SPHERICAL FUNCTIONS ON RIEMANNIAN
SYMMETRIC SPACES

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Dedicated to Professor Gestur Ólafsson on his 65th birthday.

1. Introduction

Let $X = G/K$ be a symmetric space where $G$ is a connected noncompact semisimple Lie group with finite center and $K$ a maximal compact subgroup. Let $D(X)$ denote the algebra of $G$-invariant differential operators on $X$. Let $\delta$ be a unitary irreducible representation of $K$ on a vector space $V_\delta$.

Definition 1.1. A spherical function of $K$ type $\delta$ is a $C^\infty$ function $\Phi : X \to \text{Hom}(V_\delta, V_\delta)$ satisfying the following conditions.

1. $\Phi$ is an eigenfunction of each $D \in D(X)$.
2. $\Phi(k \cdot x) = \delta(k)\Phi(x)$ for $k \in K, x \in X$.

Remark 1.2. This definition has some similarity with those of Godement [G52] and Harish-Chandra [HC72]. However, the first one is modeled after invariance under $g \to k g k^{-1}$; the second deals with bi-invariant differential operators on $G$ and a double representation of $K \times K$. Our definition by conditions (1.1) and (1.2) is thus rather different.

It is also unrelated to a definition of a spherical function by Tirao [T76] as a function characterized by the function equation (3.3) for zonal spherical functions but with the measure $dk$ replaced by $\chi(k)\,dk$ where $\chi$ is the character of a representation of $K$.

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Our definition (1.1) stresses spherical functions as functions on \( X \) rather than as \( K \)-right invariant functions on \( G \). I remark that Harish-Chandra’s major papers [HC58] on zonal spherical functions do not actually mention the space \( X = G/K \) nor the algebra \( \mathcal{D}(X) \).

This paper deals with some simple results about the functions satisfying (1.1) and (1.2), namely new integral formulas, new results about behavior at infinity and some facts about the related \( C_\sigma \) functions.

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2. Notation and background

As usual \( \mathbb{R}, \mathbb{C}, \) and \( \mathbb{Z} \) denote the sets of real numbers, the complex numbers and integers, respectively. If \( c = a + ib, a, b \in \mathbb{R} \) we write \( a = \text{Re} \, c, b = \text{Im} \, c \) and \( \bar{c} = a - ib \). If \( L \) is a Lie group with Lie algebra \( \mathfrak{l} \), \( \exp : \mathfrak{l} \to L \) denotes the exponential mapping and \( \text{ad} \) (resp. \( \text{Ad} \)) the adjoint representation of \( \mathfrak{l} \) (resp. \( L \)).

Going back to §1, our group \( G \) has a Lie algebra \( \mathfrak{g} \) with Cartan decomposition \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) where \( \mathfrak{k} \) is the Lie algebra of \( K \) and \( \mathfrak{p} \) is the orthocomplement of \( \mathfrak{k} \) relative to the Killing form \( B \) of \( \mathfrak{g} \). We fix a maximal abelian subspace \( \mathfrak{a} \) of \( \mathfrak{p} \) and fix a Weyl chamber \( \mathfrak{a}^+ \subset \mathfrak{a} \). All such choices are conjugate under \( \text{Ad}(K) \). The choice of \( \mathfrak{a}^+ \) induces Iwasawa decompositions \( G = NAK \) and \( G = KAN \) where \( A = \exp \mathfrak{a} \) and \( N \) is nilpotent. In these decompositions we write \( g = n \exp A(g)k, g = k_1 \exp H(g)n_1 \) where \( A(g) \) and \( H(g) \) are uniquely determined in \( \mathfrak{a} \) and \( A(g) = -H(g^{-1}) \).

If \( M \) is the centralizer of \( A \) in \( \hat{K} \) the “vector valued” inner product \( A(gK, kM) = A(k^{-1}g) \) is well defined and considered analog to the Euclidean \( (x, w) \) the distance from 0 to the hyperplane through \( x \) with unit normal \( w \). We put \( B = K/M \). If \( \hat{K} \) denotes the set of irreducible representations of \( K \), condition (1.2) implies that \( \delta(M) \) has a common fixed point; we denote by \( \hat{K}_M \) the set of these \( \delta \).

Let \( V_\delta \) denote the space on which \( \delta \) operates and \( V_\delta^M \) the subspace of fixed points under \( \delta(M) \). We also use the notation \( \mathcal{E}(X) \) (resp. \( \mathcal{D}(X) \)) for the space of \( C^\infty \) functions on \( X \) (resp. those of compact support).

We denote by \( \pi \) the natural map of \( G \) on \( G/K \) and put \( \tilde{f} = f \circ \pi \) for a function \( f \) on \( G/K \). We also denote by \( \circ \) the coset \( eK \).
Let $a^*$ (resp. $a^*_C$) be the space of $\mathbb{R}$-linear maps of $a$ into $\mathbb{R}$ (resp. $\mathbb{C}$). In the bijection of $a$ with $a^*$ via the Killing form of $g$ let $a^*_+$ correspond to $a^+$. Let $S(a)$ denote the symmetric algebra over $a$ and $I(a)$ the subspace of $p \in S(a)$ invariant under the Weyl group $W$. We put $\tilde{N} = \theta N$ if $\theta$ is the Cartan involution. Let $M'$ be the normalizer of $A$ in $K$. If $\sigma \in W$ and $m_\sigma$ representing $\sigma$ in $M'$ we put $\bar{N}_\sigma = \tilde{N} \cap m_\sigma^{-1} N m_\sigma$.

