[SADE] A Maple package for the Symmetry Analysis of Differential Equations

Tarcísio M. Rocha Filho\textsuperscript{a,1} Annibal Figueiredo\textsuperscript{a}

\textsuperscript{a}Instituto de Física and International Center for Condensed Matter Physics
Universidade de Brasília, CP: 04455, 70919-970 - Brasília, Brazil

Abstract

We present the package SADE (Symmetry Analysis of Differential Equations) for the determination of symmetries and related properties of systems of differential equations. The main methods implemented are: Lie, non classical, Lie-Bäcklund and potential symmetries, invariant solutions, first-integrals, Nöther theorem for both discrete and continuous systems, solution of ordinary differential equations, order and dimension reductions using Lie symmetries, classification of differential equations, Casimir invariants, and the quasi-polynomial formalism for ODE’s (previously implemented by the authors in the package QPSI) for the determination of quasi-polynomial first-integrals, Lie symmetries and invariant surfaces. Examples of use of the package are given.

Key words: Symmetry transformations; Invariant solutions; Conservation laws; Symbolic computation. 02.70.Wz; 11.30.-j; 02.30.Jr

\textsuperscript{1} Corresponding author, e-mail: marciano@fis.unb.br
PROGRAM SUMMARY

Title of the program: SADE

Catalogue identifier: None

Program obtainable from: The author by e-mail.

Operating systems under which the program has been tested: UNIX/LINUX systems and WINDOWS

Programming language used: MAPLE 13 and MAPLE 14

No. of bytes in a word: 32

No. of bytes in distributed program: 300 KB

Distribution format: zip or gzip

Card punching code: ASCII

Keywords: symmetry transformations, invariant solutions, first integrals, nöther theorem.

Nature of the physical problem: Determination of analytical properties of systems of differential equations, including symmetry transformations, analytical solutions and conservation laws.

Method of resolution: The package implements in MAPLE some algorithms (discussed in the text) for the study of systems of differential equations.

Restrictions on the complexity of the problem: Depends strongly on the system and on the algorithm required. Typical restrictions are related to the solution of a large over-determined system of linear or non-linear differential equations.

Typical running time: Depends strongly on the order, the complexity of the differential system and the object computed. Ranges from seconds to hours.

LONG WRITE-UP

1 Introduction

Natural phenomena are very often modeled by differential equations, which exhibit a plethora of dynamical behaviors. These can be classified somewhat
vaguely in two categories: regular and irregular, according to the complexity exhibited by its solutions. The notion of integrability is then used as an attempt to put a more stringent distinction between regular and irregular behavior. Usually a regular behavior is characterized by the existence of conservation laws that strongly restrict the types of solution a system can exhibit. Even for non-integrable systems some of these laws can be obtained. Also the determination of particular analytical solutions for both Ordinary (ODE’s) and Partial Differential Equations (PDE’s) is of utmost importance in many fields of physics and applied sciences. These solutions are helpful to shed some light and gain insight on the physics of the system, and are also useful as benchmarks for numerical methods. Almost all known analytical solutions in physics are solutions invariant under one or more symmetry transformations.

The theory of symmetry transformations of Differential Equations (DE’s) was introduced by Lie in the end of the XIX century [1]. Solutions invariant under symmetry transformations are called invariant solutions, and different methods are described in the literature (see References [2,3,4,5] and references therein). Olver and Rosenau showed that new solutions can be obtained by requiring that they are invariant under infinitesimal symmetry transformations while also preserving additional side conditions [3]. Non classical symmetries were introduced by Bluman and Cole and are based on the idea that the required analytical solution is invariant under symmetry transformations preserving both the form of the differential equation and the invariant solution condition [5]. This approach is less restrictive in the sense that there exist usually more non classical symmetries than Lie (classical) symmetries, the latter being a subset of the former. Other generalizations of the classical Lie method considered here are potential symmetries [6] and Lie-Bäcklund transformations [7]. For a first introduction to Lie Symmetries see [8,9], and [10,12] for a more complete and formal approach. A description of methods for solving differential equations using Symmetries is found in [11].

Different packages in computer algebra systems exist implementing Lie symmetry computations and related methods: SPDE by Schwarz [12], CRACK and LIEPDE by Wolf [13,14] and DIMSYM by Sherring and Prince [15] in REDUCE, LIE and BIGLIE by Head [16,17] in MUMATH and MATHLIE by Baumann [18] in Mathematica. For MAPLE there are also some useful packages: PDEtools by Cheb-Terrab [19] which is distributed since Release 11, DESOLV by Vu and Carminati [20,4], and GeM by Cheviakov [21]. For good reviews with a comparison between some of these packages see References [22] and [23]. The package QPSI by the authors implements the Quasi-Polynomial formalism for symmetry generators, first-integrals and invariant hyper-surfaces for ODE’s [24], now part of the present package.

In this work we present the package Symmetry Analysis of Differential Equations (SADE) in MAPLE, for the computation of Lie, Lie-Bäcklund and non-
classical symmetries, invariant solutions, first-integrals, Nöther theorem for both discrete and continuous systems, quasi-polynomial first-integrals and symmetry generators, solution and reduction of order or dimension for ODE’s, classification of differential equations, invariant surfaces and Casimir invariants \[8,10,25,24\], and some other features presented below. Our package is well suited for efficiently computing Lie symmetries of large systems, as for instance the Yang-Mills with $SU(2)$ and $SU(3)$ gauge group \[26\], and has been used in the last years in our group in different applications \[27,28,29,30,31\].

Our aim in developing this package was to implement these methods, being as user friendly as possible for researchers in many fields of pure and applied sciences, and still being capable to handle reasonably complicated systems of equations. The paper is structure in the following way: in section 2 we briefly revise the mathematical methods implemented. A discussion of the heuristics for the solution of the determining system for Lie and nonclassical symmetries of linear and non-linear overdetermined systems of PDE’s is given in section 3. The package routines are described in section 4, and some illustrative examples are given in section 5. Benchmarks for computing Lie symmetries are given in section 6. We close the paper with some concluding remarks in section 7.

2 Methods

2.1 Lie symmetries of differential equations

Let \( \{u_1, \ldots, u_n\} \equiv u \) be a set of functions (dependent variables) of the (independent) variables \( \{x_1, \ldots, x_m\} \equiv x \). A system of \( p \) differential equations satisfied by the \( n \) functions \( u_j(x) \) can be written as

\[
F_\mu(u_j, x_i, u_{jI}) = 0, \quad \mu = 1, \ldots, p, \tag{1}
\]

with

\[
u_{jI} \equiv u_{j,i_1,\ldots,i_k} = \partial^k u_j / \partial x_{i_1} \cdots \partial x_{i_k}, \quad I \equiv i_1, \ldots, i_k. \tag{2}\]

For \( m = 1 \) eq. (1) is a set of ODE’s.

