A CLT for the $L^2$ norm of increments of local times of Lévy processes as time goes to infinity

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September 6, 2009

Abstract

Let $X = \{X_t, t \in R_+\}$ be a symmetric Lévy process with local time $\{L^x_t; (x, t) \in R^1 \times R^+_1\}$. When the Lévy exponent $\psi(\lambda)$ is regularly varying at zero with index $1 < \beta \leq 2$, and satisfies some additional regularity conditions,

$$\frac{\int_{-\infty}^{\infty} (L^x_{t+1} - L^x_t)^2 \, dx - E(\int_{-\infty}^{\infty} (L^x_{t+1} - L^x_t)^2 \, dx)}{t^{\psi^{-1}(1/t)}} \quad (0.1)$$

$$\Rightarrow (8c_{\psi,1})^{1/2} \left(\int_{-\infty}^{\infty} (L^x_{\beta,1})^2 \, dx\right)^{1/2} \eta$$

as $t \to \infty$, where $L^x_{\beta,1} = \{L^x_{\beta,1}; x \in R^1\}$ denotes the local time, at time 1, of a symmetric stable process with index $\beta$, $\eta$ is a normal random variable with mean zero and variance one that is independent of $L^x_{\beta,1}$, and $c_{\psi,1}$ is a known constant that depends on $\psi$.

*The research of both authors was supported, in part, by grants from the National Science Foundation and PSC-CUNY.

Key words and phrases: Central Limit Theorem, $L^2$ norm of increments, local time, Lévy process.

AMS 2000 subject classification: Primary 60F05, 60J55, 60G51.
1 Introduction

The earliest result we know about the asymptotic behavior in time, of increments of local times in the spatial variable, is due to Dobrushin, [6]. Let \( \{S_n; n = 0, 1, 2, \ldots\} \) be a simple random walk on \( \mathbb{Z}^1 \) and let \( \ell^x_n = \sum_{j=1}^n 1\{s_j=x\} \) denote its the local time. Dobrushin shows that

\[
\frac{\ell^1_n - \ell^0_n}{n^{1/4}} \Rightarrow (2\|Z\|)^{1/2} \eta
\]

as \( n \to \infty \), where \( Z \) and \( \eta \) are independent normal random variables with mean zero and variance one. Two aspects of this result are relevant to this paper. One is that, since \( \ell^0_n \) grows like \( n^{1/2} \), see for example [18, (10.1), (9.13)], (1.1) is a result about fluctuations. The other is that the right-hand side of (1.1) is not a standard normal random variable, but is the product of a standard normal random variable and an independent random variable. Extensions of (1.1) to the local time of Brownian motion and other processes can be found in Révész, [18, (11.10), (12.17), (12.19)], Marcus and Rosen, [14, 15], Rosen, [19] and Yor, [21].

Our motivation for considering the increments of local times is our interest in the Hamiltonian for the critical attractive random polymer in one dimension, [8, 9],

\[
H_n = \sum_{x \in \mathbb{Z}^1} (\ell^{x+1}_n - \ell^x_n)^2,
\]

where \( \ell^x_n \) is as defined above. This is the square of the \( \ell^2 \) norm of the increments of the local time at time \( n \).

We began our study of expressions like (1.2) in [4], with X. Chen and W. Li, by considering the continuous version of this problem for Brownian motion. Let \( \{L^x_t; (x, t) \in \mathbb{R}^1 \times \mathbb{R}_+^1\} \) denote the local times of Brownian motion. We show that

\[
\frac{\int_{-\infty}^\infty (L^x_{t+1} - L^x_t)^2 \, dx - 4t}{t^{3/4}} \Rightarrow (64/3)^{1/2} \left( \int_{-\infty}^\infty (L^x_1)^2 \, dx \right)^{1/2} \eta \quad (1.3)
\]

as \( t \to \infty \), where \( \eta \) is a normal random variable with mean zero and variance one that is independent of \( \{L^x_t; x \in \mathbb{R}^1\} \). The proof in [4] makes extensive use of the scaling property of Brownian motion. A different proof in [20] uses stochastic integrals and the
theorem of Papanicolaou, Stroock, and Varadhan, [17, Chapter XIII]. Neither of these approaches can be used to extend (1.3) to general Lévy processes. We do that in this paper using the method of moments.

Let \( X_t, t \in R_+ \) be a symmetric Lévy process with characteristic function
\[
E \left( e^{i\lambda X_t} \right) = e^{-\psi(\lambda)t},
\]
(1.4)
and local time which we continue to denote by \( \{L^x_t; (x,t) \in R^1 \times R^1_+\} \). The behavior of a suitably scaled version of \( \int (L^{x+1}_t - L^x_t)^2 dx \) as \( t \) goes to infinity depends primarily on the behavior of \( \psi(\lambda) \) as \( \lambda \) goes to 0. This is not surprising, since large time properties of \( X \) such as transience and recurrence depend on the behavior of \( \psi(\lambda) \) as \( \lambda \) goes to 0; (see [1, Chapter 1, Theorem 17], which shows, in particular, that the processes we consider are recurrent.)

We assume that \( \psi(\lambda) \) satisfies the following conditions:

1. \( \psi(\lambda) \) is regularly varying at 0 with index \( 1 < \beta \leq 2 \); (1.5)

2. \[ \int_{-\infty}^{\infty} \frac{1}{1 + \psi(\lambda)} d\lambda < \infty; \]

3. \( \psi \) is twice differentiable almost everywhere, and there exist constants \( D_1, D_2 < \infty \) such that for \( 0 < \lambda \leq 1 \)
\[
\lambda |\psi'(\lambda)| \leq D_1 \psi(\lambda) \quad \text{and} \quad \lambda^2 |\psi''(\lambda)| \leq D_2 \psi(\lambda)
\]
(1.7)
and
\[
\int_{1}^{\infty} \frac{|\psi'(\lambda)|}{\psi(\lambda)^2} d\lambda < \infty, \quad \int_{1}^{\infty} \frac{|\psi'(\lambda)|^2}{\psi(\lambda)^2} d\lambda < \infty, \quad \int_{1}^{\infty} \frac{|\psi'''(\lambda)|}{\psi(\lambda)} d\lambda < \infty.
\]
(1.8)
(Condition 1. is substantive. Condition 2. is the necessary and sufficient condition for a symmetric Lévy process to have a local time. The criteria in Condition 3. are rather weak.)

We prove the following theorem:

**Theorem 1.1** Let \( \{L^x_t; (x,t) \in R^1 \times R^1_+\} \) be the local time of a symmetric Lévy process \( X \), with Lévy exponent \( \psi(\lambda) \), that is regularly varying at
zero with index $1 < \beta \leq 2$ and satisfies (1.6)–(1.8). Then
\[
\int_{-\infty}^{\infty}(L_{t}^{x+1} - L_{t}^{x})^{2} dx - E\left(\int_{-\infty}^{\infty}(L_{t}^{x+1} - L_{t}^{x})^{2} dx\right)
t \sqrt{\psi^{-1}(1/t)}
\xrightarrow{\mathcal{L}} (8c_{\psi,1})^{1/2} \left(\int_{-\infty}^{\infty}(L_{\beta,1}^{x})^{2} dx\right)^{1/2} \eta
\]
as $t \to \infty$, where $L_{\beta,1} = \{L_{\beta,1}^{x}; x \in \mathbb{R}\}$ is the local time, at time 1, of a symmetric stable process of index $\beta$, $\eta$ and $L_{\beta,1}$ are independent, and
\[
c_{\psi,1} = \frac{16}{\pi} \int_{0}^{\infty} \sin^{4}(p/2) \psi(p) dp.
\]  
(1.10)
Since $\psi$ is regularly varying at zero, it is asymptotic to a monotonic function at zero. We define $\psi^{-1}$ as the inverse of this function.)
It follows from Lemma 5.1, in this paper, that
\[
E\left(\int_{-\infty}^{\infty}(L_{t}^{x+1} - L_{t}^{x})^{2} dx\right) = 4c_{\psi,0}t + o\left(t\sqrt{\psi^{-1}(1/t)}\right),
\]  
(1.11)
where
\[
c_{\psi,0} := \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin^{2}(p/2)}{\psi(p)} dp.
\]  
(1.12)
Therefore, we can replace the mean in (1.9) by $4c_{\psi,0}t$.
Note that by Lemma 2.4, $\int_{-\infty}^{\infty}(L_{t}^{x})^{2} dx$ grows like $t^{2}\psi^{-1}(1/t)$, therefore (1.9) is also a fluctuation result.
In Remark 2.1 we evaluate the constants and make the necessary changes to verify that when $X$ is Brownian motion, (1.9) along with (1.11), is the same as (1.3).

The methods developed to prove Theorem 1.1 also give a Central Limit Theorem for the Hamiltonian of the critical attractive random polymer in one dimension, which is usually written as
\[
H_{n} = 2 \sum_{i,j=1}^{n} 1\{S_{i}=S_{j}\} - \sum_{i,j=1}^{n} 1\{|S_{i}-S_{j}|=1\}.
\]  
(1.13)
It is easy to see that this is the same as (1.2). In [16] we use the methods of this paper to show that
\[
\frac{H_{n} - 4n}{n^{3/4}} \xrightarrow{\mathcal{L}} (12)^{1/2} \left(\int_{-\infty}^{\infty}(L_{t}^{x})^{2} dx\right)^{1/2} \eta
\]  
(1.14)
as $n \to \infty$, where $\{L_{x}^{1}, x \in \mathbb{R}^1\}$ is the local time of Brownian motion at time 1. A similar formula holds for symmetric random walks with variance 1. In fact in [16] we obtain central limit theorems like (1.14) for a large class of random walks in the domain of attraction of stable processes of index $1 < \beta \leq 2$. (The conditions are the ones in (1.5)–(1.8) with obvious modifications.)

Another application of the techniques used to prove Theorem 1.1 is suggested by applying the scaling relationship for the local times of $\beta$ stable processes,

$$\{L_{\beta,t/\delta}^{x}; (x,t) \in \mathbb{R}^1 \times \mathbb{R}^1_{+}\} \overset{\mathcal{L}}{\Rightarrow} \{\delta^{-(\beta-1)} L_{\beta,t}^{x}; (x,t) \in \mathbb{R}^1 \times \mathbb{R}^1_{+}\},$$

(1.15)

(see e.g. [11, Lemma 10.5.2]), to (1.9). This gives

$$\frac{\int_{-\infty}^{\infty} (L_{\beta,1}^{x+h} - L_{\beta,1}^{x})^2 \, dx - 4c_{|\beta|,0} h^{\beta-1}}{h^{(2\beta-1)/2}} \overset{\mathcal{L}}{\Rightarrow} (8c_{|\beta|,1})^{1/2} \left( \int_{-\infty}^{\infty} (L_{\beta,1}^{x})^2 \, dx \right)^{1/2} \eta,$$

as $h \to 0$.

One can not use scaling to obtain limits like (1.16) for more general Lévy processes. In [13] we do this by modifying the methods developed in this paper. We show that when $\psi(\lambda)$ is regularly varying at infinity, with index $1 < \beta \leq 2$, and satisfies some additional regularity conditions,

$$\sqrt{h} \psi^2(1/h) \left\{ \int_{-\infty}^{\infty} (L_{1}^{x+h} - L_{1}^{x})^2 \, dx - E \left( \int_{-\infty}^{\infty} (L_{1}^{x+h} - L_{1}^{x})^2 \, dx \right) \right\} \overset{\mathcal{L}}{\Rightarrow} (8c_{|\beta|,1})^{1/2} \left( \int_{-\infty}^{\infty} (L_{1}^{x})^2 \, dx \right)^{1/2} \eta,$$

as $h \to 0$. It is not surprising that this limit depends on the behavior of $\psi(\lambda)$ as $\lambda$ goes to infinity, since the behavior of $\psi(\lambda)$ as $\lambda$ goes to infinity controls the small jumps of $X$.

In Section 2 we give the proof of Theorem 1.1 based on four lemmas. The remainder of the paper is devoted to the proof of these lemmas. The critical ingredient in the proof of Theorem 1.1 is Lemma 4.1 which gives moments for the $L^2$ norm of increments of local times of Lévy processes satisfying (1.5)–(1.8). In Section 3 we state Lemmas 3.1–3.3 which give properties of $p_t(x)$, $\Delta^1 p_t(x)$ and $\Delta^1 \Delta^{-1} p_t(x)$, where $p_t(x)$ is the transition
probability density of these Lévy processes. The lemmas are used in Section 4 in the proof of Lemma 4.1. They are proved in Section 6. In the proof of Theorem 1.1 we need good asymptotic estimates of the mean and variance of $\int_{-\infty}^{\infty} (L_t^{x+h} - L_t^{x})^2 \, dx$. These are stated in Section 2 and proved in Section 8. In Section 9 a short appendix, we derive the form of the Kac Moment Formula used in this paper.

We anticipate that the estimates in Section 3 of properties of the probability densities of a wide class of Lévy processes, will be useful in the study of Lévy processes and their local times.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 is long and difficult. In order to make it easier to follow we first present the main steps of the proof heuristically. We then restate them, precisely, in a series of lemmas and show how Theorem 1.1 follows from these lemmas. These lemmas are proved in Sections 3 through 8.

Let

\[ I_{j,k,t} := \int (L_t^{x+1} - L_t^{x}) \circ \theta_{jt} (L_t^{x+1} - L_t^{x}) \circ \theta_{kt} \, dx \]  

(2.1)

and

\[ \alpha_{j,k,t} := \int L_t^{x} \circ \theta_{jt} L_t^{x} \circ \theta_{kt} \, dx. \]  

(2.2)

(An integral sign without limits is to be read as $\int_{-\infty}^{\infty}$.)

For any integer $l$ set

\[ \tilde{I}_{l,t} := \sum_{\substack{j,k=0 \atop j<k}}^{l-1} I_{j,k,t/l}. \]  

(2.3)

Using the additivity property of local times we can write

\[ L_t^{x} = \sum_{j=0}^{l-1} L_{t/l}^{x} \circ \theta_{jt/l}, \]  

(2.4)

so that

\[ \int (L_t^{x+1} - L_t^{x})^2 \, dx = \sum_{\substack{j,k=0 \atop j<k}}^{l-1} I_{j,k,t/l} = 2\tilde{I}_{l,t} + \sum_{j=0}^{l-1} I_{j,j,t/l}. \]  

(2.5)
Consequently
\[\int (L_t^{x+1} - L_t^x)^2 dx - E\left(\int (L_t^{x+1} - L_t^x)^2 dx\right) = 2\tilde{I}_{l,t} + \left\{\sum_{j=0}^{l-1} I_{j,j,t/l} - E\left(\int (L_t^{x+1} - L_t^x)^2 dx\right)\right\}.\] (2.6)

Similarly we set
\[\tilde{\alpha}_{l,t} = \sum_{j<k}^{l-1} \alpha_{j,k,t/l},\] (2.7)
and write
\[\alpha_t := \int (L_t^x)^2 dx = \sum_{j,k=0}^{l-1} \alpha_{j,k,t/l} = 2\tilde{\alpha}_{l,t} + \sum_{j=0}^{l-1} \alpha_{j,j,t/l}.\] (2.8)

The main steps in the proof of Theorem 1.1 are to show that:

1. The ‘off-diagonal’ terms \(\tilde{I}_{l,t}\) and \(\sqrt{\tilde{\alpha}_{l,t}}\) are comparable asymptotically as \(t \to \infty\).

2. The diagonal term \(\sum_{j=0}^{l-1} \alpha_{j,j,t/l}\) is negligible, as \(t \to \infty\), compared to the terms in 1.

3. The diagonal term \(\sum_{j=0}^{l-1} I_{j,j,t/l}\) is such that
\[\sum_{j=0}^{l-1} I_{j,j,t/l} - E\left(\int (L_t^{x+1} - L_t^x)^2 dx\right)\] (2.9)
is negligible, as \(t \to \infty\), compared to the terms in 1.

We now explain the precise meaning of these statements, and show how they imply Theorem 1.1.

The precise meaning of step 1. is given by the following lemma. (Lemmas 2.1 2.3 are proved in Section 5.)

**Lemma 2.1** Under the hypotheses of Theorem 1.1, for each \(m\), with \(l = l(t) = [\log t]q\), for any \(q > 0\),
\[E\left((\tilde{I}_{l,t})^m\right) = \begin{cases} \frac{(2n)!}{2^n n!} (4c_{\psi,1})^n E\left\{(\tilde{\alpha}_{l,t})^n\right\} + o((t^2 \psi^{-1}(1/t))^n) & m = 2n \\ O((t^2 \psi^{-1}(1/t))^{m/2} t^{-\epsilon}) & \text{otherwise.} \end{cases}\] (2.10)
This lemma is the crux of this paper. We note that even though the summands \( I_{j,k,t/l} \) of \( \tilde{I}_{l,t} \) are not independent, the fact that \( j \neq k \) provides enough structure for a proof.

The precise meaning of step 2. is given by the next lemma.

**Lemma 2.2** Under the hypotheses of Theorem 1.1, for each \( n \), with \( l = l(t) = \lfloor \log t \rfloor \), for any \( q > 0 \),

\[
\lim_{t \to \infty} \frac{|E(2\tilde{\alpha}_{l,t})^n - E(\alpha_t)^n|}{(t^{2\psi^{-1}(1/t)})^n} = 0. \tag{2.11}
\]

Lastly, the precise meaning of step 3. is given by the next lemma.

**Lemma 2.3** Under the hypotheses of Theorem 1.1 with \( l = l(t) = \lfloor \log t \rfloor \), for \( q \) sufficiently large,

\[
\lim_{t \to \infty} \sum_{j=0}^{l-1} \left( I_{j,j,t/l} - E \left( f(\int_{l_t}^{x+1/l} - L_t^x) \right)^2 \right) = 0. \tag{2.12}
\]

in \( L^2 \).

We also need to know the limiting behavior of the moments of \( \alpha_t \). It is given by the next lemma which is proved in Section 7.

**Lemma 2.4** Under the hypotheses of Theorem 1.1 for each \( n \),

\[
\lim_{t \to \infty} E \left\{ \left( \frac{\alpha_t}{t^{2\psi^{-1}(1/t)}} \right)^n \right\} = E \{ (\alpha_{\beta,1})^n \}. \tag{2.13}
\]

**Proof of Theorem** 1.1 In (2.10) replace \( \tilde{I}_{l,t} \) by \( 2\tilde{I}_{l,t} \) and \( (4c_{\psi,1})^n E \{ (\tilde{\alpha}_{l,t})^n \} \) by \( (8c_{\psi,1})^n E \{ (2\tilde{\alpha}_{l,t})^n \} \). Then set

\[
E \{ (2\tilde{\alpha}_{l,t})^n \} = E(\alpha_t)^n + E \{ (2\tilde{\alpha}_{l,t})^n \} - E(\alpha_t)^n. \tag{2.14}
\]

Then use Lemmas 2.2 and 2.4 to see that for each integer \( m \)

\[
\lim_{t \to \infty} E \left( \left( \frac{2\tilde{I}_{l,t}}{t^{\psi^{-1}(1/t)}} \right)^m \right) = \begin{cases} \frac{(2n)!}{2^n n!} (8c_{\psi,1})^n E \{ (\alpha_{\beta,1})^n \} & \text{if } m = 2n \\ 0 & \text{otherwise.} \end{cases} \tag{2.15}
\]
Note that the right-hand side of (2.15) is the $2n$–th moment of $(8c_{\psi,1})^{1/2} \sqrt{\alpha_{1,1}} \eta$ when $\alpha_{1,1}$ and $\eta$ are independent. Furthermore, it follows from [5] (6.12) that
\[ E(\alpha_{1,1})^n \leq C_n((2n)!)^{1/(2\beta)}. \] (2.16)
Consequently, since $\sqrt{(2n)!} \leq 2^n n!$
\[ E\left((8c_{\psi,1})^{1/2} \sqrt{\alpha_{1,1}} \eta\right)^m \leq C_n^m(m!)^{(\beta+1)/(2\beta)}. \] (2.17)
This implies that $(8c_{\psi,1})^{1/2} \sqrt{\alpha_{1,1}} \eta$ is determined by its moments; (see [7, p. 227-228]). Therefore, by the method of moments, [2, Theorem 30.2]), it follows from (2.15) that
\[ \lim_{t \to \infty} \frac{2\tilde{I}_{l,t}}{t^{\psi^{-1}(1/t)}} \overset{c}{\to} (8c_{\psi,1})^{1/2} \sqrt{\alpha_{1,1}} \eta. \] (2.18)

Theorem 1.1 then follows from (2.6), (2.18) and Lemma 2.3.

**Remark 2.1** For Brownian motion $\psi(p) = p^2/2$. We have
\[ c_{p^2/2,0} = \frac{4}{\pi} \int_0^\infty \frac{\sin^2(p/2)}{p^2} dp = 1; \] (2.19)
\[ c_{p^2/2,1} = \frac{64}{\pi} \int_0^\infty \frac{\sin^4 p/2}{p^4} dp = \frac{8}{3}. \] (2.20)

Also $L_{2,1}^x \overset{c}{=} (1/2)L_2^x \overset{c}{=} (1/\sqrt{2})L_1^{x/\sqrt{2}}$ where $\{L_t^x\}$ denotes the local time of Brownian motion, so that
\[ \left(\int_{-\infty}^\infty (L_{2,1}^x)^2 \, dx\right)^{1/2} = \frac{1}{2^{1/4}} \left(\int_{-\infty}^\infty (L_t^x)^2 \, dx\right)^{1/2}. \] (2.21)

Since $\sqrt{\psi^{-1}(1/t)} = 2^{1/4}/t^{1/4}$, we see that (1.9) and (1.11) imply (1.3).

### 3 Estimates for the probability densities of certain Lévy processes

Let $p_s(x)$ denote the density of the symmetric Lévy process $X$ with Lévy exponent $\psi(\lambda)$ as described in (1.4). Let $\Delta^\gamma_x$ denote the finite difference operator on the variable $x$, i.e.
\[ \Delta^\gamma_x f(x) = f(x + \gamma) - f(x). \] (3.1)
We write $\Delta^\gamma$ for $\Delta_x^\gamma$ when the variable $x$ is clear.

Let

\[
\begin{align*}
  u(x, t) & := \int_0^t p_s(x) \, ds \quad (3.2) \\
v_s(x, t) & := \int_0^t |\Delta^\gamma p_s(x)| \, ds \quad (3.3) \\
w_s(x, t) & := \int_0^t |\Delta^\gamma \Delta^{-\gamma} p_s(x)| \, ds \quad (3.4)
\end{align*}
\]

We also write $v(x, t)$ for $v_1(x, t)$ and $w(x, t)$ for $w_1(x, t)$.

The following lemmas provide the main estimates we use in this paper. Their proofs are given in Section 6.

**Lemma 3.1** Under the hypotheses of Theorem 1.1, for all $t$ sufficiently large

\[
\begin{align*}
  \sup_{x \in \mathbb{R}^1} u(x, t) & \leq Ct \psi^{-1}(1/t); \\
  \sup_{x \in \mathbb{R}^1} v(x, t) & \leq C \log t; \\
  \sup_{x \in \mathbb{R}^1} w(x, t) & \leq C,
\end{align*}
\]

and

\[
\begin{align*}
  \int u(x, t) \, dx & = t; \\
  \int v(x, t) \, dx & \leq Ct \left( \psi^{-1}(1/t) \right) \log t; \\
  \int w(x, t) \, dx & \leq C(\log t)^2; \\
  \int w^2(x, t) \, dx & \leq C \log t; \\
  \int_{|x| \geq u} w^2(x, t) \, dx & \leq C \frac{(\log t)^2}{u}.
\end{align*}
\]

**Lemma 3.2** Under the hypotheses of Theorem 1.1, for all $t$ sufficiently large

\[
\begin{align*}
  |\Delta^3 p_t(0)| & \leq C \left( \psi^{-1}(1/t) \right)^3,
\end{align*}
\]

and

\[
\begin{align*}
  \int_0^{2t} \int_0^{2t} |\Delta^3 p_{r+s}(0)| \, dr \, ds & \leq C \left( t^2 \left( \psi^{-1}(1/t) \right)^3 + L(t) + 1 \right),
\end{align*}
\]
where \( L(t) \) is a slowly varying function at infinity.

**Lemma 3.3** Under the hypotheses of Theorem 1.1

\[
\int_0^\infty \Delta^1 p_s(0) \, ds = -c_{\psi,0}; \quad (3.15)
\]
\[
\int \left( \int_0^\infty \Delta^1 \Delta^{-1} p_s(x) \, ds \right)^2 \, dx = c_{\psi,1}; \quad (3.16)
\]

and

\[
\int \left( \int_0^t \Delta^1 \Delta^{-1} p_s(x) \, ds \right)^2 \, dx = c_{\psi,1} + O(t^{-1/3}), \quad (3.17)
\]
as \( t \to \infty \).

4 Moments of increments of local times.

The goal of this section is to establish the following lemma, in which \( I_{j,k,t} \) and \( \alpha_{j,k,t} \) are defined in (2.1) and (2.2).

**Lemma 4.1** Let \( m_{j,k}, 0 \leq j < k \leq K \) be positive integers with \( \sum_{j,k=0,j<k}^K m_{j,k} = m \). If all the integers \( m_{j,k} \) are even, then for some \( \epsilon > 0 \)

\[
E \left( \prod_{j,k=0 \atop j<k}^K (I_{j,k,t})^{m_{j,k}} \right) = \prod_{j,k=0 \atop j<k}^K \frac{(2n_{j,k})!}{2^{n_{j,k}}(n_{j,k})!} (4c_{\psi,1})^{n_{j,k}} E \left( \prod_{j,k=0 \atop j<k}^K (\alpha_{j,k,t})^{n_{j,k}} \right) + O \left( t^{(2-1/\beta)n-\epsilon} \right),
\]

where \( n_{j,k} = m_{j,k}/2 \).

If any of the \( m_{j,k} \) are odd, then

\[
E \left( \prod_{j,k=0 \atop j<k}^K (I_{j,k,t})^{m_{j,k}} \right) = O \left( t^{(2-1/\beta)m/2-\epsilon} \right). \quad (4.2)
\]

In (4.1) and (4.2) the error terms may depend on \( m \), but not on the individual terms \( m_{j,k} \).
Proof We can write

\[
E \left( \prod_{j,k=0}^{K} (I_{j,k,t})^{m_{j,k}} \right) \tag{4.3}
\]

\[
= E \left( \prod_{j,k=0}^{K} \prod_{j<k}^{m_{j,k}} \left( \left( \Delta_{x_{j,k,i}}^{1} L_{t}^{x_{j,k,i}} \circ \theta_{jt} \right) \left( \Delta_{x_{j,k,i}}^{1} L_{t}^{x_{j,k,i}} \circ \theta_{kt} \right) \right) \right)
\]

\[
= \int \left\{ \prod_{j,k=0}^{K} m_{j,k} \prod_{j<k}^{1} \left( \Delta_{x_{j,k,i}}^{1} \right) \right\} \left( \prod_{j,k=0}^{K} \prod_{j<k}^{1} \left( \left( L_{t}^{x_{j,k,i}} \circ \theta_{jt} \right) \left( L_{t}^{x_{j,k,i}} \circ \theta_{kt} \right) \right) \right) \prod_{j,k=0}^{K} \prod_{j<k}^{1} dx_{j,k,i},
\]

where the notation \( \Delta_{x_{j,k,i}}^{1} \) indicates that we apply the difference operator \( \Delta_{x_{j,k,i}}^{1} \) to \( L_{t}^{x_{j,k,i}} \circ \theta_{jt} \). Note that there are \( 2m \) applications of the difference operator \( \Delta^{1} \).

