Determining the number of factors in a large-dimensional generalised factor model

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Abstract: This paper proposes new estimators of the number of factors for a generalised factor model with more relaxed assumptions than the strict factor model. Under the framework of large cross-sections $N$ and large time dimensions $T$, we first derive the bias-corrected estimator $\hat{\sigma}_*^2$ of the noise variance in a generalised factor model by random matrix theory. Then we construct three information criteria based on $\hat{\sigma}_*^2$, further propose the consistent estimators of the number of factors. Finally, simulations and real data analysis illustrate that our proposed estimations are more accurate and avoid the overestimation in some existing works.

Keywords: Factor model; Noise estimation; Number of factors; Information criteria

Classification: C13, C38, C43

1 Introduction

With the rapid development of information technology, the factor models with large cross-sections and time-series dimensions have emerged more intensively in the fields of economy
and finance. For example, it can be used to determine the tick-by-tick transaction prices of a large number of assets, and to depict the leading eigenvalues of the coincident indexes in macroeconomics, and so on. The factor model reveals the intricate relationship of the mass of variables through several common factors and simplifies the model structure.

A critical problem of factor models is to determine the number of static factors or dynamic factors. “Dynamic” refers to whether the common factor $F_t$ itself is modeled as a dynamic process. If it is assumed that $F_t$ does not have an auto-correlation structure, i.e. $E(F_tF_s) = 0, \forall t \neq s$, the dynamic factor is transformed into a static factor. There is a lot of literature on this issue for both static and dynamic factor models. On one hand, [Bai and Ng (2002), Onatski (2006, 2010), Alessi et al. (2010), Ahn and Horenstein (2013), Caner and Han (2014)] studied on the estimation of the number of static factors. On the other hand, [Forni et al. (2000), Hallin and Liska (2007), Amengual and Watson (2007), Bai and Ng (2007), Onatski (2009), etc. investigated on the determination of the number of dynamic factors. Moreover, the factor models are closely related to principal component analysis (PCA) models and spiked models in random matrix theory. Many related works to determine the number of factors/principal components/spikes are developed, such as [Kritchman and Nadler (2008), Ulfarsson and Solo (2008), Johnstone and Lu (2009), Passemier and Yao (2012), Passemier et al. (2017), etc.

In this paper, we focus on the static factor model and propose new estimators of the number of factors by random matrix theory, as both the cross-section units $N$ and time series observations $T$ go to infinity. Within this context, a pioneering work is developed by [Bai and Ng (2002)], which provided some information criteria for estimating the number of factors and established the consistency of the estimators of the number of factors as
Determine the number of factors $N, T \to \infty$ simultaneously. Following their work, Alessi et al. (2010) improved their criteria by introducing a tuning multiplicative constant in the penalty. More recently, Passemier et al. (2017) modified the information criteria to determine the number of factors for the strict factor model.

However, these existing works are constrained by different reasons. Practical analysis shows that the information criteria developed by Bai and Ng (2002) often led to non-robust estimations, i.e. the number of factors may be overestimated (see e.g. the application on U.S. macroeconomic data in Forni et al. (2009)). Alessi et al. (2010) considered a factor model with the idiosyncratic components only being mildly cross-correlated. Passemier et al. (2017) required the idiosyncratic components $\{e_t\}_{1 \leq t \leq T}$ to be independent and the population covariance of the observations to be a finite-rank perturbation matrix on the identity matrix.

The main contributions of our work are reflected in the following points. First, we relax the independent distributed assumptions of $\{e_t\}_{1 \leq t \leq T}$ in Passemier et al. (2017), and generalise the strict factor model to a more general form. Thus, for our target model, the population covariance matrix of the observations can be regarded as a more generalised spiked population covariance matrix as mentioned in Jiang and Bai (2021a). Second, we establish the new information criteria by random matrix theory, and propose more accurate estimators of the number of factors. Compared with the existing works, our proposed estimators provide the smaller standard errors as illustrated in the simulations. Moreover, we also prove the consistency of our estimators as $N$ and $T$ approach infinity. Finally, although our method is constructed under the framework of large $N$ and $T$, the estimations of the number of factors are still robust even if both $N$ and $T$ are small. As shown in simulation study, when
N = 10 and T = 50, our estimations are much closer to the true number of factors, while the estimations in [Bai and Ng (2002)] fail for finite samples.

The arrangement of this article is as follows. First, we generalise the strict factor model and introduce the bias-corrected estimator of the noise variance in Section 2. Then we construct three new information criteria based on the above bias-corrected noise estimator, and give the new estimators of the number of factors for a generalised factor model in Section 3. As a by-product, we also prove the consistency of our proposed estimators as \( N, T \to \infty \) simultaneously. In Section 4, the Monte Carlo simulations are conducted to evaluate the performance of the proposed estimators of the number of factors. In Section 5, we apply our proposed methods to some real data sets to illustrate their feasibility in practice.

2 Estimation on the variance of the noise in a generalised factor model

2.1 A bias-corrected estimation of the noise variance

We consider the generalised factor model with the following form

\[
X_t = \Lambda F_t + e_t + \mu,
\]

(1)

where \( X_t = (X_{1t}, X_{2t}, \ldots, X_{Nt})' \) is an \( N \)-dimensional cross-section vector at time \( t \), \( \Lambda \) is an \( N \times M \) matrix of factor loading, \( F_t \) is an \( M \)-dimensional vector of common factors, \( \mu \) represents the general mean and \( e_t \) is an idiosyncratic error vector. Compared with the strict
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factor model, which assumes that the covariance matrix of \( e_t \) in the model \(^1\) is \( \sigma^2 I \), we extend this assumption to a general case of \( \sigma^2 V \), where the matrix \( V \) is a general Hermitian matrix. The matrix \( V \) satisfies the following points: First, the eigenvalues of \( V \) are scattered into spaces of several bulks of the general population eigenvalues. Second, the independent assumption of \( \{e_t\}_{1 \leq t \leq T} \) can be removed. Thus the model in \(^1\) is so-called a generalised factor model.

