Faster Approximations for Metric-TSP via Linear Programming*

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Abstract

We develop faster approximation algorithms for Metric-TSP building on recent, nearly linear time approximation schemes for the LP relaxation [Chekuri and Quanrud, 2017a]. We show that the LP solution can be sparsified via cut-sparsification techniques such as those of Benczur and Karger [2015]. Given a weighted graph $G$ with $m$ edges and $n$ vertices, and $\epsilon > 0$, our randomized algorithm outputs with high probability a $(1 + \epsilon)$-approximate solution to the LP relaxation whose support has $O(n \log n / \epsilon^2)$ edges. The running time of the algorithm is $\tilde{O}(m / \epsilon^2)$. This can be generically used to speed up algorithms that rely on the LP.

For Metric-TSP, we obtain the following concrete result. For a weighted graph $G$ with $m$ edges and $n$ vertices, and $\epsilon > 0$, we describe an algorithm that outputs with high probability a tour of $G$ with cost at most $(1 + \epsilon)^3 / 2$ times the minimum cost tour of $G$ in time $\tilde{O}\left(\frac{m}{\epsilon^3} + \frac{n^{1.5}}{\epsilon^3}\right)$. Previous implementations of Christofides’ algorithm [Christofides, 1976] require, for a $3 / 2$-optimal tour, $\tilde{O}(n^{2.5})$ time when the metric is explicitly given, or $\tilde{O}(\min\{m^{1.5}, mn + n^{2.5}\})$ time when the metric is given implicitly as the shortest path metric of a weighted graph.

1 Introduction

The traveling salesman problem (abbr. TSP) and its variants including Metric-TSP are extensively studied in discrete and combinatorial optimization. In this short paper we focus on approximation algorithms for Metric-TSP which is NP-Hard. An instance of Metric-TSP is specified by a complete undirected graph $G = (V, E, c)$ with positive edges costs $c : E \to \mathbb{R}_{>0}$ satisfying the triangle inequality (and form a metric over $V$). The goal is to find a Hamiltonian cycle in $G$ of minimum cost. Note that if $G$ is not required to be a complete graph, then the minimum cost Hamiltonian cycle is inapproximable, as deciding if an undirected graph has a Hamiltonian cycle is NP-Complete.

Definition 1.1. An instance of Metric-TSP is explicit if the input is a metric on $n$ nodes with all $\binom{n}{2}$ distances specified. An instance of Metric-TSP is implicit if it is specified implicitly as the metric completion of an underlying graph $G = (V, E, c)$.

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**Remark 1.2.** A sparse graph with \( m \) edges and \( n \) vertices generates a metric TSP problem that is inherently of size \( O(n^2) \). Given an implicit instance of Metric-TSP, it is desirable to obtain running times relative to the number of edges \( m \) in the underlying graph, which represents the true input size of the problem.

Many instances of Metric-TSP are implicitly defined, either as a sparse graph, or in other settings where there is an easy function that returns \( c(u, v) \) given nodes \( u, v \) (geometric instances in low dimensions are a good example). In this paper we focus on the setting where \( G \) is specified as a sparse weighted graph. In such a setting, finding a solution corresponds to finding a tour of the vertices of minimum total cost. This is equivalent to finding a minimum cost Eulerian multigraph in the support of the given graph \( G = (V, E, c) \).

Christofides [1976] described a \( \frac{3}{2} \)-approximation for Metric-TSP and this is still the best known. There is a well-known conjecture that a \( \frac{4}{3} \)-approximation is achievable via a solution to an LP relaxation called the subtour elimination LP due to Dantzig, Fulkerson, and Johnson [1954]; see Section 2.2 for a description of this LP referred to as (SE) and also a related LP (2ECSS) whose optimum values are equal for instances of Metric-TSP. There has been recent exciting progress on various special cases for which we refer the reader to a survey by Vygen [2012].

