The Classical $r$-Matrix for the Relativistic Ruijsenaars-Schneider System

by

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ABSTRACT

We compute the classical $r$-matrix for the relativistic generalization of the Calogero-Moser model, or Ruijsenaars-Schneider model, at all values of the speed-of-light parameter $\lambda$. We connect it with the non-relativistic Calogero-Moser $r$-matrix ($\lambda \to -1$) and the $\lambda = 1$ sine-Gordon soliton limit.

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1 Introduction

Our understanding of classical and quantum dynamical $r$-matrices has recently known a number of interesting developments. Dynamical classical $r$-matrices were found for the Calogero-Moser models [1] and their spin extension [2] in the rational, trigonometric/hyperbolic and elliptic cases [3, 4, 5]. This lead to the construction of observable algebras of $W_n$-type [6] containing in particular the classical version of the Yangian subalgebra already found as a symmetry of the spin (or Euler) Calogero-Moser [7, 8]. Moreover a quantum analogue of the classical (dynamical) Yang-Baxter equation for these $r$-matrices was found [9, 10] leading to the construction of commuting Hamiltonians for the spinless (Calogero-Moser) and spin (Calogero-Sutherland) quantum systems [11].

A relativistic-invariant extension of the Calogero-Moser model has been constructed and shown to be integrable [12]. It describes the dynamics of solitons of the sine-Gordon model for a particular choice of parameters [12, 13]. A spin extension was recently proposed [14]. This model has several remarkable features. First of all it was shown [15] to be a Hamiltonian-type reduction of a simpler model describing a free dynamics on a Heisenberg double [16] of a Lie group (instead of a cotangent bundle as in the Calogero-Moser case). This in turn explained the duality symmetry of the corresponding quantum system [17]. The trigonometric Ruijsenaars-Schneider model depends on two parameters, corresponding to the speed-of-light and the radius of a compactifying circle. In turn they are related to two parameters associated with the two copies of the group $SL(N)$ which build the original Heisenberg double with its particular Poisson structure. Hence the RS model exhibits a self-duality property under suitable interchange of these parameters. Consequently also, the limits when the speed-of-light parameter $\lambda$ goes to $-1$ (i.e. the non-relativistic trigonometric CM model) and when the compactifying radius goes to $+\infty$ (i.e. the rational relativistic RS model) are dual ( "Ruijsennars duality") [17] to each other. Finally the wave functions of this model were shown to be expressed by a path-integral formula for the $G/G$ gauged Wess-Zumino-Witten model [18], a topological field theory over a cylinder with a marked line.

Our main purpose in this paper will be to find the classical $r$-matrix of this system for all values of its parameters, particularly $\lambda$. Its existence is guaranteed [19] by the explicit but cumbersome proof of commutation of the Lax matrix adjoint invariants [12]. Its structure should reflect the Hamiltonian reduction procedure on the Heisenberg double. Moreover the Calogero-Moser $r$-matrix [3] arises from a non-relativistic limit, and the "Sine-Gordon Soliton" $r$-matrix [13] from a particular choice of the speed-of-light parameter, making this $r$-matrix into an all-containing structure for such models.
2 The Ruijsenaars-Schneider Model

The canonical variables are a set of rapidities \( \{\theta_i, i = 1 \cdots N\} \) and conjugate positions \( q_i \) such that \( \{\theta_i, q_j\} = \delta_{ij} \). The Hamiltonian is:

\[
H = mc^2 \sum_{j=1}^{N} (\cosh \theta_j) \prod_{k \neq j} f(q_k - q_j)
\]  

where

\[
f(q) = \left(1 + \frac{g^2}{q^2}\right)^{1/2} \text{ (rational)}
\]

\[
f(q) = \left(1 + \frac{\alpha^2}{\sinh^2 \frac{\nu}{2} q^2}\right)^{1/2} \text{ (hyperbolic)}
\]

\[
f(q) = (\lambda + \nu \mathcal{P}(q)) \text{ (elliptic), } \mathcal{P} = \text{Weierstrass function}
\]

