ON SOME FRACTIONAL INTEGRO–DIFFERENTIAL INCLUSIONS
WITH NONLOCAL MULTI–POINT BOUNDARY CONDITIONS

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Abstract. Existence of solutions for two classes of fractional integro-differential inclusions with nonlocal multi-point boundary conditions is investigated in the case when the values of the set-valued map are not convex.

1. Introduction

In the last years one may see a strong development of the theory of differential equations of fractional order ([4, 9, 12, 13, 14] etc.) and of the theory of fractional differential inclusions (e.g., [15]). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena.

In some recent papers [1, 3] etc. the attention was focused on special classes of boundary value problems associated to fractional differential equations; namely, nonlocal multi-point boundary conditions. This is the explanation for the study in the present paper of some fractional integro-differential inclusions with nonlocal multi-point boundary conditions.

We consider first the problem

\[ D^q x(t) \in F(t,x(t),I^\gamma x(t)) \quad a.e. \quad ([1,T]), \]
\[ x(1) = 0, \quad D^r x(T) = \sum_{i=1}^n \lambda_i D^\gamma x(\mu_i), \]

where \( D^q \) is the Hadamard fractional derivative of order \( q, q \in (1,2], r \in (0,1), I^\gamma \) is the Hadamard integral of order \( \gamma, \gamma > 0, \mu_i \in (1,T), \lambda_i \in \mathbb{R}, i = 1,n, n \geq 2 \) and \( F : [1,T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \) is a set-valued map.

If \( F \) is single-valued and does not depend on the last variable, fractional differential inclusion (1.1) reduces to the fractional differential equation

\[ D^q x(t) = f(t,x(t)), \]

where \( f : [1,T] \times \mathbb{R} \rightarrow \mathbb{R}. \)

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Existence results for problem (1.3)-(1.2) are obtained in [3] and are based on a nonlinear alternative of Leray-Schauder type and some suitable theorems of fixed point theory.

Our goal is to extend the study in [3] to the more general problem (1.1)-(1.2) and to show that Filippov’s ideas ([10]) can be suitably adapted in order to obtain the existence of solutions for this problem. Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov’s theorem ([10]) consists in proving the existence of a solution starting from a given ”quasi” solution. At the same time, the result provides an estimate between the ”quasi” solution and the solution obtained.

Secondly, we obtain similar results for problem
\[ D^q_c x(t) \in F(t,x(t),V(x)(t)) \quad \text{a.e.} \quad ([0,T]) \]
where \( q \in (1,2], \, p \in (0,1), \, \delta, a, b, \alpha_i \in \mathbb{R}, \, \sigma, \xi_1, \xi_2, \beta_i \in (0,T), \, i = 1,m-2, \) \( D^q_c \) is the Caputo fractional derivative of order \( q \), \( F : [0,1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \) is a set-valued map, \( V : C([0,1],\mathbb{R}) \rightarrow C([0,1],\mathbb{R}) \) is a nonlinear Volterra operator \( V(x)(t) = \int_0^t k(t,s,x(s))ds \) with \( k(.,.,.) : [0,1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) a given function.

In the case when \( F \) does not depend on the last variable and is single-valued, fractional differential inclusion (1.4) reduces to the fractional differential equation
\[ D^q_c x(t) = f(t,x(t)), \]
where \( f : [0,T] \times \mathbb{R} \rightarrow \mathbb{R} \) is a given mapping.

In [1] fixed point techniques are employed to obtain the existence of solutions for problem (1.6)-(1.5).

We note that existence results of the type provided in the present paper exists in the literature ([6, 7, 8] etc.), but their exposure in the framework of problems (1.1)-(1.2) and (1.4)-(1.5) is new.

The novelty of the present paper concerns several aspects. On one hand, the study in [1, 3] is extended to the set-valued framework. This allows to deduce certain existence results concerning fractional differential equations in [1, 3] as consequences of more general results. On the other hand, we consider problems whose right-hand side contains an integral term and we implement to these integro-differential inclusions Filippov techniques. For such kind of problems the usual fixed point techniques (e.g., [15]) are difficult to be adapted.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our results.

2. Preliminaries

Let \((X,d)\) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets \( A, B \subset X \) is defined by
\[ d_H(A,B) = \max\{d^+(A,B),d^+(B,A)\}, \quad d^+(A,B) = \sup\{d(a,B); a \in A\}, \]
where $d(x, B) = \inf_{y \in B} d(x, y)$.

