Finding duality and Riesz bases of exponentials on multi-tiles

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Abstract

It is known [5, 11, 17] that if $\Omega \subset \mathbb{R}^d$ belongs to a class of $k$–tilings of $\mathbb{R}^d$ when translated by a lattice $L$, there exists a Riesz basis of exponentials for $L^2(\Omega)$ constructed using $k$ translates of the dual lattice $L^*$. In this paper we give an explicit construction of the corresponding biorthogonal dual Riesz basis. In addition, we extend the iterative sampling algorithm introduced in [8], to this multivariate setting.

Keywords: Riesz bases of exponentials, frames of exponentials, multi-tiling, sampling and reconstruction, Paley-Wiener spaces, Vandermonde systems, biorthogonal systems

2019 MSC: 42B99 94A20 06D50 42A15

1. Introduction

Given a set $\Omega \subset \mathbb{R}^d$ of positive and finite Lebesgue measure, and a discrete set $\Lambda \subset \mathbb{R}^d$, we say that the complete sequence $\{e_\lambda := e^{2\pi i \lambda \cdot \cdot \cdot} \}_{\lambda \in \Lambda}$ is a Riesz basis for $L^2(\Omega)$ if there exist constants $0 < c \leq C < \infty$ such that for any sequence $\{a_\lambda\}_{\lambda \in \Lambda} \in \ell_2(\Lambda)$ we have

$$c \sum_{\lambda \in \Lambda} |a_\lambda|^2 \leq \left\| \sum_{\lambda \in \Lambda} a_\lambda e^{2\pi i \lambda \cdot \cdot \cdot} \right\|_{L^2(\Omega)}^2 \leq C \sum_{\lambda \in \Lambda} |a_\lambda|^2. \quad (1)$$

In this case, there exists a (unique) biorthogonal dual Riesz basis $\{g_\lambda\}_{\lambda \in \Lambda}$ such that

$$\langle e_\lambda, g_{\lambda'} \rangle = \delta(\lambda - \lambda') = \begin{cases} 1 & \text{if } \lambda = \lambda' \\ 0 & \text{else.} \end{cases}$$
If \( \{e_\lambda\}_{\lambda \in \Lambda} \) is an orthonormal basis (ONB) for \( L^2(\Omega) \) then \( c = C = 1 \) in (1), and conversely. However, according to the Fuglede Conjecture [10], such an ONB exists if and only if \( \Omega \) tiles \( \mathbb{R}^d \) with respect to the discrete set \( \Lambda \). As is well-known now, this conjecture has been disproved in both directions when \( d \geq 3 \), [10, 16, 18, 23], but remains open when \( d = 1, 2 \). However, it has recently been proved in [21] that the Fuglede conjecture does hold for convex domains in all dimensions. On the other hand, it is generally believed that once the rigidity of orthonormality of this set of exponential functions is removed, one can instead aim to obtain Riesz bases for \( \Omega \subset \mathbb{R}^d \). Indeed, and though no general proof of this statement exists, there are many classes of such subsets which admit a Riesz basis of exponential functions. For example, when \( \Omega \) is a finite union of co-measurable intervals or when \( \Lambda \) is a stable set of sampling for the Paley-Wiener space \( PW_\Omega \) (to be defined below), then it is known that \( \Omega \) admits a Riesz basis of exponentials [4, 13, 22, 24]. So is the case when \( \Omega \) is a union of multi-rectangles in \( \mathbb{R}^d \), [6, 19]. In addition, it was recently established in [7] that any convex polytope which is centrally symmetric and whose faces of all dimensions are also centrally symmetric, admits a Riesz basis of exponentials. For more on properties of sets of exponentials we refer to [26].

More generally, assume that \( \Omega \) is a multi-tiling subset of \( \mathbb{R}^d \), i.e., \( \Omega \subset \mathbb{R}^d \) is a bounded, Lebesgue measurable set such that for some positive integer \( k \) and a lattice \( L \) we have

\[
\sum_{t \in L} \chi_\Omega(x - t) = k \quad \text{for almost all } x \in \mathbb{R}^d .
\]  

(2)

In this case, it is known that \( \Omega \) admits a Riesz basis of exponentials [11, 17]. More specifically, there exists a set of vectors \( \{a_j\}_{j=1}^k \subset \mathbb{R}^d \) such that the exponentials

\[
\{e_\lambda(x) = e^{2\pi i (\lambda^* + a_j) \cdot x}, j = 1, \ldots k, \lambda^* \in L^* \}
\]

form a Riesz basis for \( L^2(\Omega) \), where \( L^* \) is the dual lattice of \( L \) [11, 17]. Consequently, for every function \( f \in L^2(\Omega) \) there exists a unique set of coefficients \( \{c_\lambda\}_{\lambda \in \Lambda} \in l^2(\Lambda) \) given by \( c_\lambda = \langle f, g_\lambda \rangle \) (where \( \{g_\lambda\}_{\lambda \in \Lambda} \) is the corresponding biorthogonal dual) so that \( f \) can be written as

\[
f(x) = \sum_{\lambda \in \Lambda} c_\lambda e_\lambda(x).
\]

Therefore, to reconstruct \( f \) from \( \{e_\lambda\}_\lambda \), these coefficients must be explicitly computed, or, equivalently, one must compute \( \langle f, e_\lambda \rangle \) and get

\[
f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle e_\lambda = \sum_{\lambda \in \Lambda} \langle f, e_\lambda \rangle g_\lambda.
\]  

(3)

In particular, this reconstruction formula (3) allows one to recover any function in \( L^2(\Omega) \), as long as one can find the biorthogonal Riesz basis. However, in many of the results in the literature on Riesz bases, and to the best knowledge of the authors, no explicit formulas for the dual Riesz bases are available. One
of the goals of this paper is to construct biorthogonal Riesz bases for $L^2(\Omega)$ for a class of subsets $\Omega \subset \mathbb{R}^d$ of finite, positive Lebesgue measure. In particular, we shall prove the following result, which extends a recent result obtained by the first named author in the univariate case [9].

**Theorem 1.** Given a full lattice $L$ and $\Omega \in \mathcal{C}(M)$ where $\mathcal{C}(M)$ is a corresponding class of Lebesgue measurable sets of positive and finite measure, there exists a pair of biorthogonal Riesz bases for $L^2(\Omega)$ of the form $\{e_\lambda\}_{\lambda \in \Lambda}$ and $\{g_\lambda\}_{\lambda \in \Lambda}$ where $\Lambda$ is a discrete set and $e_\lambda$, and $g_\lambda$ are explicitly given.

Investigating the existence of Riesz bases of exponentials for $L^2(\Omega)$ is equivalent to establishing sampling results in the corresponding Paley-Wiener space $PW_\Omega$, which is the space of all square integrable functions $f$, with Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$ supported in $\Omega$. In particular, suppose that $f \in PW_\Omega$, hence $\hat{f} \in L^2(\Omega)$. It follows from (3) that

$$\hat{f} = \sum_{\lambda \in \Lambda} \langle \hat{f}, e_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} f(-\lambda) g_\lambda$$

By taking the inverse Fourier transform we can write that

$$f = \sum_{\lambda \in \Lambda} f(-\lambda) \hat{g}_\lambda$$

This last formula gives a reconstruction of $f$ from its samples on $\Lambda$ and the dual Riesz basis $\{g_\lambda\}$.

The second goal of this paper is to derive from this reconstruction formula an efficient, deterministic, and iterative sampling algorithm using shifted lattices. The algorithm has the following properties:

(A1) The algorithm is deterministic;

(A2) It only involves inverting 1D Vandermonde systems;

(A3) It produces a sampling set $\Lambda$ that is optimal in the sense that $D^+(\Lambda) = D^-(\Lambda) = |\Omega|$.

As we will show in Section 4, this algorithm is an extension of the results that recently appeared in [8], and achieves an explicit formula for the sampled functions instead of the usual existence theorems and is also associated with an optimal sampling rate in the sense of Landau [20]. In [8], sets $\Omega$ of the form $\Omega = \bigcup_{z \in Z} ([-1/2, 1/2]^d + zN)$ are considered, where $N \geq 1$, and $Z = \mathbb{Z}^d \cap [-A, A]^d$ for some $A \in \mathbb{N}$. The algorithm presented here are based on extending the idea to sets $\Omega$ that form $k$-tilings of $\mathbb{R}^d$ with $k < (2A)^d$ (see Figure 1).

