AN EXTENSION OF WIENER INTEGRATION WITH THE USE OF OPERATOR THEORY

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ABSTRACT. With the use of tensor product of Hilbert space, and a diagonalization procedure from operator theory, we derive an approximation formula for a general class of stochastic integrals. Further we establish a generalized Fourier expansion for these stochastic integrals. In our extension, we circumvent some of the limitations of the more widely used stochastic integral due to Wiener and Ito, i.e., stochastic integration with respect to Brownian motion. Finally we discuss the connection between the two approaches, as well as a priori estimates and applications.

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1. INTRODUCTION

Recently there has been increase in the number of applications of stochastic integration and stochastic differential equations (SDEs). In addition to the traditional applications in physics and dynamics, stochastic processes have found uses in such areas as option pricing in finance, filtering in signal processing, computations biological models. This fact suggests a need for a widening of the more traditional approach centered about Brownian motion $B(t)$ and Wiener’s integral.

Since SDEs are solved with the use of stochastic integrals, we will focus here on integration with respect to a wider class of stochastic processes than has previously been considered. In evaluation of a stochastic integral we deal with the term $dB(t)$
by making use of the basic properties of Brownian motion, such as the fact that
\( B(t) \) has independent increments. If instead \( X(t) \) is an arbitrary stochastic process,
it is not at all clear how to make precise a stochastic integration with respect to
\( dX(t) \). We will develop a method, based on a Karhunen-Loève diagonalization, for
doing precisely that.

The theory of stochastic integrals is well developed, see e.g., [Kuo06, IM65].
For many applications, such as the solutions to stochastic differential equations
in physics and finance, it is important to have tools for evaluating integrals with
respect to \( dB \) where \( B \) is Brownian motion. The reason for the technical issues
involved in the computation of stochastic integrals can be understood this way: A
naive approach runs into difficulties, for example because the length of Brownian
paths is infinite, and because Brownian paths are discontinuous (with probability
one). Wiener and Ito offered a, by now, well known way around this difficulty. The
idea of Wiener in fact is operator theoretic: It is to establish the value of an integral
as a limit that takes place in a Hilbert space of random variables. This is successful
because of the existence of an isometry between this Hilbert space on the one hand
and a standard \( L^2 \)-Lebesgue space on the other.

In this paper we extend this operator theoretic approach to a much wider class
of stochastic integrals, i.e., integration with respect to \( dX \) where \( X \) belongs to a
rather general class of stochastic processes. And we give some applications.

In the proof of our theorem we make use of a result from two earlier papers
[JS07, JS08] by the coauthors. The idea is again operator theoretic, and it is
based on an application of von Neumann’s spectral theorem to an integral operator
directly associated with the process \( X \) under consideration.

While the applications of stochastic integrals to physics (e.g., [BC97], and their
interplay with operator theory (e.g., [AL08a]) are manifold, the idea of exploring and
extending the scope in the present direction appears to be new. The need for such
an extension is convincing: For example, physical disturbances or perturbations will
typically take you outside the particular path-space framework where the theory
was initially developed.

Earlier approaches to stochastic integrals with reproducing kernels include [AL08a,
AAL08, AL08b] and operator theory [IM08]; and papers exploring physical ramifi-
cations: [BC97, BDSG+07, Hud07b, Hud07a]. Although the papers cited here with
physics applications represent only the tip of an iceberg!

2. Notation and Definitions

To make precise the operator theoretic tools going into our construction, we must
first introduce the ambient Hilbert spaces. Since stochastic integrals take values
in a space of random variables, we must specify a fixed probability space \( \Omega \), with
sigma algebra and probability measure. In the case of Brownian motion, the proba-
bility space amounts to the standard construction of Wiener and Kolmogorov: The
essential axiom in that case is that all finite samples are jointly Gaussian, but we
will consider general stochastic processes, and so we will not make these restricting
assumptions on the sample distributions and on the underlying probability space.
For more details on this case, see section 4.

