Strict Hierarchy of Strategies for Non-asymptotic Quantum Metrology

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(Dated: March 21, 2022)

One of the main quests in quantum metrology is to determine the ultimate precision limit with given resources, where the resources are not only of the number of queries, but more importantly of the allowed strategies. With the same number of queries, the restrictions on the strategies constrain the achievable precision. In this work, we establish a framework to systematically determine the ultimate precision limit of different strategies, which include the parallel, the sequential and the indefinite-causal-order strategies. With this framework, we show there exists a hierarchy of the precision limits under different families of strategies. An efficient algorithm to design an optimal strategy compatible with given constraints is provided, which has immediate practical applications.

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Introduction.—Quantum metrology [1,2] features a series of promising applications in the near future [3]. In the prototypical setting of quantum metrology, the goal is to estimate an unknown parameter carried by a quantum channel, given $N$ queries to it. A pivotal task is to design a strategy that utilizes these $N$ queries to generate a quantum state with as much information about the unknown parameter as possible. This often involves, for example, preparing a suitable input probe state $|\rho\rangle$ and applying intermediate quantum control [7, 8] as well as quantum error correction [9–12].

In reality, strategies that we could implement are often subject to physical restrictions. For instance, constraints on the amount of admissible resources, like energy, could affect the optimal probe state that can be prepared [13]. In particular, within the noisy and intermediate-scale quantum (NISQ) era [14], we often have to adjust the strategy to accommodate the limitations on the system. For example, for systems with short coherence time it might be favorable to adopt the parallel strategy, where multiple queries of the unknown channel are applied simultaneously on a multipartite entangled state (like the NOON state [15]), as illustrated in Fig. 1(a). In contrast for systems with restricted entanglement, it would be better to query the channel sequentially with a smaller system, as shown in Fig. 1(b).

It is therefore of utmost importance to identify the ultimate precision of quantum metrology under each family of strategies and determine whether one family of strategies strictly outperforms the other. In the previous literature, a nonzero gap between the optimal performance of parallel strategies and that of sequential strategies has been found by one of the authors in a multiparameter estimation problem [15]. It was also proven that in the asymptotic limit ($N \rightarrow \infty$) [16, 17] the gap vanishes for a set of quantum channels in the single-parameter estimation. However, it remains an intriguing open problem to quantify the gap in the fundamental setup of estimating a single parameter with $N < \infty$ queries for general quantum channels, which is always the case in practice. The main difficulty is the lack of a systematic method to determine the optimal precision under the sequential strategy where, unlike the parallel strategy, infinitely many choices of intermediate control operations can be employed. In addition to the parallel and sequential strategy, it was recently discovered that the quantum SWITCH [18], a primitive where the order of making queries to the unknown channel is in a quantum superposition (see Fig. 1(c)), can be employed to generate new strategies of quantum metrology [19, 20], some of which can even break the Heisenberg limit [21]. Moreover, strategies built upon indefinite causal structures beyond the quantum SWITCH [18, 22, 23] (see Fig. 1(d) and (e)) have recently been shown to further boost the per-
formance of certain information processing tasks \[24\], \[25\]. The performance of the indefinite-causal-order strategies in quantum metrology, however, remains unknown. The main reason is also the lack of a systematic approach to identify the precision limit under these new strategies.

In this work, we develop a systematic method of evaluating the optimal precision of single-parameter quantum metrology for finite \(N\) over a family of admissible strategies. With this method, we show a strict hierarchy (see Fig. 3) of the optimal performances under different families of strategies, which include the parallel, the sequential and the indefinite-causal-order \[18\], \[22\], \[23\] ones (see Fig. 1). An efficient algorithm to obtain the optimal strategy achieving the highest precision is also presented. A strict hierarchy of similar strategies has been investigated in channel discrimination previously \[25\], but much less is known in quantum metrology.

Quantum Fisher information.—The uncertainty \(\delta \phi\) of estimating an unknown parameter \(\phi\) encoded in a quantum state \(\rho_\phi\), for any unbiased estimator \(\hat{\phi}\), can be determined via the quantum Cramér-Rao bound (QCRB) as \(\delta \phi \geq 1/\sqrt{\text{J}_Q(\rho_\phi)}\) \[20\], \[23\], where \(\text{J}_Q(\rho_\phi)\) is the quantum Fisher information (QFI) of the state \(\rho_\phi\) and \(\nu \) is the number of repeated measurements. For single-parameter estimation, the QCRB is achievable, and the QFI thus quantifies the amount of information that can be extracted from the quantum state. One way to compute the QFI is \[20\], \[30\]:

\[
\text{J}_Q(\rho_\phi) = 4 \min_{\text{Tr}_A(\{\rho_\phi\})} \langle \Psi_\phi | \Psi_\phi \rangle,
\]

where \(\rho_\phi = \rho_\phi \otimes \rho_m\) is the purification of \(\rho_\phi\) with an ancillary space \(\mathcal{H}_A\), \(\text{Tr}_A\) denotes the partial trace over \(\mathcal{H}_A\), and \(\tilde{\Psi} := \partial \Psi / \partial \phi\). When the parameter is carried by a quantum channel \(\mathcal{E}_\phi\), i.e., a completely positive trace-preserving (CPTP) map, the channel QFI can be defined as the maximal QFI of output states using the optimal input assisted by arbitrary ancillae \[20\], \[31\], \[32\]:

\[
\text{J}_Q^{\text{(chan)}}(\mathcal{E}_\phi) = \max_{\rho_\phi \in \mathcal{S}(\mathcal{H}_\phi)} [\text{J}_Q(\mathcal{E}_\phi \otimes \mathcal{I}_A)(\rho_\phi)],
\]

where \(\mathcal{S}(\mathcal{H})\) denotes the space of density operators on the Hilbert space \(\mathcal{H}\), \(\mathcal{H}_{S/A}\) denotes the Hilbert space of the system/ancilla, and \(\mathcal{I}_A\) is the identity operator on \(\mathcal{H}_A\). Due to the convexity of the QFI, the input states, \(\rho_\phi\), can be restricted to pure states for optimization.

We denote by \(\mathcal{L}(\mathcal{H})\) the set of linear operators on the finite-dimensional Hilbert space \(\mathcal{H}\), and \(\mathcal{L}(\mathcal{H}(\mathcal{H}_1), \mathcal{H}(\mathcal{H}_2))\) denotes the set of linear maps from \(\mathcal{H}(\mathcal{H}_1)\) to \(\mathcal{H}(\mathcal{H}_2)\). By the Choi–Jamiołkowski (CJ) isomorphism, a parameterized quantum channel \(\mathcal{E}_\phi \in \mathcal{L}(\mathcal{H}(\mathcal{H}_{S\phi}), \mathcal{H}(\mathcal{H}_1))\) for \(1 \leq i \leq N\) can be represented by a positive semidefinite operator (called the CJ operator) \(\mathcal{E}_\phi = \text{Choi}(\mathcal{E}_\phi) = \mathcal{E}_\phi \otimes \mathcal{I}(I) \langle I \rangle\), where \(\langle I \rangle = \sum_i |i \rangle \langle i|\). The CJ operator of \(N\) identical quantum channels is \(\mathcal{E}_\phi = \mathcal{E}_\phi^N \in \mathcal{L}(\otimes_{i=1}^{N} \mathcal{H}_i)\).

