Supersymmetry of Massive D=9 Supergravity

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abstract

By applying generalized dimensional reduction on the type IIB supersymmetry variations, we derive the supersymmetry variations for the massive 9-dimensional supergravity. We use these variations and the ones for massive type IIA to derive the supersymmetry transformation of the gravitino for the proposed massive 11-dimensional supergravity.

The prime example of massive supergravities must be massive type IIA, or Romans’ theory [1]. As of late it became clear that this theory is dual to type IIA with G-flux, as can be seen by compactifying massive IIA on a 2-torus [2]. Taking into account the possible flux, one ends up with an eight dimensional theory, where the RR mass and the 2-form flux form a vector under the duality group $Sl(2, \mathbb{R})$, enabling one to completely rotate the RR mass into flux, which is itself a manifestation of T-duality between the (massive) IIA and IIB supergravities [3, 4]. Another example of this is the compactification of massive type IIA on a K3 manifold with fluxes turned on [5, 6]. In that case the RR mass and the fluxes combine into an $O(4, 20)$ vector, and the RR-mass can once again be rotated into flux.

A further connection that was made in the last year is the one between massive and gauged supergravities. It was shown in [7] that the massive $d = 9$ theory obtained by Generalized Scherk-Schwarz reduction of IIB supergravity [8, 9], contained in a certain limit the $N = 2$ $d = 9$ gauged supergravity. The link between massive and gauged supergravities was further investigated in [10] by considering the dimensional reduction of the bosonic sector of the proposed 11-dimensional massive supergravity [11, 9] to 8 dimensions, and showing that the theory automatically has an $SU(2)$ gauge symmetry. This ‘gauged supergravity’ was not the same as the one found by Sezgin and Salam [12],

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though, but seemed to be related to it by an S-duality transformation. A priori this should not be a point of worry: Ungauged $N = 2 d = 8$ supergravity contains two triplets of vectors, related by $Sl(2,\mathbb{R})$ transformations, and either one can, in principle, be used to gauge the $SU(2)$ R-symmetry.

Although we can dualize the type IIA mass-parameter into fluxes, after which the theory can be oxidized to 11-dimensional supergravity, one would prefer an action from which to derive massive type IIA without too much effort. In case of the oxidation of massive IIA such an action was proposed in [11], and was afterwards generalized to other theories [9]. The main feature of these theories is that covariance is ‘broken’ by the explicit occurrence of killing vectors in the action. This seems necessary since by a well-known no-go theorem by Deser et. al. [13] 11-dimensional supergravity cannot be extended by a cosmological constant, when one assumes covariance. Since a (killing)vector is dimensionful, one is forced to introduce mass-parameters. Some of the parameters are then identified with the charges of the M9/KK9-branes. The M9/KK9-brane is, from the point of view of solutions, the oxidation of the D8-brane [14]. In order to discuss the BPSness of the M9-brane in the setting of massive 11-dimensional supergravity, a supersymmetry variation for the gravitino was put forward in [14], which after dimensional reduction leads to the variations in massive IIA. A drawback of the proposed supersymmetry rule is that this variation cannot be straightforwardly generalized to include more than one killing vector and part of this article is devoted to its generalization.

The plan of this letter is as follows: In section 1 we will apply generalized dimensional reduction to the type IIB supersymmetry transformations, in order to obtain the supersymmetry rules for the 9-dimensional massive supergravity theory presented in [8, 9]. Using these rules it is discussed that, contrary to the claims made in [7], 9-dimensional Minkowski space is not a supersymmetric vacuum of the 9-dimensional gauged supergravity. Then in section 2 we will generalize the form of the supersymmetry variation of the gravitino in the massive 11-dimensional supergravity. We will then check that upon dimensional reduction it leads to the correct supersymmetry variations for massive type IIA theory and to the supersymmetry rules derived in Section 1.

1 Type IIB SUSY rules and Generalized Dimensional Reduction

It is possible to formulate type IIB supergravity in an $SU(1,1)$ covariant way [15]. The bosonic field content is 2 scalars parameterizing an $SU(1,1)/U(1)$ coset, an $SU(1,1)$ doublet of 2-forms $\tilde{A}_{(2)}$, a metric $g$ and a 4-form $D_{(4)}$, both transforming as a singlet. The 4-form has self-dual 5-form field strength $F_{(5)}$. The scalars are combined into an $SU(1,1)$ group matrix $\tilde{V} = (\tilde{V}_- \tilde{V}_+)$ (see the appendix for our choice of parameterization in terms
of the usual $Sl(2, \mathbb{R})$ scalars). The $SU(1,1)$ invariant objects are then defined as

\[ \hat{P}_{(1)} = -\hat{\mathcal{V}}_+ T \varepsilon \hat{\mathcal{V}}_+ , \quad \hat{Q}_{(1)} = -i \hat{\mathcal{V}}_+ T \varepsilon \hat{\mathcal{V}}_+ , \]

\[ \hat{G}_{(3)} = -\hat{\mathcal{V}}_+ T \hat{\mathcal{F}}_{(3)} , \quad \hat{\mathcal{F}}_{(3)} = d \hat{\mathcal{A}}_{(2)} , \]