A transformation of variables

\[
x'_i = x'_i(u, x),
\]

\[
u'_j = u'_j(u, x), \tag{3}
\]
is a symmetry transformation of eq. (1) if

\[ F_\mu(u'_j, x'_i, u''_j) = 0, \]  

(4)

where \( u'_j \equiv \partial^k u'_j / \partial x'_{i_1} \cdots \partial x'_{i_k} \), whenever eq. (1) holds, i.e. if eq. (1) is form invariant under (3), or equivalently, if eq. (3) maps a solution into another solution of eq. (1). Such transformations are called Lie symmetries (point symmetries). The set of Lie symmetries of a (system of) differential equation(s) is a Lie group, and therefore can be obtained from the knowledge of the infinitesimal transformations (in fact only the subgroup of transformations connected to the identity transformation) \[10\]. The infinitesimal symmetries can be written as:

\[ x'_i = x_i + \epsilon \theta_i(u, x), \]
\[ u'_j = u_j + \epsilon \eta_j(u, x), \]

(5)

where \( \epsilon \) is an infinitesimal parameter and \( \theta_i \) and \( \eta_j \) are functions of the dependent and independent variables. The infinitesimal symmetry generator of transformation (5) is

\[ G = \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial u_j} + \sum_{i=1}^{m} \theta_i \frac{\partial}{\partial x_i}. \]

(6)

The set of all infinitesimal symmetry generators form a Lie algebra with respect to the commutation operation.

In order to determine the invariance condition of (1) under the infinitesimal transformation (5), we note that

\[ \frac{\partial u'_j}{\partial x'_i} = \frac{\partial u_j}{\partial x_i} + \epsilon \left[ \frac{\partial \eta_j}{\partial x_i} - \sum_{l=1}^{m} \frac{\partial u_j}{\partial x_l} \frac{\partial \theta_l}{\partial x_i} \right] \equiv \frac{\partial u_j}{\partial x_i} + \epsilon \eta^{(1)}_{ji}. \]

(7)

The transformation rules for higher order derivatives can be obtained similarly. In the general case we have:

\[ \frac{\partial^k u'_j}{\partial x'_{i_1} \cdots \partial x'_{i_k}} = \frac{\partial^k u_j}{\partial x_{i_1} \cdots \partial x_{i_k}} + \epsilon \eta_{j,i_1 \cdots i_k}^{(k)} \equiv u_{j,i_1 \cdots i_k} + \epsilon \eta_{j,i_1 \cdots i_k}^{(k)}, \]

(8)

with \( \eta_{j,i_1 \cdots i_k}^{(k)} \) functions of the independent and dependent variables and its derivatives \[10\]. Supposing that the highest derivative in (1) is of order \( k \) we can express its invariance under an infinitesimal transformation by

\[ G^{(k)} F_\mu = 0, \]

(9)
where \( G^{(k)} \) is the \( k \)-th prolongation of the generator \( G \) in (6) and is obtained from eq. (8) as:

\[
G^{(k)} = \sum_{i=1}^{m} \theta_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial u_j} + \sum_{l=1}^{k} \sum_{i_1,\ldots,i_l} \eta_{j_1,\ldots,j_l}^{(l)} \frac{\partial}{\partial u_{j_1,\ldots,j_l}}. 
\]

(10)

Using the orthonomic form of eq. (1) to eliminate highest order derivatives from eq. (9), and equating to zero the coefficients of the remaining derivatives, or more precisely the coefficients of linearly independent functions of the latter, we obtain the determining system for the symmetry transformations of eq. (1). The reduction to the orthonomic form of eq. (1) is performed using standard methods (see [42,43] and references therein). If the reduction is not possible SADe will issue an error message before aborting the calculations.

The symmetry generator in eq. (6) is equivalent to the following evolutionary form:

\[
\tilde{G} = \sum_{j=1}^{n} \left[ \eta_j - \sum_{i=1}^{m} \frac{\partial u_j}{\partial x_i} \theta_i \right] \frac{\partial}{\partial u_j} \equiv \sum_{j=1}^{n} Q_j \frac{\partial}{\partial u_j}. 
\]

(11)

Both forms as given in eqs. (6) and (11) describe the same transformation, in the sense that they map a given solution to the same transformed solution.

2.1.1 Quasi-Polynomial Symmetries

First order differential equations usually admit an infinite dimensional Lie symmetry group. To determine their Lie symmetries it is usually necessary to impose an ansatz on its symmetry generators. One possibility is to suppose that the coefficients \( \eta_i \) of the symmetry generator are polynomial functions of the dependent variables. A more general ansatz consists to consider the class of quasi-polynomial functions, as introduced in ref. [25], previously implemented by the authors in the package QPSI [24], and now included in SADe.

A system of ODE’s of the form

\[
\dot{x}_i = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}; \quad i = 1, \ldots, n , 
\]

(12)

is called Quasi-Polynomial (QP) [32]. In eq. (12) \( A_{ij} \) and \( B_{jk} \) are real or complex constants and \( m \) is the number of different quasi-monomials in (12). We
define a new set of variables by the Quasi-Monomial Transformation (QMT):
\[
y_i = \prod_{k=1}^{n} x_k^{\tilde{B}_{ik}},
\]  
(13)

where \( \tilde{B}_{ik} = B_{ik} \) for all \( i \) and \( k \leq n \), \( \tilde{B}_{ik} = 0 \) for \( i \leq n \) and \( k \geq m \), and \( \tilde{B}_{ik} = \delta_{ik} \) for \( n \leq i, k \leq m \), in such way that the inverse transformation is also a QMT with the exponent matrix given by \( \tilde{B}^{-1} \) (the case with \( \tilde{B} \) singular can also be handled as discussed in [32]). System (1) is cast by transformation (13) into a quadratic system of equations, the Lotka-Volterra form:
\[
\dot{y}_i = y_i \sum_{j=1}^{m} M_{ij} y_j, \quad i = 1, \ldots, m,
\]  
(14)

where the matrix \( M \) is given by:
\[
M = BA.
\]  
(15)

Lotka-Volterra equations are extensively studied in the literature. Many results obtained for this special class of equations can then be recast into the more general QP form [25,27,30,32,33,34].

The central result obtained in Ref. [25] is that any quasi-polynomial symmetry generator \( G \), such that \( \theta_i = 0 \) and \( \eta_i \) a quasi-polynomial function of the dependent variables, can be decomposed as:
\[
G = \sum_i G^{(i)},
\]  
(16)

with:
\[
G^{(i)} = y^{\xi^{(i)}} T^{(i)}, \quad \{F, G^{(i)}\} = 0,
\]  
(17)

where \( F \) is the flow associated to the quasi-polynomial system (12):
\[
F \equiv \sum_{i,j=1}^{m} M_{ij} y_i y_j \frac{\partial}{\partial y_i},
\]  
(18)

\( y^{\xi^{(i)}} \) is a quasi-monomial:
\[
y^{\xi^{(i)}} \equiv (y_1)^{\xi_1^{(i)}} \cdots (y_m)^{\xi_m^{(i)}},
\]  
(19)
with $\xi_j^{(i)}$ real numbers (see eq. (22) below) and $T^{(i)}$ a polynomial semi-invariant vector field satisfying:

$$[F, T^{(i)}] = \lambda^{(i)} T^{(i)}.$$  \hspace{1cm} (20)

The eigenvalue $\lambda^{(i)}$ is a linear function of the form

$$\lambda^{(i)} = \sum_j \lambda_j^{(i)} y_j.$$ \hspace{1cm} (21)

Both $\lambda_j^{(i)}$ and $\xi_j^{(i)}$ are solutions of the equation:

$$\sum_j \xi_j^{(i)} M_{jk} = -\lambda_k^{(i)}.$$ \hspace{1cm} (22)

It is straightforward to show that if a symmetry generator (including the flow $F$) can be written as a linear combination of the remaining generators, then the coefficients of the expansion (as functions of $x_i$) are first-integrals of system (12).

