Intermittency for the wave and heat equations with fractional noise in time

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Abstract

In this article, we consider the stochastic wave and heat equations driven by a Gaussian noise which is spatially homogeneous and behaves in time like a fractional Brownian motion with Hurst index $H > 1/2$. Using Malliavin calculus techniques, we obtain an upper bound for the moments of order $p \geq 2$ of the solution. In the case of the wave equation, we derive a Feynman-Kac-type formula for the second moment of the solution, based on the points of a planar Poisson process. This formula is an extension of the one given in [22] in the case $H = 1/2$, and allows us to obtain a lower bound for the second moment of the solution. These results suggest that the moments of the solution grow much faster in the case of the fractional noise in time than in the case of the white noise in time.

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1 Introduction

In this article, we consider the stochastic wave equation:

$$
\begin{align*}
\frac{\partial^2 u}{\partial t^2}(t, x) &= \Delta u(t, x) + u(t, x) \dot{W}(t, x), \quad (t > 0, x \in \mathbb{R}^d) \\
u(0, x) &= u_0, \\
\frac{\partial u}{\partial t}(0, x) &= v_0,
\end{align*}
$$

(SWE)
and the stochastic heat equation:

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= \Delta u(t, x) + u(t, x) \dot{W}(t, x), \quad (t > 0, x \in \mathbb{R}^d) \\
u(0, x) &= u_0
\end{align*}
\]

(SHE)

where \(\Delta\) stands for the Laplacian operator on \(\mathbb{R}^d\), and \(\dot{W}\) denotes the formal derivative of a Gaussian noise \(W\) (whose rigorous definition is given below). Intuitively, the noise \(\dot{W}\) is homogeneous in space (with spatial covariance kernel \(f\)), and behaves in time like a fractional Brownian motion (fBm) with Hurst index \(H > 1/2\). The initial conditions \(u_0\) and \(v_0\) are nonnegative constants. In the case of the wave equation (SWE), we assume that \(d \leq 3\), while for the heat equation (SHE), \(d \geq 1\) can be arbitrary.

There is a large amount of literature dedicated to the case \(H = 1/2\), when the noise behaves in time like the Brownian motion. In this case, we say that the noise is white in time. We refer the reader to the lecture notes [19] for an introduction to the subject, as well as [15, 17, 34, 37, 36, 13, 23, 26] for a sample of relevant references. The case \(H > 1/2\) has to be treated by different methods, since the noise is not a semi-martingale in time. Equations (SHE) and (SWE) with this type of noise have been studied in [5], respectively [2], using Malliavin calculus techniques. The question of the existence of the solution for the heat and wave equations with additive noise \(\dot{W}\) was treated in [6], leading to different conditions on the parameters \((H, f)\) of the noise for the two equations.

In the present article, the noise is introduced by a zero-mean Gaussian process \(W = \{W(h); h \in \mathcal{H}\}\) with covariance

\[E(W(\varphi)W(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}.\]

Here \(\mathcal{H}\) is a Hilbert space defined as the completion of the space \(C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^d)\) of infinitely differentiable functions with compact support on \(\mathbb{R}_+ \times \mathbb{R}^d\), with respect to the inner product \(\langle \cdot, \cdot \rangle_{\mathcal{H}}\) defined by:

\[\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^2} \varphi(t, x)\psi(s, y)|t - s|^{2H-2}f(x - y)dt\, ds\, dx\, dy, \quad (1)\]

and \(\alpha_H = H(2H-1)\). We assume that \(H \in (\frac{1}{2}, 1)\) and \(f\) is the Fourier transform in \(S'(\mathbb{R}^d)\) of a tempered measure \(\mu\) on \(\mathbb{R}^d\), where \(S'(\mathbb{R}^d)\) is the dual of the space \(S(\mathbb{R}^d)\) of rapidly decreasing infinitely differentiable functions on \(\mathbb{R}^d\).

Using the fact that

\[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x)\psi(y)f(x - y)dx\, dy = \int_{\mathbb{R}^d} F\varphi(\xi)F\psi(\xi)\mu(d\xi) \quad \forall \varphi, \psi \in S(\mathbb{R}^d)\]

we arrive at an alternative expression for the inner product \(\langle \cdot, \cdot \rangle_{\mathcal{H}}\):

\[\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} |t - s|^{2H-2}F\varphi(t, \cdot)(\xi)F\psi(s, \cdot)(\xi)\mu(d\xi)dt\, ds, \quad (2)\]
where \( F \) denotes the Fourier transform in the \( x \)-variable.

In the present article, we consider the following four cases:

(i) \( f(0) < \infty \) (i.e. \( \mu \) is a finite measure),

(ii) \( f(x) = |x|^{-\alpha} \) for some \( 0 < \alpha < d \) (i.e. \( \mu(d\xi) = c_{\alpha,d}|\xi|^{-(d-\alpha)}d\xi \)),

(iii) \( f(x) = \prod_{j=1}^{d} |x_j|^{-\alpha_j} \) for some \( \alpha_j \in (0,1) \) (i.e. \( \mu(d\xi) = c_{(\alpha_j),d} \prod_{j=1}^{d} |\xi_j|^{\alpha_j-1}d\xi \)),

(iv) \( d = 1 \) and \( f = \delta_0 \) (i.e. \( \mu \) is the Lebesgue measure).

Here we denote by \( |x| \) the Euclidean norm of \( x \in \mathbb{R}^d \).

Case (i) corresponds to a spatially smooth noise \( \dot{W} \). In case (ii), \( f \) is called the Riesz kernel with exponent \( \alpha \). Case (iii) with the parametrization \( \alpha_j = 2 - 2H_j \) for some \( H_j \in (\frac{1}{2},1) \) leads to a noise \( \dot{W} \) which is called a fractional Brownian sheet with indices \( (H,H_1,\ldots,H_d) \). Finally, case (iv) corresponds to a (rougher) noise \( \dot{W} \) which is “white in space”. This describes the spatial behavior of the noise in the four cases. On the other hand, in time, the noise is smoother than the white noise (the Brownian motion), since \( H > 1/2 \). We note in passing that the results of the present article can be extended to \( H = 1/2 \), recovering results which are already known for equations \( \text{SHE} \) and \( \text{SWE} \) with white noise in time. To ease the exposition, we discuss only the case \( H > 1/2 \).

The stochastic heat equation (SHE) driven by space-time white noise \( \dot{W} \) arises in different contexts and has been studied by many authors. This equation is the continuous form of the parabolic Anderson model studied by Carmona and Molchanov in [10], and plays a major role in the study of the KPZ equation in physics (see [32]). The connection between the stochastic heat equation and the KPZ equation (via the Hopf-Cole transformation) was known informally by physicists for quite some time (see e.g. [8]). Recently, this connection has been made rigorous by Hairer in [27], using the theory of rough paths (see also [7]).

Equation (SHE) with fractional noise in time has been studied in [28, 29, 5]. References [9, 38, 2] are dedicated to the wave equation with fractional noise.

In this article, we consider the Malliavin calculus approach for defining a solution to equations (SHE) and (SWE), as in [2], respectively [5]. In particular, we introduce the following assumption, known as Dalang’s condition:

\[
\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty. \quad \text{(DC)}
\]

This condition is necessary and sufficient for the existence of the solution to equations (SHE) and (SWE), when the noise is white in time (see [17]). It is also sufficient for the existence of the solution to these equations, when the noise is fractional in time and has spatial covariance given by the Riesz kernel (see [5] [2]). The necessity of (DC) in the case of (SHE) has been proved in [4].

Note that (DC) is satisfied in cases (i) and (iv). In cases (ii) and (iii), it holds if and only if \( a < 2 \), where \( a \) is defined by (8) below.
The purpose of this paper is to study intermittency properties for the solutions to equations (SHE) and (SWE). Intuitively, a space-time random field is called physically intermittent if it develops very high peaks concentrated on small spatial islands, as time becomes large. To give a formal mathematical definition of intermittency for a random-field \( u = \{ u(t, x) ; t \geq 0, x \in \mathbb{R}^d \} \), we consider the upper Lyapunov exponent:

\[
\gamma(p) := \limsup_{t \to \infty} \frac{1}{t} \log E|u(t, x)|^p
\]  

(3)

for any \( p \geq 1 \) (assuming that \( \gamma(p) \) does not depend on \( x \)). Traditionally, in the literature, the random-field \( u \) is called weakly intermittent if

\[
\gamma(2) > 0 \quad \text{and} \quad \gamma(p) < \infty \quad \text{for all} \ p \geq 2.
\]

(4)

If \( \gamma(1) = 0 \), and \( u(t, x) \geq 0 \), then weak intermittency implies full intermittency. Recall that a random field \( u \) is fully intermittent if \( p \to \gamma(p)/p \) is strictly increasing (see [10]). Intuitively, full intermittency shows that for \( p > q \),

\[
\limsup_{t \to \infty} \frac{\|u(t, x)\|_p}{\|u(t, x)\|_q} = \infty,
\]

(5)

where \( \| \cdot \|_p \) denotes the norm in \( L^p(\Omega) \). In other words, asymptotically, the \( p \)-th moment of \( u(t, x) \) is significantly larger than its \( q \)-th moment. This suggests that the random variable \( u(t, x) \) may take very large values with small (but significant) probabilities, and therefore it develops high peaks, when \( t \) is large. We refer to [8, Section 2.4] for a detailed explanation of this phenomenon.

Intermittency for the spatially-discrete heat equation was studied in [10]. In [25], Foondun and Khoshnevisan have proved weak intermittency for the solution to equation (SHE) driven by space-time white noise, assuming that the initial condition \( u_0 \) is bounded away from 0. Similar investigations have been carried out in [8, 16, 11]. The recent preprint [12] gives the exact asymptotics for the moments of the solution to equation (SHE) driven by a fractional noise in time, with spatial covariance kernel given by cases (ii)-(iv) above. Intermittency for the stochastic wave equation driven by a Gaussian noise which is white in time was studied in [21, 15].

The fractional aspect of the noise in time leads to a different notion of weak intermittency, which is obtained by a slight modification of the Lyapunov exponent. More precisely, for \( \rho > 0 \) and \( p \geq 1 \), we define the modified upper Lyapunov exponent (of index \( \rho \)) by

\[
\gamma_\rho(p) := \limsup_{t \to \infty} \frac{1}{t^\rho} \log E|u(t, x)|^p.
\]

(6)

We say that the random-field \( u \) is weakly \( \rho \)-intermittent if

\[
\gamma_\rho(2) > 0 \quad \text{and} \quad \gamma_\rho(p) < \infty, \quad \text{for all} \ p \geq 2.
\]
This definition guarantees that relation (5) still holds, and so the intuitive notion
of intermittency remains valid. A similar argument as the one developed in [8]
still applies to explain the existence of the high peaks and the islands.

The appropriate exponents \( \rho \) are different in the hyperbolic and parabolic
cases (see Section 2 below). Nevertheless, in both cases, \( \rho > 1 \). This shows that
the high peaks of the solution are typically larger in the case of the fractional
noise in time compared to the white-noise case. Since \( H > 1/2 \), the noise is
positively correlated in time, which explains why peaks build up larger values.
Indeed, the fractional noise, when large, tends to remain large for a longer period
of time, which results in a higher build-up for the random-field \( u \).

This article is organized as follows. In Section 2, we describe our main
results and introduce the exponents \( \rho \) for equations (SWE) and (SHE). Section
3 contains a review of some Malliavin calculus techniques which are needed for
the definition of the solution. In Section 4, we prove the existence of the solution
to equation (SWE) in any spatial dimension \( d \geq 1 \), and we give an upper bound
for its second moment. An upper bound for its \( p \)-th moment is given in Section
5. In Section 6, we obtain a Feynman-Kac-type representation for the second
moment of the solution of (SWE) with \( d \leq 3 \), based on the points of a planar
Poisson process. This result is used in Section 7 to obtain a lower bound for the
second moment of the solution to (SWE). Section 8 is dedicated to the equation
(SHE). An elementary estimate is given in Appendix A.

2 Main results

In this section, we discuss the two main results of this article.

The following exponents are used for the weak \( \rho \)-intermittency of the solutions
to equations (SWE), respectively (SHE):

\[
\rho_w = \frac{2H + 2 - a}{3 - a}, \quad \rho_h = \frac{4H - a}{2 - a},
\]

where

\[
a = \begin{cases} 
0 & \text{in case (i),} \\
\alpha & \text{in case (ii),} \\
\sum_{j=1}^{d} \alpha_j & \text{in case (iii),} \\
1 & \text{in case (iv).}
\end{cases}
\]

We are now ready to state the first result about equation (SWE). We refer
to (15) below for the definition of the solution.

**Theorem 2.1.** Let \( f \) be a kernel of cases (i)-(iv). Let \( \rho_w \) and \( a \) be the constants
given by (7), respectively (8). Assume that condition (DC) holds.

(a) Equation (SWE) has a solution \( \{u(t,x); t \geq 0, x \in \mathbb{R}^d\} \) for any \( d \geq 1 \).

(b) For any \( d \geq 1, p \geq 2, x \in \mathbb{R}^d \) and for any \( t > 0 \) such that \( pt^{2H+2-a} > t_1 \),

\[
E|u(t,x)|^p \leq C_1^p (u_0 + tv_0)^p \exp \left( C_1 p^{(4-a)/(3-a)} t^{\rho_w} \right),
\]

(9)
where $C_1 > 0$ and $t_1 > 0$ are constants depending on $H$ and $a$.

(c) Suppose that $d \leq 3$. Then for any $x \in \mathbb{R}^d$ and for any $t > t_2$

$$E|u(t, x)|^2 \geq u_1^2 \exp(C_2 t^p u_0),$$

(10)

where $C_2 > 0$ and $t_2 > 0$ are constants depending on $H$ and $a$.

