Spectral analysis of an abstract pair interaction model

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Abstract

We consider an abstract pair-interaction model in quantum field theory with a coupling constant \( \lambda \in \mathbb{R} \) and analyze the Hamiltonian \( H(\lambda) \) of the model. In the massive case, there exist constants \( \lambda_c < 0 \) and \( \lambda_{c,0} < \lambda_c \) such that, for each \( \lambda \in (\lambda_{c,0}, \lambda_c) \cup (\lambda_c, \infty) \), \( H(\lambda) \) is diagonalized by a proper Bogoliubov transformation, so that the spectrum of \( H(\lambda) \) is explicitly identified, where the spectrum of \( H(\lambda) \) for \( \lambda > \lambda_c \) is different from that for \( \lambda \in (\lambda_{c,0}, \lambda_c) \). As for the case \( \lambda < \lambda_{c,0} \), we show that \( H(\lambda) \) is unbounded from above and below. In the massless case, \( \lambda_c \) coincides with \( \lambda_{c,0} \).

Key words: quantum field, pair-interaction model, spectral analysis, Bogoliubov transformation.

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1 Introduction

In this paper, we consider an abstract pair-interaction model in quantum field theory. The Hamiltonian of the model is of the form

\[
H(\lambda) := d\Gamma_b(T) + \frac{\lambda}{2} \Phi_s(g)^2
\]

acting in the boson Fock space \( \mathcal{F}_b(\mathcal{H}) \) over a Hilbert space \( \mathcal{H} \) (see Subsection 2.1), where \( T \) is a self-adjoint operator on \( \mathcal{H} \), \( d\Gamma_b(T) \) is the second quantization operator of \( T \), \( \Phi_s(g) \) is...
the Segal field operator with test vector $g$ in $\mathcal{H}$ (see Subsection 2.1) and $\lambda \in \mathbb{R}$ is a coupling constant. A model of this type is called a $\phi^2$-model.

There have been many studies on massive or massless $\phi^2$-models in concrete forms or abstract forms (see, e.g., [4, 7, 8, 10, 11, 15]). In [10] and [15], the (essential) self-adjointness of the Hamiltonian of a $\phi^2$-model is proved in the case where $\lambda > 0$ or $|\lambda|$ is sufficiently small. In [10], the existence of a ground state of a $\phi^2$-model also is shown in the case where the quantum field under consideration is massive and $\lambda > 0$.

It is well known that Hamiltonians with linear and/or quadratic interactions in quantum fields may be analyzed by the method of Bogoliubov transformations (see, e.g., [1, 2, 3, 4, 6, 7, 9, 11]). A typical Bogoliubov transformation is constructed from bounded linear operators $U, V$ and a conjugation operator $J$ on $\mathcal{H}$ satisfying the following equations:

\[
\begin{align*}
U^*U - V^*V &= I, \\
U^*_jV - V^*_jU &= 0, \\
UU^* - V^*_jV &= I, \\
UV^* - V^*_jU &= 0,
\end{align*}
\]

(1.1)

where $A_j := JAJ$ and $A^*$ is the adjoint of a densely defined linear operator $A$. It is well known that there is a unitary operator $U$ on $\mathcal{F}_b(\mathcal{H})$ which implements the Bogoliubov transformation in question if and only if $V$ is Hilbert-Schmidt $[6, 12, 13, 14]$. Moreover, it is shown that, under the condition that $V$ is Hilbert-Schmidt and suitable additional conditions, the Hamiltonian under consideration is unitarily equivalent via $U$ to a second quantization operator up to a constant addition. For example, the Pauli-Fierz model with dipole approximation, which can be regarded as a kind of $\phi^2$-model, is analyzed by this method in [9].

Recently, a general quadratic form Hamiltonian with a coupling constant $\lambda \in \mathbb{R}$ has been analyzed in [11] and it is shown that, in the case of a massive quantum field, under suitable conditions, the Hamiltonian is diagonalized by a Bogoliubov transformation. In [7], the sufficient condition formulated in [11] to obtain the result just mentioned has been extended. The spectrum of the standard pair-interaction model in physics, which is a concrete realization of the abstract pair-interaction model, is formally known $[8]$ in the case where $\lambda > \lambda_{c,0}$ and $\lambda \neq \lambda_c$ for some constants $\lambda_c$ and $\lambda_{c,0} < \lambda_c$. The paper [4] gives a rigorous proof for that in the framework of the boson Fock space theory over $\mathcal{H} = L^2(\mathbb{R}^d)$ for any $d \in \mathbb{N}$ and $\lambda > \lambda_c$.

One of the motivations for the present work is to extend the theory developed in [4] with $\mathcal{H} = L^2(\mathbb{R}^d)$ to the theory with $\mathcal{H}$ being an abstract Hilbert space including the case where $\lambda < \lambda_c$. 

2
It is known that spectral properties of a pair-interaction model may depend on the range of \( \lambda \) with \( \lambda_c \) being a border point. Hence it is important to make this aspect clear mathematically. Therefore we analyze our model also for the region \( \lambda < \lambda_c \). We show that, in the massive case with \( \lambda \in (\lambda_{c,0}, \lambda_c) \) also, the method of Bogoliubov transformations can be applied to prove that the Hamiltonian \( H(\lambda) \) is unitarily equivalent to a second quantization operator up to a constant addition. Then we see that the spectrum of \( H(\lambda) \) for \( \lambda \in (\lambda_{c,0}, \lambda_c) \) is different from that for \( \lambda > \lambda_c \). In the massless case, \( \lambda_{c,0} \) coincides with \( \lambda_0 \).

The main results of the present paper include the following (1)–(3) (see Theorem 2.8 for more details): (1) Identification of the spectra of \( H(\lambda) \) for \( \lambda > \lambda_c \). (2) Identification of the spectra of \( H(\lambda) \) for \( \lambda_{c,0} < \lambda < \lambda_c \) (the massive case; in the massless case, \( \lambda_{c,0} = \lambda_c \)). In this case, bound states different from the ground state appear. (3) Unboundedness from above and below of \( H(\lambda) \) for \( \lambda < \lambda_{c,0} \).

The outline of this paper is as follows. In Section 2, we define our model and recall a fundamental fact in a general theory of Bogoliubov transformations. We prove the (essential) self-adjointness of \( H(\lambda) \) (Theorem 2.3). Then we state the main theorem of this paper (Theorem 2.10). In Section 3, we construct operators \( U \) and \( V \) which are used to define the Bogoliubov transformation we need. In Section 4, we show that \( U \) and \( V \) satisfy (1.1) and \( V \) is Hilbert-Schmidt. In Section 5, we prove Theorem 2.8 (1) and calculate the ground state energy of \( H(\lambda) \) in the case \( \lambda > \lambda_c \). In Section 6, we prove Theorem 2.8(2). In Section 7, we prove Theorem 2.8 (3). In Section 8, we consider a slightly generalized Hamiltonian of the form \( H(\eta,\lambda) := H(\lambda) + \eta \Phi_S(f) \) with \( \eta \in \mathbb{R} \) and \( f \in \mathcal{H} \). Applying the methods and results in the preceding sections, we can analyze \( H(\eta,\lambda) \) to identify the spectra of it. In Appendix, we state some basic facts in the theory of boson Fock space.

2 Preliminaries

2.1 The abstract Boson Fock Space

Let \( \mathcal{H} \) be a Hilbert space over the complex field \( \mathbb{C} \) with the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \). The inner product is linear in the second variable and anti-linear in the first one. The symbol \( \| \cdot \|_{\mathcal{H}} \) denotes the norm associated with it. We omit \( \mathcal{H} \) in \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and \( \| \cdot \|_{\mathcal{H}} \), respectively if there is no danger of confusion. For each non-negative integer \( n = 0, 1, 2, \ldots \), \( \otimes_n \mathcal{H} \) denotes the \( n \)-fold symmetric tensor product Hilbert space of \( \mathcal{H} \) with convention \( \otimes_0 \mathcal{H} := \mathbb{C} \). Then

\[
\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^\infty \otimes^n_s \mathcal{H}
\]
is called the second quantization operator of $I$. For a dense subspace $D$ in $\mathcal{H}$, $\hat{\otimes}^n_s D$ denotes the algebraic $n$-fold symmetric tensor product of $D$ with $\hat{\otimes}^0_s \mathcal{H} := \mathbb{C}$. Then

$$\mathcal{F}_{b,\text{fin}}(D) := \hat{\otimes}_{n=0}^\infty \hat{\otimes}^n_s D$$

is a dense subspace of $\mathcal{F}_b(\mathcal{H})$, where $\hat{\otimes}_{n=0}^\infty D_n$ denotes the algebraic direct sum of subspace $D_n \subset \otimes^n_s \mathcal{H}$, $n = 0, 1, 2, \ldots$. The finite particle vector subspace

$$\mathcal{F}_{b,0}(\mathcal{H}) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_b(\mathcal{H}) \left| \begin{array}{c} \psi^{(n)} \in \otimes^n_s \mathcal{H}, \ n \geq 0, \text{there is an integer } n_0 \in \mathbb{N} \\ \text{such that } \psi^{(n)} = 0, \text{for all } n \geq n_0 \end{array} \right. \right\}$$

satisfies $\mathcal{F}_{b,\text{fin}}(D) \subset \mathcal{F}_{b,0}(\mathcal{H}) \subset \mathcal{F}_b(\mathcal{H})$. It is dense in $\mathcal{F}_b(\mathcal{H})$. For a linear operator $T$ on a Hilbert space, we denote its domain by $D(T)$.

For a densely defined closable operator $T$ on $\mathcal{H}$, let $T^{(n)}_b$ be the densely defined closed operator on $\otimes^n_s \mathcal{H}$ defined by

$$T^{(n)}_b := \left\{ \begin{array}{cl} \sum_{j=1}^n \underbrace{I \otimes \cdots \otimes I}_{j-\text{th}}, & n \geq 1, \\ 0, & n = 0, \end{array} \right.$$  

where $I$ denotes the identity operator on $\mathcal{H}$, $\overline{A}$ denotes the closure of a closable operator $A$ and $A \upharpoonright \mathcal{M}$ denotes the restriction of a linear operator $A$ on a subspace $\mathcal{M}$. The operator

$$d\Gamma_b(T) := \oplus_{n=0}^\infty T^{(n)}_b$$

is called the second quantization operator of $T$. If $T$ is self-adjoint or non-negative, then so is $d\Gamma_b(T)$. For each $f \in \mathcal{H}$, there exists a unique densely defined closed operator $A(f)$ on $\mathcal{F}_b(\mathcal{H})$ such that its adjoint $A(f)^*$ is given as follows:

$$D(A(f)^*) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_b(\mathcal{H}) \left| \begin{array}{c} \sum_{n=0}^\infty n \|S_n(f \otimes \psi^{(n-1)})\|^2 < \infty \\ (A(f)^*\psi)(n) = \sqrt{n}S_n(f \otimes \psi^{(n-1)}), n \in \mathbb{N}, \ (A(f)^*\psi)(0) = 0 \text{ for } \psi \in D(A(f)^*), \end{array} \right. \right\},$$

where $S_n$ is the symmetrization operator on the $n$-fold tensor product $\otimes^n_s \mathcal{H}$ of $\mathcal{H}$. The operator $A(f)$ (resp. $A(f)^*$) is called the annihilation (resp. creation) operator with test vector $f$. We have

$$\mathcal{F}_{b,0}(\mathcal{H}) \subset D(A(f)) \cap D(A(f)^*)$$

for all $f \in \mathcal{H}$ and $A(f)$ and $A(f)^*$ leave $\mathcal{F}_{b,0}(\mathcal{H})$ invariant. Moreover, they satisfy the following commutation relations:

$$[A(f), A(g)^*] = \langle f, g \rangle, \ [A(f), A(g)] = 0, \ [A(f)^*, A(g)^*] = 0, \ \text{for all } f, g \in \mathcal{H} \quad (2.1)$$

4
on $\mathcal{F}_{b,0}(\mathcal{H})$, where $[A, B] := AB - BA$ is the commutator of linear operators $A$ and $B$. The relation (2.1) is called the canonical commutation relations (CCR) over $\mathcal{H}$. The symmetric operator

$$\Phi_s(f) := \frac{1}{\sqrt{2}} (A(f) + A(f)^*), \ f \in \mathcal{H}$$

is called the Segal-field operator with test vector $f$. We write its closure by the same symbol.

### 2.2 Bogoliubov Transformation

In this subsection, we define a Bogoliubov transformation and recall an important theorem about it. For a conjugation $J$ on $\mathcal{H}$ (i.e., $J$ is an anti-linear operator on $\mathcal{H}$ satisfying $\|Jf\| = \|f\|$ for all $f \in \mathcal{H}$ and $J^2 = I$) and a linear operator $A$ on $\mathcal{H}$, we define

$$A_J := JAJ.$$

**Definition 2.1.** Let $U$ and $V$ be bounded linear operators on $\mathcal{H}$ and $J$ be a conjugation on $\mathcal{H}$. Then, for each $f \in \mathcal{H}$, we define a linear operator $B(f)$ on $\mathcal{F}_{b}(\mathcal{H})$ by

$$B(f) := A(Uf) + A(JVf)^*.$$ 

Then the correspondence $(A(\cdot), A(\cdot)^*) \mapsto (B(\cdot), B(\cdot)^*)$ is called a Bogoliubov transformation.

By $\mathcal{F}_{b,0}(\mathcal{H}) \subset D(B(f))$, the adjoint $B(f)^*$ exists and the equation $B(f)^* = A(Uf)^* + A(JVf)$ holds on $\mathcal{F}_{b,0}(\mathcal{H})$ for each $f \in \mathcal{H}$. If the equations

$$U^*U - V^*V = I, \quad U_J^*V - V_J^*U = 0$$

hold, then the Bogoliubov transformation preserves CCR, i.e., it holds that

$$[B(f), B(g)^*] = \langle f, g \rangle, \ [B(f), B(g)] = 0, \ [B(f)^*, B(g)^*] = 0, \ \text{for all } f, g \in \mathcal{H},$$

on $\mathcal{F}_b(\mathcal{H})$. The following theorem is well-known [13, 14]:

**Theorem 2.2.** Let $\mathcal{H}$ be separable and $U$ and $V$ satisfy (1.1). Then there exists a unitary operator $U$ on $\mathcal{F}_b(\mathcal{H})$ such that

$$UB(f)U^{-1} = A(f), \ f \in \mathcal{H}$$

if and only if $V$ is Hilbert-Schmidt.
2.3 Hamiltonian

For a self-adjoint operator $T$ on $\mathcal{H}$, constants $\lambda, \eta \in \mathbb{R}$ which are called coupling constants, and vectors $f, g \in \mathcal{H}$, we define an operator

$$H(\lambda) := d\Gamma_b(T) + \frac{\lambda}{2} \Phi_s(g)^2, \quad H(\eta, \lambda) := H(\lambda) + \eta \Phi_s(f).$$

If $g \in D(T^{-1/2})$, we define the constant

$$\lambda_{c, 0} := -\|T^{-1/2}g\|^{-2}.$$

**Theorem 2.3.** Suppose that $T$ is an injective, non-negative, self-adjoint operator on $\mathcal{H}$. Let $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T)$. Then the following (1)-(3) hold:

1. Let
   $$\lambda_T(g) := \|T^{-1/2}g\|^{-1} (\|T^{-1/2}g\| + \|T^{1/2}g\|)^{-1}$$
   and $|\lambda| < \lambda_T(g)$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(d\Gamma_b(T))$ and essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover $H(\eta, \lambda)$ is bounded from below.

2. Let $|\lambda| \geq \lambda_T(g)$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover, if $\lambda \geq \lambda_T(g)$, then $H(\eta, \lambda)$ is self-adjoint.

3. Let $f \in D(T^{1/2})$. Then $\overline{H(\lambda_{c, 0})}$ is bounded from below. Moreover, if $\lambda > \lambda_{c, 0}$, then $\overline{H(\eta, \lambda)}$ is bounded from below for all $\eta \in \mathbb{R}$ and $D(d\Gamma_b(T)^{1/2}) = D(H(\eta, \lambda) + M)^{1/2}$ for a constant $M \geq 0$ satisfying $\overline{H(\eta, \lambda)} + M \geq 0$.

