In this paper, we study two dg (differential graded) operads related to the homology of moduli spaces of pointed algebraic curves of genus 0. These two operads are dual to each other, in the sense of Kontsevich [22] and Ginzburg and Kapranov [15].

This duality is analogous to the duality between commutative and Lie algebras, which goes back to Quillen [27]. Let us recall what form this duality takes. Note that all commutative algebras in this paper will be non-unital — that is, we do not take the existence of an identity as one of the axioms.

Associate to a dg commutative algebra \( A \) the chain complex \( LQA \) whose underlying graded vector space is the desuspension of the free graded Lie coalgebra generated by \( \Sigma A \), and whose differential is the sum of the internal differential of \( A \) and the coderivation induced by the product of \( A \), thought of as a map from \( \Sigma^{-1}(\Sigma A \wedge \Sigma A) \) to \( \Sigma^{-1}(\Sigma A) \). (Thus, \( LQA \) is the desuspension of the Harrison complex of \( A \).) There is a natural map from \( LQA \) to the complex of irreducibles \( QA = A/A^2 \) of \( A \).

Similarly, associate to a dg Lie algebra \( L \) the chain complex \( LQL \) whose underlying graded vector space is the desuspension of the free graded commutative coalgebra generated by \( \Sigma L \), and whose differential is the sum of the internal differential of \( L \) and the coderivation induced by the bracket of \( L \), thought of as a map from \( \Sigma^{-1}(\Sigma L \vee \Sigma L) \) to \( \Sigma^{-1}(\Sigma L) \). (Thus, \( LQA \) is the desuspension of the Chevalley-Eilenberg complex of \( L \).) There is a natural map from \( LQL \) to the complex of irreducibles \( QL = L/[L,L] \) of \( L \).

Over a field of characteristic zero, the following basic result holds; the first part is due to Koszul, and the second part is due to Barr and Quillen. (In fact, the first part holds in any characteristic, but we will restrict attention to fields of characteristic zero throughout this paper.)

**Theorem 0.1.** If \( A \) is a free dg commutative algebra, the map \( LQA \rightarrow QA \) is a homotopy equivalence (that is, a quasi-isomorphism of chain complexes). If \( L \) is a free dg Lie algebra, the map \( LQL \rightarrow QL \) is a homotopy equivalence.

In this paper, we prove an analogue of this theorem in which the rôle of commutative algebras is taken by a beautiful algebraic structure discovered by Dijkgraaf, Verlinde and Verlinde [8], which we call a hypercommutative algebra. This is a chain complex \( A \) with a sequence of graded symmetric products \( (x_1, \ldots, x_n) : A^{\otimes n} \rightarrow A \) of degree \( 2(n-2) \), which satisfy the following generalized associativity condition: if \( a, b, c, x_1, \ldots, x_n, n \geq 0 \), are elements of \( A \),

\[
\sum_{S_1 \cup S_2 = \{1, \ldots, n\}} \pm ((a, b, x_{S_1}), c, x_{S_2}) = \sum_{S_1 \cup S_2 = \{1, \ldots, n\}} \pm (a, (b, c, x_{S_1}), x_{S_2}).
\]

Here, if \( S = \{s_1, \ldots, s_k\} \) is a finite set, \( x_S \) is an abbreviation for \( x_{s_1}, \ldots, x_{s_k} \). The symbol \( \pm \) indicates the Quillen sign convention for \( \mathbb{Z}/2 \)-graded vector spaces: it equals +1 if all of the variables have even degree.

---

The author is partially supported by a fellowship of the Sloan Foundation and a research grant of the NSF.
In the cases \( n = 0 \) and \( 1 \), we obtain respectively the relations \((a, (b, c)) = ((a, b), c)\) and \((a, (b, c, d)) + (a, (b, c), d) = ((a, b), c, d) + (-1)^{|c||d|}((a, b, d), c)\). In particular, the product \((a, b)\) is a graded commutative, associative product in the usual sense.

We may think of the products in a hypercommutative algebra as the Taylor coefficients of a formal deformation of the commutative product \((a, b)\), parametrized by the double suspension \(\Sigma^{-2}A\) of \(A\):

\[
(a, b)_t = \sum_{k=0}^{\infty} \frac{1}{k!} (a, b, t, \ldots, t).
\]

The associativity of \((a, b)_t\) is equivalent to the sequence of identities satisfied by the products \((x_1, \ldots, x_n)\).

Recently, Kontsevich and Manin have shown that under certain circumstances, the quantum (or Floer) cohomology of a compact Kähler manifold is a hypercommutative algebra \(\mathbb{C}^3\); this shows that the rather heuristic calculations of Dijkgraaf, Verlinde and Verlinde may be justified by the study of the moduli spaces \(\overline{M}_{0,n}\). We refer to this reference for further details; extensions to a larger class of symplectic manifolds may be found in \(\mathbb{C}^3\) and \(\mathbb{C}^3\).

In our analogue of Theorem \(\mathbb{C}^3\), commutative algebras are replaced by hypercommutative algebras, while Lie algebras are replaced by gravity algebras, introduced by the author in \(\mathbb{C}^3\). A gravity algebra is a chain complex with graded antisymmetric products \([x_1, \ldots, x_n] : A^{\otimes n} \to A\) of degree \(2 - n\), satisfying the following relations: if \(k > 2\) and \(\ell \geq 0\), and \(a_1, \ldots, a_k, b_1, \ldots, b_\ell \in A\),

\[
\sum_{1 \leq i < j \leq k} \pm [[a_i, a_j], a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k, b_1, \ldots, b_\ell] = \begin{cases} [[a_1, \ldots, a_k], b_1, \ldots, b_\ell], \quad \ell > 0, \\ 0, \quad \ell = 0. \end{cases}
\]

For example, setting \(k = 3\) and \(\ell = 0\), we obtain the Jacobi relation for \([a, b]\).

Our proof of the duality between hypercommutative and gravity algebras is based on work of Beilinson and Ginzburg \(\mathbb{C}^3\) and Ginzburg and Kapranov \(\mathbb{C}^3\). If \(X\) is a topological space, let \(X_0^n\) be the configuration space of \(n\) distinct labelled points

\[
X_0^n = \text{embeddings of } \{1, \ldots, n\} \text{ in } X.
\]

Let \(\mathcal{M}_{0,n} = (\mathbb{P}^1)^n_0 / \text{PSL}(2, \mathbb{C})\), \(n \geq 3\), be the moduli space of smooth projective curve \(\Sigma\) of genus 0 with \(n\) marked points: it is a smooth variety of dimension \(n - 3\).

The moduli spaces \(\mathcal{M}_{0,n}\) have smooth compactifications \(\overline{\mathcal{M}}_{0,n}\), which were constructed by Deligne, Mumford and Knudsen \(\mathbb{C}^3\). The manifold \(\overline{\mathcal{M}}_{0,n}\) is a stratified space, whose closed strata \(\overline{\mathcal{M}}((T))\) are labelled by trees \(T\) with \(n\) legs. The above authors studied the duality between commutative and Lie algebras by applying mixed Hodge theory to the spectral sequence for the inclusion \(\mathcal{M}_{0,n} \hookrightarrow \overline{\mathcal{M}}_{0,n}\). Keel \(\mathbb{C}^3\) has shown that the cycles \(\overline{\mathcal{M}}((T))\) carried by these strata span the homology of \(\overline{\mathcal{M}}_{0,n}\), and building on his work, Kontsevich and Manin \(\mathbb{C}^3\) have found a complete set of relations among these cycles; in fact, these relations are in a sense exactly the relations which hold in a hypercommutative algebra. We will recover their results using mixed Hodge theory.

This paper is written in the language of operads, and may be considered to be an appendix to \(\mathbb{C}^3\). We extend their notion of Koszul operads to operads which are generated by operations with more than two operands, such as the hypercommutative and gravity operads. (The idea of generalizing Koszul duality to this setting came from conversations with Kontsevich. I wish to thank him for their permission to present those ideas here.)
As a byproduct of our work, we obtain new formulas for the characters of the $\mathbb{S}_n$-modules $H^i(M_{0,n})$ and $H^i(M_{0,n})$. These formulas illustrate the use of the Legendre transform for symmetric functions introduced in [4]. Of course, it would be very interesting if any of the ideas of the paper applied to higher genus moduli spaces of curves. However, since the purity of the mixed Hodge structure of $M_{0,n}$ plays such a central role here, and is in some sense equivalent to our main theorem, any extension to higher genus will be rather subtle.

The research of the author is partially supported by the NSF and the A.P. Sloan Research Foundation. The author is grateful to J.D.S. Jones, M.M. Kapranov and particularly M. Kontsevich for the many ways in which they assisted in the preparation of this article.

1. S-modules and operads

In this section, we recall the basic definitions of the theory of operads. For more details, see [12] and [15].

1.1. S-modules. An S-module is a collection of chain complexes (all chain complexes in this paper are over the field $\mathbb{C}$, and have finite dimensional total homology)

$$\{V(n) \mid n \geq 0\},$$

together with an action of $\mathbb{S}_n$ on $V(n)$. This definition generalizes Joyal’s notion of a linear species [17], which is an ungraded $\mathbb{S}$-module.

A chain complex $V$ may be thought of as an $\mathbb{S}$-module by setting

$$V(n) = \begin{cases} V, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

1.2. Schur functors. Given an $\mathbb{S}$-module $V$ and a finite set $S$, we define

$$V(S) = \bigoplus_{\text{bijections } f : \{1, \ldots, n\} \longrightarrow S} V(n).$$

It is clear that if $S = \{1, \ldots, n\}$, then $V(S)$ is naturally identified with $V(n)$.

To an $\mathbb{S}$-module $V$ is associated an endofunctor of the category of chain complexes, called the Schur functor of $V$, by the formula

$$V \mapsto S(V, V) = \bigoplus_{n=0}^{\infty} V \otimes_{\mathbb{S}_n} V^\otimes n,$$

here $V^\otimes n$ is the graded $n$th tensor power of $V$. Introduce a monoidal structure on the category of $\mathbb{S}$-modules, with tensor product

$$(V \otimes W)(n) = \bigoplus_{k=0}^{\infty} \left( V(k) \otimes \bigoplus_{f : \{1, \ldots, n\} \longrightarrow \{1, \ldots, k\}} W(f^{-1}(i)) \right) \otimes_{\mathbb{S}_k},$$

and unit the $\mathbb{S}$-module $\mathbb{1}$:

$$\mathbb{1}(n) = \begin{cases} \mathbb{C}, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

The peculiar formula for $V \circ W$ is justified by

$$S(V, S(W, V)) = S(V \circ W, V).$$

Note also that $S(\mathbb{1}, V) = V$. For more on this formalism, see Chapter 1 of [12].
1.3. Operads. An operad is a monoid in the category of \(S\)-modules, that is, an \(S\)-module \(A\) with product \(\rho : A \circ A \longrightarrow A\) and unit \(\eta : 1 \longrightarrow A\) satisfying the axioms of associativity and unit [28]. We denote the image under the product \(\rho\) of
\[
a \otimes b_1 \otimes \ldots \otimes b_k \in A(k) \otimes A(n_1) \otimes \ldots \otimes A(n_k)
\]
by \(a(b_1, \ldots, b_k)\) and the unit by \(1 \in A(1)\).

An operad structure on an \(S\)-module \(A\) such that \(A(n) = 0\) for \(n \neq 1\) is the same thing as an associative algebra structure on \(A(1)\). Whereas an element of an algebra has only one “input” and one “output,” an element of an operad has multiple inputs and one output.

1.4. The endomorphism operad of a chain complex. If \(V\) is a chain complex, its endomorphism operad is the \(S\)-module
\[
E_V(n) = \text{Hom}(V \otimes^n, V).
\]
This is an operad, whose product is given by composition: if \(a \in \text{Hom}(V \otimes^k, V)\) and \(b_i \in \text{Hom}(V \otimes^{n_i}, V)\), then
\[
a(b_1, \ldots, b_k) = a \cdot (b_1 \otimes \ldots \otimes b_k),
\]
where we think of \(b_1 \otimes \ldots \otimes b_k\) as an element of \(\text{Hom}(V \otimes^{n_1 + \cdots + n_k}, V \otimes^k)\).

