All-order waveforms from amplitudes

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ABSTRACT: Waveforms are classical observables associated with any radiative physical process. Using scattering amplitudes, these are usually computed in a weak-field regime to some finite order in the post-Newtonian or post-Minkowskian approximation. Here, we use strong field amplitudes to compute the waveform produced in scattering of massive particles on gravitational plane waves, treated as exact nonlinear solutions of the vacuum Einstein equations. Notably, the waveform contains an infinite number of post-Minkowskian contributions, as well as tail effects. We also provide, and contrast with, analogous results in electromagnetism.
1 Introduction

The observation of gravitational waves has brought renewed importance to the study of general relativity and its observables. Surprisingly, scattering amplitudes – one of the key outputs of quantum field theory – are providing a new way to study classical general relativity; for reviews see [1–3]. Starting from novel perspectives on [4–6], and a remarkable state-of-the-art calculation for [7], the conservative Hamiltonian of the gravitational two-body problem, a new program for providing higher-order post-Minkowskian (PM) approximations to gravitational observables has emerged based on the classical limit of scattering amplitudes. This has led to a variety of exciting new results for gravitational observables, e.g. [8–24], which build on many of the powerful structures in scattering amplitudes such as generalized unitarity and double copy, as well as techniques from effective field theory.

A key tool in this program has been the development of a formalism to systematise the extraction of classical physical observables from scattering amplitudes [25]. So far, all observables computed with this approach are valid for weak fields only: they are obtained from amplitudes at finite PM order, so truncate at a corresponding fixed order in the coupling [26–37]. This is in sharp contrast with other approaches to gravitational dynamics such as the self-force paradigm [38–43], where perturbation theory is implemented around a curved background and the weak field limit is not considered.

To address this gap, the amplitudes-based approach can be generalised to curved backgrounds by means of strong field scattering amplitudes and their classical limits [44]. This provides an alternative route to the computation of classical observables, as strong field amplitudes encode a substantial amount of information about higher-order processes [45–52] and finite size effects [53–55] in trivial backgrounds, and can also admit remarkably compact formulae [56–58]. A key aspect is that even first order perturbation theory around a curved background – which we refer to as ‘first post-background’, or 1PB, order – encodes infinitely many orders of the PM expansion. This is analogous to the relation between the PM and post-Newtonian (PN) expansions for bound orbits, where a fixed contribution of the former encodes infinitely many orders of the latter due to the virial theorem.

Here we show for the first time how classical observables encoding all-order results can be extracted from scattering amplitudes. We derive expressions for the gravitational waveform emitted by a point particle scattering on a gravitational plane wave background (an exact solution to the nonlinear Einstein equations), encoding all-order contributions in the PM expansion when the flat spacetime limit is taken, as well as tail effects which usually enter at high order in the PM approximation. These plane waves are not just good models of gravitational waves, but also provide a local description of any spacetime in the neighbourhood of a null geodesic [59]. We also perform an analogous calculation of, and compare with, electromagnetic waveforms for charged particles scattering on electromagnetic plane waves (which does not seem to appear in the literature, despite a long history of related studies [60–63]), which sheds new light on the interface between QED and gravitational observables [64–67].
1.1 Waveforms from amplitudes on curved backgrounds

Asymptotic waveforms. Let $|\Psi\rangle$ be a superposition of massive particle states describing a free particle of mass $m$ as in [44]:

$$|\Psi\rangle = \int d\Phi(p) \phi(p) e^{ipb/\hbar} |p\rangle,$$  

(1.1)

such that $\langle \Psi|\Psi\rangle = 1$, where $d\Phi(p) := \hat{d}_4 p \Theta(p_0) \delta(p^2 - m^2)$ is the Lorentz-invariant on-shell measure, while $\hat{\delta}(x) := \frac{2}{\pi} \delta(x)$ and $\hat{d}_x := \frac{d_x}{2\pi}$ throughout. The wavepacket $\phi(p)$ is a square-integrable function with a well-defined classical limit (cf., [25]). This state is evolved on an electromagnetic or gravitational plane wave background, in the latter case a solution to the fully non-linear Einstein equations. In terms of the S-matrix $S$ on that background, the time-evolved state is simply $S|\Psi\rangle$.

Our interest is in the classical gravitational or electromagnetic radiation emitted by a scalar particle as it scatters on these backgrounds, as measured by an asymptotic observer at future null infinity. This observable is encoded in the classical limit of $\langle O_{\mu}(x) \rangle := \langle \Psi|S^\dagger O_{\mu}(x)S|\Psi\rangle$ in which $O_{\mu}(x)$ represents the field strength $F_{\mu\nu}(x)$ or curvature tensor $R_{\mu\nu\sigma\rho}(x)$ at future null infinity, and where $\bar{\mu}$ denotes a number of indices appropriate to the radiated field. In terms of creation and annihilation operators $a^\dagger_\eta(k)$ and $a_\eta(k)$ for a state of helicity $\eta$, these operators have the schematic form

$$O_{\bar{\mu}}(x) = \int d\Phi(k) e^{-ik\cdot x/\hbar} C^\eta_{\bar{\mu}}(k) a_\eta(k) + c.c.,$$  

(1.2)

where $C^\eta_{\bar{\mu}}(k)$ is a placeholder for the polarisation content of the field, and a sum over helicities $\eta$ is implied. Now, with coordinates $x^\mu = (t, \mathbf{x})$, future null infinity corresponds to $r \equiv |\mathbf{x}| \to \infty$ while $u = t - r$ is held constant. The mode expansion then becomes, to leading order in $1/r$ [30],

$$O_{\bar{\mu}}(x) = -\frac{i\hbar^2}{4\pi^2} \int_0^\infty d\omega e^{-i\omega u} C^\eta_{\bar{\mu}}(k) a_\eta(k) \bigg|_{k = \hbar\omega\hat{x}} + c.c. \quad (1.3)$$

in which we define the null vector $\hat{x}^\mu = (1, \hat{x})$ and $\omega$ is a classical ($\hbar$-independent) frequency; this parametrisation will be useful later when we take the classical limit.

