CONFORMAL BLOCKS FOR ADMISSIBLE REPRESENTATIONS IN SL(2) CURRENT ALGEBRA

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ABSTRACT

A review is presented of the recently obtained expressions for conformal blocks for admissible representations in SL(2) current algebra based on the Wakimoto free field construction. In this realization one needs to introduce a second screening charge, one which depends on fractional powers of free fields. The techniques necessary to deal with these complications are developed, and explicit general integral representations for conformal blocks on the sphere are provided. The fusion rules are discussed and as a check it is verified that the conformal blocks satisfy the Knizhnik-Zamolodchikov equations.

1. Introduction

In refs. [1]-[2] H.-L. Hu and M. Yu and O. Aharony et al have proposed an equivalence between the coupling of usual conformal minimal matter to 2-d gravity using Hamiltonian reduction and the twisted SL(2)/SL(2) WZNW models. Their discussions cover comparisons of field contents and cohomologies. These works constitute our motivation for studying N-point correlation functions of 2-d conformal WZNW theories based on affine $\hat{SL}(2)_k$ since the cases of admissible representations [3] are the ones relevant for treating the minimal matter. Much attention has been paid to the N point correlators either by applying the Wakimoto free field realization [4], from which results have been given in refs. [5], or by solving the Knizhnik-Zamolodchikov (KZ) equations [6], from which results have been giving in e.g. refs. [7, 8]. The results are quite complete as far as unitary, integrable representations are concerned, but have appeared incomplete for the general case including the admissible representations. In ref. [9] we have found complete integral expressions in the case of admissible representations, based on the free field construction, for exactly the conformal blocks relevant to the above applications.

\footnote{Talk presented by J. Rasmussen}
In general the WZNW theory is characterized by the level, \( k \), or equivalently by \( t = k + 2 \) (for \( \widehat{SL}(2)_k \)). Then degenerate primary fields exist for representations characterized by spins, \( j_{r,s} \), given by \[ 2j_{r,s} + 1 = r - st \] with \( r, s \) integers. However, previous applications of the free field Wakimoto realization can be characterized as being fully complete only for the case, \( s = 0 \), which is the full case only for integrable representations. The reason for this restriction is fairly natural, since (see sect. 2) the screening charge usually employed in the free field realization is capable of screening just such primary fields. In fact, a possible second screening operator, capable of screening the general case, was proposed by Bershadsky and Ooguri \[ \text{[18]} \], but since it involved fractional powers of the free ghost fields, discussions on its interpretation have been only partly successful \[ \text{[8, 7]} \]. In ref. \[ \text{[15]} \] we have overcome this difficulty by showing how the techniques of fractional calculus \[ \text{[19]} \] naturally provide a solution. It should be mentioned that the authors of ref. \[ \text{[14]} \] have found a different class of solutions to the KZ equations than the class belonging to the integrable representations, but as we discuss in \[ \text{[15]} \] their solutions only have a rather small overlap with ours and are insufficient for solving the theory.

We refer to \[ \text{[15]} \] for more details.

2. Notation. Introduction of fractional calculus

The \( \widehat{SL}(2)_k \) affine current algebra may be written as

\[
J^+(z)J^-(w) = \frac{2}{z-w}J^3(w) + \frac{k}{(z-w)^2}
\]

\[
J^3(z)J^\pm(w) = \pm \frac{1}{z-w}J^\pm(w)
\]

\[
J^3(z)J^3(w) = \frac{k/2}{(z-w)^2}
\]

when we only consider one chirality of the fields. The Wakimoto realization \[ \text{[4]} \] is based on the free scalar field, \( \varphi(z) \), and bosonic ghost fields, \( (\beta(z), \gamma(z)) \), of dimensions \( (1, 0) \) which we take to have the following contractions

\[
\varphi(z)\varphi(w) = \log(z-w), \quad \beta(z)\gamma(w) = \frac{1}{z-w}
\]

The currents are represented as

\[
J^+(z) = \beta(z)
\]

\[
J^3(z) = - :\gamma^2\beta : (z) + k\partial\gamma(z) - \sqrt{2t}\gamma\partial\varphi(z)
\]

