FINITELY PRESENTED LATTICE-ORDERED ABELIAN GROUPS WITH ORDER-UNIT

LEONARDO CABRER ‡ AND DANIELE MUNDICI†

ABSTRACT. Let $G$ be an $\ell$-group (which is short for “lattice-ordered abelian group”). Baker and Beynon proved that $G$ is finitely presented iff it is finitely generated and projective. In the category $\mathcal{U}$ of unital $\ell$-groups—those $\ell$-groups having a distinguished order-unit $u$—only the ($\Leftarrow$)-direction holds in general. Morphisms in $\mathcal{U}$ are unital $\ell$-homomorphisms, i.e., homomorphisms that preserve the order-unit and the lattice structure. We show that a unital $\ell$-group $(G, u)$ is finitely presented iff it has a basis, i.e., $G$ is generated by an abstract Schauder basis over its maximal spectral space. Thus every finitely generated projective unital $\ell$-group has a basis $B$. As a partial converse, a large class of projectives is constructed from bases satisfying $\bigwedge B \neq 0$. Without using the Effros-Handelman-Shen theorem, we finally show that the bases of any finitely presented unital $\ell$-group $(G, u)$ provide a direct system of simplicial groups with 1-1 positive unital homomorphisms, whose limit is $(G, u)$.

1. Introduction

We refer to [4], [10] and [13] for background on $\ell$-groups. A unital $\ell$-group $(G, u)$ is an abelian group $G$ equipped with a translation-invariant lattice-order and a distinguished order-unit $u$, i.e., an element whose positive integer multiples eventually dominate each element of $G$. Unital $\ell$-groups are a modern mathematization of the time-honored euclidean magnitudes with an archimedean unit (see [17]). By [19, Theorem 3.9], the category $\mathcal{U}$ of unital $\ell$-groups is equivalent to the equational class of MV-algebras. Thus, while the archimedean property of order-units is not definable in first-order logic, $\mathcal{U}$ is endowed with all typical properties of equational classes: in particular, $\mathcal{U}$ has subalgebras, quotients and products, which in general are not cartesian products, [5].

Finitely presented $\ell$-groups (with or without unit) are an active topic of current research, because they have a basic proteiform reality, as computable algebraic structures, rational polyhedra, fans, finitely axiomatizable theories in many-valued logic, and finitely presented AF $C^*$ algebras whose Murray-von Neumann order of projections is a lattice. See [11, 20, 18, 16, 21].

Morphisms in the category of $\ell$-groups are lattice-preserving homomorphisms. Morphisms in the category of unital $\ell$-groups are also required to preserve order-units. A unital $\ell$-group $(G, u)$ is projective if whenever $\psi: (A, a) \to (B, b)$ is a...
surjective morphism and \( \phi: (G, u) \to (B, b) \) is a morphism, there is a morphism \( \theta: (G, u) \to (A, a) \) such that \( \phi = \psi \circ \theta \). For \( \ell \)-groups, Baker [1] and Beynon [2], [3, Theorem 3.1] (also see [10, Corollary 5.2.2]) gave the following characterization: An \( \ell \)-group \( G \) is finitely generated projective iff it is finitely presented. For unital \( \ell \)-groups the \(( \Rightarrow )\)-direction holds ([21, Proposition 5]). The converse direction fails in general.

Schauder bases provide the main tool to prove that an \( \ell \)-group is finitely generated projective iff it is presented by a word in the pure language of lattices, without resorting to the group structure, [16]. This strengthens the characterization by Baker-Beynon mentioned above, where lattice-group words are used, and paves the way to a full understanding of the sharp differences between measure theory in unital and in non-unital \( \ell \)-groups, [21].

For a geometric investigation of finitely presented unital \( \ell \)-groups, in [18] the notion of basis (see Definition 2.1) was introduced as a purely algebraic counterpart of Schauder bases. Specifically, in [18, Theorem 4.5] it is proved that an archimedean unital \( \ell \)-group \((G, u)\) is finitely presented iff it has a basis. The archimedean condition means that \( G \) is isomorphic to an \( \ell \)-group of real-valued functions defined on some set \( X \). In Theorem 3.1 we will prove that the archimedean assumption can be dropped, thus obtaining a characterization of finitely presented unital \( \ell \)-groups that does not mention free objects and their universal property.

As a corollary, every finitely generated projective unital \( \ell \)-group has a basis. In Section 4 we will prove a partial converse, yielding a method to construct large classes of projective unital \( \ell \)-groups.

With reference to [9] and [12], the underlying dimension group of \((G, u)\) will be considered in the final section. In Theorem 5.3 it is proved that if \((G, u)\) has a basis then its bases provide a direct system of simplicial groups with 1-1 positive unital homomorphisms, whose limit is \((G, u)\). Thus the Effros-Handelman-Shen representation theorem [6], Grillet’s theorem [15, 2.1], and Marra’s theorem [17] have a very simple proof for any such \((G, u)\).

2. Preliminaries

2.1. Unital \( \ell \)-groups and bases. A lattice-ordered abelian group (\( \ell \)-group) is a structure \((G, +, -, 0, \lor, \land)\) such that \((G, +, -, 0)\) is an abelian group, \((G, \lor, \land)\) is a lattice, and \( x + (y \lor z) = (x + y) \lor (x + z) \) for all \( x, y, z \in G \). An order-unit in \( G \) is an element \( u \in G \) with the property that for every \( g \in G \) there is \( n \in \{1, 2, 3, \ldots \} \) such that \( g \leq nu \). A unital \( \ell \)-group \((G, u)\) is an \( \ell \)-group \( G \) with a distinguished order-unit \( u \).

A map \( h: (G, u) \to (G', u') \) is said to be a unital \( \ell \)-homomorphism if it preserves the lattice as well as the group structure, and \( h(u) = u' \). By an ideal \( i \) of a unital \( \ell \)-group \((G, u)\) we mean the kernel of a unital \( \ell \)-homomorphism of \((G, u)\). We denote by MaxSpec\((G, u)\) the set of maximal ideals of \((G, u)\) equipped with the spectral topology, [4, §10]: a basis of closed sets for MaxSpec\((G)\) is given by sets of the form \( \{ p \in \text{MaxSpec}(G) \mid a \in p \} \), where \( a \) ranges over all elements of \( G \). Since \( G \) has an order-unit, MaxSpec\((G)\) is a nonempty compact Hausdorff space, [4, 10.2.2].

