TIME-VARYING CLOCK OFFSET ESTIMATION IN TWO-WAY TIMING MESSAGE EXCHANGE IN WIRELESS SENSOR NETWORKS USING FACTOR GRAPHS

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ABSTRACT

The problem of clock offset estimation in a two-way timing exchange regime is considered when the likelihood function of the observation time stamps is exponentially distributed. In order to capture the imperfections in node oscillators, which render a time-varying nature to the clock offset, a novel Bayesian approach to the clock offset estimation is proposed using a factor graph representation of the posterior density. Message passing using the max-product algorithm yields a closed form expression for the Bayesian inference problem.

Index Terms— Clock offset, factor graphs, message passing, max-product algorithm

1. INTRODUCTION

Clock synchronization in wireless sensor networks (WSN) is a critical component in data fusion and duty cycling, and has gained widespread interest in recent years [1]. Most of the current methods consider sensor networks exchanging time stamps based on the time at their respective clocks [2]. In a two-way timing exchange process, adjacent nodes aim to achieve pairwise synchronization by communicating their timing information with each other. After a round of \( N \) messages, each node tries to estimate its own clock parameters. A representative protocol of this class is the timing-sync protocol for sensor networks (TPSNs) which uses this strategy in two phases to synchronize clocks in a network [3].

The clock synchronization problem in a WSN offers a natural statistical signal processing framework [4]. Assuming an exponential delay distribution, several estimators were proposed in [5]. It was argued that when the propagation delay \( d \) is unknown, the maximum likelihood (ML) estimator for the clock offset \( \theta \) is not unique. However, it was shown in [6] that the ML estimator of \( \theta \) does exist uniquely for the case of unknown \( d \). The performance of these estimators was compared with benchmark estimation bounds in [7]. A common theme in the aforementioned contributions is that the effect of possible time variations in clock offset, arising from imperfect oscillators, is not incorporated and hence, they entail frequent re-synchronization requirements.

In this work, assuming an exponential distribution for the network delays, a closed form solution to clock offset estimation is presented by considering the clock offset as a random Gauss-Markov process. Bayesian inference is performed using factor graphs and the max-product algorithm.

2. SYSTEM MODEL

By assuming that the respective clocks of a sender node \( S \) and a receiver node \( R \) are related by \( C_R(t) = \theta + C_S(t) \) at real time \( t \), the two-way timing message exchange model at the \( k \)th instant can be represented as [5] [6]

\[
U_k = d + \theta + X_k, \quad V_k = d - \theta + Y_k
\]

(1)

where \( d \) represents the propagation delay, assumed symmetric in both directions, and \( \theta \) is offset of the clock at node \( R \) relative to the clock at node \( S \). The network delays, \( X_k \) and \( Y_k \), are the independent exponential random variables. By further defining \( \xi \triangleq d + \theta \) and \( \psi \triangleq d - \theta \), the model in (1) can be written as

\[
U_k = \xi + X_k, \quad V_k = \psi + Y_k
\]

(2)

for \( k = 1, \ldots, N \). The imperfections introduced by environmental conditions in the quartz oscillator in sensor nodes result in a time-varying clock offset between nodes in a WSN. In order to sufficiently capture these temporal variations, the parameters \( \xi \) and \( \psi \) are assumed to evolve through a Gauss-Markov process given by

\[
\xi_k = \xi_{k-1} + w_k, \quad \psi_k = \psi_{k-1} + v_k \quad \text{for} \quad k = 1, \ldots, N
\]

where \( w_k \) and \( v_k \) are i.i.d such that \( w_k, v_k \sim \mathcal{N}(0, \sigma^2) \). The goal is to determine precise estimates of \( \xi \) and \( \psi \) in the Bayesian framework using observations \( \{U_k, V_k\}_{k=1}^N \). An estimate of \( \theta \) can, in turn, be obtained as

\[
\hat{\theta} = \frac{\hat{\xi} - \hat{\psi}}{2}.
\]

