Q-balls: some analytical results

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Abstract. Motivated by the renewed interest in the role of Q-balls in cosmological evolution, we present a discussion of the main properties of Q-balls, including some new results.

1 Introduction

The existence of the Baryon asymmetry in nature is still an unsolved puzzle. As there is a general agreement that it can’t be explained within the framework of the Standard Model (SM), it has become a common practice to investigate supersymmetric extensions of the SM. However, in addition to electroweak baryogenesis, these models allow for other possible mechanisms for generating the B-asymmetry. One of these is the Affleck-Dine (AD) mechanism \cite{AD1,AD2}, where the baryon number surplus is produced in the late stages of inflation, being stored in a condensate of supersymmetric scalars (e.g. a mixture of squarks). But, as Kusenko and Shaposhnikov pointed out \cite{KS}, this homogeneous condensate is not always the best way of packing the baryon number: there are non-homogeneous states, lumps of baryonic matter, with the same B-number but less energy, which are known as Q-balls (or B-balls). The condensate is therefore expected to collapse, producing a large number of Q-balls. The B-number, stored in the Q-balls in this way, survives the sphaleron processes if they decay at temperatures below the weak scale. As shown in \cite{AD3}, in the context of models with gravity-mediated supersymmetry breaking this happens only for Q-balls with $B \gtrsim 10^{16-17}$, as it is actually realized in such models. Since their decay is due to baryon and neutralino evaporation from the surface this could explain the similarity between the present baryon density and the dark matter density.

Our intention here is not, however, to investigate the role of Q-balls in baryogenesis, but to discuss their general properties. As was shown by Coleman, the existence of Q-balls is a general feature of theories with scalars carrying a conserved U(1)-charge \cite{AD1,AD2} (e.g. B-number). They can be regarded as bound states of the scalar particles and appear as stable classical solutions of the equations of motion, i.e. they are non-topological solitons which arise due to the non-linear self-interactions of the scalar field.

The present paper (see also ref.\cite{AD3}) is a study of the main general properties of Q-balls. We will start in sec.2 by explaining why one should search for solutions of the eq. of motion of the Q-ball type (i.e. $\phi = \sigma(r)e^{i\omega t}$), and when they exist. In the case that such solutions exist, the next rational move is to investigate their stability. This is what we do in sec.3. We prove a statement which relates the classical stability of a Q-ball with the dependence of its charge on the internal frequency $\omega$. In sec.4 we discuss the limits of small and high internal frequency using the thin-wall and the thick-wall approximation, respectively. These results are then applied to the case of a model with broken symmetry. Finally we regard in sec.5 the issue of the meaning of Q-balls in the quantum theory. We close with a recapitulation and discussion of our results.

2 Non-topological solitons in scalar theories

To make our investigation as simple as possible we will neglect the interactions of the scalar sector with the rest of the world (the other sectors of the theory), and investigate U(1)-invariant scalar theories, defined in a general way by the lagrangian density

$$\mathcal{L} = |\partial_{\mu} \phi|^2 - \bar{U}(|\phi|).$$

(1)

In this case, classical dynamics is described by the equation of motion

$$\partial^2_t \phi - \nabla^2 \phi + \bar{U}'(|\phi|) \frac{\phi}{2|\phi|} = 0,$$

(2)

for which the charge
with \( \sigma \) of Q-balls, of the form that carry the U(1) charge, and have masses \( m^2 = \frac{1}{2}U''(0) \). The purpose of this section is to show that for certain potentials there are solutions of the eq. of motion, called Q-balls, of the form

\[
\phi(x, t) = \frac{\sigma(r)}{\sqrt{2}} e^{i\omega t},
\]

with \( \sigma(r) \) being a decreasing function of \( r = |x| \geq 0 \) (a ball). The point of this ansatz is, that it has the simplest form, which allows the Q-ball to carry finite energy's variation at fixed charge in the form \( \omega^2 \sigma \), that carry the U(1) charge, and have masses \( m^2 = \frac{1}{2}U''(0) \).

Since we want to find minima of the energy in sectors of fixed charge, it will be useful to rewrite the energy functional as follows

\[
E_Q[\sigma] = \frac{1}{2} I[\sigma] + \int d^3x \left[ \frac{1}{2} (\nabla \sigma)^2 + U(\sigma) \right],
\]

with

\[
I[\sigma] = \int d^3x \sigma^2.
\]

With \( E_Q \) recast in this form, we can investigate the stability of any Q-ball solution with respect to perturbations inside the subspace of fixed charge and configurations of the form \( \phi(x, t) = \sigma(x) e^{i\omega t}/\sqrt{2} \). The handling of general variations around the Q-ball solution will be postponed, because, as we will see in section 3, the only relevant configurations with regard to the stability of the Q-balls are those with the above form.

By Taylor expanding \( I^{-1}[\sigma + \delta \sigma] \), we can write the energy's variation at fixed charge in the form

\[
\Delta E_Q = \int d^3x \delta \sigma (-\nabla^2 \sigma + U' - \omega^2 \sigma) + \int d^3x \frac{1}{2} \delta \sigma (-\nabla^2 + U'' - \omega^2) \delta \sigma + 2 \omega^2 \left( \int d^3 x \sigma \delta \sigma \right)^2 + O(\delta \sigma^3).
\]

Note that we traded \( Q \) for \( \omega \) using \( Q = \omega I \) only after performing the variation. It is helpful to define the following functionals,

\[
U_\omega(\sigma) = U(\sigma) - \frac{1}{2} \omega^2 \sigma^2,
\]

\[
S_\omega[\sigma] = \int d^3x \left( \frac{1}{2} (\nabla \sigma)^2 + U_\omega \right),
\]

to recast the energy’s variation in a simpler way:

\[
\Delta E_Q = \Delta S_\omega + 2 \omega^2 \left( \int d^3 x \sigma \delta \sigma \right)^2 + O(\delta \sigma^3),
\]

where \( \Delta S_\omega \) is a variation made while keeping \( \omega \) fixed. We are now in position to make some remarks:

(i) Extrema of the functional \( S_\omega \) at fixed \( \omega \) are extrema of the energy \( E_Q \) at fixed charge \( Q \), as we can see from eqs.(8) and (13). They satisfy automatically the equation of motion eq.(5). Now, if \( \{ \sigma_\omega(r) \} \) is a set of solutions of this equation, parameterized by \( \omega \), \( E_Q(\sigma_\omega) \) can be seen as a function of only \( Q \) and \( S(\omega) \equiv S_\omega[\sigma_\omega] \) as a function of \( \omega \). These two functions are related through a Legendre transformation. To see this note that from eq.(5) it follows that

\[
E(\omega) = S(\omega) + \omega Q,
\]

which defines a Legendre transformation because

\[
\frac{dS(\omega)}{d\omega} = \int d^3r \frac{dS_\omega}{d\sigma} \frac{d\sigma}{d\omega} + \frac{dS(\omega)}{d\omega} = 0 - \int d^3r \sigma^2 \frac{d}{d\omega} \left( \frac{1}{2} \omega^2 \right) = -Q(\omega),
\]

where \( Q(\omega) \equiv Q[\sigma_\omega] \). (These expressions will prove to be useful in sec.4.)

(ii) Fortunately the problem of finding the solutions of \( \delta S_\omega = 0 \) is a well investigated one. In fact, it is known that for values of \( \omega^2 \) within a certain range, the extremum of \( S_\omega \) with the smallest value of \( S_\omega \) is a decreasing function of \( r = |x| \), the so-called bounce, which satisfies the boundary conditions \( \frac{d \sigma}{d \phi}(0) = 0 \), \( \sigma(+\infty) = 0 \).

