STRICHARTZ ESTIMATES IN WIENER AMALGAM SPACES FOR THE SCHRÖDINGER EQUATION

ELENA CORDERO AND FABIO NICOLA

ABSTRACT. We study the dispersive properties of the Schrödinger equation. Precisely, we look for estimates which give a control of the local regularity and decay at infinity separately. The Banach spaces that allow such a treatment are the Wiener amalgam spaces, and Strichartz-type estimates are proved in this framework. These estimates improve some of the classical ones in the case of large time.

1. Introduction

The study of space-time integrability properties of the solution of the Cauchy problem for the Schrödinger equation

\[
\begin{cases}
i\partial_t u + \Delta u = 0, \\
u(0, x) = u_0(x),
\end{cases}
\]

with \((t, x) \in \mathbb{R} \times \mathbb{R}^d, d \geq 1\), has been pursued by many authors in the last thirty years. The celebrated homogeneous Strichartz estimates \cite{10, 12, 13, 20} for the solution \(u(t, x) = (e^{it\Delta} u_0)(x)\) read

\[
\|e^{it\Delta} u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2_x},
\]

for \(q \geq 2, r \geq 2\), with \(2/q + d/r = d/2\), \((q, r, d) \neq (2, \infty, 2)\), i.e., for \((q, r)\) Schrödinger admissible. Here, as usual, we set

\[
\|F\|_{L^q_t L^r_x} = \left( \int \|F(t, \cdot)\|_{L^r_x}^q dt \right)^{1/q}.
\]

As a matter of fact, these estimates express a gain of local \(x\)-regularity of the solution \(u(t, \cdot)\), and a decay of its \(L^r_x\)-norm, both in some \(L^q_t\)-averaged sense.

In this paper we study similar estimates in spaces which, unlike the \(L^p\) spaces, control the local and global behaviour of a function independently (for example, spaces whose functions are locally in some \(L^p\) space whereas globally display a \(L^q\)-decay, with \(q \neq p\)). The Wiener amalgam spaces, introduced by H. Feichtinger in

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1980 [4], enjoy this property and are the means to perform such a finer analysis of local integrability and decay at infinity. For instance, the desired Wiener amalgam space in the example above is denoted by \( W(\mathcal{F}L^p, L^q) \); similarly, one can consider other Banach spaces to measure the local behaviour of a function, e.g., \( \mathcal{F}L^p \) instead of \( L^p \), the related Wiener amalgam space being then \( W(\mathcal{F}L^p, L^q) \), and so on. In general, given two (suitable) Banach function spaces \( B, C \), the Wiener amalgam space \( W(B, C) \) is the space of functions which “locally” are in \( B \) and “globally” in \( C \) (see [4] and Section 2 below for the precise definition).

Here is a brief discussion of the results proved in this paper. As usual we first establish an estimate of dispersive type. Namely, in our setting, we prove

\[
\|e^{it\Delta} u_0\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \left( \frac{1 + |t|}{t^2} \right) \|u_0\|_{W(\mathcal{F}L^\infty, L^1)}.
\]

Notice that \( L^1 = W(L^1, L^1) \hookrightarrow W(\mathcal{F}L^\infty, L^1) \) and \( W(\mathcal{F}L^1, L^\infty) \hookrightarrow W(L^\infty, L^\infty) = L^\infty \).

Comparing with the classical dispersive estimate [12]

\[
\|e^{it\Delta} u_0\|_{L^\infty} \lesssim |t|^{-d/2} \|u_0\|_{L^1},
\]

we therefore get an improvement for every fixed \( t \neq 0 \). Indeed, we start from less regular data \( u_0 \in W(\mathcal{F}L^\infty, L^1) \) (for example, a compactly supported Radon measure in \( \mathbb{R}^d \)), and we end up with a solution \( u(t, \cdot) \) locally in \( \mathcal{F}L^1 \), which is strictly smaller than \( L^\infty \). Observe that in (3) we recapture the classical time decay \( |t|^{-d/2} \) as \( |t| \to +\infty \), whereas getting a better result locally in space costs a worsening as \( |t| \to 0 \): the factor \( |t|^{-d/2} \) is replaced by \( |t|^{-d} \) as \( t \to 0 \).

Next, we focus on space-time estimates. Precisely, upon defining the Wiener amalgam norms as

\[
\|F\|_{W(L^{q_1}, L^{q_2}), W(\mathcal{F}L^{r_1}, L^{r_2})_x} := \left\| \left\| F(t) \right\|_{W(\mathcal{F}L^{r_1}, L^{r_2})_x} \right\|_{W(L^{q_1}, L^{q_2})_t}
\]

\[
= \left\| F \right\|_{W(L^{q_1}, \mathcal{F}L^{r_1}, L^{r_2}, L^{q_2})_x},
\]

(the last equality shall be proved in Section 2), our result can be stated as follows.

**Theorem 1.1.** Let \( 4 < q, \tilde{q} \leq \infty, 2 \leq r, \tilde{r} \leq \infty \), such that

\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2},
\]

and similarly for \( \tilde{q}, \tilde{r} \). Then we have the homogeneous Strichartz estimates

\[
\|e^{it\Delta} u_0\|_{W(L^{q/2}, L^q)} \lesssim \|u_0\|_{L^2},
\]

the dual homogeneous Strichartz estimates

\[
\| \int e^{-is\Delta} F(s) \, ds \|_{L^2} \lesssim \| F \|_{W(L^{q/(2q)}, L^q)} \| W(\mathcal{F}L^r, L^{r})_x \|,
\]
and the retarded Strichartz estimates

\[ \left\| \int_{s<t} e^{i(t-s)\Delta} F(s) \, ds \right\|_{W(L^{q/2},L^{q})_{t}W(FL^{r'},L^{r'})_{x}} \lesssim \left\| F \right\|_{W(L^{(q/2)'},L^{(r')'})_{t}W(FL^{r},L^{r'})_{x}}. \]

The solution of (11), with \( u_{0} \in L^{2} \), is therefore shown to be in the space \( L_{t,loc}^{q/2} W(FL^{r'},L^{r'})_{x} \) and, roughly speaking, its \( W(FL^{r'},L^{r'})_{x} \)-norm has a \( L_{t}^{q} \)-decay at infinity, for \( 2/q + d/r = d/2 \), \( q > 4 \), \( r \geq 2 \). Hence, for \( q > 4 \), the result is better than the classical one for large time, while locally we pay that improvement: the classical \( L_{t}^{q} \) regularity is replaced by \( L_{t}^{q/2} \).

In the case \((q,r) = (\infty,2)\) (included in Theorem 1.1), we recapture the usual estimate

\[ \left\| e^{it\Delta} u_{0} \right\|_{L_{t}^{q}L_{x}^{p}} \lesssim \left\| u_{0} \right\|_{L_{x}^{p}}. \]

From a local point of view, the other cases are not comparable to those of (2), since here we gain in space (being \( FL^{r'} \subset L^{r} \) for \( r \geq 2 \)) but lose in time. In fact, it is important to observe that, for any given \( s \geq 1 \), the inclusion \( L^{s} \subset (FL^{r'})_{loc} \) is always false when \( r > 2 \), so that estimate (7) cannot be deduced from (2) applied to the Schrödinger admissible pair \((q/2,s = 2dq/(dq - 8))\), \( q > 4 \). In other terms, estimate (7) contains information on the oscillations in \( x \) of the solution, which cannot be extracted from (2).

Observe that locally the \( L_{x}^{2} \)-norm would be obtained with \( q = 4 \), which corresponds to our endpoint case. Namely, let \( P \) be the endpoint

\[ P := (4,2d/(d - 1)), \quad d > 1; \]

then our version of the main result of [13] can be formulated as follows.

