THEORETICAL INTRODUCTION TO B DECAYS AND THE
SOFT-COLLINEAR EFFECTIVE THEORY

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In this talk I give an introduction to methods used to give a model independent description of QCD effects in B-decays. I discuss three useful expansions: $m_b/m_W \ll 1$ for the electroweak Hamiltonian, $\Lambda_{QCD}/m_b \ll 1$ for Heavy Quark Effective theory, and $\Lambda_{QCD}/Q \ll 1$ for the Soft-Collinear Effective Theory where $Q \sim m_b$ is the large energy of light final state hadrons. I discuss predictions that can be made with each expansion.

1 Introduction

Hadrons containing b-quarks are the heaviest known bound states that are composed entirely of fundamental standard model particles. The lightest mesons, $B^0$, $\bar{B}^0$, $B^\pm$, $B_s$ decay weakly, and the plethora of decay channels provide us with a laboratory for studying electroweak physics and the hadronic structure of QCD, as well as for searching for physics beyond the standard model (see Ref.1 for recent reviews). These goals go hand in hand, since from the point of view of studying QCD the electroweak interactions provide a clean non-interacting probe, while to test the electroweak interactions or search for new physics the hadronic uncertainties from QCD must be understood.

Theoretically we wish to simplify as much as possible the role of QCD interactions without giving up on our ability to make precise predictions. This is achieved by focusing on the interactions which are strong, $\alpha_s(\mu \simeq \Lambda) \gtrsim 1$, while expanding in those which are weak $\alpha_s(\mu \gg \Lambda) \ll 1$. This requires distinguishing the perturbative mass scales $m_W$, $m_b$, ..., from $\Lambda_{QCD}$, and is made possible by constructing effective field theories. The ratio of scales serve as small expansion parameters so that predictions can be improved to any desired precision, albeit at the cost of additional non-perturbative hadronic parameters. In this talk I focus on describing the QCD aspects of B-decays inputing bits of electroweak physics as necessary. In particular I discuss the separation of the scales $m_W$, $m_t$ (mediating the weak decays), $m_b$, $Q$, (the mass of the decaying particle and energy released to light final state particles), and $\Lambda_{QCD}$ (the scale responsible for binding in the hadrons).

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2 The Electroweak Hamiltonian, $m_b \ll m_W$

By integrating out the top quarks and $Z$ and $W$ bosons from the standard model we arrive at an effective weak Hamiltonian $H_W$ describing interactions involving the $b$ and lighter quarks. Since the expansion parameter is very small $m_b/m_W \sim 1/17$ it suffices to work to leading order (meaning first order for semileptonic decays, second order for $B$--$\bar{B}$ mixing etc.). In particular for non-leptonic $\Delta B = \pm 1$ transitions we have

$$H_{W}^{\Delta B = \pm 1} = \frac{G_F}{\sqrt{2}} \sum_{j,k,\ell=1}^{2} V_{u_j b} V_{u_k \ell}^{*} C_i(\mu) O_i(\mu) + \text{h.c..} \quad (1)$$

The operator basis can be reduced using the equations of motion to

$$O_1 = (\bar{u}_j b)_{V-A}(\bar{d}_k u_k)_{V-A}, \quad O_2 = (\bar{u}_j b^\beta)_{V-A}(\bar{d}_k^{\alpha} u_k^{\alpha})_{V-A},$$

$$O_3 = \delta_{jk} (\bar{d}_k b)_{V-A} \sum_q (\bar{q}q)_{V-A}, \quad O_4 = \delta_{jk} (\bar{d}_k^\beta b^\beta)_{V-A} \sum_q (\bar{q}^\beta q^\alpha)_{V-A},$$

$$O_5 = \delta_{jk} (\bar{d}_k b)_{V-A} \sum_q (\bar{q}q)_{V+A}, \quad O_6 = \delta_{jk} (\bar{d}_k^\beta b^\beta)_{V-A} \sum_q (\bar{q}^\beta q^\alpha)_{V+A},$$

$$O_7 = -e m_b / 16\pi^2 \delta_{jk} \bar{d}_k \sigma^{\mu\nu} F_{\mu\nu} P_R b, \quad O_8 = -g_{m_b} / 16\pi^2 \delta_{jk} \bar{d}_k \sigma^{\mu\nu} G_{\mu\nu} P_R b, \quad (2)$$

with flavors $d_1 = d$, $d_2 = s$, $u_1 = u$, $u_2 = c$, $q = \{u, d, s, c, b\}$, and color indices $\alpha, \beta$. Here $O_{1,2}$ are current-current operators, $O_{3,4,5,6}$ are QCD penguin operators, and $O_{7,8}$ are magnetic penguins. Additional electroweak penguin operators $O_{7,8,9,10}^{ew}$ are also relevant but are not shown. For the $\Delta B = \pm 2$ operators relevant to $B$--$\bar{B}$ mixing we have

$$H_{W}^{\Delta B = \pm 2} = \frac{G_F^2}{16\pi^2} \sum_{\ell=1}^{2} (V_{tb} V_{t\ell})^{2} C_{BB}(\mu) \{ [\bar{d}_b b]_{V-A}(\bar{d}_\ell b)_{V-A} \} (\mu) + \text{h.c.} \quad (3)$$

