On the near-optimality of one-shot classical communication over quantum channels

Anurag Anshu∗ Rahul Jain† Naqueeb Ahmad Warsi‡

Abstract

We study the problem of transmission of classical messages through a quantum channel in several network scenarios in the one-shot setting. We consider both the entanglement assisted and unassisted cases for the point to point quantum channel, quantum multiple-access channel, quantum channel with state and the quantum broadcast channel. We show that it is possible to near-optimally characterize the amount of communication that can be transmitted in these scenarios, using the position-based decoding strategy introduced in a prior work [1]. In the process, we provide a short and elementary proof of the converse for entanglement-assisted quantum channel coding in terms of the quantum hypothesis testing divergence (obtained earlier in [2]). Our proof has the additional utility that it naturally extends to various network scenarios mentioned above. Furthermore, none of our achievability results require a simultaneous decoding strategy, existence of which is an important open question in quantum Shannon theory.

1 Introduction

Understanding the limits of communication through various models of channels is a central aspect of classical information theory. Some landmark results in this direction are the models of point to point channel [3], multiple access channel [4, 5], channel with a state [6] and broadcast channel [7]. The diversity of scenarios in which information theory can be applied has led to various settings in which the problem of channel coding is studied. Below we discuss two settings relevant to this work.

• Asymptotic and i.i.d. setting: Here, the senders are allowed to use the channel multiple times in a memoryless fashion and the goal is to obtain bounds on the rate of transmission for an arbitrarily large number of channel uses, as the error is made to go to zero. It is highly desirable that the resulting bounds are single letter, that is, they do not require unbounded optimization in their computation. Without this restriction, it would be possible to obtain tight characterization of the capacity of all of the aforementioned channel settings [8, Section 4.3].

• One-shot setting: Here, the senders are allowed to use the channel only once, which can arise in many practical scenarios. It is desirable to obtain bounds on the amount of communication which are near optimal. That is, a communication cost (or cost region for multiple messages) $R(\varepsilon)$ may be obtained which is a converse cost for any protocol that makes an error $\varepsilon$ (in terms of the probability of incorrectly decoding the messages) and there exists a protocol that achieves the cost $R(\varepsilon')$ (where $\varepsilon'$ is of the order of $\varepsilon$) up to some additive factors.

Quantum information theory generalizes the models of classical channels in various ways, by introducing channels that can take quantum inputs and produce quantum outputs or by allowing new resources such as quantum entanglement. Several works have studied the problem of transmission of quantum information through a point to point quantum channel in the asymptotic and i.i.d. setting, both in the entanglement assisted case [9] and the entanglement unassisted cases ([10, 11] for the transmission of classical information and [12, 13, 14] for the transmission of quantum information). In the entanglement assisted case, the transmission of classical information is equivalent to the transmission of quantum information up to a factor of 2, due to the duality between quantum teleportation [15] and

∗Center for Quantum Technologies, National University of Singapore, Singapore. a0109169@u.nus.edu
†Center for Quantum Technologies, National University of Singapore and MajuLab, UMI 3654, Singapore. rahul@comp.nus.edu.sg
‡Center for Quantum Technologies, National University of Singapore and School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore and IIITD, Delhi. warsi.naqueeb@gmail.com
super-dense coding [16]. In the entanglement unassisted case, the duality is lost and we have two different aforementioned scenarios for the transmission of classical information and quantum information. In this work, we shall focus on the transmission of classical information for both of the entanglement assisted and unassisted cases.

Several quantum network scenarios have also been studied in the asymptotic and i.i.d. setting, such as the quantum multiple access channel [17][18][19][20][21], the quantum broadcast channel [22][23][24][25] and quantum channel with state [26][27]. In most of these cases (both the entanglement assisted and unassisted), a single letter characterization is not known. Some exceptions, where a single letter characterization is known, are the entanglement assisted point to point quantum channel [9], the classical-quantum multiple access channel [17] and the entanglement assisted quantum channel with state [26][27].

Classical communication over the point to point channel has been studied in several works in the one-shot setting [27][28][29][30][31][1]. These results have been extended to the quantum network scenarios in the works [26][23][27][1]. However, in all the cases except for the point to point channel (both entanglement assisted [1] and entanglement unassisted [29]), a near-optimal one-shot characterization is not known. An interesting variant, where the communicating parties are equipped with arbitrary non-local correlations, has also been considered in the works [32][33], providing improvements to the entanglement assisted case (which is a weaker non-local resource).

In this work, we provide a near-optimal one-shot characterization for many quantum network scenarios, using the position-based decoding strategy introduced in [1]. In contrast with [1], we do not require the convex-split technique [34] for our achievability results. Our results, as obtained in Sections 3 and 4, are summarized below.

- **Point to point quantum channel**: A converse bound for the entanglement assisted case has been given in [2] and a nearly matching achievability result has been obtained recently in [1]. We provide a short proof of the converse in [2]. This also considerably simplifies an alternative proof given in [1] arXiv version 2], which was inspired by the analogous asymptotic and i.i.d. result [35 Section 21.5] and used a one-shot analogue of the chain rule for the conditional quantum mutual information. We are able to avoid the use of any such chain rule in our converse proof, by considering the quantum hypothesis testing divergence between appropriate quantum states. Our proof technique has the utility that it easily extends to various network scenarios. As an application, we recover the one-way case of [36 Theorem 1.1].

  For the entanglement unassisted case, we provide a similar characterization to that given in [29]. Our characterization has the property that it is of similar form for other entanglement unassisted network scenarios.

- **Quantum channel with state**: We provide a near optimal characterization for this channel in the one-shot setting, with a tight dependence on the error of decoding. The optimization involved in our bound is comparable to the optimization involved in earlier known results [26][27][1]. It is not clear if our bound attains a single letter expression in the asymptotic and i.i.d. setting, in contrast with the asymptotic and i.i.d. form of the bounds given in [26][27][1]. On the other hand, we show as a corollary that the achievability bound given in [1] for the quantum channel with state is near optimal in the one-shot setting. Same feature is not known for the one-shot achievability bounds in [26][27].

  We also provide near-optimal bounds for this channel for the entanglement unassisted case, with the property that the registers involved in our bounds have dimension comparable to that of the input and output registers of the channel.

- **Quantum broadcast channel**: In a similar fashion to the quantum channel with state, we provide a near optimal characterization for this channel in the one-shot setting (discussing the case of one sender and two receivers), with a tight dependence on the error of decoding. The optimization involved in our bound is comparable to the optimization involved in earlier known results [23][27][1]. We note that the asymptotic and i.i.d. analogue of our converse result is implicit in [23 Theorem 3]. It is not clear if our bound attains a single letter expression in the asymptotic and i.i.d. setting, which is also the case for the asymptotic and i.i.d. form of the bounds given in [23][27][1]. On the other hand, we show as a corollary that the achievability bound given in [1] for the quantum channel with state is near optimal in the one-shot setting. Same feature is not known for the one-shot achievability bound in [23][27].

  We also provide near-optimal bounds for this channel for the entanglement unassisted case, with the property that the registers involved in our bounds have dimension comparable to that of the input and output registers of the channel.

- **Quantum multiple access channel**: We provide a new converse bound for the multiple access channel with two senders and one receiver (which can easily be extended to the case of more than two senders). We show how to achieve this bound in two different ways (both of which can easily be extended to the case of more than two senders). The first way uses the pretty good measurement technique of Hayashi and Nagaoka [37] and
has a tight dependence on the error of decoding one of the messages (at the cost of quadratic loss on the error of decoding the other message). The second way uses the sequential decoding strategy of Sen [38] (with the quantitatively improved version of [39]; see also the related works [40] [41] and the recent improvement [42]) and has a tight dependence on the error of decoding both messages up to multiplicative constants. As far as we know, this is a first instance where the sequentially decoding strategy gives a better dependence on the overall error of decoding the messages in comparison to the pretty good measurement. Furthermore, our achievability results do not need a simultaneous decoding strategy [20] [43] [44]. It is not clear if this bound leads to a single letter characterization in the asymptotic and i.i.d. setting, a situation similar to the other known bound for the entanglement assisted quantum multiple access channel in the asymptotic and i.i.d. setting [18].

We also provide near-optimal bounds for the entanglement unassisted case, with the property that the registers involved in our bounds have dimension comparable to that of the input and output registers of the channel.

2 Preliminaries

In this section we set our notations, make the definitions and state the facts that we will need later for our proofs.

Consider a finite dimensional Hilbert space \( \mathcal{H} \) endowed with an inner product \( \langle \cdot, \cdot \rangle \). The \( \ell_1 \) norm of an operator \( X \) on \( \mathcal{H} \) is \( \|X\|_1 := \text{Tr} \sqrt{X^\dagger X} \) and \( \ell_2 \) norm is \( \|X\|_2 := \sqrt{\text{Tr} X^\dagger X} \). For hermitian operators \( X, X' \), the notation \( X \succeq X' \) implies that \( X' - X \) is a positive semi-definite operator. A quantum state (or a density matrix or a state) is a positive semi-definite matrix on \( \mathcal{H} \) with trace equal to \( 1 \). It is called pure if and only if its rank is \( 1 \). A sub-normalized state is a positive semi-definite matrix on \( \mathcal{H} \) with trace less than or equal to \( 1 \). With some abuse of notation, we use \( \psi \) to represent the state and also the density matrix \( |\psi\rangle\langle\psi| \), associated with \( |\psi\rangle \).

A quantum register \( A \) is associated with some Hilbert space \( \mathcal{H}_A \). Define \( |A\rangle := \text{dim}(\mathcal{H}_A) \). Let \( \mathcal{L}(A) \) represent the set of all linear operators acting on the set of quantum states acting on the Hilbert space \( \mathcal{H}_A \). We denote by \( \mathcal{D}(A) \), the set of quantum states on the Hilbert space \( \mathcal{H}_A \). State \( \rho \) with subscript \( A \) indicates \( \rho_A \in \mathcal{D}(A) \). If two registers \( A, B \) are associated with the same Hilbert space, we shall represent the relation by \( A \equiv B \). Composition of two registers \( A \) and \( B \), denoted \( AB \), is associated with Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \). For two quantum states \( \rho, \sigma \in \mathcal{D}(A) \) and \( \sigma \in \mathcal{D}(B) \), \( \rho \otimes \sigma \in \mathcal{D}(AB) \) represents the tensor product (Kronecker product) of \( \rho \) and \( \sigma \). The identity operator on \( \mathcal{H}_A \) (and associated register \( A \)) is denoted \( I_A \).

Let \( \rho_{AB} \in \mathcal{D}(AB) \). We define
\[
\rho_B := \text{Tr}_A(\rho_{AB}) := \sum_i (|i\rangle \otimes I_B)\rho_{AB}(|i\rangle \otimes I_B),
\]
where \( \{|i\rangle\} \) is an orthonormal basis for the Hilbert space \( \mathcal{H}_A \). The state \( \rho_B \in \mathcal{D}(B) \) is referred to as the marginal state of \( \rho_{AB} \). Unless otherwise stated, a missing register from subscript in a state will represent partial trace over that register. Given a \( \rho_A \in \mathcal{D}(A) \), a purification of \( \rho_A \) is a pure state \( \rho_{AB} \in \mathcal{D}(AB) \) such that \( \text{Tr}_B(\rho_{AB}) = \rho_A \). A purification of a quantum state is not unique.