**Theorem 1.2.** For \((q,r) = P, \ d > 1\), we have

\[ \left\| e^{it\Delta} u_{0} \right\|_{W(L^{2},L^{1/2})_{t}W(FL^{r'},L^{r'})_{x}} \lesssim \left\| u_{0} \right\|_{L_{x}^{p}}; \]

\[ \left\| \int e^{-is\Delta} F(s) \, ds \right\|_{L_{x}^{2}} \lesssim \left\| F \right\|_{W(L^{2},L^{1/3})_{t}W(FL^{r'},L^{r'})_{x}}. \]

The retarded estimates (9) still hold with \((q,r)\) satisfying (10), \( q > 4, r \geq 2, (\tilde{q},\tilde{r}) = P, \) if one replaces \( FL^{r'} \) by \( FL^{r'}.2 \). Similarly, it holds for \((q,r) = P \) and \((\tilde{q},\tilde{r}) \neq P \) as above if one replaces \( FL^{r'} \) by \( FL^{r'}.2 \). It holds for both \((p,r) = (\tilde{p},\tilde{r}) = P \) if one replaces \( FL^{r'} \) by \( FL^{r'}.2 \) and \( FL^{r'} \) by \( FL^{r'}.2 \).

Here \( L^{r'.2} \) is a Lorentz space (see [16] and Section 2 below). We recall, \( L^{r'.2} \subset L^{r} \) and \( L^{r'} \subset L^{r'.2} \), for \( r > 2 \). Indeed, here we attain slightly weaker estimates than in Theorem 1.1.

Let us observe that, for \( d = 2 \), there is no estimate of the form \( \left\| e^{it\Delta} u_{0} \right\|_{L_{t}^{2}L_{x}^{p}} \lesssim \left\| u_{0} \right\|_{L_{x}^{p}}; \)
\[ \|u_0\|_{L^2}, \text{ with } s \geq 1 \text{ (see \cite{[17]}). However, Theorem 1.2 above shows that the solution } e^{it\Delta}u_0 \text{ lies in the space } L^2_{t,\text{loc}}(W(FL^{4/3,2}, L^4)) \text{ and that, roughly speaking, its } W(FL^{4/3,2}, L^4) \text{-norm has a } L^4_t \text{-decay at infinity.}

A natural question arises: can our results be extended to the case } 2 \leq q < 4? \text{ So, for instance, for } q = 2 \text{ one would obtain}

\[ \|e^{it\Delta}u_0\|_{W(L^1,L^2), W(FL^{r',r}, L^r)} \lesssim \|u_0\|_{L^2_x}, \quad r = 2d/(d - 2). \]

For sure this case cannot be treated with the techniques developed here, which use the Hardy-Littlewood-Sobolev’s singular integral theory.

The Strichartz estimates of Theorems 1.1 and 1.2 can be applied, e.g., to the well-posedness of non-linear Schrödinger equations or of linear Schrödinger equations with time-dependent potentials. As an example, in Section 6 we combine our estimates with the methods of \cite{[3]} to deduce some estimates for the Schrödinger equation with a potential \( V(t, x) \in L^q_t L^p_x \).

We point out that interesting estimates for the operator \( e^{it\Delta} \), for fixed \( t \), have been recently obtained in \cite{[1]} and \cite{[2]} using the framework of modulation spaces. Such spaces are related to the Wiener amalgam spaces considered here via Fourier transform. The overlap with our results is Proposition 3.1 below, which was first obtained there.

The paper is organized as follows. In Section 2 we recall the definition and the main properties of the function spaces used in this paper. In Section 3 we prove the dispersive estimate and other fixed time estimates for the solution of (1). In Section 4 we prove Theorem 1.1. In Section 5 we prove Theorem 1.2. Finally in Section 6 we present the above mentioned application to Schrödinger equations with time-dependent potentials.

**Notation.** We define \( |x|^2 = x \cdot x \), for \( x \in \mathbb{R}^d \), where \( x \cdot y = xy \) is the scalar product on \( \mathbb{R}^d \). The space of smooth functions with compact support is denoted by \( C_0^\infty(\mathbb{R}^d) \), the Schwartz class is \( S(\mathbb{R}^d) \), the space of tempered distributions \( S'(\mathbb{R}^d) \). The Fourier transform is normalized to be \( \hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t \omega} dt \). Translation and modulation operators (time and frequency shifts) are defined, respectively, by

\[ T_xf(t) = f(t - x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i t \omega} f(t). \]

We have the formulas \( (T_xf) = M_{-x}\hat{f}, \quad (M_\omega f) = T_\omega\hat{f} \), and \( M_\omega T_x = e^{2\pi i x \omega} T_x M_\omega \). The notation \( A \lesssim B \) means \( A \leq cB \) for a suitable constant \( c > 0 \), whereas \( A \asymp B \) means \( c^{-1}A \leq B \leq cA \), for some \( c \geq 1 \). The symbol \( B_1 \hookrightarrow B_2 \) denotes the continuous embedding of the linear space \( B_1 \) into \( B_2 \).
2. Function Spaces

Lorentz spaces (\[15, 16\]). We recall that the Lorentz space \(L^{p,q}\) on \(\mathbb{R}^d\) is defined as the space of measurable functions \(f\) such that
\[
\|f\|_{pq}^* = \left( \frac{q}{p} \int_0^\infty \left( t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} < \infty,
\]
when \(1 \leq p < \infty, 1 \leq q < \infty\), and
\[
\|f\|_{pq} = \sup_{t>0} t^{1/p} f^*(t) < \infty
\]
when \(1 \leq p \leq \infty, q = \infty\). Here, as usual, \(\lambda(s) = |\{ |f| > s \}|\) denotes the distribution function of \(f\) and \(f^*(t) = \inf\{ s : \lambda(s) \leq t \}\).

We also recall that the following equality holds:
\[
\|f\|_{p\infty}^* = \sup_{s>0} s \lambda(s)^{1/p},
\]
which gives a characterization of \(L^{p,\infty}\).

One has \(L^{p,q_1} \subset L^{p,q_2}\) if \(q_1 \leq q_2\), and \(L^{p,p} = L^p\). Moreover, for \(1 < p < \infty\) and \(1 \leq q \leq \infty\), \(L^{p,q}\) is a normed space and its norm \(\| \cdot \|_{L^{p,q}}\) is equivalent to the above quasinorm \(\| \cdot \|_{pq}^*\).

The following important result ([19, Theorem 2], page 139) will be crucial in the sequel. It generalizes the Hardy-Littlewood-Sobolev fractional integration theorem (see e.g. [14], page 119) which corresponds to the model case of convolution by \(K(x) = |x|^{-\alpha} \in L^{d/\alpha,\infty}, 0 < \alpha < d\).

**Theorem 2.1.** Let \(1 \leq p < q < \infty, 0 < \alpha < d,\) with \(1/p = 1/q + 1 - \alpha/d\). Then,
\[
L^p(\mathbb{R}^d) \ast L^{d/\alpha,\infty}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d).
\]

Wiener amalgam spaces ([4, 5, 6, 7, 8]). Let \(g \in C_0^\infty(\mathbb{R}^n)\) be a test function that satisfies \(\|g\|_{L^2} = 1\). We will refer to \(g\) as a window function.

Let \(B\) one of the following Banach spaces: \(L^p, \mathcal{F}L^p, 1 \leq p \leq \infty, L^{p,q}, 1 < p < \infty, 1 \leq q \leq \infty,\) possibly valued in a Banach space, or also spaces obtained from these by real or complex interpolation.

Moreover, let \(C\) be one of the following Banach spaces: \(L^p, 1 \leq p \leq \infty,\) or \(L^{p,q}, 1 < p < \infty, 1 \leq q \leq \infty,\) scalar valued.

For any given function \(f\) which is locally in \(B\) (i.e. \(gf \in B, \forall g \in C_0^\infty\)), we fix \(g \in C_0^\infty\) and set \(f_B(x) = \|f T_x g\|_B\). Then, the Wiener amalgam space \(W(B,C)\) with local component \(B\) and global component \(C\) is defined as the space of all functions \(f\) locally in \(B\) such that \(f_B \in C\). Endowed with the norm \(\|f\|_{W(B,C)} := \|f_B\|_C\), \(W(B,C)\) is a Banach space. Besides, different choices of \(g \in C_0^\infty\) generate the same space and yield equivalent norms. In particular, if we choose \(B = \mathcal{F}L^1\) (the Fourier algebra), then the space of admissible windows for the Wiener amalgam
spaces $W(\mathcal{F}L^1, C)$ can be enlarged to the so-called Feichtinger algebra $W(\mathcal{F}L^1, L^1)$. Let us recall that the Schwartz class $S$ is dense in $W(\mathcal{F}L^1, L^1)$.

