PRIMITIVE DECOMPOSITIONS OF DOLBEAULT HARMONIC FORMS ON COMPACT ALMOST-KÄHLER MANIFOLDS

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1. Introduction

In complex geometry, the Dolbeault cohomology plays a fundamental role in the study of complex manifolds, and a classical way to compute it on compact complex manifolds is through the use of the associated spaces of harmonic forms. More precisely, if \( X \) is a complex manifold, then the exterior derivative \( d \) splits as \( \partial + \bar{\partial} \), and such operators satisfy \( \bar{\partial}^2 = \partial^2 = \partial \bar{\partial} + \bar{\partial} \partial = 0 \). Hence, one can define the Dolbeault cohomology and its conjugate as

\[
H^{\bullet,\bullet}_{\bar{\partial}}(X) := \frac{\text{Ker } \bar{\partial}}{\text{Im } \bar{\partial}}, \quad H^{\bullet,\bullet}_{\partial}(X) := \frac{\text{Ker } \partial}{\text{Im } \partial}.
\]

If \( X \) is compact and we fix an Hermitian metric, then it turns out that these spaces are isomorphic to the kernel of two suitable elliptic operators, \( \Delta_{\bar{\partial}} \) and \( \Delta_{\partial} \), respectively. More precisely, denoting with \( \mathcal{H}^{\bullet,\bullet}_{\bar{\partial}}(X) \) and \( \mathcal{H}^{\bullet,\bullet}_{\partial}(X) \) the spaces of harmonic forms, they have a cohomological meaning, namely

\[
H^{\bullet,\bullet}_{\bar{\partial}}(X) \simeq \mathcal{H}^{\bullet,\bullet}_{\bar{\partial}}(X), \quad H^{\bullet,\bullet}_{\partial}(X) \simeq \mathcal{H}^{\bullet,\bullet}_{\partial}(X),
\]

and in particular their dimensions are holomorphic invariants.

Moreover, if the Hermitian metric is Kähler, then by the Kähler identities it turns out that \( \Delta_{\bar{\partial}} = \Delta_{\partial} \) and in particular

\[
\mathcal{H}^{\bullet,\bullet}_{\bar{\partial}}(X) = \mathcal{H}^{\bullet,\bullet}_{\partial}(X).
\]
therefore giving isomorphisms for the respective cohomologies, namely

\[ H_{\partial}^{\bullet, \bullet}(X) \simeq H_{\bar{\partial}}^{\bullet, \bullet}(X). \]

The integrability assumption on the complex structure is crucial in the proof of all these results.

Furthermore, a remarkable feature of Kähler geometry is that the primitive decomposition of differential forms passes to cohomology and leads to a primitive decomposition of de Rham cohomology (see, e.g., [18]). Kähler geometry is at the crossroad of complex and symplectic geometry. From the symplectic point of view we recall that Tseng and Yau [17] introduced natural cohomologies on (compact) symplectic manifolds, involving the symplectic co-differential and the exterior derivative, proving a primitive decomposition for them.

If \( J \) is a non-integrable almost-complex structure on a \( 2n \)-dimensional smooth manifold \( X \), then the exterior derivative splits as \( \mu + \partial + \bar{\partial} + \bar{\mu} \), and in particular \( \bar{\partial}^2 \neq 0 \). Hence, the standard Dolbeault cohomology and its conjugate are not well-defined. Recently, Cirici and Wilson [6] gave a definition for the Dolbeault cohomology in the non-integrable setting considering also the operator \( \bar{\mu} \) together with \( \bar{\partial} \). Such cohomology groups might be infinite-dimensional on compact almost-complex manifolds as shown in [7].

On the other hand, fixing an almost-Hermitian metric \( g \) on \( (X, J) \) one can develop a Hodge theory for harmonic forms on \( (X, J, g) \) without a cohomological counterpart. More precisely, setting, similarly to the integrable case,

\[ \Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\partial, \quad \Delta_{\partial} := \partial\partial^* + \partial^*\partial, \]

it turns out that they are elliptic selfadjoint differential operators. Therefore, if \( X \) is compact, their kernels, denoted again with \( \mathcal{H}_{\bar{\partial}}^{\bullet, \bullet}(X) \) and \( \mathcal{H}_{\partial}^{\bullet, \bullet}(X) \), are finite dimensional complex vector spaces. Holt and Zhang [11] answered to a question of Kodaira and Spencer [9] showing that, contrarily to the complex case, the dimensions of the spaces of \( \bar{\partial} \)-harmonic \((0,1)\)-forms on a \( 4 \)-dimensional manifold depend on the metric. Indeed they construct on the Kodaira–Thurston manifold an almost-complex structure that, with respect to different almost-Hermitian metrics, has varying \( \dim \mathcal{H}_{\bar{\partial}}^{0,1} \). With different techniques, in [16] it was shown that also the dimension of the space of \( \bar{\partial} \)-harmonic \((1,1)\)-forms depends on the metric on \( 4 \)-dimensional manifolds (for other results in this direction, see [13] and [10]).

We note that performing explicit computations of \( \bar{\partial} \)-harmonic forms is a difficult task and not much is known in higher dimensions (see [15], [3], [4] for some detailed computations).