To develop meaningful asymptotic theory in the large-dimensional setting, we assume that both \( N \) and \( T \) go to infinity proportionally, i.e. \( N/T = cT \to c > 0, \) as \( T \to \infty \). Therefore, the population covariance matrix of \( \{X_t\}_{1 \leq t \leq T} \) is

\[
\Sigma = \Lambda \Lambda' + \sigma^2 V. \tag{2}
\]

To ensure the identification of the model, we impose some assumptions on the model parameters, as mentioned in Anderson (2003):

**Assumption 1.** \( \mathbb{E}(F_t) = 0 \) and \( \mathbb{E}(F_tF_t') = I; \)

**Assumption 2.** \( \Gamma = \Lambda' \Lambda \) is diagonal matrix of \( M \) distinct diagonal eigenvalues.

Then the population covariance matrix \( \Sigma \) in \(^2\) is exactly the generalised spiked population covariance proposed in Jiang and Bai (2021a), which has the spectrum form as

\[
\text{spec}(\Sigma) = \sigma^2 (\alpha_1, \cdots, \alpha_1, \cdots, \alpha_K, \cdots, \alpha_K, \cdots, r_1, \cdots, r_1, \cdots, r_s, \cdots, r_s), \tag{3}
\]

where \( \alpha_1, \cdots, \alpha_K \) are spikes with multiplicity \( n_k, k = 1, \ldots, K \), respectively, satisfying \( n_1 + \cdots + n_K = M \) with the fixed integer \( M \). The rest \( r_1, \cdots, r_s \) are non-spiked eigenvalues, where
s is a fixed small number. Moreover, we assume that the empirical spectral distribution (ESD) of $\Sigma$ converges weakly to a nonrandom probability distribution $H$ on the real line as $N \to \infty$, which follows a probability distribution and takes the value $r_i \sigma^2$ in probability $\omega_i, i = 1, \cdots, s$, and $\omega_1 + \cdots + \omega_s = 1$.

The spiked model describes a phenomenon of a few perturbations to a positively definite matrix, see the references such as Johnstone (2001) [1], Bai and Yao (2008, 2012), Jiang and Bai (2021a), etc. By the close relationship between the factor model and the spiked model, to determine the number the factors is equivalent to find the number of the spikes in the spiked covariance matrix $\Sigma$. Thus we focus on the study of the spiked eigenvalues of the matrix $\Sigma$.

Following the works in Passemier et al. (2017), it is necessary to get an accurate estimation of $\sigma^2$ before estimating the number of factors (or spikes). Then, we refer the work of Jiang (2022), which provided a bias-corrected estimation $\hat{\sigma}^2$ based on random matrix theory.

Before introducing it, we first provide some preliminary knowledge. Decompose the population covariance matrix as $\Sigma = B_N B_N^*$, where $^*$ denotes conjugate transposition. Let $\xi_t = B_N^{-1} (X_t - \mu)$, then $B_N \xi = B_N (\xi_1, \xi_2, \cdots, \xi_T)$ can be seen as random samples from the population covariance matrix $\Sigma$. And the corresponding sample covariance matrix of $B_N \xi$ is

$$S = B_N \left( \frac{1}{T} \xi \xi^* \right) B_N^*,$$

which is the generalised spiked sample covariance matrix. It is should be noted that if the mean parameter $\mu$ is unknown, the sample covariance matrix needs to be replaced by an unbiased form, and the corresponding ratio $c_T = N/T$ should be replaced by $N/(T - 1)$.
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We denote the set of ranks of $\sigma^2 \alpha_k$ with multiplicity $n_k$ among all the eigenvalues of $\Sigma$ as $J_k = \{j_k + 1, \cdots, j_k + n_k\}$, and represent the eigenvalues of $S$ sorted in descending order as $l_1 \geq l_2 \geq \cdots \geq l_N$. According to the classical statistical theory, the maximum likelihood estimator of $\sigma^2$ can be obtained as

$$\hat{\sigma}^2_{MLE} = \frac{1}{(N - M)(\omega_1 r_1 + \cdots + \omega_s r_s)} \left( \sum_{j=1}^{N} l_j - \sum_{j \in J_k, k=1}^{K} l_j \right), \quad (5)$$

which can be viewed as an appropriate estimation of the noise variance $\sigma^2$. However, it is well known that the sample eigenvalues do not converge to the population ones when the cross-section dimension $N$ is large compared to the time dimension $T$. Therefore, the estimator (5) will underestimate the true noise variance $\sigma^2$.

To this end, Jiang (2022) established the central limit theorem of $\hat{\sigma}^2_{MLE}$ in the large-dimensional setting, and gave the corresponding bias-corrected estimation. To refer this work, we define $\beta = E |\xi_{11}|^4 - 4$ with $q = 1$ for real case and 0 for complex, where $\xi_{11}$ is the first element of $\xi_1 = (\xi_{11}, \xi_{12}, \cdots, \xi_{1T})$. We denote $F_{c,H}$ as the LSD of the sample matrix $S$, and further $m(z) \equiv m_{F_{c,H}}(z)$ as the Stieltjes Transform of $F_{c,H} \equiv (1 - c)I_{[0, \infty)} + cF_{c,H}$. Then the bias-corrected estimator of the noise variance $\sigma^2$ is given in the following proposition.

**Proposition 1.** For the factor model (1) with the spectrum in (3), the bias-corrected estimator $\hat{\sigma}^2_*$ is given as

$$\hat{\sigma}^2_* = \hat{\sigma}^2_{MLE} + \frac{b(\alpha_k, \hat{\sigma}^2_{MLE}) - \mu_x}{(N - M) \sum_{i=1}^{K} \omega_i r_i}, \quad (6)$$

where
\[ b(\alpha_k, \sigma^2) = \sum_{k=1}^{K} \sum_{i=1}^{s} \frac{n_k c\alpha_k \sigma^2 \rho_i}{\alpha_k - r_i} \]

and

\[ \mu_x = - \frac{q}{2\pi i} \oint \frac{cm^2(z) [cm(z) \int t\{1 + tm(z)\}^{-1}dH(t) - 1] \int t^2\{1 + tm(z)\}^{-3}dH(t)}{[1 - c \int m^2(z)t^2\{1 + tm(z)\}^{-2}dH(t)]^2} dz 
\]

\[ - \frac{\beta c}{2\pi i} \oint \frac{m^2(z) \left[-1 + cm(z) \int t\{1 + tm(z)\}^{-1}dH(t)\right]}{1 - c \int m^2(z)t^2\{1 + tm(z)\}^{-2}dH(t)} dz \]

2.2 Monte Carlo experiments

Since Jiang (2022) only performed the simulations for the case of equal non-spikes, we design the following simulation to verify the feasibility of Proposition 1 in our model for more complex cases. We first set up the following models:

**Model 1.** Assuming that \( \Sigma = \sigma^2 D \), where \( \sigma^2 = 4 \), \( D = \text{diag}(25, 16, 16, 9, 2, \cdots, 2, 1, \cdots, 1) \) is an \( N \times N \) matrix with spikes \( (25, 16, 16, 9) \) of the multiplicity \( (1, 2, 1) \) and non-spikes 2 of \( (N - 4)/2 \) time, non-spikes 1 of \( N/2 \) time.

**Model 2.** Assuming that \( \Sigma = \sigma^2 UDU^* \), where \( D \) is defined in Model 1, \( U \) is composed of eigenvectors of an \( N \times N \) matrix \( HH' \) with the entries of \( H \) being independently sampled from standard Gaussian population.

Moreover, for each model, the Gaussian and Gamma populations are studied to show the conclusion is extensively utilisable without the limitations of population.

**Gaussian Assumption.** \( \{\xi_{it}\} \) are i.i.d. samples from standard Gaussian population;

**Gamma Assumption.** \( \{\xi_{it}\} \) are i.i.d. samples from \( \{\text{Gamma}(2, 1) - 2\}/\sqrt{2} \).
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Next we will compare $\hat{\sigma}^2$ with $\hat{\sigma}^2_{MLE}$ and several other existing noise variance estimators. The definitions of these estimators are given below.

