A Lagrangian form of tangent forms

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Abstract

The aim of the paper is to study some dynamic aspects coming from a tangent form, i.e. a time dependent differential form on a tangent bundle. The action on curves of a tangent form is natural associated with that of a second order Lagrangian linear in accelerations, while the converse association is not unique. An equivalence relation of tangent form, compatible with gauge equivalent Lagrangians, is considered. We express the Euler-Lagrange equation of the Lagrangian as a second order Lagrange derivative of a tangent form, considering controlled and higher order tangent forms. Hamiltonian forms of the dynamics generated are given, extending some quantization formulas given by Lukierski, Stichel and Zakrzewski. Using semi-sprays, local solutions of the E-L equations are given in some special particular cases.

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1 Introduction

The second order Lagrangians are considered, for example, in [5, 6], [15], [21] etc. (see, for example, [14, 17, 19] for a study of higher order Lagrangians). The second order Lagrangians that are affine in acceleration are involved in some special problems and studied for example in [1], [3], [4], [5], [9], [12], [13], [16] etc. These are the most possible singular Lagrangians - their vertical hessian vanishes. According to [5 Sect. 6.3], some special regularity conditions can be considered. Third order Lagrangians, that are affine in the third order derivatives and possessing an acceleration-extended Galilean symmetry, are studied in [10]; they extend the second order case considered previously by the same authors and considered in a general form in this paper. It can be a model for a future development of constructions in the present paper.

The goal of this paper is to study tangent forms, i.e. differentiable one forms ω on IR × TM, where M is a manifold. Some basic aspects and motivating examples can be found in our previous paper [18]. We consider an action of a tangent form ω on differentiable curves on M, in fact the same as the action of a suitable second order Lagrangian affine in accelerations (it corresponds
canonically to $\omega$, by Proposition 2.1. Conversely, the action of a second order Lagrangian affine in accelerations can correspond to at least one tangent form (Proposition 2.2). We consider a certain equivalence relation on tangent forms such that an equivalence class corresponds to some gauge equivalent Lagrangians given by the actions (Proposition 3.1).

Considering controlled tangent forms (Proposition 3.2), higher order tangent forms, top tangent forms and Lagrange derivatives of tangent forms, then the Euler Lagrange equation of a tangent form can be obtained by (two) successive Lagrange derivatives of tangent forms (Proposition 4.1), considering also an Ostrogradski tangent form, closed related to Ostrogradski momenta. The Euler-Lagrange equation contains the second derivatives and we prove that in the case of a regular Lagrangian, the solutions are integral curves of a global second order differential equation (Proposition 4.2).

Considering a Legendre map and considering non-degenerated, hyper-non-degenerated and biregular Lagrangians, we study the dynamics given by Lagrangians affine in accelerations given by tangent forms. We prove that for a regular tangent form, the dynamics on $M$ (i.e. the solutions of E-L equations) comes from the projection of the integral curves of a vector field $X$ on $T^2M = T^*M \times_M TM$ (Proposition 5.1), while for a biregular tangent form, the dynamics on $M$ comes from the projection of the integral curves of a vector field $Y$ on $T^9M = TM \times_M TM$ (Proposition 5.2).

Important tools in describing the dynamic equations of a Hamiltonian system are offered by quantization. Following similar ideas used in [9, Section 2.], where Ostrogradski-Dirac and Fadeev-Jakiw methods are used, we use here a modified Ostrogradski-Dirac method, offered by the possibility to construct constraints slight different from the canonical ones used in Ostrogradski theory. The Ostrogradski-Dirac method was also used in [3] to a quantization of a system derived from a Lagrangian affine in accelerations, involved in the study of a Reegge-Teitelboim model. Since in the cases considered in our paper it is not necessary to express the constraints techniques explicitly, we use a symplectic formalism instead, giving here a global form of the quoted methods. In Subsection 5.2 we present a Hamiltonian description of the dynamics defined by the vector fields $X$ and $Y$ described above, proving that:

– If $\omega$ is regular and its essential part is time independent, then there are a symplectic form $\Xi'$ on $T^2M$ and a Hamiltonian $H : \mathbb{R} \times T^2M \to \mathbb{R}$ such that the Hamiltonian vector field $X_H$ is $X$ (Theorem 5.1).

– If $\omega$ is biregular and its essential part is time independent, then there are a symplectic form $\Xi''$ on $T^9M$ and a Hamiltonian $H' : \mathbb{R} \times T^9M \to \mathbb{R}$ such that the Hamiltonian vector field $X_{H'}$ is $Y$ (Theorem 5.2).

Some examples and special cases are given in Subsection 6. In the case when $\dim M = 1$, we prove in Proposition 6.2 that the generalized Euler-Lagrange equation of a regular and basic tangent form admits locally standard Lagrangian descriptions (in the sense of [2, Section 2.]). In order to describe the dynamics generated by some classes of tangent forms, we use first order semi-sprays. Following some concrete examples, we consider some special cases (Propositions 6.3 to 6.7) when families of local semi-sprays of first order are considered; their
integral curves project on (sometimes all) integral curves of the generalized Euler-Lagrange equation associated with the Lagrangian of the tangent form.

Using local calculus, certain geometrical objects on higher order tangent bundles and on general fibered manifolds are described in an Appendix.

2 Tangent forms, Lagrangians and actions on curves

A tangent form on a differentiable manifold \( M \) is a differentiable form \( \omega \in \mathcal{X}^\ast (\mathbb{R} \times TM) \). Denote by \( p_1 : \mathbb{R} \times TM \rightarrow \mathbb{R} \) and \( p_2 : \mathbb{R} \times TM \rightarrow TM \) the natural projections. The pull-backs \( p_1^\ast \) and \( p_2^\ast \) of 1–forms in \( \mathcal{X}^\ast (\mathbb{R} \times TM) \) give rise to a direct sum decomposition \( \mathcal{X}^\ast (\mathbb{R} \times TM) = \mathcal{X}^\ast (\mathbb{R} \times TM) \oplus \mathcal{X}^\ast (\mathbb{R} \times TM) \); let \( \omega = \omega_0 + \omega' \) the corresponding decomposition of \( \omega \). We say that \( \omega_0 \) is the Lagrangian component and \( \omega' \) is the essential component of \( \omega \).

There is a natural flip \( \iota : T^*TM \rightarrow T^*TM \) that is a diffeomorphism of manifolds (see the Appendix). The natural projection \( \pi_{T^*M} : T^*TM \rightarrow T^*M \) gives the morphism of vector bundles \( \pi_{T^*M} \circ \iota^{-1} : T^*TM \rightarrow T^*M \) that induces an epimorphism of vector bundles \( C_{TM} = \pi_{T^*M} \circ \iota^{-1} : T^*TM \rightarrow T^*TM \), over \( TM \). Let us call \( \ker C_{TM} \subset T^*TM \) as the co-vertical bundle of \( TM \). It is easy to see that it is canonically isomorphic with the dual of the vertical vector bundle of \( TM \). Indeed, \( \ker C_{TM} \) is canonically isomorphic with the induced vector bundle \( \pi_{T^*M}^* T^*M \) that is the dual vector bundle of \( \pi_{T^*M}^* TM \), canonically isomorphic to its turn with the vertical vector bundle \( VTM \) of \( TM \). (Notice that \( VTM \) is the kernel of the differential map of \( \pi_{T^*M} : TM \rightarrow M \).) Thus we can denote \( \ker C_{TM} = V^*TM \), without any confusion. Notice also that using the canonical isomorphism depicted above of \( \pi_{T^*M}^* T^*M \) and \( \ker C_{TM} \), then \( C_{TM} \) gives a canonical map \( J^* : T^*TM \rightarrow V^*TM \subset T^*TM \) having the property that \( \ker J^* = J^*(T^*TM) = V^*TM \), thus \( (J^*)^2 = 0 \); it is the dual counterpart of the almost tangent structure on \( TM \) (see [7]).

We can consider the time dependent counterpart, taking \( T^*(\mathbb{R} \times TM) \rightarrow \mathbb{R} \times TM \) instead of \( T^*TM \rightarrow TM \) and \( \mathbb{R} \times V^*TM \subset T^*(\mathbb{R} \times TM) \), with the base \( \mathbb{R} \times TM \), instead of \( V^*TM \subset T^*TM \) with the base \( TM \). More specifically, taking into account the canonical isomorphisms depicted above, the vector bundle \( \mathbb{R} \times V^*TM \rightarrow \mathbb{R} \times TM \) is canonically isomorphic with \( \mathbb{R} \times \pi_{T^*M}^* T^*M \rightarrow \mathbb{R} \times TM \).

A top tangent form \( \eta \) is defined as a section of the vector bundle \( \mathbb{R} \times V^*TM \rightarrow \mathbb{R} \times TM \), or, via canonical isomorphisms, a section of the vector bundle \( \mathbb{R} \times \pi_{T^*M}^* T^*M \rightarrow \mathbb{R} \times TM \). Thus we can regard \( \eta : \mathbb{R} \times TM \rightarrow \pi^* T^*M \).

Since a tangent form is actually a section \( \omega : \mathbb{R} \times TM \rightarrow \mathbb{R} \times T^*TM \), then it gives a top tangent form \( \eta = (I_{\mathbb{R} \times J^*}) \circ \omega : \mathbb{R} \times TM \rightarrow \mathbb{R} \times VTM \).

Using local coordinates (see the Appendix) a tangent form \( \omega \) has the local expression

\[
\omega = \omega_0(t, x^i, y^j)dt + (\omega_i(t, x^j, y^j))dx^i + (\omega_i(t, x^i, y^j))dy^j = \omega_0 + \omega'.
\]
A top tangent form $\gamma$ has the local expression $\gamma = \gamma(t, x^i, y^j)dx^i$; the top tangent form given by the tangent form $\Pi$ has the expression $\bar{\omega} = \bar{\omega}_i dx^i$.

A tangent form can be related to a second order dynamic form considered in [7]. According to [7] Section 2], a first order dynamic form on the bundle $Y = \mathbb{R} \times M \to M$ is a one contact and horizontal two form $\nu$ on $J^1(Y)$, having the local expression $\nu(t, x^i, y^j)dx^i\wedge dt + \tilde{\nu}_g(t, x^i, y^j)dy^j\wedge dt$. Obviously a first order dynamic form is equivalent to give a pure tangent form. An advantage to use tangent forms is having the Lagrangian forms in the same setting. Another motivation to use tangent forms is given by their action on curves, the same as the action of suitable second order Lagrangians that are affine in accelerations.

If $\gamma : [a, b] \to M$ is a curve on $M$, then for $t \in [a, b]$, consider $\gamma(t) = (x^i(t), \frac{dx^i}{dt}(t)) \in \mathbb{R} \times \gamma(t)M$ and the scalar $\omega_{\gamma(t)}(\frac{d^2x^i}{dt^2}(t))$. The action of the tangent form $\omega$ on $\gamma$ is given by the formula

$$I_{\omega}(\gamma) = \int_a^b \omega_{\gamma(t)}\left(\frac{d^2x^i}{dt^2}(t)\right) dt.$$  (2)

Using local coordinates: $t \in \mathbb{R}$, $(x^i)$ on $M$ and $(x^i, y^j)$ on $TM$, if $\omega$ has the expression $\Pi$ and a curve $\gamma$ has $t \to (x^i(t))$, then the action $\Pi$ has the expression

$$I_{\omega}(\gamma) = \int_a^b (\omega_0 + \omega_i \frac{dx^i}{dt} + \tilde{\omega}_i \frac{d^2x^i}{dt^2}) dt.$$  (3)

Let us relate the action of tangent forms on curves to the actions of Lagrangians on curves. First, the action of $L^{(1)} : \mathbb{R} \times TM \to \mathbb{R}$ on a curve $\gamma : [a, b] \to M$ is given by the formula:

$$I_{L^{(1)}}(\gamma) = \int_a^b L^{(1)}(t,\gamma(t), \frac{d\gamma}{dt}(t)) dt.$$  (4)

If $\gamma : [a, b] \to M$ is a curve on $M$, then the curves $\frac{d\gamma}{dt} : [a, b] \to TM$ (the velocity curve) and $\frac{d^2\gamma}{dt^2} : [a, b] \to T^2M \subset TT M$ (the acceleration curve) are the first order lift and the second order lift respectively, of the curve $\gamma$. A second order Lagrangian on $M$ is a differentiable map $L^{(2)} : \mathbb{R} \times T^2M \to \mathbb{R}$, where $T^2M$ is the second order tangent space of $M$ (see the Appendix). The action of $L^{(2)}$ on $\gamma$ is given by the formula:

$$I_{L^{(2)}}(\gamma) = \int_a^b L^{(2)}(t,\gamma(t), \frac{d\gamma}{dt}(t), \frac{d^2\gamma}{dt^2}(t)) dt.$$  (5)

A second order Lagrangian $L$ is affine in accelerations if its vertical Hessian vanishes; using local coordinates, $L(t, x^i, y^j, z^j) = f_0(t, x^i, y^j) + z^j \tilde{g}_i(t, x^i, y^j)$. Notice that if $g_i = 0$, then $L = f_0$ is a first order Lagrangian. In this case $f_0$ is obtained projecting $L : T^2M \to \mathbb{R}$ on $f_0 : TM \to \mathbb{R}$, by the natural projection $T^2M \to TM$; the degeneration case described in this situation is refined later in the paper.
It is easy to see that the Lagrangian action $I_{L_0}$ of a Lagrangian $L_0$ is the same as the tangent action $I_\omega$ of the Lagrangian tangent form $\omega_0 = L_0 dt \in \mathcal{X}^*(\mathbb{R} \times TM)$. It worth to remark that $\omega_0$ is a closed form only if $L_0 = L_0(t)$.

The two actions (2) and (4) are related as follows.

**Proposition 2.1** If $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ is a tangent form, then there is a second order Lagrangian $L^{(2)} : T^2 M \to \mathbb{R}$, affine in accelerations, such that $I_\omega = I_L$.

**Proof.** Let $z \in T^2 M$ and $\gamma : [a, b] \to M$ be a curve, $t' \in (a, b)$ and $z = \frac{d^2}{dt^2}(t')$. Then we define $L_\omega^{(2)} : T^2 M \to \mathbb{R}$, $L_\omega^{(2)}(z) = \omega_{\tilde{\gamma}(t)}(\frac{d^2}{dt^2}(t))$. It is easy to see that the actions of $L_\omega^{(2)}$ and $\omega$ on a curve $\gamma$ have the same form, given by the right side of the formula (2), thus the conclusion follows. □

Using coordinates $(x^i, y^j, z^i)$ on $T^2 M$ (see Appendix), if $\omega$ is given by (1), then we have:

$$L_\omega^{(2)}(t, x^i, y^j, z^i) = \omega_0(t, x^i, y^j) + \omega_i(t, x^j, y^j)y^i + \bar{\omega}_i(t, x^j, y^j)z^i. \quad (5)$$

The following result shows that the action of every second order Lagrangian, affine in accelerations, can be represented as well as an action of a suitable tangent form.

**Proposition 2.2** Let $L^{(2)} : \mathbb{R} \times T^2 M \to \mathbb{R}$ be a second order Lagrangian affine in accelerations. Then there is a tangent form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ such that $I_\omega = I_L$.

