Noncommutative instantons come in families

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Abstract. A construction of instantons in the context of noncommutative geometry, in particular SU(2) instantons on a noncommutative 4 sphere, has been recently reported. Firstly, a noncommutative principal fibration \( \mathcal{A}(S^2_0) \hookrightarrow \mathcal{A}(S^2_0) \) which ‘quantizes’ the classical SU(2)-Hopf fibration over \( S^4 \), has been constructed in [11] on the toric noncommutative four-sphere \( S^4_0 \). The generators of \( \mathcal{A}(S^2_0) \) are the entries of a projection \( p \) which describes the basic instanton on \( \mathcal{A}(S^2_0) \). That is, \( p \) gives a projective module of finite type \( p[\mathcal{A}(S^2_0)]^4 \) and a connection \( \nabla = p \circ d \) on it which has a self-dual curvature and charge 1, in some appropriate sense; this is the basic instanton. In [12] infinitesimal instantons – ‘the tangent space to the moduli space’ – were constructed using infinitesimal conformal transformations, that is elements in a quantized enveloping algebra \( U(\mathfrak{so}(5,1)) \). In [10] we looked at a global construction and obtain generic charge 1 instantons by ‘quantizing’ the action of the Lie groups SL(2,\( \mathbb{R} \)) and SO(2) on the basic instanton which enter the classical construction [1]. We review all this here.

1. Toric noncommutative manifolds

Toric noncommutative manifolds – introduced in [8] (with further elaborations in [7]) and called isospectral deformations – are deformations of classical Riemannian manifolds \( M \) satisfying all the properties of noncommutative spin geometries. The main idea consists in deforming the spectral triple of the Riemannian geometry of \( M \) isospectral deformations – are deformations of classical Riemannian manifolds \( M \) satisfying all the properties of noncommutative spin geometries. The main idea consists in deforming the spectral triple of the Riemannian geometry of \( M \) along a torus embedded in the isometry group.

Let \( (M, g) \) be an \( m \)-dimensional closed Riemannian spin manifold endowed with an isometric smooth action \( \sigma \) of an \( n \)-torus \( \mathbb{T}^n \), \( n \geq 2 \). There is a corresponding action, \( \sigma \) say, of \( \mathbb{T}^n \) by automorphisms on the algebra \( C^\infty(M) \) of smooth functions on \( M \). The latter algebra is decomposed [15] into spectral subspaces indexed by the dual group \( \mathbb{Z}^n = \mathbb{T}^n \). Given \( s = (s_1, \ldots, s_n) \in \mathbb{T}^n \), each \( r \in \mathbb{Z}^n \) labels a character \( e^{2\pi i s} \mapsto e^{2\pi i r \cdot s} \) of \( \mathbb{T}^n \), where \( r \cdot s := \sum_{j=1}^n r_j s_j \). The \( r \)-th spectral subspace for the action \( \sigma \) is made of those smooth functions \( f_r \) for which

\[
\sigma_s(f_r) = e^{2\pi i r \cdot s} f_r,
\]  

and each \( f \in C^\infty(M) \) is the sum of a unique rapidly convergent series \( f = \sum_r f_r \). With \( \theta = (\theta_{jk}) \) a real antisymmetric \( n \times n \) matrix, the \( \theta \)-deformation of \( C^\infty(M) \) is obtained by replacing the usual product by a deformed one. On spectral subspaces:

\[
f_r \times_\theta g_{r'} := f_r \sigma_{1 - r \cdot \theta} (g_{r'}) = e^{\pi i r \cdot \theta \cdot r'} f_r g_{r'},
\]  

extended linearly to the whole of \( C^\infty(M) \). We denote \( C^\infty(M_\theta) := (C^\infty(M), \times_\theta) \) which still carries an action \( \sigma \) of \( \mathbb{T}^n \) given by (1) on the homogeneous elements.
Next, let $\mathcal{H} := L^2(M, S)$ be the Hilbert space of spinors and $\slashed{D}$ the Dirac operator of the metric $g$ of $M$. Smooth functions act on spinors by pointwise multiplication, thus giving a representation $\pi : C^\infty(M) \to \mathcal{B}(\mathcal{H})$, into bounded operators on $\mathcal{H}$. There is a double cover $c : \mathbb{T}^n \to \mathbb{T}^n$ and a representation of $\mathbb{T}^n$ on $\mathcal{H}$ by unitary operators $U(s), s \in \mathbb{T}^n$, so that

$$U(s)\slashed{D}U(s)^{-1} = \slashed{D},$$

since the torus action is assumed to be isometric, and such that for all $f \in C^\infty(M)$,

$$U(s)\pi(f)U(s)^{-1} = \pi(\sigma_{c(s)}(f)).$$

Much as it was done for smooth functions, the torus action is used for a spectral decomposition of smooth elements of $\mathcal{B}(\mathcal{H})$ — operators for which the map $\mathbb{T}^n \ni s \mapsto \alpha_s(T) := U(s)TU(s)^{-1}$, is smooth for the norm topology. Any such a $T$ is decomposed as a (rapidly convergent) series $T = \sum T_r$ with $r \in \mathbb{Z}$ and each $T_r$ is homogeneous of degree $r$ under the action of $\mathbb{T}^n$, i.e.

$$\alpha_s(T_r) = e^{2\pi ir \cdot s}T_r, \quad \forall \ s \in \mathbb{T}^n. \quad (3)$$

Let $P = (P_1, P_2, \ldots, P_n)$ be the infinitesimal generators of the action of $\mathbb{T}^n$ so that we can write $U(s) = \exp 2\pi is \cdot P$. Now, with $\theta$ a real $n \times n$ anti-symmetric matrix as above, one defines a twisted representation of the smooth elements $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ by

$$L_{\theta}(T) := \sum_r T_r U(\frac{1}{2}r \cdot \theta) = \sum_r T_r \exp \left\{ \pi ir_j \theta_{jk} P_k \right\},$$

The twist $L_{\theta}$ commutes with the action $\alpha_s$ of $\mathbb{T}^n$ and preserves the spectral components of smooth operators. In particular, taking smooth functions on $M$ as elements of $\mathcal{B}(\mathcal{H})$, via the representation $\pi$, the (3) yields an algebra $L_{\theta}(C^\infty(M))$ which we think of as a representation on $\mathcal{H}$ (as bounded operators) of the algebra $C^\infty(M_\theta)$. By the definition of the product $\times_{\theta}$ in (2),

$$L_{\theta}(f \times_{\theta} g) = L_{\theta}(f)L_{\theta}(g),$$

proving that the algebra $C^\infty(M)$ equipped with the product $\times_{\theta}$ is isomorphic to the algebra $L_{\theta}(C^\infty(M))$. Thus, we can think of $L_{\theta}$ as a quantization map

$$L_{\theta} : C^\infty(M) \to C^\infty(M_\theta). \quad (4)$$

It plays a key role, allowing us to extend differential geometric techniques from $M$ to the noncommutative space $M_\theta$. It was shown in [8] that the datum $(L_{\theta}(C^\infty(M)), \mathcal{H}, D)$, with the operator $D = \slashed{D}$ just the ‘undeformed Dirac operator’ on the ‘undeformed Hilbert space of spinors’ $\mathcal{H}$, satisfies all the requirements for a noncommutative spin geometry [6]. Since $\mathbb{T}^n$ acts by isometries, each generator $P_k$ commutes with $D$ and the latter is of degree 0. This yields boundedness of the commutators $[D, L_{\theta}(f)]$ for $f \in C^\infty(M)$, being then $[D, L_{\theta}(f)] = L_{\theta}([D, f])$. There is also a grading $\gamma$ (for the even case) and a real structure $J$ obtained by ‘twisting’ the undeformed one. The noncommutative geometry of $M_\theta$ is an isospectral deformation of the classical geometry of $M$ and all spectral properties — in particular $m^+$-summability — are unchanged. Thus there is a noncommutative integral as a Dixmier trace [5]:

$$\int L_{\theta}(f) := \text{Tr}_\omega(L_{\theta}(f)|D|^{-m}), \quad f \in C^\infty(M_\theta).$$

A differential calculus on $M_\theta$ can be given in two equivalent ways, either by using the general construction in [5] by means of the Dirac operator or by extending to forms the quantization
maps. Let \( \Omega(M, d) \) be the usual differential calculus on \( M \), with \( d \) the exterior derivative. The quantization map \( L_\theta \) in (4) is extended to \( \Omega(M) \) by imposing that it commutes with \( d \). The image \( L_\theta(\Omega(M)) \) will be denoted \( \Omega(M_\theta) \). Then \( \Omega(M_\theta) \) is defined to be \( \Omega(M) \) as a vector space but equipped with an ‘exterior deformed product’ which is the extension of the product (2) to \( \Omega(M) \) by the requirement that it commutes with \( d \). Indeed, since the action of \( \mathbb{T}^n \) commutes with \( d \), an element in \( \Omega(M) \) can be decomposed into a sum of homogeneous elements for the action of \( \mathbb{T}^n \) – as was done for \( C^\infty(M) \). Then one defines a star product \( \times_\theta \) on homogeneous elements in \( \Omega(M) \) as in (2) and denotes \( \Omega(M_\theta) = (\Omega(M), \times_\theta) \). The extended action of \( \mathbb{T}^n \) from \( C^\infty(M) \) to \( \Omega(M) \) is used to endow the space \( \Omega(M_\theta) \) with the structure of a \( C^\infty(M_\theta) \)-bimodule.

