UMD BANACH SPACES AND SQUARE FUNCTIONS ASSOCIATED WITH
HEAT SEMIGROUPS FOR SCHRÖDINGER AND LAGUERRE OPERATORS

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Abstract. In this paper we define square functions (also called Littlewood-Paley-Stein func-
tions) associated with heat semigroups for Schrödinger and Laguerre operators acting on func-
tions which take values in UMD Banach spaces. We extend classical (scalar) $L^p$-boundedness
properties for the square functions to our Banach valued setting by using $\gamma$-radonifying op-
erators. We also prove that these $L^p$-boundedness properties of the square functions actually
characterize the Banach spaces having the UMD property.

1. Introduction

Suppose that $(\Omega, \mu)$ is a measure space and $\{T_t\}_{t>0}$ is an analytic semigroup on
$L^p(\Omega, \mu)$, where $1 \leq p \leq \infty$. If $k \in \mathbb{N}$, the $k$-th vertical square function
$g^k(\{T_t\}_{t>0})(f)$ of $f \in L^p(\Omega, \mu)$ is defined by

$$g^k(\{T_t\}_{t>0})(f)(x) = \left( \int_0^\infty \left| t^k \partial^k_t T_t(f)(x) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

The $L^p$-boundedness properties of $g^k$-square functions are very useful in order to describe the
behavior in $L^p$-spaces of multipliers associated to the infinitesimal generator of the semigroup
$\{T_t\}_{t>0}$ (see [26], [29] and [33]).

It is well-known ([29, p. 120]) that if $\{T_t\}_{t>0}$ is the classical heat or Poisson semigroup then,
for every $1 < p < \infty$,

$$\|g^k(\{T_t\}_{t>0})(f)\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

This property can be extended to other semigroups of operators (see [25], [29], [33], [39], amongst
others).

In the sequel we denote as usual by $\{W_t\}_{t>0}$ and $\{P_t\}_{t>0}$ the classical heat and Poisson
semigroup on $\mathbb{R}^n$, respectively. We have that, for every $t > 0$ and $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$,

$$W_t(f)(x) = \int_{\mathbb{R}^n} e^{-|x-y|^2/(4t)} \left( \frac{t}{4\pi} \right)^{n/2} f(y) dy, \quad x \in \mathbb{R}^n,$$

and

$$P_t(f)(x) = c_n \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}} f(y) dy, \quad x \in \mathbb{R}^n,$$

being $c_n = \pi^{-(n+1)/2} \Gamma((n+1)/2)$.

If $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a measurable function on $\mathbb{R}^n$, we define $\psi_t(x) = t^{-n} \psi(x/t)$, $x \in \mathbb{R}^n$ and $t > 0$. Then, it is clear that, for every $t > 0$, $W_t(f) = G_{\sqrt{t}} * f$ and $P_t(f) = P_t * f$, where
Let $\mathbb{B}$ be a Banach space and $1 < p < \infty$. Then, the following assertions are equivalent.

(i) $\mathbb{B}$ is isomorphic to a Hilbert space.

(ii) For every $f \in L^p(\mathbb{R}^n, \mathbb{B})$,

$$
\|g^k_\mathbb{B} (\{P_t\}_{t>0}) (f)(x)\|_{L^p(\mathbb{R}^n, \mathbb{B})} \sim \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}.
$$

Other authors ([19], [20], [22] and [23]) have extended the definition of the $g$-square functions to a Banach valued setting by different points of view. As one of their goals, they wanted to extend the equivalence in (1) to Banach spaces which are not isomorphic to Hilbert spaces. Hytönen [19] extended (1) to a UMD Banach space setting by using Banach-valued stochastic integration. On the other hand, Kaiser and Weis [23] generalized (4) to functions taking values in UMD Banach spaces by using $\gamma$-radonifying operators. These two approaches are closely connected (see, for instance, [38] and [37]). In this paper we use $\gamma$-radonifying operators to study $g$-square functions associated with the heat semigroups for Schrödinger and Laguerre operators in UMD Banach spaces.

The main properties of UMD Banach spaces can be encountered in [6], [7] and [27].
Suppose that $H$ is a separable Hilbert space and $B$ is a real Banach space. We take a sequence $(\gamma_k)_{k \in \mathbb{N}}$ of independent standard Gaussians. We say that an operator $T$ bounded from $H$ to $B$, shortly $T \in L(H, B)$, is $\gamma$-radonifying, written $T \in \gamma(H, B)$, when

$$\|T\|_{\gamma(H, B)} = \left( \mathbb{E} \left\| \sum_{k=1}^{\infty} \gamma_k T(h_k) \right\|_B^2 \right)^{1/2} < \infty,$$

where $\{h_k\}_{k \in \mathbb{N}}$ is an orthonormal basis in $H$. If $B$ is a Banach space not containing a copy of $c_0$ (that is the case of UMD spaces), then

$$\|T\|_{\gamma(H, B)} = \sup \left( \mathbb{E} \left\| \sum_{k=1}^{\infty} \gamma_k T(h_k) \right\|_B^2 \right)^{1/2},$$

where the supremum is taken over all the finite families $\{h_k\}$ of orthonormal functions in $H$ \cite[Theorem 5.9]{35}). In the sequel by $H$ we denote the space $L^2((0, \infty), dt/t)$.

If $f : (0, \infty) \longrightarrow B$ is a strongly $\mu$-measurable function such that, for every $L \in B^*$, $L \circ f \in H$, then there exists $T_f \in L(H, B)$ such that

$$\langle L, T_f(h) \rangle = \int_0^\infty (L, f(t))/_{B^*, B} h(t) \frac{dt}{t}, \quad h \in H \text{ and } L \in B^*.$$

We say that $f \in \gamma((0, \infty), dt/t, B)$ provided that $T_f \in \gamma(H, B)$. We identify $f$ with $T_f$. If $B$ does not contain a copy of $c_0$ then $\gamma((0, \infty), dt/t, B)$ is a dense subspace of $\gamma(H, B)$ \cite[Remark 2.16]{23}). In the sequel we assume that $B$ is UMD. Then, $B$ does not contain a copy of $c_0$.

In \cite[Theorem 4.2]{23} Kaiser and Weis gave conditions over the function $\psi$ in order to the wavelet transform $W_\psi$ satisfies the following equivalence:

$$\|W_\psi(f)\|_{L^p(\mathbb{R}^n, \gamma(H, B))} \sim \|f\|_{L^p(\mathbb{R}^n, B)},$$

for every $f \in L^p(\mathbb{R}^n, B)$ and $1 < p < \infty$. Note that, since $\gamma(H, \mathbb{C}) = H$, \cite[4]{} reduces to \cite[2]{} when $B = \mathbb{C}$. Then, \cite[4]{} can be seen as an extension of \cite[1]{} when we consider the classical heat or Poisson semigroups and functions taking values in a UMD Banach space.

In this paper we extend the equivalence \cite[1]{} to a UMD-Banach valued setting for the heat semigroup defined by Schrödinger operator in $\mathbb{R}^n$, $n \geq 3$, the Hermite operator on $\mathbb{R}^n$, $n \geq 1$, and the Laguerre operator on $(0, \infty)$. Then, we prove that these new equivalences allow us to characterize the UMD Banach spaces.

The Schrödinger operator $\mathcal{L}$ is defined by $\mathcal{L} = -\Delta + V$ in $\mathbb{R}^n$, $n \geq 3$, where $\Delta$ is the Euclidean Laplacian in $\mathbb{R}^n$ and $V$ is a nonnegative measurable function in $\mathbb{R}^n$. Here we assume that $V \in RH_s(\mathbb{R}^n)$, that is, $V$ satisfies the following $s$-reverse Hölder’s inequality: there exists $C > 0$ such that, for every ball $B$ in $\mathbb{R}^n$,

$$\left( \int_B V(x)^s dx \right)^{1/s} \leq C \int_B V(x) dx,$$

where $s > n/2$. If $E_{\mathcal{L}}$ represents the spectral measure associated with the operator $\mathcal{L}$, the heat semigroup of operators generated by $-\mathcal{L}$ is denoted by $\{W_t^{\mathcal{L}}\}_{t > 0}$, where

$$W_t^{\mathcal{L}}(f) = \int_{(0, \infty)} e^{-\lambda t} E_{\mathcal{L}}(d\lambda)f, \quad f \in L^2(\mathbb{R}^n).$$
We can write, for every \( f \in L^2(\mathbb{R}^n) \),

\[
W^\alpha_t(x,y) = \int_{\mathbb{R}^n} W^\alpha_t(x,y)f(y)dy, \quad x \in \mathbb{R}^n \text{ and } t > 0.
\]  

The main properties of the kernel function \( W^\alpha_t(x,y) \), \( t > 0, x, y \in \mathbb{R}^n \), can be encountered in [9] and [28]. Also, for every \( t > 0 \), the operator \( W^\alpha_t \) defined in (6) is bounded from \( L^p(\mathbb{R}^n) \) into itself, \( 1 \leq p \leq \infty \). Thus, \( \{W^\alpha_t\}_{t>0} \) is a positive semigroup of contractions in \( L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \).

The Hermite (also called harmonic oscillator) operator \( \mathcal{H} = -\Delta + |x|^2 \) is a special case of the Schrödinger operator. Here we consider \( \mathcal{H} \) on \( \mathbb{R}^n \), with \( n \geq 1 \). We define, for every \( k \in \mathbb{N}_0 \), the \( k \)-th Hermite function \( h_k \) by

\[
h_k(x) = (\sqrt{\pi}2^k k!)^{-1/2}e^{-x^2/2}H_k(x), \quad x \in \mathbb{R},
\]

where by \( H_k \) we denote the \( k \)-th Hermite polynomial ([31 pp. 105–106]). If \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \) the \( k \)-th Hermite function \( h_k \) is defined by

\[
h_k(x) = \prod_{j=1}^n h_{k_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]

The system \( \{h_k\}_{k \in \mathbb{N}^n} \) is orthonormal and complete in \( L^2(\mathbb{R}^n) \). Moreover, \( \mathcal{H}h_k = (2|k| + n)h_k \), where \( |k| = k_1 + \ldots + k_n \) and \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \). The operator \( -\mathcal{H} \) generates in \( L^2(\mathbb{R}^n) \) the semigroup of operators \( \{W^\mathcal{H}_t\}_{t>0} \) where, for every \( t > 0 \),

\[
W^\mathcal{H}_t(f) = \sum_{k \in \mathbb{N}^n} e^{-t(2|k|+n)}c_k(f)h_k, \quad f \in L^2(\mathbb{R}^n),
\]

being

\[
c_k(f) = \int_{\mathbb{R}^n} h_k(y)f(y)dy, \quad k \in \mathbb{N}^n \text{ and } f \in L^2(\mathbb{R}^n).
\]

According to the Mehler’s formula ([32 (1.1.36)]) we can write, for every \( t > 0 \),

\[
W^\mathcal{H}_t(f)(x) = \int_{\mathbb{R}^n} W^\mathcal{H}_t(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^n, \mathcal{B}),
\]

where, for each \( x, y \in \mathbb{R}^n \) and \( t > 0 \),

\[
W^\mathcal{H}_t(x,y) = \frac{1}{\pi^{n/2}} \left( \frac{e^{-2t}}{1 - e^{-4t}} \right)^{n/2} \exp \left[ -\frac{1}{4} \left( |x-y|^2 - 1 - e^{-2t} + |x+y|^2 1 - e^{-2t} \right) \right].
\]

By defining \( W^n_t \), for every \( t > 0 \), on \( L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), by means of (7), then the system \( \{W^n_t\}_{t>0} \) is a positive semigroup of contractions in \( L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \).

Since \( \{W^\alpha_t\}_{t>0} \) and \( \{W^n_t\}_{t>0} \) are positive, they have tensor extensions to \( L^p(\mathbb{R}^n, \mathcal{B}) \) satisfying the same \( L^p \)-boundedness properties.