1.5. Suspension of operads. If \(V\) is a chain complex, denote by \(\Sigma V\) the chain complex such that \((\Sigma V)_i = V_{i-1}\), with differential \(-\delta\). If \(V\) is an \(S\)-module, denote by \(\Lambda V\) the \(S\)-module
\[
(\Lambda V)(n) = \Sigma^{1-n} \text{sgn}_n \otimes V(n),
\]
where \(\text{sgn}_n\) is the sign character of \(S_n\). There is a natural isomorphism
\[
S(\Lambda V, V) \cong \Sigma S(V, \Sigma^{-1}V).
\]
It follows that if \(A\) is an operad, then so is \(\Lambda A\), and if \(A\) is an \(A\)-algebra, then \(\Sigma A\) is a \(\Lambda A\)-algebra.

1.6. Algebras over an operad. An algebra over an operad \(A\) is a chain complex \(A\), together with a morphism of operads \(A \longrightarrow E_A\). Thus, if \(A\) is an algebra over an operad \(A\) and \(\rho \in A(n)\), there is a product \(a_1 \otimes \ldots \otimes a_n \mapsto \rho(a_1, \ldots, a_n)\) from \(A \otimes^n\) to \(A\). These products are equivariant, under the actions of \(S_n\) on \(A(n)\) and \(A \otimes^n\), associative with respect to the product of \(A\), and \(1(a) = a\), where \(1 \in A(1)\) is the unit of \(A\).

1.7. Construction of operads. Given an algebraic structure defined by a set of multilinear relations among them, one may construct the operad \(A\) having this presentation, in such a way that an \(A\)-algebra is the same thing as an instance of the original algebraic structure.

Denote by \(V \mapsto T(V)\) the free algebra generated by the chain complex \(V\) with respect to this algebraic structure. To define \(A(n)\), we form the free algebra \(T(x_1, \ldots, x_n)\) generated by the free vector space \(\mathbb{C}^n\) of rank \(n\). The torus \((\mathbb{C}^\times)^n\) acts on this chain complex; let \(A(n)\) be the \(S_n\)-submodule on which it acts by the character \((z_1, \ldots, z_n) \mapsto z_1 \ldots z_n\). The group \(S_n\) acts on \(T(x_1, \ldots, x_n)\), and thus on \(A(n)\), by permutation of the letters \(x_i\). It is not difficult to see that the \(S\)-module thus constructed is an operad: the unit is the word \(x_1 \in A(1) \subset T(x_1)\), while the product is defined by substitution.

Let us give some examples of this construction. In the case of commutative algebras, we call the resulting operad \(\text{Com}^+\); the \(S_n\)-module \(\text{Com}^+(n)\) is spanned by the word \(x_1 \ldots x_n\), the free commutative algebra generated by letters \(\{x_1, \ldots, x_n\}\), and carries the trivial action of \(S_n\). (Here, we are dealing with non-unital commutative algebras, so that \(\text{Com}^+(0) = 0\).)
Let \( \mathcal{Ass}^+ \) be the operad associated to associative algebras; the \( S_n \)-module \( \mathcal{Ass}^+(n) \) is spanned by the words
\[
\{ x_{\sigma(1)} \ldots x_{\sigma(n)} \mid \sigma \in S_n \},
\]
and carries the regular representation of \( S_n \). (Again, we set \( \mathcal{Ass}^+(0) = 0 \).)

Finally, the operad associated to Lie algebra is denoted \( \mathcal{Lie}^+ \). The underlying \( S \)-module may be studied by means of the Poincaré-Birkhoff-Witt theorem, which implies that
\[
\mathcal{Ass}^+ \cong \mathcal{Com}^+ \circ \mathcal{Lie}^+.
\]

In Proposition 5.3, we will show how this leads to a formula for the character of \( \mathcal{Lie}^+(n) \), due to Klyachko [20].

### 1.8. The configuration spaces \( \mathbb{C}_0^n \) and the braid operad \( B \)

Let \( \mathbb{C}_0^n \) be the configuration space of \( n \) labelled points in \( \mathbb{C} \). Define the \( S \)-module \( \text{Braid} \) by
\[
\text{Braid}(n) = \begin{cases} 
H_\bullet(\mathbb{C}_0^n), & n > 0, \\
0, & n = 0.
\end{cases}
\]

We now construct a natural operad structure on \( \text{Braid} \). (We called this the braid operad in [13].)

In the definition of operads, one can replace the category of chain complexes by the category of topological spaces, and the tensor product by Cartesian product, obtaining the notion of a topological operad. If \( \mathcal{O} \) is a topological operad and \( H_\bullet(-) \) is a generalized homology theory with products, \( H_\bullet(\mathcal{O}) \) is an operad in the category of graded vector spaces; this gives a useful method of constructing operads.

Boardman and Vogt have constructed a topological operad called the little discs operad [1]. Let \( D \) be the closed unit disc in \( \mathbb{C} \), and let \( \mathcal{O}(n) \) be the topological space
\[
\mathcal{O}(n) = \left\{ (z_1, \ldots, z_n) \in \left( \frac{D^n}{\mathbb{Z}_n^2} \right) \mid \text{the discs } r_i D + z_i \text{ are disjoint subsets of } D \right\}.
\]

The symmetric group \( S_n \) acts on \( \mathbb{C} \mathbb{P}(n) \) by permuting the discs:
\[
\sigma(z_1, \ldots, z_n) = (z_{\sigma(1)}, \ldots, z_{\sigma(n)}).
\]

The product in this operad is defined by gluing of disks: if \( a = (z_1, \ldots, z_k) \) and \( b = (y_{i_1}, \ldots, y_{i_l}) \), then
\[
a(b_1, \ldots, b_k) = \left( \begin{array}{c}
\frac{r_1 y_{1,1} + z_1}{r_1 s_{1,1}} & \ldots & \frac{r_1 y_{1,n_1} + z_1}{r_1 s_{1,n_1}} & \ldots & \frac{r_k y_{k,1} + z_k}{r_k s_{k,1}} & \ldots & \frac{r_k y_{k,n_k} + z_k}{r_k s_{k,n_k}}
\end{array} \right).
\]

The map \( \mathcal{O}(n) \longrightarrow \mathbb{C}_0^n \) defined by \( (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n) \) is a homotopy equivalence, and thus the homology operad \( H_\bullet(\mathcal{O}) \) of this topological operad has \( \text{Braid} \) as its underlying \( S \)-module.

The operad \( \text{Braid} \) has the following presentation (see [3] and [13]): it is generated by two operations, a commutative product of degree 0 and a Lie bracket of degree 1, satisfying the Poisson relation:
\[
[a, bc] = [a, b]c + (-1)^{|a|+1}b[a, c].
\]

It follows that the \( S \)-module \( \text{Braid} \) is isomorphic to \( \mathcal{Com}^+ \circ \Lambda^{-1} \mathcal{Lie}^+ \). In particular, we obtain another realization of the \( S_n \)-module \( \mathcal{Lie}^+(n) \):
\[
\mathcal{Lie}^+(n) \cong \text{sgn}_n \otimes \text{H}_{n-1}(\mathbb{C}_0^n).
\]
2. Cyclic operads

All of the operads which we discuss in this article, notably the hypercommutative and gravity operads, are cyclic operads — that is, there is a notion of invariant inner product on algebras over these operads. In this section, we recall the basics of the theory of cyclic operads.

2.1. Cyclic $\mathcal{S}$-modules. A cyclic $\mathcal{S}$-module is an $\mathcal{S}$-module $\mathcal{V}$ together with an action of $\mathcal{S}_{n+1}$ on $\mathcal{V}(n)$ extending the action of $\mathcal{S}_n$. The name derives from the fact that an action of $\mathcal{S}_{n+1}$ is determined by compatible actions of $\mathcal{S}_n$ and the cyclic group $\mathcal{C}_{n+1} \subset \mathcal{S}_{n+1}$ generated by the cycle $(01 \ldots n)$. We denote the action of $(01 \ldots n)$ on $\mathcal{V}(n)$ by $v \mapsto v^*$, motivated by the fact that a cyclic $\mathcal{S}$-module structure on the $\mathcal{S}$-module associated to a chain complex $\mathcal{V}$ is just an involution $v \mapsto v^*$. It is convenient to write $\mathcal{V}(\langle n \rangle)$ for $\mathcal{V}(n+1)$.

If $\mathcal{V}$ is a cyclic $\mathcal{S}$-module, denote by $\Lambda \mathcal{V}$ the cyclic $\mathcal{S}$-module

$$(\Lambda \mathcal{V})(\langle n \rangle) = \Sigma^{2-n} \text{sgn}_n \otimes \mathcal{V}(\langle n \rangle).$$

Thus, $\Lambda$ applied to the $\mathcal{S}$-module underlying $\mathcal{V}$ is isomorphic to the $\mathcal{S}$-module underlying $\Lambda \mathcal{V}$.

A stable cyclic $\mathcal{S}$-modules is a cyclic $\mathcal{S}$-module $\mathcal{A}$ which satisfies the condition that $\mathcal{A}(\langle n \rangle) = 0$ for $n < 3$. (The word “stable” comes from the theory of algebraic curves.)

2.2. Cyclic operads. A cyclic operad $\mathcal{A}$ is a cyclic $\mathcal{S}$-module $\mathcal{A}$ with an operad structure, such that $1^* = 1$ and for all $a \in \mathcal{A}(k)$ and $b \in \mathcal{A}(l)$,

$$a(1, \ldots, 1, b)^* = (-1)^{|a||b|} b^* (a^*, 1, \ldots, 1).$$

Note that if $\mathcal{V}$ is a cyclic $\mathcal{S}$-module associated to a chain complex $\mathcal{V}$ with involution, then a cyclic operad structure on $\mathcal{V}$ is the same thing as a $\ast$-algebra structure on $\mathcal{V}$.

2.3. Stable cyclic operads. A cyclic operad $\mathcal{A}$ whose underlying $\mathcal{S}$-module is stable is called a stable cyclic operad. Because of this condition, $\mathcal{A}(\langle 2 \rangle) = 0$, so we can no longer think of $\mathcal{A}$ as having a unit. This requires the introduction of non-unital operads, following Markl. Observe that all of the products $a(b_1, \ldots, b_k)$ of a (unital) operad may be obtained by iterating the products

$$a \circ_i b = a(1, \ldots, 1, b, 1, \ldots, 1).$$

The axioms for non-unital operads may be found in [14]: they are of two types, equivariance and associativity. Finally, a non-unital operad is cyclic if $(a \circ_k b)^* = (-1)^{|a||b|} b^* \circ_1 a^*$. 

2.4. Invariant inner products and cyclic algebras. If $\mathcal{V}$ is a chain complex with inner product $\langle \cdot, \cdot \rangle$ (which we suppose to be non-degenerate), the endomorphism operad $\mathcal{E}_\mathcal{V}$ is a cyclic operad, such that if $a \in \mathcal{A}(n)$ and $v_i \in \mathcal{V}$, $0 \leq i \leq n$,

$$\langle v_0, a(v_1, \ldots, v_n) \rangle = \langle v_0, a^*(v_0, \ldots, v_{n-1}) \rangle.$$ 

A cyclic algebra over a cyclic operad is a chain complex $\mathcal{A}$ with inner product, together with a morphism of cyclic operads $\mathcal{A} \longrightarrow \mathcal{E}_\mathcal{A}$.

If $\mathcal{A}$ is an algebra over the operad underlying a cyclic operad $\mathcal{A}$, and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{A}$, we say that the inner product is invariant.

Note that if $\mathcal{A}$ is a cyclic operad, then the operad $\Lambda \mathcal{A}$ is not cyclic, but rather anticyclic: $(a \circ_k b)^* = (-1)^{|a||b|} b^* \circ_1 a^*$. This reflects the fact that if $\mathcal{V}$ is a chain complex and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{V}$, then there is induced on $\Sigma \mathcal{V}$ an antisymmetric non-degenerate bilinear form $(-1)^{|a|} \langle \Sigma v, \Sigma w \rangle$. However, $\Lambda^2 \mathcal{A}$ is again a cyclic operad, and if $\mathcal{A}$ is a cyclic $\mathcal{A}$-algebra, then $\Sigma^2 \mathcal{A}$ is a cyclic $\Lambda^2 \mathcal{A}$-algebra.
2.5. **Examples of cyclic operads.** Since associative, commutative and Lie algebras all have well-known notions of invariant inner product, it is not surprising that the corresponding operads are cyclic. For the operads $A^+$, where $A \in \{\text{Ass}, \text{Com}, \text{Lie}\}$, define the $\mathbb{S}$-module $A$ by setting $A(n) = A^+(n)$ for $n > 1$ and $A(n) = 0$ for $n \leq 1$.

