COHERENT DOUBLE COVERINGS OF VIRTUAL LINK DIAGRAMS

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Abstract. A virtual link diagram is called normal if the associated abstract link diagram is checkerboard colorable, and a virtual link is normal if it has a normal diagram as a representative. Normal virtual links have some properties similar to classical links. In this paper, we introduce a method of converting a virtual link diagram to a normal virtual link diagram. We show that the normal virtual link diagrams obtained by this method from two equivalent virtual link diagrams are equivalent. We discuss the relationship between this method and some invariants of virtual links.

1. Introduction

Virtual links correspond to stable equivalence classes of links in thickened surfaces [1, 5]. A virtual link diagram is called normal if the associated abstract link diagram is checkerboard colorable (§2). A virtual link is called normal or checkerboard colorable if it has a normal diagram as a representative. Every classical link diagram is normal, and hence the set of classical link diagrams is a subset of that of normal virtual link diagrams. The set of normal virtual link diagrams is a subset of that of virtual link diagrams. Jones polynomial is extend to virtual links [7]. It is shown in [3] that Jones polynomial of a normal virtual link has a property that Jones polynomial of a classical link has. For this property Khovanov homology is extended to normal virtual links in a natural way as stated in O. Viro [10]. The author introduced the method of converting a virtual link diagram to a normal virtual link diagram by use of the double covering technique in [4]. In this method, two converting virtual link diagrams obtained from two equivalent virtual link diagrams are related by generalized Riedemeister moves and K-flypes (See [4]).

In this paper, we introduce another method of converting a virtual link diagram to a normal virtual link diagram, which is called the coherent double covering technique. We show that two normal virtual link diagrams obtained from two equivalent virtual link diagrams by our method are equivalent as a virtual link. In Section 4 we discuss the relationship between our method and some invariants of virtual links, as the odd writhe, the invariant of even ordered virtual link which is introduced by H. Miyazawa, K. Wada and Y. Yasuhara[9].

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2. Definitions and main results

A virtual link diagram is a generically immersed, closed and oriented 1-manifold in $\mathbb{R}^2$ with information of positive, negative or virtual crossing, on each double point. A virtual crossing is an encircled double point without over-under information [7]. A virtual link is an equivalence class of virtual link diagrams under Reidemeister moves and virtual Reidemeister moves depicted in Figure 1. We call Reidemeister moves and virtual Reidemeister moves generalized Reidemeister moves.

\[ \begin{array}{ccc}
\text{I} & \text{II} & \text{III} \\
\text{Reidemeister moves} & & \\
\text{I} & \text{II} & \text{III} & \text{IV} \\
\text{Virtual Reidemeister moves}
\end{array} \]

Figure 1. Generalized Reidemeister moves

An abstract link diagram (ALD) is a pair $(\Sigma, D)$ of a compact surface $\Sigma$ and a link diagram $D$ on $\Sigma$ such that the underlying 4-valent graph $|D|$ is a deformation retract of $\Sigma$.

We obtain an ALD from a virtual link diagram $D$ by corresponding a diagram to an ALD as in Figure 2 (i) and (ii). Such an ALD is called the ALD associated with $D$. Figure 3 shows a virtual link diagram and the ALD associated with it. For details on abstract link diagrams and their relations to virtual links, refer to [5]. Let $D$ be a virtual

\[ \begin{array}{ccc}
\text{(i)} & \text{(ii)} & \text{(iii)} \
\end{array} \]

Figure 2. The correspondence from a virtual link diagrams to an ALD

\[ \begin{array}{ccc}
\text{(i)} & \text{(ii)} \\
\end{array} \]

Figure 3. A virtual link diagram and an ALD
link diagram and \((\Sigma, D_\Sigma)\) the ALD associated with \(D\). The diagram \(D\) is said to be normal or checkerboard colorable if the regions of \(\Sigma - |D_\Sigma|\) can be colored black and white such that colors of two adjacent regions are different. In Figure 4 we show an example of a normal diagram. (The orientations of the diagrams in Figure 4 are alternate orientations, which are discussed later.) A classical link diagram is normal. A virtual link is said to be normal if it has a normal virtual link diagram. Note that normality is not necessary to be preserved under generalized Reidemeister moves. For example the virtual link diagram in the right of Figure 5 is not normal and is equivalent to the trefoil knot diagram in the left which is normal.

**Figure 4.** A normal twisted link diagram and its associated ALD with a checkerboard coloring

![Figure 4](image)

**Figure 5.** A diagram of a normal virtual link which is not normal

![Figure 5](image)

H. Dye introduced the notion of cut points on a virtual link diagram in [2].

Let \((D, P)\) be a pair of a virtual link diagram \(D\) and a finite set \(P\) of points on edges of \(D\). We obtain an ALD from \((D, P)\) as in Figure 2 (i), (ii) and (iii) and call it the ALD associated with \((D, P)\). See Figure 6 (ii) and (iii). If the ALD associated with \((D, P)\) is normal, then we call the set of points \(P\) a cut system of \(D\) and call each point of \(P\) a cut point. For the virtual link diagram in Figure 6 (i) we show an example of a cut system in Figure 6 (ii) and the ALD associated with it with a checkerboard coloring in Figure 6 (iii).

