On 2D integro-differential systems. Stability and sensitivity analysis.

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Abstract In the paper a two-dimensional integro-differential system is considered. Using some variational methods we give sufficient conditions for the existence and uniqueness of a solution to the considered system. Moreover, we show that the system is stable and robust.

1 Introduction

We will denote by $Q$ the unit interval in $\mathbb{R}^2$, i.e.

$$Q = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1] \text{ and } y \in [0, 1]\}.$$ (1)

General continuous 2D differential system has the following form

$$\dot{z}_{xy}(x, y) = f(x, y, z(x, y), z_x(x, y), z_y(x, y)), \quad (2)$$

$$z(x, 0) = a(x), \quad z(0, y) = b(y), \quad a(0) = b(0). \quad (3)$$

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where \( f : Q \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( n \geq 1 \), \( a, b : [0, 1] \rightarrow \mathbb{R}^n \) are given functions.

The linear case of system (2) can be written as

\[
\begin{align*}
  z_{xy}(x, y) &= A_0(x, y)z(x, y) + A_1(x, y)z_x(x, y) + A_2(x, y)z_y(x, y) \\
                  &+ w(x, y),
\end{align*}
\]

where \( A_0, A_1, A_2 \) are some matrix functions of the dimension \( n \times n \), \( w \) is a given \( n \)-dimensional vector function and \( z \) satisfies boundary conditions (3). Continuous 2D systems correspond to the discrete model of Fornasini-Marchesini type, which has the following form (see [Fornasini and Marchesini, 1976])

\[
\begin{align*}
  z(i+1, j+1) &= A_0(i, j)z(i, j) + A_1(i, j)z(i+1, j) + A_2(i, j)z(i, j) \\
               &+ w(i, j),
\end{align*}
\]

\[
\begin{align*}
  z(i, 0) &= a(i), z(0, j) = b(j), a(0) = b(0),
\end{align*}
\]

\( i, j = 0, 1, 2, \ldots \).

Two-dimensional discrete systems (4)–(6) and continuous systems (2)–(3) play an essential role in mathematical modeling of many technical, physical, biological and other phenomena. For example, in the paper by Fornasini (see [Fornasini, 1991]) 2D space models of the form (5)–(6) were applied to the investigation of the process of pollution and self purification of the river. Application of the 2D discrete models to image processing and transmission were studied in the book of Bracewell (see [Bracewell, 1995]). The 2D continuous systems of the form (2)–(3) were adopted to investigation of the gas filtration model (see [Bors and Walczak, 2012]). Other applications of discrete and continuous 2D systems in the theory of automatic control, stability, robotics and optimization can be found in papers of Gakowski, Lam, Xu and Lin [Galkowski et al., 2003]; Paszke, Lam, Gakowski, Xu and Lin [Paszke et al., 2004]; Lomadze, Rogers and Wood [Lomadze et al., 2008]; Kaczorek [Kaczorek, 2001]; Dey and Kar [Dey and Kar, 2011]; Singh [Singh, 2008]; Idczak and Walczak [Idczak and Walczak, 2000] and in the monograph of Kaczorek [Kaczorek, 1985].

In the paper, we investigate two-dimensional integro-differential system of the form

\[
\begin{align*}
  z_{xy}(x, y) + f_1(x, y, z(x, y)) \\
  + \int_0^x \int_0^y (f_2(s, t, z(s, t)) + A_1(s, t)z_x(s, t) + A_2(s, t)z_y(s, t)) \, ds \, dt &= v(x, y)
\end{align*}
\]

with the following boundary conditions

\[
\begin{align*}
  z(x, 0) &= 0 \text{ for } x \in [0, 1] \text{ and } z(0, y) = 0 \text{ for } y \in [0, 1],
\end{align*}
\]

where \( f_1, f_2 : Q \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( A_1, A_2 : Q \rightarrow \mathbb{R}^n \) are given functions (for more details see section 3). We shall consider the above system in the space of absolutely continuous functions of two variables. The definition and basic properties of absolutely continuous functions defined on the interval \( Q \) are presented in section 2.
In the paper we prove, under assumptions (C1)–(C3) (see section 3), that for any square integrable function \( v \) system (7)–(8) possesses a unique solution \( z_v \) which continuously depends on \( v \) and the operator \( v \mapsto z_v \) is differentiable in the Fréchet sense, i.e. the considered system is well-posed and robust.

The proof of the main result is based on the global diffeomorphism theorem (Theorem 1). In the final part of the paper we give an example and compare the method used in the paper with the methods based on the contraction principle and the Schauder fixed point theorem.

2 Preliminaries

We begin with the following theorem on a diffeomorphism between Banach and Hilbert spaces

**Theorem 1** Let \( Z \) be a real Banach space, \( V \) be a real Hilbert space, \( F : Z \to V \) be an operator of \( C^1 \) class. If

1. for any \( v \in V \) the functional \( \varphi(z) = \frac{1}{2} \| F(z) - v \|_V^2 \) satisfies Palais-Smale condition ((PS)–condition),
2. for any \( v \in V \) the equation \( F'(z)h = v \) possesses a unique solution,

then

1. for any \( v \in V \) there exists exactly one solution \( z_v \in Z \) to the system \( F(z) = v \),
2. the operator \( V \ni v \mapsto z_v \in Z \) is differentiable in the Fréchet sense.

In other words, the operator \( F \) is a diffeomorphism between Banach space \( Z \) and Hilbert space \( V \).

