Holonomic modules associated with multivariate normal probabilities of polyhedra

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Abstract

The probability content of a convex polyhedron with a multivariate normal distribution can be regarded as a real analytic function. We give a system of linear partial differential equations with polynomial coefficients for the function and show that the system induces a holonomic module. The rank of the holonomic module is equal to the number of nonempty faces of the convex polyhedron, and we provide an explicit Pfaffian equation (an integrable connection) that is associated with the holonomic module. These are generalizations of results for the Schl"afi function that were given by Aomoto.

Keywords. convex polyhedron, inclusion–exclusion identity, holonomic modules, holonomic rank, Pfaffian equation

1 Introduction

A convex polyhedron $P$ is the intersection of half-spaces of the $d$-dimensional Euclidean space $\mathbb{R}^d$. We are interested in numerical calculation of the probability $P(X \in P)$, where $X$ is a random vector distributed as a $d$-dimensional normal distribution with mean $\mu$ and covariance matrix $\Sigma$. When $P$ is the orthant $\{x \in \mathbb{R}^d : x_i \geq 0, 1 \leq i \leq d\}$ in $\mathbb{R}^d$, the probability is called the orthant probability and can be regarded as a function of $\mu$ and $\Sigma$. By the inclusion–exclusion identity given in [6] and [16], the probability content of the convex polyhedron can be written as a linear combination of the orthant probabilities. For this reason, the literature includes many discussions of methods for evaluating the orthant probabilities. For example, [15] proposed a method based on recursive integration, and [8] and [10] proposed the use of the randomized quasi-Monte Carlo procedure. For details, see [9]. In [14], the probability content of a general convex polyhedron was evaluated by calculating the orthant probabilities.

The motivation of our study is to evaluate the probability content of a convex polyhedron by a completely different and novel approach that uses the holonomic gradient method (HGM) proposed in [17]. The HGM, which is based on the theory and algorithms of $D$-modules, is a method for numerically calculating
definite integrals. It can be applied to a broad class of problems. In fact, various applications of HGM have been proposed, for example, [11], [23], and [12]. In order to apply HGM, we need to regard the probability \( P(X \in P) \) as a function and provide an explicit Pfaffian equation for it. In the case where \( P \) is the orthant and \( \mu = 0 \), the orthant probability is a function of \( \Sigma \) and Schl"afli gave a recurrence formula for it in [22]. In [21], Plackett generalized Schl"afli’s result for the case where \( P \) is the orthant and \( \mu \neq 0 \). In [13], we provided a holonomic system and a Pfaffian equation that is associated with the orthant probability. Our Pfaffian equation corresponds with the reduction formula in [21] and [7]. In this paper, we generalize our previous results [13] and give a recurrence formula as a Pfaffian equation for the case of a general convex polyhedron.

Let \( \Sigma = AA^T \) be a Cholesky decomposition of the covariance matrix \( \Sigma \), and let \( Y \) be a random vector that is \( d \)-variate normally distributed with mean vector 0 and for which the covariance matrix is the identity matrix. Then, we have \( P(X \in P) = P(AY + \mu \in P) \), and the set \( \{ y \in \mathbb{R}^d : Ay + \mu \in P \} \) is also a convex polyhedron. Hence, it is enough to consider the case in which the mean vector of \( X \) is 0 and the covariance matrix of \( X \) is the identity matrix. Under this assumption, the probability \( P(X \in P) \) can be written as

\[
P(X \in P) = \frac{1}{(2\pi)^{d/2}} \int_{x \in P} \exp\left(-\frac{1}{2} \sum_{i=1}^{d} x_i^2\right) dx_1 \cdots dx_d.
\]

The polyhedron \( P \) can be written as

\[
P = \{ x \in \mathbb{R}^d : \sum_{i=1}^{d} \tilde{a}_{ij} x_i + \tilde{b}_j \geq 0, 1 \leq j \leq n \},
\]

where \( \tilde{a}_{ij} \) and \( \tilde{b}_j \) are real numbers. We wish to study this integral with the HGM, and as a first step, we will assume that the convex polyhedron is in the “general position;” the precise definition of “general position” will be given in Section 3.

Let \( a \) be a \( d \times n \) matrix, and let \( b \) be a vector with length \( n \). We are interested in the analytic properties of the function

\[
\varphi(a, b) = \int_{\mathbb{R}^d} \exp(-\frac{1}{2} \sum_{i=1}^{d} x_i^2) \prod_{j=1}^{n} H \left( \sum_{i=1}^{d} a_{ij} x_i + b_j \right) dx_1 \cdots dx_d,
\]

which is defined on a neighborhood of \( \tilde{a} = (\tilde{a}_{ij}), \tilde{b} = (\tilde{b}_j) \). Here, we denote by \( H \) the Heaviside function. Note that this function is an interesting specialization of the one studied by Aomoto in [2] and Aomoto, Kita, Orlik, and Terao in [4]. For the meaning of the specialization, see Remark [23] below. In this paper, we provide a holonomic system and a Pfaffian equation associated with this function. The Pfaffian equation is required by the HGM for \( \varphi(a, b) \). In order to explicitly provide the holonomic system, we decompose the function by the inclusion–exclusion identity associated with the polyhedron \( P \). We also show
that the holonomic rank of the system is equal to the number of nonempty faces of the polyhedron $P$. This Pfaffian equation is a generalization of the recursion formula given by Plackett [21]. In addition, the singular locus of the Pfaffian equation is compatible with that of the Schl"afli function given in [1].

This paper is constructed as follows. In Section 2 we will give a brief explanation of holonomic modules and Pfaffian equations. In Section 3 we provide an analytic continuation of the function $\varphi(a, b)$. In section 4 we prove an existence of an open neighborhood of a given point in general position. In Section 5 we give a system of linear partial differential equations with polynomial coefficients for the function $\varphi(a, b)$ and show that the system induces a holonomic module. In Section 6 we show that the holonomic rank of the module is equal to the number of nonempty faces of $P$ and explicitly provide the Pfaffian equation associated with the module.

## 2 Holonomic module and Pfaffian equation

Before starting the main discussion, we briefly review holonomic modules and Pfaffian equations. For a comprehensive presentation, see [17] and the references cited therein. We denote by $\mathcal{O}(U)$ the ring of differential operators of $n$ variables $x_1, \cdots, x_n$ with polynomial coefficients. Here, we put $\partial_{x_i} = \partial/\partial x_i$. Let us consider a system of linear partial differential equations

$$\sum_{j=1}^{m} P_{kj} g_j = 0 \quad (P_{kj} \in D_x, 1 \leq k \leq m') \quad (3)$$

for unknown functions $g_1, \ldots, g_m$. Let $(D_x)^m$ be the free $D_x$-module with the basis $\{g_1, \ldots, g_m\}$, and let $N$ be a $D_x$-submodule of $(D_x)^m$ generated by $P_k = \sum_{j=1}^{m} P_{kj} g_j \in (D_x)^m (1 \leq k \leq \ell)$. We denote by $M$ the quotient module $(D_x)^m / N$.

The set consisting of the holomorphic functions on a domain $U \subset \mathbb{C}^n$ forms a left $D_x$-module $\mathcal{O}(U)$. For a morphism $\varphi : M \to \mathcal{O}(U)$ of left $D_x$-modules, the functions $\varphi(g_1), \ldots, \varphi(g_m)$ satisfy system (3). For this reason, we call a vector-valued function $(g'_1, \ldots, g'_m)$ on $U$ a solution of $M$ when there is a morphism of $D_x$-modules $\varphi : M \to \mathcal{O}(U)$ such that $\varphi(g_j) = g'_j (1 \leq j \leq m)$.

By the theory of the Gröbner basis in Weyl algebra, the characteristic variety $\text{Char}(M)$ of $M$ can be computed explicitly. For details, see [18]. According to the Bernstein inequality, the Krull dimension of the characteristic variety is not less than $n$ (see, e.g., [3]). When the dimension of $\text{Char}(M)$ is equal to $n$, the $D_x$-module $M$ is said to be holonomic. When the system of differential equations (3) induces a holonomic $D_x$-module, we call (3) a holonomic system.

We denote by $R_x$ the ring of differential operators of $n$ variables $x_1, \cdots, x_n$ with rational function coefficients. The left $D_x$-module $\mathbb{C}(x) \otimes \mathbb{C}[x]$ $M$ is a left $R_x$-module, where $\mathbb{C}(x)$ is the field of rational functions. When the module $M$ is holonomic, $\mathbb{C}(x) \otimes \mathbb{C}[x]$ $M$ as a linear space over $\mathbb{C}(x)$ has finite dimension. This value is called the holonomic rank of $M$, and we denote it by rank $M$. Let
be the holonomic rank of $M$, and let $\{f_1, \ldots, f_r\}$ be a basis of $\mathbf{C}(x) \otimes_{\mathbf{C}[x]} M$ as a linear space over $\mathbf{C}(x)$. Then, there exist rational functions $c_{ijk} \in \mathbf{C}(x)(1 \leq i, j, k \leq r)$ such that

$$\partial_{x_i} f_j = \sum_{k=1}^{r} c_{ijk} f_k \quad (1 \leq i, j \leq r)$$

(4)

in $\mathbf{C}(x) \otimes_{\mathbf{C}[x]} M$. Moreover, the matrices $c_i = (c_{ijk})_{j,k=1}^{r} \ (1 \leq i \leq r)$ satisfy the integrability condition

$$\frac{\partial c_i}{\partial x_j} + c_i c_j = \frac{\partial c_j}{\partial x_i} + c_j c_i \quad (1 \leq i, j \leq r).$$

We call equation (4) a Pfaffian equation associated with the holonomic module $M$. The union of the zero sets of the denominators of the elements of $c_i$’s is called the singular locus of the Pfaffian equation. Note that a Pfaffian equation associated with $M$ depends on the choice of the basis of $\mathbf{C}(x) \otimes_{\mathbf{C}[x]} M$, and it is not unique.

