EXTENDING FOUR DIMENSIONAL RICCI FLOWS WITH BOUNDED SCALAR CURVATURE

MILES SIMON

Abstract. We consider solutions \((M,g(t)), 0 \leq t < T\), to Ricci flow on compact, connected four dimensional manifolds without boundary. We assume that the scalar curvature is bounded uniformly, and that \(T < \infty\). In this case, we show that the metric space \((M,d(t))\) associated to \((M,g(t))\) converges uniformly in the \(C^0\) sense to \((X,d)\), as \(t \to T\), where \((X,d)\) is a \(C^0\) Riemannian orbifold with at most finitely many orbifold points. Estimates on the rate of convergence near and away from the orbifold points are given. We also show that it is possible to continue the flow past \((X,d)\) using the orbifold Ricci flow.

1. Introduction

In this paper we consider smooth solutions to Ricci flow, \(\frac{\partial}{\partial t} g(t) = -2 \text{Rc}(g(t))\) for all \(t \in [0,T)\), on closed, connected four manifolds without boundary. We assume that \(T < \infty\) and that the scalar curvature satisfies \(\sup_{M \times (0,T)} |R| \leq 1\). In a previous paper, see Theorem 3.6 in [Si1], we showed that this implies

(i) Integral bounds for the Ricci and Riemannian curvature

\[
\sup_{t \in (0,T)} \int_M |\text{Riem}(\cdot,t)|^2 d\mu_{g(t)} \leq c_1 < \infty
\]

\[
\int_0^T \int_M |\text{Rc}(\cdot,t)|^4 d\mu_{g(t)} dt \leq c_2 < \infty
\]

for explicit constants \(c_1 = c_1(M,g(0),T)\) and \(c_2(M,g(0),T)\). An estimate of the first type was independently proved, using different methods, in a recent paper, [BZ] (see Theorem 1.8).

In this paper we show the following.

(ii) Estimates for the singular and regular regions

A point \(p \in M\) is said to be regular, if there exists an \(r > 0\) such that

\[
\int_{B_r(p)} |\text{Riem}(\cdot,t)|^2 d\mu_{g(t)} \leq \varepsilon_0
\]

for all \(t \in (0,T)\), for some fixed small \(\varepsilon_0\) (not depending on \(p\)) which is specified in the proof of Theorem 4.5. In Definition 4.7, an alternative definition of regular is

Date: April 14, 2015.

2000 Mathematics Subject Classification. 53C44.

Key words and phrases. Ricci flow, scalar curvature.
given. The singular points are those which are not regular. In Theorem 4.5 (and the Corollaries 4.9 and 4.10 thereof) and Theorem 5.1 we obtain estimates for the evolving metric in the singular and regular regions of the manifold.

(iii) **Uniform continuity of the distance function in time.**
Using the estimates mentioned in (ii) we show the following (see Theorem 5.6). For all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|d(x, y, t) - d(x, y, s)| \leq \varepsilon
\]

for all \( x, y \in M \) for all \( t, s \in [0, T] \) with \( |t - s| \leq \delta \).

(iv) **Convergence of \((M, d(g(t)))\) to a \(C^0\) Riemannian orbifold \((X, d)\) as \( t \searrow T\).**
Using the estimates mentioned in (i),(ii) and (iii), we show that \((M, d(g(t))) \searrow (X, d_X)\) in the Gromov-Hausdorff sense, where \((X, d_X)\) is a \(C^0\)-Riemannian orbifold with finitely many orbifold points, and that the Riemannian orbifold metric on \(X\) is smooth away from the orbifold points. Also: the convergence is smooth away from the orbifold points (see Lemma 6.2 and Theorems 6.5, 6.6, 8.3).

(v) **The flow may be continued past time \(T\) using the orbifold Ricci flow.**
There exists a smooth solution \((N, h(t))_{t \in [0, \tilde{T}]}\) to the orbifold Ricci flow, such that \((N, d(h(t))) \searrow (X, d_X)\) in the Gromov-Hausdorff sense as \( t \downarrow 0 \) (see Theorem 9.1).

In another paper, [BZ], which recently appeared, the authors also consider Ricci flow of four manifolds with bounded scalar curvature, and they also investigate the structure of the limiting space one obtains by letting \( t \searrow T \): see Theorem 1.8 and Corollary 1.11 of [BZ].

---

2. **Setup, background, previous results and notation**

In this paper we often consider solutions \((M^4, g(t))_{t \in [0, T]}\) which satisfy the following **basic assumptions**.

(a) \(M^4\) is a smooth, compact, connected four dimensional manifold without boundary

(b) \((M^4, g(t))_{t \in [0, T]}\) is a smooth solution to the Ricci flow \( \frac{\partial}{\partial t} g(t) = -2\text{Ricci}(g(t)) \)

(c) \( T < \infty \)

(d) \( \sup_{M^4 \times [0, T]} |\text{R}(x, t)| \leq 1 \)

If instead of (d) we only have \( \sup_{M^4 \times [0, T]} |\text{R}(x, t)| \leq K < \infty \) for some constant \( 1 < K < \infty \), then we may rescale the solution \( \tilde{g}(\cdot, \tilde{t}) := Kg(\cdot, \frac{1}{K} \tilde{t}) \) to obtain a new solution \((M, \tilde{g}(\tilde{t}))_{\tilde{t} \in [0, \tilde{T}]}\), where \( \tilde{T} := KT \), which satisfies the basic assumptions. As we mentioned in the introduction, any solution satisfying the basic assumptions
also satisfies
\begin{align}
(2.1) \quad \sup_{t \in [0, T]} \int_M |\text{Riem}(\cdot, t)|^2 d\mu_{g(t)} &\leq K_0 < \infty \\
(2.2) \quad \int_0^T \int_M |\text{Rc}(\cdot, t)| d\mu_{g(t)} dt &\leq c_2 < \infty.
\end{align}

See Theorem 3.6 in [Si1]. The estimate (2.1) was independently obtained in [BZ] (see Theorem 1.8 of that paper), using different methods to those used in [Si1].

There are many papers in which conditions are considered which imply that the solution to Ricci flow defined on $[0, T)$ may be extended. Generally, in the real case, this extension is a smooth extension, and the conditions imply that the solution may be smoothly extended to a time interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$: that is, the solution does not form a singularity as $t \nearrow T$. Here we list some of these conditions. This is by no means an exhaustive list and further references may be found in the papers we have listed here.

In the following we assume that $(M^n, g(t))_{t \in [0, T)}$ is a smooth solution to Ricci flow on a compact $n$-dimensional manifold without boundary, and we write the condition which guarantees that one can extend the solution past time $T$, followed by an appropriate reference.

- $\sup_{M^n \times [0, T]} |\text{Riem}| < \infty$ [HaThree].
- $\sup_{M^n \times [0, T]} |\text{Ricci}| < \infty$ [Sesum].
- $\limsup_{t \nearrow T} |g(t) - h| \leq \varepsilon(n)$ for some smooth metric $h$ [SimC0] (see also [KL]).
- $\sup_{(x, t) \in M^n \times [0, T]} |\text{Riem}(x, t)(T - t) + |R(x, t)| < \infty$ [TME] (see also [SesumLd]).

\[
\int_0^T \int_{M^n} |\text{Rm}|^\alpha(\cdot, t) d\mu_{g(t)} dt < \infty \quad \text{for some } \alpha \geq \frac{(n+2)}{2},
\]
\[
\int_0^T \int_{M^n} |\text{Weil}|^\alpha(\cdot, t) + |R|^\alpha(\cdot, t) d\mu_{g(t)} dt < \infty,
\]
where $\alpha \geq \frac{(n+2)}{2}$ [Wang1].

See also [Wang1], [Wang2], [ChenWang] for further results on extending Ricci flow.

If one considers solutions to the Kähler Ricci flow, $\frac{\partial}{\partial t} g_{ij} = -2 \text{Ric}_{ij}$, then the following is known: If $\sup_{M^n \times [0, T]} |R| < \infty$, then one can extend the flow smoothly past time $T$ [Zhang].

The situation in this paper is somewhat different. We consider solutions with bounded scalar curvature, and we do not rule out the possibility that singularities can form as $t \nearrow T$. However, using our integral curvature estimates (and other estimates) we show that there is a singular limiting space as $t \nearrow T$, and that this singular space is a $C^0$ Riemannian orbifold which can then be evolved by the orbifold Ricci flow: the limiting space is immediately smoothed out by the orbifold Ricci flow.

The possibility of flowing to a singular time and then continuing with another flow (for example orbifold Ricci flow or a weak Kähler Ricci flow) has been considered in other papers. In the real case, see for example [CTZ].

In the Kähler case see for example Theorem 1.1 in [SongWeinkove2] (see also [SongWeinkove1], [EGZ] and [EGZII] for related papers). Further references can be found in the papers mentioned above.

In [ChenWang], the authors investigate the moduli space of solutions to Ricci flow which have: bounded curvature in the $L^{n/2}$ sense, bounded scalar curvature and are non-collapsed.

There are examples of solutions to Ricci flow which are smooth on $[0, T)$, singular at time $T$, and then become immediately smooth again after this time: see the
neck-pinching examples given in [ACK]. See also [KLo] and [FIK]. This notion of extending the flow is once again different to the one we are considering, and different to the notion of smooth extension discussed above.

The Orbifold Ricci flow and related flows has been studied in many papers. Here is a (by no means exhaustive) list of some of them: [CTZ], [ChenYWangI], [ChenYWangII], [ChowII], [ChowWu], [HaThreeO], [KLThree], [LiuZhang], [WuFF], [Yin], [YinII].

Notation:
We use the Einstein summation convention, and we use the notation of Hamilton [HaThree].

For $i \in \{1, \ldots, n\}$, $\frac{\partial}{\partial x^i}$ denotes a coordinate vector, and $dx^i$ is the corresponding one form.

$(M^n, g)$ is an $n$-dimensional Riemannian manifold with Riemannian metric $g$.

$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ is the Riemannian metric $g$ with respect to this coordinate system.

$g^{ij}$ is the inverse of the Riemannian metric ($g^{ij}g_{jk} = \delta_{jk}$).

$\delta_{ij}$ is the Kronecker delta.

$\omega$ is the volume form associated to $g$.

$Rm(g)_{ijkl} = g^{im}g^{jn}Rm(g)_{m,jkl}$ is the Ricci curvature.

$R := R_{ij}g^{ij}$ is the Ricci curvature.

$\nabla T = \nabla T$ is the covariant derivative of $T$ with respect to $g$. For example, locally

$\nabla T_{jk} = (\nabla T)(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, dx^i)$ (the first index denotes the direction in which the covariant derivative is taken) if locally $T = T_{jk}dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$.

$|T| = g|T|$ is the norm of a tensor with respect to a metric $g$. For example for $T = T_{jk}dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$, $|T|^2 = g^{im}g^{jn}T_{ij}T_{mn}$.

Sometimes we make it clearer which Riemannian metric we are considering by including the metric in the definition. For example $R(h)$ refers to the scalar curvature of the Riemannian metric $h$.

We suppress the $g$ in the notation used for the norm, $|T| = g|T|$, and for other quantities, in the case that is is clear from the context which Riemannian metric we are considering.

A ball of radius $r > 0$ in a metric space $(X, d)$ will be denoted by

$dB_r(z) := \{x \in X \mid d(x, z) < r\}$.

An annulus of inner radius $0 \leq s$ and outer radius $r > s$ on a metric space $(X, d)$ will be denoted by

$dB_{r,s}(z) := \{x \in X \mid s < d(x, z) < r\}$.

Note then that $dB_{0,s}(z) := \{x \in X \mid 0 < d(x, z) < r\} = dB_s(z) \setminus \{z\}$.

The sphere of radius $r > 0$ and centre point $p$ in a metric space $(X, d)$ will be denoted by

$dS_r(p) := \{x \in X \mid d(x, p) = r\}$.

$D_{r,R} \subseteq \mathbb{R}^n$ is the standard open annulus of inner radius $r \geq 0$ and outer radius $R \leq \infty$, $(r < R)$ centred at 0:

$D_{r,R} = \{x \in \mathbb{R}^n \mid |x| > r, |x| < R\}$.

$D_r$ represents the open disc of radius $r$ centred at 0:

$D_r := \{x \in \mathbb{R}^n \mid |x| < r\}$.

Note $D_{0,R} = \{x \in \mathbb{R}^n \mid |x| > 0, |x| < R\} = D_R \setminus \{0\}$.

$S^{n-1}_r(c) := \{x \in \mathbb{R}^n \mid |x - c| = r\}$ is the $(n-1)$-dimensional sphere of radius $r > 0$.
and centre point $c \in \mathbb{R}^n$ in $\mathbb{R}^n$.

$\omega_n$ is the volume of a ball of radius one in $\mathbb{R}^n$ with respect to the Lebesgue measure.

If $\Gamma$ is a finite subgroup of $O(n)$ acting on $\mathbb{R}^n$, then $((\mathbb{R}^n \setminus \{0\})/\Gamma, g)$ is the quotient manifold with the induced (flat) metric coming from $\pi : \mathbb{R}^n \setminus \{0\} \to (\mathbb{R}^n \setminus \{0\})/\Gamma$, $\pi(x) := \{ x \mid x \in \mathbb{R}^n \setminus \{0\} \}$, where $|x| := \{ Gx \mid G \in \Gamma \}$.

$(B_{r,s}(0), g) \subseteq ((\mathbb{R}^n \setminus \{0\})/\Gamma, g)$ refers to the set $sB_{r,s}(0) := \{ \pi(x) \mid x \in D_{r,s} \}$ with the Riemannian metric $g$.

3. Volume control, and the Sobolev inequality

In [Ye] and [Zhang1] [Zhang2] the first inequality appearing below was proved, and in [Zhang3] (and in [ChenWang]) the second inequality appearing below was proved.

**Theorem 3.1.** (Ye, R. Ye, Zhang, Q. Zhang1, Zhang2, Zhang3 (see ChenWang also))

Let $(M^n, g(t))_{t \in [0, T)}$, $T < \infty$, be a smooth solution to Ricci flow on a closed manifold with $\sup_{M \times [0,T]} |R(x, t)| \leq 1 < \infty$. Then there exist constants $0 < \sigma_0, \sigma_1 < \infty$ depending only on $(M, g_0)$ and $T$ such that

$$
\sigma_1 \leq \frac{\text{vol}(B_r(x))}{r^n} \leq \sigma_2 \text{ for all } x \in M, 0 \leq t < T \text{ and } r \leq 1.
$$

We use the following notation in this paper which was introduced by Q. Zhang. A solution which satisfies the first inequality is said to be $\sigma_1$ non-collapsed on scales less than 1. This condition is similar to but stronger than Perelman’s non-collapsing condition (see [Pe1]), as we make no requirements on the curvature within the balls $B_r(x)$ appearing in (3.1). A solution which satisfies the second inequality is said to be $\sigma_2$ non-inflated on scales less than 1.

**Remark 3.2.** Let $(M^n, g(t))_{t \in [0, T)}$ be a smooth solution to Ricci flow which satisfies the inequalities (3.1), and define $\tilde{g}(\tilde{t}) := cg(\cdot, \frac{\tilde{t}}{c})$ for a constant $c > 0$. Then

$$
\sigma_1 \leq \frac{\text{vol}(\tilde{B}_{\tilde{r}}(x))}{\tilde{r}^n} \leq \sigma_2 \text{ for all } x \in M, 0 \leq \tilde{t} < \tilde{T} := cT \text{ and } \tilde{r} \leq \sqrt{c},
$$

that is $(M, \tilde{g}(\tilde{t}))_{\tilde{t} \in [0, \tilde{T})}$ is $\sigma_1$ non-collapsed and $\sigma_2$ non-inflated on scales less than $\sqrt{c}$. This is because: $\frac{\text{vol}(B_{r}(x, t))}{r^n} = \frac{\text{vol}(B_{\tilde{r}}(x, \tilde{t}))}{\tilde{r}^n}$ for $\tilde{r} := \sqrt{c} r$ and $\tilde{t} := c t$, and $r = \frac{\tilde{r}}{\sqrt{c}} \leq 1$ for $\tilde{r} \leq \sqrt{c}$. Hence, we can’t say if the solution $(M, \tilde{g}(\tilde{t}))_{\tilde{t} \in [0, \tilde{T})}$ is $\sigma_1$ non-collapsed and $\sigma_2$ non-inflated on scales less than 1, if we scale by a constant $c < 1$, but the scale improves if we multiply by constants $c > 1$.

In the papers [Ye] and [Zhang1] [Zhang2] it is also shown that for any Ricci flow satisfying the basic assumptions a Sobolev inequality holds in which the constants may be chosen to be time independent. Here, we only write down the four dimensional version of their theorem.

**Theorem 3.3.** (Ye, R. Ye, Zhang, Q. Zhang1, Zhang2)

Let $(M^4, g(t))_{t \in [0, T)}$, $T < \infty$, be a smooth solution to Ricci flow satisfying the basic assumptions. Then there exists a constant $A = A(M, g_0, T) < \infty$ such that

$$
\left( \int_M |f|^4 d\mu_{g(t)} \right)^{\frac{1}{2}} \leq A \left( \int_M \nabla f^2 d\mu_{g(t)} + \int_M |f|^2 d\mu_{g(t)} \right)
$$
for all smooth $f : M \rightarrow \mathbb{R}$.

Note that this Sobolev inequality is not scale invariant, as the last term scales incorrectly. However, we have a scale-invariant version for small balls, as we see in the following:

**Corollary 3.4.** Let $(M^4, g(t))_{t \in [0,T)}, T < \infty$ be a smooth solution to Ricci flow satisfying the basic assumptions. Then there exists a constant $r^2 = r^2(M,g(0),T) = \frac{1}{2\sqrt{\sigma_2}} > 0$ such that

$$
(3.4) \quad \left( \int_M |f|^4 d\mu_{g(t)} \right)^{\frac{1}{2}} \leq 2A \int_M g(t)|\nabla f|^2 d\mu_{g(t)}
$$

for all smooth $f : M \rightarrow \mathbb{R}$ whose support is contained in a ball $\tilde{B}_r(x)$, for some $x \in M$, where $A$ is the constant occurring in the Sobolev inequality (3.3) above. If $\tilde{g}(-,\tilde{t}) := cg(-, \frac{t}{\tilde{t}})$ is a scaled solution with $c \geq 1$ then the estimate

$$
(3.5) \quad \left( \int_M |f|^4 d\mu_{\tilde{g}(\tilde{t})} \right)^{\frac{1}{2}} \leq 2A \int_M \tilde{g}(t)|\nabla f|^2 d\mu_{\tilde{g}(\tilde{t})}
$$

holds for all $f : M \rightarrow \mathbb{R}$ whose support is contained in a ball $\tilde{B}_{\tilde{r}}(x)$ where $\tilde{r} := r\sqrt{c} \geq r$.

**Proof.** Let $r$ be chosen so that $r^2 \sqrt{\sigma_2} \leq \frac{1}{2A}$, where $A$ is the constant occurring in the Sobolev inequality and $\sigma_2$ is the non-inflating constant defined above. Using Hölder’s inequality and the above Sobolev inequality we get

$$
(3.6) \quad \left( \int_M |f|^4 d\mu_{g(t)} \right)^{\frac{1}{2}} \leq A \int_M |\nabla f|^2 d\mu_{g(t)} + A \int_M |f|^2 d\mu_{g(t)}
$$

$$
\leq A \int_M |\nabla f|^2 d\mu_{g(t)} + A \left( \int_M |f|^4 d\mu_{g(t)} \right)^{\frac{1}{2}} \left( \int_M (\text{vol } B_r(x))^{\frac{1}{2}} \right)
$$

$$
\leq A \int_M |\nabla f|^2 d\mu_{g(t)} + A \left( \int_M |f|^4 d\mu_{g(t)} \right)^{\frac{1}{2}} \left( \int_M (\text{vol } B_r(x))^{\frac{1}{2}} \right)
$$

which implies the result, after subtracting $\frac{1}{2} \left( \int_M |f|^4 d\mu_{g(t)} \right)^{\frac{1}{2}}$ from both sides of this inequality. The second inequality follows immediately from the fact that

$$
(3.7) \quad \left( \int_M |f|^4 d\mu_{\tilde{g}(\tilde{t})} \right)^{\frac{1}{2}} - \left( \int_M |\nabla f|^2 d\mu_{\tilde{g}(\tilde{t})} \right) = c \left( \int_M |f|^4 d\mu_{g(t)} \right)^{\frac{1}{2}} - \left( \int_M |\nabla f|^2 d\mu_{g(t)} \right)
$$

if we scale as in the statement of the theorem. □

It is well know that, for a solution satisfying the basic assumptions, the volume of $M$ is changing at a controlled rate:

$$
(3.8) \quad \text{vol}(M,g(t)) \geq - \int_M R d\mu_{g(t)} = \frac{\partial}{\partial t} \text{vol}(M,g(t)) \geq - \text{vol}(M,g(t))
$$

$$
(- \int_M R d\mu_{g(t)} = \frac{\partial}{\partial t} \text{vol}(M,g(t)) was shown in [HaThree]). Integrating in time we see that $e^T \text{vol}(M,g(0)) \geq \text{vol}(M,g(t)) \geq e^{-T} \text{vol}(M,g(0)).$
Notice that the estimates of Peter Topping (see [Topping]) and these volume bounds combined with the non-inflating estimate guarantee that the distance is bounded from above and below:

**Lemma 3.5.** (Topping, P. [Topping], Zhang, Q. Zhang1, Zhang2) Let \((M^4,g(t)))_{t \in [0,T)}\) be a solution satisfying the basic assumptions (in particular \(T < \infty\) and \(|R| \leq 1\) at all times and points). Then there exists \(d_0 = d_0(M,g_0,T) > 0\) such that

\[
\text{(3.9)} \quad \infty > d_0 \geq \text{diam}(M,g(t)) \geq \frac{1}{d_0} > 0
\]

for all \(t \in [0,T)\).

**Proof.** The diameter bound from above follows immediately from Theorem 2.4 (see also Remark 2.5 there) of [Topping] combined with the fact that \(\int_M |R|^{5/2} \leq \text{vol}(M,g(0))e^T\) for a solution satisfying the basic assumptions. The diameter bound from below is obtained as follows. Assume that there are times \(t_i \in [0,T)\) with \(\varepsilon_i := \text{diam}(M,g(t_i)) \to 0\) as \(i \to \infty\). Due to smoothness, we must have \(t_i \not\to T\). From the volume estimates above, we must have \(\text{vol}(M,g(t)) \geq e^{-T} \text{vol}(M,g(0)) =: v_0 > 0\) for all \(t \in [0,T)\). Combining this with the non-inflating estimate we get:

\[
v_0 \leq \text{vol}(M,g(t_i)) = \text{vol}(t_iB_{\varepsilon_i}(x_0)) \leq \sigma_2(\varepsilon_i)^4 \to 0
\]

as \(i \to \infty\), which is a contradiction. \(\square\)

## 4. The regular part of the flow

We wish to show that the limit as \(t \not\to T\) (in some to be characterised sense) of \((M,g(t))\) is a \(C^0\) Riemannian orbifold \((X,d_X)\) with at most finitely many orbifold points and that \((X,d_X)\) is smooth away from the orbifold points. In the static case, M. Anderson showed results of this type for sequences of Einstein manifolds whose curvature tensor is bounded in the \(L^{n/2}\) sense: see for example Theorem 1.3 in [And1]. Similar results were shown independently by [BKN] (see Theorem 5.5 in [BKN]). See also [Tian]. In the paper [AnCh], the condition that the manifolds have Ricci curvature bounded from above and below or bounded Einstein constant was replaced by the condition that the Ricci curvature is bounded from below. To deal with this situation the authors introduced the \(W^{1,p}\) harmonic radius, which we also use here.

To prove the convergence to an orbifold and to obtain information on the orbifold points we require regularity estimates for regions where \(\int_{\text{B}_r(x)} |\text{Riem}(g(t))|^2 d\mu_{g(t)}\) is small. Regularity estimates in the static case (for example the Einstein case) were shown for example in Lemma 2.1 in [And2]. We show that for certain so called good times \(t < T\), which are close enough to \(T\), that if \(\int_{\text{B}_r(x)} |\text{Riem}(g(t))|^2 d\mu_{g(t)} \leq \varepsilon_0\) is small enough, where \(r(t) = R\sqrt{T - t}\) for some large \(R > 0\), then we will have time dependent bounds on the metric on the ball \(\text{B}_{\sqrt{T/2}}(x)\) for later times \(s\), \(t \leq s < T\): see Theorem 4.5 below for the explicit bounds (the constants \(\varepsilon_0, R\) appearing above, will not depend on \(x\)). That is, we have a fixed set \(\text{B}_{\sqrt{T/2}}(x)\) where we obtain our estimates for later times \(s \in [t,T)\) (that is, the set \(\text{B}_{\sqrt{T/2}}(x)\) doesn’t depend on \(s\)). Furthermore, we show that the metric \(g(s)\) on the ball \(\text{B}_{\sqrt{T/2}}(x)\) is \(C^0\) close to the metric \(g(l)\) on \(\text{B}_{\sqrt{T/2}}(x)\) if \(s, l \in [t,T)\) and \(|s - l|\) is small enough.
In order to obtain our regularity estimates we require a number of ingredients. The estimates from the previous section, a slightly modified version of a result from [And1] and [AnCh] on the $W^{1,p}$ harmonic radius (see also Lemma 4.5 of [Petersen]), a Nash-Moser-de Giorgi argument, and the Pseudolocality result of G. Perelman (see Theorem 10.1 of [Pe1]) being the main ones. The Nash-Moser-de Giorgi argument which we use is a modified version of that given in the paper [Li]. The proofs in the paper of [Li] are written for a four dimensional setting, and can be adapted to our setting.

Before stating the theorem we introduce some notation, which we will also use in the subsequent sections of this paper.

Let $(M^4, g(t))_{t \in [0,T)}$ be a solution to Ricci flow satisfying the basic assumptions. In Theorem 3.6 of [Si1], it was shown that

$$\int_S^R \int_M |\text{Rc}|^4(\cdot, t) d\mu_{g(t)} dt \leq K_0 = K_0(M, g_0, T) < \infty$$

for $S < R \leq T$. In particular, for any $0 < r < \frac{T}{4}$, and $1 \geq \sigma > 0$, we can find a $t \in [T - (1 + \sigma)r, T - r]$ such that

$$\int_M |\text{Rc}|^4(\cdot, t) d\mu_{g(t)} \leq \frac{2K_0}{\sigma r}$$

If not, then we can find $\sigma$ and $r$ such that $\int_M |\text{Rc}|^4(\cdot, t) d\mu_{g(t)} > \frac{2K_0}{\sigma r}$ for all $t \in [T - (1 + \sigma)r, T - r]$, and hence

$$\int_{T - (1 + \sigma)r}^{T - r} \int_M |\text{Rc}|^4(\cdot, t) d\mu_{g(t)} > \sigma r \frac{2K_0}{\sigma r} = 2K_0$$

which contradicts equation (4.1).

If $t := T - r < T$ is given, where $r < \frac{T}{10}$, then the argument above shows that we can always find a (nearby) $\tilde{t} \in [T - 2r, T - r]$ such that

$$\int_M |\text{Rc}|^4(\cdot, \tilde{t}) d\mu_{g(\tilde{t})} \leq \frac{2K_0}{r} = \frac{2K_0}{T - t} \leq \frac{4K_0}{T - \tilde{t}}$$

A time $\tilde{t}$ which satisfies (4.3) will be known as a $4K_0$ good time. More generally, we make the following definition.

**Definition 4.1.** Let $(M, g(t))_{t \in [0,T)}$ be a smooth solution to Ricci flow. Any $t \in [0, T)$ which satisfies

$$\int_M |\text{Rc}|^4(\cdot, t) d\mu_{g(t)} \leq \frac{C}{T - t}$$

($C > 0$) shall be called a $C$-good time. If $C = 1$, then we call such a $t$ a good time.

By modifying the above argument we see that the following is true.

**Lemma 4.2.** Let $(M^4, g(t))_{t \in [0,T)}$ be a solution to Ricci flow satisfying the basic assumptions and let $C > 0$ be given. Then there exists an $\tilde{r} > 0$ such that for all $0 < r < \tilde{r}$ the following holds. For any $\tilde{t} \in [0, T)$ with $r := T - \tilde{t}$ there exists a $t \in [\tilde{t} - r, \tilde{t}] = [T - 2r, T - r]$ which is a $C$ good time.
Remark 4.3. \( \tilde{r} \) will possibly depend on \( C, (M, g(0)) \) and \( T \) as can be seen in the proof below.

Proof. Fix \( C > 0 \) and assume the conclusion of the theorem doesn’t hold. Then we can find a sequence \( r_i \to 0 \) and \( t_i := T - r_i \not\nearrow T \) such that every \( t \in [T - 2r_i, T - r_i] \) is \( C \) good time. That is \( \int_M |\text{Rc}|^4(\cdot, t)d\mu_{g(t)} > \frac{C}{t} \) for all \( t \in [T - 2r_i, T - r_i] \). Integrating in time from \( T - 2r_i \) to \( T - r_i \) we get

\[
\int_{T - 2r_i}^{T - r_i} \int_M |\text{Rc}|^4(\cdot, t)d\mu_{g(t)}dt > C \int_{T - 2r_i}^{T - r_i} \frac{1}{T - t}dt \\
\geq \frac{C}{2r_i} \int_{T - 2r_i}^{T - r_i} dt \\
= \frac{C}{2}.
\]

Without loss of generality the intervals \( [T - 2r_i, T - r_i] \in \mathbb{N} \) are pairwise disjoint (since \( r_i \to 0 \)). Summing over \( i \in \mathbb{N} \) we get

\[
\int_0^T \int_M |\text{Rc}|^4(\cdot, t)d\mu_{g(t)}dt \geq \sum_{i=1}^{\infty} \int_{T - 2r_i}^{T - r_i} \int_M |\text{Rc}|^4(\cdot, t)d\mu_{g(t)}dt \\
\geq \sum_{i=1}^{\infty} \frac{C}{2} = \infty
\]

which contradicts the fact that \( \int_0^T \int_M |\text{Rc}|^4(\cdot, t)d\mu_{g(t)}dt < \infty \). \( \square \)

Let \( 0 < t_i \not\nearrow T, i \in \mathbb{N} \) be a sequence of times approaching \( T \) from below. We wish to show that \( (M, g(t_i)) \to (X, d) \) as \( i \to \infty \) in some to be characterised sense, where \( (X, d) \) is a \( C^0 \) Riemannian orbifold with only finitely many orbifold points. These orbifold points will be characterised by the fact that they are points where the \( L^2 \) integral of curvature concentrates as \( t_i \not\nearrow T \). To explain this more precisely we introduce some notation.

Definition 4.4. Let \( (M^4, g(t))_{t \in [0, T)} \) be a solution to Ricci flow with \( T < \infty \) satisfying the basic assumptions. A point \( p \in M \) is a \textit{regular point in} \( M \) (or \( p \in M \) is \textit{regular}) if there exists an \( r = r(p) > 0 \) such that

\[
\int_{B_r(p)} |\text{Riem}|^2(\cdot, t)d\mu_{g(t)} \leq \varepsilon_0
\]

for all times \( t \in [0, T) \), where \( \varepsilon_0 > 0 \) is a small fixed constant depending on \( (M^4, g(0)) \) and \( T \), which will be specified in the proof of Theorem 4.5 below. A point \( p \in M \) is a \textit{singular point in} \( M \) (or \( p \in M \) is \textit{singular}) if \( p \in M \) is not a regular point. In this case, due to smoothness of the flow on \([0, T)\), there must exist a sequence of times \( s_i \not\nearrow T \) and a sequence of numbers \( 0 < r_i \not\searrow 0 \) as \( i \to \infty \) such that \( \int_{B_{r_i}(p)} |\text{Riem}|^2 > \varepsilon_0 \) for all \( i \in \mathbb{N} \). We denote the set of regular points in \( M \) by \( \text{Reg}(M) := \{ p \in M \mid p \text{ is regular} \} \) and the set of singular points in \( M \) by \( \text{Sing}(M) := \{ p \in M \mid p \text{ is singular} \} \).

In this section we obtain information about regular points. In particular we will give another characterisation of the property \textit{regular}. This characterisation is implied by the following theorem (see the Corollary directly after the statement of the Theorem).
**Theorem 4.5.** Let $k \in \mathbb{N}$ be fixed, and let $(M, g(t))_{t \in [0,T)}$ be a solution to Ricci flow satisfying the basic assumptions. There exists a (large) constant $R > 0$, and (small) constants $v, \varepsilon_0 > 0$, and constants $c_1, \ldots, c_k$ such that if

$$
\int_{B_{R-t/2}(p)} |\text{Riem}|^2 \, d\mu_g(t) \leq \varepsilon_0
$$

for a good time $t$ which satisfies $|T-t| \leq v$, then $p$ is a regular point. We also show that if $p, t$ satisfy these conditions, then

$$
\exp\left(-\frac{8|s^\frac{1}{2} - s^\frac{1}{2}|}{(T-t)^\frac{1}{2}}\right) g(r) \leq g(s) \leq \exp\left(\frac{8|s^\frac{1}{2} - s^\frac{1}{2}|}{(T-t)^\frac{1}{2}}\right) g(r), \quad \forall t \leq r, s < T, \quad \text{on } B_{\sqrt{T-t}}(p)
$$

$$
\frac{1}{2} g(r) \leq g(s) \leq 2g(t), \quad \forall t \leq r, s < T, \quad \text{on } B_{\sqrt{T-t}}(p)
$$

$$
|\nabla^j \text{Riem}(x, s)|^2 \leq \frac{c_j}{(T-t)^{j+2}}
$$

$$
\forall t + \frac{T-t}{2} \leq s < T, x \in B_{\sqrt{T-t}}(p), \quad \forall j \in \{0, \ldots, k\}
$$

The constants $\varepsilon_0, R$ and $v$ depend only on $\sigma_0, \sigma_1$ from (4.1), $A$ from (4.3), and $c(g(0), T)$ from Theorem (4.5) the constants $c_j$ depend only on $j, \sigma_0, \sigma_1, A$ and $c(g(0), T)$. That is, all constants depend only on $(M, g(0))$ and $T$.

For such $p$ and $t$ we therefore have: all $x \in B_{\sqrt{T-t/2}}(p)$ are also regular (see the proof for an explanation), and there is a limit in the smooth sense (and hence also in the Cheeger-Gromov sense) of $(B_{\sqrt{T-t}}(p), g(s))$ as $s \nearrow T$.

**Remark 4.6.** The condition $\int_{B_{R-t/2}(p)} |\text{Riem}|^2 \, d\mu_g(t) \leq \varepsilon_0$ for a good time $t$ which satisfies $|T-t| \leq v$ ($v, \varepsilon_0$ as in the statement of the Theorem above) therefore implies that $p$ is regular (see the proof for an explanation). This new condition contains however more information, namely that the estimates appearing in the statement of Theorem (4.5) hold. Furthermore: to show that a point $p \in M$ is regular, we only need to find one good time $t$ with $|T-t| < v$ for which $\int_{B_{R-t/2}(p)} |\text{Riem}|^2 \, d\mu_g(t) \leq \varepsilon_0$. We do not need to show that $\int_{B_{T-t}(p)} |\text{Riem}|^2 \, d\mu_g(t) \leq \varepsilon_0$ for all $t < T$ for some fixed $r(p) > 0$.

This characterisation is useful when it comes to showing that a limit space (in a sense which will be explained later in this paper) $(X, d_X) := \lim_{t \nearrow T}(M, g(t))$ exists and when it comes to describing its structure.

**Definition 4.7.** Let $t \in (0, T)$. We say $p \in \text{Reg}_\varepsilon(M)$ if

$$
\int_{B_{R-t/2}(p)} |\text{Riem}|^2 \, d\mu_g(t) \leq \varepsilon_0,
$$

where $\varepsilon_0, R$ are from the above theorem.

**Remark 4.8.** Notice that this condition is scale invariant: if $(M, \tilde{g}(\tilde{t}))_{\tilde{t} \in [0,\tilde{T})}$ is the solution we get by setting $\tilde{g}(\tilde{t}) := cg(\tilde{t})$, $\tilde{T} := ct$, $\tilde{\varepsilon} = ct$, then

$$
\int_{B_{R-T\tilde{t}}(p)} |\text{Riem}|^2 \, d\mu_{\tilde{g}(\tilde{t})} = \int_{B_{R-T\tilde{t}}(p)} |\text{Riem}|^2 \, d\mu_{g(t)} \leq \varepsilon_0
$$
Corollary 4.9. Theorem 4.5 above shows us that \( \text{Reg}_t(M) \subseteq \text{Reg}(M) \) for all good times \( t \in (T-v, T) \). From the definition of \( \text{Reg}(M) \) we also see: for all \( p \in \text{Reg}(M) \) there exists a \( T-v < S(p) < T \) such that \( p \in \text{Reg}_t(M) \) for all good times \( t \) with \( t \in (S(p), T) \). Furthermore, Theorem 4.5 above also tells us, that for every good time \( t \in (T-v, T) \), and for all \( \varepsilon > 0 \), there exists a \( \delta > 0 \) (depending on \( t \)), such that

\[
(1 - \varepsilon)g(p, s) \leq g(p, r) \leq (1 + \varepsilon)g(p, s)
\]

\( \forall p \in \text{Reg}_t(M), \) for all \( r, s \in (t, T) \) with \( |r - s| \leq \delta. \)

Corollary 4.10. For all good times \( t \in (T-v, v) \) for all \( p \in \text{Reg}_t(M) \), where \( v \) and \( \text{Reg}_t(M) \) are as above, we have

\[
\int_{M} |\text{Rc}(g_i(-1))|^4d\mu_{g_i(-1)} = (T-t_i)^2\int_{M} |\text{Rc}(g_i(t_i))|^4(t)d\mu_{g_i(t_i)} \leq \delta_i := (T-t_i) \to 0
\]

as \( i \to \infty. \) The scale invariant inequalities (3.1) are also valid for \( g_i(-1). \)

proof (of Theorem 4.5) : Let \( t_i \not\to T \) be a sequence of good times. We scale (blow up) and shift (in time) the solution \( g \) as follows: \( g_i(t) := \frac{1}{T-t_i}g(\cdot, T + t(T-t_i)). \) Then we have a solution which is defined for \( t \in [-A_i := -\frac{T}{T-t_i}, 0) \) and \( A_i \to \infty \) as \( i \to \infty. \) Furthermore, using the fact that the \( t_i \) are good times (for the solution before scaling), we see that

\[
\int_{M} |\text{Rc}(g_i(-1))|^4d\mu_{g_i(-1)} = (T-t_i)^2\int_{M} |\text{Rc}(g_i(t_i))|^4(t)d\mu_{g_i(t_i)} \leq \delta_i := (T-t_i) \to 0
\]

Let \( B_R(p) = g_i(-1)B_R(p) \subseteq M \) be an arbitrary ball with

\[
\int_{g_i(-1)B_R(p)} |\text{Rm}|^2d\mu_{g_i(-1)} \leq \varepsilon_0 \text{ and } R \geq 4 > 0. \]

Scaling by \( \delta = \frac{1}{R^2} < 1 \) (*), (that is \( g_\delta(t) := \delta g_i(t) \) once again \( g_i(t) \)) we see that

(a) \( \int_{g_\delta(-1)B_2(p)} |\text{Rm}|^2d\mu_{g_\delta(-1)} \leq \varepsilon_0 \) and \( \int_{M} |\text{Rc}|^4d\mu_{g_i(-1)} \leq \delta_i, \)

where \( \delta_i := \delta_i/\delta^2 = (T-t_i)/\delta^2 \to 0 \) as \( i \to \infty \)

(b) we have control over non-inflating constants and non-decreasing constants:

\[\sigma_0r^4 \leq \text{vol}^{g_\delta(-1)}B_r(x) \leq \sigma_1r^4 \text{ for all } r \leq r_i \to \infty \text{ and for all } x \in g_i(-1)B_2(p).\]

the works of Anderson [And1] and Deane Yang [YangD] imply that \( B_1(p) \) is in some \( C^{0,\alpha} \) sense close to euclidean space if \( \varepsilon_0 \) is small enough, and \( i \in \mathbb{N} \) is large enough (that is if \( \delta_i = (T-t_i) \) is small enough). This is a fact about smooth Riemannian manifolds satisfying (a) and (b), and has nothing to do with the Ricci flow. We state below a qualitative version of this fact. Our proof method is essentially the same as the method used in the proof of Main Lemma 2.2 in [And1] (see Remark 2.3 (ii) there). We also use some notions from [AnCh] on the \( W^{1,p} \) harmonic radius.

