Anomaly cancellation condition in lattice effective electroweak theory

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Abstract

We consider an effective lattice electroweak model composed by a family of elementary particles (two quarks and two leptons) with a quartic interaction describing the weak forces and an interaction with the e.m. field: the lattice step plays the role of an ultraviolet cut-off, the fermions are massless and a photon mass is an infrared cut-off. We write the anomaly as an expansion which is convergent uniformly in the infinite volume and for an inverse lattice step of the order of the gauge masses. We prove that, under a suitable condition on the charges, the anomaly vanishes up to subdominant corrections which are rigorously bounded. The result is an essential step toward the proof of the non-perturbative existence of a finite anomaly-free electroweak theory up to exponentially large cut-off in the inverse coupling.

1 Introduction and Main results

1.1 Electroweak theory

The Standard Model is expressed by series expansion which are typically non convergent. It seems unlikely that a rigorous formulation can be obtained removing all cut-offs, at least if one considers the electroweak sector alone, due to the lack of asymptotic freedom and the presence of the Landau pole. A non-perturbative formulation can be achieved in principle defining the fields on a finite lattice, with the inverse of the lattice step acting as an energy cut-off. The lattice breaks some of the symmetries which are experimentally observed, which are therefore only approximate; consistency requires therefore that the cut-off is much higher than the energy scale of experiments, so that corrections are smaller than the precision. A finite cut-off has a clear physical meaning, as one does not expect that the Quantum Field Theory description is valid at arbitrarily large scale. The choice of the regularization affects only the corrections, in the class with the same symmetries.

The problem to give a meaning to the lattice functional integrals defining the theory becomes more and more complex increasing the cut-off. According to general Renormalization Group considerations, one expects a relation between the perturbative renormalizability properties and the size of the cut-off. In the case of a non-renormalizable theory, like the Fermi theory of
weak interactions, one expects that only energy cut-offs of the order of the inverse coupling can be reached. In contrast electroweak theory is renormalizable [1,2] and the interactions have vanishing dimensions, so that a cut-off exponentially large in the inverse coupling could be reached, which is much higher than the energy in actual experiments. The realization of this program is however a non trivial issue and requires the solution of a number of technical problems. The situation is somewhat parallel (but more involved) to the infrared problem of Fermi liquids, where one expects that perturbative renormalizability implies the existence of the theory up to exponentially small temperatures, which has been proved only in 2d [3, 4] [5].

Let us recall some basic facts on the electroweak theory, see e.g. [6]. One has a family of fermionic Dirac particles $\psi_{i,c,x}$, composed by two leptons $(\nu, e)$ and two quarks $(u, d)$ denoted by $i = \nu, e, u, d$. We call $c$ the color index with $c = (r, g, b)$ for $i = (u, d)$ and $c = 0$ for $i = (\nu, e)$. $s = L, R$ is the chiral index $\psi = (\psi_L, \psi_R)$. If $j_{\mu,i,c,s,x} = \frac{1}{\sqrt{2}}\bar{\psi}_{i,c,x}\gamma_{\mu}(1 + \gamma_5)\psi_{i,c,x}$, $\epsilon_{i,L} = -\epsilon_{i,R} = 1$, the boson $B_{\mu}$ is coupled with the current $gj_{\mu,x}^B = g \sum_{i,s,c} Y_{i,s}j_{\mu,i,c,s,x} = g \sum_{i,s,c} j_{\mu,i,c,s,x}$ where $Y_{i,s}$ is the hypercharge; we can write $j_{\mu,x}^B = j_{\mu,V,x} + j_{\mu,A,x}$, where $j_{\mu,V,x}$ includes the vector currents $\bar{\psi}_{i,c,x}\gamma_{\mu}\psi_{i,c,x}$ and the axial part is $j_{\mu,A,x}^B$ includes the vector currents $\bar{\psi}_{i,c,x}\gamma_{\mu}\gamma_5\psi_{i,c,x}$. The bosons $W_k^\mu$ are coupled with the current $g j_{\mu,x}^{k,W} = \frac{1}{2}g \sum_c (\bar{\Psi}_{c,e,c,x}\tau^k\gamma_{\mu}(1 + \gamma_5)\Psi_{c,e,c,x} + \bar{\Psi}_{q,c,x}\tau^k\gamma_{\mu}(1 + \gamma_5))\Psi_{q,c,x}^*$ where $\tau^k$ pauli matrices, $\Psi_{c,e,c,x} = (\psi_{u,d,c,x}, \psi_{l,c,x})$ (the lepton doublet) and $\Psi_{q,c,x}^* = (\psi_{u,c,x}, \psi_{d,c,x})$ (the quark doublet). Again we can write $j_{\mu,x}^{k,W} = j_{\mu,V,x} + j_{\mu,A,x}^{k,W}$. The boson and fermions are massless and coupled with the Higgs particle. Two regimes can be identified.

1. A first high energy regime of scales from the cut-off up to mass scale generated by the Higgs, expressed in terms of the gauge fields $W_\mu^\pm, W_\mu^3$ and $B_\mu$ (associated to $SU(2)$ and $U(1)$ invariance) and the fermionic fields. The theory is renormalizable [2] and the $W_\mu^\pm$ becomes massive due to the interaction with the Higgs and the $W_\mu^3, B_\mu$ are combined in a massive $Z_\mu$ field and massless $A_\mu$. The $Z_\mu$ with mass $M_Z$ is coupled with the current $g j_{\mu,x}^Z = \bar{g} \sum_{i,s} (\tau^i/2 - \sin^2\theta Q_i) j_{\mu,i,s,x}$ and the massless $A_\mu$ is coupled with $e j_{\mu,x}^{e,m} = \sum_i e Q_i j_{\mu,i,x}$ with $Q_i = \frac{1}{2}(Y_i^s + \tau^3_{i,s})$ the Gell-Mann relation, $\tau^3_{L,\mu} = \tau^3_{q,\mu} = 1$, $\tau^3_{L,d} = \tau^3_{L,c} = -1$ and $\tau^3_{R,i} = 0$, $\cos \theta = g'/g$, $\bar{g} = \sqrt{g^2 + g'^2}$.

2. A second regime from mass scale to zero, in which the massive bosons can be integrated out and one gets an effective description of the form

$$\sum_\mu \left[ \frac{g^2}{2M_W^2} j_{\mu,x}^W j_{\mu,x}^W + \frac{g'^2}{2M_Z^2} j_{\mu,x}^Z j_{\mu,x}^Z + e A_{\mu,x} j_{\mu,x}^{e,m} \right]$$

(1)

In such regime the quartic terms are irrelevant and the e.m. regime is asymptotically free.

The renormalizability in the first high energy regime is a subtle issue. The generation of the mass in the bosons has the effect that the propagator has the form $\frac{1}{(x_\mu - x_\nu)^2}$ (if $x_\mu + x_\nu$). The scaling dimension is indeed $4 + (2 - z)n/2 - 3n^v/2 - (4 - z)n^G/2$, where $n$ is the order and $n^G, n^v$ is the number of bosonic or fermionic fields; if only the first part of the propagator contributes $z = 2$ and the theory is dimensionally renormalizable, but if the second term is present $z = 0$ and the theory is non renormalizable. In a gauge theory renormalizability is not simply dictated by dimensional considerations. In QED one can add a mass to the photon without loosing renormalizability [6] at least in perturbation theory with dimensional regularization; conservation of current $k_{\mu,j_{\mu}} = 0$ (in the sense of correlations) following by invariance under
\( \psi_x \to e^{i \alpha x} \psi_x \) has the effect that the contribution from the second term of the massive propagator vanishes. The reduction of the ultraviolet divergence by the conservation of current can be proved at a non-perturbative level in 2d [7]. Electroweak theory is instead a chiral gauge theory; the bosons are coupled to chiral currents which are non conserved for non vanishing fermion mass, and this requires the introduction of the Higgs particle. In addition chiral currents in the massless case can be non conserved due to the presence of anomalies, even if they apparently should be on the basis of classical considerations. The generic non conservation of the current in electroweak theory can be immediately seen considering \( B_\mu, W^\alpha_\mu \) fields as classical external fields and considering the response of the current to such fields. Considering in particular the response of the axial part one gets, \( p = p_1 + p_2 \) one gets, with momentum regularization and up to \( O(p^2) \) terms

\[
P_{\mu_1} \sum_i \left< \bar{\psi}_{B_1, A, i, s, p, \mu_2, \nu, p} \gamma^\nu \gamma^\alpha \gamma^\beta \psi_{B_1, A, i, s, p, \mu_3, \nu, p} \right> = \frac{1}{2\pi^2} \epsilon_{\mu_1 \mu_2 \alpha \beta} p^1_{\alpha \beta} \left[ Y^L_{\epsilon} + Y^L_{\nu} + 3Y^L_{\mu} + 3Y^L_{d} \right]
\]

\[
P_{\mu_1} \sum_i \left< \bar{\psi}_{B_1, A, i, s, p, \mu_2, \nu, p} \gamma^\nu \gamma^\alpha \gamma^\beta \psi_{B_1, A, i, s, p, \mu_3, \nu, p} \right> = \frac{1}{2\pi^2} \epsilon_{\mu_1 \mu_2 \alpha \beta} p^1_{\alpha \beta} \times \left[ (Y^L_{\epsilon})^3 + (Y^L_{\nu})^3 + 3(Y^L_{\mu})^3 + 3(Y^L_{d})^3 - (Y^R_{\epsilon})^3 - (Y^R_{\nu})^3 - 3(Y^R_{\mu})^3 - 3(Y^R_{d})^3 \right]
\]

The l.h.s. of the above expression is called the anomaly and means that the current is generically non conserved unless the terms in [...] in the above equation vanishes. A non conservation of current would imply the non renormalizability of the electroweak theory. Remarkably, it turns out that the values in nature verify such condition; as \( Y^L_{\nu} = Y^L_{\epsilon} = -1, Y^L_{\mu} = Y^L_{d} = 1/3, Y^R_{\nu} = 0, Y^R_{\epsilon} = -2, Y^R_{\mu} = 4/3, Y^R_{d} = -2/3 \) then \( -2 + 6 \frac{1}{3} = 0 \) and \( 6(1/3)^3 + 2(-1)^3 - 3(4/3)^3 - 3(-2/3)^3 - (-2)^3 = 0 \). In addition the condition strongly constraints the values of charges of elementary particles [11], [12] (if we require for instance the neutrality of \( \nu \) they are fixed), providing a partial explanation to charge quantization. The above argument is valid for quantum fermions and classical gauge fields; considering quantum gauge fields produces radiative corrections, and the 3-current correlations \( \left< j_{\mu_1} \gamma_{\mu_2} j_{\mu_3} \right> \) and \( \left< j_{\mu_1} \gamma_{\mu_2} j_{\mu_3} \gamma_{\mu_4} \right> \) can be written as a series expansion in the coupling, whose lowest order is just \( \left< j_{\mu_1} \gamma_{\mu_2} j_{\mu_3} \right> >0 \) and \( \left< j_{\mu_1} \gamma_{\mu_2} j_{\mu_3} \gamma_{\mu_4} \right> >0 \). This could modify the anomaly cancellation condition, but another property of anomalies is invoked to say that this is not the case; their non-renormalization [10], actually the fact that higher orders do not contribute. The proof is based on order by order cancellations based on dimensional regularization.

In this paper we compute the anomaly at a non-perturbative level in the low energy regime up to scale of the order of the gauge mass; the vanishing of the anomaly in such a regime is an essential prerequisite to the validity of a similar property in the higher energy regime. We define a model corresponding to (1) with a lattice regularization which is suitable for a non-perturbative analysis. As infrared cut-off we assume that also the photon is massive, and the inverse lattice is of the order of the inverse gauge masses; the infinite volume limit is taken. We prove that anomaly vanishes up to subdominant corrections which are rigorously bounded, under the same condition as in the free theory; the condition implies a relation on the dressed charges and we show the dressed and bare charge coincide. This is essential as if they would flow the cancellation of the anomaly at lower scales would not imply cancellation at higher scales, when is crucial for renormalizability.