Compared to the results obtained in [1, 2] where explicit formulas for iterative methods for sampling bandlimited functions on unions of two shifted lattices were given, our results provide explicit results for arbitrary finite unions of shifted lattices. Another line of investigation that compares to the present
work is considered in [12, 14] where numerical sampling algorithms for multivariate trigonometric polynomials are used to derive approximations of infinite dimensional bandlimited functions from nonuniform sampling. In particular, recent results in [14] employ numerical methods for sampling along random rank-1 lattices for a given index set $\mathcal{Z}$ to approximate multivariate periodic functions, focusing on the error in approximation. In contrast, the results presented here offer a deterministic algorithm that guarantees exact reconstruction.

The rest of the paper is organized as follows. In Section 2, we set the notations for the paper and give a brief review of relevant results on the sampling of functions in $PW_\Omega$. In Section 3 we prove our first main result Theorem 2 from which Theorem 1 follows. In addition, we give an example that demonstrates how to generate multiple Riesz bases using lattices with different densities. The construction of sampling index sets and the derivation of the the sampling algorithm are provided in Section 4, where Theorem 4 is proved.

2. Notation and Preliminaries

This section contains a brief background on notation and sampling theory needed for the main proofs, see, for example, [22] for more details.

2.1. General notation

Throughout, $\Omega$ will be a bounded, measurable subset of $\mathbb{R}^d$. The Fourier transform of $f \in L^2(\Omega)$, denoted by $\mathcal{F}(f)$ or $\hat{f}$ is $\mathcal{F}(f)(\xi) := \int_\Omega f(x)e^{-2\pi i \xi \cdot x} dx$. The inverse Fourier transform for $g \in L^2(\mathbb{R})$, denoted by $\mathcal{F}^{-1}(g)$ or $\check{g}$, is given by $\mathcal{F}^{-1}(g)(\xi) := \check{g}(\xi) := \int_{\mathbb{R}^d} g(\xi)e^{2\pi i \xi \cdot x} d\xi$. Denoted by $PW_\Omega$ is the set of functions $f \in L^2(\mathbb{R}^d)$ with Fourier transform supported on $\Omega$, i.e., $PW_\Omega = \{f \in L^2(\mathbb{R}^d) \mid \hat{f}(\xi) = 0$ for a.e. $\xi \notin \Omega\}$. If $X \subseteq \mathbb{R}^d$ is a set, then $\chi_X$ is its indicator function. The cardinality of the finite set $\mathcal{S}$ is denoted by $\# \mathcal{S}$.

If $L = \{Mz \mid z \in \mathbb{Z}^d\}$ is a lattice generated by the basis vectors $M = [m_1, \ldots, m_n] \in \mathbb{R}^{n \times n}$, its canonical dual lattice is denoted by $L^* = \{M^{-T}z, z \in \mathbb{Z}^d\}$.
\( \) From now on, when working with a lattice \( L \), we denote by \( \Pi_M \) the fundamental parallelipiped, \( \Pi_M = M[0, 1)^d := \{ Mx \mid x \in [0, 1)^d \} \).

Let \( Q_h(x) = \prod_{j=1}^d [x_j - h/2, x_j + h/2] \). The upper and lower Beurling density [3] of a set \( \Lambda \) are given, respectively, by

\[
D^+(\Lambda) = \liminf_{h \to \infty} \inf_{\beta \in \mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(\beta))}{h^d}, \quad D^-(\Lambda) = \limsup_{h \to \infty} \sup_{\beta \in \mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(\beta))}{h^d}.
\]

2.2. Multidimensional sampling on shifted lattices

Suppose \( \Omega \subset \mathbb{R}^d \) is a measurable set and \( f \) is a function in \( PW_\Omega \). Let \( M \in \mathbb{R}^{d \times d} \) be an invertible matrix with fundamental parallelipiped \( \Pi_M \) and let \( \{\alpha_s\}_{s=0}^{k-1} \subset \mathbb{Z}^d \) be a collection of distinct vectors. Associated with \( \{\alpha_s\}_{s=0}^{k-1} \), we denote by \( \Lambda \) the discrete sampling set given by the union of shifted lattices

\[
\Lambda = \bigcup_{s=0}^{k-1} \Lambda_s, \quad \Lambda_s = M^{-T}(\mathbb{Z}^d + \alpha_s), \tag{4}
\]

and the functions \( \varphi, f_s, 0 \leq s \leq k - 1 \) by

\[
\varphi(x) := \frac{1}{|\det(M)|} \int_{\Pi_M} e^{2\pi i \xi \cdot x} d\xi, \quad f_s(x) := \sum_{\lambda \in \Lambda_s} f(\lambda) \varphi(x - \lambda). \tag{5}
\]

Note that \( \hat{\varphi} \) and \( \hat{f}_s \) by construction are in the space \( L^2(\Pi_M) \), and

\[
\hat{f}_s(\xi) = \sum_{\lambda \in \Lambda_s} f(\lambda) \hat{\varphi}(\xi) e^{-2\pi i \lambda \cdot \xi}. \tag{6}
\]

Shannon’s sampling theorem and various generalizations [15, 25] produce the following equality

\[
\|\hat{f}_s\|_{L^2(\Pi_M)} = \frac{1}{|\det(M)|} \sum_{\lambda \in \Lambda_s} |f_s(\lambda)|^2 = \frac{1}{|\det(M)|} \sum_{\lambda \in \Lambda_s} |f(\lambda)|^2. \tag{7}
\]

Then, applying the Poisson summation formula to (5),

\[
f_s(x) = \int_{\Pi_M} \left[ \sum_{z \in \mathbb{Z}^d} \hat{f}(Mz + \xi)e^{2\pi i Mz \cdot M^{-T} \alpha_s} \right] e^{2\pi i \xi \cdot z} d\xi.
\]

Therefore, for a.e. \( \xi \in \Pi_M \),

\[
\hat{f}_s(\xi) = \sum_{z \in \mathbb{Z}^d} \hat{f}(Mz + \xi)e^{2\pi i z \cdot \alpha_s}. \tag{8}
\]

Note that both (6) and (8) are determined uniquely by the samples \( \{f(\lambda)\}_{\lambda \in \Lambda_s} \in \ell_2(\Lambda_s) \).
For sets $\Omega$ that form a $k$–tiling of $\mathbb{R}^d$ with respect to the lattice $L = M\mathbb{Z}^d$, it follows that for a.e. $\xi \in \Pi_M$ there is a set of $k$ distinct vectors $\{z_s(\xi)\}_{s=0}^{k-1} \subset \mathbb{Z}^d$ such that $\xi + Mz_s(\xi) \in \Omega$, $0 \leq s \leq k - 1$. Then, (8) is a finite sum and the unique reconstruction of $\hat{f}(\xi)$ is determined by the invertibility of the $k \times k$ matrix $V$ with entries $V_{st} = e^{2\pi i z_s(\xi) \cdot \alpha_t}$. For example, in one dimension, setting $\alpha_t = ts$ for a sufficiently small constant $\delta > 0$, $V$ is an invertible Vandermonde matrix. The situation is considerably more difficult in higher dimensions, where the linear systems are not guaranteed to be invertible.

3. Dual bases and Riesz bases of exponentials

To prove Theorem 1, we give an explicit construction of the corresponding biorthogonal system for a certain class of domains. To define this class, we first let $L = M\mathbb{Z}^d$ be a full lattice with basis matrix $M$, and let $\Omega \subset \mathbb{R}^d$ be a measurable set with $0 < |\Omega| < \infty$. Then for almost every $\xi \in \Pi_M$, we will define the frequency index set $Z(\xi; \Omega) := Z(\xi)$ to be the set given by

$$Z(\xi) = \{ z \in \mathbb{Z}^d \mid \xi + Mz \in \Omega \}. \tag{9}$$

We say that $\Omega$ belongs to the set $C(M)$ if both of the following conditions are satisfied:

1. There exists a natural number $k$ such that $\Omega$ forms a $k$–tiling of $\mathbb{R}^d$ with respect to the lattice $L = M\mathbb{Z}^d$;

2. $\Omega$ is admissible, meaning that there exists an element of the dual lattice, $v \in L^* = M^{-T}\mathbb{Z}^d$, and a number $n \in \mathbb{N}$ such that for almost every $\xi \in \Pi_M$, the numbers $\{ v \cdot Mz \mid z \in Z(\xi) \}$ are all distinct (mod $n$).