If \( \ell = 1, 2 \) and \( f \in L^p(\mathbb{R}^n, \mathcal{B}), 1 < p < \infty \), we define

\[
\mathcal{G}_{\mathcal{L},\ell}(f)(x,t) = t^{\ell/2} \delta_t^\ell W^\alpha_1(f)(x), \quad x \in \mathbb{R}^n, \text{ } t > 0, \text{ } n \geq 3,
\]

and

\[
\mathcal{G}_{\mathcal{H},\ell}(f)(x,t) = t^{\ell/2} \delta_t^\ell W^\mathcal{H}_1(f)(x), \quad x \in \mathbb{R}^n, \text{ } t > 0, \text{ } n \geq 1.
\]

Let \( \alpha > -1/2 \). The Laguerre operator \( \mathcal{L}_\alpha \) is defined by

\[
\mathcal{L}_\alpha = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - 1/4}{x^2} \right), \quad x \in (0, \infty).
\]

If \( k \in \mathbb{N} \) we consider the \( k \)-th Laguerre function

\[
\varphi^n_k(x) = \left( \frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)} \right)^{1/2} e^{-x^2/2}x^\alpha L_k^n(x^2), \quad x \in (0, \infty),
\]
where $L^k_\alpha$ represents the $k$-th Laguerre polynomial (31 pp. 100–102). The family $\{\varphi^\alpha_k\}_{k \in \mathbb{N}}$ is orthonormal and complete in $L^2(0, \infty)$. Moreover, for every $k \in \mathbb{N}$,

$$L^\alpha \varphi^\alpha_k = (2k + \alpha + 1) \varphi^\alpha_k.$$  

The semigroup of operators $\{W^\alpha_t\}_{t > 0}$ generated by $-L^\alpha$ in $L^2(0, \infty)$ is defined by

$$W^\alpha_t(f) = \sum_{k=0}^{\infty} e^{-t(2k+\alpha+1)} c_k^\alpha(f) \varphi^\alpha_k, \quad t > 0 \text{ and } f \in L^2(0, \infty),$$

where $c_k^\alpha(f) = \int_0^\infty \varphi^\alpha_k(y)f(y)dy$, $k \in \mathbb{N}$.

According to the Mehler’s formula ([33, (1.1.47)]) we can write, for every $t > 0$,

$$(8) \quad W^\alpha_t(f)(x) = \int_0^\infty W^\alpha_t(x,y)f(y)dy, \quad f \in L^2(0, \infty),$$

where, for each $x, y, t \in (0, \infty)$

$$W^\alpha_t(x,y) = \left(\frac{2e^{-t}}{1 - e^{-2t}}\right)^{1/2} \left(\frac{2xye^{-t}}{1 - e^{-2t}}\right)^{1/2} I_\alpha \left(\frac{2xye^{-t}}{1 - e^{-2t}}\right) \exp \left[-\frac{1}{2}(x^2 + y^2)\left(\frac{1}{1 - e^{-2t}}\right)\right],$$

and $I_\alpha$ denotes the modified Bessel function of the first kind and order $\alpha$.

If we define, for every $t > 0$, $W^\alpha_t$ on $L^p(0, \infty)$, $1 \leq p \leq \infty$ by [5], then $\{W^\alpha_t\}_{t > 0}$ is a positive semigroup of contractions in $L^p(0, \infty)$, $1 \leq p \leq \infty$. Moreover, for every $t > 0$, $W^\alpha_t$ can be extended to $L^p((0, \infty), \mathbb{B})$ preserving the $L^p$-boundedness properties.

If $\ell = 1, 2$ we consider

$$G^\ell_{\alpha, \mathbb{B}}(f)(x,t) = t^\ell \partial^\ell_t W^\alpha_t(f)(x), \quad x, t \in (0, \infty),$$

for every $f \in L^p((0, \infty), \mathbb{B})$, $1 < p < \infty$.

We now establish the main result of this paper.

**Theorem 1.1.** Let $\mathbb{B}$ be a Banach space and $\alpha > -1/2$. The following assertions are equivalent.

(a) $\mathbb{B}$ is UMD.

(b) For $\ell = 1, 2$ and for every (equivalently, for some) $1 < p < \infty$,

$$\|G^\ell_{\alpha, \mathbb{B}}(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \sim \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}), \quad n \geq 1.$$  

(c) For $\ell = 1, 2$ and for every (equivalently, for some) $1 < p < \infty$,

$$\|G^\ell_{\alpha, \mathbb{B}}(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \sim \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}), \quad n \geq 3.$$  

(d) For $\ell = 1, 2$ and for every (equivalently, for some) $1 < p < \infty$,

$$\|G^\ell_{\alpha, \mathbb{B}}(f)\|_{L^p((0, \infty), \gamma(H, \mathbb{B}))} \sim \|f\|_{L^p((0, \infty), \mathbb{B})}, \quad f \in L^p((0, \infty), \mathbb{B}).$$

Note that, since $\gamma(H, \mathbb{C}) = H$, the equivalences in Theorem 1.1 (b), (c) and (d) are Banach valued versions of the corresponding scalar equivalences (see [5], [32], [33] Chapter 4 and [38]).

In [11] we study square functions associated to the subordinated Poisson semigroup for the Hermite operator in a Banach valued setting. By using auxiliary operators and Cauchy-Riemann type equations adapted to the Hermite setting we characterized the UMD Banach spaces. We remark that, as it can be observed in [19], [25] and [39], in order to describe geometric properties of Banach spaces (UMD, $q$-martingale type and cotype,...) by using square functions, subordinated (Poisson) diffusion semigroups must be considered. Moreover, in [19], Hytönen dealt with diffusion semigroups and the semigroups $\{W^H_t\}_{t > 0}$, $\{W^\ell_t\}_{t > 0}$ and $\{W^\alpha_t\}_{t > 0}$ are not diffusion semigroups because they are not conservative. Then, in particular the results in [11] are not
covered by the ones in [19]. The results obtained by Hytönen for general diffusion semigroups in a UMD setting are weaker than the ones got for subordinated diffusion semigroups ([19, Theorem 5.1]). In order to get a better result for every diffusion semigroups Hytönen reduced the admisible class of Banach spaces. He considered the class of Banach spaces which are isomorphic to a closed subspace of a complex interpolation space \([Z, Y]_\theta\) where \(Z\) is a Hilbert space, \(Y\) is a UMD Banach space and \(0 < \theta < 1\). We write \(\zeta\) to refer this class of Banach spaces. \(\zeta\) contains all the standard UMD spaces. In [27] Rubio de Francia posed the question whether the equality \(\zeta = \text{UMD}\) holds. As far as we know this question remains open.

In contrast with the results in [19] we get Theorem 1.1 for the semigroups \(\{W^\mathcal{H}_t\}_{t>0}\), \(\{W^\mathcal{L}_t\}_{t>0}\) and \(\{W^\mathcal{L}_\alpha\}_{t>0}\) which are not diffusion semigroups and, as it was above mentioned, they are not conservative. In order to prove Theorem 1.1 we use a procedure different to the one used in [19].

For establishing that if \(\mathcal{B}\) is a UMD Banach space the equivalences in (b), (c) and (d) hold, we take advantage of the following fact: close to singularities, our operators are good perturbations of the corresponding operators associated with the Laplacian operator. The exact meaning of this idea is clear in the proof. Then, we use [23, Theorem 4.2]. To see that the equivalences in (b), (c) and (d) imply that \(\mathcal{B}\) is UMD, we have taken into account that the UMD Banach spaces are characterized by the \(L^p\)-boundedness properties of the imaginary powers \(\mathcal{H}^\gamma, \mathcal{L}^\gamma, \mathcal{L}_\alpha^\gamma, \gamma > 0\) of \(\mathcal{H}, \mathcal{L}\) and \(\mathcal{L}_\alpha\), respectively ([2, Theorem 1.2] and [3, Theorem 3]).

In the next sections we prove our result for the Hermite operator in \(\mathbb{R}^n, n \geq 1\) (Section 2), the Schrödinger operators in \(\mathbb{R}^n, n \geq 3\) (Section 3) and the Laguerre operators in \((0, \infty)\) (Section 4).

Throughout this paper by \(C\) and \(c\) we always denote positive constants that can change in each occurrence.

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### 2. Proof of Theorem 1.1 for the Hermite operator

In this section we prove \((a) \iff (b)\) in Theorem 1.1.

#### 2.1. \((a) \Rightarrow (b)\)

Let \(\ell = 1, 2, n \geq 1\) and \(1 < p < \infty\). We define \(\mathcal{G}^\ell_{-\Delta, \mathcal{B}}(f)\), for every \(f \in L^p(\mathbb{R}^n, \mathcal{B})\), as follows

\[
\mathcal{G}^\ell_{-\Delta, \mathcal{B}}(f)(x, t) = t^\ell \partial^\ell_t W_t(f)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0.
\]

Assume that \(\mathcal{B}\) is a UMD Banach space.

We start proving that

\[
\|\mathcal{G}^\ell_{H, \mathcal{B}}(f)\|_{L^p(\mathbb{R}^n, \mathcal{H}, \mathcal{B})} \leq C\|f\|_{L^p(\mathbb{R}^n, \mathcal{B})}, \quad f \in L^p(\mathbb{R}^n, \mathcal{B}).
\]

Let \(f \in L^p(\mathbb{R}^n, \mathcal{B})\). We can write

\[
\partial^\ell_t W^\mathcal{H}_t(f)(x) = \int_{\mathbb{R}^n} \partial^\ell_t W_t^\mathcal{H}(x, y)f(y)dy, \quad x \in \mathbb{R}^n \text{ and } t > 0.
\]
Lemma 2.1. Let
\[ S(x) = \frac{1}{2\pi e^{-0.04x^2}} \exp \left[ -\frac{1}{4} \left( |x-y|^2 + |y|^2 \right) \right] \]
\[ \frac{e^{-2t}}{1 - e^{-4t}} \]
and
\[ \frac{e^{-2t}}{(1 - e^{-2t})^2} \]
for every \( x, y \in \mathbb{R}^n \) and \( t > 0 \),
\[ \partial_t W_t^H(x, y) = \frac{1}{\pi^{n/2}} \left( \frac{e^{-2t}}{1 - e^{-4t}} \right)^{n/2} \exp \left[ -\frac{1}{4} \left( |x-y|^2 + |y|^2 \right) \right] \]
\[ \frac{e^{-2t}}{(1 - e^{-2t})^2} \]
and
\[ \frac{8n e^{-4t}}{(1 - e^{-4t})^2} - |x-y|^2 \]
\[ \frac{2e^{-2t}(1 + e^{-2t})}{(1 - e^{-2t})^3} \]
for every \( x, y \in \mathbb{R}^n \) and \( t > 0 \).

Estimation (12) justifies the derivation under the integral sign in (9).

We split the operators \( G_{H,B}^\ell \) and \( G_{-\Delta,B}^\ell \) as follows. We write, for \( B = H \) or \( B = -\Delta \),
\[ G_{Q,B}^\ell = G_{Q,B,loc}^\ell + G_{Q,B,\text{glob}}^\ell, \]
where
\[ G_{Q,B,loc}^\ell(f)(x,t) = G_{Q,B}^\ell(\chi_B(x,t))f(y)(x,t), \]
\[ \rho(x) = \begin{cases} 
\frac{1}{2}, & |x| \leq 1 \\
\frac{1}{1+|x|}, & |x| > 1 
\end{cases} \]
for every \( x \in \mathbb{R}^n \), \( \rho(x) \) is called the critical radius in \( x \) (see [28] p. 516).

We consider the following decomposition of the operator \( G_{H,B}^\ell \):
\[ G_{H,B}^\ell = \sum_{j=1}^3 T_{j,B}^\ell, \]
where \( T_{1,B}^\ell = G_{H,B,loc}^\ell - G_{-\Delta,B,loc}^\ell \), \( T_{2,B}^\ell = G_{H,B,\text{glob}}^\ell \), \( T_{3,B}^\ell = G_{-\Delta,B,loc}^\ell \).

Lemma 2.1. Let \( B \) be a UMD Banach space and \( j = 1, 2, 3 \). Then, there exists \( C > 0 \) verifying that
\[ \|T_{j,B}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H,B))} \leq C\|f\|_{L^p(\mathbb{R}^n, B)}, \quad f \in L^p(\mathbb{R}^n, B). \]

Proof of Lemma 2.7 for \( T_{3,B}^\ell \). We consider \( \varphi^\ell(x) = \left( \partial_t^\ell G_{j,B}^\ell(x) \right)_{t=1} \) for \( x \in \mathbb{R}^n \). Thus, \( \varphi^\ell \in S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \), where \( S(\mathbb{R}^n) \) denotes the Schwartz class.

