Lessons from the LQG String
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Abstract:

We give a non-technical description of the differences of quantisation of the bosonic string between the usual Fock-space approach and the treatment inspired by methods of loop quantum gravity termed the LCQ string. We point out the role of covariant states with continuous representations of the Weyl operators versus invariant states leading to discontinuous polymer representations. In the example of the harmonic oscillator we compare the optical absorption spectrum for the two quantisations and find that the question of distinguishability depends on the order in which limits are taken: For a fixed UV cut-off restricting the Hilbert space to a finite dimensional subspace the spectra can be made arbitrarily similar by an appropriate choice of state. However, if the states are chosen first, they differ at high frequencies.

1. Introduction

Recently, there have been a number of attempts to bridge the gap of understanding between the string and the loop quantum gravity community. After a “Strings meet Loops” conference at the AEI in Golm, there had been the paper by Thiemann[T1] discussing the quantisation of the bosonic string with loop inspired techniques with a number of surprising results. As a follow-up we published [HP] where we contrasted the construction of [T1] with the usual Fock space construction in a common notational framework focussing on the issue of the conformal anomaly or central charge.

In an attempt to make precise statements, [HP] is to some extend quite technical. As a result, there are a number of misconceptions about that work. In many discussions I learned that we should have been more careful to point out the central points of [HP] and contrasting them to mere technical details. Furthermore, my own understanding of several of the issues involved has grown in the last two years. Therefore I feel it might be profitable to publish this note to explain in a much less technical manner the differences between the two approaches to the bosonic string and possible generalisations to other theories with gauge (e.g. diffeomorphism) symmetries such as gravity. On the other hand, this means that the reader interested in details of arguments that have been left out for readability here should consult [HP].

Even before I start, I would like to point out that all the issues discussed in [HP] and here only apply at the “kinematical” level (i.e. the treatment of spacial diffeomorphisms in loop quantum gravity). As the world-sheet theory of the string splits into independent left- and right-moving (holomorphic and anti-holomorphic) sectors it becomes effectively one dimensional and this kinematical treatment is all that is required. In higher dimensional theories, after one has solved these kinematical issues, one can attack the much thornier problems of the dynamics that appear around the treatment of the Hamiltonian constraint.
I have nothing to say about these and the interested reader should consult [NPZ] and [NP] where these are discussed in detail from a stringy perspective and [T2] for the "inside" perspective. In this sense, our discussion is orthogonal to what is discussed in those papers.

The structure of this note is as follows: In the following section, we review the two approaches to the quantisation of the bosonic string. The main lesson from this is the distinction between invariance and covariance of the state to build the Hilbert space representation upon. Then, in order to understand the physical consequences of the discontinuity of the polymer representation we shift attention to the harmonic oscillator and discuss the optical absorption spectrum in the two treatments. There is a final section before conclusions where the philosophical details for the possibility of a distinction are detailed. Two appendices contain the technical details of the calculation of the absorption spectra for a single and several oscillators.

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2. The bosonic string in a nutshell

In this chapter, I will summarise the core of the construction of [T1] and [HP] for reference in the later discussion.

To understand the issues in which the two approaches to the quantisation of the bosonic string differ one has to distinguish between the abstract algebra $\mathcal{A}$ of observables and its representation as operators acting on a Hilbert space. In the first step, both approaches are parallel and only in the second, the choice of the Hilbert space, there is a difference with potentially physical consequences.

The algebra $\mathcal{A}$ is generated by elements denoted $W(f)$ where $f : S^1 \rightarrow \mathbb{R}$ is a real function on the spacial (actually: light-like) part of the world-sheet of the string which for the closed string we are discussing is a circle. Although one should think of $f$ as a test-function used to smear the current $\partial X$, we will not worry about smoothness properties of these functions as in the end the algebra will be completed to a $C^*$-algebra anyway and thus it is enough to work on a dense subset. In [T1], Thiemann uses piecewise constant functions as these can be thought of analogues of holonomies along paths (or spin networks in the higher dimensional case) which are central in the LQG framework while in the usual textbook treatment of the string, $f$ is expanded in its Fourier modes (usually denoted as $\alpha_n$). Here, we will use general functions $f$ to have a unified language.