The goal of this paper is to find fast approximation algorithms for Metric-TSP. One can easily obtain a 2-approximation in nearly linear time by traversing the minimum spanning tree (abbr. MST). Christofides’s algorithm requires the computation of a minimum-cost perfect matching (see Section 2.1 for more details). The fastest known implementation requires \( \tilde{O}(n^{2.5}) \) time when the metric is explicitly given and \( \tilde{O}(\min(m^{1.5}, mn + n^{2.5})) \) time when it is given implicitly as a weighted graph [Gabow and Tarjan, 1991]. In a recent paper we obtained the following result to compute a nearly linear time approximation scheme for the LP relaxation (2ECSS).

**Theorem 1.3** (Chekuri and Quanrud, 2017a). There is a randomized algorithm that in \( \tilde{O}(m/\varepsilon^2) \) time, with high probability, computes a feasible point \( x \in \mathbb{R}^E \) for (2ECSS) with objective value \( \sum_{e \in E} c_e x_e \leq (1 + \varepsilon) \text{OPT}(2ECSS) \).

Surprisingly the running time to solve the LP is significantly faster than the time to implement Christofides’s heuristic. Chekuri and Quanrud [2017a] raised the question of faster algorithms that yield a \( \frac{3}{2} \)-approximation. In this paper we make progress towards the question. Our first result shows that the LP solution can be sparsified as follows.

**Theorem 1.4.** There is a randomized algorithm that given a feasible solution \( x \in \mathbb{R}^E \) for (2ECSS) and \( \varepsilon > 0 \), outputs, with high probability, another feasible solution \( x' \) such that (i) support of \( x' \) is \( O(n \log n/\varepsilon^2) \) and (ii) the objective value of \( x' \) is close to that of \( x \), that is, \( \sum_{e \in E} c_e x'_e \leq (1 + \varepsilon) \sum_{e \in E} c_e x_e \).

Combining the two preceding theorems yields a randomized algorithm that, in nearly linear time, sparsifies a given graph \( G \) on \( n \) nodes and \( m \) edges to a subgraph \( H \) with \( O(n \log n/\varepsilon^2) \) edges such that the LP value on \( H \) is within a \( (1 + \varepsilon) \)-factor of the LP value on \( G \). This sparsification
can generically help any approximation algorithm that relies on the LP solution either directly or indirectly. We use cut sparsification techniques for the above. The only novelty is that, unlike most applications that we are aware of, we also need to preserve the cost of the sparsifier; we observe that importance sampling which underlies a class of sparsifiers [Benczur and Karger, 2015, Fung, Hariharan, Harvey, and Panigrahi, 2011] is suitable for this purpose. Stronger sparsification results such as the one of Batson, Spielman, and Srivastava [2012] do not appear to preserve the cost.

We utilize the two preceding theorems and the analysis by Wolsey [1980] of Christofides’ heuristic with respect to the LP, to obtain a fast \( \left( \frac{3}{2} + \epsilon \right) \)-approximation.

**Theorem 1.5.** There is a randomized algorithm that in \( \tilde{O}\left( \frac{m}{\epsilon^2} + \frac{n^{1.5}}{\epsilon^3} \right) \) time, with probability at least \( 1 - \frac{1}{\text{poly}(m)} \), returns a tour of cost at most \((1 + \epsilon)\frac{3}{2}\)OPT(SE).

2 Preliminaries

2.1 Christofides’ \( \frac{3}{2} \)-approximation algorithm

In this section, we review the approximation algorithm of Christofides [1976] for Metric-TSP.

**Definition 2.1.** Given an even set of vertices \( T \subseteq V \), a \( T \)-join is a subgraph \( H \) of \( G \) for which \( T \) is the set of vertices with odd degree. The cost of a \( T \)-join is the sum cost of all the edges in \( H \), \( \tilde{c}(H) = \sum_{e \in \partial T} c_e. \)

\( T \)-joins and polynomial-time algorithms for computing minimum \( T \)-joins are discussed later in Section 2.3.
1. Compute the minimum spanning tree $M$ of $G$.

2. Compute the minimum cost $T$-join $H$ of $G$, where $T$ is the set of odd-degree vertices in $M$.

3. The multiset $M + H$ is an Eulerian graph. Return an Eulerian tour of $M + H$.

**Theorem 2.2** (Christofides, 1976). Christofides’ algorithm returns a tour of cost at most $\frac{3}{2}$ times the optimal value.