Strategy set in quantum metrology.—A strategy is an arrangement of physical processes which, when concatenated with given queries to \(\mathcal{E}_\phi\), generates an output quantum state carrying the information about \(\phi\) (see Fig. 2 for an illustration). Mathematically, a strategy can be described by a CJ operator on \(\mathcal{L}(\mathcal{H}_F \otimes_{i=1}^{N} \mathcal{H}_i)\), where \(\mathcal{H}_F\) denotes the output Hilbert space of the strategy, referred to as the global future space. The concatenation of two processes is characterized by the link product \[24\], \[34\] of two corresponding CJ operators \(A \in \mathcal{L}(\otimes_{a\in A} \mathcal{H}_a)\) and \(B \in \mathcal{L}(\otimes_{b\in B} \mathcal{H}_b)\) as

\[
A * B := \text{Tr}_{A:B} \left[ \left( \mathbf{1}_{B:A} \otimes A^{T_A:B} \right) \left( B \otimes \mathbf{1}_{A:B} \right) \right],
\]

where \(\mathbf{T}_i\) denotes the partial transpose on \(\mathcal{H}_i\), and \(\otimes_{i\in A:B} \mathcal{H}_i\) denotes \(\mathcal{H}_{A:B}\) future. The output state lies in the global future future, which should not affect any state in the past.

Following the above formalism, a strategy set is described by a subset \(P\) of \(\mathcal{L}(\mathcal{H}_F \otimes_{i=1}^{N} \mathcal{H}_i)\) determined by the relevant physical constraints. Our goal is to identify the ultimate precision limit of parameter estimation characterized by the QFI within such constraints:

**Definition 1.** The QFI of \(N\) quantum channels \(\mathcal{E}_\phi\) given a strategy set \(P\) is

\[
\text{J}^{(P)}(\mathcal{N}_\phi) := \max_{P \in \mathcal{P}} \text{J}_Q(\mathcal{P} * \mathcal{N}_\phi),
\]

where \(\text{J}_Q(\mathcal{P})\) is the QFI of the state \(\rho\), and \(\mathcal{N}_\phi\) is the CJ operator of \(N\) channels.

In general we can write the ensemble decomposition \[24\] of the CJ operator \(\mathcal{N}_\phi\) as \(\mathcal{N}_\phi = \sum_{i=1}^{r} |N_\phi,i\rangle \langle N_\phi,i| = \mathcal{N}_\phi N_\phi^\dagger\), where \(\mathcal{N}_\phi := (|N_\phi,1\rangle, \ldots, |N_\phi,r\rangle)\) and \(r := \max_{\phi} \text{rank}(\mathcal{N}_\phi)\). We also define \(\tilde{\mathcal{N}}_\phi := \mathcal{N}_\phi - iN_\phi h\) for \(h \in \mathbb{R}\), where \(\mathbb{R}\) denotes the set of \(r \times r\) Hermitian matrices, and the performance operator \[37\]

\[
\Omega_\phi(h) := 4 \left( \tilde{\mathcal{N}}_\phi \mathcal{N}_\phi^\dagger \right)^T.
\]

With these notions, we have the following:
Lemma 1. Given a strategy set $\mathcal{P}$, the QFI of $N$ quantum channels $\mathcal{E}_\phi$ is given by

$$J^{(P)}(N_\phi) = \min \lambda \quad \lambda, Q_i, h,$$

$$\text{s.t. } \lambda Q_i \geq \Omega(h), \quad Q_i \in S^i, \quad i = 1, \ldots, K,$$

where $S^i := \{ Q \text{ is Hermitian} \mid \text{Tr}(QS) = 1, S \in S^i \}$ is the dual affine space of $S^i$.

The proof can be found in Appendix B. We remark that similar optimization ideas have been applied to other tasks, such as quantum Bayesian estimation [42], quantum network optimization [43], non-Markovian quantum metrology [37], and quantum channel discrimination [25]. The minimization problem in Theorem 1 can be further written in the form of SDP and solved efficiently. Detailed numerically solvable forms for each strategy set defined below are also explicitly given in Appendix C where the constraints in Eq. (10) can be further simplified in some cases.

We consider the evaluation of QFI for five different families of strategies. First, we introduce two types of strategies following definite causal structures: parallel and sequential ones (see Fig. 1(a) and (b)). In these strategies, the order of making $N$ queries is fixed, and the causal relations for these strategies have been derived in Ref. [35]. In all the following definitions the subscript $i$ of an operator denotes the Hilbert space $\mathcal{H}_i$ it acts on.

Parallel strategy set.—The family of parallel strategies is the first and one of the most successful examples of quantum-enhanced metrology, featuring the usage of entanglement to achieve precision beyond the classical limit [44]. By making parallel use of $N$ quantum channels together with ancillae, we can regard these $N$ channels as one single channel from $\mathcal{L}(\otimes_{i=1}^{2N} \mathcal{H}_{2i-1})$ to $\mathcal{L}(\otimes_{i=1}^{2N} \mathcal{H}_{2i})$. A parallel strategy set is

$$\mathcal{P} = \{ P \in \mathcal{L}(\mathcal{H}_F \otimes \mathcal{H}_{2i-1}) \},$$

$$\text{s.t. } P \in \mathcal{P}, \quad \text{Tr} P(1) = 1, \quad \text{Tr} P = 1_{2, 4, \ldots, 2N} \otimes P^{(1)}.$$

Note that the optimal QFI of parallel strategies can also be evaluated using the method in Refs. [29, 41].

Sequential strategy set.—A more general protocol is to allow for sequential use of $N$ channels assisted by ancillae, where only the output of the former channel can affect the input of the latter channel, and any control gates can be inserted between channels:

$$\mathcal{P} = \{ P \in \mathcal{L}(\mathcal{H}_F \otimes \mathcal{H}_{2i-1}) \},$$

$$\text{s.t. } P \in \mathcal{P}, \quad \text{Tr} P = 1_{2N} \otimes P^{(N)}, \quad \text{Tr} P^{(1)} = 1, \quad \text{Tr} P^{(k)} = 1_{2k-2} \otimes P^{(k-1)}, \quad k = 2, \ldots, N.$$

Unlike the case of parallel strategies, there is no existing way of evaluating the exact QFI using sequential strategies.

We also consider three families of strategies involving indefinite causal order (see Fig. 1(c) and (d) and (e)).

Quantum SWITCH strategy set.—The first one takes advantage of the (generalized) quantum SWITCH [45, 46], where the execution order of $N$ channels is entangled with the state of an $N!$-dimensional control system. See Appendix C for the formal definition of a quantum SWITCH strategy set, denoted by SWI.

Causal superposition strategy set.—More generally, we consider the quantum superposition of multiple sequential orders, each with a unique order of querying the $N$ channels. This can be implemented by entangling $N!$ definite causal orders with a quantum control system [47].
If \( N = 2 \) and the control system is traced out, this notion is equivalent to causal separability \([22, 23]\). A causal superposition strategy set is
\[
\text{Sup} := \{ P \in \mathcal{L} (\mathcal{H}_F \otimes \mathcal{H}_i)^N : \text{Tr}_F P = \sum q^\pi P^\pi, \sum q^\pi = 1, P^\pi \text{ is } \text{Seq}^\pi, q^\pi \geq 0, \pi \in \mathcal{S}_N, \}
\]
where each permutation \( \pi \) is an element of the symmetric group \( \mathcal{S}_N \) of degree \( N \), each \( \text{Seq}^\pi \) denotes a sequential strategy set whose execution order of \( N \) channels is \( \mathcal{E}_\phi^{(1)} \rightarrow \mathcal{E}_\phi^{(2)} \rightarrow \cdots \rightarrow \mathcal{E}_\phi^{(N)} \), having denoted by \( \mathcal{E}_\phi^k \) the channel from \( \mathcal{L}(\mathcal{H}_{2k-1}) \) to \( \mathcal{L}(\mathcal{H}_{2k}) \). Note that \( \text{Swi} \), where the intermediate control is trivial, is a subset of \( \text{Sup} \).

