ON THE ISOPERIMETRIC CONSTANT OF SYMMETRIC
SPACES OF NONCOMPACT TYPE

XIAODONG WANG

Let $N^n$ be a complete, noncompact Riemannian manifold. We consider the
isoperimetric constant $I(N)$ defined by

$$I(N) = \inf_{\Omega} \frac{A(\partial\Omega)}{V(\Omega)},$$

where $\Omega$ ranges over open submanifolds of $N$ with compact closure and smooth
boundary. This is also called Cheeger’s constant [Ch]. The importance of this
global geometric invariant is illustrated by the following fundamental inequality
relating it to another global analytic invariant

$$(0.1) \quad \lambda_0(N) \geq I(N)^2/4,$$

where $\lambda_0$ is the bottom of spectrum of the Laplace operator on $N$. It is a well
known fact that $\lambda_0$ can be characterized as

$$(0.2) \quad \lambda_0 = \inf f_N \frac{\|\nabla f\|^2}{\|f\|^2},$$

where $f$ ranges over nonzero $C^1$ functions with compact support.

For $\mathbb{R}^n$ or any Riemannian manifold with nonnegative Ricci curvature, the
isoperimetric constant $I$ is trivial. On the other hand, for Cartan-Hadamard man-
ifolds with sectional curvature bounded by a negative constant from above, Yau
proved that $I$ is always positive.

**Proposition 1.** (Yau [Y]) If $N$ is simply connected with sectional curvatures $\leq$
$\kappa < 0$, then

$$I(N) \geq (n - 1) \sqrt{-\kappa}.$$  

From this result one can easily deduce $I(\mathbb{R}^n) = n - 1$. For a detailed discussion
of Cheeger’s constant and related results, one can consult [Ch, Chapter 6]. In
general, it is very difficult to know if the isoperimetric constant is positive or not
and it is almost impossible to compute it explicitly if it is known to be positive. In
this short note, we prove that the isoperimetric constant is positive for all symmetric
spaces of noncompact type and compute it explicitly.

Let $(\widetilde{M}, g)$ be a Cartan-Hadamard manifold (i.e. complete, simply-connected
with nonpositive curvature) and $S\widetilde{M}$ its unit tangent bundle. For any $p \in M$
and $u \in S_p\widetilde{M}$ we have a nonnegative symmetric operator $R_u : T_p\widetilde{M} \to T_p\widetilde{M}$ defined by

$$R_u(X) = -R(u, X)u.$$  

Let $0 = \lambda_0(u) \leq \lambda_2(u) \leq \cdots \leq \lambda_{n-1}(u)$ be its eigenvalues. In this way we have $n$
continuous functions $\lambda_0, \cdots, \lambda_{n-1}$ on $S\widetilde{M}$. Obviously $\lambda_i(-u) = \lambda_i(u).$
From now on we assume \( \tilde{M} \) is a symmetric space of noncompact type. By this we mean that \( \tilde{M} \) is a Cartan-Hadamard manifold with parallel curvature tensor and there is no Euclidean factor in its de Rham decomposition. When \( \tilde{M} \) has rank one, it is negatively curved. But if the rank is higher, its sectional curvature vanishes on certain 2-planes. A standard reference on symmetric spaces is Helgason [H].

We fix a base point \( o \in \tilde{M} \). For \( \xi \in S_o \tilde{M} \) let \( \gamma_\xi \) be the geodesic ray with initial velocity \( \xi \). We can choose an orthonormal basis \( \{ e_1, \ldots, e_{n-1} \} \) for \( \xi^\perp \) s.t.

\[
R \xi e_i = -R(\xi, e_i) \xi = \lambda_i(\xi) e_i.
\]

Let \( E_i \) be the parallel vector field along \( \gamma_\xi \) with \( E_i(0) = e_i \). Since the curvature tensor is parallel, we have along \( \gamma_\xi \)

\[
R_{\gamma_\xi(t)} E_i(t) = -R(\gamma_\xi'(t), E_i(t)) \gamma_\xi'(t) = \lambda_i(\xi) E_i(t).
\]

Therefore \( \lambda_i(\gamma_\xi(t)) = \lambda_i(\xi) \). This proves that the \( n \) continuous functions \( \lambda_0, \ldots, \lambda_{n-1} \) on \( S \tilde{M} \) are invariant under the geodesic flow.