**Proof sketch.** As the minimum cost connected subgraph of $G$ the minimum spanning tree has cost at most $(1 - 1/n) \text{OPT}$. The minimum cost $T$-join has cost at most $1/2 \text{OPT}$, which can be seen as follows. Break the optimal tour into paths between vertices in $T$. There is an even number of these paths, and taking every other path induces a $T$-join. That is, the optimal tour can be divided into 2 $T$-joins. The smaller of the two $T$-joins has cost at most half of the optimal tour.

### 2.2 LP Relaxations for Metric TSP

A standard LP for TSP is the following *subtour elimination* LP.

\[
\begin{align*}
\text{minimize} & \quad \sum_{e \in E} c_e y_e \quad \text{over } y \in \mathbb{R}^E \\
\text{s.t.} & \quad \sum_{e \in \delta(v)} y_e = 2 \quad \text{for all } v \in V, \\
& \quad \sum_{e \in \delta(U)} y_e \geq 2 \quad \text{for all } \emptyset \subsetneq U \subsetneq V, \\
& \quad y_e \in [0, 1] \quad \text{for all } e \in E.
\end{align*}
\]

(SE)

Here $\delta(S)$ denotes the set of edges crossing a set $S \subset V$. The first set of constraints require each vertex to be incident to exactly two edges (in the integral setting), and are referred to as degree constraints. The second constraint forces two-edge connectivity. The LP provides a lower bound for TSP that coincides with the lower bound of Held and Karp [1970]. The worst-case integrality gap of this LP is conjectured to be $\frac{4}{3}$ and is a major open problem in approximation algorithms.

Wolsey [1980] showed that the integrality gap is at most $\frac{3}{2}$ by analyzing Christofides’s algorithm via the LP.

**Fact 2.3** (Wolsey [1980]). Christofides’ algorithm returns a tour of cost $\leq \frac{3}{2}$ times the cost of $\text{OPT (SE)}$. In particular, the integrality gap of (SE) is at most $\frac{3}{2}$.
To apply the lower bound to an implicit instance of Metric-TSP defined by \( G \), one needs to apply it to the metric completion of \( G \). Instead one can consider a simpler LP obtained by dropping the degree constraints and the upper bound constraints in (SE). This leads to the following LP relaxation for the 2-edge connected spanning subgraph problem (allowing multiplicities).

\[
\text{minimize } \sum_{e \in E} c_e y_e \text{ over } y \in \mathbb{R}^E \\
\text{s.t. } \sum_{e \in \delta(U)} y_e \geq 2 \text{ for all } \emptyset \subsetneq U \subsetneq V \\
\text{and } y_e \geq 0 \text{ for all } e \in E.
\]  

(2ECSS)

**Fact 2.4** (Cunningham [via Monma, Munson, and Pulleyblank, 1990], Goemans and Bertsimas, 1993). The optimum value of subtour elimination LP (SE) for the metric completion of \( G \) coincides with the optimum value of the 2-edge connected spanning subgraph LP (2ECSS).

**2.3 Perfect matchings and \( T \)-joins**

Before addressing \( T \)-joins in general, we first review the following special case where \( T = V \).

**Definition 2.5.** A matching in \( G \) is a set of edges \( M \subseteq E \) such that each vertex is incident to at most one edge in \( M \). A perfect matching is a set of edges \( M \subseteq E \) such that each vertex is incident to exactly one edge in \( M \). The cost of a matching \( M \) is \( \sum_{e \in E} c_e \).

**Problem 2.6** (Minimum cost perfect matching). Assuming \( G \) contains a perfect matching, compute the minimum cost perfect matching.

The following running times are known for computing minimum cost perfect matchings.

**Fact 2.7** (Gabow and Tarjan, 1991). If the edge costs are integers between \( -W \) and \( W \), the minimum cost perfect matching can be computed in \( \tilde{O}(m\sqrt{n}\log W) \) time. For general edge weights and any \( \epsilon > 0 \), an \((1 + \epsilon)\)-minimum perfect matching can be computed in \( \tilde{O}\left( m\sqrt{n}\log \frac{1}{\epsilon} \right) \) time.

