Abstract

Maximising a cumulative reward function that is Markov and stationary, i.e., defined over state-action pairs and independent of time, is sufficient to capture many kinds of goals in a Markov decision process (MDP). However, not all goals can be captured in this manner. In this paper we study convex MDPs in which goals are expressed as convex functions of the stationary distribution and show that they cannot be formulated using stationary reward functions. Convex MDPs generalize the standard reinforcement learning (RL) problem formulation to a larger framework that includes many supervised and unsupervised RL problems, such as apprenticeship learning, constrained MDPs, and so-called ‘pure exploration’. Our approach is to reformulate the convex MDP problem as a min-max game involving policy and cost (negative reward) ‘players’, using Fenchel duality. We propose a meta-algorithm for solving this problem and show that it unifies many existing algorithms in the literature.

1 Introduction

In reinforcement learning (RL), an agent learns how to map situations to actions so as to maximize a cumulative scalar reward signal. The learner is not told which actions to take, but instead must discover which actions lead to the most reward [65]. Mathematically, the RL problem can be written as finding a policy whose state occupancy has the largest inner product with a reward vector [56], i.e., the goal of the agent is to solve

$$\text{RL: } \max_{d_\pi \in K} \sum_{s,a} r(s, a) d_\pi(s, a),$$

(1)

where $d_\pi$ is the state-action stationary distribution induced by policy $\pi$ and $K$ is the set of admissible stationary distributions (see Definition 1). A significant body of work is dedicated to solving the RL problem efficiently in challenging domains [46, 63]. However, not all decision making problems of interest take this form. In particular we consider the more general convex MDP problem,

$$\text{Convex MDP: } \min_{d_\pi \in K} f(d_\pi),$$

(2)

where $f : K \to \mathbb{R}$ is a convex function. Sequential decision making problems that take this form include Apprenticeship Learning (AL), pure exploration, and constrained MDPs, among others; see Table 1. In this paper we prove the following claim:

We can solve Eq. (2) by using any algorithm that solves Eq. (1) as a subroutine.

In other words, any algorithm that solves the standard RL problem can be used to solve the more general convex MDP problem. More specifically, we make the following contributions.

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Firstly, we adapt the meta-algorithm of Abernethy and Wang \cite{abernethy} for solving Eq. (2). The key idea is to use Fenchel duality to convert the convex MDP problem into a two-player zero-sum game between the agent (henceforth, policy player) and an adversary that produces rewards (henceforth, cost player) that the agent must maximize \[3, 6\]. From the agent’s point of view, the game is bilinear, and so for fixed rewards produced by the adversary the problem reduces to the standard RL problem with non-stationary reward (Fig. 1).

Secondly, we propose a sample efficient policy that uses a standard RL algorithm (e.g. \cite{fiftl, omd}), and computes an optimistic policy with respect to the non-stationary reward at each iteration. In other words, we use algorithms that were developed to achieve low regret in the standard RL setup, to achieve low regret as policy players in the min-max game we formulate to solve the convex MDP. Our main result is that the average of the policies produced by the policy player converges to a solution to the convex MDP problem (Eq. (2)). Inspired by this principle, we also propose a recipe for using deep-RL (DRL) agents to solve convex MDPs heuristically: provide the agent non-stationary rewards from the environment.

Finally, we show that choosing specific algorithms for the policy and cost players unifies several disparate branches of RL problems, such as apprenticeship learning, constrained MDPs, and pure exploration into a single framework, as we summarize in Table 1.

In RL an agent interacts with an environment over a number of time steps and seeks to maximize its cumulative reward. We consider two cases, the average reward case and the discounted case. The Markov decision process (MDP) is defined by the tuple \((S, A, P, R)\) for the average reward case and by the tuple \((S, A, P, R, \gamma, d_0)\) for the discounted case. We assume an infinite horizon, finite state-action problem where initially, the state of the agent is sampled according to \(s_0 \sim d_0\), then at each time \(t\) the agent is in state \(s_t \in S\), selects action \(a_t \in A\) according to some policy \(\pi(s_t, \cdot)\), receives reward \(r_t \sim R(s_t, a_t)\) and transitions to new state \(s_{t+1} \in S\) according to the probability distribution \(P(\cdot, s_t, a_t)\).

The two performance metrics we consider are given by

\[
J_\pi^{\text{avg}} = \lim_{T \to \infty} \frac{1}{T} E \sum_{t=1}^{T} r_t, \quad J_\pi^{\gamma} = (1 - \gamma) E \sum_{t=1}^{\infty} \gamma^t r_t,
\]

for the average reward case and discounted case respectively. The goal of the agent is to find a policy that maximizes \(J_\pi^{\text{avg}}\) or \(J_\pi^{\gamma}\). Any stationary policy \(\pi\) induces a state-action occupancy measure \(d_\pi\),

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Convex objective } f & \text{Cost player} & \text{Policy player} & \text{Application} \\
\hline
\lambda \cdot d_e & \text{FTL} & \text{RL} & \text{(Standard) RL with } -\lambda \text{ as stationary reward function} \\
\|d_e - d_E\|^2 & \text{FTL} & \text{Best response} & \text{Apprenticeship learning (AL) } \cite{al, al2} \\
d_e \cdot \log(d_e) & \text{FTL} & \text{Best response} & \text{Pure exploration } \cite{pure} \\
\|d_e - d_E\|_\infty & \text{OMD} & \text{Best response} & \text{AL } \cite{al, al2} \\
\mathbb{E}[\lambda \cdot (d_e - d_E(c))] & \text{OMD} & \text{Best response} & \text{Inverse RL in contextual MDPs } \cite{contextual} \\
\lambda_1 \cdot d_e, \ldots, \lambda_k \cdot d_e \leq c & \text{OMD} & \text{RL} & \text{Constrained MDPs } \cite{constrained1, constrained2, constrained3, constrained4, constrained5, constrained6} \\
dist(d_e, C) & \text{OMD} & \text{Best response} & \text{Feasibility of convex-constrained MDPs } \cite{constraint} \\
\min_{d_e \in \mathbb{R}^d} \lambda \cdot (d_e - d_E) & \text{OMD} & \text{RL} & \text{Adversarial Markov Decision Processes } \cite{adversarial} \\
\text{KL}(d_e || d_E) & \text{FTL} & \text{RL} & \text{Online AL } \cite{online}, \text{Wasserstein GAIL } \cite{wasserstein} \\
-\mathbb{E}_d \text{KL}(d_e || d_E) & \text{FTL} & \text{RL} & \text{Diverse skill discovery } \cite{diverse1, diverse2} \\
\hline
\end{array}
\]

Table 1: Instances of Algorithm 1 in various convex MDPs. * as well as other KL divergences. † † \(C\) is a convex set. † † † \(f\) is concave. See Sections 4 & 6 for more details.

2  Reinforcement Learning Preliminaries

![Figure 1: Convex MDP as an RL problem](image-url)
Furthermore, the biconjugate \( \Lambda \) where
\[
\Lambda = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq f(y) \text{ for all } y \in \mathbb{R}^m \}
\]

which measures how often the agent visits each state-action when following \( \pi \). Let \( P_\pi(s_t = \cdot) \) be the probability measure over states at time \( t \) under policy \( \pi \), then
\[
d^\text{avg}(s, a) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} P_\pi(s_t = s) \pi(s, a), \quad d^\gamma(s, a) = (1 - \gamma) \mathbb{E} \sum_{t=1}^{\infty} \gamma^t P_\pi(s_t = s) \pi(s, a),
\]
for the average reward case and the discounted case respectively. With these, we can rewrite the RL objective in Eq. (3) in terms of the occupancy measure using the following well-known result, which for completeness we prove in Appendix B.

**Proposition 1.** For both the average and the discounted case, the agent objective function Eq. (3) can be written in terms of the occupancy measure as \( J_\pi = \sum s, a r(s, a) d_\pi(s, a) \).

Given an occupancy measure it is possible to recover the policy by setting \( \pi(s, a) = d_\pi(s, a)/\sum_a d_\pi(s, a) \) if \( \sum_a d_\pi(s, a) > 0 \), and \( \pi(s, a) = 1/|A| \) otherwise. Accordingly, in this paper we shall formulate the RL problem using the state-action occupancy measure, and both the standard RL problem (Eq. (1)) and the convex MDP problem (Eq. (2)) are convex optimization problems in variable \( d_\pi \). For the purposes of this manuscript we do not make a distinction between the average and discounted settings, other than through the convex polytopes of feasible occupancy measures, which we define next.

**Definition 1** (State-action occupancy’s polytope [56]). For the average reward case the set of admissible state-action occupancies is
\[
\mathcal{K}_{\text{avg}} = \{ d_\pi : d_\pi \geq 0, \sum_s d_\pi(s, a) = 1, \sum_a d_\pi(s, a) = \sum_{s', a'} P(s, s', a') d_\pi(s', a') \forall s \in S \},
\]
and for the discounted case it is given by
\[
\mathcal{K}_{\gamma} = \{ d_\pi : d_\pi \geq 0, \sum_a d_\pi(s, a) = (1 - \gamma) d_0(s) + \gamma \sum_{s', a'} P(s, s', a') d_\pi(s', a') \forall s \in S \}.
\]

We note that being a polytope implies that \( \mathcal{K} \) is a convex and compact set.

The convex MDP problem is defined for the tuple \( (S, A, P, f, \gamma, d_0) \) in the average cost case and \( (S, A, P, f, \gamma, d_0) \) in the discounted case. This tuple is defining a state-action occupancy’s polytope \( \mathcal{K} \) (Definition 1), and the problem is to find a policy \( \pi \) whose state occupancy \( d_\pi \) is in this polytope and minimizes the function \( f \) (Eq. (2)).

