Hydrodynamic Interactions between Two Forced Objects of Arbitrary Shape: I Effect on Alignment

Tomer Goldfriend,1,a) Haim Diamant,2,b) and Thomas A. Witten3,c)
1) Raymond & Beverly Sackler School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel
2) Raymond & Beverly Sackler School of Chemistry, Tel Aviv University, Tel Aviv 69978, Israel
3) Department of Physics and James Franck Institute, University of Chicago, Chicago, Illinois 60637, USA

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We study the properties and symmetries governing the hydrodynamic interaction between two identical, arbitrarily shaped objects, driven through a viscous fluid. We treat analytically the leading (dipolar) terms of the pair-mobility matrix, affecting the instantaneous relative linear and angular velocities of the two objects at large separation. We find that the ability to align asymmetric objects by an external time-dependent drive [Moths and Witten, Phys. Rev. Lett. 110, 028301 (2013)] is degraded by the hydrodynamic interaction. The effects of hydrodynamic interactions are explicitly demonstrated through numerically calculated time-dependent trajectories of model alignable objects composed of four stokeslets. In addition to the orientational effect, we find that the two objects generally repel each other, thus restoring full alignment at long times.

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a) Electronic mail: goldfriend@tau.ac.il
b) Electronic mail: hdiamant@tau.ac.il
c) Electronic mail: t-witten@uchicago.edu
I. INTRODUCTION

The dynamics of colloid suspensions is crucially influenced by flow-mediated correlations. These hydrodynamic interactions (HI) are long-ranged, decaying as $1/R$ with the distance $R$ between the objects. While HI have an important role in the dynamics of ambient suspensions at thermal equilibrium, their effect becomes even more pronounced for objects driven out of equilibrium. A well-known example is colloid sedimentation, where HI lead to strongly correlated motions and large-scale dynamic structures. Various types of driving, such as electrophoresis, are widely used to control the transport of colloids and other polyatomic objects. Theoretical studies of driven colloids traditionally focus on highly symmetric particle shapes such as spheres and ellipsoids. The driving of asymmetric objects is richer as it generally includes coupling between translation and rotation — when the object is subjected to a force it also rotates, and when it is under torque it also translates. The choice of a rotation sense under a unidirectional force implies a chiral response of the driven asymmetric object. Such richer responses can be exploited to obtain “steerable colloids” — objects whose orientation and transport can be controlled in much more detail. For example, applying a torque by a rotating uniform magnetic field was used to achieve efficient transport of chiral magnetic objects. Another example, which is the main issue of the present work, is the ability to achieve orientational alignment of asymmetric objects by applying an external force.

The earlier theoretical works of Refs. dealt with isolated asymmetric objects in Stokes flow. The chiral response of such an object is encoded in the off-diagonal block of its self-mobility matrix, referred to as the twist matrix. Some objects have a twist matrix that leads them to align one axis in the body with the applied force. If the twist matrix has only a single real eigenvalue, the object becomes “axially aligned” in this way, and the aligning direction is along the corresponding eigenvector. Hence, in the absence of HI and thermal fluctuations, a set of identical, axially aligning objects reach a partially aligned state, where all the objects rotate about the same axis with the same angular velocity, but with an arbitrary phase. Furthermore, it was shown that, by applying an appropriate time-dependent forcing, the system can be driven to a fully aligned state, where all the objects are phase-locked with the force and rotate in synchrony.

The theoretical groundwork for treating the HI between arbitrary objects in Stokes flow was laid by Brenner and O’Neill. The theory was subsequently applied to particles of various symmetric shapes. In addition, several numerical techniques have been introduced to treat suspensions of arbitrarily shaped objects.

In this work we focus on simple, general properties of the pair HI between two arbitrarily shaped objects at zero Reynolds number, and the resulting effect on their orientational alignment. The study of translational effects will be presented in a separate publication. We use Brenner’s analytical framework, specializing to the leading order of the HI in the distance between the objects (multipole expansion). We demonstrate the results for stokeslet objects. The properties of the instantaneous interaction are treated analytically. We provide typical examples for the time evolution of object pairs by numerical integration. We begin by presenting in Sec. III the notation to be used in the article. In Sec. III we discuss the general properties and symmetries of the pair-mobility matrix for two arbitrarily shaped objects. In Sec. IV we apply a multipole expansion to the pair-mobility matrix and obtain results for the instantaneous HI at large distances. In Sec. V we derive the resulting properties of stokeslet objects, and in Sec. VI we use them to perform numerical integrations.
time integration for the evolution of object pairs and their alignment. Finally, in Sec. VII, we discuss several consequences of our results, and conclude in Sec. VIII.

II. NOTATION

The dynamics of arbitrarily shaped objects is complex and requires an elaborate notation. We use the following notation regarding vectors, tensors, and matrices:

1. 3-vectors are denoted by an arrow, $\vec{v}$, and unit 3-vectors by a hat, $\hat{v}$.
2. 6-vectors are denoted by a calligraphic font, $\vec{F}$.
3. Matrices are marked by a blackboard-bold letter, e.g., $\mathbb{M}$, where the dimension of the matrix is understood from the context.
4. Tensors of rank 3 are denoted by a capital Greek letter, e.g., $\Phi$.
5. A set of $N$ 3-vectors, representing $N$ stokeslets, is denoted by a bold letter, e.g., $\vec{v}^a = (\vec{v}_1^a, \ldots, \vec{v}_N^a)$.
6. Subscripts with parentheses, e.g., $\mathbb{M}_{(2)}$, represent a term in a multipole expansion.
7. $I_{n \times n}$ is the $n \times n$ identity matrix.
8. Tensor multiplication—the dot notation, $\cdot$, denotes a contraction over one index. The double dot notation, $:=$, denotes a contraction over two indices. Thus, given a tensor $\mathbb{Y}$ of rank $N$ and a tensor $\Xi$ of rank $M > N$, the tensors $\mathbb{Y} \cdot \Xi$ and $\mathbb{Y} : \Xi$ are tensors of rank $N + M - 2$ and $N + M - 4$. For example, for $\mathbb{Y}$ of rank 2 and $\Xi$ of rank 3, $(\Xi \cdot \mathbb{Y})_{ikj} = \mathbb{Y}_{is} \Xi_{skj}$ and $(\mathbb{Y} : \Xi)_j = \mathbb{Y}_{ks} \Xi_{skj}$.
9. The matrix $\vec{Y} \times$ obtained from the vector $\vec{Y}$ is defined as $(\vec{Y} \times)_{ij} = \epsilon_{ijk} Y_k$, such that, for any vector $\vec{X}$, $\vec{Y} \times \vec{X} = \vec{Y} \times \vec{X}$.

III. PAIR-MOBILITY MATRIX: GENERAL CONSIDERATIONS

A. Structure of the Pair-Mobility Matrix

The kinematics of a rigid object is represented by a translational velocity $\vec{V}$, which refers to an arbitrary reference point rigidly affixed to the object, and an angular velocity $\vec{\omega}$. We designate the reference point as the origin of the object. Note that the angular velocity of the object is independent of the choice of its origin, and that the origin does not necessarily lie on the instantaneous axis of rotation of the object.