Let $I = [1, T]$, we denote by $C(I, \mathbb{R})$ the Banach space of all continuous functions from $I$ to $\mathbb{R}$ with the norm $||x(\cdot)||_C = \sup_{t \in I} |x(t)|$ and $L^1(I, \mathbb{R})$ is the Banach space of integrable functions $u(\cdot) : I \to \mathbb{R}$ endowed with the norm $||u(\cdot)||_1 = \int_I |u(t)| dt$.

The Hadamard fractional integral of order $q > 0$ of a Lebesgue integrable function $f : [1, \infty) \to \mathbb{R}$ is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_1^t \left( \ln \frac{t}{s} \right)^{q-1} \frac{f(s)}{s} ds,$$

provided the integral exists and $\Gamma$ is the (Euler’s) Gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt$.

The Hadamard fractional derivative of order $q > 0$ of a function $f : [1, \infty) \to \mathbb{R}$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left( \frac{d}{dt} \right)^n \left( \ln \frac{t}{s} \right)^{n-q-1} \frac{f(s)}{s} ds,$$

where $n = [q] + 1$, $[q]$ is the integer part of $q$.

Details and properties of Hadamard fractional derivative may be found in [11, 12].

The fractional integral of order $q > 0$ of a Lebesgue integrable function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I^q f(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

The Caputo fractional derivative of order $q > 0$ of a function $f : [0, \infty) \to \mathbb{R}$ is defined by

$$D^q_C f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{-q-n-1} f^{(n)}(s) ds,$$

where $n = [q] + 1$. It is assumed implicitly that $f$ is $n$ times differentiable whose $n$-th derivative is absolutely continuous.

The next technical result is proved in [3]. Set $\Lambda := (\ln T)^{q-r-1} - \sum_{i=1}^n \lambda_i (\ln \mu_i)^{q-r-1}$.

**Lemma 1.** ([3]) Assume that $\Lambda \neq 0$. For a given $f(\cdot) \in C(I, \mathbb{R})$, the unique solution $x(\cdot)$ of problem $D^q x(t) = f(t)$ a.e. $([1, T])$ with boundary conditions (1.2) is given by

$$x(t) = \frac{(\ln t)^{q-1}}{\Gamma(q) \Lambda} \left( \sum_{i=1}^n \lambda_i \int_1^t \left( \ln \frac{\mu_i}{s} \right)^{q-r-1} \frac{h(s)}{s} ds + \int_1^T \left( \ln \frac{T}{s} \right)^{q-r-1} \frac{h(s)}{s} ds \right)$$

$$+ \frac{1}{\Gamma(q)} \int_1^t \left( \ln \frac{t}{s} \right)^{q-1} \frac{h(s)}{s} ds.$$  \hspace{1cm} (2.1)
REMARK 1. If we denote

\[ G_1(t, s) = \frac{(\ln t)^{q-1}}{\Gamma(q)\Lambda} \left( \sum_{i=1}^{n} \lambda_i (\ln \frac{\mu_i}{s})^{q-r-1} \frac{1}{s} \chi_{[1, \mu_i]}(s) - (\ln \frac{T}{s})^{q-r-1} \frac{1}{s} \right) + \frac{1}{\Gamma(q)} (\ln t)^{q-1} \chi_{[1, t]}(s), \]

where \( \chi_{S}(\cdot) \) is the characteristic function of the set \( S \), then the solution \( x(\cdot) \) in Lemma 1 may be written as \( x(t) = \int_{1}^{T} G_1(t, s) f(s) ds \).

Using the fact that, for fixed \( t \), the function \( g(s) = (\ln \frac{s}{T})^{\alpha} \frac{1}{s} \) with \( \alpha > 0 \) is decreasing we deduce that, if \( q - r - 1 > 0 \), for any \( t, s \in I \),

\[ |G_1(t, s)| \leq \frac{(\ln T)^{q-1}}{\Gamma(q)\Lambda} \left( \sum_{i=1}^{n} |\lambda_i| (\ln \mu_i)^{q-r-1} + (\ln T)^{q-r-1} \right) + \frac{1}{\Gamma(q)} (\ln T)^{q-1} =: M_1. \]

DEFINITION 1. A function \( x(\cdot) \in C(I, \mathbb{R}) \) with its Hadamard derivative of order \( q \) existing on \([1, T]\) is called a solution of problem (1.1)-(1.2) if there exists a function \( f(\cdot) \in L^1(I, \mathbb{R}) \) that satisfies

\[ f(t) \in F(t, x(t), I^\gamma x(t)) \quad a.e. \quad (I) \]

and \( x(\cdot) \) is given by (2.1).

