MATROIDS AND THEIR DRESSIANS

MADELINE BRANDT

ABSTRACT. We study Dressians of matroids using the initial matroids of Dress and Wenzel. These correspond to cells in regular matroid subdivisions of matroid polytopes. We characterize matroids that do not admit any proper matroid subdivisions. An efficient algorithm for computing Dressians is presented, and its implementation is applied to a range of interesting matroids.

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Introduction

Let $K$ be an algebraically closed field with a non-trivial valuation, valuation ring $R$, and residue field $k$. Consider a collection of vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ spanning $K^d$. These vectors give a rank $d$ matroid $M$ on $n$ elements, whose bases are given by the bases of $K^d$ coming from the $v_1, \ldots, v_n$. If we pass these vectors to the residue field $k$, their images will generate a matroid $M'$, which is a special kind of weak image of $M$. In this setting, the tropical Grassmannian of $M$ records the possible residues of realizations of $M$. One can also expand these ideas to non-realizable matroids, and the Dressian is the tropical object which records the possible initial matroids. In this paper, we take this vantage point to study Dressians and tropical Grassmannians of matroids.

The tropical Grassmannian was first introduced by Speyer and Sturmfels [SS04]. Its connection to the space of phylogenetic trees and the moduli space of rational tropical curves is a celebrated and motivating result in studying these objects. In [Spe08], it is demonstrated that points in $\mathbb{R}^{\binom{n}{d}}$ satisfying the tropicalized Plücker relations induce subdivisions of the $(d, n)$-hypersimplex whose cells are matroid polytopes. These points also correspond to tropical linear spaces. It has been observed (e.g., [MS15, HJJS09]) that these points also give valuations on the uniform matroid, as in [DW92]. Later, in [HJJS09], the authors call this collection of points in $\mathbb{R}^{\binom{n}{d}}$ the Dressian. They introduce a Dressian for each matroid, whose points give valuations on that matroid, and also happen to induce regular matroid subdivisions of the matroid polytope. The authors compute and give a detailed description of the Dressian of the Pappus configuration.

Since then, many questions about Dressians have been studied. Bounds on the dimension of Dressians were given in [JS17, HJJS09]. Rays of the Dressian have been studied in [JS17, HJS14].
The question of when a matroid polytope has a split was studied in [MMI]. Computing Dressians of uniform matroids has also been completed up to $d = 3$ and $n = 8$ [HJJS09]. Recently, in [OPS18], the authors have studied the fan structure of Dressians and prove that the Dressian of the sum of two matroids is given by the product of their Dressians.

In this paper, we investigate the nature of Dressians of matroids further. Given a matroid $\mathcal{M}$ with valuation $v : B(\mathcal{M}) \to \mathbb{R} \cup \infty$, we define the initial matroid $\mathcal{M}_v$ (as in [DW92, MR18]) to be the matroid with basis set $B_v = \{ \sigma \in B \mid v(\sigma) \text{ is minimal} \}$. This gives a useful restriction on the notion of a weak map which is compatible with matroid valuations. In Section 2, we study initial matroids and their polytopes. The main result of this section is the following Theorem A.

**Theorem A.** Let $\mathcal{M}$ be a matroid with matroid polytope $P_\mathcal{M}$, let $v$ be a valuation on $\mathcal{M}$, let $L$ be the lineality space of the Dressian of $\mathcal{M}$, and let $\Delta_v$ be the matroid subdivision of $P_\mathcal{M}$ induced by $v$. Then,

$$\Delta_v = \{ P(\mathcal{M}_w) \mid w \in v + L \}.$$

In Section 2 we also show that points in the tropical Grassmannian of a matroid over a field $K$ give weight vectors on the matroid polytope which induce regular matroid subdivisions containing cells corresponding to matroids which are also realizable over the field $K$. We also explore failures of the converse to this, namely examples where all cells of a regular matroid subdivision are polytopes of realizable matroids, but the point of the Dressian inducing the subdivision is not contained in the Grassmannian.

In Section 3 we investigate the question of when a matroid is rigid, meaning that the matroid polytope does not have matroid subdivisions. In Proposition 3.4 we give a criterion for rigidity of a matroid in terms of the connectivity of its initial matroids. This allows us to answer Question 2 from [OPS18] with the following Theorem B.

**Theorem B.** All maximal cells in a finest matroid subdivision of a matroid polytope of a connected matroid are matroid polytopes of connected rigid matroids.

In Section 4, we turn to the problem of effectively computing Dressians, and give Algorithm 1 which reduces the number of variables and equations for computing Dressians. An implementation of Algorithm 1 can be found at https://math.berkeley.edu/~brandtm/research.html.

**Theorem C.** Let $\mathcal{M}$ be a matroid and let $G_M$ be the identified generators of its matroid Plücker ideal coming from setting the variables indexing non-bases in the three term Plücker relations to 0. Then Algorithm 1 produces a set of generators $G'$ and linear inequalities $L'$ in fewer variables such that the tropical prevariety (or variety) defined by the equations $G'$ intersected with the constraints in $L'$ is isomorphic as a polyhedral complex via a linear map to $Dr_M$ (or $Gr(\mathcal{M})$).

This yields efficiencies which speed up the computations of Dressians of matroids. This is used when the Dressian is contained in a classical linear space; geometrically the equation reduction which occurs in the algorithm corresponds to projecting the Dressian onto this linear space.

In Section 5, we use Algorithm 1 to compute the Dressians of the star $10_3$ configuration, the non-Pappus matroid, the Vámos and non-Vámos matroids, the Desargues configuration, and others. In these examples, we illustrate the features of Dressians given by the results from the previous sections.

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1. Dressians and tropical Grassmannians of matroids

We begin with some notions from tropical geometry and matroid theory. Let $K$ be an algebraically closed field with a valuation $\text{val}_K$. Let $I$ be an ideal in the Laurent polynomial ring with $n + 1$ variables $K[x_0^\pm 1, \ldots, x_n^\pm 1]$. The tropical variety associated to $I$ is defined as $\cap_{f \in I} \text{trop}(V(f))$, where the $\text{trop}(V(f))$ are the tropical hypersurfaces corresponding to polynomials $f \in I$ (See [MS15, Definition 3.3.1]). For every ideal $I$ there exists a finite subset $B \subset I$ called a tropical basis such that the tropical variety is equal to $\cap_{f \in B} \text{trop}(V(f))$. Using a tropical basis one can compute the corresponding tropical variety. In many cases, however, it is computationally difficult to find a tropical basis. Given any collection of generators $B'$ for the ideal $I$, we call the set $\cap_{f \in B'} \text{trop}(V(f))$ a tropical prevariety. The lineality space of a tropical (pre)variety $T$ is the largest linear space $L$ such that for any point $w \in T$ and any point $v \in L$, we have that $w + v \in T$.