In the present paper we study the validity of primitive decompositions on compact almost-Kähler manifolds in any dimension. More precisely, in Propositions 3.1 and 3.2, Theorem 3.4 and Corollary 3.5 we prove, on compact almost-Kähler \( 2n \)-dimensional manifolds, primitive decompositions for \( \bar{\partial} \)- and \( \partial \)-harmonic forms in bidegrees \((p,0)\), \((0,q)\), \((1,1)\), \((n,n-p)\), \((n-q,n)\) and \((n-1,n-1)\), with \( p, q \leq n \). One cannot hope to have such decompositions for any bidegree as shown in Example 5.3. For similar results in the case of Bott–Chern harmonic forms, we refer the reader to [12].
We notice that, even though the metric is almost-Kähler, the decompositions of $\bar{\partial}$- and $\partial$-harmonic forms might differ. Indeed, in Section 4 we show explicitly that, differently from the Kähler case, one can have $\Delta_{\bar{\partial}} \neq \Delta_{\partial}$, and also

$$\mathcal{H}_{\bar{\partial}}^{1,1}(X) \neq \mathcal{H}_{\partial}^{1,1}(X).$$

We observe that a key ingredient in the proof of the results in [16] (see also [11]) is indeed the primitive decomposition of $\bar{\partial}$-harmonic $(1,1)$-forms on 4-dimensional manifolds. In fact, in this dimension in Proposition 4.1 we prove the general equality $\mathcal{H}_{\bar{\partial}}^{1,1}(X) = \mathcal{H}_{\partial}^{1,1}(X)$.

All the examples we present are nilmanifolds, of dimensions 6 and 8, endowed with possibly non-left-invariant almost-Kähler structures.

We recall that if one wants to mimic and recover all the Kähler identities, the proper operator to consider is $\bar{\delta} := \bar{\partial} + \mu$ (see [5], [14]). However, considering just the operator $\bar{\partial}$ on almost-Kähler manifolds we are able to see how genuinely almost-Kähler manifolds differ from Kähler ones. More precisely, the study of the kernel of $\Delta_{\bar{\partial}}$ illuminates the purely almost-complex properties.

2. Preliminaries

In this section we recall some basic facts about almost-complex and almost-Hermitian manifolds and fix some notations. Let $X$ be a smooth manifold of dimension $2n$ and let $J$ be an almost-complex structure on $X$, namely a $(1,1)$-tensor on $X$ such that $J^2 = -\text{id}$. Then $J$ induces on the space of forms $A^\bullet(X)$ a natural bigrading, namely

$$A^\bullet(X) = \bigoplus_{p+q=\bullet} A^{p,q}(X).$$

Accordingly, the exterior derivative $d$ splits into four operators:

$$d : A^{p,q}(X) \to A^{p+2,q-1}(X) \oplus A^{p+1,q}(X) \oplus A^{p,q+1}(X) \oplus A^{p-1,q+2}(X),$$

$$d = \mu + \partial + \bar{\partial} + \bar{\mu},$$

where $\mu$ and $\bar{\mu}$ are differential operators that are linear over functions. In particular, they are related to the Nijenhuis tensor $N_J$ by

$$(\mu \alpha + \bar{\mu} \alpha)(u,v) = \frac{1}{4} \alpha (N_J(u,v)), $$

where $\alpha \in A^1(X)$. Hence, $J$ is integrable, that is, $J$ induces a complex structure on $X$ if and only if $\mu = \bar{\mu} = 0.$
In general, since $d^2 = 0$, one has
\[
\begin{cases}
\mu^2 = 0 \\
\mu \partial + \partial \mu = 0 \\
\partial^2 + \mu \bar{\partial} + \bar{\partial} \mu = 0 \\
\partial \bar{\partial} + \bar{\partial} \partial + \mu \bar{\mu} + \bar{\mu} \mu = 0 \\
\bar{\partial}^2 + \bar{\mu} \partial + \partial \bar{\mu} = 0 \\
\bar{\mu} \partial + \partial \bar{\mu} = 0 \\
\bar{\mu}^2 = 0.
\end{cases}
\]

In particular, $\bar{\partial}^2 \neq 0$, and so the Dolbeault cohomology of $X$
\[
H^*_{\bar{\partial}}(X) := \frac{\ker \bar{\partial}}{\text{Im} \bar{\partial}}
\]
is well defined if and only if $J$ is integrable. The same holds for the operator $\partial$.

If $g$ is an Hermitian metric on $(X,J)$ with fundamental form $\omega$ and $*$ is the associated $\mathbb{C}$-linear Hodge-$*$ operator, one can consider the adjoint operators
\[
d^* = - * d^*, \quad \mu^* = - * \mu^*, \quad \partial^* = - * \bar{\partial}^*, \quad \bar{\partial}^* = - * \partial^*, \quad \bar{\mu}^* = - * \mu^*,
\]
and, for $D \in \{d, \partial, \bar{\partial}, \mu, \bar{\mu}\}$, one defines the associated Laplacians
\[
\Delta_D := DD^* + D^* D,
\]
and we will denote the kernel by
\[
\mathcal{H}^{p,q}_{D}(X) := \ker \Delta_{D|\Lambda^{p,q}(X)}.
\]
These spaces will be called the spaces of $D$-harmonic forms. The operators $\Delta_{\bar{\partial}}$ and $\Delta_{\partial}$ are second-order, elliptic, differential operators; in particular, if $X$ is compact, the associated spaces of harmonic forms are finite-dimensional, and their dimensions will be denoted by $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{\partial}$.

If $X$ is compact, then we easily deduce the following relations for a $(p,q)$-form $\alpha$:
\[
\begin{cases}
\Delta_{\partial} \alpha = 0 \iff \partial \alpha = 0, \quad \bar{\partial}^* \alpha = 0, \\
\Delta_{\bar{\partial}} \alpha = 0 \iff \bar{\partial} \alpha = 0, \quad \partial^* \alpha = 0,
\end{cases}
\]
which characterize the spaces of harmonic forms.

