A Dual Zariski Topology for Modules*

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Abstract

We introduce a dual Zariski topology on the spectrum of fully coprime $R$-submodules of a given duo module $M$ over an associative (not necessarily commutative) ring $R$. This topology is defined in a way dual to that of defining the Zariski topology on the prime spectrum of $R$. We investigate this topology and clarify the interplay between the properties of this space and the algebraic properties of the module under consideration.

1 Introduction

Inspired by the interplay between the Zariski topology defined on the prime spectrum of a commutative ring $R$ and the ring theoretic properties of $R$ in [Bou1998, AM1969] (see also [LY2006, ST2010, ZTW2006]), we introduce in this paper a dual Zariski topology on the spectrum of fully coprime submodules of a given non-zero duo module $M$ over an associative ring $R$ and study the interplay between the properties of $RM$ and the topological space we obtain. The spectrum we consider was introduced for bicomodules over corings in [Abu2006] (see also [Wij2006, WW2009]) and topologized in [Abu2008] where a Zariski-like topology was investigated. Some of the results in this paper can be considered as module-theoretic versions of results in [Abu2006] and are dual to results in [Abu] which is devoted to a Zariski topology on the spectrum of fully prime submodules of a given duo module.

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This paper extends also the study of the so called top modules, i.e. modules whose spectrum of prime submodules attains a Zariski topology (e.g. [Lu1984, Lu1999, MMS1997, MMS1998, Zha2006-a, Zha2006-b]), to a notion of primeness not dealt with so far from the topological point of view (other notions appear, for example, in [Dau1978, Wis1996, RRRF-AS2002, RRW2005, Wij2006, Abu2006, WW2009]).

The paper is organized as follows: After this introduction, we collect in Section 2 some preliminaries and recall some properties and notions from Module Theory that will be needed in the sequel. In Section 3, given a non-zero duo left $R$-module $M$, we introduce and investigate a dual Zariski topology on the spectrum $\text{Spec}^c(M)$ of non-zero submodules that are fully coprime in $M$ (e.g. [Abu2006, Wij2006, WW2009]). In particular, we investigate when this space is Noetherian or Artinian (Theorem 3.24), irreducible (Theorem 3.26), ultraconnected (Proposition 3.30), compact or locally compact (Theorem 3.33), connected (Theorem 3.34), $T_1$ (Proposition 3.40) or $T_2$ (Theorem 3.41).

2 Preliminaries

In this section, we fix some notation and recall some definitions and basic results. For any undefined terminology, the reader is referred to [Wis1991] and [AF1974].

Throughout, $R$ is an associative (not necessarily commutative) ring with $1_R \neq 0_R$ and $M$ is a non-zero unital left $R$-module. By an ideal we mean a two-sided ideal and by an $R$-module we mean a left $R$-module, unless explicitly otherwise mentioned. We set $S := \text{End}(R M)^{\text{op}}$ (the ring of $R$-linear endomorphisms of $M$ with multiplication given by the opposite composition of maps) and consider $M$ as an $(R, S)$-bimodule in the canonical way. We write $L \leq_R M$ ($L \preceq_R M$) to indicate that $L$ is a (proper) $R$-submodule of $M$. For non-empty subsets $L \subseteq M$ and $I \subseteq R$ we set

$$(L :_R M) := \{r \in R | r M \subseteq L\} \text{ and } (L :_M I) := \{m \in M | I m \subseteq L\}.$$  

**Definition 2.1.** We say $L \leq_R M$ is fully invariant or characteristic iff $f(L) \subseteq L$ for every $f \in S$ (equivalently iff $L \leq M$ is an $(R, S)$-subbimodule). In this case, we write $L \leq_{fi}^R M$. We call $R M$ duo or invariant iff every $R$-submodule of $M$ is fully invariant.

Recall that the ring $R$ is said to be left duo (right duo) iff every left (right) ideal of $R$ is two-sided and to be left quasi-duo (right quasi-duo) iff every maximal left (right) ideal of $R$ is two-sided. Moreover, $R$ is said to be (quasi-) duo iff $R$ is left and right (quasi-) duo.

2.2. Examples of duo modules are:

1. uniserial Artinian modules [OHS2006].

2. self-injective self-cogenerator modules with commutative endomorphism rings (e.g. [Wis1991, 48.16]).
3. multiplication modules: \( R M \) is multiplication iff every \( L \leq_R M \) is of the form \( L = IM \) for some ideal \( I \) of \( R \), equivalently \( L = (L :_R M)M \). Multiplication modules over commutative rings have been studied intensively in the literature (e.g.\[AS2004\], [PC1995], [Smi1994]). Several results in these paper have been generalized by Tuganbaev (e.g. [Tug2003], [Tug2004]) to modules over rings close to be commutative (see [Abu] for a summary).

4. comultiplication modules: \( R M \) is comultiplication iff every \( L \leq_R M \) is of the form \( L = (0 :_M I) \) for some ideal \( I \) of \( R \), equivalently, \( L = (0 :_M (0 :_R L)) \). A commutative ring for which \( R R \) is a comultiplication module is called a dual ring. For examples and results on such modules, we refer mainly to [AS] and [A-TF2007].

**Notation.** With \( \mathcal{L}(M) \) (\( \mathcal{L}^{li.}(M) \)) we denote the lattice of (fully invariant) \( R \)-submodules of \( M \). Moreover, for every \( L \leq_R M \) we set

\[
\mathcal{K}^{li.}(L) := \{ \bar{L} \leq_R M \mid \bar{L} \subseteq L \text{ and } \bar{L} \leq^{li.}_R M \}. 
\]

**Lemma 2.3.** Let \( L \leq^{li.}_R M \). Then \( \mathcal{L}^{li.}(L) \subseteq \mathcal{K}^{li.}(L) \) with equality in case \( R M \) is self-injective. In particular, if \( R M \) is self-injective (and duo), then \( R L \) is self-injective (and duo).

2.4. By \( \mathcal{S}(M) \) (\( \mathcal{S}^{li.}(M) \)) we denote the (possibly empty) class of simple \( R \)-submodules of \( M \) (simple \( (R,S) \)-subbimodules of \( RM_S \), i.e. non-zero fully invariant \( R \)-submodules of \( M \) that have no non-zero proper fully invariant \( R \)-submodules). For every \( L \leq_R M \), we set

\[
\mathcal{S}(L) := \{ K \in \mathcal{S}(M) \mid K \subseteq L \}; \\
\mathcal{S}^{li.}(L) := \{ K \in \mathcal{S}^{li.}(M) \mid K \subseteq L \}.
\]

2.5. Let \( L \leq_R M \). We say that \( L \) is essential or large in \( M \), and write \( L \leq M \), iff \( L \cap \bar{L} \neq 0 \) for every \( 0 \neq \bar{L} \leq M \). On the other hand, we say \( L \) is superfluous or small in \( M \) and we write \( L \ll M \), iff \( L + \bar{L} \neq M \) for every \( \bar{L} \leq_R M \). With \( \text{Max}(R) \) we denote the spectrum of maximal sided ideals of \( R \) and with \( \text{Spec}(R) \) prime spectrum of \( R \); moreover, the prime radical of \( R \) is defined as

\[
\text{Prad}(R) := \bigcap_{P \in \text{Spec}(R)} P.
\]

