KNUTSON IDEALS AND DETERMINANTAL IDEALS OF HANKEL MATRICES

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Abstract. Motivated by a work of Knutson, in a recent paper Conca and Varbaro have defined a new class of ideals, namely “Knutson ideals”, starting from a polynomial \( f \) with squarefree leading term. We will show that the main properties that this class has in polynomial rings over fields of characteristic \( p \) are preserved when one introduces the definition of Knutson ideal also in polynomial rings over fields of characteristic zero. Then we will show that determinantal ideals of Hankel matrices are Knutson ideals for a suitable choice of the polynomial \( f \).

1. Introduction

Let \( \mathbb{K} \) be a field of any characteristic. Fix \( f \in S = \mathbb{K}[x_1, \ldots, x_n] \) a polynomial such that its leading term in \( \prec(f) \) is a squarefree monomial for some term order \( \prec \). We can define a new family of ideals starting from the principal ideal \( (f) \) and taking associated primes, intersections and sums. Geometrically this means that we start from the hypersurface defined by \( f \) and we construct a family of new subvarieties \( \{Y_i\}_i \) by taking irreducible components, intersections and unions.

In [CV], Conca and Varbaro called this class of ideals Knutson ideals, since they were first studied by Knutson in [Kn].

Definition 1 (Knutson ideals). Let \( f \in S = \mathbb{K}[x_1, \ldots, x_n] \) be a polynomial such that its leading term in \( \prec(f) \) is a squarefree monomial for some term order \( \prec \). Define \( \mathcal{C}_f \) to be the smallest set of ideals satisfying the following conditions:

1. \( (f) \in \mathcal{C}_f \);
2. If \( I \in \mathcal{C}_f \) then \( I : J \in \mathcal{C}_f \) for every ideal \( J \subseteq S \);
3. If \( I \) and \( J \) are in \( \mathcal{C}_f \) then also \( I + J \) and \( I \cap J \) must be in \( \mathcal{C}_f \).

If \( I \) is an ideal in \( \mathcal{C}_f \), we say that \( I \) is a Knutson ideal associated to \( f \). More generally, we say that \( I \) is a Knutson ideal if \( I \in \mathcal{C}_f \) for some \( f \).

In [Kn] Knutson proved that if \( \mathbb{K} = \mathbb{Z}/p\mathbb{Z} \), this class of ideals has some interesting properties.

Theorem 1.1. Let \( S = \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n] \) and let \( f \) be a polynomial in \( S \) such that \( \mathbf{in}_\prec(f) = \prod_i x_i \) with respect to some term order.

(i) [Kn] Theorem 4] If \( I \in \mathcal{C}_f \), then \( \mathbf{in}(I) \) is squarefree. In particular, \( I \) is radical.
(ii) [Kn] Corollary 2 If \( I, J \in \mathcal{C}_f \) then \( \mathcal{G}_{f+J} = \mathcal{G}_I \cup \mathcal{G}_J \) where \( \mathcal{G}_I \) (respectively \( \mathcal{G}_J \)) is a Gröbner basis of \( I \) (respectively \( J \)).
Remark 1. Note that if $C$ is a family of ideals closed under intersections and such that $\text{in}_<(I)$ is squarefree for every $I \in C$, then $C$ is a finite set. In fact, it is easy to check that

$$\text{in}_<(I \cap J) \subseteq \text{in}_<(I) \cap \text{in}_<(J) \subseteq \sqrt{\text{in}_<(I \cap J)}.$$ 

Since $C$ is closed under intersections, $I \cap J \in C$ and therefore $\text{in}_<(I \cap J) = \sqrt{\text{in}_<(I \cap J)}$. So, from the previous chain of subsets, we get

$$\text{in}_<(I \cap J) = \text{in}_<(I) \cap \text{in}_<(J).$$

More generally, this holds for every finite intersection:

$$\text{in}_<(\bigcap_i I_i) = \bigcap_i \text{in}_<(I_i).$$

We claim that if $I, J \in C$ and $I \neq J$, then $\text{in}_<(I) \neq \text{in}_<(J)$ and since $\text{in}_<(I)$ is squarefree for every $I \in C$, these initial ideals are a finite number. Hence $C$ is finite. To prove the claim, assume that $\text{in}_<(I) = \text{in}_<(J)$. Then

$$\text{in}_<(I \cap J) = \text{in}_<(I) \cap \text{in}_<(J) = \text{in}_<(I) = \text{in}_<(J).$$

Considering that $I \cap J \subseteq I, J$, we get $I = I \cap J = J$. This completes the proof of the claim.

From Theorem 1.1.(i) and Remark 1, one can infer that $C_f$ is finite.

Remark 2. Actually, assuming that every ideal of $C_f$ is radical, the second condition in Definition 1 can be replaced by the following:

2’. If $I \in C_f$ then $\mathcal{P} \in C_f$ for every $\mathcal{P} \in \text{Min}(I)$.

In fact, let $I \in C_f$, then 

$$I = \sqrt{I} = P_1 \cap P_2 \cap \ldots \cap P_r$$

where $P_i$ are the minimal primes of $I$. Fix $c \in (P_2 \cap \ldots \cap P_r) \setminus P_1$. Clearly $P_1 \subseteq (I : c) \subseteq P_1$, hence $P_1 = (I : c)$. The same holds for every $P_i$. Viceversa, it is easy to observe that if $I$ is radical, then the minimal primes of $I : J$ are exactly the minimal primes of $I$ that do not contain $J$.

In this paper we begin the study of this class of ideals whose properties allow us to prove interesting results on radicality and $F$-purity of certain ideals.

In Section 2, we introduce the definition of Knutson ideals in polynomial rings over any field and we show that the properties listed in the previous discussion stay unchanged. In particular, we start by proving the following:

Main Theorem 1. Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over any field and let $f$ be a polynomial in $S$ such that $\text{in}_<(f) = \prod_i x_i$ with respect to some term order. If $I \in C_f$, then $\text{in}_<(I)$ is squarefree. In particular, $I$ is radical.

To do so, we first generalize Knutson’s result [Kn, Theorem 4] to fields of positive characteristic (see Proposition 2.1) and then we use the achieved result together with reduction modulo $p$ to prove that the same holds for polynomial rings over fields of characteristic 0 (see Proposition 2.4). Once we have proved these results, the finitness of
the family $C_f$ can be inferred again from Remark 1, while the last property about Gröbner bases can be deduced by Remark 1 using the fact that

$$\text{in}_\prec (I \cap J) = \text{in}_\prec (I) \cap \text{in}_\prec (J) \iff \text{in}_\prec (I + J) = \text{in}_\prec (I) + \text{in}_\prec (J).$$

In the case of homogeneous ideals, the latter equivalence comes from the usual short exact sequence

$$0 \rightarrow S/(I \cap J) \rightarrow S/I \oplus S/J \rightarrow S/(I + J) \rightarrow 0$$

using the fact that the Hilbert function does not change when passing to the initial ideal. If $I$ and $J$ are not homogeneous, the equivalence is still true but the proof requires more work.