(a) The estimator $\hat{\sigma}^2_P$ in [Passenier et al. (2017)] is also a bias correction of the maximum likelihood estimation of the noise, but for the case where the non-spikes are all 1 and \( \{e_t\}_{1 \leq t \leq T} \) are independent, which is defined as

\[
\hat{\sigma}^2_P = \hat{\sigma}^2 + \frac{c\hat{\sigma}^2}{N - M} \left( M + \sum_{k=1}^{K} \frac{n_k}{\alpha_k - 1} \right),
\]

where $\hat{\sigma}^2$ is the maximum likelihood estimation of the noise given in their work.

(b) The estimator $\hat{\sigma}^2_{KN}$ in [Kritchman and Nadler (2008)] is described as the solution of the following system of nonlinear equations with \( m + 1 \) unknowns,

\[
\hat{\sigma}^2_{KN} - \frac{1}{N - m} \left\{ \sum_{j=m+1}^{N} l_j + \sum_{j=1}^{m} (l_j - \hat{\rho}_j) \right\} = 0
\]

and

\[
\hat{\rho}_j^2 - \hat{\rho}_j \left( l_j + \hat{\sigma}^2_{KN} - \hat{\sigma}^2_{KN} \frac{N - m}{T} \right) + l_j \hat{\sigma}^2_{KN} = 0, \quad j = 1, \cdots, m
\]

(c) The estimator $\hat{\sigma}^2_{US}$ in [Ulfarsson and Solo (2008)] is defined as the ratio of the median of the non-spike sample eigenvalues to the the median of the Marchenko-Pastur distribution $F_{\alpha,1}$,

\[
\hat{\sigma}^2_{US} = \frac{\text{median} (l_{M+1}, \cdots, l_N)}{m_{N/T,1}},
\]

where $m_{\alpha,1}$ is the median of $F_{\alpha,1}$. 
(d) The estimator $\tilde{\sigma}_{\text{median}}^2$ in Johnstone and Lu (2009) is defined as the median of the variances across all dimensions of the $T$ samples,

$$\tilde{\sigma}_{\text{median}}^2 = \text{median} \left( \frac{1}{T} \sum_{i=1}^{T} \tilde{X}_{it}^2, \ 1 \leq i \leq N \right),$$

where $\{\tilde{X}_{it}\}$ are the centralised data of the original samples $\{X_{it}\}$.

We respectively simulate the numerical logarithm transformed mean absolute error (MAE) with 1,000 replications as $N$ increases for $c = 0.5$ and $c = 1.5$. When the ratio $c$ is set to be 0.5, the value of $N$ is set to 50, 100, 150, 200, 250, 300, 350, 400, respectively.
And when the ratio is set to 1.5, the value of $N$ is set to 90, 150, 210, 270, 330, 390, 450, 510, respectively.

The results are reported in Figure 1 and 2. Under different models and distributions, the proposed bias-corrected estimator $\hat{\sigma}^2_*$ has the smallest logarithm transformed MAE. As the $N$ increases, the logarithm transformed MAE of $\hat{\sigma}^2_*$ becomes smaller and smaller. However, the estimator $\hat{\sigma}_\text{KN}^2$, $\hat{\sigma}_\text{US}^2$ and $\hat{\sigma}_\text{median}^2$ obviously fail to estimate $\sigma^2$ in our model. In addition, the numerical results of $\hat{\sigma}_\text{P}^2$ and $\hat{\sigma}_\text{KN}^2$ are so close that their corresponding curves are not easy to distinguish on the Figure. To show the results more accurately, we put the original simulation values in the Appendix.
3 Estimation on the number of factors in a generalised large-dimensional factor model

In this section, we will construct the new information criteria based on $\hat{\sigma}^2$, and propose the estimators of the number of factors for our generalised factor model. Recall the work in Bai and Ng (2002), the common factor $F$ and the factor loading $\Lambda$ can be estimated by the asymptotic principal components method in a large panel. The asymptotic principal component method minimises

$$V(m) = \min_{\Lambda^m, F^m} (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T (X_{it} - \Lambda^m_i F^m_t)^2$$

subject to the normalisation $F^m F^m / T = I_m$, where $\Lambda^m$ is the $N \times m$ factor loading matrix, $F^m$ is $T \times m$ factor matrix, respectively. To be specific, under the normalisation of $F^m F^m / T = I_m$, we adopt $\tilde{F}^m = \sqrt{T}(U_1, U_1, \cdots, U_m)$ as the estimated factor matrix minimising $V(m)$, where $U_i$ is the eigenvector corresponding to the $i$th largest eigenvalue of $XX'$. Then, applying the least square method, we can obtain the corresponding factor loading matrix, $\tilde{\Lambda}^m = (\tilde{F}^m \tilde{F}^m)^{-1} \tilde{F}^m X = \tilde{F}^m X / T$.

According to the knowledge of the linear model, the formula (7) is a decreasing function of $m$. With the increasing integer $m$, we will get the smaller squared error loss $V(m)$. But selecting the excessive number $m$ of factors will lose the efficiency of the model, and the simplicity of the model cannot be guaranteed. For this reason, Bai and Ng (2002) developed the penalty function $g(N, T)$ relied on both $N$ and $T$, and avoided over fitting. Let $V(m, \tilde{F}^m) = (NT)^{-1} \min_{\Lambda} \sum_{i=1}^N \sum_{t=1}^T (X_{it} - (\Lambda^m_i \tilde{F}^m_t)^2$, then a loss function
Determine the number of factors $V(m, \tilde{F}^m) + mg(N,T)$ can be used to determine the number of factors. In order to balance the goodness of fit and simplicity of the model, Bai and Ng (2002) generalised the $C_p$ criterion of Mallows (1973) and suggested three $PC_p$ criteria under the framework of large $N$ and $T$ as follows:

$$PC_{pj}(m) = V(m, \tilde{F}^m) + m\hat{\sigma}^2 g_j(N,T), \quad j \in \{1, 2, 3\},$$

where $V(m, \tilde{F}^m) = (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( X_{it} - \tilde{A}_i^m \tilde{F}_t^m \right)^2$, $\hat{\sigma}^2$ is a consistent estimator of $(NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} E(e_{it}^2)$, and $g_j(N,T)$'s are different penalty functions

$$g_1(N,T) = \left(\frac{N+T}{NT}\right) \ln \left(\frac{N+T}{NT}\right),$$

$$g_2(N,T) = \left(\frac{N+T}{NT}\right) \ln C_{NT}^2,$$

$$g_3(N,T) = \ln \left(\frac{C_{NT}^2}{C_{NT}^2}\right)$$

with $C_{NT} = \min\{\sqrt{N}, \sqrt{T}\}$. All of these penalty functions satisfy two conditions: (i) $g(N,T) \to 0$, (ii) $C_{NT}^2 g(N,T) \to \infty$ as $N,T \to \infty$.

