Chern-Weil calculus extended to a class of infinite dimensional manifolds

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Abstract

We discuss possible extensions of the classical Chern-Weil formalism to an infinite dimensional setup. This is based on joint work with Steven Rosenberg [PR1, PR2], joint work with Simon Scott [PS1, PS2] and joint work with Jouko Mickelsson [MP].

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Introduction

Classical Chern-Weil formalism relates geometry to topology, assigning to the curvature of a connection, de Rham cohomology groups of the underlying manifold. This theory developed in the 40’s by Shing-Shen Chern [C2] and André Weil\(^\text{1}\): which can be seen as a generalisation of the Chern-Gauss-Bonnet theorem [C1], was an important step in the theory of characteristic classes. Let \( G \) be a lie group with Lie algebra \( \text{Lie}(G) \). The Chern-Weil homomorphism assigns to an \( \text{Ad}(G) \)-invariant polynomial \( f \) on \( \text{Lie}(G) \) a de Rham cohomology class defined as follows. Let \( P \to M \) be a \( G \)-principal bundle equipped with a connection \( \nabla \), since the curvature \( \nabla^2 \) is a \( \text{Lie}(G) \)-valued two-form on \( P \), to a homogeneous \( \text{Ad}(G) \)-invariant polynomial \( f \) of degree \( j \) on \( \text{Lie}(G) \) corresponds a \( 2j \)-form \( f(\nabla^2) \) on \( P \). This form turns out to be closed with de Rham cohomology class independent of the choice of connection.

When \( G \) is a matrix group, \( \text{Ad}(G) \)-invariant monomials on \( \text{Lie}(G) \) can be built from the trace on matrices in view of the \( \text{Ad}(G) \)-invariance of the trace (section 1); the invariant polynomials are actually generated by the monomials \( X \mapsto \text{tr}(X^j) \) when \( G \) is an infinite dimensional Lie group, there is a priori a problem to define a trace and therefore to get invariant polynomials on \( \text{Lie}(G) \). We are concerned here with the Fréchet Lie group \( \text{Cl}^{0,*}(M, E) \) (and its subgroups) of invertible zero order classical pseudodifferential operators acting on smooth sections of some vector bundle \( E \to M \) over a closed Riemannian manifold \( M \) (section 5). Its Lie algebra is the Fréchet algebra \( \text{Cl}^{0}(M, E) \) of zero order classical pseudodifferential operators acting on smooth sections \( E \to M \) (section 2); it carries two types of traces [LP] together with their linear combinations (section 3), the noncommutative residue introduced by Adler, Manin,\(^{1}\) In an unpublished paper.
generalised by Guillemin [G] and Wodzicki [W1] (see [K] for a survey) and leading symbol traces used in [PR1, PR2]. An explicit example of an infinite rank bundle with non vanishing first Chern class is built in [RT] using the noncommutative residue on classical pseudodifferential operators as an Ersatz for the trace on matrices. However, generally speaking, Chern classes built from the noncommutative residue or leading symbol traces seem too coarse to capture non trivial cohomology classes so that we turn to mere linear extensions to the algebra Cl\(_0\)(\(M, E\)) of the ordinary trace on smoothing operators (section 4). The latter might not be traces since they are not expected to vanish on brackets. We refer all the same to these as regularised traces (and weighted traces later in the text); in constrast with the noncommutative residue and leading symbol traces which vanish smoothing pseudodifferential operators, regularised traces coincide with the usual trace on smoothing operators. The price to pay for choosing regularised traces instead of genuine traces is that analogues of Chern-Weil invariant polynomials do not give rise to closed forms. Implementing techniques borrowed from the theory of classical pseudodifferential calculus, one measures the obstructions to the closedness in terms of noncommutative residues (section 8). In specific situations such as in hamiltonian gauge theory (section 9) where we need to build Chern classes on pseudodifferential Grassmannians, the very locality of the noncommutative residue can provide a way to build counterterms, and thereby to renormalise the original non closed forms in order to turn them into closed ones. Loop groups [F] also provide an interesting geometric setup since obstructions to the closedness can vanish, thus leading to closed forms. On infinite rank vector bundles associated with a family of Dirac operators on even dimensional closed spin manifolds, these obstructions can be circumvented by an appropriate choice of regularised trace involving the very superconnection which gives rise to the curvature. We discuss these last two geometric setups in section 9.

The paper is organised as follows:

1. Chern-Weil calculus in finite dimensions
2. The algebra of (zero order) classical pseudodifferential operators
3. Traces on (zero order) classical \(\psi\)\(\text{do}\)s
4. Linear extensions of the trace on smoothing operators
5. The group of invertible zero order \(\psi\)\(\text{do}\)s
6. A class of infinite dimensional manifolds
7. Singular Chern-Weil forms in infinite dimensions
8. Weighted Chern-Weil forms; discrepancies
9. Renormalised Chern-Weil forms on \(\psi\)\(\text{do}\) Grassmanians
10. Regular Chern-Weil forms in infinite dimensions.

1 Chern-Weil calculus in finite dimensions

Let \(E \to X\) be a vector bundle over a \(d\)-dimensional manifold \(X\) with structure group \(G\) a subgroup of the linear group \(\text{Gl}_d(\mathbb{C})\) and let \(\mathcal{A} = \text{End}(E)\) the bundle of endomorphisms of \(E\) over \(B\). Let \(\Omega(X, \mathcal{A})\) denote the algebra of exterior forms on \(X\) with values in \(\mathcal{A}\) equipped with the product induced from the wedge product on forms and the product in \(\mathcal{A}\). If \(\sigma\) is a section of \(E\) over \(X\) and \(\alpha \in \Omega^k(X, \mathcal{A})\) then \(\alpha(\sigma) \in \Omega^k(X)\).

If \(\nabla\) is a connection on \(P\) then \(\nabla^2\) lies in \(\Omega^2(X, \mathcal{A})\). More generally, if \(\mathcal{C}(P)\) is the space of connections on \(P\), to an analytic map \(f(z)\) we assign a map

\[
\nabla \mapsto f(\nabla^2) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} \nabla^{2i}.
\]

Interestingly, the very fact that the class vanishes can be used as a starting point to define Chern-Simons classes as in [MRT] where the authors build non trivial Wodzicki-Chern-Simons classes via a transgression of Wodzicki-Chern forms.
Remark 1 This sum is actually finite since $\nabla^{2i} = 0$ $\forall i > \frac{d}{2}$.

The connection $\nabla$ extends to a map

$$C^\infty(X, TX) \times \Omega(X, A) \to \Omega(X, A)$$

$$(U, \alpha) \mapsto (\sigma \mapsto [\nabla_U, \alpha](\sigma) := \nabla_U(\alpha(\sigma)) + (-1)^{|\alpha|+1} \alpha(\nabla_U \sigma)).$$

Here $\sigma$ stands for a section of $E$ over $X$ and $|\alpha|$ for the degree of the form.

The trace $\text{tr} : \text{gl}_d(\mathbb{C}) \to \mathbb{C}$ on the algebra $\text{gl}_d(\mathbb{C})$ of $d \times d$ matrices with complex coefficients extends to a trace on $\text{End}(E)$ by

$$\text{tr} : \text{End}(E) \to X \times \mathbb{C}$$

$$(x, A) \mapsto (x, \text{tr}(A))$$

where $\text{tr}$ on the r.h.s is the ordinary trace on matrices. This is a bundle morphism since

$$\text{tr}(C^{-1} AC) = \text{tr}(A) \forall C \in \text{Gl}_d(\mathbb{C}), \forall A \in \text{gl}_d(\mathbb{C}).$$

Similarly, to a form $\alpha(x) = A(x) dx_1 \wedge \cdots \wedge dx_d$ in $\Omega(X, A)$ corresponds a form $\text{tr}(\alpha)(x) := \text{tr}(A(x)) dx_1 \wedge \cdots \wedge dx_d$ in $\Omega(X)$.

From the fact that the trace $\text{tr}$ obeys the following properties

$$[d, \text{tr}](\alpha) := d\text{tr}(\alpha) - \text{tr}(d\alpha) = 0 \forall \alpha \in \Omega(X, A)$$

and

$$\partial \text{tr}(\alpha, \beta) := \text{tr} \left( \alpha \wedge \beta + (-1)^{|\alpha|+1} \beta \wedge \alpha \right) = 0 \forall \alpha, \beta \in \Omega(X, A),$$

we infer the subsequent useful lemma.

Lemma 1 For any $\alpha \in \Omega(X, A)$

$$[\nabla, \text{tr}](\alpha) := d\text{tr}(\alpha) - \text{tr}([\nabla, \alpha]) = 0.$$  \hspace{1cm} \text{(4)}

Proof: In a local chart above an open subset $U$ of $X$,

$$[\nabla, \alpha] = d\alpha + \theta \wedge \alpha + (-1)^{|\alpha|+1} \alpha \wedge \theta$$

for some one form $\theta \in \Omega^1(U, A)$ so that we can write

$$[\nabla, \text{tr}](\alpha) = d\text{tr}(\alpha) - \text{tr}([\nabla, \alpha])$$

$$= d\text{tr}(\alpha) - \text{tr} \left( d\alpha + \theta \wedge \alpha + (-1)^{|\alpha|+1} \alpha \wedge \theta \right)$$

$$= -\text{tr} \left( \theta \wedge \alpha + (-1)^{|\alpha|+1} \alpha \wedge \theta \right) \text{ by (2)}$$

$$= 0 \text{ by (3)}.$$

\square

Combining this lemma with the Bianchi identity

$$[\nabla, \nabla^2] = 0.$$  \hspace{1cm} \text{(5)}

leads to closed Chern-Weil forms.

Proposition 1 For any analytic function $f$, the form $\text{tr}(f(\nabla^2))$ is closed with de Rham cohomology class independent of the choice of connection.
Proof: It is sufficient to carry out the proof for monomials $f(x) = x^i$ in which case we have:

$$d \text{tr} (f(\nabla^2)) = [\nabla, \text{tr}] (f(\nabla^2)) + \text{tr} ([\nabla, f(\nabla^2)])$$

$$= \text{tr} ([\nabla, \nabla^{2i}]) \quad \text{by (4)}$$

$$= \sum_{j=0}^{i} \text{tr} ([\nabla, \nabla^2] \nabla^{2(i-1)})$$

$$= 0 \quad \text{by (5)},$$

which proves the closedness of $\text{tr}(f(\nabla^2))$.

Let $\nabla_t, t \in \mathbb{R}$ be a smooth one parameter family of connections on $E$. Its derivative w.r. to $t$ is a one form $\hat{\nabla}_t = \hat{\theta}_t \in \Omega^1(X, \mathcal{A})$. Applying (2) to $X = \mathbb{R}$ yields

$$\frac{d}{dt} (\text{tr}(f(\nabla^2_t))) = \text{tr} \left( \frac{d}{dt} \nabla^{2i}_t \right)$$

$$= \sum_{j=0}^{i} \text{tr} \left( \frac{d}{dt} \nabla_t^2 \nabla^{2(j-1)}_t \right)$$

$$= \sum_{j=0}^{i} \text{tr} \left( \left[ \nabla_t, \nabla_t \right] \nabla^{2(j-1)}_t \right) \quad \text{by (5)}$$

$$= d \sum_{j=0}^{i} \text{tr} \left( \nabla_t \nabla^{2(i-1)}_t \right) \quad \text{by (4)}.$$

The variation $\frac{d}{dt} (\text{tr}(f(\nabla^2_t)))$ is therefore exact and the de Rham class of $\text{tr}(f(\nabla^2_t))$ is independent of the parameter $t$. □

2 The algebra of (zero order) classical pseudodifferential operators

In the infinite dimensional situations considered in these notes, the algebra of matrices $\text{gl}_d(\mathbb{C})$ on which lives the trace used for ordinary Chern-Weil calculus, is replaced by the algebra of zero order classical pseudodifferential operators on a closed manifold $M$ with values in $\mathbb{C}^n$. Such an algebra contains the algebra $\text{Map}(M, \text{gl}_d(\mathbb{C}))$ of smooth maps from $M$ to the algebra of matrices. One can think of $\text{gl}_d(\mathbb{C})$ as what remains of the infinite dimensional algebra of zero order classical pseudodifferential operators on $M$ when $M$ is reduced to a point $\{\ast\}$.

We briefly recall the definition of classical pseudodifferential operators (\psi\text{dos}) on closed manifolds, referring the reader to [H], [Sh], [T], [Tr] for further details.

Let $U$ be an open subset of $\mathbb{R}^n$. Given $a \in \mathbb{C}$, we consider the space of symbols $S^a(U)$ which consists of smooth functions $\sigma(x, \xi)$ on $U \times \mathbb{R}^n$ such that for any compact subset $K$ of $U$ and any two multiindices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n$ there exists a constant $C_{K\alpha\beta}$ satisfying for all $(x, \xi) \in K \times \mathbb{R}^n$
Here we write for short $\sigma$ where $a$ of order on smooth functions $f$.

The product $\star$ on symbols is defined as follows: if $\sigma_1 \in S^{a_1}(U)$ and $\sigma_2 \in S^{a_2}(U)$,

$$\sigma_1 \star \sigma_2(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{(-i)^{\lvert \alpha \rvert}}{\alpha!} \partial^{\alpha}_x \sigma_1(x, \xi) \partial^{\alpha}_x \sigma_2(x, \xi)$$

i.e. for any integer $N \geq 1$ we have

$$\sigma_1 \star \sigma_2(x, \xi) - \sum_{|\alpha| < N} \frac{(-i)^{\lvert \alpha \rvert}}{\alpha!} \partial^{\alpha}_x \sigma_1(x, \xi) \partial^{\alpha}_x \sigma_2(x, \xi) \in S^{a_1+a_2-N}(U).$$

In particular, $\sigma_1 \star \sigma_2 \in S^{a_1+a_2}(U)$.

We denote by $S^{-\infty}(U) := \bigcap_{a \in \mathbb{C}} S^a(U)$ the algebra of smoothing symbols on $U$ and let $S(U)$ be the algebra generated by $\bigcup_{a \in \mathbb{C}} S^a(U)$.

