Directional necessary optimality conditions for bilevel programs

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Abstract

The bilevel program is an optimization problem where the constraint involves solutions to a parametric optimization problem. It is well-known that the value function reformulation provides an equivalent single-level optimization problem but it results in a nonsmooth optimization problem which never satisfies the usual constraint qualification such as the Mangasarian-Fromovitz constraint qualification (MFCQ). In this paper we show that even the first order sufficient condition for metric subregularity (which is in general weaker than MFCQ) fails at each feasible point of the bilevel program. We introduce the concept of directional calmness condition and show that under the directional calmness condition, the directional necessary optimality condition holds. While the directional optimality condition is in general sharper than the non-directional one, the directional calmness condition is in general weaker than the classical calmness condition and hence is more likely to hold. We perform the directional sensitivity analysis of the value function and propose the directional quasi-normality as a sufficient condition for the directional calmness. An example is given to show that the directional quasi-normality condition may hold for the bilevel program.

Key words. bilevel programs, constraint qualifications, necessary optimality conditions, directional derivatives, directional subdifferentials, directional quasi-normality

AMS subject classification: 90C30, 91A65, 49K40.

1 Introduction

The motivation for studying bilevel optimization originated in economics under the name of Stackelberg games [28] since 1934. In economics, it is used to model interactions between a leader and its follower of a two level hierarchical system and hence is referred to as leader and follower games or principal-agent problems. In recent years, bilevel programs find wider range of applications (see e.g. [6, 19, 23, 26] and references within). In particular, bilevel programs have been used to model hyper-parameter selection in machine learning (see e.g. [20, 21]) in recent years.

In this paper, we consider bilevel programs in the following form:

\[ \begin{align*}
\text{(BP)} \quad & \min_{x,y} \ F(x, y) \\
& \text{s.t. } \ y \in S(x), \ G(x, y) \leq 0,
\end{align*} \]

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where for any given $x$, $S(x)$ denotes the solution set of the lower level program

$$(P_x) \quad \min_y f(x, y) \quad \text{s.t. } g(x, y) \leq 0,$$

and $F, f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are continuously differentiable.

To obtain an optimality condition for $(BP)$, one may reformulate it as a single-level optimization problem and apply optimality conditions to the single-level problem. There are three approaches for reformulating $(BP)$ as a single-level optimization problem in the literature. The earliest approach is the so-called first order approach or the Karush-Kuhn-Tucker (KKT) approach by which one replaces the constraint $y \in S(x)$ by its first order optimality conditions and minimizing over the original variables as well as the multipliers. The resulting single-level optimization problem is the so-called mathematical program with equilibrium constraints (MPEC), which was popularly studied over the last three decades; see e.g. [23, 29] for the general theory and [30, 32, 14] for the optimality conditions derived by using this approach. The value function approach (see e.g. [33]) replaces the constraint $y \in S(x)$ by $f(x, y) - V(x) \leq 0$, where $V(x) := \inf_y \{ f(x, y) | g(x, y) \leq 0 \}$ is the value function of the lower level program $(P_x)$. And the combined approach ([34]) not only replaces the constraint $y \in S(x)$ by $f(x, y) - V(x) \leq 0$ but also adds the first order optimality conditions. The first order approach is obviously only applicable if the first order optimality condition is necessary and sufficient for optimality; e.g. when the lower level program is convex and certain constraint qualification holds. Both the KKT approach and the combined approach suffer from the drawback that the resulting MPEC may not be equivalent to the original $(BP)$ if the local optimality is considered; see e.g. [30] and the reference within for further discussions on this issue.

In this paper by the value function approach, we reformulate $(BP)$ as the following equivalent problem:

$$(VP) \quad \min_{x, y} F(x, y)$$

$$\text{s.t. } f(x, y) - V(x) \leq 0, \quad g(x, y) \leq 0, \quad G(x, y) \leq 0.$$

Under fairly reasonable assumptions, the value function $V(x)$ is Lipschitz continuous and hence a nonsmooth Fritz John type necessary optimality condition holds at a local optimal solution. For a KKT type necessary optimality condition to hold, in general one needs to assume certain constraint qualifications. Unfortunately, it is known ([33, Proposition 3.1]) that the nonsmooth MFCQ or equivalently the no nonzero abnormal multiplier constraint qualification (NNAMCQ), a standard constraint qualification for nonsmooth mathematical programs, fails to hold at any feasible point of $(VP)$. For an optimization problem with Lipschitz continuous problem data, it is known that the necessary optimality condition holds provided that the problem is calm in the sense of Clarke [5, Definition 6.41]. Ye and Zhu [33] introduced the partial calmness condition for problem $(VP)$ which means that a local solution of problem $(VP)$ is also a local solution of the partially penalized problem for certain $\rho > 0$

$$(VP)_\rho \quad \min_{x, y} \quad F(x, y) + \rho(f(x, y) - V(x))$$

$$\text{s.t. } g(x, y) \leq 0, \quad G(x, y) \leq 0.$$

Since the most difficult constraint $f(x, y) - V(x)$ is moved to the objective, the KKT condition would hold under some constraint qualifications for the partially penalized problem
It is easy to show that the full calmness implies the partial calmness and the partial calmness plus the full calmness of the partially penalized problem $(VP)_\rho$ implies the full calmness condition for problem $(VP)$. Some sufficient conditions for partial calmness and its relationship with exact penalization were further discussed in [33, 34, 36]. Unfortunately for problem $(VP)$, the partial calmness or the full calmness condition is still a fairly strong condition. And there are very few constraint qualifications or sufficient conditions for partial calmness for $(VP)$ in the literature. Recently, [29] has extended the relaxed constant positive linear dependence constraint qualification (RCPLD) to bilevel programs and has shown that it is a constraint qualification.

Recently Gfrerer [9, Theorem 7] derived a directional version of the KKT type necessary optimality condition for mathematical programs with a generalized equation constraint induced by a set-valued map under the directional metric subregularity constraint qualification. The directional KKT condition is in general sharper than the nondirectional KKT condition and the directional metric subregularity is weaker than the nondirectional one. Inspired by this approach, in this paper we aim at developing a directional KKT condition for problem $(VP)$. First we review the following concept of directional neighborhood recently introduced by Gfrerer in [9]. Given a direction $d \in \mathbb{R}^n$, and positive numbers $\epsilon, \delta > 0$, the directional neighborhood of direction $d$ is a set defined by
\[
V_{\epsilon,\delta}(d) := \{ z \in \epsilon B | \|d\| \leq \delta \|z\| \},
\]
where $B$ denotes the open unit ball and $\| \cdot \|$ denotes the Euclidean norm. It is easy to see that the directional neighborhood of direction $d = 0$ is just the open ball $\epsilon B$ and the directional neighborhood of a nonzero direction $d \neq 0$ is a smaller subset of $\epsilon B$. Hence many regularity conditions can be extended to a directional version which is weaker than the original nondirectional one. We say that $(VP)$ is calm at a feasible solution $(\bar{x}, \bar{y})$ in direction $d \in \mathbb{R}^n$, and positive numbers $\epsilon, \delta > 0$, if there exist positive scalars $\epsilon, \delta, \rho$, such that for any $\alpha \in \epsilon B$ and any $(x, y) \in (\bar{x}, \bar{y}) + V_{\epsilon,\delta}(d)$ satisfying $\varphi(x, y) + \alpha \in \mathbb{R}^{1+p+q}$ with $\varphi(x, y) := (f(x, y) - V(x), g(x, y), G(x, y))$ one has,
\[
F(x, y) - F(\bar{x}, \bar{y}) + \rho \|\alpha\| \geq 0.
\]
It is obvious that when the direction $d = 0$, the directional calmness is reduced to the classical calmness condition [5, Definition 6.41]. When $d \neq 0$, since the directional neighborhood is in general smaller than the usual neighborhood, the directional calmness condition is in general weaker than the nondirectional calmness condition. It is obvious that if $(\bar{x}, \bar{y})$ solves $(VP)$, then under the calmness condition in direction $d$, $(\bar{x}, \bar{y})$ is also a solution of the following penalized problem
\[
(DP) \quad \min_{x,y} \quad F(x, y) + \rho \text{dist}(\varphi(x, y), \mathbb{R}^{1+p+q})
\]
\[
\text{s.t.} \quad (x, y) \in (\bar{x}, \bar{y}) + V_{\epsilon,\delta}(d).
\]
The directionally penalized problem $(DP)$ is much easier to deal with than $(VP)$ since all the inequality constraints are moved to the objective function. By using the nonsmooth calculus, one can then show that $(\bar{x}, \bar{y})$ satisfies a KKT condition provided the value function is Lipschitz continuous. In fact we can achieve more. When $d$ is a critical direction, we can show that $(\bar{x}, \bar{y})$ satisfies a directional KKT condition in which a directional Clarke subdifferential (see Definition 2.8 and [2]) of the value function $V(x)$ at $\bar{x}$ in direction $d$ is used instead of the Clarke subdifferential. Since the directional Clarke subdifferential is
a subset of the Clarke subdifferential, the directional KKT condition is sharper than the nondirectional one.

To make the directional calmness condition and the directional KKT condition useful, we have two issues to consider. First, under what conditions, the value function is directionally Lipschitz continuous and directionally differentiable and how to calculate the directional limiting subdifferential and the directional derivative of the value function which will be needed in the directional KKT condition for problem (VP). In this paper, we have derived some formulas for the directional derivative of the value function and an upper estimate for the Clarke directional subdifferential of the value function \( V(x) \). Secondly, how to derive a verifiable constraint qualification which ensures the directional calmness condition of (VP)? It is known that the first order sufficient condition for metric subregularity (FOSCMS) (introduced in Gfrerer and Klatte [12, Corollary] for the smooth case and [11, Proposition 2.2] for the nonsmooth case) is a sufficient condition for the metric subregularity of the set-valued map \( \Phi(x, y) := \varphi(x, y) - \mathbb{R}^{p+q+1} \) which in turn implies the calmness of the problem (VP). FOSCMS is in general weaker than NNAMCQ and hence it is natural to ask if FOSCMS would hold for (VP). Unfortunately in Proposition 5.1, we show that FOSCMS also fails for problem (VP) in any critical direction. We propose the directional quasi-normality as a sufficient condition for the directional calmness condition and give an example to show that the directional quasi-normality is possible to hold for (VP).

Other than deriving a weaker constraint qualification and a shaper necessary optimality condition for bilevel programs, we have also made contributions that are of independent interest as summarized below.

- We introduce the concept of directional Clarke subdifferentials and derive some useful calculus rules for directional subdifferentials; see Propositions 2.4 and 2.6.
- For an optimization problem with directionally Lipschitz continuous objective function and directionally Lipschitz and directionally differentiable inequality constraints, we derive a directional KKT condition under the directional calmness condition; see Theorem 3.1. An example of a bilevel program is given to show that the directional calmness is weaker than the classical calmness; see Example 3.1.
- The classical results for the directional derivative of the value function are improved with weaker assumptions: see Propositions 4.3 and 4.4 and Corollary 4.1. Sufficient conditions for directional Lipschitz continuity of the value function is given in Theorem 4.1 and the upper estimate of the directional subdifferential of the value function is given in Theorems 4.2 and 4.3.

We organize the paper as follows. In the next section, we provide the notations, preliminaries and preliminary results. In Section 3 we derive the directional KKT condition under the directional calmness condition for a general optimization problem with directionally Lipschitz inequality constraints. In section 4, we study directional sensitivity analysis of the value function. Finally in section 5, we apply the previous results to (VP) and derive a verifiable constraint qualification and a necessary optimality condition.

2 Preliminaries

We first give notations that will be used in the paper. We denote by \( \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\} \), while \( \mathbb{R} := (-\infty, +\infty) \). \( \langle a, b \rangle \) denotes the inner product of vectors \( a, b \). Let \( \Omega \) be a set. By
\( x^k \overset{\Omega}{\to} \bar{x} \) we mean \( x^k \to \bar{x} \) and for each \( k, x^k \in \Omega \). By \( x^k \overset{u}{\to} \bar{x} \) where \( u \) is a vector, we mean that the sequence \( \{x^k\} \) approaches \( \bar{x} \) in direction \( u \), i.e., there exist \( t_k \downarrow 0, u^k \to u \) such that \( x^k = \bar{x} + t_k u^k \). By \( o(t) \), we mean \( \lim_{t \to 0} \frac{o(t)}{t} = 0 \). We denote by \( \mathbb{B}, \overline{\mathbb{B}}, \mathbb{S} \) the open unit ball, the closed unit ball and the unit sphere, respectively. \( B_{\delta}(\bar{z}) \) denotes the open unit ball centered at \( \bar{z} \) with radius \( \delta \). We denote by \( \text{co} \Omega \) and \( \partial \Omega \) the convex hull and the closure of a set \( \Omega \), respectively. The distance from a point \( x \) to a set \( \Omega \) is denoted by \( \text{dist}(x, \Omega) := \inf\{|x - y||y \in \Omega\} \) and the indicator function of set \( \Omega \) is denoted by \( \delta_\Omega \). For a single-valued map \( \phi : \mathbb{R}^n \to \mathbb{R}^m \), we denote by \( \nabla \phi(x) \in \mathbb{R}^{m \times n} \) the Jacobian matrix of \( \phi \) at \( x \) and for a function \( \varphi : \mathbb{R}^n \to \mathbb{R} \), we denote by \( \nabla \varphi(x) \) both the gradient and the Jacobian of \( \varphi \) at \( x \). For a set-valued map \( \Phi : \mathbb{R}^n \rightharpoonup \mathbb{R}^m \) the graph of \( \Phi \) is defined by \( \text{gph} \Phi := \{(x, y)|y \in \Phi(x)\} \). For an extended-valued function \( \varphi : \mathbb{R}^n \to \mathbb{R} \), we define its domain by \( \text{dom} \varphi := \{x \in \mathbb{R}^n|\varphi(x) < \infty\} \), and its epigraph by \( \text{epi} \varphi := \{(x, \alpha) \in \mathbb{R}^{n+1} | \alpha \geq \varphi(x)\} \).

We now review some basic concepts and results in variational analysis, which will be used later on. For more details see e.g. [3, 4, 5, 7, 22, 25, 27]. Moreover we derive some preliminary results that will be needed.

**Definition 2.1 (Tangent Cone and Normal Cone) (see, e.g., [27] Definitions 6.1 and 6.3)** Given a set \( \Omega \subseteq \mathbb{R}^n \) and a point \( \bar{x} \in \Omega \), the tangent cone to \( \Omega \) at \( \bar{x} \) is defined as

\[
T_{\Omega}(\bar{x}) := \{d \in \mathbb{R}^n | \exists t_k \downarrow 0, d_k \to d \text{ s.t. } \bar{x} + t_k d_k \in \Omega \ \forall k\}.
\]

The regular normal cone and the limiting normal cone to \( \Omega \) at \( \bar{x} \) are defined as

\[
N_{\Omega}(\bar{x}) := \left\{ \zeta \in \mathbb{R}^n | \langle \zeta, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|) \ \forall x \in \Omega \right\},
\]

\[
\hat{N}_{\Omega}(\bar{x}) := \left\{ \zeta \in \mathbb{R}^n | \exists x_k \overset{\Omega}{\to} \bar{x}, \zeta_k \to \zeta \text{ such that } \zeta_k \in \hat{N}_{\Omega}(x_k) \ \forall k \right\},
\]

respectively.

For any \( y \in \mathbb{R}^p \), define the active index set \( I_y := \{i = 1, \ldots, p | y_i = 0\} \). One can easily obtain that \( N_{\mathbb{R}^p}(y) = \{\zeta \in \mathbb{R}^p_+ | \zeta_i = 0, \ i \notin I_y\} \). The property stated in the following proposition will be useful.

**Proposition 2.1** Let \( y, z \in \mathbb{R}^p_+ \) be such that \( I_y \subseteq I_z \). Then \( N_{\mathbb{R}^p}(y) = N_{\mathbb{R}^p}(z) \cap [y - z]_+ \).

**Proof.** Since \( I_y \subseteq I_z \), we have \( N_{\mathbb{R}^p}(y) \subseteq N_{\mathbb{R}^p}(z) \). Let \( \zeta \in N_{\mathbb{R}^p}(y) \). If \( \zeta_i = 0 \) we have \( \zeta_i(y_i - z_i) = 0 \). Otherwise if \( \zeta_i \neq 0 \) then \( i \in I_y \subseteq I_z \) by which we have \( (y_i - z_i) = 0 \). This implies that \( \zeta \perp (y - z) \). Now take \( \zeta \in N_{\mathbb{R}^p}(z) \cap [y - z]_+ \). Then \( \zeta \geq 0 \) and for any \( j \notin I_z, \xi_j = 0 \). Consider any \( j \in I_z \backslash I_y \), since \( (y - z)_j < 0 \) and \( \xi^T(y - z) = 0 \), we have \( \xi_j = 0 \). This proves that \( \zeta \in N_{\mathbb{R}^p}(y) \). The proof is complete. \( \square \)

**Definition 2.2 (Directional Normal Cone) ( [15] Definition 2.3 | or [4] Definition 2.1).** Given a set \( \Omega \subseteq \mathbb{R}^n \), a point \( \bar{x} \in \Omega \) and a direction \( d \in \mathbb{R}^n \), the limiting normal cone to \( \Omega \) at \( \bar{x} \) in direction \( d \) is defined by

\[
N_{\Omega}(\bar{x}; d) := \{\zeta \in \mathbb{R}^n | \exists t_k \downarrow 0, d_k \to d, \zeta_k \to \zeta \text{ s.t. } \zeta_k \in \hat{N}_{\Omega}(\bar{x} + t_k d_k) \ \forall k\}.
\]
It is obvious that $N_\Omega(\bar{x}; 0) = N_\Omega(\bar{x})$, $N_\Omega(\bar{x}; d) = \emptyset$ if $d \not\in T_\Omega(\bar{x})$ and $N_\Omega(\bar{x}; d) \subseteq N_\Omega(\bar{x})$. It is also obvious that for all $d \in T_\Omega(\bar{x}) \setminus T_{\partial_\Omega}(\bar{x})$, one has $N_\Omega(\bar{x}; d) = \{0\}$. Moreover when $\Omega$ is convex, by [11, Lemma 2.1] the directional and the classical normal cone have the following relationship

$$N_\Omega(\bar{x}; d) = N_\Omega(\bar{x}) \cap \{d\}^\perp \quad \forall d \in T_\Omega(\bar{x}). \quad (1)$$

When $u = 0$ the following definition coincides with the Painlevé-Kuratowski inner/lower and outer/upper limit of $\Phi$ as $x \to \bar{x}$ respectively; see e.g., [25].

**Definition 2.3** Given a set-valued map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ and a direction $d \in \mathbb{R}^n$, the inner/lower and outer/upper limit of $\Phi$ as $x \xrightarrow{d} \bar{x}$ respectively is defined by

$$\liminf_{x \xrightarrow{d} \bar{x}} \Phi(x) := \{y \in \mathbb{R}^m \mid \forall \text{ sequence } t_k \downarrow 0, d^k \to d, \exists y^k \to y \text{ s.t. } y^k \in \Phi(\bar{x} + t_k d^k)\}$$

$$\limsup_{x \xrightarrow{d} \bar{x}} \Phi(x) := \{y \in \mathbb{R}^m \mid \exists \text{ sequence } t_k \downarrow 0, d^k \to d, y^k \to y \text{ s.t. } y^k \in \Phi(\bar{x} + t_k d^k)\},$$

respectively.