\[
t = k + 2 \neq 0
\]
while the Sugawara energy momentum tensor is obtained as
\[
T(z) =: \beta \partial \gamma : (z) + \frac{1}{2} : \partial \varphi \partial \varphi : (z) + \frac{1}{\sqrt{2t}} \partial^2 \varphi (z)
\] (5)
with central charge
\[
c = \frac{3k}{k + 2}
\] (6)
Very conveniently one can collect the primary fields in multiplets \( \phi_j(w, x) \) (cf. [10]) parametrized by a variable \( x \) which keeps track of the \( SL(2) \) representation, \( j \)
\[
J^a(z) \phi_j(w, x) = \frac{1}{z - w} J^a_0(w) \phi_j(w, x)
\] (7)
where the \( SL(2) \) representation is provided by the differential operators
\[
\begin{align*}
J_0^a(z) \phi_j(z, x) &= [J_0^a, \phi_j(z, x)] = D_x^a \phi_j(z, x) \\
D_x^+ &= -x^2 \partial_x + 2xj \\
D_x^3 &= -x \partial_x + j \\
D_x^- &= \partial_x 
\end{align*}
\] (8)
It is a matter of direct verification to check that
\[
\phi_j(z, x) = (1 + \gamma(z)x)^{2j} : e^{-j\sqrt{2/t}\varphi(z)} :
\] (9)
is a primary field as defined above. In the case of admissible representations
\[
\begin{align*}
t &= p/q \\
2j_i + 1 &= r_i - s_it \\
1 &\leq r_i \leq p - 1 \\
0 &\leq s_i \leq q - 1
\end{align*}
\] (10)
where \( (p, q) = 1 \) and \( p, q \in \mathbb{N} \), one needs two screening currents
\[
\begin{align*}
S_1(z) &= \beta(z)e^{+\sqrt{2/t}\varphi(z)} \\
S_2(z) &= \beta(z)^{-t}e^{-\sqrt{2/t}\varphi(z)}
\end{align*}
\] (11)
The labelling of the \( j \)'s in (10) refers to the \( N \) primary fields in the correlators. Screening currents are dimension 1 fields and have total derivatives in the OPE’s not only with the energy momentum tensor but also with the affine currents. This ensures that the screening charges (integrated screening currents) can be inserted into the correlators without spoiling the affine Ward identities.
The appearance of the fractional ghost field \( \beta(z)^nF(\gamma(w)) \) leads to the necessity of generalizing the usual Wick contractions which may be written
\[
\beta(z)^nF(\gamma(w)) = : (\beta(z) + \frac{1}{z-w}\partial\gamma(w))^nF(\gamma(w)) : \\
\gamma(z)^nF(\beta(w)) = : (\gamma(z) - \frac{1}{z-w}\partial\beta(w))^nF(\beta(w)) :
\]
(12)

Our proposal to deal with entities like \( \beta(z)^{-t} \) consists in the following generalization of (12)
\[
G(\beta(z))F(\gamma(w)) = : G(\beta(z) + \frac{1}{z-w}\partial\gamma(w))^nF(\gamma(w)) :
\]
(13)

To be able to perform these we need to know how to expand the different expressions, for instance
\[
(1 + \gamma(z)x)^{2j} = \sum_{n \in \mathbb{Z}} \binom{2j}{n+\alpha} (\gamma(z)x)^{n+\alpha}
\]
which appears as the ghost content of the primary field. The different choices of asymptotic expansions have been labelled by a parameter \( \alpha \). When deciding on what expansions to adopt, we use the criterion that after all Wick contractions have been performed the powers, which are then inside normal ordering signs, are non-negative integers. Then the resulting terms have an obvious interpretation when sandwiched between bra and ket states.

For non-unitary representations as the admissible ones, \( 2j \) is not necessarily integer and we see the need for fractional calculus. We use
\[
\partial^a x^b = \frac{\Gamma(b+1)}{\Gamma(b-a+1)}x^{b-a}
\]
as the basic definition of fractional differentiation and
\[
D^a \exp(x) = \sum_{n \in \mathbb{Z}} \frac{1}{\Gamma(n-a+1)}x^{n-a}, \quad a \in \mathbb{R}
\]
is then an example of an asymptotic expansion of the exponential function.

Before considering correlators let us briefly go through our notation for mode expansions, vacuum states etc. Using the mode expansions
\[
j(z) = - : \gamma(z)\beta(z) := +\partial\phi(z) \\
\varphi(z)\varphi(z') = +\log(z-z') \\
\phi(z)\phi(z') = -\log(z-z') \\
\varphi(z) = q_\varphi + a_0 \log z + \sum_{n \neq 0} \frac{a_n}{-n} z^{-n} \\
\phi(z) = q_\phi + j_0 \log z + \sum_{n \neq 0} \frac{j_n}{-n} z^{-n}
\]
\[
[a_0, q_\varphi] = +1 \quad [j_0, q_\phi] = -1
\]
(17)
the dual vacuum state \( \langle 0 | \) is defined by

\[
\langle 0 | = \langle s l_2 | e^{-q_\phi e^{\sqrt{2/t_\phi}}}
\]

where \( \langle s l_2 | \) is the usual \( SL(2) \) invariant bra vacuum, while the ket vacuum, \(|0\rangle\), is identical to the \( SL(2) \) invariant ket vacuum \(|s l_2\rangle\). From these states we construct dual bra states of lowest \( SL(2) \) weight

\[
\langle j | = \langle 0 | e^{j\sqrt{2/t_\phi}}
\]

and similarly the highest weight ket state

\[
| j \rangle = e^{-j\sqrt{2/t_\phi}}|0\rangle
\]