Moreover, \( \varphi^\ell \) satisfies conditions
\((C1)\) and \((C2)\) in \([23]\) p. 111. Indeed, according to \([13]\) p. 121, \((23)\) we have that

\[
\tilde{\varphi}^t(y) = \int_{\mathbb{R}^n} e^{-ix \cdot y} \left[ \partial_t \left( \frac{e^{-|x|^2/(4t)}}{(4\pi t)^{n/2}} \right) \right]_{t=1} \, dx \\
= \partial_t \left[ \int_{\mathbb{R}^n} e^{-ix \cdot y} \frac{e^{-|x|^2/(4t)}}{(4\pi t)^{n/2}} \, dx \right]_{t=1} \\
= \partial_t \left( -|y|^2 \right)_{t=1} = (-|y|^2)^t e^{-|y|^2}, \quad y \in \mathbb{R}^n.
\]

Now, straightforward manipulations allow us to see that the conditions \((C1)\) and \((C2)\) in \([23]\) p. 111 are satisfied by \(\varphi^t\). On the other hand, \(\varphi^t(x) = t^{-n} \varphi^t(x/t) = \left( s^t \partial_t^r G_n(x) \right)_{|s| = t^2}, \quad x \in \mathbb{R}^n, \) and \(t > 0\). Note that if \(\{h_n\}_{n \in \mathbb{N}}\) is an orthonormal basis in \(H\), then \(\{h_n(\sqrt{t})/\sqrt{2}\}_{n \in \mathbb{N}}\) is also an orthonormal basis in \(H\). Hence,

\[
\|G^\ell_{\Delta,B}(g)(x, \cdot)\|_{\gamma(H,B)} = \sqrt{2}\|\langle \varphi^* g \rangle(x)\|_{\gamma(H,B), \B}, \quad g \in S(\mathbb{R}^n, B) \text{ and } x \in \mathbb{R}^n.
\]

Hence, by invoking \([23]\) Theorem 4.2 there exists a bounded operator \(\tilde{\mathcal{G}}^\ell_{\Delta,B}\) from \(L^p(\mathbb{R}^n, B)\) into \(L^p(\mathbb{R}^n, \gamma(H,B))\) such that

\[
\tilde{\mathcal{G}}^\ell_{\Delta,B}(g) = G^\ell_{\Delta,B}(g), \quad g \in S(\mathbb{R}^n, B).
\]

Let \(f \in L^p(\mathbb{R}^n, B)\). We are going to see that \(\tilde{\mathcal{G}}^\ell_{\Delta,B}(f) = G^\ell_{\Delta,B}(f)\). In order to do this we choose a sequence \((f_m)_{m=1}^{\infty} \subset C^\infty_c(\mathbb{R}^n) \otimes B\) such that \(f_m \rightarrow f\), as \(m \rightarrow \infty\), in \(L^p(\mathbb{R}^n, B)\). Note that \(C^\infty_c(\mathbb{R}^n) \otimes B \subset S(\mathbb{R}^n, B)\) is a dense subset of \(L^p(\mathbb{R}^n, B)\). It can be shown that

\[
(15) \quad |t^\ell \partial^\ell_{s,t} G_n(x-y)| \leq C \frac{e^{-c|x-y|^2/t}}{t^{n/2}}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.
\]

Then, for every \(N \in \mathbb{N}\) and \(x \in \mathbb{R}^n\), there exists \(C_N > 0\) for which

\[
\|G^\ell_{\Delta,B}(f)(x, \cdot) - G^\ell_{\Delta,B}(f_m)(x, \cdot)\|_{L^2((1/N,N), \text{wrt } B)} \\
\leq \int_{\mathbb{R}^n} \|f(y) - f_m(y)\| \|t^\ell \partial^\ell_{s,t} G_n(x-y)\|_{L^2((1/N,N), \text{wrt } B)} \, dy \\
\leq C_N \int_{\mathbb{R}^n} \|f(y) - f_m(y)\| \left( \int_{1/N}^{N} \frac{1}{(t + |x-y|^2)^{(n-1)/2}} \, dt \right)^{1/2} \, dy \\
\leq C_N \int_{\mathbb{R}^n} \|f(y) - f_m(y)\| \left( \frac{1}{(1/N + |x-y|^2)^{n/2}} \right) \, dy \\
\leq C_N \|f - f_m\|_{L^p(\mathbb{R}^n, B)} \left( \int_{\mathbb{R}^n} \frac{1}{(1/\sqrt{N} + |x-y|)^{np'}} \, dy \right)^{1/p'} \\
\leq C_N \|f - f_m\|_{L^p(\mathbb{R}^n, B)}, \quad m \in \mathbb{N}.
\]

Hence, for every \(N \in \mathbb{N}\) and \(x \in \mathbb{R}^n\),

\[
G^\ell_{\Delta,B}(f_m)(x, \cdot) \rightarrow G^\ell_{\Delta,B}(f)(x, \cdot), \quad \text{as } m \rightarrow \infty \text{ in } L^2((1/N,N), \frac{dt}{t}; B).
\]

On the other hand,

\[
G^\ell_{\Delta,B}(f_m) \rightarrow \tilde{G}^\ell_{\Delta,B}(f), \quad \text{as } m \rightarrow \infty \text{ in } L^p(\mathbb{R}^n, \gamma(H,B)).
\]

Then, there exists a subsequence of \((f_m)_{m=1}^{\infty}\) which we continue denoting by \((f_m)_{m=1}^{\infty}\), satisfying

\[
G^\ell_{\Delta,B}(f_m)(x, \cdot) \rightarrow \tilde{G}^\ell_{\Delta,B}(f)(x), \quad \text{as } m \rightarrow \infty \text{ in } \gamma(H,B),
\]
for every \( x \in \mathbb{N} \), where \( \mathbb{N} \subset \mathbb{R}^n \) and \(|\mathbb{R}^n \setminus \mathbb{N}| = 0\). Since \( \gamma(H,\mathbb{B}) \) is continuously contained in \( L(H,\mathbb{B}) \), we have that, for every \( x \in \mathbb{N} \),

\[
\mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f_m)(x,\cdot) \longrightarrow \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x), \quad \text{as } m \to \infty \text{ in } L(H,\mathbb{B}).
\]

Let \( x \in \mathbb{N} \). We choose \( h \in H \) such that its support is compact and contained in \((0,\infty)\). For every \( S \in \mathbb{B}^* \) we can write

\[
\langle S, [\mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x)](h) \rangle_{\mathbb{B}^*,\mathbb{B}} = \lim_{m \to \infty} \langle S, [\mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f_m)(x,\cdot)](h) \rangle_{\mathbb{B}^*,\mathbb{B}}
\]

\[
= \langle S, \int_0^\infty \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x,t)h(t) \frac{dt}{t} \rangle_{\mathbb{B}^*,\mathbb{B}} = \int_0^\infty \langle S, \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x,t) \rangle_{\mathbb{B}^*,\mathbb{B}} h(t) \frac{dt}{t}.
\]

Moreover,

\[
\left| \int_0^\infty \langle S, \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x,t) \rangle_{\mathbb{B}^*,\mathbb{B}} h(t) \frac{dt}{t} \right| = \left| \langle S, [\mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x)](h) \rangle_{\mathbb{B}^*,\mathbb{B}} \right| \leq \|S\|_{\mathbb{B}^*} \|\mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x)\|_{L(U,\mathbb{B})} \|h\|_{\mathbb{B}}.
\]

We conclude that \( \langle S, \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x,\cdot) \rangle_{\mathbb{B}^*,\mathbb{B}} \in H \) and

\[
\langle S, [\mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x)](w) \rangle_{\mathbb{B}^*,\mathbb{B}} = \int_0^\infty \langle S, \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x,t) \rangle_{\mathbb{B}^*,\mathbb{B}} w(t) \frac{dt}{t}, \quad w \in H.
\]

Thus we prove that \( \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x) = \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(f)(x,\cdot) \) as elements of \( \gamma(H,\mathbb{B}) \).

We now use the ideas developed in [12] Proposition 2.3] to see that [14] holds for \( j = 2 \). According to [9] Proposition 5, for every \( M > 0 \) there exists \( C > 0 \) such that

\[
(16) \quad \frac{1}{C} \leq \frac{\rho(x)}{\rho(y)} \leq C, \quad x \in B(y, M \rho(y)).
\]

We can find a sequence \((x_k)_{k=1}^\infty\) such that

(i) \( \bigcup_{k=1}^\infty B(x_k, \rho(x_k)) = \mathbb{R}^n \),

(ii) For every \( M > 0 \) there exists \( m \in \mathbb{N} \) such that, for each \( j \in \mathbb{N} \),

\[
\text{card} \{ k \in \mathbb{N} : B(x_k, M \rho(x_k)) \cap B(x_j, M \rho(x_j)) \neq \emptyset \} \leq m.
\]

Let \( k \in \mathbb{N} \). If \( x \in B(x_k, \rho(x_k)) \), then [16] implies that \( |y - x_k| \leq \rho(x) + \rho(x_k) \leq C_0 \rho(x_k) \), provided that \( y \in B(x, \rho(x)) \). Here \( C_0 > 0 \) does not depend on \( k \in \mathbb{N} \). We can write for every \( x \in B(x_k, \rho(x_k)) \) and \( t > 0 \),

\[
\mathcal{G}^\ell_{-\Delta,\mathbb{B},\text{loc}}(f)(x,t) = \mathcal{G}^\ell_{-\Delta,\mathbb{B}} \left( \chi_{B(x_k, C_0 \rho(x_k))} f \right)(x,t) + \mathcal{G}^\ell_{-\Delta,\mathbb{B}} \left( \left( \chi_{B(x, \rho(x))} - \chi_{B(x_k, C_0 \rho(x_k))} \right) f \right)(x,t)
\]

\[
= \mathcal{G}^\ell_{-\Delta,\mathbb{B}} \left( \chi_{B(x_k, C_0 \rho(x_k))} f \right)(x,t) - \mathcal{G}^\ell_{-\Delta,\mathbb{B}} \left( \chi_{B(x_k, C_0 \rho(x_k))} \setminus B(x, \rho(x)) \right) f(x,t).
\]

Let \( x \in B(x_k, \rho(x_k)) \). We consider the operator

\[
L_k,x(f)(t) = \mathcal{G}^\ell_{-\Delta,\mathbb{B}} \left( \chi_{B(x_k, C_0 \rho(x_k)) \setminus B(x, \rho(x))} f \right)(x,t), \quad t > 0.
\]

By [15] we have that

\[
\| \partial^\ell_\nu \mathcal{G}^\ell_{-\Delta,\mathbb{B}}(x-y) \|_H \leq C \left( \int_0^\infty e^{-c|x-y|^2/t} \frac{dt}{t^{n+1}} \right)^{1/2} \leq \frac{C}{|x-y|^n}, \quad y \in \mathbb{R}^n \setminus \{x\}.
\]
Hence, for every \( y \notin B(x, \rho(x)) \), the function \( g_{x,y}(t) = t^d G_{\gamma}(x - y) \), \( t \in (0, \infty) \), belongs to \( H \) and \( \|g_{x,y}\|_H \leq C/\rho(x)^n \). Then, \( L_{k,x}(f) \in L^2((0, \infty), dt/t; B) \) and

\[
\|L_{k,x}(f)\|_{L^2((0, \infty), dt/t; B)} \leq \frac{C}{\rho(x)^n} \int_{B(x_0, C_0 \rho(x_k))} \|f(y)\|_B dy.
\]

Hence, \( L_{k,x}(f) \in \gamma(H, B) \). Indeed, by (3) we have

\[
\|L_{k,x}(f)\|_{\gamma(H, B)} = \sup \left( \left| \mathbb{E} \left| \sum_{j=1}^m \alpha_j \int_0^\infty L_{k,x}(f)(t) h_j(t) \frac{dt}{t} \right|^2 \right|_B \right)^{1/2},
\]

where \((\alpha_j)_{j=1}^\infty\) is a sequence of independent standard Gaussian random variables and the supremum is taken over all the finite families \( \{h_j\} \) of orthonormal functions in \( H \). Suppose that \((h_j)_{j=1}^m\) is an orthonormal set in \( H \). We can write

\[
\left( \mathbb{E} \left| \sum_{j=1}^m \alpha_j \int_0^\infty L_{k,x}(f)(t) h_j(t) \frac{dt}{t} \right|^2 \right)^{1/2} = \left( \mathbb{E} \left| \sum_{j=1}^m \alpha_j \int_0^\infty \int_{B(x_0, C_0 \rho(x_k))} g_{x,y}(t) f(y) dy h_j(t) \frac{dt}{t} \right|^2 \right)^{1/2} \leq \int_{B(x_0, C_0 \rho(x_k))} \|f(y)\|_B \left( \mathbb{E} \left| \sum_{j=1}^m \alpha_j \int_0^\infty g_{x,y}(t) h_j(t) \frac{dt}{t} \right|^2 \right)^{1/2} dy
\]

\[
\leq \int_{B(x_k, C_0 \rho(x_k))} \|f(y)\|_B \int_{B(x_0, C_0 \rho(x_k))} \|g_{x,y}\|_{\gamma(H, C)} dy
\]

\[
= \int_{B(x_k, C_0 \rho(x_k))} \|f(y)\|_B \|g_{x,y}\|_H dy
\]

\[
\leq \frac{C}{\rho(x)^n} \int_{B(x_k, C_0 \rho(x_k))} \|f(y)\|_B dy.
\]