In Section 3, we discuss the case of determinantal ideals of generic Hankel matrices and we prove that they are Knutson ideals for a suitable choice of $f$ (see Theorem 3.1 and Theorem 3.2):

**Main Theorem 2.** Let $H$ be a generic Hankel matrix of size $r \times s$. Then $I_t(H)$ is a Knutson ideal for every $t \leq \min(r, s).

In particular, this implies that the determinantal ring of a generic Hankel matrix is $F$-pure (see Corollary 3.3), a result recently proved by different methods in [CMSV].

Furthermore, we characterize all the ideals belonging to the family for this choice of $f$ (see Theorem 3.4).

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2. Knutson ideals in any characteristic

The aim of this section is to try to generalize Knutson’s results first to fields of characteristic $p > 0$ (not necessarily finite) and then to fields of characteristic 0.

2.1. Fields of characteristic $p > 0$. Let $\mathbb{K}$ be a field of characteristic $p > 0$ and let $f \in S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial such that $\text{in}_\prec (f)$ is squarefree for some term order $\prec$. As in the case of $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$, we can construct the family $C_f$ as the smallest set of ideals such that:

- $(f) \in C_f$
- $I \in C_f, J \subseteq S \Rightarrow I : J \in C_f$
- $I, J \in C_f \Rightarrow I + J, I \cap J \in C_f$

We want to prove the following result.

**Proposition 2.1.** Let $\mathbb{K}$ be a field of characteristic $p > 0$ and let $f$ be a polynomial in $S = \mathbb{K}[x_1, \ldots, x_n]$ such that $\text{in}_\prec (f)$ is squarefree for some term order $\prec$. If $I \in C_f$ then $\text{in}_\prec (I)$ is squarefree.

A first step toward this result is given by the following observation.
Remark 3. If \( K \) is a perfect field of characteristic \( p > 0 \) then it is easy to generalize Theorem 1.1(i). In fact [Kn Theorem 4] is a consequence of [Kn Theorem 2] and the proof of this latter theorem relies on two lemmas, namely [Kn Lemma 2] and [Kn Lemma 5]. We observe that Knutson gives a proof of [Kn Lemma 2] in the case \( K = \mathbb{F}_p \) which easily extends to perfect fields of characteristic \( p \); the proof is actually the same, one has just to keep in mind that every element \( c \) of a perfect field \( K \) of characteristic \( p \) has a \( p \)th root in \( K \) (in the case \( K = \mathbb{F}_p \), \( c^p = c \)). Furthermore, since [Kn, Lemma 5] holds for perfect fields of characteristic \( p > 0 \), we get that [Kn, Theorem 2], and thus [Kn, Theorem 4], extends to polynomial rings over perfect fields of positive characteristic.

To prove Proposition 2.1, we reduce to the case of perfect fields of positive characteristic so that we can apply Remark 3.

In the proof we will need these well known facts.

Proposition 2.2. (see e.g. [Ma, p.46]) The extension of polynomial rings
\[ S = K[x_1, \ldots, x_n] \hookrightarrow \overline{S} = \overline{K}[x_1, \ldots, x_n] \]
is a flat extension.

Proposition 2.3. (see e.g. [Ma, Theorem 7.4]) Let \( \pi: A \longrightarrow B \) be a flat ring extension and \( I \) and \( J \) two ideals of \( A \). Then:

(i) \( (I \cap J)B = IB \cap JB \)
(ii) If \( J \) is finitely generated then \( (I : J)B = IB : JB \).

Proof of Proposition 2.1 Let \( \overline{K} \hookrightarrow \overline{K} \) be the extension of \( K \) to its algebraic closure \( \overline{K} \). Since \( \text{char}(K) = p \), then \( \overline{K} \) is a perfect field of characteristic \( p \).

Let \( \overline{S} = \overline{K}[x_1, \ldots, x_n] \) and consider the natural extension:
\[ \iota: S \longrightarrow \overline{S}. \]

So \( \overline{f} := \iota(f) \) is a polynomial in \( \overline{S} \) (we regard \( f \) as a polynomial with coefficients in \( \overline{K} \)). Again we can construct the family \( \overline{C}_f := C_f \) in \( \overline{S} \).

First of all, we claim that if \( I \in C_f \) then \( I\overline{S} \in \overline{C}_f \). Indeed, by Proposition 2.2 \( \iota: S \longrightarrow \overline{S} \) is a flat extension and we can then use Proposition 2.3 to get the following equalities:

- \( (I + J)\overline{S} = I\overline{S} + J\overline{S} \) (always true)
- \( (I \cap J)\overline{S} = I\overline{S} \cap J\overline{S} \) (true for flat extensions)
- \( (I : J)\overline{S} = I\overline{S} : J\overline{S} \) (true for flat extensions and \( J \) finitely generated).

Consider \( (f) \in C_f \), then \( (f)\overline{S} = (\overline{f}) \subseteq \overline{S} \) and \( \overline{f} \in \overline{C}_f \) by definition.

Now let \( I, J \in C_f \) such that \( I\overline{S}, J\overline{S} \in \overline{C}_f \). By definition \( I + J, I \cap J \in C_f \).

Using previous identities, we get
\[
(I + J)\overline{S} = \bigcup_{\overline{f} \in \overline{C}_f} I\overline{S} + \bigcup_{\overline{f} \in \overline{C}_f} J\overline{S} \in \overline{C}_f
\]
\[
(I \cap J)\overline{S} = \bigcap_{\overline{f} \in \overline{C}_f} I\overline{S} \cap \bigcap_{\overline{f} \in \overline{C}_f} J\overline{S} \in \overline{C}_f.
\]
Eventually, let’s consider $I \in C_f$ and $J \subseteq S$ an arbitrary ideal. By definition $I : J \in C_f$. Suppose $I\mathcal{S} \in \mathcal{C}_f$. Since $J$ is finitely generated, then

$$(I : J)\mathcal{S} = \bigcup_{\mathcal{S} \in \mathcal{C}_f} I\mathcal{S} : J\mathcal{S} \subseteq \mathcal{C}_f.$$

This proves the claim. Using this result, we can easily conclude the proof. In fact, let $I$ be an ideal in $C_f$. Then $I\mathcal{S} \in C_f$ and by Remark 3 in $\mathcal{S}$ is squarefree because we are working in a polynomial ring over a perfect field of characteristic $p > 0$. But since Buchberger’s algorithm is “stable” under base extensions, we have

$$\text{in}_\prec (I\mathcal{S}) = \text{in}_\prec (I)\mathcal{S}.$$ 

So $\text{in}_\prec (I)$ is squarefree. □

2.2. Fields of characteristic 0. Let $K$ be a field of characteristic 0 and let $f \in S = K[x_1, \ldots, x_n]$ be a polynomial such that $\text{in}_\prec (f)$ is squarefree for some term order $\prec$. As in the previous cases, we can construct the family $C_f$ as the smallest set of ideals such that:

- $(f) \in C_f$
- $I \in C_f$, $J \subseteq S \Rightarrow I : J \in C_f$
- $I, J \in C_f \Rightarrow I + J, I \cap J \in C_f$.

