Hidden teleportation power for entangled quantum states

Jyun-Yi Li,†, * Xiao-Xu Fang,‡, * Ting Zhang,§ Gelo Noel M. Tabia,†, 3, † He Lu,‡, † and Yeong-Cherng Liang†, §

1 Department of Physics and Center for Quantum Frontiers of Research & Technology (QFort), National Cheng Kung University, Tainan 701, Taiwan
2 School of Physics, Shandong University, Jinan 250100, China
3 Center for Quantum Technology, National Tsing Hua University, Hsinchu 300, Taiwan
(Dated: August 5, 2020)

Ideal quantum teleportation transfers an unknown quantum state intact from one party Alice to the other Bob via the use of a maximally entangled state and the communication of classical information. If Alice and Bob do not share entanglement, the teleportation fidelity, i.e., the maximal average fidelity between the state to be teleported and the state received, is upper bounded by a function $f_c$ that is inversely proportional to the Hilbert space dimension. If they share an entangled state $\rho$ with teleportation fidelity $f < f_c$ but upon successful local filtering, the teleportation fidelity becomes larger than $f_c$, we say that $\rho$ has hidden teleportation power. Here, we show that a non-vanishing interval of two-qudit Werner states exhibit hidden teleportation power for all dimensions larger than two. In contrast, for a family of two-qudit rank-deficient states, their teleportation power is hidden only if the Hilbert space dimension is less than or equal to three. Using hybrid entanglement prepared in photon pairs, we also provide the first experimental demonstration of the activation of teleportation power (hidden in this latter family of qubit states). The connection between hidden teleportation power with the closely-related problem of entanglement distillation is discussed.

In quantum information science, entanglement [1] arises naturally as a resource within the paradigm of local operations assisted by classical communications (LOCC). Sharing entanglement is in fact essential for exhibiting a quantum advantage over classical resources in computation [2,3], secret key distribution [4], superdense coding [5], and metrology [6], etc. Among the many possibilities that entanglement empowers, quantum teleportation [7], i.e., the transfer of quantum states with the help of shared entanglement and classical communication, is especially worth noting (see, e.g., [8,9] for a review on some advances in this topic).

Indeed, teleportation serves as a primitive in various quantum protocols such as remote state preparation [10], entanglement swapping [11] and quantum repeaters [12]. In universal quantum computing with linear optics, it enables near-deterministic two-qubit gates [13] and makes assembling cluster states more efficient [14,15]. Recently, it was used in an experiment to provably demonstrate the scrambling of quantum information [16]. It has also been employed as a theoretical tool for exploring closed timelike curves [17] and black hole evaporation [18]. In this work, we compare entangled states to classical resources for the task of teleportation.

In the original protocol [7], two remote parties, which we call Alice and Bob, share an entangled pair of qubits. By performing a joint measurement on her half of the entangled qubit and another (unknown) qubit $|\psi\rangle$ given to her, Alice may transfer $|\psi\rangle$ to Bob by transmitting only the classical measurement outcome to Bob. The quality of this state transfer is measured [19] by the teleportation fidelity $[20,21]$, which measures the average overlap between the state Alice wants to teleport and the state Bob receives.

To teleport a quantum state perfectly, sharing a maximally entangled state is imperative. However, due to decoherence, this ideal resource is often not readily shared between remote parties, thus giving rise to a nonideal teleportation fidelity. In fact, when the entanglement is too weak, the resulting teleportation fidelity can even be simulated by Alice and Bob adopting a measure-and-prepare scheme [19], without sharing any entanglement. Thus, whenever an entangled state yields a teleportation fidelity larger than the classical threshold of $f_c = \frac{2}{d^2+1}$ [22]—with $d$ being the dimension of the local state space—it is conventionally said to be useful for teleportation, but otherwise useless (see [23] for some other notion of nonclassicality).

Importantly, certain desirable features of an entangled state may be activated by means of so-called local filtering [24] operations (see, e.g., [25–31]). Here, we investigate such activation for the usefulness in teleportation: if Alice and Bob share an entangled state $\rho$ useless for teleportation and if after successfully applying an appropriate local filtering operation, they obtain a filtered state $\rho_f$ useful for teleportation, we say that $\rho$ has hidden teleportation power. Bound entangled [32] states are useless for teleportation and have no hidden teleportation power [22]. Meanwhile, all entangled isotropic states [33] are useful for teleportation. In fact, the usefulness of isotropic states for teleportation cannot be improved even if we allow the more general class of positive partial transpose-preserving (filtering) operations [34].

In contrast, all two-qubit entangled states are either already useful or can exhibit hidden teleportation power [35–37] (with the corresponding optimal filters determined in [36]). What about higher-dimensional entangled states? In this work, we show that for all local dimensions $d \geq 3$, there exist entangled Werner states [38] that exhibit hidden teleportation power. At the same time, there exist a family of rank-deficient states that show hidden teleportation power only for $d \leq 3$. Using this latter family of qubit states as an example, we give an experimental demonstration of the activation process using photon
pairs and hybrid entanglement. For that matter, we first make use of quantum state tomography (QST) [39] to certify the uselessness of the prepared state and then quantum process tomography [39] as well as QST again to certify the usefulness of the filtered entangled state.

For any two qudit state $\rho$, determining its maximal teleportation fidelity $f(\rho)$ (and hence its usefulness) is a priori not trivial, as it requires one to perform an integral over all pure states according to the Haar measure. The seminal work of Horodecki et al. [22], however, established that $f(\rho)$ is monotonically related to its fully entangled fraction (FEF), $F_d(\rho)$, also called the singlet fraction, as

$$f(\rho) = \frac{F_d(\rho)d + 1}{d + 1}, \quad F_d(\rho) = \max_{|\Psi_d\rangle} \langle \Psi_d | \rho | \Psi_d \rangle \tag{1}$$

where the optimization is to be carried out over all $d$-dimensional maximally entangled states, i.e., $|\Psi_d\rangle = (I_d \otimes U)|\Phi^+_d\rangle$, $I_d$ is the $d \times d$ identity matrix, $U$ is a unitary matrix and $|\Phi^+_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_i |i\rangle_i$. The classical (prepare-and-measure) threshold $F_c = \frac{2}{d+1}$ corresponds to $F_d = \frac{1}{2}$. Hence, a quantum state $\rho$ is useful for teleportation if and only if $F_d(\rho) > F_c$.

Boosting teleportation power.– We are interested to activate the usefulness for teleportation by means of local filtering, or equivalently stochastic LOCC (SLOCC), i.e., LOCC operations that succeed only probabilistically [40]. Formally, an SLOCC acting on a bipartite system $\rho$ gives $\tau = (A \otimes B)\rho(A \otimes B)^\dagger$, where $A, B$ are $d \times d$ matrices representing the operation of local filters. Without loss of generality, we may demand that $\|A\|_\infty \leq 1$, $\|B\|_\infty \leq 1$, where $\| \cdot \|_\infty$ is the Schatten $\infty$-norm [41]. Then, $p = \text{tr}(\tau)$ is the probability of success in performing the local filtering operations. Conditioned on this success, the resulting filtered state is $\rho_f = \frac{\tau}{p}$. Throughout, we use the subscript “$f$” to signify the filtered state.

As remarked in [36], whenever the filtering probability $p \neq 0$, the process of boosting teleportation power can be made deterministic by preparing a (pure) separable state $\rho_{\text{sep}}$ whenever the filtering operation fails. In doing so, one obtains an output state $\rho_{\text{ave}} = p\rho_f + (1 - p)\rho_{\text{sep}}$ that is the weighted average between the filtered state $\rho_f$ and the separable state $\rho_{\text{sep}}$. In Appendix A, we provide the explicit Kraus decomposition for such a process. From here, it is clear that there are two natural figures of merit to be maximized in the context of boosting the teleportation power of $\rho$: $F_d(\rho_f)$ and $F_d(\rho_{\text{ave}})$. In fact, as we illustrate in Appendix B, via an appropriate choice of filters, the maximum value of $F_d(\rho_f)$ can always be attained by setting $|\Psi_d\rangle = |\Phi^+_d\rangle$ in Eq. (1). Consequently, in the maximization of $F_d(\rho_{\text{ave}})$, called the cost function $K(\rho)$ in [36], we may set $\rho_{\text{sep}} = |0\rangle\langle 0| \otimes |0\rangle\langle 0|$, thereby giving $K(\rho) = F_d(\rho_{\text{ave}}) = pF_d(\rho_f) + \frac{1+p}{2}$. Importantly, as we show in Appendix B, so long as $p \neq 0$, the possibility for $\rho$ to exhibit hidden teleportation power is independent of which figure of merit one maximizes, even though the optimal filter(s), and hence the final FEF may be different.