The kind of integrals we consider presently are stated precisely in Definition
2.1, eq (2.5) below. I particular, initially we consider only functions of time in the
integrant, so \( f(t)dX_t \). When the stochastic process \( X \) is given, we show (Theorem
that the corresponding integrals live in a Hilbert space which is a direct sum of standard Lebesgue Hilbert spaces carrying the function \( f(t) \). In the case of Brownian motion, we show (Example 4.1) that the direct sum representation then only has one term.

We now list the symbols and the terminology.

- \( L^2 \): an \( L^2 \)-space.
- \( L^2(\mathbb{R}) \): all \( L^2 \)-functions on \( \mathbb{R} \).
- \( J \subset \mathbb{R} \) a finite closed interval.
- \( L^2(J) \): \( L^2 \) with respect to the Lebesgue measure restricted to \( J \).
- \( (\Omega, \mathcal{S}, P) \): a fixed probability space.
- \( \Omega \): sample space.
- \( \mathcal{S} \): some sigma algebra of subsets of \( \Omega \).
- \( P \): a probability measure defined on \( \mathcal{S} \).
- \( L^2(J \times \Omega, m \times P) \): the \( L^2 \)-space on \( J \times \Omega \) with respect to product measure \( m \times P \) where \( m \) denotes Lebesgue measure.

**Restricting Assumptions:**

(i) \( X \in L^2(J \times \Omega, m \times P) \) for all finite intervals \( J \subset \mathbb{R} \).

(ii) \((s, t) \mapsto \mathbb{E}(X_sX_t) \) is continuous on \( J \times J \). \( \mathbb{E}(X_t) = 0 \).

(iii) For all \( J \), and all \( s \in J \), the function,

\[
(2.1) \quad t \mapsto \mathbb{E}(X_sX_t)
\]

is of bounded variation.

For \( J \subset \mathbb{R} \) fixed, we consider partitions

\[
(2.2) \quad \pi : t_0 < t_1 < \cdots < t_{n-1} < t_n =: t, \quad J = [t_0, t];
\]

and we set

\[
(2.3) \quad |\pi| := \max_i (t_{i+1} - t_i), \quad \text{ and } \quad \Delta t_i := t_{i+1} - t_i, \quad 0 \leq i < n.
\]

If \( f : \mathbb{R} \to \mathbb{C} \) is continuous, we set

\[
(2.4) \quad S_{\pi}(f, X) := \sum_{i=0}^{n-1} f(t_i)(X_{t_{i+1}} - X_{t_i}).
\]

**Definition 2.1.** By a stochastic integral, we mean a limit

\[
(2.5) \quad \lim_{|\pi| \to 0} S_{\pi}(f, X) =: \int_{t_0}^{t} f(s)dX_s
\]

We now turn to questions of existence of this limit for a rather general family of stochastic processes \( X_t \); see (i)-(iii) below.

### 3. Statement of the Main Theorem

When the stochastic process \( X \) is given, we proved in Theorem 3.1 that the corresponding integrals live in a Hilbert space which is a direct sum of standard Lebesgue Hilbert spaces carrying the function \( f(t) \). In the case of Brownian motion, we now show that the direct sum representation then only has one term. Yet the method from section 3 still offers a Fourier decomposition of the Wiener integration.
**Theorem 3.1.** Let \((\Omega, S, P)\) be given as above, and let \(X\) be a stochastic process satisfying conditions (i) – (iii). Let \(f\) be given and continuous.

(a) Then the stochastic integral \(\int_{t_0}^t f(s)dX_s\) exists and is in \(L^2(\Omega, P)\).

(b) There is a family of bounded variation functions \(\varphi_1, \varphi_2, \ldots\) and numbers \(\lambda_1, \lambda_2, \ldots\) satisfying the following conditions:

\[ \lambda_1 \geq \lambda_2 \geq \cdots > 0, \quad \lambda_k \to 0 \]

in fact \(\sum_k \lambda_k < \infty\), such that

\[ E\left(\int_{t_0}^t f(s)dX_s\right)^2 = \sum_{k=1}^\infty \lambda_k \left| \int_{t_0}^t f(s)d\varphi_k(s) \right|^2 \]

where the terms \(\int_{t_0}^t f(s)d\varphi_k(s)\) refer to Stieltjes integration.

