The Statefinder hierarchy: An extended null diagnostic for concordance cosmology

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We show how higher derivatives of the expansion factor can be developed into a null diagnostic for concordance cosmology (ΛCDM). It is well known that the Statefinder – the third derivative of the expansion factor written in dimensionless form, $a^3/aH^3$, equals unity for ΛCDM. We generalize this result and demonstrate that the hierarchy, $a^n/aH^n$, can be converted to a form that stays pegged at unity in concordance cosmology. This remarkable property of the Statefinder hierarchy enables it to be used as an extended null diagnostic for the cosmological constant. The Statefinder hierarchy combined with the growth rate of matter perturbations defines a composite null diagnostic which can distinguish evolving dark energy from ΛCDM.

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INTRODUCTION

Despite its radical connotations, there is mounting observational evidence in support of a universe that is currently accelerating [1]. Theoretically there appear to be two distinct ways in which the universe can be made to accelerate [2]: (i) through the presence of an additional component in the matter sector, which, following [2], we call physical dark energy. Physical DE models possess large negative pressure and lead to the violation of the strong energy condition, $\rho + 3P \geq 0$, which forms a necessary condition for achieving cosmic acceleration. Prominent examples of this class of models include the cosmological constant ‘Λ’, Quintessence, the Chaplygin gas, etc. (ii) The universe can also accelerate because of changes in the gravitational sector of the theory. These models (sometimes referred to as geometrical DE or modified gravity) include $f(R)$ theories, extra-dimensional Braneworld models, etc.

Due to its elegance and simplicity the cosmological constant, with $P = -\rho$, occupies a privileged place in the burgeoning pantheon of DE models. Although the reasons behind the extremely small value of $\Lambda$ remain unclear, concordance cosmology (ΛCDM) does appear to provide a very good fit to current data (although possible departures from $P = -\rho$ have also been noted [2]). Given the success and simplicity of concordance cosmology it is perhaps natural to discuss diagnostic measures which can be used to compare a given DE model with ΛCDM. ‘Null measures’ of concordance cosmology proposed so far include the Om diagnostic [3, 4], and the Statefinders [4]. While Om involves measurements of the expansion rate, $H(z)$, the Statefinders are related to the third derivative of the expansion factor. In a spatially flat ΛCDM universe, the Statefinders and Om remain pegged at a fixed value during our recent expansion history ($z \lesssim 10^8$).

In this letter we introduce the notion of the ‘Statefinder hierarchy’ which includes higher derivatives of the expansion factor $d^n a/ dt^n, n \geq 2$. We demonstrate that, for concordance cosmology, all members of the Statefinder hierarchy can be expressed in terms of elementary functions of the deceleration parameter $q$ (equivalently the density parameter $\Omega_m$). This property singles out the cosmological constant from evolving DE models and allows the Statefinder hierarchy to be used as an extended null diagnostic for ΛCDM.

THE STATEFINDER HIERARCHY

The expansion factor of the universe can be Taylor expanded around the present epoch $t_0$ as follows

$$(1 + z)^{-1} := \frac{a(t)}{a_0} = 1 + \sum_{n=1}^{\infty} \frac{A_n(t_0)}{n!} [H_0(t - t_0)]^n \quad (1)$$

where

$$A_n := \frac{a^{(n)}}{aH^n}, \quad n \in N, \quad (2)$$

$a^{(n)}$ is the $n^{th}$ derivative of the scale factor with respect to time. Historically different letters of the alphabet have been used to describe various derivatives of the scale factor. Thus $q = -A_2$ is the deceleration parameter, while $A_3$, which was first discussed in [7], has been called the Statefinder ‘r’ [8] as well as the jerk ‘j’ [6]. $A_4$ is the snap ‘s’, $A_5$ is the jerk ‘l’, etc. (See for instance [6] and references therein.)