The product of two such operators $W(f)$ and $W(g)$ is given by $W(f + g)$ up to a phase proportional to $\sigma(f, g) = \int_{S^1} f dg$. Expanded in Fourier modes, this phase is the representation of the canonical commutation relations $[\alpha_m, \alpha_n] = m\delta_{m+n}$. Note well that the expression for the phase is given in coordinate free form. Thus it is independent of any choice of coordinates. We can reformulate this as an active rather than passive transformation as $\sigma(f, g) = \sigma(f \circ S, g \circ S)$ in terms of diffeomorphisms $S : S^1 \rightarrow S^1$ which
map the circle onto itself. This is the diffeomorphism invariance that we would like to preserve in the quantum theory as well.

In any case, these diffeomorphisms are symmetries (automorphisms more precisely) of the algebra $\mathcal{A}$. The crucial question is if they are also symmetries of the Hilbert space on which $\mathcal{A}$ is represented. Specifically, for each such $S$ we need a unitary operator $U(S)$ on the Hilbert space such that for the operators $\varrho(W(F))$ (here $\varrho : \mathcal{A} \rightarrow \text{End}(\mathcal{H})$ is the representation) we have

$$\varrho(W(f \circ S)) = U(S)^{-1} \varrho(W(f)) U(S).$$

(1)

The existence of these $U(S)$ is really what makes the Hilbert space covariant.

Of course there are many diffeomorphisms and they form a group under concatenations. For two such diffeomorphisms $S$ and $T$, (1) guaranties that $U(S)U(T) = U(S \circ T) \times \exp(i\phi(S,T))$ for some phase $\phi(S,T)$. Obviously the relation (1) only defines $U(S)$ up to a phase $\psi(S)$ and making a different choice for the $\psi$’s changes $\phi$ as well. In the following, one has to proceed to the physical Hilbert space of invariants of all $U(S)$. This should not be \{0\} and therefore one needs to find an assignment of $\psi$’s that make all the $\phi$’s vanish. Unfortunately, this is not always possible and the obstruction is the central charge or conformal anomaly.

For example, in the usual Fock space construction of the $U(S)$ (which is a bit lengthy and described in detail in [HP]) for a single target space coordinate $X$ the central charge does not vanish. However, one can take $D$ copies of the algebra $\mathcal{A}$ and represent them on the direct sum of Hilbert spaces. The central charge is additive under this addition of theories and it turns out that by taking exactly $D = 26$ copies of the theory and adding to it a similar but fermionic theory of the $bc$-ghost system one can make the total phase $\phi$ vanish. This is the way the critical dimension appears in this setup.

To understand the statement that in the LQG quantization there is no anomaly and thus no central charge one has to understand the connection between states of the algebra $\mathcal{A}$ and its representations. For Hilbert space representations of C*-algebras are constructed in a similar way (called GNS construction) as highest weight representations of Lie algebras: One starts with a state (called the vacuum but as we are not discussing dynamics there is no notion of energy and thus it should not be thought of as “lowest energy state” but just as a reference state) and acts on it with the elements of $\mathcal{A}$. This state is characterised by specifying the expectation values in this state of all the observables in the algebra $\mathcal{A}$. Formally, it is given by a linear function $\omega : \mathcal{A} \rightarrow \mathbb{C}$. It has to be positive and normalised as $\omega(W(0)) = 1$ as $W(f)$ for the function $f$ which is 0 everywhere is the unit in $\mathcal{A}$. The Hilbert space built on this $\omega$ contains besides the vacuum $|0\rangle$ all the vectors $|A\rangle$ created by acting with elements $A \in \mathcal{A}$ on that vacuum. The role of $\omega$ is to supply this Hilbert space with a scalar product as $\langle A|B \rangle = \omega(A^*B)$ where the adjoint of an $W(f)$ is given by $W(f)^* = W(-f)$.

As any element in $\mathcal{A}$ can be given as a linear combination of $W(f)$’s it suffices to give $\omega$ just on those observables. Every Hilbert space representation can be obtained in this way. However, we have not yet constructed the unitary implementers of the symmetries $U(S)$. If
the state $\omega$ happens to be invariant under the symmetries, that is $\omega(W(f)) = \omega(W(f \circ S))$ for all $f$, there is an easy choice: Just define $U(S)$ as the operator that maps $|W(f)\rangle$ to $|W(f \circ S)\rangle$. The invariance of $\omega$ implies that this map is unitary. Furthermore, with this choice of $U(S)$ the phase $\phi(S,T)$ which would lead to the central charge vanishes automatically. But we should reiterate that this simple choice is only available if $\omega$ is invariant as otherwise these operators fail to be unitary.

