Robust satisfiability check and online control synthesis for uncertain systems under signal temporal logic specifications

Pian Yu\textsuperscript{a\ast*}, Yulong Gao\textsuperscript{a}, Karl H. Johansson\textsuperscript{a}, Dimos V. Dimarogonas\textsuperscript{a}

\textsuperscript{a}Division of Decision and Control Systems, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, SE-10044 Stockholm, Sweden

Abstract

This paper studies the robust satisfiability check and online control synthesis problems for uncertain discrete-time systems subject to signal temporal logic (STL) specifications. Different from existing techniques, this work proposes an approach based on STL, reachability analysis, and temporal logic trees. Firstly, a real-time version of STL semantics and a tube-based temporal logic tree are proposed. We show that such a tree can be constructed from every STL formula. Secondly, using the tube-based temporal logic tree, a sufficient condition is obtained for the robust satisfiability check of the uncertain system. When the underlying system is deterministic, a necessary and sufficient condition for satisfiability is obtained. Thirdly, an online control synthesis algorithm is designed. It is shown that when the STL formula is robustly satisfiable and the initial state of the system belongs to the initial root node of the tube-based temporal logic tree, it is guaranteed that the trajectory generated by the controller satisfies the STL formula. The effectiveness of the proposed approach is verified by an automated car overtaking example.

Keywords: signal temporal logic, robust satisfiability check, control synthesis, and reachability analysis

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\textsuperscript{\ast\ast}Corresponding author.

Email addresses: piany@kth.se (Pian Yu ), yulongg@kth.se (Yulong Gao), kallej@kth.se (Karl H. Johansson), dimos@kth.se (Dimos V. Dimarogonas)
1. Introduction

1.1. Motivation and Related Work

Rapid growth of robotic applications, such as autonomous vehicles and service robots, has stimulated the need for new control synthesis approaches to safely accomplish more complex objectives such as nondeterministic, periodic, or sequential tasks. Temporal logics, such as linear temporal logic (LTL) \cite{1}, metric interval temporal logic (MITL) \cite{2}, and signal temporal logic (STL) \cite{3}, have shown capability in expressing such objectives for dynamical systems in the last decade. Various control approaches have been developed accordingly.

LTL focuses on the Boolean satisfaction of properties by given signals while MITL is a continuous-time extension that allows to express temporal constraints. Existing control approaches that use LTL or MITL mainly rely on a finite abstraction of the system dynamics and a language equivalent automata \cite{4} or timed-automata \cite{5} representation of the LTL or MITL specification. The controller is synthesized by solving a game over the product automata \cite{6, 7, 8}. Other control approaches include optimization-based \cite{9, 10} and sampling-based methods \cite{11, 12}. STL is a more recently developed temporal logic, which allows the specification of properties over dense-time. Due to a number of advantages, such as explicitly treating real-valued signals \cite{3}, and admitting qualitative semantics \cite{13}, control synthesis under STL specifications has gained popularity in the last few years.

Different from LTL or MITL, automata-based methods have not been developed for STL specifications to the same extent due to their complexity. Existing approaches that deal with control synthesis under STL specifications include barrier function \cite{14, 15} and optimization methods \cite{16, 17, 18}. Barrier function methods are mainly used for continuous-time systems. The idea is to transfer the STL formula into one or several (time-varying) control barrier functions, and then obtain feedback control laws by solving quadratic programs \cite{14, 15}. This method is computationally efficient. However, as the existence and design of barrier functions are still open problems, it currently mainly applies to deterministic affine systems. Optimization methods are mainly used for discrete-time systems. The idea is to encode STL formulas as mixed-integer constraints, and then the satisfying controller can be obtained by solving a series of optimization problems \cite{16, 17}. An extension of the mixed-integer formulation is investigated for linear systems with additive bounded disturbances in \cite{18}, where the controller is obtained by solving the optimization problem at each time step in a receding horizon fashion. The mixed-integer programming approach is sound but not complete for
uncertain systems. Moreover, it deals with only bounded STL specifications. Other control synthesis approaches include sampling-based [19] and learning-based methods [20, 21].

We note that although various methods exist for the control synthesis under STL specifications, guaranteeing robustness under uncertainties is still a challenging problem. One core contribution of this paper is on robust control synthesis for uncertain systems under STL specifications. In addition, we also study the satisfiability check problem, i.e., the problem to check whether or not there exists a control policy such that the resulting trajectory of the underlying system satisfies properties specified by the temporal logic formula. To the best of our knowledge, the satisfiability check problem and its robust variant for uncertain systems (i.e., check the existence of a control policy under all possible uncertainties) under STL specifications have not been solved so far.

1.2. Main Contributions and Organization

Motivated by the above considerations, this work considers the robust satisfiability check and online control synthesis problems for uncertain discrete-time systems under STL specifications. The paper is inspired by [22], where relationships between temporal operators and reachable sets are developed, and [23], where the notion of temporal logic tree is proposed for LTL. However, we note that it is far from straightforward to extend these results to general STL formulas. The contributions of our paper are summarized as follows:

(i) A real-time version of satisfaction relation and a tube-based temporal logic tree (tTLT) are proposed for STL formulas. A correspondence between STL formulas and tTLT is established via reachability analysis on the underlying systems. An algorithm is proposed for the automated construction of tTLT. Note that the tTLTs in this paper are different from the TLTs defined for LTL formulas in [23], due to the time constraints encoded in the STL formulas.

(ii) We use the tTLT to address the STL robust satisfiability check problem. A sufficient condition is obtained for uncertain systems. That is, the STL robust satisfiability check can be replaced by a tTLT robust satisfiability check. Moreover, when the underlying system is deterministic, a necessary and sufficient condition is obtained for the satisfiability check.
We solve the STL control synthesis problem for uncertain systems. An online control synthesis algorithm is proposed based on the constructed tTLT from the STL formula. When the STL formula is robustly satisfiable and the initial state of the system belongs to the initial root node of the tTLT, it is proven that the trajectory generated by the proposed online control synthesis algorithm satisfies the STL formula.

The remainder of the paper is organized as follows. In Section II, preliminaries and the problem under consideration are formulated. In Section III, definitions of real-time STL semantics and tTLT are introduced. Section IV deals with the robust satisfiability check problem and Section V with the online control synthesis problem. The results are validated by simulations in Section VI. Conclusions are given in Section VII.

Notation. Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq} := [0, \infty)$. Let $\mathbb{N}$ be the set of natural numbers. Denote $\mathbb{R}^n$ as the $n$ dimensional real vector space, $\mathbb{R}^{n \times m}$ as the $n \times m$ real matrix space. Given a vector $x \in \mathbb{R}^n$, define $\|x\|$ and $x^T$ as the Euclidean norm and the transpose of vector $x$, respectively. Given a set $\Omega$, $\overline{\Omega}$ denotes its complement, $2^\Omega$ denotes its powerset, and $|\Omega|$ denotes its cardinality. The operators $\cup$ and $\cap$ represent set union and set intersection, respectively. In addition, we use $\land$ to denote the logical operator AND and $\lor$ to denote the logical operator OR. The set difference $A \setminus B$ is defined by $A \setminus B := \{ x : x \in A \land x \notin B \}$.

2. Preliminaries and problem formulation

2.1. Systems dynamics

Consider an uncertain discrete-time control system of the form

$$x_{k+1} = f(x_k, u_k, w_k),$$

(1)

where $x_k := x(t_k) \in \mathbb{R}^n$, $u_k := u(t_k) \in U$, $w_k := w(t_k) \in W$, $k \in \mathbb{N}$ are the state, control input, and disturbance at time $t_k$, respectively. The time sequence $\{t_k\}$ can be seen as a sequence of sampling instants, which satisfy $t_0 < t_1 < \cdots$. The control input is constrained to a compact set $U \subset \mathbb{R}^n$ and the disturbance is constrained to a compact set $W \subset \mathbb{R}^l$.

Definition 2.1. A control policy $\nu = \nu_0\nu_1 \ldots \nu_k \ldots$ is a sequence of maps $\nu_k : \mathbb{R}^n \to U$, $\forall k \in \mathbb{N}$. Denote by $U_{\geq k}$ the set of all control policies that start from time $t_k$. 
Definition 2.2. A disturbance signal \( w = w_0w_1 \ldots w_k \ldots \) is called admissible if \( w_k \in W, \forall k \in \mathbb{N} \). Denote by \( W_{\geq k} \) the set of all admissible disturbance signals that start from time \( t_k \).

The solution of (1) is defined as a discrete-time signal \( x := x_0x_1 \ldots \). We call \( x \) a trajectory of (1) if there exists a control policy \( \nu \in U_{\geq 0} \) and a disturbance signal \( w \in W_{\geq 0} \) satisfying (1), i.e.,

\[
x_{k+1} = f(x_k, \nu_k(x_k), w_k), \forall k.
\]

We use \( x_{x_0}^{\nu,w}(t_k) \) to denote the trajectory point reached at time \( t_k \) under the control policy \( \nu \) and the disturbance \( w \) from initial state \( x_0 \) at time \( t_0 \).

The deterministic system is defined by

\[
x_{k+1} = f_d(x_k, u_k)
\]

and \( x_{x_0}^{\nu}(t_k) \) denotes the solution at time \( t_k \) of the deterministic system when the control policy is \( \nu \) and the initial state is \( x_0 \) at time \( t_0 \).

2.2. STL

STL [3] is a predicate logic consisting of predicates \( \mu \), which are defined through a predicate function \( g_\mu : \mathbb{R}^n \rightarrow \mathbb{R} \) as

\[
\mu := \begin{cases} 
\top, & \text{if } g_\mu(x) \geq 0 \\
\bot, & \text{if } g_\mu(x) < 0.
\end{cases}
\]

The syntax of STL is given by

\[
\varphi ::= \top | \mu | \neg \varphi | \varphi_1 \land \varphi_2 | \varphi_1 \cup \varphi_2,
\]

where \( \varphi, \varphi_1, \varphi_2 \) are STL formulas and \( I \) is a closed or half-closed interval of \( \mathbb{R} \) of the form \([a, b]\) or \([a, b)\) with \( a, b \in \mathbb{R}_{\geq 0} \cup \infty \) and \( a \leq b \).

The validity of a STL formula \( \varphi \) with respect to a discrete-time signal \( x \) at time \( t_k \), is defined inductively as follows [16]:

\[
(x, t_k) \vdash \mu \iff g_\mu(x(t_k)) \geq 0,
\]

\[
(x, t_k) \vdash \neg \varphi \iff \neg((x, t_k) \vdash \varphi),
\]

\[
(x, t_k) \vdash \varphi_1 \land \varphi_2 \iff (x, t_k) \vdash \varphi_1 \land (x, t_k) \vdash \varphi_2,
\]

\[
(x, t_k) \vdash \varphi_1 \cup [a, b] \varphi_2 \iff \exists t_{k'} \in [t_k + a, t_k + b] \text{ s.t. } (x, t_{k'}) \vdash \varphi_2 \land \forall t_{k''} \in [t_k, t_{k'}], (x, t_{k''}) \vdash \varphi_1.
\]
The signal \( x = x_0x_1 \ldots \) satisfies \( \varphi \), denoted by \( x \models \varphi \) if \( (x, t_0) \models \varphi \).

