APPROXIMATION OF EIGENVALUES OF SPOT CROSS VOLATILITY MATRIX WITH A VIEW TOWARD PRINCIPAL COMPONENT ANALYSIS

NIEN-LIN LIU AND HOANG-LONG NGO

ABSTRACT. In order to study the geometry of interest rates market dynamics, Malliavin, Mancino and Recchioni [A non-parametric calibration of the HJM geometry: an application of Itô calculus to financial statistics, Japanese Journal of Mathematics, 2, pp.55–77, 2007] introduced a scheme, which is based on the Fourier Series method, to estimate eigenvalues of a spot cross volatility matrix. In this paper, we present another estimation scheme based on the Quadratic Variation method. We first establish limit theorems for each scheme and then we use a stochastic volatility model of Heston’s type to compare the effectiveness of these two schemes.

1. INTRODUCTION

Let $X$ be a $d$-dimensional stochastic process defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ by

\[ dX(t) = A(t, w)dt + B(t, w)dW(t), \quad 0 \leq t \leq T, \]

where $W$ is a $d_1$-dimensional standard Brownian motion, $A$ is a $d$-dimensional drift process and $B$ is a $R^{d \times d_1}$-valued càdlàg volatility process. In mathematical finance it is widely accepted that processes $X$ of the form defined by (1.1) are reasonable models for the (log return of) price processes and interest rates.

The spot cross volatility matrix $\Sigma = (\Sigma_{i,j})_{1 \leq i, j \leq d}$ of process $X$ is defined by

\[ \Sigma_{i,j}(t) = \sum_{k=1}^{d_1} B_{i,k}(t)B_{j,k}(t), \quad 0 \leq t \leq T. \]

We are interested in the following problem: given a finite set of observation data \{\(X(t_k, \omega_0) : t_k = kT/n, k = 0, \ldots, n\)\} of a single trajectory $\omega_0 \in \Omega$, we want to estimate the eigenvalues of $\Sigma(t, \omega_0)$ for any $t \in [0,T]$. This problem appears in mathematical finance, especially in principal component analysis (see [2, 12, 15]). The estimation of the eigenvalues of the integrated volatility matrix was studied by Wang and Zou [24] (see also the references therein). By the time
we completed this paper, we learnt that Jacod and Podolskij [9] had previously introduced some statistics based on a random perturbation approach for ranks of volatility metric of continuous Itô process. Our approach differs from that in [9] and can be applied to Itô processes with jump components.

Our method to solve this problem is as follows: first we approximate the spot cross volatility matrix $\Sigma$ by a matrix $\tilde{\Sigma}$ using the given observations of $X$; next we approximate the eigenvalues of $\Sigma$ by those of $\tilde{\Sigma}$.

The spot volatility estimation is an important problem in mathematical finance and has been extensively studied by many authors. Up to now, there are two main approaches to this problem. The first approach called the Fourier Series method was introduced by [13] and later developed in [15, 14]. The second approach called the Quadratic Variation method was introduced by [21] and later developed in [18, 19, 17] (see also [3, 10]). It should be noted that there is a very rich literature on the problem of measuring the so called realized volatility as well as problem of estimating parameters of diffusion processes, see [22, 5, 11, 6] and the references therein.

In this paper, we present some limit theorems and a numerical study to analyze the effectiveness of the estimation of eigenvalues by using Fourier Series and Quadratic Variation methods. It should be mentioned that in reality, one cannot observe directly either cross volatility matrix or the eigenvalues. Therefore we perform a numerical study with dummy data for which we know both the volatility matrix and its eigenvalues beforehand. In particular, we show that the Fourier Series method may lead to some unexpected results when estimating small eigenvalues; this situation would never arise using the Quadratic Variation method.

Acknowledgment. The authors thank Jiro Akahori, Freddy Delbaen, Arturo Kohatsu-Higa, Maria Elvira Mancino, and Shigeyoshi Ogawa for their helpful comments. The authors are also grateful to the referee for her/his valuable comments which led to improvement of the paper.

2. The first Fourier Series estimation scheme

In a series of papers [13, 15, 14], Malliavin et al. introduced a number of Fourier Series estimation schemes for spot volatilities. Although these schemes are essentially based on a same idea, they are slightly different. As we will present later, each scheme has both advantages and disadvantages compared to the other.
In this section, we summarize the Fourier Series method presented in [13]. By a change of origin and rescaling, one can suppose that $T = 2\pi$ and the Fourier Series method reconstructs $\Sigma(t)$ for all $t \in (0, 2\pi)$. Let us denote the Fourier coefficients of $dX_j$, $j = 1, \ldots, d$, by

$$a_k(dX_j) = \frac{1}{\pi} \int_{(0,2\pi)} \cos(kt) dX_j(t), \quad b_k(dX_j) = \frac{1}{\pi} \int_{(0,2\pi)} \sin(kt) dX_j(t).$$