A similar result holds for the parabolic equation (SHE).

Theorem 2.2. Let $f$ be a kernel of cases (i)-(iv). Let $\rho_h$ and $a$ be the constants given by (7), respectively (8). Assume that condition (DC) holds. Let $d \geq 1$ be arbitrary.

(a) Equation (SHE) has a solution \{u(t, x); t \geq 0, x \in \mathbb{R}^d\}.

(b) For any $p \geq 2$, for any $x \in \mathbb{R}^d$ and for any $t > 0$ such that $pt^{(4H-a)/2} > t_1$

$$E|u(t, x)|^p \leq C_1^p u_0^p \exp \left( C_1 t^{(4-a)/(2-a) t^p} \right),$$

(11)

where $C_1 > 0$ and $t_1 > 0$ are constants depending on $H$ and $a$.

(c) For any $x \in \mathbb{R}^d$ and for any $t > t_2$,

$$E|u(t, x)|^2 \geq u_0^2 \exp(C_2 t^p u_0).$$

(12)

where $C_2 > 0$ and $t_2 > 0$ are constants depending on $H$ and $a$.

Most moment estimates for solutions to s.p.d.e.’s with white noise in time rely on martingale properties of stochastic integrals. Since the fBm is not a semimartingale, different techniques have to be used when the noise is fractional in time. In the case of equations (SWE) and (SHE), one can give explicitly the Wiener chaos representation of the solution. The upper bounds (9) and (11) are obtained directly using the equivalence of $L^2(\Omega)$- and $L^p(\Omega)$-norms on each Wiener chaos. The lower bounds require more work. For this, we follow the approach of [21], which consists in using a Feynman-Kac (FK) type representation for the second moment of the solution, based on a Poisson process. Such a representation was originally developed in [22] for equations driven by a noise that is white in time. It was extended to the heat equation driven by fractional noise in time by the first author of this article in [1]. The extension to the wave equation with fractional noise in time is given in Section 6 below.

As in [21], we focus mainly on the hyperbolic case (Theorem 2.1). The proof of Theorem 2.2 is very similar and we only point out the differences compared with the hyperbolic case in Section 8. We made this choice since the results for the wave equation are completely new, in particular the second moment FK type representation.

In [12], Chen, Hu, Song and Xing have obtained stronger results than our Theorem 2.2 by computing the exact Lyapunov exponent for the solution to (SHE), defined as the limit when $t \to \infty$, instead of the lim sup in (6). However, their method requires the additional assumptions $a < 4H - 2$ in cases (ii)-(iii), and $H > 3/4$ in case (iv), which are not needed in the present article. The proofs
of [12] rely on a Feynman-Kac representation for the solution itself (due to [30]), which can only be proved under the above-mentioned additional assumptions.

Since $H > 1/2$, $\rho_h > \rho_w > 1$. Therefore, the lower bounds in Theorems [34] and [35] imply that $\gamma(2) = \infty$, which shows that the solutions to (SWE) and (SHE) are not weakly intermittent in the classical sense. However, these solutions are weakly $\rho$-intermittent (in the sense defined above) with $\rho = \rho_w$ for the wave equation and $\rho = \rho_h$ for the heat equation. If $H = 1/2$, then $\rho_w = \rho_h = 1$ and we recover the known results of intermittency for the heat and wave equations with white noise in time, as in [24, 15].

Finally, we discuss the behavior of the Lyapunov exponent as a function of $p$. Understanding this behavior plays an important role in the study of the size and position of the high peaks (see [14, 15]). For the wave equation, we obtain that $\gamma_{\rho_w}(p) \lesssim p^{4/3}$ in case (i) (spatially smooth noise), and $\gamma_{\rho_w}(p) \lesssim p^{3/2}$ in case (iv) (spatial white-noise). For the heat equation, we obtain that $\gamma_{\rho_h}(p) \lesssim p^2$ in case (i) and $\gamma_{\rho_h}(p) \lesssim p^3$ in case (iv). We note that in these bounds, the exponents of $p$ do not depend on $H$ and coincide with the exponents obtained in [21, 15], in the case $H = 1/2$.

3 Framework

In this section, we introduce the framework and we give a brief summary of the results of [2] which are needed in the present article.

We denote by $G_w$ (resp. $G_h$) the fundamental solution of the wave equation, respectively the heat equation. In the case of the wave equation, recall that when $d \leq 2$, $G_w(t, \cdot)$ is a function given by:

$$G_w(t, x) = \frac{1}{2} \text{1}_{\{|x| \leq t\}} \text{ if } d = 1 \quad \text{and} \quad G_w(t, x) = \frac{1}{2\pi} \sqrt{\frac{1}{t^2 - |x|^2}} \text{1}_{\{|x| < t\}} \text{ if } d = 2.$$  \hspace{1cm} (13)

In both cases, $\int_{\mathbb{R}^d} G_w(t, x) dx = t$. When $d = 3$, $G_w(t, \cdot)$ is a finite measure on $\mathbb{R}^3$ given by:

$$G_w(t, \cdot) = \frac{1}{4\pi t^{d/2}} \sigma_t$$

where $\sigma_t$ is the surface measure on $\partial B(0, t)$, and $G_w(t, \mathbb{R}^d) = t$. In all three cases, the Fourier transform of $G_w(t, \cdot)$ is given by:

$$\mathcal{F}G_w(t, \cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad \xi \in \mathbb{R}^d.$$  \hspace{1cm} (14)

In the case of the heat equation, for any dimension $d \geq 1$, $G_h(t, \cdot)$ is a function (known as the heat kernel) given by:

$$G_h(t, x) = \frac{1}{(2\pi t)^{d/2}} \exp \left( -\frac{|x|^2}{2t} \right) \quad \text{and} \quad \mathcal{F}G_h(t, \cdot)(\xi) = \exp \left( -\frac{t|\xi|^2}{2} \right).$$

Below, we write $G$ when the results apply for both $G_w$ or $G_h$. 7
We now discuss the concept of solution.

An adapted square-integrable process \( u = \{u(t, x); t \geq 0, x \in \mathbb{R}^d\} \) is called a (mild) solution of (SWE) or (SHE) if it satisfies the following integral equation:

\[
u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) u(s, y) W(\delta s, \delta y) \tag{15}\]

where the stochastic integral is interpreted in the Skorohod sense and \( w = w_w \) or \( w_h \) is the solution of the homogeneous equation with the same initial condition as (SWE) (respectively (SHE)). In particular, \( w \) depends on the equation considered. Namely,

\[
w_w(t, x) = u_0 + tv_0 \quad \text{and} \quad w_h(t, x) = u_0 \quad \text{(16)}
\]

Note that \( w(t, x) \) does not depend on \( x \) in either case, and \( w(t, x) \geq u_0 \geq 0 \) for all \( t > 0 \) and \( x \in \mathbb{R}^d \).

Replacing informally \( u(s, y) \) on the right-hand side of (15) by its definition and iterating this procedure, we conclude that the solution of (SWE) or (SHE) should be given by the following series of iterated integrals:

\[
u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) W(ds, dy) + \int_0^t \int_{\mathbb{R}^d} \int_0^s \int_{\mathbb{R}^d} G(t - s, x - y) G(s - r, y - z) W(dr, dz) W(ds, dy) + \ldots \tag{17}
\]

To give a rigorous meaning to this procedure, we use an approach based on Malliavin calculus, whose basic elements we recall below (see [35] for more details). It is known that every square integrable random variable \( F \) which is measurable with respect to an isonormal Gaussian process \( W = \{W(h); h \in \mathcal{H}\} \), has the Wiener chaos expansion

\[
F = E(F) + \sum_{n \geq 1} F_n \quad \text{with} \quad F_n \in \mathcal{H}_n,
\]

where \( \mathcal{H}_n \) is the \( n \)-th Wiener chaos space associated to \( W \). Moreover, each \( F_n \) can be represented as \( F_n = I_n(f_n) \) for some \( f_n \in \mathcal{H}^{\otimes n} \), where \( \mathcal{H}^{\otimes n} \) is the \( n \)-th tensor product of \( \mathcal{H} \) and \( I_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}_n \) is the multiple Wiener integral with respect to \( W \). By the orthogonality of the Wiener chaos spaces and an isometry-type property of \( I_n \), we obtain that

\[
E|F|^2 = (EF)^2 + \sum_{n \geq 1} E|I_n(f_n)|^2 = (EF)^2 + \sum_{n \geq 1} n! \|\tilde{f}_n\|_{\mathcal{H}^{\otimes n}}^2,
\]

where \( \tilde{f}_n \) is the symmetrization of \( f_n \) in all \( n \) variables:

\[
\tilde{f}_n(t_1, x_1, \ldots, t_n, x_n) = \frac{1}{n!} \sum_{\rho \in S_n} f_n(t_{\rho(1)}, x_{\rho(1)}, \ldots, t_{\rho(n)}, x_{\rho(n)})
\]
where $S_n$ is the set of all permutations of $\{1, \ldots, n\}$.

In our case, $W$ is the isonormal Gaussian process with covariance $\langle \cdot \rangle$ and $F = u(t, x)$. By Theorem 2.8 of [2], it is known that a solution of the Wiener chaos expansion

$$u(t, x) = w(t, x) + \sum_{n \geq 1} f_n(t, \cdot, x)$$

(18)

for some kernels $f_n(\cdot, t, x)$ (defined below), provided that $f_n(\cdot, t, x) \in \mathcal{H}^\otimes n$ and the series on the right-hand side of (18) converges in $L^2(\Omega)$. Verifying that $f_n(\cdot, t, x) \in \mathcal{H}^\otimes n$ reduces to checking that $\|f_n(\cdot, t, x)\|^2_{\mathcal{H}^\otimes n} < \infty$ (see Remark 2.3 of [2]), while proving the convergence in $L^2(\Omega)$ reduces to checking that $\sum_{n \geq 1} n!\|f_n(\cdot, t, x)\|^2_{\mathcal{H}^\otimes n} < \infty$. In this case,

$$E|u(t, x)|^2 = w(t, x)^2 + \sum_{n \geq 1} \frac{1}{n!} \alpha_n(t),$$

(19)

where $\alpha_n(t) = (n!)^2 \|f_n(\cdot, t, x)\|^2_{\mathcal{H}^\otimes n}$.

We give now the definition of the kernel $f_n(\cdot, t, x)$. By Proposition 2.1 of [2], for any $0 < t_1 < \ldots < t_n < t$, $f_n(t_1, \ldots, t_n, \cdot, t, x)$ is a distribution in $\mathbb{R}^d$ whose Fourier transform (in $\mathcal{S}'(\mathbb{R}^d)$) is the function:

$$\mathcal{F}f_n(t_1, \ldots, t_n, \cdot, t, x)(\xi_1, \ldots, \xi_n) = (u_0 + t_1v_0)e^{-i(\xi_1 + \ldots + \xi_n)x}\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)$$

$$\mathcal{F}G(t_3 - t_2, \cdot)(\xi_1 + \xi_2) \ldots \mathcal{F}G(t - t_n, \cdot)(\xi_1 + \ldots + \xi_n).$$

(20)

$f_n(t_1, \ldots, t_n, \cdot, t, x)$ is defined to be 0 for $(t_1, \ldots, t_n) \in [0, t]^n \setminus T_n(t)$ where $T_n(t) = \{0 < t_1 < \ldots < t_n < t\}$. (Note that in Proposition 2.1 of [2], it was assumed that $u_0 = 1$ and $v_0 = 0$, so that $w = 1$. This result continues to hold when the function $w$ is given by (19), since $w$ does not depend on $x$.)

After examining (20), we infer that if $d \leq 2$ for the wave equation and in any dimension for the heat equation, the kernel $f_n(t_1, \ldots, t_n, \cdot, t, x)$ is a function on $\mathbb{R}^d$ given by:

$$f_n(t_1, x_1, \ldots, t_n, x_n, t, x) = G(t - t_n, x - x_n)G(t_n - t_{n-1}, x_n - x_{n-1})$$

$$\ldots$$

$$G(t_2 - t_1, x_2 - x_1)w(t_1, x_1)1_{\{0 < t_1 < \ldots < t_n < t\}}.$$  

This coincides with the informal interpretation (17). On the other hand, if $d = 3$ for the wave equation, $f_n(t_1, \ldots, t_n, \cdot, t, x)$ is a finite measure on $\mathbb{R}^d$ given by:

$$f_n(t_1, \ldots, t_n, \cdot, t, x) = G(t - t_n, x - dx_n)G(t_n - t_{n-1}, x_n - dx_{n-1})$$

$$\ldots$$

$$G(t_2 - t_1, x_2 - dx_1)w(t_1, x_1)1_{\{0 < t_1 < \ldots < t_n < t\}},$$

where for fixed $a \in \mathbb{R}^d$, we denote by $G(t, a - \cdot)\langle A \rangle = G(t, a - A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. 

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Remark 3.1. Notice that in both the hyperbolic and parabolic cases, the function \( f_n \) is stationary in the sense that, for all \( t_1, \ldots, t_n \in [0, t] \) and any \( x_1, \ldots, x_n, x \in \mathbb{R}^d \),

\[
f_n(t_1, x_1, \cdots, t_n, x_n, t, x) = f_n(t_1, x_1 - x, \cdots, t_n, x_n - x, t, 0).
\]

This remains valid for \( \tilde{f}_n \). A direct consequence is that \( \| \tilde{f}_n(\cdot, t, x) \|_{H^s} \) and, hence, \( \alpha_n(t) \) do not depend on \( x \). Since the initial conditions are constant, \( w \) does not depend on \( x \) either and the moments of \( u \) are independent of \( x \). This justifies the definition of Lyapunov exponent independent of \( x \). Also, notice that it is possible to show that the law of \( u(t, x) \) is independent of \( x \) (see for instance \cite{17} in the white noise case). These remarks are not true if the initial conditions are not constant.