Consider two arbitrarily shaped rigid objects, $a$ and $b$, with typical size $l$, subject to external forces and torques $\vec{F}^a$, $\vec{F}^b$ and $\vec{r}^a$, $\vec{r}^b$ in an unbounded, otherwise quiescent fluid of viscosity $\eta$. In the creeping flow regime, the objects respond with linear and angular velocities to the external forces and torques through a $12 \times 12$ pair-mobility matrix,

$$
\begin{pmatrix}
\vec{V}^a \\
\vec{V}^b
\end{pmatrix}
= \frac{1}{\eta l}
\begin{pmatrix}
\mathbb{M}_{aa} & \mathbb{M}_{ab} \\
\mathbb{M}_{ba} & \mathbb{M}_{bb}
\end{pmatrix}
\begin{pmatrix}
\vec{F}^a \\
\vec{F}^b
\end{pmatrix},
$$

(1)
where we define *generalized velocity* and *generalized force* 6-vectors, $\vec{V}^x = (\vec{V}^x, \mathbf{I}_6 \vec{\omega}^x)^T$ and $\vec{F}^x = (\vec{F}^x, \vec{r}^x/l)^T$ for $x = a, b$. The diagonal blocks, $M^{aa}$ and $M^{bb}$, correspond to the self-mobilities of the objects (which nevertheless depend on the configuration of both objects). The off-diagonal blocks, $M^{ab}$ and $M^{ba}$, describe the pair hydrodynamic interaction. We hereafter omit the factor $(\eta l)^{-1}$ (i.e., set $\eta l = 1$). This, together with the representation of the generalized forces and velocities, make $M$ dimensionless and dependent on the geometry alone.

Since $\vec{V}$ and $\vec{\tau}$ depend on the choice of object origins, so does the pair-mobility matrix. The transformation between pair-mobility matrices corresponding to different origins is given in Appendix A.

The pair-mobility matrix is a function of the objects’ geometries, their orientations, and the vector connecting their origins, indicated hereafter by $\vec{R}$. (We define the direction of $\vec{R}$ from the origin of object $b$ to the origin of object $a$.) The geometry of object $x$ is denoted by $r^x$. For example, if the object consists of a discrete set of $N_x$ stokeslets (see Sec. VA), then $r^x$ is a $3N_x$-vector specifying the positions of the stokeslets; otherwise, it represents the surface of the object.

The pair-mobility matrix is positive-definite and symmetric$^{1,28,29}$, hence, $M^{ab} = (M^{ba})^T$, and the self-blocks can be written as

$$M^{xx} = \begin{pmatrix} A^{xx} & (T^{xx})^T \\ T^{xx} & S^{xx} \end{pmatrix}.$$ 

As in the analysis for isolated objects$^2$, the self-mobility matrix contains the following $3 \times 3$ blocks: the alacrity matrix $A$ (translational response to force); the screw matrix $S$ (rotational response to torque); and the twist matrix $T$ (translation–rotation coupling). The twist matrix characterizes the chiral response of the object (the sense of rotation under a force).

In the present article we deal with alignable objects, whose individual $T$ is necessarily non-vanishing. Furthermore, in the case of a pair of objects, the presence of the other object makes the self-twist matrix, $T^{xx}$, differ from the single-object one. As to the off-diagonal blocks of the pair-mobility matrix, the symmetry of $M$ implies the following structure:

$$M^{ab} = \begin{pmatrix} A^{ab} & (T^{ba})^T \\ T^{ab} & S^{ab} \end{pmatrix}, \quad M^{ba} = \begin{pmatrix} (A^{ab})^T & (T^{ab})^T \\ T^{ba} & (S^{ab})^T \end{pmatrix}.$$ 

**B. Further Symmetries of the Pair-Mobility Matrix**

In what follows, we focus on the case in which the two objects are identical in shape and orientation, i.e., $r^a = r^b \equiv r$. Our goal is to understand what the instantaneous relative velocities (linear and angular) between the two objects are, when the objects are subjected to the same external forcing. The restriction to identical objects makes $M$ invariant under exchange of objects. This additional symmetry is made of two operations: interchanging the blocks $M^{aa} \leftrightarrow M^{bb}$ and $M^{ab} \leftrightarrow M^{ba}$; and inversion of $\vec{R}$. That is,

$$M(r, \vec{R}) = EM(r, -\vec{R})E^{-1},$$

where $E$ is a $12 \times 12$ matrix which interchanges the objects,

$$E = \begin{pmatrix} 0 & \mathbf{I}_{6\times6} \\ \mathbf{I}_{6\times6} & 0 \end{pmatrix}.$$
The symmetry to object exchange, when combined with the parity of $\mathcal{M}$ under $\vec{R}$-inversion, has important consequences for the effect of hydrodynamic interactions on alignment. If $\mathcal{M}$ has a definite parity one can determine what the relative response of the objects to forcing is — i.e., whether they attain the same or the opposite linear and angular velocities. If the term is symmetric to inversion, the velocities would be identical, and if it is antisymmetric, they would be opposite. This is because

$$
\begin{pmatrix}
\mathcal{M}^{aa}(\vec{R}) & \mathcal{M}^{ab}(\vec{R}) \\
\mathcal{M}^{ba}(\vec{R}) & \mathcal{M}^{bb}(\vec{R})
\end{pmatrix}
= \pm
\begin{pmatrix}
\mathcal{M}^{aa}(\vec{R}) & \mathcal{M}^{ab}(\vec{R}) \\
\mathcal{M}^{ba}(\vec{R}) & \mathcal{M}^{bb}(\vec{R})
\end{pmatrix}
= \pm
\begin{pmatrix}
\mathcal{M}^{ab}(\vec{R}) & \mathcal{M}^{ba}(\vec{R}) \\
\mathcal{M}^{ba}(\vec{R}) & \mathcal{M}^{aa}(\vec{R})
\end{pmatrix},
$$

(3)

where the second equality comes from the response to exchange of objects, Eq. (2). Consequently, under identical forcing of the two objects one finds,

$$
\vec{V}^a = (\mathcal{M}^{aa} + \mathcal{M}^{ab}) \vec{F} = \pm (\mathcal{M}^{bb} + \mathcal{M}^{ba}) \vec{F} = \pm \vec{V}^b.
$$

(4)

Thus, since any $\mathcal{M}$ can be decomposed into even and odd terms, we find that only the odd ones cause relative motions of the two objects.

The pair-mobility as a whole, however, never has a definite parity under $\vec{R}$-inversion, i.e., it is made of both even and odd terms. This becomes clear when $\mathcal{M}(\vec{r}, \vec{R})$ is expanded in small $l/R$, i.e., in multipoles. A general discussion of the parity of each multipole term is given in the next section. For now, let us consider those two leading multipoles which are independent of the objects’ shape, and therefore always exist. The monopole–monopole interaction (Oseen tensor), which is the leading term in $\mathcal{A}^{ab}$ making particle $a$ translate due to the force on particle $b$, is symmetric under $\vec{R}$-inversion. The part of the monopole–dipole interaction causing the second object to rotate due to the force on the first, i.e., the leading term in $\mathcal{T}^{ab}$, is antisymmetric. For example, even the most symmetric pair of objects — two spheres — has an $\vec{R}$-symmetric $\mathcal{A}^{ab}$, leading to zero relative velocity, and an $\vec{R}$-antisymmetric $\mathcal{T}^{ab}$, causing them to rotate with opposite senses. Thus, for a general object, the highest order which maintains $\mathcal{M}$ of definite parity is the monopole $1/R$ Oseen one, which is even. (The self-blocks are constant up to order $1/R^4$; see below.)

From this discussion we can immediately conclude that, to leading order in the objects’ separation, their hydrodynamic interaction must degrade the alignment. The leading degrading term comes from $\mathcal{T}^{ab}$, their rotational response to force, and is of order $1/R^2$.

The relation between object-exchange symmetry and the symmetry of the linear-velocity response is intimately related to the issue of hydrodynamic pseudo-potentials, which will be discussed in detail in a forthcoming publication.

