HARD–SOFT RENORMALIZATION AND THE EXACT RENORMALIZATION GROUP

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ABSTRACT

The Wilsonian exact renormalization group gives a natural framework in which ultraviolet and infrared divergences can be treated separately. In massless QED we introduce, as the only mass parameter, a renormalization scale $\Lambda_R > 0$. We prove, using the flow equation technique, that infrared convergence is a necessary consequence of any zero-momentum renormalization condition at $\Lambda_R$ compatible with the effective Ward identities and axial symmetry. The same formalism is applied to renormalize gauge-invariant composite operators and to prove their infrared finiteness; in particular we consider the case of the axial current operator and its anomaly.

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INTRODUCTION

The quantization of massless theories requires the introduction of a mass scale which, in perturbation theory, can appear for instance in the renormalization conditions as a non-zero momentum subtraction point \( \Lambda \). In these theories one has to deal both with infrared and ultraviolet divergences and it is convenient to choose a renormalization scheme allowing a separate treatment of these divergences. The frequency-splitting Wilsonian approach to renormalization \( \Lambda \) yields an elegant way of making this separation.

In \( \Lambda \) this splitting is realized in massless \( g\phi^4 \) replacing the scalar field with hard and soft fields, which roughly propagate respectively above and below the renormalization scale \( \Lambda_R \); it is shown that in the hard-soft theory the renormalization conditions can be taken at zero momentum, allowing a simpler BPHZ renormalization than using a non-zero momentum subtraction point.

Making a momentum decomposition with a continuous scale \( \Lambda \), the Wilsonian Green functions satisfy an exact renormalization flow equation \( \Lambda \), using which Polchinski \( \Lambda \) gave a simple proof of renormalizability in massive \( g\phi^4 \) for Green functions with momenta smaller than \( \Lambda_R \). In \( \Lambda \) the same equation has a complementary meaning; it describes indeed the variation of the Green functions computed with an infrared cut-off \( \Lambda \), as \( \Lambda \) varies from the ultraviolet cut-off \( \Lambda_0 \) to 0; technical simplifications of the proof in \( \Lambda \) were moreover obtained. This proof has been extended to the massless case in \( \Lambda \). In \( \Lambda \) it was noticed that the \( \Lambda \) and \( \Lambda \) approaches can be dealt together using a hard-soft field decomposition.

It is clear that the approach \( \Lambda \) is related to the one in \( \Lambda \)–\( \Lambda \); to our knowledge this relation has not been discussed in the literature. In both cases the hard-soft decomposition at scale \( \Lambda_R \) allows the choice of zero-momentum renormalization conditions, which we will call in the following hard-soft (HS) renormalization schemes.

The main differences between the two approaches are: i) the hard-soft field decomposition depends on the cut-off function chosen; in \( \Lambda \) the cut-off function is \( \frac{\Lambda^2}{p^2+\Lambda_R^2} \), so that the soft theory is super-renormalizable; in \( \Lambda \) and \( \Lambda \) a smooth compact-support cut-off function is chosen, and the corresponding soft theory is ultraviolet-finite. ii) from a technical point of view, in \( \Lambda \) \( \Lambda \) is kept fixed at \( \Lambda_R \) and the proof of renormalizability is made using the standard BPHZ results, while the analogous proof in \( \Lambda \) is made using the flow equation for varying \( \Lambda \).

Apart from renormalizability, other issues have been studied independently in these two approaches.

In \( \Lambda \) the \( \Lambda_R \)-dependence of the Green functions is controlled by a renormalization group equation, proved using the Quantum Action Principle; in the Polchinski approach it is controlled similarly by the renormalization flow equation \( \Lambda \).

The application of these methods to the case of gauge theories might seem difficult, since the hard-soft decomposition is not gauge invariant; however, in the approach of \( \Lambda \), it has been shown in \( \Lambda \) that, after this splitting, the original BRS invariance is replaced by a generalized non-local BRS symmetry in the hard-soft theory. Analogously, Polchinski’s approach has been extended to the case of gauge theories in \( \Lambda \)–\( \Lambda \). In \( \Lambda \) it has been shown that in Yang–Mills theory there are effective Ward identities for the Wilsonian action at scale \( \Lambda \). The corresponding Slavnov–Taylor identities on the effective 1-PI functional generator have been written down in \( \Lambda \)–\( \Lambda \). We will show that these identities follow
indeed from generalized non-local BRS symmetry in the hard-soft theory; in the particular case in which the cut-off function is $\frac{A^2_R}{p^2 + \Lambda_R}$, and the Feynman gauge is chosen, this is the non-local BRS symmetry discussed in [10].

In [11] the effective Ward identities have been proven in a HS scheme together with renormalizability without dealing with the infrared problem. This problem has been addressed using the flow equation in [14,16], where the renormalization conditions have been chosen more conventionally at $\Lambda = 0$ and at a non-zero momentum subtraction point; in this scheme the infrared and ultraviolet problems must be solved together, loop by loop.

Although this solution of the infrared problem guarantees, using the flow equation, its solution in the HS schemes, it would be simpler and more natural to give a direct proof in the latter schemes, relying only on the effective Ward identities of [11]: infrared finiteness and the usual Ward identities should follow simply from this requirement. Another motivation for a direct solution of the infrared problem in the HS schemes, already advocated in [3], is that it is technically easier to make subtractions at zero-momentum rather than at a non-zero subtraction point; this fact is particularly relevant when the Ward identities cannot be trivially maintained with a suitable choice of regularization.

In this perspective it is interesting to investigate further the HS schemes; to make these schemes useful tools, one should be able to reproduce the main results in QFT without making any reference to other more traditional renormalization schemes; first of all, it is necessary to prove the infrared finiteness.

In this paper we consider massless QED as a simple model to implement this program. We outline a proof of infrared finiteness in massless QED in a HS scheme with smooth cut-off, using the flow equation technique. There are only two relevant couplings which are not fixed by symmetry requirements at $\Lambda_R$: for any value of them the theory is infrared finite. In fact the proof is in some way simpler than in massless $g\phi^4$, where the relevant coupling of dimension two must be fine-tuned at $\Lambda_R$ to ensure infrared finiteness.

We discuss gauge-invariant composite operators using a variant, in the HS scheme, of the Zimmermann definition of the normal products. In particular we define the gauge-invariant axial current operator and we show how the axial anomaly appears in this context.

In the first section we discuss the hard-soft decomposition in gauge theories, showing that the gauge (or BRS) symmetry is replaced by a non-local symmetry between the hard and soft fields, along the lines of [10], leading to the effective Ward identities found subsequently in [11,17,18] on the Wilsonian action. In the second section we make one-loop computations in massless QED using a HS renormalization scheme. In the third section we use the effective Ward identities in massless QED to prove its infrared convergence in a HS scheme using the exact renormalization flow equation. We discuss the effective axial Ward identity in terms of Zimmermann-like normal operators. The last section contains remarks and the conclusion.

**I. HARD-SOFT RENORMALIZATION IN GAUGE THEORIES**

**A. Scale decomposition**

In this subsection we review the hard-soft field formalism along the lines of [3, 19] and [8]; our presentation holds also in the case of momentum cut-off functions with compact support.
Consider a massless field theory in Euclidean four-dimensional space, with classical action

\[ S_{cl}(\Phi) = \frac{1}{2} \Phi D^{-1} \Phi + S_{cl}^I(\Phi) \]

where we use a compact notation in which \( \Phi \equiv \{ \Phi_i \} \) is a vector corresponding to a collection of fields and the index \( i \) includes the space-time variables; \( D \equiv \{ D_{ij} \} \) is the propagator matrix. \( D_{ij} = (-1)^{\delta_{ij}} D_{ji} \) where \( \delta_{ij} = 0 \) (or 1) for an (anti)commuting field \( \Phi_j \). In all inner products like \( \Phi D^{-1} \Phi \equiv \Phi_i D_{ij}^{-1} \Phi_j \) the inner product symbol is understood.

The bare action has the form:

\[ S_{\Lambda_0}(\Phi) = \frac{1}{2} \Phi D_{0\Lambda_0}^{-1} \Phi + S_{\Lambda_0}^I(\Phi) \]

where \( D_{0\Lambda_0} = DK_{\Lambda_0} \); \( \Lambda_0 \) is the ultraviolet cut-off.

The cut-off function \( K_{\Lambda}(p) = K \left( \frac{p^2}{\Lambda^2} \right) \) can be defined in various ways; for some purposes it is convenient to define it on a compact support (this is the choice made by Polchinski in his proof of renormalizability of \( \phi^4 \) using the exact renormalization flow [4]) or it can be defined analytic, as long as it goes to zero at least as \( \Lambda^4/p^4 \) for large momentum and \( K(0) = 1 \); we will use such a cut-off in the next section.

At tree level \( S_{\Lambda_0}^I(\Phi) = S_{cl}^I(\Phi) \). At quantum level the theory is characterized by a renormalization mass scale \( \Lambda_R \). The usual functional generator is

\[ Z_{0\Lambda_0}[J] = e^{W_{0\Lambda_0}[J]} = \int D\Phi e^{-S_{\Lambda_0}(\Phi) + J\Phi} \]

The corresponding usual 1-PI functional generator is called \( \Gamma_{0\Lambda_0}[\Phi] \).

