STOCHASTIC ACCELERATION AND THE EVOLUTION OF SPECTRAL DISTRIBUTIONS IN SYNCHRO-SELF-COMPTON SOURCES: A SELF-CONSISTENT MODELING OF BLAZARS’ FLARES

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ABSTRACT

The broadband spectral distributions of non-thermal sources, such as those of several known blazars, are well described by a log-parabolic fit. The second-degree term in these fits measures the curvature in the spectrum. In this paper, we investigate whether the curvature parameter observed in the spectra of the synchrotron emission can be used as a fingerprint of stochastic acceleration. As a first approach, we use the multiplicative central limit theorem to show how fluctuations in the energy gain result in the broadening of the spectral shape, introducing a curvature into the energy distribution. Then, by means of a Monte Carlo description, we investigate how the curvature produced in the electron distribution is linked to the diffusion in momentum space. To get a more generic description of the problem we turn to the diffusion equation in momentum space. We first study some “standard” scenarios, in order to understand the conditions that make the curvature in the spectra significant, and the relevance of cooling during the acceleration process. We try to quantify the correlation between the curvature and the diffusive process in the pre-equilibrium stage, and investigate how the transition between the Klein–Nishina and the Thomson regimes, in inverse Compton cooling, determine the curvature in the distribution at equilibrium. We apply these results to some observed trends, such as the anticorrelation between the peak energy and the curvature term observed in the spectra of Mrk 421, and a sample of BL Lac objects whose synchrotron emission peaks at X-ray energies.

Key words: acceleration of particles – BL Lacertae objects: general – BL Lacertae objects: individual (Mrk 421, Mrk 501, 1H 1426+428, 1ES 1959+650, Mrk 180, PKS 0548−32)

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1. INTRODUCTION

A defining feature of the non-thermal emission from different types of galactic and extragalactic sources is that their spectra are described by a power law (PL) over a broad photon energy range. In several sources, however, their spectra show significant curvature that is typically milder than that expected from an exponential cutoff. In previous papers, Massaro et al. (2004, 2006) discussed the curvature observed in the broadband X-ray spectra of the two well-known HBL (High-energy peaked BL Lac) objects Mrk 421 and Mrk 501. The basic idea was that this curvature was not simply the result of radiative cooling of high energy electrons, responsible for the synchrotron and inverse Compton (IC) emission, but that it was essentially related to the acceleration mechanism. Massaro et al. (2004) showed that curved spectral distributions, in particular log-parabolic (i.e., log-normal) ones, develop when the acceleration probability is a decreasing function of the electron energy. In subsequent works, through the analysis of a large collection of X-ray observations of Mrk 421, Tramacere et al. (2007, 2009) and Tramacere (2007) pointed out that the observed anticorrelation between the peak energy and the curvature measured in the synchrotron spectral energy distribution (SED) could be used as a clear signature of a stochastic component in the acceleration process. Very recently, the log-parabolic law has also been applied to describe the spectral distribution and evolution of some gamma-ray bursts (Massaro & Grindlay 2011).

The principal aim of the present paper is to investigate this scenario by extracting information on the acceleration processes using the curvature parameter measured in the observed synchrotron and IC spectra of Synchro-Self-Compton (SSC) sources. We study the conditions in which the energy distributions of electrons, resulting from stochastic acceleration, can be approximated by a log-parabolic law and how its curvature evolves during their acceleration, and the role of IC cooling. We compare predictions from our theoretical descriptions with the curved spectra of some HBL objects.

In Section 2, we give an intuitive picture, taking into account the effect of random fluctuations on the energy gain of particles and the role these play in determining the spectral curvature, as a consequence of the multiplicative central limit theorem, and compare these results with the analytical solution of the diffusion equation, in the “hard-spheres” approximation. In Sections 3 and 4, we give a more physical description of the problem, using first a Monte Carlo (MC) approach and second by solving numerically the momentum-diffusion equation. We discuss the evolution of the curvature in the electron distribution as a result of momentum diffusion before equilibrium is reached and the role that synchrotron and IC cooling processes play in reaching the equilibrium. In Section 5, we study the peak energy, fluxes, and curvature trends in the SED of both the synchrotron and IC emission, looking for the fingerprints of the stochastic component. In Section 6, we show how our results can reproduce the spectral trends observed in some HBLs, in particular we investigate the relation between the peak energy and the curvature, and between the peak energy and the peak flux. The good agreement between predictions and observed trends confirms that the stochastic acceleration mechanism can play an important role in the physics of blazars’ jets and other SSC sources.
2. THE LOG-PARABOLA ORIGIN: ANALYTICAL APPROACH

2.1. Statistical Description

In the statistical picture, the change in energy of the particles at each acceleration step \( n_s \) is expressed as

\[
\gamma_{ns} = \varepsilon_{ns} \gamma_{ns-1} = \gamma_{ns-1}(1 + \Delta \gamma_{ns-1}/\gamma_{ns-1}), \tag{1}
\]

where \( \gamma \) is the Lorentz factor of the particle and \( \varepsilon \) is the fractional energy gain. Here we investigate the role of fluctuations of \( \varepsilon \) on the spectral shape of the accelerated particles. With this aim in mind, we express the energy gain fluctuations as

\[
\varepsilon = \bar{\varepsilon} + \chi,
\]

where the random variable \( \chi \) has a probability density function with zero mean value \( \langle \chi \rangle = 0 \) and variance \( \sigma^2_{\chi} \), and \( \bar{\varepsilon} \) represents the systematic energy gain, which we treat as a non-random variable and the probability density function of \( \varepsilon \) is defined on the range \( \varepsilon \gtrless 0 \). The particle energy at step \( n_s \) can be expressed as

\[
\gamma_{n_s} = \gamma_0 \prod_{i=1}^{n_s} \varepsilon_i,
\]

where \( \gamma_0 \) is the initial energy of the particle. This equation clearly shows that the final energy distribution \( n(\gamma) = dN(\gamma)/d\gamma \) will result from the product of the random variables \( \varepsilon_i \). The determination of an analytic expression for the distribution resulting from the multiplication of generic random variable is not an easy task (Glen et al. 2004). Using the simplifying assumption that the particles are always accelerated, namely, the acceleration probability, \( P_c \), is set to unity and applying the multiplicative case of the central limit theorem (e.g., Cowan 1998), it is possible to show that the particle energies will be distributed as a log-normal law

\[
n(\gamma) = \frac{N_0}{\gamma \sigma_{\gamma} \sqrt{2\pi}} \exp \left[ -\left( \ln \gamma - \mu \right)^2/2\sigma_{\gamma}^2 \right], \tag{4}
\]

where \( N_0 \) is the total number of particles, \( \mu = \langle \ln \gamma \rangle \), \( \sigma_{\gamma}^2 = \sigma^2(\ln \gamma) \). We can determine these two quantities by taking the logarithm of Equation (3),

\[
\ln \gamma_{n_s} = \ln \gamma_0 + \sum_{i=1}^{n_s} \ln(\bar{\varepsilon} + \chi_i)
\]

\[
= \ln(\gamma_0 \bar{\varepsilon}^{n_s}) + \sum_{i=1}^{n_s} \ln \left( 1 + \frac{\chi_i}{\bar{\varepsilon}} \right)
\]

\[
\approx \ln(\gamma_0 \bar{\varepsilon}^{n_s}) + \sum_{i=1}^{n_s} \left( \frac{\chi_i}{\bar{\varepsilon}} - \frac{\chi_i^2}{2 \bar{\varepsilon}^2} \right), \tag{5}
\]

assuming that \( \chi_i/\bar{\varepsilon} \) is not large. We obtain for the two parameters \( \mu \) and \( \sigma_{\gamma} \)

\[
\mu = \ln(\gamma_0) + n_s \ln \bar{\varepsilon} + n_s \left[ \frac{\chi_i}{\bar{\varepsilon}} - \frac{1}{2} \left( \frac{\sigma_{\chi}}{\bar{\varepsilon}} \right)^2 - \frac{\chi_i^2}{2 \bar{\varepsilon}^2} \right]
\]

\[
\sigma_{\gamma}^2 = n_s \left[ \left( \frac{\sigma_{\chi}}{\bar{\varepsilon}} \right)^2 + \left( \frac{\sigma_{\gamma}}{\bar{\varepsilon}} \right)^2 + 2 \left( \frac{\sigma_{\gamma}}{\bar{\varepsilon}} \right) \left( \frac{\chi_i}{\bar{\varepsilon}} \right) \right]. \tag{6}
\]

This equation shows that the variance increases linearly with the number of acceleration steps and is proportional to \( \sigma_{\gamma}^2 \). Substituting \( \mu \) and \( \sigma_{\gamma} \) into Equation (4),

\[
n(\gamma) = \frac{N_0}{\gamma \sigma_{\gamma} \sqrt{2\pi}} \exp \left[ -\left( \ln \gamma - \mu \right)^2/2\sigma_{\gamma}^2 \right]. \tag{8}
\]