**Werner states.–** Consider the Werner state [38]: $W(v) = \frac{2v}{d(d+1)}P_+ + \frac{2(1-v)}{d(d+1)}P_-$, $v \in [0, 1]$, where $P_\pm = (I_d \pm \sqrt{d} P) / 2$ is the projector onto the (anti)symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$ and $V = \sum_{i,j=0}^{d-1} |i\rangle_i \otimes |j\rangle_j |i\rangle_i$ is the swap operator. $W(v)$ is entangled if and only if $0 \leq v < \frac{1}{d}$. Surprisingly, it follows from the analytic form of $F_d(W(v))$ determined in [42] (see Appendix C) that for all $d > 2$, all (entangled) $W(v)$ are useless for teleportation. Here, we show that there is always a nonvanishing interval of $v$ for which these Werner states have hidden teleportation power.

Indeed, our numerical optimization suggests that the FEF of the filtered state may be maximized using the qubit filter:

$$A_F(W) = \sigma_z \oplus 0_{d-2}, \quad B_F(W) = \sigma_x \oplus 0_{d-2}, \tag{2}$$

where $\sigma_x, \sigma_z$ are Pauli matrices. In fact, local-filtering both Hilbert spaces of $W(v)$ onto the same qubit subspace span$\{|j\rangle, |k\rangle, j \neq k$ would be sufficient. The additional unitary transformation effected by the Pauli matrices, however, makes it evident that that the FEF of the filtered state $W_F(v)$ is maximized using $|\Phi^+_d\rangle$ since $W_F(v) = \frac{1}{2} \langle d+1 \rangle (1 - v)|\Phi^+_d\rangle \langle \Phi^+_d| + v(d-1)\|\Phi^+_d\|\langle \Phi^+_d| \oplus 0_{d-2}$, where the success probability is $p_F(v) = \frac{d+1}{d(d+1)}$ and $N = (d+1)(1 + v) + 3v(d-1)$ is a normalization factor. Explicitly, $W_F(v)$ has a FEF given by $F_d(W_F(v)) = \frac{2(d+1)(1-v)}{d N}$, which exceeds the classical threshold of $\frac{1}{2}$ if

$$v < v_{cr} = \frac{d + 1}{4d - 2} = \frac{1}{4} + \frac{3}{4d - 2}. \tag{3}$$

That is, for $d > 2$, $W(v)$ exhibits hidden teleportation power whenever $0 \leq v < v_{cr}$. Clearly, $v_{cr}$ decreases monotonically with $d$ but even when $d \to \infty$, $v_{cr}$ does not vanish and thus for $d \geq 3$, there is always a finite interval of $v$ where $W(v)$ shows hidden teleportation power. These results are schematically summarized in Fig. 1. In Appendix C, we also demonstrate how the increase in FEF $[F_d(W_F) - F_d(W)]$ varies with the success probability $p_F(v)$ of local filtering.

Two other remarks are in order. Firstly, even if we take into account of the success probability and maximize the cost-function $K(W(v))$, the optimal filters again appear to be those
given by Eq. (2). Secondly, although our criterion for the usefulness of teleportation of a two-qudit state is based on teleporting a qudit state of the same dimension, a little calculation shows that if the locally-filtered Werner state $W_f(v)$ is useful for teleportation (i.e., having $F_d(W_f(v)) > \frac{1}{d} \sqrt{d}$), it is also useful for teleporting a quantum state of dimension $d'$, i.e., $F_{d'}(W_f(v)) > \frac{1}{d'} \sqrt{d'}$ for all integer $d' \geq 2$. However, as we shall see below, none of these features should be taken for granted.

Rank-deficient states.-- To this end, consider the one-parameter family of two-qudit, rank-two entangled states [22, 36]:

$$\rho(q) = q|\Phi_+^d\rangle\langle\Phi_+^d| + \left(1 - q\right)|0\rangle\langle0| \otimes |1\rangle\langle1|,$$

where $0 < q < 1$. Clearly, for $q > \frac{1}{2}$, $F_d(\rho(q)) \geq q > \frac{1}{2}$ and thus the corresponding $\rho(q)$ is already useful for teleportation. We show in Appendix D that for $d \geq 4$, all entangled $\rho(q)$ are useful for teleportation whereas for $d \leq 3$, the state is useful for teleportation if and only if $q > \frac{1}{3}$.

To see that useless $\rho(q)$ may possess hidden teleportation power, note from [22] that $\rho(q)$ may be quasi-distilled into $|\Phi_+^d\rangle$ by applying the local filters $A_n = \text{diag}[1/n, 1,..., 1, 1/n]$, $B_n = \text{diag}[1, 1/n,..., 1/n, 1]$, and with $n \to \infty$. In this limit, $F_d(\rho(q))$ approaches one but the success probability $\frac{2}{n^2} + \frac{1}{n^2} - \frac{2}{n} \to 0$, and thus cost-function $K(\rho(q)) \to \frac{1}{2}$, showing that these limiting filters are suboptimal in terms of maximizing $K(\rho(q))$ (see Appendix D for details).

Indeed, since $\rho(q)$ violates the reduction criterion [33], it is possible—in contrast to the Werner state—to make the filtered state useful by applying local filtering on one side. For definiteness, let us consider filtering only on Alice’s side. Then the filter maximizing $K(\rho(q))$, which generalizes that given in [36] for the $d = 2$ case, appears to be $A_K = \kappa|0\rangle\langle0| + \sum_{j=1}^{d-1} |j\rangle\langle j|$, where $\kappa = \frac{(d-1)q}{d(d-1)}$.

The filtered state $\rho_{f,\kappa}(q)$, whose explicit expression can be found in Appendix D, has an FEF of

$$F_d(\rho_{f,\kappa}(q)) = \frac{q}{2\rho_{c}(q)} \left(\kappa + d - 1\right)^2, \quad q \in \left(0, \frac{d}{2d-1}\right),$$

and is obtained with probability $p_{c}(q) = \kappa^2 \left(\frac{q}{2} + 1 - q\right) + \frac{q}{2} \left(1 - d\right)$. That this indeed demonstrates the hidden teleportation power of useless $\rho(q)$ is illustrated for the qubit case in Fig. 2 (see also Appendix D). More generally, it can be shown that for all $d \geq 2$, $F_d(\rho_{c}(q)) > F_d(\rho(q))$ for the interval of $q$ specified in Eq. (4). At the upper limit, $A_K = 1_d$, i.e., if we increase $q$ further, the expression for $p_{c}(q)$ becomes unphysical.

Of course, we can also consider a maximization of the FEF of the filtered state by allowing only filtering on Alice’s side. In this case, the optimized local filter appears to take the same form as $A_K$ but with $\kappa$ replaced by $\kappa' = \frac{q}{2\rho_{c}(q)}$. Consequently, the success probability $p_{c}(q)$ and the FEF of the filtered state $F_{d'}(\rho_{f,\kappa'}(q))$ are similarly obtained by replacing $\kappa$ with $\kappa'$ in Eq. (4) (see more details in Appendix D).

Experimental demonstration.-- As an illustration, we experimentally prepare the rank-two states $\rho(q)$ for $q = \frac{1}{15}, \frac{2}{15}, \ldots , \frac{10}{15}$ and demonstrate how local filtering can be applied on one side to boost its teleportation power. See Fig. 3 for a summary of our experimental scheme and Fig. 4 for the detailed experimental setup. To this end, we first generate polarization-entangled photon pairs via a periodically poled potassium titanyl phosphate (PPKTP) crystal in a Sagnac interferometer [43], which is bidirectionally pumped by an ultraviolet (UV) diode laser at 405 nm. Via quantum state tomography (QST), we find that the generated entangled state $\rho_{12}$ gives a fidelity of $0.954 \pm 0.003$ with respect to an ideal Bell pair $|\Psi_+^2\rangle_{12} = \frac{1}{\sqrt{2}}(|H_1V_2\rangle + |V_1H_2\rangle)$, where the polarization $H$ (horizontal) and $V$ (vertical) encode, respectively, the state $|0\rangle$ and $|1\rangle$. "}

FIG. 2. Experimental and theoretical results illustrating the teleportation power before and after filtering for qubit $\rho(q)$. a. FEF $F_2$ estimated from quantum state tomography and Eq. (1) b. teleportation fidelity $f$ estimated from quantum process tomography. Dashed lines represent the theoretical results for $\rho(q)$ (bottom, red, corresponding to $\rho_{12}$), $\rho_{f,\kappa}(q)$ [mid-blue, corresponding to $\rho_{12}(\kappa)$], and $\rho_{f,\kappa'}(q)$ [top, turquoise, corresponding to $\rho_{12}(\kappa')$]. The classical thresholds are marked by solid lines.