Clearly, computing the minimum weight perfect matching reduces to computing the minimum cost \( T \)-join for \( T = V \). Conversely, we have the following.

**Fact 2.8** (Edmonds, 1965). If all edge weights are non-negative, then the minimum cost \( T \)-join equals the minimum weight perfect matching on the clique with vertex set \( T \), where the weight of an edge \((s, t)\) in \( T \times T \) is the length of the shortest path from \( s \) to \( t \) in \( G \).

**Fact 2.9** (Gabow and Tarjan, 1991). A minimum \( T \)-join can be computed in time; an \((1 + \epsilon)\)-minimum \( T \)-join can be computed in \( \tilde{O}\left( |T|m + |T|^{2.5}\log \frac{1}{\epsilon} \right) \).

**Proof sketch.** For non-negative edge costs, all shortest paths between vertices in \( T \) can be found in \( \tilde{O}(|T|m) \) time with Dijkstra's shortest path algorithms. Submitting these lengths to the matching algorithms of Gabow and Tarjan [1991] (Fact 2.7) gives the desired running times. ■
A different reduction from $T$-joins to perfect matching, better suited for sparse graphs and $|T|$ large, is the following.

**Fact 2.10** (Berman, Kahng, Vidhani, and Zelikovsky, 1999, Theorem 3). The minimum cost $T$-join is equivalent to the minimum cost perfect matching of an auxiliary graph with $O(m)$ nodes and $O(m)$ edges. The auxiliary graph can be computed in $O(m)$ time.

**Fact 2.11.** The minimum $T$-join can be computed in $\tilde{O}(m^{1.5} \log W)$ time if the edge costs are integers between 1 and $W$; an $(1 + \epsilon)$-minimum $T$-join can be computed in $\tilde{O}(m^{1.5} \log \frac{1}{\epsilon})$ time.

**Proof sketch.** Here we combine Fact 2.10 with the min-cost perfect matching algorithm of Gabow and Tarjan [1991] to obtain the desired running time. ■

**Remark 2.12.** The bottleneck of Christofides' algorithm is computing a $T$-join, where $T$ can have size $|T| = \Omega(n)$. Moreover, it suffices to compute an $(1 + \epsilon)$-approximate minimum $T$-join for $\epsilon = \frac{2}{n}$, since the MST has cost at most $\left(1 - \frac{1}{n}\right)$-fraction of the minimum cost tour. Combining the $(1 + \epsilon)$-approximation algorithm Fact 2.7 with alternatively the reductions of Fact 2.8 or Fact 2.10, Christofides’ algorithm can be implemented in $\tilde{O}(mn + n^{2.5})$ or $\tilde{O}(m^{1.5})$ time.

**The $T$-join polytope:**

**Definition 2.13.** The dominant of a polytope $P$ is the set $\{P + x : x \geq 0\}$.

**Fact 2.14** (Edmonds and Johnson, 1973). The dominant of the $T$-join polytope is the set of vectors $x \in \mathbb{R}^E$ such that

$$x \geq 0 \text{ and } \sum_{e \in \delta(S)} x_e \geq 1 \text{ for each } S \subseteq V \text{ with } |S \cap T| \text{ odd.}$$

*(JD)*

**Observation 2.15** (Wolsey [1980]). Suppose $x \in \mathbb{R}^n$ is feasible for the LP (2ECSS), then $\frac{x}{2}$ is in the dominant of the $T$-join polytope for any even $T \subseteq V$.

### 3 Sparsifying solutions to (2ECSS)

In this section we prove Theorem 1.4. The idea is straightforward in retrospect. Let $x$ be a feasible solution to (2ECSS). We can view $x$ as capacities on the edges w/r/t which $G$ is 2-edge-connected, and apply cut sparsification techniques to obtain another solution $x'$ where $x'$ is sparse. Cut sparsification is a standard technique with many applications. Here we also have an objective function that needs to be preserved, and a black box sparsification does not suffice. We observe that random sampling based cut sparsification techniques Benczur and Karger [2015], Fung et al. [2011] are based on importance sampling and can be adjusted to preserve the cost of the objective function. We first give the high-level details of the scheme from Benczur and Karger [2015] and then build upon it to derive our result.
**Setting 3.1.** Let \( G = (V, E, w) \) be a weighted undirected graph.

