Graduality From Embedding-Projection Pairs

Extended Version

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Gradually typed languages allow statically typed and dynamically typed code to interact while maintaining benefits of both styles. The key to reasoning about these mixed programs is Siek-Vitousek-Cimini-Boyland’s (dynamic) gradual guarantee, which says that giving components of a program more precise types only adds runtime type checking, and does not otherwise change behavior. In this paper, we give a semantic reformulation of the gradual guarantee called graduality. We change the name to promote the analogy that graduality is to gradual typing what parametricity is to polymorphism. Each gives a local-to-global, syntactic-to-semantic reasoning principle that is formulated in terms of a kind of observational approximation.

Utilizing the analogy, we develop a novel logical relation for proving graduality. We show that embedding-projection pairs (ep pairs) are to graduality what relations are to parametricity. We argue that casts between two types where one is “more dynamic” (less precise) than the other necessarily form an ep pair, and we use this to cleanly prove the graduality cases for casts from the ep-pair property. To construct ep pairs, we give an analysis of the type dynamism relation—also known as type precision or naïve subtyping—that interprets the rules for type dynamism as comositional constructions on ep pairs, analogous to the coercion interpretation of subtyping.

CCS Concepts: • Software and its engineering → General programming languages; • Social and professional topics → History of programming languages;

Additional Key Words and Phrases: Gradual typing, keyword2, keyword3

1 INTRODUCTION

Gradually typed programming languages are designed to resolve the conflict between static and dynamically typed programming styles [Tobin-Hochstadt and Felleisen 2006, 2008; Siek and Taha 2006]. A gradual language allows a smooth transition from dynamic to static typing through gradual addition of types to dynamically typed programs, and allows for safe interactions between more statically typed and more dynamically typed components. With such an enticing goal there has been extensive research on gradual typing—e.g., [Siek and Taha 2006; Gronski et al. 2006; Wadler and Findler 2009; Ina and Igarashi 2011; Swamy et al. 2014; Allende et al. 2013]—with recent work aimed at extending gradual typing to more advanced language features, such as parametric polymorphism [Ahmed et al. 2017; Igarashi et al. 2017a], effect tracking [Bañados Schwerter et al. 2014], typstate [Wolff et al. 2011], session types [Igarashi et al. 2017b], and refinement types [Lehmann and Tanter 2017].

Formalizing the idea of a “smooth transition”, a key property that every gradually typed language should satisfy is Siek, Vitousek, Cimini, and Boyland’s (dynamic) gradual guarantee, which we refer to as graduality (by analogy with parametricity). Graduality enables programmers to modify their program from a dynamically typed to a statically typed style, and vice-versa, with confidence that the program’s behavior only changes in predictable ways. Specifically, it says that

\[ \text{In this paper, we use blue to typeset our gradual cast calculus } \lambda_G \text{ and red to typeset our typed language with errors } \lambda_T, \text{.} \]

The paper will be much easier to read if viewed/printed in color.

1The same work also introduces a static gradual guarantee that says that changing the types in a program to be less dynamic means type checking becomes stricter. We do not consider this in our paper because we only consider the semantics of cast calculi, not the type systems of gradual surface languages. We discuss the relationship further in §7

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changing the types in a program to be “less dynamic”/“more precise”—i.e., changing from the dynamic type to some more precise type such as integers or functions—either produces the same behavior as the original program or causes a dynamic type error. Conversely, if a program does not error and some types are made “more dynamic”/“less precise” then the program has the exact same behavior. This is an important reasoning principle for programmers as the alternative would be quite counterintuitive: for instance, changing certain type annotations might cause a terminating program to diverge, or a program that prints your calendar to tweet your home address! This distinguishes dynamic type checking in gradual typing from exceptions: raising an exception is a valid program behavior that can be caught and handled by a caller, whereas a dynamic type error is always considered to be a bug, and terminates the program.

More formally, the notion of when a type $A$ is “less dynamic” than another type $B$ is specified by a type dynamism relation (also known as type precision or naïve subtyping), written $A \sqsubseteq B$, which is defined for simple languages as the least congruence relation such that the dynamic type $?$ is the most dynamic type: $A \sqsubseteq ?$. Then, term dynamism (also known as term precision) is the natural extension of type dynamism to terms, written $t \sqsubseteq s$. The graduality theorem is then that if $t \sqsubseteq s$, then the behavior of $t$ must be “less dynamic” than the behavior of $s$—that is, either $t$ produces a runtime type error or both terms have the exact same behavior. We say $t$ is “less dynamic” in the sense that it has fewer behaviors.

Unfortunately, for the majority of gradually typed languages, the (dynamic) gradual guarantee is considered quite challenging to prove, and there is only limited guidance about how to design new languages so that they satisfy this property. There are two notable exceptions: Abstracting Gradual Typing (AGT) [Garcia et al. 2016a] and the Gradualizer [Cimini and Siek 2016, 2017] provide systematic methods and formal tools, respectively, for deriving a gradually typed language from a statically typed language, and they both provide the gradual guarantee by construction. However, while they provide a proof of the gradual guarantee for languages produced in the respective frameworks, most gradually typed languages are not produced in this way; for instance, Typed Racket’s approach to gradual typing [Tobin-Hochstadt and Felleisen 2006, 2008] is not explained by either system. Furthermore, both Gradualizer and AGT base their semantics on static type checking itself, but this is the reverse of the semantic view of type checking. In the semantic viewpoint, type checking should be justified by a sensible semantics, and not the other way around.

Type Dynamism and Embedding-Projection Pairs. While the gradual guarantee as presented in Siek et al. [2015] makes type dynamism a central component, the semantic meaning of type dynamism is unclear. This is not just a philosophical question: it is unclear how to extend type dynamism to new language features. For instance, polymorphic gradually typed languages have been developed recently by Ahmed et al. [2017] and Igarashi et al. [2017a], but the two papers have different definitions of type dynamism, and neither attempts a proof of the (dynamic) gradual guarantee. The AGT [Garcia et al. 2016a] approach gives a systematic definition of type dynamism in terms of sets of static types, but that definition is difficult to separate from the rest of their framework, whereas we would like a definition that can be interpreted in any gradually typed language. At present, the best guidance we have comes from the gradual guarantee itself: the dynamic type should be the greatest element, and the gradual guarantee should hold.

We propose a semantic definition for type dynamism that naturally leads to a clean formulation and proof of the gradual guarantee: An ordering $A \sqsubseteq B$ should hold when the casts between the two types form an embedding-projection pair.
What does this mean? First, in order to support interaction between statically typed and dynamically typed code while still maintaining the guarantees of the static types, gradually typed languages include casts\(^2\) \(\langle B \leftarrow A \rangle\) that dynamically check if a value of type \(A\) corresponds to a valid inhabitant of the type \(B\), and if so, transform its value to have the right type. Then if \(A \subseteq B\), we say that the casts \(\langle B \leftarrow A \rangle\) and \(\langle A \leftarrow B \rangle\) form an embedding-projection pair, which means that they satisfy the following two properties that describe acceptable behaviors when casting between the two types: retraction and projection.

First, \(A\) should be a stricter type than \(B\), so anything satisfying \(A\) should also satisfy \(B\). This is captured in the retraction property: if we cast a value \(v : A\) from \(A\) to \(B\) and then back down to \(A\), we should get back an equivalent value because \(v\) should satisfy the type of \(A\) and \(B\). Formally, \(\langle A \leftarrow B \rangle \langle B \leftarrow A \rangle t \approx t\) where \(\approx\) means observational equivalence of the programs: when placed in the same spot in a program, they produce the same behavior.

Second, casts should only be doing type checking, and not otherwise changing the behavior of the term. Since \(B\) is a weaker property than \(A\), if we cast a value of \(v : B\) down to \(A\), there may be a runtime type error. However, if \(v\) really does satisfy \(A\) the cast succeeds, and if we cast back to \(B\) we should get back a value with similar behavior to \(v\). If \(B\) is a first-order type like booleans or numbers, we should get back exactly the same value. However, if \(A, B\) are higher-order types like functions or objects, then it is impossible to check if a value of one type \(B\) satisfies \(A\). For instance, if \(B = ? \rightarrow ?\) and \(A = ? \rightarrow \mathbb{N}\), then it is not decidable whether or not a value of \(B\) will always return a number on every input. Instead, following [Findler and Felleisen 2002], gradual type casts wrap the function with a cast on its outputs and if at any point it returns something that is not a number, a type error is raised. So if \(v : B\) is cast to \(A\) and back, we cannot expect to always get an equivalent value back, but the result should error more—that is, either the cast to \(A\) raises an error, or we get back a new value \(v' : B\) that has the same behavior as \(v\) except it sometimes raises a type error. We formalize this as observational error approximation and write the ordering \(t \subseteq t'\) as “\(t\) errors more than \(t'\).” We then use this to formalize the projection property: \(\langle B \leftarrow A \rangle \langle A \leftarrow B \rangle s \subseteq s\).

Notice how the justification for the projection property uses the same intuition as graduality: that casts should only be doing checking and not completely changing a program’s behavior. This is the key to why embedding-projection pairs help to formulate and prove graduality: we view graduality as the natural extension of the projection property from a property of casts to a property of arbitrary gradually typed programs.

This gives us nice properties of some casts, but what do we know about casts that are not upcasts or downcasts? In traditional formulations, gradual typing includes casts \(\langle B \leftarrow A \rangle\) between types that are shallowly compatible—i.e., that are not guaranteed to fail. For instance, we can cast a pair where the left side is known to be a number \(\mathbb{N} \times ?\) to a type where the right side is known to be a number \(? \times \mathbb{N}\) with casts succeeding on values where both sides are numbers. The resulting cast \(\langle ? \times \mathbb{N} \leftarrow \mathbb{N} \times ? \rangle\) is neither an upcast nor a downcast. We argue that the formulation based on these “general” casts is ill-behaved from a meta-theoretic perspective: you are quite limited in your ability to break casts for larger types into casts for smaller types. Most notably, the composition of two general casts is very rarely the same as the direct cast. For instance, casting from \(\mathbb{N}\) to \(? \rightarrow ?\) and back to \(\mathbb{N}\) always errors, but obviously the direct cast \(\langle \mathbb{N} \leftarrow \mathbb{N} \rangle\) is the identity. We show that upcast and downcasts on the other hand satisfy a decomposition theorem: if \(A_1 \subseteq A_2 \subseteq A_3\), then the upcast from \(A_1\) to \(A_3\) factors through \(A_2\) and similarly for the downcast.

\(^2\)It is not literally true that every gradual language uses this presentation of casts from cast calculi, but in order for a language to be gradually typed, some means of casting between types must be available, such as embedding dynamic code in statically typed code, or type annotations. We argue that the properties of casts we identify here should apply to those presentations as well.
Furthermore, if we disregard performance of the casts, and only care about the observational behavior, we show that any “general” cast is the composition of an upcast followed by a downcast. For instance, our cast from before \( (\times N \leftarrow N \times ?) \) is observationally equivalent to the composition of first upcasting to a pair where both sides are dynamically typed \(? \times \?) and then downcasting: \( (? \times N \leftarrow ? \times ?)(? \times ? \leftarrow N \times ?) \). We show that all the casts in a standard gradually typed language exhibit this factorization, which means that for the purposes of formulating and proving graduality, we need only discuss upcasts and downcasts. For implementation, it is more convenient to have a primitive notion of coercion/direct cast to eliminate/collapse casts [Herman et al. 2010; Siek and Wadler 2010], but we argue that the correctness of such an implementation should be justified by a simulation relation with a simpler semantics, meaning the implementation would inherit a proof of graduality from the simpler semantics as well.

To prove these equivalence and approximation results, we develop a novel step-indexed logical relation that is sound for observational error approximation. We also develop high-level reasoning principles from the relation so that our main lemmas do not involve any manual step-manipulation.

Finally, based on our semantic interpretation of type dynamism as embedding-projection pairs, we provide a refined analysis of the proof theory of type dynamism as a syntax for building ep pairs. We give a semantics for these proof terms analogous to the coercion interpretation of subtyping derivations. Similar to subtyping, we prove a coherence theorem which gives, as a corollary, our decomposition theorem for upcasts and downcasts.

Graduality. In Siek et al. [2015], they prove the (dynamic) gradual guarantee by an operational simulation argument whose details are quite tied to the specific cast calculus used. Using the ep pairs, we provide a more semantic formulation and proof of graduality. First, we use our analysis of type dynamism as denoting ep pairs to define graduality as a kind of observational error approximation up to upcast/downcast, building on the axiomatic semantics of graduality in New and Licata [2018]. We then prove the graduality theorem using our logical relation for error approximation. Notably, the decomposition theorem for ep pairs leads to a clean, uniform proof of the cast case of graduality.

Overview of Technical Development and Contributions. In this paper, we show how to prove graduality for a standard gradually typed cast calculus by translating it into a simple typed language with recursive types and errors. Specifically, our development proceeds as follows:

1. We present a standard gradually typed cast calculus \( \lambda_G \) and its operational semantics, using “general” casts (§2).
2. We present a simple typed language with recursive types and a type error \( \lambda_{T, U} \), into which we translate the cast calculus. Casts in \( \lambda_G \) are translated to contracts implemented in the typed language (§3).
3. We develop a novel step-indexed logical relation that is sound for our notion of observational error approximation (§4). We prove transitivity of the logical relation and other high-level reasoning principles so that our main lemmas for ep-pairs and graduality do not involve any manual step-manipulation.
4. We present a novel analysis of type dynamism as a coherent syntax for ep pairs and show that all of the casts of the gradual language can be factorized as an upcast followed by a downcast (§5).
5. We give a semantic formulation of graduality and then prove it using our error-approximation logical relation and ep pairs (§6).

3Note that this is not the same as the factorization of casts known as “threesomes”, see §7 for a comparison.
Types \[ A, B ::= ? | 1 | A \times B | A + B | A \rightarrow B \]

Tags \[ G ::= 1 | ? \times ? | ? + ? | ? \rightarrow ? \]

Terms \[ t, s ::= \mathcal{U} | x | \langle B \equiv A \rangle t | \langle \cdot \rangle | \langle t_1, t_2 \rangle | \text{let } (x, y) = t \text{ in } s \]

| \begin{align*}
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\[ [A] \overset{\text{def}}{=} G \quad \text{where } A \neq ? \]

\[
[1] \overset{\text{def}}{=} 1
\]

\[
[A \times B] \overset{\text{def}}{=} ? \times ?
\]

\[
[A + B] \overset{\text{def}}{=} ? + ?
\]

\[
[A \rightarrow B] \overset{\text{def}}{=} ? \rightarrow ?
\]

**Fig. 3. \( \lambda_G \): Tag of a (non-dynamic) Type**

\[
E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]]
\]

\[
E[\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]]
\]

| A ? | \[A] \neq A | \[A] \neq A |
|-----|----------------|----------------|
| E[\(\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]] | E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]] |
| E[\(\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]] | E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]] |

| A \neq ? | \[A] \neq A | \[A] \neq A |
|-----|----------------|----------------|
| E[\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]] | E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]] |
| E[\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]] | E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]] |

| G \neq G' | \[G] \neq \[G'] | \[G] \neq \[G'] |
|-----|----------------|----------------|
| E[\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]] | E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]] |
| E[\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]] | E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]] |

\[ \begin{array}{ll}
E[\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]] & E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]]
\end{array} \]

\[ \begin{array}{ll}
E[\text{let } (x_1, x_2) = (v_1, v_2) \text{ in } t ] \mapsto E[t[v_1/x_1, v_2/x_2]] & E[\text{case} (\text{inj } v) \text{ of } \text{inj } \cdot t ] \mapsto E[t[v/x]]
\end{array} \]

The basic sorts of dynamically typed values. Every type except \( ? \) has an “underlying” tag type we write as \( [A] \) and define in Figure 3. These tag types are the cases of the dynamic type \( ? \) seen as a sum type, which is how we model it in §3.2. For any two types \( A, B \), we can form the cast \( B \overset{\text{cast}}{\leftarrow} A \) which at runtime will attempt to coerce a term \( v : A \) into a valid term of type \( B \). If the value cannot sensibly be interpreted as a value in \( B \), the cast fails and reduces to the dynamic type error \( U \). The type error is like an uncatchable exception, modeling the fact that the program crashes with an error message when a dynamic type error is encountered. In this paper we consider all type errors to be equivalent. The calculus is based on that of Wadler and Findler [2009], but does not have blame and removes the restriction that types must be compatible in order to define a cast.

Figure 4 presents the operational semantics of the gradual language in the style of Felleisen and Hieb [1992], using evaluation contexts \( E \) to specify a left-to-right, call-by-value evaluation order. The top of the figure shows the reductions not involving casts. This includes the standard reductions for pairs, sums, and functions using the obvious notion of substitution \( t[y] \), in addition to a reduction \( E[U] \mapsto U \) to propagate a dynamic type error to the top level.

More importantly, the bottom of the figure shows the reductions of casts, specifying the dynamic type checking necessary for gradual typing. First \( (\text{DYN} \text{DYN}) \), casting from dynamic to itself is the identity. For any type \( A \) that is not a tag type (checked by \( [A] \neq A \) or the dynamic type, casting to the dynamic type first casts to its underlying tag type \( [A] \) and then tags it at that type \( (\text{TAGUP}) \).
Similarly, casting down from the dynamic type first casts to the underlying tag type (TagDN). The next two rules are the primitive reductions for tags: if you project at the correct tag type, you get the underlying value out (TagMATCH) and otherwise a dynamic type error is raised (TagMISMATCH). Similarly, the next rule (TagMISMATCH') says that if two types are incompatible in that they have distinct tag types and neither is dynamic, then the cast errors. The next three (Fun, Pair, Sum) are the standard “wrapping” implementations of contracts/casts [Findler and Felleisen 2002], also familiar from subtyping. For the function cast \(< A_2 \rightarrow B_2 \iff A_1 \rightarrow B_1 >\), note that while the output type is the same direction \(< B_2 \iff B_1 >\), the input cast is flipped: \(< A_1 \iff A_2 >\).

We note that this standard operational semantics is quite complex for such a small language. In particular, it is more complicated than the operational semantics of typed and dynamically typed languages of similar size. Typed languages have reductions for each elimination form and dynamically typed languages add only the possibility of type error to those reductions. Here on the other hand, the semantics is not modular in the same way: there are five rules involving the dynamic type and four of them involve comparing arbitrary types.