Following [30], the waveform $W_{\bar{\mu}}(u, \hat{x})$ is defined simply as the expectation value of the coefficient of this leading $1/r$ term. It is a function of $u$ and the two angular degrees of freedom encoded in $\hat{x}^\mu$. So for the Maxwell and Riemann tensors, respectively, we have

$$\langle F_{\mu\nu}(x) \rangle = \frac{1}{r} W_{\mu\nu}(u, \hat{x}) + O(r^{-2}), \quad \langle R_{\mu\nu\sigma\rho}(x) \rangle = \frac{1}{r} W_{\mu\nu\sigma\rho}(u, \hat{x}) + O(r^{-2}), \quad (1.4)$$

The amplitudes contributing to the waveform are easily identified by inserting complete sets of states into the expectation value; the leading contribution is at 1PB, meaning order $e^{\kappa}$ in QED (gravity) but all orders in the background fields, and comes from interference...
between tree-level 2-point and 3-point amplitudes since

\[ \langle \Psi | S^a \eta(k) S | \Psi \rangle = \int d\Phi(p') \langle \Psi | S^a \rangle \langle p' | k^\eta | S | \Psi \rangle \langle \Psi | \rangle + \cdots . \] (1.5)

It is easy to show that this combination of amplitudes reproduces the radiation emitted due to geodesic motion, i.e. the first contribution of self-force effects \[40\].

We stress that, unlike in vacuum, two-point amplitudes on backgrounds are not trivial even at tree-level, encoding e.g. memory effects \[44\]. Suppressing the ‘tree’ subscript from here on we arrive at, in QED

\[ W_{\mu\nu}(u, \hat{x}) = -\frac{i}{\pi} \int d\omega e^{-i\omega u} k_{[\mu} \epsilon_{\nu]}^{\eta} \int d\Phi(p') \langle \Psi | S^a \rangle \langle p' | k^\eta | S | \Psi \rangle \mid_{k=\hbar\omega \hat{x}} + \text{c.c.} \quad (1.6) \]

where \( \epsilon^{\mu}_{\mu}(k) \) is the photon polarisation, while in gravity

\[ W_{\mu\sigma\rho}(u, \hat{x}) = \frac{i\kappa}{2\pi \hbar^2} \int d\omega e^{-i\omega u} k_{[\mu} \epsilon_{\nu]}^{\eta} k_{[\sigma} \epsilon_{\rho]}^{\eta} \int d\Phi(p') \langle \Psi | S^a \rangle \langle p' | k^\eta | S | \Psi \rangle \mid_{k=\hbar\omega \hat{x}} + \text{c.c.} \quad . \] (1.7)

**Plane wave backgrounds.** Plane waves are highly symmetric vacuum solutions of the Einstein or Maxwell equations with two functional degrees of freedom. In gravity, they are described by metrics of the form \[68\]:

\[ ds^2 = 2dx^+dx^- - dx^a dx^a - \kappa H_{ab}(x^-) x^a x^b (dx^-)^2 , \] (1.8)

where \( \kappa \) is the gravitational coupling constant, Latin indices label the ‘transverse’ directions \( x^\pm = (x^1, x^2) \), while the \( 2 \times 2 \) matrix \( H_{ab}(x^-) \) is symmetric and traceless (ensuring the vacuum Einstein equations are satisfied) and compactly supported in the region \( x^-_i < x^- < x^-_f \) (ensuring the spacetime admits a well-defined S-matrix \[69\]). The metric has a covariantly constant null Killing vector \( n = \partial_- \) (or \( n_\mu = \delta^-_\mu \)) which will recur throughout. To ease notation, the gravitational coupling can be absorbed into the background by taking \( \kappa H_{ab} \rightarrow H_{ab} \), which we use from now on.

Plane wave metrics have several associated geometric structures. First, there is a zweibein \( E^a_i(x^-) \) and its inverse \( E^i_a(x^-) \), labelled by the index \( i = 1, 2 \) satisfying

\[ \ddot{E}_{i a} = H_{ab} E^b_i, \quad \dot{E}^a_i \dot{E}_{j[a} = 0 , \] (1.9)

which can be viewed as a solution to the geodesic deviation equation. It encodes gravitational (velocity) memory through the difference

\[ \Delta E^i_a = E^i_a(x^- > x^-_f) - E^i_a(x^- < x^-_i) , \] (1.10)

which compares the relative transverse positions of two neighbouring geodesics. The
zweibein defines a transverse metric $\gamma$ and deformation tensor $\sigma$,

$$\gamma_{ij}(x^-) := E^a_i E^a_j,$$  
$$\sigma_{ab}(x^-) := \dot{E}^i_a E^b_i,$$  

(1.11)

the latter encoding the expansion and shear of the null geodesic congruence associated to (1.8). These definitions are completed by the initial condition $E^a_i(x^- < x^-_i) = \delta^i_a$, which yields $\gamma_{ij}(x^- < x^-_i) = \delta_{ij}$ and $\sigma_{ab}(x^- < x^-_i) = 0$.

Turning to electromagnetism, plane waves can be defined by the potential

$$A_\mu(x) = -x^b E_0(x^-) n_\mu,$$  

(1.12)

in lightfront coordinates (defined by the flat space part of (1.8)) and $n_\mu$ is as above. The two-component electric field $E_0(x^-)$ is taken to be compactly supported. A useful quantity associated with this background is

$$a_\perp(x^-) := \int_{-\infty}^{x^-} ds E_\perp(s),$$  

(1.13)

such that $e a_\perp$ is the effective ‘work done’ on a charge. The electromagnetic velocity memory effect is encoded in the constant $e a_\perp(x^- > x^-_f)$ [70]: this is the change in transverse momentum of a particle crossing the background from the asymptotic past to the future.