In fact, a map $\omega$ such that $\omega(W(f))$ which is invariant under moving the points of the $S^1$ around cannot locally depend on $f$: The choice according to LQG (the “polymer state”) accordingly is to set $\omega(W(f))$ to 1 if $f$ is the function $f = 0$ which vanishes everywhere and to 0 otherwise. The price for this invariance however is a high one: This $\omega(W(f))$ is obviously not continuous under variations of $f$! We will explore the physical consequences of this discontinuity in the next chapter.

In contrast, in the usual Fock space quantisation of the string, the vacuum is in general not invariant under diffeomorphisms. Rather, it is only invariant under those diffeomorphisms which do not mix positive and negative frequency modes. Those diffeomorphisms (acting as Boguliubov operators) which map particles to anti-particles and vice versa change the Fock vacuum. Therefore for those $S$ one has to define $U(S)$ by different means which are detailed in [HP]. The price to pay however is the central charge which selects the critical dimension as explained above.

3. Invariance vs. Covariance

It seems, both quantisations have their disadvantages: The Fock space quantisation is anomalous while the LCQ inspired quantisation has a discontinuity.

Indeed, the anomaly of the Fock quantisation fatal: If present, the physical Hilbert space of invariants is empty. However in the critical dimension it cancels between the space-time fields and the ghosts. Thus in this case, the bosonic string provides a non-trivial, continuous and covariant (in the sense that the $U(S)$ exist for all diffeomorphisms) representation. This construction (which is in fact the Gupta-Bleuler construction in disguise) yields a realisation of gauge symmetries which is potentially anomalous and thus only works for special classical theories where the anomalies happen to vanish. The other price to pay is that (in some sense half of) the unitary implementers $U(S)$ act in a non-trivial way on the vacuum and are thus spontaneously broken in this state.

This is in contrast to the LQG quantisation where the vacuum is invariant by construction and as we have explained there is never an anomaly. Here the price is the discontinuity of the representation of the algebra which results in a quite singular scalar product: In fact, two states $|W(f)\rangle$ and $|W(g)\rangle$ only have a non-vanishing scalar product (and thus overlap) if $f = g$. This induces the discrete topology on the space of functions $f$. Not only diffeomorphisms but all maps $\tilde{S}$ which point-wise map $S^1$ to itself are symmetries. Thus any topological notion of “nearness” is lost and only “equality” is preserved.

In higher dimensions, there is a theorem [LOST] which shows that the requirement of invariance of $\omega$ is so strong that it singles out the polymer state uniquely. It seems plausible that a similar uniqueness theorem holds in the case of the string as well. This theorem is
usually cited as justification for the inevitability of the polymer state in LQG.

The existence of the Fock space is not a counter example to this possible generalisation as we argued above it is not invariant. However, as explained, the physical requirement is only the existence of the unitary implementers \( U(S) \) and those happen to be particularly simple for Hilbert spaces built on invariant states. The Fock space is an example where the \( U(S) \) are realised in a different, non-trivial way. What this example really shows is that other unitary implementers than those from invariant states can have better properties as they allow for a continuous representation of the algebra. Thus the restriction of attention to Hilbert spaces built on invariant states is at least not the most general one and might in fact be too narrow. Unfortunately, constructing \( U(S) \) in general is a very hard problem and there is no solution in sight for the more relevant case of higher dimensional diffeomorphisms needed for realistic theories of quantum gravity. What the example of the bosonic string suggests is that consequences of the discontinuity of the polymer state might be in fact due to a poor choice of Hilbert space representation which is not mathematically forced but just a choice of convenience. It might be that there are better choices where some of the diffeomorphisms are spontaneously broken.

One should compare this to the situation in classical general relativity: GR is obviously diffeomorphism invariant. However any classical state given by Cauchy data or a four-metric which satisfies Einstein’s equations breaks an infinite number of diffeomorphisms and just leaves a finite number of isometries unbroken. For the Minkowski solution there are just four translations three rotations and three boosts.