(c) If an interval \(J\) is chosen such that \(t_0\) and \(t \in J\), and if \(f\) has a weak derivative \(f'\) in \(L^2(J)\), then the following estimate holds for the RHS in (3.1):

\[ \text{RHS} \leq "\text{Const}" + \lambda_1 \int_{t_0}^t |f'(s)|^2 ds \]

where “\text{Const}” depends on certain boundary conditions, and where \(\lambda_1\) is the maximal eigenvalue see (5.4) below.

In the next corollary, we stay with the assumptions from the theorem; in particular \(X_t\) is a stochastic process subject to conditions (i)-(iii), and a compact interval \(J\) is fixed.

**Corollary 3.2.** Covariance relations:

- \(E(X_sX_t) = \sum_{k=1}^\infty \lambda_k \varphi_k(s)\varphi_k(t)\)
- \(E(X_s^2) = \sum_{k=1}^\infty \lambda_k |\varphi_k(t)|^2\)
- **Dependency of increments:** If \(s < t < u\) in \(J\), then

\[ E((X_t - X_s)(X_u - X_t)) = \sum_{k=1}^\infty \lambda_k (\varphi_k(t)\varphi_k(u) - \varphi_k(s)\varphi_k(u) + \varphi_k(s)\varphi_k(t) - |\varphi_k(t)|^2) \]

- \(E((X_{t+\Delta t} - X_t)^2) = \sum_{k=1}^\infty \lambda_k |\varphi_k(t + \Delta t) - \varphi_k(t)|^2\)

4. **An Application**

In this section we restrict the setting of Theorem 3.1 to the special case when \(X = B\), i.e., to the special case of integration with respect to Brownian motion. We then work out the eigenfunctions and eigenvalues for the covariance operator. It turns out to be the familiar Fourier basis. Actually there is a choice of bases depending on boundary conditions. A choice of the Dirichlet conditions yields the ONB of the sine functions. We further show that when the eigenvalue expansion is summed (using orthogonality) we then arrive at the familiar Wiener-Ito formula.

**Example 4.1.** \(X = B = \text{Brownian motion.}\)

- \((\Omega, S, P)\) Gaussian space;
- \(\Omega\) a space of functions, \(S := \langle \text{cylinder sets} \rangle\) \(\sigma\)-algebra; the sigma algebra generated by the cylinder-sets.
- \(J = [0, 1];\)
\[ E(X_sX_t) = \min(s,t) =: s \wedge t; \]
\[ X_t(\omega) = \omega(t), \ \omega \in \Omega; \]
\[ E((X_t + \Delta t - X_t)^2) = \Delta t. \]

We now show that the known formula

\[ E\left( \left| \int_0^t f(s) dB_s \right|^2 \right) = \int_0^t |f(s)|^2 ds \]

follows from the theorem; and in particular from (3.1).

In the case of Brownian motion for the functions \( \varphi_k \) we may take

\[ \varphi_k(t) = \sqrt{2} \sin(k\pi t); \quad k = 1, 2, \ldots. \]

Note that

\[ \varphi_k(0) = \varphi_k(1) = 0, \quad \forall k = 1, 2, \ldots; \]

and

\[ \lambda_k = \frac{1}{(k\pi)^2}. \]

Set \( t_0 = 0 \) for simplicity. Note that if

\[ g(t) := \int_0^1 t \wedge sf(s)ds, \]

then

\[ \left( \frac{d}{dt} \right)^2 g(t) = -f(t), \]

so the eigenvalue problem

\[ \int_0^1 t \wedge sf(s)ds = \lambda f(t) \]

has the solution given by (4.2) and (4.4). Note further that (4.3) is a choice of boundary conditions.

