Barrow Holographic Dark Energy with Hybrid Expansion Law

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Abstract—We investigate the Barrow holographic dark energy (BHDE) model using IR cutoff as the Hubble horizon in the background of a flat Friedmann–Lemaître–Robertson–Walker universe. The deceleration parameter exhibits the universe evolution from decelerated to accelerated phase. The EoS parameter of the BHDE model presents a nice behavior as it lies in the phantom era ($\omega_B < -1$) and the quintessence era ($\omega_B \geq -1$) for different values of the Barrow parameter $\Delta$. The squared sound velocity $v_s^2$ has been investigated for the stability of the model. In addition, a correspondence with the phantom and quintessence scalar fields has been studied for the model, which helps us to describe the accelerated expansion of the universe.

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1. INTRODUCTION

It is proven by past and latest observational data that dark matter (DM) and dark energy (DE) both fill the dark sector of our universe [1–6]. The fluid dark energy drives the recent accelerating phase of the universe, while the accountability of structure formation of the universe is taken by dark matter. The observational data also indicate that around 96% of the total energy density of the universe is coming from the united dark component, where DM contributes approximately 28%, while the contribution of DE is about 68% of the total energy allocation. Though, the origin, evolution and characteristics of these dark sectors are yet unknown [7–12]. However, indirect gravitational effects appear to also help the researchers to understand the nature of DM, though DE remains unknown. As a result, in the last couple of years, a number of cosmological models have been proposed and investigated. The interacting model, in which DE and DM are allowed to interact, is the most common type of cosmological models [13], although the noninteracting model universe is thought to be the simplest.

The holographic principle (HP) has led directly to an alternative interesting scenario called Holographic dark energy (HDE) for the DE quantitative formulation [14–18]. At astronomical scales, starting with the relationship between its UV cutoff and the longest length of a quantum field theory with [19] will lead to the holographic source vacuum energy structure of DE [20, 21]. Both the basic HDE versions of [20–31] and their modified versions [32–56] contribute to a fascinating cosmological behavior, although they are restricted by observations [57–65]. Similar to the Bekenstein–Hawking entropy of a black hole, the key step in utilization of the holographic principle in the cosmological framework is that at the largest distance (i.e., the universe horizon) the entropy is proportionate to the area. The HDE model is based on assuming that the energy density $\rho_B$ is responsible for the current acceleration of the universe with $L$ being a suitable cosmological scale.

In fact, it was soon observed that by considering $L$ as the inverse of the current Hubble rate, $\rho_B$ is of the order of the current DE density [21], even though the equation of state (EoS) would not be the appropriate one as it would not induce acceleration in an isotropic and homogeneous universe. While, Bouhmadi-Lopez et al. [66] showed that even though the holographic energy density with the Hubble horizon as the IR cutoff is not suitable for explaining the late-time acceleration of a brane-world model with induced gravity nor a 4D relativistic model, it is suitable to describe the late-time acceleration of a brane world-model with an induced gravity term on thebrane and a Gauss-
Bonnet effect in the bulk. They proposed that by including a Gauss-Bonnet term in the bulk the situation can be improved, and the brane undergoes an acceleration at late time with the Hubble horizon as the IR cutoff [66].

Being inspired by the recent Covid-19 experiments, Barrow suggested that quantum-gravitational forces could rise to elaborate fractal structures on the BH structure. This complex structure, however, leads in a limited volume and finite (or infinite) area, and so a distorted BH entropy is [67]

\[ S_B = \left( \frac{A}{A_0} \right)^{(2+\Delta)/2}, \]

where \( A_0 \) is the Planck area, and \( A \) is the standard horizon area. The new exponent \( \Delta \), contributing to the quantum-gravitational deformation, the most intricate and fractal structure corresponded by \( \Delta = 1 \), while the standard BSH entropy is with \( \Delta = 0 \). It is important to mention that the established “quantum-corrected” entropy having logarithmic corrections [68, 69] differs from the above quantum-gravitationally corrected entropy, anyway it is indistinguishable from the Tsallis non-extensive entropy [70–72], but the physical principles and foundations are entirely different.

