Optimal Weight Allocation of Dynamic Distribution Networks and Positive Semi-definiteness of Signed Laplacians

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Abstract—In this paper, we consider the robustness of a basic model of a dynamical distribution network. As the first problem, i.e., optimal weight allocation, we minimize the $\mathcal{H}_\infty$ norm of the dynamical distribution network subject to allocation of the weights on the edges. It is shown that this optimization problem can be formulated as a semi-definite program. Next we consider the semi-definiteness of the weighted graph Laplacian matrix with negative weights on the edges. A necessary and sufficient condition, using the effective resistance matrix, is established to guarantee the positive semi-definiteness of the Laplacian matrix. Furthermore, the bounded real lemma is derived for state-space symmetric systems.

Index Terms—Network Analysis and Control, $\mathcal{H}_\infty$ control, optimization, signed Laplacian.

I. INTRODUCTION

Modern societies critically rely on distribution networks of various kinds. Typically, a distribution network is depicted as a graph where resources can enter the network via supply vertices and leave the network via demand vertices, together with edges that connect the supply, demand and additional internal vertices. Often, flow capacity constraints and cost functions are assigned to the edges.

Distribution networks can be divided into two classes, depending on whether the vertices can store resources or not. If the vertices can only distribute resources without storage, we refer to this type of distribution networks as static. The study of static distribution networks is a broad research topic which has a long history and a large number of applications [3]. One celebrated result is the max-flow min-cut theorem [15]. The static distribution problem is closely related to monotropic programming problems which enjoy a complete and symmetric duality theory [26].

Differently from static distribution networks, in dynamical distribution networks vertices have storage of resources. This type of models has many applications in, e.g., communication networks [14], transportation networks [4], [11], [19], [20], hydraulic networks [27], flow networks [12], [17], and inventory and production systems [5], [6].

In this paper, we analyze the robustness of a basic dynamical distribution networks where we assign a set of single integrators to the vertices (with state variables corresponding to storage). All the integrators are controlled by the flows on the edges. On each edge, the flow is the weighted storage difference of the adjacent vertices. Furthermore, unknown in/outflows may enter or leave the network through some of the vertices. The aim here is to minimize the induced $\mathcal{L}_2$ gain from the in/outflows to the output of the network by allocating the weights on the edges, which will be called optimal weight allocation problem in this paper. The results of this problem can be relevant when designing robust multi-agent systems. Especially, our setup is similar to the setting in [24], when one considers the in/outflows as malicious attacks whose goal is to maximize the differences of the storages of the vertices. Then by solving the optimal weight allocation problem, the effect of the worst attack will be minimized. The distribution networks considered in this paper can be seen as linear time-invariant port-Hamiltonian systems [1], but also resides in the category of state-space symmetric systems [21], [23], [31], [34]. One useful property of the state-space symmetric system is that its $\mathcal{H}_\infty$-norm is attained at the zero frequency [29].

One closely related problem to the optimal weight allocation, where the connection will be clear in the primary part of the paper, is the positive semi-definiteness of weighted Laplacian with both negative and positive weights. This problem is of salient importance in distributed algorithms [2], [32], [33]. This problem was considered by many authors. In [35], the authors provided one sufficient and necessary condition, using effective resistance, for a weighted graph, namely those where the negatively weighted edges are isolated in different cycles in the graphs spanned by the positive edges. Under the same assumption, the authors of [10] re-derived the result in [35] by using geometrical and passivity-based approaches. For general weighted graphs, one sufficient and necessary condition was proposed in [8], [9] using pseudo-inverse of weighted (with negative ones) Laplacian. Here we propose a sufficient and necessary condition using the effective resistance matrix of the positive subgraph from $\mathcal{H}_\infty$ approach.

The contributions of this paper are listed as follows. First, we derive a bounded real lemma type of result for state-space symmetric systems. Second, the problem of minimizing the $\mathcal{H}_\infty$-norm of the dynamical distribution networks subject to the allocation of the flow capacities is formulated as a semi-definite program. Third, we present a necessary and sufficient condition of positive semi-definiteness of weighted Laplacians,
with negative weights, i.e., signed Laplacians.