The $\hat{\sigma}^2$ in (8) plays a role as an appropriate scaling parameter for the penalty term. Bai and Ng (2002) recommended replacing it with $V(m_0, \tilde{F}^{m_0})$, where $m_0$ is a bounded integer such that $m \leq m_0$. The estimators of the number of factors corresponding to the three information criteria are $\hat{m}_j = \arg\min_{0 \leq m \leq m_0} PC_{pj}(m), j \in \{1, 2, 3\}$.

As mentioned in the Introduction, the method proposed by Bai and Ng (2002) often overestimate the number of factors. To improve this problem, we construct the new information criteria based on the bias-corrected noise estimator $\hat{\sigma}_n^2$ in proposition 1.
\( V \left( m, \tilde{F}^m \right) \) and \( \hat{\sigma}^2 \) are indeed estimations of the noise variance in model (1). Then we substitute \( V \left( m, \tilde{F}^m \right) \) in (8) by \( \hat{\sigma}^2(m) \), where \( \hat{\sigma}^2(m) \) is the function of \( m \) with definition in Proposition 1. Moreover, we substitute \( \hat{\sigma}^2 \) by \( \hat{\sigma}^2^*(m_0) \). Thus, our proposed new information criteria and the estimators of the number are obtained in the following theorem.

**Theorem 1.** For the determination of the number of factors in the generalised factor model (1), we propose three information criteria as follows

\[
PC^*_1(m) = \hat{\sigma}^2(m) + m\hat{\sigma}^2^*(m_0) \left( \frac{N + T}{NT} \right) \ln \left( \frac{N + T}{NT} \right),
\]

\[
PC^*_2(m) = \hat{\sigma}^2(m) + m\hat{\sigma}^2^*(m_0) \left( \frac{N + T}{NT} \right) \ln C_2^2 \left( \frac{N + T}{NT} \right),
\]

\[
PC^*_3(m) = \hat{\sigma}^2(m) + m\hat{\sigma}^2^*(m_0) \left( \ln C_2^2 \left( \frac{N + T}{NT} \right) \right)
\]

and the corresponding estimators of the number of factors are

\[
\hat{m}^*_j = \arg \min_{0 \leq m \leq m_0} PC^*_j(m), \quad j \in \{1, 2, 3\},
\]

Furthermore, we establish the consistency of the corresponding estimators of \( M \) as \( N, T \to \infty \) and give the proof as follows.

**Theorem 2.** Let \( 0 \leq m \leq m_0 \) and \( \hat{m}^*_j = \arg \min_{0 \leq m \leq m_0} PC^*_j(m), \quad j \in \{1, 2, 3\} \). Then we have \( \lim_{N, T \to \infty} \Pr\{ \hat{m}^*_j = M \} = 1 \), where \( M \) is the true number of factors.

**Proof.** We are going to prove that \( \lim_{N, T \to \infty} \Pr\{ PC^*_p(m) < PC^*_p(M) \} = 0 \) for all \( m \neq M \) and \( m \leq m_0 \), where \( PC^*_p \) stands for all \( PC^*_j, \quad j = 1, 2, 3 \), because they have the same limiting
Determine the number of factors properties. Since

\[ PC_p^*(m) - PC_p^*(M) = \hat{\sigma}^2_s(m) - \hat{\sigma}^2_s(M) - (M - m)\hat{\sigma}^2_s(m_0)g(N, T), \]

it is sufficient to prove

\[ P \left[ \hat{\sigma}^2_s(M) - \hat{\sigma}^2_s(m) > (m - M)\hat{\sigma}^2_s(m_0)g(N, T) \right] \to 0 \tag{10} \]

or

\[ P \left[ \hat{\sigma}^2_s(m) - \hat{\sigma}^2_s(M) < (M - m)\hat{\sigma}^2_s(m_0)g(N, T) \right] \to 0, \tag{11} \]

as \( N, T \to \infty \).

Consider first \( m > M \). By expression (6), we have

\[ \hat{\sigma}^2_s(M) - \hat{\sigma}^2_s(m) = \{ \hat{\sigma}_{MLE}^2(M) - \hat{\sigma}_{MLE}^2(m) \} \{ 1 + o_p(1) \} \]

Moreover,

\[ (N - m) \sum_{i=1}^{s} w_i r_i \{ \hat{\sigma}_{MLE}^2(M) - \hat{\sigma}_{MLE}^2(m) \} = (N - M + M - m)\hat{\sigma}_{MLE}^2(M) \sum_{i=1}^{s} w_i r_i - (N - m)\hat{\sigma}_{MLE}^2(m) \sum_{i=1}^{s} w_i r_i \]

\[ = \sum_{j = M+1}^{N} l_j - \sum_{j = m+1}^{N} l_j - (m - M) \sum_{i=1}^{s} w_i r_i \hat{\sigma}_{MLE}^2(M) \]

\[ = \sum_{M < i \leq m} l_i - (m - M)\hat{\sigma}_{MLE}^2(M) \sum_{i=1}^{s} w_i r_i \]

\[ \leq (m - M) \{ l_{M+1} - \hat{\sigma}_{MLE}^2(M) \} \sum_{i=1}^{s} w_i r_i \] \tag{12}
Since both $l_{M+1}$ and $\hat{\sigma}_2^{MLE}(M)$ are bounded positive values, the right-hand side of (12) is bounded. The inequality (10) will hold if the penalty satisfies
\[(N - m)g(N,T) > \frac{l_{M+1} - \hat{\sigma}_2^{MLE}(M) \sum_{i=1}^s w_i r_i}{\hat{\sigma}_2^2(m_0) \sum_{i=1}^s w_i r_i}\]
for large $N$ and $T$. And we have $C_{NT}^2 g(N,T) \to \infty$, the right part of inequality expression (13) is a bounded value, and the conclusion follows.

Next, for $m < M$, we have
$$\hat{\sigma}_2^2(m) - \hat{\sigma}_2^2(M) = [\hat{\sigma}_2^2(m) - \hat{\sigma}_2^{MLE}(m)] + [\hat{\sigma}_2^{MLE}(m) - \hat{\sigma}_2^{MLE}(M)] + [\hat{\sigma}_2^{MLE}(M) - \hat{\sigma}_2^2(M)],$$
(14)
where $\hat{\sigma}_2^{MLE}(k)$ represents the maximum likelihood estimation of $\sigma^2$ when the population covariance matrix has $k$ spikes.

The first and the third terms on the right side of expression (14) are both $O_p(C_{NT}^{-2})$. Next, consider the second term. When the number of real population spikes is $m$ and $M$, there is only a difference of $(M - m)$ spikes between the two populations. We retain Assumption B about the factor loadings in Bai and Ng (2002), that the factor loadings grow to $\infty$ with the dimension $N$. It implies that the second term is a positive bounded value. Since $g(N,T) \to 0$ as $N,T \to \infty$, the inequality (11) holds.

