Are There Topologically Charged States Associated with Quantum Electrodynamics?

E.C. Marino

Joseph Henry Laboratories
Princeton University
Princeton, NJ 08544

Abstract

We present a formulation of Quantum Electrodynamics in terms of an antisymmetric tensor gauge field. In this formulation the topological current of this field appears as a source for the electromagnetic field and the topological charge therefore acts physically as an electric charge. The charged states of QED lie in the sector where the topological charge is identical to the matter charge. The antisymmetric field theory, however, admits new sectors where the topological charge is more general. These nontrivial, electrically charged, sectors contain massless states orthogonal to the vacuum which are created by a gauge invariant operator and can be interpreted as coherent states of photons. We evaluate the correlation functions of these states in the absence of matter. The new states have a positive definite norm and do interact with the charged states of QED in the usual way. It is argued that if these new sectors are in fact realized in nature then a very intense background electromagnetic field is necessary for the experimental observation of them. The order of magnitude of the intensity threshold is estimated.

1On sabbatical leave from Departamento de Física, Pontifícia Universidade Católica, Rio de Janeiro, Brazil. E-mail: marino@puhep1.princeton.edu
1) Introduction

An extremely interesting feature of many quantum field theories is the presence of topological sectors in the spectrum of quantum excitations. These, to use a broad definition, are characterized by the fact that they carry a charge which is associated to an identically conserved current rather than to one whose conservation stems from a global symmetry of the lagrangian, via Noether theorem. A general feature of this kind of excitations is the fact that their physical behavior cannot be described in terms of the lagrangian fields or even in terms of polynomials in these fields. This fact immediately rules out the possibility of any perturbative treatment of them.

There is a large diversity of systems in various number of dimensions displaying quantum topological excitations in their spectrum. Usually they are associated to finite energy classical solutions of the field equations, known generally as solitons, of which they are the quantum counterparts. In this case, the quantum solitons are massive, the quantum mass being of the order of the total energy of the classical solution. Well known examples are the kinks, Sine-Gordon solitons, vortices and magnetic monopoles [1,11] There are, however, less known examples where the system possesses quantum topological excitations even though there are no corresponding classical finite energy solutions of the field equations. These occur, for example, in Z(4) symmetric systems of field theory or quantum statistical mechanics in two dimensional spacetime, whenever there are phases where the symmetry is partially broken down to Z(2) [11,2].

Quantum Electrodynamics in 3+1 dimensions is a theory which does not possess either classical or quantum topological excitations. In this work, however, we show that QED can be formulated as a particular sector of a theory for the antisymmetric tensor gauge field $W_{\mu\nu}$. This field possesses a topological current which appears as source for the electromagnetic field. The topological charge, as a consequence, acts physically as an electric charge. The usual (matter) charged states of QED lie in the sector where the topological charge is identical to the matter charge. The theory, however, possesses new nontrivial topologically charged sectors which do not belong
to QED. The charge of this states does not have its origin in matter. It is a state of the gauge field.

We construct a gauge invariant operator which creates the topologically charged excitations in the new sectors and compute its correlation functions in the absence of charged matter. These can be taken to describe the large distance regime of the correlation functions when charged matter is present. The long distance behavior of the correlation functions indicates that the new topologically charged states are massless and orthogonal to the vacuum. From the two-point function, one can also infer that the norm of these states is strictly positive definite. The new topologically charged states interact among themselves as well as with the usual matter charged states in the usual way as described by QED. When we express their creation operator in terms of the electromagnetic field we see that they can be interpreted as a coherent state of photons.

There is a two-dimensional system which presents a very similar structure as our four-dimensional theory for the antisymmetric tensor gauge field. This is the free real massless scalar field in 1+1 D. The main features of this theory are reviewed in Section 2. An important difference of the system considered here from the two-dimensional analog is the fact that for a certain value of the topological charge there are asymptotic states interpolated by the creation operator of topological excitations. This fact allows us to determine the value of the topological charge quantum.

It may well happen that the more general formulation of QED which admits these new charged sectors is simply not realized in nature. If, on the other hand, the contrary is true, we argue that an extremely strong background electromagnetic would be required for the observation of the new states. We make an estimate of the order of magnitude of the threshold magnitude of the intensity of this background field.

In Section 3, we present the formulation of QED in terms of the antisymmetric tensor gauge field. In Section 4, we introduce the creation operator of the quantum topological excitations. In Section 5 we compute the correlation functions of this operator. Some phenomenological considerations concerning the possibility of observation of the topological states are made in Section 6. Conclusions and questions are
presented in Section 8. Four Appendixes are included in order to demonstrate useful results.

2) An Analogous System in 1+1 D

Let us review in this section an extremely simple system which presents essentially the same features as the antisymmetric tensor gauge field theory we found to be associated to QED in 3+1 D. This is the real massless scalar field in 1+1 D:

\[ L = \partial_\mu \phi \partial^\mu \phi \]  

(2.1)

This field possesses an identically conserved topological current

\[ J^\mu = \epsilon^{\mu \nu} \partial_\nu \phi \]  

(2.2)

In terms of it the field equation \( \Box \phi = 0 \) can be written as

\[ \epsilon^{\mu \nu} \partial_\mu J_\nu = 0 \]  

(2.3)

This admits the operator solution

\[ J^\mu = \partial^\mu \chi \implies \Box \chi = 0 \]  

(2.4)

No polynomial in \( \phi \) or its derivatives can create states carrying the topological charge \( Q = \int dx J^0 \). Nevertheless, if we introduce the operator

\[ \sigma(x, t) = \exp \left\{ ib \int^x \phi(z, t)dz \right\} \Rightarrow \sigma(x, t) = \exp \{ ib \chi(x, t) \} \]  

(2.5)

it is easy to see, by using canonical commutation relations, that

\[ [Q, \sigma(x)] = b \sigma(x) \]  

(2.6)

This shows that \( \sigma \) creates states bearing \( b \) units of the topological charge \( Q \) in spite of the fact that the theory possesses no classical soliton solutions. In the above expressions, \( b \) is an arbitrary real parameter.
The topologically charged states will have nontrivial interactions even though the original theory is free. We can figure out what this interaction will be by simply writing the original lagrangian in terms of $J^\mu$:

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi = \frac{1}{2} \epsilon^{\mu \alpha \beta} \partial_\alpha \phi \epsilon_{\mu \beta} \partial^\beta \phi \Rightarrow L = \frac{1}{2} J^\mu J_\mu \quad (2.7)$$

This indicates that the topological states will have a Thirring-like interaction which is known to be the correct one.

We can easily compute the $\sigma$ euclidean correlation functions to be

$$< \sigma(x) \sigma^\dagger(y) > = |x - y|^{-\frac{1}{8}} \quad (2.8)$$

The long distance behavior of this implies that $< \sigma > = 0$ and therefore the sigma operator creates nontrivial states orthogonal to the vacuum. Going to euclidean space and taking the limit $x \rightarrow y$ we can see that the norm of the $\sigma$ states is strictly positive, in spite of the fact that the quantization of the free massless scalar field in $2$ dimensional spacetime requires a Hilbert space with an indefinite metric [3]. We are going to see that the theory of the antisymmetric tensor gauge field we are going to introduce in $3+1$ D in connection to QED presents a structure quite similar to the one just described.

