COMMENTS ON QUANTUM COSMOLOGY WITH EXTRINSIC TIME

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ABSTRACT

The closed de Sitter universe is used to present a way to deal with the deparametrization and quantization of cosmological models with extrinsic time.

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1 Introduction

An essential property of gravitational dynamics is that the canonical Hamiltonian vanishes on the physical trajectories of the system, which then includes a constraint $\mathcal{H} \approx 0$. This reflects the fact that the evolution of the gravitational field is given in terms of a parameter $\tau$ which does not have physical significance. This feature leads to a fundamental difference between ordinary quantum mechanics and the quantization of gravitation, because the existence of a unitary quantum theory is related to the possibility of defining the time as an absolute parameter. The identification of a global phase time $\uparrow$ can therefore be considered as the previous step before quantization $\uparrow$.

For minisuperspace models we have an action functional of the form

$$ S[q^i, p_i, N] = \int_{\tau_1}^{\tau_2} \left( p_i \frac{dq_i}{d\tau} - N\mathcal{H} \right) d\tau $$

(1)

where $N$ is a Lagrange multiplier enforcing the quadratic Hamiltonian constraint

$$ \mathcal{H} = G^{ij} p_i p_j + V(q) \approx 0, $$

(2)

with $G^{ij}$ the reduced version of the DeWitt supermetric. The extremal condition $\delta S = 0$ gives the canonical equations

$$ \frac{dq_i}{d\tau} = N[q^i, \mathcal{H}], \quad \frac{dp_i}{d\tau} = N[p_i, \mathcal{H}]. $$

(3)

The solution of these equations describes the evolution of a spacelike hypersurface along the timelike direction; the presence the of the multiplier $N$ introduces an arbitrariness in the evolution which is associated to a multiplicity of times. From a different point of view, the constraint $\mathcal{H} \approx 0$ acts as a generator of gauge transformations which can be written

$$ \delta_\epsilon q^i = \epsilon(\tau)[q^i, \mathcal{H}], \quad \delta_\epsilon p_i = \epsilon(\tau)[p_i, \mathcal{H}], \quad \delta_\epsilon N = \frac{\partial \epsilon(\tau)}{\partial \tau}. $$

(4)
From (3) and (4) we see that the dynamical evolution can be reproduced by a gauge transformation progressing with time, that is, any two successive points on each classical trajectory are connected by a gauge transformation \[3\]. Hence, the gauge fixation can be thought not only as a way to select one path from each class of equivalent paths in phase space, but also as a reduction procedure identifying a time for the system.

Admissible gauge conditions are those which can be reached from any path in phase space by means of gauge transformations leaving the action unchanged. Under a gauge transformation defined by the parameters \(\epsilon^m\) the action of a system with constraints \(C_m\) changes by

\[
\delta_{\epsilon} S = \left[\epsilon^m(\tau) \left(\dot{p}_i \frac{\partial C_m}{\partial p_i} - C_m\right)\right]_{\tau_1}^{\tau_2}.
\]  

(5)

For ordinary gauge systems, which include constraints that are linear and homogeneous in the momenta, it is \(\delta_{\epsilon} S = 0\), and gauge conditions of the form \(\chi(q, p, \tau) = 0\) (canonical gauges) are admissible. In the case of gravitation, instead, the Hamiltonian constraint is quadratic in the momenta, and we would have \(\delta_{\epsilon} S \neq 0\) unless \(\epsilon(\tau_1) = \epsilon(\tau_2) = 0\); then gauge conditions involving derivatives of Lagrange multipliers as, for example, \(\chi \equiv dN/d\tau = 0\) (derivative gauges) should be used. These gauges cannot define a time in terms of the canonical variables. At the quantum level this has the consequence that the usual Fadeev–Popov path integral procedure for quantizing gauge systems could not be applied.