We want to prove the analogous of Proposition 2.1 in the case of polynomial rings over fields of characteristic 0.

Proposition 2.4. Let $K$ be a field of characteristic 0 and let $f$ be a polynomial in $S = K[x_1, \ldots, x_n]$ such that $\text{in}_\prec (f)$ is squarefree for some term order $\prec$. If $I \in C_f$ then $\text{in}_\prec (I)$ is squarefree.

We know that this holds in polynomial rings over fields of characteristic $p > 0$. Using Proposition 2.1 we will show that the same holds if we are working over fields of characteristic 0.

2.2.1. Reduction modulo $p$ and initial ideals. Let $K$ be a field of characteristic 0 and define $S := K[x_1, \ldots, x_n]$. Consider an ideal $I \subseteq S$. Since $I$ is finitely generated, it is always possible to construct a finitely generated $\mathbb{Z}$-algebra $A \subset K$ such that if $I' := I \cap A[x_1, \ldots, x_n]$ then $I'S = I$. To do so it suffices to take $A = \mathbb{Z}[\alpha_1, \ldots, \alpha_s]$ where the $\alpha_i$ are the coefficients of the generators of $I$ which are not integers.

Example 1. If $I = (\sqrt{2}x - \pi y) \subset \mathbb{R}[x, y]$, we can take $A = \mathbb{Z}[\sqrt{2}, \pi]$.

Let $p$ be a prime number which is not invertible in $A$ and fix $P \in \text{Min}(pA)$. The quotient ring $A/P$ is an integral domain of characteristic $p > 0$ and we can define $I'_p$ to be the image of $I'$ under the projection map

$$\pi: A[x_1, \ldots, x_n] \longrightarrow A/P[x_1, \ldots, x_n].$$

Since $A/P$ is a domain we can construct its fraction field $F \left( A/P \right)$ and we define $S_p := F \left( A/P \right)[x_1, \ldots, x_n]$. So we can consider the extended ideal $I(p) := I'_pS_p$ in the polynomial ring $S_p$. 

This is what we call a reduction modulo \( p \in \mathbb{N} \). Although the notation might be confusing, the ideal \( I(p) \) does not depend only on \( p \) and \( I \) but also on the choice of \( P \in \text{Min}(pA) \).

To summarize, we have constructed the following diagram:

\[ A[x_1, \ldots, x_n] \xrightarrow{\pi} S = \mathbb{K}[x_1, \ldots, x_n] \quad \text{char} = 0 \]

\[ (A/P)[x_1, \ldots, x_n] \xrightarrow{\pi} F(A/P)[x_1, \ldots, x_n] \quad \text{char} = p > 0 \]

Note that the lower map in the diagram is flat.

The next lemma states that taking initial ideals commutes with reduction modulo \( p \) for all sufficiently large \( p \).

**Lemma 2.5.** Let \( \mathbb{K} \) be a field of characteristic 0 and \( S = \mathbb{K}[x_1, \ldots, x_n] \) the polynomial ring over \( \mathbb{K} \) with a fixed term order \( \prec \). Take \( I_1, \ldots, I_m \) ideals in \( S \). Then for all \( p \gg 0 \) there exists a reduction modulo \( p \) such that

\[ \text{in}(I_j(p)) = \text{in}(I_j)(p) \quad \forall j = 1, \ldots, m. \]

**Proof.** It suffices to prove the result for \( m = 1 \). If \( m > 1 \) we can always choose \( p \) greater than the maximum of the \( p_j \) such that the result is true for \( I_j \) and we are done.

Consider an ideal \( I = (f_1, \ldots, f_r) \subseteq S \) and construct the finitely generated \( \mathbb{Z} \)-algebra \( A = \mathbb{Z}[\alpha_1, \ldots, \alpha_t] \subset \mathbb{K} \) where the \( \alpha_i \) are the coefficients of the generators of \( I \) which are not integers. Note that since \( A \) is a finitely generated \( \mathbb{Z} \)-algebra, all the polynomial rings we are dealing with are Noetherian.

Accordingly with the previous notation, we set

\[ I' := I \cap A[x_1, \ldots, x_n] \]

\[ I'' := I'F(A)[x_1, \ldots, x_n] = I \cap F(A)[x_1, \ldots, x_n]. \]

Using Buchberger’s algorithm we can compute a Gröbner basis for \( I'' \). Let \( G_{I''} = \{g_1, \ldots, g_s\} \) be this Gröbner basis. Since Buchberger’s algorithm is “stable” under base extensions we get \( G_{I''} = G_I \), that is \( G_{I''} \) is also a Gröbner basis for \( I \) in \( S \).

Observe that, possibly multiplying by an element of \( A \), we can assume that \( g_1, \ldots, g_s \) are polynomials in \( A[x_1, \ldots, x_n] \).

In the computation of the Gröbner basis, no new coefficients appear but we need to invert some elements \( \lambda_1, \ldots, \lambda_t \) to \( A \) to compute S-polynomials. If we find a prime number \( p \) and a minimal prime \( P \in \text{Min}(pA) \) such that \( \lambda_1, \ldots, \lambda_t \notin P \), then \( \lambda_1, \ldots, \lambda_t \) are invertible in \( F(A/P) \), so the algorithm is exactly the same also when we reduce modulo \( p \). This will imply that \( \overline{G_I} = \{\overline{g_1}, \ldots, \overline{g_s}\} \) is a Gröbner basis for \( I(p) \), hence

\[ \text{in}(I(p)) = (\text{in}(\overline{g_1}), \ldots, \text{in}(\overline{g_s})). \]

Since we are working in Noetherian domains, the principal ideal \( (pA) \) has finitely many minimal primes and by Krulls Hauptidealsatz if \( P \in \text{Min}(pA) \) then \( \text{ht}(P) = 1 \). Moreover
it’s easy to see that if p and q are two different prime numbers, then \( \min(pA) \cap \min(qA) = \emptyset \). Assume that there exists a prime ideal \( Q \in \min(pA) \cap \min(qA) \), then \( Q \supseteq (pA), (qA) \). In particular, \( p, q \in Q \) and they are coprime. This would imply that \( 1 \in Q \), a contradiction. Similarly the ideal \((\lambda_i A)\) has finitely many minimal primes of height 1, therefore there exists a prime number \( \tilde{p} \) such that 

\[
\forall p > \frac{p}{\tilde{p}}, \forall P \in \min(pA) \quad \lambda_i \notin P.
\]

Taking \( \tilde{p} := \max \{ p_i \} \), we get that 

\[
\forall p > \frac{p}{\tilde{p}}, \forall P \in \min(pA) \quad \lambda_1, \ldots, \lambda_t \notin P.
\]

This proves that \( \in_\prec(I(p)) = (\in_\prec(\overline{g}_1), \ldots, \in_\prec(\overline{g}_s)) \) for \( p > \frac{p}{\tilde{p}} \).