**Definition 3.2.** \( G \) is \( k \)-connected if the value of each cut in \( G \) is at least \( k \).

**Definition 3.3.** A \( k \)-strong component is a maximal \( k \)-connected vertex-induced subgraph of \( G \).

**Definition 3.4.** The strong connectivity or strength of an edge \( e \), denoted by \( \kappa_e \), is the maximum value of \( k \) such that a \( k \)-strong component contains (both endpoints of) \( e \).

**Fact 3.5** (Benczur and Karger, 2015, Lemma 4.11). \( \sum_{e \in E} \frac{w_e}{\kappa_e} \leq n - 1 \).

**Fact 3.6** (Benczur and Karger, 2015, Compression Theorem 6.2). Let \( p : E \to [0, 1] \) be a set of probabilities on the edges of \( G \). Let \( H = (V, E', w') \) be a random weighted graph where for each edge \( e \in E, E' \) independently samples \( e \) with weight \( u'_e = \frac{u_e}{p_e} \) with probability \( p_e \), and \( e \notin E \) with probability \( 1 - p_e \). For \( \delta \geq \Omega(\log n) \), if \( p_e \geq \min \left\{ 1, \frac{\delta}{\kappa_e} \right\} \) for all \( e \in E \), then with probability \( 1 - \exp(-\Omega(\epsilon^2 \delta)) \), every cut in \( H \) has value between \((1 - \epsilon)\) and \((1 + \epsilon)\) times its value in \( G \).

**Fact 3.7** (Benczur and Karger, 2015, Theorem 6.5). In \( O(m \log^3 n) \) time, one can compute values \( \tilde{\kappa}_e \geq 0 \) for each \( e \in E \) such that \( \tilde{\kappa}_e \leq \kappa_e \) for each \( e \in E \) and \( \sum_{e \in E} \frac{w_e}{\tilde{\kappa}_e} = O(n) \).

**Lemma 3.8.** Given a feasible solution \( x \) to (2ECSS), and a non-negative cost function \( c : E \to \mathbb{R}_{\geq 0} \), an \( \epsilon > 0 \), there is a randomized algorithm that runs in \( O(m \log^3 n) \) time and with probability at least \((1 - 1/n^2)\), outputs another feasible point \( y \) for (2ECSS) such that (i) \( \langle c, y \rangle = (1 + \epsilon) \langle c, x \rangle \), (ii) \( \text{support}(y) \subseteq \text{support}(x) \), and (iii) \( \text{support}(y) = O\left(\frac{n \log n}{\epsilon^2}\right) \).

**Proof.** For each edge \( e \in E \), let \( \kappa_e \) be the strength of edge \( e \) w.r.t. the weighted graph \((G, x)\).

By Fact 3.7, we can compute approximate strengths \( \tilde{\kappa}_e \in [0, \kappa_e] \) such that \( O\left(\sum_{e \in E} \frac{x_e}{\tilde{\kappa}_e}\right) = O(n) \) in \( O(m \log^2 n) \) time. Let \( \delta = d \log n \) where \( d \) is a sufficiently large constant. For each edge \( e \in E \), let \( p_e = \min \left\{ 1, \frac{\delta x_e}{\epsilon^2 \tilde{\kappa}_e} \right\} \), and let \( q_e = \min \left\{ 1, \frac{\delta c_e x_e}{\epsilon^2 \sum_{e' \in E} c_{e'} x_{e'}} \right\} \), and let \( r_e = \max\{p_e, q_e\} \). We have \( \sum_{e \in E} r_e \leq \sum_{e \in E} p_e + \sum_{e \in E} q_e = O\left(\frac{n \delta}{\epsilon^2}\right) + O\left(\frac{\delta}{\epsilon^2}\right) \leq O\left(\frac{n \delta}{\epsilon^2}\right) \).

Let \( H = (V', E', x') \) be the random weighted graph where each edge \( e \in E \) is independently sampled with weight \( x'_e = x_e / r_e \) with probability \( r_e \).

