SIMPLICES IN THIN SUBSETS OF EUCLIDEAN SPACES

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Abstract. Let \( \Delta \) be a non-degenerate simplex on \( k \) vertices. We prove that there exists a threshold \( s_k < k \) such that any set \( A \subseteq \mathbb{R}^k \) of Hausdorff dimension \( \dim A \geq s_k \) necessarily contains a similar copy of the simplex \( \Delta \).

1. Introduction.

A classical problem of geometric Ramsey theory is to show that a sufficiently large sets contain a given geometric configuration. The underlying settings can be the Euclidean space, the integer lattice or vector spaces over finite fields. By a geometric configuration we understand the collection of finite point sets obtained from a given finite set \( F \subseteq \mathbb{R}^k \) via translations, rotations and dilations.

If the size is measured in terms of the positivity of the Lebesgue density, then it is known that large sets in \( \mathbb{R}^k \) contain a translated and rotated copy of all sufficiently large dilates of any non-degenerate simplex \( \Delta \) with \( k \) vertices [2]. However, on the scale of the Hausdorff dimension \( s < k \) this question is not very well understood, the only affirmative result in this direction obtained by Iosevich-Liu [6].

In the other direction, a construction due to Keleti [9] shows that there exists set \( A \subseteq \mathbb{R} \) of full Hausdorff dimension which do not contain any non-trivial 3-term arithmetic progression. In two dimensions an example due to Falconer [3] and Maga [11] shows that there exists set \( A \subseteq \mathbb{R}^2 \) of Hausdorff dimension 2, which do not contain the vertices of an equilateral triangle, or more generally a non-trivial similar copy of a given non-degenerate triangle. It seems plausible that examples of such sets exist in all dimensions, but this is not currently known. See (4) for related results.

The purpose of this paper is to show that measurable sets \( A \subseteq \mathbb{R}^k \) of sufficiently large Hausdorff dimension \( s < k \) contain a similar copy of any given non-degenerate \( k \)-simplex with bounded eccentricity. Our arguments

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make use of and have some similarity to those of Lyall-Magyar \cite{10}. We also extend our results to bounded degree distance graphs. For the special case of a path (or chain), and, more generally, a tree, similar but somewhat stronger results were obtained in \cite{1} and \cite{8}.

2. Main results.

Let $V = \{v_1, \ldots, v_k\} \subseteq \mathbb{R}^k$ be a non-degenerate $k$-simplex, a set of $k$ vertices which are in general position spanning a $k-1$-dimensional affine subspace. For $1 \leq j \leq k$ let $r_j(V)$ be the distance of the vertex $v_j$ to the affine subspace spanned by the remaining vertices $v_i$, $i \neq j$ and define $r(V) := \min_{1 \leq j \leq k} r_j(V)$. Let $d(V)$ denote the diameter of the simplex, which is also the maximum distance between two vertices. Then the quantity $\delta(V) := r(V)/d(V)$, which is positive if and only if $V$ is non-degenerate, measures how close the simplex $V$ is to being degenerate.

We say that a simplex $V'$ is similar to $V$, if $V' = x + \lambda \cdot U(V)$ for some $x \in \mathbb{R}^k$, $\lambda > 0$ and $U \in SO(k)$, that is if $V'$ is obtained from $V$ by a translation, dilation and rotation.

Theorem 1. Let $k \in \mathbb{N}$, $\delta > 0$. There exists $s_0 = s_0(k, \delta) < k$ such that if $E$ is a compact subset of $\mathbb{R}^k$ of Hausdorff dimension $\dim E \geq s_0$, then $E$ contains the vertices of a simplex $V'$ similar to $V$, for any non-degenerate $k$-simplex $V$ with $\delta(V) \geq \delta$.

Remark 2.1. Note that the dimension condition is sharp for $k = 2$ as a construction due to Maga \cite{11} shows the existence of a set $E \subseteq \mathbb{R}^2$ with $\dim(E) = 2$ which does not contain any equilateral triangle or more generally a similar copy of any given triangle.

Remark 2.2. It is also interesting to note that the proof of Theorem \cite{7} above proves much more than just the existence of vertices of $V'$ similar to $V$ inside $E$. The proof proceeds by constructing a natural measure on the set of simplexes and proving an upper and a lower bound on this measure. This argument shows that an infinite "statistically" correct "amount" of simplexes $V$'s that satisfy the conclusion of the theorem exist, shedding considerable light on the structure of set of positive upper Lebesgue density.

Remark 2.3. Theorem \cite{7} establishes a non-trivial exponent $s_0 < k$, but the proof yields $s_0$ very close to $k$ and not explicitly computable. The analogous results in the finite field setting (see e.g. \cite{3}, \cite{7} and the references contained therein) suggest that it may be possible to obtain explicit exponents, but this would require a fundamentally different approach to certain lower bounds obtained in the proof of Theorem \cite{7}.

A distance graph is a connected finite graph embedded in Euclidean space, with a set of vertices $V = \{v_0, v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ and a set of edges $E \subseteq \{(i,j); \ 0 \leq i < j \leq n\}$. We say that a graph $\Gamma = (V,E)$ has degree at most $k$ if $|V_j| \leq k$ for all $1 \leq j \leq n$, where $V_j = |\{v_i: (i,j) \in E\}|$. The
graph $\Gamma$ is called proper if the sets $V_j \cup \{v_j\}$ are in general position. Let $r(\Gamma)$ be the minimum of the distances from the vertices $v_j$ to the corresponding affine subspace spanned by the sets $V_j$ and note that $r(\Gamma) > 0$ if $\Gamma$ is proper. Let $d(\Gamma)$ denote length of the longest edge of $\Gamma$ and let $\delta(\Gamma) := r(\Gamma)/d(\Gamma)$.

We say that a distance graph $\Gamma' = (V',E)$ is isometric to $\Gamma$, and write $\Gamma' \simeq \Gamma$ if there is a one-one and onto mapping $\phi : V \to V'$ so that $|\phi(v_i) - \phi(v_j)| = |v_i - v_j|$ for all $(i,j) \in E$. One may picture $\Gamma'$ obtained from $\Gamma$ by a translation followed by rotating the edges around the vertices, if possible. By $\lambda \cdot \Gamma$ we mean the dilate of the distance graph $\Gamma$ by a factor $\lambda > 0$ and we say that $\Gamma'$ is similar to $\Gamma$ if $\Gamma'$ is isometric to $\lambda \cdot \Gamma$.

