FERMION DENSITY INDUCED INSTABILITY OF THE W-BOSON PAIR CONDENSATE IN STRONG MAGNETIC FIELD

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ABSTRACT

The electroweak vacuum structure in an external magnetic field close to the lower critical value is considered at finite fermion density. It is shown that the leading effect of the fermions is to reduce the symmetry of the W-pair condensate in the direction of the magnetic field. The energy is minimized by the appearance of a helicoidal structure of the condensate along the magnetic field.

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\[1\] This work has been supported by the U.S. National Science Foundation, grant PHY-90-9619.

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1. In the presence of strong external magnetic fields the ground state of the Weinberg-Salam theory exhibits a very interesting structure. When the field is increased above the critical value $H_{c1} = m_W^2/e \sim 10^{24}$ Gauss, $W^+ - W^-$ pairs condense in an Abrikosov-type vortex lattice. Increasing the field above $H_{c2} = H_{c1}/\cos \theta_W$ leads to the vanishing of the Higgs vacuum expectation value and a restoration of SU(2) × U(1)-symmetry. It is thus possible to reach the ordered phase of the electroweak theory not only by increasing the temperature but also by increasing the magnetic field \cite{1}, \cite{2}, \cite{3}.

Previously, the development of the instability for $H > H_{c1}$ has been studied by neglecting the presence of fermions in the system. For gauge and Yukawa couplings in the perturbative regime, the effect of the Dirac sea will be to renormalize the effective action of the bosonic fields. The presence of a finite density of fermions, however, gives a qualitatively new effect on the intermediate phase, because of the fermion number nonconservation in the Weinberg-Salam theory.

The lattice of vortices, found in \cite{1}, is homogeneous in the direction of the external field. In the presence of finite fermion density, however, the development of an inhomogeneity in the field direction causes (through the $B + L$-anomaly) some occupied positive energy levels to fall into the Dirac sea, lowering the Fermi energy. This energy gain may overcome the energy loss due to the inhomogeneity, leading to the formation of a structure that is periodic in the direction of the field. The physics of this instability is reminiscent of the Peierls transition in 1-d metals \cite{4}, or more closely, of the instability of the charged-$W$ condensate in cold nonneutral fermionic matter \cite{5}.

This letter is organized as follows. In sect.2 we review the essential properties of the Ambjørn-Olesen solution. We show that the fermion number anomaly leads, at finite chemical potential, to an instability towards the formation of a helicoidal structure of the $W$-pair condensate along the field direction. In sect.3 we discuss the fermionic ground state at $H \simeq H_{c1}$ and for fermion masses and chemical potential obeying $m_f \ll \mu \ll \sqrt{eH}$. For fields close to the lower critical value, the state in a homogeneous magnetic field can then be treated as the unperturbed ground state, and the $W$-pair condensate as a perturbation. We show that under these conditions the leading effect of the fermions is the destabilization of the $W$-condensate towards inhomogeneity in the field direction. Sect.4 contains discussion of the results
2. Let us consider first the bosonic sector of the standard model. The static energy functional in the unitary gauge is \[6\]:

\[
E_0 = \frac{1}{4} \left( \partial_i W^3_{3j} \right)^2 + \frac{1}{4} \left( \partial_i B_{3j} \right)^2 + \frac{1}{2} |\partial_i W^i_{3j}|^2
- \frac{ig}{2} \partial_i W^3_{3j} W^i_{3j} W_j - \frac{g^2}{2} \left[ |W^i_{3i}|^2 - \left( W^i_{3i} W^i_{3i} \right)^2 \right]

+ ig \partial_i W^i_{3j} W^3_{3j} - ig \partial_i W^i_{3j} W^3_{3j} W^j_{3j} - g^2 W^3_{3j} W^i_{3j} W^i_{3j}

+ \left( \vec{\nabla} \phi \right)^2 + \lambda \left( \phi^2 - v^2 \right)^2 + \frac{g^2}{2} \phi^2 W^i_{3i} W^i_{3i} + \frac{g^2}{4 \cos \theta} \phi^2 Z_i Z_i.
\]

Here \( W^i \equiv (W^1_i - i W^2_i)/\sqrt{2} \) is the charged-\( W \)-field, \( Z_i = W^3_i \cos \theta - B_i \sin \theta \), the electromagnetic vector potential is \( A_i = W^3_i \sin \theta + B_i \cos \theta \), and \( \phi \) is the Higgs scalar.

