FULLY MAXIMAL AND FULLY MINIMAL ABELIAN VARIETIES

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Abstract. We introduce and study a new way to categorize supersingular abelian varieties defined over a finite field by classifying them as fully maximal, mixed or fully minimal. The type of $A$ depends on the normalized Weil numbers of $A$ and its twists over its minimal field of definition. We analyze these types for supersingular abelian varieties and curves under conditions on the automorphism group. In particular, we present a complete analysis of these properties for supersingular elliptic curves and supersingular abelian surfaces in arbitrary characteristic, and for supersingular curves of genus 3 in characteristic 2.

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1. Introduction

Suppose that $X$ is a smooth projective connected curve of genus $g$ defined over a finite field $\mathbb{F}_q$ of characteristic $p$. The curve $X$ is supersingular if the only slope of the Newton polygon of its $L$-polynomial is $\frac{1}{2}$ or, equivalently, if its normalized Weil numbers are all roots of unity. In fact, $X$ is supersingular if and only if it is minimal over some finite extension $\mathbb{F}_{q^m}$, meaning that the number of $\mathbb{F}_{q^m}$-points of $X$ realizes the lower bound in the Hasse-Weil theorem. If $p = 2$, there exists a supersingular curve over $\mathbb{F}_2$ of every genus $g$ [10]. If $p$ is odd, it is not known whether there exists a supersingular curve over $\mathbb{F}_p$ of every genus $g$.

In this paper, we study the property that a supersingular curve $X$ over $\mathbb{F}_q$ is maximal over some finite extension $\mathbb{F}_{q^m}$, meaning that the number of $\mathbb{F}_{q^m}$-points of $X$ realizes the upper bound in the Hasse-Weil theorem.

More generally, suppose that $A$ is a principally polarized abelian variety of dimension $g$ defined over $\mathbb{F}_q$. Then $A$ is supersingular if the only slope of its $p$-divisible group $A[p^\infty]$ is $\frac{1}{2}$ or, equivalently, if its normalized Weil numbers are all roots of unity. One says that $A$ is minimal (resp. maximal) if the normalized Weil numbers all equal 1 (resp. $-1$), which means that Frobenius acts on the $\ell$-adic Tate module of $A$ by multiplication by $\sqrt{q}$ (resp. $-\sqrt{q}$). Thus, $A$ is supersingular if and only if it is minimal over some finite extension of $\mathbb{F}_q$. In this paper, we study the property that a supersingular abelian variety $A$ over $\mathbb{F}_q$ is maximal over some finite extension of $\mathbb{F}_q$.

Suppose that $K = \mathbb{F}_q$ is the minimal field of definition of a supersingular principally polarized abelian variety $A$. Write $q = p^r$. It is possible that $A$ is maximal over some finite extension of $K$, but that it has a $K$-twist which is not maximal over any finite extension of $K$. We define $A$ to be (i) fully maximal, (ii) fully minimal, (iii) mixed if (i) all, (ii) none, or

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(iii) some of its $K$-twists have the property that they are maximal over some finite extension of $K$ (Definition 4.3).

In order to study these properties, we first briefly review material about supersingular abelian varieties and curves in Section 2. In Section 3, we describe the connection between the set of twists of $A/K$ and the $K$-Frobenius conjugacy classes of automorphisms $g \in \text{Aut}_k(A)$ (Proposition 3.7); we give a formula for the order of the twist in terms of the field of definition and Frobenius orbit of $g$ (Lemma 3.11); and we analyze the effect of twists on the Frobenius endomorphism (Proposition 3.14).

In Section 4, we study the fully maximal, fully minimal, and mixed behavior of supersingular principally polarized abelian varieties $A/K$ in terms of arithmetic properties of their normalized Weil numbers. These are roots of unity; the key ingredient for our analysis is the 2-divisibility of their orders, encoded in a multiset $e(A/K)$ (Definition 4.4). The first main result in Section 4 is Corollary 4.8, which gives a criterion for the mixed case in terms of the orders of the twists and $e(A/K)$. As an application, we show that $A$ is not fully minimal if $A$ is simple and $r$ is even (Proposition 4.10). In the second main result of Section 4, we give a complete characterization of the fully maximal, fully minimal, and mixed types under the hypothesis that $|\text{Aut}_k(A)| = 2$ (Corollary 4.11). We note that it is not known in general whether $|\text{Aut}_k(A)| = 2$ for a generic supersingular abelian variety $A$ over a finite field (Remark 4.13).

In Section 5, we define the fully maximal, fully minimal, and mixed properties for a supersingular curve $X$ over a finite field, and study these properties under constraints on $\text{Aut}_k(X)$.

In the rest of the paper, we analyze these types when the dimension $g$ of $A$ is small. For $g = 1$, in Section 6, we prove that a supersingular elliptic curve defined over a finite field of characteristic $p$ is fully maximal if it is defined over $\mathbb{F}_p$ and mixed otherwise (Theorem 6.2).

For $g = 2$, in Section 7, we build on a significant body of work (including [22], [12], [3], [4], and [14]) to give a complete analysis of these types for supersingular abelian surfaces. For example, if $A$ is a simple supersingular abelian surface with minimal field of definition $\mathbb{F}_{p^r}$ and $\text{Aut}_k(A) \simeq \mathbb{Z}/2\mathbb{Z}$, then $A$ is not mixed if $r$ is odd and $A$ is not fully minimal if $r$ is even (Proposition 7.2). We prove that the condition $\text{Aut}_k(A) \simeq \mathbb{Z}/2\mathbb{Z}$ holds generically for a supersingular abelian surface (Proposition 7.5), using results from [1], [14], and [17]. In Propositions 7.3 and 7.4, we determine the type for each genus 2 supersingular curve $X$ with $\text{Aut}_k(X) \not\simeq \mathbb{Z}/2\mathbb{Z}$.

In Section 8, we study the moduli space $\mathcal{M}_{3,ss} \otimes \mathbb{F}_2$ of supersingular curves of genus 3 in characteristic 2. This moduli space was parametrized in [41] by the 2-dimensional family

$$X_{a,b} : x + y + a(x^3y + xy^3) + bx^2y^2 = 0.$$  \hspace{1cm} (1)

Given $a, b \in \mathbb{F}_2$, we determine the $L$-polynomial $L(X_{a,b}, T)$ in Proposition 8.14 and use this to determine exactly when $X_{a,b}$ is fully maximal, fully minimal, or mixed (Theorem 8.1).

2. Background: L-polynomials and supersingular abelian varieties

Let $k = \mathbb{F}_p$ and let $K = \mathbb{F}_q$ be a finite field of cardinality $q = p^r$. Let $A$ be an abelian variety defined over $k$. We write $K$ instead of $\text{Spec}(K)$ when this causes no imprecision.

2.1. Field of definition.

Definition 2.1. A finite field $L \subset k$ is a field of definition of $A$ and $A$ is defined over $L$ if there exists an abelian variety $A_0$ over $L$ such that $A_0 \times_L k \simeq A$. In this case, we write $A/L$. 
A field of definition for a (smooth projective connected) curve $X/k$ is defined analogously. Throughout this paper, we assume $A$ is defined over some finite field $K$ of characteristic $p$.

2.2. Frobenius.

**Definition 2.2.** An abelian variety $A/K$ is a projective group scheme. If $R$ is a $K$-algebra and $U = \text{Spec}(R)$, then the map which sends $x \mapsto x^q$ for all $x \in R$ induces a Frobenius map $f_U$ on $U$. The absolute Frobenius endomorphism $f_A: A \times_K k \to A \times_K k$ of an abelian variety $A/K$ is defined to be the glueing of these Frobenius endomorphisms $f_U$ over all open affine subschemes of $U$ of $A$.

The relative Frobenius endomorphism $\pi = \pi_A: A \times_K k \to A \times_K k$ of $A$ is defined as the factorization of $f_A$ over the fiber product of $A \to \text{Spec}(k) \leftarrow \text{Spec}(k)$, where the second map is the absolute Frobenius $f_{\text{Spec}(k)}$ [28, 21.2].

Let $Fr_K$ denote a fixed topological generator of the absolute Galois group $G_K = \text{Gal}(k/K)$ of $K$. The endomorphism $Fr_K$ is also called the topological Frobenius endomorphism of $K$.

These three Frobenius maps are related via

$$\pi_A = f_A \otimes Fr_K^{-1}. \tag{2}$$

2.3. Weil numbers and zeta functions.

Let $A$ be an abelian variety of dimension $g$ defined over $\mathbb{F}_q$. The characteristic polynomial $P(A/\mathbb{F}_q, T)$ of the relative Frobenius endomorphism $\pi_A$ of $A$ is a monic polynomial in $\mathbb{Z}[T]$ of degree $2g$. Writing $P(A/\mathbb{F}_q, T) = \prod_{i=1}^{2g} (T - \alpha_i)$, the roots $\alpha_i \in \overline{\mathbb{Q}}$ all satisfy $|\alpha_i| = \sqrt{q}$.

**Definition 2.3.** The roots $\{\alpha_1, \ldots, \alpha_{2g}\} = \{\alpha_1, \bar{\alpha}_1, \ldots, \alpha_g, \bar{\alpha}_g\}$ of $P(A/\mathbb{F}_q, T)$ are the Weil numbers of $A$. The normalized Weil numbers of $A/\mathbb{F}_q$ are $\{z_1, \bar{z}_1, \ldots, z_g, \bar{z}_g\}$, where $z_i = \frac{\alpha_i}{\sqrt{q}}$.

**Theorem 2.4.** [25] Chapter II, Theorem 1.1,

1. The number of $\mathbb{F}_q$-points of $A$ is

$$|A(\mathbb{F}_q)| = \deg(\pi_A/\mathbb{F}_q - \text{id}) = P(A/\mathbb{F}_q, 1) = \prod_{i=1}^{2g} (1 - \alpha_i). \tag{3}$$

2. Also,

$$||A(\mathbb{F}_q)| - q^g| \leq 2gq^{\left(g - \frac{1}{2}\right)} + (2^{2g} - 2g - 1)q^{(g-1)}.$$ 

Note that $|A(\mathbb{F}_q)|$ has order of magnitude $q^g$.

**Theorem 2.5.** [25] Chapter II, Section 1, [6] Theorem 1.6, [44] §IX, 71] The zeta function of $A$ over $\mathbb{F}_q$ is

$$Z(A/\mathbb{F}_q, T) = \exp \left( \sum_{m \geq 1} |A(\mathbb{F}_q^m)| \frac{T^m}{m} \right) = \frac{P_1(T) \cdots P_{2g-1}(T)}{P_{2g}(T) \cdots P_{2g-2}(T) P_0(T)},$$

where $P_s(T) = \prod_{i=1}^{2g} (1 - \alpha_{i,s} T)$ for $\alpha_{i,s} = \alpha_1 \alpha_{i_2} \cdots \alpha_{i_s}$, $0 < i_1 < \ldots < i_s \leq 2g$.

Note that $P(A/\mathbb{F}_q, T) = T^{2g} P_1(T) \left(\frac{1}{T}\right)$. The polynomials $P_i$ describe the action of Frobenius on the $i$-th cohomology of $A$. By [37] Theorem 1, two abelian varieties $A_1$ and $A_2$ over $\mathbb{F}_q$ have the same zeta function if and only if $P(A_1/\mathbb{F}_q, T) = P(A_2/\mathbb{F}_q, T)$, which holds if and only if $A_1$ and $A_2$ are isogenous over $\mathbb{F}_q$. 

2.4. Supersingular abelian varieties.

Definition 2.6. If the Newton polygon of $P(A/F_q, T)$ is a line segment of slope $\frac{1}{2}$, then $A$ is supersingular.

There are many equivalent formulations of the supersingular property:

Theorem 2.7. Suppose that $A/F_q$ is an abelian variety of dimension $g$. The following properties are each equivalent to $A$ being supersingular:

1. $A$ is geometrically isogenous to a product of supersingular elliptic curves, i.e.,
   
   $A \times_{F_q} k \sim E^g \times k$ for an elliptic curve $E$ satisfying $E[p](k) = \{0\}$, [27, Theorem 4.2];

2. the formal group of $A$ is geometrically isogenous to $(G_1)_g$, [21, Section 1.4];

3. the only slope of the $p$-divisible group $A[p^{\infty}]$ is $\frac{1}{2}$;

4. the normalized Weil numbers are roots of unity, [23, Theorem 4.1].

Definition 2.8. The abelian variety $A/F_q$ is maximal (resp. minimal) if its normalized Weil numbers all equal $-1$ (resp. 1). Theorem 2.5 then implies that $q$ is a square (i.e., $r$ is even).

2.5. Supersingular curves.

Let $X$ be a smooth projective connected curve of genus $g$ defined over $F_q$. The curve $X$ is supersingular if its Jacobian $\text{Jac}(X)$ is supersingular.

Theorem 2.9. [43, §IV, 22], [44, §IX, 69] The zeta function of $X/F_q$ can be written as

$$Z(X/F_q, T) = \frac{L(X/F_q, T)}{(1-T)(1-qT)}$$

where the $L$-polynomial $L(X/F_q, T) \in \mathbb{Z}[T]$ of $X/F_q$ has degree $2g$ and factors as

$$L(X/F_q, T) = \prod_{i=1}^{2g} (1-\alpha_i T).$$

Then $P(\text{Jac}(X)/F_q, T) = T^{2g} L(X/F_q, T^{-1})$ is the characteristic polynomial of the relative Frobenius endomorphism of $\text{Jac}(X)$. The roots $\{\alpha_1, \alpha_1, \ldots, \alpha_g, \alpha_g\}$ of $P(X/F_q, T)$ are the Weil numbers of $X$. The normalized Weil numbers of $X/F_q$ are $\{z_1, \bar{z}_1, \ldots z_g, \bar{z}_g\}$, where $z_i = \frac{\alpha_i}{\sqrt{q}}$. (Note that $|z_i| = 1$.)

Corollary 2.10 (Hasse-Weil Bound). The number of $F_q$-points of $X$ satisfies

$$|X(F_q) - (q + 1)| \leq 2g\sqrt{q}.$$  (4)

Definition 2.11. Let $X/F_q$ be a curve of genus $g$.

1. The curve $X/F_q$ is maximal if $|X(F_q)| = q + 1 + 2g\sqrt{q}$. Equivalently, $L(X, F_q, T) = (1 + \sqrt{q}T)^{2g}$, or the normalized Weil numbers are all $-1$.

2. The curve $X/F_q$ is minimal if $|X(F_q)| = q + 1 - 2g\sqrt{q}$. Equivalently, $L(X, F_q, T) = (1 - \sqrt{q}T)^{2g}$, or its normalized Weil numbers are all $1$.

Note that if $X/F_q$ is maximal or minimal, then $q$ is a square ($r$ is even).
2.6. Basic properties.

The following facts are well-known and hold for curves as well as for abelian varieties, cf. [41, Theorem 1.9] and [35, Theorem V.1.15(f)].

Lemma 2.12. If $P(A/\mathbb{F}_q, T) = \prod_{i=1}^{q^g} (T - \alpha_i)$, then $P(A/\mathbb{F}_{q^m}, T) = \prod_{i=1}^{q^g} (T - \alpha_i^m)$.

Lemma 2.13. If $A/\mathbb{F}_q$ is minimal or maximal, then it is supersingular. Conversely, if $A/\mathbb{F}_q$ is supersingular, then it is minimal over some finite extension $\mathbb{F}_{q^m}$.

Lemma 2.14.

(1) If $A/\mathbb{F}_q$ is maximal, then $A/\mathbb{F}_{q^m}$ is maximal for odd $m$ and minimal for even $m$.

(2) If $A/\mathbb{F}_q$ is minimal, then $A/\mathbb{F}_{q^m}$ is minimal for all $m$.

3. Twists

Throughout this section, let $K = \mathbb{F}_q$ with $q = p^r$ and let $k = \overline{\mathbb{F}}_p$. For any $m \in \mathbb{N}$, let $K_m$ be the unique extension of $K$ of degree $m$. Write $Fr_K$ (Frobenius) for the topological generator of $G_K = \text{Gal}(k/K)$. As in the previous section, we write that objects are defined over $K$ rather than $\text{Spec}(K)$ if this does not cause ambiguity.

Let $A/K$ be a principally polarized abelian variety of dimension $g$. We restrict to automorphisms of $A$ that are compatible with the principal polarization $\lambda$. For ease of notation, we write $A$ instead of $(A, \lambda)$ and $\text{Aut}_k(A)$ instead of $\text{Aut}_k((A, \lambda))$.

3.1. Twists, cocycles, and Frobenius conjugacy classes.

In this section, we review the theory of twists of abelian varieties following [32] and [5].

Definition 3.1. A $(K)$-twist of $A/K$ is an abelian variety $A'/K$ for which there exists a geometric isomorphism

$$\phi: A \times_K k \xrightarrow{\sim} A' \times_K k. \quad (5)$$

A twist $A'/K$ is trivial if $A \cong_K A'$.

Definition 3.2. Let $\Theta(A/K)$ denote the set of $K$-isomorphism classes of twists $A'/K$ of $A/K$. For a field extension $K'/K$, let $\Theta(A, K'/K) \subset \Theta(A/K)$ denote the set of twists $A'/K$ of $A/K$ such that $A \times_K K' \cong A' \times_K K'$.

Write $\tilde{A} = A \times_K k$ and $\tilde{A}' = A' \times_K k$.

