Numerical modelling for elastic wave equations including rotational deformation and strain gradient

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Abstract. In this work, through introducing the first and second derivatives of strain gradient into the strain energy density function that depends only on the classical strain in conventional continuum mechanics theory to describe the microscopic interactions, the elastic wave equations including rotational deformation and strain gradient based on the Aifantis’s strain gradient theory is derived. The elastic wave equations including rotational deformation and strain gradient with consideration of the scale effects caused by the microscopic interactions will make seismograms appear obvious changes, which are reflected in amplitude attenuation and dispersion, especially P-wave and S-wave propagate in a dispersive manner based on this theory. The results of numerical modelling and theoretical analysis verify this conclusion. The closer the wavelength is to the characteristic scale of the media (lattice size and interparticle distance), the macroscopic response which results from the microscopic interactions becomes more prominent. To further study the effects of microstructure interactions, we compare the results of numerical modelling with the real data when high-speed train passed the piers in Dingxing County in both time and frequency domains, some conclusions are drawn from the comparison.

1. Introduction

While the earth model is getting closer to physical reality, scholars have conducted research on wave propagation theories. Seismology is established based on the conventional continuum mechanics theory, which considers the media as a continuous mass rather than as discrete particles and only contains the classical strain in the strain energy density function for the media. The theoretical basis of the conventional continuum mechanics theory is the Cauchy stress principle, in other words, the studied media have no characteristic scales and internal microstructures, hence, it is not possible to describe the more complex interactions occurring in generalized continua by means of the sole Cauchy stress tensor [6], such as the microstructural interactions. In fact, the earth media are extremely complex, or rather, the earth media are heterogeneous, imperfect elastic, anisotropic [4], as a result, the earth media have a complicated internal microstructure. In order to build the connection between the microstructure and the conventional continuum mechanics theory, the generalized continuum mechanics theory [2,3,5,7,8,9,10,11,12,14] have been proposed and developed as stretch of conventional continuum mechanics theory, and all the theories and approaches of generalized continuum mechanics theory are based on the couple stress theory (the Cosserat continuum theory) proposed by Cosserat brother [3].

Early in 1909, Cosserat brother [3] proposed the Cosserat continuum theory, in which consider the effects of microstructure interactions by considering each point of media as rigid body with six degrees
of freedom. Toupin [13] established constitutive relation based on the Cosserat’s theory and introduced the gradient of the rotation vector into the strain energy density function. In 1965, Mindlin [10] established ‘the Second Strain Gradient theory’ (SSG theory), in which the strain energy density is not only determined by the strain, but also depend on the second and third derivatives of the displacement. Aifantis [1] proposed a strain gradient theory including only one media-related parameter, which ignores the effect of some strain gradients and can be considered as a simplification or approximation of SSG theory. Because of the existence of microstructure interactions described by the high order terms, some new wave propagation phenomena have emerged. Suikeret al. [12] thought body waves propagate in a dispersive manner based on the generalized continuum mechanics theory. It’s worth noting that the closer the wavelength is to the characteristic scale of the media (the lattice size, the interparticle distance), the macroscopic wave field response result from the microscopic interactions becomes more prominent. Nowadays, the sensitivity of seismometers has been notably improved, hence, the macroscopic changes caused by microstructure interactions can be completely observed.

In this paper, through introducing the first and second derivatives of strain gradient into the strain energy density function that depend only on the classical strain in conventional continuum mechanics theory to describe the microscopic interactions, we derive the elastic wave equations including rotational deformation and strain gradient based on the Aifantis’s strain gradient theory which contains only one media-related parameter. Theoretical analysis shows that due to the existence of microscopic interactions, the P-wave and S-wave propagate in a dispersive manner if using the high order terms usually accompanied with additional media-related parameters (or higher order constants) in describing scale effects caused by the microscopic interactions, and the subsequent results of numerical modelling for homogeneous isotropic model verify this theory. To further study the effects of microstructure interactions, we compare the results of numerical modelling using the conventional elastic wave equations and the elastic wave equations including rotational deformation and strain gradient with the real data when high-speed train passed the piers in Dingxing county in both time and frequency domains, we notice that the existence of scale effects will make the response attenuation more prominent and lead the energy of excited seismic wave field to move to low frequency range, which is more similar to the spectra characteristics of the real data.