A quantum map \( \mathcal{E} : \mathcal{L}(A) \to \mathcal{L}(B) \) is a completely positive and trace preserving (CPTP) linear map (mapping states in \( \mathcal{D}(A) \) to states in \( \mathcal{D}(B) \)). A unitary operator \( U_A : \mathcal{H}_A \to \mathcal{H}_A \) is such that \( U_A^\dagger U_A = U_A U_A^\dagger = I_A \). An isometry \( V : \mathcal{H}_A \to \mathcal{H}_B \) is such that \( V^\dagger V = I_A \) and \( V V^\dagger = \Pi_B \), where \( \Pi_B \) is a projection on \( \mathcal{H}_B \). The set of all unitary operations on register \( A \) is denoted by \( \mathcal{U}(A) \).

We shall consider the following information theoretic quantities. Let \( \varepsilon \in (0, 1) \).

1. **Fidelity** ([45], see also [46]). For \( \rho_A, \sigma_A \in \mathcal{D}(A) \),
\[
F(\rho_A, \sigma_A) \overset{\text{def}}{=} \|\sqrt{\rho_A}\sqrt{\sigma_A}\|_1.
\]

2. **Purified distance** ([47]). For \( \rho_A, \sigma_A \in \mathcal{D}(A) \),
\[
P(\rho_A, \sigma_A) = \sqrt{1 - F^2(\rho_A, \sigma_A)}.
\]

3. **\( \varepsilon \)-ball**. For \( \rho_A \in \mathcal{D}(A) \),
\[
B^\varepsilon(\rho_A) \overset{\text{def}}{=} \{\rho'_A \in \mathcal{D}(A) \mid P(\rho_A, \rho'_A) \leq \varepsilon\}.
\]
4. **Relative entropy** ([48]). For \( \rho_A, \sigma_A \in \mathcal{D}(A) \) such that \( \text{supp}(\rho_A) \subseteq \text{supp}(\sigma_A) \),
\[
D(\rho_A\|\sigma_A) \overset{\text{def}}{=} \text{Tr}(\rho_A \log \rho_A) - \text{Tr}(\rho_A \log \sigma_A).
\]

5. **Smooth quantum hypothesis testing divergence** ([28], see also [37]). For \( \rho_A, \sigma_A \in \mathcal{D}(A) \) and \( \varepsilon \in (0, 1) \),
\[
D_H^\varepsilon(\rho_A\|\sigma_A) \overset{\text{def}}{=} \sup_{\Lambda \leq I_A, \text{Tr}(\Lambda \rho_A) \geq 1 - \varepsilon} \log \left( \frac{1}{\text{Tr}(\Lambda \sigma_A)} \right).
\]

6. **Max-relative entropy** ([49]). For \( \rho_A, \sigma_A \in \mathcal{D}(A) \) such that \( \text{supp}(\rho_A) \subseteq \text{supp}(\sigma_A) \),
\[
D_{\text{max}}(\rho_A\|\sigma_A) \overset{\text{def}}{=} \inf\{\lambda \in \mathbb{R} : \rho_A \leq 2^\lambda \sigma_A\}.
\]

We will use the following facts.

**Fact 1** (Triangle inequality for purified distance, [47, 50]). For states \( \rho_A, \sigma_A, \tau_A \in \mathcal{D}(A) \),
\[
P(\rho_A, \sigma_A) \leq P(\rho_A, \tau_A) + P(\tau_A, \sigma_A).
\]

**Fact 2** (Monotonicity under quantum operations, [51, 52]). For quantum states \( \rho, \sigma \in \mathcal{D}(A) \), and quantum operation \( \mathcal{E}(\cdot) : \mathcal{L}(A) \to \mathcal{L}(B) \), it holds that
\[
F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma) \quad \text{and} \quad D_H^\varepsilon(\rho\|\sigma) \geq D_H^\varepsilon(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)).
\]

**Fact 3** ([1]). Let \( \rho, \sigma \) be quantum states and \( 0 \leq \Pi \leq I \) be an operator. Then
\[
|\sqrt{\text{Tr}(\Pi \sigma)} - \sqrt{\text{Tr}(\Pi \rho)}| \leq P(\rho, \sigma).
\]

**Fact 4** (Gentle measurement lemma, [53, 54]). Let \( \rho \) be a quantum state and \( 0 < A < I \) be an operator. Then
\[
F(\rho, \frac{A \rho A}{\text{Tr}(A^2 \rho)}) \geq \sqrt{\text{Tr}(A^2 \rho)}.
\]

Following fact is analogous to the gentle measurement lemma (Fact 4).

**Fact 5.** Consider a quantum state \(|\rho\rangle\) and a measurement \(\{A_i\}_i\) such that \(A_i \geq 0\). Let \(\rho' = \sum_i A_i \langle \rho | A_i\rangle\). Then
\[
F^2(\rho, \rho') = \sum_i \text{Tr}(A_i \rho)^2 \geq \sum_i \text{Tr}(A_i^2 \rho)^2.
\]

**Fact 6** (Hayashi-Nagaoka inequality, [37]). Let \(0 \leq S \leq T\) be positive semi-definite operators and \(c > 0\). Then
\[
I - (S + T)^{-\frac{1}{2}} S (S + T)^{-\frac{1}{2}} \leq (1 + c)(I - S) + (2 + c + \frac{1}{c}) T.
\]

**Fact 7** ([29]). Let \(\varepsilon \in (0, 1)\) and \(\rho_A, \sigma_A \in \mathcal{D}(A)\). It holds that
\[
D_H^\varepsilon(\rho_A\|\sigma_A) \leq \frac{D(\rho_A\|\sigma_A)}{1 - \varepsilon}.
\]

**Fact 8** (Sequential measurement, [38, 39]). Let \(\rho\) be a quantum state and \(\Pi_1, \Pi_2, \ldots, \Pi_k\) be projectors. Let \(\Pi'_i = I - \Pi_i\). Then
\[
\text{Tr}(\Pi'_{k-1} \ldots \Pi'_1 \rho \Pi'_1 \Pi'_2 \ldots \Pi'_k) \geq 1 - 4 \sum_i \text{Tr}(\Pi_i \rho).
\]

**Fact 9.** Let \(\varepsilon \in (0, 1)\). Let \(\rho_{M,M'}\) be a quantum state such that \(\rho_M = \frac{1}{|M|} (\sum_m |m\rangle \langle m| \otimes |m\rangle |m\rangle \langle m|) \rho_{M,M'} \geq 1 - \varepsilon\). Then for any quantum state \(\sigma_{M'}\),
\[
D_H^\varepsilon(\rho_{M,M'}\|\rho_M \otimes \sigma_{M'}) \geq \log |M|.
\]

**Proof.** Setting \(\Lambda_{M,M'} = \sum_m |m\rangle \langle m| \otimes |m\rangle |m\rangle \langle m| \rho_{M,M'}\), consider
\[
\text{Tr}(\Lambda_{M,M'} \rho_M \otimes \sigma_{M'}) = \sum_m \langle m| \rho_M |m\rangle \langle m| \sigma_{M'} |m\rangle = \frac{1}{|M|} \sum_m \langle m| \sigma_{M'} |m\rangle = \frac{1}{|M|}.
\]
Further, \(\text{Tr}(\Lambda_{M,M'} \rho_{M,M'}) \geq 1 - \varepsilon\). The bound now follows from the definition of \(D_H^\varepsilon(\rho_{M,M'}\|\rho_M \otimes \sigma_{M'})\).\qed
Fact 10 (Neumark’s theorem, [55]). For any POVM \( \{M_i\}_{i \in \mathcal{I}} \) acting on a system \( S \), there exists a unitary \( U_{SP} \) and an orthonormal basis \( \{|i\>_p\}_{i \in \mathcal{I}} \) such that for all quantum states \( \rho_S \), we have
\[
\text{Tr} \left[ U_{SP}^\dagger (I_S \otimes |i\>_p) U_{SP} (\rho_S \otimes |0\>_p) \right] = \text{Tr} [M_i \rho_S].
\]

Fact 11. Let \( \varepsilon \in (0, 1) \) and \( |\rho\rangle\langle A|, \sigma_A \) be quantum states. Then
\[
D_H (|\rho\rangle\langle A| \| \sigma_A) = \sup_{0 \leq \Lambda \leq I_A : \text{rk}(\Lambda) = 1, \text{Tr}(\Lambda |\rho\rangle\langle A|) \geq 1 - \varepsilon} \log \left( \frac{1}{\text{Tr}(\Lambda \sigma_A)} \right),
\]
where \( \text{rk}(\Lambda) \) is the rank of the operator \( \Lambda \).

Proof. We apply Neumark’s theorem (Fact 10) to rewrite the smooth quantum hypothesis divergence as
\[
\sup_{\Pi : \Pi^2 = \Pi, \text{Tr}(\Pi |\rho\rangle\langle A| \otimes |0\>_p) \geq 1 - \varepsilon} \log \left( \frac{1}{\text{Tr}(\Pi \sigma_A \otimes |0\>_p)} \right).
\]

Fix a \( \Pi \) such that \( \text{Tr}(\Pi |\rho\rangle\langle A|) \geq 1 - \varepsilon \), define \( |\phi\rangle \) by \( |\phi\rangle = \Pi |\rho\rangle \), then
\[
\text{Tr}(|\phi\rangle\langle A| \cdot |\phi\rangle \otimes |0\>_p) = \frac{\langle \rho |A(0)p\Pi A(0)|p\rangle \cdot \langle \rho |A(0)p\Pi A(0)|p\rangle}{\langle \rho |A(0)p\Pi A(0)|0\rangle} \geq 1 - \varepsilon.
\]

Further, since \( \Pi |\phi\rangle = |\phi\rangle \), we have \( |\phi\rangle\langle A| \leq \Pi \). Thus, \( \text{Tr}(\langle \phi |\phi\rangle \leq \Pi (|0\>_p) \leq \text{Tr}(\Pi |\sigma_A \otimes |0\>_p) \leq \text{Tr}(\Pi |\sigma_A \otimes |0\>_p) \). Thus, the projector achieving the supremum in Equation 1 has rank 1. The proof now follows by defining \( \Lambda \) as defined above, which satisfies \( \text{rk}(\Lambda) = 1 \).

3 Entanglement assisted quantum coding

3.1 Point to point quantum channel

Alice wants to communicate a classical message \( M \) chosen from \( [2^R] \) to Bob over a quantum channel such that Bob is able to decode the correct message with probability at least \( 1 - \varepsilon \), for all message \( m \). To accomplish this task Alice and Bob also share entanglement between them. Let the input to Alice be given in a register \( M \). We now make the following definition.

Definition 1. Let \( |\theta\rangle E_A E_B \) be the shared entanglement between Alice (\( E_A \)) and Bob (\( E_B \)). An \((R, \varepsilon)\)-entanglement assisted code for the quantum channel \( N_{A \rightarrow B} \) consists of

- An encoding map \( U : ME_A \rightarrow A \) for Alice.
- A decoding operation \( D : BE_B \rightarrow M' \) for Bob, with \( M' \equiv M \) being the output register such that for all \( m \in [2^R], \)
  \[
  \Pr(M' \neq m | M = m) \leq \varepsilon.
  \]

The following converse was shown in [2]. We provide a simpler proof with the utility that it can be easily extended to complex network scenarios.

