Necessary and convenient number of indications in a measurement

A C Baratto\textsuperscript{1}, L V G Tarelho\textsuperscript{1}, G A Garcia\textsuperscript{1}

\textsuperscript{1} Divisão de Metrologia em Tecnologias da Informação e Telecomunicações, Inmetro, Av. Nossa Senhora das Graças, 50, 25250-020, Duque de Caxias- RJ – Brazil

E-mail: acbaratto@inmetro.gov.br

Abstract. The work addresses the problem of how to properly stipulate a suitable number of indications or repetitions to be used in a measurement. This appropriate and adequate number of indications is understood and is proposed as the minimal number that induces, when the resulting Type A uncertainty is included in the calculation, a relative change in the preexisting combined uncertainty lower than some parameter $Q$. In its turn, the parameter $Q$ is anticipated by the metrologist based on an analysis of the experimentational context in which the measurement is being made.

1. Introduction

There is a widespread and recurrent idea that getting a large number of indications is a positive and safe procedure in measurement processes, and that it is always better than taking a small number of them. But the cost of a specific process, which derives from superfluous overwork, from extra time spent in measuring and in storage, transmission and processing of data, and from the waste of energy expended to maintain the measurement system in a steady state, grows with the number of indications. Time and energy are money, but the matter will not be a significant problem for you if you have an abundance of time, energy and money. However, the subject is of great importance, and particularly delicate, in cases in which the measurement procedure is cumbersome and tiring, leading to abnormal results when extensively repeated, and when the integrity of the measurand is in some way sensitive to certain parts of the process.

The decision about the best number of indications to be used in a measurement involves a careful consideration of the “experimentational perspective that dictates the uses and applications one intends to give to (the) results” [1]. Before the beginning of a measurement it is necessary to define a measurand, or to make a careful exegesis of the elected preexisting definition. After, we need to select the method and the instruments that will be used in the measurement, all in accordance to the use and application that is intended for the results. At this time we already know (or we are in conditions to know) the values of all Type B uncertainty components. The combination of these Type B uncertainties in the “law of propagation of uncertainties” results in a minimum value for the combined uncertainty ($u_{cB}$). Now, the next step is deciding how many indications will be acquired. For this, the metrologist needs to know the minimum acceptable number of indications and, additionally, the most convenient number of indications to be used so that the measurement is carried out based on the wisest decision. The use of a number smaller than the necessary implies an undesirable large uncertainty. The
use of a number greater than the most convenient (in a “lato sensu”) implies an unwanted large cost of the measurement process.

The number of indications delivered by the physical measurement process will result in a Type A uncertainty that will increase the value of the minimum combined uncertainty \( u_{cb} \) reported above by some relative factor \( q \). The metrologist wants to establish a parameter \( Q \) as an upper limit for the value of \( q \). The problem treated in this work is to calculate the minimum number \( n \) of indications that satisfies the condition \( q < Q \). The value of \( Q \), thought as a tolerable or desirable value, is chosen by the metrologist based on experimental requirements. Of course, the number \( n \) that results from the treatment is in general not a natural one, and should be rounded up to the next integer. In practice, the use of this next integer will result in a relative variation factor \( q \) slightly smaller than \( Q \). This is valid for the expanded uncertainty as well.

2. Calculation of the minimum number of indications

For a type A evaluation, the experimental standard deviation of the mean \( s(\bar{y}) \) of a series of \( n \) observations ([2, B.2.17 NOTE 2]) in a measurement process is given by equation (1):

\[
s(\bar{y}) = \frac{s(y_i)}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sqrt{\frac{\sum_{k=1}^{n} (q_k - \bar{q})^2}{n(n-1)}} = \frac{1}{\sqrt{n}} \sqrt{\sum_{k=1}^{n} (q_k - \bar{q})^2 \frac{1}{n(n-1)}}
\]

(1)

where \( s(q_i) \) is the experimental standard deviation of the observations and \( \bar{q} \) is the mean of \( n \) effective \( q_i \) indications.

By (1) we see that \( s(\bar{y}) \) decreases when the number of indications increases. For \( n \gg 1 \) the dependence of \( s(\bar{y}) \) is just inversely proportional to \( n \). In the present treatment, and by supposition, the other uncertainty components \( u(y) \) are evaluated by Type B uncertainty evaluation methods.

