The GSI method for studying neutrino mass differences
For Pedestrians

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Abstract

New experiment studying radioactive ion before K-capture weak decay provides new method for investigating creation of coherent final state producing $\nu$ oscillations. Single radioactive ion passes through known electromagnetic interactions in storage ring before decaying by first order weak interaction conserving momentum described by Fermi’s golden rule. Initial state must contain coherent linear combinations of two states with same momentum difference producing final state oscillations. Following passage of ion through storage ring provides information about $\nu$ masses and mixing without detecting $\nu$. Observing every weak decay avoids suppression in conventional oscillation experiments by low $\nu$ absorption cross sections. Normally unobservable long wave lengths made observable by long distance circulation around storage ring. No oscillations can be observed in “missing mass” experiment where both energy and momentum are conserved. Energy-time uncertainty allows decay into same final state of two initial state components with same momentum difference and slightly different energies. Relative phase of these two components changes with time and can produce oscillations between Dicke superradiant and subradiant states. Analysis of oscillations in simple model with no fudge factors gives value for the squared neutrino mass difference in the same ball park as the KAMLAND value. Treatment consistent with quantum mechanics and causality.

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I. INTRODUCTION - THE BASIC PARADOX OF NEUTRINO OSCILLATIONS

A. Interference is possible only if we can’t know everything

Neutrino oscillations are produced from a coherent mixture of different $\nu$ mass eigenstates. If the $\nu$ is produced in a reaction where all other particles have definite momentum and energy the $\nu$ mass is determined by conservation of energy and momentum. Interference between amplitudes from different $\nu$ mass eigenstates is not observable in such a “missing mass” experiment. Something must prevent knowing the neutrino mass from conservation laws. Ignorance alone does not produce interference. Quantum mechanics must hide information. To check how coherence and oscillations can occur we investigate what is known and what information is hidden by quantum mechanics.

A recent GSI experiment [1] describes a unique opportunity for this investigation. Oscillations are observed in the decay time of a radioactive ion circulating in a storage ring. The decay by atomic electron capture emits an unobserved monoenergetic neutrino. This oscillation offers a new and very interesting method for determining neutrino masses and mixing angles [2–4]. The detailed energy-momentum analysis is much simpler here than in other oscillation experiments. The initial radioactive “Mother” ion is in a one-particle state with a definite mass moving in a storage ring. There is no entanglement [5] since no other particles are present.

The search for what is known and what is hidden leads to energy-time uncertainty. The time interval between the last time the initial ion was observed without decay and the observed decay time is sufficiently short to allow enough violation of energy conservation to prevent its use in a missing mass experiment. This is related to the line broadening of a decay observed in a time short compared to its natural line width. The decay to two final states is described by two Breit-Wigner energy distributions. These are separated at long times. But in the GSI experiment [1] the decay time is sufficiently short to make the separation negligible in comparison with their broadened widths. The transition can occur...
coherently from two components of the initial state with different energies and momenta into a same final state with a different common energy and the same momentum difference. The sum of the transition amplitudes from these two components of an initial state to the same final state depends on their relative phase. Changes in this phase can produce oscillations.

**B. Energy and momentum considerations**

The energy-time uncertainty is not covariant and defined only in the laboratory system. Covariant descriptions and transformations from the laboratory to any center-of-mass system are not valid for a description of neutrino oscillations.

Macroscopic neutrino detectors in thermal equilibrium with their environment destroy all interference between states having different energies [6]. The observation of reactor neutrino oscillations shows that the emitted neutrino wave function must contain coherent linear combinations of three mass eigenstates with the same energy, a definite relative phase and a momentum differences defined by the energy and mass differences.

The final state is a “Daughter” ion and a $\nu_e$ neutrino, a linear combination of several $\nu$ mass eigenstates. This $\nu_e$ is a complicated wave packet containing components with three different masses and a continuous spectrum of energies and momenta. It would oscillate with distance from the source in the same way that reactor neutrinos oscillate.

We now summarize what is known and what is hidden by quantum mechanics.

1. The final state state has coherent pairs of states with same energy but has neutrinos with different masses and different momenta.

2. The initial state is a one-particle state with a definite mass.

3. The initial state can contain coherent pairs with the same momentum difference present in the final state state but these must have different energies.

4. Momentum is conserved in the transition.
5. Energy-time uncertainty hides information and prevents use of energy conservation.

