Ultra-ligh dark photon and Casimir effect

Abdaljalel Alizzi
Novosibirsk State University, Novosibirsk 630 090, Russia

Z. K. Silagadze
Budker Institute of Nuclear Physics and Novosibirsk State University, Novosibirsk 630 090, Russia

We investigate the influence of a dark photon on the Casimir effect and calculate the corresponding leading contribution to the Casimir energy. For expected magnitudes of the photon - dark photon mixing parameter, the influence turns out to be negligible. The plasmon dispersion relation is also not noticeably modified by the presence of a dark photon.

I. INTRODUCTION

The true nature and composition of the dark matter is currently unknown, which remains one of the most significant unsolved problems in modern physics. Dark photons (also known as hidden, para-, or secluded photons) are well-motivated dark matter candidates [1]. Dark photon is the gauge boson of the additional $U(1)$ gauge group beyond the Standard Model. What makes such extra $U(1)$ gauge factors of particular phenomenological interest is that they are a generic feature of string theory [2].

Dark photons can kinetically mix with the Standard Model photon [3]. This makes it possible to search for them in astrophysical and cosmological observations and in laboratory experiments [1, 4]. Below the electroweak scale, the minimum coupling between a dark photon and particles of the visible sector (Standard Model particles) can be described by the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_b^2 A_{\mu\nu} A^{\mu\nu} - \frac{\epsilon}{2} F_{\mu\nu} F_{\mu\nu} - e a J_{\mu} A^{\mu} - e_b \tilde{J}_{\mu} A^{\mu},$$  (1)

where the gauge boson $A_\mu^a$ couples to the ordinary electromagnetic current $J_{\mu}$, $A_\mu^b$ couples to the dark sector analog of the electromagnetic current $\tilde{J}_\mu$ (if any), and $\epsilon$ is the dimensionless kinetic mixing parameter, which in some scenarios can be as large as $10^{-3} - 10^{-2}$ [3]. The dark boson mass, $m_b$, can arise, for example, by St"uckelberg mechanism [6].

The Lagrangian (1) can be diagonalized by introducing the physical fields (mass eigenstates) $A_{\mu}$ and $\tilde{A}_{\mu}$:

$$A_{\mu\nu} = A_{\mu} - \frac{\epsilon}{\sqrt{1 - \epsilon^2}} \tilde{A}_{\mu}, \quad A_{\mu\nu} = \frac{1}{\sqrt{1 - \epsilon^2}} \tilde{A}_{\mu},$$  (2)

in terms of which the Lagrangian (1) takes the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} \frac{m_b^2}{1 - \epsilon^2} \tilde{A}_{\mu} \tilde{A}^{\mu} - e a J_{\mu} A^{\mu} - \frac{1}{\sqrt{1 - \epsilon^2}} \left( e_b \tilde{J}_{\mu} - e_a J_{\mu} \right) \tilde{A}^{\mu}.$$  (3)

As we see, the dark photon $\tilde{A}_{\mu}$ can interact with ordinary charged particles. If the dark photon mass $\mu = \frac{m_b}{\sqrt{1 - \epsilon^2}}$ is very small (as we assume), then the measured Coulomb force between two charged particles will be $F_C = \frac{e^2}{4\pi \epsilon r^2} \left( 1 + \frac{\epsilon^2}{1 - \epsilon^2} \right) = \frac{\epsilon^2}{4\pi \epsilon r^2} \frac{1}{1 - \epsilon^2}$. Therefore the measured electric charge is $e = \frac{e}{\sqrt{1 - \epsilon^2}}$, and (3) expressed in terms of the renormalized quantities (the charge $e$ and the mass $\mu$) reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} \mu^2 \frac{e^2}{1 - \epsilon^2} \tilde{A}_{\mu} \tilde{A}^{\mu} - e J_{\mu} \left( \sqrt{1 - \epsilon^2} A^{\mu} - \epsilon \tilde{A}^{\mu} \right) - \frac{e_b}{\sqrt{1 - \epsilon^2}} \tilde{J}_{\mu} \tilde{A}^{\mu}.$$  (4)

Such kind of phenomenological model (without $\tilde{J}_{\mu}$, and without any connection with the vector bosons mixing) was first investigated by Okun [3], and consequences of the mixing of two abelian gauge bosons were first illustrated by Holdom [3].

*Electronic address: abdaljalec99@gmail.com
†Electronic address: Z.K.Silagadze@inp.nsk.su
The combination $A'_{\mu} = \sqrt{1-c^2} A_{\mu} - c \tilde{A}_{\mu}$, which has a direct coupling to the electromagnetic current, acts as an active photon. The orthogonal combination $\tilde{A}'_{\mu} = c A_{\mu} + \sqrt{1-c^2} \tilde{A}_{\mu}$ acts as a sterile photon, which has no direct coupling to the electromagnetic current. The transformation
\[
\begin{pmatrix}
A_{\mu} \\
\tilde{A}_{\mu}
\end{pmatrix} = \left( \begin{array}{cc}
\sqrt{1-c^2} & c \\
-c & \sqrt{1-c^2}
\end{array} \right) \begin{pmatrix}
A'_{\mu} \\
\tilde{A}'_{\mu}
\end{pmatrix} = \left( \begin{array}{cc}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{array} \right) \begin{pmatrix}
A'_{\mu} \\
\tilde{A}'_{\mu}
\end{pmatrix}
\]
(5)

is a rotation which leaves the kinetic terms in (3) diagonal, but induces a mass mixing between the rotated fields:
\[
\mathcal{L} = -\frac{1}{4} F'^{\mu\nu} F'^{\mu\nu} - \frac{1}{4} \tilde{F}^{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} \mu^2 \cos^2 \theta \left( \tilde{A}'_{\mu} - \tan \theta A_{\mu} \right) \left( \tilde{A}'_{\mu} - \tan \theta A_{\mu} \right) - e J_{\mu} A'^{\mu} - e_b \tilde{J}_{\mu} \left( \tilde{A}'_{\mu} - \tan \theta A_{\mu} \right).
\]
(6)

In terms of $A'_{\mu}$ and $\tilde{A}'_{\mu}$, the original $A_{a\mu}$ and $A_{b\mu}$ fields are expressed as follows
\[
A_{a\mu} = \frac{1}{\sqrt{1-c^2}} A'_{\mu}, \quad A_{b\mu} = \tilde{A}'_{\mu} - \frac{c}{\sqrt{1-c^2}} A'_{\mu}.
\]
(7)

In analogy with neutrino oscillations, $A'_{\mu}$ and $\tilde{A}'_{\mu}$ fields can be called flavor eigenstates, while $A_{\mu}$ and $\tilde{A}_{\mu}$ are mass eigenstates (propagating states).

Naively, one can expect that the resulting physics does not depend on the choice of the basis (2) or (7), and such a choice is a matter of convenience \cite{8, 9}. This is indeed the case in quantum mechanics. However, in quantum field theory, the situation is more subtle, since the Fock spaces for fields with definite mass and fields with definite flavor are not unitarily equivalent \cite{10, 12}.

In light of this inequivalence between mass and flavor vacua, it is interesting to ask how dark photons and the corresponding mixing phenomena affect the Casimir energy. At first glance, a natural choice for studying the effect of field mixing on the Casimir energy is the propagating basis (2) and the corresponding Lagrangian (4). In this note we make such a choice and try to investigate the influence of photon-dark photon mixing on the Casimir energy. Natural units ($\hbar = 1, c = 1$) are assumed, unless $\hbar$ and/or $c$ are explicitly indicated in the equation. For electromagnetic quantities we use Lorentz-Heaviside units.

II. CASIMIR EFFECT

Casimir effect is a fascinating phenomenon (Pauli initially dismissed it as “absolute nonsense” \cite{13}) in which two electrically neutral, parallel metal plates, spaced a short distance from each other, experience a force of mutual attraction. After its first discovery by Hendrik Casimir \cite{14}, there is still debate about whether the Casimir effect is a manifestation of the reality of zero-point quantum fluctuations of the electromagnetic vacuum or is it just the relativistic, retarded van der Waals force between the metal plates \cite{15}. Casimir himself, when asked the same question twice with an interval of eleven years, whether the Casimir effect is the result quantum fluctuations of the electromagnetic field, or is it caused by van der Waals forces between molecules in two objects, replied that he has not made up his mind (see foreword by I.H. Brevik in \cite{16}).

Nevertheless, since pioneering works by Casimir and Polder, the normal-mode expansion of a quantized electromagnetic field inside a cavity with suitable boundary conditions has been widely used to study dispersion interactions \cite{17} to which the Casimir force belongs. The picture of a fluctuating quantum vacuum, altered by the presence of an object, has proven to be very useful for predicting new and interesting effects such as the quantum atmosphere effect \cite{18} and the dissipationless rotation-induced axial Casimir force that emerges when a particle rotates above a plate that exhibits either time-reversal symmetry breaking or parity-symmetry breaking \cite{19}. Therefore, we consider it appropriate to remind the spirit of the Casimir's original derivation \cite{14}.

To quantize the electromagnetic field, a large cubic box of size $L$ with periodic boundary conditions is usually considered. Then the wave vector $\vec{k}$ of the electromagnetic field is quantized inside the box:
\[
k_x = \frac{2\pi}{L} n, \quad k_y = \frac{2\pi}{L} m, \quad k_z = \frac{2\pi}{L} l, \quad n, m, l = 0, \pm 1, \pm 2, \ldots
\]
(8)

Zero-point energy of the fluctuating electromagnetic vacuum inside the box is
\[
W_L = \frac{1}{2} \sum_{\vec{k}, \sigma} \hbar \omega_{\vec{k}} = \hbar c \sum_{\vec{k}} \sqrt{k_x^2 + k_y^2 + k_z^2} \xrightarrow{L \to \infty} \hbar c \frac{L^3}{(2\pi)^3} \int d\vec{k} \sqrt{k_x^2 + k_y^2 + k_z^2},
\]
(9)
where $\sigma$ labels two photon polarizations, and for $L \to \infty$ we can replace the sum by the integral according to the rule
\[
\frac{1}{L^3} \sum_k \to \int \frac{d\vec{k}}{(2\pi)^3}
\]
since the number of quantized $\vec{k}$-states per $d\vec{k}$ according to (9) is
\[
\frac{dN}{d\vec{k}} = \frac{dn}{dk_x} \frac{dl}{dk_y} \frac{dl}{dk_z} = \frac{L^3}{(2\pi)^3}.
\]

If two ideally conducting $L \times L$ plates are inserted at $z = 0$ and $x = R$ in the previous construction, the vacuum energy will change, since the plates impose Dirichlet boundary conditions at their locations, and, correspondingly, the quantization conditions of the zero-point electromagnetic field between the plates will change to
\[k_x = \frac{\pi}{L} n, \quad k_y = \frac{2\pi}{L} m, \quad k_z = \frac{2\pi}{L} l, \quad n = 0, 1, 2, \ldots, \ m, l = 0, \pm 1, \pm 2, \ldots\] (10)

As before, we can replace the summations over $k_y$ and $k_z$ with the corresponding integrals in the limit $L \to \infty$. However, the $k_x$ sum should be kept as the $R$ distance between the plates remains finite.

There is one subtlety when considering this sum (see, for example, [20]). As already mentioned, in the general case we have two independent polarizations of an electromagnetic wave. The corresponding modes are called transverse electric (TE) and transverse magnetic (TM), respectively. In TE mode, the electric field is parallel to the plates, and in TM mode, the magnetic field is parallel to the plates. There is no TE mode that corresponds to $n = 0$, since in this case the boundary conditions will force the electric field between the plates to be identically zero.

Therefore, for the zero-point energy of the fluctuating electromagnetic field between the conducting plates, we have
\[
W_R = \frac{\hbar c L^2}{(2\pi)^2} \sum_{n=(0)1}^{\infty} \int \int d\vec{k}_y \int d\vec{k}_z \sqrt{k_y^2 + k_z^2 + \left(\frac{\pi n}{R}\right)^2},
\]

where the notation $\sum_{n=(0)1}^{\infty}$ means that the term $n = 0$ in the sum must be multiplied by $1/2$ [14].