A symbol $\sigma \in S^n(U)$ is called classical of order $a \in \mathbb{C}$ if

$$\forall N \in \mathbb{N}, \quad \sigma - \sum_{j < N} \psi(\xi) \sigma_{a-j}(x, \xi) \in S^{a-N}(U),$$

where $\sigma_{a-j}(x, \xi)$ is a positively homogeneous function on $U \times \mathbb{R}^n$ of degree $a-j$, i.e. $\sigma_{a-j}(x, t\xi) = t^{a-j} \sigma_{a-j}(x, \xi)$ for all $t \in \mathbb{R}^+$. We write for short

$$\sigma(x, \xi) \sim \sum_{j=0}^{\infty} \psi(\xi) \sigma_{a-j}(x, \xi). \tag{6}$$

Here $\psi \in C^\infty(\mathbb{R}^n)$ is any cut-off function which vanishes for $|\xi| \leq \frac{1}{2}$ and such that $\psi(\xi) = 1$ for $|\xi| \geq 1$.

We call $a$ the order of the classical symbol $\sigma$ and denote by $CS^n(U)$ the subset of classical symbols of order $a$. The positively homogeneous component $\sigma_a(x, \xi)$ of degree $a$ corresponds to the leading symbol of $\sigma$.

**Example 1** A smooth function $h \in C^\infty(U)$ can be viewed as a multiplication operator $f \mapsto hf$ on smooth functions $f \in C^\infty(U)$ and hence as a zero order classical symbol.

The symbol product of two classical symbols is a classical symbol and we denote by

$$CS(U) = \langle \bigcup_{a \in \mathbb{C}} CS^a(U) \rangle$$

the algebra generated by all classical symbols on $U$.

Given a symbol $\sigma \in S(U)$, we can associate to it the continuous operator $Op(\sigma) : C^\infty_c(U) \to C^\infty(U)$ defined for $u \in C^\infty_c(U)$ - the space of smooth compactly supported functions on $U$ - by

$$(Op(\sigma)u)(x) = \int e^{ix \xi} \sigma(x, \xi) \hat{u}(\xi) d\xi,$$

where $d\xi := \frac{1}{(2\pi)^n} d\xi$ with $d\xi$ the ordinary Lebesgue measure on $T^*_x M \simeq \mathbb{R}^n$ and where $\hat{u}(\xi)$ is the Fourier transform of $u$. Since

$$|\partial^\alpha \partial^{\beta}_\xi \sigma(x, \xi)| \leq C_{K, \beta}(1 + |\xi|)^{\Re(a) - |\beta|},$$

where $|\beta| = \beta_1 + \cdots + \beta_n$. If $\Re(a_1) < \Re(a_2)$, then $S^{a_1}(U) \subset S^{a_2}(U)$.

**Remark 2** If $a \in \mathbb{R}$, $a$ corresponds to the order of $\sigma \in S^a(U)$. The notion of order extends to complex values for classical pseudodifferential symbols (see below).
Remark 3

It follows from the above discussion that

\[ (Op(\sigma)u)(x) = \int \int e^{i(x-y)\cdot \xi} \sigma(x, \xi)u(y)d\xi dy, \]

\( Op(\sigma) \) is an operator with Schwartz kernel given by \( k(x, y) = \int e^{i(x-y)\cdot \xi} \sigma(x, \xi)d\xi \), which is smooth off the diagonal.

A pseudodifferential operator \( A \) on \( U \) is an operator which can be written in the form \( A = Op(\sigma)+R \) where \( \sigma \in S(U) \) and \( R \) is a smoothing operator i.e. \( R \) has a smooth kernel. If \( \sigma \) is a classical symbol of order \( a \), then \( A \) is called a classical pseudodifferential operator \( \psi(\sigma) \) of order \( a \).

The symbol \( \sigma(A) \) of a pseudodifferential operator \( A \) of order \( a \) is only locally defined whereas the leading symbol \( \psi_\sigma(A) = \sigma_\sigma(A) \) is globally defined.

Example 2 Multiplication by \( f \in C^\infty(U) \) can be viewed as a zero order classical \( \psi(\sigma) \) on \( U \). Here the leading symbol coincides with the symbol.

The product on symbols induces a composition \( Op(\sigma_1 \star_2) = Op(\sigma_1)Op(\sigma_2) \). This in turn induces a composition on properly supported operators. A \( \psi(\sigma) \) on \( U \) is called properly supported if for any compact \( C \subset U \), the set \( \{(x, y) \in \text{Supp}(K_\lambda), \ x \in C \text{ or } y \in C \} \) is compact, where \( \text{Supp}(K_\lambda) \) denotes the support of the Schwartz kernel of \( A \) i.e. a distribution on \( U \times U \) such that, for \( u \in C^\infty_c(U) \), \( Au(x) = \int K_\lambda(x, y)u(y)dy \). A properly supported \( \psi(\sigma) \) maps \( C^\infty_c(U) \) into itself and admits a symbol given by \( \sigma(A)(x, \xi) = e^{-ix\cdot \xi}Ae^{ix\cdot \xi} \). The composition \( AB \) of two properly supported \( \psi(\sigma) \)’s is a properly supported \( \psi(\sigma) \) and \( \sigma(AB) = \sigma(A) \star \sigma(B) \).

More generally, let \( M \) be a smooth closed manifold of dimension \( n \) and \( \pi : E \to M \) a smooth vector bundle of rank \( d \) over \( M \); an operator \( P : C^\infty(M, E) \to C^\infty(M, E) \) is a (resp. classical) pseudodifferential operator of order \( a \) if given a local trivializing chart \( (U, \phi) \) on \( M \), for any localization \( P_\nu = \chi^*_\nu P \chi_\nu : C^\infty_c(U, \mathcal{E}^d) \to C^\infty_c(U, \mathcal{E}^d) \) of \( P \) where \( \chi_\nu \in C^\infty_c(U) \), the operator \( \phi_\nu(P_\nu) := \phi P_\nu \phi^{-1} \) from the space \( C^\infty_c(\phi(U), \mathcal{E}^d) \) into \( C^\infty(\phi(U), \mathcal{E}^d) \) is a (resp classical) pseudodifferential operator of order \( a \).

Example 3 A smooth section \( f \in C^\infty(M, \text{End}(E)) \) can be viewed as a multiplication operator \( u \mapsto fu \) on smooth sections \( u \) of \( E \) and hence as a zero order classical \( \psi(\sigma) \).

Let \( \text{Cl}^a(M, E) \) denote the set of classical pseudodifferential operators of order \( a \).

If \( A_1 \in \text{Cl}^{a_1}(M, E), A_2 \in \text{Cl}^{a_2}(M, E) \), then \( A_1A_2 \in \text{Cl}^{a_1+a_2}(M, E) \) and we denote by

\[ \text{Cl}(M, E) := \langle \bigcup_{a \in \mathbb{C}} \text{Cl}^a(M, E) \rangle \]

the algebra generated by all classical pseudodifferential operators acting on smooth sections of \( E \). It follows from the above discussion that

\[ C^\infty(M, \text{End}(E)) \subset \text{Cl}^0(M, E) \subset \text{Cl}(M, E). \]

Remark 3 When \( E \) is the trivial bundle \( M \times \mathbb{R} \), we drop \( E \) in the notation writing \( \text{Cl}^a(M), \text{Cl}^{-\infty}(M), \text{Cl}(M) \) instead of \( \text{Cl}^a(M, E), \text{Cl}^{-\infty}(M, E), \text{Cl}(M, E) \).

Remark 4 When \( M \) reduces to a point, then \( E \) is a vector space corresponding to the model space of the original bundle \( M \) and we have

\[ M = \{ \ast \} \implies C^\infty(M, \text{End}(E)) = \text{Cl}(M, E) = \text{Cl}^0(M, E) = \text{Cl}^{-\infty}(M, E) = \text{End}(E). \]

3 Traces on (zero order) classical \( \psi(\sigma) \)\’s

Having chosen \( \text{Cl}^0(M, E) \) as a potential infinite dimensional Ersatz for the algebra \( \text{gl}_d(\mathbb{C}) \), it remains to find linear forms on \( \text{Cl}^0(M, E) \) as an Ersatz for the trace on matrices.

The ordinary trace on matrices extends to a trace on smoothing operators:
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\[ \text{tr} : \text{Cl}^{-\infty}(M, E) \to \mathbb{C} \]

\[ A \mapsto \int_M \text{tr}_x(k_A(x, x)) \, dx = \int_{T^\ast M} \text{tr}_x(\sigma(A)(x, \xi)) \, dx \, d\xi, \]

where \( k_A \) stands for the Schwartz kernel of \( A \), \( \sigma_A \) for the symbol of \( A \) and \( \text{tr}_x \) for the fibrewise trace defined previously using the ordinary trace on matrices.

But it does not further extend to a trace on \( \text{Cl}^0(M, E) \) i.e. to a linear form \( \lambda : \text{Cl}^0(M, E) \to \mathbb{C} \) which vanishes on brackets

\[ \partial \lambda(A, B) := \lambda([A, B]) = 0 \quad \forall A, B \in \text{Cl}^0(M, E). \]

A well known result by Wodzicki [W1] (see also [W2] and [K] for a review) and proved independently by Guillemin [G] gives the uniqueness (up to a multiplicative factor) of a trace on the whole algebra \( \text{Cl}(M, E) \) of classical pseudodifferential operators.\(^3\)

Indeed, Wodzicki showed that any trace on \( \text{Cl}(M, E) \) is proportional to the noncommutative residue defined as follows. The residue density at point \( x \in M \)

\[ \omega_{\text{res}}(A)(x) := \left( \int_{S^\ast_x M} \text{tr}_x(\sigma_{-n}(A)(x, \xi)) \, dS^\ast \xi \right) \, dx \]

where \( S^\ast_x M \subset T^\ast_x M \) is the cotangent unit sphere (here \( dS^\ast \xi := \frac{dS \xi}{(2\pi)^n} \) stands for the (normalised) volume measure on the cotangent unit sphere \( S^\ast M \) induced by the canonical volume measure on the cotangent bundle \( T^\ast M \) and \( (\cdot)_{-n} \) denotes the positively homogeneous component degree \(-n\) of the symbol) is globally defined so that the noncommutative residue\(^4\)

\[ \text{res}(A) := \int_M \omega_{\text{res}}(A)(x) := \int_M dx \, \text{res}_x(A) \quad (7) \]

is well defined on \( \text{Cl}(M, E) \).

Restricting to zero order classical pseudodifferential operators allows for another type of trace, leading symbol traces associated with any linear form \( \tau \) on \( C^\infty(S^\ast M) \) introduced in [PR1, PR2] in relation to Chern-Weil forms:

\[ \text{Tr}_0^\tau(A) := \tau(\text{tr}_x \sigma_0(A)(x, \xi)). \]

Whenever \( \tau(1) \neq 0 \) we set:

\[ \text{tr}_0^\tau(A) := \frac{\text{Tr}_0^\tau(A)}{\tau(1)} = \frac{\tau(\text{tr}_x(\sigma_0(A)(x, \xi)))}{\tau(1)}. \quad (8) \]

**Theorem 1** [LP] All traces on the algebra \( \text{Cl}^0(M, E) \) are linear combinations the Wodzicki residue and leading symbol traces.

**Remark 5** When \( M \) reduces to a point so that \( n = 0 \) and \( A \) is a matrix, then both \( \text{res}(A) \) and \( \text{tr}_0^\tau(A) \) are proportional to the ordinary matrix trace.

\(^3\) Since then other proofs, in particular a homological proof on symbols in [BG] (see also [P2] for another alternative proof) and various extensions of this uniqueness result were derived, see [FGLS] for a generalisation to manifolds with boundary; see [S] for a generalisation to manifolds with conical singularities (both of which prove uniqueness up to smoothing operators), see [L] for an extension to log-polyhomogeneous operators as well as for an argument due to Wodzicki to get uniqueness on the whole algebra of classical operators, see [Po2] for an extension to Heisenberg manifolds.

\(^4\) It generalises to higher dimensions a notion of residue previously introduced by Adler and Manin in the one dimensional case.
Both the noncommutative residue and the leading symbol traces clearly vanish on smoothing operators and therefore neither of them extends the ordinary trace on smoothing operators. If we insist on building linear forms on Cl\(^0\)(M, E) that extend the ordinary trace on smoothing operators, we need to drop the requirement that it vanishes on brackets. The linear forms we are about to describe are actually defined on the whole algebra Cl(M, E).

We use the unique extension [MSS] (see also [P2] where the uniqueness of the noncommutative residue and the canonical trace are handled simultaneously), called the canonical trace, of the trace on smoothing operators to the set

\[ \text{Cl}^{\mathbb{Z}}(M, E) := \bigcup_{a \in \mathbb{Z}} \text{Cl}^a(M, E) \]

of non integer order operators in Cl(M, E). It was popularised by Kontsevich and Vishik in [KV] even though it was known long before by Wodzicki and Guillemin and is defined as follows.

For any \( A \in \text{Cl}(M, E) \), for any \( x \in M \), one can infer from (6) (see e.g. [L]) that the integral 
\[ \int_{T^*_xM} \text{tr}_x (\sigma(A)(x, \xi)) \, d\xi \] 
which clearly coincides with the ordinary integral on smoothing symbols.

**Theorem 2** [KV] Whenever the operator \( A \in \text{Cl}(M, E) \) has non integer order or has order \( < -n \) then

\[ \omega_{KV}(A)(x) := \left( \int_{T^*_xM} \text{tr}_x (\sigma(A)(x, \xi)) \, d\xi \right) \, dx \quad \forall x \in M, \]

defines a global density on \( M \) so that the canonical trace [KV] (see also [L] for an extension to log-polyhomogeneous operators):

\[ \text{TR}(A) := \int_M \omega_{KV}(A)(x) \, dx := \int_M \text{TR}_x(A) \, dx \]

makes sense 5. The canonical trace vanishes on brackets of non integer order or of order \( < -n \) [KV] (see also [L]) i.e.

\[ \text{TR}([A, B]) = 0 \quad \forall A, B \in \text{Cl}(M, E) \quad \text{s.t.} \quad [A, B] \in \text{Cl}^{\mathbb{Z}}(M, E) \cup \text{Cl}^{\leq -n}(M, E). \]

**Remark 6** For any smoothing operator

\[ \text{TR}(A) = \int_{T^*_xM} \text{tr}_x (\sigma(A)(x, \xi)) \, d\xi \, dx = \text{tr}(A) \]

so that the canonical trace indeed extends the ordinary trace on smoothing operators.