To see that (4.1) follows from (3.1) in the theorem, we proceed as follows; starting with the RHS in (3.1) and using \( d(\sin(k\pi t)) = -k\pi \cos(k\pi t)dt \):

\[ \sum_{k=1}^{\infty} \lambda_k \left| \int_0^t f(s) d\varphi_k(s) \right|^2 = \sum_{k=1}^{\infty} \frac{2}{(k\pi)^2} \left| \int_0^t f(s) d\sin(k\pi s) \right|^2 \]

by (4.3)

\[ = \sum_{k=1}^{\infty} \frac{2}{(k\pi)^2} \left| \int_0^t f(s) \cos(k\pi s) ds \right|^2 \]

by Parseval's formula

\[ = \int_0^t |f(s)|^2 ds; \]

and the desired conclusion follows.

5. Proof of Theorem 3.1

Here we give the details of proof of theorem 3.1 Since the proof is long, to help the reader our presentation is divided into two parts, A and B.

Part A is an outline of the steps in the proof itself, and part B contains the details arguments making up each part in the proof. Part A begins with the notation and the terminology, introducing an auxiliary selfadjoint operator, its matrix approximations, and its spectral resolution.
5.1. Part A.

- Select a fixed interval $J := [a, b], \ a < b$.
- From the assumptions on the process $(X_t)_{t \in \mathbb{R}}$ note that the operator

$$
(T_J f)(t) := \int_J E(X_t X_s) f(s) ds
$$

is compact and selfadjoint in the Hilbert space $L^2(J)$.

- For every partition

$$
\pi : t_0 < t_1 < \cdots < t_n, \ t_0 = a, t_n = b;
$$

the following matrix

$$
(M_{J, \pi})_{i,j} := E(X_{t_i} X_{t_j})
$$

offers a discrete approximation for the operator $T_J$ in (5.1).

- Set

$$
H(J) := L^2(J) \ominus \ker(T_J) = \{ g \in L^2(J) : \langle g, k \rangle_{L^2} = 0, \forall k \in \ker(T_J) \}.
$$

Then an application of the spectral theorem to $T_J$ yields the following sequence of orthogonal eigenfunctions $\varphi_1, \varphi_2, \cdots$ in $H(J)$, and numbers $\lambda_1, \lambda_2, \cdots \in \mathbb{R}_+$ such that $\lambda_k \to 0;$

$$
\lambda_1 \geq \lambda_2 \geq \cdots > 0, \ \lambda_k \to 0
$$

such that

$$
T_J \varphi_k = \lambda_k \varphi_k \ k = 1, 2, \cdots
$$

orthogonality relations in the $t$–domain:

$$
\langle \varphi_j, \varphi_k \rangle_{L^2(J)} = \int_J \overline{\varphi_j(t)} \varphi_k(t) dt = \delta_{j,k};
$$

and the closed span of $\{ \varphi_k \}$ is $H(J)$.

- Set

$$
Z_k(\cdot) := \frac{1}{\sqrt{\lambda_k}} \int_J \varphi_k(t) X_t(\cdot) dt,
$$

and note that each $Z_k, k = 1, 2, \cdots$ is a random variable,

$$
Z_k \in L^2(\Omega, \mathcal{S}, P).
$$

Moreover, a calculation yields (orthogonality relations in the $\omega$–domain):

$$
E(Z_j Z_k) = \delta_{j,k}
$$

- Aside; note that if $(X_t)$ is assumed Gaussian, then each $Z_k, k = 1, 2, \cdots$ is Gaussian as well.
- Karhunen-Loève, or Generalized Fourier Expansion:

In $L^2(J \times \Omega, m \times P)$, we have the following pointwise a. e. representation

$$
X(t, \omega) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \varphi_k(t) Z_k(\omega),
$$

as well as

$$
\lim_{N \to \infty} \left\| X(\cdot, \cdot) - \sum_{k=1}^{N} \sqrt{\lambda_k} \varphi_k(\cdot) Z_k(\cdot) \right\|_{L^2(m \times P)} = 0.
$$
5.2. Part B. We now turn to the details of the proof of (2.4) and (3.1).

Writing out equation (5.4), we get
\[
\int_J E(X_t X_s) \varphi_k(s) ds = \lambda_k \varphi_k(t);
\]
and so from the assumptions (i)-(iii) and eq. (2.1) we conclude that each of the eigenfunctions \( \varphi_k(\cdot) \) is continuous and of bounded variation.