## 3 Meta-Algorithm for Solving Convex MDPs via RL

To solve the convex MDP problem (Eq. (2)) we need to find an occupancy measure \( d_\pi \) (and associated policy) that minimizes the function \( f \). Since both \( f : \mathcal{K} \to \mathbb{R} \) and the set \( \mathcal{K} \) are convex this is a convex optimization problem. However, it is a challenging one due to the nature of learning about the environment through stochastic interactions. In this section we show how to reformulate the convex MDP problem (Eq. (2)) so that standard RL algorithms can be used to solve it, allowing us to harness decades of work on solving vanilla RL problems. To do that we will need the following definition.

**Definition 2** (Fenchel conjugate). For a function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{ -\infty, \infty \} \), its Fenchel conjugate is denoted \( f^* : \mathbb{R}^n \to \mathbb{R} \cup \{ -\infty, \infty \} \) and defined as \( f^*(x) := \sup_y x \cdot y - f(y) \).

**Remark 1.** The Fenchel conjugate function \( f^* \) is always convex (when it exists) even if \( f \) is not. Furthermore, the biconjugate \( f^{**} := (f^*)^* \) equals \( f \) if and only if \( f \) is convex and lower semi-continuous.

Using this we can rewrite the convex MDP problem (Eq. (2)) as
\[
f^\text{OPT} = \min_{d_\pi \in \mathcal{K}} f(d_\pi) = \min_{d_\pi \in \mathcal{K}} \max_{\lambda \in \Lambda} (\lambda \cdot d_\pi - f^*(\lambda)) = \max_{\lambda \in \Lambda} \min_{d_\pi \in \mathcal{K}} (\lambda \cdot d_\pi - f^*(\lambda)) \quad (4)
\]
where \( \Lambda \) is the closure of (sub-)gradient space \( \{ \partial f(d_\pi) | d_\pi \in \mathcal{K} \} \), which is a convex set [3, Theorem 4]. As both sets are convex, this is a convex-concave saddle-point problem and a zero-sum two-player game [55, 50], and we were able to swap the order of minimization and maximization using the minimax theorem [72].
With this we define the Lagrangian as $L(d_x, \lambda) := \lambda \cdot d_x - f^*(\lambda)$. For a fixed $\lambda \in \Lambda$, minimizing the Lagrangian is a standard RL problem of the form of Eq. (1), i.e., equivalent to maximizing a reward $r = -\lambda$. Thus, one might hope that by producing an optimal dual variable $\lambda^*$ we could simply solve $d^*_x = \arg\min_{d_x \in K} L(\cdot, \lambda^*)$ for the optimal occupancy measure. However, the next lemma states that this is not possible in general.

**Lemma 1.** There exists an MDP $M$ and convex function $f$ for which there is no stationary reward $r \in \mathbb{R}^{S \times A}$ such that $\arg\max_{d_x \in K} d_x \cdot r = \arg\min_{d_x \in K} f(d_x)$.

To see this note that for any reward $r$ there is a deterministic policy that optimizes the reward \[\bar{L} = L(\cdot, \lambda^*)\] but for some choices of $f$ no deterministic policy is optimal, e.g., when $f$ is the negative entropy function. This result tells us that even if we have access to an optimal dual-variable we cannot simply use it to recover the stationary distribution that solves the convex MDP problem in general.

To overcome this issue we develop an algorithm that generates a sequence of policies $\{\pi_k\}_{k \in \mathbb{N}}$ such that the average converges to an optimal policy for Eq. (2), i.e., $(1/K)\sum_{k=1}^K d^k \rightarrow d^*_x$, with associated average regret $R^\pi_K$ and is adapted from the meta-algorithm described in Abernethy and Wang \[3\]. It is referred to as a meta-algorithm since it relies on supplied sub-routine algorithms $\text{Alg}_{\pi}$ and $\text{Alg}_\lambda$. The reinforcement learning algorithm $\text{Alg}_\pi$ takes as input a reward vector and returns a state-action occupancy measure $d_x$. The cost algorithm $\text{Alg}_\lambda$ can be a more general function of the entire history. We discuss concrete examples of $\text{Alg}_\pi$ and $\text{Alg}_\lambda$ in Section 4.

**Algorithm 1:** meta-algorithm for convex MDPs

1: **Input:** convex-concave payoff $L : K \times \Lambda \rightarrow \mathbb{R}$, algorithms $\text{Alg}_\lambda$, $\text{Alg}_\pi$, $K \in \mathbb{N}$
2: for $k = 1, \ldots, K$ do
3: \quad $\lambda^K = \text{Alg}_\lambda(d^1_x, \ldots, d^{k-1}_x; L)$
4: \quad $d^K_x = \text{Alg}_\pi(-\lambda^K)$
5: end for
6: Return $\bar{d}^K_x = \frac{1}{K} \sum_{k=1}^K d^K_x$, $\bar{\lambda}^K = \frac{1}{K} \sum_{k=1}^K \lambda^K$

In order to analyze this algorithm we will need a small detour into online convex optimization (OCO). In OCO, a learner is presented with a sequence of $K$ convex loss functions $\ell_1, \ell_2, \ldots, \ell_K : K \rightarrow \mathbb{R}$ and at each round $k$ must select a point $x_k \in K$ after which it suffers a loss of $\ell_k(x_k)$. At time period $k$ the learner is assumed to have perfect knowledge of the loss functions $\ell_1, \ldots, \ell_{k-1}$. The learner wants to minimize its average regret, defined as

$$R^\pi_K := \frac{1}{K} \left( \sum_{k=1}^K \ell_k(x_k) - \min_{x \in K} \sum_{k=1}^K \ell_k(x) \right).$$

In the context of convex reinforcement learning and meta-algorithm $\text{Alg}_\lambda$ the loss functions for the cost player are $\ell^K_x = -L(\cdot, \lambda^K)$, and for the policy player are $\ell^K_x = L(d^K_x, \cdot)$, with associated average regrets $R^\pi_K$ and $R^\lambda_K$. This brings us to the following theorem.

**Theorem 1 (Theorem 2, [3]).** Assume that $\text{Alg}_\pi$ and $\text{Alg}_\lambda$ have guaranteed average regret bounded as $R^\pi_K \leq \epsilon_K$ and $R^\lambda_K \leq \delta_K$, respectively. Then Algorithm 1 outputs $\bar{d}^K_x$ and $\bar{\lambda}^K$ satisfying $\min_{d_x \in K} L(d_x, \bar{\lambda}^K) \geq f^{OPT} - \epsilon_K - \delta_K$ and $\max_{\lambda \in \Lambda} L(\bar{d}^K_x, \cdot) \leq f^{OPT} + \epsilon_K + \delta_K$.

This theorem tells us that so long as the RL algorithm we employ has guaranteed low-regret, and assuming we choose a reasonable low-regret algorithm for deciding the costs, then the meta-algorithm will produce a solution to the convex MDP problem (Eq. (2)) to any desired tolerance, this is because $f^{OPT} \leq f(\bar{d}^K_x) = \max_{\lambda \in \Lambda} L(\bar{d}^K_x, \cdot) \leq f^{OPT} + \epsilon_K + \delta_K$. For example, we shall later present algorithms that have regret bounded as $\epsilon_K = \delta_K \leq O(1/\sqrt{K})$, in which case we have

$$f(\bar{d}^K_x) - f^{OPT} \leq O(1/\sqrt{K}).$$

**Non-Convex** $f$. Remark 1 implies that the game $\max_{\lambda \in \Lambda} \min_{d_x \in K} (\lambda \cdot d_x - f^*(\lambda))$ is concave-convex for any function $f$, so we can solve it with Algorithm 1 even for a non-convex $f$. From weak duality the value of the Lagrangian on the output of Algorithm 1 $L(\bar{d}_x, \bar{\lambda})$, is a lower bound on the optimal solution $f^{OPT}$. In addition, since $f(\bar{d}_x)$ is always an upper bound on $f^{OPT}$ we have both an upper bound and a lower bound on the optimal value: $L(\bar{d}_x, \bar{\lambda}) \leq f^{OPT} \leq f(\bar{d}_x)$.  

(5)
4 Policy and Cost Players for Convex MDPs

In this section we present several algorithms for the policy and cost players that can be used in Algorithm [1]. Any combination of these algorithms is valid and will come with different practical and theoretical performance. In Section [9] we show that several well known methods in the literature correspond to particular choices of cost and policy players and so fall under our framework.

In addition, in this section we assume that

\[ \lambda_{\text{max}} = \max_{\lambda \in \Lambda} \max_{s,a} |\lambda(s,a)| < \infty, \]

which holds when the set \( \Lambda \) is compact. One way to guarantee that \( \Lambda \) is compact is to consider functions \( f \) with Lipschitz continuous gradients (which implies bounded gradients since the set \( K \) is compact). For simplicity, we further assume that \( \lambda_{\text{max}} \leq 1 \). By making this assumption we assure that the non stationary rewards produced by the cost player are bounded by 1 as is usually done in RL.

4.1 Cost Player

**Follow the Leader (FTL)** is a classic OCO algorithm that selects \( \lambda_k \) to be the best point in hindsight. In the special case of convex MDPs, as defined in Eq. (4), FTL has a simpler form:

\[ \lambda_k = \arg \max_{\lambda \in \Lambda} \sum_{j=1}^{k-1} L(d^j_\pi, \lambda) = \arg \max_{\lambda \in \Lambda} \left( \lambda \cdot \sum_{j=1}^{k-1} d^j_\pi - K f^*(\lambda) \right) = \nabla f(d^{k-1}_\pi), \quad (6) \]

where \( d^{k-1}_\pi = \sum_{j=1}^{k-1} d^j_\pi \) and the last equality follows from the fact that \( (\nabla f^*)^{-1} = \nabla f [57] \). The average regret of FTL is guaranteed to be \( \tilde{R}_K \leq c/\sqrt{K} \) under some assumptions [30]. In some cases, and specifically when the set \( K \) is a polytope and the function \( f \) is strongly convex, FTL can enjoy logarithmic or even constant regret; see [33, 30] for more details.