---

FIG. 3. Experimental scheme used in demonstrating the hidden teleportation power of $\rho(q)$. QST may be performed at stage 1, 1', and 1'' to estimate the density matrix corresponding, respectively, to the initial entangled state, the experimentally prepared $\rho(q)$, and the locally filtered state $\rho_{f,\kappa}(q)$ or $\rho_{f,\kappa'}(q)$.
QST requires both photons to be measured in different bases, which are achieved by getting each of them to go through a quarter-wave plate (QWP) and a half-wave plate (HWP) set at the appropriate angles before passing them through a polarized beam splitter (PBS), and eventually registered by a detector. To generate ρ, we do not measure photon 1 (see Fig. 3) but rather let it pass through a noisy channel between the two beam displacers (BDs). To determine the FEF before filtering, we again determine ρ′′(θ) on photon 2, which indicates that the filtered state ρ′′(θ) is the process matrix χexp of our teleportation channel using quantum process tomography (QPT) [39] (see Appendix E 6 for details). The teleportation fidelity f(ρ) based on the shared state ρ—which equals the average (identity) gate fidelity F(ρ)—is related to the fidelity F exp (ρ) by f(ρ) = F exp (ρ) = 1/2 ln[tr(ρ exp) + 1]/3, where χid is the process matrix of the ideal teleportation channel. As shown in Fig. 2b, f(ρ) shows the same trend as F exp (ρ) when we vary q(θ1), which indicates the linear dependence of f(ρ) on E(θ2), as required by Eq. (1). Other experimental results and further experimental details, including the coincidence count rate, could be found in Appendix E.

Discussion.— One may have noticed that the interval of v at which W(v) exhibits hidden teleportation power (HTP) coincides exactly with the interval where these states are known to be 1-distillable [49–51]. The problem of n-distillability concerns the conversion of n ≥ 1 copies of a given initial state ρ to a finite number of two-qubit maximally entangled states by means of LOCC. Since all two-qubit entangled states are distillable [52], a two-qubit state ρ is distillable if one can find qubit projections that maps ρ to a two-qubit entangled state. With the qubit projection first considered by Popescu [25], it is known [49–51] that Werner state can be locally filtered to a two-qubit entangled state for v ∈ [0, vC].
Thus, our result regarding the HTP of $W(v)$ may also be derived from [49–51] by noting that (i) the filtered two-qubit state $W_f(v)$ satisfies $F_d[W_f(v)] > \frac{1}{d}$ if and if only it is entangled and (ii) for $d > 2$, $F_d[W(v)] \leq \frac{1}{d}$ for all $v$ (see Appendix C). This observation clarifies the differences between the problem of HTP and that of 1-distillability. Firstly, for a state $\rho$ to possess HTP, we not only require that it becomes useful after filtering but also that it is useless before filtering. Secondly, to address the 1-distillability of $\rho$ by qubit projection\footnote{In general, a distillation protocol may involve LOCC that cannot be described by local filtering alone.}, one seeks for filters $A$ and $B$ such that $\rho_f = \frac{A \otimes B \rho (A \otimes B)^\dagger}{\text{tr}(A \otimes B \rho (A \otimes B)^\dagger)}$ is a normalized two-qubit entangled state. 

However, for the problem of HTP, one is interested to know whether there exist filters such that $\rho_f$ satisfies $F_d(\rho_f) > \frac{1}{d}$.

Despite these apparent differences, the two problems are indeed closely related. On the one hand, if $\rho$ is 1-distillable by qubit projection, then embedding the filtered two-qubit state $\rho_f$ in $C^d \otimes C^d$ makes it obvious that $F_d(\rho_f) > \frac{1}{d}$, i.e., the filtered state must be useful for teleporting a qudit state. On the other hand, if filters can be found such that $F_d(\rho_f) > \frac{1}{d}$, then $\rho_f$ is distillable by subjecting $\rho_f^m$ for some finite $m$ to the recurrence protocol of [33]. Numerically, we have also observed that whenever $F_d(\rho_f) > \frac{1}{d}$, qubit filter(s) can be found such that a (potentially different) two-qubit filtered state $\tilde{\rho}_f$ satisfies $F_2(\tilde{\rho}_f) > \frac{1}{2}$. A proof of this implication, i.e., $F_d(\rho_f) > \frac{1}{d} \implies F_2(\tilde{\rho}_f) > \frac{1}{2}$, is to our knowledge, lacking. If true, then the problem of exhibiting (hidden) teleportation power better than classical becomes equivalent to the problem of 1-distillability by qubit projection.

It is also worth noting that even though there has been [53] a local-filtering experiment demonstrating hidden nonlocality [25], the experiment reported there in did not show hidden teleportation power as the initial state—as introduced in [24]—is already useful for teleportation before any filtering. Indeed, the connection between Bell-nonlocality and usefulness for teleportation is intricate, see, e.g., [54, 55]. Thus, an open problem that stems from the present work is whether there exists quantum states that could exhibit hidden teleportation power but not hidden nonlocality. Both Werner states and the rank deficient that we discussed seem to be plausible candidates but a conclusive answer would require a thorough analysis would need to be carried out, e.g., using techniques from [56, 57]. Finally, what if we allow local filtering on multiple copies of the same state? For Bell-nonlocality [58], this is known [26] to be useful but its effectiveness in terms of boosting teleportation power remains to be clarified. Notice that when joint local filtering is allowed, all entangled states are capable of boosting the teleportation power of some other entangled state [27].

ACKNOWLEDGMENTS

We thank Jebarathinam Chellasamy, Huan-Yu Ku, and Marco Túlio Quintino for stimulating discussions. This work is supported by the Ministry of Science and Technology, Taiwan (Grants No. 104-2112-M-006-021-MY3, 107-2112-M-006-005-MY2, 107-2627-E-006-001, and 108-2627-E-006-001). H. L., X.-X. F. and T. Z. were supported by the National Natural Science Foundation of China (No. 11974213), National Key R & D Program of China (No. 2019YFA0308200) and Shandong Provincial Natural Science Foundation (No. ZR2019MA001).

Appendix A: CPTP Map for Boosting Teleportation Power

As remarked in [36], a non-deterministic local filtering operation that transforms a two-qubit state $\rho$ with $F_2(\rho) \leq \frac{1}{2}$ to one with $F_2(\rho_f) > \frac{1}{2}$ can always be converted to a deterministic LOCC by preparing a pure product state when the filtering operation fails. Here, we give a straightforward generalization of this observation to the qudit case and provide the explicit Kraus operators of the completely-positive trace-preserving (CPTP) map achieving this.

To this end, let us consider the case where local filters $A$ and $B$ are, respectively, applied on Alice and Bob’s subsystem, both of which are assumed, without loss of generality, to be of dimension $d$. There are four possible outcomes associated with the application of these filters, i.e., the filtering may (i) succeed on both sides, (ii) succeed on Alice’s side but fail on Bob’s side, (iii) succeed on Bob’s side but fail on Alice’s side, or it may fail on both sides. Each of these cases leads, respectively, to the resulting state $M_1 \rho M_1^\dagger$, $G_1 \rho G_1^\dagger$, $G_2 \rho G_2^\dagger$, and $G_3 \rho G_3^\dagger$, where

$$
M_1 = A \otimes B, \quad G_3 = \sqrt{I_d - A^\dagger A \otimes I_d - B^\dagger B}, \\
G_1 = A \otimes \sqrt{I_d - B^\dagger B}, \quad G_2 = \sqrt{I_d - A^\dagger A \otimes B}. 
$$

(A1)

To prepare the product state $|\psi\rangle \langle \phi|$ conditioned on the failure of any of the local filtering operation, it suffices to trace out the local subsystems in the last three cases and locally prepare, respectively $|\psi\rangle$ and $|\phi\rangle$. It is straightforward to verify that this can be achieved via the following CPTP map:

$$
\mathcal{E}(\rho) = M_1 \rho M_1^\dagger + \sum_{k=1}^{3} \sum_{i,j=0}^{d-1} M_{ijk} \rho M_{ijk}^\dagger
$$

(A2)

where $M_{ijk} = |\psi\rangle \langle i| (j) |G_k \rangle$ $(i, j = 0, 1, ..., d - 1, k = 1, 2, 3)$. In particular, if we write $M_1 \rho M_1^\dagger = pp_f$ where $p = \text{tr}(M_1 \rho M_1^\dagger)$ is the success probability of filtering, then after the map $\mathcal{E}$, Alice and Bob will end up with the average state

$$
\rho_{\text{ave}} = pp_f + (1 - p) |\psi\rangle \langle \psi| \otimes |\phi\rangle \langle \phi|. 
$$

(A3)
Appendix B: Two Figures of Merit

Let the unnormalized filtered state be
\[ \tau(A, B) = (A \otimes B)\rho(A \otimes B)\dagger, \]
then the corresponding success probability of filtering is
\[ p(A, B) = \text{tr}[\tau(A, B)]. \]
(B1)

If \( p(A, B) \neq 0 \), then conditioned on the success of filtering, one obtains the filtered state \( \rho_f(A, B) = \frac{\tau(A, B)}{p(A, B)} \).

