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Abstract. We consider an initial–boundary value problem for the continuity equation in a class of non-negative measure-valued solutions. We prove that any solution in the considered class can be represented as a superposition of elementary solutions, associated with the solutions of the corresponding ordinary differential equation.

1. Introduction
Let $b: [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel measurable uniformly bounded time-dependent vector field, where $T > 0$ and $d \in \mathbb{N}$.

Given $x \in \mathbb{R}^d$ let $\delta_x$ denote the Dirac measure concentrated at $x$. With the term measure in this work we will, by default, refer to a Borel measure on some underlying space, which will be always clear from the context.

It is well-known that if an absolutely continuous function $\gamma: [0,T] \to \mathbb{R}^d$ solves
\[
\frac{\partial}{\partial t} \gamma(t) = b(t, \gamma(t)) \quad \text{a.e. } t \in (0,T)
\] (ODE)

then the family $\left\{ \mu_t \right\}_{t \in [0,T]}$ of measures on $\mathbb{R}^d$
\[
\mu_t := \delta_{\gamma(t)}, \quad t \in [0,T],
\] (1)
solves the continuity equation
\[
\frac{\partial}{\partial t} \mu_t + \text{div}_x (b \mu_t) = 0 \quad \text{in } \mathcal{D}'(Q)
\] (CE)

where $Q = (0,T) \times \mathbb{R}^d$. Solutions of (CE) having the form (1) will be called elementary solutions (associated with solutions $\gamma$ of (ODE)).

Let
\[
\Gamma_b := \{ \gamma \in \Gamma: \gamma \text{ is absolutely continuous and solves (ODE)} \}.
\] (2)

Note that the set $\Gamma_b$ is Borel since $b$ is Borel, see [13], Proposition 2.

By linearity of (CE) any linear combination of elementary solutions will solve (CE) as well.

More generally, suppose that $\eta$ is a (possibly signed) Borel measure on $\Gamma := C([0,T]; \mathbb{R}^d)$
\[
\}
such that \( \eta \) is concentrated on the set \( \Gamma_b \) (i.e. satisfies \( |\eta|(\Gamma \setminus \Gamma_b) = 0 \), where \(| \cdot |\) denotes the total variation). Then family \( \{\mu_t\}_{t \in [0,T]} \) of measures

\[
\mu_t := \int \delta_{\gamma(t)} \, d\eta(\gamma)
\]  

solves (CE). (Eq. (4) can be understood in the following sense: for any Borel set \( E \subset \mathbb{R}^d \) it holds that \( \mu_t(E) = \int_{\Gamma} 1_E(\gamma(t)) \, d\eta(\gamma) \), where \( 1_E \) is the indicator of \( E \).)

In other words, any superposition of the elementary solutions of (CE) solves (CE) as well. It is therefore natural to ask whether any measure-valued solution of (CE) can be represented as a superposition of elementary solutions.

Recall that a family \( \{\mu_t\}_{t \in [0,T]} \) of finite measures on \( \mathbb{R}^d \) is called a measure-valued solution of (CE) if

(i) \( \{\mu_t\}_{t \in [0,T]} \) is a Borel family, i.e. the map \( t \mapsto \mu_t(\cdot) \) is Borel measurable for any Borel set \( E \subset \mathbb{R}^d \);

(ii) \( \int_0^T |\mu_t|(\mathbb{R}^d) \, dt < \infty \);

(iii) Eq. (CE) holds.

Points 1 and 2 here imply that for any bounded Borel function \( g : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) the function \( t \mapsto \int_{\mathbb{R}^d} g(t, x) \, d\mu_t(x) \) is Borel measurable (see e.g. [1]), therefore the distributional formulation of (CE) is well-defined.

For the class of non-negative measure-valued solutions of (CE) the question stated above was positively answered by Ambrosio, Gigli and Savare [11, 2]:

**Theorem 1** (Superposition principle). Let \( \{\mu_t\}_{t \in [0,T]} \) be a non-negative measure-valued solution of (CE). Then there exists a non-negative measure \( \eta \) on \( \Gamma \) such that (4) holds for a.e. \( t \in [0,T] \).

Apart from providing a precise characterisation of non-negative measure-valued solutions of (CE), this result can be used together with additional assumptions. For instance, in [5] it was involved in the proof of uniqueness of bounded weak solutions of the continuity equation when \( d = 2 \) and \( b \) is nearly incompressible, autonomous and has bounded variation. This result was recently generalized in [12] for multidimensional non-autonomous nearly incompressible BV vector fields.

In the present work we provide a generalization of Theorem 1 for the case of a bounded domain (instead of the whole space). Solvability of the initial–boundary value problem for the continuity equation with a non-smooth vector field \( b \) was studied in [6] (under some weak differentiability assumptions on \( b \)).

Let \( \Omega \subset \mathbb{R}^d \) be an open set and let \( Q := (0,T) \times \Omega \). Suppose that \( b : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a uniformly bounded Borel time-dependent vector field. We will assume that \( b \) is extended with \( 0 \) to \( \mathbb{R}^{d+1} \setminus Q \). Similarly, for any family \( \{\mu_t\}_{t \in [0,T]} \) of Borel measures on \( \Omega \) we will extend the measure \( \mu_t \) (for each \( t \in [0,T] \)) with \( 0 \) outside of \( \Omega \).

