Application of some combinatorial arrays in
coloring of total graph of a commutative ring *†

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Abstract

Let $R$ be a commutative ring with unity and $Z(R)$ and $\text{Reg}(R)$ be the set of zero-divisors and non-zero zero-divisors of $R$, respectively. We denote by $T(\Gamma(R))$, the total graph of $R$, a simple graph with the vertex set $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in Z(R)$. The induced subgraphs on $Z(R)$ and $\text{Reg}(R)$ are denoted by $Z(\Gamma(R))$ and $\text{Reg}(\Gamma(R))$, respectively. These graphs were first introduced by D.F. Anderson and A. Badawi in 2008. In this paper, we prove the following result: let $R$ be a finite ring and one of the following conditions hold: (i) The residue field of $R$ of minimum size has even characteristic, (ii) Every residue field of $R$ has odd characteristic and $\frac{R}{\text{Nil}(R)}$ has no summand isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, then the chromatic number and clique number of $T(\Gamma(R))$ are equal to $\max\{|m| : m \in \text{Max}(R)\}$. The same result holds for $Z(\Gamma(R))$. Moreover, if the residue field of $R$ of minimum size has even characteristic or every residue field of $R$ has odd characteristic, then we determine the chromatic number and clique number of $\text{Reg}(\Gamma(R))$ as well.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with unity. Let $R$ be a ring. We denote by $U(R)$, $Z(R)$, $Z^*(R)$, $\text{Reg}(R)$, $\text{Min}(R)$, $\text{Spec}(R)$ and $\text{Max}(R)$, the set of invertible elements, zero-divisors, non-zero zero-divisors, regular elements, minimal prime ideals, prime ideals and maximal ideals of $R$, respectively. The Jacobson radical and the nilradical of $R$ are denoted by $J(R)$ and $\text{Nil}(R)$, respectively. The ring $R$ is said to be reduced if it has no non-zero nilpotent element. The Krull dimension of $R$ is denoted by $\text{dim}(R)$. A local ring is a ring with exactly one maximal ideal. A ring with finitely many maximal ideals is called a semi-local ring. The set of associated prime ideals of an $R$-module $R$ is denoted by $\text{Ass}(R) = \{p \in \text{Spec}(R) : p = \text{Ann}(x), \text{ for some } x \in R\}$. For classical theorems and notations in commutative algebra, the interested reader is referred to [4] and [5].

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Let $G$ be a graph with the vertex set $V(G)$. The complement of $G$ is denoted by $\overline{G}$. If $G$ is connected, then we mean by $\text{diam}(G)$, the diameter of $G$. If $G$ is not connected, then $\text{diam}(G)$ is defined to be $\infty$. We denote by $K_X$ and $K_{X,Y}$, the complete graph with the vertex set $X$ and the complete bipartite graph with two parts $X$ and $Y$, respectively. The direct product (sometimes called Kronecker product or tensor product) of two graphs $G$ and $H$, denoted by $G \times H$, is a graph with the vertex set $V(G) \times V(H)$ and two distinct vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if $x_1$ and $x_2$ are adjacent in $G$ and $y_1$ and $y_2$ are adjacent in $H$. A clique in a graph $G$ is a subset of pairwise adjacent vertices and the supremum of the size of cliques in $G$, denoted by $\omega(G)$, is called the clique number of $G$. By $\chi(G)$, we denote the chromatic number of $G$, i.e. the minimum number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. A coloring of the vertices that any two adjacent vertices have different colors, is called a proper vertex coloring. Let $r \leq n$ be two positive integers. A Latin rectangle is an $r \times n$ matrix whose entries are $n$ distinct symbol with no symbol occurring more than once in any row or column.