3 Integral representation of the probability content of a polyhedron

In this section, we show that the function $\varphi(a, b)$ in (2) can be regarded as a real analytic function. Since the Heaviside function $H(x)$ is the hyperfunction defined by $-\frac{1}{2\pi \sqrt{-1}} \log(-z)$, we can expect the function $\varphi(a, b)$ to be expressed in terms of a logarithmic function. However, we cannot find a suitable $d$-simplex for the $d$-form obtained by replacing $H\left(\sum_{i=1}^{d} a_i x_i + b_j\right)$ in the integrand of (2) with $-\frac{1}{2\pi \sqrt{-1}} \log \left(-\left(\sum_{i=1}^{d} a_i x_i + b_j\right)\right)$. In order to overcome this difficulty, we use a decomposition of $\varphi(a, b)$, which will be given in (3), and show that $\varphi(a, b)$ can be written as a linear combination of complex integrals. This implies that $\varphi(a, b)$ is a real analytic function.

First, let us review some notions of polyhedra. In the remainder of this paper, we will assume that $d$ and $n$ are positive integers. A subset $H \subset \mathbf{R}^d$ is called a half-space if $H$ can be written as $H = \{x \in \mathbf{R}^d | \sum_{i=1}^{d} a_i x_i + a_0 \geq 0\}$ for some $a_i, a_0 \in \mathbf{R}$. A polyhedron is a finite intersection of half-spaces. An inequality $\sum_{i=1}^{d} a_i x_i + b \geq 0$ is called valid for $P$ if all the points of $P$ satisfy the inequality. We call a subset $S \subset \mathbf{R}^d$ an affine subspace if $S$ can be written as an intersection of hyperplanes. For a subset $S \subset \mathbf{R}^d$, the affine hull of $S$ is the smallest affine subspace which contains $S$, and we denote it by $\text{aff}(S)$. The dimension of a polyhedron $P$ is the dimension of $\text{aff}(P)$. For details, see [24].

In order to describe the combinatorial structure of a polyhedron, we use the notion of the abstract simplicial complex [6]. Let $\mathcal{F}$ be a set consisting of subsets of $[n] := \{1, 2, \ldots, n\}$. We call $\mathcal{F}$ an abstract simplicial complex when $J \in \mathcal{F}$ and $J' \subset J$ implies $J' \in \mathcal{F}$. Let $\mathcal{F}$ and $\mathcal{F}'$ be two abstract simplicial complexes.
We say that $\mathcal{F}$ is equal to $\mathcal{F}'$ and denote $\mathcal{F} = \mathcal{F}'$ when $\mathcal{F}$ and $\mathcal{F}'$ are equal as sets. We say that $\mathcal{F}$ and $\mathcal{F}'$ are equivalent and denote $\mathcal{F} \cong \mathcal{F}'$ when there is a bijection $\sigma : \mathcal{F} \to \mathcal{F}'$ such that $J_1 \subset J_2$ if and only if $\sigma(J_1) \subset \sigma(J_2)$.

Let $P \subset \mathbb{R}^d$ be a polyhedron, and let $F_1, \ldots, F_n$ be all the facets of $P$. For each facet $F_j$, there is a unique half-space $H_j \subset \mathbb{R}^d$ that satisfies $(\partial H_j) \cap P = F_j$ and $P \subset H_j$ (see, e.g., exercise 2.14(iv) of Lecture 2 in [24]). We call $\mathcal{H} = \{H_1, \ldots, H_n\}$ the family of the bounding half-spaces for the polyhedron $P$. The nerve of $\{F_1, \ldots, F_n\}$ is the abstract simplicial complex defined by

$$\mathcal{F} = \{J \subset [n] : F_J \neq \emptyset\}, \quad \left( F_J := \bigcap_{j \in J} F_j \right).$$

We also call $\mathcal{F}$ the abstract simplicial complex associated with the polyhedron $P$. When $\mathcal{F}$ is an abstract simplicial complex associated with a polyhedron $P$, we have $\{\} \in \mathcal{F}$ for any $j \in [n]$.

Next, we introduce the notion of a polyhedron in the “general position.” Since we need to consider information for points at infinity, we will use the idea of “homogenization.” The homogenization $\hat{\mathcal{H}}$ of a half-space $H = \{x \in \mathbb{R}^d| \sum a_i x_i + a_0 \geq 0\}$ is defined as

$$\hat{\mathcal{H}} = \{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1}| \sum_{i=0}^d a_i x_i \geq 0\}.$$

For a family of half-spaces $\mathcal{H} = \{H_1, \ldots, H_n\}$, we call $\hat{\mathcal{H}} = \{\hat{H}_0, \hat{H}_1, \ldots, \hat{H}_n\}$ the homogenization of $\mathcal{H}$. Here, we put $\hat{H}_0 = \{x_0 \geq 0\}$. We say that a family of half-spaces $\mathcal{H} = \{H_1, \ldots, H_n\}$ (or its homogenization $\hat{\mathcal{H}} = \{\hat{H}_0, \ldots, \hat{H}_n\}$) is in general position when, for $J \subset [n+1]$,

$$\hat{F}_J := \left( \bigcap_{j \in J} \partial \hat{H}_j \right) \cap \left( \bigcap_{j=0}^n \hat{H}_j \right)$$

is a $d+1-|J|$-dimensional cone (i.e., the affine hull of the cone is $d+1-|J|$-dimensional affine space) or $\{0\}$. Here, we denote by $[n+1]$ the set $\{0, 1, \ldots, n\}$. This is somewhat analogous to the “general position” for hyperplane arrangements in [3] Chap 2. Section 9, but we emphasize that they are different (see Example 1). The polyhedron $P$ is in general position when the family $\mathcal{H}$ of the bounding half-spaces of $P$ is in general position.

**Example 1.** Let $d = 2$. We define $H_j$ ($1 \leq j \leq 4$) by

$H_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}, \quad H_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 1\},$

$H_3 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}, \quad H_4 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 1\}.$

Then, the family of half-spaces $\mathcal{H} := \{H_1, H_2, H_3, H_4\}$ is in general position. However, the family of half-spaces $\mathcal{H}' := \{H_1, H_2, H_3\}$ is not in general position.
In fact, the homogenization of $\mathcal{H}$ and $\mathcal{H}'$ can be written as

$$\hat{\mathcal{H}} = \left\{ \hat{H}_0, \hat{H}_1, \hat{H}_2, \hat{H}_3, \hat{H}_4 \right\}, \quad \hat{\mathcal{H}}' = \left\{ \hat{H}_0, \hat{H}_1, \hat{H}_2, \hat{H}_3 \right\}$$

where

\[
\begin{align*}
\hat{H}_0 & := \{(x_0, x_1, x_2) \in \mathbb{R}^{2+1} : x_0 \geq 0\}, \\
\hat{H}_1 & := \{(x_0, x_1, x_2) \in \mathbb{R}^{2+1} : x_1 \geq 0\}, \\
\hat{H}_2 & := \{(x_0, x_1, x_2) \in \mathbb{R}^{2+1} : x_2 \leq x_0\}, \\
\hat{H}_3 & := \{(x_0, x_1, x_2) \in \mathbb{R}^{2+1} : x_2 \leq 0\}, \\
\hat{H}_4 & := \{(x_0, x_1, x_2) \in \mathbb{R}^{2+1} : x_2 \geq x_0\}.
\end{align*}
\]

Calculating $\hat{F}_J$ for each $J \subset \{0, 1, 2, 3, 4\}$, we can show that $\hat{\mathcal{H}}$ is in general position. For example, the set $\hat{H}_0 \cap \partial \hat{H}_1 \cap \partial \hat{H}_2 \cap \partial \hat{H}_3 \cap \partial \hat{H}_4$ is a $d+1-2$-dimensional cone, and the set $\hat{H}_0 \cap \partial \hat{H}_1 \cap \partial \hat{H}_2 \cap \hat{H}_3 \cap \hat{H}_4$ is equal to $\{0\}$. On the other hand, the family $\hat{\mathcal{H}}'$ is not in general position since the dimension of

$$\text{aff}\left(\partial \hat{H}_0 \cap \partial \hat{H}_1 \cap \partial \hat{H}_2 \cap \hat{H}_3\right) = \{x \in \mathbb{R}^{d+1} : x_0 = x_1 = 0\}$$

is not equal to $d + 1 - 3 = 0$.

Hence, the polyhedron $\bigcap_{j=1}^{4} H_j$ in Figure 1(a) is in general position, but the polyhedron $\bigcap_{j=1}^{3} H_j$ in Figure 1(b) is not in general position.

We note that the hyperplane arrangement $\{\partial H_1, \partial H_2, \partial H_3, \partial H_4\}$ is not in general position in the sense of [3], but $\mathcal{H}$ is in general position.