Depending on whether one cares about the success probability \( p(A, B) \), there are two natural figures of merit, and hence two different optimization problems to consider in terms of boosting the teleportation power of a given quantum state \( \rho \). However, as we illustrate below, whether a given \( \rho \) possesses hidden teleportation power is independent of which among these optimization problems one decides to perform.

The first of these concerns the maximization of the FEF of the filtered state \( \rho_f(A, B) \), i.e.,
\[ \max_{A, B} F_d[\rho_f(A, B)], \]
\[ \text{such that } ||A||_\infty \leq 1, \quad ||B||_\infty \leq 1 \]  
(B3)

From the definition of FEF given in Eq. (1), we see that the first line of Eq. (B3) can be expressed as
\[ \max_{A, B} \max_U \langle \Phi^+_d | (I \otimes U)\dagger (A \otimes B)\rho(A \otimes B)\dagger \text{tr}[(A \otimes B)\rho(A \otimes B)]^{-1/2} U | \Phi^+_d \rangle \]
\[ = \max_{A, B} \langle \Phi^+_d | \frac{(A \otimes B)\rho(A \otimes B)\dagger \text{tr}[(A \otimes B)\rho(A \otimes B)]^{-1/2} | \Phi^+_d \rangle \]  
(B4)

where the last equality follows by absorbing the optimizing unitary \( U \) to the definition of the Bob’s filter, i.e., \( U^1 B \rightarrow B \), and the cyclic property of trace. Hence, we may write the optimization problem of Eq. (B3) as
\[ \max_{A, B} \langle \Phi^+_d | \rho_f(A, B) | \Phi^+_d \rangle, \]
\[ \text{such that } ||A||_\infty \leq 1, \quad ||B||_\infty \leq 1, \]  
(B5)

which means, when optimizing the filters, we may assume without loss of generality that \( F_d[\rho_f(A, B)] \) is attained with respect to the maximally entangled state \( |\Phi^+_d \rangle \).

Alternatively, one could also maximize the so-called cost function [36] of \( \rho \), which is simply the FEF of the average output state \( \rho_{\text{ave}} \), cf. Eq. (A3). From the simplification to Eq. (B3) given above and the form of \( \rho_{\text{ave}} \), it is clear that in order to optimize \( K[\rho] \), we should choose \( |\psi\rangle = |\phi\rangle = |i\rangle \) for any \( i \in \{0, 1, \ldots, d-1\} \). In this case, the corresponding optimization problem simplifies to:
\[ \max_{A, B} p(A, B)|\Phi^+_d \rangle \langle \rho_f(A, B)| \Phi^+_d \rangle + \frac{1 - p(A, B)}{d}, \]
\[ \text{such that } ||A||_\infty \leq 1, \quad ||B||_\infty \leq 1 \]  
(B6)

Since this last optimization takes into account the success probability \( p(A, B) \), the resulting optimal filters \( A_K, B_K \) and hence the optimal filtered state \( \rho_f \) are generally different from those obtained by solving Eq. (B5). However, it is important to note that if the optimum value of Eq. (B5) is such that \( F_d(\rho_f) > \frac{d}{2} \) and the corresponding \( p(A, B) \neq 0 \), then by using the same set of filters in the evaluation of \( K(\rho) \), one would obtain
\[ K(\rho) = pF_d(\rho_f) + (1 - p)\frac{d}{2} > p\frac{d}{2} + (1 - p)\frac{d}{2} = \frac{d}{2}, \]  
(B7)
i.e., the resulting cost function is larger than \( \frac{d}{2} \). Conversely, if the cost-function is larger than \( \frac{d}{2} \), then \( F_d(\rho_f) \) must also be larger than \( \frac{d}{2} \). Thus, although the optimization problems of Eq. (B5) and Eq. (B6) may lead to different \( F_d(\rho_f) \), their difference is irrelevant when deciding whether a quantum state \( \rho \) can have its FEF boosted beyond \( \frac{d}{2} \).

Note that in the qubit case, it is always sufficient [36] to consider filtering on one side when maximizing \( K[\rho] \) or \( F_d(\rho_f) \). When the local Hilbert space dimension is larger than two, it is also sufficient to consider single-side filtering for states that violate the reduction criterion [33]. In general, however, to decide if a quantum state can have its FEF boosted beyond \( \frac{d}{2} \) would require the consideration of filtering on both sides.

Appendix C: Detailed results for Werner states

For ease of reference, we reproduce here the FEF of Werner state \( W(v) \) derived in [42]:
\[ F_d[W(v)] = \begin{cases} \frac{2v}{d(d+1)}, & 0 \leq v \leq \frac{d+1}{2d}, \\ \frac{2(1-v)}{d(d-1)}, & 0 \leq v \leq \frac{d+1}{2d}, \quad d \text{ even}, \\ \frac{2(1-v)^2 + 2v(d-1)}{d^2(d^2-1)}, & 0 \leq v \leq \frac{d+1}{2d}, \quad d \text{ odd}, \end{cases} \]  
(C1)

For the optimal (qubit) filter, it can be shown that the success probability of filtering is
\[ p_w(v) = \frac{2(2d+1)(1-v) + 3v(d-1)}{d(d^2-1)} \]  
(C2)

while the corresponding increase in FEF for \( 0 \leq v \leq \frac{d+1}{2d-2} < \frac{d+1}{2d} \) is:
\[ F_d[W_f(v)] - F_d[W(v)] = \begin{cases} \frac{(dp-2)(dp^2 + dp - 6)}{2(d-2)d^2p} & (\text{for even } d) \\ \frac{12 - 2d^2 + 4d - 4d + (d-1)d^2p^2}{2(d-2)d^2p} & (\text{for odd } d). \end{cases} \]  
(C3)

For \( v \in \left[\frac{d+1}{2d^2-2}, \frac{1}{2}\right] \), the qubit filter could not result in an entangled \( W_f(v) \) that beats the classical threshold \( F_c \) (they do not seem to exhibit teleportation power).

In Fig. 5, we show, for \( d = 3 \) and \( d = 4 \), the FEF of Werner states before and after filtering, as well as the corresponding cost function. In the same figure, we also show the
corresponding difference in FEF, i.e., \(F_d[W_f(v)] - F_d[W(v)]\) vs \(v\) [and hence \(p_W(v)\), which depends linearly on \(v\)]. Clearly, we see that when the success probability \(p_W(v)\) increases, the amount of FEF that can be increased by local filtering decreases, thus exhibiting some kind of trade-off between these two quantities for \(W(v)\). These latter plots for larger values of \(d\) look similar and are thus omitted.

**Appendix D: Detailed results for rank-deficient states**

1. **Fully-entangled fraction of \(\rho(q)\)**

Here we show that the family of rank-deficient states

\[
\rho(q) = q|\Phi_d^+\rangle\langle\Phi_d^+| + (1-q)|0\rangle\langle0| \otimes |1\rangle\langle1|, \quad q \in (0, 1]
\]  

is already useful for teleportation whenever \(d \geq 4\).

**Proof.** We shall first determine \(F_d[\rho(q)]\), which requires the maximization of

\[
\langle\Psi_d| \rho(q) |\Psi_d\rangle = q |\langle\Psi_d | \Phi_d^+\rangle|^2 + (1-q) |\langle\Psi_d | 01\rangle|^2 
\]

over unitary matrix \(U\) such that \(|\Psi_d\rangle = (I \otimes U)|\Phi_d^+\rangle\). Clearly, from Eq. (D2) and the form of \(|\Psi_d\rangle\), any \(U\) that maps \(|0\rangle, |1\rangle\) outside \(S_{01} = \text{span}\{|0\rangle, |1\rangle\}\) would be suboptimal, since it decreases—when compared with one that acts only nontrivially in \(S_{01}\)—the overlap \(|\langle\Psi_d | \Phi_d^+\rangle|^2\) and \(|\langle\Psi_d | 01\rangle|^2\).