The structure of the paper is as follows. Some preliminaries will be given in Section II. The considered class of dynamic distribution networks and the corresponding weights allocation problem, and the problem of positive semi-definiteness of weighted Laplacian are formulated in Section III. The main results are presented in Section IV and V. Conclusions and future work are given in Section VI.

The notations used in the current paper are collected as follows.

**Notation.** A positive definite (positive semi-definite) matrix $M$ is denoted as $M \succeq 0$ ($M \succeq 0$). The element on the $i$th row and $j$th column of a matrix $M$ is denoted as $M_{ij}$. The pseudo-inverse of $M$ is $M^\dagger$. Recall that, for any finite dimensional square matrix $M$, the induced $\ell_2$ norm, denoted by $\|M\|_2$, is the largest singular value which is denoted by $\sigma(M)$. The image of a matrix $M$ is $\text{im} M$. The identity matrix is denoted as $I$. The vectors $1_n$ represents a $n$-dimensional column vector with each entry being 1. We will omit the subscript $n$ when no confusion arises. The Euclidean norm of a vector $x$ is denoted as $\|x\|_2$. Given a set $S$, $\text{int}\{S\}$ denotes its interior.

II. PRELIMINARIES

In this section, we briefly review some essentials about graph theory [7] and robust analysis [36].

A. Graph Theory

An undirected graph $G = (\mathcal{V}, \mathcal{E})$ consists of a finite set of vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$, a set of edges $\mathcal{E} = \{E_1, \ldots, E_m\}$ that contains unordered pairs of elements of $\mathcal{V}$, and a set of corresponding edge weights $\mathcal{W} = \{w_1, \ldots, w_m\}$. The set of neighbours to vertex $i$ is $N_i = \{v_j \mid (v_i, v_j) \in \mathcal{E}\}$.

The graph Laplacian $L \in \mathbb{R}^{n \times n}$ is defined component-wise as

$$L_{w,ij} = \begin{cases} w_{ij} & \text{if } i = j, \\ -w_{ij} & \text{if } j \in N_i \setminus \{i\}, \\ 0 & \text{if } j \notin N_i, \end{cases}$$

where both positive and negative weights are allowed. Given an arbitrary orientation for each edge, the incidence matrix $B \in \mathbb{R}^{n \times n}$ is defined as

$$B_{ij} = \begin{cases} 1 & \text{if } E_j \text{ starts in vertex } v_i, \\ -1 & \text{if } E_j \text{ ends in vertex } v_i, \\ 0 & \text{else.} \end{cases}$$

These two matrices are related by $L_w = BWB^T$, where $W = \text{diag}(w_1, \ldots, w_m)$. If $W \succ 0$, i.e., there are only positive edges, then it is well-known that the eigenvalues of $L_w$ can be structured as $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, where the eigenvector corresponding to $\lambda_1 = 0$ is $1$. If $W = I$, the Laplacian is denoted without subscript as $L$.

If a graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ has both positive and negative weights, we separate the edges set $\mathcal{E}$ into $\mathcal{E}_+$ and $\mathcal{E}_-$, which contains the positive and negative edges, respectively. Thus $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+$. Accordingly, the weight matrix $W = W_+ - W_-$, where $W_+$ ($W_-$) is the absolute value of the weights corresponding the positive (negative) edges. The Laplacian matrix is referred to as signed Laplacian which can be decomposed as

$$L_w = L_{w_+} - L_{w_-} := B_+W_+B_+^T - B_-W_-B_-^T,$$

where $B_+$ and $B_-$ are incidence matrices corresponding to the positive and negative sub-graphs, respectively.

The undirected and connected graph without self-loops and with only positive weights can be associated with electrical networks [18]. One important concept is the effective resistance matrix, see e.g., [13], [18], which is defined as

$$\Gamma = B^T L_w B,$$

where $L_w^\dagger$ is the Moore-Penrose pseudo-inverse of $L_w$ and $B$ is the incidence matrix.