IV. FAR-FIELD INTERACTION: MULTIPOLAR EXPANSION

There are two characteristic length scales in our problem: the typical size of the objects, $l$, and the distance between them, $R = |\vec{R}|$. If $l \ll R$, we can write the pair-mobility matrix as a power series in $(l/R)$,

$$
\mathcal{M} = \mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots,
$$

$^a$ Parity does not mean here symmetry under full spatial inversion, as such an operation would turn the chiral objects into their enantiomers; rather, we mean here symmetry under the inversion of $\vec{R}$.
where $M_{(n)} \sim (l/R)^n$. The zeroth order, $M_{(0)}$, is a block diagonal matrix which is made of the self-mobilities of the two non-interacting objects. (These should be distinguished from $M^{aa}$ and $M^{bb}$, the self-mobilities of the interacting objects.)

The hydrodynamic multipole expansion is based on the Green’s function of Stokes flow, the Oseen tensor, given in our units ($\eta l = 1$) by

$$G_{ij}(\vec{r}) = \frac{1}{8\pi r} \left( \delta_{ij} + \frac{r_i r_j}{r^2} \right), \quad (5)$$

which is a symmetric $3 \times 3$ tensor, invariant under $\vec{r}$-inversion. A point force at $\vec{r}_0$, $\delta(\vec{r} - \vec{r}_0) \vec{f}$, generates a velocity field $\vec{u}(\vec{r}) = G(\vec{r} - \vec{r}_0) \cdot \vec{f}$.

We obtain two general results concerning the multipoles of the hydrodynamic interaction between two arbitrary objects. The two objects need not be identical. The proofs are given in Appendix C.

1. The leading interaction multipole in the self-blocks of the pair-mobility matrix is $n = 4$. That is, any response of one object to forces on itself, owing to the other object, must fall off with distance $R$ between the objects at least as fast as $R^{-4}$.

2. The $n$th multipole has self-blocks of $(-1)^n$ parity, and coupling blocks of the opposite, $(-1)^{n+1}$ parity. Thus, e.g., the leading term in $M^{aa}$, proportional to $R^{-4}$, is invariant under $\vec{R}$-inversion, and the $R^{-4}$ part of $M^{ab}$ changes sign under $\vec{R}$-inversion. Likewise for the multipole varying as $R^{-5}$, the $M^{aa}$ changes sign under $\vec{R}$-inversion while $M^{ab}$ remains invariant.

These statements pertain to the mobility matrix. As to the propulsion matrix (the inverse of the mobility matrix), the leading correction to the self-block becomes $\sim 1/R^2$, and the second statement concerning parity remains intact.

We now consider again two identical objects and specialize to the first and second multipoles, i.e., the hydrodynamic interaction up to order $1/R^2$. The discussion in the preceding and current sections implies the following form of the two leading terms in the pair-mobility matrix:

$$M_{(1)} = \begin{pmatrix} 0 & M^{ab}_{(1)} \\ M^{ab}_{(1)} & 0 \end{pmatrix}, \quad M_{(2)} = \begin{pmatrix} 0 & M^{ab}_{(2)} \\ -M^{ab}_{(2)} & 0 \end{pmatrix}. \quad (6)$$

In more detail: there are no first- and second-order corrections to the objects’ self-mobility. Hence, these two multipoles have definite parities—the first is even, and the second is odd. Consequently, the first multipole does not cause any relative motion of the two objects, whereas the second multipole makes them translate and rotate in opposite linear and angular velocities.

The first multipole arises directly from the Green’s function,

$$M^{ab}_{(1)} = \begin{pmatrix} G(\vec{R}) & 0 \\ 0 & 0 \end{pmatrix}, \quad (7)$$

where $G(\vec{R})$ is the Oseen tensor, given in Eq. (5).

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b In fact, these results are not special to the hydrodynamic interaction but can be similarly proven for any multipole expansion. As such, they were most probably derived before.
In the interaction described by the second multipole one object sees the other as a point. Accordingly, this term contains two types of interaction: (1) the response of object \(a\) to the non-uniformity of the flow due to the force monopole at object \(b\) (regarded as a point); (2) the advection of object \(a\) (regarded as a point) by the flow due to the force dipole acting at object \(b\). These two effects are both proportional to \(\nabla \mathbf{G}(\mathbf{R}) \sim 1/R^2\). Each can be written as a product of a tensor which arises from the medium alone, through derivatives of the Oseen tensor \(\nabla \mathbf{G}(\mathbf{R})\), and another tensor which depends on the objects’ geometry. The second-order correction to the velocity of object \(a\) is given by the sum of these two effects, each expressed in terms of a coupling tensor \(\Theta\) and an object tensor \(\Phi\):

\[
\vec{V}_{(2)}^a = M_{(2)}^{ab} \cdot \vec{F}^b
\]

\[
M_{(2)}^{ab} = \Phi^a : \Theta(\mathbf{R}) - \Theta^T(\mathbf{R}) : \tilde{\Phi}^b.
\]

Equation (8) contains three tensors of rank 3. The first, \(\Phi\), with dimensions \(6 \times 3 \times 3\), gives the generalized velocity of the object in linear response to the velocity gradient of the flow in which it is embedded. The second, \(\tilde{\Phi}\), having dimensions \(3 \times 3 \times 6\), gives the force dipole acting on the fluid around the object’s origin in linear response to the generalized force acting on it. Both \(\Phi\) and \(\tilde{\Phi}\) depend on the objects’ geometry alone. The third tensor, \(\Theta\), with dimensions \(3 \times 3 \times 6\), describes the coupling of these object responses through the fluid. It is given by

\[
\Theta_{skj}(\mathbf{R}) \equiv \begin{cases} 
\partial_s \mathbf{G}_{kj}(\mathbf{r}) |_{\mathbf{R}} & j = 1, 2, 3 \\
0 & j = 4, 5, 6.
\end{cases}
\]

Repeating the same procedure for \(\vec{V}^b\) in response to \(\vec{F}^a\) while using the odd parity of \(\Theta\), we get

\[
M_{(2)}^{ba} = \Theta^T(\mathbf{R}) : \tilde{\Phi}^a - \Phi^b : \Theta(\mathbf{R}).
\]

The tensors \(\Phi\) and \(\tilde{\Phi}\) are not independent. We now show that \(\Phi = \tilde{\Phi}^T\). The symmetry of \(\tilde{M}\) implies that each multipole is also a symmetric matrix. Using Eqs. (8) and (10) and equating \((M_{(2)}^{ba})^T = M_{(2)}^{ab}\), we get \(\tilde{\Phi}^a = (\Phi^a)^T\) and \(\tilde{\Phi}^b = (\Phi^b)^T\).

To summarize, the matrix \(\tilde{M}_{(2)}\) is given by

\[
\tilde{M}_{(2)} = \begin{pmatrix} 0 & \Phi^a : \Theta(\mathbf{R}) - [\Phi^b : \Theta(\mathbf{R})]^T \\
-\tilde{\Phi}^b : \Theta(\mathbf{R}) + [\Phi^a : \Theta(\mathbf{R})]^T & 0 \end{pmatrix}.
\]

By separating the tensors \(\Phi\) and \(\Theta\) into their symmetric and antisymmetric parts, the second-order term of the pair-mobility matrix can be simplified further. It should be mentioned, in addition, that the \(\Phi\) tensor depends on the origin selected for the object. These two technical issues are addressed in Appendices [D] and [B] respectively.