**Online Mirror Descent (OMD)** uses the following update [48, 9]:

\[ \lambda_k = \arg \max_{\lambda \in \Lambda} \left( (\lambda - \lambda^{k-1}) \cdot \nabla_{\lambda} \mathcal{L}(d^{k-1}_\pi, \lambda^{k-1}) + \alpha_k B_r(\lambda, \lambda^{k-1}) \right), \]

where \( \alpha_k \) is a learning rate and \( B_r \) is a Bregman divergence [14]. For \( B_r(x) = 0.5 \|x\|^2 \), we get online gradient descent [80] and for \( B_r(x) = x \cdot \log(x) \) we get multiplicative weights [24] as special cases. We also note that OMD is equivalent to a linearized version of Follow the Regularized Leader (FTRL) [44, 29]. The average regret of OMD is \( \tilde{R}_K \leq c/\sqrt{K} \) under some assumptions, see, for example [29].

4.2 Policy Players

4.2.1 Best Response

In OCO, the best response is to simply ignore the history and play the best option on the current round, which has guaranteed average regret bound of \( \tilde{R}_K \leq 0 \) (this requires knowledge of the current loss function, which is usually not applicable but is in this case). When applied to Eq. (4), it is possible to find the best response \( d^*_\pi \) using standard RL techniques since

\[ d^*_\pi = \arg \min_{d_\pi \in K} \mathcal{L}_K(d_\pi, \lambda_k) = \arg \min_{d_\pi \in K} d_\pi \cdot \lambda_k - f^*(\lambda_k) = \arg \max_{d_\pi \in K} d_\pi \cdot (-\lambda^k), \]

which is an RL problem for maximizing the reward \((-\lambda^k)\). In principle, any RL algorithm that eventually solves the RL problem can be used to find the best response, which substantiates our claim in the introduction. For example, tabular Q-learning executed for sufficiently long and with a suitable exploration strategy will converge to the optimal policy [23]. In the non-tabular case we could parameterize a deep neural network to represent the Q-values [46], and if the network has sufficient capacity then similar guarantees might hold. We make no claims on efficiency or tractability of this approach, just that in principle such an approach would provide the best-response at each iteration and therefore satisfy the required conditions to solve the convex MDP problem.
4.2.2 Approximate Best Response

The caveat in using the best response as a policy player is that in practice, it can only be found approximately by executing an RL algorithm in the environment. This leads to defining an approximate best response via the Probably Approximately Correct (PAC) framework. We say that a policy player is PAC($\epsilon, \delta$), if it finds an $\epsilon$-optimal policy to an RL problem with probability of at least $1 - \delta$. In addition, we say that a policy $\pi'$ is $\epsilon$-optimal if its state occupancy $d_{\pi'}^k$ is such that

$$\max_{d_{\pi} \in A} d_{\pi} \cdot (-\lambda_k) - d_{\pi'}^k \cdot (-\lambda_k) \leq \epsilon.$$  

For example, the algorithm in [41] can find an $\epsilon$-optimal policy to the discounted RL problem after seeing $O\left( \frac{SA}{(1 - \gamma)^2 \epsilon} \log \left( \frac{1}{\delta} \right) \right)$ samples; and the algorithm in [37] can find an $\epsilon$-optimal policy for the average reward RL problem after seeing $O\left( \frac{SA}{\epsilon^2} \log \left( \frac{1}{\delta} \right) \right)$ samples, where $t_{\text{mix}}$ is the mixing time (see, e.g., [43, 77] for a formal definition). The following Lemma analyzes the sample complexity of Algorithm [41] with an approximate best response policy player for the average reward RL problem [37]. The result can be easily extended to the discounted case using the algorithm in [41]. Other relaxations to the best response for specific algorithms can be found in [66, 45, 34, 31].

**Lemma 2** (The sample complexity of approximate best response in convex MDPs with average occupancy measure). For a convex function $f$, running Algorithm [41] with an oracle cost player with regret $R_K = O(1/K)$ and an approximate best response policy player that solves the average reward RL problem in iteration $k$ to accuracy $\epsilon_k = 1/k$ returns an occupancy measure $d_{\pi_k}^k$ that satisfies $f(d_{\pi_k}^k) - f^{\text{OPT}} \leq \epsilon$ with probability $1 - \delta$ after seeing $O\left( t_{\text{mix}}^k SA \log(2K/\epsilon \delta) / \epsilon^3 \delta^3 \right)$ samples. Similarly, for $R_k^k = O(1/\sqrt{K})$, setting $\epsilon_k = 1/\sqrt{K}$ requires $O(t_{\text{mix}}^k SA \log(2K/\epsilon \delta) / \epsilon^3 \delta^4)$ samples.

4.2.3 Non-Stationary RL Algorithms

We now discuss a different type of policy players; instead of solving an MDP to accuracy which denotes the diameter of the MDP, see [36, Definition 1] for more details. In the supplementary Algorithm 1 with an approximate best response policy player for the average reward RL problem [37]. To support this conjecture we shall prove that this is exactly the case for UCRL2 [36]. UCRL2 is an RL algorithm that was designed and analyzed in the standard RL setup, and we shall show that it is easily adapted to the non-stationary but known reward setup that we require. To make this claim more general, we will also discuss a similar result for the MDPO algorithm [62] that was given in a slightly different setup.

UCRL2 is a model based algorithm that maintains an estimate of the reward and the transition function as well as confidence sets about those estimates. In our case the reward at time $k$ is known, so we only need to consider uncertainty in the dynamics. UCRL2 guarantees that in any iteration $k$, the true transition function is in a confidence set with high probability, i.e., $P \in \mathcal{P}_k$ for confidence set $\mathcal{P}_k$. If we denote by $J_{\pi}^{\text{opt}}$ the value of policy $\pi$ in an MDP with dynamics $P$ and reward $R$ then the optimistic policy is $\pi_k = \text{arg} \max_{\pi \in \mathcal{P}_k} J_{\pi}^{\text{opt}} - \lambda_k$. Acting according to this policy is guaranteed to attain low regret. In the following results for UCRL2 we will use the constant $D$, which denotes the diameter of the MDP, see [36, Definition 1] for more details. In the supplementary material (Appendix E), we provide a proof sketch that closely follows [56].

**Lemma 3** (Non stationary regret of UCRL2). For an MDP with dynamics $P$, diameter $D$, an arbitrary sequence of known and bounded rewards $\{r^i : \max_{s,a} r^i(s,a) \leq 1\}_{i=1}^K$, such that the optimal average reward at episode $k$, with respect to $P$ and $r_k$ is $J_{\pi_k}^{\text{opt}}$, then with probability at least $1-\delta$, the average regret of UCRL2 is at most $R_K = \frac{1}{K} \sum_{k=1}^K J_{\pi_k}^{\text{opt}} - J_{\pi_k}^{\text{opt}} \leq O(\sqrt{A \log(1/\delta) / K})$.

Next, we give a PAC($\epsilon, \delta$) sample complexity result for the mixed policy $\pi^K$, that is produced by running Algorithm [41] with UCRL2 as a policy player.
We have restricted the presentation so far to unconstrained convex problems, in this section we extend the above results to the constrained case. The problem we consider is

\[
\min_{d_\pi \in \mathcal{K}} f(d_\pi) \quad \text{subject to} \quad g_i(d_\pi) \leq 0, \quad i = 1, \ldots, m,
\]

where \( f \) and the constraint functions \( g_i \) are convex. Previous work focused on the case where both \( f \) and \( g_i \) are linear. We can use the same Fenchel dual machinery we developed before, but now taking into account the constraints. Consider the Lagrangian

\[
L(d_\pi, \mu) = f(d_\pi) + \sum_{i=1}^m \mu_i g_i(d_\pi) = \max_{\nu} (\nu \cdot d_\pi - f^*(\nu)) + \sum_{i=1}^m \mu_i \max_{v_i} (d_\pi v_i - g_i^*(v_i)).
\]

over dual variables \( \mu \geq 0 \), with new variables \( v_i \) and \( \nu \). At first glance this does not look convex-concave, however we can introduce new variables \( \zeta_i = \mu_i v_i \) to obtain

\[
L(d_\pi, \mu, \nu, \zeta_1, \ldots, \zeta_m) = \nu \cdot d_\pi - f^*(\nu) + \sum_{i=1}^m (d_\pi \zeta_i - \mu_i g_i^*(\zeta_i/\mu_i)).
\]

This is convex (indeed affine) in \( d_\pi \) and concave in \((\nu, \mu, \zeta_1, \ldots, \zeta_m)\), since it includes the perspective transform of the functions \( g_i \). The Lagrangian involves a cost vector, \( \nu + \sum_{i=1}^m \zeta_i \), linearly
interacting with \( d_\pi \), and therefore we can use the same policy players as before to minimize this cost. For the cost player, it is possible to use OMD on Eq. (7) jointly for the variables \( \nu, \mu \) and \( \zeta \).

It is more challenging to use best-response and FTL for the cost-player variables as the maximum value of the Lagrangian is unbounded for some values of \( d_\pi \). Another option is to treat the problem as a three-player game. In this case the policy player controls \( d_\pi \) as before, one cost player chooses \((\nu, \zeta_1, \ldots, \zeta_m)\) and can use the algorithms we have previously discussed, and the other cost player chooses \( \mu \) with some restrictions on their choice of algorithm. Analyzing the regret in that case is outside the scope of this paper.

6 Examples

In this section we explain how existing algorithms can be seen as instances of the meta-algorithm for various choices of the objective function \( f \) and the cost and policy player algorithms \( \text{Alg}_\lambda \) and \( \text{Alg}_\pi \). We summarized the relationships in Table 1.

6.1 Apprenticeship Learning

In apprenticeship learning (AL), we have an MDP without an explicit reward function. Instead, an expert provides demonstrations which are used to estimate the expert state occupancy measure \( d_E \).