**General indefinite-causal-order strategy set.**—Finally we introduce the most general strategy set considered in this work, where the only requirement is that the concatenation of the strategy \( P \) and \( N \) arbitrary channels (each possibly acting on arbitrary ancillae) should result in a legitimate quantum state. The causal relations in this case \([23]\) are a bit cumbersome, but for our purpose what matters is the dual affine space (see Theorem 1), which is simply the space of no-signalling channels \([18, 43]\). A general indefinite-causal-order strategy set is
\[
\text{ICO} := \{ P \in \mathcal{L} (\mathcal{H}_F \otimes \mathcal{H}_i)^N : \text{Tr}_F P = 1, P \in \text{Seq}^\pi, q^\pi \geq 0, \pi \in \mathcal{S}_N, \}
\]
where \( E^j \in \mathcal{L}(\mathcal{H}_{2j-1} \otimes \mathcal{H}_{2j} \otimes \mathcal{H}_{A_j}) \) denotes the CJ operator of an arbitrary quantum channel with an arbitrary ancillary space \( \mathcal{H}_{A_j} \).

**Hierarchy of strategies.**—By substituting the definitions of different strategy sets into Theorem 1 we obtain the exact values of the optimal QFI. We find that a strict hierarchy of QFI exists quite prevalently. For demonstration purposes, here we present the result for the amplitude damping channel for \( N = 2 \), while more results are available in Appendix D. In this case, the process encoding \( \phi \) is a \( z \)-rotation \( U_z(\phi) = e^{i\phi t} \) [2], where \( t \) is the evolution time, followed by an amplitude damping channel described by two Kraus operators: \( K_{1,2}^{(AD)} = |0\rangle \langle 0| \pm \sqrt{1 - p} |1\rangle \langle 1| \) and \( K_2^{(AD)} = \sqrt{p} |0\rangle \langle 1| \), with the decay parameter \( p \).

In Fig. 3 we plot the QFI versus the decay parameter \( p \) for amplitude damping noise with all 5 strategies for \( N = 2 \). The first observation is that strict hierarchy of \( \text{Par}, \text{Seq} \) and ICO holds if \( p \) is neither 1 nor 0, i.e., \( J^{(\text{Par})} < J^{(\text{Seq})} < J^{(ICO)} \). This is in stark contrast to the asymptotic regime of \( N \rightarrow \infty \), where the relative difference between the QFI of sequential strategies and the QFI of parallel strategies vanishes for this channel \([17]\).

Moreover, assisted by the quantum SWITCH (without any additional control operations), we can beat any sequential strategies in certain cases (e.g. \( p < 0.5 \)). Besides, in this case general ICO cannot strictly outperform \( \text{Sup} \), implying that causally superposing two sequential strategies is sufficient to achieve the general optimality in this regime. However, the gap between \( J^{(\text{Sup})} \) and \( J^{(ICO)} \) could be observed for the same channel with larger \( N \) or for other channels when \( N = 2 \) (see Appendix D). In fact, by randomly sampling noise channels from CPTP channel ensembles, we find that for 984 of 1000 random channels, a strict hierarchy \( J^{(\text{Par})} < J^{(\text{Seq})} < J^{(\text{Sup})} < J^{(ICO)} \) holds for \( N = 2 \), implying that there exist more powerful strategies than causal superposition strategies in these cases.

Using our method, we can also test the tightness of existing QFI bounds in the non-asymptotic regime, which has seldom been done till this work. Here we take the commonly used upper bound for parallel strategies (see Ref. \([29] \) Theorem 4) or \([31] \text{ Eq. (16)} \), which is asymptotically tight \([17]\). As illustrated in Fig. 3 our result shows that the asymptotically tight bound is clearly not tight for small \( N \). In fact, it is even greater than \( J^{(\text{Seq})} \).

**An algorithm for optimal strategies.**—By itself, the QFI does not reveal how to implement the optimal strategy achieving the highest precision. Here, in addition to Theorem 1 we design an algorithm that yields a strategy attaining the optimal QFI for any strategy set satisfying...
The QFI is plotted as a function of the evolution time $t$ for $N = 2$, $\phi = 1.0$ and $p = 0.5$.

Eq. (9). The method, which generalizes the method of finding an optimal probe state for a single channel [17, 49], is summarised as Algorithm 1 (see Appendix F for its derivation).

Algorithm 1. Find an optimal strategy in the set $P$.

- Given $N_\phi$ the CJ operator of $N$ channels, solve an optimal value $h = h^{(\text{opt})}$ in Eq. (10) of Theorem 1 via SDP.
- Fixing $h = h^{(\text{opt})}$, solve an optimal value $P^{(\text{opt})} \in P$ in Eq. 6 of Lemma 1 for $P$ defined by Eq. (7) via SDP such that

$$\text{Re} \left\{ \text{Tr} \left\{ \hat{P}^{(\text{opt})} \left( -i N_\phi \mathcal{H} \right) (N_\phi - i N_\phi h^{(\text{opt})} )^T \right\} \right\} = 0 \text{ for all } \mathcal{H} \in \mathcal{H}_r,$$

(15)

where $N_\phi := (|N_{\phi,1}>, \ldots, |N_{\phi,N}>)$. An optimal strategy $P^{(\text{opt})} \in P$ can be taken as a purification of $\hat{P}^{(\text{opt})}$.

By this algorithm we obtain the CJ operator of a strategy that attains the optimal QFI. For strategies following definite causal order, this process can be represented in the quantum circuit formalism. In this case, an operational way of mapping the CJ operator of the strategy to a probe state and a sequence of in-between control operations with minimal memory space is given in Ref. [50]. This provides a systematic method to identify an optimal sequential strategy, one of the key problems in quantum metrology.

Conclusion.—Our results manifest that, when the number of channels to be estimated is finite, a strict hierarchy exists for commonly considered strategy sets in quantum metrology, and existing asymptotically tight QFI bounds are apparently loose. Our work also features a general tool of finding the optimal precision and an associated optimal strategy, which can be applied to a wide range of practical tasks and is expected to help design optimal experimental schemes that achieve the ultimate precision limit.

This work is supported by HKU Seed Fund for Basic Research for New Staff via Project 202107185045 and by the Research Grants Council of Hong Kong through the Grant No. 14307420.

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Appendix A: Proof of Lemma 1

The formalism of this proof has been developed for strategies following a definite causal order in Ref. [37], and the generalization to indefinite-causal-order strategies considered here is straightforward.

