A General Class of Weighted Rank Correlation Measures

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Abstract

In this paper we propose a class of weighted rank correlation coefficients extending the Spearman’s rho. The proposed class constructed by giving suitable weights to the distance between two sets of ranks to place more emphasis on items having low rankings than those have high rankings or vice versa. The asymptotic distribution of the proposed measures and properties of the parameters estimated by them are studied through the associated copula. A simulation study is performed to compare the performance of the proposed statistics for testing independence using asymptotic relative efficiency calculations.

1 Introduction

Many situations exist in which \( n \) objects are ranked by two independent sources, where the interest is focused on agreement on the top rankings and disagreements on items at the bottom of the rankings, or vice versa. For example, every year a large number of students apply for higher education. The graduate committee of the university may like to choose the best candidates based on some criteria such as GPA and the average of their grades in the major courses that they passed during their bachelors level. In such cases, to minimize the cost of interviewing all of the candidates, a measure which gives more weighted for those who have higher grades is required. Measures of rank correlation such as the Spearman’s rho and Kendall’s tau generally give a value for the overall agreement without giving explicit information about those parts of a data set which are similar. This problem motivates the definition of the Weighted Rank Correlation (WRC) measures which emphasize items having low rankings and de-emphasize those having high rankings, or vice versa. Salama and Quade (\cite{17}, \cite{15}) first studied the WRC of two sets of rankings, sensitive to agreements in the top rankings and ignore disagreements on the rest items in a certain degree. For application in the sensitivity analysis, Iman and Conover \cite{8} proposed the top-down concordance coefficient which centres on the agreement in the top rankings.
Shieh [19] studied a weighted version of the Kendall’s tau which could place more emphasis on items having low rankings than those have high rankings or vice versa. Blest [2] introduced a rank correlation measure which gives more weights to the top rankings. Pinto da Costa and Soares [14] proposed a WRC measure that weights the distance between two ranks using a linear function of those ranks, giving more importance to higher ranks than lower ones. Maturi and Abdelfattah [11] proposed a WRC measure with the different weights to emphasize the agreement of the top rankings. Coolen-Maturi [3] extended this index to the more than two sets of rankings but again the focus was only on the agreement on the top or bottom rankings. The behaviour of several WRC measures derived from Spearman’s rank correlation was investigated by Dancelli et al. [4]. Starting from the formula of the Spearman’s rank correlation measure, this paper proposed a general class of WRC measures that weight the distance between two sets of ranks. Two classes of weights, which are polynomial functions of the ranks, are considered to place more emphasis the items having low rankings than those have high rankings or vice versa. The first one which extends the Blest’s rank correlation places more emphasis to the agreement on the top rankings. The second one constructs a new class of WRC measures and places more emphasis on the bottom ranks. The rest of the paper is organized as follows. The proposed WRC measures are introduced in Section 2. The weighted correlation coefficients which are the population versions of the proposed WRC measures are introduced in Section 3. The quantiles of the proposed WRC measures for small samples and their asymptotic distributions for large samples are presented in Section 4. A simulation study is performed to compare the performance of the proposed statistics for testing independence by using the asymptotic relative efficiency and the empirical powers of the tests, in Section 5. Finally, some discussions and possible extensions are given in Section 6.

2 The Proposed WRC measures

Let \((X_1, Y_1), \ldots, (X_n, Y_n)\) be a random sample of size \(n\) from a continuous bivariate distribution and let \((R_1, S_1), \ldots, (R_n, S_n)\) denote the corresponding vectors of ranks. The well-known Spearman’s rank correlation measure is given by

\[
\rho_{n,s} = \frac{12}{n^3 - n} \sum_{i=1}^{n} R_i S_i - \frac{3(n+1)}{n-1}.
\]

A drawback of the Spearman’s rank correlation is that it generally gives a value for the overall agreement of two sets of ranks without giving explicit information about those
parts of the sets which are similar. For example consider 3 consumers A, B and C that ranked 9 aspects of a product attributing '1' to the most important aspect and '9' to the least important one. Their rankings are given in Table 1. As we see, the top ranks of (A,B) are more similar than those of (A,C) and the bottom ranks of (A,C) are more similar than those of (A,B), but the Spearman’s rank correlation gives the same value 0.416 for two sets of rankings (A,B) and (A,C). For the cases where the differences in the top ranks would seem to be more critical, Blest [2] suggests that these discrepancies should be emphasized. He proposed an alternative measure of rank correlation that attaches more significance to the early ranking of an initially given order. The Blest’s index is defined by

\[
\gamma_n = \frac{2n + 1}{n - 1} - \frac{12}{n^2 - 1} \sum_{i=1}^{n} \left( 1 - \frac{R_i}{n+1} \right)^2 S_i. \tag{2}
\]

The values of the Blest’s index for two sets of rankings (A,B) and (A,C) are given by 0.241 and 0.591, respectively. As we see, in situations such as the rankings (A,C) where the bottom ranks should be emphasized, the Blest’s index is a suitable rank correlation measure. In the following we develop a general theory for weighted rank correlation measures by giving suitable weights to the distance between two sets of ranks to place more emphasis on items having low rankings than those have high rankings, or vice versa.

Let \( D_i = S_i - R_i, \ i = 1, \ldots, n \). The most common form of the Spearman’s rank correlation coefficient between two sets of rankings \( R_1, \ldots, R_n \) and \( S_1, \ldots, S_n \) is given by (Kendall, [9])

\[
\rho_{n,s} = 1 - \frac{2 \sum_{i=1}^{n} D_i^2}{\max(\sum_{i=1}^{n} D_i^2)}, \tag{3}
\]

where \( \max(\sum_{i=1}^{n} D_i^2) = (n^3 - n)/3 \) represents the value of the summation when there is a perfect discordance between rankings, that is, \( S_i = n + 1 - R_i, \ i = 1, \ldots, n \). Throughout the rest of the paper we assume, without loss of generality, that the sample pairs are given in accordance with the increasing magnitude of \( X \) components, so that \( R_i = i, \) for \( i = 1, 2, \ldots, n \) and \( D_i = S_i - i \). According to Blest’s idea [2] if the set of points

| A | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| B | 1 | 2 | 3 | 9 | 8 | 7 | 6 | 4 | 5 |
| C | 5 | 6 | 4 | 3 | 2 | 1 | 7 | 8 | 9 |
\((0, 0), [(\sum_{i=1}^{k}(n+1-i), \sum_{i=1}^{k}(S_i-i), k=1,\ldots,n] \) determined in the coordinate plane, the Spearman’s rho is normalized version of the sum of bars made the width of the given points, i.e. \(\sum_{k=1}^{n} \sum_{i=1}^{k}(S_i-i)\) as a measure of the disarray of originally ordered data, i.e.,

\[
\rho_{n,s} = 1 - \frac{2 \sum_{i=1}^{n} \sum_{k=1}^{i}(S_i-i)}{\max(\sum_{k=1}^{n} \sum_{i=1}^{k}(S_i-i))}.
\]  

By changing the order of summation, it is easy to see that \(\sum_{k=1}^{n} \sum_{i=1}^{k}(S_i-i) = \frac{1}{2} \sum_{k=1}^{n} D_i^2\). While (4) and (5) are two different representations of ordinary Spearman’s rho, the Blest’s index is normalized version of the area which appears from connecting the mentioned points to each other. By looking again to the Blest’s index, one can imagine that the bars \(\eta_k = \sum_{i=1}^{k}(S_i-i), k=1,\ldots,n\) is assigned a certain weight, in comparison to the Spearman’s rho that does not give any weight to the mentioned bar. Now we consider a general class of WRC measures of the form

\[
v_n = 1 - \frac{2 \sum_{i=1}^{n} w_i \eta_i}{\max(\sum_{i=1}^{n} w_i \eta_i)},
\]

where the positive constants \(w_i\)s are suitable weights. To construct WRC measures which are sensitive to agreement on top rankings (lower ranks), for \(p = 1, 2, 3,\ldots\) and \(n > 1\), we choose the weights \(w_i = (n+1-i)^p - (n-i)^p\). The class of WRC measures constructed by (5) is then

\[
v_{n,p}^{(l)} = 1 + \frac{2 \sum_{i=1}^{n} (i-S_i)(n+1-i)^p}{\sum_{i=1}^{n} (n+1-2i)(n+1-i)^p}.
\]