We return now to the series \cite{19} which gives the second moment of \( u(t, x) \). An important role in the present paper is played by the \( n \)-th term of this series, which depends on \( \alpha_n(t) \). First, note that an expression similar to \cite{2} exists for the \( n \)-fold inner product \( \langle \cdot, \cdot \rangle_{H^s} \). Using this expression, we have:

\[
\alpha_n(t) = \alpha_H^n \int_{[0,t]^2} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_n(t, s) dt ds,
\]

(21)

where we denote \( t = (t_1, \ldots, t_n) \) and \( s = (s_1, \ldots, s_n) \), and we define

\[
\psi_n(t, s) = \int_{\mathbb{R}^d} \mathcal{F}g_n^{(n)}(\cdot, t, x)(\xi_1, \ldots, \xi_n) \mathcal{F}g_n^{(n)}(\cdot, t, x)(\xi_1, \ldots, \xi_n) \mu(d\xi_1) \cdots \mu(d\xi_n)
\]

(22)

with \( g_n^{(n)}(\cdot, t, x) = n! \tilde{f}_n(t_1, \cdots, t_n, \cdot, t, x) \).

An alternative calculation of the function \( \psi_n(t, s) \) is needed in Section 6 below. For this, let \( \rho, \sigma \in S_n \) be such that

\[
0 < t_{\rho(1)} < \cdots < t_{\rho(n)} < t \quad \text{and} \quad 0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t
\]

and denote \( t_{\rho(n+1)} = s_{\sigma(n+1)} = t \). Then, if \( d \leq 2 \), we have:

\[
\psi_n(t, s) = \int_{\mathbb{R}^{2nd}} \prod_{j=1}^n G(t_{\rho(j+1)} - t_{\rho(j)}, x_{\rho(j+1)} - x_{\rho(j)}) \prod_{j=1}^n G(s_{\sigma(j+1)} - s_{\sigma(j)}, y_{\sigma(j+1)} - y_{\sigma(j)}) \prod_{j=1}^n f(x_j - y_j) dxdy,
\]

with the notation \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^{nd} \), whereas if \( d = 3 \),

\[
\psi_n(t, s) = \int_{\mathbb{R}^{3nd}} \prod_{j=1}^n G(t_{\rho(j+1)} - t_{\rho(j)}, x_{\rho(j+1)} - x_{\rho(j)}) dx_{\rho(j)}w(t_{\rho(1)}, x_{\rho(1)})
\]

\[
\prod_{j=1}^n G(s_{\sigma(j+1)} - s_{\sigma(j)}, y_{\sigma(j+1)} - dy_{\sigma(j)})w(s_{\sigma(1)}, y_{\sigma(1)}) \prod_{j=1}^n f(x_j - y_j).
\]

(22)

(In both integrals above, we used the notation \( x_{\rho(n+1)} = y_{\sigma(n+1)} = x \).)

This concludes the summary of the results of \cite{2} which are needed here.
4 Hyperbolic case: existence of the solution

In this section, we prove the existence of the solution of equation (SWE) in any space dimension $d \geq 1$, when $f$ is a kernel of cases (i)-(iv). This yields the conclusion of Theorem 2.1.(a) and (b) (with $p = 2$).

We let $G = G_w$ and $w = w_w$. We introduce the following constant:

$$K(\mu) := \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi - \eta|^2} \mu(d\xi).$$  \hspace{1cm} (23)

Note that $K(\mu) < \infty$ if and only if (DC) holds (see the proof of Lemma 8 of [20]).

Note that (DC) is satisfied in cases (i) and (iv). In cases (ii) and (iii), (DC) holds if and only if $a < 2$. We define a constant $K_w$ by

$$K_w = \begin{cases} 
\mu(\mathbb{R}^d) & \text{in case (i)}, \\
4K(\mu) & \text{in cases (ii) and (iii)}, \\
\pi & \text{in case (iv)}. 
\end{cases}$$  \hspace{1cm} (24)

We have the following preliminary result.

**Lemma 4.1.** Let $f$ be a kernels of cases (i)-(iv). Assume that (DC) holds. For any $t > 0$ and for any $t = (t_1, \ldots, t_n)$ in $[0, t]^n$,

$$\psi_n(t, t) \leq (u_0 + tv_0)^2 K_w(u_1 \ldots u_n)^{2-a}$$

where $a$ is given by (8), $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for $j = 1, \ldots, n$, $t_{\rho(1)} \leq \ldots \leq t_{\rho(n)}$ for some $\rho \in S_n$, $t_{\rho(n+1)} = t$, and $K_w$ is the constant defined in (24).

**Proof:** As in the proof of Lemma 3.2 of [2], by (22), (20) and (14), we obtain:

$$\psi_n(t, t) = (u_0 + t_{\rho(1)}v_0)^2 \int_{\mathbb{R}^n} \frac{\sin^2(u_1|\xi_1|)}{|\xi_1|^2} \cdots \frac{\sin^2(u_n|\xi_1 + \ldots + \xi_n|)}{|\xi_1 + \ldots + \xi_n|^2} \mu(d\xi_1) \cdots \mu(d\xi_n).$$

We consider separately the four cases.

- **Case (i)** Using the fact that $|x^{-1}\sin x| \leq 1$, we have

  $$\psi_n(t, t) \leq (u_0 + tv_0)^2 [\mu(\mathbb{R}^d)]^n (u_1 \ldots u_n)^2.$$

- **Case (ii)** This case was treated in Lemma 3.2 of [2].
• Case (iii) Let \( c = c(\alpha_j) \). Using the change of variables \( \eta_j = \xi_1 + \ldots + \xi_j \),
\[
    \psi_n(t, t) = c^n (u_0 + t \rho(1)v_0)^2 \int_{\mathbb{R}^d} d\eta_1 \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2} \prod_{j=1}^d |\eta_{1,j}|^{\alpha_j-1}
    \times \int_{\mathbb{R}^d} d\eta_2 \frac{\sin^2(u_2|\eta_2|)}{|\eta_2|^2} \prod_{j=1}^d |\eta_{2,j} - \eta_{1,j}|^{\alpha_j-1}
    \quad \vdots
    \times \int_{\mathbb{R}^d} d\eta_n \frac{\sin^2(u_n|\eta_n|)}{|\eta_n|^2} \prod_{j=1}^d |\eta_{n,j} - \eta_{n-1,j}|^{\alpha_j-1}.
\]
where \( \eta_i = (\eta_{i,j})_{j=1}^{d} \) with \( \eta_{i,j} \in \mathbb{R} \). Note that for any \( t > 0 \) and \( \eta \in \mathbb{R}^d \),
\[
c \int_{\mathbb{R}^d} \frac{\sin^2(t|\xi|)}{|\xi|^2} \prod_{j=1}^d |\xi_j - \eta_j|^{\alpha_j-1} d\xi = ct^{2-a} \int_{\mathbb{R}^d} \frac{\sin^2(|\xi|)}{|\xi|^2} \prod_{j=1}^d |\xi_j - t\eta_j|^{\alpha_j-1} d\xi
= t^{2-a} \int_{\mathbb{R}^d} \frac{\sin^2(|\xi + t\eta|)}{|\xi + t\eta|^2} \mu(d\xi) \leq 4t^{2-a} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi + t\eta|^2} \leq 4t^{2-a} K(\mu),
\]
since \( \sin(x)/x^2 \leq 4/(1 + x^2) \) for all \( x > 0 \). Hence,
\[
    \psi_n(t, t) \leq (u_0 + tv_0)^2 (4K(\mu))^n (u_1 \ldots u_n)^{2-a}.
\]

• Case (iv) Using the change of variables \( \eta_j = \xi_1 + \ldots + \xi_j \), we have:
\[
    \psi_n(t, t) = (u_0 + t \rho(1)v_0)^2 \int_{\mathbb{R}^n} \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2} \ldots \frac{\sin^2(u_n|\eta_n|)}{|\eta_n|^2} d\eta_1 \ldots d\eta_n.
\]

Using (13), (14) and Plancherel’s theorem, we obtain that for any \( t > 0 \),
\[
    \int_{\mathbb{R}^n} \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi = \pi t.
\]
Hence,
\[
    \psi_n(t, t) = (u_0 + tv_0)^2 \pi^n u_1 \ldots u_n.
\]
\[\square\]

The following result is an extension of Proposition 3.1 of [21] to the case of the fractional noise in time.

**Proposition 4.2.** Let \( f \) be a kernel of cases (i)-(iv), and \( \rho_\omega, a, K_w \) be the constants given by [7], [8], respectively [27]. Assume that (17) holds. Then:

a) for any \( t > 0 \) and for any integer \( n \geq 1 \),
\[
    \alpha_n(t) \leq (u_0 + tv_0)^2 c^n K_w^{\frac{t(2H+2-a)n}{(n!)^{2-a}}},
    \tag{25}
\]
where \( c \) is a constant depending on \( H \) and \( a \);
b) equation (SWZ) has a solution \( u(t, x) \) which has the following property: for any \( x \in \mathbb{R}^d \) and for any \( t > 0 \) such that \( K_w t^{2H + 2 - a} > t_0 \),

\[
E|u(t, x)|^2 \leq 2(u_0 + tv_0)^2 \exp(c_0 K_w^{1/(3-a)} t^{a/2}),
\]

where \( c_0 > 0 \) and \( t_0 > 0 \) are some constants depending on \( H \) and \( a \).

**Proof:** a) We proceed as in the proof of Proposition 3.3 of [2].

For any \( t = (t_1, \ldots, t_n) \in [0, t]^n \), we define \( \beta(t) = \prod_{j=1}^n (t_{\rho(j+1)} - t_{\rho(j)}) \), where \( \rho \in S_n \) is chosen such that \( t_{\rho(1)} < \ldots < t_{\rho(n)} \), and \( t_{\rho(n+1)} = t \).

By the Cauchy-Schwarz inequality, \( \psi_n(t, s) \leq \psi_n(t, t)^{1/2} \psi_n(s, s)^{1/2} \). By Lemma [4.1] it follows that

\[
\psi_n(t, s) \leq (u_0 + tv_0)^2 K_w^n [\beta(t) \beta(s)]^{(2-a)/2}.
\]

Using definition (24) of \( \alpha_n(t) \) and (26), we obtain:

\[
\alpha_n(t) \leq (u_0 + tv_0)^2 K_w^n b_H^n \int_{[0,t]^n} \prod_{j=1}^n |t_j - s_j|^{2H-2} [\beta(t) \beta(s)]^{(2-a)/2} dt ds.
\]

We now use the fact that for any \( \varphi \in L^{1/H}(\mathbb{R}_+^n) \)

\[
\alpha_n(t) \leq (u_0 + tv_0)^2 K_w^n b_H^n \int_{[0,t]^n} \prod_{j=1}^n \beta(t)^{(2-a)/(2H)} dt
\]

for some constant \( b_H > 0 \) (see [33]). We obtain:

\[
\alpha_n(t) \leq (u_0 + tv_0)^2 K_w^n b_H^n \left( \frac{n! \int_{T_n(t)} [(t - t_n) \ldots (t_2 - t_1)]^{(2-a)/(2H)} dt}{T_n(t)} \right)^{2H}
\]

where \( T_n(t) = \{0 < t_1 < \ldots < t_n < t\} \). By Lemma 3.5 of [5], for any \( h > -1 \),

\[
\frac{n! \int_{T_n(t)} [(t - t_n) \ldots (t_2 - t_1)]^h dt}{T_n(t)} = \frac{\Gamma((1 + h)n + 1)}{\Gamma((1 + h)n + 1)} t^{(1+h)n}.
\]

By Stirling’s formula, \( \Gamma((1 + h)n + 1) \approx C_n (n!)^{1+h} \), where \( C_n \) is such that \( \lambda^{-n} \leq C_n \leq \lambda^n \) for some constant \( \lambda > 1 \) depending on \( h \). Hence,

\[
\frac{n! \int_{T_n(t)} [(t - t_n) \ldots (t_2 - t_1)]^h dt}{T_n(t)} \leq \frac{\Gamma((1 + h)n + 1)}{(n!)^{1+h}} t^{(1+h)n}
\]

for some \( c_1 > 0 \). In our case, \( h = (2-a)/(2H) \). We obtain:

\[
\alpha_n(t) \leq (u_0 + tv_0)^2 K_w^n b_H^n \left( \frac{n! \Gamma((1 + h)n + 1)}{(n!)^{1+h}} \right)^{2H}
\]

\[
\leq (u_0 + tv_0)^2 K_w^n c_1 \frac{1}{(n!)^{2-a}} t^{(2H+2-a)n},
\]

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where \( c = bH(1 + h)^{2H}c_1^{2H} \) depends on \( H \) and \( a \).

b) We use (19) and the result in part a). We obtain that for any \( t > 0 \),

\[
E|u(t, x)|^2 \leq (u_0 + tv_0)^2 \sum_{n \geq 0} \frac{c^n K_w^n (2H+2-a)^n}{(n!)^{3-a}}.
\]

Since this series is convergent for any fixed \( t > 0 \), this proves the existence result. Now, using Lemma A.1 (Appendix A), we have that for all \( t > 0 \) such that \( cK_w t^{2H+2-a} > t_1 \),

\[
E|u(t, x)|^2 \leq 2(u_0 + tv_0)^2 \exp(c_1(cK_w t^{2H+2-a})^{1/(3-a)}),
\]

where \( c_1 > 0 \) and \( t_1 > 0 \) are some constants depending on \( a \). □

5 Hyperbolic case: upper bound on the moments

In this section, we give an upper bound for the moments of order \( p > 2 \) of the solution of equation (SWE). This yields the conclusion of Theorem 2.1(b).