Of course, it is still possible to use different coordinate systems in Minkowski space such as spherical coordinates and thus the theory is still diffeomorphism invariant. The important point is however that the solution specified by the metric tensor is not invariant but the metric has to be transformed. One has to take the pull-back! Similarly, in general, the state represented by \( \omega \) has to be transformed when a diffeomorphism is applied. And this is achieved by the non-trivial \( U(S) \).

In fact, there is one choice of tensor which is invariant under all diffeomorphisms which is \( g_{\mu\nu} = 0 \). This tensor is of course not suitable as a metric as it fails to be invertible. Requiring an invariant state in classical relativity would force one to choose this pathological metric.

So, there might be other choices of covariant Hilbert spaces for quantum gravity than those built upon the polymer state just as the bosonic string has a continuous representation which is not the polymer representation. But there remains the question if one should regard the polymer state as a valid (although maybe unfortunately chosen) physical state or whether one should abandon it because of the discontinuity. In the end, [T1] proposes it as a quantisation of the string in others than the critical dimension.

Thus, the question is if one should regard weak continuity as a physical requirement of a “good” quantisation procedure or if we can do without. Experience has shown that in the past, successful quantisations of physical theories there was a continuous representation available and one did not need to resort to these more singular choices. But before a fully formulated theory of quantum gravity is known there is the possibility that for such a
4. Weak discontinuity: Harmonic Oscillator

At the current state, one can still try to understand the physical consequences of the discontinuity. In [HP] we approached this question by considering an even simpler but physically well known system where we explored the properties of a discontinuous quantisation along the lines of the polymer state: The harmonic oscillator.

At the kinematical level, we are talking about quantum mechanics of a single degree of freedom. There, the algebra of Weyl operators looks very similar to the algebra for the bosonic string, the only difference being that one has to replace $W(f)$ by $W(z)$ where $z$ is a complex number representing a point in phase space. The multiplication law is given by $W(z_1)W(z_2) = W(z_1 + z_2) \exp(i\Im(\overline{z_1}z_2))$. The usual Fock space vacuum is represented by $\omega_F(W(z)) = \exp(-|z|/4)$ while again the polymer state is 1 for $z = 0$ and vanishes elsewhere.

The first physical observation about this polymer state is that formally it can be obtained as a thermal state: The KMS-state one obtains by coupling the oscillator to a heat-bath is labelled by the inverse temperature $\beta$ (which determines the periodicity in imaginary time). Taking the limit $\beta \to 0$ yields the polymer state. Thus this state can be thought of as a thermal state of infinite temperature. It is not surprising that this state has unusual spectral properties.

The crucial difference to the string is that there are no gauge symmetries one wishes to mod out. However, the time evolution characterising the harmonic oscillator (equivalent to giving the Hamiltonian if that existed) has a very similar form as it is given by $\alpha_t(W(z)) = W(e^{it}z)$. This time, both states are invariant under this transformation and thus the unitary time evolution operator on the respective Hilbert spaces $U(t)$ is readily available.

In the Fock representation, it is continuous in $t$ and one can define an unbounded operator $H = \frac{d}{dt}U(t)|_{t=0}$ which has the well known evenly spaced spectrum that one can for example observe in Raman molecular vibration spectra. Furthermore, there are stationary states in the Hilbert space which are labelled by an integer, the excitation number.

In the polymer representation, due to the discontinuity the derivative does not exist thus there is no Hamiltonian and it does not make sense to ask for its spectrum. This has a parallel in the polymer string where one cannot take derivatives which would yield operators for the field and the field momentum or creation and annihilation operators. Only the integrated finite transformations exist.

In [HP], we defined a family of operators $H_\epsilon$ which are finite difference versions of the time derivative of the time evolution $U(t)$. In Fock space, where the derivative exists, $H_\epsilon$ becomes the Hamiltonian when $\epsilon \to 0$. In the polymer Hilbert space we showed that generically the $H_\epsilon$ have vanishing expectation value and a variance which for a fixed state scales as $1/\epsilon^2$. In no Fock state, the $H_\epsilon$ have these properties. This we took as a clear indication of different physics of the two quantisations.
This conclusion was criticised on the ground that it is physically impossible to probe arbitrarily short intervals of time (for example the Planck time being a natural cut-off) and thus taking the limit $\epsilon \to 0$ is not physical. For any finite $\epsilon$ the variance is finite and thus one does not get a sharp distinction.