3) QED Formulated as a Theory for the Antisymmetric Tensor Gauge Field

Let us show in this section that QED can be formulated as a particular sector of a theory for the antisymmetric (Kalb-Ramond) tensor gauge field. Let us start by the case in which matter is absent. Consider the following theory for the antisymmetric tensor gauge field $W_{\mu \nu}$:

$$L_W = -\frac{1}{12} W_{\mu \nu \alpha} (-\Box)^{-1} W^{\mu \nu \alpha} \quad (3.1)$$

where $W_{\mu \nu \alpha} = \partial_\mu W_{\nu \alpha} + \partial_\nu W_{\alpha \mu} + \partial_\alpha W_{\mu \nu}$ is the field intensity tensor of the antisymmetric field. The kernel in (3.1) is

$$(-\Box^{-1}) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(z-z')}}{k^2} = \frac{1}{8\pi^2 |z - z'|^2} \quad (3.2)$$
All over this paper, of course, multiplication by $\Box^{-1}$ is understood in the convolution sense. The topological current of the $W_{\mu\nu}$ field is given by

$$J^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta} \partial_\nu W_{\alpha\beta} \quad (3.3)$$

The field equation associated to (3.1) is

$$\Box^{-1} \partial_\alpha W^{\alpha\mu\nu} = 0 \quad (3.4)$$

We can rewrite this in terms of the topological current as

$$\Box^{-1} \epsilon^{\mu\alpha\beta} \partial_\alpha J_\beta = 0 \quad (3.5)$$

We immediately see that we can obtain an operator solution for $J^\mu$ in terms of a free massless scalar field

$$J^\mu = \partial^\mu \chi \quad \Box \chi = 0 \quad (3.6)$$

Since $J^\mu$ has dimension 3, it follows that $\chi$ is a noncanonical massless free field. We can express $J^\mu$ in terms of a canonical massless free field $\phi$ by using the pseudodifferential operator $(-\Box)^{1/2}$:

$$J^\mu = \partial^\mu (-\Box)^{1/2} \phi \quad (3.7)$$

Using $\Box \phi = 0$, it is clear that clear that

$$\partial_\mu J^\mu = \Box (-\Box)^{1/2} \phi = (-\Box)^{1/2} \Box \phi = 0 \quad (3.8)$$

As in our two-dimensional example, no polynomial in the field intensity $W_{\mu\nu\alpha}$ can create states carrying the topological charge $Q = \int d^3x J^0$. Nevertheless, we are going to see in the next section that it is possible to construct an operator, analogous to the $\sigma$- operator of the last section, which is going to create the topologically charged states.

As before, let us try to see what kind of interaction the topological charge carrying states will have. In order to do this, let us express the $L_W$ lagrangian in terms of the topological current. Using (3.1) and (3.3) we get

$$L_W = \frac{1}{2} J^\mu (-\Box)^{-1} J_\mu \quad (3.9)$$
which is exactly the effective electromagnetic interaction between charged particles associated with a current $J^\mu$ which would be obtained in QED upon integration over the electromagnetic field. For static point chrges, for instance, the energy corresponding to (3.9) is the 1/r Coulomb potential energy. We see that the topological charges present an electromagnetic-like interaction among themselves. This fact justifies our peculiar form of the lagrangian $L_W$ containing the kernel $\Box^{-1}$.

The fact that we can describe the properties of QED through a theory like the one given by $L_W$, however, only becomes evident when we couple the $W_{\mu\nu}$- field with the charged matter current $j^\mu$ in the following way:

$$L_W + L_I = -\frac{1}{12}W_{\mu\alpha}(-\Box)^{-1}W^{\mu\alpha} - \frac{q}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu W_{\alpha\beta}(-\Box)^{-1}j_\mu$$  \hspace{1cm} (3.10)

where $q$ is the charge coupling constant of matter.

In order to get the effective interaction of the matter current generated by (3.10), let us consider the euclidean functional integration over $W_{\mu\nu}$ in (3.10):

$$Z[j^\mu] = Z^{-1}\int D W_{\mu\nu} \exp\{-\int d^4z[L_W + L_I + L_{GFW}]\} =$$

$$\exp\left\{ -\frac{q^2}{2}\int d^4zd^4z'\left[ (-\Box)j^\mu \epsilon^{\mu\nu\alpha\beta}\partial_\nu \right]\left[ (-\Box)j^\sigma \epsilon^{\sigma\lambda\gamma\rho}\partial_\lambda \right] D_{\alpha\beta\gamma\rho}(z-z') \right\}$$  \hspace{1cm} (3.11)

In the above expression $L_{GFW}$ is the gauge fixing term, given by

$$L_{GFW} = -(\xi/8)W_{\mu\nu}K^{\mu\nu\alpha\beta}(-\Box)^{-1}W_{\alpha\beta}$$  \hspace{1cm} (3.12)

where $K^{\mu\nu\alpha\beta} = \partial^\mu \partial^\nu \delta^{\alpha\beta} + \partial^\nu \partial^\beta \delta^{\mu\alpha} - (\alpha \leftrightarrow \beta)$ and $\xi$ is gauge fixing parameter and $D^{\mu\nu\alpha\beta}$ is the euclidean propagator of the $W_{\mu\nu}$ field:

$$D^{\mu\nu\alpha\beta}(x) = (1/4)[(-\Box)\Delta^{\mu\nu\alpha\beta} + (1-\xi^{-1})K^{\mu\nu\alpha\beta}]|8\pi^2 |x|^2\]^{-1}$$  \hspace{1cm} (3.13)

where $\Delta^{\mu\nu\alpha\beta} = \delta^{\mu\alpha} \delta^{\nu\beta} - \delta^{\mu\beta} \delta^{\nu\alpha}$. Inserting (3.13) in (3.11) we see that only the first term of (3.13) contributes to (3.11). Using the fact that $j^\mu$ is a conserved current, we get

$$Z[j^\mu] = \exp\left\{ \frac{q^2}{2}\int d^4zd^4z'j^\mu(z)(-\Box)^{-1}j_\mu(z') \right\}$$  \hspace{1cm} (3.14)
The effective interaction $Z[j^\mu]$, given by (3.14) is precisely the one which is generated by Maxwell QED upon integration over the electromagnetic field. We therefore conclude that the theory given by (3.10) does describe the same interaction of the charged matter coupled to the antisymmetric field as does Maxwell QED.

The field equation associated with (3.10) is

$$\partial_\alpha W^{\mu\nu\alpha} = q \epsilon^{\mu\nu\alpha\beta} \partial_\alpha j_\beta \quad (3.15)$$

We can reexpress this in terms of the topological current as

$$\epsilon^{\mu\nu\alpha\beta} \partial_\alpha J_\beta = q \epsilon^{\mu\nu\alpha\beta} \partial_\alpha j_\beta \quad (3.16)$$

by using the identity

$$\partial_\alpha W^{\mu\nu\alpha} \equiv \epsilon^{\mu\nu\alpha\beta} \partial_\alpha J_\beta \quad (3.17)$$

From (3.16) it is clear that we can obtain an operator solution for $J^\mu$ in terms of the matter current and a free massless scalar field:

$$J^\mu = q j^\mu + \partial^\mu \chi \quad (3.18)$$

where $\square \chi = 0$. As before, we could express $J^\mu$ as well in terms of a canonical free massless field.