B. $\mathcal{L}_2$-Norm and induced $\mathcal{L}_2$-Gain

In this subsection, we recall some definitions from robust control. The notations used in this paper are fairly standard and are consistent with [36], [25]. The space of square-integrable signals $f : [0, \infty) \to \mathbb{R}^n$ is denoted by $\mathcal{L}_2[0, \infty)$. For the linear time-invariant system

$$\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx + Du,
\end{align*}$$

the transfer matrix is $G(s) = C(sI - A)^{-1}B + D$, which has the impulse response

$$g(t) = \mathcal{L}^{-1}(G(s)) = C e^{At}B 1_+(t) + D \delta(t),$$

where $\delta(t)$ is the unit impulse and $1_+(t)$ is the unit step defined as

$$1_+(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

If $x(0) = 0$, then we have $y(t) = \int_0^t g(t - \tau)u(\tau)d\tau$. Then the induced $\mathcal{L}_2$-gain is defined as

$$\|g\|_{2-\text{ind}} = \sup_{u \in \mathcal{L}_2[0, \infty)} \frac{\|y\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2[0, \infty)} \frac{\|g * u\|_2}{\|u\|_2},$$

where $\|u\|_2 = \left( \int_0^\infty |u(t)|^2 dt \right)^{1/2}$.

This induced $\mathcal{L}_2$-gain, i.e., $\|g\|_{2-\text{ind}}$ or $\|G\|_{2-\text{ind}}$, is often called the $\mathcal{H}_\infty$-norm, denoted as $\|G\|_{\infty}$. It is well-known that for stable systems we have that $\|G\|_{\infty} = \sup_{\omega \in \mathbb{R}} \sigma\{G(j\omega)\}$, where $\sigma(A)$ denotes the largest singular value of the matrix $A$.

If the matrices in (1) satisfy $A = A^T$, $D = D^T$ and $C = B^T$, the system is referred to as state-space symmetric system [29]. Here, we present a bounded real lemma type of result with respect to state-space symmetric system. Notice that for internally positive LTI system, the bounded real lemma was established in [30].

**Lemma 1.** Consider any state-space symmetric system (1) with $A \prec 0$ and $D = 0$. Then the following conditions are equivalent,
1) $\|G\|_{\infty} \leq \gamma$.

2) the inequality

$$PA + AP + BB^T + \frac{1}{\gamma^2} PBB^T P \preceq 0 \quad (2)$$

has a solution $P = \gamma I$.

The proof is given in Appendix.

**Remark 1.** In the previous lemma, we gave an explicit solution for the Riccati inequality (2) where the bounded real lemma can only guarantee the existence of the solutions.

### III. PROBLEMS FORMULATION

In this paper, we first consider the weight allocation problem in the scenario of dynamical distribution networks, which is defined on a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$. Consider the dynamic model

$$\dot{x}(t) = Bu(t) + Ed(t), \quad (3)$$

where $B$ is the incidence matrix of the graph $G$, $x \in \mathbb{R}^n$ is the system state whose components represent the storage levels in the vertices, $u \in \mathbb{R}^m$ is the controlled flows on the edges, $E \in \mathbb{R}^{n \times k}$ is an assigned matrix and $d(t) \in \mathbb{R}^k$ is an unknown external in/outflows. Here we assume that the image of $E$ is a subset of the image of $B$, i.e., the inflow is equal to the outflow. To simplify the composition, we further assume that, for all $i = 1, \ldots, k$, the $i$th column of $B$ consists of one element which is $\alpha_i > 0$ (inflow) and one element $-\alpha_i$ (outflow), while the rest of the elements are zero. Without specification, we set $\alpha_i = 1$. A port is a set of vertices (terminals) to where the external flows which enter and leave the network sum to zero. Thus, $E$ defines $k$ ports. One example of system (3) is depicted as Fig. 1.

The condition $\text{im} E \subset \text{im} B$ is a standard assumption, in order to have a stable distribution network, for example in [6] which is recalled in the following remark.

**Remark 2.** In [6], the authors considered a distribution network with constraints on the storages, flows and external in/outflows as

$$x(t) \in \mathcal{X} := \{x \in \mathbb{R}^n \mid x^- \leq x \leq x^+\}$$

$$u(t) \in \mathcal{U} := \{u \in \mathbb{R}^m \mid u^- \leq u \leq u^+\}$$

$$d(t) \in \mathcal{D} := \{d \in \mathbb{R}^k \mid d^- \leq d \leq d^+\}$$

where $x^+, x^-, u^+, u^-, d^+, d^-$ are assigned vectors and the inequalities hold component-wisely. First, it was proved that the existance of a state-feedback control $u(t) \in \mathcal{U}$ and a set of initial conditions $X_0 \subset \mathcal{X}$, such that for every $x(0) \in X_0$, the solutions of (3) satisfy

$$x(t) \in \mathcal{X}, \quad \forall d(t) \in \mathcal{D}, \quad t \geq 0$$

if and only if $ED \subset -BU$.

