Numerical analysis of the applicability of engineering linear models of inelastic behavior and fracture for the description of porous rocks under confined conditions

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Abstract. The paper is devoted to the numerical study of the features of inelastic deformation and fracture of representative mesoscopic volumes of porous brittle materials (including rocks) under the condition of triaxial compression. The study is focused on the analysis of applicability of the classical macroscopic “linear” criteria of plasticity and strength for the description of the mechanical response and assessment of the onset of fracture of representative volumes. Analysis of simulation results confirmed that conventional plasticity conditions (criteria), which take into account the contribution of the local average stress in the linear approximation, are able to adequately describe the mechanical behavior of confined mesoscopic representative volumes of brittle porous rocks only up to the stage of formation of the system of distributed relatively long cracks. We found that material work softening under confined conditions is concerned not with loss of integrity of the sample, but with subsequent formation of localized shear band. The conditions for actual fracture of confined brittle porous rocks are adequately described using "linear" failure criteria with parameters determined on the base of multiaxial compression tests.

1. Introduction
The mechanical behavior of porous rocks under compressive volumetric stress is of great interest from both fundamental and technical standpoints [1-3]. Porous rocks typically have multiscale porosity (from nanoscale isolated pores up to micrometer size pores and channels). Pores of the largest scale are typically the least strong and largely determine the mechanical characteristics of rock at the mesoscopic structural scale [4-5]. In addition, the magnitude and variation of the largest scale porosity determine the maximum attainable values of the fluid flow rate and its variations. Therefore, understanding the change in pore space of the largest scale during the course of confined shear or triaxial compression is important in many contexts, such as hydrogeology of groundwater, oil and gas reservoir engineering and so on.

It is well known that the mechanical response of rocks (including reservoirs) at all scales is largely determined by the features of stress state, including the ratio between volumetric and deviator stresses [1-3,6]. This is especially pronounced for porous rocks with brittle skeleton (consolidated sandstones, granites, diorites, basalts and so on). In particular, uniaxially compressed (unconfined) rock samples are often broken in a brittle manner by dynamically forming and propagating main cracks. At the same time, even brittle porous rocks show strongly pronounced inelastic behavior under confined loading,
starting with a certain value of the lateral pressure. In this case, fracture has a "quasi-viscous" mode and is associated with the formation of a localized shear band, or (at still greater pressures) by a volumetric cataclastic flow. Interpretation and generalization of the results of experimental studies of the behavior of confined porous materials are key issues of both rock mechanics and rock engineering.

Forecasting of macroscopic mechanical properties (including inelasticity, strength and change in pore space) of such materials under complex loading conditions is traditionally carried out with use of macroscopic engineering criteria and models of plasticity and failure. The most common in engineering practice are linear criteria and models in which contributions of invariants of stress and strain tensors are taken into account in the linear approximation. Well-known examples are Mohr-Coulomb and Mises-Schleicher (or Drucker-Prager) criteria based models [7]. An advantage of linear criteria is a small number of model parameters and the possibility of their determination on the basis of a small number of standard laboratory tests (for example, uniaxial compression and uniaxial tension or three-point bending test). It is known that linear criteria and associated models of inelastic behavior and failure of rocks well describe the mechanical response of low-porous (porosity \( \sim 1\% \) and less) consolidated brittle solids even under high ambient pressures (up to the brittle-to-ductile transition threshold, above which classical mechanisms of plastic deformation associated with the motion of atomic-scale defects can also be involved). At the same time, applicability of linear criteria to the macroscopic description of the mechanical response of confined brittle rocks with essentially non-zero porosity values (from a few percent up to tens of percent) is a subject of wide discussion. Despite the large number of works in this field [8-10], there is still no unambiguous understanding of pressure and shear strain intervals in which macroscopic models of plasticity and failure with linear criteria are applicable to describe the response of confined porous brittle materials to mechanical loading. These questions are relevant not only for the macroscopic study of the behavior of porous rocks, but also for an adequate description of the inelastic response of brittle solids at the mesoscopic scale (at the scale of the largest pores).

The present paper is devoted to the numerical study of the above discussed problem. The object of the study was mesoscopic fragments of brittle porous model material containing a sufficiently large number of mesoscopic pores. The study was carried out using computer simulation by the discrete element method.