2.2 Lie-Bäcklund symmetries

Lie symmetries are diffeomorphisms on the space of dependent $u_j$ and independent $x_i$ variables. Lie-Bäcklund, or generalized, symmetries depend also on derivatives of $u_j$. We present here a brief account of how to compute the generators of Lie-Bäcklund symmetries (for more details see [10] and [7]). The determining equations for Lie-Bäcklund symmetries are more easily obtained using the evolutionary form (11), with $\eta_j = \eta_j(u_j, x_i, u_{jI})$ and $\theta_i = \theta_i(u_j, x_i, u_{jI})$, with prolongation

$$G^{(k)} = \sum_{j=1}^{n} \sum_I D_I Q_j \frac{\partial}{\partial u_I},$$ \hspace{1cm} (23)

where $I \equiv i_1, \ldots, i_m$, $D_I \equiv d^k/dx_{i_1} \cdots dx_{i_k}$, $D_0 \equiv 1$, and the summation over $I$ is taken for all values of indices such that $|I| = i_1 + \ldots + i_k \leq k$. The invariance condition can be expressed as

$$G^{(k)} F_\mu(x, u, u_I) = 0,$$ \hspace{1cm} (24)

with $k$ the maximum differentiation order of $u_j$ in $F_\mu$. In order to equate coefficients of independent derivatives of $u_j$ in (24), we distinguish dependent and
independent derivatives of $u_j$ using the original system in the orthonomic form, and its differential consequences. In this way only independent derivatives remain, and at this step each coefficient of derivatives that are not arguments of $Q_j$ is equated to zero, yielding the determining system for Lie-Bäcklund symmetries. Its solution demands a greater computational effort than the solution of the analogous determining system for Lie symmetries.

2.3 Reduction of PDE’s and invariant solutions

Symmetries of a differential system can be used to construct analytical solutions or a reduction into a system depending on a smaller number of independent variables. A symmetry generator as given in (6) can be transformed, by a change of dependent $r_i = r_i(u, x)$ and independent variables $s_j = s_j(u, x)$, called canonical coordinates, into the form

$$G_1 = \frac{\partial}{\partial s_1}. \quad (25)$$

Solutions invariant under the symmetry generated by $G_1$ do not depend on $s_1$. In this way we obtain a reduction into a system with $n - 1$ independent variables. More generally $p < m$ symmetry generators can be used to reduce to a new system with $m - p$ independent variables, on the condition that a set of mutual canonical variables exists for the set of $p$ generators. For $p = m - 1$ we obtain a system of ODE’s. The latter, if solvable, then yields an analytical (particular) solution for the original system. In practice, considering $p$ generators $G^{(i)}$, $i = 1, \ldots, p$, we look for solutions $u_i(x)$ satisfying

$$\tilde{G}^{(i)} u_j(x) = 0, \quad j = 1, \ldots, n, \quad (26)$$

where $\tilde{G}^{(i)}$ is the evolutionary form (11). Equation (26) is a linear system usually simpler to solve than the original system using the characteristics method. Replacing its solution into the original system yields a reduced system with $n - p$ independent variables.

2.4 Nonclassical symmetries

Lie symmetries maps the set of all solutions of a differential system into itself. Invariant solutions then correspond, among all solutions, to those that are invariant under one or more symmetry transformations. More generally, nonclassical symmetries transform a solution, still to be determined, into itself.
This amounts to require that both (1) and the invariance condition:

\[ Q_i = \mathcal{G} u_i(x) = \left[ \eta_i - \sum_{j=1}^{m} \frac{\partial u_i}{\partial x_j} \theta_j \right] = 0, \quad (27) \]

are invariant under (5). This is expressed by:

\[ \mathbf{G}^{(k)} F_\mu = 0, \quad (28) \]

and

\[ \mathbf{G}^{(k)} Q_i = 0. \quad (29) \]

A more detailed account of nonclassical symmetries is given in Ref. [35]. It can be shown that eq. (29) holds whenever eq. (28) is satisfied. The resulting system (28) is non-linear in the unknowns \( \eta_i \) and \( \theta_i \) as it must be solved modulo eq. (29), and thus much harder to solve than the linear determining system for Lie symmetries. The set of all nonclassical symmetries include all Lie symmetries, and do not form a vector space (no associated Lie algebra). Another useful property is that if \( \mathbf{G} \) is the generator of a nonclassical symmetry, then \( F(u, x) \mathbf{G} \) also generates a nonclassical symmetry, for any arbitrary (sufficiently differentiable) function \( F \). As a consequence and without loss of generality, we consider the cases with \( \theta_1 = 1 \) or \( \theta_1 = 0 \). In the later case there are still two possibilities: either \( \theta_2 = 1 \) or \( \theta_2 = 0 \), and so on.

Computer algebra determination of the invariance condition for nonclassical symmetries can lead to infinite loops when replacing dependent derivatives from eq. (29) into eq. (28). This is avoided in our approach by the following steps:

1. Chose an independent variable \( x_k \).
2. Solve the invariance conditions (27) for all derivatives \( \partial u_i / \partial x_k \), \( i = 1, \ldots, n \).
3. From the result of the previous step, eliminate all derivatives with respect to \( x_k \) in (1).
4. Determine the invariance condition using the resulting differential system.
5. Replace in the determining system all derivatives of \( u_i \) with respect to \( x_k \) using step (2).

The variable \( x_k \) is chosen such that \( \text{MaxDer}(x_k) < \text{MaxDer}(x_i) \) for \( i \neq k \), with \( \text{MaxDer}(x_i) \) the maximum derivative order of any dependent variable with respect to \( x_i \) in (1). This usually results in a “simpler” determining system.
2.5 Potential symmetries

For systems in conserved form, potential symmetries can be used to construct invariant solutions that are not obtainable neither from Lie nor nonclassical symmetries \[6,37\]. A partial differential equation is said to be in a conserved form if it can be written as:

\[
\sum_{i=1}^{m} \frac{\partial}{\partial x_i} F_i(u_j, x_i, u_{j1}) = 0.
\]  

(30)

This implies that there exists \(m(m-1)\) components (potentials) \(\Psi_{ij} (i < j)\) of an antisymmetric tensor such that

\[
F_i = \sum_{i<j=1}^{m} (-1)^j \frac{\partial \Psi_{ij}}{\partial x_j} + \sum_{j<i=1}^{m} (-1)^{i-1} \frac{\partial \Psi_{ji}}{\partial x_j}.
\]

(31)

The generalization to a system of PDE’s is straightforward (each equation must be put in a conserved form). Eq. (31) is a system of \(M\) PDE’s with \(1 + m(m-1)/2\) dependent variables \((u_j\) and the potentials). For \(M \geq 3\) the system is under-determined, and some “gauge” conditions on the potentials must be given \[37\].