Notice that if $\Omega$ is a $k$–tiling of $\mathbb{R}^d$ with respect to $L$ for some natural number $k$, then $\#Z(\xi) = k$ for a.e. $\xi \in \Pi_M$. In this case we enumerate the elements by

$$Z(\xi) = \{ z_t(\xi) \}_{t=0}^{k-1} \subset \mathbb{Z}^d,$$

and we denote by $\{ \omega_t(\xi) \}_{t=0}^{k-1}$, the unique points in $\Omega$ satisfying

$$\omega_t(\xi) = \xi + Mz_t(\xi), \quad t = 0, \ldots, k - 1. \tag{10}$$

We note that the mapping $\omega_t : \Pi_M \to \Omega$ given by $\xi \to \omega_t(\xi)$ is therefore invertible.

For almost every $\xi \in \Pi_M$ we consider the function-valued $k \times k$ matrix $V(\xi)$ whose $(t, s)$ entry is $V_{ts}(\xi) := (V(\xi))_{ts} = e^{2\pi i Mz_t(\xi) \cdot \alpha_s}$ where $0 \leq t, s \leq k - 1$ and the vectors $\{a_s\}_{s=0}^{k-1} \subset \mathbb{R}^d$ will be specified. In particular, we shall show that there is a choice of this set of vectors such that $V(\xi)$ is invertible for almost every $\xi \in \Pi_M$ and the inverse matrix, which we denote by $V^{-1}(\xi)$, is the matrix whose $(t, s)$ entry is $V^{-1}_{ts}(\xi) := (V(\xi))_{ts}^{-1}$ where $0 \leq t, s \leq k - 1$. With these notations, our first main result is the following.
Theorem 2. Let $L = M\mathbb{Z}^d$ be a full lattice with basis matrix $M$ and suppose $\Omega \in \mathcal{C}(M)$. Then, there exist a number $k \in \mathbb{N}$ and vectors $\{a_s\}_{s=0}^{k-1} \subset \mathbb{R}^d$ so that the two sets of functions

$$\{e^{2\pi i (M^{-T}z + a_s) \cdot \xi} \}, \quad \{e^{2\pi i (M^{-T}z + a_s) \cdot \xi} h_s(\xi) \}, \quad z \in \mathbb{Z}^d, s = 0, \ldots k - 1$$

form a pair of biorthogonal Riesz bases for $L^2(\Omega)$, where the functions $h_s$ are defined for each $0 \leq s \leq k - 1$, by

$$h_s(\xi) = \frac{1}{\det(M)} \sum_{t=0}^{k-1} V_{ts}(\omega_t^{-1}(\xi)) V_{st}(\omega_s^{-1}(\xi)) \chi_{\omega_t(\Pi_M)}(\xi).$$

Proof. Since $\Omega \in \mathcal{C}(M)$, the vectors $\{a_s\}_{s=0}^{k-1}$ can be chosen (as in [17] if $\Omega$ is bounded, or as in [5] if $\Omega$ is admissible for $L$) so that for a.e. $\xi \in \Pi_M$ all of the matrices

$$V_{ts}(\xi) = e^{-2\pi i M z_t(\xi) \cdot a_s}, \quad s, t = 0, \ldots k - 1$$

are invertible, and as a result,

$$\{e^{2\pi i (M^{-T}z + a_s) \cdot \xi} | z \in \mathbb{Z}^d, 0 \leq s \leq k - 1\}$$

is a Riesz basis for $L^2(\Omega)$. To translate this into the sampling framework, for $0 \leq s \leq k - 1$, define the discrete sampling sets $\Lambda_s \Lambda$ by (4) for $\alpha_s = M^T a_s$:

$$\Lambda_s = M^{-T} \mathbb{Z}^d + a_s, \quad \Lambda = \bigcup_{s=0}^{k-1} \Lambda_s.$$ 

Then, the sampling functions $\varphi, f_s$ in (5) are well defined and the Fourier transform $\hat{f}_s$ has the form (8). Since $\Omega$ forms a $k$-tiling of $\mathbb{R}^d$ with respect to $\Lambda$, (8) reduces to a finite sum for a.e. $\xi \in \Pi_M$

$$\hat{f}_s(\xi) = \sum_{t=0}^{k-1} \hat{f} (M z_t(\xi) + \xi) e^{2\pi i M z_t(\xi) \cdot a_s}.$$

Then, for almost every $\xi \in \Pi_M$, (14) forms a linear system

$$F(\xi) = V(\xi) \hat{F}(\xi)$$

with

$$F(\xi) = (\hat{f}_0(\xi), \ldots, \hat{f}_{k-1}(\xi)), \quad \hat{F}(\xi) = (\hat{f}(M z_0(\xi) + \xi), \ldots, \hat{f}(M z_{k-1}(\xi) + \xi)).$$

This system is also invertible for a.e. $\xi \in \Pi_M$, and therefore all functions $f \in PW_\Omega$ can be uniquely recovered from the samples $\{f(\lambda) | \lambda \in \Lambda\}$. 

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Recall that the biorthogonal dual Riesz basis \( \{ g_\lambda(\xi) \}_{\lambda \in \Lambda} \) corresponding to \( \{ e_\lambda(\xi) = e^{2\pi i \lambda \cdot \xi} \}_{\lambda \in \Lambda} \) satisfies

\[
\langle g_\lambda, e_\lambda \rangle = \int_\Omega g_\lambda(\xi) e^{2\pi i (-\tilde{\lambda}) \cdot \xi} d\xi = \delta(\lambda - \tilde{\lambda}), \quad \lambda, \tilde{\lambda} \in \Lambda, \tag{15}
\]

implying that for a fixed \( \lambda \in \Lambda \), the samples of the function \( \mathcal{F}^{-1}(g_\lambda) \in PW_\Omega \) taken on the set \(-\Lambda\) are explicitly given by

\[
\mathcal{F}^{-1}(g_\lambda)(-\tilde{\lambda}) = \delta(\lambda - \tilde{\lambda}), \quad \tilde{\lambda} \in \Lambda. \tag{16}
\]

Then, the reconstruction of \( g_\lambda \in L^2(\Omega) \) given these samples (16), can be accomplished by solving the corresponding linear system (14).

To proceed with the reconstruction, fix \( s \) in \( \{0, \ldots, k-1\} \) and \( \lambda \in \Lambda_s \). For \( s' \in \{0, \ldots, k-1\} \) define

\[
g_{\lambda, s'}(x) = \sum_{\tilde{\lambda} \in \Lambda_{s'}} \{ \mathcal{F}^{-1}(g_\lambda) \}(-\tilde{\lambda}) \varphi(x + \tilde{\lambda}).
\]

Applying (16),

\[
g_{\lambda, s}(x) := \sum_{\tilde{\lambda} \in \Lambda_s} \delta(\lambda - \tilde{\lambda}) \varphi(x + \tilde{\lambda}) = \varphi(x + \lambda) \implies \hat{g}_{\lambda, s}(\xi) = \hat{\varphi}(\xi) e^{2\pi i \xi \cdot \lambda},
\]

and if \( s' \neq s \),

\[
g_{\lambda, s'}(x) := \sum_{\tilde{\lambda} \in \Lambda_{s'}} \delta(\lambda - \tilde{\lambda}) \varphi(x + \tilde{\lambda}) = 0 \implies \hat{g}_{\lambda, s'}(\xi) = 0.
\]

The recovery of \( g_\lambda \) using samples \( \{ \mathcal{F}^{-1}(g_\lambda)(\tilde{\lambda}) \}_{\tilde{\lambda} \in -\Lambda} \) is accomplished by solving the system \( G(\xi) = V(\xi) \hat{G}(\xi) \) for a.e. \( \xi \in \Pi_M \):

\[
\begin{pmatrix}
\hat{g}_{\lambda, 0}(\xi) \\
\vdots \\
\hat{g}_{\lambda, s-1}(\xi) \\
\hat{g}_{\lambda, s}(\xi) \\
\hat{g}_{\lambda, s+1}(\xi) \\
\vdots \\
\hat{g}_{\lambda, k-1}(\xi)
\end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0 \\
\varphi(\xi) e^{2\pi i \xi \cdot \lambda}
\end{pmatrix}
= V(\xi)
\begin{pmatrix}
g_\lambda(\xi) \\
\vdots \\
g_\lambda(Mz_{s-1}(\xi) + \xi) \\
g_\lambda(Mz_s(\xi) + \xi) \\
g_\lambda(Mz_{s+1}(\xi) + \xi) \\
\vdots \\
g_\lambda(Mz_{k-1}(\xi) + \xi)
\end{pmatrix}. \tag{17}
\]