In order to simplify the presentation of our results we make the assumption that velocity memory effects induced by our backgrounds are parametrically small, and thus negligible. Technically, this means setting $a_\perp(x^- > x^-_f) = 0$ in electromagnetism, and $E^a_i(x^- > x^-_f) = \delta^a_i$ in gravity. The main simplification which results is that the tree-level 2-point amplitudes collapse to

$$\langle p'| S | \Psi \rangle \to \int d\Phi(p)\phi(p) 2p_+ \delta^3(p' - p) e^{i\theta(p)} = e^{i\theta(p')} \phi(p'),$$  

(1.14)

where $\theta$ is theory-dependent; for our purposes it can be absorbed into a redefinition of $u$, but in general it will encode position memory effects on the scattered scalar [71].

2 Electromagnetism

In this section we construct the classical limit of the electromagnetic waveform $W_{\mu\nu}(u, \dot{x})$ from (1.6). Given our assumption of no memory, the only ingredient required is the 3-point amplitude on an electromagnetic plane wave background, to which we now turn.

Tree-level 3-point amplitude. We require the amplitude for an incoming charged scalar to emit a photon. Let the incoming momentum be $p_\mu$ (with $p^2 = m^2$), the outgoing be $p'_\mu$ (also with $p'^2 = m^2$) and the emitted photon have null momentum $k_\mu$ and helicity $\eta$. The amplitude can be calculated by evaluating the cubic part of the action on
the appropriate scattering states in a plane wave background, see e.g. [63]. The result is
\begin{equation}
\langle p', k' | S | \Psi \rangle = \int d\Phi(p) \phi(p) e^{ip \cdot b / \hbar} \hat{A}_{3}(p \rightarrow p' + k') \, ,
\end{equation}
(2.1)
where \( \int_{-\infty}^{+\infty} dy := \int y \), while \( y \) and \( z \) are integration variables denoting lightfront times. The ‘dressed’ momentum \( P_{\mu}(y) \) is the classical momentum of the particle in the background,
\begin{equation}
P_{\mu}(y) = p_{\mu} - e a_{\mu}(y) + n_{\mu} \frac{2e a_{\mu}(y) \cdot p - e^{2}a^{2}(y)}{2p_{+}} \, ,
\end{equation}
(2.3)
where \( a_{\mu}(y) = \delta_{\mu} a_{\perp}(y) \), obeying \( P^{2}(y) = m^{2} \). Note that only three components of overall momentum are conserved in (2.2), as the background breaks translation symmetry in \( y \) − \( \hat{x} \).

**Calculation of the waveform.** We assemble the waveform (1.6) from (2.2) and (1.14). The first step is to perform the sum over photon helicity using the completeness relation in lightfront gauge. Upon inserting this into the waveform all gauge-dependent pieces vanish by anti-symmetry or generate boundary terms which can be ignored [70], leaving only a contribution from \( -\eta_{\mu\nu} \). An immediate simplification is that the delta function sets \( p' = p \), and thus the wavepacket appears as \( |\phi(p)|^{2} \). This means that the impact parameter \( b \) drops out, and under the usual assumption that \( \phi \) is sharply peaked around some classical momentum, we can integrate over \( p \), localising the integrand at the on-shell momentum of the incoming particle, which we continue to write as \( p \) for simplicity. This gives
\begin{equation}
W_{\mu\nu}(u, \hat{x}) = \frac{-ie}{4\pi^{2}p_{+}} \int_{y,\omega} \omega e^{-i\omega(u - \hat{x} \cdot X(y))} \hat{x} [\mu P_{\nu}](y) \, ,
\end{equation}
(2.4)
in which \( X_{\mu}(y) \) is the classical particle orbit, obeying \( X'_{\mu}(y) = P_{\mu}(y) / p_{+} \). The frequency integral can be performed by writing it as a derivative with respect to \( y \) and integrating by parts. This yields a very compact final expression for the classical waveform:
\begin{equation}
W_{\mu\nu}(u, \hat{x}) = \frac{e}{2\pi} \int_{y} \delta(u - \hat{x} \cdot X(y)) \frac{d}{dy} \hat{x} [\mu P_{\nu}](y) = \frac{e}{2\pi} \sum_{\text{sols}} \frac{p_{+}}{\hat{x} \cdot P} \frac{d}{d\hat{x}} [\mu P_{\nu}](2.5)
\end{equation}
where the sum runs over all solutions of the delta-function constraint. It can be checked that this matches the result obtained directly from classical electrodynamics; see Appendix A.

We now highlight several properties of the waveform.

**2.1 Properties of the waveform**
First observe that, due to the derivative, the waveform is vanishing in the absence of acceleration. Indeed the final integration by parts, performed as part of the evaluation of the frequency interval, corresponds to removing Coulomb field contributions from the asymptotic waveform, i.e. restricting to the radiation field which is of interest [72].
Figure 1. Two examples of the waveform $W_{\mu \nu}(u, \hat{x})$ for a particle at rest struck by the wave $ea_1 = m\xi \text{sech}^2(\nu y^{-})$ and $a_2 = 0$, for strength $\xi$ and frequency $\nu$. We work in units where $\nu = 1$. Left: $W_1-$ as a function of $u$ for various $\theta$. We have fixed $\xi = 2$ and $\phi = 0$. At $\theta = 0$ (red/black dashed curve), the waveform is a multiple of the driving field $F_{\mu \nu}$ as in (2.6), but is very different for larger angles. Right: $W_1+$ for various $\xi$ at fixed scattering angles $\theta = \pi$, $\phi = 0$, showing a nonlinear change in the waveform as the strength $\xi$ of the background increases.