By using the negation operator \( \neg \) and the conjunction operator \( \wedge \), we can define disjunction, \( \varphi_1 \lor \varphi_2 = \neg(\neg \varphi_1 \land \neg \varphi_2) \). And by employing the until operator \( U \), we can define: (1) eventually, \( F\varphi = T U \varphi \); and (2) always, \( G\varphi = \neg F \neg \varphi \).

**Definition 2.3.** [24] The time horizon \( \| \varphi \| \) of a STL formula \( \phi \) is inductively defined as

\[
\| \varphi \| = \begin{cases} 0, & \text{if } \varphi = \mu \\ \| \varphi_1 \|, & \text{if } \varphi = \neg \varphi_1 \\ \max\{\| \varphi_1 \|, \| \varphi_2 \|\}, & \text{if } \varphi = \varphi_1 \land \varphi_2 \\ b + \max\{\| \varphi_1 \|, \| \varphi_2 \|\}, & \text{if } \varphi = \varphi_1 U_{[a,b]} \varphi_2. \end{cases}
\]

**Definition 2.4.** (Satisfiability) Consider the deterministic system \( [2] \) and the STL formula \( \varphi \). We say \( \varphi \) is satisfiable from the initial state \( x_0 \) if there exists a control policy \( \nu \) such that

\[
x_{x_0}^{\nu} \models \varphi. \tag{4}
\]

**Definition 2.5.** (Robust satisfiability) Consider the uncertain system \( [4] \) and the STL formula \( \varphi \). We say \( \varphi \) is robustly satisfiable from the initial state \( x_0 \) if there exists a control policy \( \nu \) such that

\[
x_{x_0}^{\nu, w} \models \varphi, \forall w \in W_{\geq 0}. \tag{5}
\]

Given an STL formula \( \varphi \), let

\[
S_{\varphi} := \{ x_0 \in \mathbb{R}^n | \varphi \text{ is (robustly) satisfiable from } x_0 \} \tag{6}
\]
denote the set of initial states from which \( \varphi \) is (robustly) satisfiable.

**2.3. Reachability operators**

The definitions of maximal and minimal reachable tube are given as follows.

**Definition 2.6.** Consider the system \( [7] \), three sets \( \Omega_1, \Omega_2, \mathcal{C} \subseteq \mathbb{R}^n \), and a time interval \([a, b]\). The maximal reachable tube from \( \Omega_1 \) to \( \Omega_2 \) is defined as

\[
\mathcal{R}^M(\Omega_1, \Omega_2, \mathcal{C}, [a, b], k) = \left\{ x_k \in \Omega_1 \mid \exists \nu \in U_{\geq k}, \forall w \in W_{\geq k}, \right. \\
\text{s.t. } \exists t_k' \in [\max\{a, t_k\}, b], x_{x_k}^{\nu, w}(t_k') \in \Omega_2, \\
\forall t_k'' \in [t_k, t_k'], x_{x_k}^{\nu, w}(t_k'') \in \mathcal{C}, t_k \in [0, b]. \right\}
\]
The set $\mathcal{R}^m(\Omega_1,\Omega_2,\mathcal{C},[a,b],k)$ collects all states in $\Omega_1$ at time $t_k$ from which there exists a control policy $\nu \in \mathcal{U}_{\geq k}$ that, despite the worst disturbance signals, drives the system to the target set $\Omega_2$ at some time instant $t_{k'} \in [\max\{a,t_k\},b]$ while satisfying constraints defined by $\mathcal{C}$ prior to reaching the target.

Definition 2.7. Consider the system $[4]$, two sets $\Omega_1,\Omega_2 \subseteq \mathbb{R}^n$, and a time interval $[a,b]$. The minimal reachable tube from $\Omega_1$ to $\Omega_2$ is defined as

$$\mathcal{R}^m(\Omega_1,\Omega_2,[a,b],k) = \left\{ x_k \in \Omega_1 \mid \forall \nu \in \mathcal{U}_{\geq k}, \exists w \in \mathcal{W}_{\geq k}, \exists t_{k'} \in [\max\{a,t_k\},b], x_{\nu,w}(t_{k'}) \in \Omega_2 \right\}, t_k \in [0,b].$$

The set $\mathcal{R}^m(\Omega_1,\Omega_2,[a,b],k)$ collects all states in $\Omega_1$ at time $t_k$ from which no matter what control policy $\nu$ is applied, there exists a disturbance signal that drives the system to the target set $\Omega_2$ at some time instant $t_{k'} \in [\max\{a,t_k\},b]$. In this definition, the constraint set $\mathcal{C}$ is redundant.

Lemma 2.1. [22] Consider the system $[4]$ and the STL formulas $\varphi$, $\varphi_1$, and $\varphi_2$. Then, one has

i) until: $S_{\varphi_1 U [a,b] \varphi_2} = \mathcal{R}^m(\mathbb{R}^n, S_{\varphi_2}, S_{\varphi_1}, [a,b], 0)$;

ii) always: $S_{\varphi [a,b] \varphi} = \mathcal{R}^m(\mathbb{R}^n, S_{\varphi}, [a,b], 0)$.

2.4. Problem formulation

The problems under consideration are formulated as follows.

Problem 2.1 (Robust satisfiability check). Consider the system $[4]$ and a task specification expressed as an STL formula $\varphi$. For an initial state $x_0$, determine whether the task specification $\varphi$ is robustly satisfiable or not.

Problem 2.2 (Online control synthesis). Consider the system $[4]$ and a task specification expressed as an STL formula $\varphi$. For an initial state $x_0$, find, if there exists, a control policy $\nu = \nu_0 \nu_1 \ldots \nu_k \ldots$ such that the resulting trajectory $x = x_0 x_1 \ldots x_k \ldots$ satisfies $\varphi$.

3. Real-time STL semantics and tube-based temporal logic tree

In this section, a real-time version of STL semantics and a tTLT are proposed.
3.1. Real-time STL semantics

It has been proven in [18] that each STL formula has an equivalent STL formula in positive normal form, i.e., negations only occur adjacent to predicates. The syntax of the positive normal form STL is given by

\[ \varphi ::= \top \mid \mu \mid \neg \mu \mid \varphi_1 \land \varphi_2 \mid \varphi_1 \lor \varphi_2 \mid \varphi_1 \mathbb{U} \varphi_2 \mid \mathcal{G}_t \varphi. \tag{7} \]

Before proceeding, the following definition is required.

**Definition 3.1 (Suffix and Completions).** Given a discrete-time signal \( \mathbf{x} = x_0 x_1 \ldots \), we say that a partial signal \( s = s_l s_{l+1} \ldots, l \in \mathbb{N} \), is a suffix of the signal \( \mathbf{x} \) if \( \forall k' \geq l, s_{k'} = x_k. \) The set of completions of a partial signal \( s \), denoted by \( C(s) \), is given by

\[ C(s) := \{ \mathbf{x} : s \text{ is a suffix of } \mathbf{x} \}. \]

Given a time instant \( t_k \) and a time interval \([a, b]\), define

\[ t_k + [a, b] := [t_k + a, t_k + b]. \]

The satisfaction relation of a partial signal \( s = s_l s_{l+1} \ldots \) starting from time instant \( t_l \) is defined.

**Definition 3.2.** The real-time STL semantics are recursively defined by

\begin{align*}
(s, t_k, t_l) \models \mu & \iff g_\mu(s(t_k)) \geq 0, t_l \leq t_k; \\
(s, t_k, t_l) \models \neg \mu & \iff \neg((s, t_k, t_l) \models \mu), t_l \leq t_k; \\
(s, t_k, t_l) \models \varphi_1 \land \varphi_2 & \iff (s, t_k, t_l) \models \varphi_1 \land (s, t_k, t_l) \models \varphi_2, t_l \in t_k + \mathbb{N}[\varphi_1 \land \varphi_2]; \\
(s, t_k, t_l) \models \varphi_1 \lor \varphi_2 & \iff (s, t_k, t_l) \models \varphi_1 \lor (s, t_k, t_l) \models \varphi_2, t_l \in t_k + \mathbb{N}[\varphi_1 \lor \varphi_2]; \\
(s, t_k, t_l) \models \varphi_1 \mathbb{U} \varphi_2 & \iff \begin{cases} \\
\exists t_{k'} \in \max\{t_k + a, t_l\}, t_{k'} + b \text{ s.t. } (s, t_{k'}, t_l) \models \varphi_2, \text{ if } \varphi_2 \text{ contains no temporal operator, } \\
\exists t_{k'} \in [t_k + a, t_k + b] \text{ s.t. } (s, t_{k'}, t_l) \models \varphi_2, \text{ otherwise, } \\
\end{cases} \\
& \text{ if } \varphi_1 \text{ contains no temporal operator, } \\
& \forall t_{k'} \in \max\{t_k + a, t_l\}, t_{k'} + b \text{ s.t. } (s, t_{k'}, t_l) \models \varphi_1, \\
& \forall t_{k'} \in [t_k + a, t_k + b] \text{ s.t. } (s, t_{k'}, t_l) \models \varphi_1, \text{ otherwise. } \\
& t_l \in t_k + [0, \|G_{[a,b]} \varphi_1\|].
\end{align*}
The real-time satisfaction relation \((s,t_k,t_l) \models \varphi\) denotes that the partial signal \(s\) is the suffix of a satisfying trajectory that starts from \(t_k\), i.e.,

\[
(s,t_k,t_l) \models \varphi \iff \exists x \in C(s), (x,t_k) \models \varphi.
\]

Note that when \(t_l = t_k\), the satisfaction relation \((s,t_k,t_l) \models \varphi\) degenerates to \((s,t_k) \models \varphi\).

**Definition 3.3.** Consider the deterministic system (2) and the STL formula \(\varphi\). We say \(\varphi\) is satisfiable from the state \(x_k\) at time \(t_k\) if there exists a control policy \(\nu \in U_{\geq k}\) such that

\[
(x_{x_k}^\nu,t_0,t_k) \models \varphi. \tag{8}
\]

**Definition 3.4.** Consider the uncertain system (1) and the STL formula \(\varphi\). We say \(\varphi\) is robustly satisfiable from the state \(x_k\) at time \(t_k\) if there exists a control policy \(\nu \in U_{\geq k}\) such that

\[
(x_{x_k}^\nu,w,t_0,t_k) \models \varphi, \forall w \in W_{\geq k}. \tag{9}
\]

Note that when \(t_k = t_0\), Definitions 3.4 and 3.3 degenerate to Definitions 2.5 and 2.4, respectively. Given an STL formula \(\varphi\), let

\[
S_{\varphi}(t_k) := \{x_k \in \mathbb{R}^n| \varphi \text{ is (robustly) satisfiable from } x_k \text{ at } t_k\} \tag{10}
\]

denote the set of states from which \(\varphi\) is robustly satisfiable at \(t_k\). Then, we have the following result.