For a.e. \( \xi \in \Pi_M \) this system can be explicitly solved using only the \( s \)th column of \( V^{-1}(\xi) \):

\[
\frac{1}{\det M} V^{-1}_s(\xi) e^{2\pi i \xi \cdot \lambda} = g_\lambda(Mz_t(\xi) + \xi), \quad 0 \leq t \leq k - 1.
\]
Then, since $\lambda = M^{-T} n + a_s$ for some $n \in \mathbb{Z}^d$, this simplifies to
\[
\frac{1}{\det M} V_{ts}^{-1}(\xi) V_{st}(\xi) e^{\lambda (M z_t + \xi)} = g_\lambda (M z_t + \xi), \quad 0 \leq t \leq k - 1
\]
for a.e. $\xi \in \Pi_M$.

\[\square\]

**Remark 1.** Note that the system is an orthogonal basis if $(V(\xi))^* V(\xi) = I$ or if $k = 1$, in which case $h_0(\xi) = \frac{1}{|\det(M)|}$.

**Example 1.** Suppose $\mathcal{Z} = \{z_s\}_{s=0}^3 = \{(0,0), (1,0), (2,0), (0,m-1)\}$ and
\[
\Omega = \bigcup_{s=0}^3 [0,1/m)^2 + z_s/m
\]
for an integer $m \geq 3$. Then $\Omega$ is a 4-tiling of $\mathbb{R}^2$ with respect to the lattice $L = M \mathbb{Z}^2$, $M = \frac{1}{m} \text{Id}$ (see Figure 3 for $m = 3$). Let $\delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}$, and consider the set $\{a_s = m \delta j_s\}_{s=0}^{k-1}$ where
\[
\mathcal{J} := \{j_s\}_{s=0}^{k-1} = \{(0,0), (1,0), (2,0), (0,1)\} \subset \mathbb{Z}^2.
\]

It can be verified when $\delta$ is a diagonal matrix with $\delta_{ii} \leq 1/m$, that the choice of the set $\{a_s = m \delta j_s\}_{s=0}^{k-1}$ produces an invertible matrix $V_{ts} = e^{-2\pi i M z_t + a_s} = e^{-2\pi i z_t \cdot \delta j_s}$. Denoting by $F_{j_s}$ the function $\hat{f}_s$ in (8), the linear system (14), produced by sampling, is
\[
\begin{pmatrix}
F_{(0,0)}(\xi) \\
F_{(1,0)}(\xi) \\
F_{(2,0)}(\xi) \\
F_{(0,1)}(\xi)
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 & 1 \\
e^{2\pi i \delta_{11}} & e^{2\pi i \delta_{12}} & 1 & 1 \\
e^{2\pi i \delta_{21}} & e^{2\pi i \delta_{22}} & 1 & 1 \\
e^{2\pi i (m-1) \delta_{22}} & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\hat{f}(\xi) \\
\hat{f}(z_1/m + \xi) \\
\hat{f}(z_2/m + \xi) \\
\hat{f}(z_3/m + \xi)
\end{pmatrix}.
\]
Therefore, the four sampling sets
\[ \Lambda_s = L^* + a_s = m\mathbb{Z}^2 + m\delta js, \quad s = 0, 1, 2, 3, \]
are needed to recover the four unknowns \( \{\hat{f}(z_t/m + \xi)\}_{t=0}^{k-1} \). The corresponding Riesz basis is
\[ e_\lambda(\xi) = e^{2\pi i \lambda \cdot \xi}, \quad \lambda \in \Lambda = \bigcup_{s=0}^{3} m(\mathbb{Z}^2 + \delta js), \]
with equal upper and lower Beurling densities \( D^+(\Lambda) = D^-(\Lambda) = \frac{4}{m^2} = |\Omega| \).

By Theorem 2 the dual basis is then given by
\[ g_\lambda(\xi) = e_\lambda(\xi) h_s(\xi); \quad \lambda \in \Lambda_s \]
\[ h_s(Mz_t + \xi) = m^2 V_{st}^{-1} V_{st}, \quad \text{a.e. } \xi \in \Pi_M. \]

For example, if \( m = 4 \) and \( \delta_{11} = \delta_{22} = 1/4 \), this matrix given by the matrix
\[
V_{ts}^{-1} V_{st} = \begin{pmatrix}
-0.25 - 0.25i & 0.5 & 0.25 - 0.25i & 0.5 + 0.5i \\
0.5 & 0 & 0.5 & 0 \\
0.25 - 0.25i & 0.5 & 0.25 + 0.25i & 0 \\
0.5 + 0.5i & 0 & 0 & 0.5 - 0.5i
\end{pmatrix}.
\]
The sum of each column is equal to one because
\[
V_{ts}^{-1} V_{st} = 1 = \langle e_\lambda, g_\lambda \rangle = \langle e_\lambda, e_\lambda h_s \rangle = \sum_{t=0}^{3} \langle e_\lambda, e_\lambda V_{ts}^{-1} V_{st} \rangle [0,1/m]^2 + z_t/m \quad (19)
\]
\[
= \langle e_\lambda, e_\lambda [0,1/m]^2 \rangle \sum_{t=0}^{3} V_{ts}^{-1} V_{st} = \sum_{t=0}^{3} V_{ts}^{-1} V_{st} \quad (20)
\]
Furthermore, since the exponentials \( e_\lambda \) are orthogonal on \( \Pi_M \), as long as \( g_\lambda \)
is constant on each of the translates \([0,1/m]^2 + z_t/m\), it is easy to compute directly that \( g_\lambda \) and \( e_\lambda \) are biorthogonal. If \( \lambda \in \Lambda_{s'} \) and \( \lambda \in \Lambda_s \) with \( s \neq s' \):
\[
\langle e_\lambda, g_{\bar{\lambda}} \rangle = \langle e_\lambda, e_\lambda h_s \rangle = \sum_{t=0}^{3} V_{ts}^{-1} V_{st} \langle e_\lambda, e_\lambda [0,1/m]^2 + z_t/m \rangle = 0.
\]
Notice that if \( z_3 = (3,0) \), the coefficient matrix satisfies \( V_{ts}^{-1} V_{st} = 1/4 \) for all \( 0 \leq s, t \leq 3 \), then \( V^* = V \) and the system is self dual.

4. Iterative sampling algorithm for Vandermonde systems

In this section we see that under certain conditions on \( Z(\xi), \xi \in \Pi_M \), a sampling algorithm given in with desirable properties (A1) - (A3), can be derived to invert the system (14). The derivation of the sampling algorithm
has three parts. First is the generation of sampling index sets $\mathcal{J}(Z)$ based on
the knowledge of the frequency index sets $Z$, described in Section 4.1. The
uniqueness of the reconstruction is proved in Section 4.2 along with the main
sampling algorithm and these techniques are compared to known methods for
constructing Riesz bases.

However, we begin with the following motivating example that captures the
main idea in our proposed algorithm.