**4 Linear extensions of the trace on smoothing operators**

Unfortunately, the operators one comes across in infinite dimensional geometry as well as in quantum field theory are typically integer order operators such as the Laplace operator, the Dirac operator, the Green operator...so that we cannot implement the canonical trace on such operators.

5 However, in general \( \omega_{KV}(A)(x) \) is only locally defined and does not integrate over \( M \) to a well defined linear form.
In order to match the canonical trace with our needs in spite of this apparent discrepancy, we perturb the operators holomorphically \( A \mapsto A(z) \) thereby perturbing their order \( a \mapsto \alpha(z) \) and we define regularised trace of such operators as finite parts at \( z = 0 \) of \( \text{TR}(A(z)) \).

To carry out this construction we need the notion of holomorphic family of symbols which we now recall.

**Definition 1** Let \( \Omega \) be a domain of \( \mathbb{C} \). A family \((\sigma(z))_{z \in \Omega} \subset \mathcal{CS}(U)\) is holomorphic when

(i) the order \( \alpha(z) \) of \( \sigma(z) \) is holomorphic on \( \Omega \).

(ii) For \((x,\xi) \in U \times \mathbb{R}^n\), the function \( z \mapsto \sigma(z)(x,\xi) \) is holomorphic on \( \Omega \) and \( \forall k \geq 0, \partial^k_z \sigma(z) \in \mathcal{S}^{\alpha(z)+\epsilon(U)} \) for any \( \epsilon > 0 \).

(iii) For any integer \( j \geq 0 \), the homogeneous symbol \( \sigma(z)_{\alpha(z)-j}(x,\xi) \) is holomorphic on \( \Omega \).

It leads to the following notion of holomorphic family of \( \psi\)dos.

**Definition 2** A family \( z \mapsto A(z) \in \mathcal{Cl}^{\alpha(z)}(M,E) \) of log-classical \( \psi\)dos parametrized by a domain \( \Omega \subset \mathbb{C} \) is holomorphic if in each local trivialisation of \( E \) one has

\[
A(z) = \text{Op}(\sigma_{A(z)}) + R(z)
\]

with \( \sigma_{A(z)} \) a holomorphic family of classical symbols of order \( \alpha(z) \)\(^6\) and \( R(z) \) a smoothing operator with Schwartz kernel \( R(z,x,\xi) \in C^\infty(\Omega \times M \times M, \text{End}(V)) \) holomorphic in \( z \) where \( V \) is the model space of \( E \).

A holomorphic family of classical operators of holomorphic order \( \alpha(z) \) parametrised by \( \Omega \) has integer order \( \geq -n \) on the set \( \Omega \cap \alpha^{-1}(\mathbb{Z} \cap [-n,\infty]) \). Outside that set, the canonical trace \( \text{TR}(A(z)) \) is therefore well defined.

**Theorem 3** [KV][L] Let \( z \mapsto A(z) \in \mathcal{Cl}^{\alpha(z)}(M,E) \) be a holomorphic family of classical \( \psi\)dos on a domain \( \Omega \subset \mathbb{C} \). Then the map

\[
z \mapsto \text{TR}(A(z))
\]

is meromorphic with poles of order 1 at points \( z_j \in \Omega \cap \alpha^{-1}([-n,\infty[ \cap \mathbb{Z}) \) such that \( \alpha'(z_j) \neq 0 \).

**Definition 3** A holomorphic regularisation scheme on \( \mathcal{Cl}(M,E) \) is a linear map which sends \( A \in \mathcal{Cl}(M,E) \) to a holomorphic family \( A(z) \in \mathcal{Cl}(M,E) \) such that \( A(0) = A \) and \( A(z) \) has order \( \alpha(z) \) with \( \alpha \) holomorphic and \( \alpha'(0) \neq 0 \).

In order to illustrate this with examples, it is useful to introduce the following definition.

**Definition 4** An operator \( A \in \mathcal{Cl}(M,E) \) has principal angle \( \theta \) if for every \((x,\xi) \in T^*M - \{0\}, \) the leading symbol \( \sigma_A^1(x,\xi) \) has no eigenvalues on the ray \( \overline{L_\theta} = \{re^{i\theta}, r \geq 0\}; \) in that case \( A \) is elliptic.

We call an operator \( A \in \mathcal{Cl}(M,E) \) admissible with spectral cut \( \theta \) if \( A \) has principal angle \( \theta \) and the spectrum of \( A \) does not meet the open ray \( L_\theta = \{re^{i\theta}, r > 0\} \). In particular such an operator is elliptic, i.e. it has invertible leading symbol \( \sigma_A(x,\xi) \in \text{End}(E_x) \) for all \( x \in M, \xi \in T^*_xM - \{0\} \) where \( E_x \) is the fibre of \( E \) over \( x \).

**Remark 7** When \( A \) has principal angle \( \theta \) and the spectrum of \( A \) does not meet \( L_\theta, \theta \) is called an Agmon angle of \( A \). In that case, \( A \) is invertible elliptic. We want to allow for non invertible operators hence the need for introducing admissibility.

**Example 4** \( \zeta \)-regularisation

\[
R : A \mapsto A(z) := AQ^{\frac{1}{\zeta}}_\theta
\]

with \( Q \) an admissible operator in \( \mathcal{Cl}(M,E) \) with positive order \( q \) and spectral cut \( \theta \) yields typical (and very useful) examples of holomorphic regularisations.

\(^6\) In applications the order is affine in \( z \).
On the basis of the results of the previous section, given a holomorphic regularisation \( R : A \mapsto A(z) \), we can pick the finite part in the Laurent expansion \( \text{TR} (A(z)) \) and set the following definition.

**Definition 5** A holomorphic regularisation scheme \( R : A \mapsto A(z) \) on \( \text{Cl}(M, E) \) induces a linear form:

\[
\text{tr}^R : \text{Cl}(M, E) \to \mathbb{C} \\
A \mapsto \text{tr}^R(A) := \text{fp}_{z=0} \text{TR} (A(z))
\]

called \( R \)-regularised trace\(^7\). When \( R \) is a \( \zeta \)-regularisation \((9)\) determined by an admissible operator \( Q \) with spectral cut \( \theta \) we call \( Q \)-weighted trace the linear form \( \text{tr}^R \) and denote it by \( \text{tr}^Q_{\theta} \).

The following result measures the difference between the regularised trace and the (generally non-existing) canonical trace.

**Theorem 4** Let \( R : A \mapsto A(z) \) with \( A(z) \) of order \( \alpha(z) \) be a holomorphic regularisation with order \( z \mapsto \alpha(z) \) affine in \( z \).

The linear form \( \text{tr}^R \) extends the usual trace defined on operators of order \(< -n \) as well as the canonical trace \( \text{TR} \) defined on non integer order operators) to \( \psi \)dos of all orders.

Moreover \( \text{PS1} \),

\[
\text{tr}^R(A) = \int_M dx \left( \text{TR}_x(A) - \frac{1}{\alpha'(0)} \text{res}_x(A'(0)) \right)
\]

where, in spite of the fact that \( A'(0) \) is no\( \text{more} \) expected to be classical\(^8\), its residue density is defined in a similar manner:

\[
\text{res}_x(A'(0)) := \int_{S^*M} \text{tr}_x(\sigma(A)) - n(x, \xi) \, d\xi.
\]

**Remark 8** When the residue density \( \text{res}_x(A'(0)) \) vanishes, \( \text{TR}_x(A) \, dx \) defines a global density and \( \text{tr}^R(A) = \text{TR}(A) \). In particular

\[
\text{tr}^R(A) = \text{TR}(A) \quad \forall A \in \text{Cl}^{\mathbb{Z}}(M, E)
\]

is independent of the regularisation scheme.

\( \zeta \)-regularisations provide an interesting class of examples. Let

\[
R : A \mapsto A(z) := AQ^{-z}\theta
\]

with \( Q \) an admissible operator in \( \text{Cl}(M, E) \) with positive order \( q \) and spectral cut \( \theta \). Then \( A'(0) = -A \log_\theta Q \). Here \( \log_\theta Q \) stands for the logarithm of an admissible operators \( Q \in \text{Cl}(M, E) \) with spectral cut \( \theta \) defined in terms of the derivative at \( z = 0 \) of its complex power \( \text{Se} \):

\[
\log_\theta Q = \partial_z Q^z_{\theta|z=0},
\]

where \( Q^z_{\theta} \) is the complex power of \( A \) defined using a Cauchy integral on a contour \( \Gamma_\theta \) around the spectrum of \( A \). Formula \( (10) \) therefore reads:

\[
\text{tr}^Q_{\theta}(A) = \int_M dx \left( \text{TR}_x(A) - \frac{1}{q} \text{res}_x(A \log_\theta Q) \right).
\]

We borrow the following definition from \( \text{OP} \).

\(^7\) It carries this name because it extends the ordinary trace on smoothing operators and in spite of the fact that it does not vanish on brackets as we shall soon see.

\(^8\) It is log-polyhomogeneous of log type 1 \( \text{PS1} \), meaning by this that the asymptotic expansion \( (6) \) might present a logarithmic divergence \( \log |\xi| \) in \( |\xi| \) as \( |\xi| \to \infty \).
Definition 6 We call an operator $A \in \text{Cl}(M, E)$ conditionally trace-class whenever the fibrewise trace of its symbol $\text{tr}_x(\sigma_A(x, \cdot))$ at a point $x \in M$ is of order $<-n$ in which case we set

$$\text{tr}_{\text{cond}}(A) := \int_M dx \int_{T^*_x M} \text{tr}_x(\sigma(A)(x, \xi)) \, d\xi$$

which we call the conditioned trace of $A$.

Example 5 Clearly, operators in $\text{Cl}(M, E)$ of order $<-n$ are conditionally trace-class and their conditioned trace coincides with their ordinary trace.

Example 6 Let $E = M \times \mathbb{R}^n$ be a rank $n$ trivial vector bundle over $M$ then $(Au)(x) := Au(x) \quad \forall x \in M, u \in C^\infty(M, \mathbb{R}^n)$ with $A \in o(\mathbb{R}^n)$ (the Lie algebra of the orthogonal group $O(\mathbb{R}^n)$) is not trace class since it is a multiplication operator. However, it is conditionally trace-class with zero conditioned trace since the fibrewise trace of its symbol which coincides with the trace of the matrix $A$, vanishes.

Example 7 More generally, let us consider a trivial vector bundle $E = M \times V$ with $V$ a finite dimensional space, then $\text{Cl}(M, E) \cong \text{Cl}(M) \otimes \text{Hom}(V)$ so that $\text{tr}_V(A) \in \text{Cl}(M)$ for any $A \in \text{Cl}(M, E)$ where $\text{tr}_V$ is the ordinary trace on $\text{Hom}(V)$. If $\text{tr}_V(A)$ is trace-class then $A$ is conditionally trace-class.

Proposition 2 Let $A \in \text{Cl}(M, E)$ be conditionally trace-class and $\mathcal{R}: A \mapsto A(z)$ a holomorphic regularisation. Then

1. $\text{res}(A'(0))$ vanishes,
2. $A$ has a well defined canonical trace

$$\text{TR}(A) := \int_M dx \int_{T^*_x M} \text{tr}_x(\sigma(A)(x, \xi)) \, d\xi$$

and

$$\text{tr}^{\mathcal{R}}(A) = \text{tr}_{\text{cond}}(A) = \text{TR}(A).$$

Proof: The assertions follow from the fact that the scalar symbol $\text{tr}_x(\sigma_A(x, \cdot))$ together with the fact that the derivative $^9 \text{tr}_x(\sigma_A'(0)(x, \cdot))$ are of order $<-n$.

Indeed, this implies that $\text{res}_x(A'(0))$ vanishes and hence that $\text{tr}^{\mathcal{R}}(A) = \text{TR}(A)$ by (10). \hfill $\square$

5 The group of invertible zero order $\psi$do’s

The Lie algebra $\text{Cl}^0(M, E)$ offers a natural generalisation of the algebra $\text{End}(E)$. The corresponding Lie group of invertible zero order $\psi$do’s offers a natural generalisation of the group $\text{GL}(E)$ of linear transformations of a vector space $E$.

For $a \in \mathcal{C}$, the linear space $\text{Cl}^0(M, E)$ of classical pseudodifferential operators of order $m$ can be equipped with a Fréchet topology. For this, one first equips the set $CS^a(U, W) = CS^a(U) \otimes \text{End}(W)$ of classical symbols of order $a$ on an open subset $U$ of $\mathbb{R}^n$ with values in an euclidean vector space $W$ (with norm $\| \cdot \|$) with a Fréchet structure. The following semi-norms labelled by multiindices $\alpha, \beta$ and integers $j \geq 0$, $N$ give rise to a Fréchet topology on $CS^m(U, W)$ (see [H]):

$$\sup_{x \in K, \xi \in \mathbb{R}^n} (1 + |\xi|)^{-\Re(\alpha)+|\beta|} \| \partial^\alpha_{\xi} \partial^\beta x \sigma(x, \xi) \|;$$

$$\sup_{x \in K, \xi \in \mathbb{R}^n} (|\xi|)^{-\Re(\alpha)+N+|\beta|} \| \partial^\alpha_{x} \partial^\beta x (\sigma - \sum_{j=0}^{N-1} \psi(x) \sigma_{a-j}(x, \xi)) \|;$$

$$\sup_{x \in K, |\xi|=1} \| \partial^\alpha_{x} \partial^\beta x \sigma_{m-j}(x, \xi) \|,$$  

$^9$ We recall that $A'(0)$ has the same order as $A$. 