[HP] only discusses properties of a decoupled harmonic oscillator. In this note, we take the analysis further by coupling the oscillator to its environment to “observe” its properties.

The polymer Hilbert space is much bigger (it is not separable) than Fock space as all vectors $|W(z)\rangle$ are orthogonal to each other and have no overlap. This implies that the subspace orthogonal to a given vector is in some sense much larger: Take an eigenvector of some Hamilton operator in a separable Hilbert space. Then act on it with a generic operator describing an interaction. The resulting vector typically has non-zero overlap with all other eigenvectors of the Hamiltonian unless there are selection rules forbidding a certain transition.

This is different in the polymer Hilbert space: There, the overlap can only be with a subspace of countable dimension in the whole space of uncountable dimension. Thus, generic transitions are always forbidden for a given interaction.

Even if one focusses on those special pairs of states for which interactions are possible for a given perturbation, the resulting picture differs. In the appendix, we compute the absorption spectrum for the polymer harmonic oscillator interacting with electromagnetic radiation of frequency $\Omega$. We find the rate of absorption to be given by

$$A_P = \frac{1}{8 \sin^2 \left(\frac{\pi (m'-m-\Omega)}{N}\right)}.$$  

$N$ is an arbitrary integer assumed to be large but finite. It is a property of the pair of states involved in the transition. $m$ and $m'$ are “generalised frequencies” of the two states involved with the condition that both have to divide $N$. This should be compared to the rate of absorption computed in the ordinary Fock Hilbert space:

$$A_F = \frac{1}{2(\omega_{m'} - \omega_m + \Omega)^2}.$$  

As detailed in the appendix, the question of whether the two expressions can be distinguished depends on the order of limits to be taken (there is a trivial factor of $(2\pi/N)^2$ which could be absorbed in in the “path integral” over the time of interaction): For $|\Omega| < \Omega_{max}$ for fixed $\Omega_{max}$, the two expressions can be made arbitrarily similar by taking a sufficiently large $N$. However, for any fixed $N$ (that is for fixed pair of states), the two expressions deviate for large enough $\Omega$.

This behaviour is typical for comparisons between Fock- and polymer-physics: For a specified set of measurements, it is possible to find states in the polymer prescription such that for these measurements the two approaches agree. However, by performing measurements outside the predetermined set, it is possible to differentiate the two approaches.
Whether this means that the to approaches are physically equivalent or not is thus a philosophical question as it depends on the exact rules of the comparison: Do we have to prepare the system before we decide which measurement to make to make or is it allowed to prepare the system afterwards in a state in the polymer Hilbert space that mimics the Fock state for that specific measurement?

5. Can we observationally distinguish Fock- and the polymer Hilbert-space

Remember $N$ was chosen as an arbitrary integer above. So what is its physical status? There are two basic strategies to reconcile the two results. Either $N$ is to be understood as a fixed physical constant \textit{a priori} or it is chosen \textit{a posteriori} according to circumstances. Both have parallels in the literature.

The first alludes to ideas of “space-time foam” and posits that at the Planck scale (space-) time has a fundamental granularity and it is only possible to measure time in integer multiples of Planck time. Then $N$ is the oscillatory period of the oscillator in Planck units. This would imply that it is physically impossible to access frequencies larger than $N$ and thus the periodicity of the absorption would be invisible in principle. Furthermore, all our measurements of oscillators in the past would have been in the $m, m', \Omega \ll N$ regime and thus we could not have observed a deviation from $A_F$. In this case however the spectra of oscillators with slightly different frequencies would look very different depending on the selection rule that restricts absorption to possible values for $m - m'$ in terms of divisors of $N$. This is not a very attractive scenario.

Furthermore, as time in that world would be quantised in units of $2\pi/N$, the effective Hilbert space would separate into subspaces each of dimension $N$ rendering it effectively finite dimensional. In this restriction, and time being discrete, anything becomes continuous (as for the discrete space-time there is only the discrete topology). Thus by restricting to a finite dimensional subspace it is not surprising that there are no differences between the two constructions which differ in their continuity properties. What this approach does is to build a hard UV regulator into the theory. This of course is a possible although not very popular approach which could as well be used in theories of quantum gravity immediately solving problems associated to non-renormalisability.

Even if this is not a problem in the context of the oscillator, it appears to be very difficult to implement a UV cut-off in a diffeomorphism invariant way. So this possibility might not be available in the case of the string or gravity.