For these reasons, we find the cast calculus presentation inconvenient for semantic analysis, and we choose not to develop our theory of graduality or even prove type safety directly for this language. Instead, we will translate the cast calculus into a typed language where the casts are translated to functions implemented in the language, i.e. contracts [Findler and Felleisen 2002]. This has the advantage of reducing the size of the language, making “language-level” theorems about type safety and soundness of a logical relation easier to prove. Finally, note that our central theorems are still about the gradual language, but we will prove them by lifting results about their translations using an adequacy theorem (theorem 3.8).

3 TRANSLATING GRADUAL TYPING

We now translate our cast calculus into a simpler, non-gradual typed language with errors. We then prove an adequacy theorem that enables us to prove theorems about gradual programs by reasoning about their translations.

3.1 Typed Language with Errors

The typed language we will translate into is \(\lambda T, \Upsilon\) (pronounced “lambda tee error” and typeset in bold red serif font), a call-by-value typed lambda calculus with iso-recursive types and an uncatchable error. Figure 5 shows the syntax of the language. Figure 6 shows some of the typing rules; the rest are completely standard.

The types of the language are similar to the cast calculus: they include the standard type formers of products, sums, and functions. Rather than the specific dynamic type, we include the more general, but standard, iso-recursive type \(\mu \alpha. A\), which is isomorphic to the unfolding \(A[\mu \alpha. A/\alpha]\) by the terms \(\text{roll}_{\mu \alpha. A} \cdot\) and \(\text{unroll} \cdot\). As in the source language we have an uncatchable error \(\Upsilon\).

Figure 7 presents the operational semantics of the language. For the purposes of later defining a step-indexed logical relation, we assign a weight to each small step of the operational semantics that is 1 for unrolling a value of recursive type and 0 for other reductions. We then define a “quantitative” reflexive, transitive closure of the small-step relation \(t \Rightarrow^i t'\) that adds the weights of its constituent small steps. When the number of steps is irrelevant, we just use \(\Rightarrow\) and \(\Rightarrow^i\). We can then establish some simple facts about this operational semantics.

**Lemma 3.1 (Subject Reduction).** If \(\cdot \vdash t : A\) and \(t \Rightarrow t'\), then \(\cdot \vdash t' : A\).

**Lemma 3.2 (Progress).** If \(\cdot \vdash t : A\) and \(t\) is not a value or \(\Upsilon\), then there exists \(t'\) with \(t \Rightarrow t'\).

**Proof.** By induction on the typing derivation for \(t\). \(\square\)
3.2 Translating Gradual Typing

Next we translate the cast calculus into our typed language, and prove that the cast calculus semantics is in a simulation relation with the typed language. Since the two languages share so much of their syntax, most of the translation is a simple “color change”, only the parts that are truly components of gradual typing need much translation.

Lemma 3.3 (Determinism). If $t \mapsto s$ and $t \mapsto s'$, then $s = s'$.
Graduality From Embedding-Projection Pairs

\[
\begin{align*}
\langle ? \rangle & \overset{\text{def}}{=} \mu \alpha. 1 + (\alpha \times \alpha) + (\alpha + \alpha) + (\alpha \rightarrow \alpha) \\
\langle 1 \rangle & \overset{\text{def}}{=} 1 \\
\langle A \times B \rangle & \overset{\text{def}}{=} \langle A \rangle \times \langle B \rangle \\
\langle A + B \rangle & \overset{\text{def}}{=} \langle A \rangle + \langle B \rangle \\
\langle A \rightarrow B \rangle & \overset{\text{def}}{=} \langle A \rangle \rightarrow \langle B \rangle
\end{align*}
\]

Fig. 8. Type Translation

\[
\begin{align*}
\text{where if } x_1 : A_1, \ldots, x_n : A_n + t : A & \text{ then } x_1 : \langle A_1 \rangle, \ldots, x_n : \langle A_n \rangle + \langle t \rangle : \langle A \rangle \\
\langle x \rangle & \overset{\text{def}}{=} x \\
\langle (B \equiv A) t \rangle & \overset{\text{def}}{=} E_{(B \equiv A)} \langle \langle t \rangle \rangle \\
\langle [] \rangle & \overset{\text{def}}{=} \langle \rangle \\
\langle [t_1, t_2] \rangle & \overset{\text{def}}{=} \langle \langle t_1 \rangle, \langle t_2 \rangle \rangle \\
\langle \text{let } \langle x, y \rangle = t \text{ in } s \rangle & \overset{\text{def}}{=} \langle \text{let } \langle x, y \rangle = \langle \langle t \rangle \rangle \text{ in } \langle s \rangle \rangle \\
\langle \text{inj } t \rangle & \overset{\text{def}}{=} \langle \text{inj } \langle t \rangle \rangle \\
\langle \text{inj}' t \rangle & \overset{\text{def}}{=} \langle \text{inj}' \langle t \rangle \rangle \\
\langle \text{case } t \text{ of } \text{inj } x. s | \text{inj }' x'. s' \rangle & \overset{\text{def}}{=} \langle \text{case } \langle t \rangle \text{ of } \langle \text{inj } x. \langle s \rangle | \text{inj }' x'. \langle s' \rangle \rangle \rangle \\
\langle \lambda(x : A). t \rangle & \overset{\text{def}}{=} \langle \lambda(x : \langle A \rangle). \langle t \rangle \rangle \\
\langle [t, s] \rangle & \overset{\text{def}}{=} \langle t \rangle \langle s \rangle
\end{align*}
\]

Fig. 9. Term Translation

\[
E_{(B \equiv A)} \text{ where } x : \langle A \rangle \vdash E_{(B \equiv A)} \langle x \rangle : \langle B \rangle
\]

\[
\begin{align*}
E_{(\langle ? \rangle \equiv ?)} & \overset{\text{def}}{=} \langle [\cdot] \rangle \\
E_{(\langle A_2 \times B_2 \equiv A_1 \times B_1 \rangle)} & \overset{\text{def}}{=} E_{(\langle A_2 \equiv A_1 \rangle \times E_{(\langle B_2 \equiv B_1 \rangle)}} \\
E_{(\langle A_2 + B_2 \equiv A_1 + B_1 \rangle)} & \overset{\text{def}}{=} E_{(\langle A_2 \equiv A_1 \rangle) + E_{(\langle B_2 \equiv B_1 \rangle)}} \\
E_{(\langle A_2 \rightarrow B_2 \equiv A_1 \rightarrow B_1 \rangle)} & \overset{\text{def}}{=} E_{(\langle A_2 \equiv A_1 \rangle) \rightarrow E_{(\langle B_2 \equiv B_1 \rangle)}} \\
E_{(\langle ? \rangle \equiv G)} & \overset{\text{def}}{=} \text{roll} \langle \langle [\cdot] \rangle \text{ inj}_G \langle [\cdot] \rangle \rangle \\
E_{(\langle G \equiv ? \rangle)} & \overset{\text{def}}{=} \text{case } \langle \text{unroll } \langle [\cdot] \rangle \rangle \text{ of } \langle \text{inj}_G x. x | \text{else } U \rangle \\
E_{(\langle ? \equiv A \rangle)} & \overset{\text{def}}{=} E_{(\langle ? \equiv \langle A \rangle \rangle)} \langle E_{(\langle \langle A \rangle \equiv \langle A \rangle \rangle)} \langle [\cdot] \rangle \rangle \quad \text{if } A \neq ?, \langle [A] \rangle \\
E_{(\langle A \equiv ? \rangle)} & \overset{\text{def}}{=} E_{(\langle A \equiv \langle A \rangle \rangle)} \langle E_{(\langle \langle A \rangle \equiv ? \rangle \rangle)} \langle [\cdot] \rangle \rangle \quad \text{if } A \neq ?, \langle [A] \rangle \\
E_{(\langle B \equiv A \rangle)} & \overset{\text{def}}{=} \langle \text{let } x = [\cdot] \text{ in } U \rangle \quad \text{if } A, B \neq ? \text{ and } \langle [A] \rangle \neq [B]
\end{align*}
\]

Fig. 10. Direct Cast Translation

Our translation is type preserving, so we first define a type translation in Figure 8. The dynamic type is interpreted as a recursive sum of the translations of the tag types of the gradual language. The unit, pair, sum and function types are all interpreted as the corresponding connectives in the typed language.

Next, we define the translation of terms in Figure 9, which is type preserving in that if \( x_1 : A_1, \ldots, x_n : A_n + t : A \) then \( x_1 : \langle A_1 \rangle, \ldots, x_n : \langle A_n \rangle + \langle t \rangle : \langle A \rangle \). Again, most of the translation is just a change of hue. The most important rule of the term translation is that of casts. A cast \( \langle B \equiv A \rangle \) is translated to an evaluation context \( E_{(B \equiv A)} \) of the appropriate type, which are defined in Figure 10. Each case of the definition corresponds to one or more rules of the operational semantics.
The product, sum, and function rules use the definitions of functorial actions of their types from Figure 11. We separate them because we will use the functoriality property in several definitions, theorems, and proofs later.

### 3.3 Operational Properties

Next, we consider the relationship between the operational semantics of the two languages and how to lift properties of the typed language to the gradual language. We want to view the translation of the cast calculus into the typed language as definitional, and in that regard view the operational semantics of the source language as being based on the typed language. We capture this relationship in the following forward simulation theorem, which says that any reduction in the cast calculus corresponds to (and is justified by) multiple steps in the target:

**Lemma 3.4 (Translation Preserves Values, Evaluation Contexts).**

1. For any value \( v \), \( \llbracket v \rrbracket \) is a value.
2. For any evaluation context \( E \), \( \llbracket E \rrbracket \) is an evaluation context.

**Lemma 3.5 (Simulation of Operational Semantics).**

If \( t \mapsto t' \) then there exists \( s \) with \( \llbracket t \rrbracket \mapsto s \) and \( s \Rightarrow \llbracket t' \rrbracket \).

**Proof.** By cases of \( t \mapsto t' \). The non-cast cases are clear by lemma 3.4.

1. **DynDyn**

   \[
   E_{(\varphi = ?)}\llbracket \varphi \rrbracket = \text{let } x = \llbracket \varphi \rrbracket \text{ in } x
   \]

2. **TagUp:** Trivial because \( \llbracket \varphi \leftarrow A \| A \rrbracket = \llbracket \varphi \leftarrow [A] \| [A] \leftarrow A \| A \rrbracket \).
3. **TagDn:** Trivial because \( \llbracket A \leftarrow ? \| A \rrbracket = \llbracket A \leftarrow [A] \| [A] \leftarrow ? \rrbracket \).
4. **(TagMatch) Valid because**

   \[
   \text{case unroll roll \llbracket \varphi \rrbracket of } G. x \mapsto \text{case } \text{inj}_G \llbracket \varphi \rrbracket \text{ of } G. x
   \]

   \[
   | \text{else. } x | \text{ else. } x
   \]

   \[
   \mapsto \llbracket \varphi \rrbracket
   \]

5. **(TagMismatch) Valid because**

   \[
   \text{case unroll roll \llbracket \varphi \rrbracket of } G. x \mapsto \text{case } \text{inj}_{G'} \llbracket \varphi \rrbracket \text{ of } G. x
   \]

   \[
   | \text{else. } x | \text{ else. } x
   \]

   \[
   \mapsto \llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket
   \]

6. **(TagMismatch’) Valid because**

   \[
   \text{let } x = \llbracket \varphi \rrbracket \text{ in } \llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket
   \]
(7) Pair Valid by
\[ E_{(A_2 \times B_2 \leftarrow A_1 \times B_1)}[[[v], [v']]] = \text{let } x = y \text{ in } [[v], [v']] \langle E_{(A_2 \leftarrow A_1)}[x], E_{(B_2 \leftarrow B_1)}[y] \rangle \]
\[ \quad \rightarrow \langle E_{(A_2 \leftarrow A_1)}[[v]], E_{(B_2 \leftarrow B_1)}[[v']] \rangle \]
\[ \quad = [[(A_2 \leftarrow A_1)v, (B_2 \leftarrow B_1)v']] \]

(8) Sum
\[ E_{(A_2 \times B_2 \leftarrow A_1 \times B_1)}[\text{inj } [v]] = \text{case } \text{inj } [v] \text{ of} \]
\[ \quad \text{inj} x \cdot E_{(A_2 \leftarrow A_1)}[x] \]
\[ \quad | \text{inj} x'. E_{(B_2 \leftarrow B_1)}[x'] \]
\[ \quad \rightarrow E_{(A_2 \leftarrow A_1)}[[v]] \]
\[ \quad = [[(A_2 \leftarrow A_1)v]] \]

(9) Sum’
\[ E_{(A_2 \times B_2 \leftarrow A_1 \times B_1)}[\text{inj'} [v]] = \text{case } \text{inj'} [v] \text{ of} \]
\[ \quad \text{inj} x \cdot E_{(A_2 \leftarrow A_1)}[x] \]
\[ \quad | \text{inj} x'. E_{(B_2 \leftarrow B_1)}[x'] \]
\[ \quad \rightarrow E_{(B_2 \leftarrow B_1)}[[v]] \]
\[ \quad = [[(B_2 \leftarrow B_1)v]] \]

(10) (Fun) Valid because
\[ E_{(A_2 \rightarrow B_2 \leftarrow A_1 \rightarrow B_1)}[[v]] \rightarrow \text{let } x_f = [v] \text{ in} \]
\[ \quad \lambda(x_a : A_2) \cdot E_{(B_2 \leftarrow B_1)}[[v] E_{(A_1 \leftarrow A_2)}[x_a]]] \]
\[ \quad = \lambda(x_a : A_2) \cdot (B_2 \leftarrow B_1)(v ((A_1 \leftarrow A_2)x_a))] \]
\[ \quad \Box \]

To lift theorems for the gradual language from the typed language, we need to establish an adequacy theorem, which says that the operational behavior of a translated term determines the behavior of the original source term. To do this we use the following backward simulation theorem.

**Lemma 3.6 (Translation reflects Results).**

1. If \( [t] \) is a value, \( t \rightarrow^* v \) for some \( v \) with \( [t] = [v] \).
2. If \( [t] = \mathcal{U} \), then \( t \rightarrow^* \mathcal{U} \).

**Proof.** By induction on \( t \). For the non-casts, follows by inductive hypothesis. For the casts, only two cases can be values:

1. \( (? \leftarrow ?) : t \) if \( [(? \leftarrow ?) : t] = [t] \) is a value then by inductive hypothesis, \( t \) is a value, so \( (? \leftarrow ?)v \rightarrow v \).
2. \( (? \leftarrow G) : t \) if \( \text{roll} [?] \text{ inj}_G [t] \) is a value, then \( [t] \) is a value so by inductive hypothesis \( t \rightarrow^* v \) so \( (? \leftarrow G)t \rightarrow^* (? \leftarrow G)v \).

For the error case, there is only one case where it is possible for \( [t] = \mathcal{U} \) without \( t = \mathcal{U} \):

1. For \( (? \leftarrow ?) : s \), if \( [(? \leftarrow ?) : s] = [s] \) is an error then clearly \( [s] = \mathcal{U} \) so by inductive hypothesis \( s \rightarrow^* \mathcal{U} \) and because casts are strict,
\[ (? \leftarrow ?)s \rightarrow^* s \]
\[ \Box \]

**Lemma 3.7 (Backward Simulation).** If \( [t] \rightarrow s \) then there exists \( s' \) with \( t \rightarrow s' \) and \( s \rightarrow [s'] \).
Proof. By induction on \( t \). We show two illustrative cases, the rest follow by the same reasoning.

1. if and only if \( \exists A \) such that \( \langle \Gamma \vdash A \rangle \). Then we use the same argument as the case for \( \langle [A] \rangle \).

2. \( \langle [t] \rangle \) diverges if and only if \( \forall A \) such that \( \langle \Gamma \vdash A \rangle \) does not run to a value or error.

3. \( \langle [t] \rangle \) diverges if and only if \( \forall A \) such that \( \langle \Gamma \vdash A \rangle \) diverges.

Theorem 3.8 (Adequacy).

1. if and only if \( \forall A \) such that \( \langle \Gamma \vdash A \rangle \) does not run to a value or error.

2. \( \langle [t] \rangle \) diverges if and only if \( \forall A \) such that \( \langle \Gamma \vdash A \rangle \) diverges.

3. \( \langle [t] \rangle \) diverges if and only if \( \forall A \) such that \( \langle \Gamma \vdash A \rangle \) diverges.

While this has reduced the number of primitives of the language, reasoning about the behavior of the translated casts isn’t any simpler than the original operational semantics since they have the same behavior. For simpler reasoning about cast behavior, we will move further away from a direct simulation of the source operational semantics, to a second semantics based on ep pairs that is observationally equivalent but also conceptually simpler and helps prove the gradual guarantee. However, in order to prove that the second semantics is equivalent, we first need to develop a usable theory of observational equivalence and approximation.

4 A LOGICAL RELATION FOR ERROR APPROXIMATION

Next, we define observational equivalence and error approximation of programs in the gradual and typed languages, the two properties with which we formulate embedding-projection pairs. To facilitate proofs of error approximation, we develop a novel step-indexed logical relation. Since our notion of approximation is non-standard, the use of step-indexing in our logical relation is inconvenient to use directly. So, on top of the “implementation” of the logical relation as a step-indexed relation, we prove many high-level lemmas so that all proofs in the next sections are performed relative to these lemmas, and none manipulate step indices directly.

4.1 Observational Equivalence and Approximation

A suitable notion of equivalence for programs is observational equivalence. We say \( t \) is observationally equivalent to \( t' \) if replacing one with the other in the context of a larger program produces the same result (termination, error, or divergence). We formalize this saying a program context \( C \) is a term with a single hole \([\_]\). A context is typed \( \Gamma \vdash C[\Gamma \vdash \cdot : A] : A' \) when for any term \( \Gamma \vdash t : A \), replacing the hole with \( t \) results in a well-typed \( \Gamma' \vdash C[t] : A' \).
While this notion of observational equivalence is entirely standard, the notion of approximation we use—which we call error approximation—is not the standard notion of observational approximation. Usually, we would say \( t \) observationally approximates \( t' \) if, when placing them into the same context \( C \), either \( C[t] \) diverges or they both terminate or both error. We call this form of approximation divergence approximation. However, for gradual typing we are not particularly interested in when one program diverges more than another, but rather when it produces more type errors. We might be tempted to conflate the two, but their behavior is quite distinct! We can never truly know if a black-box program will continue indefinitely: it would frustrate any programmer to use a language that runs forever when accidentally using a function as a number. The reader should keep this difference in mind when seeing how our logical relation differs from the standard treatment. In the rest of this paper, when discussing the two together we will clearly distinguish between divergence and error approximation, but when there is no qualifier, approximation is meant as error approximation.