**Example 2.** Assume $\Omega$ is given as in Example 1, and reconsider the solution
of the linear system (14). For (almost) each $\xi \in \Pi_M = [0, \frac{1}{m})^2$ the aim is to
recover the 4 unknown translates
\[
\{\hat{f}(\xi + \frac{z}{m})\}_{z \in Z}, \quad Z = \{(0, 0), (1, 0), (2, 0), (0, m - 1)\} \subset Z^2.
\]

Notice that if the set $\Omega$ is projected onto the $x-$axis, the resulting problem
only has three unknowns. This is reflected in the system formed for a.e. $\xi \in \Pi_M$
by using the first three sampling sets:
\[
\begin{pmatrix}
F_{(0,0)}(\xi) \\
F_{(1,0)}(\xi) \\
F_{(2,0)}(\xi)
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 \\
e^{2\pi i \delta_{11}} & e^{2\pi i \delta_{11}} & e^{2\pi i \delta_{11}} \\
e^{2\pi i 2\delta_{11}} & e^{2\pi i 4\delta_{11}} & e^{2\pi i 4\delta_{11}}
\end{pmatrix}
\begin{pmatrix}
F^0(\xi) := \hat{f}(\xi) + \hat{f}(z_3/m + \xi) \\
\hat{f}(z_1/m + \xi) \\
\hat{f}(z_2/m + \xi)
\end{pmatrix}.
\]

At this point, two of the four unknowns $\hat{f}(\frac{1}{m}(1, 0) + \xi)$ and $\hat{f}(\frac{1}{m}(2, 0) + \xi)$ can
be uniquely recovered. This information can be leveraged when considering the
fourth sampling set,
\[
F_{(0,1)}(\xi) = \hat{f}(\xi) + \hat{f}(z_1/m + \xi) + \hat{f}(z_2/m + \xi) + \hat{f}(z_3/m + \xi)e^{2\pi i (m-1)\delta_{22}}.
\]

The last two unknowns solve the linear system for a.e. $\xi \in \Pi_M$
\[
\begin{pmatrix}
F^0_0(\xi) \\
F^0_1(\xi)
\end{pmatrix} := F_{(0,1)}(\xi) - \sum_{q=1}^2 \hat{f}(\xi + z_q/m) =
\begin{pmatrix}
1 & 1 \\
e^{2\pi i (m-1)\delta_{22}} & e^{2\pi i (m-1)\delta_{22}}
\end{pmatrix}
\begin{pmatrix}
\hat{f}(\xi) \\
\hat{f}(\xi + z_3/m)
\end{pmatrix}.
\]

This process is equivalent to the factorization $V = W_1W_2$:
\[
\begin{pmatrix}
F_{(0,0)}(\xi) \\
F_{(1,0)}(\xi)
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
e^{2\pi i \delta_{11}} & e^{2\pi i 2\delta_{11}} & e^{2\pi i 4\delta_{11}} & 0 & 0 \\
e^{2\pi i 2\delta_{11}} & e^{2\pi i 4\delta_{11}} & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\hat{f}(z_1/m + \xi) \\
\hat{f}(z_2/m + \xi) \\
\hat{f}(z_3/m + \xi)
\end{pmatrix}.
\]

### 4.1. Novel construction of optimal sampling sets

Denote by $Z \subset \mathbb{Z}^d$ a bounded frequency index set, where, for simplicity
the dependence on $\xi$ is suppressed. The key to designing a sampling algorithm
for more general domains is finding an optimal sampling index set \( \mathcal{J}(\mathcal{Z}) = \{ j_k \}_{k=0}^{k-1} \subset \mathbb{Z}^d \) so that

\[
\Lambda = \bigcup_{s=0}^{k-1} \Lambda_s,
\Lambda_s = M^{-T} (\mathbb{Z}^d + \delta j_s),
0 \leq s \leq k - 1
\]  

produces stable sampling sets for a well-chosen matrix \( \delta \in \mathbb{R}^{d \times d} \). Again, since \( \Omega \) forms a \( k \)-tiling of \( \mathbb{R}^d \) with respect to the lattice \( L = M \mathbb{Z}^d \), the frequency index sets \( \mathcal{Z}(\xi) \) exist almost everywhere in \( \Pi_M \) so that (8) reduces to the \( k \times k \) linear system \( F = V \tilde{F} \),

\[
\hat{f}_s(\xi) = \sum_{t=0}^{k-1} \hat{f}(M z_t + \xi) e^{2\pi i z_t \cdot \delta j_s}, \quad \text{a.e. } \xi \in \Pi_M.
\]  

The construction of the sampling index set \( \mathcal{J}(\mathcal{Z}) \) is guided by two main principles. First, \( \mathcal{J}(\mathcal{Z}) \) will have the same dimensional “structure” as the frequency index set \( \mathcal{Z} \). For instance in Example 1, \( \mathcal{Z} \) contains three frequency translates in the \( x \)-direction, and therefore \( \mathcal{J}(\mathcal{Z}) \) contains three different sampling indices in the first coordinate. Second, while the index sets can have arbitrary gaps, e.g., \( \mathcal{Z} = \{0, 3, 4\} \), the sampling sets need to have consecutive integers e.g., \( \mathcal{J}(\mathcal{Z}) = \{0, 1, 2\} \) so that the linear systems produced by (23) can be factorized using Vandermonde matrices.

The set \( \mathcal{J}(\mathcal{Z}) \) is constructed by separating a frequency set \( \mathcal{Z} \), dimension by dimension, and then determining the needed variation in each dimension. In this way, the sampling sets are constructed independently of the order of the coordinates.

Let \( \mathcal{M}^d = \mathcal{Z} \). For \( l = 1, \ldots, d-1 \), denote by \( \mathcal{M}^l \subset \mathbb{Z}^l \) the set

\[
\mathcal{M}^l = \{ m \in \mathbb{Z}^l \mid (m, m') \in \mathcal{M}^d \text{ for some } m' \in \mathbb{Z}^{d-l} \}.
\]

In other words a vector in \( \mathcal{M}^l \) consists of the first \( l \) coordinates of a vector in \( \mathcal{M}^d \).

For each \( l = 2, \ldots, d-1 \) and \( m_i \in \mathcal{M}^{l-1} \), define the sets \( \mathcal{Z}_i^l \subset \mathbb{Z} \)

\[
\mathcal{Z}_i^l = \{ z_i \in \mathbb{Z} \mid (m_i, z_i) \in \mathcal{M}^l \}, \quad 1 \leq i \leq \# \mathcal{M}^{l-1}.
\]

Without loss of generality, it can be assumed that \( \mathcal{M}^{l-1} = \{ m_i \} \) is ordered so that \( \# \mathcal{Z}_i^l \leq \# \mathcal{Z}_j^l \) for \( i \leq j \). Then

\[
\sum_{i=1}^{\# \mathcal{M}^l} \# \mathcal{Z}_i^l = \# \bigcup_{i=1}^{\# \mathcal{M}^l} \{ (m_i, z_i) \mid z_i \in \mathcal{Z}_i^l \} = \# \mathcal{M}^l.
\]

Remark 2. It can be assumed that \( \# \mathcal{M}^l > 1 \) for all \( l \), otherwise, the dimension of the problem can be reduced by one. For example, if \( \mathcal{Z} = \{(0, 1), (0, 2)\} \), the number of distinct integers in the first coordinate is 1, and therefore \( \# \mathcal{M}^l = 1 \) and the sampling index sets do not need to be varied in the first coordinate.
Now sampling index sets can be created. First, given a set \( M_1 \subset \mathbb{Z} \), define the sampling index set \( J_1(M_1) := \{0, \ldots, \#M_1 - 1\} \subset \mathbb{Z} \). (24)

Then, for \( l > 1 \), given a set \( M_l \subset \mathbb{Z}^l \), define the set \( J_l(M_l) \subset \mathbb{Z}^l \)

\[
J_l(M_l) := \bigcup_{i=1}^{\#M^{l-1}} J^{l-1}(M^{l-1}_i) \times Q_i(M_l),
\]

(25)

where \( M^{l-1}_i = \{m'_j\}_{j=1}^{\#M^{l-1}} \) and

\[
Q_i(M_l) = \begin{cases} 
\{0, \ldots, \#Z^1_i - 1\}, & i = 1 \\
\{\#Z^{l-1}_i, \ldots, \#Z^1_i - 1\} & i = 2, \ldots, \#M^{l-1} \end{cases}.
\]

(26)

**Lemma 3.** For each \( l = 1, \ldots, d \), \( \#J_l(M_l) = \#M_l \).

**Proof.** By induction. This is true for \( l = 1 \) by definition. Consider \( 1 < l < d \). The induction assumption asserts that there are \( \#M^{l'} \) elements in \( J^{l'}(M^{l'}) \) for \( 1 \leq l' < l \). Therefore there are \( \#M^{l-1} \) elements in \( J^{l-1}(M^{l-1}) \) and for \( i > 1 \), \( \#M^{l-1} - i + 1 = \#M^{l-1}_i = \#J^{l-1}(M^{l-1}_i) \). Then, there are \( \#Z^1_i \) in \( Q_1(M^l) \) and for \( i > 1 \), \( (\#Z^i - \#Z^{l-1}_i) \) elements in \( Q_i(M^l) \). Summing this up,

\[
\#J^l(M^l) = \#M^{l-1} \#Z^1_i + \sum_{i=2}^{\#M^{l-1}} (\#M^{l-1} - i + 1)(\#Z^i - \#Z^{l-1}_i)
\]

\[
= \sum_{i=1}^{\#M^{l-1}} \#Z^i = \#M^l.
\]

□

This construction of \( J^d(M^d) \) generates a tree structure that gives a way to enumerate the elements in the sampling index with a labeling that informs the reconstruction procedure. The sets \( M^l \) correspond to the collection of vectors in each parent node in the \( l^{th} \) level of the tree. The ordering \( M^{l-1} = \{m_i\} \) corresponds to nodes in increasing order of the number of immediate children. The sets \( Z^l_p \) correspond to the last coordinate of the children of the \( p^{th} \) vector in \( M^l \).