A matroid of rank $d$ on $n$ elements is a collection $\mathcal{B} \subset \binom{[n]}{d}$ called the bases of $\mathcal{M}$ satisfying:

(B0) $\mathcal{B}$ is nonempty,
(B1) Given any $\sigma, \sigma' \in \mathcal{B}$ and $e \in \sigma' \setminus \sigma$, there is an element $f \in \sigma$ such that $\sigma \setminus \{f\} \cup \{e\} \in \mathcal{B}$.

A matroid $\mathcal{M}$ is called realizable over $K$ if there exist vectors $v_1, \ldots, v_n \in K^d$ such that the bases of $K^d$ from these vectors are indexed by the bases of $\mathcal{M}$:

$$\mathcal{B} = \left\{ \sigma \in \binom{[n]}{d} \mid \{v_{\sigma_1}, \ldots, v_{\sigma_d}\} \text{ is a basis of } K^d \right\}.$$ 

In this case, we write $\mathcal{M} = \mathcal{M}[v_1, \ldots, v_n]$. The uniform matroid $\mathcal{U}_{d,n}$ is the matroid with basis set $\binom{[n]}{d}$. For more information on matroids, we encourage the reader to consult [Oxl11] or [Whi86].

The Grassmannian $G(d,n) \subset \mathbb{P}(d^{-1})$ is the image of $K^{d \times n}$ under the Plücker embedding, which sends a $d \times n$-matrix to the vector of its $d \times d$ minors. This vector is called the Plücker coordinates of the matrix. The Grassmannian is a smooth algebraic variety defined by equations called the Plücker relations, which give the relations among the maximal minors of the matrix. Points of this variety correspond to $d$-dimensional linear subspaces of $K^n$. The open subset $G^0(d,n)$ of the Grassmannian parametrizes subspaces whose Plücker coordinates are all nonzero. Points in this variety correspond to equivalence classes of matrices where no minor vanishes. In other words, these are matrices which give the uniform matroid of rank $d$ on $[n]$.

We now recall the definition of the tropical Grassmannian and Dressian of a matroid, as in [MS15]. Let $\mathcal{M}$ be a matroid of rank $d$ on the set $E = [n]$. For any basis $\sigma$ of $\mathcal{M}$, we introduce a variable $p_\sigma$. Consider the Laurent polynomial ring $K[p_\sigma^\pm 1 \mid \sigma \text{ is a basis of } \mathcal{M}]$ in these variables. Let $G_\mathcal{M}$ be the collection of polynomials obtained from the three-term Plücker relations by setting all variables not indexing a basis to zero. More precisely, these are the equations

$$G_{\mathcal{M}} = \left\{ p_{Sij}p_{Skl} - p_{Sik}p_{Sjl} + p_{Sil}p_{Skj} : S \in \binom{[n]}{d-2}, i \neq j \neq k \neq l, \text{ and } p_\sigma = 0 \text{ if } \sigma \not\in \mathcal{B} \right\}.$$
Let $I_M$ be the ideal generated by $G_M$. We call $I_M$ the matroid Plücker ideal of $M$, and refer to elements of $G_M$ as matroid Plücker relations.

The points of the variety $V(I_M)$ correspond to realizations of the matroid $M$ in the following sense. Points in $V(I_M)$ give equivalence classes of $d \times n$ matrices whose maximal minors vanish exactly when those minors are indexed by a nonbasis of $M$. We will call $V(I_M)$ the matroid Grassmannian of $M$. The variety $V(I_M)$ is empty if and only if $M$ is not realizable over $K$. Its tropicalization $Gr_M = \text{trop}(V(I_M))$ is called the tropical Grassmannian of $M$. We note here that if the rank of $M$ is 2, then $G_M$ is a tropical basis for $I_M$ [MS15, Chapter 4.4].

**Definition 1.1.** The Dressian $Dr_M$ of the matroid $M$ is the tropical prevariety obtained by intersecting the tropical hypersurfaces corresponding to elements of $G_M$:

$$Dr_M = \bigcap_{f \in G_M} \text{trop}(V(f)).$$

By definition, we have that $Gr_M \subseteq Dr_M$, and equality holds if and only if the matroid Plücker relations form a tropical basis.

Let $M_1$ and $M_2$ be matroids with disjoint ground sets $E_1$ and $E_2$ respectively, and basis sets $B_1$ and $B_2$ respectively. The direct sum of $M_1$ and $M_2$ is the matroid $M_1 \oplus M_2$ with ground set $E_1 \cup E_2$ and bases $B_1 \cup B_2$ such that $B_1 \in B_1$ and $B_2 \in B_2$. A matroid is connected if it cannot be written as the direct sum of other matroids. The number of connected components of a matroid is the number of connected matroids it is a direct sum of. In [OPS18], the authors show that if $M_1$ and $M_2$ are matroids with disjoint element sets, then $Dr_{M_1 \oplus M_2} = Dr_{M_1} \times Dr_{M_2}$. For this reason, we will often assume that our matroids are connected.

The matroid polytope $P_M$ of $M$ is the convex hull of the indicator vectors of the bases of $M$:

$$P_M = \text{conv}(e_{\sigma_1} + \cdots + e_{\sigma_d} \mid \sigma \in B).$$

The dimension of $P_M$ is $n - c$, where $c$ is the number of connected components of $M$ [FS05].

**Theorem 1.2** (GGMS Theorem, 4.2.12 in [MS15]). A polytope $P$ with vertices in $\{0, 1\}^{n+1}$ is a matroid polytope if and only if every edge of $P$ is parallel to $e_i - e_i$.

Points in the Dressian of $M$ have an interesting relationship to the matroid polytope of $M$. Every vector $w$ in $\mathbb{R}^{|B|} / \mathbb{R}1$ induces a regular subdivision $\Delta_w$ of the polytope $P_M$. A subdivision of the matroid polytope $P_M$ is a matroid subdivision if all of its edges are translates of $e_i - e_i$. Equivalently, by Theorem 1.2, this implies all of the cells of the subdivision are matroid polytopes.