In Ref. [29], the QFI of a quantum state is expressed as a minimization problem

\[
J_Q(\rho) = 4 \min_{|\psi_{\phi,i}\rangle} \sum_i \Tr \left( |\psi_{\phi,i}\rangle \langle \psi_{\phi,i}| \right),
\]

where \( \{|\psi_{\phi,i}\rangle\} \) is a set of unnormalized vectors such that \( \rho = \sum_i |\psi_{\phi,i}\rangle \langle \psi_{\phi,i}| \). In the main text the QFI of \( N \) quantum channels \( E_{\phi} \) is defined as the QFI of the output state obtained from the concatenation of the CJ operator \( N_{\phi} \) of \( N \) quantum channels and an optimal strategy \( P \) in a given strategy set \( \mathcal{P} \):

\[
J^{(P)}(N_{\phi}) := \max_{P \in \mathcal{P}} J_Q(P \ast N_{\phi}).
\]

Due to the convexity of the QFI and the linearity of the link product, an optimal \( P \) can be taken as a pure process (rank-1 operator) denoted by \( |P\rangle\langle P| \). For a fixed \( P = |P\rangle\langle P| \), minimization over decompositions of \( P \ast N_{\phi} \) is equivalent to minimization over decompositions of \( N_{\phi} \).

As a positive semidefinite operator, \( N_{\phi} \) has a decomposition as:

\[
N_{\phi} = \sum_{i=1}^{r} |N_{\phi,i}\rangle \langle N_{\phi,i}|,
\]

where \( r = \max_{\phi} \text{rank}(N_{\phi}) \). Note that the decomposition is non-unique. Defining \( N_{\phi} := ([N_{\phi,1}], \ldots, [N_{\phi,r}]) \), an arbitrary alternative decomposition \( \tilde{N}_{\phi} := ([\tilde{N}_{\phi,1}], \ldots, [\tilde{N}_{\phi,r}]) \) can be related to \( N_{\phi} \) by \( \tilde{N}_{\phi} = N_{\phi} V_{\phi} \), where \( V_{\phi} \) is an \( r \times r \) unitary matrix. Then the QFI of \( N \) channels can be expressed as

\[
J^{(P)}(N_{\phi}) = \max_{|P\rangle\langle P| \in \mathcal{P}} \min_{V_{\phi}} \Tr \left( |P\rangle\langle P| [1_F \otimes \Omega_{\phi}] \right),
\]

having defined the performance operator \( \Omega_{\phi} := 4 \left( \dot{N}_{\phi} \dot{N}_{\phi}^\dagger \right)^T = 4 \sum_{i=1}^{r} \left( |\dot{N}_{\phi,i}\rangle \langle \dot{N}_{\phi,i}| \right)^T \), where the superscript \( T \) denotes the partial trace. We can further define an \( r \times r \) Hermitian matrix \( h := iV_{\phi}^\dagger V_{\phi} \) to take care of the freedom of choice for the decomposition \( \dot{N}_{\phi} \), by noting that \( \dot{N}_{\phi} = \tilde{N}_{\phi} - iN_{\phi} h \). We can rewrite \( \Omega_{\phi} \) as \( \Omega_{\phi}(h) \) to explicitly manifest its dependence on \( h \). Hence, we finally arrive at the result of Lemma 1

\[
J^{(P)}(N_{\phi}) = \max_{|P\rangle\langle P| \in \mathcal{P}} \min_{h \in \mathbb{H}_r} \Tr \left( |P\rangle\langle P| [1_F \otimes \Omega_{\phi}(h)] \right)
\]

\[
= \max_{|P\rangle\langle P| \in \mathcal{P}} \min_{h \in \mathbb{H}_r} \Tr \left( \tilde{P}_{\phi}(h) \right),
\]

where \( \tilde{P} := \{ \tilde{P} = \Tr_F (|P\rangle\langle P|) | |P\rangle\langle P| \in \mathcal{P} \} \).

Appendix B: Proof of Theorem 1

Starting from Lemma 1, we exchange the order of minimization and maximization thanks to Fan’s minimax theorem [39], since \( \Tr \left( \tilde{P}\Omega_{\phi}(h) \right) \) is concave on \( \tilde{P} \) and convex on \( h \), and \( \tilde{P} \) is assumed to be a compact set. Then the problem of QFI evaluation can be rewritten as

\[
J^{(P)}(N_{\phi}) = \min_{h} \max_{\tilde{P} \in \tilde{P}} \Tr \left( \tilde{P}\Omega_{\phi}(h) \right).
\]

Reformulating the condition of Theorem 1, we require that each operator \( \tilde{P} \in \tilde{P} \) can be written as a convex combination of positive semidefinite operators \( S_i \), \( i = 1, \ldots, K \):

\[
\tilde{P} = \sum_{i=1}^{K} q^i S_i, \text{ for } \sum_{i=1}^{K} q^i = 1, q^i \geq 0, S_i \geq 0, S_i \in S_i, \text{ } i = 1, \ldots, K,
\]
where each $S^i$ is an affine space. Thus Eq. (B1) can be reformulated as

$$J^{(p)}(N_\phi) = \min_h \max_P \text{Tr}[\hat{P}\Omega_\phi(h)],$$

subject to

$$\hat{P} = \sum_{i=1}^K q^i S^i,$$

$$\sum_{i=1}^K q^i = 1,$$

$$q^i \geq 0, \ S^i \geq 0, \ S^i \in S^i, \ i = 1, \ldots, K.$$ (B3)

For now we fix $h$ and consider the dual problem of maximization over $\hat{P}$. For each affine space $S^i$ we have defined its dual affine space $\tilde{S}^i$, whose dual affine space in turn is exactly $S^i$ [43]. Choose an affine basis $\{Q^{i,j}\}_{j=1}^L$ for $\tilde{S}^i$, the maximization problem is further expressed as

$$\max_P \text{Tr}[\hat{P}\Omega_\phi(h)],$$

subject to

$$\hat{P} = \sum_{i=1}^K p^i,$$

$$\sum_{i=1}^K p^i = 1,$$

$$p^i \geq 0, \ Tr\left(p^i Q^{i,j}\right) = q^i, \ i = 1, \ldots, K, \ j = 1, \ldots, L_i.$$ (B4)

Defining $P^i := q^i S^i$ to avoid the product of variables in optimization, we have

$$\max_P \text{Tr}[\hat{P}\Omega_\phi(h)],$$

subject to

$$\hat{P} = \sum_{i=1}^K P^i,$$

$$\sum_{i=1}^K p^i = 1,$$

$$p^i \geq 0, \ Tr\left(p^i Q^{i,j}\right) = q^i, \ i = 1, \ldots, K, \ j = 1, \ldots, L_i,$$

where the constraints $q^i \geq 0$ can be safely removed, since $Tr S^i = \prod_{j=1}^{N_i} d_{2j}$, implying that $S^i$ includes a positive operator proportional to identity for any $i = 1, \ldots, K$, having denoted $d_j := \dim(\mathcal{H}_j)$ for simplicity. The Lagrangian of the problem is given by

$$L = \sum_i \text{Tr}[P^i \Omega_\phi(h)] + \left(1 - \sum_i q^i\right) \lambda + \sum_i \text{Tr}\left[P^i \hat{Q}^i\right] + \sum_{i,j} \left[q^i - \text{Tr}\left(P^i Q^{i,j}\right)\right] \lambda^{i,j}$$

$$= \lambda + \sum_i \text{Tr}\left(P^i \left[\Omega_\phi(h) + \hat{Q}^i - \sum_j \lambda^{i,j} Q^{i,j}\right]\right) + \sum_i \left[q^i \left(\sum_j \lambda^{i,j} - \lambda\right)\right].$$ (B6)

for $\hat{Q}^i \geq 0$. Hence, by removing $\hat{Q}^i$ the dual problem is written as

$$\min_\lambda, \quad \text{s.t.} \ \sum_j \lambda^{i,j} Q^{i,j} \geq \Omega_\phi(h), \ \lambda = \sum_j \lambda^{i,j}, \ i = 1, \ldots, K, \ j = 1, \ldots, L_i.$$ (B7)

