Schwinger Boson Theory of the Magnetic Spectrum of \( \text{Ba}_3\text{CoSb}_2\text{O}_9 \)

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The spin-1/2 triangular lattice Heisenberg Hamiltonian is a prototypical model to study the enhancement of quantum fluctuations induced by geometric frustration. Nearly fifty years ago, P. W. Anderson invoked this model to propose the celebrated resonance valence bond ground state that sparked the modern interest in quantum spin liquids. While later studies demonstrated that the ground state exhibits long-range magnetic ordering when the spin exchange interaction is restricted to nearest-neighbors, recent inelastic neutron scattering experiments on the triangular antiferromagnet \( \text{Ba}_3\text{CoSb}_2\text{O}_9 \) have revealed strong deviations from the dynamical spin structure factor predicted by semi-classical approaches. This work demonstrates that key qualitative and quantitative features of the data are reproduced by a parton Schwinger-boson theory, suggesting that the observed anomalies originate from the proximity of this material to a critical quantum melting point.

INTRODUCTION

Identifying new states of matter is a central theme of condensed matter physics. Although theorists have predicted an abundance of such states, it is often difficult to find experimental realizations. This is particularly a challenge for quantum spin liquids (QSLs), where the lack of smoking gun experimental signatures is forcing the community to develop more comprehensive approaches [1–3]. The singular interest in the fractionalized quasi-particles of these highly entangled states of matter resides on their potential application to quantum information [2, 4, 5]. However, it has been frustratingly difficult to detect these quasi-particles in real materials.

Beginning with Anderson’s proposal of the resonating valence bond state [6], the triangular geometry has long been studied as a platform for finding QSLs. Although the ground state of the simplest spin-1/2 model with nearest-neighbor antiferromagnetic Heisenberg interactions \( J_1 \) exhibits a 120° long-range magnetic order, geometric frustration makes this order weak [7]. Indeed, a next-nearest-neighbor exchange coupling \( J_2 \) as small as \( \approx 0.06 J_1 \) is enough to continuously melt the magnetic order into a QSL phase [8–15]. Determining the nature of the QSL is an ongoing theoretical challenge, with proposals ranging from gapped \( \mathbb{Z}_2 \) and gapless \( U(1) \) Dirac to chiral [8–14]. To discern among QSL candidates and the corresponding low-energy parton theories, it is imperative to make contact with experiments. Since most of the known realizations of the triangular lattice Heisenberg antiferromagnet (TLHA) lie on the ordered side of the quantum critical point (QCP) at \( J_2/J_1 \approx 0.06 \) [14–19], reproducing the measured excitation spectrum of these ordered magnets is currently the most stringent test for the alternative parton theories.

The idea of describing 2D frustrated antiferromagnets by means of fractional excitations (spinons) coupled to emergent gauge fields has been around for many years [20–24]. The Schwinger Boson theory (SBT) is one of the first parton formalizations that was introduced to describe ordered and disordered phases on an equal footing [22, 25]. However, a qualitatively correct (beyond the saddle point level) computation of the dynamical structure factor of magnetically ordered phases has been achieved only recently [26–28], enabling comparisons with inelastic neutron scattering (INS) measurements.

\( \text{Ba}_3\text{CoSb}_2\text{O}_9 \) is one of the best known realizations of a spin-1/2 nearest-neighbor triangular lattice Heisenberg antiferromagnet (TLHA) [29–32]. Inelastic neutron scattering studies of this material [29, 31, 32] reveal an unusual three-stage energy structure of the magnetic spectral weight. The lowest-energy or first stage is composed of dispersive branches of single-magnon excitations. The second and third stages correspond to dispersive continua that extend up to energies six times larger than the single-magnon bandwidth [31]. These observations are qualitatively and quantitatively consistent with the NLSWT [32, 33], suggesting that magnons should be described as bound states of spinons, which are the fractionalized quasiparticles of the neighboring QSL state. Here we investigate this hypothesis by comparing INS measurements of \( \text{Ba}_3\text{CoSb}_2\text{O}_9 \) [29, 31] against the SBT described in Refs. [26–28].

The comparisons presented in this work demonstrate that a low-order SBT provides an excellent starting point to reproduce the measured spectrum of low-energy excitations, including the magnon dispersion (first stage) reported in Ref. [29] and the dispersion of the broad low-energy peak that appears in the continuum (second stage) [29, 31]. Importantly, these results shed light on the nature of the proximate QCP and of the quantum spin liquid phase that is expected for \( J_2/J_1 \approx 0.06 \) [34]. The QCP is expected to have a dynamically generated \( O(4) \) symmetry [23, 35] and the quantum spin liquid corresponds to the zero-flux gapped \( \mathbb{Z}_2 \) spin liquid introduced by Sachdev [36], which on condensation of spinons leads to the 120° ordered state [37].
MATERIAL AND MODEL

Ba$_3$CoSb$_2$O$_9$ comprises vertically stacked triangular layers of effective spin-1/2 moments arising from the $J = 1/2$ Kramers doublet of Co$^{3+}$ in a trigonally-distorted octahedral ligand field. Excited multiplets are separated by a gap of 200–300 K due to spin-orbit coupling, which is much larger than the Néel temperature $T_N = 3.8$ K. Below $T_N$, the material develops conventional 120° ordering with wavevector $Q = (1/3, 1/3, 1)$ [38].