The Fourier coefficients of each cross volatility $\Sigma_{u,v}$, $1 \leq u, v \leq d$, are defined by

$$a_k(\Sigma_{u,v}) = \frac{1}{\pi} \int_{(0,2\pi)} \cos(kt) \Sigma_{u,v}(t) dt, \quad b_k(\Sigma_{u,v}) = \frac{1}{\pi} \int_{(0,2\pi)} \sin(kt) \Sigma_{u,v}(t) dt.$$

It follows from the Fourier-Féjer inversion formula that one can reconstruct $\Sigma$ from its Fourier coefficients by

$$\Sigma_{u,v}(t) = \lim_{N \to \infty} \sum_{k=0}^{N} \left( 1 - \frac{k}{N} \right) (a_k(\Sigma_{u,v}) \cos(kt) + b_k(\Sigma_{u,v}) \sin(kt)).$$

In practice, based on the observation of $X$ at times $t_i = 2\pi i/n$, $i = 0, \ldots, n$, one can approximate $\Sigma$ as follows. We fix some positive integer $N$.

1. Fourier coefficients $a_k(dX_j)$, $b_k(dX_j)$, $k = 0, \ldots, 2N$, are approximated by

$$\hat{a}_k(dX_j) = \frac{1}{\pi} \sum_{i=1}^{n} (\cos(kt_{i-1}) - \cos(kt_i)) X_j(t_{i-1}) + \frac{1}{\pi} \left( X_j(t_n) - X_j(t_0) \right),$$

$$\hat{b}_k(dX_j) = \frac{1}{\pi} \sum_{i=1}^{n} (\sin(kt_{i-1}) - \sin(kt_i)) X_j(t_{i-1}).$$

2. Fourier coefficients of each cross volatility $\Sigma_{u,v}$, $1 \leq u, v \leq d$, are approximated by

$$\hat{a}_0(\Sigma_{u,v}) = \frac{\pi}{2(N + 1 - n_0)} \sum_{s=n_0}^{N} \left( \hat{a}_s(dX_u) \hat{a}_s(dX_v) + \hat{b}_s(dX_u) \hat{b}_s(dX_v) \right),$$

$$\hat{a}_k(\Sigma_{u,v}) = \frac{\pi}{N + 1 - n_0} \sum_{s=n_0}^{N} \left( \hat{a}_s(dX_u) \hat{a}_{s+k}(dX_v) + \hat{a}_s(dX_v) \hat{a}_{s+k}(dX_u) \right),$$

$$\hat{b}_k(\Sigma_{u,v}) = \frac{\pi}{N + 1 - n_0} \sum_{s=n_0}^{N} \left( \hat{a}_s(dX_u) \hat{b}_{s+k}(dX_v) + \hat{a}_s(dX_v) \hat{b}_{s+k}(dX_u) \right),$$

for each $k = 0, \ldots, N$. 

3
The volatilities $\Sigma_{u,v}(t)$ are approximated by

$$\hat{\Sigma}_{u,v}^{N,n}(t) = \sum_{k=0}^{N} \left( 1 - \frac{k}{N} \right) \left( \hat{a}_k(\Sigma_{u,v}) \right) \cos(kt) + \hat{b}_k(\Sigma_{u,v}) \sin(kt).$$

Sometime, it is preferable to smooth the Féjer kernel in (2.1) by replacing $(1 - k/N)$ with $\sin^2(\delta k)/(\delta k)^2$ for some appropriate parameter $\delta > 0$.

**Remark 1.** It should be noted here that although the matrix $\hat{\Sigma}_{u,v}^{N,n}(t)$ is symmetric, it is not non-negative definite in general. Therefore some of its eigenvalues may be negative, which is not expected in practice.

### 3. The second Fourier Series estimation scheme

In [14], the authors introduced another version of Fourier Series estimation scheme. Their new scheme was designed to deal with asynchronous data. In the following, we will specialize it for the case of regular sampling. We define $\delta^j_i := X^j(t_{i+1}) - X^j(t_i), \quad j = 1, \ldots, d$. For any integer $k, |k| \leq 2N$, let

$$c_k^j := \frac{1}{2\pi} \sum_{i=0}^{n-1} e^{-ikt_i} \delta^j_i, \quad j = 1, \ldots, d.$$  

For each $1 \leq j_1 \leq j_2 \leq d$, let $\alpha_k(N, j_1, j_2)$ for $|k| \leq N$ be given by

$$\alpha_k(N, j_1, j_2) := \frac{2\pi}{2N+1} \sum_{|s| \leq N} c_{s}^{j_1} c_{k-s}^{j_2}.$$  

Finally, define

$$\Sigma_{n,N}^{j_1,j_2}(t) := \sum_{|k| \leq N} \left( 1 - \frac{|k|}{N} \right) \alpha_k(N, j_1, j_2) e^{ikt}$$

$$= \alpha_0(N, j_1, j_2) + \sum_{k=1}^{N} \left( 1 - \frac{k}{N} \right) \left( \alpha_k(N, j_1, j_2) e^{ikt} + \alpha_{-k}(N, j_1, j_2) e^{-ikt} \right).$$