Definition 3.3. Given $\sigma \in G_K$ and $\phi: \tilde{A} \xrightarrow{\sim} \tilde{A}'$, let $\sigma \phi: \tilde{A} \xrightarrow{\sim} \tilde{A}'$ denote the (twisted) isomorphism which acts on $x \in \tilde{A}(k)$ via\footnote{More precisely, suppose that $\tilde{A}$ has structure morphism $\pi: \tilde{A} \to \text{Spec}(k)$. For $\sigma \in G_K$, define $(\sigma \tilde{A})$ to be $\tilde{A}$ with structure morphism $\text{Spec}(\sigma)^{-1} \circ \pi$. Then $\tilde{A}$ and $(\sigma \tilde{A})$ are isomorphic as schemes through the isomorphism $\phi_{\sigma} := \text{id}_{\tilde{A}} \times_{\text{Spec}(k)} \text{Spec}(\sigma)$, but a priori only isomorphic as $k$-varieties if the isomorphism $\text{Spec}(\sigma)$ lifts to an isomorphism $\tilde{A} \xrightarrow{\sim} \tilde{A}$. (The latter holds if $K$ is a field of moduli for $A$. This is true, since $K$ is a field of definition for $A$, and since $K$ is a finite field, which has cohomological dimension $\leq 1$.)}

$$\sigma \phi(x) = \sigma(\phi(\sigma^{-1}(x))). \quad (6)$$
Similarly, if \( A' = A \) and \( \tau \in \text{Aut}_k(A) \), let \( F_{\tau K} \) denote the (twisted) automorphism, which acts on \( x \in \bar{A}(k) \) by
\[
F_{\tau K}(x) = F_{\tau K}(\tau(F_{\tau K}^{-1}(x))).
\]

**Definition 3.4.** Two automorphisms \( g, h \in \text{Aut}_k(A) \) are \( K \)-Frobenius conjugate if there exists \( \tau \in \text{Aut}_k(A) \) such that
\[
g = \tau^{-1}h(F_{\tau K}).
\]
In particular, \( g \) is \( K \)-Frobenius conjugate to the identity if and only if \( g = \tau^{-1}(F_{\tau K}) \) for some \( \tau \in \text{Aut}_k(A) \).

**Remark 3.5.** If all automorphisms of \( A \) are defined over \( K \), then \( G_K \) acts trivially on \( \text{Aut}_k(A) \) and the \( K \)-Frobenius conjugacy classes are the same as standard conjugacy classes. By [37] Theorem 2(d)], if \( A \) is maximal or minimal over \( K \), then its automorphisms are indeed all defined over \( K \).

If \( H \subset \text{Aut}_k(A) \) is stabilized by both conjugation and \( F_{\tau K} \)-conjugation, then the number of \( K \)-Frobenius conjugacy classes in \( H \) is bounded above by the number of conjugacy classes in \( H \) by [24] Proposition 8.

**Definition 3.6.** Let \( C^1(G_K, \text{Aut}_k(A)) \) be the set of cocycles, i.e., maps \( \xi : G_K \to \text{Aut}_k(A) \) such that \( \xi(\sigma_1 \sigma_2) = \xi(\sigma_1)^{\sigma_2} \xi(\sigma_2) \). Two cocycles \( \xi_1 \) and \( \xi_2 \) are equivalent in \( H^1(G_K, \text{Aut}_k(A)) \) if and only if there is an automorphism \( h \in \text{Aut}_k(A) \) such that \( \xi_2(\sigma) = h^{-1} \circ \xi_1(\sigma) \circ \sigma \circ h \).

**Proposition 3.7.** ([32] Proposition III.5], [24] Proposition 9]) There are bijections:
\[
\Theta(A/K) \xrightarrow{\xi} H^1(G_K, \text{Aut}_k(A)) \xrightarrow{\Delta} \{ \text{\( K \)-Frobenius conjugacy classes of \( \text{Aut}_k(A) \)} \}. \tag{7}
\]

**Proof.** We sketch the proof in order to introduce key notation. Suppose that \( A'/K \) is a \( K \)-twist of \( A/K \). Then there exists a geometric isomorphism \( \phi \) as in (5). Consider the cocycle \( \xi_\phi : G_K \to \text{Aut}_k(A) \) defined by
\[
\xi_\phi(\sigma) = \phi^{-1} \circ \sigma \circ \phi.
\tag{8}
\]
If \( \phi' : \bar{A} \xrightarrow{\sim} \bar{A}' \) is another geometric isomorphism, then \( \tau = \phi^{-1} \circ \phi' \in \text{Aut}_k(A) \), so \( \phi' = \phi \circ \tau \). Then \( \xi_{\phi'}(F_{\tau K}) = \tau^{-1} \circ \xi_\phi \circ (F_{\tau K}) \), so \( \xi_\phi \) and \( \xi_{\phi'} \) are equivalent in \( H^1(G_K, \text{Aut}_k(A)) \). This yields a bijection \( \Xi : \Theta(A/K) \to H^1(G_K, \text{Aut}_k(A)) \). In fact, Serre proves that \( \Xi \) is induced from the following bijections, which exist for any finite Galois extension \( K'/K \):
\[
\Theta(A, K'/K) \to H^1(\text{Gal}(K'/K), \text{Aut}_{K'}(A)). \tag{9}
\]

Next, for any \( \xi \in C^1(G_K, \text{Aut}_k(A)) \), let
\[
g_\xi = \xi(F_{\tau K}) \in \text{Aut}_k(A). \tag{10}
\]
Conversely, for any \( g \in \text{Aut}_k(A) \), the map \( F_{\tau K} \to g \) extends uniquely to a well-defined cocycle \( \xi_g \) in \( C^1(G_K, \text{Aut}_k(A)) \) (cf. [24] Proposition 9], based on [31] Chapter XIII, 1]). Since \( \xi_g \) is determined by its value at \( F_{\tau K} \), the map \( \xi \mapsto g_\xi \) yields a bijection \( C^1(G_K, \text{Aut}_k(A)) \to \text{Aut}_k(A) \). This bijection descends to a bijection \( \Delta \) in (7) because, by Definition 3.4 and Remark 3.6, two cocycles \( \xi_1 \) and \( \xi_2 \) are equivalent in \( H^1(G_K, \text{Aut}_k(A)) \) if and only if \( \xi_1(F_{\tau K}) \) and \( \xi_2(F_{\tau K}) \) are \( K \)-Frobenius conjugate.

**Remark 3.8.** Given \( g \in \text{Aut}_k(A) \), the procedure for obtaining a twist \( A'/K \) is outlined here. Let \( K = K(A) \) be the function field of \( A \) and let \( \bar{K} = k(A) \). Then \( g \) can be viewed as a map \( g : \bar{A} \to \bar{A} \) of \( k \)-varieties or as a map \( g : \bar{K} \to \bar{K} \) of function fields; but, a priori, \( g \) is not a map of \( K \)-varieties.
Let $\sigma = Fr_K$. Define $\tilde{g} = g \circ (1 \otimes \sigma)$, a “twist of $g$ by $\sigma$”. This is an automorphism of $K$ such that $\tilde{g}|_k = \sigma$ and $\tilde{g}|_{\tilde{A}} = g$. (Note that $\tilde{g}$ is a morphism of $K$-varieties, but not of $k$-varieties.) Then $\tilde{g}$ yields a homomorphism $G_K \to \operatorname{Aut}(\tilde{A}/k)$ (i.e., as morphisms of $k$-varieties). This implies that $\sigma^T g = g$. Moreover,

$$\tilde{g}^T = g \circ (1 \otimes \sigma) \circ \cdots \circ g \circ (1 \otimes \sigma) = g(\sigma^2 g) \cdots (\sigma^{T-1} g) \circ \sigma^T = \sigma^T.$$

So $\tilde{g}^T$ is trivial as a morphism of varieties, since it acts nontrivially only on the constant field. Thus $A$ is isomorphic to $A'$ over $K_T$ (which is fixed by $\tilde{g}^T = \sigma^T$), since $(\tilde{K})^\tilde{g} = (K_T(A))^\tilde{g}$.

The value $T$ (which determines the field of definition of $\phi$) may not be the same for another automorphism $h = \tau^{-1} g(\sigma \tau)$ in the $K$-Frobenius conjugacy class of $g$. In fact, $h$ defines the same twist $A'$, since $(\tilde{K})^h \simeq (\tilde{K})^\tilde{g}$, but the geometric isomorphisms realizing the twist differ by $\tau$. Explicitly,

$$\tilde{h} = h \circ (1 \otimes \sigma) = \tau^{-1} \circ g \circ (1 \otimes \sigma) \circ \tau \circ (1 \otimes \sigma)^{-1} \circ (1 \otimes \sigma) = \tau^{-1} \tilde{g} \tau.$$

Thus $\tilde{h}^T = \tau^{-1} \tilde{g}^T \tau = \tau^{-1} \sigma^T \tau$, which is only trivial as a morphism of varieties if $\tau$ is defined over $K_T$.

**Notation 3.9.** Given $\phi : \tilde{A} \tilde{\to} A'$, let $\xi_\phi$ denote the cocycle defined in (8) and $g_\phi = g_{\xi_\phi}$ the automorphism defined in (11). Given $g \in \operatorname{Aut}_k(A)$, let $\xi_g$ be the cocycle such that $Fr_K(\xi_g) = g$ and let $\phi_g : \tilde{A} \tilde{\to} A'$ be an isomorphism such that $\xi_{\phi_g} = \xi_g$, as in Remark 3.8.

We remark that $\phi_g$ is not uniquely determined: for any $\tau \in \operatorname{Aut}_k(A)$ such that $\tau^{-1} g Fr \tau = g$, the isomorphism $\phi' = \phi_g \circ \tau : \tilde{A} \tilde{\to} A'$ also has the property that $g_{\phi'} = g$. In this case, $\tau$ must be defined over $K$, so $\phi \circ \tau$ and $\phi$ have the same field of definition.

### 3.2. Field of definition of twists.

Let $A'/K$ be a $K$-twist of $A/K$. Consider a geometric isomorphism $\phi : \tilde{A} \tilde{\to} A'$ and let $\xi_\phi$ and $g := g_{\phi}$ be the corresponding cocycle and automorphism, respectively. Let $K_{T_g}$ be the field of definition of $\phi$ (of degree $T_g$ over $K$).

**Lemma 3.10.** The degree $T_g$ is the smallest positive integer $T$ such that

$$g^{(Fr_K g)}(Fr_K^T g) \cdots (Fr_K^{T-1} g) = \operatorname{id}.$$

**Proof.** Form the base change $A \times_K K'$ and $A' \times_K K'$ to $K'$ and consider the $K$-twist as a $K'$-twist. The equivalent map on cocoys is

$$C^1(G_K, \operatorname{Aut}_k(A)) \to C^1(G_{K'}, \operatorname{Aut}_k(A)), \xi \mapsto \xi|_{G_{K'}}.$$

Thus, $\xi|_{G_{K'}}$ is trivial if and only if $\operatorname{id} = \xi|_{G_{K'}}(Fr_{K'}) = \xi(Fr_{K'}^T)$. Since $\xi(Fr_K) = g$, the result follows from the fact that

$$\xi(Fr_{K'}^T) = g^{(Fr_K g)}(Fr_{K'}^T g) \cdots (Fr_{K'}^{T-1} g). \quad (11)$$

□

**Lemma 3.11.** Let $c_g$ be the smallest positive integer $c$ such that the following expression is defined over $K_c$:

$$g^{(Fr_K g)}(Fr_K^c g) \cdots (Fr_K^{c-1} g).$$

Then $c_g$ divides $T_g$ and $T_g/c_g$ equals the order of $G := g^{(Fr_K g)}(Fr_K^c g) \cdots (Fr_K^{c-1} g)$. 
Proof. First consider the case that \( g \) is defined over \( K \), which is equivalent to \( c_g = 1 \) and \( G = g \). Then \( Fr_K \) acts trivially on \( g \). Let \( \kappa \) be the order of \( g \) (and \( G \)). Replacing \( T \) by \( \kappa \) in (11), it follows that \( \xi(Fr_K^e) = g^e = id \), so the twist is trivialized over the degree \( \kappa \) extension of \( K \). On the other hand, \( \xi(Fr_K^t) = g^t \neq id \) for any \( 1 \leq t < \kappa \). Thus, \( K_\kappa \) is the minimal field of definition of \( \phi \). So \( T_g = T_g/c_g = \kappa \).

Now suppose that \( c_g > 1 \). By assumption, the twist is an element \( A' \in \Theta(A, K_{T_g}/K) \) in the notation of Definition 3.2. The bijection

\[
\theta: \Theta(A, K_{T_g}/K) \to H^1(Gal(K_{T_g}/K), \text{Aut}_{K_{T_g}}(A))
\]
defined in the proof of Proposition 3.7 shows that \( A' \) induces and is induced from an automorphism in \( \text{Aut}_{K_{T_g}}(A) \). Since this automorphism is \( \xi_g(Fr_K) = g \) by assumption, it follows that \( g \) is defined over \( K_{T_g} \), so the same is true for \( G \). Hence, \( K_{c_g} \subset K_{T_g} \) and \( c_g | T_g \).

Consider the base changes \( A_{c_g} = A \times_K K_{c_g} \) and \( A'_g = A' \times_K K_{c_g} \) to \( K_{c_g} \). These first become isomorphic over \( K_{T_g} \cdot K_{c_g} = K_{T_g} \); the corresponding isomorphism is thus defined over an extension of \( K_{c_g} \) of degree \( T' = [K_{T_g} : K_{c_g}] = T_g/c_g \). The automorphism corresponding to the twist over \( K_{c_g} \) is \( G \) by the proof of Lemma 3.10. Hence, replacing \( g \) by \( G \), the conclusion follows from the case when \( c_g = 1 \). \( \square \)

Definition 3.12. The order of a twist \( A'/K \) is the degree of the minimal field extension \( K' \) of \( K \) over which there exists an isomorphism \( \phi: A \times_K K' \cong A' \times_K K' \).

If \( A'/K \) is a twist of order \( m \) and \( \psi: A \times_K k \cong A' \times_K k \) is an isomorphism, Definition 3.12 implies that \( \psi \circ \tau \) is defined over the degree \( m \) extension \( K_m \) of \( K \) for some \( \tau \in \text{Aut}_K(A) \).

Lemma 3.13. For a given isomorphism \( \phi: A \cong A' \), suppose that \( g_\phi \in \text{Aut}_K(A) \) has order 2. Then the twist \( A'/K \) is either quadratic or trivial. Moreover, it is trivial if and only if \( g_\phi \) is \( K \)-Frobenius conjugate to the identity.

Proof. Write \( g = g_\phi \). By hypothesis, \( c_g = 1 \), so by Lemma 3.11 \( T_g = |g| = 2 \). Hence, the order of the twist is at most 2. \( \square \)

The conclusion of Lemma 3.13 can be false if \( g_\phi \) is not defined over \( K \). For example, in Section 8 the automorphism \( \upsilon \) of \( X_{a,b} \) has order 2 but \( T_g = 4 \).

3.3. Effect of a twist on the Frobenius endomorphism.

Suppose that \( A \) is defined over \( K_c \) and \( G \in \text{Aut}_{K_c}(A) \). In this section, we study how twisting \( A/K_c \) by \( G \) affects the relative Frobenius endomorphism \( \pi = \pi_A \in \text{End}_{K_c}(A) \) of \( A \) and the normalized Weil numbers of \( A \) over \( K_c \).

By a foundational result of Tate [37, page 135], for any \( \ell \neq p \), there is a bijection

\[
\text{End}_{K_c}(A) \otimes \mathbb{Q}_\ell \to \text{End}_{G_{K_c}}(T_\ell(A) \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell)
\]

(12)

where \( T_\ell(A) \) denotes the \( \ell \)-adic Tate module of \( A \). Via this bijection, \( \pi_A \) can be viewed as a linear operator on the vector space \( T_\ell(A) \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell \). (Since \( \pi_A \) is semisimple (cf. [37, p. 138]), this linear operator is diagonalizable over \( \mathbb{Q}_\ell \).) Moreover, the characteristic polynomial \( P(A/K_c, T) \) of \( \pi_A \) (in the sense of [19, p. 110]) coincides with that of its corresponding linear operator, by e.g., [19, Chapter VII, Theorem 3].
Proposition 3.14. Suppose that \( \phi : A \times_{K_c} k \isom A' \times_{K_c} k \) is a geometric isomorphism. Suppose that \( G_\phi = \xi_\phi(Fr_{K_c}) \) is in Aut\(_{K_c}(A) \). Then the relative Frobenius endomorphism \( \pi' \) of \( A' \) satisfies
\[
\phi^{-1} \circ \pi' \circ \phi = \pi_A \circ G_\phi^{-1}.
\]

Remark 3.15. The right hand side of (13) is defined over \( K_c \), so is the left hand side is as well. Thus, over any finite extension \( L \) of \( K_c \), the endomorphisms \( \pi' \) and \( \pi_A \circ G_\phi^{-1} \) have the same characteristic polynomial.

Proof. We adapt the proof of [24, Proposition 11]. Let \( f' = f_{A'} \) be the absolute Frobenius endomorphism of \( A' \). By Equation (2),
\[
\pi_A = f_A \otimes Fr_{K}^{-1} \quad \text{and} \quad \pi' = f_{A'} \otimes Fr_{K}^{-1}.
\]
In addition, \( f \) has the property that \( f = \phi^{-1} \circ f' \circ \phi \). Furthermore, by Equation (8),
\[
G_\phi^{-1} = (\text{id}_{A} \otimes Fr_{K}) \circ \phi^{-1} \circ (\text{id}_{A} \otimes Fr_{K}^{-1}) \circ \phi.
\]
Hence,
\[
\phi^{-1} \circ \pi' \circ \phi = \phi^{-1} \circ (f' \otimes Fr_{K}^{-1}) \circ \phi = \phi^{-1} \circ ((\phi \circ f \circ \phi^{-1}) \otimes Fr_{K}^{-1}) \circ \phi = (f \otimes Fr_{K}^{-1}) \circ (\text{id}_{A} \otimes Fr_{K}) \circ \phi^{-1} \circ (\text{id}_{A} \otimes Fr_{K}^{-1}) \circ \phi = \pi_A \circ G_\phi^{-1}.
\]
\((\Box)\)

3.4. The twist by \([-1]\).

Definition 3.16. Let \( \iota \in \text{End}_{K}(A) \otimes \mathbb{Q}_\ell \) correspond to \(-1\) in \( \text{End}_{G_K}(T_\ell(A) \otimes \mathbb{Z}_\ell \otimes \mathbb{Q}_\ell) \) under the bijection in (12). If \( p \neq 2 \), then \( \iota \) is the multiplication-by-(\(-1\)) map on \( A \). Thus, \( \iota \) is defined over \( K \) and central in Aut\(_K(A) \). Let \( A_\iota \) denote the \( K \)-twist of \( A \) corresponding to \( \iota \).
By Lemma \[3.13\] \( A_\iota/K \) is either a trivial or a quadratic twist.