When calculating the zero-point energy of the fluctuating electromagnetic field outside the conducting plates (but inside the quantization volume $L^3$) $W_{L-R}$, we must specify the boundary conditions in the $x$-direction. We have a node at $x = R$ because there is a conducting plate here. At $x = L$ we also must have a node due to the assumed periodicity in $L$, since at $x = 0$ we also have a conducting plate. Therefore, when calculating $W_{L-R}$, we must take Dirichlet boundary conditions in the $x$-direction, not the periodic boundary condition [21]. However, in the $L \to \infty$ limit this difference does not matter. Indeed, under the Dirichlet boundary conditions, the continuum limit corresponds to the replacement
\[
\frac{1}{L-R} \sum_{n=0}^{\infty} \to \int_0^{\infty} \frac{dk_x}{\pi},
\]

and we will have
\[
W_{L-R} = \frac{\hbar c L^2}{(2\pi)^2} \frac{L^2}{2} (L-R) \int_0^{\infty} dk_x \int_0^{\infty} dk_y \int_0^{\infty} dk_z \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\hbar c}{(2\pi)^3} L^2 (L-R) \int d\vec{k} \sqrt{k_x^2 + k_y^2 + k_z^2}.
\]

Note that in Casimir’s original paper [14] (and late in some textbooks, see e.g. [22]), the Dirichlet boundary conditions are used in all three dimensions (assuming the volume $L^3$ bounded by perfectly conducting walls). However, more natural in the spirit of field quantization procedure in quantum field theory is the assumption of periodic boundary conditions in the $y$- and $z$-directions [21, 23]. Of course, the final result does not depend on this choice.

Only the difference $\Delta W = W_R + W_{L-R} - W_L$, which shows how the energy of electromagnetic zero-point fluctuations changes due to the introduction of conducting plates, has a definite physical meaning [14]. According to (9), (11) and (12), this difference is
\[
\Delta W = \frac{\hbar c L^2}{(2\pi)^2} \int \int d\vec{k}_y \int d\vec{k}_z \left\{ \sum_{n=(0)1}^{\infty} \sqrt{k_y^2 + k_z^2 + \left(\frac{\pi n}{R}\right)^2} - R \frac{2}{\pi} \int_0^{\infty} \int d\vec{k}_x \sqrt{k_x^2 + k_y^2 + k_z^2} \right\}.
\]

Introducing dimensionless variables $x$ and $\nu$ through
\[
k_y^2 + k_z^2 = k_\perp^2 = \left(\frac{\pi}{R}\right)^2 x, \quad k_x = \frac{\pi}{R} \nu, \quad d\vec{k}_y dq_x dq_z = 2\pi k_\perp dq_\perp = \left(\frac{\pi}{R}\right)^2 dx,
\]

(14)
we get
\[
\Delta W = \frac{\pi^2 \hbar c L^2}{4 R^3} \int_0^\infty dx \left\{ \sum_{n=0}^\infty \sqrt{x+n^2} \lambda \left( \frac{\pi}{k P R} \sqrt{x+n^2} \right) \right. - \frac{1}{2} \int_{-\infty}^\infty d\nu \sqrt{x+\nu^2} \lambda \left( \frac{\pi}{k P R} \sqrt{x+\nu^2} \right) \left\}, \tag{15}
\]
where, to take into account that real conductors are effectively transparent above a certain frequency \( \omega_p \) (plasma frequency of the metal) and, therefore, cannot provide ideal boundary conditions, we multiplied the contribution of each mode with the magnitude of the wave vector \( k_n = \sqrt{k_y^2 + k_z^2 + (\frac{\pi}{k P R})^2} = \frac{\pi}{R} \sqrt{x+n^2} \) by a cutoff function \( \lambda(k/k_P) \), with the cut-off wave vector \( k_P \sim \omega_p \), such that
\[
\lambda(x) = \begin{cases} 
1, & \text{if } x \ll 1, \\
0, & \text{if } x \gg 1.
\end{cases} \tag{16}
\]
Thanks to this cutoff function, everything nicely converges in (15) and we can calculate it using Euler-Maclaurin summation formula (see the appendix)
\[
\sum_{n=0}^\infty f(n) - \int_0^\infty f(x) \, dx = \frac{1}{2} f(0) - \sum_{n=2}^\infty \frac{B_n}{n!} \frac{d^{n-1} f(0)}{dx^{n-1}}, \tag{17}
\]
where \( B_n \) are Bernoulli numbers. To use this formula, we rewrite (15) as follows
\[
\Delta W = \frac{\pi^2 \hbar c L^2}{4 R^3} \left\{ \sum_{n=0}^\infty f(n) - \frac{1}{2} \int_{-\infty}^\infty d\nu f(\nu) \right\} = \frac{\pi^2 \hbar c L^2}{4 R^3} \left\{ -\frac{1}{2} f(0) + \sum_{n=0}^\infty f(n) - \int_0^\infty d\nu f(\nu) \right\}, \tag{18}
\]
where
\[
f(\nu) = \int_0^\infty dx \sqrt{x+\nu^2} \lambda \left( \frac{\pi}{k P R} \sqrt{x+\nu^2} \right) = \int_{\nu^2}^\infty dy \sqrt{y} \lambda \left( \frac{\pi}{k P R} \sqrt{y} \right). \tag{19}
\]
To get (18) from (15), we have interchanged the order of summation and integrals, which is justified due to the absolute convergence in the presence of the cutoff function \( \lambda \). [24]

Using the Leibniz rule for differentiating an integral (for generalizations of this useful formula to more than one dimension, see [25])
\[
\frac{d}{dt} \left( \int_{g(t)}^{h(t)} F(x,t) \, dx \right) = F[h(t),t] \frac{dh}{dt} - F[g(t),t] \frac{dg}{dt} + \int_{g(t)}^{h(t)} \frac{\partial F(x,t)}{\partial t} \, dx,
\tag{20}
\]
we get
\[
\frac{df(0)}{d\nu} = \left[ -2\nu^2 \lambda \left( \frac{\pi}{k P R} \nu \right) \right]_{\nu=0} = 0,
\]
\[
\frac{d^2 f(0)}{d\nu^2} = \left[ -4\nu \lambda \left( \frac{\pi}{k P R} \nu \right) - 2\nu^2 \frac{\pi}{k P R} \lambda' \left( \frac{\pi}{k P R} \nu \right) \right]_{\nu=0} = 0,
\]
\[
\frac{d^3 f(0)}{d\nu^3} = \left[ -4\lambda \left( \frac{\pi}{k P R} \nu \right) - 8\nu \frac{\pi}{k P R} \lambda' \left( \frac{\pi}{k P R} \nu \right) - 2\nu^2 \left( \frac{\pi}{k P R} \right)^2 \lambda'' \left( \frac{\pi}{k P R} \nu \right) \right]_{\nu=0} = -4.
\tag{21}
\]
It is clear that when \( k \gg 3, \frac{d^k f(0)}{d\nu^k} \) is proportional to \( \left( \frac{\pi}{k P R} \right)^{k-3} \) and can be neglected as far as \( k P R \gg 1 \). Therefore, applying the Euler-Maclaurin summation formula in (15), we recover the well-known result
\[
\Delta W = \frac{\pi^2 \hbar c L^2}{R^3} \frac{B_4}{4!} = -\frac{\pi^2 \hbar c L^2}{720 R^3}, \tag{22}
\]
Since the pioneering work of Casimir \([14]\), many other methods were proposed for solving the problem of zero-point fluctuations in the presence of boundaries that can alter the spectrum of zero-point oscillations (see, for example, \([16, 19, 26–29]\) and references therein).

Casimir force \(F = -\frac{\partial \Delta W}{\partial R}\) that follows from \([22]\) depends only on the fundamental constants \(\hbar\) and \(c\), and does not depend on the material constitution of metallic plates. It is this universal nature of the Casimir effect that made it so famous.

Remarkably, the Casimir force does not depend on the fine structure constant \(\alpha = e^2/(\hbar c)\). Naively, a dark photon, no matter how weakly it interacts with electrons, should double the Casimir force. However, the independence of the Casimir force from the fine structure constant is an illusion \([15]\), and this can be seen from our derivation above.

The derivation of \([22]\) assumes \(k_p R \gg 1\). But \(k_p = \omega_p/c\), where \(\omega_p\) is the plasma frequency of the metal, and by introducing Bohr radius \(a_B = \frac{\hbar}{m_e e}\), \(k_p R \gg 1\) condition can be rewritten in the following way

\[
\frac{\hbar \omega_p R}{\hbar c} = \frac{1}{\alpha a_B} \frac{\hbar \omega_p}{m_e c^2} \gg 1. \tag{23}
\]

For real metals, \(\hbar \omega_p \approx 10\ eV \ [30]\), and experiments to measure the Casimir force are usually done at separations of the order of \(R \approx 1\ \mu m\). For these numbers, the left-hand side of the inequality \(23\) is about 50, and the inequality is well satisfied. However, if we decrease the electromagnetic coupling constant \(\alpha\) (but keep the electron mass \(m_e\) intact), \(\alpha a_B = \hbar/(mc)\) will not change. However, we will see in the next section that \(\omega_p \sim \sqrt{\alpha n}\), where \(n \sim 1/\alpha^3\) is the number of conduction electrons per unit volume, and therefore \(\omega_p \sim \alpha^2\). As a result, the left-hand side of the inequality \(23\) decreases as \(\alpha^2\), and at some point the inequality will be impossible to satisfy. In fact, it was argued in \([15]\) that in the limit \(\alpha \to 0\) the Casimir force vanishes, like any other dynamical effect in quantum electrodynamics.

It is sometimes assumed \([24]\) that all derivatives of the cutoff function \(\lambda(x)\) vanish at the origin. Although such so-called flat functions do exist (for example, \(\lambda(x) = 1 - e^{-1/x^2}\)) and are frequently used in physics \([31]\), such a limitation of the permissible cutoff functions seems unnatural and devoid of physical motivation in the case of the Casimir effect.

### III. PLASMONS AND DARK PHOTON

To describe metals, we use the simple jellium model \([32]\), according to which valence electrons are detached from atoms, and the result is a plasma, consisting of a collection of electrons and ions, which, on average, is electrically neutral. Such a model, of course, does not take into account the periodicity of the crystal lattice of a true solid, but in some respects this model is a good approximation to a real metal, which is a crystalline aggregate consisting of small crystals of random orientation.

In addition, we use the self-consistent field approximation \([23, 33, 34]\). That is, we assume that electrons and ions interact with electrostatic potentials \(\phi(\vec{x})\) and \(\tilde{\phi}(\vec{x})\), which are themselves determined by the Poisson equations, which depend on the average charge densities in the plasma. Consequently, the dynamics of electrons and ions in plasma is determined by the (nonrelativistic) Hamiltonian \(23\)

\[
\hat{H} = \sum_{s=e,i} \int d\vec{x} \hat{\psi}_s^+ \left[ -\frac{\hbar^2}{2m}\nabla^2 + e_s \left( \sqrt{1 - e^2}\phi - e\tilde{\phi} \right) \right] \hat{\psi}_s = \hat{H}_0 + \hat{H}_I, \tag{24}
\]

where the \(s\) index was introduced to take into account various types of particles in the plasma (in our case, electrons and ions: \(e_e = -e, e_i = e\)). Expanding \(\hat{\psi}_s\) and \(\hat{\psi}_s^+\) in free particle wave functions

\[
\hat{\psi}_s = \sum_q b_{sq} e^{i\vec{q}\cdot\vec{x}} \sqrt{\Omega}, \quad \hat{\psi}_s^+ = \sum_q b_{sq}^* e^{-i\vec{q}\cdot\vec{x}} \sqrt{\Omega}, \tag{25}
\]

where \(\Omega\) is the quantization volume, we get

\[
\hat{H}_0 = \sum_{s,\vec{q}} \frac{\hbar^2 \vec{q}^2}{2m} b_{sq}^+ b_{sq}, \quad \hat{H}_I = \sum_{s,\vec{q}_1,\vec{q}_2} e_s b_{sq_1}^+ b_{sq_2} \phi'(\vec{q}_1 - \vec{q}_2, t), \quad \phi'(\vec{q}, t) = \int \frac{d\vec{x}}{\Omega} e^{-i\vec{q}\cdot\vec{x}} \left( \sqrt{1 - e^2}\phi(\vec{x}, t) - e\tilde{\phi}(\vec{x}, t) \right). \tag{26}
\]

The distribution function \(F_s(\vec{q}', \vec{q}, t)\) is obtained by quantum-mechanical and statistical averaging of the number operator \(b_{sq}^+ b_{sq}(t)\) \([23, 34]\):

\[
F_s(\vec{q}', \vec{q}, t) = \sum_{\alpha} P_s \langle \alpha | b_{sq}^+ (t) b_{sq}(t) | \alpha \rangle, \tag{27}
\]
where $P_\alpha$ is the probability of finding the system in the quantum state $\alpha$.