**Proof.** Let us consider a local chart $(U, \varphi)$ on $M$; we define a locally tangent form $\theta_U = \frac{\partial}{\partial x^i} dy^i$, thus $(\theta_U)_i = \frac{\partial}{\partial x^i}$ and $(\theta_U)_j = (\theta_U)_0 = 0$ for this tangent form. Let $\{f_\alpha\}_{\alpha \in \mathbb{N}}$ be a partition of unity subordinated to a locally finite open cover $\{U_\alpha\}_{\alpha \in \mathbb{N}}$ of such domains of coordinates. Then the tangent form $\theta = \sum_{\alpha \in \mathbb{N}} f_\alpha \cdot \theta_{U_\alpha}$ is a tangent form $\theta \in \mathcal{X}^*(\mathbb{R} \times TM)$ that has the top component $\bar{\theta}_i = (\theta_U)_i = \frac{\partial L}{\partial x^i}$ and $\theta = \bar{\theta}_i dy^i + \theta_i dx^i$. Since $L^{(2)}$ has the local expression $L^{(2)}(t, x^i, y^j, z^i) = \bar{\theta}_i(t, x^j, y^j)z^i + u(t, x^j, y^j)$, one also has $L^{(2)}(t, x^i, y^j, z^i) = \bar{\theta}_i(t, x^j, y^j)z^i + \theta_i(t, x^j, y^j)y^j + (u(t, x^j, y^j) - \theta i y^j)$. Then the local functions $L_0(t, x^i, y^j) = u(t, x^i, y^j) - \theta i y^j$ give a global function $L_0 : \mathbb{R} \times TM \to \mathbb{R}$ and the tangent form $\omega = \theta + L_0 dt$ has the property that $I_\omega = I_L$. □

The actions of tangent forms on curves are related to the well-known actions of the first and the second order Lagrangians on curves. Let us consider two points $x, y \in M$ and $\gamma_0 = (x^i_0(t))$ be a curve joining $x$ and $y$, i.e. $x^0_0(0) = x$ and $x^0_0(1) = y$. Let us consider variations of $\gamma_0$, as curves joining $x$ and $y$, having the local expression $\gamma_v = (x^i_v(t))$, where $x^i_v(t) = x^i_0(t) + \varepsilon h^i(t)$.

In the case of the actions of second order Lagrangians on curves, the specific variational conditions, impose:

$$h^i(a) = h^i(b) = 0, \quad (6)$$

$$\frac{dh^i}{dt}(a) = \frac{dh^i}{dt}(b) = 0. \quad (7)$$
For a second order Lagrangian $L^{(2)} : \mathbb{R} \times T^2 M \to \mathbb{R}$, the extrema curves of the action $I_{L^{(2)}}$ are given by the well-known Euler-Lagrange equations

$$\frac{\partial L^{(2)}}{\partial x^i} - \frac{d}{dt} \frac{\partial L^{(2)}}{\partial \dot{x}^i} + \frac{d^2}{dt^2} \frac{\partial L^{(2)}}{\partial \dot{z}^i} = 0.$$  

(8)

In the particular case of a Lagrangian \[5\], the Euler-Lagrange equations have the form

$$\frac{\partial \omega_0}{\partial x^i} + \frac{\partial \omega_j}{\partial x^i} \frac{dx^j}{dt} + \frac{d}{dt} \left( \frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i} \frac{dx^j}{dt} \right) + \omega_i + \frac{\partial \omega_j}{\partial y^i} \frac{d^2 x^j}{dt^2} \right) + \frac{d^2}{dt^2} \dot{\omega}_i = 0.$$  

(9)

Let us consider $\mathbb{R}^2$ with coordinates $x$ and $y$. The canonical symplectic form $\alpha = dx \wedge dy$ gives the tangent form $\omega^{(1)} = \dot{x}dy - \dot{y}dx$ and the second order Lagrangian $L_0(t, \dot{x}, \ddot{x}, \dot{y}, \ddot{y}) = \dot{x} \ddot{y} - \dot{y} \ddot{x}$ on $\mathbb{R}^2$; here $(x, y) := (x^1, x^2)$; $(\dot{x}, \dot{y}) := (y^1, y^2)$, in the previous notations. This Lagrangian was involved in \[9\], concerning its invariance to the $(2+1)$-Galilean symmetry; the authors prove in the Appendix that the general form of a one-particle Lagrangian which is at most linearly dependent on $\ddot{x}$ and $\ddot{y}$ leading to Euler-Lagrange equations of motion which are covariant with respect to the $D = 2$ Galilei group, is given, up to gauge transformations, by $L(t, x, y, \dot{x}, \ddot{x}, \dot{y}, \ddot{y}) = -k(\dot{x} \ddot{y} - \dot{y} \ddot{x}) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$. This Lagrangian is affine in accelerations, but it can come from two tangent forms:

$$\omega_1 = -k(\dot{x} \ddot{y} - \dot{y} \ddot{x}) + \frac{m}{2}(\dot{x} dx + \dot{y} dy),$$

$$\omega_2 = -k(\dot{x} \ddot{y} - \dot{y} \ddot{x}) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2) dt.$$  

(10)

In order to put together these two tangent forms, we define below an equivalence relation, ruled by their action and implicitly by their second order Lagrangians, affine in accelerations.

3 Equivalence of tangent forms

A first order Lagrangian $F : \mathbb{R} \times TM \to \mathbb{R}$ and a tangent form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ give together a tangent form $\omega' = \omega + dF$. Then

$$I_{\omega'}(\gamma) = I_\omega(\gamma) + \int_a^b \left( \frac{\partial F}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial F}{\partial \dot{x}^i} \frac{d\dot{x}^i}{dt} + \frac{\partial F}{\partial \dot{z}^i} \frac{d\dot{z}^i}{dt} \right) dt = I_\omega(\gamma) + F(x^i(b), \frac{dx^i}{dt}(b)) - F(x^i(a), \frac{dx^i}{dt}(a)).$$

According to the variation conditions \[6\] and \[7\], it is easy to see that $I_\omega$ and $I_{\omega'}$ have the same extrema curves.

Analogous considerations as made in \[8\] for the gauge equivalence of first order Lagrangians can be transposed for second order Lagrangians (see for example \[13\] Section 4.4). It reads that the second order Lagrangians $L$ and $L' = L + \frac{d}{dt} F$, where $F : \mathbb{R} \times TM \to \mathbb{R}$, are gauge equivalent, i.e. they have the same extrema curves. Here $\frac{d}{dt} F$ stands for $L_{dF}$, the second order Lagrangian associated with the tangent form $dF$. The analogous gauge form for actions of the
corresponding tangent forms, reads that the tangent forms \( \omega \) and \( \omega' = \omega + dF \) have the same extrema curves.

We notice that the Lagrangians given by \([3]\) formula (11) or \([3]\) formula (34)] are gauge equivalent, but they are studied without using this fact.

Let us consider the submodule \( \mathcal{G} \subset \mathcal{X}^*(\mathbb{R} \times TM) \) generated (as a sheaf) by the local differential forms \( \{ \delta x^i = dx^i - y^i dt \}_{i=1}^m \). A differential form \( \eta \in \mathcal{G} \) iff it has the local expression \( \eta = a_i(t, x^i, y^i)dx^i \). It is easy to see that any form \( \eta \in \mathcal{G} \) vanishes along the (second order) lift of a curve on \( M \). Thus for any tangent form \( \omega \in \mathcal{X}^*(\mathbb{R} \times TM) \), the tangent forms \( \omega \) and \( \omega' = \omega + \eta \) have the same extrema curves (see \([13]\) for other implications concerning the module \( \mathcal{G} \)).

We say that:

two tangent forms \( \omega, \omega' \in \mathcal{X}^*(\mathbb{R} \times M) \) are equivalent if there is an \( F \in \mathcal{X}^*(\mathbb{R} \times M) \) such that \( \omega' - \omega - dF \in \mathcal{G} \);

two second order Lagrangians \( L' \) and \( L \) are gauge equivalent if there is an \( F \in \mathcal{X}^*(\mathbb{R} \times TM) \) such that \( L' - L = \frac{d}{dt}F \).

It is easy to see that two equivalent tangent forms have the same extrema curves. Analogously, two second order Lagrangians \( L' \) and \( L \) that are gauge equivalents have the same extrema curves.

**Proposition 3.1** Two tangent forms \( \omega' \) and \( \omega \) are equivalent iff the corresponding second order Lagrangians \( L_{\omega'} \) and \( L_\omega \) are gauge equivalent.

**Proof.** Let \( \omega' = \bar{\omega}'_i dy^i + \omega'_i dx^i + \omega'_0 \) and \( \omega = \bar{\omega}_i dy^i + \omega_i dx^i + \omega_0 \) be equivalent. Thus there is \( F \in \mathcal{F}(\mathbb{R} \times M) \) such that \( \omega' - \omega - dF = \eta_i(dx^i - y^i dt) \). It is easy to see that \( L_{\omega'} - L_\omega = \frac{d}{dt}F \), thus \( L_{\omega'} \) and \( L_\omega \) are gauge equivalent. Conversely, let us suppose that \( L_{\omega'} \) and \( L_\omega \) are gauge equivalent, thus \( L_{\omega'} - L_\omega = \frac{d}{dt}F \).

Then \( \bar{\omega}'_i = \bar{\omega}_i + \frac{\partial F}{\partial y^i} \) and \( \omega'_i y^i + \omega'_0 = (\frac{\partial F}{\partial y^i} + \omega_i) y^i + \frac{\partial F}{\partial t} + \omega_0 \). It follows that \( \omega' - \omega - dF = (\omega'_i - \omega_i - \frac{\partial F}{\partial t}) (dx^i - y^i dt) \in \mathcal{G} \), thus \( \omega' \) and \( \omega \) are equivalent. □

It follows that the property of the above Proposition can be used as a definition of equivalent tangent forms.

**Corrolary 3.1** If two tangent forms correspond to the same second order Lagrangian affine in accelerations, then they are equivalent.

The Poincaré-Cartan form \( \theta_L = L dt + \frac{\partial L}{\partial y^i} dx^i \) of a first order Lagrangian \( L \) is obviously equivalent to the canonical Lagrangian form \( L dt \) and both correspond to the same Lagrangian \( L \), seen of second order by \( T^2 M \to TM \to \mathbb{R} \).

There are two possibilities to associate a tangent form to a pointed Lagrangian \( L(t, x^i, y^i) = y^i \nu_i : \omega_1 = \nu_i dx^i \) and \( \omega_2 = L dt \) respectively. The first is pure and the second is a Lagrangian one, but they have the same action on curves, that given by the action of the same Lagrangian. It is easy to see that \( d\omega_1 - d\omega_2 = 0 \) iff \( \nu^i = 0 \); thus \( \omega_1 \) and \( \omega_2 \) are not differential equivalent (i.e. \( \omega_1 - \omega_2 = dF \) for \( L \neq 0 \). Thus there are tangent forms that are not differential equivalent (i.e. their difference is not a exact differential), but equivalent.
Every tangent form \( \omega \) of the form (1) is locally equivalent to the local tangent form \( \omega = (\omega_0 + y^i \omega_i) dt + \bar{\omega}_i dy^i \), since \( \omega - \omega' = \omega_i \delta x^i \). But, in general \( \omega' \) is not a global tangent form.

The tangent forms \( \omega_1 = -k(\dot{x}dy - \dot{y}dx) + \frac{\partial}{\partial t} (\dot{x}dx + \dot{y}dy) \) and \( \omega_2 = -k(\dot{x}dy - \dot{y}dx) + \frac{\partial}{\partial t} (\dot{x}^2 + \dot{y}^2) \) considered previously are equivalent, since \( \omega_1 - \omega_2 = \frac{\partial}{\partial t} (\dot{x}(dx - \dot{x}dt) + \dot{y}(dy - \dot{y}dt)) \). Let us consider below two other situations.

1) Considering the canonical symplectic form in \( \mathbb{R}^2 \) given by \( (\varepsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), we obtain the pure tangent form \( \omega_1 = -k\varepsilon_{ij} y^i dy^j \), or \( \omega_1 = -k\varepsilon_{ij} \dot{x}^i \dot{x}^j \) where \( k \) is a non-null constant, that corresponds to the second order Lagrangian \( L_0(x^i, y^i, z^i) = -k\varepsilon_{ij} y^i z^j \), or \( L_0(x^i, \dot{x}^i, \dot{z}^i) = -k\varepsilon_{ij} \dot{x}^i \dot{z}^j \).

The Lagrangian \( L(x^i, y^i, z^i) = \frac{m}{2} y^i \dot{y}^i - k\varepsilon_{ij} y^i z^j \), or \( L(x^i, \dot{x}^i, \dot{z}^i) = \frac{m}{2} \dot{x}^i \dot{z}^i - k\varepsilon_{ij} \dot{x}^i \dot{z}^j \) was considered in [9, 10, 1].

In order to obtain a tangent form we have two possibilities: \( \omega = m\dot{y}^i \delta_{ij} dx^j - k\varepsilon_{ij} y^i dy^j \) and \( \omega' = \frac{m\dot{y}^i \delta_{ij}}{\parallel y^i \parallel} dt - k \varepsilon_{ij} y^i dy^j \); the first is pure and the second is a mixed one.

2) The Lagrangian \( L(x^i, y^i, z^i) = -m \parallel y^i \parallel + \varepsilon_{ij} \dot{x}^i \dot{x}^j \), where \( \parallel y^i \parallel = \sqrt{\frac{m\dot{y}^i \delta_{ij}}{\parallel y^i \parallel}} \) or \( L(x^i, \dot{x}^i, \dot{z}^i) = -m \parallel \dot{y} \parallel + \varepsilon_{ij} \dot{x}^i \dot{z}^j \), where \( \parallel \dot{y} \parallel = \sqrt{\frac{m\dot{x}^i \dot{z}^i \delta_{ij}}{2}} \) was considered in [12]. The two tangent forms, one pure and one mixed, can also be considered: \( \omega = \frac{m\dot{y}^i \delta_{ij}}{\parallel y^i \parallel} dx^j + \varepsilon_{ij} \dot{x}^i \dot{x}^j \), and \( \omega' = -m \parallel y^i \parallel dt + \frac{\varepsilon_{ij} \dot{y}^i \dot{y}^j}{\parallel y^i \parallel} dy^j \).

Unlike the first example, in the second example the tangent form \( \omega' \) is not differentiable in the points where \( y^i = 0 \).

### 3.1 Controlled and higher order tangent forms

We consider below controlled tangent form and, in particular, higher order tangent forms. The controlled top derivative, defined also in this subsection, is used in the next subsection in an accurate study of the Euler-Lagrange equation of a tangent form.

Let \( \pi_E : E \to M \) be a fiber manifold (i.e. a surjective submersion) that factorize as a composition \( \pi_E = \pi_{TM} \circ \pi_E' \) a fibered manifolds \( \pi_E' : E \to TM \) and the canonical projection \( \pi_{TM} : TM \to M \).

A controlled tangent form on \( E \) is a fibered map \( \omega : E \times E \to T^* TM \) over the base \( TM \). A controlled top tangent form on \( E \) is a bundle map \( \bar{\omega} : E \times E \to \pi_{TM}^* T^* M \cong V^* M \) over the base \( TM \).