(iii) Minima of \( E_Q \) don’t need to be minima of \( S_\omega \) (see eq.(13)):

\[
\delta^2 E_Q \geq 0 \implies \delta^2 S_\omega \geq -2 \frac{\omega^2}{T} \left( \int d^3 x \sigma \delta \sigma \right)^2.
\]

This is an important feature, because it is a well known fact in the theory of bounces that these solutions are not minima of \( S_\omega \), but rather saddle points. They have one mode \( \delta \sigma_{-1} \) with \( \frac{d^2 S_\omega}{d \sigma_{-1}^2} < 0 \). These configurations describe quantum tunneling in a real scalar field theory in (2+1)-dimensions with potential \( U_\omega(\sigma) \).
We want now to find the range of $\omega^2$, for which there are configurations of the form discussed in (ii) which satisfy eq. (6). For radial-dependent solutions this equation turns to be

$$\frac{d^2\sigma}{dr^2} = -\frac{2}{r} \frac{d\sigma}{dr} + U'_\omega(\sigma). \quad (17)$$

It is easy to recognize this as the equation of motion for a particle of unit mass, with position $\sigma$ and time $r$, under the action of the potential $-U_\omega(\sigma)$ and of a viscous term proportional to the velocity and the inverse of the time. In this mechanical analogon, the boundary conditions are that the particle starts at rest somewhere at the positive axis, and moves towards the origin, which is attained at infinite time. This is only possible if

1) $\omega^2 \leq m^2$: When the particle approaches the origin the potential behaves like

$$-U_\omega(\sigma) \simeq -U_\omega(0) - \frac{1}{2} \frac{d^2U_\omega(0)}{d\sigma^2}\sigma^2 = -\frac{1}{2}(m^2 - \omega^2)\sigma^2. \quad (18)$$

If $\omega^2 > m^2$, the particle will ultimately start oscillating around the origin as

$$\sigma(r) \sim \sin\left(\sqrt{\omega^2 - m^2}r\right), \quad (19)$$

causing $E_Q[\sigma]$ to be infinite.

2) $\omega^2 > \omega_0^2$, where $\omega_0^2$ is defined as

$$\omega_0^2 \equiv \min\left(\frac{2U(\sigma)}{\sigma^2}\right) = \frac{2U(\sigma_0)}{\sigma_0^2}. \quad (20)$$

If $\omega_0^2 < 0$ this is automatically satisfied. To understand this condition for $\omega_0^2 \geq 0$ note first that as the particle attains the origin its energy, $E_{\text{part}} = \frac{1}{2}(\frac{d\sigma}{dr})^2 - U_\omega(\sigma)$, is equal to zero. From the equation of motion we know that the particle’s energy is a decreasing function of the time $r$, in virtue of the action of the viscous force. The potential energy must therefore be positive at time $r = 0$, when the particle starts at rest with position $\sigma(0) \neq 0$. But if $\omega^2 < \omega_0^2$ the potential energy $-U_\omega$ is never positive. This is easy to see in fig. 1, but can also be shown mathematically: If for some $\sigma$ the potential is positive and $\omega^2 < \omega_0^2$ we would have

$$-U_\omega(\sigma) > 0 \Rightarrow U(\sigma) < \frac{1}{2}\omega^2\sigma^2$$

$$\Rightarrow \frac{2U(\sigma)}{\sigma^2} < \omega^2 < \min\left(\frac{2U(\sigma)}{\sigma^2}\right). \quad (21)$$

In this section we have seen that as long as $\omega_0^2 \neq m^2$, the theory contains classical solutions of the equations of motion, which are energy extrema in a sector of fixed charge. These solutions rotate in internal $U(1)$-space with frequency $\omega (\omega_0^2 < \omega^2 < m^2)$ and their modules $\sigma(r)$ have a bounce-like shape, i.e. they have a maximum at $r = 0$ and decrease monotonically towards zero at infinity.

### 3 Stability of Q-balls

In the following paragraphs we will argue that the stability of Q-ball solutions depends on the way the charge changes with the internal frequency $\omega$. If at least some deviations from the solution grow with time, we say that the solution is unstable. If any perturbation remains oscillating around the solution, this is a stable one. It’s clear that, due to charge conservation, minima of the energy for fixed charge are stable under small perturbations.

The question if a Q-ball solution is an absolute minimum of the energy at fixed charge was answered by Coleman. He proved that as long as

$$E_{Q\text{ball}} < mQ, \text{ and } \omega_0^2 > 0 \quad (22)$$

the Q-ball is the absolute minimum configuration with charge $Q$.

There is however another more useful theorem regarding the local stability of Q-balls, that also works for $\omega_0^2 \leq 0$, that is, in the case that the symmetry is broken. It states that

- if $\frac{dQ}{d\omega} < 0$, the Q-ball is a local minimum, and therefore stable;
- if $\frac{dQ}{d\omega} > 0$, the Q-ball is a saddle point of the energy, with one instability mode.

To prove this we must investigate general perturbations of the solutions around the form $\phi = \bar{\sigma}(r)e^{i\omega t}/\sqrt{2}$ which satisfy the eq. of motion (6). Since the general configurations $\sigma(x,t)e^{i\omega(x,t)/\sqrt{2}}$ depend on both space and time it seems difficult to fix the charge in an explicit way when performing variations of the energy. That is, we

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2 This theorem was stated for a similar kind of nontopological solitons in ref. [13].
would like to find an expression, like eq. (8), where the charge dependence is explicit, but which is valid for general configurations. We can solve this problem by introducing the following function,

\[ \omega(t) \equiv \frac{\int d^3x \sigma^2(x, t) \dot{\theta}(x, t)}{\int d^3x \sigma^2(x, t)} = \frac{Q[\sigma, \theta]}{T[\sigma]}, \tag{23} \]

and splitting \( \dot{\theta}(x, t) \) as

\[ \dot{\theta}(x, t) = \theta_0(t) + \theta(x, t), \tag{24} \]

where \( \theta_0(t) \) is an indefinite integral of \( \omega(t) \), which depends only on time [4]. With these definitions the energy functional reads:

\[
E[\sigma, \theta] = \int d^3x \left[ \frac{1}{2} (\dot{\theta})^2 + \frac{1}{2} \sigma^2(\dot{\theta})^2 + \frac{1}{2} \sigma^2 \nabla \theta^2 \right] + \frac{1}{2} \frac{Q^2}{T[\sigma]} + \int d^3x \left[ \frac{1}{2} (\nabla \sigma)^2 + U(\sigma) \right] \equiv K[\sigma, \theta] + E_Q[\sigma],
\]

where \( E_Q \) is the functional defined by eq. (8). For a configuration of the Q-ball type, \( \theta(x, t) = \omega t \) and \( \sigma(x, t) = r(t) \).

We have thus \( \theta(x, t) = \text{const.} \) and therefore it follows readily that \( K_{Qball} = 0 \). Since in general \( K[\sigma, \theta] \geq 0 \), Q-balls (or configurations of the Q-ball type) are minima of this functional. This means that the stability depends only on the behavior of the functional \( E_Q[\sigma] \) under variations \( \sigma(x, t) = r(t) + \delta \sigma(x, t) \). Note that for this variational problem only the spatial dependence of \( \sigma \) is relevant, the time \( t \) playing the role of a fixed parameter. This proves the assertion of Section 2 that the stability of Q-balls depends only on the energy in the subspace of configurations of the type \( \sigma(x) e^{i \omega t} \), for which the energy reduces to \( E_Q[\sigma] \).