Let

\[
\Gamma_\Omega := \{ \gamma \in \Gamma : \gamma^{-1}(\Omega) \neq \emptyset \}
\]

denote the set of curves intersecting \( \Omega \). Clearly \( \Gamma_\Omega \) is an open subset of \( \Gamma \).

In order to take into account the initial and boundary conditions for the continuity equation we will introduce some auxiliary maps. (In Lemma 4 below we prove that these maps are Borel measurable.)

**Definition 1.** Given \( \gamma \in \Gamma \) let

\[
S_\gamma := \inf(\gamma^{-1}(\Omega) \cup \{T\}), \quad E_\gamma := \sup(\gamma^{-1}(\Omega) \cup \{0\})
\]
If $\gamma \in \Gamma_{\Omega}$ then $S_{\gamma}$ and $E_{\gamma}$ are the times when $\gamma$ starts and, respectively, ends in $\Omega$.

Now we are ready to state our main result:

**Theorem 2.** Let $\{\mu_{t}\}_{t \in [0,T]}$ be a Borel family of non-negative measures on $\mathbb{R}^d$ such that $\int_0^T \mu_{t}(\Omega) \, dt < \infty$ and $\int_0^T \mu_{t}(\mathbb{R}^d \setminus \Omega) \, dt = 0$. Let $\nu$ be a signed measure on $\mathbb{R}^{d+1}$ concentrated on $\partial Q$. Suppose that

$$\partial_t \mu_t + \text{div}(b\mu_t) = \nu \quad \text{in } \mathcal{D}'(\mathbb{R}^{d+1}) \quad (5)$$

Then there exists a non-negative measure $\eta$ on $\Gamma$ such that

(i) for $\eta$-a.e. $\gamma$ we have $S_{\gamma} < E_{\gamma}$ and

- $\gamma(t) = \gamma(0)$ for $t \in [0, S_{\gamma}]$,
- $\gamma(t) \in \Omega$ for $\text{a.e. } t \in [S_{\gamma}, E_{\gamma}]$,
- $\gamma(t) = \gamma(T)$ for $t \in [E_{\gamma}, T]$,
- $\gamma$ solves the ODE $\dot{\gamma}(t) = b(t, \gamma(t))$ a.e. on $[S_{\gamma}, E_{\gamma}]$;

(ii) for a.e. $t \in [0, T]$ and any $\varphi \in \mathcal{D}(\mathbb{R}^d)

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu_t(x) = \int_{\Gamma} \varphi(\gamma(t)) \, 1_{\Omega}(\gamma(t)) \, d\eta(\gamma);$$

(iii) for any $\Phi \in \mathcal{D}(\mathbb{R}^{d+1})$

$$\int_{\mathbb{R}^{d+1}} \Phi(t, x) \, d\nu(t, x) = \int_{\Gamma} (\Phi(S_{\gamma}, \gamma(S_{\gamma})) - \Phi(E_{\gamma}, \gamma(E_{\gamma}))) \, d\eta(\gamma), \quad (6a)$$

$$\int_{\mathbb{R}^{d+1}} \Phi(t, x) \, d\mu(t, x) = \int_{\Gamma} (\Phi(S_{\gamma}, \gamma(S_{\gamma})) + \Phi(E_{\gamma}, \gamma(E_{\gamma}))) \, d\eta(\gamma). \quad (6b)$$

Note that (5) is not a definition of the solution of the initial–boundary value problem for the continuity equation, but a property of such solutions. In particular, this property (i.e. existence of the measure $\nu$) holds for the distributional solutions introduced in [6, Definition 3.4]. The motivation for such setting is the following. Let $U \subset \mathbb{R}^n$ be a bounded open set with a $C^1$ boundary. Suppose that $V: U \to \mathbb{R}^n$ is a bounded measurable vector field and $\text{div } V = 0$ (in sense of distributions) inside $U$. Let us extend $V$ with 0 to $\mathbb{R}^n \setminus U$. Then by [7] there exists a bounded Borel measurable function $g: \partial U \to \mathbb{R}$ such that

$$\text{div}(1_{U} V) = g \mathcal{H}^{n-1}|_{\partial U} \quad \text{in } \mathcal{D}'(\mathbb{R}^n),$$

where $\mathcal{H}^k$ denotes the $k$-dimensional Hausdorff measure and $\mu|_E$ denotes the restriction of a measure $\mu$ to the set $E$. From this viewpoint the measure $\nu$ in (5) represents both initial and boundary conditions.

**2. Currents and their properties**

The proof of Theorem 2 is based on the structure of normal 1-currents, therefore first we would like to recall some relevant notation.

**2.1. Polyvectors and covectors**

Let $\Lambda_m(\mathbb{R}^n) \equiv \Lambda_m \mathbb{R}^n$ denote the space of $m$-vectors (polyvectors) on $\mathbb{R}^n$. Let $\Lambda^n(\mathbb{R}^n) \equiv \Lambda^m \mathbb{R}^n$ denote the space of $m$-forms (covectors) on $\mathbb{R}^n$.

The *inner product* on $\Lambda_m \mathbb{R}^n$ is defined as follows. For simple polyvectors we set

$$(v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots \wedge w_m) := \det \| (v_i, w_j) \|$$

(7)

Then we extend bilinearly the inner product $(\cdot, \cdot)$ for non-simple polyvectors.