An infinitesimal symmetry of eq. (31)

\[
\begin{align*}
    u'_i &= u_i + \epsilon \eta_i(u, \Psi, x), \\
    \Psi'_{ij} &= \Psi_{ij} + \epsilon \xi_{ij}(u, \Psi, x), \\
    x'_i &= x_i + \epsilon \theta_i(u, \Psi, x),
\end{align*}
\]

(32)

is a potential symmetry of the original system (30) if \(\eta_i\) or \(\theta_i\) depend on \(\Psi_{ij}\). As a result the transformation for \(u\) and \(x_i\) is non-local, since it depends on the potentials, which are solutions of eq. (31). A potential symmetry can then be used to reduce eq. (30) and, in some cases, to obtain invariant solutions.

2.6 Symmetries and conservation laws

2.6.1 Quasi-polynomial first-integrals

For the special case of quasi-polynomial first-order systems, we first obtain the associated Lotka-Volterra form (14), and define a semi-invariant (Darboux
polynomial) as a polynomial function \( f(y) \) such that

\[
\dot{f} = \sum_{j=1}^{m} y_j \frac{\partial f}{\partial y_j} = \lambda f ,
\]

(33)

where the eigenvalue \( \lambda \) is a function of \( y_1, \ldots, y_m \).

For the Lotka-Volterra form the following properties were proved in \[25,33\]:

(i) \( \lambda \) is a linear function, i.e.,

\[
\lambda = \sum_{j=1}^{m} \lambda_j y_j .
\]

(34)

(ii) Any polynomial semi-invariant \( f \) can be decomposed as:

\[
f = \sum_{p} f^{(p)} ,
\]

(35)

where \( f^{(n)} \) is a homogeneous polynomial of degree \( p \). Furthermore, each \( f^{(n)} \) is also a semi-invariant with the same eigenvalue as \( f \). Any Quasi-Polynomial first-integral can be decomposed as:

\[
J = \sum_{p} y^\xi^{(p)} f^{(p)} ,
\]

(36)

where \( y^\xi^{(p)} f^{(p)} \) is a first-integral with \( \xi^{(p)} \) a solution of eq. (22).

If one of the Quasi-Monomials in (5) is a constant, \( \lambda \) may also admit a constant value with respect to the original variables \( x_k \), and a first-integral can be obtained by multiplying the corresponding semi-invariant by \( \exp(-\lambda t) \). It is easy to show that for \( f^{(1)} \) and \( f^{(2)} \) semi-invariants with respective eigenvalues \( \lambda^{(1)} \) and \( \lambda^{(2)} \), \( f^{(1)} f^{(2)} \) and \( f^{(1)}/f^{(2)} \) are also-semi-invariants with eigenvalues \( \lambda^{(1)} + \lambda^{(2)} \) and \( \lambda^{(1)} - \lambda^{(2)} \), respectively. The first integrals are then obtained by combinations of the form

\[
QP_1(x) [QP_2(x)]_{\pm 1} \exp(\rho t),
\]

(37)

such that it has a vanishing eigenvalue, and therefore has a zero time derivative. Analogously, it is straightforward to implement the computation of first-integrals of the form \( P_1(x) + \log(x^k) \) and \( P_2(x, \ln(x)) \), where \( P_1 \) and \( P_2 \) are polynomials in their arguments. A similar result also holds for quasi-polynomial symmetries \[24\].
2.6.2 Nöther theorem

Many systems of interest are described by equations that can be deduced from a variational principle, with action $S$ defined by

$$S \equiv \int_{\mathcal{C}} \mathcal{L}(u_j, u_{j,i}, x_i) \, d^m x.$$  \hspace{1cm} (38)

where $u_j \ (j = 1, \ldots, n)$ are the dependent variables, $x_i \ (i = 1, \ldots, m)$ the independent variables and $u_{j,i} \equiv \partial u_j / \partial x_i$. The lagrangian of the system is denoted by $\mathcal{L}$ and $\mathcal{C}$ is a bounded region of the $m$-dimensional space of independent variables. The differential system is obtained from the requirement that $S$ is an extremum for any solution $u_j(x)$.

Nöther theorem [38] states that every symmetry transformation of the action $S$ of the form

$$x_i' = x_i + \epsilon \theta_i(u, x),$$  
$$u_j' = u_j + \epsilon \eta_j(u, x),$$  \hspace{1cm} (39)

is related to a conservation law. The invariance of $S$ under (39) implies that

$$\mathcal{D} \mathcal{L} = \frac{d f_i}{dx_i},$$  \hspace{1cm} (40)

with

$$\mathcal{D} \equiv \sum_i \theta_i \frac{\partial}{\partial x_i} + \sum_j u_j \frac{\partial}{\partial u_j} + \sum_{j,i} \left( \frac{\partial \eta_j}{\partial x_i} - \sum_k u_{j,k} \frac{d \theta_k}{dx_i} \right) \frac{\partial}{\partial u_{j,i}} + \sum_i \frac{d \theta_i}{dx_i},$$  \hspace{1cm} (41)

where $f_i$ are also unknowns to be determined from condition (40) alongside with the $\theta_i$’s and the $\eta_{\mu}$’s. The associated first-integral or conserved current is given by

$$I_i = \mathcal{L} \theta_i + \sum_j \frac{\partial \mathcal{L}}{\partial u_{j,i}} \left( \eta_j - \sum_k u_{j,k} \theta_k \right) - f_i,$$  \hspace{1cm} (42)

which satisfies the conservation law

$$\sum_i \frac{d I_i}{dx_i} = 0.$$  \hspace{1cm} (43)
2.7 Reduction of order of an ODE

Let us consider an ODE \( x^{(k)} = f(t, x, x', \ldots, x^{(k-1)}) \) where \( x^{(k)} \) is the k-th derivative of \( x \) with respect to \( t \), admitting a symmetry generator \( G_1 \). Using the canonical coordinates \( r \) (dependent variable) and \( s \) (independent variables), we have \( G_1 = \partial / \partial r \), and consequently the original equation is cast in the form \( r^{(k)} = g(s, r', \ldots, r^{(k-1)}) \), for some function \( g \), which is an ODE of order \( k - 1 \) in \( u = r' \).

Now suppose the original equation admits another symmetry generator \( G_2 \). It can also be rewritten using the same canonical variables. If the extended generator \( G_2^{(k)} \) is such that it does not act on \( s \) directly but only on its derivatives, then it can be used for a further reduction of order. This procedure can then be iterated for any number of generators, provided that at each step the generator used acts only in the remaining derivatives. Of course if \( m = n \) the system can be completely solved. This reduction is possible iff the Lie algebra spanned by the \( k \) generators is solvable [39].