The theoretical modelling of different experimental results [32, 33, 39, 40] indicates that the magnetic properties of Ba$_3$CoSb$_2$O$_9$ are well described by a nearest-neighbor Heisenberg model:

$$\mathcal{H} = \sum_{(i,j)} J_{ij} \left( S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z \right),$$

where $(i,j)$ restricts the sum to nearest-neighbor intralayer and interlayer bonds with exchange interactions $J_{ij} = J$ and $J_{ij} = J_c$, respectively. The parameter $\Delta$ accounts for a small easy-plane exchange anisotropy. The high-symmetry structure of this material forbids Dzyaloshinskii-Moriya interactions between any pair of Co ions in the same $ab$ plane or relatively displaced along the $c$-axis. The set of in-plane Hamiltonian parameters reported in Refs. [29, 31, 33] coincide with each other within a relative error $\sim 5\%$. Correspondingly, here we adopt values $J = 1.66$ meV, and $\Delta = 0.937$ that are consistent with the reported values. As for the inter-plane exchange, we adopt the value $J_c = 0.061$ J, which is in between the values $J_c = 0.05 J$ and $J_c = 0.08$ J reported in Refs. [33] and [29, 31], respectively.

As expected for this effective spin model, experiments confirmed a one-third magnetization plateau (up-up-down phase) induced by a magnetic field parallel to the easy-plane [39–44]. While the dynamical spin structure factor of the up-up-down phase is well described by NLSW [33, 45], the observed zero field magnon dispersions cannot be described with any known semi-classical treatment [31, 32], suggesting that quantum renormalization effects in zero field are underestimated by a perturbative 1/$S$ expansion. These strong quantum fluctuations can be attributed to the proximity of the TLHA to the above-mentioned “quantum melting point” that signals a continuous $T = 0$ transition into a quantum spin liquid.

SCHWINGER BOSON THEORY

The SBT [20, 25, 26] starts from a parton representation of the spin operators expressed as bilinear forms of spin-1/2 bosons that represent the spinors of the theory: $\hat{S}_i = \frac{1}{2} \hat{b}_i^\dagger \sigma \hat{b}_i$, where $\hat{b}_i^\dagger = (b_i^1, b_i^2)$, and $\sigma \equiv (\sigma^x, \sigma^y, \sigma^z)$ is the vector of Pauli matrices. The spin-1/2 representation of the spin operator is enforced by the constraint $b_i^1 b_i^1 + b_i^2 b_i^2 = 1$.

The XXZ interaction in Eq. (1) is expressed in terms of SU(2) spin-rotation invariant bond operators [46], $A_{ij} = \frac{1}{2} (b_{ij}^1 b_{ij}^1 - b_{ij}^2 b_{ij}^2)$. $B_{ij} = \frac{1}{2} (b_{ij}^1 b_{ij}^1 + b_{ij}^2 b_{ij}^2)$, and the U(1) spin-rotation invariant bond operators $C_{ij} = \frac{1}{2} (b_{ij}^1 b_{ij}^1 - b_{ij}^2 b_{ij}^2)$ and $D_{ij} = \frac{1}{2} (b_{ij}^1 b_{ij}^1 + b_{ij}^2 b_{ij}^2)$ are required to account for the finite uniaxial anisotropy. The operator $A_{ij}^\dagger (D_{ij}^\dagger)$ creates a singlet (triplet) state on the bond $ij$. The operator $B_{ij}$ moves singlets and triplets from the bond $ij$ to the bond $ik$ preserving their character. In contrast, the operator $C_{ij}$ promotes a singlet-bond $ij$ into a triplet-bond $ik$ and vice versa. Up to an irrelevant constant, the spin-spin interaction is expressed as (see Supplementary Material [47]),

$$S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z = -2 \left( \frac{\Delta - 1}{2} - \alpha \right) A_{ij}^\dagger A_{ij} + 2 \alpha :B_{ij}^\dagger B_{ij}: + \frac{\Delta - 1}{2} (C_{ij}^\dagger C_{ij} - D_{ij}^\dagger D_{ij})$$

where the real parameter $\alpha$ fixes the decoupling scheme of a path integral formulation over coherent states. Similarly to the case of KYbSe$_2$ [34], we set $\alpha = 0.436$ and follow the procedure described in Ref. [26]. After introducing the auxiliary fields $\lambda$ (Lagrange multiplier that enforces the local constraint) and $W_{ij}^X$ with $X = A, B, C$ and $D$ (bond fields obtained via a Hubbard-Stratonovich transformation), we obtain a partition function expressed as a path integral over these fields

$$Z[j] = \int [D\overline{W}D\overline{W}] [D\lambda] e^{-S_{\text{eff}}(\overline{W}, \lambda)}$$

where the symmetry-breaking field $h$ is introduced to select a broken symmetry ground state with 120° Néel order (this field is sent to zero at the end of the calculation). The effective action can be divided into two contributions: $S_{\text{eff}}(\overline{W}, \lambda, j) = S_0(\overline{W}, \lambda) + S_{\text{bos}}(\overline{W}, \lambda, j)$, with

$$S_0(\overline{W}, \lambda) = \int_0^\beta d\tau \left( \sum_{ij, X} J_{ij} W_{ij}^{(X)\tau} W_{ij}^{(X)\tau} - i2S \sum_i \lambda_i^\dagger \right),$$

and

$$S_{\text{bos}}(\overline{W}, \lambda, j, h) = \frac{1}{2} \text{Tr} \ln \left[ \mathcal{G}^{-1}(\overline{W}, \lambda, j, h) \right].$$
over space, time, and boson indices. The bosonic degrees of freedom can be formally integrated out to obtain
\[ Z_{\text{bos}}(\mathbf{W}, W, \lambda, j, h) = \int \mathcal{D} \mathcal{D} \mathcal{D} b e^{\mathcal{L} - \mathcal{G}^{-1}(\mathbf{W}, W, \lambda, j, h) b} \]
= \text{Det}[\mathcal{G}(\mathbf{W}, W, \lambda, j, h)] . \tag{6}

