Coexistence and harvesting optimal policy in three species food chain model with general Holling type functional response

Mohammed Y. Dawed | Kiros G. Kebedow

Department of Mathematics, Hawassa University, Hawassa, Ethiopia

Correspondence
Kiros G. Kebedow, Department of Mathematics, Hawassa University, P. O. Box, 05, Hawassa, Ethiopia.
Email: kirosg@hu.edu.et

Abstract
In this paper, we have discussed harvesting of prey and intermediate predator species. Both are subjected to Holling type I–V functional response. Conditions for local and global stability of the nonnegative equilibria are verified. The permanent coexistence criterion of the model system and existence of optimal equilibrium solution of the control problem are demonstrated. Maximum sustainable yield and maximal net present revenue are determined. To confirm analytical results, numerical solution has been carried out using the Matlab™ ODE solver ODE45 and the simulations show the model system reveals complex behavior (such as oscillations), which reflects the real situation.

Recommendations for Resource Managers
From our investigation of this study, we recommend to the management the following points.

1. Coexistence of the three species with harvesting, or persistence of the model system is possible provided that good management(treatment) of some factors (such as harvesting rate, growth rate of species, etc.) are performed.
2. The dynamics reveals complex behavior (such as oscillations), which reflects the real situation and it is sensitive to the above factors, especially the growth rate of the intermediate predator.

3. The policy makers should recommend the optimal effort $h^*$ to be applied and the optimal stock $(x^*, y^*, z^*)$ to harvest. This indicates that maximum profit will attain while securing sustainability of the three species in the ecosystem.

**KEYWORDS**
bionomic equilibrium, equilibrium point, Holling type response functions, Liapunov function, maximal net revenue, optimal harvesting policy, persistence

### 1 | INTRODUCTION

The population size and coexistence of species in ecological system are dynamic and complex. One of the important objectives in mathematical ecology is the study of dynamical relationship among the population sizes of predators and the respective prey. Species interactions in an ecosystem is expressed mathematically in a nonlinear differential equation form. In this study we proposed a general nonlinear mathematical model that describes three interacting species with harvesting prey and intermediate predator population. Real life example is plant—herbivore–Omnivore. The three interacting species are two predators (a generalist we call from now onwards *intermediate* predator $Y$, a specialist we call from onwards *top* predator $Z$), and one source prey species $X$ where both are subjected with a general Holling type I–IV functional responses.

A three species predator-prey model with prey switching where, top predator and intermediate predator are competed with each other for same prey with new Holling type functional response. It demonstrates an intermediate predator is the key species in regulating the feeding mechanism between the prey and the top predator (Mortoja et al., 2020).

The dynamical behaviors of three species such as toxin-producing Phytoplankton, Zooplankton, and Fish in a fishery system is studied with Holling type II functional response (Panja & Mondal, 2015). Further they extended predator–prey interaction among Phytoplankton, Zooplankton, and Fish in the presence of Environmental toxin and includes harvesting on the Fish (Panja et al., 2017).

Transition to chaotic behavior is established (Upadhyay & Raw, 2011) via period-doubling bifurcation and some sequences of distinctive period-halving bifurcation leading to limit cycles are observed, furthermore, Holling type IV predator response function is considered. The rich dynamics in a tritrophic food chain mathematical model, consists of three species: prey, intermediate predator, and top predator while both the prey and intermediate predator are being subjected to Holling type I–IV functional response is investigated in Dawed et al. (2020). Some research have been proposed to understand the dynamics in a food chain (one predator–two
prey). Coexistence and harvesting control policy in a food chain model with mutual defense of prey is explained well in Tchepmo Djomegni et al. (2019) and in contrast to many approaches, this study considers mutualism (for defense against predators) between the two groups of prey. As it is explained in Tchepmo Djomegni et al. (2019), in a group defense when prey density is sufficiently large, predators can go extinct. To maintain the sustainability, one can harvest the abundant prey population (Chen et al., 2013). Due attention has been given to study the impact of harvesting the dynamics in predator-prey systems (Huang et al., 2013). Harvesting has social, economic, and ecological interests. Socially, it can be used for domestic feeding. Economically, it is used for food, clothes, and medicine purpose. From ecological perspective, harvesting can be used for resource renewal and sustainability. Furthermore, in the paper by Bergland et al. (2019), a predatory prey model of livestock prey density with sigmoidal harvesting rate is proposed and studied.

Functional response is an indispensable component of predation models which determines both prey death rate and the predator increase rate. C. S. Holling has suggested three classes of functions to cover the most interesting cases of functional responses (type IV came later). A Holling type functional response is described as a predator's instantaneous per capita feeding rate and is a function of prey abundance. This means that the consumption rate of an individual predator depends on the prey density. The detail about each Holling type response function is found in Dawed et al. (2020).

Dynamical complexities of a three species food web with square root functional response is investigated where the prey species shows a herd behavior (Bera et al., 2016). Furthermore, discrete time-delay mathematical models for tri-trophic food chains have been developed and studied in recent past (Das & Samanta, 2020; Maiti et al., 2008; Maiti & Samanta, 2006; Manna et al., 2017; Pathak et al., 2009). The effect of discrete time delay has been investigated in these papers.

In this paper, we consider a food chain model (a top predator, an intermediate predator, and a source prey) with intermediate predator and prey harvesting. We observe the ecological interest and investigate the dynamics in the food chain and the impact of harvesting on the persistence of the model system. The necessary conditions for the permanent coexistence of the model, and criteria for local and global stability of the nonnegative equilibria are presented. Existence of optimal equilibrium solution of the control problem, maximum sustainable yield, and maximal net present revenue are determined. Numerical simulations using Matlab programming are performed to investigate the dynamic outcome of the system. This model shows complex behavior (such as oscillations), which reflects the real situation. These dynamic outcomes are supported by numerical simulations in the paper.

2 | MATHEMATICAL MODEL

The rich dynamics in a tritrophic food chain mathematical model, consisting of three species: source prey \(X\), a generalist intermediate predator \(Y\), and top predator \(Z\) that includes a general Holling type response function is investigated in Dawed et al. (2020). Following this approach, we have considered harvesting terms \(q_1EX\) and \(q_2EY\) in the existing model system. Hence, the improved mathematical model of the three species
interaction with general Holling type functional response and harvesting prey and inter-
mediate predator is given as follows:

\[
\frac{dX}{dt} = r_X X \left(1 - \frac{X}{K_X}\right) - \Phi_X(X) Y - q_1 E X,
\]

\[
\frac{dY}{dt} = r_Y Y \left(1 - \frac{Y}{K_Y}\right) + c_1 \Phi_X(X) Y - \Phi_Y(Y) Z - q_2 E Y,
\]

\[
\frac{dZ}{dt} = -r_Z Z + c_2 \Phi_Y(Y) Z.
\]

Descriptions of the state variables and parameters are given in Table 1.

Here, variable \( E \), \( 0 < E \leq E_{\text{max}} \), called fishing effort (Fishing mortality) is introduced. If \( E_{\text{max}} \geq \frac{r_X}{q_1} \) and \( E_{\text{max}} \geq \frac{r_Y}{q_2} \), the stock would be driven to extinction (Kot, 2001).

Using the dimensionless variables, \( \tau = \frac{r_X}{r_Y}, x = \frac{X}{K_X}, y = \frac{Y}{K_Y} \) and \( z = \frac{r_Z}{r_Y} Z \), the scaled version of the model system (1)–(3) is simplified to the form

\[
\frac{dx}{d\tau} = x(1 - x) - \kappa x \varphi_x(x)y - hx,
\]

\[
\frac{dy}{d\tau} = \varepsilon y(1 - y) + \varphi_x(x)y - \varphi_y(y)z - \delta hy,
\]

\[
\frac{dz}{d\tau} = -\gamma z + \beta \varphi_y(y)z,
\]

where, \( \varepsilon = \frac{r_Y}{r_X}, \gamma = \frac{r_Y}{r_Z}, \kappa = \frac{K_Y}{c_1 K_X}, h = \frac{q_Y E}{r_Y}, \delta = \frac{q_2}{q_1}, \beta = c_2 \varepsilon, \varphi_x(x) = \frac{c_0 \Phi_X(X)}{r_Y} \) and

\[\varphi_y(y) = \frac{\Phi_Y(Y)}{r_Y}.
\]

**Table 1** Description of variables and parameters in the mathematical model system (1)–(3)

| Var/Par | Description | Unit       |
|---------|-------------|------------|
| \( X \) | Density of prey | g/m\(^2\) |
| \( Y \) | Density of intermediate predator | g/m\(^2\) |
| \( Z \) | Density of top predator | g/m\(^2\) |
| \( \Phi_X(X) \) (resp. \( \Phi_Y(Y) \)) | Predator response function | /year |
| \( r_X \) (resp. \( r_Y \)) | Intrinsic growth rate of prey (resp. intermediate predator) | /year |
| \( K_X \) (resp. \( K_Y \)) | Carrying capacity of prey (resp. intermediate predator) | g/m\(^2\) |
| \( r_Z \) | Reduction rate of top predator due to other factors | /year |
| \( c_1 \) (resp. \( c_2 \)) | Conversion proportion of prey (resp. intermediate predator) biomass, \( c_1, c_2 \varepsilon(0, 1) \) into intermediate predator (resp. top predator) biomass | Dimensionless |
| \( q_1, q_2 \) | Proportionality constant known as “catch ability” coefficients | Dimensionless |
| \( q_Y E \) (resp.\( q_Z E \)) | Mortality or reduction of source prey (resp. intermediate predator) caused due to harvesting | /year |
3 | MATHEMATICAL MODEL ANALYSIS

3.1 | Positivity, uniqueness, and boundedness of solution

3.1.1 | Positivity

**Proposition 3.1.** The solution \(x(\tau), y(\tau), z(\tau)\) of the model system (4)–(6) with positive initial condition \((x_0, y_0, z_0)\) is positive.