They are normalized such that

\[
\langle j | j \rangle = 1
\]

3. Three point functions and fusion rules

Let us now consider the evaluation of the (chiral) three point function

\[
\langle j_3 | \phi_{j_2}(z, x) | j_1 \rangle
\]

Using the free field realization (9) of \( \phi_{j_2}(z, x) \) the three point function may be evaluated only provided the "momentas" or "charges" may be screened away in the standard way \[20\], and correspondingly \( \phi_{j_2}(z, x) \) is replaced by the intertwining field, \( (\phi_{j_2}(z, x))^{j_3}_{j_1} \), which maps a \( j_1 \) highest weight module into a \( j_3 \) highest weight module. Following Felder \[21\, 5\], but using the two screening charges in (11) instead, we are led to consider the intertwining field

\[
(\phi_{j_2}(z, x))^{j_3}_{j_1} = \oint_{s} \prod_{j=1}^{s} \frac{dv_j}{2\pi i} \prod_{i=1}^{r} \frac{du_i}{2\pi i} \phi_{j_2}(z, x) P(u_1, ..., u_r; v_1, ..., v_s)
\]

\[
P(u_1, ..., u_r; v_1, ..., v_s) = \prod_{j=1}^{s} \beta^{-t}(v_j) e^{-\sqrt{2/t_\phi}(v_j)} \prod_{i=1}^{r} \beta(u_i) e^{\sqrt{2/t_\phi}(u_i)}
\]

This requires that

\[
j_1 + j_2 - j_3 = r - st
\]

with \( r \) and \( s \) non-negative integers. It is trivial using well known techniques to perform the \( \phi \) part of the Wick contractions. Hence we concentrate on explaining how to perform the ghost part. The calculations rely on the following two lemmas

**Lemma 1**

\[
(1 + \gamma(z)x)^{2j} = \Gamma(2j + 1) \int_{0}^{1} \frac{du}{2\pi i u} (u^{-1} D)^{-2j} \exp [(1 + \gamma(z)x)/u]
\]
This is an almost trivial integral representation of the ghost part of the primary field. **Lemma 2**

\[ \beta^a(w) \exp \left[ (1 + \gamma(z)x)/u \right] =: (\beta(w) + \frac{x/u}{w-z})^a D^a \exp [(1 + \gamma(z)x)/u] : \]  

(26)

This second lemma tells us how to perform the contraction between a fractional \( \beta \) field and the \( \gamma \) content of the integral representation in the first lemma. Now it is straightforward to obtain the total ghost part of the contractions. After all contractions have been carried out, the sandwiching between the dual bra and the ket results in effectively putting \( \beta = \gamma = 0 \) because of the normal ordering. In the case of admissible representations we reach an integral expression for the three point function \( W_3 \)

\[
W_3 = \frac{\Gamma(2j_2 + 1)}{\Gamma(2j_2 - r + st + 1)} \cdot \int \prod_{i=1}^{r} \frac{du_i}{2\pi i} \prod_{j=1}^{s} \frac{dv_j}{2\pi i} \prod_{1<i<j} (u_{i_1} - u_{i_2})^{2/t} \prod_{j_1<j_2} (v_{j_1} - v_{j_2})^{2/t} \prod_{i,j} (u_i - v_j)^{-2} \cdot \prod_{i=1}^{r} u_i^{(1-r_i)/t+s}(1-u_i)^{(1-r_2)/t+s-1} \prod_{j=1}^{s} v_j^{1-s_i/s}(1-v_j)^{1-(s_2-1)t} 
\]  

(27)