**Definition 4.1 (Gradual Observational Equivalence, Error Approximation).** For any well typed terms \( \Gamma \vdash t, t' : A \),

1. Define \( \Gamma \vdash t \approx \text{obs} t' : A \), pronounced "\( t \) is observationally equivalent to \( t' \)" to hold when for any \( \cdot \vdash C[\cdot : A] : B \), either \( C[t] \) and \( C[t'] \) both reduce to a value, both reduce to an error, or both diverge.
2. Define \( \Gamma \vdash t \sqsubseteq \text{obs} t' : A \), pronounced "\( t \) observationally (error) approximates \( t' \)" to hold when for any \( \cdot \vdash C[\cdot : A] : B \), either \( C[t] \) reduces to \( \bot \) or both \( C[t] \) and \( C[t'] \) reduce to a value or both diverge.

As with divergence approximation, we can prove two programs are observationally equivalent by showing each error approximates the other.

**Lemma 4.2 (Equivalence is Approximation Both Ways).** \( \Gamma \vdash t_1 \approx \text{obs} t_2 : A \) if and only if both \( \Gamma \vdash t_1 \sqsubseteq \text{obs} t_2 : A \) and \( \Gamma \vdash t_2 \sqsubseteq \text{obs} t_1 : A \).

We define typed observational equivalence \( \Gamma \vdash t \approx \text{obs} t' : A \) and observational error approximation \( \Gamma \vdash t \sqsubseteq \text{obs} t' : A \) with the exact same definition as for the gradual language above, but in **red** instead of **blue**. We rarely work with the gradual language directly, instead we prove approximation results for their translations.

**Definition 4.3 (Typed Observational equivalence, Error Approximation).** For any well typed terms \( \Gamma \vdash t, t' : A \),

1. Define \( \Gamma \vdash t \approx \text{obs} t' : A \), pronounced "\( t \) is observationally equivalent to \( t' \)" to hold when for any \( C[\cdot : A] : B \), either \( C[t] \) and \( C[t'] \) both reduce to a value, both reduce to an error, or both diverge.
2. Define \( \Gamma \vdash t \sqsubseteq \text{obs} t' : A \), pronounced "\( t \) observationally (error) approximates to \( t' \)" to hold when for any \( C[\cdot : A] : B \), either \( C[t] \) reduces to \( \bot \) or both \( C[t] \) and \( C[t'] \) reduce to a value or both diverge.

This is justified by the following lemma, a consequence of our adequacy result (theorem 3.8).

**Lemma 4.4 (Typed Observational Approximation implies Gradual Observational Approximation).** If \( [\Gamma] \models [t_1] \sqsubseteq \text{obs} [t_2] : [A] \) then \( \Gamma \vdash t_1 \sqsubseteq \text{obs} t_2 : A \)

**Proof.** For any \( C \), by compositionality of the translation, \( [C[t_1]] = [C][[t_1]] \) and \( [C[t_2]] = [C][[t_2]] \). Then we analyze \( [t_1] \sqsubseteq \text{obs} [t_2] \)

1. If \( [C][[t_1]] \mapsto^U \) then theorem 3.8 states that \( C[t_1] \mapsto^U \).
4.2 Logical Relation

Observational equivalence and approximation are extremely difficult to prove directly, so we use the usual method of proving observational results by using a logical relation that we prove sound with respect to observational approximation. Due to the non-well-founded nature of recursive types (and the dynamic type specifically), we develop a step-indexed logical relation following Ahmed [2006]. We define our logical relation for error approximation in Figure 12. Because our notion of error approximation is not the standard notion of approximation, the definition is a bit unusual, but this is necessary for technical reasons.

It is key to compositional reasoning about embedding-projection pairs that approximation be transitive and care must be taken to show transitivity for a step-indexed relation. However, for standard definitions of logical relations for observational equivalence, it is difficult to prove transitivity directly. Therefore, it is often established through indirect reasoning—e.g., by setting up a biorthogonal (TT-closed) logical relation so one can easily show it is complete with respect to observational equivalence, which in turn implies that it must be transitive since observational equivalence is easily proven transitive. The reason establishing transitivity is tricky is that a step-indexed relation is not transitive at a fixed index, i.e., if \( t_1 \sim^j e_1 \) and \( e_2 \sim^j e_3 \) it is not necessarily the case that \( e_1 \sim^j e_3 \). For instance, \( e_1 \sim^j e_2 \) might be related because \( e_1 \) terminates in less than \( i \) steps and has the same behavior as \( e_2 \) which takes more than \( i \) steps to terminate, whereas \( e_2 \sim^j e_3 \) are related because they both take \( i \) steps of reduction so cannot be distinguished in \( i \) steps but have different behavior when run for more steps. One direct method for proving transitivity, originally presented in Ahmed [2006], is to observe that two terms are observationally equivalent when \( i \) diverge approximates the other, and then use a step-indexed relation for divergence approximation. Because a conjunction of transitive relations is transitive, this proves transitivity of equivalence. A step-indexed relation for divergence approximation can be shown to have a kind of “half-indexed” transitivity, i.e., if \( t_1 <^i A t_2 \) and for every natural \( j \), we know \( e_2 <^j e_3 \) then \( e_1 <^i e_3 \).

We have a similar issue with error approximation: the naïve logical relation for error approximation is not clearly transitive. Inspired by the case of observational equivalence, we similarly “split” our logical relation into two relations that can be proven transitive by an argument similar to divergence approximation. However, unlike for observational equivalence, our two relations are not the same. Instead, one \( \sqsubseteq< \) is error approximation up to divergence on the left and the other \( \sqsubseteq> \) is error approximation up to divergence on the right.

For a given natural number \( i \in \mathbb{N} \) and type \( A \), and closed terms \( t_1, t_2 \) of type \( A \), \( t_1 \sqsubseteq<_{i, A} t_2 \) intuitively means that, if we only inspect \( t_1 \)’s behavior up to \( i \) uses of unroll \( \cdot \), then it appears that \( t_1 \) error approximates \( t_2 \). Less constructively, it means that we cannot show that \( t_1 \) does not error approximate \( t_2 \) when limited to \( i \) uses of unroll \( \cdot \). However, even if we knew \( t_1 \sqsubseteq<_{i, A} t_2 \) for every \( i \in \mathbb{N} \), it still might be the case that \( t_1 \) diverges, since no finite number of unrolling can ever exhaust \( t_1 \)’s behavior. So we also require that we know \( t_1 \sqsubseteq>_i_{i, A} t_2 \), which means that up to \( i \) uses of unroll on \( t_2 \), it appears that \( t_1 \) error approximates \( t_2 \).

The above intuition should help to understand the definition of error approximation for terms (i.e., the relations \( \sqsubseteq< \) and \( \sqsubseteq> \)). The relation \( t_1 \sqsubseteq<_{i, A} t_2 \) is defined by inspection of \( t_1 \)’s behavior: it holds if \( t_1 \) is still running after \( i + 1 \) unrolls; or if it steps to an error in fewer than \( i \) unrolls; or if
it results in a value in fewer than \( i \) unrolls and also \( t_2 \) runs to a value and those values are related for the remaining steps. The definition of \( t_1 \sqsupseteq t_2 : A \) is defined by inspection of \( t_2 \)'s behavior: it holds if \( t_2 \) is still running after \( i+1 \) unrolls; or if \( t_2 \) steps to an error in fewer than \( i \) steps then \( t_1 \) errors as well; or if \( t_2 \) steps to a value, either \( t_1 \) errors or steps to a value related for the remaining steps.

While the relations and \( \sqsubset \) on terms are different, fortunately, the relations on values are essentially the same, so we abstract over the cases by having the symbol \( \sqsim \) to range over either \( \sqsubset \) or \( \sqsupseteq \). For values of recursive type, if the step-index is 0, we consider them related, because otherwise we would need to perform an unroll to inspect them further. Otherwise, we decrement the index and check if they are related. Decrementing the index here is exactly what makes the definition of the relation well-founded. For the standard types, the value relation definition is indeed standard: pairs are related when the two sides are related, sums must be the same case and functions must be related when applied to any related values in the future (i.e., when we may have exhausted some of the available steps).

Finally, we extend these relations to \( \text{open} \) terms in the standard way: we define substitutions to be related pointwise (similar to products) and then say that \( \Gamma \vdash t_1 \sqsim t_2 : A \) holds if for every pair of substitutions \( y_1, y_2 \) related for \( i \) steps, the terms after substitution, written \( t_1[y_1] \) and \( t_2[y_2] \),
are related for \( i \) steps. Then our resulting relation \( \Gamma \vdash t_1 \subseteq t_2 \) is defined to hold when \( t_1 \) error approximates \( t_2 \) up to divergence of \( t_1 (\subseteq \prec) \), and up to divergence of \( t_2 (\subseteq \succ) \).

We need the following standard lemmas.

**Lemma 4.5 (Downward Closure).** If \( j \leq i \) then

1. If \( t_1 \vdash t_2 \) then \( t_1 \vdash t_2 \).
2. If \( \mathcal{U} \vdash \mathcal{V} \), then \( \mathcal{V} \vdash \mathcal{U} \).

**Proof.** By lexicographic induction on the pair \( (i, A) \). \( \square \)

**Lemma 4.6 (Anti-Reduction).** This theorem is different for the two relations as we allow arbitrary steps on the “divergence greater-than” side.

1. If \( t_1 \vdash t_2 \) and \( t_1 \vdash t_2 \) then \( t_1 \vdash t_2 \).
2. If \( \mathcal{U} \vdash \mathcal{V} \), then \( \mathcal{V} \vdash \mathcal{U} \).

A simple corollary that applies in common cases to both relations is that if \( t_1 \vdash t_2 \) and \( t_1 \vdash t_2 \), then \( t_1 \vdash t_2 \).

**Proof.** By direct inspection and downward closure (Lemma 4.5). \( \square \)

**Lemma 4.7 (Monadic Bind).** For any \( i \in \mathbb{N} \), if for any \( j \leq i \) and \( \mathcal{U} \vdash \mathcal{V} \), we can show \( E_1[\mathcal{U}] \vdash \mathcal{V} \) holds, then for any \( t_1 \vdash t_2 \), it is the case that \( E_1[\mathcal{U}] \vdash E_2[\mathcal{V}] \).

**Proof.** We consider the proof for \( \vdash \mathcal{U} \), the other is similar/easier. By case analysis of \( t_1 \vdash t_2 \).

1. If \( t_2 \) takes \( i + 1 \) steps, do \( E_2[t_2] \).
2. If \( t_2 \vdash \mathcal{U} \), then \( \mathcal{U} \vdash \mathcal{V} \), then \( \mathcal{V} \vdash E_2[\mathcal{U}] \).
3. Otherwise \( E_2[t_2] \vdash \mathcal{V} \) and \( E_2[t_1] \vdash \mathcal{V} \).

Assume there exist \( j \leq i \), \( \mathcal{U} \vdash \mathcal{V} \), and \( t_2 \vdash \mathcal{V} \), then by assumption, \( E_1[\mathcal{U}] \vdash E_2[\mathcal{V}] \). Then by antireduction (Lemma 4.6), \( E_1[t_1] \vdash E_2[t_2] \).
We then prove that our logical relation is sound for observational error approximation by showing that it is a congruence relation and showing that if we can prove error approximation up to divergence on the left and on the right, then we have true error approximation.

**Lemma 4.8 (Congruence for Logical Relation).** All of the congruence rules in Figure 13 are valid.

**Proof.** Each case is done by proving the implication for $\subseteq^i$ and $\subseteq^>$. Most cases follow by monadic bind (Lemma 4.7), downward closure (Lemma 4.5) and direct use of the inductive hypotheses. We show some illustrative cases.

1. Given $y_1 \subseteq^i \Gamma y_2$, we need to show $\lambda(x:A). t_1[y_1] \subseteq^i_{t,A\rightarrow B} \lambda(x:A). t_2[y_2]$. Since they are values, we show they are related values. Given any $v_1 \subseteq^j v_2$ with $j \leq i$, each side $\beta$ reduces in 0 unroll steps so it is sufficient to show $t_1[y_1, v_1/x] \subseteq^j_{t,B} t_2[y_2, v_2/x]$.

Which follows by inductive hypothesis and downward-closure and the substitution relation.

**Theorem 4.9 (Logical Relation implies Observational Error Approximation).** If $\Gamma \vdash t_1 \subseteq t_2 : A$, then $\Gamma \vdash t_1 \subseteq^{\text{obs}} t_2 : A$.

**Proof.** If $\Gamma \vdash t_1 \subseteq t_2 : A$, then for any closing context, by lemma 4.8, $\cdot \vdash C[t_1] \subseteq C[t_2] : B$ holds.

Then we do a case analysis of $C[t_1]$’s behavior.

1. If $C[t_1]$ diverges, then for any $i$, since $C[t_1] \subseteq^{>}_{t,B} C[t_2]$, only the $C[t_2]$ also diverges.
2. If $C[t_1] \Rightarrow^i \top$ we’re done.
3. If $C[t_1] \Rightarrow^i \bot$, then because $C[t_1] \subseteq^{<}_{t,B} C[t_2]$, we know $C[t_2] \Rightarrow \bot$.

**4.3 Approximation and Equivalence Lemmas**

The step-indexed logical relation is on the face of it quite complex, especially due to the splitting of error approximation into two step-indexed relations. However, we should view the step-indexed relation as an “implementation” of the high-level concept of error approximation, and we work as much as possible with the error approximation relation $\Gamma \vdash t_1 \subseteq t_2 : A$. In order to do this we now prove some high-level lemmas, which are proven using the step-indexed relations, but allow us to develop conceptual proofs of the key theorems of the paper.

First, there is reflexivity, also known as the fundamental lemma, which is proved using the same congruence cases as the soundness theorem (theorem 4.9.) Note that by the definition of our logical relation, this is really a kind of monotonicity theorem for every term in the language, the first component of our graduality proof.

**Corollary 4.10 (Reflexivity).** If $\Gamma \vdash t : A$ then $\Gamma \vdash t \subseteq t : A$.

**Proof.** By induction on the typing derivation of $t$, in each case using the corresponding congruence rule from Lemma 4.8.
Next, crucial to reasoning about ep pairs is the use of transitivity, a notoriously tedious property to prove for step-indexed logical relations. This is where our splitting of error-approximation into two pieces proves essential, adapting the approach for divergence-approximation relations introduced in Ahmed [2006]. The proof works as follows: due to the function and open-term cases, we cannot simply prove transitivity in the limit directly. Instead we get a kind of “asymmetric” transitivity: if \( t_1 \sqsubseteq^{l\cdot} t_2 \) and for any \( j \in \mathbb{N}, t_2 \sqsubseteq^{o\cdot} t_3 \), then we know \( t_1 \sqsubseteq^{l\cdot} t_3 \). We abbreviate the \( \forall j \) part as \( t_2 \sqsubseteq^{o\cdot} t_3 \) in what follows. The key to the proof is in the function and open terms cases, which rely on reflexivity, corollary 4.10, as in Ahmed [2006]. Reflexivity says that when we have \( v_1 \sqsubseteq^{l\cdot}_{v\cdot A} v_2 \) then we also have \( v_2 \sqsubseteq^{o\cdot}_{v\cdot A} v_2 \), which allows us to use the inductive hypothesis.

**Lemma 4.11 (Transitivity for Closed Terms/Values).** The following are true for any \( A \).

1. If \( t_1 \sqsubseteq^{l\cdot}_{t\cdot A} t_2 \) and \( t_2 \sqsubseteq^{o\cdot}_{t\cdot A} t_3 \) then \( t_1 \sqsubseteq^{l\cdot}_{t\cdot A} t_3 \).
2. If \( v_1 \sqsubseteq^{l\cdot}_{v\cdot A} v_2 \) and \( v_2 \sqsubseteq^{o\cdot}_{v\cdot A} v_3 \) then \( v_1 \sqsubseteq^{l\cdot}_{v\cdot A} v_3 \).

Similarly,

1. If \( t_1 \sqsupseteq^{l\cdot}_{t\cdot A} t_2 \) and \( t_2 \sqsupseteq^{o\cdot}_{t\cdot A} t_3 \) then \( t_1 \sqsupseteq^{l\cdot}_{t\cdot A} t_3 \).
2. If \( v_1 \sqsupseteq^{l\cdot}_{v\cdot A} v_2 \) and \( v_2 \sqsupseteq^{o\cdot}_{v\cdot A} v_3 \) then \( v_1 \sqsupseteq^{l\cdot}_{v\cdot A} v_3 \).

**Proof.** We prove the \( \sqsubseteq^{l\cdot}_{t\cdot A} \) and \( \sqsupseteq^{l\cdot}_{v\cdot A} \) mutually by induction on \((i, A)\). The other logical relation is similar. Most value cases are simple uses of the inductive hypotheses.

1. (Terms) By case analysis of \( t_1 \sqsubseteq^{l\cdot}_{t\cdot A} t_2 \).
   a. If \( t_1 \Rightarrow^{i\cdot} t_1' \) or \( t_1 \Rightarrow^{l\cdot} \mathcal{U} \), we have the result.
   b. Let \( j \leq i, k \in \mathbb{N} \) and \((v_1 \sqsubseteq^{l\cdot}_{v\cdot A} v_2)\) with \( t_1 \Rightarrow^{j\cdot} v_1 \) and \( t_2 \Rightarrow^{k\cdot} v_2 \). By inductive hypothesis for values, it is sufficient to show that \( t_3 \Rightarrow^{k\cdot} v_3 \) and \( v_2 \sqsubseteq^{o\cdot}_{v\cdot A} v_3 \).

Since \( t_2 \sqsubseteq^{o\cdot}_{t\cdot A} t_3 \), in particular we know \( t_2 \sqsubseteq^{k+i\cdot}_{t\cdot A} t_3 \) for every \( l \in \mathbb{N} \), so since \( t_2 \Rightarrow^{k\cdot} v_2 \), we know that \( t_3 \Rightarrow^{l\cdot} v_3 \) and \( v_2 \sqsubseteq^{l\cdot}_{v\cdot A} v_3 \), for every \( l \), i.e., \( v_2 \sqsubseteq^{o\cdot}_{v\cdot A} v_3 \).