The inner product defined above induces the *euclidean norm* on $\Lambda_m \mathbb{R}^n$:

$$|v| := \sqrt{(v, v)}. \quad (8)$$
2.2. Differential forms and currents
Let \( U \subset \mathbb{R}^n \) be an open set. Let \( \mathcal{D}^m(U) := \mathcal{D}(U; \Lambda^m \mathbb{R}^n) \) denote the space of smooth \( m \)-forms. By definition, an \( m \)-dimensional current \( T \) is a linear continuous functional on \( \mathcal{D}^m(U) \). Let

\[
\mathcal{D}_m(U) := \mathcal{D}(U; \Lambda^m \mathbb{R}^n)'
\]

denote the space of \( m \)-dimensional currents. It is clear that \( \Lambda_m(\mathbb{R}^n) \subset \mathcal{D}_m(\mathbb{R}^n) \). More generally, any polyvector field is a current.

The \textit{mass} of \( T \in \mathcal{D}_m(\mathbb{R}^n) \) is defined as

\[
M(T) := \sup\langle T, \omega \rangle,
\]

where the supremum is taken over all differential forms \( \omega \in \mathcal{D}^m(\mathbb{R}^n) \) such that

\[
|\omega|_{\infty} := \sup_{x \in \mathbb{R}^n} |\omega(x)| \leq 1.
\]

(We use here the euclidean norm of \( \Lambda^m(\mathbb{R}^n) \).)

**Definition 2.** A current \( S \) is called a subcurrent of a current \( T \) with \( M(T) < \infty \) if

\[
M(T - S) + M(S) = M(T).
\]

If (12) holds then we write \( S \subset T \).

2.3. Operations with currents
The \textit{boundary} \( \partial T \in \mathcal{D}_{m-1}(\mathbb{R}^n) \) of \( T \in \mathcal{D}_m(\mathbb{R}^n) \) is defined by

\[
\langle \partial T, \omega \rangle := \langle T, d\omega \rangle, \quad \omega \in \mathcal{D}^{m-1}(\mathbb{R}^n)
\]

if \( m > 0 \). If \( m = 0 \) then we set \( \partial T = 0 \).

Let \( U \subset \mathbb{R}^n, V \subset \mathbb{R}^k \) be open and let \( f : U \rightarrow V \subset \mathbb{R}^k \) be a smooth proper map (i.e. preimage of any compact is compact). The \textit{pushforward} \( f_#T \in \mathcal{D}_m(V) \) of a current \( T \in \mathcal{D}_m(U) \), is defined by

\[
\langle f_#T, \omega \rangle := \left\langle T, f^#\omega \right\rangle, \quad \omega \in \mathcal{D}^m(\mathbb{R}^n),
\]

where \( f^#\omega \) is the pullback of \( \omega \) with respect to \( f \).

Recall the following definition [8, Definition 3.7]:

**Definition 3.** Let \( T \in \mathcal{D}_m(\mathbb{R}^n) \). A current \( C \in \mathcal{D}_m(\mathbb{R}^n) \) is called a cycle of \( T \) if \( C \subset T \) and \( \partial C = 0 \). The current \( T \) is called acyclic if \( C = 0 \) is the only cycle of \( T \).

2.4. Examples of currents
The simplest example of a current is an oriented segment. Let \([a, b] \subset \mathbb{R}\). We define

\[
\langle [a, b], \omega \rangle := \int_a^b \omega
\]

for any \( \omega \in \mathcal{D}^1(\mathbb{R}) \) (smooth compactly supported 1-form). Clearly \([a, b] \in \mathcal{D}_1(\mathbb{R})\).

More generally, suppose that \( v \in L^1(\mathbb{R}^n; \mathbb{R}^n) \). Given \( \omega \in \mathcal{D}^1(\mathbb{R}^n) \) define

\[
\langle P, \omega \rangle := \int_{\mathbb{R}^n} \langle \omega(x), v(x) \rangle \, dx.
\]
Then $P$ is a 1-current on $\mathbb{R}^n$.

If $\theta: [a, b] \to \mathbb{R}^n$ is Lipschitz and $\theta' \neq 0$ a.e., then we define

$$[\theta] := \theta_# [a, b]$$

Clearly $[\theta] \in \mathcal{D}_1(\mathbb{R}^n)$. Let us show that if, in addition, $\theta$ is a simple curve with fixed orientation (or even if it intersects itself in finitely many points) then $[\theta]$ does not depend on the choice of the parametrization $\theta$.

Let $\mathcal{L}^n$ denote the Lebesgue measure and let $\mathcal{H}^k$ denote the $k$-dimensional Hausdorff measure on $\mathbb{R}^n$.

From the definition of the pullback of a differential form (see e.g. [3], def. 6.2.7) it follows that

$$\langle \theta^# \omega \rangle(t) = \langle \omega(\theta(t)), \theta'(t) \rangle \, dt$$

Then using Area formula we get

$$\langle [\theta], \omega \rangle = \left\langle [a, b], \theta^# \omega \rightangle$$

$$= \int_a^b \langle \omega(\theta(t)), \theta'(t) \rangle \, dt$$

$$= \int_a^b \langle \omega(x), \tau(x) \rangle \, d\mathcal{H}^1(x),$$

where $\tau(\theta(t)) = \theta'(t)/|\theta'(t)|$ for a.e. $t \in [a, b]$. In other words, we have proved that

$$[\theta] = \tau \mathcal{H}^1 \cap [a, b],$$

where $\mu \cap A$ denotes the restriction of the measure $\mu$ on the set $A$. Hence $[\theta]$ does not depend on the choice of the parametrization $\theta$ (which preserves the orientation).

