GAIN/LOSS OF DERIVATIVES FOR COMPLEX VECTOR FIELDS

LUCA BARACCO AND GIUSEPPE ZAMPIERI

Abstract. In $\mathbb{C} \times \mathbb{R}$ we consider the function $g = g(z)$, set $g_1 = \partial_z g$, $g_{1\bar{1}} = \partial_{\bar{z}} \partial_z g$ and define the operator $L_g = \partial_z + ig_1 \partial_t$. We discuss estimates with loss of derivatives, in the sense of Kohn, for the system $(\bar{L}_g, f^k L_g)$ where $(\bar{L}_g, L_g)$ is $\frac{1}{2m}$ subelliptic at 0 and $f(0) = 0$, $df(0) \neq 0$. We prove estimates with a loss $l = \frac{k-1}{2m}$ if the “multiplier” condition $|f| \gtrsim |g_{1\bar{1}}|^{\frac{1}{2(m-1)}}$ is fulfilled. (For estimates without cut-off, subellipticity can be weakened to compactness and this results in a loss of $l = \frac{k-1}{2(m-1)}$.) For the choice $(g, f^k) = (|z|^{2m}, z^k)$ this result was obtained by Kohn and Bove-Derridj-Kohn-Tartakoff for $m = 1$ and $m \geq 1$ respectively. Also, the loss $l = \frac{k-1}{2m}$ was proven to be optimal. We show that it remains optimal for the model $(g, f^k) = (|z|^{2m}, x^k)$, in which the multiplier condition is violated, the loss is not lowered by the type and must be $\geq \frac{k-1}{2}$.

MSC: 32F10, 32F20, 32N15, 32T25

Contents

1. Introduction
2. Statements and proofs
References

1. Introduction

(s1) In $\mathbb{C} \times \mathbb{R}$ with coordinates $(z, t)$, $z = x + iy$, let $g = g(z)$ be a smooth real function, set $g_1 = \partial_z g$, $g_{1\bar{1}} = \partial_{\bar{z}} \partial_z g$, assume that $g$ is subharmonic, that is, $g_{1\bar{1}} \geq 0$, and define the vector field $\hat{L}_g := \partial_z - ig_1 \partial_t$. We denote by $\tilde{L}_g$ the conjugate to $\hat{L}_g$ and sometimes write $\bar{L}$ and $L$ instead of $\hat{L}_g$ and $L_g$. For a (complex) smooth function $f = f(z)$ with $f(0) = 0$ and $\bar{\partial} f(0) \neq 0$, and for an integer $k > 0$, our interest goes to the estimates for the existence and the local regularity of the system $\{\tilde{L}_g, f^k L_g\}$ in a neighborhood $V$ of 0. In our discussion, a subellipticity, or compactness, assumption is made for $\{\tilde{L}_g, L_g\}$, but this is destroyed by the effect of the factor $f^k$. In these estimates a “loss of derivatives”, that we quantify by $l$, is expected. In detail, estimates for existence are in the form

\begin{equation}
\tag{1.1} \|u\|_s \lesssim \|\tilde{L}_g u\|_{s+l} + \|f^k L_g u\|_{s+l} + \|u\|_{-\infty} \quad \text{for any } u \in C_c^\infty (V).
\end{equation}
As for local regularity, for any pair of cut-off functions ζ and ˜ζ at (z, t) = (0, 0) with ζ ≺ ˜ζ, in the sense that ˜ζ|supp ζ = 1, these estimates are

\( \| \zeta u \|_s \leq c_{\zeta, \tilde{\zeta}} (\| \tilde{\zeta} \bar{L}_g u \|_{s+t} + \| \tilde{\zeta} f^k \bar{L}_g u \|_{s+t} + \| u \|_{-\infty}) \) for any \( u \in C^\infty(V) \).