Obviously a controlled tangent form \( \omega \) as above gives rise to a top tangent form \( \bar{\omega} = J^\omega \circ \omega \).

Let us consider some local coordinates, adapted to the fibered structures: \( (x^i) \) on \( M \), \( (x^i, y^i) \) on \( TM \) and \( (x^i, y^i, u^\alpha) = (x^i, u^\alpha) \) on \( E \). A controlled tangent form \( \omega \) has the expression \( (t, x^i, u^\alpha) \to (x^i, y^i, \omega_i(t, x^i, u^\alpha), \bar{\omega}_i(t, x^i, u^\alpha)) \); its expression on fibers is \( \omega = \omega_i(t, x^i, u^\alpha) dx^i + \bar{\omega}_i(t, x^i, u^\alpha) dy^i \). Considering the (local) differential operator \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + u^\alpha \frac{\partial}{\partial u^\alpha} \) on the sheaf of local real
functions on $TE$ having the same domain of definition, we say that
\[ \mathcal{E}'_{\omega}(t, x^i, y^i, u^\alpha, v^\alpha) = (\omega_i - \frac{d}{dt}\omega_i)dx^i \]
is the Lagrange controlled top derivative of $\omega$.

**Proposition 3.2** The Lagrange controlled top derivative of $\omega$ is a global fibered map $\mathcal{E}'_{\omega} : \mathbb{R} \times TE \to T^*M$, over $M$.

**Proof.** Considering the local expressions $\mathcal{E}'_{\omega}(t, x^i, y^i, u^\alpha, v^\alpha) = [\omega_i - (\frac{\partial \omega_i}{\partial y^j}y^j + \frac{\partial \omega_i}{\partial u^\alpha}u^\alpha + \frac{\partial \omega_i}{\partial v^\alpha}v^\alpha)]dx^i$, we have to prove that the definition does not depend on coordinates. If $\{x^i, y'^i, u'^\alpha, v'^\alpha\}$ is another set of coordinates, on the intersection domain we have the rules $\omega_i = \frac{\partial x^i}{\partial x^j}x^j$ and $\omega_i = \frac{\partial y'^i}{\partial y^j}y^j + \frac{\partial y'^i}{\partial u^\alpha}u^\alpha + \frac{\partial y'^i}{\partial v^\alpha}v^\alpha$ respectively. Analogously, a controlled top tangent form $\bar{\omega}$ has the expression $\bar{\omega}_i = \frac{\partial \bar{x}^i}{\partial x^j}x^j$ and $\bar{\omega}_i = \frac{\partial \bar{y}'^i}{\partial y^j}y^j + \frac{\partial \bar{y}'^i}{\partial u^\alpha}u^\alpha + \frac{\partial \bar{y}'^i}{\partial v^\alpha}v^\alpha$.

We have $\omega_i = \frac{\partial x^i}{\partial x^j}x^j + \frac{\partial x^i}{\partial y^j}y^j + \frac{\partial x^i}{\partial u^\alpha}u^\alpha + \frac{\partial x^i}{\partial v^\alpha}v^\alpha$ and $\bar{\omega}_i = \frac{\partial \bar{x}^i}{\partial x^j}x^j + \frac{\partial \bar{x}^i}{\partial y^j}y^j + \frac{\partial \bar{x}^i}{\partial u^\alpha}u^\alpha + \frac{\partial \bar{x}^i}{\partial v^\alpha}v^\alpha$.

It is easy to see that in the case $E = TM$, we obtain that a controlled (top) tangent form is just a (top) tangent form. The Lagrange controlled derivative of a tangent form $\omega : \mathcal{E}'_{\omega} : \mathbb{R} \times TT^*M \to T^*TM$ restricts to the Lagrange derivative of $\omega : \mathcal{E}'_{\omega} : \mathbb{R} \times T^2M \to T^*TM$ given above by formula (11); it follows using the local expression of the inclusion $T^2M \subset TT^*M$ (see the Appendix).

We define:

a *k*-order tangent form as a controlled tangent form $\omega : \mathbb{R} \times T^k M \to T^*TM$

and

a *k*-order top tangent form as a bundle map $\bar{\omega} : \mathbb{R} \times T^k M \to \pi_T^*T^*M$.

As in the general case, a *k*-order tangent form $\omega$ gives a *k*-order top tangent form $\bar{\omega} = J^* \circ \omega$.

Let us define now the Lagrange top derivative $\mathcal{E}^{(k+1)}_{\omega}$ of a *k*-order tangent form $\omega : \mathbb{R} \times T^k M \to T^*TM$. The Lagrange controlled top derivative of $\omega$ is $\mathcal{E}^{(k+1)}_{\omega} : \mathbb{R} \times TT^k M \to \pi_T^*T^*M$ restricts to the Lagrange derivative of $\omega : \mathcal{E}^{(k+1)}_{\omega} : \mathbb{R} \times T^{k+1} M \to \pi_T^*T^*M$, according to the inclusion $T^{k+1}M \subset TT^k M$. Using local coordinates, the inclusion has the form $(x^i, y^j, w^i, \bar{w}^i) \to (x^i, y^j, w^i, \bar{w}^i, \bar{y}'^i, \bar{u}'^\alpha, \bar{v}'^\alpha)$. The expression
\[ \mathcal{E}^{(k+1)}_{\omega}(t, x^i, y^j, w^i, \bar{y}'^i, \bar{u}'^\alpha, \bar{v}'^\alpha) = (\omega_i - (\frac{\partial \omega_i}{\partial y^j}Y^j + \frac{\partial \omega_i}{\partial w^i}W^i)]dx^i \]
and the restriction to \( \mathbb{R} \times \pi_T^2 M T^{k+1} M \) is
\[
E^{(k+1)}_\omega : \mathbb{R} \times T^{k+1} M \to T^* M, \quad E^{(k+1)}_\omega (t, x^i, y^i, z^j, w^i, \dot{w}^i) = \left[ \omega_i - \left( \frac{\partial \omega_i}{\partial t} + \frac{\partial \omega_i}{\partial x^j} y^j + \cdots + \frac{\partial \omega_i}{\partial z^j} w^j + \frac{\partial \omega_i}{\partial w^j} \dot{w}^j \right) \right] dx^i,
\]
or
\[
E^{(k+1)}_\omega = \left[ \omega_i - \frac{d}{dt} \omega_i \right] dx^i, \tag{12}
\]
where \( \frac{d}{dt} \) is the local operator given by \( \frac{d}{dt} = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + \cdots + w^i \frac{\partial}{\partial z^i} + \dot{w}^i \frac{\partial}{\partial w^i} \).

The first order Lagrange top derivative of a tangent form \( \omega = \omega_i dx^i + \omega_j dy^j \) is the second order top tangent form \( E^{(2)}_\omega = E^{(2)}_\omega : \mathbb{R} \times \pi_T^2 M T^2 M \to \pi_T^2 M T^* M \) given using (12).

In the case \( k = 3 \), the local operator \( \frac{d}{dt} \) is given by \( \frac{d}{dt} = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + z^i \frac{\partial}{\partial y^i} + w^i \frac{\partial}{\partial z^i} + \dot{w}^i \frac{\partial}{\partial w^i} \). In the case when \( \omega = \omega_i dx^i \) has the order 2, then \( \frac{\partial \omega_i}{\partial w^i} = 0 \) and the third order Lagrange top derivative \( E^{(3)}_\omega \) has the order at most 3, as \( \omega \). This is the case below when the Euler-Lagrange top form of a tangent form has the third order.

4 The Euler-Lagrange equation as a top tangent form

Any second order Lagrangian \( L : \mathbb{R} \times T^2 M \to \mathbb{R} \) gives rise to an at most forth order top tangent form \( E_i dx^i \), that we call the Euler-Lagrange top tangent form of \( L \), where
\[
E_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial z^i}, \tag{13}
\]
\( \frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial y^j} + w^j \frac{\partial}{\partial z^j} + \dot{w}^j \frac{\partial}{\partial w^j} \) and \((x^i, y^i, z^i, w^i, \dot{w}^i)\) are the canonical local coordinates on \( T^4 M \) induced by the local coordinates \((x^i)\) on \( M \). In the case when a second order Lagrangian \( L_{\omega} \) is affine in accelerations and it is associated with a tangent form \( \omega \), its local formula is given as in formula (15) with \( L^{(2)} = L_{\omega} \). We say that the Euler-Lagrange top tangent form \( E = E_{\omega} \) of \( L_{\omega} \) is the Euler-Lagrange top tangent form of \( \omega \). Specifically, if \( \omega \) is a tangent form given by formula (11), then \( L_{\omega}(t, x^i, y^i, z^i) = \omega_0 + y^i \omega_i + z^i \dot{\omega}_i + \ddot{\omega}_i \) thus
\[
E_i = \frac{\partial \omega_0}{\partial x^i} + \frac{\partial \omega_1}{\partial x^i} y^j + \frac{\partial \omega_2}{\partial x^i} z^j - \frac{d}{dt} \left( \frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_1}{\partial y^i} y^j + \omega_i + \frac{\partial \omega_3}{\partial y^i} z^j + \frac{d^2}{dt^2} \dot{\omega}_i \right), \tag{14}
\]
where \( \frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial y^j} + w^j \frac{\partial}{\partial z^j} \), since \( \frac{\partial L}{\partial \dot{w}^j} = \dot{\omega}_i(t, x^i, y^i, z^i) \) and the forth order coordinates \((\ddot{w}^i)\) are not involved. Thus the top tangent form \( E \) is at most third order in this case.

We prove below that the Euler-Lagrange top tangent form can be obtained using two second order tangent forms.
Proposition 4.1 Let \( \omega \) be a (first order) tangent form such that the Euler-Lagrange top tangent form \( \mathcal{E}_\omega \) is of third order. Then the following assertions hold true.

1. If \( \Omega \) is a first or a second order tangent form such that such that \( \bar{\Omega} = \bar{\omega} \), then there is a second or a third order tangent form \( \Phi \), uniquely determined by the conditions that the Lagrange top derivative of \( \Omega \) is \( \Phi \) and the Lagrange top derivative of \( \Phi \) is the Euler-Lagrange top tangent form \( \mathcal{E}_\omega \).

2. There are two second order tangent forms \( \Omega \) and \( \Phi \) such that \( \bar{\Omega} = \bar{\omega} \), the Lagrange top derivative of \( \Omega \) is \( \Phi \) and the Lagrange top derivative of \( \Phi \) is \( \mathcal{E}_\omega \).

Proof. 1. The conditions on \( \Phi \) read \( \bar{\Phi}_i = \Omega_i - \frac{d}{dt} \bar{\Omega}_i = \Omega_i - \frac{d}{dt} \bar{\omega}_i \) and \( \bar{\Phi}_i = \frac{d}{dt} \bar{\Phi}_i + \mathcal{E}_\omega \), thus \( \Omega \) is uniquely given by these conditions. If \( \Omega \) is of first order then \( \Phi \) is of second order, thus \( \Phi \) is of second or of third order. If \( \Omega \) is of second order, then \( \Phi \) is of second or third order, thus \( \Phi \) is of second or third order.

2. Let us denote \( \omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_j dy^j \) and let \( L = \omega_0 + \omega_i y^i + \bar{\omega}_j z^j \) be the associated two order Lagrangian affine in accelerations. We consider \( \Omega = \Omega_i dx^i + \bar{\Omega}_i dy^j = \left( \frac{\partial L}{\partial \dot{x}^i} - \omega_i \right) dx^i + \frac{\partial L}{\partial \dot{y}^j} dy^j = \left( \frac{\partial L}{\partial \dot{y}^j} + \frac{\partial \omega_j}{\partial y^j} y^j + \frac{\partial \bar{\omega}_{j}}{\partial y^j} z^j \right) dx^i + \bar{\omega}_j dy^j \).

We have \( \frac{\partial}{\partial y^j} = \frac{\partial}{\partial z^i} \frac{\partial y^j}{\partial z^i} + 2 \frac{\partial x'^i}{\partial z^i} \frac{\partial}{\partial y^j} \frac{\partial x'^j}{\partial z^i} \) and \( \frac{\partial}{\partial z^i} = \frac{\partial x'^i}{\partial z^i} \frac{\partial}{\partial z^i} \). It follows that

\[
\bar{\Omega}_i = \frac{\partial x'^i}{\partial x^i} \bar{\Omega}'_i, \\
\Omega_i = \frac{\partial y'^j}{\partial x^i} \bar{\Omega}'_i + \frac{\partial x'^i}{\partial x^i} \bar{\Omega}'_i, 
\]

thus the local 1-forms \( \Omega = \Omega_i dx^i + \bar{\Omega}_i dy^j \) are the restrictions of a (global) second order tangent form \( \Omega: T^* M \to T^* T M \).

According to 1., \( \bar{\Phi}_i = \Omega_i - \frac{d}{dt} \bar{\Omega}_i = \frac{d}{dt} \bar{\omega}_i - \omega_i - \frac{d}{dt} \bar{\omega}_i \) and \( \Phi_i = \frac{d}{dt} \bar{\Phi}_i + \mathcal{E}_\omega = \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial \dot{z}^i} - \omega_i, \) thus \( \Phi \) is of second order. \( \square \)

We say that the second order tangent form \( \Omega \) constructed in 2. of Proposition 4.1 is an Ostrogradski tangent form of \( \omega \).

The above construction is related to a general approach, related to the classical Ostrogradski theory.

Let \( L: \mathbb{R} \times T^2 M \to \mathbb{R} \) be a second order Lagrangian. There is a top tangent form \( \bar{\omega} = \frac{\partial L}{\partial \dot{x}^i} dx^i \) associated with this Lagrangian and having an order at most 2. Let us suppose that \( \omega \) is a first or second order tangent form \( \omega \) such that its top tangent form is \( \bar{\omega} \), i.e. \( \omega = \omega_i dx^i + \frac{\partial L}{\partial \dot{y}^j} dy^j \). One can consider for example \( \omega = \frac{1}{2} \frac{\partial L}{\partial \dot{y}^j} dx^i + \frac{\partial L}{\partial \dot{z}^i} dy^i \). Then the formula \( \Omega = \Omega_i dx^i + \bar{\Omega}_i dy^j \) defines a tangent form of order at most 2.

Then \( \eta: \mathbb{R} \times T^2 M \to T^* T M, \eta(t, x^i, y^i, z^i) = \left( \frac{\partial L}{\partial \dot{y}^j} - \omega_i \right) dx^i + \frac{\partial L}{\partial \dot{z}^i} dy^i \) is a second order tangent form. The Lagrange top derivative of \( \eta \) is \( \mathcal{E}_\eta^{(3)} \).
\[ \mathbb{R} \times T^2 M \to T^* M, \quad E''_\eta = \left( \frac{\partial L}{\partial \dot{x}^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i} \right) dx^i. \] Usually \( \left( \frac{\partial L}{\partial \dot{y}^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{z}^i} \right) \)

is denoted by \( p_i \).