Finally the \( u \) and \( v \) integrations around the Felder contours are of the Dotsenko-Fateev [20] form and may be performed explicitly

\[
W_3 = \frac{\Gamma(2j_2 + 1)}{\Gamma(j_2 + j_3 - j_1 + 1)} \cdot \int \prod_{j=1}^{r} (1-e^{2\pi i(r_1)/t}) \prod_{j=1}^{s} (1-e^{2\pi ij/t}) \prod_{i=1}^{r} \frac{\Gamma(i/t)}{\Gamma(1/t)} \prod_{i=1}^{s} \frac{\Gamma(it - s)}{\Gamma(t)} \cdot \prod_{i=0}^{r-1} \frac{\Gamma(s_1 + 1 + (1-r_1 + i)/t) \Gamma(s_2 + (1-r_2 + i)/t)}{\Gamma(s_1 + s_2 + 1 - 2s + (r - r_1 - r_2 + i + 1)/t)} \cdot \prod_{i=0}^{s-1} \frac{\Gamma(r_1 - r + (i - s_1)/t) \Gamma(r_2 - r + (1 - s_2 + i)/t)}{\Gamma(r_1 - r + 2r + (s - s_1 - s_2 + i)t)} 
\]  

(28)

The analysis of this expression in terms of fusion rules is standard [21]. The result may be written as follows

\[
1 + |r_1 - r_2| \leq r_3 \leq p - 1 - |r_1 + r_2 - p| \]

\[
|s_1 - s_2| \leq s_3 \leq q - 1 - |s_1 + s_2 - q + 1| 
\]  

(29)

The first line of these fusion rules is well known for the case, \( q = 1 \), of integrable representations, and it was obtained in the general case in [5]. The second was
obtained by Awata and Yamada [22] by considering the conditions for decoupling of null-states, and by Feigin and Malikov [23] by cohomological methods. In addition these authors provide a fusion rule ((II) for [22], (I) for [23]), which we do not get in the free field realization. We do not know if there exist conformal field theories with non-vanishing couplings respecting those.

4. N point functions

We wish to evaluate the conformal block

\[ W_N = \langle j_N | [\phi_{j_{N-1}}(z_{N-1}, x_{N-1})]^{j_N}_{i_{N-2}} \cdots [\phi_{j_n}(z_n, x_n)]^{i_n}_{i_{n-1}} \cdots [\phi_{j_2}(z_2, x_2)]^{i_2}_{i_1} | j_1 \rangle \]  

(30)

From the pictorial version

\[ j_N \quad i_{N-2} \quad \ldots \quad j_n \quad i_n \quad \ldots \quad j_3 \quad i_3 \quad j_2 \quad i_2 \quad j_1 \]  

(31)

one reads off the following screening conditions

\[ j_1 + j_2 - i_2 = \rho_2 - \sigma_2 t \]
\[ i_2 + j_3 - i_3 = \rho_3 - \sigma_3 t \]
\[ \vdots \]
\[ i_{n-1} + j_n - i_n = \rho_n - \sigma_n t \]
\[ \vdots \]
\[ i_{N-2} + j_{N-1} - j_N = \rho_{N-1} - \sigma_{N-1} t \]
\[ 2j_i + 1 = r_i - s_i t \]  

(32)

with \( \sigma_n, \rho_n \) non-negative integers, while the last line is the usual parametrization of the weights. Following the procedure outlined in the previous section one may calculate the ghost field contribution to the correlator, while the \( \varphi \) part is a matter of standard computation. Let us summarize our findings in a compact notation

\[ M = \sum_{m=2}^{N-1} (\rho_m + \sigma_m) \]
\[ w_i = 1, \ldots, M \]  

(33)

\( w_i \) collectively denote the positions of all screening charges. Furthermore we introduce

\[ k_i = \begin{cases} -1 & i = 1, \ldots, \sum_{m} \rho_{m} \\ t & i = \sum_{m} \rho_{m} + 1, \ldots, M \end{cases} \]
\[ B(w_i) = \sum_{\ell=1}^{N-1} \frac{x_{\ell}}{w_i - z_{\ell}} \]

(34)
(here $x_1 = 0$). Finally we may write down an integral representation of the $N$ point conformal block

$$W_N = \oint \prod_{i=1}^{M} \frac{dw_i}{2\pi i} \oint \prod_{m=2}^{N-1} \frac{du_m}{2\pi i} W_N^{\beta \gamma} W_N^\varphi$$ \hfill (35)

where

$$W_N^{\beta \gamma} = \prod_{i=1}^{M} B(w_i)^{-k_i} \prod_{m=2}^{N-1} \Gamma(2j_m + 1) u_m^{2j_m-1} e^{w_m}$$

$$W_N^\varphi = \prod_{m<n} (z_m - z_n)^{2j_m j_n/t} \prod_{i=1}^{M} \prod_{m=1}^{N-1} (w_i - z_m)^{2k_i j_m/t} \prod_{i<j<M} (w_i - w_j)^{2k_i d/t}$$ \hfill (36)

This is the main result.