2. (Function values) Suppose \( v_1 \sqsubseteq^{l\cdot}_{v\cdot A \rightarrow B} v_2 \) and \( v_2 \sqsubseteq^{o\cdot}_{v\cdot A \rightarrow B} v_3 \). Then, let \( j \leq i \) and \( v_1' \sqsubseteq^{l\cdot}_{v\cdot A \rightarrow B} v_2' \). We need to show \( v_1 \sqsubseteq^{l\cdot}_{v\cdot A \rightarrow B} v_2 \sqsubseteq^{o\cdot}_{v\cdot A \rightarrow B} v_3 \). By inductive hypothesis, it is sufficient to show \( v_1 \sqsubseteq^{l\cdot}_{v\cdot A \rightarrow B} v_2 \sqsubseteq^{o\cdot}_{v\cdot A \rightarrow B} v_3 \). The former is clear: The latter follows by the congruence rule for application Lemma 4.8 and reflexivity corollary 4.10 on \( v_2' \). Since \( \cdot \vdash v_2' : A \), we have \( v_2' \sqsubseteq^{o\cdot}_{v\cdot A} v_2' \).

\( \square \)

**Lemma 4.12 (Transitivity).** If \( \Gamma \vdash t_1 \sqsubseteq t_2 : A \) and \( \Gamma \vdash t_2 \sqsubseteq t_3 : A \) then \( \Gamma \vdash t_1 \sqsubseteq t_3 : A \).

**Proof.** The argument is essentially the same as the function value case, invoking the fundamental lemma corollary 4.10 for each component of the substitutions and transitivity for the closed relation Lemma 4.11.

\( \square \)

Next, we want to extract approximation and equivalence principles for open programs from syntactic operational properties of closed programs. First, obviously any operational reduction is a contextual equivalence, and the next lemma extends that to open programs. Note that we use \( \sqsubseteq \) to mean approximation in both directions, i.e., equivalence.

**Lemma 4.13 (Open \( \beta \) Reductions).** Given \( \Gamma \vdash t : A \), \( \Gamma \vdash t' : A \), if for every \( y : \Gamma, t[y] \Rightarrow t'[y] \), then \( \Gamma \vdash t \sqsubseteq t' : A \).

**Proof.** By reflexivity corollary 4.10 on \( t' \), \( y \) and anti-reduction lemma 4.6.

\( \square \)
We call this open \(\beta\) reduction because we will use it to justify equivalences that look like an operational reduction, but have open values (i.e. including variables) rather than closed as in the operational semantics. For instance,

\[
\text{let } x = y \text{ in } t \quad \equiv \quad t[y/x]
\]

and

\[
\text{let } \langle x, y \rangle = \langle x', y' \rangle \text{ in } t \quad \equiv \quad t[x'/x, y'/y]
\]

Additionally, it is convenient to use \(\eta\) expansions for our types as well. Note that since we are using a call-by-value language, the \(\eta\) expansion for functions is restricted to values.

**Lemma 4.14 (\(\eta\) Expansion).**

1. For any \(\Gamma \vdash v : A \rightarrow B\), \(v \equiv \lambda(x : A). v x\)
2. For any \(\Gamma, x : A + A', \Gamma' \vdash t : B\),
   \[
   t \equiv \text{case } x \text{ of } \text{inj } y . t[\text{inj } y'/x] \mid \text{inj } y'. t[\text{inj } y'/x]
   \]
3. For any \(\Gamma, x : A \times A', \Gamma' \vdash t : B\),
   \[
   t \equiv \text{let } (y, y') = x \text{ in } t[(y, y')/x]
   \]

**Proof.** All are consequences of lemma 4.13.

Next, with term constructors that involve continuations, we often need to rearrange the programs such as the “case-of-case” transformation. These are called commuting conversions and are presented in Figure 14.

**Lemma 4.15 (Commuting Conversions).** All of the commuting conversions in Figure 14 are equivalences.

**Proof.** By monadic bind, anti-reduction and the reflexivity (lemmas 4.6 and 4.7 and corollary 4.10).

Next, the following theorem is the main reason we so heavily use \(\textit{evaluation contexts}\). It is a kind of open version of the monadic bind lemma lemma 4.7.

**Lemma 4.16 (Evaluation Contexts are Linear).** If \(\Gamma \vdash t : A\) and \(\Gamma, x : A \vdash E[x] : B\), then

\[
\text{let } x = t \text{ in } E[x] \quad \equiv \quad E[t]
\]

**Proof.** By a commuting conversion and an open \(\beta\) reduction, connected by transitivity lemmas 4.12, 4.13 and 4.15

\[
\text{let } x = t \text{ in } E[x] \quad \equiv \quad E[\text{let } x = t \text{ in } x]
\]

\[
\equiv \quad E[t]
\]

As a simple example, consider the following standard equivalence of let and \(\lambda\), which we will need later and prove using the above lemmas:

**Lemma 4.17 (Let-\(\lambda\) Equivalence).** For any \(\Gamma, x : A \vdash t : B\) and \(\Gamma \vdash s : A\),

\[
(\lambda(x : A). t) s \equiv \text{let } x = s \text{ in } t
\]
Proof. First, we lift \( t \) using linearity of evaluation contexts, then an open \( \beta \)-reduction, linked by transitivity:

\[
(\lambda (x : A). t) s \dashv \vdash \text{let } x = s \text{ in } \\
(\lambda (x : A). t) \quad \vdash \text{let } x = s \text{ in } t
\]

The concepts of pure and terminating terms are useful because when subterms are pure or terminating, they can be moved around to prove equivalences more easily.

**Definition 4.18 (Pure, Terminating Terms).**

1. A term \( \Gamma \vdash t : A \) is terminating
term if for any closing \( y \), either \( t[y] \Rightarrow U \) or \( t[y] \Rightarrow v \) for some \( v \).
2. A term \( \Gamma \vdash t : A \) is pure if for any closing \( y \), \( t[y] \Rightarrow v \) for some \( v \).

The following terminology and proof are taken from [Fühmann 1999].

**Lemma 4.19 (Pure Terms are Thunkable).** For any pure \( \Gamma \vdash t : A \),

\[
\text{let } x = t \text{ in } \lambda (y : B). x \vdash \vdash \lambda (y : B). t
\]

**Proof.** There are two cases \( \sqsubseteq \sqsubseteq \).

1. Let \( y_1 \sqsubseteq \sqsubseteq_{\Gamma} y_2 \) and define \( t_1 = t[y_1] \) and \( t_2 = t[y_2] \). Then we know \( t_1 \sqsubseteq \sqsubseteq_{\Gamma} t_2 \).
   - (a) If \( t_1 \Rightarrow^{i+1} \Gamma \) we’re done.
   - (b) It is impossible that \( t_1 \Rightarrow U \) because \( t \) is terminating.
   - (c) If \( t_1 \Rightarrow^{i \leq j} \Gamma \) \( \nu_1 \), then we know that \( t_1 \Rightarrow \nu_2 \) with \( \nu_1 \sqsubseteq \sqsubseteq_{\nu, \Gamma} \nu_2 \). Next,

\[
\text{let } x = t_1 \text{ in } \lambda (y : B). x \Rightarrow^{l} \lambda (y : B). \nu_1
\]

Then it is sufficient to show \( \lambda (y : B). \nu_1 \sqsubseteq \sqsubseteq_{\nu, \Gamma} \lambda (y : B). t_2 \), i.e. that for any \( \nu_1' \sqsubseteq \sqsubseteq_{\nu, \Gamma} \nu_2' \) that

\[
(\lambda (y : B). \nu_1) \nu_1' \sqsubseteq \sqsubseteq_{\Gamma} (\lambda (y : B). t_2) \nu_2'
\]

The left side steps

\[
(\lambda (y : B). \nu_1) \nu_1' \Rightarrow^{0} \nu_1
\]

And the right side steps

\[
(\lambda (y : B). t_1) \nu_2' \Rightarrow^{0} t_1 \Rightarrow^{l} \nu_2
\]

And \( \nu_1 \sqsubseteq \sqsubseteq_{\nu, \Gamma} \nu_2 \) by assumption above.

2. Let \( y_1 \sqsubseteq \sqsubseteq \Gamma y_2 \) and define \( t_1 = t[y_1] \) and \( t_2 = t[y_2] \). Then we know \( t_1 \sqsupseteq \sqsupseteq_{\Gamma} t_2 \).

Since \( t \) is terminating we know \( t_1 \Rightarrow \nu_1 \) and for some \( k, t_2 \Rightarrow^{k} \nu_2 \). Then we need to show

And we need to show \( \lambda (y : B). \nu_1 \sqsupseteq \sqsupseteq_{\Gamma} \lambda (y : B). t_2 \). Given any \( \nu_1' \sqsubseteq \sqsubseteq_{\nu, \nu_2} \), we need to show

\[
(\lambda (y : B). \nu_1) \nu_1' \sqsubseteq \sqsubseteq_{\Gamma} (\lambda (y : B). t_2) \nu_2'
\]

The \( \beta \) reduction takes \( 0 \) steps, then \( t_2 \) starts running. If \( k > i \), there is nothing left to show. Otherwise, \( k \leq i \) and we know \( \nu_1 \sqsubseteq \sqsubseteq_{\Gamma} \nu_2 \) which is the needed result.

**Lemma 4.20 (Pure Terms are Essentially Values).** If \( \Gamma \vdash t : A \) is a pure term, then for any \( \Gamma, x : A \vdash s : B, \) let \( x = t \) in \( s \vdash s[t/x] \) holds.
**Proof.** First, since by open $\beta$ we have $\Gamma \vdash (\lambda(y:1). t') \equiv t$, by congruence (lemma 4.8)

$$s[t/x] \equiv s[(\lambda(y:1). t') / x]$$

And by reverse $\beta$ reduction, this is further equivalent to

$$s[(\lambda(y:1). t') / x]$$

By thunkability of $t$ and a commuting conversion this is equivalent to:

$$s[(\lambda(y:1). t') / x]$$

Which by $\beta$ reduction at each $x$ in $x$ is:

$$s[x]$$

And a final $\beta$ reduction eliminates the auxiliary $x_f$:

$$s[x]$$

Also, since we consider all type errors to be equal, terminating terms can be reordered:

**Lemma 4.21 (Terminating Terms Commute).** If $\Gamma \vdash t : A$ and $\Gamma \vdash t' : A'$ and $\Gamma, x : A, y : A' \vdash s : B$, then let $s = t$ in let $x = t'$ in $s$ $\equiv$ let $x = t'$ in let $x = t$ in $s$

**Proof.** By symmetry it is sufficient to prove one direction. Let $i, j \in \mathbb{N}$.

(1) Let $y_1 \equiv i \Gamma y_2$. We need to show

$$\begin{align*}
let x = t[y_1] & \in i \Gamma t' \quad let x' = t'[y_2] \in i \Gamma t' \\
let x' = t'[y_1] & \in i \Gamma x \\
let x = t y_2 & \in s[y_1] \\
let x = t y_2 & \in s[y_2]
\end{align*}$$

Note that this is true if the left side diverges or errors, so this is true with no conditions on $t, t'$

By corollary 4.10, we know $t[y_1] \equiv i_{t, A} t'[y_2]$ and $t'[y_1] \equiv i_{t', A'} t'[y_2]$. We do a joint case analysis on these two facts.

(a) If $t[y_1] \Rightarrow i_{t, A} t'[y_2]$, done.

(b) If $t[y_1] \Rightarrow j \leq i$, done.

(c) If $t[y_1] \Rightarrow j \leq i$, then also $t[y_2] \Rightarrow k \leq i$.

(i) If $t'[y_1] \Rightarrow (i-j)+1$, done.

(ii) If $t'[y_1] \Rightarrow k \leq (i-j)$, done.

(iii) If $t'[y_1] \Rightarrow k \leq (i-j)$, then $t'[y_2] \Rightarrow k \leq (i-j)$ with $t'[y_2] \equiv k \leq (i-j)$ $t'[y_2]'$ and the result follows by corollary 4.10 for $s$ because we know

$$s[y_1, v_1/x, v_1'/x'] \equiv i_{t, A} s[y_2, v_2/x, v_2'/x']$$
(2) Let \( y_1 \sqsupseteq^l_{v, γ} y_2 \). We need to show

\[
\begin{align*}
\text{let } x &= t[y_1] \text{ in } \sqsupseteq^l_{v, B} \text{ let } x' = t'[y_2] \text{ in } \\
\text{let } x' = t'[y_1] \text{ in } &\quad \text{let } x = t y_2 \text{ in } \\
\quad s[y_1] &\quad s[y_2] 
\end{align*}
\]

By corollary 4.10, we know \( t[y_1] \sqsupseteq^l_{v, A} t[y_2] \) and \( t'[y_1] \sqsupseteq^l_{v, A'} t'[y_2] \). We do a joint case analysis on these two facts.

(a) If \( t'[y_2] \Rightarrow^l s \) then either \( t' \models y.alt \) or \( x \models y ] \). In this case we know the right hand side errors, so we must show the left side errors. Since \( t \) is terminating, either \( t[y_1] \Rightarrow \mathcal{U} \) (done) or \( t'[y_1] \Rightarrow v_1 \). In the latter case we are also done because:

\[
\begin{align*}
\text{let } x &= t[y_1] \text{ in } &\Rightarrow \text{let } x' = t'[y_1] \text{ in } &\Rightarrow \mathcal{U} \\
\text{let } x' = t'[y_1] \text{ in } &\quad s[y_1, v/x] 
\end{align*}
\]

(c) If \( t'[y_2] \Rightarrow^l s \) then either \( t'[y_1] \Rightarrow \mathcal{U} \) or \( t'[y_1] \Rightarrow v' \sqsubseteq_{v, A'} v'' \). Next, consider \( t[y_2] \).

(i) If \( t[y_2] \Rightarrow \mathcal{U} \) done.

(ii) If \( t[y_2] \Rightarrow k \leq (i-j) \mathcal{U} \), then we know also that \( t[y_1] \Rightarrow \mathcal{U} \), there is nothing left to show.

(iii) If \( t[y_2] \Rightarrow k \leq (i-j) \mathcal{U} \), then either \( t[y_1] \Rightarrow \mathcal{U} \) or \( t'[y_1] \Rightarrow v_1 \sqsubseteq_{v, A} v_2 \). If \( t'[y_1] \) or \( t[y_1] \) errors, done, otherwise both return values and the result follows by corollary 4.10 for \( s \).

\[\square\]

5 CASTS AS EMBEDDING-PROJECTION PAIRS

In this section, we show how arbitrary casts can be broken down into embedding-projection pairs. First, we define type dynamism and show that casts between less and more dynamic types form an ep pair. Then we will show that every cast is a composition of an upcast and a downcast.

5.1 Embedding-Projection Pairs

First, we define ep pairs with respect to logical approximation. Note that since logical approximation implies observational error approximation, these are also ep pairs with respect to observational error approximation. However, in theorems where we construct new ep pairs from old ones, we will need that the input ep pairs are logical ep pairs, not just observational, because we have not proven that logical approximation is complete for observational error approximation. As with casts, we use evaluation contexts for convenience.

**Definition 5.1 (EP Pair).** A (logical) ep pair \((E_e, E_p) : A \triangleleft B\) is a pair of an embedding \(E_e[\cdot : A] : B\) and a projection \(E_p[\cdot : B] : A\) satisfying

1. Retraction: \( x : A \vdash x \sqsupseteq E_p[E_e[x]] : A \)
2. Projection: \( y : B \vdash E_e[E_p[y]] \sqsubseteq y : B \)

Next, we prove that in any embedding-projection pair that embeddings are pure (always produce a value with no effects) and that projections are terminating (either error or produce a value). Paired with the lemmas we have proven about pure and terminating programs in the previous section, we will be able to prove theorems about ep pairs more easily.

**Lemma 5.2 (Embeddings Are Pure).** If \(E_e, E_p : A \triangleleft B\) is an embedding-projection pair then \( x : A \vdash E_e[x] : B\) is pure.
A ⊆ B  
\[ \text{REFLEXIVITY} \]  
A ⊆ A  
\[ \text{TRANSITIVITY} \]  
A₁ ⊆ A₂  A₂ ⊆ A₃  
A₁ ⊆ A₃  
A₂ ⊆ A₁  
B₁ ⊆ B₂  
A₁ + B₁ ⊆ A₂ + B₂  
SUM  
A₁ × B₁ ⊆ A₂ × B₂  
PROD  
A₁ → B₁ ⊆ A₂ → B₂  
FUN  
A ⊆ ?  
\[ \text{DYN-TOP} \]  

Fig. 15. Type Dynamism

PROOF. The ep pair property states that \( x : A \vdash E_p[E_x[x]] \sqsupseteq x : A \), by Lemma 4.10, we know \( \nu \sqsubseteq^0_{\nu,B} E_p[E_x[\nu]] \) and since \( \nu \sqsubseteq^0 \nu \), this means there exists \( \nu' \) such that \( E_p[E_x[\nu]] \Rightarrow \nu' \), and since \( E_p \) is an evaluation context, this means there must exist \( \nu'' \) with \( E_p[E_x[\nu]] \Rightarrow E_p[\nu''] \Rightarrow \nu' \) .

\[ \blacksquare \]

Lem 5.3 (PROJECTIONS ARE TERMINATING). If \( E_e, E_p : A \triangleleft B \) is an embedding-projection pair then \( y : B \vdash E_p[y] : A \) is terminating.

PROOF. The ep pair property states that \( y : B \vdash E_p[y] \subseteq y : B \) Given any \( \nu : B \), by Lemma 4.10, we know \( E_e[E_p[\nu]] \sqsubseteq y : B \) so therefore either \( E_e[E_p[\nu]] \Rightarrow \nu \), which because \( E_e \) is pure means \( E_p[\nu] \Rightarrow \nu \) or \( E_e[E_p[\nu]] \Rightarrow \nu' \) which by strictness of evaluation contexts means \( E_p[\nu] \Rightarrow \nu'' \) for some \( \nu'' \).

\[ \blacksquare \]

Crucially, ep pairs can be constructed using simple function composition. First, the identity function is an ep pair by reflexivity.

Lem 5.4 (IDENTITY EP PAIR). For any type \( A, [:] : A \triangleleft A \).

Second, if we compose the embeddings one way and projections the opposite way, the result is an ep pair, by congruence.