Hence,

\[
\|L_{k,x}(f)\|_{\gamma(H, B)} \leq \frac{C}{\rho(x)^n} \int_{B(x_0, C_0 \rho(x_k))} \|f(y)\|_B dy.
\]

By using (16), we deduce that \( B(x_k, C_0 \rho(x_k)) \subset B(x, C_1 \rho(x)) \), where \( C_1 \) does not depend on \( k \) neither on \( x \). Then, we get

\[
\|L_{k,x}(f)\|_{\gamma(H, B)} \leq \frac{C}{\rho(x)^n} \int_{B(x, C_1 \rho(x))} \|f(y)\|_B dy \leq M(\|f\|_B)(x),
\]

where \( M \) denotes the Hardy-Littlewood maximal function.
According to the classical maximal theorem and the boundedness of $G_{\Delta, B}^t$ from $L^p(\mathbb{R}^n, \gamma(H, B))$ into $L^p(\mathbb{R}^n, \gamma(H, B))$, we obtain
\[
\|G_{\Delta, B}^t, \text{loc}(f)\|_{L^p(\mathbb{R}^n, \gamma(H, B))} \leq C \sum_{k=1}^{\infty} \int_{B(x_k, \rho(x_k))} \|G_{\Delta, B}^t, \text{loc}(f)\|_{L^p(\gamma(H, B))} dx
\]
\[
\leq C \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}^n} \|G_{\Delta, B}^t \chi(\mathbb{B}_{(x_k, C_0 \rho(x_k))}) (x, \cdot)\|_{L^p(\gamma(H, B))}^p dx \right)
\]
\[
+ \int_{\mathbb{R}^n} \|G_{\Delta, B}^t \chi(\mathbb{B}_{(x_k, C_0 \rho(x_k))}) \mathbb{B}_{(x, \rho(x))} (x, \cdot)\|_{L^p(\gamma(H, B))}^p dx
\]
\[
\leq C \left( \sum_{k=1}^{\infty} \int_{B(x_k, C_0 \rho(x_k))} \|f(y)\|_B^p dy + \int_{\mathbb{R}^n} |\mathcal{M}(\|f\|_B)|x|| \right)
\]
\[
\leq C \int_{\mathbb{R}^n} \|f(y)\|_B^p dy.
\]
We conclude that (14) holds for $T_{t, B}$. \hfill \square

**Proof of Lemma 2.1 for $T_{t, B}^\dag$.** By using the perturbation formula ([9, (5.25)]) we can write
\[
\partial_t [G_{\sqrt{t}}(x - y) - W_t^H(x, y)] = \int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 \left[ \partial_u G_\sqrt{t} \mathbb{H}(x-z) \right]_{u=t-s} W_t^H(z, y) dz ds
\]
\[
+ \int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 G_\sqrt{t} \left[ \partial_u W_t^H(z, y) \right]_{u=t-s} dz ds
\]
\[
+ \int_{\mathbb{R}^n} |z|^2 G_\sqrt{t} \mathbb{H}(x-z) W_t^{H/2}(z, y) dz = H_t^1(x, y, t) + H_t^2(x, y, t) + H_t^3(x, y, t), \quad x, y \in \mathbb{R}^n \text{ and } t > 0.
\]

Then, it follows that
\[
\partial_t^2 [G_{\sqrt{t}}(x - y) - W_t^H(x, y)] = \int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 \left[ \partial^2_u G_\sqrt{t} \mathbb{H}(x-z) \right]_{u=t-s} W_t^H(z, y) dz ds
\]
\[
+ \int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 G_\sqrt{t} \left[ \partial^2_u W_t^H(z, y) \right]_{u=t-s} dz ds
\]
\[
+ \int_{\mathbb{R}^n} |z|^2 \left[ \partial_u G_\sqrt{t} \mathbb{H}(x-z) \right]_{u=t/2} W_t^{H/2}(z, y) + G_\sqrt{t} \mathbb{H}(x-z) \left[ \partial_u W_t^H(z, y) \right]_{u=t/2} dz
\]
\[
= H_t^1(x, y, t) + H_t^2(x, y, t) + H_t^3(x, y, t), \quad x, y \in \mathbb{R}^n \text{ and } t > 0.
\]

The following estimation will be very useful in the sequel. For every $x \in \mathbb{R}^n$ and $t > 0$, we get
\[
\int_{\mathbb{R}^n} e^{-\epsilon |y|^2/t} |y|^2 dy \leq C \int_{\mathbb{R}^n} e^{-\epsilon |z|^2/t} (|z|^2 + |x|^2) dz \leq C t^{n/2} (t + |x|^2).
\]

Then, we obtain
\[
(17) \quad \int_{\mathbb{R}^n} e^{-\epsilon |y|^2/t} |y|^2 dy \leq C \frac{t^{n/2}}{\rho(x)^2}, \quad x \in \mathbb{R}^n \text{ and } 0 < t \leq \rho(x)^2.
\]

Minkowski’s inequality leads to
\[
\|T_{t, B}^f(f)(x, \cdot)\|_{L^2((0, \infty), \mathbb{R}, \mathfrak{B})} \leq \int_{B(x, \rho(x))} \|f(y)\|_B \left( \int_0^\infty \|t \partial_t^2 [G_{\sqrt{t}}(x-y) - W_t^H(x, y)]\|^2 dt \right)^{1/2} dy
\]
\[
\leq C \sum_{j=1}^{3} \int_{B(x, \rho(x))} \|f(y)\|_B \left( \int_0^\infty \|t H_j^t(x, y, t)^2 dt \right)^{1/2} dy, \quad x \in \mathbb{R}^n.
\]
We now study
\[ A_j^f(x, y) = \left( \int_0^{\rho(x)^2} |t^f H_j^f(x, y, t)|^2 \frac{dt}{t} \right)^{1/2}, \quad x, y \in \mathbb{R}^n \quad \text{and} \quad j = 1, 2, 3. \]

According to (12), (15), (16) and (17) we get
\[
\left( \int_0^{\rho(x)^2} |t^f H_j^f(x, y, t)|^2 \frac{dt}{t} \right)^{1/2} \leq \frac{C}{\rho(x)^2} \left( \int_0^{\rho(x)^2} \left( \int_{\mathbb{R}^n} |z|^2 e^{-c|x-y|^2/t} \frac{dz}{s^{n/2}} \right)^2 dt \right)^{1/2} \leq \frac{C}{\sqrt{\rho(x)}} |x-y|^{n-1/2}, \quad x, y \in \mathbb{R}^n, y \in B(x, \rho(x)), x \neq y.
\]

Also by taking into account (12) and again (17) it follows that
\[
\left( \int_0^{\rho(x)^2} |t^f H_3^f(x, y, t)|^2 \frac{dt}{t} \right)^{1/2} \leq \frac{C}{\rho(x)^2} \left( \int_0^{\rho(x)^2} \left( \int_{\mathbb{R}^n} |z|^2 e^{-c|x-y|^2/t} \frac{dz}{s^{n/2}} \right)^2 dt \right)^{1/2} \leq \frac{C}{\sqrt{\rho(x)}} |x-y|^{n-1/2}, \quad x, y \in \mathbb{R}^n, x \neq y,
\]

and
\[
\left( \int_0^{\rho(x)^2} |t^f H_3^f(x, y, t)|^2 \frac{dt}{t} \right)^{1/2} \leq \frac{C}{\rho(x)^2} \left( \int_0^{\rho(x)^2} \left( \int_{\mathbb{R}^n} |z|^2 e^{-c|x-y|^2/t} \frac{dz}{s^{n/2}} \right)^2 dt \right)^{1/2} \leq \frac{C}{\sqrt{\rho(x)}} |x-y|^{n-1/2}, \quad x, y \in \mathbb{R}^n, x \neq y.
\]

By combining the above estimations we obtain
\[
\sum_{j=1}^3 \int_{B(x, \rho(x))} \|f(y)\|_B A_j^f(x, y) dy \leq C \int_{B(x, \rho(x))} \frac{\|f(y)\|_B}{\sqrt{\rho(x)}} |x-y|^{n-1/2} dy \leq C \sum_{m=0}^{\infty} \frac{1}{\sqrt{\rho(x)}} \int_{2^{-m-1} \rho(x) \leq |x-y| < 2^{-m} \rho(x)} \|f(y)\|_B |x-y|^{n-1/2} dy \leq C \sum_{m=0}^{\infty} b(x)^{n/2} (2^{-m(n-1/2)}) \|f\|_B dy \leq C \sum_{m=0}^{\infty} \frac{1}{2^{m/2}} M(\|f\|_B)(x), \quad x \in \mathbb{R}^n.
\]
On the other hand, (12) and (15) lead to

\[
\int_{B(x,\rho(x))} \|f(y)\| B \left( \int_{\rho(x)^2}^{\infty} |t^\ell \partial_t^j [G\sqrt{T}(x-y) - W^2_{T}(x,y)]|^2 \frac{dt}{t} \right)^{1/2} dy \\
\leq C \int_{B(x,\rho(x))} \|f(y)\| B \left( \int_{\rho(x)^2}^{\infty} \frac{e^{-c|x-y|^2/t}}{tn+1} dt \right)^{1/2} dy \\
\leq C \left( \int_{\rho(x)^2}^{\infty} \frac{1}{tn+1} dt \right)^{1/2} \int_{B(x,\rho(x))} \|f(y)\| B dy \\
\leq \frac{1}{\rho(x)^n} \int_{B(x,\rho(x))} \|f(y)\| B dy
\]

(19)

\[C M(||f||_B(x)), \quad x \in \mathbb{R}^n.\]

From (18), (19) we conclude that

\[\|T^f_{1,\mathbb{B}}(f)(x,\cdot)\|_{L^2((0,\infty), \mathbb{F}; \mathbb{B})} \leq CM(||f||_B)(x), \quad x \in \mathbb{R}^n.\]

Then, \(T^f_{1,\mathbb{B}}(f)(x,\cdot) \in \gamma(H,\mathbb{B})\) and by proceeding as above we show that

\[\|T^f_{1,\mathbb{B}}(f)(x,\cdot)\|_{\gamma(H,\mathbb{B})} \leq CM(||f||_B)(x), \quad x \in \mathbb{R}^n.\]

Classical maximal theorems leads to

\[\|T^f_{1,\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n, \gamma(H,\mathbb{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}.\]

**Proof of Lemma 2.4 for \(T^f_{2,\mathbb{B}}\).** By taking into account the following estimation (see [4] (4.4) and (4.5))

\[\exp \left[ -c \left( |x-y|^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + |x+y|^2 \frac{1 - e^{-2t}}{1 + e^{-2t}} \right) \right] \leq C \exp \left[ -c(|x| + |y|)|x-y| \right], \quad x, y \in \mathbb{R}^n \text{ and } t > 0,
\]

we get, by using (10) and (11),

\[|t^\ell \partial_t^j W^2_T(x,y)| \leq C \frac{e^{-c(|x|+|y|)|x-y|} e^{-c|x-y|^2/t}}{tn^{1/2}}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.
\]

Hence, Minkowski’s inequality allows us to write

\[\|T^f_{2,\mathbb{B}}(f)(x,\cdot)\|_{L^2((0,\infty), \mathbb{F}; \mathbb{B})} \leq \int_{|x-y| > \rho(x)} \|f(y)\| B \left( \int_0^{\infty} |t^\ell \partial_t^j W^2_T(x,y)|^2 \frac{dt}{t} \right)^{1/2} dy \\
\leq C \int_{|x-y| > \rho(x)} \|f(y)\| B e^{-c(|x|+|y|)|x-y|} \left( \int_0^{\infty} \frac{e^{-c|x-y|^2/t}}{tn+1} dt \right)^{1/2} dy \\
\leq C \int_{|x-y| > \rho(x)} \|f(y)\| B \frac{e^{-c(|x|+|y|)|x-y|}}{|x-y|^n} dy \\
\leq C \sum_{m=0}^{\infty} \frac{1}{(2m^2 \rho(x))^n} \int_{|x-y| < 2^{m+1} \rho(x)} \|f(y)\| B e^{-c(|x|+|y|)^2/2m^2 \rho(x)} dy, \quad x \in \mathbb{R}^n.
\]

Note that if \(|x-y| > \rho(x)\), then

\[(|x| + |y|)\rho(x) \geq |x-y|\rho(x) > \rho(x)^2 = \frac{1}{4}, \quad \text{when } |x| \leq 1,
\]

and

\[(|x| + |y|)\rho(x) \geq \frac{|x|}{1 + |x|} > \frac{1}{2}, \quad \text{when } |x| > 1.
\]
Hence,
\[
\|T_{2,\mathcal{B}}^f(f)(x,\cdot)\|_{L^p((0,\infty),\mathcal{B})} \leq C \sum_{m=0}^{\infty} e^{-c2^m} \int_{|x-y| \leq 2^{m+1}r(x)} \|f(y)\|_\mathcal{B} dy \\
\leq C\mathcal{M}(\|f\|_\mathcal{B})(x), \quad x \in \mathbb{R}^n,
\]
and we get that \(T_{2,\mathcal{B}}^f(f)(x,\cdot) \in \mathcal{G}(H,\mathcal{B}), \quad x \in \mathbb{R}^n, \quad \) and
\[
\|T_{2,\mathcal{B}}^f(f)(x,\cdot)\|_{\mathcal{G}(H,\mathcal{B})} \leq C\mathcal{M}(\|f\|_\mathcal{B})(x), \quad x \in \mathbb{R}^n.
\]
Maximal theorem implies now that (14) holds for \(T_{2,\mathcal{B}}^f\).

