Stochastic ordering results in parallel and series systems with Gumble distributed random variables

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Abstract

The stochastic comparisons of parallel and series system are worthy of study. In this paper, we present some stochastic comparisons of parallel and series systems having independent components from Gumble distribution with two parameters (one location and one shape). Here, we first put a condition for the likelihood ratio ordering of the parallel systems and second we use the concept of vector majorization technique to compare the systems by the reversed hazard rate ordering, the hazard rate ordering, the dispersive ordering, and the less uncertainty ordering with respect to the location parameter.

Keywords: Gumble distribution, entropy, Schur convex, vector majorization, stochastic comparisons.

1 Introduction

Discussion about order statistics is quite important for different distributions. A wide range of application of order statistics can be found in many different areas like statistics, applied probability, reliability theory, actuarial science, auction theory, hydrology, life testing etc.

Let $X_1, \ldots, X_n$ be a set of independently distributed random variables and $X_{n:n} = \max \{X_1, \ldots, X_n\}$, $X_{1:n} = \min \{X_1, \ldots, X_n\}$. $X_{1:n}$ represents a series system and $X_{n:n}$ represents a parallel system. $X_{1:n}$ and $X_{n:n}$ are also known as 1st and n\textsuperscript{th} order statistics respectively. It is well known that in a parallel system at least one of its $n$ components needs to work and in a series system all the components of the system need to work at a time. That’s why the parallel and the series system are called as 1-out-of-$n$ and $n$-out-of-$n$ systems respectively.

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Thus, the study of stochastic comparisons of $m$-out-of-$n$ systems is similar to the study of the stochastic comparisons of the order statistics.

We use extreme value theory to study the stochastic behavior of extreme values of a process. The Gumble distribution is one of the extreme value distribution which is also known as an extreme value type-I or smallest extreme value (Type I) distribution due to Emil Julius Gumbel (1954) [10]. We can observe that the maximum of a random sample after proper randomization can only converge to one of the three possible distributions, the Gumbel distribution, the Fréchet distribution or the Weibull distribution (see [9]).

Various results on stochastic comparisons of the Weibull distribution and Fréchet distribution are already presented by Torrado et al. (see [5]), Fang et al. (see [6]) and Gupta et al. (see [1]). In this paper, we present several results for the parallel and series systems having independent Gumble distributed components. The cumulative distribution function (cdf) of the Gumble distribution ('Gum' in short) is given by

$$F(x) = e^{-e^{-\frac{x-\mu}{\sigma}}}, \quad x, \mu \in \mathbb{R}, \sigma > 0. \quad (1)$$

Here $\mu$ is the location parameter and $\sigma$ is the scale parameter. A random variable $X$ is said to follow Gumble distribution, denoted as $X \sim \text{Gum}(\mu, \sigma)$ if $X$ has the cumulative distribution function given in (1). It is very much useful for analyzing several extreme natural events like earthquakes, floods, rainfall, wind speeds, snowfall etc. For applications of the distribution distribution one may refer to ([10],[11]).

Many researchers studied order statistics in terms of stochastic comparisons. A vast literature on stochastic comparisons for the lifetimes of the series and parallel systems are already available where the component variables follow Fréchet [1], Generalized Exponential [7], Exponentiated Gamma [13], Exponentiated Scale model [14], Exponential Weibull [15] distributions etc. The comparisons are made of with respect to the usual stochastic order, hazard rate order, reversed hazard rate order, likelihood ratio order, dispersive order, etc. For further details on stochastic comparisons, one may refer to ([5],[6],[7],[8],[9]). See, Shaked M et al. [2] for more comprehensive discussion on order statistics.