Observe that this definition mixes (amalgamates) the local properties of functions in $B$ with the global properties of functions in $C$.

Now, it is straightforward to prove the norm equality in \(5\). Precisely, for a fixed window function $g \in C_0^\infty(\mathbb{R})$, we can write

\[
\|F\|_{W(L^{r_2}, L^{q_2})} = \|T_u g(t)(\|F(t)\|_{W(\mathcal{F}L^{r_1}, L^{q_1})})\|_{L^{q_2}} \\
= \|T_u g(t)\|_{W(L^{r_1}, L^{q_1})}\|F(t)\|_{L^{r_2}} = \|F\|_{W(L^{r_2}(W(\mathcal{F}L^{r_1}, L^{q_2})))}.
\]

Hereafter we shall recall some useful properties of the Wiener amalgam spaces.

**Lemma 2.1.** Let $B_i, C_i, \ i = 1, 2, 3$ be Banach spaces such that $W(B_i, C_i)$ are well defined. Then,

(i) Convolution. If $B_1 * B_2 \hookrightarrow B_3$ and $C_1 * C_2 \hookrightarrow C_3$, we have

\[W(B_1, C_1) * W(B_2, C_2) \hookrightarrow W(B_3, C_3).\]  \hspace{1cm} (13)

(ii) Inclusions. If $B_1 \hookrightarrow B_2$ and $C_1 \hookrightarrow C_2$,

\[W(B_1, C_1) \hookrightarrow W(B_2, C_2).\]

Moreover, the inclusion of $B_1$ into $B_2$ need only hold “locally” and the inclusion of $C_1$ into $C_2$ “globally”. In particular, for $1 \leq p_i, q_i \leq \infty, \ i = 1, 2$, we have

\[p_1 \geq p_2 \text{ and } q_1 \leq q_2 \implies W(L^{p_1}, L^{q_1}) \hookrightarrow W(L^{p_2}, L^{q_2}).\]  \hspace{1cm} (14)

(iii) Complex interpolation. For $0 < \theta < 1$, we have

\[W(B_1, C_1), W(B_2, C_2)]_{\theta} = W([B_1, B_2], [C_1, C_2], \theta),\]

if $C_1$ or $C_2$ has absolutely continuous norm.

(iv) Duality. If $B', C'$ are the topological dual spaces of the Banach spaces $B, C$ respectively, and the space of test functions $C_0^\infty$ is dense in both $B$ and $C$, then

\[W(B, C)' = W(B', C').\]  \hspace{1cm} (15)

The proof of all these results can be found in \([4, 5, 6, 11]\).

Here are instead some results on real interpolation theory we did not find explicitly established in the literature. We use the notation and terminology of \([19]\).
Proposition 2.2. Let \( \{A_0, A_1\} \) be an interpolation couple. For every \( 1 \leq p_0, p_1 < \infty \), \( 0 < \theta < 1 \), \( 1/p = (1 - \theta)/p_0 + \theta/p_1 \) and \( p \leq q \) we have

\[
l^p \left( (A_0, A_1)_{\theta,q} \right) \hookrightarrow (l^{p_0}(A_0), l^{p_1}(A_1))_{\theta,q}.
\]

Proof. Set \( \eta = p\theta/p_1 \) and \( q = pr \), \( r \geq 1 \). It follows from Theorem 1.4.2 of [19], page 29, that given \( c = \{c_j\} \in l^p(A_0) + l^p(A_1) \) we have

\[
\|c\|^p_{(l^{p_0}(A_0), l^{p_1}(A_1))_{\theta,q}} \geq \|t^{-\eta} \inf_{a+b=c} \|a\|_{l^{p_0}(A_0)} + t \|b\|_{l^{p_1}(A_1)}\|L^r(\mathbb{R}^d, \mathbb{F})}.
\]

Hence,

\[
\|c\|^p_{(l^{p_0}(A_0), l^{p_1}(A_1))_{\theta,q}} \geq \|t^{-\eta} \inf_{a+b=c} \|a\|_{l^{p_0}(A_0)} + t \|b\|_{l^{p_1}(A_1)}\|L^r(\mathbb{R}^d, \mathbb{F})}.
\]

By Minkowski’s inequality we deduce,

\[
\|c\|^p_{(l^{p_0}(A_0), l^{p_1}(A_1))_{\theta,q}} \leq \sum_{j \geq 0} \|t^{-\eta} \inf_{a_j+b_j=c_j} \|a_j\|_{l^{p_0}(A_0)} + t \|b_j\|_{l^{p_1}(A_1)}\|L^r(\mathbb{R}^d, \mathbb{F})}
\]

Consider now a partition of unity \(^1\) given by functions \( \phi_\alpha \in C_0^\infty(\mathbb{R}^d) \), \( \alpha \in \mathbb{Z}^d \), with \( \phi_\alpha = T_\alpha \phi \), \( \text{supp} \phi \subset [-2, 2]^d \). Thus \( \text{supp} \phi_\alpha \subset \alpha + [-2, 2]^d \). Let then \( \psi \in C_0^\infty(\mathbb{R}^d) \), \( \psi = 1 \) on \([-2, 2]^d \) and \( \psi = 0 \) away from \([-4, 4]^d \), and set \( \psi_\alpha = T_\alpha \psi \). Observe that there is a constant \( C_d \) such that,

\[
\forall \alpha \in \mathbb{Z}^d, \# \{ \beta \in \mathbb{Z}^d : \text{supp } \psi_\beta \cap \text{supp } \phi_\alpha \neq \emptyset \} \leq C_d
\]

and

\[
\forall \alpha \in \mathbb{Z}^d, \# \{ \beta \in \mathbb{Z}^d : \text{supp } \psi_\alpha \cap \text{supp } \phi_\beta \neq \emptyset \} \leq C_d.
\]

The linear operators

\[
S : L^1_{\text{loc}} \to (L^1_{\text{loc}})^{2^d}, \quad R : (L^1_{\text{loc}})^{2^d} \to L^1_{\text{loc}},
\]

defined by

\[
Sf = \{ f \phi_\alpha \}_\alpha, \quad R(\{ u_\alpha \}_\alpha) = \sum_\alpha u_\alpha \psi_\alpha,
\]

\(^1\)Such a partition of unity can be constructed as follows. Take \( \chi \in C_0^\infty(\mathbb{R}^d) \), \( 0 \leq \chi \leq 1 \), \( \chi = 1 \) on \([-1, 1]^d \), \( \chi = 0 \) away from \([-2, 2]^d \). Set \( \Phi(x) = \sum_\alpha \chi(\alpha - x) \). Since the sum is locally finite, \( \Phi \) is well defined and smooth. Moreover \( \Phi(x + \beta) = \Phi(x) \) for every \( \beta \in \mathbb{Z}^d \), and also \( \Phi \geq 1 \). Hence it suffices to take \( \phi_\alpha = T_\alpha \chi/\Phi = T_\alpha (\chi/\Phi) \).
enjoy the following properties (see [5, Remark 2.2]).

**Proposition 2.3.** We have $RS = \text{Id}$ on $L^1_{\text{loc}}$ and, for every local component $B$, as at the beginning of this section, and every $p \geq 1$, we have

\begin{equation}
S : W(B, L^p) \to L^p(B),
\end{equation}

and

\begin{equation}
R : L^p(B) \to W(B, L^p)
\end{equation}

continuously.