Since the above estimator is written with complex numbers, it may be inconvenient to do simulation. Therefore we rewrite it as below:

$$\Sigma_{n,N}^{j_1,j_2}(t) := \alpha_0(N, j_1, j_2) + \sum_{k=1}^{N} \left( 1 - \frac{k}{N} \right) \left( a_k^{j_1,j_2} \cos(kt) + b_k^{j_1,j_2} \sin(kt) \right),$$
where
\[
\alpha_{ij}^{kj} = \frac{1}{\pi(2N+1)} \left[ \sum_{s=1}^{N} \left\{ \hat{a}_s(dX^{j_1}) \hat{a}_{k-s}(dX^{j_2}) - \hat{b}_s(dX^{j_1}) \hat{b}_{k-s}(dX^{j_2}) \\
+ \hat{a}_s(dX^{j_1}) \hat{a}_{k+s}(dX^{j_2}) + \hat{b}_s(dX^{j_1}) \hat{b}_{k+s}(dX^{j_2}) \right\} + \hat{a}_k(dX^{j_2}) \left( X^{j_1}(2\pi) - X^{j_1}(0) \right) \right],
\]
\[
b_k^{ij} = \frac{1}{\pi(2N+1)} \left[ \sum_{s=1}^{N} \left\{ \hat{a}_s(dX^{j_1}) \hat{b}_{k-s}(dX^{j_2}) + \hat{b}_s(dX^{j_1}) \hat{a}_{k-s}(dX^{j_2}) \\
+ \hat{a}_s(dX^{j_1}) \hat{b}_{k+s}(dX^{j_2}) - \hat{b}_s(dX^{j_1}) \hat{a}_{k+s}(dX^{j_2}) \right\} + \hat{b}_k(dX^{j_2}) \left( X^{j_1}(2\pi) - X^{j_1}(0) \right) \right],
\]
and,
\[
\hat{a}_s(dX^{j_1}) = \sum_i \cos(st_i) \delta_i^{j_1}, \quad \hat{a}_s(dX^{j_2}) = \sum_j \cos(st_j) \delta_j^{j_2},
\]
\[
\hat{b}_s(dX^{j_1}) = \sum_i \sin(st_i) \delta_i^{j_1}, \quad \hat{b}_s(dX^{j_2}) = \sum_j \sin(st_j) \delta_j^{j_2},
\]
and
\[
\alpha_0(N, j_1, j_2) = \frac{1}{2\pi(2N+1)} \left[ \sum_{s=1}^{N} 2 \left\{ \hat{a}_s(dX^{j_1}) \hat{a}_s(dX^{j_2}) + \hat{b}_s(dX^{j_1}) \hat{b}_s(dX^{j_2}) \right\} \\
+ \left( X^{j_1}(2\pi) - X^{j_1}(0) \right) \left( X^{j_2}(2\pi) - X^{j_2}(0) \right) \right].
\]

**Remark 2.** Since the matrix \(\Sigma_{n,N}^{j_1,j_2}(t)\) is not symmetric, the eigenvalues may not be real numbers. In order to overcome this drawback we propose two symmetrization methods as follows.

3.1. **The first symmetrization.** A naive idea to symmetrize the covariance matrix is that one first calculates \(\Sigma_{n,N}^{j_1,j_2}\) using formula (3.1) for all \(1 \leq j_1 \leq j_2 \leq d\), and then puts \(\Sigma_{n,N}^{j_2,j_1} := \Sigma_{n,N}^{j_2,j_1}\).

3.2. **The second symmetrization.** Another way to symmetrize the covariance matrix is as follows: Denote
\[
\alpha_k(N, j_1, j_2) := \frac{\pi}{2N+1} \sum_{|s| \leq N} \left( c_s^{j_1} c_{k-s}^{j_2} + c_s^{j_2} c_{k-s}^{j_1} \right),
\]
for any \(|k| \leq N\) and define
\[
\Sigma_{n,N}^{j_1,j_2}(t) := \sum_{|k| \leq N} \left( 1 - \frac{|k|}{N} \right) \alpha_k(N, j_1, j_2) e^{ikt}
\]
\[
= \alpha_0(N, j_1, j_2) + \sum_{k=1}^{N} \left( 1 - \frac{k}{N} \right) \left( \alpha_k(N, j_1, j_2) e^{ikt} + \alpha_{-k}(N, j_1, j_2) e^{-ikt} \right).
\]
To simplify the simulation, we rewrite $\Sigma_{n,N}^{j_1j_2}$ as follows

$$\Sigma_{n,N}^{j_1j_2}(t) := \alpha_0(N, j_1, j_2) + \sum_{k=1}^{N} \left(1 - \frac{k}{N}\right) \left(a_k^{j_1j_2} \cos(kt) + b_k^{j_1j_2} \sin(kt)\right),$$

where

$$a_k^{j_1j_2} = \frac{1}{2\pi(2N+1)} \left[\sum_{s=1}^{N} \left\{\hat{a}_s(dX^1)\hat{a}_{k-s}(dX^2) - \hat{b}_s(dX^1)\hat{b}_{k-s}(dX^2) + \hat{a}_s(dX^1)\hat{a}_{k+s}(dX^2)
+ \hat{b}_s(dX^1)\hat{b}_{k+s}(dX^2)\right\} + \hat{a}_k(dX^2)\left(X^1(2\pi) - X^1(0)\right) + \hat{a}_k(dX^1)\left(X^2(2\pi) - X^2(0)\right)\right],$$

