}
    (\Gamma + R)_{aa} \cdot \nabla_a D_{KL}[q(\Psi|\lambda)||p^\ast(\Psi|s,a,\lambda)] &= 0 \\
    (\Gamma + R)_{\lambda\lambda} \cdot \nabla_\lambda D_{KL}[q(\Psi|\lambda)||p^\ast(\Psi|s,a,\lambda)] &= 0.
\end{align*}
\]

(33) \hspace{5cm} (34)

In words, these equations say that the Free Energy Lemma holds if any of the following three conditions (of strictly increasing strengths) are given:

1. There is a \(q(\Psi|\lambda)\) such that the partial gradients \(\nabla_a\) and \(\nabla_\lambda\) of the KL divergence between the variational density and the conditional ergodic density are elements of the nullspaces of \((\Gamma + R)_{aa}\) and \((\Gamma + R)_{\lambda\lambda}\) respectively.

2. There is a \(q(\Psi|\lambda)\) such that the gradients of the KL divergence to \(p^\ast(\Psi|s,a,\lambda)\) are equal to the nullvector:

\[
\begin{align*}
    \nabla_a D_{KL}[q(\Psi|\lambda)||p^\ast(\Psi|s,a,\lambda)] &= 0, \\
    \nabla_\lambda D_{KL}[q(\Psi|\lambda)||p^\ast(\Psi|s,a,\lambda)] &= 0.
\end{align*}
\]

(35) \hspace{5cm} (36)

Then they are always elements of the nullspaces of \((\Gamma + R)_{aa}\) and \((\Gamma + R)_{\lambda\lambda}\) respectively.

3. There is a \(q(\Psi|\lambda)\) such that \(q(\Psi|\lambda) = p^\ast(\Psi|s,a,\lambda)\) (and hence \(p^\ast(\Psi|s,a,\lambda) = p^\ast(\Psi|\lambda)\)) which implies that the KL divergence to \(p^\ast(\Psi|s,a,\lambda)\) vanishes for all \(a,\lambda\) and the two partial gradients are always nullvectors and therefore elements of the according nullspaces.

The Free Energy Lemma can then be proven by showing that one of these three cases follows from the conditions of the lemma. However, no attempt is made in Friston (2013) to establish this. Instead the given proof discusses purported consequences of the existence of a suitable \(q(\Psi|\lambda)\). These will be discussed in Steps 4 and 5.

\(^8\)Here, we keep the conditioning argument \(\lambda\), as in Friston (2013), and do not explicitly assume Condition 2, though our conclusions are unaffected by it.
Even if the Free Energy Lemma does not hold for systems obeying Eqs. (19) and (20), one might expect that systems that instead only satisfy the more general Eqs. (21) and (22) or the most general Eqs. (17) and (18). For these systems the Free Energy Lemma would require that there is a \( q(\Psi|\lambda) \) such that

\[
f_a(s, a, \lambda) = ((\Gamma_{aa} + R_{aa}) \cdot \nabla_a + R_{a\lambda} \cdot \nabla_{\lambda}) F(s, a, \lambda), \quad (37)
\]

\[
f_{\lambda}(s, a, \lambda) = (R_{a\lambda} \cdot \nabla_a + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla_{\lambda}) F(s, a, \lambda). \quad (38)
\]

or

\[
f_a(s, a, \lambda) = (R_{aa} \cdot \nabla_a + (\Gamma_{aa} + R_{aa}) \cdot \nabla_a + R_{a\lambda} \cdot \nabla_{\lambda}) F(s, a, \lambda), \quad (39)
\]

\[
f_{\lambda}(s, a, \lambda) = (R_{a\lambda} \cdot \nabla_a + R_{a\lambda} \cdot \nabla_a + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla_{\lambda}) F(s, a, \lambda), \quad (40)
\]

hold respectively. However, we find this not to be the case in general.

**Observation 4.** Given a random dynamical system obeying Eq. (1), ergodicity, Condition 1 and Condition 2, there need not exist a free energy expressed in terms of a variational density \( q(\Psi|\lambda) \) such that:

(i) Eqs. (29) and (30) hold if Eqs. (19) and (20) do;

(ii) Eqs. (37) and (38) hold if Eqs. (19) and (20) don’t hold but Eqs. (21) and (22) do;

(iii) Eqs. (39) and (40) hold if neither Eqs. (19) and (20) nor Eqs. (21) and (22) hold but Eqs. (17) and (18) do.

**Proof.** In Appendix C, we derive a set of conditions on the \( R \) and \( U \) matrices, and on the putative variational density \( q(\Psi|\lambda) \), that follow from each of the pairs of equations in cases (i-iii). We show that, in general, each pair leads to a contradiction and, in each case, provide a counterexample that falls in the according system class.

Before proceeding, we note that later works present an alternative version of the Free Energy Lemma, where the conditioning argument of \( q(\Psi|\lambda) \) is replaced by the most likely value of \( \lambda \) conditional on the \((s, a)\) coordinates (Friston, 2019; Parr et al., 2019). We here concern ourselves with the version apparent in Friston (2013), where \( q(\Psi|\lambda) \) is parameterised by the internal states themselves, but will briefly comment on the interpretation of the alternative approach in Step 6.

### 4 Vanishing gradients

As mentioned in Step 3, the proof of the Free Energy Lemma in Friston (2013) only discusses its consequences. The first proposed consequence is that expressing the vector field in terms of a free energy as in Eqs. (29) and (30) “requires” that the gradients with respect to \( a \) and \( \lambda \) of the KL divergence vanish, i.e. that Eqs. (35) and (36) hold.

We mentioned in Step 3 that the implication in the opposite direction holds. This can be seen from Eqs. (33) and (34). However, if the nullspace of \((\Gamma+R)_{aa}\) or \((\Gamma+R)_{\lambda\lambda}\) is non-trivial, then the gradient may be a non-zero element of this
subspace and Eqs. (29) and (30) will still hold. In that case the vanishing gradients would not be necessary for the Free Energy Lemma.

The conditions under which a non-trivial nullspace exists are discussed in Kwon et al. (2005). In short, the nullspace is guaranteed to be trivial in the special case where $\Gamma$ is positive definite. Whether or not ergodic systems with a Markov blanket can ever admit a non-trivial nullspace, and hence divergences in Eqs. (31) and (32) with non-vanishing gradients, is not immediately clear. However, in order to establish the necessity of Eqs. (35) and (36) this remains to be proven.