However, in recent papers \[7\]-\[10\] we have shown that if the system results separable the action can be provided with gauge invariance at the end points by means of a canonical transformation \((q^i, p_i) \rightarrow (Q^i, P_i)\) which matches the constraint \(H\) with one of the new momenta. Then canonical gauges are admissible and a global phase time can be identified by imposing \(\tau\)—dependent gauge conditions, and, simultaneously, the transition amplitude can be obtained by means of the usual path integral procedure for gauge systems. Moreover, we have shown that a transition amplitude between two given quantum
configurations in terms of only the original coordinates can be found only in the case that an *intrinsic time* exists.

A function \( t(q^i, p_i) \) is a global phase time if \([t, \mathcal{H}] > 0\). Because the supermetric \( G^{ik} \) does not depend on the momenta, a function \( t(q) \) is a global phase time if the bracket

\[
[t(q), \mathcal{H}] = [t(q), G^{ik} p_i p_k] = 2 \frac{\partial t}{\partial q^i} G^{ik} p_k
\]

is positive definite. If the supermetric has a diagonal form and one of the momenta vanishes at a given point of phase space, then no function of its conjugate coordinate only can be a global phase time. This is something to be remarked, as in some early works [11] the scale factor was used as time parameter in the obtention of a Schrödinger equation for the minisuperspaces, and this is not licit in general; examples of this are the cases of Kantowski–Sachs anisotropic universe [10] and isotropic models yielding from the low energy limit of closed bosonic string theory [12]. For a constraint whose potential can be zero for finite values of the coordinates, the momenta \( p_k \) can be all equal to zero at a given point, and \([t(q), \mathcal{H}]\) can vanish. Hence an intrinsic global phase time \( t(q) \) can be identified only if the potential in the constraint has a definite sign. In the most general case a global phase time should be a function including the canonical momenta; in this case it is said that the system has an extrinsic time \( t(q, p) \) [13], because the momenta are related to the extrinsic curvature. When only an extrinsic time exists we shall have to revise some points of the path integral quantization to which we are used: as we shall see below, a quantum description in terms of only the original coordinates may be impossible if we want to work in a theory with a clear notion of time.
2 A simple model: The closed de Sitter universe.

In order to discuss this topic we shall use a simple model which presents all the peculiarities of the models of physical interest; this model has the special feature that the transition probability is trivially known since it has one constraint and only one degree of freedom, so that there is only one physical state for the system. Despite its simplicity, it is a good example to understand the quantization with extrinsic time.

Consider the Hamiltonian constraint of the most general empty homogeneous and isotropic cosmological model:

\[ H = -\frac{1}{4} e^{-3\Omega} \pi_\Omega^2 - k e^{\Omega} + \Lambda e^{3\Omega} \approx 0. \]  

This Hamiltonian corresponds to a universe with arbitrary curvature \( k = -1, 0, 1 \) and non-zero cosmological constant; we shall suppose \( \Lambda > 0 \). In the case \( k = 0 \) we obtain the de Sitter universe; although the absence of matter makes this universe basically a toy model, it has received considerable attention because it reproduces the behaviour of models with matter or with non-zero curvature when the scale factor \( a \sim e^{\Omega} \) is great enough. The classical evolution is easy to obtain, and corresponds to an exponential expansion. In fact, for both \( k = 0 \) and \( k = -1 \) the potential is never zero, and then \( \pi_\Omega \) cannot change its sign. Instead, for the closed model \( \pi_\Omega = 0 \) is possible.

It is convenient to work with the rescaled Hamiltonian \( H = e^{-\Omega} \mathcal{H} : \)

\[ H = -\frac{1}{4} e^{-4\Omega} \pi_\Omega^2 - k + \Lambda e^{2\Omega} \approx 0. \]  

The constraints \( H \) and \( \mathcal{H} \) are equivalent because they differ only in a positive definite factor.