For a commutative ring $R$, the total graph of $R$, denoted by $T(\Gamma(R))$, is a graph with the vertex set $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in Z(R)$. This is the Cayley sum graph (or sometimes is called addition Cayley graph) $\text{Cay}^+(R, Z(R))$. For more information on Cayley sum graphs see [6], [7] and the other references there. The authors in [2] and [3] have studied $\Gamma(R)$ and $\text{Reg}(\Gamma(R))$ on $\text{Reg}(R)$ and $Z(R)$, respectively. Some other properties of $\text{Reg}(\Gamma(R))$ are studied in [4]. The authors in [1], characterize all rings $R$ such that $2 \notin Z(R)$ and $\text{Reg}(\Gamma(R))$ is a complete graph. They also prove that if $\text{Reg}(\Gamma(R))$ is a tree, then it has at most two vertices. In this paper, we investigate the clique number and the chromatic number of $T(\Gamma(R))$ and its two subgraphs $Z(\Gamma(R))$ and $\text{Reg}(\Gamma(R))$ as well.

2. Preliminaries on the Total Graph of a Ring

In this section, for a commutative ring $R$, we provide some preliminary lemmas on $T(\Gamma(R))$.

**Lemma 1.** ([8] Theorem 91) Let $R$ be a zero-dimensional ring. Then $Z(R)$ is the union of all maximal ideals of $R$. In particular, $\text{Reg}(R) = U(R)$.

**Lemma 2.** Let $R$ be a zero-dimensional ring, $x \in R$ and $a \in \text{Nil}(R)$. Then $x + a \in Z(R)$ if and only if $x \in Z(R)$.

**Proof.** Let $u$ be an arbitrary element of $R$. It is clear that $u \in U(R)$ if and only if $u + a \in U(R)$. Since $Z(R) = R \setminus \text{Reg}(R)$, the assertion immediately follows from Lemma 1.

**Lemma 3.** Let $R$ be a zero-dimensional ring and $x \in R$. Then $x + \text{Nil}(R) \in Z(\frac{R}{\text{Nil}(R)})$ if and only if $x \in Z(R)$.

**Proof.** Since $R$ is zero-dimensional, we have $\text{Nil}(R) = J(R)$ and $\text{Max}(\frac{R}{\text{Nil}(R)}) = \{ \frac{m}{\text{Nil}(R)} \mid m \in \text{Max}(R) \}$. Thus, for a maximal ideal $m$ of $R$, we have $x \in m$ if and only if $x + \text{Nil}(R) \in \frac{m}{\text{Nil}(R)}$. 

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Since $R_{\text{Nil}(R)}$ is zero-dimensional, by Lemma 1 we conclude that $x + \text{Nil}(R) \in Z(R_{\text{Nil}(R)})$ if and only if $x \in Z(R)$. The proof is complete. □

**Lemma 4.** (Theorems 2.1 and 2.2) Let $R$ be a ring and $Z(R)$ be an ideal of $R$. Then the following statements hold.

(i) If $2 \in Z(R)$, then $T(\Gamma(R))$ is a disjoint union of $K_{|Z(R)|}$ complete graphs $K_{|Z(R)|}$ on a coset of $Z(R)$.

(ii) If $2 \notin Z(R)$, then $T(\Gamma(R))$ is a disjoint union of $K_{|Z(R)|}$ and $K_{|Z(R)|-1}$ complete bipartite graphs $K_{|Z(R)|}$.

3. Coloring of the total graph of a ring

In this section we study the chromatic number of the total graph of a finite commutative ring. We conjecture that for every finite ring $R$, the following equalities hold:

$$\chi(T(\Gamma(R))) = \omega(T(\Gamma(R))) = \begin{cases} 4; & R \cong \mathbb{Z}_3 \times \mathbb{Z}_3, \\ \max\{|m| : m \in \text{Max}(R)|; \text{otherwise}. \end{cases} \quad (1)$$

**Remark 5.** In order to determine the clique number and the chromatic number of $T(\Gamma(R))$, we assume that $R$ is a finite ring because if $R$ is an infinite ring which is not a domain, then for every non-zero zero-divisor $x$ of $R$, $R_{\text{Ann}(x)} \cong Rx$, as $R$-modules. Since both $Rx$ and $\text{Ann}(x)$ form a clique for $Z(\Gamma(R))$, we deduce that $Z(\Gamma(R))$ has an infinite clique and so $T(\Gamma(R))$ does.