The aim of this paper is to present the likelihood ratio, reversed hazard rate, hazard rate, dispersive, and the less uncertainty ordering for parallel, series systems having Gumble distributed components. Now, let $X_1, \ldots, X_n$ be continuous independent random variables such that, $X_i \sim \text{Gum}(\mu_i, \sigma), \quad i = 1, 2, \ldots, n$. Furthermore, let $Y_1, \ldots, Y_n$ be another set of continuous independent random variables such that $Y_i \sim \text{Gum}(\mu_i^*, \sigma), i = 1, 2, \ldots, n$. When $\mu_i \geq \mu_i^*$, we obtain the likelihood ratio ordering for the parallel systems, and when $(\mu_1, \ldots, \mu_n) \succeq_m (\mu_1^*, \ldots, \mu_n^*)$, we proceed as follows. First, we discuss the reversed hazard rate ordering for the parallel systems. Second, we discuss the hazard rate ordering for the series systems. Finally, we discuss the dispersive and less uncertainty ordering for the series systems.

The paper has been organized in the following manner:
In Section-2 we briefly present some basic useful definitions, lemmas, and theorems which we have used throughout this paper. In the Section-3, we deal with the concept of vector majorization technique and achieve several ordering results.

2 Preliminaries

This section consists of some important definitions, theorems, and lemmas that are most pertinent to developments in Section-3. We use the notation $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, ‘log’ for usual logarithm base $e$, and we use ‘increasing’ and ‘decreasing’ for ‘non-decreasing’ and ‘non-increasing’ respectively throughout this paper.

Let $X, Y$ be two continuous random variables having the cumulative distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, the survival functions $\bar{F}_X(\cdot)$ and $\bar{F}_Y(\cdot)$, density functions $f_X(\cdot)$ and $f_Y(\cdot)$, hazard rate functions $r_X(\cdot) = \frac{f_X(\cdot)}{\bar{F}_X(\cdot)}$ and $r_Y(\cdot) = \frac{f_Y(\cdot)}{\bar{F}_Y(\cdot)}$, and reversed hazard rate functions $\tilde{r}_X(\cdot) = \frac{f_X(\cdot)}{F_X(\cdot)}$ and $\tilde{r}_Y(\cdot) = \frac{f_Y(\cdot)}{F_Y(\cdot)}$.

Let $X$ be a continuous random variable having $F_X(\cdot)$ and $f_X(\cdot)$ as cumulative distribution and density functions respectively. The measure of entropy of $X$ due to Shannon(1948) is defined by

$$\mathcal{H}(f_X) = -E[\log f_X(x)] = -\int_0^\infty f_X(x) \log f_X(x),$$

which is commonly known as Shannon information measure. This is a measure of the uncertainty of the lifetime of a system. A system having low uncertainty is more reliable than a system with great uncertainty. Ebrahimi and Pellerey (1995) defined the uncertainty of residual lifetime distributions, $\mathcal{H}(f_X, t)$, of a component by

$$\mathcal{H}(f_X, t) = \mathcal{H}(X - 1|X > t) = -\int_t^\infty \frac{f_X(x)}{\bar{F}(t)} \log \frac{f_X(x)}{\bar{F}(t)} dx$$

$$= \log \bar{F}(t) - \frac{1}{F(t)} \int_t^\infty f_X(x) \log f_X(x) dx$$

$$= 1 - \frac{1}{F(t)} \int_t^\infty f_X(x) \log r_X(x) dx,$$

where $r_X(\cdot), \bar{F}(\cdot)$ are hazard and survival functions respectively. After the component has survived up to time $t$, $\mathcal{H}(f_X, t)$ measures the expected uncertainty contained in the conditional density of $X - t$ given $X > t$ about the predictability of the remaining lifetime of the component. Now for $t = 0$, we can see that

$$\mathcal{H}(f_X, 0) = -\int_0^\infty f_X(x) \log f_X(x),$$

which is nothing but the Shannon’s measure of entropy of $X$.

Next, we present some useful definitions for different stochastic ordering.
**Definition 2.1.** *(Stochastic Order)*

Let \( X, Y \) be two random variables. We say that \( X \) is smaller than \( Y \) in

1. the likelihood ratio order (denoted by, \( X \leq_{lr} Y \)) if \( \frac{f_Y(x)}{f_X(x)} \) is increasing in \( x \).