5 Equality of $q(\Psi|\lambda)$ and $p^*(\Psi|s, a, \lambda)$

The proof of the Free Energy Lemma in Friston (2013) also proposes that the vanishing of gradients of the KL divergence, of the variational density $q(\Psi|\lambda)$ from the conditional ergodic density $p^*(\Psi|s, a, \lambda)$, implies the equality of these densities. We mentioned in Step 4 that the implication in the opposite direction holds. This can also be seen from Eqs. (33) and (34). Concerning the implication in the direction proposed by Friston (2013), let us now assume that for a given system Eqs. (19) and (20) hold, a variational density $q(\Psi|\lambda)$ does exist, and the gradients of the KL divergence of the variational and ergodic densities vanish i.e. Eqs. (35) and (36) hold. Then consider the argument by Friston (2013) in this direct quote (comments in square brackets by us):

However, equation (2.6) [Eqs. (19) and (20) above] requires the gradients of the divergence to be zero [Eqs. (35) and (36)], which means the divergence must be minimized with respect to internal states. This means that the variational and posterior densities must be equal:

\[
q(\psi|\lambda) = p^*(\psi|s, a, \lambda) \Rightarrow D_{KL} = 0 \Rightarrow \begin{cases} 
(\Gamma + R) \cdot \nabla_\lambda D_{KL} = 0, \\
(\Gamma + R) \cdot \nabla_a D_{KL} = 0.
\end{cases}
\]

In other words, the flow of internal and active states minimizes free energy, rendering the variational density equivalent to the posterior density over external states.

The first problem in the above quote is that the minimization of the divergence does not follow from the vanishing gradients. On the contrary, since Eqs. (35) and (36) must hold for all $(s, a, \lambda)$, the KL divergence

\[
D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s, a, \lambda)]
\]

cannot depend on $(\lambda, a)$; it therefore has no extremum (and thus no minimum) with respect to either of these coordinates.

The second problem pertains to the identification of the two distributions at a minimum. In general, if we try to find the minimum of a KL divergence between a given probability density $p_1(Y)$ and a family of densities $p_2(Y|\theta)$ parameterized by $\theta$, then the lowest possible value of zero is achieved only if there is a parameter $\theta_1$ such that $p_2(Y|\theta_1) = p_1(Y)$. If there is no such $\theta_1$, then the minimum value will be larger than zero. So, even if the divergence were
minimized, it would not need to be zero. More generally, the divergence $K(s)$ need not be zero for any value of $s$.

There is therefore no satisfactory reason given why the variational density $q(\Psi|\lambda)$ and the posterior density $p^*(\Psi|s,a,\lambda)$ should be equal or have low KL divergence. In fact they need not be.

**Observation 5.** Given a random dynamical system obeying Eq. (1), ergodicity, Condition 1 and Condition 2. Then if, additionally,

(i) Eqs. (19) and (20) hold and the Free Energy Lemma holds i.e. there exists a probability density $q(\Psi|\lambda)$ such that Eqs. (29) and (30) hold, or

(ii) Eqs. (21) and (22) hold and there exists $q(\Psi|\lambda)$ such that Eqs. (37) and (38) hold, or

(iii) Eqs. (17) and (18) hold and there exists $q(\Psi|\lambda)$ such that Eqs. (39) and (40) hold,

then there is no $c \geq 0$ for which it can be guaranteed that

$$D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)] < c.$$  \hspace{1cm} (41)

In particular, it does not follow from these conditions that

$$q(\Psi|\lambda) = p^*(\Psi|s,a,\lambda).$$ \hspace{1cm} (42)

**Proof.** By example; see Appendix D. To show that the implication does not generally hold for given system and densities $q(\Psi|\lambda)$ that obey Eqs. (19), (20), (29) and (30), Eqs. (21), (22), (37) and (38), or Eqs. (17), (18), (39) and (40) we only have to consider a system that obeys all three pairs of equations, Eqs. (19) and (20), Eqs. (21) and (22), and Eqs. (21) and (22), and for which suitable $q(\Psi|\lambda)$ exist. For this system we then need to show that the $q(\Psi|\lambda)$ that obey Eqs. (29) and (30) are not necessarily equal (or similar) to $p^*(\Psi|s,a,\lambda)$.

We use a variant of the model used in Appendix B as such a counterexample. This system obeys all three of Eqs. (19) and (20), Eqs. (21) and (22), and Eqs. (21) and (22) and the nullspace of the associated $\Gamma + R$ is trivial. We identify a set of possible $q(\Psi|\lambda)$ satisfying Eqs. (29) and (30) which implies that the gradients of the KL divergence between those $q(\Psi|\lambda)$ and $p^*(\Psi|s,a,\lambda)$ vanish i.e. Eqs. (35) and (36) hold. We then demonstrate that for the $q(\Psi|\lambda)$ in this set the value of the KL divergence to $p^*(\Psi|s,a,\lambda)$ can be arbitrarily large. \hfill \Box

## 6 Interpretation

Finally, we turn our attention to the interpretation in terms of Bayesian inference, i.e. Step 6. We again quote directly from Friston (2013):

Because (by Gibbs inequality) this divergence $[D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)]]$ cannot be less than zero, the internal flow will appear to have minimized the divergence between the variational and posterior density.

In other words, the internal states will appear to have solved the problem of Bayesian inference by encoding posterior beliefs about hidden (external) states, under a generative model provided by the Gibbs energy.
We have shown that, in general, there is no suitable variational density and that, even if there is one, it can be arbitrarily different from the posterior density. Since the arguments for the internal flow appearing to minimize the divergence between variational and posterior density are therefore incorrect, there is no reason why the internal states should appear to have solved the problem of Bayesian inference.

As mentioned in Step 3, some newer works (e.g., Friston (2019); Parr et al. (2019)) formulate a version of the Free Energy principle\(^9\) where the variational density of beliefs is parameterised not by the internal coordinates \(\lambda\) but by \(\lambda(s,a) = \arg \max_{\lambda} p^*(\lambda|s,a)\), the most likely value of the internal coordinates given the sensory and active ones. In this case, many of the arguments we raised in Steps 3-5 do not apply. However, the new parameters \(\lambda(s,a)\) are strictly a function of the sensory and active coordinates. This means we have a Markov chain\(^10\) \(\Lambda \rightarrow (S,A) \rightarrow \Lambda\) and, by the data processing inequality (Cover and Thomas, 2006), the mutual information between the both sensory and active coordinates and the belief parameter \(\lambda\) upper bounds that between the internal coordinates and the belief parameter. It is therefore not clear to what extent the internal coordinates \(\lambda\), rather than the active and sensory coordinates \((s,a)\) themselves, can be said to be encoding beliefs about the external coordinates. Note also that, on any given trajectory, unless the distribution \(p^*(\lambda|s,a)\) is sufficiently peaked and unimodal, the internal coordinates are not guaranteed to spend most of their time close to their most likely conditional value, and (by definition if Condition 2 holds), they will not be better predictors of the external coordinates than those in the Markov blanket.

