INTEGRABLE GEODESIC FLOWS ON NILMANIFOLDS:
NEW EXAMPLES

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Abstract. In this work we study the geodesic flow on nilmanifolds associated to graphs. We find explicit first integrals to show complete integrability of the geodesic flow on compact quotients, associated to singular Lie algebras. Also general graphs are studied and more examples of integrable geodesic flows as of non-integrable Lie algebras are shown.

1. Introduction

The study of the geodesic flow on a given Riemannian manifold \( M \) is a classical topic in geometry. The integrability of the geodesic flow imposes obstructions to the topology of the supporting manifold \( [23, 24, 19] \). However integrability is not well understood. Even under restricted conditions, for instance on locally homogeneous manifolds, there is no general theory to determine the complete integrability of the geodesic flow.

Lie groups and their compact quotients were already used to answer nice questions \([2, 3, 4, 5, 20]\). For 2-step nilpotent Lie groups Butler introduced the notion of non-integrable Lie algebras, and proved that the geodesic flow on manifolds arising from Lie groups with non-integrable Lie algebras cannot be integrable. It is not hard to see that non-integrable Lie algebras are singular. In the same work Butler \([6]\) proved non-commutative integrability for manifolds associated to almost non-singular Lie algebras. Since Bolsinov and Jovanovic \([2]\) proved that integrability in the non-commutative sense implies Liouville integrability, we get integrability on manifolds \( M = \Gamma \backslash N \) arising from almost non-singular Lie algebras. Thus we have some questions between the type of the Lie algebra \( n \), as non-singular, almost non-singular or singular, and the integrability of the geodesic flow on \( M = \Gamma \backslash N \):

\[
\begin{align*}
n \text{ almost non-singular} & \implies \text{integrability on } M; \\
n \text{ non-integrable} & \implies \text{the geodesic flow on } M \text{ non-integrable.}
\end{align*}
\]

It is natural to ask about the converse of the implications above. In this work we deal with these questions. We study a family of nilmanifolds arising from graphs. Dani and Mainkar \([7]\) introduced these 2-step nilpotent Lie algebras \( n_G \) to study automorphisms of the Lie algebra which give rise to Anosov automorphisms on nilmanifolds. More recently, in \([8]\) the authors find Heisenberg like algebras and conditions are given on the graph \( G \) and on a lattice \( \Gamma \subset N \) for which the quotient \( \Gamma \backslash M \), a compact nilmanifold, has a dense set of smoothly closed geodesics.

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We start the work by recalling the construction of the Lie groups from graphs and the existence of lattices. We offer preliminaries to study the geodesic flow on nilmanifolds. More details can be found in [15]. Summarising our results we get:

- **There exists a family of compact manifolds** $M = \Gamma \backslash N$ **with integrable geodesic flow such that the Lie algebra of $N$ is singular.** They are associated to the star graphs on $k+1$-vertices, $S_k$, for $k \leq 3$. For $k = 2$ the Lie algebra arising from the graph is the Heisenberg Lie algebra of dimension three. Topologically the compact quotients are $T^{2k}$-bundles over $S^1$.

- **Nilmanifolds $\Gamma \backslash N$ for $N$ constructed from complete graphs on $s$-vertices, $K_s$, are**
  - non-integrable for $s = 2n + 1$. This generalizes the example in Butler [6].
  - almost non-singular for $s = 2n$.

- **Let $G$ denote a connected graph on $k$ vertices with $k \leq 4$. Then except for the complete graph $K_3$, any 2-step nilpotent Lie group $N_G$ so as the corresponding compact quotient admits a completely integrable geodesic flow.**

Tools and techniques used here have no direct relationship with the previous theory developed in the 80’s. We refer to Hamiltonian sytems constructed with an algebraic data as in the Adler-Kostant-Symes scheme [16, 22], or the work by Thimm concerning the geodesic flow in [25]. In these cases the underlying manifold was related to semisimple Lie groups, where ad-invariant functions induce first integrals once there exists an ad-invariant metric on the Lie algebra. The point of having an ad-invariant metric makes the adjoint and coadjoint representation equivalent. On solvable and a more general class of Lie algebras, an ad-invariant metric does not necessarily exist and this makes difficult the application of these tools. Also the question of complete integrability on orbits on Lie algebras were considered in [12, 13, 18]. In the present paper we study invariant functions and we relate them to coadjoint orbits via the Gauss map. Notice that invariant functions are trivially induced to any compact quotient $\Gamma \backslash N$. That is the reason to search first integrals of this kind.

### 2. Graphs and 2-step nilpotent Lie groups

Let $N$ denote a 2-step nilpotent Lie group equipped with a left-invariant metric $\langle , \rangle$. Its Lie algebra $\mathfrak{n}$ decomposes as the orthogonal sum

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad \text{where} \quad \mathfrak{v} = \mathfrak{z}^\perp$$

where $\mathfrak{z}$ is the center of $\mathfrak{n}$. In this situation each element $Z \in \mathfrak{z}$ induces a skew-symmetric linear map on $\mathfrak{v}$, $j(Z) : \mathfrak{v} \to \mathfrak{v}$, given by

$$\langle j(Z)U, V \rangle = \langle [U, V], Z \rangle$$

for all $U, V \in \mathfrak{v}$. The geometry of $N$ is encoded in the maps $j(Z)$ [9].

Recall that whenever $N$ is simply connected, the exponential map, $\exp : \mathfrak{n} \to N$, is a diffeomorphism with inverse map $\log : N \to \mathfrak{n}$. Moreover one has the formula:

$$\exp(X) \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y]) \quad \text{for all} \ X, Y \in \mathfrak{n}.$$
It is not hard to see that by choosing a basis of \( n \), namely \( X_1, \ldots, X_n \) so that a general left-invariant vector field \( X \) is written as \( X = \sum_{i=1}^{n} X_i \) then the exponential maps \( X \) to \( (x_1, x_2, \ldots, x_n) \) for the usual coordinates in \( \mathbb{R}^n \).

As usual we say that a lattice in \( N \) is a discrete closed subgroup \( \Lambda \) such that the quotient space \( \Lambda \backslash N \) is compact. The resulting compact manifold \( M := \Lambda \backslash N \) is called a compact nilmanifold.

The question of existence of lattices in nilpotent Lie groups is treated in the next result.

**Theorem 2.1.** [21] A simply-connected nilpotent Lie group \( N \) admits a lattice if and only if there exists a basis \( \{X_1, \ldots, X_n\} \) of the Lie algebra \( n \) of \( N \) such that the structure constants arising in the brackets \( [X_i, X_j] = \sum_k C_{ij}^k X_k \) are rational numbers. More precisely:

(i) Let \( n \) have a basis with respect to which the structure constants are rational.

Let \( n_\mathbb{Q} \) be the vector space over \( \mathbb{Q} \) spanned by this basis. Then, if \( L \) is any lattice of maximal rank in \( n \) contained in \( n_\mathbb{Q} \), the group generated by \( \exp(L) \) is a lattice in \( N \).

(ii) If \( \Gamma \) is a lattice in \( N \), then the \( \mathbb{Z} \)-span of \( \log(\Gamma) \) is a lattice \( \mathcal{L} \) of maximal rank in the vector space \( n \) such that the structure constants with respect to any basis contained in \( \mathcal{L} \) belong to \( \mathbb{Q} \).

For a given lattice \( \Gamma \) in a connected and simply-connected nilpotent Lie group \( G \), the subset \( \log(\Gamma) \) need not to be an additive subgroup of the Lie algebra \( n \). The de Rham cohomology of a nilmanifold can be done using invariant differential forms, i.e. by the Chevalley-Eilenberg cohomology \( H^*(n) \) of \( n \). This is the case if the Mostow condition holds, namely, the algebraic closures \( \mathcal{A}(Ad_G(G)) \) and \( \mathcal{A}(Ad_G(\Gamma)) \) are equal, see for instance [21], Corollary 7.29].

**Theorem (Nomizu [36])** Let \( M = \Gamma \backslash N \) be a compact nilmanifold. Then the inclusion of left-invariant differential forms in the de Rham complex

\[ \Lambda^*n^* \hookrightarrow \mathcal{A}^*(M) \]

induces an isomorphism between the Lie-algebra cohomology of \( n \) and the de Rham cohomology of \( M \),

\[ H^*(n, \mathbb{R}) \simeq H^*_d(M, \mathbb{R}). \]

A distinguished family of 2-step nilpotent Lie algebras can be constructed starting with a graph \( G \). Let \( G \) be a directed graph with at least one edge. Denote the vertices of \( G \) by \( S = \{X_1, \ldots, X_m\} \), the edges of \( G \) by \( E = \{Z_1, \ldots, Z_q\} \).

The Lie algebra \( n_G \) is the vector space direct sum \( n_G = v \oplus \mathfrak{z} \) where we let \( E \) be a basis over \( \mathbb{R} \) for \( \mathfrak{z} \) and \( S \) be a basis over \( \mathbb{R} \) for \( v \). Define the bracket relations among elements of \( S \) according to adjacency rules:

- if \( Z_k \) is a directed edge from vertex \( X_i \) to vertex \( X_j \) then define the skew-symmetric bracket \( [X_i, X_j] = Z_k \).
- If there is no edge between two vertices, then define the bracket of those two elements in \( S \) to be zero.
Extend the bracket relation to all of $\mathfrak{v}$ using bilinearity of the bracket.

