Stochastic quantum process

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Abstract

I propose to treat quantum evolution as a stochastic process consisting from a sequence of doubly stochastic matrices, which naturally arise in the generalized quantum evolution. Then it is proved that the law of non-decreasing entropy is fulfilled and that the law characterizes doubly stochastic matrices. Finally, an application of the model to support the generalized second law of black hole thermodynamics and a relation to the quantum histories formulation of quantum physics appear.

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I focus in the article on two successful physical theories, quantum unitary and statistical physics to find a relation between them, see ref. [1]. The motivation is that the observed law of non-decreasing entropy in all physical processes can not be directly modeled in the invertible treating. In classical mechanics the analogue trouble was solved by admitting collision points of evolution in H equation. In quantum physics irreversibility seems to be caused by measurements, a discontinuity of unitary evolution [2]. The nature of measurement is not comprehended, for a mathematical model with use of Ito calculus for Schrodinger equation on reduced Hilbert space see [3], while its functional description is core of contemporary quantum theories. I do not solve the measurement problem, but extract all we know about it and use in wider context. Namely, I further assume that measurements belong to physical phenomena and they happen independently on conscious beings and the

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role of measuring devices participating in the process. For a quantum system interacting with an environment with qualitatively higher degree of freedom I assume that approximately point events of reductions violating unitarity take places. Precisely, in the place of unitary evolution I put transformation of quantum states represented by rays via Kraus representation, so I admit a nearest quantum surrounding \[4, 5\] . Whole environment influences the system in the way that reductions to pure states appear, compare \[2\] . The reduction itself is treated as a Poisson process of time events. This assumption of the model refers to uniform conditions the system is subjected to. It will appear that it is not essential for truthfulness of the non-decreasing entropy theorem.

Possessing the sequence of reduction moments for one entity \(t_0 < t_1 < t_2 < \ldots\) the quantum evolution describes the steps from \(t_i\) to \(t_{i+1}\). In Feynman integrals approach it means that admitted quantum trajectories need to cross only pure states in the pointed time moments \(t_i\). For a finite dimensional Hilbert space \(\mathcal{H}\) the skips are realized by stochastic matrices \(M \in M(n \times n, \mathbb{R})\) acting on \(\mathbb{R}^n\), where a complete set of observables (an orthonormal basis in the Hilbert space) was chosen. The normalized states are represented by points of \(\triangle_{n-1} \subset \mathbb{R}^n, \triangle_{n-1} := \{\{p_i\} | p_i \geq 0 \land \sum p_i = 1\}\). Really, the \(\{p_i\}\) corresponds to a ray in \(\mathcal{H}\) which is sufficient subject of quantum evolution, for example see ref. \[6\].

It will appear that only special stochastic matrices play a role in the quantum context, see Landé’s conjecture \[7\]. Let us remind their definition.

**Definition 1.** Let \(M\) be a stochastic matrix i.e. \(M \in M(n \times n, \mathbb{R})\), \(M_{ij} \geq 0, \sum_{i=1}^{n} M_{ij} = 1\). Then \(M\) is doubly stochastic (DS) iff \(\sum_{j=1}^{n} M_{ij} = 1\).

Stochastic matrices appear as descending objects from unitary evolution in presence of a quantum surrounding between two different time moments. They are constructed via a Kraus representation. Then in the finite dimensional case, I study in the article, a finite set of operators \(A_\alpha \in L(\mathbb{C}^n, \mathbb{C}^n), \alpha = 1, \ldots, N\) exists, which are built by contraction of an unitary higher dimensional dynamics, such that

\[
M_{ij} = \sum_{\alpha=1}^{N} Tr(P_i A_\alpha P_j A^*_\alpha) \tag{1}
\]

and

\[
\sum_{\alpha=1}^{N} A^*_\alpha A_\alpha = 1 = \sum_{\alpha=1}^{N} A_\alpha A^*_\alpha, \tag{2}
\]
where \( P_i \) are the projection operators define by the standard orthonormal basis \( \{ |i> \} \in \mathbb{R}^n \subset \mathbb{C}^n \). Using \( \sum P_i = id \) and (1),(2) one concludes that such \( M \) is a doubly stochastic matrix. From the other side having any finite doubly stochastic matrix \( M \) acting on \( \mathbb{R}^n \) to find a representation it is enough to put \( A_{i,j} = |i> \sqrt{M_{i,j}} < j| \) as an element of \( L(\mathbb{C}^n,\mathbb{C}^n) \). The doubly stochastic condition guarantees fulfilling (2).