Using standard anti-commutation relations

$$\{b_{s_1q_1}, b^+_{s_2q_2}\} = \delta_{s_1s_2} \delta^{q_1q_2}, \quad \{b_{s_1q_1}, b_{s_2q_2}\} = 0, \quad \{b^+_{s_1q_1}, b^+_{s_2q_2}\} = 0,$$

and Heisenberg equations of motion

$$i\hbar \frac{\partial}{\partial t} b^+_{s q}(t)b_{s q}(t) = -[\hat{H}, b^+_{s q}(t)b_{s q}(t)],$$

we get

$$\frac{\partial}{\partial t} F_s(q', \bar{q}; t) = \frac{i}{\hbar} (E_{s q'} - E_{s q}) F_s(q', \bar{q}; t) +$$

$$\frac{i}{\hbar} e_s \sum_{\bar{q}'} [\phi' (\bar{q} - q', t) F_s(\bar{p}, \bar{q}', t) - \phi' (\bar{q} - \bar{p}, t) F_s(q', \bar{p}, t)],$$

where $E_{s q} = \frac{\hbar^2 q^2}{2m_s}$. This is a quantum-mechanical analogue of the Vlasov equation known in plasma physics.\[34\]

In equilibrium, the distribution functions $F_s$ do not depend on time, and the potential $\phi'$ vanishes (as does the charge density induced by the distribution functions $F_s$). Then (30) is reduced to $(E_{s q'} - E_{s q}) F_s(q', \bar{q}; t)$ with the solution $F_s(q', \bar{q}; t) = F_{s0}(q') \delta_{q\bar{q}}$. Small oscillations about this equilibrium are described by distribution functions \[23, 34\]

$$F_s(q', \bar{q}; t) = F_{s0}(q') b_{s q'} + F_{s1}(q', \bar{q}) e^{-i\omega t},$$

where $F_{s1}(q', \bar{q})$, like $\phi'(\bar{q}, t) = \phi'(\bar{q}) e^{-i\omega t}$, are assumed to be small quantities. We linearize (30) with respect to these small quantities and find the corresponding solution \[34\]

$$F_{s1}(q', \bar{q}) = -e_s \frac{F_{s0}(q) - F_{s0}(q')}{\hbar \omega - (E_{s q} - E_{s q'})} \phi'(\bar{q} - q').$$

For the average number densities, using decompositions \[25\] we get

$$\bar{n}_s(\bar{x}, t) = \sum_\alpha P_\alpha \langle \psi_s^+ (\bar{x}, t) \psi_s (\bar{x}, t) | \alpha \rangle =$$

$$\sum_{q, q'} F_s(q', \bar{q}, t) \frac{e^{i(\bar{q} - q') \cdot \bar{x}}}{\Omega} = \sum_{\bar{q}, \bar{p}} F_s(\bar{p}, \bar{q}, t) \frac{e^{i\bar{q} \cdot \bar{x}}}{\Omega},$$

where in the last step we just renamed the momentum variables for future convenience.

In the self-consistent field approximation, these average number densities should be used in the Poisson equations $(\mu = m_D/\hbar$, where $m_D$ is the dark photon mass): \[34\]

$$\Delta \phi(\bar{x}, t) = -\sum_s e_s \sqrt{1 - \mu^2} \bar{n}(\bar{x}, t), \quad (\Delta - \mu^2) \bar{\phi}(\bar{x}, t) = -\sum_s e_s \epsilon \bar{n}(\bar{x}, t).$$

The part of $\bar{n}(\bar{x}, t)$ due to $F_{s0}$ corresponds to the equilibrium with zero potentials $\phi$ and $\bar{\phi}$. This leads to the condition \[35\]

$$\frac{1}{\Omega} \sum_{s, \bar{p}} e_s F_{s0}(\bar{p}) = 0.$$

The part due to $F_{s1}$ corresponds to small oscillations near the equilibrium, and when substituted into (34), along with decompositions

$$\phi(\bar{x}, t) = \sum_q \phi(q) e^{i\bar{q} \cdot \bar{x} - i\omega t}, \quad \bar{\phi}(\bar{x}, t) = \sum_q \bar{\phi}(q) e^{i\bar{q} \cdot \bar{x} - i\omega t},$$

leads to equations

$$- \sum_q \bar{\phi}^2 (\bar{q}) = \sum_{s, \bar{q}, \bar{p}} \frac{e_s^2 \sqrt{1 - \mu^2}}{\Omega} \frac{F_{s0}(\bar{q} + \bar{p}) - F_{s0}(\bar{p})}{\hbar \omega - (E_{s, \bar{q}} + \bar{p} - E_{s, \bar{p}})} \phi'(\bar{q}),$$

$$- \sum_q (\bar{q}^2 + \mu^2) \bar{\phi}(q) = - \sum_{s, \bar{q}, \bar{p}} \frac{e_s^2 \Omega}{\hbar \omega - (E_{s, \bar{q}} + \bar{p} - E_{s, \bar{p}})} \phi'(q).$$
From this system of equations it is clear that

$$\sum_{\vec{q}} \left[ \epsilon \hat{q}^2 \phi(\hat{q}) + \sqrt{1 - \epsilon^2 (\hat{q}^2 + \mu^2)} \tilde{\phi}(\hat{q}) \right] = 0. \tag{38}$$

Therefore

$$\tilde{\phi}(\hat{q}) = -\frac{\epsilon}{\sqrt{1 - \epsilon^2}} \frac{\hat{q}^2}{\hat{q}^2 + \mu^2} \phi(\hat{q}), \tag{39}$$

and

$$\phi'(\hat{q}) = \sqrt{1 - \epsilon^2} \phi(\hat{q}) - \epsilon \tilde{\phi}(\hat{q}) = \frac{1}{\sqrt{1 - \epsilon^2}} \left[ 1 - \frac{\epsilon^2 \mu^2}{\hat{q}^2 + \mu^2} \right] \phi(\hat{q}) = \frac{\hat{q}^2 + \mu^2}{\epsilon \hat{q}^2} \left[ 1 - \frac{\epsilon^2 \mu^2}{\hat{q}^2 + \mu^2} \right] \tilde{\phi}(\hat{q}). \tag{40}$$

Taking these relations into account, the system \([37]\) can be rewritten as follows

$$\phi(\hat{q}) \epsilon(\hat{q}, \omega) \approx 0, \quad \tilde{\phi}(\hat{q}) \epsilon(\hat{q}, \omega) = 0, \tag{41}$$

where \(\epsilon(\hat{q}, \omega)\), the plasma dielectric function \([23]\), has the form

$$\epsilon(q, \omega) = 1 + \sum_{s,\vec{p}} \frac{\epsilon_s^2}{q^2 \Omega} \left[ 1 - \frac{\epsilon^2 \mu^2}{q^2 + \mu^2} \right] \frac{F_{s0}(q + \vec{p}) - F_{s0}(\vec{p})}{\hbar \omega - (E_{s,\vec{q}+\vec{p}} - E_{s,\vec{p}})}. \tag{42}$$

At this point it is convenient \([23]\) to let the quantization volume \(\Omega\) become infinite and replace \(F_{s0}(\vec{p})\) by the corresponding velocity distribution function \(f_{s0}(\vec{V})\), such that

$$\frac{1}{\Omega} \sum_{\vec{p}} F_{s0}(\vec{p}) = \int d\vec{p} (2\pi)^3 F_{s0}(\vec{p}) = \int f_{s0}(\vec{V}) \, d\vec{V}. \tag{43}$$

Since \(\vec{V} = \hbar \vec{p}/m\), we will have \(d\vec{V} = \hbar^3 d\vec{p}/m^3\) and therefore

$$f_{s0}(\vec{V}) = \frac{m^3}{(2\pi)^3 \hbar^3} F_{s0}(\vec{q}), \quad \frac{1}{\Omega} \sum_{\vec{p}} \rightarrow \int \frac{m^3}{(2\pi)^3 \hbar^3} d\vec{V}. \tag{44}$$

Besides

$$E_{s,\vec{q}+\vec{p}} - E_{s,\vec{p}} = \frac{m_s}{2} \left[ \left( \vec{V} + \frac{\hbar \vec{q}}{m_s} \right)^2 - \vec{V}^2 \right] = \frac{\hbar}{2} \left[ 2\vec{V} \cdot \vec{q} + \frac{\hbar \vec{q}^2}{m_s} \right], \tag{45}$$

and \([42]\) takes the form

$$\epsilon(\vec{q}, \omega) = 1 + \sum_s \frac{\epsilon_s^2}{\hbar q^2} \left[ 1 - \frac{\epsilon^2 \mu^2}{q^2 + \mu^2} \right] \int d\vec{V} \frac{f_{s0} \left( \vec{V} + \frac{\hbar \vec{q}}{m_s} \right) - f_{s0}(\vec{V})}{\omega - \vec{V} \cdot \vec{q} - \frac{\hbar \vec{q}^2}{2m_s}}. \tag{46}$$

In the classical limit \(\hbar \rightarrow 0\), and

$$f_{s0} \left( \vec{V} + \frac{\hbar \vec{q}}{m_s} \right) - f_{s0}(\vec{V}) \approx \sum_{i=1}^3 \frac{\hbar q_i}{m_s} \frac{\partial f_{s0}}{\partial V_i} = \frac{\hbar}{m_s} \vec{q} \cdot \nabla_f s0. \tag{47}$$

Therefore, in this limit,

$$\epsilon(\vec{q}, \omega) = 1 + \sum_s \frac{\epsilon_s^2}{m_s q^2} \left[ 1 - \frac{\epsilon^2 \mu^2}{q^2 + \mu^2} \right] \int d\vec{V} \frac{\vec{q} \cdot \nabla_f s0}{\omega - \vec{V} \cdot \vec{q}}. \tag{48}$$

For the Maxwellian velocity distribution function

$$f_{s0}(\vec{V}) = \frac{n_s}{\pi^{3/2} \alpha_s^3} e^{-V^2/\alpha_s^2}, \quad \alpha_s = \sqrt{\frac{2kT}{m_s}}, \tag{49}$$
where \( n_s = n \) is the density of \( s \)-type particles in the plasma assumed to be the same for electrons and ions, we have

\[
\mathbf{q} \cdot \nabla V f_{s0} = -\frac{2n}{\pi^3/2\alpha_s^3} \mathbf{V} \cdot \mathbf{q} e^{-\mathbf{q}^2/\alpha_s^2}.
\]

(50)

Assuming that \( z \)-axis is along \( \mathbf{q} \), and using

\[
\int_{-\infty}^{\infty} e^{-x^2/\alpha^2} \, dx = \alpha \sqrt{\pi}, \quad qV_z = -1 + \frac{\omega}{\omega - qV_z},
\]

we easily get

\[
\epsilon(\mathbf{q}, \omega) = 1 + \sum_s \frac{e^2}{m_s q_s^2} \left[ 1 - \frac{e^2 \mu_s^2}{q_s^2 + \mu_s^2} \right] \frac{2n}{\alpha_s^2} - \frac{2n \omega}{\sqrt{\pi} \alpha_s^2 q} Z(z_s),
\]

where

\[
q = |\mathbf{q}|, \quad z_s = \frac{\omega}{q \alpha_s}, \quad Z(z_s) = \int_{-\infty}^{\infty} e^{-x^2} \, dx.
\]

(53)