2.8 Equations admitting a symmetry group

In many situations the equations describing a given system are not known in closed form, but some of its symmetries are known. This is the case for instance if an underlying kinematical group (e.g. the Poincaré or Lorenz group) is imposed by the physics of the system. This is frequently the case for transport equations for which no complete general theory exists [40]. One may hope that using the knowledge of symmetries may determine a class of equations for the problem at hand. In this way, Let us consider a set of \( p \) equations on the unknowns \( u_i, i = 1, \ldots, n \), of the form:

\[
\sum_{j=1}^{n} \Delta_i^j u_j = 0, \quad i = 1, \ldots, p, \tag{44}
\]

where \( \Delta_i^j \) is a differential operator which can be non-linear. The system (44) defines a class of equations if the operators \( \Delta_i^j \) depend on unknown functions. Now we impose that eq. (44) is invariant under symmetry transformations generated by \( G_i; \ i = 1, \ldots, k \), forming a system of differential equations for the unknown functions in \( \Delta_i^j \), that can be solved in some cases. This is implemented in SADe in the routine equivalence. The main shortcoming here is that the system to be solved is non-linear.
3 Heuristics for solving the determining system

No proved fully general, finite and terminating algorithm exists for the solution of linear or non-linear over-determined systems of partial differential equations, of the form obtained as determining systems for Lie, Lie-Bäcklund and nonclassical symmetries. Here we present the heuristics used in the present package. For linear systems, it was able to efficiently solve all test cases, spanning a large number of equations ranging from the simple harmonic oscillator to equations describing coupled relativistic fields.

3.1 Over-determined system of linear partial differential equations

The basic idea is to solve simpler equations first, always trying to simplify further the system. Of course the meaning of “simpler” is quite subjective and our definition will become clear below. In a few cases it is necessary to append the determining system with integrability conditions for some, or all, of its equations. This is done using the MAPLE built-in routine rifsimp, when it is applicable, or otherwise using a slightly modified version of the Kolchin-Ritt algorithm with sorting [41]. The reduction to the involutive form (see [42,43] for a proper definition) is usually very expensive in computational time, and should be done only if the system cannot be solved otherwise, and after some preliminary simplifications. Another strategy is to reduce only a subsystem with equations containing up to a prescribed number of terms.

In what follows the number of terms in an equation is the number of its summands. Parameters controlling the flow of the solution algorithm are specified in global variables that can be modified by the user, and with default values given below. These are the main steps of our algorithm:

(1) Solve all algebraic equations with a maximum of 2 terms.
(2) Solve all differential equations of the form

\[
\frac{\partial^k f}{\partial x_{i_1} \ldots x_{i_k}} = 0, \tag{45}
\]

where \( f \) is any unknown in the determining system.
(3) Solve any algebraic equation in the original unknowns \( \theta_i \) and \( \eta_k \).
(4) Reduce to involutive form the subset of equations with at most \( N_1 \) terms.
(5) If any equation can be written as an expansion in linearly independent functions, then equate each coefficient to zero. Repeat until no more such decomposition is possible.
(6) Repeat step 2. If any equations is solved, go to step 10.
(7) Completely reduce to involutive form, and in case it succeeds, go to step 10.
(8) Solve all equations with at most $N_2$ terms that can integrated as ODE’s in one of the unknowns. If no equation can be solved, then repeat with $N_2 + 3$ terms, and so on up to the maximal value $N_3$. If any equation is solved, then go to step 10.
(9) Solve one ODE with any number of terms. If it succeeds, go to step 11.
(10) Repeat step 11.
(11) Look for all equations that are expansions in linearly independent functions, and equate to zero each coefficient of the expansion.
(12) Repeat step 11.
(13) Repeat step 2.
(14) Repeat step 8.
(15) Solve all algebraic equations (with any number of terms). If any equation is solved, then go to step 10.
(16) Reduce to the involutive form. If succeeds, go to step 10.
(17) Look for one equation of the form $f_1(x_1, x_2) = f_2(x_1, x_3)$ and replace it by $f_1(x_1, x_2) = f_3(x_1)$ and $f_2(x_1, x_3) = f_3(x_1)$, where $f_1$ and $f_2$ are two unknowns in the system and $f_3$ a new unknown. If succeeds go to step 10.
(18) Repeat step 8.

The whole algorithm is repeated until the system is completely solved or no additional simplification occurs. The following global variables are used:

$$N_1 = \text{SADE}[\text{partial\_reduction}]$$
$$N_2 = \text{SADE}[\_\text{ne}]$$
$$N_3 = \text{SADE}[\_\text{nmaxeq}]$$

with default values $N_1 = N_2 = 5$ and $N_3 = 8$.

3.2 Non-linear systems

Solving non-linear overdetermined systems of PDE’s, as those resulting from the determination of nonclassical symmetries, is a very difficult task. Its implementation in SADE is still under development, but can nevertheless be used in some interesting cases. There are other more efficient algorithms, such as the one used in the REDUCE package CRACK [13]. Our algorithm can be roughly sketched as:

(1) Solve all linear equations.
(2) Reduce the resulting system to the involutive form.
(3) Solve all linear equations.
(4) Decomposes equations which are expansions in linearly independent functions.
(5) Solve a single nonlinear ODE. If any equation is solved, repeat step 4 and go to step 1.
(6) Solve all purely algebraic equations, considering multiple solutions. If any equation is solved, then go to step 1.

There are also options for reducing the determining system to involutive form before trying to solve it, and to use the MAPLE builtin routine for solving PDE’s.

4 Package Commands

Here we briefly describe each command available in SADE. The examples given in section 5 should be self explanatory and complementary to this section. The following abbreviations are used for the input arguments:

- **eqs**: a single or a set of differential equations.
- **unks**: the unknowns in eqs.
- **gen**: a symmetry generator, written in SADE notation (see section 5).
- **depvars**: list of dependent variables.
- **indepvars**: list of independent variables.
- **vars**: list of dependent and independent variables.
- **der_order**: list with the derivative order of each dependent variable in the independent variables (see section 5.8).
- **drvs**: set with derivatives of unknowns in eqs.
- **subs_rule**: a substitution rule.
- **funcs**: a set with new undetermined functions.
- **name**: a maple variable name.
- **opt**: optional arguments.
- **determining**: optional argument to return only the determining system;
- **involutive**: optional argument to reduce the determining system to involutive form.
- **params**: a set on free parameters.

Package commands and corresponding inputs are given in the following listing:

- **liesymmetries(eqs,unks,opt)**: Computes Lie symmetry generators. Optional arguments: determining, involutive, case= n, n integer, only the case with \( \theta_n = 1, \theta_i = 0 \) \( i < n \) is considered. builtin - solves the determining system using the MAPLE builtin command pdsolve.
  - **default_parameters** - the determining system is solved using default parameters reducing CPU time, although the system may not be completely
solved.

**ncsymmetries(eqs, unks, opts):** Computes nonclassical symmetry generators of DE’s. Optional arguments: determining and involutive.

**LBsymmetries(eqs, unks, opts):** Obtains Lie-Bäcklund symmetry generators. Optional arguments: determining, involutive and parameter = paramset - computes the generators with conditions on the free parameters in paramset.

**lindsolve(eqs, unks):** Solves a linear overdetermined system of PDE’s.

**nonlindsolve(eqs, unks):** Solves a non-linear overdetermined system of partial differential equations.

**casimir_invariant({gen1, gen2, ...}, depvars, indepvars, der_order):** Computes the Casimir invariants of a set of generators.

**ansatz(subs_rule, funcs):** Applies a set of ansätze to the determining equations. This routine can be used either once the determining system is obtained or if SADE could not completely solve the determining system. The elements of subs_rule must be given in the form function = expression with new undetermined functions in expression specified in funcs.

**noether(lagrangian, funcs, gen):** Computes the Noether conserved currents or first-integrals from a lagrangian function.