Finally we recall some results previously obtained in related systems, limiting to \( d = 4 \). Order by order renormalizability has been proved for QED in [13], [14], [15] and for (non chiral)
pure gauge Yang-Mills fields in [16]. \textit{QED} models in \(d = 4\) with ultraviolet cut-off have been rigorously investigated in [17], integrating out the fermions ad assuming large fermionic mass (of the order of the inverse charge). \textit{QED} with massive photons and massless fermions have been constructed in [18], by integrating out the bosons and reducing to a purely fermionic theory; such a system has a natural condensed matter interpretation [19]. The ultraviolet cut-off is of the order of the inverse electric charge; no results are known for exponentially large cut-off.

The Higgs mechanism in a bosonic theory has been established in [20] removing the ultraviolet cut-off; \textit{Yang Mills} theory with no matter has been constructed in [21]. See e.g. [22] for an extensive review. In [23] it is proved that the anomaly non-renormalization is true even with a finite lattice cut-off breaking chiral and Lorentz invariance, in the case of quartic interaction. The interaction is irrelevant in the Renormalization Group sense in that case, but this is not-essential for the anomaly non-renormalization, as shown in \(d = 2\ \textit{QED}\) where the interaction is marginal [24], [25], [26]. Finally in [27] a purely fermionic electroweak effective model was considered with momentum regularization; the breaking of gauge invariance implies that the electric charges have a non trivial flow and acquire renormalizations depending on the particle species unless a fine tuning is introduced.

1.2 The model

The effective electroweak model is defined by its generating function

\[
e^W(J,J^5,J^W,\phi) = \int P(dA) \int P(d\psi)e^{V_c(\psi)+V_{e.m.}(A+J,\psi)+V_G(\psi)+B(\psi,J,J^5,J^W,\phi)}
\]

with

\[
B(\psi, J, J^5, J^W, \phi) = \int dx \left[ \sum_{i=e,\mu,\nu,u,d} j^5_{i,\mu} x J^5_{i,\mu,\nu} x \right.
\]

\[
+ \sum_{j=1, \ne L, R} \sum_{k=1}^3 \sum_{s=L,R} Z^{W,j} k s j_{i,s,x} J^{W,j} k_{i,s,x} x + \sum_{i=e,\mu,\nu,u,d} \int dx (\psi_{i,s} x \phi_{i,s} x + \psi_{i,s} x \phi_{i,s} x)
\]

where

1. \(A_\mu(x) : \Lambda \rightarrow \mathbb{R}, \Lambda = [-L, L]^4 \cap a\mathbb{Z}^4\), \(e_\mu, \mu = 0, 1, 2, 3\) an orthonormal basis, \(A_\mu(x) = A_\mu(x + Le_\mu)\) (periodic boundary conditions) and the bosonic integration is

\[
P(dA) = \frac{1}{N_A} \prod_{x \in \Lambda} \prod_{\mu = 0}^3 dA_\mu(x) e^{-\frac{1}{2} \int dx \sum_{\mu=0}^3 A_\mu(x)(-\Delta + M^2)A_\mu(x)}
\]

where we use the convention

\[
\int dx \equiv a^4 \sum_x
\]

\(N_A\) is the normalization, \(\Delta f = \frac{1}{a^4} \sum_{\mu=0}^3 (f(x + ae_\mu) + f(x - ae_\mu) - 2f(x))\). The bosonic simple expectation

\[
\mathcal{E}_A(A_{\mu_1}(x_1)...A_{\mu_n}(x_n)) = \int P(dA)A_{\mu_1}(x_1)...A_{\mu_n}(x_n)
\]
is expressed by the Wick rule with covariance

$$g^A_{\mu,\nu}(x, y) = \delta_{\mu,\nu} \frac{1}{L^4} \sum_k \frac{e^{ik(x-y)}}{c(k)^2 + M_A^2}$$  

(7)

with $c^2(k) = \sum_{\mu} (1 - \cos k_{\mu} a) a^{-2}$, $k = 2 \pi n/L$ and $k \in [-\pi/a, \pi/a]^4$, $n_i = -L/a, (L - a)/a$. $A_\mu(x)$ is the photon fields in the Feynman gauge. We will need also formulas for the exponentials which are given by

$$\mathcal{E}_A(n \prod_{i=1}^n e^{i \epsilon_i \alpha_i A_{\mu_i}(x_i)}) = \delta \sum \epsilon_{i,0} e^{-\frac{1}{2} \sum_{i,j} \epsilon_i \epsilon_j \alpha_i \alpha_j g^A_{\mu_i,\mu_j}(x_i,x_j)}$$  

(8)

2. $\psi^+_{i,c,s,j,x}$ are Grassmann variables, with; $i = \nu, e, u, d$ the particle index; $c$ the color index $c = (r, g, b)$ for $i = (u, d)$ and $c = 0$ for $i = \nu, e$; $s = L, R$ the chiral index, $j = 1, 2$ the component index; anti periodic boundary conditions are imposed and

$$\{\psi^+_{i,c,s,j,x}, \psi^+_{i',c',s',j',x'}\} = \{\psi^-_{i,c,s,j,x}, \psi^-_{i',c',s',j',x'}\} = 0$$  

(9)

We use also the notation $\psi^+_{i,c,x} = (\psi^+_{i,c,L,x}, \psi^+_{i,c,R,x})$ and $\psi^\pm_{i,c,s,x} = (\psi^\pm_{i,c,s,1,x}, \psi^\pm_{i,c,s,2,x})$. The index $c$ will be often omitted when no ambiguities arise. We define $\psi^\pm_{i,c,s,x} = \frac{1}{\sqrt{2}} \sum_k \epsilon_{i,k} x \hat{\psi}^\pm_{i,c,s,k}$, with $\hat{\psi}^\pm_{i,c,s,k}$ another set of Grassmann variable.

The fermionic gaussian measure is defined as, $i = \nu, e, u, d, s = L, R$

$$P(d\psi) = \frac{1}{\mathcal{N}_\psi} \left[ \prod_{i,s,x} d\psi^+_{i,s,x} d\psi^-_{i,s,x} \right] e^{-S}$$  

(10)

where

$$S = \frac{1}{2a} \int dx \sum_{i=\nu, e, u, d} \left[ \psi^+_{i,x} \gamma_0 \gamma_{\mu} \psi^-_{i,x} + \gamma_j \psi^+_{i,x} \gamma_0 \gamma_{\mu} \psi^-_{i,x} \right]$$

(11)

with the gamma matrices are $\gamma_{\mu}, \gamma_5$, $\mu = 0, 1, 2, 3$

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma_j = \begin{pmatrix} 0 & i\sigma_j \\ -i\sigma_j & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$  

(12)

We set $\sigma^L = (\sigma_0, i\sigma)$ and $\sigma^R = (\sigma_0, -i\sigma)$ and the matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  

(13)

We can write therefore

$$S = \frac{1}{2a} \sum_{i=\nu, e, u, d} \int dx \left[ \sum_{s=L,R} (\psi^+_{i,s,x} \sigma^s_{\mu} \psi^-_{i,s,x} + \psi^+_{i,s,x} \sigma^s_{\mu} \psi^-_{i,s,x}) + r(\psi^+_{i,L,x} \psi^-_{i,R,x} + \psi^+_{i,R,x} \psi^-_{i,L,x} + \psi^+_{i,R,x} \psi^-_{i,L,x} - \psi^+_{i,L,x} \psi^-_{i,R,x}) \right]$$  

(14)
The e.m. gauge field is introduced via the Peierls substitution
\[ E^T_\psi(\psi_{i_1,1}^{\bar{1}} \cdots \psi_{i_n,n}^{\bar{n}}) = \int P(d\psi)\psi_{i_1,1}^{\bar{1}} \cdots \psi_{i_n,n}^{\bar{n}} \]
is expressed by the anticommutative Wick rule with covariance, \( k = \frac{2\mu}{L}(n+\frac{1}{2}) \), \( n \in \mathcal{N} \)
\[ \int P(d\psi)\psi_{i,x}^{-} \psi_{i,x}^{+} = \delta_{i,j} \frac{1}{L^2} \sum_{k} e^{ik(x-y)} \left( \sum_{\mu} i\gamma_{\mu} a^{-1} \sin(k_{\mu}a) + ra^{-1} \gamma_{0} \sum_{\mu} (1 - \cos k_{\mu}a) \right)^{-1} \]

3. The e.m. gauge field is introduced via the Peierls substitution
\[ G_{\mu,i}^{\pm} (A + J) = a^{-1}(1 + e^{\pm iQ_{\mu} \sum_{0} dsA_{\mu}(x+te_{\mu})} : -1) \]
with \( e^{\mp iQ_{\mu} \sum_{0} dsA_{\mu}(x+te_{\mu})} := e^{\pm iQ_{\mu} \sum_{0} dsA_{\mu}(x+te_{\mu})} e^{\frac{1}{2}(eQ)_{\mu,\alpha}(0,0)} \) and we define
\[ V_{e.m.}(A + J) = \frac{1}{2a} \sum_{i=0}^{u,v,u,d} \int dx \sum_{s=L,R} (\psi_{i,s,x}^{+} \sigma_{\mu}^{a} G_{\mu,i}^{\pm}(A + J) \psi_{i,s,x+e_{\mu},a} - \psi_{i,x+e_{\mu},a}^{+} \sigma_{\mu}^{a} G_{\mu,i}^{\pm}(A + J) \psi_{i,s,x} - r(\psi_{i,s,x}^{+} G_{\mu,i}^{\pm}(A + J) \psi_{i,s,x+e_{\mu},a} + \psi_{i,x+e_{\mu},a}^{+} G_{\mu,i}^{\pm}(A + J) \psi_{i,s,x} - \psi_{i,x+e_{\mu},a}^{+} G_{\mu,i}^{\pm}(A + J) \psi_{i,s,x}) \]
where \( J \) are external fields; we can write \( \varepsilon = \pm \)
\[ V_{e.m.}(A + J) = \sum_{\varepsilon,\mu} O_{\mu}^{\varepsilon}(\psi) G_{\mu}^{\varepsilon}(A + J) \]
The weak interaction is
\[ V_G = \frac{g^2}{2M_W^2} \sum_{\mu} \int dx j_{\mu,x}^{W} j_{\mu,x}^{-} + \frac{g^2}{2M_Z^2} \sum_{\mu} \int dx j_{\mu,x}^{Z} j_{\mu,x}^{-} \]
where
\[ j_{\mu,x}^{W} = (j_{\mu,L,x}^{W} + j_{\mu,R,x}^{W})/2 \]
\[ j_{\mu,x}^{-} = (j_{\mu,L,x}^{W} - j_{\mu,R,x}^{W})/2 \]
\[ j_{\mu,x}^{-} = \sum_{i,s} (\tau_{i}/2 - \sin^2 \theta_{Q_{i}}) \psi_{i,s,x}^{+} \sigma_{\mu}^{a} \psi_{i,s,x}^{-} \]
\[ j_{\mu,s,x}^{W} = j_{i,s,x}^{W} + j_{i,s,x}^{-} \]
where \( \tau_{L} = + \) for \( i = \nu, u \) and \( \tau_{L} = - \) for \( i = d, \nu, u \) and \( j_{\mu,s,x}^{W} = \psi_{i,s,x}^{+} \tau_{L} \sigma_{\mu}^{a} \psi_{i,s,x}^{-} \)
\[ j_{i,s,x}^{k,W} = \psi_{i,s,x}^{+} \tau_{L} \sigma_{\mu}^{a} \psi_{i,s,x}^{-} \]
Finally \( V_{c}(\psi) \) is the mass counterterm
\[ V_c = \sum_{i=\nu,u,d} a^{-1} \int dx (\psi_{i,L,x}^{+} \psi_{i,L,x}^{-} + \psi_{i,U,x}^{+} \psi_{i,U,x}^{-}) \]
and the axial current is defined as
\[ j_{\mu,x}^{5} = \sum_{i} (Z_{i}^{5} \psi_{x,i}^{+} L_{\mu} \psi_{x,i}^{-} - Z_{i}^{5} \psi_{x,i}^{-} L_{\mu} \psi_{x,i}^{+}) \]
\[ j_{\mu,x}^{5} = \sum_{i} (Z_{i}^{5} \psi_{x,i}^{+} L_{\mu} \psi_{x,i}^{-} - Z_{i}^{5} \psi_{x,i}^{-} L_{\mu} \psi_{x,i}^{+}) \]
4. The fermionic 2-point function is
\[
S_{i,s,s'}^\Lambda(x,y) = \frac{\partial^2}{\partial \phi^-_{i,s,x} \partial \phi^-_{i,s',y}} W(J, J^5, J^W, \phi)|_0
\] (25)
and the Fourier transform is
\[
\tilde{S}_{i,s,s'}^\Lambda(k) = \int dx S_{i,s,s'}^\Lambda(x,0)e^{-ikx}
\] (26)