We shall be interested in the trigonometric/hyperbolic case. The rational case is obtained from an easy limit procedure as we shall soon see. The elliptic case is more involved. Its integrability proof relies on specific identities for elliptic functions [21] and we expect its \( r \)-matrix structure to be very different from the \( r \)-matrices of the rational/trigonometric type. Such is the case in the non-relativistic limit (Calogero-Moser) for which the Olshanetsky-Perelomov Lax matrix without spectral parameter [22] has yet no known \( r \)-matrix and the usual ansatz [3] is known to be inconsistent [20]. This is ultimately due to the different nature of the original dynamical system from which the elliptic model is Hamiltonian-reduced [23].

The Lax matrix reads:

\[
L = \sum_{j,k=1}^{N} L_{jk} e_{jk}
\]

\[
L_{jk} = \exp \frac{g}{2} (\theta_j + \theta_k) \cdot C_{jk} (q_j - q_k) \cdot \left( \prod_{m \neq j} f(q_j - q_m) \prod_{l \neq k} f(q_l - q_k) \right)^{1/2}
\]  

where \( \{e_{jk}\} \) is the usual basis for \( N \times N \) matrices; \( f \) was given in (2) and

\[
C_{jk}(q) = \frac{\gamma}{\gamma + iq} \text{ (rational)}
\]

\[
C_{jk}(q) = \left(\cosh\frac{\nu}{2}q + ia \sinh\frac{\nu}{2}q\right)^{-1} \text{ (trigonometric)}
\]

One has denoted in (4) \( a = \sqrt{1 - \alpha^{-2}} \), and \( \gamma = \alpha/\nu \) for \( \nu \to 0, \alpha \to 0 \) in the rational limit. The Hamiltonian \( H \) in (1) is \( Tr L + Tr L^{-1} \) and the space-translation generator \( P \) is \( Tr L - Tr L^{-1} \). Connection with the more suitable notations in [13] is obtained
by introducing a “Speed-of-light” parameter $\lambda = \frac{1 - i\omega}{1 + i\omega}$ and an exponentiated variable $z_i = e^{i\nu \omega_i}$. One ends up with:

$$L_{jk} = \exp \frac{\beta}{2} (\theta_j + \theta_k) \cdot (z_j z_k)^{1/2} \cdot \frac{\lambda + 1}{2} \prod_{l \neq j, m \neq k} \left( \frac{(\lambda z_l + z_j)^{1/2}(\lambda^{-1} z_l + z_j)^{1/2}}{z_l - z_j} \cdot \frac{(\lambda z_m + z_k)^{1/2}(\lambda^{-1} z_m + z_k)^{1/2}}{z_m - z_k} \right)^{1/2}$$

(5)

Note that when $\beta, \nu \to 0$, one can reabsorb $\sqrt{z_j z_k}$ into $\exp \frac{\beta}{2} (\theta_j + \theta_k)$ by a straightforward canonical transformation. Relativistic invariance is achieved by explicit introduction of the speed of light $c$ and elimination of $\beta$. As indicated in [12] this requires a somewhat awkward redefinition of the canonical variables $\theta_i$ and $z_i$ and we shall not use this parametrization here. However it is important to know that $\alpha$ must be normalized as $\frac{\alpha}{c}$ and $\nu$ as $\frac{\nu}{c}$. Hence the non-relativistic limit $c \to \infty$ implies $\alpha \to 0$ and $\lambda \to -1$; the ultrarelativistic limit $c \to 0$ implies $\alpha \to +\infty$ and $\lambda = \frac{1 - i}{1 + i}$. It is in this sense that we sometimes call $\lambda$ the “speed-of-light parameter” although strictly speaking the compactification radius $\nu$ also contains $c$ as seen here. The since Gordon soliton Lax matrix [13] is obtained by setting $\lambda = 1$. The non-relativistic limit is obtained by setting $\gamma \equiv g(\beta)$ (rational case) and $\alpha = g(\beta)$ (trigonometric case) with $\beta \to 0$, hence $\lambda \to -1$. One then gets.

$$L_{jk} = \delta_{jk} + \beta \left\{ \theta_k \delta_{jk} + g \frac{\sqrt{z_j z_k}}{z_j - z_k} \right\} + O(\beta^2)$$

(6)

and the order $\beta$-term is precisely the Calogero-Moser trigonometric Lax matrix. Note finally that the rational limit of (5) is obtained by setting $\alpha \equiv \gamma \nu$ and $\nu = 0$.