Theorem 4.11. Let \( (M^4, g) \) be a smooth connected manifold without boundary (not necessarily complete) and \( B_2(p) \subseteq M \) be an arbitrary ball which is compactly contained in \( M \). Assume that
The inequality from (4.14) reads, in our case, non-collapsing estimates, the evolution equation \( \partial r \leq \sigma_0 r^4 \leq \sigma_1 r^4 \) for all \( r \leq 1 \), for all \( x \in B_{2}(p) \), \( \delta \ll a \) is fixed (and small), that is \( 0 < \delta = \delta(V) < a(\sigma_0, \sigma_1) > 0 \) such that \( r_h(g)(y) \geq V \text{ dist}_g(y, \partial(B_{\frac{3}{2}}(p))) \), for all \( r > 0 \), for all \( y \in B_{3/2}(p) \), where \( r_h(g)(y) \) is the \( W^{1,12} \) harmonic radius of \( (M, g) \) at \( y \) (see [B.1] in Appendix [B] for the definition of harmonic radius that we are using).

Remark 4.12. As noted above, this theorem does not require that the metrics involved are coming from a Ricci flow.

Remark 4.13. A different approach and a similar result is given in, respectively obtained in, the paper by Deane Yang [YangD].

Remark 4.14. Compare with Theorem 2.35 of [TZ].

A proof and the definition of the \( W^{1,12} \) harmonic radius is given in Appendix [B]. The inequality from (4.14) reads, in our case, \( r_h(g)(\delta)(y) \geq V \text{ dist}_{g(\delta)}(y, \partial(-\delta B_{3/2}(p))) \) for all \( y \in -\delta B_{3/2}(p) \) if \( (T-t_i)/\delta^2 \leq 1 \). In particular \( r_h(g)(\delta)(y) \geq \frac{1}{\delta} \) for all \( y \in -\delta B_1(p) \), if \( (T-t_i)/\delta^2 \leq 1 \). Comparing Perelman’s definition of \textit{almost euclidean} (see Theorem 10.1 in [Pe1] for the definition of \textit{almost euclidean}) with the definition of harmonic radius we are using, we see that there is a constant \( 1 > a = a(V) = a(\sigma_0, \sigma_1) > 0 \) such that \( g(-\delta)B_\delta(y) \) is almost euclidean if \( (T-t_i)/\delta^2 \leq 1 \). Notice that \( a \) doesn’t depend on \( \delta \) and hence, without loss of generality \( \delta << a \). \( \delta = \frac{1}{\sqrt{\delta}} \) and \( R > 0 \) was arbitrary up until this point, so we choose \( R^2 >> \frac{1}{4\sigma_0} \). Perelman’s first Pseudolocality result (Theorem 10.1 in [Pe1]) now tells us that

\[
|\text{Riem}(g_t(x,t))| \leq \frac{1}{\delta + t}, \text{ for all } t \in (-\delta, 0), \quad x \in g_t(B_{\delta}(y))
\]

for some constant \( \tilde{\alpha} = \hat{\alpha}(a) > 0 \), for all \( y \in -\delta B_1(p) \). Here we use that \( \delta << a \) that is \( 0 < \delta = \delta(V) << a(V) \) is chosen small so that the Pseudolocality Theorem applies on the whole time interval \( (-\delta, 0) \). Without loss of generality \( \delta << \tilde{\alpha} \) also. Now \( \delta = \delta(V) \) is fixed (and small), that is \( R = R(V) = \frac{\alpha}{\sqrt{\sigma_0}} >> 1 \) is fixed (and large). Scaling back to \( t = -1 \) (that is we set \( g_t(\tilde{t}) = \frac{R^2}{4} g_t(\frac{\tilde{t}}{R^2}) \) so that we are dealing with the solution we had before blowing down at the point \( (*) \) of the argument above: we call the solution \( \tilde{g}_t(\tilde{t}) \) once again \( g_t(t) \) for ease of reading) we have

\[
|\text{Riem}(g_t(x,t))| \leq \frac{1}{1+t}, \text{ for all } t \in (-1, 0), \quad x \in g_t(B_{\frac{1}{1+t}}(y))
\]

for all \( y \in g_{t-1}(B_{\frac{1}{2}}(p)) \). Using Shi’s estimates (see [Shi]), the non-inflating and non-collapsing estimates, the evolution equation \( \frac{\partial}{\partial t} g = -2\text{Rc} \), and the injectivity radius estimate of Cheeger-Gromov-Taylor (Theorem 4.3 in [CGT]), we get

\[
|\nabla^j \text{Riem}(g_t)(y,t)| \leq A_j, \text{ for all } t \in (-\frac{1}{2}, 0),
\]

for all \( 0 \leq j \leq K \) where \( K \in \mathbb{N} \) is fixed and large and \( A_j < \infty \) is a constant, for all \( y \in -B_{\frac{1}{2}}(p) \), as long as \( R\tilde{\alpha} \) is sufficiently large: as we chose \( \delta << \tilde{\alpha} \), this is
without loss of generality the case. Translating in time and scaling back to the original solution, we obtain the claimed curvature estimates (4.18).

We explain why all \( y \in g_i(-1)B_R/p \), are regular (in particular, \( p \) is regular). Choose \( t \) close to 0 and \( 0 < r \leq 1 \) small, so that \( tB^i_{10} + r(y) \subseteq g_i(-1)B_R/(p) \) : for every \( t < 0 \) such an \( r \) must exist in view of the fact that the solution is smooth. Then \(|\text{Riem}(\cdot,t)| \leq 10 |tB^i_{10} + r(y) \subseteq g_i(-1)B_R/(p) \) due to (4.16). Then \( sB_r(y) \) remains in \( tB^i_{10} + r(y) \subseteq g_i(-1)B_R/(p) \) for all \( s \in [t,0) \) due to (4.16) and the fact that the metric evolves according to the equation \( \frac{\partial}{\partial t} g = -\text{Rc}(g) \), and \( t \) is close to 0.

Hence \( \int_{B_r(y)} |\text{Riem}(g(t))^2(\cdot,s) d\mu_{g(t)} \leq \varepsilon_0 \) for all \( s \in [t,0) \), if \( r \) is small enough, in view of (4.16) and the non-expanding estimate.

Although these estimates show us that \( p \) is a regular point, they do not tell us that

\[
\int_{B_{R/2}(p)} |\text{Riem}(\cdot,t)| d\mu_{g(t)} \leq \varepsilon_0
\]

for all \( t \in (-1,0) \): as \( t \) gets closer to \(-1 \) from above, our estimates on the curvature, (4.16), blow up. However by appropriately modifying the arguments in (4.16) we can show that the Riemannian metrics remain close in a \( C^0 \) sense to one another on some fixed time independent region within these balls. This fact is useful when it comes to describing \((X,d_X)\) , the limit as \( t \to T \) (before scaling) of the solution \((M,g(t))_{t \in [0,T]} \), and how this limit is obtained.

Examining the setup considered in the first part of the paper (4.13) of Ye Li, we see that we are almost in the same setup: Scale back down to \( t = -\delta \) (that is do the step (*) in the argument above again), call the solution \( g_i \) again, and consider an arbitrary \( y \in g_i(-\delta)B_1(p) \) as above.

From the argument above we have

\[
|\text{Riem}(g_i(t,x,t))| \leq \frac{1}{\delta + t}, \quad \text{for all} \quad t \in (-\delta,0) \quad \text{for all} \quad x \in g_i(t)B_\delta(y)
\]

for some constant \( \tilde{a} = \tilde{a}(a) > 0 \), and \( \delta << \tilde{a} < a \).

In order to see that we are almost in the same situation as YeLi, we shift in time by \( \delta \): that is fix \( i \) and define \( g(t) = g_i(t + \delta) \). This means that the old time 0 (where the flow possibly becomes singular) is now time \( \delta \) and the old good time \( -\delta \) is now the good time 0. Then we have

\[
|\text{Riem}(g_i(x,t))| \leq \frac{1}{t}, \quad \text{for all} \quad t \in (0,\delta) \quad \text{for all} \quad x \in g_i(s)B_\delta(y),
\]

for all \( y \in g_i(0)B_1(p) \), where \( \tilde{a} \) depends only on \( a \) which depends only on \( \sigma_0, \sigma_1 \), and we have chosen \( \delta \) so that \( \delta << \tilde{a} \leq a \). Without loss of generality, we may assume \( \tilde{a} = 2 \) for this argument. If not, then scale so that it is: we still have \( 0 < \delta << \tilde{a} \) is still as small we we like (but fixed).

This solution also satisfies

\[
|\text{Rc}(g(t))| \leq \hat{\delta}_i, \quad \text{with} \quad \hat{\delta}_i \to 0 \quad \text{as} \quad i \to \infty \quad \text{(by scaling we have changed the constants} \quad \hat{\delta}_i \quad \text{above by a fixed factor:} \quad \hat{\delta}_i = \frac{\delta_i}{(10i)^2}).
\]
necessary modifications, the following:

we get

In our arguments, we will replace this function by a time dependent cut-off function

\[ (4.22) \]

Examining Lemma 1, Lemma 2, Lemma 3 and Theorem 2 of [Li], we see that this is exactly the setup of that paper, call \( \mu := (K_0)^{\frac{1}{4}} \), except for the condition

\[ 1/2g(y) < g(t) < 2g(s) \]

for all \( 0 < t < s < \delta \), which is also assumed there. We are considering the case that \( u \) and \( f \) of the paper by Ye Li are \( u := |\text{Riem}| \) and \( f := |\text{Rc}| \). The extra assumption \( 1/2g(y) < g(t) < 2g(s) \) for all \( 0 < t < s < \delta \) is used in [Li] to construct a time independent cut-off function (in Lemma 3 of [Li]), which is also used in Lemma 1 and Lemma 2 of [Li]) for \( 0 < r' < r \). This cut-off function \( \varphi : M \to \mathbb{R} \) is smooth and satisfies \( \varphi|_{B_r(y)} = 1 \), \( \varphi = 0 \) on \( (B_r(y))^c \), \( |\nabla \varphi|_{g(y)} \leq \frac{2}{r} \) and \( |\nabla \varphi|_{g(t)} \leq 2|\nabla \varphi|_{g(y)} \leq \frac{4}{r} \). We will only consider \( 1 \geq r, r' \geq \frac{1}{4} \).

In our arguments, we will replace this function by a time dependent cut-off function \( \varphi(x, t) \) using the method of Perelman. This new \( \varphi \) satisfies

\[ \frac{\partial}{\partial t} \varphi \leq \Delta \varphi + \frac{c}{(r - r')^2} + \frac{c\varphi}{t} \]

\[ |\nabla \varphi|_{g(t)}^2 \leq \frac{2}{(r - r')^2} \]

\[ \varphi|_{B_r(y)} = 1, \]

\[ \varphi|_{(B_r(y))^c} = 0, \]

for all \( t \leq S(c_1) \), wherever the function differentiable is, where \( S(c_1) > 0 \) and \( c = c(c_1) \), where \( c_1 \) is a constant satisfying \( |\text{Riem}| \leq \frac{2}{r} \) on \( B_{4}(y) \) : in our case \( c_1 = 1 \). Using this new \( \varphi \) in the argument given in [Li], we obtain, after making necessary modifications, the following:

\[ |\text{Rc}(:, t)| \leq \frac{\delta^4}{t^{3/4}} \]

on \( B_{3/4}(y) \),

\[ (4.22) \]

as long as \( (T - t_i) \leq \delta(\sigma_0, \sigma_1, c(g(0), T), A) \) is small enough. See Appendix [A] for the details. In particular, translating and scaling back to the solution we had before we performed the step (*), we see that \( |\text{Rc}(y, t)| \leq \frac{\delta}{t^{3/4}} \) for all \( y \in g((-1)B_{R/2}(p)) \), for all \( t \in (-1, 0) \). Hence, integrating the evolution equation \( \frac{\partial}{\partial t}g(y, s) = -2\text{Rc}(g)(y, s) \), we get

\[ (4.23) \]

\[ g(y, s)e^{-8\delta|s|^{\frac{3}{4}} - r^{\frac{1}{4}}} \leq g(y, r) \leq g(y, s)e^{8\delta|s|^{\frac{3}{4}} - r^{\frac{1}{4}}} \]
for all $r, s \in [-1, 0)$ for all $y \in \sigma(\cdot, T) = \overline{\sigma(-1)B_{R/2}(p)}$, where $\delta > 0$ is small. Translating in time and scaling back to the original solution, we obtain (4.10). Before scaling back, note that it also implies

$$
\sigma 
$$

(4.24) \frac{1}{2} g(y, s) \leq g(y, r) \leq 2g(y, s)

for all $r, s \in [-1, 0)$ for all $y \in \sigma(\cdot, T) = \overline{\sigma(-1)B_{R/2}(p)}$. This condition is scale invariant, so translating and scaling back to the original solution, we obtain (4.7).

For later, notice, that (4.23) implies that: for all $\sigma > 0$, there exists a $\tilde{\delta} > 0$ such that,

$$
g(\cdot, s)(1 - \sigma) \leq g(\cdot, r) \leq g(\cdot, s)(1 + \sigma)
$$

for all $r, s \in (-1, 0]$ such that $|r - s| \leq \tilde{\delta}$ on $\overline{B_{R/2}(p)}$. Examining the argument above, we see that the results are correct for any good time $t_i \in (0, T)$, as long as $(T - t_i) \leq v(\sigma_0, \sigma_1, A, c(g(0), T))$ is small enough. This finishes the proof.

**(End of the proof of Theorem 4.5)**

**proof of the Corollary 4.10**

Let $x, y, t, s$ be as in the statement of the corollary. Scale to the situation as in the proof of Theorem 4.5. Let $\gamma : [0, 1] \rightarrow M$ be a length minimising geodesic between $x$ and $y$ with respect to the metric $g(-1)$. The curve doesn’t leave $\overline{B_{\frac{R}{4}}(p)}$, and hence, using (4.24), $d(x, y, s) \leq L_s(\gamma) \leq 2L_{-1}(\gamma) = 2d(x, y, -1)$. Now let $\sigma : [0, 1] \rightarrow M$ be a length minimising geodesic between $x$ and $y$ with respect to $g(s)$. If $\sigma$ doesn’t leave $\overline{B_{\frac{R}{4}}(p)}$, then $d(x, y, -1) \leq L_{-1}(\sigma) \leq 2L_s(\sigma) = 2d(x, y, s)$, and hence $d(x, y, s) \geq \frac{1}{2}d(x, y, -1)$ in this case. If $\sigma$ leaves $\overline{B_{\frac{R}{4}}(p)}$, then let $m$ be the first point at which it does so: $\sigma(m) \in \partial \overline{B_{\frac{R}{4}}(p)}$, $\sigma(r) \in \overline{B_{\frac{R}{4}}(p)}$ for all $r < m$, and consider $\alpha = \sigma|_{[0, m]}$. Then $d(x, y, s) = L_s(\sigma) \geq L_s(\alpha) \geq L_{-1}(\alpha) \geq \frac{1}{10}d(x, y, -1)$. Hence $d(x, y, s) \geq \frac{1}{2}d(x, y, -1)$ in this case as well.

**End of the proof of Corollary 4.10**

5. Behaviour of the flow near singular points

In this section we examine the behaviour of the flow near singular points $p$. We consider a sequence of good times $t_i \nearrow T$. We will show that the singular set $\text{Sing}(M)$ can be covered by $L$ balls $(t_iB_{R(t_i)}(p_j))_{j=1}^L$ (L being independent of $t_i$) of radius $R(t_i) = C \sqrt{T - t_i}$ ($C$ a large fixed constant, which is determined in the proof of Theorem 5.1 below) at time $t_i$, where $t_i$ are good times close enough to $T$, and that the balls $t_iB_{R(t_i)}(p_j)$ with $t \in (t_i, T)$ also cover $\text{Sing}(M)$. We say nothing at this stage about the topology of these regions, or how they geometrically look. In the next sections we give more information on how singular regions look like in the limit (as $t \nearrow T$).

The results of this section are used at the end of this section to show that distance is uniformly continuous in the following sense: For all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|d(x, y, t) - d(x, y, s)| \leq \varepsilon$ for all $x, y \in M$ for all $t, s \in [0, T]$ with $|t - s| \leq \delta$. The singular set and the regular set were defined in the previous section: $\text{Reg}(M) := \{ p \in M \mid p \text{ is regular } \}$ was defined in Definition 4.4 and $\text{Reg}_r(M)$ was defined in Definition 4.7. $\text{Sing}(M) := \{ p \in M \mid p \text{ is not regular } \}$. The theorem that we prove in this section is
Theorem 5.1. Let \((M, g(t))_{t \in [0, T)}\) be a solution to Ricci flow satisfying the basic assumptions. Then there exist (large) constants \(0 < J_0, J_1, J_2 < \infty\), a (small) constant \(0 < w < \infty\), and a constant \(L \in \mathbb{N}\) such that for all good times \(s < T\) with \(|s - T| \leq w\), there exist \(p_1(s), \ldots, p_L(s) \in M\) such that

\[
\text{Sing}(M) = (\text{Reg}(M))^c 
\subset (\text{Reg}_{s}(M))^c 
\subset \bigcup_{j=1}^{L} t J_{0}\sqrt{T-s}(p_j(s)) 
\subset \bigcup_{j=1}^{L} s B_{J_1\sqrt{T-s}}(p_j(s)) 
\subset \bigcup_{j=1}^{L} r B_{J_2\sqrt{T-s}}(p_j(s)) 
\tag{5.1}
\]

for all \(s \leq t, r < T\).

Remark 5.2. Notice that for fixed \(s\), the sets \(B_{J_2} \sqrt{T-s}(p_j(s))\) in the statement of the theorem don’t depend on \(t\) or \(r\) \((s \leq t, r < T)\), but \(B_{J_1} \sqrt{T-s}(p_j(s))\) and \(B_{J_2} \sqrt{T-s}(p_j(s))\) do.

Remark 5.3. Using the estimates of the previous section and this covering, we will obtain as a corollary, that the distance function is uniformly continuous in time (see Theorem 5.2).

Proof. Let \((M, h)\) be a Riemannian manifold with \(\int_M |\text{Riem}(h)|^2 \leq K_0 < \infty\). Let \(R > 0\) be given fixed. Assume there is some point \(p_1\) with \(\int_{B_{R}(p_1)} |\text{Riem}|^2 \geq \varepsilon_0\). Then we look for a ball \(B_R(p_2)\) which is disjoint from \(B_R(p_1)\) and satisfies \(\int_{B_R(p_2)} |\text{Riem}|^2 \geq \varepsilon_0\). We continue in this way as long as it is possible to do so. This leads to a family of pairwise disjoint balls \((B_R(p_j))_{j \in \{1, \ldots, L\}}\) such that \(\int_{B_R(p_j)} |\text{Riem}|^2 \geq \varepsilon_0\) for all \(j \in \{1, \ldots, L\}\). We define

\[
\begin{align*}
B_R & := B_R(h) := \bigcup_{j=1}^{L} B_{2R}(p_j) \\
\Omega_R & := \Omega_R(h) := M \setminus \bigcup_{j=1}^{L} B_{2R}(p_j).
\end{align*}
\tag{5.2}
\]

From the definition of \(\Omega_R\) it follows that \(\int_{B_R(x)} |\text{Riem}|^2 \leq \varepsilon_0\) for all \(x \in \Omega_R\).

Using \(\int_M |\text{Riem}|^2(h) d\mu_h \leq K_0\), we see that we have an upper bound \(L \leq \frac{K_0}{\varepsilon_0}\) for the number of balls constructed in this way.

Notice that for fixed \(R\) this construction is not unique: by choosing the balls in the construction differently we obtain a different \(B_R\), respectively \(\Omega_R\).

If \((M, g(t))_{t \in I}\) is a solution to Ricci flow, \(I\) an interval, then \(\Omega_R(g(t))\) and \(\Omega_R(g(t))\) and \(\Omega_R(g(t))\) will denote \(\text{Reg}(M)\) and \(\text{Reg}(M)\) for any \(t \in I\). Take a sequence of good times \(t_i \uparrow T\), and assume we have scaled as in the proof Theorem 4.3, above, to obtain a solution \((M, g(t))_{t \in (-\Delta, 0)}\). Using the characterisation of the regular set given in Theorem 4.5 and using the \(R\) appearing there, we see that \(\Omega_R = \text{Reg}(M)\) and hence \(\text{Sing}(M) = M \setminus \text{Reg}(M) \subseteq M \setminus \Omega_R\).

We wish to show that distance is not changing too rapidly near and in \(\text{Reg}(M)\) and \(\Omega_R\).

In order to explain this statement more precisely, and to explain the argument which proves the statement, we assume for the moment that there is only one ball \(B_{2R}(p_j)\) coming from the above construction of \(B_R(g(-1))\) and we call this ball \(B_{2R}(p)\). Note that for each \(i\), we may obtain a different point \(p_i\) depending on
Assume that there is some time \( t \). We wish to show that dist(\( p, \partial G, t \)) \( \leq J/8 \) for all \( t \in [-1,0) \).

We define \( G := (-1)^{-1}B_{2J}(p) \), for some large \( J \gg R \) fixed, and \( H := (-1)^{-1}B_J(p) \). It follows, that \( H^c \subseteq (-1)^{-1}B_{2R}(p)^c = \{ M \setminus (-1)^{-1}B_{2R}(p) \} \subseteq \text{Reg}_{-1}(M) \subseteq \text{Reg}(M) \). Hence, using (12.3), we have \( \frac{1}{8}g(x,t) \leq g(x,-1) \leq 8g(x,t) \) for all \( x \in H^c \cap G \) for all \( t \in [-1,0) \).

We may assume that dist(\( g(t = -1) \)) is as large as we like, as we just noted. We have \( \text{dist}(g(t)) \geq \frac{J}{8} \) for all \( t \in (-1,0) \) as we explain now. Any smooth regular curve \( \gamma : [0,1] \rightarrow M \) \( J' \) which goes to infinity, and hence the diameter of the resulting solution is as large as we like at all times).

Note by construction \( G \cap H^c \neq \emptyset \) as the diameter of the solutions we are considering is as large as we like, as we just noted. We have \( J/8 \geq \text{dist}(g(t)) \geq \frac{J}{8} \) for all \( t \in (-1,0) \) as we explain now. Any smooth regular curve \( \gamma : [0,1] \rightarrow M \) \( J' \) which goes to infinity, and hence the diameter of the resulting solution is as large as we like at all times).

\[
L_{g(t)}(\gamma) = \int_0^1 g(\gamma(r), t)(\gamma'(r), \gamma'(r))dr 
\geq \frac{1}{8} \int_0^1 g(\gamma(r), t = -1)(\gamma'(r), \gamma'(r))dr
\geq \frac{1}{8} \text{dist}(g(t = -1))(\partial G, \partial H) = \frac{J}{8}
\]

in view of the definition of \( G \) and \( H \). Hence

\[
\text{dist}(g(t))(\partial G, \partial H) \geq \frac{J}{8} \tag{5.4}
\]

for all \( t \in [-1,0) \). Notice that this means

\[
\text{dist}(g(t))(\partial G, \partial G) \geq \text{dist}(g(t))(H, \partial G) = \text{dist}(g(t))(\partial H, \partial G)
\]

which is larger than or equal to \( J/8 \) in view of equation (5.4). Similarly, for \( z \in H^c \cap G \), let \( \gamma : [0,1] \rightarrow M \) be the radial geodesic with respect to the metric at time \( t = -1 \) coming out of \( p \), starting at \( z \) and stopping at \( \partial G \). We have \( \gamma([0,1]) \subseteq H^c \cap G \) and hence

\[
\text{dist}(g(t))(z, \partial G) \leq L_{g(t)}(\gamma)
\]

\[
\geq \int_0^1 \sqrt{g(\gamma(r), t)(\gamma'(r), \gamma'(r))}dr
\]

\[
\leq 8 \int_0^1 \sqrt{g(\gamma(r), t = -1)(\gamma'(r), \gamma'(r))}dr
\]

\[
\leq 8L_{g(-1)}(\gamma) \leq 8J
\]

(5.6)

That is,

\[
\text{dist}(g(t))(z, \partial G) \leq 8J \text{ for all } z \in H^c \cap G, t \in [-1,0) \text{ and }
\]

\[
\text{dist}(g(t))(\partial G, \partial H) \leq 8J \text{ for all } t \in [-1,0)
\]

(5.7)

We wish to show that dist(\( p, \partial G, t \)) is bounded by a constant independent of time.

**Claim:** dist(\( p, \partial G, t \)) \( \leq J^5 \)

Assume that there is some time \( t \in (-1,0) \) with dist(\( p, \partial G, t \)) \( \geq N \geq J^5 \). Choose
\( q \in \partial G \) such that \( d(p, q, t) = N \). This part of the argument was inspired by the argument given in the proof of Claim 5.1 in the paper \[ \text{Topping}. \] Take a distance minimising geodesic \( \gamma : [0, N] \to M \) from \( p \) to \( q \), at time \( t \), which is parameterised by arclength. Consider points

\[
z_0 := \gamma(0), z_1 := \gamma(1), z_2 := \gamma(2), \ldots, z_N := \gamma(N) = q.
\]

Without loss of generality \( J \in \mathbb{N} \). From the above, we see that the first \( N - 16J \) points \( z_0, \ldots, z_{N-16J} \) must lie in \( H \), as we now explain. If not, then let \( z_i = \gamma(i) \) be the first point with \( i \leq N - 16J \) such that \( z_i \notin H \). Then we could join the point \( z_{i-1} = \gamma(i-1) \) to \( \partial G \) by a geodesic whose length w.r.t to \( g(t) \) is less than \( 8J + 2 \), in view of \( (5.7) \). This would result in a path from \( p \) to \( \partial G \) at time \( t \) whose length is less than \( N \) which is a contradiction. Also, using equation \( (5.4) \), we see that \( tB_1(z_i) \subseteq G \) for all \( 0 \leq i \leq N - 16J - 1 \) \((z_i \in H \) for such \( i \), so to reach \( \partial G \) we must first reach \( \partial H \) and then reach \( \partial G \): any such path must have length larger than \( J/8 \gg 1 \).

For \( i \in \{1, \ldots, N-1\} \), the ball \( tB_1(z_i) \) is disjoint from all other balls \( tB_1(z_j) \), for all \( j \in \{0, \ldots, N\} \) except for its two immediate neighbours \( tB_1(z_i-1) \) and \( tB_1(z_i+1) \), since \( \gamma \) is distance minimising implies \( |\gamma|_t \) is distance minimising for all intervals \( I \subseteq [0, N] \). Hence: for \( i \neq 0 \) we have \( tB_1(z_i) \cap tB_1(z_j) = \emptyset \) as long as \( j \neq i - 1 \) and \( j \neq i + 1 \), where \( i \in 1, \ldots, N - 16J \).

Using the non-collapsing estimate we see that

\[
\text{vol}(G, g(t)) \geq \text{vol}(\bigcup_{i=1}^{N-16J-1} tB_1(z_i)) \\
\geq \text{vol}(\bigcup_{i=1}^{N-16J-1/2} tB_1(z_{2i})) \\
= \sum_{i=1}^{(N-16J-1)/2} \text{vol}(tB_1(z_{2i})) \\
\geq \sum_{i=1}^{(N-16J-1)/2} \sigma_0 = \sigma_0(N - 16J - 1)/2
\]

(5.8)

On the other hand, \( \text{vol}(G, g(t)) \leq e^2 \text{vol}(G, g(-1)) = e^2 \text{vol}(-1B_{2J}(p)) \leq e^2 \sigma_1 64J^4 \) in view of the non-expanding estimate and the fact that \( G \) is defined independently of time (here we used the fact that \( \frac{d\mu_{g(t)}}{dt} \leq d\mu_{g(t)} \)). This leads to a contradiction since, \( N = J^5 > 16J + \frac{e^2 \sigma_1 128J^4}{\sigma_0} \) if \( J \) is large enough and \( t_i \) is close enough to time \( T \) before scaling: we need \( t_i \) close to \( T \) to guarantee that the non-expanding and non-collapsing estimates hold for balls (after scaling) of radius \( 0 \leq r \leq 2J \).

**This finishes the proof of the claim.**

Note, that this estimate and \( (5.5) \) imply that

\[
tB_{J/8}(p) \subseteq G = t=1B_{2J}(p) \subseteq B_{J/4}(p)
\]

(5.9)

for all \( t, r \in [-1, 0] \). Repeating the argument for \( J/10 \) instead of \( J \), we get

\[
tB_{J/8}(p) \subseteq G = t=1B_{2J}(p) \subseteq B_{J/10J}(p) \subseteq G := t=1B_{2J}(p) \subseteq B_{J/25}(p).
\]

(5.10)

for all \( t, r \in [-1, 0] \), and \( \text{Sing}(M) \subseteq G \). This implies

\[
\text{Sing}(M) = (\text{Reg}(M))^c \subseteq (\text{Reg}_{-1}(M))^c \subseteq t=1B_{2J}(p) \subseteq B_{J/10J}(p).
\]
for all $t, r \in [-1, 0]$.

The general case is as follows. We wish to cluster those points $i^tp_k$ (the centre points of the balls appearing in the construction of $B_R(g(-1))$) together if they satisfy the condition: $\text{dist}(g(t = -1))(i^tp_k, i^tp_l) \leq \Lambda$ if $i^tp_k, i^tp_l \in {}^iT_s$ for some $s \in \{1, \ldots, \tilde{L}\}$, or

\[ \text{dist}(g(t = -1))(i^tp_k, i^tp_l) \to \infty \text{ as } i \to \infty \text{ if } i^tp_k \in {}^iT_s \text{ and } i^tp_l \in {}^iT_v \text{ and } s \neq v, s, v \in \{1, \ldots, \tilde{L}\}. \]

We explain now how the sets $T_1$ are constructed. Fix $k, l \in \{1, \ldots, L\}$. If there is a subsequence in $i$ such that after taking this subsequence dist$((g(t = -1))(i^tp_k, i^tp_l)) \to \infty$ as $i \to \infty$, then take this subsequence. Do this for all $k, l \in \{1, \ldots, L\}$. As the index set $\{1, \ldots, L\}$ is finite, after taking finitely many subsequences, we will arrive at the following situation: there exists a constant $\Lambda < \infty$ such that for all $k, l \in \{1, \ldots, L\}$ one of the following two statements is true:

- dist$((g(t = -1))(i^tp_k, i^tp_l) \rightarrow \infty$ for all $i$ or
- dist$((g(t = -1))(i^tp_k, i^tp_l) \leq \Lambda$ for all $i$

Now we define $T_1$ as the set of all $i^tp_k, k \in \{1, \ldots, L\}$, such that dist$((g(t = -1))(i^tp_k, i^tp_l) \leq \Lambda$ for all $i$. $T_2$ is the set of all $i^tp_k, k \in \{1, \ldots, L\}$, such that dist$((g(t = -1))(i^tp_k, i^tp_l) \leq \Lambda$ for all $i$. And so on. This gives us sets $T_1, \ldots, T_L$. Each set contains finitely many points, and for arbitrary $k, l \in \{1, \ldots, L\}$ either $T_k \cap T_l = \emptyset$ for all $i \in \mathbb{N}$ or $T_k = T_l$ for all $i \in \mathbb{N}$. For fixed $i \in \mathbb{N}$: if a set appears more than once, we throw away all copies of the set except for one. This completes the construction of the sets $T_1, \ldots, T_L$ (we drop the index $i$ again for the moment).

Take one of these sets, for example $T_1$. $BB_1$ will denote the union of the balls $B_{2R}(z)$ where $z \in T_1$. Let $i^tp_1 \in BB_1$ be arbitrary: we are rechoosing the points $i^tp_j$ (we choose exactly one point $i^tp_j$, arbitrarily, with $i^tp_j \in BB_1$, and we do this for each $j \in \{1, \ldots, L\}$). Define $G_1 := {}^{t=-1}B_{2J}(i^tp_1)$, $H_1 := {}^{t=-1}B_J(i^tp_1)$ where $J >> \max(\Lambda, R)$ is large but fixed (independent of $i$). Arguing as in the case of one point as above, we see that (for $i$ large enough)

$$G_1 := {}^{t=-1}B_{2J}(p) \subseteq {}^{r}B_{\frac{1}{2}J}^+(p) \subseteq \tilde{G}_1 = {}^{t=-1}B_{2J}(p) \subseteq {}^{r}B_{J}^+(p)$$

for all $p \in BB_1$ (the choice of $i^tp_1 \in BB_1$ was arbitrary), for all $t, r \in [-1, 0]$. Note that we need $i$ large enough here, to guarantee that all other sets $T_2, \ldots, T_L$ do
not interfere with the arguments presented above: that is, we can guarantee that $H_i \cap G_i \subseteq \text{Reg}_{-1}(M)$ and $\overline{H_i \cap G_i} \subseteq \text{Reg}_{-1}(M)$. Now do the same for the other sets $BB_j$, $j \in \{1, \ldots, \hat{L}\}$.

We call the constant \( L \) once again \( L \). Hence,

\[
\text{Sing}(M) \subseteq (\text{Reg}_{-1}(M))^c \\
\subseteq \bigcup_{j=1}^L (iB_{j0}(t_pj)) \\
\subseteq \tilde{G} = \bigcup_{j=1}^L (iB_{j1}(t_pj)) \\
\subseteq \bigcup_{j=1}^L (iB_{j2}(t_pj))
\]

(5.13)

for all \( t, r \in [-1, 0] \), where \( J_0 := \frac{1}{10}J^5 \), \( J_1 := 2J^5, J_2 := J^{25} \).

Note, that by construction we have \( d(-1)(-B_{J1}(t_pj), -B_{J2}(t_pk)) \to \infty \) as \( i \to \infty \) for \( j \not= k \).

Scaling and translating back to the original solution, we get

\[
\text{Sing}(M) \subseteq (\text{Reg}_s(M))^c \\
\subseteq \bigcup_{j=1}^L (iB_{j0}\sqrt{t_r}(i_pj)) \\
\subseteq \tilde{G} = \bigcup_{j=1}^L (iB_{j1}\sqrt{t_r}(t_pj)) \\
\subseteq \bigcup_{j=1}^L (iB_{j2}\sqrt{t_r}(t_pj))
\]

(5.14)

for all \( t, r \in [t_1, T) \).

The proof of the claim of the theorem is as follows. Assume the conclusion of the theorem is false. Then for any constants \( J_0, J_1, J_2 \), we can find good times \( t_i \in (T - w_i, T) \), where \( w_i \to 0 \), such that we cannot find points \( p_1(t_1), \ldots, p_L(t_1) \), with \( L \leq \frac{K_0}{s_0} \) for which \( 5.11 \) holds. Taking a subsequence, as above, and choosing \( p_1(t_i) = t_{p1}, \ldots, p_L(t_i) = t_{pL} \) leads to a contradiction if \( i \) is large enough. Note at first that it could be that \( L = L(s) \leq L \frac{K_0}{s_0} \) depends on \( s \). But by adding regular points \( p_{L(s)+1}(s), \ldots, p_{L(s)s}(s) \) which are in \( \text{Reg}_s(M) \), and satisfy \( \text{dist}(s)(p_j(s), p_{j+1}(s)) \geq \sigma_0 > 0 \), for all \( i \geq L(s) + 1 \), for all \( j \in \{1, \ldots, \frac{K_0}{s_0}\} \), the conclusion of the theorem is still correct, and the comments which follow this proof are still valid.

\( \square \)

Remark 5.4. Note, that in the construction above, \( d(-1)(-B_{J1}(t_pj), -B_{J2}(t_pk)) \to \infty \) as \( i \to \infty \) for all \( j \not= k \) (before scaling back). Hence, any smooth curve \( \gamma : [0, 1] \to M \) which lies in \( (\bigcup_{j=1}^L (-B_{J1}(t_pj))^{c} \) and has \( \gamma(0) \in \partial(-B_{J1}(t_pj)) \) and \( \gamma(1) \in \partial(-B_{J2}(t_pk)) \) must have \( L_i(t) \geq N(i) \) for all \( t \in [-1, 0] \) with \( N(i) \to \infty \) as \( i \to \infty \) (in \( (\bigcup_{j=1}^L (-B_{J1}(t_pj))^{c} \) we have \( 10g(t) \leq g(-1) \leq 10g(t) \) for all \( t \in [-1, 0] \)). Hence \( d(t)(-B_{J1}(t_pj), -B_{J2}(t_pk)) \geq N(i) \) for all \( t \in [-1, 0] \) with \( N(i) \to \infty \) as \( i \to \infty \). Hence, without loss of generality we can assume that the \( p_j(s) \) in the statement of the Theorem satisfy

\[
d(t)(iB_{J1}(\sqrt{t_r}p_j(s)), iB_{J2}(\sqrt{t_r}p_k(s))) \geq N(s)\sqrt{T-s}
\]

for all \( t \in (s, T) \) for all \( j \not= k \), where \( N(s) \to \infty \) as \( s \nearrow T \). That is: the new claim is the claim of the Theorem \( 5.1 \) but with the extra claim \( 5.15 \). The proof is: repeat the contradiction argument at the end of the proof above for this new claim, using the information mentioned at the beginning of this remark.
Remark 5.5. Note that in the conclusion of the theorem, we may also assume, that
\[ tB_{J^5\sqrt{T-s}}(p_j(s)) \subseteq sB_{16J^5\sqrt{T-s}}(p_j(s)) \]
(5.16)
for all \( r, t \in [s, T) \), for all \( j \in \{1, \ldots, L\} \) holds (not just for the union of the balls).
Repeating this part of the proof for larger \( J \), but keeping the same \( p_j(s) \), we see that in fact the following is also true:
\[ tB_{K\sqrt{T-s}}(p_j(s)) \subseteq sB_{16K\sqrt{T-s}}(p_j(s)) \subseteq tB_{K\sqrt{T-s}}(p_j(s)) \]
(5.17)
for all \( r, t \in [s, T) \), for all \( j \in \{1, \ldots, L\} \) for all \( K \geq J^5 \in \mathbb{R}^+ \) as long as \( |T-s| \leq w(K) \) is small enough, for all good times \( s \), in view of Remark 5.4 from above.

As a corollary we obtain that the distance is uniformly continuous in time. We explain this in the following.
Let \( x, y \in M \) and \( t \in [t_i, T) \) and \( \gamma : [0, 1] \to M \) be a distance minimizing curve with respect to \( g(t) \) from \( x \) to \( y \), \( d_t(x, y) = L_t(\gamma) \), \( t_i \) a good time close to \( T \). We use the notation \( L_t(\sigma) = L_{g(t)}(\sigma) \) here, to denote the length of a curve \( \sigma \) with respect to \( g(t) \).
We modify the curve \( \gamma \) to obtain a new curve \( \tilde{\gamma} : [0, 1] \to M \) in the following way: if \( \gamma \) reaches the closure of the ball \( tB_{J^5\sqrt{T-t_i}}(p_k) \) (here, \( p_k = p_k(t_i) \), \( k \in \{1, \ldots, L\} \) and \( J_0, J_1, J_2 \) are from the above construction) at a first point \( \gamma(r) \) then let \( \gamma(\tilde{r}) \) be the last point which is in the closure of the ball \( tB_{J^5\sqrt{T-t_i}}(p_k) \) (it could go out and come in a number of times). Remove \( \gamma(\tilde{r}, \ldots, \tilde{r}) \) from the curve \( \gamma \). In doing this we obtain the finite union of at most \( L + 4 \) curves \( \tilde{\gamma}_j \). Call this finite union \( \tilde{\gamma} \) and consider it as a curve with finitely many discontinuities.