**Figure 6.** Example of cut points

![Figure 6](image)

A virtual link diagram is said to admit an alternate orientation if it can be given an orientation such that an orientation of an edge switches at each classical crossing as in Figure 7. (Here an edge means an edge of \(|D|\), where endpoints are classical crossings and there might be some virtual crossings on it.) The virtual link diagram in Figure 4 admits
an alternate orientation. It is known that a virtual link diagram is normal if and only if it admits an alternate orientation [6]. Note that a finite set $P$ of points on $D$ is a cut system if and only if $(D, P)$ admits an alternate orientation such that the orientations of edges are as in Figure 7 at each classical crossing of $D$ and each point of $P$ (cf. [4, 6]).

The canonical cut system of a virtual link diagram $D$ is the set of points that is obtained by giving two points in a neighborhood of each virtual crossing of $D$ as in Figure 8 (i).

![Figure 7. Alternate orientation](image)

![Figure 8. The canonical cut system of a virtual link diagram](image)

**Proposition 1 ([4]).** The canonical cut system is a cut system.

Dye introduced cut point moves depicted in Figure 9. Then we have the following.

![Figure 9. Cut point moves](image)

**Theorem 2 ([4]).** For a virtual link diagram $D$, two cut systems of $D$ are related by a sequence of cut point moves I, II and III.

**Corollary 3 ([2], c.f. [4]).** For any virtual link diagram with a cut system, the number of cut points is even.

Let $D$ be a virtual link diagram with a cut system $P$. Assume that $(D, P)$ is on the right of the $y$-axis in the $xy$-plane and all crossings and cut points have distinct $y$-coordinates. Let $(D^*, P^*)$ be a copy of $(D, P)$ on the left of the $y$-axis in the $xy$-plane which is obtained from $(D, P)$ by sliding along the $x$-axis. Let $\{p_1, \ldots, p_k\}$ be the set of cut points of $P$ and for $i \in \{1, \ldots, k\}$, we denote by $p_i^*$ the cut point of $D^*$ corresponding to $p_i$. See Figure 10 (i). For horizontal lines $l_1, \ldots, l_k$ such that $l_i$ contains $p_i$ and the corresponding cut point $p_i^*$ of $D^*$, we replace each part of $D II D^*$ in a neighborhood of $N(l_i)$ for each $i \in \{1, \ldots, k\}$ as in Figure 11. We denote by $\phi(D, P)$ the virtual link diagram obtained this way.
For example, for the virtual link diagram $D$ with the cut system $P$ depicted as in Figure 10 (i), the virtual link diagram $\phi(D, P)$ is as in Figure 10 (ii). Then we have the following.

**Proposition 4.** For a virtual link diagram $D$ with a cut system $P$, $(D, P)$, $\phi(D, P)$ is normal.

*Proof.* Let $D$ be a virtual link diagram $D$ with a cut system $P$. The virtual link diagram $D^*$ with the cut system $P^*$ is a copy of $(D, P)$ as the previous manner as in Figure 12 (i). The ALDs obtained from $D$ and $D^*$ can be colored as in Figure 12 (ii). Then we see that $\phi(D, P)$ is normal as in the right of Figure 12 (ii). $\square$

Then we have

$$\phi : \{\text{virtual link diagrams with cut systems}\} \longrightarrow \{\text{normal virtual link diagrams}\}.$$
Remark 5. Precisely speaking, \( \phi(D, P) \) is well-defined when \((D, P)\) satisfies that all crossings and cut points have distinct \(y\)-coordinates. If we change \((D, P)\) by an isotopy of \(\mathbb{R}^2\), \(\phi(D, P)\) is preserved under an isotopy of \(\mathbb{R}^2\) and virtual Reidemeister moves.

Let \((D, P)\) be a virtual link diagram with a cut system \(P\). We call \(\phi(D, P)\) the converted normal diagram or the coherent double covering diagram of \((D, P)\). Let \(D\) be a virtual link diagram. For the canonical cut system \(P_0\) of \(D\), we denote by \(\phi_0(D)\) the converted normal diagram \(\phi(D, P_0)\). We call \(\phi_0(D)\) the canonical converted normal diagram of \(D\). Then we have

\[ \phi_0 : \{\text{virtual link diagrams}\} \longrightarrow \{\text{normal virtual link diagrams}\}. \]

The following is our main theorem.

**Theorem 6.** Let \((D, P)\) and \((D', P')\) be virtual link diagrams with cut systems. If \(D\) and \(D'\) are equivalent, then the converted normal diagrams \(\phi(D, P)\) and \(\phi(D', P')\) are equivalent. In particular, if \(D\) and \(D'\) are equivalent, then the canonical converted normal diagrams \(\phi_0(D)\) and \(\phi_0(D')\) are equivalent.

### 3. Proof of Theorem 6

Theorem 6 is obtained from Lemmas 7 and 8 stated below.

**Lemma 7.** Let \(D\) be a virtual link diagram. Suppose that \(P\) and \(P'\) are cut systems of \(D\). Then the converted normal diagrams \(\phi(D, P)\) and \(\phi(D, P')\) are equivalent.