Hereafter we will consider decimal logarithms (\( \log \equiv \log_{10} \)). \( c_e = 1/\log_{10} e \approx 2.3 \) to make easier the comparison of the curvature results from this paper with those presented in observational papers. Taking the logarithm of Equation (8) and substituting the parameters from Equation (8) we obtain

\[
\log n(\gamma) = K - \log \gamma - \frac{(c_e \ln \gamma / \gamma_0 - n_s \ln \bar{\varepsilon} + 1/2(\sigma_{\gamma}^2/\bar{\varepsilon}^2))^2}{c_e 2 n_s (\sigma_{\gamma}/\bar{\varepsilon})^2}, \tag{9}
\]

where \( K \) includes all the constant factors. This is a log-parabolic law with the curvature (second degree in log \( \gamma \)) coefficient given by

\[
r = \frac{c_e}{2 n_s (\sigma_{\gamma}/\bar{\varepsilon})^2}. \tag{10}
\]

The interesting physical insight of this equation is that the curvature of the particle energy distribution is inversely proportional to the acceleration steps \( n_s \) and to the variance of the energy gain \( (\sigma_{\gamma}^2) \). In the case of \( P_c < 1 \), the distribution at step \( n_s \) will be given by the convolution of different log-normal distributions for each acceleration step, with the distribution at \( n_s \) broader than that at \( n_s - 1 \) and containing fewer particles, as already noted in Peacock (1981).

Similar results are obtained considering a constant energy gain but a fluctuating number of acceleration steps. Assuming that after a time \( t \) the probability distribution for the number of steps undergone by a particle is given by a Poisson law, it is possible to show that the energy distribution follows a log-parabolic whose curvature term depends on the inverse of the mean number of steps multiplied by the duration of the acceleration process.

2.2. Diffusion Equation Approach

The above statistical description provides an intuitive link between the curvature in the energy distribution of accelerated particles and the presence of a randomization process, such as the dispersion in the energy gain or in the number of acceleration steps. However, this approach does not give a complete physical description of the processes responsible for the systematic and stochastic energy gain, ignoring other physical processes, such as the radiative cooling and injection rates, or the acceleration energy dependence, necessary to give a complete description of the particles energy distribution evolution. A physical self-consistent description of stochastic acceleration in a time-dependent fashion can be achieved through a kinetic equation approach. Employing the quasi-linear approximation with the inclusion of momentum-diffusion term (Ramy 1979; Becker et al. 2006), the equation governing the temporal evolution of \( n(\gamma) \) is

\[
\frac{\partial n(\gamma, t)}{\partial t} = \frac{\partial}{\partial \gamma} \left[ - [S(\gamma, t) + D(\gamma, t)] n(\gamma, t) \right] + D_p(\gamma, t) \frac{\partial n(\gamma, t)}{\partial \gamma} + \frac{n(\gamma, t)}{T_{esc}(\gamma)} \cdot Q(\gamma, t), \tag{11}
\]
where \( D_p(y, t) \) is the momentum-diffusion coefficient, \( D_A(y, t) = (2/\gamma) D_p(y, t) \) is the average energy change term resulting from the momentum-diffusion process, and \( S(y, t) = -C(y, t) + A(y, t) \) is an extra term describing systematic energy loss (C) and/or gain (A), and \( Q(y, t) \) is the injection term. In the standard diffusive shock acceleration scenario, there are several possibilities for which one can expect that energy gain fluctuations will occur, due to the momentum-diffusion term. In particular, for the case of a turbulent magnetized medium, the advection of particles toward the shock due to pitch angle scattering may be accompanied by stochastic momentum-diffusion mechanism. In this scenario, particles embedded in a magnetic field with both an ordered (\( B_0 \)) and turbulent (\( \delta B \)) component, exchange energy with resonant plasma waves, and the related diffusion coefficient is determined by the spectrum of the plasma waves. Following the approach of Becker et al. (2006), we describe the energy distribution \( W(k) \) in terms of the wave number \( k = 2\pi/\lambda \) with a PL
\[
W(k) = \frac{\delta B(k)^2}{8\pi} = \frac{\delta B(k_0)^2}{8\pi} \left( \frac{k}{k_0} \right)^{-q},
\]
with \( q = 2 \) for the “hard-sphere” spectrum, \( q = 5/3 \) for the Kolmogorov spectrum, and \( q = 3/2 \) for the Kraichnan spectrum, the total energy density in the fluctuations being
\[
U_{\delta B} = \int_{k_0}^{k_{\text{max}}} W(k) dk.
\]
Under these assumptions, the momentum-diffusion coefficient reads (O’Sullivan et al. 2009)
\[
D_p \approx \beta_A^2 \left( \frac{\delta B}{B_0} \right)^2 \left( \frac{\rho_e}{\rho_{e,\text{max}}} \right)^q \left( \frac{\gamma - 1}{\rho_A} \right)^{2-q} \frac{p^2 c^2 \rho_e c}{\rho_A},
\]
where \( \beta_A = V_A/c \) and \( V_A \) is the Alfvén waves velocity, \( \rho_A = pc/qB \) is the Larmor radius, and \( \lambda_{\text{max}} \) is the maximum wavelength of the Alfvén waves spectrum. The acceleration time for particles with Lorentz factor \( \gamma \), whose Larmor radii resonate with one particular magnetic field turbulence length scale, is dictated by the momentum-diffusion coefficient \( D_p \) as
\[
t_{\text{acc}} \approx \frac{p^2}{D_p} = \frac{\rho_e(y_0)}{c \beta_A^2} \left( \frac{B_0^2 \delta B^2}{\rho_A} \right) \left( \frac{\gamma}{y_0} \right)^{2-q} \left( \frac{\gamma - 1}{\rho_A} \right)^{2-q}.
\]
The spatial diffusion coefficient relates to the momentum-diffusion coefficient through the relation, \( D_s D_p \approx p^2 \beta_A^2 \) (Skilling 1975), hence the escape time of the particles from the acceleration region of size \( R \) depends on the spatial diffusion coefficient through the relation
\[
t_{\text{esc}} \approx \frac{R^2}{D_s} \approx \frac{R^2}{(c \beta_A)^2 t_{\text{acc}}}. \tag{16}
\]
The coefficients in Equation (11), and their related timescales, can be expressed as a PL in terms of the Lorentz factor \( \gamma \)
\[
\begin{align*}
D_p(\gamma) &= D_p(\gamma_0)^q, & t_D &= \frac{1}{\gamma_0} \left( \frac{\gamma}{\gamma_0} \right)^{2-q}, \\
D_A(\gamma) &= 2D_p(\gamma_0)^{q-1}, & t_{DA} &= \frac{1}{2D_p(\gamma_0)^2} \left( \frac{\gamma}{\gamma_0} \right)^{2-q}, \\
A(\gamma) &= A_{p0}(\gamma), & t_A &= \frac{1}{A_{p0}},
\end{align*}
\]
where \( \gamma_0 \) and \( \rho_0 \) have the dimension of the inverse of a time. Analytical solutions of the diffusion equation for relativistic electrons have frequently been discussed in the literature since the early work by Kardashev (1962), in particular for the case of the “hard-sphere” approximation. Neglecting the \( S \) and \( T_{\text{esc}} \) terms in Equation (11), and using a mono-energetic and instantaneous injection (\( n(\gamma, t) = N_0 \delta(\gamma - \gamma_0) \)), the solution of the diffusion equation is (Melrose 1969; Kardashev 1962)
\[
n(\gamma, t) = \frac{N_0}{\sqrt{4\pi D_{p0} t}} \exp \left\{ -\frac{[\ln(\gamma/\gamma_0) - (A_{p0} - D_{p0} t)]^2}{4D_{p0} t} \right\} \tag{18},
\]
i.e., a log-parabola distribution, whose curvature term is
\[
r = \frac{c_e}{4D_{p0} t} \propto \frac{1}{D_{p0} t}. \tag{19}
\]
This result is fully consistent with that found in the statistical description; indeed, Equations (18) and (8) have the same functional form in both the statistical and in the diffusion equation scenario, with \( t \) playing the role of \( n_e, D_{p0} \) the role of the variance of the energy gain \( \left( \sigma^2_{\gamma} \right) \), and \( A_{p0} \) the role of the absorption time \( t_{\text{esc}} \).

\section{3. Numerical Approach: Monte Carlo Simulation with Magnetic Turbulence}

In this section, we demonstrate explicitly how the introduction of energy fluctuations leads to curved spectral distributions of particles. This is carried out using an MC approach.