The remaining task is, therefore, to prove that the above theorem is true in this restricted subspace. To do this we will first show that in addition to the 3 translational zero-modes \( \delta \sigma \) of \( E_Q[\bar{\sigma}] \) there is only one other zero-mode if and only if \( \frac{\omega}{Q} \frac{dQ}{d\omega} = 0 \): From eq. (11) we see that any null mode \( \psi \) must satisfy

\[
\mathcal{H}\psi \equiv \frac{1}{2} \psi + \frac{2\omega^2}{T} \bar{\sigma} \int d^3x \bar{\sigma} \psi = 0, \tag{26} \]

where \( h \equiv -\nabla^2 + U''(\bar{\sigma}) \). With \( A \equiv -\frac{\omega^2}{2} \int d^3x \bar{\sigma} \psi \) we get

\[
h \psi = A \bar{\sigma}. \tag{27} \]

Differentiating the equation of motion with respect to \( \omega \) one sees that the last equation is satisfied by \( \psi = \frac{d\bar{\sigma}}{d\omega} \) with \( A = 2\omega \). For any other solution \( \psi \) we have thus

\[
h \left( \psi - \frac{A}{2\omega} \frac{d\bar{\sigma}}{d\omega} \right) = 0, \tag{28} \]

and since the \( \delta \bar{\sigma} \) are the only functions for which \( hf = 0 \) we see that \( \psi \) is always a linear combination of the \( \delta \bar{\sigma} \) and of \( \frac{\omega}{Q} \frac{dQ}{d\omega} \). Now, if \( \frac{dQ}{d\omega} \) is a solution of eq. (23) we get

\[
1 + \frac{2\omega}{T} \int d^3x \bar{\sigma} \frac{d\bar{\sigma}}{d\omega} = 0, \tag{29} \]

and therefore

\[
\frac{\omega}{Q} \frac{dQ}{d\omega} = \frac{\omega}{Q} \left[ \frac{Q}{\omega} + 2\omega \int d^3x \bar{\sigma} \frac{d\bar{\sigma}}{d\omega} \right] = 0, \tag{30} \]

as we intended to prove.

We will now show that the appearance of the extra null mode when \( \frac{dQ}{d\omega} = 0 \) really signals the change of sign of an eigenvalue of \( \mathcal{H} \) and with it the transition between stability and instability. The first step is to prove that since \( h \) has only one negative eigenvalue \( \mathcal{H} \) can have at most one negative eigenvalue: If \( \phi_i \) is a negative mode of \( \mathcal{H} \) it must satisfy

\[
\int d^3x \phi_i \delta \sigma_{-1} \neq 0, \tag{31} \]

where \( \delta \sigma_{-1} \) is the negative mode of \( h \), or else \( \int \phi_i \mathcal{H} \phi_i = \lambda_i \) would be positive. If there are more than one negative mode of \( \mathcal{H} \) we can build a suitable linear combination \( \Phi = \sum a_i \phi_i \) for which \( \int d^3x \Phi \delta \sigma_{-1} = 0 \). But this would mean that

\[
0 > \sum a_i^2 \lambda_i = \int d^3x \Phi \mathcal{H} \Phi = \frac{1}{2} \int d^3x \Phi \mathcal{H} \Phi + \frac{2\omega^2}{T} \left( \int d^3x \bar{\sigma} \Phi \right)^2 > 0, \tag{32} \]

what is impossible. There can be therefore at most one negative eigenvalue.

It remains to show what \( \lambda(\omega) \) is the extra eigenvalue of \( \mathcal{H} \) which is zero when \( \frac{dQ}{d\omega} = 0 \), we have \( \lambda(\omega) > 0 \) when \( \frac{dQ}{d\omega} < 0 \) and vice-versa. If \( \psi(\omega) \) is the eigenvector corresponding to \( \lambda(\omega) \), and \( \bar{\omega} \) is defined by \( \lambda(\bar{\omega}) = 0 \), we have (up to a multiplicative constant) \( \psi(\omega) = \frac{d\bar{\sigma}}{d\omega} \). Differentiating \( \mathcal{H} \psi = \lambda \psi \) at \( \omega = \bar{\omega} \) we get thus

\[
\frac{d\mathcal{H}}{d\omega} \frac{d\bar{\sigma}}{d\omega} + \mathcal{H}(\bar{\omega}) \frac{d\psi}{d\omega} = d\lambda \frac{d\bar{\sigma}}{d\omega} \frac{d\psi}{d\omega}. \tag{33} \]

We multiply this equation with \( \frac{d\bar{\sigma}}{d\omega} \) and integrate to obtain

\[
\int d^3x \frac{d\bar{\sigma}}{d\omega} \frac{d\mathcal{H}}{d\omega} \frac{d\bar{\sigma}}{d\omega} = d\lambda \int d^3x \left( \frac{d\bar{\sigma}}{d\omega} \frac{d\psi}{d\omega} \right)^2. \tag{34} \]

However we can rewrite the l.h.s. of this equation as

\[
\frac{d}{d\omega} \int d^3x \frac{d\bar{\sigma}}{d\omega} \mathcal{H} \frac{d\bar{\sigma}}{d\omega} - 2 \int d^3x \frac{d\bar{\sigma}}{d\omega} \mathcal{H} \frac{d\bar{\sigma}}{d\omega} \frac{d\bar{\sigma}}{d\omega} = \frac{dF}{d\omega}, \tag{35} \]

where \( F(\omega) = \int d^3x \frac{d\bar{\sigma}}{d\omega} \mathcal{H} \frac{d\bar{\sigma}}{d\omega} \) is given by (see eq. (11))

\[
F(\omega) = \omega \int d^3x \frac{d\bar{\sigma}}{d\omega} \left[ 1 + \frac{2\omega}{T} \int d^3x \bar{\sigma} \frac{d\bar{\sigma}}{d\omega} \right] \frac{dQ}{d\omega} \tag{36} \]
Since $\frac{dQ}{d\omega}(\bar{\omega}) = 0$ we see that in a small neighbourhood of $\bar{\omega}$ we have $\text{sign}(F(\omega)) = \text{sign}(\lambda(\bar{\omega})) = -\text{sign}(\frac{dQ}{d\omega}(\omega))$. But this is enough to prove that $\text{sign}(\lambda(\bar{\omega})) = -\text{sign}(\frac{dQ}{d\omega}(\omega))$ for any values of $\bar{\omega}$.

4 Thin-wall and thick-wall regime

As we have seen, if we want to know whether a Q-ball solution is stable or not, instead of solving two second order partial differential equations in $3 + 1$ dimensions, we need only to know whether $\frac{dQ}{d\omega}$ is positive or negative. There are two limiting cases where we can apply these results without making use of a computer: when $\omega^2 \to \omega_0^2 \geq 0$, and when $\omega^2 \to m^2$. The first limit is known as the thin-wall regime while the second one as the thick-wall regime.

4.1 Thin-wall approximation.

In the mechanical analogon which was described in sec. 2 the initial position must be chosen in such a way that for $r \to \infty$ it won’t undershoot or overshoot the top of the hill at $\sigma = 0$.