Integrating out the short distance physics associated with the scales $m_W, m_t$ has left us with products of CKM parameters $V_{u_j b} V_{u_k \ell}^{*}$, along with Wilson coefficients $C_i(\mu) = C_i(\mu, m_Z, m_t)$ which encode QCD interactions that take place at scales $\mu > m_b$. The operators $O_{i}$ encode the remaining long distance physics associated with the scales $\mu = m_b, m_c, m_s, m_d, m_u, \Lambda_{QCD}$. The computation of the $C_i$ can be carried out with free external quarks and light quark fields expanded about their massless limit. This follows from the fact that matching calculations are independent of our choice of states and the $C_i$ do not depend on the long distance scales. To evolve the operators from the short distance scale $\mu = m_W$ down to $\mu = m_b$ we make use of the renormalization group which simultaneously sums series of large logarithms $[\alpha_s \ln(m_W^2/m_b^2)] \sim 1$ into the coefficients $C_i(\mu)$. Finally, the matrix elements of $O_i$ between physical hadronic states still fully depend on the long distance scales, and the quarks and gluon fields in these operators interact through the full QCD Lagrangian.

The result for $H_W$ gives some useful predictions even without distinguishing the long distance scales any further. One example is in $\bar{B}^0$--$B^0$ ($\bar{B}_s$--$B_s$) mixing for $\Delta m_\ell$ ($\Delta m_s$), the difference between mass eigenstates. The box diagrams are dominated by top quarks, and

$$\Delta m_q = \frac{G_F^2 m_b^2 m_{B_q}}{6\pi^2} \left\{ [V_{tb} V_{t\ell}]^2 \left[ S(x_\ell) \eta_B f(\mu) \right] f_{B_q}^{2} B_{B_q}(\mu) + O\left( \frac{m_{\ell}^2}{m_W^2} \right) \right\} \quad (4)$$

where the Inami-Lin function $S(x_\ell)$ contains the dependence on $x_\ell = m_Z^2/m_t^2$ from integrating out the top and electroweak bosons. The product $\eta_B f(\mu)$ contains computable QCD effects from scales between $m_W$ and $m_b$. Finally, the product $f_{B_q}^{2} B_{B_q}$ arises from the long distance
matrix element $\langle B^0|O^{\Delta B=\pm 2}|\bar{B}^0 \rangle$. These matrix elements can be computed using lattice QCD without worrying about also encoding physics at the $W$-scale.

As our second example we consider the $\sin(2\beta)$ measurement of CP-violation using time dependent CP asymmetries like

$$a_{CP}(t) = \frac{\Gamma[\bar{B}^0(t) \to J/\psi K_S] - \Gamma[B^0(t) \to J/\psi K_S]}{\Gamma[B^0(t) \to J/\psi K_S] + \Gamma[\bar{B}^0(t) \to J/\psi K_S]}.$$  \hspace{1cm} (5)

In the standard model CP-violation occurs through the CKM matrix, and is often envisioned in the unitary triangle shown in Fig. 1. To see why the measurement of $\sin(2\beta)$ is theoretically clean, note that the amplitude for $\bar{B}^0(t) \to J/\psi K_S$ has the form

$$\bar{A}(B^0(t) \to J/\psi K_S) = (V_{cb}V_{cs}^* + V_{ub}V_{us}^*) P_t + V_{cb}V_{cs}^* (P_c + T) + V_{ub}V_{us}^* P_u.$$  \hspace{1cm} (6)

In running below $m_W$ the operators $O_{1,2}$ mix with the penguin operators $O_{3-6}$, but do not upset the CKM structure. At a scale $\mu \simeq m_b$ the matrix elements in Eq. (6) are

$$P_c + T = \sum_{i=1,2} C_i \langle O_i(j,k,\ell = 2,2,2) \rangle, \quad P_t = - \sum_{i=3,\ldots} C_i \langle O_i(\ell = 2, \text{ any q}) \rangle,$$

$$P_u = \sum_{i=1,2} C_i \langle O_i(j,k,\ell = 1,1,2) \rangle.$$  \hspace{1cm} (7)