For the choice \( (g, f^k) = (|z|^{2m}, \bar{z}^k) \), (1.1) and (1.2) with a loss \( l = \frac{k-1}{2m} \) have been established by Kohn in [7] when \( m = 1 \) and further extended to any \( m > 1 \) by Bove, Derridj, Kohn and Tartakoff in [1]. Keeping the same \( f^k = \bar{z}^k \), but extending the choice of \( g \) from \( |z|^{2m} \) to a general \( g \) of “type 2m” in the sense of (2.3) below, the estimates (1.1), (1.2) hold for any \( g, f^k \) where \( g \) has type \( 2m \) and \( f \) satisfies the “multiplier” condition \( f \geq \frac{g}{1_{11}^{\frac{1}{2(m-1)}}} \). This applies not only to \( (|z|^{2m}, \bar{z}^k) \) but also, for instance, to \( (x^{2m}, x^k) \). In both cases, this loss \( l = \frac{k-1}{2m} \) is optimal; (1.1) and (1.2) cannot hold for \( l < \frac{k-1}{2m} \). It is also proved that for (1.1) (differently from (1.2)) subellipticity is needless; if this is replaced by compactness, (1.1) still holds for a loss \( l = \frac{k}{2(2m-1)} \). Coming back to the previous number \( l = \frac{k-1}{2m} \), it is worth noticing that this has a deep meaning. If \( L^g \) is the Lie span of order \#, then \( m \) and \( k \) are the smallest numbers for which we have

\[
(1.3) \quad \{ T M = \mathcal{L}^{2m}\{ \bar{L}_g, L_g \}, \quad [L_g, \bar{L}_g] \in \mathcal{L}^{k+2}\{ \bar{L}_g, f^k L_g \} \}.
\]

Now, the first of (1.3) is the general condition of \( \frac{1}{2m} \)-subellipticity and the second says that \( \partial_z f^k|_0 \neq 0 \) (since \( df(0) \neq 0 \)). However, for the loss \( l \) in (1.1) or (1.2) to be \( l \leq \frac{k-1}{2m} \), the crucial point is not only (1.3) but also the “multiplier” type condition \( |f| \geq \frac{g}{1_{11}^{\frac{1}{2(m-1)}}} \).

If this is violated, a bigger loss occurs. Thus, for \( (g, f^k) = (|z|^{2m}, x^k) \), (1.1) cannot hold unless \( l \geq \frac{k-1}{2} \) (Theorem 2.5 below). Therefore, raising the type from 2 to 2m does not result into dividing \( l \) by \( m \).

This paper is inspired to work by J.J. Kohn and, especially, to specific questions he raised in his talk in Vienna ESI Conference in December 2010.

2. Statements and proofs

(s2) We first introduce stronger versions of (1.1) and (1.2). In a neighborhood \( V \) of 0, the first is in the form

\[
(2.1) \quad \| u \|_s \leq \| \bar{L}_g u \|_{s-\frac{1}{2m}} + \| f^k \bar{L}_g u \|_{s+t} + \| f^k L_g u \|_{s+t} + \| u \|_{-\infty} \quad \text{for any } u \in C^\infty_c(V),
\]

and the second is, for any pair of cut-off functions \( \zeta < \tilde{\zeta} \)

\[
(2.2) \quad \| \zeta u \|_s \leq c_{\zeta, \tilde{\zeta}} (\| \tilde{\zeta} \bar{L}_g u \|_{s-\frac{1}{2m}} + \| \zeta f^k \bar{L}_g u \|_{s+t} + \| \tilde{\zeta} f^k \bar{L}_g u \|_{s+t} + \| u \|_{-\infty}) \quad \text{for any } u \in C^\infty(V).
\]
The point here is that the loss \( l \) does not affect the \( \bar{L} \)-derivative unless this is multiplied by \( f^k \). Since \( \| f^k \bar{L} g u \|_{s+1} \lesssim \| \bar{L} g u \|_{s+1} \), these conditions are stronger than those introduced in Section \( \| \). We assume that \( g \) has “finite \( 2m \)-type” along a real curve \( S \subset \mathbb{C} \). By this we mean that, with \( d_S \) denoting the distance to \( S \), we have

\[
\left( \text{2.3) (nova) } \right) g_{11} \gtrsim d_S^{2(m-1)},
\]

which yields a \( \frac{1}{2m} \)-subelliptic estimate for \( \{ L_g, \bar{L}_g \} \). (An immediate example is provided by \( g = x^{2a}|z|^{2b} \) for \( a + b = m \).) Note that if we only assume that \( g_{11} \) vanishes at order \( 2(m-1) \), we get a subelliptic estimate but for an index which may be \( < \frac{1}{2m} \).

