Remarks on a Tropical Key Exchange System

Dylan Rudy   Chris Monico

Department of Mathematics and Statistics
Texas Tech University
e-mail: c.monico@ttu.edu

May 12, 2020

Abstract

We consider a key-exchange protocol based on matrices over a tropical semiring which was recently proposed in [2]. We show that a particular private parameter of that protocol can be recovered with a simple binary search, rendering it insecure.

Keywords: tropical algebra, public key exchange, cryptanalysis.

Mathematics subject classification:  15A80, 94A60.

1 Introduction

Let $S$ be any nonempty subset of $\mathbb{R}$ which is closed under addition. Define two operations $\oplus$ and $\otimes$ on $S$ by

$$a \oplus b = \min\{a, b\},$$
$$a \otimes b = a + b.$$ 

Both operations are associative and commutative and $\otimes$ distributes over $\oplus$, and hence $S$ is a commutative semiring, called a tropical semiring. The set $\mathcal{M} = \text{Mat}_{k \times k}(S)$ of $k \times k$ matrices over $S$ is therefore a semiring with the induced operations

$$a_{ij} \oplus (b_{ij}) = (a_{ij} \oplus b_{ij}),$$
$$a_{ij} \otimes (b_{ij}) = (c_{ij}), \quad \text{where} \quad c_{ij} = (a_{i1} \otimes b_{1j}) \oplus (a_{i2} \otimes b_{2j}) \oplus \cdots \oplus (a_{ik} \otimes b_{kj}).$$

In [1], the authors proposed two key exchange protocols based on the structure $\mathcal{M}$. Shortly after, an effective attack was given on one of those protocols in [3]. Subsequently, a new key exchange protocol was proposed in [2] (in fact, two new protocols, but they are very closely related to each other). It is this protocol that we consider in this paper.

In [2], the authors give two semigroup operations on $\mathcal{M} \times \mathcal{M}$ each arising as semidirect product induced by a specified action of these matrices on themselves. The two semigroup operations are given by

$$(M, G) \circ (S, H) = \left( M \oplus S \oplus H \oplus (M \otimes H), \ G \oplus H \oplus (G \otimes H) \right),$$

$$(M, G) \ast (S, H) = \left( (H \otimes M^T) \oplus (M^T \otimes H) \oplus S, \ G \otimes H \right).$$
Note that for each of these operations, the first component of the product does not depend on $G$. This fact plays a key role in the two key exchange protocols they then propose (one corresponding to each operation):

1. Alice and Bob agree on public matrices $M, H \in \mathcal{M}$ whose entries are integers in the range $[-N, N]$. Alice selects a private positive integer $m < 2^K$ and Bob selects a private positive integer $n < 2^K$.

2. Alice computes $(M, H)^m = (A, P_A)$ and sends $A$ to Bob.

3. Bob computes $(M, H)^n = (B, P_B)$ and sends $B$ to Alice.

4. Alice determines the first component of $(M, H)^{m+n} = (M, H)^n (M, H)^m = (B, P_B)(A, P_A)$ from her knowledge of $A, P_A$, and $B$ (knowledge of $P_B$ is not necessary for either of the operations (1.1) or (1.2)).

5. Bob similarly determines the first component of $(M, H)^{m+n} = (M, H)^m (M, H)^n = (A, P_A)(B, P_B)$ from his knowledge of $B, P_B$, and $A$.

In the next section, we show that an eavesdropper can find a positive integer $m'$ for which the first component of $(M, H)^{m'}$ is $A$; she can then use this $m'$ to compute the shared secret key in essentially the same way as Alice. Furthermore, such an $m'$ can be found using $O(K^2)$ operations (1.1) or (1.2).

2 The attack

Since addition of matrices in $\mathcal{M}$ is idempotent, i.e., $G \oplus G = G$, we have a partial order on $\mathcal{M}$ defined by

$$X \leq Y \quad \text{if} \quad X \oplus Y = X.$$ 

Clearly we have that $X \leq Y$ iff $x_{ij} \leq y_{ij}$ for all $i, j \in \{1, 2, \ldots, k\}$. Furthermore, this partial order respects both operations on $\mathcal{M}$; if $X \leq Y$ and $Z \in \mathcal{M}$, then $X \oplus Z \leq Y \oplus Z$ and $X \otimes Z \leq Y \otimes Z$.

**Proposition 2.1** Consider the semigroup $\mathcal{M} \times \mathcal{M}$ equipped with either of the two operations defined by (1.1) and (1.2). Let $(M, H) \in \mathcal{M} \times \mathcal{M}$, and for each positive integer $\ell$ let $(M_\ell, H_\ell) = (M, H)^\ell$. Then the sequence $\{M_\ell\}$ is monotonically decreasing: $M_1 \geq M_2 \geq M_3 \geq \ldots$.

**Proof:** Let $\ell \geq 2$. For the operation $\circ$ we have

$$(M_\ell, H_\ell) = (M_{\ell-1}, H_{\ell-1}) \circ (M, H)$$

$$= \left( M_{\ell-1} \oplus M \oplus H \oplus (M_{\ell-1} \otimes H), H_{\ell-1} \oplus H \oplus (H_{\ell-1} \otimes H) \right),$$

so that $M_\ell = M_{\ell-1} \oplus M \oplus H \oplus (M_{\ell-1} \otimes H)$. In particular, $M_\ell \oplus M_{\ell-1} = M_\ell$, and hence $M_\ell \leq M_{\ell-1}$. 