**Theorem 3.1.** Consider the system (1), a predicate \(\mu\), and STL formulas \(\varphi_1\), and \(\varphi_2\). Then,

\[i) \text{ negation: } S_{\neg \mu}(t_k) = \overline{S_{\mu}}(t_k) ;\]

\[ii) \text{ conjunction: } S_{\varphi_1 \land \varphi_2}(t_k) \subseteq S_{\varphi_1}(t_k) \cap S_{\varphi_2}(t_k) ;\]

\[iii) \text{ disjunction: } S_{\varphi_1 \lor \varphi_2}(t_k) \supseteq S_{\varphi_1}(t_k) \cup S_{\varphi_2}(t_k) ;\]

\[iv) \text{ until: } S_{\varphi_1[u_{a,b}]\varphi_2}(t_k) = R^M(\mathbb{R}^n, S_{\varphi_2}, S_{\varphi_1}, [a,b], k) ;\]

\[v) \text{ always: } S_{G_{[a,b]}\varphi_1}(t_k) = R^m(\mathbb{R}^n, S_{\varphi_1}, [a,b], k),\]

where \(S_{\varphi_1}\) and \(S_{\varphi_2}\) are defined in (6).
that for a state $x \in S$ and a control policy $\nu \in \mathcal{U}$, one further has $(x_{x_k}^{\nu_1}, t_0, t_k) \models \varphi_1 \land \varphi_2$, $\forall w \in W_{2_k}$. That is, $x_k \in S_{\varphi_1}(t_k), x_k \in S_{\varphi_2}(t_k)$, the other direction may not hold because it could happen that for a state $x_k$, there exist two control policies $\nu_1, \nu_2 \in \mathcal{U}_{2_k}$ such that $(x_{x_k}^{\nu_1}, t_0, t_k) \models \varphi_1 \land \varphi_2, \forall w \in W_{2_k}$.

Assume now that $x_k \in S_{\varphi_1}(t_k)$, then one has that there exists a control policy $\nu \in \mathcal{U}_{2_k}$ such that

$$(x_{x_k}^{\nu}, t_0, t_k) \models \varphi_1, \forall w \in W_{2_k} \land (x_{x_k}^{\nu}, t_0, t_k) \models \varphi_2, \forall w \in W_{2_k}.$$
• each node is either a tube node that maps from the nonnegative time axis, i.e., $\mathbb{R}_{\geq 0}$, to the subset of $\mathbb{R}^n$, or an operator node that belongs to $\{\land, \lor, U, G\}$;

• the root node and the leaf nodes are tube nodes;

• if a tube node is not a leaf node, its unique child is an operator node;

• the children of any operator node are tube nodes.

Definition 3.6. A complete path of a tTLT is a path that starts from the root node and ends at a leaf node. Any subsequence of a complete path is called a fragment of the complete path.

Then, the following result is obtained.

Theorem 3.2. For the system (1) and every STL formula $\varphi$ in positive normal form (7), a tTLT, denoted by $T_{\varphi}$, can be constructed from $\varphi$ through the reachability operators $R^M$ and $R^m$.

Proof. Firstly, we show that each predicate $\mu$ and its negation $\neg \mu$ have a corresponding tTLT. Given a predicate $\mu$, one has $S_\mu = \{x_0 : g_\mu(x_0) \geq 0\}$. Then, the tTLT $T_\mu$ (or $T_{\neg \mu}$) has only one root node, which is given by $S_\mu$ (or $S_{\neg \mu}$). Following the similar idea, we can show that $\top$ (or $\bot$) has a corresponding tTLT that only has a root node, which is given by $\mathbb{R}^n$ (or $\emptyset$).

Then, we will prove that if the STL formulas $\varphi_1$ and $\varphi_2$ have corresponding tTLTs, respectively, then the STL formulas $\varphi_1 \land \varphi_2$, $\varphi_1 \lor \varphi_2$, $\varphi_1 U_{[a,b]} \varphi_2$, and $G_{[a,b]} \varphi_1$ have their corresponding tTLTs, respectively.

Case 1: Boolean operators $\land$ and $\lor$. Consider two STL formulas $\varphi_1, \varphi_2$ and their corresponding tTLTs $T_{\varphi_1}, T_{\varphi_2}$. The root nodes of $T_{\varphi_1}, T_{\varphi_2}$ are denoted by $X_{\varphi_1}(t_k)$ and $X_{\varphi_2}(t_k)$, respectively. The tTLT $T_{\varphi_1 \land \varphi_2}$ (or $T_{\varphi_1 \lor \varphi_2}$) can be constructed by connecting $X_{\varphi_1}(t_k)$ and $X_{\varphi_2}(t_k)$ through the operator node $\land$ (or $\lor$) and taking the intersection (or union) of the two root nodes, i.e., $X_{\varphi_1}(t_k) \cap X_{\varphi_2}(t_k)$ (or $X_{\varphi_1}(t_k) \cup X_{\varphi_2}(t_k)$), to be the root node. An illustrative diagram for $\varphi_1 \land \varphi_2$ is given in Figure 1.

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Case 2: Until operator $U_{[a,b]}$. Consider two STL formulas $\varphi_1, \varphi_2$ and their corresponding tTLTs $\mathcal{T}_{\varphi_1}, \mathcal{T}_{\varphi_2}$. The root nodes of $\mathcal{T}_{\varphi_1}, \mathcal{T}_{\varphi_2}$ are denoted by $X_{\varphi_1}(t_k)$ and $X_{\varphi_2}(t_k)$, respectively. In addition, the leaf nodes of $\mathcal{T}_{\varphi_1}$ are denoted by $Y^1_{\varphi_1}(t_k), \ldots, Y^N_{\varphi_1}(t_k)$, where $N$ is the total number of leaf nodes of $\mathcal{T}_{\varphi_1}$. The tTLT $\mathcal{T}_{\varphi_1}U_{[a,b]}\varphi_2$ can be constructed by the following steps:
1) replace each leaf node $Y^i_{\varphi_1}(t_k)$ by $\mathcal{R}^M(R^n, X_{\varphi_2}(t_0), Y^i_{\varphi_1}(t_0), [a,b], k)$; 2) update $\mathcal{T}_{\varphi_1}$ from the leaf nodes to the root node with the new leaf nodes; and 3) connect each leaf node of the updated $\mathcal{T}_{\varphi_1}$ and the root node of $\mathcal{T}_{\varphi_2}$, i.e., $X_{\varphi_2}(t_k)$, with the operator node $U_{[a,b]}$. One illustrative diagram for $U_{[a,b]}$ is given in Figure 2.

Case 3: Always operator $G_{[a,b]}$. Consider an STL formula $\varphi_1$ and its corresponding tTLT $\mathcal{T}_{\varphi_1}$. The root node of $\mathcal{T}_{\varphi_1}$ is given by $X_{\varphi_1}(t_k)$. The tTLT $\mathcal{T}_{G_{[a,b]}\varphi_1}$ can be constructed by connecting $X_{\varphi_1}(t_k)$ through the operator $G_{[a,b]}$ and making the tube $\mathcal{R}^M(R^n, X_{\varphi_1}(t_0), [a,b], k)$ the root node. An illustrative diagram for $G_{[a,b]}$ is given in Figure 3. □
Based on Theorem 3.2 Algorithm 1 is designed for the construction of tTLT $T_\varphi$. It takes the syntax tree $T$ of the STL formula $\varphi$ as input. For an STL formula, the nodes of its syntax tree are either predicate or operator nodes. In addition, all the leaf nodes are predicates and all other nodes are operators.

**Algorithm 1 tTLTConstruction**

**Input:** the syntax tree of STL formula $\varphi$.

**Return:** the tTLT $T_\varphi$.

1. for each leaf node $\mu$ (or $\neg\mu$) of the syntax tree do,
2. Replace $\mu$ (or $\neg\mu$) by $S_\mu$ (or $S_{\neg\mu}$),
3. end for
4. for each operator node of the syntax tree through a bottom-up traversal, do
5. Construct $T_\varphi$ according to Theorem 3.2,
6. end for

**Example 3.1.** Consider the formula $\varphi = F_{[a_1, b_1]} G_{[a_2, b_2]} \mu_1 \land \mu_2 U_{[a_3, b_3]} \mu_3$, where $\mu_i, i = \{1, 2, 3\}$ are predicates. The syntax tree of $\varphi$ is shown on the left-hand side of Figure 4. The corresponding TLT for $\varphi$ (constructed using Algorithm 1) is shown on the right-hand side of Figure 4, where

$$X_4(t_k) = R^m([a_2, b_2], [a, b], k),$$

1 A syntax tree is a tree representation of the syntactic structure of the source code.  

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Figure 4: Example 3.1 syntax tree (left) and tTLT (right) for \( \varphi = F_{[a_1,b_1]} G_{[a_2,b_2]} \mu_1 \land \mu_2 U_{[a_3,b_3]} \mu_3 \). Recall that \( F_{[a,b]} \varphi = \top U_{[a,b]} \varphi \).

\[
\begin{align*}
  X_3(t_k) &= R^M(\mathbb{R}^n, S_{\mu_3}, S_{\mu_2}, [a_3, b_3], k), \\
  X_2(t_k) &= R^M(\mathbb{R}^n, X_4(t_0), \mathbb{R}^n, [a_1, b_1], k), \\
  X_1(t_k) &= X_2(t_k) \cap X_3(t_k).
\end{align*}
\]

Remark 3.1. Given an STL formula \( \varphi \) in positive normal form, let \( N \) denote the number of Boolean operators and \( M \) the number of temporal operators contained in \( \varphi \). Let \( T_\varphi \) be the tTLT corresponds to \( \varphi \). Then, \( T_\varphi \) has at most \( 2N \) number of complete paths. In addition, each complete path has at most \( 2(N + M) + 1 \) number of nodes, out of which at most \( N + M \) are non-root tube nodes. Thus, one can conclude that \( T_\varphi \) contains at most \( 4N(N + M) + 1 \) number of nodes, out of which at most \( 2N(N + M) + 1 \) number of tube nodes.

4. Robust satisfiability check

This section addresses the robust satisfiability check problem (Problem 2.1). Before that, we need to define the satisfaction relation between a trajectory and a tTLT.
4.1. Definitions

We first define the maximal temporal fragment (MTF) for a tTLT, which plays an important role when simplifying the tTLT.

**Definition 4.1.** A MTF of a complete path of the tTLT is one of the following types of fragment:

1) a fragment from the root node to the parent of the first Boolean operator node ($\land$ or $\lor$);

2) a fragment from one child of one Boolean operator node to the parent of the next Boolean operator node;

3) a fragment from one child of the last Boolean operator node to the leaf node.

One can conclude from Definition 4.1 that any MTF starts and ends with a tube node and contains no Boolean operator nodes.

**Definition 4.2.** A time coding of (a complete path of) the tTLT is an assignment of an activation time instant $t_{\kappa_i}, \kappa_i \in \mathbb{N}$ for each tube node $X_i$ of (the complete path of) the tTLT.

Now, we further define the satisfaction relation between a trajectory $x$ and a complete path of the tTLT.