**Proposition 1.3** (Lemma 4.4.6, [MS15]). Let $M$ be a matroid, and let $w \in \mathbb{R}^{|B|}$. Then $w$ lies in the Dressian $Dr_M$ if and only if the corresponding regular subdivision $\Delta_w$ of $P_M$ is a matroid subdivision.

All matroids admit the trivial subdivision of their matroid polytope as a regular matroid subdivision, so the Dressian $Dr_M$ is nonempty for all matroids $M$. This gives us the lineality space of the Dressian, as we see in the following proposition.

**Proposition 1.4** ([OPS18], [DW92]). Let $M$ be a matroid, and let $c$ be the number of connected components of $M$. The lineality space of $Dr_M$ has dimension $n - c$ (in $\mathbb{R}^{|B|} / \mathbb{R}1$) and is given by the image of the map $\mathbb{R}^n \to \mathbb{R}^{|B|}$ given by $e_i \mapsto \sum_{B \ni e_i} e_B$.

We now discuss valuated matroids, as in [DW92]. Let $M$ be a matroid on $E = \{1, \ldots, n\}$ of rank $d$ and bases $B$. Let $v : B \to \mathbb{R} \cup \infty$ be a vector so that the pair $(M, v)$ satisfies the following version of the exchange axiom:
(V0) for $B_1, B_2 \in B$ and $e \in B_1 \setminus B_2$, there exists an $f \in B_2 \setminus B_1$ with $B'_1 = (B_1 \setminus \{e\}) \cup \{f\} \in B$, $B'_2 = (B_2 \setminus \{f\}) \cup \{e\} \in B$, and $v(B_1) + v(B_2) \geq v(B'_1) + v(B'_2)$.

We will call $v$ a valuation on $M$, and the pair $(M, v)$ is called a valued matroid (See [DW92] for details). It is known that valuations on a matroid $M$ are exactly the points in $Dr_M$ [MS15]. Indeed, the above condition asserts exactly that the tropicalized matroid Plücker relations hold.

2. Initial matroids and their polytopes

Let $M$ be a rank $d$ matroid on $n$ elements which is realizable over a field $K$ with valuation $val_K$. Let $\Gamma$ be the value group, let $R$ be the valuation ring of $K$, let $m$ be its maximal ideal, and let $k$ be its residue field. If $K$ is an algebraically closed field and $val_K$ is a nontrivial valuation, then by the Fundamental Theorem of Tropical Geometry [MS15, Theorem 3.2.3] points on $Gr_M \cap \Gamma^{|B|}$ are all of the form $(val_K(p_b))_{b \in B}$ where $(p_b)_{b \in B} \in (K^*)^{|B|}$ is a point on the matroid Grassmannian $V(I_M)$. Possibly by multiplying $(p_b)_{b \in B}$ by an element of $R$, we may assume that $(p_b)_{b \in B} \in (R)^{|B|}$ and that some coordinate has valuation 0. Let $M$ be a $d \times n$ matrix realizing $M$ which we may assume is over $R$. Consider the reduction map $\pi : R \rightarrow k$. Then $\pi(M)$ gives a matroid $M[\pi(M)]$.

In what follows we investigate how this matroid is related to $M$, and in what way it depends on the choice of element in $Gr_M$. First, we expand this notion to nonrealizable matroids.

**Definition 2.1.** Let $M$ be a matroid with bases $B$ and let $v \in Dr_M$. Then the initial matroid $M_v$ is the matroid whose bases are $B_v = \{ \sigma \in B \mid v(\sigma) \text{ is minimal} \}$. Given a matroid $M$, the initial matroids of $M$ are the matroids $M'$ such that there exists a $v \in Dr_M$ with $M_v = M'$.

**Remark 2.2.** If $v$ and $w$ are valuations of a matroid $M$ such that $v - w = 1$, then they give the same initial matroid: $M_v = M_w$. So, we can consider $Dr_M$ and $Gr_M$ in the tropical projective space $\mathbb{R}^{|B|}/\mathbb{R}1$. However, points which are equivalent modulo lineality may give different initial matroids. We explore the relationship between such matroids in Theorem A.

We now give an example to illustrate the ideas and results in the rest of the section.

![Initial matroids of $U_{2,4}$](image)

**Example 2.3.** Let $M = U_{2,4}$, the uniform rank 2 matroid on 4 elements; $B = \{01, 02, 03, 12, 13, 23\}$. In Figure 1 we give all initial matroids of $U_{2,4}$. We now study the Dressian of $M$. In this case,
The Dressian is a 5 dimensional fan with a four dimensional lineality space. Let the basis for \( \pi \) with reduction map valuation. Let \( \text{Lemma 2.4}\).

Additionally, the matrix above reduces to a matrix over \( \mathbb{C} \) by the points of \( \text{Dr}_M \) which are the convex hulls of \( \text{Gr}_M \) of \( \text{Gr}_M \). The Plücker coordinate is \( \text{Plücker coordinate is} \).

The matroid polytope \( \mathcal{M} \) of \( \text{Dr}_M \) has 3 maximal cones which are each generated by a ray. The rays are spanned by the points \( r_{01,23} = (1, 0, 0, 0, 0, 1) \) \( r_{02,13} = (0, 1, 0, 0, 1, 0) \) \( r_{03,12} = (0, 0, 1, 0, 1, 0) \).

The matroid polytope \( P_M \) is the hypersimplex \( \Delta(2,4) \), which is an octahedron. Each of the cones of \( \text{Dr}_M \) corresponds to a subdivision of \( P_M \) in to two pyramids. Let us study points in the cell of \( \text{Gr}_M \) containing \( r_{01,23} \). The point \( r_{01,23} \) induces a subdivision where the two maximal cells are the pyramids which are the convex hulls of \( p_{01} \) and \( p_{23} \).