**Theorem 2.1. (t2.1)** Take \( (g, f^k) \) with \( g \) satisfying \( \text{2.3) \ and \ with} \)

\[
\left( \text{multiplier) } \right) |f| \gtrsim g_{11}^{(m-1)},
\]

Then the system \( \{ \bar{L}_g, f^k L_g \} \) satisfies \( \text{2.1) \ and \ \text{2.2) for } l = \frac{k-1}{2m}.} \)

It is clear from the proof that what is needed is not \( \text{2.3) itself but a } \frac{1}{2m} \)-subelliptic estimate.

As already recalled, for \( (g, f^k) = (|z|^{2m}, \bar{z}^k) \) Theorem \( \text{2.1) is obtained in } \mathbf{7} \text{ for } m = 1 \text{ and } \mathbf{1} \text{ for } m \geq 1 \text{ respectively and, for the pair } (g, \bar{z}^k) \text{ in which } g \text{ satisfies (i) above it is given in } \mathbf{9}. \text{ New models which enter in Theorem } \text{2.1) are } (x^k, x^{2m}) \text{ or else } (x^k, x^{2a}|z|^{2b}) \text{ or finally } (f^k, f^{2a}h^{2b}) \text{ for } f \text{ real with } \partial f \neq 0 \text{ and } |h| \sim |z|.

**Proof.** We only prove the harder part, that is, \( \text{2.2) \). We introduce some terminology. “Good” is a term which is controlled by the right side of an estimate. “Absorbable” is a term which comes as a fraction of the left or of a previous term. “Neglectable” is a term which comes with a smaller Sobolev index than previous terms and possibly with a slightly bigger cut-off; this becomes good through induction. Finally, sc and lc denote a small and large constant respectively.

In the microlocal decomposition \( u = u^+ + u^- + u^0 \) in the sense of Kohn \( \mathbf{6} \) Section 5 and 10, it is readily seen that \( u^0 \) enjoys elliptic estimates for \( \bar{L} \) and \( u^- \), \( \frac{1}{2m} \)-subelliptic ones (cf. for instance \( \mathbf{9} \) (3.1)–(3.4)). Thus only \( u^+ \) needs to be estimated. For this, there is coincidence of the full Sobolev norm with the partial Sobolev norm in \( t \), that is,

\[
\| u^+ \|_s \sim \| \Lambda^s_t u^+ \|_0,
\]

where \( \Lambda^s_t \) is the standard elliptic pseudodifferential operator of order \( s \) in \( t \). For this reason, we always write \( u \) for \( u^+ \) and \( \| \cdot \|_s \) for \( \| \Lambda^s_t \cdot \|_0 \). We start from subelliptic estimates for \( \{ \bar{L}, L \} \) applied to \( \zeta u \) which yield, by estimating the commutator \( [L, \zeta] = \zeta \prec \zeta \) (and similarly
for $L$):

$$
\|\zeta u\|_s \lesssim \|\zeta \bar{L} u\|_{s - \frac{1}{2m}} + \|\zeta L u\|_{s - \frac{1}{2m}} + \|\tilde{\zeta} u\|_{s - \frac{m}{2}}
$$

(2.3) (2.5)

$$
\lesssim \|\zeta L u\|_{s - \frac{1}{2m}} + \|\zeta |[L, L]|^\frac{1}{2} u\|_{s - \frac{1}{2m}} + \|\tilde{\zeta} u\|_{s - \frac{m}{2}}\text{ neglectable.}
$$

Here $\zeta = \zeta(z)\zeta(t)$ and $\tilde{\zeta} = \tilde{\zeta}(z)\tilde{\zeta}(t)$. We recall now a result about interpolation (cf. [1] Lemma 2.4 and [9] Lemma 3.2): for $h = h(z)$ bounded and satisfying $h(0) = 0$ and for real positive numbers $\rho, r, n_1, n_2$ with $0 < n_1 \leq r$ and $n_2 > 0$ we have