Alternatively, by choosing the weights \(w_i = i^p - (i-1)^p\), one can obtain measures which are sensitive to agreement on bottom rankings (upper ranks). In this case the class of WRC measures (5) is simplified to

\[
v_{n,p}^{(u)} = 1 + \frac{2 \sum_{i=1}^{n} (i-S_i)(n^p - (i-1)^p)}{\sum_{i=1}^{n} (n+1-2i)(n^p - (i-1)^p)}.
\]

Let \(\kappa_{n,p} = \sum_{i=1}^{n} i^p\). It is easily seen that

\[
\sum_{i=1}^{n} (n+1-2i)(n+1-i)^p = 2\kappa_{n,p+1} - (n+1)\kappa_{n,p},
\]

and

\[
\sum_{i=1}^{n} (n+1-2i)(n^p - (i-1)^p) = 2\kappa_{n-1,p+1} - (n-1)\kappa_{n-1,p}.
\]
The coefficients (6) and (7) could be rewritten in terms of \( \kappa_{n,p} \) as

\[
\nu_{n,p}^{(l)} = \frac{(n+1)\kappa_{n,p} - 2\sum_{i=1}^{n} S_i (n+1-i)^p}{2\kappa_{n+1} - (n+1)\kappa_{n,p}},
\]

(8)

and

\[
\nu_{n,p}^{(u)} = \frac{-(n+1)\kappa_{n-1,p} + 2\sum_{i=1}^{n} S_i (i-1)^p}{2\kappa_{n-1} - (n-1)\kappa_{n-1,p}}.
\]

(9)

Note that for \( p = 1 \), both \( \nu_{n,p}^{(l)} \) and \( \nu_{n,p}^{(u)} \) reduce to the Spearman’s rank correlation coefficient (1). For \( p = 2 \), the measure \( \nu_{n,p}^{(l)} \) reduces to the Blest’s rank correlation coefficient (2). The coefficients \( \nu_{n,p}^{(l)} \) and \( \nu_{n,p}^{(u)} \) are asymmetric WRC measures; that is, the correlation of \((X,Y)\) is not the same as those of \((Y,X)\). One can obtain the symmetrized version of (6) as follows

\[
\nu_{n,p}^{(s,l)} = \frac{\nu_{n,p}^{(l)}(X,Y) + \nu_{n,p}^{(l)}(Y,X)}{2} = \frac{(n+1)\kappa_{n,p} - \sum_{i=1}^{n} [S_i (n+1-i)^p + i(n+1-S_i)^p]}{2\kappa_{n+1} - (n+1)\kappa_{n,p}}.
\]

Similarly the symmetrized version of (7) is given by

\[
\nu_{n,p}^{(s,u)} = \frac{-(n+1)\kappa_{n-1,p} + \sum_{i=1}^{n} [S_i (i-1)^p + i(S_i-1)^p]}{2\kappa_{n-1} - (n-1)\kappa_{n-1,p}}.
\]

For \( p = 1 \) the WRC measures \( \nu_{n,p}^{(s,l)} \) and \( \nu_{n,p}^{(s,u)} \) are equal to the Spearman’s rank correlation (1). For \( p = 2 \) the measure \( \nu_{n,p}^{(s,l)} \) is the symmetrized version of the Blest’s index (2). Table 2 shows the values of \( \nu_{n,p}^{(l)}, \nu_{n,p}^{(u)}, p = 1, 2, 3, 4, 5 \) and their symmetrized versions for the rankings \((A,B)\) and \((A,C)\) given in Table 1. The result illustrates the sensitivity of these indices to the agreement on top and bottom rankings. We note that \( \nu_{n,p}^{(l)}, \nu_{n,p}^{(u)} \) and their symmetrized versions take values in \([-1,1]\). In particular, the value of these measures is equal to 1 when \( S_i = i \) (a perfect positive dependency between two sets of ranks) and they take \(-1\) when \( S_i = n+1-i \) (a perfect negative dependency between two sets of ranks).

3 Weighted correlation coefficients

In this section we introduce the weighted correlation coefficient’s \( \nu_p^{(l)} \) and \( \nu_p^{(u)} \) and their symmetrized versions \( \nu_p^{(s,l)} \) and \( \nu_p^{(s,u)} \) as the population counterparts of the WRC measures \( \nu_{n,p}^{(l)}, \nu_{n,p}^{(u)}, \nu_{n,p}^{(s,l)} \) and \( \nu_{n,p}^{(s,u)} \). Each of these coefficients can be expressed as a linear functional of the quantity \( A(u,v) = C(u,v) - \Pi(u,v) \), where \( C \) is the copula associated with the pair \((X,Y)\) and \( \Pi(u,v) = uv \) is the copula of independent random variables.
Table 2: Values of the Spearman’s rho and WRC measures for three sets of rankings in Table 1.

|   | \( (A, C) \) | \( (A, B) \) |
|---|-----------------|-----------------|
| 1 | \( 0.433 \) 0.433 0.433 0.433 | \( 0.433 \) 0.433 0.433 0.433 |
| 2 | \( 0.270 \) 0.637 0.263 0.645 | \( 0.596 \) 0.229 0.603 0.220 |
| 3 | \( 0.155 \) 0.768 0.140 0.776 | \( 0.720 \) 0.112 0.728 0.094 |
| 4 | \( 0.081 \) 0.851 0.057 0.858 | \( 0.808 \) 0.045 0.815 0.016 |
| 5 | \( 0.033 \) \( 0.905 \) 0.001 \( 0.910 \) | \( 0.869 \) 0.006 \( 0.875 \) 0.032 |

For \( p = 2, 3, \ldots \), we have

\[
\begin{align*}
\nu_p^{(l)} &= 2(p+1)(p+2) \int_0^1 \int_0^1 (1-u)^{p-1}(C(u,v)-uv)dudv, \\
\nu_p^{(u)} &= 2(p+1)(p+2) \int_0^1 \int_0^1 u^{p-1}(C(u,v)-uv)dudv, \\
\nu_p^{(s,u)} &= (p+1)(p+2) \int_0^1 \int_0^1 (u^{p-1}+v^{p-1})(C(u,v)-uv)dudv, \\
\nu_p^{(s,l)} &= (p+1)(p+2) \int_0^1 \int_0^1 ((1-u)^{p-1}+(1-v)^{p-1})(C(u,v)-uv)dudv. \quad (10)
\end{align*}
\]

Note that for \( p = 1 \), all of these coefficients reduce to the Spearman’s rho given by

\[
\rho_s = 12 \int_0^1 \int_0^1 (C(u,v)-uv)dudv.
\]

For \( p = 2 \), the coefficient \( \nu_p^{(l)} \) reduces to the Blest’s correlation coefficient \([5]\) given by

\[
\gamma = 24 \int_0^1 \int_0^1 (1-u)C(u,v)dudv - 2.
\]

**Remark 1.** A probabilistic interpretation can be made for the weighted correlation coefficients \( \nu_p^{(l)} \) and \( \nu_p^{(u)} \) and their symmetrized versions \( \nu_p^{(s,l)} \) and \( \nu_p^{(s,u)} \). We provide an illustration for \( \nu_p^{(l)} \). For \( p = 1, 2, \ldots \) consider the cumulative distribution function

\[
F_p(u,v) = (1 - (1-u)^p)v, \quad 0 \leq u, v \leq 1.
\]
Let \((U, V)\) be a random vector with the joint distribution function \(F_p\). For a copula \(C\), the coefficient \(v_p^{(l)}\) has the following representation

\[
v_p^{(l)} = \frac{2(p+1)(p+2)}{p} \int_0^1 \int_0^1 (C(u,v) - uv) dF_p(u,v)
= \frac{E_{F_p}[C(U,V) - \Pi(U,V)]}{E_{F_p}[M(U,V) - \Pi(U,V)]},
\]

where \(M(u,v) = \min(u,v)\) and \(E_{F_p}\) denotes the expectation with respect to \(F_p\). Thus, the coefficient \(v_p^{(l)}\) can be considered as an average distance between the copula \(C\) and the independent copula \(\Pi\), where the average is taken with respect to the bivariate distribution function \(F_p\). The proposed weighted correlation coefficients could be seen as average quadrant dependent (AQD) measures of association studied in [7].