In the presence of a static homogeneous magnetic field \( H > H_{c1} \), pointing in the third direction, the ground state is characterized by nonvanishing \( W \), \( Z \), \( A \), \( (\phi - v) \)-condensates. For the static solution, found by Ambjørn and Olesen \[1\]

\[ A_3 = W_3 = Z_3 = 0, \ A_0 = W_0 = Z_0 = 0, \tag{2} \]

and

\[ W_2 = i W_1 \equiv W(x_1, x_2). \tag{3} \]

The condensates are homogeneous along the third axis and periodic in the plane perpendicular to the magnetic field. The vortex-lattice structure has a unit cell with area \( S_k = 2\pi k/m_W^2 \), where \( k \) is the winding number of the phase of the \( W \)-field along the cell boundary. The \( W \)-field has \( k \) zeros inside the cell. The energy is minimized for \( k = 1 \) and a hexagonal lattice \[6\].

The properties (2),(3) of the \( W \)-condensate are sufficient to demonstrate that a finite density of fermions causes it to be unstable towards inhomogeneity in the field direction. In the presence of finite fermion density the fermion number anomaly in the standard model leads to the appearance of a Chern-Simons term in the bosonic effective action \[7\]

\[ E_{CS} = \mu \bar{n}_{CS}, \tag{4} \]

and possible applications.
where \( f \) is the number of doublets, and

\[
n_{CS} = \frac{g^2}{8\pi^2} \left[ e^{ijk} W^a_i \partial_j W^a_k + \frac{g}{3} e^{abc} W^a_i W^b_j W^c_k - \tan^2 \theta e^{ijk} B_i \partial_j B_k \right].
\]  

(5)

To understand the appearance of (4) in the effective action, it is useful to note that the introduction of the chemical potential is equivalent to coupling an external "fermion number gauge field" \( F_\alpha \) to the fermions, such that \( F_\alpha = (\mu, 0, 0, 0) \). Then calculating the term in the effective action, linear in \( \mu \) and quadratic in the background, is equivalent to computing the correlator

\[
\langle J^f_\alpha J^a_\beta J^b_\gamma \rangle,
\]

where \( J^f \) is the fermion number current, \( J^a, J^b \) are SU(2) and U(1) currents. The totally antisymmetric contribution to this correlator is determined by the anomaly equation:

\[
\partial_\mu J^f_\mu = \frac{fg^2}{32\pi^2} \left( F^a_\mu \tilde{F}^a_\mu - \tan^2 \theta F^a_\mu \tilde{F}^a_\mu \right),
\]

\( (F^a, \tilde{F}^a \text{ and } F, \tilde{F} \text{ are the SU(2) and U(1) field strengths and their duals, respectively}) \) and coincides with (4).

From (2),(3) it follows that (5) reduces to:

\[
E_{CS} = \mu f \frac{g^2}{8\pi^2} \left[ e^{ij3} W^i_1 \partial_j W^j_3 + h.c. + e^{ij3} W^i_3 \partial_j W^j_3 - \tan^2 \theta e^{ij3} B_i \partial_j B_j \right].
\]  

(6)

We see that the appearance of an inhomogeneity in the third direction leads to nonzero density of the Chern-Simons number of the condensate. \( n_{CS} \) is equal to the number density of the positive energy fermion levels which have fallen into the Dirac sea. Note that for the original solution (2),(3) \( n_{CS} = 0 \).