We shall turn the parametrized system associated to the Hamiltonian (7) into an ordinary gauge system by means of two successive canonical transformations. The \( \tau \)-independent
Hamilton–Jacobi equation for the Hamiltonian $H$ is

$$-\left(\frac{\partial W}{\partial \Omega}\right)^2 - 4ke^{4\Omega} + 4\Lambda e^{6\Omega} = 4e^{4\Omega}E.$$  \hspace{1cm} (8)

Matching $E = \overline{P}_0$ we obtain the solution

$$W(\Omega, \overline{P}_0) = 2\text{sign}(\pi_\Omega) \int d\Omega e^{2\Omega} \sqrt{\Lambda e^{2\Omega} - k - \overline{P}_0},$$  \hspace{1cm} (9)

which is the generating function of the canonical transformation $(\Omega, \pi_\Omega) \rightarrow (Q^0, \overline{P}_0)$ defined by

$$Q^0 = \frac{\partial W}{\partial \overline{P}_0} = -\text{sign}(\pi_\Omega)\Lambda^{-1}\sqrt{\Lambda e^{2\Omega} - k - \overline{P}_0}, \quad \overline{P}_0 = H.$$  \hspace{1cm} (10)

Then we define the function $F = Q^0\overline{P}_0 + f(\tau)$ which generates the second canonical transformation yielding a non vanishing true Hamiltonian $h = \partial f/\partial \tau$ and $Q^0 = \overline{Q}_0$, $\overline{P}_0 = \overline{P}_0$. This second transformation may seem to be an unnecessary sofistication, but it plays a central role in the case of cosmological models with true degrees of freedom, as it allows the new coordinates $Q$ associated to observables to be fixed at arbitrary values at the boundaries. If possible, $f$ should be chosen in such a way that the reduced Hamiltonian is conservative [14].

The variables $Q^0$ and $P_0$ describe the gauge system into which the model has been turned. Therefore the gauge can be fixed by means of a $\tau$–dependent canonical condition like $\chi \equiv Q^0 - T(\tau) = 0$ with $T$ a monotonic function of $\tau$. Then as $[Q^0, P_0] = 1$ we can define the global phase time as

$$t = Q^0|_{P_0=0} = -\text{sign}(\pi_\Omega)\Lambda^{-1}\sqrt{\Lambda e^{2\Omega} - k}.$$  \hspace{1cm} (11)

As on the constraint surface $P_0 = 0$ we have

$$\pi_\Omega = 2\text{sign}(\pi_\Omega)e^{2\Omega}\sqrt{\Lambda e^{2\Omega} - k},$$  \hspace{1cm} (12)
(so that in the case \( k = 1 \) the natural size of the configuration space is given by \( \Omega \geq -\ln(\sqrt{\Lambda}) \)) we can write

\[
t(\Omega, \pi_\Omega) = -\frac{1}{2} \Lambda^{-1} e^{-2\Omega} \pi_\Omega,
\]

which is in agreement with the time obtained by matching the model with the ideal clock [13, 8]. It is interesting to notice that this expression for the time can be also obtained by demanding that its functional form presents no ambiguities under different ways of factorizing the Hamiltonian constraint.

Now an important difference between the cases \( k = -1 \) and \( k = 1 \) arises: for \( k = -1 \) the potential has a definite sign, and the constraint surface splits into two disjoint sheets given by (12). In this case the evolution can be parametrized by a function of the coordinate \( \Omega \) only, the choice given by the sheet on which the system remains: if the system is on the sheet \( \pi_\Omega > 0 \) the time is \( t = -\Lambda^{-1} \sqrt{\Lambda e^{2\Omega} + 1} \), and if it is on the sheet \( \pi_\Omega < 0 \) we have \( t = \Lambda^{-1} \sqrt{\Lambda e^{2\Omega} + 1} \). The deparametrization of the flat model is completely analogous. For the closed model, instead, the potential can be zero and the topology of the constraint surface is no more equivalent to that of two disjoint planes. Although for \( \Omega = -\ln(\sqrt{\Lambda}) \) we have \( V(\Omega) = 0 \) and \( \pi_\Omega = 0 \), it is easy to verify that \( d\pi_\Omega/d\tau \neq 0 \) at this point. Hence, in this case the coordinate \( \Omega \) does not suffice to parametrize the evolution, because the system can go from \( (\Omega, \pi_\Omega) \) to \( (\Omega, -\pi_\Omega) \); therefore we must necessarily define a global phase time as a function of both the coordinate and the momentum.