Since \( E = \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}, \quad \frac{\partial^2 L}{\partial x \partial y} \) + \( \frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{x} \partial \dot{y}} \), \( \frac{\partial^2 L}{\partial x \partial \dot{y}} \) + \( \frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{x} \partial \dot{y}} \), \( \frac{\partial^2 L}{\partial x \partial \dot{z}} \) + \( \frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{x} \partial \dot{z}} \), it follows that \( \mu = \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) dx^i + \left( \frac{\partial L}{\partial y} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} \right) dy^i \) is at most third order tangent form and its Lagrange top derivative is \( E''_\mu = E'_\omega \), the Euler-Lagrange top tangent form of \( \omega \). This algorithm can produce tangent forms for a second order Lagrangian, taking a suitable tangent form \( \omega \).

A \( k \)-order semi-spray \( S : \mathbb{R} \times T^k M \to T^{k+1} M \) is a section \( (t, x^{(k)}) \xrightarrow{\partial} (t, S(t, x^{(k)})) \) of the affine bundle \( \mathbb{R} \times T^{k+1} M \to \mathbb{R} \times T^k M \), obtained as a product of the affine bundle \( T^{k+1} M \to T^k M \) and the identity \( \mathbb{R} \to \mathbb{R} \). Let \( \mathcal{E}_\omega : \mathbb{R} \times T^3 M \to T^* M \) be the Euler-Lagrange top tangent form of a tangent form \( \omega \). We say that a second order semi-spray \( \mathcal{S} : \mathbb{R} \times T^2 M \to \mathbb{R} \times T^3 M \) is adapted to the tangent form \( \omega \) if \( \mathcal{E}_\omega \circ \mathcal{S} = 0 \). The local expressions of \( \mathcal{S} \) and \( \mathcal{E}_\omega \) are \( (t, x^i, y^i, z^i) \xrightarrow{\partial} (t, x^i, y^i, z^i, S^i(t, x^i, y^i, z^i)) \) and \( \mathcal{E}_\omega = \mathcal{E}_i dx^i \) respectively, with \( \mathcal{E}_i \) given by \( [4] \). Denoting \( h_{ij} = \frac{\partial \omega^i}{\partial y^j} - \frac{\partial \omega^j}{\partial y^i} \), then there are local functions \( f_i(t, x^i, y^i, z^i) \) such that

\[ \mathcal{E}_i(t, x^i, y^i, z^i, w^i) = h_{ij} w^j + f_i(t, x^i, y^i, z^i). \]

More precisely:

\[ \mathcal{E}_i = \frac{\partial \omega}{\partial x} - \frac{\partial \omega}{\partial y} - \frac{\partial^2 \omega}{\partial x \partial y} y^i + \left( \frac{\partial \omega}{\partial x} - \frac{\partial \omega}{\partial y} \right) \frac{\partial^2 \omega}{\partial x \partial y} y^j \]

The condition that \( \mathcal{S} \) is adapted to the tangent form \( \omega \) reads

\[ h_{ij} S^j + f_i(t, x^i, y^i, z^i) = 0. \] (16)

Notice that the third order semi-spray \( S \) gives a system of third order equations, having the form

\[ \frac{d^3 x^i}{dt^3} + S^i(t, x^i, \frac{dx^i}{dt}, \frac{d^2 x^i}{dt^2}) = 0; \]

its solutions are the integral curves of the vector field \( S \).

We say that the tangent form \( \omega \) is regular if the local matrices \( (h_{ij} = \frac{\partial \omega}{\partial y^i} - \frac{\partial \omega}{\partial y^j}) \) are non-singular. It is easy to see that this property is free of coordinates.

**Proposition 4.2** If the tangent form \( \omega \) is regular, then the solutions of the generalized Euler-Lagrange equation \( \mathcal{E} = 0 \), where \( \mathcal{E} \) is given by \( [4] \) are the same solutions of a second order equation given by a global second order semi-spray \( \mathcal{S} : \mathbb{R} \times T^2 M \to T^3 M \).
Proof. If \( \omega \) is regular, then the matrix \((h_{ij})\) is invertible and let \((h^{ij}) = (h_{ij})^{-1}\). The equation (10) gives uniquely \( S^j = h^{ji} f_i(t, x^i, y^i, z^i) \) and a map \( S : \mathbb{R} \times T^3 M \to T^3 M, S(t, x^i, y^i, z^i) = (t, x^i, y^i, z^i, S^i) \), that is well defined and gives the second order semi-spray \( S \). \( \square \)

5 Non-degenerated and regular tangent forms

We recall that a (first order) tangent form \( \omega \) given by (1) is

regular if the local matrix \( (h_{ij} = \frac{\partial \omega_i}{\partial y^j} - \frac{\partial \omega_j}{\partial y^i}) \) is non-singular and

non-degenerated if the local matrix \( (\frac{\partial \omega_i}{\partial y^j}) \) is non-singular.

The two above regularity conditions are free of coordinates and can be related as follows.

If \( m = \dim M \) is odd, then there are not regular tangent form on \( M \), since a skew symmetric matrix in singular in this case; but there are non-degenerated tangent forms. For example, let \( g \) be a (pseudo-) Riemannian metric on \( M \), \( F(x, y) = \frac{1}{2}g_{x}(y, y) \) its energy map and \( \tilde{\omega}_i = \frac{\partial F}{\partial y^i} \). Then any tangent form that has \( \tilde{\omega} = \tilde{\omega}_i dx^i \) a top tangent form is non-degenerated. Even in even dimension, this top tangent form is non-degenerated, but never regular, since \( \frac{\partial \omega_i}{\partial y^j} - \frac{\partial \omega_j}{\partial y^i} = 0 \).

In even dimensions, there are regular tangent forms that are degenerated. For example, in \( \mathbb{R}^2 \), the top tangent form \( \tilde{\omega}(x, y, X, Y) = (X + Y)dx - (X + Y)dy \) is regular, but degenerated.

We say that a tangent \( \omega \) form is biregular if it is hyper-non-degenerated and also regular.

Let us extend the definition of a non-degenerated tangent form to higher order tangent forms and use it in the study of the Euler-Lagrange equation.

For \( k \geq 1 \), let us denote \( T^k M = T^* M \times_M T^{k-1} M \), the fibered product over the base \( M \). A \( k \)-order top tangent form \( \tilde{\omega} : \mathbb{R} \times T^k M \to \mathbb{R} \times T^* M \) gives rise to a bundle map \( \mathcal{L}_{\tilde{\omega}} : \mathbb{R} \times T^k M \to \mathbb{R} \times T^* M \), \( \mathcal{L}_{\tilde{\omega}}(t, x^{(k)}) = (t, \pi_k(x^{(k)}, \tilde{\omega}(t, x^{(k)}))) \), that we call the Legendre map of \( \tilde{\omega} \). The Legendre map \( \mathcal{L}_{\tilde{\omega}} \) of a \( k \)-order tangent form \( \omega \) is, by definition, the Legendre map of its associated top form. The condition that \( \mathcal{L}_{\tilde{\omega}} \) be a local diffeomorphism can be read as \( \tilde{\omega} \) be a non-degenerated top tangent form. We say that \( \omega \) is hyper-non-degenerated if \( \mathcal{L}_{\tilde{\omega}} \) is a global diffeomorphism. The same definitions (Legendre map, non-degenerated, hyper-non-degenerated and biregular) on a tangent form \( \omega \) are the same as referring to its top form \( \tilde{\omega} \), as above.

We recall that a \( k \)-order semi-spray on \( M \) is a section \( S : \mathbb{R} \times T^k M \to \mathbb{R} \times T^{k+1} M \) of the affine bundle \( \mathbb{R} \times T^{k+1} M \xrightarrow{\pi} \mathbb{R} \times T^k M \). It can be regarded as well as a (time dependent) vector field \( \Gamma_0 \) on the manifold \( T^k M \), since \( T^{k+1} M \subset TT^k M \).

Let \( \pi_E : E \to M \) be a fibered manifold. For \( k \geq 1 \), we say that a controlled semi-spray of degree \( k \) on \( M \) over \( E \) is a map \( S : \mathbb{R} \times E \times_M T^k M \to T^{k+1} M \) such that denoting \( \pi' : \mathbb{R} \times E \times_M T^k M \to T^k M \) the canonical projection, then
The dynamics of regular and biregular tangent forms

5.1 The dynamics of regular and biregular tangent forms

We prove in this subsection that the dynamics on M of a regular tangent form comes from the projection of the integral curves of a vector field X on T^2M = T^2M ×_M TM, while for a biregular tangent form, its dynamics comes from the projection of the integral curves of a vector field Y on T^2_2M = TM ×_M TM.

A regular (first order) tangent form ω gives rise to a 2-co-semi-spray \( \bar{\omega} \) of order 1. Considering local coordinates coming from an open \( U \)⊂ \( M \), then \( \mathcal{E}'_\omega = \Omega = \Omega \omega dx^i + \Omega_i dy^i \) and let \( X_U = \frac{\partial}{\partial x^i} + y^j \frac{\partial}{\partial x^j} + S^i(t, x^j, y^j, p_j) \frac{\partial}{\partial y^i} + \Phi_i \frac{\partial}{\partial p_i} \) be the expression of a local vector field on \( \mathcal{R} \times T^{2*}M \), where \( \Phi_i(t, x^j, y^i, p_j) = \Phi_i(t, x^j, y^i, X_S(t, x^j, y^j, p_j)) = \frac{\partial \omega}{\partial y^i} - \frac{\partial \omega}{\partial y^j} + \left( \frac{\partial \omega}{\partial x^j} - \frac{\partial \omega}{\partial x^j} \right) y^j + \left( \frac{\partial \omega}{\partial y^j} - \frac{\partial \omega}{\partial y^j} \right) z^j \). Let us denote by \( \pi_{2*} : T^{2*}M \rightarrow M \) the canonical projection.
Proposition 5.1 Let $\omega$ be a regular (first order) tangent form. Then:

1. the local vector fields $X_U$ glue together to a vector field $X$ on $T^2M$ and

2. the integral curves of $X$ projects by $\pi_{*2}$ to all the critical curves of the action of $\omega$.

Proof. We use local coordinates $(x^i)$, $(x^i, y^i)$, $(x^i, p_i)$ and $(x^i, y^i, p_i, X^i, Y^i, P_i)$ on $M$, $TM$, $T^*_M = T^*_M \times_M T^*_M$ and $T(TM \times_M T^*_M)$ respectively (see the Appendix). On the intersection domain of two fibered relations 15 for $\Phi$ are the components of a 1-co-semi-spray. The second rule follows using similar statements.

Let us consider that $\omega$ is a local diffeomorphism that is a global one iff the Legendre map is a global diffeomorphism. Then:

Along an integral curve of $X$ we have $\frac{dx^i}{dt} = y^i$, $\frac{dx^i}{dt} = S^i$ and $\frac{dp_i}{dt} = \Phi_i$. Since $p_i = \Phi_i(t, x^j, y^j, S^i)$ and $\Phi_i = \frac{d\Omega_i}{dy^j} - \omega_i - \frac{d}{dt}\bar{\omega}_i$, thus $\frac{dp_i}{dt} = (\frac{d\Omega_i}{dy^j} - \omega_i - \frac{d}{dt}\bar{\omega}_i)$, it follows that $\Phi_i = \frac{d\Omega_i}{dy^j} - \omega_i = \frac{d}{d\hat{\Omega}_i} - \omega_i$, i.e. the Euler-Lagrange equation holds along any curve $t \to x(t)$. This proves 2. □

Let us consider that $\omega$ is biregular, i.e. hyper-non-degenerated and regular. Let us denote $T^2M = T^*_M \times M$ and $\pi_{02*} : T^2M \to M$ the canonical projection and consider coordinates $(x^i, p_{(0)i}, p_{(1)i})$ on $T^2M$, induced by the local coordinates $(x^i)$ on $M$. We define a map $L''^0 : \mathbb{R} \times T^2M \to \mathbb{T} \times T^2M$, $L''_i^0(t, x^{(2)}) = (t, \bar{\omega}_i, \Phi_i)$, where $\bar{\omega}_i = \bar{\omega}$ and $\Phi_i$ are the corresponding top tangent forms.

Then $\omega$ is hyper-non-degenerated and regular iff the map $L''_i^0$ is a local diffeomorphism. The tangent form $\omega$ is hyper-non-degenerated iff $L''_i^0$ is a diffeomorphism.

Considering local coordinates as previously, the Legendre-map $L''_i^0$ has the local expression $L''_i^0(t, x^i, y^i, z^i) = (t, \bar{\Omega}_i(t, x^i, y^i, S^i))$. As above it follows that $L''_i^0$ is a local diffeomorphism that is a global one iff the Legendre map is a global diffeomorphism.

If the tangent form $\omega$ is biregular, then considering some local coordinates $(t, p_{(0)j}, p_{(1)j})$ on $\mathbb{T} \times T^2M$, coming from an open $U \subset M$, the equations $\bar{\omega}_i(t, x^i, y^i) = p_{(0)i}$ give $y^i = T^i(t, x^j, p_{(0)j})$ and the equations $\bar{\Omega}_i(t, x^i, T^j, z^i) = p_{(1)i}$ give $z^i = S^i(t, x^i, p_{(0)j}, p_{(1)j})$. If $\omega$ is hyperregular and hyper-non-degenerated, then the local functions $(T^i)$ and $(S^i)$ come from some co-semi-sprays and give some global diffeomorphisms $T : \mathbb{R} \times T^*M \to \mathbb{R} \times TM$ and $S : \mathbb{R} \times T^2M \to \mathbb{R} \times T^2M$ respectively.

Let us consider the vector field on $T^2U$, given by $Y_U = T^i \frac{\partial}{\partial x^i} + (\bar{\Omega}_i - p_{(1)i}) \frac{\partial}{\partial p_{(1)i}} + \Phi_i \frac{\partial}{\partial p_{(1)i}}$, where $\bar{\Omega}_i(t, x^i, p_{(0)j}) = \Omega_i(t, x^i, T^j)$, $\Phi_i(t, x^i, p_{(0)j}, p_{(1)i}) = \Phi_i(t, x^i, T^j, S^i(t, x^i, T^j, p_{(1)i}))$.

Proposition 5.2 Let $\omega$ be a biregular (first order) tangent form. Then:

1. the local vector fields $Y_U$ glue together to a vector field $Y$ on $T^2M$ and

2. the integral curves of $Y$ projects by $\pi_{*2}$ to all the critical curves of the action of $\omega$. 

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Proof. We use local coordinates \((x^i), (\dot{x}^i), (p_{(0)i}), (p_{(1)i})\) and \((x^i), (p_{(0)i}), (p_{(1)i}), (\ddot{x}^i), (p_{(0)i}), (p_{(1)i})\) on \(M, T_2^2M = T^*M \times_M T^*M\) and \(T(TM^* \times_M T^*M)\) respectively (see the Appendix). Then, on the intersection of two local bundle charts, the local components of \(Y\) follow the rules \(T^i = \frac{\partial x^t}{\partial x^i} T^t, (\hat{\Omega}_i - p_{(1)i}) = \frac{\partial x^t}{\partial x^i} (\hat{\Omega}_t - p_{(1)t})\). The first relation follows from the fact that \((T^i)\) are the components of a global map in the fibers of \(TM\).