We recall that functional \( \varphi \) satisfies (PS)–condition if whenever there is a sequence \( \{z_n\} \subset Z \) with \( |\varphi(z_n)| \leq \text{const} \) and \( \varphi'(z_n) \to 0 \) in \( Z^* \), then in the closure of the set \( \{z_n : n \in N\} \), there is some point \( \bar{z} \) where \( \varphi'(\bar{z}) = 0 \) (see [Aubin and Ekeland, 2006]).

From the bounded inverse theorem it follows that for any \( z \in Z \) there exists a constant \( \alpha_z > 0 \) such that \( \|F'(z)h\|_V \geq \alpha_z \|h\|_Z \). Therefore it follows easily that the above theorem is equivalent to [Idczak et al., 2012, Theorem 3.1], with \( f = F \).

Let us denote by \( AC(Q, \mathbb{R}^n) \) the space of absolutely continuous vector functions \( z = (z^1, z^2, \ldots, z^n) \) defined on the interval \( Q \). The geometrical definition of the space \( AC(Q, \mathbb{R}) \) can be found in papers [Berkson and Gillespie, 1984] and [Walczak, 1987]. In this paper we need necessary and sufficient conditions for \( z : Q \to \mathbb{R}^n \) to be absolutely continuous on \( Q \) i.e. \( z \in AC(Q, \mathbb{R}^n) \). We have the following theorem (see [Berkson and Gillespie, 1984, Walczak, 1987]).

**Theorem 2** A function \( z \) belongs to the space \( AC(Q, \mathbb{R}^n) \) if and only if there exist functions \( l \in L^1(Q, \mathbb{R}^n) \), \( l^1, l^2 \in L^1([0,1], \mathbb{R}^n) \) and a constant \( c \in \mathbb{R}^n \) such that

\[
    z(x, y) = \int_0^x \int_0^y l(s, t)dsdt + \int_0^x l^1(s)ds + \int_0^y l^2(t)dt + c.
\]
Moreover the function $z$ possesses partial derivatives $z_x$, $z_y$, $z_{xy}$, for a.e. $(x, y) \in Q$ and
\[
z_x(x, y) = \int_0^y l(x, t) \, dt + l_1(x),
\]
\[
z_y(x, y) = \int_0^x l(s, y) \, ds + l_2(y),
\]
\[
z_{xy}(x, y) = l(x, y).
\]

Theorem 2 follows directly from [Berkson and Gillespie, 1984, Theorem 4] and [Šremr, 2010, Proposition 3.5] (see also [Walczak, 1987, Theorem 2], [Walczak, 1998, Theorem 1]).

It is easy to check that if the function $z$ satisfies homogeneous boundary conditions, i.e. $z(x, 0) = 0$ for $x \in [0, 1]$ and $z(0, y) = 0$ for $y \in [0, 1]$ then $l_1 = 0$, $l_2 = 0$, $c = 0$ and consequently we can write
\[
z(x, y) = \int_0^x \int_0^y l(s, t) \, ds \, dt = \int_0^x \int_0^y z_{xy}(s, t) \, ds \, dt. \tag{9}
\]

By $AC^2_0(Q, \mathbb{R}^n)$ we shall denote the space of absolutely continuous functions on the interval $Q$ which satisfy homogeneous boundary conditions $z(x, 0) = z(0, y) = 0$ for $x, y \in [0, 1]$ and such that $z_{xy} \in L^2(Q, \mathbb{R}^n)$. The space $AC^2_0$ is a Hilbert space with the inner product given by formula
\[
(\bar{z}, \bar{z}) = \int_0^1 \int_0^1 (z_{xy}(x, y), z_{xy}(x, y)) \, dx \, dy. \tag{10}
\]

In the space $AC^2_0(Q, \mathbb{R}^n)$ we introduce two norms. The first one is a classical norm given by the formula
\[
\|z\| = \left( \int_0^1 \int_0^1 |z_{xy}(x, y)|^2 \, dx \, dy \right)^{\frac{1}{2}} = \|z_{xy}\|_{L^2} \tag{11}
\]
and the second is defined by the integral with exponential weight
\[
\|z\|_{AC^2_0,m} = \left( \int_0^1 \int_0^1 e^{-m(x+y)} |z_{xy}(x, y)|^2 \, dx \, dy \right)^{\frac{1}{2}}, \quad m > 0. \tag{12}
\]

Exponential norm (12) was introduced by Bielecki in [Bielecki, 1956]. The space $AC^2_0(Q, \mathbb{R}^n)$ with norm (12) will be denoted by $AC^2_{0,m}(Q, \mathbb{R}^n)$.

It is easy to notice that
\[
e^{-2m}\|z\| \leq \|z\|_{AC^2_0,m} \leq \|z\|.
\]

Thus the norms given by formulas (11) and (12) are equivalent.

Similarly, in the space of square integrable functions on $Q$ we introduce two equivalent norms:
\[
\|v\| = \left( \int_0^1 \int_0^1 |v(x, y)|^2 \, dx \, dy \right)^{\frac{1}{2}}
\]
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\[ \|v\|_{L^2_m} = \left( \int_0^1 \int_0^1 e^{-m(x+y)} |v(x,y)|^2 \, dx \, dy \right)^{\frac{1}{2}}. \]  

(13)

The space of square integrable functions with norm (13) will be denoted by \( L^2_m(Q, \mathbb{R}^n) \).