The system has one degree of freedom and one constraint, so that it is pure gauge. In other words, there is only one physical state, in the sense that from a given point in the phase space we can reach any other point on the constraint surface by means of a finite gauge transformation. For this reason, if the deparametrization procedure is consistent it should be possible to verify that the transition probability written in terms of the variables which include a globally well defined time is equal to unity.
The quantization proceeds as in references \[7\]-\[10\], and the observation above is reflected in the fact that we obtain

\[
< Q_0^2, \tau_2 | Q_0^1, \tau_1 > = \int DQ^0 DP_0 DN \delta(Q^0 - \tau) \exp \left( i \int_{\tau_1}^{\tau_2} \left[ P_0 \frac{dQ^0}{d\tau} - NP_0 - \frac{\partial f}{\partial \tau} \right] d\tau \right)
\]

\[
= \int DQ^0 DP_0 \delta(P_0) \delta(Q^0 - \tau) \exp \left( i \int_{\tau_1}^{\tau_2} \left[ P_0 \frac{dQ^0}{d\tau} - \frac{\partial f}{\partial \tau} \right] d\tau \right)
\]

\[
= \exp \left( -i \int_{\tau_1}^{\tau_2} \frac{\partial f}{\partial \tau} d\tau \right),
\]

(14)

and then the probability for the transition from \(Q_1^0\) at \(\tau_1\) to \(Q_2^0\) at \(\tau_2\) is

\[
| < Q_2^0, \tau_2 | Q_1^0, \tau_1 > |^2 = 1.
\]

(15)

When the model is open or flat the coordinates \(\Omega\) and \(Q^0\) are uniquely related, and the result can be easily understood in the sense that once a gauge is fixed there is only one possible value of the scale factor \(a \sim e^{\Omega}\) at each \(\tau\). But in the case of the closed model we have seen that this is not true: at each \(\tau\) there are two possible values of \(\Omega\); instead, there is only one possible value of \(\pi_\Omega\) at each \(\tau\). Hence the transition probability in terms of \(Q^0\) does not correspond to the evolution of the coordinate \(\Omega\), but rather of its derivative; then we conclude that the amplitude \(< Q_2^0, \tau_2 | Q_1^0, \tau_1 >\) corresponds to an amplitude \(< \pi_\Omega, 2 | \pi_\Omega, 1 >\), and we have

\[
| < \pi_\Omega, 2 | \pi_\Omega, 1 > |^2 = 1.
\]

(16)

The fact that \(< Q_2^0, \tau_2 | Q_1^0, \tau_1 >\) is not equivalent to \(< \Omega_2 | \Omega_1 >\) is natural, as the nonexistence of an intrinsic time makes impossible to find a globally good gauge condition giving \(\tau\) as a function of \(\Omega\) only \([4]\). But precisely for this reason, this should not be taken as a failure of the quantization procedure, because a characterization of the states in terms of only the original coordinates is not correct if we want to retain a clear notion of time on the whole evolution.
3 Discussion

It is common to regard an intrinsic time as more natural, and the necessity of defining an extrinsic time as a problematic peculiarity. However, this is perhaps only a consequence of working with simple parametrized systems like, for example, the relativistic particle; the formalism for these systems, when put in a manifestly covariant form, has the time included among the coordinates, and the evolution is given in terms of a physically meaningless time parameter. But while for these systems the time coordinate always refers to an external clock, this is clearly not the case in cosmology; for example, in the case of pure gravitational dynamics the coordinates are the elements of the metric $g_{ab}$ over spatial slices, and in principle there is not necessarily a connection between $g_{ab}$ and anything external. Rather, such a relation can be thought to exist for the derivatives $dg_{ab}/d\tau$ of the metric, as they appear in the expression for the extrinsic curvature $K_{ab}$ which describes the evolution of spacelike 3-dimensional hypersurfaces in 4-dimensional spacetime. If no matter fields are present the canonical momenta are given by

$$\pi_i \equiv p_i^{ab} = -2G_i{}^{abcd}K_{cd},$$

and then one must expect the momenta to appear in the definition of a global phase time. The existence of a time in terms of only the coordinates should therefore be understood as a sort of an accident related to the fact that, in some special cases which do not represent the general features of gravitation, there exists a relation that enables to obtain the coordinates in terms of the momenta with no ambiguities.