The author of [73] proposed another HDE using Barrow’s holographic dark energy (BHDE) with the extended Barrow relation for the horizon entropy, instead of the usual BSH one. In the \( \Delta = 0 \) case, BHDE contains the proven HDE as a limit, but overall it is a different situation with a more extravagant cosmological conduct and frame. The Standard HDE is given by the inequality \( \rho_B L^4 \leq S \), where \( L \) is the horizon length, and taking the imposition \( S \propto A \propto L^2 [20] \), so Eq. (1) of Barrow entropy will be written as

\[ \rho_B = CL^{2-\Delta}, \]

where \( C \) is a parameter of dimensions \( [L]^{-2-\Delta} [73] \). For \( \Delta = 0 \), Eq. (2) gives the standard HDE \( \rho_B = CL^{-2} \), where \( C = 3c^2 M_p^2 \), having the model parameter \( c^2 \), and \( M_p \) is the Planck mass. The BHDE will deviate from the standard one, depending on the \( \Delta \) parameter, leading to a different cosmological behavior. Considering a flat noninteracting FLRW universe with IR cutoff as the Hubble horizon \( (L = H^{-1}) \), the energy density of BHDE is

\[ \rho_B = CH^{2-\Delta}. \]

Using the Barrow entropy, Saridakis [74] gave another cosmological layout apart from the BSH one. He acquired an analytical expression for the evolution of the effective DE density parameter, and he showed the DM to DE phase of the universe. Recently, the authors of [75] used the Hubble parameter measurement from the CC (cosmic chronometers) as well as the Pantheon SNIa data to get constraints on the BHDE scenario. Srivastava and Sharma [76] proposed a new BHDE in spatially flat FLRW Universe taking the Hubble horizon as the \( R \) cutoff. Further, with any spatial curvature, Sheykhi [77] investigated corrections to the Friedmann equations for the FLRW universe considering the IR cutoff as an apparent horizon. In the recent past, various researchers explored the BHDE in different cosmological and gravitational setups [78–83].

Recently, the stability of some entropy inspired holographic DE models with different IR cutoffs were studied, and it was found that these models are unstable at the classical level [33, 36, 39, 84]. Among these models, in the absence of an interaction between the cosmic sectors, the RHDE, based on the Renyi entropy and the first law of thermodynamics, shows more stability by itself [39]. In this paper, we explore the stability of the BHDE model with the Hubble horizon as an IR cutoff. The squared speed of sound \( v^2 \) is a significant quantity for this purpose. For a density perturbation, the \( v^2 \geq 0 \) (a real value of speed) indicates a regular propagation mode. The perturbation equation becomes an irregular wave equation for \( v^2 < 0 \). As a result, with a density perturbation, the negative squared speed (imaginary velocity) shows an escalating mode. In other words, when the density perturbation increases, the pressure decreases, allowing for the development of an instability [85–87].

Now, we discuss the similarity to and difference from other works in the literature. The HDE and Tsallis HDE [88] with a hybrid expansion law have been discussed previously, but the BHDE model with a hybrid expansion law, using as an IR cut-off the Hubble horizon for a flat FLRW universe has never been studied. In this paper, we examine the BHDE with a hybrid expansion law for a flat FLRW universe without interaction, considering the Hubble horizon as an IR cutoff. Our present research paper is organized as follows: In Section 2, the metric and field equations for BHDE model are discussed. In Section 3, we investigate the cosmological evolution of BHDE and the EoS parameter. We discuss the stability of the model in Section 4. In Section 5, we discuss the correspondence between the BHDE along with phantom and quintessence scalar field model and potentials and scalar field models. We wrap up our findings in Section 6.

2. THE METRIC AND FIELD EQUATIONS

The metric of a flat FLRW universe, based on the principles of cosmology, is given by

\[ ds^2 = dt^2 - a^2(t) [dx^2 + dy^2 + dz^2], \]
where \(a(t)\) which is function of \(t\), the scale factor. In General Relativity, the Einstein field equation is

\[
R_{ij} - \frac{1}{2}g_{ij}R = -\frac{8\pi G}{c^4}(T_{ij} + \bar{T}_{ij}),
\]

(5)

where \(\bar{T}_{ij} = (\rho_B + p_B)u_i u_j + g_{ij}p_B\) and \(T_{ij} = \rho_m u_i u_j\) give the energy momentum tensor of the BHDE and matter, respectively. Here, \(\rho_B, \rho_m,\) and \(p_B\) are the energy densities of BHDE, dark matter and the BHDE pressure, respectively. Throughout the paper, we adopt natural units \((c = 1\) and \(8\pi G = 1\), where \(G\) is Newton’s gravitational constant). Using Eqs. (4) and (5), the Einstein field equations can be written as

\[
\left(\frac{\dot{a}^2}{a^2}\right) = \frac{8\pi G}{3}(\rho_m + \rho_B),
\]

(6)

\[
\left(\frac{\dot{a}^2}{a^2} + \frac{2}{3}\frac{\ddot{a}}{a}\right) = -\frac{8\pi G}{3}p_B.
\]

(7)