We will consider the evaluation of a measurand \( Y \), with estimate \( y \) and combined uncertainty \( u(y) \), which is composed by \( N-1 \) identified Type B uncertainty components \( (u_i(y) = c_iu(x_i), i = 1, \ldots, N-1) \), and just only one Type A \( (u_A(y) = c_Au(x_A) = s(\bar{y})) \) component. The associated combined variance \( u^2(y) = \sum_{i=1}^{N} c_i^2u^2(x_i) = c_A^2u^2(x_A) + \sum_{i=A}^{N-1} c_i^2u^2(x_i) \) may be written as in equation (2):

\[
u^2(y) = c_A^2u^2(x_A) + \sum_{i=A}^{N-1} c_i^2u^2(x_i) = s^2(\bar{y}) + \sum_{i=A}^{N-1} c_i^2u^2(x_i) = \frac{1}{n(n-1)} \sum_{k=1}^{n} (q_k - \bar{q})^2 + \sum_{i=A}^{N-1} c_i^2u^2(x_i)
\]

(2)

For simplicity we will, from now on, write \( u_c \) in place of \( u_c(y) \). We consider in principle that, before starting the procedures for the physical measurement, we had made a study of all the Type B uncertainties, we know all their values and, therefore, we also know the value of the Type B combined standard uncertainty \( u_{cb} \). The value of the combined variance of the measurement will be the sum of the corresponding combined B variance \( u^2_{cb} \) and the Type A variance component \( (c_A^2u^2(x_A)) \) when the last one is included in the calculation. Considering all \( N-1 \) Type B uncertainty components \( u_b(x_i) \) associated with the measurement, the combined B variance \( u^2_{cb} \) is given by equation (3) below:

\[
u^2_{cb} = \sum_{j=1}^{N-1} c_j^2u^2(x_j)
\]

(3)
The introduction of any new component \( u_N(y) = c_N u(x_N) \) will change the combined uncertainty, given in equation (3), from \( u_cB \) to a new value \( u_* \), given by equation (4):

\[
u_*^2 = u_N^2(y) + u_{cB}^2 = c_N^2 u^2(x_N) + u_{cB}^2 = c_N^2 u^2(x_N) + \sum_{j=1}^{N-1} c_j^2 u^2(x_j)
\]  

(4)

The relative change \( \delta u_c \), produced in the preexisting \( u_cB \), will be given, according to Baratto and Bezerra [3, formula (7)], by equation (5) below.

\[
\delta u_c(N) = \frac{u_* - u_{cB}}{u_{cB}} = \frac{u_*}{u_{cB}} - 1 = \frac{\sqrt{c_N^2 u^2(x_N) + \sum_{j=1}^{N-1} c_j^2 u^2(x_j)} - 1}{\sqrt{u_{cB}^2}} + 1 - 1
\]  

(5)

We look for the number of indications that is necessary and sufficient to be acquired in the measurement. This number of indications will determine the value of the Type A uncertainty, and then the final uncertainty. In the case discussed here, the new \( N^{th} \) component \( u_N(y) \) included in the calculation, as in equation (4), is the uncertainty associated to the dispersion of the repeated observations \( q_i \), more specifically the experimental standard deviation of the mean, \( s(\bar{q}) \), given in (1) above. The change of the term \( c_N^2 u^2(x_N) \) by the experimental variance of the mean \( s^2(\bar{q}) \) results in equation (6):

\[
\delta u_c(N) = \frac{s^2(\bar{q})}{u_{cB}^2} + 1 - 1
\]  

(6)

We want to find the number \( n \) of indications that, when used in a measurement, induces a relative increase \( \delta u_c \) in the combined standard uncertainty equal to some appropriate chosen value \( Q \). The \( Q \) value is previously decided by the metrologist based on the final quality planned for the results (the value of \( Q \) could be, for example, 0.3, or 0.1, or 0.05). This is done, supposedly, when all Type B components have already been optimized for the intended purposes. Equations (7) and (8) develop the equality \( Q = \delta u_c \), and equation (9) gives the expression for the desired value of \( n \).

\[
Q = \delta u_c(N) = \sqrt{\frac{s^2(\bar{q})}{u_{cB}^2}} + 1 - 1 \Rightarrow (Q + 1)^2 - 1 = \frac{s^2(\bar{q})}{u_{cB}^2} = Q(1 + 1)
\]  

(7)

\[
Q(1 + 1) = \frac{1}{u_{cB}^2} \sum_{i=1}^{n} (q_i - \bar{q})^2 = \frac{1}{n u_{cB}^2} \sum_{i=1}^{n} (q_i - \bar{q})^2 = \frac{1}{n u_{cB}^2} s^2(q_i)
\]  

(8)

Finally, the value of \( n \) is given by:

\[
n = \frac{s^2(q_i)}{u_{cB}^2 Q(1 + 1)}
\]  

(9)
The value of $n$ delivered by equation (9) gives a value to the Type A uncertainty that, when included in the uncertainty calculation (equation (4)), will provide the effective relative increase $Q$ in the preexisting Type B combined uncertainty $u_{iB}$ (given by equation (3)). This means that a measurement made with $n$ indications will result in a final combined uncertainty that is $Q$ times greater than the preexisting Type B combined uncertainty ($u_c = Qu_{iB}$).