The new \( \tilde{\gamma} \) has
\[ L_t(\tilde{\gamma}) \leq L_t(\gamma) = d(x, y, t) \]
(5.18)
Now \( (\cup_{k=1}^L tB_{J_0\sqrt{T-t_i}}(p_k))^c \subseteq \text{Reg}_{t_i}(M) \) (\( J_0 \) coming from 5.1 above), as we saw above, and the Riemannian metric is uniformly continuous (in time) on \( \text{Reg}_{t_i}(M) \) for good times \( t_i \). That is, for all \( \epsilon > 0 \) there exists a \( \delta(\epsilon, t_i) > 0 \) such that
\[ (1-\epsilon)g(y, t) \leq g(y, s) \leq (1+\epsilon)g(y, t) \]
(5.19)
for all \( y \in (\cup_{k=1}^L tB_{J_0\sqrt{T-t_i}}(p_k))^c \) for all \( t_i \leq t, s \leq T, |t-s| \leq \delta \) in view of (5.11) and the fact that \( y \in (\cup_{k=1}^L tB_{J_0\sqrt{T-t_i}}(p_k))^c \subseteq \text{Reg}_{t_i}(M) \). Hence \( L_t(\gamma) \geq L_s(\gamma) - \epsilon \) for all \( T-\delta \leq t, s \leq T \) in view of the fact that the diameter of the manifold is bounded: more precisely, \( L_t(\gamma) \geq \frac{1}{1+\epsilon} L_s(\tilde{\gamma}) = (1-\frac{\epsilon}{1+\epsilon})L_s(\gamma) \geq L_s(\gamma) - \epsilon L_s(\gamma) \), and \( L_s(\gamma) \leq (1+\epsilon)L_t(\gamma) \leq (1+\epsilon)d(x, y, t) \leq 2D \), in view of (5.18) and (5.19), and hence
\[ L_t(\tilde{\gamma}) \geq L_s(\tilde{\gamma}) - 2D \epsilon \]
(5.20)
for all \( T-\delta \leq t, s \leq T \) as claimed. Putting (5.18) and (5.20) together we get
\[
\begin{align*}
    d(x, y, t) &\geq L_t(\tilde{\gamma}) \\
              &\geq L_s(\tilde{\gamma}) - 2D \epsilon \\
              &\geq d(x, y, s) - 2L \epsilon - 2D \epsilon.
\end{align*}
\]
The last inequality can be seen as follows: when $\gamma$ reaches a ball $B_{\sqrt{T-t_i}(p_k)}$, it must also be in $B_{\sqrt{T-t_i}(p_k)}$, by estimate (5.10). So the two points of discontinuity on $\gamma$ may be joined smoothly by a curve with length (with respect to $g(s)$) at most $2\sqrt{T-t_i}$, which is without loss of generality less than $\varepsilon$. Doing this with all of the points of discontinuity (that is with all the balls), we obtain a new continuous curve $\hat{\gamma}$ from $x$ to $y$ with length

$$L_s(\hat{\gamma}) \leq 2L_s(\gamma) \leq \frac{d(x,y) - 2L + d(x,y,s) - 2L_s(\gamma)}{2\varepsilon}$$

as claimed.

Swapping $s$ and $t$ in this argument gives us

$$|d(x,y,t) - d(x,y,s)| \leq C\varepsilon (5.22)$$

for all $T - \delta \leq t, s \leq T$, where $x, y \in M$ are arbitrary, and the constant $C$ appearing here does not depend on the choice of $x, y \in M$. Smoothness of the flow (and bounded diameter of $M$) for $t < T$ implies that:

**Theorem 5.6.** Let $(M, g(t))_{t \in [0,T]}$ be a smooth solution on a compact manifold satisfying the basic assumptions. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|d(x,y,t) - d(x,y,s)| \leq \varepsilon$$

for all $x, y \in M$ for all $t, s \in [0,T]$ with $|t-s| \leq \delta$.

6. **Convergence to a length space**

The results of the previous sections imply that $(M, d(g(t))) \to (X, d_X)$ in the Gromov-Hausdorff sense as $t \to T$, where $(X, d_X)$ is a metric space, and that away from at most finitely many points $x_1, ..., x_L \in X$ we have that $X \setminus \{x_1, ..., x_L\}$ is a smooth Riemannian manifold with a natural metric and that the convergence is in the $C^k$ Cheeger-Gromov sense. Furthermore, $(X, d_X)$ is a length space (we explain all of this below).

In the paper [BZ], the authors also showed independently, with the help of estimates proved in their paper, a similar result to the result mentioned above (see Corollary 1.11 of their paper).

The previous sections of this paper give us lots of information on how well the limit $(X, d)$ will be achieved and what the limit looks like, geometrically and topologically, near singular points. We will use the results of the previous sections combined with a method of G. Tian (in [Tian]) to show somewhat more than the result mentioned at the start of this section: namely, we will show that $(X, d_X)$ is a $C^0$ Riemannian orbifold, smooth away from its singular points (this is shown in Section 8). In the last section of this paper we explain how it is possible to flow $C^0$ Riemannian orbifolds of this type using the orbifold Ricci flow and results from the paper [SimC0].

We construct the limit space $(X, d_X)$ directly using the following Lemma, which relies on the uniform continuity of the distance function (in the sense of Theorem 5.6).

**Lemma 6.1.** Let $(M, g(t))_{t \in [0,T]}$ be a solution to Ricci flow satisfying the standard assumptions. Then

$$X := \{[x] \mid x \in M\} \text{ where } [x] = [y] \text{ if and only if } d(x,y,t) \to 0 \text{ as } t \to T.$$
$X$ is well defined. Furthermore, the function $d_X : X \times X \to \mathbb{R}_0^+$,

\begin{equation}
    d_X([x],[y]) := \lim_{t \to T} d(x,y,t)
\end{equation}

is well defined and defines a metric on $X$.

**Proof.** If $d(x,y,t_i) \to 0$ for some sequence $t_i \nearrow T$, then $d(x,y,s_i) \to 0$ for all sequences $s_i \nearrow T$, in view of Theorem 5.6. This means that $[x]$ is well defined, and hence $X$ is well defined. Define $d_X([x],[y]) = \lim_{i \to \infty} d(x,y,t_i)$ where $t_i \nearrow T$ is any sequence of times approaching $T$. The limit on the right-hand side is well defined in view of the theorem on the uniform continuity of distance (Theorem 5.6) and $d_X$ is then also well defined, due to the theorem on the uniform continuity of distance (Theorem 5.6) and the triangle inequality on $d(\cdot,\cdot,t)$.

From the definition, we see that $d_X([x],[y]) = 0$ if and only if $[x] = [y]$. The triangle inequality of, and symmetry of $d_X$ follows from the triangle inequality of, and symmetry of $d(\cdot,\cdot,t)$.

This $(X,d_X)$ is the limiting metric space of $(M,d(g(t)))_{t\in[0,T]}$ in view of the theorem on the uniform continuity of distance, as we now show.

**Lemma 6.2.** Let everything be as in Lemma 6.1 above. The function $f : M \to X$ is defined by

\begin{equation}
    f(x) := [x].
\end{equation}

$f : (M,g(t)) \to (X,d_X)$ is a Gromov-Hausdorff approximation in the sense that

\begin{equation}
    |d_X(f(x),f(y)) - d(g(t))(x,y)| \leq \varepsilon(|T-t|)
\end{equation}

where $\varepsilon(r) \to 0$ as $r \searrow 0$. $f$ is continuous and surjective and hence $(X,d_X)$ is compact, precompact, connected and complete. In particular $(M,d(g(t))) \to (X,d_X)$ as $t \nearrow T$.

**Proof.** The first claim of the theorem follows immediately from the theorem on the uniform continuity of distance and the definition of $X$. Now we show that $f$ is continuous. Let $U$ be open in $X$ and $d_X B_{\varepsilon}(p) \subseteq U$. Due to the uniform continuity of the distance function, we know the following: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $f^{(d_t)} B_{\varepsilon/2}(q) \subseteq d_X B_{\varepsilon}(p)$ for all $|T-t| < \delta$ where $q$ is an arbitrary point with $f(q) = p$ (there could be lots of such points). Hence $d^{(t)} B_{\varepsilon/2}(q) \subseteq f^{-1}(U)$. Since $p \in U$ was arbitrary, and $q$ with $f(q) = p$ was arbitrary, we have shown the following: for any point $q \in f^{-1}(U)$ there exists an $\varepsilon(q)$ and a $t_{\varepsilon,q} < T$ such that $d^{(t_{\varepsilon,q})} B_{\varepsilon(q)}(q) \subseteq f^{-1}(U)$. So we can write

\begin{equation}
    f^{-1}(U) = \cup_{q \in f^{-1}(U)} d^{(t_{\varepsilon,q})} B_{\varepsilon(q)}(q).
\end{equation}

Each of the sets contained in the union is an open set in $(M,d(t))$ for any $t < T$ and hence, $f$ is continuous. Hence $(X,d_X)$ is compact, being the continuous image of a compact space, and hence complete and precompact as it is a metric space.

We have shown $f : M \to X$ is a continuous surjective map, where the topology on $X$ comes from $d_X$ and that on $M$ is the initial topology of the manifold $M$, which agrees with that coming from the metric space $(M,d(g(t)))$ for any $t < T$. Note
that the map \( f : M \to X \) is not necessarily injective: it could be that a set \( \Omega \) containing more than two points is all mapped onto one point in \( X \) by \( f \).

Let \( p_1, \ldots, p_L \) be the points constructed in the previous section (in the possibly singular region) for large \( i \). Taking a subsequence we can assume that \( f(\{p_k\}) \to x_k \) as \( i \to \infty \) for all \( k \in \{1, \ldots, L\} \) for some fixed \( x_1, \ldots, x_L \). We do not rule out the case \( x_j = x_k \) for \( j \neq k \). After renumbering the \( x_j \)s we have finitely many (we use the symbol \( L \) again) distinct points \( x_1, \ldots, x_L \), and \( x_i \neq x_j \) for all \( i \neq j, i, j \in \{1, \ldots, L\} \).

Definition 6.3. Let \([x] \in X, x \in M\). We say \([x]\) is a regular point in \( X \) if \([x]\) contains only one point, and \([x]\) is a singular point in \( X \), if \([x]\) contains more than one point.

Remark 6.4. Notice that the notion of singular point and regular point differs depending on whether the point is in \( X \) or \( M \). The following theorem gathers together properties that we have already proved and shows that there is a connection between the different notions of singular and regular.

Theorem 6.5. Let \((M, g(t))_{t \in [0, T)}\) be a solution to Ricci flow satisfying the basic assumptions, and \((X, d_X), x_1, x_2, \ldots, x_L \in X\) as above. Then

\[
\begin{align*}
(\text{i}) & \quad X\{x_1, \ldots, x_L\} \subseteq f(\text{Reg}(M)) \subseteq \text{Reg}(X) \text{ and } f(\text{Sing}(M)) \subseteq \{x_1, \ldots, x_L\}. \\
(\text{ii}) & \quad V := f^{-1}(X\{x_1, \ldots, x_L\}) \subseteq \text{Reg}(M), \text{ } V \text{ is open and } f|_V : V \to X \text{ is an open, continuous, bijective map, and hence } f|_{V} : V \to f(V) := X\{x_1, \ldots, x_L\} \text{ is a homeomorphism.}
\end{align*}
\]

Proof. (i) Take a point \( x \notin \{x_1, \ldots, x_L\} \). Then \( d_X(x, x_i) \geq \varepsilon > 0 \) for all \( j \in \{1, \ldots, L\} \) for some \( \varepsilon > 0 \). Let \([z] = x\). Remembering that \([p] \to x\) as \( i \to \infty \), we see that \( d_X([z], [p_i]) \geq \varepsilon/2 > 0 \) for all \( j \in \{1, \ldots, L\} \) if \( i \) is large enough. Fix \( i \) large. Then we can find an \( \hat{t}(i) \) such that \( d(z, [p_j, t] \geq \varepsilon/4 \) for all \( \hat{t}(i) < t < T \) near enough to \( T \), for some \( \hat{t}(i) \geq t_i \), for all \( j \in \{1, \ldots, L\} \) in view of the definition of \( d_X \). Scaling as in the proof of Theorem 4.1 (and using the notation of the proof), we see that \( d(z, [p_j, \hat{t}_i]) \to \infty \) for all \( j \in \{1, \ldots, L\} \), for some \( \hat{t}_i \leq 1 \) as \( i \to \infty \), and hence \( z \in \text{Reg}_{-1}(M) \subseteq \text{Reg}(M) \) due to (1.12), and hence \( x = [z] = f(z) \subseteq f(\text{Reg}(M)) \). This shows that \( X\{x_1, \ldots, x_L\} \subseteq f(\text{Reg}(M)) \) and hence we have shown the first inclusion of (i). Let \( z \in \text{Reg}(M) \) be arbitrary. Then \( z \in \text{Reg}(M) \) for \( t \) close enough to \( T \) by definition. Choose a good time \( t \) and scale the solution by \( \frac{1}{1-t} \) and translate the the solution in time (as in the proof of Theorem 4.1 above). Then \( z \in \text{Reg}_{-1}(M) \). Hence \( d(z, y, t) \geq \frac{1}{1-t} d(z, y, -1) \) for all \( y \in (-1, B_{\pi R}(z)) \) for all \( t \in (-1, 0) \) in view of (1.12), and \( d(z, p, t) \geq \inf_{y \in (-1, B_{\pi R}(z))} d(z, y, t) \geq \varepsilon_0 > 0 \) for all \( t \in (-1, 0) \), for all \( p \in (-1, B_{\pi R}(z)) \). for the same reason. That is \( f(z) = [z] \) is not singular, since \( \lim_{t \to 0} d(z, y, t) > 0 \) for all \( y \neq z, y \in M \). That is \( f(\text{Reg}(M)) \subseteq \text{Reg}(X) \). This shows the second inclusion of (i). Now we prove the last statement of (i). Let \( p \in \text{Sing}(M) \). Assume \( f(p) \in X\{x_1, \ldots, x_L\} \). Then we know that there exists a \( x \in \text{Reg}(M) \) such that \( f(x) = f(p) \) in view of the set inclusions just proved. But then \([x] = [p]\) and \( x \neq p \) (since \( \text{Reg}(M) \) and \( \text{Sing}(M) \) are disjoint). Furthermore \([x] \in \text{Reg}(X) \) due to the set inclusions just shown. This contradicts the definition of \( \text{Reg}(X) \). Hence, we must have \([p] = f(p) \in \{x_1, \ldots, x_L\} \). This finishes the proof of (i).
(ii) Let \( z \in f^{-1}(X \setminus \{x_1, \ldots, x_L\}) \). Then \( f(z) \in X \setminus \{x_1, \ldots, x_L\} \). If \( z \in \text{Sing}(M) \) were the case, then we would have \( f(z) \in \{x_1, \ldots, x_L\} \) from (i), which is a contradiction. Hence \( z \in \text{Reg}(M) \). That is \( V := f^{-1}(X \setminus \{x_1, \ldots, x_L\}) \subset \text{Reg}(\text{Reg}(M)) \). \( V \) is open, since \( f \) is continuous, and \( X \setminus \{x_1, \ldots, x_L\} \) is open. From the above, \( f|_V : V \to X \) is injective: assume there exists \( x, y \in V \) with \( f(x) = [x] = [y] = f(y) \). \( x \in V \) implies \( [x] \in X \setminus \{x_1, \ldots, x_L\} \) and hence \( [x] \in \text{Reg}(X) \) from (i). Combining this with \( [x] = [y] \), we see that \( x = y \) in view of the definition of \( \text{Reg}(X) \), that is \( f|_V : V \to X \setminus \{x_1, \ldots, x_L\} \) is injective. Let \( (f|_V)^{-1} : X \setminus \{x_1, \ldots, x_L\} \to V \) be the inverse of \( f|_V : V \to X \setminus \{x_1, \ldots, x_L\} \). Then \( (f|_V)^{-1} : N := X \setminus \{x_1, \ldots, x_L\} \to M \) is continuous as we now show. Assume \( [z_k] \to [z] \) in \( f(V) = X \setminus \{x_1, \ldots, x_L\} \) as \( k \to \infty \). Using the fact that \( f|_V : V \to X \) is injective, we see that there are unique points \( z_k, z \in V \) such that \( f(z_k) = [z_k] \) and \( f(z) = [z] \). Furthermore, \( z_k, z \in \text{Reg}(M) \): if \( z_k \) respectively \( z \) were in \( \text{Sing}(M) \), then we would have \( f(z_k) \) respectively \( f(z) \in X \setminus \{x_1, \ldots, x_L\} \) which is a contradiction.

Assume \( z_k \) does not converge to \( z \), \( z \in \text{Reg}(M) \) and hence we can find a good time \( t_i \) near \( \tau \) such that \( z \in \text{Reg}_{t_i}(M) \). Fix this \( t_i \). \( z_k \) doesn’t converge to \( z \) means we can find an \( \varepsilon(i) = \varepsilon(t_i) > 0 \) and a subsequence \( (z_{k,i})_{k \in \mathbb{N}} \) of \( (z_k)_{k \in \mathbb{N}} \) (depending possibly on \( i \)), such that \( d(t_i)(z_{k,i}, z) \geq \varepsilon(i) \) for all \( k \in \mathbb{N} \). Scale at a time \( t_i \) and translate as above to \( t = -1 \) (as in the proof of Theorem 4.5). Then we have \( z \in \text{Reg}_{-1}(M) \) and \( d(-1)(z_{k,i}, z) \geq \varepsilon(i) > 0 \) for all \( k \in \mathbb{N} \).

Hence (arguing as above) \( d(z, z_{k,i}, s) \geq \frac{1}{10} d(z, z_{k,i}, -1) \geq \varepsilon(i) > 0 \) for all \( z_{k,i} \in -B_{\varepsilon_0}(z) \) for all \( s \in (-1, 0) \) in view of (4.12), and \( d(z, z_{k,i}, s) \geq \inf_{y \in \partial(-B_{\varepsilon_0}(z))} d(z, y, s) \geq \varepsilon_0 > 0 \) for all \( s \in (-1, 0) \), for all \( z_{i,k} \in (-B_{\varepsilon_0}(z))^{c} \) for the same reason.

Taking a limit \( s \nearrow 0 \), we see \( d_X([z_{k,i}], [z]) \geq \varepsilon(i) > 0 \) for all \( k \in \mathbb{N} \), which contradicts the fact that \( [z_k] \to [z] \) as \( k \to \infty \).

These facts allows us to give \( X \setminus \{x_1, \ldots, x_L\} \) a natural manifold structure, as we now explain.

**Proposition 6.6.** Let everything be as in Lemma 6.5 above. \( N = X \setminus \{x_1, \ldots, x_L\} \) has a natural manifold structure and with this structure \( f|_V : V \to N \) is a diffeomorphism, \( V := f^{-1}(N) \). There is a natural Riemannian metric \( l \) on \( N \) defined by \( l := \lim_{t \nearrow \tau} f_{*}(g(t)) \).

**Proof.** For \( x \in N \), let \( \tilde{x} \in V \subset \tilde{M} \) be the unique point in \( V \) with \( f(\tilde{x}) = x \). Let \( \psi : \tilde{U} \subset V \subset \tilde{M} \to \mathbb{R}^4 \) be a smooth chart on \( M \) with \( \tilde{x} \in \tilde{U} \), and let \( U := f(\tilde{U}) \). \( U \) is open from the above. Define a coordinate chart \( \varphi : U \subset N \to \mathbb{R}^4 \) by \( \varphi = \psi \circ (f|_V)^{-1} \). Clearly these maps define a \( C^\infty \) atlas on \( N \) (the topology induced by \( f : V \to N \) on \( N \) is the same as that induced by \( d_X \) on \( N \)). Using this atlas on \( N \), \( f|_V : V \to N \) is then a smooth diffeomorphism by definition.

Also, we can define a limit metric \( l \) on \( N \) in a natural way: let \( l := \lim_{t \nearrow \tau} f_{*}(g(t)) \). This metric is well defined. Let \( [z] \in N \) and \( z \) be the corresponding point in \( V \). \( z \in \text{Reg}(M) \) because of (ii) above. Hence \( z \in \text{Reg}_{t}(M) \) for all good times \( t \) near enough \( \tau \) and hence, after rescaling as in the proof of Theorem 6.5, \( z \in \text{Reg}_{\tau}(M) \).
Fix coordinates $\psi : \hat{U} \subset \subset V \to \hat{U} \subset \subset \mathbb{R}^4$ with $\hat{U} \subseteq -1B_{R/2}(z)$. Let $g_{ij}(\cdot, t)$ refer to the metric $g(\cdot, t)$ with respect to the coordinates $\psi$. Then $g_{ij}(t) \to l_{ij}$ as $t \nearrow 0$ for some smooth metric $l$ on $\psi(\hat{U})$, in view of the estimates in the statement of Theorem 4.5 (see for example the arguments in Section 8 of [HaForm]). Noting that $f_*(g(t))_{ij}(\cdot, t) = g_{ij}(\cdot, t)$ in the coordinates $\varphi = \psi \circ (f|_V)^{-1} : U \to \mathbb{R}^4$, we see that this limit is well defined.

Notice that for each $x, y \in X$ we can find a $z$ with $d_X(x, z) = d_X(z, y) = \frac{1}{2}d_X(x, y)$: this follows by using the Gromov-Hausdorff approximation $f$, and the fact that this is true for $d(g(t))$ (for a sequence of times $t \nearrow T$), and using the compactness of $M$ and $X$. Hence, since $(X, d_X)$ is complete, we have that $(X, d_X)$ is also a length space. We include the statement of this fact and others, some of which appeared already in this section, in the following theorem.

**Theorem 6.7.** Let everything be as in Proposition 6.6, and let $p_1, \ldots, p_L \in M$ be arbitrary points with $f(p_j) = x_j$ for all $j \in \{1, \ldots, L\}$. $(X, d_X)$ is a compact length space, with length function $L_X$, and $(N, l) = (X \setminus \{x_1, \ldots, x_L\}, l)$ is a smooth Riemannian manifold with

\begin{equation}
\sup_{x, y \in M} |d(g(t))(x, y) - d_X(f(x), f(y))| \to 0 \text{ as } t \nearrow T, \text{ and hence} \end{equation}

\begin{equation}
\sup_{t \in [0, T]} d_{GH}(d(g(t))B_r(x_i), d_X B_r(p_j)) \to 0 \text{ as } t \nearrow T, \end{equation}

for arbitrary $p_j \in M$ with $f(p_j) = x_j$.

(b) Let $\hat{N}$ be a component of $N$ and $d_{\hat{N}, l}$ the metric induced by $(\hat{N}, l)$ on $\hat{N}$. Then, for all $x \in N$, there exists an open set $U \subset \subset N$ with $x \in U$, such that $d_X|U = d_{\hat{N}, l}|U$ and $\text{vol}(E \cap U) = \text{vol}_X(E \cap U)$ for all measurable $E \subseteq N$, where $d_X$ refers to $n$-dimensional Hausdorff-measure with respect to the metric space $(X, d_X)$, and $\text{vol}$ is the volume form coming from $l$ on $N$. Hence, $d_X|_N = \text{vol}$ if we restrict to measurable sets in $N$.

(c) $L_X(\gamma) = L_t(\gamma)$, in the case where $\gamma$ is a piecewise smooth curve which lies completely in $N = X \setminus \{x_1, \ldots, x_L\}$.

**Proof.** (a) follows directly from Lemmata 6.3, 6.5 and 6.6. As we mentioned above, for each $x, y \in X$ we can find a $z$ with $d_X(x, z) = d_X(z, y) = \frac{1}{2}d_X(x, y)$: this follows by using the Gromov-Hausdorff approximation $f$, and the fact that this is true for $d(g(t))$, and using the compactness of $M$ and $X$. Hence, since $(X, d_X)$ is complete, we have that $(X, d_X)$ is also a length space, see Chapter 2 and in particular Theorem 2.4.16 of [BBI]: in the proof of Theorem 2.4.6 in [BBI], it is shown, that one can construct a continuous curve $\gamma : [0, l := d_X(x, y)] \to X$ such that $d_X(\gamma(s), \gamma(t)) = |t - s|$ for all $0 < s, t \leq l$, and hence $d_X(x, y) = L_X(\gamma)$ where, for $\sigma : [a, b] \to \mathbb{R}$ a continuous curve, $L_X(\sigma)$ is the supremum of the sums $\Sigma(Y) = \sum_{i=1}^{N} d_X(\sigma(y_{i-1}), \sigma(y_i))$ over all finite partitions $Y = \{y_1, \ldots, y_N\}$, $N \in \mathbb{N}$ of $[a, b]$. Hence all points $x, y$ can be joined by a continuous geodesic curve $\gamma : [0, s] \to X$ such that $L_X(\gamma|_{[a, b]}) = d_X(\gamma(b), \gamma(a))$ for all $0 \leq a, b \leq s$. We are using the notation of [BBI]: a geodesic in a length space is a continuous curve whose length realises the distance.
Let $q \in N = X \setminus \{x_1, \ldots, x_L\}$. Then $q \in \hat{N}$, the unique connected component of $N$ containing $q$. For the proof of (b), $d_l$ will refer to $d_{l_{ij}}$ the distance function associated to $(\hat{N}, l)$.

From the above (Lemmata 6.5 and 6.6), there exists a unique $\hat{q} \in M$ such that $f(\hat{q}) = q$, and we can find a neighbourhood $Z \subset U \subset N$ and coordinates $\varphi: U \to \mathbb{R}^4$, with $x \in U$, $\hat{U} := \varphi(U)$, $\bar{Z} := \varphi(Z)$, $\varphi(q) = p$. By choosing $\varepsilon > 0$ small enough, we can guarantee that $d_{\hat{U}}B_{10\varepsilon}(q)$ and $d_X B_{10\varepsilon}(q)$ are compactly contained in $Z$. Using the fact that $g_{ij}(t) \to l_{ij}$ in the $C^k$ norm on $\hat{U} = \varphi(U)$, we see that every smooth, regular curve $\gamma: [0, 1] \to \bar{Z}$ with $\gamma(0) \in \varphi(\hat{U})$ and $\gamma(1) \in \varphi(Z)$ such that $l_{ij}(t)$ is the distance on the Riemannian manifold $(\hat{U}, l)$. Similarly, $d_{\hat{U}}(f^{-1}(x), f^{-1}(y)) = d_{\tilde{g}(t), \hat{U}}(\varphi(x), \varphi(y))$, $\tilde{g}(t) = \psi_t(g(t)) = (g_{ij}(t))_{i,j \in \{1, \ldots, n\}}$ if $|T - t| \leq \hat{\varepsilon}$, where we are using the coordinates $\varphi = \psi \circ f^{-1}$, introduced in Proposition 6.6. Without loss of generality, $|d_{\tilde{g}(t)}(f^{-1}(x), f^{-1}(y)) - d_X(x, y)| \leq \varepsilon$ for $|T - t| \leq \hat{\varepsilon}$, and hence $d_{\tilde{g}(t)}(f^{-1}(x), f^{-1}(y)) \leq 3\varepsilon$. If $\gamma$ is any curve in $M$ between $f^{-1}(x)$ and $f^{-1}(y)$ whose length is less than $4\varepsilon$, then $\gamma$ must lie in $f^{-1}(Z)$: otherwise, pushing down to $\hat{U}$ with the coordinates $\psi$, we would obtain a part of the curve having length larger than $10\varepsilon$, which is a contradiction. This shows us

$$d_X(x, y) = \lim_{t \to T} d_{\tilde{g}(t)}(f^{-1}(x), f^{-1}(y)) = \lim_{t \to T} d_{\tilde{g}(t), \hat{U}}(\varphi(x), \varphi(y)) = d_{\tilde{g}(t), \hat{U}}(\varphi(x), \varphi(y)) = d_l(x, y),$$

as claimed. Furthermore, since $l$ is smooth, we can assume that $\varepsilon > 0$ is so small that $\text{vol}_l |_{B_r(q)} = \mathcal{H}_d^m |_{B_r(q)}$, and hence $\text{vol}_l |_{B_r(q)} = \mathcal{H}_d^m |_{B_r(q)}$, since $d_X = d_{\tilde{g}(t)}$ on $d_{\hat{U}}B_{\varepsilon}(q)$, where here $\mathcal{H}_d$ is Hausdorff-measure on $(\hat{N}, d_l)$. This finishes the proof of (b).

It follows that $L_X(\sigma) = L_l(\sigma)$ for any piecewise smooth $\sigma: [0, 1] \to X \setminus \{x_1, \ldots, x_L\}$ curve: we cover the image by small balls for which on each of the balls $d_X = d_{\tilde{g}(t)}$, and use the fact that locally, $L_l(\sigma)$ is the supremum of the sums $\Sigma(\gamma) = \sum_{i=1}^N d_l(\sigma(y_{i-1}), \sigma(y_i)) = \sum_{i=1}^N d_X(\sigma(y_{i-1}), \sigma(y_i))$ over all finite partitions $\gamma: \{y_1, \ldots, y_N\}, N \in \mathbb{N}$ of $[a, b]$ (without loss of generality, $\sigma(y_i, y_{i+1} | \{y_i, y_{i+1}\}$ lies in a small ball on which $d_X = d_l$). This is (c). \hfill \Box

7. Curvature estimates on and near the limit space

Let $d\mu_X$ denote Hausdorff-measure on the metric space $(X, d_X)$. This is an outer measure and defined for all sets in $X$. See for example Chapter 2 of [AT]. Let $d\mu_l = \text{vol}_l$ refer to the measure on $N = X \setminus \{x_1, \ldots, x_L\}$ coming from the Riemannian metric $l$. From (b) in Theorem 6.7 above, we saw that $d\mu_l = (d\mu_X)|_N$ when we restrict to measurable sets in $N$. Hence for any measurable set $E$ in $N$, we have

(i) $d\mu_l(E) = d\mu_X(E) = \lim_{\varepsilon \to 0} d\mu_X(E \setminus d_X B_r(p)) = \lim_{\varepsilon \to 0} d\mu_l(E \setminus d_X B_r(p))$

(ii) By construction $l$ is the limit of the pull back of the metrics $g(t)$ by $f^{-1}$, and hence, $c_0 r^d \leq d\mu_l(d_X B_{r/2}(x_i)) \leq c_1 r^d$ for all $r \leq \text{diam}(X)$, where $c_0, c_1$ are fixed constants. This can be seen as follows. Let $U := d_X B_{r/2}(x_j)$. 

Then \( iB_{\epsilon/4}(p_j) \subseteq \hat{U} := f^{-1}(U) \subseteq iB_{2\epsilon}(p_j) \) for all \( t \) with \( |T - t| \leq \delta \) small enough, in view of the definition of \( f \), and the uniform continuity in time of the distance function (here \( p_j \) is an arbitrary point with \( f(p_j) = x_j \)), and hence \( c_0 \epsilon \leq \text{vol}_{\epsilon(t)}(\hat{U}) \leq c_1 \epsilon \).

Let \( t \rightarrow T \) and using \( \text{vol}_{\epsilon(t)}(\hat{U}) = \text{vol}_{f,\epsilon(t)}(U) \rightarrow \text{vol}(U) = d\mu(U) \) implies the claimed estimate.

(iii) Hence the non-collapsing / non-expanding estimates \( \tilde{\sigma}_0 \epsilon \leq d\mu(\partial^x B_{r}(z)) \leq \tilde{\sigma}_1 \epsilon \) must also hold on \( X \) for some constants \( 0 < \tilde{\sigma}_0, \tilde{\sigma}_1 < \infty \), for all \( r \leq \text{diam}(X) \). We denote the constants \( 0 < \tilde{\sigma}_0, \tilde{\sigma}_1 < \infty \) once again by \( \tilde{\sigma}_0, \tilde{\sigma}_1 \), and the uniform continuity in time necessary to pass to a subsequence in order to obtain the result. That is, the non-collapsing / non-expanding estimates survive into the limit.

In view of the results of the previous sections we have

**Theorem 7.1.** Let everything be as in the previous section \( (X, x_1, \ldots, x_L) \) are defined in Lemma \((b, d)\) and \( l \) is defined in Lemma \((b, d)\). Then,

(i)

\[
\int_X |\text{Riem}(l)(x)|^2 d\mu_X \leq K_0 := c_0(g_0, T)
\]

where \( c_0(g_0, T) \) is the constant appearing in \((2.1)\), and we define \(|\text{Riem}(l)(x)| = 0\) for \( x \in \{x_1, \ldots, x_L\} \) (this is a measurable function, since \( d\mu_X(S) = 0 \) for any finite set \( S \subseteq X \)).

(ii) The following flatness estimates are also true.
Let \((a_i)_{i \in \mathbb{N}}\) be any sequence with \( \epsilon_i \rightarrow \infty \), and let \( l_i = a_i^2 l, d_i = \sqrt{a_i} d_X \). Then for all \( 0 < \epsilon < N < \infty, K \in \mathbb{N} \), we have

\[
|\nabla^k \text{Riem}(l_i)(x)| \leq \epsilon(i, \sigma, N, K) \text{ on } d_i B_{\sigma, N}(x_j)
\]

where \( \epsilon(i, \sigma, N, K) \rightarrow 0 \) as \( i \rightarrow \infty \) for fixed \( N, \sigma, K \), and \( j \in \{1, \ldots, L\} \).

**Remark 7.2.** Note that we obtain the result \((\ref{eq:2})\) for all sequences. It is not necessary to pass to a subsequence in order to obtain the result.

**Remark 7.3.** Compare the estimates with those stated in Corollary 1.11 in \([BZ]\), which were obtained independently.

**Proof.** (i)
Using the theorem on monotone convergence (see for example Theorem 2 Section 1.3 in \([EG]\) and the fact that \( d\mu_X(\cup_{i=1}^{L} B_{\epsilon}(x_i)) \rightarrow 0 \) as \( \epsilon \rightarrow 0 \), we see that

\[
\int_X |\text{Riem}(l)(x)|^2 d\mu_X (x) = \lim_{\epsilon \rightarrow 0} \int_{X \setminus (\cup_{i=1}^{L} B_{\epsilon}(x_i))} |\text{Riem}(l)(x)|^2 d\mu_X
\]

\[
= \lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow T} \int_{f^{-1}(X \setminus (\cup_{i=1}^{L} B_{\epsilon}(x_i)))} |\text{Riem}(g(t))(x)|^2 d\mu_t \leq K_0
\]

This finishes the proof of (i).

(ii)
Let \( c_i := \frac{1}{t_i} \), where \( t_i \) is a sequence of good times. Scale and translate in time \((M, g(t))_{t \in [0, T]}\) as in Theorem \((1.5)\) we call the resulting solution also \((M, g(t))_{t \in (-A_1, 0]}\), and scale \( d_X \) by \( d_i = \sqrt{c_i} d_X \). Notice that \( d_i(x_k, x_l) \rightarrow \infty \) as \( i \rightarrow \infty \) and we will
only be concerned with these blow ups near one point \(x_k\): without loss of generality \(x_k = x_i\). Assume \(x_1 \in f(S\text{ing}(M))\), and let \(p_1 \in S\text{ing}(M)\) be a point with \(f(p_1) = x_1\). If \(x_1 \in f(\text{Reg}(M))\), then the theorem follows by blowing up the region around \(x_1\), which has a Riemannian manifold structure. From the estimates of Theorem 5.1 we have \(S\text{ing}(M) \subseteq (\text{Reg}_{\beta}(M))^c \subseteq \bigcup_{k=1}^L (-1)_B J_i(t_k)\) and hence \(p_1 \in (-1)_B J_i(t_k)\) for some \(k \in \{1, \ldots, L\}\): renaming the \((t_k)\)'s we can assume \(p_1 \in (-1)_B J_i(t_{p_1})\) and hence

\[
(7.4) \quad |\nabla^j \text{Riem}(g_i(\tilde{t}))| \leq C_j \quad \text{on } \bigcup_{k=1}^L (-1)_B J_i(t_k) \quad \text{if } \tilde{t} \in (-\frac{1}{2}, 0),
\]

in view of the estimates (4.11), where \(i_{p_1} = p_1\). From Remark 5.5 we see that, without loss of generality, \((t_{B_{J_1}(p_1)})_j \subseteq (-1)_B J_{2}\text{J}_1(p_1) \subseteq (-1)_B J_{2}\text{J}_1(p_1)\) for all \(t \in (-1, 0]\) and \((t_{B_{16N}(p_1)})_i \subseteq (-1)_B J_{2}\text{J}_1(p_1)\) for all \(t \in (-1, 0]\) if \(i\) is large enough. Hence, using the fact that \(d(-1)(t_{B_{J_1}(p_1)} \to \infty \text{ as } i \to \infty)\) (see Remark 5.4) for all \(j \neq k\), we see that \((t_{B_{16N}(p_1)})_i \subseteq (-1)_B J_{2}\text{J}_1(p_1)\) and

\[
(7.5) \quad \text{if } \tilde{t} \in (-\frac{1}{2}, 0),
\]

and hence, taking a limit \(t \not

\]

\[
(7.6) \quad |\nabla^j \text{Riem}(l_i)| \leq C_j \quad \text{on } \bigcup_{k=1}^L (-1)_B J_{16N}(p_1) \subseteq (-1)_B J_{2}\text{J}_1(p_1)\),
\]

where \(l_i = c_i l_i, J_i = 2^5 J_i^2\). Using that \(\int_{B_i} |\text{Riem}(l_i)|^2 d\mu_X(x) \to 0\) as \(r \to 0\), we see that

\[
(7.7) \quad \int_{B_i} |\text{Riem}(l_i)|^2 d\mu_X(x) \to 0 \quad \text{as } i \to \infty,
\]

where \(d\mu_X\) is Hausdorff-measure on \((X, d_i)\), and hence

\[
|\text{Riem}(l_i)| \quad \leq \varepsilon(i) \to 0 \quad \text{as } i \to \infty \quad \text{on } d_i B_{J_4+N-1}(x_1)
\]

in view of the fact that \(|\nabla^j \text{Riem}(l_i)| \leq C_j\) for all \(j \leq K\) on the same set \((C_j\) not depending on \(i\)). In fact we may assume smallness for all gradients up to a fixed order. This can be seen as follows. Introduce geodesic coordinates at a point \(m_i \in d_i B_{J_4+N-1}(x_1)\). The injectivity radius at \(m_i\) is larger that \(\beta > 0\) for all metrics independent of \(i\) in view of the injectivity radius estimate of Cheeger-Gromov-Taylor, Theorem 4.3 in [CGT], and the non-collapsing/non-inflating estimates. Now using Theorem 4.11 of [HaComp], and writing \(l_i\) in these geodesic coordinates, we get \(|D^{k_i}l_i|_{B_0} \leq C(K)\) for all \(k \in \{1, \ldots, K\}\). Hence taking a subsequence, we get a limit metric in \(C^{\infty}(B_{\beta+1}(0))\), which is equal to \(\delta\), by Theorem 4.10 of [HaComp]. That is, without loss of generality,

\[
(7.8) \quad |\nabla^k \text{Riem}(l_i)| \quad \leq \varepsilon(i) \to 0 \quad \text{as } i \to \infty \quad \text{on } d_i B_{J_4+N-1}(x_1)
\]

for all \(k \leq K \in \mathbb{N}\), where \(K\) is fixed but as large as we like, for \(l_i = c_i l, c_i = \frac{1}{(T-t_i)}\), where \(t_i \not

\]

\[
T \text{ is a sequence of good times, where we took various subsequences to achieve this. In fact the equation (7.8) is true for any sequence } c_i \not

\]

\[
\text{not necessary to take a subsequence, and it is not necessary that } c_i \text{ has the form } c_i = \frac{1}{(T-t_i)} \text{, where } t_i \text{ are good times. We explain this now. First, the statement is true for any sequence of the form } c_i = \frac{1}{(T-t_i)}: \text{ if not, then take a sequence for}
\]

\[
\text{and hence, taking a limit } t \not

\]
which it fails. Taking a subsequence, if necessary, in the proof above, we arrive at a contradiction.

Now let \( c_\ell \to \infty \) be arbitrary. We can always write \( c_\ell \) for some sequence of good times \( t_i \) for any \( \alpha \in (1/4, 4) \), in view of Lemma \[ \ref{lem:4.6} \]. Now \[ \ref{thm:7.3} \] holds for the metrics \( \bar{l}_i = \frac{1}{(T-t_i)} l \), as we have just shown, and hence, for \( l_i = \frac{1}{(T-t_i)} l = \alpha \bar{l}_i \), we get

\[
|\nabla^k \text{Riem}(l_i)(x)| \leq \varepsilon(i) \to 0 \quad \text{as } \ell \to \infty \quad \text{on } d_i B_{2(J_{\ell+1}), \frac{1}{4}(N-1)}(x_1)
\]

Now let \( a_i \) be an arbitrary sequence going to infinity, and \( \bar{l}_i = a_i l \). Writing \( l_i = c_i \) with \( c_i = \frac{1}{4(J_{i+1})} a_i \), we see that \( \bar{l}_i = \frac{a_i^2}{4(J_{i+1})} l_i \), and hence, using the fact that \( N \) was arbitrary (but large), we get,

\[
|\nabla^k \text{Riem}(l_i)(x)| \leq \varepsilon(i) \to 0 \quad \text{as } \ell \to \infty \quad \text{on } d_i B_{\sigma, N}(x_1)
\]

So we see that the manifold is becoming very flat away from singular points, in the sense just described, after scaling. Using these flatness estimates we will show that \( X \) is a \textit{generalised} \( C^0 \) \textit{Riemannian orbifold}. We wish also to show that at each possible orbifold point there is only one component: that is, that \( X \) is actually a \( C^0 \) \textit{Riemannian orbifold} with only finitely many orbifold points. To do this, it will be necessary to obtain approximations of the blow ups \((d_i B_{\sigma, N}(x_1))\) (constructed in the proof above) by Riemannian manifolds which have certain nice properties.

This is the content of the next theorem.