Theorem 1. Fix a quantum channel \( N_{A \rightarrow B} \) and \( \varepsilon \in (0, 1) \). For any \((R, \varepsilon)\)-entanglement assisted code for this quantum channel, it holds that
\[
R \leq \max \min_{|\psi\rangle_{AB'}} D_H (N_{A \rightarrow B} (|\psi\rangle_{AB'}) \| \sigma_B \otimes \psi_{B'}).\]

Proof. We will prove the upper bound for uniform distribution over the messages. Fix a quantum state \( \sigma_B \). Let \( \psi_{MAE_B} \) be the quantum state after Alice’s encoding. There exists a register \( F \) that purifies \( \psi_{MAE_B} \) into the pure state \( |\psi\rangle_{MAE_B F} \). Let \( \rho_{MBE_B F} \) be the quantum state after the action of the channel and \( \phi_{MM'} = D(\rho_{MBE_B F}) \). From Facts 9 and 2 we have
\[
R \leq D_H (\phi_{MM'} \| \phi_{M} \otimes D(\sigma_B \otimes \rho_{E_B})) = D_H (\rho_{MBE_B} \| \rho_{MBE_B} \otimes \sigma_B) \leq D_H (\rho_{MBE_B} \| \rho_{MBE_B} \otimes \sigma_B),
\]
where we have used the facts that $\rho_{MEB} = \rho_M \otimes \rho_E F$ and $\phi_M = \rho_M$. Since register $B$ is obtained by an action of the channel $N_{A \rightarrow B}$, we have

$$R \leq D_H^c(N_{A \rightarrow B}(\psi_{MAE_B F})\|\psi_{MEB F} \otimes \sigma_B).$$

Setting $B' \overset{\text{def}}{=} ME_B F$ and optimizing over all $\sigma_B$, we conclude the converse.

Following achievability was shown in [11], which is near optimal.

**Theorem 2 ([11]).** Fix a quantum channel $N_{A \rightarrow B}$ and $\varepsilon, \delta \in (0, 1)$. There exists an $(R, 2\varepsilon + \delta)$-entanglement assisted code for this channel if

$$R \leq \max_{|\psi\rangle_{AB'}} D_H^c(N_{A \rightarrow B}(\psi_{AB'})) \| N_{A \rightarrow B}(\psi_A) \otimes \psi_{B'}) - \log \frac{1}{\delta}.$$

The error of $2\varepsilon + \delta$ can be improved to $\varepsilon + \delta$, by tuning the parameter $c$ in Hayashi-Nagaoka inequality (Fact 6), as noted in [11].

**Success probability for entanglement assisted communication over noiseless channel.** Now, we recover the result in [56] Theorem 1.1 for one way protocols, as an application of Theorem 11.

**Corollary 1.** For any $(R, \varepsilon)$-entanglement assisted code for the identity channel $N_{A \rightarrow A}(\rho_A) = \rho_A$, it holds that

$$1 - \varepsilon \leq \frac{|A|^2}{2R}. $$

**Proof.** We apply Theorem 11 with $\sigma_A = \frac{I_A}{|A|}$ to obtain

$$R \leq \max_{|\psi\rangle_{AB'}} D_H^c(|\psi\rangle_{AB'}\| \frac{I_A}{|A|} \otimes |\psi\rangle_{B'}).$$

Let $|\psi\rangle_{AB'} = \sum_i \lambda_i |i\rangle_A |i\rangle_{B'}$ be the Schmidt decomposition of $|\psi\rangle_{AB'}$ such that $\sum_i \lambda_i^2 = 1$ and let $\Pi' \overset{\text{def}}{=} \sum_i |i\rangle_A \otimes |i\rangle_{B'}$. It holds that $\Pi' |\psi\rangle_{AB'} = |\psi\rangle_{AB'}$. From Fact 11 let $|\Pi\rangle$ (with some abuse of notation) be the rank one operator achieving the optimum for $D_H^c(|\psi\rangle_{AB'}\| \frac{I_A}{|A|} \otimes |\psi\rangle_{B'})$. We recall that $|\Pi\rangle$ need not be normalized. Since $\langle \Pi | |\psi\rangle_{AB'} = \langle \Pi | \Pi' | \psi\rangle_{AB'}$ and

$$\langle \Pi | \Pi' \frac{I_A}{|A|} \otimes |\psi\rangle_{B'} \Pi' |\Pi\rangle = \langle \Pi | \frac{I_A}{|A|} \otimes |\psi\rangle_{B'} \Pi' |\Pi\rangle \leq \langle \Pi | \frac{I_A}{|A|} \otimes |\psi\rangle_{B'} |\Pi\rangle,$$

we have that $\Pi' |\Pi\rangle = |\Pi\rangle$. Thus, we expand $|\Pi\rangle = \sum_i a_i |i\rangle_A |i\rangle_{B'}$ such that $\sum_i a_i^2 \leq 1$. The condition $\langle \Pi | |\psi\rangle_{AB'}|^2 \geq 1 - \varepsilon$ translates to $\sum_i a_i \lambda_i \geq 1 - \varepsilon$. Further,

$$\langle \Pi | \frac{I_A}{|A|} \otimes |\psi\rangle_{B'} |\Pi\rangle = \frac{1}{|A|} \sum_i |a_i \lambda_i|^2.$$

By Cauchy-Schwarz inequality,

$$1 - \varepsilon \leq \sum_i |a_i \lambda_i|^2 \leq |A| \sum_i |a_i \lambda_i|^2 \implies \frac{1 - \varepsilon}{|A|} \leq \sum_i |a_i \lambda_i|^2.$$

Hence, it holds that

$$\langle \Pi | \frac{I_A}{|A|} \otimes |\psi\rangle_{B'} |\Pi\rangle \geq \frac{1 - \varepsilon}{|A|^2},$$

for any feasible choice of $a_i, \lambda_i$. The inequality is achieved when $\lambda_i = \frac{1}{\sqrt{|A|}}$ and $a_i = \sqrt{\frac{1 - \varepsilon}{|A|}}$, which also satisfies the constraints $\sum_i \lambda_i^2 = 1$, $\sum_i a_i^2 \leq 1$, $\sum_i |a_i \lambda_i|^2 \geq 1 - \varepsilon$. Hence, we conclude that

$$R \leq \max_{|\psi\rangle_{AB'}} D_H^c(|\psi\rangle_{AB'}\| \frac{I_A}{|A|} \otimes |\psi\rangle_{B'}) \leq \log \frac{|A|^2}{1 - \varepsilon}.$$

□
3.2 Quantum channel with state

Alice wants to communicate a classical message $M$ chosen from $\mathbb{Z}_2^R$ to Bob over a quantum channel $N_{AS\rightarrow B}$ such that Bob is able to decode the correct message with probability at least $1 - \varepsilon$. Alice shares entanglement $(\tau)_{SS'}$ with the channel as well. This model in the classical setting is known as the Gel'fand-Pinsker channel.

**Definition 2.** Let $|0\rangle_E B E_B$ be the shared entanglement between Alice and Bob and let $(\tau)_{SS'}$ be the state shared between Alice and Channel. An $(R, \varepsilon)$-entanglement assisted code for the quantum channel $N_{AS\rightarrow B}$ consists of

- An encoding operation $E : ME AS' \rightarrow A$ for Alice.
- A decoding operation $D : BE B \rightarrow M'$ for Bob, with $M' \equiv M$ being the output register such that for all $m \in [2^R]$
  \[
  \Pr(M' \neq m|M = m) \leq \varepsilon.
  \]

We have the following converse.

**Theorem 3.** Fix a quantum channel $N_{AS\rightarrow B}$ with state $\tau_S$ and an $\varepsilon \in (0, 1)$. For every $(R, \varepsilon)$-entanglement assisted code for this channel, it holds that

\[
R \leq \max_{\psi_{ASB'} : \min_{\psi_{SB'}} \sigma_B} \min_{\sigma_B} D_H(N_{AS\rightarrow B}(\psi_{ASB'})\|\psi_{B'} \otimes \sigma_B).
\]

**Proof.** We will prove the upper bound for uniform distribution over the messages. Fix a quantum state $\sigma_B$. Let $\psi_{MASEB}$ be the quantum state after Alice's encoding and $\rho_{M BEB}$ be the quantum state after the action of the channel. Let $\phi_{MM'} = D(\rho_{M BEB})$. From Facts 9 and 2

\[
D_H(\phi_{MM'} \| \phi M \otimes D(\sigma_B \otimes E_B)) \leq D_H(\rho_{M BEB} \| \rho M \otimes \sigma_B \otimes \rho E_B)
\]

where we have used the facts that $\rho_{M BEB} = \rho M \otimes \rho E_B$ and $\rho M = \phi_M$. Now, observe that $N_{AS\rightarrow B}(\psi_{MASEB}) = \rho_{M BEB}$ and $\psi_{MSEB} = \psi_{MEB} \otimes \tau_S$. Setting $B' = ME B$, we conclude that

\[
R \leq D_H(N_{AS\rightarrow B}(\psi_{ASB'})\|\psi_{B'} \otimes \sigma_B),
\]

where $\psi_{SB'} = \tau_S \otimes \psi_{B'}$. \hfill \Box

As a corollary of above converse, we obtain the following converse statement, which matches (up to some constants) the achievability result given in [1] Theorem 5.

**Corollary 2.** Fix a quantum channel $N_{AS\rightarrow B}$ with state $\tau_S$ and an $\varepsilon \in (0, 1)$. For every $(R, \varepsilon)$-entanglement assisted code for this channel, it holds that

\[
R \leq \max_{\psi_{ASB'} : \min_{\psi_{SB'}} \sigma_B} \left( \min_{\sigma_B} D_H(N_{AS \rightarrow B}(\psi_{ASB'})\|\psi_{B'} \otimes \sigma_B) - D_{\max}(\psi_{SB'}\|\psi_S \otimes \psi_{B'}) \right).
\]

**Proof.** If $\psi_{SB'} = \psi S \otimes \psi_{B'}$, then $D_{\max}(\psi_{SB'}\|\psi S \otimes \psi_{B'}) = 0$. Thus, the optimization in above statement is over a larger set, as compared to that given in Theorem 5 \hfill \Box

The utility of [1] Theorem 5) is that it yields a single letter expression in the asymptotic and i.i.d. setting, as shown in [27] using different techniques. It is also possible to directly achieve the bound given in Theorem 5 as we show below. The utility of this bound is that it is of the form similar to that for multiple access channel and broadcast channel, both of which are one-shot optimal.

**Theorem 4.** Fix a quantum channel $N_{AS\rightarrow B}$ with state $\tau_S$ and $\varepsilon, \delta \in (0, 1)$. There exists an $(R, \varepsilon + 2\delta)$-entanglement assisted code for this channel, if

\[
R \leq \max_{\psi_{ASB'} : \min_{\psi_{SB'}} \sigma_B} D_H(N_{AS \rightarrow B}(\psi_{ASB'})\|N_{AS \rightarrow B}(\psi AS) \otimes \psi_{B'}) - \log \frac{4\varepsilon}{\delta^2}.
\]
Proof. Fix a quantum state $\psi_{ASB'}$ achieving the optimum above such that $\psi_{SB'} = \tau_S \otimes \psi_{B'}$. Let $|\psi'\rangle_{EB'}$ be a purification of $\psi_{B'}$. Alice and Bob share $2^{R_i}$ copies of $|\psi'\rangle_{EB'}$ in registers $E_1B'_1, \ldots, E_2B'_2$. Let $|\psi\rangle_{AUSB'}$ be a purification of $\psi_{ASB'}$. Fix a quantum channel $W : H_{SE} \to H_{AU}$ be an isometry such that $W |\psi'\rangle_{EB'} \otimes |\tau\rangle_{S'S} = |\psi\rangle_{AUSB'}$.