**Example 3.** Let \( M = 5, N = 10 \) and consider the set \( Z \subset \mathbb{Z}^4 \) given by

\[
Z = \left\{ (1,1,1,1), (2,1,1,1), (3,1,1,1), (4,1,1,1), (2,2,1,1), (3,2,1,1), (4,2,1,1), (2,2,1,2), (3,2,2,1), (4,3,1,1) \right\}
\]

The tree diagram is illustrated in Figure 3 and the sampling index sets are given in Table 1.
4.2. Uniqueness and stability of the reconstruction

The invertibility of the linear system (23) is equivalent to finding a factorization of $V$ into a product involving invertible Vandermonde matrices.

**Theorem 4.** Let $Ω ⊂ ℝ^d$ be an admissible $k$–tiling of $ℝ^d$ with respect to the lattice $L = Mℤ^d$. For (almost) each fixed $ξ ∈ Π_M$, let $M^d = Z(ξ)$, and define $M^4$ according to Section 4.1. Let the sampling index set $J^d(M^d)$ be given by (24)–(26) and define the discrete sets $Λ_s$ and $Λ$ by (22) for a fixed diagonal matrix $δ ∈ ℝ^{d×d}$.

For $1 ≤ l ≤ d$, the system of equations produced by (23),

$$F = V \tilde{F}$$

with $F = (\hat{f}_0(ξ), \ldots, \hat{f}_{k-1}(ξ))$, $\tilde{F} = (\hat{f}(ω_0(ξ)), \ldots, \hat{f}(ω_{k-1}(ξ)))$, and $V_{st} = e^{2πiξ_s(ξ)·δ}$, can be expressed in the form,

$$J^1(M^1) = \{0, 1, 2, 3\}$$

$$J^2(M^2) = J^3(M^3) = J^4(M^4) = Λ = \{0, 1, 2, 3\} × Q_1(M^1) \times Q_2(M^2) \times Q_3(M^3) \times Q_4(M^4)$$

$$J^2(M^2) × Q_1(M^1) = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (2, 0, 0), (3, 0, 0)\}$$

$$J^2(M^2) × Q_3(M^3) = \{(0, 0, 0, 0)\}$$

$$J^3(M^3) × Q_2(M^2) × Q_3(M^3) = \{(0, 0, 0, 0, 0)\}$$

$$J^4(M^4) × Q_4(M^4) = \{(0, 0, 0, 0, 0)\}$$

Table 1: Construction of sampling index sets corresponding to Example 3

| $l=1$ | $J^1(M^1) = \{0, 1, 2, 3\}$ |
| $l=2$ | $J^2(M^2) × Q_1(M^1) = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (2, 0, 0), (3, 0, 0)\}$ |
| $l=3$ | $J^3(M^3) × Q_2(M^2) × Q_3(M^3) = \{(0, 0, 0, 0)\}$ |
| $l=4$ | $J^4(M^4) × Q_4(M^4) = \{(0, 0, 0, 0, 0)\}$ |

Figure 3: Tree structure produced by the construction of the frequency subindex sets $M^l$ in Example 3.
\[
\begin{aligned}
F_{j''}(\xi) &= \sum_{m: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta' j'' - m}, \\
F_{j'}(\xi) &= \sum_{m: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta' j' - m}, \\
F_{j''}(\xi) &= \sum_{m: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta'' j'' - m}, \\
F_{j'''}(\xi) &= \sum_{m: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta''' j''' - m},
\end{aligned}
\]

where \( F_{j'}(\xi) = \hat{f}_s(\xi) \) for 0 \( \leq s \leq k - 1 \) and \( \delta'' \) is the diagonal submatrix \( \delta'' = \text{diag}(\delta_{11}, \ldots, \delta_{ll}) \). Moreover, if \( 0 < \delta_{ii} < \left( \max_{0 \leq l \leq k - 1} |(z_l(\xi),)|^{-1} \right) \) the system (27) is invertible and has the unique solution

\[
\begin{aligned}
F_{j'}(\xi) &= \sum_{m: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta' j' - m}, \\
F_{j''}(\xi) &= \sum_{m: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta'' j'' - m}, \\
F_{j'''}(\xi) &= \sum_{m: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta''' j''' - m},
\end{aligned}
\]

where \( \delta' \) is the diagonal submatrix \( \delta' = \text{diag}(\delta_{(l+1)(l+1)}, \ldots, \delta_{dd}) \).

**Proof.** Assume \( d > 1 \). Since for any \( l = 1, \ldots d - 1 \), and \( j'' \in J^l(M^l), (j'', j') \in J^d(M^d), \)

\[
F_{j''}(\xi) = \sum_{z: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta j'' - z} = \sum_{m: \xi \in J^d(M^d)} \hat{f}(Mz + \xi) e^{2\pi i \delta'' j'' - m},
\]

and it follows that a solution to (27) is given by (28). Induction can be used to prove that the solution is unique. In the case \( l = 1 \), (27) is an invertible Vandermonde system and therefore the solution is unique. Now, for \( l > 1 \), assume that (28) is the unique solution to (27) for \( l' \leq l - 1 \). Recall that for \( 1 \leq l \leq d - 1 \) the set \( M^l = \{m_p\}_{p=1}^{\#M^l} \) is ordered according to increasing \( \#Z_p \). Fix \( j' \in Z^{d-l+1} \).

1. \( (p = 1) \) Using the induction assumption for each \( j_i \in J^l(M^{l-1}) \), corresponding to samples on

\[
\{A_{j''}(j_i, j') \mid j'' \in J^l(M^{l-1})\}, \quad j_i = 0, \ldots \#Z^l - 1
\]

the solutions of (27) are uniquely obtained as

\[
\begin{aligned}
F_{m_p}(\xi) &= \sum_{z: \xi \in J^d(M^d)} \hat{f}_{m_p}(z) e^{2\pi i j_i z \delta_{ll}^{-1}} \#M^{l-1}, \\
j_i = 0, \ldots \#Z^l - 1,
\end{aligned}
\]

(30)

The system formed by (30) for \( p = 1 \), and ranging \( j_i \) is a Vandermonde system

\[
F_{j'''}(\xi) = \sum_{z: \xi \in J^d(M^d)} \hat{f}_{j'''}(z) e^{2\pi i j_i z \delta_{ll}^{-1}}, \quad j_i = 0, \ldots \#Z^l - 1.
\]

For $0 < \delta_l < (\max_{0 \leq t \leq k-1} |(z_t(\xi))|)^{-1}$ this system admits the unique solution
\[
\{F_{j'}^{(m_1, z)}(\xi)\}_{z \in \mathbb{Z}^{l-1}}.
\] (32)

II) Case $p = 2, \ldots \#M^{l-1}$. For a fixed $j'' \in \mathbb{Z}^l$, $j = (j'', j') \in \mathbb{Z}^d$, define $\tilde{F}_j^p$, the unique function satisfying
\[
\tilde{F}_j^p(\xi) = F_j(\xi) - \sum_{q=1}^{p-1} F_{j'}^{(m_q, z)}(\xi) e^{2\pi i m_q \delta'' j''} = \sum_{m \in M_p^l} F_j^m(\xi) e^{2\pi i m \delta'' j''}.
\] (33)