**Theorem 7.4.** (Approximation Theorem)

Let \( l \) and \( X, x_1, \ldots, x_L \), be as in Lemma \[ \ref{lem:7.7} \] and Lemma \[ \ref{lem:6.6} \]. There exist smooth metrics \( g_i \) on \( M \), and points \( p_j \in M \) for \( j \in \{1, \ldots, L\} \) such that

\[
d_{\text{GH}}(g_i B_{N(i)}(p_j), d_i B_{N(i)}(x_j)) \leq \alpha_i,
|\nabla^k \text{Riem}(l_i)|^2 \leq \alpha_i \quad \text{on } d_i B_{\sigma(i), N(i)}, \quad \text{and}
(d_i B_{\sigma(i), N(i)}(x_j), l_i), \quad \text{is } \alpha_i \text{ close to } (g_i B_{\sigma(i), N(i)}(p_j), g_i)
\]

in the \( C^k \) sense, and

\[
\int_M |\text{Ricci}(g_i)|^4 dt_{g_i} \to 0, \quad \text{as } \ell \to \infty,
\]

where \( d_i(\cdot, \cdot) = a_i d_X(\cdot, \cdot), \quad l_i = a_i^2 l \) and \( a_i, \sigma(i), \alpha_i, N(i) \in \mathbb{R}^+ \) are numbers satisfying \( 0 < \alpha_i, \sigma(i) \to 0 \) as \( \ell \to \infty, \quad a_i, N(i) \to \infty \) as \( \ell \to \infty \).

The condition \( \varepsilon \) close in the \( C^k \) sense, is made precise in the proof of the theorem, and the approximations are always achieved with \( f \).

**Proof.** Let \( x_j \) be fixed. If \( x_j \in f(\text{Reg}(M)) \) then the theorem follows directly using the definition of \( C^k \) close and Theorem \[ \ref{thm:4.5} \] (see below). So assume \( x_j = x_1 \notin f(\text{Reg}(M)) \) and let \( f(p_1) = x_1 \).

Let \( t_i \) be a sequence of good times and scale by \( a_i := \frac{1}{T-t_i} \) and translate as in the proof of (ii) Theorem \[ \ref{thm:7.1} \]
First we use a similar argument to that given at the end of Section 5 to show that
\[ |d_1(\cdot, \cdot) - d_2(\cdot, \cdot)| \leq C(N_j_1) \] for all \( t, s \in (\delta(N, J_1), 0] \) for all \( x, y \in iB_{N/4}(p_1) \). We use the notation from the proof of (ii) Theorem 7.1 in this argument, and we take various subsequences when necessary.

Let \( x, y \in iB_{N/4}(p_1) \) be arbitrary in, and \( \gamma \) a distance minimising curve between these two points w.r.t to \( g(t) \) (\( \gamma \) must lie in \( iB_N(p_1) \) and we have \( L_1(\gamma) \leq N \).

We modify the curve \( \gamma \) to obtain a new curve \( \tilde{\gamma} : [0, 1] \to M \) in the following way: if \( \gamma \) reaches the closure of the ball \( iB_{2J_1}(p_1) \) at a first point \( \gamma(\tilde{r}) \) then let \( \gamma(\tilde{r}) \) be the last point which is in the closure of the ball \( iB_{2J_1}(p_1) \) (it could go out and come in a number of times). Remove \( \gamma|_{(r, \tilde{r})} \) from the curve \( \gamma \). In doing this we obtain the finite union of at most 2 curves \( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \). Call this finite union \( \tilde{\gamma} \) and consider it as a curve with finitely many discontinuities.

The new \( \tilde{\gamma} \) has
\[ (7.13) \quad L_1(\tilde{\gamma}) \leq L_1(\gamma) = d(x, y, t) \leq N \]

From equation (7.13) in the proof above, we see that for all \( \varepsilon > 0 \) there exists a \( \delta(\varepsilon) > 0 \) such that
\[ (1 - \varepsilon)g(y, t) \leq g(y, s) \leq (1 + \varepsilon)g(y, t) \]
for all \( y \in iB_{\delta, N}(p_1) = iB_{2\delta, N}(p_1) \subseteq -1B_{32J_16N}(p_1) \) for all \( t, s \in (-\delta, 0] \) (\( \delta \) independent of \( i \)): use (7.5) and the evolution equation for the curvature as in Section 8 of [HaForm].

\[ L_4(\tilde{\gamma}) \geq L_4(\gamma) - \varepsilon L_1(\gamma) \geq L_4(\gamma) - \varepsilon N \]
for all \( t, s \in (-\delta, 0] \), which, when combined with (7.13), gives us
\[ d(x, y, t) \geq L_4(\tilde{\gamma}) \]
\[ \geq L_4(\gamma) - \varepsilon N \]
\[ \geq d(x, y, s) - \varepsilon N - J_4^6. \]

The last inequality can be seen as follows: when \( \tilde{\gamma} \) reaches the ball \( iB_{J_1}(p_1) \), it must also be in \( sB_{J_1}(p_1) \) in view of Remark 5.5. So the two points of discontinuity on \( \tilde{\gamma} \) may be joined smoothly by a curve with length (with respect to \( g(s) \)) at most \( 2J_4^6 \).

Call this curve \( \tilde{\gamma} \). Hence \( L_4(\tilde{\gamma}) \leq 2J_4^6 + J_4(\tilde{\gamma}) \), which implies \( L_4(\tilde{\gamma}) \leq L_4(\gamma) - 2J_4^6 \geq d(x, y, s) - J_4^6 \) as claimed. So we have \( d(x, y, t) \geq d(x, y, s) - J_4^6 \) if we choose \( \varepsilon = \frac{1}{N} \). Swapping \( s \) and \( t \), we see that
\[ (7.16) \quad |d(x, y, t) - d(x, y, s)| \leq J_4^7 \]
if \( t, s \in (-\delta, 0] \), and \( x, y \in iB_{N/4}(p_1) \) where \( \delta = \delta(N, J) \) and may depend on the solution, but does not depend on \( i \), as long as \( i \) is large enough.

In particular, \( iB_{J_{100}, N}(p_1) \subseteq iB_{J_{50}, N}(p_1) \subseteq iB_{J_2, N}(p_1) \) for all \( N > J_{100} \) and \( i \) large enough, for all \( t, s \in (-\delta, 0] \) where \( \delta = \delta(N, J) \) and may depend on the solution, but does not depend on \( i \). Notice, by taking a limit \( s \not\to 0 \), we see
\[ f(iB_{J_{100}, N}(p_1)) \subseteq d_iB_{J_{50}, N}(x_1) \]
Using (7.15), and the evolution equation for the curvature as in Section 8 of [HaForm], we see that
\[ (7.17) \quad |f_*(g(t)) - l_1|_{C^0(iB_{J_{50}, N}(x_1), g(t))} \leq \delta \]
if \( t \in (-\delta, \varepsilon, 0] \). We explain now why Inequality (7.17) is true. To see this, work with fixed geodesic coordinates \( \varphi : B_{\delta_0}(z) \to B_{\delta_0}(0) \) of radius larger \( \delta_0 > 0 \) at any
point in $z \in d_{B_{j^0,N}}(p_1)$ (these exist because of the curvature estimates of the previous theorem, Theorem 7.1, and the non-collapsing estimates). Writing $l_i$ in these coordinates, (we drop the $i$ in these coordinates and call $f \sigma g(t)$ also $g(t)$ in the coordinates) we have $l_i \delta j \leq l_i \delta j \leq C \delta j$, $\sum_{j=0}^{K} D^j(t)^2(\cdot) \leq C$ on $B_{\omega}(0)$ for some $C$ not depending on $i$, where here $D$ is the standard euclidean derivative (the manifolds are non-collapsed and satisfy the curvature bounds of Theorem 7.1, see Corollary 4.12 in [HaComp] for example). Using the evolution equation for $g(t)$ and the curvature bounds, and the fact that $g(0) - l = 0$ we see, using arguments similar to those of Section 8 in [HaForm], $C(t)|l | \leq C(t)|l | \leq C|t|$, $D^2(g(t) - l)| \leq C t$ and so on. This implies $\sum_{j=0}^{K} |g(t)\nabla^j (l - g(t))|_{g(t)}^2 + |\nabla^j (l - g(t))|_{g(t)}^2 \leq \varepsilon$ on $B_{\omega}(z)$ if $\varepsilon \leq \delta(C, \varepsilon)$, where $\delta$ is chosen near enough to 0. This finishes the explanation of why Inequality (7.17) is true. For a tensor $T$ and a metric $l$ defined on $U$, we have used the following notation:

$$|T|_{C^k(U,l)}^2 := \sum_{j=0}^{k} \sup_{x \in U} |\nabla^j T|^2_l(x),$$

where $\nabla^j$ refers to the $j$th covariant derivative with respect to $l$, if $j \in \mathbb{N}$, and $\nabla^0 T := T$. Note that this $\delta$ doesn’t depend on $i$. In fact, what we have shown is $\sum_{j=0}^{K} |g(t)\nabla^j (f^* (l_i - g(t))|_{g(t)}^2 + |\nabla^j (l_i - f \sigma g(t))|_{g(t)}^2 \leq \varepsilon$ for all $t \in (-\delta, 0]$ if $z \in d_{B_{j^0,N}}(p_1)$. Hence, using (*), we have also shown

$$\sum_{j=0}^{K} |g(t)\nabla^j (f^* (l_i - g(t))|_{g(t)}^2 + |\nabla^j (l_i - f \sigma g(t))|_{g(t)}^2 \leq \varepsilon$$

for all $t \in (-\delta, 0]$ if $w \in d_{B_{j^0,N}}(p_1)$.

Scaling the solution by $(\frac{8N_{\varepsilon}}{\sigma})^2$, and assuming $N = \frac{8N_{\varepsilon}}{\sigma}$, we see that $|g(t) - f^*(l_i)|_{C^k(U,l)} \leq \sigma$ and $|f \sigma g(t) - l_i|_{C^k(U,l)} \leq \sigma$ if $t \in (-\delta, 0]$. (the original $\varepsilon$ is as small as we like) and $|d(x, y, t) - d(x, y, s)| \leq \sigma$ if $t, s \in (-\delta, 0]$ and $x, y \in \Omega$. Choosing $i$ large enough, and a time $t_1 \in (-\delta, -\frac{\delta}{i})$ which corresponds to a good time of the original solution, we see that we may assume without loss of generality, that $g_1 := g(t_1)$ satisfies $\int_M |\text{Ricci}(g_1)|^2 \mu_g \leq 0$. $g_1$ is our first metric. It satisfies $|g_1 - f^*(l_i)|_{C^k(U,l)} \leq \alpha_1$, $|f \sigma g_1 - l_i|_{C^k(U,l)} \leq \alpha_1$ and $|d_3(x, y)|_{g_1} \leq \alpha_1$ on $g_1 B_{\omega}(p_1)$, where $\alpha_1 = \sigma$, and $\int_M |\text{Ricci}(g_1)|^2 \mu_g \leq \alpha_1$.

Repeating the procedure, but scaling by $(\frac{\sigma}{\varepsilon})^2$, at the end, with $N = \frac{2x8N_{\varepsilon}}{\sigma^2}$ leads to our second metric $g_2$, and $g_2$ satisfies (for a new larger $i$)

$|g_2 - f^*(l_i)|_{C^k(U,l)} \leq \alpha_2$, $|f \sigma g_2 - l_i|_{C^k(U,l)} \leq \alpha_2$ and $|d_3(x, y)|_{g_2} \leq \alpha_2$ on $g_2 B_{\omega}(p_1)$, where $\alpha_2 = \sigma^2$, and $\int_M |\text{Ricci}(g_2)|^2 \mu_g \leq \alpha_2$.

And so on. Choosing $\sigma_i$ to be an arbitrary sequence with $\sigma_i >> \sigma^i$ and $\sigma_i \to 0$ as $i \to \infty$ completes the proof.

\[\square\]

For convenience we introduce some notation which will help us describe the phenomenon of metric annuli being $C^k$-close, as described in the theorem above. This phenomenon occurs at a number of points in the rest of the paper.
Definition 7.5. Let \((X, d_X), (Y, d_Y)\) be complete, connected metric spaces. We assume also that these spaces have a given Riemannian structure with at most finitely many (possible) singularities in the following sense: \(N := X \setminus \{x_1, \ldots, x_L\}\) and \(V := Y \setminus \{y_1, \ldots, y_L\}\) are smooth manifolds, and \(l\) is a Riemannian metric on \(N\) and \(v\) on \(V\). For \(0 < r < R \leq \infty\), \(E \subseteq X\) an open set (\(E = X\) is allowed), and \(x_0 \in \{1, \ldots, x_L\}, y_0 \in \{y_1, \ldots, y_L\}\) we say that

\[
d_{C^k}(E \cap d_X B_{r,R}(x_0), d_Y B_{r,R}(y_0)) \leq \varepsilon
\]

(we always assume \(\varepsilon << \min(r, R - r)\)) if

(i) \(E \cap d_X B_{r,R}(x_0) \subseteq N\) and \(d_Y B_{r,R}(y_0) \subseteq V\), and

(ii) there exists a \(C^{k+1}\) map \(f : E \cap d_X B_{r,R}(x_0) \to V\), such that \(f\) is a \(C^{k+1}\) diffeomorphism onto its image, \(d_Y B_{r+\varepsilon, R-\varepsilon}(y_0) \subseteq f(E \cap d_X B_{r,R}(x_0))\)

(iii) \(|d_X(w, x_0) - d_Y(f(w), y_0)| \leq \varepsilon\) for all \(w \in E \cap d_X B_{r,R}(x_0)\): in particular \(d_Y B_{s+\varepsilon, m-\varepsilon}(y_0) \subseteq f(E \cap d_X B_{s, m}(x_0)) \subseteq d_Y B_{s-\varepsilon, m+\varepsilon}(y_0)\) for all \(0 < r \leq s < m \leq R\) with \(s + \varepsilon < m - \varepsilon\).

(iv) \(|f^*(v) - l|^2_{C^k(E \cap d_X B_{r,R}(x_0))} := \sum_{j,l=0}^k \sup_{x \in E \cap d_X B_{r,R}(x_0)} |\nabla^j (f^*(v) - l)|^2_l(x) \leq \varepsilon\) and \(|v - f_j|^2_{C^k(d_Y B_{r+\varepsilon, R-\varepsilon}(y_0), v)} \leq \varepsilon\).

Remark 7.6. Note that in the Approximation Theorem above, Theorem 7.4, the \(f\) that occurs there is also a Gromov-Hausdorff approximation when considered as a map on the balls being considered (and \(E = M\)). Here we only require condition (iii), which is weaker.

Remark 7.7. The definition of \(C^k\) close is coordinate free. This allows us to compare elements of sequences of Annuli in a coordinate invariant way.

Remark 7.8. From the definition we see, that if \((X, d_X), (Y, d_Y)\), are metric spaces of the type occurring in the theorem then \(d_{C^k}(d_X B_{r,R}(x_0), d_Y B_{r,R}(y_0)) \leq \varepsilon\) implies \(d_{C^k}(d_Y B_{r+4\varepsilon, R-4\varepsilon}(y_0), d_X B_{r+4\varepsilon, R-4\varepsilon}(x_0)) \leq \varepsilon\) (almost symmetry).

8. Orbifold structure of the limit space

The flatness estimate \(7.2\) of the previous section, along with the non-collapsing and non-expanding estimates (which survive into the limit, as explained in (iii)) at the beginning of Section 7 guarantee that \(X\) is actually a so-called generalised \(C^0\) Riemannian orbifold with only finitely many isolated orbifold points: points \(q \in X\) for which there exists a neighbourhood \(q \in U \subseteq X\) and a smooth diffeomorphism \(\varphi : U \to \mathbb{R}^4\) are called manifold points, all other points in \(X\) are called orbifold points. These objects have been studied in [Tian], [And1], [BKN]. In the papers [HM] [HM2], the authors also used generalised Riemannian orbifolds (they refer to them as multifold: see section 3 of [HM2]) to prove an orbifold compactness result for solitons. They were introduced and used in the static (for example the Einstein) setting by M. Anderson [And1] (see also [BKN]), to describe non-collapsing limits of Einstein manifolds. The estimates required to show that \(X\) is a generalised \(C^0\) Riemannian orbifold are contained in the previous section. Generalised Riemannian orbifolds can have a number of components at each orbifold type point. In our case we will see that there is exactly one component at each singular point. Before showing this, we state the general result which follows from the argument for example in [Tian] (see also [And1], [BKN]).
We use the following notation in the statement of the theorem and in the rest of the paper: $D_{r,R} \subseteq \mathbb{R}^4$ is the standard open annulus of inner radius $r \geq 0$ and outer radius $R \leq \infty$, $(r < R)$ centred at 0: $D_{r,R} = \{ x \in \mathbb{R}^4 \mid |x| > r, |x| < R \}$. $D_r$ represents the open disc of radius $r$ centred at 0: $D_r := \{ x \in \mathbb{R}^4 \mid |x| < r \}$. Note $D_{0,R} = \{ x \in \mathbb{R}^4 \mid |x| > 0, |x| < R \} = D_R \setminus \{0\}$.

**Theorem 8.1.** $X$ is a generalised $C^0$ Riemannian orbifold in the following sense.

(i) $X \setminus \{ x_1, \ldots, x_L \}$ is a manifold, with the structure explained above in Lemmata 6.5 and 6.6.

(ii) There exists an $r_0 > 0$ small, and an $N < \infty$ such that the following is true. Let $x_i \in X$ be one of the singular points. Then the number of connected components $(E_{i,j}(r))_{j \in \{1, \ldots, N_i\}}$ of $\partial B_r(x_i) \setminus \{x_i\}$ in $X \setminus \{ x_1, \ldots, x_L \}$ is finite and bounded by $N$ (that is $\bar{N}_i \leq N$) for $r \leq r_0$, where $N = N(\sigma_0, \sigma_1) < \infty$.

(iii) Fix $i \in \{1, \ldots, L\}, j \in \{1, 2, \ldots, \bar{N}_i\}$, and let $E = E_{i,j}(r_0)$ be one of the components from (ii). Then there exists a $0 < \tilde{r} \leq r_0$ and a diffeomorphism $k : D_{\tilde{r},\tilde{r}} \to k(D_{0,\tilde{r}}) \subseteq \bar{E}$ where $\bar{E}$ is the universal covering space of $E \cap (\cup_{j=1}^{\bar{N}_i} E_{i,j}(\tilde{r}(1+\varepsilon)))$, such that the covering map $\pi_E : \bar{E} \to E$ is finite and for $r \leq \tilde{r}$ we have

\[
\sup_{D_{0,r}} (\pi_E \circ k)^* l - \delta \leq \varepsilon_1(r)
\]

where $\varepsilon_1(r) \geq 0$ is a decreasing function with $\lim_{r \to 0} \varepsilon_1(r) = 0$, $\delta$ is the standard euclidean metric on $\mathbb{R}^4$ or subsets thereof, $| \cdot |_{C^0(\mathbb{R}^4)}$ is the standard euclidean norm on two tensors, $|v|_{C^0(L)}^2 := \sup_{x \in L} \sum_{i,j=1}^n |v_{ij}(x)|^2$ for any set $L \subseteq \mathbb{R}^4$ and any two tensor $v = v_{ij}dx^idx^j$.

**Proof.** (i) was shown above. (ii) follows from the non-expanding and non-collapsing estimates, exactly as in the proof of Lemma 3.4 in Tian.

(iii) Follows as in the proof of Lemma 3.6 in Tian using the flatness estimates, (7.2), and the non-collapsing and non-expanding estimates.  

**Remark 8.2.** Some of the proofs of the Lemmata mentioned here (Lemma 3.6 and Lemma 3.4 of Tian) can be simplified at certain points, by using that $\text{inj}(B_r(p)) \geq c_0r$ for all balls $B_r(p)$ which are compactly contained in $(D,h)$ where $(D,h)$ is any smooth, open flat (Riem$(h) = 0$) non-collapsed, non-inflated (on all scales) manifold without boundary: this follows from the injectivity radius estimate of Cheeger-Gromov-Taylor, Theorem 4.3 in CGT, whose proof is local.

The construction of this $k$ in Tian (see Lemma 3.6 in Tian) is achieved by pasting together maps $\varphi_i : D_{\frac{r}{\pi^2}, \frac{r}{\pi^2}} \to \pi^{-1}(B_{\frac{r}{\pi^2}, \frac{r}{\pi^2}}(x))$ where $i \in \mathbb{N}$. That is $\varphi_1, \varphi_2, \ldots$ are first constructed, and then $\varphi_1$ is pasted to $\varphi_2$ and $\varphi_2$ to $\varphi_3$ and so on. This leads to a map $k$ with the properties given in the theorem above: see the proof of Lemma 3.6 in Tian. We construct a $\varphi$ here using the method described in the proof of Lemma 3.6 in Tian with some minor modifications: the explicit construction will be used in later sections.
As shown in the proof of Lemma 3.6 of Tian: if we scale, $l_i = (2i+2)^2 l, d_i = 2i^2 + 2d x$, then

$$d_{C^k}((g) B_{1/2,4}(0), g(i)), (d_i B_{1/2,4}(x_1) \cap E, l_i) \leq \varepsilon(i) \to 0 \text{ as } i \to \infty,$$

where $(g) B_{1/4,4}(0), g(i)) \subseteq ((\mathbb{R}^4 \setminus \{0\})/\Gamma(i), g(i)),$ and $g(i)$ is the standard metric on $(\mathbb{R}^4 \setminus \{0\})/\Gamma(i),$ and $\Gamma(i)$ is some finite subgroup of $O(4)$ with finitely many elements (less than or equal to $N$ elements, $N$ independent of $i$). Hence there exists a diffeomorphism

$$(8.2) \quad v_i : (g) B_{1/2,4}(0), g(i)) \to (d_i B_{1/2-\varepsilon(i), 4+\varepsilon(i)}(0) \cap E, l_i) \subseteq (E, l_i),$$

such that

$$|v_i^* l_i - g(i)|_{C^k((g), B_{1/2,4}(0), g(i))} + |(v_i)_* g(i) - l_i|_{C^k(d_i B_{1/2-\varepsilon(i), 4+\varepsilon(i)}(0) \cap E, l_i)} \leq \varepsilon(i)$$

In the following, $\varepsilon(i) > 0$ will refer to positive numbers with the property that $\varepsilon(i) \to 0$ as $i \to \infty$. As the notation suggests, in fact this $\Gamma(i)$ (and hence $g(i)$) could depend on the sequence we take, and could depend on $i \in \mathbb{N}$. However, $\text{inj}((\mathbb{R}^4 \setminus \{0\})/\Gamma(i), g(i)) \geq |x|_{0}$ for some fixed $i_0 > 0$, where $|x| = |x|_{\mathbb{R}^4}$ is the euclidean norm of the point $x$ lifted to $\bar{x} \in \mathbb{R}^4$ (any such $\bar{x}$ has the same euclidean distance from the origin, regardless of which $\bar{x}$, covering $x$, we choose). [Explanation 1: this follows in view of the construction: for any ball $d_i B_{r}(x) \subseteq d_i B_{1/4}(x_1) \cap E$ we have $r^i \sigma_4 \geq \text{vol}(d_i B_{r}(x)) \geq r^i \sigma_0$, and the norm of the curvature tensor on $d_i B_{1/4}(x_1)$ goes to zero as $i \to \infty$. Hence $\text{inj}(d_i B_{1/100}(0), l_0(x)) \geq i_0$ for any $x \in d_i B_{2/3}(x_1),$ for some $i_0 > 0$, if $i$ is large enough, in view of the injectivity radius estimate of Cheeger-Gromov-Taylor contained in Theorem 4.3 in [CGT]). Hence, using $d_{C^k}((g), B_{1/2,4}(0), g(i)), (d_i B_{1/2,4}(x_1) \cap E, l_i)) \leq \varepsilon(i)$, we see that we have $\text{inj}(g(i) B_{1/100}(0), g(i))(x) \geq i_0/2$ for some $i_0 > 0$ for any $x \in (g(i) B_{2/3}(0), g(i))$, if $i$ is large enough.]

Let $\pi_i : \mathbb{R}^4 \setminus \{0\} \to (\mathbb{R}^4 \setminus \{0\})/\Gamma(i)$ be the standard projection, and $x \in (\mathbb{R}^4 \setminus \{0\})/\Gamma(i)$, $(\pi_i)^{-1}(x) = \{x_1, \ldots, x_N\}$. $\pi_i$ is a covering map and a local isometry, and using the fact that $\text{inj}(\mathbb{R}^4 \setminus \{0\}/\Gamma(i), g(i))(x) \geq |x|_{0}$, we see that $d_{\mathbb{R}^4}(x_k, x_l) \geq (i_0|x|)/20 > 0$ in $\mathbb{R}^4$ for $x_k, x_l \in (\pi_i)^{-1}(x), k \neq l$.

Let $\psi : D_{1,4} \to E$ be the natural map $\psi = v_i \circ \pi_i |_{D_{1,4}}$ where

$v_i : (g(i) B_{1/2,4}(0), g(i)) \to (d_i B_{1/2-\varepsilon(i), 4+\varepsilon(i)}(0) \cap E, l_i) \subseteq (E, l_i)$ is the map defined in $(8.2)$ above, and $\pi_i : \mathbb{R}^4 \setminus \{0\} \to (\mathbb{R}^4 \setminus \{0\})/\Gamma(i)$, the standard projection, is as above. Define $\varphi_i(x) = \psi_i(2^{i+2} x)$ for $x \in D_{1,4}/2^{i+2}$: this is the unscaled version of $\psi_i$. Later we will paste the $\varphi_i$'s together. To do this, it is convenient to work at the scaled level. We will require that neighbours $\varphi_i$ and $\varphi_{i+1}$ are close to one another for all $i \in \mathbb{N}$, in a $C^k$ sense (to be described) on their common domain of definition, at least at the scaled level. To show this, we have to compare neighbours $\varphi_i$ and $\varphi_{i+1}$, for all $i \in \mathbb{N}$, on their common domain of definition $D_{2^{i+2}}/2^{i+2}$. We do this at the scaled level: $\psi_i : D_{1,4} \to E$ is as defined above, $\psi_i(x) = \varphi_i(2^{i+2} x),$ and we define $\eta_{i+1} : D_{2^{i+2}} \to E$ by $\eta_{i+1}(x) = \varphi_{i+1}(2^{i+2} x)$.

Notice that in defining the $\psi_i$'s, we have the freedom to change the coverings $\pi_i$ by a deck transformation, that is by an element $A \in O(4)$. Also, in view of the definitions, and the notion of convergence introduced in Definition 7.5, we have $(\psi_i)_*(l_i)$
is \( C^k \) close to \( D_{1+\varepsilon(i), 2-\varepsilon(i)} \) and \((\eta_{i+1})^*(l_i)\) is \( C^k \) close to \( D_{1/2+\varepsilon(i), 2-\varepsilon(i)} \), in view of the fact that \((\eta_{i+1})^*(l_i)(x) = (\psi_{i+1})^*(l_{i+1})(2x)\).

**Step 1.** For all \( i \geq N \in \mathbb{N} \) the following is true: By changing the map \( \pi_{i+1} \) by an element \( A \in O(4) \), if necessary, we can assume that the pair \( \psi_i \) and \( \eta_{i+1} \) are, for sufficiently large \( i \in \mathbb{N} \), \( C^k \) close to one another on their common domain of definition, in a sense which we now describe: take any arbitrary ball \( \delta B_s(y) \subseteq D_{1+\delta/2, 2-\delta/2} \) with some fixed \( s > 0 \) \( s \leq \frac{\delta}{10} \) where \( y \in D_{1+\delta, 2-\delta} \), is in the common domain of definition of \( \psi_i \) and \( \eta_{i+1} \), where \( \delta > 0 \) is some fixed small number. Then \( d_i(\psi_i(x), \eta_{i+1}(x)) \leq \varepsilon(i) \) for all \( x \in D_{1+\delta/2, 2-\delta/2} \) and \( \psi_i(B_s(y)) \cup \eta_{i+1}(B_s(y)) \subseteq d_iB_{\delta s}(\hat{y}) \), \( \theta \circ \psi_i - \theta \circ \eta_{i+1} \in C^k(B_s(y), \mathbb{R}^4) \leq \varepsilon(i) \), where \( \theta : d_iB_{2\varepsilon}(\hat{y}) \to \delta B_0(0) \subseteq \mathbb{R}^4 \) are geodesic coordinates on \((M, l_i)\) centred at the point \( \hat{y} = \eta_{i+1}(y) \) (note these coordinates exist, in view of the fact that \( d_{C^k}((d_iB_{1, 4}(x_1) \cap E, l_i), (\varepsilon(i)B_{1, 4}(0), g(i))) \leq \varepsilon(i) \)).

**Proof of Step 1.**
Assume this is not the case. Then we find a sequence for which this is not true. Taking a subsequence (we denote the subsequence of the pairs \( \psi_i, \eta_{i+1} \) also by \( \psi_i, \eta_{i+1} \)), we see that \((\varepsilon(i)B_{1, 2}(0), g(i))\) and \((\varepsilon(i+1)B_{1, 2}(0), g(i+1))\) converge to the same limit space, \((B_{1, 2}(0), g) \subseteq (\mathbb{R}^4 \setminus \{0\}, \delta)/\Gamma \) (in the sense of \( C^k \) convergence described above in Definition 7.3), where \( \Gamma \) is a finite subgroup of \( O(4) \) with finitely many (bounded by \( N \)) elements: the argument in the beginning of the proof of Lemma 3.6 in [Tian], for example, gives us this.

Let us denote by \( Z_i : (\varepsilon(i)B_{1, 2}(0), g(i)) \to (B_{1-\varepsilon(i), 2+\varepsilon(i)}(0), g) \) and \( Z_{i+1} : (\varepsilon(i+1)B_{1, 2}(0), g(i+1)) \to (B_{1-\varepsilon(i), 2+\varepsilon(i)}(0), g) \) the natural maps which are diffeomorphisms and almost \( C^k \) local isometries onto their images: these must exist in view of this convergence.

Let us denote by \( R_i : (E \cap d_iB_{1, 2}(x_1), l_i) \to (B_{1-\varepsilon(i), 2+\varepsilon(i)}(0), g) \) the natural map, which is also a diffeomorphism onto its image and almost an isometry, that arises in this way: \( R_i = Z_i \circ (\psi_i)^{-1} \) (if \( \varepsilon(i) \) changes in the proof, but the new constant \( \tilde{\varepsilon}(i) \to 0 \) as \( i \to \infty \), then we denote \( \tilde{\varepsilon}(i) \) by \( \varepsilon(i) \) again). Then \( R_i \circ \psi_i \) converges (after taking a subsequence) to a map \( \tilde{\pi} : D_{1, 2} \to (B_{1, 2}(0), g) \subseteq (\mathbb{R}^4 \setminus \{0\}, \delta)/\Gamma \) which is a covering map, with \((\tilde{\pi})^*g = \delta \) and \( R_i \circ \eta_{i+1} \) converges (after taking a subsequence) to a map \( \tilde{\pi} : D_{1, 2} \to (B_{1, 2}(0), g) \) which is a covering map with \( \tilde{\pi}^*g = \delta \), and the convergence is in the usual \( C^k \) sense of convergence of maps between fixed smooth Riemannian manifolds [Explanation: \( R_i \circ \psi_i, R_i \circ \eta_{i+1} : D_{1+\varepsilon(i), 2-\varepsilon(i)} \to (B_{1-2\varepsilon(i), 2+2\varepsilon(i)}(0), g) \), have \((R_i \circ \psi_i)^*g \) and \((R_i \circ \eta_{i+1})^*g \) are \( \varepsilon(i) \) close in the \( C^k \) norm to \( \delta \), and hence, taking a subsequence, we obtain maps \( \tilde{\pi} : D_{1, 2} \to (B_{1, 2}(0), g) \) with \( \tilde{\pi}^*g = \tilde{\pi}^*g = \delta \). We work now with \( \tilde{\pi} \): the same argument works for \( \tilde{\pi} \). For any \( x \in D_{1, 2} \) we can find a small neighbourhood \( U \subseteq D_{1, 2} \) with \( x \in U \) such that \( \tilde{\pi}(U) \subseteq gB_s(p) \) where \( gB_s(p) \subseteq (gB_{1, 2}(0), g) \) is a geodesic ball and there exist geodesic coordinates \( \beta : gB_s(p) \to \delta B_s(0) \) (s small enough). Then \( \beta \circ \tilde{\pi} : U \to \mathbb{R}^4 \) is well defined, and has \( \det(D(\beta \circ \tilde{\pi})) = 1 \) and hence \( \tilde{\pi} : D_{1, 2} \to gB_{1, 2}(0) \) is a local diffeomorphism. The map is, per construction, surjective (here the definition of the convergence of annuli from Definition 7.3 is used). It is also proper, since by
construction, \( D_{r,s} \subseteq D_{1,2} \) is mapped onto \((B_{r,s}(0), g) \subseteq (B_{1,2}(0), g(0))\) (here the definition of the convergence of annuli from Definition 7.5 is used). Hence, \( \hat{\pi} \) is a covering map (see, for example, Proposition 2.19 in [Lee]). End of the Explanation.

Hence the two maps differ only by a deck transformation, which is an element \( A \) in \( O(4) \): \( \hat{\pi} = \hat{\pi} \circ A \). Before taking a limit, we can change \( \eta_{i+1} \) by this element, \( \hat{\eta}_{i+1} := \eta_{i+1} \circ A \). Remembering the definitions of \( \eta_{i+1} \) and \( \psi_{i+1} \), we see that we have \( \hat{\eta}_{i+1}(x) = (\eta_{i+1} \circ A)(x) = \psi_{i+1}(A(2x)) = (\psi_{i+1} \circ A)(2x) = ((\psi_{i+1}) \circ (\eta_{i+1}) \circ A)(2x) \).

That is we change the covering map \( \pi_{i+1} \) to the covering map \( \hat{\pi}_{i+1} = \hat{\pi}_{i+1} \circ A \), and then define \( \hat{\eta}_{i+1} := (\psi_{i+1} \circ \hat{\pi}_{i+1})(2x) \): we have this freedom in the choice of our \( \pi_{i+1} \)’s. Now both \( R_i \circ \psi_i \) and \( R_i \circ \eta_{i+1} = R_i \circ \eta_{i+1} \circ A \) converge to \( \hat{\pi} \) in the sense explained above. In particular, returning to \((d_i B_{1,2}(x_1), l_i)\) with \((R_i)^{-1}\) and writing things in geodesic coordinates, we see that \( \hat{\eta}_{i+1} \) is arbitrarily close to \( \psi_i \), which leads to a contradiction. Here we used the following fact. In geodesic coordinates \( \beta : B_s(p) \subseteq (B_{1+\varepsilon(i),2-\varepsilon(i)}(0), g) \rightarrow B_s(0) \subseteq \mathbb{R}^4 \), the metric is \( \delta \). Hence for geodesic coordinates \( \gamma : d_i B_{s/2}(z) \subseteq d_i B_{1/2}(x_1) \rightarrow B_{s/2}(0) \subseteq \mathbb{R}^4 \) with \( R_i(z) = p \), we see \( \beta \circ R_i \circ \gamma^{-1} : B_s(2) \rightarrow \mathbb{R}^4 \) is \( C^k \) close to an element in \( O(4) \), in view of, for example, Corollary 4.12 in [HaComp]. End of the Explanation.

We assume in the following, that we have made the necessary modifications to the \( \varphi_i \)’s (note, that in changing \( \pi_i \) by a deck transformation, we are also changing the \( \varphi_i \)’s and hence the \( \psi_i \)’s), so that the above \( C^k \) closeness of neighbours \( \psi_i, \eta_{i+1} \) for all \( i \in \mathbb{N} \) large enough is guaranteed. These modifications are made inductively: for \( i \in \mathbb{N} \) sufficiently large, first change \( \pi_{i+1} \) by a deck transformation if necessary, then \( \pi_{i+2} \) by a deck transformation if necessary, then \( \pi_{i+3} \) by a deck transformation if necessary, and so on.

**End of Step 1.**

Now, **Step 2**, we explain how to join \( \varphi_i \) and \( \varphi_{i+1} \), assuming we have made the necessary modifications to the \( \varphi_i \)’s, as explained in Step 1. The resulting map, at the unscaled level will be \( \varphi \).

For large \( i \in \mathbb{N} \), we know that \( (\psi_i^{-1} \circ \psi_i) : D_{1+\varepsilon(i),4-\varepsilon(i)} \rightarrow (B_{1,4}(0), g(i)) \) and \( (\psi_i^{-1} \circ \eta_{i+1}) : D_{1/2+\varepsilon(i),2-\varepsilon(i)} \rightarrow (B_{1,4}(0), g(i)) \) are well defined smooth maps which are \( C^k \) close to one another on the common domain of definition \( D_{1+\varepsilon(i),2-\varepsilon(i)} \) and \( C^k \) close to \( \pi_i : D_{1+\varepsilon(i),2-\varepsilon(i)} \rightarrow (B_{1,4}(0), g(i)) \) on \( D_{1+\varepsilon(i),2-\varepsilon(i)} \) in the sense just described. Lifting these maps to \( D_{0,4}(0) \subseteq \mathbb{R}^4 \) with respect to the covering \( \pi_i : D_{0,4}(0) \rightarrow (B_{0,4}(0), g(i)) \), we see that we obtain maps \( \tilde{\psi}_i : D_{1+\varepsilon(i),4-\varepsilon(i)} \rightarrow D_{1,4}(0) \) and \( \tilde{\eta}_{i+1} : D_{1/2+\varepsilon(i),2-\varepsilon(i)} \rightarrow D_{1,2}(0) \) (these maps are lifts with respect to \( \pi_i \), that is \( \pi_i \circ \psi_i = (\psi_i^{-1} \circ \psi_i), \pi_i \circ \tilde{\eta}_{i+1} = (\psi_i^{-1} \circ \eta_{i+1}) \)), and these lifts exist, since the domain of the maps we are lifting are simply connected: see Corollary 11.19 in [Lee2] which are \( C^k \) close to the same element in \( O(4) \) on \( D_{1+\varepsilon(i),2-\varepsilon(i)} \) which is without loss of generality the identity (transform the lifts \( \psi_i, \tilde{\eta}_{i+1} \) by the inverse of this element in the target space: the resulting maps are still lifts). Also \( \tilde{\psi}_i(\delta) \) and \( \tilde{\eta}_{i+1}^*(\delta) \) are \( C^k \) close to \( \delta \), on their domains of definition, and hence \( \tilde{\psi}_i \) is \( C^k \) close to an element in \( O(4) \) on \( D_{1/2+\varepsilon(i),2-\varepsilon(i)} \) and \( \tilde{\eta}_{i+1} \) is \( C^k \) close to an element in \( O(4) \) on \( D_{1/2+\varepsilon(i),2-\varepsilon(i)} \) and using the information in the previous line, this element is the identity in each case.
Defining
\[\tilde{\phi}_i : D_{1/2+\varepsilon(i),4-\varepsilon(i)} \to B_{1/2,4}(x_1),\]
(8.3)
\[\tilde{\phi}_i := v_1 \circ \tau_i \circ (\eta_i + (1-\eta)\eta_{i+1})\]
where \(\eta : \mathbb{R}^4 \to \mathbb{R}_{+}^4\) is a smooth cutoff function, with \(0 \geq \eta \leq 1, \eta = 0\) on \(D_{0,2-\delta_i}\), \(\eta = 1\) on \(D_{2-\delta,\infty}\). (***) we obtain a smooth map, which is equal to \(\eta_{i+1}\) on \(D_{1/2+\varepsilon(i),2-\delta}\) and equal to \(\psi_i\) on \(D_{2-\delta,4-\varepsilon(i)}\), and for which \((v_i)^{-1} \circ \tilde{\phi}_i : D_{1/2+2\varepsilon(i),4-2\varepsilon(i)} \to g(i)B_{1/2,4}(0)\) is \(C^k\) close to \(\tau_i\). The map \(\tilde{\phi}_i\) satisfies
\[(1 - \varepsilon(i))|x| \leq d_i(\tilde{\phi}_i(x), x_1) \leq (1 + \varepsilon(i))|x|\]
on \(D_{1/2+\varepsilon(i),4-\varepsilon(i)}\), by construction. We can now define \(\phi : D_{0,\varepsilon} \to X\setminus\{x_1\}\). For \(x \in [\frac{1-\delta}{2}, \frac{1-\delta}{2}]\) and \(i \in \mathbb{N}\) large, we define \(\phi(x) := (\tilde{\phi}_i)^{(2i+2)x}\). This map is smooth and well defined: fix \(i \in \mathbb{N}\), and let \(x \in [\frac{1-\delta}{2}, \frac{1-\delta}{2}]\). Then \(\phi(x) = \tilde{\phi}_i(2^{i+2})x = \eta_{i+1}(2^{i+2})x = \phi_{i+1}(x)\), and if \(x \in [\frac{1-\delta}{2}, \frac{1-\delta}{2}]\), then \(\phi(x) = \tilde{\phi}_i(2^{i+2})x = \psi_i(2^{i+2})x = \phi_i(x)\). This finishes Step 2.