The cyclic structure of $\text{Com}$ is simple to describe, since an inner product on a commutative algebra is invariant if and only if

$$\langle a, bc \rangle = \langle ab, c \rangle.$$  

It follows from this formula that $a = a^*$ for all $a \in \text{Com}(n)$.

In a similar way, the action of $\mathbb{S}_n$ on $\text{Ass}(n)$ is determined by the condition that an inner product on an associative algebra is invariant if and only if (3) holds. It turns out that the $\mathbb{S}_n$-module $\text{Ass}(n)$ is the induced representation $\text{Ind}_{C_n}^{S_n} \mathbb{I}$, where $C_n$ is the subgroup of $\mathbb{S}_n$ generated by $\tau_n$, and $\mathbb{I}$ is its trivial representation. We may think of $\text{Ass}(n)$ as being the $\mathbb{S}_n$-module spanned by symbols $\langle x_{\sigma(1)}, x_{\sigma(2)} \ldots x_{\sigma(n)} \rangle$, $\sigma \in \mathbb{S}_n$, representing the inner product of $x_{\sigma(2)} \ldots x_{\sigma(n)}$ with $x_{\sigma(1)}$, subject to the relations

$$\langle x_{\sigma(1)}, x_{\sigma(2)} \ldots x_{\sigma(n)} \rangle \sim \langle x_{\sigma(n)}, x_{\sigma(1)} \ldots x_{\sigma(n-1)} \rangle.$$

The cyclic structure on the Lie operad is associated with the usual notion of an invariant inner product (or Killing form) on a Lie algebra, satisfying

$$\langle [a, b], c \rangle = \langle a, [b, c] \rangle.$$  

The character of the $\mathbb{S}_{n+1}$-module $\text{Lie}(n)$ is calculated in [14]. We will obtain a realization of this representation as a homology group in the next section.

It is proved in [13] that the braid operad $\text{Braid}$ is not a cyclic operad.

3. **The moduli spaces $\overline{M}_{0,n}$**

In this section, we study the combinatorial structure of the compactified moduli spaces $\overline{M}_{0,n}$. We then define the gravity and hypercommutative operads, and introduce the fundamental exact sequences relating them, which are obtained by considering the spectral sequence associated by Deligne to the stratified space $\overline{M}_{0,n}$.

3.1. **Graphs and trees.** The strata of the compactification $\overline{M}_{0,n}$ are labelled by trees with $n$ legs, and we recall some definitions from the theory of trees in this paragraph (see also [15] and [14]).

A graph $G = (F, \pi, \tau)$ is a finite set $F = \text{Flag}(G)$, the set of flags of the graph, together with a partition $\pi$ and an involution $\tau$ of $F$. (By a partition, we mean a decomposition of $F$ into disjoint subsets, possibly empty, called its blocks.)

The vertices $\text{Vert}(G)$ of the graph $G$ are the blocks of $\pi$, the edges $\text{Edge}(G)$ are the 2-cycles of $\tau$, while the legs $\text{Leg}(G)$ are the fixed points of $\tau$. To a graph $G$ is associated a cell complex $G$ whose cells have dimension 0 and 1, and whose ends correspond to the legs $\text{Leg}(G)$. A graph is called a tree if this complex is simply connected. We will have no further use for non-simply connected graphs in this paper; however, much of the theory we describe has an analogue for general graphs [14].

The legs $\text{Leg}(v)$ of a vertex $v \in \text{Vert}(G)$ are the flags in the corresponding equivalence class, while the valence $|v|$ of a vertex is the cardinality of $\text{Leg}(v)$.

If $S$ is a finite set, let $T(\langle S \rangle)$ be the set of isomorphism classes of trees $T$ whose external edges are labelled by the elements of $S$ and such that each vertex has valence at least three.
Note that \( \mathcal{T}(\langle S \rangle) \) is finite. The set of trees is graded by the number of edges:

\[
\mathcal{T}(\langle S \rangle) = \bigcup_{i=0}^{\mid S \mid-3} \mathcal{T}_i(\langle S \rangle).
\]

In particular, \( \mathcal{T}_0(\langle S \rangle) \) has a single element, the tree with one vertex whose set of flags equals \( S \).

Denote by \( \det(S) \) the determinant line \( \Lambda^{\max} \mathbb{C}^S \), which is a representation of \( \text{Aut}(S) \). (For example, \( \det(\{1, \ldots, n\}) \) is just the sign representation \( sgn_n \).) If \( T \) is a tree, let \( \det(T) \) be the determinant line \( \det(\text{Vert}(T)) \) of the set of vertices of \( T \). There are natural isomorphisms

\[
\det(T) \cong \det(\text{Edge}(T)) \cong \det(\text{Leg}(T)) \otimes \bigotimes_{v \in \text{Vert}(T)} \det(\text{Leg}(v)).
\]

(4)

### 3.2. Stable curves.

A stable curve with \( n \) marked points is a projective curve \( \Sigma \) whose only singularities are double points, together with an embedding of \( \{1, \ldots, n\} \) in the set of smooth points of \( \Sigma \), such that there are no continuous automorphisms of \( \Sigma \) fixing the marked points and double points. Knudsen has proved that the moduli space \( \overline{\mathcal{M}}_{g,n} \) of stable curves of arithmetic genus \( g \) with \( n \) marked points is a compact orbifold, obtained by adjoining to \( \mathcal{M}_{g,n} \) a divisor with normal crossings \( \square \). We will only be interested in the genus zero case, in which case \( \overline{\mathcal{M}}_{0,n} \) is actually a projective variety. Another reference for the genus 0 case is [18].

A stable curve \( \Sigma \) determines a graph \( \Gamma_{\Sigma} \), called the dual graph of \( \Sigma \): the vertices of \( \Gamma_{\Sigma} \) are the components of the subvariety of smooth points of \( \Sigma \), the edges are the double points, and \( \text{Leg}(\Gamma_{\Sigma}) \) is the set of marked points. In particular, if \( \Sigma \) has arithmetic genus 0 and \( n \) marked points, then \( \Gamma_{\Sigma} \in \mathcal{T}(\langle n \rangle) \).

The moduli space \( \overline{\mathcal{M}}_{0,n} \) is a stratified space: it has one stratum \( \overline{\mathcal{M}}(\langle T \rangle) \) for each tree \( T \in \mathcal{T}(\langle n \rangle) \), consisting of all curves \( \Sigma \in \overline{\mathcal{M}}_{0,n} \) such that \( \Gamma_{\Sigma} = T \). The stratum \( \overline{\mathcal{M}}(\langle T \rangle) \) is isomorphic to the product

\[
\prod_{v \in \text{Vert}(T)} \mathcal{M}_{0,|v|},
\]

and has codimension equal to the number of edges of \( T \). For example \( \overline{\mathcal{M}}_{0,4} \) has the following four strata:

\[
\begin{array}{cccccccc}
0 & 2 & 0 & 3 & 0 & 1 & 0 & 1 \\
3 & 1 & 1 & 2 & 2 & 3 & 2 & \end{array}
\]

If \( T \in \mathcal{T}_k(n) \), denote by \( \overline{\mathcal{M}}(\langle T \rangle) \) the closure of the stratum \( \mathcal{M}(\langle T \rangle) \) of \( \overline{\mathcal{M}}_{0,n} \), and by \( \overline{\mathcal{M}}(\langle T \rangle) \) the corresponding cycle in \( H_{2(n-k-3)}(\overline{\mathcal{M}}_{0,n}) \). The following theorem is due to Keel [19].

**Theorem 3.3.** The cycles \( \overline{\mathcal{M}}(\langle T \rangle), T \in \mathcal{T}(\langle n \rangle) \) span \( H_*({\overline{\mathcal{M}}_{0,n}}) \).

In Section 3.13, we give a proof of this theorem which differs from Keel’s, and uses mixed Hodge theory.

### 3.4. The gravity operad.

Let \( \mathcal{G}_{\text{grav}} \) be the stable cyclic \( \mathcal{S} \)-module

\[
\mathcal{G}_{\text{grav}}(\langle n \rangle) = \begin{cases} \Sigma^{3-n} \text{sgn}_n \otimes H_*(\mathcal{M}_{0,n}), & n \geq 3, \\ 0, & n < 3. \end{cases}
\]
Note that $\mathcal{Grav}(n)$ is concentrated in degrees $3 - n \leq i \leq 0$.

There is a natural cyclic operad structure on $\mathcal{Grav}$. To define the product

$$c_i : \mathcal{Grav}(m + 1) \otimes \mathcal{Grav}(n + 1) \to \mathcal{Grav}(m + n),$$

consider the embedding $j$ of $\mathcal{M}_{0,\{0,\ldots,m\}} \times \mathcal{M}_{0,\{0',\ldots,n'\}}$ as a stratum of $\overline{\mathcal{M}}_{0,\{0,\ldots,i,m,1',\ldots,n'\}}$, corresponding to the joining of the point labelled $i$ in the curve $\Sigma_1 \in \mathcal{M}_{0,\{0,\ldots,m\}}$ to the point labelled $0'$ in the curve $\Sigma_2 \in \mathcal{M}_{0,\{0',\ldots,n'\}}$. Consider the Poincaré residue map associated to this embedding $[\hat{i}]$:

$$\text{Res} : H^\bullet(\mathcal{M}_{0,\{0,\ldots,i,m,1',\ldots,n'\}}) \to H^\bullet(\mathcal{M}_{0,\{0,\ldots,m\}} \times \mathcal{M}_{0,\{0',\ldots,n'\}})$$

Suitably suspending the adjoint of this map, we obtain the product $c_i$ of $\mathcal{Grav}$. This construction makes it quite obvious that $\mathcal{Grav}$ satisfies the equivariance and associativity axioms of a cyclic operad.

Denote by $[x_1, \ldots, x_n]$ the element of $\mathcal{Grav}(n)$ of degree $2 - n$ corresponding to the standard basis vector of $H_0(\mathcal{M}_{0,n+1})$. This operation is graded antisymmetric, since $\mathbb{S}_{n+1}$ acts on it by the sign representation, and it is proved in [11] that this sequence of operations generates $\mathcal{Grav}$, and that all relations are generated by the quadratic relations of (4). Note that in that paper, we work with the operad $\Lambda^{-1}\mathcal{Grav}$, and the generators are all in degree 1: the relationship between these two sets of generators at the level of algebras is

$$\{x_1, \ldots, x_n\} = (-1)^{(n-1)|x_1|+(n-2)|x_2|+\cdots+|x_{n-1}|}\Sigma^{-1}[\Sigma x_1, \ldots, \Sigma x_n].$$

In that paper, the operad structure is constructed in a different, though equivalent, way, using $\mathbb{C}^\infty$-equivariant homology.

3.5. $\mathcal{Grav}$ as a mixed Hodge operad. Operads (and, more specifically, stable cyclic operads) may be defined in any symmetric monoidal category with colimits. Up to this point, we have concentrated on the examples of operads in the categories of chain complexes (differential graded operads) and topological spaces (topological operads). However, the category of mixed Hodge complexes (3) is a symmetric monoidal category with colimits, with graded tensor product as the monoidal structure, and operads in this category are called mixed Hodge operads. In fact, the mixed Hodge operads which most concern us are pure and have vanishing differential.

The gravity operad is an example of a mixed Hodge operad. This carries a unique mixed Hodge structure compatible with the Poincaré residue maps which define the products in $\mathcal{Grav}$:

$$\mathcal{Grav}(n) = \Sigma^{3-n}\text{sgn}_n \otimes H_\bullet(\mathcal{M}_{0,n}, \mathbb{C}(n-3)).$$

Here, $\mathbb{C}(n-3)$ is the Tate Hodge structure, which is a line with Hodge numbers $(n-3, n-3)$.