Remark 3.17. The twist \( A_\iota/K \) is trivial if and only if \( \iota \) is Frobenius conjugate to the identity, in which case, \( A \) is self-dual. If so, write \( \iota = \tau^{-1}Fr_{K} \tau \) for some \( \tau \in \text{Aut}_k(A) \). Then \( Fr_{K} \tau = \iota \tau \) and \( Fr_{\overline{K}} \tau = \tau \). So \( \tau \) is defined over \( K_2 \), but not \( K \), and thus defines a twist of even order by Lemma \[3.11\].

Proposition 3.18. Suppose that \( \phi : A \times_{K} k \isom A' \times_{K} k \) where \( A/K \) is maximal and \( A'/K \) is minimal (or vice versa). Then \( g_\phi = \iota \) and \( A'/K \) is a quadratic twist.

Proof. By Definition \[2.8\] since \( A/K \) is maximal, \( \pi_A = \sqrt{q} \cdot \iota \) in \( \text{End}_{K}(A) \otimes \mathbb{Q}_\ell \), because the linear operator corresponding to \( \pi_A \) under (12) is diagonalizable over \( \mathbb{Q}_\ell \). Similarly, since \( A'/K \) is minimal, \( \pi_{A'} = \sqrt{q} \cdot \iota \) in \( \text{End}_{K}(A) \otimes \mathbb{Q}_\ell \). By Proposition \[3.14\] this implies that \( g_\phi = \xi_\phi(Fr_{K}) \) is \( K \)-Frobenius conjugate to \( \iota \). So \( g_\phi = \tau^{-1}Fr_{K} \tau \) for some \( \tau \in \text{Aut}_k(A) \).

Since \( A/K \) is maximal, \( \text{Aut}_k(A) = \text{Aut}_K(A) \). In particular, \( Fr_{K} \tau = \tau \). Because \( \iota \) is central in \( \text{Aut}_k(A) \), the \( K \)-Frobenius conjugacy class of \( \iota \) consists of one element, so \( g_\phi = \iota \). This proves the first statement.
Moreover, \( \iota \) satisfies the conditions of Lemma \[3.13\] Since \( A \) and \( A' \) are not isomorphic over \( K \), the twist \( A'/K \) is nontrivial and thus quadratic. \((\Box)\)
4. The period and the parity

As before, let $K = \mathbb{F}_q$ with $q = p^r$ and let $k = \mathbb{F}_p$. Let $A$ be a principally polarized supersingular abelian variety of dimension $g$ defined over $K$. (Throughout, one could replace $A$ by a smooth projective connected supersingular curve $X$ of genus $g$.)

4.1. Period, parity, and types.

Definition 4.1.

(1) The $\mathbb{F}_q$-period $\mu(A)$ of $A$ is the smallest natural number $m$ such that $q^m$ is square (if $m$ is even) and

(i) $z_i^m = -1$ for all $1 \leq i \leq g$, or

(ii) $z_i^m = 1$ for all $1 \leq i \leq g$.

(2) The $\mathbb{F}_q$-parity $\delta(A)$ is 1 in case (i) and is $-1$ in case (ii).

The following is an equivalent definition.

Definition 4.2. \cite[page 144]{36} Write $P(A/\mathbb{F}_q, T) = \prod f_i^{d_i}$ with $f_i$ pairwise relatively prime.

(1) The $\mathbb{F}_q$-period of $A/\mathbb{F}_q$ is

$$\mu(A) = \min \{ n \in \mathbb{N} | q^{n/2} \in \mathbb{Z} | \prod f_i \text{ divides } (i) (T^n + q^{n/2}) \text{ or } (ii) (T^n - q^{n/2}) \}.$$  

(2) The $\mathbb{F}_q$-parity of $A/\mathbb{F}_q$ is $\delta(A) = 1$ in case (i) and $\delta(A) = -1$ in case (ii).

By definition, $A$ is maximal (resp. minimal) over $\mathbb{F}_q$ if and only if $\mu(A) = 1$ and $\delta(A) = 1$ (resp. $\delta(A) = -1$).

Let $\Theta(A/K)$ be the set of $K$-isomorphism classes of twists $A'/K$ of $A$, see Definition 3.1.

Definition 4.3. A supersingular abelian variety $A$ with minimal field of definition $K$ is of one of the following types:

(1) fully maximal if $A'/K$ has $K$-parity $\delta = 1$ for all $A' \in \Theta(A/K)$;

(2) fully minimal if $A'/K$ has $K$-parity $\delta = -1$ for all $A' \in \Theta(A/K)$;

(3) mixed if there exist $A', A'' \in \Theta(A/K)$ with $K$-parities $\delta(A') = 1$ and $\delta(A'') = -1$.

4.2. Initial information from Weil numbers.

Suppose that $\{ z_1, z_1, \ldots, z_g, \bar{z}_g \}$ are the normalized Weil numbers of a supersingular abelian variety $A/K$. Recall that $z_1, \ldots, z_g$ are roots of unity. We measure the 2-divisibility of their orders in the next definition.

Definition 4.4. Let $e_i = \ord_2(|z_i|)$. The 2-valuation vector of $A/K$ is the multiset $\varpi = \varpi(A/K) := \{ e_1, \ldots, e_g \}$. The notation $\varpi = \{ e \}$ means that $e_i = e$ for $1 \leq i \leq g$.

In other words, $|z_i| = 2^{e_i} o_i$ with $o_i$ odd. Note that $z_i^k = -1$ for some $k \in \mathbb{N}$ if and only if $e_i \geq 1$. By definition, the $\mathbb{F}_q$-parity of $A$ satisfies $\delta(A) = 1$ if and only if $\varpi = \{ e \}$ with $e \geq 2$ when $r$ is odd and $e \geq 1$ when $r$ is even. Note that:

$$\ord_2(|z|) = 1 \text{ if and only if } \ord_2(| - z|) = 0; \text{ if } \ord_2(|z|) \geq 2, \text{ then } \ord_2(| - z|) \geq 2. \quad (15)$$

The next result compares the 2-divisibility of the period $\mu = \mu(A)$ with the values $e_i$.

Lemma 4.5. Write $\varpi = \{ e_1, \ldots, e_g \}$ and $\mu = 2^E \bar{\mu}$ where $\bar{\mu}$ is odd.

(1) If $r$ even and $\varpi = \{ 0 \}$ or $\varpi = \{ 1 \}$, then $E = 0$.

(2) If $r$ odd and $\varpi = \{ 0 \}$ or $\varpi = \{ 1 \}$, then $E = 1$. 

(3) If \( \mathbf{e} = \{e\} \) is constant with \( e \geq 2 \), then \( E = e - 1 \).
(4) If \( \mathbf{e} \) is not constant, then \( E = \max\{e_i \mid 1 \leq i \leq g\} \).

Proof. This follows from the fact that \( \mu \) is the smallest number such that \( q^{\mu/2} \in \mathbb{Z} \) and \( \{z_i^\mu\} = \{1\} \) or \( \{z_i^\mu\} = \{-1\} \). \( \square \)

4.3. The Weil numbers of twists.

Suppose that \( A/K \) is a supersingular abelian variety and \( A'/K \in \Theta(A/K) \) is a \( K \)-twist of \( A \) of order \( T \). Then there is an isomorphism \( \phi : A \times_K K_T \isom A' \times_K K_T \) defined over \( K_T \).

Denote the normalized Weil numbers of \( A/K \) by \( \{z_i, \bar{z}_i\}_{1 \leq i \leq g} \) and those of \( A'/K \) by \( \{w_i, \bar{w}_i\}_{1 \leq i \leq g} \). After possibly reordering, \( z_i^T = w_i^T \) and hence

\[
    w_i = \lambda_i z_i
\]

for some (not necessarily primitive) \( T \)-th root of unity \( \lambda_i \). Let \( t = \text{lcm}\{|\lambda_i| : 1 \leq i \leq g\} \).

By definition, \( t \mid T \). In particular, if \( A'/K \) is a trivial twist, then \( t = 1 \) and \( z_i = w_i \) for all \( i \). If \( t \neq T \), it means that \( A \) and \( A_0 \) are isogenous but not isomorphic over \( \mathbb{F}_q \). Conversely, if \( (16) \) holds, then \( A \) and \( A' \) are isogenous, but not necessarily isomorphic, over \( K_T \).

Lemma 4.6. Let \( \mathbf{e} \) be the 2-valuation vector of \( A/K \). Suppose that \( A'/K \) is a twist of \( A/K \) of order \( T \). Let \( \mathbf{e} = \text{ord}_2(T) \). If \( \mathbf{e} < \min\{e_i \mid 1 \leq i \leq g\} \), then \( \mathbf{e}(A'/K) = \mathbf{e} \).

Proof. If \( w_i = \lambda_i z_i \), then \( \text{ord}_2(|w_i|) \leq \max(\text{ord}_2(|\lambda_i|), \text{ord}_2(|z_i|)) \), with equality if the two values are not equal. The result follows because \( \text{ord}_2(|\lambda_i|) < e_i = \text{ord}_2(|z_i|) \). \( \square \)

Proposition 4.7. Suppose that \( A/K \) has \( K \)-period \( M \) and \( K \)-parity \( +1 \) and its \( K \)-twist \( A'/K \) has \( K \)-period \( N \) and \( K \)-parity \( -1 \). Let \( e_M = \text{ord}_2(M) \) and \( e_N = \text{ord}_2(N) \). If \( e_N \leq e_M \), then \( \text{ord}_2(t) = 1 + e_M \); if \( e_N > e_M \), then \( \text{ord}_2(t) = e_N \).

Proof. Write \( L = \text{lcm}(M, N) \). Recall that \( z_i^M = -1 \) and \( w_i^N = 1 \) for \( 1 \leq i \leq g \).

Suppose that \( e_N \leq e_M \). Then \( t = L/M \) is odd and \( \text{ord}_2(L) = e_M \). Then

\[
    1 = w_i^L = \lambda_i L z_i^L = \lambda_i^T (z_i^M)^{\ell_2} = \lambda_i^T (-1)^{\ell_2}.
\]

This implies that \( \lambda_i^T = -1 \) and so \( \text{ord}_2(|\lambda_i|) = 1 + e_M \) for \( 1 \leq i \leq g \).

Suppose that \( e_M < e_N \). For \( 1 \leq i \leq g \), then \( \text{ord}_2(|z_i|) = 1 + e_M \) and \( \text{ord}_2(|w_i|) \leq e_N \). The equation \( w_i = \lambda_i z_i \) implies that \( \text{ord}_2(|\lambda_i|) \leq e_N \) for \( 1 \leq i \leq g \). To show that \( \text{ord}_2(t) = e_N \), it thus suffices to show that \( \text{ord}_2(|\lambda_i|) = e_N \) for some \( i \).

When \( e_M < e_N \), \( rN/2 \) is even, because \( rM \) is even by definition of the period. So if \( r \) is odd, then \( e_N > e_M \geq 1 \). By the minimality of \( N \) (such that \( rN \) is even), it cannot hold that \( w_i^{N/2} = 1 \) for all \( i \). Thus, there is at least one value \( i_0 \) such that \( \text{ord}_2(|w_{i_0}|) = e_N \). Furthermore, since the \( K \)-parity is \( -1 \), it is not true that \( w_i^{N/2} = -1 \) for all \( i \). So there is at least one value \( i_1 \) such that \( \text{ord}_2(|w_{i_1}|) < e_N \).

Consider the equality \( z_i = \lambda_i^{-1} w_i \). If \( e_N > 1 + e_M \), then substituting \( i = i_0 \) shows that \( \text{ord}_2(|\lambda_{i_0}^{-1}|) = e_N \). If \( e_N = 1 + e_M \), then substituting \( i = i_1 \) shows \( \text{ord}_2(|\lambda_{i_1}^{-1}|) = 1 + e_M \). \( \square \)

Corollary 4.8. If \( A/K \) has \( K \)-parity \( +1 \) and its \( K \)-twist \( A'/K \) has \( K \)-parity \( -1 \), then \( t \) (and \( T \)) is even. More precisely:

1. Suppose that \( A/K \) has \( K \)-period \( M \) and \( K \)-parity \( +1 \). If \( A'/K \) is a \( K \)-twist of order \( T \) with \( \text{ord}_2(T) \leq e_M \), then \( A'/K \) also has \( K \)-parity \( +1 \).
(2) Suppose that \( A'/K \) has \( K \)-period \( N \) and \( K \)-parity \(-1 \). If \( A/K \) is a twist of order \( T \) with \( \text{ord}_2(T) < e_N \), then \( A'/K \) also has \( K \)-parity \(-1 \).

**Proof.** The first statement follows from Proposition 4.7 since \( \text{ord}_2(t) \geq 1 \) when \( e_N \leq e_M \) and when \( e_N > e_M \). For the second statement, note that \( \text{ord}_2(t) \leq \text{ord}_2(T) \). The bound on the latter is sufficient to show the twist cannot change the parity by Proposition 4.7. \( \square \)

4.4. **Relationship between types and Weil numbers.**

Let \( A \) be a supersingular principally polarized abelian variety with minimal field of definition \( K = \mathbb{F}_q \). Let \( \{z_1, z_1, \ldots, z_g, \bar{z}_g\} \) be the normalized Weil numbers of \( A/K \) and let \( e_i = \text{ord}_2(|z_i|) \).

**Lemma 4.9.** Let \( \underline{e} = \underline{e}(A/K) \).

1. If \( A \) is fully maximal, then (i) \( \underline{e} = \{e\} \) with \( e \geq 2 \);
2. If \( A \) is fully minimal, then (ii) the \( e_i \) are not all equal, or \( \underline{e} = \{e\} \) with \( e \in \{0, 1\} \) and \( r \) is odd.
3. If (iii) \( \underline{e} = \{e\} \) with \( e \in \{0, 1\} \) and \( r \) is even, then \( A \) is mixed.

**Proof.**

1. If \( A/K \) is fully maximal, then it has parity \( +1 \); so \( \underline{e} = \{e\} \) for some \( e \geq 1 \) (with \( e \geq 2 \) if \( r \) is odd). Suppose that \( r \) is even and \( \underline{e} = \{1\} \). By (15), the twist \( A_i \) has the property that \( \underline{e} = \{0\} \). In particular, the twist is nontrivial, and \( A_i \) has parity \(-1 \), which contradicts the fact that \( A/K \) is fully maximal. Thus condition (i) holds.
2. If \( A/K \) is fully minimal, then it has parity \(-1 \). Hence, either \( \underline{e} = \{0\} \), or the \( e_i \) are not all the same, or \( \underline{e} = \{1\} \) and \( r \) is odd. If \( r \) is even and \( \underline{e} = \{0\} \), then the twist by \( i \) is maximal, giving a contradiction. Thus condition (ii) holds.
3. This is the contrapositive of parts (a) and (b). \( \square \)

**Proposition 4.10.** If \( A \) is simple and \( r \) is even, then \( A/\mathbb{F}_q \) is not fully minimal.

**Proof.** If \( A/K \) is simple, the Weil numbers \( \{\sqrt{q}z_j\} \) are all conjugate over \( \mathbb{Q} \). Let \( n = |z_1| \) and \( e = v_2(n) \). Since \( r \) is even, \( \sqrt{q} \in \mathbb{Q} \), so the conjugates of \( \sqrt{q}z_1 \) are precisely the elements \( \sqrt{q}z_n^j \) for \( j \in (\mathbb{Z}/n\mathbb{Z})^* \), of which there are \( \phi(n) \). In particular, \( \underline{e} = \{e\} \). By Lemma 4.9(2), \( A \) is not fully minimal. \( \square \)

4.5. **Types of abelian varieties with small automorphism group.**

**Corollary 4.11.** Suppose that \( |\text{Aut}_k(A)| = 2 \). Then

1. \( A \) is fully maximal if and only if (i) \( \underline{e} = \{e\} \) with \( e \geq 2 \);
2. \( A \) is fully minimal if and only if (ii) the \( e_i \) are not all equal, or \( \underline{e} = \{e\} \) with \( e \in \{0, 1\} \) and \( r \) is odd;
3. \( A \) is mixed if and only if (iii) \( \underline{e} = \{e\} \) with \( e \in \{0, 1\} \) and \( r \) is even.

**Proof.** One set of implications follows from Lemma 4.9(1,2,3). Conversely, if \( |\text{Aut}_k(A)| = 2 \), then there is at most one nontrivial twist of \( A/K \), which must be \( A_i \). Thus, \( A \) is fully maximal (resp. fully minimal) if and only if \( A \) and \( A_i \) are both maximal (resp. minimal). The result follows from the fact that multiplication by \(-1 \) on \( \{z_i\} \) preserves each of the conditions (i), (ii), (iii) for \( \underline{e} \) by (15). \( \square \)
If $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \subset \text{Aut}_K(A)$, then $A$ is not simple over $K$ by [16] Theorem B. The converse implication also holds, since the automorphism group of each simple factor of $A/K$ contains a copy of $\mathbb{Z}/2\mathbb{Z}$.