**Remark 2.2.** In [17] it was proved that the two-step nilpotent Lie algebras associated with two graphs are Lie isomorphic if and only if the graphs from which they arise are isomorphic.

Choose the inner product on $\mathfrak{n}_G$ so that $S \cup E$ is an orthonormal basis for $\mathfrak{n}_G$.

The maps $j(Z) : \mathfrak{v} \to \mathfrak{v}$ are then given by Definition 1 and are linear over $Z \in \mathfrak{z}$.

Observe that if $Z_k$ is a directed edge from $X_i$ to $X_l$ then $j(Z_k)X_i = X_l$. Note that $j(Z_k)X_p = 0$ for any other $X_p \in S$ where $p \neq i, l$.

**Example 2.3.** The star graph $S_k$ has $k + 1$ vertices $V_0, V_1, \ldots, V_k$ and edges $Z_i$ that relate the vertices $V_0$ and $V_i$. So in the above construction the Lie bracket gives $[V_0, V_i] = Z_i$. Thus $j(Z_i)X_k = 0$ for all $k = 1, \ldots, n$ except for $k = i$, which shows that $j(Z_i)$ is singular whenever $i \geq 2$.

Setting $Z = a_1Z_1 + \ldots + a_nZ_n$, the matrix representation of $j(Z) : \mathfrak{v} \to \mathfrak{v}$ in the basis $V_0, V_1, \ldots, V_k$, is given by

$$
\begin{pmatrix}
0 & -a_1 & -a_2 & \ldots & -a_n \\
a_1 & 0 & 0 & \ldots & 0 \\
a_2 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & 0 & 0 & \ldots & 0
\end{pmatrix}.
$$

Therefore $j(Z)$ is singular for $k + 1 > 2$, that is for the star graph with $k + 1 > 2$ vertices.

**Example 2.4.** The complete graph on $n$ vertices $K_n$ is the graph which has an edge between every pair of distinct vertices.

The graph $K_2$ corresponds to the Heisenberg Lie algebra of dimension three, it has two vertices $V_1, V_2$ and one edge $Z$,

$$
\begin{tikzpicture}
  \node (V1) at (0,0) [circle,draw] {$V_1$};
  \node (V2) at (1,0) [circle,draw] {$V_2$};
  \node (Z) at (0.5,0.5) [circle,draw] {}; 
  \draw [->,thick] (V1) -- (Z) node [midway,above] {$Z$} -- (V2);
\end{tikzpicture}
$$

so that the corresponding map $j(Z)$ is non-singular.

On the other hand the complete graph $K_3$ has three vertices $V_1, V_2, V_3$ and three edges $Z_1, Z_2, Z_3$.

$$
\begin{tikzpicture}
  \node (V1) at (0,0) [circle,draw] {$V_1$};
  \node (V2) at (1,1) [circle,draw] {$V_2$};
  \node (V3) at (2,0) [circle,draw] {$V_3$};
  \node (Z1) at (0.5,0.5) [circle,draw] {}; 
  \node (Z2) at (1.5,0.5) [circle,draw] {}; 
  \node (Z3) at (1,0) [circle,draw] {}; 
  \draw [->,thick] (V1) -- (Z1) node [midway,above] {$Z_1$} -- (V2) node [midway,above] {$Z_2$} -- (V3); 
  \draw [->,thick] (V1) -- (Z3) node [midway,above] {$Z_3$};
\end{tikzpicture}
$$

Let $\mathfrak{n}_{K_3}$ denote the corresponding 2-step nilpotent Lie algebra where we have the Lie brackets

$$
[V_1, V_2] = Z_1 \quad [V_2, V_3] = Z_2 \quad [V_3, V_1] = Z_3.
$$
Let $\langle , \rangle$ denote the metric on $\mathfrak{n}_{K_3}$ that makes this basis into an orthonormal basis. The map $j(Z) : \mathfrak{v} \to \mathfrak{v}$ has a matrix presentation

$$j(aZ_1 + bZ_2 + cZ_3) = \begin{pmatrix} 0 & -a & -c \\ a & 0 & -b \\ c & b & 0 \end{pmatrix}$$

in the basis $\{V_1, V_2, V_3\}$ of $\mathfrak{v}$. Notice that dimension $\ker j(Z) = 1$, for all $Z \in \mathfrak{z} - \{0\}$.

More generally consider the complete graph on $n$ vertices, $K_n$. The dimension of the center is $\dim \mathfrak{z} = \binom{n}{2} = \dim \mathfrak{so}(n)$. Thus

- if $n$ is odd, every $j(Z)$ is a singular map for every $Z \in \mathfrak{z}$;
- if $n$ is even, there exists $Z \in \mathfrak{z}$ such that $j(Z)$ is non-singular. In fact, assuming $V_1, V_2, \ldots, V_2s$ are vertices and $Z_{ij} = [V_i, V_j]$, take the non-singular map $j(Z)$ with matrix

$$j(Z_{12} + Z_{34} + \ldots Z_{2s-1,2s}) = \begin{pmatrix} 0 & -1 & \cdots & 0 \\ 1 & 0 & \cdots & 1 \\ & 0 & -1 & \cdots \\ & & \ddots & \ddots \\ & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix}.$$  

The maps $j(Z)$ enable the following definitions. A 2-step nilpotent Lie algebra $\mathfrak{n}$ is said

- **non-singular** if $\text{ad}(X) : \mathfrak{n} \to \mathfrak{z}$ is surjective for all $X \notin \mathfrak{z}$. The fact of being $\mathfrak{n}$ non-singular is equivalent to asking $j(Z)$ non-singular for any $Z \in \mathfrak{z} - \{0\}$ for a (any) metric on $\mathfrak{n}$ (in fact this does not depend on the choice of the left-invariant metric, see for instance [9];
- **almost non-singular** if $j(Z)$ is non-singular for every $Z$ in an open dense subset of $\mathfrak{z}$.
- **singular** if $j(Z)$ is singular for all $Z$ in $\mathfrak{z}$.

Every 2-step nilpotent Lie algebra belongs to one and only one of the types non-singular, almost non-singular or singular [14].

In particular whenever a 2-step nilpotent Lie algebra $\mathfrak{n}$ is equipped with a metric, if there are two nonzero elements $Z, Z' \in \mathfrak{z}$ such that $j(Z)$ is non-singular and $j(Z')$ is singular, then the Lie algebra $\mathfrak{n}$ is almost non-singular.

**Remark 2.5.** Assume that $G$ is a graph with at least one edge. If the graph $G$ is isomorphic to the complete graph $K_2$ then its Lie algebra $\mathfrak{n}_G$ is non-singular (this Lie algebra is also isomorphic to the Heisenberg Lie algebra of dimension three).

Assume $G$ is not isomorphic to $K_2$. Thus $G$ contains an edge $Z$ and a vertex $X$ for which the edge $Z$ is not incident to the vertex $X$. Then $j(Z)X = 0$, hence $\mathfrak{n}_G$ is either almost non-singular or singular. See Lemma 3.3 in [8].

**Lemma 2.6.** [8] Let $G$ be a graph with at least one edge. The Lie algebra $\mathfrak{n}_G$ is non-singular if and only if $G = K_2$, the complete graph on two vertices.
In general the Lie algebra \( n_G \) arising from a graph \( G \) will be almost non-singular or singular.

**Definition 2.7.** Let \( n \) be a 2-step nilpotent Lie algebra.

(i) For \( \lambda \in n^* \), let \( n_{\lambda} := \{ X \in n : \text{ad}^*(X) \lambda = 0 \} \).

(ii) A \( \lambda \in n^* \) is called regular if \( n_{\lambda} \) has minimal dimension.

(iii) \( n \) is called non-integrable if there exists a dense open subset \( \mathcal{W} \) of \( n^* \times n^* \) such that for each \( (\lambda, \mu) \in \mathcal{W} \), both \( \lambda \) and \( \mu \) are regular and \([n_{\lambda}, n_{\mu}]\) has positive dimension.

We can read this non-integrability property at the Lie algebra level. Let \( n \) denote a Lie algebra equipped with an inner product \( \langle , \rangle \) and assume one has the splitting as direct sum of vector spaces \( n = v \oplus z \) with \( v = z^\perp \).

For any \( \lambda \in n^* \) there exists a unique \( Z \in z, V \in v \) such that \( \lambda = \ell_{V+Z} \) where \( \ell_{V+Z}(X) = \langle V + Z, X \rangle \). Thus we denote \( n_{\ell_{V+Z}} \) directly by \( n_{V+Z} \). So

\[
\begin{align*}
n_{V+Z} &= \{ X \in n : \langle V + Z, \text{ad}(X)U \rangle = 0 \text{ for all } U \in n \} \\
&= \begin{cases} 
n & \text{if } Z = 0 \\
z \oplus \text{ker } j(Z) & \text{if } Z \neq 0
\end{cases}
\end{align*}
\]

Assume \( \ell_{V+Z} \) is regular, thus for \( n \) non-singular or almost non-singular, it is clear that \( n_{V+Z} = z \). In both cases for both \( \lambda, \mu \) regular, one has \([n_{\lambda}, n_{\mu}] = 0\).

**Corollary 2.8.** Let \( n \) denote a non-integrable Lie algebra \( n \) then \( n \) must be singular.

**Example 2.9.** In [6] Butler showed an example of a non-integrable Lie algebra. Its Lie group has Lie algebra isomorphic to \( K_3 \).