The conclusion follows.
4 Simulation study

To check the improvement performance of our proposed information criteria, Monte Carlo simulations are conducted. We refer the data generating process in Bai and Ng (2002), which is expressed as

\[ X_{it} = \sum_{j=1}^{M} \lambda_{ij} F_{jt} + \sqrt{\theta} e_{it} \]

with the factors \( F_{jt} \) being \( \mathcal{N}(0,1) \) variates and \( \theta = 3 \). Different from simulated design in Bai and Ng (2002) and Passemier et al. (2017), we generalised the settings as \( (\lambda_{1j}, \lambda_{1j}, \cdots, \lambda_{Nj}) \sim \mathcal{N}(0_N, A) \) and \( \{e_{it}\} = V^{\frac{1}{2}} \{\xi_{it}\} \), where \( A = \text{diag}(5, 4, 4, 3, 0, \cdots, 0) \), \( V \) is diagonal or off-diagonal matrix listed in Model 3 to 5, and \( \{\xi_{it}\} \) are i.i.d. random variables such that \( E\xi_{11} = 0, E|\xi_{11}|^2 = 1 \).

**Model 3.** Assuming that \( V = D \), where \( D = I_N \) is an \( N \times N \) identity matrix.

**Model 4.** Assuming that \( V = D \), and \( D = \text{diag}(2, 2, \cdots, 2, 1, 1, \cdots, 1) \), where eigenvalues 2 and 1 are half and half.

**Model 5.** Assuming that \( V = UDU^* \), where \( D \) is defined in Model 4, and \( U \) is defined in Model 2.

By the generalised settings, we relax the independent or mild cross-correlated assumptions of the error sequence \( \{e_{it}\}_{1 \leq t \leq T} \) than previous works. Furthermore, we reuse the two population assumptions of \( \{\xi_{it}\} \) in subsection 2.2.

Reported in Tables 1 to 3 are the empirical means of the estimations of the number of factors over 1,000 replications, corresponding to Model 3 to 5 respectively. The standard
Table 1. Comparison between $PC_{pj}$ and $PC_{pj}^*$ for Model 3

| $(N,T)$  | $PC_{pj}$ | $PC_{pj}^*$ | $PC_{pj}$ | $PC_{pj}^*$ | $PC_{pj}^*$ | $PC_{pj}^*$ |
|----------|-----------|-------------|-----------|-------------|-------------|-------------|
| $N = 100, T = 40$ | 4.00 | 4.00 | 4.32(0.48) | 4.00 | 4.00 | 4.00 |
| $N = 100, T = 60$ | 4.00 | 4.00 | 4.16(0.37) | 4.00 | 4.00 | 4.00 |
| $N = 200, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 500, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 1000, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 2000, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 100, T = 100$ | 4.00 | 4.00 | 4.72(0.57) | 4.00 | 4.00 | 4.00 |
| $N = 40, T = 100$ | 4.00 | 4.00 | 4.27(0.46) | 4.00 | 4.00 | 4.00 |
| $N = 60, T = 100$ | 4.00 | 4.00 | 4.16(0.36) | 4.00 | 4.00 | 4.00 |
| $N = 60, T = 200$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 10, T = 50$ | 8.00 | 8.00 | 8.00 | 8.00 | 3.98(0.61) | 4.05(0.60) |
| $N = 10, T = 100$ | 8.00 | 8.00 | 8.00 | 3.92(0.36) | 3.91(0.36) | 3.93(0.33) |
| $N = 20, T = 100$ | 5.50(0.66) | 4.92(0.64) | 6.78(0.66) | 4.00 | 4.00 | 4.00 |
| $N = 100, T = 20$ | 5.53(0.65) | 4.95(0.61) | 6.79(0.65) | 4.32(0.50) | 4.11(0.32) | 5.32(0.77) |

errors are also given in parentheses following the estimations. If the standard error is 0, no further annotations will be made. Refer to Passemier et al. (2017), for these six scenarios, the predetermined maximum number $m_0$ of factors is set to 8.

As shown in the simulated results, our proposed information criteria $PC_{pj}^*$ have an overall better performance than $PC_{pj}$ in Bai and Ng (2002) for different models and populations. When min$(N,T) > 40$, both information criteria $PC_{pj}$ and $PC_{pj}^*$ can obtain satisfactory estimation of the number of factors. But our estimation are more accurate with smaller
Determine the number of factors

Table 2. Comparison between $PC_{pj}$ and $PC_{pj}^*$ for Model 4

| $(N,T)$          | $PC_{p1}$ | $PC_{p1}$ | $PC_{p1}$ | $PC_{p1}^*$ | $PC_{p2}^*$ | $PC_{p3}^*$ |
|------------------|-----------|-----------|-----------|-------------|-------------|-------------|
| $N = 100, T = 40$ | 4.00(0.03)| 4.00      | 4.00      | 4.00        | 4.00        | 4.01(0.09)  |
| $N = 100, T = 60$ | 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 200, T = 60$ | 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 500, T = 60$ | 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 1000, T = 60$| 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 2000, T = 60$| 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 100, T = 100$| 4.00      | 4.00      | 6.08(0.65)| 4.00        | 4.00        | 4.00        |
| $N = 40, T = 100$ | 4.00      | 4.00      | 5.54(0.66)| 4.00        | 4.00        | 4.00        |
| $N = 60, T = 100$ | 4.00      | 4.00      | 5.25(0.62)| 4.00        | 4.00        | 4.00        |
| $N = 60, T = 200$ | 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 10, T = 50$  | 8.00      | 8.00      | 8.00      | 4.067(0.69) | 3.96(0.62)  | 4.26(0.82)  |
| $N = 10, T = 100$ | 8.00      | 8.00      | 8.00      | 3.90(0.42)  | 3.87(0.41)  | 3.94(0.44)  |
| $N = 20, T = 100$ | 6.98(0.63)| 6.41(0.66)| 7.86(0.38)| 4.00        | 4.00(0.04)  | 4.00        |
| $N = 100, T = 20$ | 5.85(0.67)| 5.26(0.62)| 7.08(0.64)| 4.67(0.63)  | 4.31(0.49)  | 5.73(0.80)  |