### 3 Basic assumptions and lemmas

On the functions defining system (7) we assume that

(C1) the functions \( f^1(\cdot, \cdot, z), f^2(\cdot, \cdot, z) \) are measurable on \( Q \) for every \( z \in \mathbb{R}^n \) and \( f^1(x,y,\cdot), f^2(x,y,\cdot) \) are continuously differentiable on \( \mathbb{R}^n \) for a.e. \((x,y)\in Q\), the function \( A^1(x, \cdot) \) is differentiable for a.e. \( x \in [0, 1] \), the functions \( A^1, A^2, A^1_x, A^2_y \) are measurable on \( Q \) and essentially bounded on \( Q \), the function \( v \in L^2(Spring 

(C2) there exist a constant \( B > 0 \) and a function \( b \in L^2(Q, \mathbb{R}^+ \) such that

\[ |f^1(x,y,z)|, |f^2(x,y,z)| \leq B|z| + b(x,y) \]

and

\[ |A^1(x,y)|, |A^2(x,y)|, |A^1_x(x,y)|, |A^2_y(x,y)| \leq B \]

for \( z \in \mathbb{R}^n \) and a.e. \((x,y)\in Q\);

(C3) the functions \( f^1_z, f^2_z \) are bounded on bounded sets, i.e. for any \( \rho > 0 \) there exists a constant \( M_\rho \) such that

\[ |f^1_z(x,y,z)|, |f^2_z(x,y,z)| \leq M_\rho \]

for \((x,y)\in Q\) and \(|z| \leq \rho\).

In the following lemma we prove some estimates for functions from the space \( AC^2_0(Q, \mathbb{R}^n) \).

**Lemma 1** If the function \( z \in AC^2_0(Q, \mathbb{R}^n) \) then

\[ \|z\|_{L^2_m} \leq \frac{2}{m} \|z\|_{AC^2_{0,m}}, \]  

(14)

\[ \|w_0\|_{L^2_m} \leq \frac{2}{m} \|z\|_{AC^2_{0,m}}, \]  

(15)

\[ \|w_1\|_{L^2_m} \leq \frac{2}{m} \|z\|_{AC^2_{0,m}}, \]  

(16)

\[ \|w_2\|_{L^2_m} \leq \frac{2}{m} \|z\|_{AC^2_{0,m}}, \]  

(17)

where \( w_0(x,y) = \int_0^x \int_0^y |z(s,t)| \, ds \, dt, w_1(x,y) = \int_0^x \int_0^y |z_x(s,t)| \, ds \, dt, w_2(x,y) = \int_0^x \int_0^y |z_y(s,t)| \, ds \, dt \).
Remark 1 The norms $\| \cdot \|_{L_m^2}$ and $\| \cdot \|_{AC_{0,m}^2}$ are defined by (12) and (13) respectively.

**Proof** Let $z$ be an arbitrary function from the space $AC_{0}^{2}(Q,\mathbb{R}^{n})$. By (10), (12) and (13) we get

$$\| z \|_{L_m^2}^2 = \int_{0}^{1} \int_{0}^{1} e^{-m(x+y)} \left| \int_{0}^{x} \int_{0}^{y} z_{xy}(s,t) dsdt \right|^2 dxdy$$

$$\leq \int_{0}^{1} \int_{0}^{1} \left( e^{-m(x+y)} \int_{0}^{x} \int_{0}^{y} |z_{xy}(s,t)|^2 dsdt \right) dxdy$$

Integrating by parts we obtain successively

$$\int_{0}^{1} \left( \int_{0}^{1} \left( e^{-m(x+y)} \int_{0}^{x} \int_{0}^{y} |z_{xy}(s,t)|^2 dsdt \right) dx \right) dy$$

$$= \int_{0}^{1} \left( \left[ -\frac{1}{m} e^{-m(x+y)} \int_{0}^{x} \int_{0}^{y} |z_{xy}(s,t)|^2 dsdt \right]_{x=1}^{x=0} \right) dy$$

$$- \int_{0}^{1} \left( -\frac{1}{m} e^{-m(x+y)} \int_{0}^{y} |z_{xy}(x,t)|^2 dt \right) dx \right) dy$$

$$= \int_{0}^{1} \left( \left[ \frac{1}{m} e^{-m(x+y)} \int_{0}^{y} |z_{xy}(s,t)|^2 ds \right]_{y=1}^{y=0} \right) dx$$

$$= \int_{0}^{1} \left( \left[ \frac{1}{m} e^{-m(x+y)} \int_{0}^{y} |z_{xy}(x,t)|^2 dt \right]_{y=1}^{y=0} \right) dx$$

$$\leq \frac{4}{m^2} \int_{0}^{1} \int_{0}^{1} e^{-m(x+y)} |z_{xy}(s,t)|^2 dsdt$$

$$= \frac{4}{m^2} \| z_{xy} \|_{L_m^2}^2 = \frac{4}{m^2} \| z \|_{AC_{0,m}^2}^2.$$
Thus we proved inequality (14). By the above and applying the Cauchy-Schwarz inequality we get

\[ \|w_0\|_{L^2_m}^2 = \int_0^1 \int_0^1 e^{-m(x+y)} \left( \int_0^x \int_0^y |z(s,t)| ds dt \right)^2 dx dy \]

\[ \leq \int_0^1 \int_0^1 e^{-m(x+y)} \left( \int_0^x \int_0^y \left( \int_0^t |z_{xy}(s,\tau)| d\tau dt \right)^2 ds dt \right)^2 dx dy \]

\[ \leq \int_0^1 \int_0^1 \left( e^{-m(x+y)} \int_0^x \int_0^y |z_{xy}(s,t)| ds dt \right)^2 dx dy \]

\[ \leq \int_0^1 \int_0^1 \left( e^{-m(x+y)} \int_0^x \int_0^y |z_{xy}(s,t)|^2 ds dt \right)^2 dx dy \]

\[ \leq \frac{4}{m^2} \|z\|^2_{AC^2_{0,m}}. \]