**Proof.** The equality $RS = \text{Id}$ on $L^1_{\text{loc}}$ is clear, whereas (22) follows at once from Remark 4 of [4]. To prove (23) we observe that

\begin{equation}
\|R_{\{u_\alpha\}_\alpha}\|_{W(B, L^p)}^p \lesssim \sum_\beta \sum_\alpha \|u_\alpha \psi_\alpha \phi_\beta\|_{B}^p, \quad \text{(by Remark 4 of [4])}
\end{equation}

\begin{equation}
\lesssim \sum_\alpha \sum_\beta \|u_\alpha \psi_\alpha \phi_\beta\|_{B}^p \quad \text{(by (19))}
\end{equation}

\begin{equation}
\lesssim \sum_\alpha \sum_\beta \|\psi_\alpha \phi_\beta\|_{F^1}^p \|u_\alpha\|_{B}^p
\end{equation}

\begin{equation}
\lesssim \|\{u_\alpha\}_\alpha\|_{L^p(B)}^p.
\end{equation}

In the last inequality we used the fact that, by (20),

\begin{equation}
\sum_\beta \|\psi_\alpha \phi_\beta\|_{F^1}^p \lesssim \sum_{\beta: \text{supp } \psi_\alpha \cap \text{supp } \phi_\beta \neq \emptyset} \|\psi_\alpha\|_{F^1}^p \|\phi_\beta\|_{F^1}^p \leq C \|\psi_0\|_{F^1} \|\phi_0\|_{F^1}.
\end{equation}

\Box

**Proposition 2.4.** Given two local components $B_0, B_1$ as at the beginning of this section, for every $1 \leq p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1 - \theta)/p_0 + \theta/p_1$, and $p \leq q$ we have

\begin{equation}
W((B_0, B_1)_{\theta,q}, L^p) \hookrightarrow (W(B_0, L^{p_0}), W(B_1, L^{p_1}))_{\theta,q}.
\end{equation}

**Proof.** Let $R$ and $S$ be the operators defined above. Then, by Proposition 2.3

\begin{equation}
\|f\|_{(W(B_0, L^{p_0}), W(B_1, L^{p_1}))_{\theta,q}} = \|RSf\|_{(W(B_0, L^{p_0}), W(B_1, L^{p_1}))_{\theta,q}} \lesssim \|Sf\|_{L^p((B_0, B_1)_{\theta,q})},
\end{equation}

\begin{equation}
\lesssim \|Sf\|_{W((B_0, B_1)_{\theta,q}, L^p)},
\end{equation}

where for (28) we used (16). This concludes the proof. \Box
3. Fixed time estimates

In this section we study estimates for the solution \( u(t, x) \) of the Cauchy problem (1), for fixed \( t \). We take advantage of the explicit formula for the solution

\[ u(t, x) = (K_t * u_0)(x) \]

where

\[ K_t(x) = \frac{1}{(4\pi it)^d/2} e^{i|x|^2/(4t)}. \]

Precisely, we are going to show that the function in (30) is in the Wiener amalgam space \( W(FL^1, L^\infty) \) and we shall compute its norm. This goal is attained thanks to the nice choice of the Gaussian function \( e^{-\pi|x|^2} \in S(\mathbb{R}^d) \), as a fitting window function. Then we will make use of the convolution properties of Wiener amalgam spaces.

We first remind the Fourier transform of the Gaussian function (see, e.g., [9]).

**Lemma 3.1.** For \( \text{Re} \, c \geq 0, \, c \neq 0 \). Let \( \varphi_c(x) = e^{-\pi|x|^2/c} \), then

\[ \hat{\varphi}_c(\omega) = c^{d/2} \varphi_{1/c}(\omega), \quad \omega \in \mathbb{R}^d, \]

where the square root is chosen to have a positive real part.

**Proposition 3.1.** (cf. [1, 2]) For \( a \in \mathbb{R}, \, a \neq 0 \), let \( f_a(x) = (ai)^{-d/2} e^{-\pi|x|^2/(ai)} \). Then \( f_a \in W(FL^1, L^\infty) \), with

\[ \|f_a\|_{W(FL^1, L^\infty)} = \left( \frac{1 + a^2}{a^4} \right)^{d/4}. \]

**Proof.** By definition (see, e.g., [3, 11]),

\[ \|f_a\|_{W(FL^1, L^\infty)} = \sup_{x \in \mathbb{R}^d} \|f_a T_x g\|_{FL^1}, \]

for some non-zero window \( g \in W(FL^1, L^1) \) (different windows give equivalent norms). We then choose \( g = e^{-\pi|x|^2} \), and, observing that \( \hat{g} = g \), we can write

\[ \|f_a T_x g\|_{FL^1} = \|f_a \hat{T}_x g\|_{L^1} = \|f_a * M_x g\|_{L^1}. \]

Using (31) with \( c = ai \) we compute the Fourier transform of \( f \) that reveals to be

\[ \hat{f}_a(\omega) = (ai)^{-d/2} (ai)^{d/2} e^{-\pi a \omega^2} = e^{-\pi a \omega^2}. \]
Thereby,
\[
\| \hat{f}_a \ast M_x g \|_{L^1} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{-\pi ai(\omega-y)^2} e^{-2\pi ixy} e^{-\pi^2|y|^2} \, dy \right| \, d\omega \\
= \int_{\mathbb{R}^d} \left| e^{-\pi ai\omega^2} \int_{\mathbb{R}^d} e^{-2\pi i(x-a\omega)y} e^{-\pi(1+ai)|y|^2} \, dy \right| \, d\omega \\
= \int_{\mathbb{R}^d} \left| \mathcal{F}(e^{-\pi(1+ai)|y|^2})(x-a\omega) \right| \, d\omega \\
= \int_{\mathbb{R}^d} \left| (1 + ai)^{-d/2} e^{-\pi(x-a\omega)^2/(1+ai)} \right| \, d\omega,
\]
where in the last equality we use (31). Performing the change of variables \( x-a\omega = z \), hence \( d\omega = a^{-d} \, dz \) and observing that \( |(1+ai)^{-d/2}| = (1+a^2)^{-d/4} \), we can write
\[
\| \hat{f}_a \ast M_x g \|_{L^1} = (1 + a^2)^{-d/4} a^{-d} \int_{\mathbb{R}^d} \left| e^{-\pi|z|^2/(1+ai)} \right| \, dz \\
= (1 + a^2)^{-d/4} a^{-d} \int_{\mathbb{R}^d} e^{-\pi|z|^2/(1+a^2)} \, dz \\
= (1 + a^2)^{-d/4} a^{-d} (1 + a^2)^{d/2} \\
= \left( \frac{1 + a^2}{a^4} \right)^{d/4}.
\]
Since the right-hand side does not depend on \( x \), taking the supremum on \( \mathbb{R}^d \) with respect to the \( x \)-variable we attain the desired estimate. \( \square \)

**Lemma 3.2.** It turns out
\[
(33) \quad W(\mathcal{F}L^1, L^\infty) \ast W(\mathcal{F}L^\infty, L^1) \hookrightarrow W(\mathcal{F}L^1, L^\infty).
\]

**Proof.** This is a consequence of the convolution relations for Wiener amalgam spaces in Lemma 2.1 (i), being \( \mathcal{F}L^1 \ast \mathcal{F}L^\infty = \mathcal{F}(L^1 \cdot L^\infty) = \mathcal{F}L^1 \) and \( L^\infty \ast L^1 \hookrightarrow L^\infty. \) \( \square \)

**Proposition 3.3.** We have
\[
(34) \quad \| e^{it\Delta} u_0 \|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \left( \frac{1 + |t|}{t^2} \right)^{d/2} \| u_0 \|_{W(\mathcal{F}L^\infty, L^1)}.
\]

**Proof.** We use the explicit representation of the Schrödinger evolution operator \( e^{it\Delta} u_0(x) = (K_t \ast u_0)(x) \). From (32) (with \( a = 4\pi t \)) we infer
\[
\| K_t \|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \left( \frac{1 + |t|}{t^2} \right)^{d/2}.
\]
Finally, the convolution relations (33) yield the desired result.