**Encoding:** To send the message $m \in [2^{R_i}]$, Alice prepares the pure state $|\psi\rangle_{AUSB_m}$ by applying the isometry $W$ on the registers $S'E_m$ and sends register $A$ through the channel.

**Decoding and error analysis:** Bob performs the position-based decoding strategy across the $B'$ registers. Let $\Pi_{BB'}$ be the operator achieving the optimum in the definition of $D_H(N_{AS \to B}(\psi_{ASB'}) || N_{AS \to B}(\psi_{AS}) \otimes \psi_{B'})$. Define

$$\Lambda(m) \overset{\text{def}}{=} I_{B'_1} \otimes I_{B'_2} \otimes \cdots \Pi_{BB_m} \otimes \cdots \otimes I_{B'_K},$$

and

$$\Omega(m) \overset{\text{def}}{=} \left( \sum_{m' \in [1:2^{R_i}]} \Lambda(m') \right)^{-\frac{1}{2}} \Lambda(m) \left( \sum_{m' \in [1:2^{R_i}]} \Lambda(m') \right)^{-\frac{1}{2}}.$$ 

Bob applies the measurement $\{\Omega(1), \ldots, \Omega(2^{R_i}), I - \sum_m \Omega(m)\}$ to decode $m$.

**Error analysis:** Employing Hayashi-Nagaoka inequality (Fact 9), we have

$$\Pr\{M' \neq m \mid M = m\} \leq (1 + c) \text{Tr}( (I - \Pi_{BB'})N_{AS \to B}(\psi_{ASB'}) + (2 + c + \frac{1}{c}) \cdot 2^{R_i}\text{Tr}( \Pi_{BB'}N_{AS \to B}(\psi_{AS}) \otimes \psi_{B'})$$

$$\leq (1 + c)c + \frac{4}{c} 2^{R_i} - D_H(N_{AS \to B}(\psi_{ASB'}) || N_{AS \to B}(\psi_{AS}) \otimes \psi_{B'})$$

$$\leq \varepsilon + 2\delta,$$

where we choose $c = \frac{1}{2}$.

This completes the proof. \(\square\)

### 3.3 Broadcast quantum channel

Alice wishes to communicate message pair $(m_1, m_2)$ simultaneously to Bob and Charlie over a quantum broadcast channel, where $m_1$ is intended for Bob and $m_2$ is intended for Charlie, such that both Bob and Charlie output the correct message with probability at least $1 - \varepsilon$.

**Definition 3.** Let $|\theta\rangle_{E_A, E_B}$ and $|\varphi\rangle_{E_A, E_C}$ be the shared entanglement between Alice and Bob and Alice and Charlie respectively. An $(R_1, R_2, \varepsilon_1, \varepsilon_2)$ entanglement assisted code for the quantum broadcast channel $N_{A \to BC}$ consists of

- An encoding operation $E : M_1M_2E_A, E_A \to A$ for Alice.
- A pair of decoding operations $(D_B, D_C)$, $D_B : B \otimes E_B \to M'_1$ and $D_C : C \otimes E_C \to M'_2$, with $(M'_1, M'_2) \equiv (M_1, M_2)$ being the output registers, such that for all $(m_1, m_2) \in [2^{R_1}] \times [2^{R_2}]$

$$\Pr(M'_1 \neq m_1 | M_1 = m_1) \leq \varepsilon_1, \Pr(M'_2 \neq m_2 | M_2 = m_2) \leq \varepsilon_2.$$

We have the following converse.

**Theorem 5.** Fix a quantum channel $N_{A \to BC}$ and $\varepsilon_1, \varepsilon_2 \in (0, 1)$. For any $(R_1, R_2, \varepsilon_1, \varepsilon_2)$-entanglement assisted code for this channel, there exist registers $B', C'$ and a quantum state $|\varphi\rangle_{ABC'}$ satisfying $|\varphi\rangle_{B'C'} = |\varphi\rangle_{B'} \otimes |\varphi\rangle_{C'}$ such that

$$R_1 \leq \min_{\sigma_B} D_H^\varepsilon(\text{Tr}_C N_{A \to BC}(\psi_{AB'}) || \sigma_B \otimes \psi_{B'}),$$

$$R_2 \leq \min_{\tau_C} D_H^\varepsilon(\text{Tr}_B N_{A \to BC}(\psi_{AC'}) || \tau_C \otimes \psi_{C'}).$$

Proof. We will prove the upper bound for uniform distribution over the messages. Fix quantum states $\sigma_B, \tau_C$. Let $\psi_{M_1M_2AE_BEC_C}$ be the quantum state after Alice’s encoding and $\rho_{M_1M_2BCE_BEC_C}$ be the quantum state after the action of the channel. Let $\phi_{M_1M'_1}^B \overset{\text{def}}{=} D_B(\rho_{M_1BEC_B})$ and $\phi_{M_2M'_2}^C \overset{\text{def}}{=} D_C(\rho_{M_2CE_C})$. From Facts 9 and 2

$$R_1 \leq D_H^\varepsilon(\phi_{M_1M'_1}^B \phi_{M_2M'_2}^C D(\sigma_B \otimes \rho_{E_B})) \leq D_H^\varepsilon(\rho_{M_1BEC_B} || \rho_{M_1} \otimes \sigma_B) \leq D_H^\varepsilon(\rho_{M_1BEC_B} || \rho_{M_1} \otimes \rho_{E_B}) = D_H^\varepsilon(\rho_{M_1BEC_B} || \rho_{M_1} \otimes \sigma_B),$$

for all $\varepsilon_1, \varepsilon_2 \in (0, 1)$. This completes the proof.
where we have used the facts that $\rho_{M_1E_B} = \rho_{M_1} \otimes \rho_{E_B}$ and $\phi_{M_1}^{E_B} = \rho_{M_1}$. Similarly,

$$R_2 \leq D_H^2(\rho_{M_1CE_C} \|\rho_{M_2E_C} \otimes \tau_C).$$

Observe that $N_{A\rightarrow BC}(\psi_{M_1M_2AE_BE_C}) = \rho_{M_1M_2BCE_BE_C}$ and $\psi_{M_1M_2E_BE_C} = \psi_{M_1E_B} \otimes \psi_{M_2E_C}$. Define $B' \overset{\text{def}}{=} M_1E_B$ and $C' \overset{\text{def}}{=} M_2E_C$. Thus, we conclude that

$$R_1 \leq D_H^2(\text{Tr}_C N_{A\rightarrow BC}(\psi_{AB}) \| \sigma_B \otimes \psi_{B'}), \quad R_2 \leq D_H^2(\text{Tr}_B N_{A\rightarrow BC}(\psi_{AC}) \| \tau_C \otimes \psi_{C'}).$$

where $\psi_{B'C'} = \psi_{B'} \otimes \psi_{C'}$.

As a corollary, we obtain the following converse result, which shows the one-shot near optimality of the bound for quantum broadcast channel given in [1]. The fact that the following corollary follows from Theorem [5] is implicit in the asymptotic and i.i.d. converse of [23, Theorem 3].

**Corollary 3.** Fix a quantum channel $N_{A\rightarrow BC}$ and $\varepsilon_1, \varepsilon_2 \in (0, 1)$. For any $(R_1, R_2, \varepsilon_1, \varepsilon_2)$-entanglement assisted code for this channel, there exist registers $B', C'$ and a quantum state $\psi_{AB'C'}$ such that

$$R_1 \leq \min_{\sigma_B} D_H^2(\text{Tr}_C N_{A\rightarrow BC}(\psi_{AB}) \| \sigma_B \otimes \psi_{B'}),
R_2 \leq \min_{\tau_C} D_H^2(\text{Tr}_B N_{A\rightarrow BC}(\psi_{AC}) \| \tau_C \otimes \psi_{C'}),
R_1 + R_2 \leq \min_{\sigma_B} D_H^2(\text{Tr}_C N_{A\rightarrow BC}(\psi_{AB}) \| \sigma_B \otimes \psi_{B'}) + \min_{\tau_C} D_H^2(\text{Tr}_B N_{A\rightarrow BC}(\psi_{AC}) \| \tau_C \otimes \psi_{C'}) - D_\text{max}(\psi_{B'C'} \| \psi_{B'} \otimes \psi_{C'}).$$

**Proof.** The proof follows by relaxing the constraint $\psi_{B'C'} = \psi_{B'} \otimes \psi_{C'}$ in Theorem 5. \hfill \Box

We have the following achievability result.

**Theorem 6.** Fix a quantum channel $N_{A\rightarrow BC}$ and $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$. For every quantum state $\psi_{AB'C'}$ satisfying $\psi_{B'C'} = \psi_{B'} \otimes \psi_{C'}$, there exists an $(R_1, R_2, \varepsilon_1, \varepsilon_2, \delta)$-entanglement assisted code for this channel, if

$$R_1 \leq D_H^2(\text{Tr}_C N_{A\rightarrow BC}(\psi_{AB}) \| \text{Tr}_C N_{A\rightarrow BC}(\psi_A) \otimes \psi_{B'}) - \log \frac{4\varepsilon_1}{\delta^2},
R_2 \leq D_H^2(\text{Tr}_B N_{A\rightarrow BC}(\psi_{AC}) \| \text{Tr}_B N_{A\rightarrow BC}(\psi_A) \otimes \psi_{C'}) - \log \frac{4\varepsilon_2}{\delta^2}.$$

**Proof.** Let $|\psi\rangle_{UAB'C'}$ be a purification of $\psi_{AB'C'}$. Let $|\kappa\rangle_{EB'}$ be a purification of $\psi_{B'}$ and $|\kappa\rangle_{FC'}$ be a purification of $\psi_{C'}$. Let $W : \mathcal{H}_{EF} \rightarrow \mathcal{H}_{U}$ be an isometry such that $W|\kappa\rangle_{EB'} \otimes |\kappa\rangle_{FC'} = |\psi\rangle_{UAB'C'}$. Alice and Bob share $2^{R_1}$ copies of the quantum state $|\kappa\rangle_{EB'}$ in registers $E_1, B_1', \ldots, E_{2^{R_1}}, B_{2^{R_1}}$. Alice and Charlie share $2^{R_2}$ copies of the quantum state $|\kappa\rangle_{FC'}$ in registers $F_1, C_1', \ldots, F_{2^{R_2}}, C_{2^{R_2}}$.

**Encoding:** To send the message pair $(m_1, m_2)$, Alice prepares the pure state $|\psi\rangle_{UAB'_m C_m}$ by applying the isometry $W$ on the registers $E_{m_1}', F_{m_2}'$ and sends the register $A$ through the channel.