It can be seen that one necessary condition to have $ED \subset -BU$ and $ED \subset \text{int}\{BU\}$ is $\mathbb{1}^T E = 0$ which implies that the image of $E$ is a subset of the image of $B$ for connected graphs.

![Fig. 1: Distribution network (3) on the graph. The state $x_i$ is the storage at the vertex $v_i$. The flows on the corresponding edges are denoted as $u_i$. The orientations on the edges are consistent with the incidence matrix. The vertices $v_5$ has inflow $d_1$ and $d_2$, the vertices $v_4$ and $v_3$ have outflow $d_1$ and $d_2$, respectively. In this case, $E \in \mathbb{R}^{5 \times 2}$ whose first and second column are $[0, 0, 0, -1, 1]^\top$ and $[0, 0, -1, 0, 1]^\top$, respectively.](image)

In this paper, we consider the flows on the edges are proportional to the state differences of the adjacent vertices. More precisely, the flows are given as

$$u = WB^T x, \quad (4)$$

where the diagonal matrix $W \in \mathbb{R}^{m \times m}$ is the control gain. The output $y \in \mathbb{R}^k$, which measures the state difference at each port, is given as

$$y = E^T x. \quad (5)$$

This form of the output can be due to the physical constraints of the distribution network, i.e., only the state differences at the ports can be measured. Furthermore, for SISO dynamical distribution networks defined on some special graphs, it can be shown that the induced $L_2$ gain from $d$ to $y$ in (5) is the largest among all $y = C^T x$ with $C \in \mathbb{R}^{1 \times n}$ and $C \mathbb{1} = 0$. See Corollary 4 in appendix for details. Now the closed-loop is,

$$\dot{x} = -L_w x + Ed,$$

$$y = E^T x. \quad (6)$$

where $L_w$ is the graph Laplacian of $G = (W, V, E)$ and $W$ is the set of weights specifying by control gain $W$ in (4).

We are ready to introduce two problems which we shall tackle in this paper.

**Optimal Weight Allocation:** For a given graph and a
positive constant $c$,
\[
\min_W \|G\|_\infty \quad \text{s.t., } \sum w_i = c, \quad w_i \geq 0,
\]
where $G$ is the transfer function (from $d$ to $y$) of the system (6), $W = \text{diag}(w_1, \ldots, w_m)$ and $w_i$, for $i = 1, \ldots, m$, are the weights on the edges, and $c$ is a positive constant.

**Positive semi-definite Laplacian:** Given a weighted graph $G$ with both positive and negative edge weights, what are the upper bounds on the magnitudes of the negative weights in order to have the Laplacian to be positive semi-definite?

The following two sections are devoted to these two problems, respectively. Before proceeding, this section is closed with following physical interpretation of distribution networks.

**Example 1.** One physical interpretation of the system (6) is a basic model of a dynamic flow network, where there are water reservoirs on the vertices and pipes on the edges. The reservoirs are identical cylinders and the pipes are horizontal. The state $x$ is constituted by the water levels in the reservoirs and the pressures are proportional to the water levels. The flow in the pipes is passively driven by pressure difference between the reservoirs. The weights $W$ are representing the capacities of the pipes, in terms of diameter and friction. The passive flow from reservoir $i$ to reservoir $j$ is then $q_{ij} = w_{ij}(x_i - x_j)$. The external input $d$ can e.g. be interpreted as flow into pumps which are distributing water inside the network. The output $y$ is then the difference between water levels of the reservoirs which the pumps are pumping to and the reservoirs which the pumps are pumping from.

**IV. $H_{\infty}$-NORM OF THE DISTRIBUTION NETWORK**

In this section, we shall solve the optimal weight allocation problem by reformulating problem (7) as an equivalent optimization problem with LMI constraints, which can then be efficiently solved numerically using, e.g., CVX [16]. The main result of this section is presented as follows.