(2.6) (2.4) $$
\|h^ru\|_0 \lesssim sc\|h^{r-n_1}u\|_{-n_1\rho} + lc\|h^{r+n_2}u\|_{n_2\rho},
$$

where, again, the partial Sobolev norm in $t$ is meant. (We have to notice here that $h$ needs not to be smooth because only Sobolev norm with respect to $t$ is considered; it only needs to be $H^0$ so that (2.6) pointwise for almost every $z$ implies (2.6) integrated in $z$.) Remark that $[L, \bar{L}] = g_{11}^1\partial_t$ for $g_{1,1} \geq 0$ and set $l = \frac{k-1}{2m}$; it follows

$$
\|\zeta |[L, L]|^\frac{1}{2} u\|_{s - \frac{1}{2m}} \sim \|\zeta g_{11}^1 \Lambda_t^\frac{1}{2} u\|_{s - \frac{1}{2m}}
$$

$$
\lesssim \langle \zeta \rangle\langle \zeta \rangle u_{s} + lc\|g_{11}^1 \Lambda_t^\frac{k}{2(m-1)} + \Lambda_t^\frac{1}{2} u\|_{s + l} + \|\tilde{\zeta} u\|_{s - 1}\text{ absorbable}
$$

(2.5) (2.7)

$$
\lesssim \|\zeta g_{11}^1 \Lambda_t^\frac{1}{2} \tilde{h}^k u\|_{s + l}
$$

$$
= \|\zeta |[L, L]|^\frac{1}{2} \tilde{h}^k u\|_{s + l}
$$

$$
\lesssim \|\zeta L \tilde{h}^k u\|_{s + l} + \|\zeta \bar{L} \tilde{h}^k u\|_{s + l} + \|\tilde{\zeta} \tilde{h}^k u\|_{s + l},\hspace{1cm}(*)
$$

where the inequality in the second line follows from (2.6) under the choice $n_1 = m-1, n_2 = k, r = m-1, \rho = \frac{1}{2m}$ and $h = g_{11}^1 \Lambda_{11}$. We have to estimate the three terms in the bottom of (2.7). As for the first, we have

$$
\|\zeta L \tilde{h}^k u\|_{s + l} \leq \text{good} + \|\partial_2 \tilde{h}^{k-1} \zeta u\|_{s + l}
$$
and

\[ \| f^{k-1} \partial_z f \zeta u \|_{s+l} = \frac{1}{k} (\partial_z f f^{k-1} \zeta u, [L, f^k] \zeta u)_{s+l} \]

(2.5) (2.8)

\[ = \frac{1}{k} (\partial_z f f^{k-1} \zeta u, f^k \zeta u)_{s+l} + \frac{1}{k} (\partial_z f f^{k-1} \zeta u, f^k \zeta u)_{s+l} \]

(2.5) (2.8)

\[ + \frac{k-1}{k} (\partial_z f f f^{k-2} \zeta u, f^k \zeta u) + \frac{1}{k} (\partial_z f f^{k-1} \zeta u, f^k \zeta u)_{s+l} \]

(2.5) (2.8)

(Here \( \partial_z f \) and \( \partial^2_z f \) have been neglected as constants since \( f = f(z) \) and \( \Lambda^{s+l} = \Lambda^s \).)