V. NUMERICAL ANALYSIS FOR STOKESLET OBJECTS

In the preceding sections we have discussed the general properties of the hydrodynamic interaction between two arbitrarily shaped objects. We now move on to detailed examples studied numerically. Since we are interested in generic properties, we allow ourselves to

\footnote{These tensors are related to the two introduced by Brenner\cite{Brenner}. Brenner’s tensors give the force and torque exerted on an object in linear response to a flow gradient in which it is embedded. Our \(\Phi\) is related to these two via the individual self-mobility matrix.}
restrict the analysis to the simplest, even if unrealistic, objects. Arguably the simplest form of an arbitrarily shaped object is the so-called stokeslet object—a discrete set of small spheres, separated by much larger, rigid distances, where each sphere is approximated as a point force. The sparseness of these objects makes them free-draining, which may be valid for macromolecules but not for compact objects.

We treat pairs of identical objects, each made of four stokeslets. Rather than designing these objects, we create them randomly. Four points are placed at random distances ranging between 0 and 1 from an arbitrary origin. The origin is then shifted to the points’ center of mass. The radius \( \rho \) of the stokeslets is taken as 0.01. The resulting configuration is checked to be “sufficiently chiral”, in the sense that the \( T \)-matrix of the individual object is strongly asymmetric, having a single real eigenvalue of absolute value \( |\lambda_3| > 0.005 \), which makes the object alignable. (See Sec. III). Examples of the stokeslet objects we use are provided in Fig. 1.

The way to calculate the mobility of a single stokeslet object was presented in Ref. 9. First, we briefly present in Sec. V A the simple extension of this method to pair-mobilities. We calculate both the pair mobility and the tensor \( \Phi \) introduced in Secs. III I and IV. The latter allows us to calculate pair mobilities up to second order in the multipole expansion. Section V B describes how we use the pair mobility to numerically calculate the time evolution of the pair configuration.

A. Pair-Mobility and \( \Phi \) Tensor

The properties of a stokeslet object can be derived self-consistently from the linear relations which describe the stokeslets’ configuration. This is done without finding the stokeslets’ strengths explicitly. Below we find the pair-mobility matrix, and the \( \Phi \) tensor associated with a single object, given the stokeslet configuration and the size of the spheres that they represent.

Each of the two objects, \( x = a, b \), consists of \( N_x \) stokeslets, \( \mathbf{F}^x = (\tilde{F}_{1}^{x}, \ldots, \tilde{F}_{N_x}^{x}) \), in a known configuration, \( \mathbf{r}^x = (\tilde{r}_{1}^{x}, \ldots, \tilde{r}_{N_x}^{x}) \). Here, \( \tilde{r}_{n}^{x} \) indicates the position 3-vector of the \( n \)-th stokeslet in object \( x \) with respect to the object’s origin. Each stokeslet is a sphere of radius \( \rho \), where \( \rho < \min(\tilde{r}_{1}^{x}, \ldots, \tilde{r}_{N_x}^{x}) \). The boundary conditions at the sphere surface enter only through its self-mobility coefficient. The velocities of the spheres, \( \tilde{v}_{n}^{x} \), are known from the object’s linear and angular velocities,

\[
\begin{pmatrix}
\mathbf{v}^a \\
\mathbf{v}^b
\end{pmatrix} = \begin{pmatrix}
\mathbb{U}^a & 0 \\
0 & \mathbb{U}^b
\end{pmatrix} \begin{pmatrix}
\tilde{v}_{n}^a \\
\tilde{v}_{n}^b
\end{pmatrix}, \quad \text{where} \quad \mathbb{U}^x = \begin{pmatrix}
\mathbb{I}_{3 \times 3}, & -\tilde{r}_{1}^{x} \times / l \\
\vdots & \vdots \\
\mathbb{I}_{3 \times 3}, & -\tilde{r}_{N_x}^{x} \times / l
\end{pmatrix}, \quad \text{for} \quad x = a, b. \quad (12)
\]

Each stokeslet force is proportional to the relative velocity of the sphere that it represents, with respect to the flow around it as created by the other stokeslets. This gives a linear relation between the stokeslets and the velocities of the spheres \( ^d \),

\[
\begin{pmatrix}
\mathbf{v}^a \\
\mathbf{v}^b
\end{pmatrix} = \begin{pmatrix}
\mathbb{I}^{aa} & \mathbb{I}^{ab} \\
\mathbb{I}^{ba} & \mathbb{I}^{bb}
\end{pmatrix} \begin{pmatrix}
\mathbf{F}^a \\
\mathbf{F}^b
\end{pmatrix}, \quad \text{where}
\]

\( ^d \) More explicitly, consider the stokeslet at position \( \tilde{r}_{n}^x \). The flow at that point which is created by the other stokeslets, belonging to the two objects, is \( \tilde{u}(\tilde{r}_{n}^x) = \Sigma_{m \neq n} G(\tilde{r}_{n}^x - \tilde{r}_{m}^x) \cdot \tilde{F}_{m}^x + \Sigma_{m} G(R + \tilde{r}_{n}^x - \tilde{r}_{m}^x) \cdot \tilde{F}_{m}^x \). The stokeslet at that point is proportional to the velocity of the sphere relative to the local flow, \( \tilde{F}_{n}^x = \gamma(\tilde{r}_{n}^x - \tilde{u}(\tilde{r}_{n}^x)) \). This gives Eq. (13).
\[(L_{nm})_{ij} = \begin{cases} G_{ij}(r_n^a - r_m^a) & \text{if } n \neq m \\ \gamma^{-1}\delta_{ij} & \text{else} \end{cases} \]

and \(\gamma = 6\pi\rho/l\).

First we find the pair-mobility matrix as a generalization of the analysis in Ref. 9. The sum of the stokeslets and the corresponding total torque must be equal to the external generalized forces applied on the objects. In a matrix form we can write

\[
\begin{pmatrix} F_a \\ F_b \end{pmatrix} = \begin{pmatrix} (U_a)^T & 0 \\ 0 & (U_b)^T \end{pmatrix} \begin{pmatrix} F_a \\ F_b \end{pmatrix}.
\]

Using Eqs. (12) and (13), we have

\[
\begin{pmatrix} U_a \\ 0 \\ U_b \end{pmatrix}^T \begin{pmatrix} L_{aa} & L_{ab} \\ L_{ab}^T & L_{bb} \end{pmatrix}^{-1} \begin{pmatrix} U_a \\ 0 \\ U_b \end{pmatrix} \cdot \begin{pmatrix} \tilde{V}_a \\ \tilde{V}_b \end{pmatrix} = \begin{pmatrix} \tilde{F}_a \\ \tilde{F}_b \end{pmatrix}.
\]

From this expression we identify the pair-mobility matrix as

\[
\mathbb{M} = \begin{pmatrix} (U_a)^T & 0 \\ 0 & (U_b)^T \end{pmatrix} \begin{pmatrix} L_{aa} & L_{ab} \\ L_{ab}^T & L_{bb} \end{pmatrix}^{-1} \begin{pmatrix} U_a \\ 0 \\ U_b \end{pmatrix}^{-1}.
\]

This expression allows to calculate the pair-mobility matrix, with the help of Eqs. (12) and (14), based on the stokeslets’ configuration and the Oseen tensor alone.