**Theorem 2.** Let $\delta > 0$, $n \geq 1$, $1 \leq k < d$ and let $E$ be a compact subset of $\mathbb{R}^k$ of Hausdorff dimension $s < d$. There exists $s_0 = s_0(n,d,\delta) < d$ such if $s \geq s_0$ then $E$ contains a distance graph $\Gamma'$ similar to $\Gamma$, for any proper distance graph $\Gamma = (V,E)$ of degree at most $k$, with $V \subseteq \mathbb{R}^d$, $|V| = n$ and $\delta(\Gamma) \geq \delta$.

Note that Theorem 2 implies Theorem 1 as a non-degenerate simplex is a proper distance graph of degree $k - 1$.

**3. Proof of Theorem 1**

Let $E \subseteq B(0,1)$ be a compact subset of the unit ball $B(0,1)$ in $\mathbb{R}^k$ of Hausdorff dimension $s < k$. It is well-known that there is a probability measure $\mu$ supported on $E$ such that $\mu(B(x,r)) \leq C \mu r^s$ for all balls $B(x,r)$. The following observation shows that we may take $C = 4$ for our purposes. $^1$

**Lemma 1.** There exists a set $E' \subseteq B(0,1)$ of the form $E' = \rho^{-1}(F - u)$ for some $\rho > 0$, $u \in \mathbb{R}^k$ and $F \subseteq E$, and a probability measure $\mu'$ supported on $E'$ which satisfies

$$\mu'(B(x,r)) \leq 4r^s, \quad \text{for all} \quad x \in \mathbb{R}^k, \ r > 0. \quad (3.1)$$

**Proof.** Let $K := \inf(S)$, where

$$S := \{C \in \mathbb{R} : \mu(B(x,r)) \leq Cr^s, \ \forall B(x,r)\}.$$

By Frostman’s lemma $^2$ we have that $S \neq \emptyset$, $K > 0$, moreover

$$\mu(B(x,r)) \leq 2Kr^s,$$

for all balls $B(x,r)$. There exists a ball $Q = B(v,\rho)$ or radius $\rho$ such that $\mu(Q) \geq \frac{1}{2} Kr^s$. We translate $E$ so $Q$ is centered at the origin, set $F = E \cap Q$ and denote by $\mu_F$ the induced probability measure on $F$

$$\mu_F(A) = \frac{\mu(A \cap F)}{\mu(F)}.$$  

$^1$We’d like to thank Giorgis Petridis for bringing this observation to our attention.
Note that for all balls $B = B(x, r)$, 
\[ \mu_F(B) \leq \frac{2K r^s}{\frac{s}{2} K \rho^s} = 4 \left( \frac{r}{\rho} \right)^s. \]

Finally we define the probability measure $\mu'$, by $\mu'(A) := \mu_F(\rho A)$. It is supported on $E' = \rho^{-1} F \subseteq B(0, 1)$ and satisfies 
\[ \mu'(B(x, r)) = \mu_F(B(px, pr)) \leq 4r^s. \]

Clearly $E$ contains a similar copy of $V$ if the same holds for $E'$, thus one can pass from $E$ to $E'$ and hence assuming that (3.1) holds, in proving our main results. Given $\varepsilon > 0$ let $\psi_\varepsilon(x) = \varepsilon^{-k} \psi(x/\varepsilon) \geq 0$, where $\psi \geq 0$ is a Schwarz function whose Fourier transform, $\hat{\psi}$, is a compactly supported smooth function, satisfying $\hat{\psi}(0) = 1$ and $0 \leq \hat{\psi} \leq 1$.

We define $\mu_\varepsilon := \mu * \psi_\varepsilon$. Note that $\mu_\varepsilon$ is a continuous function satisfying $\|\mu_\varepsilon\|_\infty \leq C \varepsilon^{s-k}$ with an absolute constant $C = C_\psi > 0$, by Lemma 1.

Let $V = \{v_0 = 0, \ldots, v_{k-1}\}$ be a given a non-degenerate simplex and note that in proving Theorem 1 we may assume that $d(V) = 1$ hence $\delta(V) = r(V)$. A simplex $V' = \{x_0 = 0, x_1, \ldots, x_{k-1}\}$ is isometric to $V$ if for every $1 \leq j \leq k$ one has that $x_j \in S_{x_1, \ldots, x_{j-1}}$, where 
\[ S_{x_1, \ldots, x_{j-1}} = \{y \in \mathbb{R}^k : |y - x_i| = |v_j - v_i|, \ 0 \leq i < j\} \]

is a sphere of dimension $k - j$, of radius $r_j = r_j(V) \geq r(V) > 0$. Let $\sigma_{x_1, \ldots, x_{j-1}}$ denote its normalized surface area measure.

Given $0 < \lambda, \varepsilon \leq 1$ define the multi-linear expression,

\[ T_{\lambda V}(\mu_\varepsilon) := \int \mu_\varepsilon(x) \mu_\varepsilon(x - \lambda x_1) \cdots \mu_\varepsilon(x - \lambda x_{k-1}) d\sigma(x_1) d\sigma(x_2) \cdots d\sigma_{x_1, \ldots, x_{k-2}}(x_{k-1}) dx, \]

which may be viewed as a weighted count of the isometric copies of $\lambda \Delta$.

We have the following crucial upper bound

**Lemma 2.** There exists a constant $C_k > 0$, depending only on $k$, such that

\[ |T_{\lambda V}(\mu_{2\varepsilon}) - T_{\lambda V}(\mu_\varepsilon)| \leq C_k r(V)^{-\frac{3}{2}} \lambda^{\frac{1}{2}} \varepsilon^{(k-\frac{3}{2})(s-k)+\frac{1}{2}}. \]  \hfill (3.3)

As an immediate corollary we have that

**Lemma 3.** Let $k - \frac{3}{4k} \leq s < k$. There exists 
\[ T_{\lambda V}(\mu) := \lim_{\varepsilon \to 0} T_{\lambda V}(\mu_\varepsilon), \]

moreover

\[ |T_{\lambda V}(\mu) - T_{\lambda V}(\mu_\varepsilon)| \leq C_k r(V)^{-\frac{3}{2}} \lambda^{-\frac{1}{2}} \varepsilon^{(k-\frac{3}{2})(s-k)+\frac{1}{2}}. \]  \hfill (3.5)
Indeed, the left side of (3.5) can be written as telescopic sum:

\[ \sum_{j \geq 0} T_{\lambda^j}(\mu_{2\varepsilon_j}) - T_{\lambda^j}(\mu_{\varepsilon_j}) \quad \text{with} \quad \varepsilon_j = 2^{-j} \varepsilon. \]

