Concentration in the flux approximation limit of Riemann solutions to the extended Chaplygin gas equations

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Abstract

In this paper, two kinds of occurrence mechanism on the phenomenon of concentration and the formation of delta shock wave in the flux approximation limit of Riemann solutions to the extended Chaplygin gas equations are analyzed and identified. Firstly, the Riemann problem of the extended Chaplygin gas equations is solved completely. Secondly, we rigorously show that, as the pressure vanishes, any two-shock Riemann solution to the extended Chaplygin gas equations tends to a $\delta$-shock solution to the transport equations, and the intermediate density between the two shocks tends to a weighted $\delta$-measure that forms the $\delta$-shock; any two-rarefaction-wave Riemann solution to the extended Chaplygin gas equations tends to a two-contact-discontinuity solution to the transport equations, and the nonvacuum intermediate state between the two rarefaction waves tends to a vacuum state. At last, we also show that, as the pressure approaches the generalized Chaplygin pressure, any two-shock Riemann solution tends to a delta-shock solution to the generalized Chaplygin gas equations.

Key words: Extended Chaplygin gas; Delta shock wave; flux approximation limit; Riemann solutions; Transport equations; generalized Chaplygin gas.

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1. Introduction

The extended Chaplygin gas equations can be expressed as

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P)_x &= 0,
\end{align*}
\]

(1.1)

where $\rho$, $u$ and $P$ represent the density, the velocity and the scalar pressure, respectively, and

\[
P = A\rho^n - \frac{B}{\rho^\alpha}, \quad 1 \leq n \leq 3, \quad 0 < \alpha \leq 1,
\]

(1.2)

with two parameters $A, B > 0$. This model was proposed by Naji in 2014 \cite{25} to study the evolution of dark energy. When $B = 0$ in (1.2), $P = A\rho^n$ is the standard equation of state of perfect fluid. Up to now, various kinds of theoretical models have been proposed to interpret the behavior of dark energy. Specially, when $n = 1$ in (1.2), it reduces to the state equation for modified Chaplygin gas, which was originally proposed by Benaoum in 2002 \cite{2}. As an exotic fluid, such a gas can explain the current accelerated expansion of the universe. Whereas when $A = 0$ in (1.2), $P = -\frac{B}{\rho^\alpha}$ is called the pressure for the generalized Chaplygin gas \cite{27}. Furthermore, when $\alpha = 1$, $P = -\frac{B}{\rho}$ is called the pressure for (pure) Chaplygin gas which was introduced by Chaplygin \cite{9}, Tsien \cite{38} and von Karman \cite{18} as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. Such a gas own a negative pressure and occurs in certain theories of cosmology. It has been also advertised as a possible model for dark energy \cite{3, 15, 28}.

When two parameters $A, B \to 0$, the limit system of (1.1) with (1.2) formally becomes the following transport equations:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2)_x &= 0,
\end{align*}
\]

(1.3)
which was also called the zero-pressure gas dynamics, and can be derived from Boltzmann equations \(^4\) and the flux-splitting numerical schemes for the full compressible Euler equations \(^6, 23\). It can also be used to describe some important physical phenomena, such as the motion of free particles sticking together under collision and the formation of large scale structures in the universe \(^7, 14, 29\).

The transport equation \((1.3)\) has been studied extensively since 1994. The existence of measure solutions of the Riemann problem was first proved by Bouchut \(^4\) and the existence of the global weak solutions was obtained by Brenier and Grenier \(^1\) and E.Rykov and Sinai \(^14\). Sheng and Zhang \(^34\) discovered that the \(\delta\)-shock and vacuum states do occur in the Riemann solutions to the transport equation \((1.3)\) by the vanishing viscosity method. Huang and Wang \(^17\) proved the uniqueness of the weak solution when the initial data is a Radon measure. Also see \(^31, 40, 41, 42, 44\) for more related results.