Let us express now the $W_{\mu\nu}$ field directly in terms of the electromagnetic field. We see that $\mathcal{L}_I$ can be put exactly in the form of a minimal coupling of $j^\mu$ with the electromagnetic field $A_\mu$, by defining (in the Lorentz gauge, $\partial_\mu A^\mu = 0$)

$$J^\mu \equiv \partial_\nu F^{\nu\mu} \quad (3.19)$$

This can also be written in the equivalent form

$$W^{\alpha\mu\nu} \equiv \epsilon^{\mu\alpha\nu\beta} (\square) A_\beta \quad (3.20)$$

The equivalence between (3.19) and (3.20) is another form of the identity (3.17). We could also obtain (3.19) and (3.20) as field equations provided we make the substitution

$$(\square)^{-1} j^\mu \equiv A^\mu \quad (3.21)$$
in $\mathcal{L}_I$, eq. (3.10), obtaining thereby
\[ \mathcal{L}_I = \varepsilon^{\mu\nu\alpha\beta} A_\mu \partial_\nu W_{\alpha\beta} \] (3.22)

Observe that (3.21) is precisely the relation between $j^\mu$ and $A_\mu$ one would obtain from QED in the Lorentz gauge.

If we substitute (3.19) in (3.16) we get
\[ \varepsilon^{\mu\nu\alpha\beta} \partial_\alpha [\partial_\gamma F_{\gamma\beta}] = q \varepsilon^{\mu\nu\alpha\beta} \partial_\alpha [j_\beta] \] (3.23)
which can be thought of as a generalized form of the Maxwell equations. The general operator solution of (3.23) is
\[ \partial_\nu F_{\nu\mu} = q j^\mu + \partial^\mu \chi \] (3.24)
where $\Box \chi = 0$. We could also have obtained (3.24) by inserting (3.19) in (3.18).

We see from (3.23) that (3.10) provides a description of the electromagnetic interaction which is more general than the one of QED because it admits solutions of the type (3.24) which contain the additional $\chi$ dependent term. The usual states of QED lie in the sector where $J^\mu = q j^\mu$, that is, the vacuum sector of the free scalar field $\chi$. In the $W_{\mu\nu}$-field formulation, however, we have new nontrivial sectors for which the topological charge is nonzero even in the absence of matter, for instance. For these states, the charge associated with $J^\mu = \partial^\mu \chi$ would be nonzero. The topological current of the $W_{\mu\nu}$-field appears as a source for the electromagnetic field. Since the solution for $J^\mu$, eq. (3.18) contains a piece which is independent of the matter charge, we see that this formulation of QED admits new charged sectors which are not related to matter.

We can ask ourselves at this point about the interaction between these new topologically charged states and the usual charged matter states of QED. Observing that the interaction lagrangian can be written as
\[ \mathcal{L}_I = q J^\mu (-\Box)^{-1} j_\mu = J^\mu A_\mu \] (3.25)
we conclude that the topological states would interact with the charged states of QED in the usual way. Also, they would emit electromagnetic radiation in the same way as the charged states of QED.
The quantization of theories having nonlocalities of the type appearing in (3.1) has been studied in detail in [4] for an arbitrary power of $-\Box$. There it is shown that these theories are well behaved and sensible. The particular lagrangian $L_W$ considered here, where the power is -1 is actually local since it is a mass term for the transverse part of $W_{\mu\nu}$.

In the next section we are going to show that despite the absence of finite energy solutions carrying the topological charge we can construct an operator, analog to the $\sigma$-operator of our two-dimensional example, which is going to create the states in the new nontrivial topological sectors.

4) The $\mu$ Operator

In this section we are going to introduce the operator which will create the states in the nontrivial topological sectors of the $W_{\mu\nu}$-field theory. This belongs to a class of operators which was introduced in two, three, and four dimensional spacetime as the creation operators of the respective topological excitations, namely—kinks, vortices and magnetic monopoles [3, 4, 8, 9]. The method of construction of these operators relies on the fact that the operator which creates the topological excitations of a certain theory must behave as a disorder variable and therefore be dual to the basic lagrangian fields. As a consequence, these operators must satisfy a certain order-disorder or dual algebra with the lagrangian fields [11].

A basic ingredient in the construction of topological excitations creation operators is always [11] a singular external field which, when coupled to the dynamical fields produces the desired operator. In the present case, the external field is the antisymmetric tensor

$$\tilde{A}_{\mu\nu}(z, x) = \frac{b}{4\pi} \int_{T_x(S)} d^3 \xi \Phi_{\nu}(\xi - x) \delta^4(z - \xi) - (\mu \leftrightarrow \nu) \quad (4.1)$$

where $b$ is an arbitrary dimensionless real parameter and $\Phi_{\nu} = (0, 0, 0, \frac{1}{r \sin \theta})$ (in the coordinate system $(t, r, \theta, \phi)$). The integral in (4.1) is performed over the three-dimensional hypersurface $T_x(S)$ whose surface element $d^3 \xi_{\mu}$ has only the 0-component.
nonvanishing. \( T_x(S) \) is the region of the \( \mathbb{R}^3 \) space external to the surface \( S \) at \( z^0 = x^0 \). This surface is depicted in Fig. 1 and consists of a piece of sphere of radius \( \rho \) centered at \( \vec{x} \) \( (0 \leq \theta \leq \pi - \delta) \) superimposed to an infinite trunk of cone with vertex at \( \vec{x} \) and angle \( \delta \) (cut a distance \( \rho \cos \delta \) from the tip) with axis along \( \theta = \pi \). In the region \( T_x(S) \), the vector \( \vec{\Phi} \) satisfies the following useful identity

\[
\vec{\nabla} \times \vec{\Phi} \equiv \vec{\nabla} \left[ -\frac{1}{|\vec{x}|} \right]
\]  

(4.2)

The \( \mu \)-operator is constructed in the following way

\[
\mu(x) = \lim_{\rho,\delta \to 0} \exp \left\{ -\frac{i}{6} \int d^4z \tilde{A}_{\mu\alpha}(z; x)(-\Box)^{-1} W^{\mu\alpha} \right\}
\]  

(4.3)

where \( \tilde{A}_{\mu\alpha} \) is the field intensity tensor of \( \tilde{A}_{\mu\nu} \). The operator \( \mu \) is in principle nonlocal, depending on the hypersurface \( T_x(S) \). Nevertheless, as it happens in the case of the above mentioned related operators \([6, 9]\) all the hypersurface dependence of the \( \mu \) correlation functions can be eliminated by the introduction of a renormalization factor whose form is uniquely determined only by the requirement of hypersurface invariance. The parameters \( \rho \) and \( \delta \) will be used as regulators which will be eliminated at the end of the calculations.