**Definition 2.4 (Directional Lipschitz continuity)** ([3, Page 719]) We say that a single-valued map $\phi(x) : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous at $\bar{x}$ in direction $d$ if there exists a scalar $L > 0$ and a directional neighborhood $\mathcal{V}_{e,\delta}(d)$ of $d$ such that

$$\|\phi(x) - \phi(x')\| \leq L\|x - x'\| \quad \forall x, x' \in \bar{x} + \mathcal{V}_{e,\delta}(d).$$

Obviously, the directional Lipschitz continuity in direction $d = 0$ coincides with the classical Lipschitz continuity.

**Definition 2.5 (Graphical derivatives)** ([2, Page 199]) For a set-valued map $\Phi : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ and a pair $(x, y)$ with $y \in \Phi(x)$, the graphical derivative of $\Phi$ at $x$ for $y$ is the mapping $D\Phi(x|y) : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ whose graph is the tangent cone $T_{gph\Phi}(x, y)$ to $gph\Phi$ at $(x, y)$:

$$v \in D\Phi(x|y)(u) \iff (u, v) \in T_{gph\Phi}(x, y).$$

For a single-valued map $\phi : \mathbb{R}^n \to \mathbb{R}^m$, we denote its graphical derivative at $x$ for $y = \phi(x)$ as $D\phi(x)(u) := D\phi(x|y)(u)$ and it follows by definition of the tangent cone that,

$$D\phi(x)(u) = \{v \mid \exists t_k \downarrow 0, u_k \to u \text{ s.t. } v = \lim_{k \to +\infty} \frac{\phi(x + t_k u_k) - \phi(x)}{t_k}\}.$$

**Definition 2.6 (Directional derivatives)** Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and $x, u \in \mathbb{R}^n$. The usual directional derivative of $\phi$ at $x$ in the direction $u$ is

$$\phi'(x; u) := \lim_{t \downarrow 0} \frac{\phi(x + tu) - \phi(x)}{t}$$

when this limit exists.

It is easy to see that if $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous and directionally differentiable at $x$ in direction $u$, then for all sequence $\{u^k\}$ which converges to $u$, we have

$$\phi'(x; u) = \lim_{k \to +\infty} \frac{\phi(x + t_k u^k) - \phi(x)}{t_k} = D\phi(x)(u).$$

We now recall the definition of some subdifferentials below.
Definition 2.7 (Subdifferentials) \([27\,\text{Definition 8.3}]\) Let \(\varphi : \mathbb{R}^n \to \mathbb{R}\) and \(\bar{x} \in \text{dom}\varphi\). The Fréchet (regular) subdifferential of \(\varphi\) at \(\bar{x}\) is the set
\[
\partial \varphi(\bar{x}) := \{\xi \in \mathbb{R}^n | \varphi(x) \geq \varphi(\bar{x}) + \langle \xi, x - \bar{x} \rangle + o(||x - \bar{x}||)\},
\]
the limiting (Mordukhovich or basic) subdifferential of \(\varphi\) at \(\bar{x}\) is the set
\[
\partial \varphi(\bar{x}) := \{\xi \in \mathbb{R}^n \exists x_k \to \bar{x}, \xi^k \to \xi \text{ s.t. } \varphi(x_k) \to \varphi(\bar{x}), \xi^k \in \hat{\partial} \varphi(x_k)\}.
\]

Definition 2.8 (Analytic directional subdifferentials) \([9, 15, 22, 3]\) Let \(\varphi : \mathbb{R}^n \to \mathbb{R}\) and \(\bar{x} \in \text{dom}\varphi\). The analytic limiting subdifferential of \(\varphi\) at \(\bar{x}\) in direction \(u \in \mathbb{R}^n\) is defined as
\[
\partial_a \varphi(\bar{x}; u) := \{\xi \in \mathbb{R}^n | \exists t_k \downarrow 0, u^k \to u, \xi^k \to \xi \text{ s.t. } \varphi(\bar{x} + t_k u^k) \to \varphi(\bar{x}), \xi^k \in \hat{\partial} \varphi(\bar{x} + t_k u^k)\}.
\]

It is easy to see that if \(u \notin T_{\text{dom}\varphi}(\bar{x})\), then \(\partial_a \varphi(\bar{x}; u) = \emptyset\).

Recently, based on the directional limiting normal cone, Benko, Gfrerer and Outrata \([3]\) introduced a directional limiting subdifferential.

Definition 2.9 (Directional subdifferentials) Let \(\varphi : \mathbb{R}^n \to \mathbb{R}\) and \(\bar{x} \in \text{dom}\varphi\). The limiting subdifferential of \(\varphi\) at \(\bar{x}\) in direction \((u, \zeta) \in \mathbb{R}^{n+1}\) is defined as
\[
\partial \varphi(\bar{x}; (u, \zeta)) := \{\xi \in \mathbb{R}^n | (\xi, -1) \in N_{\text{epi}\varphi}(\bar{x}, \varphi(\bar{x}); (u, \zeta))\}
\[
= \left\{\xi \in \mathbb{R}^n | \exists t_k \downarrow 0, u^k \to u, \zeta^k \to \zeta, \xi^k \to \xi, \varphi(\bar{x} + t_k u^k) = \varphi(\bar{x}) + t_k \zeta^k, \xi^k \in \hat{\partial} \varphi(\bar{x} + t_k u^k)\right\}.
\]

By definition, it is clear that \(\partial \varphi(\bar{x}; (u, \zeta)) = \emptyset\) unless \((u, \zeta) \in T_{\text{gph}\varphi}(\bar{x}, \varphi(\bar{x}))\) or equivalently \(\zeta \in D\varphi(\bar{x})(u)\). In general \(D\varphi(\bar{x})(u)\) is a set-valued map. However, if \(D\varphi(\bar{x})(u)\) is a singleton, we have \(\partial \varphi(\bar{x}; (u, \zeta)) = \partial_a \varphi(\bar{x}; u)\). In particular, when \(\varphi\) is Lipschitz continuous and directionally differentiable at \(\bar{x}\) in direction \(u\) we have \(D\varphi(\bar{x})(u) = \{\varphi'(\bar{x}; u)\}\) and in this case \(\partial \varphi(\bar{x}; (u, \varphi'(\bar{x}; u))) = \partial_a \varphi(\bar{x}; u)\); see \([3\,\text{Corollary 4.1}]\).

Proposition 2.2 \([22\,\text{Theorem 5.4}]\) Given \(\varphi : \mathbb{R}^n \to \mathbb{R}\), \(\bar{x} \in \text{dom}(\varphi)\) and a direction \(u \in \mathbb{R}^n\), one has
\[
\partial_a \varphi(\bar{x}; u) = \{\xi \in \mathbb{R}^n | \exists t_k \downarrow 0, u^k \to u, \xi^k \to \xi \text{ s.t. } \varphi(\bar{x} + t_k u^k) \to \varphi(\bar{x}), \xi^k \in \partial \varphi(\bar{x} + t_k u^k)\}.
\]

Furthermore, if \(\varphi(x)\) is Lipschitz continuous near \(\bar{x}\) in direction \(u\), we define the directional Clarke subdifferential of \(\varphi\) at \(\bar{x}\) in direction \(u\) as
\[
\partial^c \varphi(\bar{x}; u) := co(\partial_a \varphi(\bar{x}; u)). \tag{2}
\]

Proposition 2.3 Let \(\varphi : \mathbb{R}^n \to R\) be Lipschitz continuous at \(\bar{x}\) in direction \(u\). Then we have
\[
\partial^c \varphi(\bar{x}; u) = \co \limsup_{x \to \bar{x}} \partial^c \varphi(x), \quad \partial^c (-\varphi)(\bar{x}; u) = -\partial^c \varphi(\bar{x}; u).
\]
Proof. By Proposition 2.2, we have $\partial_u \varphi(\bar{x}; u) = \lim_{x \to \bar{x}} \partial \varphi(x)$. It follows that

$\partial^c \varphi(\bar{x}; u) := \co(\partial_u \varphi(\bar{x}; u)) = \co(\lim_{x \to \bar{x}} \partial \varphi(x)) = \co(\lim_{x \to \bar{x}} \partial^c \varphi(x))$,

where the last equality follows from the fact that for any sequence of sets \( \{A_k\} \subseteq \Omega \) where \( \Omega \subseteq \mathbb{R}^n \) is compact, we have \( \limsup_k \co(A_k) \subseteq \co(\limsup_k A_k) \). Indeed, for any sequence \( x^k \xrightarrow{k} \bar{x} \), let \( A_k = \partial \varphi(x^k) \) and by the Lipschitz continuity of \( \varphi(x) \) and [27, Theorem 9.13], there exists \( L > 0 \) such that \( \{A_k\} \subseteq L \mathbb{B} \). And the second equality follows directly from the first equality and the scalar multiplication rule of Clarke subdifferential [5, Proposition 2.3.1].

Proposition 2.4 (Sum Rules for analytic directional differentials) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be Lipschitz at \( \bar{x} \) in direction \( u \) and \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be l.s.c., proper and finite at \( \bar{x} \). Let \( \alpha, \beta \) be nonnegative scalars and \( u \in \mathbb{R}^n \). Then \( \partial_{(\alpha u + \beta g)}(\bar{x}; u) \subseteq \alpha \partial_{u}(\bar{x}; u) + \beta \partial_{g}(\bar{x}; u) \).

Proof. Let \( \xi \in \partial_{(\alpha u + \beta g)}(\bar{x}; u) \), by Proposition 2.2 there exist sequences \( t_k \downarrow 0 \), \( u^k \to u \), \( \xi^k \to \xi \) such that \( (\alpha u + \beta g)(\bar{x} + t_k u^k) \to (\alpha u + \beta g)(\bar{x}) \) and \( \xi^k \in \partial (\alpha u + \beta g)(\bar{x} + t_k u^k) \). Since \( f \) is Lipschitz continuous at \( \bar{x} \) in direction \( u \), for all sufficiently large \( k \), \( f(x) \) is Lipschitz continuous near \( \bar{x} + t_k u^k \) and hence \( \beta g(\bar{x} + t_k u^k) \) is finite. It follows by the sum rule of limiting subdifferentials (see e.g., [27, Exercise 10.10]) that we have

$$\partial (\alpha u + \beta g)(\bar{x} + t_k u^k) \subseteq \alpha \partial_{u}(\bar{x} + t_k u^k) + \beta \partial_{g}(\bar{x} + t_k u^k).$$

That is, there exist \( \xi^k_f \in \partial_{u} f(\bar{x} + t_k u^k) \) and \( \xi^k_g \in \partial_{g} g(\bar{x} + t_k u^k) \) such that \( \xi^k = \alpha \xi^k_f + \beta \xi^k_g \).

By the continuity of \( f \), \( f(\bar{x} + t_k u^k) \to f(\bar{x}) \). Hence, \( \beta g(\bar{x} + t_k u^k) \) is finite. Since \( f(x) \) is Lipschitz continuous near \( \bar{x} + t_k u^k \) with a Lipschitz constant \( K \) for all \( k \) large enough, \( ||\xi^k|| \leq K \) (see e.g., [27, Theorem 9.13]). Hence both \( \{\xi^k_f\} \) and \( \{\beta \xi^k_g\} \) are bounded. Without loss of generality, we assume \( \xi_f := \lim_k \xi^k_f \), \( \beta \xi_g := \lim_k \beta \xi^k_g \). We have \( \xi_f \in \partial_{u} f(\bar{x}; u) \). If \( \beta > 0 \), we have \( \xi = \alpha \xi_f + \beta \xi_g \). Otherwise if \( \beta > 0 \), we have \( g(\bar{x} + t_k u^k) \to g(\bar{x}) \). By Proposition 2.2 we have \( \xi_g \in \partial_{g} g(\bar{x}; u) \) and \( \xi = \alpha \xi_f + \beta \xi_g \). For both of these two cases, we can obtain \( \xi \in \alpha \partial_{u} f(\bar{x}; u) + \beta \partial_{g} g(\bar{x}; u) \). By the choice of \( \xi \), the desired inclusion is proved.

Definition 2.10 (Directional codervatives) (see e.g., [3]) Let \( \Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) be a set-valued map and \( (\bar{x}, \bar{y}) \in \text{gph} \Phi \), \( (u, \xi) \in \mathbb{R}^n \times \mathbb{R}^m \). The limiting codervative of \( \Phi \) at \( (\bar{x}, \bar{y}) \) in direction \( (u, \xi) \) in \( \mathbb{R}^n \times \mathbb{R}^m \) is defined as

$$D^* \Phi(\bar{x}, \bar{y}; (u, \xi)) \in \partial \Phi(\bar{x}, \bar{y}; (u, \xi)) := \{\xi \in \mathbb{R}^m | (\xi, -\zeta) \in N_{\text{gph} \Phi}(\bar{x}, \bar{y}; (u, \xi))\}.$$

The symbol \( D^* \Phi(\bar{x}; (u, \xi)) \) is used when \( \Phi \) is single-valued.

Remark 2.1 By [3, Proposition 5.1], if \( \phi : \mathbb{R}^n \to \mathbb{R}^m \) is Lipschitz at \( \bar{x} \) in direction \( u \), then \( D^* \phi(\bar{x}; (u, \xi)) \notin \emptyset \) if and only if \( \xi \in D \phi(\bar{x}; u) \), in which case

$$D^* \phi(\bar{x}; (u, \xi)) \subseteq \partial \phi(\bar{x}; u).$$

If \( \phi \) is Lipschitz continuous and directionally differentiable at \( \bar{x} \) in direction \( u \), then \( D \phi(\bar{x}; u) = \phi'(\bar{x}; u) \) and

$$D^* \phi(\bar{x}; (u, \xi)) = \partial \phi(\bar{x}; u).$$

If \( \phi \) is continuously differentiable, then \( D^* \phi(\bar{x}; (u, \xi)) = \nabla \phi(\bar{x})^T \xi \).
We now give the definition of directional metric subregularity.

**Definition 2.11 (Directional Metric Subregularity)** [9, Definition 2.1] Let \( \Phi: \mathbb{R}^n \to \mathbb{R}^m \) be a set-valued map and \((\bar{x}, \bar{y}) \in \text{gph} \Phi \). Given a direction \( u \in \mathbb{R}^n \), \( \Phi \) is said to be metrically subregular in direction \( u \) at \((\bar{x}, \bar{y})\), if there are positive reals \( \epsilon > 0, \delta > 0 \), and \( \kappa > 0 \) such that

\[
\text{dist}(x, \Phi^{-1}(\bar{y})) \leq \kappa \text{dist}(\bar{y}, \Phi(x)) \quad \forall x \in \bar{x} + \mathcal{V}_{\epsilon, \delta}(u).
\]

If \( u = 0 \) in the above definition, then we say that the set-valued map \( \Phi \) is metrically subregular at \((\bar{x}, \bar{y})\). It is known that the metric subregularity of \( \Phi \) at \((\bar{x}, \bar{y})\) is equivalent to the calmness of \( \Phi^{-1} \) at \((\bar{y}, \bar{x})\) (see [7, Theorem 3H.3]). Recall that a set-valued map \( \Psi \) is said to be calm ([27]) or pseudo-upper Lipschitz continuous ([32, Definition 2.8]) at \((\bar{y}, \bar{x})\) if there exist neighborhoods \( U \) of \( \bar{x} \), \( V \) of \( \bar{y} \) and a positive scalar \( \kappa \) such that

\[
\Psi(y) \cap U \subseteq \Psi(\bar{y}) + \kappa \| y - \bar{y} \| \mathbb{B}, \quad \forall y \in V.
\]

**Proposition 2.5** Let \( C \subseteq \mathbb{R}^p \) be closed and \( \phi: \mathbb{R}^n \to \mathbb{R}^p \). The set-valued map \( \Phi(x) := -\phi(x) + C \) is metrically subregular at \((\bar{x}, 0)\) in direction \( u \) if and only if the set-valued map \( \Psi(x, \alpha) := \text{epi} \delta_C - (\phi(x), \alpha) \) is metrically subregular at \(((\bar{x}, 0), (0, 0))\) in direction \((u, r)\) \( \forall r \in \mathbb{R} \).

**Proof.** It is easy to verify that \( \text{epi} \delta_C = C \times \mathbb{R}_+ \) and \( \Phi^{-1}(0) = \Phi^{-1}(0) \times \mathbb{R}_+ \). By the equivalence of norms in Euclidean space and the triangle inequality, we can find a positive scalar \( \rho \) such that for any \( x, x' \in \mathbb{R}^p, \alpha, \alpha' \in \mathbb{R} \), it holds \( \rho(|x - x'| + |\alpha - \alpha'|) \leq ||(x, \alpha) - (x', \alpha')|| \leq ||x - x'|| + |\alpha - \alpha'| \). Therefore there exists \( \rho > 0 \) such that

\[
\rho(\text{dist}(x, \Phi^{-1}(0)) + \text{dist}(\alpha, \mathbb{R}_+)) \leq \text{dist}((x, \alpha), \Psi^{-1}(0)) \leq \text{dist}(x, \Phi^{-1}(0)) + \text{dist}(\alpha, \mathbb{R}_+). \tag{3}
\]

Similarly, there exists \( \rho' > 0 \) such that

\[
\rho'(\text{dist}(0, \Phi(x)) + \text{dist}(\alpha, \mathbb{R}_+)) \leq \text{dist}(0, \Psi(x, \alpha)) \leq \text{dist}(0, \Phi(x)) + \text{dist}(\alpha, \mathbb{R}_+). \tag{4}
\]

Since \( \Psi(x, \alpha) \) is metrically subregular at \(((\bar{x}, 0), (0, 0))\) in direction \((u, r)\) if and only if there exist positive scalars \( \kappa, \epsilon, \delta \) such that

\[
\text{dist}((x, \alpha), \Psi^{-1}(0)) \leq \kappa \text{dist}(0, \Psi(x, \alpha)), \quad \forall x \in \bar{x} + \mathcal{V}_{\epsilon, \delta}(u), \forall \alpha \in \delta \mathbb{B}, \tag{5}
\]

and \( \Phi(x) \) is metrically subregular at \((\bar{x}, 0)\) in direction \( u \) if there exist positive scalars \( \kappa', \epsilon', \delta' \) such that

\[
\text{dist}(x, \Phi^{-1}(0)) \leq \kappa' \text{dist}(0, \Phi(x)), \quad \forall x \in \bar{x} + \mathcal{V}_{\epsilon', \delta'}(u), \tag{6}
\]

by (3) and (4), it follows that (5) holds if and only if (6) holds and the proof is complete.