Next, recall from (2.3) that the dressed momentum $P$, hence the orbit $X$, is quadratic in the background $ea$: it follows immediately that the waveform contains terms of all orders in the background, and hence the coupling $e$. This is both explicit, due to the presence of $P$ in the denominator, and implicit, in that one must solve the delta-function constraint. This requires inverting $\hat{x} \cdot X(y)$ which will introduce arbitrary non-polynomial dependence on the coupling. (Even for the simple field choice of an unphysical ‘box’ electric field, solving the constraint means solving a cubic equation.) In general, there will be multiple solutions to the constraint, meaning that the waveform at any given $u$ and $\hat{x}$ is sourced at several points on the particle orbit.

For any plane wave profile, we can consider the waveform aligned with the direction of the background: $\hat{x}_\mu = \sqrt{2} n_\mu$ (the factor results from conventions). Parameterising $\hat{x}^\mu$ by azimuthal and polar angles $\phi$ and $\theta$, respectively, alignment with the background corresponds to $\theta = 0$. At this collinear point the argument of the delta function is simply $u - \sqrt{2} y$, and thus has a single point of support. Most of the structure in the waveform vanishes due to contraction or commutation with $n_\mu$, and one finds

$$W_{\mu \nu} \big|_{\theta = 0} = -\frac{e^2 F_{\mu \nu}(\frac{y}{\sqrt{2}})}{4\pi p_+ \sqrt{2}},$$

a result we will contrast with gravity in Sec. 3.1. If we consider any other point on the celestial sphere, the waveform has a far richer structure, though – see again Fig. 1.
3 Gravity

3-point amplitude in gravity  We now require the tree-level 3-point amplitude for a massive scalar emitting a graviton, on the gravitational plane wave background. Let the on-shell incoming/outgoing momentum for the scalar be $p_\mu/p_\mu'$, but let $k_\mu$ now label the emitted graviton momentum. In contrast to QED, all particles are ‘dressed’ in gravity: in scattering calculations, any particle of asymptotic momentum $l_\mu$ and mass $m$ has the dressed momentum \[73, 74\]

\[
L_\mu(y)dy^\mu = l_+ dy^+ + (l_+ E^I_\mu(y^-) + l_+ \sigma_{ab}(y^-) y^b)dy^a \\
+ \left(\frac{m^2}{2l_+} + \gamma^{ij}(y^-) \frac{l_+ l_j}{2l_+} + \frac{l_+}{2} \delta_{bc}(y^-) y^b y^c + l_+ E^I_\mu(y^-) y^h\right)dy^- ,
\]

(3.1)

which obeys $g^{\mu\nu}L_\mu L_\nu = m^2$. Note that, in contrast to the dressed momentum (2.3) in QED, the gravitational dressing depends on the perpendicular coordinates $y^a$. The outgoing graviton polarisation also becomes ‘dressed’ by the background; it is conveniently expressed in terms of a projector acting on the free polarisation:

\[
E_\eta^{\mu\nu}(k; y) = \mathbb{P}_{\mu\nu,\sigma\rho}(k; y)\varepsilon^{\sigma\rho}_\eta := \left[\mathbb{P}_{\mu\nu}(k; y)\mathbb{P}_{\rho\sigma}(k; y) - \frac{i\hbar}{k_+} n_\mu n_\nu \delta_\rho^a \delta_\sigma^b \sigma_{ab}(y)\right] \varepsilon_\eta^{\sigma\rho} ,
\]

(3.2)

where $\mathbb{P}_{\mu\nu}(k; y) = g_{\mu\nu}(y) - 2K_{\mu}(y)n_\nu/k_+$ contains the dressed momentum $K_\mu(y)$ of the graviton. With these ingredients and the simplification of negligible memory, we can write down the required amplitude [74]:

\[
A_3(p \rightarrow p' + k^0) = \frac{2i\kappa}{\hbar^{3/2}} \int_y \frac{\exp[\imath \mathcal{V}(y)]}{|E(y)|} E_\eta^{\mu\nu}(k; y)P^\mu(y)P'^\nu(y) ,
\]

(3.3)

where (2.1) still holds, the exponent is

\[
\mathcal{V}(y) := \frac{1}{\hbar} \int_y^{y^-} dz \frac{P_\mu(z)K_\nu(z)g^{\mu\nu}(z)}{p_+ - k_+} ,
\]

(3.4)

and $|E(y)|$ is the zweibein determinant. It can be checked that all contractions between dressed momenta and polarisations appearing are independent of the transverse coordinates, even though their constituents are not. Hence the integrand in (3.3) is a function of only $y^-$, and is (trivially) evaluated on the classical particle orbit parametrized by $y^-$. 

Calculation of the waveform  The calculation proceeds as in QED: assemble the waveform (1.7) from the three-point amplitude (3.3) and the two-point amplitude (1.14). Similarly to the electromagnetic case, we can restrict the sum over graviton polarisations to physical degrees of freedom using the gauge-invariant portion of the completeness relation. We only need the leading classical behaviour of the 3-point amplitude (3.3). Inspecting the powers $\hbar$ in the amplitude and in the definition of the waveform (1.7), it is again clear that all pre-factors of $\hbar$ are absent, and the the classical limit is obtained by setting $\hbar = 0$. 

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determined by the 0PB classical orbit from the polarisation sum can be simplified by observing that classical orbit. Note that \[ \hat{\bar{\nu}} \]

Some insight into the gravitational waveform is provided by observing from (3.6) that \[ \hat{\bar{\nu}} \]

\[ X \]

3.1 Properties of the waveform

general relativity; see Appendix A.