Abbeel and Ng [11] formalized the AL problem as finding a policy \( \pi \) whose state occupancy is close to that of the expert by minimizing the convex function \( f(d_\pi) = ||d_\pi - d_E|| \). The convex conjugate of \( f \) is given by \( f^*(y) = y \cdot d_E \) if \( ||y||_* \leq 1 \) and \( \infty \) otherwise, where \( || \cdot ||_* \) denotes the dual norm. Plugging \( f^* \) into Eq. (4) results in the following game:

\[
\min_{d_\pi \in K} ||d_\pi - d_E|| = \min_{d_\pi \in K} \max_{||\lambda||_* \leq 1} \lambda \cdot d_\pi - \lambda \cdot d_E. \tag{8}
\]

Inspecting Eq. (8), we can see that the norm in the function \( f \) that is used to measure the distance from the expert induces a constraint set for the cost variable, which is a unit ball in the dual norm.

\( \text{Alg}_\lambda = \text{OMD}, \text{Alg}_\pi = \text{Best Response/RL} \). The Multiplicative Weights AL algorithms [66, MWAL] was proposed to solve the AL problem with \( f(d_\pi) = ||d_\pi - d_E||_\infty \). It uses the best response as the policy player and multiplicative weights as the cost player (a special case of OMD). MWAL has also been used to solve AL in contextual MDPs [10] and to find feasible solutions to convex-constrained MDPs [45]. We note that in practice the best response can only be solved approximately, as we discussed in Section 4. Instead, in online AL [62] the authors proposed to use MDPO as the policy player, which guarantees a regret bound of \( R_K \leq c/\sqrt{K} \). They showed that their algorithm is equivalent to Wasserstein GAIL [74, 79] and in practice tends to perform similarly to GAIL.

\( \text{Alg}_\pi = \text{FTL}, \text{Alg}_\pi = \text{Best Response} \). When the policy player plays the best response and the cost player plays FTL, Algorithm 1 is equivalent to the Frank-Wolfe algorithm [23, 3] for minimizing \( f \) (Eq. 2). Pseudo-code for this is included in the appendix (Algorithm 3). The algorithm finds a point \( d^*_\pi \in K \) that has the largest inner-product (best response) with the negative gradient (i.e., FTL).

Abbeel and Ng [11] proposed two algorithms for AL, the projection algorithm and the max margin algorithm. The projection algorithm is essentially a FW algorithm, as was suggested in the supplementary [11] and was later shown formally in [76]. Thus, it is a projection free algorithm in the sense that it avoids projecting \( d_\pi \) into \( K \), despite the name. In their case the gradient is given by \( \nabla f(d_\pi) = d_\pi - d_E \). Thus, finding the best response is equivalent to solving an MDP whose reward is \( d_E - d_\pi \). In a similar fashion, FW can be used to solve convex MDPs more generally [31]. Specifically, in [31], the authors considered the problem of pure exploration, which they defined as finding a policy that maximizes entropy.

Fully Corrective FW. The FW algorithm has many variants (see [34] for a survey) some of which enjoy faster rates of convergence in special cases. Concretely, when the constraint set is a polytope, which is the case for convex MDPs (Definition 1), some variants achieve a linear rate of convergence [35, 76]. One such variant is the Fully corrective FW, which replaces the learning rate update (see line 4 of Algorithm 3 in the supplementary), with a minimization problem over the convex hull of occupancy measures at the previous time-step. This is guaranteed to be at least as good as the learning rate update. Interestingly, the second algorithm of Abbeel and Ng [11], the max margin algorithm, is...
exact equivalent to this fully corrective FW variant. This implies that the max-margin algorithm enjoys a better theoretical convergence rate than the ‘projection’ variant, as was observed empirically in [11].

6.2 GAIL and DIAYN: Alg\_\_=FTL, Alg\_\_=RL

We now discuss the objectives of two popular algorithms, GAIL [32] and DIAYN [21], which perform AL and diverse skill discovery respectively. Our analysis suggests that GAIL and DIAYN share the same objective function. In GAIL, this objective function is minimized, which is a convex MDP, however, in DIAYN it is maximized, which is therefore not a convex MDP. We start the discussion with DIAYN and follow with a simple construction showing the equivalence to GAIL.

DIAYN. Discriminative approaches [27][21] rely on the intuition that skills are diverse when they are entropic and easily discriminated by observing the states that they visit. Given a probability space (Ω, F, P), state random variables S : Ω → S and latent skills Z : Ω → Z with prior p, the key term of interest being maximized in DIAYN [21] is the mutual information:

\[ I(S; Z) = E_{z \sim p; s \sim d_z} [\log p(z | s) - \log p(z)], \quad (9) \]

where \( d_z \) is the stationary distribution induced by the policy \( z \). For each skill \( z \), this corresponds to a standard RL problem with (conditional) policy \( z \) and reward function \( r(s | z) = \log p(z | s) - \log p(z) \). The first term encourages the policy to visit states for which the underlying skill has high-probability under the posterior \( p(z | s) \), while the second term ensures a high entropy distribution over skills. In practice, the full DIAYN objective further regularizes the learnt policy by including entropy terms \( -\log p(z | s, z) \). For large state spaces, \( p(z | s) \) is typically intractable and Eq. (9) is replaced with a variational lower-bound, where the true posterior is replaced with a learned discriminator \( q_\phi(z | s) \). Here, we focus on the simple setting where \( z \) is a categorical distribution over \(|Z|\) outcomes, yielding \(|Z|\) policies \( z \), and \( q_\phi \) is a classifier over these \(|Z|\) skills with parameters \( \phi \).

We now show that a similar intrinsic reward can be derived using the framework of convex MDPs. We start by writing the true posterior as a function of the per-skill state occupancy \( q \) and reward function \( r \) with DIAYN and follow with a simple construction showing the equivalence to GAIL.

\[ f(z) = E_z E_{s,z} [\log p(z | s) - \log p(z)], \]

where KL denotes the Kullback–Leibler divergence [40].

Intuitively, finding a set of policies \( \pi^1, \ldots, \pi^z \) that minimize Eq. (10) will result in finding policies that visit similar states, measured using the KL distance between their respective state occupancies \( d_{\pi^j} \). This is a convex MDP because the KL-divergence is jointly convex in both arguments [13, Example 3.19]. We will soon show that this is the objective of GAIL. On the other hand, a set of policies that maximize Eq. (10) is diverse, as the policies visit different states, measured using the KL distance between their respective state occupancies \( d_{\pi^j} \).

We follow on with deriving the FTL player for the convex MDP in Eq. (10). We will then show that this FTL player is producing an intrinsic reward that is equivalent to the intrinsic reward used in GAIL and DIAYN (despite the fact that DIAYN is not a convex MDP). According to Eq. (6), the FTL cost player will produce a cost \( \lambda^k \) at iteration \( k \) given by

\[ \nabla_{d_{\pi}^k} KL(d_{\pi}^k || \sum_k p(k) d_{\pi}^k) = p(z) \left[ \log \frac{d_{\pi}^z p(z)}{\sum_k d_{\pi}^k p(k)} + 1 - \frac{d_{\pi}^z p(z)}{\sum_k d_{\pi}^k p(k)} \right] - p(z) \sum_{j \neq z} p(j) d_{\pi}^j \]

\[ = p(z) \left[ \log(p(z | s)) - \log(p(z)) \right], \quad (11) \]
where the equality follows from writing the posterior as a function of the per-skill state occupancy $d^*_n = p(s \mid z)$, and using Bayes rules, $p(z|s) = \frac{d^*_n(s)p(z)}{\sum_{j \neq n} d^*_j(s)p(z)}$. Replacing the posterior $p(z|s)$ with a learnt discriminator $q_\phi(z|s)$ recovers the mutual-information rewards of DIAYN.

**GAIL.** We further show how Eq. (10) extends to GAIL [32] via a simple construction. Consider a binary latent space of size $|Z| = 2$, where $z = 1$ corresponds to the policy of the agent and $z = 2$ corresponds to the policy of the expert which is fixed. In addition, consider a uniform prior over the latent variables, i.e., $p(z = 1) = \frac{1}{2}$. By removing the constant terms in Eq. (11), one retrieves the GAIL [32] algorithm. The cost $\log(p(z|s))$ is the probability of the discriminator to identify the agent, and the policy player is MDPO (which is similar to TRPO in GAIL).

## 7 Discussion

In this work we reformulated the convex MDP problem as a convex-concave game between the agent and another player that is producing costs (negative rewards) and proposed a meta-algorithm for solving it.

We observed that many algorithms in the literature can be interpreted as instances of the meta-algorithm by selecting different pairs of subroutines employed by the policy and cost players. The Frank-Wolfe algorithm, which combines best response with FTL, was originally proposed for AL [1, 76] but can be used for any convex MDP problem as was suggested in [31]. Cheung [17] further studied the combination of FW with UCRL, proposed a novel gradient thresholding scheme and applied it to complex vectorial objectives. Zhang et al. [78], unified the problems of RL, AL, constrained MDPs with linear constraints and maximum entropy exploration under the framework of convex MDPs. We extended the framework to allow convex constraints (Section 5) and explained the objective of GAIL as a convex MDP (Section 6.2). We also discussed non convex objectives (Section 3) and analyzed unsupervised skill discovery via the maximization of mutual information (Section 6.2) as a special case. Finally, we would like to point out a recent work by Geist et al. [26], which was published concurrently to ours, and studies the convex MDP problem from the viewpoint of mean field games.