2. Problem statement
We considered 2D model rectangular samples with round pores in the plane stress approximation (figure 1). The samples imitate the mesoscopic region of brittle solid with multiscale pore structure.
The pores of the largest scale (‘‘mesoscopic’’ pores) were modelled directly as stochastically distributed holes (small white circles in figure 1), while the influence of pores of smaller scales on the material response was taken into account indirectly through the parameters of failure criterion of the skeleton. In the study we varied the initial value of mesoscopic porosity from 5% up to 15% (typical interval for consolidated sandstones).

The material of skeleton was treated as isotropic and elastic-brittle. In the study we used the following values of elastic constants of the skeleton: Young’s modulus $E = 82$ GPa, Poisson’s ratio $\nu = 0.12$. Note that these values are close to macroscopic elastic moduli of typical low-porosity sandstones.

The model samples were numerically simulated by the method of movable cellular automata (MCA). MCA belongs to the class of numerical methods of discrete elements and particularly to the group of explicit methods of simply deformable elements. Here the term ‘‘explicit’’ means that evolution of an ensemble of interacting discrete elements is defined by a numerical solution of the system of Newton-Euler equations of motion using an explicit numerical scheme. In the simply deformable element approximation the state of a discrete element (movable cellular automaton) is determined by the average stress tensor $\sigma_{\text{avg}}$ and average strain tensor $\varepsilon_{\text{avg}}$, which are calculated through interaction forces, pair overlaps and relative shear displacements in the interacting pairs of the automaton with its neighbours [11,12]. The most important feature of the MCA method is the use of the original many-body formulation of the element-element interaction. It allowed us to implement various models of elasticity, plasticity and fluid affected behaviour within the MCA method [12-14].

In the study we used linear model of elasticity (generalized Hooke’s law for locally isotropic materials) to describe the mutual relation between average stresses and strains in the volume of the cellular automaton [12]. Local failure in the skeleton was modelled by changing the state of a pair of interacting automata from linked (or bonded) to unlinked. As the failure criterion we applied the two-parameter Drucker-Prager criterion in the following formulation:

$$1.5(\lambda - 1)\sigma_{\text{mean}} + 0.5(\lambda + 1)\sigma_{\text{eq}} = \sigma_c$$

(1)

where $\lambda = \sigma_c / \sigma_t$ is the strength ratio of skeleton material under uniaxial compression ($\sigma_t$) and tension ($\sigma_c$), $\sigma_{\text{mean}}$ and $\sigma_{\text{eq}}$ are local values of the mean stress and the equivalent stress. Detailed description of the features of the numerical method and model implementation can be found in the paper [11,12].

It is well known that the strength ratio of brittle materials is largely determined by the concentration and linear dimensions of discontinuities and other defects. In particular, the increase in the size/length of initial discontinuities is accompanied by a strong fall of the tensile strength $\sigma_t$ with a significantly smaller decrease in the compressive strength $\sigma_c$ (the value of $\lambda$ increases). In the study we considered two values of the parameter $\lambda$: $\lambda=1$ and $\lambda=3$ (in both cases $\sigma_c = 190$ MPa). Both selected values correspond to a fairly homogeneous and consolidated structure of the skeleton walls that do not contain mesoscopic pores. The case $\lambda=1$ can be considered as a theoretically achievable limit corresponding to the perfect internal structure of the skeleton walls (there are no low scale pores and other defects). The case $\lambda=3$ imply low scale pores in the skeleton.

We modelled axial compression of confined samples at a small constant velocity $V_y$. Mechanical confinement was assigned by applying constant compressive force $F_\tau$ to the lateral faces of the model samples (hereinafter we will characterize lateral load in term of lateral pressure $\sigma_\tau$, which is a specific value of $F_\tau$). We measured axial and lateral resistance forces and strains during the course of axial deformation. These parameters of the mechanical response were used to calculate invariants of stress and strain tensors for the considered representative volumes (model samples) and to estimate the applicability of Mises-Schleicher criterion as a criterion of plasticity and failure of confined porous brittle solids.
3. Features of fracture of confined porous samples