2
Similarly, for the operation $*$ we have that
\[
(M_\ell, H_\ell) = (M, H) * (M_{\ell-1}, H_{\ell-1}) = ((H_{\ell-1} \otimes M^T) \oplus (M^T \otimes H_{\ell-1}) \oplus M_{\ell-1}, H \otimes H_{\ell-1}),
\]
and hence $M_\ell = (H_{\ell-1} \otimes M^T) \oplus (M^T \otimes H_{\ell-1}) \oplus M_{\ell-1}$. Again, $M_\ell \oplus M_{\ell-1} = M_\ell$, so that $M_\ell \leq M_{\ell-1}$. \qed

The problem alluded to at the end of the introduction is now easily solved with a binary search. Let $M, H \in \mathcal{M}$ and $(M, H)^\ell = (M_\ell, H_\ell)$. Suppose $A \in \mathcal{M}$ satisfies $A = M_m$ for some positive integer $m < 2^K$. First, obtain an upper bound on $m$ by computing successive squares $M_1, M_2, M_4, M_8, \ldots$ until finding a positive integer $t$ for which $M_{2t} \leq A$. Since it is then known that $1 \leq m \leq 2^t$, a simple binary search will find an integer $m'$ for which $M_m = A$. The sequence $M_1, M_2, \ldots$ is generally strictly decreasing, in which case $m' = m$. However, even if $m' \neq m$, finding such an integer $m'$ is enough for the eavesdropper to recover the shared secret key. Let $\pi_1 : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be the map $\pi_1(C, D) = C$. Suppose $(M, H)^n = (B, P_B), (M, H)^m = (A, P_A)$ and $(M, H)^{m'} = (A, P_E)$. Then for each of the operations (1.1) and (1.2), the shared secret key satisfies
\[
\pi_1((M, H)^{m+n}) = \pi_1((M, H)^{m'+n}).
\]
This is clear, since this shared secret key can be expressed in terms of $A, B,$ and $P_B$ only, but it may also be explicitly verified. For example, with the operation (1.1),
\[
\pi_1((M, H)^{m+n}) = \pi_1((A, P_A) \circ (B, P_B)) = A \oplus B \oplus P_B \oplus (A \otimes P_B) = \pi_1((A, P_E) \circ (B, P_B)) = \pi_1((M, H)^{m'+n}).
\]
In particular, the eavesdropper may recover the shared secret key via
\[
\pi_1((M, H)^{m+n}) = \pi_1((M, H)^n \circ (M, H)^{m'}) = \pi_1((B, P_B) \circ (A, P_E)) = B \oplus A \oplus P_E \oplus (B \otimes P_E).
\]

Finding $t$ as described above requires at most $K$ semigroup operations in $\mathcal{M} \times \mathcal{M}$. The binary search, done in the most obvious way, would compute $K$ powers of $(M, H)$, each of which requires no more than $2K$ semigroup operations in $\mathcal{M} \times \mathcal{M}$, for a total complexity of at most $2K^2 + K$ operations in $\mathcal{M} \times \mathcal{M}$.

With the parameter sizes suggested in [2], $k = 30$, $N = 1000$, and $K = 200$, our Python implementation determines such an integer $m'$ in about 15 minutes on a single core of an i7 CPU at 3.40GHz.
We would like to make one final remark about the key sizes in this system. With the notation as above and the operation \( (1.1) \), for example, we have
\[
M_{\ell+1} = M_\ell \oplus M \oplus H \oplus (M_\ell \otimes H).
\]
Since \( M_2 \leq M \) and \( M_2 \leq H \) and \( M_{\ell+1} \leq M_2 \) for all \( \ell \geq 2 \), it follows that
\[
M_{\ell+1} = M_\ell \oplus (M_\ell \otimes H), \quad \text{for } \ell \geq 2.
\]
This means that, on average, the entries of \( M_{\ell+1} \) decrease from those of \( M_\ell \) by an approximately constant amount, proportional to the size of the entries of \( H \). With the parameter sizes suggested in [2], this means that the entries of \( A \) and \( B \) are on the order of \(-c \times 2^{200}\), or about 200 bits each, for a total of \( 30 \times 30 \times 200 = 180,000 \) bits for each of \( A \) and \( B \). Whereas \( M \) and \( H \) consist of about \( 9000 \) bits each, for an increase of around \( 170,000 \) bits; not the 20,000 bit increase claimed in [2].

3 Conclusion

The attack presented here exploits the fact that the sequence \( \{(M, H)^\ell\} \) is linearly ordered. It is quite effective and practical against the protocols described in [2]. For those protocols, Alice and Bob must do approximately \( O(K) \) operations in the semigroup \( \mathcal{M} \times \mathcal{M} \), and this attack requires about \( O(K^2) \) operations in that same semigroup, so an increase of parameter sizes does not help.

References

[1] Dima Grigoriev and Vladimir Shpilrain. Tropical cryptography. *Comm. Algebra*, 42(6):2624–2632, 2014.

[2] Dima Grigoriev and Vladimir Shpilrain. Tropical cryptography II: extensions by homomorphisms. *Comm. Algebra*, 47(10):4224–4229, 2019.

[3] Matvei Kotov and Alexander Ushakov. Analysis of a key exchange protocol based on tropical matrix algebra. *J. Math.Cryptol.*, 12(3):137–141, 2018.