**Proposition 4.12.** If $|\text{Aut}_k(A)| = 2$, $g$ is odd, and $r$ is odd, then $A$ is fully maximal.

**Proof.** If $|\text{Aut}_k(A)| = 2$, then $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \not\subset \text{Aut}_K(A)$, so $A/K$ is simple. Hence, its Weil numbers are all conjugate over $\mathbb{Q}$. This implies that the $e_i$ are all the same or equally divided between the values 0 and 1. Since $g$ is odd, the latter does not occur, so $\underline{e} = \{e\}$. If $e = 0$ (resp. $e = 1$), then there exists $c$ odd such that $z_i^c = 1$ (resp. $z_i^c = -1$) for all $i$. Since $r$ is odd, this contradicts the fact that $P(A/\mathbb{F}_{p^r}, T)$ has integer coefficients. Thus $e \geq 2$. By Corollary 4.11, $A$ is fully maximal. \qed

**Remark 4.13.** Let $S$ be an irreducible component of the supersingular locus of the moduli space of principally polarized abelian varieties of dimension $g$. Among the abelian varieties $A$ represented by $\mathbb{F}_q$-points of $S$, the typical structure of $\text{Aut}_k(A)$ is not known in general, nor is it known how this structure depends on the characteristic $p$ or the choice of component $S$. For $g = 2$, we prove in Proposition 7.2 that typically $\text{Aut}_k(A) \simeq \mathbb{Z}/2\mathbb{Z}$ when $p$ is odd, but this is false when $p = 2$ [39, Theorem 3.1]. For $g \geq 3$, one might expect that typically $\text{Aut}_k(A) \simeq \mathbb{Z}/2\mathbb{Z}$ when $p$ is odd. In Lemma 8.17 we prove that $\text{Aut}_k(A) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ on an open dense subset of $S$ when $p = 2$ and $g = 3$.

**Remark 4.14.** For $S$ as in Remark 4.13 and $K = \mathbb{F}_q$, the proportion of the $K$-points of $S$ which represent abelian varieties $A$ that are simple over $K$ is not known in general. For all $g$ and $p$, it is known that $A$ generically has $a$-number 1 [21, Section 4.9]. If $p$ is odd, one expects the proportion of $A$ with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \subset \text{Aut}_K(A)$ to be small, because this condition implies that the $a$-number of $A$ is at least two, by [7, Proposition 4].

5. **Fully maximal and minimal Jacobians**

5.1. **Types for Jacobians.**

Let $X/K$ be a smooth projective connected supersingular curve of genus $g$ and let $\text{Jac}(X)$ be its Jacobian, which is canonically principally polarized. If $X$ is hyperelliptic, its hyperelliptic involution acts as $[-1]$ on $\text{Jac}(X)$ and is thus denoted $\iota$.

The theory of twists of $X$ and definitions of the period and parity of $X$ are almost identical to those for $\text{Jac}(X)$, as studied in Sections 3 and 4. The normalized Weil numbers $\{z_i, \bar{z}_i\}_{1 \leq i \leq g}$ and the 2-adic valuations $\underline{e} = \{e_i\}_{1 \leq i \leq g}$ of the $|z_i|$ for $X$ are the same as for $\text{Jac}(X)$. The main difference is that $X$ may have fewer twists than $\text{Jac}(X)$.

By [20, Appendix], $X$ and $\text{Jac}(X)$ have the same minimal field of definition and

$$\text{Aut}_k(\text{Jac}(X)) \simeq \begin{cases} \text{Aut}_k(X) \text{ if } X \text{ is hyperelliptic,} \\ \langle \iota \rangle \times \text{Aut}_k(X) \text{ if } X \text{ is not hyperelliptic.} \end{cases} \quad (17)$$

Let $\Theta(X/K)$ denote the set of $K$-isomorphism classes of twists of $X/K$. We define the type of a supersingular curve as follows.

**Definition 5.1.** A supersingular curve $X$ with field of definition $K$ is:

1. **fully maximal** if $X'/K$ has $K$-parity $\delta = 1$ for all $X' \in \Theta(X/K)$;
2. **fully minimal** if $X'/K$ has $K$-parity $\delta = -1$ for all $X' \in \Theta(X/K)$;
(3) mixed if there exist $X', X'' \in \Theta(X/K)$ with $K$-parities $\delta(X') = 1$ and $\delta(X'') = -1$.

When $X$ is hyperelliptic, then $\Theta(Jac(X)/K) = \Theta(X/K)$, so $X$ and Jac($X$) have the same type. When $X$ is not hyperelliptic, a twist of the curve still induces a twist of its Jacobian, but the converse no longer holds.

**Lemma 5.2.** The types of $X$ and Jac($X$) are not the same if and only if the following hold: $X$ is not hyperelliptic, Jac($X$) is mixed, $r$ is even, and $e(X/K) = \{ e \}$ with $e \leq 1$.

**Proof.** Suppose that the types of $X$ and Jac($X$) are not the same. This is only possible if Jac($X$) has more twists than $X$, so [17] implies that $X$ is not hyperelliptic. Furthermore, since the extra twist corresponds to $\iota$, it follows that Jac($X$) is mixed, with Jac($X$) and Jac($X$)$_{\iota}$ having different parities.

Consider $e = e(X/K)$. If not all $e_i \in e$ are the same, then not all $e_i \in e(Jac(X)_{\iota})$ are the same. Then both Jac($X$) and Jac($X$)$_{\iota}$ would have parity $-1$, a contradiction. Thus $e = \{ e \}$.

If $e \geq 2$, then $e(Jac(X)_{\iota}) = \{ e \}$ and both Jac($X$) and Jac($X$)$_{\iota}$ would have parity $1$; thus $e \leq 1$. If $r$ is odd, then both Jac($X$) and Jac($X$)$_{\iota}$ would have parity $-1$; thus $r$ is even.

We omit the converse direction. □

The following result is the analogue of Proposition 3.18 for curves.

**Proposition 5.3.** Suppose that $\phi : X \times_K k \to X' \times_K k$ where $X/K$ is maximal and $X'/K$ is minimal (or vice versa). Then $X$ is hyperelliptic, $g_\phi = \iota$, and $X'/K$ is a quadratic twist.

**Proof.** The proofs that $g_\phi = \iota$ and $X'/K$ is a quadratic twist are the same as in Proposition 3.18 Further, the quotient of $X$ by $\iota$ has genus 0 since the trivial eigenspace for the action of $\iota$ is trivial. This implies that $X$ is hyperelliptic. □

If $X$ is not hyperelliptic, then mixed behavior can still occur, as seen in Example 5.4.

### 5.2. A supersingular non-hyperelliptic curve of mixed type.

Suppose that $q$ is odd and $a \in \mathbb{F}_q$. This plane quartic was introduced by Ciani in 1899:

$$C_a : x^4 + y^4 + z^4 = (a + 1)(x^2y^2 + y^2z^2 + x^2z^2).$$

It is smooth of genus 3 when $a \not\in \{1, 0, -3\}$. By [2] Lemma 1.1, Jac($C_a$) $\simeq_{\mathbb{F}_q} E_a^3$, where

$$E_a : (a + 3)y^2 = x(x - 1)(x - a).$$

So $C_a/\mathbb{F}_q$ is maximal (resp. minimal) if and only if $E_a/\mathbb{F}_q$ is maximal (resp. minimal). By [2] Propositions 1.3 and 4.2, $E_a$ is maximal over $\mathbb{F}_q = \mathbb{F}_{p^r}$ if $p = 3$ and $r \equiv 2 \mod 4$ or if $p \equiv 3 \mod 4$ and $\text{ord}_2(r) = 1$. If $a \neq -1$ and $a^2 - a + 16 \neq 0$, then Aut$_{\mathbb{F}_q}(C_a) \simeq S_4$.

By [24] Proposition 12, if $a$ is such that $C_a$ is maximal over $\mathbb{F}_q$, then the nontrivial twists of $C_a$ are not maximal over $\mathbb{F}_q$.

**Example 5.4.** Suppose that $a$ is such that $C_a$ is maximal over $\mathbb{F}_q$ and Aut$_{\mathbb{F}_q}(C_a) \simeq S_4$. Then $C_a$ is a supersingular curve of mixed type.

**Proof.** The automorphisms of $C_a$ are defined over $\mathbb{F}_q$, so the twists of $C_a$ are in bijection with conjugacy classes in $S_4$. There are 5 twists of $C_a$, corresponding to the 5 cycle types of automorphisms $g \in S_4$. By Lemma 3.11 since $c_g = 1$, the order of a twist corresponding to $g$ equals $|g|$. The computation below shows $C_a$ is mixed, since it has twists of parity $-1$. 


5.3. Action of involutions.

The following material is used in Section 8. Suppose that \( \tau \in \text{Aut}_k(X) \) is an involution. Assume that \( \tau \) is defined over \( K \); this is true, for example, if \( \tau = \iota \) or if \( \text{Aut}_k(X) \) has a unique element of order 2. Let \( Z = X/\tau \) be the quotient of \( X \) by \( \tau \), which is also defined over \( K \). Thus, \( X \to Z \) is a geometric cover with Galois group \( G \simeq \mathbb{Z}/2\mathbb{Z} \). Let \( \chi \) be the nontrivial character of \( G \); it satisfies \( \chi(P) = 1 \) if \( P \in Z \) is split in \( X \) and \( \chi(P) = -1 \) if \( P \) is inert in \( X \).

Consider the Artin \( L \)-series
\[
L(Z/K, T, \chi) = \prod_{P \in Z} (1 - \chi(P)|P|^{-s})^{-1}.
\]

\textbf{Lemma 5.5.} Suppose that \( \tau \in \text{Aut}_K(X) \) is an involution.

1. There is a factorization \( L(X/K, T) = L(Z/K, T)L(Z/K, T, \chi) \) in \( \mathbb{Z}[T] \).
2. The coefficient \( \rho_1 \) of \( T \) in \( L(Z/K, T, \chi) \) equals \( S_1 - I_1 \), where \( I_1 \) (resp. \( S_1 \)) is the number of \( K \)-points of \( Z \) that are inert (resp. split) in \( X \).
3. \( \tau \) negates the roots of \( L(Z/K, T, \chi) \) and fixes the roots of \( L(Z/K, T) \).

\textit{Proof.} (1) This result follows from [29, Chapter 9, page 130].
(2) The zeta function of \( X/K \) can be written as \( \zeta(X/K, T) = \prod_{Q \in X} (1 - |Q|^{-s})^{-1} \), where \( T = q^{-s} \). Similarly, \( \zeta(Z/K, T) = \prod_{P \in Z} (1 - |P|^{-s})^{-1} \). Write
\[
\zeta(X/K, T) = \prod(1 - |P_1|^{-2s})^{-1} \prod (1 - |P_s|^{-s})^{-2} \prod (1 - |P_r|^{-s})^{-1},
\]
where \( P_1, P_s, P_r \) range over points of \( Z \) that are inert, split, and ramified in \( X \), respectively. Note that \( (1 - |P|^{-2s}) = (1 - |P|^{-s})(1 + |P|^{-s}) \). The result follows by comparing Equations (18) and (19) and computing the coefficients of \( T \).
(3) Since \( Z = X/\tau \), the involution acts trivially on \( Z \) and thus fixes its (normalized) Weil numbers over \( K \). We may write \( \text{Jac}(X) \sim \text{Jac}(Z) \times V \) where \( V \) is the nontrivial eigenspace for \( \tau \). This implies that the roots of \( L(Z/K, T, \chi) \) are the normalized Weil numbers of \( L(X/K, T) \) on which \( \tau \) acts by negation, by construction.

\textbf{Remark 5.6.} Suppose that \( \text{Aut}_k(X) \) contains a subgroup \( H \) isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Write \( H = \{ \iota, \tau, \mu, \mu\tau \} \). Suppose that \( H \) is stabilized by \( K \)-Frobenius conjugation, in which case the number \( \gamma \) of nontrivial involutions in \( H \) defined over \( K \) is either three, one or zero.

1. When \( \gamma = 3 \), by [16, Theorem B], there is a decomposition \( \text{Jac}(X) \sim_K A_1 \times A_2 \times A_3 \) (and hence \( L(X/K, T) = \prod_{i=1}^3 L(A_i/K, T) \)), such that each of the nontrivial involutions in \( H \) acts by negating the normalized Weil numbers for exactly two of
the factors. Write \( e_i = e(A_i) \) and \( e(X) = \bigcup_{i=1}^{3} e_i \). Then twists corresponding to \( H \) change the parity if and only if either \( e(X) = \{1\} \) or \( e_1, e_2, e_3 \) are \( \{0\}, \{0\}, \{1\} \), after possibly rearranging.

(2) When \( \gamma = 0 \), \( K \)-Frobenius conjugation acts via a 3-cycle on the nontrivial involutions in \( H \). This implies that the twist corresponding to \( \tau \) has order 3, since \( \tau^{F_{r \cdot K}} \tau^{F_{r \cdot K}} \tau = \text{id} \). By Corollary 4.8 this does not change the parity. The same argument applies for \( \mu \) and \( \mu \tau \).

(3) When \( \gamma = 1 \), suppose that \( \tau \) is defined over \( K \) while \( \mu \) and \( \mu \tau \) are not. Let \( Z = X/\tau \). By Lemma 5.3(3), \( \tau \) negates the roots of \( L(Z/K, T, \chi) \) and fixes the roots of \( L(Z/K, T) \). Note that \( F_{r \cdot K} \mu = \mu \tau \) and \( \mu^{F_{r \cdot K}} \mu = \tau \). This implies, using the notation of Lemma 3.11, that the twist corresponding to \( \mu \) has \( c = 2 \) and \( |G| = 2 \). Moreover, the twist by \( \mu \) over \( K \) corresponds to the twist by \( \tau \) over \( K_2 \), so it negates the roots of \( L(Z/K_2, T, \chi) \) and fixes the roots of \( L(Z/K_2, T) \) by Lemma 5.5(3).

To find the action of \( \mu \) on \( e(X/K) \), it is necessary to take the square roots of the normalized Weil numbers of the twist of \( X/K_2 \) by \( \tau \). If \( e_i \leq 1 \) for any \( i \), this leads to some ambiguity in the 2-adic valuations of the square roots. This ambiguity can be partially resolved by the following observation (found also in Lemma 8.13 in a special case).

Claim: When \( \gamma = 1 \), the coefficient \( \rho_1 \) of \( T \) in \( L(Z/K, T, \chi) \) equals \( 0 \).

Proof. By Lemma 5.5(2), it suffices to prove that \( S_1 = I_1 \), where \( I_1 \) (resp. \( S_1 \)) is the number of \( K \)-points of \( Z \) that are inert (resp. split) in \( X \). To see this for \( p \) odd, note that the equation for \( X \to Z \) is of the form \( y^2 = F \). Given an arbitrary \( K \)-point \( P \) of \( Z \), it suffices to show that \( P \) is split in \( X \) if and only if \( \mu(P) \) is inert in \( X \). The point \( P \) splits in \( X \) if and only if \( F(P) \) is a square in \( \mathbb{F}_q^* \). Since \( \mu \) and \( \tau \) commute, \( \mu \) acts on both \( X \) and \( Z \); by assumption, this action is defined over \( K_2 \) but not over \( K \). The same holds for the action of \( \mu \) on the covering equation \( y^2 = F \). In particular, the \( K \)-action of \( \mu \) yields a quadratic twist of \( y^2 = F \). This implies that \( \mu(y) = wy \) for some \( w \in (\mathbb{F}_{q^2})^* \setminus (\mathbb{F}_q)^* \) such that \( z = w^2 \) is in \( \mathbb{F}_q^* \), and \( F(\mu(P)) = zy \). Thus, \( F(P) \) is a square in \( \mathbb{F}_q^* \) if and only if \( F(\mu(P)) \) is not.

The proof for \( p = 2 \) is the same, after replacing \( y^2 \) by \( y^2 - y \), \( \mu(y) = wy \) by \( \mu(y) = y + w \) for some \( w \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) such that \( z = w^2 - w \) is in \( \mathbb{F}_q \), and \( F(\mu(P)) = zF(P) \) by \( F(\mu(P)) = F(P) + z \).

5.4. Automorphism groups with prescribed structure.

Let \( X \) be a smooth projective connected supersingular curve with minimal field of definition \( K = \mathbb{F}_q \). In general, it can be hard to determine whether the order of a twist is odd. In this section, we provide some conditions on \( \text{Aut}_k(X) \) under which all the twists can be analyzed. We do not know whether these conditions occur for a supersingular curve of \( g \geq 3 \).

**Proposition 5.7.** Suppose that \( \text{Aut}_k(X) \) has odd order and that \( \text{Aut}_k(X) = \text{Aut}_K(X) \). Then \( X \) is fully maximal if and only if \( e = \{e\} \) with \( e \geq 1 \) (and \( e \geq 2 \) if \( r \) odd), and \( X \) is fully minimal otherwise.

Proof. By hypothesis, all the elements in \( \text{Aut}_k(X) \) are defined over \( K \) and have odd order, thus yield a twist of odd order. By Lemma 4.6, none of the twists can change the parity. Thus \( X \) is fully maximal if and only if it has parity 1 and \( X \) is fully minimal otherwise. \( \square \)
A result similar to Proposition 5.7 holds when $X$ has an involution $\tau$ which is defined over $K$ and in the center of $\text{Aut}_k(X)$. Write $Z = X/\tau$ and recall Lemma 5.5. Write $\mathcal{E} = \mathcal{E}(Z/K) \cup \mathcal{E}(Z/K, \chi)$ where $\mathcal{E}(Z/K, \chi)$ denotes the multiset of 2-valuations of the normalized roots of $L(Z/K, T, \chi)$. In particular, if $\tau$ is the hyperelliptic involution, then $\mathcal{E}(Z/K)$ is empty and $\mathcal{E} = \mathcal{E}(Z/K, \chi)$.

**Proposition 5.8.** Suppose that $\text{Aut}_k(X) \simeq \langle \tau \rangle \times H$ where $|H|$ is odd and all $h \in H$ are defined over $K$. Then $X$ is fully maximal if and only if $\mathcal{E} = \{e\}$ with $e \geq 2$. Also, $X$ is mixed if and only if $r$ is even, $\mathcal{E}(Z/K) = \{1\}$, and $\mathcal{E}(Z/K, \chi) = \{e\}$ where $e \leq 1$.