7 Consequences for Friston et al. (2014)

Friston et al. (2014) argues for the same interpretation as Friston (2013) but there are some differences in the argument.

The differences are the following:

- In Friston et al. (2014), Eq. (1) is formulated for “generalized states,” which we refer to here as generalized coordinates. This means that the variable \(x\) is replaced by a multidimensional variable denoted \(\tilde{x} = (x, x', x'', ...).\)

- The Markov blanket structure is not explicitly defined via Eq. (2). Formally, it is introduced directly (see Friston et al., 2014, Eq.(10)) in a less general form corresponding to Eqs. (19) and (20).\(^11\) Therefore, our observations concerning Steps 1 to 3 are not directly relevant to this paper.

- The internal coordinate \(\lambda\) is renamed to \(r\) and the role of matrix \(R\) is played by the matrix \(-Q\).

- The proof of the Free Energy Lemma given in Friston et al. (2014) is different. It (implicitly) suggests to set the variational density equal to the ergodic conditional posterior.

\(^9\) As discussed in Step 1 and Step 2, these make a slightly different set of assumptions from Friston (2013), but rely on similar arguments to the ones we disprove here.

\(^10\) With capitalisations indicating random variables associated to the corresponding lower case coordinates (or functions of coordinates).

\(^11\) At the same time Friston (2013) is referenced in connection to the Markov blanket so there seems to be no intention to replace the original definition with the stronger one.
The proof of the Free Energy Lemma no longer contains the proposition that the gradient of the KL divergence of the variational density and the ergodic conditional density vanish i.e. Step 4.

The proof also no longer contains the claim that the vanishing gradients of the KL divergence of the variational density and the ergodic conditional density imply equality of those densities i.e. Step 5 is not present.

The interpretation in terms of Bayesian inference is unchanged and still relies on the equality of the variational and the ergodic conditional density.

Since there are no explicit generalized coordinate versions of Steps 1, 2, 4 and 5 in Friston et al. (2014) we do not discuss those steps here. We only disprove the Free Energy Lemma and the claim that when the Free Energy Lemma holds the variational and ergodic conditional density become equal. For this we present a way to translate the counterexamples used in Observations 4 and 5 into counterexamples in generalized coordinates. The interpretation in terms of Bayesian inference given in Friston et al. (2014) is therefore equally as unjustified as the one in Friston (2013).

For completeness, we first state the generalized coordinate versions of the Langevin equation Eq. (1)
\[ \dot{x} = f(\tilde{x}) + \tilde{\omega}, \] (43)

the less general version of the Markov blanket structure Eq. (2)
\[
\begin{align*}
    f_\psi(\tilde{\psi}, \tilde{s}, \tilde{a}) &= (\Gamma - Q)\tilde{\psi} \nabla_\psi \ln p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) \\
    f_s(\tilde{s}, \tilde{s}, \tilde{a}) &= (\Gamma - Q)\tilde{s} \nabla_\psi \ln p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) \\
    f_a(\tilde{s}, \tilde{a}, \tilde{r}) &= (\Gamma - Q)\tilde{a} \nabla_\psi \ln p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) \\
    f_\tilde{r}(\tilde{s}, \tilde{a}, \tilde{r}) &= (\Gamma - Q)\tilde{r} \nabla_\psi \ln p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}),
\end{align*}
\] (44)

the expression of the \( \tilde{a} \) and \( \tilde{r} \) components of the vectorfield in terms of the marginalised ergodic density Eqs. (19) and (20)
\[
\begin{align*}
    f_\tilde{a}(\tilde{s}, \tilde{a}, \tilde{r}) &= (\Gamma - Q)\tilde{a} \cdot \nabla_\psi \ln p^*(\tilde{s}, \tilde{a}, \tilde{r}), \\
    f_\tilde{r}(\tilde{s}, \tilde{a}, \tilde{r}) &= (\Gamma - Q)\tilde{r} \cdot \nabla_\psi \ln p^*(\tilde{s}, \tilde{a}, \tilde{r}),
\end{align*}
\] (45, 46)

and in terms of free energy Eqs. (29) and (30):
\[
\begin{align*}
    f_\tilde{a}(\tilde{s}, \tilde{a}, \tilde{r}) &= (Q - \Gamma)\tilde{a} \cdot \nabla_\psi F(\tilde{s}, \tilde{a}, \tilde{r}), \\
    f_\tilde{r}(\tilde{s}, \tilde{a}, \tilde{r}) &= (Q - \Gamma)\tilde{r} \cdot \nabla_\psi F(\tilde{s}, \tilde{a}, \tilde{r}).
\end{align*}
\] (47, 48)

The Free Energy Lemma then requires that there exists \( q(\tilde{\psi} | \tilde{r}) \) such that the KL divergence between \( p^*(\tilde{\psi} | \tilde{s}, \tilde{a}, \tilde{r}) \) vanishes. Without going into further details of the difference between the proof in Friston et al. (2014) and that in Friston (2013), we can prove the former wrong by translating the counterexample used for the latter into generalised coordinates.

**Observation 6.** There is a general way to translate a system in ordinary coordinates into a system of generalised coordinates that corresponds to an infinite number of independent copies of the original system. This means all properties of the original system (e.g. linearity, ergodicity, the Gaussian and Markovian property of the noise, Conditions 1 and 2, properties of \( \Gamma, R, U \)) are preserved during this translation.
Proof. By construction, see Appendix E.

This implies that the counterexamples used in proving Observations 4 and 5 directly translate to the setting of the generalised coordinates. The Free Energy Lemma is therefore also wrong for generalised coordinates and the variational density \( q(\hat{\Psi} | \hat{r}) \) is not “ensured” (Friston et al., 2014) to be equal to the conditional ergodic density \( p^*(\hat{\Psi}|\hat{s}, \hat{a}, \hat{r}) \).

Conclusion

We found that the two different Markov blanket conditions proposed in Friston (2013, 2019); Parr et al. (2019) are independent from each other. We then showed that under both of those Markov blanket conditions, among the six steps contained in the argument in Friston (2013), three do not hold independently from each other. We also showed that fixing the second of those steps (Step 2) does not provide an valid alternative. The line of reasoning of Friston (2013) therefore does not support its claim that the internal coordinates of a Markov blanket “appear to have solved the problem of Bayesian inference by encoding posterior beliefs about hidden (external) [coordinates]...”. We have also shown that using generalised coordinates as in Friston et al. (2014) does not remedy the situation. Additionally, we identified a technical error in Friston (2019) and an interpretational issue resulting from possibly too strong assumptions (both Conditions 1 and 3) in Parr et al. (2019). We also remarked that the latter publications both argue that it is the most likely internal coordinates given sensory and active coordinates that encode posterior beliefs about external states instead of the internal coordinates themselves. This is a different proposal that is not subject to our technical critique.