**Theorem 2.** Consider the system (6), where $G$ is an undirected graph and each port belongs to exactly one connected component of $G$. Suppose $L_w \succeq 0$. Then
1) the $H_{\infty}$-norm of (6) is finite, and
2) the following statements are equivalent:
   - the $H_{\infty}$-norm is less than or equal to $\gamma$.
   - the following LMI is satisfied,
     \[
     \begin{bmatrix}
     L_w & E \\
     E^\top & \gamma I_k
     \end{bmatrix} \succeq 0. \tag{8}
     \]

**Proof.** We show that the theorem is true for the case where $G$ has exactly one connected component. This is without loss of generality since if $G$ has more than one connected component, the same procedure can be done for each component and the LMIs can be merged with a common $\gamma$. Denote
\[
U^T = \begin{bmatrix}
\frac{1}{\sqrt{n}} 1_n, u_2^T, \ldots, u_n^T
\end{bmatrix}
\]
and $U_2^T = [u_2^T, \ldots, u_n^T]$, for which $UL_wU^\top = \text{diag}(0, \lambda_2, \ldots, \lambda_n) =: \Lambda$. Denote $\hat{\Lambda} = \text{diag}(\lambda_2, \ldots, \lambda_n)$. Then the system (6) has equal $H_{\infty}$-norm as the system
\[
\dot{x} = -\hat{\Lambda}x + UEd, \\
z = E^\top U_2^\top \hat{x}.
\]

Notice that the first row of $UE$ is zero, thus the $H_{\infty}$-norm of the system (6) equals the $H_{\infty}$-norm of the system
\[
\dot{x} = -\hat{\Lambda}x + U_2Ed, \\
z = E^\top U_2^\top \hat{x}. \tag{10}
\]

Due to the symmetry of the system and by Theorem 6 in [29], the $H_{\infty}$-norm of the system (10) is $\|E^\top U_2^\top \hat{\Lambda}^{-1} U_2E\|_2$, which is finite. The $H_{\infty}$-norm of the system (6) is then less than or equal to $\gamma$ if and only if
\[
\|E^\top U_2^\top \hat{\Lambda}^{-1} U_2E\|_2 \leq \gamma.
\]

By the property of real symmetric matrix, we can further rewrite the previous constrain as $E^\top U_2^\top \hat{\Lambda}^{-1} U_2E \preceq \gamma I_k$. By Schur complement, we have
\[
\begin{bmatrix}
\hat{\Lambda} & U_2E \\
E^\top U_2^\top & \gamma I_k
\end{bmatrix} \succeq 0,
\]
which is equivalent to
\[
\begin{bmatrix}
\Lambda & UE \\
E^\top U^\top & \gamma I_k
\end{bmatrix} \succeq 0.
\]

By pre and post multiplication of matrix $\text{diag}(U^\top, I_k)$ and $\text{diag}(U, I_k)$, respectively, the previous inequality is transformed to
\[
\begin{bmatrix}
L_w & E \\
E^\top & \gamma I_k
\end{bmatrix} \succeq 0.
\]

Then the conclusion follows.

**Remark 3.** Notice that in Theorem 2, the weighted Laplacians $L_w$ can have both positive and negative weights. The result still holds as long as $L_w$ is positive semi-definite.

**Remark 4.** By Theorem 2, the problem (7) is equivalent to the following semi-definite programming (SDP) problem
\[
\min_W \gamma \quad \text{s.t., } \begin{bmatrix}
L_w & E \\
E^\top & \gamma I_k
\end{bmatrix} \succeq 0, \tag{11}
\]
which can be efficiently solved by e.g., CVX.

As one numerical example, we consider problem (11) defined on the graph in Fig. 1 with $c = 8$. Then the optimal weights are $w_1 = 0, w_2 = 1.0427, w_3 = 2, w_4 = 3.0427, w_5 = 0.9573, w_6 = 0.9573$, and $w_7 = 0$. Here the minimum is $\gamma = 1$. It can be seen that the flows on the first and seventh edge do not contribute to the minimization of $H_{\infty}$-norm of this network. The mechanism of this weight allocation is under investigation.