Next, we derive the \(\Phi^x\) tensor of a stokeslet object \(x\). From this tensor we may readily obtain the second multipole of the pair interaction (cf. Sec. IV). The force dipole around the origin of a forced object is given by [Eq. (8)], \((rF)^x = (\Phi^x)^T \cdot F^x\). Similar to the \(U^x\) matrix relating the stokeslets to the total generalized force, \(\tilde{F}^x = (U^x)^T \cdot F^x\), we define a tensor of rank 3, \(\Upsilon^x\), which relates the stokeslet forces to the total force dipole on the object by \((rF)^x = (\Upsilon^x)^T \cdot F^x\). (Note that no force dipole is applied on the individual stokeslets; being arbitrarily small they possess only a force monopole.) Specifically, it is made of \(N\) blocks of \(3 \times 3 \times 3\), given by \((\Upsilon_n)_{ij} = r_{n,s}^a \delta_{ij}, n = 1 \ldots N, i, j, s = 1, 2, 3\) (i.e., \(r_{n,s}^a\) is the \(s\) Cartesian coordinate of the stokeslet \(n\)). Using Eqs. (12) and (13), we have

\[
\begin{pmatrix} rF^a \\ rF^b \end{pmatrix} = \begin{pmatrix} (U^a)^T \\ 0 \end{pmatrix} \cdot \begin{pmatrix} L_{aa} & L_{ab} \\ L_{ab}^T & L_{bb} \end{pmatrix}^{-1} \cdot \begin{pmatrix} rF^a \\ rF^b \end{pmatrix}.
\]

Recalling that the matrices \(M_{\text{self}}\) and \(L\) are symmetric, we finally get

\[
\Phi^x = M_{\text{self}}^{xx} \cdot (U^x)^T \cdot (L^{xx})^{-1} \cdot \Upsilon^x.
\]

It is important to note that in the above derivation we compute \(\mathbb{M}\) and \(\Phi\) under the assumption that, for each object, the stokeslet sizes are arbitrarily small compared to the distances between them, \(\rho \ll l\) (where \(l\) is the object’s radius of gyration). However, in a more general case one can use the Rotne-Prager-Yamakawa tensor\(^{31,32}\), which corrects the Oseen tensor for force distributions with finite size\(^{23}\).

**B. Numerical Time Integration**

We present a numerical integration scheme for the dynamics of two stokeslet objects. We should first define the reference frames used in the scheme. Each rigid object is characterized
by axes which are affixed and rotate with it. We define the object reference frame (ORF) such that its \( z \) axis coincides with the object’s alignment axis (the corresponding eigenvector of the \( T \)-matrix). The other two axes are selected arbitrarily. The \( z \) axis of the laboratory frame is defined along the forcing direction. During the evolution we follow the translation and rotation of the ORF in the laboratory frame.

We represent the orientation of an object by the Euler-Rodrigues 4-parameters, defined by \((\Gamma, \tilde{\Omega}) \equiv (\cos \frac{\theta}{2}, \hat{n} \sin \frac{\theta}{2})\), where \( \hat{n} \) and \( \theta \) are the axis and angle of rotation\(^\text{a}\). The following properties hold for this 4-parameter representation: (a) The norm of \((\Gamma, \tilde{\Omega})\) in 4D-space is unity, \(\Gamma^2 + \tilde{\Omega}^2 = 1\). (b) A rotation matrix is given by Rodrigues’ rotation formula,

\[
R(\Gamma, \tilde{\Omega}) = \mathbb{I}_{3\times3} + 2\tilde{\Omega}^\top \times 2(\Gamma^\times) \tag{18}
\]

(c) The parameters are invariant under inversion, i.e., \((\Gamma, \tilde{\Omega})\) and \((-\Gamma, -\tilde{\Omega})\) correspond to the same orientation. (d) Given the angular velocity of the object, the dynamics of its orientation simply reads

\[
\begin{pmatrix}
\dot{\tilde{\Omega}} \\
\tilde{\Omega}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
0 & -\bar{\omega}^T \\
\bar{\omega} & \bar{\omega}^x
\end{pmatrix} \begin{pmatrix}
\Gamma \\
\tilde{\Omega}
\end{pmatrix} \tag{19}
\]

Since we choose the ORF such that the \( z \)-axis is the axis of alignment, the terminal orientation of an axially aligned object under a constant force along the \( z \)-axis is \((\Gamma, \tilde{\Omega}) = (\cos(\frac{\omega t + \alpha}{2}), \hat{z} \sin(\frac{\omega t + \alpha}{2}))\), where \( \alpha \) is a constant phase which depends on the object’s initial orientation at time \( t = 0 \).

The state of a pair of objects at time \( t \) is described by the position of the origins of the objects, \( \vec{R}^a(t) \) and \( \vec{R}^b(t) \), and their orientation parameters, \((\Gamma^a(t), \tilde{\Omega}^a(t))\) and \((\Gamma^b(t), \tilde{\Omega}^b(t))\). We time-integrate from the initial state, \( \vec{R}^a_0 = (0, 0, 0) \), \( \vec{R}^b_0 = \vec{R}_0 \), \( (\Gamma^a_0, \tilde{\Omega}^a_0) \) and \( (\Gamma^b, \tilde{\Omega}^b_0) \), as follows. Given the positions of the stokeslets at time \( t \), the pair-mobility matrix, \( M(t) \), is calculated as explained in Sec. \( \nabla A \) either exactly or using the multipole approximation. Then, the linear and angular velocities of the objects are given by \((\vec{V}^a(t), \vec{V}^b(t))^T = M(t) \cdot (\vec{F}^a(t), \vec{F}^b(t))^T\) from them the origins and orientations of the objects at time \( t + dt \) are derived according to

\[
\begin{align*}
\vec{R}^x(t + dt) &= \vec{R}^x(t) + \vec{V}^x(t)dt \\
(\Gamma^x(t + dt)) &\equiv \exp \left[ \frac{dt}{2} \begin{pmatrix}
0 & -\bar{\omega}^x^T \\
\bar{\omega}^x & -\bar{\omega}^{xx}
\end{pmatrix} \right] (\Gamma^x(t)) \tag{20}
\end{align*}
\]

for \( x = a, b \). During the evolution we make sure that the small stokeslet spheres do not overlap, and that the pair mobility matrix remains positive-definite. In practice we never encountered such problems when using the exact pair mobility matrices; when it did happen in the case of the multipole approximation we stopped the integration.

We define a scalar order parameter which characterizes the degree of mutual alignment of the two objects,

\[
m \equiv \left( (\Gamma^a, \tilde{\Omega}^a) \cdot (\Gamma^b, \tilde{\Omega}^b) \right)^2 = \left( \Gamma^a \Gamma^b + \tilde{\Omega}^a \cdot \tilde{\Omega}^b \right)^2 \tag{22}
\]

As required, the order parameter is invariant under 3-rotation. This can be verified by explicitly applying a 3-rotation to the laboratory frame, or alternatively, by the following

\(^a\) This is the same as the unit-quarterion representation\(\text{33}\).
argument. Since 3-rotation leaves the norm of the 4-parameter orientation unchanged (property (a) above), it is a unitary transformation in 4-space. Hence, the dot product is invariant. When the objects are aligned, \((\Gamma^a, \Omega^a) = \pm(\Gamma^b, \Omega^b)\), and \(m = 1\); otherwise \(0 \leq m < 1\). In the case of partial alignment, \(m = \cos^2(\Delta\alpha)\), where \(\Delta\alpha\) is the mutual phase difference.

Another scalar property of the two-object system is the energy dissipation rate. At time \(t\), the latter is given by \(\tilde{\mathbf{V}}^a(t) \cdot \tilde{\mathbf{F}}^a(t) + \tilde{\mathbf{V}}^b(t) \cdot \tilde{\mathbf{F}}^b(t)\). Since the pair-mobility matrix is positive definite the energy dissipation of the driven pair is positive at all times.

VI. NUMERICAL RESULTS: EFFECT ON ALIGNMENT

We present in Figs. 2–7 several examples for the numerically integrated evolution of object pairs under various conditions. In these examples we compare the dynamics under a time-dependent forcing, which for a single object achieves full alignment, with the case of constant forcing. The results are presented in a dimensionless form, using units such that \(\eta = |\omega_0| = 1\) and \(\rho = 0.01\). The distances between the stokeslets of each object are taken randomly between 0 and 1; hence, \(\rho \ll l \sim 1\). The time-dependent forcing protocol is \(\mathbf{F}_t = F_0 (\sin(\omega_0 t) \sin(\theta), \cos(\omega_0 t) \sin(\theta), -\cos(\theta))\), where \(\theta = 0.1\pi\), \(F_0 = -|\lambda_3|^{-1}\), \(\omega_0 = \text{sign}(\lambda_3)\) and \(\lambda_3\) is the real eigenvalue of the single-particle twist matrix. We examine both the trajectories of the separation vector connecting the origins of the two objects, and the corresponding evolution of the orientation order parameter.