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A Counterexamples for Observation 1

Consider a four dimensional linear system obeying Eq. (1) for which there are coordinates \( x = (\psi, s, a, \lambda) \) with \( n_\psi = n_s = n_a = n_\lambda = 1 \) and

\[
f(x) = Mx,
\]

(49)
with the parameterisation

\[
M = \begin{pmatrix}
-1 & m_1 & m_2 & m_3 \\
-1 & m_2 & m_3 & m_2 \\
m_3 & m_2 & -1 & m_2 \\
m_3 & m_2 & m_1 & -1
\end{pmatrix},
\]  

(50)

From Eq. (9), it is clear that the system obeys Condition 1 if \( m_3 = 0 \). In this case, taking \( \Gamma \) to be the identity matrix, it is possible to show that

\[
U_{\psi \lambda} = -\frac{m_2(m_1 - m_2 + 2)(m_1^2 + m_1^2m_2 - 2m_1m_2^2 - 4m_1 - 2)}{(m_1^2 + m_2^2 - 4m_2 + 4)(m_1^2 + 5m_2^2 - 4m_1m_2 + 4m_2 + 4)}.
\]

(51)

For fixed, finite \( m_2 \), this is zero only for a few discrete values of \( m_1 \), such as \( m_1 = m_2 - 2 \); that it is generically non-zero proves Eq. (11). As a concrete example, the following

\[
M = \begin{pmatrix}
-1 & -2/3 & -2/3 & 0 \\
-2/3 & -1 & -2/3 & 0 \\
0 & -2/3 & -1 & -2/3 \\
0 & -2/3 & -2/3 & -1
\end{pmatrix},
\]

(52)

has (full rank and hence ergodic)

\[
U = \begin{pmatrix}
236/255 & 127/255 & -31/85 & -12/85 \\
127/255 & 274/255 & 206/255 & 31/85 \\
31/85 & 206/255 & 274/255 & 127/255 \\
-12/85 & 31/85 & 127/255 & 236/255
\end{pmatrix},
\]

(53)

and hence ergodic density

\[
p^* (\psi, s, a, \lambda) = \sqrt{\frac{28}{2295\pi^4}} \exp \left[ -\frac{1}{255} (137(a^2 + s^2) + 118(\psi^2 + \lambda^2) + 127(\psi s + a\lambda) + 93(\psi a + s\lambda) + 206as - 36\psi\lambda) \right],
\]

(54)

which does not conditionally factorise.

Taking the same parameterisation as in Eq. (50), and fixing \( m_1 = m_2 = -1/2 \), we can search for a non-zero value of \( m_3 \) that leads to \( U_{\psi \lambda} = 0 \) (equivalent to Condition 2 through Eq. (10)). We find such a value in the real root \( c \approx -0.08 \) of the quintic equation \( 8c^5 - 4c^4 - 6c^3 + 31c^2 + 40c + 3 = 0 \). That is, with

\[
M = \begin{pmatrix}
-1 & -1/2 & -1/2 & c \\
-1/2 & -1 & -1/2 & c \\
c & -1/2 & -1 & -1/2 \\
c & -1/2 & -1/2 & -1
\end{pmatrix},
\]

(55)

which does not satisfy Condition 1, we have

\[
U = \begin{pmatrix}
0.96 \ldots & 0.43 \ldots & 0.30 \ldots & 0 \\
0.43 \ldots & 1.03 \ldots & 0.58 \ldots & 0.30 \ldots \\
0.30 \ldots & 0.58 \ldots & 1.03 \ldots & 0.43 \ldots \\
0 & 0.30 \ldots & 0.43 \ldots & 0.96 \ldots
\end{pmatrix},
\]

(56)

which has non-zero determinant (i.e., the dynamics is ergodic) and an ergodic density satisfying Condition 2. This proves Eq. (12).
Here, we consider a linear system, as in the previous appendix. We again assume \( \Gamma \) equal to the identity matrix and choose a force matrix of the form
\[
M = \begin{pmatrix}
-1 & -1 & -1 & 0 \\
\frac{1}{2\sqrt{2}} & -1 & -1 & 0 \\
0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & -1 & -1
\end{pmatrix}
\]
which explicitly satisfies Condition 1 and has full rank such that the system is ergodic. Using Eq. (5) this leads to
\[
U = \begin{pmatrix}
\frac{1023}{1057} & \frac{260\sqrt{2}}{1057} & \frac{116\sqrt{2}}{1057} & 0 \\
\frac{260\sqrt{2}}{1057} & \frac{1057}{1057} & 0 & -\frac{136\sqrt{2}}{1057} \\
\frac{1057}{1057} & \frac{0}{1057} & \frac{1057}{1057} & \frac{260\sqrt{2}}{1057} \\
0 & -\frac{136\sqrt{2}}{1057} & \frac{260\sqrt{2}}{1057} & \frac{1023}{1057}
\end{pmatrix}
\]
which shows that this system also satisfies Condition 2 since \( U_{\psi \lambda} = U_{\lambda \psi} = 0 \). We also find
\[
R = \begin{pmatrix}
0 & -\frac{17}{1786\sqrt{2}} & -\frac{479}{1786\sqrt{2}} & -\frac{62}{1786\sqrt{2}} \\
\frac{1786\sqrt{2}}{1786\sqrt{2}} & 0 & -\frac{14288}{1786\sqrt{2}} & \frac{62}{1786\sqrt{2}} \\
-\frac{1786\sqrt{2}}{1786\sqrt{2}} & \frac{14288}{1786\sqrt{2}} & 0 & \frac{1786\sqrt{2}}{1786\sqrt{2}} \\
\frac{62}{1786\sqrt{2}} & \frac{14288}{1786\sqrt{2}} & \frac{62}{1786\sqrt{2}} & 0
\end{pmatrix}
\]
which shows that all entries or \( R \) that can be non-zero for an anti-symmetric matrix are non-zero. For the marginal ergodic density we find
\[
p^*(s, a, \lambda) = \frac{239}{16\sqrt{2415\pi}^{3/2}} \exp \left[ -\frac{69a^2}{140} - \frac{37as}{70\sqrt{2}} + \frac{1}{35}\sqrt{22a\psi} - \frac{4867s^2}{9660} + \frac{74s\psi}{2415} - \frac{8429\psi^2}{19320} \right]
\]
The difference between the right hand sides of Eqs. (17) and (19) is
\[
R_{a\lambda} \nabla_s \ln p^*(s, a, \lambda) + R_{a\lambda} \nabla_a \ln p^*(s, a, \lambda) = \frac{37a + 69\sqrt{2}\lambda - 2563s}{4830} \neq 0,
\]
which shows that Eq. (19) is wrong in this example and therefore not generally equivalent to Eq. (17). Similarly, computing the difference between the right hand sides of Eqs. (18) and (20), one finds
\[
R_{\lambda s} \nabla_s \ln p^*(s, a, \lambda) + R_{\lambda a} \nabla_a \ln p^*(s, a, \lambda) = \frac{2a - \sqrt{2}\lambda + 27s}{70\sqrt{2}} \neq 0,
\]
and hence Eq. (20) is also incorrect in general.