This means that whenever \( t_0, t \in J \), i.e., \( a \leq t_0 < t \leq b \), the expression
\[
\int_{t_0}^t f(s) d\varphi_k(s)
\]
is a well defined Stieltjes integral. Moreover, if \( f \) is assumed of bounded variation,
\[
\int_{t_0}^t f(s) d\varphi_k(s) = \left[ f \varphi_k \right]_{t_0}^t - \int_{t_0}^t \varphi_k(s) f'(s) ds.
\]

We now turn to the approximation (2.4) from the theorem, and we use the Karhunen-Loève expansion (5.8) in the computation of
\[
\Delta X_{t_i} := X_{t_{i+1}} - X_{t_i}
\]
for a fixed (chosen) partition \( \pi \) as specified in (2.2).

Using condition (3)(ii) in the statement of the theorem, we note that for fixed \( J \), the operator \( T_j \) in (5.1) is trace-class. From operator theory (Mercer’s theorem), we know that
\[
\text{trace}(T_j) = \int_J E(X_t^2) dt = \sum_{k=1}^{\infty} \lambda_k < \infty
\]
i.e., integration in (5.1) over the diagonal \( s = t \). And so in particular, finiteness of \( \sum_{k=1}^{\infty} \lambda_k \) follows.

In the study of the operator \( T_j \) from (5.1) we make use of tools from Hilbert space theory of integral operators. In particular, in the estimate (5.14) we use Mercer’s theorem. However in applications to covariance kernels (2.1) one often has stronger properties. It is known that if the kernel in (2.1) is Lipschitz of degree \( \gamma \) with \( \gamma > \frac{1}{2} \) in one of the two variables (with the other fixed), then the operator \( T_j \) in (5.1) will automatically be nuclear. For the literature on this we refer to [Dos93, Küh83, LL52, Sti58]. We further note that this Lipschitz condition is indeed satisfied for the covariance kernel of fractional Brownian motion, see e.g., [LA07].

Set \( (\Delta \varphi_k)_{t_i} := \varphi_k(t + \Delta t) - \varphi_k(t) \). Using the Hilbert space \( L^2(J \times \Omega, m \times P) \) and its tensor-product representation, \( L^2(J) \otimes L^2(\Omega, S, P) \), we get
\[
\sum_{t=0}^{n-1} f(t_i) \Delta X_{t_i}(\omega) \text{ by (5.8) } \sum_{i=0}^{n-1} \sum_{k=1}^{\infty} f(t_i) \sqrt{\lambda_k} (\varphi_k(t_{i+1}) - \varphi_k(t_i)) Z_k(\omega)
= \sum_{k=1}^{\infty} \sum_{i=0}^{n-1} f(t_i) (\Delta \varphi_k)_{t_i} \sqrt{\lambda_k} Z_k(\omega)
\]

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and therefore

\begin{equation}
E(|S_n(f, X)|^2) = \sum_{k=1}^{\infty} \lambda_k \left| \sum_{i=0}^{n-1} f(t_i)(\Delta \varphi_k)_{t_i} \right|^2.
\end{equation}

Since \(f\) is assumed continuous, and each \(\varphi_k\) is of bounded variation, the following convergence holds:

\begin{equation}
\lim_{|\pi| \to 0} \sum_{i=0}^{n-1} f(t_i)(\Delta \varphi_k)_{t_i} = \int_{t_0}^{t} f d\varphi_k.
\end{equation}

Now if the function \(f\) is satisfying \(f \in L^2(J)\) and \(f' \in L^2(J)\), then we get the following estimate, relying on the boundary representation \((5.12)\) and Parseval, see also \((5.5)\): For the RHS in \((5.15)\) we have; after passing to the limit:

\begin{align*}
\sum_{k=1}^{\infty} \lambda_k \left| \int_{t_0}^{t} f(s) d\varphi_k(s) \right|^2 &\overset{\text{by} \ (5.4)}{=} (\text{boundary terms}) + \sum_{k=1}^{\infty} \lambda_k \left| \int_{t_0}^{t} \varphi_k(s)f'(s)ds \right|^2 \\
&\overset{\text{by} \ (5.4)\ \text{and Parseval}}{\leq} (\text{boundary terms}) + \lambda_1 \sum_{k=1}^{\infty} \left| \int_{t_0}^{t} \varphi_k f' ds \right|^2 \\
&\overset{\text{by} \ (5.5)}{\leq} (\text{boundary terms}) + \lambda_1 \int_{t_0}^{t} |f'(s)|^2 ds;
\end{align*}

which is the desired conclusion in part (c) of the theorem.