Next \( I = [0, T] \). The proof of the following lemma may be found in [1]. Define

\[ A := (1 - \delta)\left( \frac{a_1^{1-p} + b_2^{1-p}}{\Gamma(2 - p)} - \sum_{i=1}^{m-2} \alpha_i \xi_i - \delta \sigma \sum_{i=1}^{m-2} \alpha_i. \right) \]

LEMMA 2. ([1]) Assume that \( A \neq 0 \). For a given \( f(\cdot) \in C(I, \mathbb{R}) \), the unique solution \( x(\cdot) \) of problem \( D_0^\gamma x(t) = f(t) \quad a.e. \quad ([0, T]) \) with boundary conditions (1.5) is given by

\[
x(t) = \int_{0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds + \frac{\delta}{1 - \delta} \int_{0}^{\sigma} \frac{(\sigma-s)^{q-1}}{\Gamma(q)} f(s) ds + \frac{\delta \sigma}{A(1 - \delta)} + \frac{t}{A} \right) [(1 - \delta) \
\cdot \left( \sum_{i=1}^{m-2} \alpha_i \int_{0}^{\beta_i} \frac{(\beta_i - s)^{q-1}}{\Gamma(q)} f(s) ds \right. - a \int_{0}^{\xi_1} \frac{(\xi_1 - s)^{q-p-1}}{\Gamma(q-p)} f(s) ds \
- b \int_{0}^{\xi_2} \frac{(\xi_2 - s)^{q-p-1}}{\Gamma(q-p)} f(s) ds + \left. \sum_{i=1}^{m-2} \alpha_i \int_{0}^{\sigma} \frac{(\sigma-s)^{q-1}}{\Gamma(q)} f(s) ds \right.]
\]

\[(2.2)\]
If we denote
\[ G_2(t,s) = \frac{(t-s)^{q-1}}{\Gamma(q)} \chi_{[0,i]}(s) + \frac{\delta}{1-\delta} \frac{(\sigma-s)^{q-1}}{\Gamma(q)} \chi_{[0,\sigma]}(s) + \left[ \frac{\delta \sigma}{A(1-\delta)} + \frac{t}{A} \right] (1-\delta) \]
\[ \cdot \left( \sum_{i=1}^{m-2} \alpha_i \frac{(\beta_i-s)^{q-1}}{\Gamma(q)} \chi_{[0,\beta_i]}(s) - a \frac{(\xi_1-s)^{q-p-1}}{\Gamma(q-p)} \chi_{[0,\xi_1]}(s) \right) 
\[ - b \frac{(\xi_2-s)^{q-p-1}}{\Gamma(q-p)} \chi_{[0,\xi_2]}(s)) + \delta \sum_{i=1}^{m-2} \alpha_i \frac{(\sigma-s)^{q-1}}{\Gamma(q)} \chi_{[0,\sigma]}(s), \]
then solution \( x(\cdot) \) in Lemma 2 may be written as \( x(t) = \int_0^T G_2(t,s) f(s) ds \).
Moreover, if \( q - p - 1 > 0 \) for any \( t, s \in I \) we have
\[ |G_2(t,s)| \leq \frac{T^{q-1}}{\Gamma(q)} + \frac{\delta}{1-\delta} \frac{|T+T|^{q-1}}{\Gamma(q)} + \frac{\sigma^{q-1}}{A} (\sum_{i=1}^{m-2} |\alpha_i| \frac{\beta_i^{q-1}}{\Gamma(q)} + |a| \frac{\xi_1^{q-p-1}}{\Gamma(q-p)} + |b| \frac{\xi_2^{q-p-1}}{\Gamma(q-p)} + |\delta| \sum_{i=1}^{m-2} |\alpha_i| \frac{\sigma^{q-1}}{\Gamma(q)} ) =: M_2. \]

**Definition 2.** A function \( x(.) \in C(I, \mathbb{R}) \) with its Caputo derivative of order \( q \) existing on \([0,T]\) is called a solution of problem (1.4)-(1.5) if there exists a function \( f(.) \in L^1(I, \mathbb{R}) \) that satisfies
\[ f(t) \in F(t,x(t),V(x)(t)) \quad \text{a.e. } (I) \]
and \( x(.) \) is given by (2.2).