Lem 5.5 (COMPOSITION OF EP PAIRS). For any ep pairs \( E_{e, 1}, E_{p, 1} : A_1 \triangleleft A_2 \) and \( E_{e, 2}, E_{p, 2} : A_2 \triangleleft A_3, E_{e, 2}[E_{e, 1}], E_{p, 1}[E_{p, 2}] : A_1 \triangleleft A_3 \).

5.2 Type Dynamism

Next, we consider type dynamism and its relationship to the casts. The type dynamism relation is presented in Figure 15. The relation \( A \triangleleft B \) reads as “\( A \) is less dynamic than \( B \)” or “\( A \) is more precise than \( B \)”. For the purposes of its definition, we can say that it is the least reflexive and transitive relation such that every type constructor is monotone and \( ? \) is the greatest element. Even the function type is monotone in both input and output argument and for this reason, type dynamism is sometimes called naïve subtyping [Wadler and Findler 2009]. However, this gives us no semantic intuition about what it could possibly mean. We propose that \( A \triangleleft B \) should hold when the casts between \( A \) and \( B \) form an embedding-projection pair \( E_{(B=A)}, E_{(A=B)} : A \triangleleft B \). We can then view each of the cases of the gradual guarantee as being compositional rules for constructing ep pairs. Reflexivity and transitivity correspond to the identity and composition of ep pairs, and the monotonicity of types comes from the fact that every functor preserves ep pairs.

Taking this idea further, we can view type dynamism not just as an analysis of pre-existing gradual type casts, but by considering its simple proof theory, we can view proofs of type dynamism as synthesizing the definitions of casts. To accomplish this, we give a refined formulation of the proof theory of type dynamism in Figure 16, which includes explicit proof terms \( c : A \triangleleft B \). The methodology behind the presentation is to make reflexivity, transitivity, and the fact that \( ? \) is a
\[
\begin{align*}
\text{c : A} & \subseteq \text{B} \\
A & \in \{1, ?\} \\
id(A) & : A \subseteq A \\
A & \neq ? \\
c & : A \subseteq [A] \\
tag([A]) & : [A] \subseteq ? \\
\tag([A]) \circ c & : A \subseteq ? \\
c & : A_1 \subseteq A_2 \\
d & : B_1 \subseteq B_2 \\
c \times d & : A_1 \times B_1 \subseteq A_2 \times B_2 \\
c + d & : A_1 + B_1 \subseteq A_2 + B_2 \\
c \rightarrow d & : A_1 \rightarrow B_1 \subseteq A_2 \rightarrow B_2
\end{align*}
\]

Fig. 16. Canonical Proof Terms for Type Dynamism

greatest element into admissible properties of the system, rather than primitive rules. First, by making proofs admissible, we see in detail how bigger casts are built up from small pieces.

Second, this formulation satisfies a canonicity property: there is exactly one proof of any given derivation, which simplifies the definition of the semantics. By giving a presentation where derivations are canonical, the typical “coherence” theorem, that says any two derivations have equivalent semantics, becomes trivial. An alternative formulation would define an ep-pair semantics where reflexivity and transitivity denote identity and composition of ep pairs, and then prove that any two derivations have equivalent semantics. Instead, we define admissible constructions for reflexivity and transitivity, and then prove a decomposition lemma (lemma 5.11) that states that the ep-pair semantics interprets our admissible reflexivity derivation as identity and transitivity derivation as a composition. In short, our presentation makes it obvious that the semantics is coherent, but not that it is built out of composition, whereas the alternative makes it obvious that the semantics is built out of composition, but not that it is coherent.

We present the proof terms for type dynamism in Figure 16. As in presentations of sequent calculus, we include the identity ep pair (reflexivity) only for the base types 1, ?. The next rule \(\tag([A]) \circ c : A \subseteq ?\) states that any casts between a non-dynamic type \(A\) and the dynamic type \(?\) are the composition \(\circ\) of, first, a tagging-untagging ep-pair with its underlying tag type \(\tag([A]) : [A] \subseteq ?\) and an ep pair from \(A\) to its tag type \(c : A \subseteq [A]\). The product, sum, and function rules are written to evoke that their ep pairs use the functorial action.

As mentioned the proof terms are canonical, meaning there is at most one derivation of any \(A \subseteq B\).

Lemma 5.6 (Canonical Type Dynamism Derivations). Any two derivations \(c, d : A \subseteq B\) are equal \(c = d\).

Proof. By induction on \(c\), noting in each case that exactly one case is possible. \(\square\)

Next, we need to show that the rules in Figure 15 are all admissible in the refined system Figure 16. The proof of admissibility is given in Figure 17. First, to show reflexivity is admissible, we construct the proof \(\text{id}(A) : A \subseteq A\). It is primitive for ? and 1 and we use the congruence rule to lift the others. Second, to show transitivity is admissible, for every \(d : A_1 \subseteq A_2\) and \(c : A_2 \subseteq A_3\), we construct their composite \(c \circ d : A_1 \subseteq A_3\) by recursion on \(c\). If \(c\) is a primitive composite with a tag, we use associativity of composition to push the composite in. If \(c\) is the identity, the composite is just \(d\). Otherwise, both \(c\) and \(d\) must be between a connective, and we push the compositions in. Finally, we show that ? is the most dynamic type by constructing a derivation \(\text{top}(A) : A \subseteq ?\) for every \(A\). For ?, it is just the identity; for the remaining types, we use the tag ep pair and compose with lifted uses of top.

Next, we construct a semantics for the type dynamism proofs that justifies the intuition we have given so far; it is presented in Figure 18. Every type dynamism proof \(c : A \subseteq B\) defines a pair of an embedding \(E_{e,c}\) and a projection \(E_{p,c}\). Since many rules are the same for embeddings and projections, we use \(m \in \{e, p\}\) to abstract over the mode of the cases. We define the complement of a
Graduality From Embedding-Projection Pairs

\[
\begin{align*}
\text{id}(?) & \triangleq \text{id}(?) & (\text{tag}(A) \circ c) \circ d & \triangleq (\text{tag}(A) \circ (c \circ d)) \\
\text{id}(1) & \triangleq \text{id}(1) & (\text{id}(A)) \circ d & \triangleq d \\
\text{id}(A_1 \times A_2) & \triangleq \text{id}(A_1) \times \text{id}(A_2) & (c \times d) \circ (c' \times d') & \triangleq (c \circ c') \times (d \circ d') \\
\text{id}(A_1 + A_2) & \triangleq \text{id}(A_1) + \text{id}(A_2) & (c + d) \circ (c' + d') & \triangleq (c \circ c') + (d \circ d') \\
\text{id}(A_1 \rightarrow A_2) & \triangleq \text{id}(A_1) \rightarrow \text{id}(A_2) & (c \rightarrow d) \circ (c' \rightarrow d') & \triangleq (c \circ c') \rightarrow (d \circ d') \\
\text{top}(?) & \triangleq \text{id}(?) \\
\text{top}(1) & \triangleq \text{tag}(1) \circ \text{id}(1) \\
\text{top}(A \times B) & \triangleq \text{tag}(\text{?} \times ?) \circ (\text{top}(A) \times \text{top}(B)) \\
\text{top}(A + B) & \triangleq \text{tag}(? + ?) \circ (\text{top}(A) + \text{top}(B)) \\
\text{top}(A \rightarrow B) & \triangleq \text{tag}(? \rightarrow ?) \circ (\text{top}(A) \rightarrow \text{top}(B))
\end{align*}
\]

Fig. 17. Type Dynamism Admissible Proof Terms

\[
\begin{align*}
m & \in \{e, p\} \\
\overline{\tau} & \triangleq p & \overline{e} & \triangleq e
\end{align*}
\]

\[
\begin{align*}
E_{m,\text{id}(A)} & \triangleq [-] \\
E_{e,\text{tag}(G)} & \triangleq \text{roll} \llbracket \text{inj}_G \mid [-] \rrbracket \text{ of } \text{inj}_G \times \times \text{else } \text{U} \\
E_{p,\text{tag}(G)} & \triangleq \text{case unroll } [-] \text{ of } \text{inj}_G \times \times \text{else } \text{U}
\end{align*}
\]

Fig. 18. Type Dynamism Cast Translation

mode \(\overline{m}\) to swap between embeddings and projections; it is used in the function case. The primitive identity casts are interpreted as the identity, and the primitive composition of casts \(\text{tag}(G) \circ d\) is interpreted as the composition of ep pairs of \(\text{tag}(G)\) and \(d\). The tag type derivation is interpreted by the same definition as the cast in Figure 10: tagging puts the correct sum case and \(\text{roll}_G\), and untagging unwraps if its the correct sum case and otherwise errors. We abbreviate this as pattern matching with an “else” clause, where the else clause stands for all of the clauses that do not match the tag type \(G\). The desugaring to repeated case statements on sums should be clear. The product and sum type are just given by their functorial action with the same mode. The function type similarly uses its functorial action, but swaps from embedding to projection or vice-versa on the domain side. This shows that there is nothing strange about the function rule: it is the same construction as for subtyping, but constructing arrows back and forth at the same time. The fact that contravariant functors are covariant with respect to ep pairs in this way is precisely the reason they are used extensively in domain theory.

We next verify that these actually are embedding-projection pairs. To do this, we use the identity and composition lemmas proved before, but we also need to use functoriality of the actions of type constructors, meaning that the action of the type interacts well with identity and composition of evaluation contexts.

**Lemma 5.7 (Identity Extension).**

\[
[-] \times [-] \sqsubseteq [-] \quad \text{and} \quad [-] + [-] \sqsubseteq [-] \quad \text{and} \quad [-] \rightarrow [-] \sqsubseteq [-]
\]

**Proof.** All are instances of \(\eta\) expansion. \(\square\)

In a call-by-value language, the functoriality rules do not hold in general for the product functor, but they do for terminating programs because their order of evaluation is irrelevant. Also notice
that when composing using the functorial action of the function type \( \to \), the composition flips on the domain side, because the function type is contravariant in its domain.

**Lemma 5.8 (Functoriality for Terminating Programs).** The following equivalences are true for any well-typed, terminating evaluation contexts.

\[
\begin{align*}
(E_2 \times E'_2)[E_1 \times E'_1] &\equiv (E_2[E_1]) \times (E'_2[E'_1]) \\
(E_2 + E'_2)[E_1 + E'_1] &\equiv (E_2[E_1]) + (E'_2[E'_1]) \\
(E_2 \to E'_2)[E_1 \to E'_1] &\equiv (E_2[E_1]) \to (E'_2[E'_1])
\end{align*}
\]

**Proof.** (1) \( (\to) \) We need to show (after a commuting conversion)

\[
\begin{align*}
\text{let } x_f &= [\cdot] \text{ in } \\
\text{let } y_f &= \lambda x_a. E'_1[x_f (E_1[x_a])] \text{ in } \\
\lambda y_a. E'_2[y_f (E_2[y_a])] &\equiv \text{let } x_f &= [\cdot] \text{ in } \\
\text{let } y_f &= \lambda x_a. E'_1[x_f (E_1[x_a])] \text{ in } \\
\lambda y_a. E'_2[y_f (E_2[y_a])] &\equiv \text{let } x_f &= [\cdot] \text{ in } \\
\lambda y_a. E'_2[(\lambda x_a. E'_1[x_f (E_1[x_a])]) (E_2[y_a])] \text{ out and } \beta \text{ reduce:}
\end{align*}
\]

\[
\begin{align*}
\text{let } x_f &= [\cdot] \text{ in } \\
\text{let } y_f &= \lambda x_a. E'_1[x_f (E_1[x_a])] \text{ in } \\
\lambda y_a. E'_2[y_f (E_2[y_a])] &\equiv \text{let } x_f &= [\cdot] \text{ in } \\
\lambda y_a. E'_2[(\lambda x_a. E'_1[x_f (E_1[x_a])]) (E_2[y_a])] &\equiv \text{let } x_f &= [\cdot] \text{ in } \\
\lambda y_a. \text{let } x_a = E_2[y_a] \text{ in } \\
E'_2[(\lambda x_a. E'_1[x_f (E_1[x_a])]) x_a] &\equiv \text{let } x_f &= [\cdot] \text{ in } \\
\lambda y_a. \text{let } x_a = E_2[y_a] \text{ in } \\
E'_2[E'_1[x_f (E_1[x_a])]] &\equiv \text{let } x_f &= [\cdot] \text{ in } \\
\lambda y_a. E'_2[E'_1[x_f (E_1[x_a])]]
\end{align*}
\]

(2) \( (+) \) We need to show

\[
\begin{align*}
\text{case case } x \text{ of } &\equiv \text{case } x \text{ of } \\
\text{inj } y. \text{inj } E_1[y] &\quad \text{inj } y. \text{inj } E_2[E_1[y]] \\
| \text{inj' } y'. \text{inj' } E'_1[y'] &\quad | \text{inj' } y'. \text{inj' } E'_2[E'_1[y']]
\end{align*}
\]
First, we do a case-of-case commuting conversion, then lift the discriminees out, $\beta$ reduce and restore them.

First, we make the evaluation order explicit, then re-order using the fact that terminating programs commute lemma 4.21.

(3) ($\times$) We need to show

$$\text{let } \langle x, x' \rangle = x_\rho \text{ in } \quad \text{let } \langle y, y' \rangle = \langle E_1[x], E'_1[x'] \rangle \text{ in } \quad \langle E_2[y], E'_2[y'] \rangle$$

$$\text{let } \langle x, x' \rangle = x_\rho \text{ in } \quad \text{let } \langle y, y' \rangle = \langle E_1[x], E'_1[x'] \rangle \text{ in } \quad \langle E_2[y], E'_2[y'] \rangle$$
let \((x, x') = x_\rho\) in
let \((y, y') = \langle E_1[x], E'_1[x'] \rangle\) in
\langle E_2[y], E_2'[y'] \rangle

With these cases covered, we can show the casts given by type dynamism really are ep pairs.

**Lemma 5.9 (Type Dynamism Derivation Denotes EP Pair).** For any derivation \(c : A \sqsubseteq B\), then \(E_{e,c}, E_{p,c} : \llbracket A \rrbracket \sqsubseteq \llbracket B \rrbracket\) are an ep pair.

**Proof.** By induction on the derivation \(c\).

1. (Identity) \(E_{e, id(A)}, E_{p, id(A)} : \llbracket A \rrbracket \sqsubseteq \llbracket A \rrbracket\). This case is immediate by corollary 4.10.

2. (Composition) \(E_{e,c}, E_{p,c} : \llbracket A_1 \rrbracket \sqsubseteq \llbracket A_2 \rrbracket\), \(E_{e,c'}, E_{p,c'} : \llbracket A_2 \rrbracket \sqsubseteq \llbracket A_3 \rrbracket\). We need to show the retract property:

\[ x : \llbracket A_1 \rrbracket \vdash E_{p,c}[E_{e,c}[E_{e,c'}[E_{e,c}[x]]]] \sqsubseteq x : \llbracket A_1 \rrbracket \]

and the projection property:

\[ y : \llbracket A_3 \rrbracket \vdash E_{e,c}[E_{e,c'}[E_{e,c}[E_{p,c'}[y]]]] \sqsubseteq y : \llbracket A_3 \rrbracket \]

Both follow by congruence and the inductive hypothesis, we show the projection property:

\[ E_{e,c'}[E_{e,c}[E_{p,c'}[y]]] \sqsubseteq E_{e,c'}[E_{p,c'}[y]] \]

(inductive hyp, cong 4.8)

\[ \sqsubseteq y \]

(inductive hyp)
The retraction case follows by $\beta$ reduction
\[
\text{case unroll roll}[?] \text{ inj}_G \ x \ of \ \exists \ x \ \text{ inj}_G \ x_G \cdot x_G \ | \ \text{ else. } U
\]

For the projection case, we need to show
\[
\text{roll}[?] \ \text{ inj}_G \ \left( \ \text{ case unroll y of } \ \text{ inj}_G \ x_G \cdot x_G \ \right) \ \subseteq \ y
\]

First, on the left side, we do a commuting conversion (lemma 4.15) and then use linearity of evaluation contexts to reduce the cases to error:
\[
\text{roll}[?] \ \text{ inj}_G \ \left( \ \text{ case unroll y of } \ \text{ inj}_G \ x_G \cdot x_G \ \right) \ \exists \text{ case unroll y of } \ \text{ inj}_G \ x_G \cdot x_G \ | \ \text{ else. } \ U
\]

Next, we $\eta$-expand the right hand side
\[
y \ \exists \ \text{ roll}[?] \ \text{ unroll y}
\]

The result follows by congruence because $U \subseteq t$ for any $t$.

4. (Functions) We prove the projection property, the retraction proof is similar. We want to show
\[
y : [A_2] \rightarrow [B_2] \supseteq (E_{e,c} \rightarrow E_{e,c'})(E_{p,c} \rightarrow E_{p,c'}[y]) \subseteq y : [A_2] \rightarrow [B_2]
\]

Since embeddings and projections are terminating, we can apply functoriality Lemma 5.8 to show the left hand side is equivalent to
\[
((E_{p,c}[E_{e,c}]) \rightarrow (E_{p,c'}[E_{e,c'}]))[y]
\]
which by congruence and inductive hypothesis is ⊑:

\[ ([\cdot] \rightarrow [\cdot])[y] \]

which by identity extension lemma 5.7 is equivalent to \( y \).

(5) (Products) By the same argument as the function case.

(6) (Sums) By the same argument as the function case.

\( \square \)

Next, while we showed that transitivity and reflexivity were admissible with the \( id(A) \) and \( c \circ d \) definitions, their semantics are not given directly by the identity and composition of evaluation contexts. We justify this notation by the following theorems. First, \( id(A) \) is the identity by identity extension.

**Lemma 5.10 (Reflexivity Proofs denote Identity).** For every \( A, E_{e,id(A)} \equiv [\cdot] \) and \( E_{p,id(A)} \equiv [\cdot] \).

**Proof.** By induction on \( A \), using the identity extension lemma. \( \square \)

Second, we have our key decomposition theorem. While the composition theorem says that the composition of any two ep pairs is an ep pair, the decomposition theorem is really a theorem about the coherence of our type dynamism proofs. It says that given any ep pair given by \( c : A_1 \subseteq A_3 \), if we can find a middle type \( A_2 \), then we can decompose \( c \)'s ep pairs into a composition. This theorem is used extensively, especially in the proof of the gradual guarantee.