The vertex functions are
\[
\Gamma_{\mu,i,s}^\Lambda(z, x, y) = \frac{\partial^3}{\partial \phi^+_{\mu,i,x} \partial \phi^+_{\mu,s,x} \partial \phi^-_{\mu,s,y}} W(J, J^5, J^W, \phi)|_0
\] (27)

The Fourier transform is
\[
\tilde{\Gamma}_{\mu,i,s}^\Lambda(k, p) = \int dx \int dy S_{2,i,s}(x, 0)e^{-ipz - iky} \Gamma_{\mu,i,s}^\Lambda(z, 0, y)
\] (28)

5. We assume that the charges $Q_i$ are such that
\[
Q_u - Q_d = Q_\nu - Q_e
\] (30)
so that the generating function is invariant under the following transformation
\[
\tilde{\psi}_{i,c,s,j,x}^+ \rightarrow \tilde{\psi}_{i,c,s,j,x}^+ e^{\pm iQ_i \alpha} \quad J_{W}^+ \rightarrow J_{W}^+ e^{\pm i\alpha(Q_u - Q_d)}
\] (31)

with $J, J^5$ unchanged; we also assume that
\[
Q_i = Q_j
\] (32)
for any $i, j$. 

7
1.3 Renomalization conditions

The physical parameters of the interacting theory do not necessarily coincide with the ones (bare parameters) appearing in (3); their values are instead obtained from the correlations in the limit of vanishing momenta (dressed parameters). The dressed wave function renormalization is defined as

$$Z_{i,s}^D I_2 = \lim_{k \to 0} \frac{\hat{g}_{i,s,s}(k)}{S_{i,s,s}(k)}$$

(33)

and the dressed electric charge as

$$e_{i,s}^{D} \sigma_{\mu} = \lim_{k,p \to 0} \frac{1}{Z_{i,s}^D} \hat{S}_{i,s,s}(k)^{-1} \hat{\Gamma}_{\mu;i,s}(k,p) \hat{S}_{i,s,s}(k+p)^{-1}$$

(34)

The bare parameters have to be properly chosen to impose physical conditions, and in particular:

1. The parameters $Z_{i,s}^5$ are chosen so that, if $\varepsilon_L = -\varepsilon_R = 1$

$$e_{i,s}^{D} \sigma_{\mu} = \varepsilon_s \lim_{k,p \to 0} \frac{1}{Z_{i,s}^D} \hat{S}_{i,s,s}(k)^{-1} \hat{\Gamma}_{\mu;i,s}(k,p) \hat{S}_{i,s,s}(k+p)^{-1}$$

(35)

2. The constants $Z_{j,s}^W$, $j = l, q$ are chosen so that

$$\frac{1}{\sqrt{Z_{e,L}^D Z_{e,L}^D}} \lim_{k,p \to 0} \hat{S}_{\nu,s,s}(k)^{-1} \hat{\Gamma}_{\mu;\nu,e,s}(k,p) \hat{S}_{e,s,s}(k+p)^{-1} = \frac{1}{\sqrt{Z_{d,L}^D Z_{u,L}^D}} \lim_{k,p \to 0} \hat{S}_{\mu,s,s}(k)^{-1} \hat{\Gamma}_{\mu;q,u,d,s}(k,p) \hat{S}_{d,s,s}(k+p)^{-1} = \delta_s \sigma_{\mu}$$

(36)

with $\delta_R = 1$ and $\delta_L = 0$.

3. The fermionic mass counterterms $\nu_{i,s}$ are chosen so that the dressed masses are vanishing, in the sense that the 2-point functions decay as a power law for large distances.

Note that the dressed electric charges (34) could acquire renormalizations different for each particle and chirality but, as we will see below, e.m. chiral gauge invariance ensures that this is not the case; the renormalizations cancel out and the bare and dressed charge are equal. While the symmetry (31) dictates the form of the e.m. current (29), this is not true for the axial current (24); its definition is dictated by the fact that it must reduce to the standard definition of the chiral current in the continuum limit and by the fact that the charges carried by the axial current are the same as the one associated to the e.m. current. This is obtained by properly tuning the normalizations $Z_{i,s}^5$ in (24) so that (35) holds. The lack of protection of the charges associated to the chiral current is due to the lattice breaking of chiral symmetry, but is even true in perturbative QFT with dimensional regularization, see [8]; the absence of corrections in the anomaly is true only provided that the correct normalization for the axial current is chosen. With condition (35), one can define the chiral currents as $\frac{1}{2}(\hat{\gamma}_{\mu;i,p}^{c.m.} \pm \hat{\gamma}_{\mu;i,p}^{\text{f.m.}})$. Similarly in (36) we impose that the charges associated to the $L$-handed component of the $W$ current are equal, and the $R$-handed component is vanishing. While the first two conditions are imposed for physical reasons, the condition 3 is just for technical convenience; the dressed and bare masses are different, and fixing the value of the dressed mass equal to the non interacting one is somewhat convenient. In particular, the dressed mass of fermions is assumed vanishing, but of course all the analysis could be repeated choosing any dressed mass.
1.4 Ward Identities

We define

\[
\frac{\partial}{\partial J_{\mu,x}^{e,m.}} = \sum_i \frac{\partial}{\partial J_{i,x}^e} \tag{37}
\]

and the corresponding vertex functions

\[
\Gamma^{\Lambda,e.m.}_{\mu,i,s}(z,x,y) = \frac{\partial^3}{\partial J_{\mu,x}^{e,m.} \partial \phi_{i,x}^e \partial \phi_{i,s,y}^e} W(J,J^5,J^W,\phi) \bigg|_0 \tag{38}
\]

The correlations are connected by relations known as Ward Identities. They can be obtained by performing the change of variables

\[
\psi_{i,c,s,j,x}^+ \rightarrow \psi_{i,c,s,j,x}^+ e^{\pm i e Q_{\alpha} \phi} \tag{39}
\]

with \(\alpha_x\) is a function on \(a\mathbb{Z}^4\), with the periodicity of \(\Lambda\). Let \(Q(\psi^+,\psi^-)\) be a monomial in the Grassmann variables and \(Q_{\alpha}(\psi^+,\psi^-)\) be the monomial obtained performing the replacement (39) in \(Q(\psi^+,\psi^-)\). It holds that

\[
\int \left[ \prod_{i,s,x} d\psi_{i,s,x}^+ d\psi_{i,s,x}^- \right] Q(\psi^+,\psi^-) = \int \left[ \prod_{i,s,x} d\psi_{i,s,x}^+ d\psi_{i,s,x}^- \right] Q_{\alpha}(\psi^+,\psi^-) \tag{40}
\]

as both the left-hand side and the right-hand side of (40) are zero unless the same Grassmann field \(\psi_{i,s,x}^+\) appears once in the monomial, hence the fields \(\psi_{i,s,x}^+\), \(\psi_{i,s,x}^-\) come in pairs and the \(\alpha\) dependence cancels. By linearity of the Grassmann integration, the property (40) implies the following identity, valid for any function \(f\) on the finite Grassmann algebra:

\[
\int \left[ \prod_{i,s,x} d\psi_{i,s,x}^+ d\psi_{i,s,x}^- \right] f(\psi) = \int \left[ \prod_{i,s,x} d\psi_{i,s,x}^+ d\psi_{i,s,x}^- \right] f_{\alpha}(\psi) \tag{41}
\]

with \(f_{\alpha}(\psi)\) the function obtained from \(f(\psi)\), after the chiral gauge transformation (39).

We apply now (41) to (3); the phase in the non-local terms can be exactly compensated by modifying \(J\), that is

\[
\psi_{i,s,x}^+ \sigma^a \mu^+ G_{\mu,i}^+(A + J) \psi_{i,s,x}^- = \psi_{i,s,x}^+ \sigma^a \mu^+ e^{i e Q_{\alpha} \phi} G_{\mu,i}^+(A + J + \partial \phi) e^{-i e Q_{\alpha} \phi} \psi_{i,s,x}^- \tag{42}
\]

as

\[
\int_0^a dt \partial \phi_{\alpha x + t \mu} = \alpha_{x + \mu} - \alpha_x \tag{43}
\]

and similar identities for the other non local terms. We get therefore

\[
W(J,J^5,0,\phi) = W(J + \partial \phi, J^5,0, e^{i e Q_{\alpha} \phi}) \tag{44}
\]

where \(J + \partial \phi\) is a shorthand for \(J_{\mu,x} + \partial \phi_{\mu x}\) and \(e^{i e Q_{\alpha} \phi}\) is a shorthand for \(e^{i e Q_{\alpha} \phi_{\mu x}} \phi_{\mu x}^\epsilon\).

By differentiating with respect to \(\alpha_x, \phi_y^+, \phi_z^-\) and passing to Fourier transform we get the Ward Identity,

\[
\sum_{\mu} p_\mu \hat{\Gamma}_{\mu, i,s}^{e,m.}(k,p) = e Q_i \left( \hat{S}_{i,s,x}(k) - \hat{S}_{i,s}(k + p) \right) \tag{45}
\]
1.5 Main result

The model (3) describes the electroweak theory up to energies of the order of the gauge mass. The electromagnetic current is a non-local expression, as we require the validity of Ward Identities for it; instead the $j_\mu^W, j_\mu^Z, j_\mu^5$ currents are local and fixed by the requirement that they reduce to the continuum expression in the limit. The Wilson term is necessary to have the correct scaling limit for the fermionic sector. If $r = 0$ there is an also invariance under the transformation

$$
\psi_{i,\mu,L,j,x}^+ \rightarrow \psi_{i,\mu,L,j,x}^+ e^{ieQ_\alpha x}, \quad \psi_{i,\mu,R,j,x}^+ \rightarrow \psi_{i,\mu,R,j,x}^+ e^{-ieQ_\alpha x}
$$

which would imply the conservation of the chiral e.m. current; however the choice $r = 0$ has the effect that unphysical particle states appear at low energies.

We define the following 3-current functions

$$
\Pi^A_{B,\mu_1;B,\mu_2;B,\mu_3}(x_1,x_2,x_3) = \sum_{i,s} \frac{\partial}{\partial J_{\mu_1,i,s,x_1}} \frac{\partial}{\partial J_{\mu_3,i,s,x_2}} \frac{\partial}{\partial J_{\mu_3,i,s,x_3}} W(J,J^5,J^W,\phi) |_0
$$

(46)

where

$$
\frac{\partial}{\partial J_{\mu_1,i,s,x}} = \frac{1}{2} Y_{i,s} \varepsilon_{i,s} \frac{\partial}{\partial J_{\mu_1,i,s}}, \quad \frac{\partial}{\partial J_{\mu_3,i,s,x}} = \frac{1}{2} Y_{i,s} \varepsilon_{i,s} \frac{\partial}{\partial J_{\mu_3,i,s}}.
$$

Similarly we define

$$
\Pi^A_{B,\mu_1;W,\mu_2;W,\mu_3}(x_1,x_2,x_3) = \sum_{i} \frac{\partial}{\partial J_{\mu_1,i,L,x_1}} \frac{\partial}{\partial J_{\mu_3,i,s,x_2}} \frac{\partial}{\partial J_{\mu_3,i,s,x_3}} W(J,J^5,J^W,\phi) |_0
$$

(47)

with

$$
\frac{\partial}{\partial J_{\mu_1,i,s,x}} = \frac{1}{2} \sum_{j,s} \frac{\partial}{\partial J_{\mu_1,i,j,s}}.
$$

In the following we prove the following result.