Using a similar argument we can prove that there exists a prime number \( \tilde{p} > 0 \) such that 

\[
\in_\prec(I)(p) = (\in_\prec(g_1), \ldots, \in_\prec(g_s)) = (\in_\prec(\overline{g}_1), \ldots, \in_\prec(\overline{g}_s)) \quad \forall p > \tilde{p}.
\]

So we can conclude that \( \in_\prec(I)(p) = \in_\prec(I(p)) \) for \( p \gg 0 \). \( \square \)

2.2.2. Knutson ideals in characteristic 0. We want to prove Proposition \[2.4\] in characteristic 0. To do so, we reduce to the case of fields of positive characteristic using previous results.

As in the case of fields of positive characteristic, we first need to show that if \( I \in C_f \) then \( I(p) \in C_f(p) := C_{f(p)} \) for all prime numbers \( \tilde{p} \) large enough.

The following result simplifies our proof, allowing us to prove the result for a single ideal at time using \[2.5\]

**Lemma 2.6.** TFAE:

1. \( \exists \tilde{p} \gg 0 \) s.t. \( I \in C_f \Rightarrow I(p) \in C_f(p) \) \( \forall p \geq \tilde{p} \).
2. \( \forall I \in C_f \exists \tilde{p} \gg 0 \) s.t. \( I(p) \in C_f(p) \) \( \forall p \geq \tilde{p} \).

**Proof.** 1. \( \Rightarrow \) 2. Obvious.

2. \( \Rightarrow \) 1. If 2 holds, then \( \in_\prec(I(p)) \) is squarefree \( \forall I \in C_f \) and for \( p \geq \tilde{p} \). But we know from Lemma \[2.5\] that \( \in_\prec(I(p)) = \in_\prec(I)(p) \) for \( p \) large enough, so \( \in_\prec(I) \) is squarefree for every \( I \in C_f \). By Remark \[1\] we get that \( C_f \) is finite. Once we know that \( C_f \) is finite, we can take \( p = \max p_I \) and we are done. \( \square \)

**Proof of Proposition \[2.4\]** We begin by proving that if \( I \in C_f \) then there exists a prime number \( \tilde{p} \) such that \( I(p) \in C_f(p) = C_{f(p)} \) for all \( p \geq \tilde{p} \). By Lemma \[2.6\] this is equivalent to prove that there exists a prime number \( p \) which does not depend on the choice of the ideal, such that if \( I \in C_f \) then \( I(p) \in C_f(p) = C_{f(p)} \) for all \( p \geq \tilde{p} \).

Consider \( (f) \in C_f \), then \( (f) = (f(p)) \subseteq S_p \) and as we already explained \( (f(p)) \in C_f(p) = C_{f(p)} \) for all \( p \gg 0 \).

Now let \( I, J \in C_f \) such that \( I(p), J(p) \in C_f(p) \). By definition of \( C_f \), \( I + J, I \cap J \in C_f \) and we need to prove that \( (I + J)(p), (I \cap J)(p) \in C_f(p) \).

Obviously 

\[
(I + J)(p) = \underbrace{I(p)}_{\in C_f(p)} + \underbrace{J(p)}_{\in C_f(p)} \in C_f(p).
\]
Now consider the intersection ideal $I \cap J \in S$. It is clear that $(I \cap J)(p) \subseteq I(p) \cap J(p)$. If we show that they have the same initial ideal, we get

$$(I \cap J)(p) = \bigcup_{I(p) \cap J(p) \in C_f(p)} (I(p) \cap J(p)) \in C_f(p).$$

Using elimination theory and Buchberger’s algorithm, we can compute a Gröbner basis of $I \cap J$. In fact it is a well known fact (see e.g. [CLO, Theorem 11, p.187]) that

$$I \cap J = (tI + (1-t)J) \cap S$$

where $t$ is a new variable and $(tI + (1-t)J)$ is an ideal in $S[t]$ that we are contracting back to $S$.

In other words, a Gröbner basis of $I \cap J$ is obtained from a Gröbner basis of $tI + (1-t)J$ by dropping the elements of the basis that contain the variable $t$ (the so called first elimination ideal with respect to a suitable term order).

Therefore

$$(I \cap J)(p) = ((tI + (1-t)J) \cap S)(p)$$

$$I(p) \cap J(p) = (tI(p) + (1-t)J(p)) \cap S_p.$$  

By Lemma 2.5 $\text{in}_\prec (tI + (1-t)J)(p) = \text{in}_\prec (tI(p) + (1-t)J(p))$ for all $p \gg 0$ and we can conclude that

$$\text{in}_\prec (I \cap J)(p) = \text{in}_\prec (I(p) \cap J(p)).$$

A similar argument works for $I : J$ with $I \in C_f(p)$ and $J = (f_1, \ldots, f_l) \subset S$. In fact it is known (see e.g. [CLO, Theorem 11, p.196]) that

$$I : J = \left( \frac{1}{f_1} (I \cap (f_1)) \right) \cap \left( \frac{1}{f_2} (I \cap (f_2)) \right) \cap \ldots \cap \left( \frac{1}{f_l} (I \cap (f_l)) \right).$$

Thus, we can use again elimination theory to compute these intersections and arguing as we have done before, we get that

$$(I : J)(p) = \bigcup_{I(p) \cap J(p) \in C_f(p)} (I(p) : J(p)) \in C_f(p).$$

In conclusion, we have proved that if $I \in C_f$ then $I(p) \in C_f(p) = C_{f(p)}$ for all $p$ large enough.

Now let $I \in C_f$. Then $I(p) \in C_f(p)$ for $p \gg 0$ and by Proposition 2.1 $\text{in}_\prec (I(p))$ is square-free because we are working in a polynomial ring over a field of positive characteristic. But we know from Lemma 2.5 that

$$\text{in}_\prec (I(p)) = \text{in}_\prec (I)(p) \quad \forall p \gg 0.$$  

So $\text{in}_\prec (I)$ is squarefree. \hfill \Box
3. Determinantal ideals of Hankel matrices

Denote by $X_m^{(l,n)}$ the generic Hankel matrix with $m$ rows and entries $x_l, \ldots, x_n$, that is

$$X_m^{(l,n)} = \begin{bmatrix} x_l & x_{l+1} & x_{l+2} & \cdots & x_{n-m+1} \\ x_{l+1} & x_{l+2} & x_{l+3} & \cdots & x_{n-m+2} \\ x_{l+2} & x_{l+3} & x_{l+4} & \cdots & x_{n-m+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{l+m-1} & x_{l+m} & x_{l+m-1} & \cdots & x_n \end{bmatrix}.$$  

Note that once we have fixed $m$, $l$ and $n$, the number of columns of $X_m^{(l,n)}$ is $n - m - l + 2$.

In particular we are interested in square Hankel matrices of size $m$ and rectangular Hankel matrices of size $m \times (m + 1)$. In these cases, if we fix $m$ then $n$ is uniquely determined.