**Proof.** (1) For any $\lambda \in \mathbb{R}$, by using \cite[(2.1), (9.1), (9.2) and [5] Theorem 5.18.], one can easily see that there are constants $a, b \geq 0$ such that for all $\psi \in D(d\Gamma_b(T))$,

$$\left\| \frac{\lambda}{2} \Phi_s(g)^2 \psi \right\| \leq \frac{|\lambda|}{4} (a\|d\Gamma_b(T)\| + b\|\psi\|).$$

In particular, we can choose $a$ and $b$ which satisfy $a|\lambda|/4 < 1$ if $|\lambda| < \lambda_T(g)$. We remark that, to obtain the factor $\lambda_T(g)$, we need to deform terms $\|A(g)^2\psi\|^2$, $\|A(g)^*A(g)\psi\|^2$ and $\|A(g)^2\psi\|^2$ coming from $\|\Phi_s(g)\psi\|^2$ ($\psi \in \Phi_{b, 0}(\mathcal{H})$) to $\|A(g)A(g)^*\psi\|^2 + a$ marginal term respectively. Thus, for $|\lambda| < \lambda_T(g)$, by the Kato-Rellich theorem, $H(\lambda)$ is self-adjoint. It is well known that $\Phi_s(f)$ is infinitesimally small with respect to $d\Gamma_b(T)$. Hence, by the Kato-Rellich theorem, for $|\lambda| < \lambda_T(g)$, $H(\eta, \lambda)$ is self-adjoint.
(2) Firstly, we show that, for any $f \in D(T^{1/2})$ and $\eta, \lambda \in \mathbb{R}$, $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$. By (9.1), (9.2) and [5, Theorem 5.18.], we can see that there exists $a > 0$ such that $\|H(\eta, \lambda)\psi\| \leq a\|d\Gamma_b(T) + I\psi\|$ for all $\psi \in D(d\Gamma_b(T))$. For the first let $f \in D(T)$. Then by (2.1) and (9.3), for any $\psi \in \mathcal{F}_{b,fin}(D(T))$, we have

$$
\langle H(\eta, \lambda)\psi, (d\Gamma_b(T) + I)\psi \rangle - \langle (d\Gamma_b(T) + I)\psi, H(\eta, \lambda)\psi \rangle = \frac{\lambda}{\sqrt{2}} \langle (\Phi_s(g)\psi, A(Tg)\psi) - \langle A(Tg)\psi, \Phi_s(g)\psi \rangle + \frac{\eta}{\sqrt{2}} \langle (\psi, A(Tf)\psi) - \langle A(Tf)\psi, \psi \rangle \rangle.
$$

Thus, by (9.1) and (9.2), we have

$$
|\langle H(\eta, \lambda)\psi, (d\Gamma_b(T) + I)\psi \rangle - \langle (d\Gamma_b(T) + I)\psi, H(\eta, \lambda)\psi \rangle | \leq C\|d\Gamma_b(T) + I\|^{1/2}\psi\|^2,
$$

(2.3)

where $C := \{\|\lambda\|T^{1/2}\|g\|\|g\| + 2\|T^{-1/2}g\|\|g\|\} + \sqrt{\eta}\|T^{1/2}f\|\}$. By a limiting argument, using the fact that $\mathcal{F}_{b,fin}(D(T))$ is a core of $d\Gamma_b(T)$ and $d\Gamma_b(T)$-boundedness of $\Phi_s(g)^2$, we can show that for $f \in D(T^{1/2})$ and $\psi \in D(d\Gamma_b(T))$, (2.3) holds. Thus, by the Nelson commutator theorem, for all $\eta, \lambda \in \mathbb{R}$, $H(\eta, \lambda)$ is essentially self-adjoint and $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$. The equation $H(\eta, \lambda) \upharpoonright \mathcal{D} = H(\eta, \lambda) \upharpoonright \mathcal{D}$ holds for any core $\mathcal{D}$ of $d\Gamma_b(T)$. Hence $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta, \lambda \in \mathbb{R}$. Next we show that, if $\lambda \geq -\|T^{-1/2}g\|^{-1}(\|T^{-1/2}g\| + \|T^{1/2}g\|)^{-1}$, then $H(\eta, \lambda)$ is self-adjoint. We can show that, for $\lambda > 0$ and any $0 < \varepsilon < 1$, there is a constant $c_\varepsilon > 0$ such that

$$
(1 - \varepsilon)\|d\Gamma_b(T)\psi\|^2 + \left\| \frac{\lambda}{2} \Phi_s(g)^2\psi \right\|^2 \leq \|H(\eta, \lambda)\psi\|^2 + c_\varepsilon\|\psi\|^2, \quad \psi \in D(d\Gamma_b(T)).
$$

Hence $H(\eta, \lambda)$ is closed. In particular, it is self-adjoint.

(3) It is well known that, for any $\varepsilon > 0$, $\varepsilon d\Gamma_b(T) + \eta \Phi_s(f)$ is bounded from below. For any $\varepsilon > 0$ and $\psi \in D(d\Gamma_b(T)^{1/2})$,

$$
|\langle \psi, A(f)\psi \rangle \langle H(\eta, \lambda)\psi \|^2 \leq \|T^{-1/2}f\| \left(\varepsilon\|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{1}{4\varepsilon}\|\psi\|^2 \right).
$$

Hence if the assertion follows for $\eta = 0$, then so is for all $\eta$. Thus we show that the assertion follows for $\eta = 0$. If $\lambda > 0$, then clearly $H(\lambda) \geq 0$. Let $\lambda < 0$. For any $\psi \in D(d\Gamma_b(T)^{1/2})$,

$$
\|\Phi_s(g)\psi\|^2 \leq 2\|T^{-1/2}g\|^2\|d\Gamma_b(T)^{1/2}\psi\|^2 + \|g\|^2\|\psi\|^2.
$$
Thus for any $\psi \in D(d\Gamma_b(T))$,
\[
\langle \psi, H(\lambda)\psi \rangle = \|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{\lambda}{2}\|\Phi_s(g)\psi\|^2 \\
\ge (1 + \lambda\|T^{-1/2}g\|^2)\|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{\lambda}{2}\|g\|^2\|\psi\|^2.
\] (2.4)

Hence $H(\lambda)$ is bounded from below if $\lambda \ge \lambda_{c,0}$.

Let $\lambda \ge \lambda_{c,0}$ and $M \ge 0$ be a constant satisfying $H(\lambda) + M \ge 0$. Then for any $\psi \in D(d\Gamma_b(T)) = D(H(\lambda))$,
\[
\|(\overline{H(\lambda)} + M)^{1/2}\psi\|^2 \le (1 + |\lambda|\|T^{-1/2}g\|^2)\|d\Gamma_b(T)^{1/2}\psi\|^2 + \left(\frac{|\lambda|}{2}\|g\|^2 + M\right)\|\psi\|^2.
\] (2.5)

By the fact that $D(d\Gamma_b(T))$ is a core of $d\Gamma_b(T)^{1/2}$, we have $D(d\Gamma_b(T)^{1/2}) \subset D((\overline{H(\lambda)} + M)^{1/2})$ and (2.5) holds on $D(d\Gamma_b(T)^{1/2})$.

In the case of $\lambda > 0$, it is easy to see that $\|H(\lambda)^{1/2}\psi\| \ge \|d\Gamma_b(T)^{1/2}\psi\|$ holds for any $\psi \in D(d\Gamma_b(T))$. In the case of $0 > \lambda > \lambda_{c,0}$,
\[
\|d\Gamma_b(T)^{1/2}\psi\|^2 \le \frac{1}{1 + \lambda\|T^{-1/2}g\|^2}\left\{\|(\overline{H(\lambda)} + M)^{1/2}\psi\|^2 - \left(\frac{|\lambda|}{2}\|g\|^2 + M\right)\|\psi\|^2\right\}.
\]
holds for any $\psi \in D(d\Gamma_b(T))$ by (2.4). Hence for $\lambda > \lambda_{c,0}$ there is a constant $a, b \ge 0$ such that
\[
\|d\Gamma_b(T)^{1/2}\psi\| \le a\|(\overline{H(\lambda)} + M)^{1/2}\psi\| + b\|\psi\|.
\] (2.6)

By operational calculus, $D(d\Gamma_b(T))$ is a core of $((\overline{H(\lambda)} + M)^{1/2})$. Thus we have $D((\overline{H(\lambda)} + M)^{1/2}) \subset D(d\Gamma_b(T)^{1/2})$ and (2.6) holds on $D((\overline{H(\lambda)} + M)^{1/2})$.

\[\square\]

**Remark 2.4.** By [3, Lemma13-15], if $\mathcal{H}$ is separable, then Theorem 2.3 takes the following forms:

Let $\mathcal{H}$ be separable, $T$ be a non-negative, injective self-adjoint operator, $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2})$. Then the following (1)-(3) holds:

(1) Let $\lambda > \lambda_{c,0}$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(d\Gamma_b(T))$ and essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover $H(\eta, \lambda)$ is bounded from below.

(2) Let $\lambda \le \lambda_{c,0}$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. In particular, if $\eta = 0$ and $\lambda = \lambda_{c,0}$, then $H(\lambda_{c,0}) = H(0, \lambda_{c,0})$ is bounded from below.
Let $\lambda > \lambda_{c,0}$. Then $D(d\Gamma_b(T)^{1/2}) = D(H(\eta, \lambda) + M)^{1/2}$ for a constant $M \geq 0$ satisfying $H(\eta, \lambda) + M \geq 0$.

**Definition 2.5.** Let $T$ be a self-adjoint operator on $\mathcal{H}$ and $\{E(B) \mid B \in \mathcal{B}^1\}$ be the spectral measure associated with $T$ on the Borel field $\mathcal{B}^1$ on $\mathbb{R}$. The operator $T$ is called purely absolutely continuous if, for each $f \in \mathcal{H}$, the measure $\|E(\cdot)f\|^2$ on $\mathcal{B}^1$ is absolutely continuous with respect to the one-dimensional Lebesgue measure.

**Definition 2.6.** For a purely absolutely continuous self-adjoint operator $T$ and vectors $f, g \in \mathcal{H}$, $\psi_{g,f}$ denotes the Radon-Nikodym derivative of the finite complex Borel measure $\langle g, E(\cdot)f \rangle$ on $\mathcal{B}^1$. In particular, we set $\psi_g := \psi_{g,g}$.

### 2.4 Assumptions

To prove our main theorem stated later (Theorem [2.10]), we need some assumptions. For a closed operator $A$, $\sigma(A)$ denotes the spectrum of $A$. If $A$ is self-adjoint, then $\sigma_{ac}(A)$ (resp. $\sigma_p(A), \sigma_{sc}(A)$) denotes the absolutely continuous (resp. point, singular continuous) spectrum of $A$. For a self-adjoint operator $A$ bounded from below, $E_0(A) := \inf \sigma(A)$ is called the lowest energy of $A$. In particular, it is called the ground state energy of $A$ if $E_0(A) \in \sigma_p(A)$. In this case, any for responding eigenvector is called a ground state of $A$. The ground state is said to be unique if $\dim \ker(A - E_0(A)) = 1$. For linear operators $A$ and $B$, the symbol $A \subset B$ means that $D(A) \subset D(B)$ and $Af = Bf$ for all $f \in D(A)$, i.e., $B$ is an extension of $A$.

**Assumption 2.7.**

1. The operator $T$ is a non-negative, purely absolutely continuous self-adjoint operator;
2. The fixed vector $g \in \mathcal{H}$ satisfies $g \in D(\hat{T}^{-1/2}) \cap D(T^{1/2})$ and $Jg = g$, where $\hat{T} := T - E_0$, $E_0 := E_0(T)$ and $J$ is a conjugation on $\mathcal{H}$ satisfying $JD(T) \subset D(T)$ and $JT \psi = T J \psi$ for any $\psi \in D(T)$ (i.e., $JT \subset T J$);
3. $\sup_{E_0 < x} x^{+1} \psi_g(x) < \infty$ and $\psi_g(x) > 0$ for all $x \in (E_0, \infty)$,
4. $\psi_g \in C^1((E_0, \infty))$ and $\lim_{x \downarrow E_0} x^{-1} \psi_g'(x) = 0 = \lim_{x \to \infty} x^{-1} \psi_g'(x)$.

**Remark 2.8.** The operator $T$ is injective since it is a purely absolutely continuous self-adjoint operator. Since $T$ has no eigenvector, the inverse of $\hat{T}$ exists. Assumption 2.7 (2)
implies that $T_J = T$. In general, for a self-adjoint operator $A$ and a conjugation $J$, we can choose a vector $f \in D(A)$ satisfying $Jf = f$ if $A_J = A$. Thus the vector $g$ in Assumption 2.7 (2) exists. By Assumption 2.7 (3), one can easily show that $\sup_{x \in \sigma(T)} \psi_g(x) < \infty$ and, for each $f \in \mathcal{H}$, the functions $\psi_{g,f}, \psi_{T^{\pm1/2}g,f}$ are in $L^2(\mathbb{R})$ and the maps $f \mapsto \psi_{g,f}, \psi_{T^{\pm1/2}g,f}$ are bounded. Actually, for any $h \in \mathcal{H}$ and $B \in B^1$, the following inequality holds

$$|\langle E(B)h, f \rangle|^2 \leq \|E(B)h\|^2 \|E(B)f\|^2$$

by Schwarz’s inequality. Thus we obtain $|\psi_{h,f}(\mu)|^2 \leq \psi_h(\mu) \psi_f(\mu)$ for almost all $\mu \in \mathbb{R}$ with respect to the Lebesgue measure. Hence, by Assumption 2.7 (3), we have the boundedness of the mappings. Moreover, we see that for any $f \in \mathcal{H}$.

**Lemma 2.9.** Let $T$ be a self-adjoint operator such that $JT \subset TJ$. Then

(1) $E(B)_J = E(B)$, for all $B \in B^1$.

(2) Let $F$ be a Borel measurable function on $\mathbb{R}$. Then $F(T)_J = F^*(T)$, where $F^*$ is complex conjugation of $F$.

**Proof.** These are proved by using the spectral theorem.

### 2.5 The Main Theorem

In this subsection, we state the main theorem of the present paper. Let $\lambda_c$ be a constant defined by

$$\lambda_c := -\left( \int_{[E_0, \infty]} \frac{\mu}{\mu^2 - E_0^2} \ d\|E(\mu)g\|^2 \right)^{-1} < 0.$$ 

Then it is easy to see that $\lambda_{c,0} \leq \lambda_c$, and $\lambda_{c,0} = \lambda_c$ if and only if $E_0 = 0$.

**Theorem 2.10.** Let $\mathcal{H}$ be separable. Then the following (1)-(3) hold:

(1) Let $T$ and $g$ satisfy Assumption 2.7. If $\lambda > \lambda_c$, then there are a unitary operator $U$ on $\mathcal{F}_b(\mathcal{H})$ and a constant $E_g \in \mathbb{R}$ such that

$$UH(\lambda)U^{-1} = d\Gamma_b(T) + E_g.$$  

In particular, $U^{-1} \Omega_0$ is the unique ground state of $H(\lambda)$ up to constant multiples, and

$$\sigma(H(\lambda)) = \{E_g\} \cup [E_0 + E_g, \infty),$$

$$\sigma_{ac}(H(\lambda)) = [E_0 + E_g, \infty), \quad \sigma_p(H(\lambda)) = \{E_g\}, \quad \sigma_{sc}(H(\lambda)) = \emptyset.$$ 

\[10\]
(2) Let $T$ and $g$ satisfy Assumption 2.7 and $E_0 > 0$. If $\lambda_{c,0} < \lambda < \lambda_c$, then there exist a unitary operator $\mathcal{V}$ on $\mathcal{F}_b(\mathcal{H})$, an injective non-negative self-adjoint operator $\xi$ on $\mathcal{H}$ and a constant $E_b \geq 0$ such that $\xi$ has a ground state and

$$\mathcal{V} H(\lambda) \mathcal{V}^{-1} = d \Gamma_b(\xi) + E_g - E_b.$$ 

In particular, $\mathcal{V}^{-1} \Omega_0$ is the unique ground state of $H(\lambda)$ up to constant multiples, and

$$\sigma(H(\lambda)) = \bigcup_{n=0}^{\infty} \{ n \beta + E_g - E_b \} \cup [E_0 + E_g - E_b, \infty),$$

$$\sigma_{ac}(H(\lambda)) = [E_0 + E_g - E_b, \infty),$$

$$\sigma_p(H(\lambda)) = \bigcup_{n=0}^{\infty} \{ n \beta + E_g - E_b \}, \quad \sigma_{sc}(H(\lambda)) = \emptyset,$$

where $\beta > 0$ is the discrete ground state energy of $\xi$.

(3) Let $T$ be a non-negative, injective self-adjoint operator. If $g \in D(T^{-1/2})$ and $\lambda < \lambda_{c,0}$, then $H(\lambda)$ is unbounded from above and below.

Example 2.11. A concrete realization of the abstract model is given as follows (see Chapter 12):

$$\mathcal{H} \leftrightarrow L^2(\mathbb{R}^d), \quad T \leftrightarrow \omega, \quad g \leftrightarrow \frac{\hat{\rho}}{\sqrt{\omega}}$$

where $\omega$ is a multiplication operator associated with the function $\omega(k) := \sqrt{|k|^2 + m^2}, k \in \mathbb{R}^d$ for a fixed $m \geq 0$ and $\hat{\rho}$ is the Fourier transform of a function $\rho \in L^2(\mathbb{R}^d)$ satisfying $\hat{\rho} / \sqrt{\omega} \in L^2(\mathbb{R}^d)$. Assume that $\hat{\rho}$ is rotation invariant, i.e., there exists a function $v$ on $[0, \infty)$ such that $\hat{\rho}(k) = v(|k|)$ for all $k \in \mathbb{R}^d$. Then we have $\psi_g(s) = |S^{d-1}| \omega^{-1}(s)^{d-2} |v(\omega^{-1}(s))|^2$ for $s \geq m$, where $|S^{d-1}|$ is the surface area of the $(d-1)$-dimensional unite sphere with convention $|S^0| = 2\pi$ and $\omega_1(r) = \sqrt{r^2 + m^2}, r \geq 0$. Hence, with $J$ being the complex conjugation, the following conditions (2)'-(4)' imply that the present model satisfies Assumption 2.7.