**Proof.** We can rewrite Equation (4) as \(\frac{dx}{d\tau} = -q(\tau)x - x^2\), \(q(\tau) = \left( h + \frac{\mu y(x)^y}{x}\right)\), which is Bernoulli type differential equation. Hence, \(x(\tau) = \frac{e^{-\int_0^\tau q(s)ds}}{x(0) + \int_0^\tau e^{-\int_0^s q(t)dt}ds} > 0\). Similarly, \(y(\tau)\) and \(z(\tau)\) are also positive. □

3.1.2 | Uniqueness

The right hand side equations of the model system are continuous and possess continuous partial derivatives of all orders in a domain \(D\) of the positive octant \(\mathbb{R}^3_+ = \{(x, y, z): x \geq 0, \, y \geq 0, \, z \geq 0\}\), and the solution of the system initiating in the non-negative octant is bounded. Therefore, the solution of the model system with nonnegative initial condition exists and is unique (Allen, 2007).

3.1.3 | Boundedness

**Proposition 3.2.** All solutions \((x(\tau), y(\tau), z(\tau))\) of the model system (4)–(6) with positive initial condition \((x_0, y_0, z_0)\) are bounded in the region

\[
\Omega = \left\{(x, y, z): 0 \leq x \leq 1, \, \epsilon \leq x \leq v + \epsilon, \, 0 \leq x + xy + yz \leq \frac{M}{\gamma} \right\},
\]

where \(M\) is global maximum of function \(G(x, y) = x(1 - x) + \xi y(1 - y) + (\gamma - h)x + x(\gamma - \delta h)y\).

**Proof.** Boundedness argument for \(x\): From the first equation of the model it is true that \(\frac{dx}{d\tau} \leq x(1 - x)\). This implies that \(x(\tau) \leq \frac{x_0}{x_0 + (1 - x_0)e^{-\tau}}\) where \(x(0) = x_0\). Therefore, \(\frac{dx}{d\tau} \leq x(1 - x)\) implies that \(x(\tau) \leq \max(x_0, 1), \forall \tau \geq 0\). Boundedness argument for \(y\): From the properties of Holling response function we assume maximum value of \(H\) is \(v\) which is finite. Hence, \(\frac{dy}{d\tau} \leq \epsilon y(1 - y) + vy\). This implies \(y(\tau) \leq \frac{y_0}{e^{-\frac{\epsilon}{\gamma} + \frac{\epsilon}{\gamma} + \frac{\epsilon y_0}{\gamma}}}, \) where \(y_0 = y(0)\). Thus, \(y(\tau) \leq \max(\epsilon + v, y_0), \forall \tau \geq 0\) and therefore, it is bounded.
Boundedness argument for $z$: Consider the sum relation $m = x + \kappa y + \frac{\kappa}{\beta} z$. After differentiation with respect to $\tau$ it takes the form

$$\frac{dm}{d\tau} = x(1 - x) + \kappa x y (1 - y) - h x - \frac{\kappa y}{\beta} z - \kappa \delta h y.$$ 

Using the relation $\frac{\kappa}{\beta} z = m - x - \kappa y$ the differential equation reduces to the form as

$$\frac{dm}{d\tau} + \gamma m = x(1 - x) + \kappa x y (1 - y) + (y - h) x + \kappa (y - \delta h) y.$$ 

One can easily show that the right hand side function of the differential equation is strict concave with unique equilibrium point $(x_m, y_m) = \left( \frac{\gamma - (1 - \gamma - h) + \gamma - \frac{\epsilon - \delta h}{\beta}}{2}, \frac{\gamma - (1 - \gamma - h) + \gamma - \frac{\epsilon - \delta h}{\beta}}{2} \right)$ at which it attains global maximum say $M = \left( \frac{1 + \frac{x - (1 - \gamma - h) + \gamma - \frac{\epsilon - \delta h}{\beta}}{2}}{4} \right)$. Then the differential equation can be expressed as $\frac{dm}{d\tau} + \gamma m \leq M$. This leads to $\frac{dm}{d\tau} \leq M - \gamma m$. The foregoing inequality can be expressed as a Gronwall's inequality $\frac{dg}{d\tau} \leq g$ with $g(\tau) = m(\tau) - (M/\gamma)$ and $\beta = -\gamma$.

Hence, $\frac{dm}{d\tau} \leq M - \gamma m \Rightarrow m(\tau) \leq \frac{M}{\gamma} + (\frac{m_0 - \frac{M}{\gamma}}{\gamma}) e^{-\gamma \tau}$ where $m(0) = m_0$. This shows $0 < m(\tau) \leq \frac{M}{\gamma}$. Therefore, the solution of the model system is bounded in the region $\Omega$. □

3.2 | Existence of equilibrium points

The axial equilibrium points of the system (4)–(6) are $E_0(0, 0, 0), E_x(1 - h, 0, 0)$, and $E_y\left(0, \frac{\epsilon - \delta h}{\beta}, 0\right)$. The first equilibrium point exists with no condition but $E_x$ and $E_y$ exist if $h < 1$ and $\epsilon > \delta h$, respectively. Furthermore, additional equilibrium points of the form $E_{xy}(x^*, y^*, 0), E_{yz}(0, y^*, z^*)$ and $E_{xyz}(x^*, y^*, z^*)$ exist with some conditions. The equilibrium point $E_{xy}(x^*, y^*, 0)$ exists as the prey and intermediate predator population grow to the difference of their carrying capacities and harvesting in the absence of top predator. To obtain suitable expressions for $x^*$ and $y^*$ of $E_{xy}$, let both $x$ and $y$ be considered different from zero while $z$ is zero. The equations now reduce to

$$x(1 - x) - \kappa \varphi_x(x) y - h x = 0,\quad (7)$$

$$\gamma y (1 - y) + \varphi_x(x) y - \delta h y = 0.\quad (8)$$

Considering $y \neq 0$, solve for $y$ from (8), it can be obtained as $y^* = \frac{\epsilon + \varphi_x(x^*) - \delta h}{\gamma}$. Substituting this expression for $y^*$ into (7) resembles that $x^*$ is a positive solution of the equation

$$\epsilon x (1 - x) - \kappa \varphi_x(x) (\epsilon + \varphi_x(x) - \delta h) - \epsilon h x = 0.\quad (9)$$

Therefore, if (9) has a positive solution, say $x^*$, then the equilibrium point $E_{xy}(x^*, y^*, 0) = \left(x^*, \frac{\epsilon + \varphi_x(x^*) - \delta h}{\epsilon}, 0\right)$, provided that $\delta h < \epsilon + \varphi_x(x^*)$.

To show existence of positive solution for (9) at each Holling type functional response, we use the theorem below.
**Theorem 3.3.** Consider a polynomial of degree $n$, with $n$ odd, in the form 

$$p(x) = (-1)^n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_{n-1}$ are real parameters and $a_0 \neq 0$. If $a_0 > 0$, then $p(x)$ has at least one positive root.

The proof of the theorem is found in Dawed et al. (2020).

Now we can show existence of positive solution for (9) for each Holling type functional response.

**Case one (HT-I):** $\varphi(x) = \alpha x$, where $\alpha = \frac{c_1 a K x}{r}$. Substituting this into (9) and solving for $x^*$, we get 

$$x^* = \frac{\varepsilon (1 - h) + \kappa \alpha (\delta h - \varepsilon)}{\varepsilon + \kappa \alpha^2},$$

provided that $\varepsilon > \frac{\kappa \alpha \delta h}{h + \kappa \alpha - 1}$.

**Case two (HT-II):** $\varphi(x) = \frac{\alpha x}{b_1 + x}$, where $\alpha = \frac{a}{K_x}, b_1 = \frac{b}{K_x}$. Substituting this into (9) and making arrangements, then $x^*$ is the solution of

$$- \varepsilon x^3 + \varepsilon (1 - 2b_1 - h) x^2 + \left(2\varepsilon b_1 - \varepsilon b_1^2 - \kappa \alpha_1 (\varepsilon + \alpha_1 - \delta h)\right) - 2\varepsilon h b_1 x + \varepsilon \left(b_1^2 - \kappa \alpha_1 b_1 - h b_1^2\right) + \kappa \alpha_1 \delta b_1 h = 0,$$

which exists (by Theorem 3.3) if $\varepsilon (\kappa \alpha_1 b_1 + h b_1^2 - b_1^2) < \kappa \alpha_1 \delta b_1 h$.

**Case three (HT-III):** $\varphi(x) = \frac{\alpha x^2}{b_2 + x^2}$, where $\alpha = \frac{a}{K_x}, b_2 = \frac{b}{K_x}$. In this case, $x^*$ is solution of the equation

$$- \varepsilon x^5 + \varepsilon (1 - h) x^4 - \left(2\varepsilon b_2 + \kappa \alpha_1 (\varepsilon + \alpha_1 - \delta h)\right) x^3 + 2\varepsilon b_2 (1 - h) x^2 + \left( - \varepsilon b_2^2 - \kappa \alpha_1 \varepsilon b_2 + \kappa \alpha_1 \delta b_2 h \right) x + \varepsilon b_2^2 (1 - h) = 0,$$

which exists (by Theorem 3.3), provided that $h < 1$.