In our simulations, we considered \( 10^5 \) particles injected into the system with a cold mono-energetic distribution of Lorentz factors, with \( \gamma_0 = 1 \). To compare these results with the ones presented in Section 2, we remind the reader that in the MC approach, the duration of the acceleration process \( t \) is the equivalent of the number of acceleration steps \( n \) used in the statistical picture and that the probability of the particle to be upscattered or downscattered in the MC realizations can be expressed in the statistical approach as \( P(\varepsilon > 1) \) and \( P(\varepsilon < 1) \), respectively. The scattering probability of the particles is dictated by the intensity of resonant waves in the turbulent magnetic power spectrum. As a working hypothesis, we assume that particles interact with a turbulent magnetic field whose power spectrum is expressed by Equation (12). In each scattering, the particles have a probability of \( (1 + \beta_A)/2 \) of being upscattered and a probability of \( (1 - \beta_A)/2 \) of being downscattered. The energy dispersion of the particle due to resonant scattering with Alfvén waves will be \( \langle \Delta E^2 \rangle \propto \langle E^2 \rangle^2 t \), where \( E = m_e c^2 \gamma \). Using the very good approximation for the variance of the product of \( n \) uncorrelated random variables (Goodman 1962)
\[
\sigma^2(\Pi \chi) = \Pi(\chi_1)^2 \sum \left( \frac{\sigma^2_{\chi_1}}{\langle \chi_1 \rangle^2} \right), \tag{21}
\]
and plugging Equation (2) into the equation above, we get

\[ \langle \Delta E^2 \rangle \propto (E \beta_{\Lambda}^2)^2 \propto (E \gamma_0^2)^2 \beta_{\Lambda}^2 \sigma_{\epsilon}^2 \frac{\epsilon}{\bar{\epsilon}} \]  \hspace{1cm} (22)

since \( E \) is the particle energy at time \( t \) (namely, step \( n_t \)), we have

\[ E^2 = (m_ec^2 \gamma_0 \epsilon_{\bar{0}})^2, \]  \hspace{1cm} from which follows

\[ \beta_{\Lambda}^2 \propto \left( \frac{\sigma_{\epsilon}}{\bar{\epsilon}} \right)^2. \hspace{1cm} (23) \]

In the following two sections (Sections 3.1 and 3.2), we study the consequences of the structure in the magnetic turbulence, on the evolution of the particle spectra, following their stochastic acceleration in the turbulent field.

### 3.1. Hard-sphere Turbulence

Under the “hard-sphere” approximation (\( q = 2 \)), the spatial diffusion coefficient does not depend on the particle energy, since the exponent of Equation (15) is \( q - 2 = 0 \). Only three independent parameters exist in this description: the scattering time, the escape time, and the velocity of the scatterers. The spectra are purely determined by how many scatterings have been able to occur, the velocity of the scatterer, and what fraction of the injected particles has escaped out of the acceleration region. The scattering time relates to the spatial diffusion coefficient by \( t_{\text{scat}} \approx D_{\epsilon}/c \). Similarly, the resulting acceleration time relates to the spatial diffusion coefficient by \( t_{\text{acc}} \approx D_{\epsilon}/\beta_{\epsilon}^2 c \approx t_{\Lambda}/\beta_{\Lambda}^2 \). Thus, for “hard-sphere” turbulence, the scattering and acceleration timescales are independent of the particle energy (since there is equal energy density of scatterers which particles of all energies may resonantly scatter with).

The left panel in Figure 1 shows the resulting instantaneous evolution of spectra for the “hard-sphere” turbulence. The log-parabolic shape is maintained along the entire acceleration process, as shown by the solid lines representing the fit of the MC distributions by means of the law in Equation (9) (Cowsik & Sarkar 1984). The evolution of the curvature parameter, obtained from the \( r \) in the log-parabolic fit and plotted in Figure 2 with the red dashed line, clearly shows the trend due to the momentum diffusion, in agreement with the prediction from Equation (19) (blue line in the plot) demonstrating the connection between \( D_{\rho_0}, \sigma_\epsilon/\bar{\epsilon}, \) and \( \beta_{\Lambda} \).

### 3.2. Soft Turbulence Spectra

To account for the effects of turbulent magnetic field spectra softer than the “hard-sphere” case, we also consider acceleration in Kolmogorov- and Kraichnan-type turbulence. We have to therefore include a fourth parameter in the MC simulation, in addition to the three considered above: the turbulent field spectral slope \( q \) (see Equation (12)).

The right plot in Figure 1 shows the evolution of spectra for the “Kolmogorov” turbulence case. Similar spectra were obtained by Lemoine & Pelletier (2003) and O’Sullivan et al. (2009) who integrated the trajectories of charged particles in a turbulent magnetic field embedded in a fluid. The results are compared to the quasi-linear theory results (Becker et al. 2006; solid lines in Figure 1). We can identify two phases in the temporal evolution. In the first phase, the SEDs are more symmetric and the curvature evolves as in the case \( q = 2 \), while in the second phase they develop a low-energy PL tail. Figure 2 shows that, for the Kolmogorov (green line) and the Kraichnan (black line) turbulence, \( r \) is systematically larger than the “hard-sphere” case (red line), and that for \( t \gtrsim 2 \times t_{\text{acc}} \), \( r \) approaches an asymptotic value (\( r \approx 1.2 \) and \( r \approx 1.5 \) for \( q = 5/3 \) and \( q = 3/2 \), respectively) ruled by the exponential cutoff in the equilibrium distribution.

### 4. NUMERICAL APPROACH: DIFFUSION EQUATION WITH STOCHASTIC COMPONENT AND LOSSES

Both the MC approach and statistical description are able to explain the link between the curvature in the energy distribution of accelerated particles and the presence of a stochastic energy gain term. In order to incorporate a more complete description,
taking into account the competition between radiative losses and acceleration, and its influence on the curvature, we use the diffusion equation approach, already outlined in Section 2.2, by inserting into Equation (11) a cooling term for the synchrotron and IC radiative losses. Following Moderski et al. (2005) we can write

\[
|\dot{\gamma}_{\text{synch}}| = \frac{4\sigma_T c}{3m_e c^2} \gamma^2 U_B = C_0 \gamma^2 U_B \\
|\dot{\gamma}_{\text{IC}}| = \frac{4\sigma_T c}{3m_e c^2} \gamma^2 \int f_{\text{KN}}(4\gamma \epsilon_0) \epsilon_0 n_{\text{ph}}(\epsilon_0) d\epsilon_0 = C_0 \gamma^2 F_{\text{KN}}(\gamma) \\
C(\gamma) = |\dot{\gamma}_{\text{synch}}| + |\dot{\gamma}_{\text{IC}}| = C_0 \gamma^2 (U_B + F_{\text{KN}}(\gamma)),
\]

where \(U_B = B^2/8\pi\) is the energy density of the magnetic field, \(\epsilon_0 = h\nu_0/m_e c^2\) is the IC seed photon energy in units of \(m_e c^2\), \(n_{\text{ph}}(\epsilon_0)\) is the number density of IC seed photons with the corresponding photon energy density \(U_{\text{ph}} = m_e c^2 \int \epsilon_0 n_{\text{ph}}(\epsilon_0) d\epsilon_0\). The function \(f_{\text{KN}}\) results from the analytical integration of the Jones (1968) Compton kernel, fully taking into account Klein–Nishina (KN) effects for an isotropic seed photon field (see Moderski et al. 2005, their Appendix C), and \(F_{\text{KN}}(\gamma)\) represents its convolution with the seed photon field. We remark that \(f_{\text{KN}}\) plays a crucial role in the cooling process, depending both on the IC regime (Thomson (TH) limit for \(4\gamma \epsilon_0 \ll 1\), KN limit for \(4\gamma \epsilon_0 \gg 1\)) and on \(\epsilon_0 n_{\text{ph}}(\epsilon_0) \propto B^2/R^2\).

Since analytical solutions are possible only for a limited number of cases, to follow the complex dependence of the IC cooling term on \(n(\gamma)\), in a self-consistent way we must solve the diffusion equation numerically. For this purpose, we further developed the numerical code (Tramacere et al. 2009; Tramacere 2007) used to compute numerically the synchrotron and IC emission and introduced it into the numerical solution of the diffusion equation. In the numerical calculations, we adopted the method proposed by Chang & Cooper (1970) and used the numerical recipe given by Park & Petsian (1996). This is a finite difference scheme based on the centered difference of the diffusive term, employing weighted differences for the advective term. We use a 5000 point energy grid over the range 1.0 \(\leq \gamma \leq 10^8\), and a time grid that is finely tuned to have a temporal mesh several orders of magnitude smaller than typical cooling and acceleration timescales. The results from our code were compared, when possible, with known analytical solutions and always found good agreement.