The other possibility is to choose $N$ on an ad hoc basis depending on the measurements one is going to perform. For example you intend to match observations of absorption lines done with a certain bandwidth. Then you can choose $N$ sufficiently large to prepare the oscillator in a state $|\psi_m\rangle$ for this $N$ such that deviations from the absorption spectrum of a Fock oscillator are invisible at this bandwidth.

If $N$ is not tied to a constant of nature (like the Planck time), one could also consider transitions between states of different $N$, say $N_1$ and $N_2$. In this case however, the two scalar products involved in the computation of $\mathcal{M}$ would only be simultaneously non-vanishing if $t'$ is an integer multiple both of $2\pi/N_1$ and $2\pi/N_2$. Thus this computation
is equivalent to a transition where both initial and final state have a common $N_{\text{eff}} = \gcd(N_1, N_2)$. As the periodicity in the absorption rate becomes visible for $\Omega \sim N_{\text{eff}}$, considering transitions with different $N$ would lower the effective $N$ and thus increase the deviation from the Fock space result.

In this second approach, there is no fundamental granularity of time and $N$ is just a property of the quantum state the oscillator happens to be in. On the other hand, in this approach, as there is no fundamental limit on possible observations, one could, after the state is prepared, decide to measure outside the prespecified bandwidth and reveal the difference of that oscillator from the one using Fock space.

Again, one could object that any real world oscillator has its built in bandwidth outside which it ceases to behave like an harmonic oscillator. For example, once too much energy has been put into a vibrational mode of a molecule, the oscillatory degree of freedom is no longer decoupled from the remaining degrees of freedom (including eventually fundamental particle degrees of freedom inside the nuclei) and ceases to be harmonic. However, only slightly extending this argument against analysing idealised subsystems one should never discuss systems like the harmonic oscillator and only attempt to quantise the universe as a whole interacting system as anything else is an unphysical idealisation.

In the case of two coupled oscillators (see appendix) one sees that in order to make the eventually decreasing absorption unobservable one would have also to forbid arbitrarily long measurements. Thus in addition to a UV regulator one as well has to impose an IR regulator and choose the state accordingly. Therefore an analogous reasoning applies: What has to be chosen first, the state or the regulator on the observations?

If one decides to follow the first assumption above and assume a fundamental (Planck) frequency such that all oscillators would have to have frequencies $1/N$ times that fundamental frequency for some $N$ this would have to hold for the eigenfrequencies as well and the irrationality assumption of the eigenfrequencies would be impossible to fulfil. This however would impose quantisation conditions on the allowed range of possible couplings $\lambda$ on which the eigenfrequencies depend non-polynomially as well which appears very unnatural.

6. Conclusions

This note contrasted two approaches to quantisation, especially in their application to the world-sheet theory of the bosonic string. The polymer state is based on an invariant state while the Fock state is only covariant and thus potentially anomalous and in fact the anomaly vanishes only in the critical dimension. This example shows that requiring invariance of the state in the quantisation of a gauge system is too strict and one should in fact only ask for covariance (existence of unitary implementers of the symmetry).

As a result of this too strong invariance requirement the polymer state ceases to be continuous and thus leads to a very singular Hilbert space. The second half of this note investigated physical consequences of this discontinuity. Specifically, in the example of the harmonic oscillator, optical absorption spectra were computed in perturbation theory. It was shown that for a given finite bandwidth spectrum one can construct states in the
polymer Hilbert space which resembled the Fock result to an arbitrary finite precision. Once prepared however, the absorption differs significantly at high frequencies outside the prespecified range. Thus for any state in the polymer Hilbert space can be observationally distinguished from the usual Fock states assuming the experimenter has access to sufficiently high frequency radiation.

7. Appendix A: Absorption spectrum

Here we compute the rate of absorption of photons from a field of coherent radiation as a function of the frequency of that radiation. We do this computation in first order perturbation theory as already at this order we have good agreement with experimentally observed absorption spectra. For the same reason the radiation field is treated classically as QED effects should not be important for this simple process.

The most obvious way to observe the spectrum of a physical system is to measure an absorption spectrum. We can compute this in time dependent perturbation theory. All we need is the time evolution, the Hamiltonian is not required.