Consider

\[
E \left( \prod_{j,k=0}^{K} \prod_{j<k}^{m_{j,k}} \left( \left( L_{t}^{x_{j,k,i}} \circ \theta_{jt} \right) \left( L_{t}^{x_{j,k,i}} \circ \theta_{kt} \right) \right) \right). \tag{4.4}
\]

We collect all the factors containing \( \theta_{lt} \) and write

\[
E \left( \prod_{j,k=0}^{K} \prod_{j<k}^{m_{j,k}} \left( \left( L_{t}^{x_{j,k,i}} \circ \theta_{jt} \right) \left( L_{t}^{x_{j,k,i}} \circ \theta_{kt} \right) \right) \right) \tag{4.5}
\]

\[
= E \left( \prod_{l=0}^{K} \left\{ \prod_{j=0}^{l-1} \prod_{i=1}^{m_{j,l}} L_{t}^{x_{j,l,i}} \right\} \left( \prod_{k=l+1}^{K} \prod_{i=1}^{m_{l,k}} L_{t}^{x_{l,k,i}} \right) \circ \theta_{lt} \right)
\]

\[
= E \left( \prod_{l=0}^{K} H_{l} \circ \theta_{lt} \right),
\]

where

\[
H_{l} = \left( \prod_{j=0}^{l-1} \prod_{i=1}^{m_{j,l}} L_{t}^{x_{j,l,i}} \right) \left( \prod_{k=l+1}^{K} \prod_{i=1}^{m_{l,k}} L_{t}^{x_{l,k,i}} \right). \tag{4.6}
\]
By the Markov property
\[
E \left( \prod_{l=0}^{K} H_l \circ \theta_{lt} \right) = E \left( H_0 E^{X_t} \left( \prod_{l=1}^{K} H_l \circ \theta_{(l-1)t} \right) \right). \tag{4.7}
\]

Let
\[
m_l = \sum_{k=l+1}^{K} m_{l,k} + \sum_{j=0}^{l-1} m_{j,l}, \quad l = 0, \ldots, K - 1, \tag{4.8}
\]
and note that \(m_l\) is the number of local time factors in \(H_l\).

Let
\[
f(y) = E^y \left( \prod_{l=1}^{K} H_l \circ \theta_{(l-1)t} \right). \tag{4.9}
\]

It follows from Kac’s Moment Formula, Theorem 9.1, that for any \(z \in R^1\)
\[
E^z \left( \prod_{l=0}^{K} H_l \circ \theta_{lt} \right) = E^z \left( H_0 f(X_t) \right) \tag{4.10}
\]
\[
= \sum_{\pi_0} \int_{\{ \sum_{q=1}^{m_0} r_{0,q} \leq t \}} \prod_{q=1}^{m_0} p_{r_{0,q}}(x_{\pi_0(1)} - z) \pi_0 \prod_{q=2}^{m_0} p_{r_{0,q}}(x_{\pi_0(q)} - x_{\pi_0(q-1)})
\]
\[
\left( \int p_{t-\sum_{q=1}^{m_0} r_{0,q}}(y - x_{\pi_0(m_0)}) f(y) \, dy \right) \prod_{q=1}^{m_0} dr_{0,q},
\]
where the sum runs over all bijections \(\pi_0\) from \([1, m_0]\) to
\[
I_0 = \bigcup_{k=1}^{K} \{ (0, k, i), 1 \leq i \leq m_{0,k} \}. \tag{4.11}
\]

Clearly, \(I_0\) is the set of subscripts of the terms \(x_i\) appearing in the local time factors in \(H_0\).

By the Markov property
\[
f(y) = E^y \left( H_1 E^{X_{2t}} \left( \prod_{l=2}^{K} H_l \circ \theta_{(l-2)t} \right) \right) \tag{4.12}
\]
\[
:= E^y (H_1 g(X_{2t})).
\]
Therefore, by (4.11)–(4.12), for any \( z' \in \mathbb{R}^1 \)

\[
E^{z'} \left( \prod_{i=0}^{K} H_i \circ \theta_{it} \right) = E^{z'} \left( H_0 E^{X_t} (H_1 g(X_{2t})) \right)
\]

\[
= \sum_{\pi} \int_{\{ \sum_{q=1}^{m_0} r_{0,q} \leq t \}} \left( \prod_{q=1}^{m_0} P_{r_{0,q}} \left( x_{\pi_0(1)} - z' \right) \prod_{q=2}^{m_0} P_{r_{0,q}} \left( x_{\pi_0(q)} - x_{\pi_0(q-1)} \right) \prod_{q=1}^{m_0} d r_{0,q} \right) \int_{\{ \sum_{q=1}^{m_1} r_{1,q} \leq t \}} \left( \prod_{q=1}^{m_1} P_{r_{1,q}} \left( x_{\pi_1(1)} - y \right) \prod_{q=2}^{m_1} P_{r_{1,q}} \left( x_{\pi_1(q)} - x_{\pi_1(q-1)} \right) \prod_{q=1}^{m_1} d r_{1,q} \right) d y 
\]

where the second sum runs over all bijections \( \pi_1 \) from \([1, m_1]\) to

\[
I_1 = \{(0, 1, i), 1 \leq i \leq m_{0,1}\} \cup \{(1, k, i), 1 \leq i \leq m_{1,k}\} \quad \quad (4.14)
\]

As above, \( I_1 \) is the set of subscripts of the terms \( x \). appearing in the local time factors in \( H_1 \).

We now use the Chapman-Kolmogorov equation to integrate with respect to \( y \) to get

\[
E^{z'} \left( H_0 E^{X_t} (H_1 g(X_{t})) \right) \quad \quad (4.15)
\]

\[
= \sum_{\pi_0, \pi_1} \int_{\{ \sum_{q=1}^{m_0} r_{0,q} \leq t \}} \left( \prod_{q=1}^{m_0} P_{r_{0,q}} \left( x_{\pi_0(1)} - z' \right) \prod_{q=2}^{m_0} P_{r_{0,q}} \left( x_{\pi_0(q)} - x_{\pi_0(q-1)} \right) \prod_{q=1}^{m_0} d r_{0,q} \right) \int_{\{ \sum_{q=1}^{m_1} r_{1,q} \leq t \}} \left( \prod_{q=1}^{m_1} P_{r_{1,q}} \left( x_{\pi_1(1)} - y \right) \prod_{q=2}^{m_1} P_{r_{1,q}} \left( x_{\pi_1(q)} - x_{\pi_1(q-1)} \right) \prod_{q=1}^{m_1} d r_{1,q} \right) d y
\]
Iterating this procedure, and recalling (4.5) we see that

\[
E \left( \prod_{j,k=0}^{K} \prod_{i=1}^{m_{j,k}} \left( (L_{t}^{x_{j,k,i} \circ \theta_{jt}}) (L_{t}^{x_{j,k,i} \circ \theta_{kt}}) \right) \right) \tag{4.16}
\]

\[
= \sum_{\pi_{0},\ldots,\pi_{K}} \prod_{l=0}^{K} \int_{\{\sum_{q=1}^{m_{l}} r_{l,q} \leq t\}} \sum_{m_{l}=2}^{m_{l}} p_{t-\sum_{q=1}^{m_{l}-1} r_{l-1,q}}(x_{\pi_{l}(1)} - x_{\pi_{l-1}(m_{l-1})}) \prod_{q=1}^{m_{l}} dr_{l,q},
\]

where \(\pi_{-1}(m_{-1}) := 0\) and \(1 - \sum_{q=1}^{m_{l}-1} r_{l-1,q} := 0\). In (4.16) the sum runs over all \(\pi_{l}\) such that each \(\pi_{l}\) is a bijection from \([1, m_{l}]\) to

\[
I_{l} = \bigcup_{j=0}^{l-1} \{(j, l, i), 1 \leq i \leq m_{j,l}\} \bigcup_{k=l+1}^{K} \{(l, k, i), 1 \leq i \leq m_{l,k}\}. \tag{4.17}
\]

As in the observations about \(I_{0}\) and \(I_{1}\), we see that \(I_{l}\) is the set of subscripts of the terms \(x_{\cdot}\) terms appearing in the local time factors in \(H_{l}\). Since there are \(2 m\) local time factors we have that \(\sum_{l=0}^{K} m_{l} = 2m\).

We now use (4.16) in (4.3) and continue to develop an expression for the left-hand side of (4.3). Let \(\mathcal{B}\) to denote the set of \((K + 1)\)–tuples, \(\pi = (\pi_{0}, \ldots, \pi_{K})\), of bijections described in (4.17). Clearly

\[
|\mathcal{B}| = \prod_{l=0}^{K} m_{l}! \leq (2m)!.
\]

Also, similarly to the way we obtain the first equality in (4.5), we see that

\[
\prod_{j,k=0}^{K} \prod_{j<k}^{m_{j,k}} \Delta^{1,j}_{x_{j,k,i}} \Delta^{1,k}_{x_{j,k,i}} = \prod_{l=0}^{K} \prod_{q=1}^{m_{l}} \Delta^{1,l}_{x_{\pi_{l}(q)}}. \tag{4.19}
\]

Consequently

\[
E \left( \prod_{j,k=0}^{K} \prod_{j<k}^{m_{j,k}} (I_{j,k,l}) \right) = \sum_{\pi_{0},\ldots,\pi_{K}} \int_{\mathcal{I}_{l}(x; \pi)} dx_{j,k,i} \tag{4.20}
\]

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where we take the product over \( \{ 0 \leq j < k \leq K, 1 \leq i \leq m_{j,k} \} \), \( \pi \in B \) and 

\[
\tilde{T}_l(x; \pi) = \prod_{l=0}^{K} \prod_{q=1}^{m_l} \Delta_{x_{\pi_l(q)}}^{1} \int_{\sum_{q=1}^{m_l} r_{\lambda(q)} \leq t} p_{(t-\sum_{q=1}^{m_{l-1}} r_{\lambda(q)+r_{\lambda,1}})} \left( x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})} \right) \\
\prod_{q=2}^{m_l} p_{r_{\lambda,q}} \left( x_{\pi_l(q)} - x_{\pi_l(q-1)} \right) \prod_{q=1}^{m_l} dr_{l,q}.
\]

\hspace{1cm}

We continue to rewrite the right-hand side of (4.20).

In (4.21), each difference operators, say \( \Delta_{u}^{1} \) is applied to the product of two terms, say \( p \cdot (u-a) \cdot p \cdot (u-b) \), using the product rule for difference operators we see that 

\[
\Delta_{u}^{1} \{ p \cdot (u-a) \cdot p \cdot (u-b) \} \]

\[= \Delta_{u}^{1} p \cdot (u-a) \cdot p \cdot (u+1-b) + p \cdot (u-a) \Delta_{u}^{1} p \cdot (u-b) \]

(4.22)

Consider an example of how the term \( \Delta_{u}^{1} \Delta_{u}^{1} p \cdot (u-a) \) may appear. It could be by the application

\[
\Delta_{u}^{1} \left( \Delta_{u}^{1} p \cdot (u-a) \cdot p \cdot (v-a) \right),
\]

in which we take account of the two terms to which \( \Delta_{u}^{1} \) is applied. Using the product rule in (4.22) we see that

\[
= \left( \Delta_{u}^{1} \Delta_{u}^{1} p \cdot (u-a) \right) p \cdot (v-(a+1)) + \Delta_{u}^{1} p \cdot (u-a) \Delta_{u}^{1} p \cdot (v-a).
\]

(4.24)

Consider one more example

\[
\Delta_{u}^{1} \left( \Delta_{u}^{1} p \cdot (u-a) \Delta_{u}^{1} p \cdot (v-a) \right)
\]

\[= \left( \Delta_{u}^{1} \Delta_{u}^{1} p \cdot (u-a) \right) \Delta_{u}^{1} p \cdot (v-(a+1))
\]

\[+ \Delta_{u}^{1} p \cdot (u-a) \Delta_{u}^{1} \Delta_{u}^{1} p \cdot (v-a).
\]

(4.25)

Note that in both examples the arguments of probability densities with two difference operators applied to it does not contain a 1. This is true in general because the difference formula, (4.22), does not add a 1 to the argument of a term to which a difference operator is applied. Otherwise
we may have a \( \pm 1 \) added to the arguments of probability densities to which one difference operator is applied, as in (4.25), or to the arguments of probability densities to which no difference operator is applied, as in (4.21).

Based on the argument of the preceding paragraph we write (4.21) in the form

\[
E \left( \prod_{j,k=0}^{K} (I_{j,k,t})^{m_{j,k}} \right) = \sum_{a} \sum_{\pi_{0},...\pi_{K}} \int_{\mathcal{T}_{l}} \mathcal{T}'(x; \pi, a) \prod_{j,k,i} dx_{j,k,i},
\]

(4.26)

where

\[
\mathcal{T}'(x; \pi, a) = \prod_{l=0}^{K} \int_{\mathcal{R}_{l}} \left( (\Delta_{x_{\pi(l-1)}}^{1})^{a_{1}(l,1)} (\Delta_{x_{\pi(l-1)}^{1}}^{1})^{a_{2}(l,1)} \right) \prod_{q=2}^{m_{l}} \left( (\Delta_{x_{\pi(q-1)}}^{1})^{a_{1}(l,q)} (\Delta_{x_{\pi(q-1)}}^{1})^{a_{2}(l,q)} \right) p_{l,q}^{c} \prod_{r=1}^{m_{l}} dr_{l,q}.
\]

(4.27)

In (4.27) \( \mathcal{R}_{l} = \{ \sum_{q=1}^{m_{l}} r_{l,q} \leq t \} \). In (4.26) the first sum is taken over all

\[
a = (a_{1}, a_{2}) : \{(l, q), 0 \leq l \leq K, 1 \leq q \leq m_{l}\} \mapsto \{0, 1\} \times \{0, 1\}
\]

(4.28)

with the restriction that for each triple \( j, k, i \), there are exactly two factors of the form \( \Delta_{x_{i,j,k}}^{1} \), each of which is applied to one of the terms \( p^{c}_{r}(\cdot) \) that contains \( x_{j,k,i} \) in its argument. This condition can be stated more formally by saying that for each \( l \) and \( q = 1, \ldots, m_{l} - 1 \), if \( \pi_{l}(q) = (j, k, i) \), then \( \{a_{1}(l, q), a_{2}(l, q + 1)\} = \{0, 1\} \) and if \( q = m_{l} \) then \( \{a_{1}(l, m_{l}), a_{2}(l + 1, 1)\} = \{0, 1\} \). (Note that when we write \( \{a_{1}(l, q), a_{2}(l, q + 1)\} = \{0, 1\} \) we mean as two sets, so, according to what \( a \) is, we may have \( a_{1}(l, q) = 1 \) and \( a_{2}(l, q + 1) = 0 \) or \( a_{1}(l, q) = 0 \) and \( a_{2}(l, q + 1) = 1 \) and similarly for \( \{a_{1}(l, m_{l}), a_{2}(l + 1, 1)\} \).) Also, in (4.27) we define \( (\Delta_{x_{l}}^{1})^{0} = 1 \) and \( (\Delta_{x_{1}}^{1})^{1} = 1 \).

In (4.27), because of (4.22), \( p^{c}_{r}(z) \) can take any of the values \( p_{r}(z) \), \( p_{r}(z+1) \) or \( p_{r}(z-1) \). (We must consider all three possibilities.) Finally, it is important to emphasize that in (4.27) each of the difference operators is applied to only one of the terms \( p^{c}_{r}(\cdot) \).
Rather than (4.26), we first analyze
\[
\sum_a \pi_0, ... \pi_K \int T_t(x; \pi, a) \prod_{j,k,i} dx_{j,k,i},
\] (4.29)
where
\[
T_t(x; \pi, a) = \prod_{l=0}^K \int_{R_l} \left( \left( \Delta^1_{x_{\pi_l}(1)} \right)^{a_1(l,1)} \left( \Delta^1_{x_{\pi_{l-1}}(m_{l-1})} \right)^{a_2(l,1)} \right)
\]
\[
p(t - \sum_{q=1}^{m_l} r_{l-1,q}) + r_{l,1} \left( x_{\pi_l(1)} - x_{\pi_{l-1}}(m_{l-1}) \right)
\]
\[
\prod_{q=2}^{m_l} \left( \left( \Delta^1_{x_{\pi_l(q)}} \right)^{a_1(l,q)} \left( \Delta^1_{x_{\pi_{l-1}(q-1)}} \right)^{a_2(l,q)} p_{r_{l,q}} \left( x_{\pi_l(q)} - x_{\pi_{l-1}(q-1)} \right) \right) \prod_{q=1}^{m_l} dr_{l,q}.
\] (4.30)

The difference between \( T_t(x; \pi, a) \) and \( T'_t(x; \pi, a) \) is that in the former we replace \( p^\oplus \) by \( p \). It is easier to analyze (4.29) than (4.26). At the conclusion of this proof we show that both (4.29) than (4.26) have the same asymptotic limit as \( t \) goes to infinity.

We first obtain (4.1). Let \( m = 2n \), since \( m_{j,k} = 2n_{j,k} \), \( m_l = 2n_l \) for some integer \( n_l \). (Recall (4.8)). To begin we consider the case in which \( a = e \), where
\[
e(l, 2q) = (1, 1) \quad \text{and} \quad e(l, 2q - 1) = (0, 0) \quad \forall q.
\] (4.31)

When \( a = e \) we have
\[
T_t(x; \pi, e) = \prod_{l=0}^K \int_{R_l} \left( \left( \Delta^1_{x_{\pi_l}(1)} \right)^{a_1(l,1)} \left( \Delta^1_{x_{\pi_{l-1}}(m_{l-1})} \right)^{a_2(l,1)} \right)
\]
\[
p(t - \sum_{q=1}^{m_l} r_{l-1,q}) + r_{l,1} \left( x_{\pi_l(1)} - x_{\pi_{l-1}}(m_{l-1}) \right)
\]
\[
\prod_{q=2}^{m_l} \left( \left( \Delta^1_{x_{\pi_l(q)}} \right)^{a_1(l,q)} \left( \Delta^1_{x_{\pi_{l-1}(q-1)}} \right)^{a_2(l,q)} p_{r_{l,q}} \left( x_{\pi_l(q)} - x_{\pi_{l-1}(q-1)} \right) \right) \prod_{q=1}^{m_l} dr_{l,q}.
\] (4.32)

Here we use the following notation: \( \Delta^1 p(u - v) = p(u - v + 1) - p(u - v) \), i.e., when \( \Delta^1 \) has no subscript, the difference operator is applied to the whole argument of the function. In this notation,
\[
\Delta^1 \Delta^1_p(u - v) = \Delta^1 \Delta^{-1} p(u - v).
\] (4.33)
Consider the multigraph $G_\pi$ with vertices $\{(j,k,i), 0 \leq j < k \leq K, 1 \leq i \leq m_{j,k}\}$. Assign an edge between the vertices $\pi_l(2q - 1)$ and $\pi_l(2q)$ for each $0 \leq l \leq K$ and $1 \leq q \leq n_l$. Each vertex is connected to two edges. To see this suppose that $\pi_l(2q) = \{(j,k,i)\}$, with $j = l$ and $k = l' \neq l$, then there is a unique $q'$ such that $\pi_{l'}(2q')$ or $\pi_{l'}(2q' - 1)$ is equal to $\{(j,k,i)\}$. Therefore all the vertices lie in some cycle. Assume that there are $S$ cycles. We denote them by $C_s, s = 1, \ldots, S$. Clearly, it is possible to have cycles of order two, in which case two vertices are connected by two edges.

It is important to note that the graph $G_\pi$ does not assign edges between $\pi_l(2q)$ and $\pi_l(2q + 1)$, although these vertices may be connected by the edge assigned between $\pi_{l'}(2q' - 1)$ and $\pi_{l'}(2q')$ for some $l'$ and $q'$.

We proceed to estimate (4.30) by breaking the calculation into two cases: when $a = e$ and all the cycles of $G_\pi$ are of order two; when $a = e$ and not all the cycles of $G_\pi$ are of order two or when $a \neq e$.

4.1 $a = e$, with all cycles of order two

Let $\mathcal{P} = \{\gamma_{2v-1}, \gamma_{2v} : 1 \leq v \leq n\}$ be a pairing of the $m$ vertices

$$\{(j,k,i), 0 \leq j < k \leq K, 1 \leq i \leq m_{j,k}\}$$

of $G_\pi$, that satisfies the following special property: whenever $(j,k,i)$ and $(j',k',i')$ are paired together, $j = j'$ and $k = k'$. Equivalently,

$$\mathcal{P} = \bigcup_{j,k=0}^{K} \mathcal{P}_{j,k}$$

(4.34)

where each $\mathcal{P}_{j,k}$ is a pairing of the $m_{j,k}$ vertices

$$\{(j,k,i), 1 \leq i \leq m_{j,k}\}.$$

We refer to such a pairing $\mathcal{P}$ as a special pairing and denote the set of special pairings by $\mathcal{S}$.

Given a special pairing $\mathcal{P} \in \mathcal{S}$, let $\pi$ be such that for each $0 \leq l \leq K$ and $1 \leq q \leq n_l$,

$$\{\pi_l(2q - 1), \pi_l(2q)\} = \{\gamma_{2v-1}, \gamma_{2v}\}$$

(4.35)
for some, necessarily unique, \(1 \leq v \leq n_t\). In this case we say that \(\pi\) is compatible with the pairing \(P\) and write this as \(\pi \sim P\). (Recall that when we write \(\{\pi_t(2q-1), \pi_t(2q)\}\) = \{\(\gamma_{2v-1}, \gamma_{2v}\), we mean as two sets, so, according to what \(\pi_t\) is, we may have \(\pi_t(2q-1) = \gamma_{2v-1}\) and \(\pi_t(2q) = \gamma_{2v}\) or \(\pi_t(2q-1) = \gamma_{2v}\) and \(\pi_t(2q) = \gamma_{2v-1}\).) Clearly

\[
|S| \leq \frac{(2n)!}{2^n n!}
\]  

(4.36)

the number of pairings of \(m = 2n\) objects.

Let \(\pi \in \mathcal{B}\) be such that \(G_\pi\) consists of cycles of order two. It is easy to see that \(\pi \sim P\) for some \(P \in S\). To see this note that if \(\{(j, k, i), (j', k', i')\}\) form a cycle of order two, there must exist \(l\) and \(l'\) with \(l \neq l'\) and \(q\) and \(q'\) such that both \(\{(j, k, i), (j', k', i')\}\) = \(\{\pi_t(2q-1), \pi_t(2q)\}\) and \(\{(j, k, i), (j', k', i')\}\) = \(\{\pi_{t'}(2q'-1), \pi_{t'}(2q')\}\). This implies that \(j = j', k = k'\) and \(\{j, k\} = \{l, l'\}\). Furthermore, by (4.35) we have

\[
\{\pi_t(2q-1), \pi_t(2q)\} = \{\pi_{t'}(2q'-1), \pi_{t'}(2q')\} = \{\gamma_{2v-1}, \gamma_{2v}\}
\]  

(4.37)

When \(\pi \sim P\) and all cycles are of order two we can write

\[
\prod_{l=0}^{K} \prod_{q=1}^{n_l} \Delta^1 \Delta^{-1} p_{r_{l,2q}} (x_{\pi_i(2q)} - x_{\pi_i(2q-1)}) = \prod_{v=1}^{n} \Delta^1 \Delta^{-1} p_{r_{2v}} (x_{\gamma_{2v}} - x_{\gamma_{2v-1}}) \Delta^1 \Delta^{-1} p_{r'_{2v}} (x_{\gamma_{2v}} - x_{\gamma_{2v-1}}),
\]  

(4.38)

where \(r_{2v}\) and \(r'_{2v}\) are the rearranged indices \(r_{l,2q}\) and \(r'_{l',2q'}\). We also use the fact that \(\sum_{l=0}^{K} n_l = 2n\).

For use in (4.44) below we note that

\[
\int_0^{t} \int_0^{t} |\Delta^1 \Delta^{-1} p_{r_{2v}} (x_{\gamma_{2v}} - x_{\gamma_{2v-1}})| |\Delta^1 \Delta^{-1} p_{r'_{2v}} (x_{\gamma_{2v}} - x_{\gamma_{2v-1}})| dr_{2v} dr'_{2v}
\]

\[
= \left( \int_0^{t} |\Delta^1 \Delta^{-1} p_r (x_{\gamma_{2v}} - x_{\gamma_{2v-1}})| dr \right)^2 = w^2(x_{\gamma_{2v}} - x_{\gamma_{2v-1}}, t),
\]  

(4.39)

(see (6.7)).

We want to estimate the integrals in (4.29). However, it is difficult to integrate \(T_i(x; \pi, e)\) directly, because the variables,

\[
\{x_{\pi_t(1)} - x_{\pi_{t-1}(m_{t-1})}, x_{\pi_t(2q-1)} - x_{\pi_{t}(2q-2)}, x_{\pi_{t}(2q)} - x_{\pi_{t}(2q-1)}; \}
\]

\[
l \in [0, K], q \in [1, n_t]\},
\]

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are not independent. We begin the estimation by showing that over much of the domain of integration, the integral is negligible, asymptotically, as \( t \to \infty \). To begin, we write

\[
1 = \prod_{v=1}^{n} \left( 1_{\{ |x_{\gamma_{2v-1}} - x_{\gamma_{2v-1}}| \leq t^{(\beta-1)/(4\beta)} \}} + 1_{\{ |x_{\gamma_{2v-1}} - x_{\gamma_{2v-1}}| \geq t^{(\beta-1)/(4\beta)} \}} \right) \tag{4.40}
\]

and expand it as a sum of \( 2^n \) terms and use it to write

\[
\int T_t(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} = \int \prod_{v=1}^{n} \left( 1_{\{ |x_{\gamma_{2v-1}} - x_{\gamma_{2v-1}}| \leq t^{(\beta-1)/(4\beta)} \}} \right) T_t(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} + E_{1,t}. \tag{4.41}
\]

We now show that

\[
E_{1,t} = O \left( t^{- (\beta-1)/(5\beta)} \left( t^{2\psi^{-1}}(1/t) \right)^n \right). \tag{4.42}
\]

Note that every term in \( E_{1,t} \) can be written in the form

\[
B_t(\pi, e, D) := \int \prod_{v=1}^{n} 1_{D_v} T_t(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \tag{4.43}
\]

where each \( D_v \) is either \( \{ |x_{\gamma_{2v-1}} - x_{\gamma_{2v-1}}| \leq t^{(\beta-1)/(4\beta)} \} \) or \( \{ |x_{\gamma_{2v-1}} - x_{\gamma_{2v-1}}| \geq t^{(\beta-1)/(4\beta)} \} \), and at least one of the \( D_v \) is of the second type.