Finally, we recall a selection result ([2]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

**Lemma 3.** ([2]) Consider \( X \) a separable Banach space, \( B \) is the closed unit ball in \( X \), \( G : I \to \mathcal{P}(X) \) is a set-valued map with nonempty closed values and \( c : I \to X, r : I \to \mathbb{R}_+ \) are measurable functions. If
\[ G(t) \cap (c(t) + r(t)B) \neq \emptyset \quad \text{a.e.}(I), \]
then the set-valued map \( t \to G(t) \cap (c(t) + r(t)B) \) has a measurable selection.

### 3. The main results

In order to prove our results we need the following hypotheses.

**Hypothesis H1.** i) \( F(.,.,.) : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) has nonempty closed values and is \( \mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}) \) measurable.
ii) There exists \( l(.) \in L^1(I, (0,\infty)) \) such that, for almost all \( t \in I, F(t,.,.) \) is \( l(t) \)-Lipschitz in the sense that
\[ d_H(F(t,x_1,y_1), F(t,x_2,y_2)) \leq l(t)(|x_1 - x_2| + |y_1 - y_2|) \forall x_1, x_2, y_1, y_2 \in \mathbb{R}. \]
We use next the following notations

\[
L(t) := l(t)(1 + \frac{1}{\Gamma(\gamma)} \int_1^t \left( \frac{\ln t}{s} \right)^{\gamma-1} \frac{1}{s} ds) = l(t)(1 + \frac{(\ln t)^{\gamma}}{\Gamma(\gamma+1)}),
\]

(3.1)

\[
L_0 = \int_1^T L(t) dt.
\]

(3.2)

**Theorem 1.** Assume that Hypothesis H1 is satisfied, \( q - r - 1 > 0, \Lambda \neq 0 \) and \( M_1 L_0 < 1 \). Consider \( y(.) \in C(I, \mathbb{R}) \) with its Hadamard derivative of order \( q \) existing on \([1, T]\) such that \( y(1) = 0, D^r y(T) = \sum_{i=1}^n \lambda_i D^r y(\mu_i) \) and there exists \( p(.) \in L^1(I, \mathbb{R}) \) verifying \( d(D^q y(t), F(t, y(t), I^q y(t))) \leq p(t) \text{ a.e. } (I) \).

Then there exists \( x(.) \) a solution of problem (1.1)-(1.2) satisfying for all \( t \in I \)

\[
|x(t) - y(t)| \leq \frac{M_1}{1 - M_1 L_0} \int_1^T p(t) dt.
\]

(3.3)

**Proof.** The multifunction \( t \rightarrow F(t, y(t), I^q y(t)) \) has closed values, is measurable and from hypothesis of theorem one has

\[
F(t, y(t), I^q y(t)) \cap \{D^q y(t) + p(t)[-1, 1]\} \neq \emptyset \text{ a.e. } (I).
\]

We apply Lemma 3 to find a measurable function \( f_1(t) \in F(t, y(t), I^q y(t)) \text{ a.e. } (I) \) such that

\[
|f_1(t) - D^q y(t)| \leq p(t) \text{ a.e. } (I)
\]

(3.4)

Define \( x_1(t) = \int_1^T G(t, s) f_1(s) ds \) and one has \( |x_1(t) - y(t)| \leq M_1 \int_1^T p(t) dt \).

We point out that it is enough to construct the sequences \( x_n(.) \in C(I, \mathbb{R}) \), \( f_n(.) \in L^1(I, \mathbb{R}) \), \( n \geq 1 \), with the following properties

\[
x_n(t) = \int_1^T G(t, s) f_n(s) ds, \quad t \in I, \tag{3.5}
\]

\[
f_n(t) \in F(t, x_{n-1}(t), I^q x_{n-1}(t)) \text{ a.e. } (I), \tag{3.6}
\]

\[
|f_{n+1}(t) - f_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \frac{1}{\Gamma(\gamma)} \int_1^t \left( \frac{\ln t}{s} \right)^{\gamma-1} \frac{1}{s} |x_n(s) - x_{n-1}(s)| ds)
\]

(3.7)

for almost all \( t \in I \).