Inserting (4.1) in (4.3) and observing that the surface element in (4.1) only possesses the 0-component nonvanishing we get, after integration over \( z \)

\[
\mu(x) = \lim_{\rho,\delta \to 0} \exp \left\{ \frac{i b}{4\pi} \int_{T_x(S)} d^3\vec{\xi} \partial_i \Phi_j(\vec{\xi} - \vec{x})(-\Box)^{-1} W^{0ij}(x^0, \vec{\xi}) \right\}
\]  

(4.4)

We can also reexpress \( \mu \) in terms of the electromagnetic field by using eq. (3.20):

\[
\mu(x) = \lim_{\rho,\delta \to 0} \exp \left\{ \frac{i b}{4\pi} \int_{T_x(S)} d^3\vec{\xi} \partial_i \Phi_j(\vec{\xi} - \vec{x}) \epsilon^{ijk} A_k(x^0, \vec{\xi}) \right\}
\]  

(4.5)

The charge and topological charge densities, which are identified by (3.19) are given, respectively, by

\[
\rho(x) = \partial_i E^i = \frac{1}{2} \epsilon^{ijk} \partial_i W_{jk}
\]  

(4.6)

Using a covariant (Lorentz) gauge quantization (see \([5]\) for instance) we get the equal-time commutation relations \([A^i(x), E_j(y)] = i\delta^i_j \delta^3(x - y)\) where \( E^i = F^{i0} \) is the electric field.
The commutator between $\mu$ and the charge density can be obtained by writing $\mu \equiv e^A$, using the expansion for $e^A Be^{-A}$, Eq. (4.5) and the commutator $[A^i, E_j]$. The result is

$$[\rho(y), \mu(x)] = \frac{b}{4\pi} \mu(x) \lim_{\rho, \delta \to 0} \int_{T_x(S)} d^3\xi \ e^{ijk} \partial_k(y) \delta^3(\xi - \vec{y}) \partial_i(\xi - \vec{x})$$

where $\rho(x)$ stands either for the charge or topological charge density.

Using the identity $\partial_i(y) = -\partial_i(\xi)$ and the results of Appendix D we can immediately evaluate (4.7) to be

$$[\rho(y), \mu(x)]_{ET} = b\mu(x) \delta^3(\vec{x} - \vec{y})$$

This equation shows that the operator $\mu$ does in fact carry $b$ units of charge.

Another interesting commutator involving the operator $\mu$ is the one with the field $W_{\mu\nu}$. From (3.19), we can express this field in terms of the electromagnetic field. In the Lorentz gauge ($\partial_\mu W^{\mu\nu} = 0$), we have

$$W^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

From this and (4.5) we can obtain, along the same lines as we did for the charge commutator, the equal times relation

$$\mu(x) W_{ij}(y) = \left[ W_{ij}(y) - \frac{b}{4\pi} \epsilon_{ijk} \frac{(y - \vec{x})^k}{|y - \vec{x}|^3} \right] \mu(x)$$

$$\mu(x) W_{i0}(y) = W_{i0}(y) \mu(x)$$

This shows that the operator $\mu$ has indeed the same kind of commutation relation with the lagrangian field as other topological excitation creation operators as the vortex [7] and magnetic monopole operators, for instance [8]. Observe that through commutation with $W_{\mu\nu}$, the $\mu$ operator introduces a configuration having a nonzero topological charge.

Another interesting relation which can be obtained very in the same way as (4.10) is the commutator of $\mu$ with the electric field:

$$[E^k(y), \mu(x)]_{ET} = b \frac{(y - \vec{x})^k}{4\pi |y - \vec{x}|^3} \mu(x)$$
In Appendix A it is shown that this implies

\[
\frac{\langle \mu(x) | E^k(y) | \mu(x) \rangle_{ET}}{\langle \mu(x) | \mu(x) \rangle} = b \frac{(\vec{y} - \vec{x})^k}{4\pi|\vec{y} - \vec{x}|^3}
\]  

(4.12)

The vacuum expectation value of the electric field in the states created by \( \mu \) is the field configuration of a point electric charge of magnitude \( b \). This characterizes \( |\mu(x)\rangle \) as a coherent state of photons. Observe that the nonvanishing divergence of the above equation is guaranteed by the r.h.s. of (3.24) even in the absence of matter. This also shows that the states created by \( \mu \) for which the charge \( b \) is different from the charge \( q \) do not belong to QED in spite of the fact that we can express the \( \mu \)-operator in terms of \( A_\mu \). They belong to the additional nontrivial sectors of the \( W_{\mu\nu} \) theory.

Let us remark at this point the similarity with the two-dimensional case studied in Section 2. Even though no polynomial in the lagrangian fields can create states carrying the topological charge we have an operator - \( \mu \) in our case, \( \sigma \) in the two-dimensional example - expressed nonperturbatively in terms of the lagrangian fields which do indeed create states bearing this charge. In the next Section, we are going to evaluate the two-point correlation function of \( \mu \).

5) The \( \mu \) Correlation Functions

Let us compute in this section the euclidean correlation functions of the operator \( \mu \) introduced above. We will consider the case in which matter is not present. Taking the expression of \( \mu \), Eq. (4.3), the lagrangian \( L_W \) and going to euclidean space (in which we will work henceforth) we may write

\[
\langle \mu(x) \mu^\dagger(y) \rangle = Z^{-1} \int DW_{\mu\nu} \exp \left\{ - \int d^4z \left[ \frac{1}{12} W^{\mu\nu\alpha}(-\Box)^{-1} W_{\mu\nu\alpha} + \frac{1}{6} \tilde{A}_{\mu\nu\alpha}(z;x,y)(-\Box)^{-1} W^{\mu\nu\alpha} + L_{GF} + L_{CT} \right] \right\}
\]  

(5.1)

where \( L_{GF} \) is the gauge fixing term given by (3.12) and \( \tilde{A}_{\mu\nu\alpha}(z;x,y) = \tilde{A}_{\mu\nu\alpha}(z;x) - \tilde{A}_{\mu\nu\alpha}(z;y) \), the minus sign of the y-factor corresponding to the fact that we have \( \mu^\dagger(y) \).
\( \mathcal{L}_{CT} \) is the above mentioned hypersurface renormalization factor to be determined below.

We see that \(< \mu \mu^\dagger > = e^{F[\bar{A}_{\mu\nu}]} \) is the vacuum functional in the presence of the external field \( \bar{A}_{\mu\nu} \). This property of the correlation functions of \( \mu \) is common to all of the above mentioned topological charge bearing related operators [6, 7, 8] and follows from the general fact that topological charge carrying operators are closely related to the disorder variables of Statistical Mechanics [11]. Indeed, treating these operators as disorder variables one can demonstrate in general [11] that the \( \mu \) operator correlation functions can be expressed in terms of the coupling of the lagrangian field to an external field like \( \bar{A}_{\mu\nu} \) as in (4.3). One can also show in general [11] that the appropriate renormalization factor consists of the corresponding self-coupling of the the external field. Also here, we will see explicitly that the renormalization counterterm

\[
\mathcal{L}_{CT} = \frac{1}{12} \bar{A}^{\mu\nu\alpha}(-\Box)^{-1} \bar{A}_{\mu\nu\alpha}
\] (5.2)

will absorb all the hypersurface dependence of the correlation function, thereby making it completely local. We can understand the reason for this in the following way. Neglecting the gauge fixing term for a while, we see that we can write

\[
< \mu(x)\mu^\dagger(y) > = e^{F[\bar{A}_{\mu\nu}(T_x(S);x,y)]} = Z^{-1} \int DW_{\mu\nu} \exp \left\{ W_{\mu\nu} + \bar{A}_{\mu\nu}(T_x(S);x,y) \right\}
\] (5.3)