By Fact 3.6 and the assumption that \( x \in (2ECSS) \), with probability \( 1 - \exp(-\Omega(\delta)) \), we have

\[
\sum_{e \in E} x'_e \in (1 \pm \epsilon) \sum_{e \in E} x_e \geq (1 - \epsilon)2
\]

for all \( S \subset V \). By the multiplicative Chernoff inequality, we also have

\[
P\left[ \sum_{e \in E} c_e x'_e \geq (1 + \epsilon) \sum_{e \in E} c_e x_e \right] \leq \exp(-\Omega(\delta)), \quad \text{and} \quad P\left[ |E'| \geq (1 + \epsilon) O\left(\frac{n \delta}{\epsilon^2}\right) \right] \leq \exp(-\delta/\epsilon^2). \]
apx-Christofides($G = (V, E, c), \epsilon$)

1. Compute the minimum spanning tree $S$ in $\tilde{O}(m)$ time. Let $T$ be the set of odd-degree vertices in $S$.

2. By Theorem 1.3, compute an $(1 + \epsilon)$-approximation $x$ to $(2ECSS)$ in $\tilde{O}(m/\epsilon^2)$ time with probability $1 - 1/\text{poly}(m)$.

3. By Theorem 1.4, in time $\tilde{O}(m/\epsilon^2)$ and with probability $1 - 1/\text{poly}(m)$, compute a feasible point $y \in \mathbb{R}^E$ for $(2ECSS)$ such that
   - $\sum_{e} c_e y_e \leq (1 + \epsilon) \sum_{e} c_e x_e$
   - $|\text{support}(y)| = \tilde{O}(n/\epsilon^2)$.

   Note that $y/2$ lies in the dominant of the $T$-join polytope, $(JD)$.

4. Let $H = (V, \text{support}(y), c)$ be the subgraph of edges with nonzero values in $y$. By Fact 2.11, compute the minimum cost $T$-join $J \subseteq H$ in $\tilde{O}(n^{1.5})$ time.

5. Compute, shortcut, and return an Euler tour on the multigraph $S \cup J$.

Figure 1: Pseudocode for a randomized $(1 + O(\epsilon))^{3/2}$ approximation algorithm to Metric TSP w/r/t the shortest path metric of a weighted undirected graph (see Theorem 4).

By the union bound, we have $\sum_{e \in \text{support}(x')} x'_e \geq (1 - \epsilon)2$ for all $S \subseteq V$, $|\text{support}(x')| \leq O\left(\frac{n\delta}{\epsilon^2}\right)$, and $\sum_{e \in E} c_e x'_e \leq (1 + \epsilon) \sum_{e \in E} c_e x_e$ with probability of $\geq 1 - \exp(-\Omega(\delta))$. Then $y = (1 + \epsilon)x'$ is feasible for $(2ECSS)$, has $O\left(\frac{n\delta}{\epsilon^2}\right)$ nonzeros, and has cost $\sum_{e \in E} c_e y_e \leq (1 + \epsilon)^2 \sum_{e \in E} c_e x_e$. We obtain the desired statement by choosing $d$ sufficiently large to ensure that the success probability is at least $(1 - 1/n^2)$ and by choosing a smaller $\epsilon'$ in the above analysis so that $(1 + \epsilon')^2 \leq (1 + \epsilon)$.

Theorem 1.4 is an easy consequence of Lemma 3.8 applied to Theorem 1.3.

4 Approximation schemes for Christofides’ algorithm

Figure 1 describes our randomized algorithm that yields a $\left(\frac{3}{2} + \epsilon\right)$-approximation for Metric-TSP via the LP solution followed by sparsification. It basically implements Christofides’s algorithm on the sparsified graph.

Theorem 1.5. In $\tilde{O}\left(\frac{m}{\epsilon^2} + \frac{n^{1.5}}{\epsilon^3}\right)$ time, with probability at least $1 - \frac{1}{\text{poly}(m)}$, apx-
\textbf{Christofides}(\mathcal{G}, \epsilon) \text{ returns a tour of cost at most } (1 + \epsilon) \frac{3}{2} \text{OPT(SE)}. \\

\textit{Proof.} With probability \(1 - 1/\text{poly}(m)\), \textbf{apx-Christofides} computes a minimum spanning tree \(S\) and the minimum cost \(T\)-join on the odd degree vertices of \(S\) within a subgraph \(H\) of \(G\). Since every vertex in the multigraph \(S \cup T\) has even degree, it has an Eulerian tour, which can be shortcut can returned.