This group appears later.

### 3. Zonal spherical functions

A zonal spherical function $\phi$ on $G$ is a $C^\infty$ function on $G$ satisfying

(3.1) \[ \phi \text{ is an eigenfunction of each } D \in D_K(G). \]

(3.2) \[ \phi \text{ is bi-invariant under } K, \phi(e) = 1. \]

Here $D_K(G)$ is the algebra of differential operators on $G$ which are left invariant under $G$ and right invariant under $K$.

Properties (3.1) and (3.2) are well known to be equivalent to

(3.3) \[ \int_K \phi(xky)dk = \phi(x)\phi(y). \]

The zonal spherical functions are all given by Harish-Chandra’s formula ([HC54])

(3.4) \[ \phi(g) = \int_K e^{(i\lambda-\rho)(H(gk))}dk \]

for some $\lambda \in a^*_C$. Here $\rho$ is half the sum of the positive restricted roots with multiplicity. Writing $\phi = \phi_\lambda$ this function has from [HC58] an expansion

(3.5) \[ \phi_\lambda(a) = \sum_{s \in W} c(s\lambda)e^{(is\lambda-\rho)(\log a)} \sum_{\mu \in \Lambda} \Gamma(s\lambda)e^{-\mu(\log a)} \]

for $a$ in Weyl chamber $\exp(a^+)$, $W$ the Weyl group, $\Lambda$ the lattice

$\Lambda = \{ m_1\alpha_1 + \cdots + m_\ell\alpha_\ell \mid m_i \in \mathbb{Z}^+ \}$,

the $\alpha_1, \ldots, \alpha_\ell$ being the simple restricted roots. This is where the remarkable $c$ function first appears.
The $\Gamma$ are rational functions on $\mathfrak{a}_c^*$ and $c$ is a meromorphic function on $\mathfrak{a}_c^*$ given by Harish-Chandra [HC58] as the integral

$$c(\lambda) = \int_{\bar{N}} e^{-(i\lambda+\rho)(H(\bar{n}))} d\bar{n}. \tag{3.6}$$

Through the work of Harish-Chandra [HC58], Bhanu-Murthy [BM60] and Gindikin-Karpelevic [GK62] the $c$ function is given by

$$c(\lambda) = \prod_{\alpha \in \Sigma_+^0} \frac{2^{-(i\lambda-\rho,\alpha_0)} \Gamma\left(\frac{1}{2}(m_\alpha + m_{2\alpha} + 1)\right)}{\Gamma\left(\frac{1}{2} m_\alpha + 1 + (i\lambda, \alpha_0)\right)} \Gamma\left(\frac{1}{2} m_\alpha + m_{2\alpha} + (i\lambda, \alpha_0)\right). \tag{3.7}$$

Here $\Sigma_+^0$ denotes the set of positive, indivisible roots, $m_\alpha$ the multiplicity of $\alpha$ and $\alpha_0 = \alpha/\langle \alpha, \alpha \rangle$.

In Harish-Chandra’s work, $|c(\lambda)|^{-2}$ served as the dual measure for the spherical transform on $G$. However, formula (3.7) has many other interesting features. See e.g. [H00].

4. The spaces $X = G/K$ and its Dual $\Xi = G/MN$

As proved in [H62], p. 439 and [H70], p. 94 the modified integrand in (3.4), that is the function

$$gK \rightarrow e^{(i\lambda+\rho)(A(k^{-1}g))}, \tag{4.1}$$

is for each $k$ an eigenfunction of $D(X)$ and the eigenvalue is $\Gamma(D)(i\lambda)$ where $D \in D(X)$ and $\Gamma(D) \in I(a)$. (This is related to Lemma 3 in [HC58], I) but not contained in it). The map $\Gamma$ is spelled out in [H84], II, Theorem 5.18.

This led in [H65] to the definition of a Fourier transform $f \rightarrow \hat{f}$ for a function $f$ on $X$,

$$\hat{f}(\lambda, b) = \int_X f(x) e^{-i\lambda+\rho)(A(x,b))} \, dx, \ b \in B, \lambda \in \mathfrak{a}_c^* \tag{4.2}$$
in analogy with the polar coordinate expression

$$\tilde{F}(\lambda w) = \int_X F(x) e^{-i\lambda(x,w)} \, dx, \ |w| = 1$$

for the Fourier transform on $\mathbb{R}^n$. Here $dx$ denotes the volume element in both cases.

In addition we consider the Poisson transform
(4.3) \((P_\lambda F)(x) = \int_B e^{(i\lambda + \rho)(A(x,b))} F(b) \, db\), \(F\) a function on \(B\).

These transforms are intimately related to the c-function. By \([H70]\), p. 120, the map \(f \rightarrow \tilde{f}\) is an isometry of \(L^2(X)\) onto \(L^2(a_c^* \times B; |c(\lambda)|^{-2} \, d\lambda \, db)\). Secondly, \(P_\lambda\) is related to the denominator \(\Gamma_X^+(\lambda)\) in (3.7), called the Gamma function of \(X\).