### 2.5. Polar decomposition of a current with finite mass

Given a topological space $\mathcal{X}$ let $\mathcal{M}(\mathcal{X})$ denote the set of finite Borel measures on $\mathcal{X}$. Let $\mathcal{M}_+(\mathcal{X}) \subset \mathcal{M}(\mathcal{X})$ denote the set of non-negative measures.

Let $\mu \in \mathcal{M}_+(\mathbb{R}^n)$. Let $f: \mathbb{R}^n \to \Lambda^m(\mathbb{R}^n)$ be a Borel polyvector field on $\mathbb{R}^n$ such that $f \in L^1(\mu)$. Then one can define an $m$-current $T := f\mu$ as follows:

$$\langle T, \omega \rangle = \int_{\mathbb{R}^n} \langle \omega, f \rangle \, d\mu, \quad \forall \omega \in \mathcal{D}^m(\mathbb{R}^n).$$

**Lemma 1.** If $f \in L^1(\mathbb{R}^n, \mu; \Lambda^m\mathbb{R}^n)$ then

$$M(f\mu) = \int_{\mathbb{R}^n} |f| \, d\mu.$$  

**Proof.** First, notice that $T: \omega \mapsto \langle f\mu, \omega \rangle$ can be extended by continuity for any $\omega \in C_b(\mathbb{R}^n; \Lambda^m\mathbb{R}^n)$. Using Lusin’s theorem we extend $T$ further to any bounded Borel $\omega: \mathbb{R}^n \to \Lambda^m\mathbb{R}^n$. Then in the definition of mass (10) we can take sup over all bounded Borel differential forms.

By (10) $M(f\mu) \leq \int_{\mathbb{R}^n} |f| \, d\mu$.

Let us define a Borel differential form $\omega: \mathbb{R}^n \to \Lambda^m\mathbb{R}^n$ by

$$\langle \omega, v \rangle := \frac{\langle f, v \rangle}{|f|}.$$
Lemma 2. Suppose that \( f \in \mathcal{F} \). Clearly \( |\omega| = 1 \), and
\[
\langle f \mu, \omega \rangle = \int_{\mathbb{R}^n} \langle \omega, f \rangle \, d\mu = \int_{\mathbb{R}^n} |f| \, d\mu.
\] (23)

All currents with finite mass have the form (20) (see e.g. [3, §7.2]):

**Proposition 1** (Polar decomposition). Let \( T \in \mathcal{D}_m(\mathbb{R}^n) \) and suppose that \( M(T) < \infty \). Then there exists a Borel polyvector field \( f: \mathbb{R}^n \to \Lambda_m(\mathbb{R}^n) \) such that \( \|f(x)\| = 1 \) for all \( x \in \mathbb{R}^n \) and a finite non-negative Borel measure \( \mu \) on \( \mathbb{R}^n \) such that (20) holds.

(Proposition 1 essentially follows from the Riesz representation theorem.)

It is easy to see that this decomposition is unique in the following sense: if \( T = g\nu \) for some other non-negative Borel measure \( \nu \) on \( \mathbb{R}^n \) and Borel polyvector field \( g: \mathbb{R}^n \to \Lambda_m(\mathbb{R}^n) \) such that \( \|g(x)\| = 1 \) \( \forall x \in \mathbb{R}^n \) then \( \nu = \mu \) (as measures) and the equality \( f = g \) holds \( \mu \)-a.e. We will therefore denote
\[
|T| := \mu
\] (24)
where \( \mu \) is the measure provided by Proposition 1.

Clearly for the current \( \|\theta\| \) defined in (16) one has
\[
\|\theta\| = \mathcal{H}_1, \theta([a, b]) \quad \text{and} \quad |\partial\| = \begin{cases} \delta_{\theta(0)} + \delta_{\theta(1)}, & \theta(0) \neq \theta(1); \\ 0, & \theta(0) = \theta(1). \end{cases}
\] (25)

**Lemma 2.** Suppose that \( \mu = \mu_1 + \mu_2 \) and \( \mu_1 \perp \mu_2 \) are non-negative finite Borel measures. Let \( f \in L^1(\mathbb{R}^n, \mu; \Lambda^m \mathbb{R}^n) \). Then
\[
M(\mu f) = M(\mu_1 f) + M(\mu_2 f).
\] (26)

**Proof.** The proof follows immediately from (21). \( \square \)

The generic form of subcurrent of a current with finite mass can be described as follows:

**Proposition 2.** Let \( \mu \in \mathcal{M}_+(\mathbb{R}^n) \) and \( f \in L^1(\mathbb{R}^n, \mu; \Lambda^m \mathbb{R}^n) \). Then \( S \in \mathcal{D}_m(\mathbb{R}^n) \) is a subcurrent of \( T := f\mu \) if and only if
\[
S = g\mu
\] (27)
where \( g \in L^1(\mathbb{R}^n, \mu; \Lambda^m \mathbb{R}^n) \) satisfies
\[
g(x) \in \text{conv}\{0, f(x)\} := \{tf(x) \mid t \in [0, 1]\}
\] (28)
for \( \mu \)-a.e. \( x \in \mathbb{R}^n \).