For \(v_p^{(l)}\), it is more convenient to use the following alternative representation

\[
v_p^{(l)} = \frac{2(p+1)(p+2)}{p} \int_0^1 \int_0^1 (1-u)^p(1-v)dC(u,v) - \frac{p+2}{p}, \tag{11}\]

which follows from the fact that

\[
\int_0^1 \int_0^1 C(u,v) dF_p(u,v) = P(W \leq U, Z \leq V) = P(U \geq W, V \geq Z) = \int_0^1 \int_0^1 \tilde{F}_p(u,v) dC(u,v),
\]

where \((W,Z)\) and \((U,V)\) are two independent pairs distributed as the copula \(C\) and the joint distribution \(F_p\), respectively and \(\tilde{F}_p(u,v) = P(U > u, V > v) = (1-u)^p(1-v)\), is the survival function associated with \(F_p\). An alternative representation for \(v_p^{(u)}\) is given by

\[
v_p^{(u)} = \frac{2(p+1)(p+2)}{p} \int_0^1 \int_0^1 (1-u^p)(1-v) dC(u,v) - (p+2). \tag{12}\]

In the following examples we provide the values of the weighted correlation coefficient’s \(v_p^{(l)}\) and \(v_p^{(u)}\) for some copulas.

**Example 1.** Let \(C_\theta\) be a member of the Cuadras-Augé family of copulas [12] given by

\[C_\theta(u,v) = [\min(u,v)]^\theta [uv]^{1-\theta}, \quad \theta \in [0,1]. \tag{13}\]

This family of copula is positively ordered in \(\theta \in [0,1]\), that is, for \(\theta_1 \leq \theta_2\), we have that \(C_{\theta_1}(u,v) \leq C_{\theta_2}(u,v)\) for all \(u,v \in [0,1]\). This family of copulas has no lower tail dependence, whereas the upper tail dependence parameter is given by \(\lambda_u = \theta [12]\). For this family of copulas we have

\[
v_p^{(u)} = \frac{(p+2)(p^2 + 2p - 3 + \theta)}{p(p+3-\theta)}, \tag{14}\]
Figure 1: The values of $v_p^{(l)}$ and $v_p^{(u)}$, $p = 1, 2, 3, 4, 5, 10$, for Cuadras-Augé family of copulas.

and

$$v_p^{(l)} = \frac{\theta(p+2)(1-p(p+1)B(p,4-\theta))}{p(2-\theta)},$$

(15)

where $B(a,b) = \int_0^1 \int_0^1 x^{a-1}(1-x)^{b-1}dx$, is the beta function. For every $\theta \in [0,1]$, $v_p^{(u)}$ is increasing in $p$ and $v_p^{(l)}$ is decreasing in $p$. For $p = 1, 2, 3, 4, 5, 10$, the values of $v_p^{(l)}$ and $v_p^{(u)}$, as a function of $\theta$ is plotted in Figure 1. In particular for this family of copulas it follows that for every $\theta \in [0,1]$ and $p = 2, 3,...$, 

$$v_p^{(l)} \leq v_1^{(l)} = \rho = v_1^{(u)} \leq v_p^{(u)}.$$

**Example 2.** Let $C_\theta$ be a member of Raftery family of copulas $[12]$ given by

$$C_\theta(u,v) = \min(u,v) + \frac{1-\theta}{1+\theta} (uv)^{1-\theta} \left\{ 1 - [\max(u,v)]^{1+\theta} \right\}, \quad \theta \in [0,1].$$

This family of copulas is also positively ordered in $\theta \in [0,1]$ and has no upper tail dependence, whereas the lower tail dependence parameter is given by $\lambda_L = \frac{2\theta}{\theta+1}$ $[12]$. The values of $v_p^{(l)}$ and $v_p^{(u)}$, $p = 1, 2, 3, 4, 5, 10$, as a function of $\theta$ are plotted in Figure 2. For this family of copulas as we see for every $\theta \in [0,1]$ and $p = 2, 3,...$, 

$$v_p^{(u)} \leq v_1^{(u)} = \rho = v_1^{(l)} \leq v_p^{(l)}.$$ 

### 4 The Quantiles and Asymptotic Distributions

The asymptotic behavior of the proposed WRC measures in general can be studied by the standard results from the theory of empirical processes $[22]$. Before that, we mention the
asymptotic formula for $\kappa_{n,p} = \sum_{i=1}^{n} i^p$, that we need in the sequel. By definition of the Riemann integral, it holds that

$$\frac{\kappa_{n,p}}{n^{p+1}} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^p = \int_0^1 x^p dx + O(n^{-1}).$$

(16)

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample of size $n$ from a pair $(X, Y)$ of continuous random variables with the joint distribution function $H$, marginal distribution functions $F$ and $G$ and the associated copula $C$. Let $(1, S_1), (2, S_2), \ldots, (n, S_n)$ be the ranks of the rearranged sample. It is known that (Rüschendorf, [16]) the copula $C$ can be estimated by the empirical copula defined for all $u, v \in [0, 1]$ by

$$C_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} I\left( \frac{i}{n+1} \leq u, \frac{S_i}{n+1} \leq v \right),$$

where $I(A)$ denotes the indicator function of the set $A$. The empirical versions of the weighted correlation coefficient’s $\nu_{p}^{(l)}$ and $\nu_{p}^{(u)}$ and their symmetrized versions $\nu_{p}^{(s,l)}$ and $\nu_{p}^{(s,u)}$ defined by (10), could be written in terms of the empirical copula $C_n$. By plugging $C_n$ in (11), the empirical version of $\nu_{p}^{(l)}$ is of the form

$$\tilde{\nu}_{n,p}^{(l)} = \frac{2(p+1)(p+2)}{p} \int_0^1 \int_0^1 (1-u)^p(1-v)dC_n(u, v) - \frac{p+2}{p}.$$
By using the representation (8) and the identity (16), straightforward calculations gives:

\[ \tilde{v}_{n,p}^{(l)} = \frac{2(p+1)(p+2)}{np} \sum_{i=1}^{n} \left( 1 - \frac{i}{n+1} \right)^p \left( 1 - \frac{S_i}{n+1} \right) - \frac{p+2}{p} \]

\[ = \frac{2(p+1)(p+2)}{np(n+1)^p} \sum_{i=1}^{n} S_i(n+1-i)^p - \frac{p+2}{np(n+1)^{p+1}} \sum_{i=1}^{n} S_i \]

\[ = \frac{(p+1)(p+2)\kappa_n, p}{np(n+1)^p} + \frac{(p+1)(p+2)(2\kappa_{n,p+1} - (n+1)\kappa_n)}{np(n+1)^{p+1}} \nu_{n,p}^{(l)} - \frac{p+2}{p} \]

\[ = (1 + O(n^{-1}))\nu_{n,p}^{(l)} + O(n^{-1}). \quad (17) \]

By using (12), a similar argument shows that the empirical version of the coefficient \( v_p^{(u)} \) is given by

\[ v_{n,p}^{(u)} = \frac{2(p+1)(p+2)}{np} \sum_{i=1}^{n} \left( 1 - \left( \frac{i}{n+1} \right)^p \right) \left( 1 - \frac{S_i}{n+1} \right) - (p+2) \]

\[ = (1 + O(n^{-1}))v_{n,p}^{(u)} + O(n^{-1}). \quad (18) \]