There are also algorithms for convex MDPs that cannot be explained as instances of Algorithm 1. In particular, Zhang et al. [78] proposed a policy gradient algorithm for convex MDPs in which each step of policy gradient involves solving a new saddle point problem (formulated using the Fenchel dual). This is different from our approach since we solve a single saddle point problem iteratively, and furthermore we have much more flexibility about which algorithms the policy player can use. Moreover, for the convergence guarantee [78, Theorem 4.5] to hold, the saddle point problem has to be solved exactly, while in practice it is only solved approximately [78, Algorithm 1], which hinders its sample efficiency. Fenchel duality has also been used in off policy evaluation (OPE) in [47, 75]. The difference between these works and ours is that we train a policy to minimize an objective, while in OPE a target policy is fixed and its value is estimated from data produced by a behaviour policy.

In order to solve a practical convex MDP problem in a given domain it would be prudent to use an RL algorithm that is known to be high performing for the vanilla RL problem as the policy player. From the theoretical point of view this could be MDPO or UCRL2, which we have shown come with strong guarantees. From the practical point of view using a high performing DRL algorithm, which may be specific to the domain, will usually yield the best results. For the cost player using FTL, *i.e.*, using the gradient of the objective function, is typically the best choice.

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1 In a previous version of this work we mistakenly forgot to take the gradient of the objective in Eq. (11) w.r.t $d^*_n$ for the summands $j \neq n$. Taking this gradient gives the last term in the first row of Eq. (11), which in turn, cancels out the "gradient correction term" we introduced in the previous version. The current derivation retrieves the standard reward used in DIAYN and VIC.
References

[1] P. Abbeel and A. Y. Ng. Apprenticeship learning via inverse reinforcement learning. In Proceedings of the twenty-first international conference on Machine learning, page 1. ACM, 2004.

[2] A. Abdolmaleki, J. T. Springenberg, Y. Tassa, R. Munos, N. Heess, and M. Riedmiller. Maximum a posteriori policy optimisation. arXiv preprint arXiv:1806.06920, 2018.

[3] J. D. Abernethy and J.-K. Wang. On frank-wolfe and equilibrium computation. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information Processing Systems, volume 30. Curran Associates, Inc., 2017. URL https://proceedings.neurips.cc/paper/2017/file/7371364b3d72ac9a3ed8638e6f0be2c9-Paper.pdf.

[4] J. Achiam, H. Edwards, D. Amodei, and P. Abbeel. Variational option discovery algorithms. arXiv preprint arXiv:1807.10299, 2018.

[5] A. Agarwal, S. M. Kakade, J. D. Lee, and G. Mahajan. Optimality and approximation with policy gradient methods in markov decision processes. In Conference on Learning Theory, pages 64–66. PMLR, 2020.

[6] S. Agrawal and N. R. Devanur. Fast algorithms for online stochastic convex programming. In Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms, pages 1405–1424. SIAM, 2014.

[7] E. Altman. Constrained Markov decision processes, volume 7. CRC Press, 1999.

[8] M. G. Azar, I. Osband, and R. Munos. Minimax regret bounds for reinforcement learning. In International Conference on Machine Learning, pages 263–272, 2017.

[9] A. Beck and M. Teboulle. Mirror descent and nonlinear projected subgradient methods for convex optimization. Operations Research Letters, 31:167–175, 2003.

[10] S. Belogolovsky, P. Korsunsky, S. Mannor, C. Tessler, and T. Zahavy. Inverse reinforcement learning in contextual mdps. Machine Learning, 2021.

[11] S. Bhatnagar and K. Lakshmanan. An online actor–critic algorithm with function approximation for constrained markov decision processes. Journal of Optimization Theory and Applications, 153(3):688–708, 2012.

[12] V. S. Borkar. An actor-critic algorithm for constrained markov decision processes. Systems & Control Letters, 54(3):207–213, 2005.

[13] S. Boyd and L. Vandenberghe. Convex optimization. Cambridge university press, 2004.

[14] L. M. Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. USSR computational mathematics and mathematical physics, 7(3):200–217, 1967.

[15] Q. Cai, Z. Yang, C. Jin, and Z. Wang. Provably efficient exploration in policy optimization. In International Conference on Machine Learning, pages 1283–1294. PMLR, 2020.

[16] D. A. Calian, D. J. Mankowitz, T. Zahavy, Z. Xu, J. Oh, N. Levine, and T. Mann. Balancing constraints and rewards with meta-gradient d4{pg}. In International Conference on Learning Representations, 2021. URL https://openreview.net/forum?id=TQP98Ya7UMP

[17] W. C. Cheung. Regret minimization for reinforcement learning with vectorial feedback and complex objectives. In Advances in Neural Information Processing Systems, volume 32. Curran Associates, Inc., 2019. URL https://proceedings.neurips.cc/paper/2019/file/a02ff91cece5e7efeb46db8f10a74059-Paper.pdf.

[18] C. Dann and E. Brunskill. Sample complexity of episodic fixed-horizon reinforcement learning. In Advances in Neural Information Processing Systems, pages 2818–2826, 2015.
[19] Y. Efroni, S. Mannor, and M. Pirotta. Exploration-exploitation in constrained mdps. arXiv preprint arXiv:2003.02189, 2020.

[20] L. Espeholt, H. Soyer, R. Munos, K. Simonyan, V. Mnih, T. Ward, Y. Doron, V. Firoiu, T. Harley, I. Dunning, S. Legg, and K. Kavukcuoglu. IMPALA: Scalable distributed deep-RL with importance weighted actor-learner architectures. In Proceedings of the 35th International Conference on Machine Learning, 2018.

[21] B. Eysenbach, A. Gupta, J. Ibarz, and S. Levine. Diversity is all you need: Learning skills without a reward function. In International Conference on Learning Representations, 2019. URL https://openreview.net/forum?id=SJx63jRqlm.

[22] C. Florensa, Y. Duan, and P. Abbeel. Stochastic neural networks for hierarchical reinforcement learning. In International Conference on Learning Representations, 2016.

[23] M. Frank and P. Wolfe. An algorithm for quadratic programming. Naval research logistics quarterly, 3(1-2):95–110, 1956.

[24] Y. Freund and R. E. Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. Journal of computer and system sciences, 55(1):119–139, 1997.

[25] M. Geist, B. Scherrer, and O. Pietquin. A theory of regularized markov decision processes. In International Conference on Machine Learning, pages 2160–2169. PMLR, 2019.

[26] M. Geist, J. Pérolat, M. Laurière, R. Elie, S. Perrin, O. Bachem, R. Munos, and O. Pietquin. Concave utility reinforcement learning: the mean-field game viewpoint. arXiv preprint arXiv:2106.03787, 2021.

[27] K. Gregor, D. J. Rezende, and D. Wierstra. Variational intrinsic control. International Conference on Learning Representations, Workshop Track, 2017. URL https://openreview.net/forum?id=Skc-Fo4Yg.

[28] K. Hausman, J. T. Springenberg, Z. Wang, N. Heess, and M. Riedmiller. Learning an embedding space for transferable robot skills. In International Conference on Learning Representations, 2018. URL https://openreview.net/forum?id=rk07ZXZRb.

[29] E. Hazan. Introduction to online convex optimization. Foundations and Trends in Optimization, 2(3-4):157–325, 2016.

[30] E. Hazan, A. Agarwal, and S. Kale. Logarithmic regret algorithms for online convex optimization. Machine Learning, 69(2-3):169–192, 2007.

[31] E. Hazan, S. Kakade, K. Singh, and A. Van Soest. Provably efficient maximum entropy exploration. In International Conference on Machine Learning, pages 2681–2691. PMLR, 2019.

[32] J. Ho and S. Ermon. Generative adversarial imitation learning. arXiv preprint arXiv:1606.03476, 2016.

[33] R. Huang, T. Lattimore, A. György, and C. Szepesvári. Following the leader and fast rates in linear prediction: Curved constraint sets and other regularities. In Advances in Neural Information Processing Systems, pages 4970–4978, 2016.

[34] M. Jaggi. Revisiting frank-wolfe: Projection-free sparse convex optimization. In Proceedings of the 30th international conference on Machine learning. ACM, 2013.

[35] M. Jaggi and S. Lacoste-Julien. On the global linear convergence of frank-wolfe optimization variants. Advances in Neural Information Processing Systems, 28, 2015.

[36] T. Jaksch, R. Ortner, and P. Auer. Near-optimal regret bounds for reinforcement learning. Journal of Machine Learning Research, 11(Apr):1563–1600, 2010.

[37] Y. Jin and A. Sidford. Efficiently solving mdps with stochastic mirror descent. In International Conference on Machine Learning, pages 4890–4900. PMLR, 2020.
[38] Y. Jin and A. Sidford. Towards tight bounds on the sample complexity of average-reward mdps. 
*arXiv preprint arXiv:2106.07046*, 2021.

[39] E. Kaufmann, P. Ménard, O. D. Domingues, A. Jonsson, E. Leurent, and M. Valko. Adaptive 
reward-free exploration. In *Algorithmic Learning Theory*, pages 865–891. PMLR, 2021.

[40] S. Kullback. *Information theory and statistics*. Courier Corporation, 1997.

[41] T. Lattimore and M. Hutter. Pac bounds for discounted mdps. In *International Conference on 
Algorithmic Learning Theory*, pages 320–334. Springer, 2012.

[42] L. Lee, B. Eysenbach, E. Parisotto, E. Xing, S. Levine, and R. Salakhutdinov. Efficient 
exploration via state marginal matching. *arXiv preprint arXiv:1906.05274*, 2019.

[43] D. Levin, Y. Peres, and E. Wilmer. *Markov Chains and Mixing Times*. American Mathematical 
Society, 2017.

[44] B. McMahan. Follow-the-regularized-leader and mirror descent: Equivalence theorems and 
II regularization. In *Proceedings of the Fourteenth International Conference on Artificial 
Intelligence and Statistics*, pages 525–533. JMLR Workshop and Conference Proceedings, 
2011.

[45] S. Miryoosefi, K. Brantley, H. Daumé III, M. Dudík, and R. Schapire. Reinforcement learning 
with convex constraints. *arXiv preprint arXiv:1906.09323*, 2019.