Classically, the Hodge star operator \( * : \Omega^p(M) \to \Omega^{m-p}(M) \) depends only on the conformal class of the metric on \( M \). Acting \( \mathbb{T}^n \) by isometries, it leaves the conformal structure invariant and it commutes with \( * \). Also, with isospectral deformations the metric is not changed. Thus on \( \Omega(M_\theta) \) it makes sense to define a Hodge star map \( *_\theta : \Omega^p(M_\theta) \to \Omega^{m-p}(M_\theta) \) by

\[
*_\theta L_\theta(\omega) = L_\theta(*\omega), \quad \text{for } L_\theta(\omega) \in \Omega(M_\theta).
\]

With this Hodge operator, there is a definition of an inner product on \( \Omega(M_\theta) \). Given that \( *_\theta \) maps \( \Omega^p(M_\theta) \) to \( \Omega^{m-p}(M_\theta) \), for \( \alpha, \beta \in \Omega(M_\theta) \) we can define their inner product as

\[
(\alpha, \beta)_2 = \int *_\theta(\alpha^* *_\theta \beta),
\]

since \( *_\theta(\alpha^* *_\theta \beta) \in C^\infty(M_\theta) \). Forms of different degree are declared to be orthogonal.

2. The principal fibration on \( S^4_\theta \)

A noncommutative principal fibration \( \mathcal{A}(S^4_\theta) \to \mathcal{A}(S^7_\theta) \) was introduced in [11] and infinitesimal instantons on it were constructed in [12] using infinitesimal conformal transformations; a global version is in [10] with the construction of a noncommutative family of instantons. We refer to these papers for a detailed description of the inclusion \( \mathcal{A}(S^4_\theta) \to \mathcal{A}(S^7_\theta) \) as a noncommutative principal fibration (with classical SU(2) as structure group) and for its use for noncommutative instantons. Here we limit ourself to a brief description. The coordinate algebra \( \mathcal{A}(S^7_\theta) \) on the sphere \( S^7_\theta \) is the \( * \)-algebra generated by elements \( \{z_j, z_j^* : j = 1, \ldots, 4\} \) with relations,

\[
z_j z_k = \lambda_{jk} z_k z_j ; \quad z_j^* z_k = \lambda_{kj} z_k z_j^* ; \quad z_j^* z_k^* = \lambda_{jk} z_k^* z_j^* ;
\]

and spherical relation \( \sum z_j^* z_j = 1 \). The deformation matrix \( \{\lambda_{jk}\} \) is taken so to allow an action by automorphisms of the undeformed group SU(2) on \( \mathcal{A}(S^7_\theta) \) and so that the subalgebra of invariants under this action is identified with the coordinate algebra \( \mathcal{A}(S^4_\theta) \) of a sphere \( S^4_\theta \). With deformation parameter \( \lambda = e^{2\pi i \theta} \), the \( * \)-algebra \( \mathcal{A}(S^4_\theta) \) is generated by a central element \( x \) and elements \( \alpha, \beta, \alpha^*, \beta^* \) with the spherical relation \( \alpha^* \alpha + \beta^* \beta + x^2 = 1 \) and with commutation relations:

\[
\alpha \beta = \lambda \beta \alpha, \quad \alpha^* \beta^* = \lambda \beta^* \alpha^*, \quad \beta^* \alpha = \lambda \alpha \beta^*, \quad \beta \alpha^* = \lambda \alpha^* \beta,
\]

All this (including the relation between the deformation parameter for \( S^7_\theta \) and \( S^4_\theta \)) is mostly easily seen by taking the generators of \( \mathcal{A}(S^4_\theta) \) as the entries of a projection which determines an ‘instanton bundle’ over \( S^7_\theta \). Consider the matrix-valued function on \( S^7_\theta \) given by

\[
u = (|\psi_1\rangle, |\psi_2\rangle) = \begin{pmatrix} z_1 & -z_2^* & z_3 & -z_4^* \\ z_2 & z_1^* & z_4 & -z_3^* \end{pmatrix}^t,
\]

where \( ^t \) denotes matrix transposition, and \( |\psi_1\rangle, |\psi_2\rangle \) are elements in the right \( \mathcal{A}(S^7_\theta) \)-module \( \mathbb{C}^4 \otimes \mathcal{A}(S^7_\theta) \). They are orthonormal with respect to the \( \mathcal{A}(S^7_\theta) \)-valued Hermitian structure \( \langle \xi, \eta \rangle = \sum \xi_j^* \eta_j \) and as a consequence, \( u^* u = I_2 \). Hence the matrix

\[
p = uu^* = |\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|
\]
is a self-adjoint idempotent with entries in $A(S^4_0)$; we have explicitly:

$$p = \frac{1}{2} \begin{pmatrix} 1 + x & 0 & \alpha & \beta \\ 0 & 1 + x & -\mu \beta^* & \bar{\mu} \alpha^* \\ \alpha^* & -\bar{\mu} \beta & 1 - x & 0 \\ \beta^* & \mu \alpha & 0 & 1 - x \end{pmatrix},$$

(10)

with $\mu = \sqrt{\lambda} = e^{\pi i \theta}$. The generators of $A(S^4_0)$ are bilinear in those of $A(S^7_0)$ given by,

$$\alpha = 2(z_1 z_3^* + z_2 z_4^*) , \quad \beta = 2(-z_1 z_4^* + z_2 z_3^*) , \quad x = z_1 z_2^* - z_3 z_4^* - z_4 z_3^* .$$

(11)

The defining relation of the algebra $A(S^7_0)$ can be given on the entries of the matrix $u$ in (8). Writing $u = (u_{ia})$, with $i, j = 1, \ldots, 4$ and $a = 1, 2$, one gets

$$u_{ia} u_{jb} = \eta_{ij} u_{jb} u_{ia} .$$

(12)

with $\eta = (\eta_{ij})$ the matrix

$$\eta = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \mu & \bar{\mu} \\ \mu & \bar{\mu} & 1 & 1 \\ \bar{\mu} & \mu & 1 & 1 \end{pmatrix} .$$

(13)

The deformation matrix $\{\lambda_{ij}\}$ in (6) is just the above $\eta$ with entries rearranged.

Both spheres $S^4_0$ and $S^7_0$ are particular examples of the general construction described in Sect. 1. They carry compatible toric actions: the torus $T^2$ acts on $A(S^4_0)$ as

$$\sigma_s(x, \alpha, \beta) = (x, e^{2\pi i s_1} \alpha, e^{2\pi i s_2} \beta), \quad \text{for} \quad s \in T^2.$$  

(14)

This action lifts to a double cover action on $A(S^7_0)$ with double cover map $p : \tilde{T}^2 \to T^2$ given explicitly by $p : (s_1, s_2) \mapsto (s_1 + s_2, -s_1 + s_2).$ Then $\tilde{T}^2$ acts on the $z_j$’s as

$$\tilde{\sigma} : (z_1, z_2, z_3, z_4) \mapsto \left( e^{2\pi i s_1} z_1, e^{-2\pi i s_1} z_2, e^{-2\pi i s_2} z_3, e^{2\pi i s_2} z_4 \right).$$

(15)

Relations (11) show that $\tilde{\sigma}$ lifts to $S^7_0$ the action of $T^2$ on $S^4_0$. This compatibility is built into the construction of the fibration $S^7_0 \to S^4_0$ as a deformation of the classical fibration $S^7 \to S^4$ with respect to an action of $T^2$, a fact that also reflects the choice of deformation parameters.