Define now fields \( \Phi_S \) and \( \Phi_H \) on the supports of

\[ K_{\Lambda_R} \equiv K_S \quad K_{\Lambda_0} - K_{\Lambda_R} \equiv K_H \]

respectively, with propagators

\[ D_{0\Lambda_R} = K_{\Lambda_R} D \equiv D_S \quad D_{\Lambda_R\Lambda_0} = (K_{\Lambda_0} - K_{\Lambda_R}) D \equiv D_H \]

Using gaussian integration one can decompose the usual functional integral in the following way:

\[ Z_{0\Lambda_0}[J] = N \int D\Phi_S D\Phi_H e^{-S_{\Lambda_0}(\Phi_S, \Phi_H) + J(\Phi_S + \Phi_H)} \]

where \( N \) is a normalization constant and

\[ S_{\Lambda_0}(\Phi_S, \Phi_H) = \frac{1}{2} \Phi_S D_S^{-1} \Phi_S + \frac{1}{2} \Phi_H D_H^{-1} \Phi_H + S_{\Lambda_0}^I(\Phi_S + \Phi_H) \]

The Wilsonian path-integral, in which only the high modes \( \Phi_H \) are integrated out, is

\[ Z_{\Lambda_R\Lambda_0}[J, \Phi_S] = \exp W_{\Lambda_R\Lambda_0}[J, \Phi_S] = \int D\Phi_H e^{-S_{\Lambda_0}(\Phi_S, \Phi_H) + J\Phi_H} \]

Making a Legendre transformation from \( W_{\Lambda_R\Lambda_0} \) to \( \Gamma_{\Lambda_R\Lambda_0} \) one arrives at an expression for the effective action of the form

\[ \Gamma_{\Lambda_R\Lambda_0} = \int D\Phi_H e^{-\frac{1}{2} \Phi_H D_H^{-1} \Phi_H + \frac{1}{2} \Phi_S D_S^{-1} \Phi_S + S_{\Lambda_0}^I(\Phi_S + \Phi_H)} \]
\[ \Gamma_{\Lambda R, \Lambda_0} \Phi_S, \Phi_H = \frac{1}{2} \Phi_S D_S^{-1} \Phi_S + \frac{1}{2} \Phi_H D_H^{-1} \Phi_H + \Gamma_{\Lambda R, \Lambda_0}^{int} \Phi_S, \Phi_H \]  

(8)

where

\[ \Gamma_{\Lambda R, \Lambda_0}^{int} \Phi_S, \Phi_H = \Gamma_{\Lambda R, \Lambda_0}^{int} \Phi_S + \Phi_H \]  

(9)

In fact (for \( p \) in the intersection of the supports of \( K_H \) and \( K_S \)):

\[ 0 = \int D\Phi_H \frac{\delta}{\delta \Phi_H (p)} e^{-S_{\Lambda_0} (\Phi_S, \Phi_H)} + J \Phi_H \]

\[ = \int D\Phi_H \left[ -D_H^{-1} \Phi_H - \frac{\delta S_{\Lambda_0}^f}{\delta \Phi_S} + (-)^{\delta} J \right] e^{-S_{\Lambda_0} (\Phi_S, \Phi_H)} + J \Phi_H \]

\[ = \left[ -D_H^{-1} \frac{\delta W_{\Lambda R, \Lambda_0}}{\delta J} + (-)^{\delta} J + \frac{\delta W_{\Lambda R, \Lambda_0}}{\delta \Phi_S} + D_S^{-1} \Phi_S \right] e^{W_{\Lambda R, \Lambda_0} [J, \Phi_S]} \]

where \( \delta = 0 \) (or 1) for (anti)commuting fields. Making the Legendre transformation one obtains

\[ -D_H^{-1} \Phi_H^c + \frac{\delta \Gamma_{\Lambda R, \Lambda_0}}{\delta \Phi_H^c} + D_S^{-1} \Phi_S - \frac{\delta \Gamma_{\Lambda R, \Lambda_0}}{\delta \Phi_S} = 0 \]

and using eq. (8) one has

\[ \frac{\delta \Gamma_{\Lambda R, \Lambda_0}}{\delta \Phi_H^c} - \frac{\delta \Gamma_{\Lambda R, \Lambda_0}}{\delta \Phi_S} = 0 \]

which proves eq. (9).

**B. Effective Ward identities**

The hard-soft decomposition is not gauge-invariant. As discussed in [10] in the context of BRS quantization, the symmetry of the original theory is not lost, but it is replaced by a non-local symmetry on the hard and soft fields. Let us consider first the simpler case of an abelian gauge theory, in which it is not necessary to introduce ghosts.

Consider an abelian gauge theory with gauge-fixed classical action \( S_{cl} (\Phi) \) and infinitesimal gauge transformations

\[ \delta \Phi = R (\omega) \Phi + T (\omega) \]  

(10)

where \( \omega \) is the infinitesimal gauge parameter. For instance in electrodynamics the field content is \( \Phi = (A_\mu, \psi, \bar{\psi}) \), with gauge transformations

\[ \delta A_\mu = -\frac{1}{c} \partial_i \omega \]

\[ \delta \psi = i \omega \psi \]

\[ \delta \bar{\psi} = -i \omega \bar{\psi} \]

The classical action transforms as \( \delta S_{cl} (\Phi) = c (\omega) \Phi \) where \( c \Phi \) is the breaking term due to the (linear covariant) gauge-fixing.

After the scale decomposition described above the gauge symmetry acts non-locally on the hard and soft fields; under
\[ \delta \Phi_S = K_S [R(\Phi_S + \Phi_H) + T] \quad \delta \Phi_H = K_H [R(\Phi_S + \Phi_H) + T] \]

the action (\ref{eq:action}) transforms as

\[ \delta S_{\Lambda_0}(\Phi_S, \Phi_H) = c(\Phi_S + \Phi_H) + O_{\Lambda_0}(\Phi_S, \Phi_H; \omega) \]  \hspace{1cm} (12)

\( O_{\Lambda_0} \) depends on its arguments only through their sum; at tree level it is an irrelevant term vanishing for \( \Lambda_0 \to \infty \).

Multiplying eq. (12) by \( e^{-S_{\Lambda_0}(\Phi_S, \Phi_H)} + J \Phi_H \) and performing a functional integration over \( \Phi_H \), after an integration by parts one gets:

\[ 0 = \int D\Phi_H \left[ \delta \Phi_S \frac{\delta}{\delta \Phi_S} - J \delta \Phi_H + c(\Phi_S + \Phi_H) + O_{\Lambda_0}(\Phi_S + \Phi_H; \omega) \right] e^{-S_{\Lambda_0}(\Phi_S, \Phi_H)} + J \Phi_H \]  \hspace{1cm} (13)

The correctness of the naive procedure leading to eq. (13) is ensured, in the perturbative framework, by the Quantum Action Principle \cite{20,21}; if there are no anomalies \( O_{\Lambda_0} \) is evanescent at quantum level, being evanescent at tree level. We will prove this point in the third section using the flow equation technique.

Making the Legendre transformation to \( \Gamma_{\Lambda R, \Lambda_0} \), using eqs. (8,9) and collecting terms, the effective Ward identity depends only on the field \( \Phi = \Phi_S + \Phi_H \); one gets

\[ \left( R \Phi + T \right) \left[ K_{\Lambda_0} \frac{\delta \Gamma_{\Lambda R, \Lambda_0}^\text{int}}{\delta \Phi} + D^{-1} \Phi \right] - c \Phi = \mathcal{T}_{\Lambda R, \Lambda_0} [\Phi; \omega] + O_{\Lambda R, \Lambda_0} [\Phi; \omega] \]  \hspace{1cm} (14)

where \( O_{\Lambda R, \Lambda_0} [\Phi; \omega] \) is the functional generator of the operator insertion corresponding to \( O_{\Lambda_0} \) and

\[ \mathcal{T}_{\Lambda R, \Lambda_0} [\Phi; \omega] \equiv tr K_S R \frac{\delta^2 W_{\Lambda R, \Lambda_0}}{\delta J \delta \Phi_S} = tr D_H^{-1} K_{\Lambda R} R \left[ D_H^{T-1} + \frac{\delta^2 \Gamma_{\Lambda R, \Lambda_0}^\text{int}}{\delta \Phi^2} \right]^{-1} \]  \hspace{1cm} (15)

in which the trace includes a momentum loop and \( \left[ \frac{\delta^2 \Gamma}{\delta \Phi^2} \right]_{i_1 \ldots i_n} \equiv \frac{\delta^2 \Gamma}{\delta \Phi_{i_1} \ldots \delta \Phi_{i_n}} \); const. is an unimportant field-independent term.

\( \mathcal{T}_{\Lambda R, \Lambda_0} \) is the non-linear part of the effective Ward identity on \( \Gamma_{\Lambda R, \Lambda_0} \); it has the form of a one-loop skeleton diagram. If the proper vertices are renormalized up to loop \( l - 1 \), then \( \mathcal{T}_{\Lambda R, \Lambda_0} \) is finite at loop \( l \); in fact the loop contained in the trace has the ultraviolet cut-off \( K_{\Lambda R} \), so that \( \mathcal{T}_{\Lambda R, \Lambda_0} \rightarrow \mathcal{T}_{\Lambda R} \) finite for \( \Lambda_0 \to \infty \). If there are no anomalies, it is possible to choose \( \Gamma_{\Lambda R, \Lambda_0} |_{rel} \) such that \( O_{\Lambda R, \Lambda_0} [\Phi; \omega] \) is evanescent, namely beyond the tree level:

\[ \left[ (R \Phi + T) \frac{\delta \Gamma_{\Lambda R, \Lambda_0}^\text{int}}{\delta \Phi} \right]_{rel} = (R \Phi + T) \frac{\delta \Gamma_{\Lambda R, \Lambda_0}^{\text{int}} |_{rel}}{\delta \Phi} = \mathcal{T}_{\Lambda R} [\Phi; \omega] \]  \hspace{1cm} (16)

Since \( \Lambda_R > 0 \), the 1PI Green functions are regular functions of the momenta (in particular in the origin). Therefore in (16) the relevant terms can be defined at zero-momentum in the hard-soft renormalization scheme and the first equality is actually trivial.

In sect. III we will show that if eq. (16) is satisfied the effective Ward identities of the hard-soft theory become exact in the limit \( \Lambda_0 \to \infty \):

\[ (R \Phi + T) \left[ \frac{\delta \Gamma_{\Lambda R, \Lambda_0}^{\text{int}}}{\delta \Phi} + D^{-1} \Phi \right] - c \Phi = \mathcal{T}_{\Lambda R} [\Phi; \omega] \]  \hspace{1cm} (17)
In Sect. III it will be shown that, if eq. (14) holds, then the usual Ward
identity on \( \Gamma_{0\infty}[\Phi] \) is satisfied.