Consequently, let us consider only \(U\) of the form

\[
U = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \oplus I_{d-2}, 
\]

where \(a, b \in \mathbb{C}, \overline{a} (\overline{b})\) denotes complex conjugation of \(a (b)\), and the unitary requirement implies that \(|a|^2 + |b|^2 = 1\). Eval-
uating the overlap gives
\[ d(\Psi_{d})|\rho(q)|\Psi_{d}) = \frac{q}{d} |2a + (d - 2)Re[a] + (d - 2)^2| + (1 - q)|b|^2. \tag{D4} \]
Since \( d \geq 2 \), it is clear that in order to maximize this overlap, we may, without loss of generality, consider only real-valued \( a \) and real-valued \( b \). For convenience, let us define
\[ f(d, q, a) := \frac{q}{d^2} [2a + (d - 2)] + (1 - q)(1 - a^2). \tag{D5} \]
Then, we have \( F_d[\rho(q)] = \max_a f(d, q, a) \).

Using standard variation technique, we find that the local extremum of \( f(d, q, a) \) occurs at \( a^* = \frac{2(d - 2)q}{(d - 1)q - 4} \). Note that \( |a^*| \leq 1 \) if and only if \( q \) lies in the interval \( Q := (0, \frac{1}{2}] \cup [q_0, 1] \) where \( q_0 = \frac{d - 2}{d} \). Evaluating \( f(d, q, a) \) for \( a = a^* \) and the boundary points \( a = 0, 1 \) gives
\[ f(d, q, a^*) = \frac{(1 - q)(d - 5) + 1}{d(1 - q) - 4q}, \quad q \in Q, \tag{D6a} \]
\[ f(d, q, 0) = \frac{d^2q - 5dq + d + 4q}{d^2}, \quad q \in Q, \tag{D6b} \]
\[ f(d, q, 1) = q. \tag{D6c} \]
Taking their difference gives
\[ f(d, q, 1) - f(d, q, 0) = \frac{q(5d - 4) - d}{d^2}, \tag{D7a} \]
\[ f(d, q, a^*) - f(d, q, 0) = \frac{4d(2 - d)^2q^2}{d^2[d(1 - q) - 4q]}, \tag{D7b} \]
\[ f(d, q, a^*) - f(d, q, 1) = \frac{(1 - 3q)^2}{d(1 - q) - 4q}, \tag{D7c} \]
where we remind that the last two equations are only meaningful for \( q \in Q \).

For \( d = 2 \), Eq. (D7b) vanishes and it is easy to verify that Eq. (D7c) is non-negative if and only if \( q \in (0, \frac{1}{2}] \). For \( d = 3 \), it can be similarly verified that Eq. (D7b) is positive for \( q \in (0, \frac{1}{2}] \) while Eq. (D6c) dominates for other values of \( q \in (0, 1] \). For \( d \geq 4 \), \( Q = (0, \frac{1}{2}] \) since \( |q_0| \geq 1 \). For \( q \in Q \), one can see that \( f(d, 1 - q) > 0 \) and thus \( f(d, q, a^*) \) dominates over the other expressions in Eq. (D6).

Coming back to \( q \in (0, \frac{1}{2}] \), we see that \( F_d[\rho(q)] > \frac{1}{2} \) if and only if \( G(d, q) > 0 \). It is easy to see that \( G(d, q) > 0 \) when \( d \geq 5 \) since both numerator and denominator are positive for \( 0 < q \leq \frac{1}{2} \). Similarly, for \( d = 4 \), \( G(d, q) \) simplifies to \( \frac{q^2}{1 - 2q} \), which is strictly positive for \( 0 < q \leq \frac{1}{2} \). Hence, as claimed, \( F_d[\rho(q)] > \frac{1}{2} \) for \( d \geq 4 \) and \( q \in (0, 1] \), i.e., these states are all useful for teleportation even before any filtering.

For the case of \( d = 3 \), we have \( G(d, q) = \frac{2q(1 - 3q)}{q^2 - 1} \), which is easily verified to be non-positive for \( q \in (0, \frac{1}{2}] \). Together with Eq. (D8), we thus see that \( \rho(q) \) for \( d = 3 \) is useless for teleportation if and only if \( q \in (0, \frac{1}{2}] \). Finally, \( G(2, q) = -q < 0 \) and thus \( \rho(q) \) for \( d = 2 \) is useless for teleportation if and only if \( q \in (0, \frac{1}{2}] \).

### 2. Two-side filtering (quasidistillation)

In [22], the family of local filters \( A_n = \text{diag}[1/n, 1, ..., 1] \), \( B_n = \text{diag}[1, 1/n, ..., 1/n] \) were proposed to quasi-distill \( \rho(q) \) into \( |\Phi_d^+\rangle \). From some simple calculation, one finds that these filters yield the unnormalized state\(^2\)
\[ \tau_n = \frac{1}{n^2} \left[ q|\Phi_d^+\rangle\langle\Phi_d^+| + \left(1 - \frac{q}{n^2}\right)|0\rangle\langle0| \otimes |1\rangle\langle1| \right] \tag{D10} \]
with a success probability of \( p_n = \frac{q(n^2 - 1) + 1}{n^2} \). For sufficiently large \( n \), the FEF is attained by taking the overlap with \( |\Phi_d^+\rangle \), then
\[ F_d \left( \frac{\tau_n}{p_n} \right) = \frac{q}{n^2p_n} = 1 - \frac{1 - q}{q(n^2 - 1) + 1}. \tag{D11} \]
Thus, when \( n \to \infty \), \( F_d \left( \frac{\tau_n}{p_n} \right) \to 1 \) but the success probability \( \lim_{n \to \infty} p_n = 0 \).

### 3. Single-side filtering

Let \( \kappa = \frac{(d - 1)q}{d(1 - q)} \), then Alice’s filter that appears to maximize the cost-function \( K[\rho(q)] \) is
\[ A_{\kappa} = \kappa|0\rangle\langle0| + \sum_{j=1}^{d-1} |j\rangle\langle j|. \tag{D12} \]
Let us define the subnormalized state \( |\chi\rangle := \frac{1}{\sqrt{q}}(\kappa|0\rangle|0\rangle + \sum_{i=1}^{d-1} i|i\rangle \rangle \). Then, conditioned on successful filtering, which occurs with probability
\[ p_s(q) = \kappa^2 \left( \frac{q}{d} + 1 - q \right) + \frac{q}{d} (d - 1), \tag{D13} \]
\(^2\) Note that it was claimed in Eq. (40) in [22] that the filtered state takes the form of \( \frac{1}{n} \left[ q|\Phi_d^+\rangle\langle\Phi_d^+| + \left(\frac{d-1}{n}\right)|0\rangle\langle0| \otimes |1\rangle\langle1| \right] \), which is incorrect.
one obtains the filtered state

$$\rho_{f,k}(q) = \frac{1}{p_k(q)} (q|\chi\rangle\langle\chi| + (1 - q)\kappa^2 |0\rangle\langle 0| \otimes |1\rangle\langle 1|). \quad \text{(D14)}$$

Numerically, we have observed that the FEF of $\rho_{f,k}(q)$ is obtained by taking the overlap with $|\Phi_f^+\rangle$, thus resulting in the expression of $F_d(\rho_{f,k}(q))$ given in Eq. (4).

If we maximize instead $F_d(\rho_{f,k}(q))$ by allowing only the filtering on Alice’s side, then the optimal filter again appears to be given by Eq. (D12), but with $\kappa$ replaced by $\kappa' = \frac{q}{\sqrt{1 + d(1 - q)}}$. In a similar manner, the filtered state $\rho_{f,k'}(q)$, the success probability $p_{k'}(q)$, and the FEF of the filtered state can all be obtained if we replace $\kappa$ by $\kappa'$ in Eq. (D14), Eq. (D13), and Eq. (4).

### Appendix E: Experimental details

In this Appendix, we provide further details about our experimental setup. A schematic, simplified version of this setup that emphasizes its connection with the teleportation protocol can be found in Fig. 3 whereas an overview of the full experimental setup is given in Fig. 4. In the following subsections, we explain how each of the boxed section in Fig. 4 functions. To this end, it would be useful to bear in mind the following:

(i) A half-wave plate (HWP) @ $\theta$ performs the unitary transformation $U_{\text{HWP}} = \cos 2\theta |H\rangle\langle H| + |V\rangle\langle V| + \sin 2\theta |H\rangle\langle V| + |V\rangle\langle H|$ on a polarization state, where $\theta$ is the angle between fast axis of HWP and vertical polarization.

(ii) A beam displacer (BD) transmits a vertically polarized photon but deviates a horizontally polarized one.

(iii) A polarized beam splitter (PBS) transmits a horizontally polarized photon but reflects a vertically polarized one.

(iv) A quarter-wave plate (QWP) @ $\theta$ performs the unitary transformation $U_{\text{QWP}} = \frac{1}{2} |I_2 + i \sin 2\theta |H\rangle\langle V| + |V\rangle\langle H| \rangle + |H\rangle\langle H| + |V\rangle\langle V| \rangle$ and $\theta$ is the angle between fast axis of QWP and vertical polarization, on a polarization state.