When $\omega_0^2 \geq 0$ and $\omega^2 \to \omega_0^2$, the absolute maximum of $-U_\omega$ becomes virtually degenerate with the one at $\sigma = 0$. In this limit the particle must spend a very long time close to $\sigma_0$ (defined by eq. (34)) otherwise it would undershoot the origin $\sigma = 0$. To see this note first that if $\sigma_1$ is the zero of $U_\omega$ and $\sigma(0) < \sigma_1$, the particle doesn’t have enough energy to reach the origin, for the energy is a decreasing function of time. Now, if $\sigma(0) > \sigma_1$, $\sigma(0)$ is so close to $\sigma_0$ that we can linearize the eq. of motion:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \mu^2\right] (\sigma(r) - \sigma_0) = 0,$$

with $\mu^2 = U''_\omega(\sigma_0) > 0$. This equation is a good approximation as long as $s(r) \equiv \sigma_0 - \sigma(r)$ is not too large. To find the solution with $\frac{ds}{dr}(0) = 0$ is quite a simple task:

$$s(r) = s_0 \frac{\sinh(\mu r)}{\mu r}.$$  

We still have the freedom of choosing $\sigma(0)$ but since we know that $\sigma(0) > \sigma_1$ we see that as $\omega^2 \to \omega_0^2$, $s_0 = (\sigma_0 - \sigma(0)) \to 0$. When this happens eq. (35) implies that the particle spends more and more time close to $\sigma_0$. We can define the time $R$ that the particle spends close to $\sigma_0$ to satisfy

$$s(R) = \sigma_0 - \sigma(R) = s_R,$$

where $s_R$ is just small enough to allow for the linear approximation. Now, since $\sinh(x)/x$ is a growing function of $x$ it’s clear that as $s_0 \to 0$ (i.e. $\omega^2 \to \omega_0^2$) we must expect $R \mu \to \infty$. On the other hand $\mu^2 = -\omega^2 + \text{const.}$ is bounded in $[\omega_0^2, m^2]$, which proves that in the thin-wall limit, as $s_0 = \sigma_0 - \sigma(0)$ gets smaller, $R$ becomes larger and larger and grows to infinity. Note that the asymptotic behaviour of $R$ is independent of the definition point $s_R$.

We can also easily show that for $r = R$ the damping term is already unimportant when compared with the potential term:

$$\left| \frac{1}{R} \frac{d\sigma}{dr}(R) \right| \simeq \frac{s_0}{R} \frac{e^{\mu R}}{2 \mu R} \simeq \frac{1}{\mu R} \mu^2 s_R \ll \left| U'_\omega(\sigma(R)) \right|. \quad (40)$$

That means that we can describe the Q-ball for $\omega^2 \to \omega_0^2$ as

$$\frac{\sigma}{\sigma} = \begin{cases} \sigma_0 - s(r) & \text{if } r < R \\ \sigma(r - R) & \text{if } r > R \end{cases} \quad (41)$$

where $\sigma(x)$ is the solution of the eq. of motion without damping term, which fits to $\sigma(0) = \sigma_0 - s_R$ and $\sigma(\infty) = 0$. Back to the field-theoretical language we can describe this Q-ball as formed by a very large core with radius $R$ surrounded by a comparably thin surface. The free parameter $\sigma(0)$ (or $s_0$ or $R$) must now be adjusted as to minimize the energy at fixed $Q < S_\omega$ at fixed $\omega$. This is not a difficult thing to do.

We must first note the following: (i) for $r < R$ we have $U_\omega(\sigma) = \frac{1}{2} \mu^2 s^2 - \epsilon$, where $\epsilon = -U_\omega(\sigma_0) = \frac{1}{2} \mu^2 (\omega^2 - \omega_0^2)$ is the energy difference between the tops of the two hills; (ii) when $r > R$ the eq. of motion has no damping term and therefore $\frac{1}{2} (\frac{ds}{dr})^2 = U_\omega(\sigma_\omega)$ (i.e. the particle’s energy is conserved). A straightforward calculation gives

$$S_\omega = S_\omega^C + S_\omega^R, \quad (42)$$

where the core’s contribution is

$$S_\omega^C = -\frac{4\pi}{3} R^3 \epsilon + 2\pi R^2 s_R^2 - 2\pi R s_R^2, \quad (43)$$

and

$$S_\omega^R = 4\pi R^2 T + 2\pi A_1 + A_2, \quad T = \int_0^{\sigma R} d\sigma \sqrt{2U_\omega(\sigma)} \quad (44)$$

is the surface’s contribution. The quantities $T$ and $A_n \equiv 8\pi \int_0^\infty dx x^n U_\omega(\sigma)$ depend weakly on $\omega$ when compared with any positive power of $R$, which as we know become infinite as $\omega^2 \to \omega_0^2$. In this limit we can drop powers of $R$ of order lower than 2, obtaining therefore

$$S_\omega = -\frac{4\pi}{3} R^3 \epsilon + 2\pi (2T + s_R^2 \mu) R^2. \quad (45)$$

The condition that we can neglect the term linear in $R$ is $R \gg A_1/2\pi T$. A quick look at the definitions of $T$ and $A_1$ shows that $\delta = A_1/2\pi T$ is the thickness of the Q-ball’s wall. This is the reason for calling this the thin-wall limit.

\[ \text{We use here and in eq. (35), } \sinh(\mu R) \simeq \cosh(\mu R) \simeq \frac{1}{2} e^{\mu R}. \]
The quantity \( \frac{1}{2} \tau \equiv T + \frac{1}{2} \sigma_{s2}^2 \mu \) has also a simple interpretation: If we note that \( s_{s2}^2 \mu = 2 \int_0^{\sigma_0} ds \sqrt{s^2 \mu^2} = 2 \int_{\sigma_0}^{\sigma} d\sigma \sqrt{2U_0} \), it becomes clear that

\[
\frac{1}{2} \tau = \int_0^{\sigma} d\sigma \sqrt{2U_0} + \int_{\sigma_0}^{\sigma} d\sigma \sqrt{2U_0} \simeq \int_0^{\sigma_0} d\sigma \sqrt{2U_0}
\]

is the surface tension of the thin-walled Q-ball.

The function of eq. (43) has a maximum at \( R = \tau/\epsilon \).

Putting this back in (45) and using eq. (14), (15) we get

\[
\text{These equations are valid for both } \omega \text{ } \text{and } \text{equation (47).}
\]

Putting this back in (45) and using eq. (14), (15) we get

\[
\frac{1}{2} \tau = \int_0^{\sigma} d\sigma \sqrt{2U_0} + \int_{\sigma_0}^{\sigma} d\sigma \sqrt{2U_0} \simeq \int_0^{\sigma_0} d\sigma \sqrt{2U_0}
\]

is the surface tension of the thin-walled Q-ball.

The function of eq. (43) has a maximum at \( R = \tau/\epsilon \).

Putting this back in (45) and using eq. (14), (15) we get

\[
\text{These equations are valid for both } \omega \text{ and } \omega_0 = 0.
\]

and

\[
E(\omega) = S(\omega) + \omega Q(\omega) = \omega Q(\omega) \left[ \frac{5}{4} - \frac{\omega_0^2}{4\omega^2} \right].
\]

Assuming a weak \( \omega \)-dependence for \( \tau \), it follows from eq. (18) the stability of thin-walled Q-balls:

\[
\frac{\omega}{Q} \frac{dQ}{d\omega} = 1 - 6 \frac{\omega^2}{\omega^2 - \omega_0^2} < 0.
\]

These equations are valid for both \( \omega_0^2 > 0 \) and \( \omega_0^2 = 0 \) and show a difference in the properties of thin-walled Q-balls between models with unbroken symmetry (\( \omega_0^2 > 0 \)) and broken symmetry (\( \omega_0^2 = 0 \)):

(i) For \( \omega_0^2 > 0 \) and \( \omega^2 \simeq \omega_0^2 \) we get the Q-balls as they were first conceived by Coleman [3]. They have the property that \( E = \omega_0 Q \), i.e. they behave like a ball of Q-matter: the energy being proportional to the charge which is proportional to the volume of the (core).