6. The same final energy eigenstate can be produced from states with different energies.

7. The transition can occur coherently from two components of the initial state with different energies and momenta into a same final state with a different common energy and the same momentum difference.

8. The relative phase between components with different energies changes during the passage of the ion through the storage ring.

9. These phase changes can produce oscillations.

A treatment of neutrino oscillations without explicit violation of energy conservation describes a missing mass experiment where no neutrino oscillations of any kind are allowed.

C. The two principal difficulties of neutrino experiments

1. Ordinary neutrino oscillation experiments are difficult because

   - The tiny neutrino absorption cross section makes number of neutrino events actually used many orders of magnitude smaller than number of undetected neutrinos.
   - The oscillation wave lengths are so large that it is difficult to actually follow even one oscillation period in any experiment.

2. This experiment opens up a new line for dealing with these difficulties

   - The oscillation is measured without detecting the $\nu$. Detection of every $\nu$ creation event avoids the losses from the low neutrino absorption cross section.
   - The long wave length problem is solved when ions move a long distance circulating around in a storage ring and show many oscillations in same experiment.
II. THE BASIC PHYSICS OF THE GSI EXPERIMENT

A. A first order weak transition

The initial state wave function $|i(t)\rangle$ is a “Mother” ion wave packet containing components with different momenta. Its development in time is described by an unperturbed Hamiltonian denoted by $H_o$ which describes the motion of the initial and final states in the electromagnetic fields constraining their motion in a storage ring.

$$|i(t)\rangle = e^{iH_0 t} |i(0)\rangle$$ (2.1)

The time $t = 0$ is defined as the time of entry into the apparatus. Relative phases of wave function components with different momenta are determined by localization in space at the point of entry into the apparatus. Since plane waves have equal amplitudes over all space, these relative phases are seriously constrained by requiring that the probability of finding the ion outside the storage ring must be zero.

A first-order weak decay is described by the Fermi Golden Rule. The transition probability per unit time at time $t$ from an initial state $|i(t)\rangle$ to a final state $|f\rangle$ is

$$W(t) = \frac{2\pi}{\hbar} |\langle f | T |i(t)\rangle|^2 \rho(E_f) = \frac{2\pi}{\hbar} |\langle f | T e^{iH_0 t} |i(0)\rangle|^2 \rho(E_f)$$ (2.2)

where $T$ is the transition operator and $\rho(E_f)$ is the density of final states. The transition operator $T$ conserves momentum.

If two components of the initial state with slightly different energies can both decay into the same final state, their relative phase changes linearly with time and can produce changes in the transition matrix element. The quantitative result and the question of whether oscillations can be observed depend upon the evolution of the initial state. The neutrino is not detected in the GSI experiment [1], but the information that a particular linear combination of mass and momentum eigenstates would be created existed in the system. Thus the same final state can be created by either of three initial states that have the same momentum difference. Violation of energy conservation allows the decay and provides a new method for investigating the creation of such a coherent state.
B. Time dependence and internal clocks

A measurement of the time between the initial observation and the decay of a radioactive ion circulating in a storage ring depends upon the existence of an internal clock in the system.

1. An initial ion in a one-particle energy eigenstate has no clock. Its propagation in time is completely described by a single unobservable phase.

2. If the initial ion is in a coherent superposition of different energy eigenstates, the relative phase of any pair changes with energy. This phase defines a clock which can measure the time between initial observation and decay.

3. If the decay transition conserves energy, the final states produced by the transition must also have different energies.

4. The decay probability is proportional to the square of the sum of the transition matrix elements to all final states. There are no interference terms between final states with different energies and their relative phases are unobservable.