The matroid \( \mathcal{M}_{r_{01,23}} \) has bases \( \{02, 03, 13, 12\} \). Its matroid polytope is the square face which is shared by they pyramids \( p_{01} \) and \( p_{23} \). Over \( \mathbb{C}([t]) \), we can realize \( \mathcal{M} \) with the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 + t & 1 + 2t & t & 2t
\end{bmatrix}
\]

and the resulting Plücker vector valuates to \( r_{01,23} \). This matrix reduces to a matrix over \( \mathbb{C} \) whose matroid is \( \mathcal{M}_{r_{01,23}} \). Alternatively, we can also realize \( \mathcal{M} \) with the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 3 + t^2
\end{bmatrix}
\]

The Plücker coordinate of this matrix valuates to

\[
v = (0, 0, 0, 0, 0, 2) = r_{01,23} - (1, 0, 0, 0, 0, -1) \in r_{01,23} + L.
\]

The matroid \( \mathcal{M}_v \) is the matroid with bases \( \{01, 02, 03, 12, 13\} \), whose matroid polytope is \( p_{01} \). Additionally, the matrix above reduces to a matrix over \( \mathbb{C} \) whose matroid is exactly \( \mathcal{M}_v \).

**Lemma 2.4.** Let \( \mathcal{M} \) be a rank \( d \) matroid on \( n \) elements which is realizable over a field \( K \) with nontrivial valuation \( \text{val}_K \). Let \( R \) be the valuation ring of \( K \) and let \( m \) be its maximal ideal, and \( k \) its residue field, with reduction map \( \pi \). Let \( v \in \text{Gr}_M \) so that \( \min(v) = 0 \) and let \( \mathcal{M} \) be a matrix over \( R \) realizing \( \mathcal{M} \) whose Plücker coordinate is \( v \). Then the initial matroid \( \mathcal{M}_v \) is \( \mathcal{M}[[\pi(M)]] \).

**Proof.** The bases of \( \mathcal{M}[[\pi(M)]] \) are indices \( \sigma \) of the Plücker coordinate of \( \pi(M) \) which do not vanish. In \( \mathcal{M} \), the corresponding Plücker coordinates necessarily have valuation 0, and since this is minimal, they will be bases of \( \mathcal{M}_v \). Conversely, all Plücker coordinates of \( \mathcal{M} \) with valuation 0 index columns of \( \pi(M) \) whose Plücker coordinates do not vanish, so we have \( \mathcal{M}_v = \mathcal{M}[[\pi(M)]] \), the matroid of \( \pi(M) \).

This lemma tells us that for realizable matroids, initial matroids are reductions, and vice versa. Now, we turn our attention to how initial matroids sit inside the matroid polytope \( P_M \), and prove **Theorem A** from the introduction.
Proof of Theorem A. First, we show that $P(M_v)$ is a cell of $\Delta_v$. To that end, we must show that there is a linear functional $l$ on $\mathbb{R}^{[\mathcal{B}] + 1}$ whose last coordinate is positive such that the face of $\text{conv}((e_\sigma, v_\sigma)_{\sigma \in \mathcal{B}})$ minimized by $l$ is the matroid polytope of $M_v$. We obtain $M_v$ by taking bases $\sigma$ with $v_\sigma$ minimal; in other words, the linear functional $l = (0, \ldots, 0, 1) \in (\mathbb{R}^{[\mathcal{B}] + 1})^\vee$ works.

Now, let $P$ be a polytope in $\Delta_v$. Then, there is a linear functional $l \in (\mathbb{R}^{[\mathcal{B}] + 1})^\vee$ with last coordinate scaled to 1 such that $P = \text{conv} (e_\sigma | l \cdot (e_\sigma, v_\sigma)$ is minimal). Since $l$ is linear on the vertices of the matroid polytope $P_\sigma$, the restriction $l_{| \mathbb{R}^{[\mathcal{B}]} }$ induces the trivial subdivision on $P_{M_\sigma}$ and is therefore contained in the lineality space of the Dressian. Then, the vector $w = (l \cdot e_\sigma)_{\sigma \in \mathcal{B}} + v$ is such that $P(M_w) = P$.

Remark 2.5. If $v$ is a valuation on $M$, the identity map on the ground set $M \to M_v$ gives a weak map (see [KN86]). There are examples of weak maps which do not arise in this way [DW92, Section 3]. By Theorem A, the question of whether or not all weak maps between connected matroids arise in this way is equivalent to [OPS18, Question 1], and as far as we are aware this is an open problem. By [Spe08, Proposition 4.4], when $M$ is uniform all weak images are initial matroids.

Remark 2.6. Initial matroids as in [MS15, Definition 4.2.7] are a special case of our initial matroids. Let $M$ be a rank $d$ matroid on $n$ elements. Given a weight vector $w' \in \mathbb{R}^n$, we can make a weight vector $w \in \mathbb{R}^{[\mathcal{B}]}$ by taking $w_\sigma = -\sum_{i \in \sigma} w'_i$. Any weight vector $w$ arising in this way is in the lineality space of $\text{Dr}_M$ and induces a trivial subdivision on $P(M)$. The initial matroid $M_w$ will be the initial matroid corresponding to $w'$ by [MS15, Proposition 4.2.10]. Among the cells of matroid subdivisions of $P(M)$, these initial matroids only correspond to faces of $P(M)$, while initial matroids in general give all cells of matroid subdivisions by Theorem A.

The Dressian does not depend on the field over which it is defined. On the other hand, the Grassmannian of a matroid, which is always contained in the Dressian, does depend on the residue characteristic of the field. We now give a result which explains the dependence on the residue characteristic, and gives a criterion to distinguish whether a point in the Dressian is contained in the Grassmannian of a matroid. First, we study an example.

Example 2.7. The non-Fano matroid is the rank 3 matroid on 7 elements with nonbases $\{ 014, 025, 036, 126, 234, 456 \}$. It is depicted in Figure 2. Its Dressian has dimension 8 with a 7 dimensional

![Figure 2. The non-Fano matroid.](image)
Proposition 2.8. Let $\mathcal{M}$ be a matroid and $K$ be an algebraically closed field with nontrivial valuation $\text{val}_K$ and residue field $k$. Then,

$$\text{Gr}_M \subset \{v \in \mathbb{R}^{[S]} \mid \text{all cells of } \Delta_v \text{ are matroid polytopes of matroids which are realizable over } k.\} \subset \text{Dr}_M$$

If $\text{Gr}_M = \text{Dr}_M$, then no regular matroidal subdivision of the matroid polytope $P_M$ contains a cell which is the matroid polytope of a non-realizable matroid, and all initial matroids of $M$ are realizable. Both of the subsets above can be strict.