### 4.1. Physical Set-up: the Relations Between \(D_p\) and \(t_p\) with \(\gamma_{\text{max}}\) and \(R\)

We study the evolution of \(n(\gamma)\) and of the curvature term in a homogeneous spherical geometry, with radius \(R\) and an entangled coherent magnetic field \(B\) and a turbulent component \(\delta B\), in the two cases of impulsive and continuous injection with a quasi mono-energetic source function \(Q(\gamma_{\text{inj}}, t)\) normalized to have a fixed energy input rate:

\[
L_{\text{inj}} = \frac{4}{3} \pi R^3 \int \gamma_{\text{inj}}^3 m_e c^2 Q(\gamma_{\text{inj}}, t) d\gamma_{\text{inj}} \text{ (erg s}^{-1}) \tag{25}
\]

In our approach, we do not distinguish the acceleration region from the radiative one and during the acceleration process we take into account both synchrotron and IC cooling. According to Equation (14), to determine the order of magnitude of \(D_p\), we assume \(1 \gg \delta B/B \simeq 0.1–0.01\) and require Alfvén waves to be at least mildly relativistic, with \(\beta_A \simeq 0.1–0.5\), and their maximum wavelength to be much smaller than the accelerator size (\(\lambda_{\text{max}} < R\)). To study the effect of IC cooling on the evolution of \(n(\gamma)\), we consider two different sizes of the acceleration region, a compact one (\(R = 5 \times 10^{13} \text{ cm}\)) and a larger one (\(R = 1 \times 10^{15} \text{ cm}\)). With this choice of accelerator size, we set \(\lambda_{\text{max}} \approx 10^{12} \text{ cm}\). We stress that the choice of \(\lambda_{\text{max}}\) constrains the acceleration upper limit through \(\rho_{\text{g}} < \lambda_{\text{max}}\) leading to \(\gamma_{\text{max}} < (\lambda_{\text{max}} q B)/m_e c^2\), since particles with larger \(\rho_{\text{g}}\) (hence larger \(\gamma\)) cannot resonate with shorter wavelengths. Taking into account a coherent magnetic field of the order of 0.1 G and \(\lambda_{\text{max}} \approx 10^{12} \text{ cm}\) we found that the purely accelerative efficiency limits the particle energy to \(\gamma_{\text{max}} \lesssim 10^{7.5}\).

In the left panel of Figure 3, we plot \(D_p\), given by Equation (17), as a function of \(\lambda_{\text{max}}\) for the case of \(q = 2\), \(\delta B/B = 0.1\), and \(\beta_A = 0.5\). In this case, the acceleration time is energy independent and for \(\lambda_{\text{max}} \approx 10^{12} \text{ cm}\) it will be of the order of \(t_p = 1/D_{p0} \approx 10^4 \text{ s}\). In the case of \(q \neq 2\), the acceleration will have an energy dependence given by Equation (17), as shown in the right panel of Figure 3 for the case of \(q = 3/2\).

In this section, we focus on the evolution of the curvature as a function of the momentum-diffusion term, and therefore use only the accelerative contributions coming from the diffusion terms \((D_p(\gamma), D_A(\gamma))\), neglecting the systematic extra term \(A(\gamma)\). All the parameters and their numerical values are given in Table 1.
Figure 3. Left panel: the $t_D$ acceleration time as a function of $\lambda_{\text{max}}$ for $q = 2$, $8B/B = 0.1$, and $\beta_A = 0.5$. The vertical lines represent the Larmor radius for $\gamma = 10^5$ (red line), $\gamma = 1.5 \times 10^7$ (cyan line), and $\gamma = 10^9$ (orange line). Right panel: the $t_D$ acceleration time for the same parameters as in the left panel, for the case of $q = 3/2$ and as function of $\gamma$, for the two different cases of $\lambda_{\text{max}} = 3 \times 10^{10}$ cm (black line) and $\lambda_{\text{max}} = 1 \times 10^{15}$ cm (purple line). The thick black line shows $t_D$, for the case of $\lambda_{\text{max}} = 3 \times 10^{10}$ cm, limited to the highest acceleration energy of the particles constrained by the resonant scattering limit: $\rho_k = \lambda_{\text{max}}$.

Figure 4. Left panel: evolution of the particle spectrum with impulsive injection and no escape for the case of $R = 1 \times 10^{15}$ cm and $q = 2$. Upper panels represent the temporal evolution of $n(\gamma)$; lower panels represent the temporal evolution of $\gamma^3 n(\gamma)$. Solid lines represent the case of SSC cooling. Red and blue solid lines represent the final stage for $B = 1.0$ G and $B = 0.1$ G, respectively. Green solid lines represent the temporal evolution, for $B = 0.1$ G, with step of $0.8 \times t_D$. The dashed lines represent the temporal evolution of the only synchrotron cooling. The vertical dot-dashed lines represent the equilibrium energy in the case of only synchrotron cooling. Right panel: evolution of the curvature as a function of $\ell/t_D$. Upper panel: curvature $r$ evaluated at $\gamma_p$, for the case of SSC cooling (red and blue lines) and for the case of only synchrotron cooling (dashed red and blue lines). The solid green line represents the prediction from Equation (19). Lower panel: the same as in the upper panel, for the curvature $r_{\gamma p}$ evaluated at $\gamma_{\text{pp}}$ (open and filled circles) compared to the case of $r$ (solid lines).

(A color version of this figure is available in the online journal.)

Table 1

| Parameter | Impulsive Inj. | Cont. Inj. |
|-----------|----------------|------------|
| $R$ (cm)  | $5 \times 10^{15}, 1 \times 10^{15}$ | ... | ... |
| $B$ (G)   | 0.1, 1.0 | ... | ... |
| $L_{\text{mag}}$ (erg s$^{-1}$) | $10^{39}$ | $10^{39}$ | ... | ... |
| $q$       | 2 | 3/2 | 2 | 3/2 |
| $t_D0$ = 1/$D_{\text{p0}}$ (s) | $1 \times 10^4$ | $1 \times 10^3$ | $1 \times 10^4$ | $1 \times 10^3$ |
| $T_{\text{inj}}$ (s) | 100 | ... | $1 \times 10^4$ | ... |
| $T_{\text{esc}}$ | $R/c$ | $R/c$ | ... | $2$ |
| Duration (s) | $1 \times 10^5$ | ... | ... | ... |
| $\gamma_{\text{inj}}$ | 10.0 | 10.0 | ... | ... |

4.2. Impulsive Injection

In the left panels of Figure 4 and Figure 5, we plot the evolution of energy distribution $n(\gamma, t)$ (upper panels) and of $\gamma^3 n(\gamma, t)$ (lower panels) in the case of the impulsive injection without escape, for $q = 2$, and for two values of $R$: $1 \times 10^{15}$ cm (Figure 4) and $5 \times 10^{13}$ cm (Figure 5). We inject a quasi-monoenergetic electron distribution with $\gamma_{\text{inj}} \approx 10$. The $\gamma^3 n(\gamma, t)$ representation is useful to compare the results concerning $n(\gamma)$ presented in this section, with those regarding the synchrotron emission presented in Section 5. We denote by $\gamma_{\text{pp}}$ the peak energy of $n(\gamma)$ and by $r$ the curvature evaluated by means of a log-parabolic best fit over a one decade-broad interval centered at $\gamma_{\text{pp}}$. $\gamma_{\text{pp}}$ and $r_{\gamma p}$ represent the peak of $\gamma^3 n(\gamma)$ and its curvature, respectively. In the right panels of Figures 4 and 5, we report on the corresponding temporal evolutions of the curvatures under the effect of both momentum diffusion and cooling terms. The solid black line corresponds to $r = 0.2 \times t_{\text{acc}}$, where $t_{\text{acc}} = t_D0$ is the acceleration time due to momentum diffusion. As the time increases, the diffusion term acts on the distribution by means of both $D_A$ and $D_p$. The effect of the latter is to make the distribution broader.
One can distinguish three phases: in the first one the energy of particles increases and the curvature parameter decreases following a law $r \propto t^{-1}$ in agreement with the statistical scenario of Section 2 and with the Equation (19), independent of the magnetic field strength ($B = 1.0$ G and $B = 0.1$ G) and of the source size, because the accelerative contribution dominates over the radiative losses; in the second phase, the radiation losses become relevant and the distribution approaches the equilibrium with an increase of the curvature; and in the third phase, the balance between acceleration and radiation losses is established and the curvature reaches a stable value.