**Definition 4.3.** Consider a trajectory $x := x_0x_1 \ldots$ and a complete path $p$ of a tTLT encoded in the form of $p = X_0\Theta_1X_1\Theta_2 \ldots \Theta_{N_f}X_{N_f}$, where $N_f$ is the number of operator nodes contained in the complete path, $X_i : \mathbb{R}_{\geq 0} \rightarrow 2^\mathbb{R}, \forall i \in \{0, 1, \ldots, N_f\}$ represent tube nodes, and $\Theta_j, \forall j \in \{1, \ldots, N_f\}$ represent operator nodes. We say $x$ satisfies $p$, denoted by $x \equiv p$, if there exists a time coding for $p$ such that

i) if $\Theta_i \in \{\land, \lor\}$, then $t_{\kappa_i} = t_{\kappa_{i-1}}$;

ii) if $\Theta_i = U_I$, then $t_{\kappa_i} \in t_{\kappa_{i-1}} + I$;

iii) if $\Theta_i = G_I$, then $t_{\kappa_i} = \arg\max_{t_k \in t_{\kappa_{i-1}} + I} \{t_k\}$;

and

iv) $x_k \in X_i(t_{k-\kappa_i}), \forall k \in [\kappa_i, \kappa_{i+1}], i = 0, \ldots, N_f - 1$;

v) $x_{\kappa_{N_f}} \in X_{N_f}(t_0)$. 

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Remark 4.1. From items i)-iii) of Definition 4.3, one has that \( t_{\kappa_0} \leq t_{\kappa_1} \leq \cdots \leq t_{\kappa_N} \). This means that if a trajectory \( x \equiv p \), it must visit each tube node \( \mathcal{X}_i \) of the complete path \( p \) sequentially. In addition, we can further conclude from items i)-v) that the trajectory \( x \) has to stay in each tube node \( \mathcal{X}_i \) for sufficiently long time steps.

With Definition 4.3, the satisfaction relation between a trajectory \( x \) and a tTLT can be defined as follows.

Definition 4.4. Consider a trajectory \( x \) and a tTLT \( T_\varphi \). We say \( x \) satisfies \( T_\varphi \), denoted by \( x \models T_\varphi \), if the output of Algorithm 2 is true.

Definition 4.5. (Robust satisfiable tTLT) The tTLT \( T_\varphi \) is called robust satisfiable for the system (4) with initial state \( x_0 \) if there exists a control policy \( \nu \in \mathcal{U}_{\geq 0} \) such that \( x_{w, x_0} \models T_\varphi, \forall w \in \mathcal{W}_{\geq 0} \).

---

**Algorithm 2 tTLTSatisfaction**

**Input:** a trajectory \( x \) and a tTLT \( T_\varphi \).

**Return:** true or false.

1: \( T_\varphi^c \leftarrow\) Compression(\( T_\varphi \)),
2: set all tube nodes in \( T_\varphi^c \) with false,
3: for each complete path of \( T_\varphi \), do
4: if \( x \) satisfies the complete path then
5: set the leaf node of the corresponding complete path in \( T_\varphi^c \) with true,
6: else
7: set the leaf node of the corresponding complete path in \( T_\varphi^c \) with false,
8: end if
9: end for
10: Backtracking(\( T_\varphi^c \)).

---

We further detail the Compression algorithm (Algorithm 3) and the Backtracking algorithm (Algorithm 4) in the following. Algorithm 3 aims at obtaining a simplified tree with Boolean operator nodes only. To do so, we first encode each MTF in the form of \( \mathcal{X}_1 \Theta_1 \cdots \Theta_{N_f-1} \mathcal{X}_{N_f} \) (line 3), and then replace it with one tube node (line 4). Algorithm 4 takes the compressed tree \( T_\varphi^c \) as an input, and the output is the updated root node. This is done by updating the parent of each Boolean operator node through a bottom-up
traversal. In Algorithm 4, PA(Θ) and CH₁(Θ), CH₂(Θ) represent the parent node and the two children of Θ ∈ {∧, ∨}, respectively.

**Algorithm 3 Compression**

**Input:** a tTLT Tᵦ.
**Return:** the compressed tree Tᵦᶜ.
1: for each complete path of Tᵦ, do
2: for each MTF, do
3: encode the MTF in the form of X₁(t_k)Θ₁…Θᴺ₋₁Xᴺ₋₁(t_k),
4: replace the MTF with one tube node ∪ᵢ=₁ᴺ Xᵢ,
5: end for
6: end for

**Algorithm 4 Backtracking**

**Input:** a compressed tree Tᵦᶜ.
**Return:** the root node of Tᵦᶜ.
1: for each Boolean operator node Θ of Tᵦᶜ through a bottom-up traversal, do
2: if Θ = ∧, then
3: PA(Θ) ← PA(Θ) ∨ (CH₁(Θ) ∧ CH₂(Θ)),
4: else
5: PA(Θ) ← PA(Θ) ∨ (CH₁(Θ) ∨ CH₂(Θ)),
6: end if
7: end for

**Example 4.1.** Let us continue with Example 3.1. The tTLT Tᵦ (right of Figure 4) contains 2 complete paths, i.e.,

\[ p₁ := X₁ ∧ X₂ U_{[a₁,b₁]} X₄ G_{[a₂,b₂]} S_{μ₁} \]

and

\[ p₂ := X₁ ∧ X₃ U_{[a₃,b₃]} S_{μ₃}. \]

Let

\[ \{tₖ₁, tₖ₂, tₖ₄, tₖ₅ \} \]

be the time coding of the complete path p₁, where tₖ₁, tₖ₂, tₖ₄, and tₖ₅ are the activation time instants of the tube nodes X₁, X₂, X₄, and X₅ := S_{μ₁}, respectively. Then, we have according to Definition 4.3 that a trajectory \( x \) I
\( p \) if i) \( t_{\kappa_1} = t_{\kappa_2} \); ii) \( t_{\kappa_4} \in t_{\kappa_2} + [a_1, b_1] \); iii) \( t_{\kappa_5} = \arg\max_{t_k} \{ t_k \in t_{\kappa_4} + [a_2, b_2] \} \); iv) \( x_0 \in X_1(t_0), x_k \in X_2(t_{k-\kappa_2}), \forall k \in [\kappa_2, \kappa_4] \), \( x_k \in X_4(t_{k-\kappa_4}), \forall k \in [\kappa_4, \kappa_5] \), and v) \( x_{\kappa_5} \in X_5 \).

In addition, the tTLT \( T_{\varphi} \) contains 3 MTFs, i.e., \( X_{1} \), \( X_{2} \cup X_{4} G_{[a_2, b_2]} S_{\mu_1} \), and \( X_{3} U_{[a_3, b_3]} S_{\mu_3} \). The compressed tree \( T_{\varphi}^c \) is shown in Figure 5. If a trajectory \( x \) satisfies both of the complete paths \( p_1 \) and \( p_2 \), the output of Algorithm 2 is true, otherwise, the output is false.

\[ \begin{align*}
X_1(t_k) & \land \\
X_2(t_k) \cup X_4(t_k) \cup S_{\mu_1} & \land \\
X_3(t_k) \cup S_{\mu_3} & \land
\end{align*} \]

Figure 5: Example 4.1: compressed tree \( T_{\varphi}^c \), where \( T_{\varphi} \) is plotted in Figure 4.

4.2. Robust satisfiability check

Firstly, a sufficient condition is obtained for the robust satisfiability check of the uncertain system (1).

Theorem 4.1. Consider the uncertain system (1) with initial state \( x_0 \) and an STL formula \( \varphi \). Let \( T_{\varphi} \) be the tTLT corresponding to \( \varphi \). Then, one has that \( \varphi \) is robustly satisfiable for (1) if the tTLT \( T_{\varphi} \) is robustly satisfiable for (1).

Proof. From Definitions 2.5 and 4.5, one has that to prove Theorem 4.1, it is equivalent to prove \( x'_{w} \equiv T_{\varphi} \) for \( \varphi \), where the sub-formulas \( \varphi_1 \), \( \varphi_2 \) are assumed to contain no Boolean operators.

In the following, we will first prove

i) \( x'_{w} \equiv T_{\varphi} \iff x'_{w} \equiv \varphi \) for \( \top \), predicates \( \mu, -\mu \), and STL formulas \( \mu_1 \land \mu_2, \mu_1 \lor \mu_2 \), \( \mu_1 U [a, b] \mu_2 \), \( G [a, b] \mu_1 \), \( \varphi_1 \land \varphi_2 \);

ii) \( x'_{w} \equiv T_{\varphi} \Rightarrow x'_{w} \equiv \varphi \) for STL formula \( \varphi_1 \lor \varphi_2 \);

where the sub-formulas \( \varphi_1, \varphi_2 \) are assumed to contain no Boolean operators.
**Case 1:** For $\top$, predicates $\mu$, $\neg \mu$, and $\mu_1 \land \mu_2, \mu_1 \lor \mu_2$, it is easy to verify that $x_{\nu_0} \equiv T_{\varphi} \iff x_{\nu_0} \models \varphi$.

**Case 2:** $\varphi = \mu_1 U_{[a,b]} \mu_2$ and $\varphi = G_{[a,b]} \mu_1$. We note that the proofs of the two are similar, therefore, in the following, we only consider the case $\varphi = \mu_1 U_{[a,b]} \mu_2$. The $\tau$TLT $T_{\varphi}$ can be constructed via Algorithm 1, which is shown in Figure 6.

Assume that $x_{\nu_0} \equiv T_{\varphi}$, then one has from Definition 4.3 that $\exists \kappa_1 \in t_0 + [a,b], x_{\kappa_1} \in S_{\mu_2}$ and $\forall k \in [0,k_1], x_k \in R^M([R^n, S_{\mu_2}, S_{\mu_1}, [a,b], k]) \subseteq \mu_{\kappa_1}$, which implies $x_{\nu_0} \models \varphi$. That is, $x_{\nu_0} \equiv T_{\varphi} \Rightarrow x_{\nu_0} \models \varphi$. Assume now that $x_{\nu_0} \models \varphi$. Then, one has from STL semantics that i) $\exists \kappa' \in t_0 + [a,b], x_{\kappa'} \in S_{\mu_2}$ and ii) $\forall \kappa'' \in [t_0, k'], x_{\kappa''} \in S_{\varphi_1}$. Moreover, from Definition 2.6, one has that i) and ii) together implies $\forall \kappa'' \in [t_0, t_k'], x_{\kappa''} \in R^M([R^n, S_{\mu_2}, S_{\mu_1}, [a,b], k'\prime])$. Therefore, $x_{\nu_0} \models \varphi \Rightarrow x_{\nu_0} \equiv T_{\varphi}$.

**Case 3:** $\varphi = \varphi_1 \land \varphi_2$. Assume that $x_{\nu_0} \equiv T_{\varphi}$, then one has from Definition 4.3 that $x_{\nu_0} \equiv T_{\varphi_1}$ and $x_{\nu_0} \equiv T_{\varphi_2}$. Moreover, since $\varphi_1, \varphi_2$ contain no Boolean operators, then one can conclude from Case 2 that $x_{\nu_0} \equiv T_{\varphi_1} \Rightarrow x_{\nu_0} \models \varphi_1, i = \{1,2\}$, which implies $x_{\nu_0} \models \varphi_1 \land \varphi_2$. That is, $x_{\nu_0} \equiv T_{\varphi} \Rightarrow x_{\nu_0} \models \varphi$. The proof of the other direction is similar and hence omitted.