3. Primitive decompositions of Dolbeault harmonic forms

Let $(X,J,g,\omega)$ be a $2n$-dimensional almost-Hermitian manifold. We denote with
\[
L : \Lambda^k X \to \Lambda^{k+2} X, \quad \alpha \mapsto \omega \wedge \alpha
\]
the Lefschetz operator and with
\[
\Lambda : \Lambda^k X \to \Lambda^{k-2} X, \quad \Lambda = - * L^*
\]
its dual. A $k$-form $\alpha_k$ on $X$, for $k \leq n$, is said to be primitive if $\Lambda \alpha_k = 0$, or equivalently, $L_{n-k+1} \alpha_k = 0$. Then, the following vector bundle decomposition holds (see, e.g., [18]):

\[
\Lambda^k X = \bigoplus_{r \geq \max(k-n,0)} L^r (P^{k-2r} X),
\]

where

\[
P^s X := \ker \left( \Lambda : \Lambda^s X \to \Lambda^{s-2} X \right)
\]
is the bundle of $s$-primitive forms. Accordingly, given any $k$-form $\alpha_k$ on $X$, we can write

\[
\alpha_k = \sum_{r \geq \max(k-n,0)} \frac{1}{r!} L^r \beta_{k-2r}, \tag{3.1}
\]

where $\beta_{k-2r} \in \Gamma(P^{k-2r} X)$, that is,

\[
\Lambda \beta_{k-2r} = 0,
\]
or equivalently

\[
L_{n-k+2r+1} \beta_{k-2r} = 0.
\]

Furthermore, the decomposition above is compatible with the bidegree decomposition on the bundle of complex $k$-forms $\Lambda^k C X$ induced by $J$, that is,

\[
P^k C X = \bigoplus_{p+q=k} P^{p,q} X,
\]

where

\[
P^{p,q} X = P^k C X \cap \Lambda^{p,q} X.
\]

For any given $\beta_k \in P^k X$, we have the following formula (cf. [18, p. 23, Théorème 2]):

\[
* L^r \beta_k = (-1)^{k(k+1)/2} \frac{r!}{(n-k-r)!} L^{n-k-r} J \beta_k. \tag{3.2}
\]

In what follows we will write $P^\bullet = P^k X$ and so on.

We recall that by [5, Corollary 5.4] such decompositions in primitive forms pass to the spaces of $d$-harmonic forms whenever there exists an almost-Kähler metric. More precisely, if $(X, J, \omega)$ is a compact $2n$-dimensional almost-Kähler manifold, then, for every $p, q$,

\[
H^{p,q}_{\delta} (X) = \bigoplus_{r \geq \max(k-n,0)} L^r \left( H^{p-r,q+r}_d (X) \cap P^{p-r,q+r} \right).
\]

In fact, this holds also for the spaces of $\bar{\delta}$- and $\delta$-harmonic forms introduced in [14], where $\bar{\delta} := \bar{\partial} + \mu$ and $\delta := \partial + \bar{\mu}$. Indeed, by [14, Proposition 6.2 and Theorem 6.7], one has

\[
H^{p,q}_d (X) = H^{p,q}_{\bar{\delta}} (X) = H^{p,q}_{\delta} (X).
\]

Next, we are going to study such decompositions for $\bar{\partial}$-harmonic forms. First, notice that, since $(p,0)$-forms and $(0,q)$-forms are trivially primitive, we immediately derive the following results.
Proposition 3.1. Let \((X, J, g, \omega)\) be a compact 2n-dimensional almost-Hermitian manifold (with \(n \geq 2\)). Then the following decompositions hold for every \(p, q \leq n\):

\[
\mathcal{H}^{p,0}_{\bar{\partial}} = \mathcal{H}^{p,0}_{\partial} \cap P^{p,0}, \quad \mathcal{H}^{0,q}_{\bar{\partial}} = \mathcal{H}^{0,q}_{\partial} \cap P^{0,q},
\]

By applying to such decompositions the Hodge-* operator and formula (3.2), we obtain the following result.

Proposition 3.2. Let \((X, J, g, \omega)\) be a compact 2n-dimensional almost-Hermitian manifold (with \(n \geq 2\)). Then the following decompositions hold for every \(p, q \leq n\):

\[
\mathcal{H}^{n,n-p}_{\bar{\partial}} = L^{n-p}(\mathcal{H}^{p,0}_{\partial} \cap P^{p,0}), \quad \mathcal{H}^{n,q-n}_{\bar{\partial}} = L^{n-q}(\mathcal{H}^{0,q}_{\partial} \cap P^{0,q}).
\]

As a consequence, we derive the following corollary.

Corollary 3.3. Let \((X, J, g, \omega)\) be a compact 2n-dimensional almost-Hermitian manifold (with \(n \geq 2\)). Then,

\[
\mathcal{H}^{n,0}_{\bar{\partial}} = \mathcal{H}^{n,0}_{\partial} \quad \text{and} \quad \mathcal{H}^{0,n}_{\bar{\partial}} = \mathcal{H}^{0,n}_{\partial}.
\]

Proof. This follows taking \(p = n\) and \(q = n\) in Proposition 3.2. Otherwise, it can be proved directly. Indeed, let \(\alpha\) be an \((n,0)\)-form (the case \((0,n)\) is similar); then \(\alpha\) is primitive, and by Formula (3.2), \(*\alpha = c_n \alpha\), with \(c_n \neq 0\) a constant depending only on the dimension of \(X\). Therefore, for bidegree reasons,

\[
\alpha \in \mathcal{H}^{n,0}_{\bar{\partial}} \iff \bar{\partial} \alpha = 0 \iff \bar{\partial} * \alpha = 0 \iff \alpha \in \mathcal{H}^{n,0}_{\partial}. \quad \square
\]

We show now that primitive decompositions hold also in other suitable degrees as soon as we assume the existence of an almost-Kähler metric.