The socle of \( M \) is defined as

\[
\text{Soc}(M) := \sum_{L \in \mathcal{S}(M)} L = \bigcap_{L \leq M} L \quad (:= 0 \text{ iff } \mathcal{S}(M) = \emptyset)
\]

Call a semisimple module \( M \) completely inhomogenous iff \( M \) is a direct sum of pairwise non-isomorphic simple submodules.
Definition 2.6. We say \( R_M \) is

colocal (or cocyclic \[Wis1991\], subdirectly irreducible \[AF1974\]) iff \( M \) contains a smallest non-zero \( R \)-submodule that is contained in every non-zero \( R \)-submodule of \( M \), equivalently iff
\[
\bigcap_{0 \neq L \leq_R M} L \neq 0;
\]
uniform, iff for any \( 0 \neq L_1, L_2 \leq_R M \), also \( L_1 \cap L_2 \neq 0 \), equivalently iff every non-zero \( R \)-submodule of \( M \) is essential;
atomic iff every \( 0 \neq L \leq_R M \) contains a simple \( R \)-submodule, equivalently iff \( \text{Soc}(L) \neq 0 \) for every \( 0 \neq L \leq_R M \);
f.i.-atomic iff \( \mathcal{S}_{\text{f.i.}}(L) \neq \emptyset \) for every \( 0 \neq L \leq_{R^1} M \), equivalently iff \( R \mathcal{M}_S \) is atomic.

Examples 2.7. ([HS2010]) Cofinitely generated modules (modules with finitely generated essential socles; called also \textit{finitely related} modules), semisimple modules and Artinian modules are atomic. All left modules over right perfect (e.g. left Artinian) rings are atomic. The Abelian group \( \mathbb{Z} \) is uniform but not atomic.

Definition 2.8. We call an \( R \)-module \( M \) an \textit{S-IAD-module} iff \( R \mathcal{M} \) is self-injective, atomic and duo.

Examples 2.9. The following are classes of S-IAD-modules:

1. self-injective Artinian uniserial modules: Artinian modules are atomic by \[HS2010\] and Artinian uniserial modules are duo by \[OHS2006\];

2. finitely generated self-injective self-cogenerator modules with commutative endomorphism rings: such modules are duo and finitely cogenerated by \[Wis1991\] 48.16, whence atomic.

3. self-injective duo left modules over right perfect rings;

4. self-injective duo modules that are cofinitely generated (resp. semisimple, Artinian).

Topological Spaces

In what follows, we fix some definitions and notions for topological spaces. For further information, the reader might consult any book in General Topology (e.g. \[Bou1966\]).

Definition 2.10. We call a topological space \( X \) \textit{(countably) compact}, iff every open cover of \( X \) has a finite subcover. Countably compact spaces are also called \textit{Lindelof spaces}. Note that some authors (e.g. \[Bou1966\] \[Bou1998\]) assume that compact spaces are in addition Hausdorff.

2.11. We say a topological space \( X \) is \textit{Noetherian} (Artinian), iff every ascending (descending) chain of open sets is stationary, equivalently iff every descending (ascending) chain of closed sets is stationary.
Definition 2.12. (e.g. [Bou1966], [Bou1998]) A non-empty topological space $X$ is said to be

1. *ultraconnected*, iff the intersection of any two non-empty closed subsets is non-empty.

2. *irreducible* (or *hyperconnected*), iff $X$ is not the union of two proper closed subsets; equivalently, iff the intersection of any two non-empty open subsets is non-empty.

3. *connected*, iff $X$ is not the disjoint union of two proper closed subsets; equivalently, iff the only subsets of $X$ that are *clopen* (i.e. closed and open) are $\emptyset$ and $X$.

2.13. ([Bou1966], [Bou1998]) Let $X$ be a non-empty topological space. A non-empty subset $A \subseteq X$ is an *irreducible set* in $X$ iff it’s an irreducible space w.r.t. the relative (subspace) topology; in fact, $A \subseteq X$ is irreducible iff for any proper closed subsets $A_1, A_2$ of $X$ we have

\[ A \subseteq A_1 \cup A_2 \Rightarrow A \subseteq A_1 \text{ or } A \subseteq A_2. \]

A maximal irreducible subspace of $X$ is called an *irreducible component*. An irreducible component of a topological space is necessarily closed. The irreducible components of a Hausdorff space are just the singleton sets.

Definition 2.14. Let $X$ be a topological space and $Y \subseteq X$ be an closed set. A point $y \in Y$ is said to be a *generic point*, iff $Y = \{y\}$. If every irreducible closed subset of $X$ has a unique generic point, then we call $X$ a *Sober space*.

Definition 2.15. A collection $G$ of subsets of a topological space $X$ is *locally finite*, iff every point of $X$ has a neighborhood that intersects only finitely many elements of $G$.

3 Fully Coprime Submodules

As before, $M$ is a non-zero unital left $R$-module. In this section, we topologize the spectrum $\text{Spec}^c(M)$ of submodules that are *fully coprime in $M$*. For more information on fully coprimeness, the interested reader can consult [Abu2006] (see also [Wij2000] and [WW2009]).

Notation. For subsets $L \subseteq M$ and $I \subseteq S$, we set

\[ \text{An}(L) := \{ f \in S \mid f(L) = 0 \} \text{ and } \text{Ke}(I) = \bigcap_{f \in I} \text{Ker}(f). \]

(1)

3.1. For $R$-submodules $X, Y \leq_R M$ we set

\[ X \odot_M Y := \bigcap \{ f^{-1}(Y) \mid f \in \text{An}(X) \} = \bigcap_{f \in \text{An}(X)} \{ \text{Ker}(\pi_Y \circ f : M \to M/Y) \}. \]

If $X \leq_R M$ is fully invariant, then $X \odot_M Y \leq_R M$ is also fully invariant; and if $Y \leq_R M$ is fully invariant, then $X + Y \subseteq X \odot_M Y$. 
Lemma 3.2. (See [Abu2006, Lemma 4.9]) Let $X, Y \leq_{R^i} M$. If $RM$ is self-cogenerator, then
\[ X \odot_M Y = \text{Ke} (\text{An}(X) \circ_{\text{op}} \text{An}(Y)). \tag{2} \]

Definition 3.3. We call a non-zero fully invariant $R$-submodule $0 \neq K \leq_{R^i} M$ fully coprime in $M$ iff for any fully invariant $R$-submodules $X, Y \leq_{R^i} M$:
\[ K \leq_{R} X \odot_{M} Y \Rightarrow K \leq_{R} X \text{ or } K \leq_{R} Y. \]

In particular, we say $RM$ is a fully coprime module iff for any fully invariant $R$-submodules $X, Y \leq_{R^i} M$:
\[ M = X \odot_{M} Y \Rightarrow M = X \text{ or } M = Y. \]

Proposition 3.4. ([Wij2006, 1.7.3], [WW2009, 3.7]) The following are equivalent:
1. $RM$ is fully coprime;
2. $M$ is $M/K$-generated for any $K \leq_{R^i} M$;
3. Any $M$-generated $R$-module is $M/K$-generated for any $K \leq_{R^i} M$.