Once again, this result can be confirmed by comparison with the calculation in classical

Using the Jacobi identity for determinant derivatives, it can be checked that the term in

\[ \eta_{\gamma\delta} \hat{P}^{\alpha\beta\gamma\delta}(\hat{x}, y) P_\alpha(y) P_\beta(y) = m^2 + \frac{2i\eta_{\gamma\delta}}{\omega \hat{x}^+} \left[ \frac{2i\eta_{\gamma\delta}}{\omega \hat{x}^+} \frac{2i\eta_{\gamma\delta}}{\omega \hat{x}^+} \right] \].

Using the Jacobi identity for determinant derivatives, it can be checked that the term in brackets is exactly the \( y^- \) derivative of the entire integrand in (3.5), and hence gives a boundary term which can be dropped, leaving only the mass term.

It remains to perform the \( \omega \) integral. However, in contrast to QED, the projector \( \hat{P}^{\alpha\beta\gamma\delta}(\hat{x}, y) \) contains terms with different scaling in \( \omega \). We highlight this by defining

\[ T_{\nu\rho}^0(\hat{x}, y) := \frac{P_{\nu\alpha}(\hat{x}, y) P_\rho(\hat{y}) P^\alpha(\hat{y}) P^\beta(\hat{y})}{\sqrt{|E(\hat{y})|}} - \frac{1}{2} \eta_{\nu\rho} m^2, \quad T_{\nu\rho}^1(\hat{x}, y) := \frac{\delta^\alpha_{\nu} \delta^\beta_{\rho} \eta_{\sigma\tau}(\hat{y})}{\hat{x}^+ \sqrt{|E(\hat{y})|}} p_\sigma^+ p_\tau^+, \]

such that the integrand scales in the frequency as \( \sim \omega^2 T^0 - i\omega T^1 \). Combining the presented term in (3.5) with its complex conjugate and trading explicit \( \omega \) factors for \( y^- \) - derivatives gives our final result for the waveform:

\[ W_{\mu\nu\sigma\rho}(u, \hat{x}) = -\frac{\kappa^2}{\pi} \hat{x}_+ \hat{x}_- \int_y \delta(u - \hat{\nu}(y)) \left[ D^2 T_{\nu\rho}^0(\hat{x}, y) - \mathcal{D} T_{\nu\rho}^1(\hat{x}, y) \right], \]

in which the derivative \( \mathcal{D} \) acts as

\[ \mathcal{D} f(y) := \frac{d}{dy} \left( \frac{f(y)}{\hat{\nu}(y)} \right). \]

Once again, this result can be confirmed by comparison with the calculation in classical general relativity; see Appendix A.

3.1 Properties of the waveform

Some insight into the gravitational waveform is provided by observing from (3.6) that \( \hat{\nu} \) is determined by the 0PB classical orbit \( X^\mu(y) \) of a particle crossing the planewave spacetime.
The orbit itself goes like the integral of the transverse metric $\gamma^{ij} = E^{(i|a}E^{j|)}_a$. Reinstating explicit dependence on the gravitational coupling by taking $H_{ab} \to \kappa H_{ab}$, it is clear that the integral of $\gamma^{ij}$ will contain terms which are at least linear in $\kappa$. Since (3.9) contains terms which go like $\bar{V}^{-1}$, as well as an integral localised in terms of $\bar{V}$, it follows that the waveform will contain terms of all orders in the background and hence in $\kappa$.

While the non-linearity of general relativity makes it harder to evaluate the waveform analytically for test plane wave profiles, progress can be made in the impulsive case where $\kappa H_{ab}(x^-) = \delta(x^-) \kappa \text{diag}(\lambda, -\lambda)$. This is demonstrated in Appendix B, along with the resulting waveform which is explicitly all-orders in $\kappa \lambda$.

The structure of (3.9) indicates the presence of tail effects in the gravitational waveform. This follows from the fact that the two terms in the waveform descend directly from those in the polarization tensor (3.2). The background dressing of this polarization is directly related to the failure of Huygens’ principle for gravitational perturbations in plane wave spacetimes: initial data localized on a lightcone spreads outside of the lightcone as it evolves [73, 75, 76]. These effects are present in both the $T^0$ and $T^1$ terms of the 1PB waveform, with the $T^1$ contribution being pure tail; by comparison, in the PM expansion of the two-body problem tail effects only emerge at fourth-order (e.g., [16]).

These tail effects are a consequence of the inherent non-linearity of gravity compared to electromagnetism, and this leads to another interesting feature of the gravitational waveform which is not present in QED. Consider the case, as in (2.6), where the direction of observation $\hat{x}^\mu$ aligns with the wave direction $n^\mu$, corresponding to azimuthal angle $\theta = 0$. The background metric is not asymptotically flat in this direction, so we approach it with caution. For any $\theta \neq 0$ the gravitational waveform is well-defined, but in the limit $\theta \to 0$, it is divergent. To see this, one uses the small-$\theta$ expansion of $\hat{x}_\mu$:

$$\hat{x}_j = \sin \theta \{\cos \phi, \sin \phi\} \sim \theta, \quad \hat{x}_+ = \frac{1 - \cos \theta}{\sqrt{2}} \sim \theta^2, \quad \hat{x}_- = \frac{1 + \cos \theta}{\sqrt{2}} \sim 1. \quad (3.11)$$

With this, it is simplest to pick components of $W$, and also to focus on the pure tail term which contains the deformation tensor $\sigma$. The contribution of this term to $W_{-a-b}$ is

$$W_{-a-b} = \frac{\kappa^2 p^2}{\pi \hat{x}_+} \int_y \delta(u - \bar{V}(y)) \frac{D}{\sqrt{|E(y)|}} + \cdots \sim \frac{1}{\theta^2}. \quad (3.12)$$

in which the $1/\hat{x}_+$ term generates the divergence (it is easily seen that $\bar{V}$ and $\partial_- \bar{V}$ remain finite in the limit $\theta \to 0$). The divergence reflects the fact that it is not possible to ‘scatter’ gravitons in the $n_\mu$ direction, in which the background is not asymptotically flat; the interaction between the emitted radiation and the background never switches off.