Theorem 3.4. Let \((X, J, g, \omega)\) be a compact 2n-dimensional almost-Kähler manifold (with \(n \geq 2\)). Then the following decomposition holds:

\[
\mathcal{H}^{1,1}_{\bar{\partial}} = \mathbb{C} \cdot \omega \oplus (\mathcal{H}^{1,1}_{\partial} \cap P^{1,1}).
\]

Proof. Let \(\alpha_{1,1} \in A^{1,1}(X)\). Then the primitive decomposition (3.1) reads as

\[
\alpha_{1,1} = \beta_{1,1} + \beta \omega,
\]

where

\[
\beta_{1,1} \in A^{1,1}(X), \quad \beta_{1,1} \wedge \omega^{n-1} = 0, \quad \beta \in C^\infty(X; \mathbb{C}).
\]

The form \(\alpha_{1,1}\) belongs to \(\mathcal{H}^{1,1}_{\bar{\partial}}\) if and only if \(\alpha_{1,1}\) satisfies the equations

\[
\bar{\partial} \alpha_{1,1} = 0, \quad \partial * \alpha_{1,1} = 0. \tag{3.4}
\]

By (3.2) we compute

\[
*\alpha_{1,1} = -\frac{1}{(n-2)!} \beta_{1,1} \wedge \omega^{n-2} + \beta \frac{1}{(n-1)!} \omega^{n-1}. \tag{3.5}
\]
Therefore, by (3.3), (3.5), taking into account that \( g \) is almost-Kähler, equations (3.4) are equivalent to
\[
\begin{cases}
\bar{\partial} \beta_1,1 \cdot 1 + \bar{\partial} \beta \wedge \omega = 0 \\
- \frac{1}{(n-2)!} \partial \beta_1,1 \wedge \omega^{n-2} + \partial \beta \wedge \frac{1}{(n-1)!} \omega^{n-1} = 0.
\end{cases}
\] (3.6)

After multiplying the first equation by \( \omega^{n-2} \) and the second by \( (n-2)! \), we obtain
\[
\begin{cases}
\bar{\partial} \beta_1,1 \wedge \omega^{n-2} + \bar{\partial} \beta \wedge \omega^{n-1} = 0 \\
- \partial \beta_1,1 \wedge \omega^{n-2} + \frac{1}{n-1} \partial \beta \wedge \omega^{n-1} = 0,
\end{cases}
\]
and taking the sum of the last two equations we obtain
\[
(\bar{\partial} \beta_1,1 - \partial \beta_1,1) \wedge \omega^{n-2} + \left( \bar{\partial} + \frac{1}{n-1} \partial \beta \right) \wedge \omega^{n-1} = 0.
\]

By definition, we have
\[d^c = i(\bar{\partial} - \partial + \mu - \bar{\mu}),\]
where \(|\mu| = (2, -1), |\bar{\mu}| = (-1, 2)\). Consequently, the last equation can be written as
\[
\left( \bar{\partial} + \frac{1}{n-1} \partial \beta \right) \wedge \omega^{n-1} = i d^c \beta_1,1 \wedge \omega^{n-2}.
\]
Applying \(-id^c\) to both sides of the above equation, we obtain
\[
\left[ (\bar{\partial} - \partial + \mu - \bar{\mu}) \left( \bar{\partial} + \frac{1}{n-1} \partial \beta \right) \right] \wedge \omega^{n-1} = 0,
\]
which yields
\[
\left( \frac{1}{n-1} + 1 \right) \bar{\partial} \partial \beta \wedge \omega^{n-1} = 0,
\]
since \(\bar{\partial} \partial + \partial \bar{\partial} = 0\) on functions and the other contributions vanish by bidegree reasons when we take the wedge product with \(\omega^{n-1}\). Therefore,
\[
\bar{\partial} \partial (\beta \cdot \omega^{n-1}) = 0,
\]
from which we derive that \(\beta \equiv \beta_0 \in \mathbb{C}\) is constant (see, for instance [8], [1, Theorem 10] or [16, Proposition 3.4] for the 4-dimensional case). Hence
\[
\alpha_1,1 = \beta_1,1 + \beta_0 \omega,
\]
so from (3.6) or from
\[
\bar{\partial} \beta_1,1 = \bar{\partial} \alpha_1,1 - \bar{\partial} (\beta_0 \omega) = 0 \\
\partial * \beta_1,1 = \partial * \alpha_1,1 - \partial * (\beta_0 \omega) = 0,
\]
we have that \(\beta \in \mathcal{H}_{\bar{\partial}}^{1,1}\) and \(\beta_1,1\) is primitive. This proves that
\[
\mathcal{H}_{\bar{\partial}}^{1,1} \subset \mathbb{C} \cdot \omega + \left( \mathcal{H}_{\partial}^{1,1} \cap P^{1,1} \right).
\]
Conversely, if \(\alpha_1,1 = \beta_0 \omega + \beta_1,1\), with \(\beta_0 \in \mathbb{C}\) and \(\beta_1,1 \in \mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1}\), we easily conclude that \(\partial * \alpha_1,1 = 0\) and \(\bar{\partial} \alpha_1,1 = 0\). The decomposition is thus proved. \(\square\)
As a consequence we obtain the following primitive decompositions.

**Corollary 3.5.** Let \((X, J, g, \omega)\) be a compact \(2n\)-dimensional almost-Kähler manifold (with \(n \geq 2\)). Then the following decompositions hold:

(i) \(\mathcal{H}^{1,1}_\bar{\partial} = \mathbb{C} \cdot \omega \oplus (\mathcal{H}^{1,1}_\bar{\partial} \cap P^{1,1})\),

(ii) \(\mathcal{H}^{n-1,n-1}_\bar{\partial} = \mathbb{C} \omega^{n-1} \oplus L^{n-2}(\mathcal{H}^{1,1}_\bar{\partial} \cap P^{1,1})\),

(iii) \(\mathcal{H}^{n-1,n-1}_\bar{\partial} = \mathbb{C} \omega^{n-1} \oplus L^{n-2}(\mathcal{H}^{1,1}_\bar{\partial} \cap P^{1,1})\).