We begin with the case of a time-dependent forcing, Figs. 2 and 3. The first observation, most clearly demonstrated in Fig. 3 (right panel), is that hydrodynamic interaction degrades the alignment of the two objects, as has been generally predicted based on symmetry considerations in Sec. III B. Another conclusion, supported by additional examples not shown here, is that most objects, which start sufficiently far apart, especially if they start fully aligned, tend to repel each other (Fig. 2). Even if they are not fully aligned, the growing distance and weakening interaction make them increasingly more aligned with the forcing. Thus, the repulsion helps restore the alignment at long times. The increasing separation occurs in the \(xy\) plane, while along the \(z\) axis the separation decreases and saturates to a finite distance, dependent on initial conditions, see Fig. 2.

The repulsion is accompanied by a decrease in dissipation rate (up to small oscillations), as demonstrated in Fig. 6. When the HI is turned off, the dissipation rate reaches a constant value as the two independent objects set into their ultimate aligned state (dashed curves in Fig. 6).

Another type of behavior is observed as well: when the objects start at a sufficiently small separation, they may remain “bound” in a limit cycle, oscillating about a certain mean separation and mean orientational alignment, as demonstrated by the green curves in Figs. 2-3.

In Figs. 4 and 5 we examine the same properties under constant forcing. The alignment degradation and repulsive tendency are observed in this case as well. Yet, the repulsion does not restore the alignment. On the contrary—in the absence of a time-dependent aligning force, the long-range hydrodynamic interaction continues to degrade the mutual alignment as the objects drift apart (Fig. 5 upper right panel).

Figure 7 compares results obtained using the full pair-mobility matrix of the stokeslet objects with those obtained from the multipole (dipole) approximation. As expected, the

\footnote{If the symmetry of the objects is such that their phase difference is unobservable (e.g., two ellipsoids rotating around their major axis), then we set it to zero.}
two calculations agree for objects whose mutual distance increases with time, and disagree for objects whose trajectories reach close proximity.

Further investigation (not shown here) of the orientational dynamics of the objects suggests a possible explanation for the characteristic repulsion between two chiral objects. In the absence of HI, each object rotates along $\hat{F}$ and translates on average along $\hat{F}$. One contribution to the dipolar term of the HI comes from the effect on each object by the vorticity of the Oseen flow caused by the other object. This perturbative angular velocity is along an axis which is perpendicular to the separation vector and the force, $\dot{\omega}_{\text{flow}}^a \propto -\hat{R} \times \hat{F}$ and $\dot{\omega}_{\text{flow}}^b \propto \hat{R} \times \hat{F}$. The competition between this rotation and the aligning self-response of each individual object results in an inclination of the two objects relative to their non-interacting state. This inclination alters the average unperturbed linear velocity of the object by a small rotation about the $\hat{R} \times \hat{F}$ direction—counter-clockwise for object $a$ and clockwise for object $b$. Hence, the two objects glide away from each other, $\ddot{R}^2 = 2(\ddot{V}^a - \ddot{V}^b) \cdot \ddot{R} \propto ((-\hat{R} \times \hat{F}) \times \hat{F}) \cdot \dddot{R} = R(1 - (\hat{R} \cdot \hat{F})^2) \geq 0$, where the proportionality constant is positive, i.e., the separation increases with time (unless $\dddot{R} \parallel \hat{F}$, for which the whole argument does not hold).

VII. DISCUSSION

Asymmetric objects, unlike symmetric ones, display rich dynamics already at the level of a pair of objects, as has been demonstrated above. In the present work we have focused on the effect of the hydrodynamic interaction on the orientational alignment of asymmetric objects.

The hydrodynamic interaction, in general, degrades the alignment. The leading effect in distance is dipolar rather than monopolar; yet, it is significant—a large mutual distance
FIG. 2. Trajectories of object separation under time-dependent forcing. The three rows, from top to bottom, correspond, respectively, to the separation along the $z$ direction, its projection onto the $xy$ plane, and its total magnitude. The squares in the middle row indicate the state at the end of the simulation. The panels show results for three different objects, starting from either a random mutual orientation (left column) or their fully aligned state (right column). The green trajectory on the right panels was integrated longer than 150 time units to verify that it continues in a limit cycle.
FIG. 3. Orientation order parameter as a function of time, under time-dependent forcing, for the examples of Fig. 2. The left panel presents results for random initial orientations (examples on the left column of Fig. 2); the additional dashed gray curves correspond to non-interacting objects. The right panel shows the results for initially fully aligned object pairs (right column in Fig. 2).

(compared to the object size) is required to make the degradation negligible. More quantitatively, the degradation will be significant when the perturbation to the angular velocity due to HI, \( \delta \omega \), becomes comparable to the inverse of the time required to align a single object. The unperturbed angular velocity is given by \( \omega_0 = T_{\text{self}} F \). The dimensionless eigenvalue of the self-twist matrix is generally found to be about an order of magnitude smaller than the dimensionless self-mobility coefficient \( \omega_0 \sim 0.1 F/(8\pi l^2) \). As presented in Sec. III B, \( \delta \omega \sim T_x F/\omega \sim F/(8\pi l^2)(l/R)^2 \). The alignment time is typically \( t_{\text{al}} \sim 10/\omega_0 \) (see Fig 3). Hence, the degree of degradation is \( t_{\text{al}} \delta \omega \sim 10^2 (l/R)^2 \). The conclusion is that the separation between the objects should be larger than ten times their size to maintain alignment. In the case of many objects this implies a maximum volume fraction \((l/R)^3 \sim 10^{-3}\).

As shown in Sec. VI in most cases the hydrodynamic interaction makes the rotating objects repel each other. For the case of a finite number of objects the repulsion will help restore the alignment as the objects drift apart. A possible explanation for the repulsive effect has been suggested in Sec. VI. The resulting hydrodynamic “pseudo-potential” \( \Phi \) will be addressed in a future publication.

An important distinction between symmetric and asymmetric objects, which we have not dealt with here, concerns many-body interactions in forced systems. A pair of forced spheres does not develop any relative translational velocity \( \Phi \). The same holds for a pair of forced ellipsoids to order \( 1/R^2 \) (for an ellipsoid, the components of \( \Phi \) which corresponds to the translational velocity vanish). For a suspension of many objects this implies that two-body effects on relative motion are either absent (spheres) or negligible at low volume fraction (ellipsoids). By contrast, as we have shown here, a pair of asymmetric objects develops a relative velocity already at order \( 1/R^2 \), which should lead to significant two-body interactions in a suspension. This may bring about qualitative differences between driven suspensions of symmetric and asymmetric objects in relation to such phenomena as sedimentation.
VIII. CONCLUSION

This work shows that asymmetry in sedimenting objects leads to a wealth of hydrodynamic interaction effects not seen for spheres. This study was undertaken to assess how interactions disrupt the rotational synchronization of such objects. However it proves to have striking effects independent of this alignment. The prevalent repulsion, the occasional entrapment and the intricate quasiperiodic motions shown above are examples. These effects could have significant impacts on real colloidal dispersions, e.g., in fluidized beds of catalyst particles. Though we have studied only pairwise interactions between identical objects, many of these effects are expected to apply more generally. The Onsager formalism we have developed here should prove useful in exploring these phenomena. Our work in progress aims to achieve a more general understanding of the rich behavior reported in Sec. VI.