It is quite remarkable that, in a spatially flat universe consisting of pressureless matter and a cosmological constant (henceforth referred to as concordance cosmology or ΛCDM), all the $A_n$ parameters can be expressed as elementary functions of the deceleration parameter $q$, or the density parameter $\Omega_m$. For instance

[1] It is easy to see that successive differentiations of $\ddot{a} = a(H + H^2)$ and $R = 6(H + 2H^2)$ in conjunction with [8], can be used to express higher derivatives of the Hubble parameter, $H^{(n)}/H^{n+1}$, and the Ricci scalar, $R^{(n)}/H^{n+2}$, as polynomial expansions in $q$ (or $\Omega_m$), for concordance cosmology.
define the Statefinder hierarchy where $\Omega_m$ is pegged at unity, see figure 1. All parameters evolve with time with even $A_{2n}$ (odd $A_{2n+1}$) remaining smaller (larger) than unity, see figure 1. All $A_n$ approach unity in the distant future: $A_n \to 1$ when $\Omega_m \to 0$ and $\Omega_\Lambda \to 1$. The above expressions allow us to define the Statefinder hierarchy $S_n$:

$$A_2 = 1 - \frac{3}{2}\Omega_m,$$

$$A_3 = 1,$$

$$A_4 = 1 - \frac{3^2}{2}\Omega_m,$$

$$A_5 = 1 + 3\Omega_m + \frac{3^3}{2}\Omega_m^2,$$

$$A_6 = 1 - \frac{3^3}{2}\Omega_m - 3^4\Omega_m^2 - \frac{3^4}{4}\Omega_m^3,$$

where $\Omega_m = \Omega_m(1+z)^3/h^2(z)$ and $\Omega_m = \frac{2}{3}(1+q)$ in concordance cosmology. It is interesting to note that while $A_3$ remains pegged at unity, the remaining $A_n$ parameters evolve with time with even $A_{2n}$ (odd $A_{2n+1}$) remaining smaller (larger) than unity, see figure 1. All $A_n$ approach unity in the distant future: $A_n \to 1$ when $\Omega_m \to 0$ and $\Omega_\Lambda \to 1$. The above expressions allow us to define the Statefinder hierarchy $S_n$:

$$S_2 := A_2 + \frac{3}{2}\Omega_m,$$

$$S_3 := A_3,$$

$$S_4 := A_4 + \frac{3^2}{2}\Omega_m,$$

$$S_5 := A_5 - 3\Omega_m - \frac{3^3}{2}\Omega_m^2,$$

$$S_6 := A_6 + \frac{3^3}{2}\Omega_m + 3^4\Omega_m^2 + \frac{3^4}{4}\Omega_m^3,$$

where $\Omega_m$ is pegged at zero. Consequently the Statefinder pair $\{S_3^{(1)}, S_3^{(2)}\} = \{1, 0\}$ provides a model independent means of distinguishing evolving dark energy models from the cosmological constant $\Lambda$. In analogy with (7) we define the second member of the Statefinder hierarchy as follows:

$$S_n = S_n^{(2)} = \frac{S_n^{(1)} - 1}{\alpha(q - 1/2)}. \quad (8)$$

where $\alpha$ is an arbitrary constant. In concordance cosmology $S_3^{(1)} = 1$ while $S_3^{(2)}$ stays pegged at zero. Consequently the Statefinder pair $\{S_3^{(1)}, S_3^{(2)}\} = \{1, 0\}$ provides a model independent means of distinguishing evolving dark energy models from the cosmological constant $\Lambda$. In analogy with (7) we define the second member of the Statefinder hierarchy as follows:

$$S_n^{(2)} = \frac{S_n^{(1)} - 1}{\alpha(q - 1/2)}. \quad (8)$$

Interestingly, for $n > 3$ there is more than one way in which to define a null diagnostic. Using the relationship $\Omega_m = \frac{2}{3}(1+q)$, valid in $\Lambda$CDM, it is easy to see that the Statefinders can also be written in the alternate form

$$S_4^{(1)} := A_4 + 3(1+q)$$

$$S_5^{(1)} := A_5 - 2(4+3q)(1+q), \; \text{etc.} \quad (6)$$

Clearly both methods yield identical results for $\Lambda$CDM: $S_4^{(1)} = S_4 = 1$, $S_5^{(1)} = S_5 = 1$. For other DE models however, the two alternate definitions of the statefinder, $S_n$ & $S_n^{(1)}$, are expected to give different results.