(2)' $\hat{\rho}(k)^* = \hat{\rho}(k)$ and

$$\hat{\rho} \in L^2(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \frac{|\hat{\rho}(k)|^2}{|k|^2} dk < \infty.$$ 

(3)' $\hat{\rho}$ is rotation invariant. $\sup_{k \in \mathbb{R}^d} \omega(k)^{1/2} |k|^{(d-2)/2} |\hat{\rho}(k)| < \infty$. $\hat{\rho}(k) > 0$, for all $k \neq 0$.

(4)' $v \in C^4([0, \infty))$ and

$$\lim_{|k| \to 0} |k|^{d-4} \hat{\rho}(k) \{(d - 2)\hat{\rho}(k) + 2v'(|k|)\} = 0.$$ 

$$\lim_{|k| \to \infty} |k|^{d-4} \hat{\rho}(k) \{(d - 2)\hat{\rho}(k) + 2v'(|k|)\} = 0.$$
For example, one can easily check that the function
\[
\hat{\rho}(k) := \exp \left( -\frac{1}{|k|^2} - |k|^2 \right); \ k \in \mathbb{R}^d \setminus \{0\}, \ \hat{\rho}(0) := 0
\]
satisfies the above conditions (2)'-(4)'.

3 Definitions and properties of some functions and operators

In this section we introduce some functions and operators. We assume that \( \mathcal{H} \) is separable and Assumption 2.7 from this section to Section 6.

3.1 Functions \( D \) and \( D_\pm \)

Lemma 3.1. Let \( D : \mathbb{C} \setminus (0, \infty) \to \mathbb{C} \) be the function
\[
D(z) := 1 + \lambda \int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2 - z} d\|E(\mu)g\|^2, \quad z \in \mathbb{C} \setminus (0, \infty).
\]
Then \( D \) is well-defined and analytic in \( \mathbb{C} \setminus [0, \infty) \). Moreover, the following hold:

1. For all \( \lambda > \lambda_c \), \( D(z) \) has no zeros in \( \mathbb{C} \setminus [0, \infty) \).

2. For all \( \lambda < \lambda_c \), \( D(z) \) has a unique simple zero in the negative real axis \((-\infty, 0)\).

Proof. If \( \text{Im} z \neq 0 \) (resp. \( \text{Re} z < 0 \)), then for any \( n \in \mathbb{N} \),
\[
\int_{[E_0, \infty)} \left| \frac{\mu}{(\mu^2 - E_0^2 - z)^n} \right| d\|E(\mu)g\|^2 \leq c^{-n}\|T^{1/2}g\|^2 < \infty,
\]
where \( c = |\text{Re} z| \) (resp. \( |\text{Im} z| \)). If \( z = 0 \), then
\[
\int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 \leq \|T^{-1/2}g\|^2 < \infty.
\]
Thus, by using the Lebesgue dominated convergence theorem, \( D \) is well-defined and analytic in \( \mathbb{C} \setminus [0, \infty) \).

1. If \( \lambda = 0 \), then \( D(z) = 1 \) for all \( z \in \mathbb{C} \setminus (0, \infty) \), so it has no zeros. Let \( \lambda \neq 0 \) and \( z = x + iy \in \mathbb{C} \setminus (0, \infty) \). Then we see that
\[
\text{Im} \ D(z) = y\lambda \int_{[E_0, \infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2 + y^2} d\|E(\mu)g\|^2.
\]
Thus $\text{Im } D(z) = 0$ is equivalent to $y = 0$. Therefore $D(z) = 0$ if and only if $D(x) = 0$. Let $y = 0$. In the case $\lambda > 0$, one has $D(x) > 0$ for all $x \in (-\infty, 0]$. Thus $D$ has no zeros. Next, we consider the case $\lambda < 0$. We have for $x < 0$,

$$D' = \lambda \int_{[-\infty, 0)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2} d\|E(\mu)g\|^2 < 0.$$ 

Thus $D$ is monotone decreasing in $(-\infty, 0)$. If $\lambda > \lambda_c$, then $D(0) > 0$. Hence $D$ has no zeros.

(2) Let $\lambda < \lambda_c$. We can see that

$$D(0) = 1 + \lambda \int_{[-\infty, 0)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 = 1 - \frac{\lambda}{\lambda_c} < 0.$$

By the Lebesgue dominated convergence theorem $D(x) \to 1$ as $x \to -\infty$. Since $D$ is monotone decreasing in $(-\infty, 0)$, $D$ has a unique simple zero in $(-\infty, 0)$.

Let

$$\phi_g(x) := \psi_g(\sqrt{x})\chi_{[E_0^2, \infty)}(x), x \in \mathbb{R},$$

where $\chi_B$ is the characteristic function of $B \in \mathcal{B}^1$.

**Lemma 3.2.** The following hold:

1. The function $\phi_g$ satisfies $\phi_g \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\sup_{x \in \mathbb{R}} |\phi_g'(x)| < \infty$.

2. Let

$$A^{(1)}_\varepsilon(x) := \frac{x}{\pi(x^2 + \varepsilon^2)}, \quad A^{(2)}_\varepsilon(x) := \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}, \quad x \in \mathbb{R}, \quad \varepsilon > 0$$

be the conjugate poisson kernel and the poisson kernel respectively and $f * h$ denote the convolution of functions $f$ and $h$. Let

$$(H_\varepsilon f)(s) := \frac{1}{\pi} \int_{|x - s| \geq \varepsilon} \frac{f(x)}{x - s} dx, \quad (H f)(s) := \lim_{\varepsilon \downarrow 0} (H_\varepsilon f)(s), \quad s \in \mathbb{R}, \quad \varepsilon > 0,$$

where $Hf$ is called the Hilbert transform of $f$. Then for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} (A^{(1)}_\varepsilon * \phi_g)(x) = (H\phi_g)(x), \quad \lim_{\varepsilon \downarrow 0} (A^{(2)}_\varepsilon * \phi_g)(x) = \phi_g(x),$$

hold uniformly in $x$. 

Proof. By Assumption 2.7 (2), (3) and (4), the assertion (1) holds. Next we consider the assertion (2). By (1), in particular, \( \phi_g \) is bounded and uniformly continuous. Thus it is easy to see that \( A^{(2)}_\epsilon * \phi_g \) converges uniformly to \( \phi_g \). Moreover, by (1), Holder’s inequality, the mean value theorem and a similar estimate to the proof of [16, Theorem 92.], we can show that \( (A^{(1)}_\epsilon * \phi_g)(x) - (H_\epsilon \phi_g)(x) \) tends to 0 uniformly in \( x \) as \( \epsilon \downarrow 0 \). Hence the assertion (2) holds.

Detailed studies of the Hilbert transform are given in [16].

**Lemma 3.3.** For all \( s \geq 0 \), \( D_\pm(s) := \lim_{\epsilon \downarrow 0} D(s \pm i\epsilon) \) are uniformly convergent and continuous in \( s \geq 0 \) with

\[
D_\pm(s) = 1 + \frac{\lambda \pi}{2} (H \phi_g (E_0^2 + s) \pm i \frac{\lambda \pi}{2} \psi_g (\sqrt{E_0^2 + s})) , \quad s \geq 0 . \tag{3.1}
\]

**Proof.** For any \( s \geq 0 \) and \( \epsilon > 0 \), we have by change of variable

\[
D(s \pm i\epsilon) = \frac{\lambda \pi}{2} (A^{(1)}_\epsilon * \phi_g) (E_0^2 + s) \pm i \frac{\lambda \pi}{2} (A^{(2)}_\epsilon * \phi_g) (E_0^2 + s).
\]

Thus, by Lemma 3.2, \( D_\pm \) converge uniformly in \( s \geq 0 \) and (3.1) holds. The continuity of \( D_\pm \) is due to the uniform convergence.

**Remark 3.4.** For all \( \mu \in [E_0, \infty) \), we have

\[
i \pi \lambda \psi_g (\mu) = D_+ (\mu^2 - E_0^2) - D_- (\mu^2 - E_0^2).
\]

**Lemma 3.5.** Let \( \lambda \neq \lambda_c \), then \( \delta := \inf_{s \geq 0} |D_\pm(s)| > 0 \).

**Proof.** If \( \lambda = 0 \), then clearly \( D_\pm(s) = 1 > 0 \) for all \( s \in [0, \infty) \). Let \( \lambda \neq 0, \lambda_c \). Then \( D_\pm(0) = D(0) \neq 0 \). Hence, by the continuity of \( D_\pm \), \( D_\pm \) has no zeros near \( s = 0 \). By the property that \( \phi'_g(x) \to 0 \) as \( x \to \infty \) and some estimate of \( H \phi_g \), we can see that \( (H \phi_g)(x) \to 0 \) as \( x \to \infty \). This fact implies that \( \inf_{s_0 \leq s} \Re D_\pm(s) > 0 \) for a sufficiently large number \( s_0 > 0 \). In addition, \( \Im D_\pm(s) \) are positive for any closed interval included in \( (0, \infty) \) by Assumption 2.7 (3) and the continuity of \( \psi_g \). Hence we can see that \( \inf_{s \geq 0} |D_\pm(s)| > 0 \).

**Remark 3.6.** By Lemmas 3.3 and 3.5, we can see that there are constants \( c, d, \epsilon_0 > 0 \) with \( 0 < c < d \) and \( \epsilon_0 > 0 \) such that

\[
c \leq \left| \frac{D(s \pm i\epsilon)}{D_\pm(s)} \right| \leq d \tag{3.3}
\]

for all \( s \geq 0, 0 < \epsilon < \epsilon_0 \).
3.2 Operators $R_\pm$

**Lemma 3.7.** One can define bounded operators $R_\pm$ on $\mathcal{H}$ as follows:

$$R_\pm f := -\lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \frac{R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_\pm(\mu^2 - E_0^2)} d \langle T^{1/2}g, E(\mu')f \rangle, \quad f \in \mathcal{H},$$

where $R_z(A)$ is the resolvent of a linear operator $A$ at $z \in \rho(A)$ (the resolvent set of a linear operator $A$).

**Proof.** For a fixed $\varepsilon > 0$ and any $f \in \mathcal{H}$,

$$\int_{[E_0, \infty)} \left\| R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g \right\| \left\| E(\mu')f \right\|^2 \leq \frac{\|f\|^2\|T^{1/2}g\|}{\delta \varepsilon} < \infty$$

by Lemma 3.5 and a property of a resolvent. Thus we can define linear operators $R_\pm^{(\varepsilon)}$ on $\mathcal{H}$ by

$$R_\pm^{(\varepsilon)} f := -\lambda \int_{[E_0, \infty)} \frac{R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_\pm(\mu^2 - E_0^2)} d \langle T^{1/2}g, E(\mu')f \rangle$$

in the sense of Bochner integral with the polarization identity. For any $h, f \in \mathcal{H}$,

$$\langle h, R_\pm^{(\varepsilon)} f \rangle = -\lambda \int_{[E_0, \infty)} \frac{\langle h, R_{\mu^2 \pm i\varepsilon}(T^2)T^{1/2}g \rangle}{D_\pm(\mu^2 - E_0^2)} d \langle T^{1/2}g, E(\mu')f \rangle$$

$$= -\lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu^{1/2}}{(\mu^2 - \mu'^2 \mp i\varepsilon)D_\pm(\mu^2 - E_0^2)} d \langle h, E(\mu)g \rangle d \langle T^{1/2}g, E(\mu')f \rangle,$$

where we have used the functional calculus. By change of variables in the Lebesgue-Stieltjes integration, functional calculus and Fubini’s theorem, we have

$$\langle h, R_\pm^{(\varepsilon)} f \rangle = \frac{\lambda \pi}{2} \int_{[E_0, \infty)} (A^{(1)}_\varepsilon \ast \phi^{\pm}_{g,f})(\mu^2)\mu^{1/2} \mp i (A^{(2)}_\varepsilon \ast \phi^{\pm}_{g,f})(\mu^2)\mu^{1/2} d \langle h, E(\mu)g \rangle,$$

where $\phi^{\pm}_{g,f}(x) = \psi_{g,f}(\sqrt{x})x^{-1/4}D_\pm(x - E_0^2)^{-1} \chi_{[E_0^2, \infty)}(x), x \in \mathbb{R}$. We have $\phi^{\pm}_{g,f} \in L^2(\mathbb{R})$ by Remark 2.8, and the function $(A^{(j)}_\varepsilon \ast \phi^{\pm}_{g,f})(\mu^2)\mu^{1/2} (\mu \in \mathbb{R})$ is in $L^2(\mathbb{R})$ for each $j = 1, 2$. Thus we have

$$R_\pm^{(\varepsilon)} f \to (\pi \lambda/2)(H\phi^{\pm}_{g,f})(T^2)T^{1/2}g \mp (1/2)A_\pm f \quad \text{as} \quad \varepsilon \downarrow 0$$

by a property of Hilbert transform and the continuity of the inner product with $L^2(\mathbb{R})$, where the linear operators

$$A_\pm f := i\pi \lambda \psi_{g,f}(T)D_\pm(T^2 - E_0^2)^{-1}g, f \in \mathcal{H}.$$
are well-defined (see Remark 2.8 and Lemma 3.5). Moreover, by change of variables, the isometry of Hilbert transform and Remark 2.8 we can show that the inequalities
\[
\left\|(H\phi_{g,f})(T^2)T^{1/2}g\right\| \leq \frac{c_g}{\delta} \|f\|, \quad \|A_\pm f\| \leq \frac{2\pi|\lambda|c_g}{\delta}\|f\|
\]
hold for all \( f \in \mathcal{H} \) with constant \( c_g := \sup_{\sigma(T)} \psi_g \). Hence \( R_\pm \) are bounded.

It is easy to see that \( R_\pm^* := (R_\pm)^* \) are given as follows: for \( f \in \mathcal{H} \),
\[
R_\pm^{(e)x} f = \lambda \int_{[E_0,\infty)} R_{\mu^2 \pm i\epsilon} (T^2)D_\pm (T^2 - E_0^2)^{-1}T^{1/2}g \, d\langle T^{1/2}g, E(\mu')f \rangle.
\]
For a densely defined linear operator \( A \) on a Hilbert space, we denote by \( A^* A \) or \( A^* \).

**Lemma 3.8.** The ranges of \( R_\pm^x \) are included in \( D(T^{-1}) \cap D(T) \) and \( R_\pm^x \) map \( D(T) \) into \( D(T^2) \).

**Proof.** For any \( f, h \in \mathcal{H} \), we have
\[
\langle h, R_\pm f \rangle = \frac{\lambda\pi}{2} \int_{[E_0,\infty)} (H\phi_{g,f}^\pm)(\mu^2) \mu^{1/2} \mp i \frac{\psi_{g,f}(\mu)}{D_\pm(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle. \tag{3.5}
\]
By change variable, we have
\[
(H\phi_{g,f}^\pm)(\mu^2) = (H\psi_{T^{1/2}g,f}^\pm)(\mu) + (H\psi_{T^{-1/2}g,f}^\pm)(-\mu), \mu \in \mathbb{R}, \tag{3.6}
\]
where \( \psi_{h,f}(x) := \psi_{h,f}(x)D_\pm (x^2 - E_0)^{-1} \chi_{(E_0,\infty)}(x), x \in \mathbb{R} \) for \( h, f \in \mathcal{H} \). Thus we see by Assumption 2.7 (3) and functional calculus that \( \text{Ran}(R_\pm) \subset D(T^{-1}) \). The equation
\[
\mu (H\phi_{g,f}^\pm)(\mu^2) = (H\psi_{T^{1/2}g,f}^\pm)(\mu) - (H\psi_{T^{-1/2}g,f}^\pm)(-\mu), \mu \in \mathbb{R}, \tag{3.7}
\]
operational calculus for (3.5) and Assumption 2.7 (3) imply that \( \text{Ran}(R_\pm) \subset D(T) \). For any \( f \in D(T) \) and \( \mu \in \mathbb{R} \),
\[
\mu^2 (H\phi_{g,f}^\pm)(\mu^2) = (H\psi_{T^{1/2}g,f}^\pm)(\mu) + (H\psi_{T^{1/2}g,f}^\pm)(-\mu) - \frac{2}{\pi} \int_{[E_0,\infty)} \psi_{T^{1/2}g,f}(x) \, dx.
\]
Hence \( R_\pm f \in D(T^2) \) and the following equation holds for any \( h \in \mathcal{H} \),
\[
\langle h, T^2 R_\pm f \rangle = \frac{\lambda\pi}{2} \int_{[E_0,\infty)} \left\{ (H\psi_{T^{1/2}g,f}^\pm)(\mu) + (H\psi_{T^{1/2}g,f}^\pm)(-\mu) - \frac{2c}{\pi} \right\} \mu^{1/2} \, d\langle h, E(\mu)g \rangle
\]
\[
\mp i \frac{\lambda\pi}{2} \int_{[E_0,\infty)} \psi_{T^{1/2}g,f}^\pm(\mu) \mu^{1/2} \, d\langle h, E(\mu)g \rangle ,
\]
where \( c := \int_\mathbb{R} \psi_{T^{1/2}g,f}^\pm(x) dx \). In quite the same manner as in the case of \( R_\pm \), we can prove the statement for \( R_\pm^* \).
Lemma 3.9. The operator equations $(R_{\pm})_f = R_{\mp}$ hold.