**Case 4:** $\varphi = \varphi_1 \lor \varphi_2$. The proof of $x_{\nu_0} \equiv T_{\varphi} \Rightarrow x_{\nu_0} \models \varphi$ is similar to Case 3. The other direction does not hold because for an uncertain system, it is possible that there exists a trajectory $x_{\nu_0}$ such that $x_{\nu_0} \models \varphi$, however, the initial state $x_0 \notin X^c_{\text{root}}(t_0)$ (due to item iii) of Theorem 3.1), where $X^c_{\text{root}}$ denotes the root node of $T_{\varphi}$. In this case, $x_{\nu_0}$ does not satisfy $T_{\varphi}$.

The proof of $x_{\nu_0} \equiv T_{\varphi} \Rightarrow x_{\nu_0} \models \varphi$ for STL formulas $\varphi_1 U_{[a,b]} \varphi_2$, $G_{[a,b]} \varphi_1$, $\varphi_1 \land \varphi_2$, and $\varphi_1 \lor \varphi_2$ (here, no assumption on $\varphi_1, \varphi_2$) can be completed inductively by combining Cases 1, 2, 3 and 4. Therefore, the conclusion follows.

When the deterministic system [2] is considered, the above condition becomes a necessary and sufficient condition, as shown in the following.
Corollary 4.1. Consider the deterministic system \( [2] \) with initial state \( x_0 \) and an STL formula \( \varphi \). Let \( T_{\varphi} \) be the tTLT corresponding to \( \varphi \). Then, one has that \( \varphi \) is satisfiable for \( [2] \) if and only if the tTLT \( T_{\varphi} \) is satisfiable for \( [2] \).

Proof. For the deterministic system \( [2] \), one has from the proof of Theorem 4.1 that

\[ x_{x_0}^{\mu, w} \cong T_{\varphi} \iff x_{x_0}^{\mu, w} \models \varphi \] for \( \mu \), predicates \( \mu, \neg \mu \), and STL formulas \( \mu_1 \land \mu_2, \mu_1 \lor \mu_2, \mu_1 U_{[a,b]} \mu_2, G_{[a,b]} \mu_1, \varphi_1 \land \varphi_2 \)

holds. Next, we will prove

\[ x_{x_0}^{\mu, w} \cong T_{\varphi} \iff x_{x_0}^{\mu, w} \models \varphi \] for STL formula \( \varphi \), where the sub-formulas \( \varphi_1, \varphi_2 \) are assumed to contain no Boolean operators.

Let \( \varphi = \varphi_1 \lor \varphi_2 \). The proof of \( x_{x_0}^{\mu, w} \cong T_{\varphi} \Rightarrow x_{x_0}^{\mu, w} \models \varphi \) is given in Theorem 4.1. Assume now that \( x_{x_0}^{\mu, w} \models \varphi \), then one has from STL semantics that \( x_{x_0}^{\mu, w} \models \varphi_1 \) or \( x_{x_0}^{\mu, w} \models \varphi_2 \). Since \( \varphi_1, \varphi_2 \) contain no Boolean operators, then one can conclude from item i) that \( x_{x_0}^{\mu, w} \models \varphi_i \Rightarrow x_{x_0}^{\mu, w} \cong T_{\varphi_i}, i = \{1, 2\} \).

Moreover, one has from Corollary 3.1 that \( S_{\varphi_1 \lor \varphi_2} (t_k) = S_{\varphi_1} (t_k) \cup S_{\varphi_2} (t_k) \). Therefore, \( x_{x_0}^{\mu, w} \models \varphi \Rightarrow x_{x_0}^{\mu, w} \cong T_{\varphi} \).

The proof of \( x_{x_0}^{\mu, w} \cong T_{\varphi} \iff x_{x_0}^{\mu, w} \models \varphi \) for STL formulas \( \varphi_1 U_{[a,b]} \varphi_2, G_{[a,b]} \varphi_1, \varphi_1 \land \varphi_2, \) and \( \varphi_1 \lor \varphi_2 \) (here, no assumption on \( \varphi_1, \varphi_2 \)) can be completed inductively by combining items i) and iii). Therefore, the conclusion follows.

Given a tTLT \( T_{\varphi} \), denote by \( X_{\varphi}^{\mathbf{root}} \) the root node of \( T_{\varphi} \). The conditions given in Theorems 4.1 and Corollary 4.1 are in general difficult to check, therefore, two necessary conditions are given in the following propositions.

Proposition 4.1. Consider the uncertain system \( [4] \) with initial state \( x_0 \) and an STL formula \( \varphi \). Let \( T_{\varphi} \) be the tTLT corresponding to \( \varphi \). Then, \( T_{\varphi} \) is robustly satisfiable for \( [4] \) only if \( x_0 \in X_{\mathbf{root}}^{\varphi} (t_0) \).

Proof. It follows from Definitions 4.3 and 4.5 that if \( x \cong T_{\varphi} \), it must have \( x_0 \in X_{\mathbf{root}}^{\varphi} (t_0) \). Similar necessary condition also holds for the deterministic systems.

Proposition 4.2. Consider the deterministic system \( [2] \) with initial state \( x_0 \) and an STL formula \( \varphi \). Let \( T_{\varphi} \) be the tTLT corresponding to \( \varphi \). Then, \( \varphi \) is satisfiable for \( [2] \) only if \( x_0 \in X_{\mathbf{root}}^{\varphi} (t_0) \).
**Proof.** Consider a predicate \( \mu \) and STL formulas \( \varphi_1, \varphi_2 \). Define

\[
\begin{align*}
\hat{S}_\mu &= \{ x \in \mathbb{R}^n : g_\mu(x) \geq 0 \}, \\
\hat{S}_{\neg \mu} &= \overline{\hat{S}_\mu}, \\
\hat{S}_{\varphi_1 \land \varphi_2}(t_k) &= \hat{S}_{\varphi_1}(t_k) \cap \hat{S}_{\varphi_2}(t_k), \\
\hat{S}_{\varphi_1 \lor \varphi_2}(t_k) &= \hat{S}_{\varphi_1}(t_k) \cup \hat{S}_{\varphi_2}(t_k), \\
\hat{S}_{\varphi_1 U [a,b]}(t_k) &= \mathcal{R}^M(\mathbb{R}^n, \hat{S}_{\varphi_2}(t_0), \hat{S}_{\varphi_1}(t_0), [a,b], k), \quad \text{and} \\
\hat{S}_{G[a,b]}(t_k) &= \mathcal{R}^m(\mathbb{R}^n, \widehat{S}_\varphi(t_0), [a,b], k).
\end{align*}
\]

From the construction of tTLT (Algorithm 1), one has that \( \hat{S}_\varphi(t_0) = \mathbb{X}_{\text{root}}(t_0) \). In addition, one can conclude from Corollary 3.1 that \( \mathbb{S}_\varphi \subseteq \hat{S}_\varphi(t_0) \), where \( \mathbb{S}_\varphi \) defined in (6) is the set of initial states from which \( \varphi \) is satisfiable. Therefore, \( \mathbb{S}_\varphi \subseteq \mathbb{X}_\varphi(t_0) \). The conclusion follows. \( \square \)

5. Online Control Synthesis

This section concerns online control synthesis as defined by Problem 2.2. From Theorems 4.1 (Corollary 4.1), one can see that to guarantee the satisfaction of the STL formula \( \varphi \), it is sufficient to find a control policy \( \nu \) that guarantees the (robust) satisfaction of the corresponding tTLT \( T_{\varphi} \). In the following, the control synthesis algorithms are designed such that the tTLT \( T_{\varphi} \) is satisfied based on Definitions 4.3 and 4.4.

5.1. Definitions and notations

Before proceeding, the following definitions and notations are needed.

**Definition 5.1.** The time horizon \(|\Theta|\) of an STL operator \( \Theta \), where \( \Theta \in \{ \land, \lor, U[a,b], G[a,b] \} \) is defined as

\[
|\Theta| = \begin{cases} 
0, & \text{if } \Theta = \{ \land, \lor \}, \\
\hat{b}, & \text{if } \Theta \in \{ U[a,b], G[a,b] \},
\end{cases}
\]  

(11)

where \( \hat{b} = \text{argmax}_{t_k} \{ a \leq t_k \leq b \} \).
Definition 5.2. A fragment of the complete path of a tTLT is called a Boolean fragment if it starts and ends with a tube node and contains only Boolean operator nodes. We say a tube node \( X_j \) is reachable from \( X_i \) by a Boolean fragment if there exists a Boolean fragment that starts with \( X_i \) and ends with \( X_j \).

Definition 5.3. If each node of a tree is either a set node that is a subset of \( U \) or an operator node that belongs to \( \{\land, \lor, \land_i, \lor_i\} \), then the tree is called a control tree.

Each tube node \( X_i \) of the tTLT \( T_\varphi \) is characterized by the following two parameters:

- \( t_a(X_i) \): the activation time of \( X_i \),
- \( t_h(X_i) \): the time horizon of \( X_i \), i.e., the time that \( X_i \) is deactivated.

Denote by \( T_\varphi(t_k) \) the resulting tree of \( T_\varphi \) at time instant \( t_k \). It is obtained by fixing the value of each tube node \( X_i \) according to the activation time \( t_a(X_i) \) (i.e., \( T_\varphi(t_k) \) contains either set nodes or operator nodes). Let \( S_i(t_k) \) be the \( i \)-th set node of \( T_\varphi(t_k) \), where \( S_i(t_k) \) corresponds to the tube node \( X_i \). The relationship between \( S_i(t_k) \) and \( X_i \) can be described as follows:

\[
S_i(t_k) = \begin{cases} 
X_i(t_0), & \text{if } t_k \leq t_a(X_i), \\
X_i(t_k - t_a(X_i)), & \text{if } t_k > t_a(X_i).
\end{cases}
\] (12)

Moreover, one has that

\[
t_a(S_i(t_k)) = t_a(X_i), t_h(S_i(t_k)) = t_h(X_i), \forall k \geq 0.
\]

At each time instant \( t_k \), \( T_\varphi(t_k) \) is characterized by

- \( P(t_k) \): the set which collects all the set nodes of \( T_\varphi(t_k) \), i.e., \( P(t_k) = \cup_i S_i(t_k) \),
- \( \Theta \): the set which collects all the operator nodes of \( T_\varphi(t_k) \), which is time invariant.

For a node \( N_i(t_k) \in P(t_k) \cup \Theta \), define

- \( \text{CH}(N_i(t_k)) \): the set of children of node \( N_i(t_k) \),
- \( \text{PA}(N_i(t_k)) \): the set of parents of node \( N_i(t_k) \),
Post($N_i(t_k)$) := CH(CH($N_i(t_k)$)),

Pre($N_i(t_k)$) := PA(PA($N_i(t_k)$)).