Let us prove the next estimation. By (9) we have

\[ \|w_1\|_{L^2_m}^2 = \int_0^1 \int_0^1 e^{-m(x+y)} \left( \int_0^x \int_0^y |z_{xy}(s,t)| ds dt \right)^2 dx dy \]

\[ = \int_0^1 \int_0^1 e^{-m(x+y)} \left( \int_0^x \int_0^y \left( \int_0^t |z_{xy}(s,\tau)| d\tau dt \right)^2 ds dt \right)^2 dx dy \]

\[ \leq \int_0^1 \int_0^1 e^{-m(x+y)} \left( \int_0^x \int_0^y \left( \int_0^t |z_{xy}(s,t)| d\tau \right)^2 ds dt \right)^2 dx dy \]

\[ \leq \int_0^1 \int_0^1 e^{-m(x+y)} \left( \int_0^x \int_0^y |z_{xy}(s,t)| ds dt \right)^2 dx dy. \]

Integrating by parts as in (15), we get

\[ \|w_1\|_{L^2_m}^2 \leq \frac{4}{m^2} \|z_{xy}\|^2 = \frac{4}{m^2} \|z\|^2_{AC^2_{0,m}}. \]

The proof of (17) is similar. \(\square\)

Denote by \( F : AC^2_0(Q, \mathbb{R}^n) \rightarrow L^2(Q, \mathbb{R}^n) \) the operator:

\[ F(z)(x,y) = z_{xy}(x,y) + f^1(x,y,z(x,y)) \]

\[ + \int_0^x \int_0^y \left( f^2(s,t,z(s,t)) + A^1(s,t)z_x(s,t) + A^2(s,t)z_y(s,t) \right) ds dt. \]

We will prove that the norm of \( F \) is coercive.

**Lemma 2** If the functions \( f^1, f^2, A^1, A^2 \) satisfy assumptions (C1) and (C2) then the functional \( z \mapsto \|F(z)\|_{L^2} \) is coercive, i.e.

\[ \|F(z)\|_{L^2} \rightarrow \infty \text{ whenever } \|z\|_{AC^2_0} \rightarrow \infty. \]
Proof Let us take \( m > 8B \) (cf. (C2)). By (19) and assumptions (C1)–(C2) we have
\[
\| F(z) \|_{L^2_{\infty}} \geq \| z \|_{L^2_{\infty}} - (B \| z \|_{L^2_{\infty}} + B \| w_0 \|_{L^2_{\infty}} + B \| w_1 \|_{L^2_{\infty}} + B \| w_2 \|_{L^2_{\infty}}) - D,
\]
where \( D = 2\| b \|_{L^2_{\infty}}. \) By Lemma 1 and thanks to (12) it follows that
\[
\| F(z) \|_{L^2_{\infty}} \geq \| z \|_{AC^2_{0,m}} - \frac{8B}{m} \| z \|_{AC^2_{0,m}} - D = \| z \|_{AC^2_{0,m}} \left(1 - \frac{8B}{m}\right) - D.
\]
Inequality \( m > 8B \) implies that \( \| F(z) \|_{L^2_{\infty}} \to \infty \) if \( \| z \|_{AC^2_{0,m}} \to \infty. \) Since the pairs of the norms \( \| \cdot \|_{L^2}, \| \cdot \|_{L^2_{\infty}} \) and \( \| \cdot \|_{AC^2_0}, \| \cdot \|_{AC^2_{0,m}} \) are equivalent, we conclude that (20) holds. \( \square \)

Let \( \{z^k\}_{k=0}^\infty \subset AC^2_0 \) be an arbitrary sequence. Denote by \( \{g^k\} \subset L^2(Q, \mathbb{R}^n) \) a sequence of functions defined by
\[
g^k(x, y) = f^1(x, y, z^k(x, y)) + \int_0^x \int_0^y (f^2(s, t, z^k(s, t)) + A^1(s, t)z^k_y(s, t) + A^2(s, t)z^k_y(s, t)) \, dsdt - v(x, y), \tag{21}
\]
for \( k = 0, 1, 2, \ldots. \)

Lemma 3 If
1. the functions \( f^1, f^2, A^1, A^2 \) satisfy assumptions (C1) and (C2);
2. the sequence \( \{z^k\}_{k=0}^\infty \subset AC^2_0(Q, \mathbb{R}^n) \) tends to \( z^0 \in AC^2_0(Q, \mathbb{R}^n) \) weakly in \( AC^2_0(Q, \mathbb{R}^n) \)

then
(a) the sequence of functions \( \{z^k\} \) tends uniformly to \( z^0 \) on the interval \( Q; \)
(b) the sequence \( \{g^k\} \) tends to \( g^0 \) for \( (x, y) \in Q \) a.e.

Moreover, there exists a function \( b^0 \in L^2(Q, \mathbb{R}^+) \) such that
\[
|g^k(x, y)| \leq b^0(x, y)
\]
for a.e. \( (x, y) \in Q \) and \( k = 1, 2, \ldots. \)