The simulation results showed that under conditions of unconfined \((\sigma_x=0)\) uniaxial compression or tension, all considered porous brittle samples deform linear-eligastically up to fracture. Fracture of the samples occurs by the dynamic propagation of the main crack. As the degree of mechanical confinement of the samples characterized by the magnitude of applied lateral stress \(\sigma_x\) increases, the behavior of the samples during the course of axial compression changes from the typical elastic-brittle to essentially nonlinear and inelastic. Figure 2 shows typical diagrams of the axial compression of confined porous model samples, which are characterized by different values of the strength ratio \(\lambda\) of skeleton material. One can see that at low degrees of confinement (at \(\sigma_x\) below a certain threshold value of lateral pressure \(\sigma_{\text{trans}}\)), the samples are deformed elastically (curves 1 and 2). At \(\sigma_x>\sigma_{\text{trans}}\), the deformation pattern of porous samples changes from elastic-brittle to quasi-plastic (the latter means that integral response of the sample is inelastic). With increasing \(\sigma_x\), both the elastic limit and the length of the irreversible part of the loading diagram increase (curves 3-6). Inelastic response of the samples is provided by the formation of “damages” (short cracks that connect adjacent pores), subsequent merging of these damages into longer cracks and finally with the formation of a main crack (brittle fracture) or localized shear band (quasi-viscous fracture). Note that the shear band is a highly fragmented strip of material with low shear resistance.

![Figure 2](https://example.com/figure2.jpg)

**Figure 2.** Loading diagrams of porous samples (porosity 10%) with skeleton walls composed of structurally “perfect” material \(\lambda=1\) (a) and material with pores of small scales \(\lambda=3\) (b).

Analysis of simulation results showed that the particular value of \(\sigma_{\text{trans}}\) decreases with an increase in the porosity of the material and a decrease in the value of the parameter \(\lambda\). For the considered interval of initial porosities, the values of \(\sigma_{\text{trans}}\) are in the range 30-80 MPa, which is close to the corresponding threshold values for porous sandstones.

The sequence of stages of damage accumulation in confined samples depends on not only on lateral pressure \(\sigma_x\) but also on the structural perfection of the material of skeleton walls. At low lateral pressures \(\sigma_x\) below \(\sigma_{\text{trans}}\), fracture of the samples is dynamic and occurs by propagation of one or more main cracks without the preceding stage of damage accumulation (duration of this stage and its contribution to stress relaxation and redistribution in the sample is negligible). Loading diagrams have linear profile up to fracture (curves 1 and 2 in figure 2). Fracture of the samples at \(\sigma_x>\sigma_{\text{trans}}\) develops in several stages and has a “quasi-viscous” character (curves 3-6 in figure 2). In this range of lateral pressures, the inelastic behavior of the samples during compression is largely determined by the influence of the local mean stress \(\sigma_{\text{mean}}\) on shear strength of skeleton walls (this influence is characterized by the value of the parameter \(\lambda\)).

In particular, at \(\lambda=3\) (shear strength of skeleton material is significantly sensitive to local mean stress \(\sigma_{\text{mean}}\)), three stages of inelastic deformation of porous samples can be distinguished: (1) the
formation of "short" cracks that connect several adjacent micropores (these cracks relatively evenly distributed in the volume of the sample, figure 3a); (2) elongation and merging of short cracks (figure 3b); (3) localization stage, namely integration of some adjacent cracks in the shear band (figure 3c). The shear band is a thin zone of highly fractured material that divide the sample into two parts. Formation of a shear band is accompanied by transition from strain hardening to strain softening stage in the loading diagram.

![Figure 3](image-url)

**Figure 3.** The example of the main stages of fracture of porous sample (porosity 10%) at $\sigma_x=90$ MPa ($\lambda=3$). Figure (a) shows enlarged part of the sample.

In samples with skeleton walls comprised by “perfect” (non-porous and defect-free) material ($\lambda=1$), fracture at $\sigma_x > \sigma_{\text{eq}}$ is generally similar to that described above (for skeleton material with $\lambda=3$). However, at very high values of $\sigma_{\text{lat}} (>100$ MPa) there is no localization of deformations in the form of a shear band. Fracture of samples in this pressure range is spatially distributed (crushing of the entire volume of the sample and transition of the porous consolidated material to a structure close to the granular one). Thus, brittle porous materials with skeleton walls composed of "perfect" material show cataclastic pattern of fracture at high confining pressures.