**Proof.** Let \(D\) be a virtual link diagram with a cut system \(P\). Suppose that \(P'\) is a cut system of \(D\) obtained from \(P\) by one of cut point moves I, II or III in Figure 9. Then \(\phi(D, P')\) is related to \(\phi(D, P)\) by virtual Reidemeister moves II and III as in Figure 13. Thus \(\phi(D, P)\) and \(\phi(D, P')\) are equivalent. \(\square\)

![Figure 13](attachment:image.png)

**Figure 13.** Converted normal diagrams related by a cut point move

**Lemma 8.** Let \(D_1\) and \(D_2\) be virtual link diagrams. If \(D_1\) and \(D_2\) are equivalent then the canonical converted normal virtual link diagrams \(\phi_0(D_1)\) and \(\phi_0(D_2)\) are equivalent.
Proof. Let $D_1$ be a virtual link diagram with the canonical cut system $P_1$. Suppose that a virtual link diagram $D_2$ is obtained from $D_1$ by one of generalized Reidemeister moves and $P_2$ is the canonical cut system of $D_2$. If $D_2$ is related to $D_1$ by one of Reidemeister moves, then $\phi_0(D_1) = \phi(D_1, P_1)$ and $\phi_0(D_2) = \phi(D_2, P_2)$ are related by two Reidemeister moves. As in Figure 14 (i) (or (ii)), suppose that $D_2$ is related to $D_1$ by a virtual Reidemeister move I (or II) and let $P'_2$ be the cut system obtained from $P_2$ by cut point moves I and II as in the figure. By Lemma 7, $\phi(D_2, P_2)$ and $\phi(D_2, P'_2)$ are equivalent. On the other hand $\phi(D_1, P_1)$ and $\phi(D_2, P'_2)$ are related by two virtual Reidemeister moves I (or II) as in Figure 14 (i) (or (ii)). Thus $\phi(D_1, D_1)$ and $\phi(D_2, D_2)$ are equivalent. As in Figure 14 (iii), suppose that $D_2$ is related to $D_1$ by a virtual Reidemeister move III and let $P'_1$ (or $P'_2$) be the cut system obtained from $P_1$ (or $P_2$) by cut point moves I and II as in the figure. By Lemma 7, $\phi(D_1, P_1)$ (or $\phi(D_2, P_2)$) and $\phi(D_1, P'_1)$ (or $\phi(D_2, P'_2)$) are equivalent. On the other hand, $\phi(D_1, P'_1)$ and $\phi(D_2, P'_2)$ are related by two virtual Reidemeister moves III. As in Figure 14 (iv), suppose that $D_2$ is related to $D_1$ by a virtual Reidemeister move IV and let $P'_1$ (or $P'_2$) be the cut system obtained from $P_1$ (or $P_2$) by cut point moves I and II (or cut point moves I, II and III) as in the figure. By Lemma 7, $\phi(D_1, P_1)$ (or $\phi(D_2, P_2)$) and $\phi(D_1, P'_1)$ (or $\phi(D_2, P'_2)$) are equivalent. On the other hand, as in Figure 14 (v), $\phi(D_1, P'_1)$ and $\phi(D_2, P'_2)$ are equivalent by virtual Reidemeister moves.

In Figure 14, if the orientation of some strings of virtual link diagram $D_i$ are different from those of it, we have the result by a similar argument.

![Virtual Reidemeister moves](image_url)

**Figure 14.** Diagrams related by a virtual Reidemeister move
4. A RELATIONSHIP BETWEEN INVARIANTS AND COHERENT DOUBLE COVERING DIAGRAMS

In this section we discuss a relationship between some invariants and coherent covering diagrams.

4.1. Odd writhe. We give an interpretation of the odd writhe of a virtual knot in terms of the linking number of the converted normal diagram \( \phi(D, P) \). The argument of this subsection is similar to that of Section 4 in [4].

For a 2-component virtual link diagram \( D \), the half of the sum of signs of nonself classical crossings of \( D \) is called the linking number of \( D \). The linking number is an invariant of 2-component virtual link.

We have the following theorems (Theorems 9 and 10).

**Theorem 9.** Let \((D, P)\) be a virtual knot diagram with a cut system. Then \( \phi(D, P) \) is a 2-component virtual link diagram and the linking number of \( \phi(D, P) \) is an invariant of the virtual knot represented by \( D \).

The odd writhe is a numerical invariant of virtual knots [8]. We recall the definition of the odd writhe later.

**Theorem 10.** Let \((D, P)\) be a virtual knot diagram with a cut system. The linking number of \( \phi(D, P) \) is equal to the odd writhe of \( D \).

Let \( D \) be a virtual link diagram. The Gauss diagram of \( D \) is a set of oriented circles which are the preimage of \( D \) with oriented chords each of which corresponds to a classical crossing and its tail (or its head) indicates an overpass (or an underpass) of the classical crossing. Each chord is equipped with a sign of the corresponding classical crossing. The Gauss diagram in Figure 15 (i) is that of the virtual knot diagram in Figure 3 (i). For

![Gauss diagram](image)