3.6. The hypercommutative operad. Let $\mathcal{Hycom}$ be the stable cyclic $\mathbb{S}$-module

$$\mathcal{Hycom}(n) = \begin{cases} H_\bullet(\overline{\mathcal{M}}_{0,n}), & n \geq 3, \\ 0, & n < 3. \end{cases}$$

The cyclic $\mathbb{S}$-space $\overline{\mathcal{M}}(n) = \overline{\mathcal{M}}_{0,n}$ is a topological cyclic operad: the product is given by gluing stable curves together at marked points. It follows that $\mathcal{Hycom}$ is a cyclic operad. Kontsevich and Manin found [23] that algebras over $\mathcal{Hycom}$ are just hypercommutative algebras in the sense of (4), where the operation $(x_1, \ldots, x_n) \in \mathcal{Hycom}(n+1)$ corresponds to the fundamental class $[\overline{\mathcal{M}}_{0,n+1}] \in H_{2(n-2)}(\overline{\mathcal{M}}_{0,n+1})$. The fact that the operations $(x_1, \ldots, x_n)$ generate $\mathcal{Hycom}$ is an elegant restatement of Theorem 3.3. In Proposition 3.11, we will give a new proof (obtained jointly with Kontsevich) of the relations between the generators of $\mathcal{Hycom}$, which relies on the duality between hypercommutative algebras and gravity algebras, together with our explicit presentation (2) of the gravity operad $\mathcal{Grav}$. 
3.7. A spectral sequence of Deligne. Let $M$ be a smooth projective variety of complex dimension $n$, and let $\{D_1, \ldots, D_N\}$ be a sequence of smooth divisors with normal crossings; we denote their union by $D$. The sheaf of logarithmic differential forms $\mathcal{E}_M^\bullet(\log D)$ on $M$ is generated over the sheaf of differential forms $\mathcal{E}_M^\bullet$ by symbols $d(\log f)$, where $f$ is a section of $\mathcal{O}(D)$, subject to the relations

$$d(\log fg) = d(\log f) + d(\log g) \quad \text{and} \quad f \cdot d(\log f) = df.$$

The sheaf $\mathcal{E}_M^\bullet$ has a differential $d$, characterized by $d(d(\log f)) = 0$, and we have the fundamental isomorphism

$$H^\bullet(U) \cong H^\bullet(M, \mathcal{E}_M^\bullet(\log D)),$$

where $U = M \setminus D$.

The sheaf $\mathcal{E}_M^\bullet(\log D)$ is filtered by subsheaves

$$W_k\mathcal{E}_M^\bullet(\log D) = \text{span}_\mathcal{O}\{d(\log f_1) \wedge \ldots \wedge d(\log f_i) \mid i \leq k\}.$$

Let $j^k : D^k \hookrightarrow M$ be the embedding of the closed subvariety

$$D^k = \coprod_{i_1 < \ldots < i_k} D_{i_1} \cap \ldots \cap D_{i_k},$$

and let $\epsilon_k$ be the locally constant line bundle over $D^k$, which over the component $D_{i_1} \cap \ldots \cap D_{i_k}$ equals the determinant line $\det((i_1, \ldots, i_k))$. There is a canonical quasi-isomorphism

$$\text{gr}^W_k \mathcal{E}_M^\bullet(\log D) \simeq \Sigma^{-k} j^k_! \mathcal{E}_M^\bullet \otimes \epsilon_k.$$

The associated spectral sequence

$$E^{p,q}_1 = H^{2p+q}(D^p, \epsilon_p) \Rightarrow E^{p,q} = \text{gr}^W_p H^{-p+q}(U)$$

carries a Hodge filtration $F$, induced by the Hodge filtration of $\mathcal{E}_M^\bullet(\log D)$, and by the principal of two types, $E_2 = E_\infty$ (\cite{3}, Section 3.2). The weight filtration induced on $H^\bullet(U)$ by $W$ defines, up to translation, its mixed Hodge structure: $\text{gr}^W_p H^{-p+q}(M)$ carries a pure Hodge structure of weight $q$.

If a finite group $\Gamma$ acts on $M$, preserving $U \subset M$, this spectral sequence carries an action of $\Gamma$ compatible with its action on $H^\bullet(U)$.

3.8. The cohomology ring of $\mathcal{M}_{0,n}$. We now describe the cohomology ring of the moduli space $\mathcal{M}_{0,n}$. Our main tool is Arnold’s description of the cohomology ring of the configuration space $\mathbb{C}^\bullet_0$.

If $1 \leq j \neq k \leq n$, let $\omega_{jk}$ be the logarithmic differential form on the configuration space $\mathbb{C}^n_0$ given by the formula

$$\omega_{jk} = \frac{d \log(z_j - z_k)}{2\pi i}.$$

Note that the cohomology class of $\omega_{jk}$ is integral.

Proposition 3.9. The cohomology ring $H^\bullet(\mathbb{C}^\bullet_0, \mathbb{Z})$ is the graded commutative ring with generators $[\omega_{jk}]$, and relations $\omega_{jk} = \omega_{kj}$ and $\omega_{ij}\omega_{jk} + \omega_{ik}\omega_{kj} + \omega_{ki}\omega_{ij} = 0$. The symmetric group $S_n$ acts on $H^\bullet(\mathbb{C}^\bullet_0, \mathbb{Z})$ through its action on the generators $\sigma \cdot \omega_{ij} = \omega_{\sigma(i)\sigma(j)}$.

Proof: The Serre spectral sequence for the fibration

$$\mathbb{C} \setminus \{z_1, \ldots, z_n\} \longrightarrow \mathbb{C}^{n+1}_0 \longrightarrow \mathbb{C}^n_0$$

defined by projecting $(z_1, \ldots, z_{n+1})$ to $(z_1, \ldots, z_n)$ collapses at $E_2$, and the monodromy of $\pi_1(\mathbb{C}^n_0)$ on $H^\bullet(\mathbb{C} \setminus \{z_1, \ldots, z_n\})$ is trivial. The proof now proceeds by induction on $n$. \qed

Corollary 3.10. The cohomology ring $H^\bullet(\mathcal{M}_{0,n+1}, \mathbb{C})$ may be identified with the kernel of the differential $i$ on $H^\bullet(\mathbb{C}^n_0, \mathbb{C})$ whose action on the generators is $i \omega_{jk} = 1$. 

Proof: The isotropy group of the point $\infty \in \mathbb{C}P^1$ under the action of $\text{PSL}(2, \mathbb{C})$ is
\[ \text{Aff}(\mathbb{C}) = \left\{ \left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) \mid a \in \mathbb{C}^\times, b \in \mathbb{C} \right\} \subset \text{PSL}(2, \mathbb{C}). \]
Since $\text{PSL}(2, \mathbb{C})$ acts transitively on $\mathbb{C}P^1$, we see that $\mathcal{M}_{0,n+1} \cong \mathbb{C}^n_0 / \text{Aff}(\mathbb{C})$. But the group $\text{Aff}(\mathbb{C})$ is homotopy equivalent to the circle group
\[ \left\{ \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \mid |a| = 1 \right\}, \]
giving a homotopy equivalence $\mathbb{C}^n_0 \cong \mathcal{M}_{0,n+1} \times S^1$. This allows us to identify the cohomology of $\mathcal{M}_{0,n+1}$ with the $S^1$-equivariant cohomology of $\mathbb{C}^n_0$.

The infinitesimal generator of the circle action on $\mathbb{C}^n_0$ is the vector field
\[ T = 2\pi i \sum_{k=1}^n (z_k \partial_k - \bar{z}_k \bar{\partial}_k), \]
whose contraction with a generator $\omega_{jk}$ is $\omega_{jk}(T) = 1$. \(\square\)

The above result leads to yet another realization of the $\mathbb{S}_n$-module structure on $\mathcal{L}ie((n)) \cong \mathcal{L}ie(n-1)$:
\[ \mathcal{L}ie((n)) \cong \text{sgn}_n \otimes H_{n-3}(\mathcal{M}_{0,n}). \]

3.11. Application to $\mathcal{M}_{0,n} \subset \overline{\mathcal{M}}_{0,n}$. We now apply Deligne’s spectral sequence with $U = \mathcal{M}_{0,n}$ and $M = \overline{\mathcal{M}}_{0,n}$. (This spectral sequence is also discussed in \cite{15}, Section 3.4.5.) Denote the closure of the open stratum $\mathcal{M}(T))$ by $\overline{\mathcal{M}(T))}$. Then the divisors are the closed strata $\mathcal{M}(T), T \in T_{p}(n))$, while $D_p$ is the union of the closed strata $\overline{\mathcal{M}(T))}, T \in T_{p}(n))$. The restriction of $\epsilon_{p}$ to $\overline{\mathcal{M}(T))}$ equals $\det(\text{Edge}(T))$; as this is naturally isomorphic to $\det(T)$, we see that
\[ E_1^{-p,q} \cong \bigoplus_{T \in T_{p}(n))} H^{-2p+q}(\overline{\mathcal{M}(T))}, \det(T)). \]
The differential $d_1: E_1^{-p,q} \longrightarrow E_1^{-p+1,q}$ is easy to describe: it is the composition
\[ \bigoplus_{T \in T_{p}(n))} H^{-2p+q}(\overline{\mathcal{M}(T))}, \det(T)) \bigoplus_{T \in T_{p-1}(n))} H^{-2p+2+q}(\overline{\mathcal{M}(T))}, \det(T)) \]
\[ \bigoplus_{T \in T_{p}(n))} H_{2(n-3)-q}(\overline{\mathcal{M}(T))}, \det(T)) \longrightarrow \bigoplus_{T \in T_{p-1}(n))} H_{2(n-3)-q}(\overline{\mathcal{M}(T))}, \det(T)) \]
where the vertical isomorphisms are induced by Poincaré duality, and the bottom arrow is the map induced on the homology groups by the inclusion of the $p$-codimensional closed strata of $\overline{\mathcal{M}_{0,n}}$ into the $(p - 1)$-codimensional closed strata.

The key to unlocking this spectral sequence is the following lemma, which shows that
\[ E_2^{-k,2k} \cong H^k(\mathcal{M}_{0,n}), \]
while $E_2^{pq} = 0$ if $2p + q \neq 0$.

Lemma 3.12. The mixed Hodge structure of $H^k(\mathcal{M}_{0,n}), n \geq 3$, is pure of weight $2k$.

In degree $i$, $\text{Grav}(i)$ equals the $\mathbb{S}_n$-module $\text{sgn}_n \otimes H_{i+n-3}(\mathcal{M}_{0,n})$, with pure Hodge structure of weight $-2i$.

Proof: By Proposition 3.9, the cohomology ring of $\mathbb{C}^n_0$ is generated by the logarithmic differential forms $\omega_{ij}$; it follows that the mixed Hodge structure of $H^k(\mathbb{C}^n_0)$ is pure of weight $2k$. By Corollary 3.10, there is an injection of the cohomology of $\mathcal{M}_{0,n+1}$ into the cohomology of $\mathbb{C}^n_0$, induced by the quotient map $\mathbb{C}^n_0 \longrightarrow \mathcal{M}_{0,n+1}$, and the result follows. \(\square\)
Thus, for $q$ even, the $E_1$-term of the spectral sequence gives rise to a resolution of the graded vector space $H^\ast (\mathcal{M}_{0,n})$

\begin{equation}
0 \longrightarrow H^p(\mathcal{M}_{0,n}) \longrightarrow \bigoplus_{T \in T_p(n)} H^0(\overline{\mathcal{M}}(T)), \det(T) \longrightarrow \bigoplus_{T \in T_{p-1}(n)} H^2(\overline{\mathcal{M}}(T)), \det(T) \longrightarrow \ldots
\end{equation}

For $q$ odd, we obtain the exact sequence

\[ \ldots \longrightarrow \bigoplus_{T \in T_1(n)} H^{q-2}(\overline{\mathcal{M}}(T)), \det(T) \longrightarrow H^q(\overline{\mathcal{M}}_{0,n}) = \bigoplus_{T \in T_0(n)} H^q(\overline{\mathcal{M}}(T)), \det(T) \longrightarrow 0. \]

which shows, by induction on odd $q$, that the cohomology groups $H^q(\overline{\mathcal{M}}_{0,n})$ vanish if $q$ is odd.