At the saddle-point (SP) level (uniform and static auxiliary fields), the theory describes non-interacting spin-1/2 spinons, whose condensation leads to 120° magnetic ordering within each triangular layer and antiferromagnetic ordering between adjacent layers [20, 25, 46]. The dynamical spin susceptibility in frequency \( i\omega \) and momentum \( q \) space is given by [25]
\[ \chi_{\mu\nu}(q, i\omega) = \lim_{N_s \rightarrow \infty} \frac{1}{N_s} \text{Tr} \left[ G^{sp} \partial_{\mu} q_{i\omega} G^{sp} \partial_{\nu} q_{-i\omega} \right] . \tag{7} \]
where \( \mu, \nu = x, y, z \) and \( N_s \) is the total number of spins. Following the procedure described in detail in Ref. 26, we compute the 1/\( N_s \) correction by including the Gaussian fluctuations of the auxiliary fields \( W^{sp}, W^{ch}, W^{po} \) and \( \lambda \) (\( N \) is the number of bosonic flavors). At this level, the resulting dynamical spin susceptibility takes the form
\[ \chi_{\mu\nu}(q, i\omega) = \chi_{\mu\nu}^{sp}(q, i\omega) + \chi_{\mu\nu}^{0}(q, i\omega) , \]
where
\[ \chi_{\mu\nu}^{sp}(q, i\omega) = \frac{1}{2} \text{Tr} \left[ G^{sp} \partial_{\mu} q_{i\omega} G^{sp} \partial_{\nu} q_{-i\omega} \right] . \tag{8} \]
denotes the contribution obtained at the saddle-point level, and
\[ \chi_{\mu\nu}^{0}(q, i\omega) = \sum_{\alpha_1, \alpha_2} \frac{1}{2} \text{Tr} \left[ G^{sp} \nu_{\alpha_1} G^{sp} \partial_{\mu} q_{i\omega} \right] \]
\[ \times \frac{D_{\alpha_1, \alpha_2}(q, i\omega)}{2} \text{Tr} \left[ G^{sp} \nu_{\alpha_2} G^{sp} \partial_{\nu} q_{-i\omega} \right] . \tag{9} \]
is the contribution from fluctuations around the saddle-point solution that acts as a counter-diagram for \( \chi_{\mu\nu}^{sp}(q, i\omega) \) [28]. \( G^{sp} \) is single-spinon propagator \( \mathcal{G} \) evaluated at the saddle point solution, while the propagator of the auxiliary fields \( D_{\alpha_1, \alpha_2}(q, i\omega) \) is the inverse of the fluctuation matrix \( \mathcal{S}_{\alpha_1, \alpha_2}^{(2)} \) (see Supplementary Material [47]). The traces are taken over momentum \( k \), Matsubara frequency \( \omega \), and boson indices. The internal and external vertices, \( \nu_{\alpha} = \delta \mathcal{G}^{-1} / \delta q_{\alpha, i\omega} \) and \( u_{\mu}(q, i\omega) = \partial \mathcal{G}^{-1} / \partial q_{\mu, i\omega} \), couple the spinons to the auxiliary fields \( \phi_{\alpha} \equiv \left\{ W_{k, \omega^\prime}, W_{k, \omega^\prime}, \lambda_{k} \omega^\prime \right\} \) and to the external fields, respectively. The contributions \( \chi_{\mu\nu}^{sp}(q, i\omega) \) and \( \chi_{\mu\nu}^{0}(q, i\omega) \) are represented as Feynman diagrams in Figs. 1 (a) and (b), respectively. The Lagrange multiplier \( \lambda \) and the phases of \( W \) and \( \mathbf{W} \) are the gauge fields of the SBT. The effective spinon-spinon interactions induced by the fluctuations of these fields produce a drastic change of the excitation spectrum. Indeed, the counter diagram shown in Fig. 1 (b) exactly cancels the residues of the single-spinon poles of \( \chi_{\mu\nu}(q, i\omega) \) [28] and the true collective modes (magnons) of the antiferromagnetically ordered phase emerge as two-spinon bound states associated with the new poles of \( \chi_{\mu\nu}(q, i\omega) \). Moreover, the two-spinon continuum is strongly renormalized by this correction [26, 27].

The total neutron scattering cross-section at \( T = 0 \), including the neutron polarization factor, is given by
\[ I(q, \omega) = f^2(q) \sum_{\mu} \left( 1 - \frac{q_{\mu}^2}{q^2} \right) S_{\mu\mu}(q, \omega) . \tag{10} \]
where \( f(q) \) is the spherical magnetic form factor for Co\(^{2+} \) ions and \( S_{\mu\mu}(q, \omega) = -1/\pi \text{Im} \left[ \chi_{\mu\mu}(q, \omega) \right] \) is the dynamical spin structure factor.

**SINGLE-MAGNON DISPERSION**

As we mentioned before, the lowest-energy part (first stage) of the spectrum of magnetic excitations of Ba\(_3\)CoSb\(_2\)O\(_9\) is composed of different branches of sharp single-magnon excitations with no decay throughout the reciprocal space [29]. The failure of NLSWT has motivated an empirical parametrization of the single-magnon dispersion with more than ten fitting parameters [29]. As we will see in this section, the SBT described in the previous section reproduces the measured dispersion to a very good approximation. Remarkably, the only tuning parameter is \( \alpha = 0.436 \), which turns out to be very close to the value \( \alpha = 0.5 \) adopted in previous works [16, 26, 27, 48, 49].