3. Yang-Mills theory on $S^4_0$

Let $E = p(C^\infty(S^4_0))^N$ be a vector bundle on $S^4_0$, with $p$ a projection in $M_N(C^\infty(S^4_0))$. A connection (or a covariant derivative) on $E$ is a map from $\nabla : E \otimes \Omega(S^4_0) \to E \otimes \Omega(S^4_0)$ obeying a Leibniz rule $\nabla (\eta \omega) = \nabla (\eta \omega) + \eta \omega$. Its curvature $\nabla^2$, which is $\Omega(S^4_0)$-linear, is determined by $F = \nabla^2 : E \to E \otimes \Omega^2(S^4_0)$, for which there is a Bianchi identity [13]:

$$[\nabla, F] = 0.$$  

(16)

The collection $C(E)$ of compatible connections $\nabla$ on the module $E$ is an affine space modeled on the space $\text{Hom}(E, E \otimes C^\infty(S^4_0), \Omega^1(S^4_0))$. On the other hand, The curvature $F$ belongs $\text{Hom}_{C^\infty(S^4_0)}(E, E \otimes \Omega^2(S^4_0))$ or equivalently, it is an element of degree 2 in $\text{End}_{\Omega(S^4_0)}(E \otimes \Omega(S^4_0))$. Any $T \in \text{End}_{\Omega(S^4_0)}(E \otimes \Omega(S^4_0))$ of degree $k$ can be understood as an element in $pM_N(\Omega^k(S^4_0))p$, since $E \otimes \Omega(S^4_0)$ is a finite projective module over $\Omega(S^4_0)$. 


A trace over internal indices, together with the inner product given in (5), defines an inner product $(\cdot, \cdot)_2$ on $\text{End}_{\Omega(S^4_\rho)}(\mathcal{E} \otimes \Omega(S^4_\rho))$. The Yang-Mills action functional on $C(\mathcal{E})$ is defined by

$$\text{YM}(\nabla) = (F, F)_2 = \int \ast_\theta \text{tr}(F \ast_\theta F),$$

(17)

for any connection $\nabla$ with curvature $F$. The Yang-Mills action functional is gauge invariant, positive and quartic. The Yang-Mills equations (equations for critical points) are obtained from the Yang-Mills action functional by a variational principle. We consider a linear perturbation $\nabla_t = \nabla + t\alpha$ of a connection $\nabla$ on $\mathcal{E}$ by an element $\alpha \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes C^\infty(S^4_\rho) \Omega^1(S^4_\rho))$. The curvature $F_t$ of $\nabla_t$ is readily computed as $F_t = F + t[\nabla, \alpha] + O(t^2)$. If we suppose that $\nabla$ is an extremum of the Yang-Mills action functional, this linear perturbation should not affect the action. In other words, we should require that $\partial/\partial t|_{t=0} \text{YM}(\nabla_t) = 0$. Working out the details, the equations of motion can also be written as the more familiar Yang-Mills equations:

$$[\nabla, \ast_\theta F] = 0.$$  

(18)

Connections with a self-dual or anti-self-dual curvature $\ast_\theta F = \pm F$ are special solutions of the Yang-Mills equations: the latter equations follow from the Bianchi identity $[\nabla, F] = 0$, in eq. 16.

We have also a so-called topological action on $S^4_\rho$. Suppose $\mathcal{E} = \rho(C^\infty(S^4_\rho))^N$ is a vector bundle over $S^4_\rho$. The topological action for $\mathcal{E}$ is the pairing of the class of $p$ in K-theory with the cyclic cohomology of $C^\infty(S^4_\rho)$; it can be equivalently given via the curvature of a connection on the module $\mathcal{E}$. Let $\nabla$ be a connection on $\mathcal{E}$ with curvature $F$. The topological action is,

$$\text{Top}(\mathcal{E}) = (F, \ast_\theta F)_2 = \int \ast_\theta \text{tr}(F^2),$$

(19)

where in the second equality we have used the identity $\ast_\theta \circ \ast_\theta = \text{id}$ on $S^4_\rho$. Not surprisingly $\text{Top}(\mathcal{E})$ does not depend on the choice of a connection on $\mathcal{E}$. The Hodge star operator $\ast_\theta$ splits $\Omega^2(S^4_\rho)$ into a self-dual and anti-self-dual component, $\Omega^2(S^4_\rho) = \Omega^2_+(S^4_\rho) \oplus \Omega^2_-(S^4_\rho)$. This decomposition is orthogonal with respect to the inner product $(\cdot, \cdot)$ and allows one to writes the Yang-Mills action functional as $\text{YM}(\nabla) = (F_+, F_+)_2 + (F_-, F_-)_2$. Comparing this with the topological action, $\text{Top}(\mathcal{E}) = (F_+, F_+)_2 - (F_-, F_-)_2$, we see that $\text{YM}(\nabla) \geq |\text{Top}(\mathcal{E})|$, with equality iff $\ast_\theta F = \pm F$.

Connections which are solutions of these equations are called instantons; they are absolute minima of the Yang-Mills action functional.

### 3.1. The basic instanton

Given the projection $p$ in (10), its image $\mathcal{E} = p[\mathcal{A}(S^4_\rho)]^4$ in $[\mathcal{A}(S^4_\rho)]^4$ is clearly a right $\mathcal{A}(S^4_\rho)$-module. An equivalent description of this module comes from considering ‘equivariant maps’ for the defining representation of $\text{SU}(2)$ on $\mathbb{C}^2$, $\rho : \text{SU}(2) \to \text{Mat}_2(\mathbb{C})$, given by

$$\mathcal{A}(S^4_\rho) \otimes_{\rho} \mathbb{C}^2 := \{ \varphi \in \mathcal{A}(S^4_\rho) \otimes \mathbb{C}^2 : (\text{id} \otimes \pi(g)^{-1})(\varphi) = (\alpha_g \otimes \text{id})(\varphi) \},$$

(20)

This collection is a right $\mathcal{A}(S^4_\rho)$-module (it is in fact a $\mathcal{A}(S^4_\rho)$-bimodule) since multiplication by an element in $\mathcal{A}(S^4_\rho)$ does not affect the equivariance condition (20). Since $\text{SU}(2)$ acts classically on $\mathcal{A}(S^4_\rho)$, the equivariant maps are elements of the form $\varphi := u^* f$ for some $f \in \mathcal{A}(S^4_\rho) \otimes \mathbb{C}^2$ and $u$ is the matrix (8). In terms of the canonical basis $\{e_1, e_2\}$ of $\mathbb{C}^2$, we can write $\varphi := \sum_\alpha \varphi_\alpha \otimes e_\alpha = \sum_\alpha \langle \psi_\alpha| f \rangle \otimes e_\alpha$, for $|f\rangle \in [\mathcal{A}(S^4_\rho)]^4$. We have the isomorphism

$$p[\mathcal{A}(S^4_\rho)]^4 \simeq \mathcal{A}(S^4_\rho) \otimes_{\rho} \mathbb{C}^2,$$

$$\sigma = p|f\rangle \mapsto \varphi = u^* f = \sum_\alpha \langle \psi_\alpha| f \rangle \otimes e_\alpha,$$
with \( p = u^* u \) the projection in (10). On the projective module \( \mathcal{E} = p[\mathcal{A}(S_\theta^4)] \), there is the canonical (Grassmann) connection given by,

\[
\nabla := p \circ d : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}(S_\theta^4)} \Omega^1(S_\theta^4).
\]

(21)

When acting on equivariant maps, we can write \( \nabla \) as

\[
(\nabla \varphi)_a = d \varphi_a + \sum_b \omega_{ab} \varphi_b.
\]

The gauge potential \( \omega = (\omega_{ab}) \) is found to be given in terms of the matrix \( u \) by

\[
(\omega_{ab}) = \frac{1}{2} \sum_k ((u^*)_{ak} du_{kb} - d(u^*)_{ak} u_{kb}).
\]

(22)

One has \( \omega_{ab} = - (\omega^*)_b \) and \( \sum a \omega_{aa} = 0 \) so that \( \omega \in \Omega^1(S_\theta^2) \otimes su(2) \). Note that the entries \( \omega_{ab} \) commute with all elements in \( \mathcal{A}(S_\theta^4) \): from (8) the elements in \( \omega_{ab} \) are \( \mathbb{T}^2 \)-invariant and hence central (as one-forms) in \( \Omega(S_\theta^4) \). In other words \( L_0(\omega) = \omega \), which shows that for an element \( f \in \mathcal{E} \) as above, we have \( \nabla(\varphi)_a = d \varphi_a + \sum_b \omega_{ab} \varphi_b = d \varphi_a + \sum_b \omega_{ab} \varphi_b \) which coincides with the action of the classical connection \( d + \omega \) on \( f \). This connection is an instanton since its curvature \( F = \nabla^2 = d\omega + \omega^2 \) is an element of \( \text{End}(\mathcal{E}) \otimes_{C^\infty(S_\theta^4)} \Omega^2(S_\theta^4) \) – is self-dual: \( \star \theta F = F \).