**Proof.** By Proposition 1 there exists \( \nu \in \mathcal{M}_+(\mathbb{R}^n) \) and \( g \in L^1(\mathbb{R}^n, \mu; \Lambda^m \mathbb{R}^n) \) such that \( S = g\nu \). By Radon-Nikodym theorem there exist mutually singular \( \nu^a, \nu^s \in \mathcal{M}_+(\mathbb{R}^n) \) such that \( \nu = \nu^a + \nu^s, \nu^a \ll \mu \) and \( \nu^s \perp \mu \). Then
\[
M(f\mu) \stackrel{(12)}{=} M(f\mu - g\nu) + M(g\nu)
\]
\[
\stackrel{(26)}{=} M(f\mu - g\nu^a - g\nu^s) + M(g\nu^a) + M(g\nu^s)
\]
\[
\stackrel{(26)}{=} M(f\mu - g\nu^a) + M(g\nu^a) + 2M(g\nu^s).
\]
On the other hand by triangle inequality

\[ M(f) = M(f - g(x)) + M(g(x)) \]

hence \( 2M(g(x)) = 0 \), i.e. \( \nu^g = 0 \).

Therefore without loss of generality we can assume that \( S = g\mu \).

Again using (12) we get

\[ \int_{\mathbb{R}^n} (|f - g| + |g|) \, d\mu = \int_{\mathbb{R}^n} |f| \, d\mu. \tag{29} \]

Let \( E \subset \mathbb{R}^n \) be a Borel set. By triangle inequality \( |f(x)| \leq |f(x) - g(x)| + |g(x)| \) for any \( x \in \mathbb{R}^n \) hence

\[ \int_{\mathbb{R}^n \setminus E} (|f - g| + |g|) \, d\mu \geq \int_{\mathbb{R}^n \setminus E} |f| \, d\mu. \tag{30} \]

Subtracting (30) from (29) we get

\[ \int_E (|f - g| + |g|) \, d\mu \leq \int_E |f| \, d\mu. \tag{31} \]

Again by triangle inequality the opposite inequality holds, hence in fact (31) is an equality. By arbitrariness of \( E \) we conclude that

\[ |f(x) - g(x)| + |g(x)| = |f(x)| \tag{32} \]

for \( \mu \text{-a.e. } x \in \mathbb{R}^n \). Let \( f_1 := \frac{g(x)}{|f(x)|} f \), \( f_2 := \frac{f(x)}{|f(x)|} f = f - f_1 \). Let \( h := g - f_1 \). Then \( f - g - f_2 = -h \). Since the norm is euclidean,

\[ |f - g| + |g| = |f_1| + |f_2| + 2|h|. \]

By triangle inequality \( |f| \leq |f_1| + |f_2| \) hence using (32) and the equality above we get \( h = 0 \). Consequently \( |f_1| \leq |f| \).

Corollary 1. Suppose that \( \mu \in \mathcal{M}_+(\mathbb{R}^n) \), \( f \in L^1(\mathbb{R}^n, \mu; \Lambda_1\mathbb{R}^n) \) has the components \((f_1, \ldots, f_n)\).

If \( f_1 > 0 \) \( \mu \text{-a.e.} \) then the current \( P := f\mu \) is acyclic.

Proof. Let \( C \subset P \) be a cycle of \( P \). By Proposition 2 \( C = g\mu \) with some \( g \) such that \( g_1 \geq 0 \). For every \( k \in \mathbb{N} \) set \( \omega^k(x) := x \lor k \land (-k) \). Then \( \omega^k \) is continuous and bounded, so we can test and deduce that

\[ 0 = \langle \partial C, \omega^k \rangle = \langle C, d\omega^k \rangle = \int_{\{x_1 \in [-k, +k]\}} g_1(x) \, d\mu(x) = \int_{\mathbb{R}^n} 1_{\{x_1 \in [-k, +k]\}}(x) \omega(x) \, d\mu(x) \to \int_{\mathbb{R}^n} g_1 \, d\mu(x) \]

where the passage to the limit is allowed by dominated convergence \((g_1 \in L^1(\mu))\).  

\[ \blacksquare \]
2.6. **Decomposition of normal 1-currents**

Our main tool is the decomposition result for normal 1-currents obtained originally in [9] for finite-dimensional Euclidean spaces, and generalized in [8] for arbitrary complete separable metric spaces. Given metric spaces $M,N$ let $\text{Lip}(M,N)$ denote the metric space of bounded Lipschitz continuous maps from $M$ to $N$, endowed with the standard sup norm. We state the decomposition result for $\mathbb{R}^n$ following [8]:

**Theorem 3** (Smirnov; Paolini, Stepanov). Let $P$ be an acyclic normal 1-current on $\mathbb{R}^n$. Then there exists a finite measure $\xi \geq 0$ on $X := \text{Lip}([0,1];\mathbb{R}^n)$ such that

(i) $\xi$-a.e. $\theta$ is simple;
(ii) for any $\omega \in \mathcal{D}^1(\mathbb{R}^n)$

$$\langle P, \omega \rangle = \int_X \langle [\theta], \omega \rangle d\xi(\theta),$$

where $[\cdot]$ was introduced in (16);
(iii) for any $\varphi \in \mathcal{D}^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi d|P| = \int_X \left( \int_{\mathbb{R}^n} \varphi(x) d|[\theta]|(x) \right) d\xi(\theta),$$

where $|[\theta]|$ is the total variation of the current $[\theta]$, see (24) and (25);
(iv) for any $\varphi \in \mathcal{D}^1(\mathbb{R}^n)$

$$\langle |\partial P|, \omega \rangle = \int_X \langle |\partial[\theta]|, \omega \rangle d\xi(\theta).$$

We will refer to any measure $\xi$ given by Theorem 3 as to a decomposition of $P$.