**Theorem 1.1.** Let us consider the generating function (3) and assume (30),(32). There exists $\varepsilon_0$ such that, for $e^2, g^2, g^2 \leq \varepsilon_0(Ma)^2$, with $M = \min(M,M_W,M_Z)$, $Ma \gg 1$ constants, it is possible to choose $\nu_{i,s}, Z^W_i, Z^W_{j,s}, j = l, q$ in (3) so that, in the limit $\Lambda \rightarrow \infty$, the 2-point function (25) decays as a power law for large distances and (35), (36) holds; moreover the dressed charges (34) verify

$$
e^D_{i,s} = e
$$

(48)

and the 3 current correlations (47) can be written as, $(\alpha_2, \alpha_3) = (\tilde{B}, \tilde{B})$ or $(\alpha_2, \alpha_3) = (W, W)$

$$
\tilde{\Pi}_{B,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2) = \tilde{\Pi}_{B,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2) + \tilde{\Pi}_{B,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2)
$$

(49)

with $\tilde{\Pi}_{B,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}$ with Holder continuous derivative while $\tilde{\Pi}_{B,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}$ non differentiable and such that

$$
p_{\mu_1} \tilde{\Pi}_{B,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2) = \frac{1}{12\pi^2} \varepsilon_{\mu_1,\mu_2,\alpha,\beta} \int_{p_1}^{p_2} [Y^L_\epsilon + Y^L_\nu + 3Y^L_u + 3Y^L_d] + O(ap^3)
$$

$$
p_{\mu_1} \tilde{\Pi}_{B,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2) = \frac{1}{12\pi^2} \varepsilon_{\mu_1,\mu_2,\alpha,\beta} \int_{p_1}^{p_2} [Y^L_\epsilon + 3(Y^L_u + 3Y^L_d - (Y^R_\epsilon)^3 - 3(Y^R_u)^3 - 3(Y^R_d)^3] + O(ap^3)
$$

(50)

The anomaly is expressed by a convergent expansion uniformly in the infinite volume and for a cut-off of the order of the gauge masses. The anomaly vanishes up to subdominant corrections which are rigorously bounded, under the condition that the hyperchargers are such that the expressions in [...] in the r.h.s. of (50) vanishes. This property is an essential prerequisite for the construction of electroweak theory in the high energy regime. Note that the condition...
implies a relation on the dressed charges. By (48) the dressed and bare charge coincide; if they would flow the cancellation of the anomaly at lower scales would not imply cancellation at higher scales, when is crucial for renormalizability. The cancellation is non perfect, as in the case of non chiral theories [23],[24], but this is related to well known properties of lattice chiral theories [28], [29] [30]; it should be however still sufficient for renormalizability of the higher energy region.

Interesting extensions of this result would be the removal of the mass of the photon field, by a multiscale expansion both in the bosonic and fermionic sector; even in the case of QED, the case of massless fermions and bosons (the physically significant case as the fermion masses are extremely small) on a lattice has never been faced at a non-perturbative level. Other interesting question would be consider massive fermions and investigate the problem of fermion decoupling in the anomaly cancellation, see [31]. The crucial issue is of course to use this result to construct the high energy regime of electroweak theory up to exponentially large cut-off.

The rest of this paper is organized in the following way. In §2 we integrate the A fields reducing to a fermionic theory; this is done exploring positivity properties using a representaion of truncated expactations [32] used also in [33],[34]. In §3 the fermionic sector is analyzed by multiscalar Renormalization Group using a tree expansion [35] and determinant bounds for fermions. In §4 the flow of running coupling constants is analyzed and finally in §6 the cancellation is established; in the appendix some properties of truncared expectations are recallled.

2 Integration of the e.m. field

We set $\gamma^{N+1} = \pi/4a$, $\gamma > 1$ and we define

$$e^{V_A(\psi, J)} = \int P(dA)e^{V_{e.m.}(A+J, \psi)}$$

with

$$V_A(\psi, J) = \sum_{l, m \geq 0} \int dx \prod_{j=1}^{n} O^{\xi_j}_{\mu_j}(\psi) [\prod_{i=1}^{m} (a^{-1} \int_{0}^{a} dt_i J_{\mu_i}(x_i + t_i e_{\mu_i}) )]W_{n,m}(x)$$

where $O^{\xi_j}_{\mu_j}(\psi)$ is defined in (19). Note that

$$\bar{V}_A(\psi, J) = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{E}^{T}_A(V_{e.m.}(A + J, \psi); ..., V_{e.m.}(A + J, \psi)) =$$

$$\sum_{\varepsilon, \mu} \int dx a^{-1}(e^{i\varepsilon \xi Q_1} \int_{0}^{a} dt J_{\mu_{1}}(x + te_{\mu_{1}})) - 1)O^{\xi}_{\mu}(\psi) +$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} \int dx_{1} ... \int dx_{n} a^{-n} [\prod_{j=1}^{n} O^{\xi_j}_{\mu_j}(\psi) e^{i\varepsilon \xi Q_1} \int_{0}^{a} dt J_{\mu_{1}}(x_{1} + t_{1} e_{\mu_{1}}) ... e^{i\varepsilon \xi Q_{n}} \int_{0}^{a} dt J_{\mu_{n}}(x_{n} + t_{n} e_{\mu_{n}}) ]$$

where $\mathcal{E}^{T}_A$ is the truncated expectation.

Lemma 2.1. The kernels in (52) verify the following bound, for $n \geq 2$

$$\frac{1}{L^n} \int dx |W_{n,m}(x)| \leq C^n \gamma^{(4-3n-m)N} e^{n(Ma)^{2-2n}}$$

(53)
Proof. By definition : 

\[ e^{\pm i e Q \int_0^n ds \mu (x + t e)} := e^{\pm i e Q \int_0^n ds (\mu (x + t e) + 2 (e Q)_2 g^{A}_\mu (0,0))} \]

is bounded by a constant as \(|g^{A}_\mu (0,0)| \leq C a^{-2}\). By (8) we can write, \(\varepsilon_i = \pm 1\)

\[
\mathcal{E}_A \left( \prod_i e^{i \varepsilon_i e Q_i} \int_0^n dt_i A_{\mu_i} (x_i + t_i e_{\mu_i}) \right) = e^{-V(X)}
\]


\[
V(X) = \frac{1}{2} \sum_{i,j \in X} V_{i,j}
\]

where \(X = (1, \ldots, n)\) and

\[
V_{i,j} = e^{2 \varepsilon_i \varepsilon_j Q_i Q_j} \mathcal{E} \left( \int_0^n dt_i A_{\mu_i} (x_i + t_i e_{\mu_i}) \int_0^n dt_j A_{\mu_j} (x_j + t_j e_{\mu_j}) \right) = e^{2 \varepsilon_i \varepsilon_j Q_i Q_j} \int_0^n dt_i \int_0^n dt_j g^{A}_{\mu_i,\mu_j} (x_i + t_i e_{\mu_i}, x_j + t_j e_{\mu_j})
\]

The truncated expectation \(\mathcal{E}_A^T = e^{-V(X)}|_T\) are obtained by solving recursively

\[
e^{-V(X)} = \sum_{\pi} \prod_{Y \in \pi} e^{-V(Y)|_T}
\]

where \(\pi\) are the partitions of \(X\). An explicit expression for the connected part is, (see [32] )

\[
e^{-V(X)}|_T = \sum_{g \in G} \prod_{i,j \in g} (e^{-V_{i,j}} - 1) \prod_{i \in X} e^{-V_{i,i}/2}
\]

where \(G\) is the sum over the connected graphs in \(X\). A different representation for (57) is however more convenient [32] (see also [33],[34]) , whose derivation is recalled in App. I

\[
e^{-V(X)}|_T = \sum_{T \in \mathcal{T}} \prod_{i,j \in T} V_{i,j} \int dp_T(s) e^{-V_T(s)}
\]

where

- \(\mathcal{T}\) is the set of tree graphs \(T\) on \(X\)
- \(s \in (0,1)\) is an interpolation parameter
- \(V_T(s)\) is a convex linear combination of \(V(Y) = \sum_{i,j \in Y} \varepsilon_i \varepsilon_j V_{i,j}\), \(Y\) subsets of \(X\).
- \(dp_T\) is a probability measure

Note that \(V(Y)\) is stable, that is

\[
V(Y) = \sum_{i,j \in Y} V_{i,j} = \sum_{i,j \in Y} e^{2 \varepsilon_i \varepsilon_j Q_i Q_j} \int_0^n dt_i \int_0^n dt_j g^{A}_{\mu_i,\mu_j} (x_i + t_i e_{\mu_i}, x_j + t_j e_{\mu_j}) = \\
\mathcal{E} (\int_{i \in Y} e Q_i \varepsilon_i \int_0^n dt_i A_{\mu_i} (x_i + t_i e_{\mu_i})^2) \geq 0
\]

hence

\[
V(s) \geq 0
\]

and

\[
\int dp_T (s) e^{-V(s)} \leq 1
\]
We can therefore write
\[
C^n a^{-n} \sum_{T \in \mathcal{T}} \prod_{i,j \in T} |g^A(x_i, x_j)|_1 \leq C M^{-2n} (\frac{M}{a})^{n-1}
\] where we have used that the number of $T$ is $\leq C^n n!$. Finally we can expand $e^{i \varepsilon t e Q_i \int_0^t dt_J(x_i + t_i e_{\mu_i} a)$ in series obtaining an extra $a$ for any $J$ so that the bound is $C^2(4^{-\frac{1}{2}}a)^N$.

\section{Integration of the fermionic field}

\subsection{Multiscale decomposition}

In contrast with the integration of the $A$ fields, the $\psi$ fields are massless and a multiscale integration procedure is necessary. If $\chi_0(t)$ is a Gevray class 2 function which is $= 1$ for $t \leq 1$ and $= 0$ for $t \geq \gamma$ with $\gamma > 1$, we define
\[
1 = \chi_N(k) + f_{N+1}(k)
\]
where $\chi_N(k) = \sum_{h=-\infty}^N f_h(k)$ with $f_h(k) = \chi_0(\gamma^{-h}|k|) - \chi_0(\gamma^{-h+1}|k|)$ and $f_h(k)$ non-vanishing for $\gamma^{-h} \leq |k| \leq \gamma^{-h+1}$; therefore $\chi_N(k) = 0$ for $|k| \geq \gamma^{N+1} = \pi/4a$ then $f_{N+1}(k)$ has support for $|k| \geq \pi/4a$. We define $g^{(N+1)}_i(k) = f_{N+1}(k)\tilde{g}_i(k)$; by (16)
\[
a^{-2} \sum_\mu (\sin k_\mu a)^2 + a^{-2} \sum_\mu (1 - \cos k_\mu a)^2 \geq a^{-2} \sum_\mu (1 - \cos k_\mu a)^2 \geq C/a^2
\]
for $|k| \geq \pi/4a$, and using that the volume of the support is $O(a^{-4})$, we get
\[
|g^{(N+1)}_i(x)| \leq C \gamma^{3N} e^{-c \gamma^N |x|^\frac{1}{2}}
\]
Regarding $g^{(h)}_i(x)$ in the support of $f^h$ one has that
\[
a^{-2} \sum_\mu (\sin k_\mu a)^2 + a^{-2} \sum_\mu (1 - \cos k_\mu a)^2 \geq a^{-2} \sum_\mu (\sin k_\mu a)^2 \geq C \gamma^{2h}
\]
The multiscale integration is defined inductively in the following way; assume that we have integrated the fields \( \psi^{(N)} \), \( \psi^{(N-1)} \), ... \( \psi^{(h)} \) obtaining (in the \( \phi = 0 \) for definiteness)

\[
e^{W(J, J^5, J^W, 0)} = \int PZ_h(d\psi^{(\leq h)})e^{V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, J, J^5, J^W)}
\]

with \( P(d\psi^{(\leq h)}) \) given by

\[
\hat{g}_i^{(\leq h)}(k) = \chi_h(k)(\sum_\mu \gamma_0^h \gamma_\mu a^{-1} i \sin(k_\mu a) + a^{-1} \gamma_0^h \sum_\mu (1 - \cos(k_\mu a)) \gamma_\mu)
\]

so that

\[
|g_i^{(h)}(x)| \leq C \gamma^h e^{-(\gamma h |x|)^{1/2}}
\]  

(70)

(71)

(72)

(73)

(74)

(75)

(76)

(77)
with $\varepsilon_L = -\varepsilon_R = 1$ and $j_{a,\mu,x}^{W,k} = \Psi^+_a\sigma_L^+\Psi^-_a\epsilon^\mu_{a,x}$. Note that the local part of the electromagnetic and axial current, see the last line of (77), is equal (up to a sign), even if the corresponding expression at scale $N$ were different (the e.m. current is non local while the axial current is local).