$$b_k^{j_1j_2} = \frac{1}{2\pi(2N+1)} \left[\sum_{s=1}^{N} \left\{\hat{a}_s(dX^1)\hat{b}_{k-s}(dX^2) + \hat{b}_s(dX^1)\hat{a}_{k-s}(dX^2) + \hat{a}_s(dX^1)\hat{b}_{k+s}(dX^2)
- \hat{b}_s(dX^1)\hat{a}_{k+s}(dX^2) + \hat{a}_s(dX^2)\hat{b}_{k-s}(dX^1) + \hat{b}_s(dX^2)\hat{a}_{k-s}(dX^1) + \hat{a}_s(dX^2)\hat{b}_{k+s}(dX^1)
- \hat{b}_s(dX^2)\hat{a}_{k+s}(dX^1)\right\} + \hat{b}_k(dX^2)\left(X^1(2\pi) - X^1(0)\right) + \hat{b}_k(dX^1)\left(X^2(2\pi) - X^2(0)\right)\right],$$

and

$$\alpha_0(N, j_1, j_2) = \frac{1}{2\pi(2N+1)} \left[\sum_{s=1}^{N} 2\left\{\hat{a}_s(dX^{j_1})\hat{a}_s(dX^{j_2}) + \hat{b}_s(dX^{j_1})\hat{b}_s(dX^{j_2})\right\}
+ \left(X^{j_2}(2\pi) - X^{j_2}(0)\right)\left(X^{j_1}(2\pi) - X^{j_1}(0)\right)\right],$$

and $\hat{a}, \hat{b}$ are defined as before.

**Remark 3.** Each matrix $\Sigma_{n,N}^{j_1j_2}(t)$ is symmetric, but not necessary positive definite.

### 3.3. Limit theorem.

Since for each $t$, $\Sigma(t)$ is a symmetric non-negative definite matrix, we denote its eigenvalues by $\lambda_i(t), i = 1, \ldots, n$, such that $\lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_d(t) \geq 0$. We also denote by $\hat{\lambda}_i^n(t) \geq \lambda_2^n(t) \geq \ldots \geq \hat{\lambda}_d^n(t)$ the eigenvalues of the symmetric matrix $\Sigma_{n,N}^{j_1j_2}(t)$ defined by either the first or the second symmetrization.

Now we are in a position to state the first main result of this paper.

**Theorem 4.** Assume that $\Sigma(t)$ is continuous and for $i = 1, \ldots, d, j = 1, \ldots, d_1$, and $\frac{N}{n} \to 0$ as $n \to \infty$,

$$E\left(\int_0^{2\pi} \left[\|A_i(t)\|^2 + \|B_{i,j}(t)\|^4\right]dt\right) < \infty.$$
Then the following convergence in probability holds

\[
\lim_{n,N \to \infty} \sup_{0 \leq t \leq 2\pi} \sum_{i=1}^{d} |\hat{\lambda}_{i}^{n}(t) - \lambda_{i}(t)| = 0.
\]

Remark 5. This method has been used by Malliavin et al. in [15] to estimate the eigenvalues of the covariance matrix of a time series of Euro swap rates and Euribor rates. However, these authors did not provide any discussion on the asymptotic behaviour of the estimators.

4. Quadratic Variation method

We briefly recall the Quadratic Variation method which was proposed in Ogawa and Wakayama [21]. Let \((h_n)\) be a sequence of positive numbers satisfying \(\lim_{n \to \infty} h_n = 0\). For each \(t \in (0, T)\), \(1 \leq u, v \leq d\), we denote

\[
\tilde{\Sigma}_{u,v}^{n}(t) = \frac{1}{2h_n} \sum_{i: (t-h_n) \leq t_i < t_i + 1 \leq (t+h_n)} (X_u(t_{i+1}) - X_u(t_i)) \times (X_v(t_{i+1}) - X_v(t_{i})).
\]

We suppose that the diffusion coefficient \(B\) satisfies the following Hölder continuous condition

\[H(\alpha): \text{ For some } \alpha \in (0, 1], \text{ there exists a constant } K \text{ such that for all } s, t \in [0, T],\]

\[E\|B(s) - B(t)\|^2 \leq K|s - t|^{2\alpha}.\]

For each \(t \in (0, T)\), the approximating matrix \(\tilde{\Sigma}^{n}(t)\) is symmetric, non-negative defined. Hence all of its eigenvalues are non-negative. Let \(\lambda_{i}^{n}(t) \geq \lambda_{2}^{n}(t) \geq \ldots \geq \lambda_{d}^{n}(t)\) denote the eigenvalues of \(\tilde{\Sigma}^{n}(t)\).

Here is the second main result of this paper.