At the classical value of the deformation parameter, \( \theta = 0 \), the connection (22) is nothing but the SU(2) instanton of [3]. Its ‘topological charge’, i.e. the values of \( \text{Top}(\mathcal{E}) \) in eq. (19), clearly depends only on the class \([p]\) of the bundle and can be evaluated as the index,

\[
\text{Top}([p]) = \text{index}(D_p) = \int \gamma_5 \pi_D(ch_2(p)),
\]

(23)

of the twisted Dirac operator \( D_p = p(D \otimes 1_4) p \). The second equality follows from the vanishing – established by direct computation – of the first component \( ch_1(p) \) of the Chern character of \( p \); hence the second component \( ch_2(p) \) is a Hochschild cycle, i.e. \( b ch_2(p) = 0 \), playing the role of the round volume form on \( S_\theta^4 \). Indeed, with the isospectral geometry \( (C^\infty(S_\theta^4), D, \mathcal{H}, \gamma_5) \), the image \( \pi_D(ch_2(p)) \) as an operator on \( \mathcal{H} \) satisfies the following quartic equation in \( D \),

\[
\pi_D(ch_2(p)) = 3 \gamma_5.
\]

(24)

Then \( \text{Top}([p]) = 1 \), using that \( \int 1 = Tr_c([D]^{-4}) = \frac{1}{3} \) on \( S^4 \). This ‘basic’ noncommutative instanton above has been given a twistor decpiction in [4].

4. Noncommutative conformal transformations

Classically, charge 1 instantons are generated from the basic one by the action of the conformal group \( SL(2, \mathbb{H}) \) of \( S^4 \). Elements of the subgroup \( SO(2) \subset SL(2, \mathbb{H}) \) leave invariant the basic one, hence to get new instantons one needs to quotient \( SL(2, \mathbb{H}) \) by the spin group \( SO(2) \simeq \text{Spin}(5) \). The resulting moduli space of SU(2) instantons on \( S^4 \) modulo gauge transformations is identified (cf. [1]) with the five-dimensional quotient manifold \( SL(2, \mathbb{H})/SO(2) \).

A parallel construction of instantons on \( \mathcal{A}(S_\theta^4) \) is in [10], using a quantum group \( SL_\theta(2, \mathbb{H}) \) and its quantum subgroup \( SO_\theta(2) \). An infinitesimal construction was proposed in [12] where a deformed dual enveloping algebra \( U_\theta(so(5, 1)) \) was used to generate infinitesimal instantons (‘the tangent space to the moduli space’) by acting on the basic instanton described above.

The Hopf algebras \( \mathcal{A}(SL_\theta(2, \mathbb{H})) \) and \( \mathcal{A}(SO_\theta(2)) \) are special case of the quantization of compact Lie groups using Rieffel’s strategy in [16]. Firstly, a deformed (Moyal-type) product \( \times_\theta \) is constructed on the algebra of (continuous) functions \( \mathcal{A}(G) \) on a compact Lie group \( G \), starting
with an action of a closed connected abelian subgroup of $G$ (usually a torus). The algebra $\mathcal{A}(G)$ equipped with the deformed product is denoted by $\mathcal{A}(G_\theta)$. Keeping the classical expression of the coproduct, counit and antipode on $\mathcal{A}(G)$, but now on the algebra $\mathcal{A}(G_\theta)$, the latter becomes a Hopf algebra. It is in duality with a deformation $U_\theta(g)$ of the universal enveloping algebra $U(g)$ of the Lie algebra $g$ of $G$ obtained by leaving unchanged the algebra structure while twisting the coproduct, counit and antipode of $U(g)$. The deformation from $U(g)$ to $U_\theta(g)$ is implemented with a twist of Drinfel’d type [9] – in fact constructed in [14] for the cases at hand.

The enveloping algebra $U_\theta(so(5,1))$ was explicitly constructed in [12] while the dual Hopf algebra $\mathcal{A}(SL_\theta(2,\mathbb{H}))$ and its quotient $\mathcal{A}(SO_\theta(2))$ are given in [10]. We briefly review them here.

4.1. The quantum groups $SL_\theta(2,\mathbb{H})$ and $\mathcal{A}(SO_\theta(2))$

In the present paper, we need not only the Hopf algebra $\mathcal{A}(SL_\theta(2,\mathbb{H}))$ but also its coaction on the principal bundle $\mathcal{A}(S^2_\theta) \hookrightarrow \mathcal{A}(S^4_\theta)$ and in turn on the basic instanton connection (21) on the bundle in order to generate new instantons. We give an explicit construction of $\mathcal{A}(SL_\theta(2,\mathbb{H}))$ out of its action in a way that also shows its quaternionic nature.

We start from the algebra of a two-dimensional deformed quaternionic space $\mathbb{H}_\theta^2$. Let $\mathcal{A}(\mathbb{C}_\theta^4)$ be the $*$-algebra generated by elements $\{z_j, z_j^*: a = 1, \ldots, 4\}$ with the relations as in equation (6) but without the spherical relation that defines $\mathcal{A}(S^2_\theta)$. We take $\mathcal{A}(\mathbb{H}_\theta^2)$ to be the algebra $\mathcal{A}(\mathbb{C}_\theta^4)$ equipped with the antilinear $*$-algebra map $j : \mathcal{A}(\mathbb{C}_\theta^4) \to \mathcal{A}(\mathbb{C}_\theta^4)$ defined on generators by

$$j : (z_1, z_2, z_3, z_4) \mapsto (z_2, -z_1, z_4, -z_3).$$

This deformation of the quaternions is between the two copies of $\mathbb{H}$ while leaving the quaternionic structure within each copy of $\mathbb{H}$ undeformed. Since the second column of the matrix $u$ in (8) is the image through $j$ of the first one, we may think of $u$ as made of two deformed quaternions.

Following a general strategy [18], we now define $\mathcal{A}(M_\theta(2,\mathbb{H}))$ to be the universal bialgebra for which $\mathcal{A}(\mathbb{H}_\theta^2)$ is a comodule $*$-algebra. More precisely, we define a transformation bialgebra of $\mathcal{A}(\mathbb{H}_\theta^2)$ to be a bialgebra $B$ such that there is a $*$-algebra map

$$\Delta_L : \mathcal{A}(\mathbb{C}_\theta^4) \to B \otimes \mathcal{A}(\mathbb{C}_\theta^4),$$

which satisfies

$$(\text{id} \otimes j) \circ \Delta_L = \Delta_L \circ j.$$  \hspace{2cm} (25)

We then set $\mathcal{A}(M_\theta(2,\mathbb{H}))$ to be the universal transformation bialgebra: for any transformation bialgebra $B$ there exists a morphism of transformation bialgebras from $\mathcal{A}(M_\theta(2,\mathbb{H}))$ onto $B$.

Asking that $\mathcal{A}(\mathbb{H}_\theta^2)$ be a $\mathcal{A}(M_\theta(2,\mathbb{H}))$ comodule algebra allows us to derive the commutation relations of the latter. A coaction $\Delta_L$ is given by matrix multiplication,

$$\Delta_L : (z_1, z_2, z_3, z_4) \mapsto A_\theta \hat{\otimes} (z_1, z_2, z_3, z_4),$$

for a $4 \times 4$ matrix $A_\theta = (A_{ij})$. With the notation $j' = j + (-1)^{j+1}$, by imposing (25) we have that $(A_{jk})^* = (-1)^{j+k}A_{j'k'}$. This means that $A_\theta$ has the ‘quaternion form’

$$A_\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with} \quad a = (a_{ij}) = \begin{pmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{pmatrix},$$

and similarly for the remaining parts.