\[ \hat{F}_{(5)} = d \hat{D}_{(4)} + \frac{i}{4} \hat{\mathcal{A}}_{(2)} \hat{\mathcal{F}}_{(3)} , \]

where $\varepsilon_{12} = +1$. Remark that these objects are not invariant under the (local) $U(1)$, $Q_{(1)}$ being the gauge field of this transformation. In terms of the above objects we can write down the bosonic part of the Non-Self-Dual type IIB action \[3\],

\[ \int_{10} \sqrt{\hat{g}} \left[ \hat{R}(\hat{g}) + 2 \hat{\mathcal{P}}_{\mu} \hat{P}_{(1)} + \frac{1}{2 \cdot 3!} \hat{\mathcal{G}}_{(3)}^{\mu \nu \rho} \hat{\mathcal{G}}_{(3)}^{\mu \nu \rho} + \frac{1}{3 \cdot 4!} \hat{\mathcal{F}}_{(5)}^{2} \right] + \frac{i}{8} \int_{10} \hat{D}_{(4)} \hat{\mathcal{F}}_{(3)} \varepsilon \hat{\mathcal{F}}_{(3)} , \]

which, as is usual, has to be supplemented by the self-duality condition of the 5-form field-strength. In (2), we define the components of a p-form to be

\[ F_{(p)} = \frac{1}{p!} F_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \]

### 1.1 Ansatz for the bosonic fields

In regular Kaluza-Klein reduction, one writes the dependence of the higher dimensional fields on the compact manifold, as a Fourier series. Then, reduction means forgetting about all non-zero Fourier components. If one is reducing on tori (as is the case), the zero components do not have any dependence on the compact coordinates. However, as IIB supergravity has a global $SU(1,1)$ symmetry, one can loosen this condition, giving the ‘zero modes’ a well-defined dependence on the compact coordinate. More concretely, if the dependence is a specific $SU(1,1)$ transformation, the resulting lower-dimensional theory will not have any dependence on the circle’s coordinate. This is called generalised Scherk-Schwartz reduction \[3, 4, 9\]. Thus, we take the Ansatz for the fields to be

\[ \hat{e}_{\mu} = \begin{pmatrix} K^{3/28} e_{\mu}^{A} K^{-3/4} A_{(1)}^{\mu} \\ 0 \end{pmatrix} , \quad \hat{D}_{(4)} = D_{(4)} - C_{(3)} dy , \quad \hat{\mathcal{}_{(2)}} = \Lambda(y) \left[ \hat{\mathcal{A}}_{(2)} - \hat{A}_{(1)} dy \right] \]

\[ \hat{\mathcal{V}}_{\pm} = e^{\pm \Sigma} \Lambda(y) \hat{V}_{\pm} , \]

where the $y$ dependence is made explicit. The $\Lambda(y)$ is an $SU(1,1)$ matrix which depends on $y$ in such a way that $M \equiv \Lambda^{-1} \partial_y \Lambda$ is a constant matrix, which we take to be a

\[ M = \frac{1}{2} \begin{pmatrix} i m_3 & i m_2 - m_1 \\ -i m_2 - m_1 & -i m_3 \end{pmatrix} . \]

\[ ^4 \]In the next sections, we will constantly switch between the $SU(1,1)$ and the $Sl(2, \mathbb{R})$ formulation. We will use caligraphic letters for objects transforming under the action of $SU(1,1)$ and use normal letters for their $Sl(2, \mathbb{R})$ counterparts, the only exception being $D_{(4)}$ and its field strength.

\[ ^5 \]In this form the masses have the same labeling as in \[9\]. Therefore Romans’ is recovered when $m_3 = m_2, m_1 = 0$ and Cowdall’s gauged sugra when $m_1 = m_2 = 0, m_3 \neq 0$. 

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The matrix $\Lambda$ corresponding to the mass-matrix in Eq. (3) is determined by
\begin{equation}
    u = \cosh(\alpha y) + i \frac{m_2}{2\alpha} \sinh(\alpha y), \quad v = -\frac{m_1 - im_2}{2\alpha} \sinh(\alpha y),
\end{equation}
where $4\alpha^2 = m_1^2 + m_2^2 - m_3^2$. (See the Appendix for the meaning of $u$ and $v$.) $\Sigma$ is the parameter for the necessary\footnote{It can be seen that this transformation is necessary whenever $m_2 \neq m_3$.} compensating $U(1)$-transformation, its form is determined through Eq. (13).