5. The Knizhnik-Zamolodchikov equations

In ref. [15] several non-trivial consistency checks are provided. Except for the case of the Knizhnik-Zamolodchikov (KZ) equations we refer to this paper for a discussion of these, which include projective invariance and a verification of the equivalence of the following two correlators:

$$W_N^{(I)}(z_N = \infty, x_N = \infty, z_{N-1}, x_{N-1}, ..., z_2, x_2, z_1 = 0, x_1 = 0) = \langle j_N | \phi_{j_{N-1}}(z_{N-1}, x_{N-1}) j_{N-2} \cdots \phi_{j_2}(z_2, x_2) j_1 | j_1 \rangle$$ \hfill (37)

and

$$W_N^{(II)}(z_N, x_N, z_{N-1}, x_{N-1}, ..., z_2, x_2, z_1, x_1) = \langle 0 | \phi_{j_N}(z_N, x_N) j_N \cdots \phi_{j_{N-1}}(z_{N-1}, x_{N-1}) j_{N-2} \cdots \phi_{j_2}(z_2, x_2) j_1 \phi_{j_1}(z_1, x_1) j_0 | 0 \rangle$$ \hfill (38)

The non-triviality of this check stems from the fact that the second correlator involves more screening charges around the last field than the first one.

In the formalism using the $x$ parameters, the KZ equations (which expresses the decoupling of singular vectors) are the differential equations

$$\{t \partial_{z_{m_0}} + \sum_{m \neq m_0} \frac{2D^a_{x_{m_0}} D^a_{x_m}}{z_m - z_{m_0}} \} W_N = 0$$ \hfill (39)

where $z_{m_0}$ refers to the position of a selected field in the correlator. What we want to check is that our final expression for the $N$ point function (35) fulfils the KZ equations without using any formal properties, like associativity, of the algebra. This is then merely a consistency check of our formalism. Let us define the "God given" function

$$G(w) = \frac{1}{w - z_{m_0}} \{D^+_{x_{m_0}} G^+(w) + 2D^3_{x_{m_0}} G^3(w) + D^-_{x_{m_0}} G^-(w) \}$$ \hfill (40)
where the defining functions are (using $D_B \equiv \frac{\partial}{\partial B(w_i)}$)

$$G^+(w) = W_\beta^\gamma W_N^\varphi \left( B(w) \sum_{i=1}^M \frac{D_{B_i}}{w - w_i} + \sum_{i=1}^M \frac{k_i}{w - w_i} + \sum_{m=1}^{N-1} \frac{j_m}{w - z_m} \right) W_\beta^\gamma W_N^\varphi$$

$$G^3(w) = \left( -\sum_{i,j} B(w) \frac{D_{B_i} D_{B_j}}{(w - w_i)(w - w_j)} + (t - 2) \sum_i \frac{D_{B_i}}{(w - w_i)^2} \right) W_\beta^\gamma W_N^\varphi$$

$$G^-(w) = \left( -\sum_{i,j} B(w) \frac{D_{B_i} D_{B_j}}{(w - w_i)(w - w_j)} + (t - 2) \sum_i \frac{D_{B_i}}{(w - w_i)^2} \right) W_\beta^\gamma W_N^\varphi$$

(41)

(this is a slight abuse of notation since the right hand sides are supposed to be integrated as in (35)). On the left hand side one can show that $G(w)$ behaves like $\mathcal{O}(w^{-2})$, which means that the total sum of pole residues must vanish. On the right hand side one may carry out explicitly the computation of the pole residues, and one then finds that the vanishing condition for the sum of these is exactly the KZ equations. One might wonder why it is precisely this function (40) which apply for such an argumentation. The answer is quite simple, because the expressions

$$G^\alpha(w) = \langle J^\alpha(w) \mathcal{O} \rangle \quad (42)$$

(where $\mathcal{O}$ is the collection of free field realizations of all the chiral vertex operators and screening charges) appear in the usual proof of the KZ equations, and it is indeed this way we have found (41).

6. Outlook

We believe that the techniques developed and results obtained in the case of $SL(2)$ may be generalized to higher (super-)groups. Indeed for $SL(n)$ and other simple groups we have managed to perform many of the generalizations, and in the case of $SL(3)$ we will hopefully soon be able to write down integral expressions for at least three point functions and determine the fusion rules [24]. Then it should be possible also to generalize the discussion of Hamiltonian reduction in our recent paper [25] to $SL(3)$ and thereby shed some light on $W_3$ gravity. Generalizations to even higher (super-)groups might open up for a treatment of more general non-critical string theories.

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