Assume that this construction is done; then from (3.4)-(3.7) we have for almost all \( t \in I \)

\[
|x_{n+1}(t) - x_n(t)| \leq M_1 (M_1 L_0)^n \int_1^T p(t) dt \quad \forall n \in \mathbb{N}.
\]
Indeed, assume that the last inequality is true for $n - 1$ and we prove it for $n$. One has

$$|x_{n+1}(t) - x_n(t)| \leq \int_1^T |G_1(t, t_1)| |f_{n+1}(t_1) - f_n(t_1)| dt_1$$

$$\leq M_1 \int_1^T l(t_1)(|x_n(t_1) - x_{n-1}(t_1)| + \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left( \frac{t_1}{s} \right)^{\gamma - 1} \frac{1}{s} |x_n(s) - x_{n-1}(s)| ds)dt_1$$

$$\leq M_1 \int_1^T l(t_1)(1 + \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left( \frac{t_1}{s} \right)^{\gamma - 1} \frac{1}{s} ds)dt_1 M_1^n L_0^{n-1} \int_1^T p(t) dt$$

$$= M_1 (M_1 L_0)^n \int_1^T p(t) dt.$$

Thus, $\{x_n(\cdot)\}$ is Cauchy in the Banach space $C(I, \mathbb{R})$, therefore, converging uniformly to some $x(\cdot) \in C(I, \mathbb{R})$. Hence, by (3.7), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbb{R}$. Denote $f(\cdot)$ the pointwise limit of $f_n(\cdot)$.

At the same time, one has

$$|x_n(t) - y(t)| \leq |x_1(t) - y(t)| + \sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)|$$

$$\leq M_1 \int_1^T p(t) dt + \sum_{i=1}^{n-1} (M_1 \int_1^T p(t) dt)(M_1 L_0)^i = \frac{M_1 \int_1^T p(t) dt}{1 - M_1 L_0}. \quad (3.8)$$

Moreover, from (3.4), (3.7) and (3.8) we obtain for almost all $t \in I$

$$|f_n(t) - D^q y(t)| \leq \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - D^q y(t)| \leq L(t) \frac{M_1 \int_1^T p(t) dt}{1 - M_1 L_0} + p(t).$$

In particular, the sequence $f_n(\cdot)$ is integrably bounded and thus $f(\cdot) \in L^1(I, \mathbb{R})$.

From Lebesgue’s dominated convergence theorem and passing the limit in (3.5), (3.6) we obtain that $x(\cdot)$ is a solution of (1.1). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(\cdot)$.

In order to finish the proof it remains to realize the construction of the sequences $x_n(\cdot), f_n(\cdot)$ with the properties in (3.5)-(3.7). This will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(\cdot) \in C(I, \mathbb{R})$ and $f_n(\cdot) \in L^1(I, \mathbb{R}), n = 1, 2, \ldots N$ satisfying (3.5), (3.7) for $n = 1, 2, \ldots N$ and (3.6) for $n = 1, 2, \ldots N - 1$. The set-valued map $t \rightarrow F(t, x_N(t), I^\gamma x_N(t))$ is measurable; as well as the map $t \rightarrow L(t)(|x_N(t) - x_{N-1}(t)| + \frac{1}{\Gamma(\gamma)} \int_1^t \left( \frac{t}{s} \right)^{\gamma - 1} \frac{1}{s} |x_N(s) - x_{N-1}(s)| ds)$ is measurable. Since $F(t, \cdot, \cdot)$ is Lipschitz we have that for almost all $t \in I$

$$F(t, x_N(t), I^\gamma x_N(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \frac{1}{\Gamma(\gamma)} \int_1^t \left( \frac{t}{s} \right)^{\gamma - 1} \frac{1}{s} |x_N(s) - x_{N-1}(s)| ds)[-1, 1]\} \neq \emptyset.$$
Lemma 3 allows to find a measurable selection \( f_{N+1}(.) \) of \( F(.,x_N(.),I'x_N(.)) \) such that for almost all \( t \in I \)

\[
|f_{N+1}(t) - f_N(t)| \leq L(t)(|x_N(t) - x_{N-1}(t)| + \frac{1}{\Gamma(\gamma)} \int_1^t \left( \frac{t}{s} \right)^{\gamma-1} \frac{1}{s} |x_N(s) - x_{N-1}(s)| ds).
\]

We define \( x_{N+1}(.) \) as in (3.5) with \( n = N + 1 \). Thus \( f_{N+1}(.) \) satisfies (3.6) and (3.7) and the proof is complete. \( \square \)

The assumptions in Theorem 1 are satisfied, in particular, for \( y(.) = 0 \) and therefore with \( p(.) = l(.) \). We obtain the following consequence of Theorem 1.