Figures 2 (a) and (c) show an overview of the measured excitation spectrum of Ba\(_3\)CoSb\(_2\)O\(_9\) as a function of energy and wave-vector along a representative path in momentum space [29]. According to the notation introduced in Ref. [29], the wave vector labels \( \Gamma, M \) and \( K \) refer to the conventional high-symmetry points in the 2D hexagonal Brillouin zone (BZ), where an unprimed (primed) label indicates \( l = 0 \) (\( l = 1 \)) and numbered subscripts refer to symmetry related distinct points when reduced to the first BZ. The scattering intensity is strongest around the magnetic Bragg wave vectors \( K'_{1,2} \), from which a linearly dispersing in-plane Goldstone mode emerges. The second out-of-plane mode is gapped because of the easy-plane anisotropy. A clear roton-like minimum appears in the lower energy mode at the \( M'_1 \) point while the higher energy mode exhibits a flattened dispersion.

Figures 2 (b) and (d) show the zero-temperature neutron scattering cross section \( I(q, \omega) \) obtained from the two diagrams shown in Fig. 1. In addition, Fig. 3 shows a comparison between the measured single-magnon dispersion and the theoretical result. Both comparisons reveal that the overall single magnon dispersion is very well reproduced by the theory, that predicts a magnon velocity \( c_m \approx 1.2J \). The only noticeable discrepancies are the small roton-like anomalies near the \( M' \) and \( K'/2 \) points. Returning to Figure 2, the overall spectral weight modulation of the sharp magnons is also well reproduced over the whole Brillouin zone, except for the points that exhibit the roton-like anomaly, as we will show later in more detail.
FIG. 2. Comparison between the INS measurements of \(\text{Ba}_3\text{CoSb}_2\text{O}_9\) reproduced from Ref. [29] (left column) and the zero-temperature \(I(q, \omega)\) computed with the SBT [26–28] (right column). In all figures included in this work the calculated \(I(q, \omega)\) was multiplied by a single overall intensity scale factor to compare with the observed experimental intensity data. The brackets on the right-hand side in (b) indicate the energy cuts plotted in Fig. 5. The wave-vector path in (a) and (b) is \(\Gamma'(0,1,1)\rightarrow K'_1(1/3,1/3,1)\rightarrow M'_2(0,1/2,1)\rightarrow K'_2(-1/3,2/3,1)\rightarrow \Gamma'\), shown in the inset. In (c) and (d), the path is \(K_2(-1/3,2/3,0)\rightarrow M_2(0,1/2,0)\rightarrow \Gamma(0,1,0)\rightarrow K_2\rightarrow K'_2(-1/3,2/3,1)\rightarrow M'_2(0,1/2,1)\rightarrow \Gamma'(0,1,1)\).

The quantitative agreement between the measured single-magnon dispersion and the SBT is remarkable if we consider that NLSWT predicts a single-magnon bandwidth of 2.4meV, which is more than 40% higher than the experimental value [32]. The combination of both results indicates that a non-interacting spinon gas is a better starting point to describe the low-energy magnons of \(\text{Ba}_3\text{CoSb}_2\text{O}_9\). In other words, magnons that are obtained as two-spinon bound states can account for the strongly renormalized single-magnon dispersion of \(\text{Ba}_3\text{CoSb}_2\text{O}_9\). Another important consequence of the composite nature of the single-magnon excitations is the emergence of a two-spinon continuum with a well-defined modulation as a function of energy and wave-vector, whose analysis is the focus of the next section.

The lack of the roton-like anomalies and the corresponding renormalization of the single-magnon spectral weight are expected shortcomings of the current level of approximation, if we consider that the diagrams shown in Fig. 1 correspond to the lowest order approximation that is required to obtain the true collective modes (magnons) of the theory. In other words, these diagrams do not include self-energy corrections of the single-spinon and the auxiliary field propagators. It is well known that roton-like anomalies arise in NLSWT only after including self-energy \(1/S\) corrections to the bare single-magnon propagator [50–53]. In the case of the SBT, self-energy corrections to the single-spinon propagator also renormalize the single-magnon dispersion because magnons are two-spinon bound states. This renormalization is expected to shift the position of the magnon-peaks relative to the onset of the two-spinon continuum. As shown in Fig. 4, the overlap between the...
higher energy magnon at the M’ point and the two-spinon continuum leads to a strong reduction of the spectral weight that is not observed in the experiment, where the separation between the magnon peak and the continuum is roughly 0.3 meV [see Figures 2(a)]. Based on these observations we conjecture that, while the diagrams shown in Fig. 1 are a good starting point to reproduce the single-magnon dispersion of Ba$_3$CoSb$_2$O$_9$, the roton-like anomalies will arise from self-energy corrections of the single-spinon and/or single-magnon propagators.

CONTINUUM SCATTERING

As anticipated in the previous section, the measured INS intensity exhibits a highly structured continuum above the sharp magnon peaks. Figs. 2 (a) and (b) reveal that this continuum has strong intensity modulations and clear dispersion across the Brillouin zone. Together with the strong renormalization of the single-magnon dispersion, this particular structure of the continuum scattering poses a stringent test for various theories that aim to explain the nature of the magnetic excitations of Ba$_3$CoSb$_2$O$_9$.

To facilitate the comparison with INS data [29], Fig. 5 shows intensity maps of the measured and calculated $I(q, \omega)$ at different energies. The comparison at energies below the top of the magnon band, shown in panels (a)-(h), confirms the above-discussed overall agreement between experiment and theory, except for the missing roton-like anomalies in the theoretical calculation that explain the differences between the (c) and (g) panels near the M points.