**Proof of Lemma 2** Write \( \Delta \)

Writing \( \varepsilon \) on the sphere \( S \) thus by Cauchy-Schwarz and Placherel’s identity

\[ \left| \langle \mu_{\varepsilon} \Delta \mu_{\varepsilon} \rangle \right| \leq \varepsilon^{(k-2)(s-d)} \int \left| \int \mu_{\varepsilon}(x) \Delta \mu_{\varepsilon} *_{\lambda} \sigma_{x_1, \ldots, x_{k-2}}(x) \, dx \right| \, d\omega(x_1, \ldots, x_{k-2}) \]

where \( d\omega(x_1, \ldots, x_{k-2}) = d\sigma(x_1) \ldots d\sigma(x_{k-2}) \) for \( k \geq 3 \), while for \( k = 3 \) we have that \( d\omega(x_1) = d\sigma(x_1) \) the normalised surface area measure on the sphere \( S = \{ y : |y| = |v_1| \} \).

The inner integral is of the form

\[ \left| \langle \mu_{\varepsilon} \Delta \mu_{\varepsilon} \rangle \right| \leq \varepsilon^{s-d} \left\| \Delta \mu_{\varepsilon} *_{\lambda} \sigma_{x_1, \ldots, x_{k-2}} \right\|_2, \]

thus by Cauchy-Schwarz and Placherel’s identity

\[ \left| \Delta_{k-1} T(\mu_{\varepsilon}) \right|^2 \leq \varepsilon^{2(k-1)(s-d)} \int \left| \overline{\Delta \mu_{\varepsilon}}(\xi) \right|^2 I_{\lambda}(\xi) \, d\xi, \]

where

\[ I_{\lambda}(\xi) = \int |\hat{\sigma}_{x_1, \ldots, x_{k-2}}(\lambda \xi)|^2 \, d\omega(x_1, \ldots, x_{k-2}). \]

Since \( S_{x_1, \ldots, x_{k-2}} \) is a 1-dimensional circle of radius \( r_{k-1} \geq r(V) > 0 \), contained in an affine subspace orthogonal to \( M_{x_1, \ldots, x_{k-2}} = \text{Span}\{x_1, \ldots, x_{k-2}\} \), we have that

\[ |\hat{\sigma}_{x_1, \ldots, x_{k-2}}(\lambda \xi)|^2 \leq (1 + r(V)\lambda \text{dist}(\xi, M_{x_1, \ldots, x_{k-2}}))^{-1}. \]

Since the measure \( \omega(x_1, \ldots, x_{k-2}) \) is invariant with respect to that change of variables \( (x_1, \ldots, x_{k-2}) \to (Ux_1, \ldots, Ux_{k-2}) \) for any rotation \( U \in \)}
SO(k), one estimates

\[ I_{\lambda}(\xi) \lesssim \int \int (1 + r(V) \lambda \, \text{dist}(\xi, M_{U_{x_1, \ldots, x_{k-2}}}))^{-1} \, d\omega(x_1, \ldots, x_{k-2}) \, dU \]

\[ = \int \int (1 + r(V) \lambda \, \text{dist}(U, M_{x_1, \ldots, x_{k-2}}))^{-1} \, d\omega(x_1, \ldots, x_{k-2}) \, dU \]

\[ = \int \int (1 + r(V) \lambda \, |\eta| \, \text{dist}(\eta, M_{x_1, \ldots, x_{k-2}}))^{-1} \, d\omega(x_1, \ldots, x_{k-2}) \, d\sigma_{k-2}(\eta) \]

\[ \lesssim (1 + r(V) \lambda \, |\eta|)^{-1}, \]

where we have written \( \eta := |\xi|^2 U \xi \) and \( \sigma_{k-1} \) denotes the surface area measure on the unit sphere \( S^{k-1} \subseteq \mathbb{R}^k \).

Note that \( \Delta \mu_\varepsilon(\xi) = \hat{\mu}(\xi) (\hat{\psi}(2\varepsilon \xi) - \hat{\psi}(\varepsilon \xi)) \), which is supported on \( |\xi| \lesssim \varepsilon^{-1} \) and is essentially supported on \( |\xi| \approx \varepsilon^{-1} \). Indeed, writing

\[ J := \int |\Delta \mu_\varepsilon(\xi)|^2 I_{\lambda}(\xi) \, d\xi \]

\[ = \int_{|\xi| \lesssim \varepsilon^{-1/2}} |\Delta \mu_\varepsilon(\xi)|^2 I_{\lambda}(\xi) \, d\xi + \int_{\varepsilon^{-1/2} \leq |\xi| \leq \varepsilon^{-1}} |\Delta \mu_\varepsilon(\xi)|^2 I_{\lambda}(\xi) \, d\xi =: J_1 + J_2. \]

Using \( |\hat{\psi}(2\varepsilon \xi) - \hat{\psi}(\varepsilon \xi)| \lesssim \varepsilon^{1/2} \) for \( |\xi| \leq \varepsilon^{-1/2} \), we estimate

\[ J_1 \lesssim \varepsilon^{1/2} \int |\hat{\mu}(\xi)|^2 (\hat{\psi}(2\varepsilon \xi) + \hat{\psi}(\varepsilon \xi)) \, d\xi \lesssim \varepsilon^{1/2} + s - k, \]

as

\[ \int |\hat{\mu}(\xi)|^2 \hat{\psi}(\varepsilon \xi) \, d\xi = \int \mu_\varepsilon(x) \, d\mu_\varepsilon(x) \lesssim \varepsilon^{s-k}. \]

On the other hand, as \( I_{\lambda}(\xi) \lesssim \varepsilon^{1/2} r(V)^{-1} \lambda^{-1} \) for \( |\xi| \geq \varepsilon^{-1/2} \) we have

\[ J_2 \lesssim \varepsilon^{1/2} r(V)^{-1} \lambda^{-1} \int |\hat{\mu}(\xi)|^2 \hat{\phi}(\varepsilon \xi) \, d\xi \lesssim r(V)^{-1} \lambda^{-1} \varepsilon^{1/2} + s - k, \]

where we have written \( \hat{\phi}(\xi) = (\hat{\psi}(2\xi) - \hat{\psi}(\xi))^2 \). Plugging this estimates into (3.3) we obtain

\[ |\Delta T(\mu_\varepsilon)|^2 \lesssim r(V)^{-1} \lambda^{-1} \varepsilon^{1/2} + (2k-1)(s-d), \]

and (3.5) follows.