This \(P_\lambda\) is closely related to the dual Radon transform \(\phi \rightarrow \vee\phi\) from \(\Xi\), the space of horocycles in \(X\), to \(X\) which to a function \(\phi\) on \(\Xi\) associates \(\phi(x)\), the average of \(\phi\) over horocycles \(\xi \in \Xi\) passing through \(x \in X\) \(([H08], p. 103)\).

The element \(\lambda \in a_c^*\) is said to be simple if \(P_\lambda\) is injective. The connection with the denominator in (3.7) is \([H76]\):

**Theorem 4.1.** \(\lambda\) is non-simple if and only if \(\Gamma_X^+(\lambda)^{-1} = 0\).

On the other hand, the numerator in (3.7) is connected with analysis on the dual of \(X\), that is, the space \(\Xi = G/MN\) of horocycles in \(X\). The counterpart to the zonal spherical functions on \(X\) would be the \(MN\)-invariant eigendistributions of \(D(G/MN)\), the algebra of \(G\)-invariant differential operators on \(\Xi\). These "conical distributions" have a construction and theory in \([H70]\), using the numerator of (3.7).

While the set of zonal spherical functions is parametrized by \(a_c^* / W\) via (3.4), the set of conical distributions turned out to be essentially parametrized by \(a_c^* \times W\).

In the proof of (3.7) the following partial c-function \(c_\sigma\) enters for each \(\sigma \in W\). In analogy with (3.0), it is defined by

\[ c_\sigma(\lambda) = \int_{N_\sigma} e^{-((\lambda + \rho)(H(\bar{n}))} \, d\bar{n}, \quad N_\sigma = \bar{N} \cap N^{\sigma^{-1}}, \]

for a suitable normalization of \(d\bar{n}\) on \(\bar{N}_\sigma\). It has a formula generalizing (3.7),

\[ c_\sigma(\lambda) = \prod_{\alpha \in \Sigma^+_0 \cap \sigma^{-1} \Sigma^-_0} c_\alpha(\lambda_\alpha), \quad \lambda_\alpha = \lambda |a_\alpha, \]

where \(c_\alpha\) is the c-function for the group \(G_\alpha \subset G\) whose Lie algebra is the subalgebra of \(\mathfrak{g}\) generated by \(\mathfrak{g}_\alpha\) and \(\mathfrak{g}_{-\alpha}\).

The numerous beautiful features of (3.7) are the cause of the title in \([H00]\).
Let $V$ be a finite dimensional vector space. A function $\Phi : X \to V$ satisfying (1.1) is called a joint eigenfunction of $D(X)$. The eigenvalue $\lambda(D)$ in $D\Phi = \lambda(D)\Phi$ is a homomorphism of $I(a)$ into $C$ and thus has the form $\lambda(D) = \Gamma(D)(i\lambda)$ for some $\lambda \in a^\ast$. Corresponding eigenspaces are joint eigenspaces. The scalar version of Proposition 5.1 is from [H62], X, §7. We shall use [H84], mainly Ch. II, §4 and Ch.IV, §2.

**Proposition 5.1.** The joint eigenfunctions $\Phi : X \to V$ of $D(X)$ are the continuous functions $\Phi$ satisfying

\[(5.1) \int_K \Phi(xky \cdot o)dk = \Phi(x \cdot o)\phi(y \cdot o), \quad x, y \in G\]

for some $\lambda \in a^\ast_c$.

**Proof.** First assume (5.1). Integrating (5.1) against a test function $f$ in $y$, $\Phi(x)$ can be written in terms of derivative of $f$ in $x$ so $\Phi$ is smooth. Then applying $D_y$ to (5.1) and putting $y = e$ we find

\[(5.2) D\Phi = \Gamma(D)(i\lambda)\Phi\]

so $\Phi$ is an eigenfunction. For the converse assume (5.2) and put

\[\Psi_x(y) = \int_K \Phi(xky)\ dk\]

and note that

\[(5.3) D\Psi_x = \Gamma(D)(i\lambda)\Psi_x.\]

Let $D(G)$ denote the algebra of left invariant differential operators on $G$ and as before $D_K(G)$ the subalgebra of those which are also right $K$-invariant. For $D \in D(G)$ let

\[(5.4) D_0 = \int_K \text{Ad}(k)D\ dk\]

and recall that $D \to D_0$ maps $D(G)$ onto $D_K(G)$. Also consider the map $\mu$ given by

\[(5.5) (\mu(u)f)\tilde{\ } = u\tilde{f} \quad u \in D_K(G), \quad \tilde{f} = f \circ \pi\]

which maps $D_K(G)$ onto $D(G/K)$ ([H84], Ch. II, §4).