The cost of the tour is at most \(\sum_{e \in S} c_e + \sum_{e \in J} c_e\). Since a \(n - 1/n\)-fraction of any feasible solutions to the \((\text{SE})\) lies in the spanning tree polytope, the cost of \(S\) is at most \(\sum_{e \in S} c_e \leq \text{OPT(SE)}\).

To bound the cost of the \(T\)-join \(J\), we first observe that by Fact 2.4 and Theorem 1.3, \(\sum_{e \in \mathcal{E}} c_e x_e \leq (1 + \epsilon) \text{OPT(SE)}\). Applying Theorem 1.4 to \(x\), with probability \(1 - 1/\text{poly}(m)\), we output a vector \(y\) that also lies in \((2\text{ECSS})\), has cost \(\sum_{e} c_e y_e \leq (1 + \epsilon)^2 \text{OPT(SE)}\), and has support of size \(\|y\|_0 = O(n \ln(n)/\epsilon^2)\). Since \(y/2\) lies in the dominant of the \(T\)-join polytope, the minimum cost \(T\)-join \(J\) in the support of \(y\) has cost at most \(\sum_{e \in J} c_e \leq \sum_{e} c_e y_e / 2\). Thus, the cost of a \((1 + \epsilon)\)-approximate \(T\)-join is \((1 + O(\epsilon)) \text{OPT(SE)}\).

To bound the running time, consider the steps as enumerated in Figure 1. Step 1, computing the minimum spanning tree, takes \(\tilde{O}(m)\) time. Step 2 computes an approximate solution \(x\) to \((2\text{ECSS})\), which by Theorem 1.3 can be computed in \(\tilde{O}(m/\epsilon^2)\) time. Step 3 sparsifies \(x\) to produce a point \(y\) with \(|\text{support}(y)| = O\left(\frac{n \log n}{\epsilon^2}\right)\) time. Step 4 applies a \((1 + 1/n)\)-approximate \(T\)-join algorithm from Fact 2.11 to the subgraph induced by the support of \(y\), which contains \(\tilde{O}(n/\epsilon^2)\) edges, and takes \(\tilde{O}\left(\frac{n^{3/2}}{\epsilon^3}\right)\) -time w/r/t the input graph \(\mathcal{G}\). The two bottlenecks are approximating \((2\text{ECSS})\) and computing the \(T\)-join, and the total running time is \(\tilde{O}\left(\frac{m}{\epsilon^2} + \frac{n^{3/2}}{\epsilon^3}\right)\). \(\blacksquare\)

\textbf{Remarks:} Genova and Williamson [2017] did an experimental evaluation of several variants of “Best-of-many Christofides” heuristics for Metric-TSP. These heuristics are based on choosing a \textit{random} spanning tree \(T\) from an appropriate distribution that is based on an LP solution \(x\), and then using that tree in the Christofides’s heuristic. One of these heuristics is to decompose a scaled version of the LP solution \(x\) into a convex combination of trees, and use the technique of swap-rounding [Chekuri, Vondrak, and Zenklusen, 2010] to generate a random spanning tree from the convex combination. The work in Chekuri and Quanrud [2017b] describes a near linear time algorithm to decompose \(x\) into a \((1 - \epsilon)\)-approximate convex combination of spanning trees. The particular structure of the decomposition, we believe, should allow one to implement swap-rounding step also in near linear time. This would lead to an implementation whose running time is similar to the one we have in this paper for the basic Christofides heuristic. We also believe that the ideas in Chekuri and Quanrud [2017a], and the ones here, will extend to develop faster approximation.
algorithms for the $s$-$t$-path TSP problem that has received substantial attention recently; we refer the reader to [An, Kleinberg, and Shmoys, 2015, Vygen, 2012, Sebo and Van Zuylen, 2016].

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