Note that (33) involves the known sampled function $F_j$ and the functions $\{F_{j'}^{(m_q, z)}\}_{z \in \mathbb{Z}^l}$ uniquely determined in the previous computation for $q < p$. Then, since (33) satisfies (27)-(28) for the frequency index set $\tilde{M}^d = \{(m_q, m') \in M^d, q \geq p\} \subseteq M^d$, and noticing that $\tilde{M}^{l-1} = M_{p-1}^l$, the induction assumption is applied, using samples on
\[
\{\Lambda_{(j'', j, j')} \mid j'' \in \mathcal{J}^{l-1}(M_{p-1}^{l-1})\}, \quad j_i = \#Z_{p-1}^l, \ldots, \#Z_p^l - 1
\]
to obtain the linear systems
\[
\begin{align*}
\tilde{F}_j^p(\xi) &= \sum_{q=p}^{\#M_p^{l-1}} F_{j'}^{(m_q, z)}(\xi) e^{2\pi i m_q \delta'' j''}, \\
&\quad j'' \in \mathcal{J}^{l-1}(M_{p-1}^{l-1}).
\end{align*}
\] (34)

with unique solutions
\[
\{F_{j'}^{(m_q, z)}(\xi)\}_{q=p}^{\#M_p^{l-1}}, \quad j_i = \#Z_{p-1}^l, \ldots, \#Z_p^l - 1.
\] (35)

Combining (30) and all previous solutions $F_{(j_i, j')}^{(m_p, z)}$ to (35) for $q = p$ produces the invertible linear system
\[
F_{(j_i, j')}^{(m_p, z)}(\xi) = \sum_{z_i \in Z_p^l} F_{j'}^{(m_p, z_i)}(\xi) e^{2\pi i z_i j_i}, \quad j_i = \{0, \ldots, \#Z_p^l - 1\}.
\] (36)

with unique solution $\{F_{j'}^{(m_p, z_i)}(\xi)\}_{z_i \in Z_p^l}$. Therefore, the claim is true for $l < d$.

In the case $l = d$, assume that (27) has a unique solution (28) for $l' \leq d - 1$. Then, repeat I) and II) using the sampling sets
\[
\{\Lambda_{(j'', j, j')} \mid j'' \in \mathcal{J}^{d-1}(M_{p-1}^{d-1})\}, \quad j_d = 0, \ldots, \#Z_d^{d-1} - 1
\]
to uniquely obtain
\[
F^z(\xi) = \tilde{f}(M z + \xi), \quad z \in M^d.
\]
\[\square\]
Remark 3. For almost every $\delta_j < (\max_{0 \leq i \leq k-1} |(z_i(\xi))_i|)^{-1}$. Then, given samples $\{f(\lambda)\}_{\lambda \in \Lambda}$, the values $\hat{f}(Mz_s(\xi) + \xi), 0 \leq s \leq k-1$ can be uniquely reconstructed using the following algorithm:

**Algorithm 1.** Suppose $f \in PW_\Omega$, where $\Omega \subset C(M)$ for an invertible $d \times d$ matrix $M$. Fix $\xi \in \Pi_M$ and $M^d = Z(\xi)$, and define $\mathcal{M}$ according to Section 4.1. Let $\Lambda$ be the discrete set (22) for a fixed diagonal matrix $0 < \delta \leq (\max_{0 \leq t \leq k-1} |(z_t(\xi))_t|)^{-1}$. Then, given samples $\{f(\lambda)\}_{\lambda \in \Lambda}$, the values $\hat{f}(Mz_s(\xi) + \xi), 0 \leq s \leq k-1$ can be uniquely reconstructed using the following algorithm:

**Step 1** For each $j \in \mathcal{J}(M^d)$, define $F_j(\xi)$ by (28).

**Step 2** If $d = 1$, solve the system (27) for $l = 1$, obtaining $\{F^{m_1}(\xi)\}_{m_1 \in M^d}$. Then, skip to Step 4. If $d \geq 2$, solve (29) with $l = 1$ for $j'' \in \mathcal{J}^1(M^d)$, $(j'', j') \in \mathcal{J}(M^d)$, obtaining $\{F^{m_1}(\xi)\}_{m_1 \in M^d}$.

**Step 3** For $l = 2, \ldots, d-1$, set $M^{l-1} = \{m_p\}_{p=1}^{#M^{l-1}}$. Obtain $\{F^{(m_p, z_i)} \}_{z_i \in \mathcal{Z}_p}$ by solving the Vandermonde system (31) ($p = 1$), or by inverting (34) and solving the Vandermonde system (36) ($p = 2, \ldots, #M^{l-1}$).

**Step 4** For $l = d$, repeat Step 3, substituting the sampling indices $(j'', j_l, j') \in \mathcal{Z}^d$ with $(j'', j_d) \in \mathcal{Z}^d$ to obtain

$$\{F^{z_x}(\xi) = \hat{f}(Mz_s + \xi)\}_{z_s \in M^d}.$$

If $\mathcal{J}(Z(\xi)) = \mathcal{J}(Z(\xi'))$ for all $\xi, \xi' \in \Omega$, then the union $\Lambda$ of $k$ translates $\Lambda_\alpha$ of $M^{-T}Z$ given by (22) is a stable set of sampling for $PW_\Omega$ with optimal density.

**Corollary 5.** If $\Omega \subset \mathbb{R}^d$ is an admissible domain that forms a $k$--tiling of $\mathbb{R}^d$ with respect to the lattice $L = MZ^d$ and for all $\xi, \xi' \in \Omega$, $\mathcal{J}(Z(\xi)) = \mathcal{J}(Z(\xi'))$, then $f \in PW_\Omega$ is uniquely reconstructed by the samples $\{f(\lambda)\}_{\lambda \in \Lambda}$, where the set $\Lambda$ is given by (22) using Algorithm 1. Furthermore the upper and lower Beurling densities of $\Lambda$ equal $\#Z^{\frac{1}{\dim M}} = \frac{k}{\dim M} = |\Omega|$.

The assumption on $Z(\xi)$ implies that the same tree structure is produced for almost every $\xi \in \Omega$, which is a nontrivial assumption.

**Remark 3.** 1. Theorem 4 implies that the matrix $V$ defined by $V_{st} = e^{2\pi i \alpha_s z_t}$, $0 \leq s, t \leq k-1$ is invertible if $\alpha_s = \delta j_s$ for a diagonal matrix $\delta$ with sufficiently small entries, and therefore $\hat{f}(\xi) \in L^2(\Omega)$ can be uniquely recovered for almost every $\xi \in \Omega$ from the samples $\{f(\lambda)\}_{\lambda \in \Lambda}$ by finding the solution to the linear system

$$
\begin{pmatrix}
\hat{f}_0(\xi) \\
\hat{f}_1(\xi) \\
\vdots \\
\hat{f}_{k-1}(\xi)
\end{pmatrix}
= V
\begin{pmatrix}
\hat{f}(\xi) \\
\hat{f}(Mz_1(\xi) + \xi) \\
\vdots \\
\hat{f}(Mz_{k-1}(\xi) + \xi)
\end{pmatrix}.
$$

17
Note that the formulation (37), derived from the sampling algorithm, provides a constructive way to prove the existence of a Riesz basis that seems to be different than known results. For example, the proof given in [17] adapted to this setting relies on the existence of a set of vectors \( \{a_s\}_{s=0}^{k-1} \subset \mathbb{R}^d \) and \( M \mathbb{Z}^d \)-periodic functions \( \tilde{f}_s \in L^2(\Pi_M), 0 \leq s \leq k-1 \),

\[
\hat{f}(Mz_t(\xi) + \xi) = \sum_{s=0}^{k-1} e^{2\pi i a_s \cdot (\xi - Mz_t(\xi))} \tilde{f}_s(\xi), \quad t = 0, \ldots, k-1.
\]  

Theorem 4 provides one way to construct satisfactory \( \{a_s\}_{s=0}^{k-1} \) and \( f_j \in L^2(\Pi_M) \) for certain domains \( \Omega \). Setting \( \{a_s\}_{s=0}^{k-1} = M^{-T} \delta_j \), \( j \in J(M^d) \), the functions \( \tilde{f}_s(\xi) \) can be defined as the \( M \mathbb{Z}^d \)-periodic extension of the solutions \( \{\tilde{f}_s(\xi)\}_{s=0}^{k-1} \) to the following system :

\[
\begin{pmatrix}
\hat{f}(Mz_0(\xi) + \xi) \\
\hat{f}(Mz_1(\xi) + \xi) \\
\vdots \\
\hat{f}(Mz_{k-1}(\xi) + \xi)
\end{pmatrix} = V^* \begin{pmatrix}
e^{2\pi i a_0 \cdot \xi} \hat{f}_0(\xi) \\
e^{2\pi i a_1 \cdot \xi} \hat{f}_1(\xi) \\
\vdots \\
e^{2\pi i a_{k-1} \cdot \xi} \hat{f}_{k-1}(\xi)
\end{pmatrix}, \quad \implies (39)
\]

for a.e. \( \xi \in \Pi_M \), where \( D(\xi) = \text{diag}(e^{-2\pi i a_0 \cdot \xi}, e^{-2\pi i a_1 \cdot \xi}, \ldots, e^{-2\pi i a_{k-1} \cdot \xi}) \).