Given a state-time pair $(x_k, t_k)$, define $L : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow 2^{P(t_k)}$ as the labelling function, given by

$$L(x_k, t_k) = \{S_i(t_k) \in P(t_k) : x_k \in S_i(t_k), t_k \leq t_h(S_i(t_k))\}, \quad (13)$$

which maps $(x_k, t_k)$ to a subset of $P(t_k)$. Moreover, define the function $B : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow 2^{P(t_k)}$, which maps $(x_k, t_k)$ to a set of valid set nodes in $P(t_k)$. Function $L(x_k, t_k)$ computes the subset of set nodes of $P(t_k)$ that contains $x_k$ at time $t_k$ (without the consideration of history trajectory) while function $B(x_k, t_k)$ is further introduced to capture the fact that given the history trajectory, not all set nodes in $L(x_k, t_k)$ are valid at time $t_k$. A rule for determining $B(x_k, t_k)$ given $L(x_k, t_k)$ is detailed in Algorithm 7 in the next subsection.

5.2. Online control synthesis

In the following, we will first present the online control synthesis algorithm (and its sub-algorithms), and then an example is given to further explain how each sub-algorithm works.
Algorithm 5 onlineControlSynthesis

Input: The tTLT $\mathcal{T}_\varphi$ and $(x_0, t_0)$.
Return: NExis or $(\nu, x)$ with $\nu = \nu_0\nu_1\ldots\nu_k \ldots$ and $x = x_0x_1\ldots x_k \ldots$

1: $(t_a, t_h, \text{Post}(B(x_{-1}, t_{-1}))) \leftarrow \text{initialization}(\mathcal{T}_\varphi)$,
2: $B(x_k, t_k) \leftarrow \text{trackingSetNode}(\text{Post}(B(x_{k-1}, t_{k-1})))$,
3: for each $S_i(t_k) \in B(x_k, t_k)$, do
   4:  if $t_a(S_i(t_k)) \triangleright \infty$, then
      5:     $t_a(X_i) \leftarrow t_k$,
   6:  end if
7: end for
8: $\mathcal{T}_\varphi(t_{k+1}) \leftarrow \text{updateTLT}(\mathcal{T}_\varphi(t_k), t_a, B(x_k, t_k))$,
9: $\mathcal{T}_a(t_k) \leftarrow \text{buildControlTree}(\mathcal{T}_\varphi(t_k), B(x_k, t_k), \mathcal{T}_\varphi(t_{k+1}))$,
10: $\mathcal{T}_c^*(t_k) \leftarrow \text{Compression}(\mathcal{T}_a(t_k))$,
11: $\mathcal{U}(x_k, t_k) \leftarrow \text{Backtracking}^*(\mathcal{T}_c^*)$,
12: if $\mathcal{U}(x_k, t_k) = \emptyset$, then
13:   stop and return NExis,
14: else
15:   choose $\nu_k \in \mathcal{U}(x_k, t_k)$,
16:   implement $\nu_k$ and measure $x_{k+1}$,
17:   $\text{Post}(B(x_k, t_k)) \leftarrow \text{postSet}(B(x_k, t_k), t_a, \mathcal{T}_\varphi(t_{k+1}))$,
18:   update $k = k + 1$ and go to line 2.
19: end if

The online control synthesis algorithm is outlined in Algorithm 5. Before implementation, an initialization process (line 1) is required, which is outlined in Algorithm 6. Here, $t_a$ and $t_h$ are two functions that map each tube node $X_i$ to its activation time and time horizon, respectively. If $t_a(X_i)$ or $t_h(X_i)$ is unknown for $X_i$, its value will be set as $\infty$. Then, at each time instant $t_k$, a feasible control set $\mathcal{U}(x_k, t_k)$ is synthesized (lines 2-11). This process contains the following steps: 1) find the subset of set nodes in $P(t_k)$ that are valid at time $t_k$, i.e., $B(x_k, t_k)$, via Algorithm 7 (line 2); 2) determine the activation time of $X_i$, whose corresponding set node $S_i(t_k) \in B(x_k, t_k)$ (if $t_a(X_i)$ is unknown, i.e., being visited for the first time, it is set as $t_k$; otherwise, i.e., being visited before, it is unchanged) (lines 3-7); 3) calculate $\mathcal{T}_\varphi(t_{k+1})$ via Algorithm 8 (line 8); 4) build a control tree $\mathcal{T}_a(t_k)$ (Definition 5.3) via Algorithm 9 (line 9), compress it via Algorithm 3 (line 10), and then the feasible control set $\mathcal{U}(x_k, t_k)$ is given by backtracking the compressed control tree $\mathcal{T}_c^*(t_k)$ via Algorithm 10 (line 11). If the obtained feasible control set $\mathcal{U}(x_k, t_k) = \emptyset$, the control synthesis process stops and returns.
Algorithm 6 initialization

Input: The tTLT $T_\varphi$.
Return: $t_a, t_h, \text{Post}(B(x_{-1}, t_{-1}))$.

1: $t_a(X_\varphi^{\text{root}}) \leftarrow t_0, t_h(X_\varphi^{\text{root}}) \leftarrow t_0 + |\text{CH}(X_\varphi^{\text{root}})|,$
2: for each non-root and non-leaf tube node $X_i$ through a top-down traversal, do
3: $t_a(X_i) \leftarrow \Delta\ll , t_h(X_i) \leftarrow t_h(\text{Pre}(X_i) + |\text{CH}(X_i)|,$
4: end for
5: for each leaf node $X_i$, do
6: $t_a(X_i) \leftarrow \Delta\ll , t_h(X_i) \leftarrow \infty,$
7: end for
8: $\text{Post}(B(x_{-1}, t_{-1})) \leftarrow X_\varphi^{\text{root}}(t_0),$ 
9: for each $X_j$ that is reachable from $X_\varphi^{\text{root}}$ by a Boolean fragment (see Definition 5.2), do
10: $\text{Post}(B(x_{-1}, t_{-1})) \leftarrow \text{Post}(B(x_{-1}, t_{-1})) \cup X_j(t_0),$ 
11: $t_a(X_j) \leftarrow t_0,$
12: end for

Algorithm 7 trackingSetNode

Input: $\text{Post}(B(x_{k-1}, t_{k-1}))$.
Return: $B(x_k, t_k)$.

1: Compute $L(x_k, t_k)$ according to (13),
2: $B(x_k, t_k) \leftarrow L(x_k, t_k) \cap \text{Post}(B(x_{k-1}, t_{k-1})),$
3: for each $S_i(t_k) \in B(x_k, t_k)$ do,
4: if $\exists S_j(t_k) \in B(x_k, t_k)$ s.t. $S_j(t_k) = \text{Post}(S_i(t_k))$, then
5: $B(x_k, t_k) \leftarrow B(x_k, t_k) \setminus S_i(t_k),$ 
6: end if
7: end for
NExis (lines 12-13); otherwise, the control input \( \nu_k \) can be chosen as any element of \( U(x_k,t_k) \) (one example is to choose \( \nu_k \) as \( \min_{\nu_k \in U(x_k,t_k)} \{\|\nu_k\|\} \)) (line 15). Then, we implement the chosen \( \nu_k \), measure \( x_{k+1} \) (line 16), and finally compute the subset of set nodes that are possibly available at the next time instant \( t_{k+1} \), i.e., \( \text{Post}(B(x_k,t_k)) \), via Algorithm 11 (line 17).

**Algorithm 8 updateTLT**

**Input:** \( \mathcal{T}_\varphi(t_k) \), \( t_a \) and \( B(x_k,t_k) \).

**Return:** \( \mathcal{T}_\varphi(t_{k+1}) \).

1: for each set node \( S_i(t_k) \) of \( \mathcal{T}_\varphi(t_k) \), do
2: \( \text{if } S_i(t_k) \in B(x_k,t_k) \land t_a(S_i(t_k)) + |\text{CH}(S_i(t_k))| \geq t_{k+1}, \text{ then} \)
3: \( S_i(t_{k+1}) \leftarrow X_i(t_{k+1} - t_a(S_i(t_k))), \)
4: \( \text{else} \)
5: \( S_i(t_{k+1}) \leftarrow S_i(t_k), \)
6: end if
7: end for

We further detail the Algorithms 6-11 in the following.

- Algorithm 6 calculates the functions \( t_a \) and \( t_h \) (lines 1-7) and \( \text{Post}(B(x_{-1},t_{-1})) \) (lines 8-12).

- Algorithm 7 outlines the procedure of finding the subset of set nodes in \( P(t_k) \) that are valid at time \( t_k \), i.e., \( B(x_k,t_k) \). This is the most important step of the control synthesis, and it relates to Algorithm 11 postSet. Firstly, one needs to compute the subset of set nodes of \( P(t_k) \) that contains \( x_k \) at time \( t_k \), i.e., \( L(x_k,t_k) \) (line 1). Then, one has from Definition 4.3 that if a trajectory \( x \) satisfies one complete path of the tTLT, it must i) visit each tube node of the complete path sequentially and ii) stay in each tube node for sufficiently long time steps (Remark 4.1). Based on these two requirements, Algorithm 11 is designed to predict the subset of set nodes that are possibly available at the next time instant, i.e., \( \text{Post}(B(x_{k-1},t_{k-1})) \). \( B(x_k,t_k) \) must belong to \( L(x_k,t_k) \) and \( \text{Post}(B(x_{k-1},t_{k-1})) \) at the same time. Therefore, we let \( B(x_k,t_k) \leftarrow L(x_k,t_k) \cap \text{Post}(B(x_{k-1},t_{k-1})) \) (line 2). The rest of Algorithm 7 (lines 3-7) is to guarantee that \( B(x_k,t_k) \) contains at most one set node for each complete path of \( \mathcal{T}_\varphi(t_k) \).

- Algorithm 8 outlines the procedure of calculating \( \mathcal{T}_\varphi(t_{k+1}) \), given \( \mathcal{T}_\varphi(t_k) \), \( t_a \) and \( B(x_k,t_k) \). It is designed based on [12].
• Algorithm 9 outlines the procedure of building a control tree $T_u(t_k)$, which is then used for control set synthesis. It is initialized as $T_\varphi(t_k)$ (line 1). Then, for those set nodes $S_i(t_k)$ that belongs to $B(x_k, t_k)$, it is replaced with the feasible control set (lines 2-8), otherwise, it is replaced with $\emptyset$ (lines 9-11).

• Algorithm 10 is similar to Algorithm 4, which outlines the procedure of backtracking a compressed tree.

• Algorithm 11 outlines the procedure of finding the subset of set nodes that are possibly available at the next time instant $t_{k+1}$ given $B(x_k, t_k)$, $t_a$ and $T_\varphi(t_{k+1})$. It is designed based on Definition 4.3 where the three cases (lines 4-8, 9-12, 13-16) correspond to items i)-iii) of Definition 4.3 respectively. It guarantees that the resulting trajectory visits each tube node of $T_\varphi$ sequentially and stays in each tube node for sufficiently long time steps (as we discussed in Algorithm 7).