With the above KK Ansatz we can derive
\begin{align*}
    \hat{P}_a &= e^{i\Sigma} K^{-3/28} \hat{P}_a : \mathcal{P} = -\hat{V}_+^T \varepsilon \mathcal{D} \hat{V}_+, \\
    \hat{Q}_a &= K^{-3/28} \hat{Q}_a + \hat{e}_a^\mu \partial_\mu \Sigma : \mathcal{Q}_1 = -i \hat{V}_+^T \varepsilon \mathcal{D} \hat{V}_+, \\
    \hat{P}_9 &= -K^{3/4} \hat{V}_+^T \varepsilon M \hat{V}_+ : \hat{Q}_9 = -iK^{3/4} \hat{V}_-^T \varepsilon M \hat{V}_+ + K^{3/4} \partial_y \Sigma, \\
    \hat{G}_{(3)abc} &= e^{i\Sigma} K^{-9/28} \hat{G}_{(3)abc} : \hat{F}_3 = d\hat{A}_2 + A_1 \hat{F}_2, \\
    \hat{G}_{(3)ab9} &= e^{i\Sigma} K^{-15/28} \hat{G}_{(2)ab} : \hat{F}_2 = d\hat{A}_1 - M\hat{A}_2, \\
    \hat{F}_{(5)a_1...a_5} &= K^{-15/28} F_{(5)a_1...a_5} : \hat{F}_{(5)a_1...a_4} = -K^{9/28} F_{(4)a_1...a_4},
\end{align*}
\begin{align*}
    F_{(5)} &= dD(4) + A_1 F_4 + \hat{\chi} A_1 \hat{A}_{(1)}^T \varepsilon \hat{F}_2 + \frac{i}{4} A_1 \hat{A}_{(1)}^T \varepsilon \hat{F}_3, \\
    F_{(4)} &= dC(3) - \frac{i}{4} A_{(1)}^T \varepsilon \hat{F}_3 - \frac{i}{4} A_{(2)}^T \varepsilon \hat{F}_2 + \frac{i}{4} A_1 A_{(1)} \hat{A}_{(1)}^T \varepsilon \hat{F}_2,
\end{align*}
where we have defined $\mathcal{D} \hat{V} = (d - A_{(1)} M) \hat{V}$ and $\mathcal{G}_{(n)} = -\hat{V}_+^T \varepsilon \hat{F}_n$. Since the reduced action is, after rewriting it in terms of $SL(2, \mathbb{R})$ objects, the same as the one presented in \cite{1}, we abstain from presenting it here.

As is known, the reduced theory is invariant under local $U(1)$ transformations. In the $SU(1,1)$ notation the action of these transformations on the bosonic fields are
\begin{align}
    \hat{V}_+ &\rightarrow e^{\pm i \Omega} e^{RM} \hat{V}_+, \quad A_{(1)} \rightarrow A_{(1)} + d\hat{N}, \\
    \hat{A}_{(1)} &\rightarrow e^{RM} \hat{A}_{(1)}, \quad \hat{A}_{(2)} \rightarrow e^{RM} \left[ \hat{A}_{(2)} + d\hat{N} \wedge \hat{A}_{(1)} \right],
\end{align}
where $\hat{N}$ is an arbitrary function and the rest of the objects are invariant. Once again we stress that we need to apply a compensating transformation, with a parameter $\Omega$ which is determined by the form of the $SU(1,1)$-matrix $e^{RM}$, in order to stay in the chosen parameterization of the coset.
1.2 Reduction of the SUSY transformations

The fermionic field content of the theory consists of a chiral complex Rarita-Schwinger field \( \Psi_\mu \) and an anti-chiral complex spinor \( \chi \). The chiralities of the fermions in the theory are

\[
\Gamma_{11} \hat{\Psi}_\mu = -\hat{\Psi}_\mu, \quad \Gamma_{11} \hat{\chi} = \hat{\chi}.
\]  

(9)

The fermions are \( SU(1,1) \) singlets but transform under the local \( U(1) \) with weight 1/2 resp. 3/2. Since we have fields that are charged under the local \( U(1) \) we define the \( U(1) \) covariantized derivative \( \mathcal{D} \) by

\[
\Phi[U] \to e^{iU(x)} \Phi[U] \quad \longrightarrow \quad \mathcal{D} \Phi[U] \equiv \nabla \Phi[U] - iU \mathcal{Q}(1) \Phi[U]
\]  

(10)

Note that \( \mathcal{Q}(1) \) acts as a connection for the local \( U(1) \) since it transforms as \( \mathcal{Q}(1) \to \mathcal{Q}(1) + d\mathbb{N} \).

The type IIB supersymmetry transformations are \([15, 16]\)

\[
\delta \hat{\Psi}_\mu = \hat{\mathcal{D}}_\mu \hat{\epsilon} + \frac{i}{16 \cdot 5!} \hat{F}_{(5)} \hat{\Gamma}_\mu \hat{\epsilon} + \frac{1}{96} \left[ \hat{\Gamma}_\mu \hat{\gamma}^{\hat{\rho} \hat{\sigma}} \hat{\mathcal{G}}_{\hat{\mu} \hat{\rho} \hat{\sigma}} - 9 \hat{\Gamma}^{\hat{\rho} \hat{\sigma}} \hat{\mathcal{G}}_{\hat{\rho} \hat{\sigma}} \right] \hat{\epsilon}^*,
\]

\[
\delta \hat{\chi} = i \hat{\mathcal{P}} \hat{\epsilon}^* - \frac{i}{4 \cdot 3!} \hat{\mathcal{G}}_{(3)} \hat{\epsilon},
\]

where

\[
\hat{\mathcal{D}}_\mu \hat{\epsilon} = \partial_\mu \hat{\epsilon} - \frac{1}{4} \hat{\gamma} \mu \hat{\epsilon} - \frac{i}{2} \hat{\mathcal{Q}}_\mu \hat{\epsilon}.
\]