**Proof.** The first decomposition follows from the one proved in Theorem 3.4 by conjugation.

To prove the second, observe that the Hodge-* operator induces an isomorphism \(\mathcal{H}^{1,1}_\bar{\partial} \cong \mathcal{H}^{n-1,n-1}_\bar{\partial}\). Via this isomorphism, \(\omega\) corresponds to \(\omega^{n-1}\), while by (3.2) on primitive \((1, 1)\)-forms we have \(* = -\frac{1}{(n-2)!} L^{n-2}\). So we just have to apply * to the decomposition of the previous point.

Finally, the last point follows from the second by conjugation. \(\square\)

Recall that by [5] (see also [14]) on compact almost-Kähler manifolds we have

\[ \Delta_{\bar{\partial}} + \Delta_{\partial} = \Delta_{\bar{\partial}} + \Delta_{\partial}, \]

and so, for every \(p, q\),

\[ \mathcal{H}^{p,q}_\bar{\partial} \cap \mathcal{H}^{p,q}_\partial = \mathcal{H}^{p,q}_\bar{\partial} \cap \mathcal{H}^{p,q}_\partial. \]

In particular, if \(J\) is integrable, namely \((X, J, g, \omega)\) is a compact Kähler manifold, one recovers the well-known identities

\[ \Delta_{\bar{\partial}} = \Delta_{\bar{\partial}} \]

and

\[ \mathcal{H}^{p,q}_\bar{\partial} = \mathcal{H}^{p,q}_\partial. \]

Therefore, one could wonder if this last identity holds true also in the non-integrable case for some special bidegrees. More precisely, we want to show that the two primitive decompositions we obtained in Theorem 3.4 and Corollary 3.5 for \(\mathcal{H}^{1,1}_\bar{\partial}\) and \(\mathcal{H}^{1,1}_\partial\) are not the same.

4. **Relations between \(\Delta_{\bar{\partial}}\) and \(\Delta_{\partial}\)**

Let us start by considering the 4-dimensional case. Let \(\alpha_{1,1}\) be a primitive \((1, 1)\)-form on an almost-Kähler 4-dimensional manifold \(X\). It follows from (3.2) that \(* \alpha_{1,1} = -\alpha_{1,1}\). As a consequence we have the following result.

**Proposition 4.1.** Let \(X\) be an almost-Kähler 4-dimensional manifold. Then, on \((1, 1)\)-forms we have

\[ \Delta_{\bar{\partial}} \alpha_{1,1} = \Delta_{\partial} \alpha_{1,1}, \]

and in particular their kernels coincide:

\[ \mathcal{H}^{1,1}_{\bar{\partial}} = \mathcal{H}^{1,1}_\partial. \]
Notice that this follows also from [5], since on almost-Kähler manifolds we have \( \Delta_\partial + \Delta_\bar{\partial} = \Delta_\bar{\partial} + \Delta_\partial \), and on \((1, 1)\)-forms on 4-dimensional almost-Kähler manifolds, \( \Delta_\mu = \Delta_{\bar{\mu}} = 0 \).

We show now that in higher dimension the equality
\[
\Delta_\bar{\partial}|_{A,1,1} = \Delta_\partial|_{A,1,1}
\]
does not hold in general.

**Example 4.2.** Let \( T^6 = \mathbb{Z}^6 \setminus \mathbb{R}^6 \) be the 6-dimensional torus with coordinates \((x_1, x_2, x_3, y_1, y_2, y_3)\) on \( \mathbb{R}^6 \). Let \( f = f(x_2) \) be a non-constant \( \mathbb{Z} \)-periodic function, and we define the following non-left-invariant almost-complex structure \( J \) on \( T^6 \):
\[
J_\partial x_1 := e^{-f} \partial y_1, \quad J_\partial x_2 := \partial y_2, \quad J_\partial x_3 := \partial y_3.
\]
A global co-frame of \((1, 0)\)-forms is given by
\[
\Phi^1 := dx_1 + ie^f dy_1, \quad \Phi^2 := dx_2 + idy_2, \quad \Phi^3 := dx_3 + idy_3.
\]
The structure equations are
\[
d\Phi^1 = -\frac{1}{4} f'(x_2) \Phi^{12} - \frac{1}{4} f'(x_2) \Phi^{21} - \frac{1}{4} f'(x_2) \Phi^{12} + \frac{1}{4} f'(x_2) \Phi^{12}
\]
and \( d\Phi^2 = d\Phi^3 = 0 \). Then, the \((1, 1)\)-form
\[
\omega := \frac{i}{2} e^{-f} \Phi^{1\bar{1}} + \frac{i}{2} \Phi^{2\bar{2}} + \frac{i}{2} \Phi^{3\bar{3}}
\]
is a compatible symplectic structure, namely \((J, \omega)\) is an almost-Kähler structure on \( T^6 \).

Notice now that by a direct computation
\[
\bar{\mu} \Phi^{13} = \frac{1}{4} f'(x_2) \Phi^{1\bar{2}\bar{3}} \neq 0
\]
and
\[
\mu \Phi^{13} = 0.
\]
Therefore, from [5], we have
\[
(\Delta_\bar{\partial} - \Delta_\partial) \Phi^{13} = -\bar{\mu} \mu \Phi^{13} \neq 0.
\]
The last point follows either by direct computation or by noticing that
\[
\bar{\mu} \mu \Phi^{13} \neq 0 \iff \|\bar{\mu} \mu \Phi^{13}\|^2 \neq 0 \iff \bar{\mu} \mu \Phi^{13} \neq 0.
\]

Another example is provided by the following 8-dimensional nilmanifold with a left-invariant almost-Kähler structure.