Some symmetry considerations restrict the possible terms obtained by the $\mathcal{L}$ operation. By (31) $\psi^+_{i,s,k}\psi^-_{i',s',k} \to e^{i(Q_i - Q'_i)\alpha}\psi^+_{i,s,k}\psi^-_{i',s',k}$, $J \to J$, $J_W^\pm \to e^{\pm i\varepsilon\alpha}J_W^\pm$; this imposes $i = i'$ on the fermionic indices for the terms non proportional to $J^W$. Regarding the chiral indices in $W_{2.0}$ if $s = s'$ then $\tilde{W}_{2.0}^h(k)$ is odd; expanding the propagator in series of $r$, and noting that the $r = 0$ propagator is odd $\tilde{g}_0^h(k) = -\tilde{g}_0^h(-k)$, then there is an even number of $\nu, r$ vertices, so that the number of $\tilde{g}_0^h(k)$ is odd. Similarly the kernel with opposite chirality is even. This says that in the $\tilde{W}_{2.0}^h(0)$ terms the fields have opposite chirality and $\tilde{\sigma}_{2.0}^h(0)$ the same chirality; for the same reason in $\tilde{W}_{2.1}^h(0)$ they have the same chirality. Finally $\hat{\sigma}_{\mu}W_{2.0}^h(0), W_{2.1}^h(0)$ are proportional to $\sigma^\mu_{\mu}$.

### 3.2 Anomalous integration

We can write

$$\tilde{\mathcal{L}}V^{(h)}(\psi, J, J^5, J^W) = \mathcal{L}V^{(h)} + \sum_{i,s} z_{h,i,s} \int dx \sigma_\mu^i \psi^+_{i,s,x} \tilde{\mathcal{L}}V^{(h)}(\psi, J, J^5, J^W) \quad (78)$$

so that

$$e^{W(J,J^5,J^W,0)} = \int P_{Z_{h-1}}(d\psi^{(\leq h)}) e^{\tilde{\mathcal{L}}V^{(h)}(\sqrt{Z_h\psi^{(\leq h)}}, J, J^5, J^W) + \mathcal{L}V^{(h)}(\sqrt{Z_h\psi^{(\leq h)}}, J, J^5, J^W)} \quad (79)$$

where $P_{Z_{h-1}}(d\psi^{(\leq h)})$ has propagator given by (72) with $Z_{h,i,s}$ replaced by $Z_{h-1,i,s}(k) = Z_{h,i,s} + \chi(k)z_{h,i,s}$

$$Z_{h-1,i,s}(k) = Z_{h,i,s} + \chi(k)z_{h,i,s} \quad (80)$$

Setting $Z_{h-1,i,s}(0) = Z_{h-1,i,s}$, we can write

$$P_{Z_{h-1}}(d\psi^{(\leq h)}) = P_{Z_{h-1}}(d\psi^{(\leq h-1)})P_{Z_{h-1}}(d\psi^{(h)}) \quad (81)$$

with $P_{Z_{h-1}}(d\psi^{(\leq h-1)})$ with propagator (72) with $h$ replaced by $h - 1$ and $P_{Z_{h-1}}(d\psi^{(h)})$ has propagator which can be written as, if $|r_h|, |Z_{h,i,s}| \leq e^{C\varepsilon}, \varepsilon = \max(e^2, g^2, \bar{g}^2)$

$$g_i^{(h)}(x, y) = g_{i,\text{red}}^{(h)}(x, y) + r_i^{(h)}(x, y) \quad (82)$$

with

$$g_{i,\text{red}}^{(h)}(x, y) = \frac{1}{L^d} \sum_k e^{ik(x-y)} f_h(k) \begin{bmatrix} Z_{h-1,L,i}(k)(i\bar{k}0 + \sum_j \sigma_j k_j) & 0 \\ 0 & Z_{h-1,R,i}(k)(i\bar{k}0 - \sum_j \sigma_j k_j) \end{bmatrix}^{-1} \quad (83)$$

and

$$|g_{i,\text{red}}^{(h)}(x, y)| \leq C\frac{\gamma^3}{Z_{h-1,L,i}} e^{-(\gamma|h|x-y|)^\frac{1}{2}} \quad |r_i^{(h)}(x, y)| \leq C\gamma^3 e^{-(\gamma|h|x-y|)^\frac{1}{2}} \quad (84)$$

Finally we define

$$e^{V^{(h-1)}(\sqrt{Z_{h-1}\psi^{(\leq h-1)}})} = \int P_{Z_{h-1}}(d\psi^{(h)}) e^{\tilde{\mathcal{L}}V^{(h)}(\sqrt{Z_{h-1}\psi^{(\leq h)}}, J, J^5, J^W)} \quad (85)$$
with
\[ \tilde{C}_{\gamma}^{(h)}(\sqrt{Z_{h-1}}) = \int dx \sum_{i,s} v_{h,s,i} \sqrt{Z_{h-1,L,i}Z_{h-1,L,i}} \gamma^{h}(\psi_{i,L,x}^{+} \psi_{i,L,x}^{-} + \psi_{i,R,x}^{+} \psi_{i,R,x}^{-}) \]
\[ \sum_{j=l,q} \sum_{k=L,R} Z_{h-1,k,j,s}^{W} \int dx j_{i,s,j,x}^{k,W} \psi_{i,s,x}^{+} \psi_{i,s,x}^{-} + \sum_{i,s} Z_{i,s,h-1}^{e.m.} \int dx J_{\mu,x} \psi_{i,s,x}^{+} \sigma_{\mu} \psi_{i,s,x}^{-} \]
\[ \sum_{i,s} \varepsilon Z_{i,s,N}^{5} \int dx J_{\mu,x} \sigma_{\mu} \psi_{i,s,x}^{+} \psi_{i,s,x}^{-} \]
and the procedure can be iterated. Note that
\[ Z_{i,s,N} = 1 \quad Z_{i,s,N}^{e.m.} = 1 + O(\varepsilon^2) \] (87)
as follows from Lemma 3.1; moreover \( \nu_{N}, Z_{N}^{W}, Z_{N}^{5} \) are parameters which will be chosen below in order to verify the renormalization conditions.

### 3.3 Convergence and analyticity

We prove the following lemma, setting \( \nu_{h,i,s} = \tilde{\nu}_{h,i,s}(aM)^{-2} \).

**Lemma 3.1.** There exists a constant \( \varepsilon_{h} \) such that, for \( |r_{h}|, |Z_{h,i,s}| \leq e^{C_{\gamma}h(aM)^{2}}, \max(|\tilde{\nu}_{i,s,h}|, g^{2}, \tilde{g}^{2}, \varepsilon^{2}) \leq \varepsilon_{h}(aM)^{2} \) then
\[ \int dxdy | \tilde{W}_{x,y}^{(h)}(l,m)(x,y) | \leq C_{l+m}^{(4-(3/2)(l-m))h} \varepsilon_{h}^{\max(l/2-1,1)} \] (88)

**Proof** We write the kernels \( \tilde{W}_{x,y}^{(h)}(l,m) \) in in terms of Gallavotti trees [35], see Fig.1, defined in the following way (for details see e.g. §3 of [36]).

Let us consider the family of all trees which can be constructed by joining a point \( r \), the root, with an ordered set of \( n \geq 1 \) points, the endpoints of the unlabeled tree, so that \( r \) is not a branching point. \( n \) will be called the order of the unlabeled tree and the branching points will be called the non trivial vertices. The unlabeled trees are partially ordered from the root to the
endpoints in the natural way; we shall use the symbol \(<\) to denote the partial order. The number of unlabeled trees is \(4^n\). The set of labeled trees \(T_{h,n}\) is defined associating a label \(h \leq N - 1\) with the root; moreover we introduce a family of vertical lines, labeled by an an integer taking values in \([h, N + 1]\) intersecting all the non-trivial vertices, the endpoints and other points called trivial vertices. The set of the vertices \(v\) of \(\tau\) will be the union of the endpoints, the trivial vertices and the non trivial vertices. The scale label is \(h\), and, if \(v_1\) and \(v_2\) are two vertices and \(v_1 < v_2\), then \(h_{v_1} < h_{v_2}\). Moreover, there is only one vertex immediately following the root, which will be denoted \(v_0\) and can not be an endpoint; its scale is \(h + 1\).

There are two kinds of endpoints, normal and special, and \(n = \bar{n} + m\). The normal end-points are \(\bar{n}\) and are associated to terms in the effective potential not depending from the external fields \(J\). The \(\nu\)-endpoints are associated to the first line of (86) and have scale \(h_v \leq N + 1\) and there is the constraint that \(h_v = h_{v'} + 1\), if \(v'\) is the first non trivial vertex immediately preceding \(v\). The \(V_G\)-endpoints have scale \(h_v = N + 1\) and are associated one of the terms in (20), and to the \(V_A\)-endpoints is associated one of the terms in (52) with \(m = 0\). The special end-endpoints have associated terms with at least an external \(J\) fields; the \(Z^W, Z^{e.m.}, Z^5\) end-points have \(h_v \leq N + 1\) and there is the constraint that \(h_v = h_{v'} + 1\), and are associated with one of the terms in the second line of (86); the \(V_A\) end-points have scale \(N\) and are associated to the terms with \((n, m) \neq (1, 1)\) in (52).