3.13. Proof of Theorem 3.3. It is clear that the result holds for $n = 3$, since $\overline{\mathcal{M}}_{0,3}$ is a point. We now argue by induction on $n$. As a consequence of the surjectivity of the differential in the exact sequence (3)

\[ \bigoplus_{T \in T_1(n)} H^{2p-2}(\overline{\mathcal{M}}(T)), \det(T) \longrightarrow H^{2p}(\overline{\mathcal{M}}_{0,n}) = \bigoplus_{T \in T_0(n)} H^{2p}(\overline{\mathcal{M}}(T)), \det(T) \]

for $p \leq n - 2$, we see that all homology classes of $\overline{\mathcal{M}}_{0,n}$ except the fundamental class are supported on the closures of strata of codimension 1. Such a closed stratum is isomorphic to $\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,j}$ where $i + j = n + 2$ and $i, j \geq 3$, allowing us to apply the induction. \[\square\]

3.14. The dimension of $H^2(\overline{\mathcal{M}}_{0,n})$. Another consequence of (3) is a simple formula for the dimension $\dim H^2(\overline{\mathcal{M}}_{0,n})$ of the Picard variety of $\overline{\mathcal{M}}_{0,n}$:

\[ \dim H^2(\overline{\mathcal{M}}_{0,n}) = \sum_{k=3}^{n-1} \binom{n-1}{k} = 2^{n-1} - \frac{n^2 - n + 2}{2} = \left(\begin{array}{c} n \\ n - 4 \end{array}\right) + \left(\begin{array}{c} n \\ n - 6 \end{array}\right) + \ldots \]

Proof: When $p = 1$, the short exact sequence (2) becomes

\[ 0 \longrightarrow H^1(\mathcal{M}_{0,n}) \longrightarrow \bigoplus_{T \in T_1(n)} H^0(\overline{\mathcal{M}}(T)), \det(T) \longrightarrow H^2(\overline{\mathcal{M}}_{0,n}) \longrightarrow 0. \]

By Corollary 3.10, the dimension of $H^1(\mathcal{M}_{0,n})$ equals the coefficient of $-t$ in $(1-2t) \ldots (1-(n-2)t)$, or

\[ 2 + \cdots + (n-1) = \binom{n-1}{2} - 1. \]

(See \[\text{[1]}\] for more details of this calculation: we will actually show that $H^1(\mathcal{M}_{0,n})$ is isomorphic to the irreducible $\mathbb{S}_n$-module $V_{n-2,2}$.) Each tree in $T_1(n)$ contributes a copy of $\mathbb{C}$ to $\bigoplus_{T \in T_1(n)} H^2_{2(n-4)}(\overline{\mathcal{M}}(T))$. Let $T$ be such a tree and consider the set $S$ of the external edges attached to one vertex of $T$. We see that trees with two vertices correspond to subsets $S \subset \{1, \ldots, n\}$ where $2 \leq |S| \leq n - 2$, where we identify the trees corresponding to the subsets $S$ and $S^c$, the complement of $S$. Thus,

\[ |T_1(n)| = \frac{1}{2} \sum_{k=2}^{n-2} \binom{n}{k} = \frac{1}{2} (2^n - 2n - 2) = 2^{n-1} - n - 1. \]

The result follows easily. \[\square\]

This formula may be compared to the dimension of $H^2(\overline{\mathcal{M}}_{g,n}), g > 2$, which follows from the work of Arbarello and Cornalba \[\text{[3]}\]:

\[ \dim H^2(\overline{\mathcal{M}}_{g,n}) = 2^{n-1}(g + 1) + n + 1. \]

We see that this formula is correct for $g = 0$ up to a polynomial error.
The above dimension formula may be refined, using the realization of \( \mathcal{M}_{0,n} \) as an iterated blowup, to show that as an \( S_n \)-module, \( H^2(\mathcal{M}_{0,n}) \) is the direct sum of the suitable exterior powers of the permutation representation \( \mathbb{C}^n \) of \( S_n \).

Note that there are the same number of \( n \)-linear relations (2) among the brackets \([x_1, \ldots, x_k]\) generating the gravity operad as \( \dim H^2(\mathcal{M}_{0,n+1}) \); as we will see, this is no coincidence.

4. Koszul operads

In this section, we prove our main theorem, the duality of the hypercommutative and gravity operads. To do this, we must generalize Ginzburg and Kapranov’s theory of Koszul operads [15] so that it applies to operads which are not necessarily generated by bilinear operations. First, we recall their cobar construction for operads, an analogue of Hochschild’s bar construction for associative algebras.

The dual of an operad is only defined up to homotopy, and is represented by the cobar operad. However, there is a class of operads, the Koszul operads, for which there is a particularly nice dual, whose generators are in one-to-one correspondence with those of the original operad. A Koszul operad is quadratic (the relations among its generators are bilinear), as is its dual, and the relations in the dual operad may be characterized as the orthogonal complement of those of the original operad.

4.1. Free operads and trees. We now recall from [13] the structure of the free cyclic operad \( \mathbb{T}_+ \mathcal{V} \) generated by a cyclic \( S \)-module \( \mathcal{V} \). There is an analogous construction for operads, for which we refer to [12]. From now on, all cyclic \( S \)-modules which we discuss will be stable.

If \( \mathcal{V} \) is a (stable) cyclic \( S \)-module, let \( \mathbb{T}_+ \mathcal{V} \) be the (stable) cyclic \( S \)-module defined by

\[
\mathbb{T}_+ \mathcal{V}((n)) = \bigoplus_{T \in \mathcal{T}(n)} \mathcal{V}((T)),
\]

where \( \mathcal{V}((T)) = \bigotimes_{v \in \text{Vert}(T)} \mathcal{V}((\text{Leg}(v))) \). Note that \( \mathbb{T}_+ \mathcal{V} \) is graded by subspaces

\[
\mathbb{T}_i \mathcal{V}((n)) = \bigoplus_{T \in \mathcal{T}_i(n)} \mathcal{V}((T)).
\]

Then \( \mathbb{T}_+ \) is an endofunctor in the category of (stable) cyclic \( S \)-modules.

There is a natural structure of a triple on the functor \( \mathbb{T}_+ \):

1. since \( \mathbb{T}_+ \mathbb{T}_+ \) is a sum over trees, each vertex of which is itself a tree, the product of the triple is a natural transformation from \( \mathbb{T}_+ \mathbb{T}_+ \) to \( \mathbb{T}_+ \) obtained by gluing the trees at the vertices into the larger tree;
2. the unit of the triple is the natural transformation from the identity functor to \( \mathbb{T}_+ \) induced by the inclusion \( \mathcal{T}_0((n)) \subset \mathcal{T}(n) \).

The following theorem is a melding of results from [13] and [14].

**Theorem 4.2.** A (non-unital, stable) cyclic operad is the same thing as a \( \mathbb{T}_+ \)-algebra in the category of (stable) cyclic \( S \)-modules.

4.3. The cobar construction for operads. The cobar construction \( B \), introduced by Ginzburg and Kapranov [15], is a contravariant functor on the category of operads. We study here the slight variant of this functor which acts on the category of (non-unital, stable) cyclic operads.

The dual \( V^* \) of a chain complex \( V \) is defined as follows: \( V_i^* = (V_{-i})^* \), and \( \delta^* : V_i^* \rightarrow V_{i-1}^* \) is the adjoint of \( \delta : V_{-i+1} \rightarrow V_{-i} \).
If $V$ is a stable cyclic $S$-module, denote by $V^\vee$ the stable cyclic $S$-module

$$V^\vee((n)) = \Sigma^{n-3} \text{sgn}_n \otimes V((n))^*.$$  

This functor is an involution on the category of stable cyclic $S$-modules, that is, $(V^\vee)^\vee$ is naturally isomorphic to $V$.

The cobar operad $BA$ of a (non-unital, stable) cyclic operad is obtained by perturbing the differential of the free cyclic operad $T_+A^\vee$ by a differential $\partial$ which reflects the operad structure of $A$, and is defined as follows.

If $T \in T((n))$, and $e$ is an edge of $T$, denote by $T/e$ the tree in which $e$ is contracted to a point: thus, $T/e$ has one fewer vertices, and one fewer edges, than $T$. There is a natural map of degree $0$

$$\partial_{T/e} : A((T)) \longrightarrow A((T/e))$$

induced by composition in the operad $A$ along the edge $e$. This induces a map

$$\partial_T^e : A^\vee((T/e)) \longrightarrow A^\vee((T))$$

of degree $-1$. We now define the differential $\partial$ to be the operator whose matrix element from $A^\vee((\tilde{T})) \subset BA$ to $A^\vee((\tilde{T})) \subset BA$ is the sum of the operators $\partial_{T/e}^e$ over internal edges $e$ such that $T/e$ is isomorphic to $\tilde{T}$.

Paying careful attention to the signs coming from the suspensions, one shows that the differential $\partial$ satisfies the formulas $\partial^2 = \delta \partial + \partial \delta = 0$, and hence that $\delta + \partial$ is a differential on $T_+A^\vee$. It is also not hard to show that $\partial$ is compatible with the cyclic operad structure of $T_+A^\vee$, so that $BA = (T_+A^\vee, \delta + \partial)$ is an operad.

The properties of the resulting functor are summarized by the following theorem.

**Theorem 4.4.** (1) The cobar construction is a homotopy functor, that is, if $f : A \longrightarrow B$ is a homotopy equivalence, then so is $Bf : BA \longrightarrow BB$.

(2) There is a natural transformation from $BB$ to the identity functor, and the resulting map $BB.A \longrightarrow A$ is a homotopy equivalence for all $A$.

**Proof:** The homotopy invariance of $B$ is easy to see by a double complex argument. The natural map from $BB.A$ to $A$, which is projection onto the summand $A \cong (A^\vee)^\vee \subset T_+(T_+A^\vee)^\vee$, is shown to be a homotopy equivalence of operads in Theorem 3.2.16 of [15].

If $A$ and $B$ are operads and $\Phi : BA \longrightarrow B$ is a morphism of operads, there is a bar construction $B\Phi$ on $A$-algebras, defined for an $A$-algebra $A$ by twisting the differential on the chain complex

$$A \oplus \sum_{n=3}^{\infty} \bigoplus_{n=3}^\infty \text{Hom}_{S_n}(B(n), (\Sigma A)^{\otimes n})$$

in such a way as to reflect the $A$-algebra structure of $A$. (See [12] for details.) When $B = BA$ and $\Phi$ is the identity map, we denote the resulting functor $B$. Let $QA$ be the complex of indecomposables, obtained by taking the cokernel of the map

$$\bigoplus_{n=3}^{\infty} \rho_n : \bigoplus_{n=3}^{\infty} A(n) \otimes_{S_n} A^{\otimes n} \longrightarrow A.$$

The following theorem is proved in Chapter 3 of [13].

**Theorem 4.5.** (1) There is a natural transformation of functors $B \longrightarrow Q$, such that if $A$ is a free algebra, the morphism $BA \longrightarrow QA$ is a homotopy equivalence.

(2) The functors $B\Phi$ are homotopy functors: if $f : A \longrightarrow B$ is a homotopy equivalence, then so is $B\Phi f : B\Phi A \longrightarrow B\Phi B$. 
If $\Phi : \mathcal{B}A \rightarrow \mathcal{B}$ is a homotopy equivalence of operads, the natural morphism $\mathcal{B}A \rightarrow \mathcal{B}\Phi A$ is a homotopy equivalence for all $A$-algebras $A$. Thus, the functor $\mathcal{B}\Phi$ is a left derived functor $LQ$ of the indecomposable functor $Q$, that is, a homotopy functor homotopy equivalent to $Q$ on free $A$-algebras. It is proved in $[15]$ that there are natural homotopy equivalences of operads

$$\mathcal{B}Ass \rightarrow \mathcal{Ass} \quad \mathcal{B}Com \rightarrow \mathcal{Lie} \quad \mathcal{B}Lie \rightarrow \mathcal{Com}.$$  

The bar construction associated to the first of these homotopy equivalences is, up to a shift in degree, the Hochschild bar construction on associative algebras, while the bar constructions associated to the other two homotopy equivalences are the functors $LQ$ on commutative and Lie algebras discussed in the introduction, equal, up to a shift in degree, to the Harrison and Chevalley-Eilenberg complexes respectively. Thus, the duality result Theorem 0.1 is seen to be a special case of Theorem 4.5.

In Theorem 4.13, we will prove that $HYcom$-algebras are the same thing as hypercommutative algebras in the sense of (1). Thus the duality between hypercommutative and gravity algebras announced in the introduction follows from Theorem 4.5 combined with the following generalization of Theorem 4.25 of [15].