**Theorem 6.** Assume that assumption \(H(\alpha)\) holds for some \(\alpha \in (0, 1]\) and \(\sup_{t \in (0, T)} E(\|A(t)\|^4 + \|B(t)\|^4) < \infty\). Then we have

\[
\sup_{t \in (h_n, T-h_n)} \sum_{i=1}^{d} E|\hat{\lambda}_{i}^{n}(t) - \lambda_{i}(t)| \leq M\left(h_n^{\alpha} + \sqrt{T/h_n}\right),
\]

for some constant \(M\) which does not depend on \(n\).

In particular, if \(h_n = O(n^{-1/(2\alpha+1)})\) then

\[
\sup_{t \in (h_n, T-h_n)} n^{\frac{\alpha}{2\alpha+1}} \sum_{i=1}^{d} E|\hat{\lambda}_{i}^{n}(t) - \lambda_{i}(t)| \leq M.
\]
In the following, we will study the case where the price process $X$ contains jump components. More precisely, we suppose that $X$ is a $d$-dimensional stochastic process defined by

\begin{equation}
    dX(t) = A(t, w)dt + B(t, w)dW(t) + dJ(t), \quad 0 \leq t \leq T,
\end{equation}

where $W, A, B$ are defined as in Section 1 and $J$ is a $d$-dimensional Lévy process which may depend on $W$. The Blumenthal-Getoor index $\beta$ of $J$ is defined by

\[ \beta = \inf \{ p \geq 0 : \int |x|^p \nu(dx) < \infty \}, \]

where $\nu$ is Lévy measure of $J$. It is well-known that $\beta \in [0, 2]$.

For each $t \in (0, T)$, $1 \leq u, v \leq d$, we denote

\[ \bar{\Sigma}_{u,v}(t) = \frac{\pi}{8h_n} \sum_i \left( |\Delta_i(X_u + X_v) \Delta_{i+1}(X_u + X_v)| - |\Delta_iX_u \Delta_{i+1}X_u| - |\Delta_iX_v \Delta_{i+1}X_v| \right), \]

where the summation is taken over all indices $i$ such that $(t - h_n) \leq t_i - 1 < t_i \leq (t + h_n)$ and $\Delta_iX_s = X_s(t_i) - X_s(t_{i-1})$.

Let $\lambda^u_i(t) \geq \lambda^v_i(t) \geq \ldots \geq \lambda^d_i(t) \geq 0$ denote the eigenvalues of $\Sigma^n(t)$.

We have the following limit theorem.

**Theorem 7.** Assume that

- $H(\alpha)$ holds for some $\alpha \in (0, 1]$,
- $\forall q > 0, \sup_{t \in [0,T]} E\|A(t)\|^q + E\|B(t)\|^q < \infty$,
- $\beta < 2$ and $\int_{\|x\| \geq 1} \|x\|^2 \nu(dx) < \infty$.

Then for any $\gamma \in \left(0, \frac{\alpha}{2\alpha + 1} \wedge \frac{2-\beta}{2\beta}\right)$, there exists a constant $M$ such that

\[ \sup_n \sup_{t \in (h_n, T-h_n)} n^\gamma \sum_{i=1}^d E|\lambda^u_i(t) - \lambda^v_i(t)| \leq M, \]

provided that $h_n = O(n^{-2\gamma})$.

**Remark 8.** In [19], the authors introduce another cross volatility estimation scheme for jump diffusion processes by using a threshold parameter to reduce the effect of large size jumps. Furthermore, one can combine the threshold method with the bi-power method presented above to produce a more stable estimation (see [16]).

**Remark 9.** By following a similar argument as above, one can construct estimation schemes for eigenvalues of the cross volatility matrix of processes which are contaminated by microstructure noise (see [17, 20] for some classes of real-time schemes for the estimation of volatility in the noisy case with/without jumps).
5. Numerical Study

5.1. Complexity. The computational cost of the Quadratic Variation method is much less than that of Fourier Series method. Indeed, the cost of computing of the Quadratic Variation method is of order \( n^{2\alpha+1} \) while one of Fourier Series method is \( N^2 n \).

5.2. Dummy data. We consider a stochastic volatility model of Heston’s type defined by

\[
\begin{align*}
    dX_i(t) &= \gamma_i dt + \sum_{j=1}^{d} \lambda_{ij} \sqrt{v_j(t)} dW_j(t) \\
    dv_j(t) &= \alpha_j (b_j - v_j(t)) dt + \sigma_j \sqrt{v_j(t)} dB_j(t)
\end{align*}
\]

for \( 1 \leq i \leq d, \ 1 \leq j \leq d_1, \ t \in [0,T] \), where \( \gamma_i, \alpha_j, b_j, \sigma_j, \lambda_{ij}, 1 \leq i \leq d, 1 \leq j \leq d_1 \), are constants, \( \alpha_j, b_j, 1 \leq j \leq d_1 \), are positive; \( W_j, B_j, 1 \leq j \leq d_1 \) are mutually independent standard Brownian motions.

Remark 10. The class of square-root diffusions

\[
dv(t) = \alpha(b - v(t)) dt + \sigma \sqrt{v(t)} dB(t), \quad v(0) = v_0,
\]

with \( W \) a standard one dimensional Brownian motion, was studied in [7]. The author showed that if parameters \( \alpha, b, v_0 \) are positive, then \( v(t) \) will stay positive. And if one supposes further that \( 2\alpha b \geq \sigma^2 \), then \( v(t) \) is strictly positive for all \( t \) with probability 1.