Consider (4.43) and the representation of \( T_t(x; \pi, e) \) in (4.32). We take absolute values in the integrand in (4.32) and take all the integrals with \( r \) between 0 and \( t \) and use (4.39) to get

\[
|B_t(\pi, e, D)| \leq \int \prod_{v=1}^{n} 1_{D_v} w^2(x_{\gamma_{2v-1}}, t) \prod_{l=0}^{K} u(x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}, t) \prod_{q=2}^{n_l} u(x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)}, t) \prod_{j,k,i} dx_{j,k,i}. \tag{4.44}
\]

We now take

\[
\{ x_{\gamma_{2v-1}}, v = 1, \ldots, n \} \tag{4.45}
\]

and an additional \( n \) variables from the \( 2n \) arguments of the \( u \) terms,

\[
\bigcup_{l=0}^{K} \{ x_{\pi_l(1)} - x_{\pi_{l-1}(m_{l-1})}, x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)}, q = 2, \ldots, n_l \}. \tag{4.46}
\]
so that the chosen $2n$ variables generate the space spanned by the $2n$ variables $\{x_{j,k}\}$. There are $n$ variables in (4.46) that are not used. We bound the functions $u$ of these variables by their sup norm, which by (6.5) is bounded by $Ct\psi^{-1}(1/t)$. Then we make a change of variables and get that

$$
|B_t(\pi, e, D)| \leq C \left( t\psi^{-1}(1/t) \right)^n \int \prod_{v=1}^{n} 1_{D_v} w^2(y_v, t) \prod_{v=n+1}^{2n} u(y_v, t) \prod_{v=1}^{2n} dy_v
$$

$$
\leq C \left( t^2\psi^{-1}(1/t) \right)^n \int \prod_{v=1}^{n} 1_{D_v} w^2(y_v, t) \prod_{v=1}^{n} dy_v,
$$

$$
= O \left( t^{-(\beta-1)/(5\beta)} \left( t^2\psi^{-1}(1/t) \right)^n \right).
$$

(4.47)

Here we use (3.8) to see that the integral of a $u$ term is $t$. Then we use (3.11) and (3.12) to obtain (4.42). (Note that it is because at least one of the $D_v$ is of the second type that we can use (3.12).)

We now study

$$
\int \prod_{v=1}^{n} \left( 1_{\{|x_{\gamma_{2v}}-x_{\gamma_{2v-1}}| \leq t^{(\beta-1)/(4\beta)}\}} \right) T_t(x; \pi, e) \prod_{j,k,i} dx_{j,k,i}.
$$

(4.48)

We identify the relationships in (4.37) by setting $v = \sigma_l(q)$ so that

$$
\{\pi_l(2q-1), \pi_l(2q)\} = \{\gamma_{2\sigma_l(q)-1}, \gamma_{2\sigma_l(q)}\},
$$

(4.49)

for each $0 \leq l \leq K$ and $1 \leq q \leq n_l$. We use both (4.37) and (4.49) in what follows.

We now make a change of variables that, eventually, enables us to make the arguments of the $u$ terms and the $w$ terms independent. For $q \geq 2$ we write

$$
p_{r_{1,2q-1}}(x_{\pi_l(2q-1)} - x_{\pi_l(2q-2)})
$$

(4.50)

$$
= p_{r_{1,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}}) + \Delta h_{l,q} p_{r_{1,2q-1}}(x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}}),
$$

where $h_{l,q} = (x_{\pi_l(2q-2)} - x_{\gamma_{2\sigma_l(q-1)-1}}) + (x_{\gamma_{2\sigma_l(q-1)-1}} - x_{\pi_l(2q-2)})$. When $q = 1$ we can make a similar decomposition

$$
p_{t-\sum_{q=1}^{m_{l-1}} r_{l-1,q}+r_{l,1}}(x_{\pi_l(1)} - x_{\pi_l-1}(m_{l-1}))
$$

(4.51)

$$
= p_{t-\sum_{q=1}^{m_{l-1}} r_{l-1,q}+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_l-1}(m_{l-1}-1)})
$$

$$
+ \Delta h_{l,1} p_{t-\sum_{q=1}^{m_{l-1}} r_{l-1,q}+r_{l,1}}(x_{\gamma_{2\sigma_l(1)-1}} - x_{\gamma_{2\sigma_l-1}(m_{l-1}-1)}),
$$

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where \( h_{l,1} = (x_{\pi_1(1)} - x_{\gamma_{2\sigma_l(1)-1}}) + (x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} - x_{\pi_1(n_{l-1})}) \). Note that because of the presence of the term \( \prod_{v=1}^{n_l} \left| x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} - x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} \right| \) in the integral in (4.48) we need only be concerned with values of \( |h_{l,q}| \leq 2t^{(\beta-1)/(4\delta)} \), \( 0 \leq l \leq K \) and \( 1 \leq q \leq n_l \).

For \( q = 1, \ldots, n_l, l = 0, \ldots, K \), we substitute (4.50) and (4.51) into the term \( T_l(x; \pi, e) \) in (4.48), (see also (4.32)), and expand the products so that we can write (4.48) as a sum of \( 2\sum_{l=0}^{n_l} \) terms, which we write as

\[
\int \prod_{v=1}^{n_l} \left( \prod_{l,q=1}^{n_l} \left| x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} - x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} \right| \right) \mathcal{T}_l(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (4.52)
\]

\[
= \int \prod_{v=1}^{n_l} \left( \prod_{l,q=1}^{n_l} \left| x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} - x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} \right| \right) \mathcal{T}_{l,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} + E_{2,t},
\]

where

\[
\mathcal{T}_{l,1}(x; \pi, e) = \prod_{l=0}^{K} \int_{R_l} p_l(t-\sum_{q=1}^{m_l} r_{l,q}) \prod_{q=1}^{m_l} r_{l,q} (x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} - x_{\gamma_{2\sigma_l-1(n_{l-1})-1}})
\]

\[
= \prod_{q=1}^{m_l} \Delta^1 \Delta^{-1} p_{l,q} (x_{\pi_l(2q-1)} - x_{\pi_l(2q-1)}) \prod_{q=1}^{m_l} dr_{l,q}.
\]

Using (4.38) we can rewrite this as

\[
\mathcal{T}_{l,1}(x; \pi, e)
\]

\[
= \prod_{l=0}^{K} \int_{R_l \times \ldots \times R_K} p_l(t-\sum_{q=1}^{m_l} r_{l,q}) \prod_{q=1}^{m_l} r_{l,q} (x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} - x_{\gamma_{2\sigma_l-1(n_{l-1})-1}})
\]

\[
= \prod_{l=0}^{K} \Delta^1 \Delta^{-1} p_{l,q} (x_{\gamma_{2\sigma_l-1(n_{l-1})-1}} - x_{\gamma_{2\sigma_l-1(n_{l-1})-1}}) \prod_{q=1}^{m_l} dr_{l,q},
\]

where \( r_{2q} \) and \( r'_{2q} \) are the rearranged indices \( r_{l,2q} \) and \( r'_{l,2q} \).
The usefulness of the representations in (4.50) and (4.51) is now apparent. Since the variables \(x_{\gamma 2v}, v = 1, \ldots, n\), occur only in the last line of (4.54), we make the change of variables \(x_{\gamma 2v} \to x_{\gamma 2v-1}\) and get that

\[
\int T_{t,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (4.55)
\]

\[
= \int \int_{R_0 \times \cdots \times R_K} \left( \prod_{l=0}^{K} p_{l-t+\sum_{q=1}^{m_{l-1}} r_{l-1,q}+r_{l,1}} (x_{\gamma 2\sigma_l(q-1)} - x_{\gamma 2\sigma_l-1(q-1)}) \prod_{q=2}^{n_l} p_{r_{l,2q-1}} (x_{\gamma 2\sigma_l(q-1)} - x_{\gamma 2\sigma_l-1(q-1)}) \right) \left( \prod_{v=1}^{n} \Delta^1 \Delta^{-1} p_{r_{2v}} (x_{\gamma 2v}) \Delta^1 \Delta^{-1} p_{r_{2v}'} (x_{\gamma 2v}) \right) K \prod_{l=0}^{m_l} dr_{l,q} \prod_{j,k,i} dx_{j,k,i}.
\]

Now, since the variables \(x_{\gamma 2v}, v = 1, \ldots, n\) occur only in the last line of (4.55) and the variables \(x_{\gamma 2v-1}, v = 1, \ldots, n\) occur only in the second and third lines of (4.55), we can write (4.55) as

\[
\int T_{t,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \quad (4.56)
\]

\[
= \int \int_{R_0 \times \cdots \times R_K} \left( \prod_{l=0}^{K} p_{l-t+\sum_{q=1}^{m_{l-1}} r_{l-1,q}+r_{l,1}} (x_{\gamma 2\sigma_l(q-1)} - x_{\gamma 2\sigma_l-1(q-1)}) \prod_{q=2}^{n_l} p_{r_{l,2q-1}} (x_{\gamma 2\sigma_l(q-1)} - x_{\gamma 2\sigma_l-1(q-1)}) \right) \amp \prod_{v=1}^{n} \int \Delta^1 \Delta^{-1} p_{r_{2v}} (x_{\gamma 2v}) \Delta^1 \Delta^{-1} p_{r_{2v}'} (x_{\gamma 2v}) dx_{\gamma 2v} \amp K \prod_{l=0}^{m_l} dr_{l,q}.
\]

Note that we also use Fubini’s Theorem, which is justified since the absolute value of the integrand is integrable, (as we point out in the argument preceding (4.44)). (In the rest of this section use Fubini’s Theorem frequently for integrals like (4.56) without repeating the explanation about why it is justified.)

We now show that

\[
E_{2,t} = O \left( t^{-(\beta-1)/(3\beta)} \left( t^2 \psi^{-1}(1/t) \right)^n \right). \quad (4.57)
\]
To see this note that the terms in $E_{2,t}$ are of the form

$$
\int \prod_{v=1}^{n} \left( 1 \{ |x_{2v} - x_{2v-1}| \leq t^{(\beta - 1)/(4\beta)} \} \right)
$$

(4.58)

$$
\prod_{l=0}^{K} \int_{R^l} \tilde{p}(t - \sum_{q=1}^{m} r_{l-1,q} + r_{l,1}) \left( x_{2\gamma_{l}(1)-1} - x_{2\gamma_{l-1}(n_{l-1})-1} \right)
$$

$$
\prod_{q=2}^{m_l} \tilde{p}_{l,2q-1} \left( x_{2\gamma_{l}(q)-1} - x_{2\gamma_{l}(q-1)-1} \right)
$$

$$
\prod_{q=1}^{m_l} \Delta^{1/2} \Delta^{-1} \left( x_{\pi(2q)} - x_{\pi(2q-1)} \right) \prod_{q=1}^{m_l} dr_{l,q} \prod_{j,k,i} dx_{j,k,i},
$$

where $\tilde{p}_{l,2q-1}$ is either $p_{l,2q-1}$ or $\Delta_{h_{l,q}}p_{l,2q-1}$. Furthermore, at least one of the terms $\tilde{p}_{l,2q-1}$ is of the form $\Delta_{h_{l,q}}p_{l,2q-1}$.

As in the transition from (4.43) to (4.44) we bound the absolute value of (4.58) by

$$
\int \prod_{v=1}^{n} \left( 1 \{ |x_{2v} - x_{2v-1}| \leq t^{(\beta - 1)/(4\beta)} \} \right) w^2(x_{2v} - x_{2v-1}, t)
$$

(4.59)

$$
\prod_{l=0}^{K} \tilde{u}(x_{\gamma_{l}(1)-1} - x_{\gamma_{l-1}(n_{l-1})-1}, t) \prod_{q=2}^{m_l} \tilde{u}(x_{\gamma_{l}(q)-1} - x_{\gamma_{l}(q-1)-1}, t)
$$

$$
\prod_{j,k,i} dx_{j,k,i},
$$

where each $\tilde{u}(\cdot, t)$ is either of the form $u(\cdot, t)$ or $v_{h_{l,q}}(\cdot, t)$; (see (3.3)).

We need to introduce the following notation and estimates. The next lemma is proved in Section 6. Let

$$
v_s(x, t) := \left( \log t \land \frac{t\psi^{-1}(1/t)}{|x|} \land t^{1 + \log^+ \frac{1}{x^2}} \right),
$$

(4.60)

**Lemma 4.2** Under the hypotheses of Theorem 1.1, for all $t$ sufficiently large,

$$
v_{h_{l,q}}(x, t) \leq Ch_{l,q}^2 v_s(x, t)
$$

(4.61)

$$
\sup_{x \in R^d} v_s(x, t) \leq \log t
$$

(4.62)

$$
\int v_s(x, t) dx \leq Ct \left( \psi^{-1}(1/t) \right) \log t.
$$

(4.63)
We have \( J \) terms of the type \( v_{h_l,q}(\cdot, t) \), for some \( J \geq 1 \). It follows from (4.61) and and the fact that \( |h_{l,q}| \leq 2t^{(\beta-1)/(4J)} \), that we can bound the integral in (4.59) by

\[
C t^{J(\beta-1)/(2\beta)} \int \prod_{v=1}^{n} w^2(x_{\gamma_2v} - x_{\gamma_2v-1}, t)
\]

\[
\prod_{l=0}^{K} \bar{u}(x_{\gamma_2\sigma_l(1)-1} - x_{\gamma_2\sigma_l-1(n_{l-1})-1}, t) \prod_{q=2}^{n_l} \bar{u}(x_{\gamma_2\sigma_l(q)-1} - x_{\gamma_2\sigma_l-1(q-1)-1}, t)
\]

\[
\prod_{j,k,i} dx_{j,k,i},
\]

where \( \bar{u}(\cdot, t) \) is either \( u(\cdot, t) \) or \( v_*(\cdot, t) \), and we have precisely \( J \) of the latter.

Since the variables \( x_{\gamma_2v}, \nu = 1, \ldots, n, \) occur only in the \( w \) terms in (4.64) and the variables \( x_{\gamma_2v-1}, v = 1, \ldots, n \) occur only in the \( \bar{u} \) terms in (4.64), (refer to the change of variables arguments in (4.55) and (4.56)), we can write (4.64) as

\[
C t^{J(\beta-1)/(2\beta)} \int \left( \prod_{l=0}^{K} \bar{u}(x_{\gamma_2\sigma_l(1)-1} - x_{\gamma_2\sigma_l-1(n_{l-1})-1}, t) \right) \prod_{q=2}^{n_l} \bar{u}(x_{\gamma_2\sigma_l(q)-1} - x_{\gamma_2\sigma_l-1(q-1)-1}, t)
\]

\[
\prod_{v=1}^{n} w^2(x_{\gamma_2v}, t) \prod_{v=1}^{n} dx_{\gamma_2v}
\]

\[
\leq C t^{J(\beta-1)/(2\beta)} (\log t)^n \int \left( \prod_{l=0}^{K} \bar{u}(x_{\gamma_2\sigma_l(1)-1} - x_{\gamma_2\sigma_l-1(n_{l-1})-1}, t) \right) \prod_{q=2}^{n_l} \bar{u}(x_{\gamma_2\sigma_l(q)-1} - x_{\gamma_2\sigma_l-1(q-1)-1}, t)
\]

\[
\prod_{v=1}^{n} dx_{\gamma_2v-1}
\]

where the last inequality uses (3.11).

As we have been doing we extract a linearly independent set of variables from the arguments of the \( \bar{u} \) terms. The other \( \bar{u} \) terms we bound by their supremum. Then we make a change of variables and integrate the remaining \( \bar{u} \) terms.

Compare (3.5) with (4.62). Replacing the sup of a \( u \) term by the sup of a \( v_* \) term reduces the upper bound by a factor of \( 1/(t^\psi^{-1}(1/t)) \), (neglecting the factor of \( \log t \) which is irrelevant.) On the other hand, considering (3.8) and (4.63), we see that replacing the integral of a \( u \)
term by the integral of a \( v \) term reduces the upper bound by a factor of \( \psi^{-1}(1/t) \), (again neglecting the factor of \( \log t \).) Counting the initial factor of \( t^{(\beta-1)/(2\beta)} \) we have a reduction with is at least
\[
\left( t^{(\beta-1)/(2\beta)} \right) (t\psi^{-1}(1/t))^{-1} = o \left( \left( t^{-(\beta-1)/(3\beta)} \right)^J \right)
\]
for all \( \epsilon > 0 \). Since \( J \geq 1 \), we get \((4.57)\).

Analogous to \((4.41)\) we note that
\[
\int T_{t,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} = \int \prod_{v=1}^{n_v} \left( 1 \{ x_{\gamma_2v} - x_{\gamma_2v-1} \leq t^{(\beta-1)/(4\beta)} \} \right) T_{t,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} + \tilde{E}_{1,t},
\]
where \( \tilde{E}_{1,t} = O \left( t^{-2(\beta-1)/(5\beta)} (t^2 \psi^{-1}(1/t))^n \right) \). The proof of \((4.67)\) is the same as the proof of \((4.42)\).

Since \( \psi \) is regularly varying with index \( \beta > 1 \) we see that there exists an \( \epsilon(\beta) := \epsilon > 0 \) such that
\[
E_{1,t} + E_{2,t} + \tilde{E}_{2,t} = O \left( t^{(2-1/\beta)n-\epsilon} \right).
\]
Therefore, it follows from \((4.41)\), \((4.52)\) and \((4.67)\) that
\[
\int T_t(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} = \int T_{t,1}(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} + O \left( t^{(2-1/\beta)n-\epsilon} \right).
\]

We now obtain a sharp estimate, (asymptotically as \( t \to \infty \)), of the second integral in \((4.69)\) that leads to the \((4.1)\). Let \( \mathcal{R}_l(s) = \{ \sum_{q=1}^{n_l} r_{l,2q-1} \leq t - s \} \) and \( \tilde{\sigma}_l(q) := \gamma_{2\sigma_l(q) - 1} \). We define
\[
F_t(\bar{\sigma}, s_0, \ldots, s_K) = \int \left( \int_{\mathcal{R}_0(s_0) \times \cdots \times \mathcal{R}_K(s_K)} \prod_{l=0}^{K} P_{\left( t - \sum_{q=1}^{n_l} r_{l,2q-1} - s_{l-1} \right)} \prod_{q=2}^{n_{l-1}} \prod_{r_{l,2q-1}}^{n_l} \left( x_{\bar{\sigma}_l(n_{l-1})} - x_{\bar{\sigma}_l(q-1)} \right) \right) dx,
\]
27
where \( (t - \sum_{q=1}^{n_l} r_{1,2q-1} - s_{-1}) = 0 \) and \( \tilde{\sigma}_{-1}(n_{-1}) = 0 \). Here the generic term \( dx \) indicates integration with respect to all the variables \( x \) that appear in the integrand.

Since \( \tilde{\sigma}_l(q) = \gamma_{2\sigma_l(q)}(1) \) we can also write (4.70) as

\[
F_t(\tilde{\sigma}, s_0, \ldots, s_K) = \int (\int_{\bar{\mathcal{R}}_0(s_0) \times \cdots \times \bar{\mathcal{R}}_K(s_K)} \prod_{l=0}^K \rho_l \left( t - \sum_{q=1}^{n_l} r_{1,2q-1} x_{\gamma_{2\sigma_l(q)}} - s_{l-1} \right) + r_{l,1} \right) dx,
\]

with \( x_{\gamma_{2\sigma_l-1}(n_{l-1})} = 0 \).

Consider (4.71). By extending the time integration we have

\[
F_t(\tilde{\sigma}, s_0, \ldots, s_K) \leq \int \prod_{l=0}^K u \left( x_{\gamma_{2\sigma_l(1)}} - x_{\gamma_{2\sigma_l-1}(n_{l-1})} \right) \prod_{q=2}^{n_l} u \left( x_{\gamma_{2\sigma_l(q)-1}} - x_{\gamma_{2\sigma_l(q-1)-1}} \right) dx.
\]

Note that there are \( n \) different \( x \) variables, each one of which appears twice. Therefore, by an argument similar to the one in the paragraph containing (4.47), we see that

\[
F_t(\tilde{\sigma}, s_0, \ldots, s_K) \leq C \left( t^2 \psi^{-1}(1/t) \right)^n,
\]

for some constant depending only on \( m = 2n \).

Let \( \bar{\mathcal{R}}_l = \{ \sum_{q=1}^{n_l} r_{1,2q} \leq t \}, l = 0, \ldots, K \). We break up the integration over \( \mathcal{R}_0 \times \cdots \times \mathcal{R}_K \) in (4.30) into integration over \( \bar{\mathcal{R}}_0 \times \cdots \times \bar{\mathcal{R}}_K \) and \( \bar{\mathcal{R}}_0 \times \cdots \times \bar{\mathcal{R}}_K \); (see (4.71)). If one carefully examines the time indices in (4.30) and (4.70) and uses Fubini’s Theorem, one sees that

\[
\int T_l(x; \pi, e) \prod_{j,k,i} dx_{j,k,i}
\]
\[= \int_{\widehat{\mathbb{R}}_0 \times \cdots \times \widehat{\mathbb{R}}_K} F_t(\widehat{\sigma}, \sum_{q=1}^{n_0} r_{0,2q} \cdots \sum_{q=1}^{n_K} r_{K,2q}) \prod_{i=1}^{n} \left( \int (\Delta^1 \Delta^{-1} p_{r_i}(x)) \left( \Delta^1 \Delta^{-1} p_{r_i'}(x) \right) \, dx \right) \prod_{i=1}^{n} dr_i \, dr_i'. \]

The variables \( \{r_i, r_i' \mid i = 1, \ldots, n\} \) are simply a relabeling of the variables \( \{r_{l,2q} \mid 0 \leq l \leq K, 1 \leq q \leq n_l\} \). (The exact form of this relabeling does not matter in what follows.) Here, as always, we set \( p_r(x) = 0 \), if \( r \leq 0 \).

By Parseval’s Theorem
\[
\int (\Delta^1 \Delta^{-1} p_r(x)) \left( \Delta^1 \Delta^{-1} p_{r'}(x) \right) \, dx = \frac{1}{2\pi} \int |2 - e^{ip} - e^{-ip}|^2 e^{-r\psi(p)} e^{-r'\psi(p)} \, dp \\
= \frac{16}{\pi} \int_{0}^{\infty} \sin^4(p/2) e^{-r\psi(p)} e^{-r'\psi(p)} \, dp \geq 0.
\]

Using this, (4.73) and Fubini’s Theorem, we see that
\[
\int_{(\widehat{\mathbb{R}}_0 \times \cdots \times \widehat{\mathbb{R}}_K) \cap ([0, \sqrt{t}]^n)^c} F_t(\widehat{\sigma}, \sum_{q=1}^{n_0} r_{0,2q} \cdots \sum_{q=1}^{n_K} r_{K,2q}) \prod_{i=1}^{n} \left( \int (\Delta^1 \Delta^{-1} p_{r_i}(x)) \left( \Delta^1 \Delta^{-1} p_{r_i'}(x) \right) \, dx \right) \prod_{i=1}^{n} dr_i \, dr_i' \]
\[
\leq C \left( t^{2 \psi^{-1}(1/t)} \right)^n \int_{[0, \sqrt{t}]^n} \prod_{i=1}^{n} \left( \int (\Delta^1 \Delta^{-1} p_{r_i}(x)) \left( \Delta^1 \Delta^{-1} p_{r_i'}(x) \right) \, dx \right) \prod_{i=1}^{n} dr_i \, dr_i' \]
\[
\leq C \left( t^{2 \psi^{-1}(1/t)} \right)^n \left( \int \left( \int (\Delta^1 \Delta^{-1} p_r(x)) \, dr \right)^2 \, dx \right)^{n-1} \\
\leq \int \left\{ \int_{0}^{\infty} \int_{\sqrt{t}}^{\infty} (\Delta^1 \Delta^{-1} p_{r_i}(x)) \left( \Delta^1 \Delta^{-1} p_{r_i'}(x) \right) \, dr \, dr_i' \right\} \, dx \\
\leq C \left( t^{2 \psi^{-1}(1/t)} \right)^n \int \left\{ \int_{0}^{\infty} \int_{\sqrt{t}}^{\infty} (\Delta^1 \Delta^{-1} p_{r_i}(x)) \left( \Delta^1 \Delta^{-1} p_{r_i'}(x) \right) \, dr \, dr_i' \right\} \, dx,
\]
by (3.11). By (3.16) and (3.17) the integral in the final line of (4.76)
\[
\leq c_{\psi,1} - \int \left( \int_{0}^{\sqrt{t}} \Delta^1 \Delta^{-1} p_s(x) \, ds \right)^2 \, dx \leq O \left( t^{-1/6} \right).
\]
Therefore the first integral in (4.76) is $O(t^{(2-1/\beta)n-\epsilon})$, for some $\epsilon > 0$.

Since $\left(\hat{R}_0 \times \cdots \times \hat{R}_K\right) \supseteq [0, \sqrt{t}]^{2n}$, for $2n \sqrt{t} \leq t$, it follows from (4.74) and the preceding sentence, that

$$\int_{\hat{T}} t(x; \pi, e) \prod_{j,k,i} dx_{j,k,i}$$

(4.78)

$$= \int_{[0,\sqrt{t}]^{2n}} F_t(\bar{\sigma}, \sum_{q=1}^{n_0} r_{0,2q}, \ldots, \sum_{q=1}^{n_K} r_{K,2q}) \prod_{i=1}^{n} \left( \int (\Delta^1\Delta^{-1} p_{r_i}(x)) \left(\Delta^1\Delta^{-1} p_{r_i}(x)\right) \prod_{l=0}^{K} \prod_{q=1}^{n_l} dr_{l,2q} + O(t^{(2-1/\beta)n-\epsilon}) \right)$$

(4.79)

We use the next lemma which is proved in Subsection 4.3.

**Lemma 4.3** Under the hypotheses of Theorem 1.1, for any fixed $m$ and $s_0, \ldots, s_K \leq m\sqrt{t}$ and $1 < \beta \leq 2$, there exists an $\epsilon > 0$ such that for all $t > 0$, sufficiently large,

$$|F_t(\bar{\sigma}, s_0, \ldots, s_K) - F_t(\bar{\sigma}, 0, \ldots, 0)| \leq C \left( t^2 \psi^{-1}(1/t) \right)^{n-\epsilon}. \tag{4.79}$$

**Proof of Lemma 4.4 continued:** It follows from (4.78) and Lemmas 4.3 and 3.3, that

$$\int_{\hat{T}} t(x; \pi, e) \prod_{j,k,i} dx_{j,k,i}$$

(4.80)

$$= F_t(\bar{\sigma}, 0, \ldots, 0) \int_{[0,\sqrt{t}]^{2n}} \prod_{i=1}^{n} \left( \int (\Delta^1\Delta^{-1} p_{r_i}(x)) \left(\Delta^1\Delta^{-1} p_{r_i}(x)\right) \prod_{l=0}^{K} \prod_{q=1}^{n_l} dr_{l,2q} + O(t^{(2-1/\beta)n-\epsilon}) \right)$$

$$= (c_\psi)^n F_t(\bar{\sigma}, 0, \ldots, 0) + O(t^{(2-1/\beta)n-\epsilon}),$$

for some $\epsilon > 0$.