Model 4 under Gaussian assumption

| $(N,T)$          | $PC_{p1}$ | $PC_{p1}$ | $PC_{p1}$ | $PC_{p1}^*$ | $PC_{p2}^*$ | $PC_{p3}^*$ |
|------------------|-----------|-----------|-----------|-------------|-------------|-------------|
| $N = 100, T = 40$ | 4.02(0.13)| 4.00(0.03)| 5.22(0.64)| 4.00        | 4.00        | 4.08(0.27)  |
| $N = 100, T = 60$ | 4.00      | 4.00      | 5.19(0.61)| 4.00        | 4.00        | 4.02(0.13)  |
| $N = 200, T = 60$ | 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 500, T = 60$ | 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 1000, T = 60$| 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 2000, T = 60$| 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 100, T = 100$| 4.00      | 4.00      | 6.52(0.68)| 4.00        | 4.00        | 4.01(0.09)  |
| $N = 40, T = 100$ | 4.12(0.32)| 4.01(0.11)| 5.91(0.67)| 4.00        | 4.00        | 4.00(0.03)  |
| $N = 60, T = 100$ | 4.00(0.04)| 4.00      | 5.64(0.66)| 4.00        | 4.00        | 4.00(0.04)  |
| $N = 60, T = 200$ | 4.00      | 4.00      | 4.00      | 4.00        | 4.00        | 4.00        |
| $N = 10, T = 50$  | 8.00      | 8.00      | 8.00      | 4.30(0.83)  | 4.18(0.76)  | 4.50(0.92)  |
| $N = 10, T = 100$ | 8.00      | 8.00      | 8.00      | 3.98(0.56)  | 3.94(0.53)  | 4.04(0.62)  |
| $N = 20, T = 100$ | 7.08(0.64)| 6.56(0.69)| 7.91(0.29)| 4.00        | 4.00(0.03)  | 4.01(0.10)  |
| $N = 100, T = 20$ | 6.23(0.69)| 5.68(0.66)| 7.36(0.61)| 5.12(0.75)  | 4.66(0.63)  | 6.19(0.80)  |

standard errors. When $\min(N,T) < 40$, the original information criteria $PC_p$ almost report the predetermined maximum value 8, but the estimation of our method is closer to the true value 4. In addition, when the population assumption is gamma distribution, the efficiency of detecting the true number of factors of $PC_p$ in Bai and Ng (2002) is lower than that of Gaussian distribution, but our new information criteria $PC_p^*$ perform still well for the non-Gaussian assumptions. When the error sequence no longer satisfy the independent assumption, our method also outperforms $PC_p$ in Bai and Ng (2002).
### Table 3. Comparison between $PC_{pj}$ and $PC_{pj}^*$ for Model 5

| $(N, T)$ | $PC_{p1}$ | $PC_{p1}^*$ | $PC_{p2}^{*}$ | $PC_{p1}^{*}$ | $PC_{p2}^{*}$ | $PC_{p3}^{*}$ |
|----------|----------|-------------|--------------|--------------|--------------|--------------|
| $N = 100, T = 40$ | 4.00 | 4.00 | 4.85(0.63) | 4.00 | 4.00 | 4.01(0.11) |
| $N = 100, T = 60$ | 4.00 | 4.00 | 4.78(0.58) | 4.00 | 4.00 | 4.00 |
| $N = 200, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 500, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 1000, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 2000, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 100, T = 100$ | 4.00 | 4.00 | 6.09(0.65) | 4.00 | 4.00 | 4.00 |
| $N = 40, T = 100$ | 4.02(0.13) | 4.00 | 5.52(0.64) | 4.00 | 4.00 | 4.00 |
| $N = 60, T = 100$ | 4.00 | 4.00 | 5.26(0.64) | 4.00 | 4.00 | 4.00 |
| $N = 60, T = 200$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 10, T = 50$ | 8.00 | 8.00 | 8.00 | 4.11(0.73) | 4.00(0.65) | 4.29(0.85) |
| $N = 10, T = 100$ | 8.00 | 8.00 | 8.00 | 3.96(0.40) | 3.90(0.39) | 3.97(0.44) |
| $N = 20, T = 100$ | 6.99(0.63) | 6.39(0.63) | 7.87(0.34) | 4.00(0.03) | 4.00(0.04) | 4.00(0.03) |
| $N = 100, T = 20$ | 5.87(0.66) | 5.31(0.65) | 7.13(0.64) | 4.68(0.64) | 4.29(0.47) | 5.76(0.79) |

Model 5 under Gamma assumption

| $(N, T)$ | $PC_{p1}$ | $PC_{p1}^*$ | $PC_{p2}^{*}$ | $PC_{p1}^{*}$ | $PC_{p2}^{*}$ | $PC_{p3}^{*}$ |
|----------|----------|-------------|--------------|--------------|--------------|--------------|
| $N = 100, T = 40$ | 4.01(0.12) | 4.00 | 5.19(0.65) | 4.00 | 4.00 | 4.06(0.24) |
| $N = 100, T = 60$ | 4.00 | 4.00 | 5.15(0.64) | 4.00 | 4.00 | 4.00(0.08) |
| $N = 200, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 500, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 1000, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 2000, T = 60$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 100, T = 100$ | 4.00 | 4.00 | 6.41(0.67) | 4.00 | 4.00 | 4.00(0.05) |
| $N = 40, T = 100$ | 4.08(0.27) | 4.00(0.06) | 5.86(0.67) | 4.00 | 4.00 | 4.00 |
| $N = 60, T = 100$ | 4.00 | 4.00 | 5.58(0.64) | 4.00 | 4.00 | 4.00 |
| $N = 60, T = 200$ | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 |
| $N = 10, T = 50$ | 8.00 | 8.00 | 8.00 | 4.27(0.79) | 4.11(0.71) | 4.50(0.90) |
| $N = 10, T = 100$ | 8.00 | 8.00 | 8.00 | 3.98(0.50) | 3.95(0.49) | 4.05(0.57) |
| $N = 20, T = 100$ | 7.15(0.62) | 6.62(0.64) | 7.92(0.28) | 4.00 | 4.00 | 4.0090.06 |
| $N = 100, T = 20$ | 6.23(0.68) | 5.67(0.65) | 7.36(0.60) | 5.07(0.71) | 4.63(0.63) | 6.20(0.79) |

5 Real data analysis

The proposed information criteria seem to work rather well in the simulated experiments. We now apply the new procedure of determining the number of the factors to two real data sets. Both data sets are downloaded from [https://www.oecd.org](https://www.oecd.org). The first is the OECD Composite Leading Indicators (CLI) data set, which is constructed by weighting indicator data in various fields of the national economy according to certain standards. It is a leading
Determine the number of factors reflecting a country’s macroeconomic development cycle. And this data set consists of CLI for 32 OECD countries, 6 non-member economies and 8 zone aggregates \((N = 46)\) observed monthly from June 1998 to October 2020 \((T = 269)\). The second data set contains OECD Business Confidence Indicators (BCI) data for 36 OECD countries, 6 non-member economies and 6 zone aggregates \((N = 48)\) observed monthly from November 2003 to May 2021 \((T = 211)\).

In practice analysis, we need to pay attention to the chosen of \(m_0\), which will affect the robustness of estimations. Since the \(N \times N\) matrix \(\Lambda\Lambda'\) has only \(M\) non-zero spiked eigenvalues, but the rest tailed eigenvalues are all zero, so that we only need to select an arbitrary large integer \(m_0\), and then we can obtain the robust estimations. But in practical applications, except the \(M\) spiked eigenvalues, the tailed eigenvalues of \(\Lambda\Lambda'\) are not all zeros, but most of them are relatively small values close to zero. Due to this reason, \(m_0\) should be selected to sufficiently ensure the inclusion of the most of non-zero eigenvalues. According to practical experience, we adopt the selection of \(m_0\) in the range of \(0.6N\) to \(0.8N\).