The BHDE density and matter density conservation have the form \(\rho_B + 3H(\rho_B + p_B) = 0\), and \(\rho_m + 3H\rho_m = 0\), respectively. Also, we get the EoS parameter \(\omega_B = p_B/\rho_B\) using Eq. (3) as

\[
\omega_B = -1 + \frac{(2 - \Delta)\dot{H}}{3H^2}.
\]

(8)

3. SOLUTION OF THE FIELD EQUATIONS AND THE COSMOLOGICAL PARAMETERS

For a Universe which was decelerating in the past and is accelerating at the present time, the deceleration parameter (DP) must show signature flipping, and hence, in general, the DP is not a constant but time-variable. We consider the cosmological scale factor obeying a hybrid expansion law (HEL) \([89, 90]\]

\[
(a(t))^n = a_0\left(\frac{t}{t_0}\right)^k e^{(t/t_0 - 1)},
\]

(9)

where \(n\) and \(k\) are constants. Here \(a_0\) and \(t_0\) denote the scale factor and the age of the Universe today, respectively. Setting \(k = 0\) leads the exponential-law cosmology. This cosmology is a special case of the Hybrid expansion law cosmology.

Using the suitable transformation, Eq. (9) is reduced to the form

\[
a = (t^k e^t)^{1/n},
\]

(10)

where \(t/t_0 \rightarrow t, k \geq 0,\) and \(n > 0\) are constants. This generalized form of average scale is called the hybrid expansion law (HEL). The motivation to choose such a scale factor is that it provides a time-dependent DP. Therefore, our choice of the scale factor is physically acceptable. Using the HEL, many researchers investigated the behavior of different cosmological models in various gravitational and cosmological setups \([9–94]\).

The deceleration parameter is determined by the relation \(q = -\dot{a}/a^2\) and is obtained as:

\[
q = \frac{kn}{(k + t)^2} - 1, \quad n > 0.
\]

(11)

We can obtain the Hubble parameter \((H)\) as

\[
H = \frac{\dot{a}}{a} = \left[\frac{k + t}{nt}\right].
\]

(12)

The EoS parameter for the BHDE is obtained using Eqs. (8) and (12):

\[
\omega_B = -1 + \frac{kn(2 - \Delta)}{(t + k)^2}.
\]

(13)

We have plotted the evolutionary behavior of the deceleration parameter for the BHDE model against redshift \((z)\), which for our model is defined as \(z = -1 + a_0/a\), where \(a_0\) is the present value of the scale factor at \(z = 0\). The DP \((q)\) is totally responsible for the deceleration or acceleration of the universe. The universe is in an accelerated or decelerated phase according to \(q < 0\) or \(q > 0\), respectively. It is observed that the DP slides down in span of \(-1 \leq q < 0\), so at present the universe is accelerating. We can observe the behavior of the DP in terms of redshift in Fig. 1. It is very clear from this figure that for different pairs of \(n\) and \(k\), such as \((n, k) = (0.85, 0.2295), (0.75, 0.2020), (0.65, 0.1755),\) and \((0.55, 0.1458),\) the BHDE model is transiting from

![Fig. 1. The behaviour of the deceleration parameter vs redshift](image-url)
an early decelerated stage to the present accelerating stage.

It is possible to construct the cosmological history on the basis of one’s taste, depending on the estimate of the constants $n$ and $k$, which encourages us to use it to explain the BHDE. The constants $k$ and $n$ give the affirmation of time from decelerated to accelerated expansion. The constants $n, k$ and the definite model parameter $\Delta$ are three parameters of the current situation of the BHDE model. In the present paper, we look at the action of $\Delta$ in an unsullied manner by keeping constants $k$ and $n$ fixed, such as $n = 0.55, k = 0.1485$.

The EoS parameter $\omega_B$ as a function of the redshift $z$ is plotted for $(n, k) = (0.55, 0.1485)$ and for different values of $\Delta$ in Fig. 2. According to this figure, the EoS parameter of BHDE perceives a cordial nature, it is phantom as well as quintessence depending on the distinct estimation of $\Delta$. It is also observed from Fig. 2 that the EoS parameter is in the quintessence region for $\Delta = 1.5$ and 1.7 and in the phantom region for $\Delta = 2.1$ and 2.3. It is clear from Eq. (13) that the EoS parameter mimics the cosmological constant, i.e., $\omega_B = -1$, when $\Delta = 2$.