**Case Four (HT-IV):** $\varphi(x) = \frac{\alpha x}{\varepsilon \left( \frac{e_1}{e_1} + x + b_1 \right)}$, where $\alpha = \frac{a}{K_x}, e_1 = \frac{e}{K_x}$. Similarly, $x^*$ is solution of the equation

$$- \frac{\varepsilon}{e_1} x^5 + \left(\frac{\varepsilon}{e_1} - 2 \frac{\varepsilon}{e_1} - \frac{\varepsilon}{e_1} h\right) x^4 + \left(\frac{2\varepsilon}{e_1} - 2b_1 \varepsilon - 2h \frac{\varepsilon}{e_1}\right) x^3 + \left(2b_1 \frac{\varepsilon}{e_1} + \varepsilon - 2b_1 \varepsilon - \kappa \alpha_1 \left(\varepsilon - \delta h\right) - 2b_1 h \frac{\varepsilon}{e_1} - \varepsilon h\right) x^2 + \left(2\varepsilon b_2 - 1 - \varepsilon b_2^2 - \kappa \alpha_1 \left(\varepsilon - \delta h + \alpha_1\right) + 2b_1\right) x + \varepsilon b_2^2 (1 - h) = 0,$$

which exists (by Theorem 3.3), provided that $\varepsilon (\kappa \alpha_1 b_1 + h b_1^2 - b_1^2) < \kappa \alpha_1 \delta b_1 h$.

| Functional response | HT-I | HT-II | HT-III | HT-IV |
|---------------------|------|-------|--------|-------|
| $y^*$               | $\frac{y}{\alpha_0}$ | $\frac{\sqrt{y \beta_2}}{\alpha_0 \beta - y}$ | $\sqrt{\frac{y \beta_2}{\alpha_0 \beta - y}}$ | $(\alpha_0 \beta - y) \varepsilon_2 \sqrt{\frac{(\alpha_0 \beta - y)^2 - 4 \beta_2 \varepsilon_3 \beta_1}{\gamma}}$ |

| Condition           | $\delta h < \varepsilon (1 - y^*)$ | $\delta h < \varepsilon (1 - y^*)$ | $\delta h < \varepsilon (1 - y^*)$ | $\delta h < \varepsilon (1 - y^*)$ |
The equilibrium point $E_{xy}$ bespeaks the prospect of prey and intermediate predators to coexist certain. The stability analysis results will provide conditions for such scenario to happen. The equilibrium point in the $yz$ plane, say, $E_{yz}$, is the solution of the system of equations $x = 0, \varepsilon y (1 - y) - \varphi_x(y)z - \delta h y = 0$ and $- \gamma + \beta \varphi_y(y) = 0$. Solving this system we obtain $E_{yz}(0, y^*, z^*) = \left(0, y^*, \frac{\varepsilon \varphi_y(y^*)}{\gamma} \left(\varepsilon (1 - y^*) - \delta h\right)\right)$ which exists when $y^* \in (0, \frac{-\delta h}{\varepsilon})$ where $y^* = \varphi_y \left(\frac{\varepsilon}{\gamma}\right)$. The existence of $y^*$ for the functional response HT-I–IV is given in Table 2.

Let $\omega = \frac{aK_r}{r}, \omega_1 = \frac{\omega}{K_r}, \beta_1 = \frac{b}{K_r}, \beta_2 = \frac{\beta_1}{K_r}, e_2 = \frac{e}{K_r}$.

The coexistence equilibrium point $E_{xyz}(x^*, y^*, z^*)$ is solution of the system equations $x(1 - x) - k \varphi_x(x)y - hx = 0, \varepsilon y (1 - y) + \varphi_x(x)y - \varphi_y(y)z - \delta h y = 0, - \gamma z + \beta \varphi_y(y)z = 0$

Hence, $E_{xyz} = \left(x^*, y^*, \frac{\beta \varphi_y(y^*)}{\gamma} \left(\varepsilon (1 - y^*) - \delta h + \varphi_x(x^*)\right)\right)$, where $0 < y^* < \frac{-\varphi_x(x^*) - \delta h}{\varepsilon}, \varphi_y(y^*) = \frac{\gamma}{\beta}$ and $x^*$ is positive solution of the equation

$$x^2 - x + k \varphi_x(x)y^* + hx = 0. \quad (10)$$

We can show existence of positive solution for Equation (10) of each Holling type functional response as follows:

Case I (HT-I): $\varphi_x(x) = \alpha x$. Substituting this into Equation (10) and solving for $x^*$, we get $x^* = 1 - h - k \alpha y^*$, provided that $h + k \alpha y^* < 1$.

Case II(HT-II): $\varphi_x(x) = \frac{\alpha x}{b + x}$. In this case, $x^*$ is solution of

$$x^2 - (1 - h - b_1)x + k \alpha_1 y^* - b_1(1 - h) = 0.$$ 

This has only one positive solution if $k \alpha_1 y^* \leq b_1(1 - h)$ and $(1 - h + b_1)^2 \geq 4k \alpha_1 y^*$. Moreover, it has two positive solutions if $(1 - h + b_1)^2 \geq 4k \alpha_1 y^*, 1 - h - b_1 > 0$, and $k \alpha_1 y^* > b_1(1 - h)$.

Case III(HT-III): $\varphi_x(x) = \frac{\alpha x}{b + x^2}$. Similarly, $x^*$ is solution of

$$-x^3 + (1 - h)b_2 x^2 - b_2 x + (1 - h)b_2 - k \alpha_1 y^* = 0,$$ 

which exists (by the Theorem 3.3), provided that $(1 - h)b_2 > k \alpha_1 y^*$.

Case IV(HT-IV): $\varphi_x(x) = \frac{\alpha x}{c_1 + x + b_1}$. In this case, $x^*$ is solution of

$$\frac{-1}{e_1} x^3 - \left(1 - \frac{1 - h}{e_1}\right)x^2 - (b_1 + h - 1)x + (1 - h)b_1 - k \alpha_1 y^* = 0,$$

which exists (by the Theorem 3.3) if $(1 - h)b_1 > k \alpha_1 y^*$.

The value of $y^*$ for the functional response HT I–IV is given in Table 3. This implies that the interior equilibrium point exists when $\delta h < \varepsilon (1 - y^*) + \varphi_x(x^*)$.

| Functional response | HT-I | HT-II | HT-III | HT-IV |
|---------------------|------|-------|--------|-------|
| $y^*$               | $\frac{\gamma}{\beta \omega}$ | $\frac{\gamma \varphi_1}{\omega_1 \beta - \gamma}$ | $\sqrt{\frac{\gamma \varphi_2}{\omega_1 \beta - \gamma}}$ | $(\omega_1 \beta - \gamma)c_3 + \frac{\sqrt{3}(\omega_1 \beta - \gamma)^2 - 4\gamma^2 \epsilon_3 \beta_1}{2\gamma}$ |

TABLE 3 The values of $y^*$ in $E_{xyz}$
We also note that no equilibrium points of the form \((c^*, 0, d^*)\) and \((0, 0, z^*)\) exist. This is true as top predator feed solely on intermediate predator and do not have alternative source of food.

### 3.3 Dynamical behavior

The dynamical behavior of the equilibrium can be studied by computing the Jacobian matrix corresponding to each equilibrium point. Observing the sign of the eigenvalues of this matrix the stability of an equilibrium point is determined (Allen, 2007). The Jacobian matrix \(J(x, y, z)\) of the system (4)–(6) is

\[
J(x, y, z) = \begin{pmatrix}
1 - 2x - k\varphi'_x(x)y - h & -k\varphi'_x(x) & 0 \\
\varphi'_x(x)y & \varepsilon (1 - 2y) + \varphi'_x(x) - \varphi'_y(y)z - \delta h & -\varphi'_y(y) \\
0 & \beta \varphi'_y(y)z & -\gamma + \beta \varphi'_y(y)
\end{pmatrix}.
\]

1. **\(E_0(0, 0, 0)\):** The Jacobian matrix at \(E_0\) takes the form,

\[
J(E_0) = \begin{pmatrix}
1 - h & -k\varphi'_x(0) & 0 \\
0 & \varepsilon + \varphi'_x(0) - \delta h & -\varphi'_y(0) \\
0 & 0 & -\gamma + \beta \varphi'_y(0)
\end{pmatrix}.
\]

The three eigenvalues of this matrix are

\[
\lambda_1 = 1 - h, \lambda_2 = \varepsilon - \delta h, \lambda_3 = -\gamma.
\]

The trivial equilibrium point is saddle point which is unstable if \(h < 1\). Biologically, this implies a purely extinction is not possible. However, if \(h > 1\), and \(0 < \varepsilon < \delta h\) hold, \(E_0\) is stable, which implies biologically a pure extinction of the species.

2. **\(E_x(1 - h, 0, 0)\):** The Jacobian matrix at \(E_1\) takes the form

\[
J(E_x) = \begin{pmatrix}
h - 1 & -k\varphi'_x(1-h) & 0 \\
0 & \varepsilon + \varphi'_x(1-h) - \delta h & 0 \\
0 & 0 & -\gamma
\end{pmatrix}.
\]

The three eigenvalues are

\[
\lambda_1 = h - 1, \lambda_2 = \varepsilon + \varphi'_x(1-h) - \delta h, \lambda_3 = -\gamma.
\]

The equilibrium point \(E_x(1 - h, 0, 0)\) is asymptotically stable if \(\pi_0 = \varepsilon + \varphi'_x(1-h) - \delta h < 0\) otherwise it is a saddle point with stable manifold in the \(xz\) plane and unstable manifold in \(y\) direction.