The effective potential can be written as

\[
\mathcal{V}^{(h)}(\psi^{(\leq h)}(J, J^5, J^W)) = \sum_{n=1}^{\infty} \sum_{v \in T_{h,n}} \mathcal{V}^{(h)}(\tau),
\]

where, if \(v_0\) is the first vertex of \(\tau\) and \(\tau_1, \ldots, \tau_s\ (s = s(v_0))\) are the subtrees of \(\tau\) with root \(v_0\), \(\mathcal{V}^{(h)}\) is defined inductively by the relation, \(h \leq N - 1\)

\[
\mathcal{V}^{(h)}(\tau) = \frac{(-1)^{s+1}}{s!} \mathcal{E}_h \big[ \mathcal{V}^{(h+1)}(\tau_1) \ldots \mathcal{V}^{(h+1)}(\tau_s) \big]
\]

where \(\mathcal{E}_h\) is the truncated expectation and \(\mathcal{V}^{(h+1)}(\tau) = \mathcal{R} \mathcal{V}^{(h+1)}(\tau)\) if the subtree \(\tau_i\) contains more then one end-point. We define \(P_v\) as the set of field labels of \(v\) representing the external fields and if \(v_1, \ldots, v_{s_v}\) are the \(s_v\) vertices immediately following \(v\), then we denote by \(Q_{v_1}\) the intersection of \(P_v\) and \(P_{v_1}\); this definition implies that \(P_v = \cup_i Q_{v_i}\). The union of the subsets \(P_{v_i}\) are the internal fields of \(v\). Therefore if \(P_\tau\) is the family of all such choices and \(P\) an element we can write

\[
\mathcal{V}^{(h)}(\tau) = \sum_{P \in P_\tau} \int dx_{v_0} W^{(h+1)}_{\tau,P}(x_{v_0}) \left[ \prod_{f \in P_{v_0}} \psi^{(f)}(x_f) \right] \prod_{f} J(x_f)
\]

where \(W^{(h+1)}_{\tau,P}(x_{v_0})\) is defined inductively by the equation

\[
W^{(h+1)}_{\tau,P}(x_{v_0}) = \frac{1}{s_v!} \left[ \prod_{i=1}^{s_v} W^{(h+1)}_{\tau,P}(x_{v_i}) \right] \mathcal{E}_h \big[ \tilde{\psi}^{(h)}(P_{v_i}/Q_{v_i}); \ldots; \tilde{\psi}^{(h)}(P_{v_{s_v}}/Q_{v_{s_v}}) \big]
\]

where \(\tilde{\psi}^{(h)}(P) = \prod_{f \in P} \psi^{(f)}(x_f)\) and \(x_v\) are the coordinates associated to the vertex \(v_0\). We get

\[
W^{(h)}(x_v) = \sum_{\tau \in T_{h,n}} \sum_{P_{v_0} | P_{v_0}| = 1 + m} W^{(h)}_{\tau,P}(x_v)
\]
We use the analogue of (58) for fermionic truncated expectation,

\[ \mathcal{E}_h^T(\tilde{\psi}^{(h)}(P_1); \tilde{\psi}^{(h)}(P_2); \ldots; \tilde{\psi}^{(h)}(P_s)) = \sum_T \prod_{t \in T} g^{(h)}(x_t - y_t) \int dP_T(t) \det G^{h,T}(t) \]

where

1. \( T \) is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster.
2. \( t = \{ t_{i,i'} \in [0,1], 1 \leq i, i' \leq s \} \) and \( dP_T(t) \) is a probability measure with support on a set of \( t \) such that \( t_{i,i'} = u_i \cdot u_{i'} \) for some family of vectors \( u_i \in \mathbb{R}^s \) of unit norm.
3. \( G^{h,T}(t) \) is a \((n - s + 1) \times (n - s + 1)\) matrix, whose elements are given by \( G_{i,j,i',j'}^{h,T} = t_{i,i'} g^{(h)}(x_{ij} - y_{ij'}) \).
4. If \( \mathcal{H} = \mathbb{R}^s \otimes \mathcal{H}_0 \), where \( \mathcal{H}_0 \) is the Hilbert space of complex two dimensional vectors with scalar product \(< F, G > = \int dkF^*_k(k)G_k(k) \). It is easy to verify that

\[ G_{i,j,i',j'}^{h,T} = t_{i,i'} g^{(h)}(x_{ij} - y_{ij'}) = < u_i \otimes A^{(h_v)}_{x(f_{ij})}, u_{i'} \otimes B^{(h_v)}_{x(f_{ij'})} > \]

where \( u_i, \in \mathbb{R}^s, i = 1, \ldots, s \), are the vectors such that \( t_{i,i'} = u_i \cdot u_{i'} \) and \( A, B \) suitable functions.

By inserting the above representation we can write \( W^{(h+1)}_{\tau, P} = \sum_T W^{(h+1)}_{\tau, P, T} \) where \( T \) is the union of all the trees \( T \).

The determinants are bounded by the Gram-Hadamard inequality, stating that, if \( M \) is a square matrix with elements \( M_{ij} \) of the form \( M_{ij} = < A_i, B_j > \), where \( A_i, B_j \) are vectors in a Hilbert space with scalar product \(< \cdot, \cdot > \), then

\[ | \det M | \leq \prod_i || A_i || \cdot || B_i || \]

where \( || \cdot || \) is the norm induced by the scalar product.

The tree \( T \) connects a set of \( n \)-endpoints; to each point \( v \) are associated \( i_v, j_v \) \( \psi \)-fields and \( j_v \) \( J \)-fields; we define \( m_i^{(j)} \) the number of end-points following \( v \) with \( i \) \( \psi \) fields and \( j \) \( J \) fields. The integral over the difference of coordinates associated to propagators gives a factor \( \prod_v \gamma^{-4h_v(s_v-1)} \); to the normal \( V_G \) end-points is associated \( \gamma^{-2N(1/(Ma)^2)} \); to the \( V_A \) end-points is associated a factor \( \gamma^{4(4-3i_v/2-j_v)N}e^{i_v/2(aM)^2} \) with \( 4 - 3i_v/2 - j_v < 0 \) and \( i_v \geq 4 \) and \( (aM)^2 - i_v < e^2(aM)^{-2} \). In conclusion we get

\[ \int dx_v |W_{\tau, P, T}(x_v)| \leq L^4 \prod_{v \not\in e.p.} \frac{1}{\gamma^{h_v} \gamma^{-4h_v(s_v-1)}} \left( \prod_v \gamma^{-z_v(h_v-h_v')} \right) \left( \prod_{v \not\in e.p.} \gamma^{4(4-3i_v/2-j_v)N} \right) \]

where \( \prod_{v \not\in e.p.} \) is the product over the end-points excluded the \( \nu \) ones, \( n \) is the number of normal end-points, \( \prod_v \gamma^{-z_v(h_v-h_v')} \) is produced by the \( R \) operation and \( z_v = 2 \) if \( |P_v| = 2 \) and there are no \( J \) fields, \( z_v = 1 \) if \( |P_v| = 2 \) and there is a single \( J \) field, \( z_v = 0 \) otherwise. Note
that $\prod_{v\ e.p., \ not \ \nu} \gamma^{(4-3\iota_v/2-j_v)N}$ includes the contributions of the $Z$ special end-points, where $4 - 3\iota_v/2 - j_v = 0$. By using that

$$\sum_v (h_v - h)(s_v - 1) = \sum_v (h_v - h_v')(\sum_{i,j} m_{i,j}^v - 1)$$

$$\sum_v (h_v - h)(\sum_i |P_v| - |P_v|) = \sum_v (h_v - h_v')(\sum_{i,j} \iota m_{i,j}^v - |P_v|) \quad (97)$$

where $m_{i,j}^v$ is the number of end-points following $v$ with $i$ $\psi$ fields and $j$ $J$ fields, we get, if $\bar{n}$ is the number of normal endpoints

$$\int dx_{v_0} |W_{\tau, \mathbf{P}, T}(x_{v_0})| \leq L^4 \gamma^{h[-4 + \frac{3|P_{v_0}|}{2} - \sum_{i,j}(3i/2-4)m_{i,j}^v]} \bar{n}$$

$$\prod_{v \ not \ e.p.} \left\{ \frac{1}{s_v!} C^{\sum_{i,j} |P_v| - |P_v|} \gamma^{h[-4 + \frac{3|P_{v_0}|}{2} - \sum_{i,j}(3i/2-4)m_{i,j}^v + z_v](h_v - h_v')} \right\} \left[ \prod_{v \ e.p., \ not \ \nu} \gamma^{(4-3\iota_v/2-j_v)N} \right] \quad (98)$$

We use now that

$$\gamma^h \sum_{i,j} m_{i,j}^v \prod_{v \ not \ e.p.} \gamma^{\sum_{i,j}(h_v - h_v')(m_{i,j}^v)} = \prod_{v \ e.p.} \gamma^{h*} \quad (99)$$

so that

$$\int dx_{v_0} |W_{\tau, \mathbf{P}, T}(x_{v_0})| \leq L^4 \gamma^{h[-4 + \frac{3|P_{v_0}|}{2} + \sum_{i,j} m_{i,j}^v]} \bar{n} \quad \prod_{v \ e.p., \ not \ \nu} \gamma^{(4-3\iota_v/2-j_v)N} \quad (100)$$

Finally we use the relation

$$[\prod_{v \ e.p.} \gamma^{h*} \gamma^j] [\prod_{v \ e.p.} \gamma^{h*} \gamma j] = [\prod_{v \ e.p.} \gamma^{h*} \gamma j] \gamma^{h} \sum_{i,j} n_{i,j}^v \prod_{v \ not \ e.p.} \gamma^{-\sum_{i,j}(h_v - h_v')j_{i,j}^v} \quad (101)$$

and using that $\sum_{i,j} j_{i,j}^v = n_{i,j}^v$ we finally get ($j_v = 0$ if $v$ is a $\nu$-e.p.)

$$\int dx_{v_0} |W_{\tau, \mathbf{P}, T}(x_{v_0})| \leq L^4 \gamma^{h[-4 + \frac{3|P_{v_0}|}{2} + n_{i,j}^v]} \bar{n}$$

$$\prod_{v \ not \ e.p.} \left\{ \frac{1}{s_v!} C^{\sum_{i,j} |P_v| - |P_v|} \gamma^{h[-4 + \frac{3|P_{v_0}|}{2} + z_v + n_{i,j}^v](h_v - h_v')} \right\} \left[ \prod_{v \ e.p., \ not \ \nu} \gamma^{(4-3\iota_v/2-j_v)(N - h* \gamma)} \right] \quad (102)$$

In conclusion

$$\int dx_{v_0} |W_{\tau, \mathbf{P}, T}(x_{v_0})| \leq L^4 \gamma^{h_{v_0}} \prod_{v} \gamma^{(4-3\iota_v/2-j_v)(N - h* \gamma)} \quad (102)$$
It is a byproduct of the tree expansion defined in the previous section that \( \nu_{i,s,h} \) verify the following equation

\[
\nu_{h-1,i,s} = \gamma \nu_{h,i,s} + \beta_{\nu,i,s}^{(h)}
\]

where

\[
\beta_{\nu,i,s}^{(h)} = \sum_{n=2}^{\infty} \sum_{T} \sum_{P,T} \frac{1}{n!} \int dx_{v_0} |W_{\tau,P,T}(x_{v_0})|.
\]

where \( |P_{v_0}| = 2 \), \( h_{v_0} = h + 1 \) is a non trivial vertex and \( T^* \) is the set of trees with at least a normal end-points not of \( \nu, Z \) type; this last condition is due to the fact that in momentum space the kernels are computed at vanishing momenta and when only end-points \( \nu \) are present only chain graphs contribute; therefore they are vanishing as \( g^h(0) = 0 \) by the compact support properties of the single propagator. Therefore by (104), if \( \bar{g}^2 = \max(g^2, \bar{g}^2, e^2) \)

\[
|\beta_{\nu,i,s}^{(h)}| \leq C \gamma^{\theta(h-N)} \left[ \max(\bar{g}^2, \bar{v}_h)(Ma)^{-2} \right]^2
\]
The tree expansion is convergent provided that \((g^2, \bar{g}^2, e^2, \bar{\nu}_h)(Ma)^{-2}\) is smaller than some constant; this condition can be verified choosing \(g^2 \leq c(Ma)^2\), a condition which can be always verified. One needs however a similar condition on \(\nu_h\), which according to (110) is generically \(O(\gamma^{-h})\); it is possible however to suitably choose \(\nu \equiv \nu_N\) so that \(\nu_h\) is bounded for any \(h\). We can rewrite (110) as

\[
\nu_{h-1,i,s} = \gamma^{-h}(\nu_N,i,s + \sum_{k=h}^{N} \gamma^{k}\beta^{(h)}_{\nu,i,s}) \tag{110}
\]

We consider the system

\[
\nu_{h-1,i} = \gamma^{-h}(-\sum_{k=h}^{N} \gamma^{k}\beta^{(h)}_{\nu,i}) \tag{111}
\]