**Example 4.3.** We recall the following construction contained in [2]. Set
\[
\mathbb{H}(1, 2) := \left\{ \begin{bmatrix} 1 & 0 & x_1 & z_1 \\ 0 & 1 & x_2 & z_2 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid x_1, x_2, y, z_1, z_2 \in \mathbb{R} \right\}.
\]
Let $\Gamma$ be the subgroup of matrices with integral entries. Let $X := \Gamma \setminus \mathbb{H}(1,2)$ and define

$$M := X \times \mathbb{T}^3.$$ 

Denoting with $u, v, w$ coordinates on $\mathbb{T}^3$ we consider the following left-invariant 1-forms:

$$e^1 := dx_2, \quad e^2 := dx_1, \quad e^3 := dy, \quad e^4 := du,$$

$$e^5 := dz_1 - x_1 dy, \quad e^6 := dz_2 - x_2 dy, \quad e^7 := dv, \quad e^8 := dw,$$

and the structure equations become

$$de^1 = de^2 = de^3 = de^4 = de^7 = de^8 = 0, \quad de^5 = -e^{23}, \quad de^6 = -e^{13}.$$ 

We define the symplectic structure

$$\omega := e^{15} + e^{26} + e^{37} + e^{48},$$ 

and we take the compatible almost-complex structure defined by the following coframe of $(1,0)$-forms:

$$\psi^1 := e^1 + ie^5, \quad \psi^2 := e^2 + ie^6, \quad \psi^3 := e^3 + ie^7, \quad \psi^4 := e^4 + ie^8.$$ 

By direct computation we get

$$d\psi^{14} = -\frac{i}{4}\psi^{234} + \frac{i}{4}\psi^{324} - \frac{i}{4}\psi^{234};$$

hence

$$\mu \psi^{14} = 0, \quad \bar{\mu} \psi^{14} = -\frac{i}{4}\psi^{234}.$$ 

Therefore,

$$(\Delta_{\bar{\partial}} - \Delta_\partial)\psi^{14} = (\Delta_{\bar{\mu}} - \Delta_{\mu})\psi^{14} = \bar{\mu}^* \mu \psi^{14} \neq 0,$$

proving that

$$\Delta_{\bar{\partial}} \neq \Delta_\partial$$

on $(1,1)$-forms. However, one can show that their kernels coincide, namely $\mathcal{H}_{1,1}^{1,1} = \mathcal{H}_{\bar{\partial}}^{1,1}.$

**Remark 4.4.** We want to point out that finding explicit examples of almost-Kähler manifolds with $\Delta_{\bar{\partial}} \neq \Delta_\partial$ seems to be not so obvious. In fact, we couldn’t find any left-invariant example in dimension 6.

Even though $\Delta_{\bar{\partial}}^{1,1,1} \neq \Delta_\partial^{1,1,1}$ in general, we wonder whether their kernels coincide. Before showing that this is not the case we notice that the equality $\mathcal{H}_{\bar{\partial}}^{1,1} = \mathcal{H}_{\partial}^{1,1}$ is equivalent to $\mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1} = \mathcal{H}_{\partial}^{1,1} \cap P^{1,1}.$
Lemma 4.5. Let $(X, J, g, \omega)$ be an almost-Kähler manifold. Then $\mathcal{H}_\partial^{1,1} = \mathcal{H}_\bar{\partial}^{1,1}$ if and only if $\mathcal{H}_\partial^{1,1} \cap P^{1,1} = \mathcal{H}_\bar{\partial}^{1,1} \cap P^{1,1}$.

Proof. We prove only the non-trivial implication. Let $\alpha_{1,1} \in \mathcal{H}_\partial^{1,1}$; then we can decompose it as $\alpha_{1,1} = c\omega + \beta_{1,1}$ with $c \in \mathbb{C}$ and $\beta_{1,1} \in \mathcal{H}_\bar{\partial}^{1,1} \cap P^{1,1}$. Now,

$$\Delta_\partial \alpha_{1,1} = c \cdot \Delta_\partial \omega + \Delta_\partial \beta_{1,1} = 0 + 0 = 0,$$

so $\alpha_{1,1} \in \mathcal{H}_\partial^{1,1}$. The other inclusion is similar. □

We observe the following:

Lemma 4.6. Let $(X^{2n}, J, g, \omega)$ be a $2n$-dimensional almost-Kähler manifold. Let $k := p + q \leq n$ and let $\alpha \in P^{p,q}$. Then,

$$\bar{\partial} \alpha = 0 = \Rightarrow \partial^* \alpha = 0.$$

Similarly,

$$\partial \alpha = 0 = \Rightarrow \bar{\partial}^* \alpha = 0.$$

Proof. By (3.2) we have

$$\ast \alpha = (-1)^{k(k+1)/2} \frac{i^{p-q}}{(n-k)!} \alpha \wedge \omega^{n-k}.$$

Since $\omega$ is closed, this readily implies that $\bar{\partial} \ast \alpha = 0$. The same holds switching $\bar{\partial}$ and $\partial$. □

Lemma 4.7. Let $(X, J, g, \omega)$ be an almost-Kähler manifold. Let $\alpha_{1,1} \in \mathcal{H}_\bar{\partial}^{1,1} \cap P^{1,1}$. Then $d^* \alpha_{1,1} = 0$.