3. **\(E_y(0, \frac{\varepsilon - \delta h}{\varepsilon}, 0)\):** The Jacobian matrix at \(E_y\) takes the form

\[
J(E_y) = \begin{pmatrix}
1 - k\varphi'_x(0)y - h & 0 & 0 \\
\varphi'_x(0) & -\varepsilon - \delta h & -\varphi'_y\left(\frac{\varepsilon - \delta h}{\varepsilon}\right) \\
0 & 0 & -\gamma + \beta \varphi'_y\left(\frac{\varepsilon - \delta h}{\varepsilon}\right)
\end{pmatrix}.
\]
The three eigenvalues are
\[
\lambda_1 = 1 - k\varphi_x'(0) - h, \quad \lambda_2 = -\varepsilon - \delta h, \quad \lambda_3 = -\gamma + \beta\varphi_y\left(\frac{\varepsilon - \delta h}{\varepsilon}\right).
\]

The equilibrium point \(E_y\left(0, \frac{\varepsilon - \delta h}{\varepsilon}, 0\right)\) is stable if \(1 < k\varphi_x'(0) + h\) and \(\beta\varphi_y\left(\frac{\varepsilon - \delta h}{\varepsilon}\right) < \gamma\), otherwise it is a saddle point.

(4) \(E_{xy}(x^*, y^*, 0)\): The Jacobian matrix at \(E_{xy}\) takes the form
\[
J(E_{xy}) = \begin{pmatrix}
a & b & 0 \\
c & d & e \\
0 & 0 & f
\end{pmatrix},
\]
where
\[
a = 1 - 2x^* - k\varphi_x'(x^*)y^* - h, \quad b = -k\varphi_x(x^*) < 0, \quad c = \varphi_x'(x^*)y^*, \quad d = \varepsilon(1 - 2y^*) + \varphi_x(x^*) - \delta h, \quad e = -\varphi_y(y^*) < 0, \quad f = -\gamma + \beta\varphi_y(y^*).
\]

The characteristic equation of the above matrix \(J(E_{xy})\) is \((f - \lambda)(\lambda^2 - \lambda(a + d) + ad - bc) = 0\). The characteristic roots are \(\lambda_1, \lambda_2, \lambda_3\) such that
\[
\lambda_1 + \lambda_2 = a + d, \quad \lambda_1\lambda_2 = ad - bc, \quad \lambda_3 = f.
\]

Here, we used notations
\[
a + d = 1 + \varepsilon + \varphi_x(x^*) - (2x^* + y^*(k\varphi_x'(x^*) + 2\varepsilon)) + h(1 + \delta),
\]
\[
ad - bc = (1 - 2x^* - k\varphi_x'(x^*)y^* - h)(\varepsilon(1 - 2y^*) + \varphi_x(x^*) - \delta h) + k\varphi_x(x^*)\varphi_x'(x^*)y^*,
\]
\[
f = -\gamma + \beta\varphi_y(y^*).
\]

Hence, conditions for local stability of the equilibrium point \(E_{xy}\) are
\[
1 + \varepsilon + \varphi_x(x^*) < (2x^* + y^*(k\varphi_x'(x^*) + 2\varepsilon)) + h(1 + \delta),
\]
\[
(1 - 2x^* - k\varphi_x'(x^*)y^* - h)(\varepsilon(1 - 2y^*) + \varphi_x(x^*) - \delta h) + k\varphi_x(x^*)\varphi_x'(x^*)y^* > 0
\]
and \(\beta\varphi_y(y^*) < \gamma\).

Example: Let \(\varphi_x(x) = \alpha x\) and \(\varphi_y(y) = \alpha y\), HT-I. The equilibrium point in the \(xy\) plane exists for \(\varepsilon(1 - h) + \alpha x\delta h > \alpha x\varepsilon\) and \(\varepsilon + \alpha(1 - h) > \delta h\) and it takes the form
\[
\left(\frac{\varepsilon(1 - h) + \alpha x(\delta h - \varepsilon)}{\varepsilon + \alpha x}, \frac{\varepsilon + \alpha(1 - h) - \delta h}{\varepsilon + \alpha x}, 0\right).
\]

In this case the characteristic equation becomes
\[
(f - \lambda)(\lambda^2 - (a + d)\lambda + ad - bc) = 0,
\]
where
\[
a + d = 1 - 2x^* - \kappa\alpha y^* - h + (\varepsilon(1 - 2y^*) + \alpha x^* - \delta h),
\]
\[
ad - bc = (1 - 2x^* - \kappa\alpha y^* - h)(\varepsilon(1 - 2y^*) + \alpha x^* - \delta h) - (-\kappa\alpha x^*)(\alpha y^*),
\]
\[
f = -\gamma + \beta\alpha y^*.
\]

Expanding each term we have
The eigenvalues of the matrix are 

$$\frac{-k\varphi_0'(0)a^* - h, A_2 = \varphi_x'(0)a^*, A_3 = \varepsilon(1 - 2a^*) - \varphi_y(a^*)\frac{\beta a^*}{\gamma}(\varepsilon(1 - a^*) - \delta h), A_4 = \frac{-\gamma}{\beta}, A_5 = \beta\varphi_y(a^*)\frac{\beta a^*}{\gamma}(\varepsilon(1 - a^*) - \delta h).$$

The characteristic equation of the matrix is 

$$(A_1 - \lambda)(A_2 - \lambda) = 0.$$ 

The eigenvalues satisfy $\lambda_1, \lambda_2,$ and $\lambda_3$ of the matrix satisfy the following conditions: $\lambda_1 = A_1, \lambda_2 = A_3 - A_4, \lambda_3 = A_4 A_5.$ Hence, \(E_{xy}(0, a^*, z^*) = (0, a^*, \frac{\beta a^*}{\gamma}(\varepsilon(1 - a^*) - \delta h))^2\text{ is locally asymptotically stable if } A_1 < 0, A_3 > 0, \text{ and } A_5 > 0.

\(E_{xyz}(x^*, y^*, z^*)\): The Jacobian matrix at \(E_{xyz}\) has the form

$$J(E_{xyz}) = \begin{pmatrix}
a_1 & a_2 & 0 \\
a_3 & a_4 & a_5 \\
0 & a_6 & 0
\end{pmatrix}.$$

Here $a_1 = 1 - 2x^* - k\varphi_x(x^*)y^* - h, a_2 = -k\varphi_x(x^*), a_3 = \varphi_x(x^*)y^*, a_4 = \varepsilon(1 - 2y^*) + \varphi_x(x^*) - \varphi_y(y^*)z^* - \delta h, a_5 = -\frac{\gamma}{\beta}$ and $a_6 = \beta\varphi_y(y^*)z^*.$ The characteristic equation is $\lambda^3 + S_1 \lambda^2 + S_2 \lambda + S_3 = 0$ where $S_1 = -a_1 - a_4, S_2 = a_1 a_4 - a_2 a_3 - a_5 a_6$ and $S_3 = a_1 a_5 a_6.$ By the Routh-Hurwitz criterion the interior equilibrium point \(E_{xyz} = (x^*, y^*, z^*)\) is locally asymptotically stable if $S_1 > 0, S_2 > 0, S_3 > 0$ and $S_1 S_2 - S_3 > 0.$ It can be shown that both $S_1$ and $S_3$ are positive quantities if and only if $\frac{\gamma}{\beta}a_1 a_6 + (a_1 + a_4) < 0.$ Further, the expression $S_1 S_2 - S_3$ can be simplified so as to reduce to the form as $(a_1 + a_4)(a_2 a_3 - a_1 a_4) - \frac{\gamma}{\beta}a_4 a_6.$ Hence, the necessary conditions for $S_1 S_2 - S_3 > 0$ are the expressions $a_2 a_3 - a_1 a_4$ and $a_4 a_6$ are negative. This observation is stated in terms of a theorem as follows.

**Theorem 3.4.** Suppose that the positive interior equilibrium point \(E_{xyz}(x^*, y^*, z^*)\) exists in \(\mathbb{R}^3_+\). Then, \(E_{xyz}\) is locally asymptotically stable provided that 

$$\pi_1 = \frac{\gamma}{\beta}a_1 a_6 + (a_1 + a_4) < 0, \pi_2 = a_2 a_3 - a_1 a_4 < 0 \text{ and } \pi_3 = a_4 a_6 < 0.$$

We note that a purely extinction of all species is not possible (as \(E_0\) is a saddle point). However, when the prey and intermediate predator are harvested more than their intrinsic grow rates ($q_1 E > r_x$ and $q_2 E > r_y$) a purely extinction of all the species will happen, that is, \(E_0\) is stable. We also note that prey can survive alone when the intermediate predator is harvested.
more than it grows, as the works by (Blé et al., 2018; Ghosh et al., 2018; Kumar & Agarwal, 2017). We remark that the coexistence of prey and intermediate predator, intermediate and top predators, and all the three species is possible. In addition, we note that the type and stability of the equilibrium points could change as parameters value vary. Furthermore, oscillators behavior and limit cycles could happen which will be investigated numerically.

3.4 | Global stability analysis

In this section we show global stability of the equilibrium points $E_{xy}, E_{yz}$, and $E_{xyz}$.

3.4.1 | Global stability of $E_{xy}(x^*, y^*, 0)$ and $E_{yz}(0, y^*, z^*)$

**Theorem 3.5.**

(a) If the equilibrium point $E_{xy}$ exists and is locally asymptotically stable in the interior of positive quadrant of $xy$ plane then it is globally asymptotically stable in the region $\Omega_1 = \left\{0 < x, 0 < y, \frac{\varphi_x(x)}{x} \geq \left(\frac{\varphi_x(x^*)}{x^*}\right)\right\}$.