Assume for simplicity that $l = 1$:

$$X_m^{(1,n)} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_m \\ x_2 & x_3 & x_4 & \cdots & x_{m+1} \\ x_3 & x_4 & x_5 & \cdots & x_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_m & x_{m+1} & x_{m+2} & \cdots & x_n \end{bmatrix};$$

square Hankel matrix: $n = 2m - 1$

$$X_m^{(1,n)} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_m & x_{m+1} \\ x_2 & x_3 & x_4 & \cdots & x_{m+1} & x_{m+2} \\ x_3 & x_4 & x_5 & \cdots & x_{m+2} & x_{m+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m+1} & x_{m+2} & x_{m+3} & \cdots & x_{n-1} & x_n \end{bmatrix};$$

Hankel matrix of size $m \times (m + 1)$: $n = 2m$.

Let $X = X_m^{(1,n)}$ be a Hankel matrix and let $t \leq \min(m, n - m + 1)$, we denote by $I_t(X)$ the determinantal ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by all the $t$-minors of $X$.

Remark 4. For $t \leq m \leq n + 1 - t$ it is known (cf. [Co, Corollary 2.2]) that

$$I_t(X_m^{(1,n)}) = I_t(X_1^{(1,n)}).$$

That is $I_t(X_m^{(1,n)})$ does not depend on $m$ but only on $t$ and $n$.

We now prove that determinantal ideals of a generic square Hankel matrix are Knutson ideals for a suitable choice of $f$.

Theorem 3.1. Let $X = X_m^{(1,n)}$ be the square Hankel matrix of size $m$ with entries $x_1, \ldots, x_n$, where $n = 2m - 1$ and let $f$ be the polynomial $f = \det X \cdot \det X_m^{(2,n-1)}$ in $S = \mathbb{K}[x_1, \ldots, x_n]$. Then $I_t(X) \in C_f$ for $t = 1, \ldots, m$.

Proof. Fix a diagonal term order $\prec$ on $S$ (that is a monomial term order such that the initial term of each minor is given by the product of its diagonal terms). Then
\[ \text{in}_< (f) = \text{in}_< (\det X) \cdot \text{in}_< (\det X_{m-1}^{(2,n-1)}) = \]
\[ = (x_1 \cdot x_3 \cdot \ldots \cdot x_n) \cdot (x_2 \cdot x_4 \cdot \ldots \cdot x_{n-1}) = \prod_{i=1}^{n} x_i. \]

Hence \( \text{in}_< (f) \) is squarefree and we can construct the Knutson family of ideals associated to \( f \).

For simplicity of notation, we define

\[ P_1 := X_{m-1}^{(1,n-1)} : \text{rectangular matrix obtained by dropping the last row of } X \]
\[ P_2 := X_m^{(2,n)} : \text{rectangular matrix obtained by dropping the first column of } X \]
\[ Q := X_{m-1}^{(2,n-1)} : \text{square matrix obtained by dropping the last row and the first column of } X. \]

By Definition \( \Pi \) \((f) \in C_f \) and \((f) : J \in C_f \) for every ideal \( J \subseteq S \). Choosing \( J = (\det X) \) and \( J = (\det Q) \), we get

\[ (f) : (\det X) = (\det Q) \in C_f \]
\[ (f) : (\det Q) = (\det X) \in C_f. \]

In particular, \( I_m(X) = (\det X) \in C_f \). This proves the theorem in the case \( t = m \).

Now let \( t = m - 1 \). It is known (e.g. see \([\text{Co}]\) and \([\text{BC}]\)) that every determinantal ideal of a generic Hankel matrix \( H \) is prime and its height is given by the following formula:

\[ \text{ht}(I_s(H)) = n - 2s + 2 \]

where \( n \) is the number of variables. In this case

\[ \text{ht}(I_s(X)) = 2m - 1 - 2(m - 1) + 2 = 3. \]

From equalities \( \Pi \), taking the sum, we get

\[ I_m(X) + I_{m-1}(Q) = (\det X, \det Q) \in C_f. \]

Moreover

\[ \text{in}_< (I_m(X) + I_{m-1}(Q)) = (x_1 x_3 \cdots x_n, x_2 x_4 \cdots x_{n-1}) \]

is a complete intersection of height 2, so \( I_m(X) + I_{m-1}(Q) \) is a complete intersection of height 2 as well.

Now observe that

\[ \text{ht}(I_t(P_1)) = (n - 1) - 2t + 2 = 2(m - 1) - 2(m - 1) + 2 = \text{ht}(I_t(P_2)) \]

and

\[ I_t(P_1), I_t(P_2) \supseteq (\det X, \det Q) = I_{t+1}(X) + I_t(Q) \in C_f. \]

This means that \( I_t(P_1) \) and \( I_t(P_2) \) must be minimal primes over the ideal \( (\det X, \det Q) \in \in C_f \). Thus, they and their sum must be in \( C_f \) by definition.

Hence \( I_t(X) \) is a prime ideal of height 3 and it contains the sum of two distinct prime ideals of height 2, namely \( I_t(P_1) + I_t(P_2) \). This shows that \( I_t(X) \in C_f \), since it is a minimal
prime ideals in \( C \) two prime ideals of height one more, that is \( C \) be in \( I \) and \( I \) \( I \) This proves that prime ideals of height \( t \). Moreover

By \[2\], we know that

\[ \text{ht}(I_{t-1}(Q)) = n - 2 - 2(t - 1) + 2 = n - 2t + 2. \]

and

\[ \text{ht}(I_t(P_1)) = \text{ht}(I_t(P_2)) = n - 1 - 2t + 2 = n - 2t + 1. \]

Moreover

\[ I_{t-1}(Q) \supseteq I_t(P_1) + I_t(P_2) \]

and \( I_t(P_1) + I_t(P_2) \in C_f \) by induction. So \( I_{t-1}(Q) \) must be minimal over \( I_t(P_1) + I_t(P_2) \). This proves that \( I_{t-1}(Q) \in C_f \).

As a consequence we get that \( I_t(X) + I_{t-1}(Q) \in C_f \). This ideal is the sum of two distinct prime ideals of height \( n - 2t + 2 \) and it is contained in \( I_{t-1}(P_1) \) and \( I_{t-1}(P_2) \) which are two prime ideals of height one more, that is \( n - 2t + 3 \). Hence we have that \( I_{t-1}(P_1) \) and \( I_{t-1}(P_2) \) are minimal primes over the sum \( I_t(X) + I_{t-1}(Q) \) which is in \( C_f \) and so they must be in \( C_f \). It remains to show that \( I_{t-1}(X) \in C_f \). To do so, one can observe that

\[ I_{t-1}(X) \supseteq I_{t-1}(P_1) + I_{t-1}(P_2) \in C_f. \]

Hence \( I_{t-1}(X) \) is a prime ideal of height \( n - 2t + 4 \) that contains the sum of two distinct prime ideals in \( C_f \) of height \( n - 2t + 3 \). Thus \( I_{t-1}(X) \) must be a minimal prime over \( I_{t-1}(P_1) + I_{t-1}(P_2) \in C_f \). By definition, we get \( I_{t-1}(X) \in C_f \). This completes the proof.

A similar result holds for Hankel matrices of size \( m \times (m + 1) \).