To find the form of the condensate which minimizes the energy we have to consider the total bosonic energy functional, which includes the energy loss due to the inhomogeneity:

\[
E_{total} = E_0 + E_{CS},
\]

(7)

where \( E_0 \) is given by (4). We see that an \( x_3 \)-dependent isospin rotation of the \( W \)-condensate around the third axis

\[
W_1 \rightarrow e^{ik_3 x_3} W_1 \equiv -ie^{ik_3 x_3} W, \quad W_2 \rightarrow e^{ik_3 x_3} W_2 \equiv e^{ik_3 x_3} W,
\]

(8)
does not affect $E_0$ except for the term with two derivatives (recall that for the Ambjørn-Olesen solution all components along the third axis vanish). This two-derivative contribution represents the "elasticity" energy loss due to the inhomogeneity. On the other hand, the $E_{CS}$ contribution is linear in $k_3$, the total energy change being:

$$\delta E = k_3^2 W^\dagger W - \mu f \frac{g^2}{8\pi^2} k_3 W^\dagger W.$$  

(9)

Clearly, for small enough $k_3$ the linear term dominates and $\delta E$ is negative and minimized for a nonzero value of $k_3$:

$$k_3 = \frac{g^2 \mu f}{16\pi^2}.$$  

(10)

The last two expressions show that the finite density of fermions causes an instability of the vortex lattice towards inhomogeneity in the direction of the magnetic field and leads to the formation of a helicoidal structure of the $W$-condensate.

Being determined by the exact anomaly equation the Chern-Simons contribution $E_{CS}$ is valid for all momenta (well below the UV cut-off) of the background fields. Therefore the density-induced instability has a quite general character.

For arbitrary fermion densities and fields much stronger than the lower critical one, the $W$-condensate is not small and we are not aware of the nature of the fermionic ground state. In particular, then we cannot insist that the most important influence of the fermions on the condensate is the appearance of the inhomogeneity.

3. Let us therefore consider magnetic field strengths close to the lower critical value $H_{c1}$ and show that the instability described in the previous section is then the leading effect of the fermions.

Let us decompose $A = \vec{A} + A'$, where $\vec{A}$ is the electromagnetic vector potential, corresponding to the homogeneous part of the magnetic field background, and introduce the small parameter $\epsilon = (H - H_{c1})/H_{c1} \ll 1$. For such fields the $A', W, Z, \phi - v$ condensates are small $[3]$: 

$$|W| \sim \epsilon v$$
Their characteristic momenta in the direction perpendicular to the magnetic field are \[6\]:

\[
k_{1,2}^W \sim \epsilon gv, \tag{11}
\]

\[
k^W \gg k^{A'}, k^Z, k^\phi.
\]

Now let us couple fermions to the above background. The radius of the lowest Landau orbit \[8\] of a fermion with charge \(eq\), \(e > 0\), in a constant homogeneous magnetic field is

\[
r_H = \sqrt{2/e|q|H}.
\]

Note that for \(H \simeq m^2_W/e\) this is of the order of the unit cell size \(\sqrt{S_1}\), and that \(k_{1,2}^W r_H \ll 1\), which means that the condensates vary slowly along the Landau orbit. Therefore we will consider the state in a homogeneous magnetic field as the unperturbed ground state, i.e. treat the constant part of \(H\) exactly and consider \(A', W, Z, \phi - v\) as small perturbations.

For chemical potential \(\mu\), obeying

\[
m_f \ll \mu \ll \sqrt{eH}, \tag{12}
\]

all fermions will occupy the lowest Landau level. Their number density is then \(n_q = e|q|H\mu/(2\pi^2)\), and the energy density: \(\mathcal{E}_q = e|q|H\mu^2/(4\pi^2)\). From \(12\) we see that the Fermi energy density can be neglected in comparison with the energy density of the constant magnetic background: \(\mathcal{E}_q \ll \mathcal{E}_H = H^2/2\).

