Domain walls and M2-branes partition functions: M-theory and ABJM Theory

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Abstract
We study the BPS counting functions (free energies) of the M-string configurations. We consider separated M5-branes along with M2-branes stretched between them, with M5-branes acting as domain walls interpolating different configurations of M2-branes. We find recursive structure in the free energies of these configurations. The M-string degrees of freedom on the domain walls are interpreted in terms of a pair of interacting supersymmetric WZW models. We also compute the elliptic genus of the M-string in a toy model of the ABJM theory and compare it with the M-theory computation.

Keywords: open topological string wave function, gauged WZW models, domain walls.

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1. DOMAIN WALLS IN M-THEORY: AN INTRODUCTION

The study of classification and dynamics of the 6d Supersymmetric CFTs (SCFTs) is one of the important problems that is currently an active area of research. The maximally supersymmetric 6d CFTs are called (2,0) theories. The type IIB string theory in the background of A-type gives rise to (2,0) A-type theory. In the M-theory formulation the (2,0) A_{N-1} theory is the world volume theory of N parallel and coincident M5-branes. Away from the conformal point the CFT describes the dynamics of the self-dual strings of small tensions. In the M-theory these strings are described by the one dimensional intersections of M5-branes and M2-branes. The strings support (4,0) quiver gauge theory. One crucial result of [1][2][3][5] is that the elliptic genus of this quiver gauge theory turns out to be equal to the partition function of the bulk theory.

The superconformal group of the theory is Osp(2|6|4). Let’s denote the 11d spacetime \( R^{1,10} \) by the coordinates \( x^I, i = 0, ..., 10 \). The coincident M5-branes span the coordinates \( \{x^0, x^1, ..., x^5\} \). In the non-conformal limit the M5-branes are separated along the \( x^0 \) direction with positions denoted by \( a_i, i = 1, 2, ..., N \). The M2-brane is suspended between consecutive M5-branes and span the coordinates \( \{x^0, x^2, x^5\} \). The M2-brane couples to a 2-form field \( B \) inside the M5-brane worldvolume and the boundary of the M2-brane is what is called the M-string. If we denote by \( \Gamma^I \) the \( 32 \times 32 \) 11d gamma matrices, then the supersymmetries preserved by the M-string are given by

\[
\Gamma^{016} \epsilon = \epsilon, \quad \Gamma^{012345} \epsilon = \epsilon, \quad \Gamma^{01} \epsilon = \epsilon.
\]

where \( \epsilon \) is the 32-component spinor and \( \Gamma^{i_1 i_2 ... i_k} = \Gamma^{i_1} \Gamma^{i_2} ... \Gamma^{i_k} \).

The momentum around the \( S^1 \) defines a quantum number of the 5d BPS particles \( \frac{k}{R_1} = -\frac{1}{8g_s} \int d^4x tr(F \wedge F) \) and with the corresponding mass \( M = R_1 \delta_{ij} + \frac{k}{R_1} \), where \( \delta_{ij} \) denotes the separation between \( i \)-th and \( j \)-th M5-branes. Moreover the mass deformation also breaks the string worldsheet supersymmetry (4,4) to (4,0).

A further compactification of the theory can be considered along the compactified direction \( x^0 \). This makes the worldvolume of the M5-branes to be \( R^2 \times T^2 \). Twisting the theory as one moves around the second \( S^1 \) defines the so-called \( \Omega \)-background which makes it possible to apply the equivariant localisation to compute the partition function. To engineer 5d \( \mathcal{N} = 1^* SU(N) \) gauge theory one has to
compact M-theory on the elliptic CY$3$-fold $A_{N-1} \times T^2$. The instantons in the 4d theory are none other than the M-strings wrapped on the whole of $T^2$.

Recall that the symmetry $U(1)_{\eta_1} \times U(2)_{\eta_2} \times U(1)_{\eta_3}$ acts on the two $\mathbb{R}^4$s defined by the coordinates $R^4_1$ defined by the coordinates $z_1 = x_2 + i x_3, z_2 = x_4 + i x_5$ and $R^4_2$ defined by the coordinates $z_3 = x_7 + i x_8, z_4 = x_9 + i x_{10}$ as follows: $(z_1, z_2) \rightarrow (q_1, t^{-1} q_2) (z_3, z_4) \rightarrow (\sqrt{q} e^{\pi i (2m)} z_3, \sqrt{q} e^{\pi i (2m)} z_4)$ where $q := e^{2 \pi i \tau_1 \phi}, t := e^{-2 \pi i \tau_2 \phi}$. The toric geometry, dual to the type IIB D5-NS5-(1,1) branes web, underlying the M-string computation is given in Figure 2. The diagonal edges correspond to the mass parameter $m$, the horizontal direction is periodic with period $\tau$ and $Q$ is the fugacity corresponding to the internal vertical lines.

The target space of the gauged linear sigma model on the world-sheet of M-string is the singular space $\text{Sym}^2 \mathbb{R}^4 / S^4$ described as the configuration space of $n$ points on $\mathbb{R}^4$ moded by the permutation group $S_n$. The singularity is due to the coincidence of multiple points and resolving this singularity gives rise to the Hilbert scheme of $P$. The conditions around the 1-cycles of $T$ fermions to different bundles correspond to the different boundary conditions. The bundle of $P$ tautological bundle corresponds to the two hypermultiplets, the first of which is the $(1,1)$ branes web, underlying the M-string computation is given in Figure 2. The diagonal edges correspond to the mass parameter $m$, the horizontal direction is periodic with period $\tau$ and $Q$ is the fugacity corresponding to the internal vertical lines.

The target space of the gauged linear sigma model on the world-sheet of M-string is the singular space $\text{Sym}^2 \mathbb{R}^4 / S^4$ described as the configuration space of $n$ points on $\mathbb{R}^4$ moded by the permutation group $S_n$. The singularity is due to the coincidence of multiple points and resolving this singularity gives rise to the Hilbert scheme of $n$ points on $\mathbb{C}^2$. So instead of dealing with the ill defined sigma model on $\text{Sym}^2 \mathbb{R}^4 / S^4$, one can work with a (0,0) sigma model on the $\mathcal{M} = \text{Hilb}^3(\mathbb{C}^2)$ [1]. In the current situation we have tangent bundle $T_{\mathcal{M}}$ and the complex tautological bundle $E$. The bundle $E$ corresponds to the contribution of fundamental hypermultiplet in the gauge tensor instanton computation. In this theory the left handed fermions are the sections of tangent bundle $T_{\mathcal{M}}$, whereas the right handed fermions are the sections of $E \oplus E^*$. The combination $E \oplus E^*$, corresponding to the two hypermultiplets, signifies the fact that in the toric geometry the $\mathbb{P}^1$ of the second hypermultiplet is flipped [1].

The M-string worldsheet theory defines the target space $\text{Hilb}^3(\mathbb{C}^2)$.

The coupling of the left moving fermions and right moving fermions to different bundles correspond to the different boundary conditions around the 1-cycles of $T^2$ when computing the elliptic genus. The $(2,2)$ sigma model contains bosons $\psi^\dagger$ and the fermions $\psi_2^\dagger, \psi_2, \psi_3^\dagger, \psi_3$. Locally the bosons describe a map $\Phi : \text{world sheet} \rightarrow \mathcal{M}$. If we denote by $K$ the bundle of $(1,0)$-forms and by $R$ the bundle of $(0,1)$-forms then the fermions are defined by the following pullback maps

\[
\begin{align*}
\psi_2 & = R^{1 \dagger} \otimes \Phi^* T_{\mathcal{M}}, & \psi_3 & = R^{1 \dagger} \otimes \Phi^* T_{\mathcal{M}} \\
\psi_3^\dagger & = K^{1 \dagger} \otimes \Phi^* T_{\mathcal{M}}, & \psi_2^\dagger & = K^{1 \dagger} \otimes \Phi^* T_{\mathcal{M}}
\end{align*}
\]

If right moving fermions $\eta_2$ are also included the supersymmetry gets broken to $(2,0)$ and we define

\[
\eta^\dagger = R^{1 \dagger} \otimes \Phi^* (E \oplus E^*)
\]
string states between the D-branes wrapped on the holomorphic submanifolds. By determining the equivariant weights of the bundle $V$ the M-string partition function is determined. Intuitively each M5-brane with M2-branes ending on the left and right gives rise to a factor $Ext^1(I, J) \otimes L^{-1/2}$ in the bundle with $I$ denoting a point of $\text{Hilb}^n[C^2]$ corresponding to the M2-brane on the left and $J$ denotes a point of $\text{Hilb}^m[C^2]$ corresponding to the M2-brane on the right.