The probability $P_i(t)$ that the ion is still in its initial state at time $t$ and not yet decayed satisfies an easily solved differential equation,

$$\frac{d}{dt} P_i(t) = -W(t)P_i(t); \quad \frac{d}{dt} \log(P_i) = -W(t); \quad P_i(t) = e^{-\int W(t)dt} \quad (2.3)$$

If $W(t)$ is independent of time eq. (2.3) gives an exponential decay. The observation of a nonexponential decay implies that $W(t)$ is time dependent. A non-exponential decay can occur only if there is a violation of energy conservation. All treatments which assume energy conservation; e.g. [5] will only predict exponential decay. Time dependence can arise if the initial ion is in a coherent superposition of different energy eigenstates, whose relative phases change with time. This phase defines a clock which can measure the time between initial observation and decay. Since the time $dt$ is infinitesimal, energy need not be conserved in this transition.
$W(t)$ depends upon the unperturbed propagation of the initial state before the time $t$ where its motion in the storage ring is described by classical electrodynamics. Any departure from exponential decay must come from the evolution in time of the initial unperturbed state. This can change the wave function at the time of the decay and therefore the value of the transition matrix element. What happens after the decay cannot change the wave function before the decay. Whether or not and how the final neutrino is detected cannot change the decay rate.

C. The role of Dicke superradiance

Dicke [7] has shown that whenever two initial state components can produce amplitudes for decay into the same final state, a linear combination called “superradiant” has both components interfering constructively to enhance the transition. The orthogonal state called “subradiant” has maximum destructive interference and may even produce a cancelation.

The wave function of the initial state before the transition can contain pairs of components with a momentum difference allowing both to decay into the same final state. This wave function can be expressed as a linear combination of superradiant and subradiant states with a relative magnitude that changes with time. The variation between superradiant and subradiant wave functions affects the transition matrix element and can give rise to oscillations in the decay probability. Since the momentum difference depends on the mass difference between the two neutrino eigenstates these oscillations can provide information about neutrino masses.

III. DETAILED ANALYSIS OF A SIMPLIFIED MODEL

A. The initial and final states for the transition matrix

The final state is a daughter ion and an electron neutrino.

1. The neutrino wave packet contains different masses, energies and momenta
2. Oscillation experiments use macroscopic detectors in thermal equilibrium which kill coherence between states with different energies

3. Observed oscillations arise only from components with same energy, different masses and different momenta

The initial state before the transition is a mother ion wave packet.

1. Need to know evolution of wave packet during passage around the storage ring

2. Not easily calculated. Requires knowing path through straight sections, bending sections and focusing electric and magnetic fields

B. Dicke superradiance and subradiance in the experiment

Consider the transition from a simplified initial state for the “mother” ion with only two components denoted by $|P\rangle$ and $|P + \delta P\rangle$ having momenta $P$ and $P + \delta P$ with energies $E$ and $E + \delta E$. These can decay coherently into a final state which produces the observed oscillations. The final state has a recoil “daughter” ion and an electron neutrino which is a linear combination of two neutrino mass eigenstates. The oscillations must be produced in a final state having two components with the same energy. The recoil “daughter” ion in both components must have the same momentum $P_R$ and energy $E_R$ in order that the two final state components be coherent.

The final state $|f(E_\nu)\rangle$ has an electron neutrino with energy $E_\nu$.

$$|f(E_\nu)\rangle \equiv |P_R; \nu_e(E_\nu)\rangle = |P_R; \nu_1(E_\nu)\rangle \langle \nu_1 | \nu_e \rangle + |P_R; \nu_2(E_\nu)\rangle \langle \nu_2 | \nu_e \rangle$$

(3.1)

where $\nu_1$ and $\nu_2$ denote neutrino mass eigenstates with masses $m_1$ and $m_2$. $\langle \nu_1 | \nu_e \rangle$ and $\langle \nu_2 | \nu_e \rangle$ are elements of the neutrino mass mixing matrix, commonly expressed in terms of a mixing angle denoted by $\theta$. 

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\[
\cos \theta \equiv \langle \nu_1 | \nu_e \rangle; \quad \sin \theta \equiv \langle \nu_2 | \nu_e \rangle; \quad |f(E_\nu)\rangle = \cos \theta \left| \vec{P}_R; \nu_1(E_\nu) \right\rangle + \sin \theta \left| \vec{P}_R; \nu_2(E_\nu) \right\rangle \quad (3.2)
\]

After a very short time two components with different initial state energies can decay into a final state which has two components with the same energy and a neutrino state having two components with the same momentum difference \(\delta \vec{P}\) present in the initial state.