(3)
can be computed as follows in Fig. 1. The factor graph is cycle-free and inference by which can be rearranged as

$$\psi_k = f_k(\xi_k|x_{k-1})$$

$$(6)$$

$$(7)$$

Let $\bar{\xi}_N$ be the unconstrained maximizer of the exponent in the objective function above, i.e.,

$$\bar{\xi}_N = \arg \max_{\xi_N} \left( A_{\xi,N} \xi_N^2 + B_{\xi,N} \xi_N + C_{\xi,N} \xi_N + D_{\xi,N} \right).$$

This implies that

$$\bar{\xi}_N = \min \left( \xi_N, U_N \right).$$

If $\bar{\xi}_N > U_N$, then the estimation problem is solved, since $\bar{\xi}_N = U_N$. However, if $\bar{\xi}_N \leq U_N$, the solution is $\xi_N = \bar{\xi}_N$. Therefore, in general, we can write

$$\bar{\xi}_N = \frac{C_{\xi,N} \xi_N - D_{\xi,N}}{2A_{\xi,N}}.$$

$$(8)$$

Notice that $\bar{\xi}_N$ depends on $\xi_{N-1}$, which is undetermined at this stage. Hence, we need to further traverse the chain backwards. Assuming that $\xi_N \leq U_N$, $\bar{\xi}_N$ from $\xi_N$ can be plugged back in $\xi_N$ which after some simplification yields

$$m_{\delta_{N-1}^N} = \max_{\xi_{N-1}} \left( C_{\xi,N} \xi_{N-1} + D_{\xi,N} \right).$$

Similarly the message from the factor $\delta_{N-1}^N$ to the variable node $\xi_{N-2}$ can be expressed as

$$m_{\delta_{N-2}^N} \propto \max_{\xi_{N-2}} \left( B_{\xi,N} \xi_{N-2} + C_{\xi,N} \xi_{N-2} + D_{\xi,N} \right).$$

The message above can be compactly represented as

$$m_{\delta_{N-1}^N} \propto \max_{\xi_{N-1}} \exp \left( A_{\xi,N} \xi_{N-1}^2 + B_{\xi,N} \xi_{N-1} + C_{\xi,N} \xi_{N-1} + D_{\xi,N} \right).$$

where

$$A_{\xi,N} = \frac{1}{2\sigma^2} + B_{\xi,N} = \frac{C_{\xi,N}^2}{4A_{\xi,N}},$$

$$B_{\xi,N} = \frac{1}{2\sigma^2} + C_{\xi,N} = \frac{1}{\sigma^2},$$

$$D_{\xi,N} = \lambda - \frac{C_{\xi,N} D_{\xi,N}}{2A_{\xi,N}}.$$

$$(10)$$

3. A FACTOR GRAPH APPROACH

The posterior pdf can be expressed as

$$f(\xi, \psi|U, V) \propto f(\xi, \psi) f(U, V|\xi, \psi)$$

$$= f(\xi_0) \prod_{k=1}^N f(\xi_k|x_{k-1}) f(\psi_0) \prod_{k=1}^N f(\psi_k|x_{k-1})$$

$$\cdot \prod_{k=1}^N f(U_k|\xi_k) f(V_k|\psi_k)$$

where uniform priors $f(\xi_0)$ and $f(\psi_0)$ are assumed. Define

$$\delta_{k-1}^k \propto f(\xi_k|x_{k-1}) \sim N(\xi_{k-1}, \sigma^2),$$

$$\nu_{k-1}^k \propto f(\psi_k|x_{k-1}) \sim N(\psi_{k-1}, \sigma^2),$$

$$f_k \propto f(U_k|\xi_k),$$

$$h_k \propto f(V_k|\psi_k),$$

where the likelihood functions are given by

$$f(U_k|\xi_k) = \lambda U \exp \left( -\lambda (U_k - \xi_k) \right) I(U_k - \xi_k)$$

$$f(V_k|\psi_k) = \lambda v \exp \left( -\lambda v (V_k - \psi_k) \right) I(V_k - \psi_k).$$

The factor graph representation of the posterior pdf is shown in Fig. 1. The factor graph is cycle-free and inference by message passing is indeed optimal. In addition, the two factor graphs shown in Fig. 1 have a similar structure and hence, message computations will only be shown for the estimate $\bar{\xi}_N$. Clearly, similar expressions will apply to $\bar{\psi}_N$.