(b) If the equilibrium point $E_{yz}$ exists and is locally asymptotically stable in the interior of positive quadrant of $yz$ plane then it is globally asymptotically stable in this region $\Omega_2 = \left\{0 < y, 0 < z, \frac{\varphi_y(y)}{y + 2y} < (1 - \varepsilon)\right\}$.

**Proof.** Assume $E_{xy}(x^*, y^*, 0)$ is locally asymptotically stable. In the $xy$ plane, the model system reduces to the form

\[
\frac{dx}{d\tau} = x (1 - x) - k\varphi_x(x) y - hx,
\]

\[
\frac{dy}{d\tau} = hy (1 - y) + \varphi_x(x)y - \delta hy.
\]

Consider that

\[H(x, y) = \frac{1}{xy} > 0\]

be a Dulac’s function in the positive quadrant and

\[H_1(x, y) = x (1 - x) - k\varphi_x(x) y - hx, H_2(x, y) = hy (1 - y) + \varphi_x(x)y - \delta hy.\]

Then,

\[
\phi(x, y) = \frac{\partial}{\partial x} (HH_1) + \frac{\partial}{\partial y} (HH_2) = \frac{-1}{y} - \frac{\varepsilon}{x} - \frac{k}{x^2 y} (\varphi_x'(x) - \varphi_x(x)).
\]
The sufficient condition for $\phi(x, y) < 0$ is $x\varphi'_x(x) - \varphi_x(x) \geq 0$, which is simplified as $\varphi_x(x) \geq \frac{x}{x^*}$. Hence, $E_{xy}(x^*, y^*, 0)$ is globally asymptotically stable in the region $\{0 < x, 0 < y, \varphi_x(x) \geq \left(\frac{\varphi_x(x^*)}{x^*}\right)\}$.

Similarly for (3b) $E_{yz}(0, y^*, z^*)$ is globally asymptotically stable in the region $\{0 < y, 0 < z, \varphi_{y\gamma}(y) \leq \frac{\varphi_{y\gamma}(y^*)}{(y^* + 2y)} < (1 - \varepsilon)\}$.

3.4.2 | Global stability of $E_{xyz}(x^*, y^*, z^*)$

The Lyapunov theory is used to make conclusions about trajectories of a system (especially nonlinear) without finding the trajectories (i.e., solving the differential equation). The approach is based on the work (Upadhyay & Raw, 2011) and others.

**Theorem 3.6.** The equilibrium point $E_{xyz}(x^*, y^*, z^*)$ is globally asymptotically stable if $x^*\varphi'_x(x^*) \geq \varphi_x(x^*)$ and $y^*\varphi'_y(y^*) \geq \varphi_y(y^*)$.

**Proof.** We consider the positive definite Lyapunov function

$$V(x, y, z) = \left(x - x^* - x^*\ln\left(\frac{x}{x^*}\right)\right) + w_1\left(y - y^* - y^*\ln\left(\frac{y}{y^*}\right)\right) + \frac{w_2}{2}(z - z^*)^2,$$

where $w_1$ and $w_2$ are positives constants that will choose later. Differentiating $V$ with respect to time $\tau$,

$$\frac{dV}{d\tau} = \left(1 - \frac{x^*}{x}\right)\frac{dx}{d\tau} + w_1\left(1 - \frac{y^*}{y}\right)\frac{dy}{d\tau} + w_2(z - z^*)\frac{dz}{d\tau}$$

and substituting in the expressions for $\frac{dx}{d\tau}$, $\frac{dy}{d\tau}$, $\frac{dz}{d\tau}$ from the scaled equation of the model and taking under consideration $(x^*, y^*, z^*)$ is equilibrium point we have

$$\frac{dV}{d\tau} = -(x - x^*)^2\left(1 + \frac{xy}{x}\left(\varphi'_x(x^*) - \frac{\varphi_x(x^*)}{x^*}\right)\right) - w_1\left(\varepsilon + \frac{z}{y}\left(\varphi'_y(y^*) - \frac{\varphi_y(y^*)}{y^*}\right)\right)(y - y^*)^2$$

$$- w_2(y - \beta\varphi_y(y^*))(z - z^*)^2 + \left(1 + \frac{w_1\varphi'_x(x^*) - \frac{\varphi_x(x^*)}{x^*}}{x^*}\right)(x - x^*)(y - y^*)$$

Choosing $w_1 = \frac{x\varphi_x(x^*)}{x^*\varphi_x(x^*)}$ and $w_2 = \frac{w_1\varphi'_x(x^*)}{y^*\varphi_y(y^*)}$, we have

$$\frac{dV}{d\tau} = -(x - x^*)^2\left(1 + \frac{xy}{x}\left(\varphi'_x(x^*) - \frac{\varphi_x(x^*)}{x^*}\right)\right) - w_1\left(\varepsilon + \frac{z}{y}\left(\varphi'_y(y^*) - \frac{\varphi_y(y^*)}{y^*}\right)\right)(y - y^*)^2 - w_2(y - \beta\varphi_y(y^*))(z - z^*)^2.$$
Hence, $\frac{dV}{d\tau} < 0$ if $x^*\varphi_x'(x^*) \geq \varphi_x(x^*)$ and $y^*\varphi_y'(y^*) \geq \varphi_y(y^*)$. \hfill $\square$

## 3.5 Persistence

Persistence of a system means no state of the system approaches to zero, that is, there can be no extinction of any of the population that makes up the biological system. As it is explained, $E_0(0,0,0)$ is unstable in the $x$ direction and asymptotically stable in the $z$ direction, $E_x(1-h,0,0)$ exists on the positive $x$ axis. It is asymptotically stable in the $x$ direction. An equilibrium point $E_y(0,\frac{-\delta h}{\varepsilon},0)$ also exists and it is asymptotically stable in the $y$ direction. No equilibrium point on the $z$ axis.

In addition to the above assumptions the following hold:

A$_1$: The right hand sides of the system in (4)–(6) are $C^1$ in $(x, y, z)$.
A$_2$: All solution of the system (4)–(6) with nonnegative initial conditions are bounded in forward time according to Proposition (3.2).
A$_3$: $E_x$ and $E_y$ exist and they are hyperbolic saddle points if $\varepsilon + \varphi_x(1-h) - \delta h > 0$ and $1 - \kappa\varphi_x'(0) - h > 0$, $-\gamma + \beta\varphi_y(\frac{\varepsilon - \delta h}{\varepsilon}) > 0$. 
A$_4$: An equilibrium point $E_{xy}(x^*,y^*,0)$ in the $xy$ plane exists and it is unstable in the positive direction orthogonal to the $xy$ plane if $\beta\varphi_y(y^*) > \gamma$. An equilibrium point $E_{yz}(0,y^*,z^*)$ in the $yz$ plane exists and it is unstable in the positive direction orthogonal to the $yz$ plane if $k\varphi_y'(0)y^* \leq 1 - h$. Since $E_{xy}$ and $E_{yz}$ are locally asymptotically stable, and also they are globally asymptotically stable, there are no limit cycles around these points (Freedman & Waltman, 1984). These observations are written in theorem form as follows.

**Theorem 3.7.** The sufficient conditions for persistence of the model system (4)–(6) are $\beta\varphi_y(y^*) > \gamma$, $k\varphi_y'(0)y^* < 1 - h$, $\varepsilon + \varphi_x(1-h) - \delta h > 0$, $1 - \kappa\varphi_x'(0) - h > 0$, and $-\gamma + \beta\varphi_y(\frac{\varepsilon - \delta h}{\varepsilon}) > 0$.

## 4 HARVESTING

In this section we discuss maximum sustainable yield, bionomic equilibrium, and harvesting optimal policy.

### 4.1 Maximum sustainable yield

Assuming the equilibrium point $(x^*,y^*,z^*)$ is locally asymptotically stable, the sustainable yield $Y(h)$ during the equilibrium situation is given in Kot (2001) as

$$Y = h(x^* + \delta y^*),$$
where $x^*$ is a positive solution of

$$x^2 + (h - 1)x + k\varphi(x)y^* = 0, \quad \text{and} \quad y^* = \varphi_y^{-1}\left(\frac{\gamma}{\beta}\right).$$ \quad (12)

Here, we manipulate maximum sustainable yield for two Holling types functional response combinations:

**Case I, HT-I/HT-I:** $\varphi_x(x) = \alpha x$, and $\varphi_y(y) = \omega y$. Substituting these into Equation (12), we get $y^* = \frac{\gamma}{\beta\omega}$ and $x^* = 1 - h - k\varphi y^*$. The yield $Y$ is given as $Y(h) = h(1 - h - k\varphi y^* + \delta y^*)$. Thus, the maximum sustainable yield $MSY = \left(\frac{1 + y^* (\delta - k\alpha)}{2}\right)^2$, is attained at $h_{\text{max}} = \frac{1 + y^* (\delta - k\alpha)}{2}$ provided that $\beta\omega + \gamma\delta > k\varphi y^*$. Figure 1a shows existence of $MSY = 0.23684$ which is attained at $h_{\text{max}} = 0.48667$ using the set of parameter values given the Figure 1. Moreover, for Case I, the MSY exists if $\beta\omega > \gamma(\delta + k\alpha)$ and $\beta\omega(2\varepsilon + \alpha) + k\alpha\delta\gamma > \gamma(2\varepsilon + \alpha\delta + \alpha^2\kappa + \delta^2) + \beta\omega\delta$ conditions hold. However, as explained in Ghosh and Kar (2013), Ghosh et al. (2018), Tromeur and Loeuille (2017) harvesting at MSY leads the prey and/or top predator extinct(s) if $\beta\omega \leq \gamma(\delta + k\alpha)$ and/or $\beta\omega(2\varepsilon + \alpha) + k\alpha\delta\gamma \leq \gamma(2\varepsilon + \alpha\delta + \alpha^2\kappa + \delta^2) + \beta\omega\delta$ hold, respectively.