4. Analysis of the applicability of Mises-Schleicher criterion to confined brittle porous materials

As was mentioned in the Introduction, macroscopic (integral) description of the behavior of materials under mechanical loading is carried out using criteria and models of plasticity and failure. One of the widely used engineering criteria of plasticity of rocks is the Mises-Schleicher condition:

$$\sigma_{\text{MS}} = \alpha \sigma_{\text{eq}} + \frac{\sigma_{\text{eq}}^3}{\sqrt{3}} = Y$$

(2)

where $\alpha$ is internal friction coefficient and $Y$ is cohesion. In a general case the value of $Y$ is not a constant but an increasing function of accumulated inelastic strain or other deformation-related parameter that characterizes concentration and other parameters of accumulated damages.

We analyzed the ability of the criterion (2) to describe numerically obtained stress-strain curves at different magnitudes of lateral pressure $\sigma_x$ with use of the same (the only) value of the coefficient of internal friction $\alpha$. The determination of the value of $\alpha$ for each loading curve was carried out over the initial section of the inelastic deformation stage. The most important result of this analysis is that there is a unique value of $\alpha$ for each of the considered materials. We found that the particular magnitude of $\alpha$ insignificantly decreases with increase in the “mesoscale” porosity and strongly depends on the sensitivity of the material of skeleton to local pressure ($\alpha \approx 0.7$ at $\lambda=3$ and $\alpha \approx 0.5$ at $\lambda=1$, porosity 10%). The results of the study confirm the relevance of the Mises-Schleicher criterion as a condition for the onset of inelastic strain in mechanically confined porous materials in a wide range of lateral pressures up to brittle-to-ductile transition (>100 MPa).

Derived internal friction coefficient $\alpha$ for the particular material was used for building the unified strain hardening curve, which would be valid at different lateral pressures. In our previous paper [12]...
we suggested the following deformation parameter, which indirectly takes into account accumulated damages: 

\[ \varepsilon_{MS} = \varepsilon_{eq} \sqrt{3} + \varepsilon_{mean} \frac{K \alpha}{3M}, \]

where \( K \) and \( M \) are bulk and shear moduli, \( \varepsilon_{eq} \) and \( \varepsilon_{mean} \) – are equivalent and mean strains. Special study show that inelastic stage of deformation of porous sample under loading at different \( \sigma_{trans} \) begins at various axial strains but at the same value of \( \varepsilon_{MS} \). This holds for the different porosities and different pressure sensitivities of skeleton material (every material is characterized by its own value of \( \varepsilon_{MS} \)). Therefore the unified strain hardening curve should be formulated in the form \( Y(\varepsilon_{MS}) \).

Figure 4a shows the dependences \( Y(\varepsilon_{MS}) \) for the sample with pressure-insensitive shear strength of the skeleton material (\( \lambda=1 \)) at different \( \sigma_{trans} \). Each curve was built by recalculating the stress-strain diagrams (figure 2a) in terms of \( Y \) and \( \varepsilon_{MS} \). The curves \( Y(\varepsilon_{MS}) \) are characterized by two main sections. In the first section (up to a certain critical value \( \varepsilon_{crit}^{MS} \)) they have a profile close to linear and coincide. Thus, there is a unified strain hardening curve \( Y(\varepsilon_{MS}) \) in this section. At \( \varepsilon_{MS} > \varepsilon_{crit}^{MS} \) the curves \( Y(\varepsilon_{MS}) \) corresponding to different lateral pressures become essentially nonlinear and diverge.

**Figure 4.** Examples of the dependences \( Y(\varepsilon_{MS}) \) for porous samples (porosity 10%) with skeleton walls composed of structurally “perfect” material \( \lambda=1 \) (a) and material with pores of small scales \( \lambda=3 \) (b). Vertical dashed line in (a) denotes \( \varepsilon_{crit}^{MS} \). Oblique dashed line in (b) show common (unified) part of the curves, while dash-dot line intersects the curves in “critical” points, which correspond to actual fracture of the sample.