(12)

since \( \hat{\epsilon} \) is a \( U(1) \) charge 1/2 complex Weyl spinor \([13]\). The chirality of \( \hat{\epsilon} \) is defined by \( \Gamma_{11} \hat{\epsilon} = -\hat{\epsilon} \) and \( \hat{\epsilon}^* \) is the complex conjugate of \( \hat{\epsilon} \). In order to reduce the susy transformations we decompose the gamma matrices as

\[
\Gamma_{11} = \sigma^3 \otimes \mathbb{1}, \quad \Gamma^9 = -i \sigma^1 \otimes \mathbb{1}, \quad \Gamma^a = \sigma^2 \otimes \gamma^a,
\]

(13)

where the \( \gamma \)'s are matrices forming a real Majorana representation\([\text{[1]}]\) of the 9-dimensional Clifford algebra with \( \gamma^8 = \gamma^9 \cdots \gamma^7 \). The 10-dimensional chiral spinors reduce to ordinary spinors in 9 dimensions, i.e. to a complex combination of 2 Majorana spinors.

The KK-Ansatz for the 10-dimensional spinors is

\[
\hat{\chi} = e^{3\Sigma/2} K^{-3/56} \chi, \quad \hat{\Psi}_a = e^{i\Sigma/2} K^{-3/56} (\Psi_a - \frac{1}{4} \Gamma_a \Gamma^9 \hat{\chi}),
\]

\[
\hat{\epsilon} = e^{i\Sigma/2} K^{3/56} \epsilon, \quad \hat{\Psi}_9 = e^{i\Sigma/2} K^{-3/56} \hat{\chi}.
\]

(14)

Please note that due to the local \( U(1) \)-invariance we are forced to make this Ansatz for the fermions. It is also the same local invariance that guarantees that the final result will be, and indeed is, independent of \( \Sigma \). Also note that the above Ansatz implies some well determined \( y \)-dependence of the supersymmetry transformations parameters.

\(^7\) The signature of the tangent space metric is taken to be mostly minus.
Straightforward reduction of the gravitino and the dilatino variations (11) then gives rise to the 9-dimensional supersymmetry variations (the fermionic fields are set to zero):

$$\delta \chi = \mathcal{P} \epsilon^* - \frac{K^{6/7}}{7} \tilde{V}_+ \epsilon^* + \frac{1}{8} K^{9/14} \mathcal{G}_{(2)}(2) \epsilon - \frac{1}{4 \cdot 3!} K^{-3/14} \mathcal{G}_{(2)}(3) \epsilon,$$

$$\delta \tilde{\chi} = -\frac{3}{8} \phi \log (K) \epsilon + \frac{1}{8} K^{-6/7} F_{(2)} \epsilon - \frac{1}{2} K^{6/7} \tilde{V}_- \epsilon \tilde{M} \tilde{V}_+ \epsilon - \frac{i}{8 \cdot 4!} K^{3/7} F_{(4)} \epsilon$$

$$+ \frac{1}{96} K^{-3/14} \mathcal{G}_{(3)}(2) \epsilon^* + \frac{3}{32} K^{9/14} \mathcal{G}_{(2)}(2) \epsilon^*,$$

$$\delta \Psi = \mathcal{D}_\mu \epsilon - \frac{1}{17} K^{6/7} \tilde{V}_+ \epsilon^* + \frac{\gamma_\mu}{7} K^{6/7} \mathcal{G}_{(2)}(2) + \frac{1}{17} K^{-6/7} \mathcal{G}_{(3)}(3) \epsilon^*$$

$$- \frac{1}{78} K^{9/14} \mathcal{G}_{(2)}(2) \gamma_\mu a^b \epsilon^* + \frac{1}{8 \cdot 3!} K^{-3/14} \mathcal{G}_{(3)}(2) \epsilon^*$$

$$+ \frac{i}{17 \cdot 4!} F_{(4)} \epsilon^*$$

Note that we still have the local $U(1)$ symmetry and that the local symmetry relations of Eq. (8) are extended to the supersymmetry variation once we take into account the necessary compensating $U(1)$ transformations.

In order to deduce the transformation rules for $N = 2$ gauged supergravity, one has to set $m_1 = m_2 = 0$ in the mass-matrix $M$. In this reference, it was claimed that 10-dimensional Minkowski space survived the dimensional reduction, leading to the most uncommon of occurrences that 9-dimensional Minkowski space was a supersymmetric vacuum of 9-dimensional gauged supergravity. One can check however, by setting $K = 1$, $g_{\mu\nu} = \eta_{\mu\nu}$ and all other fields to zero while taking e.g. $i \gamma^8 \epsilon = \epsilon$, that the supersymmetry variations Eqs. (15, 16, 17) do not support this claim. Another way of seeing this is that having Minkowski space in 9-dimensions implies that the lifted solution is Minkowski space in 10 dimensions (if the compactification radius is taken to be very large). If we then consider the setup of [7], namely $\lambda = i$, $m_1 = m_2 = 0$ and $m_3 = m \neq 0$, we can see that the lifted 10-dimensional killing spinor would have to have the following $y$-dependence:

$$\hat{\epsilon}(x, y) = e^{imy/4} \hat{\epsilon}(x).$$

The killing spinors of Minkowski$_{10}$, in the Cartesian coordinates we use, do not have any coordinate dependence whatsoever, so the only consistent choice is for them to vanish. I.e. although Minkowski$_{10}$ gives rise to a solution of the gauged 9-dimensional supergravity, it will not be supersymmetric.