In [8] it was shown that a second Statefinder could be constructed from the Statefinder $S_3^{(1)} := S_3$, namely

$$S_3^{(2)} = \frac{S_3^{(1)} - 1}{3(q - 1/2)}. \quad (7)$$

In concordance cosmology $S_3^{(1)} = 1$ while $S_3^{(2)}$ stays pegged at zero. Consequently the Statefinder pair $\{S_3^{(1)}, S_3^{(2)}\} = \{1, 0\}$ provides a model independent means of distinguishing evolving dark energy models from the cosmological constant $\Lambda$. In analogy with (7) we define the second member of the Statefinder hierarchy as follows:

$$S_n^{(2)} = \frac{S_n^{(1)} - 1}{\alpha(q - 1/2)}. \quad (8)$$

where $\alpha$ is an arbitrary constant. In concordance cosmology $S_n^{(2)} = 0$ and

$$\{S_n^{(1)}, S_n^{(2)}\} = \{1, 0\}. \quad (9)$$

For other null tests including: $\Omega_m = \frac{1}{3}(1+q) + \Omega_m(z)$, $\Sigma_1 := 2(1 - A_4)/9\Omega_m$, $\Sigma_2 := A_4 + 3(A_3 - A_2)$, where $\Omega_m = 1$, $\Sigma_{1,2} = 1$ for $\Lambda$CDM.
The second statefinder $S_n^{(2)}$ serves the useful purpose of breaking some of the degeneracies present in $S_n^{(1)}$. For DE with a constant equation of state

\[
S_3^{(1)} = 1 + \frac{9w}{2}(w + 1)\Omega_{DE}
\]

\[
S_3^{(2)} = w + 1
\]

\[
S_4^{(1)} = 1 - \frac{27}{2}w(w + 1)(w + \frac{7}{6})\Omega_{DE} - \frac{27}{4}w^2(w + 1)\Omega_{DE}^2
\]

\[
S_4^{(2)} = -(w + 1)(w + \frac{7}{6}) - \frac{1}{2}w(w + 1)\Omega_{DE}
\]

where $S_4^{(2)} = \frac{S_4^{(1)} - 1}{9q - \frac{4}{2}}$ and $q - 1/2 = \frac{3w}{2}\Omega_{DE}$.

As we demonstrate in figures 2, 3, 4, 5, the Statefinder hierarchy $\{S_n^{(1)}, S_n^{(2)}\}$ provide us with an excellent means of distinguishing dynamical DE models from ΛCDM. Our discussion focuses on the following models:

1. Dark energy with a constant equation of state

\[
\frac{H(z)}{H_0} = \left[\Omega_{0m}(1 + z)^3 + \Omega_{DE}(1 + z)^3(1 + w)\right]^{1/2}
\]

where concordance cosmology (ΛCDM) corresponds to $w = -1$. For simplicity we neglect the presence of spatial curvature and radiation.

2. The Chaplygin gas [12] has the interesting EOS $p_c = -A/\rho_c$ while its density evolves as $\rho_c = \sqrt{A + B(1 + z)^6}$. The expansion rate of a universe containing the Chaplygin gas and pressureless matter is given by

\[
\frac{H(z)}{H_0} = \left[\Omega_{0m}(1 + z)^3 + \frac{\Omega_{0m}}{\kappa}\sqrt{\frac{A}{B} + (1 + z)^6}\right]^{1/2}
\]

where $\kappa$ defines the ratio between the density in cold dark matter and the Chaplygin gas at the commencement of the matter-dominated stage of expansion and

\[
A = B\left\{\kappa^2\left(1 - \frac{\Omega_{0m}}{\Omega_{0m}}\right)^2 - 1\right\}.
\]

3. The Braneworld model suggested by Dvali, Gabadadze and Porrati [13]

\[
\frac{H(z)}{H_0} = \left[\frac{1 - \Omega_{0m}}{2} + \sqrt{\Omega_{0m}(1 + z)^3 + \left(\frac{1 - \Omega_{0m}}{2}\right)^2}\right]^{1/2}
\]
FIG. 3: The left (right) panel shows the Statefinder $S_4^{(1)}$ plotted against $\Omega_m$ (left panel) and $z$ (right panel). The dark energy models are: DE with $w = -0.8$ (blue), phantom with $w = -1.2$ (green), Chaplygin gas (purple), DGP (red). The horizontal black line shows $\Lambda$CDM. The vertical band centered at $\Omega_{0m} = 0.3$ in the left panel roughly corresponds to the present epoch. $\Omega_{0m} = 0.3$ is assumed for DE models in the right panel.