Now, let us make the change of functional integration variable

\[
W_{\mu\nu} \longrightarrow W_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu
\] (5.4)

with

\[
\Lambda_\mu = \Theta_4(\Delta V)[\Phi_\mu(z - x) - \Phi_\mu(z - y)]
\] (5.5)

where \( \Phi_\mu \) was introduced above and \( \Theta_4(V) \) is the 4-dimensional Heaviside function with support on the 4-volume \( \Delta V \) which is bounded by the hypersurface \( T_x(S) \) and another arbitrary hypersurface \( \bar{T}_x(S) \), both of them limited by the surface \( S \). It is easy to see that under (5.4) and (5.5) the hypersurface \( T_x(S) \) in (5.3) will be exchanged by \( \bar{T}_x(S) \) (actually the general form of the external field \( A_\mu(z;x) \) contains
a 4-volume term which vanishes whenever we choose the hypersurface to lie in the flat \( \mathbb{R}^3 \) plane as we are doing here \([7, 8]\)). This fact clearly shows that the correlation function \((5.3)\) becomes completely hypersurface independent with our choice \((5.2)\) for the renormalization factor. When we take into account the presence of the gauge fixing term, we can still go through the same procedure above by just exchanging the singular gauge transformation \((5.4)\) by the corresponding BRST transformation \([7, 8]\).

In what follows, we are going to see by explicit computation that the correlation function \((5.1)\), with the choice \((5.2)\) made for \(\mathcal{L}_{\text{CT}}\), is indeed hypersurface independent. Let us remark that in the presence of a matter coupling, in order to attain hypersurface invariance and therefore locality, by the arguments just exposed, one would have to add a \(j^\mu\) dependent piece to the operator \(\mu\). This, of course should not change the properties of this operator studied in Section 4.

Before performing the functional integration in \((5.1)\), let us observe that we can rewrite the linear term as

\[
\frac{1}{6} \int d^4z \tilde{A}_{\mu\alpha} (-\Box)^{-1} W^\mu\nu = \frac{1}{2} \int d^4z B_{\mu\nu} (-\Box)^{-1} W^\mu\nu \tag{5.6}
\]

where

\[
B_{\mu\nu}(z; x) = \frac{b}{4\pi} \int_{T_x(S)} d^3\xi \partial_\gamma \Phi_{\beta}(\xi - x) \delta^4(z - \xi) F^{\alpha\beta\gamma}_{\mu\nu} \tag{5.7}
\]

and

\[
F^{\alpha\beta\gamma}_{\mu\nu} = \partial^\alpha \Delta^{\beta\gamma}_{\mu\nu} + \partial^\beta \Delta^{\gamma\alpha}_{\mu\nu} + \partial^\gamma \Delta^{\alpha\beta}_{\mu\nu} \tag{5.8}
\]

Integrating over \(W_{\mu\nu}\) in \((5.1)\) with the help of the euclidean propagator of this field, Eq. \((3.13)\), we obtain

\[
< \mu(x)\mu^\dagger(y) > = \lim_{\rho, \delta \to 0} \exp \left\{ \frac{1}{2} \int d^4zd^4z' B^\mu\nu(z)(-\Box)^{-1} B^{\alpha\beta}(z')(\Box)^{-1} D_{\mu\alpha\beta}(z - z') \right. \\
- S_{\text{CT}} \right\} \tag{5.9}
\]

We immediately see that only the first term of \((3.13)\) contributes to \((5.9)\). In particular all the gauge dependence disappears. This happens because of the gauge invariant way in which the external field is coupled in \((5.1)\). Using the identity

\[
F^{\mu\nu\alpha}_{\sigma\tau} F^{\gamma\rho\beta}_{\lambda\chi} = -4\epsilon^{\mu\nu\alpha\sigma} \epsilon^{\gamma\rho\beta\lambda} [-\Box \delta^{\sigma\lambda} + \partial^\sigma \partial^\lambda] \tag{5.10}
\]

15
and performing the $z$ and $z'$ integrals in (5.9), we get

\[
< \mu(x)\mu^\dagger(y) > = \lim_{\rho,\delta,m,\epsilon \to 0} \exp \left\{ \frac{\hbar^2}{32\pi^2} \sum_{i,j=1}^{2} \lambda_i \lambda_j \int_{T_{x_i}} d^3 \xi \partial_\sigma \partial_\lambda \Phi_\nu (\xi - x_i) \right. \\
\times \int_{T_{x_j}} d^3 \eta \partial_\sigma \Phi_\rho (\eta - x_j) e^{\mu \nu \sigma \lambda} \epsilon^{\rho \beta} \gamma^\lambda \right. \\
\times \left[ - \Delta_\sigma \partial_\lambda + \partial_\sigma \partial_\lambda \right] \left[ - \frac{1}{8\pi^2} \ln m \gamma [ |\xi - \eta| + |\epsilon| ] \right] - S_{CT} \right\} \quad (5.11)
\]

In this expression, $x_1 \equiv x$, $x_2 \equiv y$, $\lambda_1 \equiv +1$ and $\lambda_2 \equiv -1$. The last expression between brackets is the inverse Fourier transform of $1/k^4$. It comes from the $(-\Box)^{-1}$ terms of (4.4), (5.1) and (5.9). In (5.11), $m$ is an infrared regulator used to control the the small $k$ divergences of the inverse Fourier transform of $1/k^4$ and $\gamma$ is the Euler constant. We also introduced the ultraviolet cutoff $|\epsilon|$ in order to control the short distance singularities of $F^{-1}[1/k^4]$.

The r.h.s. in (5.11) contains two terms proportional to $\Delta_\sigma \partial_\lambda$ and $\partial_\sigma \partial_\lambda$, respectively. We are going to see that the second one is hypersurface independent and leads to a local correlation function. The first one is hypersurface dependent. We show in Appendix B that it is identical to $S_{CT}$ and therefore is exactly canceled.

Let us make now an important observation concerning the renormalization counterterm $S_{CT}$ which as we said is identical to the first term in (5.11). This contains two pieces: the crossed terms (with $i \neq j$) and the self-interaction terms (with $i = j$). In Appendix C we show that the crossed terms vanish. The self-interaction terms, on the other hand, diverge in this limit. We conclude therefore that the renormalization counterterm $S_{CT}$ contains only the unphysical self interaction terms. If the crossed terms were nonvanishing our renormalization procedure would be meaningless since it would be removing a nontrivial interaction.