Along the geodesic \( \gamma = \gamma_\xi \) the Jacobi field equation

\[
X''(t) + R(\gamma', X) \gamma' = 0.
\]

can be explicitly solved. The solution satisfying the initial condition \( X(0) = 0, X'(0) = e_i \) is given by

\[
X_i(t) = \frac{\sinh(\sqrt{\lambda_i(\xi)} t)}{\sqrt{\lambda_i(\xi)}} E_i(t).
\]

For any integer \( k \geq 1 \), define the function \( b_k \) on \( \tilde{M} \) by \( b_k(x) = d(x, p_k) - k \), where \( p_k = \gamma_\xi(k) \). The Busemann function \( b_\xi \) is the limit of \( b_k \) as \( k \to \infty \), i.e.

\[
b_\xi(x) = \lim_{k\to\infty} d(x, p_k) - k.
\]

It is well known that the limit exists. By [H], the convergence is locally uniform in \( C^2(\tilde{M}) \) and in particular \( b_\xi \in C^2(\tilde{M}) \).

**Lemma 1.** \( \Delta b_\xi \) is constant and equals \( l(\xi) := \sum_{i} \sqrt{\lambda_i(\xi)} \).

**Proof.** We fix \( x \) and denote \( l_k = d(x, p_k) \). Let \( \sigma_k : [0, l_k] \to M \) be the geodesic from \( x \) to \( p_k \). We write \( u_k = -\gamma_\xi'(k), v_k = -\sigma'(l_k) \). We have

\[
\Delta b_k(x) = \sum_{i} \sqrt{\lambda_i(v_k)} \coth \sqrt{\lambda_i(v_k)} (b_k(x) + k).
\]

Let \( \theta_k = \angle(u_k, v_k) \) be the angle between \( u_k \) and \( v_k \). By the cosine law,

\[
\cos \theta_k \geq \frac{k^2 + l_k^2 - d(o, x)^2}{2kl_k}.
\]

As \( |l_k - k| \leq d(o, x) \), it is obvious that \( \theta_k \to 0 \) as \( k \to \infty \). For each \( k \), there exists \( \phi_k \in G \) s.t. \( \phi_k(p_k) = o \). Let \( \tilde{u}_k = \phi_*(u_k), \tilde{v}_k = \phi_*(v_k) \). They are unit vectors at \( o \) and

\[
\angle(\tilde{u}_k, \tilde{v}_k) = \angle(u_k, v_k) \to 0
\]
as \( k \to \infty \). By continuity, for each \( i \) we have

\[
|\lambda_i(\tilde{u}_k) - \lambda_i(\tilde{v}_k)| \to 0
\]
as \( k \to \infty \). Since \( \lambda_i (u_k) = \lambda_i (v_k) = \lambda_i (\xi) \) and \( \lambda_i (\bar{u}_k) = \lambda_i (\bar{v}_k) \), we have \( \lambda_i (v_k) \to \lambda_i (\xi) \). As \( b_k \to b_\xi \) in \( C^K_{\text{loc}} \), we obtain from [0.4] by taking limit
\[
\Delta b_\xi (x) = \sum_{i} \sqrt{\lambda_i (\xi)}.
\]
\[\Box\]

Let \( G = I_0(\tilde{M}) \) be the connected component of the isometry group containing the identity and \( K = \{ \phi \in G : \phi a = a \} \). Thus \( G \) is a semisimple Lie group acting transitively on \( \tilde{M} \) by isometries and \( K \) a maximal compact subgroup of \( G \). Define
\[
g = \{ \text{Killing vector fields on } M \},
\]
\[
t = \{ X \in g : X(o) = 0 \},
\]
\[
p = \{ X \in g : \nabla X(o) = 0 \}.
\]
We know that \( g \) is the Lie algebra of \( G \) and \( t \) is the Lie algebra of \( K \). Moreover
\[
g = t \oplus p
\]
and \( p \) is naturally identified with \( T_o \tilde{M} \). Let \( \sigma : g \to g \) be the Cartan involution, i.e. it is the automorphism s.t. \( \sigma|_t = I, \sigma|_p = -I \). Let \( B \) be the Killing form \( g \), i.e. for \( X, Y \in g \)
\[
B(X, Y) = \text{tr} (\text{ad}_X \text{ad}_Y).
\]
Since \( \text{ad}_\sigma(X) = \sigma \circ \text{ad}_X \circ \sigma^{-1} \), \( B \) is invariant under \( \sigma \). In particular \( t \) and \( p \) are orthogonal to each other w.r.t. the Killing form of \( g \), i.e. \( B(X, Y) = 0 \) for any \( X \in t, Y \in p \). Moreover, \( B \) is negative definite on \( t \) and positive definite on \( p \). Thus
\[
\langle X, Y \rangle = -B(\sigma X, Y)
\]
is a metric on \( g \). It is easy to verify that \( t \) and \( p \) are still orthogonal and the ad-action is skew-symmetric w.r.t. this metric. To fix the scale, we assume that the Riemannian metric on \( T_o \tilde{M} = p \) coincides with the restriction of \( \langle , \rangle \) on \( p \).