[46] V. Mnih, K. Kavukcuoglu, D. Silver, A. A. Rusu, J. Veness, M. G. Bellemare, A. Graves, 
M. Riedmiller, A. K. Fidjeland, G. Ostrovski, et al. Human-level control through deep rein-
forcement learning. *nature*, 518(7540):529–533, 2015.

[47] O. Nachum, Y. Chow, B. Dai, and L. Li. Dualdice: Behavior-agnostic estimation of discounted 
stationary distribution corrections. *arXiv preprint arXiv:1906.04733*, 2019.

[48] A. S. Nemirovskij and D. B. Yudin. Problem complexity and method efficiency in optimization. 
In *Wiley-Interscience*, 1983.

[49] B. O’Donoghue. Variational Bayesian reinforcement learning with regret bounds. *arXiv preprint 
arXiv:1807.09647*, 2018.

[50] B. O’Donoghue, T. Lattimore, and I. Osband. Stochastic matrix games with bandit feedback. 
*arXiv preprint arXiv:2006.05145*, 2020.

[51] B. O’Donoghue, I. Osband, and C. Ionescu. Making sense of reinforcement learning and 
probabilistic inference. In *International Conference on Learning Representations*, 2020.

[52] I. Osband, D. Russo, and B. Van Roy. (More) efficient reinforcement learning via posterior 
sampling. In *Advances in Neural Information Processing Systems*, pages 3003–3011, 2013.

[53] I. Osband, C. Blundell, A. Pritzel, and B. V. Roy. Deep exploration via bootstrapped dqn. In 
*Proceedings of the 30th International Conference on Neural Information Processing Systems*, 
pages 4033–4041, 2016.

[54] I. Osband, Y. Doron, M. Hessel, J. Aslanides, E. Sezener, A. Saraiya, K. McKinney, T. Lattimore, 
C. Szepesvari, S. Singh, et al. Behaviour suite for reinforcement learning. In *International 
Conference on Learning Representations*, 2019.

[55] M. J. Osborne and A. Rubinstein. *A course in game theory*. MIT press, 1994.

[56] M. L. Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John 
Wiley & Sons, 1984.

[57] R. T. Rockafellar. *Convex analysis*. Princeton university press, 1970.

[58] A. Rosenberg and Y. Mansour. Online convex optimization in adversarial markov decision 
processes. In *International Conference on Machine Learning*, pages 5478–5486. PMLR, 2019.
[59] J. Schulman, S. Levine, P. Abbeel, M. Jordan, and P. Moritz. Trust region policy optimization. In *International conference on machine learning*, pages 1889–1897. PMLR, 2015.

[60] L. Shani, Y. Efroni, and S. Mannor. Adaptive trust region policy optimization: Global convergence and faster rates for regularized mdps. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 5668–5675, 2020.

[61] L. Shani, Y. Efroni, A. Rosenberg, and S. Mannor. Optimistic policy optimization with bandit feedback. In *International Conference on Machine Learning*, pages 8604–8613. PMLR, 2020.

[62] L. Shani, T. Zahavy, and S. Mannor. Online apprenticeship learning. *arXiv preprint arXiv:2102.06924*, 2021.

[63] D. Silver, J. Schrittwieser, K. Simonyan, I. Antonoglou, A. Huang, A. Guez, T. Hubert, L. Baker, M. Lai, A. Bolton, et al. Mastering the game of go without human knowledge. *nature*, 550 (7676):354–359, 2017.

[64] A. L. Strehl and M. L. Littman. An analysis of model-based interval estimation for markov decision processes. *Journal of Computer and System Sciences*, 74(8):1309–1331, 2008.

[65] R. S. Sutton and A. G. Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

[66] U. Syed and R. E. Schapire. A game-theoretic approach to apprenticeship learning. In *Advances in neural information processing systems*, pages 1449–1456, 2008.

[67] U. Syed, M. Bowling, and R. E. Schapire. Apprenticeship learning using linear programming. In *Proceedings of the 25th international conference on Machine learning*, pages 1032–1039. ACM, 2008.

[68] C. Szepesvári. Constrained mdps and the reward hypothesis, 2020. URL https://readingsml.blogspot.com/2020/03/constrained-mdps-and-reward-hypothesis.html.

[69] C. Tessler, D. J. Mankowitz, and S. Mannor. Reward constrained policy optimization. In *International Conference on Learning Representations*, 2019. URL https://openreview.net/forum?id=SkfrvsA9FX

[70] T. Zahavy, A. Cohen, H. Kaplan, and Y. Mansour. Apprenticeship learning via frank-wolfe. *AAAI*, 2020.

[71] T. Zahavy, A. Cohen, H. Kaplan, and Y. Mansour. Apprenticeship learning via frank-wolfe. The Conference on Uncertainty in Artificial Intelligence (UAI), 2020.

[72] J. Zhang, A. Koppel, A. S. Bedi, C. Szepesvari, and M. Wang. Variational policy gradient method for reinforcement learning with general utilities. *arXiv preprint arXiv:2007.02151*, 2020.
[79] M. Zhang, Y. Wang, X. Ma, L. Xia, J. Yang, Z. Li, and X. Li. Wasserstein distance guided adversarial imitation learning with reward shape exploration. In 2020 IEEE 9th Data Driven Control and Learning Systems Conference (DDCLS), pages 1165–1170. IEEE, 2020.

[80] M. Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In Proceedings of the 20th international conference on machine learning (icml-03), pages 928–936, 2003.
A Checklist

1. For all authors
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] We discussed the case where $f$ is non-convex and experimented with DIAYN as an example. We also discussed the fact that if the RL problem is a hard exploration problem, then the subroutine we use to solve it must be able to solve hard exploration problems.
   (c) Did you discuss any potential negative societal impacts of your work? [No]. This is a theoretical paper and to the best of our understanding it should not have any societal impacts.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]

3. If you ran experiments. We note that our experiments are a proof of concept for our approach. All of our experiments were performed with a basic agent, without hyper parameter tuning and evaluated on simple grid worlds. We are happy to share more details in case the reviewers will find something missing.
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No]
   (b) Did you specify all the training details (eg, data splits, hyperparameters, how they were chosen)? [Yes]
   (c) Did you report error bars (eg, with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (eg, type of GPUs, internal cluster, or cloud provider)? [No]

4. If you are using existing assets (eg, code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes]
   (b) Did you mention the license of the assets? [No]
   (c) Did you include new assets either in the supplemental material or as a URL? [No]
   (d) Did you discuss whether and how consent was obtained from people whose data you’re using/curating? [No]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [No]

5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [No]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [No]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [No]
### B  Proposition 1

**Proposition 1.** For both the average and the discounted case, the agent objective function Eq. (3) can be written in terms of the occupancy measure as $J_\pi = \sum_{s,a} r(s,a)d_\pi(s,a)$.

**Proof.** Beginning with the discounted case, the average cost is given by

$$J_\pi^\gamma = (1 - \gamma)\mathbb{E}_{t=1}^{\infty} \gamma^t r_t$$

$$= (1 - \gamma) \sum_{t=1}^{\infty} \sum_s \mathbb{P}_\pi(s_t = s) \pi(s,a) \gamma^t r(s,a)$$

$$= (1 - \gamma) \sum_{s,a} \left( \sum_{t=1}^{\infty} \gamma^t \mathbb{P}_\pi(s_t = s) \pi(s,a) \right) r(s,a)$$

$$= \sum_{s,a} d_\pi^\gamma(s,a) r(s,a).$$

Similarly, for the average reward case

$$J_\pi^{\text{avg}} = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_{t=1}^{T} r_t$$

$$= \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_s \mathbb{P}_\pi(s_t = s) \pi(s,a) r(s,a)$$

$$= \sum_{s,a} \left( \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{P}_\pi(s_t = s) \pi(s,a) \right) r(s,a)$$

$$= \sum_{s,a} d_\pi^{\text{avg}}(s,a) r(s,a).$$

$\blacksquare$
C  FW algorithms

C.1  Pseudo code

Algorithm 3: Frank-Wolfe algorithm

Input: a convex and smooth function \( f \)

2: Initialize: Pick a random element \( d_x^0 \in K \).

for \( i = 1, \ldots, T \) do

4: \( d_x^{k+1} = \arg \max_{\pi \in \Pi} d_x \cdot -\nabla f(\bar{d}_x^k) \)

\( \bar{d}_x^{k+1} = (1 - \alpha_i) d_x^k + \alpha_i d_x^{k+1} \)

6: end for

C.2  Linear convergence

Theorem 2 (Linear Convergence [35]). Suppose that \( f \) has \( L \)-Lipschitz gradient and is \( \mu \)-strongly convex. Let \( D = \{ d_x, \forall \pi \in \Pi \} \) be the set of all the state occupancy’s of deterministic policies in the MDP and let \( K = \text{Co}(D) \) be its Convex Hull. Such that \( K \) a polytope with vertices \( D \), and let \( M = \text{diam}(K) \). Also, denote the Pyramidal Width of \( D, \delta \)

Then the suboptimality \( h_t \) of the iterates of all the fully corrective FW algorithm decreases geometrically at each step, that is

\[
 f(\bar{d}_x^{k+1}) \leq (1 - \rho) f(\bar{d}_x^k), \text{ where } \rho = \frac{\mu \delta^2}{4LM^2}
\]

D  Sample complexity proofs

**Lemma** (The sample complexity of non-stationary RL algorithms in convex MDPs). For a convex function \( f \), running Algorithm [1] with an oracle cost player with regret \( \bar{R}_K^x \leq c_0/\sqrt{K} \) and UCRL2 as a policy player returns a mixed policy \( \pi^K \) that satisfies \( f(\bar{d}_x^K) - f^{\text{OPT}} \leq \epsilon \) with probability \( 1 - \delta \) after \( K = O \left( \frac{D^2 S^2 A}{\epsilon^4 \delta^4} \log^2(\frac{2D S A}{\epsilon \delta}) \right) \) steps.