Let’s do the textbook calculation first. The idea is to couple the oscillator to a radiation field. More specifically, we perturb with a “photon operator” $M(t) = \exp(i\Omega t)|m'\rangle\langle m|$. We start out with state $|m\rangle$, this evolves for a time $t'$ according to the free Hamiltonian. Then $M(t')$ acts and further on, the state evolves with the free evolution until time $t$.

$\omega$

$\Omega$

$m'$

$m$

$t'$

$t$

Fig. 1: A photon induces a transition between states $m$ and $m'$

According the “sum over paths” philosophy, we have to sum (integrate) all possible $t'$. In the end, we compute the overlap with the state $|m'\rangle$, the absolute square gives the probability of this absorption of a photon to happen. In a formula

\[
\mathcal{M} = \langle m'|U(-t)\int_0^t dt' U(t-t')\exp(i\Omega t')|m'\rangle \langle m|U(t')|m\rangle
\]

\[
= \int_0^t dt' e^{it'(\omega_m - \omega_{m'})} \exp(i\Omega t')
\]

\[
= \frac{e^{i(\omega_{m'} - \omega_m + \Omega)t} - 1}{\omega_{m'} - \omega_m + \Omega}
\]
Thus the absorption rate is given by

$$|M|^2 = \frac{\sin^2((\omega_{m'} - \omega_m + \Omega)t/2)}{(\omega_{m'} - \omega_m + \Omega)^2}.$$ 

Therefore, with an additional time average (alternatively this can be thought of as an average over the initial phase of the radiation field) we find the absorption rate behave like

$$A_F = \langle |M|^2 \rangle = \frac{1}{2(\omega_{m'} - \omega_m + \Omega)^2}.$$  

Now, we want to perform an analogous calculation for the polymer representation. The first question is to find states similar to the stationary states $|m\rangle$ in the usual treatment. Recall that the states $|z\rangle$ are orthonormal ($\langle z|z' \rangle = 1$ if $z = z'$ and 0 otherwise) for all $z \in \mathbb{C}$ and the general vector is an at most countable linear combination such that the sum of the absolute value squares of the coefficients is finite. The space of finite linear combinations is dense and we will only consider those.

For obvious reasons besides $|0\rangle$ there is no state which is an eigenstate of $U(t)$ for all $t$ (remember the time evolution acts as $U(t)|z\rangle = |e^{it}z\rangle$). However, it is easy to find states which are eigenstates at $N$ instances of time during a period $t \in [0, 2\pi)$. Namely, we want to consider the normalised states

$$|\psi_m\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{2\pi i k m / N} |e^{2\pi i k / N}\rangle.$$ 

Here $N$ is an integer which is understood to be large but finite and $m \in \{1, \ldots, N\}$. These states are periodic in time with period $2\pi / \gcd(m, N)$. Thus, the angular frequency is $\gcd(m, N)$. We can restrict ourselves to such $m$ which divide $N$ which then have (angular) frequency $m$ in analogy to the $|m\rangle$ in Fock space.

Note that in contrast to the Fock space, the $|\psi_m\rangle$ are not dense and their span (which is closed!) only forms a tiny part of the polymer Hilbert space. Thus a general state cannot be decomposed into a sum (possibly infinite) of $\psi_m$’s. Nevertheless, we will consider these states as the closest analogues of the oscillator eigenstates in Fock space.

In particular, we will concede that the radiation field couples two of these states via the operator

$$M(t) = e^{i\Omega t} |\psi_{m'}\rangle \langle \psi_m| = e^{i\Omega t} \sum_{l=1}^{N} e^{\frac{2\pi i (m' - m)}{N} l} |e^{\frac{2\pi i l}{N}}\rangle \langle e^{\frac{2\pi i l}{N}}|.$$ 

Again, at least formally, we compute

$$\mathcal{M} = \langle \psi_{m'}|U(-t) \int_0^t dt' U(t - t') M(t) U(t') |\psi_m\rangle.$$
The integrand of the $t'$ integration vanishes almost everywhere and is finite at discrete values in between. Thus, in a Lebesgue sense, the integral vanishes. However, we should be careful that this integral was a “sum over all possible times of interaction”. Thus we silently replace it by a sum over those $t' \in [0, t]$ where the integrand does not vanish. For simplicity of these now discrete expressions, we further assume that $t = 2\pi P$ for some integer $P$, the number of periods of the oscillator we shine the light on it. After some algebra, we find the rate of absorption to be

$$|\mathcal{M}_p|^2 = \frac{\sin^2(\pi p \Omega)}{\sin^2(\pi (m' - m - \Omega)/N)}$$

and again, after a time average

$$A_P = \frac{1}{8 \sin^2\left(\frac{\pi (m' - m - \Omega)}{N}\right)}.$$ 

This result should be compared to (2). For $m, m', \Omega \ll N$, the polymer rate of absorption approximates that of the Fock oscillator up to a factor due to replacing an integral over $[0, 2\pi]$ by a sum over $N$ values.