We define $Q^i := \sum_j \lambda^{i,j} Q^{i,j} / \lambda$ if $\lambda \neq 0$ ($\lambda = 0$ corresponds to a trivial case where the QFI is zero), and clearly $Q^i$ is an arbitrary operator in the set $\tilde{S}^i$. Therefore, we cast the dual problem into

$$\min_\lambda, \quad \text{s.t.} \ \lambda Q^i \geq \Omega_\phi(h), \ Q^i \in \tilde{S}^i, \ i = 1, \ldots, K.$$ (B8)

Slater’s theorem [51] implies that the strong duality holds, since the QFI is finite and the constraint can be strictly satisfied for a positive operator $\Omega_\phi(h)$, by choosing $\lambda Q^i = \mu \Omega_\phi(h) \mathbb{1}_{1,2,\ldots,2N}$ for $\mu > 1$ and any $i = 1, \ldots, K$, having denoted the operator norm by $\|\cdot\|$. Finally, by optimizing the choice of $h$ we derive the result of Theorem 1.
Appendix C: Evaluation of QFI using different strategies

In this section we provide explicit formulas of the QFI for all strategy sets considered in the main text, in the forms which can be numerically solved by SDP. Without the positivity constraints, parallel, sequential and general indefinite-causal-order strategy sets are affine spaces themselves, while quantum SWITCH and causal superposition strategy sets are convex hulls of affine spaces. In some cases the result of Theorem 1 can be simplified a bit, as it is possible to trace over certain subspace while formulating the primal problem at the beginning.

1. Parallel strategies

When definite causal order is obeyed, a strategy can be described by a quantum comb \[34, 35, 52\]. The dual affine space is the set of dual combs without the positivity constraint \[43\]. For parallel strategies the primal problem can be written as

\[
J^\text{(Par)}(N_\phi) = \min_{h \in \mathbb{H}_r} \max_P \text{Tr}\left[ \tilde{P} \Omega_\phi(h) \right],
\]

s.t. \( \tilde{P} \geq 0 \),
\( \tilde{P} = \mathds{1}_{2,4,\ldots,2N} \otimes \tilde{P}(1) \),
\( \text{Tr} \tilde{P}(1) = 1 \). \tag{C1}

Equivalently, the problem can be formulated as

\[
\min_{h \in \mathbb{H}_r} \max_P \text{Tr}\left[ P \text{Tr}_{2,4,\ldots,2N} \Omega_\phi(h) \right],
\]

s.t. \( P \geq 0 \),
\( \text{Tr} P = 1 \). \tag{C2}

The dual problem is given by

\[
\min \lambda, h
\]

s.t. \( \lambda \mathds{1}_{1,3,\ldots,2N-1} \geq \text{Tr}_{2,4,\ldots,2N} \Omega_\phi(h) \), \tag{C3}

which simplifies the result directly obtained from Theorem 1 a bit. To solve the problem via SDP, we define a block matrix

\[
A = \begin{pmatrix}
\frac{1}{2} \mathds{1} + \left( r \prod_{i=1}^N d_{2k} \right) & \langle n_{1,1} \rangle \\
\vdots & \vdots \\
\langle n_{r,1} \rangle & \langle n_{r,2} \rangle \\
|n_{1,1}\rangle & \cdots & |n_{r,2}\rangle & \mathds{1}_{1,3,\ldots,2N-1}
\end{pmatrix}, \tag{C4}
\]

wherein \( \mathds{1}(d) \) denotes a \( d \)-dim identity matrix, and

\[
|n_{i,j}\rangle = \left\langle j \right| N_{\phi,i}^j,
\]

where \( \{ |j\rangle, j = 1, \ldots, \prod_{k=1}^N d_{2k} \} \) forms an orthonormal basis of \( \otimes_{k=1}^N \mathcal{H}_{2k} \), having assumed that the identity map trivially acts on the subspace where the dual vector \( \langle j \rangle \) does not affect. Note that

\[
\sum_{i,j} |n_{i,j}\rangle \langle n_{i,j}| = \frac{1}{4} \text{Tr}_{2,4,\ldots,2N} \Omega_\phi(h). \tag{C6}
\]

By Schur complement lemma, the constraint in Eq. \[C3\] is equivalent to the requirement that \( A \geq 0 \). Finally, the QFI for parallel strategies is solved by

\[
\min \lambda, h
\]

s.t. \( A \geq 0 \),
\( h \in \mathbb{H}_r \). \tag{C7}

The problem can be solved by SDP since \( h \) is incorporated linearly in the blocks of \( A \).
2. Sequential strategies

For sequential strategies the problem can be written as (having traced over \( \mathcal{H}_{2N} \))

\[
J^{(\text{Seq})}(N, \phi) = \min_{h \in \mathbb{H}_r} \max_{P^{(h)}} \text{Tr} \left[ P^{(N)} \text{Tr}_{2N} \Omega_\phi(h) \right],
\]

s.t. \( P^{(N)} \geq 0, \)

\[
\text{Tr}_{2k-1} P^{(k)} = \mathbb{1}_{2k-2} \otimes P^{(k-1)}, \ k = 2, \ldots, N,
\]

\[
\text{Tr} P^{(1)} = 1,
\]

from which it follows that the dual problem is

\[
\min_{\lambda, Q^{(N)}, h} \lambda,
\]

s.t. \( \lambda \mathbb{1}_{2N-1} \otimes Q^{(N-1)} \geq \text{Tr}_{2N} \Omega_\phi(h), \)

\[
\text{Tr}_{2k} Q^{(k)} = \mathbb{1}_{2k-1} \otimes Q^{(k-1)}, \ k = 2, \ldots, N - 1,
\]

\[
\text{Tr}_2 Q^{(1)} = \mathbb{1}_1,
\]

where \( Q^{(N-1)} \) is Hermitian.

Similarly, in order to solve the problem via SDP we rewrite it as

\[
\min_{\lambda, Q^{(N)}, h} \lambda,
\]

s.t. \( A \geq 0, \)

\[
\text{Tr}_{2k} Q^{(k)} = \mathbb{1}_{2k-1} \otimes Q^{(k-1)}, \ k = 2, \ldots, N - 1,
\]

\[
\text{Tr}_2 Q^{(1)} = \mathbb{1}_1,
\]

\[
h \in \mathbb{H}_r,
\]

for

\[
A = \begin{pmatrix}
\frac{\lambda}{4} \mathbb{1}_{(rd_{2N})} & \langle n_{1,1} \rangle \\
\vdots & \ddots & \vdots \\
\langle n_{r,d_{2N}} \rangle & \langle n_{r,d_{2N}} \rangle & \mathbb{1}_{2N-1} \otimes Q^{(N-1)}
\end{pmatrix},
\]

wherein

\[
\langle n_{i,j} \rangle = \langle j | \hat{N}^*_{\phi,i} \rangle,
\]

where \( \{ |j\rangle, \ j = 1, \ldots, d_{2N} \} \) forms an orthonormal basis of \( \mathcal{H}_{2N} \).