**equivalence(eqs, {gen1, gen2, ...}, funcs):** Obtains the most generic form of a class of equations admitting a symmetry algebra.

**comm(gen1, gen2, vars):** Commutator of two linear operators (generators).

**com_table({gen1, gen2, ...}, vars, name):** Commutation table of a set of infinitesimal generators. name specifies a name to represent each generator.

**AdjointRep({gen1, gen2, ...}, vars, name, parameter):** Computes the table with the action of adjoint maps on each generator of a Lie Algebra. name is used to represent each generator in the table and parameter specifies the adjoint Lie group parameter.

**StructConst({gen1, gen2, ...}, vars):** Computes an array with the structure constants of a Lie algebra.

**linear_rep(operator, {gen1, gen2, ...}, vars, name):** Determines the most general linear operator representing a class of differential equations defined...
by operator admitting a symmetry algebra.

**PDEreduction(eqs, unks, \{gen_1, gen_2 \ldots\})**: Obtains the reduced form of a PDE or a PDE system from a set of symmetry generators. For a system with M independent variables K symmetry generators can be used to reduce to a new system depends with M − K (transformed) independent variables.

**invariant_sol(eqs, unks, \{gen_1, gen_2 \ldots\})**: Obtains invariant solutions of a PDE or a system of PDE’s using symmetry generators.

**issolvable({gen_1, gen_2 \ldots}, vars)**: Tests if a Lie algebra is solvable.

**canonical_basis({gen_1, gen_2 \ldots}, vars)**: Computes the canonical basis of a Lie algebra.

**derived_subalg({gen_1, gen_2 \ldots}, vars)**: Computes the generators of the derived subalgebra of a Lie algebra.

**odesolver(eqs, \{gen_1, gen_2 \ldots\}, unks, opt)**: Solves an ODE by successive reductions using a solvable Lie algebra. Optional argument: **transformation** - returns only the transformation of variables solving the system.

**reduce_ode_sist(eqs, \{gen_1, gen_2 \ldots\}, depvars_1, indepvar_1, depvars_2, indepvar_2)**: Reduces by one the dimension of a system of first order ODE’s using a symmetry generator. Note that new dependent and independent variables must be given and are represented by the index 2. Index 1 denotes original variables.

**ode_reduce_order1(eqs, gen, depvar_1, indepvar_1, depvar_2, indepvar_2)**: Reduces by one the order a a single ODE using a symmetry generator.

**ode_invsolution(eqs, unks, gen)**: Obtains invariant solutions for a single ODE.

**conserved(eqs, unks, params, n, opt)**: Obtains the QP-invariants of a QP first order system by computing a Darboux polynomial up to degree n. Optional arguments: **Groebner** - a Gröbner basis computation is used to solve the polynomial system of determining equations. **positive** - the results are simplified to the positive orthant. **surfaces** - returns the defining equations for invariant hyper-surfaces.

**QPsymmetries(eqs, unks, params, n)**: Determines QP symmetry generators with n the degree of the polynomial quasi-symmetry [24].
verif_if_inv(eqs,unks,params): Determines parameter values such that non-trivial QP first-integrals (i.e. with non-integer exponents) may exist.

constlog(eqs,unks,params,n): Computes first-integrals of the form $P_1(x) + \log(x^\xi)$ and $P_2(x, \ln(x))$, with $P_1$ and $P_2$ polynomials of degree $n$ and $x$ stands for all dependent variables.

5 Illustrative examples

We present some illustrative cases of a basic use of SADE. A whole suite of examples, including more complex problems, is given with the distribution files.

5.1 Lie symmetries

Let us consider first as a simple example the heat diffusion equation: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. The input to determine its Lie symmetries and the corresponding output are:

```plaintext
> g := liesymmetries({diff(u(x,t),t) = diff(u(x,t),x,x)},{u(x,t)});
```

\[
g := \left[ \left\{ \frac{\partial^2}{\partial x^2} F_1(t, x) - \frac{\partial}{\partial t} F_1(t, x) \right\}, \left\{ D_t, D_x, uD_u, -F_1(t, x) D_u, 2t D_t + x D_x, u x D_u - 2t D_x, t^2 D_t + 1/4 \left( -2u t - u x^2 \right) D_u + t x D_x \right\} \right]\]

The first element of the list in the output is the set of infinitesimal symmetry generators. The second element is a list of constraints on these generators, which in this case is the original equation. The element $D[\alpha] \rightarrow D_\alpha$ in the output stands for the directional derivative $\partial/\partial \alpha$, where $\alpha$ is any of the dependent or independent variables. The commutator of two generators is computed as:

```plaintext
> comm(D[x], -1/2 * u * x * D[u] + t * D[x], [u(x,t)]);
```

\[
-\frac{1}{2} u D_u
\]

The complete commutation relations for the finite dimensional algebra can be computed as follows:

```plaintext
> gen := convert(g[1] minus {_F1(t,x) * D[u]},{list});
```
5.2 Lie-Bäcklund symmetries

To illustrate how dependencies on derivatives of dependent variables are handled when computing Lie-Bäcklund symmetries, we consider the following two-dimensional PDE system [9]:

\[
\begin{align*}
\frac{\partial^2 u_2}{\partial x_1^2} &= \frac{1}{2} \frac{\partial u_2}{\partial x_2} + \frac{\partial^2 u_1}{\partial x_1^2} = \frac{\partial u_1}{\partial x_2} - \frac{u_2}{2}, \\
\frac{\partial^2 u_2}{\partial x_2^2} &= \frac{1}{2} \left( \frac{\partial}{\partial x_2} F_1(x_1, x_2) + 2 \frac{\partial}{\partial x_2} F_1(x_1, x_2) u_2 \right) D_{u_1}, \\
\frac{\partial^2 u_2}{\partial x_1 \partial x_2} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} F_1(x_1, x_2) + 2 \frac{\partial}{\partial x_1} F_1(x_1, x_2) u_2 \right) D_{u_2}, \\
\end{align*}
\]

(46)

Requiring, for instance, that the evolutionary form of the symmetry generators depends on \( \frac{\partial u_1}{\partial x_1}, \frac{\partial^2 u_1}{\partial x_1^2}, \frac{\partial u_2}{\partial x_1} \) and \( \frac{\partial^3 u_2}{\partial x_2}, \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \), implies that the independent derivatives are \( \frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2} \) and \( \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \). Here are the corresponding input and output:

> LBsymmetries(eqs, [u1(x1, x2), u2(x1, x2)],
> {diff(u1(x1, x2), x1), diff(u1(x1, x2), x1^2),
> diff(u2(x1, x2), x1),
> diff(u2(x1, x2), x1^3)});

\[
\begin{bmatrix}
0 & 0 & 0 & 2 \ G1 & -2 \ G2 & -1/2 \ G3 + \ G4 \\
0 & 0 & 0 & \ G2 & \ G3 & -1/2 \ G5 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-2 \ G1 & -\ G2 & 0 & 0 & \ G5 & 2 \ G6 \\
2 \ G2 & -\ G3 & 0 & -\ G5 & 0 & 0 \\
1/2 \ G3 - \ G4 & 1/2 \ G5 & 0 & -2 \ G6 & 0 & 0 \\
\end{bmatrix}
\]
\[ \left\{ 2 \frac{\partial^2}{\partial x_1^2} F_1(x_1, x_2) - \frac{\partial}{\partial x_2} F_1(x_1, x_2) \right\}, \]
\[ \left\{ \frac{\partial}{\partial x_1} u_1, \frac{\partial}{\partial x_2} u_1, \frac{\partial}{\partial x_1} u_2, \frac{\partial^2}{\partial x_1 \partial x_2} u_2 \right\} \]

The first element is a set with the Lie-Bäcklund symmetry generators, the second element a set with constraints on the generators (an empty set if none), and the third element the set of independent derivatives.