**Theorem 3.2.** Let \( X = X_m^{(1,n)} \) be the rectangular Hankel matrix of size \( m \times (m + 1) \) with entries \( x_1, \ldots, x_n \), where \( n = 2m \) and let \( f \) be the polynomial \( f = \det X_m^{(1,n-1)} \cdot \det X_m^{(2,n)} \) in \( S = \mathbb{K}[x_1, \ldots, x_n] \). Then \( I_t(X) \in C_f \) for \( t = 1, \ldots, m \).

**Proof.** In this case we define

\[
P_1 = X_m^{(1,n-1)}: \text{square matrix obtained by dropping the last column of } X
\]
\[
P_2 = X_m^{(2,n)}: \text{square matrix obtained by dropping the first column of } X
\]
\[
Q = X_m^{(2,n-1)}: \text{rectangular matrix obtained by dropping the first and the last column of } X.
\]

Then the proof is similar to that of the case of square Hankel matrices. □

From the previous theorems, we can derive an alternative proof of [CMSV, Theorem 4.1].
Corollary 3.3. Let \( H \) be a generic Hankel matrix of size \( r \times s \). Then
(a) \( I_t(H) \) is a Knutson ideal for every \( t \leq \min(r, s) \).
(b) If \( \mathbb{K} \) is a field of positive characteristic, then \( S/I_t(H) \) is F-pure.

Proof. (a) Using Remark [1], we may assume that the Hankel matrix \( H \) has the right size (that is \( m \times m \) or \( m \times (m+1) \)), so we can apply Theorem 3.1 or Theorem 3.2.
(b) We may assume that \( \mathbb{K} \) is a perfect field of positive characteristic. In fact, we can always reduce to this case by tensoring with the algebraic closure of \( \mathbb{K} \) and the F-purity property descends to the non-perfect case. Using Lemma 4 in [Kn], we know that the ideal \( (f) \) is compatibly split with respect to the Frobenius splitting defined by \( \Tr(f^{p-1} \bullet) \) (where \( f \) is taken to be as in the previous theorems). Thus all the ideals belonging to \( C_f \) are compatibly split with respect to the same splitting, in particular \( I_t(H) \). This implies that such Frobenius splitting of \( S \) provides a Frobenius splitting of \( S/I_t(H) \). Being \( S/I_t(H) \) F-split, it must be also F-pure. \(\square\)

Proving Theorem 3.1, it comes out that determinantal ideals of certain submatrices of Hankel matrices are Knutson ideals. Since we know that \( C_f \) is finite, it is natural to ask whether they are all the ideals belonging to the family or not.

The only way to construct new ideals in \( C_f \) starting from two ideals belonging to the family is taking their sums, their intersections and their minimal primes. So we have to control that in the algorithm we used to prove Theorem 3.1 we take all possible sums, intersections and minimal primes of ideals in \( C_f \).

The previous algorithm proceeds according to the scheme below:

\[
\begin{align*}
ht = 2 & \quad \text{minimal primes} \quad I_{m-1}(P_1), I_{m-1}(P_2) \\
ht = 3 & \quad I_m(X) + I_{m-1}(Q) \\
ht = 4 & \quad I_{m-2}(P_1), I_{m-2}(P_2) \\
\vdots & \\
\end{align*}
\]

Since two ideals of different height in the scheme are always contained one into the other, if we take their intersection or sum we do not obtain a new ideal. Moreover all the ideals of type \( I_t(P_1), I_t(P_2), I_t(X), I_t(Q) \) are prime ideals, so they are (the only) minimal primes over themselves. If we show that at each step there are no other minimal primes, it turns out that the ideals given by the above procedure are all the possible ideals belonging to the family \( C_f \), that is:

Theorem 3.4. Let \( X = X_m^{(1,n)} \) be the square Hankel matrix of size \( m \) with entries \( x_1, \ldots, x_n \) and let \( f = \det X \cdot \det X_m^{(2,n-1)} \in S = \mathbb{K}[x_1, \ldots, x_n] \). Then the only ideals belonging to \( C_f \) are those of the form
\[
I_t(P_1), I_t(P_2), I_t(X), I_t(Q), I_t(X) + I_{t-1}(Q), I_{t-1}(P_1) + I_{t-1}(P_2).
\]
By the above discussion, to prove Theorem 3.3 it is enough to prove the following:

**Proposition 3.5.** With the notation introduced before, we get the following primary decompositions:
1. \( I_t(X) + I_{t-1}(Q) = I_{t-1}(P_1) \cap I_{t-1}(P_2) \)
2. \( I_{t-1}(P_1) + I_{t-1}(P_2) = I_{t-1}(X) \cap I_{t-2}(Q) \).

The inclusion \( \subseteq \) is obvious in both cases. It remains to prove the reverse inclusion. To do so we will apply the following result which is a consequence of [BH, Corollary 4.6.8].

**Lemma 3.6.** Let \( I, J \) be two ideals in a polynomial ring \( S \) such that the following conditions hold:
1. \( \text{ht}(I) = \text{ht}(J) =: h \)
2. \( I \subseteq J \)
3. \( \text{ht}(P) = h \quad \forall P \in \text{Ass}(I) \)

then \( e(S/I) = e(S/J) \Rightarrow I = J. \)

Furthermore, in the proof of Proposition 3.5 we will need to apply recursively a result by Peskine e Szpiro (see Proposition 3.7) to prove that the ideals \( I_{t-1}(P_1) + I_{t-1}(P_2) \) and \( I_t(X) + I_{t-1}(Q) \) are Gorenstein for every \( t = 1, \ldots, m \) and that
\[
\text{ht}(I_{t-1}(P_1) + I_{t-1}(P_2)) = \text{ht}(I_t(X) + I_{t-1}(Q)) + 1.
\]

By the purity of Macaulay, this will imply that all the three conditions of Lemma 3.6 are satisfied.

**Proposition 3.7.** [PS, Remark 1.4] Let \( I \) and \( J \) be two homogeneous ideals in a polynomial ring \( S \) with no associated primes in common and suppose that \( S/(I \cap J) \) is Gorenstein. Then:
1. \( S/I \) is Cohen-Macaulay if and only if \( S/J \) is Cohen-Macaulay.
2. If \( S/I \) is Cohen-Macaulay, then \( S/(I + J) \) is Gorenstein and
\[
\text{ht}(I + J) = \text{ht}(I) + 1.
\]

Collecting together all these results, we can prove Proposition 3.5.

**Proof of Proposition 3.5.** For \( k \geq 1 \) we define:
- \( I_k := I_{m-h}(X) \) and \( J_k := I_{m-h-1}(Q) \), if \( k = 2h + 1 \)
- \( I_k := I_{m-h}(P_1) \) and \( J_k := I_{m-h}(P_2) \), if \( k = 2h \).

We want to show that \( I_k + J_k = I_{k+1} \cap J_{k+1} \) for every \( 1 \leq k \leq 2(m-2) + 1 \). We proceed by induction on \( k \), following the usual scheme.

First of all observe that all these ideals are different homogenous prime ideals of height \( k \) in \( S \) (in particular, they have no associated prime ideals in common) and since they are determinantal ideals, they are also Cohen-Macaulay.