For ordinary unitary evolution transformation to stochastic matrices, given by \( M_{ij} = <i|U|j>^2 \) and \( A_1 = U \in U(n,\mathbb{C}), N = 1 \), does not cover all doubly stochastic matrices for \( n > 2 \). Nevertheless all doubly stochastic matrices, so admitting a Kraus representation with condition (2), may be build from a higher dimensional unitary evolution. I show

**Proposition 2.** Let a finite set \( \{ A_\alpha \}_{\alpha = 1}^N \) be given, \( A_\alpha \in L(\mathbb{C}^n,\mathbb{C}^n) \) and fulfills (2). Then there is a finite dimensional complex Hilbert space \( \mathcal{H}_2 \), \( \text{dim} \mathcal{H}_2 \leq 2N \) and an unitary operator on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), \( \mathcal{H}_1 := \mathbb{C}^n \), such that the reduction to \( \mathcal{H}_1 \) leads to \( M(A_\alpha) \) defined by (1).

**Proof.** Let \( I = \{ 1,\ldots,N \} \) be a set of the indices of the operators \( A_\alpha \). I put \( \mathcal{H}_2 = \mathbb{C}^N \oplus \mathbb{C}^N \). Then \( \mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_1^I \oplus \mathcal{H}_1 \). At the beginning I define a seeking unitary operator firstly only on the diagonal of \( \mathcal{H}_1^I \), \( \Delta_I \) by

\[
U(|v> \oplus \ldots \oplus |v>) := \sqrt{N}A_1|v> \oplus \ldots \oplus \sqrt{N}A_N|v>, |v> \in \mathcal{H}_1. \quad (3)
\]

Let \( \mathcal{H}_1 = \Delta_I \oplus \Delta_I^\perp \) and \( \mathcal{H}_1^I = U(\Delta_I) \oplus U(\Delta_I)^\perp \) be the orthogonal decompositions of the Hilbert space. An isometry \( m : \Delta_I^\perp \rightarrow U(\Delta_I)^\perp \) may be chosen. And then the complete definition is as follows

\[
U|\Delta_I \oplus \Delta_I := U|\Delta_I \oplus U|\Delta_I \quad (4)
\]
\[
U : \Delta_I^\perp \oplus \{ 0 \} \ni |v> \oplus 0 \mapsto 0 \oplus m(|v>) \in \{ 0 \} \oplus U(\Delta_I)^\perp \quad (5)
\]
\[
U : \{ 0 \} \oplus \Delta_I \ni 0 \oplus |v> \mapsto m(|v>) \oplus \in U(\Delta_I)^\perp \oplus \{ 0 \} \quad (6)
\]

Then the starting \( M \) is rebuilt by \( M_{ij} = \frac{1}{2N} \sum_{\alpha,\sigma} |<i\alpha\sigma|U|j\alpha\sigma>|^2 \), where \( |i,\alpha,\sigma> := |i> \otimes (|\alpha> \otimes |\sigma>) \); \( i,j = 1,\ldots,n; \alpha \in I; \sigma = +,- \), is the orthonormal basis of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). 

It has just appeared that DS matrices play additionally essential role while put into quantum physics context. They appear also to be a link to
statistical physics. The following theorem shows that the dynamic system
\((\Delta_{n-1}, M)\) has an unique property iff \(M\) is doubly stochastic. I will use a
family of entropies, so called \(\alpha\)-entropies, \(\alpha > 0\), defined by the formula:

\[
H_{\alpha}(p) := \frac{1}{1 - \alpha} \ln(\sum_i p_i^\alpha)
\]

for \(p \in \Delta_n\). The Shannon entropy appears for the limit \(\alpha \to 1\), \(H_{\infty} = -\ln \max\{p_i\}\), for quantum \(\alpha\)-entropies see e.g. [8].

**Theorem 3.** (a) For the dynamic system \((\Delta_{n-1}, M)\) governing by the stochas-
tic matrix \(M\) and for \(\alpha \geq 1\) or \(\alpha = \infty\): \(M\) is doubly stochastic iff the entropy
\(H_{\alpha}(M^k p)\) does not decreases for each trajectory.

(b) For generic doubly stochastic matrices (for a dense and open set) the
entropy increases for all trajectories with only one exclusion, the unique sta-
tionary trajectory. Then also \(\lim_{k \to \infty} M^k p = p_e\), where \(p \in \Delta_{n-1}, p_i^e = \frac{1}{n}\).

**Proof.** (a) For \(M\) being a doubly stochastic matrix I will show that \(||M||_\alpha = 1\)
for \(\alpha > 1\), where \(||v||_\alpha := \sqrt[\alpha]{\sum_i |v_i|^\alpha}, v \in \mathbb{R}^n\), and \(||A||_\alpha := \sup||v||_\alpha=1||Av||_\alpha\).
Each doubly stochastic matrix has the form: \(M = a_i P_i\), where \(a_i \geq 0, \sum_i a_i = 1\) and \(P_i\) are the permutations. Now, \(||M||_\alpha \leq a_i ||P_i||_\alpha = 1\), where one uses
\(||P_i||_\alpha = 1\) for each \(i\).

But the condition, i.e. \(||Mv||_\alpha \leq ||v||_\alpha\), coincides with \(H_{\alpha}(p) \leq H_{\alpha}(Mp)\).
Inversely, it is enough to use a characterization of doubly stochastic matrices
through the equation

\[
M p_e = p_e
\]

While \(H_{\alpha}\) does not decrease \(p_e\) the point of reaching the greatest value is not
moved by \(M\).