### 5.3 Solving an ODE

Let us consider the equation [8]:

\[ \left( \frac{d u}{d x} \right)^5 \frac{d^3 u}{d x^3} - 3 \left( \frac{d u}{d x} \right)^4 \left( \frac{d^2 u}{d x^2} \right)^2 - \left( \frac{d^2 u}{d x^2} \right)^3 = 0, \] (47)

with \( u = u(x) \). First compute its Lie symmetries:

\[ > \text{eq} := \text{diff}(u(x), x)^5 \text{diff}(u(x), x, x, x) - \text{diff}(u(x), x, x) \cdot 3 \]
\[ - 3 \text{diff}(u(x), x)^4 \text{diff}(u(x), x, x)^2 : \]
\[ \text{liesymmetries(eq, [u(x)]);} \]
\[ \left\{ \{D_u, D_x, uD_x, uD_u + \frac{3}{2} x D_x \}, \{ \} \right\} \]

then chose a solvable subalgebra:

\[ > \text{ls} := \{D_u, D_x, uD_x\} : \]
\[ \text{issolvable(ls, [u, x]);} \]

\[ \text{true} \]

Here are the inputs for computing the solutions and displaying the first one (there are three branches):

\[ > \text{sol} := \text{odesolver(eq, \{D[u], D[x], u * D[x]\}, [u(x)]);} \]
\[ > \text{sol}[1]; \]
\[ u(x) = \frac{1}{2} \_C1 - \frac{1}{2} (\_C2 + 6 \_C3) - \_C2^3 + 2 (36 x^2 - 36 x \_C2 \_C1 + 72 x \_C3 + 6 x \_C2^3 + 9 \_C2^2 \_C1^2 - 36 \_C2 \_C1 \_C3 - 3 \_C2^4 \_C1 + 36 \_C3^2 \]
\[
+6 (C3 - C2^3)^{1/3} + 1/2 C2^2 / ((-12 x + 6 C2 - C1 - 12 C3 \\
- 2 (36 x^2 - 36 x C2 - C1 + 72 x C3 + 6 x C2^3 \\
+ 9 C2^2 C1^2 - 36 C2 C1 C3 - 3 C2^4 C1 + 36 C3^2 \\
+ 6 C3 - C2^3)^{1/3})^{1/2} C2^2)
\]

### 5.4 Invariant solutions

The Burgers equation in one dimension can be rewritten as a set of two one-dimensional equations by defining \( v = \partial u / \partial x \):

\[
\frac{\partial u}{\partial x} - v = 0; \quad \frac{\partial u}{\partial t} + uv - \frac{\partial v}{\partial x} = 0.
\]

Since the number of independent variables is two, only a single symmetry generator is necessary to obtain a group invariant solution. In this form, one of the symmetry generators admitted by the Burgers equation is:

\[
G = (ut - x) D_u + (2 vt - 1) D_v - t^2 D_t - tx D_x.
\]

The associated invariant solution is obtained using the following input in SAD:

\[
> \text{invariant\_sol(eq, [u(x, t), v(x, t)], \{(u \ast t - x) \ast D[u] \\
+ (2 \ast v \ast t - 1) \ast D[v] - t^2 \ast D[t] - t \ast x \ast D[x]\});
\]

\[
\{\{u(x, t) = -(- C1 x + \tanh(1/2 x + \frac{C2 t}{C1 t})) \ast C1^{-1} t^{-1}, \\
v(x, t) = 1/2 (2 t (\cosh(1/2 x + \frac{C2 t}{C1 t}))^2 - C1^2 - 1) t^{-2} \\
\cosh(1/2 x + \frac{C2 t}{C1 t}))^{-2} \ast C1^{-2}\}\}
\]

### 5.5 PDE reduction

Let us consider a massless nonlinear Klein-Gordon equation with a \( \lambda \phi^4/4 \) self-interaction potential:
\[
\begin{align*}
\mathbf{p} & := \phi(x,y,z,t) : \\
eq & := \text{diff}(p,x,x) + \text{diff}(p,y,y) + \text{diff}(p,z,z) - \text{diff}(p,t,t) + \lambda \ast p^3 :
\end{align*}
\]

This equation can be reduced to a PDE with two independent variables using the symmetry generators:

\[
\mathbf{G}_1 = yD_t + tD_y, \quad \mathbf{G}_2 = zD_x - xD_z,
\]

(50)
a Lorenz boost in the \( y \) direction and a spatial rotation around the \( y \) axis, respectively. The reduced equation is obtained using the SAD command:

\[
> \text{PDEreduction}(eq, [\phi(x,y,z,t)], yD_t + tD_y, zD_x - xD_z);
\]

\[
\begin{align*}
\{ & \{ \phi(x,y,z,t) = -F1(x^2 + z^2, -y^2 + t^2) \}, \\
& \{ 4 \left( \frac{\partial^2}{\partial \xi_1^2} - F1(\xi_1,\xi_2) \right) \xi_1 + 4 \frac{\partial}{\partial \xi_1} F1(\xi_1,\xi_2) - 4 \frac{\partial}{\partial \xi_2} F1(\xi_1,\xi_2) \\
& - 4 \left( \frac{\partial^2}{\partial \xi_2^2} - F1(\xi_1,\xi_2) \right) \xi_2 + \lambda \left( -F1(\xi_1,\xi_2) \right)^3 \}, \\
& \{ \xi_1 = x^2 + z^2, \xi_2 = -y^2 + t^2 \} \}
\end{align*}
\]

The output is a set with the different possible reductions (one in the present case). Each element is a list: the first element defines the relation between the original unknown(s) and the reduced dependent variable(s). The second element is the set of reduced equations, and the last element is a set with the similarity variables \( \xi_i \) used to reduce the original equation(s).

5.6 Nöther theorem

As an example, we consider the relativistic massless scalar field in 1+1 dimensions, with lagrangian density:

\[
\mathcal{L} = \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0.
\]

(51)

In this case a conserved current \( \mathbf{I} \) is a two-vector function of \( \phi, x \) and \( t \) such that \( \partial I_1/\partial x + \partial I_2/\partial t = 0 \). They can be obtained in the following way:

\[
> \text{noether(diff}(\phi(x,t),x)^2/2 - \text{diff}(\phi(x,t),t)^2/2, [\phi(x,t)]);
\]
\[
\left\{ \left[ 1/2 \left( \frac{\partial}{\partial x} \varphi \right)^2 + 1/2 \left( \frac{\partial}{\partial t} \varphi \right)^2, - \left( \frac{\partial}{\partial t} \varphi \right) \frac{\partial}{\partial x} \varphi \right], \left[ -2 \frac{\partial}{\partial t} \varphi, 2 \frac{\partial}{\partial x} \varphi \right] \right\}, [t, x]
\]

Here the output is a sequence. The first element is the set of conserved currents, and the second element the independent variables defining the component ordering used for the components of \( I \).