The support of \( \mu_\varepsilon \) is not compact, however as it is a rapidly decreasing function it can be made to be supported in small neighborhood of the support of \( \mu \) without changing our main estimates. Let \( \phi_\varepsilon(x) := \phi(c \varepsilon^{-1/2} x) \) with some small absolute constant \( c > 0 \), where \( 0 \leq \phi(x) \leq 1 \) is a smooth cut-off, which equals to one for \( |x| \leq 1/2 \) and is zero for \( |x| \geq 2 \). Define \( \tilde{\psi}_\varepsilon = \psi_\varepsilon \phi_\varepsilon \) and \( \tilde{\mu}_\varepsilon = \mu \ast \tilde{\psi}_\varepsilon \). It is easy to see that \( \tilde{\mu}_\varepsilon \leq \mu_\varepsilon \) and \( \int \tilde{\mu}_\varepsilon \geq 1/2 \), if \( c > 0 \) is chosen sufficiently small. Using the trivial upper bound, for \( k - \frac{1}{d} \leq s < k \) we have

\[ |T_{\lambda \Delta}(\mu_\varepsilon) - T_{\lambda \Delta}(\tilde{\mu}_\varepsilon)| \leq C_k \|\mu_\varepsilon\|_{\infty}^{k-1} \|\mu_\varepsilon - \tilde{\mu}_\varepsilon\|_\infty \leq C_k \varepsilon^{1/2}, \]
it follows that estimate \([3.5]\) remains true with \(\mu_\epsilon\) replaced with \(\tilde{\mu}_\epsilon\).

Let \(f_\epsilon := c\epsilon^{k-s}\tilde{\mu}_\epsilon\), where \(c = c_\phi > 0\) is a constant so that \(0 \leq f_\epsilon \leq 1\) and \(\int f_\epsilon \, dx = c\epsilon^{k-s}\). Let \(\alpha := c\epsilon^{k-s}\) and note that the set \(A_\epsilon := \{x: f_\epsilon(x) \geq \alpha/2\}\) has measure \(|A_\epsilon| \geq \alpha/2\). We apply Theorem 2 (ii) together with the more precise lower bound (18) in [10] for the set \(A_\epsilon\).

This gives that there exists an interval \(I\) of length \(|I| \geq \exp(-\epsilon^{-C_k(d-s)})\), such that for all \(\lambda \in I\), one has \(|T_{\lambda V}(A_\epsilon)| \geq c\alpha k = c\epsilon^{k(k-s)}\), where

\[
T_{\lambda V}(A_\epsilon) = \int 1_{A_\epsilon}(x) 1_{A_\epsilon}(x-\lambda x_1) \ldots 1_{A_\epsilon}(x-\lambda x_{k-1}) \, d\sigma(x_1) \ldots d\sigma(x_1,\ldots,x_{k-2}(x_{k-1}) \, dx.
\]

Since

\[
T_{\lambda \Delta}(\tilde{\mu}_\epsilon) \geq c\alpha k T_{\lambda V}(A_\epsilon),
\]

we have that

\[
T_{\lambda V}(\tilde{\mu}_\epsilon) \geq c > 0,
\]

for all \(\lambda \in I\), for a constant \(c = c(k,\psi,r(V)) > 0\).

Now, let

\[
T_{V}(\tilde{\mu}_\epsilon) := \int_0^1 \lambda^{1/2} T_{\lambda V}(\tilde{\mu}_\epsilon) \, d\lambda.
\]

For \(k - \frac{1}{4k} \leq s < k\), by \([3.5]\) we have that

\[
|T_{\lambda V}(\mu) - T_{\lambda V}(\tilde{\mu}_\epsilon)| \leq C_k r(V)^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \epsilon^{\frac{s}{k}},
\]

it follows that

\[
\int_0^1 \lambda^{1/2} |T_{\lambda V}(\mu) - T_{\lambda V}(\tilde{\mu}_\epsilon)| \, d\lambda \leq C_k r(V)^{-\frac{1}{2}} \epsilon^{\frac{1}{k}},
\]

and in particular \(\int_0^1 \lambda^{1/2} T_{\lambda V}(\mu) \, d\lambda < \infty\). On the other hand by \([3.8]\), one has

\[
\int_0^1 \lambda^{1/2} T_{\lambda V}(\tilde{\mu}_\epsilon) \, d\lambda \geq \exp(-\epsilon^{-C_k(k-s)}).
\]

Assume that \(r(V) \geq \delta\), fix a small \(\epsilon = \epsilon_{k,\delta} > 0\) and the choose \(s = s(\epsilon,\delta) < k\) such that

\[
C_k \delta^{-\frac{1}{2}} \epsilon^{\frac{s}{k}} < \frac{1}{2} \exp(-\epsilon^{-C_k(k-s)}),
\]

which ensures that

\[
\int_0^1 \lambda^{1/2} T_{\lambda V}(\mu) \, d\lambda > 0,
\]

thus there exist \(\lambda > 0\) such that \(T_{\lambda V}(\mu) > 0\). Fix such a \(\lambda\), and assume indirectly that \(E^k = E \times \ldots \times E\) does not contain any simplex isometric to \(\lambda V\), i.e. any point of the compact configuration space \(S_{\lambda V} \subseteq \mathbb{R}^{2k}\) of such simplices. By compactness, this implies that there is some \(\eta > 0\) such that
As explained in Section 6, the $\eta$-neighborhood of $E^k$ also does not contain any simplex isometric to $\lambda V$. As the support of $\tilde{\mu}_\varepsilon$ is contained in the $C_k \varepsilon^{1/2}$-neighborhood of $E$, as $E = \text{supp} \mu$, it follows that $T_{\lambda V}(\tilde{\mu}_\varepsilon) = 0$ for all $\varepsilon < c_k \eta^2$ and hence $T_{\lambda V}(\mu) = 0$, contradicting our choice of $\lambda$. This proves Theorem 1.