We examine, in the following, various properties of \(\phi\).

By construction, \(\phi : D_{0,\varepsilon} \to X\) satisfies: \(d_X(\phi(x), x_1) = |x| \leq \varepsilon(|x|x)|\), where \(\varepsilon(|x|x)| \to 0\) as \(|x|x| \to 0\): this follows from (8.3) and the definition of \(\phi\). We consider \(V := \phi^{-1}(\phi(D_{0,\varepsilon}))\) and \(V_i := \phi^{-1}(D_{0,\varepsilon})\). We claim that \(\phi_V : V \to V\) is a covering map if \(\varepsilon > 0\) is small enough. Note: we do not claim that \(V\) or \(V_i\) have smooth boundary. We first note, that the cardinality of \((\phi_V)^{-1}(x)\) for \(x \in V\) is bounded if \(\varepsilon\) is small enough. Assume there are points \(z_1, \ldots, z_K, z_s \neq z_j\) for all \(s \neq j \in \{1, \ldots, K\}\), with \(\phi(z_1) = \phi(z_2) = \ldots = \phi(z_K) = m\). We can always find an \(i \in \mathbb{N}\) with \(z_1 \in [\frac{1-\delta}{2}, \frac{1-\delta}{2}]\), and hence \((1 - \varepsilon(i))|z_1| \leq d_X(m, x_1) \leq (1 + \varepsilon(i))|z_1|\) implies \((1 - \varepsilon(i))\frac{1-\delta}{2} \leq d_X(m, x_1) \leq (1 + \varepsilon(i))\frac{1-\delta}{2}\) and hence \(\frac{(1-\varepsilon(i))(1-\delta)}{2} \leq |z_j| \leq \frac{(1-\varepsilon(i))(1-\delta)}{2}\) for \(j = 1, \ldots, K\). Hence, after scaling by \(2^{i+2}\), we have \(\tilde{z}_1, \ldots, \tilde{z}_K \in [2-11\delta, 4-19\delta]\) with \(\tilde{\phi}_i(\tilde{z}_1) = \tilde{\phi}_i(\tilde{z}_2) = \ldots = \tilde{\phi}_i(\tilde{z}_K)\). At the scaled level, we know that \((v_i)^{-1} \circ \tilde{\phi}_i : D_{1/2+2\varepsilon(i),4-2\varepsilon(i)} \to B_{1/2,4}(x_1)\) is \(C^k\) close to \(\tau_i\), the standard projection, and the pull back of \(g(i)\) with this map is \(C^k\) close to \(\delta\) on \(D_{1/2+2\varepsilon(i),4-2\varepsilon(i)}\). In fact \((v_i)^{-1} \circ \tilde{\phi}_i \circ h_i : D_{1/2+2\varepsilon(i),4-2\varepsilon(i)} \to \mathbb{R}^4\) is \(C^k\) close to the identity. In particular, \((v_i)^{-1} \circ \tilde{\phi}_i(B_{s/2}(z)) \subseteq B_{s/(v_i)^{-1} \circ \tilde{\phi}_i(z)}\) for any \(z \in [2-11\delta, 4-19\delta]\) for \(0 < s \leq \frac{1}{10}\) small and let \(\psi : g(i)B_{s/2}(\tilde{z}_j) \to B_{s/2}(0) \subseteq \mathbb{R}^4\) be geodesic coordinates in \((B_{1/2,4}, g_i(0))\), where \(\tilde{z}_j = (v_i)^{-1} \circ \tilde{\phi}_i(\tilde{z}_j)\). The map \(\psi \circ (v_i)^{-1} \circ \tilde{\phi}_i : B_{s/2}(\tilde{z}_j) \to \mathbb{R}^4\) is \(C^k\) close to an isometry \(B(i, j) = A(i, j) + r_i\) of \(\mathbb{R}^4\), where \(A(i, j) \in O(4)\) and \(r_i = \tau_j(x) = x - \tilde{z}_j\), and hence after a rotation in the geodesic coordinates and a translation, \(C^k\) close to the identity. In particular, this map is a diffeomorphism when restricted to \(B_{s/2}(\tilde{z}_j)\), and hence \(\tilde{z}_j \notin B_{s/2}(\tilde{z}_j)\) for all \(j \neq i\). Hence, \(\text{vol}(D_{1/2,4}) \geq \sum_{j=1}^{K} \text{vol}(B_{s/2}(\tilde{z}_j)) \geq K\omega_4(s/2)^4\) which leads to a contradiction if \(K\) is too large.

If we scale the map \(\phi : D_{[2-11\delta, 2-19\delta]} \to X\) to \(2^{i+2}\), that is let \(\phi : D_{[2-11\delta, 2-19\delta]} \to X\) be defined by \(\phi(x) = \phi(2^{i+2}x)\), then we obtain the map \(\tilde{\phi}_i : \phi_i(2^{i+2}, 16-14\delta)\). The argument above, shows that \((v_i)^{-1} \circ \tilde{\phi}_i|B_{s/2}(z) : B_{s/2}(z) \rightarrow \mathbb{R}^4\) is a diffeomorphism for all \(|z| \leq [2-10\delta, 4-18\delta]\) if \(s \leq \frac{1}{10}\) and \(i\) sufficiently large. That is \(\tilde{\phi}_i|B_{s/2}(z) : \phi_i(2^{i+2}, 16-14\delta)\) is a diffeomorphism for all \(|z| \leq [2-10\delta, 4-18\delta]\)
and hence $\tilde{\varphi} |_{D_{(2/\alpha)\epsilon, \epsilon - \alpha \epsilon}}$ is a local diffeomorphism, which tells us, scaling back, that $\varphi : D_{(1-\alpha \epsilon, \epsilon - (1-\alpha \epsilon) \epsilon / \alpha)} \rightarrow X$ is a local diffeomorphism.

That is, $\varphi : D_{0, \epsilon} \rightarrow X$ is a local diffeomorphism, if $\epsilon > 0$ is small enough.

Hence $V := \varphi(D_{0, \epsilon})$ is open if $\epsilon > 0$ is small enough (this corresponds to $i$ being sufficiently large), and $\varphi : \tilde{V} := \varphi^{-1}(V) \rightarrow V$ is a local diffeomorphism and an open map. $V$ is connected, as it is the image under a continuous map of a connected region. In fact $\tilde{V}$ is also connected: this will be shown below.

$\varphi : \tilde{V} \rightarrow V$ is proper: Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $K \subseteq V$, where $K$ is compact in $V$. This means, there is a subsequence of $x_i$ (also denoted $x_i$) such that $x_i \rightarrow x \in K \subseteq V$, $x = \varphi(m)$ for some $m \in D_{0, \epsilon}$. Let $z_1, z_2, \ldots, z_N \in \varphi^{-1}(x)$ be the finitely many points in $\tilde{V}$ with $\varphi(z_j) = m$. We can choose a small neighbourhood $U_j$ of each one, such that $U_i \subset \subset \tilde{V}$ and $\varphi(U_i) : U_j \rightarrow \varphi(U_j)$ is a diffeomorphism, and without loss of generality $\varphi(U_j) = U \subset \subset V$ for all $j$, and $\varphi(m) \in U$. Hence any sequence $y_k \in \varphi^{-1}(K)$ with $\varphi(y_k) = x_k$ has a convergent subsequence, $y_k \rightarrow z_i$ as $k \rightarrow \infty$ for some $z_i \in \{z_1, z_2, \ldots, z_N\}$. Hence $(\varphi|_E)^{-1}(K)$ is sequentially compact in $\tilde{V}$. That is $\varphi : \tilde{V} \rightarrow V$ is proper. That is, $\varphi : \tilde{V} \rightarrow V$ is a proper, surjective, local diffeomorphism. In particular lifts $\tilde{\gamma} : I \rightarrow \tilde{V}$ of curves $\gamma : I \rightarrow V$, $I = [a, b] \subseteq \mathbb{R}$, always exist and are uniquely determined by their starting points $\tilde{\gamma}(0)$ which is an arbitrary point in $\varphi^{-1}(\gamma(0))$.

$\tilde{V}$ is also connected. Let $\tilde{x}$ and $\tilde{y}$ be points in $\tilde{V}$ and $x = \varphi(\tilde{x}) \in V$, $y = \varphi(\tilde{y}) \in V$. $x = \varphi(\tilde{x}) \in \varphi(D_{0, \epsilon})$ implies $x = \varphi(x_0)$ for an $x_0 \in D_{0, \epsilon}$. Let $x_1$ be the point $x_1 = x_0/4$. Then $x_1 \in D_{0, \epsilon/3}$ and $\varphi^{-1}(\varphi(x_1)) \in D_{0, \epsilon/3}$ if $i$ is sufficiently large, in view of the construction of $\varphi$ (see the above).

Joining $x_0$ to $x_1$ with a ray $\alpha : I \rightarrow D_{0, \epsilon}$ (w.r.t to the euclidean metric) which points into 0 and pushing this down to $\tilde{V}$ again with $\varphi$, we obtain a continuous map $\sigma = \varphi \circ \alpha : I \rightarrow \tilde{V}$ with $\sigma(0) = \varphi(x_0) = \varphi(\tilde{x})$ and $\sigma(1) = \varphi(\tilde{x})_1$. Taking the lift of this map, and using the starting point $\tilde{x}$, we obtain a continuous curve $\tilde{\sigma} : I \rightarrow \tilde{V}$ with $\tilde{\sigma}(0) = \tilde{x}$ and $\tilde{\sigma}(1) \in \varphi^{-1}(\varphi(x_1)) \in D_{0, \epsilon/3}$. We may perform the same procedure with $y$ to get a continuous curve $\tilde{\beta} : I \rightarrow \tilde{V}$ with $\tilde{\beta}(0) = \tilde{y}$ and $\tilde{\beta}(1) \in D_{0, \epsilon/3}$. We may join $\tilde{\beta}(1)$ to $\alpha(1)$ in $D_{0, \epsilon/3} \subset \tilde{V}$ with a curve $T : I \rightarrow D_{0, \epsilon/3}$, as this space is connected. Hence, following the curve $\tilde{\sigma}$ from $\tilde{\sigma}(0) = \tilde{x}$ to $\tilde{\sigma}(1)$ in $\tilde{V}$ and then from $\tilde{\sigma}(1)$ to $\tilde{\beta}(1)$ with $T$ and then from $\tilde{\beta}(1)$ to $\tilde{\beta}(0)$ by going backwards along the curve $\tilde{\beta}$, we see that we have constructed a continuous curve in $\tilde{V}$ from $\tilde{x}$ to $\tilde{y}$ as required.

Hence $\tilde{V}$ is also connected.

That is, $\varphi : \tilde{V} \rightarrow V$ is a proper, surjective, local diffeomorphism, between two path connected spaces, and hence $\varphi : \tilde{V} \rightarrow V$ is a covering map (see Proposition 2.19 in [Lee]).

In fact, $\tilde{V}$ is simply connected if $\epsilon > 0$ is sufficiently small, and hence $\tilde{V}$ is the universal covering space of $V$. We explain this now. Let $i$ be sufficiently large, and we consider the map $\tilde{\varphi}_i : D_{1/2 + \epsilon(i), \epsilon - \epsilon(i)} \rightarrow X$ from above. $\varphi_i \circ \tilde{\varphi}_i : D_{1/2 + \epsilon(i), \epsilon - \epsilon(i)} \rightarrow \varphi_i(B_{1/2 + \epsilon(i)}(g(i)))$ is $C^k$ close to $\varphi(i)$ as shown above. In particular, $\tilde{\varphi}_i |_{B_{s/4}(x)} : B_{s/4}(x) \rightarrow X$ is a diffeomorphism onto its image and $\tilde{\varphi}_i(B_{s/8}(x)) \subseteq B_{s/4}(z)$ and $B_{s/4}(z) \subseteq \tilde{\varphi}_i(B_{s/8}(x))$ for all $x \in D_{2s/5, 2s/5}$ for all $z$ with $z = \tilde{\varphi}_i(x)$ for a fixed
Also, as we noted above, \(V_γ < r\) for all \(j = 1, \ldots, N\). \(θ_j := (\tilde{ν}_1)^{-1}_{B_j(p)} \circ \tilde{ν}_1 : B_\gamma(p) \to B_j(p)\) is \(C^k\) close to an element in \(O(4)\), and has \(θ_j(p_j) = p\). \(θ_j\) is \(C^k\) close to an element in \(O(4)\) means \(θ_j(x) = A_j \cdot x + β_{i,j}(x)\) for all \(x \in B_\gamma(p_j)\), where \(|β_{i,j}| \leq ε(i)|\) and \(A_j \in O(4)\), and hence \(A_j(p_j) = p + β_{i,j}(p_j)\) where \(|β_{i,j}(p_j)| \leq ε(i)|\). In particular,
\[
\partial_r(θ_j((1 - r)p_j)) = -Dθ_j((1 - r)p_j) \cdot p_j = -A_j \cdot p_j - Dβ_{i,j}((1 - r)p_j) \cdot p_j = -p + v_j(r),
\]
(8.5)
where \(|v_j(r)| \leq ε(i)|\). That is, using \(θ_j(p_j) = p \in D_{2,5/2}(0)\), we see that \(θ_j((1 - r)p_j) \in D_{2-s,5/2}(0)\) for all \(r \in [0, s/100]\).

That is \((1 - r)p_j \in (θ_j)^{-1}(D_{2-s,5/2}(0)) \subseteq (\tilde{ν}_1)^{-1}(\tilde{ν}_1(D_{2-s,5/2}))\) for all \(r \in [0, s/100]\):
\[
θ_j((1 - r_0)p_j) - p = θ_j((1 - r_0)p_j) - θ_j(p_j) = \int_{r_0}^0 \partial_r(θ_j((1 - r)p_j)) dr = -r_0p + r_0v_j,
\]
with \(v_j \leq ε(i)|\) implies \(\tilde{ν}_1((1 - r_0)p_j) = (1 - r_0)p + r_0v_j\) and hence \(|\tilde{ν}_1((1 - r)p_j)| = |(1 - r_0)p + r_0v_j| \leq (5/2)(1 - r_0) + ε(i)r_0 < (5/2)|\)(respectively \(≥ (1 - r_0)2 - r_0ε(i) \geq 2 - s\))

As \(p \in D_{2,5/2}\) was arbitrary, we see \((1 - r)q \in (\tilde{ν}_1)^{-1}(\tilde{ν}_1(D_{2-s,5/2}))\) for all \(r \in [0, s/100]\) for all \(q \in (\tilde{ν}_1)^{-1}(\tilde{ν}_1(D_{2,5/2}))\) for large enough \(i\). Furthermore \((1 - s/100)q \in D_{2-s,5/2} - (s/200) \subseteq D_{2-s,5/2}\) for large enough \(i\). We assume that this \(i\) corresponds to \(ε\) : that is \(ε = (\tilde{ν}_1)^{-1}(\tilde{ν}_1(D_{2,5/2}))\). Then, we have just shown that \((1 - r)\tilde{ν} \in φ^{-1}(φ(D_0,ε))\) for all \(r \in [0, s/100]\), for all \(q \in φ^{-1}(φ(D_0,ε))\), and we also know that \((1 - s/100)q \in D_0,ε \subseteq \tilde{V} := φ^{-1}(φ(D_0,ε))\). That is, there exists a smooth map \(c : [0, s/100] \times V \to V, c(r, x) = (1 - r)x + (1 - η(x))x,\) where \(η\) is a rotationally symmetric cut off function on \(D_0,ε\) with, \(0 \leq η \leq 1,\) \(η = 1\) on \(D_{ε/2,ε}\) and \(η = 0\) on \(D_{0,ε/4}\), such that \(c(0, \cdot) = Id\) and \((c(ε/100, \cdot),(V)) \subseteq D_0,ε,\) and \(D_{0,ε}\) is a simply connected space. Hence \(V\) is itself simply connected.

Notice also, that \(E \cap B_r(x_1)\) is contained in \(V\) for \(r\) small enough (**). We explain this now. There is some \(x \in E \cap B_r(x_1)\) with \(x \in V\) by construction. Let \(γ : [0, 1] \to B_r(x_1)\) be a smooth path of finite length in \(B_r(x_1)\) with \(γ(0) = x \in E \cap V \cap B_r(x_1)\), and \(|γ'(t)| ≤ C\). Let \(s\) be a value for which \(γ(t) \in E \cap V\) for all \(t < s\) and \(γ(s) \in E \cap (V)^c\). Let \(γ : [0, s) \to X \cap V\) with \(φ\), we get a curve \(γ : [0, s) \to D_0,ε\) with \(r < ε\). Clearly, \(γ(t) \to d \in D_0,ε\) as \(t \nearrow s\). Hence \(γ(t) = φ(γ(t)) \to φ(d) \in V\). On the other hand, \(γ(t) \to (γ(s)\) as \(t \to s\). Hence \(γ(s) = φ(d) \in V\), per definition of \(V\), which is a contradiction. Hence \(γ\) is also a curve in \(V\), that is \(E \cap B_r(x_1)\) is contained in \(V\).

Also, \(V \subseteq E\) if \(ε > 0\) is small enough in the definition of \(V := φ(D_0,ε)\). As we noted above, \(V\) is connected. Furthermore, \(V \cap E \neq \emptyset\) by definition of \(V\), and \(E\) is a connected component of \(B_{r_0}(x_1)\), and, without loss of generality, \(ε \ll r_0\). This means that we have: \(V\) is connected, \(V \subseteq B_{r_0}(x_1)\), and \(E\) is a connected component of \(B_{r_0}(x_1)\), and \(V \cap E \neq \emptyset\). Hence \(V\) is contained in \(E\).

We will see that for \(r_0\) small enough in the above theorem, that in fact \(dx B_r(x_1) \subseteq X\) has exactly one component for all \(r \leq r_0\). This will follow
by considering the manifolds \((M, g_i, p_1)\), which approximate a blow up \((X, d_i := \sqrt{g_i}d_X, x_1)\) in the sense explained above in the Approximation Theorem, Theorem 7.4.

The approximations and the blow ups of \(X\) itself will converge to a metric cone of the form \(\mathbb{R}^4 \setminus \{0\}/\Gamma\) for some \(\Gamma\), where \(\Gamma\) is a finite subgroup of \(O(4)\), and the number of elements in \(\Gamma\) is bounded by \(C(\sigma_0, \sigma_1) < \infty\). That is, each blow up near a singular point consists of exactly one cone. This will show us that for each \(i, B_{r_0}(x_i) \setminus \{x_i\} \subseteq X\) has exactly one component. These facts are collected in the following theorem.

**Theorem 8.3.** \(X\) is a \(C^0\) Riemannian orbifold in the following sense.

\begin{enumerate}[(i)]
\item \(X \setminus \{x_1, \ldots, x_L\}\) is a manifold, with the structure explained above in Lemmata 6.3 and 6.6.
\item There exists an \(r_0 > 0\) small such that the following is true. Let \(x_1 \in X\) be one of the singular points. Then \(B_{r}(x_1) \setminus \{x_1\}\) is connected for all \(r \leq r_0\).
\item There exists a \(0 < \hat{r} \leq r_0\) and a smooth map \(\varphi : D_{0, \hat{r}} \to X \setminus \{x_1, \ldots, x_L\}\) such that \(\varphi : \hat{V} \to V\) is a covering map, \(V\) and \(\hat{V}\) are connected sets, \(\hat{V}\) is simply connected, and, for all \(r \leq \hat{r}\), we have \(\varphi(\Sigma(0)) \subseteq d_sB_{r(1 - \varepsilon_1(r)), r(1 + \varepsilon_1(r))}\) and
\[
\sup_{D_{0, r}} |(\varphi)^*l - \delta| \leq \varepsilon_1(r)
\]
where \(\varepsilon_1(r) \leq \frac{\epsilon}{100}\) is a decreasing function with \(\lim_{r \to 0} \varepsilon_1(r) = 0\), and \(V := \varphi(D_{0, \frac{\epsilon}{100}}), \hat{V} := \varphi^{-1}(V) \subseteq D_{0, \hat{r}}, \text{ and } \Sigma(0) := \{x \in \mathbb{R}^4 \mid |x| = r\}\), and here \(\delta\) is the standard euclidean metric on \(\mathbb{R}^4\) or subsets thereof.
\end{enumerate}

**Remark 8.4.** Using the facts *** mentioned at the end of the construction of \(\varphi\), we see that \(B_{r}(x_1) \subseteq V \cup \{x_1\}\) for all \(r \leq r_0\) small enough, and hence \(V \cup \{x_1\}\) is an open neighbourhood of \(x_1\) in \(X\).

**Proof.** Fix \(x_1 \in \{x_1, \ldots, x_L\}\) and assume that \(B_{r_0}(x_1) \setminus \{x_1\}, r_0\) as above, contains more than one component: \(B_{r_0}(x_1) \setminus \{x_1\} = \bigcup_{i=1}^N E_i\) with \(E_i \cap E_j = \emptyset\) for all \(i, j \in \{1, \ldots, N\}, i \neq j\), and \(N \geq 2\). Let \(E, G\) denote two distinct components, \(E := E_1 \neq E_2 := G\). We use the following notation: for \(p \in E \cap B_{r_0/4}(x_1)\) and \(q \in G \cap B_{r_0/4}(x_1)\), \(\hat{q}, \hat{p}\) will denote the unique points in \(M\) with \(f(\hat{q}) = q, f(\hat{p}) = p\); these points are unique since \(p, q\) are not singular in \(X\).

Our proof is essentially a modified version of the Neck Lemma, Lemma 1.2 of [AnCh2], of M. Anderson and J. Cheeger adapted to our situation. Note that we do not have Ricci bounded from below (as they do) for our approximating sequences, but we do know that they all satisfy \(\int_M |\text{Rc}|(g_i) d\mu_{g_i} \to 0\) as \(i \to \infty\). Hence we can use the volume estimates of P. Petersen G.-F. Wei, [PeWe], in place of the Bishop-Gromov volume estimates. The estimates we require do not appear in [PeWe], although they follow after making minor modifications to the proof of their estimates. We have included the estimates and a proof thereof in Appendix C.

Let \((M, g_i)\) and \((X, d_i)\) be as in the Approximation Theorem, Theorem 7.4. Let \(E\) be as above, and let \(z_i \in E \cap B_{1/4, 10}(x_1)\) satisfy \(d_i(x_1, z_i) = 1\) and \(v_i \in T_{z_i}E\) be a vector such that there is a length minimizing geodesic \(\gamma_i : [0, 1] \to X\) on \((X, d_i)\)
with \( \gamma_i(0) = z_i, \gamma_i(1) = x_1, \gamma_i'(0) = v_i, \) and \( |\gamma_i'(t)|_{l_i} = 1 \) for all \( t \in [0, 1) \) (\( \gamma_i' \) makes sense on \( E \), since \( x_1 \notin E \) for all \( i \in \{x_1, \ldots, x_L\} \), and \( (X \setminus \{x_1, \ldots, x_L\}, l_i) \) is a smooth Riemannian manifold). We define \( \hat{z}_i := f^{-1}(z_i) \in M \), the corresponding point in \( M \), and \( \hat{v}_i := f^*v_i \), the corresponding vector in \( T_{\hat{z}_i}M \), where \( (M, g_i) \) are as in the Approximation Theorem. (TT)

We remember, that \( \text{inj}(b) > i_0/1000 \) for all \( b \in E \cap d_iB_{1/10}(x_1) \) due to the injectivity radius estimate of Cheeger-Gromov-Taylor (Theorem 4.3 in [CGT]) and the non-inflating/non-collapsing estimates. For any \( i \), \( (T_{\hat{z}_i}M, g_i(\hat{z}_i) = g(i)) \) is isometric (as a vector space) to \( (\mathbb{R}^4, \delta) \). We will make this identification in the following, sometimes without further mention.

Let \( S_i \subseteq S_i^3(0) \) denote the set of vectors \( \hat{w} \) in \( S_i^3(0) \subseteq (\mathbb{R}^4, \delta) = (T_{\hat{z}_i}M, g(\hat{z}_i)) \) (using the isometry above) which satisfy \( \angle(\hat{v}_i, \hat{w}) \leq \alpha \) with respect to the euclidean metric, where \( \alpha > 0 \) is a small but positive angle. We claim

**Claim 1:** there exists a small \( \varepsilon(\alpha) > 0 \) such that any geodesic \( \exp(g_i)z_i, (m) : [0, 100] \to M \) does not go through \( g_iB_2(x_i) \) if \( m \in S_i^3(0) \cap (S_i)^{\varepsilon} \) and \( 0 < \varepsilon \leq \varepsilon(\alpha) \) is small enough, and \( i \geq N \) large enough.

**Proof of Claim 1.**

Let \( \alpha > 0 \) be fixed. We assume we can find \( \tilde{w}_i \in (S_i)^{\varepsilon} \cap S_i^3(0) \subseteq \mathbb{R}^4 = T_{\tilde{z}_i}M \) and \( r_i \in (0, 100) \) such that \( g_i(\tilde{z}_i)(\tilde{w}_i, \tilde{v}_i) > \alpha \), and \( \exp(g_i)(r_i \tilde{w}_i) \in \partial(g_iB_2(x_i)) \) but \( \exp(g_i)(s \tilde{w}_i) \notin \partial(g_iB_2(x_i)) \) for all \( 0 \leq s < r_i \) for \( i \) arbitrarily large. We shall see, that this leads to a contradiction, if \( \varepsilon \leq \varepsilon(\alpha) \) is chosen small enough. Let \( w_i \) be the push forward back to \( X \), \( w_i := f_*(\tilde{w}_i) \).

For any \( \delta > 0 \), we know that \( E \cap d_iB_{4,1/\delta}(x_1) \) converges, after taking a subsequence if necessary, to \( (gB_4) / \Gamma \cap \mathbb{R}^4 \subseteq \mathbb{R}^4 \) in the sense of convergence given in Definition 7.3 in [Tian]: there exist diffeomorphisms \( F_i : d_iB_{4,1/\delta}(x_1) \cap \Gamma \to (\mathbb{R}^4 \setminus \{0\}) / \Gamma \) for \( i \) large enough, such that \( (F_i)_*l_i \to g \) in the \( C^k \) sense, where \( g \) is the Riemannian metric on \( (\mathbb{R}^4 \setminus \{0\}) / \Gamma \) and \( d_1(F_1^{-1}(x), x_1) - |x| \leq \varepsilon(i) \to 0 \) as \( i \to \infty \) for all \( x \in B_{2\delta,\varepsilon}(0) \), where \( |x| = d([x], [0]) \) here refers to the standard norm in \( \mathbb{R}^4 \) of \( x \), and \( |x| = \Gamma(x) \cap \Gamma \in \Gamma \). In particular, the curves \( F_i \circ \exp(\tilde{v}_i) : [0, 1 - 3\delta] \to (\mathbb{R}^4 \setminus \{0\}) / \Gamma \) and \( F_i \circ \exp(\tilde{w}_i) : [0, r_i] \to (\mathbb{R}^4 \setminus \{0\}) / \Gamma \), are well defined and converge smoothly to geodesic curves \( \gamma : [0, 1 - 3\delta] \to (\mathbb{R}^4 \setminus \{0\}) / \Gamma \) respectively \( \hat{\gamma} : [0, r_i] \to (\mathbb{R}^4 \setminus \{0\}) / \Gamma \). If \( \gamma \) converges to \( \hat{\gamma} \) along \( \gamma(1) = \hat{\gamma}(0) = z \) with \( d([z], [0]) = 1 \), and \( g(\gamma'(0), \gamma'(0)) \geq \alpha \) and \( g(1 - 3\delta) \in gB_{0,3\delta}(0) \). Here, we can choose \( \delta > 0 \) arbitrarily small. By considering the lift of the curve \( \gamma \) to \( \mathbb{R}^4 \setminus \{0\} \) (which must be a straight line in \( \mathbb{R}^4 \setminus \{0\} \)), and using that \( \delta \) is arbitrarily small, we see that \( \gamma : [0, 1 - 3\delta] \to (\mathbb{R}^4 \setminus \{0\}) / \Gamma \) is arbitrarily close to the projection of a ray coming out of \( 0 \) (in \( \mathbb{R}^4 \) on \( [0, 1 - \sigma] \) for \( \sigma > 0 \) as small as we like (choose \( \delta << \sigma \)). Now lifting \( \hat{\gamma} \) to a curve in \( \mathbb{R}^4 \setminus \{0\} \) (which is also a straight line in \( \mathbb{R}^4 \setminus \{0\} \)), and using the fact that \( g(\hat{\gamma}'(0), \hat{\gamma}'(0)) \geq \alpha \) (which is also true for the lift), we see that \( \hat{\gamma}(r) \in (B_{0,2\varepsilon(\alpha)}(0))^{\varepsilon} \), for some \( \varepsilon(\alpha) > 0 \). This leads to a contradiction to the fact that \( \hat{\gamma}(r) \in gB_{0,3\delta}(0) \) if \( \varepsilon > 0 \) is chosen smaller than say \( \varepsilon(\alpha)/6 \).

This finishes the proof of Claim 1.
Claim 2: For all \( z \in f^{-1}(E \cap d_i B_{\frac{4}{5}}(x_1)) \) and \( w \in f^{-1}(G \cap d_i B_{\frac{4}{5}}(x_1)) \), any length minimising geodesic from \( z \) to \( w \) must go through \( g_i B_{\varepsilon(i)}(p_1) \), where \( \varepsilon(i) \to 0 \) as \( i \to \infty \).

Proof of Claim 2.
Assume we can find \( i \) arbitrarily large, and points \( \hat{z}_i \in f^{-1}(E \cap d_i B_{\frac{4}{5}}(x_1)) \) and \( \hat{w}_i \in f^{-1}(G \cap d_i B_{\frac{4}{5}}(x_1)) \) and a length minimising geodesic \( \hat{\gamma}_i : [0, r_i] \to M \) (w.r.t. \( g_i \)), parameterised by arclength, such that \( \hat{\gamma}_i(0) = \hat{z}_i \) and \( \hat{\gamma}_i(r_i) = \hat{w}_i \), for which \( \hat{\gamma}_i \) doesn’t go through \( d_i B_{\varepsilon(i)}(p_1) \) for some \( \sigma > 0 \) (*) .

Note, the Approximation Theorem, Theorem 7.4 guarantees that once again using the Approximation Theorem, \( \gamma \) minimising geodesic from \( \hat{z}_i \) and goes from \( \gamma \) to \( \gamma \), such that \( \gamma_i(0) = \hat{z}_i \) and \( \gamma_i(r_i) = \hat{w}_i \), for which \( \gamma_i \) doesn’t go through \( d_i B_{\varepsilon(i)}(p_1) \) for some \( \sigma > 0 \) .

Proof of Claim 3.
Let \( \gamma_i : [0, r_i] \to X \) be the curve \( \gamma_i := f \circ \hat{\gamma}_i \). The Approximation Theorem guarantees that \( \gamma_i([0, r_i]) \subseteq d_i B_{41}(x_1) \) as we just noted (*).

There must be a first value \( r_0(i) \in [0, r_i] \) with \( \gamma_i(r_0(i)) = x_1 \): the curve is continuous and goes from \( E \) to \( G \), and so there must be some point \( r_0(i) \) with \( \gamma_i(r_0(i)) \in \partial E \).

\( \gamma_i(r_0(i)) \) must be equal to \( x_1 \), since \( d_i(\partial E \setminus \{x_1\}, x_1) \to \infty \) as \( i \to \infty \) and \( \gamma_i([0, r_i]) \subseteq d_i B_{41}(x_1) \).

By assumption, \( \gamma_i(r) \notin g_i B_{\varepsilon(i)}(p_1) \) for all \( r \in [0, r_i] \). But then, once again by the Approximation Theorem, \( f \circ \gamma_i([0, r_i]) \cap d_i B_{\varepsilon(i)}(x_1) = \emptyset \), which contradicts the fact that \( f \circ \gamma_i(r_0(i)) = x_1 \).

End of the proof of Claim 2.

Let \( z_i \in E \cap (d_i B_{\frac{4}{5}}(x_1)) \), \( \hat{z}_i \in S_i \), \( v_i \), \( \hat{v}_i \) be as above (see (TT) above): \( S_i \subseteq S^3(0) \) denotes the set of vectors \( \hat{v} \) in \( S(0) \subseteq \mathbb{R}^4 = T_{\hat{z}_i} M \) which satisfy \( \angle(\hat{v}, \hat{w}) \leq \alpha \), where we have identified vectors \( T_{\hat{z}_i} M \) and vectors in \( \mathbb{R}^4 \) using the isometry between \((T_{\hat{z}_i} M, g_i(0))\) and \((\mathbb{R}^4, \delta)\) explained above.

Let \( W_r := \{ \exp(g_i)_{\hat{z}_i}(t\hat{w}) | t \in [0, r], \hat{w} \in S_i \} \), \( V_r := \{ \exp(g_i)_{\hat{z}_i}(s\hat{w}) | s \in [0, r], \hat{w} \in S_i \} \) and \( \exp(g_i)_{\hat{z}_i}(\hat{w}) : [0, s] \to M \) is a minimising geodesic \}. \( E_r \) is the set in Euclidean space which corresponds to \( W_r \): \( E_r := \{ t\beta | \beta \in S_i, \angle(\beta, e_1) \leq \alpha, t \leq r \} \)

Claim 3: Let \( \hat{Z} := f^{-1}(G \cap d_i B_{1/2,1}(x_1)) \). Then \( \hat{Z} \subseteq V_3 \), if \( i \) is large enough.

Proof of Claim 3. Let \( \gamma(\cdot) := \exp(g_i)_{\hat{z}_i}(\cdot m_i) : [0, r_i] \to M \) be a length minimising geodesic from \( \hat{z}_i \) to a point \( \hat{a}_i \in f^{-1}(G \cap d_i B_{1/2,1}(x_1)) \) parameterised by arclength. Using the Approximation Theorem, Theorem 7.4 we must have \( \hat{a}_i, \hat{z}_i \in g_i B_{\varepsilon(i)}(p_1) \), since \( d_i(\hat{z}_i, x_1) = 1 \) and hence we must have \( r_i = d(g_i)(\hat{a}_i, \hat{z}_i) \leq \delta/2 \).
Assume $m_i \in (S_i)^c$. Claim 1 tells us that the curve does not go through $B_{\varepsilon(\alpha)}(p_1)$ for some $\varepsilon(\alpha) > 0$ if $i$ is large enough. But this contradicts Claim 2, if $i$ is large enough. Hence $m_i \in S_i$ and hence $\tilde{Z} \subseteq V_3$ in view of the definition of these two sets.

**End of the proof of Claim 3.**

Note for later, that $\text{vol}(g_i)(\tilde{Z}) \geq \theta > 0$ for $i$ large enough, where this $\theta$ is independent of $\alpha, i$, and independent of which subsequence we take, in view of the fact that $(\tilde{Z}_i, g_i)$ converges to $(B_{1/2}(0)/\Gamma)$ in the sense of $C^k$ manifold convergence given in Definition 7.5 (this follows from the Approximation Theorem 7.4 and Lemma 3.6 of [Tim]). Since $\text{vol}(\tilde{Z}) \geq \theta$ for $i$ large enough, we may write

$$\text{vol}(\tilde{Z}) = \text{vol}(\tilde{Z}_i, g_i) \leq c_{\alpha, i} \text{vol}(\Gamma)$$

where $c_{\alpha, i}$ is independent of $\alpha$ and $i$. Using this in equation (8.8) we get

$$\text{vol}(\tilde{Z}) \leq c_{\alpha, i} \text{vol}(\Gamma) \leq \text{vol}(\tilde{Z}) \geq \theta$$

for some fixed $\theta > 0$ since on each component the metric approaches the euclidean metric divided out by a finite subgroup of $O(4)$. Recall that $d_i B_{\frac{\pi}{4}} \cap E$ converges to $(g B_{\frac{\pi}{4}}, g) \subseteq (\mathbb{R}^4\{0\})/\Gamma$ in the sense of Definition 7.5 using a map $F_i : d_i B_{\frac{\pi}{4}} \cap E \to g B_{\frac{\pi}{4}}$, and $g_i B_{\frac{\pi}{4}}(p_1)$ is $\varepsilon(i)$ $C^k$ close to $d_i B_{\frac{\pi}{4}}$ in the sense of Definition 7.5 using the map $f$, in view of the Approximation Theorem, Theorem 7.4. Since $V_r \subseteq W_r$, we have $\text{vol}(V_{1/2}) \leq \text{vol}(W_{1/2}) \leq (\frac{1}{2})^3$ which goes to zero as $\alpha \to 0$ [Explanation].

Let $F_i \circ f(z_i) : x_i$ is at a distance $1 \pm \varepsilon(i)$ away from $0$. We use the fact that $f(W_{1/2}) \subseteq E$ in the following without further mention: this follows from the fact that $f(W_{1/2}) \cap \{x_1, \ldots, x_k\} = \emptyset$, which follows from the Approximation Theorem.

Using the fact that $(F_i \circ f) \ast (g_i) \to g$ on $g B_{\frac{\pi}{4} + \varepsilon(i)}$ as $i \to \infty$, we see that $(F_i \circ f) \ast (S_i) \subseteq \tilde{S}_i$, where $\tilde{S}_i := \{v \in T_{x_i}(\mathbb{R}^4\{0\})/\Gamma) \mid g(x_i)(n_i, v) \leq \alpha + \varepsilon(i), |v|_g \in
where \( \tilde{\theta} \) is the standard projection from \( \mathbb{R}^4 \) to \( \mathbb{R}^4 \). This leads to a contradiction if \( \alpha \) is chosen small enough and then \( i \) is chosen large enough, since \( (\int_M |\text{Rc}|^4)^{1/2} \) goes to zero as \( i \to \infty \).

That is, there cannot be two distinct components \( E \) and \( G \) as described above. \( \square \)

9. Extending the flow

Since \((X,d_X)\) is a \( C^0 \) Riemannian orbifold, it is possible to extend the flow past the singularity using the orbifold Ricci flow. We have

**Theorem 9.1.** Let everything be as above. Then there exists a smooth orbifold, \( \bar{X} \), with finitely many orbifold points, \( v_1, \ldots, v_L \), and a smooth solution to the orbifold Ricci flow, \((\bar{X}, h(t))_{t \in (0, \varepsilon)}\) for some \( \varepsilon > 0 \), such that \((\bar{X}, \bar{d}(h(t))) \to (X, d_X)\) in the Gromov-Hausdorff sense as \( t \searrow 0 \).

**Proof:** Fix \( x_i \in \{x_1, \ldots, x_L\} \subseteq X \), where \( \{x_1, \ldots, x_L\} \) are defined in Theorem 6.5. On \( d_X B_{\varepsilon}(x_i) \) we have a potentially non-smooth orbifold structure given by the map \( \varphi \): the non-smoothness may also be present without considering the Riemannian metric, as we now explain. As explained above, if we consider \( \bar{V} := \varphi^{-1}(\varphi(D_0,\varepsilon)) \) and \( \bar{V} := \varphi(D_0,\varepsilon) \), then \( \varphi|_{\bar{V}} : \bar{V} \to V \) is a covering map, \( \bar{V} \) is connected, and \( \bar{V} \) is simply connected, if \( \varepsilon > 0 \) is small enough.

Let \( x \in \bar{V} \) be fixed, and \( G_1, \ldots, G_N : \bar{V} \to \bar{V} \) the deck transformations, which are uniquely determined by \( G_i(x) = x_i \), where \( x_1, x_2, \ldots, x_N \in \bar{V} \) are the distinct points with \( \varphi(x_i) = \varphi(x_j) \) for all \( i, j \in \{1, \ldots, N\} \).