For general moduli space $\mathfrak{M}_{k_1,\ldots, k_N} = \text{Hilb}^k[C^2] \times \text{Hilb}^k[C^2] \times \cdots \times \text{Hilb}^{k_{N-1}}[C^2]$ the fiber of the corresponding bundle $V$ over $(I_1,\ldots, I_N-1) \in \mathfrak{M}_{k_1,\ldots, k_N}$ is given by

$$V|_{(I_1,\ldots, I_{N-1})} = \left( \prod_{a=0}^{N-1} Ext^1(I_a, I_{a+1}) \otimes L^{-1/2} \right)$$

(9)

The fixed points are in one-to-one correspondence with the set of partitions $(\nu_1,\ldots, \nu_{N-1})$ and the equivariant weights of $V$ over the fixed point are

$$\{Q_m q^{-i/2} \theta^{-j/2} (i,j) \in \nu_1 \} \cup \{Q_m q^{-i/2} \theta^{-j/2} (i,j) \in \nu_{N-1} \} \cup \left( \prod_{a=1}^{N-2} \left( \prod_{i,j} \theta(\tau_{\nu_a \nu_b}^a) \theta(\tau_{\nu_b \nu_c}^b) \right) \right)$$

(10)

Using these weights at the fixed points the partition function turns out to be

$$Z_N(\tau, m, t_{f_1}, t_{f_2}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2) = \sum_{k_1, k_2, \ldots, k_N} \left( \prod_{a=1}^{N-1} (-Q_{f_a})^{\nu_a} \right) \sum_{|\nu_i| = k_i, \nu_1| = k_{N-1}} \times \prod_{a=1}^{N-1} \prod_{(i,j) \in \nu_a} \theta(\tau_{\nu_a \nu_b}^a) \theta(\tau_{\nu_b \nu_c}^b)$$

(11)

where

$$\epsilon_{u_{i,j}}^a = -m + \epsilon_1 (v_{u_{i,j}} - v_{u_{i,j}} + i - \frac{1}{2}) - \epsilon_2 (v_{u_{i,j}} + j - \frac{1}{2})$$

$$v_{u_{i,j}}^a = -m + \epsilon_1 (v_{u_{i,j}} - v_{u_{i,j}} + i + \frac{1}{2}) - \epsilon_2 (v_{u_{i,j}} + j + \frac{1}{2})$$

$$w_{u_{i,j}}^a = \epsilon_1 (v_{u_{i,j}} - v_{u_{i,j}} + i + \frac{1}{2}) - \epsilon_2 (v_{u_{i,j}} + j + \frac{1}{2})$$

$$w_{u_{i,j}} = 0, v_{\nu} = 0.$$  

(12)

and $Q_{f_a} := Q_a = e^{2\pi i \theta_a}, a = 1, \ldots, N-1$ are the fugacities in terms of the coulomb branch parameters $t_{f_a}$ that determine the distance between consecutive M5-branes. From the expression (11) we can isolate the following part

$$Z_{k_1, k_2, \ldots, k_N} = \sum_{|\nu_i| = k_i, \nu_1| = k_{N-1}} \prod_{a=1}^{N-1} \prod_{(i,j) \in \nu_a} \theta(\tau_{\nu_a \nu_b}^a) \theta(\tau_{\nu_b \nu_c}^b)$$

(13)

which can be interpreted [7] as the partition function of the following configuration of the wrapped M2-branes: $k_1$ M2-branes between 1st and 2nd M5-branes, $k_2$ M2-branes between 2nd and 3rd M5-branes and so on up to $k_{N-1}$ M2-branes between $(N-2)$-th and $(N-1)$-th M5-branes.

After stripping off the gauge theory $U(1)$ part $NP\log Z_f$ from the free energy

$$\Omega_N(\tau, m, t_{f_1}, \epsilon_1, \epsilon_2) = NP\log Z_f + PL \log Z_N$$

(14)

one can expand the remaining free energy $\tilde{\Omega}(\tau, m, t_{f_1}, \epsilon_1, \epsilon_2) := PL \log Z_N$ in terms of the fugacities $Q_{f_a}$ as follows

$$\tilde{\Omega}(\tau, m, t_{f_1}, \epsilon_1, \epsilon_2) := \sum_{k_1, k_2, \ldots, k_N} Q_{f_1}^{k_1} \cdots Q_{f_N}^{k_N} F_{k_1, k_2, \ldots, k_N}(\tau, m, \epsilon_1, \epsilon_2)$$

(15)

where the multi-index function $F_{k_1, k_2, \ldots, k_N}(\tau, m, \epsilon_1, \epsilon_2)$ counts the degeneracies of the M-strings bound states.

PRESENTATION OF THE ARTICLE

After briefly introducing the M-strings and the corresponding elliptic genus of its worksheet theory in section 2 we discuss recursive structure in the expressions for free energies corresponding to various configuration of the M2-M5 branes. A general configuration consists of an array of multiple M2-branes sandwiched between M5-branes. The M2-branes vacua are labelled by the tuple of integer partitions that correspond to the Young diagrams transforming in different representations. We discuss M2-M5 branes configurations in which the M2-branes are labelled by antisymmetric representations and symmetric representations. For these representations the free energies enjoy a partial recursive structure. For mixed representations the recursive structure is lost except for the configuration shown in section 3. In section 3 it is discussed that the open topological string wave function for the configuration M2-M5-M2 of branes can be described in terms of two WZW models coupled together. In section 4 we compute the elliptic genus for the M-strings that arise in ABJM model. We compare it to the M-string elliptic genus as computed in M-theory framework.

2. RECURSIVE STRUCTURE IN THE M-STRINGS PARTITION FUNCTION

For the M-string configuration in which a single M2-brane is stretched between consecutive M5-branes, the free energies show interesting recursive structure [3]. For more complicated configurations the recursive structure is not apparent in the expression for free energies and the correct objects to decompose are the components $Z_{\nu_1, \nu_2, \nu_3}$. In doing the following computations we will often use the following symmetry of the elliptic genera indices $\{\mu_1, \ldots, \mu_n, \theta_1, \theta_2, \ldots, \theta_m\}$

$$Z_{\text{permutation}}{\mu_1, \ldots, \mu_n, \theta_1, \theta_2, \ldots, \theta_m} = Z_{\mu_1, \ldots, \mu_n, \theta_1, \theta_2, \ldots, \theta_m}$$

(16)

where $\theta_i = \theta_j$ for all $i, j, m$ can be less than, equal to or greater than $n$ and permutation denotes any possible permutation of the given indices.

For the configuration of partitions $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

$$Z_{\alpha_1, \alpha_2, \ldots, \alpha_n} = \text{all possible ways of factoring} + Z_{\alpha} W_{\alpha}^{k-1}$$

(17)

1-configurations of M5-M2 branes in which an M2-brane may carry a symmetric or an antisymmetric representation.
where \( W_\alpha \) can be thought of as a universal factor corresponding to removing a single M5-brane. The meaning of the phrase “all possible ways of factorizing” is the following: we will see explicitly in the next section that there is a recursive structure in the expansion coefficients \( Z_{\alpha_1\ldots\alpha_k} \) of the elliptic genera; for example

\[
Z_{222} - 2Z_{220}Z_{200} + Z_{200}^3 = Z_{200}^2W_2(\tau, m, e_1, e_2)
\]

where the explicit expressions for the factors \( Z_{222}, Z_{220}, Z_{200} \) and \( W_2(\tau, m, e_1, e_2) \) are given in the next section. We can also write the last expression as

\[
Z_{222} = 2Z_{220}Z_{200} - Z_{200}^3 + Z_{200}W_2(\tau, m, e_1, e_2)
\]

In general, if there are \( k \) partitions \( \alpha_1, \ldots, \alpha_k \) of the same size then it is the case that:

\[
Z_{\alpha_1\ldots\alpha_k} = all\ possible\ ways\ of\ factorizing
\]

\[
+ Z_{\beta}W_{\alpha_1\ldots\alpha_k\ldots\alpha_{k-1}}W_{\alpha_k\ldots\alpha_{k-1}} (20)
\]

where \( \beta \) is the result of fusing [8] the partitions \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \). The universal factors \( W_{\alpha\beta} \) can be thought of as the effect of removing an M5-brane which fuses partitions \( \alpha \) and \( \beta \). It should be possible to see (section [3]) to obtain \( W_{\alpha\beta} \) from the M2-brane perspective as some kind of partition function associated with the domain wall represented by the M5-brane between the vacua labeled by \( \tau \) and \( \beta \). This should also be possible to do using the ABJM theory.