Proof. This follows from Assumption 2.7 (1) and Theorem 2.9.

Lemma 3.10. The operator equation $R_- = R_+ \gamma + A_-$ holds, where

$$\gamma := D_+(T^2 - E_0^2)D_-(T^2 - E_0^2)^{-1}$$

is a bounded operator.

Proof. The first resolvent formula gives that, for any $\mu', \mu'' \in \mathbb{R}, \varepsilon > 0$,

$$R_{\mu^2 + i\varepsilon}(T^2) - R_{\mu'^2 - i\varepsilon}(T^2) = -2i\varepsilon R_{\mu^2 - i\varepsilon}(T^2)R_{\mu'^2 + i\varepsilon}(T^2).$$

Then, for any $f \in \mathcal{H}$,

$$R^{(\varepsilon)} f = -\lambda \int_{[E_0, \infty)} \frac{R_{\mu^2 + i\varepsilon}(T^2)T^{1/2}g}{D_-((\mu^2 - E_0^2)^2 + i\varepsilon)} d \langle{T^{1/2}g, E(\mu')f}\rangle$$

$$+ 2i\lambda\varepsilon \int_{[E_0, \infty)} \frac{R_{\mu^2 + i\varepsilon}(T^2)R_{\mu'^2 - i\varepsilon}(T^2)T^{1/2}g}{D_-((\mu^2 - E_0^2)^2 + i\varepsilon)} d \langle{T^{1/2}g, E(\mu')f}\rangle.$$ 

Thus, by change of variable, we have for any $h \in \mathcal{H}$

$$\langle h, R^{(\varepsilon)} f \rangle = \langle h, R_+ \gamma f \rangle + 2i\lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu)g \rangle d \langle{T^{1/2}g, E(\mu')f}\rangle$$

$$\times \{((\mu^2 - \mu'^2)^2 + \varepsilon^2)D_-((\mu^2 - E_0^2)^2 + i\varepsilon)\}$$

$$= \langle h, R_+ \gamma f \rangle + i\pi\lambda \int_{[E_0, \infty)} \left(A_{\varepsilon}^{(2)}(\mu^2) \phi_{g, f}^{-}(\mu^2)\right)^{1/2} d \langle h, E(\mu)g \rangle.$$ 

By a property of the Poisson kernel, the function $\left(A_{\varepsilon}^{(2)}(\mu^2) \phi_{g, f}^{-}(\mu^2)\right)(\mu \in \mathbb{R})$ converges to $\psi_{g, f}(\mu)/D_-((\mu^2 - E_0^2)^2)$ as $\varepsilon \to +0$ in the sense of $L^2(\mathbb{R})$. Hence the continuity of inner product with $L^2(\mathbb{R})$ implies that

$$\langle h, R_- f \rangle = \langle h, R_+ \gamma f \rangle + i\pi\lambda \int_{[E_0, \infty)} \frac{\psi_{g, f}(\mu)}{D_-((\mu^2 - E_0^2)^2)} d \langle h, E(\mu)g \rangle$$

$$= \langle h, R_+ \gamma f \rangle + \langle h, A_- f \rangle.$$ 

Since $f$ and $h$ are arbitrary, one obtains the conclusion.

It is easy to see that

$$(A_-)^* = -A_+.$$
Lemma 3.11. For any Borel measurable function $F$ on $\mathbb{R}$, $A_\pm F(T) \subset F(T)A_\pm$.

Proof. It is easy to see that for any $f \in D(F(T))$, $\psi_{g,F(T)f} = F\psi_{g,f} \in L^2(\mathbb{R})$. This fact and Lemma 3.5 imply that $\psi_{g,f}(T)g \in D(F(T))$ and $F(T)\psi_{g,f}(T)g = \psi_{g,F(T)f}(T)g$. Hence $A_\pm f \in D(F(T))$ and $F(T)A_\pm f = A_\pm F(T)f$.

Lemma 3.12. The following operator equations hold:

$$A_- R_\pm^* = (\gamma - I) R_\pm^*, \quad A_-(A_-)^* = - A_- - (A_-)^*.$$ 

Proof. By applying Lemma 3.11 to the case $F = \chi_B$, one can easily see that $A_\pm E(B) = E(B)A_\pm$ holds for any $B \in B^1$. For any $f, h \in \mathcal{H}$, we have

$$\left\langle (A_-)^* h, R_\pm^{(\varepsilon)*} f \right\rangle = i \pi \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu^{1/2} \psi_g(\mu)}{\left( \mu^2 - \mu'^2 \mp i \varepsilon \right) D_+ (\mu^2 - E_0^2) D_- (\mu^2 - E_0^2)} d \langle h, E(\mu) \rangle d \langle T^{1/2} g, E(\mu') \rangle.$$

Then, since $\gamma$ and $E(B)$ commute on $\mathcal{H}$ for any $B \in B^1$, (3.2) gives

$$\left\langle (A_-)^* h, R_\pm^{(\varepsilon)*} f \right\rangle = \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu^{1/2}}{\left( \mu^2 - \mu'^2 \mp i \varepsilon \right) D_+ (\mu^2 - E_0^2)} d \langle h, E(\mu) (\gamma - 1) \rangle d \langle T^{1/2} g, E(\mu') \rangle = \left\langle h, (\gamma - 1) R_\pm^{(\varepsilon)*} f \right\rangle.$$

Thus, by a limit argument, we obtain $A_- R_\pm^* = (\gamma - 1) R_\pm^*$. Moreover, (3.2) and the equation $(A_-)^* = - A_+$ imply that

$$\langle h, A_-(A_-)^* f \rangle = -(i \pi \lambda^2) \int_{[E_0, \infty)} \frac{\psi_{g,f}(\mu) \psi_g(\mu)}{D_+ (\mu^2 - E_0^2) D_- (\mu^2 - E_0^2)} d \langle h, E(\mu) \rangle$$

$$= -i \pi \lambda \int_{[E_0, \infty)} \frac{(D_+ (\mu^2 - E_0^2) - D_- (\mu^2 - E_0^2)) \psi_{g,f}(\mu)}{D_+ (\mu^2 - E_0^2) D_- (\mu^2 - E_0^2)} d \langle h, E(\mu) \rangle$$

$$= - \langle h, (A_-)^* f + A_- f \rangle.$$

Hence the equation $A_-(A_-)^* = - A_- - (A_-)^*$ holds.

3.3 Operators $\Omega_\pm$

In this subsection we consider the bounded operators

$$\Omega_\pm := I + R_\pm.$$
Let $x_0 < 0$ be the zero of $D(z)$ given in Lemma 3.1 (2) and

$$U_b := \sqrt{\frac{\lambda}{D'(x_0)}} R_{E_0^2 + x_0} (T^2) T^{1/2} g, \quad P := \langle U_b, \cdot \rangle U_b.$$ 

Then it is easy to see that $\|U_b\| = 1, U_b \in D(T^{-1}) \cap D(T^2)$ and

$$TU_b = \sqrt{\frac{\lambda}{D'(x_0)}} T^{-1/2} g + (E_0^2 + x_0) T^{-1} U_b.$$ 

Hence $P$ is a projection operator.

**Lemma 3.13.** Let $\lambda \neq \lambda_c$. Then the following equations hold:

$$\Omega_+^* \Omega_+ = I, \quad \Omega_-^* \Omega_- = I - \theta(\lambda_c - \lambda) P,$$

where $\theta$ is the Heaviside function:

$$\theta(t) = \begin{cases} 
1 & \text{if } t > 0, \\
0 & \text{if } t < 0.
\end{cases}$$

**Remark 3.14.** Lemma 3.13 implies that $\Omega_{\pm}$ are unitary operators if $\lambda > \lambda_c$ and partial isometries with their final subspace $\text{Ran}(I - P)$ if $\lambda < \lambda_c$.

**Proof.** (1) We first prove (3.8).

It is sufficient to prove that $R_{\mp}^* R_{\pm} = -(R_{\pm} + R_{\mp}^*)$ hold. For any $f, h \in \mathcal{H}$ and $\varepsilon > 0$,

$$\langle R_{\pm}^{(\varepsilon)} h, R_{\pm}^{(\varepsilon)} f \rangle = \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu') T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu'') f \rangle \times \left\langle R^{(\mu') T^{1/2} g}, R^{(\mu'') T^{1/2} g} \right\rangle \right\rangle.

By the definition of the function $D$, we have

$$\lambda \langle T^{1/2} g, R_{\pm} (T^2) T^{1/2} g \rangle = D(z - E_0^2) - 1, \; z \in \mathbb{C} \setminus (E_0^2, \infty).$$

By this formula and a resolvent identity, we obtain

$$\langle R_{\pm}^{(\varepsilon)} h, R_{\pm}^{(\varepsilon)} f \rangle = \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu') T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu'') f \rangle \times \frac{D(\mu^2 - E_0^2 \mp i \varepsilon) - D(\mu''^2 - E_0^2 \pm i \varepsilon)}{(\mu^2 - \mu''^2 \mp 2i \varepsilon) D_+ (\mu^2 - E_0^2) D_+ (\mu''^2 - E_0^2)}.$$

$$= - \langle E_{\pm}^{(\varepsilon)} h, R_{\pm}^{(2 \varepsilon)} f \rangle - \langle R_{\pm}^{(2 \varepsilon)} h, E_{\pm}^{(\varepsilon)} f \rangle,$$
where the operators $E_{\pm}^{(e)}$ on $\mathcal{H}$ are given as follows:

$$
\langle h, E_{\pm}^{(e)} f \rangle := \int_{[E_0, \infty)} \frac{D(\mu^2 - E_0^2 \pm i\varepsilon)}{D_{\pm}(\mu^2 - E_0^2)} \, d \langle h, E(\mu) f \rangle, \quad h, f \in \mathcal{H}.
$$

The inequality (3.3) implies that $E_{\pm}^{(e)}$ are bounded for all $0 < \varepsilon < \varepsilon_0$. Thus, by the Lebesgue dominated convergence theorem, we have $\lim_{\varepsilon \downarrow 0} E_{\pm}^{(e)} = I$. Hence we obtain that $R_\varepsilon R_\varepsilon^* = -(R_\varepsilon + R_\varepsilon^*)$.

(2) We next prove (3.9) for $\lambda \neq \lambda_c$.

It is sufficient to prove that $R_\varepsilon R_\varepsilon^* = -(R_\varepsilon + R_\varepsilon^*) - \theta(\lambda_c - \lambda)P$ hold. For any $f, h \in \mathcal{H}$ and a fixed $\varepsilon > 0$, (3.4) implies

$$
\langle R_\varepsilon R_\varepsilon^* h, R_\varepsilon R_\varepsilon^* f \rangle
= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu)T^{1/2}g \rangle d \langle T^{1/2}g, E(\mu')f \rangle
\times \langle R_{\mu^2 \pm i\varepsilon}(T^2)D_{\mp}(T^2 - E_0^2)^{-1}T^{1/2}g, R_{\mu^2 \pm i\varepsilon}(T^2)D_{\mp}(T^2 - E_0^2)^{-1}T^{1/2}g \rangle
= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu)T^{1/2}g \rangle d \langle T^{1/2}g, E(\mu')f \rangle d\|E(\mu'')g\|^2
\times \frac{1}{\mu^2 - \mu'^2 \mp 2i\varepsilon} J_{\varepsilon}^\pm(\mu, \mu') \langle h, E(\mu)T^{1/2}g \rangle \langle T^{1/2}g, E(\mu')f \rangle, \quad (3.10)
$$

where, for any $\mu, \mu' \in [E_0, \infty)$,

$$
J_{\varepsilon}^\pm(\mu, \mu')
= \int_{[E_0, \infty)} \frac{\lambda \mu''}{D_{\pm}(\mu'^2 - E_0^2)D_{\mp}(\mu^2 - E_0^2)} \left( \frac{1}{\mu^2 - \mu'^2 \pm i\varepsilon} - \frac{1}{\mu'^2 - \mu^2 \mp i\varepsilon} \right) d\|E(\mu'')g\|^2.
$$

Then, by change of variable and (3.2), one can show that

$$
J_{\varepsilon}^\pm(\mu, \mu') = \lim_{R \to \infty} \frac{1}{2\pi i} I_{\varepsilon, R}^\pm(\mu, \mu'),
$$

where, for $R > 0$,

$$
I_{\varepsilon, R}^\pm(\mu, \mu') = \int_{0}^{R} \left( \frac{1}{D_{\mp}(s)} - \frac{1}{D_{\pm}(s)} \right) G_{\mu, \mu'}^\varepsilon(s) ds
$$

and

$$
G_{\mu, \mu'}^\varepsilon(z) := \frac{1}{z - \mu'^2 + E_0^2 \mp i\varepsilon} - \frac{1}{z - \mu^2 + E_0^2 \pm i\varepsilon}, \quad z \in \mathbb{C}.
$$
For $0 < \eta < \varepsilon$ and $R > 0$, let $C_i \ (i = 1, 2, 3)$ be the curve given as follows:

\[
C_1 : \ \theta_1(t) = R - t - i\eta, \ t : 0 \to R,
\]
\[
C_2 : \ \theta_2(t) = \eta e^{-it}, \ t : \pi/2 \to (3\pi)/2,
\]
\[
C_3 : \ \theta_3(t) = t + i\eta, \ t : 0 \to R.
\]

Then, for $C = C_1 + C_2 + C_3$, we have by the Lebesgue dominated convergence theorem,

\[
I^{\pm}_{\varepsilon,R}(\mu, \mu') = \lim_{\eta \downarrow 0} \int_C \frac{1}{D(z)} G^{\varepsilon, \pm}_{\mu, \mu'}(z) dz.
\]

We take $R$ such that $R > \max\{\mu^2 - E_0^2, \mu'^2 - E_0^2\}$ and define a curve $C_4 : \theta_4(t) = \sqrt{\eta^2 + R^2 e^{-it}}, t : t_s \to t_f$, for $t_s := \arctan(\eta/R)$ and $t_f = 2\pi - t_s$. We consider two cases separately.

(i) The case $\lambda > \lambda_c$. In this case, the function $G^{\varepsilon, \pm}_{\mu, \mu'}(z)/D(z), z \in \mathbb{C}\setminus(0, \infty)$ has two simple poles at $z = \mu^2 - E_0^2 \mp i\varepsilon, z = \mu'^2 - E_0^2 \mp i\varepsilon$. Then, by the residue theorem, we have

\[
\int_C \frac{1}{D(z)} G^{\varepsilon, \pm}_{\mu, \mu'}(z) dz = 2\pi i \left( \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} - \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)} \right)
\]

Thus, as $\eta$ tends to 0, we have

\[
I^{\pm}_{\varepsilon,R}(\mu, \mu') = 2\pi i \left( \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} - \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)} \right)
\]

\[
- \lim_{\eta \downarrow 0} \int_{C_4} \frac{1}{D(z)} G^{\varepsilon, \pm}_{\mu, \mu'}(z) dz.
\]

The definition of line integral implies

\[
\int_{C_4} \frac{1}{D(z)} G^{\varepsilon, \pm}_{\mu, \mu'}(z) dz = -i \int_{t_s}^{2\pi - t_s} G^{\varepsilon, \pm}_{\mu, \mu'}(\sqrt{\eta^2 + R^2 e^{-it}}) \sqrt{\eta^2 + R^2 e^{-it}} \frac{dt}{D(\sqrt{\eta^2 + R^2 e^{-it}})}
\]

By the triangle inequality, for any $t \in [t_s, t_f]$,

\[
|G^{\varepsilon, \pm}_{\mu, \mu'}(\sqrt{\eta^2 + R^2 e^{-it}})| \leq \frac{|\mu^2 - \mu'^2 \pm 2i\varepsilon|}{(R - |\mu^2 - E_0^2 \mp i\varepsilon|)(R - |\mu'^2 - E_0^2 \mp i\varepsilon|)}
\]

On the other hand, by Lemma 3.3 and the Lebesgue dominated convergence theorem, there are constants $\tilde{R} > 0$ and $c_0 > 0$ such that $|D(z)| \geq c_0$ for all $|z| \geq \tilde{R}$. Thus we have

\[
I^{\pm}_{\varepsilon,R}(\mu, \mu') = 2\pi i \left( \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} - \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)} \right) + O(R^{-1}) \ (R \to \infty),
\]

21
where $O(\cdot)$ stands for the well known Landau symbol. Therefore we have

$$J_\varepsilon^{\pm}(\mu, \mu') = \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)}$$

for each $\mu, \mu' \in [E_0, \infty)$. Thus, by (3.10), we have

$$\langle R^{(\varepsilon)}_\pm h, R^{(\varepsilon)*}_\pm f \rangle = -\left( \left( R^{(2\varepsilon)}_\pm \right)^* h, \left( E^{(\varepsilon)}_\pm \right)^{-1} f \right) - \left( \left( E^{(\varepsilon)}_\pm \right)^{-1} h, \left( R^{(2\varepsilon)}_\pm \right)^* f \right).$$

As in the proof in (1), we obtain $s\text{-}\lim_{\varepsilon \downarrow 0} \left( E_\varepsilon^{(\varepsilon)} \right)^{-1} = I$. Therefore we obtain

$$\lim_{\varepsilon \downarrow 0} \langle R^{(\varepsilon)*}_\pm h, R^{(\varepsilon)*}_\pm f \rangle = -\langle R^*_\pm h, f \rangle - \langle h, R^*_\pm f \rangle.$$ 

Thus we obtain the desired result.