We conclude that there exists \(C > 0\) independent of \(f\) for which
\[
\|G_{\mathcal{H},\mathcal{B}}^f(f)\|_{L^p(\mathbb{R}^n,\mathcal{G}(H,\mathcal{B}))} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathcal{B})}.
\]

Our next objective is to establish that
\[
\|f\|_{L^p(\mathbb{R}^n,\mathcal{B})} \leq C\|G_{\mathcal{H},\mathcal{B}}^f(f)\|_{L^p(\mathbb{R}^n,\mathcal{G}(H,\mathcal{B}))},
\]
where \(C > 0\) does not depend on \(f\).

In order to show this, we prove the following polarization formula.

**Proposition 2.1.** Let \(\mathcal{B}\) be a UMD Banach space, \(1 < q < \infty\) and \(k \in \mathbb{N}\). For every \(f \in L^q(\mathbb{R}^n) \otimes \mathcal{B}\) and \(g \in L^q(\mathbb{R}^n) \otimes \mathcal{B}^*\), we have that
\[
\int_{\mathbb{R}^n} \int_0^{\infty} \langle t^k \partial_t^k W_t^H(g)(x), t^k \partial_t^k W_t^H(f)(x) \rangle_{\mathcal{B}^*} \frac{dt}{t} dx = \frac{\Gamma(2k)}{2^{2k}} \int_{\mathbb{R}^n} g(x) f(x) dx.
\]

**Proof.** This property can be proved by using standard spectral arguments. Indeed, if \(f, g \in \text{span}\{h_m\}_{m \in \mathbb{N}_n}\), we have that
\[
\int_{\mathbb{R}^n} \int_0^{\infty} t^k \partial_t^k W_t^H(g)(x) t^k \partial_t^k W_t^H(f)(x) \frac{dt}{t} dx = \frac{\Gamma(2k)}{2^{2k}} \int_{\mathbb{R}^n} g(x) f(x) dx.
\]
Since \(\text{span}\{h_m\}_{m \in \mathbb{N}_n}\) is dense in \(L^q(\mathbb{R}^n)\), by taking into account that, for every \(1 < r < \infty\),
\[
\|g^k(\{W_t^H\}_{t>0})(f)\|_{L^r(\mathbb{R}^n)} = \|G_{\mathcal{H},\mathcal{B}}^k(f)\|_{L^r(\mathbb{R}^n,\mathcal{B})} \leq C\|f\|_{L^r(\mathbb{R}^n)}, \quad f \in L^r(\mathbb{R}^n),
\]
we conclude that
\[
\int_{\mathbb{R}^n} \int_0^{\infty} t^k \partial_t^k W_t^H(f)(x) t^k \partial_t^k W_t^H(g)(x) \frac{dt}{t} dx = \frac{\Gamma(2k)}{2^{2k}} \int_{\mathbb{R}^n} g(x) f(x) dx,
\]
for every \(f \in L^q(\mathbb{R}^n)\) and \(g \in L^q(\mathbb{R}^n)\).

From (22) we can immediately deduce that (21) holds for every \(L^p(\mathbb{R}^n) \otimes \mathcal{B}\) and \(g \in L^q(\mathbb{R}^n) \otimes \mathcal{B}^*\).

Assume now that \(F \in L^p(\mathbb{R}^n) \otimes \mathcal{B}\). According to [10, Lemma 2.3] we have that
\[
\|F\|_{L^p(\mathbb{R}^n,\mathcal{B})} = \sup_{g \in L^q(\mathbb{R}^n) \otimes \mathcal{B}^*, \|g\|_{L^q(\mathbb{R}^n,\mathcal{B})} \leq 1} \left\|\int_{\mathbb{R}^n} \langle g(x), F(x) \rangle_{\mathcal{B}^*} dx\right\|.
\]
Then, since $\mathcal{B}^*$ is also a UMD Banach space, by using Proposition 2.1 [21] Proposition 2.2 and (20) we obtain

$$
\|F\|_{L^p(\mathbb{R}^n, \mathcal{B})} = \frac{2^{\text{e}}}{\Gamma(2^\text{e})} \sup_{\|g\|_{L^p(\mathbb{R}^n, \mathcal{B}^*)} \leq 1} \left| \int_0^\infty \int_{\mathbb{R}^n} (G_{\mathcal{H}, \mathcal{B}^*}^t(g)(x, t), G_{\mathcal{H}, \mathcal{B}^*}^t(F)(x, t))_{\mathcal{B}^*} \, dx \, dt \right|
$$

$$
\leq C \sup_{\|g\|_{L^p(\mathbb{R}^n, \mathcal{B}^*)} \leq 1} \int_{\mathbb{R}^n} \left| \left| G_{\mathcal{H}, \mathcal{B}^*}^t(g)(x, \cdot) \right|_{\gamma(\mathcal{H}, \mathcal{B}^*)} \right| \left| G_{\mathcal{H}, \mathcal{B}^*}^t(F)(x, \cdot) \right|_{\gamma(\mathcal{H}, \mathcal{B})} \, dx
$$

$$
\leq C \sup_{\|g\|_{L^p(\mathbb{R}^n, \mathcal{B}^*)} \leq 1} \|G_{\mathcal{H}, \mathcal{B}^*}^t(g)\|_{L^p(\mathbb{R}^n, \gamma(\mathcal{H}, \mathcal{B}^*))} \|G_{\mathcal{H}, \mathcal{B}^*}^t(F)\|_{L^p(\mathbb{R}^n, \gamma(\mathcal{H}, \mathcal{B}))}
$$

$$
\leq C \|G_{\mathcal{H}, \mathcal{B}^*}^t(F)\|_{L^p(\mathbb{R}^n, \gamma(\mathcal{H}, \mathcal{B}))^*}.
$$

By taking into account that $L^p(\mathbb{R}^n) \otimes \mathcal{B}$ is dense in $L^p(\mathbb{R}^n, \mathcal{B})$ and (20) we conclude that

$$
\|f\|_{L^p(\mathbb{R}^n, \mathcal{B})} \leq C \|G_{\mathcal{H}, \mathcal{B}^*}^t(f)\|_{L^p(\mathbb{R}^n, \gamma(\mathcal{H}, \mathcal{B}))^*},
$$

where $C > 0$ does not depend on $f$.

### 2.2. $(b) \Rightarrow (a)$

Let $\gamma \in \mathbb{R} \setminus \{0\}$. We define the imaginary power $\mathcal{H}^\gamma$ of $\mathcal{H}$ on $L^2(\mathbb{R}^n)$ as follows

$$
\mathcal{H}^\gamma f = \sum_{k \in \mathbb{N}^n} (2|k| + n)^\gamma c_k(f) \mathcal{h}_k, \quad f \in L^2(\mathbb{R}^n).
$$

Plancherel theorem implies that $\mathcal{H}^\gamma$ is bounded from $L^2(\mathbb{R}^n)$ into itself. Moreover, $\mathcal{H}^\gamma$ is a spectral multiplier of Laplace transform type ([29, p. 121]) associated with the Hermite operator and $\mathcal{H}^\gamma$ can be extended from $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ as a bounded operator from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$ ([4 Theorem 1.1], [3 Theorem 3]). Let $\mathcal{B}$ be a Banach space. If $1 < p < \infty$ we can define in a natural way $\mathcal{H}^\gamma$ on $L^p(\mathbb{R}^n) \otimes \mathcal{B}$ as a linear operator from $L^p(\mathbb{R}^n) \otimes \mathcal{B}$ into itself. In [2 Theorem 1.2] (see also [3 Theorem 3]) it was established that $\mathcal{B}$ is UMD if and only if $\mathcal{H}^\gamma$, $\gamma \in \mathbb{R} \setminus \{0\}$, can be extended from $L^p(\mathbb{R}^n) \otimes \mathcal{B}$ to $L^p(\mathbb{R}^n, \mathcal{B})$ as a bounded operator from $L^p(\mathbb{R}^n, \mathcal{B})$ into itself for some (equivalently, for every) $1 < p < \infty$.

Suppose now that $(b)$ holds. In order to see that $\mathcal{B}$ is UMD we prove the following vector valued version of an inequality in [29, p. 63].

**Proposition 2.2.** Let $\mathcal{B}$ be a Banach space and $\gamma \in \mathbb{R} \setminus \{0\}$. There exists $C > 0$ such that, for every $f \in \text{span}\{\mathcal{h}_k\}_{k \in \mathbb{N}^n} \otimes \mathcal{B}$,

$$
\|G_{\mathcal{H}, \mathcal{B}^*}^t(\mathcal{H}^\gamma f)(x, \cdot)\|_{\gamma(\mathcal{H}, \mathcal{B})} \leq C \|G_{\mathcal{H}, \mathcal{B}^*}^t(f)(x, \cdot)\|_{\gamma(\mathcal{H}, \mathcal{B})}, \quad x \in \mathbb{R}^n.
$$

**Proof.** Let $f \in \text{span}\{\mathcal{h}_k\}_{k \in \mathbb{N}^n} \otimes \mathcal{B}$. Then, $f = \sum_{k \in I} b_k \mathcal{h}_k$, where $I$ is a finite subset of $\mathbb{N}^n$ and $b_k \in \mathcal{B}$, $k \in I$. We introduce the operator $U \in L(\mathcal{B})$ defined by $U(b) = -b$, $b \in \mathcal{B}$, and the operator $T_\gamma$ on $\mathcal{H}$, given by

$$
T_\gamma(h)(t) = \frac{1}{\text{e}} \int_0^t \phi_\gamma(t - s) h(s) ds, \quad h \in \mathcal{H} \text{ and } t > 0,
$$
where $\phi_{\gamma}(u) = u^{i\gamma}/\Gamma(1 - i\gamma)$, $u > 0$. The operator $T_\gamma \in L(H)$ and $\|T\|_{L(H)} \leq 1/\Gamma(1 - i\gamma)$.