The equilibrium distribution reached through stochastic acceleration, is described by a relativistic Maxwellian (Schlickeiser 1985; Stawarz & Petrosian 2008),

$$n(\gamma) \propto \gamma^2 \exp \left[-\frac{1}{f(q, \gamma)} \left(\frac{\gamma}{\gamma_{eq}}\right)^{q(q, \gamma)}\right],$$

(26)

where $f(q, \gamma)$ is a function depending on the exponent of the diffusion coefficient and on the cooling process and $\gamma_{eq}$ is the Lorentz factor that satisfies the condition $U_{cool}(\gamma) = t_{acc}(\gamma)$ and is given by

$$\gamma_{eq} = \frac{1}{t_{acc} C_0 (U_B + F_{KN}(\gamma))},$$

(27)

with $t_{acc}$ equal to the fastest acceleration timescale among $t_A$, $t_D$, and $t_{DA}$. In the case of Compton-dominated cooling we have $\gamma_{eq} \propto (R^2/t_{acc} B^2 f_{KN})$, while in the case of strong KN regime, or in general for synchrotron-dominated cooling, we have $\gamma_{eq} \propto (1/t_{acc} B^2)$. Using a PL form for the acceleration terms, and in the case of only synchrotron losses (or any cooling process that can be expressed as a PL function of $\gamma$), it is possible to give an analytic expression of $f(q, \gamma)$ (Katarzynski et al. 2006; Stawarz & Petrosian 2008). The expectation for synchrotron and IC/TH cooling processes and for $q = 2$ is $f(q, \gamma) = 3 - q = 1$. The curvature resulting from a log-parabolic fit over a decade centered on $\gamma_p$ is $r \approx 2.5$ and $r_{3p} \approx 6.0$ in the case of $\gamma_{3p}$.

We first discuss the case of $R = 10^{15}$ cm (Figure 4) with only synchrotron cooling (dashed lines, left panels). In terms of behavior, we note that for the larger value of $B$ (1.0 G; red lines, right panels), the $r$--$t$ trend departs from the purely accelerative one ($r \propto t^{-1}$; green lines, right panels) early (relative to the $B = 0.1$ G case; blue lines in the right panels). This happens because the synchrotron equilibrium energy (vertical dot-dashed lines, left panels) is lower in the case of $B = 1.0$ G. For both values of $B$, the final values of $r$ are close to the synchrotron equilibrium value of $\approx 2.5$. When IC cooling is also taken into account, the final values of the curvature in $n(\gamma)$ are $r \approx 2.5$ and $r \approx 0.6$ for $B = 0.1$ G and $B = 1.0$ G, respectively. This difference is due to the different IC cooling regimes for the two cases. To show clearly the complexity of the transition from the TH to the KN regime, and its dependence on $R$ and $B$, in Figure 6 we plot the ratio $\gamma_{IC}/\gamma_{Synch}$ (solid lines), and $U_{ph}/U_B$ (dashed lines), as a function of $\gamma$ and normalized to unity, for the case of $q = 2$, for the final step of the temporal evolution. As long as the ratio $U_{ph}/U_B$ is close to $\gamma_{IC}/\gamma_{Synch}$, electrons cool in the full TH regime, and $C(\gamma) = C_0 \gamma^2(U_B + U_{ph})$. On the contrary, when the electrons radiate in the full KN regime $\gamma_{IC}/\gamma_{Synch} < U_{ph}/U_B$. In this case, due to the inefficient KN cooling regime we have $\gamma_{Synch} \gg \gamma_{IC}$, and the cooling term is dominated by the synchrotron component: $C(\gamma) \approx C_0 \gamma^2 U_B$. In the intermediate cases, it is difficult to estimate analytically the ratio $\gamma_{IC}/\gamma_{Synch}$.

For $B = 1.0$ G, the SSC equilibrium is reached at $\gamma \approx 3 \times 10^4$ and the SSC cooling occurs between the KN and TH regimes (see the top right panel in Figure 6), hence the value of $f$ is different from unity, as predicted for the case of full IC/TH or synchrotron cooling. When $B = 0.1$ G, the equilibrium energy is $\gamma \approx 10^4$ and electrons are in extreme KN cooling (see the top left panel in Figure 6), synchrotron losses are much higher than those due to IC scattering, and again $r$ reaches the previous value of $\approx 2.5$. It is also interesting to note the difference in the trends of $r$--$t$ and $r_{3p}$--$t$. In the latter case, the trend departs from the purely accelerative regime earlier (see Figure 4, lower right panel) since the electrons with energies close to $\gamma_{3p}$ are more energetic than those close to $\gamma_p$, and thus have much shorter cooling times.

The results for the compact region ($R = 5 \times 10^{13}$ cm) are plotted in Figure 5. Considering that the injected electron luminosity is the same (see Table 1), we expect a different...
response from the IC cooling, due to the higher photon density \( n_{ph}(\epsilon) \). The \( r \) evolution for the synchrotron cooling case is similar to the previous case, while for the SSC emission, both for the case of \( B = 1.0 \text{ G} \) and \( B = 0.1 \text{ G} \), the final value of \( r \) is about 2.5. This is due to the larger photon density which moves the IC scattering into the TH regime also for the case of \( B = 0.1 \text{ G} \) (compare bottom left to top left panels in Figure 6), hence \( n(\gamma) \) approaches the solution of Equation (26) in the case of \( f = 1 \).

In Figure 7, we show the temporal evolution for the case of \( q = 3/2 \) (\( R = 1.0 \times 10^{15} \text{ cm}, B = 0.1 \text{ G} \)). In this case, contrary to the \( q = 2 \) case, the acceleration time \( t_D \) is energy dependent, hence we study the evolution of \( r \) as a function of \( t/t_D(\gamma_{inj}) \), where \( t_D(\gamma_{inj}) \) is the diffusive acceleration time evaluated at the injection energy \( \gamma_{inj} \). The energy dependence of \( t_D \) affects the evolution of \( r \), and the shape of the equilibrium distribution, indeed, the \( r \sim t \) and \( r \sim t^3 \) trends show different behavior compared to the case of \( q = 2 \). The equilibrium curvature is reached for \( t \gtrsim 1 \times t_D(\gamma_{inj}) \), and the two equilibrium curvature values are \( r \approx 1.2 \) and \( r_3 \approx 3.0 \), roughly half of those found for the case of \( q = 2 \), and in agreement with the result from the MC. We note that the curvature obtained by means of a log-parabolic fit of Equation (26), for the case \( q = 3/2 \) (namely, \( f = 1.5 \)), is \( r \approx 3.7 \). Hence, both the MC and the numerical solution of the diffusion equation give a result different from that predicted by the analytical solution in Equation (26).

### 4.3. Continuous Injection

The case of continuous injection (see Figure 8) is more complex. The distribution develops a low-energy PL tail, but a log-parabolic bending, driven by the diffusion, is still present.
at high energies, hence we evaluate the curvature only at $\gamma_3$ (i.e., the representation useful to compare it to the synchrotron SED emission). Spectral curvatures are generally milder than the impulsive injection. In the left panel of Figure 8, we plot the $r-t$ trend both for the case of impulsive (red lines) and continuous (blue lines) injections, the curvature in the continuous injection case are systematically lower in the pre-equilibrium phases and in the acceleration-dominated stage the trend is again consistent with the “hard-sphere” approximation and statistical approaches. The slope of the electron distribution in the PL tail is $\approx 1.0$, in good agreement with the predicted one $\approx 1 + \log_{10} \frac{(t_{\text{min-acc}}/2t_{\text{acc}})}{10^3} = 1.075$, consistent with the results of Katarzyński et al. (2006).

5. EVOLUTION OF THE SPECTRAL PARAMETERS OF SYNCHROTRON AND IC EMISSION

The most relevant parameters describing the SED of SSC sources provided by observations are the peak energies and curvatures of the synchrotron and IC components. We denote these curvature parameters by $b_s$ and $b_c$, respectively, and by $E_s$, $E_c$, and $S_s$, $S_c$; we denote the corresponding SED peak energies and flux values. We use $\nu_s$ and $\nu_c$ to indicate the corresponding SED peak frequencies. In the following, we describe the results of the relations between these parameters assuming that electrons are injected into the acceleration region with a quasi mono-energetic spectrum with $\gamma_{\text{min}} \approx 10$ and using an injection time of $10^4$ s. We use the same working hypothesis for the momentum-diffusion coefficient as in Section 4.1 and add a systematic acceleration time for the first-order process $t_A = 1.5 \times 10^3$ s, in order to produce $E_s$ values ranging between optical and hard X-ray energies. We set the radius of the region at $R = 2 \times 10^{15}$ cm and the same duration for the injection and acceleration processes, namely, $10^4$ s. We varied the other parameters of the model, $B$, $q$, and $D_{p\delta}$ to verify how they affect the relation between the observable ones. All the parameters and their variation ranges are summarized in Table 2.