Proof. Let $v$ be a point in $\text{Gr}_M$. By Lemma 2.4, the initial matroid $\mathcal{M}_v$ is realizable. By Theorem A, $P_{\mathcal{M}_v}$ is a cell of the regular matroid subdivision induced by $v$, and all cells arise in this way.

There are indeed examples of regular matroidal subdivisions where all cells correspond to realizable matroids, but a weight vector inducing them is not necessarily contained in the Grassmannian. In his thesis [Spe], Speyer gives two examples of this behavior. In Example 4.5.6, he gives two matroids of rank 3 on 12 elements which are both cells of a regular matroid subdivision of $U(3,12)$ such that the cross ratios of four of the points 5, 6, 7, and 8 are designed to be two different values. Therefore any weight vector inducing this subdivision cannot be contained in the Grassmannian. In Example 4.5.8, he gives examples of two weight vectors inducing the same subdivision, where one weight vector is contained in the Grassmannian and the other is not. □

3. Rigidity of matroids

We say a matroid is rigid if the only regular matroidal subdivision of the matroid polytope is the trivial subdivision. Using the ideas from the previous sections, we are able to answer some previously open questions about matroid rigidity. In general, it is a difficult problem to classify when matroids are rigid, however rigidity is known for some classes of matroids. In [OPS18, Conjecture 35] the authors conjecture that all finite projective spaces are rigid. For projective lines over $\mathbb{F}_q$ this is false whenever $q > 2$ because these are uniform matroids $U(2, q+1)$, which are not rigid. In [DW92, Theorem 5.11] it is shown that all finite projective spaces of dimension at least two have the property that all of their valuations are, up to translation by an element of the lineality space, equivalent to the valuation $v_0 = (1, \ldots, 1)$. Hence, all valuations induce the trivial subdivision, and so these matroids are rigid.

Proposition 3.1 ([OPS18],[DW92]). All binary matroids are rigid [OPS18, DW92] and every finite projective space of dimension at least two is rigid [DW92].

Proposition 3.2. Let $\mathcal{M}$ and $\mathcal{M}'$ be matroids of rank $d$ on $n$ elements. We say $\mathcal{M}' \prec \mathcal{M}$ if $\mathcal{M}'$ is an initial matroid of $\mathcal{M}$. Then, the relation $\prec$ on matroids gives a partial order on the set of connected matroids of rank $d$ on $n$ elements.

Proof. We must show that $\prec$ is reflexive, antisymmetric, and transitive. Let $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}''$ be connected matroids of rank $d$ on $n$ elements. Then $\mathcal{M} \prec \mathcal{M}$ because $v = 1 \in \mathbb{R}^{[S]}$ is a valuation on $\mathcal{M}$, since $\mathcal{M}$ satisfies basis exchange. If $\mathcal{M} \prec \mathcal{M}'$ and $\mathcal{M}' \prec \mathcal{M}$, then $\mathcal{M} = \mathcal{M}'$ because they will have the same set of bases. Lastly, suppose that $\mathcal{M}'' \prec \mathcal{M}'$ and $\mathcal{M}' \prec \mathcal{M}$. By Theorem A, we may select $v \in \text{Dr}_M$ so that $\mathcal{M}_v = \mathcal{M}'$ and we may select $v' \in \text{Dr}_{\mathcal{M}'}$ so that $\mathcal{M}_{v'} = \mathcal{M}''$. Then by [DLRS10, Lemma 2.3.16] there exists a regular refinement $\Delta$ of the subdivision $\Delta_v$ by the subdivision induced by $v'$. We see that $\Delta$ is matroidal because each of its cells are cells of $\Delta_v$ or $\Delta_{v'}$, which are each matroidal. This implies that $\Delta$ is induced by some $w \in \text{Dr}_M$. The matroid
Figure 3. Connected initial matroid poset for $\mathcal{U}_{3,6}$, with all connected matroids of rank 3 on 6 elements. We indicate loops with an "x", parallel elements with concentric circles, and rank 2 flats with 3 or more elements with lines.

polytope $P(M'')$ is a cell of $\Delta$, so by Theorem A there is a weight vector $w' \in w + L$ which induces $\Delta$ such that $M_{w'} = M''$. Therefore, $M'' \prec M$.

**Example 3.3.** This is an example from [DW92, Section 3] which demonstrates that transitivity in Proposition 3.2 does not hold if we remove the connected assumption. Consider the matroid $M$ of the projective plane over $\mathbb{F}_3$. It is a rank 3 matroid on 13 elements. By Proposition 3.1, this matroid is rigid. However, as observed in [OPS18], it has octahedral faces. These correspond to $U_{2,4}$ initial matroids, which are not rigid.

**Proposition 3.4.** Let $M$ be a rank $d$ matroid on $n$ elements. Then $M$ is rigid if and only if every matroid $M'$ different from $M$ with $M' \prec M$ has more components than $M$.

**Proof.** Suppose $M$ is rigid. If $M' \prec M$, by Theorem A, the matroid polytope $P_{M'}$ is a cell in some subdivision of $M$. Since all subdivisions of $M$ are trivial, and $M' \neq M$, we have that $M'$ must be a face of $M$. Since the dimension of $P_{M'}$ is $n - \#\{\text{components of } M'\}$, we have that $M'$ must have strictly more components than $M$. 

On the other hand, if every matroid \( M' \prec M \) has more components than \( M \), then all cells of all subdivisions of \( M \) other than \( P_M \) have dimension smaller than \( M \) (by Theorem A), so \( P_M \) only has the trivial subdivision. Therefore, \( M \) is rigid. \( \square \)

We now give the proof of Theorem B from the introduction, answering Question 2 from [OPS18].

**Proof of Theorem B.** Let \( \Delta \) be a finest matroid subdivision of a matroid polytope \( P \) of a connected matroid \( M \). Let \( P' \) be any maximal cell in \( \Delta \). Then \( P' \) corresponds to a connected matroid \( M' \), where \( M' \prec M \) by Theorem A. Let \( P'' \) be a maximal cell of any matroid subdivision of \( P' \). Then \( P'' \) corresponds to a connected matroid \( M'' \), with \( M'' \prec M' \). Then by Proposition 3.2, we have by transitivity that \( M'' \prec M \). By the proof of Proposition 3.2 there is a refinement of \( \Delta \) containing \( P'' \subset P' \) as a cell. Since \( \Delta \) is already a finest subdivision, this implies that \( P'' = P' \), and so \( P' \) is a rigid matroid. \( \square \)

**Example 3.5.** In Figure 3 we give the poset of all connected matroids of rank 3 on 6 elements ordered by \( \prec \). This contains all connected initial matroids of the uniform matroid \( U_{3,6} \). The ones corresponding to rigid matroids with one component, i.e. those which appear in finest matroid subdivisions of \( \Delta(3,6) \), are \( M_6, M_{15}, M_{14}, M_{16}, \) and \( M_{21} \). These names come from the order given on the Matroid Database [MMI].