Recall that the solution has a Wiener chaos expansion given by

\[
u(t, x) = \sum_{n \geq 0} J_n(t, x),
\]

where \( J_n(t, x) \) is in the \( n \)-th Wiener chaos \( \mathcal{H}_n \) associated to the noise \( W \), and

\[
E|\nu(t, x)|^2 = \sum_{n \geq 0} E|J_n(t, x)|^2 = \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t),
\]

where \( \alpha_n(t) \) is defined in (19) and is estimated by (25).

The following result is an extension of Theorem 3.2 of [21] to the case of the fractional noise in time.

**Proposition 5.1.** Let \( f \) be one of the kernels (i)-(iv), and \( \rho_w, a, K_w \) be the constants given by (7), (8), respectively (24). Assume that \( (DC) \) holds. Let \( u(t, x) \) be the solution of (SWE). Then for any \( p \geq 2 \), for any \( x \in \mathbb{R}^d \) and for any \( t > 0 \) such that \( pK_w t^{2H+2-a} > t_0 \),

\[
E|u(t, x)|^p \leq 2^p(u_0 + tv_0)^p \exp(c_0 K_w^{1/(3-a)} p^{(4-a)/(3-a)} t_0^p),
\]

where \( c_0 > 0 \) and \( t_0 > 0 \) are some constants depending on \( H \) and \( a \).

**Proof:** When \( p = 2 \), the result is given by Theorem 4.2.

When \( p > 2 \), we use the same idea as in the proof of Theorem 4.1 of [2]. We denote by \( \| \cdot \|_p \) the \( L^p(\Omega) \)-norm. We use the fact that for elements in a fixed Wiener chaos \( \mathcal{H}_n \), the \( \| \cdot \|_p \)-norms are equivalent (see the last line of page 62 of [35] with \( q = p \) and \( p = 2 \)). More precisely,

\[
\|J_n(t, x)\|_p \leq (p - 1)^{n/2} \|J_n(t, x)\|_2 = (p - 1)^{n/2} \left( \frac{1}{n!} \alpha_n(t) \right)^{1/2}.
\]
Using (25), we obtain:
\[
\| J_n(t, x) \|_p \leq (u_0 + tv_0) C_{p,K_w}^n t^{n(2H+2-a)/2} \frac{1}{(n!)^{(3-a)/2}}
\]

where \( C_{p,K_w} = (p-1)^{1/2} c^{1/2} K_w^{1/2} \) and \( c \) depends on \( H \) and \( a \).

Recall the Minkowski’s inequality for integrals: (see Appendix A.1 of [41])
\[
\left[ \int_X \left( \int_Y |F(x, y)| \mu(dx) \right)^p \nu(dy) \right]^{1/p} \leq \int_X \left( \int_Y |F(x, y)|^p \nu(dy) \right)^p \mu(dx).
\]

We use this inequality for \((X, \mathcal{X}) = (\mathbb{N}, 2^{\mathbb{N}})\) with \( \mu \) the counting measure, \((Y, \mathcal{Y}, \nu) = (\Omega, \mathcal{F}, P)\) and \( F(n, \omega) = J_n(\omega, t, x) \). We have:
\[
\| u(t, x) \|_p = \| \sum_{n \geq 0} J_n(t, x) \|_p \leq \sum_{n \geq 0} \| J_n(t, x) \|_p \leq (u_0 + tv_0) \sum_{n \geq 0} C_{p,K_w}^n t^{n(2H+2-a)/2} \frac{1}{(n!)^{(3-a)/2}}.
\]

Using Lemma A.1, we infer that for any \( t > 0 \) such that \( C_{p,K_w} t^{(2H+2-a)/2} > t_1 \)
\[
\| u(t, x) \|_p \leq 2(u_0 + tv_0) \exp \left\{ c_1 \left( C_{p,K_w} t^{(2H+2-a)/2} \right)^{2/(3-a)} \right\},
\]

where \( c_1 \) and \( t_1 \) depend on \( a \). The result follows since \( \frac{2H+2-a}{2} \cdot \frac{2}{3-a} = \rho_w \) and
\[
pC_{p,K_w}^{2/(3-a)} = p(p-1)^{1/(3-a)} c^{1/(3-a)} K_w^{1/(3-a)}.
\]

\[\square\]

6 Hyperbolic case: FK representation for the second moment

In this section, we develop a Feynman-Kac (FK) representation for the second moment of the solution \( u(t, x) \) of the wave equation (SWE), similar to the one obtained in [22] in the case of white noise in time. Due to the fractional component of the noise, our representation is based on a Poisson random measure on \( \mathbb{R}_+^2 \), rather than a simple Poisson process. This extension follows the approach of [1] for the parabolic case. The FK representation will be used in Section 7 to obtain a lower bound for the second moment of the solution in cases (i)-(iii). Case (iv) will be treated differently using an approximation based on case (ii).

Here is the main result of this section.
Theorem 6.1. Suppose that equation (SWE) has a solution $u(t,x)$. Then for any $t > 0$ and $x \in \mathbb{R}^d$,

$$E|u(t,x)|^2$$

$$= e^2 \sum_{n \geq 0} \sum_{i_1, \ldots, i_n \text{ distinct}} E_x \left[ w_w(t - \tau_n, X^1_{\tau_n}) w_w(t - \tau'_n, X^2_{\tau'_n}) \prod_{j=1}^n (\tau_j - \tau_{j-1}) \right.$$ \hspace{1cm}

$$\times \prod_{j=1}^n (\tau'_j - \tau'_{j-1}) \prod_{j=1}^n f(X^1_{\tau_{ij}} - X^2_{\tau_{ij}}) \alpha_H \prod_{j=1}^n |T_{ij} - S_{ij}|^{2H-2} B_{i_1, \ldots, i_n}(t) \right],$$

where, by convention, the term for $n = 0$ is taken to be $w_w(t,x)^2$. Here

- $N = \sum_{i \geq 1} \delta_{P_i}$ is a Poisson random measure $\mathbb{R}_+^2$ of intensity the Lebesgue measure, with $P_i = (T_i, S_i)$;
- $B_{i_1, \ldots, i_n}(t)$ is the event that $N$ has points $P_{i_1}, \ldots, P_{i_n}$ in $[0,t]^2$;
- $\tau_1 \leq \ldots \leq \tau_n$ and $\tau'_1 \leq \ldots \leq \tau'_n$ are the points $T_{i_1}, \ldots, T_{i_n}$, respectively $S_{i_1}, \ldots, S_{i_n}$ arranged in increasing order;
- the processes $X^1 = (X^1_s)_{s \in [0,t]}$ and $X^2 = (X^2_s)_{s \in [0,t]}$ are defined by (29) and (30) below, and we denote by $P_x$ a probability measure under which $X^1_0 = X^2_0 = x$. ($E_x$ stands for the expectation with respect to $P_x$.)

Remark 6.2. A similar formula can be obtained for $E[u(t,x)u(s,y)]$ using the points of $N$ in $[0,t] \times [0,s]$ and assuming that $X^1_0 = x$ and $X^2_0 = y$.

Remark 6.3. Note that $|T_{ij} - S_{ij}|^{2H-2} = \infty$ if the point $(T_{ij}, S_{ij})$ falls on the diagonal $D = \{(s,s); 0 \leq s \leq t\}$ of the square $[0,t]^2$. This is not a problem since with probability 1, $N$ has no points in $D$: $P(N(D) = 0) = e^{-\text{Leb}(D)} = 1$.

Remark 6.4. Without loss of generality we may assume that $\tau_1 \ldots \ldots \neq \tau_n$ and $\tau'_1 \ldots \ldots \neq \tau'_n$ since the event $B_{i_1, \ldots, i_n}(t)$ for which $\tau_j = \tau_{j-1}$ (or $\tau'_{j} = \tau'_{j-1}$) for some $j = 1, \ldots, n$ has probability zero: with probability 1, no vertical (or horizontal) line contains two distinct points of $N$ (see page 223 of [39]).

Remark 6.5. Theorem 6.1 is valid for any function $f$, not necessarily as in one of the cases (i)-(iii). In fact, this representation remains valid if we replace $\alpha_H |t - s|^{2H-2}$ in (1) by a function $\eta(t,s)$, provided that $(\cdot, \cdot)_H$ defines an inner product. We only need to assume that the solution of (SWE) exists. In the new representation, $\alpha_H |T_{ij} - S_{ij}|^{2H-2} = \eta(t - T_{ij}, t - S_{ij})$.

We now introduce the necessary ingredients for the proof of Theorem 6.1. Recall first that if $(N_t)_{t \geq 0}$ is a Poisson process on $\mathbb{R}_+$ of rate 1 with jump times $\tau_1 < \tau_2 < \ldots$, then the conditional distribution of $(\tau_1, \ldots, \tau_n)$ given $N_n = n$ coincides with the distribution of the order statistics of a sample of size $n$ from the uniform distribution on $[0,t]$. This property lies at the core of the
For any measurable function $F$, Corollary 6.7. the event $B_{i_1,...,i_n}(t)$, both vectors $(P_{i_1},...,P_{i_n})$ and $(t-P_{i_1},...,t-P_{i_n})$ have a uniform distribution on $[0,t]^{2n}$, where $t = (t_1,t_2) \in \mathbb{R}_+^2$ a.s. (see Exercise 2.4 of [31]). This means that with probability 1, the points $(P_{i})_{i \geq 1}$ are distinct. For any $t > 0$ fixed, we consider the event $B_{i_1,...,i_n}(t)$ for distinct indices $i_1,...,i_n \geq 1$.

The following result plays an important role in the present paper.

**Lemma 6.6.** Given $B_{i_1,...,i_n}(t)$, both vectors $(P_{i_1},...,P_{i_n})$ and $(t-P_{i_1},...,t-P_{i_n})$ have a uniform distribution on $[0,t]^{2n}$, where $t = (t,t) \in \mathbb{R}_+^2$ a.s. (see Exercise 2.4 of [31]). This means that with probability 1, the points $(P_{i})_{i \geq 1}$ are distinct. For any $t > 0$ fixed, we consider the event $B_{i_1,...,i_n}(t)$ for distinct indices $i_1,...,i_n \geq 1$.

The following result plays an important role in the present paper.

**Proof:** The restriction of $N$ to $[0,t]^2$ can be constructed as $N = \sum_{i=1}^{Y} \delta_{X_i}$, where $(X_i)_{i \geq 1}$ are i.i.d. random variables with a uniform distribution on $[0,t]^2$ and $Y$ is an independent random variable with mean $t^2$. If $N$ has points $P_{i_1},...,P_{i_n}$ in $[0,t]^2$, the vector $(P_{j_1},...,P_{j_n})$ of the $n$ points coincides with a vector $(X_{i_1},...,X_{i_n})$ for some distinct indices $j_1,...,j_n$, which clearly has a uniform distribution on $[0,t]^{2n}$. The argument for the vector $(t-P_{i_1},...,t-P_{i_n})$ is similar (see Lemma 2.1 of [11] for an alternative proof).

As a consequence of the previous lemma, any $n$-fold integral over $([0,t]^2)^n$ of a deterministic function $F$ has a stochastic representation based on the points of $N$ (see page 257 of [11] for the proof).

**Corollary 6.7.** For any measurable function $F : [0,t]^{2n} \to \mathbb{R}$ which is either bounded or non-negative, we have:

$$\int_{[0,t]^{2n}} F(t_1, s_1, ..., t_n, s_n) dt ds = n! e^{t^2} \sum_{i_1,...,i_n \text{ distinct}} E[F(t - T_{i_1}, t - S_{i_1}, ..., t - T_{i_n}, t - S_{i_n}) 1_{B_{i_1,\ldots,i_n}(t)}],$$

where $t = (t_1, ..., t_n)$ and $s = (s_1, ..., s_n)$ with $t_i \in [0,t]$ and $s_i \in [0,t]$.

The next result gives a stochastic representation for the $n$-th term of the series $[19]$.
Lemma 6.8. For any $t > 0$ and for any integer $n \geq 1$, we have:

$$\alpha_n(t) = n! e^{t^2} \sum_{T_1, \ldots, T_n \text{ distinct}} E \left[ \prod_{j=1}^{n} |T_i - S_i|^{2H-2} \times \psi_n(t - T_i, \ldots, t - T_i, t - S_i, \ldots, t - S_i) 1_{B_1, \ldots, B_n(t)} \right].$$

Proof: The $n$-fold integral on the right-hand side of (21) can be represented in the desired form by applying Corollary 6.7 to the function:

$$F(t_1, s_1, \ldots, t_n, s_n) = \prod_{j=1}^{n} \eta(t_j, s_j) \psi_n(t, s).$$

The next result will be used to evaluate the term $\psi_n(t - T_i, \ldots, t - T_i, t - S_i, \ldots, t - S_i)$ which appears in Lemma 6.8. For simplicity, we work first with some non-random points $(t_1, s_1), \ldots, (t_n, s_n)$ in $[0, t]^2$. These points will be replaced later by $(T_1, S_1), \ldots, (T_n, S_n)$.