In the sequel, we prove (1) for many families of rings. In this direction, first we reduce the problem of coloring of the total graph of a finite ring to the problem of coloring of total graph of a finite reduced ring.

**Definition 6.** Let $G$ be a graph, $V(G) = \{v_1, \ldots, v_n\}$, $H_1, \ldots, H_n$ be $n$ graphs of order $m$. Let $G(H_1, \ldots, H_n)$ be a graph with the vertex set $\cup_{i=1}^n V(H_i)$ such that:

(i) Any vertex of $H_i$ is adjacent to any vertex of $H_j$ if and only if $v_i$ is adjacent to $v_j$ in $G$.

(ii) Two vertices of $H_i$ are adjacent in $G(H_1, \ldots, H_n)$ if and only if they are adjacent in $H_i$.

If for every $i$, $1 \leq i \leq n$, $H_i$ has no edge, then the resulting graph is called a balanced blow-up. If for every $i$, $H_i \cong K_m$, then we denote $G(H_1, \ldots, H_n)$ by $\hat{G}(m)$. We state the following simple lemma without proof.

**Lemma 7.** Let $G$ be a graph and $H_1, \ldots, H_n$ be simple graphs of order $m$. Then

$$\chi(G(H_1, \ldots, H_n)) \leq m\chi(G).$$

**Lemma 8.** If (1) holds for every finite reduced ring, then it holds for every finite ring.
**Proof.** Let \( R \) be a finite ring, \( |R/\mathfrak{m}_{(R)}| = k \) and \( |R/\mathfrak{m}_{1}, \ldots, \mathfrak{m}_{t}| = \{ f_{1} + J(R), \ldots, f_{k} + J(R) \} \). For \( 1 \leq i \leq k \), we define \( H_{i} \) as \( K_{i,J(R)} \) if \( 2f_{i} \in Z(R) \) and as \( \mathbb{K}_{i,J(R)} \), otherwise. By Lemma 3

\[
T(\Gamma(R)) \cong T\left( \Gamma\left( \frac{R}{J(R)} \right) \right)(H_{1}, \ldots, H_{k}).
\]

Thus, by Lemma 7

\[
\chi\left( T(\Gamma(R)) \right) \leq |R|\chi\left( T\left( \Gamma\left( \frac{R}{J(R)} \right) \right) \right).
\]

(2)

Now, we are ready to prove the assertion. Suppose that (1) holds for every finite reduced ring and let \( R \) be a finite ring such that \( \frac{R}{\mathfrak{m}} \) is not isomorphic to \( \mathbb{Z}_{3} \times \mathbb{Z}_{3} \). Since

\[
\chi(T(\Gamma(R))) = \max\{|m| : m \in \text{Max}(\frac{R}{J(R)})\},
\]

by (2), we obtain that

\[
\chi(T(\Gamma(R))) \leq \max\{|m| : m \in \text{Max}(R)\}.
\]

On the other hand, by Lemma 1 \( \omega(T(\Gamma(R))) \geq \max\{|m| : m \in \text{Max}(R)\} \). Hence

\[
\chi(T(\Gamma(R))) = \omega(T(\Gamma(R))) = \max\{|m| : m \in \text{Max}(R)\}.
\]

Now, assume that \( \frac{R}{J(R)} \cong \mathbb{Z}_{3} \times \mathbb{Z}_{3} \). If \( J(R) = \{0\} \), then using Figure 1, \( \chi(T(\Gamma(R))) = \omega(T(\Gamma(R))) = 4 \).