**Decoding:** Bob and Charlie apply the position based decoding strategy. Let $\Pi_{BB'}$ be the operator that achieves the optimum in $D_H^2(\text{Tr}_C N_{A\rightarrow BC}(\psi_{AB}) \| \text{Tr}_C N_{A\rightarrow BC}(\psi_A) \otimes \psi_{B'})$ and $\Pi_{CC'}$ be the operator that achieves the optimum in $D_H^2(\text{Tr}_B N_{A\rightarrow BC}(\psi_{AC}) \| \text{Tr}_B N_{A\rightarrow BC}(\psi_A) \otimes \psi_{C'})$. Define

$$\Lambda_B(m_1) \overset{\text{def}}{=} I_{B_1'} \otimes I_{B_2'} \otimes \cdots \otimes I_{B_{2^{R_1}}'},
\Lambda_C(m_2) \overset{\text{def}}{=} I_{C_1'} \otimes I_{C_2'} \otimes \cdots \otimes I_{C_{2^{R_2}}'},$$

and

$$\Omega_B(m_1) \overset{\text{def}}{=} \left( \sum_{m_1' \in [1:2^{R_1}]} \Lambda_B(m_1') \right)^{-\frac{1}{2}} \Lambda_B(m_1) \left( \sum_{m_1' \in [1:2^{R_1}]} \Lambda_B(m_1') \right)^{-\frac{1}{2}},$$

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\[ \Omega_C(m_2) \overset{\text{def}}{=} \left( \sum_{m_2 \in [1:2^{R_2}]} \Lambda_C(m_2) \right)^{-\frac{1}{2}} \Lambda_C(m_2) \left( \sum_{m_2 \in [1:2^{R_2}]} \Lambda_C(m_2) \right)^{-\frac{1}{2}}. \]

Bob applies the measurement \( \{ \Omega_B(1), \ldots, \Omega_B(2^{R_1}), I - \sum_{m_1} \Omega_B(m_1) \} \) to decode \( m_1 \). Charlie applies the measurement \( \{ \Omega_C(1), \ldots, \Omega_C(2^{R_2}), I - \sum_{m_2} \Omega_C(m_2) \} \) to decode \( m_2 \).

**Error analysis:** Employing Hayashi-Nagaoka inequality (Fact 3), we have

\[ \Pr\{ M'_1 \neq m_1 \mid M_1 = m_1 \} \leq (1 + c) \Tr((I - \Pi_{BB}) \Tr_C N_{A\rightarrow BC}(\psi_{AB})) + (2 + c) \cdot 2^{R_1} \Tr(\Pi_{BB} \Tr_C N_{A\rightarrow BC}(\psi_A \otimes \psi_B)) \]

\[ \leq (1 + c) \varepsilon_1 + \frac{4c}{c} 2^{R_1 - D_{\Pi}^1(\Tr_C N_{A\rightarrow BC}(\psi_B) \| \Tr_C N_{A\rightarrow BC}(\psi_B))} \]

\[ \leq \varepsilon_1 + 2\delta, \]

where we choose \( c = \frac{\delta}{\varepsilon_1} \). Similarly, we have

\[ \Pr\{ M'_2 \neq m_2 \mid M_2 = m_2 \} \leq \varepsilon_2 + 2\delta. \]

This completes the proof.

### 3.4 Multiple access channel

Alice wants to communicate a classical message \( m_1 \) chosen from \([2^{R_1}]\) to Charlie and Bob wants to communicate a classical message \( m_2 \) chosen from \([2^{R_2}]\) to Charlie, over a channel \( \mathcal{N}_{AB\rightarrow C} \). Alice shares entanglement with Charlie and Bob shares entanglement with Charlie. Alice and Bob do not share entanglement. This is the multiple access channel.

**Definition 4.** Let \( |\theta\rangle_{E_A E_C_1} \) and \( |\theta\rangle_{E_B E_C_2} \) be the shared entanglement between Alice and Charlie and Bob and Charlie, respectively. An \((R_1, R_2, \varepsilon_1, \varepsilon_2)\)-entanglement assisted code for the quantum multiple access channel \( \mathcal{N}_{AB\rightarrow C} \) consists of

- **Encoding operations** \( \mathcal{E}_1 : M_1 E_A \rightarrow A \) and \( \mathcal{E}_2 : M_2 E_B \rightarrow B \).
- A decoding operation \( \mathcal{D} : E_C_1 E_C_2 C \rightarrow M'_1 M'_2 \), with \( M'_1 \equiv M_1 \) and \( M'_2 \equiv M_2 \) such that for all \((m_1, m_2) \in [2^{R_1}] \times [2^{R_2}]\),

\[ \Pr\{ M'_1 \neq m_1 \mid M_1 = m_1 \} \leq \varepsilon_1, \Pr\{ M'_2 \neq m_2 \mid M_2 = m_2 \} \leq \varepsilon_2. \]

We note that the definition of error above is closely related to the definition of error as \( \Pr\{ M'_1, M'_2 \neq m_1, m_2 \mid M_1, M_2 = m_1, m_2 \} \leq \varepsilon \) through

\[ \max\{\Pr\{ M'_i \neq m_i \mid M_i = m_i \}\} \leq \Pr\{ M'_1, M'_2 \neq m_1, m_2 \mid M_1, M_2 = m_1, m_2 \} \leq \Pr\{ M'_1 \neq m_1 \mid M_1 = m_1 \} + \Pr\{ M'_2 \neq m_2 \mid M_2 = m_2 \}. \]

We have the following converse.

**Theorem 7.** Fix a quantum channel \( \mathcal{N}_{AB\rightarrow C} \) and \( \varepsilon_1, \varepsilon_2 \in (0, 1) \). For every \((R_1, R_2, \varepsilon_1, \varepsilon_2)\)-entanglement assisted code for this channel, there exist registers \( A', A'', B', B'' \) and a quantum state \( \psi_{ABA'B''A''B''} = \psi_{A'A''} \otimes \psi_{B'B''} \otimes \psi_{A'B''} \otimes \psi'_{A'} \otimes \psi_{B'} \otimes \psi_{A''} \) satisfying \( \psi_{ABA'B''A''B''} = \psi_{A'A''} \otimes \psi_{B'B''} \otimes \psi_{A'B''} \otimes \psi'_{A'} \otimes \psi_{B'} \otimes \psi_{A''} \) such that

\[ R_1 \leq D_{H}^1(\rho_{CA'A''B''} \| \rho_{CA'A''B''} \otimes \rho_{A'}), \]

and

\[ R_2 \leq D_{H}^2(\rho_{B'B''A''B''} \| \rho_{B'B''A''B''} \otimes \rho_{B'}), \]

where \( \rho_{CA'B'} \overset{\text{def}}{=} \mathcal{N}_{AB\rightarrow C}(\psi_{ABA'B'}). \)
Proof. We will prove the upper bound for uniform distribution over the messages. Let \( \psi_{M_1M_2}ABE_{C_1}E_{C_2} \) be the quantum state after the operations of Alice and Bob. It holds that \( \psi_{M_1M_2}ABE_{C_1}E_{C_2} = \psi_{M_1}AE_{C_1} \otimes \psi_{M_2}BE_{C_2} \). Let \( \rho_{M_1M_2}CE_{C_1}E_{C_2} \) be the quantum state after the action of the channel. Let \( \phi_{M_1M_2}M[M_2] = D(\rho_{M_1M_2}CE_{C_1}E_{C_2}) \). From Facts 9 and 2

\[
R_1 \leq D^I_H(\phi_{M_1M_1}M_2 || \phi_{M_1} \otimes D(\rho_{CE_{C_1}E_{C_2}})) \leq D^{II}_H(\rho_{M_1} \otimes \rho_{CE_{C_1}E_{C_2}}).
\]

Similarly,

\[
R_2 \leq D^{II}_H(\rho_{M_2}E_{C_2} || \rho_{CE_{C_1}}).
\]

We observe the relations \( N^{AB \rightarrow C}(\rho_{M_1M_2}ABE_{C_1}E_{C_2}) = \rho_{M_1M_2}CE_{C_1}E_{C_2} \), and define \( A' \equiv M_1, A'' \equiv E_{C_1}, B' \equiv M_2, B'' \equiv E_{C_2} \). This concludes the proof. \( \square \)

It is also possible to obtain a one-shot version of the converse given in [18], as follows.

**Theorem 8.** Fix a quantum channel \( N^{AB \rightarrow C} \) and \( \epsilon_1, \epsilon_2 \in (0, 1) \). For every \( (R_1, R_2, \epsilon_1, \epsilon_2) \)-entanglement assisted code for this channel, there exist registers \( A', B' \) and a pure quantum state \( |\psi\rangle_{ABA'B'} \) satisfying \( |\psi\rangle_{ABA'B'} = |\psi\rangle_{AA'} \otimes |\psi\rangle_{BB'} \), such that

\[
R_1 \leq D^I_H(\rho_{CA'B'} \otimes \rho_{CB'} \otimes \rho_{A'}),
\]

\[
R_2 \leq D^{II}_H(\rho_{CA'B'} \otimes \rho_{C'A'} \otimes \rho_{B'}).
\]

and

\[
R_1 + R_2 \leq D^{II}_H(\rho_{CA'B'} \otimes \rho_{C} \otimes \rho_{A'} \otimes \rho_{B'}),
\]

where \( \rho_{CA'B'} \equiv N^{AB \rightarrow C}(\psi_{ABA'B'}). \)

Proof. We will prove the upper bound for uniform distribution over the messages. Let \( \psi_{M_1M_2}ABE_{C_1}E_{C_2} \) be the quantum state after the operations of Alice and Bob. It holds that \( \psi_{M_1M_2}ABE_{C_1}E_{C_2} = \psi_{M_1}AE_{C_1} \otimes \psi_{M_2}BE_{C_2} \). Let \( F_1, F_2 \) be registers such that \( |\psi\rangle_{M_1}AE_{C_1}F_1 \) purifies \( \psi_{M_1AE_{C_1}} \) and \( |\psi\rangle_{M_2BE_{C_2}F_2} \) purifies \( \psi_{M_2BE_{C_2}} \). Let \( \rho_{M_1M_2}CE_{C_1}F_1E_{C_2}F_2 \) be the quantum state after the action of the channel. Let \( \phi_{M_1M_2}M_1M_2' = D(\rho_{M_1M_2}CE_{C_1}E_{C_2}) \). From Facts 9 and 8

\[
R_1 \leq D^I_H(\phi_{M_1M_1}M_2 || \phi_{M_1} \otimes D(\rho_{CE_{C_1}E_{C_2}})) \leq D^I_H(\rho_{M_1CE_{C_1}E_{C_2}} || \rho_{M_1CE_{C_1}E_{C_2}} \otimes \rho_{CE_{C_1}E_{C_2}})
\]

\[
\leq D^I_H(\rho_{M_1CE_{C_1}F_1E_{C_2}F_2} || \rho_{M_1CE_{C_1}F_1E_{C_2}F_2} \otimes \rho_{CE_{C_1}E_{C_2}}) \leq D^I_H(\rho_{M_1CE_{C_1}F_1M_2E_{C_2}F_2} || \rho_{M_1CE_{C_1}F_1} \otimes \rho_{CM_2E_{C_2}}),
\]

where we have used \( \rho_{M_1E_{C_1}} = \rho_{M_1} \otimes \rho_{E_{C_1}} \). Similarly,

\[
R_2 \leq D^{II}_H(\rho_{M_2E_{C_2}F_2CM_1E_{C_1}F_1} || \rho_{M_2E_{C_2}F_2} \otimes \rho_{CM_1E_{C_1}F_1}).
\]