Proposition 3: Let $K$ be any compact set in $U$. Given a vector bundle based on a closed manifold $M$, the set $\text{Cl}^0(M, E)$ of classical pseudodifferential operators acting on sections of $E$, namely pseudodifferential operators $A$ acting on sections of $E$ that have local classical symbols $\sigma^U(A) \in CS(U, W)$ in a local trivialization $E_{|U} \simeq U \times W$, inherits a Fréchet structure via the Fréchet structure on classical symbols. Given an atlas $(U_i, \phi_i)_{i \in I}$ on $M$ and corresponding local trivializations $E_{|U_i} \simeq U_i \times W$ where $W$ is the model fibre of $E$, in any local chart $U_i$, we can equip $\text{Cl}^0(M, E)$ with the following family of seminorms labelled by multiindices $\alpha, \beta$ and integers $j \geq 0, i \in I, N \geq 0$

\[
\sup_{x \in K, \xi \in \mathbb{R}^n} (1 + |\xi|)^{-\text{Re}(\alpha)+|\beta|} \| \partial_x^\alpha \partial^\beta_\xi \sigma^U_i(A)(x, \xi) \|;
\]

\[
\sup_{x \in K, \xi \in \mathbb{R}^n} (|\xi|)^{-\text{Re}(\alpha)+N+|\beta|} \| \partial_x^\alpha \partial^\beta_\xi \left( \sigma^U_i(A) - \sum_{j=0}^{N-1} \psi(\xi) \sigma_{a-j}^U_i(A) \right)(x, \xi) \|;
\]

\[
\sup_{x \in K, \xi \in \mathbb{R}^n} \| \partial_x^\alpha \partial^\beta_\xi \sigma^{U_j}(A)(x, \xi) \|,
\]

where $K$ is any compact subset of $\phi_i(U_i) \subset \mathbb{R}^n$.

**Proposition 4:** $\text{Cl}^0(M, E)$ is a Fréchet Lie algebra and the traces of Theorem 1 are continuous for the Fréchet topology.

**Proof:** The continuity of the traces can easily be seen from their very definition. Let us discuss the continuity of the bracket. Since $\sigma(AB) \sim \sigma(A) \ast \sigma(B)$ for two operators $A, B$ with symbols $\sigma(A), \sigma(B)$, the product map on $\text{Cl}^0(M, E)$ is smooth as a consequence of the smoothness of the symbol product $\sigma \ast \tau \sim \sum a_{\alpha \beta} \partial^\alpha \sigma \partial^\beta \tau$ on $CS(U, V)$ for any vector space $V$. It follows that the bracket is a continuous bilinear map on $\text{Cl}^0(M, E)$. \(\square\)

Let

\[
\text{Cl}^{0,\ast}(M, E) := \{ A \in \text{Cl}^0(M, E), \exists A^{-1} \in \text{Cl}^0(M, E) \}
\]

be the group of invertible zero order classical pseudodifferential operators which is strictly contained in the intersection $\text{Cl}^0(M, E) \cap \text{Cl}^\ast(M, E)$ where

\[
\text{Cl}^\ast(M, E) = \{ A \in \text{Cl}(M, E), \exists A^{-1} \in \text{Cl}(M, E) \}
\]

is the group of invertible classical pseudodifferential operators.

**Remark 9:** It is useful to note that $\text{Cl}^\ast(M, E)$ acts on $\text{Cl}^\ast(M, E)$ for any $a \in \mathbb{C}$ by the adjoint action defined for $P \in \text{Cl}^\ast(M, E)$ by

\[
\text{Cl}^\ast(M, E) \to \text{Cl}^\ast(M, E)
\]

\[
A \mapsto \text{Ad}_P A := P^{-1} A P
\]

and specifically on the algebra $\text{Cl}^0(M, E)$.

**Proposition 4:** $\text{Cl}^{0,\ast}(M, E)$ is a Fréchet Lie group with Lie algebra $\text{Cl}^0(M, E)$.

**Proof:** We only discuss the continuity of the inverse map, referring the reader to [KV] for further details.

As an open subset in the Fréchet space $\text{Cl}^0(M, E)$, $\text{Cl}^{0,\ast}(M, E)$ is a Fréchet manifold modelled on $\text{Cl}^0(M, E)$.

We already know that the product map is smooth. The smoothness of the inversion $A \mapsto A^{-1}$ on $\text{Cl}^{0,\ast}(M, E)$ follows from the fact that for an operator $A \in \text{Cl}^{0,\ast}(M, E)$ with symbol $\sigma(A)$ and order $a$, the positively homogeneous components of its inverse $A^{-1}$ of order $-a$ are given by...
\[ (\sigma(A^{-1}))_{-a-j} = \frac{i}{2\pi} \int_{\Gamma} \lambda^{-1} \tau_{-a-j}(A) \, d\lambda \]

where \( \Gamma \) is a contour around the spectrum of \( A \) and where
\[
\tau_{-a}(A) = (\sigma(A) - \lambda)^{-1},
\]
\[
\tau_{-a-j}(A) = -\tau_{-a}(A) \sum_{k+l+|\alpha|=j,d,j} i^{-|\alpha|} \frac{1}{\alpha!} \partial^\alpha_{\xi} \sigma_{a-k}(A) \partial^\alpha_{x} \tau_{-a-l}(A).
\]

\[ \square \]

Following [KM] we say that a Lie group \( \mathcal{G} \) admits an exponential mapping if there exists a smooth mapping
\[ \text{Exp}: \text{Lie}(\mathcal{G}) \to \mathcal{G} \]
such that \( t \mapsto \text{Exp}(tX) \) is a one-parameter subgroup with tangent vector \( X \). \( \text{Exp}(0) = e_\mathcal{G} \) and \( \text{Exp} \) induces the identity map \( \text{Id}_{\text{Lie}(\mathcal{G})} \) on the corresponding Lie algebra.

All known smooth Fréchet Lie groups and in particular the group \( \text{Cl}^{0,*}(M, E) \) (see [KV]) admit an exponential mapping although it is not known, according to [KM], whether any smooth Fréchet Lie group does admit an exponential mapping.

The topology of \( \text{Cl}^{0,*}(M, E) \) has been investigated in various contexts.

Recall (see e.g. [Ka]) that the fundamental group \( \pi_1(GL_d(\mathbb{C})) \) is generated by the homotopy classes \([l] \) of the loops \( l(t) = e^{2\pi it} \pi \)

where \( \pi: \mathbb{C}^d \to \mathbb{C}^d \) is a projector.

A similar statement holds for the fundamental group of \( \text{Cl}^{0,*}(M, E) \) with the projectors \( \pi \) replaced by pseudodifferential operators introduced by Burak [Bu], later used by Wodziński [W1] and further investigated by Ponge [Po1] which encode the spectral asymmetry of elliptic classical pseudodifferential operators:
\[
\Pi_{\theta,\theta'}(Q) := \frac{1}{2\pi} \int_{C_{\theta,\theta'}} \lambda^{-1} Q (Q - \lambda)^{-1} \, d\lambda
\]

where
\[
C_{\theta,\theta'} := \{ \rho e^{i\theta}, \infty > \rho \geq r \} \cup \{ r e^{it}, \theta \leq t \leq \theta' \} \cup \{ \rho e^{i\theta'}, r \leq \rho < \infty \},
\]
with \( Q \in \text{Cl}(M, E) \) elliptic with positive order and whereby \( r \) is chosen small enough so that no non-zero eigenvalue of \( Q \) lies in the disc \( |\lambda| \leq r \). It turns out that \( \Pi_{\theta,\theta'}(Q) \) is a bounded \( \psi \)-do projection on \( L^2(M, E) \) (see [BL] and [Po1]) and either a zero’th order pseudodifferential operator or a smoothing operator. For \( Q \) of order \( q \) with leading symbol \( \sigma_L(Q) \), the leading symbol of \( \Pi_{\theta,\theta'}(Q) \) reads:
\[
\pi_{\theta,\theta'}(\sigma_L(Q)) := \frac{1}{2\pi} \int_{C_{\theta,\theta'}} \lambda^{-1} \sigma_L(Q) (\sigma_L(Q) - \lambda)^{-1} \, d\lambda.
\]

The following proposition (see [KV], see also [LP]) shows that these pseudodifferential projectors generate the fundamental group \( \pi_1(\text{Cl}^{0,*}(M, E)) \). Let \( \text{GL}_\infty(A) \) be the direct limit \(^{10}\) of linear groups \( \text{GL}_n(A) \).

\(^{10}\) A natural embedding \( \text{GL}_n(A) \to \text{GL}_{n+1}(A) \) of an \( n \times n \) matrix \( g \in \text{GL}_n(A) \) in \( \text{GL}_{n+1}(A) \) is obtained inserting \( g \) in the upper left corner, 1 in the lower right corner and filling the other slots in the last line and column with zeroes.
Proposition 5
\[ \pi_1(\text{GL}_\infty(\text{Cl}^{0,*}(M, E))) \text{ is generated by the homotopy class of loops} \]
\[ L^Q_{\theta, \theta'}(t) := e^{2\pi i t \Pi_{\theta, \theta'}(Q)} \]
where \( Q \in \text{Cl}(M, E) \) is any elliptic operator with positive order.

Remark 10 When \( M \) reduces to a point \( \{x\} \) then \( \sigma_L(Q) \) reduces to a \( d \times d \) matrix \( q \) with \( d \) the rank of \( E \), and \( \Pi_{\theta, \theta'} \) reduces to \( \pi_{\theta, \theta'} = \frac{1}{2\pi} \int_{C_{\theta, \theta'}} \lambda^{-1} q(q - \lambda)^{-1} d\lambda \) which is a finite dimensional projector. Hence, the generators \( [L^Q_{\theta, \theta'}(t)] \) reduce to generators \( [\Pi^Q_{\theta, \theta'}(t)] = [e^{2\pi i \tau \pi_{\theta, \theta'}(q)}] \) built from projectors \( \pi_{\theta, \theta'}(q) \).

Proof: We take the proof from [LP] \(^11\). For any algebra \( A \), let \( K_0(A) \) denote the group of formal differences of homotopy classes of idempotents in the direct limit \( \text{gl}_\infty(A) \) of matrix algebras \( \text{gl}_n(A) \)\(^12\). When \( A \) is a good topological algebra [Bo], the Bott periodicity isomorphism:
\[ K_0(A) \longrightarrow \pi_1(\text{GL}_\infty(A)) \]
\[ [P] \longrightarrow e^{2\pi i t P} \]  \( (14) \)
holds. Since for any vector bundle \( E \) over \( M \), the algebra \( \text{Cl}^0(M, E) \) is a good topological algebra (which essentially boils down to the fact that the inverse of a classical pseudodifferential operator remains a classical pseudodifferential operator), applying (14) to \( A = \text{Cl}^0(M, E) \) reduces the proof down to checking that \( K_0(\text{Cl}^0(M, E)) \) is generated by idempotents \( \Pi_{\theta, \theta'}(Q) \).

The exact sequence
\[ 0 \longrightarrow \text{Cl}^{-1}(M, E) \longrightarrow \text{Cl}^0(M, E) \xrightarrow{\sigma_L^t} C^\infty(S^*M, p^*(\text{End}E)) \longrightarrow 0 \]
where \( p : S^*M \rightarrow M \) is the canonical projection of the cotangent sphere to the base manifold \( M \), gives rise to a long exact sequence in \( K \)-theory:
\[ K_0(\text{Cl}^{-1}(M, E)) \xrightarrow{\text{Ind}} K_0(\text{Cl}^0(M, E)) \xrightarrow{\sigma_L^t} K_0(C^\infty(S^*M, p^*(\text{End}E))) \xrightarrow{0} \]
\[ K_1(C^\infty(S^*M, p^*(\text{End}E))) \xrightarrow{\sigma_L^t} K_1(\text{Cl}^0(M, E)) \xrightarrow{0} K_1(\text{Cl}^{-1}(M, E)) = 0. \]  \( (15) \)

On the other hand, on the grounds of results of Wodzicki, \( K_0(C^\infty(S^*M, p^*(\text{End}E))) \) is generated by the classes \( \pi_{\theta, \theta'}(\sigma_L(Q)) \) where as before \( \sigma_L(Q) \) is the leading symbol of an elliptic operator \( Q \in \text{Cl}(M, E) \); this combined with the surjectivity of the map \( \sigma_L^t \) in the diagram (15) yields the result. \( \square \)

Higher homotopy groups were derived in [BW] and [R] from which we quote some results without proofs \(^13\).

Proposition 6 For odd \( k \)
\[ 1. [BW] (Proposition 15.4) \]
\[ \pi_k(\text{Cl}^{0,*}(M, E)) \simeq \mathbb{Z}, \]
where \( \text{Cl}^{0,*}(M, E) := \{ A \in \text{Cl}^{0,*}(M, E), \ \sigma_L(A) = Id \} \).

\(^{11}\) As pointed out to us by R. Ponge, the proof can probably be shortened using results of [BL] to show directly that \( K_0(\text{Cl}^0(M, E)) \) is generated by idempotents \( \Pi_{\theta, \theta'}(Q) \).

\(^{12}\) A natural embedding \( \text{gl}_n(A) \rightarrow \text{gl}_{n+1}(A) \) of an \( n \times n \) matrix \( a \in \text{gl}_n(A) \) in \( \text{gl}_{n+1}(A) \) is obtained inserting \( a \) in the upper left corner and filling the last line and column with zeroes.

\(^{13}\) Even homotopy groups were also described in [R] leading to further results which we do not report on here. Also, the statement we quote here holds provided one allows for bundles with arbitrary large rank.
6 A class of infinite dimensional manifolds

We consider a class of infinite dimensional manifolds and vector bundles inspired from the geometric setup of index theory [B], [BGV] and close to those introduced in [P1] (under the name of weighted manifolds and bundles) and further used in [CDMP], [PR1, PR2] (under the name of do-manifolds and bundles). It consists of Fréchet vector bundles with typical fibre \( C^\infty(M, E) \) for some reference finite rank vector bundle \( \pi : E \to M \) over \( M \).

Consider a smooth fibration \( \mathfrak{M} \to X \) of smooth manifolds modelled on a closed manifold \( M \) and a fibre bundle \( \pi : E \to \mathfrak{M} \) over \( \mathfrak{M} \) with typical fibre \( E \to M \).