#### 1. Entangled Photon Source

We start by describing how polarization-entangled photon pairs are produced in our setup by bidirectionally pumping a periodically poled potassium titanyl phosphate (PPKTP) crystal (placed in a Sagnac interferometer [43]) with an ultraviolet (UV) diode laser at 405 nm. Specifically, as shown in Fig. 7, the power of the pump light is first adjusted through a HWP and a PBS. Then, at the second HWP set at 22.5°, the horizontal polarization $|H_p\rangle$ is rotated to $|H_p\rangle = \frac{1}{2}(|H_p\rangle + |V_p\rangle)$. Via two lenses L₁ (with focal length 75 mm and 125 mm), the pump beam is subsequently focused into a beam waist of 74 µm and arrives at a dual-wavelength PBS (after passing through a dichroic mirror). The pump beam is split on the PBS, and coherently pumped through the PPKTP in the clockwise and counterclockwise direction. The PPKTP crystal, with dimensions of 10 mm (length) × 2 mm (width) × 1 mm (thickness) and a poling period of $\Lambda = 10.025$ µm, is housed in a copper oven and temperature controlled by a homemade temperature controller set at 29°C to realize the optimum type-II phase matching at 810 nm. The clockwise and counterclockwise photons are then recombined on the dual-wavelength PBS to generate entangled photons with an ideal form of $|\Psi_{12}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_2\rangle + |V_1H_2\rangle)$. Photon 1 and 2 are filtered by a narrow band filter (NBF) with a full width at half maximum (FWHM) of 3 nm, and coupled into single-mode fiber (SMF) by lenses of focal length 200 mm (L₂ and L₃) and objective lenses (not shown in Fig. 7). During our experiment, the pump power is set at 5 mw, and we observe a two-fold coincidence count rate of $7.3 \times 10^4$/s.

#### 2. Noisy Channel $E(\theta_1)$

In this part of the experimental setup, which does not involve photon 2 (as can be seen in Fig. 7), photon 1 goes through a noisy channel $E(\theta_1)$ that eventually results in a two-photon polarization state given by $\rho(q)$ (in the ideal scenario). To this end, photon 1 is first guided to an unbalanced Mach-Zehnder interferometer (MZI) after passing a PM. Then, BS₁ transforms an ideal maximally entangled two-qubit state $|\Psi_{12}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_2\rangle + |V_1H_2\rangle)$ to $\sqrt{2}(|H_1V_2\rangle + |V_1H_2\rangle) \otimes (\sin^2\theta_1 |s_1\rangle + |l_1\rangle)$ with $|s_1\rangle$ and $|l_1\rangle$ denote, respectively, the short and long arm of the unbalanced MZI. On the long arm, the PBS only transmits $|H_{l1}\rangle$ and filters away the $|V_{l1}\rangle$ component. On the short arm, the two BDs and a HWP (at angle $\theta_1$) work together as an attenuator so that $|s_1\rangle \rightarrow \sin^2 2\theta_1 |s_1\rangle$. Indeed, from the property of a BD and the calculation shown in Eq. (E1), we see that a photonic state that goes through the short arm is attenuated by a factor of $\sin^2 2\theta_1$. Since photons that travel through the long arm and those that travel through the short arm are distinguishable, the two spatial modes $|s_{1}\rangle$ and $|l_{1}\rangle$ are incoherently recombined at BS₂. In the experiment, we keep only photons exiting from the output port 1’, thus obtaining the state $\rho_{1:2} = $
FIG. 6. (Top row) Comparison between the FEF of the filtered state $\rho_f$ obtained by employing different filtering schemes on $\rho(q)$ for $d = 2$ and $d = 3$. Included in the plots are the two-side filtering schemes introduced in [22] for $n = 2, 3, 5, \text{ and } 10$ (see Appendix D 2) as well as the single-side filtering schemes discussed in Appendix D 3. (Middle row) Comparison of the corresponding success probabilities as a function of the parameter $q$. (Bottom row) Comparison of the corresponding change in FEF as a function of the parameter $q$. 
To illustrate the effect of this setup, we provide in Eq. (E2) a step-by-step calculation showing how a general input polarization may implement any of the filters (for Appendix E). This part of the setup consists also of two BDs in addition to three HWPs. For photons encoded in the polarization degree of freedom (DOF), we have

\[ q(\theta_1)|\Phi^+_2\rangle = \frac{1}{\sqrt{2}} (|H_1V_2\rangle + |V_1H_2\rangle) \]

with \( q(\theta_1) = \frac{2\sin^2 \theta_1}{1 + 2\sin^2 \theta_1} \). With this experimental setup, \( q(\theta_1) \) can be tuned in the range from 0 to \( \frac{2}{3} \). A step-by-step calculation detailing the evolution of the two-photon state through this setup is given in Eq. (E1).

\[
|\Psi^+_2\rangle = \frac{1}{\sqrt{2}} (|H_1V_2\rangle + |V_1H_2\rangle) \xrightarrow{\text{BS}_1} \frac{1}{2} (|H_1V_2\rangle + |V_1H_2\rangle) \otimes (|s_1\rangle + |l_1\rangle) \\
\xrightarrow{\text{PBS at long arm}} \frac{1}{\sqrt{3}} (|H_1\rangle|V_2\rangle|l_1\rangle + |H_1\rangle|V_2\rangle|s_1\rangle + |V_1\rangle|H_2\rangle|s_1\rangle) \\
\xrightarrow{\text{BD}_1 at short arm} \frac{1}{\sqrt{3}} (|H_1\rangle|V_2\rangle|l_1\rangle + |H_1\rangle|V_2\rangle|l_1\rangle + |V_1\rangle|H_2\rangle|v_1\rangle) \\
\xrightarrow{\text{HWP} \otimes \theta_1 at short arm} \frac{1}{\sqrt{3}} (|H_1\rangle|V_2\rangle|l_1\rangle + (\cos 2\theta_1|H_1\rangle + \sin 2\theta_1|V_1\rangle)|V_2\rangle|l_1\rangle + (\sin 2\theta_1|H_1\rangle - \cos 2\theta_1|V_1\rangle)|H_2\rangle|v_1\rangle) \\
\xrightarrow{\text{BD}_2 at short arm} \frac{1}{\sqrt{1 + 2\sin^2 2\theta_1}} (|H_1\rangle|V_2\rangle|l_1\rangle + \sin 2\theta_1|H_1\rangle|H_2\rangle|s_1\rangle + \sin 2\theta_1|V_1\rangle|V_2\rangle|s_1\rangle) \\
\xrightarrow{\text{BS}_2 incoherently combined} \frac{2\sin^2 2\theta_1}{1 + 2\sin^2 2\theta_1} |\Phi^+_2\rangle \langle \Phi^+_2| + \frac{1}{1 + 2\sin^2 2\theta_1} |H_1\rangle|V_2\rangle \langle H_1|V_2| \\
\]

(E1)

3. Local Filtering

Our experimental setup for implementing the local filter \( A = \text{diag}[\kappa, 1] \) is shown in Fig. 8. As with the attenuator discussed in Appendix E, this part of the setup consists also of two BDs in addition to three HWPs. For photons encoded in the polarization DOF, filter \( A \) attenuates the horizontal component \( |H\rangle \) by a factor of \( \kappa \) while keeping the vertical component \( |V\rangle \) unchanged. To illustrate the effect of this setup, we provide in Eq. (E2) a step-by-step calculation showing how a general input polarization pure state \( \alpha|H_1\rangle + \beta|V_1\rangle \) transforms. Note that \( \kappa \) is related to the angle of HWP @ \( \theta_2 \) by \( \kappa = \sin 2\theta_2 \). Thus, by tuning \( \theta_2 \), we may implement any of the filters (for \( d = 2 \)) given in Eq. (D12). With some thought, it is easy to see that the same effect applies to every term in the convex decomposition of an input mixed density matrix.