![Figure 1](image-url)

**Figure 1.** A coloring of \( T(\Gamma(\mathbb{Z}_{3} \times \mathbb{Z}_{3})) \) using 4 colors.

Thus, assume that \( |J(R)| \geq 2 \). Since by Chinese Reminder Theorem (see [4, Proposition 1.10]), \( \frac{R}{J(R)} \cong \frac{R}{\mathfrak{m}_{1}} \times \frac{R}{\mathfrak{m}_{2}} \), where \( \text{Max}(R) = \{ m_{1}, m_{2} \} \), we may assume that \( R \) is partitioned into \( (0,0) + J(R), (0,1) + J(R), (0,-1) + J(R), (1,0) + J(R), (1,1) + J(R), (1,-1) + J(R), (-1,0) + J(R), (-1,1) + J(R), (-1,-1) + J(R) \). Now, we color the the complete graph induced on \( (0,0) + J(R) \) by colors \( a_{1}, \ldots, a_{|J(R)|} \), the complete graphs induced on \( (0,1) + J(R) \) and \( (1,0) + J(R) \) by \( a'_{1}, \ldots, a'_{|J(R)|} \) and the complete graphs induced on \( (0,-1) + J(R) \) and \( (-1,0) + J(R) \) by \( a''_{1}, \ldots, a''_{|J(R)|} \). Now, the remaining vertices form a complete 4-partite graph with parts \( (1,1) + J(R), (1,-1) + J(R), (-1,1) + J(R), (-1,-1) + J(R) \). We color the vertices in part
(1, 1) + J(R) by \(a'_1\), the vertices in part \((-1, -1) + J(R)\) by \(a''_1\), the vertices in part \((1, -1) + J(R)\) by \(a_1\) and the vertices in part \((-1, 1) + J(R)\) by \(a_2\). Using Lemma 3 it can be seen that this is a proper coloring of \(T(\Gamma(R))\) with \(3|J(R)| = \max\{|m| : m \in \text{Max}(R)\}\) colors. The proof is complete. \(\square\)

**Definition 9.** Suppose that \(F_1\) and \(F_2\) are two finite fields, \(|F_1| \leq |F_2|\), \(A \subseteq F_1\), \(B \subseteq F_2\) and \(|A| \leq |B|\). Let \(L\) be a table of size \(|A| \times |B|\) over alphabet \(C\) whose rows are labelled by the elements of \(A\) and columns are labelled by the elements of \(B\). The table \(L\) is called a \((|A| \times |B|)\) Latin-sum array over \(C\) if for every two distinct pairs \((x_1, y_1)\) and \((x_2, y_2)\) of elements of \(A \times B\) both following conditions hold:

(i) If \(x_1 + x_2 = 0\) in \(F_1\), then \(L_{x_1,y_1} \neq L_{x_2,y_2}\).

(ii) If \(y_1 + y_2 = 0\) in \(F_2\), then \(L_{x_1,y_1} \neq L_{x_2,y_2}\).

**Remark 10.** If both fields have characteristic 2, then every Latin rectangle has this property. The problem is the existence of these arrays for the other characteristics.

**Lemma 11.** For every two finite fields \(F_1\) and \(F_2\) with \(|F_1| \leq |F_2|\), there exists a \(|F_1| \times |F_2|\) Latin-sum array over alphabet \(C\), where

\[
|C| = \begin{cases} 
4; & F_1 = F_2 = \mathbb{Z}_3, \\
|F_2|; & \text{otherwise}. 
\end{cases}
\]

**Proof.** We consider three following cases.

**Case 1.** First suppose that both \(F_1\) and \(F_2\) are of characteristic 2. Thus, as we mentioned above, every Latin rectangle over \(F_2\) has this property.