The “T” and “P” notation denotes the fact that we often picture the $b \to c\bar{c}s$ transition that takes place in the matrix elements as occurring through “tree” or “penguin” diagrams as shown in Fig. 1. Since $|V_{ub}V_{us}^*|/|V_{cb}V_{cs}^*| \sim 1/50$ the amplitude $\bar{A}$ is dominated by a single combination of strong matrix elements, $P_t + P_c + T$. In the ratio of $\bar{A}$ to its CP-conjugate amplitude $A$ any strong phase generated in $P_t + P_c + T$ cancels exactly. This uses the fact that QCD is known to preserve CP at the level of one part in $10^{12}$. Thus, the asymmetry $a_{CP}(t) = \sin(2\beta) \sin(\Delta m_d t)$ is dominated by the weak phase $\beta$.

3 The Heavy Quark Effective Theory, $\Lambda_{QCD} \ll m_b$

To improve our description of matrix elements we can consider separating the scale $m_b$ (and $m_c$) from the lighter scales $m_{u,d,s}$ and $\Lambda_{QCD}$ using the heavy quark effective theory (HQET). This is accomplished by integrating out the heavy anti-quarks and only keeping heavy quarks with fluctuations close to their mass-shell, $p_b = m_b v + k$ with $k \sim \Lambda_{QCD}$ and $v^2 = 1$. The QCD heavy quark fields $Q = b (Q = c)$ are traded for HQET fields, $Q(x) = \sum_v e^{-imv \cdot x} h_v(Q)(x)$ with

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This notation can be confusing. For instance the $O_1(j,k,\ell) = O_1(2,2,2)$ operator is generated by a tree topology at lowest order at $m_W$ but has both tree and penguin contractions in the long distance matrix element. On the other hand $O_1(1,1,2)$ is generated by a tree topology but has only penguin contractions for $B \to J/\psi K_S$. 

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Figure 1: Unitarity triangle, and examples of standard model tree and penguin diagrams.
\( \phi h_v^{(Q)} = \tilde{h}_v^{(Q)} \), and all operators involving a \( b \) (or \( c \)) develop an expansion in \( 1/m_Q \). The Lagrangian describing the heavy quarks is also expanded,

\[
\mathcal{L}_{\text{HQET}} = \sum_{v,Q} \tilde{h}_v^{(Q)} i v \cdot D \tilde{h}_v^{(Q)} + \frac{1}{2m_Q} \tilde{h}_v^{(Q)} \left[ (i D_\perp)^2 + \frac{c_F(\mu)}{2} g \sigma_{\mu\nu} G_{\mu\nu} \right] \tilde{h}_v^{(Q)} + \ldots \tag{8}
\]

In HQET Wilson coefficients like \( c_F(\mu) \) now incorporate QCD effects from scales \( m_Q \gg \mu \gg \Lambda_{\text{QCD}} \). At lowest order \( \mathcal{L}_{\text{HQET}} \) has a \( SU(4) \) spin-flavor symmetry not present in QCD which leads to additional predictions (such as relating the states \( B, B^*, D, \) and \( D^* \)). Constructing HQET allows us to separate perturbative \( \alpha_s(m_b) \) effects and define universal hadronic parameters. An example is \( \bar{\Lambda} \), \( \alpha \) HQET allows us to separate perturbative parameters. An example is \( \bar{\Lambda} \), \( \alpha \) which appear in the heavy meson mass formula \( m_{B(s)} = m_b + \bar{\Lambda} - \lambda_1/(2m_b) + d_b c_F(\mu) \lambda_2(\mu)/(2m_b) + \ldots \), and are defined by matrix elements of terms in the series in Eq. (5).

The HQET expansion is very useful in making predictions. In fact we have already encountered one example, for \( f_B^2 B_B \) lattice calculations are more practical using HQET for \( b \) quark fields (or the related Non-Relativistic QCD effective theory). This makes it possible to use larger lattice spacings \( a \gg 1/m_b \) and obtain more statistics to probe the non-perturbative effects at distance scales \( \sim 1/\Lambda_{\text{QCD}} \). Another well known application of HQET is measuring \( |V_{cb}| \) with \( B \to D^* \ell \bar{\nu}_\ell \). The exclusive differential decay rate is

\[
\frac{d\Gamma(B \to D^* \ell \bar{\nu}_\ell)}{d\omega} = \frac{G_F^2|V_{cb}|^2m_B^5}{48\pi^4} \sqrt{\omega^2 - 1} f(r^*, \omega) F^2_\ast(\omega) \tag{9}
\]