We now derive a chain rule for the analytic directional subdifferential of the composition function of an indicator function and a smooth map.
**Proposition 2.6** Let $\phi : \mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable and $C \subseteq \mathbb{R}^p$ be closed. Suppose $\phi(\bar{x}) \in C$ and the set-valued map $\Phi(x) := -\phi(x) + C$ is metrically subregular at $(\bar{x},0)$ in direction $u$. Then
\[
\partial_u(\delta_C \circ \phi)(\bar{x}; u) \subseteq \nabla \phi(\bar{x})^T N_C(\phi(\bar{x}); \nabla \phi(\bar{x})u).
\]

**Proof.** Define $\varphi(x) := \delta_C \circ \phi(x)$. If $u \notin T_{\text{dom} \varphi}(\bar{x})$ with $\text{dom} \varphi = \{x|\phi(x) \in C\}$, then $\partial_u \varphi(\bar{x}; u) = \emptyset$. And the proposition holds trivially. Otherwise if $u \in T_{\text{dom} \varphi}(\bar{x})$, there exist sequences $t_k \downarrow 0, u_k^k \to u$ with $\bar{x} + t_ku_k \in \text{dom} \varphi$. Then it follows that for all such sequences we have $\varphi(\bar{x} + t_ku_k) \equiv 0$ for all $k$. Hence, $D \varphi(\bar{x})(u) = \{0\}$. By the comments after Definition 2.9 we have $\partial_u \varphi(\bar{x}; u) = \partial \varphi(\bar{x}; u, 0)$. Since the set-valued mapping $\Phi(x)$ is metrically subregular at $(\bar{x},0)$ in direction $u$, by Proposition 2.5, the set-valued map given by $\Psi(x, \alpha) := epi \delta_C - (\phi(x), \alpha)$ is metrically subregular at $((\bar{x},0), (0,0))$ in direction $(u,0)$. Since $\phi$ is continuously differentiable by [3] Theorem 4.1 and Remark 2.1 we have
\[
\partial_u(\delta_C \circ \phi)(\bar{x}; u) = \partial(\delta_C \circ \phi)(\bar{x}; u, 0) \subseteq \nabla \phi(\bar{x})^T \partial \delta_C(\phi(\bar{x}); \nabla \phi(\bar{x})u).
\]

The desired result follows from the fact that $N_C(z; d) = \partial \delta_C(z; d)$ by virtue of [22] Theorem 5.5.

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**3 Directional KKT conditions under directional calmness condition**

In this section we derive directional KKT condition for the optimization problem
\[
(P) \quad \min_z \varphi(z) \quad \text{s.t. } \phi(z) \leq 0,
\]
where $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}^q$.

The concept of (Clarke) calmness for a mathematical program is first defined by Clarke [5] Definition 6.41. We now introduce a directional version of the calmness condition for (P).

**Definition 3.1 (Directional Clarke calmness)** Suppose $\bar{z}$ solves (P). We say that (P) is (Clarke) calm at $\bar{z}$ in direction $u$ if there exist positive scalars $\epsilon, \delta, \rho$, such that for any $\alpha \in \epsilon B$ and any $z \in \bar{z} + V_{\epsilon, \delta}(u)$ satisfying $\phi(z) + \alpha \leq 0$ one has,
\[
\varphi(z) - \varphi(\bar{z}) + \rho \|\alpha\| \geq 0.
\]

We now prove that the directional metric subregularity implies the directional calmness of problem (P) provided the objective function is directional Lipschitz continuous.

**Lemma 3.1** Let $\bar{z}$ solve (P) and $\varphi(z)$ be Lipschitz continuous at $\bar{z}$ in direction $u$. Suppose that the set-value map $\Phi(z) : -\phi(z) + \mathbb{R}^q_-$ is metrically subregular at $(\bar{z},0)$ in direction $u$. Then (P) is calm at $\bar{z}$ in direction $u$.

**Proof.** Since $\Phi(z)$ is metrically subregular at $(\bar{z},0)$ in direction $u$, by Definition 2.11 there exist positive scalars $\epsilon, \delta, \kappa$ such that
\[
\text{dist}(z, \Phi^{-1}(0)) \leq \kappa \text{dist}(\phi(z), \mathbb{R}^q_-) \quad \forall z \in \bar{z} + V_{\epsilon, \delta}(u).
\]
Let $\tilde{z}$ be the projection of $z$ onto $\Phi^{-1}(0)$. Since $\varphi(z)$ is directionally Lipschitz continuous, without loss of generality, taking $\epsilon, \delta$ small enough, there exists $L > 0$ such that $|\varphi(z) - \varphi(z')| \leq L \rho \parallel z - z'\parallel$ for any $z, z' \in \tilde{z} + \mathcal{V}_{\epsilon, \delta}(u)$. Then we have for any $\alpha \in c\Phi$ satisfying $\phi(z) + \alpha \leq 0$,

$$\varphi(z) - \varphi(\tilde{z}) + L \rho \parallel \alpha \parallel \geq \varphi(z) - \varphi(\tilde{z}) + L \kappa \operatorname{dist}(\varphi(z), \mathbb{R}^n)$$

$$\geq \varphi(z) - \varphi(\tilde{z}) + L \kappa \operatorname{dist}(z, \Phi^{-1}(0))$$

$$= \varphi(z) - \varphi(\tilde{z}) + L \kappa \parallel z - \tilde{z} \parallel$$

$$\geq \varphi(z) - \varphi(\tilde{z}) + L \kappa \parallel z - \tilde{z} \parallel$$

$$\geq 0,$$

where the third inequality follows from the optimality of $\varphi(z)$ at $\tilde{z}$ and the last inequality follows from the directional Lipschitz continuity of $\varphi(z)$ at $\tilde{z}$. Let $\rho := L \kappa$. The proof is complete.

Let $\tilde{z}$ be a feasible solution to problem (P). We denote by $\bar{I}_\phi := I_\phi(\tilde{z}) := \{j = 1, \ldots, q | \phi_j(\tilde{z}) = 0\}$ the set of indexes of active constraints at $\tilde{z}$. If $\varphi$ is continuously differentiable and $\phi$ is Lipschitz and directionally differentiable, we define the linearized cone by $L(\tilde{z}) := \{u \in \mathbb{R}^n | \phi_j'(\tilde{z}; u) \leq 0, j \in I_\phi(\tilde{z})\}$ and the critical cone by $C(\tilde{z}) := \{u \in L(\tilde{z}) | \nabla \varphi(z) u \leq 0\} = \{u \in \mathbb{R}^n | \phi_j'(\tilde{z}; u) \leq 0, j \in I_\phi(\tilde{z}), \nabla \varphi(z) u \leq 0\}$.

The following definition lists some sufficient conditions for the directional metric subregularity, hence are sufficient for directional calmness.

**Definition 3.2** Let $\phi(\tilde{z}) \leq 0$ and $u \in \mathbb{R}^n$.

- Suppose that $\phi$ is Lipschitz at $\tilde{z}$. We say that the no-nonzero abnormal multiplier constraint qualification (NNAMCQ) holds at $\tilde{z}$ if

$$0 \in \partial(\zeta, \phi)(\tilde{z}) \text{ and } 0 \leq \zeta \perp \phi(\tilde{z}) \implies \zeta = 0.$$

- Suppose that $\phi$ is Lipschitz and directionally differentiable at $\tilde{z}$ in direction $u$. We say that the first order sufficient condition for metric subregularity (FOSCMS) holds at $(\tilde{z}, 0)$ in direction $u$ if there exists no $\zeta \neq 0$ satisfying $0 \leq \zeta \perp \phi(\tilde{z}), \zeta \perp \phi'(\tilde{z}; u)$ and

$$0 \in \partial_a(\zeta, \phi)(\tilde{z}; u). \quad (7)$$

- Suppose that $\phi$ is Lipschitz and directionally differentiable at $\tilde{z}$ in direction $u$. We say that the directional quasi-normality holds at $\tilde{z}$ in direction $u$ if there exists no $\zeta \neq 0$ satisfying $0 \leq \zeta \perp \phi(\tilde{z}), \zeta \perp \phi'(\tilde{z}; u)$ such that (7) holds and there exists sequences $t_k \downarrow 0$, $u^k \to u$ satisfying

$$\phi_j(\tilde{z} + t_k u_k) > 0, \text{ if } j \in I_\phi \text{ and } \zeta_j > 0. \quad (8)$$

It is easy to see that for any given direction $u$, if $\phi$ is Lipschitz and directionally differentiable at $\tilde{z}$ in direction $u$ then the following implications hold:

NNAMCQ $\Rightarrow$ FOSCMS in direction $u$ $\Rightarrow$ quasi-normality in direction $u$. 

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Proposition 3.1 Let \( \phi(\bar{z}) \leq 0 \) and suppose that \( \phi \) is Lipschitz and directionally differentiable at \( \bar{z} \) in direction \( u \in L(\bar{z}) \). If the directional quasi-normality holds at \( \bar{z} \) in direction \( u \) for the inequality system \( \phi(z) \leq 0 \). Then the set-valued map \( \Phi(z) := -\phi(z) + \mathbb{R}_+^q \) is metrically subregular at \((\bar{z},0)\) in direction \( u \).

Proof. Since \( \phi \) is Lipschitz and directionally differentiable at \( \bar{z} \) in direction \( u \), we have \( D\phi(\bar{z})(u) = \{ \phi'(\bar{z};u) \} \). By Remark 2.1 and the comment below Definition 2.9

\[
D^*\phi(\bar{z}; (u, \phi'(\bar{z};u))) (\zeta) = \partial (\zeta, \phi)(\bar{z}; (u, \phi'(\bar{z};u))) = \partial \alpha(\zeta, \phi)(\bar{z}; u).
\]

By equality (1), we have

\[
N_{\mathbb{R}_+^q}(\phi(\bar{z}); \phi'(\bar{z};u)) = \{ \mu \in \mathbb{R}^q | 0 \leq \mu \perp \phi(\bar{z}), \mu \perp \phi'(\bar{z};u) \}.
\]

For any sequences \( \{\zeta^k\} \) and \( \{s^k\} \), if \( \zeta_j > 0 \) and \( \hat{N}_{\mathbb{R}_+^q}(s^k) \ni \zeta^k_j \to \zeta_j \), then for large enough \( k \), \( \zeta^k_j > 0 \) and hence \( s^k_j = 0 \). Hence the condition (8) is equivalent to the sequential condition in [1] Definition 4.1(a)). Therefore the quasi-normality in direction \( u \in L(\bar{z}) \) means that there exists no \( \zeta \neq 0 \) such that

\[
0 \in D^*\phi(\bar{z}; (u, \phi'(\bar{z};u))) (\zeta), \quad \zeta \in N_{\mathbb{R}_+^q}(\phi(\bar{z}); \phi'(\bar{z};u))
\]

and there exist sequences \( t_k \downarrow 0 \), \( u^k \to u \) such that (8) holds.

From the proof of [1] Lemma 3.1 and Corollary 4.1) and [10] Corollary 1, one can easily obtain that the quasi-normality at \( \bar{z} \) in direction \( u \) implies that \( \Phi(z) \) is metrically subregular at \((\bar{z},0)\) in direction \( u \). ■

In the following theorem, we derive the directional KKT condition under the directional calmness condition.

Theorem 3.1 Let \( \bar{z} \) be a local minimizer of \( (P) \). Suppose that \( \varphi(z) \) is continuously differentiable at \( \bar{z} \) and \( \phi(z) \) is Lipschitz and directionally differentiable at \( \bar{z} \) in direction \( u \in C(\bar{z}) \). Suppose that the \( (P) \) is calm at \( \bar{z} \) in direction \( u \). Then there exists a vector \( \lambda_\phi \in \mathbb{R}^q \) such that \( 0 \leq \lambda_\phi \perp \phi(\bar{z}), \lambda_\phi \perp \phi'(\bar{z};u) \) and

\[
0 \in \nabla \varphi(\bar{z}) + \partial \alpha(\lambda_\phi, \phi)(\bar{z};u).
\]

Proof. Since \((P)\) is calm at \( \bar{z} \) in direction \( u \), there exist positive scalars \( \epsilon, \delta, \rho \) such that

\[
\varphi(z) + \rho \text{dist}(\phi(z), \mathbb{R}_+^q) \geq \varphi(\bar{z}) \quad \forall z \in \bar{z} + \mathcal{V}_{\epsilon,\delta}(u).
\]

(9)

Since \( u \in C(\bar{z}) \), we have \( \phi(\bar{z}) + t\phi'(\bar{z};u) \in \mathbb{R}_-^q \) and hence

\[
0 \leq \frac{\text{dist}(\phi(\bar{z} + tu), \mathbb{R}_+^q)}{t} \leq \frac{\phi(\bar{z} + tu) - \phi(\bar{z}) - t\phi'(\bar{z};u)}{t}.
\]

Since \( \nabla \varphi(\bar{z})u \leq 0 \), it follows that \( \lim_{t \downarrow 0} \frac{\text{dist}(\phi(\bar{z} + tu), \mathbb{R}_+^q)}{t} = 0 \).

Since \( \bar{z} + tu \in \bar{z} + c(\mathcal{V}_{\epsilon,\delta}(u)) \) for \( t \) sufficiently small, by (9),

\[
\varphi(\bar{z} + tu) + \rho \text{dist}(\phi(\bar{z} + tv), \mathbb{R}_+^q) \geq \varphi(\bar{z})
\]

for all \( t \) small enough. Together with \( \nabla \varphi(\bar{z})u \leq 0 \) we have

\[
\lim_{t \downarrow 0} \frac{\varphi(\bar{z} + tu) + \rho \text{dist}(\phi(\bar{z} + tu), \mathbb{R}_+^q) - \varphi(\bar{z})}{t} = 0.
\]

(10)
For each $k = 0, 1, \ldots$, define $\sigma_k := 2(\varphi(\bar{z} + \frac{u}{k}) + \rho \text{dist}(\phi(\bar{z} + \frac{u}{k}), \mathbb{R}^q) - \varphi(\bar{z}))$. If $\sigma_k \equiv 0$, then for each large enough $k$, by (9), $\bar{z} + \frac{u}{k}$ is a global minimizer of the function $\varphi(z) + \rho \text{dist}(\phi(z), \mathbb{R}^q) + \delta_{\overline{z} + \text{cl}(\mathcal{V}_{\epsilon, \delta}(u))}(z)$. Since for each large enough $k$, $\bar{z} + \frac{u}{k}$ is an interior point of $\bar{z} + \text{cl}(\mathcal{V}_{\epsilon, \delta}(u))(u)$, by the well-known Fermat’s rule and the calculus rule (see e.g., [27, Corollary 10.9]),

$$0 \in \nabla \varphi(\bar{z} + \frac{u}{k}) + \rho \partial(\text{dist}_{\mathbb{R}^q} \circ \phi)(\bar{z} + \frac{u}{k}).$$

(11)

Otherwise, without loss of generality, we assume that for all $k$, $\sigma_k > 0$. Then by definition of $\sigma_k$ we have for $k$ sufficiently large,

$$\varphi(\bar{z} + \frac{u}{k}) + \rho \text{dist}(\phi(\bar{z} + \frac{u}{k}), \mathbb{R}^q) + \delta_{\overline{z} + \text{cl}(\mathcal{V}_{\epsilon, \delta}(u))}(z) < \varphi(\bar{z}) + \sigma_k.$$

Define $\lambda_k := \frac{2\|u\|}{\|k\|} \sqrt{\frac{\sigma_k}{\|k\|}}$. By Ekeland’s variation principle, there exists $\tilde{z}^k$ satisfying that $\|\tilde{z}^k - (\bar{z} + \frac{u}{k})\| \leq \lambda_k$, and $\varphi(z) + \rho \text{dist}(\phi(z), \mathbb{R}^q) + \delta_{\overline{z} + \text{cl}(\mathcal{V}_{\epsilon, \delta}(u))}(z) + \frac{\sigma_k}{\lambda_k} \|z - (\bar{z} + \frac{u}{k})\|$ attains its global minimum at $\tilde{z}^k$. Since $\frac{\sigma_k}{\lambda_k}$ is an interior point of $\text{cl}(\mathcal{V}_{\epsilon, \delta}(u))$, there exists $r \in (0, \epsilon/2)$ such that $\frac{\sigma_k}{\lambda_k} r \mathbb{B} \subset \text{cl}(\mathcal{V}_{\epsilon, \delta}(u))$. It is obvious that the following implication holds

$$z \in \text{cl}(\mathcal{V}_{\epsilon, \delta}(u)), 0 \leq \alpha \leq 1 \Rightarrow \alpha z \in \text{cl}(\mathcal{V}_{\alpha \epsilon, \delta}(u))$$

Hence $\frac{\sigma_k}{\lambda_k} r \mathbb{B} \subset \text{cl}(\mathcal{V}_{\epsilon, \delta}(u))$ and hence $\tilde{z} + \frac{u}{k} + \frac{2\|u\|}{\|k\|} r \mathbb{B} \subset (\bar{z} + \text{cl}(\mathcal{V}_{\epsilon, \delta}(u)))$ and since $\sigma_k = o(\frac{1}{k})$ by (10), $\tilde{z}^k$ is in the interior of $\bar{z} + \text{cl}(\mathcal{V}_{\epsilon, \delta}(u))$. Then by the well-known Fermat’s rule, we obtain

$$0 \in \nabla \varphi(\tilde{z}^k) + \rho \partial(\text{dist}_{\mathbb{R}^q} \circ \phi)(\tilde{z}^k) + \frac{\sigma_k}{\lambda_k} \tilde{z}^k.$$

(12)

Since $\phi$ is Lipschitz continuous near $\bar{z}$ in direction $u$, it is Lipschitz continuous at $\tilde{z}^k$ for $k$ large enough. So by the chain rule for limiting subdifferential [25, Corollary 3.43], we have

$$\partial(\text{dist}_{\mathbb{R}^q} \circ \phi)(\tilde{z}^k) \subseteq \bigcup_{\zeta' \in \partial \text{dist}_{\mathbb{R}^q}(\phi(\tilde{z}^k))} \partial \langle \zeta', \phi \rangle(\tilde{z}^k).$$

Therefore by (11) or (12), $\exists \zeta^k \in \partial \text{dist}_{\mathbb{R}^q}(\phi(\bar{z} + \frac{u}{k}))$ or $\exists \zeta^k \in \partial \text{dist}_{\mathbb{R}^q}(\phi(\tilde{z}^k))$ such that

$$0 \in \nabla \varphi(\bar{z} + \frac{u}{k}) + \rho \partial(\zeta^k, \phi)(\bar{z} + \frac{u}{k}), \text{ or } 0 \in \nabla \varphi(\tilde{z}^k) + \rho \partial(\zeta^k, \phi)(\tilde{z}^k) + \frac{\sigma_k}{\lambda_k} \tilde{z}^k.$$

(13)

Since distance functions are Lipschitz, by [27, Theorem 9.13], $\{\zeta^k\}$ is bounded. Without loss of generality, there exists $\zeta := \lim_k \zeta^k$. By the way, one can easily obtain that $\lim_k (\bar{z} + u/k - \bar{z})/\frac{1}{k} = \lim_k (\tilde{z}^k - \bar{z})/\frac{1}{k} = u$. Since $\sigma_k = o(\frac{1}{k})$, $\lim_k \frac{\sigma_k}{\lambda_k} = 0$. Taking the limit of (13) as $k \to \infty$, by Proposition [21, Corollary 4.2], $\zeta \in \partial \text{dist}_{\mathbb{R}^q}(\phi(\bar{z}) \cup \mathcal{N}_{\mathbb{R}^q}(\phi(\bar{z}) \cap \phi'((\bar{z}, u)))$. The desired result holds by taking $\lambda := \rho \zeta \in \mathcal{N}_{\mathbb{R}^q}(\phi(\bar{z}) \cap \phi'((\bar{z}, u))) \subseteq \{\xi \in \mathbb{R}^q | 0 \leq \xi \perp \phi(\bar{z}), \xi \perp \phi'(\bar{z}, u)\}$. 