A phenomenological approach, based on the $\delta$-function approximation (Tramacere et al. 2007, 2009; Massaro et al. 2006, 2004), is useful to address the expected relation between the curvature parameters and their connections with the peak energies and flux values. According to the standard synchrotron theory (e.g., Rybicki & Lightman (1986)), in the $\delta$-function approximation, the synchrotron SED peak value and the corresponding peak energy can be expressed by the following relations:

$$\frac{S_s(E_s)}{E_s} \propto n(\gamma_3) \gamma_3^3 \delta^{\alpha/4}$$

$$E_s \propto \gamma_3^2 b_s \delta,$$

which implies

$$S_s \propto (E_s)^{\alpha/4},$$

where $\alpha = 1.5$ applies for changes of $\gamma_3$ leaving constant $n(\gamma_3)$, $\alpha = 2$ for variations of $B$ only, and $\alpha = 4$ when the main driver is $\delta$. For a log-parabolic-shaped $n(\gamma)$ we have

$$\log(\gamma_3) = \log(\gamma_p) + \frac{3}{2\beta}$$

and, using the relation $b_s \approx r/5$ (Massaro et al. 2004), or, more precisely, from the analysis presented in Section 4.2, $b_s \approx r_3/5$. It follows that

$$\log(E_s) \propto \log(\gamma_p) + \frac{3}{5b_s}.$$  

The relation between $b_s$ and $E_s$ is

$$b_s = \frac{a}{\log(E_s/E_0)},$$

with $a = 3/5 = 0.6$.

| Parameter | Range | Table 2 Parameters’ Values Adopted in the Numerical Solutions of the Diffusion Equation for the Cases Studied in Section 5 |
|-----------|-------|-------------------------------------------------------------------------------------------------------------------|
| $R$ (cm)  | $2 \times 10^{15}$ |                                                                                                                   |
| $B$ (G)   | $[0.01, 1.0]$ |                                                                                                                   |
| $L_{\text{inj}}$ (erg s$^{-1}$) | $10^{38}$ |                                                                                                                   |
| $q$       | $[3/2, 2]$ |                                                                                                                   |
| $t_A$ (s) | $1.8 \times 10^3$ |                                                                                                                   |
| $t_D_{0} = 1/D_{p\delta}$ (s) | $[1.5, 25] \times 10^3$ |                                                                                                                   |
| $t_{\text{inj}}$ (s) | $10^4$ |                                                                                                                   |
| $T_{\text{esc}}$ (R/c) | $2.0$ |                                                                                                                   |
| Duration (s) | $10^4$ |                                                                                                                   |
| $\gamma_{\text{inj}}$ | $10.0$ |                                                                                                                   |
The spectral properties of the IC emission are more complex, depending on the transition from the TH to the KN regime (see Massaro et al. 2006 for a detailed discussion). In the former case, the curvature is close to that of the synchrotron emission but systematically smaller due to the energy redistribution by the scattering process. In the transition to the KN regime, the energy of IC photons will approach $\gamma m_e c^2$, hence the IC spectral shape will reflect that of the high-energy tail of $\gamma(n)$ and the curvature $b_c$ will be closer to that of the electrons. Then, provided the IC scattering happens in the TH regime, the trends involving $b_c$ are expected to be similar to those of $b_s$, but systematically show $b_c < b_s$. As the KN regime is approached, $b_c$ changes differently from $b_s$, converging toward $r$.

### 5.1. Temporal Evolution of $b_s$ and $b_c$

We compute the evolution of $b_s$ and $b_c$, as a function of the time, for the case of $t_{\text{acc}} = 1.5 \times 10^4$ s, $B = 0.1$ G, and $q = 2$, using a temporal mesh of 2 s. We plot in the top left panel of Figure 9 the instantaneous SEDs at steps of 200 s: the solid lines represent the synchrotron and IC SEDs averaged over the full duration of the acceleration process (10$^4$ s). As the time is increased, the peak energy of both the synchrotron and IC SEDs moves toward higher energies with a broadening of the spectral distribution. The corresponding evolution of curvature parameters is reported in the right top panel: $b_s$ has a trend similar to that of the electron distribution, with $b_s \propto (t/t_{\text{acc}})^{-\alpha}$ and $\alpha \approx 0.6$ (for comparison the cyan solid line represents the $r_{3p}/5$ trend, as predicted by the S $\delta$-approximation). The trend of $b_c$, as expected, is more complex because of the transition from TH to KN regime. For $t/t_{\text{acc}} \lesssim 0.4$, it follows the same trend of $b_s$ but with systematically lower values. For $t/t_{\text{acc}} \gtrsim 0.4$, when the TH–KN transition occurs, $b_c$ increases with time, approaching toward the electron curvature $r$ value. This transition starts for values of $E_{\text{ic}} \approx 5 \times 10^{-3}$ keV ($\nu_0 \approx 10^{14}$ Hz) and $E_{\text{c}} \approx 0.05$ GeV ($\nu_0 \approx 10^{22}$ Hz), and the corresponding SEDs are plotted by blue thick-dashed lines in the left panel of Figure 9.

In the bottom panels of Figure 9, we show the case of $q = 3/2$. The synchrotron curvature quickly approaches the equilibrium value of $b_s \approx 0.6$, consistent with the equilibrium value $r_{3p} \approx 3.0$ discussed in Section 4.2. In this case we do not observe the TH/KN transition in the IC curvature, since the lower values of $E_{\text{c}}$ and $E_{\text{ic}}$ keep the IC scattering mainly in the TH regime.

### 5.2. $E_{\text{ic}}-S_{\text{c}}$ and $E_{\text{ic}}-b_c$ as a Function of $D_{\rho_0}$ and $q$

The other parameter affecting the evolution of the spectral distributions is the diffusion coefficient $D_{\rho_0}$ (see Equation (15)),
which we assume to vary in the range $[1.5 \times 10^2, 2.4 \times 10^5]$ s$^{-1}$, studying how the main spectral parameters change. In the top left panel of Figure 10, we plot averaged SEDs for each different value of $D_{\rho_0}$. The top right panel shows the trend of $b_c$ versus $D_{\rho_0}$. As expected, for larger values of $D_{\rho_0}$, the curvature measured at the peak energy is smaller. The trend is described by a PL with an exponent of about $-0.6$ for $D_{\rho_0} \lesssim 2 \times 10^{-5}$ s$^{-1}$ and with an exponent of about $-0.25$ for $D_{\rho_0} \gtrsim 2 \times 10^{-5}$ s$^{-1}$.

This break clearly shows the transition between the TH and KN regimes (marked by a vertical dashed line); indeed it happens for the same values of $D_{\rho_0}$ corresponding to the TH/KN transition in both the $D_{\rho_0}$--$b_c$ trend and the $E_c$--$b_c$ plot (occuring at $E_c \approx 1$ GeV; see the bottom right panel in Figure 10). The break in the $D_{\rho_0}$--$b_c$ trend happens when electrons radiating at $E_c$ enter the KN cooling region, hence, due to the lower cooling level (compared to the TH cooling regime, on the left side of the vertical dashed line), the curvature decreases.

Blue filled circles in the top left panel represent the peak positions for both SED components. For the synchrotron component, according to Equation (29), the exponent $\alpha$ in the case of $n(\gamma_{\nu}) = \text{const.}$ should be 1.5, while the results of the computations give $\alpha = 0.6$. This difference is due to the fact that we inject in the mono-energetic initial distribution always the same total power that corresponds to the same number of particles. When the peak energy increases the distribution becomes broader, implying that the same total number of particles is spread over a larger energy interval and the number of particles contributing to the synchrotron peak emission decreases. Consequently, the $S_c$--$E_c$ trend gets softer compared to the predicted value of 1.5.

We verified quantitatively this effect by computing the trend $n(\gamma_{\nu_{\nu}})$ versus $\gamma_{\nu_{\nu}}^2$, and found a PL relation with an exponent of about 0.98, in nice agreement with the difference between the exponent of 1.5 and that resulting in our simulations. In the bottom panels of Figure 10, we plot $b_c$ versus $E_c$ (left) and $b_c$ versus $E_c$ (right). The $S_c$--$E_c$ relation can be fitted by a PL (orange line, top left panel in Figure 10) with the same exponent of the $E_c$--$S_c$ relation, as long as the IC scattering, at $E_c$ and above, happens in TH regime. When the KN suppression becomes relevant (green line, top left panel in Figure 10), the exponent is larger and is close to unity.

The synchrotron trend (the bottom left panel in Figure 10) clearly shows the expected anti-correlation between the peak energy and the spectral curvature, which is well fit by the function given in Equation (32), with $a = 0.68$, not very different from 0.6, obtained for the $\delta$-function approximation of the synchrotron emission, and assuming that $n(\gamma)$ has a purely log-parabolic shape. A simple PL fit of the same points returns an exponent $-0.14$.