Provided that processes \( v \)'s can be simulated at discretized time-point \( t_k = k\Delta = kT/N_0, \ k = 0, \ldots, N_0 \), one can simulate \( X \) by using a simple Euler - Maruyama’s scheme as follows

\[
X_i(t_{k+1}) = X_i(t_k) + \gamma_i \Delta + \sqrt{\Delta} \sum_{j=1}^{d} \lambda_{ij} \sqrt{v_j(t_k)} Z_{jk},
\]

where \( Z \)'s are independent standard normal distribution random variables.

The simulation of \( v \)'s is more involved because the values of \( v_j(t_k) \) produced by Euler - Maruyama discretization may become negative. We will simulate \( v \)'s by sampling from the exact transition laws of the processes (see [8]).

In the following, we choose \( d = 5, \ d_1 = 3, \ \gamma_i = b_j = v_j(0) = i/100, \ X_i(0) = 1, \ \alpha_j = 2, \lambda_{ij} = (-1)^{i+j} \sin(ij), \ 1 \leq i \leq d, 1 \leq j \leq d_1 \). We choose \( \sigma_j = \sqrt{2b_j \alpha_j} \) and \( T = 2\pi \) to make simulation easier.

Based on the sample data of \( X \), we use both the Quadratic Variation and Fourier Series methods, as stated in the previous sections, to estimate the cross volatility matrices of \( X \) as a function of time and after
that we calculate the eigenvalues of each estimated matrix. In particular, since the volatility coefficients of \( X \) satisfy assumption (4.1) with \( \alpha = 1/2 \), we choose \( h_n = T N_0^{-1/2} \) for the Quadratic Variation scheme. Besides, for the Fourier Series method, we calculate the Fourier coefficients of the cross volatilities up to the Nyquist frequency \( 2N = N_0/2 \) (see [23]).

We observe the mean square pathwise errors \( MSE \) and \( mSE \) defined as follows: Suppose that for each \( k = 0, \ldots, N_0 \), \( \hat{\Sigma}(t_k) \) is an estimator of matrix \( \Sigma(t_k) \). We denote by \( \hat{\lambda}_1(t_k) \) and \( \hat{\lambda}_d(t_k) \) the maximum and minimum eigenvalues of \( \hat{\Sigma}(t_k) \). We also denote by \( \lambda_1(t_k) \) and \( \lambda_d(t_k) \) the maximum and minimum eigenvalues of \( \Sigma(t_k) \). Then we measure the errors of the estimations on the whole paths by

\[
MSE(\Sigma, \hat{\Sigma}) = \frac{1}{N_0} \sum_{k=1}^{N_0} |\hat{\lambda}_1(t_k) - \lambda_1(t_k)|^2,
\]

and

\[
mSE(\Sigma, \hat{\Sigma}) = \frac{1}{N_0} \sum_{k=1}^{N_0} |\hat{\lambda}_d(t_k) - \lambda_d(t_k)|^2.
\]

5.2.1. The results of the first Fourier Series method. The simulations show that the Fourier Series estimate does not work well near 0 and \( T \). In order to have a better understanding of errors of each estimation methods at “normal” time, we eliminate 10 percent of the estimated cross volatilities near the two end points 0 and \( T \) when we calculate the mean square pathwise errors for each symmetrization of the Fourier Series method and the Quadratic Variation method. The means of \( mSE \) and \( MSE \) of each method are showed in Table [1] (Note that in all tables we use \( \epsilon \) for value less than \( 10^{-10} \)). Here QV and FS stand for Quadratic Variation and Fourier Series methods, respectively. FS\( i \), \( i = 1, 2, 3, 4 \), stand for Fourier Series estimation using smooth kernel with \( \delta = T N_0^{-0.1(i+2)} \), respectively. Figures [1] and [2] show the estimations of \( \lambda_M \) and \( \lambda_m \) during \((0, T)\) with \( N_0 = 10^3 \) and \( N_0 = 10^4 \).

Remark that we remove the graph of FS since it oscillates violently making the whole picture difficult to see. The Fourier Series scheme using the modified Féjer kernel is able to produce a good estimate provided that one can choose a correct value for the parameter \( \delta \). However, Table [1] together with Figure [1] shows that this estimation is very sensitive to the choice of \( \delta \). And to the best of our knowledge, there is still no effective way to select a good \( \delta \).