Among $D$ in (5.2) is the Laplace-Beltrami operator so $\Phi$ is an analytic function. If $F \in C^\infty(G)$ is bi-invariant under $K$ we have by [H84] (3), p. 400,

\[(5.6) (D_0F)(e) = (DF)(e).\]
This applies both to $F(y) = \Psi_x(y \cdot \circ)$ and $F(y) = \phi_{\lambda}(y \cdot \circ)$ and to
\[ f(y \cdot \circ) = \phi_{\lambda}(e)\Psi_x(y \cdot \circ) - \Psi_x(e)\phi_{\lambda}(y \cdot \circ). \]

For $D \in \mathcal{D}(G)$ arbitrary we take $u = D_0$ in (5.5). Then using (5.6), $(D\tilde{f})(e) = 0$. Since $f$ is analytic and $f(\circ) = 0$ we have $f \equiv 0$ which is formula (6.1). □

With $V$ as before we consider joint eigenfunctions for the algebra $\mathcal{D}(\Xi) = \mathcal{D}(G/MN)$. If $\mathcal{D}(A)$ denotes the left invariant differential operators on $A$ each $U \in \mathcal{D}(A)$ induces an $D_U \in \mathcal{D}(\Xi)$ by
\[ (D_U \phi)(kaMN) = U_a(\phi(kaMN)) \]
and $U \to D_U$ is an isomorphism of $\mathcal{D}(A)$ onto $\mathcal{D}(\Xi)$ ([H70], I, §2). We denote by $\hat{\Gamma}$ its inverse. Let $E_{\lambda}(\Xi)$ denote the joint eigenspace
\[ E_{\lambda}(\Xi) = \left\{ \psi \in \mathcal{E}(\Xi) : D\psi = \hat{\Gamma}(D)(i\lambda - \rho)\psi \right\}. \]

**Proposition 5.2.** The joint eigenfunctions $\Psi$ of $\Xi$ into $V$ are the smooth functions satisfying
\[ (5.7) \quad \Psi(gaMN) = \Psi(gMN)e^{(i\lambda - \rho)(\log a)}, \quad a \in A, \]
for some $\lambda \in a^*_c$.

For proof see [H08], II, §2, also for distributions on $\Xi$.

6. **Spherical functions of a given $K$-type**

Let $\Phi$ be a spherical function of type $\delta$ as in §1. Thus we take $V = \text{Hom}(V_\delta, V_\delta)$ in §5 and assume (1.2). As mentioned in §5 there exists a $\lambda \in a^*_c$ such that
\[ (6.1) \quad D\Phi = \Gamma(D)(i\lambda)\Phi, \quad D \in \mathcal{D}(X). \]

This $\lambda$ is unique up to conjugacy by $W$ and since each $\lambda \in a^*_c$ is $W$ conjugate to one which is simple we can take $\lambda \in a^*_c$ to be simple. By definition, $P_\lambda$ is injective.

Let $v_1, \ldots, v_{d(\delta)}$ be an orthonormal basis of $V_\delta$ such that $v_1, \ldots, v_{\ell(\delta)}$ span $V_\delta^M$. Then
\[ (6.2) \quad \Phi(x)v_j = \sum_{i=1}^{d(\delta)} \phi_{ij}(x)v_i \]
Then condition (1.2) implies

$$\phi_{rj}(k \cdot x) = \sum_i \delta_{ri}(k) \phi_{ij}(x),$$

where $\delta_{ri}(k)$ is the expression of $\delta(k)$ in the basis $(v_i)$. Let $\mathcal{E}_\lambda(X)$ denote the space of joint eigenfunctions of $D(X)$ with eigenvalues $\Gamma(D)(i\lambda)$. Let $\bar{\delta}$ denote the contragredient to $\delta$, $d(\bar{\delta}) = d(\delta)$ its dimension and $\chi_{\bar{\delta}} = \bar{\chi}_\delta$ its character. Let $\pi$ denote the representation of $K$ on $\mathcal{E}_\lambda(X)$ given by $\pi(k) : f(x) \mapsto f(k^{-1} \cdot x)$. By [H84], IV, §1, the map

$$d(\bar{\delta}) \pi(\bar{\chi}_{\bar{\delta}})$$

is the projection of $\mathcal{E}_\lambda(X)$ onto the space $\mathcal{E}_{\lambda,\bar{\delta}}(X)$ of $K$-finite elements in $\mathcal{E}_\lambda(X)$ of type $\bar{\delta}$.

**Lemma 6.1.** Each function $\phi_{ij}$ in (7.2) belongs to $\mathcal{E}_{\lambda,\bar{\delta}}(X)$.

**Proof.** We have

$$d(\bar{\delta}) \pi(\bar{\chi}_{\bar{\delta}})(\phi_{rj})(x) = d(\bar{\delta}) \int_K \chi_{\bar{\delta}}(k) \phi_{rj}(k^{-1} \cdot x) dk = d(\delta) \int_K \chi_{\bar{\delta}}(k) \sum_i \delta_{ri}(k) \phi_{ij}(x) dk$$

which by Schur’s orthogonality relations reduces to $\phi_{rj}(x)$. Thus $\phi_{ij} \in \mathcal{E}_{\lambda,\bar{\delta}}(X)$ for all $i, j$. □