The message updates in factor graph using max-product can be computed as follows

$$m_{f_N} = f_N,$$

$$m_{\delta_{N-1}^N} = \max_{\xi_N} \delta_{N-1}^N \cdot m_{\xi_{N-1}} \cdot \delta_{N-1}^N$$

$$= \max_{\xi_N} \frac{1}{2\pi \sigma^2} \exp \left( -\frac{(\xi_N - \xi_{N-1})^2}{2\sigma^2} \right)$$

$$\cdot \exp (\lambda \xi_N) I(U_N - \xi_N)$$

which can be rearranged as

$$m_{\delta_{N-1}^N} \propto \max_{\xi_N \leq U_N} \exp \left( A_{\xi,N} \xi_N^2 + B_{\xi,N} \xi_N + C_{\xi,N} \xi_N + D_{\xi,N} \right).$$

$$(9)$$

$$(10)$$
Proceeding as before, the unconstrained maximizer \( \tilde{\xi}_{N-1} \) of the objective function above is given by

\[
\tilde{\xi}_{N-1} = -\frac{C_{\xi,N-1}\xi_{N-2} + D_{\xi,N-1}}{2A_{\xi,N-1}}
\]

and the solution to the maximization problem (10) is expressed as

\[
\hat{\xi}_{N-1} = \min \left( \xi_{N-1}, U_{N-1} \right).
\]

Again, \( \hat{\xi}_{N-1} \) depends on \( \xi_{N-2} \) and therefore, the solution demands another traversal backwards on the factor graph representation in Fig. 1. By plugging \( \hat{\xi}_{N-1} \) back in (10), it follows that

\[
m_{\xi_{N-2}\rightarrow\xi_{N-1}} = \frac{1}{\exp} \exp \left\{ \left( B_{\xi,N-1} - \frac{C_{\xi,N-1}}{4A_{\xi,N-1}} \right) \xi_{N-2}^2 - \frac{C_{\xi,N-1}D_{\xi,N-1}}{2A_{\xi,N-1}} \xi_{N-2} \right\}
\]

(11)

which has a form similar to (9). It is clear that one can keep traversing back in the graph yielding messages similar to (9) and (11). In general, for \( i = 1, \ldots, N - 1 \), we can write

\[
A_{\xi,N-i} \triangleq -\frac{1}{2\sigma^2} + B_{\xi,N-i+1} - \frac{C_{\xi,N-i+1}}{4A_{\xi,N-i+1}},
\]

\[
B_{\xi,N-i} \triangleq -\frac{1}{2\sigma^2},
\]

\[
D_{\xi,N-i} \triangleq \lambda_{\xi} - \frac{C_{\xi,N-i+1}D_{\xi,N-i+1}}{2A_{\xi,N-i+1}}
\]

(12)

and

\[
\hat{\xi}_{N-i} = -\frac{C_{\xi,N-i}\hat{\xi}_{N-i-1} + D_{\xi,N-i}}{2A_{\xi,N-i}}.
\]

\[
\hat{\xi}_{N-i} = \min \left( \xi_{N-i}, U_{N-i} \right).
\]

(13)

(14)

Using (13) and (14) with \( i = N - 1 \), it follows that

\[
\hat{\xi}_1 = \frac{-C_{\xi,1}\xi_0 + D_{\xi,1}}{2A_{\xi,1}}, \quad \hat{\xi}_1 = \min \left( \xi_1, U_1 \right).
\]

(15)