**Figure 15.** The Gauss diagrams

a virtual link diagram \( D \) with a cut system \( P \), the Gauss diagram with points, denoted by \( G(D, P) \), of \((D, P)\) is obtained from the Gauss diagram of \( D \) by adding points on arcs which correspond to the points of \( P \). We denote by \( G(D) \) the Gauss diagram of \( D \) and by \( G(P) \) the points of \( G(D, P) \) corresponding to \( P \). Then \( G(D, P) = (G(D), G(P)) \). In Figure 15 (ii) we show the Gauss diagram with points of the virtual link diagram with points in Figure 6 (ii). Let \((D^*, P^*)\) be the virtual link diagram with a cut system which
is a copy of \((D, P)\). The Gauss diagram with points \(G(D^*, P^*)\) is a copy of \(G(D, P)\). The Gauss diagram of \(\phi(D, P)\) is obtained from the Gauss diagrams of \(D \amalg D^*\) by a local replacement around each point \(p \in G(P)\) of the Gauss diagram \(G(D)\) and around the corresponding point \(p^* \in G(P^*)\) of \(G(D^*)\) as in Figure 16. For a virtual knot diagram of \(D\) with a set of points on edges \(P\), we denote the Gauss diagram of \(\phi(D, P)\) by \(G(\phi(D, P))\). The Gauss diagram in Figure 15 (iii) is \(G(\phi(D, P))\) for \((D, P)\) depicted in Figure 16 (ii) by the map \(\phi\). For the Gauss diagram of \((D \amalg D^*, P \amalg P^*)\), suppose that \(p_1, \ldots, p_n\) are points in \(P\) such that the point \(p_{i+1}\) follows the point \(p_i\) along the orientation of \(D\) and the point \(p^*_i\) in \(P^*\) corresponds to the point \(p_i\). In what follows, we denote by the same symbol \(p_i\) (or \(p^*_i\)) for the point \(p_i\) of \(P\) (or \(p^*_i\) of \(P^*\)) and the corresponding point of \(G(P)\) (or \(G(P^*)\)).

Let \(A_i\) (or \(A^*_i\)) be the arc of the Gauss diagram of \(G(D \amalg D^*, P \amalg P^*)\) between two points \(p_i\) and \(p_{i+1}\) (or \(p^*_i\) and \(p^*_{i+1}\)), and the arc between two points \(p_n\) and \(p_1\) (or \(p^*_n\) and \(p^*_1\)) is \(A_n\) (or \(A^*_n\)). Note that \(A_i\) corresponds to \(A^*_i\). We also denote an arc of \(G(\phi(D, P))\) which corresponds to \(A_i\) or \(A^*_i\) of \(G(D \amalg D^*, P \amalg P^*)\) by \(\tilde{A}_i\) or \(\tilde{A}^*_i\), respectively. Here \(\tilde{A}_i\) (or \(\tilde{A}^*_i\)) is the arc in \(G(\phi(D, P))\) which is obtained from \(A_i\) (or \(A^*_i\)) by removing a regular neighborhood of \(p_i\) and \(p_{i+1}\) (or \(p^*_i\) and \(p^*_{i+1}\)).

**Lemma 11.** Let \((D, P)\) be a virtual knot diagram with points \(P\). Suppose \(P = \{p_1, \ldots, p_{2n}\}\) for a positive integer \(n\). Then \(\phi(D, P)\) is a 2-component virtual link diagram \(D_1 \cup D_2\). Furthermore if an arc \(\tilde{A}_i\) is in \(G(\phi(D, P))|_{D_1}\) (or \(G(\phi(D, P))|_{D_2}\)), then \(\tilde{A}_{i+1}\) and \(\tilde{A}^*_i\) are in \(G(\phi(D, P))|_{D_2}\) (or \(G(\phi(D, P))|_{D_1}\)).

**Proof.** We use the induction on \(n\). Suppose that \(n = 1\), i.e. \(D\) is a virtual knot diagram with 2 points \(p_1\) and \(p_2\). The Gauss diagram \(G(\phi(D, P))\) is depicted as in Figure 17, where the bold line and the thin line indicate the different components and we dropped all chords in the figure. In this case \(\phi(D, P)\) is a 2-component virtual link diagram. Two arcs \(\tilde{A}_1\) and \(\tilde{A}^*_2\) are in one component of \(G(\phi(D, P))\), and \(\tilde{A}_2\) and \(\tilde{A}^*_1\) are in the other. Suppose that the statement is hold if the number of points is less than \(2n\). We assume that \(D\) is a virtual knot diagram with \(2n\) points \(p_1, \ldots, p_{2n}\). First, apply the replacement as in Figure 15 to \(2n - 2\) points \(p_1, \ldots, p_{2n-2}\) and \(p^*_1, \ldots, p^*_{2n-2}\). Then we obtain a Gauss diagram \(G\) with 4 points \(p_{2n-1}, p_{2n}\) and \(p^*_{2n-1}, p^*_{2n}\). By the hypothesis, the Gauss diagram \(G\) is depicted as in Figure 15 (i), where two arcs \(A_{2n-1}\) and \(A_{2n}\) (or two points \(p_{2n-1}\) and \(p_{2n}\)) are in one component of \(G\) and two arcs \(A^*_{2n-1}\) and \(A^*_{2n}\) (or two points \(p^*_{2n-1}\) and \(p^*_{2n}\)) are in the other. If an arc \(\tilde{A}_i\) is in one component of \(G\), \(\tilde{A}^*_i\) (or \(\tilde{A}_{i+1}\)) is in the other.
for \( i \neq 2n - 1 \) by the induction hypothesis. By applying the replacement in Figure 16 to two pairs of points \( p_{2n-1} \) and \( p_{2n-1}^* \) and \( p_{2n} \) and \( p_{2n}^* \) of the Gauss diagram \( G \), we have a Gauss diagram as in Figure 18 (ii). Therefore we have the result.