**Case II, HT-I/HT-II:** $\varphi_x(x) = \alpha x$, and $\varphi_y(y) = \frac{\omega_1 y}{\beta_1 + y}$. Similarly, we obtain $y^* = \frac{\gamma\beta_1}{\beta_1 + \gamma}$ and $x^* = 1 - h - k\varphi y^*$. The maximum sustainable yield $MSY = \left(\frac{1 + y^* (\delta - k\alpha)}{2}\right)^2$, is attained at $h_{\text{max}} = \frac{1 + y^* (\delta - k\alpha)}{2}$, provided that $\beta_1 + \gamma\beta_1 > k\varphi\gamma\beta_1$. The MSY exists if $\beta_1 > \gamma(1 + \beta_1(\delta + k\alpha))$ and $(\beta_1 - \gamma)(2\varepsilon + \alpha - \delta) + k\alpha\gamma\beta_1 > \gamma\beta_1(2\varepsilon + \alpha\delta + \alpha^2\kappa - \delta^2)$ conditions hold, which is demonstrated in Figure 1b. Using the set of parameter values in Figure 1 $MSY = 0.24404$ which is attained at $h_{\text{max}} = 0.49400$. But harvesting at MSY leads either prey or predator extinct if $\beta_1 \leq \gamma(1 + \beta_1(\delta + k\alpha))$ and $(\beta_1 - \gamma)(2\varepsilon + \alpha - \delta) + k\alpha\gamma\beta_1 \leq \gamma\beta_1(2\varepsilon + \alpha\delta + \alpha^2\kappa - \delta^2)$ conditions hold, respectively.

![Graph](a) ![Graph](b)

**FIGURE 1** Yield-effort curve and prey, intermediate predator, and top predator for varying effort.

(a) Yield effect with HT-I/HT-I and (b) yield effect with HT-I/HT-II. We select the parameters as $\delta = 0.2, \kappa = 0.3, \gamma = 0.1, \varepsilon = 0.5, \beta = 0.3, \alpha = 0.8\omega = 0.6, \omega_1 = 0.6, \beta_1 = 0.2$
### 4.2 Bionomic equilibrium and harvesting optimal policy

The economic equilibrium means that the total revenue from harvesting of populations equals to the total cost of harvesting efforts together with biological equilibrium point \((x, y, z)\). As in Abid et al. (2019), Chakraborty et al. (2012), Rojas-Palma and González-Olivares (2012), Tchepmo Djomegni et al. (2019), we consider the total cost is proportional to the harvesting effort, written as \(C_E = CE\). We also assume that the total profit from the harvesting of population is proportional to the harvesting yield, written as \(P_Y = PY(E)\), where \(P\) is the unit price of the stock and \(Y(E) = qEX\) denotes the harvesting yield function. The profit function denoted \(\Pi\) is then given by \(\Pi = T_P - T_c\). Let \(C\) states harvesting cost per unit effort of population \(X\) and \(Y\) and \(P_1\) and \(P_2\) denotes the price per unit biomass of population \(X\) and \(Y\), respectively. The profit function of harvesting for the intermediate predator and prey populations in the free fishing zone is given by:

\[
\Pi(X, Y, E) = (P_1q_X + P_2q_Y - C)E. \tag{13}
\]

After scaling Equation (13) we get

\[
\pi(x, y, h) = (p_1x + \delta p_2y - c)h, \tag{14}
\]

\[
p_1 = r_X P_1 K_X, p_2 = r_Y P_2 K_Y, \text{and } c = \frac{r_X C}{q_X}.
\]

A bionomic equilibrium of the model implies both biological and economic equilibrium. Biological equilibrium occurs when interior equilibrium point of the model system exists and economic equilibrium is achieved when the economic rent is completely degraded, that is, \(\pi(h) = 0\). Thus, the bionomic equilibrium point \((x_0, y_0, z_0, h_0)\) satisfies the following conditions:

\[
x(1 - x) - \kappa \varphi_x(x)y - hx = 0, \tag{15}
\]

\[
y(1 - y) + \varphi_x(x)y - \varphi_y(y)z - \delta hy = 0 \tag{16}
\]

\[
-\gamma z + \beta \varphi_y(y)z = 0, \tag{17}
\]

\[
h(p_1x + \delta p_2y - c) = 0. \tag{18}
\]

From these equations we get

\[
y_0 = \varphi_y^{-1}\left(\frac{\gamma}{\beta}\right),
\]

\[
x_0 = \frac{c - \delta p_2y_0}{p_1},
\]

\[
h_0 = \frac{x_0(1 - x_0) - \kappa \varphi_x(x_0)y_0}{x_0},
\]

\[
z_0 = \frac{\beta}{\gamma} \left(\varepsilon y_0(1 - y_0) + \varphi_x(x_0)y_0 - \delta hy_0\right),
\]

provided that \(h_0\) exists, \(h_0\) is the critical threshold effort that determines the profitability of the investment in the harvest. Of course, \(h = h_0\) means no profit. If \(h > h_0\), the total cost of the harvest will exceed the total selling price from the harvest. Then the investment will result in a loss and may be terminated. However, if \(h < h_0\), the investment is profitable. In this situation, policy must be drawn to protect the sustainability of species while maximum profits are
targeted. Thus, we will define an optimal control problem and examine the optimal effort to apply where will produce maximum profit and secure continuity as it is explained in Tchepmo Djomegni et al. (2019).

We assume nonzero profit. Hence, the present profit is defined as

$$J_{xyh}(\pi) = \pi(x, y, h)$$

where

$$\pi(x, y, h) = \int_0^\infty e^{-\delta_1 \tau} (p_1 x + \delta p_2 y - c) h d\tau,$$

$$\delta_1$$ is the discount rate of the net revenue. We want to solve the following optimal control problem

$$\max_{h} J(x, y, h)$$

subject to (4) – (6)

and to $0 \leq h(\tau) \leq h_{\text{max}}$.

Since the system Equations (4)–(6) and the objective functional $J$ are linear in the control variable $h$, the optimal solution of the control problem Equation (21) is either a bang-bang or a singular control (Lenhart & Workman, 2007). But we will show that it cannot be a combination of a bang-bang and singular control which is similar to (Tchepmo Djomegni et al., 2019). For the singular control, the switching function

$$Q(\tau) = \frac{\partial H}{\partial h}$$

is zero on nontrivial time intervals, where the Hamiltonian $H$ is given in equation (24). Hence the optimal harvest policy (on the effort) will be (Lenhart & Workman, 2007)

$$h(\tau) = \begin{cases} h_{\text{max}} & \text{if } Q(\tau) > 0, \\ h^* & \text{if } Q(\tau) = 0, \\ 0 & \text{if } Q(\tau) < 0. \end{cases}$$

The corresponding Hamiltonian function of the problem becomes

$$H = e^{-\delta_1 \tau} ((p_1 x^* + \delta p_2 y^* - c) h + \lambda_1 (x(1 - x) - \kappa x(y) y - hx) + \lambda_2 (\gamma y (1 - y) + \varphi(x) y - \varphi(y) z - \delta hy) + \lambda_3 (-\gamma z + \beta \varphi(y) z)).$$

where the functions $\lambda_1, \lambda_2$ and $\lambda_3$ are the adjoint variables. The condition $Q(\tau) = 0$ implies.

$$e^{-\delta_1 \tau} (p_1 x + \delta p_2 y - c) - \lambda_1 x - \lambda_2 \delta y = 0.$$ 

**Theorem 4.1.** The optimal equilibrium solution $(x^*, y^*, z^*, h^*)$ of the control problem equation (21) exists and is solution of Equations (15)–(17) and (40). Moreover, the maximum net present revenue is

$$J = \frac{(p_1 x^* + \delta p_2 y^* - c)h^*}{\delta_1}.$$ 

**Proof.** Standard results established the existence of the optimal control $h^*$ and the corresponding optimal steady state $(x^*, y^*, z^*, h^*)$ (Fleming & Rishel, 1975). The adjoint equations are
\[
\frac{d\lambda_1}{d\tau} = -[e^{-\delta_1 \tau} p_1 h + \lambda_1 (1 - 2x - \kappa \varphi_x'(x)y - h) + \lambda_2 \varphi_x'(x)y],
\]
\[
\frac{d\lambda_2}{d\tau} = -[e^{-\delta_1 \tau} p_2 \delta h - \lambda_1 \kappa \varphi_x(x) + \lambda_2 (\varepsilon (1 - 2y) + \varphi_x(x) - \varphi_y'(y)z - \delta h) + \lambda_3 \beta \varphi_y'(y)z],
\]
\[
\frac{d\lambda_3}{d\tau} = -[-\lambda_2 \varphi_y(y) + \lambda_3 (-\gamma + \beta \varphi_y(y))].
\]

