SOLUTIONS WITH NEGATIVE MASS FOR THE \( SU(2) \) EINSTEIN–YANG–MILLS EQUATIONS

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The aim of this note is to clarify the structure of nontrivial asymptotically flat solutions with nonpositive ADM mass for the static, spherically symmetric Einstein–Yang–Mills equations with \( SU(2) \) gauge group. The presented numerical results demonstrate that these solutions have zero number of nodes of the gauge field function. This is a feature that is present neither for particlelike nor for black hole solutions in this model.

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1 Version 2: a statement concerning masses of black hole solutions on page 2 qualified, reference 12, and an acknowledgment added. Results unchanged.
1. Introduction

Static, spherically symmetric Einstein–Yang–Mills (EYM) equations with gauge group $SU(2)$ attract considerable attention after the celebrated discovery of the particlelike solutions made by Bartnik and McKinnon in 1988 [1] and the subsequent generalization of this result to the black hole case [2–4]. A review of the results obtained in 1988–1998 and an extensive bibliography can be found in [5]. Some more recent results are presented in [6–11]. But though a great amount of investigations of the EYM equations has already been performed, some questions still remain open. To introduce the subject of this paper, recall some basic facts.

First, the space-time metric for the static, spherically symmetric EYM equations can be written as

$$ds^2 = \sigma^2 N dt^2 - N^{-1} dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$

where $N$ and $\sigma$ depend on $r$, and the Yang–Mills gauge field reads as

$$A = (T_2 d\vartheta - T_1 \sin \vartheta d\varphi) w + T_3 \cos \vartheta d\varphi,$$

where $T_i = \frac{1}{2} \tau_i$ are the $SU(2)$ group generators and $\tau_i$ are the Pauli matrices, $i = 1, 2, 3$ (see, e.g., [5]).

It is convenient to write down the EYM equations in this framework in the form of two ordinary differential equations for the metric function $u = r^2 N$ and the gauge field function $w$:

$$ru' - (1 - 2w^2)u + (1 - w^2)^2 - r^2 = 0,$$

$$ruw'' - \left[u + (1 - w^2)^2 - r^2\right] w' + (1 - w^2) rw = 0,$$

and a decoupled equation for $\sigma$:

$$r\sigma' - 2\sigma w'^2 = 0.$$  \(1\)

Since (1) do not involve $\sigma$, one can use these to obtain $u$ and $w$, and then solve the equation for $\sigma$. Thus the following analysis is restricted to Eqs. (1). Notice also that (1) are invariant under the transformation $r \to -r$; hence only the case $r \geq 0$ will be discussed.

The mass function is usually defined as $m = (r - u/r)/2$; then the ADM mass is $M = \lim_{r \to \infty} m$.

Now, recall that the EYM equations (1) admit two explicit solutions: the Schwarzschild solution

$$w \equiv \pm 1, \quad u = r^2 - 2Mr,$$  \(2\)
and the Reissner–Nordström solution
\[ w \equiv 0, \quad u = r^2 - 2Mr + 1, \]  
where \( M \) is an arbitrary constant, representing the ADM mass of the corresponding solution. In the first case, \( m(r) \equiv M = \text{const} \), and in the second one, \( m = M - \frac{1}{2r} \). Thus, both exact solutions admit arbitrary nonpositive values of the ADM mass.

Next, both particlelike and black hole solutions of (1) are asymptotically flat and belong to the two-parameter family
\[ u = r^2 - 2Mr + o(r), \]
\[ w = w_\infty + br^{-1} + o(r^{-1}) \]  
as \( r \to \infty \), where \( M \) and \( b \) are arbitrary constants, and \( w_\infty = \pm 1 \). More than this, it has recently been proved by Wasserman [10] that all real solutions of the EYM equations (1) defined at spatial infinity either have \( w \equiv 0 \) or belong to the family (4). But while (4) includes solutions with nonpositive ADM mass, both particlelike and black hole solutions in this model have only strictly positive values of mass \([1,4]\). Thus, there appears a natural question: what are the nontrivial solutions of the family (4) that possess nonpositive ADM mass? This question is especially interesting in view of the results presented in \([12,7,13]\).