For the fields under consideration the shift of the Dirac sea energy can also be neglected \[9\]: For a fermion with mass \(m_f\) and charge \(qe\) in magnetic field \(H \gg m^2_f/(|q|e)\) it equals:

\[
\mathcal{E}^\text{vac}_H - \mathcal{E}^\text{vac}_{H=0} = \frac{|q|^2e^2H^2}{24\pi^2} \ln \frac{|q|eH}{m^2_f}
\]

and becomes comparable to \(\mathcal{E}_H = H^2/2\) only for field strengths of the order of

\[
\frac{m^2_f}{|q|e} \frac{e^{12\pi^2}}{|q|e} \gg H_{c1} = \frac{m^2_W}{e}.
\]

Finally, higher derivative terms in the effective bosonic action can be neglected if the momenta of the background in the direction perpendicular
to \( H \) obey \( k_{1,2} \ll \sqrt{eH} \), and the ones along \( H \): \( k_3 \ll \mu \). The first condition is obeyed for fields close to the lower critical one, as follows from \((11)\); the second is seen to hold from \((10)\).

The above considerations show that for small coupling and field strengths \( \sim H_{c1} \) quantum corrections due to fermions lead to renormalization of the coefficients of the bosonic energy functional only. Therefore, the Chern-Simons term generated by the anomaly is the leading contribution to the bosonic effective action, and the ground state is characterized by the helicoidal structure \((8)\) with wave vector \((14)\).

Let us note that we cannot prove that this is the only instability occurring. All we have shown is that \((14)\) minimizes the energy within the Ansatz \((8)\).

The \( x_3 \)-dependent phase of the \( W \)-condensate leads to a current flow along the third axis, which in its turn will affect the system, but in the small coupling regime these effects are suppressed by additional powers of \( e \).

The scale of the inhomogeneity \( k_3^{-1} \) is much larger than the unit cell size, for the densities for which the above analysis is valid. If however \((10)\) is taken at face value, they become comparable only for \( \mu \sim m_W / g^2 \). At these densities however, the system is classically unstable towards transition to the vacuum with different fermion number. Note that in our case there is no true fermion number violation - if the magnetic field is turned off the levels appear again and the initial number of fermions is restored.

4. Turning to possible applications, we note that the conditions necessary for the realization of the fermion-induced instability might be present only in the early Universe. The values of the critical fields \( H_{c1}, H_{c2} \) go to zero as \( T \rightarrow T_c \), where \( T_c \sim 100 \text{GeV} \) is the critical temperature of the electroweak phase transition. Therefore, slightly below \( T_c \), fields much smaller than \( 10^{24} \) Gauss could drive the system in the intermediate phase \((4)\). Such fields might be present before the phase transition, being generated by quantum fluctuations during inflation \((10)\). They could also be created during the electroweak transition itself, realizing thus a ”vacuum dynamo” effect \((4), (11)\).

The result for the fermion generated \( E_{CS} \) holds at finite temperature as well, to leading order in the high temperature expansion \( \mu, m_f \ll T \). If there were some ”seed” magnetic field \((10)\) prior to the electroweak transition, then below \( T_c \) the mixed phase might be energetically favourable for
some period of time. As we saw above it has two distinct scales of inhomogeneity - \((m_W)^{-1}\) in the direction perpendicular to the magnetic field, and 
\[4eH/(g^2n_f) \sim 4m_W^2/(g^2n_f) \gg (m_W)^{-1}\] in the longitudinal direction. This strong anisotropy would affect the gravitational wave background \[12\], but it is not clear whether its amplitude is strong enough to lead to any observable effect.

To conclude, we have shown that a finite fermion density destabilizes the \(W\)-pair condensate in strong magnetic field and breaks the translational invariance in the direction of the field, leading to the appearance of a helicoidal structure.

It is a pleasure to acknowledge helpful discussions with J. Bagger, G. Feldman and T. Gould.
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