Remark 3.5. The definition of fully coprime modules we adopt is a modification of the definition of prime modules introduced by Bican et. al. [BJKN80], where arbitrary submodules are replaced by fully invariant ones (we call such modules B-coprime). In fact, $RM$ is B-coprime if and only if $M$ is generated by each of its non-zero factor modules. Clearly, every B-coprime module is fully coprime. A duo module is B-coprime if and only if it is fully coprime. For more details on fully coprime modules, the reader is referred to [Abu2006] (see also [Wij2006] and [WW2009]).

Example 3.6. The Abelian group $\mathbb{Q}$ is fully coprime since it has no non-trivial fully invariant subgroups. Notice that $\mathbb{Z}/\mathbb{Q}$ is not B-coprime since $\mathbb{Q}$ is not generated by $\mathbb{Q}/\mathbb{Z}$.

3.7. We define
\[ \text{Spec}^{fc}(M) := \{ 0 \neq K \leq_{R^i} M \mid K \text{ is fully coprime in } M \}. \]

We say $RM$ is fc-coprimless, iff $\text{Spec}^{fc}(M) = \emptyset$. Moreover, for every $R$-submodule $L \leq_{R} M$ we set
\[ \mathcal{V}^{fc}(L) := \{ K \in \text{Spec}^{fc}(M) \mid K \subseteq L \}, \mathcal{K}^{fc}(L) := \{ K \in \text{Spec}^{fc}(M) \mid K \nsubseteq L \} \]
and
\[ \text{Corad}_{M}^{fc}(L) := \sum_{K \in \mathcal{V}^{fc}(L)} K \quad ( := 0, \text{ if } \mathcal{V}^{fc}(L) = \emptyset). \]

We say $L \leq_{R^i} M$ is fc-croradical iff $\text{Corad}_{M}^{fc}(L) = L$. In particular, we call $RM$ an fc-croradical module iff $\text{Corad}_{M}^{fc}(M) = M$.6
Remark 3.8. For any $L_1 \leq_R L_2 \leq_R M$ we have $\text{Corad}^f_M(L_1) \subseteq \text{Corad}^f_M(L_2)$. Moreover, for any $L \leq_R M$ we have
\[
\text{Corad}^f_M(\text{Corad}^f_M(L)) = \text{Corad}^f_M(L).
\]

3.9. A non-zero fully invariant $R$-submodule $L \leq_{fi}^f M$ will be called E-prime iff the ideal $\text{An}(K) \leq S$ is prime. With $\text{EP}(M)$ we denote the class of E-prime $R$-submodules of $M$.

Recall that $RM$ is said to be intrinsically injective iff $\text{AnKe}(I) = I$ for every finitely generated right ideal $I \leq S$. Every self-injective $R$-module is intrinsically injective (e.g. [Wis1991, 28.1]).

Proposition 3.10. ([Abu2006 Proposition 4.12]) If $RM$ is self-cogenerator, then $\text{EP}(M) \subseteq \text{Spec}^f(M)$ with equality if $RM$ is intrinsically injective. If moreover $S$ is right Noetherian, then
\[
\text{Prad}(S) = \text{An}(\text{Corad}^f(M)) \text{ and } \text{Corad}^f(M) = \text{Ke}(\text{Prad}(S)).
\]

Proposition 3.11. ([Abu2006 Proposition 4.7]) Let $0 \neq L \leq_{fi}^f M$. Then
\[
L^{fi}(L) \cap \text{Spec}^f(M) \subseteq \text{Spec}^f(L),
\]
with equality if $RM$ is self-injective. In particular, if $RM$ is self injective then $L$ is fully coprime in $M \iff RL$ is a fully coprime module.

Remark 3.12. Every $L \in S_{fi}(M)$ is trivially a fully coprime $R$-module. If $RM$ is self-injective, then $S_{fi}(M) \subseteq \text{Spec}^f(M)$ by Proposition 3.11, hence if, in addition, $M$ is f.i.-atomic, then for every $L \leq_{fi}^f M$ we have $\emptyset \neq S_{fi}(L) \subseteq \text{Spec}^f(L) = \mathcal{V}^f(L) \subseteq \text{Spec}^f(M)$.

Definition 3.13. Let $K \in \text{Spec}^f(M)$. We say that $K$ is maximal under $L$, where $0 \neq L \leq_{fi}^f M$ iff $K$ is a maximal element of $\mathcal{V}^f(L)$, equivalently $K \subseteq L$ and there is no $\tilde{K} \in \text{Spec}^f(M)$ that is contained in $L$ and contains $K$ strictly. We say that $K$ is maximal in $\text{Spec}^p(M)$ iff $K$ is maximal under $M$.

Lemma 3.14. Let $M$ be self-injective and f.i.-atomic. For every $0 \neq L \leq_{fi}^f M$ there exists $K \in \text{Spec}^f(M)$ which is maximal under $L$. In particular, $\text{Spec}^f(M)$ has a maximal element.

Proof. Let $0 \neq L \leq_{fi}^f M$. By Remark 3.12 $\emptyset \neq S_{fi}(L) \subseteq \mathcal{V}^f(L)$. Let
\[
K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq K_{n+1} \subseteq \cdots
\]
be an ascending chain in $\mathcal{V}^f(L)$ and set $\tilde{K} := \bigcup_{i=1}^{\infty} K_i$. Suppose that there exist two fully invariant $R$-submodules $L_1, L_2 \leq_{fi}^f M$ with $\tilde{K} \subseteq L_1 \cap_M L_2$ but $\tilde{K} \nsubseteq L_1$ and $\tilde{K} \nsubseteq L_2$. Then $K_{n_1} \nsubseteq L_1$ for some $n_1$ and $K_{n_2} \nsubseteq L_2$ for some $n_2$. Setting $n := \max\{n_1, n_2\}$, we have $K_n \subseteq L_1 \cap_M L_2$ while $K_n \nsubseteq L_1$ and $K_n \nsubseteq L_2$, a contradiction. So, $\tilde{K} \in \mathcal{V}^f(L)$. By Zorn’s Lemma, $\mathcal{V}^f(L)$ has a maximal element. In particular, $\text{Spec}^f(M) = \mathcal{V}^f(M)$ has a maximal element. \n
We introduce now a notion for modules which will prove to be useful in the sequel. Moreover, this notion seems to be of independent interest (see the survey in [Smi]).