Remark 11 In the context of the family index theorem, \( \mathfrak{E} = \Phi \otimes |A_\pi| \) for some vector bundle \( \Phi \to \mathfrak{M} \) and \( A_\pi \) is the vertical density bundle which, when restricted to the fibres of \( \mathfrak{M} \) may be identified with the bundle of densities along the fibre.

Let us denote by \( \pi_* \mathfrak{E} \to X \) the infinite dimensional Fréchet bundle with fibre \( C^\infty(M, E|_{M, \gamma}) \) over \( x \in X \) modelled on \( C^\infty(M, E) \) with \( M \) the model fibre of \( \mathfrak{M} \) and \( E \) the model fibre of \( \mathfrak{E} \).

Definition 7 We call a Fréchet vector bundle admissible if it is of the form \( \pi_* \mathfrak{E} \) for some finite rank vector bundle \( \mathfrak{E} \to \mathfrak{M} \) over a smooth fibration \( \mathfrak{M} \to X \) of smooth closed manifolds.

We call a Fréchet manifold admissible if its tangent bundle is an admissible vector bundle.

Remark 12 Locally, over an open subset \( U \subset X \),

\[
\mathfrak{E}|_U \simeq U \times M \times V
\]

where \( V \) is a finite vector space and \( M \) a closed manifold. A change of local trivialisation of the fibration \( \mathfrak{M}|_U \simeq U \times M \) induces a diffeomorphism \( f : M \to M \) in \( \mathcal{D}(M) \) whereas a change of local trivialisation of the finite rank vector bundle \( \mathfrak{E}|_{U \times M} \simeq U \times M \times V \) induces a transformation in \( \text{Gl}(V) \).

Example 8 Let \( N \) be a Riemannian manifold, then the space \( X := C^\infty(M, N) \) of smooth maps from \( M \) to \( N \) is a Fréchet manifold with tangent space at point \( \gamma \) given by \( C^\infty(M, \gamma^* TN) \). The tangent bundle \( TC^\infty(M, N) \) can therefore be realised as \( \pi_* \mathfrak{E} \) where \( \mathfrak{M} \to X \) is the trivial fibration with fibre \( M \) and \( \mathfrak{E} \) the vector bundle over \( X \times M \) with fibre at \( (\gamma, m) \in C^\infty(M, N) \times M \) given by the vector space

\[
\pi_* \mathfrak{E}_\gamma = \gamma^* T_{\gamma(m)} N
\]

so that \( \pi_* \mathfrak{E}_\gamma = C^\infty(M, \gamma^* T N) \).

Hence \( C^\infty(M, N) \) is an admissible manifold.

In passing note that this manifold, which is modelled on \( C^\infty(M, \mathbb{R}^n) \) where \( n \) is the dimension of \( N \) can be equipped with an atlas induced by the exponential map \( \exp^N \) on \( N \), a local chart being of the type \( \phi_\gamma(u)(x) = \exp^N_{\gamma(x)}(u(x)) \). The transition maps are multiplication operators.

Example 9 In particular, mapping groups \( C^\infty(M, G) \) with \( G \) a finite dimensional Lie group are admissible Fréchet manifolds.

The left action \( L_g : y \mapsto g \cdot y \) on \( G \) induces a left action \( L_g : \gamma \mapsto g \cdot \gamma \) on \( C^\infty(M, G) \) and a vector field \( V(\gamma) \in C^\infty(M, \gamma^* TG) \) is left-invariant if \( (L_g)_* V(\gamma) = V(g \cdot \gamma) \) for all \( \gamma \in C^\infty(M, G) \). Left invariant vector fields on \( C^\infty(M, G) \) can be identified with elements of the Lie algebra \( C^\infty(M, \text{Lie}(G)) \).
Example 10 The group
\[ D(M) := \{ f \in C^\infty(M, M), \ \exists f^{-1} \in C^\infty(M, M) \} \]
of smooth diffeomorphisms of \( M \) is a Fréchet Lie group \([O]\) (see \([N]\) for a review) with Lie algebra \( C^\infty(M, TM) \) where \( TM \) is the tangent bundle to \( M \). It is an admissible Fréchet manifold since its tangent bundle \( \bigcup_{f \in D(M)} C^\infty(M, f^*TM) \) can be realised as a bundle \( \pi^*E \rightarrow X \) with \( X = D(M) \) and where \( E \) is a vector bundle over the trivial fibration \( \mathbb{M} = X \times M \) with fibre above \( (f, M) \) given by the bundle \( f^*TM \).

We also introduce a class of connections inspired from the ones arising in the family index geometric setup and similar to the ones considered in \([P1]\) later named \( \psi \)-do-connections in \([PR1, PR2]\).

Following \([Sc]\), \([PS2]\), let for any \( a \in \mathfrak{C} \), \( Cl^a(M, E) \) denote the bundle over \( X \) with fibre over \( x \) given by \( Cl^a(M_x, E_{|M_x}) \) so that locally, above an open subset \( U \) of \( X \) we have
\[ Cl^a(M, E)|_U \simeq U \times Cl^a(M, E). \]

Let \( Cl(M, E) \) be the bundle of algebras generated by \( \bigcup_{a \in \mathfrak{C}} Cl^a(M, E) \).

Remark 14 When \( M \) reduces to a point \( \{ * \} \) and \( E \rightarrow M \) reduces to a finite rank vector bundle \( E \rightarrow X \), then \( Cl^a(M, E) = Cl(M, E) = \text{End}(E) \) the endomorphism bundle over \( M \).

Example 11 For mapping spaces as described in Example 8 the fibre of the bundle \( Cl(M, E) \)
above \( \gamma \) is given by
\[ Cl(M, E)_\gamma = Cl(M, \gamma^*TN). \]

Example 12 For mapping groups as described in Example 9, we can specialise to left-invariant \( \psi \)-do\-s, \( A(\gamma) \in Cl(M, \gamma^*TG) \) such that \( A(g \cdot \gamma) \circ L_{g_*} = L_{g_*} \circ A(\gamma) \) for all \( g \in C^\infty(M, G) \) and \( \gamma \in C^\infty(M, G) \).

Above an open set \( U \subset X \), a connection on a finite rank vector bundle \( E \rightarrow X \) is locally of the form:
\[ \nabla_V \sigma = d\sigma(V) + \theta^U(V)\sigma, \ \forall V \in T_xX, \ \forall \sigma \in C^\infty(U, E) \text{ with } \theta^U(V) \in \text{End}(E_x). \]

In view of the generalisation from \( \text{End}(E_x) \) to \( Cl^0(M_x, E_{|M_x}) \), we introduce a class of connections locally of the form:
\[ \nabla_V \sigma = d\sigma(\tilde{V}) + \theta^U(\tilde{V}) \ \forall V \in T_xX \ \forall \sigma \in C^\infty(X, \pi_*E) \text{ with } \theta^U(\tilde{V}) \in Cl^0(M_x, E_{|M_x}) \] (16)
with \( E \) a finite rank bundle over a fibration \( \pi : \mathbb{M} \rightarrow X \) of manifolds equipped with some horizontal distribution \( V \in T_xX \rightarrow \tilde{V} \in T_{(x,m)}\mathbb{M} \).

Similar classes of connections were considered in \([P1]\) and later in \([PR1, PR2]\) under the name of \( \psi \)-do-connection.

Definition 8 A connection \( \nabla \) on an admissible Fréchet vector bundle \( \pi_*E \rightarrow X \) with \( E \) a finite rank bundle over a fibration \( \pi : \mathbb{M} \rightarrow X \) of manifolds is admissible whenever
1. the connection \( \nabla \) is induced by a connection on a finite rank vector bundle:
\[ \nabla_V = \nabla^E_V \] (17)

for some connection \( \nabla^E \) on the finite rank vector bundle \( E \rightarrow \mathbb{M} \) and some horizontal distribution \( V \in T_xX \rightarrow \tilde{V} \in T_{(x,m)}\mathbb{M} \) on \( \mathbb{M} \),
2. or when the fibration \( \mathbb{M} = X \times M \) is trivial, if locally over an open subset \( U \subset X \)
\[ \nabla = d + \theta^U \text{ with } \theta^U \in \Omega^1(U, Cl^0(M, E)). \]
Remark 15. The two conditions have a non void intersection; indeed, in the case of a horizontal distribution on a trivial fibration $\mathbb{M} \cong X \times M$ the first condition locally reads $\nabla = d + \theta^U$ with $\theta^U$ given by a multiplication operator valued one form on $U$.

Remark 16. Admissible connections fulfill condition (16); we choose the trivial horizontal lift in the case of a trivial connection.

Remark 17. When $M$ reduces to a point $\{ \ast \}$ then any connection is admissible since $E \rightarrow \mathbb{M}$ boils down to a bundle $E \rightarrow X$ and locally $\nabla = d + \theta^U$ with $\theta^U \in \Omega^1(U, \text{End}(E))$.

Lemma 2. An admissible connection on an admissible bundle $\pi \colon E \rightarrow X$ has curvature in $\Omega(X, \text{Cl}^1(\mathbb{M}, E))$. When the fibration $\mathbb{M} \rightarrow X$ is trivial with trivial distribution then it lies in $\Omega(X, \text{Cl}^0(\mathbb{M}, E))$.

Proof:

1. If locally, $\nabla = d + \theta^U$ with $\theta^U \in \Omega^1(U, \text{Cl}^0(\mathbb{M}, E))$ then $\Omega = d\theta^U + \theta^U \wedge \theta^U$ lies in $\Omega^2(U, \text{Cl}(\mathbb{M}, E))$. Since the curvature is a globally defined two form, $\Omega$ lies in $\Omega^2(X, \text{Cl}^0(\mathbb{M}, E))$.

2. If $\nabla_V = \nabla^E_V$ then

$$\Omega(U, V) = [\nabla_U, \nabla_V] - \nabla_{[U, V]} = [\nabla^E_U, \nabla^E_V] - \nabla^E_{[U, V]} = \Omega^E(\tilde{U}, \tilde{V}) - \nabla^E_{[U, V]}[\theta^V] - \nabla^E_{[U, V]}[\theta^U] = \Omega^E(\tilde{U}, \tilde{V}) - \nabla^E_{T(U, V)}$$

where $T(U, V) = [\tilde{U}, \tilde{V}] - [\tilde{U}, \tilde{V}]$ is the curvature of the connection on $\mathbb{M}$. Since $\Omega^E(\tilde{U}, \tilde{V})$ is a multiplication operator, it follows that $\Omega(U, V)$ is a first order differential operator which therefore lies in $\text{Cl}^1(\mathbb{M}, E)$. When the distribution on $\mathbb{M}$ is trivial, then $T(U, V) = 0$ and $\Omega(U, V)$ lies in $\text{Cl}^0(\mathbb{M}, E)$.

Example 13. On mapping groups considered in Example 9, we restrict to left-invariant connections i.e. connections $\nabla$ such that if $V, W$ are left-invariant then so is $\nabla_V W$ left-invariant. An admissible left-invariant connection is defined by a left-invariant one-form $\theta_0 \in \Omega^1(X, \text{Cl}^0(M, \text{Lie}(G)))$.

With the notations of Example 9, let $(M, g)$ be a closed Riemannian manifold and let $G$ be a semisimple Lie group of compact type $14$.

Let $Q_0 := \Delta \otimes 1_{\text{Lie}(G)}$ where $\Delta$ stands for the Laplace Beltrami operator on $M$. D. Freed in $[F]$ introduces a family of left-invariant one-forms parametrised by $s \in \mathbb{R}$ on the $H^\ast$-Sobolev closure $H^\ast(M, G)$ of the the mapping group $C^\infty(M, G)$ (see formula (1.9) in $[F]$):

$$\theta_0^s(V) := \frac{1}{2} \left( \text{ad}_V + (Q_0 + \pi_0)^{-s} \text{ad}_V(Q_0 + \pi_0)^s \right) \forall V \in C^\infty(M, \text{Lie}(G)).$$

Here $\pi_0$ stands for the orthogonal projection onto the kernel of $Q_0$ which is finite dimensional. These give rise to left-invariant connections $\nabla^s$ on $H^\ast(M, G)$ which in turn induce connections $15$.

---

14. This ensures that the Killing form is non degenerate and that the adjoint representation $\text{ad}$ on the Lie algebra is antisymmetric for this bilinear form.

15. They are weak since they are defined by weak metrics on $L^2(M, \text{Lie}(G))$; they are not determined by the usual six term formula $[F]$. 
on the mapping group $C^\infty(M, G)$. Since $\theta_s^0 \in \Omega^1(C^\infty(M, G), \Cl^0(M, \Lie(G))$, these define \(\psi\)do-
connections; only if $s = 0$ does this one-form correspond to a multiplication operator.

The curvature $\Omega^s$ is given by a left-invariant two-form:

\[
\Omega^0_0(U, V) = [\theta^0_0(U), \theta^0_0(V)] - \theta^0_0([UV])
\] (18)

and by [F] (Proposition 1.14), the map

\[
R^s(U, V) : W \mapsto \Omega^s(W, U)
\]

is a pseudodifferential operator of order $\max(-1, -2s)$. It lies in $\Omega^2(M, \Cl^0(M, \Lie(G)))$.

**Remark 18** These results were extended to loop spaces in [MRT], where it was shown that the curvature operator $R^s$ on $C^\infty(S^1, N)$ built in a similar manner has order at most $-\frac{3}{2}$ for $s > \frac{3}{2}$.

**Example 14** Connections of the type (17) arise in the geometric setup underlying the index theorem for families [B] and in determinant bundles associated with families of Dirac operators [BF]. Such connections are in fact slightly perturbed by adding the divergence of the horizontal lift w.r. to the Riemannian volume element on the fibres of $\mathbb{M} \to X$ in order to produce unitary connections:

\[
\tilde{\nabla}_V = \nabla^\tau_V + \text{div}_M(V).
\]

7 SINGULAR CHERN-WEIL FORMS IN INFINITE DIMENSIONS

We aim at generalising Chern-Weil formalism to admissible vector bundles, defining when possible traces $\lambda(\nabla^{2i})$ of even powers of admissible connections. In view of the two types of admissible connections we distinguish two cases:

1. the curvature lies in $\Omega^2(\mathbb{X}, \Cl^0(\mathbb{M}, \mathbb{E}))$,
2. the curvature lies in $\Omega^2(\mathbb{X}, \Cl^1(\mathbb{M}, \mathbb{E}))$.