In the following we provide the asymptotic distribution of the WRC measures \( v_{n,p}^{(l)}, v_{n,p}^{(u)} \) and their symmetrized versions \( v_{n,p}^{(s,l)}, v_{n,p}^{(s,u)} \). As shown by Segers [18], \( C_n \) converges weakly to \( C \) as \( n \to \infty \), whenever \( C \) is regular, that is, the partial derivatives \( C_1(u,v) = \partial C(u,v)/\partial u \) and \( C_2(u,v) = \partial C(u,v)/\partial v \) exist everywhere on \([0,1]^2\) and \( C_1 \) and \( C_2 \) are continuous on \((0,1) \times [0,1] \) and \([0,1] \times (0,1) \), respectively. Moreover, the empirical copula process \( \mathcal{C}_n = \sqrt{n}(C_n - C) \) converges weakly, as \( n \to \infty \), to a centered Gaussian process \( \hat{\mathcal{C}} \) on \([0,1]^2\), defined for all \( u, v \in [0,1] \) by

\[ \hat{\mathcal{C}}(u,v) = \mathcal{C}(u,v) - \frac{\partial}{\partial u} \mathcal{C}(u,v) \mathcal{C}(u,1) - \frac{\partial}{\partial v} \mathcal{C}(u,v) \mathcal{C}(1,v), \quad (19) \]

where \( \mathcal{C}(u,v) \) is Brownian bridge on \([0,1]^2\) with the covariance function

\[ E(\mathcal{C}(u,v)\mathcal{C}(s,t)) = C(\min(u,s), \min(v,t)) - C(u,v)C(s,t). \]

**Theorem 1.** Suppose that \( C \) is a regular copula. Then \( \sqrt{n}(v_{n,p}^{(l)} - v_p^{(l)}), \sqrt{n}(v_{n,p}^{(u)} - v_p^{(u)}), \sqrt{n}(v_{n,p}^{(s,l)} - v_p^{(s,l)}) \) and \( \sqrt{n}(v_{n,p}^{(s,u)} - v_p^{(s,u)}) \) are asymptotically centered normal with the asymptotic variances, given by

\[ (\sigma_p^{(l)})^2 = 4(p+1)^2(p+2)^2 \int_{[0,1]^4} (1-u)^{p-1}(1-s)^{p-1} E(\hat{\mathcal{C}}(u,v)\hat{\mathcal{C}}(s,t)) \, du dv ds dt, \quad (20) \]

\[ (\sigma_p^{(u)})^2 = 4(p+1)^2(p+2)^2 \int_{[0,1]^4} u^{p-1}s^{p-1} E(\hat{\mathcal{C}}(u,v)\hat{\mathcal{C}}(s,t)) \, du dv ds dt, \]

\[ (\sigma_p^{(s,l)})^2 = (p+1)^2(p+2)^2 \int_{[0,1]^4} (u^{p-1} + v^{p-1})(s^{p-1} + t^{p-1}) E(\hat{\mathcal{C}}(u,v)\hat{\mathcal{C}}(s,t)) \, du dv ds dt, \]

\[ (\sigma_p^{(s,u)})^2 = (p+1)^2(p+2)^2 \int_{[0,1]^4} ((u-1)^{p-1} + (v-1)^{p-1})(s-1)^{p-1} + (t-1)^{p-1}) \times E(\hat{\mathcal{C}}(u,v)\hat{\mathcal{C}}(s,t)) \, du dv ds dt, \]

10
where \( \hat{C} \) is the Gaussian process defined by (19).

**Proof.** We prove the result for \( \sqrt{n}(v_{n,p}^{(l)} - v_p^{(l)}) \), similar arguments hold for \( \sqrt{n}(v_{n,p}^{(u)} - v_p^{(u)}) \), \( \sqrt{n}(v_{n,p}^{(s,u)} - v_p^{(s,u)}) \) and \( \sqrt{n}(v_{n,p}^{(s,l)} - v_p^{(s,l)}) \). From (17) we have

\[
\sqrt{n}(v_{n,p}^{(l)} - v_p^{(l)}) = (1 + O(n^{-1})) \sqrt{n}(\hat{v}_n^{(l)} - v_p^{(l)}) + O(n^{-1/2})
\]

\[
= 2(1 + O(n^{-1}))(p + 1)(p + 2) \int_0^1 \int_0^1 (1 - u)^{p-1} \left[ \sqrt{n}(C_n(u,v) - C(u,v)) \right] du dv + O(n^{-1/2}).
\]

Since the integral on the right side is a linear and continuous functional of the empirical copula process, then the left hand side is asymptotically centered normal with asymptotic variance \( (\sigma_p^{(l)})^2 \), as stated in the theorem. \( \square \)

**Corollary 2.** Assume the null hypothesis of independence i.e. \( C(u, v) = uv \). Then \( \sqrt{n}v_{n,p}^{(l)} \)

\( \sqrt{n}v_{n,p}^{(u)}, \sqrt{n}v_{n,p}^{(s,l)} \) and \( \sqrt{n}v_{n,p}^{(s,u)} \) are asymptotically centered normal with the asymptotic standard deviations \( \sigma_p^{(l)} = \sigma_p^{(u)} = \frac{(p+2)}{\sqrt{3(2p+1)}} \) and \( \sigma_p^{(s,l)} = \sigma_p^{(s,u)} = \sqrt{\frac{p^2 + 10p + 7}{6(2p+1)}} \).

**Proof.** For \( C(u, v) = uv \) the covariance function of the limiting Gaussian process \( \hat{C} \) takes the form \( E(\hat{C}(u,v)\hat{C}(s,t)) = (\min(u,s) - us)(\min(v,t) - vt) \). An application of Theorem 4.1 and a routine integration gives the result. \( \square \)

In order to use the proposed WRC measures for testing independence, one needs to find their distributions or the quantiles of the distribution under the hypothesis of independence. The following result provides the expectation and variance of \( v_{n,p}^{(l)} \). Similar result could be found for \( v_{n,p}^{(u)}, v_{n,p}^{(s,l)} \) and \( v_{n,p}^{(s,u)} \).

**Theorem 3.** Under the hypothesis of independence between two sets of ranks

\[
E(v_{n,p}^{(l)}) = 0, \quad \text{var}(v_{n,p}^{(l)}) = \frac{n(n+1)}{3} \frac{\kappa_{n,2p} - \frac{1}{n}((\kappa_{n,p})^2)}{(2\kappa_{n,p+1} - (n+1)\kappa_{n,p})^2}.
\]

**Proof.** We note that the WRC measure \( v_{n,p}^{(l)} \) by (6) can be written as a linear combination of the linear rank statistic of the form \( a_n + b_n \sum_{i=1}^n a(i, S_i) \), where

\[
a_n = 1 + \left( \sum_{i=1}^n i(n+1-i)^p \right) \left( \sum_{i=1}^n (n+1-2i)(n+1-i)^p \right)^{-1},
\]

\[
b_n = -2 \left( \sum_{i=1}^n (n+1-2i)(n+1-i)^p \right)^{-1},
\]

and \( a(i, S_i) = S_i(n+1-i)^p \). The mean and the variance of the quantity \( S = \sum_{i=1}^n a(i, S_i) \) can be obtained, for example, by using Theorem 1 in p. 57 in [20]. See, also [6]. \( \square \)
Table 3: Variance of the normalized WRC measures \( \sqrt{n}v_{n,p}^{(f)} \), \( \sqrt{n}v_{n,p}^{(u)} \), \( \sqrt{n}v_{n,p}^{(s,f)} \) and \( \sqrt{n}v_{n,p}^{(s,u)} \), under the assumption of independence, for \( p = 1, 2, 3, 4, 5 \).