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Appendix A: Pair-mobility: Change of object origin

Here we derive the transformation of the pair-mobility matrix under change of objects’ origins. Consider a new choice of origins given by \( \vec{R}_a' = \vec{R}_a + \vec{h}_a \) and \( \vec{R}_b' = \vec{R}_b + \vec{h}_b \), and denote the objects’ properties with respect to the new origins with \( \prime \). Following Ref. 9, the transformations for the generalized velocities and forces can be written as \( \vec{V}_x' = [I_{6\times6} - (\mathbb{B}_x)^T] \vec{V}_x \) and \( \vec{F}_x' = [I_{6\times6} + \mathbb{B}_x] \vec{F}_x \) for \( x = a, b \), where

\[
\mathbb{B}^a = \begin{pmatrix}
0 & 0 \\
-\vec{h}_a \times 0
\end{pmatrix} \quad \text{and} \quad \mathbb{B}^b = \begin{pmatrix}
0 & 0 \\
-\vec{h}_b \times 0
\end{pmatrix}.
\]

Using \( [I_{6\times6} + \mathbb{B}_x]^{-1} = [I_{6\times6} - \mathbb{B}_x] \) we have

\[
\begin{pmatrix}
M^{aa} & M^{ab} \\
M^{ba} & M^{bb}
\end{pmatrix} = \begin{pmatrix}
[I_{6\times6} - (\mathbb{B}^a)^T] & 0 \\
0 & [I_{6\times6} - (\mathbb{B}^b)^T]
\end{pmatrix} \begin{pmatrix}
M^{aa} & M^{ab} \\
M^{ba} & M^{bb}
\end{pmatrix} \begin{pmatrix}
[I_{6\times6} - \mathbb{B}^a] & 0 \\
0 & [I_{6\times6} - \mathbb{B}^b]
\end{pmatrix}.
\]

Appendix B: Properties of the tensor \( \Phi \)

Below we provide a more detailed discussion regarding the tensor \( \Phi \) introduced in Sec. IV. We consider its symmetries and its dependence on the choice of origin. We separate \( \Phi \) into a translational part—linear velocity response to a flow gradient, denoted by \( \Phi_{\text{tran}} \), and a rotational part—angular velocity response to a flow gradient, denoted by \( \Phi_{\text{rot}} \). We show that \( \Phi_{\text{tran}} \) is symmetric with respect to its last two indices while \( \Phi_{\text{rot}} \) has also an antisymmetric part which is the Levi-Civita tensor. In addition, we show that \( \Phi_{\text{tran}} \) depends on the choice of the object’s origin whereas \( \Phi_{\text{rot}} \) does not, and derive the transformation of the former under change of origins.
In order to prove the symmetry properties of $\Phi$ we consider its transpose tensor $\Phi^T = \tilde{\Phi}$ which gives the force dipole around the object when subjected to external forcing, $(rF) = \tilde{\Phi} \cdot \vec{F} = \tilde{\Phi}_{\text{tran}} \cdot \vec{F} + \tilde{\Phi}_{\text{rot}} \cdot \vec{\tau}$. We write the force dipole as a sum of symmetric and anti-symmetric terms, $\frac{1}{2} [(rF) + (rF)^T + \epsilon \cdot \vec{\tau}] = \tilde{\Phi}_{\text{tran}} \cdot \vec{F} + \tilde{\Phi}_{\text{rot}} \cdot \vec{\tau}$, where $\epsilon$ is the Levi-Civita tensor. The last equality implies that $(\tilde{\Phi}_{\text{tran}})_{ski}$ is symmetric with respect to $s$ and $k$ and that the anti-symmetric part of $(\tilde{\Phi}_{\text{rot}})_{ski}$ is $\frac{1}{2} \epsilon_{ski}$.

Next we consider the transformation of $\Phi$ under change of origins. Let us assume that an object is given in a constant, arbitrary shear flow $\vec{u}(\vec{r}) = S \cdot \vec{r}$, where $S$ is not necessarily a symmetric matrix. The object’s linear velocities measured about $\vec{R}$ and $\vec{R}' = \vec{R} + \vec{h}$ are $\vec{V} = S \cdot \vec{R} + \Phi_{\text{tran}} : S$ and $\vec{V}' = S \cdot (\vec{R} + \vec{h}) + \Phi'_{\text{tran}} : S$ respectively. The tensor $\Phi_{\text{rot}}$ does not depend on the choice of origin since the angular velocity of the object is independent of that choice, $\vec{\omega} = \Phi_{\text{rot}} : S = \Phi'_{\text{rot}} : S$. Using the relation $\vec{V}' = \vec{V} - \vec{h} \times \vec{\omega}$ we find

$$\Phi'_{\text{tran}} : S = (\Phi_{\text{tran}} - \vec{h} \times \Phi_{\text{rot}}) : S - S \cdot \vec{h}.$$  

In general, with analogy to Eq. (A.1), we can write

$$\Phi' = [\mathbb{I}_{6 \times 6} - (\mathbb{B})^T] \cdot \Phi + \Delta,$$

where

$$\mathbb{B} = \begin{pmatrix} 0 & 0 \\ -\vec{h} \times & 0 \end{pmatrix} \quad \text{and} \quad \Delta_{iks} = \begin{cases} -\delta_{is} h_k, & i = 1 \ldots 3 \\ 0, & i = 4 \ldots 6 \end{cases}.$$  

Appendix C: Proofs of general properties of interaction multipoles

Here we prove the two general results presented in Sec. [IV concerning the interaction multipoles.

Multipole expansions are constructed by repeated projections ("reflections"), between the two objects, of the Green’s function and its derivatives. The self-blocks of the mobility matrix result from even projections, and the coupling blocks from odd projections. In our case $G$, the Oseen tensor, has even parity and scales as $1/R$. The Green’s function $G$ itself appears only once in the expansion, in the first $(1/R)$ multipole. This is because the force monopoles acting on the particles are prescribed. This monopolar (odd) interaction appears only in the coupling blocks. The leading multipole appearing in the self-blocks is constructed by projecting the induced force dipole on object 2 (proportional to $\nabla G$) back onto object 1 (by another $\nabla G$). Thus, this leading multipole is of 4th order, proportional to $1/R^4$. This proves the first result in Sec. [IV] Its particular manifestation for two spheres is well known.

Now, consider the $n$th multipole, proportional to $1/R^n$. Assume that it contains $k$ $G$’s and $n-k$ derivatives. Its parity is $(-1)^{n-k}$. As explained above, for self-blocks $k$ is even, and for coupling blocks it is odd. Hence, the parity of the $n$th multipole is $(-1)^n$ in the self-blocks and $(-1)^{n+1}$ in the coupling blocks. This proves the second result.