\(\delta\)-shock is a kind of nonclassical nonlinear waves on which at least one of the state variables becomes a singular measure. Korochinski \(^19\) firstly introduced the concept of the \(\delta\)-function into the classical weak solution in his unpublished Ph. D. thesis. Tan, Zhang and Zheng \(^37\) considered some 1-D reduced system and discovered that the form of \(\delta\)-functions supported on shocks was used as parts in their Riemann solutions for certain initial data. LeFloch et al. \(^20\) applied the approach of nonconservative product to consider nonlinear hyperbolic systems in the nonconservative form. We can also refer to \(^4, 22, 34\) for related equations and results. Recently, the weak asymptotic method was widely used to study the \(\delta\)-shock wave type solution by Danilov and Shelkovich et al. \(^12, 13, 30\).

As for delta shock waves, one research focus is to explore the phenomena of concentration and cavitation and the formation of delta shock waves and vacuum states in solutions. In \(^10\), Chen and Liu considered the Euler equations for isentropic fluids, i.e., in \((1.1)\) they took the prototypical pressure function as follows:

\[
P = \varepsilon \frac{\rho^\gamma}{\gamma}, \quad \gamma > 1.
\]

They analyzed and identified the phenomena of concentration and cavitation and the formation of \(\delta\)-shocks and vacuum states as \(\varepsilon \to 0\), which checked the numerical observation for the 2-D case by Chang, Chen and Yang \(^7, 8\). They also pointed out that the occurrence of \(\delta\)-shocks and vacuum states in the process of vanishing pressure limit can be regarded as a phenomenon of resonance between the two characteristic fields. In \(^11\), they made a further step to generalize this result to the nonisentropic fluids. Specially, for \(\gamma = 1\) in \((1.4)\), the vanishing pressure limit has been studied by Li \(^21\). Besides, the results were extended to the relativistic Euler equations for polytropic gases by Yin and Sheng \(^47\), the perturbed Aw-Rascle model by Shen and Sun \(^33\) and the modified Chaplygin gas equations for by Yang and Wang \(^15, 16\). For other related works, we can also see \(^24, 48\).

In this paper, we focus on the extended Chaplygin gas equations \((1.1)\) to discuss the phenomena of concentration and cavitation and the formation of delta shock waves and vacuum states in solutions as the double parameter pressure vanishes wholly or partly, which corresponds to a two parameter limit of solutions in contrast to the previous works in \(^14, 11, 24, 33, 47\). Equivalently, we will study the limit behavior of Riemann solutions to the extended Chaplygin gas equations as the pressure vanishes, or tends to the generalized Chaplygin pressure.

It is noticed that, When \(A, B \to 0\), the system \((1.1)\) with \((1.2)\) formally becomes the transport equations \((1.3)\). For fixed \(B\), When \(A \to 0\), the system \((1.1)\) with \((1.2)\) formally becomes the generalized Chaplygin gas equations

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 - \frac{B}{\rho^\gamma})_x &= 0.
\end{align*}
\]

When \(\alpha = 1\), it is just the Chaplygin gas equations. In 1998, Brenier \(^5\) firstly studied the 1-D Riemann problem and obtained the solutions with concentration when initial data belong to a certain domain in the phase plane. Recently, Guo, Sheng and Zhang \(^10\) abandoned this constrain and constructively obtained the general solutions of the 1-D Riemann problem in which the \(\delta\)-shock wave developed. Moreover, in that paper, they also systematically studied the 2-D Riemann problem for isentropic Chaplygin gas equations. In \(^32\), Wang solved the Riemann problem of \((1.7)\) by the weak asymptotic method. It has been shown that, in their results, \(\delta\)-shocks do occur in the Riemann solutions, but vacuum states do not. For more results about Chaplygin gas, one can refer to \(^20, 32, 36, 43\).