Then, we apply the information criteria \(PC_p\) of Bai and Ng (2002) and our proposed criteria \(PC_p^*\) to estimate the number of factors for the real data sets. The estimate results on the two data sets are shown in Table 4 and 5. The three rows of the tables correspond to the estimations when \(m_0\) is selected as \(0.6N\), \(0.7N\) and \(0.8N\), respectively. It suggests that the original criteria \(PC_p\)’s seriously overestimate for the two data sets. In contrast, when \(m_0 = 0.7N\), our proposed information criteria \(PC_p^*\)’s estimate the number of factors for both data sets to be 10 or 11, and the corresponding cumulative contribution rate can reach 95%. It implies that 10 or 11 is a reasonable estimation of the number of factors for both data sets.
Table 4. Comparison between $PC_{pj}$ and $PC_{p_j}^*$ on the first data set

|       | $PC_{p1}$ | $PC_{p2}$ | $PC_{p3}$ | $PC_{p1}^*$ | $PC_{p2}^*$ | $PC_{p3}^*$ |
|-------|-----------|-----------|-----------|-------------|-------------|-------------|
| $m_0$ | 28        | 24        | 23        | 27          | 8           | 8           |
| $m_0$ | 32        | 32        | 32        | 32          | 10          | 10          |
| $m_0$ | 37        | 37        | 37        | 37          | 13          | 13          |

Table 5. Comparison between $PC_{pj}$ and $PC_{p_j}^*$ on the second data set

|       | $PC_{p1}$ | $PC_{p2}$ | $PC_{p3}$ | $PC_{p1}^*$ | $PC_{p2}^*$ | $PC_{p3}^*$ |
|-------|-----------|-----------|-----------|-------------|-------------|-------------|
| $m_0$ | 29        | 23        | 22        | 27          | 8           | 8           |
| $m_0$ | 34        | 34        | 34        | 34          | 10          | 10          |
| $m_0$ | 38        | 38        | 38        | 38          | 12          | 12          |

6 Conclusion

This paper aimed to determine the number of factors in a large-dimensional generalised factor model with more relaxed assumptions than that of previous works. For the target model, we introduced the bias-corrected noise estimator $\hat{\sigma}_p^2$ by random matrix theory, further construct the information criteria $PC_{p_j}^*$ based on $\hat{\sigma}_p^2$, and estimate the number of factors consistently. The good performance of our method is demonstrated by simulations and empirical applications. This paper only focused on the static factor models. Further we will improve the information criteria to accommodate the dynamic factor models in the future work.

A The logarithm transformed MAEs among several noise estimators
Determine the number of factors 23

Table A1. The logarithm transformed MAEs among several noise estimators in Model 1

| Estimators | $\hat{\sigma}^2_\epsilon$ | $\hat{\sigma}^2_{MLE}$ | $\hat{\sigma}^2_P$ | $\hat{\sigma}^2_{KN}$ | $\hat{\sigma}^2_{US}$ | $\hat{\sigma}^2_{\text{median}}$ |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $N = 50, T = 100$ | -0.95594 | -0.21996 | 1.079281 | 1.061465 | 0.448425 | 1.023963 |
| $N = 100, T = 200$ | -1.29808 | -0.51528 | 1.13894 | 1.105469 | 0.535754 | 1.04767 |
| $N = 150, T = 300$ | -1.48663 | -0.68761 | 1.124752 | 1.119236 | 0.565924 | 1.059354 |
| $N = 200, T = 400$ | -1.60601 | -0.81515 | 1.130371 | 1.129887 | 0.582757 | 1.067715 |
| $N = 250, T = 500$ | -1.70851 | -0.90724 | 1.133389 | 1.133012 | 0.59349 | 1.074012 |
| $N = 300, T = 600$ | -1.78126 | -1.05624 | 1.13551 | 1.135202 | 0.600825 | 1.082972 |
| $N = 350, T = 700$ | -1.86527 | -1.11401 | 1.137166 | 1.136905 | 0.605761 | 1.08972 |
| $N = 400, T = 800$ | -1.92219 | -1.11401 | 1.138295 | 1.138069 | 0.609859 | 1.086277 |

Under Gaussian assumption and $c = 0.5$

| Estimators | $\hat{\sigma}^2_\epsilon$ | $\hat{\sigma}^2_{MLE}$ | $\hat{\sigma}^2_P$ | $\hat{\sigma}^2_{KN}$ | $\hat{\sigma}^2_{US}$ | $\hat{\sigma}^2_{\text{median}}$ |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $N = 90, T = 60$ | -0.80542 | -0.2007 | 1.089998 | 1.062031 | 0.571411 | 0.967804 |
| $N = 150, T = 100$ | -0.98767 | -0.40775 | 1.112526 | 1.095778 | 0.638532 | 0.992417 |
| $N = 210, T = 140$ | -1.10124 | -0.55065 | 1.122464 | 1.110508 | 0.664882 | 1.010448 |
| $N = 270, T = 180$ | -1.18679 | -0.66006 | 1.128019 | 1.11872 | 0.679631 | 1.024954 |
| $N = 330, T = 220$ | -1.27469 | -0.74316 | 1.131205 | 1.123598 | 0.690476 | 1.039137 |
| $N = 390, T = 260$ | -1.35075 | -0.81195 | 1.13339 | 1.12695 | 0.695654 | 1.039021 |
| $N = 450, T = 300$ | -1.41206 | -0.87197 | 1.135046 | 1.129465 | 0.700718 | 1.043806 |
| $N = 510, T = 340$ | -1.44716 | -0.93001 | 1.136531 | 1.131607 | 0.704886 | 1.050749 |

Under Gamma assumption and $c = 0.5$

| Estimators | $\hat{\sigma}^2_\epsilon$ | $\hat{\sigma}^2_{MLE}$ | $\hat{\sigma}^2_P$ | $\hat{\sigma}^2_{KN}$ | $\hat{\sigma}^2_{US}$ | $\hat{\sigma}^2_{\text{median}}$ |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $N = 50, T = 100$ | -0.7978 | -0.20039 | 1.074523 | 1.056506 | 0.465334 | 0.972263 |
| $N = 100, T = 200$ | -1.13773 | -0.50752 | 1.113048 | 1.104576 | 0.547863 | 1.003191 |
| $N = 150, T = 300$ | -1.31941 | -0.68424 | 1.124514 | 1.118972 | 0.57605 | 1.017331 |
| $N = 200, T = 400$ | -1.45258 | -0.80535 | 1.129859 | 1.129375 | 0.589596 | 1.031058 |
| $N = 250, T = 500$ | -1.548 | -0.9029 | 1.133208 | 1.132832 | 0.597788 | 1.038072 |
| $N = 300, T = 600$ | -1.64719 | -0.97789 | 1.135253 | 1.134946 | 0.605367 | 1.044539 |
| $N = 350, T = 700$ | -1.70155 | -1.05285 | 1.137067 | 1.136807 | 0.61079 | 1.050343 |
| $N = 400, T = 800$ | -1.74588 | -1.11015 | 1.138197 | 1.137971 | 0.614225 | 1.055111 |