More generally assume that \( n \) is odd, \( n = 2k+1 > 2 \), and \( G = K_n \) is the complete graph. As mentioned above the Lie algebra \( n_G \) is singular. Roughly speaking the graph \( K_{2k+1} \) can be constructed from the complete graph \( K_{2k} \) by adding one vertex and all the edges joining the added vertex with the previous ones.

Let \( S \) denote the set of vertices, assume \(|S| = n \) with \( n = 2k+1 \) and choose the vertex \( V_{2k+1} \in S \). Denote by \( Z_{ij} \) (or \( Z_{i,j} \)) the basis element in \( z \) such that \( Z_{ij} := [V_i, V_j] \).

Take \( Z \in z \) defined as \( Z := \sum_{i=1}^{k} Z_{2i-1,2i} \in z \). Then the restriction of \( j(Z) \) to the vector subspace \( v_1 \) spanned by \( V_1, V_2, \ldots, V_{2k} \) - see matrix in Example 2.4 -, is a non-singular linear map. And \( V_{2k+1} \in \text{Ker } j(Z) \) since \( j(Z_{2i-1,2i})V_{2k+1} = 0 \) for all \( i \).

Analogously take \( V_1 \) in the kernel of \( j(\bar{Z}) \) for \( \bar{Z} = \sum_{i=1}^{k} Z_{2i,2i+1} \) so that the restriction of \( j(\bar{Z}) \) is non-singular on the vector subspace \( v_2 \) of dimension \( 2k \) spanned by \( V_2, V_3, \ldots V_{2k+1} \).

It is clear that \( \ell_{Z} \) and \( \ell_{\bar{Z}} \) are regular in \( n_G^{\Phi} \). Moreover \( V_1 \in n_{\bar{Z}}, V_{2k+1} \in n_Z \) and \([V_1, V_{2k+1}] = Z_{1,2k+1} \). This implies that \( K_{2k+1} \) is non-integrable.

**Proposition 2.10.** Let \( n = 2k+1 \in \mathbb{N} \) with \( k \geq 1 \) and let \( G = K_n \) denote the complete graph on \( n \) vertices. The corresponding Lie algebra \( n_G \) is non-integrable.

We summarize some results concerning the structure of a graph in relation of the question of singularity or not, which was investigated in [8].
**Proposition 2.11.** Let \( G \) be a graph. Then the associated 2-step nilpotent Lie algebra \( n_G \) is either singular or almost non-singular.

Further if \( G \) has more than one connected component, \( n_G \) is almost non-singular if and only if each connected component is either non-singular or almost non-singular.

Let \( G \) be a graph, where \( S \) is the set of vertices. Assume \(|S| > 1\) is odd. Let \( n_G \) be the associated 2-step nilpotent Lie algebra. For any \( Z \in \mathfrak{z} \) the corresponding map \( j(Z) \) is skew symmetric on an odd dimensional vector space \( v \). Thus it always has 0 as eigenvalue of \( j(Z) \) and \( n_G \) is non-singular.

Assume \(|S| > 2\) is even. Then the associated 2-step nilpotent Lie algebra is either singular or almost non-singular. Let \( Z_1, \ldots, Z_n \) denote the set of edges of \( G \). By construction if \(|S| > 2\) the \( j(Z_i) \) is non-singular. However it could be possible to have a \( Z \in \mathfrak{z} \) such that \( j(Z) \) is non-singular. And so \( n_G \) would be almost non-singular.

Let \( G \) be a graph with vertices \( V_1, V_2, \ldots, V_{2n} \). We say that \( G \) has a vertex covering by \( n \) disjoint copies of \( K_2 \) if there a permutation \( \sigma \in S_{2n} \) such that \( X_{\sigma(2i−1)}X_{\sigma(2i)} \) is an edge of \( G \), for all \( i = 1, 2, \ldots, n \) where \( S_{2n} \) denotes the symmetric group.

**Proposition 2.12.** Let \( G \) be a graph with \( 2n \) vertices and \( n > 1 \). The associated 2-step nilpotent Lie algebra \( n_G \) is almost non-singular if and only if \( G \) has a vertex covering by \( n \) disjoint copies of \( K_2 \).

**Example 2.13.** Connected graphs in low number of vertices and its type.

(i) \( n = 2 \): the complete \( K_2 \): non singular,
(ii) \( n = 3 \): the complete \( K_3 \): singular, the star graph \( S_3 \): singular;
(iii) \( n = 4 \): The star graph \( S_4 \) and the (corresponding to) almost-non-singular: the complete \( K_4 \), the cycle \( C_4 \), the path in four vertices \( P_4 \), \( G_1 \) and \( G_2 \). See [8]. The graphs \( G_1 \) and \( G_2 \) are the two non-isomorphic graphs one obtains as subgraphs of \( K_4 \), by erasing one or two edges.

3. The geodesic flow on nilmanifolds

Here we consider the geodesic flow on Lie groups equipped with a left-invariant metric. In the first part we recall the general setting and in the second part we concentrate in the study of invariant first integrals. For a given Lie group \( N \) equipped
with a left-invariant metric with Lie algebra \( n \), the symplectic structure on the tangent bundle \( T N \) is related to that one on coadjoint orbits.

Let \( T N \) denote the tangent bundle of the Lie group \( N \). The geodesic field arises as the Hamiltonian vector field of the energy function \( E : T N \to \mathbb{R} \),

\[
E(p, Y) = \frac{1}{2} \langle Y, Y \rangle.
\]

Recall that \( T N \) as a Lie group is the direct product of \( N \) with the abelian Lie group \( n \). The pair \( (p, Y) \in N \times n \) corresponds to the tangent vector \( dL_p(Y) \in T_p N \), for \( L_p : N \to N \) the multiplication on the left by \( p \). A tangent vector \( v \in T_p N \) is associated to \( (p, V) \in T_p N \), where \( V \) is the left-invariant vector field with \( V_e = (dL_p^{-1})_p v \). So the tangent bundle \( T(T N) \cong N \times n \times n \times n \) is

\[
\{(p, Y, U, V) : p \in N, Y, U, V \in n \}.
\]

The geodesic field on \( T N \) is the vector field associated with the geodesic flow \( \Phi_t(p, Y) = \gamma'(t) \)

where \( \gamma(t) \) is the geodesic on \( N \) with initial conditions \( \gamma(0) = p, \gamma'(0) = (p, Y) \). To study the integrability of the geodesic flow one introduces a Poisson bracket on \( C^\infty T N \), denoted by \( \{, \} \):

\[
\{f, g\} = \Omega(X_f, X_g), \text{ for } f, g \in C^\infty T N
\]

where \( \Omega \) is the canonical symplectic form on \( T N \), and \( X_f, X_g \) are the Hamiltonian vector fields of \( f \) and \( g \) respectively. Recall that for a smooth function \( h : T N \to \mathbb{R} \), the Hamiltonian vector field of \( h \), denoted by \( X_h \) is implicitly given by

\[
dh_{(p, Y)}(U, V) = \Omega_{(p, Y)}(X_h(p, Y), (U, V)).
\]

On the other hand, the gradient field of \( h \), denoted by \( \text{grad} h \) is given by

\[
dh_{(p, Y)}(U, V) = \langle \text{grad}_{(p, Y)} h, (U, V) \rangle,
\]

where the metric on \( T N \) is the product metric:

\[
\langle (U, V), (U', V') \rangle_{(p, Y)} = \langle U, U' \rangle + \langle V, V' \rangle.
\]

We say that a smooth function \( f : T N \to \mathbb{R} \) is a first integral of the geodesic flow if \( f \) is constant along the integral curves of the geodesic field, or equivalently if

\[
\{f, E\} = 0.
\]

Notice that the gradient field of the energy function for a left-invariant metric is \( \text{grad} E(p, Y) = (0, Y) \). In fact \( \frac{d}{ds}_{s=0} \frac{1}{2} \langle Y + sV, Y + sV \rangle = \langle Y, V \rangle \).

The proof of the following proposition arises from definitions above. See [15].

**Proposition 3.1.** Let \( N \) denote a Lie group equipped with a left-invariant metric. Let \( f, g \in C^\infty(T N) \) be smooth functions with \( \text{grad}_{(p, Y)} f = (U, V) \) and \( \text{grad}_{(p, Y)} g = (U', V') \). Then

(i) the Hamiltonian vector field for \( f : T N \to \mathbb{R} \) smooth is

\[
X_f(p, Y) = (V, \text{ad}^t(V)(Y)U),
\]

where \( \text{ad}^t(V) \) denotes the transpose of \( \text{ad}(V) \) with respect to the metric on \( n \).
(ii) the Poisson bracket follows
\[
\{f, g\}(p, Y) = \langle U, V' \rangle - \langle V, U' \rangle + \langle Y, [V', V] \rangle.
\]
In particular, for a 2-step nilpotent Lie group one has:

- For the energy function \( E : TN \to \mathbb{R} \), its Hamiltonian vector field is
  \[
  X_E(p, Y) = (Y, j(Y)Y_v).
  \]

- A function \( f : TN \to \mathbb{R} \) with gradient \( \nabla f(p, Y) = (U, V) \) is a first integral of the geodesic flow on \( TN \) if and only if
  \[
  \langle Y, U \rangle = \langle j(Y)V_v, Y_v \rangle.
  \]

Let \( f, g : TN \to \mathbb{R} \) be smooth functions. We say that they are in involution whenever they Poisson commute, that is
\[
\{f, g\} = 0.
\]

Thus first integrals of geodesic flow are those functions in involution with the energy function. Also the definitions above imply that \( \{f, g\} = 0 \) if and only if \( df(X_g) = 0 \) if and only if \( X_g(f) = 0 \) (and also \( X_f(g) = 0 \)). Therefore \( f \) is constant along integral curves of \( X_g \) (analogously for \( X_f \)).