Proof. Since $\ast \alpha_{1,1}$ is an $(n-1, n-1)$-form, by the previous lemma we have

$$d \ast \alpha_{1,1} = (\partial + \bar{\partial}) \ast \alpha_{1,1} = \partial \ast \alpha_{1,1} + \bar{\partial} \ast \alpha_{1,1} = 0.$$ □

Lemma 4.8. Let $(X, J, g, \omega)$ be an almost-Kähler manifold. Let $\alpha_{1,1} \in \mathcal{H}_\bar{\partial}^{1,1} \cap P^{1,1}$. Then $d \alpha_{1,1}, \mu \alpha_{1,1}, \partial \alpha_{1,1}, \bar{\partial} \alpha_{1,1}$ and $\bar{\mu} \alpha_{1,1}$ are primitive.

Proof. From the previous lemma and (3.2) we deduce that

$$0 = d \ast \alpha_{1,1} = -\frac{1}{(n-2)!} d(\alpha \wedge \omega^{n-2}) = -\frac{1}{(n-2)!} d\alpha \wedge \omega^{n-2}.$$ So $d \alpha_{1,1}$ is primitive, and by decomposition in types we deduce that also $\mu \alpha_{1,1}, \partial \alpha_{1,1}, \bar{\partial} \alpha_{1,1}$ and $\bar{\mu} \alpha_{1,1}$ are primitive. □

We finally show that, in general, on compact almost-Kähler manifolds we have

$$\mathcal{H}_\partial^{1,1} \neq \mathcal{H}_\bar{\partial}^{1,1}.$$ By Lemma (4.5) this will be done using primitive forms.
Example 4.9. Using the same notations as in Example 4.2 we consider $T^6 = \mathbb{Z}^6 \setminus \mathbb{R}^6$. Let $g = g(x_3, y_3)$ be a function on $T^6$. We define an almost-complex structure $J$ setting as global co-frame of $(1, 0)$-forms

$$\varphi^1 := e^g dx_1 + ie^{-g} dy_1, \quad \varphi^2 := dx_2 + idy_2, \quad \varphi^3 := dx_3 + idy_3.$$ 

The structure equations are

$$d\varphi^1 = V_3(g)\varphi^{31} - \bar{V}_3(g)\varphi^{13},$$

where $\{V_1, V_2, V_3\}$ is the global frame of vector fields dual to $\{\varphi^1, \varphi^2, \varphi^3\}$, and $d\varphi^2 = d\varphi^3 = 0$. Assume finally that $g$ satisfies $V_3(g) \neq 0$.

Then, the $(1, 1)$-form

$$\omega := \frac{i}{2} \varphi^{11} + \frac{i}{2} \varphi^{22} + \frac{i}{2} \varphi^{33}$$

is a compatible symplectic structure, namely $(J, \omega)$ is an almost-Kähler structure on $T^6$.

Notice now that

$$\bar{\partial}\varphi^{12} = V_3(g)\varphi^{312} \neq 0,$$

namely, $\varphi^{12} \notin \mathcal{H}^{1,1}_\bar{\partial}$ but $\varphi^{12} \in \mathcal{H}^{1,1}_\partial$. Indeed, $\partial\varphi^{12} = 0$, and since $\varphi^{12}$ is primitive and $\omega$ is closed,

$$\bar{\partial} \ast \varphi^{12} = \bar{\partial}(-\omega \wedge \varphi^{12}) = -\omega \wedge \bar{\partial}\varphi^{12} = -\omega \wedge (V_3(g)\varphi^{312}) = 0.$$ 

Hence, $\partial^* \varphi^{12} = -\ast \bar{\partial} \ast \varphi^{12} = 0$.

5. Primitive decompositions in dimension 6

Notice that in view of Propositions 3.1, 3.2, Theorem 3.4 and Corollary 3.5 we have a full primitive description of all $\bar{\partial}$-harmonic forms on compact 4-dimensional almost-Kähler manifolds. It is therefore natural to ask what happens for bidegrees different from $(p, 0)$, $(0, q)$, $(n, q)$, $(p, n)$, $(1, 1)$ and $(n-1, n-1)$ in higher dimension. The first interesting dimension to consider is 6, and in this case the only bidegrees left are $(2, 1)$ and $(1, 2)$. Let us focus, for instance, on bidegree $(2, 1)$. The primitive decomposition of forms is

$$A^{2,1}(X) = P^{2,1} \oplus L(A^{1,0}(X)).$$

Passing to $\bar{\partial}$-harmonic forms, it follows that

$$\mathcal{H}^{2,1}_\bar{\partial} \supseteq (\mathcal{H}^{2,1}_\bar{\partial} \cap P^{2,1}) \oplus L(\mathcal{H}^{1,0}_\bar{\partial});$$

indeed, on compact almost-Kähler manifolds, for bidegree reasons and [5] one has

$$\mathcal{H}^{1,0}_\bar{\partial} = \mathcal{H}^{1,0}_\bar{\partial} \cap \mathcal{H}^{1,0}_\partial = \mathcal{H}^{1,0}_\partial \cap \mathcal{H}^{1,0}_\bar{\partial}.$$

Therefore, it is natural to wonder whether such inclusion is indeed an identity. In fact, this is not the case in general, as shown by the following proposition.
Proposition 5.1. There exists a compact almost-Kähler 6-dimensional manifold \((X, J, \omega)\) such that
\[ \mathcal{H}^2,1_\partial \neq (\mathcal{H}^2,1_\partial \cap P^{2,1}) \oplus L(\mathcal{H}^{1,0}_\partial). \]

Proof. We refer the reader to Example 5.3 for the proof. □

First we need the following lemma, which will allow us to work only with left-invariant forms.

Lemma 5.2. Let \(X^6 = \Gamma \backslash G\) be the compact quotient of a 6-dimensional, connected, simply-connected Lie group by a lattice and let \((J, \omega)\) be a left-invariant almost-Kähler structure on \(X\). Let \(\eta \in A^{2,1}(X)\) be a left-invariant \((2,1)\)-form on \(X\) with primitive decomposition \(\eta = \alpha + L\beta\).