3. Quantum SWITCH strategies

We first formally define a quantum SWITCH strategy set as

\[
\text{SW}! := \left\{ P \in \mathcal{L} \left( \mathcal{H}_F \otimes_{i=1}^{2N} \mathcal{H}_i \right) \mid \right. \right. \left. \right.
\]

s.t. \( P = (\rho_{T,A,C}) \ast |P^{(SW)}\rangle \langle P^{(SW)}|, \)

\[
\rho_{T,A,C} \geq 0,
\]

\[
\text{Tr} \rho_{T,A,C} = 1,
\]

where \( |P^{(SW)}\rangle := |I\rangle_{A,F} \sum_{\pi \in S_N} \left[ |\pi\rangle_C |I\rangle_{T,2\pi(1)-1} (\otimes_{i=1}^{N-1}) |I\rangle_{2\pi(i),2\pi(i+1)-1} |I\rangle_{2\pi(N),F_T} |\pi\rangle_{F_C} \right] \) corresponds to a (generalized) quantum SWITCH for \( N \) operations, each permutation \( \pi \) is an element of the symmetric group \( S_N \) whose order is \( N \). We suppose each \( \mathcal{H}_i \) for \( i = 1, \ldots, 2N \) has the same dimension \( d \). \( \mathcal{H}_T = \mathcal{H}_i \) denotes the input space of
the target system, $\mathcal{H}_A$ the ancillary space, and $\mathcal{H}_C$ the space of the control system. Correspondingly, $\mathcal{H}_{F_r}$, $\mathcal{H}_{F_A}$ and $\mathcal{H}_{F_C}$ denote the future output spaces of each part. The global future $\mathcal{H}_{F} = \mathcal{H}_{F_r} \otimes \mathcal{H}_{F_A} \otimes \mathcal{H}_{F_C}$.

Using the quantum SWITCH strategy set, after tracing over the global future space $\mathcal{H}_F$ the QFI evaluation problem is written as

$$J^{(SW)}(N_\phi) = \min_{\rho \in \mathcal{H}_F} \max_P \text{Tr} \left[ \hat{P} \Omega_\phi(h) \right],$$

subject to:

$$\hat{P} = \sum_{\pi \in S_N} q^\pi \rho^\pi_{2\pi(1)-1} \left( \otimes_{i=1}^{N-1} |I\rangle_{2\pi(i), 2\pi(i+1)-1}\langle I|_{2\pi(i), 2\pi(i+1)-1} \right) \otimes \mathbb{I}_{2\pi(N)},$$

$$\sum_{\pi \in S_N} q^\pi = 1,$n

$$\rho^\pi_{2\pi(1)-1} \geq 0, \text{ Tr } \rho^\pi_{2\pi(1)-1} = 1, q^\pi \geq 0, \pi \in S_N,$$

where the superscript $\pi$ of an operator denotes a permutation label, and the subscript denotes the subspace it lies in. Note that the primal set of $P$ is a convex hull of affine spaces. Equivalently the problem can be rewritten as

$$\min_{\rho \in \mathcal{H}_F} \max_{\pi \in S_N} \sum_{\pi \in S_N} \text{Tr} \left[ q^\pi \rho^\pi_{2\pi(1)-1} \left( \otimes_{i=1}^{N-1} |I\rangle_{2\pi(i), 2\pi(i+1)-1}\langle I|_{2\pi(i), 2\pi(i+1)-1} \right) \text{Tr}_{2\pi(N)} \Omega_\phi(h) \left( \otimes_{j=1}^{N-1} |I\rangle_{2\pi(j), 2\pi(j+1)-1} \right) \right],$$

subject to:

$$\sum_{\pi \in S_N} q^\pi = 1,$n

$$\rho^\pi_{2\pi(1)-1} \geq 0, \text{ Tr } \rho^\pi_{2\pi(1)-1} = 1, q^\pi \geq 0, \pi \in S_N.$$

Following the method in the proof of Theorem 1, the dual problem is given by

$$\min_{\lambda} \max_{\rho \in \mathcal{H}_F} \lambda \sum_{\pi \in S_N} \text{Tr} \left[ q^\pi \rho^\pi_{2\pi(1)-1} \left( \otimes_{i=1}^{N-1} |I\rangle_{2\pi(i), 2\pi(i+1)-1}\langle I|_{2\pi(i), 2\pi(i+1)-1} \right) \text{Tr}_{2\pi(N)} \Omega_\phi(h) \left( \otimes_{j=1}^{N-1} |I\rangle_{2\pi(j), 2\pi(j+1)-1} \right) \right],$$

subject to:

$$\lambda \sum_{\pi \in S_N} q^\pi \geq 0, \lambda \text{Tr} \rho^\pi_{2\pi(1)-1} = 1, q^\pi \geq 0, \pi \in S_N.$$

Equivalently in an SDP form the problem is written as

$$\min_{\lambda, h} \max_{\rho \in \mathcal{H}_F} \lambda \sum_{\pi \in S_N} \text{Tr} \left[ q^\pi \rho^\pi_{2\pi(1)-1} \left( \otimes_{i=1}^{N-1} |I\rangle_{2\pi(i), 2\pi(i+1)-1}\langle I|_{2\pi(i), 2\pi(i+1)-1} \right) \text{Tr}_{2\pi(N)} \Omega_\phi(h) \left( \otimes_{j=1}^{N-1} |I\rangle_{2\pi(j), 2\pi(j+1)-1} \right) \right],$$

subject to:

$$\lambda I_{2\pi(1)-1} \geq \Omega_\phi(h), \Omega_\phi(h) := \left( \otimes_{i=1}^{N-1} |I\rangle_{2\pi(i), 2\pi(i+1)-1}\langle I|_{2\pi(i), 2\pi(i+1)-1} \right) \text{Tr}_{2\pi(N)} \Omega_\phi(h) \left( \otimes_{j=1}^{N-1} |I\rangle_{2\pi(j), 2\pi(j+1)-1} \right), \pi \in S_N.$$

Having defined

$$A^\pi = \begin{pmatrix}
\frac{\lambda}{4} & \mathbb{I}(rd_{2\pi(N)}) \\
\vdots & \vdots \\
|n_{l,1}^\pi \rangle & \cdots & |n_{r,d}^\pi \rangle & \mathbb{I}_{2\pi(N)-1}
\end{pmatrix},$$

where

$$|n_{l,i}^\pi \rangle = \langle j^\pi | \left( \otimes_{i=1}^{N-1} |I\rangle_{2\pi(i), 2\pi(i+1)-1}\langle I|_{2\pi(i), 2\pi(i+1)-1} \right) \hat{N}_{\phi,i}^\pi,$$

with the set of $\{ |j^\pi \rangle \}$ forming an orthonormal basis of $\mathcal{H}_{2\pi(N)}$.