A main trouble is to find functions which are in involution. There exist some constructions. For instance if \( M \) is a Riemannian manifold and \( X^* \) is a Killing field on \( M \), then the function \( f_{X^*} : TM \to \mathbb{R} \) defined as \( f_{X^*}(v) = \langle X^*(\pi(v)), v \rangle \) is a first integral of the geodesic flow. But in general it is not clear if one can produce enough functions in involution for proving the complete integrability.

**Definition 3.2.** Let \((M, \langle , \rangle)\) be a Riemannian manifold. It has completely integrable geodesic flow (in the sense of Liouville) if there exist \( n \) first integrals of the geodesic flow, \( f_i : TM \to \mathbb{R} \), where \( n = \dim M \), such that \( \{f_i, f_j\} = 0 \) for all \( i, j \) and the gradients of \( f_1, \ldots, f_n \) are linear independent on an open dense subset of \( TM \).

One can prove that a given system cannot be completely integrable. For 2-step nilpotent Lie groups one has an algebraic condition which implies the non-integrability.

Let \( N \) denote a Lie group equipped with a left invariant metric \( g \). We say that \( \Lambda \subset N \) is a lattice if \( \Lambda \) is a discrete subgroup such that the quotient \( \Lambda \backslash N \) is compact.

**Theorem 3.3.** ([6, Theorem 1.3]) Let \( \mathfrak{n} \) be a non-integrable 2-step nilpotent Lie algebra with associated simply connected Lie group \( N \). Assume that there exists a discrete, cocompact subgroup \( \Lambda \) of \( N \). Then for any such \( \Lambda \) and any left-invariant metric \( g \) on \( N \), the geodesic flow of \((\Lambda \backslash N, g)\) is not completely integrable.

**Example 3.4.** Proposition [2.10] and the previous theorem imply that starting with the complete graph \( K_n \), with \( n \) odd, and equipped with any left-invariant metric, the corresponding geodesic flow is not completely integrable on any compact quotient.
3.1. Invariant functions and the coadjoint action. The goal now is the study of invariant functions. We clarify the relationship between the tangent space and the coadjoint orbits with help of the Gauss map. The importance of invariant functions in relation with the complete integrability problem is that they descend to any compact quotient \( \Lambda \setminus N \) for any lattice \( \Lambda \subset N \).

Let \( N \) denote a Lie group. The natural action of \( N \) on \( TN \cong N \times \mathfrak{n} \) is given by \( n \cdot (p,Y) = (np,Y) \). We say that \( f : TN \to \mathbb{R} \) is invariant if \( f(p,Y) = f(e,Y) \) for all \( p \in N, Y \in \mathfrak{n} \).

As an example notice that the energy function on \( TN \) is invariant whenever the metric is left-invariant.

Concerning invariant functions in \( C^\infty(TN) \) for a Lie group \( N \), we can see the following.

(i) The gradient field for an invariant function \( F : TN \to \mathbb{R} \) has the form \( \text{grad} p, Y \rangle = (0, V) \) for some \( V \in \mathfrak{n} \). Thus the corresponding Hamiltonian vector field is given by

\[
X_F(p,Y) = (V, \text{ad}^t(V)Y)
\]

where \( \text{ad}^t(V) \) denotes the transpose of \( \text{ad}(V) \) relative to the metric on \( \mathfrak{n} \).

(ii) The set of invariant functions \( \{ F : TN \to \mathbb{R} : F \text{ invariant} \} \) is in correspondence with the set of functions on \( \mathfrak{n} \): \( \{ f : \mathfrak{n} \to \mathbb{R} \} \).

In fact, given an invariant function \( F : TN \to \mathbb{R} \) define \( f : \mathfrak{n} \to \mathbb{R} \) as \( f(Y) = F(e,Y) \) and conversely given \( f : \mathfrak{n} \to \mathbb{R} \) define an invariant function \( F : TN \to \mathbb{R} \) by

\[
F(p,Y) = f(Y) \quad \text{for all } p \in N, Y \in \mathfrak{n}.
\]

(iii) Let \( F_1, F_2 : TN \to \mathbb{R} \) be invariant functions with corresponding gradients \( \text{grad}(F_i)(p,Y) = (0, V_{F_i}) \) for \( i = 1, 2 \). Then their Poisson bracket is

\[
\{ F_1, F_2 \}(p,Y) = -\langle Y, [V_{F_1}, V_{F_2}] \rangle
\]

In fact one has

\[
\begin{align*}
\{ F_1, F_2 \}(p,Y) &= \langle V_{F_1}, \text{ad}^t(V_{F_2})Y \rangle - \langle V_{F_2}, \text{ad}^t(V_{F_1})Y \rangle + \langle Y, [V_{F_1}, V_{F_2}] \rangle \\
&= -\langle Y, [V_{F_1}, V_{F_2}] \rangle.
\end{align*}
\]

Notice that \( V_{F_i} = \text{grad}_n F_i \) where we think the invariant function \( F_i \) as \( F_i : \mathfrak{n} \to \mathbb{R} \) due to the correspondence.

The next proposition specifies some invariant functions which are first integrals of the geodesic flow. See the proof in [15].

**Proposition 3.5.** Let \((N, \langle \cdot, \cdot \rangle)\) be a Lie group with a left-invariant metric.

(i) The function \( f_{Z_0} : TN \to \mathbb{R} \), defined by \( f_{Z_0}(p,Y) = \langle Y, Z_0 \rangle \), is a first integral of the geodesic flow for all \( Z_0 \in \mathfrak{z} \). Moreover, the family \( \{ f_{Z_0} \}_{Z_0 \in \mathfrak{z}} \) is a Poisson-commutative family of first integrals.

(ii) Let \( A : \mathfrak{n} \to \mathfrak{n} \) be a symmetric endomorphism of \( \mathfrak{n} \) and let \( g_A : TN \to \mathbb{R} \) denote the quadratic polynomial given by \( g_A(p,Y) = \frac{1}{2} \langle Y, AY \rangle \). Then \( g_A \) is a first integral of the geodesic flow if and only if \( \langle Y, [AY, Y] \rangle = 0 \) for all \( Y \in \mathfrak{n} \).
Many symplectic manifolds are the orbits of the coadjoint actions translated via de metric. A given Lie group $N$ acts on the left on the dual space $n^*$ by the coadjoint action: $(g \cdot \xi)(Y) = \xi \circ Ad(g^{-1})(Y)$. Assuming a metric on the Lie algebra $n$, the coadjoint action is induced to $n$ by:

$$g \cdot X = Ad^t(g^{-1})(X)$$

for all $g \in N, X \in n$,

where $Ad^t(h)$ denotes the transpose of the Adjoint map relative to the metric, $(Ad^t(h)(X), Y) = (X, Ad(h)Y)$ for all $h \in N, X, Y \in n$. Any orbit of this action has the form

$$N \cdot Y = \{Ad^t(g^{-1})(Y), \quad g \in N\} \simeq N/N_Y$$

where $N_Y$ denotes the isotropy subgroup at $Y$, that is $N_Y = \{g \in N : Ad^t(g^{-1})(Y) = Y\}$. Also tangent vectors to the orbit are induced by vector fields on $N$:

$$\tilde{X}(Y) = \frac{d}{ds}_{|s=0} \exp(sX) \cdot Y = -ad^t(X)(Y).$$

A vector $X$ belongs to the isotropy algebra $n_Y$ if and only if $Ad^t(\exp(-X))(Y) = Y$ equivalently $ad^t(X)(Y) = 0$, thus the Lie algebra of the isotropy subgroup at $Y$ consists of

$$n_Y = \{X \in n : ad^t(X)(Y) = 0\}.$$

**Example 3.6.** Symplectic orbits have different dimensions. As example consider a 2-step nilpotent Lie algebra $n$ with orthogonal decomposition $n = v \oplus \mathfrak{z}$, where $v = \mathfrak{z} \perp$. Take an element $Y \in v$, thus $[Y, [X, U]] = 0$ for all $X, U \in n$ since $C(n) \subset \mathfrak{z} = v \perp$. Therefore $N_Y = \{Y\}$ for all $Y \in v$.

For a general element $Y \in n$, take $Y = V + Z \in v \oplus \mathfrak{z}$. Thus $ad^t(X)(Y) = j(Z)X$ so that the dimension of $N_Y$ equals the dimension of the vector subspace $\{j(Z)X : X \in n\}$ the image of $j(Z)$, for $Y = V + Z \in v \oplus \mathfrak{z}$.

A symplectic structure is defined on coadjoint orbits. In fact for $Y \in n$ take and $\tilde{X}, \tilde{U}$ vector fields at $Y$ take

$$\omega_Y(\tilde{X}, \tilde{U}) = -\langle Y, [X, U]\rangle$$

for all $X, U \in n$.