The effective Ward identities for a composite operator \( \mathcal{O} \) can be obtained easily by
introducing a source term \( f \, d\tau \eta(x) \mathcal{O}(x) \) in the action \( (10) \), and then by differentiating with
respect to \( \eta(x) \) in \( \eta = 0 \) the extended version of eq. (14). We will consider the case of
a gauge-invariant definition of the composite operator \( i \psi(x) \gamma_5 \bar{\psi}(x) \) with an associated
source \( \eta_{5\mu}(x) \). The action with this source term is invariant at tree level under the local
axial transformation:

\[
\begin{align*}
\delta_5 A_\mu(x) &= 0 \\
\delta_5 \psi(x) &= i \omega_5(x) \gamma_5 \bar{\psi}(x) \\
\delta_5 \eta_5(x) &= -\partial_\mu \omega_5(x)
\end{align*}
\] (18)

In a compact notation we shall write the previous formula \( \delta_5 \Phi = R_5 \Phi \) and \( \delta_5 \eta_5 = T_5 \)
Proceeding as in eqs. (13,14) one arrives at

\[
(R_5 \Phi + T_5) \left[ K_{\Lambda_0} \frac{\delta \Gamma_{\text{int}}}{\delta \Phi} + D^{-1} \Phi \right] + T_5 \frac{\delta \Gamma_{\text{int}}}{\delta \eta_5} = 0
\] (19)

\[
= tr D_{--}^{-1} K_{\Lambda_5} R_5 \left[ D_H^{--1} + \frac{\delta_5 \Gamma_{\text{int}}}{\delta \Phi_5} \right]^{-1} + \mathcal{O}_{5\Lambda_5\Lambda_0}[\Phi; \omega]
\]

where \( \mathcal{O}_{5\Lambda_5\Lambda_0}[\Phi; \omega] \) is evanescent at tree level, but not at quantum level as we will see in
sections II and III.

The fact that in the hard-soft decomposition the symmetries of the original lagrangian
are not lost, but become non-local symmetries which are well-defined at the quantum level
has been first shown in \( (10) \); it is true not only in the abelian but also in the non-abelian
case.

Consider an action \( S_{\Lambda_0}(\Phi) \) of the form \( (11) \) which modulo irrelevant terms is invariant
under the BRS transformations \( \delta \Phi = \epsilon P(\Phi) \); \( \Phi \) denotes physical and ghost fields, \( \epsilon \) is an
anticommuting parameter and \( P(\Phi) \) is polynomial in the fields. After the hard-soft
decomposition the BRS symmetry is replaced by the non-local BRS transformations \( (10) \):

\[
\delta \Phi_S = \epsilon K_S P(\Phi_S + \Phi_H) \\
\delta \Phi_H = \epsilon K_H P(\Phi_S + \Phi_H)
\] (20)

which is analogous to \( (14) \). The action \( (13) \) transforms as

\[
\delta S_{\Lambda_0}(\Phi_S, \Phi_H) = \epsilon \mathcal{O}_{\Lambda_0}(\Phi_S + \Phi_H)
\] (21)

where \( \mathcal{O}_{\Lambda_0} \) is evanescent at tree level. Adding to \( J \Phi_H \) the source term \( \eta P(\Phi_S + \Phi_H) \) and
proceeding as in eqs. (13,14,15) we obtain:

\[
- \frac{\delta \Gamma_{\text{int}}}{\delta \eta} \left[ K_{\Lambda_0} \frac{\delta \Gamma_{\text{int}}}{\delta \Phi} + D^{-1} \Phi \right] = \mathcal{T}_{\Lambda_5\Lambda_0}[\Phi, \eta] + \mathcal{O}_{\Lambda_5\Lambda_0}[\Phi, \eta]
\] (22)

where \( \mathcal{O}_{\Lambda_5\Lambda_0}[\Phi, \eta] \) is the functional generator of an operator insertion which is irrelevant at
tree level and

\[
\mathcal{T}_{\Lambda_5\Lambda_0}[\Phi, \eta] \equiv tr K_S \frac{\delta^2 W_{\Lambda_5\Lambda_0}}{\delta \eta \delta \Phi_S} = -tr D_H^{--1} K_{\Lambda_5} \frac{\delta^2 \Gamma_{\text{int}}}{\delta \eta \delta \Phi} \left[ D_H^{--1} + \frac{\delta^2 \Gamma_{\text{int}}}{\delta \Phi^2} \right]^{-1}
\] (23)
Eq. (22) is the effective Slavnov-Taylor identity which has been derived in the context of the exact renormalization flow in [11,17,18].

Let us finally remark on the choice of the cut-off function. In [3,10] the cut-off \( K = \frac{\Lambda^2}{p^2 + \Lambda^2} \) is used. This cut-off does not eliminate completely the divergences in the soft theory, which has soft propagator and Wilsonian vertices; loops containing a single soft propagator can be divergent; renormalization conditions on the two-point functions must be imposed after integrating over \( \Phi_S \). The non-local BRS transformations introduced in [10] have not the same form as eq. (20); however in the Feynman gauge they coincide with the \( \Lambda \to \infty \) limit of eq. (20) provided the above-mentioned cut-off function is chosen.

II. QED AT ONE LOOP

A. Ward identities at one loop

Let us illustrate a hard-soft renormalization scheme in the case of massless QED at one loop. At tree level the action is, in the Feynman gauge,

\[
S^{(0)} = \int \frac{1}{2} A_\mu(-p) p^2 A_\mu(p) + \bar{\psi}(-p) i \not{\partial} \psi(p) + \int_{p_1p_2} \bar{\psi}(p_1) i e \gamma^\mu \psi(p_2) A_\mu(-p_1 - p_2) \tag{24}
\]

where \( \int_p \equiv \int \frac{d^4p}{(2\pi)^4} \). The propagators are

\[
S(p) = \frac{-i}{p} \quad D_{\mu\nu}(p) = \frac{1}{p^2} \delta_{\mu\nu} \tag{25}
\]

To renormalize the theory in the HS scheme it is sufficient to renormalize the Wilsonian theory with hard propagators \( D_H \) (see eq. (4)) and bare interacting action, whose \( l \)-th term in the loop expansion is

\[
S^{(l)}_{\Lambda_0} = \int \frac{1}{2} A_\mu(-p) \left[ c_1^{(l)} \delta_{\mu\nu} + c_2^{(l)} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) + c_3^{(l)} p^2 \delta_{\mu\nu} \right] A_\nu(p) + c_4^{(l)} \bar{\psi}(-p) i \not{\partial} \psi(p) + \\
\int_{p_1p_2} c_5^{(l)} \bar{\psi}(p_1) i e \gamma^\mu \psi(p_2) A_\mu(-p_1 - p_2) + \int_{p_1p_2p_3} \frac{c_6^{(l)}}{8} A_\mu(p_1) A_\mu(p_2) A_\nu(p_3) A_\nu(-\sum_{i=1}^3 p_i) \tag{26}
\]

The renormalization conditions are chosen loop by loop imposing eq. (16); they depend on two renormalization constants and on the cut-off. To be able to determine these conditions analytically we will choose a simple cut-off function:

\[
K_\Lambda(p) = \frac{\Lambda^4}{(p^2 + \Lambda^2)^2} \tag{27}
\]

The usual Green functions are then computed using the previously determined bare action and the propagator \( DK_{\Lambda_0} \). As a consequence of eq. (5), this is equivalent to computing the Green functions with the action (6), \( S_{\Lambda_0}^{(l)} \) being given in (26), with \( A_\mu = A_{S\mu} + A_{H\mu} \) and \( \psi = \psi_S + \psi_H \).

At one loop the effective Ward identity (14) on the photon two-point function is
\[
\frac{1}{e} p_\nu \Gamma^{(1)\Lambda R \infty}_{\nu \mu}(p) = \mathcal{T}^{(1)\Lambda R \infty}_\mu(p)
\]  
\tag{28}

where

\[
\mathcal{T}^{(1)\Lambda R \infty}_\mu(p) = 2ie \int_q K_{\Lambda R}(q) T \gamma S^{\Lambda R \infty}(q + p) \gamma_\mu
\]  
\tag{29}

Using the Lorentz invariant decomposition \( \Gamma_{\mu\nu}(p) = A(p^2)(p^2 \delta_{\mu\nu} - p_\mu p_\nu) + B(p^2) \delta_{\mu\nu} \) the renormalization conditions on the photon two-point function compatible with the effective Ward identities are

\[
A^{(1)\Lambda R \infty}(0) = z_3^{(1)} \quad B^{(1)\Lambda R \infty}(0) = \frac{5}{24} \frac{e^2 \Lambda^2}{\pi^2} \quad \partial_{\mu^2} B^{(1)\Lambda R \infty}(p^2)|_{p^2=0} = -\frac{e^2}{24\pi^2}
\]  
\tag{30}

The renormalization conditions on the electron two-point function and on the electron-photon vertex are

\[
\Sigma^{(1)\Lambda R \infty}(0) = 0 \quad \frac{\partial \Sigma^{(1)\Lambda R \infty}(p)}{\partial p_\mu}|_{p=0} = i\gamma_\mu z_2^{(1)} \quad \Gamma^{(1)\Lambda R \infty}_\mu(0,0) = ie\gamma_\mu z_1^{(1)}
\]  
\tag{31}

(the first condition is required by the rigid axial invariance).

Using the effective Ward identity (14) we obtain (see also [12])

\[
\frac{1}{e} (p - q)_\mu \Gamma^{(1)\Lambda R \infty}_\mu(p,-q) + \Sigma^{(1)\Lambda R \infty}(q) - \Sigma^{(1)\Lambda R \infty}(p) = \mathcal{T}^{(1)\Lambda R \infty}(p,-q)
\]  
\tag{32}

where

\[
\mathcal{T}^{(1)\Lambda R \infty}(p,-q) = -e^2 \int_l K_{\Lambda R}(l) \left[ \gamma_\mu S^{\Lambda R \infty}(p - q + l) \gamma_\nu D^{\Lambda R \infty}_{\mu\nu}(l - q) - (q \leftrightarrow p) \right]
\]  
\tag{33}

The renormalization conditions (31) are compatible with the effective Ward identity (32) provided

\[
\frac{1}{e} \left( z_2^{(1)} - z_1^{(1)} \right) = -\frac{1}{e} \left. \frac{\partial \mathcal{T}^{(1)\Lambda R \infty}(p,0)}{\partial p_\mu} \right|_{p=0} = i\gamma_\mu \frac{53}{960} \frac{e}{2\pi^2}
\]  
\tag{34}

The last non-vanishing renormalization condition is determined by the effective Ward identities to be (see also [12])

\[
\Gamma^{(1)\Lambda R \infty}_{\mu\nu\rho\sigma}(0,0,0) = -\frac{e^4}{12\pi^2} (\delta_{\mu\nu} \delta_{\rho\sigma} + \delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho})
\]  
\tag{35}

Having fixed the renormalization conditions, we can determine the counterterms in the bare action

\[
\begin{align*}
c_1^{(1)} &= \frac{e^2 \Lambda^2}{24\pi^2} \\
c_2^{(1)} &= \frac{e^2}{12\pi^2} \left( \ln \frac{\Lambda^2}{\Lambda_0^2} + \frac{19}{15} \right) + z_1^{(1)} \\
c_3^{(1)} &= -\frac{e^2}{24\pi^2} \\
c_4^{(1)} &= \frac{e^2}{16\pi^2} \left( \ln \frac{\Lambda^2}{\Lambda_0^2} + \frac{51}{20} \right) + z_1^{(1)} \\
c_5^{(1)} &= \frac{e^2}{16\pi^2} \left( \ln \frac{\Lambda^2}{\Lambda_0^2} + \frac{5}{2} \right) + z_1^{(1)} \\
c_6^{(1)} &= -\frac{e^4}{12\pi^2}
\end{align*}
\]  
\tag{36}
These counterterms include the finite parts needed to satisfy the effective Ward identities at \( \Lambda_R \); they are easily computed due to the choice of the HS scheme and of the cut-off (27). A partial determination of the one-loop counterterms in QED using the exact renormalization group approach and renormalization conditions at \( \Lambda = 0 \) can be found in [12] and [22]; in [22] the cut-off (27) is also used.