Figure 4b show analogous curves for the sample with pressure-sensitive skeleton material (\( \lambda=3 \)). Like in figure 4a, the curves \( Y(\varepsilon_{MS}) \) have a common linear section at the initial stage of inelastic deformation. However, unlike pressure insensitive skeleton material (\( \lambda=1 \)), these curves diverge from different points. Moreover, an analysis of the dynamics of damage accumulation showed that the deviation of the curves \( Y(\varepsilon_{MS}) \) from the unified curve is associated not with the formation of the shear band, but with the beginning of the previous stage, namely the stage of formation of relatively long internal cracks (merging of “short” cracks). Actual fracture of the sample (its separation into parts) occurs at much larger strains.

We also analyzed the applicability of Mises-Schleicher criterion to forecasting actual fracture of confined porous brittle materials. We found that within each “family” of curves \( Y(\varepsilon_{MS}) \) all the values of \( Y \), corresponding to the moment sample fracture (that is, dividing it into fragments), lie on a straight line. Dash-dot straight line in Fig. 4b intersects the curves \( Y(\varepsilon_{MS}) \) in “critical” points corresponding
to sample fracture (hereinafter these points will be labeled $Y_c$). The existence of linear dependencies $Y_c(\varepsilon_{\text{MS}}) = Y_c^0 + A \cdot \varepsilon_{\text{MS}}$ confirms that the critical cohesion $Y_c$ is a linear function of the magnitude of mean stress in the sample at the moment of loss of its integrity. This means that the condition of actual fracture (loss of integrity) of porous brittle materials can be described using failure criterion of Drucker and Prager (1). The parameters of this criterion ($\lambda'$ and $\sigma_c'$) has to be determined from the results of computer modeling of the loading of confined samples. Special study has shown that quantitative values of these parameters significantly differ from the values derived from uniaxial (unconfined) compression and tension tests of the same samples. The difference increases with decrease in initial porosity of the sample.

Simulation results also showed that accumulation of inelastic volume strain in porous samples under confined loading conditions cannot be adequately described using constant value of the rate of dilatancy. Moreover, the dependence of inelastic volume strain rate on equivalent strain rate is non-monotonous. In particular, the first stage of inelastic deformation (formation of short pore connecting cracks) is accompanied by overall compaction due to local subsidence of skeleton walls into pores connected by cracks. The maximum value of compaction at this stage decreases with decrease in both initial porosity and lateral pressure. Elongation and merging of short cracks (the second deformation stage) is accompanied by local sliding of crack surfaces, which results in total dilatancy of the sample. The rate of dilatancy at this stage is not a constant and increases with increase in the concentration and length of cracks. Formation of shear band (the third deformation stage) is accompanied by collapse of mesoscopic pores and hence by repeated compaction of the mesoscopic sample.

5. Concluding remarks
The results show that conventional models of plasticity of rocks with “linear” yield criterion and linear relationship between inelastic volume strain rate and inelastic shear strain rate are limited to describe the mechanical behavior of confined porous brittle solids. The area of their application covers the initial stages of inelastic strain up to the stage of formation of relatively long cracks (with a length of 10-15 characteristic distances between pores). Moreover, such models adequately describe inelastic volumetric changes (dilatancy) of only low-porous materials, in which the initial stage of compaction is negligibly small.

The results confirm that classical linear failure criteria are able to adequately predict the condition of actual fracture of confined porous solids even at high hydrostatic pressures. Note that despite the widespread use of linear criteria of failure in engineering practice, many experimental studies indicate their limited ability to predict the onset of fracture of brittle materials (including porous rocks) under triaxial non-hydrostatic compression.

The main reason for this is called the oversimplified dependence of such criteria on principal stresses. In particular, linear dependence of the criterion of Drucker and Prager on intermediate principal stress leads to the overestimation of the ultimate strength under the loading conditions, which are characterized by large difference between the values of intermediate and minimal principal stresses. Simulation results show that this criterion allows adequate assessment of the strength of porous brittle solids (including rocks) even under the condition of essentially non-hydrostatic compression. We suggest that reported in some papers unsatisfactory results of application of the Drucker-Prager criterion for describing the experimentally determined values of strength are explained by the use of the criterion parameters $\lambda$ and $\sigma_c$ derived from the laboratory tests on unconfined or “low confined” samples.

The obtained results are important for deepening the understanding of the general laws of inelastic deformation and fracture of porous rocks (including oil and gas reservoirs) under large compressive stresses.
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