**Proof of Theorem 9.** Let \((D, P)\) be a virtual knot diagram with a cut system. By Lemma 11, \( \phi(D, P) \) is a 2-component virtual link diagram. Let \((D', P')\) be a virtual knot diagram with a cut system such that \( D \) and \( D' \) present the same virtual knot. By Theorem 6, \( \phi(D, P) \) and \( \phi(D', P') \) are equivalent. Hence they have the same linking numbers.

For a virtual knot \( K \) and its diagram \( D \) with the cut system \( P \), we denote the linking number of \( \phi(D, P) \) by \( \text{lk}_N(K) \) or \( \text{lk}_N(D) \). Theorem 10 states that \( \text{lk}_N(D) \) is equal to the odd writhe of \( D \).

It is shown in [8] that the odd writhe is an invariant of virtual knots.

**Proof of Theorem 10.** Let \( D \) be a virtual knot diagram and \( G \) be a Gauss diagram of \( D \). For a classical crossing \( c \), we denote by \( \gamma_c \) the chord of \( G \) corresponding to \( c \). The endpoints of \( \gamma_c \) divides the circle of \( G \) into 2 arcs. We denote the arcs by \( I_c \) and \( I'_c \) where \( I_c \) is the arc which starts from the tail of \( \gamma_c \) and terminates at the head. A classical crossing \( c \) of \( D \) is said to be odd if there are an odd number of endpoints of chords of \( G \) on \( I_c \). The odd writhe of \( D \) is the sum of signs of odd crossings of \( D \). Note that if a virtual knot diagram is normal, all classical crossings are not odd and hence the odd writhe is zero.

It is shown in [8] that the odd writhe is an invariant of virtual knots.
$G(D, P)$ of $(D, P)$ admits an alternate orientation such that one endpoint of each chord is a sink of the orientations and the other is a source. This implies the following condition:

(∗) For any classical crossing $c$ of $(D, P)$, the sum of the number of cut points on the arc $I_c$ and that of endpoints of chords appearing on $I_c$, is even.

Let $A_1, A_2, \ldots$ be the arcs obtained by cutting the circle of $G(D, P)$ along the cut points. We assume that the arc $A_{i+1}$ appears after the arc $A_i$ along the orientation of $D$. We also denote by $\tilde{A}_i$ the arc of $G(\phi(D, P))|_D$ which corresponds to $A_i$. For a classical crossing $c$ of $(D, P)$, the classical crossing corresponding to $c$ in $\phi(D, P)|_D$ is denoted by $\tilde{c}$. Let $c$ be an odd crossing of $(D, P)$. Suppose that one endpoint of $\gamma_c$ is on $A_k$ and the other endpoint is on $A_j$. By definition, there are an odd number of endpoints of chords on $I_c$ in $G(D, P)$. By the condition (∗) above, we see that there are an odd number of cut points on $I_c$ in $G(D, P)$. Then we see that $k$ is not congruent to $j$ modulo 2. By Lemma 11 $\tilde{A}_k$ and $\tilde{A}_j$ in $G(\phi(D, P))|_D$ are in distinct components of $G(\phi(D, P))$. Thus the classical crossing $\tilde{c}$ is a nonself classical crossing of $\phi(D, P)$. By a similar argument, we see that if $c$ is not an odd crossing of $(D, P)$, then $\tilde{c}$ is a self classical crossing of $\phi(D, P)$. □

We show some properties of $\text{lk}_N(D)$. They are also obtained from the properties of the odd writhe.

**Corollary 12 ([8]).** Let $K$ be a normal virtual knot. Then $\text{lk}_N(K)$ is zero.

**Proof.** Let $D_N$ be a normal knot diagram of $K$. The empty set $\emptyset$ is a cut system. The virtual link diagram $\phi(D_N, \emptyset)$ is the disjoint union, $D_N \sqcup D_N^*$ of $D_N$ and $D_N^*$. Thus $\text{lk}_N(K)$ is zero. □

Let $D$ be a virtual knot diagram. The virtual knot diagram obtained from $D$ by switching the over-under information of all classical crossing (or by reflection) is denoted by $D^\sharp$ (or $D^\dagger$).

**Corollary 13 ([8]).** Let $D$ be a virtual knot diagram. If $\text{lk}_N(D)$ is not zero, then $D$ is not equivalent to $D^\sharp$ (or $D^\dagger$).

**Proof.** It is clear that $\text{lk}_N(D^\sharp) = \text{lk}_N(D^\dagger) = -\text{lk}_N(D)$. This implies the result. □

For example, the virtual knot presented by the diagram $D$ in Figure 19 is not normal by Corollary 12 since $\text{lk}_N(D) = -2$. By Corollary 13 $D$ is not equivalent to $D^\sharp$ (or $D^\dagger$).

### 4.2. Linking invariants of even virtual links

H. Miyazawa K. Wada and A. Yasuhara defined an invariant of even virtual links by use of the virtual orientation. We give another definition of their invariant.

Let $D$ be a virtual link diagram. $D$ is said to be even, if on each circle of a Gauss diagram of $D$ there is an even number of endpoints of chords. A virtual knot diagram is even. If a virtual link diagram $D$ is even, any virtual link diagram which is equivalent to $D$ is even. Note that when we go around each component of an even virtual link diagram,
we meet an even number of virtual crossings. A normal virtual link diagram is even, and any a virtual link diagram which presents a normal virtual link is even.