The free energy \( F = \text{ln}(Z) \) constructed from the partition function contains information about both the single particle and multi particle BPS states. The plethysm summation is used to project out the multi-particle states. Hence the function \( F_{k_1 k_2 \ldots k_n} \) counts single particle BPS states and can be expanded as

\[
F_{k_1 k_2 \ldots k_n}(\tau, m, e_1, e_2) = coefficient\ of\ \frac{\epsilon^{f_1}}{f_1!} \ldots \frac{\epsilon^{f_{k_n-1}}}{f_{k_n-1}!} in \sum_{\mu(\ell)} \mu(\ell)_{l_1, l_2} \log(Z_{\ell 1, l_2, l_{f_2}, l_{f_1}, l_{e_1}, l_{e_2}}) (21)
\]

2.1. Recursive structure for the configuration of fully antisymmetric Young diagrams

Below we give examples for a few configurations of the M5-M2 branes system. These examples show that there is no recursive structure for the full expressions of the free energies. Only a part of the expression of the free energy shows the recursive structure. This part is what is alluded to before as

\[
\text{all possible ways of factorizing}
\]

\[
F_{200}, F_{220}, F_{222}
\]

\[
F_{200}(\tau, m, e_1, e_2) = -\frac{1}{2}Z_{100}(\tau, m, e_1, e_2)^2 + \frac{1}{2}Z_{200}(\tau, m, e_1, e_2)
\]

\[
- \left( Z_{100}(2\tau, 2m, 2e_1, 2e_2) \right)
\]

\[
F_{220}(\tau, m, e_1, e_2) = \left( Z_{200}(\tau, m, e_1, e_2) - Z_{220}(\tau, m, e_1, e_2) \right) - \left( F_{110}(2\tau, 2m, 2e_1, 2e_2) \right) + other\ terms
\]

and finally

\[
F_{222}(\tau, m, e_1, e_2) = \left( Z_{200}^3(\tau, m, e_1, e_2) - 2Z_{220}(\tau, m, e_1, e_2)Z_{200}(\tau, m, e_1, e_2) + Z_{222}(\tau, m, e_1, e_2) \right) - \left( F_{111}(2\tau, 2m, 2e_1, 2e_2) \right) + other\ terms
\]

we now show that the terms in the square brackets form a recursive structure. First we consider instanton number \( k_1 = 2 \) and the following Young diagrams

\[
v_1 = \{1, 1\}, v_2 = \{1, 1\}, \ldots \text{ and so on.} \]

Using the notation \( \theta_1(x \pm y) := \theta_1(x + y) \theta_1(x - y) \) we find

\[
Z_{220}(\tau, m, e_1, e_2) - Z_{200}(\tau, m, e_1, e_2)W_{2}(\tau, m, e_1, e_2) = Z_{200}(\tau, m, e_1, e_2)
\]

\[
W_{2}(\tau, m, e_1, e_2) := \frac{1}{2} \left( \theta_1(-m \pm \frac{e_1}{2} + \frac{e_2}{2}) - \theta_1(-m \pm \frac{e_1}{2} - \frac{e_2}{2}) \right)
\]

To next order

\[
Z_{222}(\tau, m, e_1, e_2) + Z_{220}(\tau, m, e_1, e_2) - 2Z_{220}(\tau, m, e_1, e_2)Z_{200}(\tau, m, e_1, e_2) = Z_{200}(\tau, m, e_1, e_2)W_{2}(\tau, m, e_1, e_2)
\]

\[
F_{300}, F_{330}, F_{333}
\]

\[
F_{300}(\tau, m, e_1, e_2) = \left( Z_{300}(\tau, m, e_1, e_2) - Z_{100}(\tau, m, e_1, e_2)Z_{200}(\tau, m, e_1, e_2) \right) - \left( Z_{100}(\tau, m, e_1, e_2) \right)^3
\]

\[
F_{330}(\tau, m, e_1, e_2) = \left( Z_{330}(\tau, m, e_1, e_2) - Z_{300}(\tau, m, e_1, e_2) \right)^2 + other\ terms\ (30)
\]

\[
F_{333}(\tau, m, e_1, e_2) = \left( Z_{333}(\tau, m, e_1, e_2) - 2Z_{330}(\tau, m, e_1, e_2) \right) + other\ terms\ (31)
\]

Now we consider the terms in square brackets for instanton number \( k_1 = 3 \) and the following Young diagrams

\[
v_1 = \{1, 1, 1\}, v_2 = \{1, 1, 1\}, \ldots \text{ and so on.} \]

\[
Z_{330}(\tau, m, e_1, e_2) - Z_{300}(\tau, m, e_1, e_2)^2 = Z_{300}(\tau, m, e_1, e_2)W_{3}(\tau, m, e_1, e_2)
\]
where
\[ W_3(\tau, m, e_1, e_2) = \frac{1}{\theta_1(e_1 - e_2) \theta_1(-2e_2) \theta_1(e_1) \theta_1(-3e_2) \theta_1(e_1 - 2e_2)} \]
\[ \times \left[ \theta_1(-m + \frac{\epsilon_1}{2} \pm \frac{\epsilon_2}{2}) \theta_1(-m - \frac{\epsilon_1}{2} \pm \frac{\epsilon_2}{2}) \theta_1(-m \pm \epsilon_2) \right. \\
\left. - \theta_1(-m + \frac{\epsilon_1}{2} \pm \frac{\epsilon_2}{2}) \theta_1(-m + \frac{\epsilon_1}{2} \pm \frac{3\epsilon_2}{2}) \theta_1(-m \pm \epsilon_2) \right] \]

To next order

\[ Z_{333}(\tau, m, e_1, e_2) - 2Z_{330}(\tau, m, e_1, e_2)Z_{300}(\tau, m, e_1, e_2) + Z_{300}(\tau, m, e_1, e_2)^3 = Z_{300}(\tau, m, e_1, e_2)W_3(\tau, m, e_1, e_2)^2 \]

(35)

2.2. Observation

We observe that the \( W_i(\tau, m, e_1, e_2), i = 1, \ldots, N \) follow a pattern

\[ W_1(\tau, m, e_1, e_2) = \frac{\theta_1(-m \pm \epsilon_2) \theta_1(-e_1 - e_2) \theta_1(-e_1 + (k - 1)e_2)}{\theta_1(e_1) \theta_1(-e_1 - e_2) \theta_1(-e_1 + (k - 1)e_2)} \]

(37)

The above simple observation leads to the following generalisation

\[ W_N(\tau, m, e_1, e_2) = \prod_{k=1}^{N} \left[ \frac{\theta_1(-m \pm \epsilon_2 \pm (k - 1)e_2)}{\theta_1(-e_1 - e_2) \theta_1(e_1 - (k - 1)e_2)} \right] - \prod_{k=1}^{N} \left[ \frac{\theta_1(-m \pm \epsilon_2 - (k - 1)e_2)}{\theta_1(-e_1 - e_2) \theta_1(e_1 - (k - 1)e_2)} \right] \]

It is curious to note that \( W_N(\tau, m, e_1, e_2) \) can be written in terms of \( W_i(\tau, m, e_1, e_2) \). We can rewrite \( W_i(\tau, m, e_1, e_2) \) as

\[ W_i(\tau, m, e_1, e_2) = \frac{\theta_1(-m + \epsilon_2) - \theta_1(-m - \epsilon_2)}{\theta_1(e_1 - e_2) \theta_1(e_1 - (k - 1)e_2)} \]

(38)

Then \( W_i(\tau, m, e_1, e_2) \) can be rewritten in terms of \( W_i^+(\tau, m + \epsilon_2, e_1, e_2) \) and \( W_i^-(\tau, m - \epsilon_2, e_1, e_2) \) as

\[ W_N(\tau, m, e_1, e_2) = \prod_{k=1}^{N} \left[ W_i^+(\tau, m + \epsilon_2, e_1, e_2 - (k - 1)e_2) \right] - \prod_{k=1}^{N} \left[ W_i^-(\tau, m - \epsilon_2, e_1, e_2 - (k - 1)e_2) \right] \]

(39)

Note that under the modular transformation \( (\tau, m, e_1, e_2) \rightarrow (-\frac{1}{\tau}, \frac{m}{\tau}, \frac{e_1}{\tau}, \frac{e_2}{\tau}) \), \( W_N(\tau, m, e_1, e_2) \) transforms as

\[ W_N(-\frac{1}{\tau}, \frac{m}{\tau}, \frac{e_1}{\tau}, \frac{e_2}{\tau}) = \prod_{k=1}^{N} e^{\frac{\tau}{8} (2m^2 + 2e_1 (k - 1)e_2^2 - k^2 e_2^2)} \frac{\theta(-m \pm \epsilon_2 \pm (k - 1)e_2)}{\theta(-e_1 - e_2) \theta(e_1 - (k - 1)e_2)} \]

(40)

This shows that it is not modular covariant. However in the NS limit \( e_2 \rightarrow 0 \), \( W_N(\tau, m, e_2) \) reduces to the following expression

\[ W_N(\tau, m, e_2)^{\text{NS}} = -\frac{1}{\eta(\tau)^2} \theta_1(-e_2) \prod_{k=1}^{N} \left[ \theta_1(-k \epsilon_2) \theta_1(-k(1-\epsilon_2)) \right] \]

\[ \times \left\{ \sum_{j=1}^{N} \theta_1^j(-m + 2\epsilon_2 - m) \theta_1(-m - 2\epsilon_2) \prod_{k=1, k \neq j}^{N} \theta_1(-m + 2\epsilon_2 - 2\epsilon_2) \right\} \]

(41)

Multi-wrappings contribution

if \( n > 1 \) denotes the number of wrappings there are two choices

(a) \( n \) contains repeated prime factors

In this case there will be no contributions to the Free energy from multiwrappings.