**Theorem 4.6.** There is a natural homotopy equivalence of operads $\mathcal{B}HYcom \rightarrow \mathcal{Grav}$.  

**Proof:** Let $V$ be the $S$-module obtained by summing the short exact sequences (5) (minus the terms $H^p(M_{0,n})$), placing the summand

$$\bigoplus_{T \in T_p(n)} H^q(M_{\{T\}}, det(T))$$

of $V((n))$ in degree $2(n - 3) - p - q$. Using Poincaré duality, we see that

$$V((n)) \cong \bigoplus_{p=0}^{n-3} \bigoplus_{T \in T_p(n)} \Sigma^p det(T) \otimes HYcom((T)).$$

Furthermore, there is a natural homotopy equivalence $\Sigma^{2(n-3)}H^* (M_{0,n}) \rightarrow V((n))$, which induces a homotopy equivalence $V^V \rightarrow \mathcal{Grav}$.

By the isomorphism (1), the $S$-module $V^V$ may be rewritten as

$$V^V((n)) \cong \bigoplus_{p=0}^{n-3} \Sigma^{n-3-p} \bigoplus_{T \in T_p(n)} \bigotimes_{v \in Vert(T)} \det(Leg(v)) \otimes HYcom^*((Leg(v)))$$

$$\cong \bigoplus_{p=0}^{n-3} \Sigma^{n-3-p} \bigoplus_{T \in T_p(n)} \bigotimes_{v \in Vert(T)} \Sigma^{3-|v|}HYcom^V((Leg(v)))$$

$$\cong \bigoplus_{p=0}^{n-3} \Sigma^{n-3-p+\sum_{v \in Vert(T)} (3-|v|)} \bigoplus_{T \in T_p(n)} HYcom^V((T)).$$

If $T \in T_p(n)$, we have

$$\sum_{v \in Vert(T)} (3 - |v|) = 3 - n + p,$$

showing that

$$V^V((n)) \cong \bigoplus_{p=0}^{n-3} \bigoplus_{T \in T_p(n)} HYcom^V((T)) \cong \mathcal{B}HYcom((n)).$$

A little diagram chasing shows that the differentials of the $S$-modules $V^V$ and $\mathcal{B}HYcom$ are the same, and that the resulting homotopy equivalence between $\mathcal{Grav}$ and $\mathcal{B}HYcom$ is compatible with the operad structures. $\square$
4.7. The cobar construction for mixed Hodge operads. The free operad functor \( \mathcal{V} \mapsto T \mathcal{V} \) and the functor \( \mathcal{V} \longrightarrow \mathcal{V}^\vee \) have analogues in the category of mixed Hodge \( S \)-modules, defined in precisely the same way as in the category of \( S \)-modules. (We recall that the dual \( \mathcal{V}^\vee \) in the category of mixed Hodge complexes reverses the weight filtration, sending complexes of weight \( k \) to complexes of weight \(-k\).) This allows us to extend the cobar construction to the category of mixed Hodge operads, by the same definition as in the category of dg operads.

If we follow through the proof of Theorem 4.6 paying attention to the mixed Hodge structures, we see that the homotopy equivalence \( B \mathcal{Hycom} \longrightarrow \mathcal{Grav} \) is compatible with the Hodge structures of \( \mathcal{Grav} \) and \( \mathcal{Hycom} \), where \( \mathcal{Hycom} \) carries the natural (pure) Hodge structure coming from its realization as the cohomology of the smooth Kähler manifold \( \overline{M}_{0,n} \). This observation will be essential in our calculation of the defining relations of the operad \( \mathcal{Hycom} \).

4.8. Quadratic operads. We now generalize Ginzburg and Kapranov’s notion of a Koszul operad to operads whose generators are not necessarily bilinear operations. Once more, we restrict attention to stable cyclic operads.

An ideal \( B \subset A \) of a cyclic operad is a cyclic \( S \)-submodule such that for all operations \( o_i, a o_i b \) is in \( B \) if either \( a \) or \( b \) is. The intersection of two ideals is obviously an ideal. An ideal is generated by a cyclic \( S \)-submodule \( \mathcal{R} \subset B \) if \( B \) is the intersection of all ideals of \( A \) containing \( \mathcal{R} \).

Let \( A \) be an operad, generated by a cyclic \( S \)-submodule \( \mathcal{V} \). The pair \((A, \mathcal{V})\) is a cyclic quadratic operad if the ideal \( \ker(\mathbb{T}_1 \mathcal{V} = \bigoplus_{T \in \mathbb{T}_1} \mathcal{V}(T) \longrightarrow A) \) in the free cyclic operad \( \mathbb{T}_+ \mathcal{V} \) is generated by \( \ker(\mathbb{T}_1 \mathcal{V} \longrightarrow A) \).

The word quadratic is used here because \( \mathcal{V}(T) \) is quadratic in \( \mathcal{V} \) if \( T \) has one internal edge and hence two vertices. Thus, \( \mathcal{R} \) is itself quadratic in \( \mathcal{V} \).

The cyclic operads \( A = \mathcal{Ass}, \mathcal{Com} \) and \( \mathcal{Lie} \) are all quadratic, with generating cyclic submodule \( \mathcal{V} \), where

\[
\mathcal{V}(n) = \begin{cases} 
\mathcal{A}(3), & n = 3, \\
0, & n \neq 3.
\end{cases}
\]

For example, \( \mathcal{Lie}(3) \) is one-dimensional, spanned by \([a_1, a_2]\), and the cyclic \( S \)-module of relations \( \mathcal{R} \) is given by the formula

\[
\mathcal{R}(n) = \begin{cases} 
\text{span}\{[a_1, [a_2, a_3]], [a_2, [a_3, a_1]]\}, & n = 4, \\
0, & n \neq 4.
\end{cases}
\]

4.9. The naive dual of a quadratic operad. If \( A \) is a cyclic quadratic operad, the naive dual \( A^! \) of \( A \) is the cokernel of the composition

\[
\psi : B A \longrightarrow B A \longrightarrow T_+ \mathcal{V}^\vee,
\]

where the second arrow is the surjection of cyclic \( S \)-modules \( B A \longrightarrow T_+ \mathcal{V}^\vee \) induced by the inclusion of cyclic \( S \)-modules \( \mathcal{V} \subset A \).

Definition 4.10. If \( A \) is a cyclic quadratic operad, there is a natural morphism of operads \( \Phi : B A \longrightarrow A^! \), induced by the surjection \( B A \longrightarrow T_+ \mathcal{V}^\vee \). The operad \( A \) is Koszul if the surjection of operads \( B A \longrightarrow A^! \) is a homotopy equivalence, or equivalently, if \( B A^! \longrightarrow A \) is.
Proposition 4.11. If $A$ is a cyclic quadratic operad with generators $V$ and relations $R$, let $R^\perp$ be the kernel of the natural map from $T_1 V^\vee \cong (T_1 V^\vee)^{\vee}$ to $R^\vee$. Then $A^!$ is a cyclic quadratic operad with generators $V^\vee$ and relations $R^\perp$.

Proof: It suffices to show that the image of $B \cdot A$ in $T_+ V^\vee$ under the above composition is the ideal generated by $R^\perp$. Denote by $A_k \subset A$ the image of $T_k V$ in $A$ under the quotient map $T_+ V \longrightarrow A$. Thus, $A_0 = V$ and $A_1 = T_1 V / R$. Observe that $A^!_2 \cong R^\perp$.

If $T \in T((n))$, the summand $A^\vee((T))$ of $B \cdot A((n))$ may be thought of as the vector space spanned by decorations of the tree $T$, in which each vertex of $T$ is assigned an element of $A^\vee$ of appropriate valence. The map $\psi : B \cdot A \longrightarrow T_+ V^\vee$ vanishes on such a decorated tree unless the vertex decorations lie in $V$ at all but one vertex $v$, which is decorated by $a \in A^!_2$. The map $\psi$ applied to this decorated tree produces a new decorated tree in which the vertex $v$ is replaced by the tree underlying $a$ (and thus having one additional edge). Thus, the image of $\psi$ is the ideal generated by $R^\perp$. \(\square\)

Corollary 4.12. A cyclic quadratic operad $A$ is Koszul if and only if $A^!$ is, and $(A^!)^! \cong A$.

As examples of naive duals, we have $Ass^! \cong Ass$, $Com^! \cong Lie$ and $Lie^! \cong Com$. It is proved in [12] that the operads $Ass$, $Com$ and $Lie$ are Koszul. A non-cyclic example of a Koszul operad is $Braid$ [12], which satisfies $Braid^! \cong \Lambda^{-1} Braid$.

The proof of the following theorem occupies the remainder of this section. This theorem is joint work of the author and M. Kontsevich.

Theorem 4.13. Let $V \subset Hycom$ be the cyclic $S$-submodule spanned by the fundamental classes

$$\overline{M}_{0,n} \in H_{2(n-3)}(\overline{M}_{0,n}) \subset Hycom((n)).$$

The operad $Hycom$ is Koszul, with generators $V$, and $Hycom^! \cong Grav$.

Proof: This theorem is proved using the duality between the mixed Hodge operads $Grav$ and $Hycom$, and the fact that the operad $Grav$ is quadratic.

If $A$ is a mixed Hodge operad, the natural homotopy equivalence $BB \cdot A \longrightarrow A$ of Theorem 4.4 is a morphism of mixed Hodge operads. It follows that we have a diagram in the category of mixed Hodge operads

$$
\begin{array}{ccc}
BBHycom & \longrightarrow & BGrav \\
\downarrow & & \\
Hycom & & \\
\end{array}
$$

in which both arrows are homotopy equivalences. The homology of the weight $-2p$ summand of the complex $BGrav((n))$, which by this argument is isomorphic to $H_{2p}(\overline{M}_{0,n})$, must be concentrated in degree $2p$; from this, we see that this subcomplex is exact except at the last term, giving a long exact sequence

$$(6) \quad \cdots \longrightarrow \bigoplus_{T \in T_{n-2-p}(n)} Grav^\vee((T))_{2p+1} \longrightarrow \bigoplus_{T \in T_{n-3-p}(n)} Grav^\vee((T))_{2p} \longrightarrow Hycom((n))_{2p} \longrightarrow 0.$$

Let $V^\vee \subset Grav$ be the cyclic $S$-module spanned by the generators of $Grav$; for each $n \geq 3$, there is one generator, of degree $3 - n$ and weight $2(n - 3)$, in $Grav((n))$. Thus $V((n))$ is spanned by an element of degree $2(n - 3)$ and weight $2(3 - n)$. Taking $p = n - 3$ in the long exact sequence (6), we see that $V((n))$ may be identified with $H_{2(n-3)}(\overline{M}_{0,n})$.

Let $R^\vee \subset Grav$ be the cyclic $S$-module spanned by the elements of degree $4 - n$ in $Grav$. An $S_n$-module, $R^\vee((n)) \cong H_1(\overline{M}_{0,n}) \otimes \sgn_n$; it is concentrated in degree $4 - n$, has weight
2(n − 4), and dimension \(^{(n-1)}2\) − 1. The case \(p = n − 4\) of the long exact sequence \(\mathcal{R}\) is the short exact sequence

\[
0 \longrightarrow \mathcal{R}((n)) \longrightarrow \bigoplus_{T \in \mathcal{T}_n((n))} \mathcal{V}(T) \longrightarrow H_{2(n-4)}(\mathcal{M}_{0,n}) \longrightarrow 0,
\]

showing that the \(\mathcal{S}\)-module \(\mathcal{R}\) is a subset of the set of relations for \(\mathcal{H}_{yc}\). Furthermore, there are no further relations, as may be seen from \(\mathcal{R}\): for general \(p \leq n − 3\), there is an exact sequence

\[
\cdots \longrightarrow \bigoplus_{T \in \mathcal{T}_{n-2-p}((n))} \bigoplus_{v \in \text{Vert}(T)} \mathcal{R}((\text{Leg}(v))) \otimes \bigotimes_{w \in \text{Vert}(T) \setminus \{v\}} \mathcal{V}((\text{Leg}(w))) \longrightarrow \bigoplus_{T \in \mathcal{T}_{n-3-p}((n))} \mathcal{V}((T)) \longrightarrow H_{2p}((n))_{2p} \longrightarrow 0.
\]

This shows that the operad \(\mathcal{H}_{yc}\) is quadratic, with generators \(\mathcal{V}\) and relations \(\mathcal{R}\).