It is worth mentioning that in a recent work [22], the authors considered a $H_{\infty}$ design problem for system (6) with
grounded Laplacian with respect to the topology, instead of weight allocation.

In Theorem 2, we proved that the inequality (8) is satisfied if and only if the $H_{\infty}$ norm is less than or equal to $\gamma$. Moreover, by the bounded real lemma for state-space symmetric systems, i.e., Lemma 1, we have that the following two statements are equivalent:

1. $\|G\|_{\infty} \leq \gamma$.
2. The Riccati inequality

$$-PL_w - L_w^TP + EE^T + \frac{1}{\gamma^2} PEE^TP \preceq 0.$$ (12)

is satisfied with the solution $P = \gamma I$.

In this case, (12) is simplified as

$$-L_w + \frac{EE^T}{\gamma} \preceq 0.$$ (13)

In next section, we shall focus on the positive semi-definiteness of weighted Laplacian, which is a key assumption in Theorem 2. It turns out that the inequality (13) plays a crucial role.

V. POSITIVE SEMIDEFINENESS OF SIGNED LAPLACIANS

In this section, we consider the positive semidefiniteness of signed Laplacian matrices. The main result of this section is formulated in the following theorem, we establish the relation between the magnitude of the negative weights and the effective resistance matrix of subgraph $G_\pm$. In [35], the authors assumed that for any $(i, j) \in \mathcal{E}_-$ and $(i', j') \in \mathcal{E}_-$ being two distinct pairs of vertices, there is no cycle in $G_\pm$ containing $i, j, i', j'$. Here we relax the condition to general graphs.

**Theorem 3.** The Laplacian matrix $L$ is positive semidefinite if and only if

1. for any $e_- = (i, j) \in \mathcal{E}_-$, $i, j$ belong to one connected component of $G_\pm$, and
2. the magnitude of the negative weights satisfies

$$W_{-1} \succ B_+^T L_w^+ B_-.$$ (14)

**Proof.** Sufficiency: Since for any $e_- = (v_i, v_j) \in \mathcal{E}_-$, $i, j$ belong to one connected component of $G_\pm$, by Theorem 2, we have the system

$$\dot{x} = -L_w^+x + B_- \sqrt{W_-}d$$
$$y = \sqrt{W_-}x$$ (15)

has finite $H_{\infty}$ norm. Furthermore, by Lemma 1, we have that $L_w = L_w^+ - B_- \sqrt{W_-} \sqrt{W_-} B_- \succ 0$ is equivalent to the induced $\mathcal{L}_2$ gain from $d$ to $y$ of system (15) is less than or equal to 1.

To prove that $W_{-1} \succ B_+^T L_w^+ B_-$ implies $L_w \succeq 0$, we only focus on the case that $G_\pm$ is connected, i.e., there is only one connected component, without loss of generality. Since

$$\sqrt{W_-} B_+^T L_w^+ B_- \sqrt{W_-} = \sqrt{W_-} B_+^T U_2^\top \Lambda^\top_+ U_2 B_- \sqrt{W_-},$$

where $U_2$ is given as in (9) such that $U_2 L_w U_2^\top = \Lambda_+$, and the induced $\mathcal{L}_2$ gain from $d$ to $y$ of system (15) is

$$\|\sqrt{W_-} B_+^T \hat{L}_w^+ \Lambda^\top_+ U_2 B_- \sqrt{W_-}\|_2 < 1,$$

we have

$$W_{-1} \succeq B_+^T L_w^+ B_-$$
$$\iff \|\sqrt{W_-} B_+^T \hat{L}_w^+ B_- \sqrt{W_-}\|_2 < 1$$
$$\iff \|\sqrt{W_-} B_+^T U_2^\top \Lambda^\top_+ U_2 B_- \sqrt{W_-}\|_2 < 1.$$ (16)

Then the conclusion follows.