The second relation follows from \(p_{(1)i} = \frac{\partial x^t}{\partial x^i} p_{(1)t}\) and \(\hat{\Omega}_i = \frac{\partial x^t}{\partial x^i} \hat{\Omega}_t + \frac{\partial x^t}{\partial x^i} p_{(0)i}\) (see the Appendix and definition of \(\hat{\Omega}\)). The third relation follows using the definition of \(\Theta\), the second relation \((1)\) and the fact that \(\Phi = p_{(1)i}\). In order to prove 2., along an integral curve of \(Y\) we have \(y^i = \frac{d x^i}{dt} = T_i, \frac{dp_{(0)i}}{dt} = -\hat{\Omega}_i - p_{(1)i}\) and \(\frac{dp_{(1)i}}{dt} = \hat{\Omega}_i\). According to the definitions, it is easy to prove 2. \(\Box\)

A full interpretation of the two vector fields \(X\) and \(Y\) is given in the next subsection, where we prove that the two vectors are the Hamiltonian vector fields of two suitable Hamiltonians.

5.2 Hamiltonian descriptions of biregular tangent forms

Important tools in describing the dynamic equations of a Hamiltonian system are offered by quantization. Following similar ideas used in \([9, Section 2]\), one can use Ostrogradski-Dirac and Fadeev-Jakiw methods, but also a modified Ostrogradski-Dirac method, according to the possibility to construct constraints slight different from the canonical ones used in Ostrogradski theory. The Ostrogradski-Dirac method was also used in \([3]\) to a quantization of a system derived from a Lagrangian affine in accelerations, involved in the study of a Regge-Teitelboim model. We give below a global form of these results. More specifically, we prove in this subsection that:

- if \(\omega\) is regular and its essential part is time independent, then there is a symplectic forms \(\Omega^1\) on \(T^*M\) and a Hamiltonian \(H : \mathbb{R} \times T^*M \to \mathbb{R}\) such that the Hamiltonian vector field \(X_H\) gives by projection the dynamics of \(\omega\) on \(M\);

- if \(\omega\) is biregular and its essential part is time independent, then there is a symplectic forms \(\Omega^1\) on \(T^*M\) and a Hamiltonian \(H : \mathbb{R} \times T^*M \to \mathbb{R}\) such that the Hamiltonian vector field \(X_H\) gives by projection the dynamics of \(\omega\) on \(M\).

Let us consider the map \(\Phi : \mathbb{R} \times T^*M \to \mathbb{R} \times T^*TM, \Phi(t, x^i, y^i, p_i) = (t, x^i, y^i, p_i + \omega_i, \hat{\omega}_i).\) Let us denote by \(\Phi_t : T^*M \to T^*TM\) the map \(\Phi_t(p^{(2)}) = \Phi(t, p^{(2)}),\) where \(t \in \mathbb{R}\) is given, and by \(\Xi\) the canonical symplectic 2-form on \(T^*TM\). Then we can consider the induced 2-form \(\Phi^*_t \Xi\) on \(T^*M\), that has the local expression \(\Phi^*_t \Xi = dx^i \wedge (dp_i + d\omega_i) + dy^i \wedge d\hat{\omega}_i\), where the differential \(d\) is considered on \(T^*M\).

**Proposition 5.3** Let \(\omega\) be a (first order) tangent form. For every \(t \in \mathbb{R}\) the form \(\Xi^t = \Phi^*_t \Xi\) is closed on \(T^*M\) and it is non-degenerated iff \(\omega\) is a regular tangent form.
where \(A\) above matrix is non-degenerate iff the matrix suitable Hamiltonian, namely the Hamiltonian \(H = \sum_{i,j} A_{ij} \partial_{x^i} \partial_{x^j} \). Thus, using the local base \(\{dx^i \wedge dx^j, dx^i \wedge dy^j, dx^i \wedge dp_j, dy^i \wedge dy^j, dy^i \wedge dp_j, dp_i \wedge dp_j\}_{i<j}\), then \(\Phi^*_{\nu} \Xi\) has the matrix

\[
\begin{pmatrix}
A & B & I \\
-B & C & 0 \\
-I & 0 & 0
\end{pmatrix}
\]

where \(A = \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}\right)\), \(B = \left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \omega_j}{\partial y^i}\right)\), \(I = (\delta_i^j)\), \(C = \left(\frac{\partial \omega_i}{\partial p^j} - \frac{\partial \omega_j}{\partial p^i}\right)\). The above matrix is non-degenerate iff the matrix \(C\) is non-degenerate, i.e. iff \(\omega\) is a regular tangent form.

In the case when the essential part of a tangent form on \(M\) is time independent, we can avoid the use of parameter \(t\) and then consider the induced 2-form \(\Xi = \Phi^*_{\nu} \Xi\) on \(T^2 M\), where \(\Phi(t, p^{(2)}) = (t, \Phi(p^{(2)}))\). As in Proposition 5.3, the two form \(\Xi\) is symplectic on \(T^2 M\). We prove now that \(\Xi\) can be used to quantify the Hamiltonian system derived from a Lagrangian affine field \(X\) is \(X\) from Proposition 5.1.

**Theorem 5.1** Let \(\omega\) be a regular (first order) tangent form on \(M\) such that its essential part is time independent. Then there are a symplectic form \(\Xi\) on \(T^2 M\) and a Hamiltonian \(H: \mathbb{R} \times T^2 M \to \mathbb{R}\) such that the Hamiltonian vector field \(X_H\) is \(X\) from Proposition 5.1.

**Proof.** According to Proposition 5.1, it suffices to prove that the vector field \(X = \frac{\partial}{\partial y^i} + y^i \frac{\partial}{\partial x^i} + S^i(t, x^1, y^1, p_i) \frac{\partial}{\partial y^i} + \Phi \frac{\partial}{\partial p^i}\) is the Hamiltonian vector field of a suitable Hamiltonian, namely the Hamiltonian \(H = -p_i y^i + \omega_0\). It reads that \(i_X \Xi = dH\). Indeed, using local coordinates, we have:

\[
\left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \omega_j}{\partial y^i}\right) y^j + \left(\frac{\partial \omega_i}{\partial p^j} - \frac{\partial \omega_j}{\partial p^i}\right) p_i + \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}\right) y^j + \left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \omega_j}{\partial y^i}\right) p_i - \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}\right) y^j = -p_i + \frac{\partial \omega_i}{\partial y^i} - \frac{\partial \omega_i}{\partial p^i} + \left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \omega_j}{\partial y^i}\right) y^j = -\delta_i^j y^j = -y^i = \frac{\partial H}{\partial p^i}. \]

In the case when the essential part \(\tilde{\omega} = \omega_1 dx^1 + \omega_0 dy\) of \(\omega\) is not necessarily time independent, the general formula reads \(i_X \Xi = dH - \frac{\partial}{\partial t} \tilde{\omega}\), where \(\frac{\partial}{\partial t} \tilde{\omega} = \frac{\partial \omega_i}{\partial x^1} dx^1 + \frac{\partial \omega_i}{\partial y} dy^i\) is a 1-form on \(T^2 M\) induced by the canonical projection \(T^2 M \to TM\), by a 1-form given by the same formula.

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Let \( \bar{\omega} = \bar{\omega}_i dx^i \) be a hyper-non-degenerated (first order) top tangent form, i.e. the Legendre map \( L_{\bar{\omega}} : \mathbb{R} \times TM \to \mathbb{R} \times T^*M \) is a global diffeomorphism. Then \( L_{\bar{\omega}}^{-1} : \mathbb{R} \times T^*M \to \mathbb{R} \times TM \) has the form \( (t, x^i, p_i) \overset{L_{\bar{\omega}}^{-1}}{\to} (t, x^i, T^i(t, x^i, p_i)) \). Considering the non-degenerated matrices \( \left( h_{ij} = \frac{\partial \bar{\omega}_i}{\partial p_j} \right) \) and its inverse \( \left( \bar{h}^{ij} = \frac{\partial \bar{\omega}_j}{\partial p_i} \right) \), we say that \( \bar{\omega} \) is co-regular if the matrix \( \left( \bar{h}^{ij} = \frac{\partial \bar{\omega}_i}{\partial p_j} - \frac{\partial \bar{\omega}_j}{\partial p_i} \right) \) is non-singular in every point of \( \mathbb{R} \times T^*M \). We recall that \( \bar{\omega} \) is regular if the matrix

\[
\left( \bar{h}^{ij} = \frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} \right)
\]

is non-singular in every point of \( \mathbb{R} \times TM \).

We say that a tangent form \( \omega \) is co-regular if its top tangent form \( \bar{\omega} \) is co-regular.

**Proposition 5.4** If a tangent form \( \omega \) is non-degenerated then \( \omega \) is co-regular iff \( \omega \) is regular.

**Proof.** Denoting \( H = (h_{ij}), \; \bar{H} = (\bar{h}^{ij}) = H^{-1} \), then \( \left( \bar{h}^{ij} \right) = \bar{H} - \bar{H}^t = \bar{H}(H^t - H)\bar{H}^t \), thus \( \left( \bar{h}^{ij} \right) = \bar{H} - \bar{H}^t \) is invertible iff \( (h_{ij} - h_{ji}) = H - H^t \) is invertible; this prove the assertion. \( \square \)

Let us suppose that the tangent form \( \omega \) is biregular, i.e. hyper-non-degenerated and regular. Thus there are some global co-semi-sprays that give some global diffeomorphisms \( T : \mathbb{R} \times T^*M \to \mathbb{R} \times TM \) and \( S : \mathbb{R} \times T^2*M \to \mathbb{R} \times T^2M \) respectively. We consider the local functions \((T^i)\) and \((S^i)\) that come from these co-semi-sprays.

Let us consider the diffeomorphism \( \Psi : \mathbb{R} \times T^2_2M \to \mathbb{R} \times T^*TM \),

\[
\Psi(t, x^i, p_{(0)i}, p_{(1)i}) = (t, x^i, T^i(t, x^i, p_{(0)i}), p_{(1)i} + \omega_i(t, x^i, p_{(0)i}), p_{(0)i}),
\]

where \( \omega_i(t, x^i, p_{(0)i}) = \omega_i(t, x^i, T^i(t, x^i, p_{(0)i})). \) Let us denote by \( \Psi_t : T^2_2M \to T^*TM \) the map \( \Psi_t(x^i, p_{(0)i}, p_{(1)i}) = \Psi(t, x^i, p_{(0)i}, p_{(1)i}) \), where \( t \in \mathbb{R} \) is given, and by \( \Xi \) the canonical symplectic 2-form on \( T^*TM \). Then we can consider the induced 2-form \( \Psi^*_t \Xi \) on \( T^2_2M \), that has the local expression \( \Phi^*_t \Xi = dx^i \wedge (p_{(1)i} + d\bar{\omega}_i) + dt^i \wedge dp_{(0)i} \), where the differential \( d \) is considered on \( T^2_2M \).

Let us denote by \( F : \mathbb{R} \times T^2_2M \to \mathbb{R} \times T^2M \) the diffeomorphism given by \( F(t, x^i, p_{(0)i}, p_{(1)i}) = (t, x^i, T^i(t, x^i, p_{(0)i}), p_{(1)i}) \), provided that \( \omega \) is hyper-non-degenerated. It is easy to see that \( \Psi_t = \Phi_t \circ F \), thus \( \Psi^*_t \Xi = F^*_t \Phi^*_t \Xi = F^*_t \Xi'_t \).

In a similar way as Proposition 5.3, the following statement holds true.

**Proposition 5.5** Let \( \omega \) be a biregular tangent form. For every \( t \in \mathbb{R} \) the two form \( \Xi''_t = \Psi^*_t \Xi = F^*_t \Xi'_t \) is a symplectic form on \( T^2_2M \).

Using local coordinates as above, we have:

\[
\Psi^*_t \Xi = dx^i \wedge (dp_{(1)i} + \frac{\partial \bar{\omega}_i}{\partial x^j} dx^j + \frac{\partial \bar{\omega}_i}{\partial p_{(0)j}} dp_{(0)j}) + (\frac{\partial R'}{\partial x^i} dx^j + \frac{\partial T'}{\partial p_{(0)i}} dp_{(0)j}) \wedge dp_{(0)i} = \]

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\[ dx^i \wedge dp_{(1)i} + \frac{2 \partial \omega_i}{\partial x^j} dx^i \wedge dx^j + \left( \frac{\partial \omega_i}{\partial p_{(0)j}} - \frac{\partial T^i_j}{\partial x^j} \right) dx^i \wedge dp_{(0)j} + \frac{\partial T^i_j}{\partial p_{(0)i}} dp_{(0)i} \wedge dp_{(0)j} . \]

Thus, using the local base \( \{ dx^i \wedge dx^j, dx^i \wedge dp_{(0)j}, dx^i \wedge dp_{(1)j}, dp_{(0)i} \wedge dp_{(0)j}, dp_{(0)i} \wedge dp_{(1)j} \}_{i<j}, \) then \( \Phi_1^* \Xi \) has the matrix
\[
\begin{pmatrix}
A' & B' & I \\
-B' & C' & 0 \\
-I & 0 & 0
\end{pmatrix},
\]
where \( A' = \left( \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right), \) \( B' = \left( \frac{\partial \omega_i}{\partial p_{(0)j}} - \frac{\partial T^i_j}{\partial x^j} \right), \) \( C' = \left( \frac{\partial T^i_j}{\partial p_{(0)i}} - \frac{\partial T^i_j}{\partial p_{(0)j}} \right) \) and \( I = (\delta_{ij}) \). The above matrix is non-degenerated iff the matrix \( C' \) is non-degenerated i.e. iff \( \omega \) is a biregular tangent form.

In the case when the essential part of a tangent form on \( M \) is hyper-non-degenerated and time independent, we can avoid the use of parameter \( t \) and then consider the diffeomorphisms \( \Psi : T^0_0 M \rightarrow T^* TM \) and \( f' : T^0_0 M \rightarrow T^2_2 M \), induced by \( \Psi \) and \( F \), as in the previous case of \( \Phi \) induced by \( \Phi' \). We have \( \Psi = \Phi \circ F \), thus \( \Psi^* \Xi = f^* \Phi^* \Xi = f^* \Xi' \). Notice that \( \Psi(t, x^i, p_{(0)i}, p_{(1)i}) = (t, \Psi'(x^i, p_{(0)i}, p_{(1)i})) \Phi(t, p^{(2)}) = (t, \Phi'(p^{(2)})) \). As in Proposition 5.5, the two form \( \Xi'' = \Psi^* \Xi = f^* \Xi' \) is a symplectic form on \( T^2_2 M \). We prove now that \( \Xi'' \) can be used also to quantify the Hamiltonian system derived from a Lagrangian affine in accelerations that comes from a non-degenerate tangent form. Using Theorem 5.3 we can prove the following statement.

**Theorem 5.2** Let \( \omega \) be a biregular (first order) tangent form on \( M \) such that its essential part is time independent. Then there are a symplectic form \( \Xi'' \) on \( T^0_0 M \) and a Hamiltonian \( H' : \mathbb{R} \times T^0_0 M \rightarrow \mathbb{R} \) such that the Hamiltonian vector field \( X_{H'} \) is \( Y \) from Proposition 5.2.