First, (c) is absorbable by (a) since (c) = \( \frac{k-1}{k} (a) \). Second, (b) is absorbable by (a); in fact, \( f(0) = 0 \) implies \( f^k = sc \partial_z f f^{k-1} \). This concludes the estimate of the first term in the last line of (2.7) apart from the terms marked by (*). The second can be estimated in the same way. To conclude the proof of the theorem, it only remains to estimate the term (*) which occurs in (2.8) and also in the bottom of (2.7). For this we use subelliptic estimates and iteration

\[ \| \tilde{\zeta} f^k u \|_{s+l} \leq \| \tilde{\zeta} \tilde{L} f^k u \|_{s+l-\frac{1}{2m}} + \| \tilde{\zeta} \tilde{L} f^k u \|_{s+l-\frac{1}{2m}} \]

\[ \leq \text{good} + \| \partial_z f f^{k-1} \zeta u \|_{s+l-\frac{1}{2m}} + \text{good + neglectable}. \]

\[ \text{Weaker than subelliptic are compactness estimates. By this we mean} \]

(2.9)

(\text{supernova}) \[ \| u \|_0 \lesssim \delta (\| \tilde{L} u \|_0 + \| Lu \|_0 + c_3 \| u \|_{-1}) \quad \text{for any } u \in C^\infty_c (V), \text{for any } \delta \text{ and for suitable } c_3. \]

\textbf{Theorem 2.2. (t2.1,5) Consider the pair } \( (g, f^k) \) \text{ for which (2.9) and (2.4) are satisfied. Then the system } \{ \tilde{L}_g, f^k \tilde{L}_g \} \text{ satisfies (2.1) for } l = \frac{k}{2(2m-1)}. \]

\textbf{Proof.} Differently from Theorem 2.1, we have not to control the commutators with the cut-off \( [L, \zeta] \) and \( [\tilde{L}, \zeta] \). We start from the compactness estimate (2.9) applied to \( \Lambda^s u \) and replace the term containing \( Lu \) by the aid of

\[ \| [L, \tilde{L}] \frac{1}{2} u \|_s = \| g^{\frac{1}{2}} \Lambda^\frac{1}{2} u \|_s \]

\[ \lesssim sc \| u \|_s + \| L f^k u \|_{s+\frac{k}{2(m-1)}} + \| \tilde{L} f^k u \|_{s+\frac{k}{2(m-1)}}. \]
where we have used Sobolev interpolation for $h = g^{\frac{1}{2(m-1)}}$, $n_1 = \frac{k}{2}$, $n_2 = k$, $\rho = 2(m-1)$.

We then estimate

$$\|Lf^k u\|_{s + \frac{k}{2(m-1)}} + \|\tilde{L} f^k u\|_{s + \frac{k}{2(m-1)}} \lesssim \text{good} + \|\partial_y f^{k-1} u\|_{s + \frac{k}{2(m-1)}}$$

$$\lesssim \text{good} + \text{absorbable} + \left(\frac{k-1}{k}\|\partial_y f^{k-1} u\|_{s + \frac{k}{2(m-1)}} + \|f^k u\|_{s + \frac{k}{2(m-1)}}\right)$$

As for the constraint $l \geq \frac{k-1}{2m}$ for the loss in (1.1) or (1.2) (and thus a fortiori in (2.1) or (2.2)), this is proved in [1] for the pair $(g, f^k)$ or $(h, f^k)$. The simplest examples are $h = x$ or $|h| \sim |z|$.

**Proof.** We prove the statement for $h = x$; we will specify at the end of the proof the slight modification which is needed for the general case. Following the idea of [7] and [1], we set $u_\lambda = e^{-\lambda(x^{2m} - it) - (x^{2m} - it)^2}$. We have

$$u_\lambda \sim e^{-\lambda(x^{2m} + t^2)}.$$

Assume first (2.1) and apply it for $u = \zeta u_\lambda$ where $\zeta$ is a cut-off of product type $\zeta = \zeta(x)\zeta(y)\zeta(t)$. After rescaling, we may assume that $V$ is unitary and the cut-off is supported by $V$ and is 1 in a half of it. We rewrite the terms in the right of (2.1). Now,

$$\|u_\lambda\|_0 \sim \int_{|x| \leq 1} e^{-\lambda x^{2m}} dx \int_{|y| \leq 1} dy \int_{|t| \leq 1} e^{-\lambda t^2} dt$$

$$\sim \lambda^{-\frac{1}{2m} - \frac{1}{2}}. \tag{101}$$

Also,

$$x^k L u_\lambda = (\lambda x^{2m})^{\frac{2m+k-1}{2m}} \lambda^{-\frac{k-1}{2m}} e^{-\lambda x^{2m}} e^{-\lambda t^2}.$$