Plasmon oscillations (Langmuir waves) correspond to the situation when \( z_s \gg 1 \). Since \( z_i/z_e = \alpha_e/\alpha_i = \sqrt{m_i/m_e} \), \( z_e \gg 1 \) necessarily implies \( z_i \gg 1 \), and we can use the following asymptotic expansion of the Fried-Conte function \( Z(z_s) \):

\[
Z(z_s) \approx \frac{1}{z_s} \int_{-\infty}^{\infty} e^{-x^2} \left( 1 + \frac{x}{z_s} + \frac{x^2}{z_s^2} + \frac{x^3}{z_s^3} + \frac{x^4}{z_s^4} \right) dx = \sqrt{\pi} \left[ \frac{1}{z_s} + \frac{1}{2z_s^3} + \frac{3}{4z_s^5} \right] .
\]

(54)

As a result, we get

\[
\epsilon(\mathbf{q}, \omega) = 1 - \sum_s \frac{e^2}{m_s q_s^2} \left[ 1 - \frac{e^2 \mu_s^2}{q_s^2 + \mu_s^2} \right] \frac{n}{\alpha_s^2} \left[ \frac{1}{z_s} + \frac{3}{2z_s^3} \right] = 1 - \sum_s \frac{\omega^2}{\omega_s^2} \left[ 1 - \frac{e^2 \mu_s^2}{q_s^2 + \mu_s^2} \right] \left[ 1 + \frac{3\alpha_s^2}{2\omega_s^2} q_s^2 \right] ,
\]

where

\[
\omega_{ps} = \sqrt{\frac{m_e e_i^2}{m_s}}
\]

(56)

are characteristic frequencies for electrons (\( s = e \)), and ions (\( s = i \)). Using \( \omega_{pe}^2 = \omega_{pi}^2 (m_e/m_i) \) and \( \alpha^2 = \alpha_e^2 (m_e/m_i) \), we finally get

\[
\epsilon(\mathbf{q}, \omega) = 1 - \left[ 1 - \frac{e^2 \mu_e^2}{q_e^2 + \mu_e^2} \right] \frac{\omega_{pe}^2}{\omega^2} \left[ \frac{m_e}{m_i} + \frac{3\omega_{pe}^2 \alpha_e^2 q_e^2}{2\omega^4} \left( 1 + \frac{m_e^2}{m_i^2} \right) \right].
\]

(57)

A similar result is obtained if instead of the Maxwellian velocity distribution function \( f_{s0} \) we use the Fermi-Dirac distribution function

\[
f_{s0}(\mathbf{V}) = \begin{cases} \frac{e^{-3V_{Fs}}}{\pi V_{Fs}}, & \text{if } V < V_{Fs} \\ 0, & \text{if } V > V_{Fs}. \end{cases}
\]

(58)

In this case the thermal velocity \( \alpha_e \) will be replaced by the Fermi velocity \( V_{Fe} \) and the final result will look like (for \( \epsilon = 0 \) this result was obtained in \([23, 33]\))

\[
\epsilon(\mathbf{q}, \omega) = 1 - \left[ 1 - \frac{e^2 \mu_e^2}{q_e^2 + \mu_e^2} \right] \frac{\omega_{pe}^2}{\omega^2} \left[ \frac{m_e}{m_i} + \frac{3\omega_{pe}^2 V_{Fe}^2 q_e^2}{5\omega^4} \left( 1 + \frac{m_e^3}{m_i^3} \right) \right].
\]

(59)

\[1 \] Remember that the \( e_s \) charge is expressed in Heaviside-Lorentz units. If \( e_s \) is expressed in Gaussian units common in plasma physics, we must replace \( e_s \) with \( \sqrt{4\pi\epsilon_e} \) in the formulas.
Equation (41) shows that $\phi(q)$ (as well as $\tilde{\phi}(q)$) does not vanish only when $\epsilon(q, \omega) = 0$. In the long-wavelength approximation ($\alpha^2 q^2 \ll \omega_{pe}^2$), an approximate solution of this equation has the form

$$\omega^2 \approx \left(1 - \frac{e^2 \mu^2}{q^2 + \mu^2} \right) \left(1 + \frac{m_e}{m_i} \right) \omega_{pe}^2 + \frac{3}{2} \frac{\left(1 + \frac{m_e^3}{m_i^2} \right)}{\left(1 + \frac{m_e}{m_i} \right)} \alpha^2 q^2. \quad (60)$$

As we see, for ultralight dark photon ($\mu^2 \ll q^2$), modification of the plasmon dispersion relation is negligible. In fact it is negligible for other values of $\mu^2$ also, since the natural magnitude of the mixing parameter $\epsilon$ is $\epsilon \ll 10^{-2} - 10^{-3}$.

The Fried-Conte function is defined in (53) by a singular integral, and its full specification requires a rule how to handle this singularity. Above we assumed that the integral is given by the Cauchy principal value. Landau argued [36] that a more correct way to deal with singularity in (53) is to assume that the frequency $\omega$ (and hence $z_s$) has an infinitesimal positive imaginary part: $z_s \to z_s + i\epsilon$. This imaginary part leads to the so-called Landau damping [37].

Using the Sokhotski-Plemelj formula

$$\lim_{\epsilon \to 0} \frac{1}{x \pm i\epsilon} = P \frac{1}{x} \mp i\pi\delta(x), \quad (62)$$

we see that in our case the damping is proportional to $e^{-z_s^2}$ and, thus, it can be neglected, since $z_s \gg 1$.

**IV. DARK LIGHT PENETRATION DEPTH IN METALS**

Euler-Lagrange equations that follow from the Lagrangian (41) are

$$\partial_\nu F^{\nu\mu} = -e\sqrt{1 - \epsilon^2} J^\mu, \quad \partial_\nu \tilde{F}^{\nu\mu} = \mu^2 \tilde{A}^\mu + e\epsilon J^\mu - \frac{eb}{\sqrt{1 - \epsilon^2}} \tilde{j}^\mu. \quad (63)$$

We assume Lorenz conditions $\partial_\mu A^\mu = \partial_\mu \tilde{A}^\mu = 0$ for $A^\mu$ and $\tilde{A}^\mu$ four-potentials. For $A^\mu$ this is just a gauge fixing, while for $\tilde{A}^\mu$ it follows from the corresponding Euler-Lagrange equation in (63), if the currents $J^\mu$ and $\tilde{j}^\mu$ are conserved and $\mu \neq 0$.

After the gauge fixing, the equations of motion (63) for fields $A^\mu$ and $\tilde{A}^\mu$ become

$$\Box A^\mu = e\sqrt{1 - \epsilon^2} J^\mu, \quad (\Box + \mu^2) \tilde{A}^\mu = -e\epsilon J^\mu + \frac{eb}{\sqrt{1 - \epsilon^2}} \tilde{j}^\mu. \quad (64)$$

Since electrons carry both the ordinary charge $-\sqrt{1 - \epsilon^2} e$ and the dark charge $\epsilon e$, Ohm’s law inside a metal with conductivity $\sigma$ takes the form [41]

$$eJ^\mu = \sigma \left( \sqrt{1 - \epsilon^2} \tilde{E} - \epsilon \tilde{E} \right). \quad (65)$$

Consider a transverse dark light plane wave propagating along the $z$ axis and entering a metal. We assume that the polarization of the electric field is along the $x$ axis, so that

$$A^\mu(x, t) = (0, A(z) e^{-i\omega t}, 0, 0), \quad \tilde{A}^\mu(x, t) = \left(0, \tilde{A}(z) e^{-i\omega t}, 0, 0 \right) \quad (66)$$

2 It seems, A.A. Vlasov was the first who anticipated the possibility of damping in a collisionless plasma [38–40].
Then
\[ \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi = (i\omega A(z) e^{-i\omega t}, 0, 0), \]
\[ \vec{\phi} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi = (i\omega \tilde{A}(z) e^{-i\omega t}, 0, 0), \]
and the system (64) takes the form (we have assumed that the dark current \( \tilde{J}^\mu \) vanishes inside the metal)
\[ (\partial_z^2 + \omega^2) A(z) = -i\sigma \omega \sqrt{1 - \epsilon^2} \left[ \sqrt{1 - \epsilon^2 A(z)} - \epsilon \tilde{A}(z) \right], \]
\[ (\partial_z^2 + \omega^2 - \mu^2) \tilde{A}(z) = i\sigma \omega \epsilon \left[ \sqrt{1 - \epsilon^2 A(z)} - \epsilon \tilde{A}(z) \right]. \]

The eigenmodes of this coupled system have the form \( A(z) = C e^{ikz}, \tilde{A}(z) = \tilde{C} e^{ikz} \), where the eigenvalues \( k \) are such that the homogeneous system
\[ \begin{align*}
[-k^2 + \omega^2 + i\sigma \omega(1 - \epsilon^2)] C - i\sigma \omega \epsilon \sqrt{1 - \epsilon^2} \tilde{C} &= 0, \\
-i\sigma \omega \epsilon \sqrt{1 - \epsilon^2} C + [-k^2 + \omega^2 - \mu^2 + i\sigma \omega^2] \tilde{C} &= 0. 
\end{align*} \]

has non-zero solutions for \( C, \tilde{C} \). Therefore, the determinant of the coefficient matrix of this system must be zero, and this condition leads to the equation
\[ (\omega^2 - k^2)^2 - [\mu^2 - i\sigma \omega] (\omega^2 - k^2) - i\sigma \omega \mu^2 (1 - \epsilon^2) = 0. \]

Consequently, the eigenvalues are
\[ k_1^2 = \omega^2 - \frac{1}{2} (\mu^2 - i\sigma \omega) + \frac{1}{2} \sqrt{(\mu^2 + i\sigma \omega)^2 - 4i\sigma \omega \epsilon^2 \mu^2}, \]
\[ k_2^2 = \omega^2 - \frac{1}{2} (\mu^2 - i\sigma \omega) - \frac{1}{2} \sqrt{(\mu^2 + i\sigma \omega)^2 - 4i\sigma \omega \epsilon^2 \mu^2}. \]

Since \( \epsilon \) is assumed to be small, we have approximately
\[ k_1^2 \approx \omega^2 + i\sigma \omega - \frac{i\sigma \omega \epsilon^2 \mu^2}{\mu^2 + i\sigma \omega} = \omega^2 \left( 1 - \frac{\sigma^2 \epsilon^2 \mu^2}{\mu^4 + \sigma^2 \omega^2} \right) + i\sigma \omega \left( 1 - \frac{\epsilon^2 \mu^4}{\mu^4 + \sigma^2 \omega^2} \right), \]
and
\[ k_2^2 \approx \omega^2 - \mu^2 + \frac{i\sigma \omega \epsilon^2 \mu^2}{\mu^2 + i\sigma \omega} = \omega^2 \left( 1 + \frac{\sigma^2 \epsilon^2 \mu^2}{\mu^4 + \sigma^2 \omega^2} \right) - \mu^2 + \frac{i\sigma \omega \epsilon^2 \mu^4}{\mu^4 + \sigma^2 \omega^2}. \]

The corresponding eigenvectors are
\[ \begin{pmatrix} A_1 \\ \hat{A}_1 \end{pmatrix} = C \begin{pmatrix} \frac{\sigma \omega}{\sigma \omega + i\mu^2} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} A_2 \\ \hat{A}_2 \end{pmatrix} = \tilde{C} \begin{pmatrix} -\frac{\sigma \omega}{\sigma \omega + i\mu^2} \\ 1 \end{pmatrix}. \]

Therefore, inside the metal \( A(z) \) and \( \hat{A}(z) \) fields propagate according to
\[ A(z) = Ce^{ikz} - \tilde{C} \frac{\sigma \omega \epsilon}{\sigma \omega + i\mu^2} e^{ikz}, \quad \hat{A}(z) = C - \frac{\sigma \omega \epsilon}{\sigma \omega + i\mu^2} e^{ikz} + \tilde{C} e^{ikz}. \]