**Proof.** It suffices to prove that the vector fields \( X = X_{H} \in \mathcal{X}(T^2 T^* M) \) used in Theorem 5.1 and \( Y \in \mathcal{X}(T^0_0 M) \) used in Proposition 5.2 are related by the diffeomorphisms \( f' : T^0_0 M \rightarrow T^2_2 M \), i.e. \( (f')_* Y \circ (f'^{-1}) = X \), or \( (f'^{-1})_* X \circ (f') = Y \).

Indeed, using local coordinates, \( (f'^{-1})_* \) has the local matrix
\[
\begin{pmatrix}
I & 0 & 0 \\
D & E & 0 \\
0 & 0 & I
\end{pmatrix},
\]
where \( I = (\delta_{ij}), \) \( D = (\frac{\partial \omega_i}{\partial x^j}), \) \( E = (\frac{\partial \omega_i}{\partial p_{(0)j}}) \). Then \( (f'^{-1})_* X \) and \( Y' = (f'^{-1})_* X \circ (f') \) have the local expressions
\[
(f'^{-1})_* X = y^i \frac{\partial}{\partial x^i} + (y^i \frac{\partial \omega_i}{\partial x^j} + S^j_i \frac{\partial \omega_i}{\partial p_{(0)j}}) \frac{\partial}{\partial p_{(0)i}} + \Phi_{1j} \frac{\partial}{\partial p_{(0)i}} \text{ and}
\]
\[
Y' = T^i \frac{\partial}{\partial x^i} + (T_i \frac{\partial \omega_i}{\partial x^j} + S^j_i \frac{\partial \omega_i}{\partial p_{(0)i}}) \frac{\partial}{\partial p_{(0)j}} + \Phi_{1j} \frac{\partial}{\partial p_{(0)j}}.
\]

Since
\[
y^i \frac{\partial \omega_i}{\partial x^j} + S^j_i \frac{\partial \omega_i}{\partial p_{(0)i}} = \frac{\partial \omega_i}{\partial x^j} - \omega_i - p_i = \Omega_i - p_i,
\]
Thus \( Y' = Y \in \mathcal{X}(T^0_0 M) \), used in Proposition 5.2.
Notice that the pull-back of the Hamiltonians \( H = -p_i y^i + \omega_0 \) by \( F_i \) is the Hamiltonian \( H' : \mathbb{R} \times T^2 \mathcal{M} \to \mathbb{R}, H'(t,x^i,p(0)_i,p(1)_i) = -p(1)_i T^2(x^i,p(0)_i) + \omega_0(t,x^i,T^i) \). □

6 Some examples and special cases

We say that a tangent form \( \omega \in \mathcal{X}^*(\mathbb{R} \times T\mathcal{M}) \) is singular if it is locally equivalent to a local Lagrangian.

**Proposition 6.1** A tangent form is singular iff its top component \( \bar{\omega} \), viewed as a vertical form, is vertical closed.

**Proof.** The tangent form \( \omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i = (\omega_0 + y^i \omega_i) dt + \omega_i dx^i + \bar{\omega}_i dy^i \) is locally equivalent to a local Lagrangian form iff locally its top component \( \bar{\omega}_i \) has the form \( \bar{\omega}_i = \frac{\partial \mu}{\partial y^i} \). Using Poincaré Lemma, this condition is equivalent to \( \frac{\partial \omega_0}{\partial y^i} - \frac{\partial \omega_i}{\partial y^i} = 0 \), i.e. \( \bar{\omega} = \bar{\omega}_i dx^i \) is vertically closed. □

We say that \( \omega \) is:

- **globally singular** if there are two Lagrangians \( L_0, L_1 : \mathbb{R} \times T\mathcal{M} \to \mathbb{R} \) and a top Lagrangian form \( \mu = \mu_i dx^i \), such that \( \omega - L_0 dt = \mu_i \delta x^i + dL_1 \);

- **locally singular** if there is a Lagrangian \( L_0 : \mathbb{R} \times T\mathcal{M} \to \mathbb{R} \), a closed form \( \omega_0 \in \mathcal{X}^*(\mathbb{R} \times T\mathcal{M}) \) and a top Lagrangian form \( \mu = \mu_i dx^i \), such that \( \omega - L_0 dt = \mu_i \delta x^i + \omega_0 \).

It is easy to see that if the tangent form \( \omega \) is globally or locally singular it is also singular.

A (global) non-Lagrangian system is given by a tangent form \( \omega \) for which there are two Lagrangians \( L_i, \mu_0 : T\mathcal{M} \to \mathbb{R} \) and a top tangent form \( \mu = \mu_i dx^i \), such that \( \omega - dL = \mu_0 dt + \mu_i dx^i \), thus \( \omega_0 = \frac{\partial \mu}{\partial x^i} + \mu_0, \omega_i = \frac{\partial \mu}{\partial x^i} + \mu_i \) and \( \bar{\omega}_i = \frac{\partial \mu}{\partial y^i} \).

Since \( \omega - (\mu_0 + y^i \mu_i) dt = \mu_i \delta x^i + dL \), it follows that a (global) non-Lagrangian system is equivalent to give a global singular tangent form.

We can relax the above condition defining a local non-Lagrangian system as a tangent form \( \omega \) such that \( \omega - \bar{\omega} = \mu_0 dt + \mu_i dx^i \), where \( \bar{\omega} \) is a closed form and \( \mu, \mu_0 \) are as previously. In the same way, it follows that a local non-Lagrangian system is equivalent to give a locally singular tangent form.

If \( \omega \) is a local non-Lagrangian system on \( T\mathcal{M} \), then it can be proved that it is a global one.

In the case when \( \omega \) is differentiable only on \( T\mathcal{M}_* = T\mathcal{M} \setminus \{0\} \), where \( \{0\} \) is the image of the null section, then it makes sense to mark the difference between a local and a global tangent form.

For example, the tangent form \( \omega = \frac{XY}{\sqrt{(x^2+y^2)^2}} dx - \frac{x^2}{\sqrt{(x^2+y^2)^2}} dy \) is a local non-Lagrangian on \( \mathbb{R}^2 \times \mathbb{R}^2_+ \), but not a global one.

Instead of \( T\mathcal{M}_* \) one can consider another open submanifold of \( T\mathcal{M} \).

An other example: the tangent form \( \omega = L dt \), associated with a non-constant Lagrangian \( L \), defines a non-Lagrangian system.
Some important classes of tangent forms are:

- When $\omega_0 = 0$; for example, this is the case of time independent Lagrangians $L = L(x^i, y^j)$, since $\omega_0 = \frac{dL}{dt}$.
- When $\omega_0 = \omega_i = 0$; for example, this is the case of Lagrangians that depend only on direction: $L = L(y^i)$.

If $\omega = \omega_j(\frac{dy^j}{dt})$, then the equation (9) has the form
\[
\frac{d}{dt} \left( \frac{\partial \omega_j}{\partial y^j} \right) + \frac{d^2\omega_j}{dt^2} = 0.
\]

or
\[
\frac{\partial \omega_j}{\partial y^i} \frac{d^2x^i}{dt^2} - \frac{d}{dt} \omega_i = c_i \Rightarrow \left( \frac{\partial \omega_j}{\partial y^i} - \frac{\partial \omega_i}{\partial y^j} \right) \frac{\partial^2 x^j}{\partial t^2} = c_i.
\]

**Example 1.** Let us consider coordinates $(x, y)$ on $\mathbb{R}^2$ and $(x, y, X, Y)$ on $\mathbb{R}^4 = T\mathbb{R}^2$. Let $\omega = YdX - XdY$. The equations (9) have the form:
\[
- \frac{d}{dt} \left( \frac{d^2y}{dt^2} \right) + \frac{d^2y}{dt^2} \frac{dy}{dt} = 0, \text{ or } \frac{d^3y}{dt^3} = 0, \text{ and } \frac{d}{dt} \left( \frac{d^2x}{dt^2} \right) + \frac{d^2x}{dt^2} \frac{dx}{dt} = 0, \text{ or } \frac{d^3x}{dt^3} = 0.
\]

The exact solution is: $x(t) = C_1 + C_2t + C_3t^2$, $y(t) = C_4 + C_5t + C_6t^2$.

**Example 2.** In $\mathbb{R}^2$, as in Example 1, above, let $\omega = -ydx + xdy + YdX - XdY$. The equations (9) have the form
\[
\frac{\partial \omega_j}{\partial y^i} \frac{d^2x^i}{dt^2} - \frac{d}{dt} \left( \omega_j + \frac{\partial \omega_j}{\partial y^i} \frac{d^2x^i}{dt^2} \right) = 0.
\]

For $j = 1, \frac{dy}{dt} - \frac{d}{dt} \left( -y - \frac{d^2y}{dt^2} \right) = 0, \text{ or } \frac{dy}{dt} + \frac{d^2y}{dt^2} = 0$ and

For $j = 2, -\frac{dx}{dt} - \frac{d}{dt} \left( x + \frac{d^2x}{dt^2} \right) = 0, \text{ or } \frac{dx}{dt} + \frac{d^2x}{dt^2} = 0$.

The general solution is $x(t) = c_1 \cos t + c_3 \sin t + c_5$, $x(t) = c_2 \cos t + c_4 \sin t + c_6$. The integral curves are ellipses and straight lines. If $t_1 < t_2 < t_3$ are given, then for every three distinct points $A_\alpha(x_\alpha, y_\alpha) \in \mathbb{R}^2$, $\alpha = 1, 3$, there is a unique integral curve in the family that contains the three points, i.e. $t \to (x(t), y(t))$, $x(t_\alpha) = x_\alpha, y(t_\alpha) = y_\alpha, \alpha = 1, 3$.

This feature characterizes the dynamics generated by a third order differential equation, when in general, an integral curve is determined by three distinct points.

Let us notice that for a second order differential equation, an integral curve is determined, in general, by two distinct points.

Let us consider now the case $\dim M = 1$. In this case, since the only skew-symmetric matrix of first order is the null matrix, the equation (9) is always of second order, for every tangent form $\tilde{\omega} = \omega_0 dt + \omega dx + \tilde{\omega} dy$, having the form
\[
\left( \frac{\partial^2 \omega}{\partial \alpha \partial \gamma} - 2 \frac{\partial \omega}{\partial \alpha} \frac{\partial \omega}{\partial \gamma} + \frac{\partial^2 \omega}{\partial \alpha^2} \right) \frac{dx^2}{dt^2} + \left( \frac{\partial^2 \omega}{\partial \alpha \partial \gamma} - 2 \frac{\partial \omega}{\partial \alpha} \frac{\partial \omega}{\partial \gamma} + \frac{\partial^2 \omega}{\partial \alpha^2} \right) \frac{dy}{dt} = 0.
\]

In the case when the local functions $\omega_0, \omega$ and $\tilde{\omega}$ do not depend on $y$, the above equation becomes
\[
2 \frac{\partial \omega}{\partial x} \frac{dx}{dt} \frac{d^2x}{dt^2} + \frac{\partial^2 \omega}{\partial x^2} \left( \frac{dx}{dt} \right)^2 + 2 \frac{\partial \omega}{\partial y} \frac{dy}{dt} \frac{d^2x}{dt^2} + \frac{\partial \omega_0}{\partial x} \frac{dx}{dt} + \frac{\partial \omega}{\partial t} + \frac{\partial \tilde{\omega}}{\partial t} = 0. \tag{18}
\]

We can give a global description of this fact. It well-known that any one dimensional manifold is diffeomorphic with $\mathbb{R}$ or $S^1$. On $\mathbb{R}$ one can take a single global chart, while on $S^1$ one can take two charts, where the coordinate functions follow the rule $\frac{dx}{dt} = \pm 1$. Using the rules that coordinates follow, it follows that if $\omega_0, \omega$ and $\tilde{\omega}$ do not depend on $y$ on the domains of the two local charts, this is true on the intersection domain; we call a such tangent form $\tilde{\omega}$ as a basic tangent form. We suppose also that $\frac{dx}{dt} \neq 0$ in every point, thus $\tilde{\omega}$ is regular.
According to \cite[Section 2.]{2} a standard Lagrangian has the form

\[ L(t, x, y) = \frac{1}{2} P(x, t) y^2 + Q(x, t) y + R(x, t). \tag{19} \]

Its Euler-Lagrange equation is \( 2P x'' + P_x (x')^2 + 2P_t x' + 2(Q_t - R_x) = 0 \), where subscripts \( x, t \) denote partial derivatives and \( x' = \frac{dx}{dt}, \ x'' = \frac{d^2x}{dt^2} \). In \cite[Proposition 2.1.]{2} one proves that a second order equation

\[ x'' + a(t, x) (x')^2 + b(t, x)x' + c(t, x) = 0 \]

admits a standard Lagrangian description \((19)\) if \( b_x = 2a_t \); then \( P = \exp(2 \int^s a(t, s) ds) \) and \( R = \int^s (Q_t(t, s) - c(t, s) P(t, s)) ds \), where \( Q = Q(x, t) \) is an arbitrary function. The following result can be proved by a straightforward verification.

**Proposition 6.2** The generalized Euler-Lagrange equation of a regular and basic tangent form on a one dimensional manifold admits locally standard Lagrangian descriptions.

**Proof.** We can prove by a straightforward computation that the equation \((18)\) admits a standard Lagrangian description with

\[ a(t, x) = \frac{\partial^2 \omega}{\partial x^2}, b(t, x) = \frac{\partial^2 \omega}{\partial x^2}, c(t, x) = \frac{\partial^2 \omega}{\partial t^2} + \frac{\partial \omega}{\partial x} \square . \]

A top tangent form \( \tilde{\alpha} \) and a first order semi-spray \( \tilde{S} : \mathbb{R} \times TM \rightarrow \mathbb{R} \times T^2 M \), having the local expressions \( \tilde{\alpha} = \tilde{\alpha}_i(t, x^i, y^j)dx^i \) and \( \tilde{S}(t, x^i, y^j) = (t, x^i, y^j, \tilde{S}_i(t, x^i, y^j)) \) respectively, give rise to a second order Lagrangian \( L \), affine in accelerations, by the formula

\[ L(t, x^i, y^j, z^k) = \tilde{\alpha}_i \left( z^j - \tilde{S}_i^j \right) . \]

Let us suppose that there is a map \( u : \mathbb{R} \times TM \rightarrow \mathbb{R} \times TM \) of fibered manifolds over \( \mathbb{R} \times M \), having the form \( u(t, x^i, y^j) = (t, x^i, u^i(t, x^i, y^j)) \), such that the semi-spray \( \tilde{S} \) has the local expression \( \tilde{S}_i(t, x^i, y^j) = u^i_j(t, x^i, y^j)y^j + u^i(t, x^i, y^j) \). Then we can consider the tangent form \( \omega \) given by the formula

\[ \omega = \tilde{\alpha}_i dy^i - \tilde{\alpha}_j u^i_j dx^i - \tilde{\alpha}_j u^j . \]

For example, a 2–form \( \alpha \in \mathcal{X}^{*}(M) \land \mathcal{X}^{*}(M) \), having the local expression \( \alpha = \frac{1}{4} a_{ij}(x^k) dx^i \land dx^j \), gives rise to a top tangent form \( \tilde{\alpha} = \alpha_{ij} y^i dx^j \). Adding a supplementary structure, one can consider a tangent form. For example, if \( \nabla \) is a linear connection on \( M \), then one can \( \tilde{S} \) the spray associated with \( \nabla \). Using local coordinates, if \( \{ \Gamma^i_{jk} \} \) are the local coefficients of \( \nabla \), then \( \tilde{S}_i = \frac{1}{2} \Gamma^i_{jk} y^j y^k \) are the local coefficients of the first order spray. Then

\[ \omega = \alpha_{ij} y^i dy^j - \alpha_{ij} y^i y^k \Gamma^i_{jk} dx^i . \]

A Riemannian metric \( g \) on \( M \) gives rise to the Levi-Civita connection \( \nabla \). Using local coordinates, if \( g = \frac{1}{2} g_{ij}(x^k) dx^i \otimes dx^j \), then \( \Gamma^i_{jk} = g^{ij} \Gamma_{ijk} \), where
(\Gamma_{ijl}) are the first order Christoffel coefficients \( \Gamma_{klj} = \frac{1}{2} \left( \frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^l} \right) \)

and \((g^{ij}) = (g_{ij})^{-1}\). Then \( \bar{S}^i(t, x^i, y^i) = g^{ij} \Gamma_{klj} y^k y^l = u^i_j y^i, \) where \( u^i_j = g^{ij} \Gamma_{klj} y^k \).