Lemma 6.9. Let $(t_1, s_1), \ldots, (t_n, s_n) \in [0, t]^2$. Let $\rho, \sigma \in S_n$ be such that:

$$0 < t_{\rho(n)} < \ldots < t_{\rho(1)} < t \quad \text{and} \quad 0 < s_{\sigma(n)} < \ldots < s_{\sigma(1)} < t.$$

If $d \leq 2$ then

$$\psi_n(t - t_1, \ldots, t - t_n, t - s_1, \ldots, t - s_n) =$$

$$\int_{\mathbb{R}^{2nd}} dz dw \prod_{j=1}^{n} f \left( \sum_{k=1}^{n+1-\rho^{-1}(j)} z_k - \sum_{k=1}^{n+1-\sigma^{-1}(j)} w_k \right)$$

$$\times G_w(t_{\rho(n)}, z_1) G_w(t_{\rho(n-1)} - t_{\rho(n)}, z_2) \ldots G_w(t_{\rho(2)} - t_{\rho(1)}, z_n)$$

$$\times G_w(s_{\sigma(n)}, w_1) G_w(s_{\sigma(n-1)} - s_{\sigma(n)}, w_2) \ldots G_w(s_{\sigma(1)} - s_{\sigma(2)}, w_n)$$

$$\times w\left(t - t_{\rho(1)}, x + \sum_{k=1}^{n} z_k\right) w\left(t - s_{\sigma(1)}, x + \sum_{k=1}^{n} w_k\right),$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ with $z_i \in \mathbb{R}^d, w_i \in \mathbb{R}^d$. A similar relation holds for $d = 3$, replacing $G_w(t_{\rho(n)}, z_1) dz_1$ by $G_w(t_{\rho(n)}, dz_1)$, etc.

Proof: Assume first that $d \leq 2$. We proceed as in the first part of the proof of Lemma 2.2 of [1]. Denote $t_{\rho(n+1)} = s_{\sigma(n+1)} = 0$ and $x_{\rho(n+1)} = y_{\sigma(n+1)} = x$. Note that

$$0 < t - t_{\rho(1)} < \ldots < t - t_{\rho(n)} < t \quad \text{and} \quad 0 < t - s_{\sigma(1)} < \ldots < t - s_{\sigma(n)} < t$$
and \( G_w(t, x) = G_w(t, -x) \). By definition, \( \psi_n(t - t_1, \ldots, t - t_n, t - s_1, \ldots, t - s_n) \) is equal to
\[
\int_{\mathbb{R}^{2n}} dx dy \prod_{j=1}^{n} G_w(t_{\rho(j)} - t_{\rho(j+1)}, x_{\rho(j)} - x_{\rho(j+1)}) w(t - t_{\rho(1)}, x_{\rho(1)})
\times \prod_{j=1}^{n} G_w(s_{\sigma(j)} - s_{\sigma(j+1)}, y_{\sigma(j)} - y_{\sigma(j+1)}) w(t - s_{\sigma(1)}, y_{\sigma(1)}) \prod_{j=1}^{n} f(x_j - y_j).
\]
The result follows by the change of variables: \( x_{\rho(j)} - x_{\rho(j+1)} = z_{n+1-j} \) and \( y_{\sigma(j)} - y_{\sigma(j+1)} = w_{n+1-j} \) for \( j = 1, \ldots, n \).

The same argument works also for \( d = 3 \). To see this, assume for simplicity that \( n = 2, 0 < t_1 < t_2 < t \) and \( 0 < s_2 < s_1 < t \). (The same argument applies in the general case.) Then \( \psi_2(t - t_1, t - t_2, t - s_1, t - s_2) \) is equal to
\[
\int_{\mathbb{R}^{4d}} h(x_1, x_2, y_1, y_2) G_w(t_1, dx_1 - x) G_w(t_2 - t_1, dx_2 - x_1)
\times G_w(s_2, dy_2 - x) G_w(s_1 - s_2, dy_1 - y_2)
\]
where \( h(x_1, x_2, y_1, y_2) = f(x_1 - y_1)f(x_2 - y_2)w(t - t_2, x_2)w(t - s_1, y_1) \) and we used the fact that \( G_w(t, a - dx) = G_w(t, dx - a) \). We claim that for any non-negative measurable function \( \varphi : \mathbb{R}^{4d} \to \mathbb{R} \),
\[
\int_{\mathbb{R}^{4d}} \varphi(x_1, x_2, y_1, y_2) G_w(t_1, dx_1 - x) G_w(t_2 - t_1, dx_2 - x_1)
\times G_w(s_2, dy_2 - x) G_w(s_1 - s_2, dy_1 - y_2)
= \int_{\mathbb{R}^{4d}} \varphi(x + z_1, x + z_2, x + w_1, x + w_2, x + w_1)
\times G_w(t_1, dz_1) G_w(t_2 - t_1, dz_2) G_w(s_2, dw_1) G_w(s_1 - s_2, dw_2).
\]
(This means that we can apply informally the change of variables \( x_1 - x = z_1, x_2 - x_1 = z_2 \) and \( y_2 - x = w_1, y_1 - y_2 = w_2 \).) Assuming that \( \varphi(x_1, x_2, y_1, y_2) = \phi_1(x_1) \phi_2(x_2) \psi_1(y_1) \psi_2(y_2) \), relation (27) follows using the fact that for any non-negative measurable function \( \phi : \mathbb{R}^d \to \mathbb{R} \),
\[
\int_{\mathbb{R}^d} \phi(x) G_w(t, dx - a) = \int_{\mathbb{R}^d} \phi(a + y) G_w(t, dy).
\]
The case of an arbitrary function \( \varphi \) follows by approximation. The conclusion follows applying (27) to the function \( \varphi = h \).

**Remark 6.10.** In the case of the heat equation, \( G_h(t, a \cdot) \) is the density of \( B^1_t - B^1_s \), where \( (B^1_t)_{t \geq 0} \) is a \( d \)-dimensional Brownian motion, and the product
\[
G_h(t_{\rho(n)}, z_1) G_h(t_{\rho(n-1)} - t_{\rho(n)}, z_2) \ldots G_h(t_{\rho(1)} - t_{\rho(2)}, z_n)
\]
which appears in Lemma 6.9 is the density of the random vector
\[
(B^1_{t_{\rho(n)}}, B^1_{t_{\rho(n-1)} - t_{\rho(n)}}, \ldots, B^1_{t_{\rho(1)} - t_{\rho(2)}}).
\]

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Applying a similar argument for the other n-term product (depending on s) and using an independent Brownian motion \((B^2_t)_{t \geq 0}\), we infer that:

\[
\psi_n(te - t, te - s) = E \left[ w(t - t_{\rho(1)}, x + B^1_{t_{\rho(1)}}) w(t - s_{\sigma(1)}, x + B^2_{s_{\sigma(1)}}) \prod_{j=1}^n f(B^1_j - B^2_j) \right]
\]

where \(e = (1, \ldots, 1) \in \mathbb{R}^n, t = (t_1, \ldots, t_n)\) and \(s = (s_1, \ldots, s_n)\). Something similar will happen in the case of the wave equation, conditionally on \(N\).

Recall that on the event \(B_{i_1, \ldots, i_n}(t)\), we arrange the two sets of points \(\{T_{i_1}, \ldots, T_{i_n}\}\) and \(\{S_{i_1}, \ldots, S_{i_n}\}\) in increasing order as \(\tau_1 \leq \ldots \leq \tau_n\), respectively \(\tau'_1 \leq \ldots \leq \tau'_n\). More precisely, if we denote \(U_j = T_j\) and \(V_j = S_j\) for \(j = 1, \ldots, n\), then there exist some permutations \(\rho\) and \(\sigma\) of \(\{1, \ldots, n\}\) such that

\[
U_{\rho(n)} \leq U_{\rho(n-1)} \leq \ldots \leq U_{\rho(1)} \quad \text{and} \quad V_{\sigma(n)} \leq V_{\sigma(n-1)} \leq \ldots \leq V_{\sigma(1)}.
\]

We let \(\tau_j = U_{\rho(n+j-1)}\) and \(\tau'_j = V_{\sigma(n+j-1)}\) for any \(j = 1, \ldots, n\).

We now explain how to import into our framework the method of [22]. We let \((\Theta^1_j)_{j \geq 1}\) and \((\Theta^2_j)_{j \geq 1}\) be two independent i.i.d. collections of random variables with the same law as \(\Theta_0\), where \(\Theta_0\) is a random variable with values in \(\mathbb{R}^d\) such that, if \(d \leq 2\), \(\Theta_0\) has density function \(G_w(1, \cdot)\) and if \(d = 3\), \(\Theta_0\) has distribution \(G_w(1, \cdot)\). The importance of the variable \(\Theta_0\) stems from the fact that for any \(t > 0\),

\[
\frac{G_w(t, \cdot)}{t} \quad \text{is the density/distribution of} \quad t\Theta_0.
\]

As in [22], using the points \(\tau_1 \leq \ldots \leq \tau_n\) and the variables \((\Theta^1_j)_{j \geq 1}\), we construct a process \(X^1 = (X^1_s)_{s \in [0, t]}\) by setting:

\[
X^1_s = X^1_{\tau_1} + (s - \tau_1)\Theta^1_{n+1} \quad \text{if} \quad \tau_1 \leq s \leq \tau_{n+1}
\]

for any \(1 \leq i \leq n\), where \(\tau_0 = 0, \tau_{n+1} = t\) and \(X^1_0 = 0\). We use a similar construction for the process \(X^2 = (X^2_s)_{s \in [0, t]}\) using the points \(\tau'_1 \leq \ldots \leq \tau'_n\) and the variables \((\Theta^2_j)_{j \geq 1}\), i.e. \(\tau'_0 = 0, \tau'_{n+1} = t, X^2_0 = 0\) and for any \(1 \leq i \leq n\),

\[
X^2_s = X^2_{\tau'_1} + (s - \tau'_1)\Theta^2_{n+1} \quad \text{if} \quad \tau'_1 \leq s \leq \tau'_{n+1}.
\]

**Remark 6.11.** Due to (28), when \(d \leq 2\), the product

\[
\frac{G_w(\tau_1, z_1)}{\tau_1}, \frac{G_w(\tau_2 - \tau_1, z_2)}{\tau_2 - \tau_1}, \ldots, \frac{G_w(\tau_n - \tau_{n-1}, z_n)}{\tau_n - \tau_{n-1}}
\]

is the conditional density of \(Y^1 = (X^1_{\tau_1}, X^1_{\tau_2} - X^1_{\tau_1}, \ldots, X^1_{\tau_n} - X^1_{\tau_{n-1}})\) given \(N\). Let \(Y^2 = (X^2_{\tau'_1}, X^2_{\tau'_2} - X^2_{\tau'_1}, \ldots, X^2_{\tau'_n} - X^2_{\tau'_{n-1}})\). Since \(X^1\) and \(X^2\) are conditionally independent given \(N\),

\[
\prod_{j=1}^n \frac{G_w(\tau_j - \tau_{j-1}, z_j)}{\tau_j - \tau_{j-1}} \prod_{j=1}^n \frac{G_w(\tau'_j - \tau'_{j-1}, w_j)}{\tau'_j - \tau'_{j-1}}
\]
is the conditional density of \((Y^1, Y^2)\) given \(N\). A similar thing happens when \(d = 3\). Therefore, for the wave equation, the processes \(X^1, X^2\) play the same role (conditionally on \(N\)), as the Brownian motions \(B^1, B^2\) for the heat equation (see Remark 6.10).

**Proof of Theorem 6.1.** By applying Lemma 6.9 to the points \((t_j, s_j) = (T_{ij}, S_{ij})\) we obtain that

\[
\psi_n(t - T_{i1}, \ldots, t - T_{in}, t - S_{i1}, \ldots, t - S_{in}) = \int_{\mathbb{R}^{2n,d}} dzdw \prod_{j=1}^{n+d} f \left( \sum_{k=1}^{n+1-\rho^{-1}(j)} z_k - \sum_{k=1}^{n+1-\sigma^{-1}(j)} w_k \right) \\
\times G_w(\tau_1, z_1)G_w(\tau_2 - \tau_1, z_2) \ldots G_w(\tau_n - \tau_{n-1}, z_n) \\
\times G_w(\tau'_1, w'_1)G_w(\tau'_2 - \tau'_1, w'_2) \ldots G_w(\tau'_{n-1} - \tau'_{n-2}, w'_{n-1}) \\
\times w_w \left( t - \tau_n, x + \sum_{k=1}^{n} z_k \right) w_w \left( t - \tau'_n, x + \sum_{k=1}^{n} w_k \right),
\]

assuming that \(d \leq 2\). A similar identity holds for \(d = 3\) replacing \(G_w(\tau_1, z_1)dz_1\) by \(G_w(\tau_1, dz_1)\), and so on. Inside this integral, we multiply and divide by \(\prod_{j=1}^{n}(\tau_j - \tau_{j-1})\) \(\prod_{j=1}^{n}(\tau'_j - \tau'_{j-1})\).