| Index | The exact variance | Asymptotic variance |
|-------|-------------------|---------------------|
| \( \sqrt{n}v_{n,1}^{(f)} \) | \( \frac{n((2n+1)(8n+11)}{(n+1)(n-1)} \) | 1 |
| \( \sqrt{n}v_{n,2}^{(f)} \) | \( \frac{3n}{(n+1)(n-1)} \) | 16 |
| \( \sqrt{n}v_{n,3}^{(f)} \) | \( \frac{n}{3} \) | 31 |
| \( \sqrt{n}v_{n,4}^{(f)} \) | \( \frac{25n^3+84n^2+69n^2-8}{(n-1)(9n^2+15n+4)^2} \) | 25 |
| \( \sqrt{n}v_{n,5}^{(f)} \) | \( \frac{n}{3} \) | 3 |
| \( \sqrt{n}v_{n,1}^{(u)} \) | \( \frac{3n^2+119n^2+100n^2-65n^2-62n+31}{(2n+1)} \) | 4 |
| \( \sqrt{n}v_{n,2}^{(u)} \) | \( \frac{64n^2-270n^2+319n^2+30n^2-189n^2+31}{(n+1)(n-1)} \) | 4 |
| \( \sqrt{n}v_{n,3}^{(u)} \) | \( \frac{n}{3} \) | 6 |
| \( \sqrt{n}v_{n,4}^{(u)} \) | \( \frac{n}{3} \) | 6 |
| \( \sqrt{n}v_{n,5}^{(u)} \) | \( 4900n^6+25872n^6+35396n^6-6468n^6-49359n^6+353967n^6-5880n \) | 49 |
| \( \sqrt{n}v_{n,1}^{(s,f)} \) | \( \frac{n}{3} \) | 33 |
| \( \sqrt{n}v_{n,2}^{(s,f)} \) | \( \frac{3n-33)(10n^4+28n^3+17n^2-7n-4)}{(n+1)(n-1)} \) | 33 |
| \( \sqrt{n}v_{n,3}^{(s,f)} \) | \( \frac{n}{3} \) | 41 |
| \( \sqrt{n}v_{n,4}^{(s,f)} \) | \( \frac{n}{3} \) | 39 |
| \( \sqrt{n}v_{n,5}^{(s,f)} \) | \( 4100n^8+22176n^8+38244n^8+10164n^8-27789n^8-7623n^8+15298n^2+924n-2676 \) | 41 |

The exact and asymptotic variances of the normalized WRC measures \( \sqrt{n}v_{n,p}^{(f)} \), \( \sqrt{n}v_{n,p}^{(u)} \), \( \sqrt{n}v_{n,p}^{(s,f)} \) and \( \sqrt{n}v_{n,p}^{(s,u)} \), under the assumption of independence are provided in Table 3 for \( p = 1, 2, 3, 4, 5 \). According to the results of Table 4 it seems that the variance of the symmetric versions of WRC measures are less than that of their asymmetric versions. They are more appropriate for testing independence of two random variables. Under the assumption of independence, all \( n! \) orderings of a set of rank \( (S_1, S_2, \ldots, S_n) \) are equally likely to occur. After calculating the value of the WRC measures between the two rankings \( (1, 2, \ldots, n) \) and \( (S_1, S_2, \ldots, S_n) \), their quantiles are obtained by considering that the atom of the discrete distribution is \( \frac{1}{n!} \). Here the \( r \)-quantiles in the data with \( x_i = \) the value of the specific WRC measure in the \( i \)-th paired samples, \( i = 1, 2, \ldots, n! \) are calculated by using \( (1 - \gamma)x(j) + \gamma x(j+1) \), where \( \frac{i}{n!} \leq r < \frac{i+1}{n!} \), \( x(j) \) is the \( j \)th order statistic and \( \gamma = 0.5 \) if \( nr = j \), and 1 otherwise. This algorithm discussed in Hyndman and Fan [7] as one of the common methods for calculating quantiles for discontinuous sample in statistical pack-
ages. The quantiles of normalized WRC measures $\sqrt{n}\nu_{n,p}^{(l)}$, $\sqrt{n}\nu_{n,p}^{(u)}$ and their symmetrized versions $\sqrt{n}\nu_{n,p}^{(s,l)}$, $\sqrt{n}\nu_{n,p}^{(s,u)}$ for $p = 1, \ldots, 5$ and $n = 5, 6, \ldots, 10$, are given in Tables 4-5.
Table 4: The quantiles of normalized WRC measures $\sqrt{n}V_{n,p}^{(l)}$ and $\sqrt{n}V_{n,p}^{(s,l)}$ for $p = 1, \ldots, 5$ and
$n = 5, 6, \ldots, 10.$

| $n$  | $p$ | 90%   | 95%   | 97.5% | 99%   | 90%   | 95%   | 97.5% | 99%   |
|------|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| 5    | 1   | 1.565 | 1.789 | 2.012 | 2.012 | 1.565 | 1.789 | 2.012 | 2.012 |
|      | 2   | 1.565 | 1.845 | 2.012 | 2.124 | 1.565 | 1.826 | 2.012 | 2.124 |
|      | 3   | 1.600 | 1.909 | 2.045 | 2.185 | 1.618 | 1.931 | 2.030 | 2.185 |
|      | 4   | 1.658 | 1.984 | 2.116 | 2.214 | 1.636 | 1.984 | 2.097 | 2.214 |
|      | 5   | 1.730 | 2.041 | 2.161 | 2.226 | 1.676 | 2.041 | 2.147 | 2.226 |
| 6    | 1   | 1.470 | 1.890 | 2.030 | 2.170 | 1.470 | 1.890 | 2.030 | 2.170 |
|      | 2   | 1.550 | 1.830 | 2.040 | 2.230 | 1.550 | 1.850 | 2.030 | 2.210 |
|      | 3   | 1.599 | 1.897 | 2.103 | 2.275 | 1.599 | 1.934 | 2.101 | 2.275 |
|      | 4   | 1.674 | 1.974 | 2.177 | 2.340 | 1.628 | 1.974 | 2.161 | 2.340 |
|      | 5   | 1.791 | 1.985 | 2.227 | 2.368 | 1.727 | 1.972 | 2.233 | 2.368 |
| 7    | 1   | 1.465 | 1.795 | 1.984 | 2.268 | 1.465 | 1.795 | 1.984 | 2.268 |
|      | 2   | 1.500 | 1.831 | 2.079 | 2.268 | 1.488 | 1.819 | 2.079 | 2.268 |
|      | 3   | 1.569 | 1.897 | 2.129 | 2.323 | 1.551 | 1.877 | 2.112 | 2.323 |
|      | 4   | 1.656 | 1.980 | 2.196 | 2.378 | 1.620 | 1.954 | 2.204 | 2.366 |
|      | 5   | 1.725 | 2.078 | 2.253 | 2.436 | 1.707 | 2.019 | 2.248 | 2.424 |
| 8    | 1   | 1.414 | 1.751 | 2.020 | 2.290 | 1.414 | 1.751 | 2.020 | 2.290 |
|      | 2   | 1.467 | 1.818 | 2.073 | 2.312 | 1.452 | 1.818 | 2.065 | 2.312 |
|      | 3   | 1.541 | 1.897 | 2.144 | 2.372 | 1.514 | 1.885 | 2.129 | 2.367 |
|      | 4   | 1.617 | 1.991 | 2.214 | 2.443 | 1.572 | 1.968 | 2.198 | 2.431 |
|      | 5   | 1.704 | 2.067 | 2.291 | 2.500 | 1.644 | 2.035 | 2.275 | 2.481 |
| 9    | 1   | 1.400 | 1.750 | 2.050 | 2.300 | 1.400 | 1.750 | 2.050 | 2.300 |
|      | 2   | 1.445 | 1.800 | 2.070 | 2.330 | 1.430 | 1.795 | 2.070 | 2.330 |
|      | 3   | 1.523 | 1.884 | 2.152 | 2.404 | 1.488 | 1.861 | 2.137 | 2.395 |
|      | 4   | 1.611 | 1.973 | 2.238 | 2.479 | 1.548 | 1.938 | 2.212 | 2.466 |
|      | 5   | 1.694 | 2.061 | 2.314 | 2.550 | 1.612 | 2.019 | 2.284 | 2.532 |
| 10   | 1   | 1.399 | 1.744 | 2.012 | 2.319 | 1.399 | 1.744 | 2.012 | 2.319 |
|      | 2   | 1.434 | 1.789 | 2.072 | 2.350 | 1.416 | 1.779 | 2.065 | 2.343 |
|      | 3   | 1.511 | 1.873 | 2.156 | 2.429 | 1.469 | 1.844 | 2.137 | 2.416 |
|      | 4   | 1.599 | 1.965 | 2.245 | 2.511 | 1.528 | 1.920 | 2.216 | 2.491 |
|      | 5   | 1.683 | 2.054 | 2.333 | 2.587 | 1.589 | 1.998 | 2.298 | 2.561 |
| $\infty$ | 1 | 1.282 | 1.645 | 1.960 | 2.326 | 1.282 | 1.645 | 1.960 | 2.326 |
|      | 2 | 1.324 | 1.699 | 2.024 | 2.403 | 1.303 | 1.672 | 1.992 | 2.365 |
|      | 3 | 1.398 | 1.795 | 2.138 | 2.538 | 1.341 | 1.721 | 2.051 | 2.435 |
|      | 4 | 1.480 | 1.899 | 2.263 | 2.686 | 1.384 | 1.777 | 2.117 | 2.513 |
|      | 5 | 1.562 | 2.004 | 2.388 | 2.835 | 1.428 | 1.833 | 2.185 | 2.593 |
5  Efficiency of the Tests of Independence