Definition 2.1. Let \((G, u)\) be a unital \( \ell \)-group. A basis of \((G, u)\) is a set \( B = \{b_1, \ldots, b_n\} \) of nonzero elements of the positive cone \( G^+ = \{ g \in G \mid g \geq 0 \} \) such that
(i) \( B \) generates \( G \);
(ii) for each \( k = 1,2,\ldots \) and \( k \)-element subset \( C \) of \( B \) with \( 0 \neq \bigcup \{ b \mid b \in C \} \), the set \( \{ m \in \text{MaxSpec}(G) \mid m \supseteq B \setminus C \} \) is homeomorphic to a \((k-1)\)-simplex;
(iii) it follows that the multiplicity \( m_i \) of each \( b_i \in B \) is uniquely determined.

This is an equivalent simplified reformulation of [18, Definition 4.3]. From (ii)-(iii) it follows that the multiplicity \( m_i \) of each \( b_i \in B \) is uniquely determined.

For \( n = 1,2,\ldots \) we let \( M_n \) denote the unital \( \ell \)-group of all continuous functions \( f: [0,1]^n \to \mathbb{R} \) having the following property: there are (affine) linear polynomials \( p_1,\ldots,p_m \) with integer coefficients, such that for all \( x \in [0,1]^n \) there is \( i \in \{1,\ldots,m\} \) with \( f(x) = p_i(x) \). \( M_n \) is equipped with the pointwise operations \(+,-,\wedge,\vee\) of \( \mathbb{R} \), and with the constant function 1 as the distinguished order-unit. The characteristic universal property of \( M_n \) is as follows:

**Proposition 2.2.** ([19, 4.16]) \( M_n \) is generated by the coordinate maps \( \pi_i: [0,1]^n \to \mathbb{R} \) together with the order-unit 1. For every unital \( \ell \)-group \((G,u)\) and elements \( g_1,\ldots,g_n \) in the unit interval \([0,u]\) of \( G \), if \( g_1,\ldots,g_n \) and \( u \) generate \( G \), then there is a unique unital \( \ell \)-homomorphism \( \psi \) of \( M_n \) onto \( G \) such that \( \psi(\pi_i) = g_i \) for each \( i = 1,\ldots,n \).

We say that \((G,u)\) is finitely presented if for some \( n = 1,2,\ldots \), \((G,u)\) is isomorphic to the quotient of \( M_n \) by a finitely generated (= singly generated = principal) ideal.

Given \( f \in M_n \) we denote \( Zf = a^{-1}(0) \) the zero set of \( f \). More generally, for every ideal \( J \) of \( M_n \) we will write

\[
ZJ = \bigcap \{ Zg \mid g \in J \}.
\]

In the particular case when \( J \) is maximal, \( ZJ \) is a singleton (because the functions in \( M_n \) separate points, [19, 4.17]), and we write

\[
\hat{J} = \text{the only element of } ZJ.
\]

For later use we record here a classical result, whose proof follows from the Hion-Hölder theorem [8, p.45-47], [4, 2.6]:

**Lemma 2.3.** For every unital \( \ell \)-group \((G,u)\) and ideal \( m \in \text{MaxSpec } G \) there is exactly one pair \((i_m,R_m)\) where \( R_m \) is a unital \( \ell \)-subgroup of \((\mathbb{R},1)\), and \( i_m \) is a unital \( \ell \)-isomorphism of the quotient \((G,u)/m \) onto \( R_m \). Upon identifying \((G,u)/m \) with \( R_m \) every element \( g/m \in (G,u)/m \) becomes a real number, and we can unambiguously write \( g/m \in \mathbb{R} \).

**Corollary 2.4.** Let \( i \) be an ideal of \( M_n \) and \( \text{MaxSpec}_{\geq i} M_n \) denote the compact set of all maximal ideals of \( \text{MaxSpec } M_n \) containing \( i \). Then the map \( \hat{Z} \) of (2) yields a homeomorphism of \( \text{MaxSpec}_{\geq i} M_n \) onto the compact set \( Zi \subseteq [0,1]^n \). The inverse of \( \hat{Z} \) is the map \( x \in Zi \mapsto m_x = \{ f \in M_n \mid f(x) = 0 \} \), and we have identical real numbers

\[
f/m = f(\hat{Z}(m)), \quad \forall f \in M_n, \quad \forall m \in \text{MaxSpec}_{\geq i} \subseteq M_n.
\]

**Proof.** As a matter of fact, for each \( x \in Zi \), \( m_x \) is a maximal ideal of \( M_n \). Further, for each \( f \in i \), from \( f(x) = 0 \) we get \( f \in m_x \), whence \( m_x \supseteq i \) and \( \hat{Z} m_x = x \). Let \( p \in \text{MaxSpec}_{\geq i} M_n \). Then \( Zp \subseteq Zi \) and for every \( f \in p \) with \( f(\hat{Z} p) = 0 \) we have \( p \subseteq m_{Z(p)} \) and \( \hat{Z} p \in Zi \). The assumed maximality of \( p \) is to the effect that
p = m_{\not\in (p)}$; whence $\hat{Z}$ is a one-one map from $\operatorname{MaxSpec}_2 \mathcal{M}_n$ onto $\mathbb{Z}$. By definition of spectral topology, $\hat{Z}$ is a homeomorphism. An application of Lemma 2.3 now settles the result. \hfill $\Box$

**Corollary 2.5.** The quotient map $\kappa: \mathcal{M}_n \to \mathcal{M}_n / i$ determines the homeomorphism $m \mapsto m / i$ of $\operatorname{MaxSpec}_2 \mathcal{M}_n$ onto $\operatorname{MaxSpec} \mathcal{M}_n / i$. The inverse map is given by $\kappa^{-1}(n) = \{ f \in \mathcal{M}_n \mid f / i \in n \}$ for each $n \in \operatorname{MaxSpec} \mathcal{M}_n / i$.