2. Theory

In conventional continuum mechanics theory, the strain energy density for the media is only determined by the strain and the media is modeled to be a continuous mass rather than as discrete particles [15], as a result, it’s difficulty in describing the more complex interactions between particles. In 1965, Mindlin [10] established ‘the Second Strain Gradient theory’ (SSG theory), in which the strain energy density is not only determined by the strain, but also depend on the second and third derivatives of the displacement. SSG theory is considered as one of the most general gradient elasticity theories. Aifantis [1] proposed a strain gradient theory including only one media-related parameter, which ignores the effect of some strain gradients and can be considered as a simplification or approximation of Mindlin’s theory.

The linear elasticity theory establishes a relationship between the displacements and the infinitesimal strain tensors $\varepsilon$:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$  \hspace{1cm} (1)

where, are the symmetric strain tensors under small deformation assumption, and are the spatial derivative of the displacement components.

The symmetric strain tensors are related to the symmetric stress tensors through constitutive relation as seen in equation (2).

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$  \hspace{1cm} (2)

where, are fourth-order stiffness tensors.

According to the Aifantis’s strain gradient theory, the constitutive relation of the strain gradient
theory contains only one media-related parameter, which can reflect the microstructure interactions. The constitutive relation can be expressed as follow:

\[
\sigma_{ij} = \lambda \delta_{ij} e_m + 2\mu e_{ij} - c \nabla^2 \left( \lambda \delta_{ij} e_m + 2\mu e_{ij} \right) \tag{3}
\]

where, and are Lame constants, are Kronecker delta, is the characteristic length of media, and the strain energy density is written as:

\[
W = \frac{1}{2} \lambda e_{ij} e_{ij} + \mu e_{ij} e_{ij} + c \left( \frac{1}{2} \lambda \eta_{ab} \eta_{ab} + \mu \eta_{ab} \eta_{ab} \right) \tag{4}
\]

where, are the strain gradient tensors, which is defined as.

The equation of motion ignoring body force is given as equation (5):

\[
\rho \ddot{u}_i = \sigma_{ij} \tag{5}
\]

where, is the material density, and are the second-order temporal derivative of the displacement components.

Substituting equation (1) and equation (3) into equation (5), we can obtain elastic wave equations including rotational deformation and strain gradient for isotropic media as below:

\[
(\lambda + \mu) u_{,i} + \mu u_{,ij,j} - c \left[ (\lambda + \mu) u_{,ij} + \mu u_{,ij} \right]_{,k,k} = \rho \ddot{u}_i \tag{6}
\]

Using equation (6), we can obtain the equations of P-wave and S-wave in the isotropic media based on the Aifantis’s strain gradient theory, and P-wave and S-wave propagate in a dispersive manner based on this theory.

3. Numerical modelling

3.1. Homogeneous Isotropic Media

We first perform numerical modelling for homogeneous isotropic media. The horizontal and vertical samplings of the media are \( n_x = n_z = 401 \), and grid intervals are \( dx = dz = 8 \) m. The P-wave velocity and S-wave velocity are 2.0 km/s and 1.1547 km/s, respectively, and the density is 2600 \( \text{km/m}^3 \). A Ricker wavelet with the dominant frequency of 25 Hz is placed in the center of the media. The time step is 0.001 s. Figure 1 shows the relative position relationships between shots and receivers. In order to better compare the differences of seismograms generated by the different elastic wave equations, the subsequent comparisons are performed under the premise of suppressing the numerical dispersion in advance and will not be explained later.

![Figure 1. The relative position relationships between shots and receivers.](image)

As shown in Figure 2 and 3, the shot records using different elastic equations at receiver1 and receiver2 are given. Different from the results of the conventional elastic equations, the results of the elastic wave equations including rotational deformation and strain gradient show obvious different components, whether in P-waves or in S-waves. Combining theoretical derivation and numerical modelling, we think these different components are related to physical dispersion caused by the rotational deformation and strain gradient.
3.2. Bridge Pier Model

In this section, we perform numerical modelling for a simplified bridge pier model and compare the results of modelling with the real data when high-speed train passed the piers in Dingxing county. The model size is set to 400*100. The grid spacing is $dx = 4 \text{ m}$, $dz = 1 \text{ m}$. The velocity model is divided into low velocity layer and high velocity layer, the first layer is low-speed layer with the thickness of 10 m which corresponds to the soil, and the density, P-wave velocity are respectively 400 kg/m$^3$ and 0.5 km/s. The second layer is high-speed layer with the thickness of 90 m which corresponds to the rock, and the density, P-wave velocity are 1400 kg/m$^3$ and 1.6 km/s, respectively. The S-wave velocity is a scaled version of the P-wave velocity with $v_s/v_p = 1.7$.