(b) By taking limit of \(||Mv||_\alpha \leq ||v||_\alpha\) for \(\alpha \to \infty\), where \(M\) is a doubly
stochastic matrix, one concludes that \(||M|| = 1\). The generic \(M\) is defined
as \(||Mp|| < 1\) for \(p \in \Delta_{n-1}, p \neq p_e\). Then, one estimates that \(||M(p - p_e)|| =
||M(p - p_e)|| < ||p - p_e||\) for \(p \neq p_e\) and the proof is completed. \(\square\)

**Remark 4.** The result is also kept for \(\alpha = 1\) and \(\alpha = \infty\). The extension
for \(\alpha = 1\) contains von Neumann inequality and its generalization [5]. The
theorem may be directly extended to quantum case and infinite dimensional
spaces with help of [3, 8].
Returning to the beginning evolution proposal one may note that theorem 3 is immediately generalized for a dynamics constructing with any chain of doubly stochastic matrices. For generic one it is that 
\[ ||M_1|_0 \circ M_2|_0 \circ \cdots ||_\alpha \leq ||M_1|_0||_\alpha \cdot ||M_2|_0||_\alpha \cdot \cdots = 0, \]
where \( M|_0 \) is restriction of stochastic matrix \( M \) to the subspace \( \{v \in \mathbb{R}^n | \sum v^i = 0 \} \).

Such abstract framework without all information about background process appears to be powerful to explain some unsolved points connected with the generalized second law of thermodynamics in black holes spacetimes [10], also [11, 12, 13, 14]. I formally adopt the system-bath model from condensed matter physics [4] on \( \mathcal{H}_s \otimes \mathcal{H}_b \) Hilbert space, where \( \mathcal{H}_s \) is a space of states of black holes and their neighborhood, \( \mathcal{H}_b \) - of the bath (surrounding). The only assumption I need is thermodynamic equilibrium of the bath. The assumption is implicit present in the construction of dynamics via Kraus representations. After a reduction to pure states one switches on evolution with the same beginning vector state of the bath, the identity. Additionally, the set of black holes, even gluing one with another in time, and radiation and particles around are treated in quantum field theory fashion as a quantum being with one Hilbert space. Then \( \mathcal{H}_s \otimes \mathcal{H}_b \) may be treated as a constant in time arena for quantum evolution. No knowledge about theory of quantum dynamics of black holes is needed. Now, the main result extended to the stochastic quantum evolution is read as a support of the generalized second law of thermodynamics admitting black holes as subsystems. The environment presence, the last element needed in Patrovi model, may be connected with structure of infinity of space-time bearing during evolution some constants like ADM mass, charge or angular momentum, a macroscopic parameters of inner quantum world.

The direct trial of finding a differential equation for \( M(t) \) from the Schrödinger equation or its generalization in the presence of the bath is unsuccessful. There are no differential equation of the form
\[ M^{(n)}(t) = f_{\hat{H}}(t, M(t), \ldots, M^{(n-1)}(t)) \]
where \( \hat{H} \) is a hamiltonian. For a comparison with an approximation by the constant coefficients Lindblat equation see ref. [4]. Really, it appears that the proposed model leads to a non-Markovian evolution equation [15].

To summarize it needs to be stressed that theorem 3 precisely states that invertible quantum dynamics is after the pointing viewed as a different side of entropy nondecreasing law.
The approach is closely related to the consistent history formulation of Quantum Theory (QH) of Griffiths, Gell-Mann and Hartle [16], see also [17, 18, 19], its methods and motivations. The above sequence of stochastic matrices is equivalent to a consistent histories set with a given choice of orthogonal, complete, projective, 1-dimensional operators for all pointed time moments. From the presented results follow that the unitary evolution in QH may be replaced by the most general transformation for a density operator of a quantum subsystem in thermal bath: \( \hat{\rho} \rightarrow \hat{\rho}' = \sum_\alpha A_\alpha \hat{\rho} A_\alpha^* \), where \( \sum_\alpha A_\alpha A_\alpha^* = \sum_\alpha A_\alpha^* A_\alpha = 1 \). The theorem is directly rewritten for any separable Hilbert space with the \( \alpha \)-entropies defined by

\[
H_\alpha(\hat{\rho}) = \frac{1}{1 - \alpha} \text{Tr} \hat{\rho}^\alpha
\]

after identification of the eigenvalues of \( \hat{\rho} \) and \( \hat{\rho}' \) with \( p, p' \in \Delta_{n-1}, n = 1, 2, \ldots, \infty \). The reduction moments \( t_i \) are admitted to possess wider gates of reduction, i.e. the complete, orthogonal set of projective operators \( \{P_l(t_i)\} \) i.e. \( \sum_k P_k(t_i)^2 = 1, P_k(t_i)^* = P_k(t_i) \). The measurement is a discontinuous moment evolution, but still realized by the doubly sum Kraus representation. It may be even replaced by any doubly sum transformation.

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