Another disadvantage of the Fourier Series method is evident from Figure [2]. One can see that FS3 and FS4 schemes may produce a
Figure 1. Maximum eigenvalue correspond to the Quadratic Variation method and the first Fourier Series method (left: \(N_0 = 10^3\), right: \(N_0 = 10^4\))

Figure 2. Minimum eigenvalue correspond to the Quadratic Variation method and the first Fourier Series method (left: \(N_0 = 10^3\), right: \(N_0 = 10^4\))
Table 1. Means of $MSE$ and $mSE$ ($\times 10^{-4}$) correspond to the Quadratic Variation method and the first Fourier Series method

| $N_0$ | QV | FS | FS1 | FS2 | FS3 | FS4 |
|-------|----|----|-----|-----|-----|-----|
| MSE   | 10^2 | 23 | 100 | 21  | 21  | 23  | 31  |
| mSE   | $\epsilon$ | 6.882 | $\epsilon$ | 0.006 | 0.071 | 0.391 |

| MSE   | 10^3 | 7  | 88  | 15  | 11  | 9   | 12  |
| mSE   | $\epsilon$ | 7.889 | $\epsilon$ | $\epsilon$ | 0.01 | 0.077 |

| MSE   | 10^4 | 2  | 93  | 9   | 5   | 4   | 6   |
| mSE   | $\epsilon$ | 7.609 | $\epsilon$ | $\epsilon$ | $\epsilon$ | 0.007 |

Table 2. Means of $MSE$ and $mSE$ ($\times 10^{-4}$) correspond to the Quadratic Variation method and the second Fourier Series method with first symmetrization

| $N_0$ | QV | FS | FS1 | FS2 | FS3 | FS4 |
|-------|----|----|-----|-----|-----|-----|
| MSE   | 10^2 | 18 | 399 | 456 | 391 | 164 | 106 |
| mSE   | $\epsilon$ | 3  | 10  | 32  | 96  | 1721 |

| MSE   | 10^3 | 8  | 298 | 43  | 10  | 65  |
| mSE   | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ |

| MSE   | 10^4 | 3  | 31  | 3   | 5   | 67  |
| mSE   | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $\epsilon$ |

5.2.2. The results of the second Fourier Series method. We use the same notations as above. Table 2 and Table 3 show the means of $mSE$ and $MSE$ of each method while Fourier Series method modified by first symmetrization and second symmetrization, respectively. Figures 3 and 4 show the estimations of $\lambda_M$ and $\lambda_m$ during $(0, T)$ with $N_0 = 10^3$ and $N_0 = 10^4$ with first symmetrization, and Figures 5 and 6 show the estimations of $\lambda_M$ and $\lambda_m$ during $(0, T)$ with $N_0 = 10^3$ and $N_0 = 10^4$ with second symmetrization.

Base on two symmetrization methods, FS3 and FS4 schemes give us a good result which is shown in Figures 3 and 4. Although the Fourier Series estimators are still not non-negative definite, a negative value of the estimate of eigenvalue of the cross volatility matrix is not significant as shown in Figures 5 and 6.
In our simulation, the Quadratic Variation method works quite well. Its mean square pathwise error is strictly less than the ones of Fourier Series method. In addition, because the estimated cross volatility matrices using the Quadratic Variation method are always symmetric and non-negative definite, all of their eigenvalues are non-negative. Finally, the computation time of the Quadratic Variation scheme is less than 1/100 of the Fourier Series scheme.
6. Proofs

In this section, we sketch the proofs of the main results in Sections 2 and 3. First we need the following auxiliary inequality (see [4]).

Lemma 11 (Hoffman-Wielandt). Let $A, B$ be $N \times N$ symmetric matrices, with eigenvalues $\lambda_1^A \leq \lambda_2^A \leq \ldots \leq \lambda_N^A$ and $\lambda_1^B \leq \lambda_2^B \leq \ldots \leq \lambda_N^B$. Then

$$\sum_{i=1}^{N} |\lambda_i^A - \lambda_i^B|^2 \leq tr(A - B)^2.$$  

6.1. Proof of Theorem 4. By using a similar argument as in the proof of Theorem 3.4 ([14]), one can show that the following convergence in probability holds

$$\lim_{n,N \to \infty} \sup_{0 \leq t \leq 2\pi} \| \tilde{\Sigma}^N_n(t) - \Sigma(t) \| = 0.$$  

Hence, one also has

$$\lim_{n,N \to \infty} \sup_{0 \leq t \leq 2\pi} tr(\tilde{\Sigma}^N_n(t) - \Sigma(t))^2 = 0,$$  

in probability. By applying Lemma [14], we get the desired result.
6.2. Proof of Theorem 6. After some elementary calculations, one gets from Lemma 11 that

$$\sum_{i=1}^{d} |\tilde{\lambda}_i^n(t) - \lambda_i(t)| \leq \sqrt{d} \sum_{i,j=1}^{d} |\tilde{\Sigma}_{ij}^n(t) - \Sigma_{ij}(t)|,$$

for all $t \in (0, T)$. On the other hand, it follows from Proposition 3.3 ([21]) that there exists a constant $M > 0$ such that for all $n$ and all $t \in (h_n, T - h_n)$, one has

$$\left(h_n^\alpha + \sqrt{\frac{T}{n h_n}}\right)^{-1} \sum_{i,j=1}^{d} \mathbb{E} |\tilde{\Sigma}_{ij}^n(t) - \Sigma_{ij}(t)| \leq M,$$

which concludes Theorem 6.