Let us consider now the second term in (5.11). The derivatives $\partial_\sigma (\xi)$ and $\partial_\lambda (\xi) = -\partial_\lambda (\eta)$ can be made total because of the $e^{\mu \nu \sigma \lambda}$ factors. Using the Gauss theorem we can transform the hypersurface integrals in surface integrals. Then, using the results
of Appendix D, we immediately get
\[
< \mu(x)\mu^\dagger(y) >= \lim_{m,\epsilon \to 0} \exp \left\{ \frac{b^2}{8\pi^2} \left[ -\ln m\gamma |x - y| + \ln m\gamma |\epsilon| \right] \right\} \quad (5.12)
\]
Note that the \( m\gamma \) factors cancel out. In a charge selection rule violating correlation function (like \(< \mu\mu >\), for instance) we would have the sign of the \( \ln |x - y| \) term reversed and the \( m\gamma \) factors would no longer cancel, actually forcing the correlation function to vanish in the limit \( m \to 0 \) and thereby enforcing the charge selection rule. The ultraviolet divergence at \( |\epsilon| \to 0 \) can be eliminated by a multiplicative renormalization of the field \( \mu \), namely
\[
\mu_R(x) = \mu(x)|\epsilon|^{-b^2/16\pi^2} \quad (5.13)
\]
Using this we finally get
\[
< \mu_R(x)\mu_R^\dagger(y) >= |x - y|^{-b^2/8\pi^2} \quad (5.14)
\]
This is our final expression for the \( \mu \) field euclidean two-point correlation function. Observe that it is completely local. It is remarkably similar to the \( \sigma \)-operator correlation function \((2.8)\) of the two-dimensional analog system. From the long distance behavior \( \lim_{|x - y| \to \infty} < \mu_R(x)\mu_R^\dagger(y) >= 0 \), we can infer that \(< \mu_R(x) >= 0 \) and therefore that the states \( |\mu(x) > , \) created by \( \mu \) are orthogonal to the vacuum and therefore nontrivial. The power law decay of the correlation function on the other hand implies that these states are massless.

The form of the correlation function \((5.14)\) implies that the norm of the states created by \( \mu \) is positive definite. Indeed, going back to the Minkowski space, we have
\[
\| |\mu_R > \|^2 \equiv \lim_{x \to y; (x-y)^2 < 0} < \mu_R(x)|\mu_R(y) > = \\
\lim_{x \to y; (x-y)^2 < 0} \left[ |\vec{x} - \vec{y}|^2 - (x^0 - y^0)^2 \right]^{-b^2/8\pi^2} > 0 \quad (5.15)
\]
This property shows that the states created by \( \mu \) belong to the physical sector of the Hilbert space of the \( W_{\mu\nu} \) theory even though they do not belong to QED which, in the absence of matter, constitutes the zero topological charge sector. Here again we
find the analogy with the 1+1 D massless scalar field where the corresponding above mentioned $\sigma$ operator creates states of positive norm in spite of the fact that the Hilbert space has indefinite metric.

Observe that the renormalization process used to obtain local correlation functions starting from the bare $\mu$ of course does not change the commutation rules of this field. We would like to stress that although the expression (4.3) of $\mu$ in terms of the electromagnetic field is well suited for the obtainment of commutation rules, the natural form of $\mu$ which combines with the renormalization counterterm $\mathcal{L}_{\text{CT}}$ and the lagrangian $\mathcal{L}_{W}$ in order to produce local correlation functions is the one expressed in terms of $W_{\mu\nu}$, Eq. (4.4).

An arbitrary $2n$-point correlation function could be obtained in a straightforward manner by just inserting additional external fields $\tilde{A}_{\mu\nu}$ in an expression like (5.1). They would be given by products of monomials of the type we found in $<\mu\mu^\dagger>$.

In the presence of matter we would no longer be able to obtain an exact expression for the $\mu$ correlation functions because the integration over the matter fields could no longer be done exactly. Nevertheless, due to the well known infrared asymptotic freedom of QED we still may expect that the long distance behavior of $<\mu\mu^\dagger>$ will be the same. The result that the operator $\mu(x)$ creates massless states orthogonal to the vacuum, therefore, also holds in the presence of matter fields.

Before finishing this section, let us mention a well known operator which shares some of the properties of $\mu$ as the commutator with the charge operator, for instance. This is the Wilson line, $\exp\{-i(b/4\pi)\int x A_{\mu}dx^\mu\}$. This operator is not appropriate because for it we would have nonvanishing string dependent crossed ($i \neq j$) interaction terms (see the remarks after Eq. (5.11) and Appendix C) which would inevitably lead to a nonlocal correlation function.
6) Phenomenological Estimates

In this Section, let us try to infer something about the observational characteristics of the topological excitations we have just described. Of course all that follows would only be valid in the event the more general description of QED, based on the antisymmetric tensor gauge field, would in fact be realized in nature. This is by no means guaranteed by the present formalism.

6.1) The Quantum of Charge

The charge of the states created by $\mu$ is in principle not quantized since the parameter $b$, which determines the value of the charge of the states created by $\mu$ is arbitrary, in the same way as in the two-dimensional analogous system. There is, however, an important difference from the 1+1 D case. Considering the spectral representation of the $\mu$ correlation function

$$
< \mu_R(x)\mu_R^\dagger(y) >= |x - y|^{-b^2/8\pi^2} = \int_0^\infty dM^2 \rho(M^2) \int \frac{d^4k}{(2\pi)^4} \frac{\epsilon^{ik(x-y)}}{k^2 + M^2}
$$

we immediately see that the spectral density is of the form

$$
\rho(M^2) = \begin{cases} 
\lambda (M^2)^{\sigma} & b \neq 4\pi \\
\lambda' \delta(M^2) & b = 4\pi 
\end{cases}
$$

where $\lambda$, $\lambda'$ and $\sigma$ are constants.

The spectral density is in general a power function except for $b = 4\pi$ when it is a delta function with support on a mass equal to zero. We therefore conclude that only for this value of $b$ the $\mu$-field really interpolates asymptotic states. We see that in this case, the $\mu$- correlation function behaves at long distances as a free massless scalar field correlation function. In the above mentioned 1+1 D system, on the other hand, the spectral density of the $\sigma$-field correlation function has no delta function singularity for any value of $b$ and therefore the asymptotic states associated with $\sigma$ are absent from the spectrum.

Based on the above considerations and imposing the condition that the charged topological states should be observable asymptotically, we can fix their charge to be
\( b = 4\pi \). This, of course, is in the natural units system where the fine structure constant is \( e^2/4\pi = 1/137 \) and the electron charge is \( e = (4\pi/137)^{1/2} \). The charge of these states, therefore can be expressed in terms of the electric charge as \( b = (4\pi 137)^{1/2} e \).

It would be interesting to investigate whether some additional coupling to the \( W_{\mu\nu} \) field — as it happens with the Sine-Gordon coupling in the case of the scalar field in 1+1 D — would produce a charge quantization for the topologically charged states. This would have interesting consequences in the presence of matter where some insight could possibly be obtained on the quantization of matter charge since, as we saw, in this case there are sectors where the topological charge is identified with the matter charge.