Since we are looking for the optimal equilibrium solution, we treat \(x, y, z, \) and \(h\) as constants. Moreover, we consider the interior equilibrium \((x^*, y^*, z^*, h^*)\) (as the sustainability of species must be preserved), that is, solution of Equations (15)–(17). Hence the adjoint equations are reduced to

\[
\lambda_1' = -\left[ e^{-\delta_1 \tau} p_1 h + \lambda_1 \left( \kappa y \left( \frac{\varphi_x(x)}{x} - \varphi_x'(x) \right) - x \right) + \lambda_2 \varphi_x'(x)y \right],
\]
\[
\lambda_2' = -\left[ e^{-\delta_1 \tau} p_2 \delta h - \lambda_1 \kappa \varphi_x(x) + \lambda_2 \left( \frac{\varphi_y(y)}{y} - \varphi_y'(y) \right) - \varepsilon y \right] + \lambda_3 \beta \varphi_y'(y)z \right],
\]
\[
\lambda_3' = \lambda_2 \varphi_y(y).
\]

Solving Equation (25) for \(\lambda_1\) and substituting the result into Equation (31), we obtain

\[
\lambda_2' = -a_1 e^{-\delta_1 \tau} - a_2 \lambda_2 - \lambda_3 \beta \varphi_y'(y)z,
\]

where \(a_0 = p_1 x + \delta p_2 y - c, a_1 = p_2 \delta h - \frac{a_0 \kappa \varphi_x(x)}{x},\) and \(a_2 = \frac{\delta \kappa \varphi_x(x)}{x} + z \left( \frac{\varphi_y(y)}{y} - \varphi_y'(y) \right) - \varepsilon y.\) Solving Equation (32) for \(\lambda_2\) and substituting into Equation (33), then \(\lambda_3\) satisfies the second order ordinary differential equation.

\[
\dot{\lambda}_3 + a_2 \lambda_3' + \beta \varphi_y'(y)z \lambda_3 = -\varphi_y(y)a_1 e^{-\delta_1 \tau}.
\]

This leads to the solution

\[
\lambda_3(\tau) = \mathcal{N}_0 e^{\lambda_1 \tau} + \mathcal{N}_1 e^{\lambda_2 \tau} - \frac{a_1 \varphi_y(y)}{\delta_1^2 - a_2 \delta_1 + \beta \varphi_y'(y)z} e^{-\delta_1 \tau},
\]

where \(\mathcal{N}_0\) and \(\mathcal{N}_1\) are constants of integration, \(\Lambda_1 = \frac{-a_2 + \sqrt{a_2^2 - 4a_1 \varphi_y'(y)z}}{2},\) and \(\Lambda_2 = \frac{-a_2 - \sqrt{a_2^2 - 4a_1 \varphi_y'(y)z}}{2}.\)

We note that \(\Lambda_1 + \Lambda_2 = -a_2 = \varepsilon y + z \varphi_y'(y) - \frac{\delta \kappa \varphi_x(x)}{x} - \frac{\varphi_y(y)z}{y} > 0\) and \(\Lambda_1 \Lambda_2 = \beta \varphi_y'(y)z.\)

This implies both \(\Lambda_1\) and \(\Lambda_2\) are positive. Imposing the transverse condition \(\lambda_3(\tau) = 0\) as \(\tau \to \infty,\) we get \(\mathcal{N}_0 = \mathcal{N}_1 = 0.\) Therefore,

\[
\lambda_3(\tau) = -\frac{a_1 \varphi_y(y)}{\delta_1^2 - a_2 \delta_1 + \beta \varphi_y'(y)z} e^{-\delta_1 \tau}.
\]

Solving \(\lambda_2\) from Equation (32) and then \(\lambda_1\) from Equation (25), we obtain

\[
\lambda_2(\tau) = \delta_1 a_3 e^{-\delta_1 \tau},
\]

\[
\lambda_1(\tau) = \delta_1 a_4 e^{-\delta_1 \tau}.
\]
\[ \lambda_1(\tau) = \frac{a_4 e^{-\delta_1 \tau}}{x}, \]  

(38)

where \( a_3 = \frac{a_0 \phi_x(y)}{\delta_1^2 - a_0 \delta_1 + \beta_0 y}, \) and \( a_4 = a_0 + \delta_1 a_3 \delta y. \)

Hence the shadow prices \( e^{\delta_1 \tau} \lambda_i, i = 1, 2, 3 \) are bounded and constant over time in optimal equilibrium when they satisfy the transversality condition at \( \infty \) (Rojas-Palma & González-Olivares, 2012; Tchepmo Djomegni et al., 2019).

Thus the switching function \( Q(\tau) \) becomes

\[ e^{-\delta_1 \tau}(a_0 - a_4 - a_3 \delta y) = 0, \quad \forall \, \tau. \]  

(39)

Therefore, the optimal solution is either a bang-bang or singular control (not a combination). Substituting the values of \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) into Equation (30), we obtain the relation

\[ \delta_1 a_4 + a_0 \left( qy \left( \frac{\phi_x(x)' - \phi_y(x)}{x} \right) + x \right) - \delta_1 a_3 \phi'_x(x)y - p_1 h = 0. \]  

(40)

Hence, the optimal equilibrium solution \( (x^*, y^*, z^*, h^*) \) of the control problem is solution of Equations (15)–(17) and (40), and the maximal net present revenue is

\[ J(x^*, y^*, h^*) = \int_0^\infty e^{-\delta_1 \tau}(p_1 x^* + \delta p_2 y^* - c)h^* d\tau = \frac{(p_1 x^* + \delta p_2 y^* - c)h^*}{\delta_1}. \]  

(41)

5 | NUMERICAL SIMULATION AND DISCUSSION

The interaction of the three species described by the four Holling types together with harvesting on prey and intermediate predator given by the model system (4) – (6), exhibits a variety of dynamical behavior. We take here some supposed parameter values with the solitary intention of lucubrate, the theoretical discussion that we have established in the previous sections using some of the 16 combinations of Holling type functional response types.

We have solved the model system (4) – (6) using the Matlab™ ODE solver ODE45. To examine the stability of the equilibrium points as a result of the complex nature of the species, we have used \((0.2, 0.4, 0.3), (0.2, 0.5, 0.4), (0.2, 0.6, 0.5), (0.4, 0.4, 0.3), (0.4, 0.5, 0.4), (0.4, 0.6, 0.5), (0.6, 0.4, 0.3), (0.6, 0.5, 0.4), (0.6, 0.6, 0.5)\), initial values for Figures 2 and 3. The solutions for the system are given in time series plots and corresponding phase portrait as shown in Figures 2 and 3. The values of the remaining parameters and Holling type combinations are given in the respective captions.

Now, for the data in Figure 2 and Holling type combination HT- I/III, it is observed that \( \pi_1 = -0.72372 < 0, \pi_2 = -0.015701 < 0, \) and \( \pi_3 = -0.0010003 < 0. \) Therefore, according to Theorem 3.4, the coexistence equilibrium point \( E_{xyz} = (0.699, 0.071, 0.06) \) is locally asymptotically stable. Figure 2a also shows stability of the equilibrium point \( E_{xyz}. \) As clearly shown in the figure, the prey population approaches to the difference of the carrying capacity and harvesting efforts. The reason is we have taken lower attack rate and large carrying capacity \( K_X \) relative to \( K_Y. \)

Again, for the same set of parameter values and Holling type HT- III/IV, it is observed that \( \pi_0 = \varepsilon + \alpha_1(1 - h)^2/(b_2 + (1 - h)^2) - \delta h = -0.11571 < 0, \) according to the local stability
condition for $E_x, E_x = (0.7, 0, 0)$ is locally asymptotically stable. Stability of $E_x$ is also shown in Figure 2b.

Figure 3, the time series plot indicates the prey population is stable while the predators populations are periodic. The phase portrait shows there is a limit cycle near the equilibrium point $E_{xyz} = (0.69866, 0.20000, 0.34650)$.

In Figures 4–6, we observe the influence of the parameter $\varepsilon$ in the model system. When $\varepsilon \leq 0.15$, keeping other parameter values constant, the equilibrium point $E_x$ is stable, and also if $\varepsilon \in [0.16, 0.25]$, the equilibrium point $E_{xy}$ is stable. In addition, for $\varepsilon \geq 0.25$, we observe that the interior equilibrium point $E_{xyz}$ is asymptotically stable. This implies the three species coexist in
Moreover, we noticed that a bifurcation point exists in between 0.15 and 0.16, and also another bifurcation point exists in between 0.38 and 0.4.

**Figures 7 and 8** show the effect of $h$. As $h$ increases each population size decreases as expected. Moreover, as $h$ varies in $[0, 0.75]$, we observe coexistence of the species, otherwise, both predator population are extinct. If $h \geq 1$ extinction of the species happen. However, it is not always the case, increasing harvesting rate can raise population of the species (Pal et al., 2019).
In this paper we have developed a food chain mathematical model where the prey and intermediate predator population are harvested and both are under the effect of Holling types I–IV functional response. To simplify the model, we performed scaling to reduce the number of parameters from 10 to 6. The model system has six equilibrium points $E_0, E_x, E_y, E_{xy}, E_{yz}$, and $E_{xyz}$.

FIGURE 5 Effect of $\varepsilon$ in species with HT-III/HT-I using $\kappa = 0.02, \alpha_1 = 0.04, c_2 = 0.8, \gamma = 0.04, h = 0.5$ and $\delta = 0.4, b_2 = 0.08, \omega = 0.2, \varepsilon \in [0.16, 0.38]$ with initial condition $(0.7, 0.5, 0.4)$. (a) Prey, (b) intermediate predator, (c) top predator, and (d) phase portrait.