(ii) For \( \omega_0^2 = 0 \) and in the thin-wall limit, we get from eq. (18) \( Q \sim \omega^{-5} \) and

\[
E = \frac{5}{4} \left( \frac{32 \pi^3}{3 \sigma_0^4} \right)^{1/5} Q^{4/5}.
\]

This shows that such Q-balls can’t be seen as Q-matter [14]. The charge doesn’t grow with the volume but as \( Q \sim R^{5/2} \) and the energy as \( E \sim R^2 \). The reason for this behaviour is simple: The core of the Q-ball is nearly in the asymmetric vacuum and it’s energy density decreases exponentially with its radius \( R \). The energy of the core becomes thus less important than the surface’s tension.

(iii) Finally, if \( \omega_0^2 > 0 \) but \( m^2 \gg \omega_0^2 \) the approximations we made are still valid if \( m^2 \gg \omega_0^2 \gg \omega_0^2 \). In this regime, although we have \( \omega_0^2 \neq 0 \), we can use the results of (ii) since \( \omega_0^2/\omega^2 \simeq 0 \). The results of (i) naturally still apply as \( \omega^2 \to \omega_0^2 \).

4.2 Thick-wall approximation.

We want now to look at theories which can be put in the form

\[
U(\sigma) = \frac{1}{2} m^2 \sigma^2 - A \sigma^n + \sum_{p>0} B_p \sigma^{n+p},
\]

with \( A \) a positive quantity. In the limit \( \omega^2 \to m^2 \), the Q-ball cannot in general be approximated by a step function as before. This limit is called, therefore, the thick-wall limit. Some authors (see ref. 13) proposed that since \( \sigma(0) \to 0 \) we may neglect the terms with powers higher than \( n \) in eq. (52), when calculating the properties of the Q-ball. In this way the energy and the charge of the Q-balls get a simple dependence on the relevant parameters of the theory, and the Q-balls stability can be analyzed. This is what we are going to investigate in the following lines.

Define

\[
\varepsilon_\omega \equiv m^2 - \omega^2.
\]

and rescale the field as

\[
\sigma' \equiv \frac{\sigma}{(\varepsilon/A) \pi^{-3/2}}.
\]

Changing the variable \( r \) as \( r \to r \varepsilon^{-1/2} \) we obtain (eq. 12)

\[
S_\omega(\sigma) = A^{-1} \varepsilon \pi^{1/2} S'_\varepsilon[\sigma'],
\]

where

\[
S'_\varepsilon[\sigma'] \equiv \int d^3r \left[ \frac{1}{2} \left( \frac{d\sigma'}{dr} \right)^2 + \frac{1}{2}(\sigma')^2 - (\sigma')^n \right]
\]

\[
+ \int d^3r \sum_p B_p \left( \frac{\varepsilon}{A} \right)^{n/2} (\sigma')^{n+p}.
\]

Suppose we know that the Q-ball solution \( \bar{\sigma} \) is an extremum of \( S_\omega(\sigma) \). To this solution corresponds an extremum of \( S'_\varepsilon[\sigma'] \), which is obtained by using eq. (54). We now hope that \( \bar{\sigma}(0) \) as a function of \( \varepsilon_\omega \) falls quickly enough as \( \varepsilon_\omega \to 0 \) so that the terms of order higher than \( n \) become irrelevant to the calculation of \( \bar{\sigma}' \). In that case \( S'_\varepsilon[\bar{\sigma}'] \equiv S_n \) doesn’t depend on \( \varepsilon \), and we get the following simple expression for \( S(\omega) \equiv S_\omega[\bar{\sigma}] \)

\[
S(\omega) = A^{-1} \varepsilon^{3/2} (m^2 - \omega^2)^{3/2} S_n.
\]

We can now calculate the charge and the energy from this expression. We have thus (for \( n > 2 \)):

\[
Q(\omega) = -\frac{dS(\omega)}{d\omega} = A^{-1} \varepsilon^{3/2} S_n \left( \frac{2}{n-2} - \frac{1}{2} \right) 2\omega(m^2 - \omega^2)^{3/2} - \frac{3}{4},
\]

(58)
and
\[ E(\omega) = A^{-1/2} S_n(m^2 - \omega^2)^{\sigma^2 - 1/2} \left[ (m^2 - \omega^2) + 2\omega^2 \left( \frac{2}{n-2} - \frac{1}{2} \right) \right]. \] (59)

Something must be wrong for \( n \geq 6 \): If \( n > 6 \) and \( \omega > 0 \), one sees that \( Q, E \to -\infty \). That’s obviously impossible because both the energy and the charge should be positive. Also for \( n = 6 \) these equations give \( E(\omega) = A^{-1/2} S_6 \neq 0 \) and \( Q(\omega) = 0 \), but this last expression implies that \( \sigma(r) = 0 \) with the consequence that \( E(\omega) = 0! \)

We must conclude that, at least for \( n \geq 6 \), this so-called thick-wall approximation is a bad approximation. The reason is simple: The approximation we made leads us to a functional with no extrema for \( n \geq 6 \), so that as \( \omega \to \infty \), the approximation is useful only for \( n \geq 6 \).

Thus we have proved that the so-called thick-wall approximation is useful only for \( n < 6 \). From eq. (58) we get
\[ \frac{\omega}{Q} \cdot \frac{dQ}{d\omega} = 1 - 2 \left( \frac{2}{n-2} - \frac{3}{2} \right) \frac{\omega^2}{m^2 - \omega^2}. \] (64)
so that as \( \omega^2 \to m^2 \) we have thus
\[ \frac{\omega}{Q} \cdot \frac{dQ}{d\omega} > 0 \text{ for } n = 4, 5, \]
\[ \frac{\omega}{Q} \cdot \frac{dQ}{d\omega} < 0 \text{ for } n = 3. \]
This means that thick-walled Q-balls are known to be stable only if \( n = 3 \).

### 4.3 An application: Q-balls in the false vacuum.

As an illustration of the use of the approximations we have just discussed, we will investigate what happens when the symmetric vacuum becomes unstable, i.e. for \( \omega_0^2 < 0 \). Consider the following potential
\[ U(\sigma) = \frac{1}{2} m^2 \sigma^2 - \frac{\alpha}{3} \sigma^3 + \frac{\lambda}{4} \sigma^4. \] (65)

We explained in sec. 2 that the energy spectrum includes Q-balls if \( \omega_0^2 = m^2 - \frac{\delta}{8} \sigma^2 < m^2 \) (see eq. (50)), i.e. if \( \lambda > 0 \). For \( \omega_0^2 \geq 0 \), the results of sec. 4 and 5 show that both thick-walled and thin-walled Q-balls are stable, which is enough to show, using Coleman’s theorem, that all Q-balls are absolutely stable configurations in the unbroken phase.