Consider the mappings $\bar{\sigma}_l$ that are used in (4.70). Recall that $\sigma_l(q)$ is defined by the relationship $\{\pi_l(2q-1), \pi_l(2q)\} = \{\gamma_{2\sigma_l(q)-1}, \gamma_{2\sigma_l(q)}\}$. Therefore, since $\bar{\sigma}_l(q) = \gamma_{2\sigma_l(q)-1}$ we can have that either $\bar{\sigma}_l(q) = \pi_l(2q-1)$ or $\bar{\sigma}_l(q) = \pi_l(2q)$. However, since the terms $\bar{\sigma}_l(q)$ are subscripts of the
terms \( x \), all of which are integrated, it is more convenient to define \( \tilde{\sigma}_l \) differently.

Recall that \( \mathcal{P} \), (see (4.34)), is a union of pairings \( \mathcal{P}_{j,k} \) of the \( m_{j,k} \) vertices

\[
\{ (j, k, i), 1 \leq i \leq m_{j,k} \}.
\]

Each \( \mathcal{P}_{j,k} \) consists of \( n_{j,k} \) pairs, that can ordered arbitrarily. Consider one such ordering. If \( \{ \gamma_2 \sigma_l(q) - 1, \gamma_2 \sigma_l(q) \} \) is the \( i \)-th pair in \( \mathcal{P}_{j,k} \), we set \( \tilde{\sigma}_l(q) = (j, k, i) \). (Necessarily, \( l \) will be either \( j \) or \( k \), as we point out in the paragraph containing (4.37)). Thus, each \( \tilde{\sigma}_l \) is a bijection from \([1, n_l]\) to \( \tilde{I}_l = \bigcup_{l=0}^{K} \{ (l, k, i), 1 \leq i \leq n_{l,k} \} \bigcup \{ (j, l, i), 1 \leq i \leq n_{j,l} \} \). (4.81)

Let \( \tilde{B} \) denote the set of \( K + 1 \) tuples, \( \tilde{\sigma} = (\tilde{\sigma}_0, \ldots, \tilde{\sigma}_K) \) of such bijections. Note that with this definition of \( \tilde{\sigma}_l(q) \), (4.70) remains unchanged since we have simply renamed the variables of integration.

By interchanging the elements in any of the \( 2n \) pairs \( \{ \pi_l(2q-1), \pi_l(2q) \} \) we obtain a new \( \pi' \sim \mathcal{P} \). In fact we obtain \( 2^{2n} \) different permutations \( \pi \), in this way, all of which are compatible with \( \mathcal{P} \), and all of which give the same \( \tilde{\sigma} \) in (4.70). Furthermore, by permuting the pairs \( \{ \pi_l(2q - 1), \pi_l(2q) \} \), \( 1 \leq q \leq n_l \), for each \( l \), we get all the possible permutation \( \tilde{\pi} \sim \mathcal{P} \), and these give all possible mappings \( \tilde{\sigma} \in \tilde{B} \). Note that \( |\tilde{B}| = \prod_{l=0}^{K} n_l! \leq (2n)! \).

We now use the notation introduced in the paragraph containing (4.81), and the fact that there are \( 2^{2n} \) permutations that are compatible with \( \mathcal{P} \), to see that

\[
\sum_{\pi \sim \mathcal{P}} \int \mathcal{T}_t(x; \pi, \varepsilon) \prod_{j,k,i} dx_{j,k,i} \quad (4.82)
\]

\[
= (4c_{\psi,1})^n \sum_{\sigma \in \tilde{B}} F_t(\tilde{\sigma}, 0, \ldots, 0) + O \left( t^{(2-1/\beta)n-\varepsilon} \right).
\]

Since \( |\tilde{B}| \leq (2n)! \), we see that the error term only depends on \( m \). Consider (4.82) and the definition of \( F_t(\tilde{\sigma}, 0, \ldots, 0) \) in (4.70) and use (4.16), with \( m_{j,k} \) replaced by \( n_{j,k} \), to see that

\[
\sum_{\pi \sim \mathcal{P}} \int \mathcal{T}_t(x; \pi, \varepsilon) \prod_{j,k,i} dx_{j,k,i} \quad (4.83)
\]
Recall the definition of \( S \), the set of special pairings, given in the first paragraph of this subsection. Since there are \( \frac{(2n_{j,k})!}{2^{n_{j,k}}n_{j,k}!} \) pairings of the \( 2n_{j,k} \) elements \( \{1, \ldots, m_{j,k}\} \), we see that when we sum over all the special pairings we get

\[
\sum_{P \in S} \sum_{\pi \sim P} \int_T t(x; \pi, e) \prod_{j,k,i} dx_{j,k,i} \]

It follows from (4.36) that the error term, still, only depends on \( m \).

The right-hand side of (4.84) is precisely the desired expression in (4.1). Therefore, to complete the proof of Lemma 4.1, we show that for all the other possible values of \( a \), the integral in (4.26) can be absorbed in the error term.

### 4.2 \( a = e \) but not all cycles are of order two and \( a \neq e \)

We show that when \( a = e \) but not all cycles are of order two or when \( a \neq e \)

\[
\left| \int T_t(x; \pi, a) \prod_{j,k,i} dx_{j,k,i} \right| = O \left( t^{2-1/\beta} t^{-\epsilon} \right) ,
\]

for some \( \epsilon = \epsilon_\beta > 0 \). In this subsection we do not assume that \( m \) is even.

Consider the basic formula (4.30). Since we only need an upper bound, we take absolute values in the integrand and extend all the time integrals to \([0, t]\), as we have done several times above. We refer to this integral as the extended integral. We take the time integrals and get an upper bound for (4.30) involving the terms \( u, v \) and \( w \). As we have
done several times above, we choose \( m \) of the \( u, v \) and \( w \) terms with arguments that span \( R^m \). We then bound the remaining \( u, v \) and \( w \) terms and then make a change of variables and integrate the \( u, v \) and \( w \) terms with the chosen arguments. Since we want to find the smallest possible upper bound for the extended integral, it is obvious that we first integrate as many of the \( w \) terms as possible, since such integrals are effectively bounded. (We continue to ignore slowly varying functions of \( t \)). We then try to integrate as many of the \( v \) terms as possible.

In order to do this efficiently, we divide the \( v \) and \( w \) terms into sets. As we construct the sets of \( v \) and \( w \) terms, we also choose a subset \( I \) of the \( v \) and \( w \) terms with arguments that are linearly independent. The cardinality of this subset is a lower bound on the number of \( v \) and \( w \) terms that we can integrate.

This is how we divide the \( v \) and \( w \) terms into sets. For each \( \pi \) and \( a \) we define a multigraph \( G_{\pi,a} \) with vertices \( \{(j,k,i) : 0 \leq j < k \leq K, 1 \leq i \leq m_{j,k}\} \), and an edge between the vertices \( \pi_l(q-1) \) and \( \pi_l(q) \) whenever \( (a_1(l,q), a_2(l,q)) = (1,1), l = 0, \ldots, K, 2 \leq q \leq m_l \), and an edge between the vertices \( \pi_l(1) \) and \( \pi_{l-1}(m_l) \) whenever \( (a_1(l,1), a_2(l,1)) = (1,1), 1 \leq l \leq K \).

This graph divides the \( w \) terms into cycles and chains. Assume that there are \( S \) cycles. We denote them by \( C_s = \{\phi_{s,1}, \ldots, \phi_{s,l(s)}\} \), written in cyclic order, where the cycle length \( l(s) = |C_s| \geq 1 \) and \( 1 \leq s \leq S \). For each \( 1 \leq s \leq S \) we take the set of \( l(s) \) terms

\[
G_{s}^{\text{cycle}} = \{w(x_{\phi_{s,2}} - x_{\phi_{s,1}}), \ldots, w(x_{\phi_{s,l(s)}} - x_{\phi_{s,l(s)-1}}), w(x_{\phi_{s,1}} - x_{\phi_{s,l(s)}})\}. 
\] (4.86)

Let

\[
y_{\phi_{s,i}} = x_{\phi_{s,i}} - x_{\phi_{s,i-1}}, \quad i = 2, \ldots, l(s). \quad (4.87)
\]

It is easy to see that \( \{y_{\phi_{s,i}} : i = 2, \ldots, l(s)\} \), are linearly independent. We put the corresponding \( w \) terms, \( w(x_{\phi_{s,2}} - x_{\phi_{s,1}}), \ldots, w(x_{\phi_{s,l(s)}} - x_{\phi_{s,l(s)-1}}) \) into \( I \). (On the other hand, since

\[
\sum_{i=2}^{l(s)} y_{\phi_{s,i}} = -(x_{\phi_{s,1}} - x_{\phi_{s,l(s)}}), \quad (4.88)
\]

we see that we can only extract \( l(s) - 1 \) linearly independent variables from the \( l(s) \) arguments of \( w \) for a given \( s \).
A cycle of length 1 consists of a single point $\phi_{s,1} = \phi_{t(s),1}$ in the graph, so in this case
\[ G_{s}^{\text{cycle}} = \{ w(0) \}. \] (4.89)

We explain below how this can occur. Obviously, $w(0)$ is not put into $\mathcal{I}$.

Next, suppose there are $S'$ chains. We denote them by $C_s' = \{ \phi_{s,1}', \ldots, \phi_{s,l'(s)}' \}$, written in order, where $l'(s) = |C_s'| \geq 2$ and $1 \leq s \leq S'$. Note that there are $l'(s) - 1$, $w$ terms corresponding to $C_s'$. Then for each $1 \leq s \leq S'$ we form the set of
\begin{equation}
G_{s}^{\text{chain}} = \{ v(x_{\phi_{s,1}'}, x_{a(s)}), w(x_{\phi_{s,2}'}, x_{\phi_{s,1}'}, \ldots, \ldots, w(x_{\phi_{s,l'(s)}'}, x_{b(s)} - x_{\phi_{s,l'(s)}'}) \}
\end{equation}
where $v(x_{\phi_{s,1}'}, x_{a(s)})$ is the unique $v$ term associated with $\Delta_{x_{\phi_{s,1}'}}^1$, and similarly, $v(x_{b(s)} - x_{\phi_{s,l'(s)'}})$ is the unique $v$ term associated with $\Delta_{x_{\phi_{s,l'(s)'}}}$.

(This deserves further clarification. There may be other $v$ terms containing the variable $x_{\phi_{s,1}'}$ in the extended integral, but there is only one $v$ term of the form
\[ \int_0^1 \left| \Delta_{x_{\phi_{s,1}'}}^1 p_s(x_{\phi_{s,1}'}, u) \right| ds, \] (4.91)
where $u$ is some other $x$. variable which we denote by $x_{a(s)}$. This is because one operator $\Delta_{x_{\phi_{s,1}'}}^1$ is associated with $w(x_{\phi_{s,2}'}, x_{\phi_{s,1}'})$ and there are precisely two operators $\Delta_{x_{\phi_{s,1}'}}^1$ in (4.30).

It is easy to see that variables $y_{\phi_{s,1}'}, y_{\phi_{s,2}'}, \ldots, y_{\phi_{s,l(s)'}} = x_{\phi_{s,1}'}, x_{\phi_{s,2}'}, \ldots, x_{\phi_{s,l'(s)'}}$ are linearly independent. We put the $w$ terms, $w(x_{\phi_{s,2}'}, x_{\phi_{s,1}'}, \ldots, w(x_{\phi_{s,l'(s)'}} - x_{\phi_{s,l'(s)'}})$ into $\mathcal{I}$. We leave the $v$ terms in $G_{s}^{\text{chain}}$ out of $\mathcal{I}$.

At this stage we emphasize that the terms we have put in $\mathcal{I}$ from all cycles and chains have linearly independent arguments. If fact, the set of $x$’s appearing in the different chains and the cycles are disjoint. This is obvious for the cycles and the interior of the chains since there are exactly two difference operators $\Delta_{x'}^h$ for each $x$. It also must be true for the endpoints of the chains, since if this is not the case they could be made into larger chains or cycles.

For the same reason, if a $v$ term involving $\Delta_{x'}^h$ is not in any of the sets of chains, then $x'$ will not appear in the arguments of the terms that are put in $\mathcal{I}$ from all the cycles and chains.
Suppose, after considering the \( w \) terms and the \( v \) terms associated with the chains of \( w \) terms, that there are \( p \) pairs of \( v \) terms left, each pair corresponding to difference operators \( \Delta_{z_j}^1, j = 1, \ldots, p \); (\( p \) may be 0). Let

\[
Z := \{z_1, \ldots, z_p\}
\]  

(4.92)

A typical \( v \) term is of the form

\[
v^{(j)}(z_j - u'_j) := v(z_j - u'_j) = \int_0^t |\Delta_z^h p_t(z_j - u'_j)| \, dt,
\]  

(4.93)

where \( u'_j \) is some \( x \) term. We use the superscript \( (j) \) to keep track of the fact that this \( v \) term is associated with the difference operator \( \Delta_{z_j}^1 \). We distinguish between the variables \( z_j \) and \( u'_j \) by referring to \( z_j \) as a marked variable. Note that if \( u'_j \) is also in \( Z \), say \( u'_j = z_k \), then \( u'_j \) is also a marked variable but in a different \( v \) term. (In this case, in \( v^{(k)}(z_k - u'_k) \), where \( u'_k \) is some other \( x \) variable.)

Thus \( Z \) is the collection of marked variables. Consider the corresponding terms

\[
v^{(j)}(z_j - u_j) \quad \text{and} \quad v^{(j)}(z_j - v_j), \quad j = 1, \ldots, p
\]  

(4.94)

where \( u_j \) and \( v_j \) represent whatever terms \( x \) and \( x' \) are coupled with the two variables \( z_j \).

There may be some \( j \) for which \( u_j \) and \( v_j \) in (4.94) are both in \( Z \). Choose such a \( j \). Suppose \( u_j = v_j = z_k \). We set

\[
G_{j,1}^Z = \{v^{(j)}(z_j - z_k), v^{(j)}(z_j - z_k), v^{(k)}(z_k - u_k), v^{(k)}(z_k - v_k)\}
\]  

(4.95)

and put \( v^{(j)}(z_j - z_k) \) into \( I \). Here \( u_k \) and \( v_k \) are whatever two variables appear with the two marked variables \( z_k \).

On the other hand, suppose \( u_j \) and \( v_j \) are both in \( Z \) but \( u_j = z_k \) and \( v_j = z_l \) with \( k \neq l \). We set

\[
G_{j,2}^Z = \{v^{(j)}(z_j - z_k), v^{(j)}(z_j - z_l), v^{(k)}(z_k - u_k), v^{(k)}(z_k - v_k), v^{(l)}(z_l - u_l), v^{(l)}(z_l - v_l)\}
\]  

(4.96)

and put both \( v^{(j)}(z_j - z_k) \) and \( v^{(j)}(z_j - z_l) \) into \( I \).
We then turn to the elements in \( \mathcal{Z} \) which have not yet appeared in the arguments of the terms that have been put into \( \mathcal{I} \). If there is another \( j' \) for which \( u_{j'} \) and \( v_{j'} \) are both in \( \mathcal{Z} \), choose such a \( j' \) and proceed as above. If there are no longer any such elements in \( \mathcal{Z} \), choose some remaining element, say, \( z_i \). Set

\[
G^\mathcal{Z},3_i = \{v^{(i)}(z_i - u_i), v^{(i)}(z_i - v_i)\}
\]  

(4.97)

and if \( u_i \notin \mathcal{Z} \), place \( v^{(i)}(z_i - u_i) \) into \( \mathcal{I} \). If \( u_i \in \mathcal{Z} \), so that \( v_i \notin \mathcal{Z} \), place \( v^{(i)}(z_i - v_i) \) into \( \mathcal{I} \).

We continue until we have exhausted \( \mathcal{Z} \).

The \( v \) and \( w \) terms in \( \mathcal{I} \) have linearly independent arguments. We choose an additional \( \mathcal{I}' = m - |\mathcal{I}| \) terms from the remaining \( u, v \) and \( w \) terms so that the arguments of the \( m \) terms are a spanning set of \( \mathbb{R}^m \). We bound the remaining terms by their supremum. We then make a change of variables and integrate separately each of the \( m \) terms in \( \mathcal{I} \cup \mathcal{I}' \). Our goal is to integrate as few \( u \) terms as necessary.

Let \( S_1 \) denote the number of cycles of length 1. The number of \( w \) terms in \( \mathcal{I} \) from cycles is

\[
\sum_{s=1}^{S}(l(s) - 1) \geq \frac{1}{2} \sum_{s=1}^{S} l(s) - \frac{S_1}{2}.
\]  

(4.98)

The number of \( w \) terms in \( \mathcal{I} \) from chains is

\[
\sum_{s=1}^{S'}(l'(s) - 1) \geq \frac{1}{2} \sum_{s=1}^{S'} l'(s) - \frac{S'}{2}.
\]  

(4.99)

The number of \( w \) terms may be less than \( m \). (In general it is, but we see below that it is possible that the number of \( w \) terms may be equal to \( m \).) Suppose there are \( \rho \) terms of type \( w \). Then the number of \( v \) terms must be \( 2(m - \rho) \), and consequently, the number of \( u \) terms must be \( \rho \).

We note that

\[
\rho = \sum_{s=1}^{S} l(s) + \sum_{s=1}^{S'}(l'(s) - 1).
\]  

(4.100)

Since the total number of \( w \) terms is \( \rho \), we see from (4.98) and (4.99) that the number of \( w \) terms in \( \mathcal{I} \) is at least

\[
\frac{\rho}{2} + \frac{S'}{2} - \frac{S_1}{2}.
\]  

(4.101)

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This shows that for a given \( \rho \) the number of \( w \) terms in \( I \) is minimized when there are no chains.

We now turn to the number of integrated \( v \) terms. Since the total number of \( v \) terms is \( 2(m - \rho) \), and there are also two \( v \) terms in each set \( G^\text{chain}_s \) we see that

\[
2(m - \rho) = \sum_{i,j} |G^{Z,i}_j| + 2S'.
\]  

(4.102)

It is easily seen that we place in \( I \) at least \( 1/4 \) the number of \( v \) terms in the sets \( G^{Z,i}_j \) for all \( i, j \). Consequently, the number of \( v \) terms with arguments in \( I \) is at least

\[
\frac{1}{4} \sum_{i,j} |G^{Z,i}_j| = \frac{m}{2} - \frac{\rho}{2} - \frac{S'}{2}.
\]  

(4.103)

Combined with (4.101) we see that the number of \( w \) and \( v \) terms in \( I \) is at least

\[
\left( \frac{m}{2} - \frac{\rho}{2} - \frac{S'}{2} \right) + \left( \frac{\rho}{2} + \frac{S'}{2} - \frac{S_1}{2} \right) = \frac{m}{2} - \frac{S_1}{2}.
\]  

(4.104)

Since \( \frac{m}{2} - \frac{S_1}{2} \) is an integer, it is at least

\[
\frac{m}{2} - \frac{S_1}{2} + \frac{\bar{I}}{2},
\]  

(4.105)

where \( \bar{I} = 0 \), if \( m - S_1 \) is even, and \( \bar{I} = 1 \), if \( m - S_1 \) is odd.

Suppose that \( \rho \geq \frac{m}{2} + \frac{\bar{S}_1}{2} - \frac{\bar{I}}{2} \). Then, since there are \( m \) terms that are integrated, the upper bound of the extended integral will be greatest if we integrate \( \frac{m}{2} + \frac{\bar{S}_1}{2} - \frac{\bar{I}}{2} \) terms of the form \( u \), and bound the remaining \( u \) terms by their supremum. (Note that \( \frac{m}{2} + \frac{\bar{S}_1}{2} - \frac{\bar{I}}{2} \) is also an integer.) This gives a bound for the \( u \) terms of

\[
\frac{m}{2} + \frac{\bar{S}_1}{2} - \frac{1}{2} \left( t\psi^{-1}(1/t) \right)^{\rho - \frac{m}{2} - \frac{\bar{S}_1}{2} + \frac{\bar{I}}{2}}.
\]  

(4.106)

(We ignore slowly varying function of \( t \).)

Note that when we integrate \( \frac{m}{2} + \frac{\bar{S}_1}{2} - \frac{\bar{I}}{2} \) terms of the form \( u \), we only integrate \( \frac{m}{2} - \frac{\bar{S}_1}{2} + \frac{\bar{I}}{2} \) terms of the form \( v \) and \( w \). By Lemma 3.1 integrated \( v \) terms are much larger than integrated \( u \) terms. What is the maximum number of \( v \) terms that can be integrated? 

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The maximum number of \(v\) terms that can be integrated occurs when all the \(w\) terms are in cycles of length 1 or 2, in which case \((\rho - S_1)/2\) terms of the form \(w\) are integrated. This is easy to see, since in this case the right-hand side of (4.98) is realized. (We point out in the paragraph containing (4.101) that to minimize the number of \(w\) terms that are integrated there should be no chains.)

We are left with \(m^2 - \rho^2 + \bar{b}/2\) terms of the form \(v\) that are integrated. Since, by (3.6), the supremum of the \(v\) terms are effectively bounded this gives a contribution from all \(v\) terms of \(\left(\frac{t}{\psi - 1}(1/t)\right)^{\frac{m}{2} - \rho/2 + 1/2}\). Combining the bounds for \(u\) and \(v\) terms we obtain

\[
t^m \left(\frac{t}{\psi - 1}(1/t)\right)^{\frac{m}{2} - \rho/2 + \bar{b}/2}
= \left(t^2 \psi^{-1}(1/t)\right)^{\frac{m}{2}} \left(t \left(\psi^{-1}(1/t)\right)^3\right)^{-\rho/2} \left(\psi^{-1}(1/t)\right)^{\frac{m}{2}} \left(t \psi^{-1}(1/t)\right)^{\frac{m}{2}}.
\]

It follows from (III, 4.77)] that

\[
t(\psi^{-1}(1/t))^2 \leq C \quad \forall t \geq t_0.
\]

Therefore, when \(\rho \geq \frac{m}{2} + \frac{S_1}{2} - 1/2\), (4.107) is bounded by

\[
C \left(t^2 \psi^{-1}(1/t)\right)^{\frac{m}{2}} \left(t \left(\psi^{-1}(1/t)\right)^3\right)^{-\rho/2} \left(\psi^{-1}(1/t)\right)^{\frac{m}{2}} \left(t \psi^{-1}(1/t)\right)^{\frac{m}{2}}.
\]

On the other hand, when \(\rho < \frac{m}{2} + \frac{S_1}{2} - 1/2\), we get the largest upper bound for the extended integral when we integrate all \(\rho\) of the \(u\) terms. As above, to get the most \(v\) terms integrated, we only integrate \((\rho - S_1)/2\) terms of the form \(w\). Consequently, since \(m\) terms are integrated, \(m - \frac{3\rho}{2} + \frac{S_1}{2}\) of the \(v\) terms are integrated. (The remaining \(v\) terms are bounded by their supremum, which is effectively bounded.) Combining the bounds for \(u\) and \(v\) terms we obtain

\[
t^\rho \left(t \psi^{-1}(1/t)\right)^{m - \frac{3\rho}{2} + \frac{S_1}{2}}
= \left(t^2 \psi^{-1}(1/t)\right)^{\frac{m}{2}} \left(t \left(\psi^{-1}(1/t)\right)^3\right)^{-\rho/2} \left(\psi^{-1}(1/t)\right)^{\frac{m}{2}} \left(t \psi^{-1}(1/t)\right)^{\frac{m}{2}}.
\]
We now show that we obtain (4.85) when $S_1 = 0$. Consider the case when $\rho < \frac{m}{2} - \bar{1}/2$ and refer to (4.110). Note that

$$
\left( t \left( \psi^{-1}(1/t) \right)^3 \right)^{-\rho/2} \left( \psi^{-1}(1/t) \right)^{\frac{m}{2}} = t^{(3/\beta - 1)\rho/2 - m/(2\beta)} L(t) < t^{(3/\beta - 1)m/4 - m/(2\beta)} L(t) = t^{-(\beta - 1)m/(4\beta)} L(t),
$$

where $L(t)$ is slowly varying at infinity and we use the facts that $(3/\beta) - 1 > 0$ and $\rho < \frac{m}{2}$.

Now we consider the case when $\rho \geq \frac{m}{2} - \bar{1}/2$. In dealing with (4.109) we also have that $\rho \leq m - 1$, since we arrived at this inequality by assuming that all cycles are of order two, but are excluding the case when the graph $G_{\pi,a}$ consists solely of cycles of order two. Therefore $\frac{m}{2} - (\rho/2)$ in (4.109) is strictly positive. This observation and (4.111) gives (4.85) when $S_1 = 0$.

We now eliminate the restriction that $S_1 = 0$. This requires additional work since the estimates on the right-hand side of (4.107) and (4.110) are larger in this case. Actually we show that the bounds in (4.107) and (4.110) that we obtained when $S_1 = 0$, remain the same when $S_1 \neq 0$.

The only way there can be cycles of length one is in terms of the type

$$
\Delta^1 \Delta^{-1} p_{(t-\sum_{q=1}^{m_l-1} r_{1-1,q})+r_{1,1}} (x_{\gamma_{2q_r(1)}-1} - x_{\gamma_{2q_{l-1}(n_{l-1})}-1})
$$

when $\gamma_{2q_r(1)-1} = \gamma_{2q_{l-1}(n_{l-1})-1}$. In this case

$$
\int_0^t |\Delta^1 \Delta^{-1} p_s (x_{\gamma_{2q_r(1)}-1} - x_{\gamma_{2q_{l-1}(n_{l-1})}-1})| ds = w(0, t). \quad (4.113)
$$

Note that

$$
|\Delta^1 \Delta^{-1} p_{(t-\sum_{q=1}^{m_l-1} r_{1-1,q})+r_{1,1}} (0)| = 2|\Delta^1 p_{(t-\sum_{q=1}^{m_l-1} r_{1-1,q})+r_{1,1}} (0)|. \quad (4.114)
$$

This is how we bound the right-hand side of (4.85) when $G_{\pi,a}$ contains cycles of length one. We return to the basic formulas (4.29) and (4.30). We obtain an upper bound for (4.30) by taking the absolute value of the integrand. However, we do not, initially extend the region of integration with respect to time. Instead we proceed as follows: Let $l'$ be the largest value of $l$ for which $\gamma_{2q_r(1)-1} = \gamma_{2q_{l-1}(n_{l-1})-1}$. We extend the integral with
respect to \( r_{l,q} \) to \([0,t]\) for all \( l > l' \), and also for \( l = l' \) and \( q > 1 \), and bound these integrals with terms of the form \( u(\cdot, t) \), \( v(\cdot, t) \) and \( w(\cdot, t) \).

We then consider the integral of the term in (4.114) with respect to \( r_{l',1} \).

Clearly

\[
\int_0^t |\Delta^1 p_{t-\sum_{q=1}^{m_{l'}} r_{l'-1,q} + r_{l',1}}(0)| \, dr_{l',1} \leq \int_t^{2t-\sum_{q=1}^{m_{l'}} r_{l'-1,q}} |\Delta^1 p_s(0)| \, ds
\]

If \( \sum_{q=1}^{m_{l'-1}} r_{l-1,q} \leq t/2 \), we use (3.13) to bound the left-hand side of (4.115) by

\[
\int_{t/2}^{2t} |\Delta^1 p_r(0)| \, dr \leq C t \left( \psi^{-1}(1/t) \right)^3 \leq C \psi^{-1}(1/t),
\]

by (4.108).