**Case 2.** Now, suppose that both have odd characteristic. Let \(n = \frac{|F_1| - 1}{2}\) and \(m = \frac{|F_2| - 1}{2}\). Let \(F_1 = \{0, x_1, -x_1, \ldots, x_n, -x_n\}\) and \(F_2 = \{0, y_1, -y_1, \ldots, y_m, -y_m\}\). Now, we construct a \(|F_1| \times |F_2|\) table over \(\{0, \pm 1, \ldots, \pm m\}\), say \(L\), whose rows are labelled by 0, \(x_1, -x_1, \ldots, x_n, -x_n\) and columns are labelled by 0, \(y_1, -y_1, \ldots, y_m, -y_m\), respectively. To construct \(L\) as a Latin-sum array over a set \(C\), we should arrange the elements of \(C\) in such a way that the following conditions are satisfied:

**C1:** The entries of the first row are pairwise distinct,

**C2:** The entries of the first column are pairwise distinct,

**C3:** The rows corresponding to \(x_i\) and \(-x_i\) have no common entry, for \(1 \leq i \leq n\),

**C4:** The columns corresponding to \(y_j\) and \(-y_j\) have no common entry, for \(1 \leq j \leq m\).

To satisfy these four conditions, it suffices to consider one of the following \(|F_1| \times |F_2|\) arrays. Note that Table 1 involves the case \(F_1 = F_2 = \mathbb{Z}_3\). In this case, the alphabet \(C\) has 4 elements but for \(|F_2| > 3\), it has exactly \(|F_2|\) elements.
Case 3. Now, assume that exactly one $F_i$, for $i = 1, 2$, is of characteristic 2. First suppose that $\text{char}(F_1) = 2$ and $F_2$ is of odd characteristic. Let $|F_1| = n$, $F_1 = \{x_1, x_2, \ldots, x_n\}$ and $F_2 = \{0, y_1, -y_1, \ldots, y_m, -y_m\}$. To continue the proof, we prove the following stronger assertion: we construct a $|F_1| \times |F_2|$ table $L$ whose rows are labeled by $x_1, \ldots, x_n, \ldots, x_{2m+1}$ and columns are labeled by $0, y_1, -y_1, \ldots, y_m, -y_m$. We arrange the entries of $L$ in such a way that the following conditions are satisfied:

**D1:** The entries of each row are pairwise distinct,

**D2:** The entries of the first column are pairwise distinct,

**D3:** The columns corresponding to $y_j$ and $-y_j$ have no common entry, for $1 \leq j \leq m$.

Regarding **D1**, we define $1, 2, \ldots, 2m + 1$ as the first row. We construct the $i$-th row from the $(i - 1)$-th row by changing two symbols in the $(i - 1)$-th row such that the conditions **D1**, **D2** and **D3** are satisfied for the $j$-th row, $j = 1, \ldots, i$. To construct the second row, we replace
the symbol 1 with 2 in the first row and consider the new $1 \times (2m + 1)$ vector as the second row. To construct the third row, we replace the first entry of the second row by 3 and then change two symbols 2 and 4 with each other (as $L_{1,2} = 2$, to satisfy $D3$). It can be easily checked that the entries in each row are pairwise distinct and $D3$ is satisfied in each step. We construct the fourth row from the third row by replacing the first entry of the third row by 4. For the fifth row we do similarly except that we make two changes. We can continue this procedure to construct the $(2m)$-th row, i.e. in the Step $2i$ we make just two changes and in the Step $2i + 1$ we make three changes, for $1 \leq i \leq m$. For the $(2m + 1)$-th row, we first replace the first entry of the $(2m)$-th row with the $(2m + 1)$. Now, $(2m)$ appears in the two last columns. Now, we change $(2m)$ and 1. We obtain the desired Latin-sum array. In the below, we can see the resulting Latin-sum array for $|F_2| = 7$.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| 2 | 1 | 3 | 4 | 5 | 6 | 7 |
| 3 | 1 | 4 | 2 | 5 | 6 | 7 |
| 4 | 1 | 3 | 2 | 5 | 6 | 7 |
| 5 | 1 | 3 | 2 | 6 | 4 | 7 |
| 6 | 1 | 3 | 2 | 5 | 4 | 7 |
| 7 | 6 | 3 | 2 | 5 | 4 | 1 |

Now, suppose that $F_1$ has odd characteristic and $F_2$ has characteristic 2. Let $L$ be the above $|F_2| \times |F_2|$ array in which conditions $D1$, $D2$ and $D3$ hold. Thus, the first $|F_1|$ rows of the transpose of $L$ is the desired Latin-sum array. The proof is complete. □

Now, we determine the chromatic number of the total graph of a direct product of finitely many fields.