Panels (i)-(q) show the intensity maps arising from the continuum scattering just above the top of the one magnon dispersion. As observed in the experiment, the continuum intensity obtained from the SBT is centered at the K points with a clear three-fold symmetric pattern. The ring patterns around K that become apparent at slightly higher energies and transform into triangular contours with corner touchings at the M-points are also reproduced by the SBT. However, the ring patterns of the theoretical calculation are rotated by an angle $\pi/3$ relative to the ring patterns of the experimental data [see panels (i) and (m)]. Once again, we attribute this difference to the absence of the roton-like anomaly in the theoretical calculation. The spectral weight of the measured upper magnon peak near the M points is significantly larger than the calculated spectral weight. It is then expected that self-energy corrections to the single-spinon propagator, which account for the roton like anomaly, should transfer spectral weight from the low-energy continuum (around $E \approx 2$ meV) to the upper magnon peak. The excess of continuum spectral weight near the M points at the current level of approximation explains the $\pi/3$ rotation of the ring patterns and the “bridges” that connect adjacent rings in panel (j), which do not have a counterpart in the experimental data shown in panel (n). Finally, as it is clear from the comparison between panels (l) and (q), the relatively large spectral weight of the measured continuum scattering in the high energy interval ranging from 3.6 meV to 3.8 meV is not reproduced by the SBT at the current level of approximation. Indeed, the two-stage structure of the continuum scattering becomes evident in Fig. 4, which shows an average along $l$ of the neutron scattering cross section at $(1/2,1/2,1)$, which is almost independent of $l$ [31]. Because the calculated $I((1/2,1/2,1),\omega)$ is also practically insensitive to $l$, we are comparing the experimental data against the calculated $I(M’,\omega)$. We note that the SBT predicts a ratio between the spectral weight of the continuum and the weight of the magnon peaks that is approximately equal to 3. This ratio is significantly higher than the value $\approx 0.66$ that
FIG. 5. Intensity maps of $S(q, \omega)$ as a function of momentum in the $hk$ plane at a series of constant energies. The results have been integrated over an energy range that is indicated at the top of each panel to facilitate the comparison with INS data reproduced from Ref. [29].

is obtained from NLSWT [33] and more consistent with the experimental data. It is also clear from this figure that the measured continuum scattering extends up to at least $E \approx 6$ meV, which is roughly equal to four times the single-magnon bandwidth [29, 31]. The two diagrams included in our SBT only account for the lower energy stage. Once again, we attribute this discrepancy to the lack of 4-spinon contributions arising from self-energy corrections to the single-spinon propagator not included in Fig. 1.

**DISCUSSION**

In summary, our detailed comparison between the SBT and the measured neutron scattering cross section of Ba$_3$CoSb$_2$O$_9$ reveals that, in contrast to the traditional semi-classical approaches, a low-order expansion in the control parameter $(1/N)$, that recovers the linear spin wave result in the large $S$ limit [27, 28], provides a very good framework to describe the magnetic excitations of quasi-2D TLHA. We attribute the failure of the large-$S$ expansion to the proximity of Ba$_3$CoSb$_2$O$_9$ to a quantum melting point or QCP that signals the onset of a QSL. Since the elementary excitations of the QSL phase are quasi-free single-spinons, a free spinon gas is a better starting point than a free magnon gas close enough to the QCP. Magnons are then recovered on the magnetically ordered side of the QCP as two-spinon bound states (poles of the RPA propagator) induced by fluctuations of the gauge fields.

The eventual success of an improved SBT has important implications regarding the nature of the QSL state. Indeed, the gapped $Z_2$ QSL state proposed by Sachdev [36] is the only spin liquid state compatible with the SBT theory, which
does not break any symmetries and can be continuously connected with 120° Néel ordered state, since it has lowest energy modes at the K-points [37]. (The low energy excitations of the other alternative π-flux state, are gapped at the K-points and gapless at the M-points.) The resulting quantum critical point is expected to have a dynamically generated $O(4)$ symmetry [23, 35]. It is important to note that alternative parton theories with fermionic matter fields lead to different spin liquid state on the other side of the QCP, such as gapless $U(1)$ spin liquid [13, 54]. However, while existing attempts to reproduce the unusual excitation spectrum of the ordered phase using fermionic partons seem to account for the roton-like anomaly, they have not been compared against the available experimental data [55, 56].

We note that a large continuum scattering has also been observed in ladder and spatially anisotropic triangular systems—experimentally realized in CaCu$_2$O$_3$ and Cs$_2$CuCl$_4$ compounds [57, 58], respectively. This continuum has been attributed to 1D-spinons which are confined by the inter-chain interactions. This is in sharp contrast with the 2D character of the spinons invoked in this work, which are the building blocks of the SBT and interact via emergent gauge fields consisting of the Lagrange multiplier and phases of the bond fields $W_{ij}$.

Our results have implications for other quantum magnets that are described by a similar model. For instance, the delafossite triangular lattice materials, such as CsYbSe$_2$ [59] and NaYbSe$_2$ [60], could lie even closer to the quantum melting point and the triangular layers of Ba$_3$CoTeO$_6$ [61] exhibit an INS spectrum that is remarkably similar to the one of Ba$_3$CoSb$_2$O$_9$. More generally, the SBT presented in this work can in principle be applied to other highly frustrated magnets, such as the $J_1-J_2$ square and honeycomb antiferromagnets, near continuous or quasi-continuous quantum phase transitions between magnetically ordered and QSL states.

Note: After completion of this work we became aware of an alternative interpretation of the origin of the high-energy excitation of the triangular antiferromagnet [62]. Furthermore, a very recent tensor network study of the triangular XXZ model [63] reinforces the validity this model to quantitatively describe the magnetic excitations of Ba$_3$CoSb$_2$O$_9$.

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Supplementary Material: Schwinger Boson Theory of the Magnetic Spectrum of Ba$_3$CoSb$_2$O$_9$

Here we compute the dynamical structure factor at $T = 0$ for the XXZ model of Eq. (1). In Section I, we express the XXZ model in terms of the bond operators that are the main building blocks of our theory. In Section II, we start from the partition function expressed as the functional integral over SU(2) coherent states and we derive the effective action by introducing Hubbard-Stratonovich transformations that decouple the quartic terms of the Hamiltonian. Finally, in Section III, we obtain the expression of the dynamical spin susceptibility –at the Gaussian level– and the dynamical structure factor that is compared with the inelastic neutron scattering data in the main text.