We assume that \(X^1_0 = X^2_0 = 0\). Using Remark 6.11 we infer that \(\psi_n(t - T_{i1}, \ldots, t - T_{in}, t - S_{i1}, \ldots, t - S_{in})\) is equal to the conditional expectation of

\[
\prod_{j=1}^{n} f \left( \sum_{k=1}^{n+1-\rho^{-1}(j)} (X^1_{\tau_k} - X^1_{\tau_{k-1}}) - \sum_{k=1}^{n+1-\sigma^{-1}(j)} (X^2_{\tau_k} - X^2_{\tau_{k-1}}) \right) \\
\times w_w \left( t - \tau_n, x + \sum_{k=1}^{n} (X^1_{\tau_k} - X^1_{\tau_{k-1}}) \right) w_w \left( t - \tau'_n, x + \sum_{k=1}^{n} (X^2_{\tau_k} - X^2_{\tau_{k-1}}) \right) \\
\times \prod_{j=1}^{n}(\tau_j - \tau_{j-1}) \prod_{j=1}^{n}(\tau'_j - \tau'_{j-1})
\]

given \(N\). Note that

\[
\sum_{k=1}^{n+1-\rho^{-1}(j)} (X^1_{\tau_k} - X^1_{\tau_{k-1}}) = X^1_{\tau_{n+1-\rho^{-1}(j)}} \quad \text{and} \quad \sum_{k=1}^{n} (X^1_{\tau_k} - X^1_{\tau_{k-1}}) = X^1_{\tau_n}
\]

(these are telescopic sums whose first term is \(X^1_{\tau_0} = 0\)). Recall that \(\tau_k = U_{\rho(n+1-k)}\) for any \(k = 1, \ldots, n\) (where \(U_j = T_{ij}\)). Hence

\[
\tau_{n+1-\rho^{-1}(j)} = U_{\rho(n+1-n+\rho^{-1}(j))} = U_{\rho(\rho^{-1}(j))} = U_j = T_{ij}.
\]
A similar argument applies to the terms depending on $X^2$. We obtain that

$$
\psi_n(t - T_{i_1}, \ldots, t - T_{i_n}, t - S_{i_1}, \ldots, t - S_{i_n}) =
\begin{array}{c}
E \left[ \prod_{j=1}^n f(X_{T_{ij}}^1 - X_{S_{ij}}^2) w_w(t - \tau_n, x + X_{T_{ij}}^1) w_w(t - \tau_n', x + X_{T_{ij}}^2) \right. \\
\times \left. \prod_{j=1}^n (\tau_j - \tau_j) \prod_{j=1}^n (\tau_j' - \tau_j' - 1) \right] N .
\end{array}
$$

Looking now back at the representation of $\alpha_n(t)$ (Lemma 6.8), we obtain:

$$
\frac{1}{n!} \alpha_n(t) = e^{2} \sum_{i_1, \ldots, i_n \text{ distinct}} E \left[ 1_{B_{i_1, \ldots, i_n}}(t) \prod_{j=1}^n |T_{ij} - S_{ij}|^{2H - 2} \right.
\times \left. \prod_{j=1}^n f(X_{T_{ij}}^1 - X_{S_{ij}}^2) w_w(t - \tau_n, x + X_{T_{ij}}^1) w_w(t - \tau_n', x + X_{T_{ij}}^2) \right.
\times \left. \prod_{j=1}^n (\tau_j - \tau_j) \prod_{j=1}^n (\tau_j' - \tau_j' - 1) \right] N .
$$

Note that $1_{B_{i_1, \ldots, i_n}}(t) \prod_{j=1}^n |T_{ij} - S_{ij}|^{2H - 2}$ is measurable with respect to $N$, and so, this term goes inside the conditional expectation with respect to $N$. The result follows using the fact that $E[E[\cdot |N]] = E[\cdot]$ and taking the sum over $n \geq 1$. In the final step, the values $x + X_{T_{ij}}^1$ and $x + X_{T_{ij}}^2$ are replaced by $X_{T_{ij}}^1$, respectively $X_{T_{ij}}^2$, under the probability measure $P_x$. □

7 Hyperbolic case: lower bound on the moment of order 2

In this section, we give a lower bound for the second moment of the solution to \( (SWE) \), using the FK formula developed in Section 6. Theorems 7.1, 7.2 and 7.6 give the conclusion of Theorem 2.1.(c) in cases (i), (ii)-(iii), respectively (iv).

We follow the approach of [21]. For any $x, y \in \mathbb{R}^d$ with $x \neq y$, we denote by $C(x, y)$ the solid (infinite) cone in $\mathbb{R}^d$ with vertex $y$, axis oriented in the direction of the vector $x - y$, and an angle of $\pi/4$ between the axis and any lateral side. This cone has the following properties:

(i) If $|z - y| < \delta$, $|y - x| < \delta$ and $z \in C(x, y)$, then $|z - x| < \delta$;

(ii) $y + z \in C(x, y)$ if and only if $y + rz \in C(x, y)$ for any $r > 0$;
(iii) \( C(x, y) + z = C(x + z, y + z) \).

The first result corresponds to Theorem 2.1(c), in case (i).

**Theorem 7.1.** Let \( f \) be a kernel of case (i). Then, for any \( x \in \mathbb{R}^d \) and for any \( t > 0 \) such that \( K_w t^{2H+2} > t_1 \),

\[
E|u(t, x)|^2 \geq u_0^2 \exp(c_1 K_w^{1/3} t^{\rho_w}),
\]

where \( c_1 > 0 \) and \( t_1 > 0 \) are some constants depending on \( H \), and the constants \( \rho_w \) and \( K_w \) are given by \([7]\), respectively \([21]\).

**Proof:** We proceed as in the proof of Theorem 4.1 of \([21]\). To facilitate the comparison with the proof of these authors, we use the same notation, i.e. we denote \( n \) by \( k \) in the statement of Theorem 6.1 above. We let \( N_k = N([0, t]^2) \).

Note that \( \lim_{x \to 0} f(x) = f(0) = \mu(\mathbb{R}^d) = K_w \) and therefore \( f \) satisfies Assumption C of \([21]\) with \( \alpha_0 = K_w/2 \), i.e. there exists \( \delta > 0 \) such that \( f(x) > \alpha_0 \) for all \( x \in \mathbb{R}^d \) with \( |x| < 2\delta \). We assume that \( \delta \) is a rational number.

**Step 1.** First step for the lower bound of \( E|u(t, x)|^2 \)

Let \( k \in \mathbb{Z}_+ \) be a large enough value (depending on \( t \)) such that

\[
m := k\delta \in \mathbb{Z}_+.
\]

(The precise value of \( k \) will be given in Step 8 below.) Notice that for all \( t > 0 \) and \( x \in \mathbb{R}^d \), \( w_\omega(t, x) = u_0 + v_0 t \geq u_0 \), since \( v_0 \geq 0 \). Hence, by Theorem 6.1

\[
E|u(t, x)|^2 \geq u_0^2 e^{t^2} \sum_{i_1, \ldots, i_k} E_x \left[ \prod_{j=1}^{k} (\tau_j - \tau_{j-1}) \prod_{j=1}^{k} (\tau'_j - \tau'_{j-1}) \right] \prod_{j=1}^{k} f(X^1_{\tau_j} - X^2_{S_{\tau_j}}) \prod_{j=1}^{k} |T_{ij} - S_{ij}|^{2H-2} 1_{B_{i_1, \ldots, i_k}}(t).
\]

**Step 2.** The event \( D(t) \)

We consider the event \( D(t) = D^1(t) \cap D^2(t) \), where

\[
D^1(t) = \{ X^1_{\tau_j} + \Theta^1_{j+1} \in C(x, X^1_{\tau_j}) \text{ for all } j = 1, \ldots, k-1, B_{i_1, \ldots, i_k} \},
\]

\[
D^2(t) = \{ X^2_{\tau_j} + \Theta^2_{j+1} \in C(x, X^2_{\tau_j}) \text{ for all } j = 1, \ldots, k-1, B_{i_1, \ldots, i_k} \}.
\]

On the event \( D^1(t) \), if we assume that \( \tau_j - \tau_{j-1} \leq \delta \) for all \( j = 1, \ldots, k \), then \( |X^1_{\tau_j} - x| \leq \delta \) for all \( j = 1, \ldots, k \). (This can be proved by induction on \( j \), using the properties of the cone.) Recall that \( \tau_j = U_{\rho(k+1-j)} \) for some permutation \( \rho \) of \( \{1, \ldots, k\} \), where \( U_j = T_{ij} \). As \( j \) runs through the set \( \{1, \ldots, k\} \), so does the value \( \rho(k+1-j) \). Therefore, on the event \( D^1(t) \), if we assume that \( \tau_j - \tau_{j-1} \leq \delta \) for all \( j = 1, \ldots, k \), then \( |X^1_{\tau_j} - x| \leq \delta \) for all \( j = 1, \ldots, k \). A similar property
holds for $X^2$ on the event $D^2(t)$. Hence, on the event $D(t)$, if we assume that
\[ \tau_j - \tau_{j-1} \leq \delta \text{ and } \tau'_j - \tau'_{j-1} \leq \delta \text{ for all } j = 1, \ldots, k, \]
then
\[ |X^1_{T_{ij}} - X^2_{S_{ij}}| \leq 2\delta, \text{ for all } j = 1, \ldots, k, \]
and so,
\[ f(X^1_{T_{ij}} - X^2_{S_{ij}}) \geq \alpha_0, \text{ for all } j = 1, \ldots, k. \]  
(33)

**Step 3.** The islands $(I_{j,l})_{1 \leq j,l \leq k}$

The idea of the proof is to build some small islands around the $k$ points of the process $N$ in the region $[0, t]^2$. To define these islands, we let $\varepsilon = \frac{4\delta}{m+1}$ and $t_j = j\varepsilon$ for any $j = 1, \ldots, k$. Due to (31), we have
\[ t_k = k\varepsilon = \frac{m}{m+1}t \approx t \text{ if } m \text{ is large.} \]

We consider the intervals $I_j = [a_j, b_j]$ with $j = 1, \ldots, k$, where $a_j = t_j - \varepsilon/4$, $b_j = t_j + \varepsilon/4$ if $j \leq k - 1$, and $b_k = t$. For any $j, l = 1, \ldots, k$, we define
\[ I_{j,l} = I_j \times I_l. \]

The area of each square island $I_{j,l}$ is greater than $(\varepsilon/4)^2$. In both the horizontal and vertical direction, the islands are separated by intervals of length $\varepsilon/2$.

**Step 4.** The event $C_{i_1, \ldots, i_k}(t)$

Let $C_{i_1, \ldots, i_k}(t)$ be the event that $N$ has points $P_{i_1}, \ldots, P_{i_k}$ in $[0, t]^2$ located on the islands $I_{1,i_1}, \ldots, I_{k,i_k}$, for some permutation $(i_1, \ldots, i_k)$ of $\{1, \ldots, k\}$.

Clearly, $C_{i_1, \ldots, i_k}(t)$ is included in $B_{i_1, \ldots, i_k}(t)$. Notice that on the event $C_{i_1, \ldots, i_k}(t)$, it is not possible to have two points $(T_{i_p}, S_{i_p})$ and $(T_{i_q}, S_{i_q})$ of $N$ in $[0, t]^2$ such that $T_{i_p}, T_{i_q}$ are in the same interval $I_j$ or $S_{i_p}, S_{i_q}$ are in the same interval $I_l$. Therefore, on the event $C_{i_1, \ldots, i_k}(t)$, for any $j = 1, \ldots, k$, we have $\tau_j \in I_j$, $\tau'_j \in I_j$, and hence
\[ \frac{\varepsilon}{2} \leq \tau_j - \tau_{j-1} \leq 2\varepsilon \quad \text{and} \quad \frac{\varepsilon}{2} \leq \tau'_j - \tau'_{j-1} \leq 2\varepsilon. \]  
(34)

In particular, if
\[ m \geq m_0(t) := [2t - 1] \]  
(35)
then $\tau_j - \tau_{j-1} \leq \delta$ and $\tau'_j - \tau'_{j-1} \leq \delta$ for all $j = 1, \ldots, k$. It follows that:

inequality (33) holds on the event $D(t) \cap C_{i_1, \ldots, i_k}(t)$,  
(36)

provided that $m \geq m_0$.

**Step 5.** Second step for the lower bound of $E|u(t, x)|^2$

On the event $B_{i_1, \ldots, i_k}(t)$, we define $\tilde{Z}_t = \prod_{j=1}^k (\tau_j - \tau_{j-1}) \prod_{j=1}^k (\tau'_j - \tau'_{j-1})$. Using
Figure 1: The islands $I_{j,l}$ (for $n = 4$) with points situated on the islands $I_{11}, I_{24}, I_{32}, I_{43}$ corresponding to the permutation $(l_1, l_2, l_3, l_4) = (1, 4, 2, 3)$.