where \( r^* = m_{D^*}/m_B \), phase space produces \( f(r^*, \omega) = r^3(1 - r^2)^2(\omega + 1)^2[1 + 4\omega(1 - 2\omega r^* + r^2)/(1 + \omega)/(1 - r^2)] \), and \( F_\ast(\omega) \) contains the dependence on four QCD form factors. At leading order in \( \Lambda_{\text{QCD}}/m_{b,c} \) this dependence collapses to just the Isgur-Wise function, \( F_\ast(\omega) = \xi(\omega) \) which is normalized at zero recoil due to heavy quark symmetry, \( \xi(1) = 1 \). Experimentally extrapolating Eq. (9) to zero recoil therefore gives us a method of measuring \( |V_{cb}| \). Corrections to \( F_\ast(1) = 1 \) include calculable \( \alpha_s(m_b) \) terms in Wilson coefficients, power corrections of order \( 1/m_Q^2 \) which are amenable to determination on the lattice (order \( 1/m_Q \) effects vanishing by Luke’s theorem), and effects of the shape of \( F_\ast(\omega) \) which are constrained by analyticity.\(^{11}\) HQET can also be used for inclusive decays such as \( B \to X_\ell \ell \bar{\nu}_\ell \) and \( B \to X_u \ell \bar{\nu}_\ell \).\(^{54}\) The decay rates can be computed using an operator product expansion and beyond leading order depend on the matrix elements of HQET operators, \( \bar{\Lambda} \), \( \lambda_1 \), and \( \lambda_2 \). Together with \( |V_{cb}| \) these parameters must be simultaneously fit to the data (the fit to \( m_{B^*} - m_B \) gives \( \lambda_2 = 0.128 \pm 0.010 \text{ GeV}^2 \)). Taking the average of the CLEO and BaBar results,\(^{53}\) I find

\[
\bar{\Lambda}_{\text{MS}} = 440 \pm 80 \text{ MeV}, \quad \lambda_1^{\text{MS}} = -0.299 \pm 0.069 \text{ GeV}^2 = -(547 \pm 64 \text{ MeV})^2. \tag{10}
\]

The current averages for \( |V_{cb}| \) from the heavy flavor averaging group\(^{17}\) are

\[
|V_{cb}|_{\text{excl}} = (42.6 \pm 1.2 \pm 1.9) \times 10^{-3}, \quad |V_{cb}|_{\text{incl}} = (41.9 \pm 0.7 \pm 0.6) \times 10^{-3}, \tag{11}
\]

where the first errors are experimental and the second are an estimate of the remaining theoretical uncertainty.

A heavy quark expansion is also useful in determinations of \( |V_{ub}| \) from semileptonic \( b \to u \) decays. For these decays the cuts to exclude the large \( b \to c \) background make things more challenging. For \( B \to X_u \ell \bar{\nu}_\ell \) there are four regions where different theoretical tools apply:

I) \( m^2_X \gg E_X \Lambda \gg \Lambda^2 \) \quad local OPE in \( \Lambda/m_b \),

II) \( m^2_X \sim E_X \Lambda \gg \Lambda^2 \) \quad shape function region,

III) \( m^2_X \sim \Lambda^2, E_X \sim \Lambda \) \quad resonance region (\( B \to \pi \ell \bar{\nu}_\ell, \ldots \)), slow exclusive \( X \),

IV) \( m^2_X \sim \Lambda^2, E_X \gg \Lambda \) \quad resonance region, energetic exclusive \( X \).
Region I is the most theoretically clean, and involves a local OPE as in $B \to X_c \ell \bar{\nu}_\ell$ decays. Cuts on $m_{X_c}^2$ and $q^2$ of the leptons can be made to reject the $b \to c$ background and remain in this region. Another possibility is to experimentally reconstruct the total rate. The simplest cut which rejects $b \to c$ is on the charged lepton energy and puts us in region II. Typically cuts which reject $b \to c$ and leave us in II have more events. In this region the rate becomes sensitive to a non-perturbative shape function and has an expansion in $\Lambda/E_X$. The shape function can be measured in $B \to X_s \gamma$, which then allows for a model independent extraction of $|V_{ub}|$ in the endpoint region. For regions III and IV we are dealing with exclusive decays and a heavy quark expansion alone does not provide enough information for a model independent extraction of $|V_{ub}|$. The theoretical description of regions II and IV can be improved by using the soft-collinear effective theory which is discussed in the next section. Currently for exclusive decays the experimental analysis uses form factor models and QCD sum rules and the theoretical uncertainty is still quite large (but should improve as lattice results become more accurate). Averaging the BaBar, Belle, and CLEO results separately for each region avoids dealing with their quantitatively different uncertainties and gives

$$
|V_{ub}|_{\text{II}} = (4.21 \pm 0.29 \pm 0.55) \times 10^{-3},
|V_{ub}|_{\text{endpt}} = (4.17 \pm 0.14 \pm 0.6) \times 10^{-3},
|V_{ub}|_{\text{III+IV}} = (3.4 \pm 0.2 \pm 0.5) \times 10^{-3}.
$$

The uncertainties are experimental and an estimate of the theoretical uncertainty (the latter values are taken from those quoted by the experimental analyses with no reduction for averaging).