Indeed, by using Hölder’s inequality and Fubini’s theorem we get

$$\|T_\gamma(h)\|_H \leq \|\phi_{\gamma}\|_{L^\infty(0,\infty)} \left\{ \int_0^\infty \left( \frac{1}{t} \int_0^t |h(s)|ds \right)^2 \frac{dt}{t} \right\}^{1/2} \leq \frac{1}{\Gamma(1 - i\gamma)} \left\{ \int_0^\infty |h(s)|^2 \int_s^\infty \frac{dt}{t^2} ds \right\}^{1/2} = \frac{1}{\Gamma(1 - i\gamma)} \|h\|_H, \quad h \in H.$$

Let $x \in \mathbb{R}^n$. By considering $G_{H,\mathbb{B}}^1(\mathcal{H}^\gamma f)(x,\cdot)$ and $G_{H,\mathbb{B}}^2(f)(x,\cdot)$ as elements of $\gamma(H,\mathbb{B})$ we have that

$$G_{H,\mathbb{B}}^1(\mathcal{H}^\gamma f)(x,\cdot)(h) = UG_{H,\mathbb{B}}^2(f)(x,\cdot)T_\gamma(h), \quad h \in H.$$  

In fact, for every $h \in H$ and $S \subseteq \mathbb{B}^*$, by using well-known properties of Laplace transform, we can write

$$\langle S,UG_{H,\mathbb{B}}^2(f)(x,\cdot)T_\gamma(h) \rangle = \langle S,G_{H,\mathbb{B}}^2(f)(x,\cdot)T_\gamma(h) \rangle = -\int_0^\infty \langle S, \sum_{k \in I} b_k t^2(2|k| + n)^2 e^{-t(2|k| + n)}h_k(x) \rangle T_\gamma(h)(t) \frac{dt}{t}$$

$$= -\sum_{k \in I} \langle S, b_k \rangle (2|k| + n)^2 h_k(x) \int_0^\infty t e^{-t(2|k| + n)}T_\gamma(h)(t)dt$$

$$= \left( S, -\sum_{k \in I} b_k (2|k| + n)^{\gamma+1} h_k(x) \int_0^\infty e^{-t(2|k| + n)} h(t) dt \right).$$

Hence,

$$UG_{H,\mathbb{B}}^2(f)(x,\cdot)T_\gamma(h) = -\sum_{k \in I} b_k (2|k| + n)^{\gamma+1} h_k(x) \int_0^\infty e^{-t(2|k| + n)} h(t) dt, \quad h \in H.$$  

In a similar way we can see that

$$G_{H,\mathbb{B}}^1(\mathcal{H}^\gamma f)(x,\cdot)(h) = -\sum_{k \in I} b_k (2|k| + n)^{\gamma+1} h_k(x) \int_0^\infty e^{-t(2|k| + n)} h(t) dt, \quad h \in H.$$  

Thus [23] is established.

By taking into account the ideal property for the $\gamma$-radonifying operators ([35] Theorem 6.2) we conclude that

$$\|G_{H,\mathbb{B}}^1(\mathcal{H}^\gamma f)(x,\cdot)\|_{\gamma(H,\mathbb{B})} \leq \frac{1}{\Gamma(1 - i\gamma)} \|G_{H,\mathbb{B}}^2(f)(x,\cdot)\|_{\gamma(H,\mathbb{B})}.$$  

Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $1 < p < \infty$. From (b) and Proposition [22] it follows, for every $f \in \text{span}\{h_k\}_{k \in \mathbb{N}^n} \otimes \mathbb{B}$,

$$\|\mathcal{H}^\gamma(f)\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq C\|G_{H,\mathbb{B}}^1(\mathcal{H}^\gamma f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leq C\|G_{H,\mathbb{B}}^2(f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}.$$  

Since $\text{span}\{h_k\}_{k \in \mathbb{N}^n} \otimes \mathbb{B}$ is a dense subspace in $L^p(\mathbb{R}^n,\mathbb{B})$, $\mathcal{H}^\gamma$ can be extended to $L^p(\mathbb{R}^n,\mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}^n,\mathbb{B})$ into itself. From [2] Theorem 1.2] [3 Theorem 3]) we deduce that $\mathbb{B}$ is UMD.
3. Proof of Theorem 1.1 for the Schrödinger operator

In this section we prove (a) $\Leftrightarrow$ (c) in Theorem 1.1. We assume that $n \geq 3$ and that the potential function $V$ satisfies the reverse Hölder’s inequality \cite{5} where $s > n/2$. In the proof of (c) $\Rightarrow$ (a) we will use \cite{23} Theorem 3] where UMD Banach spaces are characterized by the $L^p$-boundedness properties of the imaginary power $L^{i\gamma}$, $\gamma \in \mathbb{R} \setminus \{0\}$, of the Schrödinger operator $L$.

3.1. $(a) \Rightarrow (c)$ In order to show this result we can proceed as in the proof of $(a) \Rightarrow (b)$ by using in each moment the suitable property for the heat kernel $W_t^L(x,y)$, $x, y \in \mathbb{R}^n$ and $t > 0$, of the Schrödinger semigroup.

As it is showed in the papers of Dziubański and Zienkiewicz \cite{11}, \cite{11} and \cite{12}, Dziubański, Garrigós, Martínez, Torrea and Zienkiewicz \cite{9} and Shen \cite{28}, the function $\rho$ defined by

$$
\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n,
$$

plays an important role in the develop of the harmonic analysis in the Schrödinger setting. In the special case of the Hermite operator, we can see that

$$
\rho(x) \sim \begin{cases} 
\frac{1}{2}, & |x| \leq 1, \\
\frac{1}{1+|x|}, & |x| \geq 1.
\end{cases}
$$

This function $\rho$ is usually called “critical radius” of $x$, and we use it to split the operators in the local and global parts (see \cite{13}). The main properties of the function $\rho$ can be encountered in \cite{28} Lemma 1.4. We must apply repeatedly that, for every $M > 0$, $\rho(x) \sim \rho(y)$ provided that $x, y \in \mathbb{R}^n$ and $|x - y| \leq M \rho(x)$, where the equivalence constants depend only on $M$. Also, according to \cite{9} Proposition 5], we can find a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ such that:

(i) $\bigcup_{k \in \mathbb{N}} B(x_k, \rho(x_k)) = \mathbb{R}^n$;

(ii) for every $M > 0$ there exists $m \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$,

$$
\text{card} \{ j \in \mathbb{N} : B(x_j, M \rho(x_j)) \cap B(x_k, M \rho(x_k)) \neq \emptyset \} \leq m.
$$

To complete the proof we need to use the following properties of $W_t^L(x,y)$, $x, y \in \mathbb{R}^n$ and $t > 0$. All of them can be found, for instance in \cite{9} Section 2 and \cite{11} Section 2.

Lemma 3.1. Assume that $V \in RH_s$, where $s > n/2$. Then,

(i) For every $k, N \in \mathbb{N}$, there exist $C, c > 0$ for which

$$
|t^k \partial_t^k W_t^L(x,y)| \leq C \frac{e^{-c|x-y|^2/t}}{\rho(t)^{n/2}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.
$$

(ii) There exists a nonnegative function $w \in S(\mathbb{R}^n)$, the Schwartz functions space, and $\delta > 0$, such that

$$
|G_{\sqrt{t}}(x - y) - W_t^L(x,y)| \leq C \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta w_{\sqrt{t}}(x - y), \quad 0 < t \leq \rho(x)^2, \quad x, y \in \mathbb{R}^n.
$$
(iii) If \( w \in S(\mathbb{R}^n) \) there exist \( \delta, \beta > 0 \) such that

\[
\int_{\mathbb{R}^n} G_{\sqrt{t}}(x - y)V(y)dy \leq C \begin{cases} 
\frac{1}{t} \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta, & 0 < t \leq \rho(x)^2, \\
\left( \frac{\sqrt{t}}{\rho(x)} \right)^{\beta + 2 - n}, & t > \rho(x)^2.
\end{cases}
\]

The polarization equality (see [21]) can be shown in the Schrödinger setting by using spectral arguments.

3.2. \( (c) \Rightarrow (a) \) Assume that \( (c) \) holds for a certain \( 1 < p < \infty \).

We denote by \( E_L(d\lambda) \) the spectral measure associated to the Schrödinger operator \( \mathcal{L} \). Then, we have that

\[ W_t^\mathcal{L}(f) = \int_{[0, \infty)} e^{-\lambda t} E_L(d\lambda)f \quad f \in L^2(\mathbb{R}^n). \]

We can also write

\[ W_t^\mathcal{L}(f)(x) = \int_{\mathbb{R}^n} W_t^\mathcal{L}(x, y)f(y)dy, \quad f \in L^2(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n. \]

Let \( f, g \in L^2(\mathbb{R}^n) \). Then,

\[ \langle \partial_t W_t^\mathcal{L}(f)(x), g(x) \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_t W_t^\mathcal{L}(x, y)f(y)g(x)dx = \partial_t \langle W_t^\mathcal{L}(f), g \rangle, \quad x \in \mathbb{R}^n \text{ and } t > 0. \]

Note that by using Lemma 3.1 (i), we can justified the derivation under the integral sign.

Indeed, Lemma 3.1 (ii), implies that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\partial_t W_t^\mathcal{L}(x, y)||f(y)||g(x)|dydx \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-c|x-y|^2/s} \frac{1}{t^{n/2 + 1}} |f(y)||g(x)|dydx \leq \frac{C}{t} \int_{\mathbb{R}^n} \sup_{s > 0} \left( \int_{\mathbb{R}^n} e^{-c|x-y|^2/s} \frac{1}{s^{n/2}} |f(y)|dy \right)^2 \leq \frac{C}{t}, \quad t > 0,
\]

because the maximal operator \( W_* \) defined by

\[ W_*(F) = \sup_{s > 0} |W_s(F)|, \quad F \in L^2(\mathbb{R}^n), \]

is bounded from \( L^2(\mathbb{R}^n) \) into itself.

On the other hand, by defining

\[ D_t W_t^\mathcal{L}(f) = \lim_{h \to 0} \frac{W_{t+h}^\mathcal{L}(f) - W_t^\mathcal{L}(f)}{h}, \quad \text{on } L^2(\mathbb{R}^n), \]

we have that

\[ D_t W_t^\mathcal{L}(f) = -\int_{[0, \infty)} \lambda e^{-\lambda t} E_L(d\lambda)f, \quad t > 0. \]

Hence, we conclude that, for every \( t > 0 \),

\[ D_t W_t^\mathcal{L}(f)(x) = \int_{\mathbb{R}^n} \partial_t W_t^\mathcal{L}(x, y)f(y)dy, \quad \text{a.e. } x \in \mathbb{R}^n. \]

Then, for every \( f \in L^2(\mathbb{R}^n) \cap \mathbb{B}, \ell = 1, 2, \) and \( t > 0 \),

\[ G_{\mathcal{L}, \mathbb{B}}^\ell(f)(\cdot, t) = \int_{[0, \infty)} (-\lambda)^\ell e^{-\lambda t} E_L(d\lambda)f, \]

where the right hand side has the obvious meaning.
Proposition 3.1. Let $\gamma \in \mathbb{R} \setminus \{0\}$. The imaginary power $L^{i\gamma}$ of the operator $L$ is defined by

$$L^{i\gamma}(f) = \int_{[0,\infty)} \lambda^{i\gamma} E_{\lambda}(d\lambda) f, \quad f \in L^2(\mathbb{R}^n),$$

and we extend $L^{i\gamma}$ to $L^2(\mathbb{R}^n) \otimes \mathbb{B}$ in the natural way.

It is clear that, for every $t > 0$,

$$G_{1,\mathbb{B}}^1(L^{i\gamma}) f(\cdot, t) = -\int_{[0,\infty)} t\lambda^{1+i\gamma} e^{-\lambda t} E_{\lambda}(d\lambda) f, \quad f \in L^2(\mathbb{R}^n) \otimes \mathbb{B}.$$

In the following we establish the analogous property of Proposition 2.2 but in the Schrödinger setting.