Boundary conditions at \( z = 0 \) determine the coefficients \( C \) and \( \tilde{C} \). In particular, if the boundary conditions are \( A(0) = 0, A(0) = A_0 \), we get
\[ A(z) = \hat{A}_0 \frac{\sigma \omega \epsilon}{(\sigma \omega + i\mu^2)^2 + \sigma^2 \omega^2 \epsilon^2} e^{ikz} \approx \hat{A}_0 \frac{\sigma \omega \epsilon}{(\sigma \omega + i\mu^2)^2} [e^{ikz} - e^{-ikz}] \approx \hat{A}_0 e^{ikz}, \]
\[ \hat{A}(z) = \hat{A}_0 \frac{(-\sigma \omega + i\mu^2)^2}{(-\sigma \omega + i\mu^2)^2 + \sigma^2 \omega^2 \epsilon^2} \frac{\sigma \omega \epsilon}{\sigma \omega + i\mu^2} e^{ikz} \approx \hat{A}_0 [e^{ikz} + \frac{\sigma^2 \omega^2 \epsilon^2}{(-\sigma \omega + i\mu^2)^2} e^{ikz}] \approx \hat{A}_0 e^{ikz}. \]
and if the boundary conditions are \( A(0) = A_0 \), \( \tilde{A}(0) = 0 \), then

\[
A(z) = A_0 \frac{-(\sigma \omega + i \mu^2)^2}{(\sigma \omega + i \mu^2)^2 + \sigma^2 \omega^2 e^2} \left[ e^{ik_1z} + \frac{\sigma^2 \omega^2 e^2}{-(\sigma \omega + i \mu^2)^2} e^{ik_2z} \right] \approx A_0 e^{ik_1z},
\]

\[
\tilde{A}(z) = A_0 \frac{\sigma \omega e^{-(\sigma \omega + i \mu^2)^2}}{(\sigma \omega + i \mu^2)^2 + \sigma^2 \omega^2 e^2} \left[ e^{ik_1z} - e^{ik_2z} \right] \approx A_0 \frac{\sigma \omega e^{-(\sigma \omega + i \mu^2)^2}}{-(\sigma \omega + i \mu^2)^2} \left[ e^{ik_1z} - e^{ik_2z} \right].
\]

(77)

As we can see, up to the leading terms in the \( \epsilon \) expansion, if the incident plane wave is a pure dark light, ordinary vector-potential \( A \) in the metal will have the order of \( \epsilon^2 A \), and vice versa, if the incident plane wave is a pure ordinary light, the dark vector-potential \( \tilde{A} \) in metal will be of the order of \( \epsilon A \).

Equations (72), (73) show that \( k_1^2 \) and \( k_2^2 \) have positive imaginary parts. Then \( e^{ik_2z} \) decreases exponentially in the metal. If the imaginary part is large, penetration depth (the so-called skin depth \( 42, 43 \)) will be small. If \( k_2^2 = \rho e^{i\varphi} = \rho \cos \varphi + i \rho \sin \varphi = a + ib \), with \( \rho = |k^2| = \sqrt{a^2 + b^2} \), then \( k = \sqrt{\rho e^{i\varphi}/2} = \sqrt{\rho \cos(\varphi/2) + i \sqrt{\rho} \sin(\varphi/2)} = \alpha + i \beta \). In our case \( b > 0 \) and, therefore, \( \varphi < \pi \). Hence \( \cos(\varphi/2) = \sqrt{(1 + \cos \varphi)/2} \), \( \sin \varphi/2 = \sqrt{(1 - \cos \varphi)/2} \), and we get \( 42 \)

\[
\alpha = \sqrt{1/2 \left( \sqrt{a^2 + b^2 + a} \right)}, \quad \beta = \sqrt{1/2 \left( \sqrt{a^2 + b^2 - a} \right)}.
\]

(78)

For the first mode \( k_1^2 = a_1 + ib_1 \) with \( a_1 \approx \omega^2 \) and \( b_1 \approx \sigma \omega \). For good conductors \( \sigma \gg \omega \) \( 42 \). Therefore \( b_1 \gg a_1 \) and from (78) we get \( \beta_1 \approx b_1/2 \), which leads to the usual skin depth \( \delta_1 = 1/\beta_1 \approx \sqrt{2/\sigma \omega} \).

In the case of the second mode \( k_2^2 = a_2 + ib_2 \), if ultralight dark photon \( \mu^2 \ll \omega^2 \) is assumed, \( a_2 \approx \omega^2 \) and \( b_2 \approx \frac{\omega^4}{\sigma \omega^2} \ll a_2 \). Therefore, \( \beta_2 \approx \frac{\omega^4}{\sigma \omega^2} = \frac{\delta_2}{2 \sigma \omega^4} \). The corresponding penetration depth is enormous:

\[
\delta_2 = \frac{\sigma \omega^2}{\epsilon^2 \mu^4} = \frac{\sigma_1}{\epsilon^2} \left( \frac{\sigma}{\mu} \right) \left( \frac{\omega}{\mu} \right)^2 \sqrt{\frac{2 \sigma \omega}{\mu^2}} \gg \delta_1.
\]

(79)

Consequently, the necessary boundary conditions cannot be realized, and we conclude that the transverse polarizations of the ultralight dark photon do not make any significant contribution to the Casimir effect.

It remains to consider the propagation of the longitudinally polarized dark light. In this case we can take \( 41 \)

\[
A^\mu (\vec{x}, t) = \left( \phi(z) e^{-i\omega t}, 0, 0, A(z) e^{-i\omega t} \right),
\]

\[
\tilde{A}^\mu (\vec{x}, t) = \left( \tilde{\phi}(z) e^{-i\omega t}, 0, 0, \tilde{A}(z) e^{-i\omega t} \right).
\]

(80)

Taking into account the Lorenz conditions

\[
-i \omega \phi(z) + \frac{dA(z)}{dz} = 0, \quad -i \omega \tilde{\phi}(z) + \frac{d\tilde{A}(z)}{dz} = 0,
\]

(81)

we get for electric fields

\[
E_z = -\frac{\partial}{\partial t} \left( A(z) e^{-i\omega t} \right) - \frac{\partial}{\partial z} \left( \phi(z) e^{-i\omega t} \right) = -\frac{1}{i \omega} \left( \omega^2 + \partial_z^2 \right) A(z) e^{-i\omega t},
\]

\[
\tilde{E}_z = -\frac{\partial}{\partial t} \left( \tilde{A}(z) e^{-i\omega t} \right) - \frac{\partial}{\partial z} \left( \tilde{\phi}(z) e^{-i\omega t} \right) = -\frac{1}{i \omega} \left( \omega^2 + \partial_z^2 \right) \tilde{A}(z) e^{-i\omega t}.
\]

(82)

Then from (64), using (61) and assuming \( A(z) = C e^{ik_2}, \tilde{A}(z) = \tilde{C} e^{ik_2} \), we get the following system

\[
(k^2 - \omega^2) \left[ 1 + i \frac{\sigma}{\omega} (1 - \epsilon^2) \right] C - i \frac{\sigma}{\omega} \epsilon \sqrt{1 - \epsilon^2} \left( k^2 - \omega^2 \right) \tilde{C} = 0,
\]

\[
-i \frac{\sigma}{\omega} \epsilon \sqrt{1 - \epsilon^2} \left( k^2 - \omega^2 \right) C + \left[ (k^2 - \omega^2) \left( 1 + i \frac{\sigma}{\omega} \epsilon^2 \right) + \mu^2 \right] \tilde{C} = 0.
\]

(83)

\[3\] In plasma, photons acquire an effective mass \( 44 \), and therefore \( A^\mu \) in a metal can have a nonzero longitudinal component.
The corresponding eigenvalues are
\[
k_1 = \omega, \quad k_2 = \omega^2 - \mu^2 \left(1 - \frac{\epsilon^2 \sigma^2}{\sigma^2 + \omega^2}\right) + i\sigma \omega \frac{\mu^2 \epsilon^2}{\sigma^2 + \omega^2},
\]
(84)
with eigenvectors
\[
\begin{pmatrix} A_1 \\ \hat{A}_1 \end{pmatrix} = C \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} A_2 \\ \hat{A}_2 \end{pmatrix} = \tilde{C} \begin{pmatrix} i\sigma \sqrt{\frac{1-\epsilon^2}{2}} \\ 1 \end{pmatrix}.
\]
(85)
Consequently, for \(A(0) = 0, \hat{A}(0) = \hat{A}_0\) boundary conditions, inside the metal
\[
A(z) = \hat{A}_0 \frac{i\sigma \epsilon \sqrt{1-\epsilon^2}}{\omega + i\sigma(1-\epsilon^2)} (e^{ik_2z} - e^{-ik_1z}), \quad \hat{A}(z) = \hat{A}_0 e^{ik_2z}.
\]
(86)
In realistic case \(\sigma \gg \omega \gg \mu, k_2^2 \approx \omega^2 + i\sigma \omega \mu^2\) and (78) will give for the imaginary part of \(k_2\) the result \(\beta \approx \frac{\mu^2 \epsilon^2}{\sigma^2}\).
Therefore, the penetration depth for the longitudinal polarization is
\[
\delta_L = \frac{2\sigma}{\mu^2 \epsilon^2} = \delta_1 \left(\frac{\sigma}{\mu}\right) \sqrt{\frac{2\sigma \omega}{\mu^2}} \gg \delta_1.
\]
(87)
For an ultralight dark photon, \(\delta_L \ll \delta_2\). However, the penetration depth of longitudinal polarization is still too large for the presence of longitudinal polarization of a dark photon to have any appreciable effect on the Casimir force.

V. LEADING CONTRIBUTION OF ULTRALIGHT DARK PHOTON TO THE CASIMIR ENERGY

It follows from the previous discussion that the leading contribution of ultralight dark photon to the Casimir energy arises from the modification of the plasma dielectric function \(\sim \epsilon^2 \mu^2\) and its effect on the propagation of \(k_1^2 \approx \omega^2 + i\sigma \omega \mu^2\) transverse mode.

One might think that the contribution of the longitudinal mode is also of the same order, since its penetration depth is of the order of \(\sim \epsilon^2 \mu^2\). However, it was shown in [45] that in the small-mass case the contribution of longitudinal modes is actually of the order of \(\sim \mu^4\) and, thus, is insignificant in the present context.

To calculate the contribution to the Casimir energy (per unit area) due to modification of the plasma dielectric function, we use the Lifshitz formula at zero temperature as an integral over the imaginary frequencies, in the form given in [46, 47] (remember, by default we use natural units \(\hbar = 1, c = 1\):
\[
w = \frac{1}{8\pi^2} \int_0^\infty dk \int_{-\infty}^\infty d\xi \left[ \ln \Delta_1(k, i\xi) + \ln \Delta_2(k, i\xi) \right],
\]
(88)
where
\[
\Delta_1(k, \omega) = 1 - \frac{(K_\epsilon - \epsilon(\omega)K)^2}{(K_\epsilon + \epsilon(\omega)K)^2} e^{-2KR},
\]
\[
\Delta_2(k, \omega) = 1 - \frac{(K_\epsilon - K)^2}{(K_\epsilon + K)^2} e^{-2KR},
\]
with
\[
K^2 = k^2 - \omega^2, \quad K_\epsilon^2 = k^2 - \epsilon(\omega) \omega^2.
\]
(90)
According to [48], the plasma dielectric function at zero temperature has the form
\[
\epsilon(\vec{q}, \omega) = 1 - \left[1 - \frac{\epsilon^2 \mu^2}{q^2 + \mu^2}\right] \frac{\omega_{pe}^2}{\omega^2}.
\]
(91)
Taking \(q^2 = k_1^2 \approx \omega^2 + i\sigma \omega \approx i\sigma \omega\) in this formula, we get
\[
\epsilon(i\xi) = 1 + \left(\frac{\omega_{pe}}{\xi}\right)^2 + \chi \left(\frac{\omega_{pe}}{\xi}\right)^3 \frac{1 - \frac{\mu^2}{\sigma^2}}{1 - \frac{\mu^2}{\sigma^2}},
\]
(92)
where
\[ \chi = \frac{e^2 \mu^2}{\sigma \omega_{pe}} \ll 1 \]  
(93)
is a small parameter that determines the order of magnitude of the correction due to dark photons to the Casimir energy.