### 4. Reduction algorithms for matroid Plücker equations

Using software (for instance, Gfan [Jen]), we may compute tropical prevarieties and varieties. However, these computations become unfeasible for inputs with many equations or variables. In this section we give a reduction algorithm for matroid Plücker relations, which we use in the computations in the remainder of the paper. In the described coordinates, the Dressian \( DR_M \) of a matroid \( M \) will have a large linearity space and lineality space (see Figure 4). The linearity space is the affine span of \( Gr_M \). If the linearity space is a proper affine subspace of \( \mathbb{R}^{|E|} \), then Algorithm 1 can be used to reduce the number of variables and equations by giving equations whose prevariety is equivalent via projection onto the linearity space. The generators \( G_M \) described above will typically have many binomials because they are obtained from trinomials by setting some of the variables to 0. As we will see, binomials introduce linearity into \( Gr_M \).

![Figure 4. Linearity and Lineality](image)

**Lemma 4.1.** Let \( K \) be a field with valuation \( \text{val}_K \). Let \( G \subset K[x_1^{\pm 1}, y_1^{\pm 1}, \ldots, y_d^{\pm 1}, z_1^{\pm 1}, \ldots, z_k^{\pm 1}] \) be finite with only binomials and trinomials, and suppose \( f \in G \) is a binomial in which \( x \) has degree 1. Then there
is a collection $G' \subset K[y_1^{\pm 1}, \ldots, y_d^{\pm 1}, z_1^{\pm 1}, \ldots, z_k^{\pm 1}]$ such that from the tropical prevariety defined by the $G'$ one can recover the tropical prevariety defined by the $G$.

Proof. Suppose $f = x y_1^{m_1} \cdots y_d^{m_d} + c z_1^{n_1} \cdots z_k^{n_k}$. Then, the tropical hypersurface of $f$ is defined by the equation

$$x = \text{val}_K(c) + n_1 z_1 + \cdots + n_k z_k - m_1 y_1 - \cdots - m_d y_d.$$

This equation defines a classical hyperplane, which introduces linearity in to the tropical prevariety defined by the $G$. To obtain $G'$, we substitute $x = -c - z_1^{n_1} \cdots z_k^{n_k} (y_1^{m_1} \cdots y_d^{m_d})^{-1}$ in every equation where $x$ appears in $G$, and the following 3 situations can arise. Let $g \in G$, and denote the substitution map from $K[x^{\pm 1}, y_1^{\pm 1}, \ldots, y_d^{\pm 1}, z_1^{\pm 1}, \ldots, z_k^{\pm 1}] \to K[y_1^{\pm 1}, \ldots, y_d^{\pm 1}, z_1^{\pm 1}, \ldots, z_k^{\pm 1}]$ by $\phi$.

For any polynomial $g$, let $t(g)$ be the number of terms of $g$.

1. If $t(g) = t(\phi(g))$, then we add $\phi(g)$ to $G'$.
2. If $t(\phi(g)) < 2$, then trop$(g)$ asserts that the minimum of two or more identical linear forms is attained twice, so we do not add $\phi(g)$ to $G'$. An example of this is given in 4.2.
3. If $t(g) = 3$ and $t(\phi(g)) = 2$, then tropically this asserts an inequality. We do not add $\phi(g)$ to $G'$, but we record this inequality.

Then, the tropical prevariety defined by the $G'$ and intersected with any inequalities arising from (3) is the projection of $\text{Gr}_M$ onto the $(y_1, \ldots, y_d, z_1, \ldots, z_k)$ plane. Indeed, for any point $w'$ in the tropical prevariety of $G'$, we can recover a point $w$ in the tropical prevariety of $G$ by adding the coordinate $x = v(c) + n_1 z_1 + \cdots + n_k z_k - m_1 y_1 - \cdots - m_d y_d$. \qed

Algorithm 1 Equation reduction for Matroid Plücker Ideals

**Input:** $\mathcal{M}$ a matroid.

**Output:** $G'$ and $L'$ as in Theorem C.

Let $B$ be the binomials in $G_M$ having a variable of degree 1.

Let $G_{\text{old}} = G_M$.

Let $L' = \{\}$. 

while $|B| > 0$ do

Let $G' = \{\}$. 

Pick $f \in B$, and a variable $x$ which has degree 1 in $f$. Then we may write $x = m$ for some monomial $m$.

for $g \in G_{\text{old}}$ do

Replace $x$ by $m$ in $g$ to obtain $g'$. 

if $t(g) = t(g')$ then

Add $g'$ to $G'$.

else

if $t(g') = 2$ and $t(g) = 3$ then

Add the corresponding inequality to $L'$.

end if

end if

end for

Set $B$ to be the binomials in $G'$ having a variable of degree 1. 

Set $G_{\text{old}} = G'$.

end while
Example 4.2. We will now give an example that illustrates case (2) from above. Consider the ideal \( I \subset K[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}, w^{\pm 1}] \), with generating set \( G = \{xy - zw, xy + zw\} \). Then \( I \) is the unit ideal, but if we wish to apply Lemma 4.1 to compute the tropical prevariety, the following happens. We replace \( x \) by \( zw/y \). The equations of \( G \), after substitution, are \( \{0, 2zw\} \). This gives an empty prevariety. However, if we instead substitute \( x = z + w - y \) in to \( \text{trop}(xy + zw) \), we obtain the condition that \( \min(z + w, z + w) \) is attained twice for both equations. This is a vacuous constraint, and so any point \((y, z, w)\) can be lifted to a point in the tropical prevariety. This is why we remove equations from \( G \) which, after substitution, have fewer than two terms. This exact issue appears often when computing Dressians of non-realizable matroids, and never occurs for realizable matroids (because there is no monomial in \( I \)).