Similarly, by observing the form of (9) and (11), it follows that

\[
m_{\lambda_{\xi}\rightarrow\xi_0} \propto \exp \left\{ \left( B_{\xi,1} - \frac{C_{\xi,1}}{4A_{\xi,1}} \right) \xi_0^2 - \frac{C_{\xi,1}D_{\xi,1}}{2A_{\xi,1}} \xi_0 \right\}.
\]

(16)

The estimate \( \hat{\xi}_0 \) can be obtained by maximizing (16).

\[
\hat{\xi}_0 = \hat{\xi}_0 = \max_{\xi_0} m_{\lambda_{\xi}\rightarrow\xi_0} = \frac{C_{\xi,1}D_{\xi,1}}{2A_{\xi,1}D_{\xi,1} - C_{\xi,1}^2}.
\]

(17)

The estimate in (17) can now be substituted in (15) to yield \( \hat{\xi}_1 \), which can then be used to solve for \( \hat{\xi}_2 \). Clearly, this chain of calculations can be continued using recursions (13) and (14).

Define

\[
g_{\xi,k}(x) \triangleq \frac{-C_{\xi,k}x + D_{\xi,k}}{2A_{\xi,k}}.
\]

(18)

**Lemma 1** For real numbers \( a \) and \( b \), the function \( g_{\xi,k}(.) \) defined in (18) satisfies

\[
g_{\xi,k} \left( \min(a, b) \right) = \min \left( g_{\xi,k}(a), g_{\xi,k}(b) \right).
\]

Proof: The constants \( A_{\xi,k}, C_{\xi,k} \) and \( D_{\xi,k} \) are defined in (7) and (12). The proof follows by noting that \( \frac{C_{\xi,k}}{2A_{\xi,k}} > 0 \) which implies that \( g_{\xi,k}(.) \) is a monotonically increasing function.

Using the notation \( g_{\xi,k}(.) \), it follows that

\[
\hat{\xi}_1 = g_{\xi,1} \left( \hat{\xi}_0 \right), \quad \hat{\xi}_1 = \min \left( U_1, g_{\xi,1} \left( \hat{\xi}_0 \right) \right)
\]

\[
\hat{\xi}_2 = g_{\xi,2} \left( \hat{\xi}_1 \right), \quad \hat{\xi}_2 = \min \left( U_2, g_{\xi,2} \left( \hat{\xi}_1 \right) \right)
\]

where

\[
g_{\xi,2} \left( \hat{\xi}_1 \right) = g_{\xi,2} \left( \min \left( U_1, g_{\xi,1} \left( \hat{\xi}_0 \right) \right) \right)
\]

\[
= \min \left( g_{\xi,2}(U_1), g_{\xi,2} \left( g_{\xi,1} \left( \hat{\xi}_0 \right) \right) \right)
\]

(19)

where (19) follows from Lemma 1. The estimate \( \hat{\xi}_2 \) can be expressed as

\[
\hat{\xi}_2 = \min \left( U_2, \min \left( g_{\xi,2}(U_1), g_{\xi,2} \left( g_{\xi,1} \left( \hat{\xi}_0 \right) \right) \right) \right)
\]

\[
= \min \left( U_2, g_{\xi,2}(U_1), g_{\xi,2} \left( g_{\xi,1} \left( \hat{\xi}_0 \right) \right) \right)
\]

Hence, one can keep estimating \( \hat{\xi}_k \) at each stage using this strategy. Note that the estimator only depends on functions of data and can be readily evaluated. For \( m \geq k \), define

\[
G_{\xi,k}^m(.) \triangleq g_{\xi,m} \left( g_{\xi,m-1}(\ldots g_{\xi,k}(.) \ldots) \right).
\]

(20)

The estimate \( \hat{\xi}_N \) can, therefore, be compactly represented as

\[
\hat{\xi}_N = \min \left( U_N, G_{\xi,N}^N \left( U_{N-1}, \ldots, G_{\xi,2} \left( U_1 \right), G_{\xi,1} \left( \hat{\xi}_0 \right) \right) \right).
\]