**Lemma 5.11 (Decomposition of Upcasts, Downcasts).** For any derivations \( c : A_1 \subseteq A_2 \) and \( c' : A_2 \subseteq A_3 \), the upcasts and downcasts given by their composition \( c' \circ c \) are equivalent to the composition of their casts given by \( c, c' \):

\[
\begin{align*}
x & : [A_1] \models E_{e,c \circ c}[x] \equiv E_{e,c}[E_{e,c}[x]] : [A_3] \\
y & : [A_3] \models E_{p,c \circ c}[y] \equiv E_{p,c}[E_{p,c}[y]] : [A_1]
\end{align*}
\]

**Proof.** By induction on the pair \( c, c' \), following the recursive definition of \( c \circ c' \).

1. \((tag(G) \circ c) \circ d \overset{\text{def}}{=} tag(G) \circ (c \circ d)\). By inductive hypothesis and strict associativity of composition of evaluation contexts.
2. \(id(A_1) \circ d \overset{\text{def}}{=} d \) reflexivity.
3. \((c \times d) \circ (c' \times d') \overset{\text{def}}{=} (c \circ c') \times (d \circ d') \) By inductive hypothesis and functoriality lemma 5.8.
4. \((c + d) \circ (c' + d') \overset{\text{def}}{=} (c \circ c') + (d \circ d') \) By inductive hypothesis and functoriality lemma 5.8.
5. \((c \rightarrow d) \circ (c' \rightarrow d') \overset{\text{def}}{=} (c \circ c') \rightarrow (d \circ d') \) By inductive hypothesis and functoriality lemma 5.8.

\( \square \)

Finally, now that we have established the meaning of type dynamism derivations and proven the decomposition theorem, we can dispense with direct manipulation of derivations. So we define the following notation for ep pairs that just uses the types:

**Definition 5.12 (EP Pair Semantics).** Given \( c : A \subseteq B \), we define \( E_{m,A,B} = E_{m,c} \).

### 5.3 Casts Factorize Into EP Pairs

Next, we show how the upcasts and downcasts are sufficient to construct all the casts of \( \lambda_G \).

First, when \( A \subseteq B \), the ep pair semantics and the cast semantics coincide:
Lemma 5.13 (Upcasts and Downcasts are Casts). If \( A \subseteq B \) then \( E_{(B \Rightarrow A)} \sqsupseteq E_{e,A,B} \) and \( E_{(A \Leftarrow B)} \sqsupseteq E_{p,A,B} \).

Proof. By induction following the recursive definition of \( E_{(B \Rightarrow A)} \):

1. \( E_{(\emptyset \Rightarrow ?)} \) def \([\emptyset] \) By reflexivity.
2. \( E_{(A \times B \Rightarrow A_1 + B_1)} \) def \( E_{(A_2 \Rightarrow A_1)} + E_{(B_2 \Rightarrow B_1)} \) By inductive hypothesis and congruence.
3. \( E_{(A \times B \Rightarrow A_1 \times B_1)} \) def \( E_{(A_2 \Rightarrow A_1)} \times E_{(B_2 \Rightarrow B_1)} \) By inductive hypothesis and congruence.
4. \( E_{(A \Rightarrow B_2 \Rightarrow A_1 \Rightarrow B_1)} \) def \( E_{(A \Leftarrow A_1)} \rightarrow E_{(B_2 \Rightarrow B_1)} \) By inductive hypothesis and congruence.
5. \( E_{(\emptyset \Rightarrow \emptyset)} \) def \( \text{roll} \{ \text{inj}_G \} \) By reflexivity.
6. \( E_{(G \Rightarrow ?)} \) def \( \text{case unroll} [\cdot] \) of \( \text{inj}_G \) \( \cdot \) \text{else} \( \emptyset \) By reflexivity.
7. \( (A \neq ?, [A]) \) \( E_{(\emptyset \Rightarrow A)} \) def \( E_{(\emptyset \Rightarrow [A])} \) \( E_{\{[A] \Rightarrow \emptyset \}} \) By inductive hypothesis and decomposition of ep pairs.
8. \( (A \neq ?, [A]) \) \( E_{(\emptyset \Rightarrow \emptyset)} \) def \( E_{(\emptyset \Rightarrow [A])} \) \( E_{\{[A] \Rightarrow \emptyset \}} \) By inductive hypothesis and decomposition of ep pairs.
9. \( (A, B \neq ? \land [A] \neq [B]) \) \( E_{(B \Rightarrow A)} \) def \( \text{let} \ x = [\cdot] \text{in} \ \emptyset \) Not possible that \( A \subseteq B \).

Next, we show that the "general" casts of the gradual language can be factorized into a composition of an upcast followed by a downcast. First, we show that factorizing through any type is equivalent to factorizing through the dynamic type, as a consequence of the retraction property of ep pairs.

Lemma 5.14 (Any Factorization is Equivalent to Dynamic). For any \( A_1, A_2, A' \) with \( A_1 \subseteq A' \) and \( A_2 \subseteq A' \), \( E_{p,A_2,[E_{e,A_1},?]} \sqsupseteq E_{p,A_2,A'}[E_{e,A_1},A'] \).

Proof. By decomposition and the retraction property:

\[
E_{p,A_2,[E_{e,A_1},?]} \sqsupseteq E_{p,A_2,[E_{p,A',[E_{e,A_1},?]}]} \sqsupseteq E_{p,A_2,A'}[E_{e,A_1},A']
\]

By transitivity of equivalence, this means that factorization through one \( B \) is as good as any other. So to prove that every cast factors as an upcast followed by a downcast, we can choose whatever middle type is most convenient. This lets us choose the simplest type possible in the proof. For instance, when factorizing a function cast \( (A_2 \rightarrow B_2 \Leftarrow A_1 \rightarrow B_1) \), we can use the function tag type as the middle type \( ? \rightarrow ? \) and then the equivalence is a simple use of the inductive hypothesis and the functionality principle.

Lemma 5.15 (Every Cast Factors as Upcast, Downcast). For any \( A_1, A_2, A' \) with \( A_1 \subseteq A' \) and \( A_2 \subseteq A' \), the cast from \( A_1 \) to \( A_2 \) factors through \( A' \): \( x : [A] \Rightarrow E_{(A_2 \Leftarrow A_1)}[x] \sqsupseteq E_{p,A_2,A'}[E_{e,A_1},A'][x] : [A_2] \).

Proof. (1) If \( A_1 \subseteq A_2 \), then we choose \( A' = A_2 \) and we need to show that \( E_{(A_2 \Leftarrow A_1)} \sqsupseteq E_{p,A_2,A'}[E_{e,A_1},A'] \) this follows by lemma 5.13 and lemma 5.10.
2. If \( A_2 \subseteq A_1 \), we use a dual argument to the previous case. We choose \( A' = A_1 \) and we need to show that

\[
E_{(A_2 \Leftarrow A_1)} \sqsupseteq E_{p,A_2,A'}[E_{e,A_1},A']
\]

this follows by lemma 5.13 and lemma 5.10.
(3) $E_{\langle A_2 \times B_2 \Rightarrow A_1 \rightarrow B_1 \rangle} \overset{\text{def}}{=} E_{\langle A_2 \Rightarrow A_1 \rangle} \rightarrow E_{\langle B_2 \Rightarrow B_1 \rangle}$. We choose $A' = ? \rightarrow ?$. By inductive hypothesis,

$E_{\langle A_1 \Rightarrow A_2 \rangle} \ni E_{p, A_1, ?[E_e, A_2, ?]}$ and $E_{\langle B_2 \Rightarrow B_1 \rangle} \ni E_{p, B_2, ?[E_e, B_1, ?]}$

Then the result holds by functoriality:

$E_{\langle A_2 \Rightarrow B_2 \Rightarrow A_1 \rightarrow B_1 \rangle} = E_{\langle A_1 \Rightarrow A_2 \rangle} \rightarrow E_{\langle B_2 \Rightarrow B_1 \rangle}$

$\ni (E_{p, A_1, ?[E_e, A_2, ?]} \rightarrow (E_{p, B_2, ?[E_e, B_1, ?]}))$

$\ni (E_{e, A_2, ? \rightarrow E_{p, B_2, ?[E_e, B_1, ?]}[E_e, A_1, ? \rightarrow E_e, B_1, ?])$

$= E_{p, A_2 \rightarrow B_2, ? \rightarrow ?[E_e, A_1 \rightarrow B_2, ? \rightarrow ?]}$

(4) (Products, Sums) Same argument as function case.

(5) $(A_1, A_2 \neq ? \land [A_1] \neq [A_2])$ $E_{\langle A_2 \Rightarrow A_1 \rangle} \overset{\text{def}}{=} \text{let } x = [\cdot] \text{ in } U$ We choose $A' = ?$, so we need to show: $\text{let } x = [\cdot] \text{ in } U \ni E_{p, A_2, ?[E_e, A_1, ?]}$. By embedding, projection decomposition this is equivalent to

$\text{let } x = [\cdot] \text{ in } U \ni E_{p, A_2, [A_2]}[E_{p, [A_2], ?[E_e, A_1, [A_1]]}]$

Which holds by open $\beta$ because the embedding $E_{e, A_1, [A_1]}$ is pure and $[A_1] \neq [A_2]$. □

6 GRADUALITY FROM EP PAIRS

We now define and prove graduality of our cast calculus. Graduality, briefly stated, means that if a program is changed to make its types less dynamic, but otherwise the syntax is the same, then the operational behavior of the term is “less dynamic” in that either the new term has the same behavior as the old, or it raises a type error, hiding some behavior of the original term. Graduality, like parametricity, says that a certain type of syntactic change (making types less dynamic) results in a predictable semantic change (make behavior less dynamic). We define these two notions as syntactic and semantic term dynamism.

We present syntactic term dynamism in Figure 19, based on the rules of Siek et al. [2015]. Syntactic term dynamism captures the above idea of changing a program to use less dynamic types. If $t_1 \subseteq t_2$, we think of $t_2$ as being rewritten to $t_1$ by changing the types to be less dynamic. While we will sometimes abbreviate syntactic term dynamism as $\vdash t_1 \subseteq t_2$, the full form is $\Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1 \subseteq A_2$ and is read as “$t_1$ is syntactically less dynamic than $t_2$.” The syntax evokes the invariant that if you rewrite $t_2$ to use less dynamic types $t_1$, then its inputs must be given less dynamic types $\Gamma_1 \subseteq \Gamma_2$ and its outputs must be given less dynamic types $A_1 \subseteq A_2$. We extend type dynamism to environment dynamism in Figure 20 to say $\Gamma_1 \subseteq \Gamma_2$ when $\Gamma_1, \Gamma_2$ have the same length and the corresponding types are related. The rules of syntactic term dynamism capture exactly the idea of “types on the left are less dynamic”. Viewed order-theoretically, these rules say that all term constructors are monotone in types and terms.

The second piece of graduality is a semantic formulation of term dynamism. The intuition described above is that $t_1$ should be semantically less dynamic than $t_2$ when it has the same behavior as $t_2$ except possibly when it errors. Note that if $\Gamma_1 = \Gamma_2$ and $A_1 = A_2$, this is exactly what observational error approximation formalizes. Of course, since we can cast between any two types, we can cast any term to be of a different type. Our definition for semantic term dynamism will then be contextual approximation up to cast:

---

Here we invoke the meaning of dynamic as “active”: less dynamic terms are less active in that they kill the program with a type error where a more dynamic program would have continued to run.
Definition 6.1 (Observational Term Dynamism). We say \( \Gamma_1 \vdash t_1 : B_1 \) is observationally less dynamic than \( \Gamma_2 \vdash t_2 : B_2 \), written \( \Gamma_1 \subseteq \Gamma_2 \vdash t_1 \overset{\text{obs}}{=} t_2 : B_1 \subseteq B_2 \) when

\[
\Gamma_1 \vdash (B_2 \iff B_1) t_1 \overset{\text{obs}}{=} \text{let } x_{2,1} = \langle A_{2,1} \iff A_{1,1} \rangle x_{1,1} \text{ in } : B_2 \\
\vdots \\
\text{let } x_{2,n} = \langle A_{2,n} \iff A_{1,n} \rangle x_{1,n} \text{ in } t_2
\]

where \( \Gamma_1 = x_{1,1} : A_{1,1}, \ldots, x_{1,n} : A_{1,n} \) and \( \Gamma_2 = x_{2,1} : A_{2,1}, \ldots, x_{2,n} : A_{2,n} \). Or, abbreviated as:

\[
\Gamma_1 \vdash (B_2 \iff B_1) t_1 \overset{\text{obs}}{=} \text{let } t_2 = \langle (\Gamma_2 \vdash \Gamma_1) t_1 \text{ in } t_2 : B_2
\]

Note that we have chosen to use the two \textit{upcasts}, but there are three other ways we could have inserted casts to give \( t_1, t_2 \) the same type: we can use upcasts or downcasts on the inputs and we can use upcasts or downcasts on the outputs. We will show based on the ep-pair property of upcasts and downcasts that all of these are equivalent (lemma 6.7).

We then define graduality to mean that syntactic term dynamism implies semantic term dynamism:

Theorem 6.2 (Graduality). If \( \Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1 \subseteq A_2 \), then \( \Gamma_1 \subseteq \Gamma_2 \vdash t_1 \overset{\text{obs}}{=} t_2 : A_1 \subseteq A_2 \)

Proof. By lemma 6.4, theorems 3.8 and 6.9, and corollary 4.10. \( \square \)
Next, we present our logical relations method for proving graduality. First, to prove an approximation result for terms in \( \lambda_C \), we will prove approximation for their translations in \( \lambda_T, \Pi \), justified by our adequacy theorem. Second, to prove observational approximation, we will use our logical relation, justified by our soundness theorem. For that we use the following “logical” formulation of term dynamism.

**Definition 6.3 (Logical Term Dynamism).** For any \( \llbracket \Gamma_1 \rrbracket \vdash t_1 : [A_1] \) and \( \llbracket \Gamma_2 \rrbracket \vdash t_2 : [A_2] \) with \( \Gamma_1 \subseteq \Gamma_2 \) and \( A_1 \subseteq A_2 \), we define \( \Gamma_1 \vdash E_e, A_1, A_2 [t_1] \subseteq \llbracket \Gamma_2 \rrbracket = E_e, \Gamma_1, \Gamma_2 [\llbracket \Gamma_1 \rrbracket] \) in \( t_2 : [A_2] \)

where the right hand side is defined analogous to the environment cast \( \langle \Gamma_2 \rangle \).

**Lemma 6.4 (Logical Term Dynamism implies Observational Term Dynamism).** For any \( \Gamma_1 \vdash t_1 : A_1 \) and \( \Gamma_2 \vdash t_2 : A_2 \) with \( \Gamma_1 \subseteq \Gamma_2 \) and \( A_1 \subseteq A_2 \), if \( \Gamma_1 \vdash \llbracket t_1 \rrbracket \subseteq \llbracket t_2 \rrbracket : A_1 \subseteq A_2 \) then \( \Gamma_1 \vdash E_e, A_1, A_2 [t_1] \subseteq \llbracket \Gamma_2 \rrbracket \).

**Proof.** By theorem 4.9 and lemma 4.4. \( \Box \)

Now that we are in the realm of logical approximation, we have all the lemmas of §4.3 at our disposal, and we now start putting them to work. First, as mentioned before, we show that at least with logical term dynamism, the use of upcasts was arbitrary; we could have used downcasts instead. The property we need is that the upcast and downcast are adjoint instead. The property we need is that the upcast and downcast are adjoint instead. The property we need is that the upcast and downcast are adjoint instead. The property we need is that the upcast and downcast are adjoint instead. The property we need is that the upcast and downcast are adjoint instead.

**Lemma 6.5 (EP Pairs are Adjoint).** For any ep pair \( (E_e, E_p) \) : \( A_1 \prec A_2 \), and terms \( \Gamma \vdash t_1 : A_1, \), \( \Gamma \vdash t_2 : A_2 \),

\[ \Gamma \vdash E_e[t_1] \subseteq t_2 : A_2 \quad \text{iff} \quad \Gamma \vdash t_1 \subseteq E_p[t_2] : A_1 \]

**Proof.** The two proofs are dual.

\[ \begin{array}{c}
\Gamma \vdash t_1 \subseteq E_p[E_e[t_1]] : A_1 \\
\Gamma \vdash E_e[t_1] \subseteq t_2 : A_2 \\
\Gamma \vdash E_e[t_1] \subseteq E_p[t_2] : A_1 \\
\end{array} \]

\[ \begin{array}{c}
\Gamma \vdash t_1 \subseteq E_p[t_2] : A_1 \\
\Gamma \vdash t_1 \subseteq E_e[t_1] : A_1 \\
\Gamma \vdash E_e[t_1] \subseteq t_2 : A_2 \\
\end{array} \]

\[ \begin{array}{c}
\Gamma \vdash E_e[t_1] \subseteq E_p[E_p[t_2]] : A_1 \\
\Gamma \vdash E_e[t_1] \subseteq E_p[t_2] : A_1 \\
\Gamma \vdash E_e[t_1] \subseteq t_2 : A_2 \\
\end{array} \]

\[ \Box \]

**Lemma 6.6 (Adjointness on Inputs).** If \( \Gamma, x_1 : A_1 \vdash t_1 : B \) and \( \Gamma, x_2 : A_2 \vdash t_2 : B \), and \( E_e, E_p : A_1 \prec A_2 \), then

\[ \Gamma, x_1 : A_1 \vdash t_1 \subseteq \llbracket t_2 \rrbracket = E_e[x_1] \quad \text{iff} \quad \Gamma, x_2 : A_2 \vdash \llbracket t_1 \rrbracket = E_p[x_2] \in t_2 : B \]

**Proof.** By a similar argument to lemma 6.5. \( \Box \)

**Lemma 6.7 (Alternative Formulations of Logical Term Dynamism).** The following are equivalent

(1) \( \llbracket \Gamma_1 \rrbracket \vdash E_e, A_1, A_2 [t_1] \subseteq \llbracket \Gamma_2 \rrbracket = E_e, \Gamma_1, \Gamma_2 [\llbracket \Gamma_1 \rrbracket] \) in \( t_2 : [A_2] \).
\( \Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1 \subseteq A_2 \quad A_1 \subseteq B_2 \quad (A_2 \subseteq B_2 \lor B_2 \subseteq A_2) \) \quad \text{Cast-Right}

\( \Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1 \subseteq A_2 \quad B_1 \subseteq A_2 \quad (A_1 \subseteq B_1 \lor B_1 \subseteq A_1) \) \quad \text{Cast-Left}

Fig. 21. Term Dynamism Upcast, Downcast Rules

\[ [\Gamma_1] \models t_1 \not\subseteq \text{let } [\Gamma_2] = E_{e, r_1, r_2}[\Gamma_1] \text{ in } E_{p, A_1, A_2}[t_2] : [A_2] \]

\[ [\Gamma_1] \models \text{let } [\Gamma_1] = E_{p, r_1, r_2}[\Gamma_2] \text{ in } t_1 \not\subseteq E_{p, A_1, A_2}[t_2] : [A_2] \]

\[ [\Gamma_1] \models E_{e, A_1, A_2}[\text{let } [\Gamma_1] = E_{p, r_1, r_2}[\Gamma_2] \text{ in } t_1] \not\subseteq t_2 : [A_2] \]

Proof. By induction on \( \Gamma_1 \), using lemma 6.5 and Lemma 6.6 \( \square \)

Finally, to prove the graduality theorem, we do an induction over all the cases of syntactic term dynamism. Most important is the cast case \((B_1 \Leftarrow A_1)\ t_1 \subseteq (B_2 \Leftarrow A_2)\ t_2\) which is valid when \( A_1 \subseteq A_2\) and \( B_1 \subseteq B_2\). We break up the proof into 4 atomic steps using the factorization of general casts into an upcast followed by a downcast (lemma 5.15): \( E_{A_2 \Leftarrow A_1} \supseteq E_{A_2 \Leftarrow A_1} \). The four steps are upcast on the left, downcast on the left, upcast on the right, and downcast on the right. These are presented as rules for logical dynamism in Figure 21. Each of the inference rules accounts for two cases. The Cast-Right rule says first that if \( t_1 \not\subseteq t_2 : A_1 \subseteq A_2\) that it is OK to cast \( t_2\) to \( B_2\), as long as \( B_2\) is more dynamic than \( A_1\), and the cast is either an upcast or downcast. Here, our explicit inclusion of \( A_1 \subseteq B_2\) in the syntax of the term dynamism judgment should help: the rule says that adding an upcast or downcast to \( t_2\) results in a more dynamic term than \( t_1\), whenever it is even sensible to ask: i.e., if it were not the case that \( A_1 \subseteq B_2\), the judgment would not be well-formed, so the judgment holds whenever it makes sense! The Cast-Left rule is dual.