**Example 2.** Let $H_5 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq 0\}$. Using the notation in Example 1, the family of the half-space $\{H_j : 1 \leq j \leq 5\}$, which is shown in Figure 1(c), is not in general position. However, the polyhedron $\bigcap_{j=1}^{5} H_j$ is in general position since the family of the bounding half-spaces is $\{H_j : 1 \leq j \leq 4\}$.

**Remark 3.** Note that our definition of general position is more restrictive than that of [15], and it is less restrictive than that of [14]. For example, the polyhedron $\bigcap_{j=1}^{3} H_j$ in Example 1 is in general position by the definition in [15]; and the polyhedron $\bigcap_{j=1}^{4} H_j$ in Example 1 is not in general position by the definition in [14].

Figure 1: Examples of polyhedra
Let $P \subset \mathbb{R}^d$ be a polyhedron. Suppose the family of bounding half-spaces for $P$ is given by
\[
\left\{ x \in \mathbb{R}^d : \sum_{i=1}^{d} \tilde{a}_{ij} x_i + \tilde{b}_j \geq 0 \right\} \quad (1 \leq j \leq n).
\]
We denote by $\tilde{a}$ the $d \times n$ matrix $(\tilde{a}_{ij})$, and by $\tilde{b}$ the vector $(\tilde{b}_1, \ldots, \tilde{b}_n)$. Let $F_j$ be the intersection of $P$ and the hyperplane $\{ x \in \mathbb{R}^d : \sum_{i=1}^{d} \tilde{a}_{ij} x_i + \tilde{b}_j = 0 \}$. The sets $F_1, \ldots, F_n$ are all of the facets of $P$. Let $\mathcal{F}$ be the nerve of $\{ F_1, \ldots, F_n \}$, which is the abstract simplicial complex of $P$.

Edelsbrunner showed the inclusion–exclusion identity for the indicator function of a polyhedron [6, Lemma 5.1].

**Proposition 4** (Edelsbrunner). If $\mathcal{F}$ is the abstract simplicial complex associated with a polyhedron $P$, then the indicator function of $P$ can be written as
\[
1_P = \sum_{J \in \mathcal{F}} \prod_{j \in J} (1_{H_j} - 1).
\]

**Example 5.** For the polyhedron $\bigcap_{j=1}^{4} H_j$ in Example 4, the inclusion–exclusion identity can be written as follows:
\[
1_{\bigcap_{j=1}^{4} H_j} = 1 + (1_{H_1} - 1) + (1_{H_2} - 1) + (1_{H_3} - 1) + (1_{H_4} - 1)
+ (1_{H_1} - 1)(1_{H_2} - 1) + (1_{H_3} - 1)(1_{H_1} - 1)
+ (1_{H_2} - 1)(1_{H_3} - 1) + (1_{H_2} - 1)(1_{H_4} - 1).
\]
The first term of the right-hand side corresponds to the empty set.

With the Heaviside function $H$, the Edelsbrunner’s identity can be written as
\[
\prod_{j=1}^{n} H \left( \sum_{i=1}^{d} \tilde{a}_{ij} x_i + \tilde{b}_j \right) = \sum_{J \in \mathcal{F}} \prod_{j \in J} \left( H \left( \sum_{i=1}^{d} \tilde{a}_{ij} x_i + \tilde{b}_j \right) - 1 \right).
\]
Under the general position assumption, this identity can be generalized as follows.

**Theorem 6.** In the notation above, if the polyhedron $P$ is in general position, then there exists a neighborhood $U$ of $(\tilde{a}, \tilde{b})$ such that the equation
\[
\prod_{j=1}^{n} H \left( \sum_{i=1}^{d} a_{ij} x_i + b_j \right) = \sum_{J \in \mathcal{F}} \prod_{j \in J} \left( H \left( \sum_{i=1}^{d} a_{ij} x_i + b_j \right) - 1 \right)
\]
holds for all $(a, b, x) \in U \times \mathbb{R}^d$.

Our proof of Theorem 6 is technical, and it is independent from the other parts of this paper; thus, it is given in Section 4.

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Consider \( n \) polynomials \( f_j(a, b, x) = \sum_{i=1}^d a_{ij} x_i + b_j \) \((1 \leq j \leq n)\) with variables \( a_{ij}, b_j, x_i \) \((1 \leq i \leq d, 1 \leq j \leq n)\), and let
\[
\chi_F(a, b, x) = \prod_{j \in F} (H(f_j(a, b, x)) - 1)
\]
for \( F \in \mathcal{F} \). Note that \( \chi_{\emptyset}(a, b, x) = 1 \). We put
\[
\varphi_F(a, b) = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^d x_i^2\right) \chi_F(a, b, x) dx \quad (F \in \mathcal{F}).
\]
By Theorem 6, the function \( \varphi(a, b) \) in (2) can be decomposed as
\[
\varphi(a, b) = \sum_{F \in \mathcal{F}} \varphi_F(a, b) \quad (5)
\]
on a neighborhood of \((\tilde{a}, \tilde{b})\) if the polyhedron \( P \) is in general position.

In order to give analytic continuations of the function \( \varphi(a, b) \), it is enough to consider \( \varphi_F(a, b) \). For \( F \in \mathcal{F} \), let \( \alpha_F(a) = (\alpha_{ij}(a))_{i,j \in F} \) be an \(|F| \times |F|\) matrix, where \(|F|\) is the number of the elements in \( F \) and
\[
\alpha_{ij}(a) = \sum_{k=1}^d a_{ki} a_{kj} \quad (1 \leq i, j \leq n).
\]
This is a submatrix of the Gram matrix of \( a \). The matrices \( \alpha_F(a) \) are symmetric and positive semidefinite. Since the function \( \varphi_F(a, b) \) can be written as
\[
\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} (-1)^{|F|} \exp\left(-\frac{1}{2} \sum_{i=1}^d x_i^2\right) \prod_{j \in F} H(-f_j(a, b, x)),
\]
we can expect that \( \varphi_F(a, b) \) is written by a \( d \)-simplex and the \( d \)-form
\[
\frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^d z_i^2\right) \prod_{j \in F} \left(\frac{\log(f_j(\tilde{a}, \tilde{b}, z))}{2\pi\sqrt{-1}}\right) dz.
\]
In fact, we can find a suitable \( d \)-simplex and thus have the following lemma.

**Lemma 7.** If the polyhedron \( P \) is in general position, then for \( F \in \mathcal{F} \), the value of \( \varphi_F(\tilde{a}, \tilde{b}) \) can be written as
\[
\sum_{\lambda \in \{\pm 1\}^{|F|}} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^d z_i^2\right) \prod_{j \in F} \left(\frac{\log(f_j(\tilde{a}, \tilde{b}, z))}{2\pi\sqrt{-1}}\right) dz. \quad (6)
\]
Here, for \( \lambda \in \{\pm 1\}^{|F|}, \gamma^\lambda \) is a smooth map from \( \mathbb{R}^d \) to \( \mathbb{C}^d \). We suppose the multivalued function \( \log \) satisfies \( \log(1) = 0 \) and the branch cut is \( \{ z \in \mathbb{C} : \Re(z) \leq 0 \} \).
Proof. Let \( s \) be the number of elements in \( F \). By the general position assumption, \( s \) is not greater than \( d \). We denote by \( j(1), \ldots, j(s) \) all of the elements of \( F \). Since the polyhedron \( P \) is in general position, the vectors \( u_k = (\tilde{a}_{ij(k)}, \ldots, \tilde{a}_{dj(k)})^\top (1 \leq k \leq s) \) are linearly independent (see Corollary 12 in Section 4). Consequently, the determinant \( \alpha_F(\tilde{a}) \) is not zero. Let \( u_k(s < k \leq d) \) be an orthonormal basis of the orthogonal complement of the subspace \( \sum_{k=1}^s \mathbb{R}u_k \subset \mathbb{R}^d \). We denote the vector \( u_k \) by \( u_k = (u_{ik}, \ldots, u_{dk})^\top \).

The matrix \( u = (u_{ij}) \) is regular, and we set \( u^{-1} = (u^{ij}) \). Without loss of generality, we can assume \( \det u > 0 \). Under this assumption, we have \( \det u = \sqrt{\det \alpha_F(\tilde{a})} \). Let \( \varepsilon(t) \) be a positive bounded function on \( \mathbb{R} \), i.e., \( \inf_{t \in \mathbb{R}} \varepsilon(t) > 0 \) and \( \sup_{t \in \mathbb{R}} \varepsilon(t) < \infty \). For a vector \((\lambda_1, \ldots, \lambda_s) \in \{-1, 1\}^s \), we define \( \gamma^\lambda: \mathbb{R}^d \to \mathbb{C}^d \) by

\[
\gamma^\lambda_j(t) = \sum_{i=1}^d u^{ij} (\lambda_i t_i + \lambda_i \sqrt{-1} \varepsilon (\lambda_i t_i)) \ (t \in \mathbb{R}^d).
\]

Here, we put \( \lambda_j = 1 \) for \( s < j \leq d \).