In this section we compare the Pitman asymptotic relative efficiency (or, Pitman ARE) and the empirical power of tests of independence constructed based on the proposed WRC measures.

5.1 Pitman Efficiency

Consider a parametric family \( \{ C_\theta \} \) of copulas with \( \theta = \theta_0 \) corresponding to the independence case. Let \( T_{1n} \) and \( T_{2n} \) be two test statistics for testing \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta > \theta_0 \) that reject null hypothesis for large values of \( T_{1n} \) and \( T_{2n} \). Suppose that \( T_{1n} \) and \( T_{2n} \) satisfy the regularity conditions:

(1) There exist continuous functions \( \mu_i(\theta) \) and \( \sigma_i(\theta) \), \( \theta > \theta_0 \), \( i = 1, 2 \) such that for all sequences \( \theta_n = \theta_0 + \frac{h}{\sqrt{n}} \), \( h > 0 \), it holds

\[
\lim_{n \to \infty} P_{\theta_n}\left( \frac{\sqrt{n}(T_{in} - \mu_i(\theta_n))}{\sigma_i(\theta_n)} < z \right) = \Phi(z), z \in \mathbb{R}, \quad i = 1, 2,
\]

where \( \Phi(z) \) is the standard normal distribution function;

(2) The function \( \mu_i(\theta) \) is continuously differentiable at \( \theta = \theta_0 \) and \( \mu'_i(\theta_0) > 0, \sigma_i(\theta_0) > 0, i = 1, 2 \).

Under these conditions, the Pitman ARE of \( T_{n1} \) relative to \( T_{n2} \) is equal to

\[
\text{ARE}(T_{1n}, T_{2n}) = \left[ \frac{\partial}{\partial \theta} \mu_1|_{\theta=\theta_0} \cdot \frac{\sigma_2(\theta_0)}{\sigma_1(\theta_0)} \right]^2.
\]

For more detail see, \cite{13}. In the following we compare the ARE of proposed WRC measures, relative to the Spearman’s rho for the Cuadras-Augé family of copulas given by \cite{13}.

Example 3. Suppose that the copula of \((X, Y)\) be a member of the Cuadras-Augé family of copulas given by \cite{13} in Example 3.1. Let \( T_{1n} = \nu^{(l)}_{n,p} \) and \( T_{2n} = \rho_{ns} \). By Theorem 4.1, for the test statistics based on WRC measure \( \nu^{(l)}_{n,p} \), regularity conditions (1) and (2) are satisfied with \( \theta_0 = 0, \mu_p(\theta) = \nu^{(l)}_{p} \) and \( \sigma_p^l \) given in Corollary 4.1 (for \( \theta_0 = 0 \)). By using \cite{15} and differentiation with respect to \( \theta \) one gets

\[
\frac{d\nu^{(l)}_{p}}{d\theta}(0) = \frac{p + 5}{2(p + 3)}.
\]

Since \( \nu^{(l)}_{1} = \rho_{s} \) and \( \nu^{(l)}_{n,1} = \rho_{n,s} \), then from Corollary 4.1 we have

\[
\text{ARE}(\nu^{(l)}_{n,p}, \rho_{ns}) = \frac{4(p + 5)^2(2p + 1)}{3(p + 2)^2(p + 3)^2}.
\]
Table 5: The quantiles of normalized WRC measures $\sqrt{n}\nu_{n,p}^{(u)}$ and $\sqrt{n}\nu_{n,p}^{(s,u)}$ for $p = 1, \ldots, 5$ and $n = 5, 6, \ldots, 10.$

| n  | p | 90%  | 95%  | 97.5% | 99%  | 90%  | 95%  | 97.5% | 99%  |
|----|---|------|------|------|------|------|------|------|------|
| 5  | 1 | 1.565| 1.789| 2.012| 2.012| 1.565| 1.789| 2.012| 2.012|
|    | 2 | 1.593| 1.873| 2.012| 2.180| 1.565| 1.901| 2.012| 2.180|
|    | 3 | 1.663| 1.982| 2.120| 2.222| 1.633| 1.982| 2.098| 2.222|
|    | 4 | 1.749| 2.053| 2.176| 2.232| 1.690| 2.053| 2.162| 2.232|
|    | 5 | 1.865| 2.102| 2.205| 2.235| 1.814| 2.102| 2.197| 2.235|
| 6  | 1 | 1.470| 1.890| 2.030| 2.170| 1.470| 1.890| 2.030| 2.170|
|    | 2 | 1.582| 1.834| 2.058| 2.254| 1.582| 1.862| 2.072| 2.254|
|    | 3 | 1.664| 1.976| 2.158| 2.332| 1.623| 1.973| 2.141| 2.332|
|    | 4 | 1.794| 1.992| 2.223| 2.368| 1.722| 1.983| 2.232| 2.368|
|    | 5 | 1.818| 2.090| 2.288| 2.395| 1.786| 2.022| 2.300| 2.395|
| 7  | 1 | 1.465| 1.795| 1.984| 2.268| 1.465| 1.795| 1.984| 2.268|
|    | 2 | 1.528| 1.858| 2.095| 2.284| 1.528| 1.858| 2.087| 2.284|
|    | 3 | 1.611| 1.940| 2.182| 2.365| 1.579| 1.936| 2.187| 2.357|
|    | 4 | 1.715| 2.070| 2.249| 2.427| 1.676| 2.012| 2.240| 2.410|
|    | 5 | 1.818| 2.090| 2.288| 2.395| 1.786| 2.022| 2.300| 2.395|
| 8  | 1 | 1.414| 1.751| 2.020| 2.290| 1.414| 1.751| 2.020| 2.290|
|    | 2 | 1.491| 1.847| 2.097| 2.338| 1.472| 1.838| 2.088| 2.338|
|    | 3 | 1.578| 1.976| 2.183| 2.417| 1.546| 1.973| 2.141| 2.413|
|    | 4 | 1.675| 2.044| 2.270| 2.491| 1.620| 2.023| 2.238| 2.472|
|    | 5 | 1.764| 2.132| 2.354| 2.543| 1.713| 2.078| 2.332| 2.461|
| 9  | 1 | 1.400| 1.750| 2.050| 2.300| 1.400| 1.750| 2.050| 2.300|
|    | 2 | 1.469| 1.825| 2.094| 2.356| 1.444| 1.812| 2.094| 2.350|
|    | 3 | 1.563| 1.925| 2.195| 2.442| 1.516| 1.897| 2.176| 2.434|
|    | 4 | 1.662| 2.027| 2.289| 2.527| 1.587| 1.990| 2.260| 2.507|
|    | 5 | 1.745| 2.123| 2.374| 2.603| 1.660| 2.073| 2.338| 2.584|
| 10 | 1 | 1.399| 1.744| 2.012| 2.319| 1.399| 1.744| 2.012| 2.319|
|    | 2 | 1.450| 1.812| 2.093| 2.374| 1.429| 1.795| 2.081| 2.366|
|    | 3 | 1.548| 1.912| 2.195| 2.467| 1.493| 1.874| 2.171| 2.450|
|    | 4 | 1.646| 2.014| 2.294| 2.558| 1.560| 1.964| 2.262| 2.531|
|    | 5 | 1.737| 2.111| 2.390| 2.637| 1.630| 2.046| 2.351| 2.609|
| $n = \infty$ | 1 | 1.282| 1.645| 1.960| 2.326| 1.282| 1.645| 1.960| 2.326|
|    | 2 | 1.324| 1.699| 2.024| 2.403| 1.303| 1.672| 1.992| 2.365|
|    | 3 | 1.398| 1.795| 2.138| 2.538| 1.341| 1.721| 2.051| 2.435|
|    | 4 | 1.480| 1.899| 2.263| 2.686| 1.384| 1.777| 2.117| 2.513|
|    | 5 | 1.562| 2.004| 2.388| 2.835| 1.428| 1.833| 2.185| 2.593|
Similarly, for $\nu^{(s,l)}_{n,p}$, $\nu^{(u)}_{n,p}$ and $\nu^{(s,u)}_{n,p}$ we have