4. The configuration space of isometric distance graphs.

Let $\Gamma_0 = (V_0, E)$ be a fixed proper distance graph, with vertex set $V_0 = \{v_0 = 0, v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ of degree $k < d$. Let $t_{ij} = |v_i - v_j|^2$ for $(i, j) \in E$. A distance graph $\Gamma = (V, E)$ with $V = \{x_0 = 0, x_1, \ldots, x_n\}$ is isometric to $\Gamma_0$ if and only if $\mathbf{x} = (x_1, \ldots, x_n) \in S_{\Gamma_0}$, where

$$S_{\Gamma_0} = \{(x_1, \ldots, x_n) \in \mathbb{R}^{dn}; |x_i - x_j|^2 = t_{ij}, \forall \ 0 \leq i < j \leq n, (i, j) \in E\}$$

We call the algebraic set $S_{\Gamma_0}$ the configuration space of isometric copies of the $\Gamma_0$. Note that $S_{\Gamma_0}$ is the zero set of the family $F = \{f_{ij}; (i, j) \in E\}$, $f_{ij}(\mathbf{x}) = |x_i - x_j|^2 - t_{ij}$, thus it is a special case of the general situation described in Section 5.

If $\Gamma \cong \Gamma_0$ with vertex set $V = \{x_0 = 0, x_1, \ldots, x_n\}$ is proper then $\mathbf{x} = (x_1, \ldots, x_n)$ is a non-singular point of $S_{\Gamma_0}$. Indeed, for a fixed $1 \leq j \leq n$ let $\Gamma_j$ be the distance graph obtained from $\Gamma$ by removing the vertex $x_j$ together with all edges emanating from it. By induction we may assume that $\mathbf{x}' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ is a non-singular point i.e the gradient vectors $\nabla_{\mathbf{x}'} f_{ik}(\mathbf{x})$, $(i, k) \in E$, $i \neq j, k \neq j$ are linearly independent. Since $\Gamma$ is proper the gradient vectors $\nabla_{\mathbf{x}} f_{ij}(\mathbf{x}) = 2(x_i - x_j)$, $(i, j) \in E$ are also linearly independent hence $\mathbf{x}$ is a non-singular point. In fact we have shown that the partition of coordinates $\mathbf{x} = (y, z)$ with $y = x_j$ and $z = \mathbf{x}'$ is admissible and hence (6.4) holds.

Let $r_0 = r(\Gamma_0) > 0$. It is clear that if $\Gamma \cong \Gamma_0$ and $|x_j - v_j| \leq \eta_0$ for all $1 \leq j \leq n$, for a sufficiently small $\eta = \eta(r_0) > 0$, then $\Gamma$ is proper and $r(\Gamma) \geq r_0/2$. For given $1 \leq j \leq n$, let $X_j := \{x_i \in V; (i, j) \in E\}$ and define

$$S_{X_j} := \{x \in \mathbb{R}^d; |x_i - x_i|^2 = t_{ij}, \text{ for all } x_i \in X_j\}.$$  

As explained in Section 6, $S_{X_j}$ is a sphere of dimension $d - |X_j| \geq 1$ with radius $r(X_j) \geq r_0/2$. Let $\sigma_{X_j}$ denote the surface area measure on $S_{X_j}$ and write $\nu_{X_j} := \phi_j \sigma_{X_j}$, where $\phi_j$ is a smooth cut-off function supported in an $\eta$-neighborhood of $v_j$ with $\phi_j(v_j) = 1$.

Write $\mathbf{x} = (x_1, \ldots, x_n)$, $\phi(\mathbf{x}) := \prod_{j=1}^n \phi_j(x_j)$, then by (6.4) and (6.5), one has

$$\int g(\mathbf{x}) \phi(\mathbf{x}) d\omega_F(\mathbf{x}) = c_j(\Gamma_0) \int \int g(\mathbf{x}) \phi(\mathbf{x}') d\nu_{X_j}(x_j) d\omega_F(\mathbf{x}'), \quad (4.1)$$

where $\mathbf{x}' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and $F_j = \{f_{il}; (i, l) \in E, l \neq j\}$. The constant $c_j(\Gamma_0) > 0$ is the reciprocal of volume of the parallelotope.
5. Proof of Theorem 2

Let $d > k$ and again, without loss of generality, assume that $d(\Gamma) = 1$ and hence $\delta(\Gamma) = r(\Gamma)$. Given $\lambda, \varepsilon > 0$ define the multi-linear expression,

$$T_{\Lambda_0}(\mu_\varepsilon) := \int \cdots \int \mu_\varepsilon(x)\mu_\varepsilon(x - \lambda x_1) \cdots \mu_\varepsilon(x - \lambda x_n) \phi(x_1, \ldots, x_n) d\omega_F(x_1, \ldots, x_n) dx.$$ 

(5.1)

Given a proper distance graph $\Gamma_0 = (V, E)$ on $|V| = n$ vertices of degree $k < n$ one has the following upper bound;

Lemma 4. There exists a constant $C = C_{n,d,k}(r_0) > 0$ such that

$$|T_{\Lambda_0}(\mu_{2\varepsilon}) - T_{\Lambda_0}(\mu_\varepsilon)| \leq C \lambda^{-1/2} \varepsilon^{(n+\frac{d}{2})(s-d)+\frac{1}{2}}. \tag{5.2}$$

(5.2)

This implies again that in dimensions $d - \frac{1}{2} \leq s \leq d$, there exists the limit $T_{\Lambda_0}(\mu) := \lim_{\varepsilon \to 0} T_{\Lambda_0}(\mu_\varepsilon)$. Also, the lower bound (3.8) holds for distance graphs of degree $k$, as it was shown for a large class of graphs, the so-called $k$-degenerate distance graphs, see [10]. Thus one may argue exactly as in Section 3, to prove that there exists a $\lambda > 0$ for which

$$T_{\Lambda_0}(\mu) > 0, \tag{5.3}$$

and Theorem 2 follows from the compactness of the configuration space $S_{\Lambda_0} \subseteq \mathbb{R}^{dn}$. It remains to prove Lemma 4.