By the coordinate transformation \( w_j = \sum_{i=1}^d u_{ij} z_i \), the integral [4] can be written as

\[
\frac{1}{(2\pi)^{d/2}} \frac{1}{|\alpha_F(\tilde{a})|^{1/2}} \sum_{\lambda \in \{\pm 1\}^s} \int \gamma^\lambda \exp\left(-\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d u^{ij} w_j^2 \right) \prod_{k=1}^s \left( \frac{\log(w_k + \tilde{b}_k)}{2\pi \sqrt{-1}} \right) dw,
\]

where \( \gamma^\lambda_j(t) = \lambda_j t_j + \lambda_j \sqrt{-1} \varepsilon (\lambda_j t_j) \ (t \in \mathbb{R}^d) \). Calculating this integral recursively, we have

\[
\frac{1}{(2\pi)^{d/2}} \frac{1}{|\alpha_F(\tilde{a})|^{1/2}} \int_E \exp\left(-\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d u^{ij} y_j^2 \right) dy,
\]

where \( E = \{ y \in \mathbb{R}^d : y_j + \tilde{b}_j \leq 0, j \in F \} \). By the coordinate transformation \( x_i = \sum_{j=1}^d u^{ij} y_j \), the above integral is

\[
\frac{1}{(2\pi)^{d/2}} \int_{f_j(\tilde{a}, \tilde{b}, x) \leq 0, j \in F} \exp\left(-\frac{1}{2} \sum_{i=1}^d x_i^2 \right) dx,
\]

which is equal to \( \varphi_F(\tilde{a}, \tilde{b}) \).

Moreover, we have the following proposition.

**Proposition 8.** Let \( U_F \) be a domain \( \{(a, b) : \det \alpha_F(a) \neq 0\} \). The function \( \varphi_F(a, b) \) can be written as

\[
\sum_{\lambda \in \{\pm 1\}^{|F|}} \int_{\gamma^\lambda} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^d z_i^2 \right) \prod_{j \in F} \left( \frac{-\log(f_j(a, b, z))}{2\pi \sqrt{-1}} \right) dz \quad (7)
\]

on a connected open neighborhood of \((\tilde{a}, \tilde{b})\) in \( U_F \). Here, we suppose the multivalued function log satisfies \( \log(1) = 0 \) and the branch cut is \( \{ z \in \mathbb{C} : \Re(z) \leq 0 \} \).
Proof. Since \( \det(\alpha_F(a)) \neq 0 \), by arguments similar to those in the proof of Lemma 7, \( \varphi_F(a, b) \) can be written as
\[
\sum_{\lambda \in \{\pm 1\}^{\vert F \vert}} \int_{\gamma_{\lambda}'} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{d} z_i^2 \right) \prod_{j \in F} \left( \frac{\log(f_j(a, b, z))}{2\pi \sqrt{-1}} \right) dz.
\]
The matrix \( u' \) and the integral path \( \gamma_{\lambda}' \) can be constructed similarly.

We need to show that the above integral is equal to (7). There is a smooth path \( u(t) (t \in [0, 1]) \) in the general linear group of degree \( d \) over \( \mathbb{C} \) such that \( u(0) = u \) and \( u(1) = u' \). The homotopy between \( \gamma_{\lambda} \) and \( \gamma_{\lambda}' \) is given by
\[
\gamma_{\lambda,j}(s)(t) = \sum_{k=1}^{d} u_{jk}(s) \left( \lambda_k t_k + \lambda_k \sqrt{-1} \epsilon(\lambda_k t_k) \right) \quad (t \in \mathbb{R}^d).
\]
Consequently, the value of the integral on the right-hand side of (7) does not change when we change the integral path with \( \gamma_{\lambda}' \).

Corollary 9. The function \( \varphi(a, b) \) is a real analytic function, and it has an analytic continuation along every path in \( \bigcap_{F \in \mathcal{F}} U_F \).

4 Proof of Theorem 6

In this section, we prove Theorem 6. For this purpose, we need to present some notation and some lemmas, most importantly, Theorem 15. The argument in the proof of Lemma 10 is also important, since analogous arguments will appear repeatedly in the proof of Theorem 15.

Let \( a = (a_1, \ldots, a_n) \in \mathbb{R}^{(d+1)\times n} \) be a matrix where we denote by \( a_j \) the \( j \)-th column vector of \( a \). Put \( a_0 := (1, 0, \ldots, 0)^\top \in \mathbb{R}^{d+1} \). For a matrix \( a \), we put
\[
H_j(a) := \left\{ x \in \mathbb{R}^d : \sum_{i=1}^{d} a_{ij} x_i + a_{0j} \geq 0 \right\} \quad (j \in [n]),
\]
\[
\mathcal{H}(a) := \{ H_1(a), \ldots, H_n(a) \},
\]
\[
P(a) := \bigcap_{j=1}^{n} H_j(a),
\]
\[
F_j(a) := \partial H_j(a) \cap P(a) \quad (j \in [n]),
\]
\[
F_J(a) := \bigcap_{j \in J} F_j(a) \quad (J \subset [n]),
\]
\[
\mathcal{F}(a) := \{ J \subset [n] : F_J(a) \neq \emptyset \}.
\]
Note that \( F_j(a) \) is not necessarily a facet of \( P(a) \), and \( \mathcal{F}(a) \) is not necessarily equivalent to the abstract simplicial complex associated with \( P(a) \). For this difficulty, we need the notion of families of half-spaces in general position. In
fact, in Lemma 14, we will show that the abstract simplicial complex associated with \( P(a) \) is equivalent to \( F(a) \) under the general position assumption, which is required by the proof of Theorem 6.

In order to consider combinatorial structures at the point at infinity, we introduce the following notion: for the abstract simplicial complex \( F \) associated with a polyhedron, the homogenization of \( F \) is the abstract simplicial complex defined by

\[
\hat{F} := \left\{ J \subset [n + 1] : \hat{F}_J \neq \{0\} \right\} \quad ([n + 1] = \{0, 1, \ldots, n\}).
\]

Since \( F \subset \hat{F} \), we have \( \{j\} \in \hat{F} \) for \( j \in [n] \). Note that \( \{0\} \in \hat{F} \) does not hold in general. The following are the “homogenization” of the notations given in the previous paragraph. For a matrix \( a \), we put

\[
\hat{H}_j(a) := \left\{ x \in \mathbb{R}^{d+1} : \sum_{i=0}^{d} a_{ij}x_i \geq 0 \right\} \quad (j \in [n + 1]),
\]

\[
\hat{H}(a) := \left\{ \hat{H}_j(a) : j \in [n + 1] \right\},
\]

\[
\hat{P}(a) := \bigcap_{j=1}^{n+1} \hat{H}_j(a),
\]

\[
\hat{F}_j(a) := \partial \hat{H}_j(a) \cap \hat{P}(a) \quad (j \in [n + 1]),
\]

\[
\hat{F}(a) := \left\{ J \subset [n + 1] : \hat{F}_J(a) \neq \{0\} \right\},
\]

\[
\hat{F}_j(a) := \bigcap_{j \in J} \hat{F}_j(a) \quad (J \subset [n + 1]).
\]

For all \( J \in [n + 1] \), \( \hat{F}_J(a) \) includes the zero vector. Analogous to the case of the abstract simplicial complex associated with polyhedra, we have \( F(a) \subset \hat{F}(a) \).

For a family of half-spaces in general position, we have the following lemma, which is required by the proof of Lemma 14.

**Lemma 10.** Suppose \( \mathcal{H}(a) \) is in general position. Let \( J \subset [n] \). If \( J \in F(a) \), then the set \( F_J(a) \) is a \( d - |J| \)-dimensional face of \( P(a) \). If \( J \notin F(a) \), then the set \( F_J(a) \) is empty. In particular, \( P(a) \) is a \( d \)-dimensional polyhedron.

**Proof.** If \( J \notin F(a) \), it is obvious that \( F_J(a) = \emptyset \). Let us consider the case where \( J \in F(a) \). Put \( J = \{j(0), \ldots, j(s)\}, J^c := [n + 1] \setminus J \). Since \( \mathcal{H}(a) \) is in general position, we have \( F_j(a) = \{0\} \) or

\[
\hat{F}_j(a) \supseteq \hat{F}_j(a) \cap \partial \hat{H}_k(a)
\]

for any \( k \in J^c \). In fact, the equality of \( \hat{F}_j(a) \neq \{0\} \) contradicts the assumption about the dimension when \( \hat{F}_j(a) \neq \{0\} \). However, we have \( \hat{F}_j(a) \neq \{0\} \), since \( J \in F(a) \) implies \( J \in \hat{F}(a) \). Hence, condition \( \hat{F}_j(a) \) holds for any \( k \in J^c \). For \( k \in J^c \), take
\( x^{(k)} \in \hat{F}_J(a) \setminus \partial \hat{H}_k(a) \). Then the affine combination
\[
\frac{1}{n + 1 - |J|} \sum_{k \in J^c} x^{(k)}
\]
is an element of
\[
\hat{F}_J(a) \cap \left( \bigcap_{k \in J^c} \text{int}(\hat{H}_k(a)) \right),
\]
where \( \text{int}(S) \) denotes the interior of \( S \). Since \( 0 \in J^c \), we have
\[
\hat{F}_J(a) \cap \{ x_0 = 1 \} \cap \left( \bigcap_{k \in J^c} \text{int}(\hat{H}_k(a)) \right) \neq \emptyset.
\]