\[
|\psi_2\rangle = \frac{1}{\sqrt{2}} (|H_1\rangle|V_2\rangle + |V_1\rangle|H_2\rangle) \xrightarrow{\text{BS}_1} \frac{1}{2} (|H_1\rangle|V_2\rangle + |V_1\rangle|H_2\rangle) \otimes (|s_1\rangle + |l_1\rangle) \\
\xrightarrow{\text{PBS at long arm}} \frac{1}{\sqrt{3}} (|H_1\rangle|V_2\rangle|l_1\rangle + |H_1\rangle|V_2\rangle|s_1\rangle + |V_1\rangle|H_2\rangle|s_1\rangle) \\
\xrightarrow{\text{BD}_1 at short arm} \frac{1}{\sqrt{3}} (|H_1\rangle|V_2\rangle|l_1\rangle + |H_1\rangle|V_2\rangle|l_1\rangle + |V_1\rangle|H_2\rangle|v_1\rangle) \\
\xrightarrow{\text{HWP} \otimes \theta_1 at short arm} \frac{1}{\sqrt{3}} (|H_1\rangle|V_2\rangle|l_1\rangle + (\cos 2\theta_1|H_1\rangle + \sin 2\theta_1|V_1\rangle)|V_2\rangle|l_1\rangle + (\sin 2\theta_1|H_1\rangle - \cos 2\theta_1|V_1\rangle)|H_2\rangle|v_1\rangle) \\
\xrightarrow{\text{BD}_2 at short arm} \frac{1}{\sqrt{1 + 2\sin^2 2\theta_1}} (|H_1\rangle|V_2\rangle|l_1\rangle + \sin 2\theta_1|H_1\rangle|H_2\rangle|s_1\rangle + \sin 2\theta_1|V_1\rangle|V_2\rangle|s_1\rangle) \\
\xrightarrow{\text{BS}_2 incoherently combined} \frac{2\sin^2 2\theta_1}{1 + 2\sin^2 2\theta_1} |\Phi^+_2\rangle \langle \Phi^+_2| + \frac{1}{1 + 2\sin^2 2\theta_1} |H_1\rangle|V_2\rangle \langle H_1|V_2| \\
\]

(E2)
\[ \alpha|H_{1'}\rangle + \beta|V_{1'}\rangle \xrightarrow{\text{BD}} \alpha|H_{1'}\rangle|h_{1'}\rangle + \beta|V_{1'}\rangle|v_{1'}\rangle \]
\[ \xrightarrow{\text{HWP } @ \theta_2 \text{ on path } h} \alpha \cos 2\theta_2|H_{1'}\rangle|h_{1'}\rangle + \alpha \sin 2\theta_2|V_{1'}\rangle|h_{1'}\rangle + \beta|V_{1'}\rangle|v_{1'}\rangle \]
\[ \xrightarrow{\text{HWP } @ 45^\circ \text{ on path } v} \alpha \cos 2\theta_2|H_{1'}\rangle|h_{1'}\rangle + \alpha \sin 2\theta_2|V_{1'}\rangle|h_{1'}\rangle + \beta|H_{1'}\rangle|v_{1'}\rangle \]
\[ \xrightarrow{\text{BD}_2} \frac{1}{\sqrt{|\alpha|^2 \sin^2 2\theta_2 + |\beta|^2}} \left( \alpha \sin 2\theta_2|V_{1'}\rangle + \beta|H_{1'}\rangle \right) \]
\[ \xrightarrow{\text{HWP } @ 45^\circ \text{ on path } 1''} \frac{1}{\sqrt{|\alpha|^2 \sin^2 2\theta_2 + |\beta|^2}} \left( \alpha \sin 2\theta_2|H_{1''}\rangle + \beta|V_{1''}\rangle \right) \]

\[ \text{(E2)} \]

4. Preparation of the State to be Teleported

Our teleportation experiment is realized on a two-photon hybrid system. In the following, we show how this scheme works for an ideal two-photon polarization entangled state \( |\Phi^+\rangle_{1'2} = \frac{1}{\sqrt{2}} (|H_{1'}H_{2'}\rangle + |V_{1'}V_{2'}\rangle) \) shared between Alice and Bob. First, as shown in Fig. 9a and Eq. (E3), the polarization-polarization entangled state \( |\Phi^+\rangle_{1'2} \) is mapped to a two-photon path-polarization-polarization entangled Greenberger-Horne-Zeilinger state using a BD. Then, a HWP @ 45° placed at the spatial mode \( v \) disentangles the polarization DOF of photon 1" from this two-photon hybrid system. Finally, the state to be teleported is encoded in the polarization DOF of photon 1" by having a HWP or a QWP set at the appropriate angle and placed across both path \( v \) and \( h \). The process is described as

\[ |\Phi^+\rangle_{1'2} = \frac{1}{\sqrt{2}}(|H_{1'2}\rangle + |V_{1'2}\rangle) \xrightarrow{\text{BD}} \frac{1}{\sqrt{2}}(|H_{1'2}\rangle|H_{2'}\rangle|H_{1'}\rangle + |V_{1'2}\rangle|V_{2'}\rangle|V_{1'}\rangle) \]
\[ \xrightarrow{\text{HWP } @ 45^\circ \text{ on path } v} \frac{1}{\sqrt{2}}(|H_{1'2}\rangle|H_{2'}\rangle|H_{1'}\rangle + |V_{1'2}\rangle|V_{2'}\rangle|V_{1'}\rangle) = |H_{1'2}\rangle \otimes \frac{1}{\sqrt{2}}(|H_{2'}\rangle|H_{1'}\rangle + |V_{2'}\rangle|V_{1'}\rangle) \]
\[ \xrightarrow{\text{HWP or QWP \ across both paths}} (\alpha|H_{1'}\rangle + \beta|V_{1'}\rangle) \otimes \frac{1}{\sqrt{2}}(|H_{2'}\rangle|H_{1'}\rangle + |V_{2'}\rangle|V_{1'}\rangle). \]

\[ \text{(E3)} \]

Experimentally, we choose \(|H_{1'}\rangle, |V_{1'}\rangle, |+_{1'}\rangle = \frac{1}{\sqrt{2}}(|H_{1'}\rangle + |V_{1'}\rangle)\) and \(|R_{1'}\rangle = \frac{1}{\sqrt{2}}(|H_{1'}\rangle + i|V_{1'}\rangle)\) as the four states to be teleported. The corresponding waveplate settings are shown in Fig. 9.

![Fig. 9](image-url)

**Fig. 9.** Experimental setup (zoom in view of the “state preparation” part of Fig. 4) that prepares the four pure states to be teleported \( |\psi\rangle = \alpha|H_{1'}\rangle + \beta|V_{1'}\rangle). \) We set HWP, respectively, at 0°, 45° and 22.5° to prepare \(|H_{1'}\rangle, |V_{1'}\rangle\) and \(|+_{1'}\rangle\), and QWP at 45° to prepare \(|R_{1'}\rangle\).

5. Bell-State Measurement (BSM)

A crucial step of the teleportation protocol is to apply a Bell-state measurement on the state to be teleported together with one half of the shared entangled resource. In our case, this amounts to applying a BSM between the polarization and path DOF of photon 1". In contrast with the BSM on two photons, since this measurement is to act on two different DOFs of a single photon, all four Bell states can in principle be distinguished deterministically in a single shot. Our experimental setup for implementing this measurement is shown in Fig. 10, while the associated theoretical calculations are shown in Eq. (E4).
The path and the polarization DOF of photon \( \nu \). The last step then amounts to implementing the Hadamard gate. As such, to implement Bell-state measurement between the path and the polarization DOF of photon \( \nu \), we set Pauli observables) as a basis set for linear operators acting on qubit states. The CPTP map can then be expressed as [39]

\[
\chi_{mn} A_m \rho A^\dagger_n,
\]

where the expansion coefficient \( \chi_{mn} \) defines the \((m, n)\) element of the so-called process matrix \( \chi \) (see, e.g., [59]).

For an ideal teleportation process \( \chi_{id} \), \( \mathcal{E}(\rho) = \rho \), thus except \( \chi_{II} = 1 \), all other elements of \( \chi_{id} \) are 0. Experimentally, we set \( q(\theta) \) in the range of \( \frac{1}{15} \) to \( \frac{1}{3} \) in steps of \( \frac{1}{15} \). For each \( q(\theta) \), we perform a teleportation experiment and reconstruct the corresponding process matrix \( \chi_{exp} \) for the shared state \( \rho_{1 \nu 2} \) and \( \rho_{1 \nu 2}(\kappa') \), respectively. These experimentally determined \( \chi_{exp} \)'s then give, via Eq. \( \text{E5} \), a full description of the corresponding teleportation channel based on the various shared entangled resource.

From the point of view of a process matrix, the goal of local filtering is to make the value of \( \chi_{II} \) greater, which therefore results in a better teleportation fidelity. In Fig. 11, we show the real parts of \( \chi_{exp} \) based on the shared states \( \rho_{1 \nu 2} \) and \( \rho_{1 \nu 2}(\kappa') \), which...
clearly illustrates that the experimentally determined $\chi_{II}$ becomes more dominant after local filtering. Notice also that $\chi_{\text{exp}}$ for $\rho_{1/2}(\kappa')$ looks similar to that of $\rho_{1/2}(\kappa)$ but with $\rho_{1/2}(\kappa')$ giving a more pronounced increase in $\chi_{II}$. The corresponding plots of $\chi_{\text{exp}}$ for $\rho_{1/2}(\kappa)$ are therefore omitted.