(21)

By a similar reasoning, the estimate \( \hat{\psi}_N \) can be analogously expressed as

\[
\hat{\psi}_N = \min \left( V_N, G_{\psi,N}^N \left( V_{N-1}, \ldots, G_{\psi,2}(V_1), G_{\psi,1}(\hat{\psi}_0) \right) \right)
\]

and the factor graph based clock offset estimate (FGE) \( \hat{\theta}_N \) is given by

\[
\hat{\theta}_N = \frac{\hat{\xi}_N - \hat{\psi}_N}{2}.
\]

(22)

It only remains to calculate the functions of data \( G(.) \) in the expressions for \( \hat{\xi}_N \) and \( \hat{\psi}_N \) to determine the FGE estimate \( \hat{\theta}_N \). With the constants defined in (7), it follows that

\[
G_{\xi,N}(U_{N-1}) = -\frac{C_{\xi,N}U_{N-1} + D_{\xi,N}}{2A_{\xi,N}} = U_{N-1} + \lambda_\xi \sigma^2.
\]
Similarly it can be shown that

\[ G_{ξ,N}^i(U_{N-1}) = U_{N-1} + 2λ_ξσ^2 \]

and so on. Using the constants defined in (22) for \( i = N-1 \), it can be shown that \( G_{ξ,1}(ξ_0) = +∞ \). This implies that \( G_{ξ,1}(ξ_0) = +∞ \). Plugging this in (21) yields

\[ \hat{ξ}_N = \min(U_N, U_{N-1} + λ_ξσ^2, \ldots, U_1 + (N-1)λ_ξσ^2) . \]

Similarly, the estimate \( \hat{ψ}_N \) is given by

\[ \hat{ψ}_N = \min(V_N, V_{N-1} + λ_ψσ^2, \ldots, V_1 + (N-1)λ_ψσ^2) \]

and the estimate \( \hat{θ}_N \) can be obtained using (22) as

\[ \hat{θ}_N = \frac{1}{2} \min(U_N, U_{N-1} + λ_ξσ^2, U_{N-2} + 2λ_ξσ^2, \ldots, U_1 + (N-1)λ_ξσ^2) - \]
\[ \frac{1}{2} \min(V_N, V_{N-1} + λ_ψσ^2, V_{N-2} + 2λ_ψσ^2, \ldots, V_1 + (N-1)λ_ψσ^2) . \]

(23)

As the Gauss-Markov system noise \( σ^2 \to 0 \), (23) yields

\[ \hat{θ}_N \to \hat{θ}_{ML} = \frac{\min(U_N, \ldots, U_1) - \min(V_N, \ldots, V_1)}{2} \]

(24)

which is the ML estimator proposed in [6].

4. SIMULATION RESULTS

With \( λ_ξ = λ_ψ = 10 \) and \( σ = 10^{-2} \), Fig. 2 shows the MSE performance of \( \hat{θ}_N \) and \( \hat{θ}_{ML} \), compared with the Bayesian Chapman-Robbins bound (BCHRBR). It is clear that \( \hat{θ}_N \) exhibits a better performance than \( \hat{θ}_{ML} \) by incorporating the effects of time variations in clock offset. As the variance of the Gauss-Markov model accumulates with the addition of more samples, the MSE of \( \hat{θ}_{ML} \) gets worse. Fig. 3 depicts the MSE of \( \hat{θ}_N \) in (23) with \( N = 25 \). The horizontal line represents the MSE of the ML estimator (24). It can be observed that the MSE obtained by using the FGE for estimating \( \theta \) approaches the MSE of the ML as \( σ < 10^{-3} \).

5. CONCLUSION

The estimation of a possibly time-varying clock offset is studied using factor graphs. A closed form solution to the clock offset estimation problem is presented using a novel message passing strategy based on the max-product algorithm. This estimator shows a performance superior to the ML estimator proposed in [6] by capturing the effects of time variations in the clock offset efficiently.

6. REFERENCES

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