In practice, this is quite a useful trick when computing Dressians of matroids. In the examples we will observe that already in small cases, this trick reduces a problem with hundreds of variables to tens of variables. The reduction is implemented in Mathematica and can be downloaded at https://math.berkeley.edu/~brandtm/research.html.

5. Examples of Dressians

In this section we compute Dressians for interesting matroids analogous to the computation done in [HJJS09, Section 5] for the Pappus matroid.

5.1. The Star 103. Consider the rank 3 matroid \( M_* \) on \{0, 1, \ldots, 9\} with nonbases given by

\[
\left( \begin{array}{c} 10 \\ 3 \end{array} \right) \setminus B_{M_*} = \{026, 039, 058, 173, 145, 169, 248, 257, 368, 479\}.
\]

A realization of this matroid is depicted in Figure 5. It is a 103 configuration, meaning that it is a configuration of 10 points and 10 lines in the plane such that each point is contained in 3 lines and each line contains 3 points. Up to isomorphism, there are ten such configurations [Grü09, Table 2.2.7]. In this table, \( M_* \) is configuration \((10_3)_3\). The 103 configurations were first determined by Kantor [Kan83]. Among them is the Desargues configuration, \((10_3)_1\), which we discuss in Section 5.6. The configuration \( M_* \) is astral and has orbit type \([2, 2]\), meaning that under the action of its symmetry group there are two orbits of points and two orbits of lines, and this is the minimal number that an \( n_3 \) configuration may have [Grü09].

![Figure 5. The star matroid \( M_* \).](https://math.berkeley.edu/~brandtm/research.html)
Proposition 5.1. Modulo lineality and intersecting with a sphere, the Dressian $\text{Dr}_{M_*}$ is a 2 dimensional polyhedral complex with 30 vertices, 65 edges, and 20 triangles. It is depicted in Figure 6. In characteristic 0, the Grassmannian $\text{Gr}_{M_*}$ is a graph with 30 vertices and 55 edges. It is depicted in Figure 6 in the darker color.

Proof. Using Algorithm 1, we take the generators $G_{M_*}$ and make a new generating set whose tropical prevariety will not have linearity. Initially, we are working with 1260 equations in 110 unknowns. After applying Algorithm 1, we have 73 equations in 17 unknowns. Let $I$ be the ideal generated by these generators. Using the command tropicalintersection in gfan [Jen], we obtain the Dressian as claimed.

For each cone $\sigma$ in the Dressian we select a random $w \in \sigma$. Then, we compute $\text{In}_w(I)$. If it contains a monomial, then we conclude that the cone is not contained in $\text{Dr}_{*}$. Doing so demonstrates that no triangle is contained in $\text{Dr}_{*}$, and neither are the edges which are contained in two triangles. Lastly, to verify that everything else is contained in $\text{Gr}_{*}$, and to ensure that there was no lower-dimensional cell within a triangle, we check the balancing condition at each ray. We find that the balancing condition holds, and this concludes the proof. □

Figure 6. The Dressian and the Grassmannian of $M_*$. The Dressian is the full picture, and the Grassmannian is the darkened part.

Let us study the matroid subdivisions arising from rays in the Dressian $\text{Dr}_{M_*}$. They come in three tiers, with each tier containing 10 rays.

Tier I contains the outermost rays in Figure 6. Each of these rays induces a subdivision of the matroid polytope with 9 cells, where each cell has the following number of vertices: \{33,33,33,41,41,49,57,67,77\}. Tier II contains the middle rays in Figure 6. Each of these rays induces a subdivision of the matroid polytope with 6 cells, where each cell has the following number of vertices: \{33,41,43,61,68,81\}. Tier III contains the innermost rays in Figure 6. Each
Figure 7. This is a matroid which is an initial matroid of $M_\star$. The nonbases are indicated by gray lines and parallel elements are indicated by concentric circles. This matroid appears in subdivisions of the matroid polytope of $M_\star$ which come from points of $\text{Dr}_M \setminus \text{Gr}_M$. It is not realizable over any field because it has the Fano matroid (which is only realizable over characteristic 2) and the uniform matroid $U_{2,4}$ (which is not realizable over characteristic 2) as minors.

of these rays induces a subdivision of the matroid polytope with 5 cells, where each cell has the following number of vertices: $\{33, 33, 53, 53, 96\}$. Each of the polytopes with 33 vertices arising in these subdivisions comes from setting six of the points parallel. The resulting matroids are rank 3 matroids on 5 elements with two nonbases, which intersect at a point.

In the non-realizable edges of $\text{Gr}_{M_\star}$, subdivisions contain the matroid in Figure 7 which is not realizable over any field. This provides an example for Proposition 2.8, and demonstrates that these points are not in $\text{Gr}_{M_\star}$.

5.2. Non-Pappus. In [HJJS09] the authors study the Dressian of the Pappus matroid. They show that as a simplicial complex, it has $f$-vector $f = (18, 30, 1)$. In [MS15, Page 213], Exercise 23, the authors ask for the Dressian of the non-Pappus matroid $M_{nP}$. This matroid is not realizable over any field, as this would contradict the Pappus Theorem, which says that the points 6, 7, 8 in Figure 8 will always be collinear as long as the other collinearities hold.
**Proposition 5.2** ([MS15], Chapter 4, Exercise 23). *Modulo lineality and intersecting with a sphere, the Dressian $\text{Dr}_{M_n}$ is a 3 dimensional polyhedral complex with f-vector $(19,48,31,1)$. The Grassmannian $\text{Gr}_{M_n}$ is empty.*

*Proof.* Using Algorithm 1, we take the generators $G_{M_n}$ and make a new generating set whose tropical prevariety will not have linearity. Initially, we are working with 630 equations in 76 unknowns. After applying Algorithm 1, we have 171 equations and 29 unknowns. Let $I$ be the ideal generated by these generators. Using the command `tropicalintersection` in gfan [Jen], we obtain the Dressian as claimed. □

The projection of this Dressian along the $p_{678}$ axis yields the Dressian of the Pappus matroid. Indeed, if $f$ is the f-vector of the Pappus matroid, we see that $(19,48,31,1) = (f_0 + 1, f_1 + f_0, f_2 + f_1, f_2)$.

### 5.3. The Vámos Matroid

The Vámos Matroid is a rank 4 matroid on 8 elements which is not realizable over any field. We depict it in Figure 9. All four-element subsets of the eight elements are bases except \{0134, 0125, 2345, 3467, 2567\}.