4 Conclusions

We have derived the gravitational waveform emitted by a massive particle when it scatters off a gravitational plane wave background, a solution to the fully non linear Einstein equation. Analogous formulae have been presented for the electromagnetic case. In con-
contrast to existing results, these waveforms are manifestly all-orders in the coupling, and exhibit a rich structure including tail effects that usually enters to higher order in the PM expansion. Our results underline the power of using strong field amplitudes to study classical physics [44]. In future work we aim to go to higher orders in the PB expansion, including higher points and loops. There is no conceptual obstacle in pushing higher order calculations, and we expect this to provide easier access to observables of interest in classical gravity. It would also be interesting to consider other physically interesting strong backgrounds, like black holes or beams of gravitational radiation.

Acknowledgments

We thank Tom Heinzl for interesting conversations. TA is supported by a Royal Society University Research Fellowship and by the Leverhulme Trust (RPG-2020-386). AC is supported by the Leverhulme Trust (RPG-2020-386). SK is supported by an EPSRC studentship.

A Classical checks

This appendix contains a classical derivation of the waveforms in electromagnetism and gravity. Schematically, these stem from radiation fields ‘$A$’ generated by sources ‘$J$’ representing particles moving on a background, which take the form

$$ A_\sigma(x) := \int d^4 y \, G_{\text{ret}}(x, y) \, J_\sigma(y), \quad (A.1) $$

in which the subscript $\sigma$ is a placeholder for any number of vector indices or spin labels.

The retarded Green’s function is the inverse of $\nabla^2$ in a flat or curved background, and is therefore theory-dependent:

$$ G_{\text{ret}}^{\text{EM}}(x, y) = \frac{i \Theta(x^- - y^-)}{2k_+} \Theta(k_+) \, e^{-ik_-(x-y)}, \quad (A.2) $$

$$ G_{\text{ret}}^{\text{GR}}(x, y) = \frac{i \Theta(x^- - y^-)}{\sqrt{|E(x)|}} \int \frac{d\bar{k}_+ \, d^2 \bar{k}_\perp}{2k_+} \Theta(\bar{k}_+) \, e^{-i\bar{F}(x,y)} \sqrt{|E(y)|}, \quad (A.3) $$

in which $\bar{k}_\mu$ is on-shell and

$$ \bar{F}(x, y) = \bar{k}_+(x-y)^+ + \bar{k}_i (E^i_\alpha(x)x^\alpha - E^i_\alpha(y)y^\alpha) $$

$$ + \frac{\bar{k}_+}{2} (\sigma_{ab}(x)x^ax^b - \sigma_{ab}(y)y^ay^b) + \frac{\bar{k}_i \bar{k}_j}{2k_+} \int_{x^-}^{y^-} ds \, \gamma^{ij}(s). \quad (A.4) $$

We measure the waveform at future null infinity, hence we write the coordinate $x^\mu$ in the coordinate system $(u, r, \hat{x})$ where $r = |x|$, $x = r \hat{x}$ and $u = t - r$; the asymptotic limit is reached by taking $r \to \infty$ at fixed $u$ and angular coordinates $\hat{x}$. As long as our measurement device is not in the beam of the wave (corresponding to $\hat{x}^3 = 1$), then we can set the initial
$\sqrt{|E(x)|} = 1$ in (A.3) to unity. The step function can also be set to unity in the limit. With this, the ‘Fourier transformed’ version of (A.1) is

$$A_\sigma(x) = i \int \frac{\hat{d}_\perp \hat{d}_\perp}{2k_+} e^{-i k_+ x} \tilde{J}_\sigma(\hat{k}) ,$$

(A.5)

where $\tilde{J}_\sigma(\hat{k})$ theory-dependent. In the $r \to \infty$ limit the leading behaviour of this quantity is, performing a saddle point calculation of the $k_\perp$ integrals as in the text,

$$A_\sigma(x) \sim \frac{1}{4 \pi r} \int_0^\infty d\omega e^{-i \omega u} \tilde{J}_\sigma(\omega \hat{x}_\mu) + \text{c.c.} ,$$

(A.6)

where $\hat{x}_\mu = (1, \hat{x})$ in Cartesian coordinates. We now turn to specifics in electromagnetism and gravity.

**Electromagnetism**

In electromagnetism $J$ is the vector current for a particle moving in a background field,

$$\tilde{J}_\mu(\hat{k}) = -e \int_y e^{i \hat{k} \cdot X} \frac{\partial X'_\mu(y)}{\partial y}$$

(A.7)

where $X_\mu(y)$ is the particle orbit, and dashes represent derivatives with respect to $y^-$. Note that this form of the current generates the radiation field; the Coulomb fields from outside the wave have been subtracted. Substituting into (A.6) we obtain the gauge potential of the radiation field:

$$A_\mu(u, \hat{x}) \sim \frac{ie}{4 \pi r} \int_0^\infty d\omega \int_y e^{-i \omega(u - \hat{x} \cdot X)} \frac{1}{\omega} \frac{\partial \hat{x}_\mu}{\partial y} \frac{X'_\mu(y)}{\hat{x} \cdot X'(y)} + \text{c.c.}$$

(A.8)

The radiated field strength is $F_{\mu\nu} = 2 \partial_{[\mu} A_{\nu]}$ which, up to subleading corrections in $1/r$, we can obtain by adding factors of $-i \omega \hat{x}$ to the integrand of (A.8). The factors of $\omega$ outside the exponential cancel in $F_{\mu\nu}$ allowing us to perform the $\omega$-integral to find

$$F_{\mu\nu}(u, \hat{x}) \sim \frac{1}{r} W_{\mu\nu}(u, \hat{x}) = \frac{e}{2 \pi r} \int_y \delta(u - \hat{x} \cdot X(y)) \frac{\partial \hat{x}_\mu}{\partial y} \frac{X'_\mu(y)}{\hat{x} \cdot X'(y)} ,$$

(A.9)

where ‘$\sim$’ denotes equality up to subleading terms in $r^{-1}$. This exactly the waveform derived in the text from the classical limit of the quantum result.