Let \( x \) be an element of this set, and let \( B \) be an open ball centered at \( x \) whose closure is included in \( \bigcap_{k \in J^c} \text{int}(\hat{H}_k(a)) \). Let \( x' \) be an arbitrary point in \( \{ x_0 = 1 \} \cap \bigcap_{j \in J} \partial \hat{H}_j(a) \), and let \( x'' \) be an intersection point of \( \partial B \) and the line between \( x \) and \( x' \). Since \( x'' \in \hat{F}_J(a) \) and \( x' \) can be written as an affine combination of \( x \) and \( x'' \), we have \( x' \in \text{aff}(\hat{F}_J(a)) \). By the arbitrariness of \( x' \), we have
\[
\text{aff}(\hat{F}_J(a)) \cong \text{aff} \left( \hat{F}_J(a) \cap \{ x_0 = 1 \} \right) = \bigcap_{j \in J} \partial \hat{H}_j(a) \cap \{ x_0 = 1 \}.
\]

Analogously, we have
\[
\text{aff}(\hat{F}_J(a)) = \bigcap_{j \in J} \partial \hat{H}_j(a).
\]

Since \( \hat{H}(a) \) is in general position, the left-hand side of (10) is a \( d + 1 - |J| \)-dimensional cone. Consequently, the vectors \( a_{j(1)}, \ldots, a_{j(s)} \) are linearly independent. Moreover, the vectors \( a_0, a_{j(1)}, \ldots, a_{j(s)} \) are linearly independent. In fact, if there are \( c_j \) (\( j \in J \)) such that \( a_0 = \sum_{j \in J} c_j a_j \), then we have
\[
0 = \sum_{j \in J} c_j \left( \sum_{i=1}^d a_{ij} x_i + a_{0j} \right) = 1 \text{ for } x \in F_J. \quad \text{Hence, the dimension of } \text{aff}(F_J) \text{ is equal to } d - |J|. \]

By the argument in the proof of Lemma 10 when \( \mathcal{H}(a) \) is in general position, \( J \in \mathcal{F}(a) \) implies \( F_J(a) \neq \emptyset \). Hence, we have the following corollary.

**Corollary 11.** When \( \mathcal{H}(a) \) is in general position, for \( J \subset [n] \), \( J \in \mathcal{F}(a) \) holds if and only if \( J \in \mathcal{F}(a) \) holds.

Similarly, by the argument in the proof of Lemma 10, we have the following.

**Corollary 12.** When \( \mathcal{H}(a) \) is in general position and \( J = \{ j(1), \ldots, j(s) \} \in \mathcal{F}(a) \), the column vectors \( a_{j(1)}, \ldots, a_{j(s)} \) are linearly independent.

We now review the Farkas lemma in \([24]\).
Proposition 13 (Farkas lemma). The inequality \( \sum_{i=1}^{d} c_i x_i + c_0 \geq 0 \) is valid for \( P(a) \) if and only if one of the following conditions holds:

(i) There exist \( \lambda_j \geq 0 \) \((j \in [n])\) such that

\[
\sum_{j=1}^{n} a_{ij} \lambda_j = c_i, \quad \sum_{j=1}^{n} a_{0j} \lambda_j \leq c_0 \quad (1 \leq i \leq d);
\]

(ii) There exist \( \lambda_j \geq 0 \) \((j \in [n])\) such that

\[
\sum_{j=1}^{n} a_{ij} \lambda_j = 0, \quad \sum_{j=1}^{n} a_{0j} \lambda_j < 0 \quad (1 \leq i \leq d).
\]

Note that condition (ii) in the Farkas lemma implies \( P(a) = \emptyset \).

Lemma 10 and Proposition 13 imply the following lemma, which is not required by the proof of Theorem 15 but is required by that of Theorem 6.

Lemma 14. If \( \mathcal{H}(a) \) is in general position, then the abstract simplicial complex associated with \( P(a) \) is equivalent to \( F(a) \).

Proof. Note that we assume \( d \geq 1 \) and \( P(a) \) is a \( d \)-dimensional polyhedron. Let \( F \) be a facet of \( P(a) \); then there is an inequality \( \sum_{i=1}^{d} c_i x_i + c_0 \geq 0 \) valid for \( P(a) \) such that

\[
F = P(a) \cap \left\{ x \in \mathbb{R}^d : \sum_{i=1}^{d} c_i x_i + c_0 = 0 \right\}.
\]

Since \( P(a) \) is not empty, condition (ii) in the Farkas lemma does not hold. By the Farkas lemma, there exist \( \lambda_j \geq 0 \) \((j \in [n])\) such that

\[
\sum_{j} a_{ij} \lambda_j = c_i, \quad \sum_{j} a_{0j} \lambda_j \leq c_0 \quad (1 \leq i \leq d).
\]

Moreover, there is a unique \( \lambda_j \) which is greater than 0. In fact, if there is not such a \( j \), then we have \( c_i = 0 \) \((1 \leq i \leq d)\). This implies \( F = P(a) \) or \( F = \emptyset \). This is a contradiction.

If there are two indexes \( j(1) \) and \( j(2) \) such that \( \lambda_{j(1)} > 0, \lambda_{j(2)} > 0 \), and \( j(1) \neq j(2) \), then we have \( F \subset P(a) \cap \partial H_{j(1)}(a) \cap \partial H_{j(2)}(a) \). This contradicts the fact that \( F \) is \( d - |J| \)-dimensional.

Therefore, there is a unique \( j \) such that \( a_{ij} \lambda_j = c_i, a_{0j} \lambda_j \leq c_0 \). Since we have

\[
0 \leq \lambda_j \left( \sum_{i=1}^{d} a_{ij} x_i + a_{0j} \right) \leq \sum_{i=1}^{d} c_i x_i + c_0 = 0
\]

for \( x \in F \), we have \( c_0 = a_{0j} \lambda_j \). Consequently, \( F = F_j(a) \).

If \( F_j(a) \neq \emptyset \), then, by Lemma 10, \( F_j(a) \) is a facet of \( P(a) \).

Therefore, all the facets of \( P(a) \) are given by \( F_j(a) \{(j) \in \mathcal{F}(a) \} \). Consequently, the abstract simplicial complex \( \mathcal{F}(a) \) associated with \( P(a) \) is equivalent to \( \mathcal{F}(a) \). \( \square \)
Theorem 15. Let $P \subset \mathbb{R}^d$ be a polyhedron in general position, let $n$ be the number of facets of $P$, and let $\mathcal{F}$ be the abstract simplicial complex associated with $P$. Then, the set

$$U = \left\{ a \in \mathbb{R}^{(d+1) \times n} : H(a) \text{ is in general position and } \hat{F}(a) = \hat{\mathcal{F}} \right\}$$

is an open set of $\mathbb{R}^{(d+1) \times n}$.

Proof. Our proof is given by writing $U$ as an intersection of finite open sets $V_J (J \subset [n+1])$:

$$U = \bigcap_{J \subset [n+1]} V_J. \quad (11)$$

STEP 1. First, we define $V_J$. For $J = \{j(1), \ldots, j(s)\} \in \hat{\mathcal{F}}$, let $V_J$ consist of $a \in \mathbb{R}^{(d+1) \times n}$ such that the vectors $a_{j(1)}, \ldots, a_{j(s)}$ are linearly independent and the set $[9]$ includes a nonzero element. For $J \subset [n+1]$ such that $J \notin \hat{\mathcal{F}}$, let $V_J$ be the intersection of

$$\left\{ a \in \mathbb{R}^{(d+1) \times n} : \hat{F}_J(a) = \{0\} \right\}$$

and

$$\bigcap_{j=1}^n \left\{ a \in \mathbb{R}^{(d+1) \times n} : a_j \neq 0 \right\}. \quad (12)$$

STEP 2. Next, we show that the right-hand side of (11) includes the left-hand side. Let $J = \{j(1), \ldots, j(s)\} \subset [n+1]$.

Suppose $J \in \hat{\mathcal{F}}$. Let $a \in U$; then we have $\hat{F}(a) = \hat{\mathcal{F}}$ by the definition of $U$. When $J^c = \emptyset$, the set $[9]$ is equal to $\hat{F}_J(a)$ and includes a nonzero element. When $J^c \neq \emptyset$, by an argument analogous to that in the proof of Lemma [10] we can show that the set $[9]$ includes a nonzero element. Consequently, equation (10) holds. Since the left-hand side of (10) is a $d + 1 - |J|$-dimensional affine subspace, the vectors $a_{j(1)}, \ldots, a_{j(s)}$ are linearly independent. Hence, we have $U \subset V_J (J \in \mathcal{F})$.

Suppose $J \notin \hat{\mathcal{F}}$. For $a \in U$, it is obvious that $\hat{F}_J(a) = \{0\}$. By the assumption for $\mathcal{F}$, we have $\{j\} \in \hat{\mathcal{F}}$ for arbitrary $j \in [n]$. By the argument in the previous paragraph, $a \in U$ implies $a_j \neq 0$. Hence, we have $U \subset V_J (J \notin \mathcal{F})$.

Therefore, the right-hand side of (11) includes the left-hand side.

STEP 3. We show that the left-hand side of (11) includes the right-hand side. Suppose a matrix $a$ is included in the right-hand side of (11). Let $J \subset [n]$. When $J \in \mathcal{F}$, $a \in V_J$ implies that the set $[9]$ includes a nonzero element. Hence, we have $\hat{F}_J(a) \neq \{0\}$. When $J \notin \mathcal{F}$, $a \in V_J$ implies $\hat{F}_J(a) = \{0\}$. Consequently, we have $\hat{F}(a) = \hat{\mathcal{F}}$.