$$\text{ARE}(\nu^{(s,l)}_{n,p}, \rho_{n,s}) = \frac{8(p+5)^2(2p+1)}{3(p+3)^2(p^2+10p+7)},$$

$$\text{ARE}(\nu^{(u)}_{n,p}, \rho_{n,s}) = \frac{16(2p+1)}{3(p+3)^2},$$

and

$$\text{ARE}(\nu^{(s,u)}_{n,p}, \rho_{n,s}) = \frac{32(p+2)^2(2p+1)}{3(p+3)^2(p^2+10p+7)}.$$  

Table 8 shows the ARE of the test of independence based on WRC measures compared to the test based on Spearman’s rho for the Clayton copula, as a family of copulas with lower tail dependence and the Cuadras-Augé family of copulas, as a family of copulas with upper tail dependence. As we see, for the Clayton family of copulas the measure $\nu^{(s,l)}_{n,11}$ and for the Cuadras-Augé family of copulas, the measure $\nu^{(s,u)}_{n,3}$ has the largest Pitman ARE. The same results is still true by using Kendall’s $\tau$ instead of Spearman’s $\rho$ since $\text{ARE}(T_\tau, T_\rho) = 1$.

5.2 Comparing the Power of Tests

In the following we compare the power of tests based on the WRC measures $\nu^{(s,l)}_{n,p}$ and $\nu^{(s,u)}_{n,p}$ for $p = 2, 3, 4, 5$ with the tests based on Kendall’s tau and Spearman’s rho for testing independence against the positive quadrant dependence \[10\], i.e.,

$$H_0 : C(u,v) = uv \quad \text{against} \quad H_1 : C(u,v) > uv.$$  

Monte Carlo simulations carry out for Gumbel, Clayton, Frank and Normal copulas with various degrees of dependence, with the sample of size $n = 50$ at significance level 0.05. The first column and the second column of the Tables 7-10 indicate, the various values of the copula parameter $\theta$ and the value of the corresponding spearman’s $\rho$ (as the level of dependence). Tables 7-10, show the power of tests that obtained under alternatives, defined by Gumbel, Clayton, Frank and normal copulas. The Clayton copula has lower tail dependence, the Gumbel copula has upper tail dependence and the normal copula has neither. We see that for the Clayton’s family of copulas for all degree of dependence in terms of Spearman’s rho ($\rho_s$), the test based on $\nu^{(s,l)}_{n,5}$ has the maximum power. For the Gumbel family of copulas, the test based on $\nu^{(s,u)}_{n,3}$ has the maximum power. For the
Table 6: ARE of test of independence based on WRC measures \( \nu_{n,p}^{(l)}, \nu_{n,p}^{(u)}, \nu_{n,p}^{(s,l)} \) and \( \nu_{n,p}^{(s,u)} \), for \( p = 1, 2, \ldots, 13 \), relative to the Spearman’s rho for the Cuadras-Augé and Clayton family of copulas.

| \( p \) | Cuadras-Augé | | | | Clayton | | | | | |
|---|---|---|---|---|---|---|---|---|---|
|   | \( \nu_{n,p}^{(l)} \) | \( \nu_{n,p}^{(u)} \) | \( \nu_{n,p}^{(s,l)} \) | \( \nu_{n,p}^{(s,u)} \) | \( \nu_{n,p}^{(l)} \) | \( \nu_{n,p}^{(u)} \) | \( \nu_{n,p}^{(s,l)} \) | \( \nu_{n,p}^{(s,u)} \) |
| 1  | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 2  | 0.816 | 1.066 | 0.843 | 1.101 | 1.157 | 0.740 | 1.194 | 0.764 |
| 3  | 0.663 | 1.037 | 0.721 | **1.127** | 1.217 | 0.583 | 1.322 | 0.634 |
| 4  | 0.551 | 0.979 | 0.629 | 1.119 | 1.235 | 0.480 | 1.411 | 0.548 |
| 5  | 0.467 | 0.916 | 0.558 | 1.095 | 1.233 | 0.407 | 1.474 | 0.486 |
| 6  | 0.404 | 0.856 | 0.502 | 1.063 | 1.221 | 0.353 | 1.518 | 0.439 |
| 7  | 0.355 | 0.800 | 0.457 | 1.028 | 1.204 | 0.312 | 1.548 | 0.401 |
| 8  | 0.316 | 0.749 | 0.419 | 0.992 | 1.184 | 0.279 | 1.569 | 0.370 |
| 9  | 0.285 | 0.703 | 0.387 | 0.956 | 1.163 | 0.253 | 1.582 | 0.344 |
| 10 | 0.258 | 0.662 | 0.360 | 0.922 | 1.142 | 0.231 | 1.589 | 0.322 |
| 11 | 0.237 | 0.625 | 0.336 | 0.888 | 1.119 | 0.213 | **1.589** | 0.302 |
| 12 | 0.218 | 0.592 | 0.316 | 0.857 | 1.096 | 0.197 | 1.585 | 0.285 |
| 13 | 0.202 | 0.562 | 0.297 | 0.827 | 1.028 | 0.183 | 1.580 | 0.270 |
Table 7: The power of tests of independence based on the Kendall’s tau ($\tau_n$), Spearman’s rho ($\rho_{n,s}$) the WRC measures $v_{n,p}^{(s,f)}$, $v_{n,p}^{(s,u)}$, $p = 2, 3, 4, 5$, computed from 50,000 samples of size 50 for Clayton copula with different values of the parameter ($\theta$) and different level of dependence in terms of Spearman’s rho ($\rho_s$).