We now give an example of a bilevel program where the partial calmness and calmness fail but the calmness condition holds in a nonzero critical direction.
Example 3.1 Consider the following bilevel program:

\[
\text{(BP)} \quad \min \quad F(x, y) := (x - y - 1)^{\frac{5}{3}} + 4(x + y + 1)^{\frac{5}{3}}
\]

\[\text{s.t.} \quad -1 \leq x \leq 1, y \in S(x),\]

where for each \( x \), \( S(x) \) is the solution set for the lower level program:

\[
\min_y \{ f(x, y) := -(x + y)^2 + x^3(x + y - 1), \text{s.t.} \quad -y - x - 1 \leq 0, y + x - 1 \leq 0 \}.
\]

It is easy to see that the solution mapping \( S(x) \) of the lower level problem is equal to

\[
S(x) = \begin{cases} 
-x - 1, & x > 0, \\
\{-1, 1\}, & x = 0, \\
-x + 1, & x < 0
\end{cases}
\tag{14}
\]

And the global optimal solution of \( (BP) \) is \((\bar{x}, \bar{y}) = (0, -1)\). The constraints \( y + x - 1 \leq 0 \) and \(-1 \leq x \leq 1\) are inactive at \((0, -1)\). The value function

\[
V(x) = \begin{cases} 
-1 - 2x^3 & x > 0 \\
-1 & x \leq 0
\end{cases}
\tag{15}
\]

First, we prove that the partial calmness condition fails at \((\bar{x}, \bar{y})\). For any scalar \( \rho > 0 \), consider the partially penalized problem:

\[
\text{(VP)} \rho \quad \min \quad F(x, y) + \rho(f(x, y) - V(x))
\]

\[\text{s.t.} \quad g_1(x, y) := -y - x - 1 \leq 0, g_2(x, y) := y + x - 1 \leq 0, \]

\[-1 \leq x \leq 1.\]

Since \(-1 < \bar{x} < 1, g_1(\bar{x}, \bar{y}) = 0, g_2(\bar{x}, \bar{y}) < 0\), by \([14]-[15]\), the critical cone is

\[
C(\bar{x}, \bar{y}) = \{(u, v) | \nabla F(\bar{x}, \bar{y})(u, v) \leq 0, \nabla f(\bar{x}, \bar{y})(u, v) - V'(\bar{x}; u) = 0, \nabla g_1(\bar{x}, \bar{y})(u, v) \leq 0\}
\]

\[
= \{(u, v) | u + v = 0\}.
\]

Consider the sequence \((x^k, y^k) := (-\frac{1}{k}, \frac{1}{k} - 1)\) which are feasible to \((VP)\rho\) and converges to \((\bar{x}, \bar{y})\). Since \(F(x^k, y^k) = -\frac{2}{k^{\frac{5}{3}}}, f(x^k, y^k) = -1 + \frac{2}{k^{\frac{2}{3}}} \) and by \([15]\), \(V(x^k) = -1\), we have

\[
F(x^k, y^k) + \rho(f(x^k, y^k) - V(x^k)) = -(\frac{2}{k^{\frac{5}{3}}}) + \frac{2}{k^{\frac{2}{3}}}. \quad \text{Hence for } k \text{ sufficiently large, we have}
\]

\[
F(x^k, y^k) + \rho(f(x^k, y^k) - V(x^k)) < 0 = F(\bar{x}, \bar{y}) + \rho(f(\bar{x}, \bar{y}) - V(\bar{x})).
\]

This means that for any \( \rho > 0 \), \((\bar{x}, \bar{y})\) is not a local minimizer of \((VP)\rho\). Hence, the partial calmness condition fails. Since the calmness condition is in general stronger than partial calmness, the calmness condition also fails. In fact for this example since the constraint functions for \((VP)\rho\) are all affine, the partial calmness is equivalent to the fully calmness. Notice that \((x^k, y^k) \to (\bar{x}, \bar{y})\) in direction \((-1, 1)\) and so we have shown that problem \((VP)\) is not calm in direction \((-1, 1)\). Next, we prove that \((VP)\) is calm at \((\bar{x}, \bar{y})\) in direction \((1, -1)\in C(\bar{x}, \bar{y})\). Since the constraints \(g_2(x, y) \leq 0\) and \(-1 \leq x \leq 1\) are inactive at \((\bar{x}, \bar{y}) = (0, -1)\), it suffices to show that there exists a positive scalar \( \rho \) such that for any sequences \( t_k \downarrow 0 \), \((u^k, v^k) \to (u, v) := (1, -1)\), for \( k \) sufficiently large,

\[
F(\bar{x} + t_k u^k, \bar{y} + t_k v^k) + \rho \text{dist}(f(\bar{x} + t_k u^k, \bar{y} + t_k v^k) - V(\bar{x} + t_k u^k), \mathbb{R}_-) + \rho \text{dist}(g_1(\bar{x} + t_k u^k, \bar{y} + t_k v^k), \mathbb{R}_-) - F(\bar{x}, \bar{y}) \geq 0. \tag{16}
\]
Suppose that \( g_1(x + t_ku^k, y + t_kv^k) \leq 0 \), then for \( k \) sufficiently large, \( y + t_kv^k \) is a feasible solution for \( (P_{x + t_ku^k}) \) and hence \( f(x + t_ku^k, y + t_kv^k) - V(x + t_ku^k) \geq 0 \) by the definition of the value function. Moreover \( F(x + t_ku^k, y + t_kv^k) \geq F(x, y) \). Hence (15) holds. Otherwise suppose that
\[
g_1(x + t_ku^k, y + t_kv^k) = -t_k(u^k + v^k) > 0.
\]
Hence \( t_k(u^k + v^k) < 0 \). Together with \( u^k > 0, t_k > 0 \), we can verify that \( f(x + t_ku^k, y + t_kv^k) - V(x + t_ku^k) < 0 \). Also since \( t_k \downarrow 0, -(u^k + v^k) \downarrow 0 \), we have
\[
F(x + t_ku^k, y + t_kv^k) - F(x, y) + \rho \text{dist}(g_1(x + t_ku^k, y + t_kv^k), \mathbb{R}_-)
\]
\[
= t_k^2(u^k + v^k)^2 - t_k(u^k + v^k) \geq 0.
\]
Hence, we obtain (16). Consequently (VP) is calm at \((\bar{x}, \bar{y})\) in direction \((\bar{u}, \bar{v}) = (1, -1)\).

4 Directional sensitivity analysis of the value function

In this section we study the directional sensitivity analysis of the value function of the lower level program \((P_x)\). The results of this section could be of independent interest.

First we give some preliminary results that will be needed. We first introduce a directional version of the restricted inf-compactness condition which was first introduced in [3] Hypothesis 6.5.1] with the terminology introduced in [17] Definition 3.8.

**Definition 4.1 (Directional Restricted Inf-compactness)** We say that the restricted inf-compactness holds at \( \bar{x} \) in direction \( u \) with compact set \( \Omega_u \subseteq \mathbb{R}^n \) if \( V(\bar{x}) \) is finite and there exists positive numbers \( \epsilon > 0, \delta > 0 \) such that for all \( x \in \bar{x} + \mathcal{V}_{\epsilon, \delta}(u) \) with \( V(x) < V(\bar{x}) + \epsilon \), one always has \( S(x) \cap \Omega_u \neq \emptyset \). When \( u = 0 \) in the above, we say the restricted inf-compactness holds at \( \bar{x} \).

Next we introduce a directional version of the inf-compactness condition (see e.g., [1] Page 272)). It is not difficult to verify that the directional inf-compactness implies the directional restricted inf-compactness.

**Definition 4.2 (Directional Inf-compactness)** We say that the inf-compactness holds at \( \bar{x} \) in direction \( u \) if there exist a compact set \( \Lambda_u \subseteq \mathbb{R}^n \), and positive numbers \( \alpha > V(\bar{x}), \epsilon, \delta \) such that for all \( x \in \bar{x} + \mathcal{V}_{\epsilon, \delta}(u) \), one always has \( \{y | f(x, y) \leq \alpha, g(x, y) \leq 0\} \neq \emptyset \) and contained in \( \Lambda_u \). When \( u = 0 \) in the above, we say the inf-compactness holds at \( \bar{x} \).

When \( f(x, y) \) satisfies the growth condition, i.e., there exists \( \delta > 0 \) such that the set
\[
\{y \in \mathbb{R}^m | g(\bar{x}, y) \leq \alpha, f(\bar{x}, y) \leq M, \alpha \in \delta \mathbb{B}\}
\]
is bounded for each \( M \in \mathbb{R} \), the inf-compactness holds at \( \bar{x} \). Similarly, if \( f(x, y) \) is coercive or level bounded, the inf-compactness holds at \( \bar{x} \).

The following definition gives a directional version of the classical inner semi-continuity (see e.g. [23] Definition 1.63]).

**Definition 4.3 (Directional Inner Semi-continuity)** Given \( \bar{y} \in S(\bar{x}) \), we say that the optimal solution map \( S(x) \) is inner semi-continuous at \((\bar{x}, \bar{y})\) in direction \( u \), if for any sequences \( t_k \downarrow 0, u^k \rightarrow u \), there exists a sequence \( y^k \in S(\bar{x} + t_ku^k) \) converging to \( \bar{y} \). When \( u = 0 \) in the above, we say that \( S(x) \) is inner semi-continuous at \((\bar{x}, \bar{y})\).
Note that [22, Definition 4.4(i)] introduced a directional inner semicontinuity which requires $y^k \not\rightarrow y$ for some $v$. Since $y^k \rightarrow y$ implies that $y^k \rightarrow y$, their directional inner semicontinuity is stronger than ours. Given a direction we define a subset of the solution $S(\bar{x})$ as below. It coincides with the solution set when $u = 0$ and may be strictly contained in the solution set if the direction $u$ is nonzero.

**Definition 4.4 (Directional Solution)** The optimal solution in direction $u$ is the set defined by

$$S(\bar{x}; u) = \{ y \in S(\bar{x}) | \exists t_k \downarrow 0, u^k \rightarrow u, y^k \rightarrow y, y^k \in \{ \bar{x} + t_k u^k \} \}.$$  

If $y \in S(\bar{x}; u)$, then $y$ is upper stable in direction $u$ in the sense of Janin (see [18, Definition 3.4]).

It is obvious that if the optimal solution map $S(x)$ is inner semi-continuous at $(\bar{x}, \bar{y}) \in gphS$ in direction $u$, then $y \in S(\bar{x}; u)$.

Denote the feasible map of the problem $(P_x)$ by

$$\mathcal{F}(x) := \{ y \in \mathbb{R}^m | g(x, y) \leq 0 \}$$

and the active index set $I_g(x, y) := \{ i = 1, \ldots, p | g_i(x, y) = 0 \}$.

**Definition 4.5 (RCR regularity)** ([24, Definition 1]) We say that the feasible map $\mathcal{F}(x)$ is relaxed constant rank (RCR) regular at $(\bar{x}, \bar{y}) \in gph\mathcal{F}$ if there exists $\delta > 0$ such that for any index subset $K \subseteq I_g(\bar{x}, \bar{y})$, the family of gradient vectors $\nabla_y g_j(x, y), j \in K$, has the same rank at all points $(x, y) \in B_\delta(\bar{x}, \bar{y})$.

The following lemma shows that the RCR regularity condition is slightly stronger than the calmness of the feasible map $\mathcal{F}$ at $(\bar{x}, \bar{y}) \in gph\mathcal{F}$ and is needed in Proposition 4.3 and Theorem 4.2.

**Lemma 4.1** (see [24, Lemma 5]) Suppose that the feasible map $\mathcal{F}(x)$ is RCR regular at $(\bar{x}, \bar{y}) \in gph\mathcal{F}$. Then there exist $\delta > 0, \kappa > 0$ such that for any $x \in B_\delta(\bar{x})$ and $y \in B_\delta(\bar{y}) \cap \mathcal{F}(x)$, there exists $\bar{y} \in \mathcal{F}(\bar{x})$ such that $\|y - \bar{y}\| \leq \kappa \| x - \bar{x}\|$ and $g_j(x, y) \leq g_j(\bar{x}, \bar{y}) \leq 0$ for each $j \in I_g(\bar{x}, \bar{y})$.

We now define a directional version of the Robinson Stability ([13, Definition 1.1]).

**Definition 4.6 (Directional Robinson stability)** We say that the feasible map $\mathcal{F}(x)$ satisfies Robinson stability (RS) property at $(\bar{x}, \bar{y}) \in gph\mathcal{F}$ in direction $u \in \mathbb{R}^n$ if there exist positive scalars $\kappa, \epsilon, \delta$ such that

$$\text{dist}(y, \mathcal{F}(x)) \leq \kappa \text{dist}(g(x, y), \mathbb{R}^p) \quad \forall x \in \bar{x} + \mathcal{V}_{\epsilon, \delta}(u), y \in B_\epsilon(\bar{y}).$$  

(17)

If RS holds at $(\bar{x}, \bar{y})$ in direction $u = 0$, we say that RS holds at $(\bar{x}, \bar{y})$ ([13, Definition 1.1]). Note that RS in direction $u$ is equivalent to R-regularity with respect to set $\bar{x} + \mathcal{V}_{\epsilon, \delta}(u)$ as defined in [24, Definition 2].

**Proposition 4.1 (Sufficient Conditions for RS)** If the system $g(x, y) \leq 0$ satisfies one of the following conditions at $(\bar{x}, \bar{y})$, then RS holds at $(\bar{x}, \bar{y})$.

- **Linear constraint qualification:** $g(x, y)$ is an affine mapping.
• Partial linear independence constraint qualification: $g$ is continuously differentiable at $(\bar{x}, \bar{y})$ and the set $\{\nabla_y g_i(\bar{x}, \bar{y}) \mid i \in I_g(\bar{x}, \bar{y})\}$ is linearly independent.

• Partial NNAMCQ: $g$ is continuously differentiable at $(\bar{x}, \bar{y})$ and there exists no nonzero vector $\lambda \in \mathbb{R}^p_+$ such that $\lambda \perp g(\bar{x}, \bar{y})$ and $\nabla_y g(\bar{x}, \bar{y})^T \lambda = 0$.

One can refer to [24, 17, 13] and the references therein for more sufficient conditions for RS. The following proposition shows that the directional RS implies the directional metric subregularity.

**Proposition 4.2** Suppose that the feasible map $F$ satisfies RS at $(\bar{x}, \bar{y}) \in \text{gph} F$ in direction $u$. Then the metric subregularity of the system $g(x, y) \leq 0$ holds at $((\bar{x}, \bar{y}), 0)$ in direction $(u, v)$ for any $v \in \mathbb{R}^m$.

**Proof.** Since RS for $F$ holds at $(\bar{x}, \bar{y})$ in direction $u$, i.e., there exist numbers $\kappa > 0, \epsilon > 0, \delta > 0$ such that
\[
dist(y, F(x)) \leq \kappa \dist(g(x, y), \mathbb{R}^p),
\]
for all $x \in \bar{x} + V_{\epsilon, \delta}(u)$ and $y \in B_{\epsilon}(\bar{y})$. Then we obtain
\[
dist((x, y), g^{-1}(\mathbb{R}^p_+)) = \dist((x, y), \text{gph} F) \leq \dist(y, F(x)) \leq \kappa \dist(g(x, y), \mathbb{R}^p_+),
\]
for all $x \in \bar{x} + V_{\epsilon, \delta}(u)$ and $y \in B_{\epsilon}(\bar{y})$. This means that the metric subregularity of $g(x, y) \leq 0$ holds at $(\bar{x}, \bar{y}, 0)$ in direction $(u, v)$ for any $v \in \mathbb{R}^m$. \blacksquare

Recall that the lower Dini directional derivative of the feasible map $F(x)$ at a point $(\bar{x}, \bar{y}) \in \text{gph} F$ in direction $u$ is defined as
\[
D_+ F(\bar{x}, \bar{y}; u) := \liminf_{t \downarrow 0} \frac{F(\bar{x} + tu) - \bar{y}}{t} = \{v \mid \exists o(t) \text{ s.t. } \bar{y} + tv + o(t) \in F(\bar{x} + tu)\}.
\]
Define the $y$-projection of the linearization cone of $\text{gph} F$ at $(\bar{x}, \bar{y})$ in direction $u$, i.e.,
\[
\mathbb{L}(\bar{x}, \bar{y}; u) := \{v \in \mathbb{R}^m \mid \nabla g_i(\bar{x}, \bar{y})(u, v) \leq 0, i \in I_g(\bar{x}, \bar{y})\}.
\]
By definition, one always has $D_+ F(\bar{x}, \bar{y}; u) \subseteq \mathbb{L}(\bar{x}, \bar{y}; u)$. Since the directional MPEC R-regularity introduced in [17, Lemma 3.3] is weaker than our directional RS and $(P_\epsilon)$ is a special case of the problem studied in [17] when the equilibrium constraints are omitted, the following results follow from [17, Lemmas 3.3, 3.5].

**Lemma 4.2** [17] Lemmas 3.3, 3.5] Let $\bar{y} \in F(\bar{x})$. Suppose either the feasible map $F$ satisfies RS at $(\bar{x}, \bar{y})$ in direction $u$ or $D_+ F(\bar{x}, \bar{y}; u) \neq \emptyset$ and $F$ is RCR-regular at $(\bar{x}, \bar{y})$ in direction $u$. Then $D_+ F(\bar{x}, \bar{y}; u) = \mathbb{L}(\bar{x}, \bar{y}; u)$.

The following results will be needed in Corollary 4.4 and Theorem 4.1.

**Lemma 4.3** Suppose that the restricted inf-compactness holds at $\bar{x}$ in direction $u$ with compact set $\Omega_u$ and there exists $\bar{y} \in S(\bar{x})$ such that $F$ satisfies RS at $(\bar{x}, \bar{y})$ in direction $u$. Then $D_+ F(\bar{x}, \bar{y}; u) = \mathbb{L}(\bar{x}, \bar{y}; u) \neq \emptyset$, $\bar{y} \in \liminf_{x \rightarrow \bar{x}} F(x)$ and $S(\bar{x}; u) \neq \emptyset$. And for any $l > 0$, $\exists \epsilon, \delta > 0$ such that for $\forall x \in \bar{x} + V_{\epsilon, \delta}(u)$, $\exists y \in S(x) \cap \Omega_u$ satisfying $\dist(y, S(\bar{x}; u) \cap \Omega_u) < l$. 17
Suppose that \( F \) satisfies RS at \((\bar{x}, \bar{y})\) in direction \( u \), by Lemma 4.2, \( D_u \mathcal{F}(\bar{x}, \bar{y}; u) = L(\bar{x}, \bar{y}; u) \). Moreover by (17) there exist positive scalars \( \kappa, \epsilon, \delta \), such that for any \( x \in \bar{x} + V_{\epsilon, \delta}(u) \),

\[
\text{dist}(\bar{y}, \mathcal{F}(x)) \leq \kappa \text{dist}(g(x, \bar{y}), \mathbb{R}^p) \leq \kappa \|g(x, \bar{y}) - g(\bar{x}, \bar{y})\| \leq L_g \kappa \|x - \bar{x}\|, \tag{18}
\]

where \( L_g > 0 \) is the Lipschitz modulus of \( g(x, \bar{y}) \) around \( \bar{x} \). Then for any sequences \( t_k \downarrow 0, u^k \to u \), by (18) we can find a sequence \( y^k \in \mathcal{F}(\bar{x} + t_k u^k) \) such that \( \|\bar{y} - y^k\| \leq L_g \kappa \|\bar{x} + t_k u^k - \bar{x}\| \), which implies that \( y^k \to \bar{y} \). By Definition 2.3, this means that \( \bar{y} \in \liminf_{x \to \bar{x}} \mathcal{F}(x) \).