![Figure 9. A depiction of the Vámos Matroid.](image)

**Proposition 5.3.** *Modulo lineality and intersecting with a sphere, the Dressian of the Vámos Matroid is an 8 dimensional polyhedral complex with f-vector $(201,2014,6810,9581,5425,896,72,18,2)$. *

*Proof.* Using Algorithm 1, we take the generators $G_{M}$ and make a new generating set whose tropical prevariety will not have linearity. Initially, we are working with 420 equations in 65 unknowns. After applying Algorithm 1, we have 169 equations and 33 unknowns. Let $I$ be the ideal generated by these generators. Using the command `tropicalintersection` in gfan [Jen], we obtain the Dressian as claimed. This computation took two days to compute in gfan. □

We now study subdivisions of the matroid polytope induced by elements of the two maximal cells. In each case, points from the interior of the cell induce a matroid subdivision which has 9 polytopes with 17 vertices and one polytope with 56 vertices. The large polytopes are the matroid polytopes of the matroids $M_1, M_2$ with the following two collections of 14 nonbases:

$\mathcal{B}_0 = \langle 0134, 0125, 2345, 3467, 2567, 0167, 0246, 0356, 1247, 1357, 0237, 0457, 1236, 1456\rangle$, from Vámos
Each of these is the collection of 12 planes in a cube, together with extra nonbases 2345 and 0167. The two labellings of the cube are given in Figure 10. The matroid polytope of this matroid has no nontrivial matroid subdivisions. This matroid is not realizable over any field, since $\text{Gr}_{\mathcal{M}_1} = \text{Gr}_{\mathcal{M}_2} = \langle 1 \rangle$. The other 9 polytopes in the subdivision each correspond to one of the above 9 new nonbases. They are each matroids in which all of the elements of the corresponding nonbasis have been parallelized in the Vámos matroid.

![Figure 10](image-url) The two cubes whose planes give 12 of the 14 nonbases for the two matroids arising from the Vámos matroid

The non-Vámos matroid, which has the additional nonbasis 0167, is realizable. Its Dressian (modulo lineality and intersecting with the sphere) is a 7 dimensional polyhedral complex and has $f$-vector

$$f = (200, 1814, 4996, 4585, 840, 56, 16, 2).$$

Like in the case of the Pappus and non-Pappus matroids, we also have here that the Dressian of the non-Vámos matroid is the projection of the Dressian of the Vámos matroid along the $p_{0167}$ axis. Indeed, the $f$-vector in Proposition 5.3 is

$$(1 + f_0, f_0 + f_1, f_1 + f_2, f_2 + f_3, f_3 + f_4, f_4 + f_5, f_5 + f_6, f_6 + f_7, f_7).$$

It is tempting to wonder if whenever $\mathcal{M}$ and $\mathcal{M}'$ are matroids of rank $d$ on $n$ elements such that $B' = B \cup \sigma$, whether their Dressians are related by projection along the axis $e_{\sigma}$. By [DW92, Proposition 3.1] there is a way to extend $v \in \text{Dr}_{\mathcal{M}}$ to a valuation $v'$ on $\mathcal{M}'$. Indeed, these give subdivisions of the matroid polytope which contain a maximal cell given by the new vertex together with the closest face to it in $P_{\mathcal{M}}$. However, there could be more subdivisions than this, such as subdivisions in which the vertex corresponding to $\sigma$ is contained in more than one maximal cell. At present the author does not know of any examples of this behavior.

5.4. Cube. Consider the matroid $\mathcal{M}_{\square}$ defined by the cube on the left side of Figure 10, whose twelve planes define the nonbases. Its Dressian $\text{Dr}_{\mathcal{M}_{\square}}$, modulo lineality, consists of two points joined by a line segment. The two points each induce a subdivision with two cells, where the matroid corresponding to one cell is one in which one great tetrahedron (i.e., either 0167 or 2345) is collapsed. The points on the segment joining these two points induce subdivisions with three cells, where the largest cell corresponds to the subdivision in which both great tetrahedra have been collapsed. These matroids are not realizable, so by Proposition 2.8, the Grassmannian $\text{Gr}_{\mathcal{M}_{\square}}$ simply consists of the lineality space.
5.5. **Twisted Vámos.** We now study the Dressian of the rank 4 matroid on 8 elements arising from the polytope depicted in Figure 11 and listed at [Pol]. Its nonbases are given by \(\{0123, 0145, 2345, 2567, 3467\}\). Modulo its lineality space and intersecting with a sphere, this is a five dimensional polyhedral complex with f vector \((120, 1196, 3377, 2985, 397, 8)\).

5.6. **Desargues.** We now study the Desargues configuration. It is named after Gerard Desargues, and the Desargues Theorem proves the existence of this configuration. A depiction of the Desargues configuration is given in Figure 12. This is another example of a \(10_3\) configuration. It is the rank 3 matroid \(\mathcal{M}_D\) on \(\{0, 1, \ldots, 9\}\) with nonbases given by

\[
\binom{10}{3} \backslash B_{\mathcal{M}_D} = \{027, 036, 058, 135, 149, 168, 234, 259, 467, 789\}.
\]

**Proposition 5.4.** Modulo lineality and intersecting with a sphere, the Dressian \(\text{Dr}_{\mathcal{M}_D}\) is a 3 dimensional polyhedral complex with 70 vertices, 370 edges, 510 two dimensional cells, and 150 three dimensional cells.

**Proof.** Using Algorithm 1, we take the generators \(G_{\mathcal{M}}\), and make a new generating set whose tropical prevariety will not have linearity. Initially, we are working with 630 equations in 74 unknowns. After applying Algorithm 1, we have 69 equations and 24 unknowns. Let \(I\) be the ideal generated by these generators. Using the command `tropicalintersection` in gfan [Jen], we obtain the Dressian as claimed. \(\square\)

Of the two dimensional cells, all of the ones which are not contained in a larger cell are triangles. Of the three dimensional cells, 5 are cubes, 30 are pyramids with square bases, and 115 are tetrahedra. The square bases of the pyramids are faces of the cubes. Each pyramid shares a square base with another pyramid. In total, there are 10 vertices which are the tops of pyramids. The graph on these vertices whose edges correspond to pyramids who share bases is a Petersen graph.

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Figure 12. The Desargues Configuration, with the nonbases represented as gray lines.

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Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720
E-mail address: brandtm@berkeley.edu

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