Let \( a \subset p \) be a maximal abelian subspace. For any \( \alpha \in a^\ast \) we define
\[
g_\alpha = \{ X \in g : \text{ad}_H X = \alpha(H) X \ \text{for all } H \in a \}.
\]
If \( g_\alpha \neq 0 \), then \( \alpha \) is called a root of \( a \) and \( m_\alpha := \dim g_\alpha \) is called its multiplicity. The set of all nonzero roots is denoted by \( \Delta \). If \( \alpha \in \Delta \), then \( -\alpha \in \Delta \). Moreover, \( \sigma \) defines an isomorphism from \( g_\alpha \) onto \( g_{-\alpha} \). The connected components of \( a \setminus \cup_{\alpha \in \Delta} \ker \alpha \) are the Weyl chambers of \( a \). Pick one of them to be the positive Weyl chamber and denote it by \( a^+ \). A root is positive if it is positive on \( a^+ \). Let \( \Delta^+ \) denote the set of positive roots. We have the following orthogonal decomposition
\[
g = g_0 + \sum_{\alpha \in \Delta} g_\alpha.
\]
By definition we have \( a \subset g_0 \). In fact \( g_0 = g_0 \cap t \oplus g_0 \cap p \). Since \( a \) is maximal, \( a = g_0 \cap p \). Moreover \( [g_0, g_\beta] \subset g_{\alpha + \beta} \).

It is known that for any two maximal abelian subspaces \( a, \tilde{a} \subset p \) and Weyl chambers \( a^+ \subset a, \tilde{a}^+ \subset \tilde{a} \) there exists \( \phi \in K \) s.t. \( \phi \) maps \( a \) to \( \tilde{a} \) and \( a^+ \) to \( \tilde{a}^+ \).
Therefore, we may assume $\xi \in \mathfrak{a}^\perp$. Consider the linear map $T = \text{ad}_\xi : \mathfrak{t} \to \mathfrak{p}$ and its adjoint $T^* = \text{ad}_\xi : \mathfrak{p} \to \mathfrak{t}$. Indeed, for $u \in \mathfrak{t}, v \in \mathfrak{p}$

$$\langle Tu, v \rangle = B(\text{ad}_\xi u, v) = - B(u, \text{ad}_\xi v) = \langle u, \text{ad}_\xi v \rangle.$$ 

How do we calculate those eigenvalues $\lambda_i(\xi)$? Recall that they are the eigenvalues of the curvature operator $R_\xi : \mathfrak{p} \to \mathfrak{p}$ defined by $R_\xi v = - [[\xi, v], \xi] = \text{ad}_\xi \text{ad}_\xi v = TT^* v$. The curvature operator naturally extends to an endomorphism on $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, to be denoted by the same symbol $R_\xi$. In terms of the decomposition it is given by

$$\begin{bmatrix} 0 & T \\ T^* & 0 \end{bmatrix}^2 = \begin{bmatrix} TT^* & 0 \\ 0 & T^*T \end{bmatrix}.$$ 

Therefore

$$\sum_i \sqrt{\lambda_i(\xi)} = \text{tr} \sqrt{R_\xi} = \frac{1}{2} \text{tr} \sqrt{R_\xi}.$$ 