**Lemma 12.** Let $n \geq 2$ be a positive integer and $F_1, \ldots, F_n$ be finite fields such that $|F_1| \leq \cdots \leq |F_n|$. Suppose that one of the following conditions holds:

(i) The field $F_1$ has even characteristic.
(ii) Every $F_i$ has odd characteristic and $F_1 \times F_2 \not\cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Then $\chi(T(\Gamma(F_1 \times \cdots \times F_n))) = |F_2| \cdots |F_n|$.

**Proof.** For $i = 2, \ldots, n$, let $L_{F_i}^{F_1}$ be the Latin-sum array deduced from Lemma 11. We define the following map $f$ on $F_1 \times \cdots \times F_n$ as follows:

$$f((x_1, \ldots, x_n)) = (L_{x_1}^{F_1, F_2}, \ldots, L_{x_n}^{F_1, F_n}),$$

where $L_{x_i}^{F_1, F_k}$ is the $(x_i, x_j)$-entry of $L_{x_1}^{F_1, F_k}$. Now, we prove that $f$ is a proper vertex coloring for $T(\Gamma(F_1 \times \cdots \times F_n))$ in both Cases (i) and (ii). First, suppose that Case (i) holds. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two adjacent vertices of $T(\Gamma(F_1 \times \cdots \times F_n))$ with the same color. Hence, there exists $1 \leq j \leq n$, such that $x_j + y_j = 0$. First assume that $j = 1$. Since $L_{x_1}^{F_1, F_1}$ is a Latin-sum array and

$$f((x_1, \ldots, x_n)) = f((y_1, \ldots, y_n)),$$

we conclude that $x_i = y_i$, for $i = 2, \ldots, n$. Thus,
Let $R$ be a finite ring such that one of the following conditions holds:

(i) The residue field of $R$ of minimum size has even characteristic,

(ii) Every residue field of $R$ has odd characteristic and $\frac{R}{\mathfrak{m}}$ has no summand isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Then the following equalities hold:

$$\chi(T(\Gamma(R))) = \omega(T(\Gamma(R))) = \omega(Z(\Gamma(R))) = \max\{|m| : m \in \text{Max}(R)|\}.$$  

**Proof.** By Lemma 8 Chinese Reminder Theorem (see 4 Proposition 1.10) and Lemma 12 we find that $\chi(T(\Gamma(R))) = \omega(T(\Gamma(R))) = \max\{|m| : m \in \text{Max}(R)|\}$. On the other hand, by Lemma 1 $\omega(Z(\Gamma(R))) \geq \max\{|m| : m \in \text{Max}(R)|\}$. The proof is complete. $\Box$

**Remark 14.** Here, as an example of the correctness of (1), we will provide a proper vertex coloring of $T(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3))$ using 9 colors which is not deduced from the previous theorems. The coloring classes are as follow.

$$\begin{align*}
\{(1,0,1), (1,1,0), (0,1,1), (1,1,1)\} & \quad \{-1,0,1\}, \{1,1,1\}, \{0,1,0\} \\
\{-1,0,-1\}, \{0,-1,0\}, \{-1,-1,-1\} & \quad \{-1,0,0\}, \{0,-1,1\} \\
\{1,0,0\}, \{0,1,-1\}, \{1,1,-1\} & \quad \{-1,1,0\}, \{0,0,-1\}, \{-1,1,-1\} \\
\{-1,-1,0\}, \{-1,-1,1\}, \{0,0,1\} & \quad \{0,0,0\}, \{1,-1,1\} \\
\{1,1,0\}, \{0,1,-1\}, \{1,-1,-1\}, \{1,0,1\} & \quad \{0,1,0\}, \{1,1,1\}
\end{align*}$$

4. The induced subgraph on regular elements

In [II], for a Noetherian ring $R$ with $2 \notin Z(R)$, the clique number and the chromatic number of $\text{Reg}(\Gamma(R))$ are studied. Indeed, the following result was proved.