Finally, recall that for both particlelike and black hole solutions the gauge field function \( w \) reaches spatial infinity being within the strip \((-1,1)\), and thus having \( \text{sgn} \, b = \text{sgn} \, w_\infty \). This gives rise to another question: what are the solutions that have \( \text{sgn} \, b = -\text{sgn} \, w_\infty \), so that \( w \) reaches infinity staying outside the strip \((-1,1)\)? It is clear that in this case \( w \) should have zero number of nodes, since it cannot have local maxima for \( w > 1 \) and local minima for \( w < -1 \) \([2,4]\).

To find an answer to both questions, one should study Reissner–Nordström-like (RN-like) solutions, introduced by Smoller and Wasserman \([15]\). These solutions satisfy the following property: they are defined for all \( r > 0 \), and \( \lim_{r \to +0}(N, w, w') = (\infty, w_0, 0) \), \( |w_0| < \infty \). It is important that they may have zero number of nodes of the gauge field function \([15]\).

Notice that black hole solutions with the Reissner–Nordström interior structure, found numerically in \([16]\), do also belong to the class of RN-like solutions. Hence, the class of
RN-like solutions is too rich for the purposes of this investigation. Thus, in what follows I consider only those RN-like solutions that have no horizons, i.e., the metric function $u$ is positive for all $r > 0$; let us call them “horizonless” for brevity. In other words, only RN-like solutions that have naked singularity at $r = 0$ will be discussed below.

2. Main Results

In order to find an answer to the posed questions, one may perform numerical integration of (1) using expansion (4) at spatial infinity. But it seems to be more interesting to start integration from the vicinity of the origin, $r = 0$. Recall that RN-like solutions belong to one of the following disjoint families of local solutions as $r \to 0$ [15,6]:

1. The two-parameter family of solutions with the Schwarzschild type singularity:

$$
    u = u_1 r + r^2 + o(r^2),
    w = \pm 1 + w_2 r^2 + o(r^2),
$$

(5)

where $u_1$ and $w_2$ are arbitrary constants. Solutions of this family correspond to the vacuum value of the Yang–Mills field $|w(0)| = 1$ and generalize the Schwarzschild solution (2). For the RN-like solutions, $u_1 > 0$.

2. The three-parameter family of solutions with the Reissner–Nordström type singularity:

$$
    u = (1 - w_0^2)^2 + u_1 r + r^2 + o(r^2),
    w = w_0 + \frac{w_0}{2(1 - w_0^2)} r^2 + w_3 r^3 + o(r^3),
$$

(6)

where $w_0$, $u_1$, and $w_3$ are arbitrary constants, $w_0 \neq \pm 1$. These solutions correspond to the metric of the mass $m_0 = -u_1/2$ and the (magnetic) charge $Q_0^2 = (1 - w_0^2)^2$. Obviously, they generalize the Reissner–Nordström solution (3).

First, consider solutions that belong to the family (5). It is convenient to divide the set of all possible initial data $(w_0, u_1, w_3)$, $|w_0| \neq 1$, into two subsets: $|w_0| < 1$ and $|w_0| > 1$. Let us begin with the first one.

Take $w_0 = -\frac{1}{2}$ and $u_1 = 1$ for convenience. For this set of initial parameters one can find two discrete families of horizonless RN-like solutions with $n = 0, 1, 2, 3, \ldots$ nodes of the gauge field function, so that there are two branches of $w_3$ that give rise to these solutions, see Tables 1, 2 and Figures 1, 2. Notice that one of the nodeless solutions possesses negative ADM mass, namely, a solution defined by $w_3$ from the upper branch.
Notice also that for the family shown in Figure 2 (Table 2) solutions with \( n > 0 \) have an extra local minimum in comparison with the similar solutions from another family. This, in particular, results in a more complicated behavior of the effective charge function \( Q^2 = 2r(M - m) \), cf. Figures 3 and 4.