Performing the same comparison for the difference between the general expression in Eqs. (17) and (18) and the expressions taken from Friston (2019), one finds
\[
R_{a\lambda} \nabla_s \ln p^*(s, a, \lambda) = \frac{73 (296a + 552\sqrt{2}\lambda - 8429s)}{1154370} \neq 0
\]
for the difference between the right hand sides of Eqs. (17) and (21), and

\[ R_{\lambda s} \nabla_s \ln p^*(s,a,\lambda) = -\frac{53(296a + 552\sqrt{2}\lambda - 8429s)}{1154370\sqrt{2}} \neq 0, \] (64)

for the difference between the right hand sides of Eqs. (18) and (22). Therefore, Eqs. (21) and (22) are also incorrect in general, even when Condition 1 and Condition 2 both hold.

C Counterexamples for Step 3

We saw in Appendix B that Eqs. (19) and (20) are not generally equivalent to Eq. (4), even when Condition 1 and Condition 2 hold simultaneously. We now show that if we instead use Eqs. (17) and (18), which are generally equivalent to Eq. (4), the Free Energy Lemma does not hold in general.

The original Free Energy Lemma requires that (see Eqs. (31) and (32))

\[
\begin{align*}
(\Gamma + R)_{aa} \cdot \nabla_a D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)] &= 0 \quad (65) \\
(\Gamma + R)_{\lambda\lambda} \cdot \nabla_\lambda D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)] &= 0. \quad (66)
\end{align*}
\]

replacing the partial gradient in Eqs. (29) and (30) with the full gradient and including the entire matrix \((\Gamma + R)\) leads to the corresponding requirement for the more general case:

\[
\begin{align*}
(R_{aa} \cdot \nabla_a + (\Gamma_{aa} + R_{aa}) \cdot \nabla_a + R_{a\lambda} \cdot \nabla_\lambda) D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)] &= 0 \quad (67) \\
(R_{\lambda s} \cdot \nabla_s + R_{\lambda a} \cdot \nabla_a + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla_\lambda) D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)] &= 0. \quad (68)
\end{align*}
\]

Similarly, the version based on the equations taken from Friston (2019) implies

\[
\begin{align*}
((\Gamma_{aa} + R_{aa}) \cdot \nabla_a + R_{a\lambda} \cdot \nabla_\lambda) D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)] &= 0 \quad (69) \\
(R_{\lambda a} \cdot \nabla_a + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \cdot \nabla_\lambda) D_{KL}[q(\Psi|\lambda)||p^*(\Psi|s,a,\lambda)] &= 0. \quad (70)
\end{align*}
\]

Using the rules of Gaussian integration, we can write the logarithm of the conditional ergodic density as

\[
\ln p^*(\psi|s,a,\lambda) = -\frac{1}{2} U_{\psi\psi}^\frac{1}{2} \psi + U_{\psi s}^\frac{1}{2} U_{\psi s} s + U_{\psi a}^\frac{1}{2} U_{\psi a} a + U_{\psi\lambda}^\frac{1}{2} U_{\psi\lambda} \lambda^2 + C, \quad (71)
\]

with C a constant (and remembering each of \(\psi, s, a\) and \(\lambda\) is a vector of coordinates in general). We can then expand out the derivatives of the KL divergence...
to express them in terms of the coordinates:

\[
\nabla_s D_{KL}[q(\Psi|\lambda)||p^*(\psi|s, a, \lambda)] = -\int \psi q(\psi|\lambda) \nabla_s \ln p^*(\psi|s, a, \lambda) \nabla_s \ln p^*(\psi|s, a, \lambda)
\]

\[
= U_s \psi U_{\psi\psi}^{-1}(U_{\psi s} s + U_{\psi a} a)
\]

\[
\quad + U_{\psi a} a + U_{\psi a} (\psi|q(\Psi|\lambda)), \tag{72}
\]

\[
\nabla_a D_{KL}[q(\Psi|\lambda)||p^*(\psi|s, a, \lambda)] = -\int \psi q(\psi|\lambda) \nabla_a \ln p^*(\psi|s, a, \lambda)
\]

\[
= U_a U_{\psi\psi}^{-1}(U_{\psi s} s + U_{\psi a} a)
\]

\[
\quad + U_{\psi a} a + U_{\psi a} (\psi|q(\Psi|\lambda)), \tag{73}
\]

\[
\nabla_{\lambda} D_{KL}[q(\Psi|\lambda)||p^*(\psi|s, a, \lambda)] = \int \psi \left[ (\ln q(\psi|\lambda) - \ln p^*(\psi|s, a, \lambda) + 1) \nabla_{\lambda} q(\psi|\lambda) \right]
\]

\[
\quad - q(\psi|\lambda) \nabla_{\lambda} \ln p^*(\psi|s, a, \lambda)
\]

\[
= U_{\lambda \psi} U_{\psi\psi}^{-1}(U_{\psi s} s + U_{\psi a} a)
\]

\[
\quad + U_{\psi a} a + U_{\psi a} (\psi|q(\Psi|\lambda))
\]

\[
\quad + \nabla_{\lambda} (\psi q(\Psi|\lambda)) (U_{\psi s} s + U_{\psi a} a + U_{\psi a} \lambda)\]

\[
\quad + \nabla_{\lambda} \left( (\psi^T U_{\psi\psi} \psi) q(\Psi|\lambda) - H[q(\Psi|\lambda)] \right). \tag{74}
\]

with \(\langle g(\psi) \rangle_q(\Psi|\lambda) := \int \psi q(\psi|\lambda) g(\psi)\) and \(H\) the Shannon entropy.