The transition \( \omega_0^2 \geq 0 \to \omega_0^2 < 0 \) changes the energy spectrum in a dramatic way [14,16]. If before we had stable Q-balls with all possible charges and energies, there is now a maximum charge \( Q_c \) and a maximum energy \( E_c \) which Q-balls can have. One way of seeing this is using the thin-wall approximation of sec. 4. As we will show below, we can still use this approximation when \( -\omega_0^2 \ll m^2 \) and \( \omega^2 \ll m^2 \). Eqs. (55) ff show that there is a frequency \( \omega_c^2 = \frac{1}{6} \omega_0^2 \) such that \( Q(\omega) \) attains a maximum at \( \omega = \omega_c \) and only Q-balls with \( \omega^2 > \omega_c^2 \) are stable. Within this approximation we get
\[ Q_c = \frac{32\pi}{3\sqrt{5}} \left( \frac{5}{18} \right)^{3/2} \frac{\sqrt{m^2 + |\omega_0|^2}}{|\omega_0|^3}, \] (66)
\[ E_c = \frac{\sqrt{5}}{2} |\omega_0| Q_c, \] (67)
and
\[ R_c = \frac{5}{9} \frac{\sqrt{m^2 + |\omega_0|^2}}{|\omega_0|^2}, \] (68)
where we used \( \tau = \frac{1}{3} \sqrt{m^2 + |\omega_0|^2} \omega_0^2 \). As we know, this approximation is valid as long as the radius of the critical Q-ball, \( R_c \), is much larger than the wall’s thickness, \( \delta \sim \sqrt{m^2 - \omega_0^2} \), that is for \( m^2 \gg |\omega_0|^2 \), as we said above.

The question is now: What happens as \( |\omega_0|^2 \) becomes larger, that is, for a deeper true vacuum? We will show that for \( |\omega_0|^2 > 4m^2 \) the critical Q-ball is already a thick-walled Q-ball. For the thick-wall approximation to be valid, the term \( (\sigma')^{n+1} = (\sigma')^4 \) in eq. (64) should be negligible when compared to the term \( (\sigma')^n = (\sigma')^3 \). Using the numerical value \( \sigma'(0) \sim 2 \) we get the following condition on the parameters: \( m^2 + |\omega_0|^2 \gtrsim 10(m^2 - \omega_0^2) \). This shows that if \( |\omega_0|^2 > 4m^2 \), the approximation is valid for \( \omega^2 \geq \frac{1}{2} m^2 \). Now, a short look at eq. (14) shows that the critical Q-ball is thick-walled, the critical frequency is precisely \( \omega_c^2 = \frac{1}{2} m^2 \). Furthermore we have
\[ Q_c = \frac{3}{2} \sqrt{3m^2 \frac{\sigma_0^2}{(m^2 + |\omega_0|^2)^2}}, \] (69)
In conclusion, both thin-wall and thick-wall approximations show, for $|\omega_0|^2 < m^2/100$ and $|\omega_0|^2 > 4m^2$, resp., that there is a maximum charge $Q_c$ which stable Q-balls can have. Although we cannot calculate $Q_c$ and $\omega_c^2$ analytically in the range $|\omega_0|^2 \in [m^2/100, 4m^2]$, using scaling properties of the model it can be shown that the picture remains the same.

5 Quantum corrections

This section is meant to explain why Q-balls, which are classical configurations, are important to quantum theory. As we will see, in certain circumstances the energy of a Q-ball of charge $q$ is the zeroth-order contribution in a semi-classical expansion to the energy of the lowest lying state of charge $q$. To show this it is useful to investigate the following partition function:

$$Z(T) = \text{tr}[e^{-HT}] = \int d\phi \langle \phi | e^{-HT} | \phi \rangle,$$  \hspace{1cm} (71)

for U(1)-invariant scalar theories. It is possible to perform a separation of the contributions of sectors of different charge to $Z(T)$

$$Z(T) = \sum_q \int d\phi \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \langle \phi | e^{i\alpha Q} e^{-HT} | \phi \rangle.$$  \hspace{1cm} (72)

This can be justified in a heuristic way [7,13], by noting that

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{i\alpha (Q-q)} = \delta_{Q,q}. \hspace{1cm} (73)$$

(For a more formal derivation see ref. [12]) In the limit $T \to +\infty$, we have

$$\lim_{T \to \infty} Z(T) = \sum_q e^{-E_q^0 T} \equiv \sum_q Z_q^0(T),$$  \hspace{1cm} (74)

where $E_q^0$ is the lowest energy eigenvalue in the sector of charge $q$. As is shown in the appendix, $Z_q^0(T)$ turns out to be given by

$$Z_q^0(T) = \int [d\phi] \delta \left( (q - Q)/\sqrt{T} \right) \exp \left( - \int dt E \right) \cdot \cos \left( q \int \frac{dt}{T} - q \alpha \right),$$  \hspace{1cm} (75)

where the integration is made over configurations whose values at $t = T/2$ and $t = -T/2$ differ only by a global phase, i.e. $\phi(T/2) = \phi(-T/2)e^{i\alpha}$ and $Q[\phi]$ and $I[|\phi|]$ are the functionals defined in sec. 3.

The important feature in this expression is that the $\delta$-function constrains the integration to be over configurations of charge $q$. As we know, there is, in certain cases, a configuration which minimizes $E[\phi]$ for the given charge: the Q-ball. Furthermore, this configuration also makes the argument of the cosine zero, because $\alpha = \omega T = \int Q I^{-1} dt$. To conclude: the (stable) Q-ball configuration makes the integrand of the above integral a maximum. But this is not enough an argument for performing a semi-classical expansion around the Q-ball. The point is that if there are configurations with nearly the same energy, and with the same charge, as the Q-ball for which $q \int dt Q/I - \alpha = 2\pi k$, $k \in \mathbb{Z} \setminus \{0\}$, the oscillating part of the integrand erases the contribution of the neighbourhood of the Q-ball. This can be the case, for instance, for thin-walled Q-balls, as these have a low-lying mode, with eigenvalue $(6\omega^2/(\omega^2 - \omega_0^2))R^{-1}$, which becomes a zero-mode as $R \to \infty$.

Only in the case the cosine is not oscillating that fast in the neighbourhood of the Q-ball we can perform the semi-classical expansion around this configuration, in which case the classical energy is just the zero-order contribution to $E_q^0 = T^{-1} \ln Z_q^0(T)$. We can then set the cosine equal to 1, rewrite the $\delta$-function as

$$\prod_t \delta((q - Q)/\sqrt{T}) \sim \lim_{a \to 0} \exp \left( - \frac{1}{2a} \int dt \frac{(q - Q)^2}{T} \right),$$  \hspace{1cm} (76)

and then expand the energy functional around the Q-ball

$$E[\phi] = E_{Q\text{ball}} + E_{fl}[\psi],$$  \hspace{1cm} (77)

where $E_{fl}[\psi]$ is the energy contained in the fluctuations $\psi = \phi - \bar{\sigma} e^{i\omega t}/\sqrt{2}$. In this way we get finally

$$e^{-E_q^0 T} = e^{-E_{Q\text{ball}}^0 T} \lim_{a \to 0} \int [d\psi] \exp \left[ - \int dt \left( E_{fl} + Q_{fl}^2/2aI_{fl} \right) \right],$$  \hspace{1cm} (78)

where $Q_{fl}[\psi] \equiv Q[\phi] - q$ and $I_{fl}[\psi] \equiv I[|\sigma e^{i\omega t} + \sqrt{2}\psi|]$. Starting from this expression we can calculate the quantum corrections to the energy of the Q-ball by performing common perturbation theory.