2. the reversed hazard rate order (denoted by, \( X \leq_{rh} Y \)) iff \( \tilde{r}_X(x) \leq \tilde{r}_Y(x) \), \( x \in \mathbb{R} \). Equivalently \( F_Y(x) \) \( F_X(x) \) is increasing in \( x \).

3. the hazard rate order (denoted by, \( X \leq_{hr} Y \)) iff \( r_Y(x) \leq r_X(x) \), \( x \in \mathbb{R} \).
   Equivalently if \( \bar{F}_Y(x) \bar{F}_X(x) \) is increasing in \( x \).

4. the dispersive order (denoted by, \( X \leq_{disp} Y \)) if for \( 0 \leq \alpha < \beta \leq 1 \) we have
   \[
   F_Y^{-1}(\beta) - F_Y^{-1}(\alpha) \leq F_X^{-1}(\beta) - F_X^{-1}(\alpha).
   \]
   Equivalently \( X \leq_{disp} Y \) iff \( f_Y(F_Y^{-1}(p)) \leq f_X(F_X^{-1}(p)) \) for all \( p \in (0,1) \).

5. the less uncertainty order (denoted by \( X \leq_{LU} Y \)) if for any \( t > 0 \)
   \[
   H(f_X; t) \leq H(f_Y; t).
   \]

The following relation is well-known, that is

\[
X \leq_{lr} Y \implies X \leq_{hr} Y(X \leq_{rh} Y) \implies X \leq_{st} Y.
\]

**Definition 2.2.** *(Majorization)*

Let \( u = (u_1, ..., u_n) \) and \( v = (v_1, ..., v_n) \) be two real vectors from \( \mathbb{R}^n \) with the order components, \( u(n) \leq .... \leq u(1) \) and \( v(n) \leq .... \leq v(1) \), respectively. Then we say \( u \) majorizes \( v \), denoted by \( u \succeq_{m} v \) if

\[
\sum_{i=1}^{k} u_i \leq \sum_{i=1}^{k} v_i
\]

\( k = 1, 2, ..., n-1 \), and \( \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i \).

**Definition 2.3.** Let \( u = (u_1, ..., u_n) \) and \( v = (v_1, ..., v_n) \) be two vectors from \( \mathbb{R}^n \). A real valued function \( \sigma(u) : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be Schur-concave and Schur-convex if for all \( u \succeq_{m} v \) we have \( \sigma(u) \leq \sigma(v) \) and \( \sigma(u) \geq \sigma(v) \), respectively.

The theorem, stated below is very useful for our results.

**Theorem 2.4.** *(Marshall et al., p.84, [17]):* Let \( I \subset \mathbb{R} \) be an open interval and let \( \sigma : I^n \rightarrow \mathbb{R} \) be continuously differentiable function. The necessary and sufficient conditions for \( \sigma \) to be Schur-convex(Schur-concave) on \( I^n \) are \( \sigma \) is symmetric on \( I^n \) and, for all \( i \neq j \)

\[
(z_i - z_j) \left( \frac{\partial \sigma}{\partial z_i}(z) - \frac{\partial \sigma}{\partial z_j}(z) \right) \geq 0 \leq 0
\]
for all \( z \in I^n \). Where, \( \frac{\partial \sigma}{\partial z_i} \) is partial derivative of \( \sigma \) with respect to the \( i^{th} \) component of \( z \).

**Lemma 2.5.** Let the function \( \phi(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) be defined as

\[
\phi(t) = \frac{t}{e^t - 1}.
\]

Then,

1. \( \phi(t) \) is a convex function in \( \mathbb{R}^+ \);
2. \( \phi(t) \) is decreasing with respect to \( t \).