We consider a simplified model with 5 piers which are inserted into the ground at a depth of 50 m reach the bedrock, and the distance between the two piers is 28 m. We set a point source every 10 m on each bridge pier. The P-wave velocity of each pier is 4 km/s. The dominant frequency of the Ricker wavelet is 20 Hz. The time step is 0.005 s. A simplified bridge pier model of the source based on high-speed trains as seen in Figure 4.

![Figure 4. A simplified model of the source based on high-speed trains.](image-url)
Figure 5. Records using different elastic equations. (Z component). (a) Records using the conventional elastic wave equations; (b) Records using elastic wave equations including rotational deformation and strain gradient; (c) Real data recorded when high-speed train passed the piers; (d), (e), (f) correspond to (a), (b), (c) are the comparison in frequency domain.

Figure 6. Records with different methods (Z component). (a) the comparison in time domain; (b) the comparison in frequency domain.

Figure 5(a) and 5(b) are the results of numerical modeling using the conventional elastic wave equations and the elastic wave equations including rotational deformation and strain gradient respectively. Figure 5(c) is the record when high-speed train passed the piers in Dingxing county. Figure 5(d), (e), (f) correspond to Figure 5(a), (b), (c) are the comparison in frequency domain.

Figure 5 shows that the results of the numerical modeling and the real data are very similar whether in the time domain or the frequency domain. And when the high-speed rail has passed the piers at a speed of 300 km/h, the excited seismic wave field received at the piers is mainly dominated by low-frequency around 10 Hz. The results of numerical modeling are similar to the real data, which proves that the bridge pier model we build basically accords with the physical facts.

In order to further study this theory, we compare the results of numerical modelling using the conventional elastic wave equations and the elastic wave equations including rotational deformation and strain gradient in both time and frequency domains. As seen in Figure 6(a) and 6(b), the comparisons of the results of numerical modelling using two different elastic wave equations in both time and frequency domains are given. We can note that adding some media-related parameters (or higher order constants) into elastic wave equations will cause the differences of seismic wave field, and these differences are related to the scale effects which results from the microstructure interactions. Due to microstructure interactions, the response attenuation is more prominent, at the same time, the energy of excited seismic wave field moves to low frequency range, which is more similar to the spectra characteristics of the real data.
4. Conclusions

The elastic wave equations including rotational deformation and strain gradient based on the Aifantis’s strain gradient theory with consideration of the scale effects caused by the microscopic interactions will make seismograms appear obvious changes, which are reflected in amplitude attenuation and dispersion. We introduce the first and second derivatives of strain gradient into the strain energy density function that depend only on the classical strain in conventional continuum mechanics theory to describe the microscopic interactions. The numerical modelling and theoretical analysis show that P-wave and S-wave propagate in a dispersive manner if using the high order terms usually accompanied with additional media-related parameters (or higher order constants) in describing scale effects caused by the microscopic interactions, at the same time, wave attenuation is enhanced.

We believe that the closer the wavelength is to the characteristic scale of the media (the lattice size, the interparticle distance), the macroscopic response which results from the microscopic interactions becomes more prominent. To further verify our work, we compare the results of numerical modelling using the conventional elastic wave equations and the elastic wave equations including rotational deformation and strain gradient with the real data when high-speed train passed the piers in Dingxing county in both time and frequency domains, and we have some conclusions: Because of the existence of microstructure interactions, introducing the first and second derivatives of strain gradient to make the response attenuation more prominent, meanwhile, we notice that the existence of scale effects will lead the energy of excited seismic wave field to move to low frequency range, which is more similar to the spectra characteristics of the real data.

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