6 Conclusions

In the body of this paper we presented an analytical investigation of the main properties of Q-balls. We want now to underline part of the results, some of them because they are new, others because they seem to be unknown in the recent literature thus leading to some incorrect statements and others because of their relevance. As we have seen in sec. 3 the existence of Q-balls, solutions of non-zero charge, is a general property of scalar models with an U(1)-symmetry: They exist in the case...
that $\omega_0^2 = \min(2U(\sigma)/\sigma^2) < m^2$. As we also said, there is a theorem by Coleman which states the absolute stability of those Q-balls for which $\omega_0^2 > 0$ and $E_{Qb} < mQ$. This is however of restricted use. For instance, the gaussian-shaped B-balls which arise in models with gravity-mediated SUSY-breaking have $\omega_0^2 = 0$ (the potential grows too slow at the infinity). For this case and the one with $\omega_0^2 < 0$ we can use the theorem of sec.3 which regards the local stability of Q-balls. But also for $\omega_0^2 > 0$ this theorem is useful, as it shows that there can be also Q-balls with more energy than Q free mesons, but locally stable. At the quantum level this means that they can only decay through tunneling. It is however not clear how this should take place.

In sec.4.1 we introduced the thin-wall approximation from a somehow unusual point of view - as the consequence of a linearization of the eq. of motion for the core of large Q-balls. The way we did this makes clear, for a given model, where the approximation breaks. Another advantage of our method is that it gives the results obtained in ref.[5,14] in an unified and quantitative way. It also becomes clear that the effective potentials discussed in ref.[3,21,4] are extrema of the action or the energy. In this spirit we can’t linearize the eq. of motion as we did in sec.4.1.

In what concerns the other limit, $\omega_0^2 \rightarrow m^2$, we showed that the thick-wall approximation is useful only when the potential is of the form $U_3(\sigma) = 1/2 m^2 \sigma^2 - \alpha \sigma^3 + \cdots$, or $U_4(\sigma) = 1/2 m^2 \sigma^2 - \lambda \sigma^4 + \cdots$ - in the first case thick-walled Q-balls are stable, while in the second one unstable. For more general potentials, like $U(\sigma) = m^2 \sigma^2 - g_4 \sigma^4 + \cdots$, with $n \geq 6$, we don’t know the thick-wall behaviour. In this we disagree with the authors of ref.[13], which used the thick-wall approximation for such potentials.

Sec.4.3 discussed the existence of Q-balls living in a false vacuum. It is well known that in the limit of nearly degenerate vacua thin-walled Q-balls larger than a certain radius are unstable and can therefore induce a phase transition [4]. We shown, for the $U_3(\sigma)$ theory, that for a very deep true vacuum there still are stable Q-balls in the spectrum, although only thick-walled ones.

Finally, we revisited the proof, made in ref.[1], of the stability of thick-walled Q-balls in the model $U_3(\sigma)$. The author, trying to determine the second variation of the energy at fixed charge, used the method of lagrangean multipliers, which indeed is adequate only for the first order variations. The expression obtained in this way (eq.(18) of [1]),

$$\delta^2 E_Q \equiv \int d^4x \frac{1}{2} \delta^2 (\sigma) = \int d^4x \frac{1}{2} \delta^2 (\sigma) (-\nabla^2 + U'' + 3\omega^2) \delta \sigma,$$

misses the important non-local term that we found in eq.(10) and can be shown to be larger than our result for all values of $\omega$. For instance, with the above expression one would get the result that the translational modes, $\delta \sigma$, are not zero-modes. As we said above, with our expression we confirm that thick-walled Q-balls are stable in the model $U_3(\sigma)$.

The expression of ref.[1] was later used in ref.[22] for potentials of the form $U_4(\sigma)$ in 1+1-dimensions, as a basis for a numerical calculation of the vibration spectrum, and for the determination of the parameter regions of stable Q-balls and unstable Q-balls. To verify the validity of their results the authors observed numerically the evolution of a Q-ball belonging to the unstable parameter region, and saw that it really decays into plane waves. This positive test, however, is not a convincing argument. The reason is that this Q-ball was picked up exactly from the boarding line that separates the real unstable region and the supposed unstable region in parameter space: they chose a point in parameter space which lies on the line which separates unbroken from broken symmetry and observed a thin-walled Q-ball which, as we have seen in sec.4.3, is an unstable configuration for $\omega_0^2 < 0$. Would one have picked up a configuration lying in the middle of the supposed unstable region, one wouldn’t have observed any instability.

Our intention in sec.4.4 (and in the appendix) was to show how the classical Q-balls are related with the properties of the quantum theory. Usually classical stable configurations are supposed to appear in the quantum theory as dominant contributions to some functional integral, since they are extremas of the action or the energy. In this spirit we did an investigation of a functional integral which furnishes the lowest lying energy state for a given charge $q$. Although the Q-ball does not maximize the integrand in the original integral, since the integration runs over all possible configurations, we were able, after some cosmetics, to rewrite the integral in such a way that the integration runs only over configurations of charge $q$ and the Q-ball of charge $q$ maximizes the integrand. But, there was a collateral effect - an oscillating term appeared which can erase the contribution of the neighborhood of the Q-ball to the integral. As we remarked there, we must compare the period of the oscillation and the thickness of the gaussian around the Q-ball configuration, to see whether it makes sense to perform a semi-classical expansion or not. In the case of an affirmative answer, the energy of the Q-ball is the zeroth order contribution to the lowest lying energy eigenstate of the given charge and we can use eq.(78) to calculate radiative corrections to it.

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A Functional Integral for fixed charge

The purpose of this appendix is to derive the expression for $Z_0^\alpha(T)$, eq. (75), starting from (see eq. (72)):

$$Z_0^\alpha(T) = \int d\phi \int_0^{2\pi} \frac{d\sigma}{2\pi} e^{-i\alpha q} e^{-HT}\phi|\phi\rangle. \quad (79)$$

A great part of the this derivation follows closely the one by Rajaraman and Weinberg in ref. [17], whose intention was to get to expression (87), although in the real time (i.e. Minkowski) formalism.

Our first task is to calculate $\langle \phi|e^{i\alpha Q}e^{-HT}|\phi\rangle$. The effect of applying $e^{i\alpha Q}$ in $\langle \phi|\phi(x) = |\phi(x)|e^{i\alpha}$. We have thus

$$\int d\phi(\phi)e^{i\alpha Q} e^{-HT}|\phi\rangle = \int d\phi(\phi)e^{i\alpha}|e^{-HT}|\phi\rangle$$

$$= \int[d\phi] \exp \left(-\int dt \mathcal{L}_E[\phi] \right),$$

where the integration is made over paths with $\phi(T/2) = (T/2)e^{i\alpha}$.

We now change to the polar coordinates $\sigma \equiv \sqrt{2}\phi$ and $\vartheta \equiv \arg{\phi}$ we already used in sec. 2. With these coordinates the euclidean lagrangian reads

$$\mathcal{L}_E[\sigma, \vartheta] = \int d^3x \left[\frac{1}{2}\sigma^2 + \frac{1}{2}(\sigma\dot{\vartheta})^2\right] + \int d^3x \left[\frac{1}{2}\nabla(\sigma)^2 + \frac{1}{2}\sigma^2(\nabla \vartheta)^2 + U(\sigma)\right], \quad (81)$$

which turns to be the energy $E[\sigma, \vartheta]$ of the configuration $\sigma e^{\vartheta}/\sqrt{2}$. In the following $E$ will therefore be used in place of $L_E$. Also, the measure is now $d\sigma = |\sigma d\sigma|d\vartheta$ and the integration is performed over paths with $\sigma(x, T/2) = (x, T/2)$ and $\vartheta(x, T/2) = (x, T/2) + \alpha$. It is useful to extend the range of $\vartheta$ from $[0, 2\pi]$ to $(-\infty, +\infty]$, a step which poses no problem as its only effect is to multiply the integral with an infinite constant. The same applies to the integration over $\alpha$.