We can extend \( G_1, \ldots, G_N \) to maps \( G_1, \ldots, G_N : \bar{V} \cup \{0\} \to \bar{V} \cup \{0\} \) by defining \( G_i(0) = 0 \) for all \( i \in \{1, \ldots, N\} \). Then the maps \( G_i : \bar{V} \cup \{0\} \to \bar{V} \cup \{0\} \) are
homeomorphisms, but not necessarily smooth at 0. In this sense, the structure of
the orbifold may not be smooth. Also, as we saw above, we can extend the metric
to a continuous metric on \( \tilde{V} \cup \{0\} \) by defining \( g_{ij}(0) = \delta_{ij} \), but this extension is not
necessarily smooth. In order to do Ricci flow of this \( C^0 \) orbifold, we will proceed
as follows: \textbf{Step 1.} modify the metric \( g \) and the maps \( G_1, \ldots, G_L : \tilde{V} \to V \) inside
\( D_0, \frac{1}{2} \) to obtain a new metric \( \hat{g} \) on \( V \) and new maps \( \hat{G}_1, \ldots, \hat{G}_L : \tilde{V} \to V \), which
are isometries of \( \tilde{V} \) with respect to \( \hat{g} \), and such that these new objects can be
smoothly extended to 0. We do this in a way, so that the metric and maps are only
slightly changed (see below for details). With the help of \( \hat{g} \) and \( \hat{G}_1, \ldots, \hat{G}_L \) we will
define a new smooth Riemannian orbifold: essentially this construction smooths
out the \( G_i \)'s near the cone tips (the points \( x_1, \ldots, x_L \in X \)) in such a way, that
a group structure is preserved, and the rest of the orbifold is not changed. For
\( i \in \mathbb{N}, i \to \infty \), we denote the smooth Riemannian orbifolds which we obtain in
this way by \( (X_i, d_i) \). The construction will guarantee that \( (X_i, d_i) \to (X, d) \) in
the Gromov-Hausdorff sense, actually in the Riemannian \( C^0 \) sense: see below. In
\textbf{Step 2}, we flow each of these spaces \( (X_i, d_i) \) by Ricci flow, and we will see, that
the solution exists on a time interval \([0, T)\) with \( T > 0 \) being independent of \( i \),
and that each of the solutions satisfies estimates, independent of \( i \). In \textbf{Step 3},
we take an orbifold limit of a subsequence of the solutions constructed in \textbf{Step 2}
to obtain a limiting smooth orbifold solution to Ricci flow \( (\tilde{X}, h(t))_{t \in (0, T)} \) which
satisfies \( (\tilde{X}, d(h(t))) \to (X, d_X) \) as \( t \searrow 0 \), in the Gromov-Hausdorff sense.

Now for the details.

Let \( G_1, \ldots, G_N : \tilde{V} \to V \) be the deck transformations of \( \varphi : \tilde{V} \to V \), let \( g := \varphi^*(l) \),
and \( \varphi : \tilde{V} \to V \) be \( \varphi(\tilde{x}) = \varphi(\tilde{e}) \). We use, in the following, the
notation \( \tilde{x} = c \tilde{e} \). Then \( \tilde{x} \) is a covering map, with deck transformations \( H_1, \ldots, H_N : \tilde{V} \to V, H_i(\tilde{x}) = cG_i(\tilde{e}) \). We know that \( G_1, \ldots, G_N : \tilde{V} \to V \) are isometries
with respect to \( g \). Let \( \ell := c^2 t \) and \( \hat{g} := (\varphi^*\ell) \). Then \( \hat{g}_{ij}(\tilde{x}) = g_{ij}(x) \), and
\( H_1, \ldots, H_N : \tilde{V} \to V \) are local isometries w.r.t. \( \hat{g} \), and hence global isometries
w.r.t. \( \hat{g} \):

\[
\hat{g}(\tilde{x})(DH_i(\tilde{e})(v), DH_i(\tilde{e})(w)) = g(x)(DG_i(x)(v), DG_i(x)(w)) = g(x)(v, w).
\]

Scaling with \( c = 2^{i+2} \) we see \( \hat{g}|_{[2^{-14}, 2^{-16}]} = \varphi^*|_{[2^{-14}, 2^{-16}]} \), as shown above.

We go back to the construction of the map \( \tilde{\varphi}_i \). Remember that \( \tilde{\varphi}_i : D_{1/2+\varepsilon(i)} \delta_4, -\varepsilon(i) \to X \setminus \{x_1\} \) was defined by
\( \tilde{\varphi}_i := \psi_i \circ \eta \psi_i(1 - \eta)\tilde{h}_{i+1} : D_{1/2+\varepsilon(i), 4-\varepsilon(i)} \to d*B_{1/2, 4}(x_1) \), where \( \varphi_i : \mathbb{R}^4 \to \mathbb{R}^4_0 \) is a smooth cutoff function, with \( \eta = 1 \) on
\( D_{2-\delta} \) and \( \eta = 0 \) on \( D_{0.2-\delta} \) and \( \psi_i(1 - \eta)\tilde{h}_{i+1} \) is \( C^k \) close to the identity
on \( D_{1/2+\varepsilon(i), 4-\varepsilon(i)} \) (see [33]). As we pointed out during the construction of \( \tilde{\varphi}_i \),
this means that \( \tilde{\psi}_i^{-1} \circ \tilde{\varphi}_i : D_{1/2+\varepsilon(i), 4-\varepsilon(i)} \to (g(i)B_{1/2, 4}(0), g(i)) \) is \( C^k \) close to
\( \tilde{\varphi}_i : D_{1/2+\varepsilon(i), 4-\varepsilon(i)} \to (g(i)B_{1/2, 4}(0), g(i)) \) in \( C^k \) close to
\( \pi_i : D_{1/2+\varepsilon(i), 4-\varepsilon(i)} \to (g(i)B_{1/2, 4}(0), g(i)) \). We define

\[
\alpha_i : D_{0,4-\varepsilon(i)} \to (g(i)B_{0,4}(0), g(i))
\]

\[
\alpha_i := \pi_i \circ (\eta(1 - \eta)\tilde{h}_{i+1} \circ g(i))
\]

Then \( \alpha_i \) is \( C^k \) close to \( \pi_i : D_{0,4-\varepsilon(i)} \to (g(i)B_{0,4-\varepsilon(i)}(0), g(i)) \), and equal \( \pi_i \) on
\( D_{0,2-\delta} \). Hence, using the same argument we used above to show that \( \varphi : \tilde{V} \to V \) was a covering map, and \( \tilde{V} \) is simply connected, we have \( \alpha_i : \tilde{Z} := (\alpha_i)^{-1}(\alpha_i(D_{0,4-\varepsilon(i)})) \to Z \) is a covering map, if \( i \) is large enough (\( \varepsilon > 0 \) fixed and small),

\[
\alpha_i : \tilde{Z} \to Z
\]

\[
\alpha_i := (\alpha_i)^{-1}(\alpha_i(D_{0,4-\varepsilon(i)})) \to Z
\]
\( \hat{Z} \) is simply connected, and \( Z, \hat{Z} \) are connected. We also have \( v_i \circ \alpha_i = \tilde{\varphi}_i \) on the set \( D_{2-\delta,4-\varepsilon} \). In particular, \( \alpha_i \) has the same number of deck transformations as \( \tilde{\varphi}_i \) and hence as \( \varphi \) [Explanation: \( \hat{\alpha}_i := v_i \circ \alpha_i : \hat{Z} \to v_i(Z) \) is a covering map. Choose \( w \in D_{5/2,3} \) and let \( w = w_1, w_2, \ldots, w_N \) be the distinct points in \( \hat{Z} \) with \( \alpha_i(w_j) = \alpha_i(w) \) for all \( j = 1, \ldots, N \). Then \( w_1, \ldots, w_N \in D_{2,7/2} \) and furthermore \( \hat{\alpha}_i(w_j) = \hat{\alpha}_i(w) \) for all \( j = 1, \ldots, N \), and hence \( \hat{\varphi}_i(w_j) = \hat{\varphi}_i(w) \) for all \( j = 1, \ldots, N \). Hence \( \hat{N} \leq N \). Similarly, by considering the distinct points \( w = \tilde{w}_1, \ldots, \tilde{w}_N \in D_{5/2,3} \) such that \( \hat{\varphi}_i(w) = \hat{\varphi}_i(\tilde{w}_j) \) for all \( j = 1, \ldots, \tilde{N} \), we see \( \tilde{N} \geq N \).]

The Riemannian metric \( l_i \) on \( X \) can be pulled back to \( (B_{1/2+\varepsilon(i),4-\varepsilon(i)}, g(i)) \) with \( v_i^*: (v_i)^*(l_i) \). This metric \( h_i \) is \( C^k \) close to \( g(i) \). We interpolate between \( h(i) \) and \( g(i) \) on \( (B_{1+\delta,2-4\delta}, g(i)) \) by

\[
\beta(i) := \hat{\eta}h_i + (1 - \hat{\eta})g(i)
\]

where \( \hat{\eta} \geq 0 \) is a smooth cut-off function on \( (B_{1,4}, g(i)) \) with \( \hat{\eta} = 0 \) on \( B_{0,1+2\delta} \) and \( \hat{\eta} = 1 \) on \( (B_{1+4\delta,\infty}, g(i)) \). Note that \( \beta(i) = h_i \) on \( D_{2-\delta,4-\varepsilon} \).

Let \( \tilde{H}_1, \ldots, \tilde{H}_N : \hat{Z} \to \hat{Z} \) be the deck transformations of the covering map \( \alpha_i : \hat{Z} \to Z \). These maps are isometries w.r.t. \( \tilde{k}(i) := (\alpha_i)^*(\beta(i)) \) on \( \hat{Z} \). Scaling these maps leads to maps \( k_h : \hat{Z} \to \hat{Z}, k_h(x) := \frac{1}{\sqrt{\varepsilon}}H_k(x^{2+\varepsilon}) \) for \( k \in \{1, \ldots, N\}, \hat{Z} := \{\frac{x}{\sqrt{\varepsilon}} \mid x \in Z\} \). These maps are isometries w.r.t. \( k(x) := k(i)(x) = \tilde{k}(i)(\hat{x}) \) on \( \hat{Z} \) (see the beginning of the proof).

Note that \( k(x) := k(i)(x) = \tilde{k}(i)(\hat{x}) = (\alpha_i)^*(\beta(i))(\hat{x}) = (\alpha_i)^*(h_i) \hat{x} \)

\[
= (v_i \circ \alpha_i)^*(l_i) \hat{x} = (\tilde{\varphi}_i)^*(l_i) \hat{x} = \varphi^*(l)(x) = g(x) \text{ on } \hat{Z} \cap D_{2-\delta,4-\varepsilon}. \]

Where we used the fact that \( v_i \circ \alpha_i \) is equal to \( \tilde{\varphi}_i \) on the set \( D_{2-\delta,4-\varepsilon} \). Hence the Riemannian metric \( \hat{g} \), which is defined to be the metric \( k \) on \( D_{0,1+2\delta} \) and \( g \) on \( D_{1+4\delta,\infty} \cap \hat{V} \), is smooth and well defined. It satisfies: \( \hat{g}(x) = k(x) = \hat{k}(x) = \delta \) for \( |x| \leq c(i) \) small enough. Furthermore, \( |\hat{g} - \delta|_{C^\sigma(\hat{V})} \leq \sigma \) where \( \sigma > 0 \), can be made as small as we like, by choosing \( \varepsilon > 0 \) (in the definition of \( \hat{V} \)) small.

Using the fact that \( v_i \circ \alpha_i \) is equal to \( \tilde{\varphi}_i \) on the set \( D_{2-\delta,4-\varepsilon} \) again, we see that \( \tilde{G}_1, \ldots, \tilde{G}_N \) are the same as \( \tilde{H}_1, \ldots, \tilde{H}_N \) when all of these transformations are restricted to \( D_{2-\delta+\delta,4-\varepsilon} \) (we assume \( \delta << \delta \)). Let \( w \in D_{2-\delta+\delta,4-\varepsilon} \) and \( w = w_1, w_2, \ldots, w_N \in D_{2-\delta+\delta,4-\varepsilon} \) be the distinct points with \( \tilde{\varphi}_i(w_1) = \ldots = \tilde{\varphi}_i(w_N) \).

Let \( 0 < s << \min(\delta, i_0/100) \) be a fixed small number and \( i \) large enough. Then we have

\[
\tilde{G}_k|_{B_{s}(w)} = ((\tilde{\varphi}_i)|_{B_{s}(w)})^{-1} \circ (\tilde{\varphi}_i)_{B_{s}(w)} = (\tilde{\varphi}_i|_{B_{s}(w)})^{-1} \circ (v_i|_{B_{s}(w)})^{-1} \circ v_i \circ (\tilde{\varphi}_i)_{B_{s}(w)} = (v_i \circ \tilde{\varphi}_i|_{B_{s}(w)})^{-1} \circ (v_i \circ (\tilde{\varphi}_i)_{B_{s}(w)}) = (\alpha_i|_{B_{s}(w)})^{-1} \circ \alpha_i|_{B_{s}(w)} = \tilde{H}_k|_{B_{s}(w)}
\]

This means the maps \( H_i \) can be extended smoothly to all of \( \hat{V} \cup \{0\} \), by defining \( H_i = G_i \) on \( \hat{V} \cap (\hat{Z})^c \) and \( H_i(0) = 0 \) : call these new maps \( \tilde{G}_i \). Note that these
maps are now smooth. Near $0$, $k(x) = \delta$, and $H_j(D_{0,s}) \subseteq D_{0,2s}$, $H_j : \tilde{Z} \to \tilde{Z}$ are isometries, and hence $H_j|_{D_{0,s}} \in O(4)$ for $s$ small enough.

Note also, that for $x \in \tilde{Z}$, we always have $\tilde{G}_j(x) = H_j(x) \in \tilde{Z}$, and for $y \in \tilde{V} \cap (\tilde{Z})^c$, we have $\tilde{G}_j(y) \in \tilde{V} \cap (\tilde{Z})^c$. To see that the last statement is true, assume that $\tilde{G}_j(y) \in \tilde{Z}$ holds for some $y \in \tilde{V} \cap (\tilde{Z})^c$. Then we must have $y = (\tilde{G}_j)^{-1}(\tilde{G}_j(y)) \in \tilde{Z}$ in view of the fact that $(\tilde{G}_j)^{-1}(\tilde{Z}) \subseteq \tilde{Z}$, and this is a contradiction to the fact that $y \in \tilde{V} \cap (\tilde{Z})^c$. This shows also that the $\tilde{G}_j^*$'s are diffeomorphisms, with $(\tilde{G}_j)|_{\tilde{Z}} = H_i$ and $(\tilde{G}_j)|_{\tilde{V} \cap \tilde{V}} = G_i|_{(\tilde{Z})^c \cap \tilde{V}}$ for all $i \in \{1, \ldots, N\}$. In particular, $\{\tilde{G}_1, \ldots, \tilde{G}_N\}$ forms a subgroup of the family of diffeomorphisms on $\tilde{V} \cup \{0\}$. The metric $\tilde{g}$ agrees with $k$ on $\tilde{Z}$ and agree with $g$ on $(\tilde{Z})^c \cap \tilde{V}$. Also, the $\tilde{G}_i$'s are isometries on $(\tilde{Z}, k) = (\tilde{Z}, \tilde{g})$, since $\tilde{G}_i = H_i$ on $\tilde{Z}$, and the $\tilde{G}_i$'s are isometries on $(\tilde{Z})^c \cap \tilde{V}, g) = (\tilde{Z})^c \cap \tilde{V}, \tilde{g})$ since $\tilde{G}_i = G_i$ on $(\tilde{Z})^c \cap \tilde{V}$. Hence $\{\tilde{G}_1, \ldots, \tilde{G}_N\}$ are global isometries on $\tilde{V} \cup \{0\}$, each with one fixed point, $0$. The orbifold structure can now be defined as follows: Let $\tilde{W} := \tilde{V} \cup \{0\}$. $(\tilde{W}, \tilde{G}_1, \ldots, \tilde{G}_N)$ determines one orbifold chart $\psi : \tilde{W} \to \tilde{W}/\{(\tilde{G}_1, \ldots, \tilde{G}_N)\}$, where $\psi(x) := [x] = \{\tilde{G}_i(x) \mid i = 1, \ldots, N\}$. On $X \setminus (dx B_{\epsilon/100}(x_1) \cup dx B_{\epsilon/100}(x_2) \cup \ldots \cup dx B_{\epsilon/100}(x_L))$, we take a covering by the inverse of $K$ manifold charts, for example, geodesic coordinates: $(\theta_\alpha) : B_{\epsilon/4}(0) \to B_{\epsilon/4}(y_0) \subseteq (X \setminus (dx B_{\epsilon/100}(x_1) \cup dx B_{\epsilon/100}(x_2) \cup \ldots \cup dx B_{\epsilon/100}(x_L)))$, $\alpha \in \{1, \ldots, K\}$ (for orbifold charts the maps always go from an open set in $\mathbb{R}^2$ to an open set in the orbifold). These are fixed for this construction and don't depend on $i$. Since we don't change anything on $X \setminus (dx B_{\epsilon/100}(x_1) \cup dx B_{\epsilon/100}(x_2) \cup \ldots \cup dx B_{\epsilon/100}(x_L))$, these charts, along with $\tilde{g}$, define an Riemannian orbifold $(\tilde{X}, \tilde{g})$.

To be a bit more specific: define $X = X \setminus (dx B_{\epsilon/100}(x_1) \cup dx B_{\epsilon/100}(x_2) \cup \ldots \cup dx B_{\epsilon/100}(x_L)) \cup \tilde{W}/\{(\tilde{G}_1, \ldots, \tilde{G}_N)\}$, where we identify points $z \in X \setminus (dx B_{\epsilon/100}(x_1) \cup dx B_{\epsilon/100}(x_2) \cup \ldots \cup dx B_{\epsilon/100}(x_L))$ with points $[v] \in \tilde{W}/\{(\tilde{G}_1, \ldots, \tilde{G}_N)\}$ if $z \in \varphi^{-1}(\tilde{V})$ and $[\varphi^{-1}(v)] = [v]$. The topology is defined by saying $x_i \to x$ in $\tilde{X}$ if and only if $x_i \to x$ in $\tilde{W}/\{(\tilde{G}_1, \ldots, \tilde{G}_N)\}$ for all $i \geq N(x) \in \mathbb{N}$ and $x_1 \to x$ in $X \setminus (dx B_{\epsilon/100}(x_1) \cup dx B_{\epsilon/100}(x_2) \cup \ldots \cup dx B_{\epsilon/100}(x_L))$ for all $i \geq N(x) \in \mathbb{N}$ and $x_1 \to x$ in $X \setminus (dx B_{\epsilon/100}(x_1) \cup dx B_{\epsilon/100}(x_2) \cup \ldots \cup dx B_{\epsilon/100}(x_L))$. The charts are given above.

Call the resulting orbifold space $(X, \tilde{g}_i)$.

This finishes the construction of the modified orbifolds and metrics.

**Step 2.**

Now we have a smooth orbifold and a smooth metric, so we may evolve it with the orbifold Ricci flow, to obtain a smooth solution $(X_i, Z_i(t))_{t \in (0, T)}$ to the orbifold Ricci flow: see Section 2 of [HaThreeO] and Section 5 [KLTThree]. The new metric $g_i(0)$ at time zero on $D_\sigma$ is $\varepsilon$ away from $\delta$, and smooth. In particular,

$$|g_i(0) - g_j(0)|_{C^0(D_{\sigma}, g_i(0))} \leq 2\varepsilon \text{ for all } i, j \in \mathbb{N}. \tag{9.3}$$

if $\sigma > 0$ is small enough. One method to construct a solution to the orbifold Ricci flow is using the so called DeTurck trick ([DeT]). We can use any valid smooth background metric $h$ to do this: taking $h = g_j(0)$ for a fixed $j \in \mathbb{N}$, we have $|g_i(0) - h|_{C^0(D_{\sigma}, h)} \leq \varepsilon$ on the whole of $(X_i, g_i(0))$. Now we use the $h$-flow in place
of the Ricci-flow, that is locally the equation looks like,
\[
\frac{\partial}{\partial t} g_i = (g_i)^{\alpha\beta} \nabla^\alpha g_i + \text{Riem}(h) * (g_i)^{\alpha} * (g_i)^{-1} * (h)^{-1} \\
+ (g_i)^{-1} * (g_i)^{-1} * (\nabla g_i) * (\nabla g_i),
\]
(9.4)
where here, \( \nabla = h \nabla \). Using the estimates contained in the proof of Theorem 5.2 in [SimC0], we see that the solution \( g_i(t) \in [0,T_i] \) can be extended to \( g_i(t) \in [0,S] \) for some fixed \( S = S(h) > 0 \) and that the solution satisfies
\[
|g_i(t) - h|_{C^0(X_i)} \leq 2 \varepsilon \\
\left| \nabla^k g_i(t) \right|_{C^0(X_i)} \leq \frac{c(K,h)}{t^k}
\]
(9.5)
for all \( k \leq K \in \mathbb{N} \), as long as \( t \leq S \), where \( c(K,h) \) doesn’t depend on \( i \in \mathbb{N} \). We also have
\[
|g_i(t) - g_i(0)|_{C^0(X_i)} \leq c(h,t) \leq 2 \varepsilon \text{ for all } 0 \leq t \leq S
\]
where \( c(h,t) \to 0 \) as \( t \searrow 0 \), and \( c(h,t) \) doesn’t depend on \( i \in \mathbb{N} \), in view of the inequalities (5.5) and (5.6) in [SimC0] (the \( \varepsilon > 0 \) appearing in (5.5) and (5.6) there is arbitrary: see the proof of Theorem 5.2 in [SimC0]). In particular,
\[
d_{GH}((X_i, d(g_i(t))), (X_i, d(g_i(0)))) \leq c(t)
\]
with \( c(t) \to 0 \) as \( t \searrow 0 \). Using the smooth time dependent orbifold vector fields \( V^k(\cdot,t) = -g_i(\cdot,t) \text{sm}(\Gamma^k_{sm}(\cdot,t) - \Gamma^k_{sm}(\cdot)) \text{ and the orbifold diffeomorphisms } \varphi_i : X_i \to X_i \text{ with } \frac{\partial}{\partial t} \varphi_i = V, \varphi_0 = Id \) we obtain a solution to the orbifold Ricci flow, \( Z_i(t) := \varphi_i^* g_i(t) \) which satisfies
\[
d_{GH}((X_i, d(Z_i(t))), (X_i, d(Z_i(0)))) \leq c(t)
\]
(9.7)
\[
|\nabla^j \text{Riem}(Z_i)|(\cdot,t) \leq \frac{c(j,h)}{t^{1+j/2}} \text{ for all } 0 \leq t \leq S,
\]
(9.8)
see for example [Sim] for details. This finishes Step 2. In Step 2 we obtained various estimates which are necessary for Step 3.

Step 3.

Using the Ricci flow orbifold compactness theorem, see [Li] and Section 5.3 in [KLThree], we can now take a limit in \( i \to \infty \) for \( t \in (0,S) \), and we obtain an orbifold solution \((\bar{X}, Z(t))_{t \in (0,S)}\) to the Ricci flow with
\[
d_{GH}((X, d_X), (\bar{X}, d_{Z(t)})) \leq c(t)
\]
(9.9)
\[
|\nabla^j \text{Riem}(Z)|(\cdot,t) \leq \frac{c(j,h)}{t^{1+j/2}} \text{ for all } 0 < t \leq S
\]
where \( c(t) \to 0 \) as \( t \searrow 0 \). Here we used, that \((X_i, d(Z_i(0))) \to (X, d_X)\) in the Gromov-Hausdorff sense, which follows by the construction of the spaces \((X_i, d(Z_i(0)))\). Hence we have a found a solution \((\bar{X}, Z(t))_{t \in (0,S)}\) to the orbifold Ricci flow, with initial value \((X, d_X(0))\) in the sense that
\[
d_{GH}((X, d_X), (\bar{X}, d_{Z(0)})) \to 0 \text{ as } t \searrow 0.
\]
In this sense we have extended the flow \((M, g(t))_{t \in (0,T)}\) through the singular limit \((X, d_X)\).

\[\square\]

Remark 9.2. Some of the estimates above can be obtained using Perelman’s first pseudolocality theorem and Shi’s estimates. However, the estimate on the Gromov-Hausdorff distance, which we require when showing that the initial value of the limit solution is \((X, d_X)\), does not immediately follow from the pseudolocality theorem.
We use the estimates given in [SimC0] to show that the initial value of the solution is \((X, d_X)\).

**APPENDIX A. CUT OFF FUNCTIONS AND YE LI’S RESULT**

Our new time dependent cut-off function \(\varphi\) will satisfy:

\[
\begin{align*}
\frac{\partial}{\partial t} \varphi &\leq \Delta \varphi + \frac{c}{(r - r')^2} + \frac{\varphi}{t} \\
|\nabla \varphi|_{g(t)}^2 &\leq \frac{(r - r')^2}{c} \\
\varphi|_{\mathcal{B}_{r',y}} &\equiv e^{ct}, \\
\varphi|_{(\mathcal{B}_{r',y})^c} &\equiv 0,
\end{align*}
\]

(A.1)

for all \(t \leq S\), for some fixed universal constants, \(S, c\), wherever it is differentiable (and as long as the solution is defined). We explain here in more detail, how to construct this function, where here \(\frac{1}{2} < r' < r \leq \frac{3}{2}\). The function \(\varphi\) is constructed using a method of G. Perelman. As explained in the paper [SimSmoo], Perelman’s work shows us that

\[
\frac{\partial}{\partial t} d_t - \Delta d_t(x) \geq -\frac{c_1}{\sqrt{t}}
\]

at points in space and time where this function is smoothly differentiable, for some \(c_1 = c_1(c_0)\) for all \(t \leq S(c_0)\), if \(|\text{Riem}| \leq \frac{2}{T}\) on \(B_2(p) \cap \{t \neq p\}\). In our situation \(c_0 = 1\). Choose a standard cut-off function \(\psi : [0, \infty) \rightarrow \mathbb{R}\) with \(\psi' \leq 0, \psi|_{[0,r')} = 1, \psi|_{[r,\infty)} = 0\) and \(|\psi'|^2 \leq \frac{200}{(r-r')^2}, \psi \text{ and } \psi'' \geq -\frac{200}{(r-r')^2}\). For example Lemma 3.2 in [SimSmoo], where now \(k_0(A, B) = \frac{1}{(r-r')^2}\). Now we define \(\varphi(x,t) = \psi(d_t(x))\). Away from the cut locus we have

\[
\left(\frac{\partial}{\partial t} - \Delta\right) \varphi(x,t) = \psi'(d_t(x)) \left(\frac{\partial}{\partial t} - \Delta\right) d_t(x) - \psi''(d_t(x))
\]

\[
\leq \frac{c_1}{\sqrt{t}} + \frac{200}{(r-r')^2}
\]

\[
\leq \frac{\varphi(x,t)}{t} + \frac{c}{(r-r')^2}
\]

(A.2)

as required. This \(\varphi\) is continuous, Lipschitz in space and time, and smoothly differentiable in space and time on \(M \times [0,T] \setminus \text{CUT}_p\), where \(\text{CUT}_p := \{(x,t) \in M \times [0,T) \mid x \in \text{cut}(g(t))(p)\}\), where cut\((g(t))(p) = \{x \in M \mid x \text{ is a cut point of } p \text{ w.r.t to } g(t)\}\). \(\text{CUT}_p\) is closed in \(M \times [0,T]\), and \(D := M \times [0,T] \setminus \text{CUT}_p\) is open in \(M \times [0,T]\) (see the proof of Lemma 5 of [MT]). On \(M \times [0,T]\) the forward and backward time difference quotients of \(\varphi\) are bounded in the following sense: for all \(t \in (0,T)\) there exists an \(\delta > 0\) such that

\[
\left|\frac{\varphi(\cdot,t+h) - \varphi(\cdot,t)}{h}\right| \leq C,
\]

(A.3)

for some constant \(C = C(r,r',M, g(r), r \in [0,S])\) for all \(h \in \mathbb{R} \mid h \leq \delta (h < 0\ is allowed)\) for all \(t \in [0,S]\).
This can be seen as follows. Arguing as in the proof of Lemma 17.3 and Theorem 17.1 of [HaForm] respectively Lemma 3.5 of [HaFour], we see that \( d(x, t) \) is Lipschitz in time and that \( |d(x, y, t) - d(x, y, s)| \leq c|s - t| \) for some fixed \( c \) for all \( x, y \in M \) for all \( t, s \in [0, S] \).

This gives us the required estimates \((A.4)\). In particular this shows us that the time derivative of \( f(t) := \int_M \varphi(x, t) l(y) \, d\mu_t(x) \) is well defined for any smooth function \( l : M \to \mathbb{R} \), as we now show. For any open set \( U \subseteq M \), we have

\[
\frac{f(t + h) - f(t)}{h} = \int_U l(x) \frac{\partial}{\partial t} \varphi(x, t + h) \, d\mu_t(x) + \int_{M \setminus U} l(x) \frac{\partial}{\partial t} \varphi(x, t) \, d\mu_t(x) + \int_M \varphi(x, t + h) l(x) \, d\mu_{t+h}(x) - \int_M \varphi(x, t) l(x) \, d\mu_t(x).
\]

\[(A.5)\]

The last term in brackets converges to \( \int_M \varphi(x, t) l(x) \, d\mu_t(x) \). Choose \( V \) to be an open, star shaped set in \( T_p^k M = \mathbb{R}^4 \), so that \( U := \exp(g(t))(p) \subseteq M \setminus \text{cut}(t)(p) \), and so that \( d\mu_{g(t)}(M \setminus U) \leq \varepsilon \) (see for example the book [Chav] for a proof that this is possible along with Section 3 of [Wg] for a proof of the fact that the cut locus has measure zero). Due to the fact that \( M \times (0, T) \setminus \text{CUT}_p \) is open, we can find a small \( \delta > 0 \), such that \( U \times (t - \delta, t + \delta) \subseteq (M \times (0, T)) \setminus \text{CUT}_p \). This implies that \( U \subseteq \text{cut}(s)(p) \) for all \( s \in (t - \delta, t + \delta) \). Due to the continuity of volume, we may also assume that \( d\mu_{g(t)}(M \setminus U) \leq 2\varepsilon \) for all \( s \in (t - \delta, t + \delta) < T \) for some \( \delta = \delta(t, U, \sigma_1, \sigma_2, T, M_0) \): this follows from the facts that \( \frac{\partial}{\partial t} \int_U d\mu_{g(t)} \) and \( \frac{\partial}{\partial t} \int_M d\mu_{g(t)} \leq c \int_U d\mu_{g(t)} \) and \( \frac{\partial}{\partial t} \int_M d\mu_{g(t)} \leq c \int_M d\mu_{g(t)} \) for \( c = c(S, T, \sigma_1, \sigma_0) \) for any open set \( U \subseteq M \) as long as \( t \leq S < T \). Using these facts, and taking the \( \limsup_{h \to 0} \) respectively \( \liminf_{h \to 0} \) of the above, we get

\[
\limsup_{h \to 0} \frac{f(t + h) - f(t)}{h} = \int_U l(x) \frac{\partial}{\partial t} \varphi(x, t) \, d\mu_t(x) + C_1(\varepsilon, l, \varphi, t) + \int_M \varphi(x, t) l(x) \frac{\partial}{\partial t} \, d\mu_t(x)
\]

\[
= \int_M l(x) \frac{\partial}{\partial t} \varphi(x, t) \, d\mu_t(x) + \int_M \varphi(x, t) l(x) \frac{\partial}{\partial t} \, d\mu_t(x) + \tilde{C}_1(\varepsilon, l, \varphi, t) + \int_M \varphi(x, t) l(x) \frac{\partial}{\partial t} \, d\mu_t(x)
\]

respectively

\[
\liminf_{h \to 0} \frac{f(t + h) - f(t)}{h} = \int_M l(x) \frac{\partial}{\partial t} \varphi(x, t) \, d\mu_t(x) + \tilde{C}_2(\varepsilon, l, \varphi, t) + \int_M \varphi(x, t) l(x) \frac{\partial}{\partial t} \, d\mu_t(x)
\]

\[(A.7)\]
where $|\hat{C}_1(\varepsilon, l, \varphi)|, |\hat{C}_2(\varepsilon, l, \varphi)| \to 0$ as $\varepsilon \to 0$, and we have defined $\frac{\partial}{\partial \nu} \varphi(\cdot, t) = 0$ on $\operatorname{cut}(t)(p)$ (using this definition, $\frac{\partial}{\partial \nu} \varphi : M \to \mathbb{R}$ is a bounded measurable function). Letting $\varepsilon \to 0$, we obtain

$$(A.8) \quad \frac{\partial}{\partial t} f(t) = \int_M \frac{\partial}{\partial t} \varphi(x, t) l(x) d\mu_t + \int_M \varphi(x, t) l(x) \frac{\partial}{\partial t} d\mu_t(x).$$

Examining the argument above, we see that for any Lipschitz (that is $W^{1, \infty}(M)$) function $l : M \to \mathbb{R}$ we have

$$(A.9) \quad \int_M \frac{\partial}{\partial t} \varphi(x, t) l(x) d\mu_t = \int_M \frac{\partial}{\partial t} \varphi(x, t) l(x) d\mu_t + C(\varepsilon)$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, that is, as we choose better open starshaped sets $U$ contained in $M \setminus \operatorname{cut}(t)(p)$ to approximate $M$ (we drop the dependence on $\varphi$ and $l$ in the notation, as they will be fixed for this argument). Assume now that $l \geq 0$. The evolution inequality for $\varphi$, Inequality $(A.3)$, combined with $(A.9)$, tells us that

$$\int_M \frac{\partial}{\partial t} \varphi(x, t) l(x) d\mu_t(x)$$

$$\leq \int_U \nabla \varphi d\mu_t(x) + C(\varepsilon) + \int_{B_r(y)} \frac{Cl}{(r - r')^2} + \frac{C \varphi l}{t} d\mu_t(x)$$

$$= \int_U l(x) \frac{\partial \varphi}{\partial \nu} (x, t) d\sigma_{\partial U, t} - \int_U g(t)(\nabla \varphi(x, t), \nabla l(x)) d\mu_t(x)$$

$$+ \int_{B_r(y)} \frac{Cl}{(r - r')^2} + \frac{C \varphi l}{t} d\mu_t(x) + C(\varepsilon)$$

$$\leq - \int_U g(t)(\nabla \varphi(x, t), \nabla l(x)) d\mu_t(x) + \int_{B_r(y)} \left( \frac{C \varphi l}{t} + \frac{Cl}{(r - r')^2} \right) d\mu_t(x) + C(\varepsilon)$$

$$\leq - \int_M g(t)(\nabla \varphi(x, t), \nabla l(x)) d\mu_t(x) + \int_{B_r(y)} \left( \frac{C \varphi l}{t} + \frac{Cl}{(r - r')^2} \right) d\mu_t(x) + \tilde{C}(\varepsilon)$$

where $\tilde{C}(\varepsilon)$ goes to zero as $\varepsilon \to 0$, and hence

$$(A.10) \quad \int_M \frac{\partial}{\partial t} \varphi(x, t) l(x) d\mu_t(x) \leq - \int_M g(t)(\nabla \varphi(x, t), \nabla l(x)) d\mu_t(x) + \int_{B_r(y)} \left( \frac{C \varphi l}{t} + \frac{Cl}{(r - r')^2} \right) d\mu_t(x).$$

Here we used

$$\frac{\partial \varphi}{\partial \nu}(x, t) = g(t)(\nabla \varphi(x), \nu(x, t))$$

$$= \varphi'(d_t(x)) g(t)(\nabla d_t(x), \nu(x, t))$$

$$\leq 0$$

which holds in view of the fact that $\varphi' \leq 0$ and $g(t)(\nabla d_t(x), \nu(x, t)) \geq 0$ on $\partial U$ (by construction of $U$).

Using this $\varphi$ in Lemma 1 of [Li], we obtain the following estimate

$$\frac{\partial}{\partial t} \int_M \varphi^2 f^p \leq c \int_M \nabla \varphi^2 f^p$$

$$+ \int_{B_r(y)} \frac{C}{(r - r')^2} f^p + 100(p + c)^3 \mu^2 \rho^{-1} \int_M \varphi^2 f^p.$$
Theorem of Fubini freely for $|\nabla \gamma|$. We choose \( \int M f = \int M f(\cdot, t)d\mu_{\gamma(t)} \). The proof of this estimate is the same as that of the proof of Lemma 1 in [Li], except that we must estimate the extra terms $\int_M (\partial_t (\phi^2) f^p)$ coming from the time derivative of \( \phi \). In the following we use the fact that $\partial_t \phi, \nabla \phi : (M \times (0, T)) \setminus \text{CUT}_\rho \to \mathbb{R}$ are smooth, and $\int_0^s \int_M |\nabla \phi|^2 < \infty$ for $s < T$. In particular, this means that we may use the Theorem of Fubini freely for $|\nabla \phi|^2$ (see Theorem 1 in Section 1.4 of [EG]), and we do so without further comment.

This can be done as follows (note that $p > 2$ will always be assumed)

\[
\int_M \left( \frac{\partial}{\partial t} \phi^2 \right) f^p = \int_M 2\phi \frac{\partial}{\partial t} \phi f^p \\
\leq \int_M -2g(\nabla \phi, \nabla (\phi f^p)) + \int_{B_{r}(x)} \frac{C}{(r-r')^2} f^p + \frac{C\phi^2 f^p}{t} \\
= \int_M -2f^p|\nabla \phi|^2 - 2pf^{p-1}\phi g(\nabla \phi, \nabla f) + \int_{B_{r}(x)} \frac{C}{(r-r')^2} f^p + \frac{C\phi^2 f^p}{t} \\
\leq -2p \int_M f^{p-1}\phi g(\nabla \phi, \nabla f) + \int_{B_{r}(x)} \frac{C}{(r-r')^2} f^p + \frac{C\phi^2 f^p}{t} \\
\leq \frac{4}{\gamma} \int |\nabla \phi|^2 f^p + \gamma p^2 \int \phi^2 f^{p-2}|\nabla f|^2 + \int_{B_{r}(x)} \frac{C}{(r-r')^2} f^p + \frac{C\phi^2 f^p}{t}.
\]

(A.13)

We choose $\gamma = \frac{1}{1000}$. We estimate the second term. First note that:

\[
|\nabla (\phi f^{p/2})|^2 = |\phi f^{p/2} \nabla \phi + \phi \nabla (\phi f^{p/2})|^2 \\
= |f^p|\nabla \phi|^2 + 2\phi f^{p/2} g(\nabla \phi, \nabla (\phi f^{p/2})) + \phi^2 |\nabla (\phi f^{p/2})|^2 \\
= |f^p|\nabla \phi|^2 + 2f^{p/2} (\nabla \phi, \nabla (\phi f^{p/2})) - 2f^p|\nabla \phi|^2 + \phi^2 |\nabla (\phi f^{p/2})|^2 \\
= -f^p|\nabla \phi|^2 + 2f^{p/2} g(\nabla \phi, \nabla (\phi f^{p/2})) + \frac{p^2}{4} f^{p-2}|\nabla f|^2
\]

and hence

\[
p^2 f^{p-2}\phi^2|\nabla f|^2 = 4|\nabla (\phi f^{p/2})|^2 + 4f^p|\nabla \phi|^2 - 8f^{p/2} g(\nabla \phi, \nabla (\phi f^{p/2})) \\
\leq 8|\nabla (\phi f^{p/2})|^2 + 8f^p|\nabla \phi|^2
\]

(A.15)

and hence

\[
\gamma p^2 \int_M \phi^2 f^{p-2}|\nabla f|^2 \leq 8\gamma \int |\nabla (\phi f^{p/2})|^2 + 8\gamma \int f^p|\nabla \phi|^2.
\]

(A.16)

Substituting this into (A.13) we get

\[
\int_M \left( \frac{\partial}{\partial t} \phi^2 \right) f^p \leq (8\gamma + \frac{4}{\gamma}) \int |\nabla \phi|^2 f^p + 8\gamma \int |\nabla (\phi f^{p/2})|^2 \\
\leq \int_{B_{r}(x)} \frac{C}{(r-r')^2} f^p + \frac{C\phi^2 f^p}{t}.
\]

(A.17)

The first and last two terms are of the required form. In the last line of the proof of Lemma 1 of [Li] we choose $\varepsilon$ (appearing in his proof) such that $2\varepsilon^2 + (p+c)A = 1$ (instead of his choice of $\varepsilon^2 + (p+c)A = 1$), then his estimate becomes

\[
\frac{\partial}{\partial t} \int \phi^2 f^p + \int |\nabla (\phi f^{p/2})|^2 \leq 2 \int |\nabla \phi|^2 f^p - \frac{1}{2} \int |\nabla (\phi f^{p/2})|^2
\]
Let \((M, g)\) be a smooth, Riemannian manifold without boundary, and \(\varphi : V \to U\) be a smooth coordinate chart on \(M\), such that \(U\) is compactly contained in \(M\). The metric \(g\) is given in coordinate form by \(g_{ij} : U \to \mathbb{R}, i,j \in \{1, \ldots, n\}\) where...

\[ (A.18) \quad +10(p + c)^3 \mu^3 A^2 t^{-1} \int \varphi^2 f^p. \]

(in \([L]\), this estimate occurs with a different constant: that is the term \(-\frac{1}{2} \int |\nabla (\varphi f^{p/2})|^2\) doesn’t appear in \([L]\)). We use this second last term to absorb the term \(8\gamma \int |\nabla (\varphi f^{p/2})|^2 = \frac{8}{c^2} \int |\nabla (\varphi f^{p/2})|^2\) appearing in \((A.17)\). This finishes the proof of the claimed estimate \((A.12)\).