By [H70], IV, §1 and [H76] §7, invoking the Paley-Wiener theorem for (1.2), each $K$-finite joint eigenfunction of $D(X)$ is the Poisson transform of a $K$-finite member of $\mathcal{E}(B)$. We apply this to $\phi_{ij}(x)$. As is well known the functions

$$\langle \delta(k) v_j, v_i \rangle \quad 1 \leq i \leq d(\delta), \quad 1 \leq j \leq \ell(\delta)$$

form a basis of $\mathcal{E}_{\bar{\delta}}(B)$, the space of $K$-finite functions in $\mathcal{E}(B)$ of type $\bar{\delta}$. The corresponding images under $\mathcal{P}_\lambda$

$$\int_{K/M} e^{(i\lambda + \rho)(A(x,kM))} \langle \delta(k) v_j, v_i \rangle \ dk_M$$

(6.3)
will by [H73, H76] span the space $E_{\lambda, \delta}(X)$. Consider the Eisenstein integral

\[ \Phi_{\lambda, \delta}(x) = \int_{K} e^{(i\lambda + \rho)(A(x, kM))} \delta(k) \, dk, \]

whose matrix entries are given by (6.3).

Changing from $\lambda$ to $s\lambda$ ($s \in W$) only changes $\Phi_{\lambda, \delta}$ by a factor independent of $x$ [H73]. This proves following result.

**Theorem 6.2.** Each spherical function of type $\delta$ has each of its matrix entries linear combinations of the functions

\[ \langle \Phi_{\lambda, \delta}(x) v_j, v_i \rangle \quad 1 \leq j \leq \ell(\delta), 1 \leq i \leq d(\delta). \]

These functions all turn out to be suitable derivative of the zonal spherical functions [H76].

**Theorem 6.3.** Fix $v, w \in V_\delta$ and assume $-\lambda$ simple. Then there exists a right invariant differential operator $D$ on $G$ such that

\[ \langle v, \Phi_{\lambda, \delta}(gK), w \rangle = (D\phi_{\lambda})(g). \]

This in return implies a series expansion of $\Phi_{\lambda, \delta}$ on $exp a^+$, generalizing Harish-Chandra’s expansion (3.5) which introduced the c function.

**Theorem 6.4.** There exist meromorphic functions $C_\sigma(\sigma \in W)$ and rational functions $\Gamma_\mu(\mu \in \Lambda)$ all with values in $\text{Hom}(V_\delta^M, V_\delta^M)$ such that for $H \in a^+, v \in V_\delta^M$,

\[ \Phi_{\lambda, \delta}(\exp H \cdot \circ) v = \sum_{\sigma \in W, \mu \in \Lambda} e^{(i\sigma\lambda - \rho - \mu)(H)} \Gamma_\mu(\sigma\lambda) C_\sigma(\lambda) v \]

The $\Gamma_\mu$ are given by explicit recursion formulas. Note that in contrast to (3.5) the order of the factors in each term of the series is important. Also by [H70, H73]

\[ \Phi_{\sigma, \lambda, \delta}(x) = \Phi_{\lambda, \delta}(x) \Gamma_{\sigma, \lambda}, \]

where $\Gamma_{\sigma, \lambda}$ is meromorphic on $a^*_c$ with values in $\text{Hom}(V_\delta, V_\delta)$ and

\[ \Gamma_{\sigma, \lambda} v = \frac{C_{\sigma^{-1}}(\sigma\lambda)}{c(\lambda)} v \quad \text{for } v \in V_\delta^M. \]

We could also consider the analog of this for the dual space $\Xi = G/MN$. For this consider a representation $\sigma$ of $MN$ on a finite dimension space $V_\sigma$ and then take $V = \text{Hom}(V_\sigma, V_\sigma)$ in §5 and replace (1.2) by

\[ \Psi(h \cdot \xi) = \sigma(h) \Psi(\xi) \quad h \in MN, \xi \in \Xi, \]
extending \( \Psi \) to distributions.

This leads to Whittaker functions and Whittaker distributions, studied for example in Goodman-Wallach [80] where relations with conical distributions is also established.

7. The asymptotics of \( \Phi_{\lambda, \delta} \)

In this section we limit ourselves to the case \( \text{rank } X = 1 \). Then by Kostant [K69] \( \ell(\delta) = \dim V^M_\delta = 1 \). We fix \( v \in V^M_\delta \) of norm 1 and take \( H \in a^+ \) such that \( \alpha(H) = 1 \). We put \( a_t = \exp tH \) and

\[
\phi_{\lambda, \delta}(x) = \langle \Phi_{\lambda, \delta}(x)v, v \rangle.
\]

Then by Theorem 6.4

\[
\phi_{\lambda, \delta}(a_t \circ \cdot) = e^{(i\lambda - \rho)(tH)} \sum_{n=0}^{\infty} e^{-nt}\Gamma_n(\lambda)\langle C_e(\lambda)v, v \rangle
\]

\[
+ e^{-(i\lambda + \rho)(tH)} \sum_{n=0}^{\infty} e^{-nt}\Gamma_n(-\lambda)\langle C_{\sigma}(\lambda)v, v \rangle
\]

Multiply by \( e^{-(i\lambda - \rho)(tH)} \), \( \lambda = \xi + i\eta \).