3.15. We say that $R M$ has the min-property iff for any simple $R$-submodule $L \in S(M)$ we have $L \nsubseteq L_e$, where

$$L_e := \sum_{K \in S(M) \setminus \{L\}} K \quad (:= 0, \text{if } S(M) = \{L\}).$$

Since simple modules are cyclic, $R M$ has the min-property if and only if for any $L \in S(M)$ and any finite subset $\{L_1, \cdots, L_n\} \subseteq S(M) \setminus \{L\}$, we have $L \nsubseteq \sum_{i=1}^n L_i$. It is obvious that $R M$ has the min-property if and only if the class $S(M)$ of simple $R$-submodules is independent in the sense of [CLVW2006, page 8]. By [Smi, Theorem 2.3], $R M$ has the min-property if and only if all distinct simple $R$-submodules of $M$ are non-isomorphic (i.e. $\text{Soc}(M)$ is completely inhomogenous).

Examples 3.16. 1. Every $R$-module with at most one simple $R$-submodule (e.g. a colocal $R$-module) has the min-property.

2. Let $R$ have the property that $R/P$ is left Artinian for every primitive left ideal $P$ of $R$ (e.g. $R$ is a commutative ring, or a PI-ring, or a semilocal ring). Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be any collection of pairwise non-isomorphic simple $R$-modules and consider $M := \prod_{\lambda \in \Lambda} U_\lambda$. Then $\text{Soc}(R M) = \bigoplus_{\lambda \in \Lambda} U_\lambda$ is completely inhomogenous and so $R M$ has the min-property [Smi, Proposition 2.6].

3. Any finitely generated Artinian module over a commutative ring has the min-property if and only if it is a comultiplication module [AS, Theorem 3.11].

Lemma 3.17. If $R M$ is self-injective and duo, then $R M$ has the min-property.

Proof. Assume, without loss of generality, that $|S(M)| \geq 2$. Let $L \subseteq_R M$ be simple and suppose that $L \subseteq \sum_{i=1}^n L_i$ for $\{L_1, \cdots, L_n\} \subseteq S(M) \setminus \{L\}$. Then $L \subseteq L_1 \cap \cdots \cap L_n$. Since every simple $R$-submodule of $M$ is fully coprime in $M$ (see Remark 3.12), induction yields that $L = L_i$ for some $i = 1, \cdots, n$, a contradiction.

Top$^f$-modules

Notation. Set

$$\xi^f(M) := \{\mathcal{V}(L) \mid L \subseteq_R M\}; \quad \xi^f_{f,i}(M) := \{\mathcal{V}(L) \mid L \subseteq_{f,i} R M\};$$

$$\tau^f(M) := \{\mathcal{X}(L) \mid L \subseteq_R M\}; \quad \tau^f_{f,i}(M) := \{\mathcal{X}(L) \mid L \subseteq_{f,i} R M\};$$

$$Z^f(M) := (\text{Spec}^f(M), \tau^f(M)); \quad Z^f_{f,i}(M) := (\text{Spec}^f(M), \tau^f_{f,i}(m)).$$

Lemma 3.18. 1. $\mathcal{V}(0) = \emptyset$ and $\mathcal{V}(M) = \text{Spec}^f(M)$. 

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2. If \( \{L_\lambda \}_\Lambda \subseteq \mathcal{L}(M) \), then
\[
\bigcap_{\lambda} \mathcal{V}^{\text{fc}}(L_\lambda) = \mathcal{V}^{\text{fc}}(\bigcap_{\lambda} L_\lambda).
\]

3. If \( L, \widetilde{L} \in \mathcal{L}^{\text{f.i.}}(M) \), then
\[
\mathcal{V}^{\text{fc}}(L) \cup \mathcal{X}^{\text{fc}}(\widetilde{L}) = \mathcal{V}^{\text{fc}}(L + \widetilde{L}) = \mathcal{V}^{\text{fc}}(L \odot_M \widetilde{L}).
\]

**Proof.** Statements “1” and “2” and the inclusions \( \mathcal{V}^{\text{fc}}(L) \cup \mathcal{X}^{\text{fc}}(\widetilde{L}) \subseteq \mathcal{V}^{\text{fc}}(L + \widetilde{L}) \subseteq \mathcal{V}^{\text{fc}}(L \odot_M \widetilde{L}) \) in (3) are obvious. Let \( K \in \mathcal{V}^{\text{fc}}(L \odot_M \widetilde{L}) \), so that \( K \subseteq L \odot_M \widetilde{L} \). Since \( K \) is fully coprime in \( M \), we have \( K \subseteq L \) whence \( K \in \mathcal{V}^{\text{fc}}(L) \), or \( K \subseteq \widetilde{L} \) whence \( K \in \mathcal{V}^{\text{fc}}(\widetilde{L}). \)

For an arbitrary \( R \)-module \( M \), the set \( \xi^{\text{fc}}(M) \) is not necessarily closed under finite unions. This motivates the following

**Definition 3.19.** We call \( R^\text{fc}M \) an \( \textit{topfc-module} \) iff \( \xi^{\text{fc}}(M) \) is closed under finite unions.

**Notation.** For any \( \mathcal{A} \subseteq \text{Spec}^{\text{fc}}(M) \) set
\[
\mathcal{H}(\mathcal{A}) := \sum_{K \in \mathcal{A}} K \quad (:= 0, \text{ if } \mathcal{A} = \emptyset).
\]

As a direct consequence of Lemma 3.18 we get

**Theorem 3.20.** \( \mathcal{Z}_{\text{f.i.}}^{\text{fc}}(M) := (\text{Spec}^{\text{fc}}(M), \tau_{\text{f.i.}}^{\text{fc}}) \) is a topological space. In particular, if \( R^\text{fc}M \) is duo, then \( M \) is an \( \textit{topfc-module} \) and \( \mathcal{Z}_{\text{fc}}^{\text{fc}}(M) := (\text{Spec}^{\text{fc}}(M), \tau_{\text{fc}}^{\text{fc}}(M)) \) is a topological space.

**Lemma 3.21.** Let \( R^\text{fc}M \) be a \( \textit{topfc-module} \). The closure of any subset \( \mathcal{A} \subseteq \text{Spec}^{\text{fc}}(M) \) is
\[
\overline{\mathcal{A}} = \mathcal{V}^{\text{fc}}(\mathcal{H}(\mathcal{A})).
\]

**Proof.** Let \( \mathcal{A} \subseteq \text{Spec}^{\text{fc}}(M) \). Since \( \mathcal{A} \subseteq \mathcal{V}^{\text{fc}}(\mathcal{H}(\mathcal{A})) \) and \( \mathcal{V}^{\text{fc}}(\mathcal{H}(\mathcal{A})) \) is a closed set, we have \( \overline{\mathcal{A}} \subseteq \mathcal{V}^{\text{fc}}(\mathcal{H}(\mathcal{A})) \). On the other hand, suppose that \( H \in \mathcal{V}^{\text{fc}}(\mathcal{H}(\mathcal{A})) \) \( \setminus \mathcal{A} \) and let \( \mathcal{X}^{\text{fc}}(L) \) be a neighborhood of \( H \), so that \( H \not\subseteq L \). Then there exists \( W \in \mathcal{A} \) with \( W \not\subseteq L \) (otherwise \( H \subseteq \mathcal{H}(\mathcal{A}) \subseteq L \), a contradiction), i.e. \( W \in \mathcal{X}^{\text{fc}}(L) \cap (\mathcal{A} \setminus \{H\}) \) is a cluster point of \( \mathcal{A} \). Consequently, \( \overline{\mathcal{A}} = \mathcal{V}^{\text{fc}}(\mathcal{H}(\mathcal{A})). \)

**Remarks 3.22.** Let \( M \) be a \( \textit{topfc-module} \) and consider the Zariski topology \( \mathcal{Z}^{\text{fc}}(M) := (\text{Spec}^{\text{fc}}(M), \tau_{\text{fc}}^{\text{fc}}(M)) \).