**Proof.** In Theorem 1 and the discussion below it, we showed that \( f(\bar{d}_x^K) - f^{\text{OPT}} \leq \bar{R}_K^x + \bar{R}_K^y \). From Lemma 3, we have that with probability \( 1 - \delta \), a "positive event" happens, and the regret of the UCRL2 player, \( \epsilon_K \), is upper bounded by \( \bar{R}_K^x = \frac{1}{K} \sum_{k=1}^{K} J^*_k - J^K_k \leq c_1 D S \sqrt{A \log(K/\delta')/K} \) for some constant \( c_1 \). Recall that the function \( f \) has bounded gradients and therefore, the non-stationary reward is upper bounded by 1. Thus, when the "positive event" does not happen (with probability \( \delta' \)), we can always upper bound the regret by \( \bar{R}_K^x = \frac{1}{K} \sum_{k=1}^{K} J^*_k - J^K_k \leq 1 \). Using Markov’s inequality, we have that

\[
 \Pr(f(\bar{d}_x^K) - f^{\text{OPT}} \geq \epsilon) \leq \frac{\mathbb{E}(f(\bar{d}_x^K) - f^{\text{OPT}})}{\epsilon} \leq \frac{\mathbb{E}(\bar{R}_K^x + \bar{R}_K^y)}{\epsilon} \\
\leq \frac{1}{\epsilon} \left( (1 - \delta') c_1 D S \sqrt{A \log(K/\delta')/K} + \delta' + c_0 \sqrt{1/K} \right) \\
\leq \frac{1}{\epsilon} \left( (c_2 D S \sqrt{A \log(K/\delta')/K} + \delta') \right),
\]

thus, if we choose \( \delta' = \frac{\epsilon^2}{2} \), we have that in order for \( \Pr(f(\bar{d}_x^K) - f^{\text{OPT}} \geq \epsilon) \) to be smaller than \( \delta \) after \( K \) steps, it is enough to find a value for \( K \) such that \( \frac{1}{\epsilon} c_2 D S \sqrt{A \log(2K/\epsilon \delta)}/K \leq \delta/2 \), which is achieved for \( K = \frac{c_2 D S^2 A \log(2D S A/\epsilon \delta)}{\epsilon^4 \delta^4} \) .

**Lemma** (The sample complexity of approximate best response in convex MDPs with average occupancy measure). For a convex function \( f \), running Algorithm [2] with an oracle cost player with regret \( \bar{R}_K^x = O(1/K) \) and an approximate best response policy player that solves the average reward RL problem in iteration \( k \) to accuracy \( \epsilon_k = 1/k \) returns an occupancy measure \( \bar{d}_x^K \) that satisfies \( f(\bar{d}_x^K) - f^{\text{OPT}} \leq \epsilon \) with probability \( 1 - \delta \) after seeing \( O(t_{\text{mix}} S^2 A \log(2K/\epsilon \delta)/\epsilon^3 \delta^3) \) samples. Similarly, for \( \bar{R}_K^x = O(1/\sqrt{K}) \), setting \( \epsilon_k = 1/\sqrt{K} \) requires \( O(t_{\text{mix}} S^2 A \log(2K/\epsilon \delta)/\epsilon^4 \delta^4) \) samples.
To solve an MDP to accuracy $\epsilon_k$, it is sufficient to run an RL algorithm for $O(1/\epsilon_k^2)$ iterations. This is a lower bound and an upper bound in $\epsilon$, see, for example [37] for an upper bound of $O \left( \frac{t_{\text{mix}}SA}{\epsilon^2} \log(1/\delta) \right)$ and a lower bound [38] of $O \left( \frac{t_{\text{mix}}SA}{\epsilon^3} \log(1/\delta) \right)$ for the average reward case.

We continue the proof using the algorithm of [18] as the approximate best response player, i.e., we invoke their algorithm at iteration $k$ to find an $\epsilon_k$-optimal solution with probability $1 - \delta_k = 1 - \delta'/K$. Applying the union bound over the iterations gives us that with probability of $1 - \delta'$, the regret of the policy player is $\tilde{R}_K^\pi = \frac{1}{R} \sum \epsilon_k$.

Then, via Markov inequality we get that

$$\Pr(\bar{f}(\tilde{d}_n^K) - f^{\text{OPT}} \leq \epsilon) \leq \frac{\mathbb{E}[f(\tilde{d}_n^K) - f^{\text{OPT}}]}{\epsilon} \leq \frac{\mathbb{E}\tilde{R}_K^\pi + \tilde{R}_K^\lambda}{\epsilon} \leq \frac{1}{\epsilon} \left( \frac{c_4}{\sqrt{K}} + \delta' \right).$$

Setting $\delta' = \epsilon \delta / 2$ implies that it is enough to run the algorithm for $K = c_5/\epsilon^2 \delta^2$ iterations to find an $\epsilon$-optimal solution with probability of $1 - \delta$.

In each iteration $k$, in order to find an $\epsilon_k = c_1/\sqrt{K}$ optimal solution w.p. $1 - \delta'/K$, we need to collect $c_2 k^{t_{\text{mix}}^2} S A \log(K/\delta')$ samples. Thus, the total number of samples is

$$\sum_{k=1}^{c_5/\epsilon^2 \delta^2} c_2 k^{t_{\text{mix}}^2} S A \log(K/\delta') = c_2 t_{\text{mix}}^2 S A \log(K/\delta') \sum_{k=1}^{c_5/\epsilon^2 \delta^2} k \leq c_3 t_{\text{mix}}^2 S A \log(2K/\epsilon \delta)/\epsilon^4 \delta^4.$$

In the second scenario we have a cost player with constant regret, and therefore average regret of $\tilde{R}_K^\lambda \leq c_1/K$, which is possible to achieve under some assumptions [33]. We set $\epsilon_k = c/k$ and get that

$$\tilde{R}_K = \tilde{R}_K^\pi + \tilde{R}_K^\lambda \leq \frac{1}{K} \sum_{k=1}^{K} c_2/k + c_1/K \leq \frac{\log(K)}{K} c_3 \sqrt{K} + c_1/K \leq c_4 \frac{\log(K)}{K}.$$

Then, via Markov inequality we get that

$$\Pr(\bar{f}(\tilde{d}_n^K) - f^{\text{OPT}} \leq \epsilon) \leq \frac{\mathbb{E}[f(\tilde{d}_n^K) - f^{\text{OPT}}]}{\epsilon} \leq \frac{\mathbb{E}\tilde{R}_K^\pi + \tilde{R}_K^\lambda}{\epsilon} \leq \frac{1}{\epsilon} \left( \frac{c_4 \log(K)}{K} + \delta' \right).$$

Setting $\delta' = \epsilon \delta / 2$ implies that it is enough to run the algorithm for $K = c_5/\epsilon \delta$ iterations to find an $\epsilon$-optimal solution with probability of $1 - \delta$.

In each iteration $k$, in order to find an $\epsilon_k = c_1/k$ optimal solution w.p. $1 - \delta'/K$, we need to collect $c_2 k^{t_{\text{mix}}^2} S A \log(K/\delta')$ samples, which leads to a total number of samples of

$$\sum_{k=1}^{c_5/\epsilon \delta} c_2 k^{t_{\text{mix}}^2} S A \log(K/\delta') = c_2 t_{\text{mix}}^2 S A \log(K/\delta') \sum_{k=1}^{c_5/\epsilon \delta} k^2 \leq c_3 t_{\text{mix}}^2 S A \log(2K/\epsilon \delta)/\epsilon^3 \delta^3. \quad \blacksquare$$

**Discussion on how to choose the schedule for $\epsilon_k$**

The overall regret of the game is the sum of the regret of the policy player and the cost player, and the regret of the game is asymptotically

$$\tilde{R}_K = \tilde{R}_K^\pi + \tilde{R}_K^\lambda = O \left( \max(\tilde{R}_K^\pi, \tilde{R}_K^\lambda) \right)$$

(12)

Note that the iterations are independent from each other from the perspective of the approximate best response player so it is possible to apply the union bound. This is because each iteration involves solving a new MDP, and the upper bound does not make any assumptions about the structure of the reward in this MDP.
Consider the general case of \( \epsilon_k = 1/k^p \). Note that for the average regret \( \frac{1}{K} \sum_{k=1}^{K} 1/k^p \) to go to zero as \( K \) grows, the sum \( \frac{1}{K} \sum_{k=1}^{K} 1/k^p \) must be smaller than \( K \), so \( p \) must be positive. In addition, for larger values of \( p \), \( \epsilon_k \) is smaller. Thus the regret is smaller, but at the same time, it requires more samples to solve each RL problem. Inspecting the maximum in Eq. (12), we observe that it does not make sense to choose a value for \( p \) for which \( \frac{1}{K} \sum_{k=1}^{K} 1/k^p < R^\lambda_K \), since it will not improve the overall regret and will require more samples, than, for example, setting \( p \) such that \( \frac{1}{K} \sum_{k=1}^{K} 1/k^p = R^\lambda_K \).

Thus, in the case that the agent has constant regret, \( R^\lambda_K = O(1/K) \), we set \( p \in (0, 1) \), and in the case that the cost player has regret of \( R^\lambda_K = O(1/\sqrt{K}) \), we set \( p \in (0, 0.5) \).

We now continue and further inspect the regret. We have that \( \frac{1}{K} \sum_{k=1}^{K} \epsilon_k = \frac{1}{K} \sum_{k=1}^{K} 1/k^p = O(k^{-p}) \) for \( p \in (0, 1) \), and \( \log(K)/K \) for \( p = 1 \). Neglecting logarithmic terms, we continued with \( O(k^{-p}) \) for both cases. In other words, it is sufficient to run the meta-algorithm for \( K = 1/\epsilon^p \) iterations to guarantee an error of at most \( \epsilon \) for the convex MDP problem.