**Proof.** The routine proof follows by combining Lemma 2.3 with [4, 2.3.8]. \hfill $\Box$

### 2.2. Rational polyhedra and unimodular triangulations

We will make use of a few elementary notions and techniques of polyhedral topology. We refer to the first chapters of [7] for background. By a *rational polyhedron* we understand a finite union of simplexes $P = S_1 \cup \cdots \cup S_t$ in $\mathbb{R}^n$ such that the coordinates of the vertices of every simplex $S_i$ are rational numbers. For every simplicial complex $\Sigma$ the point-set union of the simplexes of $\Sigma$ is called the *support* of $\Sigma$ and is denoted $|\Sigma|$. The support of any rational point $v \in \mathbb{R}^n$ is called the *denominator* of $v$, denoted $\operatorname{den}(v)$. The integer vector $\hat{v} = \operatorname{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$ is called the *homogeneous correspondent* of $v$. An $m$-simplex $U = \operatorname{conv}(w_0, \ldots, w_m) \subseteq [0, 1]^n$ is said to be *unimodular* if it is rational and the set of integer vectors $\{\hat{w}_0, \ldots, \hat{w}_m\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$. A simplicial complex is said to be a *unimodular triangulation* (of its support) if all its simplexes are unimodular.

As a remainder of the relevance of unimodular triangulations, recall that the homogeneous correspondent of a unimodular triangulation is known as a regular (or, nonsingular) fan [7, 22].

The following results show the connection among rational polyhedra zero-sets of McNaughton maps and ideals in $\mathcal{M}_n$.

**Proposition 2.6.** [18, 4.1.5.1] Letting $P \subseteq [0, 1]^n$, the following are equivalent:

- (i) $P$ is a rational polyhedron.
- (ii) $P = |\Delta|$ for some unimodular triangulation $\Delta$.
- (iii) there is unimodular triangulation $\nabla$ of $[0, 1]^n$ such that $P = \bigcup\{S \in \nabla: S \subseteq P\}$.
- (iv) $P = Zf$ for some $f \in \mathcal{M}_n$.

**Lemma 2.7.** Let $i$ be an ideal of $\mathcal{M}_n$. Then the following are equivalent:

- (i) $i$ is principal.
- (ii) there exists $f \in i$ such that $Zi = Zf$.

**Proof.** For the non trivial direction, let $f \in i$ such that $Zi = Zf$. It is no loss of generality to suppose $0 \leq f$. We must verify that, for all $0 \leq g \in \mathcal{M}_n$, $g \in i \iff g \leq kf$ for some $k = 1, 2, \ldots$. The $\Rightarrow$-direction follows from the fact that $f \in i$. For the $\Leftarrow$-direction, let $\Lambda$, be a rational triangulation of $[0, 1]^n$, $f$ and $g$ are linear over each $S \in \Lambda$. Let $\{v_1, \ldots, v_s\}$ be the vertices of $\Lambda$. Since $Zf = Zi \subseteq Zg$, $f(v_i) = 0$ implies $g(v_i) = 0$. Then there exists an integer $m_i > 0$ such that $m_i f(v_i) \geq f(v_i)$ for each $i = 1, \ldots, s$. Letting $m = \max(m_1, \ldots, m_s)$, the desired result follows from the linearity of $f$ and $g$ over each simplex of $\Lambda$. \hfill $\Box$
3. FINITELY PRESENTED UNITAL ℓ-GROUPS AND BASIS

Theorem 3.1. A unital ℓ-group \((G, u)\) is finitely presented iff it has a basis.

The \((\Rightarrow)\)-direction of Theorem 3.1 is proved in [18, 5.2]. To prove the \((\Leftarrow)\)-direction let \(B = \{b_1, \ldots, b_n\}\) be a basis of \((G, u)\), with multiplicities \(m_1, \ldots, m_n\). Let \(\kappa: M_n \rightarrow (G, u)\) be the unique unital ℓ-homomorphism extending the map \(\pi_i \mapsto b_i\), as given by Proposition 2.2. Let the ideal \(i\) of \(M_n\) be defined by \(i = \ker(\kappa)\). By Definition 2.1(i), \(\kappa\) is onto \(G\), thus \((G, u) \cong M_n / i\).

For any \(E \subseteq B\) we define the simplex \(T_E \subseteq [0,1]^n\) by

\[
T_E = \text{conv}\{e_i/m_i \mid b_i \in E\},
\]

where \(e_i\) is the \(i\)th standard basis vector of \(\mathbb{R}^n\). From Definition 2.1(ii) it follows that \(\kappa(\sum_i m_i \pi_i) = \sum_i m_i \kappa(\pi_i) = \sum_i m_i b_i = u\) whence, defining the function \(a \in M_n\) by \(a = |1 - \sum_i m_i \pi_i|\),

\[
0 \leq a \in i, \text{ and } Za = T_B.
\]

Let \(k = 1, 2, \ldots\). Then by a \(k\)-cluster of \(B\) we understand a \(k\)-element subset \(C\) of \(B\) such that \(\bigwedge C \neq 0\). We denote by \(\mathcal{B}^{\leq k}\) the set of all clusters of \(B\). For each \(C \in \mathcal{B}^{\leq k}\), displaying the complementary set \(B \setminus C\) as \(\{b_{j_1}, \ldots, b_{j_s}\}\), we define the function \(a_C \in M_n\) by

\[
a_C = \pi_{j_1} \lor \ldots \lor \pi_{j_s}, \quad (a_C = 0 \text{ in case } C = \emptyset).\]

We then have

\[
T_B \cap Za_C = T_C.
\]

We next observe

\[
\bigwedge_{C \in \mathcal{B}^{\leq k}} a_C \in i.
\]

By (7), the result is trivial if \(B\) is a cluster in \(\mathcal{B}^{\leq k}\). If this is not the case, let \(b_{i_C} \in B \setminus C\) for each \(C \in \mathcal{B}^{\leq k}\). If \(D = \{b_{i_C} : C \in \mathcal{B}^{\leq k}\} \in \mathcal{B}^{\leq k}\), then \(b_{i_D} \in D\), which is a contradiction. Therefore,

\[
\kappa(\bigwedge_{C \in \mathcal{B}^{\leq k}} \pi_{i_C}) = \bigwedge_{C \in \mathcal{B}^{\leq k}} b_{i_C} = 0,
\]

i.e., \(\bigwedge_{C \in \mathcal{B}^{\leq k}} \pi_{i_C} \in i\). Since each \(b_{i_C} \in B \setminus C\) is arbitrary, (9) now follows from the distributivity of the underlying lattice of \((G, u)\).