**Proof.** Let $\tau \in \text{Aut}_k(X)$ be the element of order 2; then $\tau$ is defined over $K$ by the hypothesis on the structure of $\text{Aut}_k(X)$. Every $g \in \text{Aut}_k(X)$ is of the form $h \tau h$, where $h$ is defined over $K$ and has odd order. Then twisting by $h$ does not change the parity as seen in the proof of Proposition 5.7. Twisting by $\tau$ changes the parity exactly when $r$ is even, $\mathcal{E}(Z/K) = \{1\}$, and $\mathcal{E}(Z/K, \chi) = \{e\}$ where $e \leq 1$. \(\square\)

The information in Remark 5.6 can be used to determine the type of $X$ under conditions on $\mathcal{E}$ when $\text{Aut}_k(X) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times H$ for $|H|$ odd, with all automorphisms defined over $K$.

## 6. Analysis in Low Dimension: Elliptic Curves

### 6.1. Preliminaries.

Let $q = p^r$ and let $k = \overline{\mathbb{F}}_p$. If $E/\mathbb{F}_q$ is an elliptic curve, then $L(E/\mathbb{F}_q, T) = T^2 - \beta T + q$ for some $\beta \in \mathbb{Z}$. Either $L(E/\mathbb{F}_q, T)$ is irreducible over $\mathbb{Q}_q$ or it is of the form $(T - b)^2$ for some $b \in \mathbb{Z}$. Moreover, $E$ is supersingular if and only if $p \nmid \beta$.

The number of isomorphism classes of supersingular elliptic curves $E/k$ is $|\mathbb{F}_q| + \varepsilon$ (with $\varepsilon = 0, 1, 1, 2$ if $p \equiv 1, 5, 7, 11 \pmod{12}$) [33, Theorem V.4.1(c)]. The supersingular $j$-invariants are all in $\mathbb{F}_{p^2}$.

By Honda-Tate theory (cf. [37], [11], [38]), the $\mathbb{F}_q$-isogeny class of $E$ is determined by $\beta$. The number $N(\beta)$ of $\mathbb{F}_q$-isomorphism classes of elliptic curves in the $\mathbb{F}_q$-isogeny class determined by $\beta$ is determined in [30, Theorem 4.6].

In the next result, for each supersingular isogeny class, we list the normalized Weil numbers $z$ and $\bar{z}$, the period, the parity, and the 2-adic valuation $e = v_2(|z|)$.

**Proposition 6.1.** Let $q = p^r$. A supersingular elliptic curve $E/\mathbb{F}_q$ with $L$-polynomial $L(E/\mathbb{F}_q, T) = T^2 - \beta T + q$ is in one of the following cases.

| Case | Conditions on $r$ and $p$ | $\beta$ | $N\mathbb{W}/\mathbb{F}_q$ | $v_2(|z|)$ | Period | Parity |
|------|--------------------------|--------|----------------|----------|--------|--------|
| 1a   | $r$ even                 | $2\sqrt{q}$ | (1, 1) | 0 | 1 | -1 |
| 1b   | $r$ even                 | $-2\sqrt{q}$ | (-1, -1) | 1 | 1 | 1 |
| 2a   | $r$ even, $p \not\equiv 1 \pmod{3}$ | $\sqrt{q}$ | $(-\zeta_3, -\bar{\zeta}_3)$ | 1 | 3 | 1 |
| 2b   | $r$ even, $p \not\equiv 1 \pmod{3}$ | $-\sqrt{q}$ | $(\zeta_3, \bar{\zeta}_3)$ | 0 | 3 | -1 |
| 3    | $r$ even, $p \equiv 3 \pmod{4}$ | 0 | $(i, -i)$ | 2 | 2 | 1 |
|      | or $r$ odd               |        |                |        |      |      |
| 4a   | $r$ odd, $p = 2$         | $\sqrt{2q}$ | $(\zeta_8, \bar{\zeta}_8)$ | 3 | 4 | 1 |
| 4b   | $r$ odd, $p = 2$         | $-\sqrt{2q}$ | $(\bar{\zeta}_8, \zeta_8)$ | 3 | 4 | 1 |
| 4c   | $r$ odd, $p = 3$         | $\sqrt{3q}$ | $(\zeta_{12}, \bar{\zeta}_{12})$ | 2 | 6 | 1 |
| 4d   | $r$ odd, $p = 3$         | $-\sqrt{3q}$ | $(\bar{\zeta}_{12}, \zeta_{12})$ | 2 | 6 | 1 |
Proof. The values of $\beta$ are determined in the classification of isogeny classes by Waterhouse [42, Theorem 4.1]. The other columns follow from a short calculation. \qed

6.2. Main result for elliptic curves.

The next result shows that there are no elliptic curves that are fully minimal.

**Theorem 6.2.** Let $E$ be a supersingular elliptic curve defined over a finite field of characteristic $p$. If $E$ is defined over $\mathbb{F}_p$, then it is fully maximal; otherwise, it is mixed.

**Proof.** If $p = 2$, the result is proven in Lemma 6.3 (below). If $p \geq 3$ and $\text{Aut}_k(E) \neq \mathbb{Z}/2\mathbb{Z}$, the result is proven in Lemma 6.3 (below).

In the remaining case, $p \geq 5$ and $\text{Aut}_k(E) \cong \mathbb{Z}/2\mathbb{Z}$. Let $K$ be the minimal field of definition of $E$. If $K = \mathbb{F}_p$, then $E$ and its twist by $\iota$ are both in case 3 of Table 6.1 thus $E$ is fully maximal. If $K = \mathbb{F}_{p^2}$, then $E$ and its twist by $\iota$ are either in cases (1a) and (1b) or in case (2a) and (2b) of Table 6.1 thus $E$ is mixed. \qed

**Lemma 6.3.** If $p = 2$, then the unique supersingular elliptic curve $E/\mathbb{F}_2$ is fully maximal.

**Proof.** The uniqueness statement can be found in [33] Appendix A, Proposition 1.1]. Since $p = 2$, $E$ is defined over $\mathbb{F}_p$ with equation $y^2 + y = x^3$. Then $|E(\mathbb{F}_2)| = p + 1$, so $\beta = 0$, and $E$ is in case 3 of Table 6.1. The $\mathbb{F}_2$-twists are also defined over $\mathbb{F}_2$, thus are in case 3, 4a or 4b of Table 6.1 which all have parity +1. \qed

6.3. Elliptic curves with extra automorphisms.

**Lemma 6.4.** Let $p \geq 3$. If $\text{Aut}_k(E) \neq \mathbb{Z}/2\mathbb{Z}$, then $E$ is fully maximal.

**Proof.** If $\text{Aut}_k(E) \neq \mathbb{Z}/2\mathbb{Z}$, then $E$ is isomorphic over $k$ to either:

1. $E : y^2 = x^3 - x$, the elliptic curve with $j$-invariant 1728, which is supersingular if and only if $p \equiv 3 \pmod{4}$; or
2. $E : y^2 = x^3 + 1$, the elliptic curve with $j$-invariant 0, which is supersingular if and only if $p \equiv 2 \pmod{3}$.

In both cases, $\{z, \bar{z}\} = \{i, -i\}$ (case 3 of Table 6.1) with $\underline{e}(E/\mathbb{F}_p) = \{2\}$.

For case (1), let $g \in \text{Aut}_k(E)$ be the order 4 automorphism defined by $g(x, y) = (-x, iy)$.

(a) If $p > 3$, then $\text{Aut}_k(E) \cong \langle g \rangle$. Then $E/\mathbb{F}_p$ has only one nontrivial twist because the $\mathbb{F}_p$-Frobenius conjugacy classes in $\text{Aut}_k(E)$ are $\{\text{id}, \iota\}$ and $\{g, g^3\}$. The latter of these yields a quadratic twist since $c = 2$ and $G = g^{Fr}g = \text{id}$. By Lemma 4.6, the twist has $\underline{e} = \{2\}$ as well.

(b) If $p = 3$, then $|\text{Aut}_k(E)| = 12$ [33, Appendix A, Proposition 1.2]. Then $\text{Aut}_k(A) = \langle g, \sigma \rangle$ where $\sigma(x, y) = (x + 1, y)$. The $\mathbb{F}_p$-Frobenius conjugacy classes are $\{\text{id}, \iota\}$, $\{\sigma^2, \sigma^2\iota\}$, $\{\sigma, \sigma^2\iota\}$, and $\{g, g^3, \sigma g, \sigma g^3, \sigma^2 g, \sigma^2 g^3\}$. The first (resp. last) of these corresponds to the trivial (resp. quadratic) twist already analyzed in part (a). Since $\sigma$ and $\sigma^2$ are order 3 automorphisms defined over $\mathbb{F}_p$, the other two twists have order 3. By Lemma 4.6 the order 3 twists also have $\underline{e} = \{2\}$.
For case (2), $\text{Aut}_L(E) = \langle h \rangle$ where $h$ is the order 6 automorphism defined by $h(x, y) = (\zeta_3 x, -y)$. The two $\mathbb{F}_p$-Frobenius conjugacy classes are $\{\text{id}, h^2, h^4\}$ and $\{h, h^3, h^5\}$. Since $h^3 = \zeta$, the latter of these is a quadratic twist. By Lemma 4.6, the twist has $\epsilon = \{2\}$ as well.

Thus, in both case (1) and case (2), the elliptic curve is fully maximal. $\square$

7. Analysis in low dimension: abelian surfaces

7.1. Parity table for simple abelian surfaces.

The results in this section build on the work of Maisner and Nart [22]. Let $q = p^e$ and $k = \mathbb{F}_p$. Suppose that $A/\mathbb{F}_q$ is an abelian surface, which is not necessarily principally polarized. The $\mathbb{F}_q$-isogeny class of $A$ is determined by the conjugacy class of its Weil numbers or, equivalently, by the coefficients $(a_1, a_2)$ of

$$P(A/\mathbb{F}_q, T) = T^4 + a_1 T^2 + a_2 T + q^2 \in \mathbb{Z}[T].$$

Suppose that $A/\mathbb{F}_q$ is simple. Then $P(A/\mathbb{F}_q, T)$ is irreducible over $\mathbb{Q}$. To see this, recall that if $A$ is simple then $P(A/\mathbb{F}_q, T) = h_A(T)^e$ for some irreducible $h_A(T) = T^2 - \beta T + q \in \mathbb{Z}[T]$. By [22, Corollary 2.8], this implies that $r$ is even and either $\beta = \pm \sqrt{q}$ (for $p \equiv 1 \mod 3$) or $\beta = 0$ (for $p \equiv 1 \mod 4$). However, these cases are not supersingular by Proposition 6.1.

Let $L$ be the minimal field extension of $\mathbb{F}_q$ over which $A$ is not simple. Then $A \cong E_1 \times E_2$, where $E_1/L$ and $E_2/L$ are supersingular elliptic curves.

**Proposition 7.1.** The following table classifies all $(a_1, a_2)$ which occur as the coefficients of $P(A/\mathbb{F}_q, T)$ for a simple supersingular abelian surface $A/\mathbb{F}_q$, together with the data:

- $t_0 = \deg(L/\mathbb{F}_q)$,
- $n_E = n_{E_1} = n_{E_2}$, labeling $E_1/L$ and $E_2/L$ as in the first column of Table 6.1,
- $z/L$, one of the normalized Weil numbers $(z, \overline{z}, \overline{z})$ of $A/L$,
- $\text{NWN}/\mathbb{F}_q$, the normalized Weil numbers of $A/\mathbb{F}_q$,
- $P$ and $\delta$, the period and parity respectively of $A/\mathbb{F}_q$.

| $(a_1, a_2)$ | Conditions on $r$ and $p$ | $t_0$ | $n_E$ | $z/L$ | $\text{NWN}/\mathbb{F}_q$ | $P$ | $\delta$ |
|--------------|--------------------------|-------|-------|-------|----------------------|-----|--------|
| 1a $(0, 0)$  | $r$ odd, $p \equiv 3 \mod 4$ or $r$ even, $p \not\equiv 1 \mod 4$ | 2     | 3     | $i$   | $(\zeta_8, \zeta_8^5, \zeta_8^6, \zeta_8^8)$ | 4   | 1      |
| 1b $(0, 0)$  | $r$ odd, $p \equiv 1 \mod 4$ or $r$ even, $p \equiv 5 \mod 8$ | 4     | 1     | $-1$  | $(\zeta_8, \zeta_8^7, \zeta_8^3, \zeta_8^5)$ | 4   | 1      |
| 2a $(0, q)$  | $r$ odd, $p \not\equiv 1 \mod 3$ | 2     | 2     | $\zeta_3$ | $(\zeta_6, \zeta_5^2, \zeta_6^3, \zeta_6^4)$ | 6   | -1     |
| 2b $(0, q)$  | $r$ odd, $p \equiv 1 \mod 3$ | 6     | 1     | $-1$  | $(\zeta_0, \zeta_0^5, \zeta_0^3, \zeta_0^7)$ | 6   | 1      |
| 3a $(0, -q)$ | $r$ odd and $p \not\equiv 3 \mod 4$ or $r$ even and $p \not\equiv 1 \mod 3$ | 3     | 2     | $-\zeta_3$ | $(\zeta_0, \zeta_0^5, \zeta_0^3, \zeta_0^7)$ | 6   | 1      |
| 3b $(0, -q)$ | $r$ odd and $p \equiv 1 \mod 3$ or $r$ even and $p \equiv 4, 7, 10 \mod 12$ | 3     | 2     | $i$   | $(\zeta_0, \zeta_0^5, \zeta_0^3, \zeta_0^7)$ | 6   | 1      |
| 4a $(\sqrt{q}, q)$ | $r$ even and $p \not\equiv 1 \mod 5$ | 5     | 1     | $1$   | $(\zeta_5, \zeta_5^4, \zeta_5^2, \zeta_5^3)$ | 5   | -1     |
| 4b $(-\sqrt{q}, q)$ | $r$ even and $p \not\equiv 1 \mod 5$ | 5     | 1     | $-1$  | $(\zeta_0, \zeta_0^9, \zeta_0^5, \zeta_0^7)$ | 5   | 1      |
| 5a $(\sqrt{q}, 3q)$ | $r$ odd and $p = 5$ | 5     | 1     | $\pm 1$ | $(\zeta_0, \zeta_0^9, \zeta_0^5, \zeta_0^7)$ | 10  | -1     |
| 5b $(-\sqrt{q}, 3q)$ | $r$ odd and $p = 5$ | 5     | 1     | $\pm 1$ | $(\zeta_0, \zeta_0^9, \zeta_0^5, \zeta_0^7)$ | 10  | -1     |
| 6a $(\sqrt{q}, q)$ | $r$ odd and $p = 2$ | 4     | 2     | $-\zeta_3$ | $(\zeta_{13}, \zeta_{11}, \zeta_{24}, \zeta_{24})$ | 12  | 1      |
| 6b $(-\sqrt{q}, q)$ | $r$ odd and $p = 2$ | 4     | 2     | $-\zeta_3$ | $(\zeta_{13}, \zeta_{11}, \zeta_{24}, \zeta_{24})$ | 12  | 1      |
| 7a $(0, -2q)$ | $r$ odd | 2     | 1     | $1$   | $(1, 1, -1, -1)$ | 2   | -1     |
| Case | Equation | Condition | $\text{Aut}_k(X)$ | $|\Theta|$ | Type |
|------|----------|-----------|----------------|--------|-------|
| 7b   | $y^2 = x^4 - 1$ | $p \equiv 1 \mod 4$ | $\mathbb{Z}/10\mathbb{Z}$ | 2 | $f_{\text{max}}$ |
| 8a   | $y^2 = x^6 - 1$ | $p \equiv 2 \mod 3$ | $2D_{12}$ | 7 | mixed |
| 8b   | $y^2 = x^6 - x$ | $p \equiv 5, 7 \mod 8$ | $S_4$ | 6 | mixed |

Proof. The list of $(a_1, a_2)$, conditions on $r$ and $p$, and $t_0$ are found in [22, Table 1, page 325]. Applying [22, Lemma 2.13, Theorem 2.9], we compute the coefficients of $P(A/L, T)$ where $L = \mathbb{F}_{q^0}$ and determine $n_E$. Then the values of $z/L$, the period, and the parity can be found using Table 6.1. The period is the product of $t_0$ and the period of $E$ and the parities of $A$ and $E$ are the same. To determine the normalized Weil numbers of $A$ over $\mathbb{F}_q$, we solve $T^4 + a_1T^3 + a_2T^2 + qa_1T + q^2 = 0$ directly.

For example, in Case 5, $L = \mathbb{F}_q$ by [22, Theorem 2.9 and Table 1]. We use Mathematica to find the roots of the Weil polynomial $t^4 + \sqrt{5}qt^3 + 3qt^2 + \sqrt{5}qt + q^2$ over $\mathbb{F}_q$. The normalized Weil numbers over $L$ are $(-1, -1, 1, 1)$, so $A \sim_L E_1 \times E_2$ where $E_1$ (resp. $E_2$) has normalized Weil numbers $(-1, -1)$ (resp. $(1, 1)$). Note that $E_1$ is in case (1b) and $E_2$ in case (1a) of Table 6.1. Over $\mathbb{F}_{q^0}$, both $E_1$ and $E_2$ are minimal; thus $A$ has parity $-1$. □

Restricting now to principally polarized abelian surfaces, we obtain the following result.

**Proposition 7.2.** Let $A$ be a supersingular simple principally polarized abelian surface with minimal field of definition $\mathbb{F}_q$. Assume that $\text{Aut}_k(A) \simeq \mathbb{Z}/2\mathbb{Z}$. In Proposition 7.1:

1. if $r$ is odd, then $A$ is not mixed; Cases (1), (2b), (3a), (6) are fully maximal and Cases (2a), (5), (7a) are fully minimal.

2. if $r$ is even, then $A$ is not fully minimal; Cases (1), (3a), and (7b) are fully maximal and Cases (4) and (8) are mixed.