Since \( \{y^k - \bar{y} \} \) is bounded, taking a subsequence if necessary, we can find \( v \in \mathbb{R}^m \) such that \( v^k := \frac{y^k - \bar{y}}{t_k} \) converges to \( v \). Since for each \( i \in I_g(\bar{x}, \bar{y}) \), \( g_i(\bar{x} + t_k u^k, y^k) \leq 0 \), it follows that \( v \in \mathbb{L}(\bar{x}, \bar{y}; u) \). We also have \( \limsup_k V(\bar{x} + t_k u^k) \leq \liminf_k f(\bar{x} + t_k u^k, y^k) = V(\bar{x}) \). It follows that since the restricted inf-compactness holds at \( \bar{x} \) in direction \( u \), for each \( k \) large enough, there exists \( \bar{y}^k \in S(\bar{x} + t_k u^k) \cap \Omega_u \). By the compactness of \( \Omega_u \), the sequence \( \{\bar{y}^k\} \) is bounded. Without loss of generality, assume \( \bar{y} := \lim_k \bar{y}^k \in \Omega_u \). Since

\[
f(\bar{x}, \bar{y}) = \lim_k f(\bar{x} + t_k u^k, y^k) = \lim_k V(\bar{x} + t_k u^k) \leq \lim_k f(\bar{x} + t_k u^k, \bar{y} + t_k u^k) = f(\bar{x}, \bar{y}) = V(\bar{x}) \]

we can obtain \( \bar{y} \in S(\bar{x}) \). Consequently, \( \bar{y} \in S(\bar{x}; u) \cap \Omega_u \).

We prove the last statement by contradiction. Assume there exist \( l > 0 \) and for \( k \) large enough, \( \bar{x} + t_k u^k \in \bar{x} + V_{\epsilon, \delta}(u) \) and \( \bar{y}^k \in S(\bar{x} + t_k u^k) \cap \Omega_u \) such that \( \text{dist}(\bar{y}^k, S(\bar{x}; u) \cap \Omega_u) \geq l \). Taking the limit as \( k \to \infty \), \( \text{dist}(\bar{y}, S(\bar{x}; u) \cap \Omega_u) \geq l \), which contradicts \( \bar{y} \in S(\bar{x}; u) \cap \Omega_u \).

The proof is complete. \( \blacksquare \)

In general there may not exist relationship between RCR-regularity and RS condition. However under the inner semicontinuity of \( S(x) \), we can show that RCR-regularity implies RS/R-regularity.

**Lemma 4.4** Let \( \bar{y} \in S(\bar{x}) \) and \( S(x) \) is inner semi-continuous at \((\bar{x}, \bar{y})\) in direction \( u \). Suppose that \( \mathcal{F} \) is RCR regular at \((\bar{x}, \bar{y})\) in direction \( u \). Then \( \mathcal{F} \) satisfies RS at \((\bar{x}, \bar{y})\) in direction \( u \).

**Proof.** We approve the lemma by contradiction. Assume RS does not hold at \((\bar{x}, \bar{y})\) in direction \( u \). Then there exist sequences \( x^k \xrightarrow{u} \bar{x} \) and \( y^k \to \bar{y} \) satisfying that

\[
\text{dist}(y^k, \mathcal{F}(x^k)) > k \text{dist}(g(x^k, y^k), \mathbb{R}^p). \tag{19}
\]

Since \( S(x) \) is inner semi-continuous at \((\bar{x}, \bar{y})\) in direction \( u \), we have

\[
\bar{y} \in \liminf_{x \to \bar{x}} S(x) \subseteq \liminf_{x \to \bar{x}} \mathcal{F}(x).
\]

Then for sufficiently large \( k \) there exists a sequence \( \bar{y}^k \in \mathcal{F}(x^k) \) such that \( \bar{y}^k \to \bar{y} \). Let \( \bar{y}^k \) be the projection of \( y^k \) onto \( \mathcal{F}(x^k) \). We obtain

\[
\|y^k - \bar{y}^k\| \leq \|y^k - \bar{y}\| \to 0 \text{ as } k \to \infty.
\]

Then following the proof of [17, Lemma 3.5] for the case when the number of complementarity constraints is 0, we can find some scalar \( M > 0 \) such that

\[
\text{dist}(y^k, \mathcal{F}(x^k)) \leq M \text{dist}(g(x^k, y^k), \mathbb{R}^p)
\]

18
contradicting \([13]\). Hence, the assumption is false and RS for \(F\) holds at \((\bar{x}, \bar{y})\) in direction \(u\). 

Define the Lagrange function of \((P_x)\) by
\[
\mathcal{L}(x, y; \lambda) := f(x, y) + g(x, y)^T \lambda.
\]

From now on in this section we assume that the functions \(f, g\) are continuously differentiable. Then the set of Lagrange multiplier associated with \(y \in F(x)\) is
\[
\Lambda(x, y) := \{\lambda \in \mathbb{R}^p | \nabla_y \mathcal{L}(x, y; \lambda) = 0, \ g(x, y)^T \lambda = 0, \ \lambda \geq 0\}.
\]

### 4.1 Directional derivative of the value function

In this subsection, we study the directional differentiability of the value function. In the following proposition we derive the formula for the directional derivative of the value function. Our result improves the corresponding classical results in \([24, \text{Theorem 5}]\) and \([17, \text{Theorem 3.9}]\) in that weaker assumptions are required and in the formula the directional solution instead of the solution set is used.

**Proposition 4.3** Let \(u\) be a direction such that \(S(\bar{x}; u) \neq \emptyset\) and \(D_+ F(\bar{x}, y; u) \neq \emptyset \ \forall y \in S(\bar{x}; u)\). Suppose that the restricted inf-compactness holds at \(\bar{x}\) in direction \(u\). Moreover assume that \(F(x)\) is RCR-regular at each \(y \in S(\bar{x}; u)\). Then the value function \(V(x)\) is directionally differentiable at \(x = \bar{x}\) in direction \(u\) and
\[
V'(\bar{x}; u) = \min_{y \in S(\bar{x}; u)} \min_{v \in L(\bar{x}, y; u)} \nabla f(\bar{x}, y)(u, v) = \min_{y \in S(\bar{x}; u)} \max_{\lambda \in \Lambda(\bar{x}, y)} \nabla_x \mathcal{L}(\bar{x}, y; \lambda) u. \tag{20}
\]

**Proof.** Since for any given \(y \in S(\bar{x}; u)\), \(D_+ F(\bar{x}, y; u) \neq \emptyset\), there is \(v \in D_+ F(\bar{x}, y; u)\). It follows that there exists \(o(t)\) such that \(y + tv + o(t) \in F(\bar{x} + tu)\) for \(t \geq 0\). Thus we have
\[
V'_+(\bar{x}; u) := \limsup_{t \downarrow 0} \frac{V(\bar{x} + tu) - V(\bar{x})}{t} \leq \limsup_{t \downarrow 0} \frac{f(\bar{x} + tu, y + tv + o(t)) - f(\bar{x}, y)}{t} = \nabla f(\bar{x}, y)(u, v). \tag{21}
\]

On the other hand, let \(t_k \downarrow 0\) be the sequence satisfying
\[
V'(\bar{x}; u) := \liminf_{t \downarrow 0} \frac{V(\bar{x} + tu) - V(\bar{x})}{t} = \lim_{k \to \infty} \frac{V(\bar{x} + t_k u) - V(\bar{x})}{t_k}.
\]

By \([21]\), for any \(\epsilon > 0\) and any sequence \(t_k \downarrow 0\), \(V(\bar{x} + t_k u) < V(\bar{x}) + \epsilon\) for \(k\) large enough. Since the restricted inf-compactness holds at \(\bar{x}\) in direction \(u\) with a compact set \(\Omega_u\), there exists a sequence \(y^k \in S(\bar{x} + t_k u) \cap \Omega_u\) for \(k\) large enough. Without loss of generality, define \(\bar{y} := \lim k y^k\). Then
\[
f(\bar{x}, \bar{y}) = \lim_{k \to \infty} f(\bar{x} + t_k u, y^k) = \lim_{k \to \infty} V(\bar{x} + t_k u) \leq V(\bar{x})
\]
\[
g(\bar{x}, \bar{y}) = \lim_{k \to \infty} g(\bar{x} + t_k u, y^k) \leq 0.
\]

This means \(\bar{y} \in S(\bar{x}) \cap \Omega_u\). Moreover it is clear that \(\bar{y} \in S(\bar{x}; u) \cap \Omega_u\).
Since $F$ is RCR regular at each $y \in S(\bar{x}; u)$ and $D_+F(\bar{x}, y; u) \neq \emptyset$, by Lemma 4.2 we have

$$D_+F(\bar{x}, y; u) = L(\bar{x}, y; u) \quad \forall y \in S(\bar{x}; u).$$

Moreover by Lemma 4.1 for sufficiently large $k$, there exist $\kappa > 0$ independent of $k$ and a sequence $y^k \in F(\bar{x})$ such that

$$\|y^k - \bar{y}^k\| \leq \kappa \|\bar{x} + t_k u - \bar{x}\|, \quad g_j(\bar{x} + t_k u, y^k) - g_j(\bar{x}, \bar{y}^k) \leq 0, \quad j \in I_g(\bar{x}, \bar{y}).$$

Consequently, $\{\frac{y^k - \bar{y}^k}{t_k}\}$ is bounded. Taking a subsequence if necessary, we assume that $\bar{v} := \lim_{k \to \infty} \frac{y^k - \bar{y}^k}{t_k}$ and then $y^k = \bar{y}^k + t_k \bar{v} + o(t_k)$. Thus, we obtain $\nabla g_i(\bar{x}, \bar{y})(u, \bar{v}) \leq 0, \quad i \in I_g(\bar{x}, \bar{y})$. This implies that $\bar{v} \in L(\bar{x}, \bar{y}; u)$. Furthermore, since $\bar{y}^k \in F(\bar{x})$, we have

$$V'(\bar{x}; u) = \lim_{k \to \infty} V(\bar{x} + t_k u) - V(\bar{x})$$

$$\geq \lim_{k \to \infty} \frac{f(\bar{x} + t_k u, y^k) - f(\bar{x}, \bar{y}^k)}{t_k}$$

$$\geq \lim_{k \to \infty} \frac{f(\bar{x} + t_k u, \bar{y}^k + t_k \bar{v} + o(t_k)) - f(\bar{x}, \bar{y}^k)}{t_k}$$

$$= \nabla f(\bar{x}, \bar{y})(u, \bar{v}).$$

It follows that

$$V'(\bar{x}; u) \geq \nabla f(\bar{x}, \bar{y})(u, \bar{v}) \geq \min_{y \in S(\bar{x}; u) \cap \Omega_u} \inf_{v \in L(\bar{x}, y; u)} \nabla f(\bar{x}, y)(u, v).$$

Since (21) holds for any $y \in S(\bar{x}; u) \subseteq S(\bar{x})$ and $v \in D_+F(\bar{x}, y; u) = L(\bar{x}, y; u)$, where the equality follows from (22), we have

$$V'(\bar{x}; u) \leq \inf_{y \in S(\bar{x}; u)} \inf_{v \in L(\bar{x}, y; u)} \nabla f(\bar{x}, y)(u, v) \leq \min_{y \in S(\bar{x}; u) \cap \Omega_u} \inf_{v \in L(\bar{x}, y; u)} \nabla f(\bar{x}, y)(u, v).$$

(24) and (25) imply that

$$V'(\bar{x}; u) \geq \inf_{y \in S(\bar{x}; u)} \inf_{v \in L(\bar{x}, y; u)} \nabla f(\bar{x}, y)(u, v) \geq V'_+(\bar{x}; u).$$

Hence $V(x)$ is directionally differentiable at $\bar{x}$ in direction $u$ with the first equality in (20) holds. And the minimum with respect to $y$ in (21) can be attained on the set $S(\bar{x}; u) \cap \Omega_u$. By the linear programming duality theorem, the second equality in (20) holds and the minimum with respect to $v$ can be attained.

In Proposition 4.3, the sets $S(\bar{x}; u)$ and $D_+F(\bar{x}, y; u)$ are required to be both nonempty. However by Lemma 4.3, this condition can be guaranteed if in addition $F$ satisfies RS in direction $u$. Consequently we have the following corollary. It improves the result of [17, Theorem 3.11] in that the NNAMCQ holding at each $y \in S(\bar{x})$ is replaced by the directional RS which is in general weaker.

**Corollary 4.1** Assume that $F$ is RCR-regular at each $(\bar{x}, y) \in \text{gph}S$. Suppose that the restricted inf-compactness holds at $\bar{x}$ in direction $u$ and RS is satisfied at each $(\bar{x}, y) \in \text{gph}S$ in direction $u$. Then the value function is directionally differentiable in direction $u$ and (20) holds.
In general, according to Corollary 4.1, one needs to ensure both RS and RCR regularity for the existence of the directional derivative. However thanks to Lemma 4.4, if the solution set $S(x)$ is inner semi-continuous, only RCR-regularity is needed.

**Proposition 4.4** Suppose that the solution set $S(x)$ is inner semi-continuous at $(\bar{x}, \bar{y}) \in \text{gph}S$ in direction $u$. Moreover assume that $\mathcal{F}$ is RCR-regular at $(\bar{x}, \bar{y})$. Then the value function $V(x)$ is directionally differentiable at $\bar{x}$ in direction $u$ and

$$V'(\bar{x}; u) = \min_{v \in L(\bar{x}, \bar{y}; u)} \nabla f(\bar{x}, \bar{y})(u, v) = \max_{\lambda \in \Lambda(\bar{x}, \bar{y})} \nabla \lambda L(\bar{x}, \bar{y}, \lambda) u.$$ 

**Proof.** Since $S(x)$ is inner semi-continuous at $(\bar{x}, \bar{y}) \in \text{gph}S$ in direction $u$ we have that the restricted inf-compactness holds at $(\bar{x}, \bar{y})$ in direction $u$ holds and by Lemma 4.4, both RCR and RS holds at $(\bar{x}, \bar{y})$ in direction $u$. By definition of the directional inner semi-continuity of $S(x)$, we can always choose $\tilde{y} = \bar{y}$ in the proof of Proposition 4.3. Hence the result follows from Corollary 4.1. 

### 4.2 Directional Lipschitz continuity of the value function

In this subsection we study sufficient conditions for the directional Lipschitz continuity of $V(x)$.

The classical criterion for guaranteeing the Lipschitz continuity of the value function, is a combination of the uniform compactness condition and MFCQ holding at each $y \in S(\bar{x})$, see e.g. [8] Theorem 5.1. The following theorem gives sufficient conditions for the directional Lipschitz continuity of the value function under weaker assumptions. When $u = 0$, it recovers the result in [21] Theorem 5.5.

**Theorem 4.1** (i) Suppose that $S(\bar{x}; u) \neq \emptyset$, the restricted inf-compactness holds at $\bar{x}$ in direction $u$ with compact set $\Omega_u$ and the feasible map $\mathcal{F}(x) := \{ y | g(x, y) \leq 0 \}$ satisfies RS at $(\bar{x}, \bar{y})$ for each $y \in S(\bar{x}; u) \cap \Omega_u$. Then $V(x)$ is Lipschitz continuous at $\bar{x}$ in direction $u$.

(ii) Suppose there exists $\bar{y} \in S(\bar{x})$ such that $S(x)$ is inner semi-continuous at $(\bar{x}, \bar{y})$ in direction $u$, and the feasible map $\mathcal{F}(x) := \{ y | g(x, y) \leq 0 \}$ satisfies RS at $(\bar{x}, \bar{y})$ in direction $u$. Then $V(x)$ is Lipschitz continuous at $\bar{x}$ in direction $u$.

Furthermore, if (i) or (ii) holds in direction $u = 0$ then $V(x)$ is Lipschitz around $\bar{x}$.

**Proof.** Since RS is satisfied at each $(\bar{x}, y)$ for $y \in S(\bar{x}; u) \cap \Omega_u$ in direction $u$, by the compactness of $\Omega_u$ and Borel-Lebesgue covering theorem, there exist positive scalars $\epsilon, \delta, \kappa$ such that

$$\text{dist}(y, \mathcal{F}(x)) \leq \kappa \text{dist}(g(x, y), \mathbb{R}^n_+) \quad \forall x \in \bar{x} + \mathcal{V}_{\epsilon, \delta}(u), y \in (S(\bar{x}; u) \cap \Omega_u) + \epsilon \mathbb{B}. \quad (26)$$

By Lemma 4.3, choosing $\epsilon, \delta$ small enough, we have for any $x, x' \in \bar{x} + \mathcal{V}_{\epsilon, \delta}(u)$, there exist $y \in S(x) \cap \Omega_u$, $y' \in S(x') \cap \Omega_u$ close enough to $S(\bar{x}; u) \cap \Omega_u$. Without loss of generality assume $x, x' \in \bar{x} + \mathcal{V}_{\epsilon, \delta}(u)$ and $y, y' \in (S(\bar{x}; u) \cap \Omega_u) + \epsilon \mathbb{B}$. Then by (26) we can find $\bar{y} \in \mathcal{F}(x)$, $\bar{y}' \in \mathcal{F}(x')$ such that

$$\| y - \bar{y}' \| \leq \kappa \| g(x', y) - g(x, y) \| \leq 2\kappa \| \nabla_x g(x, y) \| \| x - x' \| ,$$

$$\| y' - \bar{y} \| \leq \kappa \| g(x, y') - g(x', y') \| \leq 2\kappa \| \nabla_x g(x', y') \| \| x - x' \| .$$
Since $\nabla_x g(x, y)$ is continuous and $\{\bar{x} + \mathcal{V}_{e, \delta}(u)\} \times (S(\bar{x}; u) \cap \Omega_u)$ is bounded, by Weierstrass extreme value theorem, there exists a positive scalar $M$ such that $2\kappa \|\nabla_x g(x, y)\| \leq M$ for any $(x, y) \in \{\bar{x} + \mathcal{V}_{e, \delta}(u)\} \times (S(\bar{x}; u) \cap \Omega_u)$. Similarly, since $\nabla f(x, y)$ is continuous, hence, locally bounded. Choosing $M'$ large enough, we have
\[
\|f(x, y) - f(x', y')\| \leq M'\|(x, y) - (x', y')\| \leq M'(1 + M)\|x - x'\|,
\]
\[
\|f(x, \bar{y}) - f(x', y')\| \leq M'\|(x, \bar{y}) - (x', y')\| \leq M'(1 + M)\|x - x'\|.
\]
Then since $f(x, y) - f(x', \bar{y}) \leq V(x) - V(x') \leq f(x, \bar{y}) - f(x', y')$, we have
\[
\|V(x) - V(x')\| \leq \max\{\|f(x, y) - f(x', \bar{y})\|, \|f(x, \bar{y}) - f(x', y')\|\} \leq M'(1 + M)\|x - x'\|.
\]
This means $V(x)$ is Lipschitz continuous at $\bar{x}$ in direction $u$ and (i) is proved.