### 6.2) The Intensity Threshold

Let us show here that the eventual observation of the charged topological states of the gauge field would require the presence of an extremely intense background electromagnetic field. We can understand this by examining the expression (4.5) of \( \mu \) in terms of the electromagnetic field and the correlation function (5.14). First of all, let us remark that we can add in (5.12) a \( \ln \ell_0 \) to both terms in the exponent. This will fix the scale of the large distance regime of the correlation function. We can therefore write and expand the \( \mu \) correlation function (5.14) as

\[
< \mu_R(x)\mu_R^\dagger(y) > = \left[ \frac{|x - y|}{\ell_0} \right]^{-b^2/(8\pi^2)} = \exp \left[ -\frac{b^2}{8\pi^2} \ln \frac{|x - y|}{\ell_0} \right] = \\
\sum_{n=0}^{\infty} \frac{(-1)^n b^2 \ln \frac{|x - y|}{\ell_0}^n}{n!} 
\]

(6.3)

If we look at expression (4.3) for \( \mu \) and expand the exponential which appears there, we immediately realize that the order \( n \) in the above expansion of the \( \mu \) correlation function is nothing but the number of photons contributing to it. Hence, we can discover the minimal number of photons which is necessary to build up the coherent topological states by determining until what order \( N \) the terms in the expansion (6.3) give an important contribution.
Observe now that in the presence of matter — which is always the real situation, because at least the vacuum fluctuations of matter we can never remove — expression (6.3) represents the large distance regime of the $\mu$ correlation function. The only scale which fixes the magnitude of large distances is the electron Compton wavelegth $\lambda_e$ and therefore we make the identification $\ell_0 \equiv \lambda_e$ in (6.3).

Using the value $b = 4\pi$ which we found above and choosing $|x - y|/\lambda_e \sim 10$ (6.4) for the onset of the large distance regime we find that for $n \simeq 14$ the summand in (6.3) is of the order of 0.02 and decreases fastly for higher values of $n$. We conclude, therefore, that $N \simeq 14$ is the order of magnitude of the minimal number of photons needed to produce the topological states. These photons, however, must be within a region of linear dimension of the order of $10\lambda_e$, according to our choice (6.4).

The intensity of the background field is now readily estimated by using the formula

$$I = \frac{NEc}{V}$$

where $E$ is the photon energy, $c$ is the speed of light and $V$ is the volume where the minimal number of photons should be. As we argued above, $V \simeq 10^3 \lambda_e^3$ and $N \simeq 14$. Taking the photon wavelength to be of the order $\lambda \simeq 1\mu m$ [12] we get

$$I \simeq 6 \times 10^{18} \text{watt/cm}^2$$

(6.6)

To have an idea of how large this intensity is, we compare it to the intensity of the electromagnetic radiation at the surface of the sun which is of the order $I_\odot \simeq 6 \times 10^3 \text{watt/cm}^2$. The fact that the threshold intensity is so high would preclude the production of the $\mu$-states in ordinary processes like electron-positron annihilation, for instance. Laser fields of intensities of the order of the threshold we estimated here, however, can be constructed [12]. There is actually a projected experiment at SLAC in which high energy electrons are scattered by such a laser field [12]. This is precisely the kind of situation in which the charged topological excitations we studied here should be observed if it is indeed true that the formulation we introduced here is realized in nature.
We have seen that it is possible to introduce a new formulation of QED in terms of an antisymmetric tensor gauge field whose topological current appears as a source for the electromagnetic field. The charged states of QED lie in the sector where the topological current is identical to the charged matter current. There are, however, new nontrivial topologically charged sectors whose charge is not associated to matter but only to the gauge field. This charge would interact with the electromagnetic field and with the charged matter states of QED in the usual way described by QED. The topologically charged states are created by a gauge invariant operator and the long distance behavior of the correlation functions of this operator indicates that these states are massless. Imposition of the existence of the asymptotic states interpolated by the topological excitation creation operator fixes the value of their charge.

Of course it may well be that this more general formulation of QED is just not realized in nature. If, on the other hand, should it turn out to provide a correct description – in which event the topologically charged states would be possibly observed – an extremely intense background electromagnetic field ($I \simeq 10^{18}$ watt/cm$^2$) would be needed in order to produce them. Laser fields of this order of intensity can be obtained \[\text{[12]}\]. Scattering in the presence of such fields would be the ideal framework to test the validity of the formulation introduced here. We are presently computing the cross section for the photoproduction of a $\mu^+\mu^-$ in the presence of such strong background fields \[\text{[13]}\].

There are some interesting questions which arise in connection with these new topologically charged states: is there any mechanism of mass generation for them? What happens when we couple other fields like in the Electroweak theory, for instance? What about the usual sectors where the topological charge is identical to the matter charge? Could this formulation shed some light on the problem of quantization of matter charge?

The results we found in this work indicate that electric charge, which is a physical quantity usually associated with matter can be obtained as an attribute of some
coherent states of the gauge field itself. It is not inconceivable that other quantities like spin, mass, flavor, color and so on could be generated as well as properties of some peculiar states of the gauge fields in general. This would lead to the outstanding possibility of describing both matter and the fields which mediate its interactions within the same unified framework. We are sure this possibility is very far from what has been presented here but we hope it could be a step towards this end.

**Acknowledgements**

I would like to thank the Physics Department of Princeton University and especially C.Callan and D.Gross for the kind hospitality. I have benefited from many stimulating and enlightening discussions with several people. I am especially grateful to C.Callan, C.Teitelboim, K.McDonald, A.Polyakov, S.MacDowell, I.Kogan and S.Dalley. I am also grateful to the Brazilian National Research Council (CNPq) for financial support.
A) Appendix A

Let us demonstrate here eq. (4.12). Starting from (4.11) and applying it on < µ | to the left and on the vacuum to the right, we get

\[< \mu(x) | E^k(y) | \mu(x) >_{ET} = < \mu(x) | \mu(x) E^k(y) | 0 > + \frac{b}{4\pi |y - x|^3} < \mu(x) | \mu(x) > \] (A.1)

Since

\[\lim_{x \to y} [\mu^\dagger(x) \mu(y), E^k(z)] = 0 \] (A.2)

it follows that the second term in (A.1) vanishes and (4.12) is immediately established.

B) Appendix B

Let us demonstrate here that the first term of (5.11) is identical to \( S_{CT} \). To begin with, let us show that

\[ I = \int d^4z \partial_\alpha A_{\mu\nu}(z; x) T^{\alpha\mu\nu}(z) = \frac{b}{4\pi} \int_{T_z(S)} d^3\xi [\partial_\alpha \Phi^{(\xi)}(\xi - x)] T^{\alpha\mu\nu}(\xi) \] (B.1)

where \( A_{\mu\nu} \) is given by (4.1) and \( T^{\mu\nu\alpha}(z) \) is an arbitrary tensor. According to (4.1) we have

\[ \partial_\alpha^{(z)} A_{\mu\nu}(z; x) = \frac{b}{4\pi} \int_{T_z(S)} d^3\xi \partial_\alpha \Phi^{(\xi)}(\xi - x) \delta^4(z - \xi) - \hat{n}^\mu \int_S d^2\xi \Phi^{(\xi)}(\xi - x) \delta^4(z - \xi) - (\mu \leftrightarrow \nu) \] (B.2)

where \( \hat{n}^\mu \) is the unit vector in the direction of \( d^3\xi^\mu \). Using Gauss’ theorem in the last term and integrating in \( d^4z \) with \( T^{\alpha\mu\nu} \) we get