It follows

$$\|x^k L u_\lambda\| = \lambda^{-\frac{k-1}{2m}} \int_{|x| \leq 1} (\lambda x^{2m})^{\frac{2m+k-1}{2m}} e^{-\lambda x^{2m}} dx \int_{|t| \leq 1} e^{-\lambda t^2} dt$$

$$\sim \lambda^{-\frac{k-1}{2m}} \lambda^{-\frac{1}{2m}} \lambda^{-\frac{1}{2}}.$$ 

On the other hand

$$[L, \zeta] \sim \dot{\zeta}(x) \zeta(y) \zeta(t) + \zeta(x) \dot{\zeta}(y) \zeta(t) + \zeta(x) \zeta(y) \dot{\zeta}(t);$$
moreover
\[
\begin{aligned}
\dot{\zeta}(x)\zeta(y)\zeta(t)u_\lambda &\sim e^{-\lambda} \\
\zeta(x)\zeta(y)\zeta(t)u_\lambda &\sim e^{-\lambda}.
\end{aligned}
\]

Instead,
\[
(2.11) \quad (100) \quad \zeta(x)\dot{\zeta}(y)\zeta(t)u_\lambda \sim u_\lambda \quad \text{on supp } \zeta.
\]

This does not look as absorbable but it will be indeed thanks to $\frac{1}{2m}$-gain in Sobolev index which follows from subellipticity. In fact, we have to remark that
\[
(2.12) \quad (102) \quad \partial_l^# u_\lambda = \lambda^# u_\lambda;
\]
thus the Sobolev norm $-\frac{1}{2m}$ and $l$ have the effect of producing a factor $\lambda^{\frac{1}{2m}}$ and $\lambda^l$ respectively. We also remark that $\|x^k[L, \zeta]u_\lambda\|_l$ and $\|x^k[L, \zeta]u_\lambda\|_l$ are errors with respect to $\|\zeta x^k L u_\lambda\|_l$ (and that $\|\zeta u_\lambda\|_{-\infty}$ is absorbed by the left of (2.1)). Thus (2.1) turns into
\[
(2.13) \quad (2.6) \quad \lambda^{-\frac{1}{2m}-\frac{1}{2}} \sim (e^{-\lambda} + \lambda^{-\frac{1}{2m}})\lambda^{-\frac{1}{2m}} + \lambda^{-\frac{1}{2m} - \frac{1}{2}} \lambda^{\frac{k-1}{2m}} \lambda^l.
\]

Finally, (2.13) forces $l \geq \frac{k-1}{2m}$.

The proof for (2.1) replaced by (2.2) is the same.

Finally for a general $h$ in place of $x$, we have just to replace $\lambda_x 2m$ by $\lambda|h|^{2m}$ in the definition of $u_\lambda$ and to substitute the integration $\int_{|x|\leq 1} e^{-\lambda |x|^{2m}} dx \int_{|y|\leq 1} dy$ by $\int \int_{|z|\leq 1} e^{-\lambda |h(z)|^{2m}} dx dy \sim \lambda^{-\frac{k}{2m}}$.

\[\Box\]

Remark 2.4. If we use (1.1) or (1.2) instead of (2.1) or (2.2), we have to estimate $\|[L, \zeta]u_\lambda\|_{\frac{1}{2m}}$, but this is not an error term since, instead, it is $\sim \lambda^{-\frac{1}{2m}} \lambda^l$ (combining (2.10), (2.11) and (2.12)). When $g = |z|^2$, we have $[L, \zeta] \sim e^{-\lambda}$ and hence the problem is overcome. It is in this sense that our argument differs from [7] and [1].

When the hypothesis (ii) of Theorem 2.1 is missing, (2.1) cannot hold for $l = \frac{k-1}{2m}$.

**Theorem 2.5.** (2.3) Let $(g, f^k) = ([h]^{2m}, f^k)$ with $|h| \sim |z|$ and $f$ real with $\partial f \neq 0$; if (2.1) holds, then
\[
l \geq \frac{k-1}{2}.
\]

The theorem applies, for example, to the pair $(g, f^k) = (|z|^{2m}, x^k)$.