It remains to identify the \(\mathcal{S}\)-module \(\mathcal{R}\) with the set of relations \(\mathcal{R}\) which hold in a hypercommutative algebra. This is done in two parts: we first show that these relations are in the orthogonal complement of the relations which define a gravity algebra, and thus form a subset of \(\mathcal{R}\), and then show that they form a subspace of \(\mathcal{R}((n))\) of dimension at least \(^{(n-1)}2\) − 1. Since \(\mathcal{R}((n))\) itself has dimension \(^{(n-1)}2\) − 1, this completes the proof.

It is simple to check that the relations \(\mathcal{R}\) are orthogonal to those which hold in a gravity algebra. Consider the relation

\[G_0 = \pm[[a_i, a_j], a_1, \ldots, \hat{a_i}, \ldots, \hat{a_j}, \ldots, a_k].\]

The inner product of this relation with the relation

\[H = \sum_{S_1 \cup S_2 = \{1, \ldots, n\}} \pm((a, b, x_{S_1}), c, x_{S_2}) - \sum_{S_1 \cup S_2 = \{1, \ldots, n\}} \pm(a, (b, c, x_{S_1}), x_{S_2})\]

vanishes, since only the terms with \(S_1 = \emptyset\) can contribute: they each contribute a term 1, but with opposite sign.

Turning now to the relation

\[G_\ell = \pm[[a_i, a_j], a_1, \ldots, \hat{a_i}, \ldots, \hat{a_j}, \ldots, a_k] - [[a_1, \ldots, a_k], b_1, \ldots, b_\ell], \quad \ell > 0,
\]

we see that there are three cases to consider:

1. none of the letters \(a, b, c\) lie in the set \(\{b_1, \ldots, b_\ell\}\), in which case the inner product of relation \(H\) with the above relation again vanishes, for the same reason as when \(\ell = 0\);
2. one of the letters, say \(c\), lies in the set \(\{b_1, \ldots, b_\ell\}\), in which case the only terms having a non-zero inner product with \(H\) are \([[a, b], c, x_S]\) and \([[a_1, \ldots, a_k], b_1, \ldots, b_\ell]\), whose contributions, each equal to 1, cancel;
3. two or three of the letters \(a, b, c\) lie in the set \(\{b_1, \ldots, b_\ell\}\), in which case the inner product of the above relation with each term of \(H\) vanishes.

Finally, we check that the space of relations among \(n\) letters in a hypercommutative algebra has dimension at least \(^{(n)}2\) − 1. Consider the projection of these relations into the space \(\mathcal{A}(n)\) of all quadratic words in the generators of the hypercommutative operad of the form

\[((x_i, x_j), x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_n).\]

(Note that this subspace of \(TV^\vee((n+1))\) is not \(S_{n+1}\)-invariant, but only \(S_n\)-invariant.) The dimension of \(\mathcal{A}(n)\) is \(^{(n)}2\), and the relations \(\mathcal{R}\) project in \(\mathcal{A}(n)\) to relations

\[((a, b), c, x_S) = ((a, c), b, x_S).\]

Clearly, the quotient of \(\mathcal{A}(n)\) by these relations is one-dimensional. This completes the proof that \(\mathcal{R}\) are all the relations in the operad \(\mathcal{H}_{yc}\). \(\square\)
5. The equivariant Poincaré polynomials of $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$

In this section, we use the results of [14] to calculate the character of the $S_n$-modules $H_i(\mathcal{M}_{0,n})$ and $H_i(\overline{\mathcal{M}}_{0,n})$. By and large, the results of this section are independent of the rest of this paper.

5.1. Symmetric functions. Let $\Lambda$ be the ring of symmetric functions: this is the limit

$$\Lambda = \lim_{\leftarrow} \mathbb{Z}[x_1, \ldots, x_k]^{S_k}.$$ 

Then $\Lambda$ is the ring $\mathbb{Z}[h_1, h_2, \ldots]$ of power series in the complete symmetric functions

$$h_n(x_i) = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n},$$

and $\Lambda_\mathbb{Q} = \Lambda \otimes \mathbb{Q}$ is a power series ring $\mathbb{Q}[p_1, p_2, \ldots]$ in the power sums

$$p_n(x_i) = \sum_i x_i^n.$$

If $\sigma \in S_n$ has cycles of length $\lambda_1 \geq \cdots \geq \lambda_\ell$, its cycle index $\psi(\sigma)$ is the monomial $p_{\lambda_1} \cdots p_{\lambda_\ell}$. If $V$ is an $S_n$-module, its characteristic is the symmetric function

$$\text{ch}_n(V) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{Tr}_V(\sigma) \psi(\sigma).$$

It may be shown that $\text{ch}_n(V) \in \Lambda \subset \Lambda_\mathbb{Q}$, and that the characteristics of the irreducible representations of the symmetric groups $S_n$, $n \geq 0$, form a basis of $\Lambda$ over $\mathbb{Z}$, called the Schur functions [24]. For example, $h_n$ is the characteristic of the trivial representation of $S_n$.

Define the Poincaré characteristic of an $S$-module to be

$$\text{ch}_t(V) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-t)^i \text{ch}_n(V_i(n)) \in \Lambda(\langle t \rangle).$$

Setting $t = 1$, we obtain the (Euler-Frobenius) characteristic $\text{ch}(V)$. For example,

$$\text{ch}(\text{Com}^+) = \sum_{n=1}^{\infty} h_n = \exp(\sum_{n=1}^{\infty} \frac{p_n}{n}) - 1, \quad \text{and} \quad \text{ch}(\text{Ass}^+) = \sum_{n=1}^{\infty} p_n^n = \frac{p_1}{1 - p_1}.$$

5.2. Plethysm. Consider the ring $\Lambda(\langle t \rangle)$ of power series in a variable $t$ with coefficients in $\Lambda$. There is an associative product on $\Lambda(\langle t \rangle)$, called plethysm and denoted $f \circ g$, characterized by the formulas

1. $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$;
2. $(f_1 f_2) \circ g = (f_1 \circ g)(f_2 \circ g)$;
3. if $f = f(t, p_1, p_2, \ldots)$, then $p_n \circ f = f(t^n, p_n, p_{2n}, \ldots)$, and $t \circ f = t$.

The following formula generalizes its analogue for ungraded $S$-modules, proved in [24]:

$$\text{ch}_t(V \circ W) = \text{ch}_t(V) \circ \text{ch}_t(W).$$

The operation

$$\text{Exp}(f) = \sum_{n=0}^{\infty} h_n \circ f$$

plays the role for symmetric functions that exponentiation does for power series. The inverse of Exp is the operation

$$\text{Log}(f) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(p_n \circ f),$$

where $\mu(n)$ is the Möbius function.
Using this formula and the Poincaré-Birkhoff-Witt theorem $\mathcal{Ass}^+ = \mathcal{Com}^+ \circ \mathcal{Lie}^+$, we may calculate $\text{ch}(\mathcal{Lie}^+)$. 

**Proposition 5.3.** $\text{ch}(\mathcal{Lie}^+) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 - p_n)$

**Proof:** We know from the Poincaré-Birkhoff-Witt theorem that $\text{ch}(\mathcal{Ass}^+) = \text{Exp}(\text{ch}(\mathcal{Lie}^+))$; it follows that

$$\text{ch}(\mathcal{Lie}^+) = \log(1 + \text{ch}(\mathcal{Ass}^+)) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 + p_n \circ \text{ch}(\mathcal{Ass}^+)).$$

Since

$$1 + p_n \circ \text{ch}(\mathcal{Ass}^+) = 1 + \frac{p_n}{1 - p_n} = \frac{1}{1 - p_n},$$

the result follows. □

It follows from this formula that

$$\text{ch}_n(\mathcal{Lie}^+(n)) = \frac{1}{n} \sum_{d|n} \mu(d) p_n^{n/d}.$$ 

This is the characteristic of the induced representation $\text{Ind}_{C_n}^{S_n} \chi$, where $\chi$ is a primitive character of the cyclic group.

We now turn to calculating the characteristic of the braid operad $\mathcal{Braid}$. First, we need a lemma.

**Lemma 5.4.** $\text{ch}_t(\Lambda V) = -t \text{ch}_t(V)(-t^{-1}p_1, -t^{-2}p_2, -t^{-3}p_3, \ldots)$

**Proof:** Tensoring with $\text{sgn}_n$ has the effect of replacing $p_n$ by $(-1)^{n-1}p_n$. Applying $\Sigma^{-n}$ to $V(n)$ then has the effect of replacing $p_n$ by $(-t)^{-n}p_n$. □

**Proposition 5.5.** For each $n \geq 1$, let

$$P_n(t) = \frac{1}{n} \sum_{d|n} \frac{\mu(n/d)}{t^d}.$$ 

Then

$$\text{ch}_t(\mathcal{Braid}) = \prod_{n=1}^{\infty} \left(1 + t^n p_n\right)^{P_n(t)} - 1.$$ 

**Proof:** The $S$-module $\Lambda^{-1}\mathcal{Lie}^+$ has Poincaré characteristic

$$\text{ch}_t(\Lambda^{-1}\mathcal{Lie}^+) = \frac{1}{t} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 + t^n p_n).$$

It follows that

$$\text{ch}_t(\mathcal{B}) = \text{Exp} \circ \text{ch}_t(\Lambda^{-1}\mathcal{Lie}^+) - 1 = \text{exp}\left(\sum_{k=1}^{\infty} \frac{p_k}{k} \circ \left(t^{-1} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 + t^n p_n)\right)\right) - 1$$

$$= \text{exp}\left(\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} t^{-k} \frac{\mu(n)}{kn} \log(1 + t^{kn} p_{kn})\right) - 1$$

$$= \prod_{n=1}^{\infty} (1 + t^n p_n)^{P_n(t)} - 1.$$ □
In particular, setting \( p_1 = x \) and \( p_n = 0, \ n > 1 \), we see that the Poincaré polynomial of the space \( \mathbb{C}_0^n \) is

\[
\sum_{i=0}^{n-1} (-t)^n \dim H^i(\mathbb{C}_0^n) = \text{coefficient of } x^n/n! \text{ in } (1+tx)^{t-1} - 1
\]

\[
= \left( \frac{t-1}{n} \right) t^n = \prod_{i=1}^{n-1} (1-it).
\]

5.6. The characteristic of a cyclic \( S \)-module. If \( V \) is a stable cyclic \( S \)-module, we define \( \text{Ch}_t(V) \) in a similar way to \( \text{ch}_t(V) \):

\[
\text{Ch}_t(V) = \sum_{n=3}^{\infty} \sum_{i=0}^{\infty} (-t)^i \text{ch}_n(V_i(n)) \in \Lambda(t).
\]

If \( \text{ch}_t(V) \) denotes the Poincaré characteristic of the \( S \)-module underlying \( V \), then we have the formula

\[
\text{ch}_t(V) = \frac{\partial \text{Ch}_t(V)}{\partial p_1}.
\]

We now calculate \( \text{Ch}_t(m) \), where \( m \) is the cyclic \( S \)-module

\[
m((n)) = \begin{cases} H_\bullet(M_{0,n}), & n \geq 3, \\ 0, & n < 3. \end{cases}
\]

Note that the Poincaré polynomial of \( M_{0,n} \) is much easier to calculate than the Poincaré characteristic: it is obtained by dividing the Poincaré polynomial \([7]\) of \( \mathbb{C}_0^{n-1} \) by \( 1-t \):

\[
\sum_{i=0}^{n-2} (-t)^n \dim H^i(M_{0,n}) = \prod_{i=2}^{n-2} (1-it).
\]

Theorem 5.7.

\[
\text{Ch}_t(m) = \frac{1}{1-t^2} \left( (1+tp_1) \prod_{n=1}^{\infty} (1+t^n p_n) P_n(t) - 1 - (1+t)h_1 - (h_2 + te_2) \right)
\]

Proof: From the \( S_n \)-equivariant homotopy equivalence \( \mathbb{C}_0^n \simeq M_{0,n+1} \times S^1 \), which holds for \( n \geq 2 \), we see that

\[
\frac{\partial \text{Ch}_t(m)}{\partial p_1} = \text{ch}_t(m) = \frac{t(\text{ch}_t(Braid) - p_1)}{t-1}.
\]

The Serre spectral sequence for the \( S_n \)-equivariant fibration

\[
\mathbb{C} \setminus \{1, \ldots, n\} \longrightarrow M_{0,n+1} \longrightarrow M_{0,n}
\]
collapses at \( E^2 \), so \( H_\bullet(M_{0,n+1}) \cong H_\bullet(\mathbb{C} \setminus \{1, \ldots, n\}) \otimes H_\bullet(M_{0,n}) \). Furthermore, this isomorphism is \( S_n \)-equivariant, where \( S_n \) acts on \( H_\bullet(\mathbb{C} \setminus \{1, \ldots, n\}) \) by the monodromy of the Gauss-Manin connection.