The transformations induced on the generators of $\mathcal{A}(\mathbb{C}_\theta^4)$ reads

$$\begin{align*}
\delta_1 &:= \Delta_L(z_1) = a_1 \otimes z_1 - a_2 \otimes z_2^* + b_1 \otimes z_3 - b_2 \otimes z_4^* \\
\delta_2 &:= \Delta_L(z_2) = a_1 \otimes z_2 + a_2 \otimes z_1^* + b_1 \otimes z_4 + b_2 \otimes z_3^* \\
\delta_3 &:= \Delta_L(z_3) = c_1 \otimes z_1 - c_2 \otimes z_2^* + d_1 \otimes z_3 - d_2 \otimes z_4^* \\
\delta_4 &:= \Delta_L(z_4) = c_1 \otimes z_2 + c_2 \otimes z_1^* + d_1 \otimes z_4 + d_2 \otimes z_3^*.
\end{align*}$$  \hspace{2cm} (28)
with $\Delta_L(z_j^*) = (\Delta_L(z_j))^*$. The condition for $\Delta_L$ to be an algebra map determines the commutation relations among the generators of $\mathcal{A}(M_\theta(2,\mathbb{H}))$: the algebra generated by the $a_{ij}$ is commutative, as well as the algebras generated by the $b_{ij}$, $c_{ij}$ and the $d_{ij}$. However, the whole algebra is not commutative and there are not trivial relations. A straightforward computation allows one to concisely write them as,

$$A_{ij}A_{kl} = \eta_{ki}\eta_{jl}A_{kl}A_{ij},$$

(29)

with $\eta = (\eta_{ki})$ the deformation matrix in (13). Indeed, imposing that the map (26) defines a $*$-algebra map on the generators of $\mathcal{A}(\mathbb{C}_\theta^4)$, and using the relations (12), we have that $\sum_{kl}(A_{ik}A_{jl} - \eta_{ji}\eta_{kl}A_{jl}A_{ik}) \otimes u_{ka}u_{lb} = 0$. Since for $a \leq b$ the elements $u_{ka}u_{lb}$ could be taken to be all independent, relations $A_{ik}A_{jl} - \eta_{ji}\eta_{kl}A_{jl}A_{ik}$ are $0$ hold, for all values of $a, b$.

It is not difficult to see that $\mathcal{A}(M_\theta(2,\mathbb{H}))$ is indeed the universal transformation bialgebra, since the commutation relations (29) and the quaternionic structure of $\mathcal{A}_\theta$ in (27) are derived from the minimal requirement of $\Delta_L$ to be a $*$-algebra map such that (25) holds.

In order to define the quantum group $\text{SL}_\theta(2,\mathbb{H})$ we need a determinant. This is most naturally introduced via the coaction on forms. There is a canonical differential calculus $\Omega(\mathbb{C}_\theta^4)$ generated in degree 1 by elements $\{dz_j, a = 1, \ldots, 4\}$ and relations,

$$z_jdz_k - \lambda_{jk}dz_kz_j = 0, \quad z_jdz^*_k - \lambda_{kj}dz^*_kz_j = 0, \quad z_j^*dz_k - \lambda_{kj}dz_kz^*_j = 0,$$

$$dz_jdz_k + \lambda_{jk}dz_kdz_j = 0, \quad dz_jdz^*_k + \lambda_{kj}dz^*_kdz_j = 0.$$

The forms $\Omega(\mathbb{C}_\theta^4)$ could be obtained from the general procedure mentioned at the end of Sect. 2. The result is also isomorphic to the one obtained for the general construction [3] which uses the Dirac operator to implement the exterior derivative as a commutator.

The coaction $\Delta_L$ is extended to forms by requiring it to commute with $d$. Having the action (26), it is natural to define a determinant element by setting

$$\Delta_L(dz_1dz_2dz_3dz^*_4) =: \det(A_\theta) \otimes dz_1dz_2dz_3dz^*_4.$$

A compact form for $\det(A_\theta)$ is found to be,

$$\det(A_\theta) = \sum_{\sigma \in S_4} (-1)^{[\sigma]} \varepsilon^{\sigma} A_{1,\sigma(1)}A_{2,\sigma(2)}A_{3,\sigma(3)}A_{4,\sigma(4)},$$

with $\varepsilon^{\sigma} = \varepsilon^{(1)(2)(3)(4)}$. The tensor $\varepsilon^{ijkl}$ has components $\varepsilon^{1234} = \varepsilon^{\text{cycl}} = \mu$, $\varepsilon^{1423} = \varepsilon^{\text{cycl}} = \bar{\mu}$, and equal to 1 otherwise. In the limit $\theta \to 0$, the element $\det(A_\theta)$ reduces to the determinant of the matrix $A_\theta = 0$ as it should. The particular form of the deformation matrix $\eta_{ij}$ defining the relations in $\mathcal{A}(S_\theta^4)$ implies that $\det(A_\theta)$ is (not surprisingly) a central element in the algebra $\mathcal{A}(M_\theta(2,\mathbb{H}))$ generated by the entries of $A_\theta$. Hence we can take the quotient of this algebra by the two-sided ideal generated by $\det(A_\theta) - 1$, which we will denote by $\mathcal{A}(\text{SL}_\theta(2,\mathbb{H}))$. The image of the elements $A_{ij}$ in the quotient algebra will again be denoted by $A_{ij}$.

On the algebra $\mathcal{A}(\text{SL}_\theta(2,\mathbb{H}))$ there are additional structures which turn it into a Hopf algebra ($\mathcal{A}(\text{SL}_\theta(2,\mathbb{H})), \Delta, \varepsilon, S$). The coaction $\Delta_L$ in (26) passes to a coaction of $\mathcal{A}(\text{SL}_\theta(2,\mathbb{H}))$ on $\mathcal{A}(\mathbb{H}_\theta^2)$ and it is still a $*$-algebra map. However, the spherical relation $\sum_j z_j^2z_j = 1$ is no longer invariant under $\Delta_L$. Thus, the algebra $\mathcal{A}(S_\theta^2)$ is not an $\mathcal{A}(\text{SL}_\theta(2,\mathbb{H}))$-comodule algebra but only a $\mathcal{A}(\text{SL}_\theta(2,\mathbb{H}))$-comodule. We shall elaborate more on this in Sect. 4.2 below.

In order to define the symplectic group $\mathcal{A}(\text{SO}_\theta(2))$, consider the two-sided $*$-ideal $I$ in $\mathcal{A}(\text{SL}_\theta(2,\mathbb{H}))$ generated by the elements $\sum_k (A_{ki})^*A_{kj} - \delta_{ij}$ for $i, j = 1, \ldots, 4$. It is easy to
check that \( I \) is a Hopf ideal in \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \). The quotient \( \mathcal{A}(\text{SO}_\theta(2)) := \mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))/I \) is a Hopf algebra with the induced Hopf algebra structures for which we still use the symbols \((\Delta, \epsilon, S)\). The 'defining matrix' \( A_\theta \) of \( \mathcal{A}(\text{SO}_\theta(2)) \) has the form (27) with the additional condition that \( A_\theta^* A_\theta = A_\theta A_\theta^* = 1 \) – equivalent to the statement that \( S(A_\theta) = A_\theta^* \).

4.2. Conformal transformations

There is a natural coaction \( \Delta_L \) of \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \) on the \( SU(2) \) noncommutative principal fibration \( \mathcal{A}(\text{S}^7_\theta) \leftarrow \mathcal{A}(\text{S}^7_\theta) \) of Sect. 2. Since the matrix \( u \) in (8) consists of two deformed quaternions, the left coaction \( \Delta_L \) of \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \) in (28) can be written on \( \mathcal{A}(\text{S}^7_\theta) \) as

\[
\Delta_L : \mathcal{A}(\text{S}^7_\theta) \rightarrow \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(\text{S}^7_\theta), \quad u \mapsto \tilde{u} := \Delta_L(u) = A_\theta \otimes u,
\]

or, in components,

\[
u_{ia} \mapsto \tilde{\nu}_{ia} := \Delta_L(u_{ia}) = \sum_j A_{ij} \otimes u_{ja}.
\]

As already mentioned, the left coaction \( \Delta_L \) of \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \) as in (28) does not leave invariant the spherical relation: \( \Delta_L(\sum_j z_j^* z_j) \neq 1 \otimes 1 \). We will denote by \( \mathcal{A}(\text{S}^7_\theta) \) the image of \( \mathcal{A}(\text{S}^7_\theta) \) under the left coaction of \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \): it is a subalgebra of \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H}) \otimes \mathcal{A}(\text{S}^7_\theta) \). We think of \( \mathcal{A}(\text{S}^7_\theta) \) as a \( \theta \)-deformation of a family of 'inflated' spheres. Since \( \sum_j z_j^* z_j \) is central in \( \mathcal{A}(\text{S}^7_\theta) \) its image

\[
\rho^2 := \Delta_L(\sum_j z_j^* z_j),
\]

is a central element in \( \mathcal{A}(\text{S}^7_\theta) \) that parametrizes a family of noncommutative 7-spheres \( \tilde{S}^7_\theta \). By evaluating \( \rho^2 \) as any real number \( r^2 \in \mathbb{R} \), we obtain an algebra \( \mathcal{A}(\text{S}^7_\theta_r) \) which is a deformation of the algebra of polynomials on a sphere of radius \( r \).