Substituting Eqs. (73) and (74) into Eqs. (65) and (66) leads to

\[
(\Gamma_{aa} + R_{aa}) U_{aa} U_{\psi\psi}^{-1} (U_{\psi s} s + U_{\psi a} a + U_{\psi a} \lambda + U_{\psi a} (\psi|q(\Psi|\lambda)) = 0, \tag{75}
\]

and

\[
0 = (\Gamma_{a\lambda} + R_{a\lambda}) U_{a\lambda} U_{\psi\psi}^{-1} (U_{\psi s} s + U_{\psi a} a + U_{\psi a} \lambda + U_{\psi a} (\psi|q(\Psi|\lambda))
\]

\[
\quad + (\Gamma_{a\lambda} + R_{a\lambda}) \nabla_{a} (\psi q(\Psi|\lambda)) (U_{\psi s} s + U_{\psi a} a + U_{\psi a} \lambda)
\]

\[
\quad + (\Gamma_{a\lambda} + R_{a\lambda}) \nabla_{\lambda} \left( (\psi^T U_{\psi\psi} \psi) q(\Psi|\lambda) - H[q(\Psi|\lambda)] \right). \tag{76}
\]

Since these must hold for all values of the coordinates, they put strong requirements on the \(U\) and \(R\) matrices. Specifically,

\[
(\Gamma_{aa} + R_{aa}) U_{aa} U_{\psi\psi}^{-1} U_{\psi s} = 0, \tag{77}
\]

\[
(\Gamma_{aa} + R_{aa}) U_{aa} U_{\psi\psi}^{-1} U_{\psi a} = 0, \tag{78}
\]

\[
(\Gamma_{aa} + R_{aa}) U_{aa} U_{\psi\psi}^{-1} U_{\psi a} = 0. \tag{79}
\]

In other words, since \(U_{\psi\psi}\) and \(\Gamma_{aa}\) must be nonzero for the dynamics to be ergodic, it must be that \(U_{\psi a} = 0.\) Specifically, consider the system specified by the force matrix

\[
M = \begin{pmatrix}
-1 & 0 & \frac{1}{2} & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \tag{80}
\]

\[12\text{This is equivalent to } p^*(\Psi|s, a, \lambda) = p^*(\Psi|s, \lambda). \text{ So if Condition 2 also holds we must have } p^*(\Psi|s, a, \lambda) = p^*(\Psi|s) \text{ in order for there to be a suitable } q(\Psi|\lambda).\]
leads to

\[ R = \begin{pmatrix} 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]  

(81)

and

\[ U = \begin{pmatrix} \frac{16}{17} & 0 & -\frac{1}{17} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{17} & 0 & \frac{16}{17} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \].  

(82)

Here \( M \) is full rank so the system is ergodic, clearly it also satisfies Condition 1 due to the structure of \( M \). Since \( R_{as} = R_{a\lambda} = R_{\lambda s} = 0 \) it obeys Eqs. (19) and (20) and since \( Q_{\psi\lambda} = 0 \) it also obeys Condition 2. Additionally, we find \( U_{\psi a} = -\frac{1}{17} \) which is a contradiction.

For the more general version, substituting Eqs. (72) to (74) into Eq. (67), one finds

\[
0 = \left( (R_{as} U_{s\psi} + (R_{aa} + \Gamma_{aa}) U_{a\psi} + R_{a\lambda} U_{\lambda\psi}) U_{\psi s}^{-1} + R_{a\lambda} \nabla_{\lambda} \langle \psi \rangle_{q(\psi|\lambda)} \right) U_{\psi s s} + \\
+ \left( (R_{as} U_{s\psi} + (R_{aa} + \Gamma_{aa}) U_{a\psi} + R_{a\lambda} U_{\lambda\psi}) U_{\psi s}^{-1} + R_{a\lambda} \nabla_{\lambda} \langle \psi \rangle_{q(\psi|\lambda)} \right) U_{\psi s a} + \\
+ \left( (R_{as} U_{s\psi} + (R_{aa} + \Gamma_{aa}) U_{a\psi} + R_{a\lambda} U_{\lambda\psi}) U_{\psi s}^{-1} + R_{a\lambda} \nabla_{\lambda} \langle \psi \rangle_{q(\psi|\lambda)} \right) U_{\psi s \lambda} + \\
+ \left( (R_{as} U_{s\psi} + (R_{aa} + \Gamma_{aa}) U_{a\psi} + R_{a\lambda} U_{\lambda\psi}) \langle \psi \rangle_{q(\psi|\lambda)} \right) + \\
+ R_{a\lambda} \nabla_{\lambda} \left( (\psi^T U_{\psi\psi} \psi)_{q(\psi|\lambda)} - H[q(\psi|\lambda)] \right),
\]

(83)

which, considering that the coordinates can take any values, implies that

\[
(R_{as} U_{s\psi} + (R_{aa} + \Gamma_{aa}) U_{a\psi} + R_{a\lambda} U_{\lambda\psi}) U_{\psi s}^{-1} + R_{a\lambda} \nabla_{\lambda} \langle \psi \rangle_{q(\psi|\lambda)} = 0
\]

(84)

lies in a common (left) nullspace of \( U_{\psi s}, \ U_{\psi a} \) and \( U_{\psi \lambda} \). However, the existence of such a nontrivial nullspace would imply that the corresponding subspace of \( \psi \) coordinates is independent of the \( s, a \) and \( \lambda \) coordinates (to see this, consider marginalising over their complement in Eq. (71)). In other words, if only \( \psi \) coordinates that play a nontrivial role in the dynamics are considered, then Eq. (67) must imply the quantity in Eq. (84) is zero, and hence that

\[
R_{a\lambda} \nabla_{\lambda} \langle \psi \rangle_{q(\psi|\lambda)} = -(R_{as} U_{s\psi} + (R_{aa} + \Gamma_{aa}) U_{a\psi} + R_{a\lambda} U_{\lambda\psi}) U_{\psi s}^{-1}.
\]

(85)

However, through a similar procedure, one finds that Eq. (68) is equivalent
to

\[ 0 = \left( (R_{\lambda s} U_{s\psi} + R_{\lambda a} U_{a\psi} + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) U_{\lambda\psi}) U_{\psi\psi}^{-1} \right. \]
\[ + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} \right) U_{\psi\psi} \]
\[ + \left( (R_{\lambda s} U_{s\psi} + R_{\lambda a} U_{a\psi} + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) U_{\lambda\psi}) U_{\psi\psi}^{-1} \right. \]
\[ + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} \right) U_{\psi\psi} A \]
\[ + \left( (R_{\lambda s} U_{s\psi} + R_{\lambda a} U_{a\psi} + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) U_{\lambda\psi}) U_{\psi\psi}^{-1} \right. \]
\[ + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} \right) U_{\psi\psi} \]
\[ + (R_{\lambda s} U_{s\psi} + R_{\lambda a} U_{a\psi} + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) U_{\lambda\psi}) \langle \psi \rangle_{q(\Psi|\lambda)} \]
\[ + \left( (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \nabla_{\lambda} \left( \langle \psi \rangle_{q(\Psi|\lambda)} - H[q(\Psi|\lambda)] \right) \right), \tag{86} \]

implying that

\[ (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} = - (R_{\lambda s} U_{s\psi} + R_{\lambda a} U_{a\psi} + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) U_{\lambda\psi}) U_{\psi\psi}^{-1}. \tag{87} \]