### 5.7 Equations admitting a symmetry algebra

Let us consider the following family of PDE's:

\[
\frac{\partial u}{\partial t} + \frac{\partial(f u)}{\partial x} + \frac{\partial^2(du^2)}{\partial x^2} = 0,
\]

where \( f = f(x, t) \) and \( d = d(x, t) \) are functions to be determined. Requiring that eq. (52) admits \( G_1 = \partial/\partial t, G_2 = \partial/\partial x \) and \( G_3 = (x/2)\partial/\partial x + t\partial/\partial t \) as symmetry generators restricts the postulated generic form (52):

\[
> \text{gen} := \{ D[t], D[x], (1/2) \ast x \ast D[x] + t \ast D[t] \} ;
\]

\[
> \text{eq} := \text{diff}(u(x, t), t) + \text{diff}(f(x) \ast u(x, t), x) + \text{diff}(d(x) \ast u(x, t)^2, x, x) ;
\]

\[
> \text{equivalence(eq, [u(x, t)], gen, \{f(x, t), d(x, t)\})};
\]

\[
\left\{ \left[ \frac{\partial}{\partial t} u(x, t) + 2 - C1 \left( \frac{\partial}{\partial x} u(x, t) \right)^2 + 2 - C1 \ u(x, t) \frac{\partial^2}{\partial x^2} u(x, t), \{\} \right], \left[ \frac{\partial}{\partial t} u(x, t) + (\frac{d}{dx} f(x)) u(x, t) + f(x) \frac{\partial}{\partial x} u(x, t), \{\} \right] \right\}
\]

The output is a set of lists. In each list, the first element is a restricted form for the class of equations, the second element is the set of remaining equations still to be solved, none in this case.

### 5.8 Casimir invariants

The order of the derivatives in a Casimir invariant is specified by a list such that each element is a list with the maximum derivative order of the corresponding dependent variable in each independent variable. Note that the order of the derivatives in the invariants are one order higher than specified in the input (this is due to the specific algorithm used in the computation):
Table 1
Benchmarks for Lie Symmetry determination.

| System                        | \(N_{eq}\) | SADE | DESOLV | PDEtools |
|-------------------------------|------------|------|--------|----------|
| Heat equation [10]            | 9          | 0.4  | 0.13   | 0.4      |
| Klein-Gordon [45]             | 32         | 1.1  | 0.69   | 0.9      |
| Magneto-Hydro-Dynamics [45]   | 39         | 14.4 | 12.9   | 13.4     |
| Navier-Stokes [6]             | 125        | 3.9  | 4.8    | 2.9      |
| Dirac [9]                     | 352        | 14.7 | 20.1   | —        |
| Gross-Neveu [46]              | 352        | 15.4 | 211    | —        |
| Maxwell-Dirac [9]             | 2,621      | 27.8 | 1,121. | —        |
| Yang-Mills SU(2) [47]         | 12,361     | 82   | 188    | —        |
| Yang-Mills SU(3) [48]         | 175,042    | 13,847 | —    | —        |

\[ \text{> casimir\_invariant}(x \ast D[t] + t \ast D[x], v \ast D[v] + u \ast D[u] - x \ast D[x], [u], [x, t], [[2, 2], [2, 2]]); \]

\[
\begin{aligned}
&\left[ -\left( \frac{\partial^2}{\partial x^2} u(x, t) \right) \left( \frac{\partial^2}{\partial t^2} u(x, t) \right) - \left( \frac{\partial^2}{\partial x \partial t} u(x, t) \right)^2 \right] u(x, t)^4,
&\frac{\partial}{\partial x} u(x, t), \frac{\partial}{\partial t} u(x, t), \frac{\partial^3}{\partial x^3} u(x, t), \frac{\partial^3}{\partial t^3} u(x, t),
&2 x t \left( \frac{\partial^2}{\partial x \partial t} u(x, t) \right) + x^2 \left( \frac{\partial^2}{\partial x^2} u(x, t) \right) + t^2 \left( \frac{\partial^2}{\partial t^2} u(x, t) \right) \right] u(x, t)\
\end{aligned}
\]

6 Lie Symmetries Benchmarks

It is beyond the scope of the present work to do an exhaustive comparison of SADE to other similar packages (see References [22,23] for an assessment for previous packages). We present only a comparison for the computation of Lie symmetries (the core of SADE), and leave a more thorough comparison to be presented in a future work. Since the release 11 of MAPLE it became possible to compute Lie symmetries using its native package PDEtools. The package GEM relies on the solution of the determining system using MAPLE built-in routines, leading to large computational effort in CPU time and memory when solving the determining system, becoming intractable for systems such as Maxwell-Dirac or SU(2) and SU(3) Yang-Mills field equations. In this way, and considering only MAPLE packages, table 1 shows CPU times for the
determination of Lie symmetries (solving the determining system and obtain
the symmetry generators in explicit form) of some representative systems us-
using SADE, DESOLV and PDEtools. All computations were performed on a
i5 2.40 GHz computer, and using the corresponding automated routines for
computing Lie symmetries in each package: symmetry and genvec in DESOLV,
liesymmetries in SADE and Infinitesimals in PDEtools. The absence of a value
in the table means that either the computation was not completed after a very
long time, or that the package was unable to obtain all symmetries of the equa-
tion. From table 1 we see that PDEtools handles only simpler systems (the
three first lines in the table), with DESOLV being the fastest and PDEtools
the slowest. Nevertheless, for the remaining cases in table 1 with a number
of equations in the determining system ranging from a few hundreds to hun-
dreds of thousands, SADE performs better than DESOLV. For the Yang-Mills
$SU(3)$ field equations, SADE took a little less than 6 hours while DESOLV
produced no output after more than 60 hours. On its turn, PDEtools missed
some symmetries of the Navier-Stokes and all of the Dirac equations, and for
the remaining equations was unable to terminate the computations after a
considerable long time. We used MAPLE 14 for timings, except for DESOLV
which performs better in MALE 13 (for the latest version available to the
authors).

7 Concluding Remarks and Perspectives

The present package implements symmetry methods for differential equations
in MAPLE, including Lie, nonclassical, Lie-Bäcklund and potential symme-
tries, the Quasi-Polynomial formalism and the computation of invariant solu-
tions and reduction of ODE’s and PDE’s. SADE is well suited to handle more
complicated systems, such as Maxwell-Dirac and Yang-Mills $SU(2)$ and $SU(3)$
equations, for which optimization is crucial. It also obtains all symmetries for
the “difficult” Vaidya and Jacob-Jones systems [22]. We performed no direct
performance comparisons besides those for Lie symmetries determination. The
implementation of Markus algebras [34,25] as a tool for the computation of
quasi-polynomial first-integrals, symmetries and invariant hyper-surfaces, and
a better algorithm for solving overdetermined systems of non-linear PDE’s are
currently under implementation. We also hope to implement a more complete
set of routines for the computation of conservation laws for PDE’s.
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