4 The Soft-Collinear Effective Theory, $\Lambda_{QCD} \ll Q$

To improve the theoretical description of $B$-decay processes with energetic hadrons we can consider separating the scales $Q \sim E_X, m_b$ from the lighter scales $m_{u,d,s}, \Lambda_{QCD}$ using the soft-collinear effective theory (SCET). This theory includes all the degrees of freedom of HQET, but adds collinear quarks and gluons with large energy $p^2 \sim Q^2 \lambda^2$ where $\lambda \ll 1$. SCET describes decays or scattering processes with energetic hadrons or jets, and can be used to describe $B$-decays to light hadrons as well as more traditional QCD processes like DIS or form factors. The analog of the scale separation formula in Eq. (1) are now more complicated and are referred to as factorization theorems. They involve convolutions between perturbative and non-perturbative functions. The convolutions arise because the separation of scales $Q^2 \gg Q \Lambda \gg \Lambda^2$ allows perturbative and non-perturbative momenta $p^2$ to communicate in some components. For example $p^2$ collinear momenta with hard $p^2 \sim Q^2$ momenta. The goal of model independent analysis in SCET is the same as that in HQET, namely to measure the non-perturbative functions in one process and then use them in another.

In the remainder I give a brief overview of the SCET formalism focusing on two sets of degrees of freedom, SCET$_I$ (collinear $p^2 \sim Q \Lambda, \text{usoft } p^2 \sim \Lambda^2$) and SCET$_{II}$ (collinear $p^2 \sim \Lambda^2, \text{soft } p^2 \sim \Lambda^3$). The former theory describes processes with inclusive jets of size $p_T^2 \sim Q \Lambda$ and soft hadrons, while the latter describes exclusive processes with both energetic and soft hadrons. Using light cone coordinates $(p^+ = n \cdot p, p^- = \bar{n} \cdot p, p^\perp)$, the momenta of collinear fields scale as $p^\mu \sim Q(\lambda^2, 1, \lambda), \lambda = \sqrt{\Lambda/Q}$ in SCET$_I$, while $p^\mu \sim Q(\eta^2, 1, \eta)$ with $\eta = \Lambda/Q$ in SCET$_{II}$.

In SCET$_I$ an important ingredient is the lowest order collinear quark Lagrangian

$$
\mathcal{L}_{\xi\xi}^{(0)} = \xi_{n,p} \left\{ i n \cdot D + i D^\perp_\ell \frac{1}{i n \cdot D_c} \frac{i D^\perp_c}{2} \right\} \frac{\bar{\eta}}{2} \xi_{n,p},
$$

where $inD = in \cdot \partial + gn \cdot A_n + gn \cdot A_{us}, in \cdot D_c = \bar{\mathcal{P}} + gn \cdot A_n, i D^\perp_c = \overline{\mathcal{P}}^\perp + g A_n^\perp$. It has interactions with all components of the collinear gluons and includes collinear quark pair production and annihilation. For interactions with usoft gluons $A_{us}$ only the $n \cdot A_{us}$ components interact with
quarks at leading order and the collinear propagators reduce to eikonal propagators when the external collinear quarks are put on-shell.

In deriving the free propagator from Eq. (13) it naively appears that only a single pole would appear in the complex plane, \(1/(n \cdot p + p_\perp^2/\bar{n} \cdot p + i\epsilon)\), which would indicate that \(\mathcal{L}^{(0)}_{\xi_\xi}\) does not contain both particles and antiparticles. To arrive at the correct result\(^{13}\) we recall that the field \(\xi_{n,p} = \xi^+_{n,p} + \xi^-_{n,-p}\) so a negative \(\bar{n} \cdot p\) momentum label corresponds to an antiparticle with \(-\bar{n} \cdot p > 0\). Taking label momentum \((\bar{n} \cdot p, p_\perp)\) and residual momentum \(n \cdot p\) both in the direction of the fermion arrow, the full SCET propagator is

\[
i \frac{g}{2} \left[ \frac{\theta(\bar{n} \cdot p)}{n \cdot k + p_\perp^2/\bar{n} \cdot p + i\epsilon} - \frac{\theta(-\bar{n} \cdot p)}{-n \cdot p - p_\perp^2/\bar{n} \cdot p + i\epsilon} \right].
\]