These four rules, combined with our factorization of casts into upcast followed by downcast suffice to prove the congruence rule for casts (we suppress the context \( \Gamma_1 \subseteq \Gamma_2 \vdash\), which is the same in each line):

\[ [t_1] \subseteq [t_2] : A_1 \subseteq A_2 \]

\[ [t_1] \subseteq E_{e, A_1, A_2}[t_2] : A_1 \subseteq ? \]

\[ E_{e, A_1, A_2}[t_1] \subseteq E_{e, A_1, A_2}[t_2] : ? \subseteq ? \]

\[ E_{e, B_1, A_2}[E_{e, A_1, A_2}[t_1]] \subseteq E_{e, B_1, A_2}[E_{e, A_1, A_2}[t_2]] : B_1 \subseteq ? \]

\[ E_{e, B_1, A_2}[E_{e, A_1, A_2}[t_1]] \subseteq E_{e, B_1, A_2}[E_{e, A_1, A_2}[t_2]] : B_1 \subseteq B_2 \]

Lemma 5.15

Next, we show the 4 rules are valid, as simple consequences of the ep pair property and the decomposition theorem. Also note that while there are technically 4 cases, each comes in a pair where the proofs are exactly dual, so conceptually speaking there are only 2 arguments.

Lemma 6.8 (Upcast, Downcast Dynamism). The four rules in Figure 21 are valid.

Proof. In each case we choose which case of lemma 6.7 is simplest.

1. Cast-Left with \( A_1 \subseteq B_1 \subseteq A_2\). We need to show \( E_{e, B_1, A_2}[E_{e, A_1, B_1}[t_1]] \subseteq \text{let } [\Gamma_2] = E_{e, r_1, r_2}[\Gamma_1] \text{ in } t_2 \)

By decomposition and congruence, \( E_{e, B_1, A_2}[E_{e, A_1, B_1}[t_1]] \equiv E_{e, A_1, A_2}\) so the conclusion holds by transitivity and the premise.
(2) **CAST-RIGHT** with \( A_1 \subseteq B_2 \subseteq A_2 \). We need to show \( \text{let } [\Gamma_1] = E_{p,\Gamma_1,\Gamma_2}[[\Gamma_2]] \text{ in } E_{p,A_1,B_1}[E_{p,B_2,A_2}[t_2]] \).

By decomposition and congruence, \( E_{p,A_1,B_1}[E_{p,B_2,A_2}[t_2]] \supseteq E_{p,A_1,A_2}[t_2] \), so the conclusion holds by transitivity and the premise.

(3) **CAST-LEFT** with \( B_1 \subseteq A_1 \subseteq A_2 \). We need to show \( E_{p,B_1,A_1}[t_1] \supseteq E_{p,B_1,A_1}[E_{p,A_1,A_2}[t_2]] \).

By decomposition, \( E_{p,B_1,A_2}[t_2] \supseteq E_{p,B_1,A_1}[E_{p,A_1,A_2}[t_2]] \), so by transitivity it is sufficient to show

\[
E_{p,B_1,A_1}\text{[let } [\Gamma_1] = E_{p,\Gamma_1,\Gamma_2}[[\Gamma_2]] \text{ in } t_1] \subseteq E_{p,B_1,A_1}[E_{p,A_1,A_2}[t_2]]
\]

which follows by congruence and the premise.

(4) **CAST-RIGHT** with \( A_1 \subseteq A_2 \subseteq B_2 \). We need to show \( E_{c,A_1,B_2}[t_1] \subseteq E_{c,A_1,B_1}[E_{c,A_1,A_2}[t_1]] \).

By decomposition, \( E_{c,A_1,B_2}[t_1] \supseteq E_{c,A_1,B_2}[E_{c,A_1,A_2}[t_1]] \), so by transitivity it is sufficient to show

\[
E_{c,A_1,B_2}[E_{c,A_1,A_2}[t_1]] \subseteq E_{c,A_1,B_1}[E_{c,A_1,A_2}[t_1]] \text{[let } [\Gamma_2] = E_{c,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in } t_2]
\]

which follows by congruence and the premise.

\[\square\]

Finally, we prove the graduality theorem by induction on syntactic term dynamism derivations, finishing the proof of theorem 6.2.

**Theorem 6.9 (Logical Graduality).** If \( \Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1 \), then \( \Gamma_1 \subseteq \Gamma_2 \vdash [t_1] \subseteq [t_2] : A_1 \subseteq A_2 \).

**Proof.** By induction on syntactic term dynamism rules.

(1) To show \( \Gamma_1 \subseteq \Gamma_2 \vdash [t_1] \subseteq [t_2] : B_1 \subseteq B_2 \) we use lemma 6.8 and the argument above.

(2) \( \Gamma_1 \subseteq \Gamma_2 \quad A_1 \subseteq B_1 \quad \Gamma'_1 \subseteq \Gamma'_2 \) We need to show:

\[
[t_1] = E_{e,\Gamma_1,A_2}[x_1] \subseteq \text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in } x_2
\]

Since embeddings are pure lemmas 4.20 and 5.2 we can substitute them in and then the two sides are literally the same.

\[
\text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in } \supseteq x_2 E_{e,\Gamma_1,A_2}'[x_1'] = E_{e,\Gamma_1,A_2}[x_1]
\]

(3) \( \Gamma_1 \subseteq \Gamma_2 \vdash \text{inj } t_1 \subseteq \text{inj } t_2 : A_1 \subseteq A_2 \quad A_1' \subseteq A_2' \)

Expanding definitions, we need to show:

\[
\text{case } \text{inj } [t_1] \text{ of } \text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in } \text{inj } x \text{. inj } (E_{e,\Gamma_1,A_2}[x]) \text{[t_2]} \\
\text{[inj' } x \text{. inj'} (E_{e,\Gamma_1,A_2}'[x'])
\]

\[\square\]
By open $\beta$ (lemma 4.13), the left side can be reduced, which we can then substitute into due to linearity of evaluation contexts (lemma 4.16):

\[
\text{case inj } [t_1] \text{ of inj } x. \text{ inj } (E_{e,A_1,A_2}[x]) | \text{ inj }' x. \text{ inj }' (E_{e,A_1',A_2'}[x']) \supseteq \text{ let } x = [t_1] \text{ in inj } (E_{e,A_1,A_2}[x]) \\
\supseteq (\text{ inj } (E_{e,A_1,A_2}[x]))([t_1]/x) = \text{ inj } (E_{e,A_1,A_2}([t_1]))
\]

So by transitivity it is sufficient to show

\[
\text{ inj } (E_{e,A_1,A_2}([t_1])) \subseteq \text{ inj } \left( \text{ let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in } \right)
\]

which follows by congruence (lemma 4.8).

(4) \[
\Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1' \subseteq A_2' A_1 \subseteq A_2 \quad \text{essentially the same as the previous case.}
\]

(5) \[
\Gamma_1 \subseteq \Gamma_2 \vdash \text{ case } t_1 \text{ of inj } x_1 : A_1 \cdot s_1 | \text{ inj }' x'_1 : A_1' \cdot s'_1 \subseteq \text{ case } t_2 \text{ of inj } x_2 : A_2 \cdot s_2 | \text{ inj }' x'_2 : A_2' \cdot s'_2 : B_1 \subseteq B_2
\]

Expanding definitions, we need to show

\[
\text{ inj } \left( \text{ case } t_1 \text{ of inj } x_1 : A_1 \cdot s_1 | \text{ inj }' x'_1 : A_1' \cdot s'_1 \right) \subseteq \text{ case } t_2 \text{ of inj } x_2 : A_2 \cdot s_2 | \text{ inj }' x'_2 : A_2' \cdot s'_2 : B_1 \subseteq B_2
\]

First, we do some simple rewrites: on the left side, we use a commuting conversion to push the embedding into the continuations:

\[
\text{ let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in case } [t_2] \text{ of case } t_1 \text{ of inj } x_1. [s_1] | \text{ inj }' x'_1. [s'_1]
\]

And on the right side we use the fact that embeddings are pure and so can be moved freely:

\[
\text{ let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in } \supseteq \text{ case } \text{ let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in of case } [t_2] \text{ of inj } x_2. [s_2] | \text{ inj }' x'_2. [s'_2]
\]

Next as with many of the elim forms, we "ep-expand" the discriminant on the left side, and then simplify based on the definition of $E_{e,\Gamma_1+\Gamma_1',\Gamma_2+\Gamma_2'}[\cdot]$, using the case-of-case commuting
conversion and open $\beta$ lemmas 4.13 and 4.15:

\[
\begin{array}{l}
\text{case } [t_1] \text{ of} \\
\quad \text{inj } x_1, E_{e,B_1,B_2}[[s_1]] \\
\quad | \text{inj'} x'_1, E_{e,B_1,B_2}[[s'_1]]
\end{array}
\]

(definition) = \[
\begin{array}{l}
\text{case } [t_1] \text{ of} \\
\quad \text{inj } x_2, \text{let } x_1 = E_{p,A_1,A_2}[x_2] \text{ in} \\
\quad E_{e,B_1,B_2}[[s_1]] \\
\quad | \text{inj'} x'_1, \text{let } x'_1 = E_{p,A'_1,A'_2}[x'_2] \text{ in} \\
\quad E_{e,B_1,B_2}[[s'_1]]
\end{array}
\]

Then the final step follows by congruence and adjointness on inputs lemmas 4.8 and 6.6:

\[
\begin{array}{l}
\text{case } E_{e,A_1+A'_1,A_2+A'_2}[t_1] \text{ of} \\
\quad \text{inj } x_2, \text{let } x_1 = E_{p,A_1,A_2}[x_2] \text{ in} \\
\quad E_{e,B_1,B_2}[[s_1]] \\
\quad | \text{inj'} x'_1, \text{let } x'_1 = E_{p,A'_1,A'_2}[x'_2] \text{ in} \\
\quad E_{e,B_1,B_2}[[s'_1]]
\end{array}
\]

Expanding we need to show

\[
E_{e,1,1}[\langle \rangle] \subseteq \text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in} \\
\langle \rangle
\]

By definition, the left side is just $\langle \rangle$ and the right side after a substitution, valid because embeddings are pure lemmas 4.20 and 5.2.

\[
\begin{array}{l}
\Gamma_1 \subseteq \Gamma_2 \Rightarrow \langle t_1, s_1 \rangle \subseteq \langle t_2, s_2 \rangle: A_1 \times B_1 \subseteq A_2 \times B_2
\end{array}
\]

Expanding definitions, we need to show

\[
E_{e,A_1 \times B_1,A_2 \times B_2}[\langle \langle t_1, s_1 \rangle \rangle] \subseteq \text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in} \\
\langle \langle t_2, s_2 \rangle \rangle
\]

On the right, we duplicate the embeddings, justified by lemmas 4.20 and 5.2, to set up congruence:

\[
\text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in } \forall \langle t_2, s_2 \rangle: \langle t_2 \rangle = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in } \\
\langle t_2 \rangle = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]] \text{ in } \\
\langle s_2 \rangle = E_{e,\Gamma_1,\Gamma_2}[[\Gamma_1]]
\]
On the left, we use linearity of evaluation contexts to lift the terms out, then perform some open $\beta$ reductions and put the terms back in:

\[
E_{e,A_1\times B_1,A_2\times B_2}([\{t_1\},[s_1]]) \pseq \text{let } x = [t_1] \text{ in } \\
\quad \text{let } y = [s_1] \text{ in } \\
\quad \text{let } (x, y) = (x, y) \text{ in } \langle E_{e,A_1,A_2}[x], E_{e,B_1,B_2}[y] \rangle
\]

(open $\beta$, 4.13) $\forall \exists x = [t_1] \text{ in } \\
\text{let } y = [s_1] \text{ in } \\
\langle E_{e,A_1,A_2}[x], E_{e,B_1,B_2}[y] \rangle$

(linearity, 4.16) $\forall \langle E_{e,A_1,A_2}([t_1]), E_{e,B_1,B_2}([s_1]) \rangle$

With the final step following by congruence (lemma 4.8) and the premise:

\[
\langle E_{e,A_1,A_2}([t_1]), E_{e,B_1,B_2}([s_1]) \rangle \subseteq \langle \text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in }, \text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in } \rangle
\]

\[
\Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1 \times A_1' \subseteq A_2 \times A_2'
\]

\[
\Gamma_1, x_1 : A_1, x_1' : A_1' \subseteq \Gamma_2, x_2 : A_2, x_2' : A_2' \vdash t_1' \subseteq t_2' : B_1 \subseteq B_2
\]

Expanding definitions, we need to show

\[
[\Gamma_1] \models E_{e,B_1,B_2}[\text{let } (x_1, x_1') = [t_1] \text{ in } [t_1']] \subseteq \text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in } : [B_2] \\
\text{let } (x_2, x_2') = [t_2] \text{ in } [t_2']
\]

On the right side, in anticipation of a use of congruence, we push the embeddings in lemmas 4.20 and 5.2:

\[
\text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in } \forall \exists \text{let } (x_2, x_2') = \left( \text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in } \right) \text{ in } \\
\text{let } [\Gamma_2] = E_{e,\Gamma_1,\Gamma_2}([\Gamma_1]) \text{ in } [t_2']
\]
On the left side, we perform a commuting conversion, ep expand the discriminatee and do some open β reductions to simplify the expression.

\[
E_{e, B_1, B_2} \left[ \langle x_1, x'_1 \rangle = \llbracket t_1 \rrbracket \ \triangleright \ \left[ \llbracket t'_1 \rrbracket \right] \right] \equiv \left[ \llbracket t_1 \rrbracket \right] \in E_{e, B_1, B_2} \left[ \llbracket t'_1 \rrbracket \right]
\]

(ev. pair, 5.9) \equiv \left[ \llbracket t_1 \rrbracket \right] \in E_{e, B_1, B_2} \left[ \llbracket t'_1 \rrbracket \right]

(definition) = \left[ \llbracket t_1 \rrbracket \right] \in E_{e, B_1, B_2} \left[ \llbracket t'_1 \rrbracket \right]

(linearity, 4.16) \equiv \left[ \llbracket t_1 \rrbracket \right] \in E_{e, B_1, B_2} \left[ \llbracket t'_1 \rrbracket \right]

(comm. conv., 4.15) \equiv \left[ \llbracket t_1 \rrbracket \right] \in E_{e, B_1, B_2} \left[ \llbracket t'_1 \rrbracket \right]

(open β, 4.13) \equiv \left[ \llbracket t_1 \rrbracket \right] \in E_{e, B_1, B_2} \left[ \llbracket t'_1 \rrbracket \right]

The final step is by congruence and adjointness on inputs (lemmas 4.8 and 6.6):

\[
\text{let } \langle x_2, x'_2 \rangle = E_{e, A_1 \times A_2, A_2 \times A_2} \left[ \llbracket t_1 \rrbracket \right] \text{ in } \equiv \left[ \llbracket t_2 \rrbracket \right] = E_{e, B_1, B_2} \left[ \llbracket t'_2 \rrbracket \right]
\]

\[
\Gamma_1, x_1 : A_1 \subseteq \Gamma_2, x_2 : A_2 \vdash t_1 \subseteq t_2 : B_1 \subseteq B_2 \quad \text{Expanding definitions, we need to show}
\]

\[
\llbracket \Gamma_1 \rrbracket \vdash \text{let } x_f = \lambda(x_1 : [A_1]). \llbracket t_1 \rrbracket \quad \text{in } \lambda(x_2 : A_2). E_{e, B_1, B_2} \left[ x_f \left( E_{p, A_1, A_2} \left[ x_2 \right] \right) \right]
\]