(ii) The case $\lambda < \lambda_c$. In this case, $G^{\varepsilon, \pm}_{\mu, \mu'}(z)/D(z)$ has a simple pole at $z = x_0$ in addition to $z = \mu^2 - E_0^2 \mp i\varepsilon, z = \mu'^2 - E_0^2 \pm i\varepsilon$. The residue $R_0$ of $G^{\varepsilon, \pm}_{\mu, \mu'}(z)/D(z)$ at $z = x_0$ is given by

$$R_0 = \frac{1}{D'(x_0)} \frac{\mu^2 - \mu'^2 \pm 2i\varepsilon}{(x_0 - \mu'^2 + E_0^2 \mp i\varepsilon)(x_0 - \mu^2 + E_0^2 \pm i\varepsilon)}.$$ 

Thus we have

$$J^{\pm}_\varepsilon(\mu, \mu') = \frac{1}{D(\mu'^2 - E_0^2 \mp i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} + R_0$$

and also

$$\lambda \frac{\mu^2 - \mu'^2 \mp 2i\varepsilon}{D'(x_0)} R_0 = -\frac{\lambda}{D'(x_0)} \frac{\mu'^2 - E_0^2}{(\mu^2 - E_0^2 \mp x_0 \pm i\varepsilon)(\mu^2 - E_0^2 \mp x_0 \mp i\varepsilon)}.$$ 

This implies that

$$\lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu^2 - \mu'^2 \mp 2i\varepsilon} R_0 d \langle h, E(\mu)T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu') f \rangle$$

$$= -\langle h, U_b \rangle \langle U_b, f \rangle = -\langle h, Pf \rangle,$$

Thus we obtain the desired result.
3.4 Operators $U$ and $V$

In this subsection, we investigate the operators $U$ and $V$ defined as follows:

$$U := \frac{1}{2}(T^{-1/2}\Omega_+ T^{1/2} + T^{1/2}\Omega_+ T^{-1/2}), \quad V := \frac{1}{2}(T^{-1/2}\Omega_+ T^{1/2} - T^{1/2}\Omega_+ T^{-1/2}),$$

which are used to construct a Bogoliubov transformation. Then, by Lemma 3.8 one can easily see that $D(U) = D(V) = D(T^{-1/2}) \cap D(T^{1/2})$.

Lemma 3.15. The operators $U$ and $V$ are bounded.

Proof. By (3.5) and Lemma 3.8 we have

$$\langle h, T^{-1/2}R_+ T^{1/2}f \rangle = \frac{\lambda\pi}{2} \int_{(E_0, \infty)} \left( H\phi_{T^{1/2}g,f}^\pm \right)(\mu^2) \mp i \frac{\psi_{g,f}(\mu)}{D_\pm(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle, \quad (3.11)$$

$$\langle h, T^{1/2}R_+ T^{-1/2}f \rangle = \frac{\lambda\pi}{2} \int_{(E_0, \infty)} \left( H\phi_{T^{-1/2}g,f}^\pm \right)(\mu^2) \mu \mp i \frac{\psi_{g,f}(\mu)}{D_\pm(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle. \quad (3.12)$$

By Assumption 2.7 (3), (3.6), (3.7) and a property of Hilbert transform, we can show that

$$\|T^{-1/2}R_+ T^{1/2}f\|, \|T^{1/2}R_+ T^{-1/2}f\| \leq \frac{|\lambda|\pi(C_g + \delta_f)}{2\delta} \|f\|,$$

where $C_g := (\sup_{E_0<\xi} x^{-1}\psi(x))^{1/2}(\sup_{E_0<\xi} x\psi(x))^{1/2}$. Hence the operators $T^{-1/2}R_+ T^{1/2}$ and $T^{1/2}R_+ T^{-1/2}$ are bounded.

In the same way as in the proof of Lemma 3.15, we see that $T^{-1/2}R_+ T^{1/2}$ and $T^{1/2}R_+ T^{-1/2}$ are bounded on each domain $D(T^{1/2})$ and $D(T^{-1/2})$. In what follows, we write the bounded extension of $U$ and $V$ by the same symbol respectively. Then

$$U^* = \frac{1}{2}(T^{-1/2}\Omega_+^* T^{1/2} + T^{1/2}\Omega_+^* T^{-1/2}).$$

Lemma 3.16. The operators $U^*$ and $V^*$ leave $D(T^{-1/2})$ (resp. $D(T^{1/2}), D(T)$) invariant.

Proof. By applying Lemma 3.8 and using the equations

$$U^* = I + \frac{1}{2} \left( T^{-1/2}R_+^* T^{1/2} + T^{1/2}R_+^* T^{-1/2} \right),$$

one can easily see that the assertion for $U^*$ is true. Similarly one can prove the statement for $V^*$.

Lemma 3.17. Let $F(x) = x^{\pm 1/2}, x^{\pm 1}, a.e. \ x \in (0, \infty)$. Then

$$\Omega_+ F(T)\Omega_+^* = (\Omega_+)_JF(T)(\Omega_+^*)_J \quad \text{on} \ D(F(T)). \quad (3.13)$$
Proof. By Lemma 3.8, the domain of each side of (3.13) includes $D(F(T))$. By Lemmas 3.11 and 3.12, we have

\[
(\Omega_+)JF(T)(\Omega^*_+) = R_+F(T)R^*_+ + R_+\{(A_-)^* + I\}F(T)\gamma + F(T)\gamma^*(A_- + I)R^*_+ + F(T)\{A_-(A_-)^* + A_- + (A_-)^* + I\} = R_+F(T)R^*_+ + F(T)R^*_+ + F(T) + F(T)R^*_+ + F(T) = \Omega_+F(T)\Omega^*_+.
\]

$$\square$$

4 Commutation relations

In this section, we prove that the pair $(U, V)$ satisfies the condition (1.1), $V$ is Hilbert-Schmidt and

\[ B(f) := A(Uf) + A(JVf)^*, \quad f \in \mathcal{H} \]

satisfies some commutation relations with $H(\lambda)$. We denote the closure of $B(f)$ by the same symbol. By Lemma 3.16, we have $D(d\Gamma_b(T)^{1/2}) \subset D(B(f)) \cap D(B(f)^*)$ for all $f \in D(T^{-1/2})$.

Theorem 4.1. The following commutation relations hold:

1. For any $f \in D(T)$ and $\psi \in \mathcal{F}_{b,\text{fin}}(D(T))$,

\[
[H(\lambda), B(f)]\psi = -B(Tf)\psi. \quad (4.1)
\]

2. For any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(d\Gamma_b(T))$,

\[
\langle H(\lambda)\phi, B(f)\psi \rangle - \langle B(f)^*\phi, H(\lambda)\psi \rangle = -\langle \phi, B(Tf)\psi \rangle. \quad (4.2)
\]

3. For any $f \in D(T^{-1/2}) \cap D(T)$, $B(f)$ maps $D(d\Gamma_b(T)^{3/2})$ into $D(d\Gamma_b(T))$ and for any $\psi \in D(d\Gamma_b(T)^{3/2})$,

\[
[H(\lambda), B(f)]\psi = -B(Tf)\psi. \quad (4.3)
\]

The both sides of (4.1), (4.2) and (4.3) have meaning by Lemma 3.16. To prove this theorem, we prove the following lemma:

Lemma 4.2. For any $f \in D(T)$, the following equations hold:

\[
[U, T]f = (VT + TV)f = \frac{\lambda}{2} \langle D_-(T^2 - E_0^2)^{-1}g, f \rangle g, \quad (4.4)
\]

\[
(V^*J - U^*)g = -D_-(T^2 - E_0^2)^{-1}g. \quad (4.5)
\]
Proof. For any \( f, h \in D(T^{-1/2}) \cap D(T^{3/2}) \), we obtain
\[
\langle h, [U, T]f \rangle = \frac{1}{2} \left( \langle T^{1/2} R_\pm^{(\varepsilon)} T^{-1/2} h, T f \rangle - \langle T h, T^{1/2} R_\pm^{(\varepsilon)} T^{-1/2} f \rangle \right).
\]
Then, for each \( \varepsilon > 0 \), we have
\[
\begin{align*}
\langle T^{1/2} R_\pm^{(\varepsilon)} T^{-1/2} h, T f \rangle - \langle T h, T^{1/2} R_\pm^{(\varepsilon)} T^{-1/2} f \rangle &= \lambda \int_{E_0, \infty} \int_{[E_0, \infty]} \frac{\mu^2 - \mu^2}{(\mu^2 - \mu^2 \pm i \varepsilon) D_\pm(\mu^2 - E_0^2)} d \langle h, E(\mu)g \rangle d \langle g, E(\mu)f \rangle \\
&= \lambda \int_{E_0, \infty} \int_{[E_0, \infty]} \frac{1}{D_\pm(\mu^2 - E_0^2)} d \langle h, E(\mu)g \rangle d \langle E(\mu)f, g \rangle + i \varepsilon \left( \langle T^{-1/2} h, R_\pm^{(\varepsilon)} T^{-1/2} f \rangle \right).
\end{align*}
\]
Taking the limit \( \varepsilon \downarrow 0 \), we have
\[
\langle T^{1/2} R_\pm T^{-1/2} h, T f \rangle - \langle T h, T^{1/2} R_\pm T^{-1/2} f \rangle = \langle h, \lambda \langle D_\pm(T^2 - E_0^2)^{-1} g, f \rangle g \rangle.
\]
Thus we have
\[
\langle h, [U, T]f \rangle = \frac{\lambda}{2} \langle h, \langle D_-(T^2 - E_0^2)^{-1} g, f \rangle g \rangle.
\]
Since \( D(T^{-1/2}) \cap D(T^{3/2}) \) is a core of \( T \), the equation (4.4) holds for \( f \in D(T) \). To prove (4.5), we note that
\[
(V^* J - U^*)g = \frac{1}{2}(T^{1/2} \Omega_+^* T^{-1/2} J - T^{-1/2} \Omega_+ T^{1/2} J - T^{1/2} \Omega_+^* T^{-1/2} - T^{-1/2} \Omega_+ T^{1/2})g
\]
where we have used \( Jg = g \). Thus, for any \( f \in \mathcal{H} \), we obtain
\[
\begin{align*}
\langle f, (V^* J - U^*)g \rangle &= -\langle f, g \rangle - \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty]} \int_{[E_0, \infty]} \frac{\mu'}{\mu^2 - \mu^2 + i \varepsilon} dE(\mu') g^2 d\mu^2 \|E(\mu')T^{1/2}g\|^2 \\
&= -\langle f, g \rangle + \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty]} \int_{[E_0, \infty]} \frac{1}{D_-(\mu^2 - E_0^2)} dE(\mu) g^2 d\mu^2 \frac{1}{D_-(\mu^2 - E_0^2)} \langle f, E(\mu)g \rangle \\
&= -\langle f, g \rangle + \int_{[E_0, \infty]} \frac{D_-(\mu^2 - E_0^2) - 1}{D_-(\mu^2 - E_0^2)} d \langle f, E(\mu)g \rangle \\
&= -\langle f, D_-(T^2 - E_0^2)^{-1} g \rangle.
\end{align*}
\]
Hence (4.5) holds. \( \square \)

Proof of Theorem 4.1

\[ \text{Proof of Theorem 4.1} \]
(1) By Lemma 3.16 for any \( f \in D(T) \), \( B(f) \) leaves \( \mathcal{F}_{b,\text{fin}}(D(T)) \) invariant and \( H(\lambda) \) maps \( \mathcal{F}_{b,\text{fin}}(D(T)) \) into \( \mathcal{F}_{b,\text{fin}}(\mathcal{H}) \subset D(B(f)) \). Thus, by using (2.1), (9.3), we have for any \( \psi \in \mathcal{F}_{b,\text{fin}}(D(T)) \),

\[
[H(\lambda), B(f)]\psi = \left\{-A(TUf) + A(TJVf)^* - \frac{\lambda}{\sqrt{2}} \langle f, (V^*J - U^*)g \rangle \Phi_n(g) \right\} \psi.
\]

Hence by Lemma 4.2, (4.1) holds.

(2) By Lemma 3.16 and fundamental properties of the annihilation operators and creation operators, we can see that, for any \( f \in D(T^{-1/2}), D(d\Gamma_b(T)^{1/2}) \subset D(B(f)) \). For any \( \psi, \phi \in D(d\Gamma_b(T)) \), there are sequences \( \psi_n, \phi_n \in \mathcal{F}_{b,\text{fin}}(D(T)), n \in \mathbb{N} \) such that \( \psi_n \to \psi, \phi_n \to \phi, d\Gamma_b(T)\psi_n \to d\Gamma_b(T)\psi, d\Gamma_b(T)\phi_n \to d\Gamma_b(T)\phi \) as \( n \to \infty \), since \( \mathcal{F}_{b,\text{fin}}(D(T)) \) is a core of \( d\Gamma_b(T) \). By (1), we have

\[
\langle H(\lambda)\phi_n, B(f)\psi_k \rangle - \langle B(f)^*\phi_n, H(\lambda)\psi_k \rangle = -\langle \phi_n, B(Tf)\psi_k \rangle
\]

for all \( n, k \in \mathbb{N} \) and \( f \in D(T^{-1/2}) \cap D(T) \). By the fundamental inequalities (9.1) and (9.2) and the \( d\Gamma_b(T) \)-boundedness of \( \Phi_n(g)^2 \), we can see that \( \{B(f)\psi_n\}_{n=1}^{\infty}, \{B(f)^*\phi_n\}_{n=1}^{\infty}, \{\Phi_n(g)^2\psi_n\}_{n=1}^{\infty}, \{\Phi_n(g)^2\phi_n\}_{n=1}^{\infty} \) and \( \{B(Tf)\psi_n\}_{n=1}^{\infty} \) converge. Hence we obtain (4.2).

(3) By Lemma 3.16 and fundamental properties of the annihilation operators and creation operators, we see that, for any \( f \in D(T^{-1/2}) \cap D(T) \), \( B(f) \) maps \( D(d\Gamma_b(T)^{3/2}) \) into \( D(d\Gamma_b(T)) \). Therefore, by (4.2) and the density of \( D(d\Gamma_b(T)) \), we have (4.3). \( \square \)

### 4.1 Relations of \( U \) and \( V \)

**Lemma 4.3.** Let \( \lambda \neq \lambda_c \). Then the following equations hold:

\[
\begin{align*}
U^*U - V^*V &= I, \\
U^*_jV - V^*_jU &= 0, \\
UU^* - V_jV_j^* &= I - \theta(\lambda_c - \lambda)Q_+, \\
UV^* - V_jU_j^* &= \theta(\lambda_c - \lambda)Q_-,
\end{align*}
\]

where

\[
Q_\pm := \frac{1}{2} \left( \langle T^{1/2}U_b, \cdot \rangle T^{-1/2}U_b \pm \langle T^{-1/2}U_b, \cdot \rangle T^{1/2}U_b \right)
\]

are bounded operators on \( \mathcal{H} \).

**Proof.** It is sufficient to prove (4.6) on \( D(T^{-1/2}) \cap D(T^{1/2}) \). Using (3.8), one can show that the first equation in (3.8) hold. We have

\[
U^*_jV - V^*_jU = \frac{1}{2} (-T^{1/2}(\Omega^+_j)\Omega_+T^{-1/2} + T^{-1/2}(\Omega^+_j)\Omega_+T^{1/2}).
\]
Multiplying the equation by $(\Omega_+)_J$ from the left, and using Lemma 3.17 we obtain

$$(\Omega_+)_J(U^*_jV - V^*_jU) = (\Omega_+)_J(-T^{1/2}(\Omega^*_+)_J(\Omega^+_+T^{-1/2} + T^{-1/2}(\Omega^*_+)_J(\Omega^+_+T^{1/2}) = 0.$$  

By (3.8), this implies that $U^*_jV - V^*_jU = 0$. By Lemma 2.9 and Lemma 3.17, we have

$$V^*_jV^*_j = \frac{1}{4}T^{-1/2}(\Omega_+^T\Omega^*_+)_JT^{-1/2} - T^{-1/2}(\Omega_+^T\Omega^*_+)_JT^{1/2}$$  

$$= \frac{1}{4}(T^{-1/2}\Omega_+^T\Omega^*_+T^{-1/2} - T^{-1/2}\Omega_+^T\Omega^*_+T^{1/2})$$  

$$= VV^*.$$  

Hence, by direct calculations and (3.9), one obtains $UU^*-V^*_jV^*_j = I - \theta(\lambda_c - \lambda)Q_+$. Similarly, one can prove the last equation in (4.6) (note that $P_J = P$).