(14)

The second term corresponds to the collinear antiquarks. In a more compact form we have

\[
i \frac{g}{2} \left[ \frac{\theta(\bar{n} \cdot p)}{n \cdot p + p_\perp^2/\bar{n} \cdot p + i\epsilon} + \frac{\theta(-\bar{n} \cdot p)}{-n \cdot p + p_\perp^2/\bar{n} \cdot p - i\epsilon} \right] = i \frac{g}{2} \frac{\bar{n} \cdot p}{n \cdot p \bar{n} \cdot p + p_\perp^2 + i\epsilon}.
\]

(15)

Thus there are indeed two poles in the collinear quark propagator. An alternative way of writing the propagator in Eq. (15) is

\[
\frac{1}{n \cdot p + p_\perp^2/\bar{n} \cdot p + i\epsilon \text{sign}(\bar{n} \cdot p)}.
\]

(16)

This form is useful for considering interactions with ultrasoft gluons. If a collinear quark propagator has momentum \(p + k\) where \(k\) is ultrasoft and \(p\) external, then onshell \(n \cdot p + p_\perp^2/\bar{n} \cdot p = 0\) and the propagator reduces to

\[
\frac{1}{n \cdot k + i\epsilon \text{sign}(\bar{n} \cdot p)},
\]

(17)

which is eikonal as promised. If SCET is formulated in position space\(^{15,17}\), then separate fields for quarks and antiquarks can be used to construct \(\mathcal{L}^{(0)}_{\xi_\xi}\) and obtain this pole structure.

The form of operators and interactions in SCET are constrained by symmetries. For the spin structure note that \(\xi_n\) are two-component fields with \(\not{n} \xi_n = 0\). The spinor components which are small for motion in the \(n\) direction have been integrated out. The \(\mathcal{L}^{(0)}_{\xi_\xi}\) Lagrangian has a global helicity spin symmetry\(^{12}\) with generator \(h_n = \xi_n^\mu \gamma^\mu /4\). \(h_n\) is independent of individual collinear particle momenta since \(\xi_n,p\) only describe particles moving close to the \(n\) direction. SCET also has a rich set of gauge symmetries\(^{14}\). In particular both the collinear and usoft gluons have their own set of gauge transformations which have support over the corresponding momenta, \(i\partial^\mu U_c \sim (\lambda^2, 1, \lambda)\) and \(i\partial^\mu U_u \sim (\lambda^2, \lambda^2, \lambda^2)\). The longer distance \(A_{ns}^\mu\) fields act like background fields to the shorter distance collinear gluon \(A_{n}^\mu\) fields. Lorentz symmetry is also realized in an interesting way in SCET. The introduction of the basis vectors \(n\) and \(\bar{n}\) breaks some of the symmetry, which is then restored by reparameterization transformations which leave \(n \bar{n} = 2\) and \(n^2 = \bar{n}^2 = 0\). These transformations fall into three classes, I) \(n \rightarrow n + \Delta_{\perp}, \bar{n} \rightarrow \bar{n}, \) II) \(n \rightarrow n, \bar{n} \rightarrow \bar{n} + \epsilon_{\perp},\) and III) \(n \rightarrow e^{\alpha} n, \bar{n} \rightarrow e^{-\alpha} \bar{n},\) where \(\Delta_{\perp} \sim \lambda\) and \(\epsilon_{\perp}, \alpha \sim \lambda^0\). Demanding invariance under these transformations order by order in \(\lambda\) eliminates some collinear operators and constrains the form of others, see for example\(^{15,16}\). A final constraint on operators which applies only to SCET\(I\) is locality. SCET\(I\) has non-local objects such as the Wilson line \(W\) built out of \(\bar{n} \cdot A_n\) fields and \(\bar{n} \cdot p\) momenta which are \(\sim Q,\) however no inverse powers of smaller momenta appear in Feynman rules. For SCET\(II\) we integrate out the scale \(Q\lambda,\) and operators involving soft and collinear fields are more non-local as emphasized in Ref.\(^{17}\).
An important aspect of SCET is the factorization of distance scales. Hard-collinear factorization is codified in the relation \( C(i\bar{n} \cdot D_c) = WC(\bar{\mathcal{P}}) W^\dagger \), where \( \mathcal{P} \) is a momentum operator that picks out \( \bar{n} \cdot p \) momenta of collinear fields. For example suppressing power corrections it allows us to write the QCD heavy-to-light current
\[
\bar{u} \Gamma b = \xi_\alpha \bar{c} C(-i\bar{n} \cdot \bar{\mathcal{D}}_c) \Gamma(W h_v) = [\xi_\alpha W h_v] C(\bar{\mathcal{P}}) \int d\omega (\xi_\alpha W) \delta(\omega - \bar{\mathcal{P}}) \Gamma h_v C(\omega) = \int d\omega J(\omega) C(\omega),
\]
which separates the perturbatively calculable coefficient \( C(\omega) \) from the longer distance SCET current \( J(\omega) \). Another type of factorization, usoft-collinear, occurs through interactions within SCET. By making the field redefinitions \( \xi_\alpha \rightarrow Y \xi_\alpha \), \( A_{us}^\mu \rightarrow Y A_{us}^\mu \) the coupling of usoft gluons is removed from the leading quark and gluon Lagrangians \( \mathcal{L}_{\xi \xi}^{(0)} \), \( \mathcal{L}_{\xi \eta}^{(0)} \). Here \( Y \) is a path ordered exponential along the \( n \) direction from \( -\infty \) to \( x^\mu \) of \( n \cdot A_{us} \) gluons. After the field redefinition, factors of \( Y \) appear in operators involving usoft fields such as the heavy-to-light current \( (\xi_\alpha W) \Gamma Y^\dagger h_v \). They also show up in power suppressed operators such as the order \( \lambda \) collinear quark Lagrangian
\[
\mathcal{L}_{\xi \xi}^{(1)} = (\xi_\alpha W)(Y^\dagger D_{us}^{-1} Y) \frac{1}{p^2} (W^\dagger D_{us}^{-1} W)(W^\dagger \xi_\alpha) + \text{h.c.}
\]
Using relations such as \( (Y^\dagger D_{us}^{-1} Y) = i\partial_{us} + [1/(i\bar{n} \cdot \partial)]Y^\dagger [i\bar{n} \cdot D_{us}, D_{us}^{-1} Y] \) it always possible to collect the collinear and usoft fields in separate matrix elements even at subleading order in \( \lambda \).