1. \( \mathcal{Z}^{\text{fc}}(M) \) is a \( T_0 \) (Kolmogorov) space.

2. If \( R^\text{fc}M \) is duo, then \( \mathcal{B} := \{\mathcal{X}^{\text{fc}}(L) \mid L \leq_R M \text{ is finitely generated} \} \) is a basis of open sets for \( \mathcal{Z}^{\text{fc}}(M) \) : any \( K \in \text{Spec}^{\text{fc}}(M) \) is contained in some \( \mathcal{X}^{\text{fc}}(L) \) for some finitely generated \( R \)-submodule \( L \leq_R M \) (e.g. \( L = 0 \)). Moreover, if \( L_1, L_2 \leq_R M \) are finitely generated and \( K \in \mathcal{X}^{\text{fc}}(L_1) \cap \mathcal{X}^{\text{fc}}(L_2) \), then \( L := L_1 + L_2 \leq_R M \) is also finitely generated and we have \( K \in \mathcal{X}^{\text{fc}}(L) = \mathcal{X}^{\text{fc}}(L_1) \cap \mathcal{X}^{\text{fc}}(L_2). \)
3. If $L \in \text{Spec}^{fc}(M)$, then $\overline{\{L\}} = \mathcal{V}^{fc}(L)$. In particular, for any $K \in \text{Spec}^{fc}(M)$:

$$K \in \overline{\{L\}} \iff K \subseteq L.$$ 

4. If $R M$ is self-injective and duo, then

$$\mathcal{X}^{fc}(L) = \emptyset \Rightarrow \text{Soc}(M) \subseteq L.$$ 

The converse of (5) holds if, for example, $S(M) = \text{Spec}^{fc}(M)$.

5. Let $R M$ be a S-IAD-module. For every $L \leq_R M$ we have $\mathcal{V}^{fc}(L) = \emptyset$ if and only if $L = 0$.

6. Let $R M$ be self-injective and $0 \neq L \leq^{fi}_{R} M$. The embedding

$$\iota : \text{Spec}^{fc}(L) \to \text{Spec}^{fc}(M)$$

is continuous: this follows from Proposition 3.11, which implies that $\iota^{-1}(\mathcal{V}^{fc}(N)) = \mathcal{V}^{fc}(N \cap L)$ for every $R$-submodule $N \leq_R M$.

7. Let $M \cong N$ be an isomorphism of non-zero $R$-modules. Then we have a bijection

$$\text{Spec}^{fc}(M) \longleftrightarrow \text{Spec}^{fc}(N).$$

In particular, we have $\theta(\text{Corad}^{fc}_M(M)) = \text{Corad}^{fc}_{N}(N)$. Moreover, we have a homeomorphism $Z^{fc}(M) \approx Z^{fc}(N)$.

Notation. Set

$$\text{CL}(Z^{fc}(M)) := \{ \mathcal{A} \subseteq \text{Spec}^{fc}(M) \mid \mathcal{A} = \overline{\mathcal{A}} \} \text{ and } \mathcal{CR}^{fc}(M) := \{ L \leq_R M \mid \text{Corad}^{fc}_M(L) = L \}.$$

Theorem 3.23. Let $M$ be a top$^{fc}$-module.

1. We have an order-preserving bijection

$$\mathcal{CR}^{fc}(M) \longleftrightarrow \text{CL}(Z^{fc}(M)), \ L \mapsto \mathcal{V}^{fc}(L).$$

2. $Z^{fc}(M)$ is Noetherian if and only if $R M$ satisfies the DCC condition on fc-coradical submodules.

3. $Z^{fc}(M)$ is Artinian if and only if $R M$ satisfies the ACC condition on fc-coradical submodules.
Proof. Notice that for any $L \leq_R M$ we have $\mathcal{V}^c(L) \in \mathbf{CL}(\mathcal{Z}^c(M))$ and $\text{Corad}_M^c(L) \in CR^c(M)$ by Remark 3.8. Consider now
\[
\psi : \mathbf{CL}(\mathcal{Z}^c(M)) \to CR^c(M), \quad \mathcal{V}^c(L) \mapsto \text{Corad}_M^c(L).
\]
For every $L \in CR^c(M)$ we have
\[
\psi(\mathcal{V}^c(L)) = \text{Corad}_M^c(L) = L.
\]
Moreover, for every $\mathcal{A} := \mathcal{V}^c(K) \in \mathbf{CL}(\mathcal{Z}^c(M))$ we have
\[
\mathcal{V}^c(\psi(\mathcal{A})) = \mathcal{V}^c(\psi(\mathcal{V}^c(K))) = \mathcal{V}^c(\text{Corad}_M^c(K)) = \mathcal{V}^c(\mathcal{H}(\mathcal{A})) = \overline{\mathcal{A}} = \mathcal{A}.
\]
Notice now that (2) and (3) follow directly from (1) and so we are done. ■

**Theorem 3.24.** Let $M$ be a top$^c$-module. If $R M$ is Artinian (Noetherian), then $\mathcal{Z}^c(M)$ is Noetherian (Artinian).

**Proposition 3.25.** Let $R M$ be duo. Then $\mathcal{A} \subseteq \text{Spec}^c(M)$ is irreducible if and only if $\mathcal{H}(\mathcal{A})$ is fully coprime in $M$.

**Proof.** Let $R M$ be duo and $\mathcal{A} \subseteq \text{Spec}^c(M)$.

$(\Rightarrow)$ Assume that $\mathcal{A}$ is irreducible. By definition, $\mathcal{A} \neq \emptyset$ and so $\mathcal{H}(\mathcal{A}) \neq \emptyset$. Suppose that $\mathcal{H}(\mathcal{A})$ is not fully coprime in $M$, so that there exist $R$-submodules $X, Y \leq_R M$ with $\mathcal{H}(\mathcal{A}) \subseteq X \oplus M Y$ but $\mathcal{H}(\mathcal{A}) \nsubseteq X$ and $\mathcal{H}(\mathcal{A}) \nsubseteq Y$. It follows that $\mathcal{A} \subseteq \mathcal{V}^c(X \oplus M Y) = \mathcal{V}^c(X) \cup \mathcal{V}^c(Y)$ a union of two proper closed subsets, a contradiction. Consequently, $\mathcal{H}(\mathcal{A})$ is fully coprime in $M$.