Let the function \(f^* \in M_n\) be defined by

\[
f^* = a \lor \bigwedge_{C \in \mathcal{B}^{\leq k}} a_C.
\]

From (6) and (9) it follows that

\[
0 \leq f^* \in i,
\]

and from (8),

\[
Zf^* = Za \cap \bigcup_{C \in \mathcal{B}^{\leq k}} Za_C = \bigcup_{C \in \mathcal{B}^{\leq k}} T_C.
\]
From (11) we immediately have
\[ Z f^* \supseteq \mathcal{Z} i. \] (13)

To prove the converse inclusion, for each cluster \( K \) of \( \mathcal{B} \) we set
\[ \text{apo}(K) = \{ n \in \text{MaxSpec} M_n / i \mid n \supseteq \mathcal{B} \setminus K \}. \] (14)
For each \( n \in \text{MaxSpec} M_n / i \), letting \( C_n \) be the cluster of all \( b \in \mathcal{B} \) such that \( b \notin n \), it follows that \( \mathcal{B} \setminus C_n \subseteq n \), whence \( n \in \text{apo}(C_n) \). Thus, \( \bigcup_{C \in \mathcal{B}^{\geq}} \text{apo}(C) \supseteq \text{MaxSpec} M_n / i \). Since the converse inclusion holds by definition, we have
\[ \text{MaxSpec} M_n / i = \bigcup_{C \in \mathcal{B}^{\geq}} \text{apo}(C). \] (15)
For each \( K \in \mathcal{B}^{\geq} \), we denote by \( \text{apo}_K(K) \) the inverse image of \( \text{apo}(K) \) under the composition of the homeomorphisms \( x \mapsto m_x \mapsto m_x / i \) of Corollaries 2.4 and 2.5, where \( m_x = \{ f \in M_n \mid f(x) = 0 \} \). In other words,
\[ \text{apo}_K(K) = \{ x \in \mathcal{Z} i \mid m_x / i \in \text{apo}(K) \}. \] (16)
From (12)-(15) we get
\[ \bigcup_{C \in \mathcal{B}^{\geq}} \text{apo}_K(C) = \mathcal{Z} i \subseteq Z f^* = \bigcup_{C \in \mathcal{B}^{\geq}} T_C. \] (17)

This inclusion can be refined as follows:

**Claim 1:** For each cluster \( C \) of \( \mathcal{B} \), \( \text{apo}_K(C) \subseteq T_C \).

As a matter of fact, by (14) and condition (iii) of Definition 2.1 we have
\[ \text{apo}(C) = \{ n \in \text{MaxSpec} M_n / i \mid b / n = 0, \forall b \in \mathcal{B} \setminus C \} \]
\[ = \{ n \in \text{MaxSpec} M_n / i \mid m_i b_{i_1} + \cdots + m_i b_{i_t} = 1 \}, \] (18)
for each cluster \( C = \{ b_{i_1}, \ldots, b_{i_t} \} \) of \( \mathcal{B} \). By Lemma 2.3, for each \( m \in \text{MaxSpec} \mathcal{Z} M_n \) the unital \( \ell \)-group \( \frac{\mathcal{M}_n}{m} \) and its isomorphic copy \( \frac{\mathcal{M}_n}{m / i} \) are canonically isomorphic to the same unital \( \ell \)-subgroup of \( (\mathbb{R}, 1) \). Thus for each \( f \in M_n \) we have identical real numbers \( \frac{f / m}{n / i} \), \( \forall n \in \text{MaxSpec} M_n / i \),
\[ f / n / i = f(\mathcal{Z}(k^{-1}(n))), \ \forall n \in \text{MaxSpec} M_n / i, \] (19)
or equivalently,
\[ f(x) = \frac{f / m_x}{n / i}, \ \forall x \in \mathcal{Z} i. \] (20)
Combining (16) with (18), we obtain \( (y_1, \ldots, y_n) \in \text{apo}_K(C) \) if and only if
\[ m_{i_1} b_{i_1} + \cdots + m_{i_t} b_{i_t} \]
\[ = \frac{(m_{i_1} y_{i_1} + \cdots + m_{i_t} y_{i_t}) / i}{m_x / i} = 1. \]
Now recalling (5), by (20) we obtain
\[ \text{apo}_K(C) = \{ (y_1, \ldots, y_n) \in \mathcal{Z} i \mid m_{i_1} y_{i_1} + \cdots + m_{i_t} y_{i_t} = 1 \} \subseteq T_C, \] (21)
thus settling Claim 1.

Actually, a stronger result holds:

**Claim 2:** For every cluster \( C \) of \( \mathcal{B} \), \( \text{apo}_K(C) = T_C \).
The proof is by induction on the number $l$ of elements of $C$.

**Base case:** $l = 1$. Then for a unique $j \in \{1, \ldots, n\}$ we have $C = \{b_j\} = \{\pi_j/i\}$. Condition (ii) in Definition 2.1 is to the effect that apo($C$) contains exactly one element $n$. By Lemma 2.3, $n$ is the only maximal ideal of $M_{n/i}$ such that $0 = b/n$ for all $b \neq b_j$; by (18), $n$ is uniquely determined by the condition $1 = m_jb_j/n = (m_j\pi_j/i)/n$. Letting $z \in Z$ be the image of $n$ in apo$_R(C)$, by (5) and Claim 1 we have $z = e_j/m_j$. We conclude that apo$_R(C) = \{e_j/m_j\} = \{e_j\}$. Therefore, the proof of Theorem 3.1 is thus complete.

**Induction Step:** $l + 1$. Let us write $C = \{b_{i_0}, \ldots, b_{i_t}\}$. Since every $l$-element subset $C'$ of $C$ is a cluster of $\mathcal{B}$, by induction hypothesis apo$_R(C') = T_{C'}$. $T_{C'}$ is known as a facet of $T_C$. By Claim 1, apo$_R(C)$ is a nonempty subset of $T_C$ containing all facets of $T_C$. Further, apo$_R(C)$ is homeomorphic to an $l$-simplex, because so is its homeomorphic copy apo($C$), by condition (ii) in Definition 2.1. Observe that $T_C$ is contractible (i.e., $T_C$ is continuously shrinkable to a point). By way of contradiction, suppose apo$_R(C)$ is a proper subset of $T_C$. Then a classical result in algebraic topology shows that apo($C$) is not contractible. Thus apo$_R(C)$ is not homeomorphic to any $l$-simplex, a contradiction showing apo$_R(C) = T_C$, and settling Claim 2.