Proof We first prove that the weak convergence of the sequence \( \{z^k\} \) to \( z^0 \) in the space \( AC^2_0(Q, \mathbb{R}^n) \) implies the uniform convergence of the sequence \( \{z^k\} \) to \( z^0 \) on the interval \( Q. \) By the definition of the inner product (see (10)) the weak convergence of the sequence \( \{z^k\} \) to \( z^0 \) in the space \( AC^2_0(Q, \mathbb{R}^n) \) is equivalent to the weak convergence of mixed second order derivatives \( \{z^k_{xy}\} \) to \( z^0_{xy} \) in the space \( L^2(Q, \mathbb{R}^n). \) Without loss of generality we can assume that \( z^0_y = 0. \) Suppose that \( z^k \) does not converge uniformly to \( z^0 = 0 \) while it converges to 0 weakly in \( AC^2_0(Q, \mathbb{R}^n). \) Therefore there exists \( \varepsilon_0 > 0 \) such that for any \( n \in \mathbb{N} \) there is a point \( (x^n, y^n) \in Q \) such that
\[
|z^n(x^n, y^n)| > \varepsilon_0. \tag{22}
\]
The sequence \( \{(x^n, y^n)\} \subset Q \) is compact. Passing if necessary to a subsequence we can assume, that \((x^n, y^n)\) tends to some \((\tilde{x}, \tilde{y})\) ∈ \(Q\). Denote by \(\chi_n\) the characteristic function of the interval

\[
\{(x, y) ∈ Q : 0 ≤ x < x^n, 0 ≤ y < y^n\}
\]

and by \(\tilde{\chi}\) the characteristic function of the interval

\[
\{(x, y) ∈ Q : 0 ≤ x < \tilde{x}, 0 ≤ y < \tilde{y}\}.
\]

It is easy to notice that \(\chi_n\) tends to \(\tilde{\chi}\) on \(Q\) a.e. This implies the following inequalities

\[
\lim_{n→∞} |z^n(x^n, y^n)| ≤ \lim_{n→∞} |z^n(x^n, y^n) - z^n(\tilde{x}, \tilde{y})| + \lim_{n→∞} |z^n(\tilde{x}, \tilde{y})|
\]

\[
= \lim_{n→∞} \left| \int_0^1 \int_0^1 \chi^n(s, t)z^n_{xy}(s, t)dsdt - \int_0^1 \int_0^1 \tilde{\chi}(s, t)z^n_{xy}(s, t)dsdt \right|
\]

\[
+ \lim_{n→∞} \left| \int_0^1 \int_0^1 \tilde{\chi}(s, t)z^n_{xy}(s, t)dsdt \right|.
\]

Since \(z^n_{xy}\) tends to zero weakly in \(L^2(Q, \mathbb{R}^n)\) the last limit is equal zero. Therefore

\[
\lim_{n→∞} |z^n(x^n, y^n)| ≤ \lim_{n→∞} \int_0^1 \int_0^1 |\chi^n(s, t) - \tilde{\chi}(s, t)||z^n_{xy}(s, t)|dsdt
\]

\[
≤ \lim_{n→∞} \left( \int_0^1 \int_0^1 |\chi^n(s, t) - \tilde{\chi}(s, t)|^2dsdt \right)^{\frac{1}{2}} \cdot \left( \int_0^1 \int_0^1 |z^n_{xy}(s, t)|^2dsdt \right)^{\frac{1}{2}}
\]

\[
≤ C \cdot \lim_{n→∞} \left( \int_0^1 \int_0^1 |\chi^n(s, t) - \tilde{\chi}(s, t)|^2dsdt \right)^{\frac{1}{2}} = 0,
\]

where \(C > 0\) is some constant such that \(||z^n_{xy}|| ≤ C\). Consequently, \(\lim_{n→∞} |z^n(x^n, y^n)| = 0\). This contradicts our assumption \((22)\). Thus \(z^k\) tends to \(z^0\) uniformly on \(Q\).

Next we prove the assertion \((b)\) of Lemma [5] By assumptions \((C1)\) and \((C2)\) we infer that

\[
\lim_{k→∞} f^1(x, y, z^k(x, y)) = f^1(x, y, z^0(x, y))
\]

for a.e. \((x, y) ∈ Q\),

\[
\lim_{k→∞} \int_0^1 \int_0^1 f^2(s, t, z^k(s, t))dsdt = \int_0^1 \int_0^1 f^2(s, t, z^0(s, t))dsdt
\]

for a.e. \((x, y) ∈ Q\) and that there exists a function \(b^1 ∈ L^2(Q, \mathbb{R}^+)\) such that

\[
|f^1(x, y, z^k(x, y))|, \left| \int_0^1 \int_0^1 f^2(s, t, z^k(s, t))dsdt \right| ≤ b^1(x, y)
\]
for a.e. \((x,y) \in Q\) and \(k = 1,2,\ldots\). Integrating by parts and taking into account assumption (C1) and Fubini’s theorem we obtain
\[
\int_0^x \int_0^y A^1(s,t)z^k_x(s,t)dsdt = \int_0^y \left( \int_0^x A^1(s,t)z^k_x(s,t)ds \right) dt \tag{24}
\]
\[
= \int_0^y A^1(x,t)z^k(x,t)dt - \int_0^x \int_0^y A^1_x(s,t)z^k(s,t)dsdt
\]
for \(k = 0,1,\ldots\). Since \(z^k\) tends to \(z^0\) uniformly on \(Q\) provided that \(z^k\) converges to \(z^0\) weakly in \(AC^2_0(Q,\mathbb{R}^n)\), we get thanks to (24) that
\[
\lim_{k \to \infty} \int_0^x \int_0^y A^1(s,t)z^k_x(s,t)dsdt = \int_0^y \lim_{k \to \infty} A^1_x(x,t)z^0(x,t)dt - \int_0^x \int_0^y A^1_x(s,t)z^0(s,t)dsdt \tag{25}
\]
for \((x,y) \in Q\). Similarly, we can show that
\[
\lim_{k \to \infty} \int_0^x \int_0^y A^2(s,t)z^k_y(s,t)dsdt = \int_0^x \int_0^y A^2(s,t)z^0_y(s,t)dsdt \tag{26}
\]
for \((x,y) \in Q\). By (24) it is easy to notice that
\[
\left| \int_0^x \int_0^y A^1(s,t)z^k_x(s,t)dsdt \right| \leq C^1 \tag{27}
\]
for some constant \(C^1 > 0\), all \((x,y) \in Q\) and \(k = 1,2,\ldots\). Similar estimation holds for the integral \(\int_0^x \int_0^y A^2(s,t)z^k_y(s,t)dsdt\). From (23), (25), (26) and (27) it follows that
\[
\lim_{k \to \infty} g^k(x,y) = g^0(x,y)
\]
for a.e. \((x,y) \in Q\). Moreover, there exists a function \(b^0 \in L^2(Q,\mathbb{R}^+)\) such that \(|g^k(x,y)| \leq b^0(x,y)\) for \(k = 1,2,\ldots\) and a.e. \((x,y) \in Q\). This completes the proof. \(\square\)