Assume \( k = 1 \). Then \( I_1 + J_1 = I_m(X) + I_{m-1}(Q) = (\det X, \det Q) \) and \( I_2 \cap J_2 = I_{m-1}(P_1) \cap I_{m-1}(P_2) \). We know that \( I_1 + J_1 \subseteq I_2 \cap J_2 \) and that \( I_1 + J_1 \) is a complete intersection of height 2. In particular it is Gorenstein and by the purity of Macaulay, all its associated primes \( P \) have the same height, namely \( \text{ht}(P) = \text{ht}(I_1 + J_1) = 2. \) Moreover
Lemma 3.6. If we show that they have the same multiplicity, we get the desired equality. Since \( I_1 + J_1 \) is a complete intersection, we have that \( e(I_1 + J_1) = m(m-1) \). Moreover the \( h \)-vector of the determinantal ring of a Hankel matrix \( H \) of size \( t \times s \) is well known. In fact, being \( \text{ht}(I_t(H)) = n - 2t + 2 \) and using Remark 4, the Eagon-Northcott complex provides a minimal free resolution of \( S/I_t(H) \). In particular \( S/I_t(H) \) is Cohen-Macaulay and has linear resolution. Therefore:

\[
\begin{align*}
\text{ht}(\mathbb{S}/H) &= (1, (s-t+1), \left(\frac{s-t+2}{2}\right), \ldots, \left(\frac{s-1}{t-1}\right)) \\
n(\mathbb{S}/H) &= 1 + (s-t+1) + \left(\frac{s-t+2}{2}\right) + \ldots + \left(\frac{s-1}{t-1}\right).
\end{align*}
\]

and its multiplicity is

\[
e(\mathbb{S}/I_t(H)) = 1 + (s-t+1) + \left(\frac{s-t+2}{2}\right) + \ldots + \left(\frac{s-1}{t-1}\right).
\]

Using this formula we get:

\[
e(I_2 \cap J_2) = e(I_{m-1}(P_1)) + e(I_{m-1}(P_2)) = 2e(I_{m-1}(P_1)) = 2 \left(1 + (m-m+1+1) + \left(\frac{3}{2}\right) + \left(\frac{4}{3}\right) + \ldots + \left(\frac{m-1}{m-2}\right)\right) = 2\left(1 + 2 + 3 + 4 + \ldots + (m-1)\right) = 2 \left(\frac{m(m-1)}{2}\right) = m(m-1).
\]

Hence \( e(I_1 + J_1) = e(I_2 \cap J_2) \) and by Lemma 3.6 we get \( I_1 + J_1 = I_2 \cap J_2 \). Furthermore, using Lemma 3.7 we get that \( I_2 + J_2 \) is Gorenstein and \( \text{ht}(I_2 + J_2) = \text{ht}(I_{m-1}(P_1)) + 1 = 3 \).

Now assume \( k = 2 \). Then \( I_2 + J_2 = I_{m-1}(P_1) + I_{m-1}(P_2) \) and \( I_3 \cap J_3 = I_{m-1}(X) \cap I_{m-2}(Q) \). From the previous case, we know that \( I_2 + J_2 \) is Gorenstein and it has height 3. As a consequence of the purity theorem of Macaulay we have that \( \text{ht}(P) = \text{ht}(I_2 + J_2) \) for all the associated primes \( P \) of \( I_2 + J_2 \). In addition we know that \( I_2 + J_2 \subseteq I_3 \cap J_3 \) and that they have the same height. Again \( I_2 + J_2 \) and \( I_3 \cap J_3 \) satisfy all the hypotheses of Lemma 3.6. If we show that they have the same multiplicity, we get the desired equality.

Iterating this procedure, we get the thesis. More generally, let \( k \geq 2 \). By induction we may assume that \( I_k \cap J_k = I_{k-1} + J_{k-1} \) is Gorenstein and that \( \text{ht}(I_k + J_k) = k + 1 = \text{ht}(I_{k-1} \cap J_{k-1}) \). Since \( I_k + J_k \subseteq I_{k+1} \cap J_{k+1} \), if we show that \( e(I_k + J_k) = e(I_{k+1} \cap J_{k+1}) \), by Lemma 3.6 we get \( I_k + J_k = I_{k+1} \cap J_{k+1} \) and using Lemma 3.7 we obtain that \( I_{k+1} + J_{k+1} \) is Gorenstein of height \( (k+1) + 1 \).

Therefore it is enough to show that \( e(I_k + J_k) = e(I_{k+1} \cap J_{k+1}) \) for every \( k \). In other words, we need to prove the following equalities:

- \( e(I_t(P_1) + I_t(P_2)) = e(I_t(X) \cap I_{t-1}(Q)) \)
- \( e(I_t(X) + I_{t-1}(Q)) = e(I_{t-1}(P_1) \cap I_{t-1}(P_2)) \).
To compute the multiplicity of these ideals, we first compute their $h$-vectors. Let $I := I_t(P_1)$ and $J := I_t(P_2)$ and consider the following exact sequence:

$$0 \longrightarrow S/(I \cap J) \longrightarrow S/I \oplus S/J \longrightarrow S/(I + J) \longrightarrow 0.$$ 

By additivity of Hilbert series on short exact sequence, we get:

$$HS_{S/(I+J)}(t) = HS_{S/I \oplus S/J}(t) - HS_{S/(I \cap J)}(t).$$

From the previous discussion we already know that $ht(I_t(P_1) + I_t(P_2)) = h + 1$ where $h := ht(I_t(P_1)) = ht(I_t(P_2)) = ht(I_t(P_1) \cap I_t(P_2))$.

This implies that

$$\dim S/(I_t(P_1) \cap I_t(P_2)) = \dim S/I_t(P_1) = \dim I_t(P_2) = n - h =: d$$

and

$$\dim S/(I_t(P_1) + I_t(P_2)) = d - 1.$$

Using the well known fact that the Hilbert series is a rational function (see e.g. [BH, Corollary 4.1.8]), we get:

$$\frac{h^{S/(I+J)}(z)}{(1 - z)^{d-1}} = \frac{h^{S/I \oplus S/J}(z) - h^{S/(I \cap J)}(z)}{(1 - z)^d},$$

hence

$$h^{S/(I+J)}(z) = \frac{h^{S/I \oplus S/J}(z) - h^{S/(I \cap J)}(z)}{(1 - z)}.$$

It is straightforward to see that $S/I$ and $S/J$ have the same $h$-vector, namely:

$$h^{S/I} = h^{S/J} = (h^S_0, h^S_1, \ldots, h^S_{t-1})$$

where $h^S_i = \binom{2m - 2t + i - 1}{i}$ for $i \leq t - 1$. As a consequence, we have that $e(I \cap J) = e(I) + e(J) = 2e(I)$.