**Gravity**

The gravitational radiation of a massive scalar moving in a background is sourced by the stress-energy tensor

$$T_{\mu\nu}(y) = \frac{P_\mu(y) P_\nu(y)}{p_+} \delta^3_{+\perp}(y - X(y)) ,$$

(A.10)

in which $X_\mu(y)$ is once again the particle orbit and $P_\mu(y) = p_+ X'_\mu(y)$. For convenience we define $\bar{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_\alpha^\alpha$ as shorthand for a ‘trace-reversed’ $T_{\mu\nu}$. From the Einstein field
equations, one can derive that the sourced gravitational field satisfies, imposing lightfront gauge $n^\mu h_{\mu\nu} = 0$,

$$h_{\mu\nu} = -\frac{2}{\nabla^2} t_{\mu\nu} + \frac{2}{\nabla} \eta_{\mu\nu} h^{ab} H_{ab}.$$  \hfill (A.11)

in which the modified stress-energy tensor $t_{\mu\nu}$ is defined by

$$t_{\mu\nu} := \kappa^2 \left[ \tilde{T}_{\mu\nu} - 2 \frac{\nabla(\mu \tilde{T}_{\nu})}{\partial_+} + \nabla(\mu \nabla_{\nu}) \tilde{T}_{++} \right]. \hfill (A.12)$$

In the notation of (A.1), the source ‘$J$’ is now

$$\bar{t}_{\mu\nu}(k) = 2\kappa^2 \int_{x_{i}}^{x_{f}} dy \ e^{i\mathcal{V}(y)} \times \frac{\partial}{\partial y} \left[ \frac{1}{\sqrt{|E(y)|}} \left( \mathbb{P}_{\mu\alpha} \mathbb{P}_{\nu\beta} P^\alpha P^\beta - \frac{1}{2} \eta_{\mu\nu} m^2 - \frac{i p^{\alpha} \sigma_{\alpha\beta}}{k_{+}} \right) \right], \hfill (A.13)$$

where $\mathbb{P}_{\mu\nu}(k; y^-)$ are the projectors defined in (3.2) and

$$\mathcal{V}(y) = \int_{-\infty}^{y} dz \left( \frac{g^{\mu\nu}(z) K_{\mu}(z) P_{\nu}(z)}{p_{+}} \right). \hfill (A.14)$$

To arrive at this expression one uses integration by parts to shift the derivatives present in (A.12) onto the propagator. Additionally, we ignore the second term in (A.11) since we are only interested in radiative contributions. See [74] for details. Substituting into (A.6) we obtain an expression for the asymptotic metric perturbation

$$h_{\mu\nu}(u, \hat{x}) = \frac{-i\kappa^2}{2\pi r} \int_{0}^{\infty} d\omega \int_{y^-}^{y^+} dy \ e^{-i\omega(u-\mathcal{V}(y))} \frac{\partial}{\partial y} \left[ \left( T_{\mu\nu}^0 - \frac{i}{\omega} T_{\mu\nu}^1 \right) \frac{1}{\partial_+ \mathcal{V}(y)} \right] + \text{c.c.}, \hfill (A.15)$$

in which the same ‘effective’ energy-momentum tensors defined in (3.8) have appeared, along with with the reduced exponent

$$\tilde{\mathcal{V}}(y) = \frac{1}{\omega} \mathcal{V}(y) \bigg|_{\hat{k} = \omega \hat{x}}. \hfill (A.16)$$

From here we form the linearised curvature $R_{\mu\nu\sigma\rho} = 2\partial_{[\mu} \partial_{[\sigma} h_{\nu\rho]}$, with each derivative introducing a factor of $(-i)\omega \hat{x}$ into the integrand. We can then integrate in $\omega$ to obtain

$$R_{\mu\nu\sigma\rho}(u, \hat{x}) \sim -\frac{\kappa^2}{\pi r} \hat{x}_{[\sigma} \left[ \int_{y}^{y+} \delta(u - \mathcal{V}(y^-)) \left( D^2 T_{\mu\nu}^0(u, \hat{x}) - DT_{\mu\nu}^1(u, \hat{x}) \right) \right], \hfill (A.17)$$

where $D$ is defined in (3.10). This confirms the classical limit of our QFT calculations.

**B The impulsive case**

In this appendix we calculate the waveform for impulsive plane wave backgrounds.
**QED**

An impulsive plane has electric fields $E_\perp(x^-) = E_\perp \delta(x^-)$, for $E_\perp$ constant. We write $a_\mu = \delta_\mu^\perp E_\perp \Theta(x^-) \equiv E_\perp \Theta(x^-)$. An incoming particle with momentum $p_\mu$ for $x^- < 0$ is kicked by the wave to momentum $P_\mu$ at $x^- > 0$ where

$$P_\mu = p_\mu - eE_\mu + n_\mu \frac{2eE \cdot p - e^2 E \cdot E}{2n \cdot p}. \tag{B.1}$$

This is a memory effect, which we neglected in the text. However, the addition of memory does not impact the final expression for the electrodynamics waveform, which holds for any plane wave. The waveform is most easily evaluated using (2.4) by splitting the $dx^-$ integral into two parts: $x^- \gg 0$. The remainder of the calculation is trivial, and one finds

$$W_{\mu\nu}(u, \hat{x}) = \frac{e}{2\pi} \delta'(u) \left[ \frac{\hat{x}[\mu P_\nu]}{\hat{x} \cdot P} - \frac{\hat{x}[\mu p_\nu]}{\hat{x} \cdot p} \right]. \tag{B.2}$$

The waveform manifestly contains terms all orders in the coupling $e$. It is supported on the same singularity structure in $u$ as the driving electric field is in $x^-$. The tensor structure clearly derives directly from standard soft factors for momentum transfer $p \to P$. Neglecting memory in this case amounts to assuming $E_\mu$ is parametrically small, in which case one replaces $P \to p$ and the waveform vanishes. For an impulsive background, the waveform is thus ‘pure memory.’