Proof of Lemma 4. Write $\Delta T(\mu_\varepsilon) := T_{\Lambda_0}(\mu_\varepsilon) - T_{\Lambda_0}(\mu_{2\varepsilon})$. Then we have $\Delta T(\mu_\varepsilon) = \sum_{j=1}^n \Delta_j T(\mu_\varepsilon)$, where $\Delta_j T(\mu_\varepsilon)$ is given by (5.1) with $\mu_\varepsilon(x - \lambda x_j)$ replaced by $\Delta \mu_\varepsilon(x - \lambda x_j)$ given in (3.8), and $\mu_\varepsilon(x - \lambda x_j)$ by $\mu_{2\varepsilon}(x - \lambda x_j)$ for $i > j$. Then by (4.1) we have the analogue of estimate (3.9)

$$|\Delta T(\mu_\varepsilon)| \lesssim \varepsilon^{(n-1)(s-d)} \int \left| \int \mu_\varepsilon(x) \Delta \mu_\varepsilon \ast X_j(x) dx \right| \phi(x') d\omega_F(x'), \tag{5.4}$$

(5.4)

where $\phi(x') = \prod_{i \neq j} \phi(x_j)$. Thus by Cauchy-Schwarz and Plancherel,

$$|\Delta_j T_{\varepsilon}(\mu)|^2 \lesssim \varepsilon^{2n(s-d)} \int |\hat{\Delta_j \mu}(\xi)|^2 I_{\lambda_j}(\xi) d\xi,$$

where

$$I_{\lambda_j}(\xi) = \int |\hat{\nu}_{X_j}(\lambda \xi)|^2 \phi(x') d\omega_F(x').$$

Proof of Lemma 4. Write $\Delta T(\mu_\varepsilon) := T_{\Lambda_0}(\mu_\varepsilon) - T_{\Lambda_0}(\mu_{2\varepsilon})$. Then we have $\Delta T(\mu_\varepsilon) = \sum_{j=1}^n \Delta_j T(\mu_\varepsilon)$, where $\Delta_j T(\mu_\varepsilon)$ is given by (5.1) with $\mu_\varepsilon(x - \lambda x_j)$ replaced by $\Delta \mu_\varepsilon(x - \lambda x_j)$ given in (3.8), and $\mu_\varepsilon(x - \lambda x_j)$ by $\mu_{2\varepsilon}(x - \lambda x_j)$ for $i > j$. Then by (4.1) we have the analogue of estimate (3.9)

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$$|\Delta_j T_{\varepsilon}(\mu)|^2 \lesssim \varepsilon^{2n(s-d)} \int |\hat{\Delta_j \mu}(\xi)|^2 I_{\lambda_j}(\xi) d\xi,$$

where

$$I_{\lambda_j}(\xi) = \int |\hat{\nu}_{X_j}(\lambda \xi)|^2 \phi(x') d\omega_F(x').$$
Recall that on the support of \( \phi(x') \) \( S_{X_j} \) is a sphere of dimension at least 1 and of radius \( r \geq r_0/2 > 0 \), contained in an affine subspace orthogonal to \( \text{Span} X_j \). Thus,

\[
|\hat{\mu}_{X_j}(\lambda \xi)|^2 \lesssim (1 + r_0 \lambda \text{dist}(\xi, \text{Span} X_j))^{-1}.
\]

Let \( U : \mathbb{R}^d \to \mathbb{R}^d \) be a rotation and for \( x' = (x_i)_{i \neq j} \) write \( Ux' = (Ux_i)_{i \neq j} \). As explained in Section 6, the measure \( \omega_{f_j} \) is invariant under the transformation \( x' \to Ux' \), hence

\[
I_\lambda(\xi) \lesssim \int \int (1 + r_0 \lambda \text{dist}(\xi, \text{Span} U X_j))^{-1} \, d\omega_{f_j}(x') \, dU
\]

\[
= \int \int (1 + r_0 \lambda |\xi| \text{dist}(\eta, \text{Span} X_j))^{-1} \, d\sigma_{d-1}(\eta) \, d\omega_{f_j}(x')
\]

\[
\lesssim (1 + r_0 \lambda |\xi|)^{-1},
\]

where we have written again \( \eta := |\xi|^{-1} U \xi \in S^{d-1} \).

Then we argue as in Lemma 2, noting that \( \hat{\Delta} \mu_\varepsilon(\xi) \) is essentially supported on \( |\xi| \approx \varepsilon^{-1} \) we have that

\[
|\Delta T(\mu_\varepsilon)|^2 \lesssim r_0^{-1} \lambda^{-1} \varepsilon^{2n(s-d)+\frac{1}{2}} \int |\hat{\mu}(\xi)|^2 \phi(\varepsilon \xi) \, d\xi \lesssim r_0^{-1} \lambda^{-1} \varepsilon^{(2n+1)(s-d)+\frac{1}{2}},
\]

with \( \hat{\mu}_\varepsilon = \mu_\varepsilon \) or \( \hat{\mu}_\varepsilon = \mu \ast \phi_\varepsilon \). This proves Lemma 4. \( \square \)

6. MEASURES ON REAL ALGEBRAIC SETS.

Let \( \mathcal{F} = \{f_1, \ldots, f_n\} \) be a family of polynomials \( f_i : \mathbb{R}^d \to \mathbb{R} \). We will describe certain measures supported on the algebraic set

\[
S_\mathcal{F} := \{x \in \mathbb{R}^d : f_1(x) = \ldots = f_n(x) = 0\}. \tag{6.1}
\]

A point \( x \in S_\mathcal{F} \) is called non-singular if the gradient vectors

\[
\nabla f_1(x), \ldots, \nabla f_n(x)
\]

are linearly independent, and let \( S^0_\mathcal{F} \) denote the set of non-singular points. It is well-known and is easy to see, that if \( S^0_\mathcal{F} \neq \emptyset \) then it is a relative open, dense subset of \( S_\mathcal{F} \), and moreover it is an \( d-n \)-dimensional sub-manifold of \( \mathbb{R}^d \). If \( x \in S^0_\mathcal{F} \) then there exists a set of coordinates, \( J = \{j_1, \ldots, j_n\} \), with \( 1 \leq j_1 < \ldots < j_n \leq d \), such that

\[
j_{\mathcal{F}, J}(x) := \det \left( \frac{\partial f_i}{\partial x_j}(x) \right)_{1 \leq i \leq n, j \in J} \neq 0. \tag{6.2}
\]

Accordingly, we will call a set of coordinates \( J \) admissible, if (6.2) holds for at least one point \( x \in S^0_\mathcal{F} \), and will denote by \( S_{\mathcal{F}, J} \) the set of such points. For a given set of coordinates \( x, J \) let \( \nabla_{x, J} f(x) := (\partial_{x_j} f(x))_{j \in J} \) and note that \( J \) is admissible if and only if the gradient vectors

\[
\nabla_{x, J} f_1(x), \ldots, \nabla_{x, J} f_n(x)
\]
are linearly independent at at least one point \( x \in S_{\mathcal{F}} \). It is clear that, unless \( S_{\mathcal{F},J} = \emptyset \), it is a relative open and dense subset of \( S_{\mathcal{F}} \) and is also \( d - n \)-dimensional sub-manifold, moreover \( S_{\mathcal{F}}^0 \) is the union of the sets \( S_{\mathcal{F},J} \) for all admissible \( J \).