Then for any $v \in \mathfrak{g}$ we have decomposition

$$v = v_0 + \sum_{\alpha \in \Delta} v_\alpha$$

Thus

$$R_\xi v = \text{ad}_\xi \text{ad}_\xi v$$

$$= \text{ad}_\xi \left( \sum_{\alpha} \alpha(\xi) v_\alpha \right)$$

$$= \sum_{\alpha} \alpha(\xi)^2 v_\alpha.$$ 

Therefore

$$\sum_i \sqrt{\lambda_i(\xi)} = \frac{1}{2} \text{tr} \sqrt{R_\xi}$$

$$= \frac{1}{2} \sum_{\alpha \in \Delta} |\alpha(\xi)| m_\alpha$$

$$= \sum_{\alpha \in \Delta^+} \alpha(\xi) m_\alpha.$$ 

Let $e_\alpha \in \mathfrak{a}$ be the vector s.t. $\alpha(X) = \langle e_\alpha, X \rangle$. Then

$$l(\xi) = \sum_i \sqrt{\lambda_i(\xi)} = \langle \xi, H \rangle,$$

where $H = \sum_{\alpha \in \Delta^+} m_\alpha e_\alpha \in \mathfrak{a}^\perp$.

**Lemma 2.** We have

$$I(M) \geq |H|.$$
Proof. For each $\xi \in a^+$, the corresponding Busemann function $b_\xi$ satisfies $\Delta b_\xi = \langle \xi, H \rangle$. Then for any open submanifold $\Omega \subset \tilde{M}$ with compact closure and smooth boundary, integrating over $\Omega$ yields

$$\langle \xi, H \rangle V(\Omega) = \int_{\Omega} \Delta b_\xi dv = \int_{\partial \Omega} \langle \nabla b_\xi, \nu \rangle d\sigma,$$

where $\nu$ is the outer unit normal of $\partial\Omega$. Since $|\nabla b_\xi| \equiv 1$, we obtain

$$\langle \xi, H \rangle V(\Omega) \leq A(\partial\Omega).$$

Therefore, for any $\xi \in a^+$

$$I(\tilde{M}) \geq \langle \xi, H \rangle.$$

Taking sup over $\xi$ yields $I(\tilde{M}) \geq |H|$. $\square$

We claim that equality holds: $I(\tilde{M}) = |H|$. For this purpose we need to bring in another geometric invariant. Since $\tilde{M}$ admits compact quotients by discrete subgroups of $G$, the following limit, called the volume entropy,

$$v = \lim_{r \to \infty} \frac{\log V(r)}{r},$$

where $V(r)$ is the volume of the geodesic ball $B(o, r)$ with center $o$ and radius $r$, exists and is independent of the base point $o$ (cf. [M]). This asymptotic invariant is computed explicitly in [BCG, Appendix C] for locally symmetric spaces of non-compact type. Indeed, the volume form on $\tilde{M}$ is given by $\prod_i \sinh \sqrt{\lambda_i(\xi)} dt \sigma d\xi$ in view of (0.3). Therefore

$$V(r) = \int_0^r \int_{S^{m-1}} \prod_i \frac{\sinh \sqrt{\lambda_i(\xi)} t}{\sqrt{\lambda_i(\xi)}} d\sigma d\xi dt.$$

From this formula it is easy to derive

$$v = \sup_{\xi \in a^+} \sum_i \sqrt{\lambda_i(\xi)} = |H|.$$

Now we can prove our main result.

**Theorem 1.** Let $\tilde{M}$ be a symmetric space of noncompact type. Then we have

$$I = v = |H|.$$

*Proof.* Given Lemma 2 it remains to prove $I \leq v$. This is well known. Indeed, by the definition of $I$ we have

$$V'(r) = A(r) \geq IV(r),$$

where $A(r)$ is the area of the surface of the geodesic ball $B(o, r)$. Integrating gives $V(r) \geq V(1) \exp(\int r)$. It follows that $v \geq I$. $\square$

As a corollary we get the following result originally proved by Olshanski (see the discussion in [BCG, Appendix C]).
Theorem 2. Let $\tilde{M}$ be a symmetric space of noncompact type. Then its bottom of spectrum $\lambda_0(\tilde{M})$ is given by the formula
\[
\lambda_0(\tilde{M}) = \frac{1}{4}|H|^2.
\]

Proof. We recall another fundamental inequality
\[
\lambda_0(\tilde{M}) \leq \frac{1}{4}v^2.
\]
This follows easily by taking test functions $f = \exp[(v + \varepsilon)r/2]$, with $r$ being the distance function to $o$ and $\varepsilon > 0$, in (0.2) and then letting $\varepsilon \to 0$. Combining the above inequality and (0.1) we can write
\[
\frac{1}{4}I^2 \leq \lambda_0(\tilde{M}) \leq \frac{1}{4}v^2.
\]
From this we obtain the desired identity from Theorem 1. □

References

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