First we simplify by performing some open β reductions on the left and let-λ equivalence and a commuting conversion (lemmas 4.13, 4.15 and 4.17):

\[
\text{let } x_f = \lambda(x_1 : [A_1]). \llbracket t_1 \rrbracket \quad \equiv \lambda(x_2 : A_2). E_{e, B_1, B_2} \left[ \text{let } x_1 = E_{p, A_1, A_2} \left[ x_2 \right] \text{ in } \llbracket t_1 \rrbracket \right]
\]

\[
\equiv \lambda(x_2 : A_2). E_{e, B_1, B_2} \left[ \llbracket t_1 \rrbracket \right]
\]
and on the right, we move the embedding into the body, which is justified because embeddings are essentially values (lemmas 4.20 and 5.2):

\[
\lambda(x_2 : A_2), \text{let } [Γ_2] = E_{e, r_1, r_2}[[Γ_1]] \text{ in } \lambda(x_2 : A_2), \text{let } [Γ_2] = E_{e, r_1, r_2}[[Γ_1]] \text{ in } [t_2]
\]

The final step is justified by congruence lemma 4.8 and adjointness on inputs lemma 6.6 and the premise:

\[
\lambda(x_2 : A_2), \text{let } x_1 = E_{ρ, A, A_1}[x_2] \text{ in } \lambda(x_2 : A_2), \text{let } [Γ_2] = E_{e, r_1, r_2}[[Γ_1]] \text{ in } [t_2]
\]

\[
[Γ_1] \subseteq [Γ_2] \quad t_1 \subseteq t_2 : A_1 \rightarrow B_1 \subseteq A_2 \rightarrow B_2 \quad [Γ_1] \subseteq [Γ_2] \subseteq s_1 \subseteq s_2 : A_1 \subseteq A_2
\]

(10) Expanding definitions, we need to show

\[
[Γ_1] \subseteq [Γ_2] \subseteq [t_1] \subseteq [t_2] \subseteq [s_1] \subseteq [s_2].
\]

First, we duplicate the embedding on the right hand side, justified by purity of embeddings, to set up a use of congruence later:

\[
\text{let } [Γ_2] = E_{e, r_1, r_2}[[Γ_1]] \text{ in } \exists \exists \left(\text{let } [Γ_2] = E_{e, r_1, r_2}[[Γ_1]] \text{ in } [t_2] \text{ and } \text{let } [Γ_2] = E_{e, r_1, r_2}[[Γ_1]] \text{ in } [s_2]\right)
\]

Next, we use linearity of evaluation contexts lemma 4.16 so that we can do reductions at the application site without worrying about evaluation order:

\[
E_{e, B_1, B_2}[t_1] \exists \exists \text{ let } x_f = [t_1] \text{ in } [t_2] \exists \exists \text{ let } x_a = [s_1] \text{ in } [s_2]
\]

Next, we ep-expand \( x_f \) (lemma 5.9) and perform some \( β \) reductions, use the ep property and then reverse the use of linearity.

\[
\text{let } x_f = [t_1] \text{ in } \exists \exists \text{ let } x_f = [t_1] \text{ in } [t_2] \exists \exists \text{ let } x_a = [s_1] \text{ in } [s_2]
\]

(open \( β \), 4.13) \( \exists \exists \text{ let } x_f = [t_1] \text{ in } [t_2] \exists \exists \text{ let } x_a = [s_1] \text{ in } [s_2]
\]

\[
E_{e, B_1, B_2} x_f x_a
\]

\[
E_{e, B_1, B_2} E_{e, A_1 \rightarrow B_1, A_2 \rightarrow B_2} E_{e, A_1 \rightarrow B_1, A_2 \rightarrow B_2} E_{e, A_1 \rightarrow B_1, A_2 \rightarrow B_2} x_f x_a
\]

\[
\text{ep pair, 5.9} \exists \exists \text{ let } x_f = [t_1] \text{ in } [t_2] \exists \exists \text{ let } x_a = [s_1] \text{ in } [s_2]
\]

\[
\text{linearity, 4.16} \exists \exists \left( E_{e, A_1 \rightarrow B_1, A_2 \rightarrow B_2}[[t_1]] \left( E_{e, A_1, A_2}[[x_a]] \right) \right) \left( E_{e, A_1, A_2}[[x_a]] \right)
\]

With the final step being congruence lemma 4.8:

\[
\left( E_{e, A_1 \rightarrow B_1, A_2 \rightarrow B_2}[[t_1]] \left( E_{e, A_1, A_2}[[s_1]] \right) \right) \exists \exists \left( \text{let } [Γ_2] = E_{e, r_1, r_2}[[Γ_1]] \text{ in } [t_2] \right) \exists \exists \left( \text{let } [Γ_2] = E_{e, r_1, r_2}[[Γ_1]] \text{ in } [s_2] \right)
\]

\( □ \)
7 RELATED WORK AND DISCUSSION

Our analysis of graduality as observational approximation and dynamism as ep pairs builds on the axiomatic and denotational semantics of graduality for a call-by-name language presented in [New and Licata 2018]. The semantics there gives axioms of type and term dynamism that imply that upcasts and downcasts are embedding-projection pairs. Our analysis here is complementary: we present the graduality theorem as a concrete property of a gradual language defined with an operational semantics. Our graduality logical relation should serve as a concrete model of a call-by-value version of gradual type theory, similar to the call-by-name denotational models presented there. Furthermore, we show here how this interpretation of graduality maps back to a standard cast calculus presentation of gradual typing.

**Graduality vs Gradual Guarantee.** The notion of graduality we present here is based on the dynamic gradual guarantee by Siek, Vitousek, Cimini, and Boyland [Siek et al. 2015; Boyland 2014]. The dynamic gradual guarantee says that syntactic term dynamism is an invariant of the operational semantics up to error on the less dynamic side. More precisely, if $\Gamma \vdash t_1 \subseteq t_2 : A_1 \subseteq A_2$ then either $t_1 \mapsto^* \emptyset$ or both $t_1, t_2$ diverge or $t_1 \mapsto^* \nu_1$ and $t_2 \mapsto^* \nu_2$ with $\nu_1 \subseteq \nu_2$. Observe that when restricting $A_1 = A_2 = 1$, this is precisely the relation on closed programs out of which we build our definition of semantic term dynamism. We view their formulation of the dynamic gradual guarantee as a syntactic proof technique for proving graduality of the system.

Graduality should be easier to formulate for different presentations of gradual typing because it does not require a second syntactic notion of term dynamism for the implementation language. In the proofs of the gradual guarantee in Siek et al. [2015], they have to develop new rules for term dynamism for their cast calculus, that they do not attempt to justify at an intuitive level. Additionally, they have to change their translation from the gradual surface language to the cast calculus because the traditional translation did not preserve the rigid syntactic formulation of term dynamism. In more detail, when a dynamically typed term $t : ?$ was applied to a term $s : A$, in their original formulation this was translated as

$$[t \ s] = (\langle (A \to ?) \Leftarrow ? \rangle \llbracket t \rrbracket \llbracket s \rrbracket)$$

but if the term in function position had a function type $t' : ? \to ?$ it was translated as

$$[t' \ s] = (\llbracket t \rrbracket \langle ? \Leftarrow A \rangle \llbracket s \rrbracket)$$

But if $t' \subseteq t$, we would not have $[t' \ s] \subseteq [t \ s]$ because the function position on the left has type $? \to ?$ which is more dynamic than on the right which has $A \to ?$. While changing this was perfectly reasonable to do to use their syntactic proof method, we can see that from the semantic point of view of graduality there was nothing wrong with their original translation and it could have been validated using a logical relation.

Another significant difference between our work and theirs is that we identify the central role of embedding-projection pairs in graduality, and take advantage of it in our proof. As mentioned above, they add rules to term dynamism for the cast calculus without justification. These rules are the generalization of our Cast-Right and Cast-Left without the restriction that the casts be upcasts or downcasts:

$$\Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1 \subseteq A_2 \quad A_1 \subseteq B_2 \quad \text{Cast-Right} \quad \Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq t_2 : A_1 \subseteq A_2 \quad B_1 \subseteq A_2 \quad \text{Cast-Left}$$

$$\Gamma_1 \subseteq \Gamma_2 \vdash t_1 \subseteq (B_2 \Leftarrow A_2) t_2 : A_1 \subseteq B_2 \quad \text{Cast-Right} \quad \Gamma_1 \subseteq \Gamma_2 \vdash (B_1 \Leftarrow A_1) t_1 \subseteq t_2 : B_1 \subseteq A_2 \quad \text{Cast-Left}$$

These are valid rules in our system, but by identifying the subset of upcasts and downcasts, we prove the validity of the rules from earlier, intuitive rules: decomposition, congruence, and the ep-pair properties. Furthermore, while we do not take these rules as primitive it is notable that
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these two rules imply that upcasts and downcasts are adjoint—i.e., if \( A_1 \sqsubseteq A_2 \), the following are provable for \( t : A_1 \) and \( s : A_2 \):

\[
\begin{align*}
\text{t} \sqsubseteq \langle A_1 \Leftarrow A_2 \rangle \langle A_2 \Leftarrow A_1 \rangle \text{t} & \quad \langle A_2 \Leftarrow A_1 \rangle \langle A_1 \Leftarrow A_2 \rangle \text{s} \sqsubseteq \text{s}
\end{align*}
\]

Siek et al. [2015] also present a theorem called the static gradual guarantee that pertains to the type checking of gradually typed programs. The static gradual guarantee says that if a term \( \Gamma \vdash t_1 : A_1 \) type checks, and \( t_2 \) is syntactically more dynamic, then \( \Gamma \vdash t_2 : A_2 \) with a more dynamic type, i.e., \( A_1 \sqsubseteq A_2 \). We view this as a corollary to graduality. If type checking is a compositional procedure that seeks to rule out dynamic type errors, then if \( t_1 \) is syntactically less dynamic than \( t_2 \), then it is also semantically less dynamic, meaning every type error in \( t_2 \)'s behavior was already present in \( t_1 \), so it should also type check.

Types as EP Pairs. The interpretation of types as retracts of a single domain originated in Scott [1972] and is a common tool in denotational semantics, especially in the presence of a convenient universal domain. A retraction is a pair of morphisms \( s : A \to B \), \( r : B \to A \) that satisfy the retraction property \( r \circ s = \text{id}_A \), but not necessarily the projection property \( s \circ r \subseteq \text{err} \circ \text{id}_B \). Thus ep pair semantics can be seen as a more refined retraction semantics. Retractions have been used to study interaction between typed and untyped languages, e.g., see Benton [2005]; (Favonia) et al. [2017].

Embedding-projection pairs are used extensively in domain theory as a technical device for solving non-well-founded domain equations, such as the semantics of a dynamic type. In this paper, our error-approximation ep pairs do not play this role, and instead the retraction and projection properties are desirable in their own right for their intuitive meaning for type checking.

Many of the properties of our embedding-projection pairs are anticipated in Henglein [1994] and Thatte [1990]. Henglein [1994] defines a language with a notion of coercion \( A \rightsquigarrow B \) that corresponds to general casts, with primitives of tagging \( tc! : tc(\ldots) \rightsquigarrow ? \) and untagging \( tc? : ? \rightsquigarrow tc(\ldots) \) for every type constructor "tc". Crucially, Henglein notes that \( tc! ; tc? \) is the identity modulo efficiency and that \( tc? ; tc! \) errors more than the identity. Furthermore, they define classes of "positive" and "negative" coercions that correspond to embeddings and projections, respectively, and a "subtyping" relation that is the same as type precision. They then prove several theorems analogous to our results:

1. (Retraction) For any pair of positive coercion \( p : A \rightsquigarrow B \), and negative coercion \( n : B \rightsquigarrow A \), they show that \( n; p \) is equal to the identity in their equational theory.
2. (Almost projection) Dually, they show that \( n; p \) is equal to the identity assuming that \( tc?; tc! \) is equal to the identity for every type constructor.
3. They show every coercion factors as a positive cast to ? followed by a negative cast to ?.
4. They show that \( A \leq B \) if and only if there exists a positive coercion \( A \rightsquigarrow B \) and a negative coercion \( B \rightsquigarrow A \).

They also prove factorization results that are similar to our factorization definition of semantic type precision, but it is unclear if their theorem is stronger or weaker than ours. One major difference is that their work is based on an equational theory of casts, whereas ours is based on notions of observational equivalence and approximation of a standard call-by-value language. Furthermore, in defining our notion of observational error approximation, we provide a more refined projection property, justifying their use of the term "safer" to compare \( p ; e \) and the identity.

The system presented in Thatte [1990], called "quasi-static typing" is a precursor to gradual typing that inserts type annotations into dynamically typed programs to make type checking explicit. There they prove a desirable soundness theorem that says their type insertion algorithm produces an explicitly coercing term that is minimal in that it errors no more than the original dynamic
term. They prove this minimality theorem with respect to a partial order \(\sqsubseteq\) defined as a logical relation over a domain-theoretic semantics that (for the types they defined) is analogous to our error ordering for the operational semantics. However, they do not define our operational formulation of the ordering as contextual approximation, linked to the denotational definition by the adequacy result, nor that any casts form embedding-projection pairs with respect to this ordering.

Finally, we note that neither of these papers [Henglein 1994; Thatte 1990] extends the analysis to anything like graduality.

**Semantics of Casts.** Superficially similar to the embedding-projection pair semantics are the threesome casts of Siek and Wadler [2010]. A threesome cast factorizes an arbitrary cast \(A \Rightarrow B\) through a third type \(C\) as a downcast \(A \Rightarrow C\) followed by an upcast \(C \Rightarrow B\), whereas ep-pair semantics factorizes a cast as an upcast \(A \Rightarrow ?\) followed by a downcast \(? \Rightarrow B\). Threesome casts can be used to implement gradual typing in a space-efficient manner, the third type \(C\) is used to collapse a sequence of arbitrarily many casts into just the two. In the general case, the threesome cast \(A \Rightarrow C \Rightarrow B\) is stronger (fails more) than the direct cast \(A \Rightarrow B\). This is the point of threesome casts: the middle type faithfully represents a sequence of casts in minimal space. EP pair semantics instead factorizes a cast \(A \Rightarrow B\) into an upcast \(A \Rightarrow ?\) followed by a downcast \(? \Rightarrow B\), a factorization already utilized in [Henglein 1994], and which we showed is always equivalent to the direct cast \(A \Rightarrow B\). We view the benefits of the techniques as orthogonal: the up-down factorization helps to prove graduality, whereas the down-up factorization helps implementation. The fact that both techniques reduce reasoning about arbitrary casts to just upcasts and downcasts supports the idea that upcasts and downcasts are a fundamental aspect of gradual typing.

Recently, work on dependent interoperability [Dagand et al. 2016, 2018] has identified Galois connections as a semantic formulation for casting between more and less precise types in a non-gradual dependently typed language, and conjectures that this should relate to type dynamism. We confirm their conjecture in showing that the casts in gradual typing satisfy the slightly stronger property of being embedding-projection pairs and have used it to explain the cast semantics of gradual typing and graduality. Furthermore, our analysis of the precision rules as compositional constructions on ep pairs is directly analogous to their library, which implements “connections” between, for instance, function types given connections between the domains and codomains using Coq’s typeclass mechanism.

**Pairs of Projections and Blame.** One of the main inspirations for this work is the analysis of contracts in Findler and Blume [2006]. They decompose contracts in untyped languages as a pair of “projections”, i.e., functions \(c : ? \rightarrow ?\) satisfying \(c \sqsubseteq \text{id}\). However, they do not provide a rigorous definition or means to prove this ordering for complex programs as we have. There is a close relationship between such projections and ep pairs (an instance of the relationship between adjunctions and (co)monads): for any ep pair \(e, p : A \triangleright B, e \circ p : B \rightarrow B\) is a projection. However, we think this relationship is a red herring; instead we think that a pair of projections is better understood as ep pairs themselves. The intuition they present is that one of the projections restricts the behavior of the “positive” party (the term) and the other restricts the behavior of the “negative” party (the continuation). EP pairs are similar, the projection restricts the positive party by directly checking, and the embedding restricts the negative party in the function case by calling a projection on any value received from its continuation. However, in our current formulation, it does not even make sense to ask if each component of our embedding-projection pairs is a projection because the definition of a projection assumes that the domain and codomain are the same (to define the composite \(c \circ c\)). We conjecture that this can be made sensible by using a PER semantics where types are relations on untyped values, so that the embedding and projection have “underlying” untyped terms representing them, and those are projections.
Their analysis of blame was adapted to gradual typing in Wadler and Findler [2009] and plays a complementary role to our analysis: they use the dynamism relation to help prove the blame soundness theorem, whereas we use it to prove graduality. The fact that they use essentially the same solution suggests there is a deeper connection between blame and graduality than is currently understood.

**Gradualization.** The Gradualizer [Cimini and Siek 2016, 2017] and Abstracting Gradual Typing (AGT) [Garcia et al. 2016b] both seek to make language design for gradually typed languages more systematic. In doing so they make proving graduality far easier than our proof technique possibly could: it holds by construction. Furthermore, these systems also provide a surface-level syntax for gradual typing and an explanation for gradual type checking, while we do not address these at all. However, the downside of their approaches is that they require a rigid adherence to a predefined language framework. While our gradual cast calculus as presented fits into this framework, many gradually typed languages do not. For instance, Typed Racket, the first gradually typed language ever implemented [Tobin-Hochstadt and Felleisen 2008], is not given an operational semantics in the style of a cast calculus, but rather is given a semantics by translation to an untyped language using contracts. We could prove the graduality of such a system by adapting our logical relation to an untyped setting.

We hope in the future to explore the connections between the above frameworks and our analysis of dynamism as embedding-projection pairs. We conjecture that both Gradualizer and AGT by construction produce upcasts and downcasts that satisfy the ep pair properties. The AGT approach in particular has some similarities that stand out: their formulation of type dynamism is based on an embedding-projection pair between static types and sets of gradual types. However, we are not sure if this is a coincidence or has a deeper connection to our approach.

8 CONCLUSION

Graduality is a key property for gradually typed languages as it validates programmer intuition that adding precise types only results in stricter type checking. Graduality is challenging to prove. Moreover, it rests upon the language’s definition of type dynamism but there has been little guidance on defining type dynamism, other than that graduality must hold. We have given a semantics for type dynamism: \( A \subseteq B \) should hold when the casts between \( A, B \) form an embedding-projection pair. This allows for natural proofs of graduality using a logical relation for observational error approximation.

Looking to the future, we would like to make use of our semantic formulation of type dynamism based on ep pairs to design and analyze gradual languages with advanced features such as parametric polymorphism, effect tracking, and mutable state. For parametric polymorphism in particular, we would like to investigate whether our approach justifies any of the type-dynamism definitions previously proposed [Ahmed et al. 2017; Igarashi et al. 2017a], and the possibility of proving both graduality and parametricity theorems with a single logical relation.

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