Proof. By [12, Theorem 1], an isogeny class in Table 7.1 does not contain a principally polarized surface if and only if $a_1 \neq q$ and all prime divisors of $a_2$ are congruent to 1 modulo 3. This excludes case (3b) and no other cases. Since $\text{Aut}_k(A) \simeq \mathbb{Z}/2\mathbb{Z}$, the type of $A$ is determined from $g(A/\mathbb{F}_q)$ by Corollary 4.11. This can be computed from the normalized Weil numbers found in Proposition 7.1. □

### 7.2. Curves of genus 2 with extra automorphisms.

We determine the type for all supersingular curves $X$ of genus 2 such that $\text{Aut}_k(X) \not\simeq \mathbb{Z}/2\mathbb{Z}$. By [15], there are six equations that describe all such curves, up to geometric isomorphism. Their twists are studied by Cardona in [3] and Cardona and Nart in [4].

Let $K = \mathbb{F}_{q^0}$ be the field of definition of $X$ with $q = p^f$. Let $|\Theta|$ denote the number of $K$-twists of $X$. We first analyze the three equations which have no moduli parameters.

**Proposition 7.3.** Let $p > 5$. The following three equations each define a unique geometric isomorphism class of genus 2 curve $X$ defined over a finite field of characteristic $p$ with $\text{Aut}_k(X) \not\simeq \mathbb{Z}/2\mathbb{Z}$ which is supersingular under the listed condition on $p$. We determine the type for each.

| Equation | Condition | $\text{Aut}_k(X)$ | $|\Theta|$ | Type |
|----------|-----------|----------------|--------|-------|
| 1 $y^2 = x^4 - 1$ | $p \neq 1 \mod 5$ | $\mathbb{Z}/10\mathbb{Z}$ | 2 | $f_{\text{max}}$ |
| 2 $y^2 = x^6 - 1$ | $p \equiv 2 \mod 3$ | $2D_{12}$ | 7 | mixed |
| 3 $y^2 = x^5 - x$ | $p \equiv 5, 7 \mod 8$ | $S_4$ | 6 | mixed |
Here $D_n$ is the dihedral group of order $n$ and $\tilde{S}_4$ is a 2-covering of $S_4$.

**Proof.** The equations and automorphism groups were taken from \[4, Theorem 3.1\]. The supersingular condition is found in \[14, 1.11-1.13\]. For equation (1), $|\Theta| = 2$ by \[3, Proposition 11\]. For equation (2), when $p \equiv 2 \mod 3$, then $-3 \notin (\mathbb{F}_p)^*$. For equation (4), $|\Theta| = 7$ by \[3, Proposition 16\].

For equation (3), when $p \equiv 5, 7 \mod 8$, then $-2 \notin (\mathbb{F}_p)^*$, so $|\Theta| = 6$ by \[3, Proposition 17\].

The pairs $(a_1, a_2)$ which occur for the twists of $X$ are in \[4, Sections 3.1-3.3, Tables 5,9,6,7\]. If $(a_1, a_2) = (0, 2p)$, note that $	ext{Jac}(X) \sim_{\mathbb{F}_p} E \times E$ where $E$ is in case (3) of Proposition \[6, Proposition 7.1\], which has parity 1. Also, $(a_1, a_2) = (0, -2p)$ has parity $-1$ by case $(7a)$ of Proposition \[7, Proposition 7.1\].

1. When $p \equiv 2, 3 \mod 5$, then $(a_1, a_2) = (0, 0)$ for $X$ and $X_i$; thus $X$ is fully maximal. When $p \equiv -1 \mod 5$, then $(a_1, a_2) = (0, 2p)$ for $X$ and $X_i$; thus $X$ is fully maximal.

2. When $p > 5$ and $p \equiv 2 \mod 3$, let $\epsilon = (-1/p)$. The first two rows of \[4, Table 9\] show that the parity 1 case $(a_1, a_2) = (0, 2p)$ occurs for $X$ or one of its $\mathbb{F}_p$-twists, regardless of the value of $\epsilon$. The third and fourth lines of \[4, Table 9\] show that the parity $-1$ case $(a_1, a_2) = (0, -2p)$ occurs for $X$ or one of its $\mathbb{F}_p$-twists, regardless of the value of $\epsilon$, as long as there exists $t \in \mathbb{F}_p$ such that $t^2 + 4$ is not a square in $\mathbb{F}_p^*$; the existence of such a $t$ can be verified using a Jacobi sum argument. So $X$ is mixed.

3. If $p \equiv 5, 7 \mod 8$, then both $(0, 2p)$ and $(0, -2p)$ occur as $(a_1, a_2)$ among the twists of $X$, so $X$ is mixed.

Next, we analyze the three equations with moduli parameters.

**Proposition 7.4.** Let $p > 5$. Any genus 2 curve $X$ with $\text{Aut}_K(X) \not\cong \mathbb{Z}/2\mathbb{Z}$ is geometrically isomorphic to one of equations (1)-(3) in Proposition \[7, Proposition 7.3\] or one of equations (4)-(6) below:

1. $y^2 = x^6 + ax^4 + bx^2 + 1$ where $a, b \in k$ are chosen such that $P(c, d) \neq 0$, where $c = ab$, $d = a^3 + b^3$, and $P(c, d) = (4c^3 - d^2)(c^2 - 4d + 18c - 27)(c^2 - 4d - 110c + 1125)$;

2. $y^2 = x^5 + x^3 + ax$, for $a \neq 0$, 1/4, 9/100;

3. $y^2 = x^6 + x^3 + a$ for $p \neq 3$, $a \neq 0$, 1/4, $-1/50$.

Let $\mathbb{F}_q = \mathbb{F}_{p^r}$ be the field of definition of $X$. The types for equations (4)-(6) are as follows:

| Aut$_k(X)$ | $|\Theta|$ | Type |
|------------|------------|------|
| 4          | $V_4$      | fmax if $r$ is odd  |
|            |            | mixed if $r$ is even |
| 5          | $D_8$      | fmax if $r$ is odd, $a \notin (k^*)^2$  |
|            |            | mixed otherwise    |
| 6          | $D_{12}$   | fmin if $q \equiv 2 \mod 3$ and $a \in (k^*)^2$  |
|            |            | mixed otherwise    |

**Proof.** The first statement, the equations, and the automorphism groups can be found in \[4, Theorem 3.1\]. The number $|\Theta|$ of twists of $X$ is determined in \[3, Propositions 10,12,13\]. By \[4, Section 3.6\], when equation (4) is supersingular, then $|\Theta| = 4$. The pairs $(a_1, a_2)$ for the twists of $X$ are in \[4, Sections 3.4-3.6, Tables 11-17\]. We determine the types below:

1. Since $\text{Jac}(X) \sim_k E_1 \times E_2$, the 4 twists of $X$ correspond to quadratic twists of either $E_1$ or $E_2$, or both. When $r$ is odd, $E_1$ and $E_2$ are both in case (3) of Proposition \[6, Proposition 6.1\]
so $X$ is fully maximal. When $r$ is even, $E_1$ and $E_2$ are either both in case (1a) (so $X$ is minimal) or both in case (1b) (so $X$ is maximal), depending on the $L$-polynomial of $E_1$. Then $X$ is mixed since the quadratic twist swaps the two cases.

(5) When $r$ is odd and $a \not\in (k^*)^2$, then $X$ and its twists have $(a_1, a_2)$ equal to $(0, 0)$ or $(0, 2q)$. Since both cases have parity 1, the curve $X$ is fully maximal.

When $r$ is odd and $a \in (k^*)^2$, there are two twists of $X$ with $(a_1, a_2)$ being $(0, 2q)$ (parity 1) and $(0, -2q)$ (parity $-1$), so $X$ is mixed. When $r$ is even, a similar argument shows that $X$ is mixed.

(6) When $q \equiv 2 \mod 3$, note that $p \equiv -1 \mod 3$ as well and $r$ is odd. Then $X$ and its twists have $(a_1, a_2)$ among $(0, 2q)$, $(0, 2\varepsilon q)$ or $(0, -\varepsilon q)$, where $\varepsilon = 1$ if $a \in (k^*)^2$ and $\varepsilon = -1$ otherwise. These curves have respective parities $1$, $\varepsilon = \pm 1$, and $\varepsilon = \pm 1$. So if $\varepsilon = 1$, then $X$ is fully maximal and if $\varepsilon = -1$, then $X$ is mixed.

When $q \equiv 1 \mod 3$ and $r$ is odd, then the coefficients $(a_1, a_2)$ of the twists include $(0, 2q)$ of parity 1 and $(0, -2q)$ of parity $-1$, so $X$ is mixed.

When $q \equiv 1 \mod 3$ and $r$ is even, let $\varepsilon = \left(\frac{-3}{q}\right)$. Then the possibilities for $(a_1, a_2)$ are $(-4\varepsilon \sqrt{q}, 6q)$ of parity $-\varepsilon = \mp 1$, $(2\varepsilon \sqrt{q}, 3q)$ of parity $-\varepsilon = \pm 1$, and $(0, -2q)$ of parity $-1$. So if $\varepsilon = 1$, then $X$ is fully minimal and if $\varepsilon = -1$, then $X$ is mixed.

The number of $k$-isomorphism classes of supersingular genus 2 curves is determined in each case (1) – (6) by Ibukiyama, Katsura, and Oort [14, Theorem 3.3].

7.3. The condition $\text{Aut}_k(A) \cong \mathbb{Z}/2\mathbb{Z}$ is not restrictive when $p$ is odd.

For general $p$, $r$, and $g$, the structure of the typical automorphism group of a $g$-dimensional supersingular abelian variety $A$ over $K = \mathbb{F}_{p^r}$ is unknown (cf. Remark [1,13]). In this section, we resolve this question for $g = 2$ and $p$ odd.

Let $g = 2$ and let $A = (A, \lambda)$ be a principally polarized abelian surface. For $p \geq 3$, we prove that the proportion of $A$ over $\mathbb{F}_{p^r}$ with $\text{Aut}_k(A) \not\cong \mathbb{Z}/2\mathbb{Z}$ tends to zero as $r \to \infty$.

Let $\mathcal{A}_2 = \mathcal{A}_2 \otimes \mathbb{F}_p$ denote the moduli space whose points represent the objects $(A, \lambda)$ in characteristic $p$. Let $\mathcal{A}_{2, ss} \subset \mathcal{A}_2$ denote the supersingular locus whose points represent supersingular $A$.

Recall that $A$ is superspecial if and only if $A \simeq_k E_1 \times E_2$. If $A = (A, \lambda)$ is superspecial, then $\lambda$ may or may not be the product polarization.

Proposition 7.5. If $p \geq 3$, then the proportion of $\mathbb{F}_{p^r}$-points in $\mathcal{A}_{2, ss}$ which represent $A$ with $\text{Aut}_k(A) \not\cong \mathbb{Z}/2\mathbb{Z}$ tends to zero as $r \to \infty$.

Proof. As observed in [1, Section 9], $p^r \ll |\mathcal{A}_{2, ss}(\mathbb{F}_{p^r})| \ll p^{r+2}$, where the notation $f(q) \ll g(q)$ means that there is a constant $C > 0$ such that $|f(q)| \leq C |g(q)|$ for all sufficiently large $q$. This is because each irreducible component of $\mathcal{A}_{2, ss}$ is geometrically isomorphic to $\mathbb{P}^1$ [27, proof of Corollary 4.7], and the number of irreducible components of $\mathcal{A}_{2, ss}$ equals the class number $H_2(1, p)$ [17, Theorem 5.7], which is $\ll p^2$ by [9], see also [14, Remark 2.17].

By [3, Theorem 3.1], an $\mathbb{F}_{p^r}$-point $A$ in $\mathcal{A}_{2, ss}$ is one of the following canonically principally polarized objects: (i) the Jacobian of a smooth supersingular curve $X$ over $\mathbb{F}_{p^r}$ of genus 2; (ii) the product $E_1 \times E_2$ of two supersingular elliptic curves over $\mathbb{F}_{p^r}$; (iii) the restriction of scalars $\text{Res}_{\mathbb{F}_{p^r}/\mathbb{F}_p}(E)$ of a supersingular elliptic curve $E/\mathbb{F}_{p^r}$. By [1, Section 9], the number of objects in cases (ii) and (iii) is $\ll p^2$. 

Thus, it suffices to restrict to case (i). By [17], $\text{Aut}_k(\text{Jac}(X)) \simeq \text{Aut}_k(X)$. The arithmetic Torelli map is injective on $\mathbb{F}_p^r$-points representing smooth curves [26, Corollary 12.2]. So for case (i), it suffices to bound the number of supersingular curves $X$ of genus 2 with $\text{Aut}_k(X) \not\simeq \mathbb{Z}/2\mathbb{Z}$, which are described in cases (1)-(6) of Propositions [7,3] and [7,4] when $p > 5$; the cases $p = 3$ and $p = 5$ can be handled similarly. In case (1), there is at most one $k$-isomorphism class of curves, with at most four twists over $\mathbb{F}_p^r$.

In case (2)-(6), the curves are all superspecial by [13, Proposition 1.3]. To bound the number of superspecial curves of genus 2, recall that the singularities of $A_{2,ss}$ are ordinary ($p + 1$)-points which occur precisely at the superspecial points [18, page 193]. By [17, page 154], there are $p^2 + 1$ superspecial points in each irreducible component of $A_{2,ss}$. Using again the fact that the number of irreducible components of $A_{2,ss}$ is $\ll p^2$, the number of superspecial points in $A_{2,ss}(k)$ is $\ll p^2(p^2 + 1)/(p + 1) \ll p^3$. (See [13, Theorem 2] for an exact formula in terms of class numbers.)

Applying [11, Lemma 9.1], the number of $\mathbb{F}_q$-models for superspecial curves of genus 2 is also $\ll p^3$. This completes the proof since $\lim_{r \to \infty} p^3/p^r = 0$. \hfill $\Box$

**Remark 7.6.** The conclusion of Corollary [7,5] is false when $p = 2$, as can be seen from [39, Theorem 3.1], where the authors determine the number of supersingular genus 2 curves $X/\mathbb{F}_q$ when $q = 2^r$ along with $|X(\mathbb{F}_q)|$ and $|\text{Aut}_{\mathbb{F}_q}(X)|$.

8. **Analysis in low dimension: genus 3 curves for $p = 2$**

8.1. **Results for supersingular curves of genus 3 when $p = 2$.**

Let $p = 2$ and $g = 3$. The supersingular locus of the moduli space $M_3 \otimes \mathbb{F}_2$ is irreducible of dimension 2. Viana and Rodriguez [11, pages 56-57] give a parametrization of it by the 2-dimensional family

$$X_{a,b} : x + y + a(x^3y + xy^3) + bx^2y^2 = 0. \quad (20)$$

For each supersingular curve $X_{a,b}$ of genus 3 over a finite field of characteristic 2, we determine whether $X_{a,b}$ is fully maximal, fully minimal, or mixed. This involves an analysis of twists by $g \in \text{Aut}_k(X_{a,b})$, which is a group of order either 12 or 36.

Let $K = \mathbb{F}_2^r$ be the smallest field containing given $a,b \in k^*$. Let $h \in \mathbb{F}_q^*$ be such that $h^2 + h = \frac{a}{b}$. Note that $h \in \mathbb{F}_q$ if and only if $\text{Tr}_r(\frac{a}{b}) = 0$, where $\text{Tr}_r : \mathbb{F}_2^r \to \mathbb{F}_2$ denotes the trace map. Let $K' = \mathbb{F}_q(h)$.

**Theorem 8.1.** Let $X_{a,b}$, $r$ and $h$ be as defined above.

1. If $r$ is odd, then $X_{a,b}$ is fully maximal if $h \in \mathbb{F}_q$ and mixed if $h \not\in \mathbb{F}_q$.
2. If $r \equiv 2 \mod 4$, then $X_{a,b}$ is fully minimal if $h \not\in \mathbb{F}_q$ and mixed if $h \in \mathbb{F}_q$.
3. If $r \equiv 0 \mod 4$, then $X_{a,b}$ is fully minimal.

Moreover, $\text{Jac}(X_{a,b})$ has the same type as $X_{a,b}$, unless $r \equiv 0 \mod 4$ and $h \in \mathbb{F}_q$, in which case $\text{Jac}(X_{a,b})$ is mixed.

The number of $(a, b) \in (\mathbb{F}_q^*)^2$ for which $\text{Tr}_r(\frac{a}{b}) = 1$ is $\frac{(q-1)q}{2}$. This shows that the proportion of $(a, b) \in (\mathbb{F}_q^*)^2$ for which $X_{a,b}$ is mixed is slightly greater than $\frac{1}{2}$ when $r$ is odd and slightly smaller than $\frac{1}{2}$ when $r \equiv 2 \mod 4$.

In fact, we determine $L(X_{a,b}/K, T)$ almost completely in the process of proving Theorem 8.1. In this introduction, for simplicity of exposition, we describe a result found in Proposition 8.7 on the $L$-polynomial of $X_{a,b}$ over $K'$ and refer the reader to Proposition 8.14 for...
the result on the $L$-polynomial of $X_{a,b}$ over $K$. For $K = \mathbb{F}_{2^r}$, define
\begin{equation}
L_{c,K}(T) = (1 - (\sqrt{2}i)^r T)(1 - (-\sqrt{2}i)^r T),
\end{equation}
and, when $r$ is even, define
\begin{equation}
L_{n,K}(T) = (1 - (2\zeta_6)^{r/2} T)(1 - (2\zeta_6^{-1})^{r/2} T).
\end{equation}
The normalized Weil numbers are \{(\pm i)^r\} for $L_{c,K}(T)$ and \{(\zeta_6^{r/2}, \zeta_6^{-r/2})\} for $L_{n,K}(T)$.

**Proposition 8.2.** Let $K' = \mathbb{F}_q(h)$, where $h \in \mathbb{F}_{q^2}$ is such that $h^2 + h = \frac{a}{b}$. Define
\begin{equation}
c_1 = ab, \quad c_2 = \left(\frac{1}{h+1}\right)^2 \frac{1}{b}, \quad \text{and} \quad c_3 = \left(\frac{1}{h}\right)^2 \frac{1}{b}.
\end{equation}
Then $L(X_{a,b}/K', T) = L_{c,K'}(T)^m L_{n,K'}(T)^{3-m}$, where $m = \#\{ i \in \{1, 2, 3\} \mid c_i \text{ is a cube in } (K')^*\}$.