**Proof.** Of Corollary 3.2 The essential point is formula \((5.8)\). However, in substitution of the expression on the RHS in \((5.8)\) we make use of double-orthogonality, i.e., \((5.5)\) and \((5.7)\). Specifically, we have the tensor product representation \(L^2(J \times \Omega, m \times P) = L^2(J) \otimes L^2(\Omega)\), and so \(X = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \varphi_k \otimes Z_k\) in \((6.8)\) refers to the tensor representation.

6. Entropy: Optimal Bases

In this section we compare the choice of ONB from section 3 with alternative choices of ONBs. The application of Karhunen-Loève dictates a particular choice of ONB.

Historically, the Karhunen-Loève arose as a tool from the interface of probability theory and information theory; see details with references inside the paper. It has served as a powerful tool in a variety of applications; starting with the problem of separating variables in stochastic processes, say \(X_t\); processes that arise from statistical noise, for example from fractional Brownian motion. Since the initial inception in mathematical statistics, the operator algebraic contents of the arguments have crystallized as follows: starting from the process \(X_t\), for simplicity assume zero mean, i.e., \(E(X_t) = 0\); create a correlation matrix \(T_J(s,t) = E(X_s X_t)\). (Strictly speaking, it is not a matrix, but rather an integral kernel. Nonetheless, the matrix terminology has stuck.) The next key analytic step in the Karhunen-Loève method is to then apply the Spectral Theorem from operator theory to a corresponding self-adjoint operator, or to some operator naturally associated with the integral kernel: Hence the name, the Karhunen-Loève Decomposition (KLC). In favorable cases (discrete spectrum), an orthogonal family of functions \((\varphi_n(t))\) in the time variable arise, and a corresponding family of eigenvalues. We take them to be normalized
in a suitably chosen square-norm. By integrating the basis functions \( \varphi_n(t) \) against \( X_t \), we get a sequence of random variables \( Z_n \). It was the insight of Karhunen-Loève \cite{Loe52} to give general conditions for when this sequence of random variables is independent, and to show that if the initial random process \( X_t \) is Gaussian, then so are the random variables \( Z_n \).

Below, we take advantage of the fact that Hilbert space and operator theory form the common language of both quantum mechanics and of signal/image processing. Recall first that in quantum mechanics, (pure) states as mathematical entities “are” one-dimensional subspaces in complex Hilbert space \( \mathcal{H} \), so we may represent them by vectors of norm one. Observables “are” selfadjoint operators in \( \mathcal{H} \), and the measurement problem entails von Neumann’s spectral theorem applied to the operators.

In signal processing, time-series, or matrices of pixel numbers may similarly be realized by vectors in Hilbert space \( \mathcal{H} \). The probability distribution of quantum mechanical observables (state space \( \mathcal{H} \)) may be represented by choices of orthonormal bases (ONBs) in \( \mathcal{H} \) in the usual way (see e.g., \cite{Jor06}). In the 1940s, Kari Karhunen \cite{Kar46, Kar52} pioneered the use of spectral theoretic methods in the analysis of time series, and more generally in stochastic processes. It was followed up by papers and books by Michel Loève in the 1950s \cite{Loe52}, and in 1965 by R.B. Ash \cite{Ash90}. (Note that this theory precedes the surge in the interest in wavelet bases!)

Parallel problems in quantum mechanics and in signal processing entail the choice of “good” orthonormal bases (ONBs). One particular such ONB goes under the name “the Karhunen-Loève basis.” We will show that it is optimal.