Combining Claim 2 and (17), we can write

$$Z f^* = \bigcup_{C \in \mathcal{B}^*} T_C = Z_1.$$  \hspace{1cm} (22)

Recalling Lemma 2.7 it follows that $i$ is the ideal generated by $f^*$. By (4), $(G, u)$ is finitely presented. The proof of Theorem 3.1 is thus complete. \hfill $\square$

4. A CLASS OF PROJECTIVE UNITAL $\ell$-GROUPS

In Theorem 4.2 below we will construct a large class of projective unital $\ell$-groups. For the proof we prepare

**Lemma 4.1.** Let $S = \conv(x_1, \ldots, x_k) \subseteq [0, 1]^n$ be a unimodular $(k - 1)$-simplex and $v \in \{0, 1\}^n$ a vertex of $[0, 1]^n$. Then for every $Y \subseteq \{x_1, \ldots, x_k\}$ there is a matrix $M \in \mathbb{Z}^{n \times n}$ and a vector $b \in \mathbb{Z}^n$ such that

$$M x_i + b_i = \begin{cases} v & \text{if } x_i \in Y, \\ x_i & \text{otherwise.} \end{cases} \hspace{1cm} (23)$$

**Proof.** Since $S$ is unimodular, the set $\{\bar{x}_1, \ldots, \bar{x}_k\}$ of homogeneous correspondents of $x_1, \ldots, x_k$ can be extended to a basis $\{\bar{x}_1, \ldots, \bar{x}_k, q_{k+1}, \ldots, q_{n+1}\}$ of the free abelian group $\mathbb{Z}^{n+1}$. The $(n + 1) \times (n + 1)$ matrix $D$ with column vectors $\bar{x}_1, \ldots, \bar{x}_k, q_{k+1}, \ldots, q_{n+1}$ is invertible and $D^{-1} \in \mathbb{Z}^{(n+1) \times (n+1)}$. For each $i = 1, \ldots, t$ let $c_i \in \mathbb{Z}^{n+1}$ be defined by

$$c_i = \begin{cases} \text{den}(x_i)(v, 1) & \text{if } x_i \in Y, \\ \bar{x}_i & \text{otherwise.} \end{cases}$$

Let $C \in \mathbb{Z}^{(n+1) \times (n+1)}$ be the matrix whose columns are given by the vectors $c_1, \ldots, c_k, q_{k+1}, \ldots, q_{n+1}$. Since $D$ and $C$ have the same $(n + 1)$th row,

$$CD^{-1} = \begin{pmatrix} M \\ 0, \ldots, 0 \end{pmatrix}$$

where
for some $n \times n$ integer matrix $M$ and vector $d \in \mathbb{Z}^n$. For each $i = 1, \ldots, k$ we then have $(CD^{-1})\bar{x}_i = (CD^{-1})\text{den}(x_i)(x_i,1) = \text{den}(x_i)(Mx_i + d,1)$. By definition, $(CD^{-1})\bar{x}_i = c_i = \text{den}(x_i)(v,1)$ if $x_i \in Y$ and $(CD^{-1})\bar{x}_i = \bar{x}_k = \text{den}(x_i)(x_i,1)$ otherwise. Thus (23) is satisfied. □

**Theorem 4.2.** Suppose the unital $\ell$-group $(G, u)$ has a basis $\mathcal{B}$ with $\wedge \mathcal{B} \neq 0$. Suppose at least one of the multiplicities of $\mathcal{B}$ is equal to 1. Then $(G, u)$ is projective.

**Proof.** By assumption, $\mathcal{B}$ itself is a basis of $(G, u)$ with multiplicities $1 = m_1 \leq m_2 \leq \ldots \leq m_n$. We keep the notation of the proof of Theorem 3.1. In particular, $T_\mathcal{B} = \text{conv}(e_1, e_2/m_2, \ldots, e_n/m_n)$, where, as the reader will recall, $e_i$ denotes the $i$th basis vector in $[0, 1]^n$ such that $T_\mathcal{B}$ is a union of simplexes of $\Delta$, and all vertices of (every simplex of) $\Delta$ have rational coordinates.

We next define the function $f : [0, 1]^n \to [0, 1]^n$ by stipulating that, for each vertex $v$ of $\Delta$,

$$f(v) = \begin{cases} v & \text{if } v \in T_\mathcal{B} \\ e_1 & \text{if } v \not\in T_\mathcal{B} \end{cases}$$

(24)

and $f$ is linear over each simplex of $\Delta$. Then $f$ is a continuous map and $f| T_\mathcal{B}$ is the identity map on $T_\mathcal{B}$. For any simplex $S$ of $\Delta$, let $\partial S$ denote the set of extremal points of $S$. Since $f$ is linear over $S$ and $f(v) \in T_\mathcal{B}$ for each $v \in \partial S$, we have $f(S) = f(\text{conv}(\partial S)) = \text{conv}(f(\partial S)) \subseteq \text{conv}(T_\mathcal{B}) = T_\mathcal{B}$, whence

$$f([0, 1]^n) = T_\mathcal{B}.$$  

(25)

We have thus shown that $f \circ f = f$ and $f$ is a continuous retraction of $[0, 1]^n$ onto $T_\mathcal{B}$ which is linear on each simplex of $\Delta$.

By Lemma 4.1, the coefficients of each linear piece of $f$ are integers. Therefore, the map $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ given by

$$\varphi(g) = g \circ f.$$  

(26)

is well defined. It follows straightforwardly that $\varphi$ is a unital $\ell$-homomorphism. Since $f \circ f = f$ then $\varphi \circ \varphi = \varphi$. In other words, $\varphi$ is an idempotent endomorphism of $\mathcal{M}_n$. Stated otherwise, the unital $\ell$-subgroup $\varphi(\mathcal{M}_n)$ of $\mathcal{M}_n$ is a retraction of $\mathcal{M}_n$.