**Theorem 3.2.** For $2 ≤ r ≤ \infty$ we have

$$\|e^{it\Delta}u_0\|_{W(\mathcal{F}L^r, L^r)} \lesssim \left(1 + \frac{|t|}{t^2}\right)^{d\left(\frac{1}{2} - \frac{1}{r}\right)} \|u_0\|_{W(\mathcal{F}L^r, L^r)}.$$  

**Proof.** Estimate (35) follows by interpolating (34) with the $L^2$ conservation law (36)

$$\|e^{it\Delta}u_0\|_{L^2} = \|u_0\|_{L^2}.$$  

Indeed, $L^2 = W(\mathcal{F}L^2, L^2)$. By Lemma 2.1 item (iii), for $0 < \theta = 2/r < 1$,

$$[W(\mathcal{F}L^1, L^\infty), W(\mathcal{F}L^2, L^2)]_{[\theta]} = W(\mathcal{F}L^{r'}, L^r)$$  

and

$$[W(\mathcal{F}L^\infty, L^1), W(\mathcal{F}L^2, L^2)]_{[\theta]} = W(\mathcal{F}L^r, L^r).$$

**Remark 3.3.** As well known (see e.g. (2.23) of [18]), the $L^p$ fixed time estimates for the solution of (1) read

$$\|e^{it\Delta}u_0\|_{L^r(R^d)} \lesssim |t|^{-d\left(\frac{1}{2} - \frac{1}{r}\right)} \|u_0\|_{L^r(R^d)}, \quad 2 ≤ r ≤ \infty.$$  

For $2 ≤ r ≤ \infty$, $\mathcal{F}L^r \hookrightarrow L^r$, and the inclusion relations for Wiener amalgam spaces in Lemma 2.1 item (ii), yield $W(\mathcal{F}L^r, L^r) \hookrightarrow W(L^r, L^r) = L^r$ and $L^r = W(L^r, L^r) \hookrightarrow W(\mathcal{F}L^r, L^r)$. Thereby the estimates (35) are indeed an improvement of (37) for every fixed $t \neq 0$, and also uniformly for $|t| > c > 0$.

4. **Strichartz estimates: non-endpoint case**

In this section we shall prove Theorem 1.1. To this aim, we need some preliminary results. For $0 < \alpha < 1/2$, let $\phi_\alpha$ be the non-negative real function defined by

$$\phi_\alpha(t) = |t|^{-\alpha} + |t|^{-2\alpha}, \quad t \in \mathbb{R}, t \neq 0.$$  

Of course, $\phi_\alpha \in L^1_{loc}(\mathbb{R})$. The next lemma sets $\phi_\alpha$ in a suitable Wiener amalgam space.

**Lemma 4.1.** We have

$$\phi_\alpha \in W(L^{1/(2\alpha)}, \infty, L^{1/\alpha}, \infty).$$  

**Proof.** Let

$$G(x) = \|\phi_\alpha \chi_{[x-1, x+1]}\|_{L^{1/(2\alpha)}, \infty},$$

then, it suffices to prove that

$$G(x) \leq \frac{C}{|x|^{\alpha}}.$$
We can suppose \( x > 0 \) (\( G(x) \) is an even function). Now, if \( 0 < x \leq 2 \), we have
\[
G(x) \leq \| \phi_\alpha \chi_{[-1,3]} \|_{L^{1/(2\alpha),\infty}}.
\]
On the other hand, when \( x > 2 \), we use the fact that \( \phi_\alpha(t) \leq 2|t|^{-\alpha} \) for \( |t| \geq 1 \). Hence the distribution function \( \lambda(s) \) of \( \phi_\alpha(t)\chi_{[x-1,x+1]}(t) \) satisfies the estimate
\[
\lambda(s) \leq \begin{cases} 
2 & \text{if } s < 2(x+1)^{-\alpha} \\
2^{1/\alpha}s^{-1/\alpha} - (x - 1) & \text{if } 2(x+1)^{-\alpha} \leq s \leq 2(x-1)^{-\alpha} \\
0 & \text{if } s > 2(x-1)^{-\alpha}.
\end{cases}
\]
As a consequence,
\[
G(x) \asymp \sup_{s>0} \{ s\lambda(s)^{2\alpha} \} \leq 2^{2\alpha+1}(x+1)^{-\alpha}.
\]
This proves (39).

**Lemma 4.2.** For \( 0 < \alpha < 1/2 \), let \( \phi_\alpha \) be the function defined in (38). Then,
\[
(40) \quad \| F \ast \phi_\alpha \|_{W(L^{1/\alpha},L^{2/\alpha})} \lesssim \| F \|_{W(L^{(1/\alpha)',L^{(2/\alpha)'})}}.
\]

**Proof.** Lemma 4.1 above shows that, locally, \( \phi_\alpha \in L^{1/(2\alpha),\infty} \). Using the fractional integration Theorem 2.1 we infer
\[
L^p \ast L^{1/(2\alpha),\infty} \hookrightarrow L^s, \quad \text{with} \quad \frac{1}{p} = \frac{1}{s} + 1 - 2\alpha;
\]
if we set \( p = s' \), then \( s = 1/\alpha \) and \( L^{(1/\alpha)',L^{(2/\alpha),\infty}} \hookrightarrow L^{1/\alpha} \). Globally, \( \phi_\alpha \in L^{1/\alpha,\infty} \) and the same argument gives \( L^{(2/\alpha)',L^{1/\alpha,\infty}} \hookrightarrow L^{2/\alpha} \).

Finally, the convolution relations for Wiener amalgam spaces (13) glue together the local and global properties and provide (40).

**Remark 4.1.** Condition \( \alpha < 1/2 \) is necessary if we want at least \( \phi_\alpha \in L^1_{\text{loc}} \) and solutions with local time estimate in \( L^p \) spaces rather than rougher spaces, subsets of \( S' \). This constraint will yield the threshold \( q = 4 \) in the Strichartz estimates.

**Proof of Theorem 1.1.** We first prove the estimate (7).

The case: \( q = \infty, \ r = 2 \) follows at once from the conservation law (38). Indeed, \( W(L^{\infty},L^{\infty})_t = L^\infty_t \) and \( W(FL^2,L^2)_x = L^2_x \), so that, taking the supremum over \( t \) in \( \| e^{it\Delta}u_0 \|_{L^2_x} = \| u_0 \|_{L^2_x} \), we attain the claim.

To prove the remaining cases, we can apply the usual \( TT^* \) method (or “orthogonality principle”, see [10] Lemma 2.1 or [14] page 353)), because of the Hölder’s type inequality
\[
(41) \quad |\langle F, G \rangle_{L^2_xL^2_x}| \leq \| F \|_{W(L^{r},L^{q})_xW(FL^{r'},L^{q'})_x} \| G \|_{W(L^{r'},L^{q'})_tW(FL^{r'},L^{q'})_x},
\]
which can be proved directly from the definition of these spaces.
As a consequence, it suffices to prove the estimate

\[(42) \quad \| \int e^{i(t-s)\Delta} F(s) \, ds \|_{W(L^{q/2},L^q)_{x}W(FL^{r'},L^r)_x} \lesssim \| F \|_{W(L^{(q/2)'},L^{q'})_{x}W(FL^r,L^r)_x}. \]

Now, set \( \alpha = d(1/2 - 1/r) = 2/q. \) Then, by (35) and Lemma 4.2,

\[
\| \int e^{i(t-s)\Delta} F(s) \, ds \|_{W(L^{q/2},L^q)_{x}W(FL^{r'},L^r)_x} \\
\leq \| \int \| e^{i(t-s)\Delta} F(s) \|_{W(FL^{r'},L^r)_x} \, ds \|_{W(L^{q/2},L^q)_t} \\
\lesssim \|\| F(t) \|_{W(FL^{r'},L^r)_x} * \phi_{\alpha}(t) \|_{W(L^{q/2},L^q)_t} \\
\lesssim \| F \|_{W(L^{(q/2)'},L^{q'})_{x}W(FL^r,L^r)_x}. 
\]

The estimate (38) follows from (7) by duality.

Consider now the retarded Strichartz estimate (9). By complex interpolation (Lemma 2.1 (iii)), in order to get (9) with \((1/q,1/r), (1/\tilde{q},1/\tilde{r}), (1/\infty,1/2)\) collinear it suffices to prove (9) in the three cases \((\tilde{q},\tilde{r}) = (q,r), (q,r) = (\infty,2), (\tilde{q},\tilde{r}) = (\infty,2)\), as shown in Figure 1.

![Figure 1](image_url)
Now, the case \((\tilde{q}, \tilde{r}) = (q, r)\) follows from (42) with \(\chi_{\{s<t\}} F\) in place of \(F\).

The case \((q, r) = (\infty, 2)\) follows because
\[
\| \int_{s<t} e^{i(t-s)\Delta} F(s) \, ds \|_{L^2_x} = \| \int_{s<t} e^{-is\Delta} F(s) \, ds \|_{L^2_x} 
\leq \| F \|_{W(L^{\tilde{q}/2}, L^{\tilde{r}'})} \| F \|_{W(L^{q}, L^{r'})}.
\]
where we applied (36) and then (8) with \(\chi_{\{s<t\}} F\) in place of \(F\).