4. Causal superposition strategies

Following a similar route for the causal superposition strategy set the problem can be written as

$$J^{(Sup)}(N_\phi) = \min_{\rho \in \mathcal{H}_F} \max_{\pi \in S_N} \sum_{\pi \in S_N} \text{Tr} \left[ q^\pi P^\pi(N) \text{Tr}_{2\pi(N)} \Omega_\phi(h) \right],$$

subject to:

$$\sum_{\pi \in S_N} q^\pi = 1,$n

$$q^\pi \geq 0, P^\pi(N) \geq 0, \text{ Tr } P^\pi_{2k-1} = 1, \text{ Tr } P^\pi_{2k-1} = \mathbb{I}_{2k-2} \otimes P^\pi_{2(k-1)} \text{ for } k = 2, \ldots, N, \pi \in S_N.$$
For each causal order in the superposition the dual affine space is the set of dual combs without the positivity constraint. Thus the dual problem is given by

\[
\begin{align*}
\min_{\lambda, h} & \quad \lambda \mathbb{1}_{2^{2\pi(N)-1}} \otimes Q^{\pi,(N-1)} \geq \text{Tr}_{2^{2\pi(N)}} \Omega_{\phi}(h), \quad \text{Tr}_{2^{2\pi(1)}} Q^{\pi,(1)} = \mathbb{1}_{2^{2\pi(1)-1}}, \quad \pi \in S_N, \\
\text{s.t.} & \quad \text{Tr}_{2^{2\pi(k)}} Q^{\pi,(k)} = \mathbb{1}_{2^{2\pi(k)-1}} \otimes Q^{\pi,(k-1)} \text{ for } k = 2, \ldots, N-1, \quad \pi \in S_N,
\end{align*}
\]  

where the constraints hold for any \( \pi \in S_N \). To solve the problem via SDP we can formulate it as

\[
\begin{align*}
\min_{\lambda, h} & \quad A^\pi \geq 0, \quad \text{Tr}_{2^{2\pi(1)}} Q^{\pi,(1)} = \mathbb{1}_{2^{2\pi(1)-1}}, \quad \text{Tr}_{2^{2\pi(k)}} Q^{\pi,(k)} = \mathbb{1}_{2^{2\pi(k)-1}} \otimes Q^{\pi,(k-1)} \text{ for } k = 2, \ldots, N-1, \quad \pi \in S_N,
\end{align*}
\]

having defined

\[
A^\pi = \begin{pmatrix} 
\lambda & \{n^\pi_{1,1}\} \\
\{n^\pi_{1,1}\} & \ddots & \{n^\pi_{r,d_{2^{2\pi(N)}}}\} \\
\vdots & \ddots & \ddots \\
\{n^\pi_{r,d_{2^{2\pi(N)}}}\} & \ddots & \{n^\pi_{r,d_{2^{2\pi(N)}}}\} \\
\end{pmatrix},
\]

wherein

\[
|n^\pi_{i,j}| = \left\{ j^\pi \left| \hat{N}^\pi_{\phi,i} \right. \right\},
\]

where \( \{|j^\pi\} \) forms an orthonormal basis of \( \mathcal{H}_{2^{2\pi(N)}} \).

5. General indefinite-causal-order strategies

In this case the explicit linear constraints on strategies have been derived in Ref. [23], and the dual affine space turns out to be the set of CJ operators of \( N \)-partite no-signalling quantum channels without the positivity constraint [18 43], mathematically defined by

\[
\begin{align*}
\text{Tr}_{2^{2k}} Q = \mathbb{1}_{d_{2^{2k}-1}} \otimes \text{Tr}_{2^{2k-1},2^{2k}} Q, \quad k = 1, \ldots, N, \\
\text{Tr} Q = \prod_{i=1}^{N} d_{2i-1}.
\end{align*}
\]

The intuitive interpretation for no-signaling channels is that locally the input of each channel only affects the output of this single channel, but cannot transmit any information to \( N-1 \) other channels. To solve the QFI evaluation problem via SDP we can write it in the form

\[
\begin{align*}
\min_{\lambda, Q, h} & \quad \lambda, \\
\text{s.t.} & \quad A \geq 0, \\
& \quad \text{Tr}_{2^{2k}} Q = \mathbb{1}_{d_{2^{2k}-1}} \otimes \text{Tr}_{2^{2k-1},2^{2k}} Q, \quad k = 1, \ldots, N, \\
& \quad \text{Tr} Q = \prod_{i=1}^{N} d_{2i-1}, \\
& \quad h \in \mathbb{H}_r,
\end{align*}
\]

having defined

\[
A = \begin{pmatrix} 
\frac{\lambda}{d} I (r) & \{\hat{N}^\pi_{\phi,1}\} \\
\{\hat{N}^\pi_{\phi,1}\} & \ddots & \{\hat{N}^\pi_{\phi,r}\} \\
\vdots & \ddots & \ddots \\
\{\hat{N}^\pi_{\phi,r}\} & \ddots & \{\hat{N}^\pi_{\phi,r}\} \\
\end{pmatrix},
\]

where \( \{\hat{N}^\pi_{\phi,i}\} \) forms an orthonormal basis of \( \mathcal{H}_{2^{2\pi(N)}} \).
Appendix D: Supplementary numerical results

1. Hierarchy for the $N = 3$ case

Numerical results in this work are obtained by implementing SDP using the open source Python package CVXPY \[53, 54\] with the solver MOSEK \[55\]. We plot the QFI for $N = 3$, amplitude damping noise in Fig. 5 and observe a similar hierarchy of the estimation performance using parallel, sequential and indefinite-causal-order strategies. Different from the $N = 2$ case presented in the main text, general indefinite-causal-order strategies indeed provide a small advantage over causal superposition strategies, which is presented in Table. 1.

On the other hand, analogous to the $N = 2$ case, simple quantum SWITCH strategies without any additional intermediate control operations could have advantage over any definite-causal-order strategies when the decay parameter $p$ is small, but this advantage becomes more insignificant. This should not be surprising since the control can make a bigger difference as $N$ grows.

2. Estimation of randomly sampled channels

To demonstrate the universality of the hierarchy of different families of strategies considered in the main text, we randomly sample noise channels drawn from an ensemble of CPTP maps defined by Bruzda et al. in Ref. \[56\]. In this work we only sample rank-2 qubit noise channels for $N = 2$, which is enough to show the hierarchy. The sampling process is implemented via an open source Python package QuTiP \[57, 58\]. We set an error tolerance of $10^{-8}$, i.e., we claim $J_1 > J_2$ only if the gap is no smaller than $10^{-8}$. We find that for 984 of 1000 random channels, a strict hierarchy
J^{(Par)} < J^{(Seq)} < J^{(Sup)} < J^{(ICO)} holds, implying that general indefinite-causal-order strategies can provide advantage over causal superposition strategies. In addition, we find that of the same 1000 channels $J^{(Par)} < J^{(SWI)}$ for 34 channels and $J^{(Seq)} < J^{(SWI)}$ only for 1 channel, so with a high probability quantum SWITCH strategies cannot outperform strategies following definite causal order for a random noise channel, which highlights the estimation enhancement from intermediate control in the general case.

**Appendix E: Comparison with asymptotic results**

In this section we focus on strategies following definite causal order, i.e., parallel and sequential ones, and compare our results and those of the extensively studied asymptotic theory.

1. Preliminaries

We first introduce some basic notions. If we write the operation-sum representation of the channel $E_\phi(\rho) = \sum_{i=1}^{r} K^\dagger_{\phi,i} \rho K_{\phi,i}$, where $\{K_{\phi,i}\}$ is a set of Kraus operators and $r$ is the rank of the channel, the channel QFI can be evaluated by optimization:

$$J_Q^{(chan)}(E_\phi) = 4 \min_{h \in \mathbb{H}_r} |\alpha|,$$

where $||\cdot||$ denotes the operator norm and $\alpha = \sum_i \hat{K}_{\phi,i}^\dagger \hat{K}_{\phi,i}$. Here $\hat{K}_{\phi,i} = K_{\phi,i} - i \sum_{j=1}^{r} h_{ij} K_{\phi,j}$ is nothing but the derivative of an equivalent Kraus representation, given an $r \times r$ Hermitian matrix $h$.