Now, \( H_0(\mathbb{C} \setminus \{1, \ldots, n\}) \) is the trivial \( S_n \)-module, while \( H_1(\mathbb{C} \setminus \{1, \ldots, n\}) \) is the irreducible representation \( V_{n-1,1} \), which is the kernel of the natural map \( \mathbb{C}^n \longrightarrow \mathbb{C} \) obtained by sending \( (x_1, \ldots, x_n) \) to \( x_1 + \cdots + x_n \). If \( \sigma \in S_n \) is a transitive permutation, \( \text{Tr}(\sigma | V_{n-1,1}) = -1 \); this shows that

\[
\text{Ch}_t(m)|_{p_1=0} = \frac{\text{ch}_t(Braid)}{1+t} |_{p_1=0}.
\]

The theorem now follows on solving the differential equation \([10]\) with initial condition \([11]\). \(\square\)
The first few terms of $\text{Ch}_t(m)$ are as follows:

| $n$ | $\text{Ch}_t(M_{0,n})$ |
|-----|----------------------|
| 3   | $s_3$                |
| 4   | $s_4 - ts_{22}$      |
| 5   | $s_5 - ts_{32} + t^2(s_{31}^2)$ |
| 6   | $s_6 - ts_{42} + t^2(s_{41}^2 + s_{32}^1) - t^3(s_{41} + s_{22} + s_{22}^1)$ |

The pattern emerging here, that $H_1(M_{0,n}) \cong V_{n-2,2}$, is easily verified in general using our formula for $\text{Ch}_t(m)$. We have seen that there is a natural identification between $\text{sgn}_n \otimes H^1(M_{0,n})$ and the space of relations (1) among $n-1$ letters in the hypercommutative operad; thus, we see that this space of relations is the irreducible $S_n$-module $V_{221n-4}$.

Applying l'Hôpital's rule to Theorem 5.7, we see that the Euler-Frobenius characteristic of $m$ is given by the formula

$$\text{Ch}_t(M_{0,n}) = \frac{1}{2}(1 + p_1)^2 \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log(1 + p_n) - \frac{1}{4}(2p_1 + 3p_1^2 + p_2).$$

Finally, it follow easily from the formula for $\text{Ch}_t(m)$ that

$$\text{Ch}_t(G_{\text{Grav}}) = -\frac{t^3}{1 - t^2} \left( (1 - p_1) \prod_{n=1}^{\infty} (1 - p_n)^{P_n(t)} - 1 + (1 + t^{-1})h_1 - (t^{-1}h_2 + t^{-2}e_2) \right)$$

5.8. The Poincaré characteristic of $T_+V$. In [14], a formula for the Poincaré characteristic of $T_+V$ in terms of the Poincaré characteristic of $V$ is derived. If $F = e_2 - \text{Ch}_t(V)$, define the Legendre transform $G = \mathcal{L}F$ of $F$ in the sense of symmetric functions by the formula

$$(12) \quad F \circ \frac{\partial(LG)}{\partial p_1} + G = p_1 \frac{\partial G}{\partial p_1}.$$ 

Then $G = h_2 + \text{Ch}_t(T_+V)$.

Note that (12) implies that $(\partial F/\partial p_1) \circ (\partial G/\partial p_1) = p_1$, from which it is straightforward to calculate $\partial G/\partial p_1$. Substituting $\partial G/\partial p_1$ into both sides of (12), we obtain an explicit formula for $G$.

As an application of (12), we now calculate the Poincaré characteristics of the varieties $\mathcal{M}_{0,n}$. We use a slight extension of (12), in which $V = V_0 \oplus V_1$ has an internal $\mathbb{Z}/2$-grading, and $\text{Ch}_t(V) = \text{Ch}_t(V_0) - \text{Ch}_t(V_1)$. Let $V$ and $W$ be the stable cyclic $S$-modules

$$V_i((n)) = \begin{cases} 0, & i = 0; \\ \mathcal{H}_{\text{ycom}}((n)) \otimes \text{sgn}_n, & i = 1; \\ \text{sgn}_n \otimes H_p(M_{0,n}), & i = 2(n - p - 3) \text{ and } j \equiv p + 1 \pmod{2}, \\ 0, & \text{otherwise}. \end{cases}$$

$$W_{ij}((n)) = \begin{cases} \text{sgn}_n \otimes H^p(M_{0,n}), & i = 2(n - p - 3) \text{ and } j \equiv p + 1 \pmod{2}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus,

$$\text{Ch}_t(V) = -\text{Ch}_t(\mathcal{H}_{\text{ycom}})|_{p_n \rightarrow (-1)^{n-1}p_n},$$

$$\text{Ch}_t(W) = \frac{-t^{-6} \text{Ch}_t(m)}{t \rightarrow -t^2p_n \rightarrow (-1)^{n-1}t^{2n}p_n}.$$ 

Using (14), we may rewrite (15) in the form

$$0 \rightarrow H^p(M_{0,n}) \otimes \text{sgn}_n \rightarrow \bigoplus_{T \in T_n((n))} V((T))_{2(n-p-3)} \rightarrow \bigoplus_{T \in T_{p-1}((n))} V((T))_{2(n-p-3)} \rightarrow \cdots$$

which shows that $\text{Ch}_t(W) = \text{Ch}_t(T_+V)$, and hence that $G = h_2 + \text{Ch}_t(W)$ is the Legendre transform of $F = e_2 - \text{Ch}_t(V)$. In this way, we have proved the following proposition.
Theorem 5.9. The symmetric function

\[ F = e_2 + \text{Ch}_t(\mathcal{H}_{\text{com}})|_{p_n \to (-1)^{n-1}p_n} \]

is the Legendre transform of the symmetric function

\[ G = h_2 - t^{-6} \text{Ch}_t(m)|_{p_n \to (-1)^{n-1}t^{n-1}p_n} . \]

Here are some sample calculations of \( \text{Ch}_t(\overline{\mathcal{M}}_{0,n}) \) for small \( n \):

| \( n \) | \( \text{Ch}_t(\overline{\mathcal{M}}_{0,n}) \) |
|---|---|
| 3 | \( s_3 \) |
| 4 | \( (1 + t^2)s_4 \) |
| 5 | \( (1 + t^4)s_5 + t^2(s_5 + s_{41}) \) |
| 6 | \( (1 + t^6)s_6 + (t^2 + t^4)(2s_6 + s_{51} + s_{41}) \) |

These formulas simplify if we are only interested in the dimensions of the vector spaces \( H_i(\overline{\mathcal{M}}_{0,n}) \). We have the formula

\[ g(x,t) = G'|_{p_1 \to x, p_n \to 0, n > 1} = x - \sum_{n=2}^{\infty} \frac{x^n}{n!} \sum_{i=0}^{n-2} (-1)^i i^2 (n-i-2) \dim H_i(\mathcal{M}_{0,n+1}) = x - \frac{(1 + x)t^2 - (1 + t^2x)}{t^2(t^2 - 1)}. \]

It is a corollary of Theorem 5.9 that

\[ f(x,t) = F'|_{p_1 \to x, p_n \to 0, n > 1} = x + \sum_{n=2}^{\infty} \frac{x^n}{n!} \sum_{i=0}^{n-2} t^{2i} \dim H_{2i}(\overline{\mathcal{M}}_{0,n+1}), \]

is the inverse of \( g \), in the sense that \( f(g(x,t), t) = x \). This is a reformulation of Fulton and MacPherson’s calculation of the Poincaré polynomial of \( \overline{\mathcal{M}}_{0,n} \). Note that their proof also makes use of mixed Hodge theory, in the form of the “fake Poincaré polynomial.” Our result Theorem 5.9 is an equivariant version of their calculation.

References

[1] E. Arbarello and M. Cornalba, The Picard groups of the moduli spaces of curves, Topology 26 (1987), 153–171.
[2] V.I. Arnold, The cohomology ring of the colored braid group, Mat. Zametki 5 (1969), 227–231.
[3] A. Beilinson, V. Ginzburg, Infinitesimal structure of moduli spaces of \( \mathcal{G} \)-bundles, Internat. Math. Res. Notices (appendix to Duke Math. J.) 66 (1992), 63–74.
[4] J.M. Boardman and R.M. Vogt, “Homotopy invariant algebraic structures on topological spaces,” Lecture Notes in Math. 347, 1973.
[5] J.L. Brylinski and S. Zucker, An overview of recent advances in Hodge theory, in “Several complex variables, VI,” 39–142, Encyclopaedia Math. Sci. 69, Springer Verlag, Berlin, 1990.
[6] F.R. Cohen, The homology of \( \mathcal{C}_{n+1} \)-spaces, \( n \geq 0 \), in “The homology of iterated loop spaces,” Lecture Notes in Math. 533, 1976, 207–351.
[7] P. Deligne, Théorie de Hodge II, Publ. Math. IHES 40 (1971), 5–58.
[8] R. Dijkgraaf, E. and H. Verlinde, Topological strings in \( d < 1 \), Nucl. Phys. B352 (1991), 59–80.
[9] W. Fulton and R. MacPherson, A compactification of configuration spaces, Ann. Math., 139 (1994), 183–225.
[10] E. Getzler, Batalin-Vilkovisky algebras and two-dimensional topological field theories, Commun. Math. Phys. 159 (1994), 265–285.
[11] E. Getzler, Equivariant cohomology and topological gravity, Commun. Math. Phys. 163 (1994), 473–490.
[12] E. Getzler and J.D.S. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, (hep-th/9403055).
[13] E. Getzler and M. Kapranov, Cyclic operads and cyclic homology, to appear in “Geometry, Topology, and Physics for Raoul,” ed. B. Mazur, International Press, Cambridge, MA, 1994.
[14] E. Getzler and M. Kapranov, Modular operads, MPIM-Bonn preprint 94/78, (dg-ga/9408003).
[15] V.A. Ginzburg and M.M. Kapranov, *Koszul duality for operads*, to appear, Duke. Math. J.
[16] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307–347.
[17] A. Joyal, *Foncteurs analytiques et espèces de structures*, Lecture Notes in Math. 1234 (1986) 126–159.
[18] M.M. Kapranov, *Permuto-associahedron, MacLane’s coherence theorem and asymptotic zones for the KZ equation*, J. Pure Appl. Algebra, 85 (1993), 119–142
[19] S. Keel, *Intersection theory of moduli spaces of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. 330 (1992), 545–574.
[20] A.A. Klyachko, *Lie elements in the tensor algebra*, Siberian Math. J., 15 (1974), 914–920.
[21] F.F. Knudsen, *The projectivity of the moduli space of stable curves II. The stacks \( \mathcal{M}_{g,n} \)*, Math. Scand. 52 (1983), 161–189.
[22] M. Kontsevich, *Formal (non)-commutative symplectic geometry*, in “The Gelfand mathematics seminars, 1990–1992,” eds. L. Corwin, I. Gelfand, J. Lepowsky, Birkhäuser, Boston, 1993.
[23] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Commun. Math. Phys. 164 (1994).
[24] I.G. Macdonald, “Symmetric Functions and Hall Polynomials,” Clarendon Press, Oxford, 1979.
[25] D. McDuff and D. Salamon, “J-holomorphic curves and quantum cohomology,” Amer. Math. Soc., Providence, 1994.
[26] S. Maclane, “Categories for the working mathematician,” Graduate Texts in Math. 5, 1971.
[27] D. Quillen, *Rational homotopy theory*, Ann. Math. 90 (1969), 205–295.
[28] Y. Ruan, G. Tian, *A mathematical theory of quantum cohomology*, preprint, 1994.

Department of Mathematics, MIT, Cambridge MA 02139 USA

E-mail address: getzler@math.mit.edu