**Proof.** We have to introduce now a different exponential solution of $\bar{L}$. We set, for a convenient $C > 0$
\[
u_\lambda = e^{-\lambda|z|^2 + C(g(z) - it - (g(z) - it)^2)}.
\]

We note that
\[
\begin{aligned}
\bar{L} u_\lambda &= 0 \\
|u_\lambda| &\sim e^{-\lambda|z|^2 - y^2 + Cy^{2m} + t^2}.
\end{aligned}
\]
For small $\epsilon$, we choose $C$ so that $-y^2 + Cy^{2m} \sim y^{2m}$ for $|y| = \epsilon$. To simplify notations we assume $f = x$; the proof of the general case needs no change. We apply (2.1) for $u = \zeta(x)\zeta(y)\zeta_1(t)u_\lambda$ where $\zeta_\lambda, \zeta_\epsilon$ and $\zeta_1$ are cut-off functions at $\frac{1}{\nu}, \epsilon$ and 1 respectively. To simplify notation we write also $\zeta = \zeta_\epsilon(y)\zeta_1(t)$ but keep separate $\zeta_\lambda(x)$ because it plays a distinguished role. Now, (2.1) for $s = 0$ becomes

$$
\| \zeta_\lambda \zeta u_\lambda \|_0 \leq \| \zeta_\lambda \zeta u_\lambda \| - \frac{1}{2m} + \| \zeta_\lambda \zeta u_\lambda \| - \frac{1}{2m} + \| \zeta_\lambda \zeta \bar{L} u_\lambda \| - \frac{1}{2m} + \| x^k \zeta_\lambda \zeta L u_\lambda \|_l + \| x^k \zeta_\lambda \zeta u_\lambda \|_l + \text{absorbable.}
$$

We notice that $\int_{|x| \leq \frac{1}{\nu}} e^{-\lambda x^2} dx \sim \frac{1}{\nu}$ and $\int_{|t| \leq 1} e^{-\lambda t^2} dt \sim 1$; we use the notation $\# := \int_{|y| \leq \epsilon} e^{-\lambda(y^2 + Cy^{2m})} dy$. We have

1: $\| \zeta_\lambda \zeta u_\lambda \| \sim \nu^{-1} \#.$

2: $e^{-\lambda x^2} \sim e^{-\frac{\lambda}{\nu^2}}$ in supp $\zeta_\lambda$; hence

$$
\| \zeta_\lambda \zeta u_\lambda \|_0 \sim e^{-\frac{\lambda}{\nu}} \#.
$$

3: $\| \zeta_\lambda \zeta u_\lambda \|_0 \sim e^{-\lambda \epsilon^{2m}}$ because

$$
(2.16)
$$

$$
(2.9) \begin{cases} (a) \ |u_\lambda| \leq e^{-\lambda(y^2 + Cy^{2m})} \sim e^{-\lambda \epsilon^{2m}} & \text{on supp } \zeta(y) \\ (b) \ |u_\lambda| \leq e^{-\lambda + \lambda \epsilon^2} & \text{on supp } \zeta(t) \end{cases}
$$

4: This is 0.

5, 6 and 7: We have

$$
x^k \zeta_\lambda \zeta |L u_\lambda| \sim \zeta_\lambda \zeta x^k \lambda |\zeta|^{2m-1} |u_\lambda| \leq \nu^{-k} \zeta_\lambda \zeta |u_\lambda|.
$$