The asymptotically tight upper bounds on QFI of $N$ quantum channels have been derived for both sequential and parallel strategies. For parallel strategies an asymptotically tight upper bound is \[ J^{(Par)}(N_\phi) \leq 4 \min_{h \in \mathbb{H}_r} [N|\alpha| + N(N-1)|\beta|^2], \] where $\beta = i \sum_i K^\dagger_{\phi,i} \hat{K}_{\phi,i}$. An asymptotically tight upper bound was also derived for sequential strategies \[ J^{(Seq)}(N_\phi) \leq 4 \min_{h \in \mathbb{H}_r} [N|\alpha| + N(N-1)|\beta|(|\beta| + 2\sqrt{|\alpha|})]. \]

It has been shown that the QFI follows the standard quantum limit if and only if there exists an $h$ such that $\beta = 0$ \[ J^{(Par)}(N_\phi) = J^{(Seq)}(N_\phi) = 4 \min_{h \in \mathbb{H}_r, \beta=0} |\alpha|. \]

We remark that the minimization in Eq. \[ J^{(Par)}(N_\phi) \] can be efficiently evaluated via SDP \[ J^{(Seq)}(N_\phi) \].
2. Tightness of QFI bounds in non-asymptotic channel estimation

We compare our non-asymptotic results and existing asymptotically tight bounds for two types of quantum channels. Apart from the amplitude damping noise described by $K_{1}^{(AD)} = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|$ and $K_{2}^{(AD)} = \sqrt{p}|0\rangle\langle 1|$, considered in the main text, here we also present a second example, where the noise is described by a SWAP-type interaction $V_{\text{int}} = e^{-igt}H_{\text{SWAP}}$ between a qubit system $S$ and a qubit environment $E$, where $g$ is the interaction strength, $\tau$ is the interaction time and the Hamiltonian is $H_{\text{SWAP}}(|i\rangle_S |j\rangle_E) = |j\rangle_S |i\rangle_E$. The initial environment state is $|0\rangle$, and the Kraus operators can be written as $K_{1}^{(\text{SWAP})} = \langle 0|_{E} V_{\text{int}} |0\rangle_{E} = e^{-igt} |0\rangle + \cos(g\tau) |1\rangle$, and $K_{2}^{(\text{SWAP})} = \langle 1|_{E} V_{\text{int}} |0\rangle_{E} = -i\sin(g\tau) |0\rangle$.

We plot the QFI for the two examples in Fig. 6. Both of them show the advantage of sequential strategies over parallel ones, and the gaps between exact results of QFI for sequential strategies and the parallel upper bounds given by Eq. 42.

![Comparison of our results with the existing asymptotically tight QFI bound](image)

**FIG. 6. Comparison of our results with the existing asymptotically tight QFI bound.** We compare our results to the asymptotically tight bound on the maximal QFI of parallel strategies. We plot the QFI versus the evolution time $t$ for $N = 2$, $\phi = 1.0$, $g = 1.0$, $\tau = t$ and $p = 0.5$. (a) $N = 2$ and (b) $N = 3$ for the amplitude damping noise, (c) $N = 2$ and (d) $N = 3$ for the SWAP-type noise.

3. Elusive advantage of sequential strategies in the asymptotic limit

We observe a gap between parallel and sequential strategies for amplitude damping channels and SWAP-type interactions for $N = 2$ and $N = 3$. Now we show that for both examples there is no advantage of sequential strategies asymptotically since there exists an $h$ such that $\beta = 0$.

For the amplitude damping channel, $K_{\phi,j}^{(AD)} = K_{i}^{(AD)}U_{j}(\phi)$, $i = 1, 2$. Direct calculation leads to

$$\beta^{(AD)} = \left(\frac{t}{2} + h_{11}^{(AD)}\right)|0\rangle\langle 0| + \left[h_{11}^{(AD)} - \frac{t}{2} + \left(h_{22}^{(AD)} - h_{11}^{(AD)}\right)p\right]|1\rangle\langle 1|.$$  \hspace{1cm} (E5)

To obtain $\beta^{(AD)} = 0$ we just need to take $h_{11}^{(AD)} = -t/2$ and $h_{22}^{(AD)} = (2 - p)t/2p$. 

Similarly, for the SWAP-type interaction we have

$$\beta^{(\text{SWAP})} = \left(\frac{t}{2} + h^{(\text{SWAP})}_{11}\right) \lvert 0 \rangle \langle 0 \rvert + \left[ h^{(\text{SWAP})}_{11} - \frac{t}{2} + \left( h^{(\text{SWAP})}_{22} - h^{(\text{SWAP})}_{11} \right) \sin^2(g\tau) \right] \lvert 1 \rangle \langle 1 \rvert. \quad (E6)$$

Thus there exists $h^{(\text{SWAP})}_{11} = -t/2$ and $h^{(\text{SWAP})}_{22} = \left(2 - \sin^2(g\tau)\right)t/2 \sin^2(g\tau)$ such that $\beta^{(\text{SWAP})} = 0$.

Appendix F: Proof of the validity of Algorithm

We first recall the minimax theorem:

$$\min_x \max_y f(x, y) = \max_y \min_x f(x, y) \quad (F1)$$

for a function $f(x, y)$ convex on $x$ and concave on $y$. Assume $(x_0, y_1)$ is a solution for the L.H.S. of Eq. (F1) and $(x_1, y_0)$ is a solution for the R.H.S. of Eq. (F1), it is easy to see that

$$f(x_0, y_1) \geq f(x_0, y_0) \geq f(x_1, y_0). \quad (F2)$$

In view of Eq. (F1) both equalities hold. Therefore, $(x_0, y_0)$ is a saddle point of $f(x, y)$, i.e., $x_0 = \arg \min_x f(x, y_0)$ and $y_0 = \arg \max_y f(x_0, y)$. Since the objective function $\text{Tr} \left[ \tilde{P} \Omega_\phi(h) \right]$ in the primal problem of estimating QFI is convex on $h$ and concave on $\tilde{P}$, we can substitute $x = h$ and $y = \tilde{P}$. Obviously, $h^{(\text{opt})}$ is an optimal solution for $\min_h \max_{\tilde{P}} \text{Tr} \left[ \tilde{P} \Omega_\phi(h) \right]$ and thus corresponds to a saddle point. Then $x_0 = \arg \min_x f(x, y_0)$ can be satisfied by requiring $\partial_h f(h, \tilde{P}^{(\text{opt})})|_{h = h^{(\text{opt})}} = 0$, resulting in Eq. (15) in the main text:

$$\text{Re} \left\{ \text{Tr} \left[ \tilde{P}^{(\text{opt})} \left[ (-i\mathcal{N}_\phi \mathcal{H}) \mathcal{N}_\phi - i\mathcal{N}_\phi h^{(\text{opt})} \right] \right] \right\} = 0 \quad \text{for all } \mathcal{H} \in \mathbb{H}_r. \quad (F3)$$

Meanwhile $y_0 = \arg \max_y f(x_0, y)$ corresponds to the requirement that $\tilde{P}^{(\text{opt})}$ is an optimal solution for fixed $h = h^{(\text{opt})}$. Therefore, $(h^{(\text{opt})}, \tilde{P}^{(\text{opt})})$ is a saddle point and an optimal solution for $\max_{\tilde{P}} \min_h \text{Tr} \left[ \tilde{P} \Omega_\phi(h) \right]$. By definition a purification of $\tilde{P}^{(\text{opt})}$ is an optimal physically implemented strategy.