Now, we put the system in a box and expand $\vartheta(x)$ in its Fourier modes:

$$\vartheta(x, t) = b_0(t) + \sum_{k, \neq 0} b_k(t)e^{ikx \cdot x} \equiv b_0(t) + \tilde{\vartheta}(x, t). \quad (82)$$

It’s easy to recognize that the zero-mode $b_0(t)$ is the only degree of freedom affected by the U(1) internal rotation $\vartheta(x) \rightarrow \vartheta(x) + \alpha$. This follows readily from the fact that $\int d^3x \tilde{\vartheta}(x, t) = 0$. The effect of this coordinate transformation in the path integral is thus to change the measure and the integration limits in the following way: $d\tilde{\vartheta} = |d\tilde{\vartheta}|[d\tilde{b}_0] \int_0^{2\pi} \frac{d\sigma}{2\pi} \int [d\sigma][d\tilde{\vartheta}] J e^{-\int dt E_{\text{eff}}}$, where $J$ is a path independent jacobian. We can now calculate

$$Z_0^\alpha(T) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \int [d\sigma][d\tilde{\vartheta}] [db_0] J e^{-\int dt E_{\text{eff}}}. \quad (83)$$

To do this note that $\alpha = b_0(T/2) - b_0(-T/2) = \int dt b_0$, and therefore

$$\int dt E[\sigma, \vartheta] + i\alpha q = \int dt E[\sigma, \vartheta] +$$

$$+ \int dt \left(\frac{1}{2}b_0^2 I[\sigma] + b_0 (Q[\sigma, \vartheta] + iq)\right), \quad (84)$$

where $I[\sigma]$ and $Q[\sigma, \vartheta]$ are the functionals defined in sec. 2.

Now, we see that the integration in $b_0(t)$ combined with the integration in $\alpha$ is the same as integrating over arbitrary paths in $b_0$, and that the integral is gaussian. Thus, from these integrations we only get a term:

$$\left(\prod_t \sqrt{T[\sigma]}^{-1}\right) \exp \left(\int dt \frac{(Q + iq)^2}{2T}\right), \quad (85)$$

in this way obtaining an effective energy $E_{\text{eff}}[\sigma, \vartheta] = E[\sigma, \vartheta] - \frac{(Q[\sigma, \vartheta] + iq)^2}{2I[\sigma]}$, (86)

and the following expression:

$$Z_0^\alpha(T) = \int [\sqrt{T[\sigma]}^{-1} \sigma d\sigma] [d\tilde{\vartheta}] J e^{-\int dt E_{\text{eff}}}, \quad (87)$$

where, as we said, $\tilde{\vartheta}(x)$ includes all non-static spatial Fourier modes, and the paths are cyclical. The energy $E_{\text{eff}}$ contains an imaginary part $\text{Im} E_{\text{eff}} = -qQ[\sigma, \vartheta]/\text{I}[\sigma]$, but this will not contribute to any imaginary part of the path integral, since while $\Re E_{\text{eff}}$ is an even function of $\tilde{\vartheta}$, $\text{Im} E_{\text{eff}}$ is an odd one and $Z_0^\alpha(T)$ becomes

$$Z_0^\alpha(T) = \int [\sqrt{T[\sigma]}^{-1} \sigma d\sigma] [d\tilde{\vartheta}] J \cdot e^{-\int dt E_q \cos \left(q \int dt \frac{Q[\sigma, \vartheta]}{I[\sigma]} \right)}, \quad (88)$$

with

$$E_q[\sigma, \vartheta] = \text{Re} E_{\text{eff}} = E[\sigma, \vartheta] + \frac{q^2 - Q^2[\sigma, \vartheta]}{2I[\sigma]}, \quad (89)$$

We will now introduce the following identity under the integral:

$$C = \int [db_0(t)] \delta \left(b - \frac{q - Q[\sigma, \vartheta]}{T}\right), \quad (90)$$

where we integrate over all possible paths, and $C$ is an infinite constant, which as usual will be absorbed in the measure. The advantage of this step is that one recovers...
the integral over all the degrees of freedom and therefore we can rewrite the measure in cartesian coordinates. To get rid of \( \sqrt{T^{-1}} = \prod_t (I(t))^{-1/2} \) we absorb it in the \( \delta \)-function as follows

\[
I^{-1/2} \delta \left( \dot{b} - \frac{q - Q[t]}{I} \right) = \delta \left( \frac{q - Q[t]}{\sqrt{I}} \right),
\]

(91)

where we used \( Q[t] = Q[\sigma, \theta] + \dot{b} I[t] \). Since the \( \delta \)-function imposes \( q = Q[t] \), we can readily show that

\[
E_q[t] = E[\sigma, \theta] + \frac{1}{2} b^2 (Q[\sigma, \theta] + Q[t])
\]

(92)

Finally with the definitions \( \phi = \sigma e^{i\theta}/\sqrt{2} \) and \( \alpha = \int dt \dot{b} \) we obtain

\[
\int dt \frac{Q[t]}{I[t]} = \int dt \frac{Q[\phi]}{I[|\phi|]} - \alpha.
\]

(93)

Using equations (88) to (93) we are led to eq. (75).

References

1. I.Affleck, M.Dine, Nucl.Phys. B249, (1985) 361.
2. M.Dine, L.Randall and S.Thomas, Nucl.Phys. B458, (1996) 291, hep-ph/9507453.
3. A.Kusenko, M.Shaposhnikov, Phys.Lett. B418, (1998) 46, hep-ph/9709492.
4. K.Enqvist, J.McDonald, Nucl.Phys. B538, (1999) 321, hep-ph/9803386.
5. S.Coleman, Nucl.Phys. B262, (1985) 263-283.
6. T.D.Lee, Y.Pang, Phys.Rep. 221, (1992).
7. A.Kusenko, Phys.Lett. B404, (1997) 285, hep-th/9704073.
8. F.Paccetti Correia, Diploma thesis (Heidelberg 2000).
9. S.Coleman, Phys.Rev. D15, (1977) 2929.
10. S.Coleman, Phys.Rev. D16, (1977) 1762.
11. S.Coleman, V.Glaser, A.Martin, Commun.Math.Phys. 58, (1978) 211.
12. S.Coleman, Nucl.Phys. B298, (1987) 178-186.
13. R.Friedberg, T.D.Lee, A.Sirlin, Phys.Rev. D13, (1976) 2739.
14. D.Spector, Phys.Lett. B194, (1987) 103.
15. T.Multamäki, I.Vilja, Nucl.Phys. B574, (2000) 130, hep-ph/9908446.
16. A.Kusenko, Phys.Lett. B406, (1997) 26, hep-ph/9705361.
17. R.Rajaraman, E.Weinberg, Phys.Rev. D11, (1975) 2950.
18. R.Rajaraman, Solitons and Instantons (North-holland, Amsterdam, 1982).
19. K.M.Benson, Nucl.Phys. B327, (1989) 649.
20. C.G.Callan, D.J.Gross, Nucl.Phys. B93, (1975) 29.
21. K.Enqvist, J.McDonald, Phys.Lett. B425, (1998) 309, hep-ph/9711514.
22. M.Axenides, S.Komineas, L.Perivolaropoulos, M.Floratos, Phys.Rev. D61, (2000) 085006, hep-ph/9910388.