Then, \(\alpha\) and \(\beta\) are left-invariant.

Proof. Let \(\eta \in A^{2,1}(X)\) be a left-invariant \((2,1)\)-form on \(X\). Its primitive decomposition is
\[ \eta = \alpha + L\beta, \]
with \(\alpha \in A^{2,1}(X)\) primitive, i.e., \(L\alpha = 0\) and \(\beta \in A^{1,0}(X)\). Notice that \(\beta\) is indeed primitive for bidegree reasons. We apply \(L\) to the decomposition and obtain
\[ L\eta = L^2\beta. \]

Since \(\omega\) is left-invariant, we have that \(L\eta\), and so \(L^2\beta\), are left-invariant. Now, since \(L^2 : \Lambda^1 \rightarrow \Lambda^5\) is an isomorphism at the level of the exterior algebra, it follows that also \(\beta\) is left-invariant. As a consequence, since \(L\beta\) and \(\eta\) are left-invariant, we get that also \(\alpha\) is left-invariant. □

Example 5.3. Let \(X\) be the Iwasawa manifold defined as the quotient \(X := \Gamma \backslash \mathbb{H}_3\), where
\[ \mathbb{H}_3 := \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\} \]
and
\[ \Gamma := \left\{ \begin{bmatrix} 1 & \gamma_1 & \gamma_3 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \mid \gamma_1, \gamma_2, \gamma_3 \in \mathbb{Z}[i] \right\}. \]

Then, setting \(z_j = x_j + iy_j\), there exists a basis of left-invariant 1-forms \(\{e_i\}\) on \(X\) given by
\[ \begin{align*}
    e^1 &= dx_1 \\
    e^2 &= dy_1 \\
    e^3 &= dx_2 \\
    e^4 &= dy_2 \\
    e^5 &= dx_3 - x_1 dx_2 + y_1 dy_2 \\
    e^6 &= dy_3 - x_1 dy_2 - y_1 dx_2.
\end{align*} \]
The following structure equations hold:

\[
\begin{cases}
    de^1 = 0 \\
    de^2 = 0 \\
    de^3 = 0 \\
    de^4 = 0 \\
    de^5 = -e^{13} + e^{24} \\
    de^6 = -e^{14} - e^{23}.
\end{cases}
\]

Let us consider the non integrable left-invariant almost-complex structure \( J \) given by

\[
\phi^1 = e^1 + ie^6, \quad \phi^2 = e^2 + ie^5, \quad \phi^3 = e^3 + ie^4
\]

being a global coframe of \((1,0)\)-forms. By a direct computation the structure equations become (see also [15])

\[
\begin{align*}
4d\phi^1 &= -\phi^{13} - i\phi^{23} + \phi^{31} - i\phi^{23} + i\phi^{31} - i\phi^{23}, \\
4d\phi^2 &= -i\phi^{13} + \phi^{23} - i\phi^{13} + i\phi^{31} - \phi^{23} - \phi^{32} - i\phi^{13} - \phi^{23}, \\
d\phi^3 &= 0.
\end{align*}
\]

Endow \((X,J)\) with the left-invariant almost-Kähler structure given by

\[
\omega = 2(e^{16} + e^{25} + e^{34}) = i(\phi^{11} + \phi^{22} + \phi^{33}).
\]

We want to find an element \( \eta \in A^{2,1}(X) \) which is contained in \( \mathcal{H}^{2,1}_\bar{\partial} \) but is not contained in

\[
(\mathcal{H}^{2,1}_\bar{\partial} \cap P^{2,1}) \oplus L(\mathcal{H}^{1,0}_\bar{\partial}).
\]

Thanks to Lemma 5.2 it is sufficient to work with left-invariant forms. Indeed if we find \( \eta \in \mathcal{H}^{2,1}_\bar{\partial} \) left-invariant that cannot be decomposed as \( \eta = \alpha + L\beta \), with \( \alpha \in \mathcal{H}^{2,1}_\bar{\partial} \cap P^{2,1} \) and \( \beta \in \mathcal{H}^{1,0}_\bar{\partial} \), both left-invariant forms, then \( \eta \notin (\mathcal{H}^{2,1}_\bar{\partial} \cap P^{2,1}) \oplus L(\mathcal{H}^{1,0}_\bar{\partial}) \).

A long but direct and straightforward computation shows that the space of left-invariant \( \bar{\partial}\)-harmonic \((2,1)\)-forms is

\[
\mathbb{C}(\phi^{13\overline{1}} + \phi^{23\overline{2}}, \phi^{13\overline{2}} + \phi^{23\overline{1}} - 2i\phi^{23\overline{2}}, \phi^{13\overline{3}} + \phi^{23\overline{3}}),
\]

while it is immediate to see that the space of left-invariant forms which are contained in \( L(\mathcal{H}^{1,0}_\bar{\partial}) \) is

\[
\mathbb{C}(\phi^{13\overline{1}} + \phi^{23\overline{2}}).
\]

Since, for instance, \( L(\phi^{13\overline{2}} + \phi^{23\overline{1}} - 2i\phi^{23\overline{2}}) = -2iL(\phi^{23\overline{2}}) \neq 0 \), it means that \( \phi^{13\overline{2}} + \phi^{23\overline{1}} - 2i\phi^{23\overline{2}} \) is not primitive. Therefore, \( \phi^{13\overline{2}} + \phi^{23\overline{1}} - 2i\phi^{23\overline{2}} \) is a left-invariant, \( \bar{\partial}\)-harmonic \((2,1)\)-form, but it is not contained in

\[
(\mathcal{H}^{2,1}_\bar{\partial} \cap P^{2,1}) \oplus L(\mathcal{H}^{1,0}_\bar{\partial}).
\]
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