Consequently,
\[ w(\chi) \approx w(0) + w'(0) \chi, \]  
(94)
with
\[ w(0) = \frac{1}{4\pi^2} \int_0^\infty kdk \int_0^\infty d\xi \frac{\omega_{pe}}{\xi} \left[ \frac{1}{1 + \frac{\omega_{pe}}{\xi}} + \frac{1}{1 + \frac{\omega_{pe}}{-\xi}} \right] \frac{1}{\Delta_1} \frac{\partial \Delta_1}{\partial \epsilon} + \frac{1}{\Delta_2} \frac{\partial \Delta_2}{\partial \epsilon} \bigg|_{\chi=0}, \]  
(95)
and
\[ w'(0) = \frac{1}{8\pi^2} \int_0^\infty kdk \int_0^\infty d\xi \frac{\omega_{pe}}{\xi} \left[ \frac{1}{1 + \frac{\omega_{pe}}{\xi}} + \frac{1}{1 + \frac{\omega_{pe}}{-\xi}} \right] \frac{1}{\Delta_1} \frac{\partial \Delta_1}{\partial \epsilon} + \frac{1}{\Delta_2} \frac{\partial \Delta_2}{\partial \epsilon} \bigg|_{\chi=0}, \]  
(96)
where
\[ \frac{\partial \Delta_1}{\partial \epsilon} = \frac{2K(K_\epsilon - \epsilon K)(K_\epsilon^2 + k^2)}{K_\epsilon(K_\epsilon + \epsilon K)^3} e^{-2K_R}, \quad \frac{\partial \Delta_2}{\partial \epsilon} = \frac{2\epsilon^4 K(1 - \epsilon)}{K_\epsilon(K_\epsilon + K)^3} e^{-2K_R}. \]  
(97)

In [88], considered as a double integral, we make the change of variables
\[ \xi = \frac{x}{2pR}, \quad k^2 = \xi^2(p^2 - 1) = \frac{x^2}{4R^2} \left( 1 - \frac{1}{p^2} \right), \]  
(98)
with the Jacobian
\[ \frac{\partial(k^2, \xi)}{\partial(p, x)} = \left| \begin{array}{cc} \frac{\partial k^2}{\partial p} & \frac{\partial k^2}{\partial x} \\ \frac{\partial \xi}{\partial p} & \frac{\partial \xi}{\partial x} \end{array} \right| = \frac{x^2}{4p^2 R^3}. \]  
(99)

Then \( K = p\xi, \ K_\epsilon = s\xi \), with \( s = \sqrt{\epsilon - 1 + p^2} \), and we get
\[ w(0) = \frac{1}{32\pi^2 R^3} \int_0^\infty x^2 \, dx \int_1^\infty \frac{dp}{p^2} \left[ \ln \left( 1 - \frac{(s - pe)^2}{(s + pe)^2} e^{-x} \right) + \ln \left( 1 - \frac{(s - p)^2}{(s + p)^2} e^{-x} \right) \right] \bigg|_{\chi=0}. \]  
(100)

Note that
\[ \epsilon|_{\chi=0} = 1 + \frac{1}{\alpha^2}, \]  
(101)
where
\[ \alpha = \frac{\xi}{\omega_{pe}} = \frac{x}{2pR\omega_{pe}} = \frac{\delta_0}{R} \frac{x}{2p} = \frac{1}{\omega_{pe}}, \]  
(102)
contains a small parameter \( \delta_0/R \sim 0.1 \) (for metals \( \omega_{pe} \sim 10 \text{ eV} \), typical interplate separation in Casimir effect experiments is \( R \sim 10^{-4} \text{ cm} \) and we have used \( eV \cdot \text{cm} \approx 8066 \)). In addition, due to the factor \( e^{-x} \), the main contribution to the integral [100] comes from the region of small \( x \). Therefore, we can assume that \( \alpha \) is small in the integrand of [100], and expand (up to first order in \( \alpha \))
\[
\left( \frac{s - pe}{s + pe} \right)^2 \bigg|_{\chi=0} = \left( \frac{\sqrt{1 + \alpha^2 p^2} - \alpha p}{\sqrt{1 + \alpha^2 p^2} + \alpha p} \right)^2 = \left( \frac{\sqrt{1 + \alpha^2 p^2} - \alpha p}{\sqrt{1 + \alpha^2 p^2} + \alpha p} \right)^4 \approx 1 - 4\alpha p, 
\]
\[
\left( \frac{s - p}{s + p} \right)^2 \bigg|_{\chi=0} = \frac{\alpha \sqrt{1 + \alpha^2 p^2} - (\alpha^2 + 1) p}{\alpha \sqrt{1 + \alpha^2 p^2} + (\alpha^2 + 1) p} \approx \frac{(\alpha - p)}{\alpha + p} \approx 1 - \frac{4\alpha}{p}. 
\]  
(103)
Then (we substituted $\alpha = \delta_0 R$ in the final expressions below)

$$
\ln \left( 1 - \frac{(s - pe)^2}{(s + pe)^2} e^{-x} \right) \bigg|_{\chi=0} \approx - \ln \frac{e^x}{e^x - 1} + \frac{2}{e^x - 1} p^2 \frac{\delta_0}{R},
$$

$$
\ln \left( 1 - \frac{(s - p)^2}{(s + p)^2} e^{-x} \right) \bigg|_{\chi=0} \approx - \ln \frac{e^x}{e^x - 1} + \frac{2}{e^x - 1} \frac{\delta_0}{R} x,
$$

and we get

$$
w(0) = \frac{1}{32\pi^2 R^3} \int_0^\infty dx \int_1^\infty \frac{dp}{p^2} \left[ -2 \ln \frac{e^x}{e^x - 1} + \frac{2}{e^x - 1} \frac{\delta_0}{R} x \left( \frac{1}{p^2} + 1 \right) \right] = \frac{1}{32\pi^2 R^3} \int_0^\infty dx \left[ -2 \ln \frac{e^x}{e^x - 1} + \frac{1}{e^x - 1} \frac{\delta_0}{R} \frac{8x}{3} \right].
$$

Performing integration by parts in the first term containing the logarithm, and using the integral (see [48], entry 3.411.1)

$$
\int_0^\infty \frac{x^3}{e^x - 1} dx = \Gamma(4)\zeta(4) = \frac{\pi^4}{15},
$$

we finally get

$$
w(0) = -\frac{1}{48\pi^2 R^3} \left( 1 - 4 \frac{\delta_0}{R} \right) \int_0^\infty \frac{x^3}{e^x - 1} dx = -\frac{\pi^2}{720 R^3} \left( 1 - 4 \frac{\delta_0}{R} \right).
$$

The first term corresponds to the result (22), and the second term is the first order finite conductivity correction. It was calculated long ago (in 1961) by Dzyaloshinskiy, Lifshitz and Pitaevskii with an error in the numerical coefficient corrected by Hargreaves in 1965 (see [47]). Here we reproduce the correct coefficient just to gain a confidence in our calculational scheme.

Finite conductivity corrections were calculated up to the fourth order in [46] and up to the sixth order in [49]. All of them are experimentally more important than the putative dark photon contribution, which we will now estimate. Similarly to what was done above, to calculate (96) we expand the integrand in powers of $\alpha$, leaving only the leading term. Then equations (103) indicate that

$$
\Delta_1|_{\chi=0} \approx \Delta_2|_{\chi=0} \approx \frac{e^x - 1}{e^x},
$$

while

$$
\frac{\partial \Delta_1}{\partial \epsilon} \bigg|_{\chi=0} = \frac{2p(s - ep)(s^2 + p^2 - 1)}{s(s + cp)^3} e^{-x} \approx -\frac{2\alpha^3}{p} e^{-x},
$$

$$
\frac{\partial \Delta_2}{\partial \epsilon} \bigg|_{\chi=0} = \frac{2p(1 - \epsilon)}{s(s + p)^4} e^{-x} \approx -2\alpha^3 p e^{-x}.
$$

Therefore,

$$
w'(0) \approx -\frac{1}{32\pi^2 R^3} \int_0^\infty \frac{x^2}{e^x - 1} dx \int_1^\infty \left( \frac{1}{p^3} + \frac{1}{p} \right) \left( \frac{1}{1 - \kappa p/x} + \frac{1}{1 + \kappa p/x} \right) dp,
$$

where

$$
\kappa = \frac{2\mu^2 R}{\sigma} \ll 1, \quad \chi = \epsilon^2 \frac{\delta_0}{2R} \kappa.
$$

(111)
Integrals over $p$ can be done using the partial fraction decompositions:
\[
\frac{1}{p(1 + \sigma p/x)} = \frac{1}{p} - \frac{\sigma}{x} \frac{1}{(1 + \sigma p/x)}, \\
\frac{1}{p^3(1 + \sigma p/x)} = \frac{1}{p^3} - \frac{\sigma}{x} \frac{1}{p^2} + \frac{\sigma^2}{x^2} \frac{1}{p} - \frac{\sigma^3}{x^3} \frac{1}{(1 + \sigma p/x)}. \tag{112}
\]

As a result, we get
\[
w'(0) \approx -\frac{1}{32 \pi^2 R^3} \int_0^\infty \frac{x^2 \, dx}{e^x - 1} \left[ 1 - 2 \ln \frac{\sigma^2}{x} + \left( 1 + \frac{\sigma^2}{x^2} \right) \ln \left| 1 - \frac{\sigma^2}{x^2} \right| \right] \approx \\
-\frac{1}{32 \pi^2 R^3} \int_0^\infty \frac{x^2 (1 - 2 \ln \sigma + 2 \ln x)}{e^x - 1} \, dx. \tag{113}
\]

But (the evaluation of these integrals is explained in detail in appendix [2])
\[
\int_0^\infty \frac{x^2 }{e^x - 1} \, dx = 2 \zeta(3), \quad \int_0^\infty \frac{x \ln x}{e^x - 1} \, dx = 2 \zeta'(3) + (3 - 2 \gamma) \zeta(3), \tag{114}
\]
where $\gamma \approx 0.5772$ is the Euler constant, $\zeta(3) \approx 1.2021$ is the Apéry constant, and $\zeta'(3) \approx -0.1981$. Consequently, we get
\[
w'(0) \approx -\frac{\zeta(3)}{16 \pi^2 R^3} \left( 4 - 2 \gamma + \frac{\zeta'(3)}{\zeta(3)} - 2 \ln \sigma \right) \approx -\frac{\zeta(3)}{8 \pi^2 R^3} \left( 1.34 - \ln \sigma \right) \approx \frac{\zeta(3)}{8 \pi^2 R^3} \ln \sigma. \tag{115}
\]

Therefore, our final result for the leading contribution of the ultralight dark photon to the Casimir energy is (we have restored $\hbar$ and $c$, and scaled the energy in accordance with the area of the plates, as in [22])
\[
W_D \approx \frac{\zeta(3) \hbar c L^2}{8 \pi^2 R^3} \frac{c^2 \mu^2}{\sigma \omega_{pc}} \ln \left( \frac{2 \mu^2 R}{\sigma} \right). \tag{116}
\]

In Drude’s free electron theory of metals $\sigma = \frac{n e^2 \tau}{m}$, where $\tau \approx 2 \times 10^{-14}$ sec $\approx \frac{1}{0.2 \text{ eV}}$ is the average time between two consecutive electron collisions [54]. Comparing with [54], we obtain (for $\omega_{pc} \approx 10 \text{ eV}$)
\[
\sigma = \omega_{pc}^2 \tau \approx 50 \omega_{pc} \approx 500 \text{ eV}. \tag{117}
\]

Then, for $R \approx 10^{-4} \text{ cm} \approx \frac{1}{18 \text{ eV}},$
\[
\sqrt{\frac{\sigma}{2R}} \approx 18 \text{ eV}, \tag{118}
\]
and the correction due to dark photon to the Casimir energy can be represented in the form (the much more important finite conductivity corrections and all other non-dark photon corrections are omitted for clarity)
\[
W \approx -\frac{\pi^2 \hbar c L^2}{720 R^3} \left[ 1 - e^2 \frac{90 \zeta(3)}{\pi^4} \frac{\delta_0}{2R} \frac{2\mu^2 R}{\sigma} \ln \left( \frac{2\mu^2 R}{\sigma} \right) \right] \approx -\frac{\pi^2 \hbar c L^2}{720 R^3} \left[ 1 - e^2 \frac{90 \zeta(3)}{\pi^4} \frac{\delta_0}{R} \left( \frac{\mu}{18 \text{ eV}} \right)^2 \ln \left( \frac{\mu}{18 \text{ eV}} \right) \right]. \tag{119}
\]