Finally, this latter argument applied to the adjoint operator
\[
G \mapsto \int_{t>s} e^{-i(t-s)\Delta} G(t) \, dt
\]
gives the case \((\tilde{q}, \tilde{r}) = (\infty, 2)\).

5. Strichartz estimates: endpoint case

In this section we prove the estimates in Theorem 1.2. Hence, \(q = 4\), \(r = \frac{2d}{d-1}\), or \(\tilde{q} = 4\), \(\tilde{r} = \frac{2d}{d-1}\), \(d > 1\). We follow the pattern in Keel-Tao [13]. Indeed, we study bilinear form estimates rather than operator estimates. This is achieved via a dyadic decomposition in time (see (44)), and estimates of each dyadic contribution (see (46) and (47)). Finally we conclude by a lemma of real interpolation theory. The proof will require however some different technical issues, due to the different nature of the Wiener amalgam spaces.

First we prove (10) and (11). Let therefore
\[
r = \frac{2d}{d-1}, \quad d > 1.
\]

By the same duality arguments as the ones used in the previous section, we observe that it suffices to prove (7). This is equivalent to the bilinear estimate
\[
| \int \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle \, ds \, dt | \lesssim \| F \|_{W(L^2, L^{4/3})} \| F \|_{W(L^{2}, L^{r'})} \| G \|_{W(L^2, L^{r'})} \| G \|_{W(L^2, L^{r'})}.
\]

By symmetry, it is enough to prove
\[
|T(F, G)| \lesssim \| F \|_{W(L^2, L^{4/3})} \| G \|_{W(L^2, L^{r'})} \| G \|_{W(L^2, L^{r'})}.
\]

where
\[
T(F, G) = \int \int_{s<t} \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle \, ds \, dt.
\]

The form \(T(F, G)\) can be decomposed dyadically as
\[
(43) \quad T = \hat{T} + \sum_{j \leq 0} T_j,
\]

\[
(44) \quad \hat{T} + \sum_{j \leq 0} T_j.
\]
with
\[ \tilde{T}(F, G) = \int_{s \leq t-2} \langle e^{-is\Delta}F(s), e^{-it\Delta}G(t) \rangle \, ds \, dt \]
and
\[ T_j(F, G) = \int_{t-2^{j+1} < s \leq t-2^j} \langle e^{-is\Delta}F(s), e^{-it\Delta}G(t) \rangle \, ds \, dt. \]

In the sequel we shall study the behaviour of \( \tilde{T} \) and \( T_j \) separately.

**Lemma 5.1.** We have
\[ |\tilde{T}(F, G)| \lesssim \|F\|_{W(L^{2, L^{4/3}}, W(FL^{r}, L^{r'}))} \|G\|_{W(L^{2, L^{4/3}}, W(FL^{r}, L^{r'}))}, \]

*Proof.* It follows from the duality properties (15) and the space estimate (35) that
\[ |\langle e^{-is\Delta}F(s), e^{-it\Delta}G(t) \rangle| = |\langle F(s), e^{i(s-t)\Delta}G(t) \rangle| \lesssim \|F(s)\|_{W(FL^{r}, L^{r'})} \|e^{i(s-t)\Delta}G(t)\|_{W(FL^{r}, L^{r'})} \lesssim \|F(s)\|_{W(FL^{r}, L^{r'})} \left( \frac{1}{(s-t)^2} \right)^{d(1 - \frac{1}{r})} \|G(t)\|_{W(FL^{r}, L^{r'})}. \]

Since \( d/(1/2 - 1/r) = 1/2 \), for \( |s-t| \geq 2 \) we have
\[ \left( \frac{1}{(s-t)^2} \right)^{d(1 - \frac{1}{r})} \lesssim (1 + |s-t|)^{-\frac{1}{2}}. \]

Thus, the form \( \tilde{T}(F, G) \) can be controlled by the following majorizations:
\[ |\tilde{T}(F, G)| \lesssim \int_{s \leq t-2} \|F(s)\|_{W(FL^{r}, L^{r'})} \left( 1 + |s-t|^2 \right)^{-\frac{1}{2}} \|G(t)\|_{W(FL^{r}, L^{r'})} \, ds \, dt \lesssim \|(1 + |t|)^{-\frac{1}{2}} \|F(t)\|_{W(FL^{r}, L^{r'})} \|G(t)\|_{W(L^{2, L^{4/3}})} \cdot \|G(t)\|_{W(FL^{r}, L^{r'})} \|W(L^{2, L^{4/3}}), \]

in the last estimate we used the duality property for Wiener amalgam spaces (15). Notice that \((4/3)'/4 = 4\).

Next, the convolution relation (12), for \( p = 4/3, d = 1 \) and \( \alpha = 1/2 \), reads \( L^{4/3} \ast L^{2, \infty} \hookrightarrow L^4 \), whereas Young’s inequality gives \( L^1 \ast L^2 \hookrightarrow L^2 \). Applying the former estimate globally and the latter locally, we use the convolution relations for Wiener amalgam spaces (13) and infer
\[ W(L^1, L^{2, \infty}) \ast W(L^2, L^{4/3}) \hookrightarrow W(L^2, L^4). \]

Since \( 4/3 < 4 \) the inclusion relations (14) give \( W(L^2, L^{4/3}) \hookrightarrow W(L^2, L^4) \). This argument ends our proof; indeed, it is straightforward to see that \( (1 + |t|)^{-\frac{1}{2}} \in \)
$W(L^1, L^{2,\infty})$, so that
\[ |\tilde{T}(F, G)| \lesssim \| (1 + |t|)^{-\frac{d}{2}} * F(t)\|_{W(L^2, L^{2,\infty})} \| G(t)\|_{W(L^2, L^{2,\infty})} \| G(t)\|_{W(L^2, L^{2,\infty})} \| G(t)\|_{W(L^2, L^{2,\infty})}, \]

as desired. \[\square\]

For $a, b \geq 1$, we define
\[ \beta(a, b) = d - 1 - \frac{d}{a} - \frac{d}{b}. \]

**Lemma 5.2.**

\[ (47) \quad |T_j(F, G)| \lesssim 2^{-j\beta(a, b)} \| F\|_{W(L^2, L^{2,\infty})} \| G\|_{W(L^2, L^{2,\infty})}, \]

for $(1/a, 1/b)$ in a neighborhood of $(1/r, 1/r)$.

**Proof.** Observe that here $r < \infty$. Then, the result follows by complex interpolation (Lemma 2.1 (iii)) from the following cases:

(i) $a = \infty, b = \infty$,
(ii) $2 \leq a < r, b = 2$,
(iii) $a = 2, 2 \leq b < r$.

**Case (i).** We need to show the estimate
\[ (48) \quad |T_j(F, G)| \lesssim 2^{-j(d-1)} \| F\|_{W(L^2, L^{2,\infty})} \| G\|_{W(L^2, L^{2,\infty})}. \]

By (34) and (38) we have
\[ \| \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle \| \lesssim \phi_{d/2}(t-s) \| F(s)\|_{W(L^\infty, L^1)} \| G(t)\|_{W(L^\infty, L^1)}. \]

Thereby
\[ |T_j(F, G)| \lesssim \phi_{d/2}(2^j) \| F\|_{L^1(W(L^\infty, L^1))} \| G\|_{L^1(W(L^\infty, L^1))}. \]

We can of course assume $F$ and $G$ compactly supported, with respect to the time, in intervals of duration $\sim 2^j$. As a consequence of Hölder’s inequality,
\[ \| F(s)\|_{L^1(W(L^\infty, L^1))} \lesssim 2^{j/2} \| F\|_{L^1(W(L^\infty, L^1))}, \]

and similarly for $G$. Hence
\[ |T_j(F, G)| \lesssim \phi_{d/2}(2^j) 2^{j} \| F\|_{L^1(W(L^\infty, L^1))} \| G\|_{L^1(W(L^\infty, L^1))}. \]

Since $\phi_{d/2}(2^j) \lesssim 2^{-dj}$ (recall, $j \leq 0$) and
\[ W(L^2(W(FL^\infty, L^1)_x), L^{2,\infty}) \leftrightarrow W(L^2(W(FL^\infty, L^1)_x), L^2) = L^2(W(FL^\infty, L^1)_x), \]

we attain the desired estimate (38).