Notice that $\omega_Y(\tilde{X}, \tilde{U}) = 0$ for all $U \in n$ if and only if $ad^t(X)(Y) = 0$ so that $X$ belongs to the isotropy Lie algebra of the action.

As for the cotangent space this symplectic structure on the orbits induces a Poisson bracket on the Lie algebra. In fact let $f : n \to \mathbb{R}$ be a smooth function. One has the Hamiltonian vector given in the formula

$$df_Y(\tilde{U}) = \omega_Y(\tilde{X}_f, \tilde{U}) = -\langle Y, [X_f, U]\rangle.$$

On the other hand

$$df_Y(\tilde{U}) = \langle (\text{grad } f)(Y), \tilde{U} \rangle = \langle (\text{grad } f)(Y), -ad^t(U)(Y) \rangle = -\langle [U, \text{grad } f(Y)], Y \rangle$$

$$= \langle [\text{grad } f(Y), U], Y \rangle,$$

so that comparing the expressions above it follows that

$$\tilde{X}_f(Y) = -\tilde{V}_f(Y), \quad \text{for} \quad V_f = \text{grad } f,$$
the gradient of \( f \) defined in the usual way. Making use of this, the Poisson bracket between two functions \( f, g \in C^\infty(n) \) is given by

\[
\{ f, g \}(Y) = -\langle Y, [V_f, V_g] \rangle,
\]

where \( V_f, V_g \) respectively denote the gradients of \( f, g \).

Let \( N \) denote a Lie group with Lie algebra \( n \). The Gauss map \( G : TN \to n \) is given by \( G(p, Y) = Y \). Thus its differential \( dG_{(p,Y)}(U,V) = V \) and its pullback \( \delta G : n^* \to T^*N \) extends to the exterior algebra. In particular one has

\[
\delta G(\omega)((U_1, V_1), (U_2, V_2)(p, Y)) = \omega_{G(p,Y)}(dG(U_1, V_1), dG(U, V))G(p, Y) = \omega_Y(V_1, V_2) = \Omega_{(p,Y)}((0, V_1), (0, V_2)).
\]

A smooth function \( f : n \to \mathbb{R}, \) induces a smooth function \( F : TN \to \mathbb{R} \) defined as \( F = f \circ G \). It is not hard to see that the gradient of \( F \) is given by \( \nabla F(p, Y) = (O, V_f) \) where \( V_f = \nabla f(Y) \). Thus for \( f_1, f_2 \in C^\infty(n) \)

\[
\{ f_1 \circ G, f_2 \circ G \}(p, Y) = \{ f_1, f_2 \} \circ G(p, Y) = -\langle Y, [V_{f_1}, V_{f_2}] \rangle.
\]

On the other side, given \( F : TN \to \mathbb{R} \) there exists \( f : n \to \mathbb{R} \) such that \( f \circ G = F \) if \( F \) is invariant. In fact \( F(p, Y) = f \circ G(p, Y) = f(Y) \) if and only if \( F \) is invariant (see 3.1).

In view of this, for this kind of functions the Poisson bracket on \( n \) is in correspondence with the Poisson bracket on \( TN \), \( \{ F_1, F_2 \}(p, Y) = \{ F_1, F_2 \}(e, Y) \), and via the identification \( F_i(p, Y) = f_i(Y) \) one has \( \{ F_1, F_2 \}(p, Y) = \{ f_1, f_2 \} \circ G(p, Y) \). We summarize below these results.

**Proposition 3.7.** Let \( N \) denote a Lie group with Lie algebra \( n \). Let \( G : TN \to n \) denote the Gauss map.

(i) For functions \( f_1, f_2 : n \to \mathbb{R} \) one has \( f_i \circ G : TN \to \mathbb{R} \) so that for every \( (p, Y) \in TN \):

\[
\{ f_1 \circ G, f_2 \circ G \}(p, Y) = \{ f_1, f_2 \} \circ G(p, Y) = -\langle Y, [V_{f_1}, V_{f_2}] \rangle.
\]

where \( V_{f_i} = \nabla f_i \) so that \( \nabla (f_i \circ G) = (0, V_{f_i}) \). Clearly \( f_i \circ G : TN \to \mathbb{R} \) is invariant by construction.

(ii) Given a smooth function \( F : TN \to \mathbb{R} \) there exists \( f : n \to \mathbb{R} \) so that \( f \circ G = F \) if and only if \( F \) is invariant. In this case we have the commuting diagram

\[
\begin{array}{ccc}
TN & \overset{G}{\longrightarrow} & n \\
\downarrow F & & \\
\mathbb{R} & \underset{f}{\leftarrow} & \\
\end{array}
\]

So for smooth invariant functions \( F_i : TN \to \mathbb{R}, \) with associated functions \( f_i : n \to \mathbb{R} : F_i = f_i \circ G, \) for \( i = 1, 2, \) the Poisson bracket follows as in Equation (2): \( \{ F_1, F_2 \}(p, Y) = \{ f_1, f_2 \} \circ G(p, Y) \).
In the following paragraphs we shall explain a construction of invariant functions for 2-step nilpotent Lie groups.

These functions appeared in [6], see also [10] [11]. Assume $N$ is a 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $\langle \cdot, \cdot \rangle$ denote a left-invariant metric on $N$ which gives the orthogonal decomposition

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$$

for $\mathfrak{v} = \mathfrak{z}^\perp$.

Let $\tilde{f}_i : \mathfrak{n} \to \mathbb{R}$ denote the function $\tilde{f}_i(V + Z) = \langle V, j(Z)^{2i-2}V \rangle$ for $i \geq 1$, where $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ was defined in (11) for $\dim \mathfrak{v} = 2n$.

Take $f_i : TN \to \mathbb{R}$ as $f_i(p, Y) = \tilde{f}_i(Y)$ for $Y \in \mathfrak{n}$ is $Y = V + Z \in \mathfrak{v} \oplus \mathfrak{z}$. Thus $f_i$ is invariant for every $i$ and it is not hard to see that the gradient is

$$\text{grad}_{(p,Y)} f_i = (0, (2i - 2)[V, j(Z)^{2i-1}V] + 2j(Z)^{2i-2}V).$$

Thus since $j(Z) \in \mathfrak{so}(\mathfrak{v})$ we have

$$\langle Z, 2[j(Z)^{2i-2}V, V] \rangle = 2\langle j(Z)^{2i-1}V, V \rangle = 0$$

which implies that $f_i$ is a first integral of the geodesic flow on $N$. It also holds

$$\langle Z, 2[j(Z)^{2i-2}V, 2j(Z)^{2j-2}V] \rangle = 4\langle j(Z)^{2i+2j-4+1}V, V \rangle = 0,$$

that says that they are in involution, $\{f_i, f_j\} = 0$, implying the next result [6]. Moreover if $\mathfrak{n}$ is almost non-singular, the functions $f_i$ are linearly independent. See [6].

**Proposition 3.8.** Let $N$ denote an almost non-singular 2-step nilpotent Lie group equipped with a left-invariant metric $\langle \cdot, \cdot \rangle$. Let $\mathfrak{n}$ denote its Lie algebra with orthogonal splitting $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\dim \mathfrak{v} = 2n$. For $i = 1, \ldots, n$, the invariant functions $f_i : TN \to \mathbb{R}$ given by

$$f_i(p, Y) = \langle V, j(Z)^{2i-2}V \rangle \quad Y = Z + V \in \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z};$$

are first integrals of the geodesic flow which are pairwise in involution.

**Example 3.9.** Let $P_4$ denote the graph which is the path of length three on four distinct vertices. Assume the vertices are $V_1, V_2, V_3, V_4$ and let $\mathfrak{n}_{P_4}$ be the 2-step nilpotent Lie algebra associated with this graph with the Lie brackets

$$[V_1, V_2] = Z_1 \quad [V_2, V_3] = Z_2 \quad [V_3, V_4] = Z_3.$$

Assume the metric so that $V_1, V_2, V_3, V_4, Z_1, Z_2, Z_3$ is an orthonormal set.

Then $j(Z_i)$ is singular for every $i$ but $j(Z_1 + Z_3)$ is non-singular. Thus $\mathfrak{n}_{P_4}$ is almost non-singular. Note that a general $j(aZ_1 + bZ_2 + cZ_3)$ in the basis $V_1, V_2, V_3, V_4$ has a matrix of the form

$$j(Z) = \begin{pmatrix}
0 & -a & 0 & 0 \\
a & 0 & -b & 0 \\
0 & b & 0 & -c \\
0 & 0 & c & 0
\end{pmatrix}.$$ 

It is not hard to prove that a symmetric map on $\mathfrak{v}$ inducing a invariant function on $TN_{P}$ of the form $g_A(p, Y) = \frac{1}{2}\langle AY, Y \rangle$ gives a first integral of the geodesic flow only for $A = Id$. In fact one verifies that $Id$ is the only solution for symmetric maps.
This induces the invariant first integral on $T_{N}$ commutative integrability. Liouville integrability is proved for the almost non-singular cases.