I. Schwinger Boson Theory of the XXZ model

Starting with the XXZ model

$$\mathcal{H} = \sum_{\langle i,j \rangle} J_{ij} \left[ S^x_i S^x_j + S^y_i S^y_j + \Delta S^z_i S^z_j \right], \quad (S1)$$

where $\langle i,j \rangle$ restricts the sum to intralayer and interlayer nearest-neighbor. The components of the spin-spin interactions can be rewritten in terms of the Schwinger boson as

$$S^x_i S^x_j + S^y_i S^y_j = \frac{1}{2} \left( :B^\dagger_{ij} B_{ij} : - A^\dagger_{ij} A_{ij} - C^\dagger_{ij} C_{ij} : + D^\dagger_{ij} D_{ij} : \right) \quad (S2)$$

and

$$S^z_i S^z_j = \frac{1}{2} \left( :B^\dagger_{ij} B_{ij} : - A^\dagger_{ij} A_{ij} + C^\dagger_{ij} C_{ij} : - D^\dagger_{ij} D_{ij} : \right), \quad (S3)$$

with the bond operators $[S1]$ $A_{ij} = \frac{1}{2} (b_{i\uparrow} b_{j\downarrow} - b_{i\downarrow} b_{j\uparrow}^\dagger)$, $B_{ij} = \frac{1}{2} (b_{i\uparrow}^\dagger b_{j\uparrow} + b_{j\downarrow}^\dagger b_{i\downarrow})$ $C_{ij} = \frac{1}{2} (b_{i\uparrow} b_{j\uparrow}^\dagger - b_{j\uparrow} b_{i\uparrow}^\dagger)$, $D_{ij} = \frac{1}{2} (b_{i\downarrow} b_{j\uparrow} + b_{i\uparrow} b_{j\downarrow}^\dagger)$.

The bond operators $A_{ij}$ and $B_{ij}$ are invariant under the SU(2) group of global spin rotations, while $C_{ij}$ and $D_{ij}$ are invariant under the U(1) subgroup of global spin rotations about the $z$ axis. The identity

$$A^\dagger_{ij} A_{ij} + B^\dagger_{ij} B_{ij} = S^2 \quad (S6)$$

allows us to write the XXZ model with different weights of $A_{ij}$ and $B_{ij}$ by means of the continuous parameter $\alpha$ $[S2]$

$$\mathcal{H} = \sum_{\langle i,j \rangle} J_{ij} \left[ -2 \left( \frac{1 + \Delta}{2} - \alpha \right) A^\dagger_{ij} A_{ij} + 2 \alpha :B^\dagger_{ij} B_{ij} : - \frac{1 - \Delta}{2} (C^\dagger_{ij} C_{ij} : - D^\dagger_{ij} D_{ij} : ) + 2 S^2 \left( \frac{\Delta + 1}{4} - \alpha \right) \right], \quad (S7)$$

For any value of $\alpha$, the Hamiltonian preserves the XXZ model symmetry U(1) × Z$_2$. Notice that the contributions of $C_{ij}$ and $D_{ij}$ arise from the anisotropy ($\Delta < 1$) of the XXZ model.

II. Derivation of the Effective Action

The partition function for $\mathcal{H}$ is expressed as a functional integral over coherent states $[S3]$

$$\mathcal{Z}[\beta] = \int [D\bar{b} Db] \ e^{ - \int_0^\beta d\tau \left[ \sum_i \bar{b}^\dagger_i \partial_\tau \bar{b}_i + \mathcal{H}(\bar{b}, b) + \mathcal{J}_s + \mathcal{J}_\beta \right] } \times \ e^{ - \int_0^\beta d\tau \ i \sum_i b^\dagger_{i\mu} (\sum_{\nu} \bar{b}^\dagger_i b_{i\nu} - 2 S^2) }, \quad (S8)$$

where $\mathcal{J}_s = \sum_i j^i_\tau \sigma^\mu b^\dagger_i \sigma^\mu \ b^\dagger_i \sigma^\mu$ represents the Zeeman coupling to a space ($i$) and time ($\tau$) dependent external field $j^i_\tau$ ($\mu = x, y, z$ and $\sigma^\mu$ are the Pauli matrices), while $\mathcal{J}_\beta = \sum_i b^\dagger_{i\mu} \sigma^\mu \ b^\dagger_{i\mu}$ corresponds to a linear coupling between...
the order parameter and a small symmetry breaking static field \( h = h(\cos(Q \cdot r_i), \sin(Q \cdot r_i), 0) \) where \( Q = (\frac{1}{3}, \frac{1}{3}, 1) \) is the ordering wave vector in terms of reciprocal lattice units \( (h, k, l) \), corresponding to the 120° Néel ordering in each triangular layer and antiferromagnetic ordering between adjacent layers. The integral over \( \lambda \) accounts for the local constraint, \( b_0^+ b_0 + b_4^+ b_4 = 2S \), and the integration measures are \( [D\delta Db] = \prod_{i\tau\sigma} \frac{db_i^+ db_{i\tau}^\sigma}{2\pi i} \), and \( [D\lambda] = \prod_{i\tau} \frac{d\lambda_i^\tau}{2\pi i} \).