The invertibility of (40) is guaranteed as a consequence of the invertibility of \( V^* \) and \( V^{-1} \) given by Theorem 4.

2. A straightforward derivation of the Riesz constants is accomplished using (37). Since the matrix \( V \) is square and invertible, both \( V \) and \( V^{-1} \) have bounded norm. Taking the \( \ell_2 \) norm of both sides, and then integrating over \( \Pi_M \):

\[
\det(M) \|V^{-1}\|^{-2} \|\hat{f}\|_{L^2(\Omega)}^2 \leq \sum_{\lambda \in \Lambda} |\langle \hat{f}, e_\lambda \rangle|^2 \leq \det(M) \|V\|^{-2} \|\hat{f}\|_{L^2(\Omega)}^2.
\]

The task of estimating the singular values of the matrix \( V \) increases in difficulty with growing dimension, however, since the sampling algorithm produces a factorization of \( V \), the Riesz bounds can be estimated using products of the norms of Vandermonde matrix factors.

Example 4. Consider the system (18) used in Example 1 for a.e. \( \xi \in [0, 1/m]^2 \).
The sampling algorithm approach essentially computes the factorization 
\[ V = W_1 W_2 \] via (21). Taking the \( \ell_2 \) norm and integrating over \([0, 1/m]^2\),
\[
\frac{\|W_2^{-1}\|^{-2}\|W_1^{-1}\|^{-2}}{m^2} \|\hat{f}\|_{L^2(\Omega)}^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq \frac{\|W_1\|^2\|W_2\|^2}{m^2} \|\hat{f}\|_{L^2(\Omega)}^2.
\]

We conclude this paper with an example that applies the ideas developed in this paper and raises interesting questions that prompt further investigation. It can be viewed as an application to sampling sets with different densities.

**Example 5.** An interesting approach is devised using sampling sets of different densities. Consider the domain \( \Omega \) defined in Examples 1 and 2 again, where \( m \geq 3 \). Define the sampling sets
\[
\Lambda_0 = \frac{m}{3} \mathbb{Z} \times m\mathbb{Z}, \quad \Lambda_1 = m(\mathbb{Z}^2 + \left(\frac{0}{\delta}\right)),
\]
and the functions
\[
\varphi_0(x) = \frac{m^2}{3} \int_{[0,3/m) \times [0,1/m)} e^{2\pi i \xi \cdot x} d\xi, \quad \varphi_1(x) = m^2 \int_{[0,1/m)^2} e^{2\pi i \xi \cdot x} d\xi.
\]
For \( s \in \{0, 1\} \) the definition (5) is adapted:
\[
f_s(x) = \sum_{\lambda \in \Lambda_s} f(\lambda) \varphi_s(x + \lambda).
\]

The Fourier transform \( \hat{f}_s \) satisfies:
\[
\hat{f}_0(\xi) = \sum_{n \in \mathbb{Z}^2} f(\xi + \left(\frac{3n_1, n_2}{m}\right)) e^{2\pi in_1 \delta} \in L^2([0, \frac{3}{m}) \times [0, \frac{1}{m})]
\]
\[
\hat{f}_1(\xi) = \sum_{n \in \mathbb{Z}^2} f(\xi + \left(\frac{n}{m}\right)) e^{2\pi in_2 \delta} \in L^2([0, \frac{1}{m})^2).
\]

The following equality holds for a.e. \( \xi \in [1/m, 3/m) \times [0, 1/m)\):
\[
\hat{f}(\xi) = \hat{f}_0(\xi).
\]

The following linear system corresponds to \( \xi \in [0, 1/m) \times [0, 1/m)\):
\[
\hat{f}_0(\xi) - \sum_{t=1}^{2} \hat{f}(\xi + z_t) = \hat{f}(\xi + z_3/m)
\]
\[
\hat{f}_1(\xi) - \sum_{t=1}^{2} \hat{f}(\xi + z_t) = \hat{f}(\xi + z_3/m) e^{2\pi i (m-1)\delta}.
\]
Then, a Vandermonde system, invertible for $\delta \leq 1/m$, is solved for the remaining two unknowns. Therefore, another Riesz basis for $L^2(\Omega)$ is given by:

$$e_\lambda(\xi) = e^{2\pi i \lambda \xi}, \quad \lambda \in \Lambda = \Lambda_0 \cup \Lambda_1, \quad D^+(\Lambda) = D^-(\Lambda) = \frac{4}{m^2} = |\Omega|.$$

To summarize, for a.e. $\xi \in [0,1/m]^2$, the sampling procedure solves the following sparse, invertible linear system $F = W_3 \hat{F}$,

$$
\begin{pmatrix}
\hat{f}_0(\xi) \\
\hat{f}_0(\xi + z_1/m) \\
\hat{f}_0(\xi + z_2/m)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{f}(\xi) \\
\hat{f}(\xi + z_1/m) \\
\hat{f}(\xi + z_2/m) \\
\hat{f}(\xi + z_3/m)
\end{pmatrix}.
$$

(41)

The recovery of $g_\lambda$ using samples $\{\mathcal{F}^{-1}(g_\lambda)(\hat{\lambda})\}_{\hat{\lambda} \in -\Lambda}$ is accomplished by applying (16) and solving the linear system (41) for $\lambda \in \Lambda_i$, $i = 0, 1$ with reduced left-hand side:

$$
m^2 \begin{pmatrix}
e^{2\pi i(\xi + z_1/m) \cdot \lambda} \\
e^{2\pi i(\xi + z_2/m) \cdot \lambda} \\
lm/L_2(\Omega)
\end{pmatrix} \text{ if } \lambda \in \Lambda_0, \text{ and } m^2 \begin{pmatrix}0 \\
0 \\
0 \end{pmatrix} \text{ if } \lambda \in \Lambda_1.
$$

Then,

$$g_\lambda(\xi) = \sum_{t=0}^{3/3} \chi_{[0,1/m]^2}(\xi - z_t/m) \sum_{s=0}^{3} \hat{C}_s^i f_\lambda(\omega_s(\xi)),$$

where the coefficient matrices $\hat{C}_i$ are given by

$$\hat{C}_0 = \frac{m^d}{3} \begin{pmatrix}
e^{2\pi i (m-1) \delta} & 0 & 0 \\
0 & e^{2\pi i (m-1) \delta} & 0 \\
0 & 0 & e^{2\pi i (m-1) \delta}
\end{pmatrix},$$

$$\hat{C}_1 = m^d \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.$$

Approximate Riesz bounds are derived by taking the $\ell_2$ norm, denoting by $W_3$ the matrix in (41) and integrating over $[0,1/m]^2$,

$$||W_3^{-1}||^{-2} \|\hat{f}\|^2_{L^2(\Omega)} \leq \frac{m^2}{3} \sum_{\lambda \in \Lambda_0} |f(\lambda)|^2 + m^2 \sum_{\lambda \in \Lambda_1} |f(\lambda)|^2 \leq ||W_3||^2 \|\hat{f}\|^2_{L^2(\Omega)}.$$

Comparing this with the system (18) used in Example 1, it is clear that for each fixed $m$, there is a constant $\delta < 0$ for which $W_3$ has a lower condition number.
than \(V\) for \(\delta < \delta_0\) (see Figure 4). Can this be generalized? Furthermore, can the periodicity of the condition number demonstrated in the plot be characterized? These properties warrant further study, and the development of a general framework for sampling on lattices with different densities is the subject of current work.

Figure 4: Plots of the condition number of the original matrix \(V\) in (18) and the matrix \(W_3\) in (41) for varying \(m\) and \(0 < \delta < 1/2\). The black dotted line shows \(k = 4\).

Acknowledgements

The authors are grateful for discussions with David Walnut and Karamatou Yacoubou Djima. The work of C. Frederick is partially supported by the Na-
tional Science Foundation grant DMS-1720306. K. A. Okoudjou was partially supported by the National Science Foundation under Grant No. DMS-1814253, and an MLK visiting professorship at MIT.

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