This means that we must revise some points of the path integral quantization to which we are used. Indeed, the system considered in this brief note represents an example of a cosmological model for which the characterization of the quantum states must include the momenta.

It is worth noting that at the quantum level the definition of an extrinsic time in
terms of the functional form (13) is not sufficient; in fact, it is also necessary to propose a prescription for the operatorial order between coordinates and momenta to give a precise definition of time. Indeed, it is possible to verify that an ordering which leads to define an extrinsic time for the closed de Sitter universe is given by

\[ \hat{t} \sim \hat{\pi}_\Omega e^{-2\hat{\Omega}}. \]  

(18)

We must remark that different orderings generate in the commutator \([\hat{t}, \hat{H}]\) linear terms in \(\pi_\Omega\), which lead to a non definite sign.

At this point, it could be interesting to comment that the description of the evolution of cosmological models in terms of the momenta is not a particular property of quantization; in fact, this is a common feature of classical cosmology where it is usual to deal with cosmological dynamics where the time is given in terms of the Hubble constant \(H\), namely

\[ t_H = H^{-1} \sim \frac{e^{3\Omega}}{\pi_\Omega}. \]  

(19)

Note that on the constraint surface the time (13) can be rewritten as

\[ t \sim \frac{k e^{2\Omega}}{\pi_\Omega} - \frac{\Lambda e^{4\Omega}}{\pi_\Omega}, \]  

(20)

which has an analogous form.

The problem of time in quantum gravity is an important open question. In this paper we have identified time variables for a particular minisuperspace model which is of interest in quantum cosmology by means of a systematic procedure to deparametrize the constrained system, and we have given a consistent quantization with a clear notion of time. We have pointed out the relevance of the study of the structure of the phase space in terms of the extrinsic curvature within the context of the procedure to find a global phase time for deparametrizing cosmological models which do not admit an intrinsic time.

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References

[1] R. Ferraro, Grav. Cosm. 5, 195 (1999).

[2] P. Hájícek, Phys. Rev. D34, 1040 (1986).

[3] A. O. Barvinsky, Phys. Rep. 230, 237 (1993).

[4] C. Teitelboim, Phys. Rev. D25, 3159 (1982).

[5] J. J. Halliwell, Phys. Rev. D38, 2468 (1988).

[6] M. Henneaux, C. Teitelboim and J. D. Vergara, Nucl. Phys. B387, 391 (1992).

[7] R. Ferraro and C. Simeone, J. Math. Phys. 38, 599 (1997).

[8] H. De Cicco and C. Simeone, Gen. Rel. Grav. 31, 1225 (1999).

[9] C. Simeone, J. Math. Phys. 40, 4527 (1999).

[10] C. Simeone, Gen. Rel. Grav. 32, 1835 (2000).

[11] M. P. Ryan and L. C. Shepley, Homogeneous Relativistic Cosmologies, Princeton Series in Physics, Princeton University Press, New Jersey (1975).

[12] G. Giribet and C. Simeone, Mod. Phys. Lett. A16, 19 (2001).

[13] K. V. Kuchař, in Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, edited by G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore, 1992).

[14] C. Simeone, The Deparametrization and Path Integral Quantization of Cosmological Models, World Scientific Lectures Notes in Physics, Vol 69, in preparation (World Scientific, Singapore, 2001).

[15] S. C. Beluardi and R. Ferraro, Phys. Rev. D52, 1963 (1995)