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**Figure 10.** Upper left panel: synchrotron (red lines) and IC (red lines) average SEDs for each different value of $t_{\text{th}}$ in the range reported in Table 2, with $q = 2$. Blue points represent the position of $E_{S,c}$ and $S_{S,c}$. The purple, orange, and green line represent the PL best fit of the $E_{S,S}$ and $E_{C,S}$ trends. Upper right panel: $b_c$ and $h_c$ for each average SED in the right panel, as a function of $D_{\rho_0}$. Dashed lines represent the PL best fit of the $b$--$D_{\rho_0}$ trend. Lower left panel: the $b_c$--$E_c$ trend obtained by means of a log-parabolic best fit of the averaged SEDs plotted in the upper right panel. Lower right panel: same as in the lower left panel, for $b_c$--$E_c$. (A color version of this figure is available in the online journal.)
We also investigate the role of $q$ on the spectral evolution, setting its variation range to $[3/2, 2]$, i.e., from the Kraichnan to the “hard-sphere” case. The relations between the spectral parameters are very similar to those found in the previous case with $S_{s} \propto E_{s}^{0.8}$ and $S_{c} \propto E_{c}^{0.9}$. Also in this case, the synchrotron component follows the expectation with a lower curvature for harder turbulence spectra, and the IC trend shows the transition from TH to KN regime. The PL best fit of $S(E_{s})$ versus $E_{s}$ gives $a = 0.88$, larger than that obtained for the case of $D_{p0}$. In fact, for values of $q$ lower than 2, corresponding to less turbulence and hence diffusion, the curvature gets higher values and the peak energy lower values, compared to the “hard-sphere” case. The PL fit for $b_{s}$ versus $E_{c}$ returns an exponent of $-0.16$, practically coincident with the previous one, indicating that the average properties of these parameters are the same in the both $q$ and $D_{p0}$ cases.

5.3. $E_{s,c} - S_{s,c}$ and $E_{s,c} - b_{s,c}$ as a Function of $B$

The magnetic field $B$ drives the radiative losses which affect the evolution of the spectral parameters. In Section 4, we showed that different cooling conditions, and the transition from TH to KN, can determine very different values of $\gamma_{eq}$ for the same acceleration conditions. Assuming that the acceleration timescale is independent of the magnetic field, Equation (27) shows that $\gamma_{eq} \propto 1/B^{2}$, implying that, as long as $B$ is small enough to result in $\gamma_{b} \ll \gamma_{eq}$, the evolution of $n(\gamma)$ around the peak value is dominated by the acceleration terms, while for values of $B$ resulting in $\gamma_{b} \gtrsim \gamma_{eq}$ the evolution obtains a notable contribution due to cooling. In the top left panel of Figure 11, we plot the averaged SEDs. According to Equations (28) and (29), the synchrotron peak value should scale as $S_{s} \propto (E_{s})^{2}$. Indeed, for values of $B \lesssim 0.2$ G we obtain an exponent equal to 2.04, very close to the value found with the $\delta$-approximation. For higher values of the magnetic field, $E_{s}$ is anti-correlated with $S_{s}$. This behavior represents a cooling signature due to the decreasing value $\gamma_{eq}$ for increasing $B$ values, with $\gamma_{eq}$ getting closer to $\gamma_{b}$. This is confirmed both by the shape of the synchrotron SEDs and by the $b_{s} - B$ plot (top right panel in Figure 11). Indeed, S SEDs for $B \gtrsim 0.2$ G exhibit an exponential decay, meaning that the distributions have reached, or are close to reaching, the equilibrium energy. Consistently with the S shape evolution, the $b_{s} - B$ relation shows an almost stable value of $b_{s}$ for $B \lesssim 0.2$ G and an increasing trend for $B \gtrsim 0.2$ G. This change, in both the $S_{s} - E_{s}$ and $b_{s} - B$ trends, is interesting and can provide a useful phenomenological tool for understanding the evolution of non-thermal sources. Another interesting feature is shown in the $S_{s} - E_{s}$ plot: for $B \lesssim 0.2$ G the IC peak energy is practically constant, as expected in the KN limit from the kinematical limit relating the scattered photons energy to that of the electrons: $h\nu_{ic} \approx \gamma m_{e}c^{2}$. In fact, photons at energies $\approx E_{c}$ are produced in the KN regime and for $B \lesssim 0.2$ G the electron peak energy $\gamma_{b}$ is constant, so $E_{c}$ must also be constant. For $B \gtrsim 0.2$ G, $\gamma_{b}$ decreases because of cooling, and, accordingly, $E_{c}$ also decreases. This is another interesting test that can provide
a probe for $B$-driven flares evolving to the KN regime. The $E_{\gamma} - b_{\gamma}$ plot in the bottom left panel of Figure 11 confirms the cooling signature discussed above, showing $b_{\gamma}$ uncorrelated with $E_{\gamma}$ as long as $\gamma_{3p} \ll \gamma_{eq}$ and an increasing value of $b_{\gamma}$ with $E_{\gamma}$ almost stable, when $\gamma_{3p} \gtrsim \gamma_{eq}$.

6. SPECTRAL EVOLUTION OF HIGH ENERGY FLARES OF BRIGHT HBL OBJECTS

The previous considerations on the spectral evolution of SSC sources, in which high energy electrons are accelerated in a relatively short timescale by stochastic processes, can be successfully applied to describe the behavior of some bright HBLs objects. These sources are, in fact, characterized by having the synchrotron peak in the UV/X-ray range and the IC peak in $\gamma$ rays up to TeV energies. Several flares, observed simultaneously in both these ranges, exhibited SEDs very well described by a log-parabolic law, whose parameters, particularly their curvature, are estimated with high accuracy. A similar analysis for low-energy peaked BL Lac objects is much more difficult because the peak of their synchrotron component is typically in the infrared range and the available simultaneous multifrequency data are extremely few. Tramacere et al. (2007, 2009) and Tramacere (2007) pointed out that the observed anticorrelation between $E_{\gamma}$ and $b_{\gamma}$ in the synchrotron SED of Mrk 421 can provide a clear signature of a stochastic component in the acceleration process. In the same analysis, these authors also presented an interesting correlation between $E_{\gamma}$ and $b_{\gamma}$ in the synchrotron SED of Mrk 421, which is variable. The peak of their synchrotron component is typically in the infrared range and the available simultaneous multifrequency data are extremely few. Tramacere et al. (2007, 2009) and Tramacere (2007) pointed out that the observed anticorrelation between $E_{\gamma}$ and $b_{\gamma}$ in the synchrotron SED of Mrk 421 can provide a clear signature of a stochastic component in the acceleration process. In the same analysis, these authors also presented an interesting correlation between $E_{\gamma}$ and $b_{\gamma}$ in the synchrotron SED of Mrk 421, which is variable.

In Figure 12, we report the scatter plot in the $E_{\gamma} - b_{\gamma}$ plane for the six considered sources. The left panel reports the results obtained by changing the value of $L_{\text{jet}}$, the green dashed lines describe the trend resulting from a log-parabolic fit of the synchrotron SED over a decade in energy centered on $E_{\gamma}$. The purple lines represent the same trend obtained by fitting a log-parabola in the fixed spectral window $[0.5, 100.0]$ keV. Both these trends are compatible with the data and track the predicted anticorrelation between $E_{\gamma}$ and $b_{\gamma}$. Purple data, however, give a better description, hinting that the “window” effect could be a real bias. Each of the three lines was computed for a different value of the magnetic field. It is remarkable that the variation of a single parameter, $D_{\text{ph}}$, can describe the observed behavior. The dispersion in the data is relevant and can be related to the variation of $B$ (as partially recovered by numerical computation), or by different values of the beaming factor, $R$, and $L_{\text{jet}}$, during different flares, and for different objects.

The dot-dashed thick line represents the best fit of the observed data by means of Equation (32), and returns a value of $a \approx 0.6$, as expected from theoretical predictions for the case of the $\delta$-approximation, and pure log-parabolic electron distribution. This fitted line is also compatible with the numerical trend shown by the purple lines. The observed curvature values are in the range $[0.1, 0.5]$, corresponding to $r_{3p} \sim [0.5, 3.0]$. According to the results presented in Section 4.2, the expected equilibrium curvature in the synchrotron emission, in the full KN or TH regime, and for $q = 2$, should be of $r_{3p} \approx 6.0$ and of $r_{3p} \approx 5.0$ in the intermediate regime. In the case of $q = 3/2$, the equilibrium curvature should be $r_{3p} \approx 3.0$. This is perhaps an interesting hint that, both in the flaring and the quiescent states, for $q = 2$, the distribution is always far from equilibrium. In the case of $q = 3/2$, only for $E_{\gamma} \lesssim 1.5$ keV is the curvature compatible with the equilibrium ($r_{3p} \approx 3.0$, corresponding to $b_{\gamma} \sim 0.6$). For larger values of $E_{\gamma}$, we find again curvature well below the equilibrium value. These results provide a good constraint on the values of the magnetic field $B \lesssim 0.1$ G.