Each quartic term (in \( \delta b \) and \( b \)) of \( \mathcal{H}(\delta b, b) \), is decoupled into quadratic terms by means of Hubbard-Stratonovich (HS) transformations, with two auxiliary fields for each bond operator, \( W^X \) and \( W^X \) with \( X = A, B, C, D \):

\[
e^{\text{sgn}(J_{ij})J_{ij}[X^\dagger_{ij}X_{ij}] = |J_{ij}| \int dW_{ij}^X dW_{ij}^X e^{-|J_{ij}|W_{ij}^X W_{ij}^X + |J_{ij}|(\text{sgn}(J_{ij})W_{ij}^X X_{ij}^\dagger + W_{ij}^X X_{ij}^\dagger)},
\]

After the HS transformation, the partition function becomes a path integral over the auxiliary fields

\[
Z[j] = \int [D\delta D][D\lambda] e^{-S_{\text{eff}}(W,W',j,h)},
\]

where the effective action can be divided into two contributions: \( S_{\text{eff}}(W, W, \lambda, j, h) = S_0(W, W, \lambda) + S_{\text{bos}}(W, W, \lambda, j, h) \), with

\[
S_0(W, W, \lambda) = \int_0^\beta d\tau \left( \sum_{ij,X} J_{ij} W_{ij}^X W_{ij}^{X\dagger} - i2S \sum_i \lambda_i^\tau \right),
\]

and

\[
S_{\text{bos}}(W, W, \lambda, j, h) = \frac{1}{2} \text{Tr} \ln \left[ \mathcal{G}^{-1}(W, W, \lambda, j, h) \right] = -\frac{1}{2} \ln Z_{\text{bos}}(W, W, \lambda, j, h),
\]

with integration measure \( [D\delta D] = \prod_{ij,X} |J_{ij}| dW_{ij}^X dW_{ij}^{X\dagger} \). Here, \( \mathcal{G}^{-1} = \mathcal{M} \) is the bosonic dynamical matrix, and the trace is taken over, space, time and boson indices. The bosonic partition function \( Z_{\text{bos}} \) can be formally integrated to get

\[
Z_{\text{bos}}(W, W, \lambda, j, h) = \int [D\delta Db] e^{-\delta b^\dagger \mathcal{G}^{-1}(W, W, \lambda, j, h) \delta b} = \text{Det} \left[ \mathcal{G}(W, W, \lambda, j, h) \right].
\]

III. Dynamical Spin Susceptibility

The dynamical spin susceptibility in frequency \( i\omega \) and momentum \( q \) space is given by [S4]

\[
\chi_{\mu\nu}(q, i\omega) = \lim_{N_s \to \infty} \lim_{h \to 0} \frac{\partial^2 \ln Z[j]}{\partial j_{\mu}^q i\omega \partial j_{\nu}^{q\dagger} - i\omega},
\]

where \( N_s \) is the total number of spins. Following the procedure described in detail in Ref. [S3], we compute the \( 1/N \) correction by including the Gaussian fluctuations of the auxiliary fields \( W^A, W^B, W^C, W^D \) and \( \lambda \). At this level, the resulting dynamical spin susceptibility takes the form

\[
\chi(q, i\omega) = \chi_{\mu\nu}^{\text{sp}}(q, i\omega) + \chi_{\mu\nu}^{\text{fl}}(q, i\omega),
\]

where sp and fl denote the contributions from the saddle point and the fluctuations of the effective action, respectively:

\[
\chi_{\mu\nu}^{\text{sp}}(q, i\omega) = \frac{1}{2} \text{Tr} \left[ \mathcal{G}^{\text{sp}} u^\mu(q, i\omega) \mathcal{G}^{\text{sp}} u^\nu(-q, -i\omega) \right],
\]

and

\[
\chi_{\mu\nu}^{\text{fl}}(q, i\omega) = \sum_{\alpha_1, \alpha_2} \frac{1}{2} \text{Tr} \left[ \mathcal{G}^{\text{sp}} v_{\phi_{\alpha_1}} \mathcal{G}^{\text{sp}} u^{\mu}(q, i\omega) \right] \times D_{\alpha_1, \alpha_2}(q, i\omega) \times \frac{1}{2} \text{Tr} \left[ \mathcal{G}^{\text{sp}} v_{\phi_{\alpha_2}} \mathcal{G}^{\text{sp}} u^{\nu}(-q, -i\omega) \right].
\]

Fig. 1 of the main text shows the Feynman diagrams representing the saddle-point \( \chi_{\mu\nu}^{\text{sp}}(q, i\omega) \) [Fig. 1(a)] and the fluctuation \( \chi_{\mu\nu}^{\text{fl}}(q, i\omega) \) [Fig. 1(b)] contributions to the dynamical spin susceptibility. Here we have only included the
contribution from the fluctuations of the auxiliary fields that acts as a counter diagram [Fig. 1(b)] for the saddle point contribution [Fig. 1(a)] [S2, S3]. The traces in equations (S15) and (S16) are taken over momentum $k$, Matsubara frequency $\omega'$, and boson indices. The expressions $v_{\phi_n} = \frac{\delta g_{\phi_n}}{\delta j_{q_{\mu}}}$ and $u^\mu(q, i\omega) = \partial g^{-1}/\partial j_{q_{\mu}}$ are the internal and external vertices, respectively, where $\phi_n$ denotes the auxiliary fields $\{W_{k,\delta}^{X\omega}, W_{k,\delta}^{X\omega'}, \lambda_{k}\}$. $\delta$ represents the intralayer and interlayers nearest-neighbors, and $\mu = x, y, z$. The saddle-point spinon propagator is given by

$$G_{\text{sp}}(k, i\omega') = \frac{g_{k}^{+}}{i\omega' - \varepsilon_k} + \frac{g_{k}^{++}}{i\omega' - \varepsilon_k} + \frac{g_{k}^{-}}{i\omega' - \varepsilon_k} + \frac{g_{k}^{++}}{i\omega' + \varepsilon_k},$$