**Algorithm 9 buildControlTree**

**Input:** $T_\varphi(t_k)$, $B(x_k, t_k)$, and $T_\varphi(t_{k+1})$.

**Return:** A control tree $T_u(t_k)$.

1: Initialize $T_u(t_k)$ as $T_\varphi(t_k)$,
2: for each $S_i(t_k) \in B(x_k, t_k)$ do
3:     if $S_i(t_k)$ is a leaf node then,
4:         $S_i(t_k) \leftarrow U(S_i(t_k)) := U,$
5:     else
6:         $S_i(t_k) \leftarrow U(S_i(t_k)) := \{u_k \in U : f_k(x_k, u_k, w_k) \in S_i(t_{k+1}), \forall w_k \in W\},$
7:     end if
8: end for
9: for each $S_i(t_k) \notin B(x_k, t_k)$ do
10:    $S_i(t_k) \leftarrow \emptyset,$
11: end for

Next, an example is given to illustrate one iteration of the control synthesis algorithm (Algorithm 5).

**Example 5.1.** Consider the single integrator model $\dot{x} = u + w$ with a sampling period of 1 second, then the resulting discrete-time system is given by

$$x_{k+1} = x_k + u_k + w_k,$$
**Algorithm 10 Backtracking**

**Input:** a compressed tree $T_{ac}(t_k)$.

**Return:** the root node of $T_{ac}(t_k)$.

1. **for** each Boolean operator node $\Theta$ of $T_{ac}(t_k)$ through a bottom-up traversal, **do**
2.   **if** $\Theta = \land$, **then**
3.     $PA(\Theta) \leftarrow PA(\Theta) \cup (CH_1(\Theta) \cap CH_2(\Theta))$,
4.   **else**
5.     $PA(\Theta) \leftarrow PA(\Theta) \cup (CH_1(\Theta) \cup CH_2(\Theta))$,
6. **end if**
7. **end for**

**Algorithm 11 postSet**

**Input:** $B(x_k, t_k), t_a$ and $T_\phi(t_{k+1})$.

**Return:** $Post(B(x_k, t_k))$.

1. Initialize $Post(S_i(t_k)) = \emptyset, \forall S_i(t_k) \in B(x_k, t_k)$.
2. **for** each $S_i(t_k) \in B(x_k, t_k)$, **do**
3.     **switch** the children of $S_i(t_k)$ **do**
4.       **case** $CH(S_i(t_k)) \in \{\land, \lor\}$,
5.         $Post(S_i(t_k)) \leftarrow S_i(t_{k+1})$,
6.       **for** each $S_j(t_k)$ that is reachable from $S_i(t_k)$ by a Boolean fragment, **do**
7.         $Post(S_i(t_k)) \leftarrow Post(S_i(t_k)) \cup S_j(t_{k+1})$,
8.       **end for**
9.       **case** $CH(S_i(t_k)) \in \{U_{[a,b]}\}$,
10.      if $t_k > t_a(Pre(S_i(t_k)) + a)$, **then**
11.         $Post(S_i(t_k)) \leftarrow S_i(t_{k+1}) \cup Post(S_i(t_{k+1}))$,
12.      **end if**
13.       **case** $CH(S_i(t_k)) \in \{G_{[a,b]}\}$,
14.      if $t_k > t_a(Pre(S_i(t_k)) + b)$, **then**
15.         $Post(S_i(t_k)) \leftarrow S_i(t_{k+1}) \cup Post(S_i(t_{k+1}))$,
16.      **end if**
17. **end for**
where \( x_k \in \mathbb{R}^2, u_k \in U := \{ u : ||u|| \leq 1 \} \subset \mathbb{R}^2, w_k \in W := \{ w : ||w|| \leq 0.1 \} \subset \mathbb{R}^2, \forall k \in \mathbb{N} \). The task specification \( \varphi \) is given in Example 3.1, i.e.,

\[
\varphi = F_{[a_1,b_1]}G_{[a_2,b_2]}\mu_1 \land \mu_2U_{[a_3,b_3]}\mu_3,
\]

where \([a_1,b_1] = [5,10], [a_2,b_2] = [0,10], [a_3,b_3] = [0,8], g_{\mu_1}(x) = 1 - ||x||, g_{\mu_2}(x) = 5 - ||x - [4,4]^T||, \text{ and } g_{\mu_3}(x) = 1 - ||x - [3,5]^T||. \]

Then, after running lines 2-7, one has

\[
S_{\mu_1} = \{ x_0 : ||x_0|| \leq 1 \},
\]

\[
S_{\mu_2} = \{ x_0 : ||x_0 - [4,4]^T|| \leq 5 \},
\]

\[
S_{\mu_3} = \{ x_0 : ||x_0 - [3,5]^T|| \leq 1 \}.
\]

The tTLT that corresponds to \( \varphi \) is plotted in Figure 4. Using Definitions 2.6 and 2.7, one can calculate that

\[
X_4(t_k) = \{ x_k : ||x_k|| \leq 0.9 \},
\]

\[
X_3(t_k) = \{ x_k : ||x_k - [3,5]^T|| \leq 8.1 - k \}
\]

\[
\land ||x_k - [4,4]^T|| \leq 5 \},
\]

\[
X_2(t_k) = \{ x_k : ||x_k|| \leq 9.9 - k \},
\]

\[
X_1(t_k) = X_2(t_k) \cap X_3(t_k).
\]

The initial state \( x_0 = [0.5,0.8]^T \), for which \( x_0 \in X_4(0) \). Firstly, an initialization process is required, and one can get from Algorithm 6 that

\[
t_h(X_1) = 0, t_h(X_2) = 10, t_h(X_3) = 8,
\]

\[
t_h(X_4) = 20, t_h(S_{\mu_1}) = \infty, t_h(S_{\mu_3}) = \infty,
\]

and

\[
Post(B(x_{-1}, t_{-1})) = \{ X_1(t_0), X_2(t_0), X_3(t_0) \}.
\]

Now, let us see how the feasible control set \( U(x_0,t_0) \) is synthesized at time instant \( t_0 \).

1) Find \( B(x_0,t_0) \) via Algorithm 7. Firstly, \( L(x_0,t_0) \) is computed according to [13],

\[
L(x_0,t_0) = \{ X_1(t_0), X_2(t_0), X_3(t_0), X_4(t_0), S_{\mu_1} \}.
\]

Then, after running lines 2-7, one has

\[
B(x_0,t_0) = \{ S_2(t_0), S_3(t_0) \}.
\]

2) Determine the activation time. Initially, both \( t_a(X_2) \) and \( t_a(X_3) \) are unknown, therefore, \( t_a(X_2) = t_a(X_3) = t_0 \).
3) Update the TLT (thus obtain $T_\varphi(t_1)$) via Algorithm 8. The output $T_\varphi(t_1)$ is given by

\[ S_1(t_1) = X_1(t_0), S_2(t_1) = X_2(t_1), \]
\[ S_3(t_1) = X_3(t_1), S_4(t_1) = X_4(t_0), \]

and the leaf nodes $S_{\mu_1}$ and $S_{\mu_3}$ are unchanged.

4) Build the control tree $T_u(t_0)$, compress it to obtain $T_u^c(t_0)$, and then get $U(x_0, t_0)$. This process is illustrated in Figure 7, and $U(x_0, t_0) = U(S_2(t_0)) \cap U(S_3(t_0))$.

Since $U(x_0, t_0) \neq \emptyset$, the online control synthesis continues, and we can further compute $\text{Post}(B(x_0, t_0))$ via Algorithm 11, which gives

\[ \text{Post}(B(x_0, t_0)) = \{S_2(t_1), S_3(t_1), S_{\mu_3}\}. \]

The following theorem and corollary show the applicability and correctness of Algorithm 5.

**Theorem 5.1.** Consider the uncertain system $\{1\}$ with initial state $x_0$ and an STL formula $\varphi$. Assume that $\varphi$ is robustly satisfiable for $\{1\}$ and $x_0 \in T_{\text{root}}^\varphi(t_0)$. Then, by implementing the online control synthesis algorithm (Algorithm 5), one can guarantee that

(i) the control set $U(x_k, t_k)$ is nonempty for all $k \in \mathbb{N}$;
(ii) the resulting trajectory $x \models \varphi$.

**Proof.** The proof follows from the construction of tTLT and Algorithms 5-11. The existence of a controller $\nu_k$ at each time step $t_k$, is guaranteed by the definition of maximal and minimal reachable sets (Definitions 2.6 and 2.7), and the construction of tTLT (Lemma 2.1, Theorem 3.1 and Algorithm 1). Moreover, the design of Algorithms 5-11 guarantees that the resulting trajectory $x$ satisfies the tTLT $T_\varphi$, i.e., $x \equiv T_\varphi$, which implies $x \models \varphi$ as proven in Theorem 4.1.

**Corollary 5.1.** Consider the deterministic system (2) with initial state $x_0$ and an STL formula $\varphi$. Assume that $\varphi$ is satisfiable for (1). Then, by implementing the online control synthesis algorithm (Algorithm 5), one can guarantee that

(i) the control set $U(x_k, t_k)$ is nonempty for all $k \in \mathbb{N}$;

(ii) the resulting trajectory $x \models \varphi$.

**Remark 5.1.** It can be concluded from Theorem 5.1 and Corollary 5.1 that the online control synthesis algorithm is sound for uncertain systems, and both sound and complete for deterministic systems.

**Remark 5.2.** The construction of tTLT relies on the computation of backward reachable tubes. It can be performed offline in many applications and has been widely studied in the existing literature [26, 27]. In addition, computational tools have also been developed for different kinds of systems, e.g., the Hamilton-Jacobi toolbox [28]. On the other hand, although the exact computation of backward reachable set/tube is in general nontrivial for high-dimensional nonlinear systems, efficient algorithms exist for linear systems with polygonal input and disturbance sets [26].

**Remark 5.3.** The online control synthesis algorithm (Algorithm 5) contains 7 sub-algorithms, i.e., Algorithm 3 and Algorithms 6-11. The computational complexity is determined by Algorithm 9, in which one-step feasible control sets need to be computed. The computational complexity of Algorithms 3, 6, 7, 8, 10, 11 is $O(1)$. Note that in Algorithm 8, the computation of reachable sets, which is required for set node update, is done offline when constructing the tTLT.

**Remark 5.4.** Different from the mixed-integer programming formulation for STL control synthesis [16, 17], where an entire control policy has to be
synthesized at each time step, the control synthesis in our work is reactive in the sense that only the control input at the current time step is generated at each time step. In [18], a robust model predictive approach is proposed to control discrete-time linear systems with additive bounded disturbances. In our work, we consider general uncertain discrete-time systems and the recursive feasibility is guaranteed when the STL formula $\varphi$ is robustly satisfiable and $x_0 \in T_{\text{root}}^\varphi(t_0)$. Moreover, the framework proposed is also capable of dealing with unbounded STL formulas (as opposed to [16, 17, 18]).

6. Simulation

In this section, a simulation example illustrating the theoretical results is provided. This example will specify an overtaking task as an STL formula and then show how to synthesize overtaking controller with safety guarantee.