Suppose, on the other hand, that \( \sum_{q=1}^{m_{l'-1}} r_{l-1,q} \geq t/2 \). Then for some \( q \) we have \( r_{l-1,q} \geq t/2m \). We do two things. We bound the contribution of

\[
\left| \left( \left( \frac{\Delta^1}{\Delta^1 x_{\pi_{l-1}(q)}} \right)^{a_1(l-1,q)} \left( \frac{\Delta^1}{\Delta^1 x_{\pi_{l-1}(q-1)}} \right)^{a_2(l-1,q)} \right) \right|
\]

by its supremum over \( t/2m \leq r_{l-1,q} \leq t \).

To express we use the notation

\[
\bar{u}(x,t) = \sup_{t/2m \leq r \leq t} u(x,r), \quad \bar{v}(x,t) = \sup_{t/2m \leq r \leq t} v(x,r),
\]

and

\[
\bar{w}(x,t) = \sup_{t/2m \leq r \leq t} w(x,r),
\]

so that the bound of (4.117) may be

\[
\bar{u}(x_{\pi_{l-1}(q)} - x_{\pi_{l-1}(q-1)}, t), \quad \bar{v}(x_{\pi_{l-1}(q)} - x_{\pi_{l-1}(q-1)}, t)
\]

or

\[
\bar{w}(x_{\pi_{l-1}(q)} - x_{\pi_{l-1}(q-1)}, t)
\]

according to whether there are no, one or two difference operators.

The terms in (4.120) no longer depend on \( r_{l-1,q} \) therefore we can integrate (4.114) with respect to both \( r_{l,1} \) and \( r_{l-1,1} \) and use (3.14) get

\[
\int_0^{2t} \int_0^{2t} |\Delta^1 p_{r+s}(0)| \, dr \, ds \leq C \left( t^2 \left( \psi^{-1}(1/t) \right)^3 + L(t) + 1 \right).
\]
where $L(t)$ is a slowly varying function at infinity.

Consider how (4.117) contributes to the bounds in (4.107) and (4.110). If there are no difference operators they would ultimately contribute either

$$\sup_x u(x, t) \quad \text{or} \quad \int u(x, t) \, dx$$

(4.122)

Now because of the bound in (4.120) we get a contribution of

$$\sup_x \nu(x, t) \quad \text{or} \quad \int \nu(x, t) \, dx$$

(4.123)

The following table summarizes results from Lemmas 3.1 and 6.6. It shows that each term in (4.122) is smaller than the corresponding term in (4.123) by a factor of $Ct^{-1}$. Up to factors of $\log t$ the same diminution, or more, occurs when we compare the two functions of $v(x, t)$ with those of $\nu(x, t)$ and the two functions of $w(x, t)$ with those of $\nu(x, t)$.

| $f(x, t)$ | $\sup_x f(x, t) \leq$ | $\int f(x, t) \, dx \leq$ |
|-----------|----------------------|---------------------|
| $u(x, t)$ | $Ct\psi^{-1}(1/t)$ | $t$ |
| $\nu(x, t)$ | $C\psi^{-1}(1/t)$ | $C$ |
| $v(x, t)$ | $C \log t$ | $Ct\psi^{-1}(1/t) \log t$ |
| $\nu(x, t)$ | $C (\psi^{-1}(1/t))^2 \leq C/t$ | $Ct\psi^{-1}(1/t) \log t$ |
| $w(x, t)$ | $C$ | $C(\log t)^2$ |
| $\nu(x, t)$ | $C (\psi^{-1}(1/t))^3$ | $C (\psi^{-1}(1/t))^2 \leq C/t$ |

To read the table note that the second line states that $\sup_x u(x, t) \leq C\psi^{-1}(1/t)$ and $\int u(x, t) \, dx \leq t$, and similarly for the remaining lines.

Combined with (4.121) we see that we have reduced the bounds in (4.107) and (4.110) by a factor of

$$C \left( t \left( \psi^{-1}(1/t) \right)^3 + \frac{L(t)+1}{t} \right) \leq C\psi^{-1}(1/t),$$

(4.124)
where for the last inequality we use (4.108), as we do in (4.116).

We apply a similar procedure for each $l$ in decreasing order, with one exception. If $r_{l-1,q} = r_{l-1,1}$, i.e., $q = 1$ in this case and we are also in the (4.114) with $l$ is replaced by $l - 1$, we skip this term because this it has already been modified. We then proceed to deal with remaining terms as we did when we assumed that there were no cycles of length one.

Consequently, if there are $S_1$ cycles of length 1 we have diminished the bounds in (4.107) and (4.110) by a factor of at least $C \left( \psi^{-1}(1/t) \right)^{\frac{S_1}{2}}$, if $S_1$ is even and by a factor of at least $C \left( \psi^{-1}(1/t) \right)^{\frac{S_1}{2} + \frac{1}{2}}$, if $S_1$ is odd.

In the case of (4.107) we are precisely in the case we considered when $S_1 = 0$, which gives (4.85). In the case of (4.110) the final factor is now $(t(\psi^{-1}(1/t))^2)^{\frac{S_1}{2}}$, which is bounded by a constant by (4.108). Thus we are again in the case we considered when $S_1 = 0$, which also gives (4.85).

It follows from (4.84) and (4.85) that when $m$ is even

\[
\sum_a \sum_{\pi_0, \ldots, \pi_K} \int T_{t}(x; \pi, a) \prod_{j,k,i} dx_{j,k,i} = \sum_{\pi_0, \ldots, \pi_K} \int T'_{t}(x; \pi, a) \prod_{j,k,i} dx_{j,k,i}
\]

\[
= \prod_{j,k=0}^{K} \left( 2n_{j,k}!! \right) (4c_{j,k})^{n_{j,k}} E \left\{ \prod_{j,k=0}^{K} (\alpha_{j,k,t})^{n_{j,k}} \right\} + O \left( t^{(2-1/\beta)n-\epsilon} \right).
\]

We now show that we get the same estimates when $T_{t}(x; \pi, a)$ is replaced by $T'_{t}(x; \pi, a)$; (see (4.27) and (4.30)).

We point out, in the paragraph containing (4.22) that terms of the form $\Delta^l \Delta^{-1} p^x$ in (4.27) are always of the form $\Delta^l \Delta^{-1} p$. Therefore, in showing that (4.26) and (1.29) have the same asymptotic behavior as $t \to \infty$ we need only consider how the proof of (4.125) must be modified when the arguments of the density functions with one or no difference operators applied is effected by adding $\pm 1$.

It is easy to see that the presence of these terms has no effect on the integrals that are $O \left( \left( t^2 \psi^{-1}(1/t) \right)^n t^{-\epsilon} \right)$ as $t \to 0$. This is because in evaluating these expressions we either integrate over all of $R^1$ or else use bounds that hold on all of $R^1$. Since terms with one difference operator only occur in these estimations, we no longer need to be concerned with them.

Consider the terms with no difference operators applied to them, now
denoted by $p^\sharp$. So, for example, instead of $F(\tilde{\sigma},0,\ldots,0)$ on the right-hand side of (4.80), we now have

$$
\int \left( \int_{\mathbb{R}_0(0) \times \cdots \times \mathbb{R}_K(0)} \prod_{l=0}^K p^\sharp_{(1-\sum_{q=1}^{n_l-1} r_{1-1,2q-1}-s_{l-1})+r_{l,1}} \right.
$$

$$
\left( x_{\tilde{\sigma}(1)} - x_{\tilde{\sigma}_{l-1}(n_{l-1})} \right) \prod_{q=2}^{n_l} p^\sharp_{r_{1,2q-1}} \left( x_{\tilde{\sigma}(q)} - x_{\tilde{\sigma}_{l}(q-1)} \right) \prod_{q=1}^{n_l} dr_{l,2q-1} \right) dx.
$$

(4.126)

Suppose that $p^\sharp_r(y_\sigma(i) - y_\sigma(i-1)) = p_r(y_\sigma(i) - y_\sigma(i-1) \pm 1)$. We write this term as

$$
p_r^\sharp(y_\sigma(i) - y_\sigma(i-1)) = p_r(y_\sigma(i) - y_\sigma(i-1)) + \Delta^\pm p_r(y_\sigma(i) - y_\sigma(i-1)).
$$

(4.127)

Substituting all such terms into (4.126) and expanding we get (4.125) and many other terms with at least one $p_r(y_\sigma(i) - y_\sigma(i-1))$ replaced by $\Delta^\pm p_r(y_\sigma(i) - y_\sigma(i-1))$.

Substitute (4.127) into (4.126) and write it as the sum of $2^m$ terms. One term, which contains no difference operator, is the term we analyzed when we replaced $p^\sharp$ by $p$. All the other terms contain at least one difference operator. It is easy to see that all these other terms are $O\left((t^2 \psi^{-1} p_t(x,y) 1/t)^n t^{-\epsilon}\right)$, for some $\epsilon > 0$.

By (4.73) the term with no difference operators is bounded by $C((t^2 \psi^{-1}(1/t))^n)$. This bound is obtained by extending the integrals to $[0,t]$ in (4.72) and integrating or bounding the resulting terms $u(\cdot,t)$. We are in a situation similar to the one considered in the paragraph containing (4.122). Each difference operator in the other terms replaces a $u(\cdot,t)$ term by a $v(\cdot,t)$ term. By Lemma 3.1 each replacement reduces $C((t^2 \psi^{-1}(1/t))^n)$ by a factor of at least $(t^2 \psi^{-1}(1/t))^{-1}$. Therefore, the replacement of $p$ by $p^\sharp$ doesn't change (4.125) when $m$ is even.

We now obtain (4.2). In Subsection 4.2 we do not require that $m$ is even. Therefore, (4.2) follows from (4.85) unless $G_{\pi,a}$ consists solely of cycles of order two and there are no terms with a single difference operator. Therefore, (4.2) follows from (4.85) when $m$ is odd unless we are in the situation covered in Subsection 4.1. This also holds when when $p$ is replaced by $p^\sharp$ for the reasons given in the case when $m$ is even.

However, if any of the $m_{j,k}$ are odd we can not be in the situation covered in Subsection 4.1. Consider the multigraph $G_{\pi}$ described in the
paragraph following (4.33), with vertices \(\{(j, k, i), 0 \leq j < k \leq K, 1 \leq i \leq m_{j,k}\}\), and an edge between the vertices \(\pi_l(2q - 1)\) and \(\pi_l(2q)\) for each \(0 \leq l \leq K\) and \(1 \leq q \leq n_l\). Each vertex is connected to two edges.

Suppose that \(\{(j, k, i)\} = \pi_l(2q)\), with \(j = l\) and \(k = l' \neq l\). Then there is a unique \(q'\) such that \(\pi_{l'}(2q')\) or \(\pi_{l'}(2q' - 1)\) is also equal to \(\{(j, k, i)\}\).

Suppose \(\pi_{l'}(2q') = \{(j, k, i)\}\) and consider \(\pi_{l'}(2q')\) and \(\pi_{l'}(2q' - 1)\). In order that \(G_{\pi}\) consist of cycles or order two, we must have \(\pi_{l'}(2q' - 1) = \pi_{l'}(2q')\). This shows that for \(G_{\pi}\) to consist of cycles or order two \(m_{j,k}\) must be even for each \(j, k\).

\[\square\]

### 4.3 Proof of Lemma 4.3

For any \(A \subseteq [0, 3t]^n\) we set

\[
F_A = \int \left\{ \int_A \prod_{l=0}^{K} p_{\pi_l,1} (x_{\pi_l(1)} - x_{\pi_{l-1}(n_{l-1})}) \right. \\
\left. \prod_{q=2}^{n_l} p_{\pi_{l',2q-1}} (x_{\pi_{l'}(q)} - x_{\pi_{l'}(q-1)}) \prod_{l=0}^{K} \prod_{q=1}^{n_l} dr_{l,2q-1} \right\} \prod_{q=1}^{n_l} dx_{\pi_{l'}(q)}.
\]

Rather than bound the time integral by that over \([0, 3t]^n\) as we have in the past, we have to be more careful.

It follows from (4.70), paying special attention to the time variable of \(p\) in the second line, that

\[
F(\sigma, s_0, \ldots, s_K) = F_{A_{s_0,\ldots,s_K}}
\]

where

\[
A_{s_0,\ldots,s_K} = \left\{ r \in R^n_+ \left| \sum_{\lambda=0}^{l-1} (t - \sum_{q=1}^{n_{\lambda}} r_{\lambda,2q-1} - s_\lambda) \leq \sum_{q=1}^{n_l} r_{l,2q-1} \right. \right. \\
\left. \left. \leq \sum_{\lambda=0}^{l-1} (t - \sum_{q=1}^{n_{\lambda}} r_{\lambda,2q-1} - s_\lambda) + (t - s_l); l = 0, 1, \ldots, K \right\}.
\]

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In particular

$$A_{0,\ldots,0} = \left\{ r \in [0, 3t]^n \left| \sum_{\lambda=0}^{l-1} (t - \sum_{q=1}^{n_{\lambda}} r_{\lambda, 2q-1}) \leq \sum_{q=1}^{n_{l}} r_{l, 2q-1} \right. \right\},$$

(4.131)

$$\leq \sum_{\lambda=0}^{l-1} (t - \sum_{q=1}^{n_{\lambda}} r_{\lambda, 2q-1}) + t; \ l = 0, 1, \ldots, K \right\}.$$

Let $\phi_l(r) = \sum_{r_{\lambda, 2q-1}}^{l} (t - \sum_{q=1}^{n_{\lambda}} r_{\lambda, 2q-1})$. We have

$$A_{s_0,\ldots,s_K} \Delta A_{0,\ldots,0}$$

(4.132)

$$\subseteq \bigcup_{l=1}^{K} \left\{ r \in [0, 3t]^n \left| \phi_{l-1}(r) - \sum_{\lambda=0}^{l-1} s_{\lambda} \leq \sum_{q=1}^{n_{l}} r_{l, 2q-1} \leq \phi_{l-1}(r) \right. \right\}$$

$$\bigcup_{l=0}^{K} \left\{ r \in [0, 3t]^n \left| \phi_{l-1}(r) + t - \sum_{\lambda=0}^{l-1} s_{\lambda} \leq \sum_{q=1}^{n_{l}} r_{l, 2q-1} \leq \phi_{l-1}(r) + t \right. \right\}.$$  

$$= \bigcup_{l=1}^{K} A_l \cup B_l$$

where, setting $\check{\phi}_{l-1}(r) = \phi_{l-1}(r) - \sum_{q=1}^{n_{l-1}} r_{l, 2q-1}$ we can write

$$A_l = \left\{ r \in [0, 3t]^n \left| \check{\phi}_{l-1}(r) - \sum_{\lambda=0}^{l-1} s_{\lambda} \leq r_{l, 2n_{l-1}} \leq \check{\phi}_{l-1}(r) \right. \right\}$$

(4.133)

and

$$B_l = \left\{ r \in [0, 3t]^n \left| \check{\phi}_{l-1}(r) + t - \sum_{\lambda=0}^{l-1} s_{\lambda} \leq r_{l, 2n_{l-1}} \leq \check{\phi}_{l-1}(r) + t \right. \right\}.$$  

(4.134)

(The first union in (4.132) are the points in $A_{s_0,\ldots,s_K}$ that are not in $A_{0,\ldots,0}$ and the second union are the points in $A_{0,\ldots,0}$ that are not in $A_{s_0,\ldots,s_K}$.)

Note that each time $r_{l, 2n_{l-1}}$ is contained in an interval of length $2(K + 1)n\sqrt{t}$.

We bound each $F_{A_l}$ and $F_{B_l}$ as in (4.133) except that we only integrate with respect to $r_{l, 2n_{l-1}}$ over $A_l$ or $B_l$. Therefore, instead of getting a bound of $u(x, t)$ or $\int u(x, t) \, dx$ from this term we get a smaller bound.

To see this, for fixed $a, b \geq 0$, let

$$u_{a,b}(x) = \int_{a}^{a+b} p_s(x) \, ds.$$  

(4.135)
Clearly
\[ \int u_{a,b}(x) \, dx = \int_{a}^{a+b} 1 \, ds = b. \tag{4.136} \]

In addition, by (6.5),
\[ \sup_{x} u_{a,b}(x) = \sup_{x} \int_{a}^{a+b} e^{ipx} e^{-s\psi(p)} \, dp \, ds \tag{4.137} \]
\[ = \int_{a}^{a+b} e^{-s\psi(p)} \, dp \, ds \leq \int_{0}^{b} e^{-s\psi(p)} \, dp \, ds \leq C b \psi^{-1}(b). \]

Using (4.135)–(4.137) with \( b = C \sqrt{t} \), and Lemma 3.1 we see that the bound in (4.73) is reduced by a factor of at least \( (t \psi^{-1}(1/t))^{-\frac{1}{2}} \) for any \( \epsilon' > 0 \).

5 Proof of Lemmas 2.1–2.3

Proof of Lemma 2.1 Using the multinomial theorem on the sum in (2.3) we have
\[ E \left( (\tilde{I}_{l,t})^{m} \right) = \sum_{\bar{m} \in \mathcal{M}} \left( \frac{m!}{\Pi_{j<k}^{l-1} (m_{j,k}!)} \right) E \left( \prod_{j,k=0}^{l-1} (I_{j,k,t/l})^{m_{j,k}} \right), \tag{5.1} \]
where
\[ \mathcal{M} = \left\{ \bar{m} = \{m_{j,k}, 0 \leq j < k \leq l-1 \} \left| \sum_{j,k=0}^{l-1} m_{j,k} = m \right\} \right. \]

We now use Lemma 4.1 with \( t \) replaced by \( t/l \) to compute the expectation on the right-hand side of (5.1). We get that when all the \( m_{j,k} \) are even, there exists an \( \epsilon > 0 \) such that
\[ E \left( (\tilde{I}_{l,t})^{m} \right) \]
\[ = \sum_{\bar{m} \in \mathcal{M}} \left( \frac{m!}{\Pi_{j<k}^{l-1} (m_{j,k}!)} \right) \prod_{j,k=0}^{l-1} \frac{(2n_{j,k})!}{2^{n_{j,k}} (n_{j,k}!)} (4c_{\psi,1})^{n_{j,k}} \prod_{j,k=0}^{l-1} \left( \alpha_{j,k,t/l} \right)^{n_{j,k}} \]
\[ + O(t^{m} (t^{2} \psi^{-1}(1/t))^{n} t^{-\epsilon}). \tag{5.2} \]
(Recall that when all the \(m_{j,k}\) are even, \(m_{j,k} = 2n_{j,k}\) for all \(j\) and \(k\) and \(n = m/2\).) Here we use the fact that

\[
\sum_{m \in \mathcal{M}} \left( \frac{m!}{\prod_{j \leq k} (m_{j,k}!)} \right) = l^m \tag{5.3}
\]

to compute the error term. It also follows from Lemma 4.1 that

\[
E \left( \left( I_{l,t} \right)^m \right) = O(l^m (t^2 \psi^{-1}(1/t))^{m/2} t^{-\epsilon}) \tag{5.4}
\]

if any of the \(m_{j,k}\) are odd. (Lemma 4.1 is for a fixed partition of \(m\). In (5.2) and (5.4) we include the factor \(l^m\), to account for the number of possible partitions.) Recall that \(l = \lceil \log t \rceil^q\) for some \(q > 0\).

When \(m_{j,k} = 2n_{j,k}\) for all \(j\) and \(k\),

\[
\left( \frac{m!}{\prod_{j \leq k}^{l-1} (m_{j,k}!)} \right) \prod_{j \leq k}^{l-1} (2n_{j,k}!) = \frac{(2n)!}{2^n n!} \frac{n!}{\prod_{j \leq k}^{l-1} (n_{j,k}!)} \tag{5.5}
\]

Using this in (5.2) we get

\[
E \left( \left( \tilde{I}_{l,t} \right)^m \right) = \frac{(2n)!}{2^n n!} (4c_{\psi,1})^n \sum_{\mathcal{N}} \left( \frac{n!}{\prod_{j \leq k}^{l-1} n_{j,k}!} \right) E \left\{ \prod_{j \leq k}^{l-1} \left( \alpha_{j,k,t/l} \right)^{n_{j,k}} \right\} + O(l^m (t^2 \psi^{-1}(1/t))^{n t^{-\epsilon}}),
\]

where \(\mathcal{N}\) is defined similarly to \(\mathcal{M}\). Using the multinomial theorem as in (5.1) we see that the sum in (5.6) is equal to \(E \{ (\tilde{\alpha}_{l,t})^n \}\), which completes the proof of (2.10).

**Proof of Lemma 2.2** By Kac’s moment formula

\[
E \{ (\alpha_t)^n \} = E \left( \left( \int (L_t^x)^2 dx \right)^n \right) \tag{5.7}
\]

\[
= 2^n \sum_{\pi} \int \left\{ \prod_{i=1}^{2n} \frac{2n}{p_{r_i}} (x_{\pi(i)} - x_{\pi(i-1)}) \right\} \prod_{i=1}^{2n} dr_i \prod_{i=1}^{n} dx_i,
\]

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where the sum runs over all maps $\pi : [1, 2n] \to [1, n]$ with $|\pi^{-1}(i)| = 2$ for each $i$. The factor $2^n$ comes from the fact that we can interchange each $x_{\pi(i)}$ and $x_{\pi(i-1)}$, $i = 1, \ldots, 2n$.

It is not difficult to see that we can find a subset $J = \{i_1, \ldots, i_n\} \subseteq [1, 2n]$, such that each of $x_1, \ldots, x_n$ can be written as a linear combination of $y_j := x_{\pi(i_j)} - x_{\pi(i_j-1)}$, $j = 1, \ldots, n$. For $i \in J^c$ we use the bound $p_{ri}(x_{\pi(i)} - x_{\pi(i-1)}) \leq p_{ri}(0)$, then change variables and integrate out the $y_j$, to see that

$$\int \left( \prod_{i=1}^{2n} \int_0^t p_{ri}(x_{\pi(i)} - x_{\pi(i-1)}) \, dr_i \right)^n \prod_{i=1}^n dx_i \leq C \left( \int_0^t p_{ri}(0) \, dr_i \right)^n \prod_{i=1}^n dx_i = Cu^n(0, t) \left( \prod_{i \in J} \int_0^t p_{ri}(y_i) \, dr_i \, dy_i \right) = Cu^n(0, t) \left( \int u(x, t) \, dx \right)^n \leq C \left( t^2 \psi^{-1}(1/t) \right)^n,$$

where we use (3.5) and (3.8) for the last line. This shows that

$$\| \alpha_t \|_n \leq Ct^2 \psi^{-1}(1/t),$$

for all $t$ sufficiently large, where $C$ depends only on $n$, and where $\| \cdot \|_n := (E(\cdot)^n)^{1/n}$.

It follows from (5.9) that for $l$ sufficiently large,

$$\left| \|2\tilde{\alpha}_{l,t}\|_n - \|\alpha_t\|_n \right| \leq \|2\tilde{\alpha}_{l,t} - \alpha_t\|_n = \| \sum_{j=0}^{l-1} \alpha_{j,l,t/l} \|_n$$

$$\leq l \| \alpha_{0,0,l/t} \|_n = l \| \alpha_{l/t} \|_n \leq Cl^2 \psi^{-1}(l/t).$$

We next show that when $l = l(t) = [\log t]^q$ for any $q > 0$,

$$\lim_{t \to \infty} \frac{\|2\tilde{\alpha}_{l,t}\|_n - \|\alpha_t\|_n}{t^2 \psi^{-1}(1/t)} = 0. \quad (5.11)$$

This follows from (5.10) since

$$\lim_{t \to \infty} \frac{\psi^{-1}(l/t)}{l \psi^{-1}(1/t)} = 0. \quad (5.12)$$
To obtain (5.12) we use [3, Theorem 1.5.6, (iii)] to see that for all $\delta > 0$, there exists a $t_0$, such that for all $t \geq t_0$

$$\frac{\psi^{-1}(l/t)}{\psi^{-1}(1/t)} \leq t^{(1/\beta)+\delta}. \quad (5.13)$$

Obviously, we pick $\delta$ such that $(1/\beta)+\delta < 1$.

The statement in (2.11) follows from (5.11). \qed

For the proof of Lemma 2.3 we need the following lemma which is proved in Section 8.

**Lemma 5.1** Under the hypotheses of Theorem 1.1

$$E \left( \int (L_t^{x+1} - L_t^x)^2 \, dx \right) = 4c_{\psi,0}t + O(g(t)) \quad (5.14)$$

as $t \to \infty$, where

$$g(t) = \begin{cases} 
  t^2 (\psi^{-1}(1/t))^3 & 3/2 < \beta \leq 2 \\
  L(t) & \beta = 3/2 \\
  C & 1 < \beta < 3/2 
\end{cases} \quad (5.15)$$

and $L(\cdot)$ is slowly varying at infinity. Also

$$\text{Var} \left( \int (L_t^{x+1} - L_t^x)^2 \, dx \right) \leq Ct^2 \psi^{-1}(1/t) \log t. \quad (5.16)$$

**Proof of Lemma 2.3** We prove this lemma by showing that

$$\lim_{t \to \infty} \frac{\sum_{j=0}^{l-1} E(I_{j,j+l/t}) - E \left( \int (L_t^{x+1} - L_t^x)^2 \, dx \right)}{t (\psi^{-1}(1/t))^{1/2}} = 0 \quad (5.17)$$

and

$$\lim_{t \to \infty} \frac{\sum_{j=0}^{l-1} \left( I_{j,j+l/t} - E(I_{j,j+l/t}) \right)}{t (\psi^{-1}(1/t))^{1/2}} = 0 \quad (5.18)$$

in $L^2$, where $l = l(t) = [\log t]^q$, for some $q$ sufficiently large.

Set

$$\phi(t) = t^2 \psi^{-1}(1/t). \quad (5.19)$$
It follows from (5.13) that
\[
\frac{l\phi(t/l)}{\phi(t)} = \psi^{-1}(l/t) \leq l^{1/\beta + \delta - 1}, \tag{5.20}
\]
for all \(\delta > 0\). Recall \(l = [\log t]^q\). We choose a \(\delta\) and \(q < \infty\) such that
\[
\frac{l(t)\phi(t/l(t))}{\phi(t)} = O\left(\frac{1}{\log^2 t}\right), \tag{5.21}
\]
as \(t \to \infty\).

We see from (5.14) that
\[
\lim_{t \to \infty} \sum_{j=0}^{l-1} E\left(I_{j,j,l(t)} - E\left(\int (L_t^{x+1} - L_t^x)^2 dx\right)\right)
= \lim_{t \to \infty} \frac{l(t)O(g(t/l(t))) + O(g(t))}{t(\psi^{-1}(1/t))^{1/2}} = 0.
\tag{5.22}
\]
The last equality follows from the fact that \(t(\psi^{-1}(1/t))^{1/2}\) is regularly varying as \(t \to \infty\) with index \(1 - 1/(2\beta) > 1/2\), since \(\beta > 1\), whereas \(g(t)\) is regularly varying as \(t \to \infty\) with index \((2 - 3/\beta)^+ \leq 1/2\) since, \(\beta \leq 2\), and \(l(t)\) is slowly varying.