Note that \(\beta^{(h)}_{\nu,i}\) is a function of \(g, \bar{g}, e\) and of the effective parameters \(\nu_{i,k}\) with \(k \geq h\). Therefore, we can regard the right side of (111) as a function of the whole sequence \(\nu_{i,k}\), which we can denote by \(\nu = \{\nu_k\}_{k \leq N}\) so that (111) can be read as a fixed point equation \(\nu = T(\nu)\) on the Banach space of sequences \(\nu\) such that \(|\nu|| = sup_{k \leq N} \gamma^{(h-kN)}|\nu_k| \leq C\bar{g}^2(Ma)^{-2}\). It is a corollary of the proof in Lemma 4.1 (see e.g. App A5 of [39] for details) that there is a choice of \(\nu_i\) such that the sequence is bounded for any \(h\). Therefore for a proper \(\nu\)

\[
|\nu_k| \leq C\gamma^{(h-N)}\bar{g}^2(Ma)^{-2} \tag{112}
\]

It remains to check the condition on boundedness of the effective renormalizations. They verify recursive equations, if \(Z_h = (Z_{i,s,h}, Z_{i,s,h}^c, Z_{i,s,h}^5, Z_{i,s,h}^W)\)

\[
Z_{h-1} = Z_h + \beta^{(2)}_{Z}(Z_h, ..., Z_N) \quad |\beta^{(2)}_{Z}| \leq C\gamma^{(h-N)}(g^2, \bar{g}^2, e^2)(Ma)^{-2} \tag{113}
\]

from which

\[
Z_{h-1} = 1 + \sum_{k=h}^{N} \beta^{(2)}_{Z}(Z_k, ..., Z_N) \tag{114}
\]

Therefore

\[
|Z_{-\infty} - 1| \leq C\bar{g}^2(Ma)^{-2} \tag{115}
\]

and

\[
|Z_{-\infty} - Z_h| \leq C\gamma^{(h-N)}\bar{g}^2(Ma)^{-2} \tag{116}
\]

Moreover we can choose \(Z^5_{s,i} = 1 + O(\bar{g}^2(Ma)^{-2})\) so that

\[
Z^5_{-\infty,s,i} = Z^c_{-\infty,s,i} \tag{117}
\]

and \(Z^W_{l,s,i}, Z^W_{i,s,i}\) such that

\[
Z^W_{l,s,i} = \sqrt{Z_{i,s,l}, \bar{Z}_{l,\bar{s},i}} \quad \bar{Z}_{l,\bar{s},i} = 0 \quad Z^W_{l,s,i} = 0 \tag{118}
\]

\[
Z^W_{q,s,i} = \sqrt{Z_{i,s,q}, \bar{Z}_{q,\bar{s},i}} \quad \bar{Z}_{q,\bar{s},i} = 0 \quad Z^W_{q,s,i} = 0 \tag{118}
\]

with

\[
Z^W_{l,s,i}, Z^W_{q,s,i} = 1 + O(\bar{g}^2(Ma)^{-2}) \quad Z^W_{l,s,i}, Z^W_{q,s,i} = O(\bar{g}^2(Ma)^{-2}) \tag{119}
\]

The existence of the limit \(L \to \infty\) for the kernels and expectations is an immediate consequence of the tree expansion; for an explicit derivation see e.g. App. D of [38]. Analyticity of the
correlations is an immediate consequence of the proof. Note that the denominator of the correlations (the partition function) at finite \(L\) is analytic in the whole complex plane as it is a finite dimensional Grassmann integral; on the other hand the RG analysis above provides an expansion which coincides order by order and is analytic in a finite domain, so that it fully reconstructs the partition function. The correlation is also analytic, as the denominator is non vanishing in a finite disk for small \(\varepsilon^2, g^2, \bar{g}^2\) for any \(L\) and the numerator is a finite dimensional integral; it coincides order by order with the expansion found analyzing the generating function by RG which is also analytic in the same domain so that they coincide.

5 The three current correlation

In order to compute the 2-point function we have to consider the generating function with \(\phi \equiv 0\); by a straightforward adaptation of the tree expansion, (for details see e.g. §3.D of [38]), one gets

\[
\hat{S}_i(k) = \left( \sum \tilde{\gamma}_-^{(\infty)} a^{-1} \sin(k_\mu a) + a^{-1} \gamma_0 \sum \mu (1 - \cos k_\mu a) \right)^{-1} (I + R(k))
\]  

(120)

with

\[
\tilde{\gamma}_0^{(\infty)} = \begin{pmatrix} Z_{-\infty,L,i} & 0 \\ 0 & Z_{-\infty,R,i} \end{pmatrix} \quad \tilde{\gamma}_j^{(\infty)} = \begin{pmatrix} iZ_{-\infty,L,i} & 0 \\ 0 & -iZ_{-\infty,R,i} \end{pmatrix}
\]

(121)

and

\[
|R(k)| \leq C g^2 |ka|^\theta
\]

(122)

where the extra factor \(|ka|^\theta\) follows from (104). From (33) we see therefore that \(Z_{-\infty,s,i} = Z_{s,i}^D\).

Regarding the vertex function, we can again separate the contribution from trees involving only \(Z\) vertex from the rest, which has by lemma 4.2 an improvement in the bound \(O(|ak|^\theta)\), with \(\kappa = \max(|k|, |k + p|)\). In the contribution of the \(Z\) terms the terms with \(s' \neq s\) and \(i' \neq i\) have an extra \(O(|ak|^\theta)\), so that

\[
\hat{\Gamma}_{\mu,i,i,s}^{e.m.}(k,p) = g_{i,s,s}(k)g_{i,s,s}(k + p)[Q_i e^{\sigma_s^* {Z_{s,i,-\infty}^{e.m.}}^{(\infty)}} + O(|ak|^\theta)]
\]

(123)

By combining the above expressions (120), (123) in the WI (45) we get

\[
\frac{Z_{s,i,-\infty}^{e.m.}}{Z_{s,i,-\infty}} = 1
\]

(124)

from which \(e_i^{D} = e\); using (124) and (118) in (132).

We consider now the three current correlations defined in (47); it turns out that

\[
\hat{\Pi}_{\alpha_1,\mu_1,\alpha_2,\mu_2,\alpha_3,\mu_3}(p_1, p_2) = \sum_{h=-\infty}^N dx_1 dx_2 W_{0,3}^h(0, x_2, x_3) e^{-ip_1 x_2 - ip_2 x_3}
\]

(125)

and using (88),(112), (115) we get

\[
|\hat{\Pi}_{\alpha_1,\mu_1,\alpha_2,\mu_2,\alpha_3,\mu_3}(p_1, p_2)| \leq \sum_{h=-\infty}^N C \gamma^h \leq \tilde{C}
\]

(126)
The Fourier transform is therefore bounded and, in the limit $L \to \infty$, continuous in $p_1, p_2$; it is however non differentiable.

We call $T^0_h$ the trees with only three special end-points $Z_h$ and $T^1_h$ the set of the remaining trees. The contributions from $T^1_h$ to $\hat{\Pi}_{\alpha_1,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2)$ have an extra $\gamma^{\theta(h-N)}$, by Lemma 4.2 and (12); therefore

$$\sum_{h=-\infty}^{N} \frac{1}{T^0} \sum_{\tau \in T_h} \|W_{\tau,P,T}(x_{10})\| \leq C \sum_{h=-\infty}^{N} g^2 \gamma^{\theta(h-N)} \leq Cg^2$$ \hspace{1cm} (127)

hence the contribution from trees $T^1_h$ is continuous and differentiable. We consider now the contribution from $T^0_h$; we can write the propagator as (82); the correction has again an extra $\gamma^{\theta(h-N)}$ by (116) and are therefore differentiable. We consider now

![Figure 2: Decomposition in terms of trees $T^0_h$ and $T^1_h$.](image)

the terms belonging to $T^0_h$ with only propagators $g^{(h)}_{x,y}(x,y)$; they are given by triangle graphs and to the $W$ vertices is associated $Z^W_h$, to the $B$ vertex $Z^0_h$ or $Z^{e.m.}_h$. We can replace the renormalizations $Z_h$ with $Z_{-\infty}$ and use (117), (124); the corrections has again an extra $\gamma^{\theta(h-N)}$ by (116) and are therefore differentiable. The conclusion is that

$$\hat{\Pi}_{\alpha_1,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2) = \hat{\Pi}_{\alpha_1,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2) + \hat{\Pi}_{\alpha_1,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2) \hspace{1cm} (128)$$

where $\hat{\Pi}_{\alpha_1,\mu_1;\alpha_2,\mu_2;\alpha_3,\mu_3}(p_1,p_2)$ is differentiable while, $p = p_1 + p_2$, $\tilde{\varepsilon}_{i,L} = -\tilde{\varepsilon}_{i,R} = 1$, $a_L = -a_R = 1$

$$\hat{\Pi}_{B,\mu,B,\nu,B,\bar{\nu}}(p_1,p_2) = \sum_{i,s,c,s' = L,R} \xi_{i,s,c,s'} Y^3_{i,s} \left( \frac{Z_{-\infty,i,s}}{Z_{-\infty,i,s'}} \right)^3 \sum_{h_1,h_2,h_3} \int \frac{dk}{(2\pi)^4}$$ \hspace{1cm} (129)

We can now perform the sums over $h_1, h_2, h_3$ obtaining

$$\hat{\Pi}_{B,\mu,B,\nu,B,\sigma}(p_1,p_2) = \sum_{i,s,c} Y^3_{i,s} \xi_s \int \frac{dk}{(2\pi)^4}$$ \hspace{1cm} (130)

$$\text{Tr} \frac{\gamma_5}{k} \frac{\gamma_\mu}{k+p} \frac{\gamma_\nu}{k+p^2} \frac{\gamma_\sigma}{k+p^2} \chi_{N}(k+p) (\nu, \xi) \to (\sigma, \xi)$$

and finally (see e.g. (3.155) of [23])

$$(p_{1,\mu_1} + p_{2,\mu_1}) \hat{\Pi}_{B,\mu_1;B,\nu;B,\bar{\nu}}(p_1,p_2) = -\frac{1}{12\pi^2} \int \left( p_1, \alpha p_2, \beta \varepsilon \alpha \beta \nu \sigma \right) F_{BB\bar{B}}(Y) \hspace{1cm} (131)$$
with
\[ F_{BBB}(Y) = (Y_e^L)^3 + (Y_{\nu}^L)^3 + 3(Y_u^L)^3 + 3(Y_d^L)^3 - (Y_e^R)^3 - (Y_{\nu}^R)^3 - 3(Y_u^R)^3 - 3(Y_d^R)^3 \] (132)

Proceeding in a similar way and using that
\[ \frac{Z_{-x,L,e}(Z_{-x,L,u})^2}{Z_{-x,L,e}^2} = \frac{Z_{-x,L,e}(Z_{-x,L,u})^2}{Z_{-x,L,e}^2} = \frac{Z_{-x,L,u}(Z_{-x,L,q})^2}{Z_{-x,L,u}^2} = \frac{Z_{-x,L,q}(Z_{-x,L})^2}{Z_{-x,L,u}^2} = 1 \] (133)

we get
\[ (p_{1,\mu_1} + p_{2,\mu_1}) \Pi_{B,\mu_1;W;W',\sigma}(p_1, p_2) = -\frac{1}{12\pi^2} p_{1,\alpha} p_{2,\beta} \epsilon_{\alpha\beta\nu\sigma} F_{BWV}(Y) \] (134)

with
\[ F_{BWV}(Y) = Y_e^L + Y_{\nu}^L + 3Y_u^L + 3Y_d^L \] (135)

This concludes the proof of the main Theorem.