A situation similar to the above can be observed for other \( w_0 \), \( |w_0| < 1 \). Namely, horizonless RN-like solutions do always come in pairs, and every time when there exists a solution with negative mass, it is unique [for fixed \((w_0, u_1)\)] and necessarily has zero number of nodes. The corresponding \( w_3 \) always belongs to the upper branch of its values. (As a result, no solutions with negative mass have been found for \( w_0 = 0 \).) Surprisingly enough, but not a single solution with negative ADM mass has been found for \( n > 0 \).

A natural question that appears at this point is the following: is there a continuous transition from negative to positive values of the ADM mass? To answer the question one has just to study the dependence of mass on \( u_1 \). The answer appears to be affirmative, see Figure 5.

To study the case \( |w_0| > 1 \), take \( w_0 = -\frac{3}{2} \) and \( u_1 = 1 \) for symmetry. In this case, one finds only two pairs of horizonless RN-like solutions, each pair containing solutions with \( n = 0 \) and \( n = 1 \), see Table 3. Similar to the above, one of the solutions with zero number of nodes of the gauge field function has negative ADM mass. But contrary to the above, \( w_3 \) that gives rise to this solution lies on the lower branch of its values. And again, studying the dependence of mass on \( u_1 \) one can find continuous transition from solutions with negative mass to solutions with positive mass, see Figure 6.

It is interesting to compare nodeless solutions with negative ADM mass with their counterparts that have positive ADM mass. Though in both cases the gauge field function is monotonic, its behavior considerably differs in these two cases. Namely, solutions with positive ADM mass get close to the asymptotic value much faster than their counterparts with negative mass as \( r \to \infty \), see Figure 7. It is also remarkable that the sign of mass of the corresponding solution seems to be completely determined by the behavior of the metric function \( u \) near the origin. As one can see from Figure 8, \( u \) is monotonic and lies above the parabola \( y = r^2 \) for both solutions with negative ADM mass. In contrast with this, for the solutions with positive mass, \( u \) has a local minimum not far from \( r = 0 \), deep enough to put \( u \) below the curve \( y = r^2 \). (A similar behavior of \( u \) was found for solutions with \( n > 0 \).) Compare also the effective charge functions for \( n = 0 \) in Figures 3 and 4.

Notice that solutions which start from \( w_0 = -\frac{3}{2} \) have \( \text{sgn} \, b = -\text{sgn} \, w_\infty \) (cf. (4)). Thus, the presented solutions provide an answer to both questions stated above.
**Table 1:** Parameters of horizonless RN-like solutions found for \( w_0 = -\frac{1}{2} \), \( u_1 = 1 \); the upper branch of \( w_3 \) is given.

| \( n \) | \( w_3 \)     | \( M \)     |
|-------|--------------|------------|
| 0     | 0.3547481    | -0.676226  |
| 1     | 2.3566517    | 0.620136   |
| 2     | 2.9640246    | 0.928290   |
| 3     | 3.1096809    | 0.987930   |

**Table 2:** Parameters of horizonless RN-like solutions found for \( w_0 = -\frac{1}{2} \), \( u_1 = 1 \); the lower branch of \( w_3 \) is given.

| \( n \) | \( w_3 \)     | \( M \)     |
|-------|--------------|------------|
| 0     | -8.3849815   | 0.190093   |
| 1     | -9.4975402   | 0.854186   |
| 2     | -9.7866841   | 0.975527   |
| 3     | -9.8382163   | 0.995995   |

**Table 3:** Parameters of horizonless RN-like solutions found for \( w_0 = -\frac{3}{2} \), \( u_1 = 1 \).

| \( n \) | \( w_3 \)     | \( M \)     |
|-------|--------------|------------|
| 0     | -0.6975762   | -1.578504  |
| 1     | 1.2800470    | 0.846658   |
| 1     | 5.4383727    | 0.860686   |
| 0     | 6.9693638    | 0.182617   |

It is necessary to say a few words about the dependence of the structure of horizonless RN-like solutions on the value of \( u_1 \). Begin with the case \( w_0 = -\frac{1}{2} \). As \( u_1 \) becomes negative, then soon after the ADM mass becomes positive, horizonless solutions begin to disappear. First, there disappear solutions with \( n = 0 \). Next comes the order of \( n = 1 \) solutions, etc. Finally, besides singular solutions, there remain only solutions with a regular horizon (for \( u_1 \lesssim -1.8 \)). A qualitatively similar situation takes place for other values of \( w_0 \), \( |w_0| < 1 \).