**Definition 6.1.** Let \( \mathcal{H} \) be a Hilbert space. Let \( (\psi_i) \) and \( (\varphi_i) \) be orthonormal bases (ONB). If \( (\psi_i)_{i \in I} \) is an ONB, we set \( Q_n := \text{the orthogonal projection onto span}\{\psi_1, \ldots, \psi_n\} \).

We now introduce a few facts about operators which will be needed in the paper. In particular we recall Dirac’s terminology \cite{Dir47} for rank-one operators in Hilbert space. While there are alternative notation available, Dirac’s bra-ket terminology is especially efficient for our present considerations.

**Definition 6.2.** Let vectors \( u, v \in \mathcal{H} \). Then

\[
(6.1) \qquad \langle u|v \rangle = \text{inner product} \in \mathbb{C},
\]

\[
(6.2) \qquad |u\rangle\langle v| = \text{rank-one operator} , \mathcal{H} \rightarrow \mathcal{H},
\]

where the operator \( |u\rangle\langle v| \) acts as follows

\[
(6.3) \qquad |u\rangle\langle v|w = |u\rangle\langle v|w = \langle v|w \rangle u, \quad \text{for all } w \in \mathcal{H}.
\]

Dirac’s bra-ket and ket-bra notation is is popular in physics, and it is especially convenient in working with rank-one operators and inner products. For example, in the middle term in eq (6.3), the vector \( u \) is multiplied by a scalar, the inner product; and the inner product comes about by just merging the two vectors.

**Definition 6.3.** If \( S \) and \( T \) are bounded operators in \( \mathcal{H} \), in \( B(\mathcal{H}) \), then

\[
(6.4) \qquad S|u\rangle\langle v|T = |Su\rangle\langle T^*v|
\]
If \((\psi_i)_{i \in \mathbb{N}}\) is an ONB then the projection
\[
Q_n := \text{proj span}\{\psi_1, \ldots, \psi_n\}
\]
is given by
\[
(6.5) \quad Q_n = \sum_{i=1}^{n} |\psi_i\rangle\langle \psi_i|;
\]
and for each \(i\), \(|\psi_i\rangle\langle \psi_i|\) is the projection onto the one-dimensional subspace \(C\psi_i \subset \mathcal{H}\).

**Definition 6.4.** Suppose \(X_t\) is a stochastic process indexed by \(t\) in a finite interval \(J\), and taking values in \(L^2(\Omega, P)\) for some probability space \((\Omega, P)\). Assume the normalization \(E(X_t) = 0\). Suppose the integral kernel \(E(X_tX_s)\) can be diagonalized, i.e., suppose that
\[
\int_J E(X_tX_s)\varphi_k(s)ds = \lambda_k \varphi_k(t)
\]
with an ONB \((\varphi_k)\) in \(L^2(J)\). If \(E(X_t) = 0\) then
\[
X_t(\omega) = \sum_k \sqrt{\lambda_k} \varphi_k(t)Z_k(\omega), \quad \omega \in \Omega
\]
where \(E(Z_jZ_k) = \delta_{j,k}\), and \(E(Z_k) = 0\). The ONB \((\varphi_k)\) is called the KL-basis with respect to the stochastic processes \(\{X_t : t \in J\}\).

**Theorem 6.5.** *(See [JS07])* The Karhunen-Loève ONB gives the smallest error terms in the approximation to a frame operator.