Appendix D: General Form of $M_{(2)}^{ab}$

Below we provide a general form of the matrix $M_{(2)}^{ab}$, the 2nd-order multipole of the coupling block in the pair-mobility matrix, and point out the number of its independent
components. This is done by decomposing the tensors \( \Phi \) and \( \Theta \) to their symmetric and anti-symmetric parts. Without loss of generality we choose the separation vector between the two objects to be along the \( x \) axis, \( \hat{R} = \hat{x} \). For two not necessarily identical objects the matrix \( M_{ab}^{(2)} \) has the form

\[
M_{ab}^{(2)} = \left( \frac{l}{R} \right)^2 \begin{pmatrix}
A_{xx}^a - A_{xx}^b & -A_{yx}^b - A_{zx}^b & -T_{xx}^a - T_{yx}^a - T_{zx}^a \\
A_{yx}^a & 0 & 0 \\
A_{zx}^a & 0 & 0 \\
T_{xx}^a & 0 & 0 \\
T_{yx}^a & 0 & 1 \\
T_{zx}^a & -1 & 0
\end{pmatrix},
\]

where the \( A_{ij}^x \) and \( T_{ij}^x \) are functions of \( \hat{R} \) and the shape and orientation of object \( x \), \( (x = a, b) \). For two identical (in shape and orientation) objects we have

\[
M_{ab}^{(2)} = \left( \frac{l}{R} \right)^2 \begin{pmatrix}
0 & -A_{yx} & -A_{zx} \\
A_{yx} & 0 & 0 \\
A_{zx} & 0 & 0 \\
T_{xx} & 0 & 0 \\
T_{yx} & 0 & 1 \\
T_{zx} & -1 & 0
\end{pmatrix}.
\]

REFERENCES

1. J. Happel and H. Brenner, *Low Reynolds number hydrodynamics: with special applications to particulate media* (Martinus Nijhoff, The Hague, 1983).
2. W. B. Russel, D. A. Saville, and W. R. Schowalter, *Colloidal Dispersions* (Cambridge University Press, 1989).
3. S. Ramaswamy, “Issues in the statistical mechanics of steady sedimentation,” *Adv. Phys.* 50, 297 (2001).
4. M. Makino and M. Doi, “Sedimentation of a particle with translation-rotation coupling,” *J. Phys. Soc. Jpn.* 72, 2699 (2003).
5. O. Gonzalez, A. B. A. Graf, and J. H. Maddocks, “Dynamics of a rigid body in a Stokes fluid,” *J. Fluid Mech.* 519, 133 (2004).
6. M. Doi and M. Makino, “Motion of micro-particles of complex shape,” *Prog. Polym. Sci.* 30, 876 (2005), plenary Lectures World Polymer Congress, 40th IUPAC International Symposium on Macromolecules.
7. M. Makino and M. Doi, “Migration of twisted ribbon-like particles in simple shear flow,” *Phys. Fluids* 17, 103605 (2005).
8. K. I. Morozov and A. M. Leshansky, “The chiral magnetic nanomotor,” *Nanoscale* 6, 1580 (2014).
9. N. W. Krapf, T. A. Witten, and N. C. Keim, “Chiral sedimentation of extended objects in viscous media,” *Phys. Rev. E* 79, 056307 (2009).
10. B. Moths and T. A. Witten, “Full alignment of colloidal objects by programmed forcing,” *Phys. Rev. Lett.* 110, 028301 (2013).
11. B. Moths and T. A. Witten, “Orientational ordering of colloidal dispersions by application of time-dependent external forces,” *Phys. Rev. E* 88, 022307 (2013).
12. H. Brenner, “The Stokes resistance of an arbitrary particle II: An extension,” Chem. Eng. Sci. 19, 599 (1964).
13. H. Brenner and M. E. O’Neill, “On the Stokes resistance of multiparticle systems in a linear shear field,” Chem. Eng. Sci. 27, 1421 (1972).
14. A. Goldman, R. Cox, and H. Brenner, “The slow motion of two identical arbitrarily oriented spheres through a viscous fluid,” Chem. Eng. Sci. 21, 1151 (1966).
15. S. Wakiya, “Mutual interaction of two spheroids sedimenting in a viscous fluid,” J. Phys. Soc. Jpn 20, 1502 (1965).
16. B. Felderhof, “Hydrodynamic interaction between two spheres,” Physica A 89, 373 (1977).
17. D. J. Jeffrey and Y. Onishi, “Calculation of the resistance and mobility functions for two unequal rigid spheres in low-Reynolds-number flow,” J. Fluid Mech. 139, 261 (1984).
18. W. Liao and D. A. Krueger, “Multipole expansion calculation of slow viscous flow about spheroids of different sizes,” J. Fluid Mech. 96, 223 (1980).
19. S. Kim, “Sedimentation of two arbitrarily oriented spheroids in a viscous fluid,” Int. J. Multiphas. Flow 11, 699 (1985).
20. S. Kim, “Singularity solutions for ellipsoids in low-Reynolds-number flows: With applications to the calculation of hydrodynamic interactions in suspensions of ellipsoids,” Int. J. Multiphas. Flow 12, 469 (1986).
21. S. J. Karrila, Y. O. Fuentes, and S. Kim, “Parallel computational strategies for hydrodynamic interactions between rigid particles of arbitrary shape in a viscous fluid,” J. Rheol. 33, 913 (1989).
22. T. Tran-Cong and N. Phan-Thien, “Stokes problems of multiparticle systems: A numerical method for arbitrary flows,” Phys. Fluids A-Fluid 1, 453 (1989).
23. B. Carrasco and J. G. de la Torre, “Hydrodynamic properties of rigid particles: Comparison of different modeling and computational procedures,” Biophys. J. 76, 3044 (1999).
24. R. Kutteh, “Rigid body dynamics approach to stokesian dynamics simulations of non-spherical particles,” J. Chem. Phys. 132, 174107 (2010).
25. B. Cichocki, B. U. Felderhof, K. Hinsen, E. Wajnryb, and J. Bławdziewicz, “Friction and mobility of many spheres in Stokes flow,” J. Chem. Phys. 100, 3780 (1994).
26. H. Brenner, “The Stokes resistance of an arbitrary particle,” Chem. Eng. Sci. 18, 1 (1963).
27. H. Brenner, “The Stokes resistance of an arbitrary particle IV: Arbitrary fields of flow,” Chem. Eng. Sci. 19, 703 (1964).
28. D. W. Condiff and J. S. Dahler, “Brownian motion of polyatomic molecules: The coupling of rotational and translational motions,” J. Chem. Phys. 44, 3988 (1966).
29. L. Landau and E. Lifshitz, Statistical Physics, Part 1 (third edition) (Pergamon Press, 1980).
30. T. M. Squires, “Effective pseudo-potentials of hydrodynamic origin,” J. Fluid Mech. 443, 403 (2001).
31. J. Rotne and S. Prager, “Variational treatment of hydrodynamic interaction in polymers,” J. Chem. Phys. 50, 4831 (1969).
32. H. Yamakawa, “Transport properties of polymer chains in dilute solution: Hydrodynamic interaction,” J. Chem. Phys. 53, 436 (1970).
33. L. D. Favro, “Theory of the rotational brownian motion of a free rigid body,” Phys. Rev. 119, 53 (1960).
34. T. Squires and M. Brenner, “Like-charge attraction and hydrodynamic interaction,” Phys. Rev. Lett. 85, 4976 (2000).
FIG. 4. Trajectories of particle separation under constant forcing. The meaning of the various panels is the same as in Fig. [2]
FIG. 5. Orientation order parameter as a function of time, under constant forcing, for the examples of Fig. 4. The upper left panel presents results for random initial orientations (examples on the left column of Fig. 4); the dashed gray curves correspond to non-interacting objects. The upper right panel shows the results for initially fully aligned object pairs (right column in Fig. 4). The lower panel presents results for objects with initial partial alignment (rotating around the same axis with random initial phases).
FIG. 6. Dissipation rate as a function of time for object pairs starting from arbitrary orientations, under time-dependent forcing (left) and constant forcing (right). Solid curves correspond to the examples of the same colors in the preceding figures. Dashed curves show the results in the absence of HI.

FIG. 7. Comparison between the evolution of pair separations obtained using the full pair-mobility matrix (solid curves) and its multipole approximation (dashed curves). The left and right panels present each three examples of pairs under time-dependent (left) and constant forcing (right). All pairs start from a fully aligned state. The multipole approximation includes the monopolar and dipolar terms.