Let $D$ be an ordered unoriented even $r$-component virtual link diagram such that $D = D^1 \cup \cdots \cup D^r$ where $D^i$ is a component of $D$. The \textit{virtual orientation} is an orientation of $D$ such that they switch an opposite direction at each virtual crossing of $D$ as in Figure 20. (The orientation is not switched at classical crossings.)

![Figure 20. Virtual orientation](image)

We give $D$ a virtual orientation $O_v(D)$. The sum of signs of classical crossings between $D^i$ and $D^j$ whose overpass of $D^i$ is denoted by $\lambda_D(i, j)$ under the virtual orientation $O_v(D)$.

\textbf{Theorem 14 ([9])}. $|\lambda_D(i, j)|$ is an invariant of ordered unoriented even virtual links.

\textbf{Proposition 15}. The number of cut points of each component of an even virtual link diagram with a cut system is even.

\textit{Proof}. Let $(D, P_0)$ be even virtual link diagram $D$ with the canonical cut system $P_0$. Each component of $D$ has an even number of cut points, since we meet an even number virtual crossings when we go around each component of it. Any cut system of $D$ is obtained from $P_0$ by cut point moves I, II and III. Then we have the same result in another cut system of $D$ since by a cut point move II or III changes the number of cut points on a component of $D$ by $\pm 2$ or $\pm 4$. \hfill \Box

Let $D$ be an even $r$-component virtual link diagram such that $D = D^1 \cup \cdots \cup D^r$ where $D^i$ is a component of $D$. Suppose that $P$ is a cut system of $D$. A \textit{cut orientation} is an orientation of $(D, P)$ such that they switch an opposite direction at each cut point of $D$ as in Figure 21. We denote it $O_c(D, P)$. (For a cut orientation, the orientation is not switched at classical crossings and virtual crossings.)
Let \((D, P)\) be an even virtual link diagram \(D\) with a cut system \(P\) such that \(D = D^1 \cup \cdots \cup D^r\) where \(D^i\) is a component of \(D\). We give \(D\) a cut orientation \(O_c(D, P)\). The sum of signs of classical crossings between \(D^i\) and \(D^j\) whose overpass of \(D^i\) is denoted by \(\nu(D, P)(i, j)\) under the cut orientation \(O_c(D, P)\).

**Proposition 16.** For two cut systems \(P\) and \(P'\) of \(D\), let \(O_c(D, P)\) (or \(O_c(D, P')\)) be a cut orientation of \((D, P)\) (or \((D, P')\)). Then we have \(|\nu(D, P)(i, j)| = |\nu(D, P')(i, j)|\).

**Proof.** Suppose that \(P'\) is obtained from \(P\) by one of cut point moves I, II and III to a component \(D^i\) (or two components \(D^i\) and \(D^j\)) of \(D\) as in Figure 22(i), (ii) (or (iii)). The cases of a cut point moves I (or II), we give a cut orientation \(O_c(D, P)\) to \((D, P)\) as in Figure 22(i) (or (ii)) and \(O_c(D, P')\) to \((D, P')\) as in Figure 22(i) (or (ii)) (a) or (b). In both cases, if a sign of each classical crossing \(c\) between \(D^i\) and \(D^j\) of \(D\) with respect to \(O_c(D, P)\) is \(\epsilon\), the sign of \(c\) with respect to \(O_c(D, P')\) is \(\pm \epsilon\). The case of a cut point moves III, we give a cut orientation \(O_c(D, P)\) to \((D, P)\) as in Figure 22(iii) and \(O_c(D, P')\) to \((D, P')\) as in Figure 22(iii) (a), (b), (c) or (d). In each case, if a sign of each classical crossing \(c\) between \(D^i\) and \(D^j\) of \(D\) with respect to \(O_c(D, P)\) is \(\epsilon\), the sign of \(c\) with respect to \(O_c(D, P')\) is \(\pm \epsilon\). Thus we have the result.

\[\begin{array}{c}
\text{(i)} \\
\text{(ii)} \\
\text{(iii)}
\end{array}\]

**Figure 22.** Cut orientation and cut moves

We denote \(|\nu(D, P)(i, j)|\) by \(|\nu_D(i, j)|\).

**Theorem 17.** \(|\nu_D(i, j)|\) coincides to \(|\lambda_D(i, j)|\). Thus \(|\nu_D(i, j)|\) is an invariant of ordered unoriented virtual link.

**Proof.** Let \(D\) be an ordered unoriented even \(r\)-component virtual link diagram such that \(D = D^1 \cup \cdots \cup D^r\) where \(D^i\) is a component of \(D\). We give a virtual orientation of \(D\), \(O_v(D)\) to \(D\). Let \(P_0\) be the canonical cut system of \(D\). There is a cut orientation of \((D, P_0)\) whose orientation of each arc of \(D\) corresponds to that of \(D\) in \(O_c(D)\) as in Figure 23. □