To construct SCET\textsubscript{II} it is useful to make use of results for SCET\textsubscript{I}, essentially because SCET\textsubscript{I} describes how the soft-collinear modes of SCET\textsubscript{II} communicate. After factorizing the usoft-collinear interactions in SCET\textsubscript{I} we match SCET\textsubscript{I} to SCET\textsubscript{II} by integrating out \( p^2 \sim Q \Lambda \) fluctuations. This is done using states involving only \( p^2 \sim \Lambda^2 \) particles. In many cases this matching is trivial, for instance \( \mathcal{L}_{\xi \xi}^{(0)} \rightarrow \mathcal{L}_{\xi \xi}^{(0)} \) and \( \xi_\alpha W T Y^\dagger h_v \rightarrow \xi_\alpha W T S h_v \) (where \( S \) is an identical Wilson line to \( Y \) but built out of soft fields). In external operators and at subleading order in the power counting matching calculations involving time ordered products of operators with both collinear and usoft fields appear. These can generate additional Wilson coefficients, and are referred to as jet functions \( J(\Lambda Q) \).

For power counting operators in SCET\textsubscript{II} it is useful to have a general formula that applies both to external operators and soft-collinear Lagrangians, much like the formula\[^{14} \]
\[
\delta = 4 + 4u + \sum_k (k - 4) V_k^C + (k - 8) V_k^U \quad \text{in SCET}\textsubscript{I}.\]
Since the full SCET\textsubscript{II} formula does not appear in the literature I take this opportunity to derive it. Counting the factors \( \eta = \Lambda / Q \) in an arbitrary diagram we find it scales as \( \sim \eta^\delta \) where
\[
\delta = \sum_k (V_k^C + V_k^{SC} + V_k^S) + 4L^C + 4L^S + 5L^{SC} - 4I^C - 4I^S.
\]
Here \( V_k^j \) counts the number of type-\( j \) operators of order \( \eta^k \) (from the scaling of fields and derivatives). For instance \( (\xi_\alpha W) \Gamma(S^\dagger h_v) \) counts as \( V_5^{SC} = 1 \). The \( L^j \)'s count factors of \( \eta \) in loop measures, and the \( I^j \)'s correct for the \( \eta \)'s from internal propagators. Note that in general momentum conservation requires that some loops will be neither collinear or soft, but instead have momenta \( (p^+, p^-, p^+ \sim (\eta^2, \eta, \eta) \). These momenta are counted by \( L^{SC} \) and occur despite the fact that no degrees of freedom (or individual propagators) have this scaling. Using the Euler identity \( \sum_k (V_k^C + V_k^{SC} + V_k^S) + L^S + L^C + L^{SC} - L^S - L^C = 1 \) then leaves
\[
\delta = 4 + \sum_k (k - 4)(V_k^C + V_k^{SC} + V_k^S) + L^{SC}.\]
This formula is still inconvenient since it depends on \( L^{SC} \) and not solely on the vertices. This can be circumvented by using two more Euler identities ala Ref.\[^{14} \]. In any diagram if we erase
Table 1: A few examples of processes that can be described by SCET.