Continuing as in the paper \([L]\), we get (only adding the extra terms we obtained in our estimates)

\[ \int_M \varphi^2 f^p d\mu_g(t) + \int_\tau^t \int_M |\nabla (\varphi f^{p/2})|^2(\cdot, s)d\mu_g(s)ds \]

\[ (A.19) \leq c \int_\tau^T \int_M (|\nabla \varphi|^2 + \chi B_r(y) \frac{C}{(r - r')^2}) f^p + \left( \hat{C}(p, \tau') + \frac{1}{\tau' - \tau} \right) \int_\tau^T \int_M \varphi^2 f^p, \]

for all \(0 < \tau < \tau' < t \leq T\), where, using the notation of \([L]\), \(\hat{C}(p, s) := \frac{100(p + c)^3 c}{s^2} \leq \tilde{c} \mu^3\) where \(c, C, \tilde{c}\) are constants independent of \(s\) and \(p, r, r'\). Define

\[ (A.20) \quad H(p, \tau, r) = \int_\tau^T \int_{B_r(x)} f^p \]

where \(\frac{1}{2} < r^2 < 1\). Now using our estimates on \(\varphi\) we get (just as in \([L]\), except the first constant \(\hat{A}\) appearing on the right hand side of the estimate below is perhaps larger than that appearing in \([L]\))

\[ (A.21) \quad H(\frac{3}{2} p, \tau', r') \leq \hat{A} \left( \hat{C}(p, \tau') + \frac{1}{\tau' - \tau} + \frac{1}{(r - r')^2} \right) \frac{3}{2} H(p, \tau, r) \]

for all \(0 < \tau < \tau' < T\), for all \(\frac{1}{2} \leq r' < r < 1\), which is Lemma 3 of \([L]\).

Now Theorem 2 of \([L]\) is also valid, up to a constant: \(f = |\text{Rc}|\) satisfies (take \(p_0 = 4\) in Theorem 2 of \([L]\))

\[ |\text{Rc}(x, t)| = |f(x, t)| \leq C(1 + \frac{1}{t^4})^{1/2} \left( \int_0^T \int_{B_1(p)} |\text{Rc}|^4 d\mu_g(t) \right)^{1/4} \]

\[ (A.22) \leq C \frac{1}{t^4} \]

for all \(x \in B_{1/2}(y)\). The proof is the same as that in \([L]\), where we have used here that in our setting

\[ (A.23) \quad \int_0^T \int_M |\text{Rc}|^4 d\mu_g(t) \leq K_0 < \infty. \]

In fact, we may assume that \(\int_0^T \int_M |\text{Rc}|^4 d\mu_g(t) \leq \delta^5\) is small, as we have scaled (and translated) the original solution by large constants: if \(\tilde{g}(\tilde{t}) := cg(\frac{\tilde{t}}{c})\), \(\tilde{T} = cT\), \(\tilde{t} = ct\), then \(\int_0^T \int_M |\text{Rc}|^4 d\mu_{\tilde{g}(\tilde{t})}d\tilde{t} = \frac{1}{c} \int_0^T \int_M |\text{Rc}|^4 d\mu_{g(t)}dt \leq \frac{K_0}{c} \).

**Appendix B. Harmonic coordinates Theorem**

Let \((M, g)\) be a smooth, Riemannian manifold without boundary, and \(\varphi : V \to U\) be a smooth coordinate chart on \(M\), such that \(U\) is compactly contained in \(M\). The metric \(g\) is given in coordinate form by \(g_{ij} : U \to \mathbb{R}, i,j \in \{1, \ldots, n\}\) where...
$g_{ij}(x) = g_{ij}(x) := g(\varphi^{-1}(x))(\partial_i(\varphi^{-1}(x)), \partial_j(\varphi^{-1}(x)))$. We define the quantity $\|Dg\|_{L^{12}(U)}$ by

$$\|Dg\|_{L^{12}(U)} := \left( \int_U \sum_{i,k,r=1}^n |\partial_ig_{kr}|^{12}dx \right)^{1/12},$$

where $dx$ refers to Lebesgue measure, $\partial_i(q) \in T_qM$ denotes a coordinate vector, and $\partial_ig_{kr}$ refers to the standard euclidean partial derivative in the $i$th direction of the function $g_{kr}$. Clearly this quantity is dependent on the chosen coordinates.

**Definition B.1.** Let $q \in M^4$. $B_S(q)$ will be a fixed reference ball, which is compactly contained in $M$. We define $r_h(p) :=$ supremum over all $r \geq 0$ such that there exists a smooth ($C^\infty$) chart $\psi : V \rightarrow \psi(V) = B_r(0)$ where $V \subseteq B_S(q)$ is open in $(M, g)$ with the following properties (here $g_{ij}$ is the metric in these coordinates, $g_{ij}(y) := g(\psi^{-1}(y))(\partial_i(\psi^{-1}(y)), \partial_j(\psi^{-1}(y)))$, and we use the notation used above)

(i) $1/2 \delta_{ij} < g_{ij} < 2 \delta_{ij}$ on $B_r(0)$

(ii) $r^{2/3}\|Dg\|_{L^{12}(B_r(0))} < 2$

(iii) $\psi : V \rightarrow B_r(0)$ is harmonic: $\Delta_g \psi^k = 0$ on $V$ for all $k \in \{1, \ldots, n\}$.

**Remark B.2.** Notice in the definition, that $\psi : V \rightarrow B_r(0)$ refers to a coordinate chart, that is, it must also be a smooth homeomorphism, whose inverse is $C^\infty$. Note also, that nowhere in the definition do we require that the image $\psi^{-1}(B_r(0))$ be a geodesic ball: it is simply an open set in $M$ which is diffeomorphic to a ball.

**Remark B.3.** $r^{2/3} = r^{1-\frac{4}{n}}$ with $n = 4, p = 12$.

**Remark B.4.** We are using the definition given in Petersen, as we will use the notation from that paper below to state and prove the theorem that we require. This differs from the original definition of Anderson, And, where the harmonic radius is defined similarly, but using geodesic balls in the manifold.

**Remark B.5.** As explained in Petersen, $\Delta_g \psi = 0$ implies that $\Gamma^i_{kl}(g)g^{kl} = 0$ everywhere in $B_r(0)$: this is crucial when it comes to the regularity theory for the transition functions associated to these coordinates (the regularity of metrics in harmonic coordinates was considered in many papers, for example [JK], SS, DeTK just to name some). Assume $\hat{\varphi} : \hat{V} \rightarrow B_r(0)$ and $\tilde{\varphi} : V \rightarrow B_r(0)$ are harmonic coordinates with $\tilde{V} \cap V \neq \emptyset$, and $\hat{\varphi}$ and $\tilde{\varphi}$ refer to the metric in the coordinates $\hat{\varphi}$ respectively $\tilde{\varphi}$. Then, on $B_\varepsilon(\hat{v}) \subseteq B_r(0)$ with $Z := \hat{\varphi}^{-1}(B_r(\varepsilon)) \subseteq \tilde{V} \cap V$ we have for $s := (\hat{\varphi} \circ (\hat{\varphi})^{-1})$

$$0 = \Delta_g(\hat{\varphi})^k = \Delta_{\hat{\varphi}}(\hat{\varphi} \circ (\hat{\varphi})^{-1})^k = \Delta_{\tilde{\varphi}}s^k = \hat{g}^{kl}\partial_a\partial_bg_{kl},$$

on $B_\varepsilon(\hat{v})$ where we used $\hat{g}^{kl}\Gamma^i_{kl}(\hat{g})g^{kl} = 0$ on $B_r(0)$. Notice that the derivatives of the metric do not appear in this equation.

**Remark B.6.** The other important fact about harmonic coordinates, is that the Ricci tensor satisfies

$$g^{ab}\partial_a\partial_bg_{kl} = (g^{-1} \ast g^{-1}\partial g \ast \partial g)_{kl} - 2\text{Ricci}(g)_{kl}$$
in harmonic coordinates. This star notation will be more explicitly described below: it refers to a combination of the quantities involved. This is a quasi-linear elliptic equation of second order.

We state the theorem that we require in this paper here once again

**Theorem B.7.** Let $(M^4, g)$ be a smooth manifold without boundary (not necessarily complete) and $B_3(q) \subseteq M$ be an arbitrary ball which is compactly contained in $M$. Assume that

\[
\begin{align*}
(a) & \quad \int_{B_3(q)} |\text{Riem}^2| d\mu_g \leq \varepsilon_0, \quad \int_{B_3(q)} |\text{Rc}|^4 d\mu_g \leq 1, \\
(b) & \quad \sigma_0 r^4 \leq \text{vol}(B_r(x)) \leq \sigma_1 r^4 \quad \text{for all } r \leq 1, \quad \text{for all } x \in B_3(q),
\end{align*}
\]

where $\varepsilon_0 = \varepsilon_0(\sigma_0, \sigma_1) > 0$ is small enough. Then there exists a constant $V = V(\sigma_0, \sigma_1 > 0)$ such that

\[
r_h(g)(y) \geq V \text{ dist}_g(y, \partial(B_1(q)))
\]

for all $y \in B_1(q)$. Here $B_3(q)$ is the reference ball used in the definition of the harmonic radius.

**Proof.** Proof by contradiction. The proof method is essentially that given in the proof of Main Lemma 2.2 in [AnCh] (see Remark 2.3 (ii)) using some notions from [AnCh] on the $W^{1,p}$ harmonic radius.

Assume the result is false. Then we can find smooth Riemannian manifolds without boundary $(M_i, g(i))$ and balls $B_3(p_i)$ which are compactly contained in $(M_i, g(i))$ such that $\int_{B_3(p_i)} |\text{Riem}^2| \leq \frac{1}{i} \to 0$ as $i \to \infty$, and we can find points $y_i \in B_i := B_1(p_i)$ such that the following holds for all $y \in B_i$:

\[
\frac{r_h(g(i))(y)}{\text{dist}_{g(i)}(y, \partial B_i)} \geq \frac{r_h(g(i))(y_i)}{\text{dist}_{g(i)}(y_i, \partial B_i)} \to 0 \quad \text{as } i \to \infty.
\]

We define $\mu_i^2 := (r_h(g(i))(y_i))^{-2}$. Notice that $\mu_i \to \infty$ as $i \to \infty$, since $\text{dist}_{g(i)}(y_i, \partial B_i) \leq 1$ for all $i \in \mathbb{N}$. We rescale our solution by $\tilde{g}(i) := \mu_i^2 g(i)$ which leads to

\[
\begin{align*}
&\int_{B_{3\mu_i}(p_i)} |\text{Rc}|^4 d\tilde{u}_{\tilde{g}(i)} = \frac{1}{\mu_i^4} \int_{B_{3}(p_i)} |\text{Rc}|^4 d\tilde{u}_{g(i)} \to 0 \text{ as } i \to \infty \\
&\int_{B_{2\mu_i}(p_i)} |\tilde{\text{Riem}}|^2 d\tilde{u}_{\tilde{g}(i)} \leq \frac{1}{i}
\end{align*}
\]

Here we used the fact that the harmonic radius scales like: $r_h(\tilde{g})(y) = c r_h(g)(y)$ if $\tilde{g} = c^2 g$, where it is to be understood that we use $\tilde{B} := B_{cS}(q)$ as our reference ball for the definition of harmonic radius for $\tilde{g}$ if $B = B_{S}(q)$ was the reference ball for the initial definition of $r_h$. Notice also that the quantity $\text{dist}_{g(i)}(x, \partial g^{(i)} B_1(p_i))$ scales similarly: $\text{dist}_{\tilde{g}(i)}(x, \partial \tilde{g}^{(i)} B_1(p_i)) = \mu_i \text{dist}_{g(i)}(x, \partial g^{(i)} B_1(p_i))$. Hence, for $B_i = g^{(i)} B_1(p_i)$ and $\tilde{B}_i = \tilde{g}^{(i)} B_1(p_i)$, we have

\[
\frac{r_h(\tilde{g}(i))(y)}{\text{dist}_{\tilde{g}(i)}(y, \partial \tilde{B}_i)} \geq \frac{r_h(\tilde{g}(i))(y_i)}{\text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i)} \frac{1}{\text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i)} \to 0 \quad \text{as } i \to \infty,
\]
for all \( y \in \tilde{B}_i \). In particular, \( y_i \in \tilde{B}_i \) satisfies \( \text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i) \to \infty \) as \( i \to \infty \).

Furthermore, for any fixed \( \rho > 0 \), we get

\[
\frac{r_H(\tilde{g}(i))(y)}{\text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i)} \geq \frac{\text{dist}_{\tilde{g}(i)}(y, \partial \tilde{B}_i)}{\text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i)} \geq \frac{\text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i) - \text{dist}_{\tilde{g}(i)}(y, y_i)}{\text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i)} \geq \frac{\text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i) - \rho}{\text{dist}_{\tilde{g}(i)}(y_i, \partial \tilde{B}_i)} \geq \frac{1}{2}
\]

(B.8)

for all \( y \in \tilde{B}_i \) with \( \text{dist}_{\tilde{g}(i)}(y, y_i) < \rho \) as long as \( i >> 1 \) is large enough. For ease of reading we remove the tildes from the \( g(i)'s \) in that which follows, and simply write \( g(i) again.

Take a maximal disjoint subset of balls \( (\tilde{g}(i)B_{1/2000}(y_{i,s}))_{s=1}^{N} \) whose centres are in \( g(i)B_L(y_i) \). By maximal, we mean: if we take any other ball \( g(i)B_{1/2000}(y) \) whose centre is in \( g(i)B_L(y_i) \), then it must intersect the collection of balls \( (g(i)B_{1/2000}(y_{i,s}))_{s=1}^{N} \). Then, \( (g(i)B_{1/1000}(y_{i,s}))_{s=1}^{N} \) must cover \( g(i)B_L(y_i) \), and each of the \( y_{i,s} \) satisfies \( r_H(g(i))(y_{i,s}) \geq \frac{1}{2} \), as explained above. Also, due to the non-inflating/non-collapsing estimates, we see that \( N \) is bounded by \( N(L, \sigma_0, \sigma_1) \).

For the same reason, the intersection number of \( (g(i)B_{1/100}(y_{i,s}))_{s=1}^{N} \) is bounded by \( Z(\sigma_0, \sigma_1) \): any subcollection, \( (g(i)B_{1/100}(y_{i,s}))_{s=1}^{Z} \), which intersects must be contained in \( g(i)B_1(p) \) for any \( p \) which is the centre of some (any) ball contained in this subcollection, and hence,

\[
\sigma_1 \geq \frac{\text{vol}(g(i)B_1(p))}{Z} \geq \frac{\text{vol}(\cup_{k=1}^{Z} \text{vol}(g(i)B_{1/100}(y_{i,s,k})))}{Z} \geq \text{vol}(g(i)B_{1/2000}(y_{i,s,k})) \geq Z\sigma_0 c_1
\]

(B.9)

Let \( \varphi_{i,s} := \psi_{i,s}^{-1}|_{B_1(0)} : B_1(0) \to U_{i,s} := \psi_{i,s}^{-1}(B_{1/100}(0)) \) where \( \psi_{i,s} : V_{i,s} \to B_{1/2}(0) \subseteq \mathbb{R}^4 \) is a harmonic coordinate chart centred at a point \( y_{i,s} \) (that is \( \varphi_{i,s}(y_{i,s}) = 0 \)). Since \( g \) satisfies (i) and (ii) in the coordinates \( \psi_{i,s}^{-1} : B_{1/2}(0) \to V_{i,s} \), we see that \( \varphi_{i,s} := (\psi_{i,s})^{-1}B_{1/100}(0) \to U_{i,s} := \psi_{i,s}^{-1}(B_{1/100}(0)) \) satisfies \( B_1/400(y_{i,s}) \subseteq U_{i,s} \subseteq B_4(y_{i,s}) \), and hence the intersection number of the collection of sets \( (U_{i,s})_{s=1}^{Z} \) is bounded by \( Z \).

Using these facts, we see that Fact 1 of [Petersen] is true for our charts (in view of the (i) in the definition of Harmonic radius above), Fact 2 is true for our charts (if \( i \) is large enough), Fact 3 is true for fixed \( l \) if \( i \) is large enough, and Fact 4. is true: Fact 3 is used in [Petersen] to show Fact 4. We obtain Fact 4 using our non-inflating non-collapsing arguments: there exists a limit space \( (X, d_X, p) = \lim_{GH}(M_i, (d(g(i)), p_i) \) where the limit is the pointed Gromov-Hausdorff sense (see Theorem 7.4.15 in [BB]). In order to obtain Fact 5, we need to show that a condition like (n4) in [Petersen] is satisfied for our coordinate transition functions (compare Section 4 of [Petersen]). We use the equation for the transition functions,
We know from the construction that $Ricci(g, B)_{(B.2)}$, mentioned above to show that such a condition holds. Let $B_{2\epsilon}(v) \subseteq B_{1/100}(0)$ be a small ball for which $s_{i,r,t} := (\varphi_{i,r}^{-1})^{-1} \circ \varphi_{r,i}^{-1} : B_{2\epsilon}(v) \to \mathbb{R}^4$ is well defined on $B_{2\epsilon}(v)$. Then Equation (B.2), the Schauder theory, and the fact that $g(i)$ is bounded in $C^{\alpha_0}(B_\epsilon(v))$ by a constant which is independent of $i$, show us that $\|s_{i,r,t}\|_{C^{\alpha_0}(B_\epsilon(v))} \leq C (**)$ for some constant independent of $i$.

Now arguing exactly as in [Petersen] after the proof of Fact 4 and at the beginning of Fact 5, we see first that $X$ is a $C^0$ manifold, with inverse coordinate charts $(\varphi_{i,r}^{-1} : V_r \to B_{1/100}(0))$, and their construction implies the following: if $\varphi_{i}(B_{2\epsilon}(v)) \subseteq V_r \cap V_{r'}$, then $s_{i,r,t} := \varphi_{i,r}^{-1} \circ \varphi_{r,i}^{-1} : B_{2\epsilon}(v) \to \mathbb{R}^4$ converges with respect to the $C^0$ norm to $s_{r,t} := (\varphi_{r,i}^{-1})^{-1} \circ \varphi_{i,r}^{-1} : B_{2\epsilon}(v) \to \mathbb{R}^4$. Then using the estimate (**) from above and the theorem of Arzela-Ascoli, we see that $s_{i,r,t} \to s_{r,t}$ in $C^{2,\beta}(B_{2\epsilon}(v), \mathbb{R}^4)$ (and hence in $W^{2,12}(B_{2\epsilon}(v))$ weakly: see (9) of Section 8.2.1 (b) of [Evans]) and in particular, $s_{s,t} \in C^{2,\beta}(B_{2\epsilon}(v), \mathbb{R}^4)$. That is the manifold $X$ is $C^{2,\beta}$. Now we may argue as in the rest of Fact 5 and Fact 6 of [Petersen] to see that $(X, d, p) = (X, h, p)$ where $h$ is a $C^\alpha$ metric.

Now we examine the convergence of the metrics $g(i)$ to $h$ in various Sobolev spaces.

Let us denote the metric $g(i)$ in the local coordinates given by $\varphi_{i,r}$ by $g(i)_{kl}$, and $h$ in the local coordinates given by $\varphi_r$ by $h_{kl}$: $r$ is fixed for the moment. By construction (see [Petersen]), we have: $g(i)_{kl} \to h_{kl}$ in $C^{0,\alpha}(B_\epsilon(u))$ for any $B_{2\epsilon}(u) \subseteq B_{1/100}(0) \subseteq \mathbb{R}^4$. Using the fact that (ii) holds, we see that $g(i)_{kl} \to h_{kl}$ weakly in $W^{1,12}(B_\epsilon(u))$, after taking a subsequence (see for example (9) of Section 8.21 (b) of [Evans]).

Note, from the theorem of Rellich/Kondrachov (see for example [Evans], Theorem 1 of Section 5.7), this tells us that $g(i) \to h$ strongly in $L^{12}(K)$. The metric $g_{kl} = g(i)_{kl}$ satisfies

\begin{equation}
(B.10) \quad g^{ab} \partial_a \partial_b g_{kl} = (g^{-1} \ast g^{-1} \ast \partial g \ast \partial g)_{kl} - 2Rc(g)_{kl}
\end{equation}

smoothly in $B_{1/100}(0)$, since the coordinates are harmonic. Here the $(0,2)$ tensor $(g^{-1} \ast g^{-1} \ast \partial g \ast \partial g)$ can be written explicitly, but in order to make the argument more readable we use this star notation. This tensor has the property that is linear in all of its terms: for $Z, \bar{Z}, W, \bar{W}$ symmetric and positive definite local $(2,0)$ Tensors, and $R, \bar{R}, S, \bar{S}$ local $(0,3)$ Tensors, we have $Z \ast W \ast R \ast S$ is a local $(0,2)$ Tensor, with the property that $(Z + \bar{Z}) \ast W \ast R \ast S = Z \ast W \ast R \ast S + \bar{Z} \ast W \ast R \ast S$ and $Z \ast (W + \bar{W}) \ast R \ast S = Z \ast W \ast R \ast S + Z \ast \bar{W} \ast R \ast S$ and so on. Furthermore $|Z \ast W \ast R \ast S| \leq c|Z||W||R||S|$, where $c$ depends only on $n, n = 4$ here. We know from the construction that $Ricci(g) \to 0$ in $L^4$ and the other terms on the right hand side are bounded in $L^4$ (because $\partial g$ is bounded in $L^{12}$ and $g, g^{-1}$ are bounded). Hence the right hand side is bounded in $L^4$ by a constant $c$ which doesn’t depend on $i$, if $i \in \mathbb{N}$ is sufficiently large. Also the terms $g^{ab}$ in front of the first and second derivatives are continuous, bounded and satisfy $0 < c_0 \|\psi\|^2 \leq g^{ij} \psi_i \psi_j \leq c_1 \|\psi\|^2$ for $c_0, c_1$ independent of $i \in \mathbb{N}$. Hence, using the $L^p$ theory (see for example [GT] Theorem 9.11), $|g(i)|_{W^{2,4}(K)} \leq \int_{B_{1/100}(0)} |g(i)|^4 + c \leq \tilde{c}$ on any smooth compact subset $K \subseteq B_{1/100}(0)$, where $\tilde{c}$ is a constant which is independent of $i$ (but does depend on $K$). In particular $g(i) \to h$ strongly in $W^{1,4}(K)$ on smooth compact subsets $K$ of $B_{1/100}$, for any $p \in (1, \infty)$ in view of the Theorem of Rellich/Kondrachov (see for example [Evans], Theorem 1 of Section
5.7). We also have
\[ h^{ab} \partial_a \partial_b g_{kl} = \left( h^{ab} - g^{ab} \right) \partial_a \partial_b g_{kl} + g^{ab} \partial_a \partial_b g_{kl} \]
\[ = \left( h^{ab} - g^{ab} \right) \partial_a \partial_b g_{kl} + (g^{-1} \ast g^{-1} \ast \partial g \ast \partial g)_{kl} - 2\text{Ricci}(g)_{kl} \]

Hence, for \( g = g(i) \) (written in the coordinates given by \( \phi_{i,r} \)) and \( \hat{g} = g(j) \) (written in the coordinates given by \( \phi_{r,s} \)), we have
\[ h^{ab} \partial_a \partial_b (g - \hat{g})_{kl} = \mathcal{L}_{kl} := \left( h^{ab} - g^{ab} \right) \partial_a \partial_b g_{kl} - \left( h^{ab} - \hat{g}^{ab} \right) \partial_a \partial_b \hat{g}_{kl} \]
\[ + (g^{-1} \ast g^{-1} \ast \partial g \ast \partial \hat{g})_{kl} - ((\hat{g})^{-1} \ast (\hat{g})^{-1} \ast \partial \hat{g} \ast \partial \hat{g})_{kl} \]
\[ + \text{Ricci}(g)_{kl} - \text{Ricci}(\hat{g})_{kl}. \]

The right hand satisfies: for all \( \varepsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( |\mathcal{L}|_{L^3(K)} \leq \varepsilon \) if \( i, j \geq N \), as we now show. The first term in \( \mathcal{L} \) is estimated by
\begin{align*}
\int_K |(h^{ab} - g^{ab}) \partial_a \partial_b g_{kl}|^3 \, dx &\leq (\int_K |h - g|^{12})^{1/4} (\int_U |\partial^2 g|^3)^{3/4} \\
&\leq (\int_K |h - g|^{12})^{1/4} C \\
&\to 0,
\end{align*}

as \( i \to \infty \), since \( g = g(i) \) is bounded in \( W^{2,4} \) on smooth compact subsets of \( B_{1/100}(0) \), as we showed above, and \( g(i) \to h, g(i)^{-1} \to h^{-1} \) in \( C^\alpha \) on smooth compact subsets of \( B_{1/100}(0) \). The second term may be estimated in a similar fashion: \( \int_K |(h^{ab} - \hat{g}^{ab}) \partial_a \partial_b \hat{g}_{kl}|^3 \, dx \to 0 \) as \( j \to \infty \). The third plus the fourth term of \( \mathcal{L} \) converges to 0 on \( L^p(K) \) for any \( p \in (1, \infty) \) since \( \partial g \) and \( \partial \hat{g} \) converge to \( h \) on \( L^p(K) \) for any \( p \in (1, \infty) \) and \( g(i) \) converges to \( h \) in \( C^{0,\alpha}(K) \). The sum of the last two terms converge to 0 in \( L^3(K) \), since they converge to 0 in \( L^4(K) \).

Hence, we can rewrite (B.12) as
\[ h^{ab} \partial_a \partial_b (g(i) - g(j))_{kl} = f(i, j)_{kl} \]
with \( \int_K |f(i, j)|^3 \leq \varepsilon(i, j) \), and \( \varepsilon(i, j) \leq \varepsilon \) for arbitrary \( \varepsilon > 0 \) if \( i, j \geq N(\varepsilon) \) is large enough.

Hence, using the \( L^p \) theory again (Theorem 9.11 in [GT]), we get
\begin{align*}
|g(i) - g(j)|_{W^{2,3}(\hat{K})} &\leq c(\int \mathcal{L}^3_{K} + \int |g(i) - g(j)|^3_{K}) \\
&\leq \delta(i, j)
\end{align*}
(B.15)
on any compact subset \( \hat{K} \subset K \subset B_{1/100}(0) \) where \( \delta : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+ \) satisfies: for all \( \varepsilon > 0 \) there exists an \( N = N(\varepsilon) \in \mathbb{N} \) such that \( \delta(i, j) \leq \varepsilon \) if \( i, j \geq N \). This implies \( g(i) \) is a Cauchy sequence in \( W^{2,3}(\hat{K}) \) and hence \( g(i) \to h \) strongly in \( W^{2,3}(\hat{K}) \) and \( h \) is in \( W^{2,3}(\hat{K}) \). Using these facts in (B.10), and taking a limit as \( i \to \infty \) in \( L^3(K) \), we also see that
\[ h^{ab} \partial_a \partial_b (h_{kl}) + (\partial h \ast \partial h)_{kl} = 0 \]
must be satisfied in the \( L^3 \) sense, and hence in the \( L^2 \) sense. In particular, we have
\[ h^{ab} \partial_a \partial_b (h_{kl}) = P_{kl} \]
(B.16)
in the usual \( W^{1,2} \) sense where \( P_{kl} = - (\partial h \ast \partial h)_{kl} \) is an \( L^p \) function on smooth compact subsets \( K \), for any \( p \in (1, \infty) \), since \( h \in W^{1,p}(K) \) for any \( p \in (1, \infty) \). The \( L^p \) theory tells us, that \( h \in W^{2,p} \) on smooth compact subsets of \( B_{1/100} \), and hence
$P \in W^{1,p}$ on smooth compact subsets of $B_{1/100}$. The $L^2$ theory (see for example Theorem 8.10 in [GT]) then tells us, that $h$ is in $W^{3,2}$ on smooth compact subsets. So we may differentiate the equation (B.16) to get a new equation,

$$h^{ab}\partial_a\partial_b(h_{kl}) = \hat{P}_{skl}$$

where $\hat{P}_{skl} = \partial_s P_{kl} - \partial_s h^{ab}\partial_s \partial_b(h_{kl})$ is in $L^p$ on any compact subset. Continuing in this way, we obtain $h$ is $C^\infty$ and bounded in $C^k$ on any smooth compact subset of $B_{1/100}$.

Let $x \in U_r \cap U_t$ in $X$. As explained above, this means $s_{i,r,t} = (\varphi_{r,t})^{-1} \circ \varphi_{i,t} : B_{2\varepsilon}(v) \to \mathbb{R}^4$ is well defined with $x = \varphi_{i,t}(v)$ for some small $\varepsilon > 0$, and satisfies $s_{i,r,t} \to s_{r,t} = (\varphi_{r})^{-1} \circ \varphi_t$ weakly in $W^{2,12}(B_{\varepsilon}(v))$. The equation satisfied by $s_i := s_{i,r,t}$ (see (B.2)) is

$$g(i)^{ab}\partial_a\partial_b s_i = 0$$

on $B_{2\varepsilon}(v)$. Using the $L^p$ theory, we see that $s_i$ is bounded in $W^{2,p}(B_{\varepsilon/2}(v))$ for all $p \in (1, \infty)$, independently of $i$. Hence, we have

$$(B.17) \quad g(i)^{ab}\partial_a\partial_b(s_i - s_j) = -g(i)^{ab}\partial_a\partial_b s_j = (g(i) - g(i)^{ab}\partial_a\partial_b s_j)$$

and the right hand side goes to $0$ in $L^p(B_{\varepsilon/2}(v))$, as does $s_i - s_j$, as $i, j \to \infty$, and hence using the $L^p$ theory again, $s_i \to s$ in $W^{2,p}(B_{\varepsilon/4}(v))$. Hence, Equation (B.17) holds in the limit,

$$(B.19) \quad h^{ab}\partial_a\partial_b s = 0.$$  

Using the regularity theory for elliptic equations (for example the argument we used above to show that $h$ is smooth), we see that $s$ is smooth. That is $(X,h,p)$ is a $C^\infty$ Riemannian manifold. It also holds, that $(M,(g(i),p_i) \to (X,h,p)$ in the pointed Gromov-Hausdorff $W^{1,12}$ sense as $i \to \infty$: for all $l$, there exists a map $F_{i,l} : U_l \to M$, such that $F_{i,l}(p_i) = p_l$ and $F_{i,l} : U_l \to M$ is a $C^\infty$ diffeomorphism onto its image, $h B_l(p) \subseteq U_l$, and $F_{i,l}^{*}(g(i)) \to h$ in $W^{1,12}(K)$ strongly for any compact set $K \subseteq U_l$, where $U_l$ is open in $X$. In fact here, this convergence will be in $W^{1,1}(K)$ for any fixed $p \in (1, \infty)$.

As soon as one knows, that the transition functions converge in $W^{2,p}$, and the metric converges on coordinate neighbourhoods (of the type above) in $W^{1,p}$ to $h$, $(X,h,p)$ is $C^\infty$ and $p > n$, then it is always possible, as explained in the introduction of the paper [AnCh], to construct diffeomorphisms of this type: see [Ka] or [Ch1], [Ch2] for earlier works. We give some more detail for the readers convenience. We use mainly the reference [Ka], and the construction of the diffeomorphisms given here is somewhat different from that presented in [Petersen]. In the following $0 < \varepsilon << r$ is fixed. Let $\cup_{j=1}^N \hat{U}_{i,j} := \varphi_{i,j}(B_{r-50\varepsilon})$ be a covering of $g^* B_{2\varepsilon}(p_i)$, $\cup_{j=1}^N \hat{U}_{j} := \varphi_j(B_{r-50\varepsilon})$ be a covering of $h^* B_{2\varepsilon}(p)$, with $\varphi_{i,j} : \hat{U}_{i,j} \to \mathbb{R}^4$ converging to $\varphi_j : \hat{U}_j \to \mathbb{R}^4$ as described by [Petersen] (see above). Here $U_j = \varphi_j(B_r)$, and $U_{i,j} = \varphi_{i,j}(B_r)$. Define $\hat{\Omega}_i := \cup_{j=1}^N (\hat{U}_{i,j} := \varphi_{i,j}(B_{r-50\varepsilon}))$, $\hat{\Omega}_i := \cup_{j=1}^N (\hat{U}_{i,j} := \varphi_{i,j}(B_{r-20\varepsilon}))$, $\hat{\Omega}_i := \cup_{j=1}^N (U_{i,j} := \varphi_{i,j}(B_{r-60\varepsilon}))$, $\hat{\Omega}_i := \cup_{j=1}^N (\hat{U}_{j} := \varphi_j(B_{r-20\varepsilon}))$, $\hat{\Omega} := \cup_{j=1}^N (U_{j} := \varphi_j(B_{r-60\varepsilon}))$, $\hat{\Omega} := \cup_{j=1}^N (\hat{U}_{j} := \varphi_j(B_{r-20\varepsilon}))$. We assume that $g^* B_{2\varepsilon}(p_j) \subseteq \hat{\Omega}_i \subseteq \hat{\Omega}_i \subseteq \hat{\Omega} \subseteq \hat{\Omega}$ and $h^* B_{2\varepsilon}(p) \subseteq \Omega \subseteq \Omega^* \subseteq \hat{\Omega} \subseteq \hat{\Omega}$.
Let \( \eta : [0, \infty) \to \mathbb{R}_0^+ \) be a smooth rotationally symmetric cut-off function, \( 0 \leq \eta \leq 1, \eta(x) = 1 \) for \( x \in (0, r - 6\varepsilon), \eta(x) \geq 1 - C\delta \) for \( x \in (r - 6\varepsilon, r - 5\varepsilon), \eta(x) = 0 \) for all \( x \in [r - 4\varepsilon, \infty) \). Define, as in \([Ka]\), the smooth functions \( \eta_{i,j} : \Omega_i \to \mathbb{R}_0^+ \) by

\[
\eta_{i,j}(x) = \eta((\varphi_{i,j}^{-1})(x)), \quad j = 1, \ldots, N.
\]

The embeddings, \( \theta_i : \Omega_i \to \mathbb{R}^k \), of \([Ka]\) are then defined by

\[
\theta_i(\cdot) := \left( \eta_{i,1}(\cdot)(\varphi_{i,1})^{-1}(\cdot), \eta_{i,2}(\cdot)(\varphi_{i,2})^{-1}(\cdot), \ldots \right),
\]

(B.20)

\[
\eta_{i,N}(\cdot)(\varphi_{i,N})^{-1}(\cdot), \eta_{i,1}(\cdot), \eta_{i,2}(\cdot), \ldots, \eta_{i,N}(\cdot) \right).
\]

The maps \( \theta_i : \tilde{\Omega}_i \to \mathbb{R}^k, \ i \in \{1, \ldots, N\} \) are embeddings, see \([Ka]\), and locally \( \theta_i \circ \varphi_{i,1} \) (for example) is a graph: \( \theta_i \circ \varphi_{i,1} : B_{r-20\varepsilon}(0) \to \mathbb{R}^k \) is given by

\[
\theta_i \circ \varphi_{i,1}(x) = (x, f_{i,2}(x), f_{i,3}(x), \ldots, f_{i,k}(x)) = \eta(|F_{i,j}(\cdot)|),
\]

where \( F_{i,j} := (\varphi_{i,j})^{-1} \circ \varphi_{i,1}, \ f_{i,j}(\cdot) := \eta(|F_{i,j}(\cdot)|) \) and the maps \( U_{i,1} : B_{r-20\varepsilon}(0) \to \mathbb{R}^{k-n} \) are bounded in \( W^{2,p} \) and converge in \( W^{2,p}(B_{r-20\varepsilon}(0), \mathbb{R}^k) \) to the smooth function \( U_1 : B_{r-20\varepsilon}(0) \to \mathbb{R}^{k-n} \), where \( U_1 := (f_2, f_3, \ldots, f_n, f_1, \ldots, f_N) \) and \( F_j := (\varphi_j)^{-1} \circ \varphi_1, \ f_j(\cdot) := \eta(|F_j(\cdot)|) \). Defining \( \theta : \tilde{\Omega} \to \mathbb{R}^k \) similarly to \( \theta_i \),

\[
\theta(\cdot) := \left( \eta_1(\cdot)(\varphi_1)^{-1}(\cdot), \eta_2(\cdot)(\varphi_2)^{-1}(\cdot), \ldots \right),
\]

(B.21)

\[
\eta_N(\cdot)(\varphi_N)^{-1}(\cdot), \eta_1(\cdot), \eta_2(\cdot), \ldots, \eta_N(\cdot) \right).
\]

where \( \eta_j : \tilde{\Omega} \to \mathbb{R}_0^+ \) are defined \( \eta_j(x) = \eta((\varphi_j)^{-1}(x)) \), for \( j = 1, \ldots, N \), we see that \( \theta \) is also an embedding, and hence \( \tilde{\Omega} := \cup_{j=1}^N \{ (t_j(x, u_j(x)) \mid x \in B_{r-20\varepsilon}(0) \} = \theta(\tilde{\Omega}) \) and \( \tilde{M} := \cup_{j=1}^N \{ (t_j(x, u_j(x)) \mid x \in B_{r-50\varepsilon}(0) \} = \theta(\tilde{\Omega}) \) are \( C^\infty \) embedded submanifolds of \( \mathbb{R}^k \), where \( t_j : \mathbb{R}^k \to \mathbb{R}^k \), is the function (which swaps position of coordinates) defined by

\[
t_j(x, m_1, \ldots, m_{N-1}, y_1, \ldots, y_N)
\]

:= \( (m_{j,1}, m_{j,2}, \ldots, m_{j-1}, x, m_{j+1}, \ldots, m_{N-1}, y_1, \ldots, y_N) \)

for \( m_i \in \mathbb{R}^n, \ i \in \{1, \ldots, N-1\} \), \( y_k \in \mathbb{R}, \ k \in \{1, \ldots, N\} \), where \( j \in \{1, \ldots, N\} \). We define the \( C^\infty \) submanifolds, \( M_i := \theta_i(\tilde{\Omega}_i), \ M^* := \theta(\Omega^*) \), \( M^* := \theta(\Omega^*) \), \( M_i := \theta_i(\tilde{\Omega}_i) \) analogously. To see that these maps are embeddings one shows the following. Let \( x : \varphi_1(\cdot) \in M, \ x \in B_{r-20\varepsilon}(0) \) for example, and \( (x, u_1(\cdot)) = (x, y) = \theta \circ \varphi_1(\cdot) \). We claim \( B_\delta(x) \times B_\delta(y) \cap \tilde{M} = \{ (z, u_2(\cdot)) \mid z \in B_{r-20\varepsilon}(0) \} \) for some \( s, \delta \) small enough. We know \( |D\theta(\cdot)| \leq C \) and \( |D\theta(\cdot)| \leq C \) on \( B_{r-4\varepsilon}(0) \) for a constant \( C \). Hence, \( \{(z, u_1(\cdot)) \mid z \in B_{r-20\varepsilon}(0) \} \subseteq (B_\delta(x) \times B_\delta(y)) \cap \tilde{M} \). Now let \( m \) be arbitrary with \( m \in (B_\delta(x) \times B_\delta(y)) \cap \tilde{M} \), and \( \tilde{m} := \theta^{-1}(m) \). Then \( |\theta(\tilde{m}) - \theta(\varphi_1(\cdot))| = |\theta(\tilde{m}) - \theta(\varphi_1(\cdot))| = |\theta(\tilde{m}) - \theta(\varphi_1(\cdot))| \leq C\delta \) and hence, using the definition of \( \theta \), \( |\eta_1(\tilde{m}) - 1| \leq C\delta \) which tells us that \( \tilde{m} \in U_1 \), and \( |v := (\varphi_1(\cdot))^{-1}(\tilde{m})| \leq r - 5\varepsilon \). If \( v \in B_{r-6\varepsilon}(0) \) then \( m = \theta(\tilde{m}) = \theta \circ \varphi_1(v) = (v, u_1(\cdot)) \) that is \( m \in B_\delta(x) \times B_\delta(y) \cap \tilde{M} \), can be written \( m = (v, u_1(\cdot)) \) with \( v \in B_\delta(x) \) and \( u_1(\cdot) \in B_\delta(y) \) as required.

If \( |v| \in (r - 6\varepsilon, r - 5\varepsilon) \), then we know that \( \theta(\tilde{m}) = \theta \circ \varphi_1(v) = (\eta(\cdot)|v|, u_1(\cdot)) \in B_\delta(x) \times B_\delta(y) \) which is Riemannian to a contradiction, since \( |(\eta(\cdot)|v| - (1 - \delta)(r - 6\varepsilon) |(r - 5\varepsilon) \delta < |v| < (r - 7\varepsilon) \delta < < \varepsilon \) and hence \( \eta(\cdot)|v| \notin B_\delta(x) \), since \( x \in B_{r-20\varepsilon}(0) \) (\( s, \delta < < \varepsilon \)).