Accordingly, to build traces of powers of the curvature along the lines of the first section, by the results of section 3 we have at our disposal two types of (singular) traces, namely traces that vanish on smoothing operators:

1. leading symbol traces (8) and the noncommutative residue (7) on $\Cl^0(M, \mathbb{E})$
2. the noncommutative residue (7) on $\Cl(M, \mathbb{E})$.

Let $V$ be the model space of $\mathbb{E} \to \mathbb{M}$. Since a change of trivialisation $((x, m), v) \in \mathbb{M} \times V \mapsto ((x, m), C v) \in \mathbb{M} \times V$, $C \in \text{Gl}(V)$ of $\mathbb{E} \to \mathbb{M}$ induces a change of trivialisation

\[
(x, A) \in \mathbb{X} \times \Cl(M, \mathbb{V}) \mapsto (x, C^{-1} AC) \in \mathbb{X} \times \Cl(M, \mathbb{V}), \quad C \in C^\infty(\mathbb{M}, \text{Gl}(\mathbb{V}))
\]

of the bundle $\Cl(\mathbb{M}, \mathbb{E}) \to \mathbb{X}$, we need to make sure the traces we implement are invariant under the adjoint action of $C^\infty(\mathbb{M}, \text{Gl}(\mathbb{V}))$. The following lemma gives more, namely their invariance under the action of invertible zero order $\psi$dos.

**Lemma 3** Let $E \to M$ be a finite rank vector bundle over a closed manifold $M$. For any $C \in \Cl^{0,*}(M, \mathbb{E})$

\[
\text{res}(C^{-1} AC) = \text{res}(A), \quad \forall A \in \Cl(M, \mathbb{E})
\] (19)

and

\[
\text{tr}^r(C^{-1} AC) = \text{tr}^r_0(A) \quad \forall A \in \Cl^0(M, \mathbb{E}).
\] (20)

**Remark 19** Here, as pointed out in the introduction, the group $\Cl^{0,*}(M, \mathbb{E})$ of invertible classical $\psi$dos generalises the structure group $\text{GL}(\mathbb{V})$. 
Proof: Both properties follow from the cyclicity of the respective traces. □

Just as the ordinary trace on matrices induces a trace on the endomorphism bundle End(E) of any finite rank vector bundle $E \to X$, the noncommutative residue (resp. and the leading symbol traces) induce a noncommutative residue (resp. and leading symbol traces) on bundles $\mathrm{Cl}(\mathbf{M}, E)$ (resp. on $\mathrm{Cl}^0(\mathbf{M}, E)$) associated to any finite rank vector bundle $E \to \mathbf{M}$ over a fibration of manifolds $\mathbf{M} \to X$.

As in the finite dimensional case, to a form $\alpha(x) = A(x) dx_1 \wedge \cdots \wedge dx_d$ in $\Omega(X, \mathrm{Cl}(M, E))$ (resp. $\Omega(X, \mathrm{Cl}^0(M, E))$) corresponds a form $\mathrm{res}(\alpha)(x) := \mathrm{res}(A(x)) dx_1 \wedge \cdots \wedge dx_d$ (resp. and $\mathrm{tr}^0(\alpha)(x) := \mathrm{tr}^0(A(x)) dx_1 \wedge \cdots \wedge dx_d$ for any $\tau \in C^\infty(S^*M)'$ in $\Omega(X)$. As in the finite dimensional case (see (2) and (3)), we first check that the linear form $\lambda = \mathrm{res}$ (resp. $\lambda = \mathrm{tr}^0$ for any $\tau \in C^\infty(S^*M)'$ on $\Omega(X, \mathcal{A})$ with $\mathcal{A} = \mathrm{Cl}(\mathbf{M}, E)$ (resp. $\mathcal{A} = \mathrm{Cl}^0(\mathbf{M}, E)$) obeys the following properties

$$[d, \lambda](\alpha) := d\lambda(\alpha) - \lambda(d\alpha) = 0 \quad \forall \alpha \in \Omega(X, \mathcal{A})$$

and

$$\partial\lambda(\alpha, \beta) := \lambda(\alpha \wedge \beta + (-1)^{||\alpha||\beta} \beta \wedge \alpha) = 0 \quad \forall \alpha, \beta \in \Omega(X, \mathcal{A}).$$

The first property is easily checked from the very definition of the two types of traces which involve integrals over the cotangent unit sphere of the manifold $M$ of the trace of a homogeneous part of the symbol of the operator. The second property is a direct consequence of their cyclicity.

As in the finite dimensional case, we then infer the following lemma.

**Lemma 4** For any $\alpha \in \Omega(X, \mathrm{Cl}(\mathbf{M}, E))$

$$[\nabla, \mathrm{res}](\alpha) := d\mathrm{res}(\alpha) - \mathrm{res}([\nabla, \alpha]) = 0.$$  \hspace{1cm} (23)

For any $\alpha \in \Omega(X, \mathrm{Cl}^0(\mathbf{M}, E))$, for any $\tau \in C^\infty(S^*M)'$

$$[\nabla, \mathrm{tr}^0](\alpha) := d\mathrm{tr}^0(\alpha) - \mathrm{tr}^0([\nabla, \alpha]) = 0.$$ \hspace{1cm} (24)

**Proof:** The proof goes as for Lemma 1 using properties (21) and (22). □

The constructions carried out in section 1 then go through leading to the following result.

**Theorem 5** Let $\mathcal{E} = \pi^*E \to X$ be an admissible vector bundle with $E \to \mathbf{M}$ a finite rank vector bundle over a fibration $\mathbf{M} \to X$, equipped with a connection with curvature $\Omega$. For any $i \in \mathbb{N}$, the $i$-th residue Chern-Weil form

1. $\mathrm{res}(\Omega^i)$ is closed with de Rham cohomology class independent of the choice of connection,

2. if $\Omega \in \Omega^2(X, \mathrm{Cl}^0(\mathbf{M}, E))$ then $\mathrm{tr}^0(\Omega^i)$ is also closed with de Rham cohomology class independent of the choice of connection.

We call these singular Chern-Weil classes.

**Proof:** The proof goes as in Proposition 1 using Lemma 4. □

**Remark 20** Singular Chern-Weil forms are clearly insensitive to smooth perturbations of the connection.

**Example 15** We refer to [RT] where an explicit example of an infinite rank bundle with non vanishing first residue Chern class is built, i.e. where the noncommutative residue is used as an Ersatz for the usual trace on matrices to build a first Chern form with non vanishing de Rham class; it is a bundle over the two-dimensional sphere $S^2$ with fiber modelled on Sobolev sections of a trivial line bundle over the three dimensional torus $T^3$. 
Example 16 Going back to Example 14, we saw that $\Omega^{\pi. \mathbf{E}}$ is a differential operator valued two form so that its noncommutative residue vanishes. When the fibration is trivial $\mathbb{M} \simeq M \times X$, $\Omega^{\pi. \mathbf{E}}$ reduces to a multiplication operator so that its leading symbol traces are well-defined and the two forms

$$(U, V) \mapsto \text{tr}_0 \left( \tilde{\Omega}^{\pi. \mathbf{E}}(U, V) \right) = \text{tr}_0 \left( \Omega^{\mathbf{E}}(\tilde{U}, \tilde{V}) \right)$$

can give rise to non trivial singular Chern-Weil classes.

But unfortunately, singular Chern-Weil classes generally seem too coarse to capture interesting information since most examples lead to vanishing singular Chern-Weil classes.

Example 17 Going back to Example 13, it was shown in [F] that $W \mapsto \Omega^s_0(W, U) V$ is conditionally trace-class, i.e. that $\text{tr}_{\text{Lie}(G)} \Omega^s_0(\cdot, U) V$ is trace-class, from which we infer that the residue vanishes. For the same reason, the leading symbol traces also vanish.

Thus, singular Chern-Weil forms vanish in the case of mapping groups.

Remark 21 In [MRT], the authors actually used the fact that residue Pontryagin forms vanish on loop spaces $C^\infty(S^1, N)$ as the starting point to build singular Chern-Simons classes. They focus on manifolds $N$ with stably trivial tangent bundle.

8 Weighted Chern-Weil forms; discrepancies

We now address the issue of how to build analogs of Chern-Weil forms using extensions of the ordinary trace of the type discussed in section 4 instead of singular traces on $\psi$dos. We show how one stumbles on various discrepancies inherent to the fact that these extensions do not actually define traces; however, it is instructive to describe the obstructions to carrying out these constructions in order to later find ways to circumvent them.

We first recall a result of Wodzicki, Guillemin and popularised by Kontsevich and Vishik in [KV] which relates the complex residue of the canonical trace of a holomorphic family at a pole with the noncommutative residue of the family at this pole: for any $C \in \text{Cl}(M, E)$ and any holomorphic family $C(z) \in \text{Cl}(M, E)$ with order $\alpha(z)$ such that $C(0) = C$ and $\alpha'(0) \neq 0$, the following identity holds:

$$\text{Res}_{z=0} \text{TR} \left( C(z) \right) = -\frac{1}{\alpha'(0)} \text{res}(C).$$

This formula provides ways to measure various defects of weighted traces.

8.1 The Hochschild coboundary of a weighted trace

The first defect is an obstruction to its cyclicity. The Hochschild coboundary of a linear form $\lambda$ on $\text{Cl}(M, E)$ is defined by

$$\partial \lambda(A, B) := \lambda \left( [A, B] \right).$$

The following proposition says this coboundary is local (see e.g. [CDMP], [Mi1, Mi2], [MN]).

**Proposition 7** Let $Q \in \text{Cl}(M, E)$ be an admissible operator of positive order $q$ with spectral cut $\theta$. Let $A \in \text{Cl}(M, E)$, $B \in (M, E)$ then

$$\partial \text{tr}_0^Q (A, B) = -\frac{1}{q} \text{res} \left( A [B, \log_\theta Q] \right).$$
Proof: Using the vanishing of the canonical trace on non integer order brackets we can write
\[
\text{TR}(\{A, B\} Q^{-z}) = \text{TR}(A B Q^{-z} - B A Q^{-z}) = \text{TR}(A B Q^{-z} - A Q^{-z} B) = \text{TR}(A [B, Q^{-z}]).
\]
The family \(C(z) := \frac{A[B, Q^{-z}]}{z} \in \text{Cl}(M, E)\) is a holomorphic family of order \(a - b - q z\) and \(C(0) = -A [B, \log Q]\). By (25) we get:
\[
\text{tr}_Q([A, B]) = \text{fp}_{z=0} \text{TR}(A B Q^{-z} - B A Q^{-z}) = \text{Res}_{z=0} \text{TR}(A [B, Q^{-z}]).
\]
\[
= -\frac{1}{q} \text{res } (A [B, \log Q]).
\]
\[\square\]

8.2 Dependence on the weight

Formula (25) also provides a way to measure the dependence on the weight \(Q\). We first need a technical lemma.

Lemma 5 Let \(Q \in \text{Cl}(M, E)\) be admissible of order \(q > 0\) and spectral cut \(\alpha\) and let \(A \in \text{Cl}(M, E)\). Then
\[
\text{tr}_Q^t([A, B]) = \text{tr}_Q^t(A) \quad \forall t > 0.
\]
Proof: We write \(\text{TR}(A Q^{-z}) = \frac{a_1}{z} + a_0 + o(z)\) in which case we have \(\text{TR}(A (Q^{-z}_t)^{-t z}) = \text{TR}(A (Q^{-z}_t)^{-t z}) = \frac{a_1}{t z} + a_0 + o(t z)\) so that
\[
\text{tr}_Q^t(A) = \text{fp}_{z=0} \text{TR}(A (Q^{-z}_t)^{-t z}) = a_0 = \text{tr}_Q^t(A).
\]
\[\square\]

The following proposition provides a well-known expression of the dependence on the weight [KV], [Ok].

Proposition 8 Let \(Q_1, Q_2 \in \text{Cl}(M, E)\) be two admissible operator with positive orders \(q_1, q_2\) and spectral cuts \(\theta_1, \theta_2\). Let \(A \in \text{Cl}(M, E)\), then
\[
\text{tr}_{Q_1}^t(A) - \text{tr}_{Q_2}^t(A) = \text{res } (A \left( \frac{\log Q_1}{q_2} - \frac{\log Q_1}{q_1} \right)).
\]
Proof: For simplicity, we leave out the explicit mention of the spectral cut. Applying formula (25) to the family \(C(z) := \frac{A}{z} \left( Q_1^{-z/\theta_1} - Q_2^{-z/\theta_2} \right)\) which is a holomorphic family of classical operators of order \(a - z\) with \(C(0) = A \left( \frac{\log Q_2}{q_2} - \frac{\log Q_1}{q_1} \right)\) we write
\[
\text{tr}^{Q_1}(A) - \text{tr}^{Q_2}(A) = \text{tr}^{Q_1}_{\theta_1}(A) - \text{tr}^{Q_2}_{\theta_2}(A)
\]
\[
= \text{Res}_{z=0} \text{TR} \left( \frac{A \left( Q_1^{-z/\theta_1} - Q_2^{-z/\theta_2} \right)}{z} \right)
\]
\[
= \text{res } (A \left( \frac{\log Q_2}{q_2} - \frac{\log Q_1}{q_1} \right)).
\]
\[\square\]
8.3 Exterior differential of a weighted trace

**Proposition 9 [CDMP], [P1]** Let $Q \in C^\infty(X, Cl(M, E))$ be a differentiable family (for the topology described previously) of operators of fixed order $q$ and spectral cut $\theta$ parametrised by a manifold $X$. Let $A \in Cl(M, E)$, then the trace defect $[d, \text{tr}^Q] := d\text{tr}^Q - \text{tr}^Q \circ d$ is local as a noncommutative residue:

$$[d, \text{tr}^Q] (A) = -\frac{1}{q} \text{res} (A \log \theta^Q).$$

**(Proof:** Again, for convenience, we drop the explicit mention of the spectral cut. Let $h \in C^\infty(X, TX)$ be a smooth vector field then by Proposition 8 we have

$$d\text{tr}^Q(A)(h) = \lim_{t \to 0} \frac{\text{tr}^Q + dQ(\theta h) - \text{tr}^Q(A)}{t} = \frac{1}{q} \lim_{t \to 0} \frac{\text{res} (A (\log(Q + dQ(\theta h)) - \log Q))}{t} = -\frac{1}{q} \text{res} (A \log Q(h)),$$

where we have used the continuity of the noncommutative residue on operators of order $a = \text{ord} A$ since $A \left(\frac{\log(Q + dQ(\theta h)) - \log Q}{t}\right)$ has order $a$ for any $t > 0$. □

8.4 Weighted traces extended to admissible fibre bundles

The covariance property (1) generalises to weighted traces as follows.