(36)

(b) \( n \) is equal to the product of \( k \) distinct prime factors

For this case the generalised expression for \( W_N(\tau, m, e_1, e_2) \) is given as

\[ W_{n,N}(\tau, m, e_1, e_2) = \prod_{k=1}^{N} \left[ \frac{\theta(-m \pm \epsilon_2 \pm n(k - 1)e_2)}{\theta(-e_1 - e_2) \theta(-e_1 - n(k - 1)e_2)} \right] \]

This result confirms that fact that the correct objects to decompose for multiple M2-branes configurations are the components \( Z_{W_3, W_4, \ldots} \) and not the free energies or BPS degeneracies \( F_i \).

2.3. Mixed Partitions

\[ F_{120000}, F_{12120}, F_{21212} \]

\[ Z_{121200}(\tau, m, e_1, e_2) - Z_{212000}(\tau, m, e_1, e_2) = Z_{120000}(\tau, m, e_1, e_2) W_{12}(\tau, m, e_1, e_2) \]

(42)

where

\[ \theta_1 \left( \frac{1}{2} (-2m + e_1 + 3e_2) \right) \theta_1 \left( \frac{1}{2} (-m - e_1 - 3e_2) \right) \]

\[ \theta_1 \left( \frac{1}{2} (-2m - e_1 + e_2) \right) \theta_1 \left( \frac{1}{2} (-2m + e_1 + e_2) \right) \times \theta_1 \left( \frac{1}{2} (-2m + e_1 - e_2) \right) \]

\[ \theta_1 \left( \frac{1}{2} (-m - e_1 - 3e_2) \right) \theta_1 \left( \frac{1}{2} (-m - e_1 - 3e_2) \right) \]

(43)

\[ Z_{V_1, V_2} \quad \text{for} \quad V_1 = \{1, 1, 1\}, V_2 = \{3\}, V_3 = \{2, 1\} \]

A non-trivial example of mixed Young diagrams case is \( Z_{V_1, V_2} \), where \( V_1 \) is the fully symmetric Young diagram, \( V_2 \) is the fully antisymmetric Young diagram, and \( V_3 = \{2, 1\} \). Note that \( |V_1| = |V_2| = |V_3| = 3 \) in
this case we get

$$Z_{V_1V_2V_3}(\tau, m, e_1, e_2) = Z_{V_1}(\tau, m, e_1, e_2)Z_{V_2}(\tau, m, e_1, e_2)$$

$$- Z_{V_1V_2}(\tau, m, e_1, e_2)Z_{V_3}(\tau, m, e_1, e_2) + Z_{V_1}(\tau, m, e_1, e_2)Z_{V_2}(\tau, m, e_1, e_2)Z_{V_3}(\tau, m, e_1, e_2) =$$

$$\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\times \theta_i(\frac{1}{2}(5e_1 - 3e_2 - 2m))\theta_i(\frac{1}{2}(5e_1 - 3e_2 - 2m))\theta_i(\frac{1}{2}(5e_1 - 3e_2 - 2m))\theta_i(\frac{1}{2}(5e_1 - 3e_2 - 2m))\theta_i(\frac{1}{2}(5e_1 - 3e_2 - 2m))\theta_i(\frac{1}{2}(5e_1 - 3e_2 - 2m))$$

$$\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\times \theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\times \theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\times \theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\times \theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\times \theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

$$\times \theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))\theta_i(\frac{1}{2}(3e_1 - 2e_2 - 2m))$$

This is in line with the statement given in [20] that if there are \( k \) partitions \( \alpha_1, ..., \alpha_k \) of the same size i.e. \( |\alpha_1| = \ldots = |\alpha_k| = n \) then it is the case that:

$$Z_{\alpha_1\alpha_2...\alpha_k} = \text{all possible ways of factoring}$$

$$+ Z_\beta W_{\alpha_1\alpha_2...\alpha_k\beta}$$

where \( \beta \) is the result of fusing [8] the partitions \( \{\alpha_1, \alpha_2, ..., \alpha_k\} \).

**General configuration**

In the general configuration we consider \( r + 1 \) M5-branes with \( k_i, i = 1, \ldots, r M2\)-branes between \( i\)-th and \((i+1)\)-th M5-branes. The Young diagrams labelling the M2-branes are in antisymmetric representations. The corresponding components of the elliptic genus \( Z_{k_1k_2...k_r\ldots k_2k_1}^{\text{XXX}000...0} \) have following recursive pattern:

$$Z_{k_1k_2...k_r\ldots k_2k_1}(\tau, m, e_1, e_2) = Z_{k_1k_2...k_r\ldots k_2k_1}(\tau, m, e_1, e_2) + Z^{\text{XXX}000...00}_{k_1k_2...k_r\ldots k_2k_1}(\tau, m, e_1, e_2)$$

(46)
\[ + \left\{ \prod_{l=1}^{k} \theta_1 \left( -m + \frac{q}{2} - e_2(k_2 - l + \frac{1}{2}) \right) \theta_1 \left( -m + \frac{q}{2} - e_2(l - \frac{1}{2}) \right) \prod_{l=1}^{l} \theta_1 \left( -m + \frac{q}{2} - e_2(k_3 - l + \frac{1}{2}) \right) \theta_1 \left( -m + \frac{q}{2} - e_2(k_3 - l + \frac{1}{2}) \right) \right\}^{3} \]

\[ = \left\{ \prod_{l=1}^{k} \theta_1 \left( -m + \frac{q}{2} - e_2(k_2 - l + \frac{1}{2}) \right) \theta_1 \left( -m + \frac{q}{2} - e_2(l - \frac{1}{2}) \right) \prod_{l=1}^{l} \theta_1 \left( -m + \frac{q}{2} - e_2(k_3 - l + \frac{1}{2}) \right) \theta_1 \left( -m + \frac{q}{2} - e_2(k_3 - l + \frac{1}{2}) \right) \right\}^{2} \]

\[ = Z_{k_1, k_2, \ldots, k_l} \langle \tau, m, e_1, e_2 \rangle W_{k_1, \ldots, k_l}^{2} \langle \tau, m, e_1, e_2 \rangle \]  

(47)

2.4. Fully symmetric configuration of Young diagrams

The symmetric configuration of Young diagram is the complex conjugate of the antisymmetric configuration and this amounts to replacing \( e_1 \) with \( e_2 \) and vice versa. The results obtained in the previous section remains valid in this situation if we make the replacement

\[ e_2 \leftrightarrow e_1 \]

(48)

\[ W_N(\tau, m, e_1, e_2) = \prod_{k=1}^{N} \left[ \frac{\theta_{1}(m + \frac{e_1}{2} + \frac{k-1}{2} e_l)}{\theta_{1}(k-1)\theta_{1}(e_1)} \right] - \prod_{k=1}^{N} \left[ \frac{\theta_{1}(m + \frac{e_1}{2} + \frac{k-1}{2} e_l)}{\theta_{1}(k-1)\theta_{1}(e_1)} \right] \]

(49)

For finite \( N \), in the NS limit \( e_2 \to 0 \) we get

\[ W_N(\tau, m, e_2)^{NS} = - \frac{1}{\eta(\tau)^{3} \theta_{1}(e_1)} \prod_{k=1}^{N} \theta_{1}(k-1) \theta_{1}(\tau) \theta_{1}(\tau - \frac{1}{2} e_1) \]

\[ \times \left( \sum_{k=1}^{N} \theta_{1}(m + \frac{2k-1}{2} e_1) \theta_{1}(m - \frac{2k-1}{2} e_1) \right) \]

(50)

2.5. Remark: Some comments on the algebra of holomorphic curves and the recursive structure

Gopakumar and Vafa reformulated [9, 10] the topological string amplitudes focussing on the target space perspective. The 5d \( N = 1 \) supersymmetric gauge theory, for a given M-theory CY3-fold compactification, has BPS particles. These BPS particles correspond to the M2-branes wrapped on holomorphic curves in the CY3-fold. The quantum numbers of the BPS particles are given by the curve class \( \Sigma \in H_2(CY3, Z) \) and the spin content of the 5d little group of the massive particles \( (j_R, j_L) = SU(2)_R \times SU(2)_L \). The particle content with charge \( \Sigma \) and spins \( (j_R, j_L) \) is an invariant for a non-compact CY3-fold and is denoted by \( N_{\Sigma}^{(j_R, j_L)} \). The moduli space furnished by the D2-branes wrapped on \( \Sigma \) is topologically non-trivial and the number of its cohomology classes is equal to \( N_{\Sigma}^{(j_R, j_L)} \). The explicit form

\[ \ldots \]

\[ \ldots \]

\[ \ldots \]

\[ \ldots \]
of the topological string partition function in given by
\[
Z(\omega, \epsilon) = \prod_{\Sigma \in H_{ CY3}} \prod_{\mu} \left(1 - q^{2k+1}(m+1)N_{\Sigma}^{(\mu, \mu)}\right)
\times \prod_{k=-\mu}^{\mu} \prod_{m=0}^{m} (1 - q^{2k+1}(m+1)N_{\Sigma}^{(\mu, \mu)})^{2}
\]
(50)

where \(Q^L = e^{- Tig} - 1\) and \(Q^R = e^{- Tig}\) and \(\omega\) is the Kähler form on the CY3-fold. It was shown in [10] that for \(j_{\mu} = 0\) the partition function \(Z(\omega, \epsilon)\) counts the states in a Hilbert space. It is interesting to note that for a given particle content \(N_{\Sigma}^{(\mu, \mu)}\), the above partition function \(Z(\omega, \epsilon)\) can be written as an index as
\[
Z = Tr_{\mathcal{H}}(1)^2(j_{\mu} + j_{\nu}) q^2 \epsilon e^{- Tig}
\]
(51)

where \(\mathcal{H}\) is the quantized Hilbert subspace containing holomorphic modes of the BPS fields and \(T_5\) is the Hamiltonian of the theory. In other words the topological string partition function can be interpreted as the counting the holomorphic (components of the) BPS states in the quantized Hilbert space.