$(\Leftarrow)$ Assume that $\mathcal{H}(\mathcal{A}) \in \text{Spec}^c(M)$. In particular, $\mathcal{H}(\mathcal{A}) \neq \emptyset$ and so $\mathcal{A} \neq \emptyset$. Suppose that $\mathcal{A} \subseteq \mathcal{V}^c(L_1) \cup \mathcal{V}^c(L_2) = \mathcal{V}^c(L_1 \oplus M L_2)$ for some $R$-submodules $0 \neq L_1, L_2 \leq_R M$. It follows that $\mathcal{H}(\mathcal{A}) \subseteq \mathcal{V}^c(L_1 \oplus M L_2)$ and it follows from our assumption that that $\mathcal{H}(\mathcal{A}) \subseteq L_1$ so that $\mathcal{A} \subseteq \mathcal{V}^c(L_1)$ or $\mathcal{H}(\mathcal{A}) \subseteq L_2$ so that $\mathcal{A} \subseteq \mathcal{V}^c(L_2)$. Consequently, $\mathcal{A}$ is not the union of two proper closed subsets, i.e. $\mathcal{A}$ is irreducible. ■

**Theorem 3.26.** Let $R M$ be duo.

1. $\text{Spec}^c(M)$ is irreducible if and only if $\text{Corad}_M^c(M)$ is fully coprime in $M$.

2. If $R M$ is self-injective, then $\mathcal{S}(M)$ is irreducible if and only if $\text{Soc}(M)$ is fully coprime in $M$.

**Example 3.27.** If $\mathcal{A} \subseteq \text{Spec}^c(M)$ is a chain, then $\mathcal{A}$ is irreducible. In particular, if $R M$ is uniserial, then $\text{Spec}^c(M)$ is irreducible.

**Proposition 3.28.** Let $R M$ be duo. The bijection \( \overline{D} \) restricts to bijections:
\[
\text{Spec}^c(M) \leftrightarrow \{ \mathcal{A} \mid \mathcal{A} \subseteq \text{Spec}^c(M) \text{ is an irreducible closed subset} \}
\]
and
\[
\text{Max}(\text{Spec}^c(M)) \leftrightarrow \{ \mathcal{A} \mid \mathcal{A} \subseteq \text{Spec}^c(M) \text{ is an irreducible component} \}.
\]
Proof. Recall the bijection $CR_{fc}(M) \xrightarrow{\psi_{fc}(\cdot)} CL(Z_{fc}(M))$.

Let $K \in \text{Spec}_{fc}(M)$. Then $K = H(V_{fc}(K))$ and it follows that the closed set $V_{fc}(K)$ is irreducible by Proposition 3.25. On the other hand, let $A \subseteq \text{Spec}_{fc}(M)$ be a closed irreducible subset. Then $A = V_{fc}(L)$ for some $L \subseteq R_M$. Notice that $H(A)$ is fully coprime in $M$ by Proposition 3.25 and that $A = \overline{A} = V_{fc}(H(A))$.

On the other hand, notice that $\text{Spec}_{fc}(M)$ has maximal elements by Lemma 3.14. If $K$ is maximal in $\text{Spec}_{fc}(M)$, then clearly $V_{fc}(K)$ is an irreducible component of $\text{Spec}_{fc}(M)$ by “1”. Conversely, let $Y$ be an irreducible component of $\text{Spec}_{fc}(M)$. Then $Y$ is closed. By “1”, $Y = V_{fc}(L)$ for some $L \in \text{Spec}_{fc}(M)$. Suppose that $L$ is not maximal in $\text{Spec}_{fc}(M)$, so that there exists $K \in \text{Spec}_{fc}(M)$ such that $L \subsetneq K \subseteq M$. It follows that $V_{fc}(L) \subsetneq V_{fc}(K)$, a contradiction since $V_{fc}(K) \subseteq \text{Spec}_{fc}(M)$ is irreducible by “1”. We conclude that $L$ is maximal in $\text{Spec}_{fc}(M)$.

Corollary 3.29. Let $R_M$ be duo. Then $\text{Spec}_{fp}(M)$ is a Sober space.

Proof. Let $A \subseteq \text{Spec}_{fc}(M)$ be a closed irreducible subset. By Proposition 3.28, $A = V_{fc}(K)$ for some $K \in \text{Spec}_{fp}(M)$. It follows that

$$A = \overline{A} = V_{fc}(H(A)) = V_{fc}(K) = \{K\},$$

i.e., $K$ is a generic point for $A$. If $L$ is a generic point of $A$, then it follows that $V_{fc}(K) = V_{fc}(L)$ whence $K = L$ since $K, L \in \text{Spec}_{fc}(M)$.

Proposition 3.30. Let $R_M$ be a $S$-IAD-module. Then $R_M$ is uniform if and only if $\text{Spec}_{fc}(M)$ is ultraconnected.

Proof. Assume that $R_M$ is uniform. If $V_{fc}(L_1), V_{fc}(L_2) \subseteq \text{Spec}_{fc}(M)$ are any two non-empty closed subsets, then $L_1 \neq 0 \neq L_2$ and so $V_{fc}(L_1) \cap V_{fc}(L_2) = V_{fc}(L_1 \cap L_2) \neq \emptyset$, since $L_1 \cap L_2 \neq 0$ contains by assumption some simple $R$-submodule whence which is indeed fully coprime in $M$. Conversely, assume that the intersection of any two non-empty closed subsets of $\text{Spec}_{fc}(M)$ is non-empty. Let $0 \neq L_1, L_2 \subseteq R_M$, so that $V_{fc}(L_1) \neq \emptyset \neq V_{fc}(L_2)$. By assumption $V_{fc}(L_1 \cap L_2) = V_{fc}(L_1) \cap V_{fc}(L_2) \neq \emptyset$, hence $L_1 \cap L_2 \neq 0$. Consequently, $R_M$ is uniform.

Theorem 3.31. Let $R_M$ be a $S$-IAD-module.

1. If $S(M)$ is countable, then $Z_{fc}(M)$ is countably compact.
2. If $S(M)$ is finite, then $Z_{fc}(M)$ is compact.

Proof. We prove only “1”, since “2” can be proved similarly. Assume that $S(M) = \{N_{\lambda_k}\}_{k \geq 1}$ is countable. Let $\{X_{fc}(L_\alpha)\}_{\alpha \in I}$ be an open cover of $\text{Spec}_{fc}(M)$, i.e. $\text{Spec}_{fc}(M) \subseteq \bigcup_{\alpha \in I} X_{fc}(L_\alpha)$. Since $S(M) \subseteq \text{Spec}_{fc}(M)$, we can pick for each $k \geq 1$, some $\alpha_k \in I$ such
that $N_{\lambda_k} \not\subseteq L_{\alpha_k}$. Suppose $\bigcap_{k \geq 1} L_{\alpha_k} \neq 0$. Since $RM$ is atomic, there exists some simple $R$-submodule $0 \neq N \subseteq \bigcap_{k \geq 1} L_{\alpha_k}$, a contradiction since $N = N_{\lambda_k} \not\subseteq L_{\alpha_k}$ for some $k \geq 1$.