Since \(I_{j,j,l(t)}\) are independent and identically distributed, we obtain (5.18) by showing that
\[
\lim_{t \to \infty} l(t)\text{Var}\left(\frac{I_{j,j,l(t)}}{t(\psi^{-1}(1/t))^{1/2}}\right) = 0. \tag{5.23}
\]
Using (5.16) and (5.19) we see that
\[
l(t)\text{Var}\left(\frac{I_{j,j,l(t)}}{t(\psi^{-1}(1/t))^{1/2}}\right) = O\left(\frac{l(t)\phi(t/l(t))}{\phi(t)} \log t\right). \tag{5.24}
\]
as \(t \to \infty\). Thus (5.23) follows from (5.21). \(\Box\)
Since the Lévy processes, $X$, that we are concerned with satisfy (1.6), it follows from the Riemann Lebesgue Lemma that they have transition probability density functions, which we designate as $p_s(\cdot)$. Taking the inverse Fourier transform of the characteristic function $X_s$, and using the symmetry of $\psi$, we see that

$$p_s(x) = \frac{1}{2\pi} \int e^{ipx} e^{-s\psi(p)} dp = \frac{1}{\pi} \int_0^\infty \cos(px) e^{-s\psi(p)} dp. \quad (6.1)$$

Our basic hypothesis is that $\psi(\lambda)$ is regularly varying at 0 with index $1 < \beta \leq 2$. Therefore $\psi(\cdot)$ is asymptotic to an increasing function near zero. Considering the way we use $\psi(\cdot)$ in the estimates below, we can assume that $\psi(\lambda)$ is strictly increasing for $0 \leq \lambda \leq \lambda_0$, for some $\lambda_0 > 0$, and that $\psi^{-1}(\lambda)$ is well defined for $0 \leq \lambda \leq \lambda_0$. Actually, we are really interested in $\psi^{-1}(1/s)$ as $s \to \infty$. Therefore, there exists an $s_0$ such that $\psi^{-1}(1/s)$, as a function of $s$ is regularly varying with index $-1/\beta$ for $s \geq s_0$.

The next two lemmas give fundamental estimates that are used in the proofs of the lemmas in Section 3.

**Lemma 6.1** Let $X$ be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at 0 with index $1 < \beta \leq 2$ and satisfies (1.6)–(1.8). Then for all $\gamma \geq 1$ and for all $s$ sufficiently large and all $x \in \mathbb{R}$,

$$p_s(x) \leq C \left( \psi^{-1}(1/s) \wedge \frac{1}{\psi^{-1}(1/s) x^2} \right); \quad (6.2)$$

$$|\Delta^\gamma p_s(x)| \leq C \gamma^2 \left( \left( \psi^{-1}(1/s) \right)^2 \wedge \frac{1 + \log^+ |x|}{x^2} \right); \quad (6.3)$$

$$|\Delta^\gamma \Delta^{-\gamma} p_s(x)| \leq C \gamma^2 \left( \left( \psi^{-1}(1/s) \right)^3 \wedge \frac{\psi^{-1}(1/s)}{x^2} \right). \quad (6.4)$$

**Lemma 6.2** Let $X$ be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at 0 with index $1 < \beta \leq 2$ and satisfies (1.6)–(1.8). Then for all $\gamma \geq 1$ and for all $s$ sufficiently large and all $x \in \mathbb{R}$,
Then for all $t$ sufficiently large and all $x \in \mathbb{R}^1$

$$u(x,t) := \int_0^t p_s(x) \, ds \leq C \left( t \psi^{-1}(1/t) \wedge \frac{t + \log^+ |x|}{|x|} \right), \quad (6.5)$$

$$v_\gamma(x,t) := \int_0^t |\Delta^\gamma p_s(x)| \, ds \leq C \gamma^2 \left( \log t \wedge \frac{t \psi^{-1}(1/t)}{|x|} \wedge t \frac{1 + \log^+ |x|}{x^2} \right) \quad (6.6)$$

and

$$w_\gamma(x,t) := \int_0^t |\Delta^\gamma \Delta^{-\gamma} p_s(x)| \, ds \leq C \gamma^2 \left( 1 \wedge \frac{\log t}{|x|} \wedge t \psi^{-1}(1/t) \right). \quad (6.7)$$

We use the following lemma repeatedly.

**Lemma 6.3** For all $p \in \mathbb{R}^1$ and $s,q > 0$,

$$e^{-s\psi(p)} \leq \frac{C}{s^q \psi^q(p)}. \quad (6.8)$$

**Proof** This is elementary since for all $q > 0$

$$e^{-s\psi(p)} \leq \frac{\sup_{s\geq 0} s^q \psi^q(p)e^{-s\psi(p)}}{s^q \psi^q(p)}. \quad (6.9)$$

□

**Proof of Lemma 6.1** We first note that by (6.8) with $q = 1$, and (1.6)

$$\int_0^\infty e^{-s\psi(p)} \, dp \quad (6.10)$$

$$\leq \left( \int_0^{\psi^{-1}(1/s)} e^{-s\psi(p)} \, dp + \int_{\psi^{-1}(1/s)}^1 e^{-s\psi(p)} \, dp + \int_1^\infty e^{-s\psi(p)} \, dp \right)$$

$$\leq \left( \psi^{-1}(1/s) + \frac{1}{s} \int_{\psi^{-1}(1/s)}^1 \frac{1}{\psi(p)} \, dp + \frac{1}{s} \int_1^\infty \frac{1}{\psi(p)} \, dp \right)$$

$$\leq C \left( \psi^{-1}(1/s) + \frac{1}{s} \right) \leq C \psi^{-1}(1/s),$$

for all $s$ sufficiently large. Therefore, it follows from (6.11), that for all $s$ sufficiently large

$$p_s(x) \leq C \left( \psi^{-1}(1/s) \right). \quad (6.11)$$
By integration by parts

\[ p_s(x) = \frac{1}{\pi x} \int_0^\infty e^{-s\psi(p)} d\sin px \]  
\[ = -\frac{1}{\pi x} \int_0^\infty \sin px \left( \frac{d}{dp} e^{-s\psi(p)} \right) dp \]
\[ = -\frac{1}{\pi x^2} \int_0^\infty \cos px \left( \frac{d^2}{dp^2} e^{-s\psi(p)} \right) dp. \]

where the last line uses the fact that \( \psi'(0) = 0 \), which follows from (1.5) and the first inequality in (1.7).

We have

\[ \frac{d^2}{dp^2} e^{-s\psi(p)} = \left( s^2 (\psi'(p))^2 - s \psi''(p) \right) e^{-s\psi(p)} \]  
(6.13)

By (6.8) and (1.7) for \( p \leq 1 \)

\[ s^2 (\psi'(p))^2 e^{-s\psi(p)} \leq Cs \frac{\psi''(p)}{\psi(p)} \]
\[ \leq C \frac{s \psi(p)}{p^2} \]  
(6.14)

and

\[ s |\psi''(p)| e^{-s\psi(p)} \leq C \frac{s \psi(p)}{p^2}. \]  
(6.15)

Therefore, for all \( s \) sufficiently large

\[ \left| \int_0^{\psi^{-1}(1/s)} \cos px \left( \frac{d^2}{dp^2} e^{-s\psi(p)} \right) dp \right| \leq Cs \int_0^{\psi^{-1}(1/s)} \frac{\psi(p)}{p^2} dp \]
\[ \leq C \frac{1}{\psi^{-1}(1/s)}. \]  
(6.16)

By (6.8), (6.13) and (1.7), for \( p \leq 1 \)

\[ \left| \frac{d^2}{dp^2} e^{-s\psi(p)} \right| \leq C \left\{ \left( \frac{\psi'(p)}{\psi(p)} \right)^2 + \frac{|\psi''(p)|}{\psi(p)} \right\} \leq C \frac{1}{p^2}. \]  
(6.17)

Therefore, for all \( s \) sufficiently large

\[ \left| \int_{\psi^{-1}(1/s)}^1 \cos px \left( \frac{d^2}{dp^2} e^{-s\psi(p)} \right) dp \right| \leq C \int_{\psi^{-1}(1/s)}^1 \frac{1}{p^2} dp \]
\[ \leq C \frac{1}{\psi^{-1}(1/s)}. \]  
(6.18)
By (6.17) and (1.8)
\[ \left| \int_1^\infty \cos px \left( \frac{d^2}{dp^2} e^{-s\psi(p)} \right) dp \right| \leq C \int_1^\infty \left\{ \left( \frac{\psi'(p)}{\psi(p)} \right)^2 + \left| \frac{\psi''(p)}{\psi(p)} \right| \right\} dp \leq C. \] (6.19)

Using (6.11), (6.12), (6.16), (6.18) and (6.19) we get (6.2).

We now obtain (6.4).

\[ \Delta^\gamma \Delta^{-\gamma} p_s(x) = 2p_s(x) - p_s(x + \gamma) - p_s(x - \gamma) \] (6.20)

Therefore, by (6.8)
\[ \left| \Delta^\gamma \Delta^{-\gamma} p_s(x) \right| \leq C \int_0^\infty \sin^2 (p \gamma/2) e^{-s\psi(p)} dp \leq C \gamma^2 \left( \psi^{-1}(1/s) \right)^3. \] (6.21)

We next show that
\[ \Delta^\gamma \Delta^{-\gamma} p_s(x) = \frac{8}{\pi} \frac{K_\gamma(s, x)}{x^2} \] (6.22)

where
\[ K_\gamma(s, x) := \int_0^\infty \sin^2 (px/2) \left( \sin^2 (p \gamma/2) e^{-s\psi(p)} \right)'' dp. \] (6.23)

To get this we integrate by parts in (6.20),
\[ \int_0^\infty \cos px \sin^2 (p \gamma/2) e^{-s\psi(p)} dp \] (6.24)

\[ = -\frac{1}{x} \int_0^\infty \sin px \left( \sin^2 (p \gamma/2) e^{-s\psi(p)} \right)' dp \]
\[ = -\frac{1}{x} \int_0^\infty \left( \sin^2 (p \gamma/2) e^{-s\psi(p)} \right)' d \left( \int_0^p \sin rx dr \right) \]
\[ = -\frac{1}{x^2} \int_0^\infty \left( \sin^2 (p \gamma/2) e^{-s\psi(p)} \right)' d (1 - \cos px) \]
\[ = \frac{2}{x^2} \int_0^\infty \sin^2 (px/2) \left( \sin^2 (p \gamma/2) e^{-s\psi(p)} \right)'' dp. \]
which gives (6.22).

Let \( g(p) = e^{-s\psi(p)} \) and note that

\[
\left( 2 \sin^2(p\gamma/2) e^{-s\psi(p)} \right)' = \gamma g(p) \sin(p\gamma) + 2g'(p) \sin^2(p\gamma/2)
\]  

(6.25)

and

\[
\left( 2 \sin^2(p/2) e^{-s\psi(p)} \right)'' = \gamma^2 g(p) \cos(p\gamma) + 2\gamma g'(p) \sin(p\gamma) + 2g''(p) \sin^2(p\gamma/2).
\]

(6.26)

Substituting (6.26) in (6.23) we write

\[
K_{\gamma}(s,x) = I + II + III.
\]

Note that

\[
|I| = \gamma^2 \left| \int_0^\infty \cos(p\gamma) \sin^2(px/2) e^{-s\psi(p)} \, dp \right| \leq C \gamma^2 \psi^{-1}(1/s)
\]

by (6.10).

By (6.26)

\[
|II| = 2\gamma \left| \int_0^\infty \sin(p\gamma) \sin^2(px/2) g'(p) \, dp \right| \leq C \gamma^2 \int_0^{\psi^{-1}(1/s)} ps|\psi'(p)| e^{-s\psi(p)} \, dp
\]

(6.28)

By (1.7) and (6.8) the first of these last three integrals

\[
\leq C \int_0^{\psi^{-1}(1/s)} s\psi(p) \, dp \leq C \int_0^{\psi^{-1}(1/s)} \, dp \leq C \psi^{-1}(1/s).
\]

(6.29)

By (1.7) and (6.8) the second of the last three integrals in (6.28)

\[
\leq \frac{C}{s} \int_0^1 \frac{s^2 \psi^2(p) \, dp}{\psi(p)} \leq \frac{C}{s} \int_0^1 \frac{dp}{\psi(p)} \leq C \psi^{-1}(1/s).
\]

(6.30)

By (1.8) and (6.8) the third of the last three integrals in (6.28)

\[
\frac{1}{s} \int_1^\infty \frac{s^2 \psi^2(p) \left| \psi'(p) \right| e^{-s\psi(p)} \, dp}{\psi^2(p)} \leq \frac{1}{s} \int_1^\infty \frac{\left| \psi'(p) \right|}{\psi^2(p)} \, dp \leq \frac{C}{s}.
\]

(6.31)
Since $1/s < \psi^{-1}(1/s)$ for all $s$ sufficiently large, and $\gamma \geq 1$, we see that
\[ |II| \leq C\gamma^2 \psi^{-1}(1/s) \quad \forall x \in R^1. \tag{6.32} \]

Similarly,
\[
|III| = 2\left| \int_0^\infty \sin^2(p\gamma) \sin^2(px/2) g''(p) \, dp \right| \tag{6.33}
\]
\[
\leq C\gamma^2 \int_0^\psi \psi^{-1}(1/s) p^2 \left( s|\psi''(p)| + s^2|\psi'(p)|^2 \right) e^{-s\psi(p)} \, dp
\]
\[
+ C\gamma^2 \int_{\psi}^1 (s|\psi''(p)| + s^2|\psi'(p)|^2) e^{-s\psi(p)} \, dp
\]
\[
+ C \int_1^\infty (s|\psi''(p)| + s^2|\psi'(p)|^2) e^{-s\psi(p)} \, dp
\]
\[
\leq C\gamma^2 \int_0^\psi \psi^{-1}(1/s) \left( s\psi(p) + s^2\psi^2(p) \right) e^{-s\psi(p)} \, dp
\]
\[
+ \frac{C\gamma^2}{s} \int_{\psi}^1 \frac{1}{\psi(p)} \left( s^2\psi^2(p) + s^3\psi^3(p) \right) e^{-s\psi(p)} \, dp
\]
\[
+ \frac{C}{s} \int_1^\infty \left( s^2\psi^2(p) \frac{|\psi''(p)|}{\psi^3(p)} + s^3\psi^3(p) \frac{|\psi'(p)|^2}{\psi^4(p)} \right) e^{-s\psi(p)} \, dp
\]
\[
\leq C\gamma^2 \psi^{-1}(1/s).
\]

Note that $\lim_{\lambda \to \infty} \psi(\lambda) = \infty$, see e.g. [11, Lemma 4.2.2], so that (1.8) implies that
\[
\int_1^\infty \frac{|\psi'(\lambda)|^2}{\psi^3(\lambda)} \, d\lambda < \infty, \quad \int_1^\infty \frac{|\psi''(\lambda)|}{\psi^2(\lambda)} \, d\lambda < \infty. \tag{6.34}
\]

We use this to bound the next to last line in (6.33).

Combining (6.21), (6.22), (6.27), (6.32), and (6.33) we get (6.4).

We now obtain (6.3). Note that
\[
\Delta^\gamma p_s(x) = p_s(x + \gamma) - p_s(x) = \frac{1}{\pi} \int_0^\infty (\cos p(x + \gamma) - \cos px) e^{-s\psi(p)} \, dp
\]
\[
= -\frac{2}{\pi} \int_0^\infty \cos(px) \sin^2(p\gamma/2) e^{-s\psi(p)}
\]
\[
- \frac{1}{\pi} \int_0^\infty \sin(px) \sin(p\gamma) e^{-s\psi(p)} \, dp
\]

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Thus

\[ \Delta^\gamma p_s(x) = -\frac{1}{2} \Delta^\gamma \Delta^{-\gamma} p_s(x) - \frac{1}{\pi} \int_0^\infty \sin(px) \sin(p\gamma) e^{-s\psi(p)} dp. \]  

(6.36)

The second order difference is bounded in (6.4). We deal with the second integral which is bounded by

\[ \int_0^\infty |\sin(p\gamma)| e^{-s\psi(p)} dp \]

(6.37)

\[ \leq \gamma \left( \int_0^{\psi^{-1}(1/s)} p dp + \int_{\psi^{-1}(1/s)}^1 p e^{-s\psi(p)} dp + \int_1^\infty e^{-s\psi(p)} dp \right) \]

\[ \leq C\gamma \left( \left( \psi^{-1}(1/s) \right)^2 + \frac{1}{s^2} \int_{\psi^{-1}(1/s)}^1 \frac{p}{\psi^2(p)} dp + \frac{1}{s} \int_1^\infty \frac{1}{\psi^2(p)} dp \right) \]

\[ \leq C\gamma \left( \psi^{-1}(1/s) \right)^2. \]

This gives us the first bound in (6.3). To obtain the second bound we integrate by parts twice to get

\[ \int_0^\infty \sin(px) \sin(p\gamma) e^{-s\psi(p)} dp \]

(6.38)

\[ = -\frac{1}{x} \int_0^\infty \sin(p\gamma) e^{-s\psi(p)} d(cos px) \]

\[ = \frac{1}{x} \int_0^\infty \cos(px) \left( \sin(p\gamma) e^{-s\psi(p)} \right)' dp \]

\[ = \frac{1}{x^2} \int_0^\infty \left( \sin(p\gamma) e^{-s\psi(p)} \right)' \left( \sin(p\gamma) e^{-s\psi(p)} \right)'' dp. \]

Since

\[ \left( \sin(p\gamma) e^{-s\psi(p)} \right)'' = \left( -\gamma^2 \sin(p\gamma) - 2s\gamma \cos(p\gamma) \psi'(p) \right) \]

(6.39)

\[ - \sin(p\gamma) \left( s \psi''(p) - s^2 (\psi'(p))^2 \right) e^{-s\psi(p)}, \]

we can write

\[ G = G_1 + G_2 + G_3, \]  

(6.40)
where
\[
|G_1| = \gamma^2 \left| \int_0^\infty \sin(px) \sin(p\gamma) e^{-s\psi(p)} \, dp \right| \leq C\gamma^2 \left( \psi^{-1}(1/s) \right),
\] (6.41)
for all s sufficiently large, by (6.10).

Using (6.8), (1.7) and (1.8), we see that
\[
|G_2| = 2\gamma \left| \int_0^\infty \sin px \cos(p\gamma) \left( \psi'(p) s e^{-s\psi(p)} \right) \, dp \right| 
\leq C\gamma \left( 1 + \log^+ x + (1/s) \right),
\] (6.42)
where we use
\[
\int_0^1 \frac{\sin px}{p} \, dp = \int_0^{\lvert x \rvert} \frac{\sin p}{p} \, dp \leq C\gamma \left( 1 + \log^+ \lvert x \rvert \right).
\] (6.43)
Therefore, for s sufficiently large
\[
|G_2| \leq C\gamma \left( 1 + \log^+ \lvert x \rvert \right).
\] (6.44)

Similarly,
\[
|G_3| = \left| \int_0^\infty \sin px \sin p\gamma \left( s\psi''(p) - s^2(\psi'(p))^2 \right) e^{-s\psi(p)} \, dp \right|
\leq C\gamma \left( \int_1^\infty \left( s\lvert \psi''(p) \rvert + s^2(\psi'(p))^2 \right) e^{-s\psi(p)} \, dp \right)
+ C\gamma \left( \int_0^1 \frac{\lvert \psi''(p) \rvert \psi(p)}{\psi^2(p)} \, dp \right)
\leq C\gamma \left( \int_0^1 \frac{\lvert \sin px \rvert}{p} \left( \frac{s\psi'(p)}{p} + \frac{s^2\psi^2(p)}{p} \right) e^{-s\psi(p)} \, dp \right)
+ C\gamma \left( \int_1^\infty \frac{\lvert \psi''(p) \rvert \psi(p)}{\psi^2(p)} + \frac{(\psi'(p))^2}{\psi^2(p)} \right) \, dp
\leq C\gamma \left( \int_0^1 \frac{\lvert \sin px \rvert}{p} \, dp + C \right) \leq C\gamma \left( 1 + \log^+ \lvert x \rvert \right).
\] (6.45)
Combining (6.41)–(6.45) we get the second bound in (6.3).

We use the next two lemmas in the proof of Lemma 6.2.

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Lemma 6.4 Let $X$ be a symmetric Lévy process with Lévy exponent $\psi(\lambda)$ that is regularly varying at zero with index $1 < \beta \leq 2$ and satisfies (1.6). Then for any $r \geq 0$ and $t > 0$,

$$\int_0^t s^r e^{-s\psi(p)} \, ds \leq C_k \left( t \wedge \frac{1}{\psi(p)} \right)^{r+1}; \quad (6.46)$$

for all $t \geq 0$, where $C_k < \infty$, is a constant depending on $k$. Furthermore, for any $r \geq 0$ and all $t$ sufficiently large,

$$\int_0^\infty \psi^r(p) \int_0^t s^r e^{-s\psi(p)} \, ds \, dp \leq Ct\psi^{-1}(1/t). \quad (6.47)$$

Proof The first part of the bound in the first inequality in (6.46) comes from taking $e^{-s\psi(p)} \leq 1$; the second from letting $t = \infty$. Since

$$\psi^r(p)s^r e^{-s\psi(p)} = 2^r \psi^r(p) \left( \frac{s}{2} \right)^r e^{-s\psi(p)/2} e^{-s\psi(p)/2}, \quad (6.48)$$

it follows from (6.8) and (6.10) that

$$\int_0^\infty \psi^r(p) s^r e^{-s\psi(p)} \, dp \leq C \int_0^\infty e^{-s\psi(p)/2} \, dp \leq C\psi^{-1}(1/s) \quad (6.49)$$

for all $s$ sufficiently large. On the other hand for any fixed $t_0$,

$$\int_0^\infty \int_0^{t_0} e^{-s\psi(p)/2} \, ds \, dp = 2 \int_0^\infty \frac{1 - e^{-t_0\psi(p)/2}}{\psi(p)} \, dp < \infty, \quad (6.50)$$

by (1.6). Putting these two together, and using the fact that $\psi^{-1}(1/t)$ is regularly varying at infinity, gives (6.47).

Lemma 6.5 Under the hypotheses of Theorem 1.1, for $r = 0, 1, \ldots$

$$\int_0^1 \frac{|\sin px|}{p} \psi^r(p) \left( t \wedge \frac{1}{\psi(p)} \right)^{r+1} \, dp \leq Ct(1 + \log^+ |x|); \quad (6.51)$$

$$\int_0^1 \psi^r(p) \left( t \wedge \frac{1}{\psi(p)} \right)^{r+1} \, dp \leq Ct\psi^{-1}(1/t); \quad (6.52)$$

$$\int_0^t \int_0^\infty |\sin p\gamma| e^{-s\psi(p)} \, dp \, ds \leq C\gamma \log t, \quad (6.53)$$

for all $t$ sufficiently large.
Proof We first note that for \( r = 0, 1, \ldots \)
\[
\psi^{r+1}(p) \left( t \wedge \frac{1}{\psi(p)} \right)^{r+2} \leq \psi^r(p) \left( t \wedge \frac{1}{\psi(p)} \right)^{r+1}
\] (6.54)
So we need only prove (6.51) and (6.52) for \( r = 0 \). In this case, (6.51)
follows immediately from (6.43).

For (6.52) we have
\[
\int_0^1 \left( t \wedge \frac{1}{\psi(p)} \right) \, dp 
\leq t \int_0^{\psi^{-1}(1/t)} \frac{1}{\psi(p)} \, dp + \int_1^{\psi^{-1}(1/t)} \frac{1}{\psi(p)} \, dp 
\leq Ct\psi^{-1}(1/t),
\]
for all \( t \) sufficiently large.

By (6.50) and (6.37), there exists a \( t_0 \) such that for all \( t \geq t_0 \),
\[
\int_0^t \int_0^\infty |\sin p\gamma| e^{-s\psi(p)} \, dp \, ds 
\leq \int_0^{t_0} \int_0^\infty e^{-s\psi(p)} \, dp \, ds + \int_0^t \int_0^{\psi^{-1}(1/s)} |\sin p\gamma| e^{-s\psi(p)} \, dp \, ds 
\leq C + C\gamma \int_{t_0}^t \left( \psi^{-1}(1/s) \right)^2 \, ds 
\leq C + C\gamma \log t,
\]
where for the last bound we use (4.108). (This bound can not be smaller since we may have \( \psi(p) = p^2 \).) \( \square \)

Proof of Lemma 6.2 For the first bound in (6.5) we use (6.1) and
(6.47) with \( r = 0 \) to get
\[
\int_0^t p_s(x) \, ds \leq \frac{1}{\pi} \int_0^\infty \int_0^t e^{-s\psi(p)} \, ds \, dp = O(t\psi^{-1}(1/t)),
\] (6.56)
as \( t \to \infty \). For the second bound in (6.5) we use (1.7), (6.46) and (6.51),
to see that
\[
\int_0^t p_s(x) \, ds 
= \frac{1}{\pi} \int_0^\infty \cos px \int_0^t e^{-s\psi(p)} \, ds \, dp 
= \left| \frac{1}{\pi x} \int_0^\infty \int_0^t e^{-s\psi(p)} \, ds \, d(sin px) \right|
\] (6.57)
\[ \frac{1}{\pi|x|} \int_0^\infty |\sin px| \left| \frac{d}{dp} \int_0^t e^{-s\psi(p)} ds \right| dp \leq \frac{C}{|x|} \int_0^\infty |\sin px| |\psi'(p)| \left( t \wedge \frac{1}{\psi(p)} \right)^2 dp \]
\[ \leq \frac{C}{|x|} \left( \int_0^1 \frac{|\sin px|}{p} \psi(p) \left( t \wedge \frac{1}{\psi(p)} \right)^2 dp \right. 
+ \int_1^\infty \frac{|\psi'(p)|}{\psi^2(p)} dp \left. \right) \]
\[ \leq Ct \left( 1 + \log^+ |x| \right). \]

Thus we get (6.5).

We next obtain (6.7). Consider (6.20). For \( \gamma > 1 \)
\[ \int_0^\infty \frac{\sin^2(p\gamma/2)}{\psi(p)} dp \leq C \gamma^2 \int_0^1 \frac{p^2}{\psi(p)} dp + \int_1^{1/\gamma} \frac{1}{\psi(p)} dp + \int_1^\infty \frac{1}{\psi(p)} dp \]
\[ \leq C \left( \frac{1}{\gamma \psi(1/\gamma)} + 1 \right), \]
and for \( \gamma = 1 \) the integral is a constant. It follows from this and (4.108) that
\[ \sup_{x \in \mathbb{R}^3} \int_0^\infty |\Delta^\gamma \Delta^{-\gamma} p_s(x)| ds \leq C \gamma^2. \]
(6.59)
This gives the first bound in (6.7).