Moreover as shown in section 3 in the computation of the M-strings partition function one has to use fixed point theorems. For that purpose it is necessary to determine the equivariant weights of certain vector bundles on the M-strings moduli space. In the description of the vector bundles the Ext-groups appear as an appearance. These groups appear in the counting problem of open string states between the D-branes wrapped on the holomorphic submanifolds.

It was shown in [11] that the free energies \(F^1, F^2, \ldots, F^n\) for a configuration of finitely separated \(N + 1\) M5-branes with a single M2-brane stretched between consecutive M5-branes, is reducible and recursive such that
\[
F^{1,1,\ldots,1} (\tau, m, e_1, e_2) = W (\tau, m, e_1, e_2) Z^{N-1} F^{1,0,\ldots,0} (\tau, m, e_1, e_2)
\]
(52)

This shows that the factor of \(W (\tau, m, e_1, e_2)\) appears every time a single M5-brane is removed from the configuration \(- \tau - M2 - M5 - M2 \ldots\). This recursive structure, as shown in [12], indicates that the degrees of freedom corresponding to the \(F^{1,1,\ldots,1} (\tau, m, e_1, e_2)\) can be obtained from \(F^{1,0,\ldots,0} (\tau, m, e_1, e_2)\) up to the universal factor \(W (\tau, m, e_1, e_2)\). A similar interpretation is expected for the generalised \(W_N (\tau, m, e_1, e_2)\) computed in section 2.

The M-string configuration when lifted to the higher dimensional F-theory corresponds [14] to an elliptic CY3-fold in which a D3-brane wraps a \(P^1\) whose normal bundle is \(\mathcal{O}(-2)\). The configuration of multiple parallel M5-branes with M2-branes stretching between them corresponds to D3-branes wrapping a chain of \(P^1\)’s with \(\mathcal{O}(-2)\) normal bundles. The elliptic CY3-fold is a resolved \(A_{N-1}\) fibration over the \(T^2\). In this set up the M5-branes correspond to a holomorphic \((-2)\) curve. In the case under consideration the holomorphic cycles correspond to the positive roots of the gauge group \(SU(N)\) of the 5d \(\mathcal{N} = 1^+\) gauge theory. Recalling that for two holomorphic curves \(C_1, C_2\) one can compute the \((C_1 + C_2)^2 = C_1 C_1 + C_2 C_2 + 2 C_1 C_2\) when their self intersections \(C_1 C_1\) and \(C_2 C_2\) is known. For the special case of \(C_1 C_1 = -2\) and \(C_2 C_2 = -2\) and \(C_1\) intersecting \(C_2\) at a single point i.e. \(C_1 C_2 = 1\) we get \((C_1 + C_2)^2 = -2 - 2 + 2(1) = -2\). Generalizing this to a chain of \((-2)\) curves \(C_1, C_2, \ldots, C_N\) in which \(i\)-th curve intersects only \((i - 1)\)-th and \((i + 1)\)-th curves i.e. \(C_i C_{i+1} = -2, i = 1, \ldots, N\) and \(C_1 C_N = 1\) with all other intersections equal to zero it is easy to see that
\[
(C_1 + C_2 + \ldots + C_N)^2 = -2
\]
(53)

It is interesting to speculate that this property of the \((-2)\) curves mimics the result [12] with the universal factor \(W (\tau, m, e_1, e_2)\) playing the role of the identity element.

As shown in the previous sections, for multiple M2-branes between consecutive M5-branes, instead of the free energies it is the elliptic genera that carry the recursive structure. It will be interesting to elaborate on this phenomena in the framework of F-theory.

3. DOMAIN WALL DEGREES OF FREEDOM: COUPLED SUPERSYMMETRIC WZW MODELS

We begin this section by reviewing the supergroup WZW models. The supergroup WZW model [16,17] is described by the maps \(f : \Sigma \rightarrow (supergroup)\) from a two dimensional Euclidean Riemann surface \(\Sigma\) to the supergroup \(SG\) and its dynamics is given by the action
\[
S [f] = \frac{k}{8\pi} \int_{\Sigma} d^2 z (f^{-1} \partial^a f \partial_a f) - \frac{ik}{2\pi} \int_{\Sigma} d^2 z \epsilon^{\mu \nu \lambda} (f^{-1} \partial^\mu f, f^{-1} \partial^\nu f, f^{-1} \partial^\lambda f)
\]
(54)

where \(M\) is a three-manifold with \(\Sigma\) as its boundary and \(k \in \mathbb{Z}\) is the level. The symmetry group \(SG(\tau) \times SG(\bar{\tau})\) that generates the left and right actions is defined by
\[
f(z, \bar{\tau}) \rightarrow A(z)f(z, \bar{\tau}) A^{-1}(z)
\]
(55)

where \(A(z)\) and \(A(\bar{\tau})\) denote the arbitrary SU(2)-valued functions of the complex variables \(z\) and \(\bar{\tau}\). Note that the equation (55) is invariant under the transformation given by (55). The conserved currents for this symmetry are given by
\[
J (z) = J^a (\tau) T^a = -ik \partial_z f^{-1}
\]
(56)

with the generators of the Lie superalgebra \(SG\) denoted by \(T^a\). The OPE of the generators \(J^a\) is given by
\[
J^a (z) J^b (w) \sim k T^a T^b \frac{(z-w)^2}{(z-w)^2} + \frac{T^a T^b J^c (w)}{z-w}
\]
(57)

As a consequence of the OPE we have the following commutation relations which define the affine Lie superalgebra \(SG\)
\[
[J^a_{\tau+m}, J^b_{\tau+m}] = [T^a, T^b] J^c_{\tau+m} + m (T^a, T^b) \partial_{\tau+m} k
\]
(58)

On the boundary of an M2-brane the description by the ABJM model gives rise to the WZW model. The content of the ABJM model can be described in terms of type IIB branes configurations. To this end, see Fig. 5. note that the T-duality operation on D3-branes wrapped on a circle gives rise to D2-branes in type IIA. These D2-branes can be lifted to the M2-branes in M-theory. To get the required contents of Chern-Simons description of the ABJM theory, the D3-branes are arranged so as to intersect two NS5-branes along the circle. Moreover the \(k\) D5-branes are added to this configuration as summarised in the table given in fig.5. The \(x^6\) direction is compact with period \(2\pi R\), with the two NS5-branes located at \(x^6 = 0\) and \(x^6 = \pi R\). \(x^6 = 0\) is the locus of D5-branes. Resolving the intersections of the NS5-brane with the \(k\) D5-branes produces a \((p,q)\) 5-brane web. The resulting theory is super Yang-Mills with massive chiral multiplets. Integrating out the chiral fields gives rise to the Chern-Simons theory. Finally the T-duality operation along \(x^6\) followed by the lift to \(11\)-dimensions gives rise to M2-branes spanning \((x^0, x^1, x^2)\). Under the T-duality the M-branes
turn into KK-monopoles and D6-branes. The low energy description is thus give by M2-branes probing $\mathbb{C}^4 / \mathbb{Z}_k$.

In summary, the configuration of N $D^3$-branes and N $D^{-3}$-branes that are stretched between coincident NS5- and NS5'--branes can be lifted to the M-theory configuration M5-N M2-M5' to give a GL(1|N) WZW model. In the same way the configuration Fig.4 M5-N M2-M5' will give GL(2|N) × GL(1|M) WZW model with additional bi-fundamental matter content.

\[ \text{Figure 5: toric diagram for the open string wave function } \mathcal{W}_{V_{i+1}}, \]

Red lines denote the periodicity of the vertical side.