Let $\overline{S/(I \cap J)}$ be the Artinian reduction of $S/(I \cap J)$. Since $I$ and $J$ are generated in degree $t$, for $i < t$ we have

$$h^S_i \overline{S/(I \cap J)} = \overline{h^S_i} = \dim \overline{S_i} = \left(\binom{2m - 2t + i - 1}{i}\right) = h^S_i.$$ 

Thus $h^S_i \overline{S/(I \cap J)} = h^S_i$ for $i < t$. But $I \cap J$ is a Gorenstein ideal, so its $h$-vector must be symmetric and we already know that $e(I \cap J) = 2e(I)$. This implies that

$$h^{S/I \cap J} = (h^S_0, h^S_1, \ldots, h^S_{t-2}, h^S_{t-1}, h^S_{t-1}, h^S_{t-2}, \ldots, h^S_1, h^S_0).$$

Substituting in (6), we get:
\[ h_{S/(I+J)}(z) = \frac{2h_{S/I}(z) - h_{S/(I \cap J)}(z)}{(1 - z)} = 
\] 
\[ = \frac{2 \sum_{i=0}^{t-1} h_i S/I z^i - \sum_{i=0}^{t-1} h_i S/I z^i - 2t-1 h_{2t-1} S/I z^i}{1 - z} = 
\] 
\[ = \frac{h_0 S/I + h_1 S/I z + \cdots + h_{t-1} S/I z^{t-1} - h_{t-1} S/I z^t - \cdots - h_1 S/I z^{2t-2} - h_0 S/I z^{2t-1}}{1 - z}. 
\]

Dividing by \(1 - z\), we finally obtain

\[ h_{S/(I+J)} = \left( h_0 S/I, h_0 S/I + h_1 S/I, \ldots, \sum_{i=0}^{t-2} h_i S/I, \sum_{i=0}^{t-2} h_i S/I, \ldots, h_0 S/I + h_1 S/I, h_0 S/I \right). 
\]

Note that a similar argument shows that if we consider \( I = I_t(X) \) and \( J = I_{t-1}(Q) \), then

\[ h_{S/(I+J)} = \left( h_0 S/I, h_0 S/I + h_1 S/I, \ldots, \sum_{i=0}^{t-2} h_i S/I, \sum_{i=0}^{t-2} h_i S/I, \ldots, h_0 S/I + h_1 S/I, h_0 S/I \right). 
\]

Now we can compute the multiplicities.

In fact, using the relation \( \binom{j}{k} + \binom{j}{k+1} = \binom{j+1}{k+1} \) and the identity \( (3) \), we get the relation

\[ h_i^{S/I_t(P)} = h_i^{S/I_t(P)} = h_i^{S/I_t(X)} - h_{i-1}^{S/I_t(X)}. 
\]

From the \( h \)-vector of \( S/(I+J) \), using the fact that \( h_i^{S/I_t(X)} = h_i^{S/I_{t-1}(Q)} \), we have:

\[ e(I_t(P_1) + I_t(P_2)) = h_0^{S/I_t(P)} + \left( h_0^{S/I_t(P)} + h_1^{S/I_t(P)} \right) + \cdots 
\]

\[ + \sum_{i=0}^{t-2} h_i^{S/I_t(P)} + \sum_{i=0}^{t-2} h_i^{S/I_t(P)} + \sum_{i=0}^{t-2} h_i^{S/I_{t-1}(P)} + \cdots 
\]

\[ + \left( h_0^{S/I_t(P)} + h_1^{S/I_t(P)} \right) + h_0^{S/I_t(P)} = 
\]

\[ = h_0^{S/I_t(X)} + h_1^{S/I_t(X)} + \cdots + h_{t-2}^{S/I_t(X)} + h_{t-1}^{S/I_t(X)} + 
\]

\[ = e(I_t(X)) + h_0^{S/I_t(X)} = e(I_t(X)) + e(I_{t-1}(Q)) = e(I_t(X)) \cap I_{t-1}(Q). 
\]

So the first equality has been proved.

For the second equality, one can argue in a similar way observing that

\[ h_i^{S/I_t(X)} = h_i^{S/I_{t-1}(P)} - h_{i-1}^{S/I_{t-1}(P)}. 
\]

Computing the multiplicity of \( I_t(X) + I_{t-1}(Q) \) from its \( h \)-vector, we get
\[
e(I_t(X) + I_{t-1}(Q)) = h_0^{S/I_t(X)} + \left( h_1^{S/I_t(X)} + h_0^{S/I_t(X)} \right) + \cdots + \sum_{i=0}^{t-2} h_i^{S/I_t(X)} +
\]
\[
+ \sum_{i=0}^{t-2} h_i^{S/I_t(X)} + \cdots + \left( h_0^{S/I_t(X)} + h_1^{S/I_t(X)} \right) + h_0^{S/I_t(X)} =
\]
\[
= 2 \left( h_0^{S/I_{t-1}(P_1)} + h_1^{S/I_{t-1}(P_1)} + \cdots + h_{t-2}^{S/I_{t-1}(P_1)} \right) =
\]
\[
= 2 \left( e(I_{t-1}(P_1)) = e(I_{t-1}(P_1) \cap I_{t-1}(P_2)) \right).
\]

\[\square\]

Remark 5. It is worth noticing that, while proving Proposition 3.5, we also found out that the ideals \(I_t(X) + I_{t-1}(Q) = I_{t-1}(P_1) \cap I_{t-1}(P_2)\) and \(I_{t-1}(P_1) + I_{t-1}(P_2) = I_{t-1}(X) \cap I_{t-2}(Q)\) are Gorenstein ideals for every \(t\).

Furthermore, we have computed the following \(h\)-vectors:

(a) Let \(I = I_t(P_1)\) and \(J = I_t(P_2)\). Then:
\[
h^{S/(I+J)} = \left( h_0^{S/I_0}, h_0^{S/I_1} + h_1^{S/I_1}, \ldots, \sum_{i=0}^{t-2} h_i^{S/I_1}, \sum_{i=0}^{t-1} h_i^{S/I_1}, \sum_{i=0}^{t-2} h_i^{S/I_1}, \ldots, h_0^{S/I_1} + h_1^{S/I_1}, h_0^{S/I_1} \right).
\]

(b) Let \(I = I_t(X)\) and \(J = I_{t-1}(Q)\). Then:
\[
h^{S/(I+J)} = \left( h_0^{S/I_0}, h_0^{S/I_1} + h_1^{S/I_1}, \ldots, \sum_{i=0}^{t-2} h_i^{S/I_1}, \sum_{i=0}^{t-1} h_i^{S/I_1}, \sum_{i=0}^{t-2} h_i^{S/I_1}, \ldots, h_0^{S/I_1} + h_1^{S/I_1}, h_0^{S/I_1} \right).
\]

Note that these \(h\)-vectors are unimodal, as \(h_i^{S/I_1}\) is non-negative for every \(i\). It should be stressed that this is expected by the \(g\)-conjecture since we have proved that \(I + J\) is always Gorenstein.

Remark 6. We have shown that if \(f = \det X \det Q\) then \(S/I\) is Cohen-Macaulay for every ideal \(I \in C_f\). We want to point out that this fact is proper of this specific choice of \(f\): if we consider for example \(f = x_1 \cdots x_n\) then \(C_f\) is the family of all the squarefree monomial ideals of \(S\) and most of them are not Cohen-Macaulay.

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