Under Gamma assumption and $c = 1.5$

| Estimators | $\hat{\sigma}^2_\epsilon$ | $\hat{\sigma}^2_{MLE}$ | $\hat{\sigma}^2_P$ | $\hat{\sigma}^2_{KN}$ | $\hat{\sigma}^2_{US}$ | $\hat{\sigma}^2_{\text{median}}$ |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $N = 50, T = 100$ | -0.81382 | -0.17868 | 1.084367 | 1.056048 | 0.5545 | 0.920996 |
| $N = 100, T = 200$ | -0.98342 | -0.3935 | 1.110467 | 1.093598 | 0.628851 | 0.935026 |
| $N = 150, T = 300$ | -1.09768 | -0.53948 | 1.121347 | 1.109331 | 0.658228 | 0.958855 |
| $N = 200, T = 400$ | -1.18749 | -0.64851 | 1.12714 | 1.117806 | 0.675449 | 0.973028 |
| $N = 250, T = 500$ | -1.262 | -0.73747 | 1.130854 | 1.123223 | 0.685124 | 0.982414 |
| $N = 300, T = 600$ | -1.32379 | -0.80877 | 1.133225 | 1.126769 | 0.692978 | 0.99098 |
| $N = 350, T = 700$ | -1.38851 | -0.86749 | 1.134844 | 1.12925 | 0.697477 | 1.000551 |
| $N = 400, T = 800$ | -1.43185 | -0.92462 | 1.136319 | 1.131384 | 0.701558 | 1.00541 |
Table A2. The logarithm transformed MAEs among several noise estimators in Model 2

| Estimators   | $\hat{\sigma}_0^2$ | $\hat{\sigma}_{MLE}^2$ | $\hat{\sigma}_{\overline{P}}^2$ | $\hat{\sigma}_{KN}^2$ | $\hat{\sigma}_{US}^2$ | $\hat{\sigma}_{\text{median}}^2$ |
|--------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $N = 50, T = 100$ | -0.95278 | -0.2181 | 1.07884 | 1.061015 | 0.448869 | 1.208373 |
| $N = 100, T = 200$ | -1.30545 | -0.51772 | 1.114157 | 1.105739 | 0.535016 | 1.180092 |
| $N = 150, T = 300$ | -1.4831 | -0.68824 | 1.124797 | 1.119282 | 0.566147 | 1.169562 |
| $N = 200, T = 400$ | -1.60578 | -0.81383 | 1.130279 | 1.129795 | 0.582211 | 1.16414 |
| $N = 250, T = 500$ | -1.70975 | -0.91001 | 1.133503 | 1.133127 | 0.592704 | 1.160719 |
| $N = 300, T = 600$ | -1.77951 | -0.98689 | 1.135565 | 1.135257 | 0.600801 | 1.158251 |
| $N = 350, T = 700$ | -1.84042 | -1.0515 | 1.137028 | 1.136767 | 0.605395 | 1.156618 |
| $N = 400, T = 800$ | -1.926 | -1.11542 | 1.13833 | 1.138105 | 0.609938 | 1.155565 |

Under Gaussian assumption and $c = 0.5$

| $N = 90, T = 60$ | -0.81416 | -0.19829 | 1.0894 | 1.061424 | 0.572178 | 1.178899 |
| $N = 150, T = 100$ | -0.971 | -0.41135 | 1.113034 | 1.096288 | 0.640573 | 1.16716 |
| $N = 210, T = 140$ | -1.09756 | -0.5519 | 1.122587 | 1.11063 | 0.666064 | 1.161346 |
| $N = 270, T = 180$ | -1.20135 | -0.65616 | 1.127725 | 1.118427 | 0.680227 | 1.157876 |
| $N = 330, T = 220$ | -1.2806 | -0.74153 | 1.131106 | 1.123497 | 0.689871 | 1.155746 |
| $N = 390, T = 260$ | -1.33833 | -0.81492 | 1.133544 | 1.127105 | 0.696498 | 1.154434 |
| $N = 450, T = 300$ | -1.40564 | -0.874 | 1.135137 | 1.129555 | 0.70147 | 1.1532 |
| $N = 510, T = 340$ | -1.45006 | -0.9301 | 1.136534 | 1.13161 | 0.703742 | 1.152568 |

Under Gamma assumption and $c = 0.5$

| $N = 50, T = 100$ | -0.85259 | -0.20869 | 1.076574 | 1.058641 | 0.442791 | 1.204876 |
| $N = 100, T = 200$ | -1.17927 | -0.50421 | 1.112681 | 1.104226 | 0.533345 | 1.176478 |
| $N = 150, T = 300$ | -1.34406 | -0.68147 | 1.127725 | 1.118427 | 0.625941 | 1.161346 |
| $N = 200, T = 400$ | -1.49078 | -0.80743 | 1.129969 | 1.129485 | 0.656031 | 1.162066 |
| $N = 250, T = 500$ | -1.58779 | -0.90644 | 1.133536 | 1.132979 | 0.594179 | 1.159008 |
| $N = 300, T = 600$ | -1.67897 | -0.98786 | 1.135598 | 1.13529 | 0.601305 | 1.156902 |
| $N = 350, T = 700$ | -1.74402 | -1.04957 | 1.136971 | 1.136711 | 0.606059 | 1.155449 |
| $N = 400, T = 800$ | -1.81112 | -1.10866 | 1.138159 | 1.137933 | 0.61054 | 1.154233 |

Under Gamma assumption and $c = 1.5$

| $N = 50, T = 100$ | -0.81058 | -0.18644 | 1.086393 | 1.058178 | 0.548338 | 1.168817 |
| $N = 100, T = 200$ | -0.96384 | -0.40608 | 1.11229 | 1.095475 | 0.625941 | 1.159911 |
| $N = 150, T = 300$ | -1.08621 | -0.54748 | 1.12215 | 1.110156 | 0.656867 | 1.156134 |
| $N = 200, T = 400$ | -1.17893 | -0.65703 | 1.127791 | 1.118473 | 0.675113 | 1.154124 |
| $N = 250, T = 500$ | -1.27892 | -0.73578 | 1.130749 | 1.123124 | 0.683333 | 1.15202 |
| $N = 300, T = 600$ | -1.34049 | -0.80886 | 1.133229 | 1.126779 | 0.691245 | 1.151347 |
| $N = 350, T = 700$ | -1.38994 | -0.87447 | 1.135158 | 1.129571 | 0.696488 | 1.150754 |
| $N = 400, T = 800$ | -1.44007 | -0.9276 | 1.136436 | 1.131505 | 0.700241 | 1.150307 |
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