| Process                              | Frame | Degrees of Freedom ($p^2$)                                      | Non-Pert. functions | Refs. |
|--------------------------------------|-------|----------------------------------------------------------------|--------------------|-------|
| $B^0 \rightarrow D^+ \pi^-$, ...     | $B$ rest | collinear ($\Lambda^2$), soft ($\Lambda^2$)           | $\xi(w), \phi_\pi$ | 22    |
| $\bar{B}^0 \rightarrow D^0 \pi^0$, ... | $B$ rest | collinear ($\Lambda^2$), soft ($\Lambda^2$)           | $S(k_f^+), \phi_\pi$ | 23    |
| $B \rightarrow X_s^{endpt} \gamma^*$ | $B$ rest | collinear ($\Lambda^2$), usoft ($\Lambda^2$)          | $f(k^+)$            | 11112 |
| $B \rightarrow \pi \ell \nu, ...$   | $B$ rest | collinear ($\Lambda^2$), usoft ($\Lambda^2$)          | $\phi_B(x), \zeta_\pi(x)$ | 19146 |
| $B \rightarrow \pi \pi$             | $B$ rest | collinear ($\Lambda^2$), soft ($\Lambda^2$)            | $\phi_B, \phi_\pi, \zeta_\pi(x)$ | 25    |
| $B \rightarrow K^* \gamma$          | $B$ rest | collinear ($\Lambda^2$), soft ($\Lambda^2$)            | $\phi_B, \phi_K, \zeta_{K^*}^2(x)$ | 27    |
| $e^- p \rightarrow e^- X$           | $p$ Breit | collinear ($\Lambda^2$)                                  | $f_{i/p}(\xi), f_{g/p}(\xi)$ | 28    |
| $e^- e^- \rightarrow j_1 + j_2$     | $\pi$ Breit | collinear ($\Lambda^2$), soft ($\Lambda^2$)          | $\phi_\pi$            | 2829  |
| $e^+ e^- \rightarrow j_1 + j_2$     | $e^+ e^- \text{ CM}$ | collinear ($\Lambda^2$), soft ($\Lambda^2$)          | $A^q_1, A^q_1$        | 30    |
| $\gamma^* M \rightarrow M'$         | $\pi$ Breit | collinear ($\Lambda^2$), soft ($\Lambda^2$)            | $\phi_M, \phi_M'$     | 2829  |
| $e^+ e^- \rightarrow J/\psi X^{endpt}$ | $\gamma$ rest | collinear ($\Lambda^2$), usoft ($\Lambda^2$) | $S^{(8,n)}(k^+)$ | 31    |

all collinear lines then we are left with $N_S$ connected components (or $N_C$ if we instead erase all soft lines). Thus, $\sum_k (V^S_k + V^{SC}_k) + L^S - I^S = N_S$ and $\sum_k (V^C_k + V^{SC}_k) + L_C - I_C = N_C$, which taken together with our original Euler identity give

$$L^{SC} = 1 - N_C - N_S + \sum_k V^{SC}_k.$$  \hspace{1cm} (22)

Note that diagrams with more than one soft-collinear vertex are required for $L^{SC} \geq 1$. Combining Eqs. 21 and 22 gives the final result

$$\delta = 5 - N_C - N_S + \sum_k (k-4)(V^S_k + V^C_k) + \sum_k (k-3)V^{SC}_k.$$  \hspace{1cm} (23)

The factor of $(k-3)$ for subleading mixed soft-collinear Lagrangians indicates that we should subtract 3 from the scaling of the operator in determining the power to use for these Lagrangians. This -3 subtraction agrees with Ref. 17. With the prefactor in Eq. 23 the result also works for external operators. For instance for $\bar{B}^0 \rightarrow D^+ \pi^-$ the electroweak Hamiltonian matches onto an SCET$_H$ operator that counts $V^{SC}_5 = 1$, $5 - N_C - N_S = 3$, so the leading order diagrams all have $\delta = 5$, as in 20.

SCET has rich applications to phenomenology which I do not have room to discuss. A few examples of processes which have been considered are summarized in Table 1 along with references to the literature. Also shown are the degrees of freedom and non-perturbative functions that describe each process at leading order in the power counting.
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