**4.2 Hilbert-Schmidtness of V**

In this subsection, we show that $V$ is Hilbert-Schmidt. Then we can use Theorem 2.2 in the case of $\lambda > \lambda_c$.

**Lemma 4.4.** The operator $V$ is Hilbert-Schmidt.

**Proof.** On $D(T^{-1/2}) \cap D(T^{1/2})$, $V^*V$ is calculated as follows:

$$V^*V = \frac{1}{4}(T^{-1/2}R_+^T T^{-1/2} + T^{1/2}R_+^T T^{-1/2} + T^{1/2}[R_+^T, T]R_+T^{1/2}$$  

$$+ T^{1/2}R_+T^{-1/2} + T^{-1/2}R_+^T T^{1/2} + T^{-1/2}[R_+^T, T]R_+T^{-1/2}$$  

$$+ T^{1/2}R_+T^{-1/2} + T^{-1/2}R_+^T R_+T^{1/2})$$  

where we have used the formula $R_+^* R_+ = -(R_+ + R_+^*)$ in the proof of Lemma 3.13 and Lemma 3.8. Thus, for any $f \in D(T^{-1/2}) \cap D(T^{1/2})$ and $\varepsilon > 0$, we have

$$\left\langle f, (T^{1/2}[R_+^{(\varepsilon)}, T]R_+^{(\varepsilon)} T^{1/2} + T^{-1/2}[R_+^{(\varepsilon)}, T]R_+^{(\varepsilon)} T^{-1/2}) f \right\rangle$$

$$= \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu'}{(\mu'^2 - \mu^2 + i\varepsilon)D_+(\mu'^2 - E_0^2)} d\left\langle [T^{-1}, R_+^{(\varepsilon)}]T^{1/2} f, E(\mu)T^{1/2} g \right\rangle d\left\langle (E(\mu'))^T g, f \right\rangle$$

$$+ \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu'}{(\mu'^2 - \mu^2 + i\varepsilon)D_+(\mu'^2 - E_0^2)} d\left\langle [T, R_+^{(\varepsilon)}]T^{-1/2} f, E(\mu)T^{-1/2} g \right\rangle d\left\langle (E(\mu'))^T g, f \right\rangle.$$

27
Then, for any $B \in \mathcal{B}^1$, we can see
\begin{equation}
\left\langle [T^{-1}, R^\varepsilon_+]T^{1/2} f, E(B)T^{1/2} g \right\rangle = \lambda \int_B \int_{[E_0, \infty)} \frac{\mu'' - \mu}{(\mu''^2 - \mu^2 - i\epsilon)D_-(\mu''^2 - E_0^2)} \, d\langle f, E(\mu'')g \rangle \, d\|E(g)\|^2.
\end{equation}
(4.7)

Similarly, we obtain
\begin{equation}
\left\langle [T, R^\varepsilon_+]T^{-1/2} f, E(B)T^{-1/2} g \right\rangle = \lambda \int_B \int_{[E_0, \infty)} \frac{\mu - \mu''}{(\mu''^2 - \mu^2 - i\epsilon)D_-(\mu''^2 - E_0^2)} \, d\langle f, E(\mu'')g \rangle \, d\|E(g)\|^2.
\end{equation}

Thus, by a formula of change of variable in Lebesgue-Stieltjes integration and Fubini’s theorem, we have
\begin{align*}
\left\langle f, (T^{1/2}[R^\varepsilon_+^*, T^{-1}]R^\varepsilon_+^*) T^{1/2} + T^{-1/2}[R^\varepsilon_+^*, T]R^\varepsilon_+^* T^{-1/2}) f \right\rangle &= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \, d\|E(g)\|^2 \, d\langle f, E(\mu'')g \rangle \, d\langle E(\mu')g, f \rangle \\
&\quad \times \frac{(\mu - \mu')(\mu - \mu'')}{(\mu''^2 - \mu^2 + i\epsilon)(\mu''^2 - \mu^2 - i\epsilon)D_+(\mu''^2 - E_0^2)D_-(\mu''^2 - E_0^2)}.
\end{align*}

Then it is easy to see that for any $\mu, \mu', \mu'' \in [E_0, \infty)$
\begin{align*}
\lim_{\varepsilon \downarrow 0} \frac{(\mu - \mu')(\mu - \mu'')}{(\mu''^2 - \mu^2 + i\epsilon)(\mu''^2 - \mu^2 - i\epsilon)D_+(\mu''^2 - E_0^2)D_-(\mu''^2 - E_0^2)} &= \frac{1}{(\mu' + \mu)(\mu'' + \mu)D_+(\mu''^2 - E_0^2)D_-(\mu''^2 - E_0^2)}.
\end{align*}

For any $\varepsilon > 0$ and $\mu, \mu', \mu'' \in [E_0, \infty)$, we have by Lemma 3.3 and the arithmetic-geometric mean inequality
\begin{align*}
\left| \frac{(\mu - \mu')(\mu - \mu'')}{(\mu''^2 - \mu^2 + i\epsilon)(\mu''^2 - \mu^2 - i\epsilon)D_+(\mu''^2 - E_0^2)D_-(\mu''^2 - E_0^2)} \right| &\leq \frac{1}{4\delta^2 \mu \sqrt{\mu'} \mu''}.
\end{align*}

On the other side, for any $\alpha, \beta \in \mathbb{C}$, we see
\begin{align*}
\int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu \mu' \mu''} \, d\|E(g)\|^2 \, d\langle f + \alpha g \rangle \, d\|E(g)\|^2 \, d\langle f + \beta g \rangle \, d\|E(g)\|^2 &\leq \|T^{-1/2} g\|^2 \|T^{-1/4} (f + \alpha g)\|^2 \|T^{-1/4} (f + \beta g)\|^2 < \infty.
\end{align*}
Thus, by the Lebesgue dominated convergence theorem, we have

\[
\lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\|E(\mu)g\|^2 d\|E(\mu')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2 \\
\times \frac{\mu - \mu'}{\mu^2 - \mu^2 + i\varepsilon} \frac{\mu - \mu''}{\mu^2 - \mu^2 - i\varepsilon} D_+(\mu^2 - E_0^2) D_-(\mu^2 - E_0^2) = \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\|E(\mu')g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2 \\
\times \frac{1}{(\mu' + \mu)(\mu'' + \mu) D_+(\mu^2 - E_0^2) D_-(\mu^2 - E_0^2)}.
\]

In particular, for each \( \alpha, \beta = \pm 1, \pm i \), the polarization identity and Fubini’s theorem give

\[
\langle f, V^* Vf \rangle = \frac{\lambda^2}{4} \int_{[E_0, \infty)} \left| \langle f, R_{-\mu}(T)D_-(T^2 - E_0^2)^{-1}g \rangle \right|^2 d\|E(\mu)g\|^2.
\]

Let \( \{e_n\}_{n=1}^\infty \subset D(T^{-1/2}) \cap D(T^{1/2}) \) be a CONS of \( \mathcal{H} \). The termwise integration implies that

\[
\sum_{n=1}^\infty \langle e_n, V^* V e_n \rangle = \frac{\lambda^2}{4} \int_{[E_0, \infty)} \| R_{-\mu}(T)D_-(T^2 - E_0^2)^{-1}g \|^2 d\|E(\mu)g\|^2 \\
= \frac{\lambda^2}{4} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{(\mu' + \mu)^2 |D_+(\mu^2 - E_0^2)|^2} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 \quad (4.8)
\]

\[
\leq \frac{\lambda^2}{16\delta^2} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu' \mu} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 < \infty,
\]

where we have used the arithmetic-geometric mean and Lemma 3.5. Hence \( V \) is Hilbert-Schmidt. \( \square \)

**Lemma 4.5.** If \( \lambda > \lambda_c \), then there is a unitary operator \( U \) on \( \mathcal{F}_b(\mathcal{H}) \) such that for all \( f \in \mathcal{H} \),

\[
UB(f)U^{-1} = A(f).
\]

**Proof.** By Lemma 4.3 and Lemma 4.4 we can apply Theorem 2.2. \( \square \)

### 5 Analysis in the case \( \lambda > \lambda_c \)

In this section we prove Theorem 2.10 (1). Before starting the proof, we need to know a property of the Hamiltonian \( H(\lambda) \).
5.1 Time evolution

Theorem 5.1 (Time evolution). If \( \lambda > \lambda_{c,0} \), then for all \( f \in D(T^{-1/2}) \), \( \psi \in D(d\Gamma_b)(T)^{1/2}) \) and \( t \in \mathbb{R} \),

\[
e^{i \lambda H(t)} B(f) e^{-i \lambda H(t)} \psi = B(e^{i t T}) \psi, \tag{5.1}
\]
\[
e^{i \lambda H(t)} B(f)^* e^{-i \lambda H(t)} \psi = B(e^{i t T})^* \psi. \tag{5.2}
\]

Proof. It is sufficient to prove (5.1), because (5.2) follows from taking the adjoint of (5.1).

We define a function \( v : \mathbb{R} \to \mathbb{C} \) by \( v(t) := \langle \phi, e^{i \lambda H(t)} B(e^{-i t T} f) e^{-i \lambda H(t)} \psi \rangle, t \in \mathbb{R} \) for any \( f \in D(T^{-1/2}) \cap D(T) \) and \( \psi, \phi \in D(d\Gamma_b(T)) \). Then \( v \) is well-defined by operational calculus and Theorem 2.3. The function \( v \) is differentiable and, by Theorem 4.1 (2), we have for any \( t \in \mathbb{R} \),

\[
\frac{d}{dt} v(t) &= i \langle H(\lambda) e^{-it H(\lambda)} \phi, B(e^{-it T} f) e^{-it H(\lambda)} \psi \rangle - i \langle B(e^{-it T} f)^* e^{-it H(\lambda)} \phi, H(\lambda) e^{-it H(\lambda)} \psi \rangle \\
& \quad + i \langle e^{-it H(\lambda)} \phi, B(T e^{-it T} f) e^{-it H(\lambda)} \psi \rangle \\
&= 0.
\]

Hence \( v(t) = v(0) \) for all \( t \in \mathbb{R} \). Hence the equation

\[
\langle \phi, e^{i \lambda H(t)} B(e^{-i t T} f) e^{-i \lambda H(t)} \psi \rangle = \langle \phi, B(f) \psi \rangle
\]

holds for all \( t \in \mathbb{R} \). By replacing \( f \) with \( e^{i t T} f \), one has for all \( \psi \in D(d\Gamma_b(T)) \),

\[
e^{i \lambda H(t)} B(f) e^{-i \lambda H(t)} \psi = B(e^{i t T} f) \psi.
\]

Since \( D(d\Gamma_b(T)) \) is a core of \( (H(\lambda) + M)^{1/2} \) and \( D(H(\lambda) + M)^{1/2} = D(d\Gamma_b(T)^{1/2}) \) by Theorem 2.3 (3), we obtain (5.1) for \( f \in D(T^{-1/2}) \cap D(T) \) and \( \psi \in D(d\Gamma_b(T)^{1/2}) \). Finally we extend (5.1) for all \( f \in D(T^{-1/2}) \) and \( \psi \in D(d\Gamma_b(T)^{1/2}) \). Then we set \( f_n := E((-\infty, n]) f \) for each \( n \in \mathbb{N} \). Then \( f_n \in D(T^{-1/2}) \cap D(T) \) for all \( n \in \mathbb{N} \) and one can easily show that \( f_n \to f \), \( T^{-1/2} f_n \to T^{-1/2} f \) as \( n \to \infty \) by using functional calculus and the Lebesgue dominated convergence theorem. Thus we have \( U f_n \to U f \), \( JV f_n \to JV f \) as \( n \to \infty \) by the boundedness of \( U \) and \( V \). By using the linearity of the Hilbert transform and that of the map \( f \mapsto \psi_{g,f} \), (3.11) and (3.12), and (3.6), we can show that \( T^{-1/2} U f_n \to T^{-1/2} U f \), \( T^{-1/2} JV f_n \to T^{-1/2} JV f \) as \( n \to \infty \). Therefore we obtain \( B(f_n) \phi \to B(f) \phi \) and \( B(e^{i t T} f_n) \phi \to B(e^{i t T} f) \phi \) as \( n \to \infty \) for any \( \phi \in D(d\Gamma_b(T)^{1/2}) \) by [3, Lemma4-28]. By the preceding result, we have for any \( n \in \mathbb{N} \),

\[
B(f_n) e^{-i \lambda H(\lambda)} \psi = e^{-i \lambda H(\lambda)} B(e^{i t T} f_n) \psi.
\]
The equation $D(d\Gamma_b(T)^{1/2}) = D((H(\lambda) + M)^{1/2})$ in Theorem 2.3 (3) implies that

$$e^{-itH(\lambda)}D(d\Gamma_b(T)^{1/2}) = D(d\Gamma_b(T)^{1/2}).$$

Hence, taking the limit $n \to \infty$, we obtain (5.1) for $f \in D(T^{-1/2})$, $\psi \in D(d\Gamma_b(T)^{1/2})$. □

5.2 Proof of Theorem 2.10 (1)

In this subsection, we assume that $\lambda > \lambda_c$.

**Lemma 5.2.** Let $\Omega := \mathbb{U}^{-1}\Omega_0$, where $\mathbb{U}$ is the unitary operator in Lemma 4.3 and $\Omega_0 := (1,0,0,\ldots) \in \mathcal{F}_b(\mathcal{H})$ is the Fock vacuum. Then there is an eigenvalue $E_g$ of $H(\lambda)$ and $\Omega$ is a corresponding eigenvector: $H(\lambda)\Omega = E_g\Omega$.

**Proof.** In general, by [3, Proposition 4-4] for a dense subspace $\mathcal{D} \subset \mathcal{H}$, if $\psi \in \cap_{f \in \mathcal{D}}D(A(f))$ satisfies $A(f)\psi = 0$ for all $f \in \mathcal{D}$, then there is a constant $\alpha \in \mathbb{C}$ such that $\psi = \alpha\Omega_0$. Thus, by Lemma 4.5, $B(f)\phi = 0$ for all $f \in D(T^{-1/2})$. Hence there is a constant $\alpha \in \mathbb{C}$ such that $\phi = \alpha\Omega$. For any $f \in D(T^{-1/2})$ and $t \in \mathbb{R}$,

$$B(f)e^{-itH(\lambda)}\Omega = e^{-itH(\lambda)}B(e^{itf})\Omega = 0$$

by Lemma 5.1. Thus, for each $t \in \mathbb{R}$, there is a constant $\alpha(t) \in \mathbb{C}$ such that $e^{-itH(\lambda)}\Omega = \alpha(t)\Omega$. Then we have $|\alpha(t)| = 1$, $\alpha(t + s) = \alpha(t)\alpha(s)$ for all $t, s \in \mathbb{R}$, since $\{e^{-itH(\lambda)}\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group. Thus there exists a constant $E_g \in \mathbb{R}$ such that $\alpha(t) = e^{-itE_g}, t \in \mathbb{R}$. The differentiation of the equation $e^{-itH(\lambda)}\Omega = e^{-itE_g}\Omega$ in $t$ implies $\Omega \in D(H(\lambda))$ and $\Omega \in \text{Ker}(H(\lambda) - E_g)$. □

**Proof of Theorem 2.10 (1).**

The subspace $\mathcal{U} := \mathcal{L}\{B(f_1)^* \cdots B(f_n)^*\Omega, \Omega \mid f_j \in D(T^{-1/2}), j = 1, \ldots, n, n \in \mathbb{N}\}$ is dense in $\mathcal{F}_b(\mathcal{H})$ by the fact that $\mathcal{U} = \mathbb{U}^{-1}\mathcal{F}_{b,\text{fin}}(D(T^{-1/2}))$, where $\mathcal{L}(\mathcal{D})$ denotes the subspace algebraically spanned by the vectors in $\mathcal{D}$. By Lemma 5.1 and Lemma 9.3, for any $t \in \mathbb{R}, f_j \in D(T^{-1/2}), j = 1, \ldots, n$, we have

$$e^{itH(\lambda)}B(f_1)^* \cdots B(f_n)^*\Omega = B(e^{itf_1})^* \cdots B(e^{itf_n})^*e^{itH(\lambda)}\Omega$$

$$= B(e^{itf_1})^* \cdots B(e^{itf_n})^*e^{itE_g}\Omega$$

$$= e^{itE_g}\mathbb{U}^{-1}e^{itd\Gamma_b(T)}A(f_1)^* \cdots A(f_n)^*\Omega_0$$

$$= \mathbb{U}^{-1}e^{it(d\Gamma_b(T)+E_g)}\mathbb{U}B(f_1)^* \cdots B(f_n)^*\Omega.$$
By this equation and a limiting argument, we obtain $U e^{itH(\lambda)} U^{-1} = e^{it(d\Gamma_b(T) + E_g)}$. By the unitary covariance of functional calculus, we have

$$U e^{itH(\lambda)} U^{-1} = e^{itU H(\lambda) U^{-1}}, \quad t \in \mathbb{R}.$$ 

Hence (2.7) holds. The equation (2.7) and the well-known spectral properties of $d\Gamma_b(T)$ imply that $E_g$ is the ground state energy of $H(\lambda)$ and $\Omega$ is the unique ground state of $H(\lambda)$.