The symplectic analogous version can be considered on a Fedosov manifold, i.e. a triple \((M, \alpha, \nabla)\), where \((M, \alpha)\) is a symplectic manifold and \(\nabla\) is a symplectic linear connection on \(M\), i.e. \(\alpha\) is parallel according the \(\nabla\).

Let us consider the canonical symplectic form on \(\mathbb{R}^{2r} \times \mathbb{R}^{2r}\), \(\alpha^{(r)} = \epsilon_i e^i \wedge e^{i+n}\), where \((\epsilon_{ij}) = \left( \begin{array}{cc} 0 & I_r \\ -I_r & 0 \end{array} \right)\) is the Levi-Civita tensor on \(\mathbb{R}^{2r}\). Then the second order Lagrangian, affine in accelerations given by \(L^{(2)} = \epsilon_{ij} y^i z^i + k \| y \|^2\), where \(\| y \|^2 = \frac{1}{2} \delta_{ij} y^i y^j\). This Lagrangian corresponds to some equivalent tangent forms \(\omega = \epsilon_{ij} y^i dy^j + k \delta_{ij} y^i dx^i\) and \(\omega' = \epsilon_{ij} y^i dy^i + k \| y \|^2\) (according to Proposition 3.1). The tangent form \(\omega\) is obtained using the symplectic form \((\epsilon_{ij})\) and the semi-spray \(\bar{S}\) on \(\mathbb{R}^{2r}\) having the form \((t, x^i, y^i) \xrightarrow{\bar{S}} (t, x^i, y^i, \bar{S}^i = -y^i \delta_{kj} \epsilon^{ij})\), where \((\epsilon^{ij}) = (\epsilon_{ij})^{-1}\).

The second order Lagrangian, affine in accelerations given by \(L^{(2)} = \epsilon_{ij} y^i z^i + k \| y \|^2\), where \(\| y \|^2 = \frac{1}{2} \delta_{ij} y^i y^j\). This Lagrangian corresponds to some equivalent tangent forms \(\omega = \epsilon_{ij} y^i dy^i + k \delta_{ij} y^i dx^i\) and \(\omega' = \epsilon_{ij} y^i dy^i + k \| y \|^2\) (according to Proposition 3.1). The tangent form \(\omega\) is obtained using the symplectic form \((\epsilon_{ij})\) and the semi-spray \(\bar{S}\) on \(\mathbb{R}^{2r}\) having the form \((t, x^i, y^i) \xrightarrow{\bar{S}} (t, x^i, y^i, \bar{S}^i = -ky^i \delta_{kj} \epsilon^{ij})\), where \((\epsilon^{ij}) = (\epsilon_{ij})^{-1} = -(\epsilon_{ij})\).

### 6.1 Tangent forms and first order semi-sprays

We show below that in some special cases, the solutions of the generalized Euler-Lagrange equation of a tangent form can be given by the integral curves of certain local first order semi-sprays.

**Example 3.** Let us consider coordinates \((x, y)\) on \(\mathbb{R}^2\) and \((x, y, X, Y)\) on \(\mathbb{R}^4\). Let \(\omega = (X + Y)dx + YdY\). As in Example 1, the equations [31] have the solutions \(z''(t) = y''(t) = 0\). Using the notations \(x = x^1\), \(y = x^2\), \(X = y^1\), \(Y = y^2\), then \(\omega = (y^1 + y^2)dy^1 + y^2 dy^2 = \omega_1 dy^1 + \omega_2 dy^2\), \((h_{ij}) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)\) and \((h^{ij}) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)\). The integral solutions of the vector field \(X\) are \(\frac{dy}{dx} = y^1\), \(\frac{dy}{dt} = \omega^1 = h^{ij} p_j = 0\). It follows that \(p_i(t) = p^0_i\), thus \(\frac{dx}{dt} = y^1\), \(\frac{dy}{dt} = c, \) where \(c_1 = -p^0_1 \) and \(c_2 = p^0_2\). Finally we obtain all the solutions \(\frac{dx}{dt} = 0\). In conclusion, considering arbitrary semi-sprays on \(\mathbb{R}^2\), with constant coefficients, then we obtain all the solutions of \(\bar{S}\) as integral solutions of these first order semi-sprays.

The above example can be extended as follows.

**Proposition 6.3** Let us suppose that there are some coordinates such that the local coefficients of a regular tangent form \(\omega\) depend only on \((y^i)\). Then there is a family of local semi-sprays of first order whose local coefficients depend only
on \((y^i)\), such that their integral curves project on all the integral curves of the generalized Euler-Lagrange equation of \(\omega\).

**Proof.** The integral solutions of the vector field \(X\) are \(\frac{dx^i}{dt} = y^i, \frac{dx^j}{dt} = S^j = h^{ij}\left(p_j - \frac{\partial \omega_i}{\partial y_j} - \frac{\partial \omega_j}{\partial y_i} y^j\right)\), \(\frac{dp_i}{dt} = \bar{\Phi}_i = -\frac{\partial \omega_i}{\partial y_j} S^j\). Using the second equation in the expression of the third, it follows that \(\frac{dp_i}{dt} = -\frac{\partial \omega_i}{\partial y_j}(y^k)\frac{dx^j}{dt}\), thus \(p_i + \omega_i = c_i\) along every solution. It follows that if considering local semi-sprays having as local component functions \(S^i(y^k) = h^{ij}(y^k)\left(c_j - \omega_j(y^k) - \frac{\partial \omega_j}{\partial y_j}(y^k) - \frac{\partial \omega_i}{\partial y_j}(y^k)y^j\right)\), we obtain all the integral curves of the generalized Euler-Lagrange equation of \(\omega\).

Since the tangent form \(\omega' = \omega + dF\) has the same extrema curves as \(\omega\), the extrema curves of the tangent forms \(\omega' = (\omega_0(y^i) + \frac{\partial F}{\partial t}) dt + (\omega_1(y^i) + \frac{\partial F}{\partial t}) dx^i + \left(\tilde{\omega}_i(y^i) + \frac{\partial F}{\partial t}\right) dy^i\) and \(\omega = \omega_0(y^i) dt + \omega_1(y^i) dx^i + \tilde{\omega}_i(y^i) dy^i\) (used in Proposition above) are the same. In order to detect when one can apply the Proposition above, we prove the following result.

**Proposition 6.4** Let us consider a tangent form \(\mu\), a point \(x_0 \in M\) and a local system of coordinates \((U, \varphi)\), where \(x_0 \in U\). Then the following statements are equivalent:

1. There is a local tangent form \(\omega = \mu - dF\) on a \(TU'\), \(x_0 \in U' \subset U\), such that the local components of \(\omega\) does depend only on \((y^i)\).

2. The local components of \(d\mu\) depend only on \((y^i)\) and the components of \(
\{dx^i \land dt, dx^i \land dx^j\}\) vanish.

**Proof.** If the property 1. holds for \(\mu\), then \(d\mu = d\omega\), thus 2. follows. Conversely, let us suppose that 2. holds, thus \(d\mu = a_i(y) dy^i \land dt + b_{ij}(y)dx^i \land dy^j + \frac{1}{2}c_{ij}(y^k)dy^i \land dy^j\). Then we have \(0 = dd\mu = \frac{\partial a_i}{\partial y^j}dy^k \land dy^j \land dt + \frac{\partial b_{ij}}{\partial y^k}dy^k \land dx^i \land dy^j + \frac{1}{2}\frac{\partial c_{ij}}{\partial y^k}dy^k \land dy^j \land dy^j\). Thus using the Poincaré Lemma, it follows that \(a_i = \frac{\partial f}{\partial y^i}, b_{ij} = \frac{\partial g}{\partial y^i}, c_{ij} = \frac{\partial h}{\partial y^i} - \frac{\partial h}{\partial y^j}\) on \(\mathbb{R}^m\), where \(f, g, h : \mathbb{R}^m \to \mathbb{R}\) are functions that depend only on \((y^i)\). Let us consider the form \(\omega = f dt + g_i dx^i + h_i dy^i\) on \(TU = U \times \mathbb{R}^m\). Then \(d\mu = d\omega\), or \(d(\mu - \omega) = 0\), thus for a sufficiently small \(U' \subset U\), \(x_0 \in U'\), one have \(\mu - \omega = dF\) on \(TU'\). □

**Example 4.** Consider the tangent form \(\omega = -ydx + xdy + YdX - XdY\) on \(\mathbb{R}^2\) used in Example 2., with coordinates \((x, y)\) on \(\mathbb{R}^2\) and \((x, y, X, Y)\) on \(\mathbb{R}^4 = T\mathbb{R}^2\). We use below also the notations \(x = x^1, y = x^2, X = y^1, Y = y^2\), then \(\omega = -x^2 dx^1 + x^1 dx^2 + y^2 dy^1 - y^1 dy^2 = \omega_1 dx^1 + \omega_2 dx^2 + \tilde{\omega}_1 dy^1 + \tilde{\omega}_2 dy^2\), \((h_{ij}) = \begin{pmatrix}
0 & 2 \\
-2 & 0
\end{pmatrix}\) and \((h^{ij}) = \begin{pmatrix}
0 & 1/2 \\
-1/2 & 0
\end{pmatrix}\). The integral solutions of the vector field \(X\) are \(\frac{dx^i}{dt} = y^i, \frac{dx^2}{dt} = S^i = h^{ij}p_j, \frac{dp_i}{dt} = \bar{\Phi}_i = \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_i}{\partial x^j}\right)y^j\). Specifically, \(\frac{dx^1}{dt} = 2y^2 = 2dx^2_1\) and \(\frac{dp_2}{dt} = -2y^1 = -2dx^2_2\). Thus \(p_1 = 2x^2 + 2c_1\) and \(p_2 = -2x^2 + 2c_2\). Considering the local first order semi-sprays \(S^1(x^1, y^1) = \frac{dx^2}{dt} = \frac{dx^3}{dt} = \frac{dx^4}{dt} = 0\), we obtain all the integral curves of the generalized Euler-Lagrange equation of \(\omega\) on \((y^i)\).
$x^2 + c_1$ and $S^2(x^i, y^j) = -x^1 + c_2$, we obtain the system $\frac{dx^i}{dt} = y^i$, $\frac{dy^i}{dt} = S^i$.

Taking into account the Example 2., the integral curves of all semi-sprays $S$ having this form give all the solutions of the generalized Euler-Lagrange equation \( \text{of } \omega \).

The above example can be extended as follows.

**Proposition 6.5** Let us suppose that there are some coordinates such that the local expression of a regular tangent form $\omega$ is $\omega = \omega_0(y^j) + \omega_1(x^j)dx^i + \omega_2(y^i)dy^i$ and $\frac{\partial \omega_1}{\partial x^j} + \frac{\partial \omega_2}{\partial y^i} = 0$. Then there is a family of local semi-sprays of first order, such that their integral curves project on all the integral curves of the generalized Euler-Lagrange equation of $\omega$.

**Proof.** The integral solutions of the vector field $X$ are $\frac{dx^i}{dt} = y^i$, $\frac{dy^i}{dt} = S^i = h^{ij} \left( p_i - \frac{\partial \omega_1}{\partial x^j} \right)$, $\frac{dp_i}{dt} = \Phi_i = \left( \frac{\partial \omega_1}{\partial x^j} - \frac{\partial \omega_2}{\partial y^i} \right) y^j = -2 \frac{\partial \omega_1}{\partial x^j} y^j$. Using Lemma \ref{lem:0.1} below, \{\omega\} have the form $\omega_i = c_{ij} x^j + d_i$, thus $p_i = -2c_{ij} x^j + e_j$, where $c_{ij} = -c_{ji}$, $d_i$ and $e_i$ are constants. It follows that considering local semi-sprays having as local component functions $S^i(y^k) = h^{ij}(y^k) \left( -2c_{ij} x^j + e_j - \frac{\partial \omega_2}{\partial y^i}(y^k) \right)$, for all constants \{e_j\}, we obtain all the integral curves of the generalized Euler-Lagrange equation of $\omega$. \( \square \)

**Lemma 6.1** Given the set \{\omega_i(x^j)\}_{i=1}^m of real functions on $\mathbb{R}^m$, then the following conditions are equivalent:

1. $\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i} = 0$, $i, j = 1, m$;
2. there are constants \{e_{ij}, d_i\}_{i,j=1}^m, $c_{ij} = -c_{ji}$ such that $\omega_i = c_{ij} x^j + d_i$;
3. there is a set \{\varphi_i\}_{i=1}^m of real functions on $\mathbb{R}^m$ such that $\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = \frac{\partial \varphi_i}{\partial x^j}$.

**Proof.** Obviously 2. implies 1. and 3. Let us suppose that 1. holds. We have $\omega_{ij} = \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 2 \frac{\partial \varphi_i}{\partial x^j}$; then $\omega_{ij} = 2 \frac{\partial \varphi_i}{\partial x^j} = \frac{\partial^2 \varphi_i}{\partial x^j \partial x^i}$, thus $\frac{\partial^2 \varphi_i}{\partial x^j \partial x^i} - \frac{\partial^2 \varphi_i}{\partial x^i \partial x^j} = \frac{\partial^2 \omega_{ij}}{\partial x^j \partial x^i} = \frac{\partial^2 \omega_{ij}}{\partial x^i \partial x^j}$, that gives $\frac{\partial \omega_{ij}}{\partial x^j} = 0$, thus 2. holds. Let us suppose that 3. holds. We have $\frac{\partial^2 \varphi_i}{\partial x^j \partial x^i} = \frac{\partial^2 \omega_{ij}}{\partial x^j \partial x^i} - \frac{\partial^2 \omega_{ij}}{\partial x^i \partial x^j}$, thus 3. holds as previously. \( \square \)

The tangent form $\omega' = \omega + dF$ has the same extrema curves as $\omega$. Thus the extrema curves of the tangent forms $\omega' = \frac{\partial F}{\partial t}dt + (\omega_i(x^j) + \frac{\partial F}{\partial x^i})dx^i + \left( \omega_i(y^j) + \frac{\partial F}{\partial y^i} \right)dy^i$ and $\omega$ from Proposition above are the same. In particular, one can relax the hypothesis of Proposition above, asking the existence of a local function $F(x^i)$ such that $\frac{\partial \omega_{ij}}{\partial x^j} + \frac{\partial \omega_{ij}}{\partial x^i} = -2 \frac{\partial^2 F}{\partial x^j \partial x^i}$; more precisely, $\omega_i + \frac{\partial F}{\partial x^i} = c_{ij} x^j + d_i$, where $c_{ij} = -c_{ji}$ and $d_i$ are constants. In order to apply the result from Proposition above, we prove the following result.