**Lemma 6 [P1], [PS2]** Order, ellipticity, admissibility and spectral cuts are preserved under the adjoint action. For any admissible operator $Q \in Cl(M, E)$ with spectral cut $\alpha$, for any operator $A \in Cl(M, E)$ we have

$$\text{tr}^{\text{ad}_C Q}(\text{ad}_C A) = \text{tr}^Q(A) \quad \forall C \in Cl^*(M, E).$$

**(Proof:** For simplicity, we drop the explicit mention of the spectral cut. Since the leading symbol is multiplicative we have

$$\sigma_L(\text{ad}_C Q) = \text{ad}_{\sigma_L(C)} \sigma_L(Q)$$

from which it follows that order, ellipticity, admissibility and spectral cuts are preserved by the adjoint action.

Let us observe that

$$(\text{ad}_C Q)^{-z} = \frac{1}{2\pi i} \int_g \lambda^{-z}(\lambda - \text{ad}_C Q)^{-1} = \frac{1}{2\pi i} \int_g \lambda^{-z} \text{ad}_C(\lambda - \text{ad}_C Q)^{-1} = \text{ad}_C Q^{-z}$$

Consequently,

$$\text{tr}^{C^{-1}QC}(C^{-1}AC) = \text{fp}_{z=0} \text{TR} \left( C^{-1}AC \left( C^{-1}QC \right)^{-z} \right) = \text{fp}_{z=0} \text{TR} \left( C^{-1}AC C^{-1}Q^{-z}C \right) = \text{fp}_{z=0} \text{TR} \left( C^{-1}AQ^{-z}C \right) = \text{fp}_{z=0} \text{TR} \left( AQ^{-z} \right) = \text{tr}^Q(A)$$
where we have used the fact that the canonical trace vanishes on non integer order brackets. □

Since the adjoint action \( \text{ad}_C : A \mapsto C^{-1}AC \) of \( \text{Cl}^{0,*}(M,E) \) on \( \text{Cl}(M,E) \) preserves the spectrum and the invertibility of the leading symbol, it makes sense to define the subbundle \( \text{Ell}^{\text{adm}}(\text{Cl}(M,E)) \) of \( \text{Cl}(M,E) \) of fibrewise admissible elliptic \( \psi \text{dos} \) with spectral cut \( \theta \); since it also preserves the order we can define \( \mathcal{Q} \) to be a smooth admissible elliptic section of order \( q \) of \( \text{Cl}(M,E) \) in which case \( \mathcal{Q}(x) \in \text{Cl}(M_x, E_{|M_x}) \) and we set:

\[
\text{tr}^\mathcal{Q}_\theta(A)(x) := \text{tr}^\mathcal{Q}(x)(A(x)) \quad \forall A \in C^\infty(X, \text{Cl}(M,E)), \quad \forall x \in X.
\]

\( \mathcal{Q} \)-weighted traces can further be extended to forms \( \alpha(x) = A(x) dx_1 \wedge \cdots \wedge dx_d \) in \( \Omega(X, \text{Cl}(M,E)) \) by \( \text{tr}^\mathcal{Q}_\theta(\alpha)(x) := \text{tr}^\mathcal{Q}_\theta(A(x)) dx_1 \wedge \cdots \wedge dx_d \) and using linearity.

### 8.5 Obstructions to closedness of weighted Chern-Weil forms

**Theorem 6** \cite{CDMP} An admissible connection \( \nabla \) on an admissible vector bundle \( \pi, E \) induces a connection \( [\nabla, A] := \nabla \circ A - A \circ \nabla \) on \( \text{Cl}(M,E) \). For any \( \alpha \in \Omega^p(X, \text{Cl}(M,E)) \) and any admissible section \( \mathcal{Q} \in C^\infty(X,A) \) with constant spectral cut \( \theta \) and constant order \( q > 0 \), the trace defect \( [\nabla, \text{tr}^\mathcal{Q}_\theta]\) := \( d \text{tr}^\mathcal{Q}_\theta(\alpha) - \text{tr}^\mathcal{Q}_\theta([\nabla, \alpha]) \) is local and explicitly given by:

\[
[\nabla, \text{tr}^\mathcal{Q}_\theta](\alpha) = \frac{(-1)^p}{q} \text{res} (\alpha [\nabla, \log \mathcal{Q}]) \quad \forall \alpha \in \Omega^p(X, \text{Cl}(M,E)).
\]

**Proof:** We prove the result for a zero form and drop the explicit mention of the spectral cut for simplicity. The result easily extends to higher order forms. In a local trivialisation of \( \mathcal{E} \) over an open subset \( U \) of \( X \) we write \( \nabla = d + \theta \) so that \( [\nabla, \cdot] = d + [\theta, \cdot] \). In this local trivialisation we have for any \( \alpha \in \Omega^p(X, \text{Cl}(M,E)) \):

\[
[\nabla, \text{tr}^\mathcal{Q}_\theta](\alpha) = d \left( \text{tr}^\mathcal{Q}_\theta(\alpha) \right) - \text{tr}^\mathcal{Q}_\theta([\nabla, \alpha])
\]

\[
= d \left( \text{tr}^\mathcal{Q}_\theta(\alpha) \right) - \text{tr}^\mathcal{Q}_\theta( d \alpha) - \text{tr}^\mathcal{Q}_\theta([\theta, \alpha])
\]

\[
= - \frac{1}{q} \text{res} (A d \log \mathcal{Q} - \frac{1}{q} \text{res} (A [\theta, \log \mathcal{Q}])
\]

\[
= - \frac{1}{q} \text{res} (A [\nabla, \log \mathcal{Q}])
\]

where we have combined \( (26) \) and \( (27) \).

□

The obstruction \( [\nabla^{\text{ad}}, \text{tr}^\mathcal{Q}] \) described in Theorem 6 prevents a straightforward generalisation of the Chern-Weil formalism to an infinite dimensional setup where the trace on matrices is replaced by a weighted trace provided the connection are admissible connections.

**Corollary 1** Let \( \nabla \) be an admissible connection on \( \mathcal{E} = \pi_* E \to X \) with curvature \( \Omega \) and let \( \mathcal{Q} \) be an admissible section of \( \text{Cl}(M,E) \) with spectral cut \( \theta \) and constant positive order \( q \). Then

\[
d \text{tr}^\mathcal{Q}_\theta(\Omega^p) = \frac{1}{q} \text{res} (\Omega^p [\nabla, \log \mathcal{Q}])
\]

**Proof:** We follow the finite dimensional proof (see Proposition 1).

\[
d \text{tr}^\mathcal{Q}_\theta(\Omega^p) = [d \text{tr}^\mathcal{Q}_\theta](\Omega^p) + \text{tr}^\mathcal{Q}_\theta([\nabla, \Omega^p])
\]

\[
= [d \text{tr}^\mathcal{Q}_\theta](\Omega^p) \quad \text{by the Bianchi identity}
\]

\[
= \frac{1}{q} \text{res} (\Omega^p [\nabla, \log \mathcal{Q}]) \quad \text{by (29)}.
\]

□
9 Renormalised Chern-Weil forms on $\psi$do Grassmannians

In view of Corollary 1 which tells us that a weighted trace of a power of the curvature is generally not closed, it seems hopeless to use weighted traces as a substitute for ordinary traces in order to extend finite dimensional Chern-Weil formalism to infinite dimensions. However, there are different ways to circumvent this difficulty, one of which is to introduce counterterms in order to compensate for the lack of closedness measured in Corollary 1 by a noncommutative residue. Such a renormalisation procedure by the introduction of counterterms can be carried out in a hamiltonian approach to gauge theory as it was shown in joint work with J. Mickelsson [MP] on which we report here.

Let us first review a finite dimensional situation which will serve as a model for infinite dimensional generalisations.

We consider the finite-dimensional Grassmann manifold $\text{Gr}(n,n)$ consisting of rank $n$ projections in $\mathbb{C}^{2n}$, which we parametrise by grading operators $F = 2P - 1$, where $P$ is a finite rank projection.

**Lemma 7** The even forms
\[
\omega_{2j} = \text{tr} \left( F (dF)^{2j} \right),
\]
where $j = 1, 2, \ldots$ are closed forms on $\text{Gr}(n,n)$.

**Remark 22** The cohomology of $G_n(\mathbb{C}^{2n}) := \text{Gr}(n,n)$ is known to actually be generated by even (nonnormalized) forms of the type $\omega_{2j}, j = 1, \cdots, n$. This follows from the fact (see Proposition 23.2 in [BT]) that the Chern classes of the quotient bundle $Q$ over $G_n(\mathbb{C}^n)$ defined by the exact sequence $0 \rightarrow S \rightarrow G_n(\mathbb{C}^{2n}) \times \mathbb{C}^n \rightarrow Q \rightarrow 0$ where $S$ is the universal bundle over $G_n$ with fibre $V$ above $V$, generate the cohomology ring $H^*(G_n(\mathbb{C}^n))$. Then the $j$-th Chern class of $Q$ turns out to be proportional to $\text{tr}(F (dF)^{2j})$ where $P(V)$ stands for the orthogonal projection on $V$.

**Proof:** By the traciality of $\text{tr}$ we have
\[
d\omega_{2j} = d\text{tr} \left( F (dF)^{2j} \right)
= \text{tr} \left( (dF)^{2j+1} \right)
= \text{tr} \left( F^2 (dF)^{2j+1} \right)
\text{since } F^2 = 1
= -\text{tr} \left( F (dF)^{2j+1} F \right)
\text{since } F dF = -dF F
= -\text{tr} \left( (dF)^{2j+1} F^2 \right)
\text{since } \text{tr}(\{A, B\}) = 0
= -\text{tr} \left( (dF)^{2j+1} \right)
= 0.
\]

\[\Box\]

We now want to extend these constructions to $\psi$do Grassmannians.

Let us consider a finite rank bundle $E$ over a trivial fibration $\pi: M = M \times X \rightarrow X$ with typical fibre a closed (Riemannian) spin manifold $M$. Let $D_x \in \text{Cl}(M, E), x \in X$ be a smooth family of Dirac operators parametrised by $X$.

On each open subset $U_\lambda := \{ x \in x, \lambda \notin \text{spec}(D_x) \} \subset X$ there is a well defined map
\[
F: X \rightarrow \text{Cl}^0(M, E)
\]
x $\mapsto F_x := (D_x - \lambda I)/||D_x - \lambda I||.

Since $F_x^2 = F_x, P_x := \frac{I + F_x}{2}$ is a projection, the range $\text{Gr}(M, E) := \text{Im} F$ of $F$ coincides with the Grassmannian consisting of classical pseudodifferential projections $P$ with kernel and cokernel of
infinite rank, acting in the complex Hilbert space \( H := L(M, E) \) of square-integrable sections of the vector bundle \( E \) over the compact manifold \( M \).

Since the map \( x \mapsto F_x \) is generally not contractible we want to define from \( F_x \) cohomology classes on \( X \) in the way we built Chern-Weil classes in finite dimensions. This issue usually arises in Hamiltonian quantization in field theory, when the physical space \( M \) is an odd dimensional manifold.

In this infinite dimensional setup traces are generally ill-defined, so that we use weighted traces as in the previous section. As expected, there are a priori obstructions to the closedness of the corresponding weighted forms.

**Proposition 10** Let \( Q \in \text{Cl}(M, E) \) be a fixed admissible elliptic operator with positive order. The exterior differential of the form

\[
\omega^Q_{2j}(F) = \text{tr}^Q (F(dF)^{2j})
\]

on \( \text{Gr}(M, E) \):

\[
d\omega^Q_{2j} = \frac{1}{2q} \text{res} ([\log Q, F](dF)^{2j+1} F).
\]

is a local expression which only depends on \( F \) modulo smoothing operators.

**Proof:** The locality and the dependence on \( F \) modulo smoothing operators follow from the expression of the exterior differential in terms of a Wodzicki residue. To derive this expression, we mimic the finite dimensional proof, taking into account that this time \( \text{tr}^Q \) is not cyclic:

\[
d\omega^Q_{2j} = d\text{tr}^Q (F(dF)^{2j})
\]

\[
= \text{tr}^Q ((dF)^{2j+1})
\]

\[
= \text{tr}^Q (F^2(dF)^{2j+1})
\]

\[
= -\text{tr}^Q (F(dF)^{2j+1} F)
\]

since \( F dF = -dF F \)

\[
= \frac{1}{q} \text{res} ([\log Q, F](dF)^{2j+1} F) - \text{tr}^Q ((dF)^{2j+1} F^2)
\]

\[
= \frac{1}{q} \text{res} ([\log Q, F](dF)^{2j+1} F) - \text{tr}^Q ((dF)^{2j+1}),
\]

where we have used (26) to write

\[
\text{tr}^Q ([F, (dF)^{2j+1} F]) = -\frac{1}{q} \text{res} (F [(dF)^{2j+1} F, \log Q]) = \frac{1}{q} \text{res} ([F, \log Q] (dF)^{2j+1} F).
\]

Hence

\[
\text{tr}^Q (F^2(dF)^{2j+1}) = \frac{1}{2q} \text{res} ([\log Q, F](dF)^{2j+1} F)
\]

from which the result then follows. \( \square \)

Let us consider the map

\[
\sigma : X \to \text{Cl}^0(M, E)/\text{Cl}^{-\infty}(M, E)
\]

\[
x \mapsto \bar{F}(x) := p \circ F(x)
\]

where \( p : \text{Cl}^0(M, E) \to \text{Cl}^0(M, E)/\text{Cl}^{-\infty}(M, E) \) is the canonical projection map.