As expected, the coaction of the quantum subgroup \( \mathcal{A}(\text{SO}_\theta(2)) \) does not ‘inflate the spheres’, i.e. \( \rho^2 = 1 \otimes 1 \) in this case. Indeed, if \( A_\theta = (A_{ij}) \) is the defining matrix of \( \mathcal{A}(\text{SO}_\theta(2)) \), one gets

\[
(u^* u)_{ab} \mapsto \sum_{ijl}(A^*)_{li} A_{ij} \otimes (u^*)_{al} u_{jb} = \sum_{ijl} \delta_{ij} \otimes (u^*)_{al} u_{jb} = 1 \otimes (u^* u)_{ab},
\]

giving \( \sum_j z_j^* z_j = 1 \otimes \sum_j z_j^* z_j \); both \( \mathcal{A}(\text{S}^7_\theta) \) and \( \mathcal{A}(\text{S}^7_\theta) \) are \( \mathcal{A}(\text{SO}_\theta(2)) \) comodule \( \ast \)-algebras.

Next, we define a right action of \( SU(2) \) on \( \mathcal{A}(\text{S}^7_\theta) \) in such a way that the corresponding algebra of invariants describes a family of noncommutative 4-spheres. It is natural to require that this action commutes with the above left coaction of \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \) on \( \mathcal{A}(\text{S}^7_\theta) \).

The algebra \( \mathcal{A}(\text{S}^7_\theta) \) is generated by elements \( \{w_j, w_j^*, j = 1, \ldots, 4\} \), the \( w_j \)'s being as in (28) but with 'coefficients' in \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \). Clearly, \( \sum_j w_j^* w_j = \rho^2 \). The algebra of invariants of the action of \( SU(2) \) on \( \mathcal{A}(\text{S}^7_\theta) \) is generated by

\[
\tilde{x} = w_1 w_4^* + w_2 w_3^* - w_3 w_2^* - w_4 w_1^*, \quad \tilde{\alpha} = 2(w_1 w_3^* + w_2 w_4^*), \quad \tilde{\beta} = 2(-w_1 w_4 + w_2 w_3),
\]

(33)

together with \( \rho^2 \). This is so because the elements (33) correspond to the elements (11) that generate the algebra of invariants under the action of \( SU(2) \) on \( \mathcal{A}(\text{S}^7_\theta) \). The correspondence also gives for their commutation relations the same expressions as the ones in (7) for the generators of \( \mathcal{A}(\text{S}^7_\theta) \). Clearly, there is not the sphere relation of \( \mathcal{A}(\text{S}^7_\theta) \) any longer but rather we find that

\[
\tilde{\alpha}^* \tilde{\alpha} + \tilde{\beta}^* \tilde{\beta} + \tilde{x}^2 = (\sum_j w_j^* w_j)^2 = \rho^4.
\]

(34)

We denote by \( \mathcal{A}(\text{S}^7_\theta) \subset \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(\text{S}^7_\theta) \) the algebra of invariants and conclude that the coaction \( \Delta_L \) of \( \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \) on the \( SU(2) \) principal fibration \( \mathcal{A}(\text{S}^7_\theta) \rightarrow \mathcal{A}(\text{S}^7_\theta) \) generates a family
of SU(2) principal fibrations $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(\tilde{S}^4_\theta)$. Evaluating the central element $\rho^2$, for any $r \in \mathbb{R}$ we get an SU(2) principal fibration $\mathcal{A}(S^4_{\theta,r}) \hookrightarrow \mathcal{A}(\tilde{S}^4_{\theta,r})$ of spheres of radius $r^2$ and $r$ respectively.

By construction, the generators $\tilde{\alpha}, \tilde{\beta}, \tilde{x}$ of $\mathcal{A}(\tilde{S}^4_\theta)$ are the images under $\Delta_L$ of the corresponding $\alpha, \beta, x$ of $\mathcal{A}(S^4_\theta)$. An explicit computation shows that in the transformation the elements of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ appear only quadratically. Hence on $\mathcal{A}(S^4_\theta)$, rather than a coaction of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ there is a coaction of the $\mathbb{Z}^2$-invariant subalgebra. This is denoted $\mathcal{A}(\text{SO}_4(5,1))$ and is a deformation of the classical conformal group $\text{SO}_4(5,1)$. For more details we refer to [10].

Motivated by the interpretation of $\text{SL}_\theta(2, \mathbb{H})$ as a parameter space (see next section), we extend to $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta)$ the differential of $\mathcal{A}(S^4_\theta)$ as $\text{(id} \otimes \ast_\theta)$. Then one finds that

$$\Delta_L(\ast_\theta \omega) = (\text{id} \otimes \ast_\theta) \Delta_L(\omega), \quad \forall \omega \in \Omega(S^4_\theta).$$

Hence, the algebra $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ coacts on $\Omega(S^4_\theta)$ by conformal transformations.

5. A noncommutative family of instantons on $S^4_\theta$

We mentioned in Sect. 2 that out of the matrix valued function $u$ in (8) one gets a projection $p = u^\ast u$, given explicitly in (10), whose Grassmannian connection $\nabla = p \circ \text{d}$ is self-dual. On equivariant maps the connexion is given as $\nabla(f_a) = \text{d}f_a + \sum_k \omega_{ab} f_k$, with the connection 1-form $\omega = (\omega_{ab})$ given – in terms of $u$ – in (22). Out of the coaction of the quantum group $\text{SL}_\theta(2, \mathbb{H})$, we shall get a family of such connections in the sense that we explain in the next sections.

5.1. A family of projections

We first describe a family of vector bundles over $S^4_\theta$, via a family of suitable projections. We know from (30) or (31) the transformation of the matrix $u$ to $\tilde{u}$ for the coaction of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$: $\tilde{u}_{ia} = \Delta_L(u_{ia}) = \sum_j A_{ij} \otimes u_{ja}$, with $A_\theta = (A_{ij})$ the defining matrix of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$. The fact that the latter does not preserve the spherical relations is also the statement that

$$\sum_k (\tilde{u}^\ast)_{ak} \tilde{u}_{kb} = \Delta_L \left( \sum_k (u^\ast)_{ak} u_{kb} \right) = \Delta_L \left( \sum_k z_{ak}^\ast z_k \right) \delta_{ab} = \rho^2 \delta_{ab}, \quad (35)$$

or $(\tilde{u})^\ast \tilde{u} = \rho^2 I_2$. Then, we define $P = (P_{jk}) \in \text{Mat}_4(\mathcal{A}(\tilde{S}^4_\theta))$ by

$$P := \tilde{u} \rho^{-2} (\tilde{u}^\ast)^\ast, \quad \text{or} \quad P_{ij} = \rho^{-2} \sum_a \tilde{u}_{ia} (\tilde{u}^\ast)_{aj}. \quad (36)$$

The condition $(\tilde{u})^\ast \tilde{u} = \rho^2 I_2$ gives that $P$ is an idempotent; being $\ast$-selfadjoint it is a projection.

For the definition of $P$ we enlarge the algebra $\mathcal{A}(\tilde{S}^4_\theta)$ by adding the element $\rho^{-2}$, the inverse of the positive self-adjoint central element $\rho^2$. Later on we shall also need the element $\rho^{-1} = \sqrt{\rho^{-2}}$. At the smooth level this is not problematic. The algebra $\mathcal{C}^\infty(\tilde{S}^4_\theta)$ is defined as a fixed point algebra [7] and one finds that the spectrum of $\rho^2$ is positive and does not contain the point 0.