The situation with \( w_0 = -\frac{3}{2} \) is in a sense opposite to the above. As we have seen, for \( u_1 = 1 \) there exist only solutions with \( n = 0, 1 \). But the number of horizonless RN-like solutions increases as \( u_1 \) grows. First, there appears a pair of \( n = 2 \) solutions, then a
pair of $n = 3$ solutions, etc., and finally one finds two families of solutions each having an arbitrary number of nodes of the gauge field function. Conversely, the horizonless solutions disappear as $u_1$ becomes smaller. First, there disappear solutions with $n = 1$ and then with $n = 0$. A qualitatively similar picture can be found in a wide range of $w_0$, $|w_0| > 1$. In all cases, only solutions with $n = 0$ possess nonpositive mass.

For solutions with negative ADM mass, the value of $M$ decreases unboundedly as $u_1$ grows, e.g., for the case $w_0 = -\frac{3}{2}$ the dependence is nearly linear and reads as follows: $M \approx -u_1/2 + \text{const.}$

In fact, the dependence of the structure of RN-like solutions and their ADM masses on the values of $(w_0, u_1)$ is interesting enough to be the subject of a separate investigation, but goes far beyond the scope of the present note (vita brevis, fisica longa).

It is necessary to stress that for all values of $w_1$ and $u_1$ considered above one can find not only horizonless (and singular) solutions, but also solutions with a regular horizon. They are not discussed here since they are either singular (i.e., cannot be continued towards $r = \infty$) or correspond to black hole solutions.

Finally, consider solutions with the Schwarzschild type singularity (3). In this case, one can also find horizonless RN-like solutions. The numerical results presented above demonstrate that only solutions with zero number of nodes of $w$ can have nonpositive values of the ADM mass. But nontrivial solutions that satisfy expansion (3) at $r = 0$ cannot have $n = 0$. Hence, it seems quite natural that solutions with nonpositive ADM mass and the Schwarzschild type singularity were not found.

The performed numerical investigation reveals that a situation qualitatively similar to the discussed above can be found for other initial values $w_0$ and $u_1$. This allows us to put forward the following conjecture: For each fixed pair $(w_0, u_1)$ in (3), there exists at most one global horizonless solution with nonpositive ADM mass; each such solution has zero number of nodes of the gauge field function.

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Fig. 1: The gauge field function for the solutions defined by the upper branch of $w_3$, see Table 1.

Fig. 2: The gauge field function for the solutions defined by the lower branch of $w_3$, see Table 2.
Fig. 3: The effective charge function for the solutions shown in Figure 1. Numbers in the body of the plot denote the number of nodes of $w$.

Fig. 4: The effective charge function for the solutions shown in Figure 2. Numbers in the body of the plot denote the number of nodes of $w$. 
**Fig. 5:** The dependence of $w_3$ and the ADM mass $M$ on $u_1$ near $M = 0$ for solutions with zero number of nodes of the gauge field function. Here $w_0 = -\frac{1}{2}$, and the upper branch of $w_3$ values is taken.

**Fig. 6:** The same as in Figure 5, but $w_0 = -\frac{3}{2}$, and the lower branch of $w_3$ values is taken.
Fig. 7: The gauge field function for the solutions with zero number of nodes. “Outer” solutions have negative ADM mass, “inner” solutions have positive ADM mass. Parameters of the solutions are given in Tables 1, 2, and 3.

Fig. 8: The metric function \( u \) for the solutions shown in Figure 7. Parameters of the solutions are given in Tables 1, 2, and 3. Dashed line: \( y = r^2 \).
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