Then if \( \eta < 0 \) we have by [H08], II, Theorem 3.16,

\[
\lim_{t \to \infty} e^{-(i\lambda - \rho)(tH)}\phi_{\lambda, \delta}(a_t \circ \cdot) = \mathbf{c}(\lambda)\langle v, v \rangle.
\]

The left hand side is

\[
\sum_{0}^{\infty} e^{-nt}\Gamma_n(\lambda)\langle C_e(\lambda)v, v \rangle + e^{-2i\lambda(tH)} \sum_{0}^{\infty} e^{-nt}\Gamma_n(-\lambda)\langle C_{\sigma}(\lambda)v, v \rangle
\]

Since \( \Gamma_n \) grows at most exponentially \( < \frac{1}{2}n \) (see [H08], III, \S 5) and \(-2i\lambda = -2i\xi + 2\eta\) the limit for \( t \to \infty \) equals \( C_e(\lambda) \). Since both sides are meromorphic in \( \lambda \) we conclude

\[
(7.2) \quad C_e(\lambda) = \mathbf{c}(\lambda).
\]

We can also multiply the expansion by \( e^{(i\lambda + \rho)(tH)} \) and for suitable \( \eta \) deduce

\[
(7.3) \quad \lim_{t \to \infty} e^{(i\lambda + \rho)(tH)}\phi_{\lambda, \delta}(a_t \circ \cdot) = C_{\sigma}(\lambda).
\]

This in itself does not give much information.
On the other hand, \( \phi_{\lambda, \delta} \) is an eigenfunction of the Laplacian and by [H76] or [H08], p. 328, given by

\[
(7.4)
\phi_{\lambda, \delta}(a_t \cdot \circ) = c_{\lambda, \delta} \tanh^s t \cosh^l t \times \mathcal{F} \left( \frac{1}{2}(s + r - l), \frac{1}{2}(s - r - l + 1 - m_{2\alpha}), s + \frac{1}{2}(m_{\alpha} + m_{2\alpha} + 1), \tanh^2 t \right)
\]

where \( l = (i\lambda - \rho)(H) \), \( m_{\alpha} \) and \( m_{2\alpha} \) are the multiplicities of \( \alpha \) and \( 2\alpha, r \) and \( s \) are integers, \( r \leq s \) given by

\[
r(r + m_{2\alpha} - 1) = -\frac{1}{4}d_{2\alpha}
\]

\[
s(s + m_{\alpha} + m_{2\alpha} - 1) = -d_{\alpha} - \frac{1}{4}d_{2\alpha}
\]

and \( d_{\alpha} \) and \( d_{2\alpha} \) determined by

\[
\delta(\omega_{\alpha})|V_{\delta}^M = d_{\alpha}(2(m_{\alpha} + 4m_{2\alpha}))^{-1}
\]

\[
\delta(\omega_{2\alpha})|V_{\delta}^M = d_{2\alpha}(2(m_{\alpha} + 4m_{2\alpha}))^{-1}.
\]

The operators \( w_{\alpha} \) and \( w_{2\alpha} \) are defined in [H76], §4.

Also by [H76]

\[
(7.5)
c_{\lambda, \delta} = \frac{\Gamma \left( \frac{1}{2}(i\lambda + \rho, \alpha_0) + 1, \rho, \alpha_0 \right) \Gamma \left( \frac{1}{2}(i\lambda + \rho, \alpha_0) + 1 - m_{2\alpha}, \rho, \alpha_0 \right)}{\Gamma \left( \frac{1}{2}(i\lambda + \rho, \alpha_0) \right) \Gamma \left( \frac{1}{2}(i\lambda + \rho, \alpha_0) + 1 - m_{2\alpha} \right)}
\]

Consider the factor \( (\cosh t)^{l} \), \( l = i\xi(H) - (\eta + \rho)(H) \) and choose \( \eta > 0 \).

Multiply \( \phi_{\lambda, \delta}(a_t \cdot \circ) \) by \( (2 \cosh t)^{-l} \) and let \( t \to +\infty \). Then

\[
\lim_{t \to \infty} (2 \cosh t)^{-l} \phi_{\lambda, \delta}(a_t \cdot \circ) = c_{\lambda, \delta} \frac{1}{2} \frac{\Gamma(c - a - b) \Gamma(c)}{\Gamma(c - a) \Gamma(c - b)}.
\]

by the limit formula \( \mathcal{F}(a, b, c, \infty) \), where

\[
a = \frac{1}{2}(s + r - l), \quad b = \frac{1}{2}(s - r - l + 1 - m_{2\alpha}), \quad c = s + \frac{1}{2}(m_{\alpha} + m_{2\alpha} + 1).
\]

Then

\[
c - a - b = i\lambda(H)
\]

\[
c - a = \frac{1}{2}s - \frac{1}{2}r + \frac{1}{2}i\lambda(H) + \frac{1}{2}m_{\alpha} + \frac{1}{2}
\]

\[
c - b = \frac{1}{2}s + \frac{1}{2}r + \frac{1}{2}i\lambda(H) + \frac{1}{2}m_{\alpha} + \frac{1}{2}m_{2\alpha}
\]