This point is comparable to the discussion in the literature of why some supersymmetric solutions will give rise to non-supersymmetric solutions after T-duality. The most
striking example of this is Minkowski with an $\mathbb{R}^2$ written in polar coordinates, $r$ and $\varphi$. Applying T-duality in the angular direction, it was found that the T-dual theory was not supersymmetric, whereas the original obviously was maximally supersymmetric. The reason is that in polar coordinates, IIB admits a Killing spinor

$$\hat{\epsilon} = \exp \left( -\frac{\kappa}{2} \Gamma^{\mathcal{M}} \right) \epsilon_0$$

where $\epsilon_0$ is some arbitrary spacetime independent spinor satisfying $\Gamma_{11} \epsilon_0 = -\epsilon_0$, and $x$ means the tangent-space direction associated to the coordinate $x$. Now usual KK reduction prohibits any dependence on the compact coordinate, so that the only consistent choice for $\epsilon_0$ is for it to vanish. Thus explaining how T-duality can connect a supersymmetric and a non-supersymmetric solution.

To conclude, we derived the supersymmetry transformation rules for the massive nine dimensional supergravity. As this theory reduces in a certain limit to $N = 2 \ d = 9$ gauged supergravity, this permitted us to show that the latter does not admit a supersymmetric Minkowski type of solution.

## 2 Supersymmetry of Massive 11-dimensional Sugra

As was said in the introduction, massive 11-dimensional sugra was proposed in order to have a way of obtaining massive type IIA sugra by means of ordinary KK-reduction. The price to pay is the explicit occurrence of killing vectors in the action, which is supported by the investigations of the M-theory origin of massive D-branes \cite{19,20}. These vector fields do not change the field content of the ‘original’ 11-dimensional sugra, as they are not dynamical. The occurrence of the killing vectors forces the introduction of mass parameters, gathered into a symmetric mass-matrix $Q$, into the action. These masses are associated to the charges of the KK9/M9-branes. The KK9-brane is the oxidation of the D8-brane, and a KK9-brane action was put forward in \cite{21}.

\cite{14} discussed the solution corresponding to the KK9-brane and put forward the relevant part of the supersymmetry variation of the 11-dimensional gravitino, in order to discuss the BPSness of the KK9-brane. The proposed form however, cannot be straightforwardly generalized to include more killing vectors. Therefore in the spirit of \cite{14,3} we will propose a generalized variation of the 11-dimensional gravitino and check that upon dimensional reduction it matches with massive type IIA and the variations derived in Section 1.2.

As soon as we introduce more than one killing direction in our system, the number of possible terms one can write down that are linear in the mass matrix and respect the tensor form of the massless variation, is rather limited. In fact the gravitino variation can be written as

$$\delta \hat{\psi}_{\hat{\mu}} = \nabla_{\hat{\mu}} (\hat{w}) \hat{\epsilon} + \frac{i}{288} \left[ \hat{\Gamma}^{\hat{a}\hat{b}\hat{c}\hat{d}}_{\hat{\mu}} - 8 \hat{\Gamma}^{\hat{b}\hat{c}\hat{d}}_{\hat{\mu}} \right] \hat{\epsilon} \hat{G}^{\hat{a}\hat{b}\hat{c}\hat{d}} - \frac{i}{12} \hat{k}(n)_{\hat{\mu}} Q^{nm} k_{(m)}^{\hat{\nu}} \hat{\Gamma}_{\hat{\mu}} \hat{\epsilon} + \frac{i}{2} \hat{k}(n)_{\hat{\mu}} Q^{nm} k_{(m)}^{\hat{\nu}} \hat{\Gamma}\hat{\nu} \hat{\epsilon},$$

(20)
where $k^\mu_{(m)}$ are the $M$ killing vectors ($m = 1, \ldots, M$),

$$
\hat{G} = d\hat{C} - \frac{1}{2}i\hat{k}_{(n)} \hat{C}Q^{nm}\hat{k}_{(m)} \hat{C},
$$

and where the torsion, the contorsion and the connection are defined by

$$
T_{ab}^c = -\left[i\hat{k}_{(n)} \hat{C}\right]_{ab} Q^{mn}\hat{k}_{(n)}^c, \quad 2K_{abc} = -T_{abc} + T_{bca} - T_{cab}, \quad \hat{\omega} = \omega + K.
$$

Note that the definition of the field-strength (21) automatically takes care of the fact that the field-strengths in lower dimensions have the correct form [11, 9].

### 2.1 Towards Romans’ theory

As was said before, the idea for massive 11-dimensional supergravity was first introduced [11] in order to resolve the question of the 11-dimensional origin of Romans’ theory. Therefore, the first thing to do would be to compare the susy rules for Romans’ theory [1] with the ones we obtain from eq. (20) by dimensional reduction over the killing direction, which we will denote by $z$. As there is only one killing vector in this setting, we obviously have to take $\hat{k}^a \partial_\mu = \partial_z$. In short, we decompose the bosonic fields as [11]

$$
\hat{C} = C + Bdz, \quad \left(\hat{e}_\mu^a\right) = \begin{pmatrix} e^{-\phi/3}e_\mu^a & e^{2\phi/3}C^{(1)}_\mu \\ 0 & e^{2\phi/3} \end{pmatrix},
$$

and the fermionic objects as

$$
\hat{\psi}_z = \frac{i}{3}e^{\phi/6}\Gamma_{11}\lambda, \quad \hat{\Gamma}^a = \Gamma^a, \\
\hat{\psi}_a = e^{\phi/6}(\psi_a - \frac{1}{6}\Gamma_a\lambda), \quad \hat{\Gamma}^{10} = i\hat{\Gamma}^{1\ldots9} = -i\Gamma_{11}, \\
\hat{\epsilon} = e^{-\phi/6}\epsilon,
$$

where $a = 0, \ldots, 9$, the tangent-space direction associated to $z$ is called $z$ and $\epsilon$ is a 10-dimensional Majorana spinor.