**Proof.**

1. The proof of this lemma is already available in [1], and therefore skipped for sake of brevity.

2. Taking derivative of \( \phi(t) \) with respect to \( t \) we have

\[
(e^t - 1)^2 \frac{d\phi}{dt} = e^t - 1 - te^t.
\]

Now, let \( g(t) = e^t - 1 - te^t \), then \( \frac{dg}{dt} = e^t - e^t - te^t \). Which implies that \( \frac{dg}{dt} = -te^t \leq 0 \) for any \( t \in \mathbb{R}^+ \). So, \( g(t) \leq g(0) = 0 \). Therefore, we have \( \frac{d\phi}{dt} \leq 0 \), this means that \( \phi(t) \) is decreasing in \( t \).

**Lemma 2.6.** Let \( I \subset \mathbb{R} \) be an open interval such that if a function \( \gamma(x) : I \to \mathbb{R} \) is convex then the function \( h(x) \) defined as

\[
h(x) = \sum_{i=1}^{n} \gamma(x_i)
\]

is a Schur-convex function in \( I^n \), where \( x = (x_1, ..., x_n) \). Consequently for \( x = (x_1, ..., x_n) \), \( y = (y_1, ..., y_n) \in I^n \) if \( (x_1, ..., x_n) \succeq^m (y_1, ..., y_n) \) then \( h(x) \geq h(y) \).

**Proof.** Proof of this lemma can be found in Marshall et al. [17].

**Theorem 2.7.** Let \( X, Y \) be two independent random variables having density functions \( f_X, f_Y \), cumulative distribution functions \( F_X, F_Y \), survival functions \( \bar{F}_X, \bar{F}_Y \), respectively. Let \( T \) be a random variable with density function \( h \) and distribution function \( H \). \( T \) is independent of \( X Y \). Then if

1. \( X \succeq_{rh} Y \) and either \( X \) or \( Y \) is IRHR then \( X^T \succeq_{rh} Y^T \).
2. \( X \preceq_{hr} Y \) and either \( X \) or \( Y \) is DHR then \( X^T \preceq_{hr} Y^T \).

**Proof.** The proof is similar to the proof in [4]. Sometimes it may not be possible to find dispersive ordering directly from the definition. In some particular cases, we can use the following theorem to have dispersive ordering.
Theorem 2.8. Let $X$ and $Y$ be two random variables. If $X \leq_{hr} Y$ and $X$ or $Y$ is DHR, then $X \leq_{disp} Y$.

Proof. For the proof see Shaked M et. al.\[2\]

The result of the Theorem-2.3 from Ebrahimi et.al.\[3\] can be strengthened for the hazard rate ordering as follows

Theorem 2.9. Let $X$ and $Y$ be two random variables. If $X \leq_{hr} Y$ and $X$ or $Y$ is DHR, then $X \leq_{LR} Y$.

Proof. Proof can be found in \[3\].

3 Results

In this section, we work on several results by comparing the lifetimes of parallel and series systems having independent Gumble distributed components. These results are presented for the location parameter $\mu$ by using vector majorization technique.

In the following theorem, we put a sufficient condition on the location parameter $\mu$ for the likelihood ratio ordering of the parallel systems where the components of the systems follow the Gumble distribution.

Theorem 3.1. Let $X_1, \ldots, X_n$ ($Y_1, \ldots, Y_n$) be the set of continuous independent random variables having $X_i \sim \text{Gum} (\mu_i, \sigma)$ ($Y_i \sim \text{Gum} (\mu^*_i, \sigma)$), $i = 1, 2, \ldots, n$. Then, if $\mu_i \geq \mu^*_i$, we have

$$X_{n:n} \geq_{lr} Y_{n:n}.$$ 

Proof. The cumulative distribution function of $X_{n:n}$ is given by

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} e^{-e^{-(x-\mu_i)/\sigma}} \quad x, \mu_i \in \mathbb{R}, \sigma > 0, i = 1, \ldots, n.$$ 

Therefore, the density function of $X_{n:n}$ is

$$f_{X_{n:n}} = \frac{d}{dx} \prod_{i=1}^{n} e^{-e^{-(x-\mu_i)/\sigma}} = \frac{F_{X_{n:n}}(x)}{\sigma} \sum_{i=1}^{n} e^{-(x-\mu_i)/\sigma}, \quad i = 1, \ldots, n.$$ 