Next, we prove (ii). If there exists $\bar{y} \in S(\bar{x})$ such that $S(x)$ is inner semi-continuous at $(\bar{x}, \bar{y})$ in direction $u$, $\bar{y} \in S(\bar{x}; u) \neq \emptyset$ and the restricted inf-compactness holds at $\bar{x}$ in direction $u$. Then one can easily replace $S(\bar{x}; u) \cap \Omega_u$ by $\{\bar{y}\}$ in the proof above and obtain the Lipschitz continuity of $V(x)$ under RS at $(\bar{x}, \bar{y})$ in direction $u$.

\subsection{Directional subdifferentials of the value function}

In this subsection, we study the analytic directional subdifferential of the value function of $(P_2)$. First, we derive an upper estimate for the analytic directional subdifferential of the value function in terms of the problem data. For any $x, y, u$, suppose $V'(x; u)$ exists. We denote by
\[
\Sigma(x, y, u) := \{v \in \mathbb{L}(x, y; u) | V'(x; u) = \nabla f(x, y)(u, v)\}. \tag{27}
\]

\textbf{Theorem 4.2} Let $u \in \mathbb{R}^n$.

(i) Suppose that the restricted inf-compactness holds at $\bar{x}$ in direction $u$ with compact set $\Omega_u$. Suppose that $V(x)$ is directionally differentiable at $\bar{x}$ in direction $u$. Then $S(\bar{x}; u) \neq \emptyset$. Moreover suppose that the feasible map $\mathcal{F}(x) := \{y | g(x, y) \leq 0\}$ satisfies RS at $(\bar{x}, \bar{y})$ in direction $u$ for each $y \in S(\bar{x}; u) \cap \Omega_u$. Then $V(x)$ is Lipschitz at $\bar{x}$ in direction $u$ and
\[
\emptyset \neq \partial_u V(\bar{x}; u) \subseteq \Theta(\bar{x}; u) \tag{28}
\]
where
\[
\Theta(\bar{x}; u) := \bigcup_{y \in S(\bar{x}; u) \cap \Omega_u} \left( \bigcup_{v \in \Sigma(\bar{x}, y, u)} \left\{ \nabla_x f(\bar{x}, \bar{y}) + \nabla_x g(\bar{x}, \bar{y})^T \lambda_g | \lambda_g \in \Lambda(\bar{x}, \bar{y}) \cap \{\nabla g(\bar{x}, \bar{y})(u, v)\}^\perp \right\} \bigcup \bigcup_{v \in \Sigma(\bar{x}, \bar{y}; 0) \cap \mathbb{S}} \left\{ \nabla_x f(\bar{x}, \bar{y}) + \nabla_x g(\bar{x}, \bar{y})^T \lambda_g | \lambda_g \in \Lambda(\bar{x}, \bar{y}) \cap \{\nabla g(\bar{x}, \bar{y})(0, v)\}^\perp \right\} \right). \tag{29}
\]

(ii) Suppose that there exists $\bar{y} \in S(\bar{x})$ such that $S(x)$ is inner semi-continuous at $(\bar{x}, \bar{y})$ in direction $u$ and the feasible map $\mathcal{F}(x) := \{y | g(x, y) \leq 0\}$ satisfies RS at $(\bar{x}, \bar{y})$ in direction $u$. Suppose that $V(x)$ is directionally differentiable at $\bar{x}$ in direction $u$. Then $V(x)$ is Lipschitz at $\bar{x}$ in direction $u$. And (28) holds with the union over $\bar{y} \in S(\bar{x}; u) \cap \Omega_u$ superfluous and $\bar{y} = \bar{y}$.\]
(iii) Suppose that there exists \( \bar{y} \in S(\bar{x}) \) such that \( S(x) \) is inner semi-continuous at \((\bar{x}, \bar{y})\) in direction \( u \) and \( F \) is RCR-regular at \((\bar{x}, \bar{y})\) in direction \( u \), then \( V(x) \) is Lipschitz at \( \bar{x} \) in direction \( u \) and

\[
\emptyset \neq \partial_v V(\bar{x}; u) \subseteq \bigcup_{v \in \Sigma(\bar{x}, \bar{y}, u)} \left\{ \nabla_x f(\bar{x}, \bar{y}) + \nabla_y f(\bar{x}, \bar{y})^T \lambda_y \mid \lambda_y \in \Lambda(\bar{x}, \bar{y}) \cap \{ \nabla g(\bar{x}, \bar{y})(u, v) \}^\perp \right\}.
\]

**Proof.** (i) Since \( V(x) \) is diffeirentiable in direction \( u \), for any sequence \( \epsilon_k \downarrow 0 \), we can find a sequence \( t_k \downarrow 0 \) such that for \( k \) large enough, we have \( V(\bar{x} + t_k u) < V(\bar{x}) + \epsilon_k \). Then by the assumption of the directional restricted inf-compactness, for \( k \) large enough, there exists \( \bar{g}^k \in S(\bar{x} + t_k u) \cap \Omega_u \). Then \( \{ \bar{g}^k \} \) is bounded. Without loss of generality, there exists \( \bar{y} = \lim_k \bar{g}^k \). And we know that \( f(\bar{x}, \bar{y}) = \lim_k V(\bar{x} + t_k u) \leq V(\bar{x}) \). Hence, \( \bar{y} \in S(\bar{x}; u) \neq \emptyset \).

Since the directional restricted inf-compactness holds at \( \bar{x} \) in direction \( u \) and RS is satisfied at \((\bar{x}, \bar{y})\) in direction \( u \) for each \( y \in S(\bar{x}; u) \cap \Omega_u \), by Theorem 4.1, \( V(x) \) is Lipschitz continuous at \( \bar{x} \) in direction \( u \). Then by the well-known Rademacher's Theorem and [27, Theorem 9.13], \( \partial_u V(\bar{x}; u) \neq \emptyset \).

Let \( \zeta \in \partial_u V(\bar{x}; u) \). Then by definition, there exist sequences \( t_k \downarrow 0 \), \( u^k \to u \), \( \zeta^k \to \zeta \) such that \( V(\bar{x} + t_k u^k) \to V(\bar{x}) \) and \( \zeta^k \in \partial V(\bar{x} + t_k u^k) \). It follows that \( V(\bar{x} + t_k u^k) < V(\bar{x}) + \epsilon \) for all \( k \) large enough and hence by the directional restricted inf-compactness, there exists \( y^k \in S(\bar{x} + t_k u^k) \cap \Omega_u \). Passing to a subsequence if necessary, we may assume that \( y^k \to \bar{y} \). Hence, by the continuity of \( f(x, y) \), \( \bar{y} \in S(\bar{x}; u) \cap \Omega_u \).

For each \( k \), since \( \zeta^k \in \partial V(\bar{x} + t_k u^k) \), there exists a neighborhood \( U^k \) of \( \bar{x} + t_k u^k \) satisfying

\[
V(x) - V(\bar{x} + t_k u^k) \geq 0 \forall x \in U^k.
\]

It follows from the fact \( V(x) = \inf_y \{ f(x, y) + \delta_{R^m}(g(x, y)) \} \) and \( y^k \in S(\bar{x} + t_k u^k) \), that

\[
f(x, y) - (\zeta^k, x - (\bar{x} + t_k u^k)) + \frac{1}{k} \| x - (\bar{x} + t_k u^k) \| + \delta_{R^m}(g(x, y)) \geq f(\bar{x} + t_k u^k, y^k),
\]

for any \( (x, y) \in U^k \times \mathbb{R}^m \). Hence the function

\[
\phi_k(x, y) := f(x, y) - (\zeta^k, x - (\bar{x} + t_k u^k)) + \frac{1}{k} \| x - (\bar{x} + t_k u^k) \| + \delta_{R^m}(g(x, y))
\]

attains its local minimum at \((x, y) = (\bar{x} + t_k u^k, y^k)\). Thus, by the well known Fermat’s rule and the sum rule ([27, Exercise 10.10]),

\[
0 \in \nabla f(\bar{x} + t_k u^k, y^k) - (\zeta^k, 0) + \frac{1}{k} \mathbb{B} \times \{ 0 \} + \partial (\delta_{R^m} \circ g)(\bar{x} + t_k u^k, y^k).
\]

Now we consider two cases.

Case (a): \( \{ \frac{y^k - \bar{y}}{t_k} \} \) is bounded. Define \( v^k := \frac{y^k - \bar{y}}{t_k} \). Passing to a subsequence if necessary, there exists \( v \in \mathbb{R}^m \) such that \( v^k \to v \). Since \( y^k \in S(\bar{x} + t_k u^k) \), it follows that

\[
\nabla f(\bar{x}, \bar{y})(u, v) = \lim_k \frac{f(\bar{x} + t_k u^k, y^k) - f(\bar{x}, \bar{y})}{t_k} = \lim_k \frac{V(\bar{x} + t_k u^k) - V(\bar{x})}{t_k} = V'(\bar{x}; u)
\]

and \( \nabla g_i(\bar{x}, \bar{y})(u, v) \leq 0, \forall i \in I_g(\bar{x}, \bar{y}) \). This means \( v \in \Sigma(\bar{x}, \bar{y}; u) \). Since for each \( k \), \( \delta_{R^m}(g(\bar{x} + t_k u^k, y^k)) = 0 = \delta_{R^m}(g(\bar{x}, \bar{y})) \), taking limits as \( k \to \infty \) in [41], by Proposition 2.2, we have

\[
0 \in \nabla f(\bar{x}, \bar{y}) - (\zeta, 0) + \partial_a (\delta_{R^m} \circ g)(\bar{x}, \bar{y}; u, v).
\]
Since RS is satisfied at \((\bar{x}, \bar{y})\) in direction \(u\), by Proposition 4.2 the metric subregularity for the system \(g(x, y) \in \mathbb{R}^p\) holds at \((\bar{x}, \bar{y}, 0)\) in direction \((u, v)\), then by Proposition 2.6 we have

\[
\partial_u (\delta_{\mathbb{R}^p} \circ g)(\bar{x}, \bar{y}; u, v) \subseteq \nabla g(\bar{x}, \bar{y})^T N_{\mathbb{R}^p_+}(g(\bar{x}, \bar{y}); \nabla g(\bar{x}, \bar{y}))(u, v).
\]

With \(g\), we obtain there exists \(\lambda_g \in N_{\mathbb{R}^p_+}(g(\bar{x}, \bar{y}); \nabla g(\bar{x}, \bar{y}))(u, v)\) with \(0 \in \nabla g_f(\bar{x}, \bar{y}) + \nabla g(\bar{x}, \bar{y})^T \lambda_g\) such that 
\[
\lambda = \nabla x f(\bar{x}, \bar{y}) + \nabla x g(\bar{x}, \bar{y})^T \lambda_g.
\]

The proof follows from

\[
N_{\mathbb{R}^p_+}(g(\bar{x}, \bar{y}); \nabla g(\bar{x}, \bar{y}))(u, v) = \{ \lambda \in \mathbb{R}^p | 0 \leq \lambda \perp g(\bar{x}, \bar{y}), \lambda \perp \nabla g(\bar{x}, \bar{y})(u, v) \}.
\]

Case (b): \(\{ \frac{y^k - \bar{y}}{\tau_k} \}\) is unbounded. Without loss of generality, assume \(\lim_{k \to \infty} \frac{\|y^k - \bar{y}\|}{\tau_k} = \infty\). Define \(\tau_k := \|y^k - \bar{y}\|\). Then \(\frac{y^k}{\tau_k} \downarrow 0\). Since the sequence \(\{ \frac{y^k - \bar{y}}{\tau_k} \}\) is bounded, passing to a subsequence if necessary, assume there exist \(v \in S\) and a sequence \(v^k \to v\) such that 
\[
y^k = \bar{y} + \tau_k v^k.
\]

Define \(\bar{u}^k := \frac{u}{\tau_k} u^k\). Then \(\bar{x} + t_k u^k = \bar{x} + \tau_k \bar{u}^k\) and \(\bar{u}^k \to 0\). Since \(y^k \in S(\bar{x} + t_k u^k)\), it follows that

\[
0 = \lim_k \frac{V(\bar{x} + \tau_k \bar{u}^k) - V(\bar{x})}{t_k} = \lim_k \frac{f(\bar{x} + \tau_k \bar{u}^k, \bar{y} + \tau_k v^k) - f(\bar{x}, \bar{y})}{\tau_k} = \nabla f(\bar{x}, \bar{y})(0, v)
\]

and \(\nabla g_2(\bar{x}, \bar{y})(0, v) \leq 0\), \(\forall i \in I_g(\bar{x}, \bar{y})\). Since \(V'(\bar{x}; 0) = 0\), we obtain \(V'(\bar{x}; 0) = \nabla f(\bar{x}, \bar{y})(0, v)\). Hence, \(v \in S(\bar{x}, \bar{y}) \cap \Sigma\). Taking limits as \(k \to \infty\) in \(g\), following a similar process as in Case (a) we have

\[
0 \in \nabla f(\bar{x}, \bar{y}) - (\zeta, 0) + \partial(\delta_{\mathbb{R}^p} \circ g)(\bar{x}, \bar{y}; 0, v)
\]

\[
\subseteq \nabla f(\bar{x}, \bar{y}) - (\zeta, 0) + \nabla g(\bar{x}, \bar{y})^T N_{\mathbb{R}^p_+}(g(\bar{x}, \bar{y}); \nabla g(\bar{x}, \bar{y}))(0, v)).
\]

So there exists \(\lambda_g \in N_{\mathbb{R}^p_+}(g(\bar{x}, \bar{y}); \nabla g(\bar{x}, \bar{y}))(0, v)\) with \(0 = \nabla y f(\bar{x}, \bar{y}) + \nabla y g(\bar{x}, \bar{y})^T \lambda_g\) such that 
\[
\zeta = \nabla x f(\bar{x}, \bar{y}) + \nabla x g(\bar{x}, \bar{y})^T \lambda_g.
\]

This completes the proof.

(ii) When \(S(x)\) is inner semi-continuous at some point \(\bar{y} \in S(\bar{x})\) in direction \(u\), one can choose \(\bar{y} = \bar{y}\). And the results follows similarly as the proof of (i).

(iii) Let \(\zeta \in \partial_a V(\bar{x}; u)\). As in the proof of (i) and taking into account the inner semi-continuity of \(S(x)\) at \((\bar{x}, \bar{y})\) in direction \(u\), we obtain \(t_k \downarrow 0\), \(u^k \to u\), \(\zeta^k \to \zeta\), \(y^k \in S(\bar{x} + t_k u^k)\), \(y^k \to \bar{y}\) satisfying \(g\). By Lemma 4.4 and Theorem 4.1, RS holds at \(\bar{x}\) in direction \(u\) and \(V(x)\) is Lipschitz continuous at \(\bar{x}\) in direction \(u\). Then by Proposition 4.2 we have metric subregularity for the system \(g(x, y) \in \mathbb{R}^p\) holds at \(\bar{x}\) in direction \(u\). Hence, by Proposition 2.6 for sufficiently large \(k\), we have

\[
\partial(\delta_{\mathbb{R}^p} \circ g)(\bar{x} + t_k u^k, y^k) \subseteq \nabla g(\bar{x} + t_k u^k, y^k)^T N_{\mathbb{R}^p_+}(g(\bar{x} + t_k u^k, y^k)).
\]

Hence

\[
0 \in \nabla f(\bar{x} + t_k u^k, y^k) - (\zeta^k, 0) + \frac{1}{k} \mathbb{B} \times \{0\} + \nabla g(\bar{x} + t_k u^k, y^k)^T N_{\mathbb{R}^p_+}(g(\bar{x} + t_k u^k, y^k)) = (31)
\]

Since RCR-regularity holds at \((\bar{x}, \bar{y})\) and \(y^k \in S(\bar{x} + t_k u^k)\), by Lemma 4.1 for sufficiently large \(k\), there exist \(\kappa > 0\) independent of \(k\) and a sequence \(\bar{y}^k \in F(\bar{x})\) such that

\[
\|y^k - \bar{y}^k\| \leq \kappa \|\bar{x} + t_k u^k - \bar{x}\|, \quad g_j(\bar{x} + t_k u^k, y^k) \leq g_j(\bar{x}, \bar{y}^k), \quad j \in I_g(\bar{x}, \bar{y}).
\]
Then $I_g(\bar{x} + t_k u^k, y^k) \subseteq I_g(\bar{x}, y^k)$ and by Proposition 2.1,

$$N_{R^p}(g(\bar{x} + t_k u^k, y^k)) = N_{R^p}(g(\bar{x}, y^k)) \cap [g(\bar{x}, y^k) - g(\bar{x} + t_k u^k, y^k)]_{\perp}$$

$$= N_{R^p}(g(\bar{x}, y^k)) \cap [\frac{g(\bar{x}, y^k) - g(\bar{x} + t_k u^k, y^k)}{t_k}]_{\perp}.$$ 

Define $v^k := \frac{2}{t_k} g(\bar{x}, y^k) - t_k v^k$. By (22), $\{v^k\}$ is bounded. Without loss of generality, there exists $v = \lim_k v^k$. Then $\lim_k \frac{g(\bar{x} + t_k u^k, y^k) - g(\bar{x}, y^k)}{t_k} = \nabla g(\bar{x}, y^k)(u, v)$. By (22), $y^k \to y$ and $v \in L(\bar{x}, y; u)$. Taking the limit in (31), we have

$$0 \in \nabla f(\bar{x}, y) - (\zeta, 0) + \nabla g(\bar{x}, y)^T (N_{R^p}(g(\bar{x}, y)) \cap [\nabla g(\bar{x}, y)(u, v)]_{\perp}).$$

We obtain the existence of $\lambda_y \in N_{R^p}(g(\bar{x}, y)) \cap [\nabla g(\bar{x}, y)(u, v)]_{\perp}$ such that $\zeta = \nabla_x f(\bar{x}, y) + \nabla_x g(\bar{x}, y)^T \lambda_y$. Furthermore, by Proposition 2.1 $V'(\bar{x}; u) = \min_{\nu \in L(\bar{x}, y; u)} \{\nabla f(\bar{x}, y)(u, \nu)\}$. It follows from a similar process as (24), we have $V'(\bar{x}; u) = \nabla f(\bar{x}, y)(u, v)$. The proof is complete.

Theorem 5.10 and 5.11 also gave an upper estimate of the value function of constrained programs in terms of the coderivatives of the constraint mapping $F$ under a stronger version of directional inner semicontinuity [22, Definition 4.4(i)] of $S(x)$. Our result cannot be obtained from [22, Theorems 5.10 and 5.11] and is in a more explicit form.