Again, $l$-Sobolev norm produces a factor $\lambda^l$ which yields

$$
\| x^k \zeta_\lambda \zeta \|_l \sim \nu^{-(k+1)} \lambda^{l+1} \#,
$$

and similarly

$$
(2.17) \quad 6. \sim \lambda^l \nu^{-k} e^{-\frac{\lambda}{\nu^2}} \# \quad \text{by the analogous of } (2.15),
$$

$$
(2.18) \quad 7. \sim e^{-\lambda \epsilon^{2m}} \quad \text{by the analogous of } (2.16).
$$
Eventually, \([2.14]\) is equivalent to
\[
\nu^{-1} \# < \lambda^{-\frac{k+1}{2m}} e^{-\frac{\lambda}{m}} \# + e^{-\lambda e^{2m}} + 0 + \nu^{-(k+1)} \lambda^{l+1} \# + \lambda^{l} \nu^{-k} \# e^{-\frac{\lambda}{m}} \# + e^{-\lambda e^{2m}}.
\]
Now, \(3\), and \(7\) can be disregarded. Next, if we choose \(\nu = \lambda^{\frac{1}{2m}}\), then \(2 \sim e^{-\lambda} \# \) and \(6 \sim \lambda e^{-\lambda} \#\); these are therefore errors of \(1\). Thus, only \(5\) survives and, under the choice \(\nu = \lambda^{\frac{1}{2m}}\), in order to have inequality
\[
1 \sim \nu^{-(k+1)} \lambda^{l+1} \sim \lambda^{l+1-k+\frac{k+1}{2}+\frac{k+1}{2}},
\]
we must require \(l > \frac{k-1}{2}\). This concludes the proof.

\(\square\)

**Remark 2.6.** When, instead of \(g = |z|^{2m}\), we have \(g = x^{2m}\), then the inequality (a) in the estimate of \(3\) above is not true and therefore \(3\) is not an error term. This explains why, for \(g = x^{2m}\) and \(f^{k} = x^{k}\), we have \(2.1\) for \(l = \frac{k-1}{2m}\) (Theorem 2.1 above).

**References**

[1] A. Bove, M. Derridj, J.J. Kohn and D.S. Tartakoff—Sums of squares of complex vector fields and (analytic-) hypoellipticity, *Math. Res. Lett.* 13 n.5 (2006), 683–701

[2] M. Christ—Hypoellipticity of the Kohn Laplacian for three-dimensional tubular Cauchy-Riemann structures, *J. of the Inst. of Math. Jussieu* 1 (2002), 279–291

[3] V.S. Fedi—A certain criterion for hypoellipticity, *Mat. Sb.* 14 (1971), 15–45

[4] G.B. Folland and J.J. Kohn—The Neumann problem for the Cauchy-Riemann complex, *Ann. Math. Studies, Princeton Univ. Press, Princeton N.J.* 75 (1972)

[5] L. Hörmander—Hypoelliptic second order differential equations, *Acta Math.* 119 (1967), 147–171

[6] J.J. Kohn—Superlogarithmic estimates on pseudoconvex domains and CR manifolds, *Annals of Math.* 156 (2002), 213–248

[7] J.J. Kohn—Hypoellipticity and loss of derivatives, *Annals of Math.* 162 (2005), 943–986

[8] T.V. Khanh and G. Zampieri—Regularity of the \(\bar{\partial}\)-Neumann problem at a point of infinite type, *J. of Funct. Analysis* 259 (2010), 2760–2775

[9] T.V. Khanh, S. Pinton and G. Zampieri—Loss of derivatives for systems of complex vector fields and sums of squares, *Proc. of AMS* 140 n. 2 (2012), 519–530

[10] J.J. Kohn and L. Nirenberg—Non-coercive boundary value problems, *Comm. Pure Appl. Math.* 18 (1965), 443–492

[11] S. Kusuoka and D. Stroock—Applications of Mallavain calculus II, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.* 32 (1985), 1–76

[12] Y. Morimoto—Hypoellipticity for infinitely degenerate elliptic operators, *Osaka J. Math.* 24 (1987), 13–35

[13] C. Parenti and A. Parmeggiani—On the hypoellipticity with a big loss of derivatives, *Kyushu J. Math.* 59 (2005), 155–230
[14] **E.M. Stein**—An example on the Heisenberg group related to the Lewy operator, *Invent. Math.* **69** (1982), 209–216

[15] **D.S. Tartakoff**—Analyticity for singular sums of squares of degenerate vector fields, *Proc. Amer. Math. Soc.* **134** n. 11 (2006), 3343–3352