| $\theta$ | $\rho_s$ | $v_{n,5}^{(s,f)}$ | $v_{n,4}^{(s,f)}$ | $v_{n,3}^{(s,f)}$ | $v_{n,2}^{(s,f)}$ | $v_{n,5}^{(s,u)}$ | $v_{n,4}^{(s,u)}$ | $v_{n,3}^{(s,u)}$ | $v_{n,2}^{(s,u)}$ | $\rho_{n,s}$ | $\tau_n$ |
|----------|----------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|----------|
| 0.000    | 0.000    | 0.052            | 0.051            | 0.050            | 0.050            | 0.051            | 0.051            | 0.050            | 0.050            | 0.049            | 0.050    |
| 0.050    | 0.036    | 0.095            | 0.092            | 0.090            | 0.087            | 0.071            | 0.072            | 0.075            | 0.077            | 0.082            | 0.082    |
| 0.110    | 0.078    | 0.167            | 0.163            | 0.156            | 0.147            | 0.101            | 0.105            | 0.110            | 0.118            | 0.134            | 0.135    |
| 0.200    | 0.136    | 0.309            | 0.301            | 0.289            | 0.271            | 0.158            | 0.168            | 0.183            | 0.204            | 0.242            | 0.244    |
| 0.350    | 0.221    | 0.571            | 0.560            | 0.541            | 0.512            | 0.282            | 0.306            | 0.338            | 0.384            | 0.458            | 0.461    |
| 0.750    | 0.397    | 0.937            | 0.934            | 0.928            | 0.915            | 0.641            | 0.688            | 0.742            | 0.806            | 0.880            | 0.881    |
| 1.800    | 0.652    | 0.999            | 0.999            | 0.999            | 0.999            | 0.982            | 0.989            | 0.994            | 0.998            | 0.999            | 0.999    |
| 3.200    | 0.800    | 1.000            | 1.000            | 1.000            | 1.000            | 0.999            | 0.999            | 1.000            | 1.000            | 1.000            | 1.000    |
| 5.600    | 0.900    | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000    |
| 30.000   | 0.993    | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000    |

Table 8: The power of tests of independence based on the Kendall’s tau ($\tau_n$), Spearman’s rho ($\rho_{n,s}$) the WRC measures $v_{n,p}^{(s,f)}$, $v_{n,p}^{(s,u)}$, $p = 2, 3, 4, 5$, computed from 50,000 samples of size 50 for Gumbel copula with different values of the parameter ($\theta$) and different level of dependence in terms of Spearman’s rho ($\rho_s$).

| $\theta$ | $\rho_s$ | $v_{n,5}^{(s,f)}$ | $v_{n,4}^{(s,f)}$ | $v_{n,3}^{(s,f)}$ | $v_{n,2}^{(s,f)}$ | $v_{n,5}^{(s,u)}$ | $v_{n,4}^{(s,u)}$ | $v_{n,3}^{(s,u)}$ | $v_{n,2}^{(s,u)}$ | $\rho_{n,s}$ | $\tau_n$ |
|----------|----------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|----------|
| 1.000    | 0.000    | 0.052            | 0.051            | 0.050            | 0.050            | 0.051            | 0.051            | 0.050            | 0.050            | 0.049            | 0.050    |
| 1.030    | 0.041    | 0.083            | 0.084            | 0.086            | 0.103            | 0.101            | 0.098            | 0.094            | 0.090            | 0.092            | 0.092    |
| 1.070    | 0.095    | 0.136            | 0.141            | 0.146            | 0.155            | 0.199            | 0.195            | 0.189            | 0.181            | 0.169            | 0.172    |
| 1.150    | 0.193    | 0.283            | 0.298            | 0.317            | 0.344            | 0.439            | 0.434            | 0.426            | 0.412            | 0.385            | 0.390    |
| 1.250    | 0.295    | 0.504            | 0.530            | 0.562            | 0.604            | 0.708            | 0.707            | 0.703            | 0.690            | 0.660            | 0.666    |
| 1.400    | 0.412    | 0.776            | 0.804            | 0.834            | 0.867            | 0.917            | 0.920            | 0.919            | 0.917            | 0.902            | 0.906    |
| 1.700    | 0.576    | 0.974            | 0.981            | 0.987            | 0.992            | 0.996            | 0.996            | 0.996            | 0.996            | 0.996            | 0.996    |
| 2.200    | 0.731    | 0.999            | 0.999            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000    |
| 3.000    | 0.848    | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000    |
| 4.500    | 0.930    | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000            | 1.000    |
Table 9: The power of tests of independence based on the Kendall’s tau ($\tau_n$), Spearman’s rho ($\rho_n$), the WRC measures $\nu_n^{(s,l)}$, $\nu_n^{(s,u)}$, $p = 2, 3, 4, 5$, computed from 50,000 samples of size 50 for Normal copula with different values of the parameter ($\theta$) and different level of dependence in terms of Spearman’s rho ($\rho_s$).

| $\theta$ | $\rho_s$ | $\nu_n^{(s,l)}_{n,5}$ | $\nu_n^{(s,l)}_{n,4}$ | $\nu_n^{(s,l)}_{n,3}$ | $\nu_n^{(s,l)}_{n,2}$ | $\nu_n^{(s,u)}_{n,5}$ | $\nu_n^{(s,u)}_{n,4}$ | $\nu_n^{(s,u)}_{n,3}$ | $\nu_n^{(s,u)}_{n,2}$ | $\rho_n/s$ | $\tau_n$ |
|--------|--------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|--------|--------|
| 0.000  | 0.000  | 0.052             | 0.051             | 0.051             | 0.050             | 0.052             | 0.051             | 0.051             | 0.050             | 0.050  | 0.050  |
| 0.040  | 0.038  | 0.083             | 0.083             | 0.083             | 0.083             | 0.083             | 0.083             | 0.083             | 0.083             | 0.084  |        |
| 0.070  | 0.066  | 0.115             | 0.115             | 0.117             | 0.115             | 0.115             | 0.116             | 0.116             | 0.117             | 0.118  |        |
| 0.120  | 0.114  | 0.185             | 0.187             | 0.190             | 0.193             | 0.184             | 0.187             | 0.189             | 0.192             | 0.195  | 0.196  |
| 0.200  | 0.191  | 0.343             | 0.351             | 0.358             | 0.366             | 0.342             | 0.350             | 0.357             | 0.366             | 0.373  | 0.374  |
| 0.300  | 0.287  | 0.594             | 0.607             | 0.620             | 0.635             | 0.590             | 0.604             | 0.618             | 0.632             | 0.646  | 0.646  |
| 0.400  | 0.384  | 0.818             | 0.831             | 0.844             | 0.857             | 0.816             | 0.829             | 0.842             | 0.856             | 0.868  | 0.867  |
| 0.550  | 0.532  | 0.978             | 0.982             | 0.985             | 0.988             | 0.978             | 0.982             | 0.985             | 0.988             | 0.990  | 0.990  |
| 0.750  | 0.734  | 1.000             | 1.000             | 1.000             | 1.000             | 1.000             | 1.000             | 1.000             | 1.000             | 1.000  | 1.000  |
| 0.950  | 0.945  | 1.000             | 1.000             | 1.000             | 1.000             | 1.000             | 1.000             | 1.000             | 1.000             | 1.000  | 1.000  |

In this paper we have presented a class of weighted rank correlation measures extending the Spearman’s rank correlation coefficient. The proposed class was constructed by giving suitable weights to the distance between two sets of ranks to place more emphasis on items having low rankings than those have high rankings, or vice versa. The asymptotic distributions of the proposed measures in general and under the null hypothesis of independence are derived. We also carried out a simulation study to compare the performance of the proposed measures with the Spearman’s and Kendall’s rank correlation measures.

Another line of research is the extension of the result to the situations where $n$ objects are ranked by $m > 2$ independent sources and the interest is focused on agreement on the bottom or top rankings.

6 Discussion

In this paper we have presented a class of weighted rank correlation measures extending the Spearman’s rank correlation coefficient. The proposed class was constructed by giving suitable weights to the distance between two sets of ranks to place more emphasis on items having low rankings than those have high rankings, or vice versa. The asymptotic distributions of the proposed measures in general and under the null hypothesis of independence are derived. We also carried out a simulation study to compare the performance of the proposed measures with the Spearman’s and Kendall’s rank correlation measures.

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