**Proposition 6.6** Let us consider a tangent form $\mu$, a point $x_0 \in M$ and a local system of coordinates $(U, \varphi)$, where $x_0 \in U$. Then the following statements are equivalent:
1. There is a local tangent form \( \omega = \mu - dF \) on a \( TU' \), \( x_0 \in U' \subset U \), such that \( \omega = \omega_i(x^i)dx^i + \bar{\omega}_i(y^j)dy^j \) and \( \frac{\partial \omega_i}{\partial x^j} + \frac{\partial \bar{\omega}_j}{\partial y^i} = 0 \).

2. The local components of \( d\mu \) have the properties that the components of \( \{dx^i \wedge dt, dy^j \wedge dt, dx^i \wedge dy^j\} \) vanish, the components of \( \{dy^i \wedge dy^j\} \) depend only on \( (y^i) \) and the components \( \mu_{ij} \) of \( \{dx^i \wedge dx^j\} \) are constants.

**Proof.** If the property 1. holds for \( \mu \), then \( d\mu = d\omega \), thus 2. follows. Conversely, let us suppose that 2. holds, thus \( d\mu = \frac{1}{2}\mu_{ij}dx^i \wedge dx^j + \frac{1}{2}\nu_{ij}(y^k)dy^i \wedge dy^j \), with \( \mu_{ij} = -\mu_{ji} \) constants. Using \( dd\mu = 0 \) and the Poincaré Lemma, it follows that \( \nu_{ij} = \frac{\partial g_i}{\partial y^j}(y^k) - \frac{\partial g_j}{\partial y^i}(y^k) \), where \( g_i : \mathbb{R}^m \rightarrow \mathbb{R} \). Let us denote \( f_i(x^k) = \mu_{ij}x^j \) and consider the local differential form \( \omega = f_i(x^k)dx^i + g_i(y^k)dy^i \) on \( TU = U \times \mathbb{R}^m \). Then \( d\mu = d\omega \), or \( d(\mu - \omega) = 0 \), thus for a sufficiently small \( U' \subset U \), \( x_0 \in U' \), one have \( \mu - \omega = dF \) on \( TU' \). □

The tangent forms \( \omega = \omega_i(x^i)dx^i + \bar{\omega}_i(y^j)dy^j \) and \( \omega' = y^i\omega_i(x^j)dt + \bar{\omega}_i(y^j)dy^j \) are equivalent. If \( \omega_i(x^i) = c_{ij}x^j \), then \( d\omega' = c_{ij}x^jdy^i \wedge dt + c_{ij}y^idy^j \wedge dt + \frac{\partial \omega_j}{\partial y^i}(y^j)dy^i \wedge dy^j \). Then the following result follows in the same line as the previous ones.

**Proposition 6.7** Let us consider a tangent form \( \mu \), a point \( x_0 \in M \) and a local system of coordinates \((U, \varphi)\), where \( x_0 \in U \). Then the following statements are equivalent:

1. There is a local tangent form \( \omega = \mu - dF \) on a \( TU' \), \( x_0 \in U' \subset U \), such that \( \omega = \omega_i(x^i)y^jdt + \bar{\omega}_i(y^j)dy^j \) and \( \frac{\partial \omega_i}{\partial x^j} + \frac{\partial \bar{\omega}_j}{\partial y^i} = 0 \).

2. The local components of \( d\mu \) have the properties that the components of \( \{dx^i \wedge dx^j, dx^i \wedge dy^j\} \) vanish, the components of \( \{dy^i \wedge dy^j\} \) depend only on \( (y^i) \) and the components \( f_i \) of \( \{dx^i \wedge dt\} \) and \( g_j \) of \( \{dy^j \wedge dt\} \) have the property that \( \frac{\partial f_i}{\partial y^j} = c_{ij} \) are constants.

### 7 Appendix

As a manifold, \( T^2M \subset TTM \) is the submanifold of the vectors \( X_\alpha \) that project according to the double vector bundle structure \( \pi^{(2)} : TTM \rightarrow TM \), as tangent bundle of \( TM \) and \( \pi^{(1)} : TTM \rightarrow TM \), as the differential of the canonical projection \( \pi^{(1)} : TM \rightarrow M \). As a manifold, the point in \( T^2M \) can be defined also as the equivalent classes of curves on \( M \) having a 2–contact in a point (see, for example \([5, 7, 14]\)).

A slashed (first order) Lagrangian on \( M \) is a differentiable map \( L : TM \rightarrow \mathbb{R} \), where \( TM = TM \setminus \{0\} \) and \( \{0\} \) is the image of the null section \( M \rightarrow TM \). Analogously, a slashed second order Lagrangian on \( M \) is a differentiable map \( L^{(2)} : T^2M \rightarrow \mathbb{R} \), where \( T^2M = T^2M \setminus \{0\} \) and \( \{0\} \) is the image of the „null” section \( M \rightarrow T^2M \) given by \( (x^i) \rightarrow (x^i, y^i = 0, z^i = 0) \).

Coordinates \((x^i, y^i)\) on \( M \), \((x^i, y^i, X^i, Y^i)\) on \( T^2M \), \((x^i, y^i, X^i, Y^i)\) on \( TM \) and \((x^i, y^i, z^i)\) on \( T^2M \) follow the rules \( x^i = x^i(x^i), y^i = \frac{\partial x^i}{\partial x^j}y^j, X^i = \frac{\partial x^i}{\partial z^j}X^j, Y^i = \)
of tangent form to top tangent form this semi-spray of order $k$ Proposition 7.1

\[ T M \text{ tangent bundle} \]

is a section $S \omega$ is pointed, then order Lagrangian, affine in accelerations, then $\bar{\omega}$

\[ \text{to two couples of coordinates } (x, y) \text{ and } (x', y', \ddot{y}') \text{ are } x' = x, \dot{y}' = y, \text{ but } \bar{x}' = 2z'. \]

Notice that using local coordinates, the inclusion $T^2 M \subset TT M$ has the expression $(x', y', z') \rightarrow (x', y', x, z, z')$. There are affine bundles structures $\pi^2_k : T^2 M \rightarrow TM$ and $\pi^3_k : T^3 M \rightarrow T^2 M$; in general $\pi^{k+1}_k : T^k M \rightarrow T^{k-1} M, k \geq 2$. A (time independent) semi-spray of order $k$ is a section $S : T^k M \rightarrow T^{k+1} M$. Considering the product bundle $\mathbb{R} \times T^k M \rightarrow \mathbb{R} \times T^k M, k \geq 1$, then a (time dependent) semi-spray of order $k$ is a section $S : \mathbb{R} \times T^k M \rightarrow \mathbb{R} \times T^{k+1} M$, such that $S(t, \bar{x}) = (t, \bar{x}, (k+1)S'((t, \bar{x}))$; this semi-spray of order $k$ is considered in the paper.

The integral curves of a $k$-order semi-spray $S$ are exactly the integral curves of $S$ regarded as a vector field on $T^k M$. Using coordinates $(x^i, y^{(1)i}, \ldots, y^{(k)i})$ on $T^k M$, the local expression of a $k$-order semi-spray is $S = y^{(1)i}, \frac{\partial}{\partial x^i} + 2y^{(2)i}, \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}, \frac{\partial}{\partial y^{(k-1)i}} - (k+1)S'(x^i, y^{(1)i}, \ldots, y^{(k)i})$. We say that $S'$ are the local functions that give $S'$.

Consider now a tangent form $\omega \in X^* (\mathbb{R} \times TM)$ given in local coordinates by (1). Then $\omega : \mathbb{R} \times TM \rightarrow \mathbb{R}$ gives a (global defined) real function. According to two couples of coordinates $(x^i, y^i)$ and $(x^i, y^i)$ on the common domain, the local components $\omega_i$ and $\bar{\omega}_i$ follow the rules $\bar{\omega}_i = \frac{\partial \omega'}{\partial x'} \bar{\omega}'$ and $\omega_i = \frac{\partial \omega'}{\partial x} \omega'$, respectively. We can consider the top components $\bar{\omega}_i$ defining a section $\bar{\omega} : \mathbb{R} \times TM \rightarrow \pi^* TM$, $\bar{\omega} = \bar{\omega}_i(t, x^i, y^i)dx^i$, of the induced vector bundle $\pi_1 = \pi^*(\pi') : \pi^* TM \rightarrow \mathbb{R} \times TM$, where $\pi : \mathbb{R} \times TM \rightarrow M$ comes from the tangent bundle $TM \rightarrow M$ and $\pi' : T^* M \rightarrow M$ is the cotangent bundle of $M$. In general, a section $\bar{\omega} : \mathbb{R} \times TM \rightarrow \pi^* TM$ is a top tangent form (on $M$).

We say that a Lagrangian $L : \mathbb{R} \times TM \rightarrow \mathbb{R}$ is pointed if $L(t, x^i, y^i) = 0$.

**Proposition 7.1** A Lagrangian $L : \mathbb{R} \times TM \rightarrow \mathbb{R}$ is a pointed one iff there is to top tangent form $\nu = \nu_i(t, x^i, y^i)dx^i$, such that $L(t, x^i, y^i) = y^i \nu_i$. Proof. The sufficiency is obvious, we prove only the necessity. Indeed, if $L$ is pointed, then $L(t, x^i, y^i) = y^i \int_0^1 \frac{\partial L}{\partial \dot{y}^i}(t, x^i, y^i) d\tau = y^i \nu_i$. It can be easily checked that $\nu = \nu_i dx^i$ is a global top tangent form. □

An other example of a top tangent form: if $L^{(2)} : T^2 M \rightarrow \mathbb{R}$ is a second order Lagrangian, affine in accelerations, then $\bar{\omega} = \frac{\partial L^{(2)}}{\partial \dot{y}^i} dx^i$ is a top tangent form. Notice that a top tangent form $\bar{\omega} = \bar{\omega}_i(t, x^i, y^i)dx^i$ is a degenerated tangent form. Since $\bar{\omega} = \bar{\omega}_i dt + \bar{\omega}_i (dx^i - y^i dt)$, it follows that $\bar{\omega}$ is equivalent to the first order (pointed) Lagrangian $L_0 = \bar{\omega}_y y^i$. Conversely, it is easy to see
that a pointed Lagrangian \( L_0 = \bar{\omega}_i(t, x^i, y^j) \) is equivalent to the top tangent form \( \bar{\omega} = \bar{\omega}_i dx^i \).

An analogous object considered in the paper is a pure tangent form that can be considered as a section \( \omega': \mathcal{R} \times TM \to \pi_0^*T^*TM \). \( \omega' = \bar{\omega}_i(t, x^i, y^j) dy^j + \omega_i(t, x^i, y^j) dx^i \), of the induced vector bundle \( \pi_2 = \pi_0^*(\pi'') : \pi_0^*T^*TM \to \mathcal{R} \times TM \), where \( \pi'' : T^*TM \to TM \) is the cotangent bundle of \( TM \) and \( \pi_0 = \mathcal{R} \times TM \to TM \) is the trivial projection.

A fibered manifold is a surjective submersion \( \pi_E : E \to M; E_x = \pi_0^{-1}(x) \) is the fiber of \( x \in M \). There are local coordinates, called as adapted to the submersion (or to the fibered manifold structure), giving the local form \( \omega \circ \pi \). Considering coordinates \( (x^i, u^\alpha) \) and \( (x^i, u^\alpha) \) on an intersection domain, then \( x^i = x^i(x^j) \) and \( u^\alpha = u^\alpha(x^i, u^\alpha) \).

A fibered map of two fibered manifolds \( \pi_E : E \to M \) and \( \pi_F : F \to M' \) is a couple \( f_0 : M \to M', f : E \to F' \) of maps that send fibers to fibers, i.e. \( f_0 \circ \pi_E = \pi_F \circ f \). If \( M = M' \) and \( f_0 = id_M \), then \( f \) is called simply a fibered map (of fibered manifolds over the same base). Let \( \pi_E : E \to M \) and \( \pi_F : F \to M \) be fibered manifolds over the same base. The fibered manifold product \( P = E \times_M F \) is \( P = \bigcup P_x \subset E \times F \), where \( P_x = \{(e, f) \in E \times F : \pi_E(e) = \pi_F(f)\} \) is a new fibered manifold \( \pi_P : P \to M \), but also over \( E \) and \( F \). The tangent space of \( P \) is locally the sum of two subspaces, each tangent to two foliations. Using coordinates, we explicit in two cases, useful in the paper. First is when \( E = TM \) and \( F = T^*TM \) are the tangent and the cotangent space of \( M \) respectively. In this case, considering \( (x^i), (y^j), (x^i, p_i), (x^i, y^j, p_i) \) and \( (x^i, y^j, p_i, X^i, Y^i, P_1) \) local coordinates on \( M, TM, T^*M, TM \times T^*M, T(TM \times T^*M) \) respectively, then these coordinates follow the next rules on a common domain: \( x'^i = x^{\prime i}(x^i), \quad y'^j = \frac{\partial x'^i}{\partial x^i} y^j \), \( p_i = \frac{\partial x'^i}{\partial x^i} p_i \), \( X^i = \frac{\partial x'^i}{\partial x^i} X^i + \frac{\partial x'^i}{\partial x^i} Y^i \) and \( P_1 = \frac{\partial x'^i}{\partial x^i} p_i + \frac{\partial x'^i}{\partial x^i} P_1 \) respectively.

A second case is when \( E = F = T^*M \) and \( T^*M \times_M T^*M = T^2_M \); considering some local coordinates \( (x^i, p_{(0)}i, p_{(1)}i, y^j, P_{(0)}i, P_{(1)}i) \) on \( TT^2_M \), then \( p_{(0)}i = \frac{\partial x'^i}{\partial x^i} P_{(0)}i, \quad p_{(0)}i = \frac{\partial x'^i}{\partial x^i} P_{(0)}i, \quad y'^j = \frac{\partial x'^i}{\partial x^i} y^j, \quad P_{(0)i} = \frac{\partial y'^j}{\partial x^i} P_{(0)i} + \frac{\partial x'^i}{\partial x^i} P_{(0)i}i, \quad P_{(1)i} = \frac{\partial y'^j}{\partial x^i} P_{(0)i} + \frac{\partial x'^i}{\partial x^i} P_{(1)i}i \).

If \( \pi_E : E \to M \) is a fibered manifold and \( f_0 : M' \to M \) is a differentiable map, then \( f_0 \circ \pi_0 \circ \pi_E \) is a differentiable manifold. The canonical projections \( \pi_{f_0} = \pi_1 : f_0^* E \to M' \) and \( f = \pi_2 : f_0^* E \to E \) give a fibered manifold \( (f_0^* E, \pi_{f_0}, M') \) and a fibered map \( f_0, f \).

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