The momentum conserving transition matrix elements between the two initial momentum components to final states with the same energy and momentum difference \(\delta \vec{P}\) are

\[
\langle f(E_\nu) | T | \vec{P} \rangle = \cos \theta \left\langle \vec{P}_R; \nu_1(E_\nu) \right| T \left| \vec{P} \right\rangle; \quad \langle f(E_\nu) | T | \vec{P} + \delta \vec{P} \rangle = \sin \theta \left\langle \vec{P}_R; \nu_2(E_\nu) \right| T \left| \vec{P} + \delta \vec{P} \right\rangle
\]

(3.3)

We neglect transverse momenta and set \(\vec{P}_r \cdot \vec{p}_\nu \approx Pp_\nu\) where \(P\) and \(p_\nu\) denote the components of the momenta in the direction of the incident beam. The Dicke superradiance analog [7] is seen by defining superradiant and subradiant states.

\[
|\text{Sup}(E_\nu)\rangle \equiv \cos \theta |P\rangle + \sin \theta |P + \delta P\rangle; \quad |\text{Sub}(E_\nu)\rangle \equiv \cos \theta |P + \delta P\rangle - \sin \theta |P\rangle \quad (3.4)
\]

The transition matrix elements for these two states are then

\[
\frac{\langle f(E_\nu) | T | \text{Sup}(E_\nu) \rangle}{\langle f | T | P \rangle} = [\cos \theta + \sin \theta]; \quad \frac{\langle f(E_\nu) | T | \text{Sub}(E_\nu) \rangle}{\langle f | T | P \rangle} = [\cos \theta - \sin \theta] \quad (3.5)
\]

where we have neglected the dependence of the transition operator \(T\) on the small change in the momentum \(P\). The squares of the transition matrix elements are

\[
\frac{\langle f(E_\nu) | T | \text{Sup}(E_\nu) \rangle}{\langle f | T | P \rangle} = [1 + \sin 2\theta]; \quad \frac{\langle f(E_\nu) | T | \text{Sub}(E_\nu) \rangle}{\langle f | T | P \rangle} = [1 - \sin 2\theta] \quad (3.6)
\]

For maximum neutrino mass mixing, \(\sin 2\theta = 1\) and

\[
\langle f(E_\nu) | T | \text{Sup}(E_\nu) \rangle = 0 \quad (3.7)
\]

This is the standard Dicke superradiance in which all the transition strength goes into the superradiant state and there is no transition from the subradiant state.
C. Kinematics for a simplified two-component initial state.

We first consider the transition for each component of the wave packet which has a momentum $\vec{P}$ and energy $E$ in the initial state. The final state has a recoil ion with momentum denoted by $\vec{P}_R$ and energy $E_R$ and a neutrino with mass $m$, energy $E_\nu$ and momentum $\vec{p}_\nu$. If both energy and momenta are conserved,

$$E_R = E - E_\nu; \quad \vec{P}_R = \vec{P} - \vec{p}_\nu; \quad M^2 + m^2 - M_R^2 = 2EE_\nu - 2\vec{P} \cdot \vec{p}_\nu \quad (3.8)$$

We again neglect transverse momenta and consider the simplified two-component initial state for the “mother” ion having momenta $P$ and $P + \delta P$ with energies $E$ and $E + \delta E$. The final state has two components having neutrino momenta $p_\nu$ and $p_\nu + \delta p_\nu$ with energies $E_\nu$ and $E_\nu + \delta E_\nu$ together with a recoil ion having the same momentum and energy for both components. The changes in these variables required to produce a small change $\Delta(m^2)$ in the squared neutrino mass are seen from eq. (3.8) to satisfy the relation

$$\frac{\Delta(m^2)}{2} = E\delta E_\nu + E_\nu\delta E - P\delta p_\nu - p_\nu\delta P = -E\delta E \cdot \left[ 1 - \frac{\delta E_\nu}{\delta E} + \frac{p_\nu}{P} - \frac{E_\nu}{E} \right] \approx -E\delta E \quad (3.9)$$

where we have noted that the two final neutrino components have the same energy; i.e. $\delta E_\nu = 0$, momentum conservation in the transition requires $P\delta p_\nu = P\delta P = E\delta E$, $E$ and $P$ are of the order of the mass $M$ of the ion and $p_\nu$ and $E_\nu$ are much less than $M$.