**Case (ii).** We have to show
\[ (49) \quad |T_j(F, G)| \lesssim 2^{-j(d-1-\frac{d}{a})} \| F\|_{W(L^2, L^{2,\infty})} \| G\|_{W(L^2, L^{2,\infty})}. \]
Using similar arguments to the previous case we obtain

\begin{equation}
|T_j(F, G)| \lesssim \sup_t \| \int_{t-2^{j+1} < s \leq t} e^{-is\Delta} F(s) \, ds \|_{L_T^2} \| G \|_{L_T^p L_x^q}, \tag{50}
\end{equation}

and

\begin{equation}
\| G \|_{L_T^1 L_x^q} \lesssim 2^{j/2} \| G \|_{W(L^2, L^{4/3})}, L_x^q. \tag{51}
\end{equation}

For \( a \geq 2 \), let now \( \tilde{q} = \tilde{q}(a) \) be defined by

\begin{equation}
\frac{2}{\tilde{q}(a)} + \frac{d}{a} = \frac{d}{2}. \tag{52}
\end{equation}

The non-endpoint case of (50), written for \( \tilde{r} = a \) and the \( \tilde{q} \) above, gives

\begin{equation}
\sup_t \| \int_{t-2^{j+1} < s \leq t} e^{-is\Delta} F(s) \, ds \|_{L_T^2} = \sup_t \| \int_{\mathbb{R}} e^{-is\Delta} (T_{-t}(\chi_{[-2^{j+1}, -2^{j}]}) F(s) \, ds \|_{L_T^2} \lesssim \| F \|_{W(L^{(q/2)', L^{q/2} W(F L^a, L^{a'})}}, \end{equation}

for every \( 2 \leq a < r \). In what follows we apply Hölder’s inequality with the triple of indices \( 2, p, (\tilde{q}/2)' \), so that \( \frac{1}{2} + \frac{1}{p} = \frac{1}{(q/2)'} \). This gives, for \( g \in C_0^\infty(\mathbb{R}) \),

\begin{equation}
\| FT_z g \|_{L_t^{\tilde{q}/2'}(W(F L^a, L^{a'})_x)} \lesssim \| FT_z g \|_{L_t^{\tilde{q}/2'}(W(F L^a, L^{a'})_x)} \lesssim \| FT_z g \|_{L_t^2(W(F L^a, L^{a'})_x)}(2^j)^{1/p} = \| FT_z g \|_{L_t^2(W(F L^a, L^{a'})_x)} 2^{-j(\theta/2 - d/a - 1/2)},
\end{equation}

where we used

\begin{equation}
\frac{1}{p} = \frac{1}{(q/2)'} - \frac{1}{2} = 1 - \frac{2}{\tilde{q}} = 1 - \frac{d}{2} + \frac{d}{a}.
\end{equation}

Since the support of \( F \) with respect to the time is contained in an interval of duration \( \asymp 2^j \leq 1 \), the support of the function \( \mathbb{R} \ni z \mapsto \| FT_z g \|_{L_t^2(W(F L^a, L^{a'})_x)} \) is contained in an interval of duration \( \asymp 1 \). This allows us to apply Hölder’s inequality with respect to the global component, too, and we end up with

\begin{equation}
\| F \|_{W(L^{(q/2)', L^{q/2}}_t W(F L^a, L^{a'})_x)} \lesssim 2^{-j(\theta/2 - d/a - 1/2)} \| FT_z g \|_{L_t^2(W(F L^a, L^{a'})_x)} \| L_t^p \lesssim 2^{-j(\theta/2 - d/a - 1/2)} \| FT_z g \|_{L_t^2(W(F L^a, L^{a'})_x)} \| L_t^{1/3}.
\end{equation}

This estimate, together with (50) and (51), yields the estimate (49).

Case (iii). Use the same arguments as in case (ii).

Since \( F L^{r, 2} \hookrightarrow L^r \) and in view of (46) and (47), in order to prove (13) it suffices to prove

\begin{equation}
\sum_{j \leq 0} |T_j(F, G)| \lesssim \| F \|_{W(L^2, L^{5/3})_t W(F L^{r, 2}, L^r)_x} \| G \|_{W(L^2, L^{5/3})_t W(F L^{r, 2}, L^r)_x}.
\end{equation}
Now, this can be achieved from (47) by the same interpolation arguments as in [13, Par. 6]. Precisely, we take $a_0, a_1, b_0, b_1$ such that $(1/r, 1/r)$ is inside a small triangle with vertices $(1/a_0, 1/b_0)$, $(1/a_1, 1/b_0)$ and $(1/a_0, 1/b_1)$ (see Figure 2), so that

$$
\beta(a_0, b_1) = \beta(a_1, b_0) \neq \beta(a_0, b_0).
$$

Then, we apply Lemma 6.1 of [13] with $T = \{T_j\}$ (upon setting $T_j = 0$ for $j > 0$), $p = q = 2$, $r = 1$, $C_0 = l^{\beta(a_0, b_0)}$, $C_1 = l^{\beta(a_0, b_1)}$ and, for $k = 0, 1$, we take

$$
A_k = W(L^2(W(FL^{a_k}, L^{a_k})), L^{4/3}), \quad B_k = W(L^2(W(FL^{b_k}, L^{b_k})), L^{4/3}).
$$

Here we choose $\theta_0, \theta_1$, so that

$$
1/r = (1 - \theta_0)/a_0 + \theta_0/a_1, \quad 1/r = (1 - \theta_1)/b_0 + \theta_1/b_1,
$$

(recall, $r = 2d/(d-1)$). This gives at once the desired result, since $\beta(r, r) = 0$ and

$$
W(L^2(W(FL^{r,2}, L^{r})), L^{4/3}) \subset (A_0, A_1)_{\theta_0,2} \cap (B_0, B_1)_{\theta_1,2},
$$

as one sees by applying (in order) Proposition 2.2 of [19], page 129 (with $p = p_0 = p_1 = 2$), and again Proposition 2.4 to the Wiener spaces with respect to $x$. This concludes the proof of (10) and (11).

The corresponding retarded estimates can be obtained as follows. The case $(\tilde{q}, \tilde{r}) = (q, r) = P$ is exactly (43). The case $(\tilde{q}, \tilde{r}) = P$, $(q, r) \neq P$, can be obtained by a repeated use of H"older’s inequality to interpolate from the case $(\tilde{q}, \tilde{r}) = (q, r) = P$, and the case $(\tilde{q}, \tilde{r}) = P$, $(q, r) = (\infty, 2)$ (that is clear from
Precisely, we want to prove
\[
\left\| \int_{s<t} e^{i(t-s)\Delta} F(s) \, ds \right\|_{W(L^{n/2}, L^q)_x} \lesssim \left\| F \right\|_{W(L^{2}, L^{4/3})_x} \lesssim \left\| F \right\|_{W(L^{2}, L^{2})_x},
\]
for \( \tilde{r} = 2d/(d-1) \), 2/q + d/r = d/2, q > 4.

We know that such an estimate holds for \((q, r) = (\infty, 2)\), as well as
\[
\left\| \int_{s<t} e^{i(t-s)\Delta} F(s) \, ds \right\|_{W(\tilde{L}^{\tilde{r}}, \tilde{L}^q)_x} \lesssim \left\| F \right\|_{W(\tilde{L}^{2}, \tilde{L}^{4/3})_x} \lesssim \left\| F \right\|_{W(\tilde{L}^{2}, \tilde{L}^{2})_x}.
\]

Hence, upon setting \( I = \int_{s<t} e^{i(t-s)\Delta} F(s) \, ds \), it suffices to prove that
\[
\left\| I \right\|_{W(L^{n/2}, L^q)_x} \lesssim \left\| I \right\|_{W(\tilde{L}^{2}, \tilde{L}^{4/3})_x} \left\| I \right\|_{\tilde{L}^2},
\]
with \( 1/r = (1 - \theta)/\tilde{r} + \theta/2 \), 1/q = (1 - \theta)/4 + \theta/\infty, 0 < \theta < 1 \). To this end, we start with the inequality
\[
\|u\|_{L^r} \lesssim \|u\|_{L^{r,2}}^{\frac{\theta}{\tilde{r}}} \|u\|_{L^2}^{\frac{1}{r}}.
\]

(54)

which follows, e.g., from Theorem 1.3.3 (g) of [19], page 25, since \((L^{\tilde{r},2}, L^{2})_{\theta,1} = L^{r,1} \rightarrow L^{r'}\).