We construct a diffeomorphism \( \tilde{F}_i : \tilde{\Omega}_i \to \widetilde{M} \) for which \( \tilde{M} \subseteq \tilde{F}_i(\tilde{\Omega}_i) \), and \( \tilde{F}_i := \theta^{-1} \circ \tilde{F}_1 \) will have the desired properties. Since \( \tilde{M} \) is a \( C^\infty \) embedded submanifold
of \( \mathbb{R}^k \), there exists a neighbourhood \( Z \) of the zero section in the normal bundle \( \tilde{M} \downarrow \) of \( \tilde{M} \) (\( \tilde{M} \) is open, without boundary, and not necessarily complete) in \( \mathbb{R}^k \) such that \( \exp_{\perp}|_Z:Z \to O:=\exp_{\perp}(Z) \subset \mathbb{R}^k \) is a diffeomorphism onto its image, and \( \tilde{M} \subset O \) by definition (see for example Section 7, Proposition 26 in [On] for the existence of \( Z \) and the definition of \( \exp_{\perp} \) the exponential map from the normal bundle). Using the fact that the closure of \( \tilde{M} \) is contained in \( \tilde{M} \), we see that the closure of \( \tilde{M} \) is a compact subset of \( O \). That is \( B_\sigma(\tilde{M}) \subset O \) for \( \sigma > 0 \) sufficiently small.

Hence the natural projection map \( \pi: B_\sigma(\tilde{M}) \to \tilde{M} \) is well defined and smooth, \( \pi(x):=\exp_{\perp}|_Z^{-1}(x) \) for all \( x \in B_\sigma(\tilde{M}) \subset O \) (see for example Section 7, Proposition 26 in [On]). By choosing \( \sigma \) smaller in necessary, we can assume that \( |\pi(y)−y| \leq \epsilon \) for all \( y \in B_\sigma(\tilde{M}) \). We show that \( \hat{F}_i:=\pi \circ \theta_i: \hat{\Omega}_i \to \tilde{M} \) is well defined and a diffeomorphism onto its image, for large enough \( i \). Using the fact that \( u_{i,j}:B_{r−\epsilon}(0) \to u_j \) in \( W^{2,p}(B_{r−\epsilon}(0)) \cap C^{1,\alpha}(B_{r−\epsilon}(0)) \) as \( i \to \infty \), we see that \( d_H(\hat{M}_i, \hat{M}) \to 0 \) as \( i \to \infty \): \( \hat{M}_i \subset B_\varepsilon(\hat{M}) \) and \( \hat{M} \subset B_\varepsilon(\hat{M}_i) \). Hence, for \( m \in \hat{U}_{i,1} \subset \hat{\Omega}_i \), we have \( \theta_i(m) = \theta_i \circ \varphi_{i,1}(x) = (x, u_{i,1}(x)) \in \hat{M}_i \subset B_\varepsilon(\hat{M}) \) and hence \( \pi \circ \theta_i \) is well defined for such \( m \). Similarly, \( \pi \circ \theta_i \) is well defined for all \( m \in \hat{\Omega}_i \). Let \( m \in \hat{\Omega} \). Without loss of generality, \( m \in \hat{U}_1 \), \( x:=(\varphi_1)^{-1}(m) \), \( \theta_1(m) = (x, u_1(x)) \in \hat{U}_1 \). \( \theta_1(\hat{U}_1) \) is an open set in \( \tilde{M} \), so we can find a small \( s \) such that \( B_\varepsilon(x) \times B_{cs}(y) \cap \hat{M} \subset \theta_1(\hat{U}_1) \), and \( \pi(B_\varepsilon(x) \times B_{cs}(y)) \subset \theta_1(\hat{U}_1) \). Then \( \theta_1(\hat{U}_1) \) is a diffeomorphism onto its image, for large enough \( i \). Hence \( \theta_1 \circ \varphi_{i,1}(v) \in B_\varepsilon(x) \times B_{cs}(y) \subset \theta_1(\hat{U}_1) \) for all \( i \geq N \) large enough, for all \( v \in B_\varepsilon(x) \), if \( s \) is small enough, and hence \( (\varphi_1)^{-1} \circ \theta_1 \circ \pi \circ \theta_i \circ \varphi_{i,1}: B_\varepsilon(x) \to B_{r−4\epsilon}(0) \) is well defined for \( i \) large enough. By construction, and the Theorem of Vitali (the \( L^p \) version), we have

\[
(\varphi_1)^{-1} \circ (\theta_1)^{-1} \circ \pi \circ \theta_i \circ \varphi_{i,1} = (\theta \circ \varphi_1)^{-1} \circ \pi \circ (\cdot, u_{i,1}(\cdot)) \rightarrow (\theta \circ \varphi_1)^{-1} \circ \pi \circ (\cdot, u_1(\cdot)) = (\theta \circ \varphi_1)^{-1} \circ \theta \circ \varphi_1 = Id
\]

in \( W^{2,p}(B_\varepsilon(x)), \mathbb{R}^4 \). Hence, \( (\varphi_1)^{-1} \circ (\theta_1)^{-1} \circ \pi \circ \theta_i \circ \varphi_{i,1}: B_\varepsilon(x) \to \mathbb{R}^4 \) is a diffeomorphism onto its image for large enough \( i \). Hence \( \theta_1 \circ \pi \circ \theta_i: \hat{\Omega}_i^* \to X \) is a local diffeomorphism, and \( \theta_1 \circ \pi \circ \theta_i(\hat{U}_{i,1}^*) \subset \hat{U}_{i,1} \) for \( i \) large enough, and

\[
\varphi_1 \circ \theta_1 \circ \pi \circ \theta_i \circ \varphi_{i,1}: B_{r−55\varepsilon}(0) \to B_{r−50\varepsilon}(0) \text{ is well defined, and}
\]

\[
\varphi_1 \circ \theta_1 \circ \pi \circ \theta_i \circ \varphi_{i,1} \to Id \text{ in } W^{2,p}(B_{r−55\varepsilon}(0), \mathbb{R}^4)
\]

as \( i \to \infty \).

Hence \( \theta_1 \circ \pi \circ \theta_i: \hat{\Omega}_i^* \to X \) is a diffeomorphism onto its image for large enough \( i \), and \( \hat{\Omega} \subset \theta_1 \circ \pi \circ \theta_i(\hat{\Omega}_i^*) \) for large enough \( i \), as we now show. Assume \( \theta_1 \circ \pi \circ \theta_i(m) = \theta_1 \circ \pi \circ \theta_i(n) \) for \( m, n \in \hat{\Omega}_i^* \). Without loss of generality, \( m \in \hat{U}_{i,1}^* \) and hence \( \theta_1 \circ \pi \circ \theta_i(m) \in \hat{U}_1 \) and hence \( (\varphi_1)^{-1} \circ \theta_1 \circ \pi \circ \theta_i(m) = (\varphi_1)^{-1} \circ \theta_1 \circ \pi \circ \theta_i(n) \), and hence \( (\varphi_1)^{-1} \circ \theta_1 \circ \pi \circ \theta_i \circ \varphi_{i,1}(\tilde{x}) = (\varphi_1)^{-1} \circ \theta_1 \circ \pi \circ \theta_i \circ \varphi_{i,1}(\tilde{z}) \), where \( \varphi_{i,1}(\tilde{x}) = m \) and \( \varphi_{i,1}(\tilde{z}) = n \), where \( \tilde{x}, \tilde{z} \in B_{r−65\varepsilon}(0) \). But this contradicts (B.22) for \( i \) large enough. That is \( \theta_1 \circ \pi \circ \theta_i: \hat{\Omega}_i^* \to X \) is a diffeomorphism, and

\[
B_{2\ell}(p) \subset \bigcup_{j=1}^N \varphi_1(B_{r−65\varepsilon}(0)) = \hat{\Omega} \subset \theta_1 \circ \pi \circ \theta_i(\hat{\Omega}_i^*) \text{ as required.}
\]
Also,
\[
((\varphi_1)^{-1} \circ F_1 \circ \varphi_{i,1})_* (g(i)) = (\theta \circ \varphi_1)^{-1} \circ \pi \circ (\cdot, U_{i,1}(\cdot))_* g(i) \rightarrow h
\]
\[
\text{(B.24)}
\]
in view of (B.23) and the Theorem of Vitali, as required (note: the inverse of 
\([\varphi_1]^{-1} \circ (\theta)^{-1} \circ \pi \circ \theta \circ \varphi_{i,1}\) also converges to the identity in \(W^{2,p}(B_{r-56e}(0), \mathbb{R}^4)\) in view of (B.23) and for example Cramer’s law).

The condition \(\int_X |\text{Riem}(h)|^2 = 0\) must hold for the limiting space, as we now show. For fixed \(r\),
\[
\int_{B_r} |\text{Riem}(h)|^2 \, d\mu_h = \int_{B_r} |h \ast (h)^{-1} \ast D^2 h + h \ast h^{-1} \ast Dh \ast Dh|^2 \, d\mu_h
\]
\[
= \lim_{i \to \infty} \int_{B_{i/100}} |g_{i,r} \ast (g_{i,r})^{-1} \ast D^2 g_{i,\alpha} |
\]
\[
+ |g_{i,r} \ast (g_{i,r})^{-1} \ast Dg_{i,\alpha} \ast Dg_{i,\alpha l}|^2 \, d\mu_{g(i)}
\]
\[
= \lim_{i \to \infty} \int_{B_{i,r}} |\text{Riem}(g(i))|^2 \, d\mu_{g(i)}
\]
\[
\leq \frac{1}{i}
\]
\[
\text{(B.25)}
\]
since the metrics converge in \(W^{2,3}\) locally. Here \(*D^2 h \ast (g(i, \alpha))\) and so on are combinations of the second partial derivatives of \(h\) and can be explicitly written down, and satisfies \(|\ast D^2 h| \leq \alpha(n)|D^2 h|^2\) pointwise where the second derivative is defined.

Hence \(\int_{U_r} |\text{Riem}(h)|^2 \, dvol_h = 0\). \(r\) was arbitrary, and \(h\) is smooth implies that \(|\text{Riem}(h)| = 0\) everywhere.

Hence, \((X, h)\) is flat, and hence must be the standard Euclidean space, since \((X, h)\) has euclidean growth, \(\text{vol}(B_r(x)) \geq \sigma_0 r^4\) for all \(r > 0\) (from the non-collapsing estimate). This leads to a contradiction, using the same argument given in the proof of Main Lemma 2.2 in [And1].

Here, for the readers convenience, we explain the argument which leads to a contradiction. Push the metrics \(g(i)\) forward to \(X = \mathbb{R}^4\) with the \(C^\infty\) diffeomorphisms \(F_i : B(p_k) \subseteq (M, g(i)) \rightarrow X = \mathbb{R}^4\) just constructed. Call these metrics \(\hat{g}(i)\). We know, that \(\hat{g}(i) \rightarrow h = \delta\) on \(B_1(0) \subseteq \mathbb{R}^4\), \(i\) is fixed but large. Solve \(\Delta_{g(i)} \varphi(i)^k \cdot 0\), on \(B_R(0)\), with the boundary conditions \(\varphi(i)|_{\partial B_R(0)} = \text{Id}\). Then \(\Delta_{g(i)}(\varphi(i) - \text{Id})^k = \Gamma(g(i))^{m} \cdot (\hat{g}(i))^k \rightarrow 0\) in \(L^P(B_R(0))\), and \((\varphi(i) - \text{Id})(\cdot) = 0\) on \(\partial B_R(0)\). Using the \(L^p\) theory (Lemma 9.17 of [GT] for example), with the fact that \(\varphi(i) - \text{Id} = 0\) on the boundary, we see that
\[
(\text{B.26}) \quad |\varphi(i) - \text{Id}|_{W^{2,p}(B_R(0))} \rightarrow 0, \text{ as } i \rightarrow \infty.
\]
for \(i\) large enough, since \(\hat{g}(i) \rightarrow h\) in \(W^{1,p}(B_R(0))\) strongly. Hence, \(\varphi(i) : B_{R/2}(0) \rightarrow V_i := \varphi(i)(B_{R/2}(0)) \subseteq \mathbb{R}^4\) is a diffeomorphism for \(i\) sufficiently large, and \(\varphi(i) \rightarrow \text{Id}\) in \(W^{1,p}(B_{R/2}(0)) \cap C^{1,\alpha}(B_{R/2}(0))\), as \(i \rightarrow \infty\). The inverse \(\psi(i) : V_i \rightarrow B_{R/2}(0)\) converges to the identity in \(W^{1,p}(B_{R/4}(0)) \cap C^{1,\alpha}(B_{R/4}(0))\), \((B_{R/4}(0) \subseteq V_i\) for \(i\) large enough), once again in view of Vitali’s Theorem. Hence, \(\psi(i)_* (\hat{g}(i)) =: \hat{g}_i\) converges to \(\delta\), once again in view of Vitali’s Theorem. But then : \(v_i := (F_i)^{-1} \circ
ψ(i) : B_{R/4}(0) → W_i ⊆ M_i is a diffeomorphism, and \( w_i := (u_i)^{-1} : W_i → B_{R/4}(0) \) satisfies
\[
\Delta_{g(i)} w_i(x) = \Delta_{g(i)} (\varphi(i) \circ F_i)(x) = \Delta_{g(i)} (\varphi(i))(F_i(x)) = 0,
\]
and \( (w_i)_* (g(i)) \) is in \( C^{1,\alpha}(B_{R/4}(0)) \cap W^{1,p}(B_{R/4}(0)) \) as close as we like to \( δ \). That is the harmonic radius at \( x \) is larger than \( R/4 \). This contradicts the fact that, \( r_h(y_i)(g(i)) = 1 \).

This finishes the proof.

\[\square\]

APPENDIX C. VOLUME ESTIMATES OF P. PETERSON AND G-F. WEI

Let \((M^n, g)\) be a smooth complete Riemannian manifold without boundary satisfying and \( p > n/2 \) and \( q \in M \) is fixed. We choose coordinates so that \( g(q) = δ \) on \( T_q M = \mathbb{R}^n \), \( δ \) the Euclidean metric on \( T_q M = \mathbb{R}^n \). Let \( S \) be an open set contained in the sphere \( S_q \subseteq T_q M \) with \( d\theta(S) = μ > 0 \), \( d\theta \) the standard measure on \( \mathbb{S}_q^{n-1}(0) \), where \( S_q M = \mathbb{S}_q^{n-1}(0) \) is the sphere of radius one, \( S_q M := \{ v \in T_q M \text{ such that } |v| = 1 \} \). So \( S \subseteq \mathbb{S}_q^{n-1}(0) \subseteq \mathbb{R}^n \). Let \( W_r = \{ \exp(sv) \mid v \in S, s \leq r \} \), and \( V_r = \{ \exp(sv) \mid v \in S, s \leq r \text{ and } \exp(v) : [0, s] → M \text{ is length minimising} \} \). We consider the corresponding set \( E_r \) in \( \mathbb{R}^n \): \( E_r = \{ sv \mid s \leq r, v \in S \} \). The estimates of P. Peterson and G-F. Wei in this setup that we require are as follows.

\[
\left( \frac{\text{vol} V_R(x)}{\text{vol}(E_R)} \right)^{1/2p} - \left( \frac{\text{vol} V_r(x)}{\text{vol}(E_r)} \right)^{1/2p} \leq c(n, R, p) \frac{1}{\mu^{1/(2p)}} \left( \int_{B_R(p)} |Rc|^p \right)^{1/(2p)}
\]

where \( \mu = d\theta(S) > 0 \) and \( r < R \).

**Remark C.1.** Compare with Theorem 2.1 in [TZ].

**Proof.** We argue as in [PeWe], but we replace their \( S \) (the sphere in \( \mathbb{R}^n \)) by our set \( S \), and we replace their \( λ \) by \( λ = 0 \). We denote with \( \hat{C}_p \) the cut locus, \( \hat{C}_p := \{ \exp(sv) \mid v \in \mathbb{S}_q^{n-1} \text{ and } \exp|_{[0, s]}(v) : [0, s] → M \text{ is distance minimising, but } \exp|_{[0, s+\varepsilon]}(v) : [0, s + \varepsilon] → M \text{ is not distance minimising for all } \varepsilon > 0 \} \).

Let \( \hat{D}_p \subseteq M^n \) be the set \( \hat{D}_p = \{ \exp(rv) \mid v ∈ \mathbb{S}_q^{n-1}, r < c(v) \} \).

Here, \( c : \mathbb{S}_q^{n-1} → \mathbb{R}^+ \) is the function which tells us how far we have to travel along a geodesic, pointing in a given direction, before we hit the cut locus: \( c(v) := s \) where \( \exp|_{[0, s]}(v) : [0, s] → M \text{ is distance minimising, but } \exp|_{[0, s+\varepsilon]}(v) : [0, s + \varepsilon] → M \text{ is not distance minimising for all } \varepsilon > 0 \) \( D_p, C_p \) will denote the corresponding sets in \( \mathbb{R}^n \): \( C_p := \{ c(v) v \mid v ∈ \mathbb{S}_q^{n-1} \} \), \( D_p := \{ rv \mid v ∈ \mathbb{S}_q^{n-1}, r < c(v) \} \). The set \( D_p \) is star shaped, and the function \( c \) is continuous: see for example the book [Chav] for a proof of these facts and other related facts. Define \( V_r := \{ \exp(sv) \mid v ∈ S, s \leq r \} \), and \( \exp(v)|_{[0, s]} : [0, s] → M \text{ is distance minimising} \). That is \( V_r \) is the set of points obtained by going along a distance minimising geodesic in the direction \( v \) for a distance \( s \) less than or equal to \( r \), where \( v ∈ S \) is arbitrary. Notice that \( V_r ⊆ W_r = \{ \exp(sv) \mid v ∈ S, s \leq r \} \).
On $D_p$, we can write the metric with respect to spherical coordinates as $d\mu_p(r,\theta) = \omega(r,\theta)dr \wedge d\theta$ where $d\theta$ are the standard coordinates on $S^{n-1}_1(0)$, and $\omega(0,\cdot) = 0$. Let us denote the corresponding volume form in Euclidean space by $d\mu_E(r,\theta) = r^{n-1}dr \wedge d\theta = \omega_E(r,\theta)dr \wedge d\theta$ that is $\omega_E(r,\theta) = r^{n-1}$ doesn’t depend on $\theta$. $\omega$ is a smooth function on $D_p \backslash \{0\}$, and $\omega(\cdot, v) : (0, c(v)) \to \mathbb{R}^+$ is smooth and satisfies $rac{\partial}{\partial t} \frac{\omega(v,t)}{\omega_E(t)} = \psi(t,v)\omega(v,t)$ where $\psi(t,v) := h(t,v) - h_E(t)$ and $h(t,v)$ is the mean curvature at $(t,v)$ of the geodesic sphere at distance $t$ from $p$ in $(M,g)$ at the point $\exp(tv)$, and $h_E(t)$ is the mean curvature of the sphere of radius $t$ in euclidean space, that is $h_E(t) = \frac{(n-1)}{t}$; see [CLN]. One uses here, that $\frac{\partial}{\partial t} \omega = h\omega$ and $\frac{\partial}{\partial t} \omega_E = h_E\omega_E$ as shown in [CLN] (see equation 1.132 there). Integrating with respect to the $r$ direction we get

$$(C.2) \quad \frac{\omega(v,r)}{\omega_E(r)} - \frac{\omega(v,t)}{\omega_E(t)} \leq \int_t^r \frac{\omega(s,v)}{\omega_E(s)} ds$$

for all $0 < t \leq r \leq c(v)$. We extend $\psi$ and $\omega$ to all of $\mathbb{R}^n$ by defining them to be zero on $(D_p)^c$. These are then measurable functions since $D_p$ can be approximated by smooth open sets $D_c$ contained in $D_p$ and so we can approximate any of these functions pointwise (call it $f$) by $\eta f$ where $\eta$ is a cut off function with $\eta = 1$ if we are a distance (euclidean) $\varepsilon$ away from $\partial D_c$ but in $D_p$. Furthermore $\omega$ is bounded on any ball $B_R(0)$; see Theorem III.3.1 in [Chav] and use that the solution of linear ordinary differential equations with smooth coefficients are themselves smooth. The equation $(C.2)$ holds for all $v \in S^{n-1}$ and all $0 \leq t \leq r < \infty$ as one readily verifies: the only new case one needs to consider is $r > c(v)$, and in this case the left hand side is less than or equal to zero and the right hand side is always larger than or equal to zero, and so the equation holds trivially. All functions are measurable and non-negative and so we may apply Fubini to them. That is, we may integrate $(C.2)$ over $S$ and change the order of integrals. We do so in the following, sometimes without any further comment.

$$(C.3) \quad \int_S \frac{\omega(\theta,r)}{\omega_E(r)} d\theta - \int_S \frac{\omega(\theta,t)}{\omega_E(t)} d\theta \leq \int_t^r \int_S \psi(s,v) \frac{\omega(v,s)}{\omega_E(s)} d\theta ds$$

Dividing by $d\theta(S) = \mu$ we get

$$(C.4) \quad \frac{\int_S \omega(\theta,r)d\theta}{\int_S \omega_E(r)d\theta} - \frac{\int_S \omega(\theta,t)d\theta}{\int_S \omega_E(t)d\theta} \leq \frac{1}{d\theta(S)} \int_t^r \int_S \psi(s,v) \frac{\omega(v,s)}{\omega_E(s)} d\theta ds.$$

This corresponds to the second inequality at the top of page 1037 of [PeWe] (with there $S^{n-1}$ replaced by our $S$). We continue on now as in [PeWe], to obtain

$$\begin{align*}
\left( \int_S \omega(r,\theta)d\theta \right) \left( \int_S \omega_E(t,\theta)d\theta \right) - \left( \int_S \omega_E(r,\theta)d\theta \right) \left( \int_S \omega(t,\theta)d\theta \right) \\
\leq d\theta(S)\omega_E(r) \left( \int_0^r \int_S \psi^{2p}(s,\theta)\omega(s,\theta)d\theta \wedge ds \right)^{1/2p} \\
\times \left( \int_0^r \int_S \omega(s,\theta)d\theta \wedge ds \right)^{1-1/(2p)} \\
(C.5) = d\theta(S)\omega_E(r)\left( \text{vol}(V_r) \right)^{1-1/(2p)} \left( \int_0^r \int_S \psi^{2p}(s,\theta)\omega(s,\theta)d\theta \wedge ds \right)^{1/2p}
\end{align*}$$

Since $\omega$ is bounded on any ball $B_R(0)$, we see that $f(r) := \text{vol}(V_r) = \int_0^r \int_S \omega(t,\theta)d\theta \wedge dt$ is Lipschitz continuous in $r$. Note we use here, that for $K_{S,r} := \{mv \mid v \in S, m \leq$
\( r \}, \ V_r = \text{exp}(K_{S, r} \cap D_p) \cup \text{exp}(K_{S, r} \cap C_p) \) and that the measure of the second set is zero, since the cut-locus \( \tilde{C}_p = \text{exp}(C_p) \) has measure zero: see Section 3 of [Wec] for a proof of the fact that the cut locus has measure zero. Notice also that this also shows that \( V_r \) is measurable: the second set has measure zero (and so is measurable) and the first set is \( \text{exp}(K_{S, r} \cap D_p) = f^{-1}(K_{S, r} \cap D_p) \) where \( f : D_p \to D_p \) is the smooth inverse of \( \text{exp}(p) : D_p \to \tilde{D}_p \).

Hence, the function \( f : [0, \infty) \to \mathbb{R}_0^+ \) is in \( W^{1,\infty} \) and the derivative exists almost everywhere and is equal to the weak derivative whenever it exists: see proof of Theorem 5.4.2.3 in [EG]. Also, using the Lebesgue-Besicovitch differentiation Theorem (see Theorem 1 in 1.7 of [EG]), we see that the derivative \( f'(r) \) of \( f \) exists almost everywhere and is equal to \( f'(r) = \int_S \omega(r, \theta) d\theta \land dr \). Using this notion of derivative, we argue as in [PeWe] again to obtain (5th line of page 1038 of [PeWe]):

\[
\frac{\partial}{\partial r} \frac{\text{vol}(V_r)}{v_r} \leq \frac{d\theta(S) r \omega_E(r)(\text{vol}(V_r))^{1-1/(2p)} (\int_{B_r(0)} \psi^{2p} \mu_\partial g)^{1/(2p)}}{v_r} \\
\leq \frac{d\theta(S) r \omega_E(r)(v_r)^{-1/(2p)} (\text{vol}(V_r))^{1-1/(2p)} (\int_{B_r(0)} \psi^{2p} \mu_\partial g)^{1/(2p)}}{v_r}
\]

(C.6)

where

\[
\begin{align*}
v_r := \int_S \int_0^r \omega_E(s) d\theta \land ds = \int_0^r s^{n-1} ds = \frac{1}{n} (\frac{d\theta(S)}{r})^n
\end{align*}
\]

(C.7)

since \( \omega_E(r) = r^{n-1} \). The first term \( \frac{d\theta(S) r \omega_E(r)}{v_r} \) is equal to \( \frac{n \, d\theta(S)}{r^n} = n \). The term \( \int_{B_r(0)} \psi^{2p} \mu_\partial g \) is shown in Lemma 2.2 of [PeWe] to be bounded by \( \int_{B_R(0)} |Rc|^{1/p} \mu_\partial g \). Hence we get:

\[
\begin{align*}
\frac{\partial}{\partial r} \frac{\text{vol}(V_r)}{v_r} &\leq n(v_r)^{-1/(2p)} (\text{vol}(V_r))^{1-1/(2p)} \frac{\Lambda}{v_r} \\
&= n \left( \frac{d\theta(S)}{r^n} \right)^{1/(2p)} (\text{vol}(V_r))^{1-1/(2p)} \frac{\Lambda}{v_r} \\
&= \frac{c(n, p)}{\mu_\partial g} (\text{vol}(V_r))^{1-1/(2p)} \frac{1}{r^{n/(2p)}} \Lambda
\end{align*}
\]

(C.8)

for all \( r \leq R \), where \( \Lambda = (\int_{B_R(0)} |Rc|^{1/p} \mu_\partial g)^{1/(2p)} \).

That is

\[
\frac{\partial}{\partial t} f(t) \leq \frac{c(n, p)}{\mu_\partial g} \Lambda f^{1-1/(2p)}(t) g(t)
\]

where \( f(t) := \frac{\text{vol}(V_r)}{v_r} \) and \( g(t) := \frac{1}{r^{n/(2p)}} \). This implies (using the chain rule for weakly differentiable functions) that

\[
\frac{\partial}{\partial t} f^{1/(2p)}(t) \leq \Lambda \frac{c(n, p)}{\mu_\partial g} g(t)
\]

(C.10)

and hence

\[
\frac{\partial}{\partial t} \left( f^{1/(2p)}(t) - \frac{1}{1 - n/(2p)} \Lambda \frac{c(n, p)}{\mu_\partial g} t^{1-n/(2p)} \right) \leq 0.
\]

(C.11)

But this means that the function \( f(t) := f^{1/(2p)}(t) - \frac{1}{1 - n/(2p)} \Lambda \frac{c(n, p)}{\mu_\partial g} t^{1-n/(2p)} \) is non-increasing: let \( \psi \) be the \( W^{1,\infty} \) function given by \( \psi(x) = (1/\varepsilon)(x-r) \) for \( x \in (r, r+\varepsilon) \) and \( \psi(x) = 1 \) for \( x \in (r+\varepsilon, s-\varepsilon) \), \( \psi(x) = (1/\varepsilon)(s-x) \) for \( x \in (s-\varepsilon, s) \), and
$\psi(x) = 0$ for all other $x$, where $0 < r < s \leq R$. Mollifying this $\psi$, we obtain a smooth function $\tilde{\psi} \geq 0$ with compact support such that

$$l(s) - l(r) \sim -\int_0^R l(t)(\psi)'(t)dt$$

$$\sim -\int_0^R l(t)(\tilde{\psi})'(t)dt$$

$$= \int_0^R l'(t)\tilde{\psi}(t)dt$$

$$\leq 0$$

(C.12)

and taking a limit with $\varepsilon \to 0$ gives us the claimed monotonicity of $l$ (use also that $l$ is Lipschitz continuous here). The monotonicity of $l$ gives us:

$$(C.13) \quad (\frac{\text{vol}(V_r(p))}{\text{vol}(E_r)})^{1/(2p)} - (\frac{\text{vol}(V_s)}{\text{vol}(E_s)})^{1/(2p)} \leq 2^{n/p} \frac{c(n,p)}{\mu^{1/(2p)}} R^{1-n/(2p)}$$

for all $0 \leq r < s \leq R$, which is the claimed estimate. \hfill \Box

REFERENCES

[AT] Ambrosio, L., Tilli, P., Topology on analysis in metric spaces, Oxford Lecture Series in Math. and its Applications, (2004).

[And1] Anderson, M., Convergence and rigidity of manifolds under Ricci curvature bounds Invent. math. 102, 429-445 (1990)

[And2] Anderson, M., Ricci curvature bounds and Einstein metrics on compact manifolds Journal American Math. Soc., Vol. 2, No. 3 (1989) 455-490

[AnCh] C$^\infty$ compactness for manifolds with Ricci curvature and injectivity radius bounded below, J. Differential Geom. Volume 35, Number 2 (1992), 265-281.

[AnCh2] Anderson, M., and J. Cheeger, J., Diffeomorphism finiteness for manifolds with Ricci curvature and $L^p$-norm of curvature bounded, Geom. Funct. Anal., 1(3):231-252, (1991)

[ACK] Angenent, S., Caputo C., Knopf, D., Minimally invasive surgery for Ricci flow singularities. J. Reine Angew. Math. (Crelle) 672 (2012) 39-87.

[BZ] Bamler, R., Zhang, Q., Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature arXiv:1501.01291 (2015)

[BBI] Burago,D., Burago, Y., Ivanov, S. A Course in Metric Geometry, Graduate Studies in Mathematics, vol.33. A.M.S., Providence, RI, 2001.

[CaoX] Cao, X., Curvature Pinching Estimate and Singularities of the Ricci Flow, Comm. Anal. Geom., Vol. 19 (5): 975-990, (2011).

[BKN] Bando, S.; Kasue, A.; Nakajima, H. On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth, Inv. math., 97, pp. 313 - 349, (1989)

[Ch1] Cheeger, J., Comparison and finiteness theorems for Riemannian manifolds, Ph.D. Thesis, Princeton University, 1967

[Ch2] Cheeger, J., Finiteness theorems for Riemannian manifolds, Amer. J. Math., 92 (1970), 61-74

[ChowII] Chow, B., On the entropy estimate for the Ricci flow on compact 2-orbifolds, J. Diff. Geom. 33, p. 597-600 (1991)

[ChowWu] Chow, B., Wu,L.-F., The Ricci flow on compact 2-orbifolds with curvature negative somewhere, Comm. Pure Appl. Math. 44, p. 275-286 (1991)

[CGT] Cheeger,J., Gromov,M., and Taylor,M. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds Journal Differential Geom. Volume 17, Number 1 (1982), 15-53.

[CTZ] Chen, B.-L., Tang,S.-H., Zhu, X.-P., Complete classification of compact four manifolds with positive isotropic curvature, J. Differential Geom. Volume 91, Number 1 (2012), 41-80.
[ChenYWangI] Chen, X., Wang, Y. Bessel functions, heat kernel and the conical Kaehler-Ricci flow. arXiv:1305.0255 (2013)

[ChenYWangII] Chen, X., Wang, Y. On the long time behavior of the conical Kaehler-Ricci flows. arXiv:1402.6689 (2014)

[ChenWang] Chen, X., Wang, B. On the conditions to extend the Ricci flow III. Int. Math. Res. Notices (2013) 2013 (10): 2349-2367. doi: 10.1093/imrn/rns117

[CLN] Chow, B., Lu, P., Ni, L. Hamilton’s Ricci flow. Graduate studies in Mathematics, Vol. 77, AMS Science Press, (2000)

[DeT] DeTurck, D. Deforming metrics in the direction of their Ricci tensors. J. Differential Geom. 18 (1983), no. 1, 157-162.

[DeTK] DeTurck, D., Kazdan, J. Some regularity theorems in Riemannian geometry. Annales Sci. de l’École Norm. Sup. 14 (3): 249-260, ISSN 0012-9593, MR 644518. (1981)

[Evans] Evans, L., Partial differential equations, Second edition, Graduate Studies Vol. 19, AMS (2010)

[EG] Evans, L., Gariepy, R., Measure Theory and fine properties of functions, Studies in advanced mathematics, CRC Press (1992)

[EGZ] Eyssidieux, Guedj, V., Zeriahi, A. Weak solutions to degenerate complex monge-Ampere flows I. Math. Ann. DOI 10.1007/s00208-014-1141-4, (2014)

[EGZII] Eyssidieux, Guedj, V., Zeriahi, A. Weak solutions to degenerate complex monge-Ampere flows II. arXiv:1407.2504 (2014)

[FIK] Feldman, M., Ilmanen, T., Knopf, D. Rotationally symmetric shrinking and expanding gradient Kaehler-Ricci solitons. J. Differential Geom. 65 (2003), no. 2, 169-209.

[GT] Gilbarg, D., Trudinger, N., Elliptic Partial Differential Equations of Second Order, Third revised edition, Springer, (2000)

[HM] Haslhofer, R., Mueller, R. A compactness theorem for complete Ricci shrinkers. Geom. Funct. Anal. 21(5):1091–1116, (2011)

[HM2] Haslhofer, R., Mueller, R. A note on the compactness theorem for 4d Ricci shrinkers arXiv:1407.1683 (2014)

[HaComp] Hamilton, R.S. A compactness property of the Ricci Flow American Journal of Mathematics, 117, 545–572, (1995)

[HaThree] Hamilton, R.S., Three-manifolds with positive Ricci curvature, Journal of Differential Geometry 17 (2):255-306(1982)

[HaThreeO] Hamilton, R.S., Three-orbifolds with positive Ricci curvature, in Collected papers on Ricci flow, Ser. Geom. Top. 37, International Press, (2003)

[HaFour] Hamilton, R.S., Four-manifolds with positive curvature operator, Journal of Differential Geometry 24: 153-179, (1986)

[HaForm] Hamilton, R.S., The formation of singularities in the Ricci flow, Collection: Surveys in differential geometry, Vol. II (Cambridge, MA), 7-136, (1995).

[JK] Jost, J., Karcher, H., Geometrische methoden zur Gewinnung von a-priori Schranken für harmonische Abbildungen, manuscrypta math. 40, 27 - 77 (1982)

[Ka] Kase, A. A convergence theorem for Riemannian manifolds and some applications, Nagoya Math. J., Vol. 114 (1989),21-51.

[KL] Koch, H., Lamm, T., Geometric flows with rough initial data, Asian J. Math. Volume 16, Number 2 (2012), 209-235.

[Lee] Lee, J., Introduction to smooth manifolds, Graduate texts in Math., Springer (2002)

[Lee2] Lee, J., Introduction to topological manifolds, 2nd. Edition, Springer (2010)

[Li] Li, Y. Smoothing Riemannian metrics with bounded Ricci curvature in dimension four, II Ann. Glob. Anal. Geom. (2012) 41:407-421

[LiuZhang] Liu, J., and Zhang, X., The conical Kaehler-Ricci flow on Fano manifolds, arXiv:1402.1892 (2014)

[Lu] Lu, P., A compactness property for solutions of the Ricci flow on Orbifolds, American Journal of Mathematics, Vol. 123, No. 6, Dec., 2001

[MT] McCann R., Topping, P., Ricci flow, entropy and optimal transportation, American Journal of Math., 132 (2010) 711-730
[MRS], Mazzeo, R., Rubinstein, Y., Sesum, N., *Ricci flow on surfaces with conic singularities*, arXiv:1306.6688 (2013)

[On] O’Neill, Barrett, *Semi-Riemannian geometry with applications to relativity*, Academic Press (1983)

[Petersen] Peter Petersen, *Convergence theorems in Riemannian geometry*, Comparison Geometry, MSRI Publications, Volume 30, 1997

[PeWe] Petersen, P., Wei, G.-F. *Relative volume comparison with integral curvature bounds*, GAFA 7 (1997) 1031-1045

[Pe1] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, MathArxiv link: math.DG/0211159 (2002)

[SS] Sabitov, I., Sefel, S. *Connections between the order of smoothness of a surface and that of its metric*, Akademija Nauk SSSR. Sibirskoe Otdelenie. Sibirskii Matematicheskii Zhurnal 17 (4): 916-925, ISSN 0037-4474, MR 0425855.

[Sesum] Sesum, N., *Curvature tensor under the Ricci flow*, American Journal of Mathematics, Vol. 127, (2005).

[SesumLe] Sesum, N., Le, N.-Q., *Remarks on the curvature behaviour at the first singular time of the Ricci flow*, Pacific Journal of Mathematics, Vol. 255, No. 1, (2012)

[Shi] Shi, W.-X., *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geom. 30 (1989) 303-394.

[Sim0] Simon, M., *Some integral curvature estimates for the Ricci flow in four dimensions*, Arxiv Preprint.

[SimC0] Simon, M. *Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature*, Comm. Anal. Geom. 10 (2002), no. 5, 1033-1074.

[SimSmoo] Simon, M. *Local smoothing results for the Ricci flow in dimensions two and three*, Miles Simon. Geometry and Topology 17 (2013) 22632287.

[SongWeinkove1] Song, W.-Y., Weinkove, B. *Contracting exceptional divisors by the Kaehler-Ricci flow*, Duke Math. J. 162 (2013), no. 2, 367-415

[SongWeinkove2] Song, W.-Y., Weinkove, B. *Contracting exceptional divisors by the Kaehler-Ricci flow II*, Proc. Lond. Math. Soc. (3) 108 (2014), no. 6, 1529-1561

[Tian] Tian, G. *On Calabi's conjecture for complex surfaces with positive first Chern class*, Inventiones mathematicae, Volume: 101, Issue: 1, page 101-172 (1990).

[TZ] Tian, G., Zheng, Z. *Regularity of Kaehler Ricci flows on manifolds*, arXiv:1310.5897

[TME] Topping, P., Mueller, R., Enders, J., *On Type I Singularities in Ricci flow* Communications in Analysis and Geometry, 19 (2011) 905-922

[Wang1] Wang, B. *On the conditions to extend the Ricci flow*, Int. Math. Res. Notices, (2008) 2008 doi: 10.1093/imrn/rnn012

[Wang2] Wang, B. *On the conditions to extend the Ricci flow (II)*, Int Math Res Notices (2012) 2012 (14): 3192-3223. doi: 10.1093/imrn/rnn141

[WangY] Wang, Y. *Smooth approximations of the conical Kaehler-Ricci flows*, arXiv:1401.5040 (2014)

[Wei] Wei, Guofang, *Manifolds with a lower Ricci curvature bound*, Surveys in Differential Geometry XI eds. J. Cheeger and K. Grove, International Press, Somerville, MA, pp. 203-228, (2007)

[WuLF] Wu, L.-F., *The Ricci flow on 2-orbifolds with positive curvature*, J. Diff. Geom. 33, p. 575-596 (1991)

[YangD] Yang, D., *Convergence of Riemannian manifolds with integral bounds on curvature I*, Ann. Sci. de l’ENS, 4, tome 25, no 1(1992), pp.77-105

[Ye] Ye, Rugang, *The logarithmic Sobolev inequality along the Ricci flow*, arXiv:0707.2124

[Yin] Yin, H., *Ricci flow on surfaces with conical singularities*, J. Geom. Anal. 20, p. 970-995 (2010)

[YinII] Yin, H., *Ricci flow on surfaces with conical singularities II*, arXiv:1305.3355 (2013)

[Zhang] Zhang, Z., *Scalar curvature behaviour for finite-time singularity of Kaehler Ricci flow*, Michigan Math. J. Volume 59, Issue 2 (2010), 419-433.

[Zhang1] Zhang, Q. *A Uniform Sobolev Inequality Under Ricci Flow*, Int. Math. Research Notices, Vol. 2007, Article ID rnm056, 17 pages. doi:10.1093/imrn/rnm056
[Zhang2] Zhang, Q. S. Erratum to: A Uniform Sobolev Inequality Under Ricci Flow, Int. Math. Research Notices, Vol. 2007, Article ID rnm096, 4 pages. doi:10.1093/imrn/rnm096

[Zhang3] Zhang, Q. S. Bounds on volume growth of geodesic balls under Ricci flow Mathematical Research Letters, 2012; 19 (1): 245-253

Miles Simon: Otto von Guericke University, Magdeburg, IAN, Universitätspalz 2, Magdeburg 39104, Germany

E-mail address: msimon@gmx.de