To obtain the third bound in (6.7), consider (6.22)–(6.26). By (6.47) with \( r = 0 \), we have
\[ \int_0^t |I| ds = \gamma^2 \int_0^t \left| \int_0^\infty \cos p \gamma \sin^2(px/2) e^{-s\psi(p)} dp \right| ds \]
\[ \leq \gamma^2 \int_0^\infty \left( \int_0^t e^{-s\psi(p)} ds \right) dp \leq C \gamma^2 t \psi^{-1}(1/t), \]
for all \( t \) sufficiently large. Using (1.7), (1.8) and (6.47) with \( r = 1 \) we get
\[ \int_0^t |II| ds = 2\gamma \int_0^t \left| \int_0^\infty \sin p \gamma \sin^2(px/2) g'(p) dp \right| ds \]
(6.61)
\[ \leq 2\gamma \int_0^1 \left| \sin(p\gamma) \psi'(p) \right| \left( \int_0^t se^{-s\psi(p)} \, ds \right) \, dp \\
+ 2\gamma \int_1^\infty \left| \sin(p\gamma) \psi'(p) \right| \left( \int_0^t se^{-s\psi(p)} \, ds \right) \, dp \\
\leq C\gamma^2 \int_0^1 \psi(p) \left( \int_0^t se^{-s\psi(p)} \, ds \right) \, dp \\
+ C\gamma \int_1^\infty \left| \psi'(p) \right| \psi^2(p) \, dp \\
\leq C\gamma^2 \left( t\psi^{-1}(1/t) \right) + C\gamma \int_1^\infty \left| \psi'(p) \right| \psi^2(p) \, dp \\
\leq C\gamma^2 \left( t\psi^{-1}(1/t) \right) + C\gamma. \]

Similarly,

\[ \int_0^t |III| \, ds = 2 \int_0^t \left| \int_0^\infty \sin^2(p\gamma/2) \sin^2(px/2) g''(p) \, dp \right| \, ds \] (6.62)
\[ \leq C\gamma^2 \int_0^1 p^2 \left( \int_0^t \left( s|\psi''(p)| + s^2|\psi'(p)|^2 \right) e^{-s\psi(p)} \, ds \right) \, dp \\
+ C \int_1^\infty \left( \int_0^\infty \left( s|\psi''(p)| + s^2|\psi'(p)|^2 \right) e^{-s\psi(p)} \, ds \right) \, dp \\
\leq C\gamma^2 \int_0^1 \left( \int_0^t \left( s\psi'(p) + s^2\psi^2(p) \right) e^{-s\psi(p)} \, ds \right) \, dp \\
+ C \int_1^\infty \left( \frac{\psi''(p)}{\psi^2(p)} + \frac{\left( \psi'(p) \right)^2}{\psi^3(p)} \right) \, dp \\
\leq C\gamma^2 t\psi^{-1}(1/t) + C. \]

Combining (6.60)-(6.62) with (6.22) we get the third bound in (6.7).

To get the second bound in (6.7) we use the third integral in (6.24) to see that

\[ \Delta^\gamma \Delta^\gamma p_x(x) = -\frac{4}{\pi} \frac{L(s, x)}{x} \] (6.63)

where

\[ L(s, x) = \int_0^\infty \sin px \left( \sin^2(p\gamma/2) e^{-s\psi(p)} \right)' \, dp. \] (6.64)

Using (6.25) and (1.7) we see that

\[ \int_0^t |L| \, ds \] (6.65)
\[ \leq C\gamma \int_0^t \int_0^\infty |\sin p\gamma| g(p) \, dp \, ds + C \int_0^t \int_0^\infty \sin^2(p\gamma/2) |g'(p)| \, dp \, ds. \]
By (6.53) the first term on the right-hand side is bounded by $C\gamma^2 \log t$.

For the second term we note that

$$
\int_0^t \int_0^\infty \sin^2 \left(\frac{p\gamma}{2}\right) |g'(p)| \, dp \, ds
\leq C\gamma^2 \int_0^1 p^2 |\psi'(p)| \int_0^t s e^{-s\psi(p)} \, ds \, dp
\leq C \int_0^\infty |\psi'(p)| \int_0^t s e^{-s\psi(p)} \, ds \, dp
= IV + V.
$$

By (6.46)

$$
IV \leq C\gamma^2 \int_0^1 p\psi(p) \left(t^2 \wedge \frac{1}{\psi^2(p)}\right) \, dp
\leq Ct^2 \gamma^2 \int_0^{\psi^{-1}(1/t)} p\psi(p) \, dp + C\gamma^2 \int_0^1 \frac{p}{\psi(p)} \, dp
\leq C\gamma^2 t (\psi^{-1}(1/t))^2 + C\gamma^2 \int_0^{\psi^{-1}(1/t)} \frac{1}{p} \, dp
\leq C\gamma^2 t \log t,
$$

where we use (4.108) which implies that $p/\psi(p) \leq C/p$ for $p \in [0, 1]$. The integral $V \leq C$ by (6.46) and (1.8). Using all the material from (6.63) to this point we get the second bound in (6.7). This completes the proof of (6.7).

Using (6.36), (6.59), (6.7) and (6.53) we get the first bound in (6.6).

We now obtain the third bound in (6.6). Considering (6.36) and (6.7), it suffices to show that

$$
\int_0^t \int_0^\infty \sin(px) \sin(p\gamma) e^{-s\psi(p)} \, dp \, ds
\leq Ct\gamma^2 \frac{1 + \log^+ |x|}{x^2}.
$$

Consider (6.38)–(6.40). We have

$$
\int_0^t |G_1| \, ds = \gamma^2 \int_0^t \int_0^\infty \sin px \sin p\gamma e^{-s\psi(p)} \, dp \, ds
\leq \gamma^2 \int_0^t \int_0^\infty e^{-s\psi(p)} \, dp \leq C\gamma^2 t \psi^{-1}(1/t),
$$

by (6.47).
Using \((6.51)\) and \((1.8)\), we see that
\[
\int_0^t \left| G_2 \right| ds = 2\gamma \int_0^1 \left| \int_0^\infty \sin px \cos p\gamma \left( \psi'(p) e^{-s\psi(p)} \right) dp \right| ds
\]
\[
\leq 2\gamma \int_0^1 \left| \sin px \right| \psi(p) \left( \int_0^t s e^{-s\psi(p)} ds \right) dp + 2\gamma \int_1^\infty \frac{\psi(p)}{\psi^2(p)} dp
\]
\[
\leq C\gamma \int_0^1 \frac{\sin px}{p} \psi(p) \left( \int_0^t \frac{1}{\psi(p)} \right)^2 dp + C\gamma
\]
\[
\leq C\gamma t \left( 1 + \log^+ |x| \right).
\]

Similarly,
\[
\int_0^t \left| G_3 \right| ds
\]
\[
= \int_0^t \left| \int_0^\infty \sin px \sin p\gamma \left( s\psi''(p) - s^2(\psi'(p))^2 \right) e^{-s\psi(p)} dp \right| ds
\]
\[
\leq C\gamma \int_0^1 \left| \sin px \right| \left( \int_0^t \left( s\psi''(p) + s^2(\psi'(p))^2 \right) e^{-s\psi(p)} ds \right) dp
\]
\[
+ C\gamma \int_1^\infty \left( \int_0^t \left( s\psi''(p) + s^2(\psi'(p))^2 \right) e^{-s\psi(p)} ds \right) dp
\]
\[
\leq C\gamma \int_0^1 \frac{\sin px}{p} \psi(p) \left( \int_0^t \frac{1}{\psi(p)} \right)^2 dp
\]
\[
+ C\gamma \int_0^1 \frac{\sin px}{p} \psi^2(p) \left( \int_0^t \frac{1}{\psi(p)} \right)^3 dp
\]
\[
+ C\gamma \int_1^\infty \frac{s\psi''(p)}{\psi^2(p)} dp + C\gamma \int_1^\infty \frac{\psi'(p)}{\psi^3(p)} dp
\]
\[
\leq C\gamma \left( 1 + \log^+ |x| \right) t.
\]

This completes the proof of \((6.68)\) and gives us the third bound in \(6.6\).

The second bound in \(6.6\) follows from the third line of \(6.38\) and the observation that
\[
\left| \int_0^t \int_0^\infty \cos px \left( \sin p\gamma e^{-s\psi(p)} \right)' dp \right|
\]
\[
\leq \gamma \int_0^\infty \int_0^t e^{-s\psi(p)} ds dp + \int_0^\infty \int_0^t \left| \sin p\gamma \left| \psi'(p) e^{-s\psi(p)} \right| ds dp
\]
\[
\leq C\gamma(t\psi^{-1}(1/t)) + \gamma \int_0^1 \frac{\psi(p)}{\psi(p)} \left( t \wedge \frac{1}{\psi(p)} \right)^2 dp \\
+ \int_1^\infty \frac{\left| \psi'(p) \right|}{\psi^2(p)} dp \\
\leq C(1 \vee \gamma)(t\psi^{-1}(1/t)).
\]

In this chain of inequalities we use (6.47), (1.7), (1.8) and (6.52). \qed

**Proof of Lemma 3.1** The inequalities in (3.5)–(3.7) follow immediately from Lemma 6.2.

The inequality in (3.8) is trivial, since \( p_s(x) \) is a probability density for all \( s > 0 \).

To obtain (3.9) we use (6.6) with \( \gamma = 1 \) to get

\[
\int_0^\infty v(x,t) dx \leq C \left( \log t \int_0^{t\psi^{-1}(1/t)} dx + t\psi^{-1}(1/t) \int_{t\psi^{-1}(1/t)}^{2c/\psi^{-1}(1/t)} \frac{1}{x} dx \\
+ Ct \int_{2c/\psi^{-1}(1/t)}^{\infty} \frac{\log x}{x^2} dx \right) \\
\leq C(t\psi^{-1}(1/t) \log t),
\]

for all \( t \) sufficiently large. Note that when \( \beta < 2 \) it is clear that \( t\psi^{-1}(1/t) < 1/\psi^{-1}(1/t) \) for all \( t \) sufficiently large. In general we use (4.108) with \( c \) representing the constant.

For (3.11) we use (6.7) to see that

\[
\int_0^\infty \left( \int_0^t \left| \Delta^1 \Delta^{-1} p_s(x) \right| ds \right)^2 dx \tag{6.74}
\leq C \int_0^{\log t} \left( \int_0^t \left| \Delta^1 \Delta^{-1} p_s(x) \right| ds \right)^2 dx \\
+ 2 \int_{\log t}^{\infty} \left( \int_0^t \left| \Delta^1 \Delta^{-1} p_s(x) \right| ds \right)^2 dx \\
\leq C \log t + C \int_{\log t}^{\infty} \frac{(\log t)^2}{x^2} dx \leq C \log t.
\]

A similar argument gives (3.12) since

\[
\int_u^\infty \left( \int_0^t \left| \Delta^1 \Delta^{-1} p_s(x) \right| ds \right)^2 dx \leq C \int_u^\infty \frac{(\log t)^2}{x^2} dx = C \frac{(\log t)^2}{u}. \tag{6.75}
\]
Finally, to obtain (6.10) we use (6.7) to get

\[
\int_0^\infty \int_0^t |\Delta^{1} \Delta^{-1} p_s(x)| \, ds \, dx
\]

\[
= \int_0^1 \int_0^t |\Delta^{1} \Delta^{-1} p_s(x)| \, ds \, dx + \int_1^{t\psi^{-1}(1/t)} \int_0^t |\Delta^{1} \Delta^{-1} p_s(x)| \, ds \, dx
\]

\[
+ \int_{t\psi^{-1}(1/t)}^\infty \int_0^t |\Delta^{1} \Delta^{-1} p_s(x)| \, ds \, dx
\]

\[
\leq C \int_0^1 1 \, dx + C \log t \int_1^{t\psi^{-1}(1/t)} \frac{1}{|x|} \, dx + C \int_{t\psi^{-1}(1/t)}^\infty \frac{t\psi^{-1}(1/t)}{|x|^2} \, dx
\]

\[
\leq C + C(\log t)^2 + C.
\]

\[\square\]

**Lemma 6.6** Under the hypotheses of Theorem 1.1, for all \( t \) sufficiently large and all \( x \in R^1 \)

\[
\begin{align*}
\overline{u}(x,t) &:= \sup_{\delta t \leq s \leq t} p_s(x) \leq C \left( \psi^{-1}(1/t) \wedge \frac{1}{\psi^{-1}(1/t)x^2} \right); \\
\overline{v}(x,t) &:= \sup_{\delta t \leq s \leq t} |\Delta^{1} p_s(x)| \leq C \left( \left( \psi^{-1}(1/t) \right)^2 \wedge \frac{1 + \log^+ |x|}{x^2} \right); \\
\overline{w}(x,t) &:= \sup_{\delta t \leq s \leq t} |\Delta^{1} \Delta^{-1} p_s(x)| \leq C \left( \left( \psi^{-1}(1/t) \right)^3 \wedge \frac{\psi^{-1}(1/t)}{x^2} \right).
\end{align*}
\]

In addition

\[
\int \overline{u}(x,t) \, dx \leq C;
\]

\[
\int \overline{v}(x,t) \, dx \leq C\psi^{-1}(1/t) \log t;
\]

\[
\int \overline{w}(x,t) \, dx \leq C \left( \psi^{-1}(1/t) \right)^2 \leq \frac{C}{t}.
\]

**Proof** By (6.2)

\[
\sup_{\delta t \leq s \leq t} p_s(x) \leq C \left( \psi^{-1}(1/\delta t) \wedge \frac{1}{\psi^{-1}(1/t)x^2} \right),
\]

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and by the regular variation property $\psi^{-1}(1/\delta t) \leq C\psi^{-1}(1/t)$. (The constant depends on $\delta$ but that doesn’t matter.) The inequalities in (6.78) and (6.79) follow similarly from (6.3) and (6.4).

The inequalities in (6.80)–(6.82) follow easily from (6.77)–(6.79). For (6.80) we write

$$\int \pi(x,t) \, dx \leq C \int_0^a \psi^{-1}(1/t) \, dx + \int_a^\infty \frac{1}{\psi^{-1}(1/t)} x^2 \, dx,$$

(6.84)

where $a = 1/\psi^{-1}(1/t)$. For (6.81) and (6.82) we proceed similarly with $a = 1/\psi^{-1}(1/t)$ in both cases.

**Proof of Lemma 3.2** The inequality in (3.13) follows from (6.36) and (6.4).

To obtain (3.14) we write

$$\int_0^{2t} \int_0^{2t} |\Delta^1 p_{r+s}(0)| \, dr \, ds = \int_0^{2t} u|\Delta^1 p_u(0)| \, du + \int_{2t}^{4t} (4t-u)|\Delta^1 p_u(0)| \, du.$$  

(6.85)

By (6.36) and (6.20)

$$\int_0^{2t} u|\Delta^1 p_u(0)| \, du \leq \frac{2}{\pi} \int_0^2 u \int_0^\infty \sin^2(p/2) e^{-u\psi(p)} \, dp \, du$$  

(6.86)

$$= \frac{2}{\pi} \int_0^\infty \sin^2(p/2) \int_0^{2t} u e^{-u\psi(p)} \, du \, dp.$$

In addition

$$\int_0^{2t} u e^{-u\psi(p)} \, du \leq \left( \frac{1}{\psi^2(p)} (1 - e^{-2t\psi(p)}) \wedge \frac{Ct}{\psi(p)} \right),$$  

(6.87)

where, for the final inequality we use Lemma 6.3. Consequently, for all $t$ sufficiently large,

$$\int_0^\infty \sin^2(p/2) \int_0^{2t} u e^{-u\psi(p)} \, du \, dp \leq Ct \int_0^{\psi^{-1}(1/t)} \frac{p^2}{\psi(p)} \, dp + \frac{1}{4} \int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)} \, dp + \int_1^\infty \frac{1}{\psi^2(p)} \, dp$$  

(6.88)

$$\leq C \left( t^2 \left( \psi^{-1}(1/t) \right)^3 + 1 \right) + \frac{1}{4} \int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)} \, dp.$$
Note that
\[
\int_{\psi^{-1}(1/t)}^{1} \frac{p^2}{\psi^2(p)} \, dp \leq \begin{cases} 
C t^2 \left( \psi^{-1}(1/t) \right)^3 & \text{if } \beta > 3/2 \\
L(t) & \text{if } \beta = 3/2 \\
C & \text{if } \beta < 3/2,
\end{cases}
\]
where \( L(t) \) is a slowly varying function at infinity. Therefore
\[
\int_{0}^{2t} u |\Delta^1 p_u(0)| \, du \leq C \left( t^2 \left( \psi^{-1}(1/t) \right)^3 + L(t) + 1 \right). \tag{6.89}
\]
In addition
\[
\int_{2t}^{4t} (4t - u) |\Delta^1 p_u(0)| \, du \leq Ct \int_{2t}^{4t} |\Delta^1 p_u(0)| \, du \tag{6.90}
= Ct \int_{0}^{2t} |\Delta^1 p_{v+2t}(0)| \, dv
\]
Note that by (6.36), (6.20) and (6.79)
\[
|\Delta^1 p_{v+2t}(0)| = \frac{1}{2} |\Delta^1 \Delta^{-1} p_{v+2t}(0)| \leq \frac{1}{2} |\Delta^1 \Delta^{-1} p_{2t}(0)| \leq C \left( \psi^{-1}(1/t) \right)^3.
\tag{6.91}
\]
Here we also use the fact that \( \Delta^1 \Delta^{-1} p_{s}(0) \) is decreasing in \( s \), and the regular variation of \( \psi \). Consequently
\[
\int_{2t}^{4t} (4t - u) |\Delta^1 p_u(0)| \, du \leq Ct^2 \left( \psi^{-1}(1/t) \right)^3. \tag{6.92}
\]
Thus we obtain (3.14). \( \square \)

**Proof of Lemma 3.3** The equality in (3.15) follows easily from (6.1).
The equality in (3.16) follows from (4.75) integrated with respect to \( r \) and \( r' \).
For (3.17) we use Parseval’s Theorem, (see (4.75)) to get
\[
\int \left( \int_{0}^{t} \Delta^1 \Delta^{-1} p_s(x) \, ds \right)^2 \, dx = \frac{16}{\pi} \int_{0}^{\infty} \sin^4 \left( \frac{p}{2} \right) \frac{1 - e^{-t\psi(p)}}{\psi^2(p)} \, dp. \tag{6.93}
\]
To complete the proof of (3.17) we note that by (6.8)
\[
\int_{0}^{\infty} \frac{\sin^4(p/2)}{\psi^2(p)} e^{-t\psi(p)} \, dp \leq \frac{C}{t^{1/3}} \int_{0}^{\infty} \frac{\sin^4(p/2)}{\psi^{7/3}(p)} \, dp \leq \frac{C}{t^{1/3}}.
\]
(6.94)

\[\square\]

**Proof of Lemma 4.2** The first inequality is given in (6.6). The second inequality follows from the definition of \(v_*\) in (4.60). For (4.63) we note that in the proof of (3.9), on page 65, we are actually integrating \(v_*(x,t)\).

\[\square\]

7 Proof of Lemma 2.4

Set
\[
h = \psi^{-1}(1/t), \quad \text{so that} \quad \psi(h) = 1/t.
\]
(7.1)

**Proof of Lemma 2.4** By Kac’s moment formula, (see (5.7)),
\[
E \left( \left( \int (L^x_t)^2 \, dx \right)^n \right)
\]
(7.2)
\[
= \sum_{\sigma} \int \left( \int_{\{\sum_{i=1}^{2n} s_i \leq t\}} \prod_{i=1}^{2n} p_{s_i}(y_{\sigma(i)} - y_{\sigma(i-1)}) \, ds_i \right) \prod_{i=1}^{n} dy_i
\]
\[
= \sum_{\sigma} \frac{1}{(2\pi)^{2n}} \int \left( \int_{\{\sum_{i=1}^{2n} s_i \leq t\}} \prod_{i=1}^{2n} \int e^{is_i(y_{\sigma(i)} - y_{\sigma(i-1)})} e^{-s_i\psi(p_i)} \, dp_i \, ds_i \right) \prod_{i=1}^{n} dy_i
\]
\[
= \left( \frac{t}{2\pi} \right)^{2n} \sum_{\sigma} \int \left( \int_{\{\sum_{i=1}^{2n} s_i \leq t\}} \prod_{i=1}^{2n} \int e^{is_i(y_{\sigma(i)} - y_{\sigma(i-1)})} e^{-s_i\psi(p_i)} \, dp_i \, ds_i \right) \prod_{i=1}^{n} dy_i.
\]

Here the sum in the second line runs over all maps \(\sigma\) of \(\{1, \ldots, 2n\}\) into \(\{1, \ldots, n\}\), such that \(|\sigma^{-1}(j)| = 2\) for each \(1 \leq j \leq n\), and we set \(\sigma(0) = 0\). Thus, by (7.1) and many changes of variables
\[
(2\pi)^{2n} (t^2\psi^{-1}(1/t))^{-n} E \left( \left( \int (L^x_t)^2 \, dx \right)^n \right)
\]
(7.3)
\[ (2\pi)^{2n}t^{-2n}h^{-n}E \left( \left( \int (L_x)^2 \, dx \right)^n \right) \]

\[ = h^{-n} \sum_{\sigma} \int \left( \int_{\left\{ \sum_{i=1}^{2n} s_i \leq 1 \right\}} \prod_{i=1}^{2n} \int e^{ip_i(y_{\sigma(i)}-y_{\sigma(i-1)})} e^{-s_i t \psi(p_i)} \, dp_i \, ds_i \right) \prod_{i=1}^{n} dy_i \]

\[ = h^n \sum_{\sigma} \int \left( \int_{\left\{ \sum_{i=1}^{2n} s_i \leq 1 \right\}} \prod_{i=1}^{2n} \int e^{ip_i h(y_{\sigma(i)}-y_{\sigma(i-1)})} e^{-s_i t \psi(p_i h)} \, dp_i \, ds_i \right) \prod_{i=1}^{n} dy_i \]

\[ = \sum_{\sigma} \int \left( \int_{\left\{ \sum_{i=1}^{2n} s_i \leq 1 \right\}} \prod_{i=1}^{2n} \int e^{ip_i (y_{\sigma(i)}-y_{\sigma(i-1)})} e^{-s_i \psi(p_i h)/\psi(h)} \, dp_i \, ds_i \right) \prod_{i=1}^{n} dy_i. \]

Using the regular variation of \( \psi \) at zero the proof follows once we justify interchanging the limit and the integrals.

For \( \sigma \) fixed let

\[ f_h(y) = \int_{\left\{ \sum_{i=1}^{2n} s_i \leq 1 \right\}} \prod_{i=1}^{2n} \int e^{ip_i y_i} e^{-s_i \psi(p_i h)/\psi(h)} \, dp_i \, ds_i \]

\[ = 2^{2n} \int_{\left\{ \sum_{i=1}^{2n} s_i \leq 1 \right\}} \prod_{i=1}^{2n} \int_0^\infty \cos p_i y_i e^{-s_i \psi(p_i h)/\psi(h)} \, dp_i \, ds_i. \]

Considering (7.3) it suffices to show that for each fixed \( y = (y_1, \ldots, y_n) \)

\[ \lim_{h \to 0} f_h(y) = 2^{2n} \int_{\left\{ \sum_{i=1}^{2n} s_i \leq 1 \right\}} \prod_{i=1}^{2n} \int_0^\infty \cos p_i y_i e^{-s_i \psi(p_i h)/\psi(h)} \, dp_i \, ds_i. \] (7.5)

and \( f_h(y) \) is bounded and integrable in \( y \), uniformly in \( h \leq h_0 \), for some \( h_0 > 0 \), sufficiently small. In fact we show that

\[ \sup_{h \leq h_0} |f_h(y)| \leq C \prod_{i=1}^{2n} \left( 1 + \frac{1}{y_i^2} \right). \] (7.6)

We first obtain (7.5). For \( M \) large, write

\[ 1 = \prod_{i=1}^{2n} \left( 1 \{0 \leq p_i \leq M\} + 1 \{p_i \geq M\} \right) \] (7.7)

and

\[ f_h(y) = 2^{2n} \int_{\left\{ \sum_{i=1}^{2n} s_i \leq 1 \right\}} \prod_{i=1}^{2n} \int_0^M \cos p_i y_i e^{-s_i \psi(p_i h)/\psi(h)} \, dp_i \, ds_i + G_h. \] (7.8)
Here $G_h$ is a sum of many terms, in each of which $p_i \geq M$, for at least one $1 \leq i \leq 2n$. Suppose there are $k$ terms with $p_i \geq M$. We bound these terms by

$$2^{2n} \left( \int_0^1 \int_0^M e^{-s \psi(p)/\psi(h)} \, dp \, ds \right)^{2n-k} \left( \int_0^1 \int_M^\infty e^{-s \psi(p)/\psi(h)} \, dp \, ds \right)^k. \quad (7.9)$$

By (1.5), for any $\epsilon > 0$, (see also [3, Theorem 1.5.6]),

$$\int_0^1 \int_0^M e^{-s \psi(p)/\psi(h)} \, dp \, ds \leq 1 + \int_0^1 \int_1^M e^{-s \psi(p)/\psi(h)} \, dp \, ds \leq 1 + \int_0^1 \int_1^M e^{-s C \psi^{-\epsilon}} \, dp \, ds, \quad (7.10)$$

which is bounded by a constant independent of $M$. Using the regular variation of $\psi$ at zero, we have

$$\int_0^1 \int_M^\infty e^{-s \psi(p)/\psi(h)} \, dp \, ds \leq \psi(h) \int_M^\infty \frac{1}{\psi(hp)} \, dp = \frac{\psi(h)}{h} \int_h^\infty \frac{1}{\psi(s)} \, ds$$

$$= \frac{\psi(h)}{h} \int_{hM}^{\infty} \frac{1}{\psi(s)} \, ds + \frac{\psi(h)}{h} \int_1^\infty \frac{1}{\psi(s)} \, ds$$

$$\leq C \frac{\psi(h)M}{\psi(hM)} + C \frac{\psi(h)}{h},$$

for all $h$ sufficiently small. Therefore, as in (7.10),

$$\limsup_{h \to 0} \int_0^1 \int_M^\infty e^{-s \psi(p)/\psi(h)} \, dp \, ds \leq \frac{C}{M^{\beta-1}}. \quad (7.11)$$

Thus

$$\limsup_{h \to 0} |G_h| \leq \frac{C}{M^{\beta-1}}. \quad (1.13)$$

Now consider the integral in (7.8). By the regular variation of $\psi$ at zero and the Dominated Convergence Theorem,

$$\lim_{h \to 0} \int_{(\sum_{i=1}^{2n} s_i \leq 1)} \prod_{i=1}^{2n} \int_0^M \cos p_i y_i e^{-s_i \psi(p_i)/\psi(h)} \, dp_i \, ds_i \quad (7.14)$$

$$= \int_{(\sum_{i=1}^{2n} s_i \leq 1)} \prod_{i=1}^{2n} \int_0^M e^{i p_i y_i} \cos p_i y_i e^{-s_i p_i^\beta} \, dp_i \, ds_i.$$
Thus we get (7.15).

We show below that for any $J \subseteq \{1, \ldots, 2n\}$ we have

$$\sup_{h \leq h_0} |f_h(y)| \leq C \prod_{i \in J} \left( \frac{1}{y_i^2} \right). \quad (7.15)$$

In particular, (7.15) also holds when $J$ is the empty set, so that $\sup_{h \leq h_0} |f_h(y)| \leq C$. Using this it is easy to see that (7.6) holds.