The open topological string wave function $\mathcal{W}_{V_{i+1}}$ which is a building block of the partition function \([11]\) is also the counting function for BPS excitations corresponding to the intersecting configurations of M2- and M5-branes. The topological open string wavefunction takes the form \([11]\)

\[ \mathcal{W}_{V_{i+1}}(Q, Q_0, t, q) = \frac{|\nu_{m+1}|^2}{2} q^{-|\nu_m|^2/2} \sum_{\nu_k \in \mathbb{Z}_k} \mathcal{Z}_{V_{i+1}}(t^{-1}, q^{-1}) \prod_{j=1}^{\nu_k} (1 - Q_j^{-1})^{-1} \]

We can rewrite the factor $W_{k_1 k_2 \ldots k_n}$, given in eq. \([47]\), in terms of the open topological string wave function $\mathcal{W}_{V_{i+1}}$. Using the definition of $Z_{k_1 k_2 \ldots k_n}$ \([4]\) as

\[ Z_{k_1 k_2 \ldots k_n} = (-1)^{k_1 + k_2 + \ldots + k_n} \sum_{V_{i+1} = V_{i+1}} \mathcal{W}_{V_1} \mathcal{W}_{V_2} \mathcal{W}_{V_3} \ldots \mathcal{W}_{V_n} \] (60)
Moreover the non-zero modes contribution of the fermions as right moving fermionic fields and bosonic zero modes as respectively. The dynamics is given by the Lagrangian
\[
S = \frac{k}{8\pi} \int d^2z \sqrt{\text{det} g} \left( \partial_\sigma \bar{\psi} \gamma^\sigma \partial_\sigma \psi - \text{det} g \right)
\]
where B is the 3-manifold whose boundary is the 2d worldsheet. \( h_{ij} \) is the worldsheet metric, \( D_z = \partial + [A_z, \cdot] \) and \( D_\tau = \partial_t + [A_\tau, \cdot] \) are the covariant derivatives and the integer k is the level. After the identification of a global U(1) that is part of the left moving \( N = 2 \) algebra, the charge assignment of the fields is given by
\[
\bar{\psi}_+ \to e^{i\pi \tau/2} \bar{\psi}_+,
\]
\[
\bar{\psi}_- \to e^{i\pi \tau/2} \bar{\psi}_-,
\]
\[
g \to -i \frac{\gamma}{(k+2)} (Ug + gU),
\]
\[
A_\tau \to A_\tau
\]
where \( U \in \text{Lie SU}(2) \) denotes the generator of the U(1) \( \subset \text{SU}(2) \) which is gauged. This allows to show that the elliptic genus of the gauged. This allows to show that the elliptic genus of the

- **contribution of the fermionic and bosonic zero modes:**
\[
(1 - Q_m q^{\nu_m + \frac{i}{2}} q^{\nu_m + \frac{i}{2}}) \frac{(1 - q^{-\nu_m + \frac{i}{2}} q^{\nu_m + \frac{i}{2}})}{(1 - q^{-\nu_m + \frac{i}{2}} q^{\nu_m + \frac{i}{2}})}
\]

- **contribution of the fermionic and bosonic non-zero modes:**
\[
(1 - Q_m q^{\nu_m + \frac{i}{2}} q^{\nu_m + \frac{i}{2}}) \frac{(1 - q^{-\nu_m + \frac{i}{2}} q^{\nu_m + \frac{i}{2}})}{(1 - q^{-\nu_m + \frac{i}{2}} q^{\nu_m + \frac{i}{2}})}
\]

- **contribution from the phase factors:**
\[
(1 - \frac{|\nu_m + 1|^2}{Q_m})
\]

Note that for the second WZW model the phase contribution changes from \( q^{-\frac{1}{2}} \) to \( q^{-\frac{1}{2}} \) i.e. from a t-factor to a q-factor. Similarly the second factor in eq.(62) also describes a WZW model. Recall that the partition function \( Z_N(\tau, m, t_f, t_f, \ldots t_f, \epsilon_1, \epsilon_2) \) of \( N \) parallel and separated M5-branes with M2-branes stretched between them can alternatively be written in terms of the normalised open topological string wavefunctions \( D_{\nu_1, \nu_2, \tau} (Q, t, Q, t) \) as
\[
Z_N(\tau, m, t_f, t_f, \ldots t_f, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \ldots, \nu_N} (\prod_{i=1}^{N-1} (1 - Q_m \nu_i)) \times D_{\nu_1, \nu_2, \nu_3, \nu_4} (Q_t, Q_t, t, Q_t) \times D_{\nu_2, \nu_3} (Q_t, Q_t, t, Q_t) \times D_{N-1, \nu_1} (Q_t, Q_t, t, Q_t)
\]

This form of the partition function allows an interpretation in terms of \( N \) domain walls interpolating between the M2-branes vacua. In terms of the supersymmetric WZW model, we can say that the partition function is a superposition of the wavefunctions of a chain of coupled supersymmetric WZW models. The centre of mass motion of the multiple M-strings as well as their mutual dynamics is encoded in this wave function. For example for the case of two M-strings it involves their centre of mass motion as well as their motion relative to each other. The components of the elliptic genus \( Z_{\nu_1, \ldots, \nu_N} \) are related to the open topological string wave function. For example
\[
Z_{22} - Z_{20} = \nu_2 \nu_2 (\nu_2 - \nu_2 \nu_2)
\]
\[
Z_{33} - Z_{23} = \nu_3 \nu_3 (\nu_3 - \nu_3 \nu_3)
\]
\[
Z_{1212} - \nu_2 (Z_{20} \nu_1, \nu_2) (Z_{21} \nu_1, \nu_2) (Z_{20} \nu_1) (Z_{21} \nu_1)
\]

In other words the universal factors \( W_2 (\tau, m, \epsilon_1, \epsilon_2) \), \( W_5 (\tau, m, \epsilon_1, \epsilon_2) \) and \( W_{12} (\tau, m, \epsilon_1, \epsilon_2) \) for these M5-M2 branes configurations can be expressed in terms of open topological wavefunction in eq.(59) as
\[
W_2 (\tau, m, \epsilon_1, \epsilon_2) = (\nu_2 - \nu_2 \nu_2)
\]
\[
W_5 (\tau, m, \epsilon_1, \epsilon_2) = (\nu_3 - \nu_3 \nu_3)
\]
\[
W_{12} (\tau, m, \epsilon_1, \epsilon_2) = (\nu_1) \nu_2 (\nu_2 - \nu_2 \nu_2)
\]
Similarly we can write for $W_{\beta}(\tau, m, \epsilon_1, \epsilon_2)$

$$W_{\beta}(\tau, m, \epsilon_1, \epsilon_2) = \left( W_{\beta N} - W_{\alpha N} W_{\beta \alpha} \right)$$

(77)

Recall that in a given M-theory vacuum the coupling constant $\tau$ is related to the radius of the circle $S^1$ parallel to the M5-brane worldvolume. Formally we can consider different coupling constants $\tau_i$ for different domain walls. Each $\tau_i$ is related to the circle $S^1$ parallel to the $i$-th M5-brane worldvolume. This M-theory set up can be dualized in type IIB strings to a 5d $\mathcal{N} = 1^*$ supersymmetric gauge theory living on a particular $(p, q)$ D5-NS5-branes web. The $\tau_i$ correspond to the gauge coupling constant of the supersymmetric gauge theories dual to corresponding M5-brane-M2-brane-M-string configurations. For these general cases for instance we can write

$$Z_{1212} - Z_{1220}Z_{0120} = \left( W_{\beta 1}^1(\tau_1) W_{12}(\tau_4) W_{20}(\tau_5) - W_{\alpha 1}^1(\tau_1) W_{12}(\tau_5) W_{20}(\tau_4) \right)$$

(78)

Comparing the last expression with equation (20), we see that the first $\tau$ term corresponds to the Chern-Simons theory with matter coupling given by the bi-vector $\nu$.

$$Z_{222} - Z_{220} Z_{020} Z_{002} - Z_{022} Z_{000} - Z_{020} Z_{002} - Z_{220} Z_{020} Z_{002} = \left( W_{\beta 2}^1(\tau_1) W_{20}(\tau_4) \left( W_{22}(\tau_2) - W_{02}(\tau_2) W_{20}(\tau_2) \right) \times (W_{22}(\tau_3) - W_{02}(\tau_3) W_{20}(\tau_3)) \right)$$

(79)

More generally we can write for $v_1 = v_2 = v_3$ the recursive relation for $Z_{v_1 v_2 v_3}$ as

$$Z_{v_1 v_2 v_3} = Z_{v_1 v_2} Z_{v_3} + Z_{v_1 v_3} Z_{v_2} + Z_{v_2 v_3} Z_{v_1} - W_{v_1 v_2} W_{v_3} - W_{v_1 v_3} W_{v_2} - W_{v_2 v_3} W_{v_1}$$

(80)

Comparing the last expression with equation (49), we see that the first factor $W_{v_1 v_2} W_{v_3}$ is the result of fusing all the partitions, the second factor $W_{v_1 v_3} W_{v_2}$ appears when we fuse the partitions $v_1, v_2$ along with the removal of the second M5-brane and the third factor $W_{v_2 v_3} W_{v_1}$ appears when we remove the third M5-brane fusing $v_2, v_3$.

