The motion of elliptic cylinder under free surface

V K Kostikov\textsuperscript{1,2} and N I Makarenko\textsuperscript{1,2}

\textsuperscript{1} Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia
\textsuperscript{2} Novosibirsk State University, Novosibirsk, Russia
E-mail: makarenko@hydro.nsc.ru

Abstract. A problem on generation of unsteady nonlinear waves on the surface of an infinitely deep ideal fluid due to the motion of a submerged elliptical cylinder is considered. It is supposed that the cylinder can rotate in addition to translational two-dimensional motion. The initial formulation of the problem is reduced to an integrodifferential system of equations for the functions defining the free surface shape, the normal and tangential components of velocity on the free boundary. The small-time asymptotics of the solution is constructed in the case of the cylinder that moves with a constant acceleration from rest.

1. Introduction
In this paper we consider analytically the nonlinear problem on unsteady free surface motion of an ideal fluid generated by submerged elliptic cylinder. The initial stage of the flow forced by a circular cylinder under a free surface was studied in detail in [1, 2]. Unique solvability of a two-dimensional problem of motion of a circular cylinder under a free surface was established in the paper [3], where an asymptotics of the solution in terms of a small parameter characterizing the body size was justified. The small-time asymptotic solution was constructed in the case of translational motion of an elliptic cylinder in [4, 5]. The 3D-problem on the motion of a sphere under the free surface was considered in [6]. The formation of a cavity behind the submerged circular and elliptic cylinder moving with high acceleration was studied semi-analytically in [7, 8]. The problem under consideration was studied numerically in the papers [9]-[13]. Experimental results on unsteady motion of a cylinder near the free surface was presented in the paper [14].

Our analysis uses reduction of the Euler equations to the nonlinear boundary integro-differential system of equations for the wave elevation together with normal and tangential fluid velocities at a free surface. In this method the key role is played by the integral equation for a normal fluid velocity that describes interaction of the cylinder with the free surface. This approach was also applied to the piston mechanism of the generation of tsunami-type waves due to a fast deformation of a compact bottom area [15].

2. Formulation of the problem
Two-dimensional potential flow of infinitely deep water is considered in the Cartesian coordinate system $Oxy$ (see Figure 1). Initially the fluid is at rest. The line $y = 0$ corresponds to the undisturbed free surface. Non-dimensional variables are used below. The initial submersion depth $h_0$ of the cylinder center from undisturbed free surface is taken as the length scale of the problem and a characteristic speed of the body $u_0$ as the velocity scale of the flow. Free surface elevation $\Gamma(t)$ is described by the equation $y = \eta(x,t)$, where $\eta(x,0) = 0$. The elliptic
cylinder can rotate around its center. The non-dimensional coordinates of the center of the cylinder are given as \( x = x_{\text{cyl}}(t) \) and \( y = y_{\text{cyl}}(t) \), where \( y_{\text{cyl}}(0) = -1 \). In Figure 1, \( a \) and \( b \) are the non-dimensional semi-axes of the cylinder, \( a \geq b \).

![Diagram of motion](image)

**Figure 1.** Scheme of motion.

Unsteady flow of incompressible fluid is described by the Euler equations for the velocity of the flow \( \mathbf{u} = (U, V) \) and hydrodynamic pressure \( p \)

\[
\begin{aligned}
U_t + UU_x + VU_y + p_x &= 0, \\
V_t + UV_x + VV_y + p_y &= -\lambda, \\
U_x + V_y &= 0, \\
U_y - V_x &= 0,
\end{aligned}
\]

(1)

where \( \lambda = gh_0/u_0^2 \) is the square of the inverse Froude number and \( g \) is the gravity acceleration. The boundary conditions on the free surface have the form

\[
\eta_t + U\eta_x = V, \quad p = 0 \quad (y = \eta(x, t)).
\]

(2)

and boundary condition on the cylinder surface \( S_{\text{cyl}}(t) \) has the form

\[
(\mathbf{u} - \mathbf{u}_{\text{cyl}}) \cdot \mathbf{n} = 0, \quad (x, y) \in S_{\text{cyl}}(t),
\]

(3)

where \( \mathbf{n} \) is the unit normal. Further we consider the translational motion of the cylinder together with rotation around its center. We assume that the coordinates of the cylinder center \( (x_{\text{cyl}}(t), y_{\text{cyl}}(t)) \) and the angle between the main axis of the cylinder and horizontal axis \( \alpha(t) \) are known at each point of time \( t \). We assume that the fluid is at rest at infinity, so we have \( U, V, \eta \to 0 \) as \( |x| \to \infty \). We assume also that the initial velocity field \( \mathbf{u}_{|t=0} = \mathbf{u}_0 \) satisfies the compatibility conditions

\[
U_{0x} + V_{0y} = 0, \quad U_{0y} - V_{0x} = 0, \quad (\mathbf{u}_0 - \mathbf{u}_{\text{cyl}}(0)) \cdot \mathbf{n}_0 = 0, \quad (x, y) \in S_{\text{cyl}}(0).
\]