Further using Facts 9 and 2

\[
R_1 + R_2 \leq D^{II}_{H^2}(\phi_{M_1M_1}M_2 || \phi_{M_1} \otimes D(\rho_{CE_{C_1}E_{C_2}})) \leq D^I_H(\rho_{M_1CE_{C_1}E_{C_2}} || \rho_{M_1} \otimes \rho_{CE_{C_1}E_{C_2}})
\]

\[
\leq D^I_H(\rho_{M_1CE_{C_1}F_1} || \rho_{M_1CE_{C_1}F_1} \otimes \rho_{CE_{C_1}E_{C_2}}) = D^I_H(\rho_{M_1CE_{C_1}F_1} || \rho_{M_1CE_{C_1}F_1} \otimes \rho_{CE_{C_1}E_{C_2}})
\]

\[
= D^I_H(\rho_{M_1CE_{C_1}F_1} \otimes \rho_{M_2E_{C_2}F_2} \otimes \rho_{CM_2E_{C_2}}).
\]

We observe the relations \( N^{AB \rightarrow C}(\psi_{M_1M_2}ABE_{C_1}F_1E_{C_2}F_2) = \rho_{M_1M_2}CE_{C_1}F_1E_{C_2}F_2 \), and define \( A' \equiv M_1E_{C_1}F_1, B' \equiv M_2E_{C_2}F_2 \). This concludes the proof. \( \square \)

While above converse has the utility that it involves optimization over registers of bounded dimensions, in contrast to converse in Theorem 7 it is not clear how to achieve it without an appropriate notion of simultaneous decoding. On the other hand, we have the following achievability result, which is near-optimal with respect to the converse given in Theorem 7 for either one of the error parameters. Furthermore, it does not require a simultaneous decoding strategy.
Theorem 9. Fix a quantum channel $N_{AB\rightarrow C}$ and $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$. Let there be a quantum state $\psi_{A'B'B''}^m$ satisfying $\psi_{A'B'B''} = \psi_{AA'A''} \otimes \psi_{BB'B''}$ and $\psi_{A'B'A''B''} = \psi_{A'} \otimes \psi_{A''} \otimes \psi_{B'} \otimes \psi_{B''}$. There exists an $(R_1, R_2, \varepsilon_1 + 2\delta, \varepsilon_2 + 2\delta + 3\sqrt{\varepsilon_2 + 2\delta})$-entanglement assisted code and an $(R_1, R_2, \varepsilon_1 + 2\delta)$-purification of this code.

Proof. Introduce registers $G_1, G_2$ such that $|\psi\rangle_{AA'A''} G_1$ and $|\psi\rangle_{BB'B''} G_2$ purify $\psi_{A'A'}$ and $\psi_{B'B''}$. Let $\rho_{CA'B'A''B''} \defeq N_{AB\rightarrow C}(\psi_{A'B'B''}^m)$. Let $[|\psi\rangle_{AB'} = |\psi\rangle_{A'B'} \otimes |\psi\rangle_{A''B''}^m, \varphi\rangle_{AB'} = |\varphi\rangle_{A'B'} \otimes |\varphi\rangle_{A''B''}^m]$ be the purifications of $\psi_{A'A'}$, $\psi_{A''A''}$, $\psi_{B'B'}$, $\psi_{B''B''}$. Let $V_A : H_{AB'} \rightarrow HA_{G_1}$ be an isometry such that $V_A |\psi\rangle_{A'B'} \otimes |\psi\rangle_{A''B''}^m = |\psi\rangle_{A'A''} G_1$ and $V_B : H_{AB'} \rightarrow HB_{G_2}$ be an isometry such that $V_B |\psi\rangle_{A'B'} \otimes |\psi\rangle_{A''B''}^m = |\psi\rangle_{B'B''} G_2$.

Alice and Charlie share one copy of $|\psi\rangle_{E'A'}^m$, where Alice holds $E''$ and Charlie holds $A''$. Let Bob and Charlie share one copy of $|\psi\rangle_{E'B''}^m$, where Bob holds $B''$ and Charlie holds $B''$. Bob and Charlie share one copy of $|\psi\rangle_{E'B''}^m$, where Bob holds $F''$ and Charlie holds $B''$. Let $\Lambda_{A'} G_1 \defeq |\psi\rangle_{A'A''} G_1$. She sends $A$ through the channel.

To send the message $m_1 \in [2^R_1]$, Alice applies the isometry $V_A$ on registers $E'' E_{m_1}$ to prepare the purification $|\psi\rangle_{A'A''} G_1$. She sends $A$ through the channel.

Decoding: Charlie applies the position based decoding strategy [11] independently across $A''$ registers and then $B''$ registers as follows. Let $\Pi_{CA''B''A''}$ be the operator achieving the optimum in $D_{H}^m(\rho_{CA''B''} \otimes \rho_{A''})$ and $\Pi_{CA''B''A''}$ be the operator achieving the optimum in $D_{H}^m(\rho_{B''CA''} \otimes \rho_{B''})$. Define

$$\Lambda_{A}(m_1) \defeq I_{A_1} \otimes I_{A_2} \otimes \cdots \Pi_{CA''B''A''} \otimes \cdots \otimes I_{A_{2R_1}}$$

$$\Lambda_{B}(m_1) \defeq I_{B_1} \otimes I_{B_2} \otimes \cdots \Pi_{CA''B''A''} \otimes \cdots \otimes I_{B_{2R_2}}$$

and

$$\Omega_{A}(m_1) \defeq \left( \sum_{m'_1 \in [1:2^{R_1}]} \Lambda_{A}(m'_1) \right)^{-\frac{1}{2}} \Lambda_{A}(m_1) \left( \sum_{m'_1 \in [1:2^{R_1}]} \Lambda_{A}(m'_1) \right)^{-\frac{1}{2}}$$

$$\Omega_{B}(m_1) \defeq \left( \sum_{m'_2 \in [1:2^{R_2}]} \Lambda_{B}(m'_2) \right)^{-\frac{1}{2}} \Lambda_{B}(m_1) \left( \sum_{m'_2 \in [1:2^{R_2}]} \Lambda_{B}(m'_2) \right)^{-\frac{1}{2}}$$

Charlie applies first the measurement $\{\Omega_{A}(1), \ldots, \Omega_{A}(2^{R_1}), I - \sum_{m_1} \Omega_{A}(m_1)\}$ to decode $m_1$. Then he applies the measurement $\{\Omega_{B}(1), \ldots, \Omega_{B}(2^{R_2}), I - \sum_{m_2} \Omega_{B}(m_2)\}$ to decode $m_2$.

Error analysis: Following the argument in [11] and employing Hayashi-Nagaoka inequality (Fact 6), we have

$$\Pr\{M_i \neq m_1 | M_1 = m_1\} \leq (1 + c) \text{Tr}(I - \Pi_{CA''B''A''} \otimes \rho_{CA''B''} \otimes \rho_{A''}) + 2(1 + \frac{1}{c}) \cdot 2^{R_1} \text{Tr}(\Pi_{CA''B''A''} \otimes \rho_{CA''B''} \otimes \rho_{A''})$$

$$\leq (1 + c) \varepsilon_1 + \frac{4}{c} 2^{R_1} D_{H}^m(\rho_{CA''B''} \otimes \rho_{CA''B''} \otimes \rho_{A''})$$

$$\leq \varepsilon_1 + 2\delta,$$

where we choose $c = \frac{\delta}{\varepsilon_1}$.
Let $M'_2$ be the output if Charlie first performed the measurement \( \{ \Omega_B(1), \ldots, \Omega_B(2R_2), I - \sum_{m_2} \Omega_B(m_2) \} \). Then we would have \( \Pr \{ M'_2 \neq m_2 \mid M_2 = m_2 \} \leq \varepsilon_2 + 2\delta \). From Fact 5 we conclude that the purified distance between the global quantum states before and after Charlie’s first measurement is at most \( \sqrt{2\varepsilon_1 + 4\delta} \). From Fact 3 we thus conclude that

\[
\Pr \{ M'_2 \neq m_2 \mid M_2 = m_2 \} \leq (\sqrt{\Pr \{ M'_2 \neq m_2 \mid M_2 = m_2 \}} + \sqrt{2\varepsilon_1 + 4\delta})^2 \leq \varepsilon_2 + 2\delta + 3\sqrt{\varepsilon_1 + 2\delta}.
\]

By decoding $m_2$ before $m_1$, an alternate protocol can be obtained. This completes the proof.

Above theorem has the limitation that the overall error scales as $O(\varepsilon_1 + \sqrt{\varepsilon_2})$ or $O(\varepsilon_2 + \sqrt{\varepsilon_1})$. We improve it in the following theorem, using the sequential decoding technique (Fact 3).

**Theorem 10.** Fix a quantum channel $N_{AB \to C}$ and $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$. Let there be a quantum state $|\psi\rangle_{ABA'B'A''B''}$ satisfying $\psi_{ABA'B'A''B''} = \psi_{AA'A'\otimes BB'B''}$ and $\psi_{A'B'A''B''} = \psi_{A'\otimes A'' \otimes B' \otimes B''}$. There exists an $(R_1, R_2, 4(\varepsilon_1 + \varepsilon_2 + 2\delta), 4(\varepsilon_1 + \varepsilon_2 + 2\delta))$-entanglement assisted code for this channel, if

\[
R_1 \leq D^e_H(\rho_{CA''B''B'} \otimes \rho_{A''B''}) - \log \frac{1}{\delta},
\]

and

\[
R_2 \leq D^e_H(\rho_{B''C''A''} \otimes \rho_{B''B''}) - \log \frac{1}{\delta},
\]

where $\rho_{CA''B''B'} \overset{\text{def}}{=} N_{AB \to C}(\psi_{ABA'B'A''B''})$. In fact, the following upper bound holds for all $(m_1, m_2)$ in $[2^{R_1}] \times [2^{R_2}]$.

\[
\Pr(M'_1, M'_2 = m_1, m_2 \mid M_1, M_2 = m_1, m_2) \leq 4 \cdot (\varepsilon_1 + \varepsilon_2 + 2\delta).
\]

**Proof.** Introduce registers $G_1, G_2$ such that $|\psi\rangle_{AA'A'G_1}$ and $|\psi\rangle_{BB'B''G_2}$ purify $\psi_{AA'A'}$ and $\psi_{BB'B''}$. Let $\rho_{CA''B''B'} \overset{\text{def}}{=} N_{AB \to C}(\psi_{ABA'B'A''B''})$. Let $|\psi\rangle_{E'\otimes E''}$, $|\psi\rangle_{F'\otimes F''}$, $|\psi\rangle_{G'\otimes G''}$ be purifications of $\psi_{A'}, \psi_{A''}, \psi_{B'}, \psi_{B''}$. Let $V_A : \mathcal{H}_{E'\otimes E''} \to \mathcal{H}_{A'\otimes G_1}$ be an isometry such that $V_A|\psi\rangle_{E'\otimes E''} = |\psi\rangle_{AA'A''G_1}$ and $V_B : \mathcal{H}_{F'\otimes F''} \to \mathcal{H}_{B'\otimes G_2}$ be an isometry such that $V_B|\psi\rangle_{F'\otimes F''} = |\psi\rangle_{BB'B''G_2}$.