**Theorem 15.** ([II, Theorem 6]) Let $R$ be a ring and $2 \notin Z(R)$. If $Z(R) = \bigcup_{i=1}^{n} P_i$, where $P_1, \ldots, P_n$ are prime ideals of $R$ and $Z(R) \neq \bigcup_{i \neq j} P_i$, for $j = 1, \ldots, n$, then $\chi\left(\text{Reg}(\Gamma(R))\right) = \omega\left(\text{Reg}(\Gamma(R))\right) = 2^{\max\{|m| : m \in \text{Max}(R)|\}}$.

**Corollary 16.** Let $R$ be a finite ring. If every residue field of $R$ has odd characteristic, then $\chi\left(\text{Reg}(\Gamma(R))\right) = \omega\left(\text{Reg}(\Gamma(R))\right) = 2^{\max\{|m| : m \in \text{Max}(R)|\}}$.
Proof. Since $R$ is finite, by Lemma 1, $Z(R) = \bigcup_{i=1}^{n} m_{i}$, where $\text{Max}(R) = \{m_{1}, \ldots, m_{n}\}$. Since every residue field of $R$ has odd characteristic, we deduce that $2 \notin \bigcup_{i=1}^{n} m_{i}$. Now using the Prime Avoidance Theorem ([9, Theorem 3.61]), the assertion immediately follows from Theorem 15. □

Similar to the previous results, we are going to determine the clique number and the chromatic number of $\text{Reg}(\Gamma(R))$, for a finite ring $R$ with $2 \in Z(R)$. We start with the following lemma.

Lemma 17. Let $R$ be a finite ring. Then the following holds.

\[
\chi\left(\text{Reg}(\Gamma(R))\right) \leq |J(R)| \chi\left(\text{Reg}(\Gamma\left(\frac{R}{J(R)}\right))\right).
\]

Moreover, if $2 \notin Z(R)$, then \(\chi\left(\text{Reg}(\Gamma(R))\right) = \chi\left(\text{Reg}(\Gamma\left(\frac{R}{J(R)}\right))\right)\).

Proof. The proof of the first statement is like to the proof of Lemma 8. To prove the second statement, note that if $2 \notin Z(R)$, then $\text{Reg}(\Gamma(R))$ is a balanced blow-up of $\text{Reg}(\Gamma\left(\frac{R}{J(R)}\right))$. Hence, \(\chi\left(\text{Reg}(\Gamma(R))\right) = \chi\left(\text{Reg}(\Gamma\left(\frac{R}{J(R)}\right))\right)\). □

Lemma 18. Let $F_1$ and $F_2$ be two finite fields with $|F_1| \leq |F_2|$. If $\text{char}(F_1) = 2$, then there exists a $((|F_1| - 1) \times (|F_2| - 1)$ Latin-sum array over $F_2 \setminus \{0\}$.

Proof. If both $F_1$ and $F_2$ have characteristic 2, then it suffices to consider a $((|F_1| - 1) \times (|F_2| - 1)$ Latin rectangle. Otherwise, construct a $((|F_1| - 1) \times (|F_2| - 1)$ table whose every row is $a_1, \ldots, a_{|F_2|-1}$, where $F_2 = \{0, a_1, \ldots, a_{|F_2|-1}\}$. In this table, the rows are labelled by the elements of $F_1 \setminus \{0\}$ and the columns are labelled by the elements of $F_2 \setminus \{0\}$. □

The proof of following result is similar to the proof of Lemma 12.

Lemma 19. Let $n \geq 2$ be a positive integer and $F_1, \ldots, F_n$ be finite fields with $|F_1| \leq \cdots \leq |F_n|$. If $\text{char}(F_1) = 2$, then \(\chi(\text{Reg}(\Gamma(F_1 \times \cdots \times F_n))) = \omega(\text{Reg}(\Gamma(F_1 \times \cdots \times F_n))) = (|F_2| - 1) \cdots (|F_n| - 1)\).