After integrating out the soft modes, the usual Ward identities are automatically satisfied. Let us check them in two cases.

The vacuum polarization is transverse:

\[
A^{(1)0\infty}(p^2) = \frac{e^2}{12\pi^2} \left( -\ln \frac{p^2}{\Lambda_R^2} + \frac{7}{15} + z_3^{(1)} \right)
\]

\[
B^{(1)0\infty}(p^2) = 0
\]

The self-energy of the electron

\[
\Sigma^{(1)0\infty}(p) = i\frac{e^2}{16\pi^2} \left( -\ln \frac{p^2}{\Lambda_R^2} + \frac{103}{60} + z_1^{(1)} \right)
\]

and the vertex

\[
\Gamma^{(1)0\infty}_\mu(p, -q) = ie\gamma_\mu \left[ z_1^{(1)} - \frac{e^2}{8\pi^2} \left( \frac{777}{120} + \int_0^1 dx \int_0^{1-x} dy \ln \frac{Q^2}{\Lambda_R^2} \right) \right]
\]

\[
+ \frac{ie^3}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy \left[ -\phi x + \phi (1 - y) \right] \gamma_\mu \left[ -\phi y + \phi (1 - x) \right] \frac{Q^2}{Q^2} + c \epsilon^{(1)} \frac{e^3}{2\pi^2} \left( p - q \right)_\sigma
\]

satisfy the relation: \( (p - q)_\mu \Gamma^{(1)0\infty}_\mu(p, -q) + e \left[ \Sigma^{(1)0\infty}(q) - \Sigma^{(1)0\infty}(p) \right] = 0 \).

**B. Axial Ward identities at one loop**

The effective axial Ward identities can be discussed analogously to the effective gauge Ward identities. The HS renormalization conditions are chosen compatible with the effective gauge Ward identities; we will verify now that it follows that the effective axial Ward identities are anomalous; only zero-momentum graphs at \( \Lambda_R \) need to be evaluated to determine the anomaly. For a computation of the axial anomaly in QED using the flow equation and imposing the usual gauge Ward identity see [12].

At one loop the fermion-axial vector vertex is equal to the fermion-photon vertex, multiplied by \( \gamma_5 \); the same is true for the corresponding effective axial Ward identity, which is equivalent to the effective gauge Ward identity eq. (32).

The matrix element of \( \bar{\psi} \gamma^\mu \gamma_5 \psi(x) \) in the two-photon sector must include a counterterm due to the possible mixing with \( \epsilon_{\mu\nu\rho\sigma} F^{\nu\sigma} A^\rho \):

\[
\Gamma^{(1)A R A_0}_5(p, -q) = 2ie^2 \int \mathcal{T} r_\mu \gamma_5 S^{A R A_0}(l - q) \gamma_\mu S^{A R A_0}(l) \gamma_\rho S^{A R A_0}(l + p) + c^{(1)} \epsilon_{\mu\nu\rho\sigma} (p - q)_\sigma
\]

It satisfies the gauge effective Ward identity [14].
\[ q_\nu \Gamma^{(1)\Lambda_R \Lambda_0}_{\mu \nu \rho}(q, p) = 2e^2 \int T \gamma_\mu \gamma_5 \left[ K_{\Lambda_R}(l - q) S^{\Lambda_R \Lambda_0}(l) - S^{\Lambda_R \Lambda_0}(l - q) K_{\Lambda_R}(l) \right] \gamma_\rho S^{\Lambda_R \Lambda_0}(l + p) \]
\[ + \mathcal{O}^{(1)\Lambda_R \Lambda_0}_{5\mu \rho}(q, p) \]  
and the axial effective Ward identity \([\text{13}]\)

\[ (q + p)_\mu \Gamma^{(1)\Lambda_R \Lambda_0}_{5\mu \nu \rho}(q, p) = 2e^2 \int T \gamma_\mu \gamma_5 \left[ K_{\Lambda_R}(l - q) S^{\Lambda_R \Lambda_0}(l + p) - (p \leftrightarrow -q) \right] \gamma_\nu S^{\Lambda_R \Lambda_0}(l) \]
\[ + \mathcal{O}^{(1)\Lambda_R \Lambda_0}_{5\mu \rho}(q, p) \]

The renormalization conditions on \( \Gamma^{(1)\Lambda_R \Lambda_0}_{5\mu \nu \rho}(q, p) \) are chosen such that \( \mathcal{O}^{(1)\Lambda_R \Lambda_0}_{5\mu \rho}(q, p) \) is evanescent:
\[ \frac{\partial^2}{\partial q_\alpha \partial p_\beta} \mathcal{O}^{(1)\Lambda_R \infty}_{5\mu \rho}(q, p)|_{p=0} = 0 \]
which implies
\[ \frac{\partial}{\partial p_\sigma} \Gamma^{(1)\Lambda_R \infty}_{5\mu \nu \rho}(0, p)|_{p=0} = 4e^2 \int T \frac{\partial K_{\Lambda_R \gamma_5 \gamma_\mu \gamma_\rho}}{\partial \nu} S^{\Lambda_R \infty} \gamma_\rho \frac{\partial S^{\Lambda_R \infty}}{\partial \nu} = \frac{4e^2}{3} \epsilon_{\mu \nu \rho \sigma} \int l_\alpha \left[ (1 - K_{\Lambda_R})^3 \frac{l_\alpha}{(l^2)^2} \right] = \frac{e^2}{6\pi^2} \epsilon_{\mu \nu \rho \sigma} \]
This condition determines the value of \( c^{(1)} \) of eq. \([\text{10}]\). It follows that \( \tilde{O} \) is fixed by the effective axial Ward identity eq. \([\text{12}]\) to satisfy
\[ \frac{\partial^2}{\partial q_\alpha \partial p_\beta} \tilde{O}^{(1)\Lambda_R \infty}_{5\mu \rho}(q, p)|_{p=0} = \frac{e^2}{2\pi^2} \epsilon_{\nu \rho \alpha \beta} \]
which is the anomaly of the axial Ward identity. Observe that in this formulation the anomaly is evaluated in terms of explicitly zero momentum finite integrals, which do not depend on the explicit form of the cut-off function. In fact the integral in \([43]\) is a total divergence. It is interesting to observe that the triangle graph does not give any contribution to the anomaly for \( \Lambda_R \neq 0 \), so that \( c^{(1)} = \frac{e^2}{6\pi^2} \); in fact
\[ \frac{\partial}{\partial p_\beta} \Gamma^{(1)\Lambda_R \Lambda_0}_{5\mu \nu \rho}(0, p)|_{p=0} = \frac{8e^2}{3} \epsilon_{\mu \gamma \rho \delta} \int l_\beta \left[ (K_{\Lambda_0} - K_{\Lambda_R})^3 \right] \frac{l_\gamma}{(l^2)^2} = 0 \]  
For example in the limit in which \( K_\Lambda \) is the step function the integral in eq. \([43]\) is proportional to
\[ \left( \int_{\Sigma_{\Lambda_0}} d\Sigma_\beta - \int_{\Sigma_{\Lambda_R}} d\Sigma_\beta \right) \frac{\partial}{\partial l_\nu} \left( \frac{l_\gamma}{l^2} \right) l_\beta = 0 \]
The two surface contributions, coming from the spheres \( l^2 = \Lambda_0^2 \) and \( l^2 = \Lambda_R^2 \) cancel each other.

The \( AAA \) vertex satisfies at one loop an effective Ward identity which is formally the same as eq. \([\text{12}]\), but now the vertex \( \Gamma^{\Lambda_R \Lambda_0}_{\mu \nu \rho} \) is completely symmetric, so that it admits no counterterm. Therefore \( \tilde{O}^{rel} = -\mathcal{T}^{rel} \), and the \( AAA \) anomaly is one-third of the \( AVV \) anomaly.
III. EXACT RENORMALIZATION GROUP

In the first section we considered a fixed scale decomposition, in which the soft and hard modes are separated at the renormalization scale $\Lambda_R$. Changing the separation point to an arbitrary scale $\Lambda > 0$ and changing appropriately the renormalization conditions, i.e. choosing the parameters $z_1\Lambda$ and $z_3\Lambda$, which are not constrained by the effective Ward identities, the physical quantities are unchanged. It is possible to write an exact renormalization flow equation describing the continuous change of the Wilsonian effective action $W_{\Lambda\Lambda_0}$.

The generating functional of the connected (interacting) amputated Green functions $L_{\Lambda\Lambda_0}[\phi, \Phi_S]$ satisfies

$$W_{\Lambda\Lambda_0}[J, \Phi_S] = \frac{1}{2} J D_{\Lambda\Lambda_0}^T J - \frac{1}{2} \Phi_S D_{0\Lambda}^{-1} \Phi_S - L_{\Lambda\Lambda_0}[D_{\Lambda\Lambda_0}^T J, \Phi_S]$$

The same proof which led to eqs. (8,9) shows that actually $L_{\Lambda\Lambda_0}$ depends on its two arguments only through their sum $\Phi_S + \phi \equiv \Phi$ so we will consider the simpler functional $L_{\Lambda\Lambda_0}[\Phi]$ (see also [8]). Another way to see this fact exploits the following representation of the functional $L_{\Lambda\Lambda_0}[\Phi]$:

$$e^{-L_{\Lambda\Lambda_0}[\phi + \Phi_S]} = e^{\Delta_{\Lambda\Lambda_0}[\phi]} e^{-L_{\Lambda_0}[\phi + \Phi_S]}$$

(47)

where $L_{\Lambda_0}$ is equal to the bare lagrangian, apart from the tree-level kinetic term, and $\Delta_{\Lambda\Lambda_0}$ is the functional Laplacian:

$$\Delta_{\Lambda\Lambda_0} = \frac{1}{2} \frac{\delta}{\delta \phi_i} D_{\Lambda\Lambda_0}^{ji} \frac{\delta}{\delta \phi_j}$$

$L_{\Lambda_0}$ depends on $\phi$ and on the background $\Phi_S$ through their sum, therefore $L_{\Lambda\Lambda_0}$ will depend only on $\Phi_S + \phi$. The above proof shows that, in case of propagators $D_{\Lambda\Lambda_0}$ with compact support, the amputated Green functions of the theory, a priori defined only for momenta in this support, can be continued to arbitrary momenta by performing functional derivatives with respect to the background field.