Next, we show that $H(a)$ is in general position. It is enough to show that $\hat{F}_J(a)$ is a $d + 1 - |J|$-dimensional cone or that it is equal to $\{0\}$ for $J =$
\{j(1), \ldots, j(s)\} \subset [n + 1]. When \(J \in \mathcal{F}\), set \([9]\) is not empty and equation \([10]\) holds. Since the vectors \(a_{j(1)}, \ldots, a_{j(s)}\) are linearly independent, the right-hand side of \([10]\) is a \(d + 1 - |J|\)-dimensional affine subspace. When \(J \notin \mathcal{F}\), we have \(\hat{\mathcal{F}}_J(a) = \{0\}\). Hence, \(\mathcal{H}(a)\) is in general position. Therefore, equation \([11]\) holds.

**STEP 4.** Let \(J \subset [n + 1]\). We show that \(V_J\) is an open set.

Suppose \(J \in \mathcal{F}\). Let \(a \in V_J\). Since the vectors \(a_j(j \in J)\) are linearly independent, there is a subset \(I \subset [n]\) such that \(|I| = |J|\) and the \(|I| \times |J|\) matrix \((a_{ij})_{i \in I, j \in J}\) has the inverse matrix \((c_{ij})_{i \in I, j \in J}\). Let \(\sigma : I \to J\) be a bijection, then

\[
y_k = \begin{cases} 
\sum_{i \in I} a_{i\sigma(k)}x_i & (k \in I) \\
x_k & (k \in I^c)
\end{cases}
\]

defines a coordinate transformation. By this coordinate transformation, the condition that set \([9]\) includes a nonzero element can be written as follows: there exists \(y \in \mathbb{R}^d\) such that

\[
\begin{align*}
\sum_{i \in I} a_{i\sigma(j)}x_i &= 0 & (j \in J), \\
\sum_{i \in I} k_{j,i}a_{i\sigma(j)} + \sum_{i \in I^c} c_{i\sigma(j)}a_{ij} &= y & (j \in J^c), \\
y & \neq 0.
\end{align*}
\]

Let \(V_{IJ}\) be the set consisting of \(a\) such that the matrix \((a_{ij})_{i \in I, j \in J}\) is invertible and there exists \(y\) satisfying condition \([13]\). Then, we have

\[
V_J = \bigcup_{I \subseteq [n], |I| = |J|} V_{IJ}.
\]

In equation \([13]\), the value of \(y_j(j \in J)\) is determined uniquely by the value of the other variables. We can regard \([13]\) as a condition for the variables \(a, y_j(j \notin J)\). The set \(V_{IJ}\) is the projection of the following open subset of \(\mathbb{R}^{(d+1)n} \times \mathbb{R}^{n-|J|}\): the set consisting of \((a, y_j)_{j \notin J} \in \mathbb{R}^{(d+1)n} \times \mathbb{R}^{n-|J|}\) that satisfies \([13]\) and in which \((a_{ij})_{i \in I, j \in J}\) is invertible. This implies that \(V_{IJ}\) is open. Consequently, \(V_J\) is an open set.

Suppose \(J \notin \mathcal{F}\). There is the isomorphism between the set \([12]\) and \(\mathbb{R}_{>0}^n \times (\mathbb{S}^d)^n\). Here, \((\mathbb{S}^d)^n\) denotes the \(n\)-th direct product of a \(d\)-dimensional sphere \(\mathbb{S}^d\). Since \(V_J\) is closed under the transformation which multiplies column vectors by positive numbers,

\[
(a_{ij}) \mapsto (\lambda_j a_{ij}), \quad (\lambda_j \in \mathbb{R}_{>0}),
\]

the image of \(V_J\) under the isomorphism is \(\mathbb{R}_{>0}^n \times ((\mathbb{S}^d)^n \cap V_J)\). It is enough to show \((\mathbb{S}^d)^n \cap V_J\) is an open set. Since the projection

\[
(\mathbb{S}^d)^n \times \mathbb{S}^d \to (\mathbb{S}^d)^n, \quad (a, x) \mapsto a
\]

is a continuous map from a compact set to a Hausdorff space, it is a closed map. The set \((\mathbb{S}^d)^n \setminus V_J\) is the projection of the following closed set of \((\mathbb{S}^d)^n \times \mathbb{S}^d:\)
the intersection of \((S^d)^n \times S^d\) and the subset of \(\mathbb{R}^{(d+1)\times n} \times \mathbb{R}^{d+1}\) consisting of \((a, x)\) satisfying
\[
\begin{align*}
\sum_{i=0}^d a_{ij} x_i &= 0 \quad (j \in J) \\
\sum_{i=0}^d a_{ij} x_i &\geq 0 \quad (j \in J^c).
\end{align*}
\]
Hence, \((S^d)^n \cap V_J\) is open.

Proof of Theorem 6. With the notation of Theorem 15, it is enough to show that for \(a \in U\), the abstract simplicial complex associated with \(P(a)\) is equivalent to \(\mathcal{F}\). Let \(a \in U\). By Lemma 14, the abstract simplicial complex associated with \(P(a)\) is equivalent to \(\tilde{F}(a)\). By Corollary 11, \(\tilde{F}(a) = \tilde{F}\) implies \(F(a) = F\).

5 Holonomic modules

In this section, we explicitly give a system of differential equations for the function
\[
\chi_P(a, b, x) = \sum_{F \in \mathcal{F}} \chi_F(a, b, x),
\]
and show that the system is holonomic. Since the equation
\[
\varphi(a, b) = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^d x_i^2\right) \chi_P(a, b, x) dx
\]
holds on a neighborhood of \((\tilde{a}, \tilde{b})\), a holonomic system for \(\varphi(a, b)\) can be given as the integration module of a holonomic module associated with
\[
\exp\left(-\frac{1}{2} \sum_{i=1}^d x_i^2\right) \chi_P(a, b, x).
\]

For more about the integration module, see [5] and [19].

For \(J \subset [n]\), we define a hyperfunction \(\chi^J\) by
\[
\chi^J = \partial^J_P \chi_P \quad (\partial^J_P := \prod_{i \in J} \partial_{b_i}).
\]

Note that \(\chi^\emptyset = \chi_P\).

Lemma 16. If \(J \subset [n]\) is not an element of \(\mathcal{F}\), then we have \(\partial^J_P \chi_P = 0\) for \(F \in \mathcal{F}\). Consequently, we have \(\chi^J = 0\).

Proof. Since \(\mathcal{F}\) is an abstract simplicial complex, we have \(J \not\subset F\). Take \(k \in J \setminus F\), then we have \(\partial^J_P \chi_P = 0\), since \(\partial_{b_k} \prod_{j \in F} (H(f_j(a, b, x)) - 1) = 0\).
We now provide a system of differential equations for the $\chi^J$'s. Let $g = (g^J)_{J \in F}$ be a vector whose elements are functions indexed by the set $F$. Let us consider the system defined by the following:

\[
\begin{align*}
\partial_{x_i} g^J &= \sum_{j=1}^{n} a_{ij} \partial_{b_j} g^J \quad (1 \leq i \leq d, \ J \in F), \\
\partial_{a_{ij}} g^J &= x_i \partial_{b_j} g^J \quad (1 \leq i \leq d, 1 \leq j \leq n, \ J \in F), \\
\partial_{b_j} g^J &= g^{J \cup \{j\}} \quad (j \in J^c, \ J \in F), \\
f_j g^J &= 0 \quad (j \in J, \ J \in F),
\end{align*}
\]

(15) - (18)

where $g^{J \cup \{j\}} = 0$ for $J \cup \{j\} \notin F$.

**Lemma 17.** Let $g = (\partial_{b_{J}} \chi_F)_{J \in F}$ for $F \in F$. Then the function $g$ satisfies equations (15), (16), (17), and (18).

**Proof.** When $F = \emptyset$, it is obvious that equations (15), (16), (17), and (18) hold, since $\chi_\emptyset = 1$. Suppose $F$ is not the empty set.

We first check equation (15). Since $\partial_{b_j} \chi_F(a, b, x) = 0$ for $j \notin F$, we have

\[
\partial_{x_i} \chi_F(a, b, x) = \sum_{j \in F} a_{ij} \partial_{b_j} \chi_F(a, b, x) = \sum_{j=1}^{n} a_{ij} \partial_{b_j} \chi_F(a, b, x).
\]

Here, we apply the chain rule for hyperfunctions. The equation above implies (15).

We now show equation (16). When $j \notin F$, both sides of (16) are equal to 0. When $j \in F$, we have $\partial_{a_{ij}} g^J = x_i \partial_{b_j} g^J$ by the chain rule.

Equation (17) holds by Lemma 16. For $J \subset F$, we have

\[
g^J = \prod_{k \in J} \delta(f_k(a, b, x)) \prod_{k \in F \setminus J} (H(f_k(a, b, x)) - 1).
\]

This implies (18).

By (13), we have $\chi^J = \sum_{F \in F} \partial_{b_{J}} \chi_F$. By this equation and Lemma 17, we have the following proposition.