Similarly to the proof of Lemma 1 one can observe that the equalities (1)–(3) of Theorem 3 hold not only for the smooth test functions and forms $\varphi$ and $\omega$, but also for merely bounded and Borel ones.

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\omega := d\varphi$. By Theorem 3(2)

$$\langle \partial P, \varphi \rangle = \int_X \langle \partial[\theta], \varphi \rangle d\xi(\theta).$$

The following result was obtained in [10]:

**Proposition 3.** Let $\mu \in \mathcal{M}_+(\mathbb{R}^n)$. Suppose that $B: \mathbb{R}^n \to \mathbb{R}^n$ is Borel measurable, $B \neq 0$ on $\mathbb{R}^n$ and $B \in L^1(\mathbb{R}^n,\mu)$. Suppose that 1-current $P := B\mu$ is normal. Then for any decomposition $\xi \in \mathcal{M}(X)$ of $P$ (provided by Theorem 3) for $\xi$-a.e. $\theta \in X$ it holds that

$$\dot{\theta}(s) = \frac{|\dot{\theta}(s)|}{|B(\theta(s))|} B(\theta(s))$$

for a.e. $s \in [0,1]$. (34)

For convenience of the reader let us present the proof of this result.

**Proof.** We apply Theorem 3 (2) with $\omega := |B|^{-1} B_1 dx$:

$$\int_{\mathbb{R}^n} 1 d\mu = \int_X \int_0^1 \theta'(s) \cdot \frac{B(\theta(s))}{|B|} ds d\xi(\theta).$$

Since $|P| = \mu$, from Theorem 3 (3) with $\varphi \equiv 1$ it follows that

$$\int_{\mathbb{R}^n} 1 d\mu = \int_X \int_0^1 |\theta'(s)| ds d\xi(\theta).$$
The equalities above imply that
\[
\int_X \int_0^1 (|\theta'(s)| - \theta'(s) \cdot \frac{B}{|B|}(\theta(s))) \, ds \, d\xi(\theta) = 0.
\]
By Cauchy–Bunyakovsky inequality the integrand is non-negative. Hence for \( \xi \)-a.e. \( \theta \in X \)
\[
|\theta'(s)| - \theta'(s) \cdot \frac{B}{|B|}(\theta(s)) = 0
\]
holds for a.e. \( s \in [0, 1] \). Clearly this equality is only possible when \( \theta'(s) = |\theta'(s)|\frac{B}{|B|}(\theta(s)) \).

**Corollary 2.** In addition to the assumptions of Proposition 3 suppose that \( B_j = 1 \) on \( \mathbb{R}^n \) and there exists \( C > 0 \) such that \( |B_j| \leq C \) for any \( j \in 2, 3, \ldots, n \). Then for any \( \tau, \tau' \in [0, 1] \) and any \( j \in 2, 3, \ldots, n \)
\[
|\theta_1(\tau') - \theta_1(\tau)| \geq \frac{1}{C}|\theta_j(\tau') - \theta_j(\tau)|
\]
(35)
Proof. Since \( B_1 = 1 \) and \( |B_j| \leq C \) the following inequalities hold:
\[
\begin{cases}
B_1(z) + \frac{1}{C}B_j(z) \geq 0, \\
B_1(z) - \frac{1}{C}B_j(z) \geq 0.
\end{cases}
\]
Substituting \( z = \theta(s) \) and multiplying these inequalities by \( |\theta(s)|\frac{1}{|B(\theta(s))|} \) in view of (34) we get
\[
\begin{cases}
\dot{\theta}_1(s) + \frac{1}{C}\dot{\theta}_j(s) \geq 0, \\
\dot{\theta}_1(s) - \frac{1}{C}\dot{\theta}_j(s) \geq 0.
\end{cases}
\]
Integrating with respect to \( s \) from \( \tau \) to \( \tau' \) (observe that \( B_1 \geq 0 \)) we get the desired result.

3. Continuity equation and 1-currents
In this section, similar to section 2.6, we denote \( X := \text{Lip}([0, 1]; \mathbb{R}^n) \), where \( n = d + 1 \) is the dimension of the spacetime.

Before proving Theorem 2 it remains to state several technical propositions.

Let
\[
X_Q := \{ \theta \in X : \theta^{-1}(Q) \neq \emptyset \}.
\]
Clearly \( X_Q \) is an open subset of \( X \).