### 8.2. The geometry of $X_{a,b}$

To prove Theorem 8.1, we study the geometry of $X_{a,b}$ and $\text{Aut}_K(X_{a,b})$.

Let $Q_0 \in X_{a,b}$ be the point $(x, y) = (0, 0)$. Let $t_1, t_2$ be the two distinct roots of $f_{a,b}(t) = t^2 + \frac{b}{a} t + 1$, which are defined over either $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$.

**Lemma 8.3.** The embedding of $X_{a,b}$ in $\mathbb{P}^2$ intersects the line at infinity in 4 points: $[1 : 0 : 0]$, $[0 : 1 : 0]$, $P_{\infty_1} = [t_1 : 1 : 0]$, and $P_{\infty_2} = [t_2 : 1 : 0]$.

**Proof.** The embedding of $X_{a,b}$ in $\mathbb{P}^2$ is
\begin{equation}
0 = xz^3 + yz^3 + a(x^3y + xy^3) + bx^2y^2.
\end{equation}
When $z = 0$, then $0 = xy(ax^2 + ay^2 + bxy)$, giving the 2 points $[1 : 0 : 0]$ and $[0 : 1 : 0]$. If $xy \neq 0$, then $ax^2 + bxy + ay^2$ simplifies to $f_{a,b}(t) = t^2 + \frac{b}{a} t + 1 \in \mathbb{F}_q[t]$ where $t = \frac{x}{y}$. Note that $f_{a,b}(t)$ is separable since $f_{a,b}'(t) = \frac{b}{a} \neq 0$. Thus $f_{a,b}(t)$ has two distinct roots $t_1, t_2 \in \mathbb{F}_{q^2}$.  

Let $E_{a,b}$ denote the supersingular elliptic curve with affine equation
\begin{equation}
E_{a,b} : R^2 + R = (ab)S^3.
\end{equation}
Let $P_0 \in E_{a,b}$ be the point $(R, S) = (0, 0)$ and let $P_{\infty} \in E_{a,b}$ be the point at infinity.

**Lemma 8.4.**

1. The curve $X_{a,b}$ has an involution $\tau$ with exactly one fixed point $Q_0$.
2. The quotient of $X_{a,b}$ by $\tau$ is isomorphic to $E_{a,b}$.
3. An equation for the $\mathbb{Z}/2\mathbb{Z}$-cover $\phi : X_{a,b} \to E_{a,b}$ is
\begin{equation}
Z^2 + Z = \frac{a}{b} R.
\end{equation}
4. The cover $\phi$ is branched only over the point $P_{\infty}$. Above $P_{\infty}$, the filtration of higher ramification groups for $\phi$ breaks at index $j = 3$.

**Proof.**

1. Consider the involution $\tau$ of $X_{a,b}$ given by $x \mapsto y$ and $y \mapsto x$. If $\tau$ fixes $(x, y)$ then $x = y$ which implies $bx^4 = 0$. So $Q_0$ is the only affine fixed point of $\tau$. Also, $\tau$ permutes the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ and permutes the other two points $P_{\infty_1}$ and $P_{\infty_2}$ from Lemma 8.3.

2. One can check that the $\tau$-invariant functions $R$ and $S$ below satisfy (24):
\begin{equation}
R = 1 + \frac{1}{axy(x+y)}, \quad \text{and} \quad S = \frac{1}{a(x+y)}.
\end{equation}
(3) Let \( Z = \frac{x}{x+y} \) and compute directly.

(4) By part (3), \( \phi \) is branched only at \( P_\infty \). Also, \( j = 3 \) because \( R \) has a pole of order 3 at \( P_\infty \). For a second proof that \( j = 3 \), note that \( 4 = 2g_X - 2 = 2(2g_E - 2) + \deg(\text{Ram}) \) by the Riemann-Hurwitz formula. Above the unique branch point \( P_\infty \), the cover is wildly ramified with inertia \( Z/2Z \). If the higher ramification groups are all trivial after index \( j \), then \( \deg(\text{Ram}) = (2 - 1)(j + 1) \). Thus \( j = 3 \).

The points \([0 : 1 : 0]\) and \([1 : 0 : 0]\) have \( Z = 0 \) and \( Z = 1 \) and lie over \((R, S) = (0, 0)\); the point \( Q_0 \) lies over \( P_\infty \); and the points \( P_\infty_1 \) and \( P_\infty_2 \) lie over the point \((R, S) = (1, 0)\) on \( E_{a,b} \).

**Lemma 8.5.** The splitting behavior of \( P \in E_{a,b}(\mathbb{F}_{q^m}) \) in the cover \( \phi : X_{a,b} \to E_{a,b} \) is:

1. \( P_\infty \) is ramified and \((0, 0)\) is split and \((1, 0)\) is split if and only if \( \text{Tr}_r(\frac{a}{b} R) = 0 \).
2. For an affine point \( P = (R, S) \in E_{a,b}(\mathbb{F}_{q^m}) \) with \( S \neq 0 \): \( P \) is split if and only if \( \text{Tr}_r(\frac{a}{b} R) = 0 \); \( P \) is inert if and only if \( \text{Tr}_r(\frac{a}{b} R) = 1 \).

**Proof.**

1. In the cover \( X_{a,b} \to E_{a,b} \): the point \( P_\infty \) is ramified since \( R \) has a pole there; the point \((0, 0)\) is split because two points lie above it; and the point \((1, 0)\) is split or inert by the same argument as in part (2).

2. By Hilbert’s Theorem 90, for a point \( P = (R, S) \in E_{a,b}(\mathbb{F}_{q^m}) \), \( P \) is split if and only if \( \text{Tr}_r(\frac{a}{b} R) = 0 \) and \( P \) is inert if and only if \( \text{Tr}_r(\frac{a}{b} R) = 1 \).

Recall the definition of \( c_1, c_2, c_3 \) from (23). We define

\[
E_1 : R^2 + R = c_1 S^3, \quad E_2 : T^2 + T = c_2 (aS)^3, \quad E_3 : U^2 + U = c_3 (aS)^3.
\]

(26)

The involution \( \nu : R \mapsto R + 1 \) on \( E_{a,b} \) lifts to \( X_{a,b} \), using the formula \( \nu(Z) = Z + h \).

**Lemma 8.6.**

1. The cover \( X_{a,b} \to E_{a,b} \to \mathbb{P}^1_S \) is Galois with group \( S_0 = \langle \tau, \nu \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and generalized Artin-Schreier equation

\[
Z^4 + (1 + \frac{a}{b})Z^2 + \frac{a}{b} = Z = \frac{1}{b} a^3 S^3.
\]

(27)

2. Over \( K' \), the quotients of \( X_{a,b} \) by \( \tau, \nu \) and \( \tau \nu \) are respectively \( E_1, E_2, \) and \( E_3 \).

3. Finally, \( \text{Jac}(X_{a,b}) \sim K' \) \( E_1 \oplus E_2 \oplus E_3 \).

**Proof.**

1. The involution \( \tau \in \text{Aut}_{\mathbb{F}_2}(X_{a,b}) \) satisfies \( \tau(Z) = Z + 1 \). The last equation follows from (24) and (25).

2. By Lemma 8.4 \( X_{a,b}/\langle \tau \rangle \simeq E_{a,b} = E_1 \). The involutions \( \nu \) and \( \tau \nu \) fix the functions \( T_1 = Z(Z + h) \) and \( U_1 = Z(Z + (h + 1)) \) respectively. A direct calculation shows that

\[
T_1^2 + (h + 1)T_1 = Z^4 + h^2 Z^2 + (h + 1)(Z^2 + hZ)
\]

\[
\quad = (Z^2 + Z)^2 + (h^2 + h)(Z^2 + Z) = (\frac{a}{b})^2 (R^2 + R) = \frac{1}{b} (aS)^3.
\]

Similarly, one can show that \( U_1^2 + h U_1 = \frac{1}{b} (aS)^3 \). Consider the changes of variables \( T_1 = (h + 1)T \) and \( U_1 = hU \), which are defined over \( K' \). Then

\[
T^2 + T = (\frac{1}{h + 1})^2 \frac{1}{b} (aS)^3, \quad \text{and} \quad U^2 + U = (\frac{1}{h})^2 \frac{1}{b} (aS)^3.
\]

(28)

3. This is immediate from part (2) and [16, Theorem B].
8.3. The $L$-polynomial of $X_{a,b}$ over $K'$.

We prove the next result (which is equivalent to Proposition 8.2) using the Galois action on $X_{a,b}$ and the structure of the $L$-polynomial $L(E_{a,b}/\mathbb{F}_q, T)$.

Proposition 8.7. Let $K' = \mathbb{F}_q$ if $\text{Tr}(\alpha) = 0$ and $K' = \mathbb{F}_{q^2}$ if $\text{Tr}(\alpha) = 1$. Let $c_1 = ab$, $c_2 = (\frac{1}{3})^2b$ and $c_3 = (\frac{1}{3})^2b$. Let $L_i = L_{c,K'}(T)$ from (21) if $c_i$ is a cube in $(K')^*$ and $L_i = L_{a,K'}(T)$ from (22) if $c_i$ is not a cube in $(K')^*$. Then $L(X_{a,b}/K', T) = L_1L_2L_3$.

Proof. Recall that $S_0 = \langle \tau, v \rangle \subset \text{Aut}_{K'}(X_{a,b})$ and $S_0 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $X_{a,b}/S_0 \simeq \mathbb{P}^1$. It follows from Lemma 8.11 that $L(X_{a,b}/K', T) = L_1L_2L_3$ where $L_i = L(E_i/K', T)$. By Lemma 8.11 (below), $E_1 \simeq_{K'} E_1, \text{if } c_1$ is a cube in $(K')^*$ and $E_1 \simeq_{K'} E_1, \text{for some } \epsilon \in \mathbb{F}_4 - \mathbb{F}_2$ if $c_1$ is not a cube in $(K')^*$. The result follows for $L_1$ from Lemmas 8.11 and 8.12 (below). When $i = 2, 3$, the proof for $L_i$ is almost identical; compare (28) for $E_i$ to (24) for $E_1$. □

Notation 8.8. Let $\mathbb{F}_q = \mathbb{F}_{2^r}$ be the smallest field containing $a$ and $b$. Fix a labeling of the elements $\alpha, \alpha' \in \mathbb{F}_4 - \{0, 1\}$, which are the roots of $x^2 + x + 1$. Define $\epsilon \in \{1, \alpha, \alpha'\}$ and $c \in \mathbb{F}_q^*$ as follows. When $q \equiv 2 \mod 3$ (r odd), then every element of $\mathbb{F}_q^*$ is a cube; let $\delta$ be such that $ab = \delta^3$ and let $\epsilon = 1$. When $q \equiv 1 \mod 3$ (r even), then there are 3 cosets of $(\mathbb{F}_q^*)^3$ in $\mathbb{F}_q^*$; let $\delta \in \mathbb{F}_q^*$ and $\epsilon \in \{1, \alpha, \alpha'\}$ be such that $ab = \delta^3\epsilon$.

The next lemma allows us to work on $E_{1,\epsilon}$ without loss of generality.

Lemma 8.9. (1) There is an isomorphism $\phi_\epsilon : E_{a,b} \rightarrow E_{1,\epsilon}$ defined over $\mathbb{F}_q$.
(2) Let $P = (R, S) \in E_{a,b}(k)$ and let $P_\epsilon = \phi_\epsilon(P) = (R_\epsilon, S_\epsilon) \in E_{1,\epsilon}(k)$. Then $P$ is defined over $\mathbb{F}_{q^m}$ if and only if $P_\epsilon$ is.

Proof. (1) Write $E_{1,\epsilon} : R_\epsilon^2 + R_\epsilon = \epsilon S_\epsilon^3$. Given $(R, S) \in E_{a,b}$, let $\phi_\epsilon(R, S) = (R_\epsilon, S_\epsilon)$ where $R_\epsilon = R$ and $S_\epsilon = S\delta$. One can check that this gives an isomorphism.
(2) This is true since $\phi_\epsilon$ is defined over $\mathbb{F}_q$.
□

Lemma 8.10. (1) The curve $E_{1,1} : R^2 + R = S^3$ is maximal over $\mathbb{F}_{2^r}$ and $L(E_{1,1}/\mathbb{F}_2, T) = 1 + 2T^2 = (1 - (\sqrt{2}i)T)(1 - (-\sqrt{2}i)T)$.
(2) Let $\epsilon \in \{\alpha, \alpha'\} = \mathbb{F}_4 - \{0, 1\}$. Then $|E_{1,\epsilon}(\mathbb{F}_4)| = 3$, the trace of Frobenius is 2 and $L(E_{1,\epsilon}/\mathbb{F}_4, T) = 1 - 2T + 4T^2 = (1 - (2\zeta_6)T)(1 - (2\zeta_6^{-1})T)$.

Proof. Omitted. □

Lemma 8.11. Suppose that $a, b \in \mathbb{F}_q$ with $q = 2^r$. Suppose that $ab$ is a cube in $\mathbb{F}_q^*$. Then $E_{a,b}$ has normalized Weil numbers $i^r$ and $-i^r$ and $L(E_{a,b}/\mathbb{F}_{2^r}, T) = L_{c,\mathbb{F}_q}(T)$ from (21).

Proof. By hypothesis, $ab = \delta^3$ and $\epsilon = 1$. By Lemma 8.9, $\phi_1 : E_{a,b} \rightarrow E_{1,1}$ is an isomorphism defined over $\mathbb{F}_q$, so the result is immediate from Lemma 8.10.(1). □

Lemma 8.12. Suppose that $a, b \in \mathbb{F}_q$ with $q = 2^r$ and $ab$ is not a cube in $\mathbb{F}_q$. Then $E_{a,b}$ has normalized Weil numbers $\{\zeta_6^{r/2}, \zeta_6^{-r/2}\}$ and $L(E_{a,b}/\mathbb{F}_{2^r}, T) = L_{c,\mathbb{F}_q}(T)$ from (22).

Proof. By hypothesis, $ab = \delta^3\epsilon$ for some $\epsilon \in \mathbb{F}_4 - \mathbb{F}_2$. By Lemma 8.9, $\phi_\epsilon : E_{a,b} \rightarrow E_{1,\epsilon}$ is an isomorphism defined over $\mathbb{F}_q$, so the result is immediate from Lemma 8.10.(2). □
8.4. The $L$-polynomial of $X_{a,b}$ over $K$.

When $h \not\in \mathbb{F}_q$, Proposition 8.7 is not quite strong enough to prove Theorem 8.1 because it only gives information about the $L$-polynomial over $\mathbb{F}_q$. In this section, we determine more information using the Artin $L$-series $L(E_{a,b}/\mathbb{F}_q, T, \chi)$, where $\chi$ is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$. By [20, Chapter 9, page 130],

$$L(X_{a,b}/\mathbb{F}_q, T) = L(E_{a,b}/\mathbb{F}_q, T)L(E_{a,b}/\mathbb{F}_q, T, \chi). \quad (29)$$

Let $\rho_1$ be the coefficient of $T$ in $L(E_{a,b}/K, T, \chi)$. Let $I_1$ (resp. $S_1$) be the number of $K$-points of $E_{a,b}$ that are inert (resp. split) in $X_{a,b}$. By Lemma 8.15(2), $\rho_1 = S_1 - I_1$. The next lemma is a special case of Remark 5.6 but we prove it directly for clarity.

**Lemma 8.13.** If $Tr_r(\frac{a}{b}) = 1$, then $\rho_1 = 0$.

**Proof.** It suffices to work on $E_{1,\epsilon}$ by Lemma 8.22. By Lemma 8.5(1), in the cover $X_{a,b} \to E_{a,b}$, the point $P_\infty$ is ramified, $(0,0)$ is split, and $(1,0)$ is inert because $Tr_r(\frac{a}{b}) = 1$. So it suffices to restrict to the set $Y_{1,\epsilon}$ of affine points of degree 1 on $E_{1,\epsilon}$ with $S \neq 0$. The points of $Y_{1,\epsilon}$ can be arranged in pairs $(P_e, v(P_e))$. Then

$$Tr_r(\mathcal{F}_e(P_e)) = Tr_r(\frac{a}{b}R_e) \text{ while } Tr_r(\mathcal{F}_e(v(P_e))) = Tr_r(\frac{a}{b}R_e) + 1.$$ 

By Lemma 8.3(2), in $(P_e, v(P_e))$, one point is split and one is inert. So $\rho_1 = S_1 - I_1 = 0$. \quad $\square$

**Proposition 8.14.** Let $K = \mathbb{F}_q$ where $q = p^r$. Let $K' = K(h)$ where $h$ is such that $h^2 + h = \frac{a}{b}$. Let $m$ be the number of $c_1, c_2, c_3$ from (23) that are cubes in $(K')^*$. The normalized Weil numbers and the 2-valuation vector $\varepsilon(X_{a,b}/K) = \{e_1, e_2, e_3\}$ are determined below.

| Case for $h$ | Case for $r$ | NWNs/K | $\varepsilon$ |
|-------------|-------------|--------|--------------|
| $h \in \mathbb{F}_q$ | $r$ odd | $\pm i$ (3 times) | $\{2, 2\}$ |
| $h \in \mathbb{F}_q$ | $r \equiv 2 \mod 4$ | $-1, -1$ (m times), $\zeta_6, \zeta_6^{-1}$ (3 - m times) | $\{1, 1\}$ |
| $h \in \mathbb{F}_q$ | $r \equiv 0 \mod 4$ | $1, 1$ (m times), $\zeta_3, \zeta_3^{-1}$ (3 - m times) | $\{0, 0\}$ |
| $h \not\in \mathbb{F}_q$ | $r \equiv 3 \mod 6$ | $\pm i$ (3 times) | $\{2, 2\}$ |
| $h \not\in \mathbb{F}_q$ | $r \equiv 1, 5 \mod 6$ | $\pm i$ (m times), $\pm \zeta_{12}, \pm \zeta_{12}$ (3 - m)/2 times | $\{2, 2\}$ |
| $h \not\in \mathbb{F}_q$ | $r \equiv 2 \mod 4$ | $-1, -1$ or $\zeta_6, \zeta_6^{-1}$, and either $\pm 1, \pm 1$ or $\pm \zeta_3, \pm \zeta_3$ | $\{1, 0, 1\}$ |
| $h \not\in \mathbb{F}_q$ | $r \equiv 0 \mod 4$ | $1, 1$ or $\zeta_3, \zeta_3$, and either $\pm 1, \pm 1$ or $\pm \zeta_3, \pm \zeta_3$ | $\{0, 0\}$ |

**Remark 8.15.** Let $m$ be the number of $c_1, c_2, c_3$ that are cubes in $(K')^*$. When $h \in \mathbb{F}_q$ and $r$ is odd, then $m = 3$. When $h \not\in \mathbb{F}_q$, then $c_1 = ab$ is a cube in $(K')^*$ and $c_2$ and $c_3$ are $\mathbb{F}_q$-Frobenius conjugate, so $m = 1$ or $m = 3$.