In \( (7.5) \) recall that \( \alpha_0 = \alpha/\langle \alpha, \alpha \rangle \)

\[
\langle \rho, \alpha_0 \rangle = \langle \frac{1}{2}m_{\alpha} \alpha + m_{2\alpha} \alpha, \alpha_0 \rangle = \frac{1}{2}m_{\alpha} + m_{2\alpha}
\]

\[
\langle i\lambda + \rho, \alpha_0 \rangle = i\lambda(H) + \frac{1}{2}m_{\alpha} + m_{2\alpha}
\]
Thus the right hand side of (7.6) equals
\[
\Gamma \left( \frac{1}{2} (i\lambda(H) + \frac{1}{2}m_\alpha + m_{2\alpha} + s + r) \right) \Gamma \left( \frac{1}{2} (i\lambda(H) + \frac{1}{2}m_\alpha + 1 + s - r) \right) \\
\times \Gamma \left( \frac{1}{2} \left( i\lambda(H) \right) \right) \Gamma \left( s + \frac{1}{2} (m_\alpha + m_{2\alpha} + 1) \right)
\]
\[
\Gamma \left( \frac{1}{2} \left( i\lambda(H) \right) \right) \Gamma \left( s + \frac{1}{2} (m_\alpha + m_{2\alpha} + 1) \right)
\]

Here top of the first fraction cancels against the bottom of second fraction. Thus (7.6) becomes (considering (3.7)),
\[
\frac{1}{2l} \Gamma \left( i\lambda(H) \right) \Gamma \left( s + \frac{n}{2} \right) c(\lambda) \Gamma \left( \frac{n}{2} \right) c(-\lambda) c(\lambda)
\]
if \( n = \dim X \).

**Theorem 7.1.** For \( \lambda = \xi + i\eta \), \( \eta > 0 \),
\[
\lim_{t \to \infty} (2 \cosh t)^{-t} \phi_{\lambda,\delta}(a_t \cdot \circ) = \frac{\Gamma \left( s + \frac{n}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} c(\lambda).
\]

So far we have studied the behavior of \( \phi_{\lambda,\delta} \) for large \( t \). For behavior for small \( t \) we just use (7.4) and conclude
\[
\lim_{t \to 0} \frac{\phi_{\lambda,\delta}(a_t \cdot \circ)}{\phi_{-\lambda,\delta}(a_t \cdot \circ)} = \frac{c_{\lambda,\delta}}{c_{-\lambda,\delta}}.
\]

On the other hand as mentioned in §6
\[
\Phi_{s\lambda,\delta}(x) = \Phi_{\lambda,\delta}(x) \Gamma_{s,\lambda},
\]
where
\[
\Gamma_{s,\lambda} v = \frac{C_{s-1}(s\lambda)}{c(\lambda)} v, \quad v \in V_\delta^M.
\]

In the rank one case where \( C_\sigma \) is a scalar this implies
\[
(7.8) \quad C_{\sigma}(-\lambda) = \frac{c_{-\lambda,\delta}}{c_{\lambda,\delta}} c(\lambda).
\]

**Theorem 7.2.** Formulas (7.2) and (7.8) determine the \( C \) functions in the rank one case.

**Remark 7.3.** By [H73], Lemma 6.1 and Lemma 6.5 we have
\[
(7.9) \quad C_{\sigma}(-\lambda) = \int_N e^{-(i\lambda + \rho)(H(\bar{n}))} \delta(k(\bar{n})^{-1}m^*) d\bar{n}
\]
where $m^* \in W$ is $-I$ on $a$. Thus (7.8) gives an evaluation of the unwieldy integral (7.9). In the papers [J76] and [JW72], [JW77] Johnson and Wallach determined (7.9) by using a classification of $\hat{K}_M$ related to a parametrization from Kostant [K69]. In each case they have a formula in the spirit of (7.8). Their models for the $\delta \in \hat{K}_M$ are spaces of homogeneous harmonic polynomials on $p$, restricted by different conditions, according to the multiplicities $m_\alpha$ and $m_{2\alpha}$. This leads to a determinations of the integers $s, r$ for each $\delta$ in (7.4) ([H74], p. 336–337). See also [C74] for the simplest cases.

8. The case of higher rank

Using a method of Schiffman [S71] we shall now investigate the $C$-functions for $X$ of higher rank.

As in [H73] we consider the endomorphism $A(\lambda, \sigma)$ of $V_\delta^M$ given by

$$A(\lambda, \sigma)v = \int_{\tilde{N}} e^{-(\lambda + \rho)(H(\tilde{n}))} \delta(m_\sigma k(\tilde{n}))v d\tilde{n},$$

where $m_\sigma$ is a representative of $\sigma \in W$ in the normalizer $M'$ of $A$ in $K$. Under conjugation $g \rightarrow mgm^{-1}$ by an $m \in M$ we have $(k(\tilde{n}))^m = k(\tilde{n}^m)$ so $A(\lambda, \sigma)$ in (8.1) is independent of the choice of $m_\sigma \in M'$ representing $\sigma \in W$. Let $\sigma = \sigma_1 \ldots \sigma_p$ be a reduced expression of $\sigma$, that is each $\sigma_i$ is a reflection in the plane $\alpha_i = 0$ where $\alpha_i$ is a simple root and $p$ is as small as possible.

Then by [H73], following a method by [S71],

$$A(\lambda, \sigma) = A(\sigma^{(1)}\lambda, m_{\sigma_1}) \cdots A(\sigma^{(p)}\lambda, m_{\sigma_p}),$$

where $\sigma^{(q)} = \sigma_{q+1} \cdots \sigma_p$.