Given \( c > 0 \) let \( X_c \) denote the closed subspace of \( X \) consisting of \( \theta : [0, 1] \to \mathbb{R}^{d+1} \) with the components \( (\theta_1, \ldots, \theta_{d+1}) \) such that

- \( s \mapsto \theta_1(s) \) is non-decreasing;
- for any \( \tau, s \in [0, 1] \) and any \( j \in 2, \ldots, d+1 \)
\[
|\theta_j(\tau) - \theta_j(s)| \leq c|\theta_1(\tau) - \theta_1(s)|.
\]
(36)
One can say that the elements of $X_c$ are *timelike* curves in the spacetime, which are confined by a family of cones with a fixed shape.

Consider $\theta \in X_c$ with the components $(\theta_1, \ldots, \theta_{d+1})$. By definition of $X_c$ for any $t \in \theta_1([0,1])$ the equation

\[ \theta_1(s) = t \]

has at least one solution, i.e. $\theta_1^{-1}(t) \neq \emptyset$. Let

\[ s_\theta(t) := \begin{cases} 0, & t < \theta_1(0); \\ \min \theta_1^{-1}(t), & t \in \theta_1([0,1]); \\ 1, & t > \theta_1(1). \end{cases} \]

Let us define the function $\gamma_\theta: [0, T] \rightarrow \mathbb{R}^d$ as the composition

\[ \gamma_\theta := (\theta_2, \ldots, \theta_{d+1}) \circ s_\theta. \]

In general $\theta_1$ may be not injective and therefore $s_\theta$ might be discontinuous. Nevertheless $\gamma_\theta$ is Lipschitz continuous by (36).

**Lemma 3.** The map

\[ g: X_c \ni \theta \mapsto \gamma_\theta \in \Gamma \]

is continuous.

**Proof.** Consider $\theta, \vartheta \in X_c$ and suppose that $\varepsilon := \|\gamma_\theta - \gamma_\vartheta\| > 0$. Let $t \in [0, T]$ be such that $|\gamma_\theta(t) - \gamma_\vartheta(t)| = \varepsilon$. Let $z := s_\theta(t)$. By (36) the image $\theta([0,1])$ is contained in the cone

\[ K_t := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \|\xi - \gamma_\theta(t)\| \leq c|\tau - t|\}. \]

Hence

\[ \|\vartheta - \theta\| \geq \text{dist}(\vartheta(z), \theta(z)) \geq \text{dist}(\vartheta(z), K_t) \geq \varepsilon \sin \varphi, \]

where $\varphi = \arctan \frac{1}{c}$. Therefore $g$ is Lipschitz continuous with the Lipschitz constant $\frac{1}{\sin \varphi}$. \qed

**Lemma 4.** The maps $\Gamma_\Omega \ni \gamma \mapsto S_\gamma \in \mathbb{R}$ and $\Gamma_\Omega \ni \gamma \mapsto E_\gamma \in \mathbb{R}$ are Borel measurable.

**Proof.** Let $\tilde{\gamma} \in \Gamma_\Omega$. For any $\varepsilon > 0$ there exists $t \in [S_{\tilde{\gamma}}, S_{\tilde{\gamma}} + \varepsilon)$ such that $\gamma(t) \in \Omega$. Since $\Omega$ is open we can always find $\delta > 0$ such that $B_\delta(\gamma(t)) \subset \Omega$. Hence for any $\gamma \in B_\delta(\tilde{\gamma})$ it holds that

\[ S_{\gamma} \leq t < S_{\tilde{\gamma}} + \varepsilon. \]

Therefore $\gamma \mapsto S_\gamma$ is upper semicontinuous. By an analogous argument the map $\gamma \mapsto E_\gamma$ is lower semicontinuous. Consequently, both maps are Borel measurable. \qed

Now we are ready to prove our main result:

*of Theorem 2.* Let $B: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ be the vector field with components

\[ B(z) := (1, b_1(z), b_2(z), \ldots, b_d(z)), \quad z \in \mathbb{R}^{d+1}. \]

Let us introduce the measure $\mu$ on $\mathbb{R}^{d+1}$ as follows:

\[ \mu(E) = \int_0^T \int_{\mathbb{R}^d} 1_E(t, x) d\mu_t(x) dt \]  \hspace{1cm} (37)
for any Borel set $E \subset \mathbb{R}^{d+1}$, where $1_E(z) = 1_E(z_1, z_2, \ldots, z_{d+1})$ is the indicator function for the set $E$. Notice that $\mu$ is a Borel measure, because the family $\{\mu_t\}_{t \in [0, T]}$ is Borel (see e.g. [1]).

It is easy to check that by (5)

$$\text{div}(B\mu) = \nu \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^{d+1}),$$ (38)

where the divergence is taken with respect to the space-time $(t, x) = (t, x_1, \ldots, x_d)$. (Recall that both $\mu_t$ and $b$ are extended with 0 outside of $Q$.)

Therefore by Corollary 1 the current $P := B\mu$ is acyclic. Hence by Theorem 3 there exists a decomposition $\xi$ of the current $P$.

By Proposition 3 the measure $\xi$ is concentrated on the solutions of (34). In view of Corollary 2 there exists a constant $C > 0$ such that $\xi$ is concentrated on $X_C$ (which was introduced above in Section 3). Therefore for $\xi$-a.e. $\theta$ we can define $\gamma_{\theta} := g(\theta)$, where $g$ is the map introduced in Lemma 3.