Hence $\bigcap_{k \geq 1} L_{\alpha_k} = 0$ and we conclude that $\text{Spec}^{fc}(M) = \mathcal{X}^{fc}(\bigcap_{k \geq 1} L_{\alpha_k}) = \bigcup_{k \geq 1} \mathcal{X}^{fc}(L_{\alpha_k})$, i.e. $\{\mathcal{X}^{fc}(L_{\alpha_k}) \mid k \geq 1\} \subseteq \{\mathcal{X}^{fc}(L_{\alpha})\}_{\alpha \in I}$ is a countable subcover.

\textbf{Proposition 3.32.} Let $RM$ be duo and assume that $\text{Spec}^{fc}(M) = S(M)$.

1. If $RM$ has the min-property, then $\text{Spec}^{fc}(M)$ is discrete.

2. $M$ has a unique simple $R$-submodule if and only if $RM$ has the min-property and $\text{Spec}^{fc}(M)$ is connected.

\textbf{Proof.} 1. If $RM$ has the min-property, then for every $K \in \text{Spec}^{fc}(M) = S(M)$ we have $\{K\} = \mathcal{X}((K)_e)$, i.e. an open set. Since every singleton set is open, $\text{Spec}^{fc}(M)$ is discrete.

2. ($\Rightarrow$) Assume that $RM$ has the a unique simple $R$-submodule. Clearly, $RM$ has the min-property and $\text{Spec}^{fc}(M)$ is connected since it consists of only one point.

($\Leftarrow$) Assume that $RM$ has the min-property and that $\text{Spec}^{fc}(M)$ is connected. By "1", $\text{Spec}^{fc}(M)$ is discrete and so $S(M) = \text{Spec}^{fc}(M)$ has only one point since a discrete connected space cannot contain more than one-point.

\textbf{Theorem 3.33.} Let $RM$ be a S-IAD-module and assume that every fully coprime $R$-submodule of $M$ is simple.

1. $\text{Spec}^{fc}(M)$ is countably compact if and only if $S(M)$ is countable.

2. $\text{Spec}^{fc}(M)$ is compact if and only if $S(M)$ is finite.

As a direct consequence of Theorem 3.31 and Proposition 3.32 we obtain:

\textbf{Theorem 3.34.} Let $RM$ be a S-IAD-module and assume that every fully coprime $R$-submodule of $M$ is simple. Then $RM$ is colocal if and only if $\text{Spec}^{fc}(M)$ is connected.

\textbf{Lemma 3.35.} Let $RM$ self-injective self-cogenerator duo and $S$ be Noetherian with every prime ideal maximal. Then $S(M) = \text{Spec}^{fc}(M)$.

\textbf{Proof.} Notice that $S(M) \subseteq \text{Spec}^{fc}(M)$ (see Remark 3.12). If $K \in \text{Spec}^{fc}(M)$, then $\text{An}(K) \leq S$ is a prime ideal by Proposition 3.10 whence a maximal ideal by our assumption on $S$. It follows that $K = \text{Ke}(\text{An}(K))$ is simple: if $0 \neq K_1 \not\subseteq K$, for some $K_1 \leq_R M$, then $\text{An}(K) \not\subseteq \text{An}(K_1) \not\leq S$ since $\text{Ke}(-)$ is injective, a contradiction.
Lemma 3.37. Let \( R^M \) be a top \( R \)-module. If \( n \geq 2 \) and \( A = \{ K_1, ..., K_n \} \subseteq \text{Spec}^{lc}(M) \) is a connected subset, then for every \( i \in \{ 1, \ldots, n \} \), there exists \( j \in \{ 1, \ldots, n \} \setminus \{ i \} \) such that \( K_i \leq_R K_j \) or \( K_j \leq_R K_i \).

Proof. Without loss of generality, suppose \( K_1 \notin K_j \) and \( K_j \notin K_1 \) for all \( 2 \leq j \leq n \) and set 
\[
F := \sum_{i=2}^{n} K_i, \quad W_1 := A \cap X^{lc}(K_1) = \{ K_2, ..., K_n \} \quad \text{and} \quad W_2 := A \cap X^{lc}(F) = \{ K_1 \} \quad \text{(if} \ n = 2, \text{then clearly} \ W_2 = \{ K_1 \}; \text{if} \ n > 2 \text{and} \ K_1 \notin W_1, \text{then} \ K_1 \subseteq \sum_{i=2}^{n} K_i \subseteq (K_2 \otimes_M \sum_{i=3}^{n} K_i) \text{and it follows that} \ K_1 \subseteq \sum_{i=3}^{n} K_i. \text{One can show by induction that} \ K_1 \leq_R K_n, \text{a contradiction).}
\]

So \( A = W_1 \cup W_2 \), a disjoint union of proper non-empty open subsets, a contradiction.\( \blacksquare \)

Proposition 3.38. Let \( R^M \) be a S-IAD-module and let \( \emptyset \neq \mathcal{K} = \{ K_{\lambda} \}_{\Lambda} \subseteq S(M) \). If \( |S(L)| < \infty \) for every \( L \in \text{Spec}^{lc}(M) \), then \( \mathcal{K} \) is locally finite.

Proof. Let \( L \in \text{Spec}^{lc}(M) \) and set 
\[
\mathcal{F} := \sum_{K \in \mathcal{K} \cap X^{lc}(L)} K \quad \text{(} := 0, \text{if} \ \mathcal{K} \cap X^{lc}(L) = \emptyset \text{)}
\]

Notice first that \( L \notin \mathcal{F} : \) If \( L \subseteq F \), then there exists a simple \( R \)-submodule \( 0 \neq \tilde{K} \subseteq L \subseteq F \). Since \( M \) has the min-property by Lemma 3.17 we conclude that \( \tilde{K} = K \) for some \( K \in \mathcal{K} \cap X^{lc}(L) \), a contradiction. So, \( L \in X^{lc}(F) \). Since \( |S(L)| < \infty \), there exists (if any) a finite number of simple \( R \)-submodules \( \{ K_{\lambda_1}, \ldots, K_{\lambda_n} \} = \mathcal{K} \cap X^{lc}(L) \). It is clear that \( \{ K_{\lambda_1}, \ldots, K_{\lambda_n} \} = \mathcal{K} \cap X^{lc}(F) \) and we are done.\( \blacksquare \)

Lemma 3.39. If \( R^M \) be a S-IAD-module, then the following are equivalent for any \( L \leq_R M \):

1. \( L \in S(M) \);
2. \( L \) is fully coprime in \( M \) and \( \mathcal{Y}^{lc}(L) = \{ L \} \);
3. \( \{ L \} \) is closed in \( Z^{lc}_M \).

Proof. Notice that, by Remark 3.12 \( S(M) \subseteq \text{Spec}^{lc}(M) \). Let \( L \leq_R M \).

(1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are obvious.

(3) \( \Rightarrow \) (1) Assume that \( \{ L \} \) is closed in \( Z^{lc}_M \). Then \( \{ L \} = \mathcal{Y}^{lc}(K) \) for some \( K \leq_R M \).