The energy density of matter and the BHDE, and the Barrow pressure are written as:

$$\rho_m = \frac{3(t + k)^2}{n^2 t^2} - C \left( \frac{t + k}{nt} \right)^{2-\Delta}, \quad (14)$$

$$\rho_B = C \left( \frac{t + k}{nt} \right)^{2-\Delta}, \quad (15)$$

$$p_B = -3(t + k)^2 + 2nk \quad \frac{3(t + k)^2 + 2nk}{n^2 t^2}. \quad (16)$$

The BHDE density parameter $\Omega_B$ and $\Omega_m$ (the energy density parameter of matter) are described as

$$\Omega_B = \frac{C}{3} \left( \frac{nt}{t + k} \right)^{2-\Delta}, \quad (17)$$

$$\Omega_m = 1 - \frac{C}{3} \left( \frac{nt}{t + k} \right)^{2-\Delta}. \quad (18)$$

Hence,

$$\Omega = \Omega_m + \Omega_B = 1. \quad (19)$$

We have plotted the behavior of BHDE density in Fig. 3a for particular values of $n$ and $k$ and different values of $\Delta$. The BHDE density increases from high to low redshift regions, i.e., from the past to the future. So we can say that the BHDE energy density parameter $\Omega_B$ is an increasing function with lower redshift. We have delineated the behavior of matter energy density in Fig. 3b for particular values of $n$ and $k$ and different values of $\Delta$. The BHDE density decreases from high to low redshift regions, i.e., from the past to the future. Genuinely, it follows that DE will dominate the universe in the future.

The behavior of the matter density parameter $\Omega_m$ and the BHDE density parameter $\Omega_B$ for quintessence ($\Delta = 1.5$) and for a phantom ($\Delta = 2.1$) is shown in Figs. 4a and 4b, respectively. We can see that the sum of these two parameters is always 1, which is also indicated in Eq. (19). So, we can say that as the universe is dominated by DE, it shows the isotropic nature.

As we know, the BHDE has two parameters, the Barrow exponent $\Delta$, and the constant $C$ (similar to the parameter $c^2$ of standard holographic DE) which incorporates the initial inequality validation [73]. We fixed $C = 3$, which is the value required if we want standard holographic DE to be an exact limit for $\Delta = 0$, and we looked at its pure role in the cosmic history. This was shown to be sufficient for a successful description that agrees with observations, which is a significant advantage over the standard holographic DE, in which case the value of the constant $c^2$ must be adjusted to fit the data. Changing the value of $C$ would undoubtedly result in an even better cosmic behavior, which reveals the capabilities of the scenario.

### 4. Stability of the BHDE Model

The physical values of the squared sound speed of DE should be in the range $0 \leq v_s^2 \leq 1$ [95]. One thing to keep in mind is that in order to analyze the effect of the DE sound speed, the values of the DE EoS $\omega_B \neq -1$ must be considered too, since DE consists of no perturbations and so a cosmology with DE in the form of a cosmological constant lacks of sensitivity to
the DE sound speed. The effect of DE sound speed on the cosmic microwave background power spectrum has been studied in detail in [96]. Here it was found that when the DE EoS satisfies \( \omega_B > -1 \), a decrease of \( v_s^2 \) from 1 to 0 suppresses the LISW effect, but when the DE EoS satisfies \( \omega_B < -1 \), a decrease of the sound speed leads in an intensification of the LISW effect. This can be understood heuristically, at least for the case where \( \omega_B > -1 \), as follows: the lower \( v_s^2 \), the more DE can cluster and effectively function as "cold" DE. Clustering increases the DE perturbations, which protect the potentials from decaying as previously described, resulting in a reduced contribution [95] to the LISW effect.
As stated in [97] we can test the stability of our BHDE model against perturbations by using the squared speed of sound \( v_s^2 = \frac{d \rho_B}{d \rho_T} \). The squared sound speed is generally given as

\[
v_s^2 = \frac{dp_B}{d\rho_B} = -\frac{12}{Ct} \left( \frac{2t - \Delta}{k + t} \right)^{1-\Delta}. \tag{20}
\]

The squared speed of sound \( v_s^2 \) given by Eq. (20) is plotted against \( z \) in Fig. 5. We can see that as \( z \rightarrow -1 \) i.e., in the future, the stability of our BHDE model occurs for \( \Delta = 1.5 \) and 1.7, while for \( \Delta = 2.1 \) and \( \Delta = 2.3 \) an instability persists.