For each simple root $\alpha$ consider the rank-one symmetric space $G_\alpha/K_\alpha$ where $G_\alpha$ is the analytic subgroup of $G$ whose Lie algebra is generated by the root spaces $g_\alpha$ and $g_{-\alpha}$ and $K_\alpha = G_\alpha \cap K$. If $a_\alpha = R H_\alpha$ and $A_\alpha = \exp a_\alpha$, $\tilde{N}_\alpha = \tilde{N}_{s_\alpha}$ then $G_\alpha = K_\alpha A_\alpha \tilde{N}_\alpha$ is an Iwasawa decomposition of $G_\alpha$. If $c_\alpha$ is the $c$ function for $G_\alpha/K_\alpha$ we have by (8.2) (for $\delta$ trivial)

$$c_\sigma(\lambda) = \prod_{1}^{p} c_{\sigma_j}(\lambda_j)$$

where $\lambda_j = (\sigma^{(j)}\lambda)|a_{\alpha_j}$. 

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Lemma 8.1. With $\delta \in \hat{K}_M$ arbitrary let $V$ denote the $K_\alpha M$ invariant subspace of $V_\delta$ generated by $V_\delta^M$ and $V = \bigoplus_{i=1}^l V_i$ a decomposition into $K_\alpha M$-irreducible subspaces. Then $l = l(\delta)$, the dimension of $V_\delta^M$ and $\dim(V_\delta^M \cap V_i) = 1$ for each $i$.

Proof. See [H73], p. 469.

In this lemma take $\alpha = \sigma_j$ and let $\delta_i$ denote the representation of $K_{\sigma_j}M$ on $V_i$ given by $\delta$. We choose a unit vector $v_i$ in $V_\delta^M \cap V_i$. Since $K_{\sigma_j}$ maps $V_\delta^M \cap V_i$ into itself the operator $A(\sigma_j^i, m_{\sigma_j})$ does too and operates by multiplication with the scalar

$$
\int_{\mathcal{N}_{\sigma_j}} \langle \delta_i(m_{\sigma_j} k(\bar{n})v_i, v_i) \rangle e^{-(i\lambda_j + \rho_{\sigma_j})(H(\bar{n}))} \, d\bar{n},
$$

where $\lambda_j = (\sigma_j^i) a_{\sigma_j}$. Here we have used the fact that the restriction $\rho|a_{\sigma_j}$ equals the $\rho$-function for $G_{\sigma_j}/K_{\sigma_j}$ [H84] (34) page 446. By [H73], Lemma 6.5 this number is equal to

$$
\int_{\mathcal{N}_{\sigma_j}} e^{-(i\lambda_j + \rho_{\sigma_j})(H(\bar{n}))} \langle \delta_i(k(\bar{n})^{-1}m_{\sigma_j})v_i, v_i \rangle \, d\bar{n},
$$

which we have calculated in (7.8), (7.9). Thus the value of (8.5) equals

$$
c_{\lambda_j, \delta_i} c_{\sigma_j}(\lambda_j).
$$

The root multiplicities are now the ones in $G_{\sigma_j}$ and the integers $r$ and $s$ are the ones which belong to $\delta_i$. Also

$$
V_\delta^M = \bigoplus_i C V_i.
$$

For a fixed $j$ this represents diagonalization of $A(\sigma_j, m_{\sigma_j})$, $1 \leq i \leq l(\delta)$. The determinant of this endomorphism of $V_\delta^M$ is then

$$
\prod_{i=1}^{l(\delta)} \left( \frac{c_{-\lambda_j, \delta_i} c_{\sigma_j}(\lambda_j)}{c_{\lambda_j, \delta_i}} \right).
$$

Changing to another $j$ will change the basis $(v_i)$ giving new representations $\delta(\bar{i}, j)$. Consequently, by (8.3)

$$
\det(A(\lambda, \sigma))|_{V_\delta^M} = c_{\sigma}(\lambda)^{l(\delta)} \prod_{j=1}^p \left( \prod_{i=1}^{l(\delta)} \left( \frac{c_{-\lambda_j, \delta(\bar{i}, j)}}{c_{\lambda_j, \delta(\bar{i}, j)}} \right) \right).
$$
On the other hand, the adjoint of \( A(\bar{\lambda}, \sigma) \) is given by

\[
(A(\bar{\lambda}, \sigma))^* v = \int_{N_\sigma} e^{-(\bar{\lambda} + \rho)(H(\bar{n}))} \delta(k(\bar{n})^{-1}m_{\sigma}^{-1}) \, d\bar{n} \, v,
\]

which by \([H73]\) equals

\[
\frac{1}{c(-\lambda)} c_{\sigma}(-\lambda) C_{\sigma^{-1}}(-\sigma \lambda) v.
\]

This by \((8.9)\) determines the determinant of \( C_{\sigma}(\lambda) \).

Formula \((8.9)\) has some resemblance to Theorem 5 in \([C74]\) but it is not clear whether there is a connection.

We do not deal with the problem of determining \( C_{\sigma} \) itself but observe that the Hilbert-Schmidt norm is given by \([H73]\):

\[
||C_{\sigma}(\lambda)||^2 = |c(\lambda)|^2 l(\delta).
\]

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