**Proposition 18.** The vector-valued function $g = (\chi^J)_{J \in F}$ satisfies equations (15), (16), (17), and (18).

Next, we show that the system defined by (15), (16), (17), and (18) for $\chi^J$ is a holonomic system. In the remainder of this paper, we will frequently use the following rings:

\[
\begin{align*}
D_{ab} &= C(a_{ij}, b_j, x_i, \partial_{a_{ij}}, \partial_{b_j}, \partial_{x_i} : 1 \leq i \leq d, 1 \leq j \leq n), \\
D_{ab} &= C(a_{ij}, b_j, \partial_{a_{ij}}, \partial_{b_j} : 1 \leq i \leq d, 1 \leq j \leq n), \\
C[a, b, x, \xi_a, \xi_b, \xi_x] &= C[a_{ij}, b_j, x_i, \xi_a, \xi_b, \xi_x : 1 \leq i \leq d, 1 \leq j \leq n].
\end{align*}
\]

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We also use the free modules \((D_{abx})^{[F]}, (D_{ab})^{[F]},\) and \(C[a, b, x]^{[F]}\), whose basis is \(\{g^J : J \in F\}\).

**Proposition 19.** Let \(N_\chi\) be the sub left \(D_{abx}\)-module of \((D_{abx})^{[F]}\) generated by

\[
\left(\partial_{x_i} - \sum_{j=1}^n a_{ij} \partial_{b_j}\right) g^J \quad (1 \leq i \leq d, J \in F),
\]

(19)

\[
\left(\partial_{a_{ij}} - x_i \partial_{b_j}\right) g^J \quad (1 \leq i \leq d, 1 \leq j \leq n, J \in F),
\]

(20)

\[
\partial_{b_j} g^J - g^{J \cup \{j\}} \quad (j \in J^c, J \in F),
\]

(21)

\[
f_j g^J, \quad (j \in J, J \in F).
\]

(22)

Then, the quotient module \(M_\chi = (D_{abx})^{[F]} / N_\chi\) is holonomic.

**Proof.** The principal symbols of (19), (20), (21), and (22) are the following:

\[
(\xi_{x_i} - \sum_{j=1}^n a_{ij} \xi_{b_j}) g^J \quad (1 \leq i \leq d, J \in F),
\]

(23)

\[
(\xi_{a_{ij}} - x_i \xi_{b_j}) g^J \quad (1 \leq i \leq d, 1 \leq j \leq n, J \in F),
\]

(24)

\[
f_j g^J, \xi_{b_k} g^J \quad (j \in J, k \in J^c, J \in F).
\]

(25)

For \(J \in F\), let \(V_J\) be an algebraic variety defined by

\[
\xi_{x_i} - \sum_{j=1}^n a_{ij} \xi_{b_j}, \xi_{a_{ij}} - x_i \xi_{b_j} \quad (1 \leq i \leq d, 1 \leq j \leq n),
\]

(26)

\[
f_j, \xi_{b_k} \quad (j \in J, k \in J^c).
\]

(27)

By [18, Proposition 1], the union \(\bigcup_{J \in F} V_J\) includes \(\text{Char}(M)\). Since the rank of the Jacobian matrix of (26) and (27) is \(nd + n + d\), the Krull dimension of \(V_J\) is equal to \(nd + n + d\). Hence, the dimension of \(\text{Char}(M)\) is not greater than \(nd + n + d\).

Let \(N_q\) be a \(D_{abx}\)-submodule of \((D_{abx})^{[F]}\) generated by (20), (21), (22), and

\[
\left(\partial_{x_i} - \sum_{j=1}^n a_{ij} \partial_{b_j}\right) g^J \quad (1 \leq i \leq d, J \in F).
\]

(28)

Proposition 19 implies that the quotient module \(M_q = (D_{abx})^{[F]} / N_q\) is a holonomic module. Moreover, the function

\[
q^J(a, b, x) = \exp\left(-\frac{1}{2}\sum_{i=1}^d x_i^2\right) \chi^J \quad (J \in F)
\]
is a solution of $N_q$ (see [20]). By Lemma [17] the function
\[ \exp(-\frac{1}{2} \sum_{i=1}^{d} x_i^2) \partial_J^I \chi_F \quad (J \in \mathcal{F}) \]
is also a solution of $N_q$ for $F \in \mathcal{F}$.
Calculating the integration module of $N_q$ with respect to $x$, we have the following theorem.

**Theorem 20.** Let $N$ be the sub left $D_{ab}$-module of $(D_{ab})^{|\mathcal{F}|}$ generated by
\[\left( \partial_{a_{ij}} - \sum_{k=1}^{n} a_{ik} \partial_{b_k} \partial_{b_j} \right) g^J \quad (1 \leq i \leq d, 1 \leq j \leq n, J \in \mathcal{F}), \quad (29)\]
\[\partial_{b_i} g^J - g^J \cup \{j\} \quad (j \in J^c, J \in \mathcal{F}), \quad (30)\]
\[ (b_j + \sum_{k=1}^{n} \sum_{i=1}^{d} a_{ij} a_{ik} \partial_{b_k}) g^J \quad (j \in J, J \in \mathcal{F}). \quad (31)\]

Then the quotient module $M = (D_{ab})^{|\mathcal{F}|}/N$ is isomorphic to the integration module $M_q/\sum_{i=1}^{d} \partial x_i^I M_q$. Consequently, $M$ is a holonomic module.

**Proof.** We denote by $\iota$ the canonical morphism from $(D_{ab})^{|\mathcal{F}|}$ to the integration module $M_q/\sum_{i=1}^{d} \partial x_i^I M_q$. Let $P_i = x_i + \partial x_i - \sum_{k=1}^{n} a_{ik} \partial_{b_k}$. Since we have
\[ (\partial_{a_{ij}} - x_i \partial_{b_j}) g^J + \partial_{b_i} P_i g^J = \left( \partial_{x_i} \partial_{b_j} + \partial_{a_{ij}} - \sum_{k=1}^{n} a_{ik} \partial_{b_k} \partial_{b_j} \right) g^J, \quad (32)\]
\[ f_j g^J - \sum_{i=1}^{d} a_{ij} P_i g^J = (b_j + \sum_{k=1}^{n} \sum_{i=1}^{d} a_{ij} a_{ik} \partial_{b_k}) g^J, \quad (33)\]
the $D_{abx}$-module $N_q$ is generated by (29), (21), and
\[ \left( \partial_{x_i} \partial_{b_j} + \partial_{a_{ij}} - \sum_{k=1}^{n} a_{ik} \partial_{b_k} \partial_{b_j} \right) g^J \quad (1 \leq i \leq d, J \in \mathcal{F}), \]
\[ (b_j + \sum_{k=1}^{n} \sum_{i=1}^{d} a_{ij} a_{ik} \partial_{b_k}) g^J \quad (j \in J, J \in \mathcal{F}). \]
Consequently, we have $N \subset \ker \iota$.

Next, we show the opposed inclusion $\ker \iota \subset N$. Regarding $(D_{ab})^{|\mathcal{F}|}$ as a subset of $(D_{abx})^{|\mathcal{F}|}$, the left $D_{ab}$-module $N_q + \sum \partial x_i (D_{abx})^{|\mathcal{F}|}$ includes $\ker \iota$. Since the module $N_q$ is generated by (20), (21), (22), and (28), the module $N_q + \sum \partial x_i (D_{abx})^{|\mathcal{F}|}$ can be written as
\[ \sum_{\lambda \in \Lambda} D_{abx} Q_\lambda + \sum_{J \in \mathcal{F}} \sum_{i=1}^{d} D_{abx} P_i g^J + \sum_{i=1}^{d} \partial x_i (D_{abx})^{|\mathcal{F}|}. \quad (34)\]
Here, we denote by \( Q_\lambda (\lambda \in \Lambda) \) the differential operators in (20), (21), and (22). The left \( D_{ab} \)-module (34) is equal to

\[
\sum_{\lambda \in \Lambda} D_{ab} Q_\lambda + \sum_{J \in \mathcal{F}} \sum_{i=1}^{d} D_{abx} P_i g^J + \sum_{i=1}^{d} \partial_{\xi_i}(D_{abx})^{[J]},
\]  

(35)

Note that the first term of (35) is different from that of (34). In fact, for any \( P_i \) and the differential operator in (20), (21), and (22), we have

\[
P_i \left( \partial_{\alpha x} - x_k \partial_{b_i} \right) g^J = (\partial_{\alpha x} - x_k \partial_{b_i}) P_i g^J,
\]

\[
P_i \left( \partial_{b_k} g^J - g^{J \cup \{k\}} \right) = \partial_{b_k} P_i g^J - P_i g^{J \cup \{k\}},
\]

\[
P_i \left( \sum_{k=1}^{d} a_k x_k + b_i \right) g^J = \left( \sum_{k=1}^{d} a_k x_k + b_i \right) P_i g^J.
\]

These equations imply that for any differential operator \( Q_\lambda \) in (20), (21), and (22), the operator \( P_i Q_\lambda \) is an element of \( \sum_{\alpha, i} D_{ab} P_i g^J \). Consequently, module (36) includes \( x_i Q_\lambda - \sum_{\alpha} a_i \partial_{b_i} Q_\lambda + P_i Q_\lambda - \partial_{\alpha x} Q_\lambda \). By induction on the multi-index \( \alpha \in \mathbb{N}_0^d \), module (35) includes \( x^\alpha Q_\lambda \) for any \( \alpha \in \mathbb{N}_0^d \). Hence, module (35) includes (34). The opposite inclusion is obvious.