\[ I = \frac{b}{4\pi} \int_{T_z(S)} d^3\xi [\partial_\alpha \Phi^{(\xi)}(\xi - x)] (-\partial_\alpha^{(\xi)}) T^{\alpha\mu\nu}(\xi) + \frac{b}{4\pi} \int_{T_z(S)} d^3\xi [\partial_\alpha^{(\xi)} \Phi^{(\xi)}(\xi - x)] \] (B.3)

thus establishing (B.1). Now consider the first term of (5.11). Antisymmetrizing in \( \mu, \nu \) and \( \gamma, \rho \) and using the fact that the last expression between brackets is \((-\Box)^{-2}\) as well as (B.3), we can write the first term of (5.11) as

\[ \frac{1}{8} \int d^4z d^4z' \epsilon^{\mu\nu\alpha\sigma} \partial_\alpha A_{\mu\nu}(z) \frac{1}{(-\Box)^2} \epsilon^{\gamma\rho\beta\sigma} \partial_\beta A_{\gamma\rho} = \]
\[
\frac{1}{12} \int d^4z A_\alpha \Delta (\Box)^{-1} A^\alpha
\] (B.4)

which is precisely \( S_{CT} \).

**C) Appendix C**

Let us show here that the \( i \neq j \) terms of \( S_{CT} \) vanish in the limit when \( \rho \rightarrow 0 \). From (5.11), we see that these terms are proportional to

\[
I = \int_{T_x} d^3\xi \frac{\partial}{\partial \alpha} \Phi (\vec{\xi} - \vec{x}_i)
\]

\[
\times \int_{T_y} d^3\eta \frac{\partial}{\partial \alpha} \Phi (\vec{\eta} - \vec{x}_i) \epsilon^{\mu\nu\alpha \lambda} \epsilon^{\sigma\rho\beta \lambda} \mathcal{F}^{-1} \left[ (\Box)^{-1} \right] (C.1)
\]

Using the fact that \( d^3\xi / / d^3\eta \) // \( \hat{n}^4 \) and the identity (5.2), we can write (C.1) as

\[
I = \int_{T_x} d^3\xi \frac{\partial}{\partial \alpha} \Phi (\vec{\xi} - \vec{x}_i)
\]

\[
\times \int_{T_y} d^3\eta \frac{\partial}{\partial \alpha} \Phi (\vec{\eta} - \vec{x}_i) \epsilon^{\mu\nu\alpha \lambda} \epsilon^{\sigma\rho\beta \lambda} \mathcal{F}^{-1} \left[ (\Box)^{-1} \right] (C.2)
\]

Taking the same steps which led us to (B.1), we can write

\[
I = \int d^4z d^4z' \partial^\alpha C^\mu (z; x) [(-\Box)^{-1}] \partial_\alpha C^\mu (z'; y) (C.3)
\]

where

\[
C^\mu (z; x) = \int_{T_x(S)} d^3\xi \frac{1}{|\vec{\xi} - \vec{x}|} \delta^4 (z - \xi) (C.4)
\]

Integrating by parts the derivatives in (C.3), we get a delta function. Subsequent integration over \( z' \) gives

\[
I = \int C^\mu (z; x) C_\mu (z; y) =
\]

\[
\int_{T_x(S)} d^3\xi \int_{T_y(S)} d^3\eta \frac{1}{|\vec{\xi} - \vec{x}|} \delta^4 (\xi - \eta) \frac{1}{|\vec{\eta} - \vec{y}|} (C.5)
\]

This expression vanishes due to the presence of the delta function and because of the fact that the hypersurfaces \( T_x \) and \( T_y \) can always be chosen so as to have an empty intersection.

For the \( i = j \) terms we would have \( T_x = T_y \) and the analogous integral would be seen to diverge.
Let us demonstrate here how to obtain results (4.8) and (5.12). In both cases, we have one or more integrals of the type

\[ \lim_{\rho,\delta \to 0} \int_{T_x(S)} d^3 \xi \epsilon^{ijk} \partial_i^{(\xi)} \Phi_j (\xi - \bar{x}) \partial_k^{(\xi)} F(\xi) \] (D.1)

for some function \( F(\xi) \). The derivative \( \partial_k \) can be made total because of the rotational. Then, we can use Gauss’ theorem to get

\[ \lim_{\rho,\delta \to 0} \oint_{S_x(\bar{x})} d^2 \xi k \epsilon^{ijk} \partial_i^{(\xi)} \Phi_j (\xi - \bar{x}) F(\xi) \] (D.2)

Now, making use of the identity (4.2), we see that the cone piece of \( S_x(\bar{x}) \) does not contribute to (D.2) and we can already take the limit \( \delta \to 0 \). Inserting (4.2) in (D.2) we get

\[ \lim_{\rho \to 0} \int_{\text{Sphere}(\bar{x})} d\Omega \rho^2 \left[ \frac{1}{\rho^2} \right] F(\xi) = 4\pi F(\bar{x}) \] (D.3)

This result leads immediately to (4.8) and (5.12).

References

[1] S.Coleman, “Aspects of Symmetry”, Cambridge, (1985).

[2] R.Köberle and E.C.Marino, Phys. Lett. 126B (1983) 475.

[3] A.S.Wightman, “Introduction to Some Aspects of the Relativistic Dynamics of Quantum Fields” in Cargèse Lectures in Theoretical Physics 1964, M.Lévy editor, Gordon and Breach, NY (1967).

[4] R.L.P.G. Amaral and E.C.Marino, J. of Phys. A25 (1992) 5183.

[5] D.M.Gitman and I.V.Tyutin, “Quantization of Fields with Constraints” Springer-Verlag, Berlin (1990).

[6] E.C.Marino and J.A.Swieca, Nucl.Phys. B170 [FS1], 175 (1980). E.C.Marino, B.Schroer and J.A.Swieca, Nucl.Phys. B200 [FS4], 473 (1982).
[7] E.C.Marino, *Phys.Rev.* **D38**, 3194 (1988).

[8] E.C.Marino and J.E.Stephany Ruiz, *Phys.Rev.* **D39**, 3690 (1989)

[9] E.C.Marino, “Duality, Quantum Vortices and Anyons in Maxwell-Chern-Simons-Higgs Theories”, Princeton University report PUPT-1330, *Annals of Physics* (1993), in press.

[10] L.P.Kadanoff and H.Ceva, *Phys.Rev.* **B3**, 3918 (1971). E.Fradkin and L.Susskind, *Phys.Rev.* **D17**, 2637 (1978). J.B.Kogut, *Rev.Mod.Phys.* **51**, 659 (1979).

[11] E.C.Marino, “Dual Quantization of Solitons” in Proceedings of the NATO Advanced Study Institute “Applications of Statistical and Field Theory Methods to Condensed Matter”, D.Baeriswyl, A.Bishop and J.Carmelo , editors (Plenum,New York)(1990)

[12] K.T.McDonald, private communication.

[13] E.C.Marino, “The Cross Section for Photoproduction of Topologically Charged Gauge Field States Associated with QED”, to appear.
Figure Caption

Fig. 1 - Three dimensional hypersurface used in the definition of the external field $A_{\mu\nu}(z; x)$ and the operator $\mu$ (actually a cut of it).