We apply (54) with \( u = \hat{I}(t)T_z g \), where \( g \in C_0^\infty(\mathbb{R}^d) \) is a non-zero window. We obtain
\[
\|I(t)\|_{W(\tilde{L}^{r'}, \tilde{L}^r)_x} = \left\| \|I(t)T_z g\|_{L_{\tilde{r}'}^r} \right\|_{L_{\tilde{r}}^r} \leq \left\| \|I(t)T_z g\|_{L_{\tilde{r}'}^r}^{1-\theta} \|I(t)T_z g\|_{L_{\tilde{r}}^r}^{\theta} \right\|_{L_{\tilde{r}}^r} \leq \|I(t)\|_{W(\tilde{L}^{r'}, \tilde{L}^r)_x} \|I(t)\|_{L_{\tilde{r}}^r}^{\theta},
\]

where in the last inequality we applied Hölder’s inequality with respect to \( z \). Hence, given any non-zero window \( h \in C_0^\infty(\mathbb{R}_t) \), it follows that
\[
\|I(t)\|_{W(\tilde{L}^{r'}, \tilde{L}^r)_x} \|T_s h\|(t) \leq \left( \|I(t)\|_{W(\tilde{L}^{r'}, \tilde{L}^r)_x} \|T_s h\|(t) \right)^{1-\theta} (\|I(t)\|_{L_{\tilde{r}}^r} \|T_s h\|(t))^{\theta}.
\]

(55)

We then take the \( L_{\tilde{r}}^2 L_{\tilde{r}}^{\theta/2} \) norm of the expression in the left hand side of (55) and we apply again Hölder’s inequality. This gives the desired result (53).

Eventually one can prove the retarded estimate in the case \((q, r) = P\), \((\bar{q}, \bar{r}) \neq P\) showing, by the arguments above, that the dual inequality holds for the adjoint operator \( G \mapsto \int_{t>s} e^{-i(t-s)\Delta} G(t) \, dt \).

This concludes the proof of Theorem 1.1.
Consider the Cauchy problem
\[
\begin{aligned}
&i\partial_t u + \Delta u = V(t, x)u, \quad t \in [0, T], \ x \in \mathbb{R}^d, \ (d \geq 2), \\
&u(0, x) = u_0(x),
\end{aligned}
\]
with a potential,
\[
V \in L^\alpha(I_T; L^p_x), \quad \frac{1}{\alpha} + \frac{d}{2p} \leq 1, \ 1 \leq \alpha < \infty, \ \frac{d}{2} < p \leq \infty.
\]
It is proved in [3, Theorem 1.1, Remark 1.3] (see also [20, Theorem 1.1]) that under the assumption (57) the Cauchy problem (56) is well-posed in $L^2$ and admits a unique solution $u \in C(I_T; L^2(\mathbb{R}^d)) \cap L^q(I_T, L^r)$, for all Schrödinger admissible pairs $(q, r)$.

We now illustrate an application of our estimates, by deducing a similar result with the solution being controlled in terms of Wiener amalgam norms. We consider the subclass of potentials
\[
V \in L^\alpha(I_T; L^p_x), \quad \frac{1}{\alpha} + \frac{d}{p} \leq 1, \ 1 \leq \alpha < \infty, \ d < p \leq \infty.
\]
Here is our result.

**Theorem 6.1.** Assume (58). Then the Cauchy problem (56) has a unique solution $u \in C(I_T; L^2(\mathbb{R}^d)) \cap L^q/2(I_T, W(FL^r, L^r)) \cap L^2(I_T, W(FL^{2d/(d+1)}, L^{2d/(d-1)}))$, for all $(q, r)$ such that $2/q + d/r = d/2$, $q > 4$, $r \geq 2$.

**Proof.** The proof closely follows the one of [3, Theorem 1.1, Remark 1.3] (based on the classical Strichartz estimates). However we present at least the main steps of the proof for the convenience of the reader who is not familiar with that result.

First of all, since the interval $I_T$ is bounded, by Hölder’s inequality it suffices to prove the theorem when $1/\alpha + d/p = 1$. Hence we assume this, and we prove the case $2 \leq \alpha < \infty$ and $1 \leq \alpha < 2$ separately.

Let $J = [0, \delta]$ be a small time interval and set, for $q \geq 2$, $q \neq 4$, $r \geq 1$,
\[
Z_{q/2,r} = L^{q/2}(J; W(FL^r, L^r)_x),
\]
\[
Z_{2,2d/(d-1)} = L^2(J, W(FL^{2d/(d+1)}, L^{2d/(d-1)})_x),
\]
and $Z = C(J; L^2) \cap Z_{2,2d/(d-1)}$, with the norm $\|v\|_Z = \max\{\|v\|_{C(J; L^2)}, \|v\|_{Z_{q/2,r}}, \|v\|_{Z_{2,2d/(d-1)}}\}$.

Notice that, by the arguments at the end of Section 3 we have $Z \subset Z_{q/2,r}$ for all $(q, r)$ as in the statement of the theorem. Consider now the integral formulation of
the Cauchy problem, namely $u = \Phi(u)$, where
\[
\Phi(v) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}V(s)v(s)\,ds.
\]
From Theorems 1.1 and 1.2 it is easy to see that the following estimates hold:
\begin{equation}
\|\Phi(v)\|_{Z_{q/2,r}} \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{Z_{(q/2)'r'}}
\end{equation}
for all $(q, r)$ such that $2/q + d/r = d/2$, $q \geq 4, r \geq 2$, and similarly for $(\tilde{q}, \tilde{r})$. Since $1/\alpha + d/p = 1$ and $\alpha \geq 2$, among such pairs there is a pair $(\tilde{q}, \tilde{r})$ such that $1/(\tilde{q}/2) = 1/2 - 1/\alpha$ and $1/\tilde{r} = (d+1)/(2d) - 1/p$. We choose such a pair in (59) and, after using the inclusion $L^{(\tilde{q}/2)'}(J; L^{\tilde{r}}) \subset Z_{(\tilde{q}/2)',\tilde{r}'}$ and $Z_{2,2d/(d-1)} \subset L^2(J; L^{2d/(d-1)})$, we apply Hölder’s inequality. We obtain
\begin{equation}
\|\Phi(v)\|_{Z_{q/2,r}} \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{L^\alpha(J; L^p)}\|v\|_{Z_{2,2d/(d-1)}}.
\end{equation}
By taking $(q, r) = (\infty, 2)$ and $(q, r) = (4, 2d/(d-1))$ one deduces that $\Phi : Z \to Z$ (the fact that $\Phi(u)$ is continuous in $t$ when valued in $L^2_x$ follows from a classical limiting argument). Also, since $\alpha < \infty$, if $J$ is small enough, $C_0\|V\|_{L^\alpha_x L^p_t} < 1/2$, and $\Phi$ is a contraction. This gives a unique solution in $J$. By iterating this argument a finite number of times one obtains a solution in $[0, T]$.

In the case $1 \leq \alpha < 2$, one starts instead from the estimate
\[
\|\Phi(v)\|_{Z_{q/2,r}} \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{L^\alpha(J; L^p)}\|v\|_{Z_{2,2d/(d-1)}}.
\]
which follows again from Theorems 1.1 and 1.2 with $((\tilde{q}/2)', \tilde{r}') = (\alpha, 2p/(p+2))$. In fact, from $1 \leq \alpha < 2$ and $1/\alpha + d/p = 1$, it follows $\tilde{q} > 4$ and $2/\tilde{q} + d/\tilde{r} = d/2$. Then, again by Hölder’s inequality, one obtains
\[
\|\Phi(v)\|_{Z_{q/2,r}} \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{L^\alpha(J; L^p)}\|v\|_{Z_{\infty,2}}.
\]
From here, proceeding as above yields the desired result.

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Department of Mathematics, University of Torino, Italy

Dipartimento di Matematica, Politecnico di Torino, Italy