FIGURE 6 Effect of $\varepsilon$ in species with HT-III/HT-I using $\kappa = 0.02, \alpha_1 = 0.04, c_2 = 0.8, \gamma = 0.04, h = 0.5$ and $\delta = 0.4, b_2 = 0.08, \omega = 0.2, \varepsilon \in [0.4, 1]$ with initial condition $(0.7, 0.5, 0.4)$. (a) Prey, (b) intermediate predator, (c) top predator, and (d) phase portrait.

6 CONCLUSION

In this paper we have developed a food chain mathematical model where the prey and intermediate predator population are harvested and both are under the effect of Holling types I–IV functional response. To simplify the model, we performed scaling to reduce the number of parameters from 10 to 6. The model system has six equilibrium points $E_0, E_x, E_y, E_{xy}, E_{yz}$, and $E_{xyz}$. 
FIGURE 7  Time series plot and phase diagram which shows harvesting effect on species with HT-II/III when $\kappa = 0.15, \varepsilon = 0.5, \beta = 0.3, \gamma = 0.01, \delta = 0.9, \alpha_1 = 0.3, b_1 = 0.2, \omega_1 = 0.3$, and $\beta_2 = 0.15$, with initial condition $(0.5, 0.4, 0.3)$. (a) Harvesting effect on prey, (b) harvesting effect on intermediate predator, (c) harvesting effect on top predator, and (d) phase diagram.

FIGURE 8  3D plots which shows harvesting effect on species with holing type HT-II/III when $\kappa = 0.15, \varepsilon = 0.5, \beta = 0.3, \gamma = 0.01, \delta = 0.9, \alpha_1 = 0.3, b_1 = 0.2, \omega_1 = 0.3$, and $\beta_2 = 0.15(0.5, 0.4, 0.3)$. (a) Harvesting effect on prey, (b) harvesting effect on intermediate predator, (c) harvesting effect on top predator, and (d) phase portal.
The conditions for coexistence of prey in the absence of predators, coexistence of intermediate predator in the absence of prey and top predator, coexistence of prey and intermediate predator in the absence of top predator, coexistence of intermediate predator and top predator in the absence of prey and top predator, and of the three species are mentioned. These conditions are vital to make certain sustainability. The stability behaviors of this three-dimensional model system have discussed around its equilibrium points. Criteria for global stability of the coexistence equilibrium and persistence of the model system are well explained in this study. Furthermore, a purely extinction of all species is not possible (as $E_0$ is a saddle point). However, when the prey and intermediate predator are harvested more than their intrinsic grow rates a purely extinction of all the species will happen, that is, $E_0$ is stable. We also note that the prey can survive alone when the intermediate predator is harvested more than it grows, as the works by Blé et al. (2018), Ghosh et al. (2018), Kumar and Agarwal (2017).

By using the Pontryagin Maximum Principle, the optimal control theory is derived analytically. Existence of optimal equilibrium solution of the control problem is verified and maximum sustainable yield and maximal net present revenue are well discussed. The numerical simulation shows the model system has rich dynamics. The time series plot and phase portrait of the mathematical model system has drawn to show the behavior of the solution near to the equilibrium points. The impact of harvesting on the population species is also elucidated in figures numerically.

**AUTHOR CONTRIBUTIONS**
Mohammed Y. Dawed and Kiros G. Kebedow: conceptualization (equal); data curation (equal); formal analysis (equal); funding acquisition (equal); investigation (equal); methodology (equal); project administration (equal); resources (equal); software (equal); supervision (equal); validation (equal); visualization (equal); writing original draft (equal); writing review & editing (equal).

**DATA AVAILABILITY STATEMENT**
All of the values of the parameters (the data we have been used) have been given under the caption of each figure/result.

**ORCID**
Kiros G. Kebedow https://orcid.org/0000-0002-6072-4444

**REFERENCES**
Abid, W., Yafia, R., Aziz-Alaoui, M., & Aghriche, A. (2019). Dynamics analysis and optimality in selective harvesting predator-prey model with modified Leslie-Gower and Holling-type ii. *Nonautonomous Dynamical Systems, 6*(1), 1–17.
Allen, L. J. (2007). *Introduction to mathematical biology*. Pearson/Prentice Hall.
Bera, S., Maiti, A., & Samanta, G. P. (2016). Dynamics of a food chain model with herd behaviour of the prey. *Modeling Earth Systems and Environment, 2*(3), 1–9.
Bergland, H., Wyller, J., & Burlakov, E. (2019). Pasture-livestock dynamics with density-dependent harvest and changing environment. *Natural Resource Modeling, 32*(4), e12213.
Blé, G., Castellanos, V., & Dela-Rosa, M. A. (2018). Coexistence of species in a tritrophic food chain model with Holling functional response type iv. *Mathematical Methods in the Applied Sciences, 41*(16), 6683–6701.
Chakraborty, K., Jana, S., & Kar, T. K. (2012). Global dynamics and bifurcation in a stage structured prey-predator fishery model with harvesting. *Applied Mathematics and Computation, 218*(18), 9271–9290.
Chen, J., Huang, J., Ruan, S., & Wang, J. (2013). Bifurcations of invariant tori in predator-prey models with seasonal prey harvesting. *SIAM Journal on Applied Mathematics, 73*(5), 1876–1905.

Das, M., & Samanta, G. P. (2020). A delayed fractional order food chain model with fear effect and prey refuge. *Mathematics and Computers in Simulation, 178*, 218–245.

Dawed, M. Y., TchepmoDjomegni, P. M., & Krogstad, H. E. (2020). Complex dynamics in a tritrophic food chain model with general functional response. *Natural Resource Modeling, 33*(2), e12260.

Fleming, W. H., & Rishel, R. W. (1975). *Deterministic and stochastic optimal control*. 1, Springer Verlag.

Freedman, H., & Waltman, P. (1984). Persistence in models of three interacting predator-prey populations. *Mathematical Biosciences, 68*(2), 213–231.

Ghosh, B., & Kar, T. (2013). Possible ecosystem impacts of applying maximum sustainable yield policy in food chain models. *Journal of Theoretical Biology, 329*, 6–14.

Ghosh, B., Pal, D., Legović, T., & Kar, T. (2018). Harvesting induced stability and instability in a tri-trophic food chain. *Mathematical Biosciences, 304*, 89–99.

Huang, J., Gong, Y., & Ruan, S. (2013). Bifurcation analysis in a predator-prey model with constant-yield predator harvesting. *Discrete & Continuous Dynamical Systems-B, 18*(8), 2101–2121.

Kot, M. (2001). *Elements of mathematical ecology*. Cambridge University Press.

Kumar, A., & Agarwal, M. (2017). Dynamics of food chain model: Role of alternative resource for top predator. *International Journal of Mathematical Modelling & Computations, 7*(2 (SPRING)), 115–128.

Lenhart, S., & Workman, J. T. (2007). *Optimal control applied to biological models*. CRC Press.

Maiti, A., Pal, A., & Samanta, G. P. (2008). Effect of time delay on a food chain model. *Applied Mathematics and Computation, 200*(1), 189–203.

Maiti, A., & Samanta, G. P. (2006). Rich dynamics of a time-delayed food chain model. *Journal of Biological Systems, 14*(03), 387–412.

Manna, D., Maiti, A., & Samanta, G. P. (2017). A Michaelis-Menten type food chain model with strong allee effect on the prey. *Applied Mathematics and Computation, 311*, 390–409.

Mortoja, S. G., Panja, P., Paul, A., Bhattacharya, S., & Mondal, S. K. (2020). Is the intermediate predator a key regulator of a tri-trophic food chain model?: An illustration through a new functional response. *Chaos, Solitons & Fractals, 132*, 109613.

Pal, D., Ghosh, B., & Kar, T. K. (2019). Hydra effects in stable food chain models. *Biosystems, 185*, 104018.

Panja, P., & Mondal, S. K. (2015). Stability analysis of coexistence of three species prey-predator model. *Nonlinear Dynamics, 81*(1-2), 373–382.

Panja, P., Mondal, S. K., & Jana, D. K. (2017). Effects of toxicants on phytoplankton-zooplankton-fish dynamics and harvesting. *Chaos, Solitons & Fractals, 104*, 389–399.

Pathak, S., Maiti, A., & Samanta, G. P. (2009). Rich dynamics of a food chain model with Hassell-Varley type functional responses. *Applied Mathematics and Computation, 208*(2), 303–317.

Rojas-Palma, A., & González-Olivares, E. (2012). Optimal harvesting in a predator-prey model with allee effect and sigmoid functional response. *Applied Mathematical Modelling, 36*(5), 1864–1874.

Tchepmo Djomegni, P. M., DoungmoGoufo, E. F., Sahu, S. K., & Mbehou, M. (2019). Coexistence and harvesting control policy in a food chain model with mutual defense of prey. *Natural Resource Modeling, 32*(4), e12230.

Tromeur, E., & Loeuille, N. (2017). Balancing yield with resilience and conservation objectives in harvested predator-prey communities. *Oikos, 126*(12), 1780–1789.

Upadhyay, R. K., & Raw, S. N. (2011). Complex dynamics of a three species food-chain model with Holling type iv functional response. *Nonlinear Analysis: Modelling and Control, 16*(3), 353–374.

---

**How to cite this article:** Dawed, M. Y., Kebedow, K. G. (2021). Coexistence and harvesting optimal policy in three species food chain model with general Holling type functional response. *Natural Resource Modeling*, e12316. [https://doi.org/10.1111/nrm.12316](https://doi.org/10.1111/nrm.12316)