**Proof.** When $h \in \mathbb{F}_q$, the normalized Weil numbers over $K = K'$ are $(\pm i)^r$ with multiplicity $m$ and $\zeta_6^r, \zeta_6^{-r}$ with multiplicity $3 - m$ by Proposition 8.2. If $r$ is odd, then $m = 3$. If $r$ is even, the value of $m$ does not affect $\varepsilon(X_{a,b}/K)$.

When $h \not\in \mathbb{F}_q$, the normalized Weil numbers of $X_{a,b}$ over $K$ are square roots of those over $K'$, which are $(\pm i)^{2r} = (-1)^r$ with multiplicity $m$ and $\zeta_6^{2r}, \zeta_6^{-2r}$ with multiplicity $3 - m$ by Proposition 8.2. If $r$ is odd with $r \equiv 3 \mod 6$, then the square roots are $\pm i$ with multiplicity 3. If $r$ is odd with $r \equiv 1, 5 \mod 6$, then the square roots are $\pm i$ with multiplicity $m$ and $3 - m$ conjugate pairs of roots of unity of order 12, which all sum to zero by Lemma 8.13.

If $r$ is even, taking square roots introduces some ambiguity, which is partially resolved by Lemma 8.13. The four normalized Weil numbers for $L(E_{a,b}/\mathbb{F}_q, T, \chi)$ sum to zero and thus
must be either \( \{1,1,-1,-1\} \) or \( \{\zeta_3, \bar{\zeta}_3, \zeta_6, \bar{\zeta}_6\} \). The normalized Weil numbers for \( E_{a,b}/K \) are found in Lemmas 8.11 and 8.12. In all cases, these data determine \( \zeta(X_{a,b}/K) \).

The next example was done by calculation. Parts of cases (1) and (4) are in 41, page 57. Cases (3)-(5) show that numerous outcomes are possible when \( h \not\in \mathbb{F}_q \) and \( r \equiv 2 \mod 4 \).

**Example 8.16.** Write \( \mathbb{F}_q = \{0,1,\alpha,\alpha^2\} \). Let \( P \) be the period and \( \delta \) the parity of \( X_{a,b}/K \).

Write \( V_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

| Case | Conditions on \( r, a, \) and \( b \) | NWN’s | \( P \) | \( \delta \) | \( \text{Aut}_k(X_{a,b}) \) | Type |
|------|---------------------------------|--------|--------|--------|-----------------|------|
| 1    | \( r = 1, a = b = 1 \)         | \( \pm i, \pm \zeta_12, \pm \zeta_12 \) | 6      | 1      | \( V_4 \times \mathbb{Z}/9\mathbb{Z} \) | fmax |
| 2    | \( r = 2, a = b \in \{\alpha,\alpha^2\} \) | \( -1, -1, \zeta_6, \bar{\zeta}_6, \zeta_3, \bar{\zeta}_3 \) | 3      | 1      | \( V_4 \times \mathbb{Z}/9\mathbb{Z} \) | mixed |
| 3    | \( r = 2, \{a,b\} = \{\alpha,\alpha^2\} \) | \( -1, -1, \zeta_6, \bar{\zeta}_6, \zeta_3, \bar{\zeta}_3 \) | 6      | -1     | \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \) | fmin |
| 4    | \( r = 2, a \in \{\alpha,\alpha^2\}, b = 1 \) | \( \zeta_6, \bar{\zeta}_6, \zeta_3, \bar{\zeta}_3, \zeta_6, \bar{\zeta}_6 \) | 6      | -1     | \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \) | fmin |
| 5    | \( r = 2, a = 1, b \in \{\alpha,\alpha^2\} \) | \( \zeta_6, \bar{\zeta}_6, 1, 1, -1, -1 \) | 6      | -1     | \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \) | fmin |

**8.5. The automorphism group of \( X_{a,b} \).**

Let \( G = \text{Aut}_k(X_{a,b}) \). Recall that \( G \) contains the commuting involutions \( \tau \) and \( v \), where

\[
\tau : (S, R, Z) \mapsto (S, R, Z + 1) \quad \text{and} \quad v : (S, R, Z) \mapsto (S, R + 1, Z + h).
\]

(30)

Now \( \tau \) is defined over \( K = \mathbb{F}_q \) and \( v \) is defined over \( K' \). Let \( S_0 = \langle \tau, v \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

Consider the order 3 automorphism of \( X_{a,b} \), given by

\[
\sigma : (x, y) \mapsto (\zeta_3 x, \zeta_3 y) \quad \text{or} \quad (S, R, Z) \mapsto (\zeta_3^2 S, R, Z).
\]

(31)

Then \( \sigma \) fixes \( Q_0 = (0,0) \) and each of the four points from Lemma 8.3, so \( X_{a,b}/\langle \sigma \rangle \simeq \mathbb{P}^1 \). Note that \( \sigma \) is defined over \( \mathbb{F}_q \) if \( r \) is even and over \( \mathbb{F}_q^2 \) if \( r \) is odd. Also, \( \sigma \) centralizes \( S_0 \).

**Lemma 8.17.** If \( a \neq b \), then \( G = S_0 \times \langle \sigma \rangle \) is an abelian group of order 12. If \( a \neq b \), then \( G \) is a semidirect product of the form \( S_0 \rtimes H \) where \( H \) is a cyclic group of order 9.

**Proof.** The degree 4 equation \( [27] \) for \( X_{a,b} \) is of the type whose automorphism group is studied in [34], see also [10] Section 12.1. We briefly sketch the argument. By [10] Theorem 12.11, \( G \) fixes the point \( Q_0 \) (the point lying above \( S = \infty \)). Thus \( G \simeq S_1 \rtimes H \) where \( S_1 \) is the normal Sylow 2-subgroup of \( G \) and \( H \) is a cyclic group of odd order. By [10] Theorem 12.7, \( |S_1| = 4 \) (so \( S_1 = S_0 \)) and \( |H| \) divides 9. Then \( |H| = 3 \) or 9 since \( \sigma \in G \).

If \( H \) contains an element \( \kappa \) of order 9, then \( \kappa(S) = \zeta_9 S \). Hence, \( \kappa \) acts on the right hand side of \( [27] \) by multiplication by \( \zeta_3 \). However, \( \kappa \) can only act on the left hand side of \( [27] \) by multiplication by \( \zeta_3 \) if the monomial \( (1 + \frac{a}{b}) x^2 \) vanishes. Thus, \( \kappa \) lifts to an automorphism of \( X_{a,b} \) if and only if \( \frac{a}{b} = 1 \), in which case \( \kappa(Z) = \zeta_3 Z \) and

\[
\kappa : (S, R, Z) \mapsto (\zeta_9 S, R, \zeta_3 Z).
\]

(32)

\( \square \)

If \( \frac{a}{b} = 1 \) and \( |H| = 9 \), note that \( \kappa^3 = x^2 \); also \( G \) is non-abelian, since \( \kappa \tau \kappa^{-1}(Z) = Z + \zeta_3^{-1} \), so \( \kappa \tau \kappa^{-1} \) is either \( v \) or \( v \tau \), depending on the choice of \( h \in \{\zeta_3, \zeta_3^3\} \). In this case, \( \kappa \) permutes the three quotients \( E_1, E_2, E_3 \) of \( X_{a,b} \) by the involutions in \( S_0 \).

Let \( Fr = Fr_K \) where \( K = \mathbb{F}_q \). We now determine the \( K \)-Frobenius conjugacy classes of \( G \).

**Lemma 8.18.** Let \( f \) be the number of \( K \)-Frobenius conjugacy classes in \( G \).

(1) Suppose that \( a \neq b \). Then \( G \) is an abelian group of order 12.

(2) Suppose that \( a = b \). Then \( G \) is a semidirect product of the form \( S_0 \rtimes H \) where \( H \) is a cyclic group of order 9.
(a) If \( r \) is even and \( h \in \mathbb{F}_q \), then \( f = 12 \).
(b) If \( r \) is even and \( h \notin \mathbb{F}_q \), then \( f = 6 \).
   The classes are \( \{\text{id}, \tau\}, \{v, v\tau\}, \{\sigma, \sigma\tau\}, \{\sigma v, \sigma v\tau\}, \{\sigma^2, \sigma^2\tau\}, \{\sigma^2 v, \sigma^2 v\tau\} \).
(c) If \( r \) is odd and \( h \in \mathbb{F}_q \), then \( f = 4 \).
   The classes are \( \{\text{id}, \sigma^2\}, \{v, v\sigma, v\sigma^2\}, \{\tau, \tau\sigma, \tau\sigma^2\}, \{\text{id}, \tau\sigma, \tau\sigma^2\} \).
(d) If \( r \) is odd and \( h \notin \mathbb{F}_q \), then \( f = 2 \).
   The classes are \( \{\text{id}, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\} \) and \( \{v, v\sigma, v\sigma^2, \tau, \tau\sigma, \tau\sigma^2\} \).

(2) If \( a = b \), then \( G \) is a non-abelian group of order 36 and \( h \in \mathbb{F}_4 - \mathbb{F}_2 \).
(a) If \( r \) is even, then \( h \in \mathbb{F}_q \) and \( f = 10 \).
   The classes are \( \{\text{id}, \{v, v\tau\}, \{\{\kappa^j, v\kappa^j, \tau\kappa^j, v\tau\kappa^j\}\} \) for \( j = 1, \ldots, 8 \).
(b) If \( r \) is odd, then \( h \notin \mathbb{F}_q \) and \( f = 2 \). Also, \( v \) is not conjugate to \( \text{id} \).
   The first class is \( \{\text{id}, \kappa, \kappa^2, \ldots, \kappa^8, v\kappa, v\kappa^2, \tau\kappa^3, v\tau\kappa^4, v\kappa^5, \tau\kappa^6, v\tau\kappa^7, v\kappa^8\} \).

Proof. We omit most of the long calculation. Cases (1a) and (2a) follow from Remark 3.5 which states that \( K \)-Frobenius conjugacy classes coincide with standard conjugacy classes when all automorphisms are defined over \( K \).

For other cases, note that \( Fr\tau = \tau \). If \( h \in \mathbb{F}_q \), then \( Frv = v \). If \( h \notin \mathbb{F}_q \), then \( h^2 = h + 1 \) and \( Frv = v\tau \); in this case, \( v^{-1} \tau (Frv) = \text{id} \), showing that \( \tau \) is \( K \)-Frobenius conjugate to \( \text{id} \), and \( v \) is \( K \)-Frobenius conjugate to \( v\tau \).

Also, \( Fr\kappa = \kappa^8 \). If \( r \) is even, then \( Fr\sigma = \sigma \). If \( r \) is odd, then \( Fr\sigma = \sigma^{-1} \); in this case, \( \sigma^{-1}\text{id}(Fr\sigma) = \sigma \), showing that \( \sigma \) is \( K \)-Frobenius conjugate to \( \text{id} \). \( \square \)

8.6. Proof of Theorem 8.1

Proof of Theorem 8.1. First, note that the results from Remark 5.6 apply here, by setting \( H = S_0 \). By Lemma 8.6(3), \( \text{Jac}(X_{ab}) \cong K' E_1 \oplus E_2 \oplus E_3 \). The automorphisms in \( G \) act on the normalized Weil numbers of \( X_{ab} \) over \( K' \). By Lemma 8.6(2), \( \tau \) acts trivially on \( E_1 \) and by \([-1]\) on \( E_2 \) and \( E_3 \). Similarly, \( v \) fixes \( E_2 \) and acts nontrivially on \( E_1 \) and \( E_3 \), and \( v\tau \) fixes \( E_3 \) and acts nontrivially on \( E_1 \) and \( E_2 \).

When \([K' : K] = 2\), the strategy in the proof below is to analyze the situation for the base change to \( K' \), where the automorphism \( g \) acts via \( g^{Fr\kappa}g \). The ambiguity caused by descending to \( K \) can be resolved using Lemma 8.13.

In each case below, the information on the normalized Weil numbers of \( X_{ab} \) over \( K = \mathbb{F}_q \) and their 2-adic valuations \( \epsilon = \{e_1, e_2, e_3\} \) is from Proposition 8.14. The data on the number and representatives of the \( K' \)-twists of \( X_{ab} \) is found in Lemma 8.18.

(1) Let \( r \) be odd. Then \( \epsilon = \{2, 2, 2\} \) so \( X_{ab} \) is maximal.
   (a) If \( h \in \mathbb{F}_q \), then there are three nontrivial twists, each of order 2. By Lemma 4.6 none of these change \( \epsilon \), so \( X_{ab} \) is fully maximal.
   (b) If \( h \notin \mathbb{F}_q \), the nontrivial \( K \)-twist is represented by \( v \). Also, \( K' = \mathbb{F}_{q^2} \), since \( v \) is not defined over \( \mathbb{F}_q \). Then \( \epsilon(X_{ab}/K') = \{1, 1, 1\} \). Over \( K' \), the nontrivial twist corresponds to \( v^{Fr\kappa}v = \tau \), which negates the two conjugate pairs of normalized Weil numbers for \( E_2 \) and \( E_3 \), thus the twist has \( \epsilon(X_{ab}/K') = \{1, 0, 0\} \). By Lemma 8.13 \( \epsilon(X_{ab}/K) = \{2, 0, 1\} \), of parity \(-1\). Thus, \( X_{ab} \) is mixed.
   In addition, \( \text{Jac}(X_{ab}) \) and \( X_{ab} \) have the same type, since the extra twist by \([-1]\) negates all three pairs of Weil numbers over \( \mathbb{F}_q \), which does not change \( \epsilon \).

(2) Let \( r \equiv 2 \mod 4 \).
(a) If \( h \in \mathbb{F}_q \), then \( \mathcal{E} = \{1,1,1\} \), so \( X_{a,b} \) has parity +1. There are either twelve \( K \)-twists (if \( a \neq b \)) or ten \( K \)-twists (if \( a = b \)). In both cases, the \( K \)-twist by \( v \) has \( \mathcal{E} = \{1,0,0\} \) and parity −1. Hence, both \( X_{a,b} \) and \( \text{Jac}(X_{a,b}) \) are mixed.

(b) If \( h \notin \mathbb{F}_q \), then \( \mathcal{E} = \{1,0,1\} \), so \( X_{a,b} \) has parity −1. Since \( a \neq b \), there are six \( K \)-twists, represented by id, \( v \), \( \sigma \), \( v\sigma \), \( \sigma^2 \), and \( v\sigma^2 \). The decomposition \( \text{Jac}(X_{a,b}) \sim K E_{a,b} \oplus A \) is stabilized by \( v \). The twist of \( E_{a,b} \) by \( v \) does not change \( e_1 = 1 \). The twist of \( A/K \) by \( v \) corresponds to the twist of \( A/K' \) by \( \tau \), which changes \( \mathcal{E}(A/K') = \{0,0\} \) to \( \{1,1\} \). So the \( K \)-twist for \( v \) has \( \mathcal{E} = \{1,2,2\} \) and parity −1. Further, \( \sigma, \sigma^2 \) have odd order and are defined over \( K \), so twisting by them does not change the parity, by Corollary 4.8. Since \( v \) and \( \sigma \) commute, all twists have parity −1 and \( X_{a,b} \) is fully minimal. The twist by \([-1]\) has \( \mathcal{E} = \{0,1,0\} \), thus \( \text{Jac}(X_{a,b}) \) is fully minimal as well.

(3) Let \( r \equiv 0 \mod 4 \).

(a) If \( h \in \mathbb{F}_q \), then \( \mathcal{E} = \{0,0,0\} \), so \( X_{a,b} \) has parity −1. There are either twelve \( K \)-twists (if \( a \neq b \)) or ten \( K \)-twists (if \( a = b \)). The nontrivial elements of \( S_0 \) yield twists such that \( \mathcal{E} = \{1,1,0\} \), of parity −1. As in case (2b), the elements \( \sigma^j \) do not change the parity. If \( a = b \), then the twists by \( \kappa^j \) permute \( E_1, E_2, E_3 \) and thus do not change the parity either. Hence, \( X_{a,b} \) is fully minimal. Since \( \text{Jac}(X_{a,b}) \) has a twist with \( \mathcal{E} = \{1,1,1\} \) and parity +1, it is mixed.

(b) If \( h \notin \mathbb{F}_q \), then \( \mathcal{E} = \{0,0,1\} \), so \( X_{a,b} \) has parity −1. The proof is the same as for case (2b), except that the \( K \)-twist for \( v \) has \( \mathcal{E} = \{0,2,2\} \) and the twist by \([-1]\) has \( \mathcal{E} = \{1,1,0\} \). Both \( X_{a,b} \) and \( \text{Jac}(X_{a,b}) \) are fully minimal.

\[ \square \]

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