The $q$-driven trend (right panel) is also compatible with the data, but for values of $E_{\gamma} \lesssim 1$ keV, the $D_{\text{ph}}$-driven case seems to take into account this possible bias in the observed data when $E_{\gamma}$ is variable.

### Table 3

| Parameter | $D$ Trend | $q$ Trend |
|-----------|-----------|-----------|
| $R$ (cm)  | $3 \times 10^{15}$ | ... |
| $B$ (G)   | [0.05, 0.2] | ... |
| $L_{\text{diss}} (E_{\gamma}, b_{\gamma}$ trend) (erg s$^{-1}$) | $5 \times 10^{39}$ | ... |
| $L_{\text{diss}} (E_{\gamma}, b_{\gamma}$ trend) (erg s$^{-1}$) | $5 \times 10^{38}, 5 \times 10^{39}$ | ... |
| $q$       | 2         | [3/2, 2] |
| $\tau_{\text{d}}$ (s) | $1.2 \times 10^{3}$ | ... |
| $\tau_{\text{dp}} = 1/D_{\text{ph}}$ (s) | $[1.5 \times 10^{4}, 1.5 \times 10^{5}]$ | $1.5 \times 10^{4}$ |
| $t_{\text{inj}}$ (s) | $10^{4}$ | ... |
| $T_{\text{esc}} (R/c)$ | 2.0 | ... |
| Duration (s) | $10^{4}$ | ... |
| $\gamma_{\text{inj}}$ | 10.0 | ... |
describe better the observed behavior, but any firm conclusion is not possible because of the dispersion of the data.

6.2. $E_s$--$L_s$ Trend

As a last benchmark for the stochastic acceleration model, we reproduce the observed correlation between $E_s$ and $S_s$, which follows naturally from the variations of $D_{\rho\theta}$ and $q$. Considering that the redshifts of the six considered HBL objects are different, we prefer to use their peak luminosity $L_s = S_s4\pi D_t^2$, where $D_t$ is the luminosity distance. To account for the different jet power of sources, we considered two data subsets, and we assumed $L_{\text{inj}} = 5 \times 10^{39}$ erg s$^{-1}$ for the first subset (top panels of Figure 13) and $L_{\text{inj}} = 5 \times 10^{38}$ for the second (bottom panels of Figure 13). In the left panels of Figure 13, we report the $D_{\rho\theta}$-driven trend and in the right panels we show the $q$-driven trend. Solid lines represent the trend obtained by deriving $L_s$ from the log-parabolic best fit of the numerically computed SEDs, centered on $E_s$; dashed lines are the trends obtained by fitting the numerical results in the fixed energy window [0.5, 100] keV.

Both results give a good description of the observed data and their shapes are similar. Solid lines follow well a PL with an exponent of about 0.6, while the windowed trends (dashed lines) show a break around 1 keV and the exponent below this energy turns to about 1.5. A similar break at the same energy can be noticed in the points of Mrk 421 in the $E_s$--$S_s$ plot presented by Tramacere et al. (2009), who found an exponent of $\sim$1.1 and of $\sim$0.4 below and above 1 keV, respectively. This could again be an indication that the observed values are actually affected by the bias.

7. DISCUSSION

Broadband observations of non-thermal sources have shown that the spectral curvature at the peaks of their SEDs can now be measured with good accuracy. In this paper, we have presented, using different approaches, the relevance of these data for the understanding of the competition between statistical acceleration and radiation losses. First, using a simple statistical approach and MC calculations, we have shown that the log-parabolic energy distribution of the relativistic electron is a good picture in the first phases before equilibrium is reached. In this case, the curvature decreases with time and, therefore, with increasing peak energies. This evolution is confirmed by numerical solutions of the diffusion equation taking properly into account both stochastic acceleration and radiative SSC cooling. The major results can be summarized as follows.

The evolution of the electron energy distributions (Section 4) shows that:

1. In the case of synchrotron and SSC cooling, and for all the values of $B$ and $R$, as long as the distribution is far from equilibrium, the trend on $r$ is dictated by $D_p$ and is well described by Equation (19).

2. When the distributions approach equilibrium, the value of $r$ is determined by the shape of the equilibrium distribution, which is a relativistic Maxwellian, with the sharpness of the cutoff determined by both $q$ and the IC cooling regime.

3. In the case of $q = 2$, and for equilibrium energies implying that IC cooling happens either in the TH regime or in the extreme KN regime (IC cooling negligible compared to the synchrotron one), the numerical solution of the diffusion equation follows the analytical prediction ($f = 1$, that holds for any $\gamma \propto \gamma^2$), and the corresponding equilibrium curvature is $r_{3p} \approx 6.0$ ($b_1 \approx 1.2$). In the case of $q = 3/2$, the equilibrium curvature is $r_{3p} \approx 3.0$ ($b_2 \approx 0.6$). These limiting values could be a useful observational test to find cooling-dominated flares with the distribution approaching to the equilibrium.

4. When cooling is in the intermediate regime between TH and KN and for the $q = 2$ case, the condition $f = 1$ fails, and the end values of $r$ decrease, strongly depending on the balance between $U_p$ and the seed IC photon energy ($U_{ph}$) numerical computations are necessary to evaluate the right value of $r$ at equilibrium.

The analysis of the spectral evolution of SSC emission (Section 5) shows that:

1. Changes of $D_{\rho\theta}$ (or $q$) imply that the curvature and peak energy of the synchrotron emission are anticorrelated; the $E_s$--$b_s$ trend can be phenomenologically described by Equation (32).

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4 We used a flat cosmology model with $H_0 = 73$ km s$^{-1}$ Mpc$^{-1}$, $\Omega_{\text{matter}} = 0.27$, and $\Omega_{\text{vacuum}} = 0.73$. 

Figure 12. Left panel: the $E_s$--$b_s$ trend observed for the six HBLs in our sample. The dashed green lines represent the trend reproduced by the stochastic acceleration model, for the parameters reported in Table 3 and for the $D$ trend; the different lines corresponding to three different values of $B$ reported in Table 3. The purple lines represent the trend obtained by fitting the numerically computed SED over a fixed spectral window in the range 0.5–100 keV. Right panel: the same as in the left panel for the case of the $q$ trend.

(A color version of this figure is available in the online journal.)
2. The $E_{\gamma}$-$b_5$ trend presents a clear signature of the transition from the TH to the KN regime. In particular, when the IC scattering approaches the KN regime we observe a sharp change in the $b_5$, with a positive correlation with $E_{\gamma}$, while in the TH regime the correlation is negative as in the case of the $E_{\gamma}$-$b_2$.

3. The magnetic field plays a relevant role on the cooling process and B-driven variations present relevant differences compared to those due to $D_{\rho 0}$ (and $q$).

In particular, for the B-driven case, we note first that the $E_{\gamma}$-$S_x$ correlation follows the prediction of the synchrotron theory and shows the PL relationship with $E_{\gamma} \propto (S_x)^{2.0}$. On the contrary, in the case of $D_{\rho 0}$ and $q$ changes, we find $E_{\gamma} \propto (S_x)^{0.5}$. Another relevant difference in the B-driven case is the evolution of $S_x$. For the case of $D_{\rho 0}$- and $q$-driven trends, $S_x$ relates to $E_{\gamma}$ through a PL with exponent of about $[0.7, 0.8]$. On the contrary, for the B-driven case with IC scattering in the full KN regime, the value of $E_{\gamma}$ is almost constant and uncorrelated with $S_x$ (see Figure 11) due to the kinematical limit of the KN regime. $E_{\gamma}$ starts to decrease when $B$ is enough large to make the cooling process dominant. This is an interesting signature that could be easily checked in the observed data.

The comparison of the $E_{\gamma}$-$b_5$ and $E_{\gamma}$-$S_x$ trends, obtained through several X-ray observations of six HBL objects spanning a period of many years, with those predicted by the stochastic acceleration model, shows very good agreement. We are able to reproduce these long-term behaviors by changing the value of only one parameter ($D_{\rho 0}$ or $q$). Interestingly, the $E_{\gamma}$-$S_x$ relation follows naturally from that between $E_{\gamma}$ and $b_5$. This result is quite robust and hints at a common accelerative scenario acting in the jets of HBLs.

As a last remark, we note that very recently Massaro & Grindlay (2011) also find that in the case of GRBs a $E_{\gamma}$-$b_5$ trend similar to that observed in the case of HBL objects. They measured values of the curvature up to 1.0, typically higher than in HBLs. It is interesting to note that the value of 1.0 is close to the limit of $\sim 1.2$ that we predict in the case of distributions approaching the equilibrium in either TH or KN regime for $q = 2$.

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