In accordance with our choice of $Q^{mn}$ in the next subsection, we take $Q = -m$ and can see that

$$
\hat{k}^a = 0, \quad \hat{k}^z = e^{2\phi/3} \rightarrow \hat{k}_{(n)}\hat{k}^{(m)} = m e^{4\phi/3}.
$$

Using the bosonic decomposition we can find

$$
\hat{G}_{abcd} = e^{4\phi/3}G_{(4)abcd}, \quad G_{(4)} = dC_{(3)} - HC_{(1)} + \frac{m}{2}B^2, \\
\hat{G}_{abcz} = e^{\phi/3}H_{abc}, \quad H = dB, \\
\hat{\omega}_{abc} = \frac{1}{2}e^{4\phi/3}G_{(2)ab}, \quad G_{(2)} = dC_{(1)} + mB,
$$

and the rest of the objects as usual [1].
Plugging everything into the supersymmetry variation (20), we find
\[ \delta \lambda = \partial \phi \epsilon + \frac{1}{2 \cdot 3} \Gamma_{11} \eta \epsilon + \frac{i}{4} \epsilon \phi \left\{ 5 \eta + \frac{3}{2} \Phi_{(2)} (-\Gamma_{11}) + \frac{1}{4} \Phi_{(4)} \right\} \epsilon , \]
\[ \delta \Psi_\mu = \nabla_\mu \epsilon - \frac{1}{8} \Gamma_{11} \eta \mu \epsilon + \frac{i}{8} \epsilon \phi \left[ m \Gamma_\mu \epsilon + \frac{1}{2} \Phi_{(2)} \Gamma_\mu (-\Gamma_{11}) \epsilon + \frac{1}{4} \Phi_{(4)} \mu \epsilon \right] , \]
which are, in our conventions, the string-frame supersymmetry variations of massive type IIA (See e.g. [23]).

2.2 Comparison with the results in Sec 1.2

We now reduce (21) to nine dimensions on a two-torus (using regular Kaluza-Klein truncation) and check if we find the tranformation rules for the massive nine dimensional supergravity. Following the conventions of [9] our Ansatz for the metric is
\[ \hat{T}^{\hat{\mu} \hat{\nu}} \text{ and check if we find the transformation rules for the massive nine dimensional supergravity. Following the conventions of [9] our Ansatz for the metric is} \]
\[ \hat{T}^{\hat{\mu} \hat{\nu}} = \left( \begin{array}{cc} K^{-1/7} e^{-\phi/2} & K^{1/2} v_m^i A_{(m)}^i \\ 0 & K^{1/2} \nu_i^m \end{array} \right) , \hat{e}_a^\mu = \left( \begin{array}{cc} K^{1/7} e_a^\mu & -K^{1/7} A_{(m)}^i \\ 0 & K^{-1/2} \nu_i^m \end{array} \right) \]
where \( i, j, m, n = 1, 2, K = (\text{det} \hat{\gamma}_{mn})^{1/2} \) and \( v_m^i \nu_i^j \delta_{ij} = M_{mn} \). Or, put differently, \( M = VV^T \) with \( V = v_m^i \). Since \( V \) is an element of \( SL(2, \mathbb{R}) \), we also have \( V^{-1} = \epsilon V^T \epsilon^T = v_i^m \).