We define a measure, near a point \( x_0 \in S_{\mathcal{F},J} \) as follows. For simplicity of notation assume that \( J = \{1, \ldots, n\} \) and let

\[
\Phi(x) := (f_1, \ldots, f_n, x_{n+1}, \ldots, x_d).
\]

Then \( \Phi : U \to V \) is a diffeomorphism on some open set \( x_0 \in U \subseteq \mathbb{R}^d \) to its image \( V = \Phi(U) \), moreover \( S_{\mathcal{F}} = \Phi^{-1}(V \cap \mathbb{R}^{d-n}) \). Indeed, \( x \in S_{\mathcal{F}} \) if and only if \( \Phi(x) = (0, \ldots, 0, x_{n+1}, \ldots, x_d) \in V \). Let \( I = \{n + 1, \ldots, d\} \) and write \( x_I := (x_{n+1}, \ldots, x_d) \). Let \( \Psi(x_I) = \Phi^{-1}(0, x_I) \) and in local coordinates \( x_I \) define the measure \( \omega_{\mathcal{F}} \) via

\[
\int g \, d\omega_{\mathcal{F}} := \int g(\Psi(x_I)) \, \text{Jac}^{-1}_{\Phi}(\Psi(x_I)) \, dx_I,
\]

for a continuous function \( g \) supported on \( U \). Note that \( \text{Jac}_{\Phi}(x) = J_{\mathcal{F},J}(x) \), i.e. the Jacobian of the mapping \( \Phi \) at \( x \in U \) is equal to the expression given in (6.2), and that the measure \( d\omega_{\mathcal{F}} \) is supported on \( S_{\mathcal{F}} \). Define the local coordinates \( y_j = f_j(x) \) for \( 1 \leq j \leq n \) and \( y_j = x_j \) for \( n < j \leq d \). Then

\[
dy_1 \wedge \ldots \wedge dy_d = df_1 \wedge \ldots \wedge df_n \wedge dx_{n+1} \wedge \ldots \wedge dx_d = \text{Jac}_{\Phi}(x) \, dx_1 \wedge \ldots \wedge dx_d,
\]

thus

\[
dx_1 \wedge \ldots \wedge dx_d = \text{Jac}_{\Phi}(x)^{-1} df_1 \wedge \ldots \wedge df_n \wedge dx_{n+1} \wedge \ldots \wedge dx_d = df_1 \wedge \ldots \wedge df_n \wedge d\omega_{\mathcal{F}}.
\]

This shows that the measure \( d\omega_{\mathcal{F}} \) (given as a differential \( d - n \)-form on \( S_{\mathcal{F}} \)) is independent of the choice of local coordinates \( x_I \). Then \( \omega_{\mathcal{F}} \) is defined on \( S_{\mathcal{F}}^0 \) and moreover the set \( S_{\mathcal{F}}^0 \setminus S_{\mathcal{F},J} \) is of measure zero with respect to \( \omega_{\mathcal{F}} \), as it is a proper analytic subset on \( \mathbb{R}^{d-n} \) in any other admissible local coordinates.

Let \( x = (z, y) \) be a partition of coordinates in \( \mathbb{R}^d \), with \( y = x_{J_2} \), \( z = x_{J_1} \), and assume that for \( i = 1, \ldots, m \) the functions \( f_i \) depend only on the \( z \)-variables. We say that the partition of coordinates is admissible, if there is a point \( x = (z, y) \in S_{\mathcal{F}} \) such that both the gradient vectors \( \nabla_z f_1(x), \ldots, \nabla_z f_m(x) \) and the vectors \( \nabla_y f_{m+1}(x), \ldots, \nabla_y f_n(x) \) for a linearly independent system. Partition the system \( \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \) with \( \mathcal{F}_1 = \{f_1, \ldots, f_m\} \) and \( \mathcal{F}_2 = \{f_{m+1}, \ldots, f_n\} \). Then there is set \( J_1' \subseteq J_1 \) for which

\[
j_{\mathcal{F}_1, J_1'}(z) := \det \left( \frac{\partial f_i}{\partial x_j}(z) \right)_{1 \leq i \leq m, j \in J_1'} \neq 0,
\]

and also a set \( J_2' \subseteq J_2 \) such that

\[
j_{\mathcal{F}_2, J_2'}(z, y) := \det \left( \frac{\partial f_i}{\partial x_j}(z, y) \right)_{m+1 \leq i \leq n, j \in J_2'} \neq 0.
\]
Since $\nabla_y f_i \equiv 0$ for $1 \leq i \leq m$, it follows that the set of coordinates $J' = J'_1 \cup J'_2$ is admissible, moreover

$$j_{J',r'}(y,z) = j_{J'_1,r'}(z) \cdot j_{J'_2,r'}(y,z).$$

For fixed $z$, let $f_{i,z}(y) := f_i(z,y)$ and let $\mathcal{F}_{2,z} = \{f_{m+1,z}, \ldots, f_n,z\}$. Then clearly $j_{J'_2,r'}(y,z) = j_{J'_2,r'}(y)$ as it only involves partial derivatives with respect to the $y$-variables. Thus we have an analogue of Fubini’s theorem, namely

$$\int g(x) \, d\omega_{\mathcal{F}}(x) = \int \int g(z,y) \, d\omega_{\mathcal{F}_{2,z}}(y) \, d\omega_{\mathcal{F}_1}(z). \quad (6.4)$$