Thus, to solve an MDP to accuracy \( \epsilon_k = 1/k^p \) it requires \( k^{2p} \) iterations, and the overall sample complexity is therefore \( \sum_{k=1}^{1/\epsilon} k^{2p} = O(1/\epsilon^{2p+1}) \).

The function \( 1/\epsilon^{2p+1} \) is monotonically increasing in \( p \), so it attains minimum for the highest value of \( p \) which is 0.5 or 1, depending on the cost player. We conclude that the optimal sample complexity with approximate best response is \( O(1/\epsilon^3) \) for the cost player that has constant regret and \( O(1/\epsilon^4) \) for a cost player with average regret of \( R^\lambda_K = O(1/\sqrt{K}) \).

### E Proof sketch for Lemma 3

We denote by \( r_k^* \) the optimal average reward at time \( k \) in an MDP with dynamics \( P \) and reward \( r_k = -\lambda_k \). We want to show that

\[
R_k = \sum_k r_k^* - r_k(s_k, a_k) \leq c/\sqrt{K},
\]

that is, that the total reward that the agent collects has low regret compared to the sum of optimal average rewards.

To show that, we make two minor adaptations to the UCRL2 algorithm and then verify that its original analysis also applies to this non-stationary setup. The first modification is that the nonstationary version of UCRL2 uses the known reward \( r_k \) at time \( k \) (which in our case is the output of the cost player) instead of estimating the unknown, stochastic, stationary, extrinsic reward. Since the current reward \( r_k \) is known and deterministic, there is no uncertainty about it, and we only have to deal with uncertainty with respect to the dynamics. The second modification is that we compute a new optimistic policy (using extended value iteration) in each iteration. This optimistic policy is computed with the current reward \( r_k \), and the current uncertainty set about the dynamics \( \mathcal{P}_k \). This also means that all of our episodes are of length 1.

After making these two clarifications, we follow the proof of UCRL2 and make changes when appropriate. We note that the analysis, basically, does not require any modifications, but we repeat the relevant parts for completeness. We begin with the definition of the regret at episode \( k \), which is now just the regret at time \( k \):

\[
\Delta_k = \sum_{s,a} v_k(s,a)(r_k^* - r_k(s,a)),
\]

where \( v_k(s,a) \) in our case is an indicator on the state action pair \( s_k, a_k \), and \( R_k = \sum_k \Delta_k \).

The instantaneous regret \( \Delta_k \) measures the difference between the optimal average reward \( r_k^* \), with respect to reward \( r_k \), and the reward \( r_k(s,a) \) that the agent collected at time \( k \) by visiting state \( s \) and taking action \( a \) from the reward that is produced by the cost player.

Section 4.1 in the UCRL2 paper is the first step in the analysis. It bounds possible fluctuations in the random reward. This step is not required in our case since our reward at time \( k \) is the output of the cost player, which is known in all the states and deterministic.
Section 4.2 considers the regret that is caused by failing confidence regions, that is, the event that the true dynamics and true reward are not in the confidence region. In our case there is only confidence region for the dynamics (since the reward is known), which we denote by \( \mathcal{P}_k \). Summing the expected regret from episodes in which \( P \not\in \mathcal{P}_k \) results in a \( \sqrt{K} \) term in the regret,

\[
\Delta_k \leq \sum_{s,a} v_k(s,a)(r_k^* - r_k(s,a)) + \sqrt{K},
\]

where from now on, we continue with the event that \( P \in \mathcal{P}_k \).

Next, we denote the optimistic policy and optimistic MDP as the solution of the following problem \( \tilde{\pi}_k, \tilde{P}_k = \arg \max_{\pi \in \Pi, P' \in \mathcal{P}_k} J^P_{\pi_k} \). In addition, we denote by \( \hat{r}_k \) the optimistic average reward, that is, the average reward of the policy \( \tilde{\pi}_k \) in the MDP with the optimistic dynamics \( \tilde{P}_k \) and reward \( r_k \).

We now continue with the case that \( P \in \mathcal{P}_k \). The next step is to bound the difference between the optimal average reward \( r^*_k \) and the optimistic average reward \( \hat{r}_k \). We note that both \( \hat{r}_k \) and \( r^*_k \) are average rewards that correspond to \( r_k \). The difference between them is that \( r^*_k \) is the optimal average reward in an MDP with the true dynamics \( P \) and \( \hat{r}_k \) is the optimal average reward in an MDP with the optimistic dynamics \( \tilde{P}_k \). Thus, the fact that the reward is known, in our case, does not change the fact that the optimistic reward is a function of the dynamics uncertainty set \( \mathcal{P}_k \).

To compute the optimistic policy and dynamics, UCRL2 uses the extended value iteration procedure of [64] to efficiently compute the following iterations:

\[
u_0(s) = 0 \tag{13}\]

\[
u_{i+1}(s) = \max_{a \in A} \left\{ r_k(s,a) + \max_{P \in \tilde{P}_k} \sum_{s' \in S} P(s'|s,a)\nu_i(s') \right\}, \tag{14}\]

Using Theorem 7 from [36] we have that running extended value iteration to find the optimistic policy in the optimistic MDP for \( t_k \) iterations guarantees that \( \hat{r}_k \geq r^*_k - 1/\sqrt{t_k} \). Thus, we have that:

\[
\Delta_k \leq \sum_{s,a} v_k(s,a)(r^*_k - r_k(s,a)) + \sqrt{K} \leq \sum_{s,a} v_k(s,a)(\hat{r}_k - r_k(s,a)) + 1/\sqrt{t_k} + \sqrt{K}
\]

Using Eq. (13), we write the last iteration of the extended value iteration procedure as:

\[
u_{i+1}(s) = r_k(s_k, \tilde{\pi}_k(s)) + \sum_{s' \in S} \tilde{P}_k(s'|s, (\tilde{\pi}_k(s)))u_i(s') \tag{15}\]

Theorem 7 from [36] guarantees that after running extended value iteration for \( t_k \) we have that

\[
\|u_{i+1}(s) - u_i(s) - \tilde{r}_k\| \leq 1/\sqrt{t_k}.
\]

Plugging Eq. (15) in Eq. (16) we have that:

\[
\|r_k(s_k, \tilde{\pi}_k(s)) - \tilde{r}_k + \sum_{s' \in S} \tilde{P}_k(s'|s, (\tilde{\pi}_k(s)))u_i(s') - u_i(s)\| \leq 1/\sqrt{t_k},
\]

and therefore

\[
\tilde{r}_k - r_k(s_k, a_k) = \tilde{r}_k - r_k(s_k, \tilde{\pi}_k(s)) \leq v_k(\tilde{P}_k - I)u_i + 1/\sqrt{t_k}.
\]

In the next step in the proof, the vector \( u_i \) is replaced with \( w_k \), which is later upper bounded by the diameter of the MDP \( D \). To conclude, we have that

\[
\Delta_k \leq \sum_{s,a} v_k(s,a)(\tilde{r}_k - r_k(s,a)) + 1/\sqrt{t_k} + \sqrt{K} \leq v_k(\tilde{P}_k - I)w_k + 2/\sqrt{t_k} + \sqrt{K}.
\]

From this point on, the proof follows by bounding the term \( v_k(\tilde{P}_k - I)w_k \), which is only related to the dynamics, and combines all of the previous results into the final result, thus, it is possible to follow the original proof without any modification. Since the leading terms in the original proof come from uncertainty about the dynamics, we obtain the same bound as in the original paper.
F Experiments: Entropy constrained RL.

Above, we presented a principled approach to using standard RL algorithms to solve convex MDPs. We also suggested that DRL agents can use this principle and solve convex MDPs by optimizing the reward from the cost player. We now demonstrate this by performing experiments with Impala [20], a distributed actor-critic DRL algorithm. Our main message is that in domains where Impala can solve RL problems (e.g., problems without hard exploration), it can also solve convex MDPs.

Here we focus on an MDP with a convex constraint, where the goal is to maximize the extrinsic reward provided by the environment with the constraint that the entropy of the state-action occupancy measure must be bounded below. In other words, the agent must solve \( \max_{d_{\pi} \in \mathcal{K}} \sum_{s,a} r(s,a) d_{\pi}(s,a) \) subject to \( H(d_{\pi}) \geq C \), where \( H \) denotes entropy and \( C > 0 \) is a constant. The policy that maximizes the entropy over the MDP acts to visit each state as close to uniformly often as is feasible. So, a solution to this convex MDP is a policy that, loosely speaking, maximizes the extrinsic reward under the constraint that it explores the state space sufficiently. The presence of the constraint means that this is not a standard RL problem in the form of Eq. (1). However, the agent can solve this problem using the techniques developed in this paper, in particular those discussed in Section 5.

We evaluated the approach on the bsuite environment ‘Deep Sea’, which is a hard exploration problem where the agent must take the exact right sequence of actions to discover the sole positive reward in the environment; more details can be found in [54]. In this domain, the features are one-hot state features, and we estimate \( d_{\pi} \) by counting the state visitations. For these experiments we chose \( C \) to be half the maximum possible entropy for the environment, which we can compute at the start of the experiment and hold fixed thereafter. We equipped the agent with the (non-stationary) Impala algorithm, and the cost-player used FTL. We present the results in Figure 2, where we compare the basic Impala agent, the entropy-constrained Impala agent and bootstrapped DQN [53]. As made clear in [51] algorithms that do not properly account for uncertainty cannot in general solve hard exploration problems. This explains why vanilla Impala, considered a strong baseline, has such poor performance on this problem. Bootstrapped DQN accounts for uncertainty via an ensemble, and consequently has good performance. Surprisingly, the entropy regularized Impala agent performs approximately as well as bootstrapped DQN, despite not handling uncertainty. This suggests that the entropy constrained approach, solved using Algorithm 1, can be a reasonably good heuristic in hard exploration problems.

![Figure 2: Entropy constrained RL](image-url)