4 Main result and example

Let us consider a functional \(\varphi : AC^2_0 \to \mathbb{R}\) given by the formula
\[
\varphi(z) = \frac{1}{2} \|F(z) - v\|_{L^2}^2, \tag{28}
\]
where \(F\) is the operator defined by (3.6) and \(v\) is a fixed function from the space \(L^2(Q,\mathbb{R}^n)\). We begin by proving some lemmas.
Lemma 4 If the functions $f^1, f^2, A^1, A^2$ satisfy assumptions (C1)–(C2), then the functional $\varphi$ given by (28) satisfies (PS)–condition.

Proof Let $\{z^k\} \subset AC^2_0$ be an arbitrary (PS)–sequence for the functional $\varphi$. By Lemma 2 $\varphi$ is coercive. It implies that the sequence $\{z^k\}$ is weakly compact in $AC^2_0$. Passing if necessary to a subsequence we can assume, that $z^k$ tends to some $z^0$ weakly in $AC^2_0$. We claim that $\{z^k\}$ is compact with respect to the norm topology of the space $AC^2_0$. Thanks to assumptions (C1)–(C2) it is easy to check that the functional $\varphi$ is Frchet differentiable and

$$
\langle \varphi' (z^k), h \rangle = \int_0^1 \int_0^1 \langle h_{xy} (x, y) + f^1_z (x, y, z^k (x, y)) h (x, y) + f^2_z (s, t, z^k (s, t)) h (s, t) + A^1 (s, t) h_x (s, t) + A^2 (s, t) h_y (s, t) \rangle dsdt, \quad z^k_{xy} (x, y) + g^k (x, y) \rangle dxdy,
$$

where the sequence $\{g^k\} \subset L^2 (Q, \mathbb{R}^n)$ is given by formula (21). Let us put $h^k = z^k - z^0$, $k = 1, 2, \ldots$. From (29) it follows that

$$
\langle \varphi' (z^k) - \varphi' (z^0), z^k - z^0 \rangle = \langle z^k_{xy} - z^0_{xy}, h^k_{xy} \rangle + \sum_{i=1}^5 V_i (z^k) \geq \left\| z^k - z^0 \right\|^2_{AC^2_0} + \sum_{i=1}^5 V_i (z^k) ,
$$

where

$$
V^1 (z^k) = \langle z^k_{xy} - z^0_{xy}, g^k - g^0 \rangle = \int_0^1 \int_0^1 \langle z^k_{xy} (x, y) - z^0_{xy} (x, y), g^k (x, y) - g^0 (x, y) \rangle dxdy,
$$

$$
V^2 (z^k) = \int_0^1 \int_0^1 \langle f^1_z (x, y, z^k (x, y)) (z^k (x, y) - z^0 (x, y)) + f^2_z (s, t, z^k (s, t)) (z^k (s, t) - z^0 (s, t)) dsdt, \quad z^k_{xy} (x, y) + g^k (x, y) \rangle dxdy,
$$

$$
V^3 (z^k) = -\int_0^1 \int_0^1 \langle f^1_z (x, y, z^0 (x, y)) (z^k (x, y) - z^0 (x, y)) + f^2_z (s, t, z^0 (s, t)) (z^k (s, t) - z^0 (s, t)) dsdt, \quad z^0_{xy} (x, y) - g^0 (x, y) \rangle dxdy,
$$

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\[ V^4 (z^k) = \int_0^1 \int_0^1 \left( \int_0^x \int_0^y A^1 (s, t) \left( z^k_x (s, t) - z^0_x (s, t) \right) ds dt, \right. \]
\[ z^k_{xy} (x, y) + g^k (x, y) \right) dx dy \]
\[ + \int_0^1 \int_0^1 \left( \int_0^x \int_0^y A^2 (s, t) \left( z^k_y (s, t) - z^0_y (s, t) \right) ds dt, \right. \]
\[ z^k_{xy} (x, y) + g^k (x, y) \right) dx dy. \]

\[ V^5 (z^k) = - \int_0^1 \int_0^1 \left( \int_0^x \int_0^y A^1 (s, t) \left( z^k_x (s, t) - z^0_x (s, t) \right) ds dt, \right. \]
\[ z^0_{xy} (x, y) + g^0 (x, y) \right) dx dy \]
\[ - \int_0^1 \int_0^1 \left( \int_0^x \int_0^y A^2 (s, t) \left( z^0_y (s, t) - z^0_y (s, t) \right) ds dt, \right. \]
\[ z^0_{xy} (x, y) + g^0 (x, y) \right) dx dy. \]

By the Cauchy-Schwarz inequality we have the following estimation
\[ |V^1 (z^k)|^2 \leq \int_0^1 \int_0^1 |z^k_{xy} (x, y) - z^0_{xy} (x, y)|^2 dx dy \]
\[ \cdot \int_0^1 \int_0^1 |g^k (x, y) - g^0 (x, y)|^2 dx dy. \]