Similarly, the density function of $Y_{n:n}$ is

$$f_{Y_{n:n}} = \frac{F_{Y_{n:n}}(x)}{\sigma} \sum_{i=1}^{n} e^{-(x-\mu^*_i)/\sigma}, \quad i = 1, \ldots, n.$$ 

Now, Let us consider

$$lr(x) = \frac{f_{X_{n:n}}(x)}{f_{Y_{n:n}}(x)} = \frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} \left[ \sum_{i=1}^{n} e^{-(x-\mu_i)/\sigma} \right] / \left[ \sum_{i=1}^{n} e^{-(x-\mu^*_i)/\sigma} \right].$$

6
Taking derivative with respect to $x$ we get

$$
\frac{d}{dx} \ln r(x) = \frac{1}{\sigma} \frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} \left[ \sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}} \right] \left[ \sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}} - e^{-\frac{(x-\mu_i)}{\sigma}} \right], \quad i = 1, \ldots, n.
$$

Now, by our assumption $\mu_i \geq \mu_i^*$ and since $e^{-\frac{(x-\mu_i)}{\sigma}}$ is increasing in $\mu$, so, we have $e^{-\frac{(x-\mu_i)}{\sigma}} \geq e^{-\frac{(x-\mu_i^*)}{\sigma}}$, $i = 1, \ldots, n$. Using this observation and since all the other terms in the right hand side of the above equation is always greater than 0.

So, we finally conclude that $\frac{d}{dx} \ln r(x) \geq 0$, which implies that $lr(x)$ is increasing in $x$ and this proves our theorem.

The following theorem deals with the reversed hazard rate ordering of a parallel system. The components of the system are having fixed scale parameter $\sigma$ and a varying location parameter $\mu$ of the Gumble distribution.

**Theorem 3.2.** Let $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be the set of continuous independent random variables having $X_i \sim \text{Gum}(\mu_i, \sigma)(Y_i \sim \text{Gum}(\mu_i^*, \sigma))$, $i = 1, 2, \ldots, n$. Then, if $(\mu_1, \ldots, \mu_n) \succgeq_m (\mu_1^*, \ldots, \mu_n^*)$, we have

$$
X_{n:n} \geq_{rh} Y_{n:n}.
$$

**Proof.** The cumulative distribution function of $X_{n:n}$ is

$$
F_{X_{n:n}}(x) = \left( \prod_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}} \right) x, \mu_i \in \mathbb{R}, \sigma > 0, \quad i = 1, \ldots, n.
$$

Now, we know that, in a parallel system, the sum of the reversed hazard rate of the lifetime of each component is equal with the reversed hazard rate of the lifetime of the system. Therefore, the reversed hazard rate function of $X_{n:n}$ is

$$
\tilde{r}_{X_{n:n}} = \frac{1}{\sigma} \sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}} , \quad i = 1, \ldots, n.
$$

It is clear that $e^{-\frac{(x-\mu_i)}{\sigma}}$ is convex in $\mu_i$ for $i = 1, \ldots, n$. Therefore, from Lemma-2.6 we obtain $\sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}}$ is Schur-convex with respect to $\mu_i, i = 1, \ldots, n$, which means that $\tilde{r}_{X_{n:n}} \geq \tilde{r}_{Y_{n:n}}$ i.e. $X_{n:n} \geq_{rh} Y_{n:n}$, as desired.