It is clear from (111) that our approximations are valid as far as $\mu \ll 18 \text{ eV}.$

The function $\sigma^2 \ln \sigma^2$ reaches its minimum $-e^{-1}$ at $\sigma^2 = e^{-1}$ (which corresponds to $\mu \approx 10 \text{ eV}$ for $\sigma \approx 500 \text{ eV}$ and $R \approx 10^{-4} \text{ cm}$). Therefore, it follows from (119) that the magnitude of the dark photon correction is less than $2 \times 10^{-2} e^2$. If dark photons comprise all of the dark matter, hidden-photon search experiments indicate $\epsilon < 10^{-12}$ in the $\sim \text{eV}$ mass range [51, 52]. Although it is possible that dark photons make up only a small fraction of the dark matter [53], the requirement that the Sun does not emit more than 10% of its photon luminosity in a dark channel still drastically constrains the kinetic mixing parameter: $\epsilon < 4 \times 10^{-12} \frac{\epsilon}{\mu}$ [54, 55]. Therefore, we conclude that the dark photon contribution to the Casimir energy is negligible from the experimental point of view.
VI. CONCLUSIONS

The Casimir effect for two mixed scalar fields was analyzed in [56]. It was shown that if the zero-point energy is evaluated for the vacuum of fields with definite mass, then the result is independent of the mixing parameters (in this case, the Casimir force is simply the sum of the Casimir forces for two unmixed fields). In contrast, if the Casimir force is evaluated using the flavor vacuum, the result shows an explicit dependence on the mixing parameters. Thus, these two approaches give different, albeit numerically close, results for the Casimir force, thus demonstrating a nonequivalence between mass and flavor representations for mixed fields [56].

However, in [56] ideal boundary conditions were assumed. Our results in this work show that the expected penetration depths for dark photons into real metals are very large, so great that the necessary boundary conditions for the Casimir effect cannot be met. As a result, the presence of dark photons cannot significantly affect the Casimir force.

At about $R \sim 10^{-4}$ cm interplate distances, the leading contribution to the Casimir energy arises from the modification of the frequency-dependent dielectric function of metals. We have estimated this contribution, which for expected dark photon parameters turned out to be negligible.

An indirect way of how dark photons could affect the Casimir force could be a change in the properties of plasmons associated with the presence of dark photons, since it is known that at short distances the interaction between plasmon excitations at the surface of Casimir mirrors dominates in the real Casimir force [58]. However, our results show that the expected modification of the plasmon dispersion relation is insignificant and, therefore, such an indirect effect of dark photons on the Casimir force is also negligible.

Theoretically, the problem of the influence of field mixing on the Casimir force is very interesting, especially in light of the unitary nonequivalence of the mass and flavor vacua. It’s a pity that the expected effects are negligible and, therefore, the Casimir effect does not allow an experimental study of this interesting phenomenon.

In the course of this work, we assumed an ultralight dark photon. Note, however, that the main result, namely the insensitivity of the Casimir force to the presence of a dark photon for all practical purposes, remains valid for a massive dark photon.

The Casimir effect for massive photons was investigated by Barton and Dombey many years ago [45]. As it is well-known [59], the Casimir force is exponentially suppressed by the factor $e^{-2\mu R}$, where $\mu$ is a photon mass and $R$ is the distance between the plates, for $\mu R \gg 1$. The frequencies that dominate the Casimir force are of the order of $1/R$ [15, 28, 29]. Therefore, the condition $\mu R \sim 1$ is translated into the condition $\mu \sim \omega$ and for small mixing parameter $\epsilon$ (79) and (87) show that the penetration depths remain unacceptably large.

Finally, note that there are physics scenarios beyond the Standard Model that are different from the dark photon model. The implications of the Casimir effect for models such as universal extra dimensions, Randall-Sundrum models, and scale-invariant models have been discussed in [60].

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Appendix A: Euler-Maclaurin Summation Formula

The Euler-Maclaurin summation formula was originally obtained in 1782 by Euler and independently and almost simultaneously by Maclaurin [61]. A rigorous treatment of this very important tool of numerical analysis with diverse applications can be found, for example, in [62]. There exist several elementary derivations [63–65] of this remarkable formula. However, we prefer a formal heuristic derivation [66], which in the simplest way demystifies the appearance of Bernoulli numbers in it. Note that using Banach spaces of entire functions of exponential type, the formal derivation of the Euler-Maclaurin formula can be made mathematically rigorous [67].

4 It has been experimentally demonstrated that the Casimir force between metallic films decreases significantly when the layer thickness is less than the skin depth, which for most common metals is about $10^{-8}$ m [57].
Let \( \hat{D} = \frac{d}{dx} \) be a differential operator, so that

\[
\hat{D} f(x) = \frac{df}{dx}, \quad f(x + n) = e^{n\hat{D}} f(x),
\]

where the second equation is a formal expression of the Taylor formula. Then

\[
\sum_{n=0}^{N-1} f(x + n) = \left( \sum_{n=0}^{N-1} e^{n\hat{D}} \right) f(x) = \frac{e^{N\hat{D}} - 1}{e^{n\hat{D}} - 1} f(x) = \frac{1}{e^{n\hat{D}} - 1} [f(x + N) - f(x)]. \tag{A2}
\]

The Bernoulli numbers \( B_n \) are defined by the power series expansion of their exponential generating function:

\[
x \frac{e^x - 1}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \quad B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42} \ldots \tag{A3}
\]

Comparing (A3) and (A2), we can write

\[
\sum_{n=0}^{N-1} f(x + n) = \left( \hat{D}^{-1} + \sum_{n=1}^{\infty} \frac{B_n}{n!} \hat{D}^{n-1} \right) [f(x + N) - f(x)]. \tag{A4}
\]

On the other hand,

\[
\int_0^N f(x + t) dt = \int_0^N e^{t\hat{D}} f(x) dt = \hat{D}^{-1} \left( e^{N\hat{D}} - 1 \right) f(x) = \hat{D}^{-1} [f(x + N) - f(x)]. \tag{A5}
\]

Therefore

\[
\sum_{n=0}^{N-1} f(x + n) - \int_0^N f(x + t) dt = \sum_{n=1}^{\infty} \frac{B_n}{n!} \hat{D}^{n-1} [f(x + N) - f(x)]. \tag{A6}
\]

When \( N \to \infty \), for convergence we need \( \hat{D}^k f(\infty) = 0, \quad k = 0, 1, \ldots \) and (A6) in this limit takes the form

\[
\sum_{n=0}^{\infty} f(x + n) - \int_0^\infty f(x + t) dt = -\sum_{n=1}^{\infty} \frac{B_n}{n!} \hat{D}^{n-1} f(x) = \frac{1}{2} f(x) - \sum_{n=2}^{\infty} \frac{B_n}{n!} \frac{d^{n-1} f(x)}{dx^{n-1}}. \tag{A7}
\]

The form of the Euler-Maclaurin Summation Formula used in the main text corresponds to \( x = 0 \) in (A7).

**Appendix B: Evaluation of the integrals**

First we evaluate the integral from entry 3.411.1 in [13]. From the definition of the gamma function

\[
\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} dx,
\]

and using

\[
\frac{1}{1 - e^{-x}} = \sum_{k=0}^{\infty} e^{-kx},
\]

we get after interchanging the orders of the integration and summation

\[
\int_0^\infty \frac{x^{\nu-1}}{e^x - 1} dx = \int_0^\infty \frac{x^{\nu-1} e^{-x}}{1 - e^{-x}} dx = \int_0^\infty \sum_{k=0}^{\infty} x^{\nu-1} e^{-(k+1)x} dx = \sum_{k=0}^{\infty} \int_0^\infty x^{\nu-1} e^{-(k+1)x} dx = \Gamma(\nu) \sum_{k=0}^{\infty} \frac{1}{(k+1)^\nu} = \Gamma(\nu) \zeta(\nu).
\]

\[
(B3)
\]
Now, if we differentiate the just proved identity

$$
\zeta(\nu) = \frac{1}{\Gamma(\nu)} \int_{0}^{\infty} \frac{x^{\nu-1}}{e^x - 1} \, dx
$$

with respect to $\nu$ and take into account $\frac{dx^{\nu-1}}{d\nu} = \ln x \cdot x^{\nu-1}$, we get

$$
\zeta'(\nu) = \frac{1}{\Gamma(\nu)} \int_{0}^{\infty} \frac{x^{\nu-1} \ln x}{e^x - 1} \, dx - \frac{\Gamma'()}{\Gamma(\nu)^2} \int_{0}^{\infty} \frac{x^{\nu-1}}{e^x - 1} \, dx = \frac{1}{\Gamma(\nu)} \int_{0}^{\infty} \frac{x^{\nu-1} \ln x}{e^x - 1} \, dx - \psi(\nu) \zeta(\nu),
$$

where $\psi(\nu) = \Gamma'(\nu)/\Gamma(\nu)$. Therefore,

$$
\int_{0}^{\infty} \frac{x^{\nu-1} \ln x}{e^x - 1} \, dx = \Gamma(\nu) \zeta'(\nu) + \psi(\nu) \Gamma(\nu) \zeta(\nu).
$$

Note that from $\psi(\nu + 1) = \frac{1}{\nu} + \psi(\nu)$ and $\psi(1) = -\gamma$, it follows that

$$
\psi(3) = \frac{1}{2} + 1 + \psi(1) = \frac{1}{2} (3 - 2\gamma).
$$

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[1] M. Fabbrichesi, E. Gabrielli and G. Lanfranchi, *The Physics of the Dark Photon: A Primer* (Springer Nature: Cham, 2021). [https://doi.org/10.1007/978-3-030-62519-1](https://doi.org/10.1007/978-3-030-62519-1)

[2] M. Goodsell, J. Jaeckel, J. Redondo and A. Ringwald, Naturally Light Hidden Photons in LARGE Volume String Compactifications, JHEP 11 (2009), 027. [https://doi.org/10.1088/1126-6708/2009/11/027](https://doi.org/10.1088/1126-6708/2009/11/027)

[3] B. Holdom, Two U(1)’s and Epsilon Charge Shifts, Phys. Lett. B 166 (1986), 196–198. [https://doi.org/10.1016/0370-2693(86)91377-8](https://doi.org/10.1016/0370-2693(86)91377-8)

[4] A. Filippis and M. De Napoli, Searching in the dark: the hunt for the dark photon, Rev. Phys. 5 (2020), 100042. [https://doi.org/10.1016/j.revip.2020.100042](https://doi.org/10.1016/j.revip.2020.100042)

[5] K. R. Dienes, C. F. Kolda and J. March-Russell, Kinetic mixing and the supersymmetric gauge hierarchy, Nucl. Phys. B 492 (1997), 104–118. [https://doi.org/10.1016/S0550-3213(97)00173-9](https://doi.org/10.1016/S0550-3213(97)00173-9)

[6] H. Ruegg and M. Ruiz-Altaba, The Stueckelberg field, Int. J. Mod. Phys. A 19 (2004), 3265–3348. [https://doi.org/10.1142/S0217751X04019755](https://doi.org/10.1142/S0217751X04019755)

[7] L. B. Okun, Limits of electrodynamics: paraphotons?, Sov. Phys. JETP 56 (1982), 502–505. [http://www.jetp.ac.ru/cgi-bin/e/index/e/56/3/p502?a=list](http://www.jetp.ac.ru/cgi-bin/e/index/e/56/3/p502?a=list)

[8] J. Jaeckel, A force beyond the Standard Model - Status of the quest for hidden photons, Frascati Phys. Ser. 56 (2012), 172–192. [https://arxiv.org/abs/1303.1821](https://arxiv.org/abs/1303.1821)