\[\text{□}\]
Figure 23. Virtual orientation and cut orientation

We give an interpretation of the linking invariant $|\lambda_D(i, j)| (= |\nu_D(i, j)|)$ in terms of our covering method. Let $(D, P)$ be an even $r$-component virtual link diagram with a cut system $P$ such that $D = D^1 \cup \cdots \cup D^r$ where $D^i$ is a component of $D$. We give an cut orientation $\mathcal{O}_c(D, P)$ to $D$. Let $(D^*, P^*)$ be a copy of $(D, P)$ and $\mathcal{O}_c(D^*, P^*)$ be the cut orientation of $(D^*, P^*)$ such that an orientation of each arc of $(D^*, P^*)$ is different from the corresponding arc of $(D, P)$ in $\mathcal{O}_c(D, P)$. We take the coherent double covering diagram of $(D, P)$ in the same manner as in Figure 11. We call this the coherent double covering diagram with a cut orientation $\mathcal{O}_c(D, P)$ denoted by $\phi_{\mathcal{O}_c(D, P)}(D, P)$. The virtual link diagram in Figure 24 (ii) is the coherent double covering diagram of the virtual link diagram in Figure 24 (i) with a cut orientation $\mathcal{O}_c(D, P)$. Note that the orientation of the coherent double covering diagram derived from cut orientations $\mathcal{O}_c(D, P)$ and $\mathcal{O}_c(D^*, P^*)$.

Let $(D, P)$ be an ordered unoriented even $r$-component virtual link diagram with a cut system $P$ such that $D = D^1 \cup \cdots \cup D^r$ where $D^i$ is a component of $D$. We denote the subset of the cut system $P$ whose points are on arcs of $D^i$ by $P^i$. A component of the Gauss diagram of $(D, P)$ corresponding a component $D^i$ with $P^i$ by $G(D^i, P^i)$. For the Gauss diagram of $(D \coprod D^*, P \coprod P^*)$, suppose that $p_{1i}, \ldots, p_{2ni}$ (or $(p_{1i}^*)^*, \ldots, (p_{2ni}^*)^*$) are points in $P^i$ such that the point $p_{j+1}^i$ follows the point $p_j^i$ along $D^i$ and the point $(p_j^i)^*$ in $(P^i)^*$ corresponds to the point $p_j^*$.

In what follows, we denote by the same symbol $p_j^i$ (or $(p_j^i)^*$) for the point $p_j^i$ of $P^i$ (or $(p_j^i)^*$ of $(P^i)^*$) and the corresponding point of $G(P)$ (or $G(P^*)$).

Let $A_j^i$ (or $(A_j^i)^*$) be the arc of the Gauss diagram of $G(D \coprod D^*, P \coprod P^*)$ between two points $p_j^i$ and $p_{j+1}^i$ (or $(p_j^i)^*$ and $(p_{j+1}^i)^*$), and the arc between two points $p_{2ni}^i$ and $p_{1i}^i$ (or $(p_{n_i}^i)^*$ and $(p_1^i)^*$) is $A_{2ni}^i$ (or $(A_{2ni}^i)^*$). Note that $A_j^i$ corresponds to $(A_j^i)^*$. We also denote
Lemma 18. Let \((D, P)\) be an even \(r\)-component virtual link diagram with a cut system \(P\) such that \(D = D^1 \cup \cdots \cup D^r\) where \(D^i\) is a component of \(D\). Then \(\phi(D, P)\) is a \(2r\)-component virtual link diagram such that \(\phi(D, P)|_{D^i}\) is a 2-component link diagram \(D^i_1 \cup D^i_2\). Furthermore if an arc \(A^i_j\) is in \(G(\phi(D, P))|_{D^i_1}\) (or \(G(\phi(D, P))|_{D^i_2}\)), then \(A^i_{j+1}\) and \((A^i_j)^*\) are in \(G(\phi(D, P))|_{D^i_1}\) (or \(G(\phi(D, P))|_{D^i_2}\)).

Proof. The proof is obtained by a similar argument as in Lemma 11. \(\square\)

Let \((D, P)\) be an ordered unoriented even \(r\)-component virtual link diagram with a cut system \(P\) such that \(D = D^1 \cup \cdots \cup D^r\) where \(D^i\) is a component of \(D\). We denote the subset of the cut system \(P\) whose points are on arcs of \(D^i\) by \(P^i\). For a classical crossing \(c^{i,j}\) between \(D^i\) and \(D^j\) of \((D, P)\), the classical crossing corresponding to \(c^{i,j}\) in \(\phi_{O_c}(D, P)(D, P)\) between \(D^i_k\) and \(D^j_l\) is denoted by \(\widetilde{c}^{i,j}_{k,l}\) \((k, l \in \{1, 2\})\).

Let \(c^{i,j}\) between \(D^i\) and \(D^j\) of \((D, P)\) such that an arc of \(D^i\) is an overpass of \(c^{i,j}\). Then the overpass of a classical crossing \(\widetilde{c}^{i,j}_{k_1,l_1}\) in \(\phi_{O_c}(D, P)(D, P)\) corresponding \(c^{i,j}\), is in \(D^i_{k_1}\). For the other classical crossing \(\widetilde{c}^{i,j}_{k_2,l_2}\) corresponding \(c^{i,j}\), its overpass is in \(D^i_{k_2}\), where \(k_2 \neq k_1\) and \(l_2 \neq l_1\) by Lemma 18. Then we have the following theorem.