Consider now algebraic sets given as the intersection of spheres. Let $x_1, \ldots, x_m \in \mathbb{R}^d$, $t_1, \ldots, t_m > 0$ and $\mathcal{F} = \{f_1, \ldots, f_m\}$ where $f_i(x) = |x - x_i|^2 - t_i$ for $i = 1, \ldots, m$. Then $S_{\mathcal{F}}$ is the intersection of spheres centered at the points $x_i$ of radius $r_i = t_i^{1/2}$. If the set of points $X = \{x_1, \ldots, x_m\}$ is in general position (i.e. they span an $m - 1$-dimensional affine subspace), then a point $x \in S_{\mathcal{F}}$ is non-singular if $x \notin \text{span} \, X$, i.e. if $x$ cannot be written as linear combination of $x_1, \ldots, x_m$. Indeed, since $\nabla f_i(x) = 2(x - x_i)$ we have that

$$\sum_{i=1}^m a_i \nabla f_i(x) = 0 \iff \sum_{i=1}^m a_i x = \sum_{i=1}^m a_i x_i,$$

which implies $\sum_{i=1}^m a_i = 0$ and $\sum_{i=1}^m a_i x_i = 0$. By replacing the equations $|x - x_i|^2 = t_i$ with $|x - x_1|^2 - |x - x_i|^2 = t_1 - t_i$, which is of the form $x \cdot (x_1 - x_i) = c_i$, for $i = 2, \ldots, m$, it follows that $S_{\mathcal{F}}$ is the intersection of sphere with an $n-1$-codimensional affine subspace $Y$, perpendicular to the affine subspace spanned by the points $x_i$. Thus $S_{\mathcal{F}}$ is an $m$-codimensional sphere of $\mathbb{R}^d$ if $S_{\mathcal{F}}$ has one point $x \notin \text{span} \{x_1, \ldots, x_m\}$ and all of its points are non-singular. Let $x'$ be the orthogonal projection of $x$ to $\text{span} \, X$. If $y \in Y$ is a point with $|y - x'| = |x - x'|$ then by the Pythagorean theorem we have that $|y - x| = |x - x_i|$ and hence $y \in S_{\mathcal{F}}$. It follows that $S_{\mathcal{F}}$ is a sphere centered at $x'$ and contained in $Y$.

Let $T = T_X$ be the inner product matrix with entries $t_{ij} := (x-x_i) \cdot (x-x_j)$ for $x \in S_{\mathcal{F}}$. Since $(x-x_i) \cdot (x-x_j) = 1/2(t_i + t_j - |x_i - x_j|^2)$ the matrix $T$ is independent of $x$. We will show that $d\omega_{\mathcal{F}} = c_T \, d\sigma_{S_{\mathcal{F}}}$ where $d\sigma_{S_{\mathcal{F}}}$ denotes the surface area measure on the sphere $S_{\mathcal{F}}$ and $c_T = 2^{-m} \text{det}(T)^{-1/2} > 0$, i.e for a function $g \in C_0(\mathbb{R}^d)$,

$$\int_{S_{\mathcal{F}}} g(x) \, d\omega_{\mathcal{F}}(x) = c_T \int_{S_{\mathcal{F}}} g(x) \, d\sigma_{S_{\mathcal{F}}}(x). \quad (6.5)$$

Let $x \in S_{\mathcal{F}}$ be fixed and let $e_1, \ldots, e_d$ be an orthonormal basis so that the tangent space $T_x S_{\mathcal{F}} = \text{Span} \{e_{m+1}, \ldots, e_d\}$ and moreover we have that
\(\text{Span}\{\nabla f_1, \ldots, \nabla f_m\} = \text{Span}\{e_1, \ldots, e_m\}\). Let \(x_1, \ldots, x_n\) be the corresponding coordinates on \(\mathbb{R}^d\) and note that in these coordinates the surface area measure, as a \(d - m\)-form at \(x\), is
\[
d\sigma_{S_F}(x) = dx_{m+1} \wedge \ldots \wedge dx_d.
\]
On the other hand, in local coordinates \(x_I = (x_{m+1}, \ldots, x_d)\), it is easy to see from (6.2)-(6.3) that \(J_{F,J}(x) = 2^m \text{vol}(x - x_1, \ldots, x - x_m)\) and hence
\[
d\omega_F(x) = 2^{-m} \text{vol}(x - x_1, \ldots, x - x_m)^{-1} dx_{m+1} \wedge \ldots \wedge dx_d,
\]
where \(\text{vol}(x - x_1, \ldots, x - x_m)\) is the volume of the parallelootope with side vectors \(x - x_j\). Finally, it is a well-known fact from linear algebra that
\[
\text{vol}(x - x_1, \ldots, x - x_m)^2 = \det(T),
\]
i.e. the volume of a parallelootope is the square root of the Gram matrix formed by the inner products of its side vectors.

References

[1] M. Bennett, A. Iosevich, K. Taylor, Finite chains inside thin subsets of \(\mathbb{R}^d\). Analysis PDE, 9(3), (2016) pp.597-614.

[2] J. Bourgain, A Szemerédi type theorem for sets of positive density in \(\mathbb{R}^k\). Israel J. Math., 54(3), (1986), 307-316.

[3] K.J. Falconer, Some problems in measure combinatorial geometry associated with Paul Erdős, [http://www.renyi.hu/conferences/erdos100/slides/falconer.pdf](http://www.renyi.hu/conferences/erdos100/slides/falconer.pdf)

[4] R. Fraser, Robert, M. Pramanik, Large sets avoiding patterns. Analysis and PDE 11, no. 5 (2018): 1083-1111.

[5] D. Hart and A. Iosevich, Ubiquity of simplices in subsets of vector spaces over finite fields, Anal. Math. 34 (2008), no. 1, 29-38.

[6] A. Iosevich, B. Liu, Equilateral triangles in subsets of \(\mathbb{R}^d\) of large Hausdorff dimension. Israel J. Math. 231, no. 1 (2019) 123-137.

[7] A. Iosevich and H. Parshall, Embedding distance graphs in finite field vector spaces, J. Korean Math. Soc. 56 (2019), no. 6, 1515-1528.

[8] A. Iosevich, K. Taylor, Finite trees inside thin subsets of \(\mathbb{R}^d\). Modern methods in operator theory and harmonic analysis, 51-56, Springer Proc. Math. Stat., 291, Springer, Cham, (2019).

[9] T. Keleti, Construction of 1-dimensional subsets of the reals not containing similar copies of given patterns. Anal. PDE 1 (2008), no. 1, 29-33.
[10] N. Lyall, Á. Magyar, *Distance Graphs and sets of positive upper density in $\mathbb{R}^d$*. Analysis and PDE (to appear)

[11] P. Maga, *Full dimensional sets without given patterns*. Real Analysis Exchange 36, no. 1 (2011): 79-90.

[12] P. Mattila, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Vol. 44. Cambridge University Press, (1999)

[13] T. Ziegler, *Nilfactors of $\mathbb{R}^m$ and configurations in sets of positive upper density in $\mathbb{R}^n$*. Journal d'Analyse Mathematique 99.1 (2006), 249-266