$L_{\Lambda\Lambda_0}[\Phi]$ satisfies the flow equation

$$\partial_\Lambda L_{\Lambda\Lambda_0} = (\partial_\Lambda \Delta_{\Lambda\Lambda_0}) L_{\Lambda\Lambda_0} - \frac{1}{2} L_{\Lambda\Lambda_0}' \left( \partial_\Lambda D_{\Lambda\Lambda_0}^T \right) L_{\Lambda\Lambda_0}'$$

(48)

where $L_{\Lambda\Lambda_0}' \equiv \frac{\delta L_{\Lambda\Lambda_0}}{\delta \phi}$. 

In the Polchinski approach $L_{\Lambda_0}$ is the boundary condition for the flow equation (48). The coefficients of the relevant polynomials in the fields, which appear in $L_{\Lambda_0}$, must depend in a suitable way on the ultraviolet cut-off. This dependence is fixed by imposing the renormalization conditions on the relevant components of the functional $L_{\Lambda R\Lambda_0}$. Perturbatively the relations between the renormalization conditions and the bare boundary condition is invertible and no ambiguities are involved.

Using a cut-off function with compact support, Polchinski gave a simple proof of the power-counting renormalization theorem; the proof has been generalized and further simplified in [3,22,23,24]. Analogous results have been obtained using an exponential cut-off in [25]; a proof of renormalizability with a Pauli-Villars cut-off, like that of eq. (27), has not yet been given using only the flow equation.
An expression analogous to eq. (47) holds for the generating functional of the connected and amputated graphs with an insertion of a composite operator \( M \):

\[
M_{ΛΛ_0}[\phi, Φ_S] = e^{L_{ΛΛ_0}[\phi, Φ_S]} e^{Δ_{ΛΛ_0}(\frac{δ}{δ})} M_{Λ_0}[\phi, Φ_S] e^{-L_{Λ_0}[\phi, Φ_S]} \tag{49}
\]

where \( M_{Λ_0} \) is the bare insertion. If \( M_{Λ_0}, \) as well as \( L_{Λ_0}, \) depends on \( Φ + Φ_S \) eq. (49) shows that also \( M_{ΛΛ_0} \) depends on the fields only through \( Φ \). \( M_{ΛΛ_0}[Φ] \) satisfies, by construction, the linear differential equation for the connected insertion of an operator:

\[
∂_Λ M_{ΛΛ_0} = (∂_Λ Δ_{ΛΛ_0}) M_{ΛΛ_0} - L'_{ΛΛ_0} \left( ∂_Λ D^T_{ΛΛ_0} \right) M'_{ΛΛ_0} \tag{50}
\]

Eq. (50) is the starting point to prove the renormalizability of a composite operator of dimension \( d \), defined by a bare boundary condition \( M_{Λ_0} \) at \( Λ = Λ_0 \) (a polynomial of dimension \( d \) in the fields, compatible with the rigid symmetries) and by the renormalization conditions at \( Λ = Λ_R \). In particular if at \( Λ = Λ_R \) the relevant part of \( M_{ΛRΛ_0} \) is zero (or suitably vanishing as \( Λ_0 \to ∞ \)) and if the irrelevant part of \( M_{Λ_0} \) fulfills suitable bounds in the \( Λ \) dependence then \( \lim_{Λ_0 \to ∞} M_{ΛΛ_0} = 0 \) [23].

A. Effective Ward identities and infrared convergence

The validity of the Ward identities in QED has been studied using the flow equation in [15,16] choosing renormalization conditions at \( Λ = 0 \) compatible with the usual Ward identities.

We want to show that, choosing a HS scheme satisfying eq. (16), the exact effective Ward identities (17) are satisfied; moreover the usual Ward identities and the infrared finiteness of the theory follow without further constraints for \( Λ \to 0 \).

The effective Ward identity on \( L_{ΛΛ_0} \) is obtained defining

\[
O_{ΛΛ_0}[Φ; ω] = i \int dx ω(x) O_{ΛΛ_0}[x, Φ] = e^{L_{ΛΛ_0}[Φ]} e^{Δ_{ΛΛ_0}(\frac{δ}{δ})} \left\{ − \left[ (RΦ + T) K_{Λ_0} L'_{Λ_0}[Φ] + Φ D^{-1} RΦ \right] e^{-L_{Λ_0}[Φ]} \right\} \tag{51}
\]

and proving that \( O_{ΛΛ_0} \to 0 \) for \( Λ_0 \to ∞ \).

\( Φ \) is the multiplet of independent fields \( (A_μ, ψ, \bar{ψ}) \) and \( R \) and \( T \) are the parameters describing the gauge transformations, introduced in the first section. Notice that the argument in square brackets on the r.h.s. of this equation is equal to \( O_{Λ_0}[Φ; ω] \) defined in eq. (12), with \( Φ = Φ_S + Φ_H \).

One wants to show that it is possible to choose the renormalization conditions on \( L_{ΛRΛ_0} \) in such a way that the operator defined in eq. (51) is evanescent. Indeed the relevant parts of \( O_{ΛRΛ_0} \) can be related to those of \( L_{ΛRΛ_0} \) by the interpolating effective Ward identity which can be obtained simply by commuting the argument in the square brackets in (51) with the exponential of the functional Laplacian:

\[
O_{ΛΛ_0}[Φ; ω] = −TK_{Λ_0} L'_{ΛΛ_0}[Φ] − Φ D^{-1} RΦ + Φ D^{-1} RD_{ΛΛ_0} L'_{ΛΛ_0}[Φ] \tag{52}
\]

\[
−Φ R^T K_{Λ} L'_{ΛΛ_0}[Φ] + L'_{ΛΛ_0}[Φ] D^T_{ΛΛ_0} R^T K_{Λ} L'_{ΛΛ_0}[Φ] − T_{ΛΛ_0}[Φ; ω]
\]

where
\[ T_{\Lambda \Lambda_0} [\Phi; \omega] = i \int d x \omega (x) T_{\Lambda \Lambda_0} [x, \Phi] = tr K_\Lambda R D_{\Lambda \Lambda_0} L''_{\Lambda \Lambda_0} [\Phi] \] (53)

We realize that only the first three terms in the r.h.s. of eq. (54) are present in the broken Ward identity of ref. [13], [16] all the others terms being contained in a redefinition of \( O_{\Lambda \Lambda_0} \), which then does not satisfy the linear equation of the insertions (50). These three addenda are the only ones surviving in the formal limit \( \Lambda \to 0, \Lambda_0 \to \infty \), giving the usual Ward identities.

From now on we consider explicitly the case of the cut-off function \( K \left( \frac{p^2}{\Lambda^2} \right) \) with \( K = K (x) \) a \( C^\infty \) function with compact support (equal to 1 for \( x \leq 1 \) and equal to zero for \( x \geq 4 \)).

In order to connect the relevant parts of \( L_{\Lambda R \Lambda_0} \) and \( O_{\Lambda R \Lambda_0} \), it is sufficient to consider the functional derivatives of eq. (52) with respect to the fields \( \bar{\psi}_{\alpha_1} (p_1) \ldots \psi_{\beta_n} (q_n) \ldots A_{\mu_m} (k_m) \) and to the gauge parameter \( \omega \), in a suitable neighbourhood of the origin of momentum space. Noticing that the terms \( \Phi D^{-1} R D_{\Lambda \Lambda_0} L'_{\Lambda \Lambda_0} [\Phi] \) and \( L'_{\Lambda \Lambda_0} [\Phi] D_{\Lambda \Lambda_0} R^T K_{\Lambda} L'_{\Lambda \Lambda_0} [\Phi] \) do not give contribution in this region, we obtain for the \( l \)-th term in the loop expansion:

\[
\frac{1}{n!} \sum_{j=1}^n (-1)^{n-j} \left[ L_{2n \, m}^{(l) \, \Lambda \Lambda_0} (P_j) - L_{2n \, m}^{(l) \, \Lambda \Lambda_0} (Q_j) \right] - T_{2n \, m}^{(l) \, \Lambda \Lambda_0} (\alpha_1 p_1, \ldots, \alpha_n p_n; \beta_1 q_1, \ldots, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m)
\]

where the multi-indices \( P_j \) and \( Q_j \) are defined by:

\[
P_j \equiv (\alpha_1 p_1, \ldots, \alpha_{j-1} p_{j-1}, \alpha_{j+1} p_{j+1}, \ldots, \alpha_n p_n, \alpha_j P + p_j; \beta_1 q_1, \ldots, \beta_{n-1} q_{n-1}, \beta_n q_n, \beta_j; \mu_1 k_1, \ldots, \mu_m k_m)
\]

\[
Q_j \equiv (\alpha_1 p_1, \ldots, \alpha_n p_n; \beta_1 q_1, \ldots, \beta_{j-1} q_{j-1}, \beta_{j+1} q_{j+1}, \ldots, \beta_n q_n, \beta_j; \mu_1 k_1, \ldots, \mu_m k_m)
\]

and

\[
( -1)^{n+1} \int P K_\Lambda (p) \left[ S_{\beta \alpha} (p + P) L_{2n \, m}^{(l-1) \, \Lambda \Lambda_0} (\alpha_1 p_1, \ldots, \alpha_n p_n, \alpha - p; \beta_1 q_1, \ldots, \beta_n q_n; \beta_j; \mu_1 k_1, \ldots, \mu_m k_m) \right. \\
- L_{2n+2 \, m}^{(l-1) \, \Lambda \Lambda_0} (\alpha_1 p_1, \ldots, \alpha_n p_n, \alpha + P; \beta_1 q_1, \ldots, \beta_n q_n; \beta; \mu_1 k_1, \ldots, \mu_m k_m) ]
\]

with \( P = - \left( \sum_{j=1}^n p_j + \sum_{j=1}^n q_j + \sum_{r=1}^m k_r \right) \).