Proposition 3.1. Let $\mathbb{B}$ be a Banach space and $\gamma \in \mathbb{R} \setminus \{0\}$. There exists $C > 0$ such that, for every $f \in S(\mathbb{R}^n) \otimes \mathbb{B}$,

$$\|G_{1,\mathbb{B}}^2(L^{i\gamma})(f, \cdot)\|_{\gamma(H,\mathbb{B})} \leq C \|G_{1,\mathbb{B}}^2(f)(\cdot, \cdot)\|_{\gamma(H,\mathbb{B})}, \quad a.e. \ x \in \mathbb{R}^n.$$

Proof. Let $h \in L^2((0,\infty), dt/t)$ such that $\text{supp } h \subset (a, b)$, $0 < a < b < \infty$, and let $f, g \in L^2(\mathbb{R}^n)$. According to Lemma 3.1 (i), we get as above

$$\int_0^\infty \int_{\mathbb{R}^n} |G_{1,\mathbb{B}}^2(L^{i\gamma})(f)(x, t) g(x)| dx |h(t)| \frac{dt}{t}$$

$$\leq C \int_0^\infty \int_{\mathbb{R}^n} \sup_{s > 0} \left( \int_{\mathbb{R}^n} e^{-c|x-y|^2/s} |L^{i\gamma} f(y)| dy \right) |g(x)| dx |h(t)| \frac{dt}{t}$$

$$\leq C \|g\|_{L^2(\mathbb{R}^n)} \|W_\ast(L^{i\gamma} f)\|_{L^2(\mathbb{R}^n)} \int_a^b |h(t)| \frac{dt}{t} < \infty.$$

We can write

$$\int_{\mathbb{R}^n} \int_0^\infty G_{1,\mathbb{B}}^2(L^{i\gamma})(f)(x, t) h(t) \frac{dt}{t} g(x) dx$$

$$= \int_0^\infty h(t) \int_{\mathbb{R}^n} G_{1,\mathbb{B}}^2(L^{i\gamma})(f)(x, t) g(x) dx \frac{dt}{t}$$

$$= -\int_0^\infty h(t) \int_{\mathbb{R}^n} \left( \int_{[0,\infty)} t\lambda^{1+i\gamma} e^{-\lambda t} E_{\lambda}(d\lambda) f \right) (x) g(x) dx \frac{dt}{t}$$

$$= -\int_0^\infty \int_{[0,\infty)} \lambda^{1+i\gamma} e^{-\lambda t} d\mu_{f,g}(\lambda) h(t) dt,$$

where $\mu_{f,g}$ represents the complex measure defined by

$$\mu_{f,g}(A) = \langle E_{\lambda}(A) f, g \rangle,$$

for every Borel set $A$ in $[0,\infty)$. If $|\mu_{f,g}|$ denotes the total variation measure of $\mu_{f,g}$, then

$$\int_0^\infty \int_{[0,\infty)} |\lambda^{1+i\gamma}| e^{-\lambda t} d|\mu_{f,g}|(\lambda) |h(t)| dt \leq C |\mu_{f,g}|([0,\infty)) \int_0^\infty |h(t)| \frac{dt}{t} < \infty.$$

Hence, we have that

$$\int_{\mathbb{R}^n} \int_0^\infty G_{1,\mathbb{B}}^2(L^{i\gamma})(f)(x, t) h(t) \frac{dt}{t} g(x) dx$$

$$= -\int_{[0,\infty)} \lambda^{1+i\gamma} \int_0^\infty e^{-\lambda t} h(t) dt d\mu_{f,g}(\lambda)$$

$$= -\int_{[0,\infty)} \lambda^{2} \int_0^\infty \lambda^{2} e^{-\lambda t} T_{\gamma}(h)(t) \frac{dt}{t} d\mu_{f,g}(\lambda),$$
where
\[ T_\gamma(h)(t) = \frac{1}{t} \int_0^t \phi_\gamma(t-s)h(s)ds, \quad t \in (0, \infty), \]
and \( \phi_\gamma(u) = u^{-\gamma}/(1 - i\gamma), \ u \in (0, \infty). \) Since
\[ \int_{[0,\infty]} \int_0^\infty (\lambda t)^2 e^{-\lambda t}|T_\gamma(h)(t)|\frac{dt}{t}d\mu_\gamma(\lambda) < \infty, \]
we can write
\[ \int_{\mathbb{R}^n} \int_0^\infty G_{\gamma,\mathbb{B}}(\mathcal{L}^{\gamma} f)(x,t)h(t)\frac{dt}{t}g(x)dx = -\int_0^\infty T_\gamma(h)(t) \int_{[0,\infty]} (\lambda t)^2 e^{-\lambda t}d\mu_\gamma(\lambda)\frac{dt}{t} \]
\[ = -\int_0^\infty T_\gamma(h)(t) \int_{\mathbb{R}^n} g(x)G_{\gamma,\mathbb{C}}(f)(x,t)dx\frac{dt}{t} \]
\[ = -\int_{\mathbb{R}^n} g(x) \int_0^\infty T_\gamma(h)(t)G_{\gamma,\mathbb{C}}(f)(x,t)\frac{dt}{t}dx. \]
The last interchange is justified because
\[ \int_0^\infty \int_{\mathbb{R}^n} |g(x)||G_{\gamma,\mathbb{C}}(f)(x,t)|dxT_\gamma(h)(t)\frac{dt}{t} \leq C\|h\|_H \int_a^\infty \int_{\mathbb{R}^n} |g(x)||W_\gamma(f)|(x)dx\frac{dt}{t} < \infty. \]
We have taken into account that, since \( \text{supp} \ h \subset (a,b), \) it follows that \( T_\gamma(h)(t) = 0, \) when \( t \in (0,a). \)
We conclude that
\[ \int_0^\infty G_{\gamma,\mathbb{C}}(\mathcal{L}^{\gamma} f)(x,t)h(t)\frac{dt}{t} = -\int_0^\infty T_\gamma(h)(t)G_{\gamma,\mathbb{C}}(f)(x,t)\frac{dt}{t}, \quad \text{a.e. } x \in \mathbb{R}^n. \]
It is well-known that the space \( C_c(0,\infty) \) of continuous functions with compact support is dense in \( H. \) Moreover, since \( H \) is separable, there exists a numerable set \( A \subset C_c(0,\infty) \) that is dense in \( H. \)
We define \( N \subset \mathbb{R}^n \) consisting on those \( x \in \mathbb{R}^n \) for which
\[ \int_0^\infty G_{\gamma,\mathbb{C}}(\mathcal{L}^{\gamma} f)(x,t)h(t)\frac{dt}{t} = -\int_0^\infty T_\gamma(h)(t)G_{\gamma,\mathbb{C}}(f)(x,t)\frac{dt}{t}, \quad h \in A. \]
We have that \( |\mathbb{R}^n \setminus N| = 0. \) then, for every \( h \in H, \)
\[ \int_0^\infty G_{\gamma,\mathbb{C}}(\mathcal{L}^{\gamma} f)(x,t)h(t)\frac{dt}{t} = -\int_0^\infty T_\gamma(h)(t)G_{\gamma,\mathbb{C}}(f)(x,t)\frac{dt}{t}, \quad x \in N. \]
Hence, if \( f \in L^2(\mathbb{R}^n) \otimes \mathbb{B}, \) there exists \( \Omega \subset \mathbb{R}^n \) such that \( |\mathbb{R}^n \setminus N| = 0 \) and
\[ \int_0^\infty G_{\gamma,\mathbb{B},\mathbb{B}}(\mathcal{L}^{\gamma} f)(x,t)h(t)\frac{dt}{t} = -\int_0^\infty T_\gamma(h)(t)G_{\gamma,\mathbb{B},\mathbb{B}}(f)(x,t)\frac{dt}{t}, \quad x \in \Omega, \]
for every \( h \in H. \) By defining \( Ub = -b, \ b \in \mathbb{B}, \) we have that, as elements of \( \gamma(H,\mathbb{B}), \)
\[ G_{\gamma,\mathbb{B}}(\mathcal{L}^{\gamma} f)(x,\cdot) = UG_{\gamma,\mathbb{B}}(f)(x,\cdot)T_\gamma, \quad \text{a.e. } x \in \mathbb{R}^n, \]
for every \( f \in S(\mathbb{R}^n) \otimes \mathbb{B}. \)
By taking into account the ideal property of \( \gamma(H,\mathbb{B}) \) ([35 Theorem 6.2]), and that the operators \( U \) and \( T_\gamma \) are bounded in \( \mathbb{B} \) and \( H, \) respectively, we conclude the proof of this proposition. \( \square \)
Finally, from the equivalences in (c) and Proposition 3.1, we have that, for every \( f \in S(\mathbb{R}^n) \otimes \mathbb{B}, \)
\[ \|\mathcal{L}^{\gamma}\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq C\|G_{\gamma,\mathbb{B}}(\mathcal{L}^{\gamma} f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leq C\|G_{\gamma,\mathbb{B}}^2(f)\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}. \]
Since \( S(\mathbb{R}^n) \otimes \mathbb{B} \) is dense in \( L^p(\mathbb{R}^n,\mathbb{B}), \) we have proved that the operator \( \mathcal{L}^{\gamma} \) can be extended to \( L^p(\mathbb{R}^n,\mathbb{B}) \) as a bounded operator from \( L^p(\mathbb{R}^n,\mathbb{B}) \) into itself. Then, according to [3] Theorem 3], \( \mathbb{B} \) is UMD.
4. Proof of Theorem 1.1 for Laguerre operators

In this section we prove the equivalence (a) ⇔ (d) in Theorem 1.1.

Suppose that $\mathbb{B}$ is a UMD Banach space. Let $\ell = 1, 2$ and $1 < p < \infty$.

We are going to see that

\begin{equation}
\|G_{L_{\alpha},B}(f)\|_{L^p((0,\infty),\gamma(H,B))} \leq C\|f\|_{L^p((0,\infty),\mathbb{B})}, \quad f \in L^p((0,\infty),\mathbb{B}).
\end{equation}

In order to show [24], we take advantage from (a) ⇒ (b) after connecting $G_{L_{\alpha},B}$ and $G_{H,B}$ in a suitable way.

Note firstly that

\begin{equation}
W_{t}^{\mathcal{L}_{\alpha}}(x,y) = W_{t}^{\mathcal{H}/2}(x,y)g_{\alpha}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right), \quad x, y, t \in (0, \infty),
\end{equation}

where $W_{t}^{\mathcal{H}/2}(x,y)$ denotes the heat kernel associated with the operator $\mathcal{H}/2$ in dimension one, that is, for every $x, y \in \mathbb{R}$ and $t > 0$,

\begin{equation}
W_{t}^{\mathcal{H}/2}(x,y) = \left(\frac{e^{-t}}{\pi(1-e^{-2t})}\right)^{1/2} \exp\left[\frac{1}{4}\left((x-y)^2 \frac{1+e^{-t}}{1-e^{-t}} + (x+y)^2 \frac{1-e^{-t}}{1+e^{-t}}\right)\right],
\end{equation}

and $g_{\alpha}$ is defined by

\begin{equation}
g_{\alpha}(z) = \sqrt{2\pi}e^{-z}I_{\alpha}(z), \quad z \in (0, \infty).
\end{equation}

To make the reading of the following lines easier, from now on we consider $\xi = \xi(x,y,t) = \frac{2xye^{-t}}{1-e^{-2t}}$, $x, y, t \in (0, \infty)$.

We have, for every $x, y, t \in (0, \infty)$,

\begin{equation}
\partial_{t}W_{t}^{\mathcal{L}_{\alpha}}(x,y) = \partial_{t}W_{t}^{\mathcal{H}/2}(x,y)g_{\alpha}(\xi) - W_{t}^{\mathcal{H}/2}(x,y)\left(\frac{d}{dz}g_{\alpha}(z)\right)_{z=\xi}\frac{\xi(1+e^{-2t})}{1-e^{-2t}},
\end{equation}

and

\begin{equation}
\partial_{t}^{2}W_{t}^{\mathcal{L}_{\alpha}}(x,y) = \partial_{t}^{2}W_{t}^{\mathcal{H}/2}(x,y)g_{\alpha}(\xi) - 2\partial_{t}W_{t}^{\mathcal{H}/2}(x,y)\left(\frac{d}{dz}g_{\alpha}(z)\right)_{z=\xi}\frac{\xi(1+e^{-2t})}{1-e^{-2t}}
\end{equation}

\begin{equation}
+ W_{t}^{\mathcal{H}/2}(x,y)\left\{\left(\frac{d^2}{dz^2}g_{\alpha}(z)\right)_{z=\xi}\frac{\xi^2(1+e^{-2t})^2}{(1-e^{-2t})^2} + \left(\frac{d}{dz}g_{\alpha}(z)\right)_{z=\xi}\frac{\xi(1+6e^{-2t}+e^{-4t})}{(1-e^{-2t})^2}\right\}.
\end{equation}

By taking into account that $\frac{d}{dz}(z^{-\alpha}I_{\alpha}(z)) = z^{-\alpha}I_{\alpha+1}(z)$, $z \in (0, \infty)$ ([24] p. 110), we get

\begin{equation}
\frac{d}{dz}g_{\alpha}(z) = -g_{\alpha}(z) + \frac{2\alpha + 1}{2z}g_{\alpha}(z) + g_{\alpha+1}(z), \quad z \in (0, \infty),
\end{equation}

and

\begin{equation}
\frac{d^2}{dz^2}g_{\alpha}(z) = \left(1 - \frac{2\alpha + 1}{z} + \frac{4\alpha^2 - 1}{4z^2}\right)g_{\alpha}(z) + 2\left(\frac{\alpha - 1}{z} - 1\right)g_{\alpha+1}(z) + g_{\alpha+2}(z), \quad z \in (0, \infty).
\end{equation}

Since $I_{\alpha}(z) \sim z^{\alpha}$, as $z \to 0^{+}$ ([24] p. 108), we deduce from (28) and (29) that, for $k = 0, 1, 2$,

\begin{equation}
\left|z^{k}\frac{d^k}{dz^k}g_{\alpha}(z)\right| \leq C, \quad z \in (0, 1).
\end{equation}