Looking at (9:2.15) we see that
\[ v_m^i = \left( \begin{array}{c} e^{-\phi/2} \\ 0 \end{array} \right) , \nu_i^m = \left( \begin{array}{c} e^{\phi/2} C(0) \\ 0 \end{array} \right) , \]
\[ v_1^i = \left( \begin{array}{c} e^{\phi/2} C(0) \\ 0 \end{array} \right) , \nu_1^m = \left( \begin{array}{c} -e^{-\phi/2} C(0) \\ 0 \end{array} \right) . \]
And as before we have \( A_{(m)}^i = -\epsilon_{mn} A_{(m)}^i \). Doublets of fields like \( A_{(m)}^i \) will be written as vectors (like \( \hat{A}_\mu \)). The 3-form will be split as
\[ \hat{C} = C(3) - \frac{i}{2} \hat{A}^m \epsilon \hat{A}_m + \frac{1}{2} C(1) \epsilon \hat{A}_m \epsilon \hat{A}_m + \frac{1}{2} A_{(1)} \hat{A}^m \epsilon \hat{A}_m , \]
where we defined \( \hat{\theta} = d\bar{y} - \epsilon \hat{A} \). A small calculation then immediately shows that
\[ \hat{G}_{abij} = K^{5/7} F_{(2)ab} \epsilon_{ij} , \quad \hat{G}_{abci} = K^{-1/14} v_i^m F_{(3)abcm} , \quad \hat{F}_{(2)} = dA_{(1)} , \quad \hat{F}_{(3)} = dA_{(2)} + A_{(1)} F_{(2)} , \]
\[ \hat{G}_{abcd} = K^{1/7} G_{(4)abcd} , \quad \hat{G}_{(4)} = dC_{(3)} + \frac{i}{2} A_{(1)} \epsilon dA_{(2)} - \frac{1}{2} A_{(2)} \epsilon F_{(2)} , \]
\[ \hat{\omega}_{abi} = \frac{1}{8} K^{11/14} v_{im} \epsilon_{mn} F_{(2)abm} , \quad \hat{\omega}_{aij} = \frac{1}{8} K^{1/7} v_i^m \epsilon_{mj} D_a v_{mj} , \quad \hat{d} v_{mj} = dv_{mj} - A_{(1)} M^m v_{mj} , \]
\[ \hat{\bar{\omega}}_{mj} = K^{1/7} \left[ \frac{1}{2} \eta_{ij} \partial_a \log(K) + v_{mi} D_a v_{mj} \right] . \]
where \( \eta_{ij} = -\delta_{ij} \).

The Clifford algebra will be split as
\[ \Gamma^a = \sigma^2 \otimes \gamma^a , \quad \Gamma^9 = i \sigma^1 \otimes \mathbb{I} , \quad \Gamma^{10} = -i \sigma^3 \otimes \mathbb{I} , \quad \Gamma^{32} = \mathbb{I} \otimes \mathbb{I} \],
and the Ansatz for the 11-dimensional gravitino is

$$\tilde{\Psi}_i = K^{1/14} \chi_i , \quad \Psi_a = K^{1/14} \left\{ \Psi_a - \frac{1}{7} \Gamma_a \chi_i \right\} , \quad \hat{\epsilon} = K^{-1/14} \epsilon ,$$

(33)

which means that we combine the 9-dimensional spinors into a vector. After a small calculation we then end up with

$$\delta \chi_i = -\frac{1}{4} \nabla \log (K) \sigma^2 \Gamma_i \epsilon - \frac{1}{4} \nu_{(i}^m \nabla v_{m]j}) \sigma^2 \Gamma^j \epsilon + \frac{i}{12} K^{3/7} \hat{F}^{(4)}_{(4)} \Gamma_i \epsilon$$

$$- \frac{1}{8} K^{9/14} v_{m} \hat{F}^{m}_{(2)} \epsilon + \frac{i}{24} K^{-3/4} v_{j} \hat{F}^{j}_{(3)} v_{i} \sigma^2 \Gamma^j \epsilon + \frac{i}{12} K^{-3/4} v_{i} \hat{F}^{i}_{(3)} \sigma^2 \epsilon$$

$$- \frac{i}{24} K^{-6/7} \hat{F}^{(2)}_{(2)} \epsilon \Gamma_j \epsilon + \frac{i}{24} K^{-6/7} T \epsilon (Q M) \hat{\Gamma}_i \epsilon + \frac{i}{24} K^{-6/7} v_{i} \epsilon (Q M) \hat{\Gamma}_j \epsilon ,$$

(34)

$$\delta \Psi_{\mu} = \mathcal{D}_{\mu} \epsilon + \frac{i}{4} K^{-6/7} T \epsilon (Q M) \gamma_{\mu} \epsilon - \frac{K^{9/14}}{4} v_{m} \hat{F}^{m}_{(2)} \left[ \gamma_{\mu}^{ab} - 12 e_{\mu}^{a} \gamma_{b} \right] \sigma^2 \hat{\Gamma}_i \epsilon$$

$$+ \frac{i}{24} K^{-3/4} v_{m} \hat{F}^{m}_{(3)} \left[ 2 \gamma_{\mu}^{abc} - 15 e_{\mu}^{a} \gamma_{bc} \right] \hat{\Gamma}_i \epsilon ,$$

(35)

where we have defined

$$\mathcal{D} \epsilon = \nabla \epsilon + \frac{1}{4} v_{i}^m \mathcal{D} v_{m} \epsilon \Gamma^i \epsilon = \nabla \epsilon - \frac{i}{2} Q(1) \sigma^2 \epsilon ,$$

(36)

For the last step in the above identification, we made use of the fact that

$$T \epsilon (Q M) = 2 i \tilde{V}_T \epsilon M \tilde{V}_+ = \frac{e^{\phi}}{2} \left( m_3 + m_2 + 2 m_1 C(0) + (m_3 - m_2) \lambda \lambda \right) .$$

(37)

Note that when we consider the case with \( Q = 0 \), Eqs. (34,35) reduce to the expressions given in [23] for \( N = 2, d = 9 \) supergravity.