6 Appendix I. Truncated expectations

For completeness we recall the proof of (57), referring for more details to App. B of [32] (see also [38], [36] or [40]). If \( X = (1, 2, \ldots, n) \), we call \( V(X) = \sum_{i,j \in X} V_{i,j} = \sum_{i \leq j} V_{i,j} \) with \( V_{i,i} = V_{i,i} \) and \( V_{i,j} = (V_{i,j} + V_{j,i})/2 \). The starting point is the following formula
\[ e^{-V(X)} = \sum_{Y \subset X} K(Y) e^{-V(X/Y)} \] (136)

with \( Y = X_1, |X_1| = 1 \) then \( K(X_1) = e^{-V(X_1)} \) and for \( r \geq 2 \)
\[ K(X_r) = \sum_{T} \prod_{l \in T} |V_l| \sum_{t_{2,2} X_1, \ldots, X_r-1} \int_0^1 dt_1 \ldots \int_0^1 dt_r \ldots \int_0^1 \frac{\prod_{k=1}^{r-1} t_k(l)}{t_n(l)} e^{-W_X(X_1, \ldots, X_{r-1}; t_1, \ldots, t_{r-1})} \] (137)

where \( T \) is a tree connecting the points \( (1, \ldots, n) \), \( l \) are bonds in \( T, X_1 \subset X_2 \subset \ldots \subset X_{r-1} \) are sets such that \( |X_i| = i \) and each boundary \( \partial X_i \) is crossed by at least a \( l \in T \),
\[ W_X(X_1, \ldots, X_r; t_1, \ldots, t_r) = \sum_{l} t_1(l) t_2(l) \ldots t_r(l) V_l \] (138)

with \( t_i(l) = t_i \) if \( l \) crosses \( \partial X_i \) and \( t_i(l) = 1 \) otherwise, \( n(l) \) is the max over \( k \) such that \( l \) crosses \( \partial X_k \). In order to prove (136) we start noting that we can reverse the sum over \( T \) and \( X \)
\[ \sum_{T} \sum_{X_1, \ldots, X_{r-1}} = \sum_{X_1, \ldots, X_{r-1}} \sum_{T} \] (139)

where in the r.h.s. \( T \) is a tree composed by \( r-1 \) lines \( l \) such that all the boundaries \( \partial X_k \) are intersected at least by a line \( l \). We write by (138), if \( X_1 = \{1\} \)
\[ W_X(X_1; t_1) = \sum_{l} t_1(l) V_l \] (140)

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with \( t_1(l) = t_1 \) if \( l \) crosses \( \partial X_1 \) and \( = 1 \) otherwise; that is

\[
W_X(X_1, t_1) = V_{1,1} + t_1 \sum_{k \geq 2} V_{1,k} + \sum_{2 \leq k < k'} V_{k,k'} =
\]

\[
t_1(V_{1,1} + \sum_{k \geq 2} V_{1,k} + \sum_{2 \leq k < k'} V_{k,k'}) + (1 - t_1)(V_{1,1} + \sum_{2 \leq k < k'} V_{k,k'}) =
\]

\[
t_1V(X) + (1 - t_1)(V(X_1) + V(X/X_1))
\]

(141)

We get \( W_X(X_1, 0) = V(X_1) + V(X/X_1) \) and \( \partial_t W(X_1, t_1) = \sum_{k \geq 2} V_{1,k} = \sum_{t_1} V_{1,t_1} \) so that

\[
e^{-V(X)} = \int_0^1 dt_1 \partial_t e^{-W_X(X_1,t_1)} + e^{-W_X(X_1,0)} = \int_0^1 dt_1 \sum_{k \geq 2} V_{1,k} e^{-W_X(X_1,t_1)} + e^{-V(X_1)} e^{-V(X/X_1)}
\]

(142)

Therefore \( e^{-V(X)} \) is decomposed in the sum of two terms; in the first there is a bond \( (1,k) \) between \( X_1 \) and the rest, in the second \( X_1 \) is decoupled. If \( n = 2 \) coincides with (136), \( Y = X_1, X_2 \) with \( X_2 = X \), \( e^{-V(X/X_2)} = 1 \).

If \( X_2 \neq X \) we further decompose the first term in the r.h.s of (142); we write \( X_2 = \{1,k\} \) and

\[
\int_0^1 dt_1 \sum_{k \geq 2} V_{1,k} e^{-W_X(X_1,t_1)} =
\]

\[
\int_0^1 dt_1 \sum_{k \geq 2} V_{1,k} \int_0^1 dt_2 \partial_t e^{-W_X(X_1,X_2,t_1,t_2)} + \int_0^1 dt_1 \sum_{k \geq 2} V_{1,k} e^{-W_X(X_1,X_2,t_1,0)}
\]

(143)

where

\[
W_X(X_1, X_2, t_1, t_2) = (1 - t_2)[W_{X_2}(X_1, t_1) + V(X/X_2)] + t_2 W_{X_1}(X_1, t_1)
\]

(144)

If for instance \( X_2 = \{1,2\} \) then

\[
W_X(X_1, X_2, t_1, t_2) = V_{1,1} + V_{2,2} + t_1 t_2 \sum_{k \geq 3} V_{1,k} + t_1 V_{1,2} + t_2 \sum_{k \geq 3} V_{2,k} + \sum_{3 \leq k < k'} V_{k,k'}
\]

(145)

Suppose that \( X = \{1,2,3\} \) and \( X_2 = \{1,2\} \), then

\[
\int_0^1 dt_1 V_{1,2} e^{-W_X(X_1,t_1)} =
\]

\[
\int_0^1 dt_1 V_{1,2} \int_0^1 dt_2 (t_1 V_{1,3} + V_{2,3}) e^{-W_X(X_1,X_2,t_1,t_2)} + [\int_0^1 dt_1 V_{1,2} e^{-W_{X_2}(X_1,t_1)}] e^{-V(X/X_2)}
\]

The first term is \( Y = X_3 \) and the trees are \( l_1 = (1,2), l_2 = (2,3) \) so that \( t_1(l_1) = t_1, t_1(l_2) = 1, t_2(l_2) = t_2; \) and \( l_1 = (1,2), l_2 = (1,3) \) so that \( t_1(l_1) = t_1 \) and \( t_1(l_2) = t_1, t_2(l_2) = t_2; \) the second \( Y = X_2 \) and \( K(X_2) e^{-V(X/X_2)} \). If \( X \) is larger than \( X_3 \), we further proceed

\[
e^{-W_X(X_1,X_2,t_1,t_2)} = \int_0^1 dt_3 \partial_t e^{-W_X(X_1,X_2,X_3,t_1,t_2,t_3)} + e^{-W_{X_3}(X_1,X_2,t_1,t_2)} e^{-V(X/X_3)}
\]

(147)

and in particular if \( X_3 = \{1,2,3\} \) one writes

\[
W(X_1, X_2, X_3, t_1, t_2, t_3) = t_1 t_2 V_{1,3} + t_2 V_{2,3} +
\]

\[
V_{1,1} + V_{2,2} + V_{3,3} + t_1 t_2 t_3 \sum_{k \geq 4} V_{1,k} + t_2 t_3 \sum_{k \geq 4} V_{2,k} + t_3 \sum_{k \geq 4} V_{3,k} + \sum_{k,k' \geq 4} V_{k,k'}
\]

(148)
If \( X = X_4 = \{1, 2, 3, 4\} \) we get

\[
\int_0^1 dt_1 V_{1,2} \int_0^1 dt_2 (t_1 V_{1,3} + V_{2,3}) \int_0^1 dt_3 (t_1 t_2 V_{1,4} + t_2 V_{2,4} + V_{3,4}) e^{-W_X(X_1, X_2, X_{3t_1}, t_2, t_3)}
\]

(149)

and the trees \( T \) are; \( l_1 = (1, 2), l_2 = (1, 3), l_3 = (1, 4) \) with \( t_1(l_1) = t_1, \ t_1(l_2) = t_1, t_2(l_2) = t_2, t_1(l_3) = t_1, t_2(l_3) = t_2, t_3(l_3) = t_3; \ l_1 = (1, 2), l_2 = (1, 3), l_3 = (2, 4), \) with \( t_1(l_2) = t_1, \ t_1(l_3) = 1, t_2(l_3) = t_2, t_3(l_3) = t_3; \ l_1 = (1, 2), l_2 = (2, 3), l_3 = (1, 4) \) with \( t_1(l_2) = 1, t_2(l_2) = t_2 \) and so on. They are all possible trees compatible with \( X_1 = 1, X_2 = 1, 2, X_3 = 1, 2, 3; \) then one has to sum over all the possible \( X_i \). Proceeding in this way one gets (136).

Note finally that, see e.g. [32] or [36]

\[
\sum_{X_1, \ldots, X_{r-1}} \int_0^1 dt_1 ... \int_0^1 dt_{r-1} \prod_{k=1}^{r-1} t_k(l) \prod_{l} \frac{1}{t_n(l)} = 1
\]

(150)

In order to prove (58) we note that by iterating (136) we get

\[
e^{-V(X)} = \sum_{\pi} \prod_{Y \in \pi} K(Y)
\]

(151)

where \( \pi \) us the set of all possible partitions of \( X \); on the other hand from (54) we have \( \mathcal{E}_A(X) = e^{-V(X)} \) and

\[
\mathcal{E}_A(X) = \sum_{\pi} \prod_{Y \in \pi} \mathcal{E}^T(Y)
\]

(152)

hence \( K = \mathcal{E}^T \) from which (58) follows.

Regarding the truncated fermionic expectations (93) we proceed in a similar way. We can write the simple expectations as

\[
\mathcal{E}(\tilde{\psi}(P_1) ... \tilde{\psi}(P_r)) = \int \prod_{T} \prod_{l} \prod_{V_l} \prod_{n} \prod_{t} t_{n(t)} \prod_{k=1}^{n-1} t_k(l) e^{-W_{X_r}(X_{1, \ldots, X_{r-1}; t_1, \ldots, t_{r-1})}
\]

(153)

with \( V_{jj} = \sum_{i=1}^{P_j} \sum_{j=1}^{P_j} \eta^{+\downarrow}_{ij} g(x_{ij}, x_{j\downarrow}) \eta^{\downarrow+}_{ij} \) and \( \eta_{ij} \) is a set of Grassmann variables. We can write therefore

\[
\mathcal{E}(\tilde{\psi}(P_1) ... \tilde{\psi}(P_r)) = \int \prod_{T} \prod_{l} \prod_{V_l} \prod_{n} \prod_{t} t_{n(t)} \prod_{k=1}^{n-1} t_k(l) e^{-W_{X_r}(X_{1, \ldots, X_{r-1}; t_1, \ldots, t_{r-1})}
\]

(154)

with \( V_{jj} = \sum_{i} \sum_{j} \eta^{+\downarrow}_{ij} g(x_{ij}, x_{j\downarrow}) \eta^{\downarrow+}_{ij} \) for each \( T \) we divide the \( \eta \) in the ones appearing in \( T \) and \( \eta' \) is the rest; note that

\[
\int \prod_{T} \prod_{l} \prod_{V_l} \prod_{n} \prod_{t} t_{n(t)} \prod_{k=1}^{n-1} t_k(l) e^{-W_{X_r}(X_{1, \ldots, X_{r-1}; t_1, \ldots, t_{r-1})}
\]

(155)

where we have used that

\[
\sum_{l} t_{1(l)} t_{2(l)} ... t_{r(l)} V_l = \sum_{l} t_{n'(l)} ... t_{n(l)-1(l)} V_l
\]

(156)

where \( n(l) \) is the max \( k \) such that \( l \) intersects \( \partial X_k \) and \( n'(l) \) is the min \( k \) such that \( l \) intersects \( \partial X_k \); moreover \( G \) is the matrix with elements \( t_{n'(jj')} ... t_{n(jj')-1} g(x_{ij}, x_{i\downarrow}, x_{j\downarrow}) \); for each sequence \( X_k \).
we can relabel the points so that $X_1 = \{1\}, X_2 = \{1, 2\}$ and so on; therefore given a line $j, j'$ then $n'(jj') = j, n(jj') = j'$ so that $t_{j...j'-1}$; one can find a family of vectors $u_1 = v_1, u_2 = t_1u_1 + v_1\sqrt{1-t_1^2}, u_3 = t_2u_2 + v_2\sqrt{1-t_2^2}, ... v_i$ orthonormal, such that $t_{j...j'-1} = u_ju_{j'}$.

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