FIG. 4: The left (right) panel shows the Statefinder $S_4^{(2)}$ ($S_3^{(2)}$) plotted against $\Omega_m$ (left panel). The vertical band centered at $\Omega_{0m} = 0.3$ in the left panel corresponds to the present epoch. Comparing figure 4 with figure 3 we find that $S_4^{(2)}$ does not appear to perform as well as $S_4^{(1)}$, or even $S_3^{(2)}$, in distinguishing between the DE models considered in this paper.

GROWTH RATE OF PERTURBATIONS

The Statefinders can be usefully supplemented by the fractional growth parameter $\epsilon(z)$ \[18\]

$$
\epsilon(z) := \frac{f(z)}{f_{\Lambda CDM}(z)}.
$$

where $f(z) = d\log \delta/d\log z$ describes the growth rate of linearized density perturbations \[14\]

$$
f(z) \simeq \Omega_m(z)^\gamma
$$

$$
\gamma(z) = \frac{3}{5 - \frac{w}{1 - w}} + \frac{3}{125} \frac{(1 - w)(1 - \frac{3}{8}w)}{(1 - \frac{5}{8}w)^3} (1 - \Omega_m(z)) + \mathcal{O}(1 - \Omega_m(z))^2.
$$
The statefinders $S_4^{(1)}$ and $S_5^{(1)} = S_5$ are shown for the DE models discussed in the previous figures. The fixed point at \{1, 1\} is $\Lambda$CDM. The arrows show time evolution and the present epoch in the different models is shown as a dot. $\Omega_{\text{m0}} = 0.3$ is assumed.

The above approximation works reasonably well for physical DE models in which $w$ is either a constant, or varies slowly with time. For instance $\gamma \simeq 0.55$ for $\Lambda$CDM \[14, 15\]. It may be noted that in physical DE models such as Quintessence, the development of perturbations can be reconstructed from a knowledge of the expansion history \[16\]. This is not the case in modified gravity theories in which perturbation growth contains information which is complementary to that contained in the expansion history. (Eqn. \[11\] is still a valid approximation for DGP cosmology but with $\gamma \simeq 2/3$ \[17\].)

For this reason, the \emph{fractional growth parameter} $\epsilon(z)$, can be used in conjunction with the Statefinders to define a \emph{composite null diagnostic} $\text{CND} \equiv \{S_n, \epsilon\}$, where $\{S_n, \epsilon\} = \{1, 1\}$ for $\Lambda$CDM. Figure 6 demonstrates how DE models are distinguished by means of CND. (One can also use the $\Omega m$ diagnostic in place of the Statefinder in CND.)

CONCLUSIONS

In this paper we have shown that a simple series of relationships links the Statefinder hierarchy in concordance cosmology with the deceleration/density parameters. These relationships can be used to define \emph{null tests} for the cosmological constant. Including information pertaining to the growth rate of perturbations increases the effectiveness of this hierarchy of null diagnostics. Our results demonstrate that lower order members of the Statefinder hierarchy already differentiate quite well between concordance cosmology on the one hand, and braneworld models and the Chaplygin gas, on the other. Since, for $n \geq 3$, the $n^{th}$ Statefinder $S_n$ contains terms proportional to $w^{(n-2)}/H^{n-2}$, higher members of the hierarchy will contain progressively greater information about the evolution of the equation of state of DE. From the observational perspective, however, one might note that a determination of the $S_n$ statefinders involves prior knowledge of the $(n-1)^{th}$ derivative of $H(z)$. Thus only lower order Statefinders, $S_n$, $n \leq 4$, together with the $\Omega m$ diagnostic, may prove compatible with the quality of observational data expected in the near future.

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