These conditions are satisfied, in particular, when the cylinder starts moving smoothly from rest.
3. Integro-differential equations on the free surface
Let us introduce the tangential velocity \( u(x, t) = (U + i\eta V)_{y=0} \) and normal velocity \( v(x, t) = (V - \eta_x U)_{y=0} \) of the fluid particles on the free surface \( \Gamma(t) \). The equations (1)-(2) can be reduced to an equivalent system of one-dimensional equations for the unknown functions \( u, v, \eta \):

\[
\eta_t = v, \quad u_t + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{u^2 - 2\eta uv - v^2}{1 + \eta_x^2} \right) + \lambda \eta_x = 0. \tag{4}
\]

The differential equations (4) should be considered together with the boundary integral equation

\[
\pi v(x) + v.p. \int_{-\infty}^{+\infty} A(x, s) v(s) \, ds = v.p. \int_{-\infty}^{+\infty} B(x, s) u(s) \, ds + v_{dip}(x), \tag{5}
\]

which is the boundary equation for the analytic function \( F(z, t) = U - iV \), where \( z = x + iy \). The time \( t \) is not shown in kostik:IntegralBoundaryEquation because it plays the role of a parameter in this equation. The kernels \( A \) and \( B \) can be split as

\[
A = A_f + r^2 A_r, \quad B = B_f + r^2 B_r,
\]

where \( r = (a + b)/2 \) and \( c = \sqrt{a^2 - b^2} \) is proportional to the eccentricity of the body contour \( S_{cyl} \). The kernels \( A_f \) and \( B_f \) depend on the deflection \( \eta \) as

\[
A_f(x, s) + iB_f(x, s) = \frac{i[1 + i\eta_x(x)]}{x - s + i[\eta(x) - \eta(s)]}.
\]

Note that the simplified version of the system (4)-(5) with truncated kernels \( A = A_f \) and \( B = B_f \) in (5) and \( v_{dip} = 0 \) corresponds to the model of hydroelastic waves (1)-(2) without any submerged body. Correspondingly, the kernels \( A_r \) and \( B_r \) describe the interaction of the body with free surface. To define these kernels, we introduce an auxiliary complex-valued function

\[
\tau(z) = \left\{ \begin{array}{ll}
\frac{1}{2} \left( z - z_{cyl} + \sqrt{(z - z_{cyl})^2 - c^2} \right), & -\frac{\pi}{2} < \text{Arg}(z) < \frac{\pi}{2}, \\
\frac{1}{2} \left( z - z_{cyl} - \sqrt{(z - z_{cyl})^2 - c^2} \right), & \frac{\pi}{2} < \text{Arg}(z) < \frac{3\pi}{2},
\end{array} \right.
\]

where \( z_{cyl}(t) = x_{cyl}(t) + iy_{cyl}(t) \). The function \( \tau(z) \) presents analytic branches of inverse mapping by the Joukowski function \( z(\tau) = z_{cyl} + \tau + c^2/(4\tau) \) which maps the exterior of the ellipse with semi-axes \( a \) and \( b \) onto the exterior \( |\tau| > r \) of the circle of radius \( r = (a + b)/2 \). By using this mapping we can avoid the integrals along the body contour in kostik:IntegralBoundaryEquation by transforming them to the integrals along the interface \( \Gamma \). The kernels \( A_r \) and \( B_r \) are given as

\[
A_r(x, s) + iB_r(x, s) = \left( \frac{1}{\tau(s + i\eta(s)) - \tau_s(x + i\eta(x))} + \frac{k}{\tau(s + i\eta(s)) - k\tau_s(x + i\eta(x))} \right) \left( \frac{i}{\tau(x + i\eta(x))} \right)_x, \tag{6}
\]

where bar denotes complex conjugate, and \( \tau_s = r^2/\tau \). Parameter \( k = \frac{a-b}{a+b} \) signifies the deviation of the elliptic cylinder from the circular cylinder. The function \( v_{dip} \) in the right-hand side of (5) is given by

\[
v_{dip}(x) = \Re 2\pi r^2 \left[ \frac{k e^{i\alpha_0} z_{cyl}(t) - z_{cyl}(t) \tau(x + i\eta(x))}{\tau(x + i\eta(x))} + r^2 \frac{i k e^{i\alpha_0} \eta_{cyl}(t)}{\tau^2(x + i\eta(x))} \right]_x,
\]
The velocity \( v_{dip} \) is a normal fluid velocity induced at \( \Gamma \) by conformal image of dipole concentrated at the origin \( \tau = 0 \) of auxiliary \( \tau \)-plane. The integral equation (5) closing the system of differential equations (4) contains all information about interaction of the cylinder with the free surface. When parameter \( k \to 0 \) the kernels \( A_r \) and \( B_r \) and the function \( v_{dip} \) converge to the kernels of integral operators and integrated terms obtained for a circular cylinder [3]. The integro-differential system (4), (5) is equivalent to the original equations (1)–(3).