However, as soon as we drop this restriction, there is a large deviation and, in fact, the rate of absorption is periodic in $\Omega$ with period $N$. Thus, for fixed $N$ there is a large deviation between the two expressions at large frequencies.

Another difference between the two cases is that while all natural numbers $m$ appear as frequencies in the Fock case, for fixed $N$, only its divisors appear as “frequencies” (inverse periods) of $|\psi_m\rangle$. Thus the total spectrum for transitions between all possible $|\psi_m\rangle$ for fixed $N$ depends on number theoretic properties of $N$. You would see an absorption line in the spectrum only for frequencies $\Omega$ which can be written as the difference of two divisors of $N$ (plus multiples of $N$). For example the spectrum would have only one transition line (up to periodicity $\Omega$ with period $N$) for $N$ prime compared to an $N$ which has many small divisors.

We should point out that the states $|\psi_m\rangle$ are not the most general vectors in the polymer Hilbert space. However, one does not gain “resemblance” with the Fock space situation by studying more general vectors: What is important for the (time averaged) absorption rate is just the periodicity in time of the support of the state and in this respect the vectors $|\psi_m\rangle$ cover already all possibilities.

8. Appendix B: Several oscillators

Our result above that at least for large $N$ the polymer absorption rate can look like the Fock rate can be traced back to the fact that for the harmonic oscillator the time evolution is essentially a $U(1)$ which can be approximated sufficiently by a $\mathbb{Z}_N$. This however is not possible for finite subgroups of $\mathbb{R}$ which becomes relevant for the time evolution once several oscillators are involved.
Thus now we want to consider two oscillators with not necessarily identical frequencies. Furthermore we assume there is a linear coupling $\lambda x_1 x_2$ where $\lambda$ is the coupling constant. The combined system can be diagonalised in terms of eigenmodes with eigenfrequencies depending on $\lambda$. If the ratio of these eigenfrequencies is irrational the time-evolution is no longer periodic but only quasi-periodic.

In the polymer description the total Hilbert space is now the tensor product of two polymer spaces for the two eigenmodes and the time evolution acts as $U(t)|z_1\rangle \otimes |z_2\rangle = |e^{i\omega_1 t}z_1\rangle \otimes |e^{i\omega_2 t}z_2\rangle$. If we couple the radiation field only to the first oscillator, say, it couples to both eigenmodes unless $\lambda = 0$. As we saw in the treatment of the single oscillator, the coupling only happens at discrete times $t'$ when there is an overlap between the time evolved initial and final state. For two coupled oscillators there has to be an overlap for both eigenmodes, otherwise the matrix element vanishes. If the two eigenfrequencies have an irrational ratio, the mutual overlap happens only a finite number of times (over all $t'$ not just one period) and thus the absorption eventually has to stop. Thus all absorption lines of these coupled oscillators in any state will eventually go away in the polymer description.

Strictly speaking, one should not only consider states which are supported at finitely many $|z\rangle$ as one could as well have countably many. In that case, the coefficients have to be square summable which also means that even if there could be absorption at arbitrarily late times the strength will eventually have to go to zero.

This conclusion hinges on the assumption of incommensurable frequencies. Otherwise the time evolution is periodic and thus similar to the case of a single oscillator. One might object that the question if $\omega_1/\omega_2$ is in $\mathbb{Q}$ or in $\mathbb{R} \setminus \mathbb{Q}$ cannot be established by an experiment of finite precision. Note however that the scalar product in the polymer Hilbert space exactly requires such infinite precision as otherwise two prepared states would overlap only with probability zero. Thus to have interactions at all in the polymer case one has to concede infinite precision in preparation of the experiment. Furthermore, the absorption will be discontinuous in $\lambda$ as the ratio of eigenfrequencies depends on it.

9. References

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