4.1. Star graphs. Let $S_{k}$ be the star graph on $k+1$ vertices introduced in Example 2.3. Let $N_{S_{k}}$ denote the simply connected 2-step nilpotent Lie group for the Lie algebra associated to it. Consider its presentation by the underlying manifold $\mathbb{R}^{2k+1}$ as follows. Let $v = (x_{0}, x_{1}, \ldots, x_{k}), v' = (x'_{0}, x'_{1}, \ldots, x'_{k}) \in \mathbb{R}^{k+1}$, the group operation on $\mathbb{R}^{2k+1}$ is given by

$$(v, z_{1}, z_{2}, \ldots, z_{k})(v', z'_{1}, z'_{2}, \ldots, z'_{k}) = (x_{0} + x'_{0}, x_{1} + x'_{1}, \ldots, x_{k} + x'_{k}, z_{1} + z'_{1} + \frac{1}{2}(x_{0}x'_{1} - x'_{0}x_{1}), z_{2} + z'_{2} + \frac{1}{2}(x_{0}x'_{2} - x'_{0}x_{2}), \ldots, z_{k} + z'_{k} + \frac{1}{2}(x_{0}x'_{k} - x'_{0}x_{k})).$$

Denote by $\partial_{u}$ the derivation on $\mathbb{R}^{2k+1}$ with respect to the variable $u$. It is clear that a basis of left invariant vector fields is given by

$$V_{0}(p) = \partial_{x_{0}} - \frac{1}{2}x_{1}\partial_{z_{1}} - \frac{1}{2}x_{2}\partial_{z_{2}} \ldots - \frac{1}{2}x_{k}\partial_{z_{k}}, \quad V_{i}(p) = \partial_{x_{i}} + \frac{1}{2}x_{0}\partial_{z_{i}},$$

$$Z_{i}(p) = \partial_{z_{i}} \quad \text{for all } i = 1, \ldots, k,$$
where \( p = ((x_0, x_1, \ldots, x_k, z_1, \ldots, z_k) \in N \). These vector fields satisfy the non-trivial Lie bracket relations
\[
[V_0, V_i] = Z_i \quad \text{for all } i = 1, 2, \ldots, k.
\]
Consider the metric on \( \mathbb{R}^{2k+1} \) which makes of this basis an orthonormal basis. In canonical coordinates it is given by
\[
g = (1 + \frac{1}{4} \sum_{j=1}^{k} x_j^2) dx_0^2 + \sum_{j=1}^{k} (\frac{x_0 x_j}{4} dx_j + \frac{x_j}{2} dz_j) dx_0 + \sum_{i=1}^{k} \left(1 + \frac{x_0^2}{4}\right) dx_i^2 - \frac{1}{2} \sum_{i=1}^{k} x_0 dz_i dx_i.
\]
Notice that the exponential map \( \exp : n_{S_k} \to N_{S_k} \) is
\[
\exp \left( \sum_{i=0}^{k} x_i V_i + \sum_{j=1}^{k} z_j Z_j \right) = (x_0, x_1, \ldots, x_k, z_1, z_2, \ldots, z_k)
\]
where \( V_i, Z_j \) are the left-invariant vector fields above for all \( i = 0, 1, \ldots, k, j = 1, \ldots, k. \)
Any right-invariant vector field on \( N_{S_k} \) may be regarded as a Killing vector field (associated with a 1-parameter subgroup of left-translations). In particular, we have the following basis of right-invariant vector fields
\[
V_0^*(p) = \partial_{x_0} + \frac{1}{2} x_1 \partial_{z_1} + \frac{1}{2} x_2 \partial_{z_2} \ldots + \frac{1}{2} x_k \partial_{z_k} \quad V_i^*(p) = \partial_{x_i} - \frac{1}{2} x_0 \partial_{z_i}
\]
\[
Z_i^*(p) = \partial_{z_i} \quad \text{for all } i = 1, \ldots, k.
\]
Notice that \( Z^* = Z \) for every \( Z \in \mathfrak{z} \). We induce smooth functions on \( TN \) given by
\[
\begin{align*}
f_{V_0^*}(p, Y) &= \langle V_0 + \sum_{i=1}^{n} x_i Z_i, Y \rangle, \\
f_{V_j^*}(p, Y) &= \langle V_j - \langle W, V_0 \rangle Z_j, Y \rangle, \\
f_{Z_j}(p, Y) &= \langle Z_j, Y \rangle,
\end{align*}
\]
which are first integrals of the geodesic flow, for all \( j = 1, \ldots, n \) and for \( \exp(W) = p \in n \). It is not hard to see that for \( j = 1, \ldots, n \) the corresponding gradient fields are given by
\[
\begin{align*}
\text{grad } f_{V_j^*}(p, Y) &= (-\langle Y, Z_j \rangle V_0, V_j - \langle W, V_0 \rangle Z_j) \\
\text{grad } f_{Z_j}(p, Y) &= (0, Z_j)
\end{align*}
\]
which are linearly independent whenever \( \langle Y, V_0 \rangle - \sum_{j=1}^{n} \langle Y, V_j \rangle \langle Y, Z_j \rangle \neq 0 \), therefore on a dense subset. This proves the following result.

**Lemma 4.1.** Let \( S_k \) denote the star graph. Take notations as in Equation \(^7\). Thus, the smooth functions on \( TS_k \) denoted by \( f_{V_j^*} \) are pairwise in involution for \( j = 1, \ldots, n \).
Moreover the geodesic flow on \( N_{S_k} \) is completely integrable (in the Liouville sense) since the set of first integrals \( \{E, f_{Z_j}, f_{V_j^*}\}_{j=1}^{k} \) satisfies that any pair of first integrals is in involution and the corresponding gradients are linearly independent on an open dense set.
We only have to prove that \( \{f_{V^*}, f_{V'}\} = 0 \) for all \( i, j = 1, \ldots, k \). In fact for \( i \neq j \)

\[
\{f_{V^*}, f_{V'}\}(p, Y) = \langle (Y^i) V_i, V_j \rangle = 0.
\]

Recall that \( V_0 \) is orthogonal to every \( V_i \) for \( i = 1, \ldots, k \). This proves the lemma.

Note that all first integrals above are polynomials on the coordinates of \( (p, Y) \). In fact writing \( W = \sum_{i=0}^k w_i V_i + \sum_{i=1}^k u_i V_i \) and \( Y = \sum_{j=0}^k y_j V_j + \sum_{j=1}^k z_j Z_j \), the first integrals follow:

\[
\begin{align*}
g(p, Y) &= \frac{1}{2}(y_0^2 + y_1^2 + \ldots + y_k^2 + z_1^2 + \ldots + z_k^2) \\
f_{V^*}(p, Y) &= y_1 - w_0 z_1 \\
&\vdots \\
f_{V^*}(p, Y) &= y_k - w_0 z_k \\
f_{Z_1}(p, Y) &= z_1 \\
&\vdots \\
f_{Z_k}(p, Y) &= z_k
\end{align*}
\]

**Remark 4.2.** Take coordinates \((x_0, x_1, \ldots, x_k, z_1, \ldots, z_k)\) for \( p \in N \) and coordinates \((y_0, y_1, \ldots, y_k, t_1, \ldots, t_k)\) on \( \mathfrak{n} \) relative to a basis of left-invariant vector fields. Take \( F : TN \to \mathbb{R}^{2k+1} \) given by \( F(p, Y) = (E, f_{Z_1}, \ldots, f_{Z_k}, f_{V^*}, \ldots, f_{V^*}) \). Let \( c = (C_0, T_1, \ldots, T_n, Y_1, \ldots, Y_n) \), the set \( F^{-1}(c) \) gives the symplectic leaves on \( TN \).

In fact

\[
E(p, Y) = \frac{1}{2}(t_0^2 + t_n^2 + y_1^2 + \ldots + y_n^2) = C_0 \\
z_i = T_i \text{ for all } i = 1, \ldots, n \text{ and } y_j - x_0 t_j = Y_j,
\]

so that coordinates \( x_1, \ldots, x_k, z_1, \ldots, z_k \) have no restriction in the preimages \( F^{-1}(c) \).

Let \( \gamma \) denote a geodesic on \( N_{S_k} \). Set \( \gamma(t) = \exp(X(t) + Z(t)) \) where \( X(t) \in \mathfrak{v} \) and \( Z(t) \in \mathfrak{z} \) with initial condition \( X_0 + Z_0 \), satisfy the following system of equations (see [9])

\[
\begin{align}
x''_0 &= -a_1 x'_1 - a_2 x'_2 - \ldots - a_k x'_k \\
x''_1 &= a_1 x'_0 \\
&\vdots \\
x''_k &= a_k x'_0 \\
z'_1 &= a_1 + \frac{1}{2}(x_0 x'_1 - x_0 x_1) \\
&\vdots \\
z'_k &= a_k + \frac{1}{2}(x_0 x'_k - x_0 x_k)
\end{align}
\]

where \( Z_0 = a_1 Z_1 + a_2 Z_2 + \ldots + a_k Z_k \), and \( X(t) = \sum x_i(t) V_i \) and \( Z(t) = \sum z_j(t) Z_j \).