4. Small-time asymptotic solution

We consider the unsteady flow which starts from rest, \( \eta(x,0) = u(x,0) = v(x,0) = 0 \), and is caused by arbitrary two-dimensional motion of the cylinder with a constant acceleration. The solution of equations (4)-(5) is derived by the small-time expansion method (see [1]-[6] and [16]) in the form

\[
\eta(x,t) = t^2 \eta_2(x) + t^3 \eta_3(x) + t^4 \eta_4(x) + \ldots, \\
u(x,t) = t u_1(x) + t^2 u_2(x) + t^3 u_3(x) \ldots, \\
v(x,t) = tv_1(x) + t^2 v_2(x) + t^3 v_3(x) + \ldots.
\]

The coefficients \( \eta_n \) and \( u_n \) are evaluated explicitly by recursive formulae which follow from the differential equations (4) once the coefficients \( v_n \) are known. The integral equation (5) provides recursive equations for the coefficients \( v_n(x) \) of the form

\[
\pi v_n(x) + \int_{-\infty}^{+\infty} A_0(x,s) v_n(s)ds = \varphi_n(x), \quad (n \geq 1)
\]

where the kernel \( A_0 \) consists of the lower-order terms of the kernels \( A \) and \( B \) with respect to the deflection \( \eta \). Further simplifications are available by using a perturbation procedure with some appropriate small parameters. If the nondimensional semi-axes \( a \) and \( b \) are small, we obtain the non-linear version of dipole approximation for submerged elliptic cylinder. Combining this approach with the method of successive approximations one can obtain analytically explicit formulae for the leading-order solution.

We consider here wave patterns that are formed by an arbitrary plane motion of the cylinder. The trajectory of the center and angle of rotation can be taken in dimensionless form as:

\[
z_{cyl}(t) = -i + t^2(\cos \theta + i \sin \theta), \quad \alpha_{cyl}(t) = \alpha_0 + \varepsilon t^2.
\]

The leading-order solution has the following coefficients for the free surface elevation \( \eta(x,t) \):

\[
\eta_2(x) = 2r^2 \left\{ (\sin \theta + k \sin(\theta - 2\alpha_0)) Q_x + (\cos \theta + k \cos(\theta - 2\alpha_0)) P_x \right\} + \\
+ 2kr^4 \left\{ \cos 2\alpha_0 (QQ_x - PP_x) - \sin 2\alpha_0 (PQ_x + QP_x) \right\}, \quad (7)
\]

Here \( P(x) \) is even and \( Q(x) \) is odd functions of \( x \). These functions are defined for \( x \geq 0 \) by the formulae

\[
P(x) = \frac{1}{2\sqrt{2}kr^2} \left( -\sqrt{2} + \sqrt{(x^2 - 4kr^2 - 1)^2 + 4x^2 - x^2 + 4kr^2 + 1} \right),
\]
\[
Q(x) = \frac{1}{2\sqrt{2}kr^2} \left( \sqrt{2} - \sqrt{(x^2 - 4kr^2 - 1)^2 + 4x^2 + x^2 - 4kr^2 - 1} \right).
\]
5. Analysis of the solution

Figures 2-3 show typical shapes of the free surface corresponding to the solution (7) in the case of motion of elliptic cylinder with a semi-axes ratio $a/b = 2/1$ with rotation ($\lambda = 5$). One can see on the figure 2 that the formation of an asymmetric splash jet is affected by rotation of the cylinder during its vertical submersion. Figure 3 demonstrates how the cylinder generates the travelling wave by it’s horizontal accelerated motion together with rotation. Figure 4 shows the evolution of the free surface when the cylinder rotates without it’s translation. One can see that free surface follows rotary motion of the body.

**Figure 2.** Submersion of elliptic cylinder with rotation at the time points $t = 0.4$ (thin line), $t = 0.6$ (dashed line) and $t = 0.8$ (thick solid line).

**Figure 3.** Horizontal motion of elliptic cylinder with rotation at the time points $t = 0.4$ (thin line), $t = 0.6$ (dashed line) and $t = 0.8$ (thick solid line).

**Figure 4.** Rotation of elliptic cylinder without translational motion at the time points $t = 0.6$ (dashed line) and $t = 0.8$ (solid line).

Thus, the fully nonlinear integro-differential system on the free boundary was constructed for non-stationary motion of submerged body under the free surface. Small-time asymptotic solution was constructed in the case of uniformly accelerated elliptic cylinder that can rotate around its center. We can conclude from the analysis of this solution that the rotation of the cylinder causes considerable asymmetry of the wave patterns generated on the free surface.

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