A generalization of $Z_{v_1 v_2 v_3 v_4 \ldots v_k}$ for $v_1 = v_2 = v_3 = \ldots = v_k$ can be expressed as

$$Z_{v_1 v_2 v_3 v_4 \ldots v_k} = Z_{v_1 v_2 v_3 v_4} + Z_{v_1 v_2 v_3 v_4} + Z_{v_1 v_2 v_3 v_4} + \ldots + Z_{v_1 v_2 v_3 v_4}$$

$$+ Z_{v_1 v_2 v_3 v_4 v_5} + Z_{v_1 v_2 v_3 v_4 v_5} + \ldots + Z_{v_1 v_2 v_3 v_4 v_5}$$

$$= - W_{v_1 v_2} W_{v_3} W_{v_4} \ldots W_{v_k}$$

(81)

4. ABJM MODEL VS M-THEORY

The ABJM model is defined by a 3d $\mathcal{N} = 6$ supersymmetric $U(N)_k \times U(N)_{-k}$ Chern-Simons theory with matter coupling given by the bi-fundamental scalars $Z_\alpha$ and spinors $\Psi^a$ with $SU(4)$ R-symmetry index $\alpha$. The low energy 2d gauge theory corresponds to the reduction of the worldvolume theory of M2-branes to two dimensions with the boundary conditions provided by the M5-branes. For details we refer the reader to [23]. This 2d theory is termed as ABJM slab and is identical to the $\mathcal{N} = (4,4)$ super Yang-Mills having $SU(2)^3$ R-symmetry and the gauge coupling $g^2_{\beta \alpha}$ is determined by the distance between the M2-branes stack and the $Z_\alpha$ orbifold singularity of the transverse space $C^2/Z_\alpha$. This 2d theory is special in the sense that the M2-branes do not sit on top of the $Z_\alpha$ singularity. This avoids the appearance of Nahm poles. Moreover the Ramond-Ramond boundary conditions used in the definition of elliptic genus project out the massive modes corresponding to the KK modes. The elliptic genus of the 2d gauge theory with non-zero coupling constant $g^2_{\beta \alpha} > 0$ turns out to be the same as that of $\mathcal{N} = (4,4)$ super Yang Mills.

The M2-M5 branes intersection is described by the boundary conditions that preserve six supercharges and the $SU(2) \times SU(2) \times U(1)$ subgroup of the full R-symmetry group $SU(4)$. The two scalars $Z_1, Z_2$ are longitudinal to the M5-brane and form a doublet under one of the two $SU(2)$s. The other two scalars $Z_3, Z_4$ are transverse and form a doublet under the second $SU(2)$. Moreover under the $U(1)$ group the two doublets are oppositely charged. The boundary conditions on bosons and fermions are given as

$$\Psi^1 = \Psi^1 = \Psi^1 = \Psi^1 = 0, \quad \Psi^2 = \Psi^2 = \Psi^2 = \Psi^2 = 0$$

$$D_\mu Z_A = 0, \quad D_\nu Z_A = 2\pi i Z_1 Z_2^\kappa Z_1 Z_2 Z_1 Z_2 Z_1 Z_2$$

(82)

The boundary forces the gauge fields of the two $U(N)$s to be related as

$$F_{\mu \nu} Z_A = Z_4 F_{\mu \nu}$$

(83)

Moreover the variation of the ABJM action gives rise to a boundary term

$$\delta S = \frac{k}{4\pi} \int_{\text{boundary}} Tr(\alpha d\lambda - \alpha d\bar{\lambda})$$

(84)

which vanishes only if $A = \bar{\lambda}$ and $\alpha = \bar{\alpha}$. However if one takes $\lambda \neq \bar{\lambda}$ then the anomalous boundary term can be cancelled by introducing boundary fermions of one chirality coupled to $A$ and boundary fermions of the opposite chirality coupled to $\bar{\lambda}$. This effectively gives rise to WZW model degrees of freedom at the boundary and the gauge anomaly they generate cancels the anomalous term [84].

We first write down the expression of the elliptic genus of the 2d gauge theory obtained from the dimensional reduction of the ABJM model for the case $k = 1$ as considered in [23]

$$Z_{ABJM}^{\mathcal{T}^6} = \prod_{i=1}^{N} \frac{d\bar{w}_i d\bar{\epsilon}_i}{8\pi} \prod_{\epsilon, j} \frac{\theta_1(w_1 - w_j + m + \epsilon_+)}{\theta_1(w_1 - w_j + m - \epsilon_+)} \frac{\theta_1(w_1 - w_j + m + \epsilon_+)}{\theta_1(w_1 - w_j + m - \epsilon_+)}$$

(85)

However in our case the integral is finite with respect to the integration variables and no special regularization is required. We will use instead a prescription given in [23] for the case of $N = 2$.

For ABJM theory and for $N = 1$ we get the expression

$$Z_{ABJM}^{\mathcal{T}^2} = \frac{\theta_1(m + \epsilon_+)}{\theta_1(\epsilon_1)} \frac{\theta_1(m - \epsilon_+)}{\theta_1(\epsilon_2)}$$

(86)

This expression matches with its $Z_{M^{\text{string}}}^{\mathcal{T}^2}$ [1].
The $w_i, \bar{w}_i$ integrals are well defined and do not require any regularisation. So we can write normalised elliptic genus in the limit $Q_m = \sqrt{q}$ as

$$\hat{Z}_{ABJM} = \int \prod_{\ell=1}^{\infty} \frac{d\zeta}{2\pi i} \prod_{\ell \neq \ell'} \theta_i \left( \frac{\zeta + \frac{\ell}{\tilde{q}}}{\zeta - \frac{\ell}{\tilde{q}}} \right) \theta_i \left( \frac{\zeta + \frac{\ell'}{\tilde{q}}}{\zeta - \frac{\ell'}{\tilde{q}}} \right)$$

(88)

For 2d gauge theories containing the adjoint matter the poles contributing to the elliptic genus were found to be [25,26]

$$z_0 = t^q q^{-\gamma}$$

(89)

where $(x, y)$ are the coordinates of the $a$-th box in the Young diagram $\mu$ such that $|\mu| = N$ for all $N \geq 0$. Evaluating the residue of (88) on these poles we get

$$\hat{Z}_{ABJM} = \sum \Pi_{\ell=x,y}^{\infty} \Pi_{\ell=1}^{\infty} \theta_i \left( \frac{x - \frac{\ell}{\tilde{q}}} {y - \frac{\ell}{\tilde{q}}} \right) \theta_i \left( \frac{y - \frac{\ell}{\tilde{q}}} {x - \frac{\ell}{\tilde{q}}} \right) \prod_{\ell=1}^{\infty} \frac{\theta_i \left( \frac{\ell}{\tilde{q}} \right)} {\theta_i \left( \frac{\ell}{\tilde{q}} \right)}$$

(91)

This can be written in the canonical form by using the theorem (2.11) of reference [27]

$$\sum_{(x_1, y_1) \in Y} \prod_{i \neq j} t^{1-x_i} t^{1-y_j}$$

as

$$\hat{Z}_{ABJM} = \Pi_{(x_1, y_1) \in Y} \theta_i \left( \frac{\ell}{\tilde{q}} \right) \prod_{(x_2, y_2) \in Y} \theta_i \left( \frac{\ell}{\tilde{q}} \right) \prod_{(x_3, y_3) \in Y} \theta_i \left( \frac{\ell}{\tilde{q}} \right)$$

(92)

It is interesting to compare $\hat{Z}_{T^{\mu}}$ with $Z_{ABJM}$ given by

$$\hat{Z}_{T^{\mu}} = \sum_{(x_1, y_1) \in Y} \Pi_{(i,j) \in \mathcal{E}} \theta_i \left( \frac{\ell}{\tilde{q}} \right) \prod_{(x_2, y_2) \in Y} \theta_i \left( \frac{\ell}{\tilde{q}} \right) \prod_{(x_3, y_3) \in Y} \theta_i \left( \frac{\ell}{\tilde{q}} \right)$$

(93)

5. CONCLUSIONS

We have studied the structure of the free energies of M-strings. An interesting recursive structure in the free energies (BPS counting functions) was observed [4] for the configuration—M2-M5-M5-M5— of M2-M5 branes. We show that for configurations containing multiple M2-branes sandwiched between M5-branes the recursive structure in free energies is lost. Instead the coefficients $Z_{A_1A_2...A_n}$ in the expansion of partition function enjoy the recursive structure.

For completeness we also describe the M2-branes configurations with symmetric representations and mixed representations.