Eq. (54) holds for \( n \geq 0 \) and \( l > 0 \). For \( n = 0 \) the second term in the r.h.s. is absent. For \( l = 0 \) the r.h.s. of eq. (54), which now includes for \( m = 0 \) and \( n = 1 \) the term \([-S_{\beta \alpha}^{-1} (p_1) + S_{\alpha \beta}^{-1} (q_1)] \), is zero due to the gauge invariance of the classical action.

A little digression on the notation is in order: because of translation invariance, the \( C^\infty \) functions \( L_{2n \, m}^{\Lambda \Lambda_0} \) depend, for \( \Lambda > 0 \), only on \( 2n + m - 1 \) momenta. In the case \( n > 0 \) we consider by convention \( L_{2n \, m}^{\Lambda \Lambda_0} \) depending on \( n - 1 \) momenta \( q \) of the fermionic fields \( \psi (q) \); therefore eq. (54) (as well as other similar in the following) appears to be asymmetric in the fermionic variables. For \( n = 0 \) the functions \( L_{0 \, m}^{\Lambda \Lambda_0} \) depend on \( m - 1 \) bosonic momenta and eq. (54) changes accordingly.

As a consequence of charge conservation the r.h.s. of eq. (54) is equal to zero for \( P = 0 \) and thus the renormalization conditions for \( O_{\Lambda R \Lambda_0} \), which do not include some derivative are vanishing. We are then interested in considering derivatives of order \( z > 0 \) with respect to momenta.
Let us discuss briefly the various relevant sectors of eq. (54).
i) $m = 1$, $n = 0$, $z = 1, 2, 3$: 
we can analyze all these cases by considering the Mc Laurin expansion up to the third order in the momenta. The first addendum on the r.h.s. of eq. (54) yields:

\[-\frac{1}{e^2}k_\mu L^{(l)\Lambda_R\Lambda_0}_{\mu\nu}(k)|_{rel} = -\frac{1}{e^2}k_\mu \left[ \xi_3^{(l)}(k^2\delta_{\mu\nu} - k_\mu k_\nu) + (\xi_1^{(l)} + \xi_2^{(l)}k^2)\delta_{\mu\nu} \right] = -\frac{1}{e^2}k_\mu(\xi_1^{(l)} + \xi_2^{(l)}k^2).\]

$O(4)$ symmetry and smoothness imply that $T^{(l)\Lambda_R\Lambda_0}_{\mu\nu}(k)|_{rel}$ has the same tensorial structure as the r.h.s. of the previous equation, so that by a suitable choice of the renormalization constants $\xi_1^{(l)}$ and $\xi_2^{(l)}$ one imposes $O^{(l)\Lambda_R\Lambda_0}_{\mu\nu}(k)|_{rel} = 0$ (at one loop see eq. (51)).

ii) $m = 2$, $n = 0$:

invariance under charge conjugation (Furry theorem) leads to $L^{(l)\Lambda_R\Lambda_0}_{\mu\nu}(k_1, k_2) = 0$ and $T^{(l)\Lambda_R\Lambda_0}_{\mu\nu}(k_1, k_2) = 0$: the r.h.s. of eq. (54) vanishes identically.

iii) $m = 3$, $n = 0$, $z = 1$:

\[-\frac{1}{e^2}(k_1 + k_2 + k_3)_{\mu}L^{(l)\Lambda_R\Lambda_0}_{\nu\rho\sigma}(k)|_{rel} = -\frac{1}{e^2}(k_1 + k_2 + k_3)\mu\delta_{\rho\sigma}\xi_4^{(l)}\]

$T^{(l)\Lambda_R\Lambda_0}_{\rho\sigma\tau}(k_1, k_2, k_3)|_{rel}$ has the same structure as the r.h.s. of the previous equation; indeed by applying $\partial_{\delta k_{\rho\sigma}}$ for $k_1 = k_2 = k_3 = 0$ to the r.h.s. of eq. (53) for $n = 0$ and $m = 3$, one notices that the only non vanishing contributions arise when the derivative act on the $P = -(k_1 + k_2 + k_3)$ variable. Namely to the first order $T^{(l)\Lambda_R\Lambda_0}_{\rho\sigma\tau}$ depends on the momenta only through $P$, then the bosonic and $O(4)$ symmetries lead easily to the conclusion. Therefore with a suitable choice of the renormalization coefficient $\xi_4^{(l)}$ (at one loop see eq. (55)) it is possible to set $O^{(l)\Lambda_R\Lambda_0}_{\rho\sigma\tau}|_{rel} = 0$.

iv) $m = 0$, $2n = 2$, $z = 1$:

for $l > 0$ the first two addenda on the r.h.s. of eq. (54) give

\[\left\{ -\frac{1}{e^2}(p + q)_{\mu}L^{(l)\Lambda_R\Lambda_0}_{\alpha\beta\mu}(p; -p - q) + L^{(l)\Lambda_R\Lambda_0}_{\alpha\beta\mu}(q) - L^{(l)\Lambda_R\Lambda_0}_{\alpha\beta\mu}(p) \right\}|_{rel} = -i(z_1^{(l)} - z_2^{(l)})(p + q)_{\alpha\beta}\]

where $z_1^{(l)}$ and $z_2^{(l)}$ are the renormalization constants already introduced for $l = 1$ in sect. II. Using charge conjugation invariance it is easy to show that $T^{(l)\Lambda_R\Lambda_0}_{\alpha\beta\mu}(p, q)$ has the same structure so that $O^{(l)\Lambda_R\Lambda_0}_{\alpha\beta\mu}|_{rel} = 0$ can be satisfied with a suitable choice of $z_1^{(l)} - z_2^{(l)}$ (see eq. (52)).

Observe that the effective Ward identities determine the renormalization conditions up to the two arbitrary constants $z_1$ and $z_3$.

After this discussion a formal proof of ultraviolet and infrared finiteness of QED can be made along these lines: one wants to prove a series of suitable bounds concerning the ultraviolet and infrared behavior of $L^{(l)\Lambda_R\Lambda_0}_{\nu\rho\sigma}(k)$, in an inductive scheme in the loop index $l$. Because of our choice of the cut-off function $K$ we can use the results of \cite{10, 11}. The infrared finiteness of a theory with massless fermions and photons has been proven in \cite{10}, independently of any Ward identity, provided the renormalization conditions

\[L^{(l)\Lambda_R\Lambda_0}_{\alpha\beta}(0) = L^{(l)\Lambda_R\Lambda_0}_{\mu\nu}(0) = 0\]  

are imposed. We shall see that, for arbitrary values of $z_1$ and $z_3$, eqs. (56) are satisfied.

The ultraviolet part of the proof is standard, ultraviolet finiteness following, for $\Lambda > 0$, from arbitrary renormalization conditions at $\Lambda = \Lambda_R$, not necessarily compatible with the Ward identities. From the U-V bounds for $L^{(l)\Lambda_R\Lambda_0}_{\nu\rho\sigma}$ \cite{10} one obtains as a consequence the suitable power-counting bounds for the irrelevant components of $O^{(l)}_{\Lambda_R\Lambda_0}$ of eq. (51) that, together
with $O_{\Lambda R_0 \Lambda_0}^{(l)}|_{rel} = 0$, lead to $\lim_{\Lambda_0 \to \infty} O_{\Lambda R_0}^{(l)}[\Phi] = 0$. Therefore for $\Lambda > 0$ and loop $l$ we can perform the limit $\Lambda_0 \to \infty$ in eq. (52), so that the effective Ward identities are satisfied.

We have to discuss now at loop $l$ the $\Lambda \to 0$ limit. From the infrared bounds on the vertices [72,75], which are valid by induction hypothesis at loop order $l' \leq l - 1$, and using eq. (55) one has $\lim_{\Lambda_0 \to 0} \partial_{\mu} T_{\nu}^{(l) \Lambda R_0 = \infty} (k)|_{k=0} = 0$. Using eq. (54) with $m = 1$, $n = 0$, $z = 1$ one gets $\lim_{\Lambda_0 \to 0} L_{\mu\nu}^{(l) \Lambda R_0 = \infty}(0) = 0$. Moreover the renormalization conditions on $L_{\alpha\beta}^{(l) \Lambda R_0}(0)$ has not been involved in the effective Ward identity (54), so that we can choose $L_{\alpha\beta}^{(l) \Lambda R_0}(0) = 0$; from rigid axial symmetry it follows that $L_{\alpha\beta}^{(l) \Lambda R_0}(0) = 0$ for any $\Lambda$. Therefore eqs. (56) are satisfied and one can prove all the I-R bounds at loop $l$ as required by the induction scheme, so that finiteness together with all the effective Ward identities are proved; moreover for non-exceptional momenta the $T^{(l)}$ functions of eq. (53) go to zero as $\Lambda \to 0$, and one recovers the usual Ward identities.

Notice that this renormalization procedure is not sufficient in massless scalar QED to ensure infrared finiteness, since there is no chiral symmetry protecting the scalar from getting a mass. Imposing renormalization conditions compatible with the effective Ward identities, the theory is ultraviolet finite and satisfies the effective Ward identities for $\Lambda > 0$ and $\Lambda_0 \to \infty$, but infrared finiteness requires that the renormalization conditions on $L_{\alpha\beta}^{(l) \Lambda R_0}(0)$ must be chosen in such a way that $\lim_{\Lambda_0 \to 0} L_{\alpha\beta}^{(l) \Lambda R_0}(0) = 0$. This fine tuning is typical for theories with massless scalars.