3. Discussion and conclusion

For some combinations of the values of $u_{iB}$ and $s(q_i)$, and the values stipulated for $Q$, the result given by equation (9) may not be an integer. In these cases the result will be round up to the next integer $n$. In practice, it is acceptable (in most cases it is mandatory) to use a number $n_a$ greater than the number $n$ derived from equation (9). The number $n_a$ to be used can be obtained from the knowledge of the experimental standard deviation $s(q_i)$ associated with the measurement. The uncertainty associated to this knowledge depends on the number of indications used in the determination of $s(q_i)$.

If we have a well-characterized measurement and we know a pooled estimate of the experimental standard deviation $s(q_i)$ with good confidence from $m_p$ indications obtained in previous measurements (for $m_p = 50$ observations the relative uncertainty in $s(\bar{q})$ is 10 percent, for example [2, GUM, E.4.3, Table E.1]), and if we had collected $m_q$ indications ($m_q \geq n$) in the current measurement, then $m_q$ will be used to calculate $u_c^2(y)$ in equation (2), and $m_p$ will be used to calculate the effective number of degrees of freedom using the Welch-Satterthwaite formula ([2], GUM, G.4). If the measurement is being taken for the first time (in which case we do not have a pooled variance), and we had made $m$ observations, than the value of $n$ calculated in (9) (or other value $m > n$) will be used for the two calculations.

The value $n_a$ to be actually used in the measurement should certainly be somewhat greater than the value $n$ given by equation (9). It may be applied to $n$ an increase factor consistent with the degree of confidence by which the experimental standard deviation $s(q_i)$ is known. This degree of confidence can be given by the relative standard deviation of the experimental standard deviation of the mean, “which is given by the ratio $\sigma(s(\bar{q}))/\sigma(\bar{q})$ and which can be taken as a measure of the relative uncertainty of $s(\bar{q})$” (of $K$ independent observations $q_k$) [2, GUM E.4.3].

From equation (1), we can see that the relative uncertainty deviation of $s(q_i)$ is equal to the relative uncertainty of $s(\bar{q})$. As the dependence of $n$ to $s(q_i)$ is quadratic (see equation (9)), the relative uncertainty of $n$ will be given by twice the relative uncertainty of $s(q_i)$ that, as we have supposed, had previously been obtained. So, if $s(q_i)$ is obtained from $K = 10$ indications, this “uncertainty of the uncertainty” is 24% (= 0.24) [2, GUM, Table E.1]. The value of $n$ given by equation (9) would then be multiplied by a factor equal to 1.48 (that is, the value to be effectively used will be $n_a = 1.48n$). This way, the consequent relative increase ($q$) in the value of the combined uncertainty $u_{iB}$ will likely satisfy the condition $q < Q$, as anticipated.

The present derivation is valid for a measurement of a single measurand that has only one uncertainty component of Type A. When the measurement involves a measurand that depends on other measurands (input quantities in a measurement model [4, VIM3: 2.50]) we need to use equation (9) above to calculate a specific value of $n$ associated to each input quantity. This will be done, of course, only for those input quantities that have a concomitant Type A uncertainty component. And only for those that contribute the most to $u_c$. The “contribution” of each component to $u_c$ may be evaluated using the results (7) or (12) given in [3].

We need to consider that this treatment is correct only in the (very restricted) GUM context, in which the definitional uncertainty is considered to be zero (or negligible), the measurand being characterized by a unique value. In the more general context addressed in [1] the inherent uncertainty is not necessarily zero. When this is the case, in general it will make little sense to spend considerable
effort in the intent of limiting the relative influence of the Type A contribution to a predetermined value Q. In a sense, this is equivalent to the procedure of increase the number n of indications in an attempt to make irrelevant the Type A uncertainty. This is not possible, because the inherent uncertainty establishes a deterministic minimum limit to the value of the uncertainty associated to the dispersion of the indications: this value cannot be smaller than the inherent or definitional uncertainty. So, it is not correct to reduce it artificially to zero. In these cases, in which the inherent uncertainty is dominant, we need to consider the use of the standard deviation, in place of the standard deviation of the mean, in the evaluation of the dispersion associated to the definitional uncertainty. This is the value to be used in the calculation of the uncertainty of the measurement. Of course, there will be some difficulties in separating the dispersion due to the inherent uncertainty from that dispersion linked to the fluctuations of the influence quantities, which are the exclusive responsible for the genuine Type A uncertainty.

Electronic Supplementary Material can be solicited through the e-mail listed in the first page. It contains a self-explanatory Excel Worksheet with a comprehensive calculation of the value of n, according to equation (9), for hypothetical measurement examples. The worksheet treats general measurement contexts with six Type B uncertainty components, whose values are chosen by the operator. It also includes the calculation of the degree of freedom and the relative increase in the expanded uncertainty which, in most cases, is the most relevant information the operator wants.

4. References
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