On the other hand, according to [24] p. 123, for every $m \in \mathbb{N}$,

\begin{equation}
g_{\alpha}(z) = \sum_{r=0}^{m} \frac{(-1)^r[\alpha, r]}{(2z)^r} + O\left(\frac{1}{z^{m+1}}\right), \quad z \in (0, \infty),
\end{equation}

where $[\alpha, 0] = 1$ and

$[\alpha, r] = \frac{(4\alpha^2 - 1)(4\alpha^2 - 3^2)\cdots(4\alpha^2 - (2r - 1)^2)}{2^{2r}\Gamma(r + 1)}$, \quad r \in \mathbb{N}, \quad r \geq 1.$
Then, for $k = 1, 2,$

\begin{equation}
\left| z^k \frac{d^k}{dz^k} g_\alpha(z) \right| \leq \frac{C}{z^2}, \quad z \in (0, \infty).
\end{equation}

Indeed, by using property \((31)\) in \((28)\) and \((29)\) we get

\[ \frac{d}{dz} g_\alpha(z) = \left[ \frac{\alpha}{2} + \frac{\alpha + 1, 1}{2} \right] + O\left( \frac{1}{z^2} \right), \quad z \in (0, \infty). \]

and

\[ \frac{d^2}{dz^2} g_\alpha(z) = \left( 1 - \frac{\alpha, 1}{2} + \frac{\alpha + 1, 1}{2} \right) \frac{1}{z} \]

\[ + \left( \frac{2}{4} + (2\alpha + 3) \right) - (\alpha + 1)[\alpha + 1, 1] = \frac{\alpha + 1, 2}{2} + \frac{\alpha + 2, 2}{4} \frac{1}{z^2} \]

\[ + O\left( \frac{1}{z^2} \right), \quad z \in (0, \infty). \]

Then, \((32)\) holds for $k = 1, 2.$

From \((26)\) and \((27)\) by using \((12)\) (note that estimate \((12)\) also holds for $\mathcal{H}/2$ instead of $\mathcal{H}$) we obtain, for every $x, y, t \in (0, \infty),$ \((30)\) implies that

\begin{equation}
\left| t^\ell \partial^\ell_t W^{c_\alpha}_t(x, y) \right| \leq C \left\{ \left| t^\ell \partial^\ell_t W^{\mathcal{H}/2}_t(x, y) \right| \left| g_\alpha(\xi) \right| + \left| t^{\ell - 1} \partial^{\ell - 1}_t W^{\mathcal{H}/2}_t(x, y) \right| \left( \frac{d}{dz} g_\alpha(z) \right) \right\}_{z = \xi} \left( 1 - e^{-2t} \right) \frac{\xi}{(1 - e^{-2t})^2} \left\{ \left| \frac{t \xi}{1 - e^{-2t}} \right| + (\ell - 1) \frac{(\xi)^2}{(1 - e^{-2t})^2} \left( \frac{d^2}{dz^2} g_\alpha(z) \right) \right\}_{z = \xi}. \end{equation}

(33)

Now, \((30)\) implies that

\begin{equation}
\left| t^\ell \partial^\ell_t W^{c_\alpha}_t(x, y) \right| \leq C \frac{e^{-c(x - y)^2/t}}{\sqrt{t}}, \quad x, y, t \in (0, \infty) \quad \text{and} \quad \xi \leq 1.
\end{equation}

We also observe that

\begin{equation}
\exp \left[ -c \left( \frac{1 + e^{-t}}{1 - e^{-t}} |x - y|^2 + \frac{1 - e^{-t}}{1 + e^{-t}} |x + y|^2 \right) \right] = \exp \left[ -2c \left( \frac{1 + e^{-2t}}{1 - e^{-2t}} (x^2 + y^2) - 8 \frac{e^{-t}}{1 - e^{-2t}} xy \right) \right]
\end{equation}

\[ \leq \exp \left[ -c \frac{1 + e^{-2t}}{1 - e^{-2t}} (x^2 + y^2) \right], \quad x, y, t \in (0, \infty) \quad \text{and} \quad \xi \leq 1. \]

Then we get

\begin{equation}
\left| t^\ell \partial^\ell_t W^{\mathcal{H}/2}_t(x, y) \right| \leq C \frac{e^{-c(x^2 + y^2)/t}}{\sqrt{t}}, \quad x, y, t \in (0, \infty), \quad \xi \leq 1,
\end{equation}

and

\begin{equation}
\left| t^\ell \partial^\ell_t W^{c_\alpha}_t(x, y) \right| \leq C \frac{e^{-c(x^2 + y^2)/t}}{\sqrt{t}}, \quad x, y, t \in (0, \infty) \quad \text{and} \quad \xi \leq 1.
\end{equation}

On the other hand, from \((32)\) and \((33)\) it follows that,

\begin{equation}
\left| t^\ell \partial^\ell_t W^{c_\alpha}_t(x, y) \right| \leq C \frac{x y e^{-c(x - y)^2/t}}{t^{3/2}}, \quad x, y, t \in (0, \infty) \quad \text{and} \quad \xi \geq 1.
\end{equation}

\begin{equation}
\left| t^\ell \partial^\ell_t W^{c_\alpha}_t(x, y) \right| \leq C \frac{x y e^{-c(x - y)^2/t}}{t^{3/2}}, \quad x, y, t \in (0, \infty) \quad \text{and} \quad \xi \geq 1.
\end{equation}
Moreover, by taking into account (12), (25), (31) and (32), we obtain that
\[
\left| t^\ell \partial_t^\ell \left[ W_t^{L_n}(x,y) - W_t^{H/2}(x,y) \right] \right| \\
\leq \left| t^\ell \partial_t^\ell [W_t^{H/2}(x,y)](g_n(\xi) - 1) \right| + \left| t^\ell \partial_t^{\ell-1} W_t^{H/2}(x,y) \right| \left( \frac{d}{dz} g_n(z) \right)_{z=\xi} \frac{\xi}{1 - e^{-2t}} \\
+ (\ell - 1) t^{2\ell} W_t^{H/2}(x,y) \left\{ \left( \frac{d^2}{dz^2} g_n(z) \right)_{z=\xi} \frac{\xi^2}{(1 - e^{-2t})^2} + \left( \frac{d}{dz} g_n(z) \right)_{z=\xi} \frac{\xi}{(1 - e^{-2t})^2} \right\} \\
\leq C \frac{e^{-t/3} e^{-c|x-y|^2/t}}{\xi^{1/4}}, \quad x, y, t \in (0, \infty).
\]

Hence, we get
(38)
\[
\left| t^\ell \partial_t^\ell [W_t^{L_n}(x,y) - W_t^{H/2}(x,y)] \right| \leq C \frac{e^{-t/3} e^{-c|x-y|^2/t}}{\xi^{1/4} \sqrt{t}} \leq C \frac{e^{-c|x-y|^2/t}}{(x y t)^{1/4}}, \quad x, y, t \in (0, \infty) \text{ and } \xi \geq 1.
\]

Let now $f \in L^p((0, \infty), \mathbb{R})$. Let us denote by $\tilde{f}$ the extension of $f$ to $\mathbb{R}$ which satisfies $\tilde{f}(x) = 0$, $x \leq 0$. By defining $G_{t/2, \mathbb{R}}(\tilde{f})$ in the obvious way, we have that
\[
\left\| G_{t/2, \mathbb{R}}(f)(x, \cdot) - G_{t/2, \mathbb{R}}(\tilde{f})(x, \cdot) \right\|_{L^2((0, \infty), \mathbb{R}, \mathbb{R})} \\
\leq \int_{(0, x/2) \cup (2x, \infty)} \|f(y)\|_\mathbb{R} \left( \left\| t^\ell \partial_t^\ell W_t^{L_n}(x,y) \right\|_H + \left\| t^\ell \partial_t^\ell W_t^{H/2}(x,y) \right\|_H \right) dy \\
+ \int_{x/2}^{2x} \|f(y)\|_\mathbb{R} \left| t^\ell \partial_t^\ell \left[ W_t^{L_n}(x,y) - W_t^{H/2}(x,y) \right] \right|_H dy \\
= T_1(f)(x) + T_2(f)(x), \quad x \in (0, \infty).
\]

By using (12), (34) and (37) we obtain, when $x \in (0, \infty)$ and $y \in (0, x/2) \cup (2x, \infty)$,
\[
\left\| t^\ell \partial_t^\ell W_t^{L_n}(x,y) \right\|_H + \left\| t^\ell \partial_t^\ell W_t^{H/2}(x,y) \right\|_H \leq C \left( 1 + \frac{xy}{|x-y|^2} \right) \left( \int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^{3/2}} dt \right)^{1/2} \\
\leq C \left( \int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^{3/2}} dt \right)^{1/2} \leq C \frac{1}{|x-y|} \leq C \begin{cases} 
1, & 0 < y < \frac{x}{2} \\
\frac{1}{y}, & 0 < 2x < y
\end{cases},
\]
because $|x-y| \sim x$, when $y \in (0, x/2)$ and $|x-y| \sim y$, when $y \in (2x, \infty)$.

Hence,
(40)
\[
T_1(f)(x) \leq C \left[ H_0(\|f\|_\mathbb{R}) \right](x) + H_\infty(\|f\|_\mathbb{R})(x) < \infty, \quad x \in (0, \infty),
\]
where $H_0$ and $H_\infty$ represents the classical Hardy operators given by
\[
H_0(g)(x) = \frac{1}{x} \int_0^x g(y) dy \quad \text{and} \quad H_\infty(g)(x) = \int_x^\infty \frac{g(y)}{y} dy, \quad x \in (0, \infty).
\]

On the other hand, by taking into account (35), (36) and (38), we can write,
\[
\left\| t^\ell \partial_t^\ell [W_t^{L_n}(x,y) - W_t^{H/2}(x,y)] \right\|_H \leq \left\{ \left( \int_{0, \xi \leq 1}^{\infty} + \int_{0, \xi \geq 1}^{\infty} \right) \left| t^\ell \partial_t^\ell [W_t^{L_n}(x,y) - W_t^{H/2}(x,y)] \right|^2 dt \right\}^{1/2} \\
\leq C \left( \int_0^\infty \frac{e^{-c(x^2+y^2)/t}}{t^{3/2}} dt + \frac{1}{(xy)^{1/2}} \int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^{3/2}} dt \right)^{1/2} \\
\leq C \left( \frac{1}{\sqrt{x^2 + y^2}} + \frac{1}{(xy)^{1/4} \sqrt{|x-y|}} \right) \leq C \frac{1}{y} \left( 1 + \sqrt{\frac{y}{|x-y|}} \right), \quad 0 < \frac{x}{2} < y < 2x.
Hence,

\[ T_2(f)(x) \leq C N(\|f\|_B)(x) < \infty, \quad x \in (0, \infty), \tag{41} \]

where

\[ N(g)(x) = \int_{x/2}^{2x} \frac{1}{y} \left(1 + \sqrt{\frac{y}{|x-y|}}\right) g(y) \, dy, \quad x \in (0, \infty). \]

From (39), (40) and (41) we conclude that, for every \( x \in (0, \infty) \),

\[ \|G_{\mathcal{L}_\alpha}^f(\cdot) - G_{\mathcal{H}/2,3}^f(\cdot)\|_{\gamma(H, B)} \leq C [H_0(\|f\|_B)(x) + H_\infty(\|f\|_B)(x) + N(\|f\|_B)(x)] < \infty. \]

It is well-known that the Hardy operators \( H_0 \) and \( H_\infty \) are bounded from \( L^p(0, \infty) \) into itself (see [18, p. 244, (9.9.1) and (9.9.2)]). Moreover, Jensen’s inequality allows us to show that the operator \( N \) is also bounded from \( L^p(0, \infty) \) into itself. Hence,

\[ \|G_{\mathcal{L}_\alpha}^f(\cdot) - G_{\mathcal{H}/2,3}^f(\cdot)\|_{L^p((0, \infty)\gamma(H, B))} \leq C \|f\|_{L^p((0, \infty)\gamma(H, B))}. \]

Since, as it was seen in Section 2 (b) holds provided that \( B \) is UMD, it follows that

\[ \|G_{\mathcal{L}_\alpha}^f(\cdot)\|_{L^p((0, \infty)\gamma(H, B))} \leq C \|f\|_{L^p((0, \infty), B)}. \]

Note that we can obtain results analogous to Propositions 2.1 and 2.2 for the operator \( \mathcal{L}_\alpha \) instead of \( \mathcal{H} \). The remainder of the proof of (a) \( \Rightarrow \) (d) and the proof of (d) \( \Rightarrow \) (a) can be made by proceeding as in the proof of the corresponding properties in Section 2.

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