Unless \( R_{a\lambda} \) and \((\Gamma_{\lambda\lambda} + R_{\lambda\lambda})\) share a common nullspace, or the \( U \) and \( R \) matrices are finely tuned, then Eqs. (85) and (87) contradict one another. In this case, there cannot exist a \( q(\Psi|\lambda) \) that satisfies both Eqs. (67) and (68), and hence the modified Free Energy Lemma is invalid in general. In particular, using the example from Appendix B, if we solve Eq. (85) for \( \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} \) we find

\[ \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} = - \frac{53}{5}, \tag{88} \]

and from Eq. (87) we get

\[ \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} = \frac{29}{239}, \tag{89} \]

which is a contradiction.

If we now perform the same procedure for Eqs. (69) and (70), we arrive at the following conditions on the gradient of the variational density:

\[ R_{a\lambda} \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} = - (R_{aa} + \Gamma_{aa}) U_{a\psi} + R_{a\lambda} U_{\lambda\psi}) U_{\psi\psi}^{-1}. \tag{90} \]

and

\[ R_{a\lambda} \nabla_{\lambda} \langle \psi \rangle_{q(\Psi|\lambda)} = - (R_{aa} U_{a\psi} + (\Gamma_{\lambda\lambda} + R_{\lambda\lambda}) U_{\lambda\psi}) U_{\psi\psi}^{-1}. \tag{91} \]

Even when Condition 2 holds and \( U_{\psi\lambda} = 0 \), these will be inconsistent in general. As a specific counterexample, take the system with force matrix

\[ M = \begin{pmatrix}
-1 & 0 & -\frac{1}{2} & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & \frac{\sqrt{3}}{2} \\
0 & 0 & 0 & -\frac{1}{2}
\end{pmatrix}, \tag{92} \]
with corresponding

\[
R = \begin{pmatrix}
0 & 0 & \frac{1}{4} & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & 0 & -\frac{1}{\sqrt{3}} \\
-\frac{1}{3\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\
\end{pmatrix},
\]

(93)

and

\[
U = \begin{pmatrix}
\frac{1}{10} & 0 & \frac{1}{10} & 0 \\
0 & 1 & 0 & 0 \\
\frac{1}{10} & 0 & \frac{1}{10} & -\frac{\sqrt{3}}{4} \\
0 & 0 & -\frac{\sqrt{3}}{4} & \frac{3}{4} \\
\end{pmatrix}.
\]

(94)

This model is ergodic (full rank \(U\)), and it satisfies both Condition 1 and Condition 2. Moreover, the forces satisfy Eqs. (21) and (22). However, substituting the relevant elements of \(U\) and \(R\) matrices into Eq. (90), we find

\[
\nabla_{\lambda} \langle \psi \rangle_{q(\psi|\lambda)} = \frac{1}{\sqrt{3}},
\]

(95)

but doing the same for Eq. (91) gives

\[
\nabla_{\lambda} \langle \psi \rangle_{q(\psi|\lambda)} = \frac{1}{3},
\]

(96)

which is a contradiction.

### D Counterexample for Step 5

Here we provide an example system for which Conditions 1 and 2 as well as Steps 1 to 4 are valid but Step 5 fails. We use a system with

\[
f(x) = Mx
\]

(97)

where

\[
M := \begin{pmatrix}
-1 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -1 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & -1 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & -1 \\
\end{pmatrix}.
\]

(98)

This system is ergodic, satisfies Condition 1 and as we will see satisfies Eqs. (19) and (20) as well. Using Eq. (5) we find

\[
R = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

(99)

and from Eq. (8)

\[
U = -M
\]

(100)
which means that Condition 2 is also satisfied. This leads to the ergodic density

$$p^*(\psi, s, a, \lambda) = \frac{\sqrt{5}}{16\pi^2} e^{-\frac{1}{2}(\psi^2 - s + \frac{s^2}{2} - sa + a^2 - a\lambda + \lambda^2)}$$

(101)

which can be used to check that Eqs. (19) and (20) hold for this example. The conditional ergodic density is

$$p^*(\psi|s, a, \lambda) = p^*(\psi|s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\psi - \frac{s}{2})^2}.$$ (102)

If we now define \( q(\psi|\lambda) = q(\psi) = \exp(- (\psi - \mu)^2 / 2) / \sqrt{2\pi} \) as a Gaussian distribution with mean \( \mu \) and variance one, we can compute the KL divergence to get:

$$D_{KL}[q(\Psi)||p^*(\Psi|s, a, \lambda)] = K(s) = \frac{1}{2} \left( \mu - \frac{1}{2} s \right)^2.$$ (103)

Clearly, for this choice of \( q(\psi|\lambda) \) the gradients with respect to \( a \) and \( \lambda \) of the KL divergence vanish everywhere (Eqs. (35) and (36) hold). This also means we can express \( f_a, f_\lambda \) in terms of a free energy i.e. the Free Energy Lemma holds for this system. However, for any proposed bound \( c \geq 0 \) on the KL divergence, there is a value of \( s \) for which it is exceeded, whatever the choice of \( \mu \). Moreover, we can choose a \( \mu \) such that the KL divergence is larger than any given \( c \), even when \( s = 0 \).

E Translating systems into generalized coordinates systems

We show how to get a generalized coordinate system from a finite dimensional system. By definition the generalized coordinates are infinite dimensional. For all \( n \in \mathbb{N} \) and a coordinate \( x \) they also include the \( n \)-th time derivative of \( x \).