Alice and Charlie share one copy of $|\psi\rangle_{E'\otimes E''}$, where Alice holds $E''$ and Charlie holds $A''$, and $2^{R_1}$ copies of $|\psi\rangle_{E'\otimes E''}$ in registers $E'_1, E'_2, \ldots, E'_{2^{R_1}}$, where Alice holds $E'_1, \ldots, E'_{2^{R_1}}$ and Charlie holds $A'_1, \ldots, A'_{2^{R_1}}$. Bob and Charlie share one copy of $|\psi\rangle_{F'\otimes F''}$, where Bob holds $F''$ and Charlie holds $B''$, and $2^{R_2}$ copies of $|\psi\rangle_{F'\otimes F''}$ in registers $F'_1, F'_2, \ldots, F'_{2^{R_2}}$, where Bob holds $F'_1, \ldots, F'_{2^{R_2}}$ and Charlie holds $A'_1, \ldots, A'_{2^{R_1}}$.

**Encoding:** To send the message $m_1 \in [2^{R_1}]$, Alice applies the isometry $V_A$ on registers $E''E'_1m_1$ to prepare the purification $|\psi\rangle_{AA'A''m_1G_1}$ of $|\psi\rangle_{AA'\otimes A''m_1G_1}$. She sends $A$ through the channel.

To send the message $m_2$, Bob applies an isometry $V_B$ on the registers $F''F'_2m_2$ to prepare the purification $|\psi\rangle_{BB''B''m_2G_2}$ of $|\psi\rangle_{BB'\otimes B''m_2G_2}$. He sends $B$ through the channel.

**Decoding:** Let $\Pi_{CA''B''B'}$ be the operator achieving the optimum in $D^e_H(\rho_{CA''B''B'} \otimes \rho_{A'A'} \otimes \rho_{B'B''})$ and $\Pi_{CA''B''B'}$ be the operator achieving the optimum in $D^e_H(\rho_{CA''B''B'} \otimes \rho_{B'B''})$. By Stinespring dilation theorem, we introduce a register $J$ in the state $|0\rangle_J$ and consider the projectors $\Pi_{CA''B''B'\otimes A'J}$ and $\Pi_{CA''B''B'\otimes B'J}$.

Charlie sequentially applies the measurement \( \{ \Pi_{CA''B''B'\otimes A'J}, I - \Pi_{CA''B''B'\otimes A'J} \} \) for $m_1$ ranging from $[1 : 2^{R_1}]$. He outputs the first $m_1$ for which he obtains the outcome corresponding to $\Pi_{CA''B''B'\otimes A'J}$. Then he sequentially applies the measurement $\{ \Pi_{CA''B''B'\otimes B'J}, I - \Pi_{CA''B''B'\otimes B'J} \}$, for $m_2$ ranging from $[1 : 2^{R_2}]$. He outputs the first $m_2$ for which he obtains the outcome corresponding to $\Pi_{CA''B''B'\otimes B'J}$.

**Error analysis:** We compute the probability of obtaining the correct outcome. Let $\omega_{m_1, m_2}$ denote the overall quantum
state with Charlie, conditioned on messages \( m_1, m_2 \). Let \( \Pi \) denote the projector orthogonal to \( \Pi \). We have

\[
\Pr(M_1', M_2' = m_1, m_2 \mid M_1, M_2 = m_1, m_2) = \text{Tr} \left( \left( \prod_{j} \Pi_{CA^a'B^a'B_j A_{m_j} J} \right) \right)
\]

where in the first inequality, we use Fact 8 and in the last step, we use the bound on \( R_1, R_2 \). Thus, we conclude that

\[
\Pr(M_1', M_2' = m_1, m_2 \mid M_1, M_2 = m_1, m_2) \leq 4 \cdot (\varepsilon_1 + \varepsilon_2 + 2\delta).
\]

This completes the proof. \( \square \)

4 Entanglement unassisted quantum coding

Similar bounds can be obtained for entanglement unassisted quantum coding. For brevity, we consider the average case error, although all of the achievability results below also hold worst case over the messages.

4.1 Point to point quantum channel

Alice wants to communicate a classical message \( M \) chosen uniformly from \( \{2^R\} \) to Bob over a quantum channel such that Bob is able to decode the correct message with probability at least \( 1 - \varepsilon \), for all message \( m \). Let the input to Alice be given in a register \( M \). We now make the following definition.

Definition 5. An \((R, \varepsilon)\)-code for the quantum channel \( \mathcal{N}_{A \to B} \) consists of

- An encoding map \( U : M \to A \) for Alice, where \( M \) takes value uniformly over the set \( \{2^R\} \).
- A decoding operation \( D : B \to M' \) for Bob, with \( M' \equiv M \) being the output register such that

\[
\Pr(M' \neq M) \leq \varepsilon.
\]

We have the following achievability and converse, obtaining results similar to that in [29]. In below, \( \psi_{UA} \) is a classical-quantum state with \( U \) being the classical register.

Theorem 11. Fix a quantum channel \( \mathcal{N}_{A \to B} \) and \( \varepsilon \in (0, 1) \). For any \((R, \varepsilon)\)-code for this quantum channel, it holds that

\[
R \leq \max_{\psi_{UA} | |U| \leq |B|, \sigma_B} \frac{\min_{\psi_U} D_{\text{H}}^\varepsilon(\mathcal{N}_{A \to B}(\psi_{UA})||\sigma_B \otimes \psi_U)}{|B|}.
\]
Further, for every $\delta \in (0, 1)$, there exists an $(R, \varepsilon + \delta)$-code for this quantum channel, if

$$R \leq \max_{\psi_U:A \mid |U| \leq |B|} \min_{\sigma_B} D_H(\mathcal{N}_{A \to B}(\psi_{U:A}) \| \mathcal{N}_{A \to B}(\psi_A) \otimes \psi_U) - \log \frac{4\varepsilon}{\delta^2}.$$  

Proof. We first show the converse for uniform distribution over the message. Fix a quantum state $\sigma_B$. Let $\psi_{MA}$ be the quantum state after Alice’s encoding and $\phi_{MM'}$ be the quantum state after Bob’s decoding. From Facts 9 and 2

$$R \leq D_H(\phi_{MM'} \| \phi_B \otimes D(\sigma_B)) \leq D_H(\mathcal{N}_{A \to B}(\psi_{MA}) \| \psi_M \otimes \sigma_B).$$

Further, from Fact 7

$$R \leq D_H(\mathcal{N}_{A \to B}(\psi_{MA}) \| \psi_M \otimes \mathcal{N}_{A \to B}(\psi_A)) \leq \frac{D(\mathcal{N}_{A \to B}(\psi_{MA}) \| \psi_M \otimes \mathcal{N}_{A \to B}(\psi_A))}{1 - \varepsilon} \leq \frac{\log |B|}{1 - \varepsilon}.$$

The converse now follows by setting $M = U$ and the fact that the state on register $M$ is uniform. The achievability follows similar to the proof of Theorem 4. Alice and Bob share $(as \ shared \ randomness)$. For sending $m \in [2^R]$, Alice inputs the register $A$ generated from $U_m$ according to the state $\psi_{UA}$. Bob performs the position-based decoding strategy to recover the message $m$. A protocol without randomness assistance is obtained since there exists a string $u_1, \ldots, u_{2^R}$ for which the error probability is maintained. \qed

### 4.2 Quantum channel with state

Alice wants to communicate a classical message $M$ chosen from $[2^R]$ to Bob over a quantum channel $\mathcal{N}_{AS \to B}$ such that Bob is able to decode the correct message with probability at least $1 - \varepsilon$. Alice shared entanglement $|\tau\rangle_{SS'}$ with the channel.

**Definition 6.** Let $|\tau\rangle_{SS'}$ be the state shared between Alice and channel. An $(R, \varepsilon)$-code for the quantum channel $\mathcal{N}_{AS \to B}$ consists of

- An encoding operation $E : M \cdot S' \to A$ for Alice, where $M$ takes values uniformly over $[2^R]$.
- A decoding operation $D : B \to M'$ for Bob, with $M' \equiv M$ being the output register such that

$$\Pr(M' \neq M) \leq \varepsilon.$$

We have the following theorem. Below, $\psi_{ASU}$ is a classical-quantum state with $U$ being the classical register.

**Theorem 12.** Fix a quantum channel $\mathcal{N}_{AS \to B}$ with state $\tau_S$ and an $\varepsilon \in (0, 1)$. For every $(R, \varepsilon)$-code for this channel, it holds that

$$R \leq \max_{\psi_{ASU} : \psi_{SU} = \tau_S \otimes \psi_S} \min_{\sigma_B} D_H(\mathcal{N}_{AS \to B}(\psi_{ASU}) \| \psi_U \otimes \sigma_B).$$

Further for every $\delta \in (0, 1)$, there exists an $(R, \varepsilon + 2\delta)$-code for this channel, if

$$R \leq \max_{\psi_{ASU} : \psi_{SU} = \tau_S \otimes \psi_S} D_H(\mathcal{N}_{AS \to B}(\psi_{ASU}) \| \psi_U \otimes \mathcal{N}_{AS \to B}(\psi_A)) - \log \frac{4\varepsilon}{\delta^2}.$$  

Proof. Let $\psi_{ASM'}$ be the state after Alice’s encoding. Fix a quantum state $\sigma_B$. Observe that $\psi_{MS} = \frac{|\tau\rangle_{MM'} \otimes |\tau\rangle_{SS'}}{\| |\tau\rangle_{MM'} \|}_{MM'}$ be the state after Bob’s decoding. Then,

$$R \leq D_H(\phi_{MM'} \| \phi_B \otimes D(\sigma_B)) \leq D_H(\rho_{MB} \| \rho_M \otimes \sigma_B).$$

Further, from Fact 7

$$\log |M| = R \leq D_H(\rho_{MB} \| \rho_M \otimes \rho_B) \leq \frac{\log |B|}{1 - \varepsilon}.$$  

Setting $U = M$, we obtain the converse. The achievability follows similar to the proof of Theorem 4 Alice and Bob share $2^R$ perfectly correlated copies of the state $\psi_U$ (as shared randomness). To send the message $m$, Alice considers the register $U_m$ and applies an isometry on the register $S'$ of $|\tau\rangle_{SS'}$ to obtain the state $\psi_{ASU_m}$. She sends the register $A$ through the channel. Bob performs the position based decoding strategy to decode the message $m$. A protocol without randomness assistance is obtained since there exists a string $u_1, \ldots, u_{2^R}$ for which the error probability is maintained. \qed
4.3 Broadcast quantum channel

Alice wishes to communicate message pair \((m_1, m_2)\) simultaneously to Bob and Charlie over a quantum broadcast channel, where \(m_1\) is for Bob and \(m_2\) is for Charlie, such that both Bob and Charlie output the correct message with probability at least \(1 - \varepsilon\).