Lemma 5.3. The ground state energy $E_g$ is given as follows:

$$E_g = \frac{\lambda}{4} \|g\|^2 - \text{Tr}(T^{1/2}V^*VT^{1/2}), \quad (5.3)$$

$$\text{Tr}(T^{1/2}V^*VT^{1/2}) = \frac{\lambda^2}{4} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu}{(\mu + \mu')^2 |D_-(\mu^2 - E_0^2)|^2} d\|E(\mu)g\|^2 d\|E(\mu')g\|^2. \quad (5.4)$$

Proof. The operator $U$ leaves $D(d\Gamma_b(T))$ invariant by Theorem 2.10 (1). In particular, $U \Omega_0 \in D(d\Gamma_b(T)^{1/2})$. Thus, by Lemma 9.4, the isometry of $U$ and the definition of $B(\cdot)$, we have

$$\langle \Omega_0, (H(\lambda) - E_g) \Omega_0 \rangle = \text{Tr}(T^{1/2}V^*VT^{1/2}).$$

It is easy to see that $\langle \Omega_0, H(\lambda) \Omega_0 \rangle = \lambda \|g\|^2/4$. Hence (5.3) holds. Formula (5.4) can be proved in the same way as in (4.8).

6 Analysis in the case $\lambda_{c,0} < \lambda < \lambda_c$

In Section 5, we proved Theorem 2.10 (1). But the proof is valid only for the case $\lambda > \lambda_c$. Therefore it is necessary to find another pair of operators $U$ and $V$ if one wants to use a Bogoliubov transformation for the spectral analysis of $H(\lambda)$ in the case $\lambda \leq \lambda_c$. In this section we assume that $T$ and $g$ satisfy Assumption 2.7, $E_0 > 0$ and $\lambda_{c,0} < \lambda < \lambda_c$. Under these conditions, we can define operators $\xi, X, Y$ and $T_{\pm}$ as follows:

$$\xi := \Omega_+ T \Omega_+^* + \beta P,$$

$$X := U \Omega_+^* + T_+ P, \quad Y := V \Omega_+^* + T_- P,$$

$$T_{\pm} := \frac{1}{2} (\beta^{1/2} T^{-1/2} \pm \beta^{-1/2} T^{1/2}),$$

where $\beta := (E_0^2 + x_0)^{1/2}$.

Remark 6.1. The definition of $x_0$ implies the following:

$$E_0^2 + x_0 \begin{cases} > 0, & \text{if } \lambda_{c,0} < \lambda < \lambda_c, \\ = 0, & \text{if } \lambda = \lambda_{c,0}, \\ < 0, & \text{if } \lambda < \lambda_{c,0}. \end{cases}$$
Thus, in the case $\lambda_{c,0} \leq \lambda \leq \lambda_c$, we see that the inequality $0 < \beta < E_0$ holds. Let

$$C(f) := A(Xf) + A(JYf)^*, f \in \mathcal{H}.$$  

Then $C(f)$ is a densely defined closable operator. We denotes its closure by the same symbol.

### 6.1 Properties of $X, Y$ and $\xi$

In this subsection we study operators $X, Y$ and $\xi$. First, we consider $\xi$. Let be

$$\tilde{T} := \Omega_+ T \Omega_+^*.$$

**Lemma 6.2.** The operator $\tilde{T}$ is a self-adjoint operator with $D(\tilde{T}) = D(T)$.

**Proof.** By Lemma 3.8 we see that $D(\tilde{T}) = D(T)$. Hence $\tilde{T}$ is symmetric. For any $\phi \in \mathcal{D}(\tilde{T}^*)$ and $\psi \in D(\tilde{T}) = D(T)$, we have $\langle \Omega_+^* (\tilde{T}^*)^* \phi, \psi \rangle = \langle \Omega_+^* \phi, T \psi \rangle$. This implies that $\Omega_+^* \phi \in D(T)$. Hence $\tilde{T}$ is self-adjoint. \hfill \Box

**Lemma 6.3.** The spectra of $\tilde{T}$ are as follows:

$$\sigma(\tilde{T}) = \{0\} \cup \sigma(T), \sigma_{ac}(\tilde{T}) = \sigma(T), \sigma_p(\tilde{T}) = \{0\}, \sigma_{sc}(\tilde{T}) = \emptyset.$$

**Proof.** We define a family of projection operators $\{E_P(B) \mid B \in \mathcal{B}^1\}$ on $\mathcal{H}$ as follows: $E_P(B) = 0$ if $0 \notin B$ and $E_P(B) = P$ if $0 \in B$ for each $B \in \mathcal{B}^1$. It is easy to see that $\{E_T(B) := \Omega_+ E(B) \Omega_+^* + E_P(B) \mid B \in \mathcal{B}^1\}$ is a spectral measure. Using functional calculus, we see that $E_T(\cdot)$ is the spectral measure of $\tilde{T}$. It is easy to see that the absolutely continuous part (resp. singular part) of $\tilde{T}$ is $\tilde{T} \upharpoonright \text{Ran}(I - P)$ (resp. $\tilde{T} \upharpoonright \text{Ran}(P)$) since $T$ is absolutely continuous and $\Omega_\pm$ are partial isometries. Thus we see $\sigma(\tilde{T}) = \{0\} \cup \sigma_{ac}(\tilde{T}), \sigma_p(\tilde{T}) = \{0\}, \sigma_{sc}(\tilde{T}) = \emptyset$.

We next show that $\sigma_{ac}(\tilde{T}) = \sigma(T)$. For any $\mu \in \sigma(T)$, there is a sequence $\psi_n \in D(T), n \in \mathbb{N}$ such that $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \|(T - \mu)\psi_n\| = 0$. For each $n \in \mathbb{N}$, there is a $\phi_n \in \text{Ran}(I - P)$ such that $\psi_n = \Omega_+^* \phi_n$. Then $\|\phi_n\| = \|\Omega_+ \psi_n\| = \|\psi_n\| = 1$ and $\|(\tilde{T} - \mu)\phi_n\| = \|(T - \mu)\psi_n\| \to 0$ as $n \to \infty$. Thus we have $\mu \in \sigma(\tilde{T} \upharpoonright \text{Ran}(I - P)) = \sigma_{ac}(\tilde{T})$.

For any $\mu \in \sigma_{ac}(\tilde{T})$, there is a sequence $\eta_n \in D(\tilde{T}) \cap \text{Ran}(I - P)$ such that $\|\eta_n\| = 1$ and $\lim_{n \to \infty} \|(\tilde{T} - \mu)\eta_n\| = 0$. Then we easily see that $\Omega_+^* \eta_n \in D(T)$ for all $n \in \mathbb{N}$. The equation $\Omega_+ \Omega_+^* \eta_n = \eta_n$ implies that $\|\Omega_+^* \eta_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|(T - \mu) \Omega_+^* \eta_n\| = \|(\tilde{T} - \mu) \eta_n\| \to 0, \quad n \to \infty.$$

Thus $\mu \in \sigma(T)$. Hence $\sigma_{ac}(\tilde{T}) = \sigma(T)$. \hfill \Box
Lemma 6.4. The operator $\xi$ is an injective, non-negative self-adjoint operator with $D(\xi) = D(T)$ and we have the following equations:

$$\sigma(\xi) = \{\beta\} \cup \sigma(T), \sigma_{ac}(\xi) = \sigma(T), \sigma_p(\xi) = \{\beta\}, \sigma_{sc}(\xi) = \emptyset.$$  \hfill (6.1)

In particular, $\beta$ is the ground state energy of $\xi$, which is an isolated eigenvalue of $\xi$, and $U_b$ is the unique ground state of $\xi$.

Proof. By Lemma 6.3 and a spectral property of direct sum of self-adjoint operators, we have the equation (6.1). Thus $\beta$ is an isolated ground state energy by Remark 6.1. It is easy to see that $U_b$ is a ground state of $\xi$. Assume that $f \in \text{Ker}(\xi - \beta)$ satisfies $(I - P)f \neq 0$. Then $\Omega^*_+, f \neq 0$ by Lemma 3.13. This implies $T\Omega^*_+, f = \beta \Omega^*_+, f$, but this contradicts Assumption 2.7 (1). Hence $(I - P)f = 0$, implying that the ground state of $\xi$ is unique.

Lemma 6.5. The operators $\xi^{\pm 1/2}$ are given by

$$\xi^{1/2} = \Omega^+_+ T^{1/2} \Omega^*_+ + \beta^{1/2} P,$$  \hfill (6.2)

$$\xi^{-1/2} = \Omega^+_+ T^{-1/2} \Omega^*_+ + \beta^{-1/2} P$$  \hfill (6.3)

with $D(\xi^{\pm 1/2}) = D(T^{\pm 1/2})$.

Proof. We can show in the same way as in the proof of Lemma 6.4 that the right hand side of (6.2) is non-negative, self-adjoint operator with its domain $D(T^{1/2})$. We have $\xi \subset (\Omega^+_+ T^{1/2} \Omega^*_+ + \beta^{1/2} P)^2$. Since a self-adjoint operator has no non-trivial symmetric extension, (6.2) holds. In the same way as in the case of (6.2), we can show that the right hand side of (6.3) is a self-adjoint operator. We have $D(\Omega^+_+ T^{-1/2} \Omega^*_+ + \beta^{-1/2} P) \subset \text{Ran}(\xi^{1/2})$ and $\xi^{1/2}(\Omega^+_+ T^{-1/2} \Omega^*_+ + \beta^{-1/2} P) = I$ on $D(\Omega^+_+ T^{-1/2} \Omega^*_+)$. Hence $\Omega^+_+ T^{-1/2} \Omega^*_+ + \beta^{-1/2} P \subset \xi^{-1/2}$. Thus the equation (6.3) holds.

Next, we study $X$ and $Y$.

Lemma 6.6. The operators $X^\sharp$ and $Y^\sharp$ leave $D(T^{-1/2})$ (resp. $D(T^{1/2}), D(T)$) invariant.

Proof. The assertion follows from Lemma 3.8, Lemma 3.16, Lemma 6.5 and the definition of $X$ and $Y$.

Lemma 6.7. The following equations hold:

$$\begin{cases} 
X^* X - Y^* Y = I, \\
X^*_j Y - Y^*_j X = 0, \\
XX^* - Y_j Y^*_j = I, \\
XY^* - Y_j X^*_j = 0. 
\end{cases}$$  \hfill (6.4)
Proof. The operator $P$ (resp. $T_\pm$) satisfies $P_f = P$ (resp. $(T_\pm)_f = T_\pm$). By (3.9), we have $\Omega_2 U_b = 0$. Hence we obtain $(U^* \pm V^*) T^{\pm 1/2} U_b = 0$ and $(U^* T_\pm - V^* T_\mp) U_b = 0$. The equations $T_+ T_+ - T_- T_- = I$ and $T_+ T_- - T_- T_+ = 0$ hold on $D(T^{-1}) \cap D(T)$. By (4.6) and direct calculations, we have $X^* X - Y^* Y = I$ and $X^* J - Y^* J = 0$. By similar calculations, we have $XX^* - Y J Y^* J = I$ and $XY^* - Y J X^* J = 0$. Then, by a limiting argument, we obtain (6.4).

Lemma 6.8. The operator $Y$ is Hilbert-Schmidt.

Proof. We can easily show that the assertion follows from Lemma 4.4, Lemma 6.6 and the choice a CONS $\{e_n\}_{n=0}^\infty \subset D(T^{1/2}) \cap D(T)$ with $e_0 = U_b$.

Lemma 6.9. There is a unitary operator $\mathcal{V}$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $f \in \mathcal{H}$,

$$\mathcal{V} C(f) \mathcal{V}^{-1} = A(f).$$

Proof. By Theorem 2.2 (6.4) and Lemma 6.8, we can prove this assertion.

6.2 Commutation relations

Theorem 6.10. The following commutation relations hold:

1. For any $f \in D(T)$ and $\psi \in \mathcal{F}_b(\mathcal{H})$,

$$[H(\lambda), C(f)] \psi = -C(\xi f) \psi.$$

2. For any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(d\Gamma_b(T))$,

$$\langle H(\lambda) \phi, C(f) \psi \rangle - \langle C(f)^* \phi, H(\lambda) \psi \rangle = - \langle \phi, C(\xi f) \psi \rangle.$$

3. For any $f \in D(T^{-1/2}) \cap D(T)$, $C(f)$ maps $D(d\Gamma_b(T)^{3/2})$ into $D(d\Gamma_b(T))$ and for any $\psi \in D(d\Gamma_b(T)^{3/2})$,

$$[H(\lambda), C(f)] \psi = -C(\xi f) \psi.$$

Theorem 6.10 follows, in the same manner as in the proof of Theorem 4.1 from Lemma 3.16, Lemma 6.5 and the next lemma:

Lemma 6.11. For any $f \in D(T)$ the following equations hold:

$$-TXf + \frac{\lambda}{2} \langle (Y^* J - X^*) g, f \rangle g = -X\xi f,$$

$$TJYf + \frac{\lambda}{2} \langle f, (Y^* J - X^*) g \rangle g = -JY\xi f.$$
Remark 6.12. By Lemma 3.16 and the definition of $\xi$, the both sides of (6.5) and (6.6) have meaning.

Proof. Let be $a := \sqrt{\lambda/D(x_0)}$. Then we can see by the definition of $x_0$ and (4.5),
\[(Y^*J - X^*)g = -\Omega_+D_-(T^2 - E_0^2)^{-1}g + \frac{\beta^{-1/2}a}{\lambda}U_b.\]

We have
\[TT_\pm U_b = \frac{1}{2}(\beta^{1/2}T^{1/2}U_b \pm \beta^{-1/2}T^{3/2}U_b)\]
\[= \frac{1}{2}(\beta^{1/2}T^{1/2}U_b \pm \beta^{3/2}T^{-1/2}U_b \pm \beta^{-1/2}ag). \tag{6.7}\]

Thus, for any $f \in D(T)$, we have
\[-TXf + \frac{\lambda}{2}\langle(Y^*J - X^*)g, f\rangle g = -TU\Omega^*_+f - \frac{\lambda}{2}\langle D_-(T^2 - E_0^2)^{-1}g, \Omega_+^*f\rangle g - TT_+Pf + \frac{\beta^{-1/2}a}{2} \langle U_b, f \rangle g.\]

Then, by (4.4) and (6.7), we have
\[-TXf + \frac{\lambda}{2}\langle(Y^*J - X^*)g, f\rangle g = -UT\Omega^*_+f - \beta \langle U_b, f \rangle T_+U_b\]
\[= -X(\Omega_+T\Omega_+^* + \beta P)f.\]

Thus we obtain (6.5). Similarly one can prove (6.6). \(\square\)

6.3 Proof of Theorem 2.10 (2)

Theorem 6.13. For all $f \in D(T^{-1/2}), \psi \in D(d\Gamma_b(T)^{1/2})$ and $t \in \mathbb{R}$,
\[e^{itH(\lambda)}C(f)e^{-itH(\lambda)}\psi = C(e^{it\xi}f)\psi,\]
\[e^{itH(\lambda)}C(f)^*e^{-itH(\lambda)}\psi = C(e^{it\xi}f)^*\psi.\]

Proof. These are proved in the same way as in the proof of Theorem 5.1 and Theorem 6.10. \(\square\)

Lemma 6.14. Let $\Omega := \nabla^{-1}\Omega_0$ where $\nabla$ is the unitary operator in Lemma 6.9. Then:

(1) There is an eigenvalue $\tilde{E}_g$ of $H(\lambda)$ and $\Omega$ is an eigenvector of $H(\lambda)$ with eigenvalue $\tilde{E}_g$. 36
The following equation holds:

\[ V H(\lambda) V^{-1} = d\Gamma_b(\xi) + \tilde{E}_g. \]

(3) The constant \( \tilde{E}_g \) is given as follows:

\[ \tilde{E}_g = E_g - \beta \| T-U_b \|^2. \]  

(6.8)

Proof. Parts (1) and (2) can be proved in the same way as in the proof of Theorem \ref{thm:2.10} (1). We have

\[ \tilde{E}_g = \frac{\lambda}{4} \| g \|^2 - \text{Tr}(\xi^{1/2}Y^*Y\xi^{1/2}) \]

in the same way as in the proof of Lemma \ref{lem:5.2}. Then, by Lemma \ref{lem:6.5}, we have

\[ \xi^{1/2}Y^*Y\xi^{1/2} = \Omega_+ T^{1/2}V^*VT^{1/2}+\Omega_+ T^{1/2}V^*\beta^{1/2}T-P+\beta^{1/2}PTVT^{1/2}+\beta PT. \]

We choose a CONS \( \{e_n\}_{n=0}^{\infty} \subset D(T) \) satisfying \( e_0 = U_b \). Then it is easy to see that \( \{\Omega^*_+e_n\}_{n=1}^{\infty} \) is a CONS for \( \mathcal{H} \) by Lemma \ref{lem:3.13}. Hence we have

\[ \text{Tr}(\xi^{1/2}Y^*Y\xi^{1/2}) = \sum_{n=1}^{\infty} \langle e_n, \Omega_+ T^{1/2}V^*VT^{1/2}+\Omega_+ T^{1/2}V^*\beta^{1/2}T-P+\beta^{1/2}PTVT^{1/2}+\beta PT, e_n \rangle + \beta \| T-U_b \|^2 \]

Thus we obtain (6.8).