Explicitly, the projection $P$ is given by,

$$P = \frac{1}{2} \rho^{-2} \begin{pmatrix}
\rho^2 + \tilde{x} & 0 & \tilde{\alpha} & \tilde{\beta} \\
0 & \rho^2 + \tilde{x} & -\mu \tilde{\beta} & \mu \tilde{\alpha} \\
\tilde{\alpha}^\ast & -\tilde{\mu} \tilde{\beta} & \rho^2 - \tilde{x} & 0 \\
\tilde{\beta}^\ast & -\mu \tilde{\alpha} & 0 & \rho^2 - \tilde{x}
\end{pmatrix},$$

a matrix strikingly similar to the matrix (10) for the basic projection.
The entries of the projection $P$ are in $\mathcal{A}(\widetilde{S}_4^4)$, that is $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta)$: we interpret $P$ as a noncommutative family of projections parametrized by the noncommutative space $\mathrm{SL}_\theta(2, \mathbb{H})$. This is the analogue for projections of the noncommutative families of maps that were introduced and studied in [19, 17]. The interpretation as a noncommutative family is justified by the classical case: at $\theta = 0$, there are evaluation maps $ev_x : \mathcal{A}(\mathrm{SL}(2, \mathbb{H})) \rightarrow \mathbb{C}$ and for each point $x$ in $\mathrm{SL}(2, \mathbb{H})$, $(ev_x \otimes \mathrm{id})P$ is a projection in $\mathcal{M}_4(\mathcal{A}(S^4))$, that is a bundle over $S^4$. Although there need not be ‘enough’ evaluation maps available in the noncommutative case (due to a lack of points) we can still work with the family at once. Having these, out of the projection $P$ one gets a noncommutative family of instantons. The family of connections $\nabla := P \circ (\mathrm{id} \otimes d)$ is self-dual:

$$(\mathrm{id} \otimes *_{\theta}) (P((\mathrm{id} \otimes d)P)^2 = P((\mathrm{id} \otimes d)P)^2.$$ 

We know already that $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ coacts by conformal transformations and the curvature $P((\mathrm{id} \otimes d)P)^2$ of $\nabla$ is the image of the curvature $\nabla^2 = p(dp)^2$ under the coaction of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$. It was shown in [8] that the charge of the basic instanton $p$ is 1. This charge was given as a pairing between the second component of the Chern character of $p$ – an element in the cyclic homology group $HC_4(\mathcal{A}(S^4_\theta))$ – with the fundamental class of $S^4_\theta$ in the cyclic cohomology $HC^4(\mathcal{A}(S^4_\theta))$. The zeroth and first components of the Chern character were shown to vanish identically in $HC_0(\mathcal{A}(S^4_\theta))$ and $HC_2(\mathcal{A}(S^4_\theta))$, respectively. We will reduce the computation of the Chern characters for the family of projections $P$ to this case by proving that $P$ is equivalent to the projection $1 \otimes p$. Hence, we can conclude that $P$ represents the same class as $1 \otimes p$ in the K-theory of the algebra $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta)$. Recall that two projections $p, q$ are Murray-von Neumann equivalent if there exists a partial isometry $V$ such that $p = VV^*$ and $q = V^*V$.

The projection $P$ is Murray-von Neumann equivalent to the projection $1 \otimes p$ in the algebra $\mathcal{M}_4(\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$. To this end, define $V = (V_{ik}) \in \mathcal{M}_4(\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$ by

$$V_{ik} = \rho^{-1} A_{ij} \otimes p_{jk} = \rho^{-1} A_{ij} \otimes u_{ja}(u^*)_{ak} = \rho^{-1} \tilde{u}_{ia}(1 \otimes (u^*)_{ak}),$$

with $\tilde{u} = (\tilde{u}_{ia})$ as in (30). Its adjoint is $(V^*)_{ik} = \rho^{-1}(1 \otimes u_{ia})(\tilde{u}^*)_{ak}$. Using (35), one obtains

$$(V^*V)_{il} = \sum_k (V^*)_{ik} V_{kl} = \rho^{-2} \sum_{kab} (1 \otimes u_{ia})(\tilde{u}^*)_{ak} \tilde{u}_{kb}(1 \otimes (u^*)_{bl}) = \rho^{-2} \sum_{ab} (1 \otimes u_{ia})(\rho^2 \delta_{ab})(1 \otimes (u^*)_{bl}) = 1 \otimes \sum_{a} u_{ia}(u^*)_{al} = 1 \otimes p_{il},$$

and

$$(VV^*)_{il} = \sum_k V_{ik}(V^*)_{kl} = \rho^{-2} \sum_{kab} \tilde{u}_{ia}(1 \otimes (u^*)_{ak})(1 \otimes u_{kb})(\tilde{u}^*)_{bl} = \rho^{-2} \sum_{kab} \tilde{u}_{ia}(1 \otimes (u^*)_{ak}u_{kb})(\tilde{u}^*)_{bl} = \rho^{-2} \sum_{ab} \tilde{u}_{ia}(1 \otimes \delta_{ab})(\tilde{u}^*)_{bl} = \rho^{-2} \sum_{a} \tilde{u}_{ia}(\tilde{u}^*)_{al} = P_{il},$$

which establishes the claim. It follows from this result that the components of the Chern character of $P$, $\chi_n(P) \in HC_{2n}(\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$, with $n = 0, 1, 2$, coincide with the pushforwards $\phi_* \chi_n(p)$ of $\chi_n(p) \in HC_{2n}(\mathcal{A}(S^4_\theta))$ under the algebra map

$$\phi : \mathcal{A}(S^4_\theta) \rightarrow \mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta), \quad a \mapsto 1 \otimes a.$$ 

Hence, both $\chi_0(P)$ and $\chi_1(P)$ are zero, from the vanishing of $\chi_0(p)$ and $\chi_1(p)$ proved in [8]. Then, one can use the map $\phi$ to pull back the fundamental class $[S^4_\theta] \in HC^4(\mathcal{A}(S^4_\theta))$ to a class $\phi^*[S^4_\theta]$ in $HC^4(\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$. When paired with $\chi_2(P)$ it gives

$$\langle \chi_2(P), \phi^*[S^4_\theta] \rangle = \langle \phi_* \chi_2(p), \phi^*[S^4_\theta] \rangle = \langle \chi_2(p), [S^4_\theta] \rangle = 1,$$

with the last equality proved in [8] too.
5.2. A family of connections

When transforming $u$ by the coaction of $\text{SL}_\theta(2, \mathbb{H})$ in (30), one transforms the instanton connection 1-form $\omega$ in (22) as well to $\tilde{\omega} = (\tilde{\omega}_{ab})$ with

$$\tilde{\omega}_{ab} := \Delta_L(\omega_{ab}) = \frac{1}{2} \sum_{kij} (A^*)_{ik} A_{kj} \otimes \left( (u^*)_ai du_{jb} - d(u^*)_ai u_{jb} \right). \quad (37)$$

Since $\Delta_L$ is linear, $\tilde{\omega}$ is still traceless ($\sum_a \tilde{\omega}_{aa} = 0$) and skew-hermitian ($\tilde{\omega}_{ab} = -(\tilde{\omega}^*)_{ba}$).

The connection 1-form $\omega$ is invariant under the coaction of the quantum group $\text{SO}_\theta(2)$. For the defining matrix of $\text{SO}_\theta(2)$ one has that $\sum_k (A^*)_ik A_{kj} = \delta_{ij}$ and (37) reduces to

$$\Delta_L(\omega_{ab}) = 1 \otimes \omega_{ab}. $$

Hence, the relevant space that parametrizes the connection one-forms is not $\text{SL}_\theta(2, \mathbb{H})$ but rather the quotient of $\text{SL}_\theta(2, \mathbb{H})$ by $\text{SO}_\theta(2)$. Denoting by $\pi$ the quotient map from $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ to $\mathcal{A}(\text{SO}_\theta(2))$, the algebra of the quotient is the algebra of coinvariants of the natural left coaction $\delta_L := (\pi \otimes \text{id}) \circ \Delta$ of $\text{SO}_\theta(2)$ on $\text{SL}_\theta(2, \mathbb{H})$:

$$\mathcal{A}(\mathcal{M}_\theta) := \{ a \in \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \mid \delta_L(a) = 1 \otimes a \}. $$

Since $\text{SO}_\theta(2)$ is a quantum subgroup of $\text{SL}_\theta(2, \mathbb{H})$ the quotient is well defined. In fact, from this the algebra $\mathcal{A}(\mathcal{M}_\theta)$ is a quantum homogeneous space and the inclusion $\mathcal{A}(\mathcal{M}_\theta) \hookrightarrow \mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$ is a noncommutative principal bundle with $\mathcal{A}(\text{SO}_\theta(2))$ as structure group.