The relative phase $\delta \phi$ at a time $t$ between the two states $|P\rangle$ and $|P + \delta P\rangle$ is given by $\delta E \cdot t$. Equation (3.9) relates $\delta E$ to the difference between the squared masses of the two neutrino mass eigenstates. Thus

$$E \cdot \delta E = -\frac{\Delta(m^2)}{2}; \quad \delta \phi \approx -\delta E \cdot t = -\frac{\Delta(m^2)}{2E} \cdot t = -\frac{\Delta(m^2)}{2\gamma M} \cdot t \quad (3.10)$$

where $\gamma$ denotes the Lorentz factor $E/M$.

Thus from eq. (3.4) the initial state at time $t$ varies periodically between the superradiant and subradiant states. The period of oscillation $\delta t$ is obtained by setting $\delta \phi \approx -2\pi$,

$$\delta t \approx \frac{4\pi \gamma M}{\Delta(m^2)}; \quad \Delta(m^2) = \frac{4\pi \gamma M}{\delta t} \approx 2.75 \Delta(m^2)_{exp} \quad (3.11)$$
where the values of $\delta t$ and $\Delta(m^2)_{\text{exp}}$ are obtained from the GSI experiment and neutrino oscillation experiments [2].

The theoretical value (3.11) obtained with minimum assumptions and no fudge factors is in the same ball park as the experimental value obtained from completely different experiments. Better values obtained from better calculations can be very useful in determining the masses and mixing angles for neutrinos.

D. A tiny energy scale

The experimental result sets a scale in time of seven seconds. This gives a tiny energy scale for the difference between two waves which beat with a period of seven seconds.

$$\Delta E \approx 2\pi \cdot \frac{\hbar}{\tau} = 2\pi \cdot \frac{6.6 \cdot 10^{-16}}{7} \approx 0.6 \cdot 10^{-15} \text{eV} \quad (3.12)$$

This tiny energy scale must be predictable from standard quantum mechanics using a scale from another input. The only other input available is in the propagation of the initial state through the storage ring during the time before the decay. One tiny scale available in the parameters that describe this experiment is the mass-squared difference between two neutrino mass eigenstates. This gives a prediction (3.11) which differs from the exact value by less than a factor of three in the simplest approximation.

That these two tiny energy scales obtained from completely different inputs are within an order of magnitude of one another suggests some relation obtainable by a serious quantum-mechanical calculation. We have shown here that the simplest model relating these two tiny mass scales gives a result that differs by only by a factor of less than three.

Many other possible mechanisms might produce oscillations. The experimenters [1] claim that they have investigated all of them. These other mechanisms generally involve energy scales very different from the scale producing a seven second period.

The observed oscillation is seen to arise from the relative phase between two components of the initial wave function with a tiny energy difference (3.12). These components travel
through the electromagnetic fields required to maintain a stable orbit. The effect in these fields on the relative phase depends on the energy difference between the two components. Since the energy difference is so tiny the effect on the phase is expected to be also tiny and calculable.

E. Effects of spatial dependence

The initial wave function travels through space as well as time. In a storage ring the ion moves through straight sections, bending sections and focusing fields. All must be included to obtain a reliable estimate for $\Delta(m^2)$. That this requires a detailed complicated calculation is seen in examining two extreme cases

1. Circular motion in constant magnetic field. The cyclotron frequency is independent of the momentum of the ion. Only the time dependent term contributes to the phase and $\delta \phi^{\text{cyc}}$ is given by eq. (3.10)

2. Straight line motion with velocity $v = (P/E) \cdot t$. The phase of the initial state at point $x$ in space and time $t$, its change with energy and momentum changes $\delta P$ and $\delta E$ are

$$\phi^{SL} = P \cdot x - E \cdot t; \quad \delta \phi^{SL} = (\delta P \cdot v - \delta E) \cdot t = \frac{P \delta P - E \delta E}{E} \cdot t = 0 \quad (3.13)$$

The large difference between the two results (3.10 and (3.13) indicate that a precise determination of the details of the motion of the mother ion in the storage ring is needed before precise predictions of the squared neutrino mass difference can be made.

IV. CONCLUSIONS

A new oscillation phenomenon providing information about neutrino mixing is obtained by following the initial radioactive ion before the decay. Difficulties introduced in conventional $\nu$ experiments by tiny neutrino absorption cross sections and very long oscillation wave lengths are avoided. Measuring the decay time enables every $\nu$ event to be observed
and counted without the necessity of observing the $\nu$ via the tiny absorption cross section. The confinement of the initial ion in a storage ring enables long wave lengths to be measured within the laboratory.

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