Theorem 19. Let \((D, P)\) be an ordered unoriented even \(r\)-component ordered unoriented virtual link diagram with a cut system \(P\) such that \(D = D^1 \cup \cdots \cup D^r\) where \(D^i\) is a component of \(D\). For a cut orientation \(O_c(D, P)\) of \((D, P)\), the absolute value of the sum of the classical crossings between \(D^i_k\) and \(D^j_l\) whose overpasses are in \(D^i_k\) in the coherent double covering diagram \(\phi_{O_c}(D, P)(D, P)\) with a cut orientation \(O_c(D, P)\) of \((D, P)\), coincides to \(|\nu_D(i, j)|\).

Proof. Let \(c^{i,j}\) be a classical crossing of \(D\) between \(D^i\) and \(D^j\) whose overpass is in \(D^i\). One of two classical crossings of \(\phi_{O_c}(D, P)(D, P)\) corresponding to \(c^{i,j}\) is between \(D^i_k\) and \(D^j_{l_1} \cup D^j_{l_2}\), and its overpass is in \(D^i_k\). The sign of such a classical crossing is equal to that of \(c^{i,j}\). \(\square\)

For an ordered unoriented even virtual link diagram \(D = D^1 \cup D^2\) with a virtual orientation in Figure 25 (i), \(|\lambda_D(1, 2)| = |\lambda_D(2, 1)| = 2\). In Figure 25 (ii), the diagram \(D\) with a cut system \(P\) and a cut orientation \(O_c(D, P)\) and an coherent double covering diagram of \((D, P)\) with a cut orientation \(O_c(D, P)\), \(\phi_{O_c}(D, P)(D, P) = D^1_1 \cup D^1_2 \cup D^2_1 \cup D^2_2\) are depicted, where \(D^i\) correspond to a component \(D^i\) of \(D\). The absolute value of the sum of the classical crossings between \(D^i_1\) (or \(D^i_2\)) and \(D^j_1 \cup D^j_2\) whose overpasses belong to \(D^i_1\) (or \(D^i_2\)) coincides to \(|\lambda_D(i, j)|\) for \(i \neq j\).

Let \((D, P)\) be an ordered oriented even \(r\)-component virtual link diagram with a cut system \(P\) such that \(D = D^1 \cup \cdots \cup D^r\) where \(D^i\) is a component of \(D\). For a cut system \(P\)
Figure 25. Example: Linking invariant and coherent double covering of a virtual link diagram with a cut orientation

of $D$, we denote a 2-component sub-diagram $\phi(D, P)_{|D_i}$ of the coherent double covering diagram $\phi(D, P)$ which corresponds to $D^i$, by $D^i_1 \cup D^i_2$. Let $\tilde{lk}(D^i_k, D^i_l)$ be the linking number between $D^i_k$ and $D^i_l$ for $k, l = 1, 2$.

The set $\{\tilde{lk}(D^i_1, D^j_1), \tilde{lk}(D^i_1, D^j_2)\}$ is denoted by $Q_{ij}(D)(i \neq j)$. For a virtual knot diagram $D$, $\tilde{lk}_N(D)$ is $\tilde{lk}(D_1, D_2)$, where $\phi(D, P) = D_1 \cup D_2$.

Corollary 20. The set $Q_{ij}(D)$ is an invariant of ordered oriented even virtual links. Furthermore $Q_{ij}(D)$ is equal to the set $\{\tilde{lk}(D^i_1, D^j_1), \tilde{lk}(D^i_2, D^j_1)\}$, $\{\tilde{lk}(D^i_1, D^j_2), \tilde{lk}(D^i_2, D^j_2)\}$. The linking number $\tilde{lk}(D^i_1, D^j_2)$ is an invariant of an ordered oriented even virtual link.

Proof. Let $(D, P)$ be an ordered oriented even $r$-component virtual link diagram with a cut system $P$ such that $D = D^1 \cup \cdots \cup D^r$ where $D^i$ is a component of $D$. Let $c^{i,j}$ be a classical crossing between $D^i$ and $D^j$ of $(D, P)$. If one classical crossing $c^{i,j}_{k_1, l_1}$ in $\phi(D, P)$ corresponding $c^{i,j}$ is between $D^i_{k_1}$ and $D^j_{l_1}$ and, then the other is a classical crossing $c^{i,j}_{k_2, l_2}$ between $D^i_{k_2}$ and $D^j_{l_2}$ for $k_1 \neq k_2$ and $l_1 \neq l_2$. The signs of two classical crossings $c^{i,j}_{k_1, l_1}$ and $c^{i,j}_{k_2, l_2}$ are the same. Then we have $\tilde{lk}(D^i_1, D^j_1) = \tilde{lk}(D^i_2, D^j_2)$ and $\tilde{lk}(D^i_1, D^j_2) = \tilde{lk}(D^i_2, D^j_1)$.

Let an ordered oriented even virtual link diagrams $D = D^1 \cup D^2$ with a cut system $(P = \emptyset)$ and $D' = D^{1'} \cup D^{2'}$ with a cut system $P'$ be as in Figure 26 (i). In Figure 26 (ii) we show a coherent double covering diagram $\phi(D', P') = D^{1'}_1 \cup D^{1'}_2 \cup D^{2'}_1 \cup D^{2'}_2$. We have $\tilde{lk}(D^1_1, D^2_1) = 0$ and $\tilde{lk}(D^{1'}_1, D^{2'}_1) = 2$. So $D$ is not equivalent to $D'$.

Figure 26. Example: Linking number of coherent double covering of a virtual link diagram
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