By identifying

$$\epsilon = \left( \begin{array}{c} \epsilon_1 \\ -\epsilon_2 \end{array} \right) , \quad \Psi_{\mu} = \left( \begin{array}{c} \Psi_1^\mu \\ -\Psi_2^\mu \end{array} \right) , \quad \tilde{\chi} = \left( \begin{array}{c} \tilde{\chi}_1 \\ -\tilde{\chi}_2 \end{array} \right) , \quad \chi = \left( \begin{array}{c} \chi_1 \\ \chi_2 \end{array} \right) ,$$

(38)

introducing the combination \( \chi_{ij} = \sigma^2 \Gamma_{i} \chi_{j} - \frac{1}{2} \sigma^2 \eta_{ij} \Gamma_{i} \chi_{j} \) and introducing the complex combinations \( \xi = \xi_1 + i \xi_2 \), e.g. \( \epsilon = \epsilon_1 + i \epsilon_2 \), we can achieve a complete correspondence if we take

$$\tilde{\chi} = -\frac{1}{4} \sigma^2 \Gamma_i \chi_i , \quad \chi = \chi_{11} + i \sigma^2 \chi_{12} .$$

(39)

Thus, regular Kaluza-Klein reduction on a two-torus of massive 11-dimensional supergravity gives rise to the massive \( d = 9 \) supergravity.
3 Conclusions

In this letter we have reduced the IIB supergravity in a generalized way as to find the supersymmetry variations of the $\text{Sl}(2, \mathbb{R})$ family of 9-dimensional supergravity theories presented in [8, 9]. This enabled us to find the supersymmetry variations of the 9-dimensional gauged supergravity, proposed recently in [7]. Contrary to the claims made in [7] 9-dimensional Minkowski space is not a supersymmetric solution of the 9-dimensional $\text{SO}(2)$ gauged supergravity, mainly because the consistency of the KK-Ansatz on the killing spinors requires them to vanish.

We gave the possible form of the gravitino variations of what is called massive 11-dimensional supergravity and have shown that it gives rise to the correct result when compared to Romans’ theory and the supersymmetry variations we obtained by generalized dimensional reduction from type IIB.

Needless to say, there still is a long way to go. It would be nice to understand the underlying structure of the 11-dimensional massive supergravity, and use this structure to derive the complete 11-dimensional supergravity. It would also be nice to see if the 11-dimensional supergravity when compactified to lower dimensions will give rise to known theories or new theories which are related to the known ones by some S-duality relation as happens in [10]. Surely, the question whether 11-dimensional massive supergravity is a consistent construct, which might be called M(assive) theory, deserves further investigation.

Acknowledgments

The authors would like to thank T. Ortín and A. van Proeyen for fruitful discussions and comments. This work was supported in part by the F.W.O.-Vlaanderen, and the E.U. RTN programmes HPRN-CT-2000-00131 and HPRN-CT-2000-00148.

A From $\text{Sl}(2, \mathbb{R})$ to $\text{SU}(1, 1)$ and back

A defining matrix in $\text{Sl}(2, \mathbb{R})$, $V$, and $\text{SU}(1, 1)$, $\mathcal{V}$, are parametrized by

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R}) \ , \quad \mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \in \text{SU}(1, 1) .$$ (40)

The explicit mapping from $\text{Sl}(2, \mathbb{R})$ to $\text{SU}(1, 1)$ can be found to be

$$u = \frac{d + a}{2} + i \frac{b - c}{2} , \quad v = \frac{d - a}{2} + i \frac{b + c}{2} ,$$ (41)

which obviously can be inverted. Splitting the $\text{SU}(1, 1)$ matrix into two vectors as $\mathcal{V} = (\mathcal{V}_{-}, \mathcal{V}_{+})$, we parametrize the coset $\text{SU}(1, 1)/U(1)$ as

$$\mathcal{V}_{+} = \frac{1}{2 \text{Im}(\lambda)^{1/2}} \begin{pmatrix} 1 + i \lambda \\ 1 - i \lambda \end{pmatrix} , \quad \mathcal{V}_{-} = \sigma^{1} \mathcal{V}_{+}^{*} ,$$ (42)
with \( \lambda = C(0) + ie^{-\phi} \) the so-called axidilaton. In order to stay in our choice of parameterization, an \( SU(1,1) \)-transformation has to be accompanied by a local \( U(1) \)-transformation, \( i.e. \) the \( SU(1,1) \)-transformation is \( \tilde{V}_\pm = e^{\pm i \Sigma} V_\pm \). The parameter \( \Sigma(x) \) is readily found to be defined by

\[
e^{2i\Sigma} = \frac{c\bar{\lambda} + d}{c\lambda + d},
\]

(43)

The other \( SU(1,1) \) fields are given in terms of their \( Sl(2,\mathbb{R}) \) counterparts by

\[
D_{(4)} = C_{(4)} - \frac{1}{2} B \wedge C_{(2)}, \quad A^1_{(2)} = B + i C_{(2)}, \quad A^2_{(2)} = \left( A^1_{(2)} \right)^*.
\]

(44)

where \( C_{(4)} \) is the usual string field. With the above identifications one finds

\[
\mathcal{P}_{(1)} = \frac{1}{2} d\phi + \frac{i}{2} e^{\phi} dC(0),
\]

(45)

\[
\mathcal{Q}_{(1)} = -\frac{1}{2} e^{\phi} dC(0),
\]

(46)

\[
\mathcal{G}_{(3)} = e^{-\phi/2} H + i e^{\phi/2} G(3),
\]

(47)

where \( G(3) = dC_{(2)} - H C(0) \) is the standard field-strength for the RR 2-form.

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