The partition functions of M2-branes configuration that enter the M-strings elliptic genera can also be interpreted as the vacuum-$n$ to vacuum-$(n+1)$ amplitude with M5-branes acting as the domain wall. The M5-brane domain wall act as the duality transformation that interpolates between the two vacua [3]. ABJM formulation of M2-branes theories allows a more direct study of the domain walls partition functions. We compute the elliptic genus of a dimensionally reduced 2d theory of ABJM slab model and compare it with the M-string computations. We find an interesting mismatch that can be explained in terms of the centre of mass motion of M2-M5 branes in the transverse space. The factor corresponding to the mismatch accounts for this centre of mass motion in the transverse space.

It will be interesting to study the WZW-topological string correspondence for more general backgrounds in M-theory.

CONFLICT OF INTEREST STATEMENT

The author declares no conflicts of interest.

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Appendix A DENOMINATOR FACTORS

$$D_{V_1,...,V_K}(\tau, M, E_1, E_2)$$

$$D_{134134134}(\tau, m, e_1, e_2) = \theta_1 \left( e_1 - m \right)^3 \theta_1 \left( m - 4e_2 \right)^3 \theta_1 \left( m - 3e_2 \right)^6$$

$$\theta_1 \left( m + e_1 - 3e_2 \right)^5 \theta_1 \left( m - 2e_2 \right)^6 \times \theta_1 \left( m + e_1 - 2e_2 \right)^6$$

$$\theta_1 \left( m - e_2 \right)^9 \theta_1 \left( m + e_1 - e_2 \right)^6$$

$$D_{134134000}(\tau, m, e_1, e_2) = \theta_1 \left( e_1 - m \right)^6 \theta_1 \left( m - 4e_2 \right)^2 \theta_1 \left( m - 3e_2 \right)^4$$

$$\theta_1 \left( m + e_1 - 3e_2 \right)^2 \theta_1 \left( m - 2e_2 \right)^4 \times \theta_1 \left( m + e_1 - 2e_2 \right)^4$$

$$\theta_1 \left( m - e_2 \right)^6 \theta_1 \left( m + e_1 - e_2 \right)^4$$

$$D_{134000000}(\tau, m, e_1, e_2) = \theta_1 \left( e_1 - m \right)^3 \theta_1 \left( m - 4e_2 \right)\theta_1 \left( m - 3e_2 \right)^2$$

$$\theta_1 \left( m + e_1 - 3e_2 \right) \theta_1 \left( m - 2e_2 \right)^2 \times \theta_1 \left( m + e_1 - 2e_2 \right)^2$$

$$\theta_1 \left( m - e_2 \right)^3 \theta_1 \left( m + e_1 - e_2 \right)^2$$

$$D_{232323}(\tau, m, e_1, e_2) = \theta_1 \left( e_1 - m \right)^6 \theta_1 \left( m - 3e_2 \right)^3 \theta_1 \left( m - 2e_2 \right)^6$$

$$\theta_1 \left( m + e_1 - 2e_2 \right)^3 \theta_1 \left( m - e_2 \right)^6$$

$$\theta_1 \left( m + e_1 - e_2 \right)^6$$
\[ D_{232308}(\tau, m, e_1, e_2) = \theta_1(e_1 - m)^4 \theta_1(-m - 3e_2)^2 \theta_1(-m - 2e_2)^4 \]
\[ \theta_1(-m + e_1 - 2e_2) \times \theta_1(-m - e_2)^4 \]
\[ \theta_1(-m + e_1 - e_2)^4 \]

\[ D_{232323}(\tau, m, e_1, e_2) = \theta_1(e_1 - m)^2 \theta_1(-m - 3e_2) \theta_1(-m - 2e_2)^2 \]
\[ \theta_1(-m + e_1 - 2e_2) \times \theta_1(-m - e_2)^2 \]
\[ \theta_1(-m + e_1 - e_2)^2 \]

References

[1] B. Haghighat, A. Iqbal, C. Kozçaz, G. Lockhart, and C. Vafa, “M-Strings,” Commun. Math. Phys. 334 no. 2, (2015) 779–842, arXiv:1305.6322 [hep-th].

[2] B. Haghighat, C. Kozçaz, G. Lockhart, and C. Vafa, “Orbifolds of M-strings,” Phys. Rev. D 89 no. 4, (2014) 046003, arXiv:1310.1185 [hep-th].

[3] S. Hohenegger and A. Iqbal, “M-strings, elliptic genera and \( N = 4 \) string amplitudes,” Fortsch. Phys. 62 (2014) 155–206, arXiv:1310.1325 [hep-th].

[4] S. Hohenegger, A. Iqbal, and S.-J. Rey, “M-strings, monopole strings, and modular forms,” Phys. Rev. D 92 no. 6, (2015) 066005, arXiv:1503.06983 [hep-th].

[5] S. Hohenegger, A. Iqbal, and S.-J. Rey, “Instanton-monopole correspondence from M-branes on \( S^1 \) and little string theory,” Phys. Rev. D 93 no. 6, (2016) 066016, arXiv:1511.02787 [hep-th].

[6] S. H. Katz and E. Sharpe, “D-branes, open string vertex operators, and Ext groups,” Adv. Theor. Math. Phys. 6 (2003) 979–1030, arXiv:hep-th/0208104.

[7] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk, and H. Verlinde, “A Massive Study of M2-brane Proposals,” JHEP 09 (2008) 113, arXiv:0807.1074 [hep-th].

[8] D. Gaio and E. Witten, “S-Duality of Boundary Conditions in N=4 Super Yang-Mills Theory,” Adv. Theor. Math. Phys. 13 no. 3, (2009) 721–896, arXiv:0807.3720 [hep-th].

[9] R. Gopakumar and C. Vafa, “M theory and topological strings. 1,” arXiv:hep-th/9809187.

[10] R. Gopakumar and C. Vafa, “M theory and topological strings. 2,” arXiv:hep-th/9812127.

[11] T. J. Hollowood, A. Iqbal, and C. Vafa, “Matrix models, geometric engineering and elliptic genera,” JHEP 03 (2008) 069, arXiv:hep-th/0310272.

[12] E. Sharpe, “Lectures on D-branes and sheaves,” 7, 2003, arXiv:hep-th/0307245.

[13] S. Hohenegger, A. Iqbal, and S.-J. Rey, “Self-Duality and Self-Similarity of Little String Orbifolds,” Phys. Rev. D 94 no. 4, (2016) 046006, arXiv:1605.02891 [hep-th].

[14] N. Del Zotto, J. J. Heckman, A. Tomasiello, and C. Vafa, “6d Conformal Matter,” JHEP 02 (2015) 054, arXiv:1407.6359 [hep-th].

[15] D. R. Morrison and W. Taylor, “Classifying bases for 6D F-theory models,” Central Eur. J. Phys. 10 (2012) 1072–1088, arXiv:1201.1943 [hep-th].

[16] T. Okazaki and D. J. Smith, “Topological M-strings and supergroup Wess-Zumino-Witten models,” Phys. Rev. D 94 no. 6, (2016) 065016, arXiv:1512.06646 [hep-th].

[17] T. Okazaki and D. J. Smith, “Mock modular index of M2-M5 brane systems,” Phys. Rev. D 96 no. 2, (2017) 026017, arXiv:1612.07565 [hep-th].

[18] E. Witten, “Elliptic Genus and Quantum Field Theory,” Commun. Math. Phys. 109 (1987) 525.

[19] E. Witten, “On the Landau-Ginzburg description of N=2 minimal models,” Int. J. Mod. Phys. A 9 (1994) 4783–4800, arXiv:hep-th/9304026.

[20] P. Berglund and M. Henningson, “Landau-Ginzburg orbifolds, mirror symmetry and the elliptic genus,” Nucl. Phys. B 433 (1995) 311–332, arXiv:hep-th/9401029.

[21] M. Henningson, “N=2 gauged WZW models and the elliptic genus,” Nucl. Phys. B 413 (1994) 73–83, arXiv:hep-th/9307040.

[22] D. Nemeschansky and N. P. Warner, “The Refined elliptic genus and Coulomb gas formulations of N=2 superconformal coset models,” Nucl. Phys. B 442 (1995) 623–654, arXiv:hep-th/9412187.

[23] K. Hosomichi and S. Lee, “Self-dual Strings and 2D SYM,” JHEP 01 (2015) 076, arXiv:1406.1802 [hep-th].

[24] A. Gadde and S. Gukov, “2d Index and Surface operators,” JHEP 04 (2013) 080, arXiv:1305.0266 [hep-th].

[25] A. Gadde, B. Haghighat, J. Kim, S. Kim, G. Lockhart, and C. Vafa, “6d String Chains,” JHEP 02 (2018) 143, arXiv:1504.04614 [hep-th].

[26] F. Benini, G. Bonelli, M. Poggio, and A. Tanzini, “Elliptic non-Abelian Donaldson-Thomas invariants of C^3,” arXiv:1807.0482 [hep-th].

[27] H. Nakajima and K. Yoshioka, “Instanton counting on blowup. 1.” Invent. Math. 162 (2005) 313–355, arXiv:math/0306198.