6.3. Proof of Theorem 7. It is not hard to see from Propositions 3.13 and 3.14 in [17] that for any $\gamma \in \left(0, \frac{\alpha}{2\alpha + 1} \land \frac{2-\beta}{2\beta}\right)$, there exists a
constant $M$ such that

$$n^\gamma \sum_{i,j=1}^d \mathbb{E}|\Sigma_{ij}^n(t) - \Sigma_{ij}(t)| \leq M,$$

for all $n$ and $t \in (0, T)$. This fact together with estimate (6.1) yields the desired result.

7. Conclusions

In this paper we studied two methods to estimate the eigenvalues of spot cross volatility matrix. The empirical studies show that in comparison with the Fourier Series method, the Quadratic Variation method is easier to implement, is much faster and is able to avoid the negative eigenvalue problem. The Quadratic Variation method is also applicable to diffusion processes with jumps for which the Fourier Series method is unsuitable.

References

[1] Y. Ait-Sahalia, L.P. Hansen, Handbook of Financial Econometrics, volume 1,2 of Finance, North-Holland, 2010.
[2] J. Akahori, N.L. Liu, On a type i error of a random walk hypothesis on interest rates, ICIC International 7 (2011) 115–131.
[3] A. Alvarez, F. Panloup, M. Pontier, N. Savy, Estimation of the instantaneous volatility, Stat. Inference Stoch. Process. 15 (2012) 27–59.
[4] G.W. Anderson, A. Guionnet, O. Zeitouni, An Introduction to Random Matrices, volume 118 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2010.
[5] O.E. Barndorff-Nielsen, N. Shephard, Econometric analysis of realized covariance: high frequency based covariance, regression, and correlation in financial economics, Econometrica 72 (2004) 885–925.
[6] L. Bauwens, C. Hafner, S. Laurent, Handbook of Volatility Model and their Applications, Financial Engineering and Econometrics, Wiley, New Jersey, 2012.
[7] W. Feller, Two singular diffusion problems, Ann. of Math. (2) 54 (1951) 173–182.
[8] P. Glasserman, Monte Carlo Methods in Financial Engineering, volume 53 of Applications of Mathematics (New York), Springer-Verlag, New York, 2004.
[9] J. Jacod, M. Podolskij, A test for the rank of the volatility process: the random perturbation approach, Preprint (2012).
[10] J. Jacod, M. Rosenbaum, Estimation of volatility functionals: the case of a square root n window, Preprint (2012).
[11] R. Litterman, J. Scheinkman, Common factors affecting bond returns, The Journal of Fixed Income 1 (1991) 54–61.
[12] N.L. Liu, Numerical study on a type I error of a random walk hypothesis on interest rates, in: Proceedings of the 41st ISCIE International Symposium on Stochastic Systems Theory and its Applications, Inst. Syst. Control Inform. Engrs. (ISCIE), Okayama, 2010, pp. 89–95.
[13] P. Malliavin, M.E. Mancino, Fourier series method for measurement of multivariate volatilities, Finance Stoch. 6 (2002) 49–61.
[14] P. Malliavin, M.E. Mancino, A Fourier transform method for nonparametric estimation of multivariate volatility, Ann. Statist. 37 (2009) 1983–2010.
[15] P. Malliavin, M.E. Mancino, M.C. Recchioni, A non-parametric calibration of the HJM geometry: an application of Itô calculus to financial statistics, Jpn. J. Math. 2 (2007) 55–77.
[16] H.L. Ngo, Parametric estimation for discretely observed stochastic processes with jumps, Electron. J. Stat. 4 (2010) 1443–1469.
[17] H.L. Ngo, S. Ogawa, A central limit theorem for the functional estimation of the spot volatility, Monte Carlo Methods Appl. 15 (2009) 353–380.
[18] S. Ogawa, Real-time scheme for the volatility estimation in the presence of microstructure noise, Monte Carlo Methods Appl. 14 (2008) 331–342.
[19] S. Ogawa, H.L. Ngo, Real-time estimation scheme for the spot cross volatility of jump diffusion processes, Math. Comput. Simulation 80 (2010) 1962–1976.
[20] S. Ogawa, S. Sanfelici, An improved two-step regularization scheme for spot volatility estimation, Economic Notes 40 (2011) 107–134.
[21] S. Ogawa, K. Wakayama, On a real-time scheme for the estimation of volatility, Monte Carlo Methods Appl. 13 (2007) 99–116.
[22] B.L.S. Prakasa Rao, Statistical Inference for Diffusion Type Processes, volume 8 of Kendall’s Library of Statistics, Edward Arnold, London, 1999.
[23] M.B. Priestley, Spectral Analysis and Time Series. Vol. 1, 2, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1981. Univariate series, Probability and Mathematical Statistics.

[24] Y. Wang, J. Zou, Vast volatility matrix estimation for high-frequency financial data, Ann. Statist. 38 (2010) 943–978.

(Nien-Lin Liu and Hoang-Long Ngo) Research Organization of Science and Engineering, Ritsumeikan University, 1-1-1 Nojihigashi - Kusatsu - Shiga - Japan

(Hoang-Long Ngo) Japan Science and Technology Agency

(Hoang-Long Ngo) Hanoi National University of Education, 136 Xuan Thuy - Cau Giay - Hanoi - Vietnam