Assume as given an ergodic, linear, random dynamical system described by

$$\dot{x} = Mx + \omega$$

(104)

where \( x = (x_1, ..., x_k) \) is a \( k \)-dimensional vector, \( M \) is a \( k \times k \) real valued matrix, and \( \dot{x} := \frac{d}{dt}x \). We can look at the second time derivative of the state by differentiating both sides:

$$\frac{d}{dt} \dot{x} = \frac{d}{dt}(Mx + \omega)$$

(105)

$$\ddot{x} = M\dot{x} + \ddot{x}$$

(106)

Similarly for the third time derivative:

$$\frac{d}{dt} \ddot{x} = \frac{d}{dt}(M\dot{x} + \ddot{x})$$

(107)

$$= M\ddot{x} + \dddot{x}$$

(108)
Similarly for all higher derivatives:

\[
\frac{d^n}{dt^n} x = M \frac{d^{n-1}}{dt^{n-1}} x + \frac{d^n}{dt^n} \omega.
\]  

(109)

Now define the generalized coordinates \( \tilde{x} = (x, x', x'', \ldots) \) as

\[
x = x
\]

(110)

\[
x' = \frac{d}{dt} x
\]

(111)

\[
x'' = \frac{d^2}{dt^2} x
\]

(112)

\[\vdots\]

(113)

\[
x^{(n)} = \frac{d^n}{dt^n} x
\]

(114)

\[\vdots\]

(115)

Define also

\[
\tilde{\omega} := \left( \omega, \frac{d}{dt} \omega, \frac{d^2}{dt^2} \omega, \ldots, \frac{d^n}{dt^n} \omega, \ldots \right).
\]

(116)

Without further clarification, the derivatives of \( \omega \) are not well defined when the latter is a Gaussian white noise process, as explicitly assumed in writing the vector field \( f(x) \) in terms of the ergodic density (Ao, 2004; Kwon et al., 2005; Kwon and Ao, 2011). As discussed in van Kampen (1981), delta-correlated Markovian noise is always a limiting approximation of noise with a finite correlation time. Meaningfully taking the derivatives requires first choosing a functional form for the (co)variance whose limit is a delta function.\(^{13}\) However, different choices can lead to vastly different central moments of the generalized noise distribution, including those that vanish or diverge at all orders. In the former case, the process in terms of generalized coordinates may not be ergodic (Cornfeld et al., 1982); in the latter case, the process is not well defined. In general, the Kramers-Moyal coefficients will not vanish beyond second order, meaning that an approach for which there is an equivalent Fokker-Planck equation, and hence Eq. (4), is not valid (Risken, 1996).

Here, we can therefore assume that the noise is such that the derivatives in Eq. (116) can be treated as Markov and Gaussian. We also assume that \( \frac{d^n}{dt^n} \omega \) is independently and identically distributed to \( \frac{d^{n-1}}{dt^{n-1}} \omega \) for all \( n \). Finally, we can then define the (infinite) matrix \( \tilde{M} \) as the block diagonal matrix with all blocks equal to \( M \):

\[
\tilde{M} := \begin{pmatrix}
M & 0 & \cdots \\
0 & M & \cdots \\
\vdots & \ddots & \ddots
\end{pmatrix}
\]

(117)

The time derivative of \( \omega \) is independent of \( \omega \), as the changes are independent of the value of \( \omega \). So we actually get an infinite number of independent and

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13Another, more direct approach would be in terms of generalized functions, but here too additional information is required to specify the derivatives (Oberguggenberger, 1996).
identically distributed systems. Using these definitions we have:

\[ \dot{x} = M\tilde{x} + \tilde{\omega}. \] (118)

These equations describe a random dynamical system composed of an infinite number of independent linear random dynamical systems, all governed by the same matrix \( M \) and driven by independently and identically distributed noise. Since the first of these systems (for the variables \( x \)) is ergodic by assumption, all of the subsystems are also ergodic and, therefore, the whole system is ergodic with the ergodic density equal to a product of the original ergodic density:

\[ \bar{p}^*(\tilde{x}) = p^*(x)p^*(x')p^*(x'')\cdots p^*(x^{(n)})\cdots. \] (119)

Additionally, if \( M \) is such that

\[ M_{\psi} (\psi, s, a, r)^T = f_{\psi}(\psi, s, a) = (\Gamma - Q)_{\psi} \nabla_{\psi} \ln p^*(\psi, s, a, r) \]

\[ M_{s} (\psi, s, a, r)^T = f_{s}(\psi, s, a) = (\Gamma - Q)_{s} \nabla_{s} \ln p^*(\psi, s, a, r) \]

\[ M_{a} (\psi, s, a, r)^T = f_{a}(\psi, s, a) = (\Gamma - Q)_{a} \nabla_{a} \ln p^*(\psi, s, a, r) \]

\[ M_{r} (\psi, s, a, r)^T = f_{r}(\psi, s, a) = (\Gamma - Q)_{r} \nabla_{r} \ln p^*(\psi, s, a, r), \] (120)

(which is the case for the \( M \) in the counterexample to Step 5) then for

\[ (x_1, x_2, x_3, x_4) := (\psi, s, a, r) \]

\[ (x'_1, x'_2, x'_3, x'_4) := (\psi', s', a', r') \]

\[ (x''_1, x''_2, x''_3, x''_4) := (\psi'', s'', a'', r'') \]

\[ \vdots \]

\[ (x^{(n)}_1, x^{(n)}_2, x^{(n)}_3, x^{(n)}_4) := (\psi^{(n)}, s^{(n)}, a^{(n)}, r^{(n)}) \]

\[ \vdots, \] (121)

\[ \bar{Q} := \begin{pmatrix} Q & 0 & \cdots \\ 0 & Q & \cdots \\ \vdots & & \ddots \end{pmatrix}, \] (122)

\[ \bar{\Gamma} := \begin{pmatrix} \Gamma & 0 & \cdots \\ 0 & \Gamma & \cdots \\ \vdots & & \ddots \end{pmatrix}, \] (123)

and using Eq. (8) and that the inverse of a block diagonal matrix is block diagonal

\[ \bar{U} := \begin{pmatrix} U & 0 & \cdots \\ 0 & U & \cdots \\ \vdots & & \ddots \end{pmatrix}, \] (124)
we also have:

\[
\begin{align*}
\mathcal{M}_\psi \cdot (\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r})^\top &= f_\psi(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) = (\bar{\Gamma} - \bar{Q})_{\psi\psi} \nabla_\psi \ln p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) \\
\mathcal{M}_s \cdot (\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r})^\top &= f_s(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) = (\bar{\Gamma} - \bar{Q})_{\psi s} \nabla_s \ln p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) \\
\mathcal{M}_a \cdot (\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r})^\top &= f_a(\tilde{s}, \tilde{a}, \tilde{r}) = (\bar{\Gamma} - \bar{Q})_{aa} \nabla_a \ln p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) \\
\mathcal{M}_r \cdot (\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r})^\top &= f_r(\tilde{s}, \tilde{a}, \tilde{r}) = (\bar{\Gamma} - \bar{Q})_{rr} \nabla_r \ln p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}).
\end{align*}
\]

(125)

The ergodic density of such a system is a product of ergodic densities of the original system Eq. (102):

\[
p^*(\tilde{\psi}, \tilde{s}, \tilde{a}, \tilde{r}) = p^*(\psi, s, a, r)p^*(\psi', s', a', r')p^*(\psi'', s'', a'', r'') \cdots.
\]

(126)

Thus any property of the original system is also a property of the generalized coordinate system.

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