Being the relations in the quotient $\mathcal{A}(\text{SO}_\theta(2))$ quadratic in the matrix elements $A_{ij}$ and $(A_{ij})^*$, the generators of $\mathcal{A}(\mathcal{M}_\theta)$ have to be at least quadratic in them. For the first leg of the tensor product $\Delta(a)$ to involve these relations in $\mathcal{A}(\text{SO}_\theta(2))$, we need to take $a = \sum_i (A_{ik})^* A_{il}$, so that

$$(\pi \otimes \text{id})\Delta(a) = \sum_{imm} \pi((A_{im})^* A_{in}) \otimes (A_{mk})^* A_{nl} = \sum_{imm} \pi((A_{im})^*) \pi(A_{in}) \otimes (A_{mk})^* A_{nl} = \sum_{mn} \delta_{mn} \otimes (A_{mk})^* A_{nl}. $$

We conclude that the quantum quotient space $\mathcal{A}(\mathcal{M}_\theta)$ is generated as an algebra by the elements

$$m_{ij} := \sum_k (A^*)_{ik} A_{kj} = \sum_k (A_{ki})^* A_{kj}. $$

We will think of the transformed $\tilde{\omega}$ in (37) as a family of connection one-forms parametrized by the noncommutative space $\mathcal{M}_\theta$. At the classical value $\theta = 0$, we get the moduli space $\mathcal{M}_{\theta=0} = \text{SL}(2, \mathbb{H})/\text{SO}(2)$ of instantons of charge 1. For each point $x$ in $\mathcal{M}_{\theta=0}$, the evaluation map $ev_x : \mathcal{A}(\mathcal{M}_{\theta=0}) \rightarrow \mathbb{C}$ gives an instanton connection (i.e. one with self-dual curvature) $(ev_x \otimes \text{id})\tilde{\omega}$ on the bundle over $S^4$ described by $(ev_x \otimes \text{id})P$.

5.3. The space $\mathcal{M}_\theta$ of connections and its geometry

The structure of the algebra $\mathcal{A}(\mathcal{M}_\theta)$ is deduced from that of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$. We collect the generators $m_{ij} = \sum_k (A_{ki})^* A_{kj}$ into a matrix $M := (m_{ij})$. Explicitly, one finds

$$M = \begin{pmatrix}
  m & 0 & g_1 & g_2 \\
  0 & m & -\bar{\mu} & g_2 \\
  g_1 & -\mu & g_2^* & n \\
  g_2 & \bar{\mu} & g_1 & 0
\end{pmatrix} \quad (38)$$

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with its entries related to those of the defining matrix $A_\theta$ in (27) of $\mathcal{A}(\text{SL}_\theta(2,\mathbb{H}))$ by,

\begin{align}
m &= m^* = a_1^*a_1 + a_2^*a_2 + c_1^2 + c_2^2, \\
n &= n^* = b_1^*b_1 + b_2^*b_2 + d_1^*d_1 + d_2^*d_2, \\
g_1 &= a_1^*b_1 + \mu b_2^*a_2 + c_2^*d_1 + \mu d_2^*c_2, \\
g_2 &= b_2^*a_1 - \mu a_2^*b_1 + d_2^*c_1 - \mu c_2^*d_1.
\end{align}

As for the commutation relations, one finds that both $m$ and $n$ are central:

\begin{align}
m \times x &= x \times m, \quad n \times x = x \times n \quad \forall x \in \mathcal{M}_\theta; \tag{40a}
\end{align}

that $g_1$ and $g_2$ are normal:

\begin{align}
g_1 g_1^* &= g_1^* g_1, \quad g_2 g_2^* = g_2^* g_2; \tag{40b}
\end{align}

and that

\begin{align}
g_1 g_2 &= \mu^2 g_2 g_1, \quad g_1 g_2^* = \mu^2 g_2^* g_1, \quad g_2 g_1 &= \mu^2 g_1 g_2, \quad g_2 g_1^* = \mu^2 g_1^* g_2. \tag{40c}
\end{align}

There is also a quadratic relation,

\begin{align}
m n - (g_1^* g_1 + g_2^* g_2) &= 1, \tag{41}
\end{align}

coming from the condition $\det(A_\theta) = 1$. Elements $(m_{ij})$ of the matrix $M$ enter the the expression for $\rho^2$. With $p_{kl}$ the components of the defining projector $p$ in (9), one finds that

\begin{align}
\rho^2 &= \frac{1}{2} \sum_{ij} \eta_{ij} m_{ij} \otimes p_{ji} \\
&= \frac{1}{2} [(m + n) \otimes 1 + (m - n) \otimes x + \mu \tilde{g}_1 \otimes \alpha + \tilde{g}_2 \otimes \beta + \bar{\mu} g_1 \otimes a^* + \mu g_2 \otimes b^*].
\end{align}

In particular, for $A_{ij} \in \mathcal{A}(\text{SO}_\theta(2))$ one gets $\rho^2 = \frac{1}{2}(1 \otimes \text{tr}(p)) = 1 \otimes 1$, as already observed.

### 5.4. The boundary of $\mathcal{M}_\theta$

The defining matrix $M$ of $\mathcal{M}_\theta$ in (38), with the commutation relations among its entries, is strikingly similar to the defining projection $p$ of $\mathcal{A}(S_\theta^4)$ in (10) with the corresponding commutation relations. Clearly, the crucial difference is that while for $\mathcal{A}(S_\theta^4)$ we have a spherical relation, for $\mathcal{M}_\theta$ we have the relation (41) which makes $\mathcal{M}_\theta$ a $\theta$-deformation of a hyperboloid in 6 dimensions. This becomes clearer if we introduce two central elements $w$ and $y$ by

\begin{align}
w := \frac{1}{2}(m + n); \quad y := \frac{1}{2}(m - n).
\end{align}

Relation (41) then reads

\begin{align}
w^2 - (y^2 + g_1^* g_1 + g_2^* g_2) &= 1, \tag{42}
\end{align}

making evident the hyperboloid structure. Let us examine its structure at ‘infinity’. We first adjoin the inverse of $w$ to $\mathcal{A}(\mathcal{M}_\theta)$, and stereographically project onto the coordinates,

\begin{align}
Y := w^{-1} y, \quad G_1 := w^{-1} g_1, \quad G_2 := w^{-1} g_2.
\end{align}

The relation (42) becomes,

\begin{align}
Y^2 + G_1^* G_1 + G_2^* G_2 &= 1 - w^{-2}.
\end{align}

Evaluating $w$ as a real number, and taking its ‘limit to infinity’ we get a spherical relation,

\begin{align}
Y^2 + G_1^* G_1 + G_2^* G_2 &= 1.
\end{align}

By combining this with relations (40), we can conclude that at the ‘boundary’ of $\mathcal{M}_\theta$, we re-encounter the noncommutative 4-sphere $\mathcal{A}(S_\theta^4)$ via the identification

\begin{align}
Y \leftrightarrow x, \quad G_1 \leftrightarrow \alpha, \quad G_2 \leftrightarrow \beta.
\end{align}

The above construction is the analogue of the classical structure, in which 4-spheres are found at the boundary of the moduli space.
5.5. Some additional remarks

We have constructed a noncommutative family of instantons of charge 1 on the noncommutative 4-sphere $S^4_\theta$. The family is parametrized by a noncommutative space $\mathcal{M}_\theta$ which reduces to the moduli space of charge 1 instantons on $S^4$ in the limit when $\theta \to 0$. Although this means that $\mathcal{M}_\theta$ is a quantization of the moduli space $\mathcal{M}_{\theta=0}$, it does not imply that it is itself a space of moduli. In order to call this the moduli space of charge 1 instantons on $S^4_\theta$ a few things must be clarified. We mention in particular two points that for the moment lack a proper understanding.

First of all, we are confronted with the difficulty of finding a proper notion of gauge group and gauge transformations. A naive dual of the undeformed construction would lead one to consider the group of $\mathcal{A}(SU(2))$-coequivariant algebra maps from the algebra $\mathcal{A}(SU(2))$ to $\mathcal{A}(S^4_\theta)$, equipped with the convolution product. However, since the algebra $\mathcal{A}(SU(2))$ is commutative as opposed to $\mathcal{A}(S^4_\theta)$, one quickly realizes that there are not so many elements in this group.

The second open problem is related to the fact that one would need some sort of universality for the noncommutative family of instantons. A possible notion of universality could be defined as follows. A family of instantons parametrized by $\mathcal{A}(M_\theta)$ is said to be universal if for any other noncommutative family of instantons parametrized by, say, an algebra $B$, there exists an algebra map $\phi: M_\theta \to B$ such that this family can be obtained from the universal family via the map $\phi$. Again, this is the analogue of the notion of universality for noncommutative families of maps as in [19, 17]. But it appears that, in order to prove universality for the actual family that we have constructed in the present paper, an argument along the classical lines – involving a local construction of the moduli space from its tangent bundle [2] – fails here, due to the fact that there is no natural notion of a tangent space to a noncommutative space.

Acknowledgments

I would like to thank the organizers of the International Conference ‘Non commutative geometry and physics’, held at the Université Paris-Sud 11, Orsay, April 23-27 2007, for the kind invitation.

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