**Necessity:** First, it can be verified that if there exists an edge $e_- = (v_i, v_j) \in \mathcal{E}_-$ such that $v_i, v_j$ belong to different connected components of $G_\pm$, $L$ cannot be positive semidefinite. More precisely, suppose $G_\pm$ has $N$ connected components, and the vertices set $\mathcal{V}$ can be divided as $\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_N$ with $|\mathcal{V}_i| = n_i$ and $\sum_{i=1}^{N} n_i = n$. Furthermore, w.l.o.g., suppose $e_- = (v_i, v_j) \in \mathcal{E}_-$ such that $v_i$ and $v_j$ belongs to the first and second component, respectively. Denote the Laplacian of the graph $(\mathcal{V}, \mathcal{E} \setminus \{(i, j)\})$ as $L_{w_-}$. Then by choosing $v = (1_{n_1}, \frac{1}{\sqrt{L_n} n_2}, 0, \ldots, 0)^\top$, we have

$$v^\top L_w v \preceq -\frac{1}{4} W_{-ij} < 0$$

where $W_{-ij} > 0$ is the magnitude of the negative weights of $(v_i, v_j)$, which is contradicts to the positive semi-definiteness of $L_w$.

With the item 1) holding, the necessity of (14) follows directly from the sufficiency part of the proof.

**Remark 5.** Notice that the matrix $B_+^T L_w^+ B_-$ is a submatrix of the effective resistance matrix of $G_\pm$. When there is only one negative edge, the condition (14) is equivalent to Theorem III.3 in [35]. However, for the multiple negative edges, the result in Theorem 3 is more general than Theorem III.4 in [35] in the sense that there are no constraints on the positions of negative edges.

The intuition of Theorem 3 is illustrated in the following example.

**Example 2.** Consider a weighted graph with two negative edges given as in Fig. 2. Suppose the negative weights of $(v_5, v_3)$ and $(v_5, v_4)$ are $-w_5$ and $-w_3$, respectively. Recall that the network in Fig. 1 represents a dynamical distribution network with two ports and in/outflows $d_1$ and $d_2$, respectively, and only positive edge weights. By setting

$$E^\top = \begin{bmatrix} 0 & 0 & 0 & -\sqrt{w_5} \\
0 & 0 & -\sqrt{w_3} & \sqrt{w_5}
\end{bmatrix}$$ (16)

and by (13), we have the Laplacian of the graph in Fig. 2 is positive semi-definite, if and only if the dynamical distribution network in Fig. 1 with $E$ defined as (16) has $H_{\infty}$-norm no larger than 1.

As an numerical example, we consider the positive weights of the graph in Fig. 2 are identical to one. In this case the submatrix of the effective resistance matrix

$$B_+^T L_w^+ B_- = \begin{bmatrix} 1.1429 & 0.7143 \\
0.7143 & 0.9048
\end{bmatrix}.$$ (17)

It can be verified that by choosing $w_3 = w_5 = 0.5$, we have that (14) holds. In this case, the eigenvalues
of $L_w$ are $0, 0.2, 2.6, 4.2, 5$. However, by choosing $w_k = 0.7, w_9 = 0.5$, which violates (14), the eigenvalues of $L_w$ are $-0.04, 0, 2.4, 4.2, 5$.

![Diagram](image)

Fig. 2: The network used in Example 2 where the graph has two negative edges $(v_1, v_2)$ and $(v_1, v_3)$ (red colored).

VI. CONCLUSIONS

For a basic dynamic distribution networks, we have derived an optimization set up with LMIs as constraints, which minimizes the $H_{\infty}$-norm with respect to the allocation of the weights on the edges. Furthermore, by using a bounded real lemma for state-space symmetric systems, we have interpreted the Riccati inequality for distribution networks as a definiteness criterion of a Laplacian to a graph containing both positive and negative edges. Thus, we have provided a sufficient and necessary condition, using effective definiteness criterion of a Laplacian to a graph containing both positive and negative edges.

A related future topic is the problem of minimizing the $H_{\infty}$-norm of dynamic flow networks with respect to topology, more precisely, a limited amount of edges is to be allocated in a graph with fixed vertices. Another future topic is to consider a fixed graph (both topology and weights), but consider saturation of the flow on the edges. The problem is then to minimize the induced $L_2$-gain with respect to allocation of the saturation limits.

VII. ACKNOWLEDGMENT

The first author would like to acknowledge Dr. Mohammad Pirani for the valuable discussions.

VIII. APPENDIX

Proof of Lemma 1. Notice that (2) implies (1) is guaranteed by the bounded real lemma. Hence we only show that (1) implies (2).