The map \( j(0) \) shown in Example 2.3 is singular. Its kernel is

\[
\{V \in \mathfrak{v} : V = \sum_{j=0}^k v_j V_j, \text{ where } v_0 = 0 \text{ and } (v_1, \ldots, v_k) \in (a_1, \ldots, a_k)^{-1} \},
\]

where we are thinking in the usual inner product for vectors in \( \mathbb{R}^k \). Thus if \( \{g^t\} \) denotes the geodesic flow in \( TN \), for every \( n \in N \), and \( X_0 \in \mathfrak{v}, Z_0 \in \mathfrak{z} \), then

\[
g^t(dL_n(X_0 + Z_0)) = dL_{\gamma(t)}(e^{t j(Z_0)} X_0 + Z_0),
\]

where \( \gamma(t) \) denotes the unique geodesic with \( \gamma'(0) = dL_n(X_0 + Z_0) \).
A Riemannian compact manifold arises as a quotient $\Lambda \backslash N_{S_k}$ where $\Lambda$ is a discrete cocompact subgroup of $N_{S_k}$. In fact $\Lambda \backslash N_{S_k}$ becomes a Riemannian manifold with the metric that makes the projection $\pi : N_{S_k} \rightarrow \Lambda \backslash N_{S_k}$ a Riemannian submersion.

Each $2k$-tuple $(r, m) = (r, r_1, \ldots, r_k, m_1, \ldots, m_k) \in (\mathbb{Z}^+)^{2k}$ defines a lattice in $N_{S_k}$ by
\begin{equation}
\Lambda_{(r,m)} = rm_0\mathbb{Z} \times 2r_1\mathbb{Z} \times \ldots \times 2r_k\mathbb{Z} \times m_1\mathbb{Z} \times m_2\mathbb{Z} \times \ldots \times m_k\mathbb{Z},
\end{equation}
for $m_0 = m_1m_2 \ldots m_k$.

Note that there are non-isomorphic lattices in this family so that we get many non-diffeomorphic compact manifolds.

Since the quotient projection $\pi : N_{S_k} \rightarrow \Lambda_{(r,m)} \backslash N_{S_k}$ is a Riemannian submersion and furthermore a local isometry, we can identify the tangent bundle $T\Lambda_{(r,m)}N_{S_k}$ with $(\Lambda_{(r,m)}N_{S_k}) \times n_{S_k}$. The projection $\pi$ maps geodesics into geodesics and the energy function $\bar{E}$ on $T(\Lambda_{(r,m)}N_{S_k})$ is related to the energy function $E$ on $TN_{S_k}$ by
\begin{equation}
\bar{E}(\Lambda_{(r,m)}p, Y) = E(p, Y) = \frac{1}{2}\langle Y, Y \rangle
\end{equation}
and clearly it is well defined.

All invariant first integrals on $TN_{S_k}$ descend to the quotients, since they do not depend on the coordinates in the manifold, so
\begin{equation}
\hat{f}_{Z_j}(\Lambda_{(r,m)}p, Y) = f_{Z_j}(p, Y), \quad \text{for all } j = 1, \ldots, k
\end{equation}
of the geodesic flow of $T(\Lambda_{(r,m)}N_{S_k})$. Moreover, such first integrals are in involution, since for all $f, g \in C^\infty(T(\Lambda_{(r,m)}N_{S_k}))$ we have
\begin{equation}
\{f \circ \pi, g \circ \pi\} = \{f, g\} \circ \pi.
\end{equation}

Note that the integrals $f_{V_j^*}, j = 1, \ldots, k$ do not descend to the quotient. However one can construct first integrals on the quotient with the following argument. Let $(p, Y) \in TN_{S_k}$ and $q \in \Lambda_{(r,m)}$. Take $W, W' \in n_{S_k}$ such that $\exp W = p, \exp W' = q$. Observe that $(W + W')_v = W_v + W'_v$, where $U_v$ denotes the orthogonal projection of $U \in n_{S_k}$ over $v = 3^\perp$. So we get
\begin{equation}
f_{V_j^*}(qp, Y) = \langle Y, V_j \rangle - \langle Z_j, Y \rangle \langle (W + W'), V_0 \rangle
\end{equation}
\begin{equation}
= f_{V_j^*}(p, Y) - f_{Z_j}(p, Y)\langle W', V_0 \rangle.
\end{equation}
Since $\langle W', V_0 \rangle \in \mathbb{Z}$ we have that
\begin{equation}
f_{V_j^*}(qp, Y) = f_{V_j^*}(p, Y) \mod f_{Z_j}(p, Y)\mathbb{Z}
\end{equation}
for every $j = 1, \ldots, k$ and since $f_{Z_j}$ is a first integral of the geodesic flow, we have that the function
\begin{equation}
\hat{f}_{V_j^*}(p, Y) = \sin \left(\frac{2\pi f_{V_j^*}(p, Y)}{f_{Z_j}(p, Y)}\right)
\end{equation}
descends to $\Lambda_{(r,m)}N_{S_k}$ and is constant along the integral curves of the geodesic vector field in $T(\Lambda_{(r,m)}N_{S_k})$. In order to get a smooth first integral let
\begin{equation}
\hat{F}_j(p, Y) = e^{-1/f_{Z_j}(p, Y)^2}\hat{f}_j(p, Y)
\end{equation}
and let us define

\[ \tilde{F}_j(\Lambda_r p, Y) = \bar{F}_j(p, Y). \]

So the functions \( \tilde{F}_k \) are smooth (non-analytic) first integrals for the geodesic flow on \( T(\Lambda_{(r,m)} \backslash N_{S_k}) \). It follows from a direct calculation that the families \( f_{Z_i}, \tilde{F}_j, \) \( i = 1, \ldots, k \) are in involution. So the geodesic flow in \( T(\Lambda_{(r,m)} \backslash N_{S_k}) \) is completely integrable in the sense of Liouville.

**Theorem 4.3.** Let \( N_{S_k} \) be the 2-step nilpotent Lie group attached to the star graph in \( k + 1 \) vertices \( S_k \), endowed with the standard metric and let \( \Lambda_{(r,m)} \) denote the lattice in (9). If \( \Lambda_{(r,m)} \backslash N_{S_k} \) is the corresponding compact manifold with the induced metric, then the geodesic flow in \( T(\Lambda_{(r,m)} \backslash N_{S_k}) \) is completely integrable with smooth first integrals \( \{ E, f_{Z_i}, \tilde{F}_j \} \), for \( i = 1, \ldots, k \).

Let \( \tilde{H} \subset N_{S_k} \) be the normal subgroup of dimension \( 2k \) defined as \( \tilde{H} = \{ g \in N_{S_k} / g = (v, z) \text{ and } v = (0, x_1, \ldots, x_k) \} \). Note that \( \tilde{H} \) is a abelian and \( \Lambda_{(r,m)} = \tilde{H} \cap \Lambda_{(r,m)} \) is a lattice in \( \tilde{H} \). So \( \Lambda_{(r,m)} \backslash \tilde{H} \cong T^{2k} \).

Note that \( N_{S_1} \) is (isomorphic to) the Heisenberg Lie group. In fact this is modeled on \( \mathbb{R}^3 \) with the group operation given by

\[ (x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')). \]

But there is another presentation which is given in terms of matrices and \( H_3 \) can be seen as the set

\[
\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \text{ for } x, y, z \in \mathbb{R} \right\}
\]

with the usual product of matrices. The map

\[ \varphi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \to (x, y, z - \frac{1}{2}xy) \]

shows an isomorphism between the two models.

The subgroup of matrices \( \Gamma_r \) of the form

\[
\Gamma_r = \left\{ \begin{pmatrix} 1 & rn & q \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \text{ for } m, n, q \in \mathbb{Z} \right\}
\]

for a fixed \( r \in \mathbb{N} \) gives rise to a cocompact lattice in \( H_3 \). Moreover denoting by \( \bar{\Gamma}_r \) the subgroup isomorphic to \( \mathbb{Z} \) and by \( \bar{\Gamma} \) the subgroup isomorphic to \( \mathbb{Z}^2 \) given respectively by

\[ \bar{\Gamma}_r = \{(rn, 0, 0) : n \in \mathbb{Z} \} \quad \bar{\Gamma} = \{(0, m, q) : m, q \in \mathbb{Z} \} \]

we get the semidirect product group \( \Gamma_r \cong \bar{\Gamma}_r \rtimes \bar{\Gamma} \) where the action is given by \( rn \cdot (m, q) = (m, q + rnm) \). So the map \( \Psi : \bar{\Gamma}_r \rtimes \bar{\Gamma} \to \Gamma_r \) given as \( (rn, (m, q)) \to (rn, m, q) \) is an isomorphism.
Now the action of $\tilde{\Gamma}$ on $H_3$ gives
\[
(0, m, q) \cdot (x, y, z) = (x, y + m, z + q)
\]
so that $\tilde{\Gamma}/H_3 \simeq \mathbb{R} \times T^2$.

One also has
\[
(rn, 0, 0) \cdot (0, m, q) \cdot (x, y, z) = (0, m, q) \cdot (rn, 0, 0) \cdot (x, y, z).
\]

Finally the action of $\bar{\Gamma}$ on $\mathbb{R} \times T^2$ gives the action of $S^1$ on $\mathbb{R} \times T^2$ and $\Gamma_r/H_3$ is a $T^2$-bundle over $S^1$:
\[
S^1 \to \Gamma_r/H_3 \to T^2.
\]

It is known that a similar procedure on $H_{2n+1}$ shows that Heisenberg nilmanifolds $\Gamma/H_{2n+1}$, as topological spaces are $T^{n+1}$-bundles over $T^n$ see [2].