![Filter with $\kappa'$](image)

![No Filter](image)

![Filter with $\kappa'$](image)

![No Filter](image)

**FIG. 11.** The real parts of $\chi_{\text{exp}}$ based on the shared states $\rho_{1/2}$ and $\rho_{1/2}(\kappa')$. The imaginary parts are omitted here as their experimentally determined values are tiny. The wire grids represent the theoretical values of the elements.

For completeness, we provide in Table I the two-fold coincidence count rates of $\rho_{1/2}$, $\rho_{1/2}(\kappa)$ and $\rho_{1/2}(\kappa')$.

7. Other Experimental Results

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[2] R. Jozsa and N. Linden, Proc. R. Soc. Lond. A. 459, 2011 (2011 (2003)).
[3] G. Vidal, Phys. Rev. Lett. 91, 147902 (2003).
[4] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[5] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[6] G. Tóth and I. Apellaniz, J. Phys. A 47, 424006 (2014).
[7] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[8] S. Pirandola, J. Eisert, C. Weedbrook, A. Furusawa, and S. L. Braunstein, Nat. Photonics 9, 641 (2015).
[9] N. Korolkova, in Quantum Information: From Foundations to Quantum Technology Applications, edited by D. Bruß and G. Leuchs (Wiley-VCH, New Jersey, 2019) Chap. 15, pp. 333–352.
[10] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. 87, 077902 (2001).
[11] M. Zukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, Phys. Rev. Lett. 71, 4287 (1993).
### TABLE I

The two-fold coincidence count rates of $\rho_{1'2}$, $\rho_{1''2}(\kappa)$ and $\rho_{1''2}(\kappa')$. For comparison, note that the two-fold coincidence count rate just before the photons enter the fibers are $7.3 \times 10^4$/s.

| $q(\theta_1)$ | $\rho_{1'2}$ | $\rho_{1''2}(\kappa)$ | $\rho_{1''2}(\kappa')$ |
|---------------|-------------|-------------------------|-------------------------|
| 1/15          | 7475/s      | 205/s                   | 225/s                   |
| 2/15          | 8077/s      | 531/s                   | 510/s                   |
| 3/15          | 8624/s      | 914/s                   | 939/s                   |
| 4/15          | 9649/s      | 1523/s                  | 1414/s                  |
| 5/15          | 10160/s     | 2256/s                  | 2026/s                  |
| 6/15          | 11454/s     | 3316/s                  | 2955/s                  |
| 7/15          | 13141/s     | 4922/s                  | 3927/s                  |
| 8/15          | 14683/s     | 7420/s                  | 5606/s                  |
| 9/15          | 17183/s     | 12340/s                 | 7498/s                  |
| 10/15         | 20514/s     | 20427/s                 | 10699/s                 |

FIG. 12. Experimentally determined success probability of filtering $p_\kappa(q)$ and $p_{\kappa'}(q)$ for filter $A_\kappa$ and $A_{\kappa'}$.

[12] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 81, 5932 (1998).
[13] D. Gottesman and I. L. Chuang, Nature 402, 390 (1999).
[14] M. A. Nielsen, Phys. Rev. Lett. 93, 040503 (2004).
[15] D. E. Browne and T. Rudolph, Phys. Rev. Lett. 95, 010501 (2005).
[16] K. A. Landsman, C. Figgatt, T. Schuster, N. M. Linke, B. Yoshida, N. Y. Yao, and C. Monroe, Nature 567, 61 (2019).
[17] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, Y. Shikano, S. Pirandola, L. A. Rozema, A. Darabi, Y. Soudagar, L. K. Shalm, and A. M. Steinberg, Phys. Rev. Lett. 106, 040403 (2011).
[18] S. Lloyd and J. Preskill, J. High Energy Phys. 2014, 126 (2014).
[19] S. Popescu, Phys. Rev. Lett. 72, 797 (1994).
[20] R. Jozsa, J. Mod. Opt. 41, 2315 (1994).
[21] Y.-C. Liang, Y.-H. Yeh, P. E. M. F. Mendonça, R. Y. Teh, M. D. Reid, and P. D. Drummond, Rep. Prog. Phys. 82, 076001 (2019).
[22] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60, 1888 (1999).
[23] D. Cavalcanti, P. Skrzypczyk, and I. Šupić, Phys. Rev. Lett. 119, 110501 (2017).
[24] N. Gisin, Phys. Lett. A 210, 151 (1996).
[25] S. Popescu, Phys. Rev. Lett. 74, 2619 (1995).
[26] A. Peres, Phys. Rev. A 54, 2685 (1996).
[27] L. Masanes, Phys. Rev. Lett. 96, 150501 (2006).
[28] L. Masanes, Y.-C. Liang, and A. C. Doherty, Phys. Rev. Lett. 100, 090403 (2008).
[29] Y.-C. Liang, L. Masanes, and D. Rosset, Phys. Rev. A 86, 052115 (2012).
[30] T. Pramanik, Y.-W. Cho, S.-W. Han, S.-Y. Lee, Y.-S. Kim, and S. Moon, Phys. Rev. A 99, 030101 (2019).
[31] R. V. Nery, M. M. Taddei, P. Sahium, S. P. Walborn, L. Aolita, and G. H. Aguilar, Phys. Rev. Lett. 124, 120402 (2020).
[32] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 80, 5239 (1998).
[33] M. Horodecki and P. Horodecki, Phys. Rev. A 59, 4206 (1999).
[34] E. M. Rains, IEEE Trans. Inf. Theory 47, 2921 (2001).
[35] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 78, 574 (1997).
[36] F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003).
[37] P. Badziag, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 62, 012311 (2000).
[38] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[39] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2010).
[40] G. Vidal, J. Mod. Opt. 47, 355 (2000).
[41] J. Watrous, The Theory of Quantum Information (Cambridge University Press, 2018).
[42] M.-J. Zhao, Z.-G. Li, S.-M. Fei, and Z.-X. Wang, J. Phys. A 43, 275203 (2010).
[43] T. Kim, M. Fiorentino, and F. N. C. Wong, Phys. Rev. A 73, 012316 (2006).
[44] K. A. G. Fisher, R. Prevedel, R. Kaltenbaek, and K. J. Resch, New J. Phys. 14, 033016 (2012).
[45] X.-M. Jin, J.-G. Ren, B. Yang, Z.-H. Yi, F. Zhou, X.-F. Xu, S.-K. Wang, D. Yang, Y.-F. Hu, S. Jiang, T. Yang, H. Yin, K. Chen, C.-Z. Peng, and J.-W. Pan, Nat. Photonics 4, 376 (2010).
[46] X.-H. Jiang, P. Chen, K.-Y. Qian, Z. zhong Chen, S.-Q. Xu, Y.-B. Xie, S.-N. Zhu, and X.-S. Ma, “Quantum teleportation mediated by surface plasmon polariton,” (2019), arXiv:1912.08345 [quant-ph].
[47] M. A. Nielsen, Phys. Lett. A 303, 249 (2002).
[48] J. L. O’Brien, G. J. Pryde, A. Gilchrist, D. F. V. James, N. K. Langford, T. C. Ralph, and A. G. White, Phys. Rev. Lett. 93, 080502 (2004).
[49] M. Horodecki, Quantum Inf. Comput. 1, 3 (2001).
[50] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and A. V. Thapliyal, Phys. Rev. A 61, 062312 (2000).
[51] W. Dür, J. I. Cirac, M. Lewenstein, and D. Bruß, Phys. Rev. A 61, 062313 (2000).
[52] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 78, 574 (1997).
[53] P. G. Kwiat, S. Barraza-Lopez, A. Stefanov, and N. Gisin, Nature 409, 1014 (2001).
[54] R. Horodecki, M. Horodecki, and P. Horodecki, Phys. Lett. A 222, 21 (1996).
[55] D. Cavalcanti, A. Acín, N. Brunner, and T. Vértesi, Phys. Rev. A 87, 042104 (2013).
[56] D. Cavalcanti, L. Guerini, R. Rabelo, and P. Skrzypczyk, Phys. Rev. Lett. 117, 190401 (2016).
[57] F. Hirsch, M. T. Quintino, T. Vértesi, M. F. Pusey, and N. Brunner, Phys. Rev. Lett. 117, 190402 (2016).
[58] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. 86, 419 (2014).
[59] A. G. White, A. Gilchrist, G. J. Pryde, J. L. O’Brien, M. J. Bremner, and N. K. Langford, J. Opt. Soc. Am. B 24, 172 (2007).