**COROLLARY 1.** Assume that Hypothesis H1 is satisfied, \( d(0,F(t,0,0) \leq L(t) \text{ a.e. (I)} \), \( q - r - 1 > 0 \), \( \Lambda \neq 0 \) and \( M_1L_0 < 1 \). Then there exists \( x(.) \) a solution of problem (1.1)-(1.2) satisfying for all \( t \in I \)

\[
|x(t)| \leq \frac{M_1}{1 - M_1L_0} \int_1^T l(t) dt.
\]

If \( F \) does not depend on the last variable, Hypothesis H1 becomes

**HYPOTHESIS H2.** i) \( F(.,.) : I \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \) has nonempty closed values and is \( \mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R}) \) measurable.

ii) There exists \( l(.) \in L^1(I,(0,\infty)) \) such that, for almost all \( t \in I \), \( F(.,.) \) is \( L(t) \)-Lipschitz in the sense that

\[
d_H(F(t,x_1),F(t,x_2)) \leq l(t)|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}.
\]

Denote \( M_0 = \int_1^T l(t) dt \) and consider the fractional differential inclusion

\[
D^q x(t) \in F(t,x(t)) \quad \text{a.e. ([1, T])}, \quad (3.9)
\]

**COROLLARY 2.** Assume that Hypothesis H2 is satisfied, \( d(0,F(t,0,0) \leq L(t) \text{ a.e. (I)} \), \( q - r - 1 > 0 \), \( \Lambda \neq 0 \) and \( M_1M_0 < 1 \). Then there exists \( x(.) \) a solution of problem (3.9)-(1.2) satisfying for all \( t \in I \)

\[
|x(t)| \leq \frac{M_1M_0}{1 - M_1M_0}.
\]

**REMARK 2.** If in (3.9) \( F \) is single-valued, then a similar result to the one in Corollary 2 may be found in [3]; namely, Theorem 3.3.

We are concern next with problem (1.4)-(1.5). In what follows \( I = [0,T] \) and we make the following notations

\[
N(t) := l(t)(1 + \int_0^t l(u) du), \quad t \in I, \quad N_0 = \int_0^T N(t) dt.
\]
**THEOREM 2.** Assume that Hypothesis H1 is satisfied, \( q - p - 1 > 0 \), \( A \neq 0 \) and \( M_2N_0 < 1 \). Consider \( y(.) \in C(I, \mathbb{R}) \) with its Caputo derivative of order \( q \) existing on \([0,T]\) such that \( y(0) = \delta y(\sigma) \), \( aD_q^p y(\xi_1) + bD_q^p y(\xi_2) = \sum_{i=1}^{m-2} \alpha_i q(y(\beta_i)) \) and there exists \( q(.) \in L^1(I, \mathbb{R}_+) \) verifying \( d(D_q^p y(t), F(t,y(t), V(y)(t))) \leq q(t) \) a.e. (I).

Then there exists \( x(.) \) a solution of problem (1.4)-(1.5) satisfying for all \( t \in I \)

\[
|x(t) - y(t)| \leq \frac{M_2}{1 - M_2N_0} \int_0^T q(t) dt.
\]

**Proof.** The proof is similar to the proof of Theorem 1. \( \square \)

If in Theorem 2, \( y(.) = 0 \) and \( q(.) = l(.) \) we get the following consequence of Theorem 2.

**COROLLARY 3.** Assume that Hypothesis H1 is satisfied, \( d(0,F(t,0,0) \leq L(t) \) a.e. (I), \( q - p - 1 > 0 \), \( A \neq 0 \) and \( M_2N_0 < 1 \). Then there exists \( x(.) \) a solution of problem (1.4)-(1.5) satisfying for all \( t \in I \)

\[
|x(t)| \leq \frac{M_2}{1 - M_2N_0} \int_0^T l(t) dt.
\]

Next \( F \) does not depend on the last variable. Set \( K_0 = \int_0^T l(t) dt \) and consider the fractional differential inclusion

\[
D_q^p x(t) \in F(t,x(t)) \quad a.e. \; ([0,T]),
\]

(3.10)

**COROLLARY 4.** Assume that Hypothesis H2 is satisfied, \( d(0,F(t,0) \leq L(t) \) a.e. (I), \( q - p - 1 > 0 \), \( A \neq 0 \) and \( M_2K_0 < 1 \). Then there exists \( x(.) \) a solution of problem (3.10)-(1.2) satisfying for all \( t \in I \)

\[
|x(t)| \leq \frac{M_2K_0}{1 - M_2K_0}.
\]

**REMARK 3.** If in (3.10), \( F \) is single-valued, then a similar result to the one in Corollary 4 is Theorem 1 in [1].

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