Applying now the universal property of $\mathcal{M}_n$, (Proposition 2.2) one sees that $\mathcal{M}_n$ is projective. A routine exercise using the fact that $\varphi(\mathcal{M}_n)$ is a retraction of $\mathcal{M}_n$ shows that $\varphi(\mathcal{M}_n)$ is projective.

To conclude the proof it is enough to show that $\varphi(\mathcal{M}_n)$ is unitally $\ell$-isomorphic to $(G, u)$. In proving the ($\Leftarrow$)-direction of Theorem 3.1 we have seen that $(G, u)$ is unitally $\ell$-isomorphic to $\mathcal{M}_n/i$, for some ideal $i$ having following characterization:

$$i = \left\{ g \in \mathcal{M}_n \mid Zg \supseteq \bigcup_{C \in \mathcal{B} \setminus \{1\}} T_C \right\} = \{ g \in \mathcal{M}_n \mid Zg \supseteq T_\mathcal{B} \}.$$  

Letting $\ker(\varphi)$ be the kernel of $\varphi$, by (25) and (26) we have

$$g \in \ker(\varphi) \iff g \circ f = 0 \iff g(f([0, 1]^n)) = \{0\} \iff g(T_\mathcal{B}) = \{0\} \iff Zg \supseteq T_\mathcal{B} \iff g \in i.$$  

Therefore, $(G, u) \cong \mathcal{M}_n/i = \mathcal{M}_n/\ker(\varphi) \cong \varphi(\mathcal{M}_n)$, and the proof is complete. □
5. The underlying dimension group of a unital $\ell$-group with a basis

In the category $\mathcal{P}$ of partially ordered abelian groups with order-unit, [13, p.12] objects are pairs $(G, u)$, where $G$ is a partially ordered abelian group and $u$ is an order-unit of $G$. A morphism $\phi: (G, u) \to (H, v)$ of $\mathcal{P}$ is a unital (i.e., unit-preserving) positive (in the sense that $\phi(G^+ \subseteq H^+)$ homomorphism.

Following [13, p.47], by a unital simplicial group we understand an object of $\mathcal{P}$ that is isomorphic (in $\mathcal{P}$) to the free abelian group $\mathbb{Z}^n$ for some integer $n > 0$ equipped with the product ordering: $(x_1, \ldots, x_n) \geq 0$ iff $x_i \geq 0 \ \forall i = 1, \ldots, n$.

A unital dimension group $(G, u)$ is an object of $\mathcal{P}$ such that $G = G^+ - G^-$, sums of intervals are intervals, and $kg \in G^+ \Rightarrow g \in G^+$, for any $g \in G$ and integer $k > 0$. For short, $G$ is directed, Riesz, and unperforated, [13, p.44]. In [9, §2] one can find several characterizations of the Riesz property. By Elliott classification theory [12], countable unital dimension groups are complete classifiers of AF algebras, the norm limits of ascending sequences of finite-dimensional $C^*$-algebras, all with the same unit.

Given a unital $\ell$-group $(G, u)$ let $(G, u)_{\text{dim}}$ denote the underlying group of $(G, u)$ equipped with the same positive cone $G^+ +$ and order-unit $u$, but forgetting the lattice structure of $(G, u)$. Then $(G, u)_{\text{dim}}$ is a unital dimension group. Thus in particular, every unital simplicial group is a unital dimension group. Since the properties of directedness, Riesz, and unperforatedness are preserved by direct limits, then direct limits of unital simplicial groups are unital dimension groups.

The celebrated Effros-Handelman-Shen theorem [6], [13, 3.21] (also see Grillet’s theorem [15, 2.1] in the light of [14, Remark 3.2]) states the converse: for every unital dimension group $(G, u)$ we can write

$$(G, u) \cong \varinjlim \{ \phi_{ij}: (\mathbb{Z}^n, u_i) \to (\mathbb{Z}^n, u_j) \mid i, j \in I \}$$

for some direct system of unital simplicial groups and unital positive homomorphisms in $\mathcal{P}$. For dimension groups of the form $(G, u)_{\text{dim}}$, with $(G, u)$ a unital $\ell$-group, Marra [17] proved that the maps $\phi_{ij}$ can be assumed to be 1-1.

A further simplification occurs when $(G, u)$ has a basis: as a matter of fact, in Theorem 5.3 below we will prove that the set of bases of $(G, u)$ is rich enough to provide a direct system of unital simplicial groups and 1-1 unital homomorphisms such that $(G, u)_{\text{dim}}$ is the limit of this system in the category $\mathcal{P}$. To this purpose, given a basis $B = \{b_1, \ldots, b_n\}$ of a unital $\ell$-group $(G, u)$ we let

$$\text{grp} B = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_n$$

denote the group generated by $B$ in (the underlying group of) $G$. Similarly,

$$\text{sgr} B = \mathbb{Z}_{\geq 0}b_1 + \cdots + \mathbb{Z}_{\geq 0}b_n$$

will denote the semigroup generated by $B$ together with the zero element.

Assuming, as we are doing throughout the rest of this paper, that the elements of $B$ are listed in some prescribed order, by definition of $B$ the $n$-tuple of multiplicities $m_B = (m_1, \ldots, m_n)$ is uniquely determined by the $n$-tuple $(b_1, \ldots, b_n)$.

**Proposition 5.1.** Let $B = \{b_1, \ldots, b_n\}$ be a basis of a unital $\ell$-group $(G, u)$. Let

$$G_B = (\text{grp} B, \text{sgr} B, u)$$
denote the group \( \text{grp} \mathcal{B} \) equipped with the positive cone \( \text{sgr} \mathcal{B} \) and with the distinguished element \( u = \sum m_i b_i \). Let
\[
\mathbb{Z}_B = (\mathbb{Z}^n, (\mathbb{Z}^+)^n, m_B)
\]
be the standard simplicial group of rank \( n \), with the \( n \)-tuple \( m_B \) as a distinguished element. Then

1. \( \mathcal{B} \) is a free generating set of the free abelian group \( \text{grp} \mathcal{B} \) of rank \( n \).
2. \( G^+ \cap \text{grp} \mathcal{B} = \text{sgr} \mathcal{B} \).
3. The map \( b_i \mapsto e_i \) uniquely extends to an isomorphism \( \psi_B : \text{grp}_B \cong \mathbb{Z}^n \).
4. \( \psi_B \) is in fact an isomorphism (in the category \( \mathcal{P} \)) of \( G_B \) onto \( \mathcal{Z}_B \), whence \( G_B \) is a unital simplicial group, called the basic group of \( \mathcal{B} \); further, \( \mathcal{B} \) is the set of atoms (= minimal positive nonzero elements) of \( G_B \); thus if \( \mathcal{B}' \neq \mathcal{B} \) is another basis of \( (G, u) \) then \( G_B \neq G_{B'} \).