If \( R^L \) is not simple then, since \( R^M \) is atomic, there exists some simple \( R \)-submodule \( \bar{L} \) of \( M \) such that \( \bar{L} \notin L \). It follows that \( \{ L, \bar{L} \} \subseteq \mathcal{Y}^{lc}(K) = \{ L \} \), a contradiction.\( \blacksquare \)
A topological space is $T_1$ if and only if every singleton set is close. In light of this, the previous lemma yields:

**Proposition 3.40.** If $R^M$ is an $S$-IAD-module, then $\text{Spec}^{fc}(M) = S(M)$ if and only if $\mathcal{Z}^{fc}(M)$ is $T_1$ (Fréchet space).

Combining the previous results we obtain

**Theorem 3.41.** Let $R^M$ be a $S$-IAD-module. The following are equivalent:

1. $\text{Spec}^{fc}(M) = S(M)$;
2. $\mathcal{Z}^{fc}(M)$ is discrete;
3. $\mathcal{Z}^{fc}(M)$ is $T_2$ (Hausdorff space);
4. $\mathcal{Z}^{fc}(M)$ is $T_1$ (Fréchet space).

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**References**

- [Abu] J. Y. Abuhlail, *A Zariski topology for modules*, preprint (arXiv: math.RA 1007.3149).
- [Abu2008] J. Y. Abuhlail, *A Zariski topology for bicomodules and corings*, Appl. Categ. Structures 16 (1-2) (2008), 13-28.
- [Abu2006] J. Y. Abuhlail, *Fully coprime comodules and fully coprime corings*, Appl. Categ. Structures 14 (5-6) (2006), 379-409.
- [AS] Y. Al-Shaniafi and P. F. Smith, *Comultiplication modules over commutative rings*, to appear in Comm. Algebra.
- [AF1974] F. W. Anderson and K. Fuller, *Rings and Categories of Modules*, Springer-Verlag (1974).
- [AS2004] M. Ali and D. J. Smith, *Some remarks on multiplication and projective modules*, Comm. Algebra 32 (1) (2004), 3897–3909.
- [AM1969] M. Atiyah and I. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co. (1969).
[A-TF2007] H. Ansari-Toroghy and H. Farshadifar, *The dual notion of multiplication modules*, Taiwanese J. Math. 11 (4) (2007), 1189–1201.

[BJKN80] L. Bican, P. Jambor, T. Kepka, P. Němec, *Prime and coprime modules*, Fund. Math. 107 (1) (1980), 33-45.

[Bou1998] N. Bourbaki, *Commutative algebra*, Springer-Verlag (1998).

[Bou1966] N. Bourbaki, *General Topology, Part I*, Addison-Wesley (1966).

[CLVW2006] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules*, Birkhäuser Verlag, Basel, 2006.

[Dur1994] T. Duraivel, *Topology on spectrum of modules*, J. Ramanujan Math. Soc. 9 (1) (1994), 25-34.

[Dau1978] J. Dauns, *Prime modules*, J. Rein. Ang. Math. 298 (1978), 165-181.

[Hir1978] Y. Hirano, *Some studies on strongly π-regular rings*, Math. J. Okayama Univ. 20 (1978), 141–149.

[HS2010] V. A. Hiremath and Poonam M. Shanbhag, *Atomic modules*, Int. J. Algebra, 4 (2) (2010), 61-69.

[Joh53] R. Johnson, *Representations of prime rings*, Trans. Amer. Math. Soc. 74 (1953), 351-357.

[Lu1984] Chin-Pi Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Paul. 33(1) (1984), 61-69.

[Lu1999] C-P. Lu, *The Zariski topology on the prime spectrum of a module*, Houston J. Math. 25 (3) (1999), 417-432.

[LY2006] D. Lu and W. Yu, *On prime spectrum of commutative rings*, Comm. Algebra 34 (2006), 2667–2672.

[MMS1997] R. McCasland, M. Moore and P. Smith, *On the spectrum of a module over a commutative ring*, Comm. Algebra 25 (1997), 79-103.

[MMS1998] R. McCasland, M. Moore and P. Smith, *An introduction to Zariski spaces over Zariski topologies*, Rocky Mountain J. Math. 28(4) (1998), 1357-1369.

[NT2001] R. Nekooei and L. Torkzadeh, *Topology on coalgebras*, Bull. Iran. Math. Soc. 27(2) (2001), 45-63.

[OHS2006] A. Özcan, A. Harmanci and P. F. Smith, *Duo modules*, Glasgow Math. J. 48 (2006), 33—545.
[PC1995] Y. S. Park and C. W. Choi, *Multiplication modules and characteristic submodules*, Bull. Korean Math. Soc. 32 (2) (1995), 321–327.

[RRRF-AS2002] F. Raggi, J. Ríos, H. Rincón, R. Fernández-Alonso, C. Signoret, *The lattice structure of preradicals. II*, Partitions, J. Algebra Appl. 1(2) (2002), 201-214.

[RRRF-AS2005] F. Raggi, J. Ríos, H. Rincón, R/ Fernández-Alonso and C. Signoret, *Prime and irreducible preradicals*, J. Alg. Appl. 4 (4) (2005), 451–466.

[RRW2005] F. Raggi, J. Ríos Montes and R. Wisbauer, *Coprime preradicals and modules*, J. Pur. App. Alg. 200 (2005), 51-69.

[Smi] P. Smith, *Modules with coinddependent maximal submodules*, to appear in J. Alg. Appl.

[Smi1994] P. Smith, *Multiplication modules and projective modules*, Period. Math. Hungar. 29 (2) (1994), 163–168.

[ST2010] N. Schwartz and M. Tressl, *Elementary properties of minimal and maximal points in Zariski spectra*, J. Algebra 323 (2010), 698–728.

[Tug2003] A. A. Tuganbaev, *Multiplication modules over noncommutative rings*, Sb. Math. 194 (11-12) (2003), 1837–1864.

[Tug2004] A. A. Tuganbaev, *Multiplication modules*, J. Math. Sci. (N.Y.) 123(2) (2004), 3839-3905.

[Wij2006] I. Wijayanti, *Coprime modules and comodules*, Ph.D. Dissertation, Heinrich-Heine Universität, Düsseldorf (2006).

[Wis1991] R. Wisbauer, *Foundations of module and ring theory. A handbook for study and research*. Gordon and Breach Science Publishers (1991).

[Wis1996] R. Wisbauer, *Modules and algebras : Bimodule structure and group action on algebras*, Addison Wesely Longman Limited (1996).

[WW2009] I. Wijayanti and R. Wisbauer, *Coprime modules and comodules*, Commun. Algebra 37 (4) (2009), 1308–1333.

[Zha2006-a] Guoyin Zhang, *Properties of top modules*, Int. J. Pure Appl. Math. 31 (3) (2006), 297–306.

[Zha2006-b] Guoyin Zhang, *Multiplication modules over any rings*, Nanjing Daxue Xuebao Shuxue Bannian Kan 23 (1) (2006), 59–69.

[ZTW2006] G. Zhang, W. Tong and F. Wang, *Spectrum of a noncommutative ring*, Comm. Algebra 34 (8) (2006), 2795–2810.