Since \( z^k_{xy} - z^0_{xy} \) converges weakly to zero in \( L^2 (Q, \mathbb{R}^n) \), therefore there exists a constant \( C > 0 \) such that
\[ |V^1 (z^k)|^2 \leq C \int_0^1 \int_0^1 |g^k (x, y) - g^0 (x, y)|^2 dx dy. \]

By Lemma [2] and Lebesgue dominated convergence theorem it follows that
\( V^1 (z^k) \to 0 \) as \( k \to \infty \). We have proved that \( z^k (x, y) \) tends to \( z^0 (x, y) \)
uniformly on \( Q \) (see Lemma [2]). Therefore, it is easy to notice that \( V^2 (z^k) \)
and \( V^3 (z^k) \) converge to zero as \( k \to \infty \).

Let us consider the functional \( V^4 \). By (24) we have
\[ V^4 (z^k) = \int_0^1 \int_0^1 \left( \int_0^x \int_0^y A^1 (x, t) \left( z^k (x, t) - z^0 (x, t) \right) dt \right. \]
\[ - \int_0^x \int_0^y A^1_s (s, t) \left( z^k (s, t) - z^0 (s, t) \right) ds dt, \]
\[ z^k_{xy} (x, y) + g^k (x, y) \right) dx dy \]
\[ + \int_0^1 \int_0^1 \left( \int_0^x \int_0^y A^2 (s, y) \left( z^k (s, y) - z^0 (s, y) \right) ds \right. \]
\[ - \int_0^x \int_0^y A^2_s (s, t) \left( z^k (s, t) - z^0 (s, t) \right) ds dt, \]
\[ z^k_{xy} (x, y) + g^k (x, y) \right) dx dy. \]
Using the Cauchy-Schwarz inequality and Lemma 2 it is easy to show that 
\( V^4(z^k) \to 0 \) as \( k \to \infty \).

Similar considerations can be applied to \( V^5(z^k) \). Thus \( \lim_{k \to \infty} \sum_{i=1}^{5} V^i(z^k) = 0 \).

Now, let us observe that 
\[
\lim_{k \to \infty} \varphi'(z^k) (z^k - z^0) = 0
\]
because \( \{z^k\} \) is the (PS)–sequence for the functional \( \varphi \) and the sequence \( \{z^k - z^0\} \) is bounded. Moreover,
\[
\lim_{k \to \infty} \varphi'(z^0) (z^k - z^0) = 0
\]
since \( z^k \) tends weakly to \( z^0 \) in \( AC^2_0 \). Combining these equalities and (30) we conclude that 
\[
\lim_{k \to \infty} \|z^k - z^0\|_{AC^2_0} = 0.
\]
This gives us the desired conclusion that the functional \( \varphi \) given by (28) satisfies (PS)–condition.

Next, we prove the following

**Lemma 5** If the functions \( f^1, f^2, A^1, A^2 \) satisfy assumptions (C1)–(C3) then for any \( v \in L^2(Q, \mathbb{R}^n) \) there exists a unique solution \( h_v \in AC^2_0 \) to the system
\[
F'(z^0) h = v,
\]
where the operator \( F : AC^2_0 \to L^2(Q, \mathbb{R}^n) \) is given by (19) and \( z^0 \in AC^2_0 \) is an arbitrary function.

**Proof** Let us put
\[
h(x, y) = \int_0^x \int_0^y g(s, t) \, ds \, dt,
\]
where \( g \in L^2(Q, \mathbb{R}^n) \). Substituting the above into (31) we obtain
\[
Ha = v,
\]
where
\[
Ha(x, y) = g(x, y) + f^1(x, y, z^0(x, y)) \cdot \int_0^x \int_0^y g(s, t) \, ds \, dt
\]
\[+ \int_0^x \int_0^y \left( f^2(s, t, z^0(s, t)) \int_0^s \int_0^t g(\sigma, \tau) \, d\sigma \, d\tau \right)
\]
\[+ A^1(s, t) \int_0^t g(s, \tau) \, d\tau + A^2(s, t) \int_0^s g(\sigma, t) \, d\sigma \right) \, ds \, dt.
\]

Let us denote by \( \hat{H} \) the operator defined by
\[
\hat{H} g = H g - g - v.
\]

(32)
We will restrict our investigation of the operator $\tilde{H}$ to the space $L^2_m(Q, \mathbb{R}^n)$. We prove that for sufficiently large $m > 0$ the mapping $\tilde{H}$ is contracting with respect to the norm $\|\cdot\|_{L^2_m}$ defined by (13). Under assumptions (C2) and (C3), there exists a constant $\tilde{d} > 0$ such that

$$\left\| \tilde{H} (g^1 - g^2) \right\|_{L^2_m}$$

$$\leq \tilde{d} \left( \int_0^1 \int_0^1 \left( e^{-m(x+y)} \int_0^x \int_0^y \left| g^1 (s, t) - g^2 (s, t) \right|^2 \, ds \, dx \right) \, dy \right)^{\frac{1}{2}}$$

$$+ d \left( \int_0^1 \int_0^1 \left( e^{-m(x+y)} \int_0^x \int_0^y \left( \int_0^t \int_0^\tau \left| (g^1 (s, \tau) - g^2 (s, \tau)) \right|^2 \, d\sigma \, d\tau \right) \, ds \, dx \right) \, dy \right)^{\frac{1}{2}}$$

$$+ d \left( \int_0^1 \int_0^1 \left( e^{-m(x+y)} \int_0^x \int_0^y \left( \int_0^s \int_0^\sigma \left| (g^1 (\sigma, t) - g^2 (\sigma, t)) \right|^2 \, d\tau \, d\sigma \right) \, ds \, dx \right) \, dy \right)^{\frac{1}{2}}$$

$$\leq 4d \left( \int_0^1 \int_0^1 \left( e^{-m(x+y)} \int_0^x \int_0^y \left| g^1 (s, t) - g^2 (s, t) \right|^2 \, ds \, dx \right) \, dy \right)^{\frac{1}{2}}.$$