Proof. (1) By condition (ii) in the definition of \( \mathcal{B} \), no nonzero linear combination of the elements of \( \mathcal{B} \) is zero in (the \( \mathbb{Z} \)-module) \( G \). It is well known that \( G \) is torsion-free. Thus \( \mathcal{B} \) is a free generating set in \( \text{grp} \mathcal{B} \), and \( \text{grp} \mathcal{B} \) is free abelian of rank \( n \).

To prove (2), suppose \( g \in G^+ \cap \text{grp} \mathcal{B} \), and write \( g = \sum_{i=1}^n l_i b_i \) for suitable integers \( l_1, \ldots, l_n \). Fix now \( j \in \{1, \ldots, n\} \) and let \( n_j \) be the only maximal ideal of \( G \) such that \( b_k \in n_j \) for all \( k \neq j \), as given by condition (ii) in the definition of \( \mathcal{B} \).

By condition (iii) we have
\[
0 \leq \sum_{i=1}^n l_i b_i \Rightarrow 0 \leq \sum_{i=1}^n l_i b_i = \sum_{j=1}^n \frac{l_j b_j}{n_j} = \sum_{j=1}^n \frac{m_j}{n_j},
\]
whence \( 0 \leq l_j \) for all \( j \), and \( g \in \text{sgr} \mathcal{B} \). The converse inclusion is trivial.

To prove (3) it is enough to note that the map \( b_i \mapsto e_i \) is a one-one correspondence between the free generating set \( \mathcal{B} \) of \( \text{grp} \mathcal{B} \) and the free generating set \( \{e_1, \ldots, e_n\} \) of \( \mathbb{Z}^n \).

Concerning (4). It is easy to see that \( \mathcal{B} \) is the set of atoms of \( G_B \), and \( \{e_1, \ldots, e_n\} \) is the set of atoms of the simplicial group \( (\mathbb{Z}^n, (\mathbb{Z}^+)^n) \). Thus \( \psi_B \) is an isomorphism of \( G_B \) onto \( (\mathbb{Z}^n, (\mathbb{Z}^+)^n) \), and \( G_B \) is simplicial. Trivially, \( \psi_B \) preserves the order-unit. So \( G_B \) is a unital simplicial group which is isomorphic (in \( \mathcal{P} \)) to \( \mathcal{Z}_B \). The rest is clear.

Given two bases \( \mathcal{B}' \) and \( \mathcal{B} \) of a unital \( \ell \)-group \( (G, u) \) we say that \( \mathcal{B}' \) refines \( \mathcal{B} \) if \( \mathcal{B} \subseteq \text{sgr} \mathcal{B}' \).

From the foregoing proposition we immediately obtain.

**Proposition 5.2.** Let \( \mathcal{B}' = \{b'_1, \ldots, b'_{n'}\} \) and \( \mathcal{B} = \{b_1, \ldots, b_n\} \) be bases of a unital \( \ell \)-group \( (G, u) \) such that \( \mathcal{B}' \) refines \( \mathcal{B} \). We then have:

1. For each \( i = 1, \ldots, n \), the element \( b_i \) is expressible as a linear combination \( b_i = m_{1i} b'_1 + \cdots + m_{ni} b'_{n'} \), for uniquely determined integers \( m_{ki} \geq 0 \), \( k = 1, \ldots, n' \).
2. The \( n' \times n \) integer matrix \( M_{BB'} \), whose entries are the \( m_{ki} \), has rank equal to \( n \).
(3) The inclusion map $G_B \to G_{B'}$ induces the unital positive 1-1 homomorphism
\[ \phi_{BB'} : (y_1, \ldots, y_n) \in \mathbb{Z}^n \mapsto (z_1, \ldots, z_n') = M_{BB'} (y_1, \ldots, y_n) \in \mathbb{Z}^{n'} \]
of $(\mathbb{Z}_B, m_B)$ into $(\mathbb{Z}_{B'}, m_{B'})$, and we have a commutative diagram
\[
\begin{array}{ccc}
G_B & \xrightarrow{\text{inclusion}} & G_{B'} \\
\downarrow{\psi_B} & \nearrow{\phi_{BB'}} & \downarrow{\psi_{B'}} \\
(\mathbb{Z}_B, m_B) & \xrightarrow{} & (\mathbb{Z}_{B'}, m_{B'})
\end{array}
\] (27)

**Theorem 5.3.** Suppose the unital $\ell$-group $(G, u)$ has a basis. We then have:

1. Any two basic groups $G_B, G_F$ of $(G, u)$ are jointly embeddable (by unit preserving, order preserving inclusions) into some basic group $G_{B'}$ of $(G, u)$.
2. We then have a direct system $\{\phi_{BB'} : (\mathbb{Z}_B, m_B) \to (\mathbb{Z}_{B'}, m_{B'})\}$ of unital simplicial groups and unital positive 1-1 homomorphisms in $P$, indexed by all pairs $B, B'$ of bases of $(G, u)$ such that $B \subseteq \text{sgr} B'$.
3. Further, $\lim \{\phi_{BB'} : (\mathbb{Z}_B, m_B) \to (\mathbb{Z}_{B'}, m_{B'})\} \cong (G, u)_{\dim}$. \[ \text{Proof.} \]

(1) By Theorem 3.1, $(G, u)$ is finitely presented, and for some $n = 1, 2, \ldots$ we have
\[ (G, u) \cong M_n / j, \quad \text{for some principal ideal } j \text{ of } M_n. \] (28)
Suppose $j$ is generated by $f \in M_n$. Recalling the notation $ZF$ for the zero set of $f$, a variant of [10, 5.2] shows that $M_n / j \cong M_n / Zf$. A fortiori, $(G, u)$ is archimedean. From the abstract De Concini-Procesi lemma [18, 5.4] it follows that $B$ and $F$ have a joint refinement $B'$. Direct inspection of the proof therein, shows that $B'$ is obtained from $B$ by finitely many applications of the following operation: replace a 2-cluster $\{b, c\}$ of a basis $A$, by the three elements $b \wedge c, b - (b \wedge c), c - (b \wedge c)$. The result is a basis $A'$ such that $A \subseteq \text{sgr} A'$. Thus $B \subseteq \text{sgr} B'$. The desired conclusion now follows from Proposition 5.2.