5. CORRESPONDENCE OF THE BHDE MODEL WITH PHANTOM AND QUINTESSENCE FIELDS

In this section, we examine the correspondence between the BHDE, quintessence and phantom scalar fields. The potentials and dynamics for phantom and quintessence scalar fields are reconstructed. We can find the correspondence by comparing Eq. (3) (energy density of the BHDE model) with energy densities of the scalar field models. Also, we analyze the EoS of the BHDE, given by Eq. (13) along with the EoS parameters of the scalar fields (phantom and quintessence). By using the scalar field models, we can well describe both the canonical and non-canonical DE. In this paper, both phantom and quintessence fields are chosen to be canonical. As is known, for a phantom fields \( \omega_B < -1 \) [98–102], and for a quintessence scalar field \( \omega_B > -1 \) [103–107]. Hoyle and Narlikar [108] proposed \( c \)-fields of phantom type as scalar fields with negative kinetic energy.

In [109], the energy density and pressure of a quintessence scalar field and a phantom scalar field are given by

\[
p = \frac{\dot{\phi}^2}{2} - V(\phi), \quad \rho = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{21}
\]

\[
p = \frac{\dot{\phi}^2}{2} - V(\phi), \quad \rho = -\frac{\dot{\phi}^2}{2} + V(\phi), \tag{22}
\]

where \( \dot{\phi} \) and \( V(\phi) \) are a scalar field and its potential. So we can write the EoS parameter for the quintessence and phantom scalar fields as

\[
\omega_B = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \tag{23}
\]

Using Eqs. (23) and (13), we get:

\[
\frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} = -1 + \frac{kn(2-\Delta)}{(t+k)^2}. \tag{24}
\]

We can calculate the kinetic energy and the scalar potential, \( \dot{\phi}^2 \) and \( V(\phi) \):

\[
\dot{\phi}^2 = C \left( \frac{t+k}{n} \right)^{2-\Delta} + \frac{2kn - 3(t+k)^2}{n^2 t^2}, \tag{25}
\]

and

\[
V(\phi) = \frac{C}{2} \left( \frac{t+k}{nt} \right)^{2-\Delta} + \frac{3(t+k)^2 - 2kn}{2n^2 t^2}. \tag{26}
\]

The kinetic energy for the quintessence and phantom scenarios are plotted in Figs. 6a and 6b. From Fig. 6a we can see that \( \dot{\phi}^2 \) (kinetic energy) increases initially from high to low redshift, reaches a specific point \((z = 0)\) and then starts to decreases for \( \Delta = 2.1 \) and 2.3. Figure 6b gives the reverse behavior even as the phantom, i.e., \( \dot{\phi}^2 \) decreases initially from high to low redshifts and reaches a specific point \((z = 0)\), then starts to increase for \( \Delta = 1.5 \) and 1.7.

In Fig. 7 we have plotted the scalar field potential \( V(\phi) \) for both quintessence and phantom scenarios. It can be seen that the potential \( V(\phi) \) increases through high redshifts and reaches a particular point and afterwards diminishes. In the future the potential becomes flat.

From Figs. 6 and 7 we can see that the evolution of the scalar potential has the same form for both quintessence and phantom scenarios, while the kinetic energies are opposite.
6. CONCLUSION

In this paper, we have studied the BHDE model in the flat FLRW universe using the Hubble horizon as the IR cutoff. We obtained solutions of Einstein’s field equations by considering the time-dependent deceleration parameter. Different cosmological parameters have been investigated for the derived BHDE model to explain the accelerated expansion of the universe. Also, to describe the late cosmic accelerated expansion of universe, the correspondence between the quintessence and phantom scalar field models were reconstructed. We summarize our results as follows:

- We observed that the deceleration parameter of our model transmits from the early decelerated stage to present accelerating phase of the Universe.
- The new Barrow exponent \( \Delta \) significantly affects the DE EoS, and according to its value, it can lead it to lie in the quintessence region and in the phantom region. The above behaviors were obtained by changing only the value of \( \Delta \). Additional adjusting of the parameter \( C \) will significantly enhance the capabilities of the scenario.
- The energy density parameter of BHDE and matter showed that the BHDE scenario can describe the universe thermal history, with the sequence of matter and dark energy eras.
- According to the findings of this paper, the reconstruction is useful in portraying the key characteristics of the potential for the phantom and quintessence models. It is necessary to understand and analyze the theoretical root of the BHDE density, even though the BHDE boosts the reconstruction quickly and unambiguously. While observing the future of the
universe, our research in this work presents delightful possibilities to experience the nature of the BHDE.

During the whole cosmic evolution, we can see that for all values of $z$ the model is not stable when we are taking $\Delta > 2$. It can be fixed by taking other IR cutoffs, probable interactions between the universe sectors, diverse entropy modifications or even a mix of these situations.

In fact, these considerations can also increase and modify the predictions and behavior of the BHDE model. These are subjects to be studied in the future to become closer to different properties of the BHDE, and hence the origin of DE.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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