The above theorem leads us to the following corollary

**Corollary 3.3.** Let $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be the set of continuous independent random variables having $X_i \sim \text{Gum}(\mu_i, \sigma)(Y_i \sim \text{Gum}(\mu_i, \sigma))$, $i = 1, 2, \ldots, n$. Then, if $(\mu_1, \ldots, \mu_n) \succgeq_m (\mu_1^*, \ldots, \mu_n^*)$, and let $T$ be any random variable which is independent of $X$ and $Y$. Then we have

$$
X_{n:n}^T \geq_{rh} Y_{n:n}^T.
$$
Proof. To prove $X_{n:n}^T \succeq_{rh} Y_{n:n}^T$. By Theorem-2.7(1), it is sufficient if we can show $X_{n:n} \succeq_{rh} Y_{n:n}$ and $X_{n:n}$ is IRHR. Now, we know the reversed hazard rate of $X_{n:n}$ with respect to $\mu_i$, $i = 1, \ldots, n$, is

$$\tilde{r}_{X_{n:n}} = \frac{1}{\sigma} \sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}}, \quad i = 1, \ldots, n.$$ 

Therefore, it is easy to check that $\sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}}$ is increasing in $\mu_i$, $i = 1, \ldots, n$. So, $\tilde{r}_{X_{n:n}}$ is IRHR. Using this observation and the result in Theorem-3.2, we get $X_{n:n}^T \succeq_{rh} Y_{n:n}^T$, which completes the proof.

Next theorem discusses the comparison of the series system with respect to hazard rate ordering, having $n$ independent Gumble distributed components with varying location parameter $\mu$.

**Theorem 3.4.** Let $X_1, \ldots, X_n$ ($Y_1, \ldots, Y_n$) be the set of continuous independent random variables having $X_i \sim \text{Gum}(\mu_i, \sigma)$ ($Y_i \sim \text{Gum}(\mu_i^*, \sigma)$), $i = 1, 2, \ldots, n$. Then, if $(\mu_1, \ldots, \mu_n) \succeq_m (\mu_1^*, \ldots, \mu_n^*)$, we have

$$X_{1:n} \succeq_{hr} Y_{1:n}.$$ 

**Proof.** The survival function of $X_{1:n}$ is

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n} \left(1 - e^{-e^{-\frac{(x-\mu_i)}{\sigma}}}\right), \quad x, \mu_i \in \mathbb{R}, \sigma > 0, \quad i = 1, \ldots, n.$$ 

It is well known that the sum of the hazard rate functions of the components of a series system is equal to the hazard rate function of that system. Therefore, we have

$$r_{X_{1:n}} = -\frac{d}{dx} \log[\bar{F}_{X_{1:n}}(x)]$$

$$= \frac{1}{\sigma} \sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}} \left[ e^{\frac{(x-\mu_i)}{\sigma}} - 1 \right]^{-1}$$

$$= \frac{1}{\sigma} \sum_{i=1}^{n} \phi(e^{-\frac{(x-\mu_i)}{\sigma}}), \quad i = 1, \ldots, n.$$ 

Where $\phi(t) = \frac{t}{e^t - 1}$. Let $t = e^{-\frac{(x-\mu_i)}{\sigma}} \in \mathbb{R}^+$, $i = 1, \ldots, n$. Therefore, from Lemma-2.5(1) we say that $\phi(e^{-\frac{(x-\mu_i)}{\sigma}})$ is convex in $e^{-\frac{(x-\mu_i)}{\sigma}}$ for $i = 1, \ldots, n$. Finally, using Lemma-2.6 we conclude that $r_{X_{1:n}}$ is Schur-convex in $\mu_i$ for $i = 1, \ldots, n$. Hence, the theorem follows.

According to the above result, we immediately obtain the following corollary

**Corollary 3.5.** Let $X_1, \ldots, X_n$ ($Y_1, \ldots, Y_n$) be the set of continuous independent random variables having $X_i \sim \text{Gum}(\mu_i, \sigma)$ ($Y_i \sim \text{Gum}(\mu_i^*, \sigma)$), $i = 1, 2, \ldots, n$. Then, if $(\mu_1, \ldots, \mu_n) \succeq_m (\mu_1^*, \ldots, \mu_n^*)$, and let $T$ be any random variable which is independent of $X$ and $Y$. Then we have
Theorem 3.3. We observe that

\[ X_{1:n}^T \leq_{hr} Y_{1:n}^T. \]