The following theorem builds from the original forms \( \omega^Q_{2j} \) new “renormalised” forms which are closed in contrast with the original ones.
Theorem 7 When \( \sigma(X) \) is contractible, there are even forms \( \theta_{2j}^{Q} \) such that
\[
\omega_{2j}^{\text{ren}, Q} := \omega_{2j}^{Q} - \theta_{2j}^{Q}
\]
is closed. The forms \( \theta_{2j}^{Q} \) vanish when the order of \((dF)^{2j+1}\) is less than \(-\dim M\). This holds in particular if the order of \((dF)^{2j}\) is less than \(-\dim M\) in which case \( \omega_{2j}^{\text{ren}, Q} = \omega_{2j}^{Q} = \tr(F(dF)^{2j}) \) is independent of \( Q \).

**Proof:** The form \( d\omega_{2j}^{Q} \) being a Wodzicki residue, it is insensitive to smoothing perturbations and is therefore a pull-back by the projection map \( p \) of a form \( \beta_{2j}^{Q} \). The pull-back of \( \beta_{2j}^{Q} \) with respect to \( \sigma \) is a closed form \( \theta_{2j+1}^{Q} \) on \( X \) which is exact since \( \sigma \) is contractible. Indeed, selecting a contraction \( \sigma_t \) with \( \sigma_1 = \sigma \) and \( \sigma_0 = \) a constant map, we have the standard formula
\[
d\theta_{2j}^{Q} = \theta_{2j+1}^{Q},
\]
with
\[
\theta_{2j} = \frac{1}{2j+1} \int_0^1 t^{2j} \iota_X \theta_{2j+1}^{Q}(\sigma_t) dt.
\]
where \( \iota_X \) is the contraction by a vector field \( X \) and the dot means differentiation with respect to the parameter \( t \).

When the order of \((dF)^{2j+1}\) is less than \(-\dim M\) the correction terms \( \theta_{2j}^{Q} \) vanish and if the order of \((dF)^{2j}\) is less than \(-\dim M\), the weighted trace \( \tr^{Q} \) coincides with the usual trace so that the naive expression \( \omega_{2j}^{Q} \) is a closed form independent of \( Q \).

This way, one builds renormalised Chern classes \([\omega_{2j}^{\text{ren}, Q}]\). We refer the reader to [MP] for the two form case which arises in the quantum field theory gerbe [CMM].

10 Regular Chern-Weil forms in infinite dimensions

We describe further geometric setups for which weighted traces actually do give rise to closed Chern-Weil forms.

Mapping groups studied by Freed [F] and later further investigated in e.g. [CDMP], [M], [MRT] provide a first illustration of such a situation.

Going back to Example 9, we specialise to the circle \( M = S^1 \); the Sobolev based loop group \( H_{\infty}^{\mathbb{C}}(S^1, G) \) can be equipped with a complex structure and its first Chern form was studied by Freed [F]. We saw in Example 17 that the corresponding curvature is conditionally trace-class which leads to the following result.

**Proposition 11** [F] Theorem 2.20 (see also [CDMP] Proposition 3) Let \( Q_0 \in \text{Cl}(S^1) \) be an admissible elliptic operator on \( S^1 \) with spectral cut \( \theta \). Let \( \nabla^\ast \) be a left-invariant connection on \( C^\infty(S^1, G) \) with curvature given by a two form \( \Omega^\ast_0 \) as in (18), then by (6) we have
\[
\tr^{Q_0}(\Omega^\ast_0) = \tr_{\text{cond}}(\Omega^\ast_0) = \text{TR}(\Omega^\ast_0).
\]
It defines a closed form which coincides with Freed’s conditioned first Chern form.

**Remark 23** It was observed by Freed in [F] that this weighted first Chern form \( \tr^{Q_0}(\Omega^\ast_0) \) relates to the Kähler form on the based loop group \( H_{\infty}^{\mathbb{C}}(S^1, G) \). See also [CDMP] for further interpretations of this two-form.

Another way around the obstructions described previously is to choose a weight \( Q \) and a connection \( \nabla \) such that the bracket \( [\nabla, \log_0 Q] \) vanishes; this can be achieved using superconnections, leading to a second geometric setup in which regularised traces do give rise to closed Chern-Weil forms.
**Definition 9** A superconnection (introduced by Quillen [Q], see also [B], [BGV]) on an admissible vector bundle $\pi, \mathcal{E}$ where $\pi : M \rightarrow X$ is a fibration of manifolds, adapted to a smooth family of formally self-adjoint elliptic $\psi$-dors $D \in C^\infty \left( X, \Cl^d(\mathcal{M}, \mathcal{E}) \right)$ with odd parity is a linear map $R$ acting on $\Omega(X, \pi_* \mathcal{E})$ of odd parity with respect to the $\mathbb{Z}_2$-grading such that:

$$R(\omega \cdot \sigma) = d\omega \wedge \sigma + (-1)^{|\omega|} \omega \wedge R(\sigma) \quad \forall \omega \in \Omega(X), \sigma \in \Omega(X, \pi_* \mathcal{E})$$

and

$$R_{[0]} := D,$$

where we have written $R = \sum_{i=0}^{\dim B} R_{[i]}$ and $R_{[i]} : \Omega^i(X, \mathcal{E}) \mapsto \Omega^{i+1}(X, \mathcal{E})$.

**Example 18** An admissible connection $\nabla$ as in (17) gives rise to a superconnection

$$R := \nabla + D.$$

The curvature of a superconnection $R$ is a $\psi$-do-valued form $R^2 \in \Omega^2(X, \Cl(\mathcal{M}, \mathcal{E}))$; it actually is a differential operator valued two form. Since $R^2 = D^2 + R^2_{[>0]}$, where $R^2_{[>0]}$ is a $\psi$-do-valued form of positive degree, just as $D^2$, $R^2$ is elliptic and admissible.

Following [Sc] and [PS2], we call a $\psi$-do-valued form $\omega = \sum_{i=0}^{\dim B} \omega_{[i]}$ with $\omega_{[i]} : \Omega^i(X, \pi_* \mathcal{E}) \mapsto \Omega^{i+1}(X, \pi_* \mathcal{E})$ elliptic, resp. admissible, resp. with spectral cut $\alpha$ whenever $\omega_{[0]} \in \Cl(X, \pi_* \mathcal{E})$ has these properties. We refer the reader to [PS2] for detailed explanations on this point.

Since $R^2_{[0]} = D^2$, $R^2$ is an elliptic $\psi$-do-valued form with spectral cut $\pi$, and hence an admissible $\psi$-do-valued form. Its complex powers and logarithm can be defined as for ordinary admissible $\psi$-dors. With these conventions, weighted traces associated with fibrations of $\psi$-do-algebras can be generalised to weights given by admissible $\psi$-do-valued forms such as the curvature $R^2$ of the superconnection. Along the lines of the proof of the previous theorem one can check that the trace defect $[R, \str R^2]$ vanishes:

**Proposition 12** [PS2] For any $\omega \in \Omega(X, \Cl(\mathcal{E}))$ we have

$$d \str R^2(\omega) = \str R^2([R, \omega]).$$

**Proof:** Since

$$d \str R^2(\omega) = \str R^2([R, \omega]) + [R, \str R^2](\omega)$$

this follows from Theorem 6 combined with the fact that $[R, \log R^2] = 0$. □

**Theorem 8** The form $\str R^2(\mathcal{R}^{2j})$ defines a closed form called the $j$-th Chern form associated with the superconnection $R$ and a de Rham cohomology class independent of the choice of connection.

**Proof:** This follows from the Bianchi identity $[R, \mathcal{R}^{2j}] = 0$ combined with Proposition 12. □

**Remark 24** Let $R = D + \nabla$ be a superconnection associated with a family of Dirac operators $D$ on a trivial fibration of manifolds. It was observed in [MP] that the expression

$$\tr D^j(\nabla^{2j}) - \tr R^2(\mathcal{R}^{2j})_{[2j]}$$

-which compares the naive infinite dimensional analog $\tr D^j(\nabla^{2j})$ of the finite dimensional Chern form $\tr(\nabla^{2j})$ and the closed form $\tr R^2(\mathcal{R}^{2j})_{[2j]}$ built from the super connection- is local in as far as it is insensitive to smoothing perturbations of the connection. The weighted Chern-Weil form $\tr R^2(\mathcal{R}^{2j})_{[2j]}$ is therefore interpreted as a renormalised version of $\tr D^j(\nabla^{2j})$. This is similar to the formula derived in the previous section where a residue correction term was added to the naive weighted form involving the curvature.
If we specialise to a fibration \( \pi : M \to X \) of even-dimensional closed spin manifolds with the Bismut superconnection \( \mathbf{R} := D + \nabla + c(T), \) \( c \) the Clifford multiplication and \( T \) the curvature of the horizontal distribution on \( M, \) we get an explicit description of the \( j \)-th Chern form \( \str^{R^2}(R^{2j}). \)

Indeed, as a consequence of the local index theorem for families [B] (see also [BGV]), the component of degree \( 2j \) of the form \( \str^{R^2}(R^{2j}) \) can be expressed in terms of the \( \hat{A} \)-genus \( \hat{A}(M/X) \) on the vertical fibre of \( M \to X \) and the Chern character \( \ch(E_{M/X}) \) on the restriction of \( E \) to the vertical fibre.

\textbf{Theorem 9 [MP]} Let \( \mathbf{R} \) be a superconnection adapted to a family of Dirac operators on even dimensional spin manifolds parametrised by \( X, \) then

\[
\str^{R^2}(R^{2j}) = \left( \frac{(-1)^{j!}}{(2\pi)^{n/2}} \int_{M/X} \hat{A}(M/X) \wedge \ch(E_{M/B}) \right)_{[2j]}.
\]

\textbf{Proof:} Along the lines of the heat-kernel proof of the index theorem (see e.g. [BGV]) we introduce the kernel \( k_\epsilon(R^2) \) of \( e^{-\epsilon R^2} \) for some \( \epsilon > 0. \) Since \( D \) is a family of Dirac operators, we have (see e.g. chap. 10 in [BGV])

\[
k_\epsilon(R^2)(x,x) \sim \epsilon \to 0 \frac{1}{(4\pi \epsilon)^{n/2}} \sum_{j=0}^{\infty} \epsilon^j k_j(R^2)(x,x).
\]

We observe that the \( j \)-th Chern form associated with \( \mathbf{R} \) is given by an integration along fiber of \( M:\)

\[
\str^{R^2}(R^{2j}) = \left( \frac{(-1)^{j!}}{(4\pi)^{n/2}} \int_{M/B} \str(k_j + \frac{n}{2}(R^2)) \right)
\]

and proceed to compute \( \str(k_j + \frac{n}{2}(R^2)). \)

Let us introduce Getzler’s rescaling which transforms a homogeneous form \( \alpha[i] \) of degree \( i \) to the expression

\[
\delta_t \cdot \alpha[i] \cdot \delta_t^{-1} = \frac{\alpha[i]}{\sqrt{t}},
\]

so that a superconnection \( \mathbf{R} = R[0] + R[1] + R[2] \) transforms to

\[
\tilde{R}_t = \delta_t \cdot \mathbf{R} \cdot \delta_t^{-1} = R[0] + \frac{R[1]}{\sqrt{t}} + \frac{R[2]}{t}.
\]

As in [BGV] par. 10.4, in view of the asymptotic expansion (35) we have:

\[
\ch(R_t) = \delta_t \left( \str(e^{-t R^2}) \right)
\]

\[
\sim t \to 0 \left( 4\pi t \right)^{-\frac{n}{2}} \sum_j t^j \int_{M/B} \delta_t \left( \str(k_j(R^2)) \right)
\]

\[
\sim t \to 0 \left( 4\pi \right)^{-\frac{n}{2}} \sum_{j,p} t^{j-(n+p)/2} \left( \int_{M/B} \str(k_j(R^2)) \right)_{[p]},
\]

so that

\[
\lim_{t \to 0} \ch(R_t) = \left( 4\pi \right)^{-\frac{n}{2}} \left( \sum_{j,p} \int_{M/B} \str(k_{j+p}(R^2)) \right)_{[p]}
\]

\textbf{Theorem 10} [BGV] yields the existence of the limit as \( t \to 0 \) and

\[
\lim_{t \to 0} \ch(R_t) = (2i\pi)^{-\frac{n}{2}} \int_{M/B} \hat{A}(M/B) \wedge \ch(E_{M/B}).
\]
Combining these two facts leads to:

\[
\left( \int_{\mathcal{M}/B} \text{str} \left( \frac{k_{2j+2}}{2\pi} (\mathbb{R}^2) \right) \right)_{[2j]} = \frac{(4\pi)^{\frac{n}{2}}}{(2i\pi)^{\frac{n}{2}}} \left( \int_{\mathcal{M}/B} \hat{A}(\mathcal{M}/B) \wedge \text{ch}(\mathcal{E}_{\mathcal{M}/B}) \right)_{[2j]}.
\]

It follows that

\[
\text{str} \mathbb{R}^2 (\mathbb{R}^{2j})_{[2j]} = \frac{(-1)^{j!}}{(2i\pi)^{\frac{n}{2}}} \left( \int_{\mathcal{M}/B} \hat{A}(\mathcal{M}/B) \wedge \text{ch}(\mathcal{E}_{\mathcal{M}/B}) \right)_{[2j]}.
\]

\(\square\)

As could be expected, \(\mathbb{R}^2\)-weighted Chern forms therefore relate to the Chern character \((2i\pi)^{\frac{n}{2}} \int_{\mathcal{M}/B} \hat{A}(\mathcal{M}/B) \wedge \text{ch}(\mathcal{E}_{\mathcal{M}/B})\) of a family of Dirac operators associated with the fibration \(\mathcal{M} \rightarrow X\) [BGV].
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