Let us see the fibration we get from the nilmanifolds arising from every star graph. First notice that a presentation of $N_{S_k}$ is given by the $(k + 2) \times (k + 2)$-matrices of the form
\[
\begin{pmatrix}
1 & x_0 & z_1 & z_2 & \ldots & z_k \\
0 & 1 & x_1 & x_2 & \ldots & x_k \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]
where we think in the usual matrix multiplication. The map
\[
\Psi : \begin{pmatrix}
1 & x_0 & z_1 & z_2 & \ldots & z_k \\
0 & 1 & x_1 & x_2 & \ldots & x_k \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1
\end{pmatrix} \to (x_0, x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_k)
\]
gives an isomorphism between both presentations.

Now take the lattice $\Gamma_r \subset N_{S_k}$ given by matrices of the form
\[
\Gamma_r = \left\{ \begin{pmatrix}
1 & rn & q_1 & q_2 & \ldots & q_k \\
0 & 1 & m_1 & m_2 & \ldots & m_k \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1
\end{pmatrix} \right\} \quad \text{for } r, m_i, q_j \in \mathbb{Z}, \forall i, j
\]

The action of the abelian subgroup $\tilde{\Gamma} = \{(0, m_1, m_2, \ldots, m_k, q_1, q_2, \ldots, q_k) : m_i, q_j \in \mathbb{Z}\}$ on $N_{S_k}$ given as
\[
(0, m_1, m_2, \ldots, m_k, q_1, q_2, \ldots, q_k) \cdot (x_0, x_1, \ldots, x_k, z_1, \ldots, z_k) = (x_0, x_1 + m_1, \ldots, x_k + m_k, z_1 + q_1, \ldots, z_k + q_k)
\]
shows that \( \tilde{\Gamma} \setminus N_{S_k} \simeq \mathbb{R} \times T^{2n} \). Denote by \( X_0 \) the element \( X_0 = (x_0, x_1 + m_1, \ldots, x_k + m_k, z_1 + q_1, \ldots, z_k + q_k) \) and take the action of \( \Gamma_r \) defined by \( \Gamma_r = \{(rn, 0, \ldots, 0) : n \in \mathbb{Z} \} \):

\[
(rn, 0, \ldots, 0) \cdot X_0 = (x_0 + rn, x_1 + m_1, \ldots, x_k + m_k,
\quad z_1 + q_1 + rnx_1 + rnm_1, \ldots, z_k + q_k + rnx_k + rnm_k).
\]

On the other hand

\[
(rn, 0, \ldots, 0)(x_0, x_1, \ldots, x_k, z_1, \ldots, z_k) = (x_0 + rn, x_1, \ldots, x_k, z_1 + rnx_1, \ldots, z_k + rnx_k)
\]

from which left multiplication by the element \( (0, m_1, m_2, \ldots, m_k, q_1, q_2, \ldots, q_k) \) gives the result above in Equation (11). So we get that the compact manifold \( \Gamma_r \setminus N_{S_k} \) is a \( T^{2n} \)-fibration over \( S^1 \):

\[
S^1 \rightarrow \Gamma_r \setminus N_{S_k} \rightarrow T^{2n}.
\]

The action of \( S^1 \) on each torus \( T^2 \) is given by the induced matrix

\[
\begin{pmatrix}
1 & r \\
0 & 1
\end{pmatrix}.
\]

As above \( \Gamma_r \simeq \tilde{\Gamma}_r \ltimes \tilde{\Gamma} \). We have

- \( r\mathbb{Z} \simeq \tilde{\Gamma}_r = \{u \in \Gamma_r : u = (rn, 0, \ldots, 0), n \in \mathbb{Z} \} \) and
- \( \tilde{\Gamma} = \{\gamma \in \Gamma_r : \gamma = (0, m_1, \ldots, m_k, q_1, \ldots, q_k)\} \).
- \( \tilde{\Gamma}_r \) acts on \( \tilde{\Gamma} \) by left multiplication induced from the group:
  \[
  rn \cdot (0, m_1, \ldots, m_k, q_1, \ldots, q_k) = (0, m_1, \ldots, m_k, q_1 + rnm_1, \ldots, q_k + rnm_k).
  \]
- In fact \( (u, \gamma) \in (\Gamma_r) \ltimes \tilde{\Gamma} \rightarrow (u, \gamma) \in \Gamma_r \) is a Lie group isomorphism.

**Remark 4.4.** If the smoothly closed geodesics in a nilmanifold \( \Gamma \setminus N \) are dense, then the nilmanifold has the density of closed geodesics property (DCG). In [8] the authors give conditions on the graph \( G \) and on a lattice \( \Gamma \subset N \) for which the quotient \( \Gamma \setminus N \), a compact nilmanifold, has a dense set of smoothly closed geodesics. In fact Theorem 5.20 in [8] proves that for the nilmanifold \( \Gamma \setminus N \), where \( N \) arises from \( K_3 \) and \( \Gamma \) is a lattice, then the DCG property holds, however the geodesic flow cannot be integrable as proved by Butler.

**4.2. Non-commutative integrability and graphs.** Here we prove the integrability of geodesic flows on manifolds associated to graphs in a low number of vertices, \( k \) where \( k \leq 4 \).

Let \( H : TN \rightarrow \mathbb{R} \) denote a smooth function. One says that it is *integrable in the non-commutative sense of Nekhorosev* or simply *integrable* if one has the following conditions. Assume \( F = (H = f_1, \ldots, f_{nk}, g_1, \ldots, g_{2k}) \) is a smooth map on \( TN \) where \( \dim N = n \) and \( k \geq 0 \), that satisfies the three conditions:

(i) \( \text{rank} \, dF = n + k \) on an open, dense subset of \( TN \);
(ii) for all \( a, b = 1, \ldots, nk \) and all \( c = 1, \ldots, 2k : \{f_a, f_b\} = \{f_a, g_c\} = 0 \);
(iii) for each regular value \( c \in \mathbb{R}^{n+k} \), each connected component of \( F^1(c) \) is compact.

In this situation the Theorem of Nekhorosev describes the level sets of \( F \), and the flow for the Hamiltonian \( X_H \) in terms of a flow on the torus. In [8] Butler proves in the non-commutative sense of the geodesic flow for \( D \setminus N \) where \( D \) is a cocoamptact lattice on \( N \) and \( N \) is 2-step nilpotent Lie group whose Lie algebra is
almost non-singular. In fact one can find $3s + t$ first integrals, where $\dim \mathfrak{v} = 2s$ and $\dim \mathfrak{z} = t$. These functions can be obtained by the two ways studied in this section, those invariant functions from Proposition 3.8 and the $n$ functions arising from a basis a right-invariant vector fields.

Since Bolsinov and Jovanovic [2] proved that integrability in the non-commutative sense implies Liouville integrability, the previous results of Butler give the Liouville integrability for an important family of 2-step nilpotent Lie groups and their compact quotients.

Let us explain the construction. Let $N$ denote a Lie group equipped with a left-invariant metric and with Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with $\dim \mathfrak{v} = 2n, \dim \mathfrak{z} = m$. Making use of Killing vectors we construct $2n + m$ first-integrals of the geodesic low. With Proposition 3.8 we construct $n$ invariant first integrals. Assume this is a Lie algebra. In this Lie algebra we have $n + m$ which are in involution. So that this gives: $2n + m + n + n + m = 2m + 4n = \dim TN$. This gives the complete integrability for almost non-singular Lie algebras. As corollary one gets the next result.

**Corollary 4.5.** Let $G$ denote a connected graph on $k$ vertices with $k \leq 4$. Then except for the complete graph $K_3$, any 2-step nilpotent Lie group $N_G$ so as the corresponding compact quotient admits a completely integrable geodesic flow.

Take the graphs in Example 2.13. For two vertices, we have the Heisenberg Lie algebra of dimension three whose geodesic flow is completely integrable with any left-invariant metric. For three vertices, the connected graphs are $S_3$ and $K_3$ which were explained above. We need to concentrate in algebras coming from graphs with four vertices.

**Example 4.6.** Let $P_4$ the path in four vertices. Let $N_{P_4}$ denote the 2-step nilpotent Lie group associated to $P$. Then $N_p$ has dimension seven. Take $p = \exp(W)$ and the functions on $TN_p$ given by

\[
\begin{align*}
    f_{V_1}(p, Y) &= \langle V_1 + \langle W, V_2 \rangle Z_1, Y \rangle, \\
    f_{V_2}(p, Y) &= \langle V_2 - \langle W, V_1 \rangle Z_1 + \langle W, V_3 \rangle Z_3, Y \rangle, \\
    f_{V_3}(p, Y) &= \langle V_3 - \langle W, V_2 \rangle Z_2 + \langle W, V_4 \rangle Z_3, Y \rangle, \\
    f_{V_4}(p, Y) &= \langle V_4 - \langle W, V_3 \rangle Z_3, Y \rangle.
\end{align*}
\]

In this situation we do not need the functions of Proposition 3.8. In fact Killing vectors give rise to a subalgebra of first integrals of dimension 7. Among these first integrals, we have 3 from the center which are in involution and also two more functions, for instance $f_{V_1}$ and $f_{V_2}$ which are in involution. So we have $7 + 1 + 5 + 1 = 14$, and applying the results in [2] we get the complete integrability in this case. Here we did not make use of invariant first integrals from Proposition 3.8. In fact, sometimes we are able to replace these $n$ functions by other $n$-functions coming from Killing vectors, as for $\mathfrak{n}_{P_4}$.

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