**Gravity**

The situation is gravity is rather more intricate, even for an impulsive background, and the structures provide an interesting contrast with electrodynamics. The impulsive metric is given by taking, in (1.8), $\kappa H_{ab}(x^-) \to \kappa \delta(x^-) H_{ab}$ with $H_{ab}$ now constant (though still symmetric and traceless). We can, without loss of generality, choose coordinates to diagonalise $H_{ab} = \text{diag}(\lambda, -\lambda)$. In contrast to QED, the memory effects present in this metric cannot, in general, be directly treated with the expressions in the text. In order to present the full impulsive result, we compute the relevant amplitudes directly, using the wavefunctions in [73].

The momentum kick is given by replacing $eE_\mu \to \kappa p_\mu \delta_\mu^a H_{ac}b^c$ in (B.1), in which $b$ is the transverse impact parameter (a dependence not present in electrodynamics). From here there are, as in electrodynamics, two contributions: one from before the impulse ($x^- < 0$) and one from after the impulse ($x^- > 0$). That from $x^- > 0$ yields a term similar to (B.2), while that from $x^- < 0$ is more complicated. One finds

$$W_{\mu\nu\sigma\rho}(u, \hat{x}) = \frac{\kappa^2}{4\pi} \delta'(u) \left[ \frac{\hat{x}[\mu \hat{x}[\nu \rho \sigma]}{\hat{x} \cdot p} - \frac{\hat{x}[\mu p_\nu]}{\hat{x} \cdot p} \right] + \frac{i\kappa^2}{4\pi} \hat{x}[\mu \hat{x}[\nu [\rho \sigma]]] \int_0^\infty d\omega \omega^2 e^{-i\omega u} \int d^2\ell_\perp (\epsilon_\eta(\ell) \cdot p)^2 \frac{1}{\ell \cdot p} F(\ell - k) + \text{c.c.}, \tag{B.3}$$

in which $k_\mu \equiv \omega \hat{x}_\mu$, $\epsilon_\mu$ without argument is $\epsilon_\mu(\hat{x})$, $\ell_\mu$ is a null vector with fixed longitudinal
Consider the first line in (B.3); the tensor structure is a double copy of the soft factor structure in electrodynamics, but the singularity is now $\delta'(u)$ rather than $\delta(u)$. The second line of (B.3) is, in a sense, ‘pure tail’ and we have not yet found a very compact expression for the remaining integrals for general $\hat{x}^\mu$ and $u$. Nevertheless, the result as presented is clearly of all orders in $\kappa$.

Moreover, we can consider a special case which allows a direct, if tedious, calculation of all terms in the impulsive waveform. First, we choose $p^\perp = 0$, so that the wave-particle collision is ‘head on’. Second, we choose the impact parameter as $b^\perp = 0$; this has the effect of turning off memory, since the kicked momentum $P$ collapses back to incoming $p$. Finally, we choose a particular point of observation on the celestial sphere, $\theta = \pi$ (antipodal to the direction in which the background is not asymptotically flat), which sets $\hat{x}_\mu \to \sqrt{2} \delta^\perp_{\mu}$. In this case, the first line of (B.3) vanishes – this is clearly due to the assumption of no memory, as in QED. The second line remains and simplifies considerably. By performing the angular integration in $\ell^\perp$, one finds that the remaining integrals are independent of helicity and the helicity sum can then be performed directly.

From here one writes the $\omega$ factors as derivatives with respect to $u$ and combines the presented term in (B.3) with its complex conjugate. This gives, writing $q \equiv |\ell^\perp|$, \begin{equation}
W_{\mu\nu\sigma\rho} = \kappa^2 \bf{p}^3 \overline{\kappa \lambda} \delta^\perp_{[\sigma} (1) \delta^\perp_{\rho]} \delta^\perp_{\nu]} \delta^\perp_{\mu]} \partial^2 \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \hat{d}q \, \hat{d}q^3 \, \hat{d}q^2 \, \hat{d}q^2 \, \cos(\omega u), \tag{B.5}
\end{equation}
in which a sum over $a \in (1, 2)$ is implied and $J_1$ is a Bessel function of the first kind. The remaining integrals may be evaluated using [77], resulting in \begin{equation}
W_{\mu\nu\sigma\rho} = -\frac{\kappa^2 \bf{p}^3}{\pi^2 \sqrt{8}} \delta^\perp_{[\sigma} (1) \delta^\perp_{\rho]} \delta^\perp_{\nu]} \delta^\perp_{\mu]} \frac{\partial^2}{\partial u^2} \left( \frac{\nu \log(\nu + \sqrt{\nu^2 - 1})}{\sqrt{\nu^2 - 1}} \right), \tag{B.6}
\end{equation}
for $\nu := \kappa \lambda \sqrt{2} \bf{p}^3 / m^2 |u|$. Thus, unlike the ‘soft’ terms in the first line of (B.3), the ‘pure tail’ terms are not localised. Taking the derivatives in (B.6), one finds that they do include a localised piece at the origin, proportional to $\delta(u)$ like in electrodynamics, rather than $\delta'(u)$ as in the first line of (B.3). This is not unexpected if the waveform is supported entirely on $T^1$; recalling the discussion around (3.8), in Fourier space the contribution from $T^1$ carries the same frequency dependence as electrodynamics.
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