It follows from integrating by parts twice that

$$\int_0^\infty \cos(py) e^{-s\psi(p)/\psi(h)} \frac{dp}{\psi(p)/\psi(h)} = \frac{-1}{y^2} \int_0^\infty \cos(py) \left( e^{-s\psi(p)/\psi(h)} \right)^{\prime\prime} \frac{dp}{\psi(p)/\psi(h)} \quad (7.16)$$

where we use the fact that $\psi'(0) = 0$, which follows from (1.5) and the first inequality in (1.7). Applying this for all $i \in J$ we see that

$$f_h(y) = \prod_{i \in J} \left( \frac{1}{y_i^2} \right) \int_{\{\sum_{i=1}^{2n} s_i \leq 1\}} \prod_{i \in J} \int_0^\infty \cos p_i y_i \left( e^{-s_i \psi(p)/\psi(h)} \right)^{\prime\prime} \frac{dp_i}{\psi(p)/\psi(h)} ds_i \prod_{i \in J} \int_0^\infty \cos p_i y_i e^{-s_i \psi(p)/\psi(h)} \frac{dp_i}{\psi(p)/\psi(h)} ds_i. \quad (7.17)$$

Therefore

$$|f_h(y)| \leq \prod_{i \in J} \left( \frac{1}{y_i^2} \right) \left( \int_0^1 \int_0^\infty \left| \left( e^{-s\psi(p)/\psi(h)} \right)^{\prime\prime} \right| \frac{dp}{\psi(p)/\psi(h)} ds \right)^{|J|} \left( \int_0^1 \int_0^\infty e^{-s\psi(p)/\psi(h)} \frac{dp}{\psi(p)/\psi(h)} ds \right)^{|J^c|}. \quad (7.18)$$

It is easily seen that

$$\int_0^1 \int_0^\infty e^{-s\psi(p)/\psi(h)} \frac{dp}{\psi(p)/\psi(h)} ds \leq C \sup_h \int \left( 1 \wedge \frac{\psi(h)}{\psi(hp)} \right) \frac{dp}{\psi(p)/\psi(hp)}. \quad (7.19)$$

Therefore, for $h \leq 1$,

$$\int \left( 1 \wedge \frac{\psi(h)}{\psi(hp)} \right) \frac{dp}{\psi(p)/\psi(hp)} \leq \int_0^1 1 \frac{dp}{\psi(hp)} + \int_1^{1/h} \frac{\psi(h)}{\psi(hp)} \frac{dp}{\psi(hp)} + \int_{1/h}^\infty \frac{\psi(h)}{\psi(hp)} \frac{dp}{\psi(hp)} \leq 1 + \frac{\psi(h)}{h} \left( \int_0^1 \frac{1}{\psi(p)} \frac{dp}{\psi(p)} + \int_{1/h}^\infty \frac{1}{\psi(p)} \frac{dp}{\psi(p)} \right) \leq C. \quad (7.20)$$
Consequently, to obtain (7.15) we need only show that, for $h \leq 1$,

$$
\int_{0}^{1} \int_{0}^{\infty} \left| \left( e^{-s\psi(ph)/\psi(h)} \right)'' \right| \, dp \, ds < \infty. \tag{7.21}
$$

We have

$$
\left| \left( e^{-s\psi(ph)/\psi(h)} \right)'' \right| \leq \left| \frac{h^2(\psi'(hp))^2}{\psi^2(h)} \right| \left| s^2 e^{-s\psi(ph)/\psi(h)} + \frac{h^2 \psi''(hp)}{\psi(h)} \right| e^{-s\psi(ph)/\psi(h)}. \tag{7.22}
$$

Using (1.7) and (1.8) we see that, for $h \leq 1$,

$$
\int_{0}^{1} \int_{0}^{\infty} \left| \frac{h^2(\psi'(hp))^2}{\psi^2(h)} \right| s^2 e^{-s\psi(ph)/\psi(h)} \, dp \, ds \leq C \int_{0}^{1} \left| \frac{h^2(\psi'(hp))^2}{\psi^2(h)} \right| \left( 1 \wedge \frac{\psi(h)}{\psi(h)} \right)^3 \, dp
$$

$$
\leq \frac{Ch^2}{\psi^2(h)} \int_{0}^{1} (\psi'(hp))^2 \, dp + Ch^2 \psi(h) \int_{1}^{\infty} \frac{(\psi'(hp))^2}{\psi^3(h)} \, dp
$$

$$
\leq \frac{Ch}{\psi^2(h)} \int_{0}^{h} (\psi'(s))^2 \, ds + Ch \psi(h) \int_{h}^{\infty} \frac{(\psi'(s))^2}{\psi^3(s)} \, ds
$$

$$
\leq C \left( \frac{h}{\psi^2(h)} \int_{0}^{h} \frac{\psi^2(s)}{s^2} \, ds + h \psi(h) \int_{h}^{1} \frac{1}{s^2 \psi(s)} \, ds + C \right)
$$

$$\leq C'.
$$

Similarly,

$$
\int_{0}^{1} \int_{0}^{\infty} \left| \frac{h^2 \psi''(hp)}{\psi(h)} \right| s e^{-s\psi(ph)/\psi(h)} \, dp \, ds \leq C \int_{0}^{1} \left| \frac{h^2 \psi''(hp)}{\psi(h)} \right| \left( 1 \wedge \frac{\psi(h)}{\psi(h)} \right)^2 \, dp
$$

$$
\leq \frac{Ch^2}{\psi(h)} \int_{0}^{1} \psi''(hp) \, dp + Ch^2 \psi(h) \int_{1}^{\infty} \frac{\psi''(hp)}{\psi^2(h)} \, dp
$$

$$
\leq \frac{Ch}{\psi(h)} \int_{0}^{h} \psi''(s) \, ds + Ch \psi(h) \int_{h}^{\infty} \frac{\psi''(s)}{\psi^2(s)} \, ds
$$

$$
\leq C \left( \frac{h}{\psi(h)} \int_{0}^{h} \frac{\psi(s)}{s^2} \, ds + h \psi(h) \int_{h}^{1} \frac{1}{s^2 \psi(s)} \, ds + C \right)
$$

$$\leq C'.
$$

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Thus we obtain (7.21).

8 Estimates for the mean and variance

Proof of Lemma 5.1 By the Kac moment formula

\[
E \left( \int (L_{t}^{x+1} - L_{t}^{x})^2 \, dx \right) = 2 \int \int_{\{ \sum_{i=1}^{2} r_i \leq t \}} \Delta^1 p_{r_1}(x) \Delta^1 p_{r_2}(0) \, dr_1 \, dr_2 \, dx
\]

\[
+ 2 \int \int_{\{ \sum_{i=1}^{2} r_i \leq t \}} p_{r_1}(x) \Delta^1 \Delta^{-1} p_{r_2}(0) \, dr_1 \, dr_2 \, dx.
\]

When we integrate with respect to \( x \) we get zero in the first integral and one in the second. Consequently, by (6.1)

\[
E \left( \int (L_{t}^{x+1} - L_{t}^{x})^2 \, dx \right) = 2 \int \int_{\{ \sum_{i=1}^{2} r_i \leq t \}} \Delta^1 \Delta^{-1} p_{r_2}(0) \, dr_1 \, dr_2 \, dx.
\]

(8.1)

Note that

\[
\int_0^t (t - r) e^{-r \psi(p)} \, dr = \frac{t}{\psi(p)} - \frac{1 - e^{-t \psi(p)}}{\psi^2(p)}.
\]

(8.2)

By (1.12)

\[
\frac{8t}{\pi} \int_0^\infty \frac{\sin^2(p/2)}{\psi(p)} \, dp = 4c_{\psi,0} t.
\]

(8.3)

Therefore the absolute value of the error term in (5.14) is

\[
\frac{8}{\pi} \int_0^\infty \frac{\sin^2(p/2)}{\psi^2(p)} \left( 1 - e^{-t \psi(p)} \right) \, dp \leq \frac{8}{\pi} \int_0^\infty \frac{\sin^2(p/2)}{\psi^2(p)} \left( 1 \wedge t \psi(p) \right) \, dp.
\]

(8.4)

We break this last integral into three parts and see that it is bounded by

\[
C \left( t \int_0^{\psi^{-1}(1/t)} \frac{p^2}{\psi(p)} \, dp + \int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)} \, dp + \int_1^\infty \frac{1}{\psi^2(p)} \, dp \right)
\]

(8.5)
We have
\[ t \int_0^{\psi^{-1}(1/t)} \frac{p^2}{\psi(p)} \, dp \leq C t^2 \left( \psi^{-1}(1/t) \right)^3. \] (8.7)
In addition
\[ \int_1^{\infty} \frac{1}{\psi^2(p)} \, dp \leq C. \] (8.8)
If \( \beta > 3/2 \)
\[ \int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)} \, dp \leq C t^2 \left( \psi^{-1}(1/t) \right)^3. \] (8.9)
If \( \beta = 3/2 \)
\[ \int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)} \, dp \leq C L(t) \] (8.10)
for some function \( L \) that is slowly varying at infinity. If \( \beta < 3/2 \)
\[ \int_{\psi^{-1}(1/t)}^1 \frac{p^2}{\psi^2(p)} \, dp \leq C. \] (8.11)
Using (8.5)–(8.11) we get (5.15).

Let
\[ Z = \int (L_t^{x+1} - L_t^x)^2 \, dx. \] (8.12)
We get an upper bound for the variance of \( Z \) by finding an upper bound for \( EZ^2 \) and using (5.14) to estimate \((EZ)^2\). We proceed as in the beginning of the proof of Lemma 4.1, however there are enough differences that it is better to repeat some of the arguments.

By the Kac Moment Theorem
\[
E \left( \prod_{i=1}^2 \left( \Delta_{x_i} L_{i}^{x_i} \right) \left( \Delta_{y_i} L_{i}^{y_i} \right) \right) \\
= \prod_{i=1}^2 \left( \Delta_{x_i} \Delta_{y_i} \right) \sum_{\sigma} \int_{\sum_{r_i \leq t}} \prod_{i=1}^4 p_{r_i} (\sigma(i) - \sigma(i - 1)) \prod_{i=1}^4 dr_i
\]
where the sum runs over all bijections \( \sigma : [1, 4] \mapsto \{x_1, y_i, 1 \leq i \leq 2\} \) and we take \( \sigma(0) = 0 \). We rewrite (8.13) so that each \( \Delta^1 \) applies to a single \( p \) factor and then set \( y_i = x_i \) and then integrate with respect to
\[ E \left( \int (L^x_{i+1} - L^x_i)^2 \, dx \right)^2 \]  
\[ = 4 \sum_{\pi, a} \int \int_{\sum_{i=1}^4 r_i \leq t} \prod_{i=1}^4 \left( \Delta_{x\pi(i)}^1 \right)^{a_1(i)} \left( \Delta_{x\pi(i-1)}^1 \right)^{a_2(i)} \]
\[ p_{r_i}^2 (x_{\pi(i)} - x_{\pi(i-1)}) \prod_{i=1}^4 dr_i \prod_{i=1}^2 dx_i. \]

In (8.14) the sum runs over all maps \( \pi : [1, 4] \mapsto [1, 2] \) with \( |\pi^{-1}(i)| = 2 \) for each \( i \) and over all \( a = (a_1, a_2) : [1, \ldots, 4] \mapsto \{0, 1\} \times \{0, 1\} \) with the property that for each \( i \) there are exactly two factors of the form \( \Delta_{x\pi}^1 \).

The factor 4 comes from the fact that we can interchange each \( y_i \) and \( x_i \), \( i = 1, 2 \). As usual we take \( \pi(0) = 0 \).

As we did in Section 4, we continue the analysis with \( p^\# \) replaced by \( p \).

Note that in (8.14) it is possible to have ‘bound states’, that is values of \( i \) for which \( \pi(i) = \pi(i - 1) \). We first consider the terms in (8.14) with two bound states. There are two possible maps. They are \((\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 1, 2, 2)\) and \((\pi(1), \pi(2), \pi(3), \pi(4)) = (2, 2, 1, 1)\).

The terms in (8.14) for the map \((\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 1, 2, 2)\) are of the form

\[ \prod_{i=1}^4 \left( \Delta_{x\pi(i)}^1 \right)^{a_1(i)} \left( \Delta_{x\pi(i-1)}^1 \right)^{a_2(i)} p_{r_i} (x_{\pi(i)} - x_{\pi(i-1)}), \]  
(8.15)

where the density terms have the form

\[ p_{r_1}(x_1) p_{r_2}(y_1 - x_1) p_{r_3}(x_2 - y_1) p_{r_4}(y_2 - x_2), \]  
(8.16)

and where \( y_i - x_i = 0, 1 = 1, 2 \).

The value of the integrals of the terms in (8.15) depend upon how the difference operators are distributed. In many cases the integrals are equal to zero. For example suppose we have

\[ \Delta_{x_1}^1 p_{r_1}(x_1) \Delta_{x_2}^1 p_{r_2}(0) \Delta_{x_2}^1 p_{r_3}(x_2 - x_1) \Delta_{x_2}^1 p_{r_4}(0), \]  
(8.17)

which we obtain by setting \( y_1 = x_1 \). Written out this term is

\[ (p_{r_1}(x_1 + 1) - p_{r_1}(x_1)) \Delta_{x_1}^1 p_{r_2}(0) \]
\[ (p_{r_3}(x_2 - x_1 + 1) - p_{r_3}(x_2 - x_1)) \Delta_{x_2}^1 p_{r_4}(0) \]  
(8.18)

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By a change of variables one sees that the integral of this term with respect to $x_1$ and $x_2$ is zero.

The only non-zero integrals in (8.15) come from

$$p_{r_1}(x_1) \Delta^1 \Delta^{-1} p_{r_2}(0) p_{r_3}(x_2 - x_1) \Delta^1 \Delta^{-1} p_{r_4}(0). \quad (8.19)$$

The integral of this term with respect to $x_1$ and $x_2$ is

$$\Delta^1 \Delta^{-1} p_{r_2}(0) \Delta^1 \Delta^{-1} p_{r_4}(0). \quad (8.20)$$

We get the same contribution when $(\pi(1), \pi(2), \pi(3), \pi(4)) = (2, 2, 1, 1)$. Consequently, the contribution to (8.14) of maps with two bound states is

$$8 \int_{\{ \sum_{i=1}^4 r_i \leq t \}} \Delta^1 \Delta^{-1} p_{r_2}(0) \Delta^1 \Delta^{-1} p_{r_4}(0) \prod_{i=1}^4 dr_i \quad (8.21)$$

$$= 32 \int_{\{ \sum_{i=1}^4 r_i \leq t \}} (p_{r_2}(0) - p_{r_2}(1)) (p_{r_4}(0) - p_{r_4}(1)) \prod_{i=1}^4 dr_i$$

$$= 16 \int_{\{u + v \leq t\}} (t - u - v)^2 (p_u(0) - p_u(1)) (p_v(0) - p_v(1)) \, du \, dv.$$

$$\leq 16t^2 \left( \int_0^\infty (p_u(0) - p_u(1)) \, du \right)^2 = (4c_{\psi,0}t)^2,$$

(see (3.15)).

We next consider the contribution from terms with exactly one bound state. These come from maps of the form $(\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 2, 2, 1)$ or $(\pi(1), \pi(2), \pi(3), \pi(4)) = (2, 1, 1, 2)$. These terms give non-zero contributions of the form

$$Q_2 := \int \int_{\{ \sum_{i=1}^4 r_i \leq t \}} p_{r_1}(x) \Delta^1_x p_{r_2}(y - x) \Delta^1_y \Delta^{-1} p_{r_3}(0) \Delta^1_x p_{r_4}(x - y) \prod_{i=1}^4 dr_i \, dx \, dy \quad (8.22)$$

$$= \int \int_{\{ \sum_{i=1}^4 r_i \leq t \}} \Delta^{-1} p_{r_2}(y) \Delta^1 \Delta^{-1} p_{r_3}(0) \Delta^{-1} p_{r_4}(y) \prod_{i=1}^4 dr_i \, dy;$$

$$Q_3 := \int \int_{\{ \sum_{i=1}^4 r_i \leq t \}} p_{r_1}(x) \Delta^1_x \Delta^1_y \Delta^{-1} p_{r_2}(y - x) \Delta^1_x \Delta^1_y \Delta^{-1} p_{r_4}(x - y)$$

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\[
\prod_{i=1}^{4} dr_i \, dx \, dy \quad \text{(8.23)}
\]

\[
= \int \int \left\{ \sum_{i=1}^{4} r_i \leq t \right\} \Delta^1 \Delta^{-1} p_{r_2}(y) \, p_{r_3}(0) \, \Delta^1 \Delta^{-1} p_{r_4}(y) \prod_{i=1}^{4} dr_i \, dy;
\]

and

\[
Q_4 := \int \int \left\{ \sum_{i=1}^{4} r_i \leq t \right\} p_{r_1}(x) \Delta^1 \Delta^{-1} p_{r_2}(y-x) \, \Delta^1 \Delta^{-1} p_{r_3}(0) \, \Delta^1 \Delta^{-1} p_{r_4}(x-y) \prod_{i=1}^{4} dr_i \, dx \, dy \quad \text{(8.24)}
\]

\[
= \int \int \left\{ \sum_{i=1}^{4} r_i \leq t \right\} \Delta^1 \Delta^{-1} p_{r_2}(y) \, \Delta^1 \Delta^{-1} p_{r_3}(0) \, \Delta^1 \Delta^{-1} p_{r_4}(y) \prod_{i=1}^{4} dr_i \, dy.
\]

For further explanation consider \( Q_2 \). This arrangement comes from the sequence \((x_1, y_2, x_2, y_1)\). The expression it is equal to comes by making the change of variables, \( y-x \rightarrow y \) and then integrating with respect to \( x \).

Integrating and using (3.6), (3.7) and (3.9) we see that

\[
|Q_2| \leq t \left( \int_0^t |\Delta^1 \Delta^{-1} p_s(0)| \, ds \right) \left( \int_0^t |\Delta^{-1} p_r(y)| \, dr \right)^2 dy \quad \text{(8.25)}
\]

\[
\leq t \, w(0, t) \sup_x v(x, t) \int v(y, t) \, dy
\]

\[
\leq C t^2 \psi^{-1}(1/t) (\log t)^2.
\]

To obtain a bound for \( Q_3 \) we use (3.5) and (3.11) to see that it is bounded in absolute value by

\[
\int_0^t p_s(0) \, ds \left( \int_0^t |\Delta^1 \Delta^{-1} p_r(y)| \, dr \right)^2 dy
\]

\[
= tu(0, t) \int w^2(y, t) \, dy
\]

\[
\leq C t^2 \psi^{-1}(1/t) \log t.
\]

Integrating \( Q_4 \) we see that it is bounded in absolute value by

\[
t \int_0^t \left| \Delta^1 p_r(0) \right| \, dr \left( \int_0^t \left| \Delta^1 \Delta^{-1} p_r(y) \right| \, dr \int_0^t \left| \Delta^1 \Delta^{-1} p_r(y) \right| \, dr \right) \, dy
\]

\[
\leq tw(0, t) \sup_x v(x, t) \int w(y, t) \, dy
\]

\[
\leq C t (\log t)^3, \quad \text{(8.27)}
\]

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Finally, we consider the contribution from terms in (8.14) with no bound states. These have to be from \( \pi \) of the form \((\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 2, 1, 2)\) or of the form \((\pi(1), \pi(2), \pi(3), \pi(4)) = (2, 1, 2, 1)\). They give contributions of the form

\[
Q_5 \quad := \int \int_{\{\sum_{i=1}^{4} r_i \leq t\}} p_{r_1}(x) \Delta_x^1 p_{r_2}(y - x) \Delta_y^1 \Delta_x^1 p_{r_3}(x - y) \Delta_y^1 p_{r_4}(y - x) \prod_{i=1}^{4} dr_i \, dx \, dy
\]

and

\[
Q_6 \quad := \int \int_{\{\sum_{i=1}^{4} r_i \leq t\}} p_{r_1}(x) \Delta_x^1 \Delta_y^1 p_{r_2}(y - x) p_{r_3}(x - y) \Delta_x^1 \Delta_y^1 p_{r_4}(x - y) \prod_{i=1}^{4} dr_i \, dx \, dy
\]

Clearly

\[
|Q_5| \leq t \left( \int_0^t |\Delta^{-1} p_r(y)| \, dr \right) \left( \int_0^t |\Delta^1 p_r(y)| \, dr \right) \left( \int_0^t |\Delta^1 \Delta^{-1} p_r(y)| \, dr \right) dy \leq t \sup_x v^2(x, t) \int w(y, t) \, dy \leq Ct(\log t)^3,
\]

by (3.6) and (3.10).

The term \(Q_6\) is bounded the same way we bounded \(Q_3\) and has the same bound.
We can now obtain an upper bound for the variance. Note that by (5.14)
\[(EZ)^2 = \left( E \left( \int (L_{x+1}^z - L_x^z)^2 \, dx \right) \right)^2 = (4c_{\psi,0}t)^2 + O \left( tg(t) \right). \tag{8.31} \]
Therefore, it follows from (8.14) and (8.21) that
\[
\text{Var } Z \leq EZ^2 - (EZ)^2 = \sum_{j=2}^{6} |Q_j| + Ctg(t) \tag{8.32}
\]
as \( t \to \infty \). Thus we see that
\[
\text{Var } Z \leq C \left( t\psi(t) + t^2\psi^{-1}(1/t)\log t \right). \tag{8.33}
\]
Note that for all \( t \) sufficiently large
\[
tg(t) \leq (t\psi^{-1}(1/t))^3 \leq Ct^2\psi^{-1}(1/t), \tag{8.34}
\]
where we use (4.108). Thus we get (5.16).

\section{Appendix: Kac Moment Formula}

Let \( X = \{ X_t, t \in \mathbb{R}_+ \} \) denote a symmetric Lévy process with continuous local time \( L = \{ L_t^x; (x,t) \in \mathbb{R}^1 \times \mathbb{R}_+ \} \). Since \( L \) is continuous we have the occupation density formula,
\[
\int_0^t g(X_s) \, ds = \int g(x)L_t^x \, dx, \tag{9.1}
\]
for all continuous functions \( g \) with compact support. (See, e.g. \cite[Theorem 3.7.1]{1}.)

Let \( f(x) \) be a continuous function on \( \mathbb{R}^1 \) with compact support with \( \int f(x) \, dx = 1 \). Let \( f_{\epsilon,y}(x) := \frac{1}{\epsilon}f\left( \frac{x-y}{\epsilon} \right) \). I.e., \( f_{\epsilon,y}(x) \) is an approximate \( \delta \)-function at \( x \). Set
\[
L_{t,\epsilon}^x = \int_0^t f_{\epsilon,x}(X_s) \, ds. \tag{9.2}
\]
It follows from (9.1) that
\[
L_t^x = \lim_{\epsilon \to 0} L_{t,\epsilon}^x \quad \text{a. s.} \tag{9.3}
\]
Let \( p_t(x,y) \) denote the probability density of \( X_t \).
Theorem 9.1 (Kac Moment Formula) Let \( X = \{X_t, t \in \mathbb{R}_+\} \) denote a symmetric Lévy process with continuous local time \( L = \{L^x_t; (x, t) \in \mathbb{R}^1 \times \mathbb{R}_+\} \). For any fixed \( 0 < t < \infty \), bounded continuous \( g \), and any \( x_1, \ldots, x_m, z \in \mathbb{R}^1 \),

\[
E^z \left( \prod_{i=1}^m L^x_{t_i} g(X_t) \right) = \sum_{\pi} \int_{\left\{ \sum_{j=1}^m r_j \leq t \right\}} \prod_{j=1}^m p_{r_j}(x_{\pi(j)-1}, x_{\pi(j)}) \left( \int p_{t-r_m}(x_m, y) g(y) \, dy \right) \prod_{j=1}^m dr_j, \tag{9.4}
\]

where the sums run over all permutations \( \pi \) of \( \{1, \ldots, m\} \) and \( \pi(0) := 0 \) and \( x_0 := z \).

**Proof**

Let

\[
F_t(x_1, \ldots, x_m) = \int_{\left\{ \sum_{j=1}^m r_j \leq t \right\}} \prod_{j=1}^m p_{r_j}(x_{j-1}, x_j) \left( \int p_{t-r_m}(x_m, y) g(y) \, dy \right) \prod_{j=1}^m dr_j, \tag{9.5}
\]

Then

\[
E^z \left( \prod_{i=1}^m L^x_{t_i} g(X_t) \right) \tag{9.6}
\]

\[
= \sum_{\pi} \int_{\left\{ 0 \leq t_{\pi(1)} \leq \cdots \leq t_{\pi(m)} \leq t \right\}} E^z \left( \prod_{j=1}^m f_{\epsilon, x_j} \left( X_{t_{\pi(j)}} \right) g(X_t) \right) \prod_{j=1}^m dt_{\pi(j)}

= \sum_{\pi} \int_{\left\{ 0 \leq t_1 \leq \cdots \leq t_m \leq t \right\}} E^z \left( \prod_{j=1}^m f_{\epsilon, x_{\pi(j)}} \left( X_{t_j} \right) g(X_t) \right) \prod_{j=1}^m dt_j

= \sum_{\pi} \int_{\left\{ \sum_{j=1}^m r_j \leq t \right\}} \prod_{j=1}^m f_{\epsilon, x_{\pi(j)}}(y_j) p_{r_j}(y_{j-1}, y_j) \left( \int p_{t-r_m}(y_m, y) g(y) \, dy \right) \prod_{j=1}^m dr_j \, dy_j \tag{9.7}

= \sum_{\pi} \int F_t(y_0, \ldots, y_m) \prod_{j=1}^m f_{\epsilon, x_{\pi(j)}}(y_j) \, dy_j
\]

where \( y_0 := z \).
Since the integrand in (9.5) is dominated by \((2\pi)^{-m/2} \prod_{j=1}^{m} r_j^{-1/2}\) it follows from the Dominated Convergence Theorem that \(F_t(x_1, \ldots, x_m)\) is a continuous function of \((x_1, \ldots, x_m)\) for all \(0 \leq t < \infty\) and all \(m\). It then follows immediately from (9.6) and the fact that \(\prod_{j=1}^{m} f_{\epsilon,x_j}(y_j)\) has compact support that
\[
\lim_{\epsilon \to 0} E_{\epsilon} L_{x,\epsilon}^{x_i} g(X_t) = \sum_{\pi} F_t(x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(m)}). \tag{9.8}
\]
A repetition of the above proof shows that \(E_{\epsilon} \left( \prod_{i=1}^{m} L_{x,\epsilon}^{x_i} \right)^2\) is bounded uniformly in \(\epsilon > 0\). This fact and (9.3) show that
\[
\lim_{\epsilon \to 0} E_{\epsilon} \left( \prod_{i=1}^{m} L_{x,\epsilon}^{x_i} \right) = E \left( \prod_{i=1}^{m} L_{x}^{x_i} \right). \tag{9.9}
\]
Obviously (9.8) and (9.9) imply (9.4). \(\square\)

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