In particular, \( H(\lambda) \) have eigenvectors as follows:

\[ \phi_n := V^{-1}A(U_b)^n\Omega_0, \quad H(\lambda)\phi_n = (n\beta + \tilde{E}_g)\phi_n, \quad n \in \mathbb{N} \cup \{0\}. \]

Hence the spectral properties of \( H(\lambda) \) as stated in Theorem \ref{thm:2.10} (2) follow.

7 Analysis in the case \( \lambda < \lambda_{c,0} \).

In this section, we show that \( H(\lambda) \) is unbounded from above and below.

**Theorem 7.1.** Let \( g \in D(T^{-1/2}) \). Then \( H(\lambda) \) is unbounded above for any \( \lambda \in \mathbb{R} \). If \( \lambda < \lambda_{c,0} \), then \( H(\lambda) \) is unbounded below too.
Proof. For any $f \in D(T) \setminus \{0\}$, we set $\psi_n := a_n A(f)^* \Omega_0$, $a_n \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N} \cup \{0\}$. Then we have the following equations:

$$\begin{align*}
d\Gamma_b(T) \psi_n &= n \frac{a_n}{a_{n-1}} A(T f)^* \psi_{n-1}, \\
\|\psi_n\|^2 &= |a_n|^2 n! \|f\|^{2n}, \\
\|A(g) \psi_n\|^2 &= \|g\| \|\psi_n\|^2 + \|A(g) \psi_n\|^2.
\end{align*}$$

Then we have

$$\langle \psi_n, H(\lambda) \psi_n \rangle = \|\psi_n\|^2 \left( \frac{\lambda}{4} \|g\|^2 + n \frac{2 \|T^{1/4} f\|^2 + \lambda \langle g, f \rangle^2}{2 \|f\|^2} + \frac{\lambda}{2} \Re \frac{a_{n-2}}{a_n} \langle g, f \rangle^2 \right).$$

We take $f$ such that $\langle g, f \rangle = 0$. Then we have $\langle \psi_n, H(\lambda) \psi_n \rangle / \|\psi_n\|^2 \to \infty$ as $n \to \infty$ for any $\lambda \in \mathbb{R}$. Thus $H(\lambda)$ is unbounded above for any $\lambda \in \mathbb{R}$.

Let $\phi_N := \sum_{n=0}^{N} \psi_n$, $N = 0, 1, 2, \ldots$. Then we have $\|\phi_N\|^2 = \sum_{n=0}^{N} \|\psi_n\|^2$ and

$$\langle \phi_N, H(\lambda) \phi_N \rangle = \sum_{n=2}^{N} \|\psi_n\|^2 \left( \frac{\lambda \|g\|^2}{4} + n \frac{2 \|T^{1/4} f\|^2 + \lambda \langle g, f \rangle^2}{2 \|f\|^2} + \frac{\lambda}{2} \Re \frac{a_{n-2}}{a_n} \langle g, f \rangle^2 \right) + \|\phi_1\|^2 \left( \frac{\lambda \|g\|^2}{4} + \frac{\|T^{1/4} f\|^2 + \lambda \langle g, f \rangle^2}{2 \|f\|^2} \right) + \frac{\lambda}{4} \|\psi_0\|^2 \|g\|^2.$$

Let $a_0 := 1$, $a_n := n^{-3/4} n^{-1/2}$, $n \in \mathbb{N}$ and, for any $0 < \delta$, $0 < \varepsilon < 1$,

$$f = f_\delta := \frac{T^{-1} E((\delta, \infty)) g}{\|T^{-1} E((\delta, \infty)) g\|},$$

$$c_\lambda(\varepsilon, \delta) := \|T^{1/4} f_\delta\|^2 \left( 1 + \frac{\lambda}{2} (2 - \varepsilon) \|T^{-1/4} E((\delta, \infty)) g\|^2 \right).$$

Then $\sum_{n=0}^{\infty} \|\psi_n\|^2$ converges and, for any $N \in \mathbb{N}$,

$$\langle \phi_N, H(\lambda) \phi_N \rangle = \sum_{n=2}^{N} \|\psi_n\|^2 n c_\lambda(\varepsilon, \delta) + \frac{\lambda}{2} \sum_{n=2}^{N} \|\psi_n\|^2 \left( \frac{a_{n-2}}{a_n} - n (1 - \varepsilon) \right) \langle g, f_\delta \rangle^2 + C_N, \tag{7.1}$$

where

$$C_N := \frac{\lambda \|g\|^2}{4} \sum_{n=0}^{N} \|\psi_n\|^2 + \|\phi_1\|^2 \left( \|T^{1/4} f_\delta\|^2 + \frac{\lambda}{2} \langle g, f_\delta \rangle^2 \right).$$

For all $0 < \delta, 0 < \varepsilon < 1$, we have

$$2 \left( \frac{1}{\|T^{-1/4} E((\delta, \infty)) g\|^2 (2 - \varepsilon)} \right) < \lambda_{c,0}. \tag{7.2}$$

The left hand side of (7.2) tends to $\lambda_{c,0}$ as $\varepsilon, \delta \downarrow 0$. Since $\lambda < \lambda_{c,0}$, we can take a pair $(\varepsilon, \delta)$ satisfying $c_\lambda(\varepsilon, \delta) < 0$. We fix such a pair. There is a $n_0 \in \mathbb{N}$ such that $a_{n-2}/a_n - n (1 - \varepsilon) > 0$ for all $n \geq n_0$. Hence we can see that $\langle \phi_N, H(\lambda) \phi_N \rangle / \|\phi_N\|^2$ tends to $-\infty$ as $N \to \infty$, because the first term on the right hand side of (7.1) tends to $-\infty$ as $N \to \infty$. \hfill \Box
8 Generalization of the $\phi^2$-model

In this section we consider $H(\eta, \lambda)$ defined in Subsection 2.3.

**Assumption 8.1.** We need the following assumptions:

1. $f \in D(T^{1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2})$,
2. $f \in D(T^{-1})$ and $\text{Re} \langle T^{-1}f, g \rangle = 0$,
3. $f, g \in D(T^{-1})$ and $\text{Re} \langle T^{-1}f, g \rangle \neq 0$.

We can prove a slight generalization of Theorem 2.10.

**Theorem 8.2.** Let $\mathcal{H}$ be separable. Then the following (1)-(5) hold:

1. Suppose that Assumption 2.7 and Assumption 8.1 (2) or (3) hold. Let $\lambda > \lambda_c$. Then there is a unitary operator $U$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta \in \mathbb{R}$,
   \[
   U H(\eta, \lambda) U^{-1} = d\Gamma_b(T) + E_g + E_{f,g},
   \]
   where the constant $E_{f,g} \in \mathbb{R}$ is defined by
   \[
   E_{f,g} = -\frac{\eta^2}{2} \| T^{-1/2} f \|^2 + \frac{(\text{Re} \langle T^{-1}f, g \rangle)^2 \eta^2 \lambda}{2(1 + \lambda \| T^{-1/2} g \|^2)}.
   \]
2. Suppose that Assumption 2.7 and Assumption 8.1 (2) or (3) hold. Let $E_0 > 0$ and $\lambda > \lambda_c$. Then there are a unitary operator $V$ on $\mathcal{F}_b(\mathcal{H})$ and a non-negative, injective self-adjoint operator $\xi$ on $\mathcal{H}$ such that, for all $\eta \in \mathbb{R}$,
   \[
   V H(\eta, \lambda) V^{-1} = d\Gamma_b(\xi) + E_g - E_b + E_{f,g}.
   \]
3. Let $T$ be a non-negative, injective self-adjoint operator and suppose that $f$ and $g$ satisfy Assumption 8.1 (1) and (2). Then there is a unitary operator $W$ on $\mathcal{F}_b(\mathcal{H})$ such that, for all $\eta \in \mathbb{R}$,
   \[
   W H(\eta, \lambda, 0) W^{-1} = H(\lambda, 0) - \frac{\eta^2}{2} \| T^{-1/2} f \|^2.
   \]
4. Let $T$ be a non-negative, injective self-adjoint operator and suppose that $f$ and $g$ satisfy Assumption 8.1 (1) and (3). Then, for all $\eta \in \mathbb{R}\{0\}$,
   \[
   \sigma(H(\eta, \lambda, 0)) = \mathbb{R}, \quad \sigma_p(H(\eta, \lambda, 0)) = \emptyset.
   \]
(5) Let $T$ be a non-negative, injective self-adjoint operator and suppose that $f$ and $g$ satisfy Assumption \ref{assumption} (1). Moreover, suppose that Assumption \ref{assumption} (2) or (3) holds. Let $\lambda < \lambda_{c,0}$. Then, for all $\eta \in \mathbb{R}$, $H(\eta, \lambda)$ is unbounded above and below.

It is easy to see that Theorem \ref{thm} is proved by the following lemma and facts in Theorem \ref{fact}. 

**Lemma 8.3.** Let $T$ be a non-negative, injective self-adjoint operator, $f \in D(T^{-1})$ and $g \in D(T^{-1/2}) \cap D(T)$.

1. Let $\text{Re} \langle T^{-1}f, g \rangle = 0$. Then there is a unitary operator $U_1$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta, \lambda \in \mathbb{R}$,
   \[ U_1 H(\eta, \lambda) U_1^{-1} = H(\lambda) - \frac{\eta^2}{2} \|T^{-1/2}f\|^2. \]  
   \[ (8.1) \]

2. Let $\text{Re} \langle T^{-1}f, g \rangle \neq 0$ and $g \in D(T^{-1})$.
   \[ (i) \] If $\lambda \neq \lambda_{c,0}$, then there is a unitary operator $U_2$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta \in \mathbb{R}$,
   \[ U_2 H(\eta, \lambda) U_2^{-1} = H(\lambda) + E_{f,g}. \]
   \[ (ii) \] If $\lambda = \lambda_{c,0}$, then for all $\eta \in \mathbb{R} \setminus \{0\}$,
   \[ \sigma(H(\eta, \lambda_{c,0})) = \mathbb{R}, \quad \sigma_p(H(\eta, \lambda_{c,0})) = \emptyset. \]  
   \[ (8.2) \]

**Proof.** Let $U_1 := e^{-i \Phi_s(i \eta T^{-1}f)}$ for all $\eta \in \mathbb{R}$. Then, by direct calculation, we obtain
   \[ U_1 H(\eta, \lambda) U_1^{-1} = H(\lambda) - \frac{\eta^2}{2} \|T^{-1/2}f\|^2 - \lambda \eta \Phi_s(g) + \frac{\lambda}{2} \eta^2 \kappa^2 \]  
   \[ (8.3) \] on $\mathcal{F}_{b, \text{fin}}(D(T))$ for all $\eta, \lambda \in \mathbb{R}$, where $\kappa := \text{Re}(T^{-1}f, g)$. In the case of \[ (1) \], we have \ref{assumption} by $\kappa = 0$ and a limit argument. Next, we prove \ref{assumption}. We assume that $g \in D(T^{-1})$ and $\text{Re} \langle T^{-1}f, g \rangle \neq 0$. Let $V_1 := e^{i \Phi_s(\alpha T^{-1}g)}$ for any $\alpha \in \mathbb{R}$ and define a unitary operator $U_2 := V_1 U_1$. Then
   \[ U_2 H(\eta, \lambda) U_2^{-1} = H(\lambda) + \left( \alpha + \lambda \alpha \|T^{-1/2}g\|^2 - \lambda \eta \kappa \right) \Phi_s(g) \]
   \[ - \frac{\eta^2}{2} \|T^{-1/2}f\|^2 + \frac{\lambda}{2} \eta^2 \kappa^2 + \frac{\alpha}{2} \|T^{-1/2}g\|^2 \left( \alpha + \lambda \alpha \|T^{-1/2}g\|^2 - 2 \lambda \eta \kappa \right) \]
   on $\mathcal{F}_{b, \text{fin}}(D(T))$ in the same way as in \ref{assumption}. For $\lambda \neq \lambda_{c,0}$, let $\alpha = \lambda \eta \kappa (1 + \lambda \|T^{-1/2}g\|^2)^{-1}$. Then we obtain
   \[ U_2 H(\eta, \lambda) U_2^{-1} = H(\lambda) - \frac{\eta^2}{2} \|T^{-1/2}f\|^2 + \frac{\lambda \eta^2 \kappa^2}{2(1 + \lambda \|T^{-1/2}g\|^2)} \]  
   \[ (8.4) \]

40
by a limit argument. If \( \lambda = \lambda_{c,0} \), then, for all \( \eta, \alpha \in \mathbb{R} \), we have

\[
\mathbb{U}_2 H(\eta, \lambda_{c,0}) \mathbb{U}_2^{-1} = H_g(-\kappa \eta \lambda_{c,0}, \lambda) - \frac{\eta^2}{2} \|T^{-1/2} f\|^2 + \frac{\lambda_{c,0} \eta^2 \kappa^2}{2} + \kappa \eta \alpha
\]

in the same way as in (8.4), where \( H_g(\nu, \lambda_{c,0}) := H(\lambda_{c,0}) + \nu \Phi_s(g) \) for all \( \nu \in \mathbb{R} \). It is easy to see that \( \sigma(H_g(\nu, \lambda_{c,0})) = \mathbb{R} \) and \( \sigma_p(H_g(\nu, \lambda_{c,0})) = \emptyset \) for all \( \nu \in \mathbb{R} \setminus \{0\} \), because \( \mathcal{V}_1 H_g(\nu, \lambda_{c,0}) \mathcal{V}_1^{-1} = H_g(\nu, \lambda_{c,0}) + \nu \alpha \|T^{-1/2} g\|^2 \) and \( \alpha \in \mathbb{R} \) is arbitrary. Hence we have (8.2).

\[\square\]

**Remark 8.4.** If \( \mathcal{H} \) is separable, then the condition \( g \in D(T^{-1/2}) \cap D(T) \) in the above lemma is weakened to the condition \( g \in D(T^{-1/2}) \cap D(T^{1/2}) \).

## 9 Appendix

In this section, we recall some known facts in Fock space theory. Let \( T \) be a non-negative, injective self-adjoint operator on \( \mathcal{H} \).

**Lemma 9.1.** \([5, \text{Theorem 5.16.}]\) Let \( f \in D(T^{-1/2}) \) and \( \psi \in D(\text{d} \Gamma_b(T)^{1/2}) \). Then \( \psi \in D(A(f)) \cap D(A(f)^*) \) and the following inequalities hold:

\[
\|A(f)\psi\| \leq \|T^{-1/2} f\| \|\text{d} \Gamma_b(T)^{1/2}\| \psi\|, \tag{9.1}
\]

\[
\|A(f)^*\psi\|^2 \leq \|T^{-1/2} f\|^2 \|\text{d} \Gamma_b(T)^{1/2}\| \psi\|^2 + \|f\|^2 \|\psi\|^2. \tag{9.2}
\]

**Lemma 9.2.** \([5, \text{Proposition 5.10.}]\) For any \( f \in D(T) \), the following commutation relations on \( \mathcal{F}_{b, \text{fin}}(D(T)) \) :

\[
[\text{d} \Gamma_b(T), A(f)] = -A(T f), \quad [\text{d} \Gamma_b(T), A(f)^*] = A(T f)^*. \tag{9.3}
\]

**Lemma 9.3.** \([5, \text{Lemma 5.21.}]\) For any \( t \in \mathbb{R}, f \in \mathcal{H} \), the following equation holds:

\[
e^{it \text{d} \Gamma_b(T)} A(f)^* e^{-it \text{d} \Gamma_b(T)} = A(e^{itT} f)^*.\]

**Lemma 9.4.** \([5, \text{Theorem 5.21.}]\) Assume that \( \mathcal{H} \) is separable. Let \( T \) be a non-negative, injective self-adjoint operator and \( \{e_n\}_{n=1}^\infty \subset D(T^{1/2}) \) be a CONS of \( \mathcal{H} \). Then, for any \( \psi \in D(\text{d} \Gamma_b(T)^{1/2}) \), \( \sum_{n=1}^\infty \|A(T^{1/2} e_n) \psi\|^2 \) converges and following equation holds:

\[
\sum_{n=1}^\infty \|A(T^{1/2} e_n) \psi\|^2 = \|\text{d} \Gamma_b(T)^{1/2}\| \psi\|^2.
\]

41
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