B. Gauge invariance of composite operators

In this subsection we will shortly discuss the gauge invariance of composite operators. One associates to the gauge variation of a local composite operator $J(x)$ the insertion:

$$O^{J}_{\Lambda R_0} [x; \Phi; \omega] = i \int dx' \omega(x') O^{J}_{\Lambda R_0} [x', x; \Phi]$$

$$\quad \quad \quad \quad = e^{L_{\Lambda R_0} [\Phi]} e^{L_{\Lambda R_0} (\frac{1}{4})} \left( (R\Phi + T) K_{\Lambda_0} \frac{\delta}{\delta \Phi} - \Phi D^{-1} R\Phi \right) \left( J_{\Lambda_0} [x; \Phi] e^{-L_{\Lambda_0} [\Phi]} \right) \quad (57)$$

$J_{\Lambda_0} [x; \Phi]$ being the bare composite operator, boundary condition of eq. (54). By commuting the argument of the square brackets with the exponential of the Laplacian we easily get the following expression in terms of the functional $J_{\Lambda R_0}[x; \Phi]$, solution of eq. (54):

$$O^{J}_{\Lambda R_0} [x; \Phi; \omega] = T K_{\Lambda_0} J'_{\Lambda R_0} [x; \Phi] - \Phi D^{-1} R D_{\Lambda R_0} J'_{\Lambda R_0} [x; \Phi] + \Phi R^T K_{\Lambda R D_{\Lambda R_0}} J'_{\Lambda R_0} [x; \Phi]$$

$$\quad \quad \quad \quad \quad - J_{\Lambda R_0} [x; \Phi] \left( D_{\Lambda R_0} R^T K_{\Lambda} + K_{\Lambda R D_{\Lambda R_0}} \right) L'_{\Lambda R_0} [\Phi] + T'_{\Lambda R_0} [x; \Phi; \omega] + O_{\Lambda R_0} [\Phi; \omega] J_{\Lambda R_0} [x; \Phi] \quad (58)$$

where

$$T'_{\Lambda R_0} [x; \Phi; \omega] = tr K_{\Lambda R D_{\Lambda R_0}} J''_{\Lambda R_0} [x; \Phi] \quad (59)$$

The primes mean as usual differentiation with respect to $\Phi$; $O_{\Lambda R_0}$ is the functional corresponding to the evanescent operator discussed in the previous subsection.

To show that a composite operator $J$ is gauge invariant, one has to prove that it is possible to impose the renormalization conditions on $J$ in such a way that $O^{J}_{\Lambda R_0}$ is evanescent. In
particular if $J_{\Lambda_0}$ is gauge invariant at classical level one checks easily from eq. (37) that $O^{J}_{\Lambda_0}$ is evanescent at tree level. We stress that by gauge invariant operator we mean an operator satisfying the effective Ward identities; we do not address the question of its independence from the gauge fixing parameter.

Taking the functional derivative with respect to the fields and the gauge parameter $\omega$, in a suitable neighbourhood of the origin of the momentum space, eqs. (58) and (59) yield for $l > 0$ (modulo evanescent terms):

$$O^{(l)\Lambda_0}_{2n,m}(\alpha_1 p_1, \ldots, \alpha_n p_n; n\beta_1 q_1, \ldots, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m; Q) =$$

$$- \frac{1}{c} P \mu J^{(l)\Lambda_0}_{2n,m+1}(\alpha_1 p_1, \ldots, \alpha_n p_n; n\beta_1 q_1, \ldots, \beta_{n-1} q_{n-1}, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m, \mu P)$$

$$- \sum_{j=1}^{n} \left[ J^{(l)\Lambda_0}_{2n,m}(p_j) - J^{(l)\Lambda_0}_{2n,m}(Q_j) \right] + T^{(l)\Lambda_0}_{2n,m}(\alpha_1 p_1, \ldots, \alpha_n p_n; n\beta_1 q_1, \ldots, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m; Q)$$

where the multi-indices $p_j$ and $Q_j$ are defined by

$$p_j \equiv (\alpha_1 p_1, \ldots, \alpha_{j-1} p_{j-1}, n\beta_j p_j, \alpha_{j+1} p_{j+1}, \ldots, \alpha_n p_n; n\beta_1 q_1, \ldots, \beta_{n-1} q_{n-1}, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m)$$

$$Q_j \equiv (\alpha_1 p_1, \ldots, \alpha_n p_n; n\beta_1 q_1, \ldots, \beta_{j-1} q_{j-1}, \beta_j p_j + q_j, \beta_{j+1} q_{j+1}, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m)$$

and

$$T^{(l)\Lambda_0}_{2n,m}(\alpha_1 p_1, \ldots, \alpha_n p_n; n\beta_1 q_1, \ldots, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m; Q) = (-1)^{n+1} \int_p \left\{ K_{\Lambda}(p) \timesight.$$}

$$\times \left[ S^{\Lambda_0}_{\beta\alpha}(p + P) J^{(l-1)\Lambda_0}_{2n+2,m}(\alpha_1 p_1, \ldots, \alpha_n p_n, -p; n\beta_1 q_1, \ldots, \beta_n q_n, \beta P + P; \mu_1 k_1, \ldots, \mu_m k_m) \right.$$

$$- J^{(l-1)\Lambda_0}_{2n+2,m}(\alpha_1 p_1, \ldots, \alpha_n p_n, \alpha p + p; n\beta_1 q_1, \ldots, \beta_n q_n, \beta - p; \mu_1 k_1, \ldots, \mu_m k_m) S^{\Lambda_0}_{\beta\alpha}(-p - P) \right\}$$

with $P = - (\sum_{j=1}^{n} p_j + \sum_{j=1}^{n} q_j + \sum_{r=1}^{m} k_r + Q)$. In eqs. (34) and (41) $Q$ is the momentum conjugated to the $x$ variable in eq. (58); notice that the arguments of the $J$ functions refer to the fields variables, their sum is therefore always equal to $-Q$.

We shall discuss the cases $J(x) = i \bar{\psi} \gamma^5 \gamma_5 \psi(x) \equiv J_5^r(x)$ and $J(x) = F \bar{F}(x)$, the former renormalized as an operator of dimension 3, the latter as an operator of dimension 4.

For $J_5^r$ the relevant projections of $O^J$ are:

$n = 0, m = 1, z = 0, 1, 2; n = 0, m = 2, z = 0, 1; n = 0, m = 3, z = 0; n = 1, m = 0, z = 0$.

The rigid symmetries constrain $J_{5\Lambda_0}^r$ to be a linear combination of $i \bar{\psi} \gamma^5 \gamma_5 \psi$ and $\epsilon_{\tau\nu\rho\sigma} F^{\nu\rho} A^\sigma$. The $z = 0$ projections are identically satisfied because of charge conservation. For $m = 1, n = 0, z = 1$ all the addenda of eq. (60) vanish, since no pseudotensor with three indices exists.

For $m = 1, n = 0, z = 2$ there is one possible structure, the pseudotensor $\epsilon_{\tau\rho\mu\nu}$. Thus the completely antisymmetric tensor $\frac{\partial}{\partial k_1} J^{(l)\Lambda_0}_{5\nu\rho\mu}(k_1, k_2)|_{k_1 = k_2 = 0}$ apart from some trivial numerical constant is to be chosen equal to $\frac{\partial}{\partial k_1} \frac{\partial}{\partial q_1} T^{(l)\Lambda_0}_{5\nu\rho\mu}(k_1; Q)|_{k_1 = Q = 0}$ which does not vanish as shown in sect. II.

The projections $m = 2, n = 0$ involving $(q_1 + q_2 + Q) \mu J_{5\rho\mu}^{(l)\Lambda_0}(q_1, q_2, -q_1 - q_2 - Q)$ and $T_{5\rho\mu}^{(l)\Lambda_0}$ and both, because of charge conjugation invariance, are identically zero. The identity for $m = 3, n = 0, z = 0$ is satisfied, as well as for $2n = 2, m = 0, z = 0$.
The conclusion of our analysis is that $J_{5\Lambda\Lambda_0}^\phi[x; \Phi]$ is unambiguously determined up to a multiplicative constant which is fixed by the renormalization condition
\begin{equation}
J_{5\alpha\beta}^{(l)T\Lambda R\Lambda_0}(0) = i\sigma^{(l)}(\gamma^\tau\gamma^5)_{\alpha\beta}
\end{equation}
the mixing between the two possible bare operators being determined loop by loop by the requirement of gauge invariance.

One can introduce at scale $\Lambda_R$ for an operator of dimension $d$ an analogue of the Zimmermann normal product, which we call $N_{d}^{\Lambda R c}[J]$ (remember connected rather than proper), by choosing for $l > 0$ vanishing renormalization conditions at $\Lambda = \Lambda_R$ and zero momentum $26,11,23$. Notice that for $J_5^\phi$ the renormalization recipe $N_{3}^{\Lambda R c}[J_5^\phi]$ is not gauge invariant, as we saw explicitly in eq. (43).

The composite operator so defined is infrared finite, namely the limit $\Lambda \to 0$ exists for non-exceptional momenta; in fact, as already stated, the positive dimension Green functions $J_{\mu\nu}^{\Lambda\Lambda_0}(0)$, $J_{\mu\nu}^{\tau\Lambda\Lambda_0}(0,0)$ and $\partial_\mu J_{5\alpha\beta}^{\Lambda\Lambda_0}(0)$ are zero due to the rigid symmetries of theory (the condition analogue of eq. (56) is then satisfied); as a consequence $\mathcal{T}_{5\Lambda\Lambda_0}[x;\Phi] \to 0$ as $\Lambda \to 0$ and in this limit we recover the usual Ward identity for the composite operators.

As a last remark we note that due to linearity of eq. (54), $\frac{\partial}{\partial x} J_{5\Lambda\Lambda_0}[x, \Phi]$ fulfills the effective Ward identity too, in the limit $\Lambda_0 \to \infty$.

Similar considerations can be repeated for the renormalization of $F \tilde{F}$ (which mixes with $\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)$) but now the result is simpler: indeed $N_4^{\Lambda R c}[F \tilde{F}]$ yields a gauge invariant renormalization.

C. Effective axial Ward identity

Let us consider the insertion $\mathcal{O}_{\Lambda\Lambda_0}^5[x; \Phi]$ defined by eq. (51) with $T = 0$ and $R = R_5$ parameter of a local axial transformation. At tree level, by the naive Noether construction, it is equal to $-i\partial_\mu J_{5\Lambda\Lambda_0}^\mu[x, \Phi]$; at quantum level
\[ i\mathcal{O}_{\Lambda\Lambda_0}^5[x; \Phi] - \partial_\mu J_{5\Lambda\Lambda_0}^\mu[x; \Phi] \equiv i\mathcal{A}_{\Lambda\Lambda_0}[x; \Phi] \]
represents the anomaly term. Proceeding as in (51) and (52) we get:
\begin{align}
&i \int dx \omega_5(x) \mathcal{A}_{\Lambda\Lambda_0}[x; \Phi] = -i \int dx \omega_5(x) \partial_\mu J_{5\Lambda\Lambda_0}^\mu[x; \Phi] - \Phi D^{-1} R_5 \Phi + \Phi D^{-1} R_5 D_{\Lambda\Lambda_0} L'_{\Lambda\Lambda_0}[\Phi] \\
&- \Phi R_5^T K_A L''_{\Lambda\Lambda_0}[\Phi] - L'_{\Lambda\Lambda_0}[\Phi] D_{\Lambda\Lambda_0} R_5^T K_A L''_{\Lambda\Lambda_0}[\Phi] - \mathcal{T}_{5\Lambda\Lambda_0}[\Phi; \omega_5]
\end{align}
(63)
where
\[ \mathcal{T}_{5\Lambda\Lambda_0}[\Phi; \omega_5] = tr K_A R_5 D_{\Lambda\Lambda_0} L''_{\Lambda\Lambda_0}[\Phi] \]
(64)
We stress that the composite operator $\mathcal{A}_{\Lambda\Lambda_0}[x; \Phi]$ satisfies the flow equation (54), indeed both $\mathcal{O}_{\Lambda\Lambda_0}^5[x; \Phi]$ and $\partial_\mu J_{5\Lambda\Lambda_0}^\mu[x; \Phi]$ solve this linear equation.