The following theorem provides an estimate of the directional Clarke subdifferential of the value function which will be used in the necessary optimality condition for bilevel programs. We give some notations. For any given $(x, y, u, v)$ we define the set

$$W(x, y, u, v) := \{\nabla_x f(x, y) + \nabla_x g(x, y)^T \lambda_g \mid \lambda_g \in \Lambda(x, y) \cap \{\nabla g(x, y)(u, v)\}_{\perp}\}.$$

**Theorem 4.3** Under the assumptions of Theorem 4.2(i), we have

$$\partial^+ V(\bar{x}; u) \subseteq \text{co} \bigcup_{y \in S(\bar{x}; u)} (W(\bar{x}, y, u, v) \cup W(\bar{x}, y, 0, v)) \quad \forall v \in \Sigma(\bar{x}, y, u), v \in \Sigma(\bar{x}, y, 0) \cap S.$$

Under the assumptions of Theorem 4.2(ii), we have

$$\partial^+ V(\bar{x}; u) \subseteq \{\mu \zeta + (1 - \mu)\xi \mid 0 \leq \mu \leq 1, \zeta \in W(\bar{x}, y, u, v), \xi \in W(\bar{x}, y, 0, v)\}$$

for $\forall v \in \Sigma(\bar{x}, y, u), v \in \Sigma(\bar{x}, y, 0) \cap S$.

Under the assumptions of Theorem 4.2(iii), we have

$$\partial^+ V(\bar{x}; u) \subseteq W(\bar{x}, y, u, v), \quad \forall v \in \Sigma(\bar{x}, y, u).$$

**Proof.** Since $\partial^+ V(\bar{x}; u) = \text{co} \partial a V(\bar{x}; u)$, by Theorem 1.2 we only need to show that $W(x, y, u, v_1) = W(x, y, u, v_2)$ for any $v_1, v_2 \in \Sigma(x, y, u)$. Let

$$C(x, y, u, v) := \{\lambda_g \nabla_y f(x, y) + \lambda_g \nabla_y g(x, y) - 0, 0 \leq \lambda_g \perp g(x, y), \lambda_g \perp \nabla g(x, y)(u, v)\}.$$ 

It suffices to show that $C(x, y, u, v_1) = C(x, y, u, v_2)$ for any $v_1, v_2 \in \Sigma(x, y, u)$. By $\nabla_y f(\bar{x}, y) + \nabla_y g(\bar{x}, y)^T \lambda_y = 0$, we have $\lambda_T^y \nabla_y g(x, y) v_1 = -\nabla_y f(x, y) v_1$ for $i = 1, 2$. And since $\nabla_y f(x, y) v_1 = \nabla_y f(x, y) v_2 = V'(x)(u) - \nabla_x f(x, y) u, \lambda_T^y \nabla_y g(x, y) v_1 = \lambda_T^y \nabla_y g(x, y) v_2$. Hence, $C(x, y, u, v_1) = C(x, y, u, v_2)$. This implies $C(x, y, u, v_1) = C(x, y, u, v_2)$.
5 Necessary optimality conditions for bilevel programs

The main purpose of this section is to apply Theorem 3.1 to problem (VP) and the result of the directional sensitivity analysis of the value functions in Section 4 to derive a sharp necessary optimality condition for (VP) under a weak and verifiable constraint qualification.

We first try to answer the question on whether it is possible for FOSCMS to hold at a feasible point of (VP). Given \((u, v) \in \mathbb{R}^{n+m}\), define the set-valued map \(M_{(u,v)} : \mathbb{R}^n \times \mathbb{R}^m \Rightarrow \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^q\) by

\[
M_{(u,v)}(x,y) := (f(x,y) - V(x) + \langle u, x - \bar{x} \rangle^3 + \langle v, y - \bar{y} \rangle^3 - \mathbb{R}_-, g(x,y) - \mathbb{R}_+, G(x,y) - \mathbb{R}_-).
\]

Assume that the value function is directionally differentiable at \(\bar{x}\) in direction \(u\). Define the linearization cone of (VP) at \((\bar{x}, \bar{y})\) by

\[
\mathbb{L}(\bar{x}, \bar{y}) := \left\{(u,v) \mid \nabla f(\bar{x}, \bar{y})(u,v) \leq V'(\bar{x}; u), \nabla g_i(\bar{x}, \bar{y})(u,v) \leq 0 \forall i \in I_g(\bar{x}, \bar{y}), \nabla G_i(\bar{x}, \bar{y})(u,v) \leq 0 \forall i \in I_G(\bar{x}, \bar{y}) \right\}.
\]

Note that although \(f(x,y) - V(x) \leq 0\) is an inequality, it is in fact an equality constraint by the definition of the value function. Hence under the Abadie constraint qualification, one always have \(\nabla f(\bar{x}, \bar{y})(u,v) \geq V'(\bar{x}; u)\) for all \((u,v)\) satisfying \(\nabla g_i(\bar{x}, \bar{y})(u,v) \leq 0 \forall i \in I_g(\bar{x}, \bar{y})\). Therefore if the Abadie constraint qualification holds, in the linearization cone the inequality \(\nabla f(\bar{x}, \bar{y})(u,v) \leq V'(\bar{x}; u)\) can be equivalently replaced by the equality.

**Lemma 5.1** Let \((\bar{x}, \bar{y})\) be a feasible point of (VP). Assume that the value function is directionally differentiable at \(\bar{x}\) in any direction \(u\) and \(0 \neq (u,v) \in \mathbb{L}(\bar{x}, \bar{y})\). Then \(M_{(u,v)}(x,y)\) is not metrically subregular at \(((\bar{x}, \bar{y}), (0,0)) \in \text{gph}M_{(u,v)}\) in direction \((u,v)\).

**Proof.** To concentrate on the main idea we omit the upper level constraint \(G(x,y) \leq 0\) in the proof. To the contrary, suppose that \(M_{(u,v)}(x,y)\) is metrically subregular at \(((\bar{x}, \bar{y}), (0,0))\) in the nonzero direction \((u,v) \in \mathbb{L}(\bar{x}, \bar{y})\). Then by definition of metric subregularity in direction \((u,v)\), \(\exists \kappa > 0\), for all sequences \(t_k \downarrow 0, u^k \to u, v^k \to v\), we have for sufficiently large \(k\)

\[
\text{dist} \left( (\bar{x} + t_k u^k, \bar{y} + t_k v^k), M_{(u,v)}^{-1}(0,0) \right) \leq \kappa \text{dist}(0, M_{(u,v)}(\bar{x} + t_k u^k, \bar{y} + t_k v^k))
\]

\[
\leq \kappa \left( \text{dist}(f(\bar{x} + t_k u^k, \bar{y} + t_k v^k) - V(\bar{x} + t_k u^k) + t_k^3 \langle u, u^k \rangle^3 + t_k^3 \langle v, v^k \rangle^3, \mathbb{R}_-) + \text{dist}(g(\bar{x} + t_k u^k, \bar{y} + t_k v^k), \mathbb{R}_+) \right).
\]

Since \((u,v) \in L(\bar{x}, \bar{y})\), we have \(g(\bar{x}, \bar{y}) + t_k \nabla g(\bar{x}, \bar{y})(u,v) \leq 0\). Hence,

\[
\lim_{k \to \infty} \frac{\text{dist}(g(\bar{x} + t_k u^k, \bar{y} + t_k v^k), \mathbb{R}_+)}{t_k} \leq \lim_{k \to \infty} \frac{\|g(\bar{x} + t_k u^k, \bar{y} + t_k v^k) - t_k \nabla g(\bar{x}, \bar{y})(u,v) - g(\bar{x}, \bar{y})\|}{t_k} = 0.
\]

Similarly, since \(f(\bar{x}, \bar{y}) - V(\bar{x}) = 0\), we have \(\nabla f(\bar{x}, \bar{y})(u,v) - V'(\bar{x}; u) \leq 0\),

\[
\lim_{k \to \infty} \frac{\text{dist}(f(\bar{x} + t_k u^k, \bar{y} + t_k v^k) - V(\bar{x} + t_k u^k) + t_k^3 \langle u, u^k \rangle^3 + t_k^3 \langle v, v^k \rangle^3, \mathbb{R}_-)}{t_k} = 0.
\]

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Proposition 5.1 Assume that the value function is directionally differentiable at \( \bar{x} \) in any direction \( u \) and \((u,v) \in C(\bar{x}, \bar{y}) \). Then there exists a nonzero vector \((\lambda, \mu, \nu) \in \mathbb{R}^{1+p+q} \) such that \( \lambda \geq 0, 0 \leq \mu \perp g(\bar{x}, \bar{y}), \mu \perp \nabla g(\bar{x}, \bar{y})(u,v), 0 \leq \nu \perp G(\bar{x}, \bar{y}), \nu \perp \nabla G(\bar{x}, \bar{y})(u,v) \) and

\[
0 \in \lambda \partial_a (f - V)(\bar{x}, \bar{y}; (u,v)) + \nabla g(\bar{x}, \bar{y})^T \mu + \nabla G(\bar{x}, \bar{y})^T \nu.
\]

Hence FOSCMS fails at any feasible solution of (VP) in any critical direction.

Proof. Since by Lemma 3.1 \( M_{(u,v)}(x, y) \), hence \(-M_{(u,v)}(x, y)\), is not metrically subregular at \((\bar{x}, \bar{y}, 0, 0)\) in direction \((u,v)\) and metric subregularity is weaker than FOSCMS, FOSCMS for the inequality system

\[
\psi(x, y) := (f(x, y) - V(x) + \langle u, x - \bar{x} \rangle^3 + \langle v, y - \bar{y} \rangle^3), g(x, y), G(x, y) \leq 0
\]

must fail at \((\bar{x}, \bar{y})\) in direction \((u,v)\). By the sum rule [22 Theorem 5.6] of analytic directional subdifferential,

\[
\partial_a (f(x, y) - V(x) + \langle u, x - \bar{x} \rangle^3 + \langle v, y - \bar{y} \rangle^3)(\bar{x}, \bar{y}; (u,v)) = \partial_a (f - V)(\bar{x}, \bar{y}; (u,v)).
\]

Hence by Definition 3.2(2) the FOSCMS for the inequality system \(\psi(x, y) \leq 0\) at \(((\bar{x}, \bar{y}), (0, 0))\) is the same as the \(\psi\) which means that FOSCMS for (VP) at \((\bar{x}, \bar{y})\) in direction \((u,v)\) fails.

We now apply Lemma 3.1, Proposition 3.1 and Theorem 3.1 to (VP) and obtain the following necessary optimality condition for the bilevel program (BP).
**Theorem 5.1** Let \((\bar{x}, \bar{y})\) be a local minimizer of (BP). Suppose that the value function \(V(x)\) is Lipschitz continuous and directionally differentiable near \(\bar{x}\) in direction \(u\) and \((u, v) \in C(\bar{x}, \bar{y})\). Moreover suppose that the directional quasi-normality holds at \((\bar{x}, \bar{y})\) in direction \((u, v)\), i.e., there exists no nonzero vector \((\alpha, \nu_g, \nu_G) \in \mathbb{R}_+^{1+p+q}\) and

\[
0 \in \alpha \nabla f(\bar{x}, \bar{y}) - \alpha \partial^f V(\bar{x}; u) \times \{0\} + \nabla g(\bar{x}, \bar{y})^T \nu_g + \nabla G(\bar{x}, \bar{y})^T \nu_G, \tag{37}
\]

\[
\nu_g \perp g(\bar{x}, \bar{y}), \quad \nu_g \perp \nabla g(\bar{x}, \bar{y})(u, v), \quad \nu_G \perp G(\bar{x}, \bar{y}), \quad \nu_G \perp \nabla G(\bar{x}, \bar{y})(u, v), \tag{38}
\]

and there exists sequences \(t_k \downarrow 0\), \((u^k, v^k) \to (u, v)\) such that

\[
\alpha (f(\bar{x} + t_k u^k, \bar{y} + t_k v^k) - V(\bar{x} + t_k u^k)) > 0, \text{ if } \alpha > 0, \tag{39}
\]

\[
g_i(\bar{x} + t_k u^k, \bar{y} + t_k v^k) > 0, \text{ if } (\nu_g)_i > 0, i \in I_g, \tag{40}
\]

\[
G_i(\bar{x} + t_k u^k, \bar{y} + t_k v^k) > 0, \text{ if } (\nu_G)_i > 0, i \in I_G.
\]

Then the directional KKT condition holds. That is, there exists \((\lambda_V, \lambda_g, \lambda_G)\) such that

\[
0 \in \nabla F(\bar{x}, \bar{y}) + \lambda_V \nabla f(\bar{x}, \bar{y}) - \lambda_V \partial^f V(\bar{x}; u) \times \{0\} + \nabla g(\bar{x}, \bar{y})^T \lambda_g + \nabla G(\bar{x}, \bar{y})^T \lambda_G,
\]

\[
\lambda_V \geq 0, \quad 0 \leq \lambda_g \perp g(\bar{x}, \bar{y}), \quad \lambda_g \perp \nabla g(\bar{x}, \bar{y})(u, v), \quad 0 \leq \lambda_G \perp G(\bar{x}, \bar{y}), \quad \lambda_G \perp \nabla G(\bar{x}, \bar{y})(u, v).
\]

**Proof.** Define \(\phi(x, y) := (f(x, y) - V(x), g(x, y), G(x, y))\) and \(\lambda_\phi := (\alpha, \nu_g, \nu_G)\). Then by assumption, \(\phi(x, y)\) is Lipschitz continuous and directionally differentiable at \((\bar{x}, \bar{y})\) in direction \((u, v)\). Since \((u, v) \in C(\bar{x}, \bar{y})\), we have \(\nabla \phi(\bar{x}, \bar{y})(u, v) \leq 0\). Then since \(f(\bar{x}, \bar{y}) - V(\bar{x}) = 0, g(\bar{x}, \bar{y}) \leq 0, G(\bar{x}, \bar{y}) \leq 0\), this means \(0 \leq \lambda_\phi \perp \phi(\bar{x}, \bar{y})\) and \(\lambda_\phi \perp \nabla \phi(\bar{x}, \bar{y})(u, v)\).

Thus, we have

\[
\begin{align*}
\partial_a (f - V)(\bar{x}, \bar{y}; (u, v)) &= \nabla f(\bar{x}, \bar{y}) + \partial_a (-V)(\bar{x}; u) \times \{0\} \\
&\subseteq \nabla f(\bar{x}, \bar{y}) + \partial^f (-V)(\bar{x}; u) \times \{0\} \\
&\subseteq \nabla f(\bar{x}, \bar{y}) - \partial^f V(\bar{x}; u) \times \{0\},
\end{align*}
\]

where the first equation follows from [22] Theorem 5.6 and the second inclusion follows from Proposition 228, (37)-(40) imply that the directional quasinormality defined in Definition 5.23 holds. Applying Theorem 5.1, the proof is complete. \(\blacksquare\)

When the conditions in Corollary 4.1 and Theorems 4.2(i) hold, one can apply the formulas of \(V'(\bar{x}; u)\) and the upper estimates for \(\partial^f V(\bar{x}; u)\) obtained in section 4 and derive the directional KKT condition in terms of the problem data under the directional quasinormality as below.

**Theorem 5.2** Let \((\bar{x}, \bar{y})\) be a local minimizer of (BP) and \(u \in \mathbb{R}^p\). Suppose that the feasible map \(F(x) := \{y | g(x, y) \leq 0\}\) is RCR-regular at each \((\bar{x}, \bar{y}) \in gphS\) and satisfies RS at each \((\bar{x}, \bar{y}) \in gphS\) in direction \(u\). Moreover assume that the restricted inf-compactness holds at \(\bar{x}\) in direction \(u\). Then the value function \(V(x)\) is Lipschitz continuous and directionally differentiable at \(\bar{x}\) in direction \(u\) with

\[
V'(\bar{x}; u) = \min_{y \in S(\bar{x}; u)} \max_{\lambda \in A(\bar{x}, \bar{y})} \nabla_x \mathcal{L}(\bar{x}, y; \lambda)u,
\]

\[
\partial_a V(\bar{x}; u) \subseteq \Theta(\bar{x}; u),
\]

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where $\Theta(x;u)$ is defined as in $[24]$. Suppose that the directional quasi-normality holds at $(\bar{x}, \bar{y})$ in direction $(u, v) \in C(\bar{x}, \bar{y})$ in Theorem 5.1 with $\partial^c V(\bar{x}; u)$ replaced by $\text{co}(\Theta(\bar{x}; u))$. Then the directional KKT condition and Theorem 5.7 holds with $\partial^c V(\bar{x}; u)$ replaced by $\text{co}(\Theta(\bar{x}; u))$.

When the solution map $S(x)$ is directionally inner semi-continuous at the point of interest, we can obtain the directional quasi-normality condition and the KKT condition of (VP) in the following more verifiable forms.

**Theorem 5.3** Let $(\bar{x}, \bar{y})$ be a local minimizer of (BP). Suppose that the feasible map $F(x)$ is RCR-regular at $(\bar{x}, \bar{y})$ and $S(x)$ is inner semi-continuous at $(\bar{x}, \bar{y})$ in direction $u$. Then the value function is Lipschitz continuous and directional differentiable at $\bar{x}$ in direction $u$ and $V'(\bar{x}; u) = \max_{\lambda \in \Lambda(\bar{x}, \bar{y})} \nabla_x L(\bar{x}, \bar{y}; \lambda) u$. Suppose that there exists $v$ such that $(u, v) \in C(\bar{x}, \bar{y})$. Furthermore suppose that there exists no nonzero vector $(\alpha, \nu_g, \nu_G) \in \mathbb{R}^{1+p+q+p}$ satisfying

$$0 < \alpha \nabla f(\bar{x}, \bar{y}) - \alpha W(\bar{x}, \bar{y}, u, v) \times \{0\} + \nabla g(\bar{x}, \bar{y})^T \nu_g + \nabla G(\bar{x}, \bar{y})^T \nu_G,$$

$$\alpha \geq 0, \ 0 \leq \nu_g \perp g(\bar{x}, \bar{y}), \ \nu_g \perp \nabla g(\bar{x}, \bar{y})(u, v), \ 0 \leq \nu_G \perp G(\bar{x}, \bar{y}), \ \nu_G \perp \nabla G(\bar{x}, \bar{y})(u, v),$$

where $W(\bar{x}, \bar{y}, u, v) = \{\nabla_x f(\bar{x}, \bar{y}) + \nabla_x g(\bar{x}, \bar{y})^T \lambda | \lambda \in \Lambda(\bar{x}, \bar{y}) \cap \{\nabla g(\bar{x}, \bar{y})(u, v)\}^0\}$ and there exists sequences $t_k \downarrow 0, (u^k, v^k) \to (u, v)$ such that (39)-(40) hold. Then there exists a vector $(\lambda^V, \lambda_g, \lambda_G, \lambda) \in \mathbb{R}^{1+p+q+p}$ satisfying

$$0 = \nabla_x F(\bar{x}, \bar{y}) - \lambda^V \nabla_x g(\bar{x}, \bar{y})^T \lambda + \nabla g(\bar{x}, \bar{y})^T \lambda_g + \nabla G(\bar{x}, \bar{y})^T \lambda_G,$$

$$0 = \nabla_y F(\bar{x}, \bar{y}) + \lambda^V \nabla_y f(\bar{x}, \bar{y}) + \nabla y g(\bar{x}, \bar{y})^T \lambda_g + \nabla G(\bar{x}, \bar{y})^T \lambda_G,$$

$$\lambda^V \geq 0, \ 0 \leq \lambda_g \perp g(\bar{x}, \bar{y}), \ \lambda_g \perp \nabla g(\bar{x}, \bar{y})(u, v), \ 0 \leq \lambda_G \perp G(\bar{x}, \bar{y}), \ \lambda_G \perp \nabla G(\bar{x}, \bar{y})(u, v)$$

$$0 = \nabla_y f(\bar{x}, \bar{y}) + \nabla y g(\bar{x}, \bar{y})^T \lambda, \ 0 \leq \lambda \perp g(\bar{x}, \bar{y}), \ \lambda \perp \nabla g(\bar{x}, \bar{y})(u, v).$$

**Proof.** By Proposition 4.4, since $F(x)$ is RCR-regular at $(\bar{x}, \bar{y})$ and $S(x)$ is inner semi-continuous at $(\bar{x}, \bar{y})$ in direction $u$, $V(x)$ is directional differentiable at $\bar{x}$ in direction $u$ and $V'(\bar{x}; u) = \min_{v \in L(\bar{x}, \bar{y}; u)} \nabla f(\bar{x}, \bar{y})(u, v)$. Since $(u, v) \in C(\bar{x}, \bar{y})$, we have $\nabla f(\bar{x}, \bar{y})(u, v) - V'(\bar{x}; u) = 0$. Hence $v \in \Sigma(\bar{x}, \bar{y}, u) = \{v \in L(x, y; u)|V'(x; u) = \nabla f(x, y)(u, v)\}$. Then by Theorem 4.3(iii), $V(x)$ is Lipschitz continuous at $\bar{x}$ in direction $u$ and $\partial^c V(\bar{x}; u) \subseteq W(\bar{x}, \bar{y}; u, v)$. The rest of result follows from Theorem 5.1.