Using the properties of the cone and the independence of $(\Theta_i^1)_{i \geq 1}$, it can be shown that $P_x[D^1(t)|N] = \gamma^{N_t-1}$, where $\gamma = P(y + \Theta_0 \in C(0,y)) \in (0,1)$ does
not depend on $y \in \mathbb{R}^d$. Note that $\gamma$ depends on $d$. A similar property holds for 
$D^2(t)$. Hence,

$$P_x[D(t)\mid N] = \gamma^{2(N_t-1)} > \gamma^{2N_t}.$$  

Combining this with the previous lower bound for $E[|u(t, x)|^2]$, we obtain:

$$E[u(t, x)]^2 \geq u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k} \sum_{i_1, \ldots, i_k \text{ distinct}} E_x \left[ \mathcal{Z}_t \prod_{j=1}^k \left| T_{i_j} - S_{i_j} \right|^{2H-2} c^{i_1, \ldots, i_k(t)} \right].$$

We define the conditional expectation of a random variable $X$ with respect to an event $B$ by $E[X \mid B] = E[X \mid B]/P(B)$. (This is not the same as $E[X \mid \mathcal{G}]$, where $\mathcal{G} = \sigma(\{B\}) = \{\emptyset, B, B^c, \Omega\}$ since $E[X \mid \mathcal{G}] = E[X \mid B]_{1_B} + E[X \mid B^c]_{1_{B^c}}$.) In our case, $X$ is the random variable appearing in the expectation above and $B = B_{i_1, \ldots, i_k(t)}$. We obtain:

$$E[u(t, x)]^2 \geq u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k}$$

$$\times \sum_{i_1, \ldots, i_k \text{ distinct}} E_x \mathcal{Z}_t \prod_{j=1}^k \left| T_{i_j} - S_{i_j} \right|^{2H-2} c^{i_1, \ldots, i_k(t)} \mathcal{B}_{i_1, \ldots, i_k(t)} P_x(B_{i_1, \ldots, i_k(t)}).$$

Note that by (34), on the event $C_{i_1, \ldots, i_k(t)}$, we have $\mathcal{Z}_t \geq (\varepsilon/2)^{2k}$. Using the fact that $\delta = m/k$ (by the definition (31) of $m$), we see that

$$\frac{\varepsilon}{2} = \frac{\delta t}{2(m+1)} = \frac{m}{m+1} \left( \frac{ct}{k} \right) \geq \frac{ct}{k}$$

with $c = 1/8$. Hence $\mathcal{Z}_t \geq (ct/k)^{2k}$ and

$$E[u(t, x)]^2 \geq e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k} \left( \frac{ct}{k} \right)^{2k}$$

$$\times \sum_{i_1, \ldots, i_k \text{ distinct}} E_x \mathcal{Z}_t \prod_{j=1}^k \left| T_{i_j} - S_{i_j} \right|^{2H-2} c^{i_1, \ldots, i_k(t)} P_x(B_{i_1, \ldots, i_k(t)}).$$

Since both $T_{i_j}$ and $S_{i_j}$ are in $[0,t]$, we obviously have $|T_{i_j} - S_{i_j}| < t$. Thus, since $2H-2 < 0$,

$$\prod_{j=1}^k \left| T_{i_j} - S_{i_j} \right|^{2H-2} > t^{(2H-2)k}.$$  

This turns out to be enough for our purposes. With this bound, we have:

$$E[u(t, x)]^2 \geq u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k} \left( \frac{ct}{k} \right)^{2k} t^{(2H-2)k}$$

$$\times \sum_{i_1, \ldots, i_k \text{ distinct}} P_x(C_{i_1, \ldots, i_k(t)} \mid B_{i_1, \ldots, i_k(t)}) P_x(B_{i_1, \ldots, i_k(t)}).$$  

(38)
Step 6. The conditional probability $P_x(C_{i_1,\ldots,i_k}(t)|B_{i_1,\ldots,i_k}(t))$

Let $S_k$ be the set of all permutations $(l_1,\ldots,l_k)$ of $\{1,\ldots,k\}$. By the definition of the event $C_{i_1,\ldots,i_k}(t)$,

$$P_x(C_{i_1,\ldots,i_k}(t)|B_{i_1,\ldots,i_k}(t)) = \sum_{(l_1,\ldots,l_k)\in S_k} P_x(A_{i_1,\ldots,i_k}(t, (l_1,\ldots,l_k))|B_{i_1,\ldots,i_k}(t))$$

where $A_{i_1,\ldots,i_k}(t, (l_1,\ldots,l_k))$ is the event that $N$ has points $P_{i_1},\ldots,P_{i_k}$ in $[0,t]^2$ located on the islands $I_{i_1,l_1},\ldots,I_{i_k,l_k}$. Note that

$$A_{i_1,\ldots,i_k}(t, (l_1,\ldots,l_k)) = \bigcup_{(j_1,\ldots,j_k)\in S_k} \{P_{i_1} \in I_{j_1,l_1},\ldots,P_{i_k} \in I_{j_k,l_k}\}$$

Given $B_{i_1,\ldots,i_k}(t)$, $(P_{i_1},\ldots,P_{i_k})$ has a uniform distribution on $[0,t]^{2k}$. Hence,

$$P_x(P_{i_1} \in I_{j_1,l_1},\ldots,P_{i_k} \in I_{j_k,l_k}|B_{i_1,\ldots,i_k}(t)) = \frac{\text{Leb}(I_{j_1,l_1} \times \ldots \times I_{j_k,l_k})}{\text{Leb}([0,t]^{2k})} \geq \frac{1}{t^{2k}} \left(\frac{\varepsilon}{4}\right)^{2k}$$

Since the last quantity does not depend on the permutations $(j_1,\ldots,j_k)$ and $(l_1,\ldots,l_k)$, we obtain that:

$$P_x(C_{i_1,\ldots,i_k}(t)|B_{i_1,\ldots,i_k}(t)) = (k!)^2 \frac{1}{t^{2k}} \left(\frac{\varepsilon}{4}\right)^{2k} \geq (k!)^2 \left(\frac{e}{k}\right)^{2k}$$

(39)

using (37) for the inequality. Relation (39) is the analogue of (4.7) of [21] (with $n = 2$) for the fractional noise.

Step 7. Third step for the lower bound of $E|u(t,x)|^2$

Combining (38) and (39), we get:

$$E|u(t,x)|^2 \geq u_0^2 e^{t^2} \alpha_H^2 \alpha_0^{2k} \gamma^{2k} \left(\frac{ct}{k}\right)^{2k} \left(\frac{e}{k}\right)^{2k} t^{(2H-2)k(k)} \frac{(k!)^2}{(\frac{e}{k})^{2k}} \times \sum_{i_1,\ldots,i_k \text{ distinct}} P_x(B_{i_1,\ldots,i_k}(t)).$$

We now use the fact that $\{N_t = k\}$ is the disjoint union of all events $B_{i_1,\ldots,i_k}(t)$ for all sets $\{i_1,\ldots,i_k\}$ of cardinality $k$. Moreover, $N_t$ has a Poisson distribution with mean $t^2$. Hence $P(N_t = k) = e^{-t^2} t^{2k}/k!$ and

$$E|u(t,x)|^2 \geq u_0^2 e^{t^2} \alpha_H^2 \alpha_0^{2k} \gamma^{2k} \left(\frac{ct}{k}\right)^{2k} \left(\frac{e}{k}\right)^{2k} t^{(2H-2)k(k)} \frac{(k!)^2}{(\frac{e}{k})^{2k}} \times \frac{1}{k^{2k}}$$

$$= u_0^2 (\alpha_0 \alpha_H \gamma^2 e^t)^k t^{(2H+2)k} \frac{1}{k^{4k}}$$

By Stirling’s formula, there exists some $k_0 \geq 1$ such that $k! \geq e^{-k} k^k$ for all $k \geq k_0$. It follows that if $k \geq k_0$, then

$$E|u(t,x)|^2 \geq u_0^2 \left(\alpha_0 \alpha_H \frac{t^{2H+2}}{k^k}\right)^k.$$
where \(c_H = \alpha_H \gamma^2 c^4 e^{-1}\) depends on \(H\). (\(c_H\) depends also on \(d\), through \(\gamma\).)

**Step 8. The choice of \(k\)**

Let

\[
k = \left[ e^{-1/3} \alpha_0 \frac{1}{3} c_H^{1/3} t^{(2H+2)/3} \right],
\]

where \([x] = k \in \mathbb{Z}\) if \(k \leq x < k + 1\). Since \(k \leq e^{-1/3} \alpha_0 \frac{1}{3} c_H^{1/3} t^{(2H+2)/3}\), it follows that \(e \leq \alpha_0 c_H t^{2H+2}/k^3\). On the other hand, letting

\[
k_1 = \frac{1}{2} (e^{-1} \alpha_0 c_H)^{1/3} = \alpha_0^{1/3} \cdot \frac{1}{2} (\alpha H \gamma^2 c^4 e^{-2})^{1/3} =: \alpha_1^{1/3} c_1
\]

we have \(k > 2k_1 t^{(2H+2)/3} - 1 \geq k_1 t^{(2H+2)/3}\) if \(k_1 t^{(2H+2)/3} \geq 1\). Using (40), we infer that:

\[
E|u(t, x)|^2 \geq u_0^2 e^k \geq u_0^2 \exp \left( k_1 t^{(2H+2)/3} \right) \quad \text{if} \quad \alpha_0 t^{2H+2} \geq t_1'' := c_3^{-3}.
\]

Note that \(k \geq k_0\) if \(\alpha_0 t^{2H+2} \geq t_1'' := c_3^{-3} e H\). We take \(t_1 = t_1' \vee t_1''\).

The next result corresponds to Theorem 2.1(c), in cases (ii)-(iii).

**Theorem 7.2.** Let \(f\) be a kernel of either case (ii) or (iii). If condition (DC) holds, then for any \(x \in \mathbb{R}^d\) and for any \(t \geq t_2\),

\[
E|u(t, x)|^2 \geq u_0^2 \exp(c_2 t^{p_e}),
\]

where \(c_2 > 0\) and \(t_2 > 0\) are some constants depending on \(H\) and \(a\), and the constants \(\rho_e\) and \(a\) are given by (7), respectively (5).

**Proof:** We use the same argument as in the proof of Theorem 4.1 but with a different method of specifying the parameters.

More precisely, we let \(k \in \mathbb{Z}_+\) be a large enough value (depending on \(t\)) which will be chosen later. Note that for any \(\delta > 0\), for any \(x \in \mathbb{R}^d\) with \(|x| \leq 2\delta\), we have \(f(x) \geq \alpha_0 (\delta) := (2\delta)^{-a}\), where \(a\) is given by (5).

We choose \(\delta = m/k\) where \(m = [2t]\). This ensures that (31) and (55) are satisfied. Note that \(\delta\) depends on \(t/k\).

Let \(c_H = \alpha H \gamma^2 c^4 e^{-1}\). Relation (40) says that if \(k \geq k_0\), then

\[
E|u(t, x)|^2 \geq u_0^2 \left(c_H (2\delta)^{-a} \frac{t^{2H+2}}{k^3} \right)^k = u_0^2 \left(c_H 2^{-a} \left(\frac{m}{k} \right)^{-a} \frac{t^{2H+2}}{k^3} \right)^k \geq u_0^2 \left(c_H 2^{-a} \left(\frac{2t}{k} \right)^{-a} \frac{t^{2H+2}}{k^3} \right)^k = u_0^2 \left(c_H 2^{H+2-a} \left(\frac{t^{2H+2} \cdot a}{k^3} \right)^k \right),
\]

where \(e^* = c_H 4^{-a}\). We let

\[
k = \left[ (e^{-1} c_H t^{2H+2-a})^{1/(3-a)} \right].
\]

Then \(e \leq c_H t^{2H+2-a}/k^{3-a}\). On the other hand, letting

\[
c_2 = \frac{1}{2} (e^{-1} c_H^{1/(3-a)} \]

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we have \( k > 2 c_2 t^{\rho_w} - 1 \geq c_2 t^{\rho_w} \) if \( c_2 t^{\rho_w} \geq 1 \). Hence

\[
E|u(t, x)|^2 \geq u_0^2 e^k \geq \exp(c_2 t^{\rho_w}) \quad \text{if} \quad t \geq t_2,
\]

where

\[
t_2 = (c c_H^{-1} n^{3+a})^{1/(2H+2-a)}.
\] (42)

Note that \( \delta \) is small since \( \delta \approx 2t/k = C t^{1-\rho_w} \) and \( \rho_w > 1 \). \( \Box \)

The remaining part of this section is dedicated to case (iv). For this, we will use an approximation technique based on case (ii).

For any \( a \in (0, 1) \), let \( W_a = \{W_a(\varphi) ; \varphi \in \mathcal{H}_a\} \) be an isonormal Gaussian noise with covariance \( E[W_a(\varphi) W_a(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}_a} \) where \( \langle \cdot, \cdot \rangle_{\mathcal{H}_a} \) is given by (II) with \( f(x) \) replaced by \( f_a(x) = |x|^{-a} \). Note that \( f = \mathcal{F} \mu_a \) where \( \mu_a(\xi) = (2\pi)^{-1} |\xi|^{a-1} d\xi \). Let \( u_a(t, x) \) be the solution of the equation

\[
\frac{\partial^2 u}{\partial t^2} = \Delta u + u \hat{W}_a, \quad (t > 0, x \in \mathbb{R})
\]

with initial conditions \( u(0, x) = u_0 \) and \( \frac{\partial u}{\partial t}(0, x) = v_0 \). This solution has the Wiener chaos expansion \( u_a(t, x) = \sum_{n>0} I_{n,a}(f_a(\cdot, t, x)) \) where \( I_{n,a} \) denotes the multiple Wiener integral with respect to \( W_a \). Hence,

\[
E|u_a(t, x)|^2 = \sum_{n>0} \frac{1}{n!} \alpha_{n,a}(t)
\]

where

\[
\alpha_{n,a}(t) = \alpha_H^n \int_{[0,t]^2n} \prod_{j=1}^{n} [t_j - s_j]^{2H-2} \psi_{n,a}(t, s) dt ds
\]

(43)

and

\[
\psi_{n,a}(t, s) = \int_{\mathbb{R}^{2n}} g_a^{(n)}(x_1, \ldots, x_n, t, x) g_a^{(n)}(y_1, \ldots, y_n, t, x) \prod_{j=1}^{n} f_a(x_j - y_j) dx dy
\]

(44)

and

\[
g_a^{(n)}(x_1, \ldots, x_n, t, x) = \prod_{j=1}^{n} G_w(t_{\rho(j+1)} - t_{\rho(j)}, x_{\rho(j+1)} - x_{\rho(j)}) w(t_{\rho(1)}, x_{\rho(1)})
\]

if \( t_{\rho(1)} < \cdots < t_{\rho(n)} \).