### 3 The Classical $r$-matrix

We now solve the $r$-matrix equation for the trigonometric Lax operator (3). The generic $r$-matrix structure is (for a Lax matrix $L$ in a Lie algebra $g$):

$$\{L \otimes L\} \equiv \sum_{ijkl=1}^N \{L_{ij}, L_{kl}\} e_{ij} \otimes e_{kl} \in g \otimes g$$

$$= [r, L \otimes 1]_{g \otimes g} - [r^\pi, 1 \otimes L]_{g \otimes g}$$

$$r \equiv \sum_{ijkl=1}^n r_{ijkl} e_{ij} \otimes e_{kl} ; \quad r^\pi = \sum_{ijkl=1}^n r_{ijkl} e_{jk} \otimes e_{ij}$$

(7)

(8)

The Poisson brackets of our Lax matrix (5) read:

$$\{L_{ij}, L_{kl}\} = \frac{\nu \beta}{8} L_{ij} L_{kl} \left\{ S_{il} (1 - \delta_{il}) + S_{ik} (1 - \delta_{ik}) + S_{jl} (1 - \delta_{jl}) + S_{jk} (1 - \delta_{jk}) + 2 (S_{ij} - G_{kl}) \delta_{ik} + (G_{ij} + G_{kl}) \delta_{il} + (-G_{ij} + G_{kl}) \delta_{jk} + (-G_{ij} + G_{kl}) \delta_{jl} \right\}$$

(9)
where:

\[ G_{ij} = \frac{\lambda z_i - z_j}{\lambda z_i + z_j} \quad F_{ij} \equiv \frac{z_i z_j}{z_i - z_j} (1 - \delta_{ij}) \]

\[ S_{ij} = G_{ij} - G_{ji} + 2F_{ij} \quad (10) \]

The Poisson bracket structure (9) has a number of features which will be helpful in finding the \( r \)-matrix. First of all, it exhibits a quadratic behavior in the Lax matrix, dressed by the quantities \( S \) and \( G \). This leads us to conjecture for the \( r \)-matrix a linear behavior, similarly dressed by \( z \)-dependent functions.

Then the indices carried by the Lax elements \( L \) and the dressing functions \( S, G \) on the r.h.s. of (8) are only the original indices \( i, j, k, l \) and no extra index ever occurs. Moreover the dressing functions \( S \) and \( G \) are rational functions of the sole \( z \) variable of which they carry the index. This “locality” property of (9) leads us, in a first step towards solving (7), to set restrictions on the algebraic structure of \( r \) in (8) from the following argument:

The \( r \)-matrix structure (7) generates a priori a Poisson bracket \( \{ L_{ij}, L_{kl} \} \) containing a summation over one extra “free” index from the products \( r \cdot L \otimes 1 \) and \( L \otimes 1 \cdot r \cdot \ldots \). However as we have just seen (9) does not exhibit such a summation.