The proof of (2) now immediately follows from Proposition 5.2.

Concerning (3), in view of (27) it is sufficient to prove that $G = \bigcup \{\text{grp} B | B \text{ a basis of } (G, u)\}$ and that $G^+ = \bigcup \{\text{sgr } B | B \text{ a basis of } (G, u)\}$. Since $G = G^+ - G^+$, only the latter identity must be proved. In other words, we must prove:

For every $p \in G^+$, $(G, u)$ has a basis $B$ such that $p \in \text{sgr } B$. \[ \text{As remarked above, we have a unital } \ell\text{-isomorphism } \omega : (G, u) \cong M_n / Zf. \] By [18, 4.5], $\omega$ induces a 1-1 correspondence between bases of the archimedean unital $\ell$-group $(G, u)$ and Schauder bases $H_\Delta$ of $M_n / Zf$, where $\Delta$ ranges over unimodular triangulations of the rational polyhedron $Zf$. Trivially, $B \subseteq \text{sgr } B'$ iff $\omega(B) \subseteq \omega(B')$. Thus (29) boils down to proving that for every $0 \leq g \in M_n / Zf$ there is a unimodular triangulation $\Delta$ of $Zf$ such that $g \in \text{sgr } H_\Delta$. Let $B$ be a unimodular triangulation of $Zf$ such that $g$ is linear over every simplex of $\Delta$. The existence of $\Delta$ is ensured by [20, 1.2]. Since every linear piece of $g$ has integer coefficients, for each vertex $v$ of $\Delta$ we can write $g(v) = n_v / \text{den}(v)$ for some $0 \leq n_v \in \mathbb{Z}$. As in the final part of the proof of Theorem 3.1, let $h_v : \Delta \to \mathbb{R}$ denote the Schauder hat of $\Delta$ at $v$. Let the function $\overline{g} \in \text{sgr } H_\Delta \subseteq M_n / Zf$ be defined by
\[ \overline{g} = \sum \{n_v h_v | v \text{ a vertex of } \Delta\}. \]
Then $\overline{g}(v) = g(v)$ for each vertex $v$ of $\Delta$ and $\overline{g}$ is linear over each simplex of $\Delta$. It follows that $\overline{g} = g$, which completes the proof. □

References

[1] K. A. Baker, Free vector lattices, Canad. J. Math., 20:58–66, 1968.
[2] W. M. Bebeyon, Duality theorems for finitely generated vector lattices, Proc. London Math. Soc., (3) 31:114–128, 1975.
[3] W. M. Bebeyon, Applications of duality in the theory of finitely generated lattice-ordered abelian groups, Canad. J. Math., 29(2):243–254, 1977.
[4] A. Bigard, K. Keimel, S. Wolfenstein, Groupes et Anneaux Réticulés, Springer Lecture Notes in Mathematics, Vol. 608, 1977.
[5] A. Dvurečenskij, C. W. Holland, Top varieties of generalized MV-algebras and unital lattice-ordered groups, Comm. Algebra, 35: 3370–3390, 2007.
[6] E. G. Effros, D.E. Handelman, C.-L. Shen, Dimension groups and their affine representations, American J. Math., 102: 385–407, 1980.
[7] G. Ewald, Combinatorial Convexity and Algebraic Geometry, Springer-Verlag, New York, 1996.
[8] L. Fuchs, Partially ordered algebraic systems, Pergamon Press, Oxford, 1963.
[9] L. Fuchs, Riesz groups, Ann. Scuola Normale Superiore Pisa, (3) 19: 1-34, 1965.
[10] A.M.W. Glass, Partially ordered groups, World Scientific, Singapore, 1999.
[11] A.M.W. Glass, F. Point. Finitely presented abelian lattice-ordered groups, In: Algebraic and Proof-Theoretical Aspects of Non-classical Logics, Springer Lecture Notes in Artificial Intelligence, 4460, (2007), pp.160–193.
[12] K. R. Goodearl, Notes on Real and Complex $C^*$-Algebras, Shiva Math. Series, Vol. 5, Birkhäuser, Boston, 1982.
[13] K. R. Goodearl, Partially Ordered Abelian Groups with Interpolation, AMS Math. Surveys and Monographs, Vol. 20, 1986.
[14] K. R. Goodearl, F. Wehrung, Representations of distributive semilattices in ideal lattices of various algebraic structures, Algebra Universalis, 45: 71–102, 2001.
[15] P.A. Grillet, Directed colimits of free commutative semigroups, J. Pure and Applied Algebra, 9: 73–87, 1976.
[16] C. Manara, V. Marra, D. Mundici Lattice-ordered abelian groups and Schauder bases of unimodular fans, Trans. Amer. Math. Soc. 359(4): 1593-1604.
[17] V. Marra, Every abelian $\ell$-group is ultrahomomorphic, Journal of Algebra, 225:872-884, 2000.
[18] V. Marra, D. Mundici, The Lebesgue state of a unital abelian lattice-ordered group, Journal of Group Theory, 10 : 655-684, 2007.
[19] D. Mundici, Interpretation of AF $C^*$-algebras in Łukasiewicz sentential calculus, J. Functional Analysis, 65: 15–63, 1986.
[20] D. Mundici, Farey stellar subdivisions, ultrahomomorphic groups, and $K_0$ of AF $C^*$-algebras, Advances in Mathematics, 68:23–39, 1988.
[21] D. Mundici, The Haar theorem for lattice-ordered abelian groups with order unit, Discrete and continuous dynamical systems, 21: 537-549, 2008.
[22] T.Oda, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties, Springer-Verlag, New York, 1988.