Let us show that, by a suitable choice of the renormalization constant $\sigma^{(l)}$ of eq. (62)
\begin{equation}
\lim_{\Lambda_0 \to \infty} \mathcal{A}_{\Lambda\Lambda_0}[x; \Phi] = a \lim_{\Lambda_0 \to \infty} [N_4^{\Lambda R c}[F \tilde{F}]]_{\Lambda\Lambda_0}[x; \Phi]
\end{equation}
(65)
where $a$ is the coefficient of the anomaly, which is independent from $\Lambda$ and $\Lambda_R$. The operator $A_{A\Lambda_0}$ of dimension 4 is defined by its renormalization conditions, which from eq. (53) are determined by those of $L_{\Lambda_R\Lambda_0}$ and $J_{5\Lambda_R\Lambda_0}^\mu$.

Consider eq. (53) in a suitably small neighbourhood of the origin of the momenta and for $\Lambda = \Lambda_R$ and $l > 0$:

$$A_{2n,m}^{(l)\Lambda_R\Lambda_0}(\alpha_1 p_1, \ldots, \alpha_n p_n; \beta_1 q_1, \ldots, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m) =$$

$$= \sum_{j=1}^{n} (-1)^{n-j} [\gamma_5^\alpha_1 \alpha_\Lambda 2n,m(P)_{j\alpha} + L_{2n,m}^{(l)\Lambda_R\Lambda_0}(Q_{j\beta}) \gamma_5^\beta]$$

$$- [\gamma_5^\alpha_{2n,m} + P/R_{2n,m}(\alpha_1 p_1, \ldots, \alpha_n p_n; \beta_1 q_1, \ldots, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m)]$$

where $P_{j\alpha}$ and $Q_{j\beta}$ are defined by

$$P_{j\alpha} \equiv (\alpha_1 p_1, \ldots, \alpha_j p_j-1, \alpha_j p_j, \ldots, \alpha_n p_n, \alpha P + p_j; \beta_1 q_1, \ldots, \beta_{n-1} q_{n-1}, \beta_n; \mu_1 k_1, \ldots, \mu_m k_m)$$

and

$$Q_{j\beta} \equiv (\alpha_1 p_1, \ldots, \alpha_n p_n; \beta_1 q_1, \ldots, \beta_{j-1} q_{j-1}, \beta_j q_j, \ldots, \beta_n q_n, \beta; \mu_1 k_1, \ldots, \mu_m k_m)$$

and

$$T^{(l)\Lambda_R\Lambda_0}_{2n,m}(\alpha_1 p_1, \ldots, \alpha_n p_n; \beta_1 q_1, \ldots, \beta_n q_n; \mu_1 k_1, \ldots, \mu_m k_m) = (-1)^{n+1} \int_p \left\{ K_\Lambda_R(p) \times \right.$$}

$$\times [S^{\Lambda_R\Lambda_0}_\alpha (p + P) \Lambda^{(l-1)\Lambda_R\Lambda_0}_{2n,m}(\alpha_1 p_1, \ldots, \alpha_n p_n, \alpha' - p; \beta_1 q_1, \ldots, \beta_n q_n, \beta; \mu_1 k_1, \ldots, \mu_m k_m)]$$

$$+ L^{(l-1)\Lambda_R\Lambda_0}_{2n+2,m}(\alpha_1 p_1, \ldots, \alpha_n p_n, \alpha P + p; \beta_1 q_1, \ldots, \beta_n q_n, \beta; \mu_1 k_1, \ldots, \mu_m k_m)) \Lambda^{(l)\Lambda_R\Lambda_0}_\alpha (p + P) \right\}$$

with $P = - \left( \sum_{j=1}^{n} p_j + \sum_{j=1}^{n} q_j + \sum_{r=1}^{m} k_r \right)$. As regards the momentum dependence in eq. (66) considerations similar to those after eq. (54) hold. When we consider the relevant projections in eq. (66) we find, using the rigid symmetries of $L_{\Lambda_0}$ and of $J_{5\Lambda_0}$, that all the renormalization conditions of $A_{\Lambda_R\Lambda_0}$ are zero identically (i.e. independently of the renormalization conditions on $J^\mu_5$) but the following two:

i) $2n = 2$, $m = 0$, $z = 1$:

the divergence of the current on the r.h.s. gives $i(\not{q} + \not{\gamma}) \gamma_5 \sigma^{(l)}$, where $\sigma^{(l)}$ is the (arbitrary for the moment) renormalization constant in eq. (52). The first addendum in the r.h.s. of eq. (66) gives an analogous term $[\gamma_5^\alpha \gamma_5^{\Lambda_R\Lambda_0} (p + q) \Lambda^{(l)\Lambda_R\Lambda_0}_{2n+2,m}(\alpha_1 p_1, \ldots, \alpha_n p_n, \alpha' - P; \beta_1 q_1, \ldots, \beta_n q_n, \beta; \mu_1 k_1, \ldots, \mu_m k_m)] \Lambda^{(l)\Lambda_R\Lambda_0}_\alpha (p + P)$, where $\gamma_5^{(l)}$ depends on $L^{(l')}_{\Lambda_0}$ at loop $l' < l$. Fixing loop by loop $\sigma^{(l)}$ in terms of $z_2^{(l)}$ we can set to zero the corresponding renormalization condition on $A^{(l)}_{\Lambda_R\Lambda_0}$.

ii) $m = 2$, $n = 0$, $z = 2$:

the divergence term $\partial_{k_{2\mu}} \partial_{k_{2\nu}} (k_1 + k_2) T^{(l)\Lambda_R\Lambda_0}_{5\mu\nu}(k_1, k_2) |_{k_1 = k_2 = 0} = 2 \partial_{k_{2\mu}} J^{(l)\Lambda_R\Lambda_0}_{5\mu\nu}(k_1, k_2) |_{k_1 = k_2 = 0}$ as previously discussed is not vanishing because of gauge invariance, but also the term $\partial_{k_{2\mu}} \partial_{k_{2\nu}} (k_1 + k_2) T^{(l)\Lambda_R\Lambda_0}_{5\mu\nu}(k_1, k_2) |_{k_1 = k_2 = 0}$ is not vanishing, both the quantities for $O(4)$ symmetry and parity being proportional to $\epsilon_{\rho\mu\nu}$. At one loop the two terms, both proportional to the same integral, were computed in sect.II; they sum up to give the well-known coefficient of the anomaly. By direct inspection on the renormalization conditions of $A$ and comparison with those of $N_{\Lambda_0}^{A\Lambda_0} [F \bar{F}]$ we can state that for $\Lambda_0 \to \infty$ the two functional are proportional. Since $A_{\Lambda_0}$ and $[N_{\Lambda_0}^{A\Lambda_0} [F \bar{F}]]_{\Lambda_0}$ satisfy eq. (54), $a$ is independent of $\Lambda$; for dimensional reasons it is also $\Lambda_R$-independent.
CONCLUSIONS

Polchinski has shown that the Wilsonian renormalizaton group can be applied to get a rigorous proof of renormalizability, which avoids the subleties involved in the BPHZ technique.

In this paper we show that this method provides a natural generalization of the hard-soft renormalization program suggested in [3], in which the renormalization scale $\Lambda_R$ substitutes the momentum subtraction scale $\mu$ as the only dimensional parameter introduced in the renormalization of massless theories. To prove that physical quantities are independent from $\Lambda_R$, it is necessary to find the Gell-Mann and Low renormalization group equation in the HS schemes, discussed in [3,9]. In this sense the coupling constants in HS schemes are as ‘physical’ as in the non-zero momentum subtraction scheme. Our modification of the HS scheme introduced in [3] has the advantage of requiring renormalization conditions only at $\Lambda_R$, while in [3] there are also renormalization conditions at $\Lambda = 0$; furthermore, renormalizability is investigated using the method of the flow equation instead of BPHZ.

As in the usual BPHZ treatment, the validity of Ward identities is established associating to them a composite operator which, if the Ward identities are not anomalous, can be proven to be evanescent; otherwise it is a local operator, the anomaly. The HS scheme has the advantage that this can be done computing simply zero-momentum graphs.

As an example of the application of this method, we consider the case of massless QED. Using the flow equation and some results of [16], we prove the validity of the effective Ward identities for massless QED at quantum level; then we show that, for any HS scheme compatible with the effective Ward identities and the rigid symmetries, the theory is infrared finite. We define in the same formalism the gauge-invariant axial current operator and its anomaly.

In the case of Yang-Mills the effective Ward identities have been proven in [11] in a HS scheme; it would be interesting to prove infrared finiteness in the same scheme.

Note Added in Proof

To prove the evanescence of $O'_{\Lambda_{\Lambda_0}}$ in eq.(57), one cannot use directly its flow equation (50), since it is a disconnected insertion; however its connected part $O'_{\Lambda_{\Lambda_0}} - O_{\Lambda_{\Lambda_0}}J'_{\Lambda_{\Lambda_0}}$ satisfies a modified flow equation, which differs from eq.(50) by an inhomogeneous term $O'_{\Lambda_{\Lambda_0}} \partial_\Lambda D_{\Lambda_{\Lambda_0}}' J'_{\Lambda_{\Lambda_0}}$. Using this modified flow equation and the fact that $O_{\Lambda_{\Lambda_0}}$ is evanescent (see Subsection III.A) the proof of evanescence of $O'_{\Lambda_{\Lambda_0}} - O_{\Lambda_{\Lambda_0}}J_{\Lambda_{\Lambda_0}}$, and hence of $O'_{\Lambda_{\Lambda_0}}$, is essentially the same as in the standard one for a connected operator insertion satisfying eq.(50).
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