**Proof.** Given the operator \(T_J\) which is trace class and positive semidefinite, we may apply the spectral theorem to it. What results is a discrete spectrum, with the natural order \(\lambda_1 \geq \lambda_2 \geq \ldots\) and a corresponding ONB \((\varphi_k)\) consisting of eigenvectors, i.e.,
\[
(6.6) \quad T_J\varphi_k = \lambda_k \varphi_k, \quad k \in \mathbb{N}
\]
called the Karhunen-Loève data. The spectral data may be constructed recursively starting with
\[
(6.7) \quad \lambda_1 = \sup_{\varphi \in \mathcal{H}, \|\varphi\| = 1} \langle \varphi | T_J \varphi \rangle = \langle \varphi_1 | T_J \varphi_1 \rangle
\]
and
\[
(6.8) \quad \lambda_{k+1} = \sup_{\substack{\varphi \in \mathcal{H}, \|\varphi\| = 1 \atop \varphi \perp \varphi_1, \varphi_2, \ldots, \varphi_k}} \langle \varphi | T_J \varphi \rangle = \langle \varphi_{k+1} | T_J \varphi_{k+1} \rangle
\]
Now an application of [ArKa06]; Theorem 4.1 yields
\[
(6.9) \quad \sum_{k=1}^{n} \lambda_k \geq \text{tr}(Q_n^\psi T_J) = \sum_{k=1}^{n} \langle \psi_k | T_J \psi_k \rangle \quad \text{for all } n,
\]
where \(Q_n^\psi\) is the sequence of projections from (6.5), deriving from some ONB \((\psi_i)\) and arranged such that
\[
(6.10) \quad \langle \psi_1 | T_J \psi_1 \rangle \geq \langle \psi_2 | T_J \psi_2 \rangle \geq \ldots
\]
Hence we are comparing ordered sequences of eigenvalues with sequences of diagonal matrix entries.
Finally, we have

\[
\text{tr} (T_J) = \sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} \langle \psi_k | T_J \psi_k \rangle < \infty.
\]

The assertion in Theorem 6.5 is the validity of

(6.11) \quad E_n^\psi \leq E_n^\phi

for all \((\psi_i) \in ONB(\mathcal{H})\), and all \(n = 1, 2, \ldots\); and moreover, that the infimum on the RHS in (6.11) is attained for the KL-ONB \((\varphi_k)\). But we see that (6.11) is equivalent to the system (6.9) in the Arveson-Kadison theorem. \(\square\)

The Arveson-Kadison theorem is the assertion (6.9) for trace class operators, see e.g., refs [Arv06] and [ArKa06]. That (6.11) is equivalent to (6.9) follows from the definitions.

Our next theorem gives Karhunen-Loève optimality for sequences of entropy numbers.

**Theorem 6.6.** (See [JS07]) The Karhunen-Loève ONB gives the smallest sequence of entropy numbers in the approximation.

**Proof.** We begin by a few facts about entropy of trace-class operators \(T_J\). The entropy is defined as

(6.12) \quad S(T_J) := -\text{tr}(T_J \log T_J).

The formula will be used on cut-down versions of an initial operator \(T_J\). In some cases only the cut-down might be trace-class. Since the Spectral Theorem applies to \(T_J\), the RHS in (6.12) is also

(6.13) \quad S(T_J) = -\sum_{k=1}^{\infty} \lambda_k \log \lambda_k.

For simplicity we normalize such that 1 = \text{tr}T_J = \sum_{k=1}^{\infty} \lambda_k, and we introduce the partial sums

(6.14) \quad S_{KL}^n(T_J) := -\sum_{k=1}^{n} \lambda_k \log \lambda_k.

and

(6.15) \quad S_n^\psi(T_J) := -\sum_{k=1}^{n} \langle \psi_k | T_J \psi_k \rangle \log \langle \psi_k | T_J \psi_k \rangle.

Let \((\psi_i) \in ONB(\mathcal{H})\), and set \(d_k^\psi := \langle \psi_k | T_J \psi_k \rangle\); then the inequalities (6.9) take the form:

(6.16) \quad \text{tr}(Q_n^\psi T_J) = \sum_{i=1}^{n} d_i^\psi \leq \sum_{i=1}^{n} \lambda_i, \quad n = 1, 2, \ldots

where as usual an ordering

(6.17) \quad d_1^\psi \geq d_2^\psi \geq \ldots

has been chosen.
Now the function $\beta(t) := t \log t$ is convex. And application of Remark 6.3 in [ArKa06] then yields

$$\sum_{i=1}^{n} \beta(d_{i}^{\psi}) \leq \sum_{i=1}^{n} \beta(\lambda_{i}), \quad n = 1, 2, \ldots.$$  

Since the RHS in (6.18) is $-\text{tr}(T_{J} \log T_{J}) = -S_{n}^{KL}(T_{J})$, the desired inequalities follow, i.e., the KL-data minimizes the sequence of entropy numbers.

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