**Definition 7.** An \((R_1, R_2, \varepsilon_1, \varepsilon_2)\) entanglement assisted code for the quantum broadcast channel \(\mathcal{N}_{A \rightarrow BC}\) consists of:

- An encoding operation \(E : M_1 M_2 \rightarrow A\) for Alice, where \(M_1, M_2\) take values uniformly over the sets \([2^{R_1}], [2^{R_2}]\) respectively.
- A pair of decoding operations \((D_B, D_C)\), \(D_B : B \rightarrow M'_1\) and \(D_C : C \rightarrow M'_2\), with \((M'_1, M'_2) \equiv (M_1, M_2)\) being the output registers, such that

\[
\Pr(M'_1 \neq M_1) \leq \varepsilon_1, \Pr(M'_2 \neq M_2) \leq \varepsilon_2.
\]

We have the following theorem.

**Theorem 13.** Fix a quantum channel \(\mathcal{N}_{A \rightarrow BC}\) and \(\varepsilon_1, \varepsilon_2 \in (0, 1)\). For any \((R_1, R_2, \varepsilon_1, \varepsilon_2)\)- code for this channel, there exist registers \(U, V\) such that \(|U||V| \leq (|B||C|)^{1/4-1/2}\) and a classical-quantum state \(\psi_{UV}\) satisfying \(\psi_{UV} = \frac{\rho_{UV}}{\Tr_U} \otimes \frac{\rho_U}{\Tr_U}\), with registers \(U, V\) classical, such that

\[
R_1 \leq \min_{\sigma_B} D_1^\varepsilon(\Tr_C \mathcal{N}_{A \rightarrow BC}(\psi_{AU}) || \sigma_B \otimes \psi_U),
\]

\[
R_2 \leq \min_{\sigma_C} D_2^\varepsilon(\Tr_B \mathcal{N}_{A \rightarrow BC}(\psi_{AV}) || \sigma_C \otimes \psi_V).
\]

Furthermore, for every \(\delta \in (0, 1)\) and classical-quantum state \(\psi_{UVC}\) satisfying \(\psi_{UV} = \psi_U \otimes \psi_V\), with registers \(U, V\) being classical, there exists an \((R_1, R_2, \varepsilon_1 + 2\delta, \varepsilon_2 + 2\delta)\)- code for this channel, if

\[
R_1 \leq D_1^\varepsilon(\Tr_C \mathcal{N}_{A \rightarrow BC}(\psi_{AU}) || \Tr_C \mathcal{N}_{A \rightarrow BC}(\phi_A) \otimes \psi_U) - \log \frac{4\varepsilon_1}{\delta^2},
\]

\[
R_2 \leq \min_{\sigma_C} D_2^\varepsilon(\Tr_B \mathcal{N}_{A \rightarrow BC}(\psi_{AV}) || \Tr_B \mathcal{N}_{A \rightarrow BC}(\phi_A) \otimes \psi_V) - \log \frac{4\varepsilon_2}{\delta^2}.
\]

**Proof.** We first show the converse for uniform distribution over messages. Fix quantum states \(\sigma_B, \tau_C\). Let \(\psi_{AM_1 M_2}\) be the quantum state after Alice’s encoding, \(\rho_{BM_1 M_2}\) be the quantum state after the action of the channel and \(\phi_{M'_1 M'_2 M_1 M_2}\) be the quantum state after Bob’s and Charlie’s decoding. Observe that \(\psi_{M_1 M_2} = \phi_{M_1 M_2} = \frac{I_{M_1} \otimes I_{M_2}}{\Tr_{M_1} \otimes \Tr_{M_2}}\).

From Facts 9 and 2,

\[
R_1 \leq D_1^\varepsilon(\phi_{M_1 M'_1} \otimes \sigma_B) \leq D_1^\varepsilon(\rho_{M_1 B} \otimes \sigma_B).
\]

Similarly,

\[
R_2 \leq D_2^\varepsilon(\rho_{M_2 C} \otimes \tau_C).
\]

Finally, from Fact 7,

\[
\log |M_1||M_2| = R_1 + R_2 \leq D_1^{\varepsilon_1 + \varepsilon_2}(\phi_{M_1 M_2 M'_1 M'_2} \otimes \phi_{M_1 M_2} \otimes \phi_{M'_1 M'_2}) \leq D_1^{\varepsilon_1 + \varepsilon_2}(\psi_{M_1 M_2} \otimes \psi_{M'_1 M'_2}) \leq \frac{\log |B||C|}{1 - \varepsilon_1 - \varepsilon_2}.
\]

Setting \(U = M_1, V = M_2\), the converse follows.

The achievability follows similar to the proof of Theorem 6. Alice and Bob share \(2^{R_1}\) perfectly correlated copies of the state \(\psi_U\) (as shared randomness). Alice and Charlie share \(2^{R_2}\) perfectly correlated copies of the state \(\psi_V\) (as shared randomness). To send the messages \((m_1, m_2)\), Alice inputs the register \(A\) obtained from \(U_{m_1}, V_{m_2}\) according to the quantum state \(\psi_{AM_1 M_2}\). Bob and Charlie respectively perform the position based decoding strategy to obtain the messages \(m_1, m_2\). A protocol without randomness assistance is obtained since there exists a string \(u_1, \ldots, u_{2^R_1}, v_1, \ldots, v_{2^R_2}\) for which the error probability is maintained.
4.4 Multiple-access channel

Alice wants to communicate a classical message $m_1$ chosen uniformly from $[2^{R_1}]$ to Charlie and Bob wants to communicate a classical message $m_2$ chosen uniformly from $[2^{R_2}]$ to Charlie, over a channel $\mathcal{N}_{AB\rightarrow C}$. This is the multiple access channel.

**Definition 8.** An $(R_1, R_2, \varepsilon_1, \varepsilon_2)$-code for the quantum multiple access channel $\mathcal{N}_{AB\rightarrow C}$ consists of

- Encoding operations $E_1 : M_1 \rightarrow A$ and $E_2 : M_2 \rightarrow B$, where $M_1, M_2$ take values uniformly over the sets $[2^{R_1}], [2^{R_2}]$ respectively.
- A decoding operation $D : C \rightarrow M'_1 M'_2$, with $M'_1 \equiv M_1$ and $M'_2 \equiv M_2$ such that for all $(m_1, m_2)$,
  \[ \Pr(M'_1 \neq M_1) \leq \varepsilon_1, \Pr(M'_2 \neq M_2) \leq \varepsilon_2. \]

We have the following result.

**Theorem 14.** Fix a quantum channel $\mathcal{N}_{AB\rightarrow C}$ and $\varepsilon_1, \varepsilon_2 \in (0, 1)$. For every $(R_1, R_2, \varepsilon_1, \varepsilon_2)$-code for this channel, there exist registers $U, V$ satisfying $|U||V| \leq |C|^{1-1/\varepsilon_1-\varepsilon_2}$ and classical-quantum states $\psi_{UA}, \psi_{VB}$ satisfying $\psi_U = \frac{1}{|U|}$ and $\psi_{V'} = \frac{1}{|V|}$ such that

\[ R_1 \leq \min_{\sigma_C} D_H(\rho_{CU} \parallel \sigma_C \otimes \rho_U), \]

and

\[ R_2 \leq \min_{\tau_C} D_H(\rho_{CV} \parallel \tau_C \otimes \rho_V), \]

where $\rho_{CUV} \equiv \mathcal{N}_{AB\rightarrow C}(\psi_{UA} \otimes \psi_{VB})$.

Furthermore, for every classical-quantum state $\psi_{UA}, \psi_{VB}$, there exists a $(R_1, R_2, 4(\varepsilon_1 + \varepsilon_2 + 2\delta), 4(\varepsilon_1 + \varepsilon_2 + 2\delta))$-code for this channel, if

\[ R_1 \leq D_{\overline{H}}(\rho_{CU} \parallel \rho_C \otimes \rho_U) - \log \frac{1}{\delta}, \]

and

\[ R_2 \leq D_{\overline{H}}(\rho_{CV} \parallel \rho_C \otimes \rho_V) - \log \frac{1}{\delta}, \]

where $\rho_{CUV} \equiv \mathcal{N}_{AB\rightarrow C}(\psi_{UA} \otimes \psi_{VB})$. In fact, the following upper bound holds,

\[ \Pr(M'_1, M'_2 \neq M_1, M_2) \leq 4 \cdot (\varepsilon_1 + \varepsilon_2 + 2\delta). \]

**Proof.** We first show the converse. Fix quantum states $\sigma_C, \tau_C$. Let $\psi_{M_1 A} \otimes \psi_{M_2 B}$ be the quantum state after Alice’s encoding and $\rho_{M_1 M_2 C}$ be the state after the action of the channel. Let $\phi_{M_1 M_2 M'_1 M'_2}$ be the state after Charlie’s decoding. From Facts 9 and 2

\[ R_1 \leq D_{\overline{H}}(\phi_{M_1 M'_1 M'_2} \parallel \phi_{M_1} \otimes D(\sigma_C)) \leq D_{\overline{H}}(\rho_{M_1 C} \parallel \rho_{M_1} \otimes \sigma_C). \]

Similarly,

\[ R_2 \leq D_{\overline{H}}(\rho_{M_2 C} \parallel \rho_{M_2} \otimes \tau_C). \]

Further, from Fact 7

\[ \log |M_1| |M_2| = R_1 + R_2 \leq D_{\overline{H}}(\rho_{M_1 M_2 C} \parallel \rho_{M_1 M_2} \otimes \rho_C) \leq \frac{\log |C|}{1 - \varepsilon_1 - \varepsilon_2}. \]

We set $U = M_1, V = M_2$, which proves the converse.

The achievability is similar to the proof of Theorem 10. Alice and Charlie share $2^{R_1}$ perfectly correlated copies of the state $\psi_U$ (as shared randomness). Bob and Charlie share $2^{R_2}$ perfectly correlated copies of $\psi_V$ (as shared randomness). To send message $m_1$, Alice inputs the register $A$ generated from $U_{m_1}$ according to the quantum state $\psi_{AU_{m_1}}$. To send message $m_2$, Bob inputs the register $B$ generated from $V_{m_2}$ according to the quantum state $\psi_{BV_{m_2}}$. Charlie performs the sequential position-based decoding, decoding message $m_1$ and then $m_2$. Another protocol is obtained where Charlie decodes $m_2$ and then $m_1$. The error analysis follows in a similar fashion. A protocol without randomness assistance is obtained since there exists a string $u_1, \ldots, u_{2^{R_1}}, v_1, \ldots, v_{2^{R_2}}$ for which the error probability is maintained. □
Conclusion

We have obtained a near-optimal one-shot characterization of the amount of communication for a wide family of quantum channels in the one-shot setting. Our one-shot bounds for the entanglement-unassisted case (Section 4) have the property that the register sizes involved in the bounds are bounded. We leave the task of obtaining similar near-optimal bounds for entanglement-assisted cases with bounded register dimensions (except for the point to point case) in Section 5 for future work.

We stress that similar results could also be obtained for the classical case. But this would lead to formulas that are not single letter in the asymptotic and i.i.d. setting [8, Section 4.3] (except for the point to point classical channel). Interestingly, there are alternative characterizations known for the multiple access classical channel and classical channel with state, which lead to single letter optimal bounds in the asymptotic and i.i.d. setting and near optimal bounds for the one-shot setting. Unfortunately, as discussed earlier, analogous bounds are not known to be single letter for the quantum multiple access channel (nor are they known to be near-optimal in the one-shot setting). Finding a single letter optimal rate region is an important open question in the asymptotic and i.i.d. theory of quantum channel coding over networks.

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