Proof. For $i = 2, \ldots, n$, let $L_{F_1,F_i}^{F_i}$ be the $((|F_1| - 1) \times (|F_i| - 1)$ Latin-sum array over $F_i \setminus \{0\}$ deduced from Lemma 18. We define the following map $g$ on $F_1 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}$ as follows:

\[
g((x_1, \ldots, x_n)) = (L_{x_1,x_2}^{F_1,F_2}, \ldots, L_{x_1,x_n}^{F_1,F_n}),
\]

where $L_{x_1,x_j}^{F_1,F_k}$ is the $(x_1, x_j)$-entry of $L_{F_1,F_i}^{F_i}$. Similar to the proof of Lemma 12, one may prove that $g$ is a proper vertex coloring for $\text{Reg}(\Gamma(F_1 \times \cdots \times F_n))$. On the other hand, $\{1\} \times \cdots \times F_n \setminus \{0\}$ is a clique for $\text{Reg}(\Gamma(F_1 \times \cdots \times F_n))$. Thus, \(\chi(\text{Reg}(\Gamma(F_1 \times \cdots \times F_n))) = (|F_2| - 1) \cdots (|F_n| - 1)\). The proof is complete. □

Now, we are in a position to prove the following result.
Theorem 20. Let $R$ be a finite ring and $m$ be a maximal ideal of $R$ of maximum size. If $\frac{R}{m}$ has characteristic 2, then

$$\chi\left(\text{Reg}(\Gamma(R))\right) = \omega\left(\text{Reg}(\Gamma(R))\right) = \frac{|\text{Reg}(R)|}{|\frac{R}{m}| - 1}.$$ 

Proof. First assume that $m$ is the unique maximal ideal of $R$. Since $\frac{R}{m}$ has characteristic 2, we deduce that $2 \in m$. Thus, $\text{Reg}(\Gamma(R))$ is a disjoint union of complete graphs with $|m|$ vertices. Hence $\chi\left(\text{Reg}(\Gamma(R))\right) = \omega\left(\text{Reg}(\Gamma(R))\right) = |m|$. Since $\text{Reg}(R) = R \setminus m$, the assertion follows. Therefore, one may assume that $|\text{Max}(R)| \geq 2$. Let $\text{Max}(R) = \{m_1, \ldots, m_n\}$, where $n \geq 2$ and $m_1 = m$. Since $m$ has the maximum size and $\frac{R}{m}$ has characteristic 2, by Chinese Reminder Theorem (see [4, Proposition 1.10]) and Lemma 19, we obtain

$$\chi\left(\text{Reg}(\Gamma\left(\frac{R}{J(R)}\right))\right) = (\frac{|R|}{|m_2|} - 1) \cdots (\frac{|R|}{m_n} - 1).$$

Moreover, Lemma 3 implies that $|\text{Reg}(R)| = |J(R)||\text{Reg}(\frac{R}{J(R)})|$. Hence, by Chinese Reminder Theorem we obtain

$$|\text{Reg}(R)| = |J(R)|(|\frac{R}{m_1} - 1)(|\frac{R}{m_2} - 1| \cdots (|\frac{R}{m_n} - 1|.$$ 

Therefore, we conclude that

$$|J(R)|\chi\left(\text{Reg}(\Gamma\left(\frac{R}{J(R)}\right))\right) = \frac{|\text{Reg}(R)|}{|\frac{R}{m}| - 1}.$$ 

On the other hand, by Lemma 3 and Lemma 19 we deduce that

$$\omega\left(\text{Reg}(\Gamma(R))\right) = |J(R)|\omega\left(\text{Reg}(\Gamma\left(\frac{R}{J(R)}\right))\right) = \frac{|\text{Reg}(R)|}{|\frac{R}{m}| - 1}.$$ 

Now, Lemma 17 completes the proof. \qed

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