(S17)

whose inverse, in the representation $\tilde{b}_{k}^{'} = (b_{k}^{1}, b_{-k}^{1}, b_{k}^{2}, b_{-k}^{2})$, takes the form

$$(G_{\text{sp}})^{-1} = \begin{pmatrix}
(i\omega' + \gamma_k^{BC} e^{i\omega'0^+}) & -\gamma_k^{AD} & 0 \\
0 & -\frac{h}{2} & 0 \\
0 & 0 & -\gamma_k^{BC} e^{i\omega'0^+} \\
-\gamma_k^{AD} & 0 & -\frac{h}{2}
\end{pmatrix}$$

(S18)

with

$$\gamma_k^{AD} = \sum_{\delta > 0} J_\delta (1 + \Delta - 2\alpha) A_\delta \sin(k \cdot \delta) - \sum_{\delta > 0} J_\delta \frac{1 - \Delta}{2} D_\delta \cos(k \cdot \delta)$$

(S19)

$$\gamma_k^{BC} = \sum_{\delta > 0} J_\delta 2\alpha B_\delta \cos(k \cdot \delta) - \sum_{\delta > 0} J_\delta \frac{1 - \Delta}{2} C_\delta \sin(k \cdot \delta).$$

(S20)

The convergence factors $e^{\pm i\omega'0^+}$ arise from the time ordering of the boson fields $\bar{b}$ and $b$ in the action.

In Eq. (S19) and (S20), $A_\delta, B_\delta, C_\delta, D_\delta$ and $\lambda$ are the real mean-field parameters that are related to the saddle-point solutions of the auxiliary fields through the static and homogeneous ansatz [S3]: $W_A^\delta|_{\text{sp}} = iA_\delta$, $\overline{W_A^\delta}|_{\text{sp}} = -iA_\delta$, $W_B^\delta|_{\text{sp}} = -B_\delta$, $\overline{W_B^\delta}|_{\text{sp}} = -iC_\delta$, $W_C^\delta|_{\text{sp}} = -iC_\delta$, $W_D^\delta|_{\text{sp}} = D_\delta$, $\overline{W_D^\delta}|_{\text{sp}} = D_\delta$, $\lambda_{\text{sp}} = i\lambda$. These mean-field parameters are found by numerically solving the saddle-point condition $\partial S_{\text{eff}}/\partial \phi_{\alpha}|_{\text{sp}} = 0$, that leads to the following set of self-consistent equations:

$$A_\delta = -\frac{1}{N_\delta} \sum_k [(g_{k}^{-})^{12} + (g_{k}^{+})^{12}] \sin(k + Q/2) \cdot \delta,$$

(S21)

$$B_\delta = -\frac{1}{N_\delta} \sum_k [(g_{k}^{-})^{11} + (g_{k}^{+})^{11}] \cos(k + Q/2) \cdot \delta,$$

(S22)

$$C_\delta = -\frac{1}{N_\delta} \sum_k [(g_{k}^{-})^{11} + (g_{k}^{+})^{11}] \sin(k + Q/2) \cdot \delta,$$

(S23)

$$D_\delta = -\frac{1}{N_\delta} \sum_k [(g_{k}^{-})^{12} + (g_{k}^{+})^{12}] \cos(k + Q/2) \cdot \delta,$$

(S24)

$$S = -\frac{1}{N_\delta} \sum_k [(g_{k}^{+})^{11} + (g_{k}^{-})^{11}] \sin(k + Q/2) \cdot \delta.$$  

(S25)

These mean-field parameters determine the saddle-point spinon propagator $G_{\text{sp}}$. The RPA propagator of the fluctuation fields is given by

$$D_{\alpha_1 \alpha_2} = \frac{1}{2^{(2)}} \int [D\phi D\bar{\phi}] \phi_{\alpha_1} \bar{\phi}_{\alpha_2} e^{-\sum_{\alpha} \bar{\phi}_{\alpha} S_{\alpha}^{(2)} \phi_{\alpha}} = \left[(S^{(2)})^{-1}\right]_{\alpha_1 \alpha_2},$$

(S26)
where

\[ Z^{(2)} = \int [D\tilde{\phi} D\phi] e^{-\sum_{\alpha\alpha'} \tilde{\phi}_\alpha S^{(2)}_{\alpha\alpha'} \phi_{\alpha'}}. \]  \quad (S27)

The fluctuation matrix \( S_{\alpha_1\alpha_2}^{(2)} \) at the saddle-point solution:

\[ S_{\alpha_1\alpha_2}^{(2)} = \left. \frac{1}{2} \frac{\partial^2 S_{\text{eff}}}{\partial \tilde{\phi}_{\alpha_1} \partial \tilde{\phi}_{\alpha_2}} \right|_{\text{sp}} = \frac{1}{2} \left\{ \frac{\partial^2 S_0}{\partial \tilde{\phi}_{\alpha_1} \partial \tilde{\phi}_{\alpha_2}} - \frac{1}{2} \text{Tr} \left[ G_{\text{sp}}^{\text{pp}} v_{\phi_{\alpha_1}} G_{\text{sp}}^{\text{pp}} v_{\phi_{\alpha_2}} \right] \right\}. \quad (S28)\]

Finally, the analytic continuation \( i\omega \rightarrow \omega + i\eta \) yields the dynamical spin susceptibility, whose imaginary part determines the dynamical structure factor at \( T = 0 \):

\[ S(q, \omega) = -\frac{1}{\pi} \text{Im} \left[ \chi(q, \omega) \right]. \quad (S29)\]

This expression must be inserted in Eq. (10) of the main text to obtain inelastic neutron scattering cross section.

\[ \begin{align*}
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\end{align*} \]