The following example verifies Theorem 5.3. For this example, $S(x)$ is not inner semi-continuous at $\bar{x}$ but it is directional inner semi-continuous, the classical quasi-normality fails but the directional quasi-normality holds.

**Example 5.1** Consider the following bilevel program

$$\min_{x,y} F(x, y) := (\sqrt{3} x - y - \sqrt{3})^2 + x + \sqrt{3} y + 3$$

$$\text{s.t.} \quad y \in S(x) := \arg \min_y \{1 - (x - y)^2 : (x - 1)^2 + y^2 - 4 \leq 0, -\sqrt{3} x - y - \sqrt{3} \leq 0\}.$$
It is easy to verify that

\[
S(x) = \begin{cases} \sqrt{4 - (x - 1)^2}, & -1 \leq x < 0, \\ \{-\sqrt{3}, \sqrt{3}\}, & x = 0, \\ -\sqrt{4 - (x - 1)^2}, & 0 < x \leq 3. \end{cases}
\]

(41)

\[
V(x) = \begin{cases} 1 - (x - \sqrt{4 - (x - 1)^2})^2, & -1 \leq x < 0, \\ -2, & x = 0, \\ 1 - (x + \sqrt{4 - (x - 1)^2})^2, & 0 < x \leq 3. \end{cases}
\]

(42)

Note that the value function is Lipschitz continuous at \( \bar{x} = 0 \) but not smooth. The global optimal solution of the bilevel program is \( (\bar{x}, \bar{y}) = (0, -\sqrt{3}) \). By (41), \( S(x) \) is inner semi-continuous at \( \bar{y} \) in any direction \( u > 0 \). Indeed, for any sequence \( x \rightarrow \bar{x} \) in direction \( u > 0 \), \( S(x) \rightarrow \bar{y} \). It follows that \( S(\bar{x}; u) = \{\bar{y}\} \). Note that since for any sequence \( x \rightarrow \bar{x} \) in direction \( u < 0 \), \( S(x) \not\rightarrow \bar{y} \), \( S(x) \) is not inner semi-continuous at \( \bar{x} \).

Denote by \( f(x, y) := 1 - (x - y)^2 \), \( g_1(x, y) := (x - 1)^2 + y^2 - 4 \), \( g_2(x, y) := -\sqrt{3}x - y - \sqrt{3} \). Then

\[
\nabla F(\bar{x}, \bar{y}) = \left[ \frac{1}{\sqrt{3}} \right], \quad \nabla f(\bar{x}, \bar{y}) = \left[ \frac{-2\sqrt{3}}{2\sqrt{3}} \right], \quad \nabla g_1(\bar{x}, \bar{y}) = \left[ \frac{-2}{-2\sqrt{3}} \right], \quad \nabla g_2(\bar{x}, \bar{y}) = \left[ \frac{-\sqrt{3}}{-1} \right].
\]

It is easy to see that the rank of the gradient vectors \( \{\nabla_y g_1(x, y), \nabla_y g_2(x, y)\} \) is always equal to 1 around \((\bar{x}, \bar{y})\) and hence, RCR-regularity holds at \((\bar{x}, \bar{y})\). Since \( g_1(\bar{x}, \bar{y}) = 0, g_2(\bar{x}, \bar{y}) = 0 \),

\[
\Lambda(\bar{x}, \bar{y}) := \{(\lambda^1, \lambda^2) \in \mathbb{R}_+^2 : 2\sqrt{3} - 2\sqrt{3}\lambda^1 - \lambda^2 = 0\}.
\]

Then by Theorem 5.3, \( V(x) \) is Lipschitz continuous and directionally differentiable in direction \( u > 0 \) and

\[
V'(\bar{x}; u) = \max\{\nabla_x \mathcal{L}(\bar{x}, \bar{y}; \lambda^1, \lambda^2) u : (\lambda^1, \lambda^2) \in \Lambda(\bar{x}, \bar{y})\}
= \max\{(-2\sqrt{3} - 2\lambda^1 - \sqrt{3}\lambda^2)u : (\lambda^1, \lambda^2) \in \mathbb{R}_+^2, 2\sqrt{3} - 2\sqrt{3}\lambda^1 - \lambda^2 = 0\}
= \max\{(-2\sqrt{3} + 4\lambda^1 - 6)u : 0 \leq \lambda^1 \leq 1\}
= -(2\sqrt{3} + 2)u.
\]

Moreover we can verify that this statement is correct by the expression (43). Now we prove that the directional quasi-normality holds at \((\bar{x}, \bar{y})\). The critical cone can be calculated as

\[
C(\bar{x}, \bar{y}) := \{(u, v) | \nabla F(\bar{x}, \bar{y})(u, v) \leq 0, \nabla f(\bar{x}, \bar{y})(u, v) - V'(\bar{x}; u) = 0, \nabla g(\bar{x}, \bar{y})(u, v) \leq 0\}
= \{(u, v) | u + \sqrt{3}v = 0, \sqrt{3}u + v \geq 0\}.
\]

Let \( \bar{u} = \sqrt{3} \) and \( \bar{v} = -1 \), we have \((\bar{u}, \bar{v}) \in C(\bar{x}, \bar{y}) \). Since \( g_1(\bar{x}, \bar{y}) = g_2(\bar{x}, \bar{y}) = 0, \nabla g_1(\bar{x}, \bar{y})(\bar{u}, \bar{v}) = 0, \nabla g_2(\bar{x}, \bar{y})(\bar{u}, \bar{v}) = -\sqrt{3}\bar{u} - \bar{v} \neq 0, \) we have

\[
W(\bar{x}, \bar{y}, \bar{u}, \bar{v}) := \{\nabla_x f(\bar{x}, \bar{y}) + \nabla_x g(\bar{x}, \bar{y})^T \eta_g(\eta_g) \in \Lambda(\bar{x}, \bar{y}) \cap \{\nabla g(\bar{x}, \bar{y})(\bar{u}, \bar{v})\}^\perp\}
= \{-2\sqrt{3} - 2\lambda^1_{\eta_g} | \lambda^1_{\eta_g} \geq 0, 2\sqrt{3} - 2\sqrt{3}\lambda^1_{\eta_g} = 0\}
= \{-2\sqrt{3} - 2\}.
\]
Since \((\bar{u}, \bar{v})\) ∈ \(C(\bar{x}, \bar{y})\), by \([27]\) we have \(\bar{v} \in \Sigma(\bar{x}, \bar{y}, \bar{u})\). Therefore by Theorem \([4.3]\), we have \(\partial^2 V(\bar{x}; \bar{u}) \subseteq W(\bar{x}, \bar{y}, \bar{u}, \bar{v})\). Since \(V(x)\) is a function of one variable, we can verify by the expression of the value function \([42]\) that

\[
\partial^2 V(\bar{x}; \bar{u}) = W(\bar{x}, \bar{y}, \bar{u}, \bar{v}) = \{-2\sqrt{3} - 2\}.
\]

Let \(\alpha, \nu_1, \nu_2\) be such that

\[
0 \in \alpha(\nabla_x f(\bar{x}, \bar{y}) - W(\bar{x}, \bar{y}, \bar{u}, \bar{v})) + \nu_1 \nabla_x g_1(\bar{x}, \bar{y}) + \nu_2 \nabla_x g_2(\bar{x}, \bar{y}),
\]

\[
0 = \alpha \nabla_y f(\bar{x}, \bar{y}) + \nu_1 \nabla_y g_1(\bar{x}, \bar{y}) + \nu_2 \nabla_y g_2(\bar{x}, \bar{y}),
\]

\[
\nu_2 \nabla g_2(\bar{x}, \bar{y})(\bar{u}, \bar{v}) = 0, \alpha \geq 0, \nu_1 \geq 0, \nu_2 \geq 0
\]

and there exist sequences \(t_k \downarrow 0\), \((u^k, v^k) \to (\bar{u}, \bar{v})\), such that

\[
f(\bar{x} + t_k u^k, \bar{y} + t_k v^k) - V(\bar{x} + t_k u^k) > 0 \text{ if } \alpha > 0,
\]

\[
g_1(\bar{x} + t_k u^k, \bar{y} + t_k v^k) > 0 \text{ if } \nu_1 > 0,
\]

\[
g_2(\bar{x} + t_k u^k, \bar{y} + t_k v^k) > 0 \text{ if } \nu_2 > 0.
\]

\([43]\) implies that \(\nu_2 = 0\) and \([48]\) will not be needed. We now show the conditions \([43)-(47]\) can only hold if \(\alpha = \nu_1 = \nu_2 = 0\). By \([41]\), \(2\sqrt{3} \alpha - 2\sqrt{3} \nu_1 = 0\). Hence \(\alpha = \nu_1\). To the contrary, assume \(\alpha > 0\). Then \(\nu_1 = \alpha > 0\). Let \(t_k \downarrow 0\), \((u^k, v^k) \to (\bar{u}, \bar{v})\) be arbitrary and suppose that \([47]\) holds. Then \(g_1(x^k, y^k) > 0\) for \((x^k, y^k) := (\bar{x} + t_k u^k, \bar{y} + t_k v^k)\). It follows that \(y^k < -\sqrt{4 - (x^k - 1)^2}\). Since \(\nabla_y f(x^k, \sqrt{4 - (x^k - 1)^2}) = 2(x^k + \sqrt{4 - (x^k - 1)^2}) > 0\) and \(y^k < -\sqrt{4 - (x^k - 1)^2}\) we have \(f(x^k, y^k) < f(x^k, -\sqrt{4 - (x^k - 1)^2}) = V(x^k)\), where the last equality follows from \([41]\). Hence \([46]\) does not hold. The contradiction show that \((\alpha, \nu_1, \nu_2) = (0, 0, 0)\) and directional quasi-normality holds at \((\bar{x}, \bar{y})\) in direction \((\bar{u}, \bar{v})\).

By now, the conditions in Theorem \([5, 6]\) are all verified and so the directional KKT condition should hold at \((\bar{x}, \bar{y})\). That is, there exists a nonzero vector \((\lambda_V, \lambda, \lambda_g) \in \mathbb{R}^{1+2+2}\) such that

\[
0 = 1 - \lambda_V(-2\lambda_1 - \sqrt{3}\lambda_2^2) - 2\lambda_1^2 - \sqrt{3}\lambda_2^2,
\]

\[
0 = \sqrt{3} + \lambda_V 2\sqrt{3} - 2\sqrt{3}\lambda_1^2 - \lambda_g^2,
\]

\[
\lambda_g, \lambda \in \Lambda(\bar{x}, \bar{y}), \lambda_g \perp \nabla g(\bar{x}, \bar{y})(\bar{u}, \bar{v}), \lambda \perp \nabla g(\bar{x}, \bar{y})(\bar{u}, \bar{v}).
\]

Obviously the vectors \((\lambda_V, \lambda, \lambda_g) := (\frac{1}{2}, (1, 0), (1, 0))\) satisfies the above conditions.

As we have mentioned before, NNAMCQ and FOSCMS always fail for (BP). In this example, the quasi-normality also fails at \((\bar{x}, \bar{y})\). Indeed, let \((\alpha, \nu_1, \nu_2) = (1, 1, 0)\). We have \((\alpha, \nu_1, \nu_2)\) satisfies \([43]\) and \([41]\). And choose \((x^k, y^k) := (-1/k - \sqrt{4 - (1/k + 1)^2} - 1/k)\), which converges to \((\bar{x}, \bar{y})\). By \([41]\), we have

\[
f(x^k, y^k) = 1 - \left(\sqrt{4 - (1/k + 1)^2}\right)^2 > 1 - \left(\frac{1}{k} + \sqrt{4 - (1/k + 1)^2}\right)^2 = V(x^k),
\]

\[
g_1(x^k, y^k) = (1/k + 1)^2 + \left(\sqrt{4 - (1/k + 1)^2} + 1/k\right)^2 - 4 > 0.
\]

By the definition of the classical quasi-normality defined in \([16]\), (one can refer to Definition \([3.2]\) for the case \(u = 0\)), this means that the quasi-normality fails at \((\bar{x}, \bar{y})\).
References

[1] K. Bai, J. J. Ye and J. Zhang, Directional quasi-/pseudo-normality as sufficient conditions for metric subregularity, SIAM J. Optim., 29(2019), pp. 2625-2649.

[2] E. M. Bednarczuk, L. I. Minchenko and K. E. Rutkowski, On Lipschitz continuity of a class of set-valued mappings, to appear in Optimization.

[3] M. Benko, H. Gfrerer and J. V. Outrata, Calculus for directional limiting normal cones and subdifferentials, Set-Valued Var. Anal., 27(2019), pp. 713–745.

[4] J. F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, New York, 2000.

[5] F. H. Clarke, Optimization and Nonsmooth Analysis, Classics Appl. Math. 5, SIAM, Philadelphia, PA, 1990.

[6] S. Dempe, Foundations of bilevel programming, Kluwer Academic, Dordrecht 2002.

[7] A. L. Dontchev, R. T. Rockafellar, Implicit Functions and Solution Mappings, Springer, New York, 2009.

[8] J. Gauvin and F. Dubeau, Differential properties of the marginal function in mathematical programming. Math. Programming Stud., 19(1982), pp. 101-119.

[9] H. Gfrerer, On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs. Set-Valued Var. Anal., 21(2013), pp. 151-176.

[10] H. Gfrerer, On metric pseudo-(sub) regularity of multifunctions and optimality conditions for degenerated mathematical programs, Set-Valued Var. Anal., 22(2014), pp. 79-115.

[11] H. Gfrerer, Optimality conditions for disjunctive programs based on generalized differentiation with application to mathematical programs with equilibrium constraints, SIAM J. Optim., 24(2014), pp. 898-931.

[12] H. Gfrerer and D. Klatte, Lipschitz and Hölder stability of optimization problems and generalized equations, Math. Program., 158(2016), pp. 35–75.

[13] H. Gfrerer and B. Mordukhovich, Robinson stability of parametric constraint systems via variational analysis, SIAM J. Optim. 27(2017), pp. 438–465.

[14] H. Gfrerer and J. J. Ye, New sharp necessary optimality conditions for mathematical programs with equilibrium constraints, Set-Valued Var. Anal., to appear.

[15] I. Ginchev and B.S. Mordukhovich, On directionally dependent subdifferentials, C.R. Bulg. Acad. Sci., 64(2011), pp. 497-508.

[16] L. Guo, J. J. Ye and J. Zhang Mathematical programs with geometric constraints in Banach spaces: enhanced optimality, exact penalty, and sensitivity, SIAM J. Optim., 23(2013), pp. 2295-2319.
[17] L. Guo, G-H. Lin, J. J. Ye, and J. Zhang, *Sensitivity analysis of the value function for parametric mathematical programs with equilibrium constraints*, SIAM J. Optim. 24(2014), pp. 1206-1237.

[18] R. Janin, *Directional derivative of the marginal function in nonlinear programming*, Math. Program. Study, 21(1984), pp. 110–126.

[19] V. Kalashnikov, S. Dempe, G. Prez-Valds, N. Kalashnykova and J. Camacho-Vallejo, *Bilevel programming and applications*. Math. Probl. Eng., 2015.

[20] G. Kunapuli, K. P. Bennet, J. Hu and J. S. Pang, *Classification model selection via bilevel programming*. Optim. methods Softw., 23(2008), pp. 475-489.

[21] K. Lachhwani and A. Dwivedi, *Bi-level and Multi-Level Programming Problems: Taxonomy of Literature Review and Research Issues*, Arch. Compt. Method E., 4(2018), pp. 847-877.

[22] P. Long, B. Wang and X. Yang, *Calculus of directional subdifferentials and coderivatives in Banach spaces*, Positivity, 21(2017), pp. 223-254.

[23] Z. Q. Luo, J. S. Pang and D. Ralph, *Mathematical programs with equilibrium constraints*, Cambridge University Press, 1996.

[24] L. Minchenko and S. Stakhovski, *Parametric nonlinear programming problems under the relaxed constant rank condition*, SIAM J. Optim. 21(2011), pp. 314-332.

[25] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I. Basic Theory*. Ser. Comprehensive Stud. Math. 330, Springer, Berlin, 2006.

[26] J. V. Outrata, M. Kočvara and J. Zowe, *Nonsmooth approach to optimization problems with equilibrium constraints: theory, applications and numerical results*, Kluwer, Dordrecht, The Netherlands, 1998.

[27] R. T. Rockafellar and R. J-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.

[28] H. von Stackelberg, *Marktform und Gleichgewicht*, Springer-Verlag, Berlin, 1934, engl. transl.: *The theory of the market economy*, Oxford University Press, Oxford, England, 1954.

[29] M. Xu and J. J. Ye, *Relaxed constant positive linear dependence constraint qualification and its application to bilevel programs*, J. Global Optim., to appear, arXiv(1912.12372).

[30] J. J. Ye, *Constraint qualifications and necessary optimality conditions in bilevel optimization*, Preprint, arXiv (1910.04072).

[31] J. J. Ye, *Constraint qualifications and necessary optimality conditions for optimization problems with variational inequality constraints*, SIAM J. Optim.,10(2000), pp. 943-962.

[32] J. J. Ye and X. Y. Ye, *Necessary optimality conditions for optimization problems with variational inequality constraints*, Math. Oper. Res., 22(1997), pp. 977-997.
[33] J. J. Ye and D. L. Zhu, *Optimality conditions for bilevel programming problems*, Optimization, 33(1995), pp. 9-27.

[34] J. J. Ye and D. L. Zhu, *A note on optimality conditions for bilevel programming problems*, Optimization, 39(1997), pp. 361-366.

[35] J. J. Ye and D. L. Zhu, *New necessary optimality conditions for bilevel programs by combining MPEC and value function approaches*, SIAM J. Optim., 20(2010), pp. 1885-1905.

[36] J. J. Ye, D. L. Zhu and Q. J. Zhu, *Exact penalization and necessary optimality conditions for generalized bilevel programming problems*, SIAM J. Optim., 2(1997), pp. 481-507.