By using Theorem 6 in [29], we have that $\|G\|_{\infty} = \| - BA^{-1} B \|_2$. Hence, by Schur complement, $\|G\|_{\infty} \leq \gamma$ implies

$$\begin{bmatrix} -A & B \\ B^T & \gamma I \end{bmatrix} \succeq 0,$$

which is equivalent to $A + \frac{1}{\gamma} BB^T \preceq 0$ and $\gamma \geq 0$. Then it is straightforward to see that $P = \gamma I$ is a solution to (2).

Corollary 4. Consider a SISO dynamical distribution network

$$\dot{x} = -L_w x + Ed,$$

$$y = Cx,$$

defined on a connected graph, with $E \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$ satisfying $E^T E = C^2 = 0$, then the $H_{\infty}$-norm is upper bounded by

$$\|C\|_2 \|B\|_2 \lambda_2.$$  \hspace{1cm} (18)

Furthermore, suppose that the eigenvalues of $L_w$ satisfy $0 = \lambda_1 < \lambda_2 = \cdots = \lambda_n$, then we have

$$E^T = \arg \max_C \|G\|_{\infty},$$

s.t. $\|C\|_2 = \|E\|_2$.

Proof. The notations in this proof are consistent with the ones in the proof of Theorem 2.

The $H_{\infty}$ norm of system (17) is

$$\sup_{\omega \in \mathbb{R}} |CU^T_j (j \omega I + \hat{L})^{-1} U_2 E|$$

$$= \sup_{\omega \in \mathbb{R}} \left| \sum_{i=1}^{n} \hat{C}_i \hat{E}_i \right|$$

$$\leq \sup_{\omega \in \mathbb{R}} \left| \sum_{i=1}^{n} \frac{\hat{C}_i \hat{E}_i}{\sqrt{\omega^2 + \lambda_i^2}} \right|$$

$$\leq \frac{\|C\|_2 \|E\|_2}{\lambda_2}$$

where $\hat{C}_i$ and $\hat{E}_i$ are the $i$th components of the vectors $CU^T_j$ and $U_2 E$, respectively, and the last inequality is based on the fact that $\|CU^T_j\|_2 = \|C\|_2$ and $\|U_2 E\|_2 = \|E\|_2$.

If we further have $\lambda_2 = \cdots = \lambda_n$, i.e., the previous upper bound can be achieved if and only if $U_2 C^T = U_2 E$. Then since $U_2^T U_2 = I - \frac{1}{\lambda_2} I$ and $E^T E = C^2 = 0$, we have $C = E^T$. Thus the conclusion follows.

REFERENCES

[1] A.J. van der Schaft and B.M. Maschke. Port-Hamiltonian systems on graphs. SIAM J. Control and Optimization, 51(2):906–937, 2013.

[2] C. Altafini. Consensus problems on networks with antagonistic interactions. IEEE Transactions on Automatic Control, 58(4):935–946, 2013.

[3] J. Aronson. A survey of dynamic network flows. Annals of Operations Research, 2011:1–66, 1989.

[4] J.C. Moreno Banos and M. Papageorgiou. A linear programming approach to large-scale linear optimal control problems. IEEE Transactions on Automatic Control, 40(5):935–946, 2003.

[5] D. Bertsimas and A. Thiele. A robust optimization approach to inventory theory. Operations Research, 54(1):150–168, 2006.

[6] F. Blanchini, S.Miani, and W. Kowtovich. Control of production-distribution systems with unknown inputs and system failures. Automatic Control, IEEE Transactions on, 45(6):1072–1081, 2000.

[7] B. Bollobas. Modern Graph Theory, volume 184 of Graduate Texts in Mathematics. Springer, New York, 1998.

[8] W. Chen. J. Liu, Y. Chen, S. Z. Khong, D. Wang, T. Basar, L. Qiu, and J. H. Johansson. Characterizing the positive semifiniteness of signed laplacians via effective resistances. In 55th IEEE Conference on Decision and Control, pages 985–990, 2016.

[9] W. Chen, D. Wang, J. Liu, T. Basar, and L. Qiu. On spectral properties of signed laplacians for undirected graphs. In 56th IEEE Conference on Decision and Control, pages 1999–2002, 2017.