This in particular strongly precludes the existence in \( r \) of terms with four distinct indices. In fact, these considerations may be extended to lower index terms, leading to the Ansatz:

\[ r_{ijkl} = \frac{\nu z_i}{8} \left\{ A_{ikl} L_{kl} \delta_{ij} + B_{ij} (L_{jl} \delta_{jk} + L_{kj} \delta_{il}) + C_{ij} (L_{ki} \delta_{jl} + L_{jl} \delta_{ik}) \right\} \quad (11) \]

when \( A, B \) and \( C \) are furthermore assumed (as a consequence of the similar property in (9)) to be rational functions of the sole dynamical variables \( \{ z_i \} \) of which they carry the explicit index. The quadratic behavior of (9) in \( L \) is also taken into account by this linear-in-\( L \) ansatz. The second step of our reasoning takes advantage of the pole structure of (9):

Plugging (11) into (7) and comparing it with the explicit expression (9) at the particular points \( \lambda z_i = -z_j \) leads us to setting:

\[ A_{ikl} = \frac{1}{2} \left( S_{il} + S_{ki} \right) + 2G_{il} \left( 1 - \delta_{ik} \right) - 2G_{ki} \left( 1 - \delta_{il} \right) \quad (12) \]

It finally turns out from a careful inspection of the remaining terms that the Poisson structure (9) is fully reproduced by (11) provided one also sets:

\[ B_{ij} = -2F_{ij} \quad C_{ij} = 0 \quad (13) \]

The explicit checking of the consistency of this form for the \( r \)-matrix is considerably simplified by using a number of functional identities connecting \( L, F, G \) and
allowing some permutations of indices in the quadratic expressions \( L_{ij}L_{kl} \). For instance one has:

\[
(F_{ik} + F_{jl}) L_{il} L_{kj} = (F_{ik} + F_{jl} + (G_{kj} - G_{il}) (1 - \delta_{ik} - \delta_{jl})) L_{ij} L_{kl} \tag{14}
\]

We interpret both the “quadratic” form of \( r \) in (11) and the role played by such functional equations as (14) in checking the \( r \)-matrix structure as a reflection of the Hamiltonian reduction procedure from a Heisenberg double. The quadratic \( r \)-matrix, in particular, is characteristic of such integrable systems and the “dressing” terms \( A, B, C \) may viewed as generated by the Hamiltonian reduction, in the same way as the dynamical terms in the \( r \)-matrix of Calogero-Moser were generated by the Marsden-Weinstein reduction procedure \[5\]. Clarifying these issues will be left for further studies.

### 4 Limits of the \( r \)-Matrix: Calogero-Moser and Sine Gordon

The non-relativistic limit of the Lax matrix (3) yielding the trigonometric Calogero-Moser model was described in the introduction (6). Using the same reparametrization \( \alpha = g\beta, \beta \to 0 \), the \( r \)-matrix (11) becomes:

\[
r_{ijkl} = -\nu \beta \left\{ \frac{1}{4} \delta_{ij} (\delta_{il} + \delta_{ik}) (1 - \delta_{kl}) \frac{1}{\sinh \frac{\nu}{2} (q_k - q_l)} + \frac{1}{2} (1 - \delta_{ij}) \delta_{il} \delta_{jk} \cotanh \frac{\nu}{2} (q_i - q_j) \right\} + O(\beta^2) \tag{15}
\]

which is precisely the well-known trigonometric C.M. matrix \[3\].

The sine Gordon limit is not immediately identifiable with the \( r \)-matrix obtained in \[13\] which is given by setting \( A = 0, B = C = F \) in (11). Of course an \( r \)-matrix is not unique and has actually a large “moduli space”. In this case, the difference \( \tilde{r} \) between the RS matrix (11-13) and the particular sine-Gordon soliton \( r \)-matrix in \[13\] obeys:

\[
[\tilde{r}, L(\lambda = 1) \otimes 1] - [\tilde{r}^\pi, 1 \otimes L] = 0
\]

but \( [\tilde{r}, L \otimes 1] \neq 0 \) \tag{16}

At this moment we do not know how to generalize \( \tilde{r} \) to \( (\lambda \neq 1) \). In fact \( \lambda = 1 \) is the particular point where \( L \) is symmetric, and the rational fractions \( S, G \) and \( F \) simplify dramatically. It seems actually that the 3-index ansatz (11) is the only generic one and the reduction to 2-index functions is particular to the symmetric point \( \lambda = 1 \).
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