[9] V. V. Flambaum, I. B. Samsonov and H. B. Tran Tan, Interference-assisted detection of dark photon using atomic transitions, Phys. Rev. D 99 (2019), 115019. [https://doi.org/10.1103/PhysRevD.99.115019](https://doi.org/10.1103/PhysRevD.99.115019)

[10] M. Blasone and G. Vitiello, Quantum field theory of fermion mixing, Annals Phys. 244 (1995), 283–311. [erratum: Annals Phys. 249 (1996), 363–364]. [https://doi.org/10.1006/aphy.1995.1115](https://doi.org/10.1006/aphy.1995.1115)

[11] M. Blasone, A. Capolupo, O. Romei and G. Vitiello, Quantum field theory of boson mixing, Phys. Rev. D 63 (2001), 125015. [https://doi.org/10.1103/PhysRevD.63.125015](https://doi.org/10.1103/PhysRevD.63.125015)

[12] C. R. Ji and Y. Mishchenko, The General theory of quantum field mixing, Phys. Rev. D 65(2002), 096015. [https://doi.org/10.1103/PhysRevD.65.096015](https://doi.org/10.1103/PhysRevD.65.096015)

[13] S. K. Lamoreaux, Casimir forces: Still surprising after 60 years, Physics Today 60N2 (2007), 40–45. [https://doi.org/10.1063/1.2711635](https://doi.org/10.1063/1.2711635)

[14] H. B. G. Casimir, On the Attraction Between Two Perfectly Conducting Plates, Indag. Math. 10 (1948), 261–263. [https://www.dwc.knaw.nl/DL/publications/PU00018547.pdf](https://www.dwc.knaw.nl/DL/publications/PU00018547.pdf)

[15] R. L. Jaffe, The Casimir effect and the quantum vacuum, Phys. Rev. D 72 (2005), 021301. [https://doi.org/10.1103/PhysRevD.72.021301](https://doi.org/10.1103/PhysRevD.72.021301)

[16] S. Y. Buhmann, *Dispersion Forces I: Macroscopic Quantum Electrodynamics and Ground-State Casimir, Casimir-Polder and van der Waals Forces* (Springer-Verlag: Berlin, 2012). [https://doi.org/10.1007/978-3-642-32484-0](https://doi.org/10.1007/978-3-642-32484-0)

[17] S. Y. Buhmann and D. G. Welsch, Dispersion forces in macroscopic quantum electrodynamics, Prog. Quant. Electron. 31 (2007), 51–130. [https://doi.org/10.1016/j.pquantelec.2007.03.001](https://doi.org/10.1016/j.pquantelec.2007.03.001)

[18] Q.-D. Jiang and F. Wilczek, Quantum Atmospherics for Materials Diagnosis, Phys. Rev. B 99 (2019), 201104. [https://doi.org/10.1103/PhysRevB.99.201104](https://doi.org/10.1103/PhysRevB.99.201104)
[19] Q.-D. Jiang and F. Wilczek, Axial Casimir Force, Phys. Rev. B 99 (2019), 165402. [https://doi.org/10.1103/PhysRevB.99.165402]

[20] L. Pálová, P. Chandra and P. Coleman, The Casimir effect from a condensed matter perspective, Am. J. Phys. 77 (2009), 1055–1060. [https://doi.org/10.1119/1.3194050]

[21] F. J. Belinfante, The Casimir effect revisited, Am. J. Phys. 55 (1987), 134–138. [https://doi.org/10.1119/1.16230]

[22] K. Huang, Quantum Field Theory: From Operators to Path Integrals (Wiley-Interscience: New York, 1998). [https://doi.org/10.1002/9783527617371]

[23] E. G. Harris, A Pedestrian Approach to Quantum Field Theory (Wiley-Interscience: New York, 1972).

[24] C. Itzykson and J.-B. Zuber, Quantum field theory (Mcgraw-hill: New York, 1980)

[25] H. Flanders, Differentiation Under the Integral Sign, Am. Math. Monthly 80 (1973), 615–627. [https://doi.org/10.2307/2319163]

[26] L. Pálová, P. Chandra and P. Coleman, The Casimir effect from a condensed matter perspective, Am. J. Phys. 77 (2009), 1055–1060. [https://doi.org/10.1119/1.3194050]

[27] G. Plunien, B. Müller and W. Greiner, The Casimir Effect, Phys. Rept.

[28] H. Flanders, Differentiation Under the Integral Sign, Am. Math. Monthly 80 (1973), 615–627. [https://doi.org/10.2307/2319163]

[29] V. Mostepanenko and N. N. Trunov, The Casimir Effect and its Applications (Clarendon Press: Oxford, 1997).

[30] A. Abramcht and S. Reynaud, Casimir force between metallic mirrors, Eur. Phys. J. D 8 (2000), 309–318. [https://doi.org/10.1007/s100530050041]

[31] P. Glaister, A “Flat” Function with Some Interesting Properties and an Application, Math. Gazette 75 (1991), 438–440. [https://doi.org/10.2307/3618627]

[32] E. N. Economou, The Physics of Solids: Essentials and Beyond (Springer-Verlag: Berlin, 2010). [https://doi.org/10.1007/978-3-642-02069-8_4]

[33] H. Ehrenreich and M. H. Cohen, Self-Consistent Field Approach to the Many-Electron Problem, Phys. Rev. 115 (1959), 786–790. [https://doi.org/10.1103/PhysRev.115.786]

[34] A. Andrianavalomahefa et al., Hidden-Photon Dark Matter in the Visible and Near-Ultraviolet Wavelength Range, Phys. Rev. D 102 (2020), 042001. [https://doi.org/10.1103/PhysRevD.102.042001]

[35] A. A. Rukhadze and V. E. Semenov, On a simplified description of waves in non-collision plasmas, Izvestiya VUZ. Applied Mathematics, 29 (2009), 3−13. [https://doi.org/10.2307/2319163]

[36] V. B. Bezerra, G. L. Klimchitskaya and C. Romero, Perturbation expansion of the conductivity correction to the casimir force, Int. J. Mod. Phys. A 16 (2001), 3103–3115. [https://doi.org/10.1142/S0217751X01004426]

[37] T. Tsang, Classical Electrodynamics (World Scientific: Singapore, 1997).

[38] A. A. Vlasov, Theory of Vibrational Properties of Electron Gas and Its Applications, Uch. Zapiski MGU, Fizika, 75 (1945), book II. part 1 [in Russian].

[39] A. A. Rukhadze and V. E. Semenov, On a simplified description of waves in non-collision plasmas, Izvestiya VUZ. Applied Mathematics, 29 (2009), 3−13. [https://doi.org/10.2307/2319163]

[40] V. Yu. Popov and V. P. Silin, Vlasov Modes in the Theory of Ion-Acoustic Turbulence, Plasma Phys. Rep. 40 (2014), 298–305. [https://doi.org/10.1134/S1063780X14040060]

[41] W. Greiner, Quantum Mechanics: Special Chapters (Springer-Verlag: Berlin, 1998).

[42] W. K. H. Panofsky and M. Phillips, Classical Electricity and Magnetism (Addison-Wesley: New York, 1962).

[43] A. Garg, Conductors in quasistatic electric fields, Am. J. Phys. 76 (2008), 615–620. [https://doi.org/10.1119/1.2894525]

[44] P. Robles and F. Claro, Can there be massive photons? A pedagogical glance at the origin of mass, Eur. J. Phys. 33 (2012), 1217–1225. [https://doi.org/10.1088/0143-0807/33/5/1217]

[45] G. Barton and N. Dombey, The Casimir Effect With Finite Mass Photons, Annals Phys. 309, 75(2000), 014102. [https://doi.org/10.1016/S0003-4916(99)00162-9]

[46] V. B. Bezerra, G. L. Klimchitskaya and V. M. Mostepanenko, Higher-order conductivity corrections to the Casimir force, Phys. Rev. A 62 (2000), 014102. [https://doi.org/10.1103/PhysRevA.62.014102]

[47] M. Bordag, G. L. Klimchitskaya, U. Mohideen and V. M. Mostepanenko, Advances in the Casimir Effect (Oxford University Press: Oxford, 2009).

[48] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Seventh Edition (Academic Press: Amsterdam, 2007).

[49] V. B. Bezerra, G. L. Klimchitskaya and C. Romero, Perturbation expansion of the conductivity correction to the casimir force, Int. J. Mod. Phys. A 16 (2001), 3103–3115. [https://doi.org/10.1142/S0217751X01004426]

[50] T. Tsang, Classical Electrodynamics (World Scientific: Singapore, 1997).

[51] J. Chiles, I. Charaev, R. Lasenby, et al., First Constraints on Dark Photon Dark Matter with Superconducting Nanowire Detectors in an Optical Haloscope, [arXiv:2110.01582 [hep-ex]]. [https://arxiv.org/abs/2110.01582]

[52] A. Andrianavalomahefa et al., FUNK Experiment, Limits from the Funk Experiment on the Mixing Strength of Hidden-Photon Matter in the Visible and Near-Ultraviolet Wavelength Range, Phys. Rev. D 102 (2020), 042001. [https://doi.org/10.1103/PhysRevD.102.042001]

[53] A. Alizii and Z. K. Silagadze, Dark photon portal into mirror world, Mod. Phys. Lett. A 36 (2021), no.30, 2150215. [https://doi.org/10.1142/S0217732321502151]

[54] J. Redondo and G. Raffelt, Solar constraints on hidden photons re-visited, JCAP 08 (2013), 034. [https://doi.org/10.1088/1475-7516/2013/08/034]

[55] H. An, M. Pospelov and J. Pradler, New stellar constraints on dark photons, Phys. Lett. B 725 (2013), 190–195.
[56] M. Blasone, G. G. Luciano, L. Petruzziello and L. Smaldone, Casimir effect for mixed fields, Phys. Lett. B 786 (2018), 278–282. https://doi.org/10.1016/j.physletb.2013.07.008

[57] M. Lisanti, D. Iannuzzi and F. Capasso, Observation of the skin-depth effect on the Casimir force between metallic surfaces, Proc. Natl Acad. Sci. USA 102 (2005), 11989–11992. https://doi.org/10.1073/pnas.0505614102

[58] F. Intravaia and A. Lambrecht, Surface Plasmon Modes and the Casimir Energy, Phys. Rev. Lett. 94 (2005), 110404. https://doi.org/10.1103/PhysRevLett.94.110404

[59] C. Farina, The Casimir effect: Some aspects, Braz. J. Phys. 36 (2006), 1137–1149. https://doi.org/10.1590/S0103-97332006000700006

[60] L. Mattioli, A. M. Frassino and O. Panella, Casimir-Polder interactions with massive photons: implications for BSM physics, Phys. Rev. D 100 (2019), 116023. https://doi.org/10.1103/PhysRevD.100.116023

[61] P. L. Butzer, P. J. S. G. Ferreira, G. Schmeisser and R. L. Stens, The Summation Formulæ of Euler-Maclaurin, Abel-Plana, Poisson, and their Interconnections with the Approximate Sampling Formula of Signal Analysis, Results. Math. 59 (2011), 359–400. https://doi.org/10.1007/s00025-010-0083-8

[62] G. H. Hardy, Divergent Series (Oxford University Press: Oxford, 1949).

[63] J. P. Dowling, The Mathematics of the Casimir Effect, Math. Magazine 62 (1989), 324–331. https://doi.org/10.2307/2689486

[64] T. M. Apostol, An Elementary View of Euler’s Summation Formula, Am. Math. Monthly 106 (1999), 409–418. https://doi.org/10.2307/2689145

[65] I. Roman, An Euler Summation Formula, Am. Math. Monthly 43 (1936), 9–21. https://doi.org/10.2307/2301097

[66] V. Kac and P. Cheung, Quantum Calculus (Springer: New York, 2002). https://doi.org/10.1007/978-1-4613-0071-7

[67] M. Sugihara, Justification of a Formal Derivation of the Euler-Maclaurin Summation Formula. In: S. Saitoh, N. Hayashi and M. Yamamoto (eds), Analytic Extension Formulas and their Applications. International Society for Analysis, Applications and Computation, vol 9. (Springer: Boston, 2001), pp. 251–261. https://doi.org/10.1007/978-1-4757-3298-6_14