As shown in Figure 8, we consider a scenario where an automated vehicle Veh$_1$ plans to move to a target set $S_{\mu_1}$ within 80 seconds. Since there is a broken vehicle Veh$_2$ in front of Veh$_1$ and there is another vehicle Veh$_3$ that moves in an opposite direction in the other lane, Veh$_1$ must overtake Veh$_2$ for reaching $S_{\mu_1}$ and avoid Veh$_3$ for safety.

We describe the dynamics of the vehicle Veh$_1$ as in [29]:

$$x_{k+1} = 
\begin{bmatrix}
1 & 0 & \delta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} x_k +
\begin{bmatrix}
0 & 0 \\
0 & \delta \\
0 & \delta
\end{bmatrix} u_k + w_k,
$$

(14)

where $x_k = [p^x(k), p^y(k), v^x(k)]^T$, $u_k = [v^y(k), a^x(k)]^T$, and $\delta$ is the sampling period. The working space is $X = \{ z \in \mathbb{R}^3 \mid [0, -5, -3]^T \leq z \leq [120, 5, 3]^T \}$, the control constraint set is $U = \{ z \in \mathbb{R}^2 \mid [-1, -1]^T \leq z \leq [1, 1]^T \}$, the disturbance set is $W = \{ z \in \mathbb{R}^3 \mid [-0.05, -0.05, -0.05]^T \leq z \leq [0.05, 0.05, 0.05]^T \}$, and the target region is $S_{\mu_1} = \{ z \in \mathbb{R}^2 \mid [115, -5, 0.5]^T \leq z \leq [120, 0, 0.5]^T \}$.

Figure 8: Scenario illustration: an automated vehicle plans to reach a target set $S_{\mu_1}$ while overtaking a broken vehicle Veh$_2$ in front of it in the same lane and avoiding Veh$_3$ moving in an opposite direction in the other lane.
We use $S_{\mu_2} = \{z \in \mathbb{R}^3 \mid [45, -5, -\infty]^T \leq z \leq [50, 0, \infty]^T\}$ to denote the state set that contains the occupancy of Veh$_2$. We describe the dynamics of the vehicle Veh$_3$ as

\[
\bar{x}_{k+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix} \bar{u}_k, \tag{15}
\]

where $x_k = [\bar{p}^x(k), \bar{p}^y(k)]^T$, $\bar{u}_k = [\bar{v}^x(k), \bar{v}^y(k)]^T$. We assume that it moves at a constant velocity $\bar{u}_k = [\bar{v}^x, 0]^T$. The initial state of Veh$_3$ is $\bar{x}_0 = [\bar{p}^x_{ini}, 2.5]^T$. Then, we have that its position of x-axis is $\bar{p}^x_k = \bar{p}^x_{ini} + \delta \times (k - 1) \times \bar{v}^x$.

To formulate the overtaking task, we define the following three sets as shown in Figure 8: $S_{\mu_3} = \{z \in \mathbb{R}^3 \mid [0, -5, -3]^T \leq z \leq [35, 0, 3]^T\}$, $S_{\mu_4} = \{z \in \mathbb{R}^3 \mid [35, -5, -3]^T \leq z \leq [60, 5, 3]^T\}$, and $S_{\mu_5} = \{z \in \mathbb{R}^3 \mid [60, -5, -3]^T \leq z \leq [120, 0, 3]^T\}$.

Let us choose the sampling period as $\delta = 0.2$s (seconds). To respect the time constraint and the input constraint for Veh$_1$, we consider two possible solutions to the previous reachability problem: (1) quick overtaking: overtake Veh$_2$ before Veh$_3$ passes Veh$_2$; (2) slow overtaking: wait until Veh$_3$ passes Veh$_2$ and then overtake Veh$_2$. The quick overtaking can be encoded into an STL formula:

\[
\varphi_1 = (\mu_3 U_{[0,16]} (\mu_4 \land \neg \mu_2) U_{[0,15]} \mu_5 U_{[0,30]} G_{[0,2]} \mu_1) \land G_{[0,80]} \neg \mu_6, \tag{16}
\]

where $S_{\mu_6} = \{z \in \mathbb{R}^6 \mid [\bar{p}^x(16), 0, -\infty]^T \leq z \leq [\bar{p}^x(0), 5, \infty]^T\}$. Note that $S_{\mu_6}$ denotes the reachable set for the vehicle Veh$_3$ within the time interval $[0, 16]$ seconds and 16 (that corresponds to the sampling index $k = 80$) is the maximal time instant that the vehicle Veh$_1$ can reach the set $S_{\mu_5}$ in the sprit of $\varphi_1$. The slow overtaking can be encoded into an STL formula

\[
\varphi_2 = (\mu_3 U_{[16,32]} (\mu_4 \land \neg \mu_2) U_{[15]} \mu_5 U_{[30]} G_{[0,2]} \mu_1) \land G_{[0,80]} \neg \mu_7, \tag{17}
\]

where $S_{\mu_7} = \{z \in \mathbb{R}^2 \mid [-\infty, 0, -\infty]^T \leq z \leq [\bar{p}^x(16), 5, \infty]^T\}$. Note that $S_{\mu_7}$ denotes the reachable set for the vehicle Veh$_3$ within the time interval $[16, +\infty)$ and 16 (that corresponds to the sampling index $k = 80$) is the minimal time instant that the vehicle Veh$_1$ can reach the set $S_{\mu_5}$ in the sprit of $\varphi_2$. The overall specification is written as $\varphi = \varphi_1 \lor \varphi_2$. Using Algorithm 1, one can construct the tTLT $T_\varphi$ (see Figure 9), where

- $X_7(t_k) = X_{12}(t_k) = R^m(X, S_{\mu_1}, [0, 2], k)$,
- $X_6(t_k) = R^M(X, X_7(t_0), S_{\mu_5}, [0, 30], k)$,
$X_{11}(t_k) = R^M(X, X_{12}(t_0), S_{\mu_5}, [0, 30], k),$
$X_5(t_k) = R^M(X, X_6(t_0), S_{\mu_4} \cap \overline{S_{\mu_2}}, [0, 15], k),$
$X_{10}(t_k) = R^M(X, X_{11}(t_0), S_{\mu_4} \cap \overline{S_{\mu_2}}, [0, 15], k),$
$X_4(t_k) = R^M(X, X_5(t_0), S_{\mu_3}, [0, 16], k),$
$X_9(t_k) = R^M(X, X_{10}(t_0), S_{\mu_3}, [16, 32], k),$
$X_8(t_k) = R^m(X, S_{\mu_6}, [0, 80], k),$
$X_{13}(t_k) = \overline{R^m}(X, S_{\mu_7}, [0, 80], k),$
$X_2(t_k) = X_4(t_k) \cap X_8(t_k),$
$X_3(t_k) = X_9(t_k) \cap X_{13}(t_k),$ and
$X_1(t_k) = X_2(t_k) \cup X_3(t_k).$

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Figure 9: The constructed tTLT $T_\varphi$, where the left and right blue boxes are the tTLTs $T_{\varphi_1}$ and $T_{\varphi_2}$, respectively.

In the following, two simulation cases are considered and the online control synthesis algorithm is implemented.

In the fast overtaking, we choose the initial position $\bar{p}_{x_{ini}} = 95$ and the moving velocity $\bar{v}^x = -2$ for the vehicle Veh_3 and the initial position $x_0 = [0.5, -2.5, 2]^T$ for Veh_1. By using the results for the satisfiability check,
the specification $\varphi_1$ is robustly satisfiable while $\varphi_2$ is infeasible. Figure 10 (a) shows the position trajectories, from which we can see that the whole specification is completed. The blue region denotes the set $S_{\mu_6}$. Figure 10 (b) shows the velocity trajectory of $v^x$ and Figure 10 (c)–(d) show the corresponding control inputs, where the dashed lines denote the control bounds. The cyan regions represent the synthesized control sets and the blue lines are the control trajectories.

![Figure 10: Trajectories for one realization of disturbance signal in the fast overtaking: (a) position trajectory; (b) velocity trajectory of x-axis; (c) control trajectory of x-axis; (d) control trajectory of y-axis.](image)

In the slow overtaking, we choose the initial position $\bar{p}_{\text{ini}} = 80$ and the moving velocity $\bar{v}^x = -3$ for the vehicle Veh$_3$ and the same initial position $x_0 = [0.5, -2.5]^T$ for Veh$_1$. In contrary to the fast overtaking, the specification $\varphi_2$ is robustly satisfiable while $\varphi_1$ is infeasible. Figure 11 (a) shows the position trajectories, from which we can see that the whole specification is completed. The blue region denotes the intersection between the set $X$ and the set $S_{\mu_7}$. Figure 11 (b) shows the velocity trajectory of $v^x$ and Figure 11 (c)–(d) show the corresponding control input trajectories of $a^x$ and $v^y$. 

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Figure 11: Trajectories for one realization of disturbance signal in the slow overtaking: (a) position trajectory; (b) velocity trajectory of $x$-axis; (c) control trajectory of $x$-axis; (d) control trajectory of $y$-axis.

Although the position trajectories in the two cases are similar as shown in Figs. 10(a)–11(a), we highlight their difference through the evolution of the position of $x$-axis along the time in Figure 12. We use $k_1$, $k_2$, and $k_3$ (or $k_1'$, $k_2'$, and $k_3'$) to denote the minimal time instants that Veh$_1$ reaches the sets $S_{\mu_4}$, $S_{\mu_5}$, and $S_{\mu_1}$ in the fast overtaking (or the slow overtaking), respectively. We can see that these two position trajectories satisfy the time intervals encoded in the $\varphi_1$ and $\varphi_2$, respectively. Furthermore, in order to show the robustness, we run 100 realizations of the disturbance trajectories in the fast overtaking. The position trajectories for such 100 realizations are shown in Figure 13.
Finally, we report the computation time of this example, which was run in Matlab R2016a with MPT toolbox \cite{mpt} on a Dell laptop with Windows 7, Intel i7-6600U CPU 2.80 GHz and 16.0 GB RAM. We perform reachability analysis for constructing the tTLT offline, which takes 59.10 seconds. For online control synthesis, the minimal computation time at a single time step over 100 realizations is 0.23 seconds, while the maximal computation time is 1.07 seconds. The average time of each time step is 0.31 seconds. We remark that the mixed-integer formulation is difficult to implement in this example. This is because the computational complexity of mixed-integer programming grows exponentially with the horizon of the STL formula, which in this example reaches up to 400 sampling instants, much longer than the horizons considered in the simulation examples of \cite{16, 17, 18}.

7. Conclusion

A novel approach for the robust satisfiability check and online control synthesis of uncertain discrete-time systems under STL specifications was
proposed in this paper. Using the notion of tTLT, a sufficient condition was obtained for the robust satisfiability check of the uncertain systems. Moreover, when the underlying system is deterministic, a necessary and sufficient condition was further obtained for satisfiability. An online control synthesis algorithm was proposed. The soundness of the algorithm was proven when the system is uncertain while the completeness was further proven when the system is deterministic. In the future, the control synthesis for multi-agent systems under local and/or global STL specifications is of interest.

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