**Proof.** By Theorem-2.7(2), we have to show \( X_{1:n} \leq_{hr} Y_{1:n} \) and \( X_{1:n} \) is DHR for proving \( X_{1:n}^T \leq_{hr} Y_{1:n}^T \). Now, we know the hazard rate of \( X_{1:n} \) with respect to \( \mu_i, i = 1, \ldots, n \), is

\[
r_{X_{1:n}} = \frac{1}{\sigma} \sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}} e^{-\frac{(x-\mu_i)}{\sigma}} - 1 = \frac{1}{\sigma} \sum_{i=1}^{n} \phi(e^{-\frac{(x-\mu_i)}{\sigma}}).
\]

Where \( \phi(t) = \frac{t}{e^t - 1} \). Let \( t = e^{-\frac{(x-\mu)}{\sigma}} \in \mathbb{R}^+ \), \( i = 1, \ldots, n \). Therefore, using Lemma-2.5(2) we say that \( \phi(e^{-\frac{(x-\mu_{i})}{\sigma}}) \) is decreasing in \( e^{-\frac{(x-\mu_{i})}{\sigma}} \) for \( i = 1, \ldots, n \). Equivalently, \( r_{X_{1:n}} \) is decreasing in \( \mu_i \) for \( i = 1, \ldots, n \). So, we see that \( X_{1:n} \) is DHR. Therefore, from Theorem-3.4, and since \( X_{1:n} \) is DHR, we conclude that \( X_{1:n}^T \leq_{hr} Y_{1:n}^T \), this completes the proof.

In the next result, we present the dispersive ordering and the less uncertainty ordering of the series system having independent Gumble distributed components. We find the ordering with respect to the location parameter \( \mu \).

**Theorem 3.6.** Let \( X_1, \ldots, X_n \) (\( Y_1, \ldots, Y_n \)) be the set of continuous independent random variables having \( X_i \sim \text{Gum}(\mu_i, \sigma)(Y_i \sim \text{Gum}(\mu_i', \sigma)) \), \( i = 1, \ldots, n \). Then, if \((\mu_1, \ldots, \mu_n) \succeq_{m} (\mu_1', \ldots, \mu_n')\), we have

1. \( X_{1:n} \leq_{\text{disp}} Y_{1:n} \);
2. \( X_{1:n} \leq_{\text{LU}} Y_{1:n} \).

**Proof.** The survival function and hazard rate function of \( X_{1:n} \), respectively, are

\[
F_{X_{1:n}}(x) = \prod_{i=1}^{n} \left( 1 - e^{-\frac{(x-\mu_i)}{\sigma}} \right),
\]

and

\[
r_{X_{1:n}} = \frac{1}{\sigma} \sum_{i=1}^{n} e^{-\frac{(x-\mu_i)}{\sigma}} e^{-\frac{(x-\mu_i)}{\sigma}} - 1.
\]

For proving \( X_{1:n} \leq_{\text{disp}} Y_{1:n} \), and \( X_{1:n} \leq_{\text{LU}} Y_{1:n} \), by Theorem-2.8 and Theorem-2.9, it is enough if we can show that, under the same condition \( X_{1:n} \leq_{hr} Y_{1:n} \) and \( X_{1:n} \) is DHR or \( r_{X_{1:n}} \) is decreasing in \( \mu_i, i = 1, \ldots, n \), hold. Now, from the Theorem-3.3, we observe that \( X_{1:n} \leq_{hr} Y_{1:n} \). Next, our claim is \( r_{X_{1:n}} \) is decreasing in \( \mu_i, i = 1, \ldots, n \). Now, in the Corollary-3.5, we have already proved that \( r_{X_{1:n}} \) is decreasing with respect to \( \mu_i, i = 1, \ldots, n \), i.e. \( X_{1:n} \) is DHR, which establishes our claim. Thus, the theorem is proved.

**4 Disclosure**

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