We denote by \( Q'_\lambda (\lambda \in \Lambda) \) the differential operators in (29), (30), and (31). By (32) and (33), the left \( D_{ab} \)-module

\[
\sum_{\lambda \in \Lambda} D_{ab} Q'_\lambda + \sum_{J \in \mathcal{F}} \sum_{i=1}^{d} D_{abx} P_i g^J + \sum_{i=1}^{d} \partial_{\xi_i}(D_{abx})^{[J]},
\]

is equal to module (35). Obviously, this module is equal to the left \( D_{ab} \)-module

\[
\sum_{\lambda \in \Lambda} D_{ab} Q'_\lambda + \sum_{\alpha} \sum_{J \in \mathcal{F}} \sum_{i=1}^{d} x^\alpha D_{ab} P'_i g^J + \sum_{i=1}^{d} \partial_{\xi_i}(D_{abx})^{[J]},
\]

(36)

where \( P'_i := x_i - \sum_{k=1}^{d} a_i k \partial_{b_k} \). Note that the module \( N_q + \sum \partial_{\xi_i}(D_{abx})^{[J]} \) is equal to (36). Since \( \ker \iota \) is a subset of the intersection of \( (D_{ab})^{[J]} \) and (36), we have

\[
\ker \iota \subset \sum_{\lambda \in \Lambda} D_{ab} Q'_\lambda + \sum_{\alpha} \sum_{J \in \mathcal{F}} \sum_{i=1}^{d} x^\alpha D_{ab} P'_i g^J.
\]

Let \( P \in \ker \iota \). Then the element \( P \) can be written as \( Q + R \), where \( Q \in \sum_{\lambda \in \Lambda} D_{ab} Q'_\lambda \) and \( R \in \sum_{\alpha} \sum_{J \in \mathcal{F}} \sum_{i=1}^{d} x^\alpha D_{ab} P'_i g^J \). The element \( P - Q = R \) is an element of the \( D_{ab} \)-module \( L := \sum_{J \in \mathcal{F}} \sum_{i=1}^{d} D_{ab} P'_i g^J \). Let \( \prec \) be a lexicographic order which satisfies \( x_i \prec m \) for \( 1 \leq i \leq d \) and any monomial \( m \in D_{ab} \). Since the Gröbner basis for \( L \) with respect to \( \prec \) is given by \( \{ P'_i g^J : 1 \leq i \leq d, J \in \mathcal{F} \} \), the leading term of \( P - Q \) must be divided by some \( x_i g^J (1 \leq i \leq d, J \in \mathcal{F}) \). Since \( P - Q \in (D_{ab})^{[J]} \), we have \( P - Q = 0 \). Therefore, we have \( \ker \iota \subset \sum D_{ab} Q'_\lambda \). □
6 Pfaffian equation and holonomic rank

We will evaluate the holonomic rank of $M$ and derive a Pfaffian equation associated with $M$. The following lemma gives a lower bound of the holonomic rank.

**Lemma 21.** The real analytic functions $\varphi_F(a, b)$, where $F$ runs over the abstract simplicial complex $F$ associated with $P$, are linearly independent solutions of $M$. Consequently, the holonomic rank of $M$ is not less than the number of the nonempty faces of $P$.

**Proof.** By Proposition 8 and Theorem 20, it is obvious that the function $\varphi_F$ is a solution of $M$.

Suppose a matrix $\tilde{a} = (\tilde{a}_{ij})$ and a vector $\tilde{b} = (\tilde{b}_j)$ satisfy equation (1). Note that $\tilde{a}_{ij}$ and $\tilde{b}_j$ are real numbers. Let $U \subset \mathbb{C}^{d \times n}$ be a sufficiently small neighborhood of $\tilde{a}$. Note that the function $\varphi_F$ is defined on $\{(a, b) : a \in U, b \in \mathbb{C}^n\}$, by Proposition 8. We prove the linear independence of these functions.

Suppose $F = \{F_1, \ldots, F_s\}$ and $|F_1| \leq |F_2| \leq \cdots \leq |F_s|$. We denote $\varphi_{F_j}$ by $\varphi_j$. Suppose $\sum_{i=1}^s c_i \varphi_j = 0$ for some complex numbers $c_j (1 \leq i \leq s)$. Take an arbitrary $k$ such that $1 \leq k \leq s$, and suppose $c_1 = \cdots = c_{k-1} = 0$. It is enough to show that $c_k = 0$.

Note that $F_\ell \not\subset F_k$ for $\ell > k$, since $|F_\ell| \geq |F_k|$. Define $a(t)$ and $b(t)$ as

$$a_{ij}(t) = \tilde{a}_{ij}, \quad b_j(t) = \begin{cases} \tilde{b}_j & j \in F_k, \\ \tilde{b}_j - t & j \notin F_k. \end{cases}, \quad t \in [0, \infty).$$

Since there is an element of $F_\ell$ which is not included in $F_k$ for $\ell > k$, we have $\lim_{t \to \infty} \prod_{j \in F_k} H(f_j(a(t), b(t), x)) = 0$ for all $x \in \mathbb{R}^d$. By the Lebesgue convergence theorem, we have

$$\lim_{t \to \infty} \varphi_{F_k}(a(t), b(t)) = 0 \quad (\ell > k).$$

Since we have $\lim_{t \to \infty} \prod_{j \in F_k} H(f_j(a(t), b(t), x)) = 1$ for all $x \in \mathbb{R}^d$, again by the Lebesgue convergence theorem, we have $\lim_{t \to \infty} \varphi_{F_k}(a(t), b(t)) = 1$. By the assumption of the induction, we have $\sum_{i=k}^s c_i \varphi_j(a(t), b(t)) = 0$. Taking the limit of both sides as $t \to \infty$, we have $c_k = 0$. \qed

**Theorem 22.** The holonomic rank of $M$ is equal to the number of nonempty faces of $P$, i.e.,

$$\text{rank}(M) = |\mathcal{F}|.$$
In addition, a Pfaffian equation associated with $M$ is given by

\[ \partial_{a_{ij}} g^J = \sum_{k=1}^{n} a_{ik} \partial_{b_k} \partial_{b_j} g^J \quad (1 \leq i \leq d, 1 \leq j \leq n, J \in \mathcal{F}), \quad (37) \]

\[ \partial_{b_j} g^J = g^{J \cup \{j\}} \quad (j \in \mathcal{J}, J \in \mathcal{F}), \quad (38) \]

\[ \partial_{b_j} g^F = -\sum_{k \in \mathcal{F}} \alpha_{jk}^F (a) \left( b_k g^J + \sum_{\ell \in \mathcal{F}^c} \alpha_{k\ell}(a) g^{J \cup \ell} \right) \quad (j \in J, J \in \mathcal{F}). \quad (39) \]

Here, $(\alpha_{ij}^F(a))_{i,j \in \mathcal{F}}$ is the inverse matrix of $\alpha_F(a)$. Note that the right-hand side of (37) can be reduced by (38) and (39).

**Proof.** We have (37) and (38) by (29) and (30), respectively. By (30) and (31), the right-hand side of (39) can be written as

\[ -\sum_{k \in \mathcal{F}} \alpha_{jk}^F \left( b_k g^J + \sum_{\ell \in \mathcal{F}^c} \alpha_{k\ell} g^{J \cup \ell} \right) = \sum_{k \in \mathcal{F}} \alpha_{jk}^F \left( \sum_{\ell=1}^{d} \alpha_{k\ell} \partial_{b_k} g^J - \sum_{\ell \in \mathcal{F}^c} \alpha_{k\ell} \partial_{b_k} g^J \right) \]

\[ = \sum_{k \in \mathcal{F}} \alpha_{jk}^F \sum_{\ell \in \mathcal{F}^c} \alpha_{k\ell} \partial_{b_k} g^J = \partial_{b_j} g^J. \]

Consequently, we have (39).

By (37), (38), and (39), the module $C(a,b) \otimes_{C[a,b]} M$ is spanned by $g^J (J \in \mathcal{F})$ as a linear space over $C(a,b)$, and we have $\text{rank}(M) \leq |\mathcal{F}|$. By this inequality and Lemma 21, we have $\text{rank}(M) = |\mathcal{F}|$.

**Remark 23.** We note that the Heaviside function $H \left( \sum_{i=1}^{d} a_{ij} x_i + b_j \right)$ is equal to $\lim_{\lambda \to 0} \left( \sum_{i=1}^{d} a_{ij} x_i + b_j \right)^{\lambda}$ as a Schwartz distribution. Thus we may expect that our integral representation is a “specialization” of the integral considered in the cohomology groups in [1]. However, we have no rigorous understanding of this limiting procedure of the twisted cohomology. An interesting observation is that the holonomic rank of $M$ can be smaller than the dimension of the twisted cohomology group $H^d(G, \nabla)$, where $G := C^d - V \left( \prod_{j=1}^{d} \left( \sum_{i=1}^{d} a_{ij} x_i + b_j \right) \right)$ and $\nabla$ is a connection defined in [1]. In our case, the dimension of the twisted cohomology is $\sum_{i=0}^{d} \binom{n}{i}$. The number of faces of the polyhedron in general position with $n$ facets can be smaller than this value.

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