**Lemma 7.3.** For any integer \( n \geq 1 \) and for any \( t, s \in [0, t]^n \),

\[
\lim_{a \to 1} \psi_{n,a}(t, s) = \psi(t, s).
\]

**Proof:** Note that for any \( g, h \in L^{1}(\mathbb{R}) \cap L^{2}(\mathbb{R}) \),

\[
\lim_{a \to 1} \int_{\mathbb{R}} \int_{\mathbb{R}} g(x) h(y) f_a(x - y) dx dy = \lim_{a \to 1} \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}g(\xi) \mathcal{F}h(\xi) |\xi|^{a-1} d\xi =
\]

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from Lemma 4.1, it follows that for any $t > 0$, the integrand $|Fg(\xi)||Fh(\xi)||\xi|^{\alpha - 1}$ is bounded by the integrable function:
\[
\|g\|_1\|h\|_1|\xi|^{-1/2}1_{\{|\xi| \leq 1\}} + |Fg(\xi)||Fh(\xi)|1_{\{|\xi| \geq 1\}}.
\]

From here we infer that for any $g, h \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,
\[
\lim_{a \to 1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x)h(y) \prod_{j=1}^n f_a(x_j - y_j) dy dx = \int_{\mathbb{R}^n} g(x)h(x) dx.
\]

We apply this to $g = g^{(n)}_t(\cdot, t, x)$ and $h = g^{(n)}_s(\cdot, t, x)$.

**Lemma 7.4.** For any $t > 0$ and for any integer $n \geq 1$,
\[
\lim_{a \to 1} \alpha_{n,a}(t) = \alpha_n(t).
\]

**Proof:** This follows by Lemma 7.3 and the Dominated Convergence Theorem. It remains to justify the application of this theorem. For this, we note that $\psi_{n,a}(t, s) \leq \psi_{n,a}(t, t)^{1/2} \psi_{n,a}(s, s)^{1/2}$. Let $u_j = t_{\rho(j+1)} - t_{\rho(j)}$. As in the proof of Lemma 4.1, it follows that for any $t \geq 1$,
\[
\psi_{n,a}(t, t) \leq (u_0 + t v_0)^2 \frac{1}{(2\pi)^n} (4K_n)^n (u_1 \ldots u_n)^{2-a}
\]
\[
\leq (u_0 + t v_0)^2 \frac{1}{(2\pi)^n} (4K_n)^n t^{n(1-a)} u_1 \ldots u_n
\]
\[
\leq (u_0 + t v_0)^2 \frac{t^n}{(2\pi)^n} (4K_n)^n u_1 \ldots u_n,
\]
where $K_n := K(\mu_a)$ is given by (23). We now prove that
\[
K_n = L_n := \int_{\mathbb{R}} \frac{1}{1 + |\xi|^2} \mu_a(d\xi).
\]

To see this, note that for any $\eta \in \mathbb{R}$,
\[
\int_{\mathbb{R}} \frac{1}{1 + |\xi - \eta|^2} \mu_a(d\xi) = \int_{\mathbb{R}} e^{i\eta \xi} p(x)|x|^{-a} dx
\]
where $p(x) = (2\pi)^{-1/2} \int_0^\infty e^{-u_1/2} e^{-|x|^2/(2u_1)} du$ (see (3.4) of [20]). Taking the modulus on both sides and using (3.5) of [20], we obtain that for any $\eta \in \mathbb{R}$,
\[
\int_{\mathbb{R}} \frac{1}{1 + |\xi - \eta|^2} \mu_a(d\xi) \leq \int_{\mathbb{R}} p(x)|x|^{-a} dx = L_n.
\]
This concludes the proof of (45).
By considering separately the regions \( \{ |\xi| \leq 1 \} \) and \( \{ |\xi| \geq 1 \} \), we see that \( L_a \leq 2(a^{-1} + (2 - a)^{-1}) \). Hence \( L_a \leq 6 \) if \( a > 1/2 \).

Denote \( \beta(t) = \prod_{j=1}^n (t_{\rho(j+1)} - t_{\rho(j)}) \). It follows that for any \( a \in (1/2, 1) \),

\[
\psi_{n,a}(t, s) \leq (u_0 + tv_0)^2 \frac{t^n}{(2\pi)^n} 24^n |\beta(t)\beta(s)|^{1/2}.
\]

(46)

The claim is justified since \( \int_{[0,\delta]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} |\beta(t)\beta(s)|^{1/2} dt \) \( ds < \infty \) (see the proof of Theorem 4.2).

\[ \square \]

Lemma 7.5. For any \( t > 0 \) and for any \( x \in \mathbb{R}^d \),

\[
\lim_{a \to 1} E|u_a(t, x)|^2 = E|u(t, x)|^2.
\]

\[ \text{Proof:} \] The result follows by Lemma 7.4 and the Dominated Convergence Theorem. We justify the application of this theorem. By (43) and (46),

\[
\alpha_{n,a}(t) = (u_0 + tv_0)^2 \frac{t^n}{(2\pi)^n} 24^n \int_{[0,\delta]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} |\beta(t)\beta(s)|^{1/2} dt \) \( ds < \infty \) (see the proof of Theorem 4.2).

Since \( \sum_n e^n t^{(2H+1)n}/(n!)^2 < \infty \), the proof is complete. \[ \square \]

The next result corresponds to Theorem 2.1(c), in case (iv).

Theorem 7.6. Let \( f \) be the kernel of case (iv). Then, for any \( x \in \mathbb{R} \) and for any \( t \geq t_3 \), we have:

\[
E|u(t, x)|^2 \geq u_0^2 \exp(c_3 t^{\rho_w}),
\]

where \( c_3 > 0 \) and \( t_3 > 0 \) are some constants depending on \( H \), and \( \rho_w \) is given by (7).

\[ \text{Proof:} \] By Theorem 7.2 for any \( x \in \mathbb{R}^d \) and for any \( t \geq t_a \)

\[
E|u_a(t, x)|^2 \geq u_0^2 \exp(c_a t^{(2H+2-a)/(3-a)}),
\]

(47)

where the constants \( c_a > 0 \) and \( t_a > 0 \) are given by (11) and (12), i.e.

\[
c_a = \frac{1}{2} (e^{-1} c_H^{-4-a})^{1/(3-a)} \quad \text{and} \quad t_a = (e c_H^{-1} 2^{3+a})^{1/(2H+2-a)}.
\]

Let \( c_3 = \lim_{a \to 1} c_a \) and \( t_3' = \lim_{a \to 1} t_a \). Then \( t_a \leq 2t_3' =: t_3 \) for all \( a \in (a_0, 1) \). Fix \( t \geq t_3 \). We let \( a \to 1 \) in (47). The conclusion follows by Lemma 7.5 \[ \square \]
8 Parabolic case: proof of Theorem 2.2

In this section, we examine equation (SHE). We state and sketch the proof of two results, which together give the conclusion of Theorem 2.2. The proofs are similar to those presented above in the hyperbolic case, taking \( G = G_h \) and \( w = w_h \). For the lower bound, we use a FK representation similar to the one given in [1], except that here we work with processes \( X^1, X^2 \) defined by (49) and (50) below, instead of Brownian motions \( B^1, B^2 \).

We define a different constant \( K_h \) than in the hyperbolic case, namely

\[
K_h = \begin{cases} 
\mu(\mathbb{R}^d) & \text{in case (i),} \\
K(\mu) & \text{in cases (ii) and (iii),} \\
\sqrt{\pi} & \text{in case (iv).}
\end{cases}
\]

We remind that in the parabolic case, the spatial dimension \( d \geq 1 \) is arbitrary.

The first result gives the existence of the solution and an upper bound for its moments of order \( p \geq 2 \).

**Proposition 8.1.** Let \( f \) be a kernel of cases (i)-(iv), and \( \rho_h, a, K_h \) be the constants given by (7), (8), respectively (48). Assume that (DC) holds. Then:

a) for any \( t > 0 \) and for any integer \( n \geq 1 \),

\[
\alpha_n(t) \leq u_0^2 K_h^n e^{n(a/2)t((4H-a)n/2)}
\]

where \( c \) is a constant depending on \( H \) and \( a \);

b) equation (SHE) has a solution \( u(t, x) \) which has the following property: for any \( p \geq 2 \), for any \( x \in \mathbb{R}^d \) and for any \( t > 0 \) such that \( pK_h t((4H-a)/2) > t_0 \),

\[
E|u(t, x)|^p \leq (2u_0)^p \exp(c_0 K_h^{2/(2-a)}p^{(1-a)/(2-a)}t^{pa}),
\]

where \( c_0 > 0 \) and \( t_0 > 0 \) are some constants depending on \( H \) and \( a \).

**Proof:** a) Similarly to Lemma 4.1, it can be shown that:

\[
\psi_n(t, t) = u_0^2 \int_{\mathbb{R}^d} \exp(-u_1|\xi_1|^2) \cdots \exp(-u_n|\xi_1 + \cdots + \xi_n|^2) \mu(d\xi_1) \cdots \mu(d\xi_n)
\leq u_0^2 K_h^n (u_1 \cdots u_n)^{-a/2}.
\]

To prove this in cases (ii) and (iii), one uses the following inequality:

\[
\int_{\mathbb{R}^d} \exp(-t|\xi - \eta|^2) \mu(d\xi) \leq K(\mu)t^{-a/2}.
\]

The conclusion follows as in the proof of Proposition 4.2 a). Note that

\[
\psi_n(t, s) \leq u_0^2 K_h^n [\beta(t)\beta(s)]^{-a/4}.
\]
b) The conclusion follows as in the proof of Proposition 4.2.b) (case \( p = 2 \)), respectively Proposition 5.1 (case \( p > 2 \)). □

For the lower bound, we use the following representation for the second moment of the solution to \((SHE)\), which can be obtained as in Section 6:

\[
E|u(t, x)|^2 = e^{t^2 u_0^2} \sum_{n \geq 0} \sum_{i_1, \ldots, i_n \text{ distinct}} \mathbb{E}_x \left[ \prod_{j=1}^{n} f(X_{T_{i_j}}^1 - X_{S_{i_j}}^2) \alpha_H^n \prod_{j=1}^{n} (T_{i_j} - S_{i_j})^{2H-2} \right]_1 \mathbb{B}_{i_1 \ldots i_n}(t).
\]

Here, the event \( \mathbb{B}_{i_1 \ldots i_n}(t) \) and the points \((T_{i_j}, S_{i_j})\) are defined as in Section 6, but the processes \(X_1^i, X_2^i\) are given by:

\[
X_{s}^1 = X_{\tau_i}^1 + \sqrt{s - \tau_i} \Theta_{i+1}^{1} \quad \text{if} \quad \tau_i \leq s \leq \tau_{i+1} \\
X_{s}^2 = X_{\tau_i}^2 + \sqrt{s - \tau_i} \Theta_{i+1}^{2} \quad \text{if} \quad \tau_i \leq s \leq \tau_{i+1}
\]

where \((\Theta_{i}^1)_{i \geq 1}\) and \((\Theta_{i}^2)_{i \geq 1}\) are two independent collections of i.i.d. random variables with values in \(\mathbb{R}^d\) with the same law as \(\Theta_0\), and \(\Theta_0\) has a \(d\)-dimensional standard normal distribution. Note that in this case,

\[
G_h(t, \cdot) \quad \text{is the density of} \quad \sqrt{t} \Theta_0.
\]

(Alternatively, \(X^1, X^2\) can be two independent \(d\)-dimensional standard Brownian motions; see Remark 6.11 and [1].)

**Proposition 8.2.** Let \(f\) be a kernel of cases (i)-(iv), and \(\rho_h\) be the constant given by (7). Then for any \(x \in \mathbb{R}^d\) and for any \(t > t_0\)

\[
E|u(t, x)|^2 \geq u_0^2 \exp(c_0 t^\rho_h),
\]

where \(c_0 > 0\) and \(t_0 > 0\) are some constants depending on \(H\) and \(a\).

**Proof:** In case (i), the argument is similar to the one used in Theorem 7.1. One difference is that we replace \(\delta\) by \(\delta^2\). This is essentially due to the use of a parabolic rather than hyperbolic scaling (compare (51) with (28)). In addition, in the events \(D^1(t), D^2(t)\), we add the condition \(|\Theta_{j+1}^1| \leq 1\), respectively \(|\Theta_{j+1}^2| \leq 1\), for all \(j = 1, \ldots, k - 1\). Note that the variable \(\bar{Z}_t\) (in Step 5) is replaced by 1. Instead of (40), we obtain that for all \(k \geq k_0\),

\[
E|u(t, x)|^2 \geq u_0^2 \left( c_0 e^{t^{2H} k} \right)^k.
\]

The argument for cases (ii)-(iii) is similar to the proof of Theorem 7.2, leading to the following lower bound:

\[
E|u(t, x)|^2 \geq u_0^2 \left( c e^{t^{2H-a/2} k^{1-a/2}} \right)^k.
\]

The argument for case (iv) is similar to the proof of Theorem 7.6. Choosing \(k\) appropriately concludes the proof. □
A An elementary result

Lemma A.1. For any $a > 0$, we have

$$\sum_{n \geq 0} \frac{x^n}{(n!)^a} \leq 2 \exp(c_0 x^{1/a}) \quad \text{for all} \quad x \geq x_0,$$

where $c_0 > 0$ and $x_0 > 0$ are some constants depending on $a$.

Proof: By Stirling’s formula, $(n!)^{-a} \leq 2C_n/\Gamma(an+1)$ for $n$ large enough. Here $C_n > 0$ is such that $\lambda^{-n} \leq C_n \leq \lambda^n$ for some constant $\lambda > 1$ depending on $a$. The conclusion follows from the asymptotic behavior of the Mittag-Leffler function (see [24]): there exist some $c_0 > 0$ and $x_0 > 0$ such that

$$\sum_{n \geq 0} \frac{x^n}{\Gamma(an+1)} \leq \exp\left(c_0 x^{1/a}\right) \quad \text{for all} \quad x > x_0.$$

□

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