Integrating by parts twice, in much the same way as in the proof of inequality (13), we obtain

$$\left\| \tilde{H} (g^1 - g^2) \right\|_{L^2_m} \leq \frac{4d}{m^2} \left\| g^1 - g^2 \right\|_{L^2_m}.$$

Hence for sufficiently large $m$, i.e. $m > 2\sqrt{d}$, the operator $\tilde{H}$ is contracting and, consequently, has a unique fixed point. It means that, there exists exactly one point $g^0 \in L^2(Q, \mathbb{R}^n)$ such that $g^0 = \tilde{H} g^0$. By (32) we get $Hg^0 = v$ and it follows easily that a function $h_v$ given by

$$h_v (x, y) = \int_0^x \int_0^y g^0 (s, t) \, ds \, dt$$

is a solution of (31) for fixed $v \in L^2(Q, \mathbb{R}^n)$. \hfill \Box

We are now in a position to show the main result of the work.

**Theorem 3** If the functions $f^1, f^2, A^1, A^2$ satisfy assumptions (C1)–(C3) then for any $v \in L^2(Q, \mathbb{R}^n)$ the integro-differential system (7)–(8) has a unique solution $z_v \in AC^2_0$. The solution $z_v$ continuously depends on $v$ with respect to the norm topology in the spaces $L^2(Q, \mathbb{R}^n)$ and $AC^2_0$. Moreover, the operator

$$L^2(Q, \mathbb{R}^n) \ni v \mapsto z_v \in AC^2_0$$

is differentiable (in Fréchet sense).

**Proof** If follows from Lemmas 4 and 5 that the operator $F$ given by (19) meets assumptions of Theorem 1. Thus system (7)–(8) has a solution $z_v$ which satisfies the requirements of our theorem. \hfill \Box
We now give an example of integro-differential system of the form (7)–(8) which satisfies assumptions of Theorem 3. For simplicity we put \( n = 1 \).

**Example 1** Consider 2D integro-differential system

\[
\begin{align*}
    z_{xy} (x,y) + w^1 (x,y) \left( \frac{z^3 (x,y)}{1 + z^2 (x,y)} + \psi^1 (z(x,y)) \right) & \\
    \quad + \int_0^x \int_0^y \left( w^2 (s,t) \frac{z(s,t) - 1}{1 + z^2 (x,y)} + \psi^2 (z(x,y)) \right) & \\
    + A^1 (s,t) z_x (s,t) + A^2 (s,t) z_y (s,t) \, dsdt \right) & = v(x,y), \\
\end{align*}
\]

(33)

where \( w^1, w^2, A^1, A^2 \) are some polynomials, \( v \in L^2 (Q, \mathbb{R}) \) and \( \psi^1, \psi^2 \) are some \( C^1 \)-class functions with unbounded derivatives. For example one can take \( \psi^1 (z) = \cos z^k \) and \( \psi^2 (z) = \sin z^l \), where \( k, l > 1 \). This simple and theoretical example allows us to emphasize the difference between our work and some other methods of nonlinear analysis.

It is easy to see that system (33) satisfies assumptions (C1)–(C3). Hence by Theorem 3 for any \( v \in L^2 (Q, \mathbb{R}) \) there exists a solution \( z_v \in AC^2_{0} \) to the system (33) with the following properties:

1. the solution \( z_v \) is unique,
2. \( z_v \) continuously depends on \( v \) with respect to the norm topology of the spaces \( L^2 (Q, \mathbb{R}^n) \) and \( AC^2_{0} \), i.e. system (33) is stable,
3. the operator \( L^2 (Q, \mathbb{R}) \ni v \mapsto z_v \in AC^2_{0} \) is differentiable in Fréchet sense, i.e. system (33) is robust.

Let us notice that the functions \( f^1 (x,y,z) = w^1 (x,y) \left( \frac{z^3}{1 + z^2} + \psi^1 (z) \right) \) and \( f^2 (x,y,z) = w^2 (x,y) \frac{z - 1}{1 + z^2} + \psi^2 (z) \) are not Lipschitz functions (\( \sin z^l \) and \( \cos z^l \) with \( k, l \geq 1 \) have “fast variation” when \( |z| \to \infty \) and consequently we cannot apply the Banach contraction principle. In this case the Schauder fixed point theorem may be applicable. But even using sophisticated fixed point theorems we get only the existence of a solution to system (33) and can hardly say anything related to properties (1)–(3).

### 5 Concluding remarks

In the paper two-dimensional integro-differential system was investigated. The main result of this work is theorem 3 on the stability and robustness of a solution to considered system (7)–(8). As far as we know 2D integro-differential systems have not been studied before. One-dimensional integro-differential systems described by ordinary differential operators were examined in many works (see monograph [Lakshmikantham, 1995] and references therein). It is important to notice that integro-differential operators can be used in mathematical modeling of systems with “memory”, i.e. systems where the state at each moment \( t \) depends on its behavior on some interval \( [t_0, t] \). In our opinion 2D integro-differential systems have the potential to play a similar role.
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