The measure $\mu$ by definition is concentrated on $Q$. Hence by the equality (3) of Theorem 3 for $\xi$-a.e. $\theta$ it holds that $||\theta||(\mathbb{R}^{d+1} \setminus Q) = 0$. For any such $\theta$ let $S := \{s \in [0, 1] : \theta(s) \notin Q\}$ and $\gamma := g(\theta)$. Then $\mathcal{H}^{1}(\theta(S)) = 0$. It is obvious that $\{t \in \theta_1([0, 1]) : \gamma(t) \notin Q\} \subset \pi(\theta(S))$, where $\pi : [0, T] \times \mathbb{R}^d \ni (t, x) \mapsto t$ is the standard projection. Consequently $\mathcal{L}^1(t \in \theta_1([0, 1]) : \gamma(t) \notin Q) \leq \mathcal{H}^1(\pi(\theta(S))) \leq \mathcal{H}^1(\theta(S)) = 0$ (see e.g. (1.4.6) in [4]). Therefore for a.e. $t \in \theta_1([0, 1])$ it holds that $\gamma(t) \in Q$. Hence $S_{\gamma} = \theta_1(0)$ and $E_{\gamma} = \theta_1(1)$. Since $\xi$-a.e. $\theta$ is injective and belongs to $X_C$, we also have $S_{\gamma} < E_{\gamma}$.

By Area formula and (34) for any $t \in [S_{\gamma}, E_{\gamma}]$ we have

$$\int_{S_{\gamma}}^{t} b_j(\tau, \gamma(\tau)) \, d\tau = \int_{\theta_1(0)}^{\theta_1(t)} b_j(\theta_1(s), \gamma_1(\theta_1(s))) \, \theta_1'(s) \, ds$$
$$= \int_{\theta_1(0)}^{\theta_1(t)} b_j(\theta_1(s), \gamma_1(\theta_1(s))) \frac{|\theta'_1(s)|}{B(\theta_1(s))} \, ds = \int_{\theta_1(0)}^{\theta_1(t)} b_j(\theta_1(s)) \frac{|\theta'_1(s)|}{B(\theta_1(s))} \, ds$$
$$= \int_{\theta_1(0)}^{\theta_1(t)} \theta'_j(s) \, ds = \theta_j(\theta_1(t)) - \theta_j(\theta_1(S_{\gamma})) = \gamma_j(t) - \gamma_j(S_{\gamma}),$$

where $j = 1, 2, \ldots, d$. Hence $\gamma$ satisfies the ODE $\gamma'(t) = b(t, \gamma(t))$ for a.e. $t \in [S_{\gamma}, E_{\gamma}]$.

Let $\Phi = (1 Q \varphi, 0, \ldots, 0)$, where $\varphi : \mathbb{R}^{d+1} \to \mathbb{R}$ is a bounded Borel function. Then

$$\int_0^T \int_{\mathbb{R}^d} \varphi \, d\mu_t \, dt = \int_Q B \cdot \Phi \, d\mu = \langle P, \Phi \rangle = \int_{X_C} \langle [\theta], \Phi \rangle \, d\xi(\theta)$$
$$= \int_{X_C} \int_0^1 \Phi(\theta(s)) \cdot \theta'(s) \, ds \, d\xi(\theta)$$
$$= \int_{X_C} \int_0^1 (1 Q \cdot \varphi)(\theta(s)) \theta'_1(s) \, ds \, d\xi(\theta)$$
$$= \int_{X_C} \int_0^1 (1 Q \cdot \varphi)(\theta(s), \gamma_0(\theta_1(s))) \theta'_1(s) \, ds \, d\xi(\theta)$$
$$= \int_{X_C} \int_0^T (1 Q \cdot \varphi)(t, \gamma_0(t)) \, dt \, d\xi(\theta) \quad \text{(by Area formula)}$$
$$= \int_0^T \int_{X_C} (1 Q \cdot \varphi)(t, \gamma_0(t)) \, d\xi(\theta) \, dt \quad \text{(by Fubini’s theorem)}$$
$$= \int_0^T \int_{X_C} (1 Q \cdot \varphi)(t, \gamma(t)) \, d\eta(\gamma) \, dt,$$
where $\eta := g\#\xi$ is the pushforward of the measure $\xi$ under the map $g$. The measure $\eta$ is well-defined by Lemma 3. Hence the claim (3) of Theorem 2 is proved.

It remains to prove claim (4) of Theorem 2. Arguing as above and using (33) we obtain (6a). By Theorem 3(4)

$$\int_{\mathbb{R}^{d+1}} \varphi \, d\nu = \langle |\partial P|, \varphi \rangle$$
$$= \int_{X_C} [\varphi(\theta(0)) + \varphi(\theta(1))] \, d\xi(\theta)$$
$$= \int_{X_C} [\varphi(\theta_1(0), \gamma_0(\theta_1(0))) + \varphi(\theta_1(1), \gamma_0(\theta_1(1)))] \, d\xi(\theta)$$
$$= \int_{X_C} [\varphi(S_{\gamma_0}, \gamma(\theta) S_{\gamma_0})) + \varphi(E_{\gamma_0}, \gamma(\theta) E_{\gamma_0}))] \, d\xi(\theta)$$
$$= \int_{\Gamma} [\varphi(S_{\gamma}, \gamma(S_{\gamma})) + \varphi(E_{\gamma}, \gamma(E_{\gamma}))] \, d\eta(\gamma)$$

Hence (6b) is proved.

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