In this paper, we first solve the Riemann problem of system \((1.1)\) with Riemann initial data

\[
(\rho, u)(x, 0) = (\rho\pm, u\pm), \quad \pm x > 0,
\]
where $\rho_+ > 0$, $u_+$ are arbitrary constants. With the help of analysis method in phase plane, we constructed the Riemann solutions with four different structures: $R_1R_2$, $R_1S_2$, $S_1R_2$ and $S_1S_2$.

Then we analyze the formation of $\delta$-shocks and vacuum states in the Riemann solutions as the pressure vanishes. It is shown that, as the pressure vanishes, any two-shock Riemann solution to the extended Chaplygin gas equations tends to a $\delta$-shock solution to the transport equations, and the intermediate density between the two shocks tends to a weighted $\delta$-measure that forms the $\delta$-shock; by contrast, any two-rarefaction-wave Riemann solution to the extended Chaplygin gas equations tends to a two-contact-discontinuity solution to the transport equations, and the nonvacuum intermediate state between the two rarefaction waves tends to a vacuum state, even when the initial data stay away from the vacuum. As a result, the delta shocks for the transport equations result from a phenomenon of concentration, while the vacuum states results from a phenomenon of cavitation in the vanishing pressure limit process. These results are completely consistent with that in [10], and also cover those obtained in [45, 46].

In addition, we also proved that as the pressure tends to the generalized Chaplygin pressure ($A \to 0$), any two-shock Riemann solution to the extended Chaplygin gas equations tends to a $\delta$-shock solution to the generalized Chaplygin gas equations, and the intermediate density between the two shocks tends to a weighted $\delta$-measure that forms the $\delta$-shock; by contrast, any two-rarefaction-wave Riemann solution to the extended Chaplygin gas equations tends to the two-rarefaction-wave (two-contact-discontinuity for $\alpha = 1$) solution to the transport equations, and the intermediate state between the two rarefaction waves (two contact discontinuities) is a nonvacuum state. Consequently, the delta shocks for the generalized Chaplygin gas equations result from a phenomenon of concentration in the partly vanishing pressure limit process.

From the above analysis, we can find two kinds of occurrence mechanism on the phenomenon of concentration and the formation of delta shock wave. On one hand, since the strict hyperbolicity of the limiting system (1.3) fails, see Section 4, the delta shock wave forms in the limit process as the pressure vanishes. This is consistent with those results obtained in [10, [11, 24, 33, 45, 47]. On the other hand, the strict hyperbolicity of the limiting system (1.5) is preserved, see Section 5, the formation of delta shock waves still occur as the pressure partly vanishes. In this regard, it is different from that in [10, 11, 24, 33, 45, 47]. In any case, the phenomenon of concentration and the formation of delta shock wave can be regarded as a process of resonance formation between two characteristic fields.

The paper is organized as follows. In Section 2, we restate the Riemann solutions to transport equations (1.3) and the generalized Chaplygin gas equations (1.5). In Section 3, we investigate the Riemann problem of the extended Chaplygin gas equations (1.1)-(1.2) and examine the dependence of the Riemann solutions on the two parameters $A, B > 0$. In Section 4, we analyze the limit of Riemann solutions to the extended Chaplygin gas equations (1.1)-(1.2) with (1.8) as the pressure vanishes. In Section 5, we discuss the limit of Riemann solutions to the extended Chaplygin gas equations (1.1)-(1.2) with (1.8) as the pressure approaches to the generalized Chaplygin pressure. Finally, conclusions and discussions are drawn in Section 6.

2. Preliminaries

2.1. Riemann problem for the transport equations

In this section, we restate the Riemann solutions to the transport equations (1.3) with initial data (1.0). See [34] for more details.

The transport equations (1.3) have a double eigenvalue $\lambda = u$ and only one right eigenvectors $\vec{r} = (1, 0)^T$. Furthermore, we have $\nabla \lambda \cdot \vec{r} = 0$, which means that $\lambda$ is linearly degenerate. The Riemann problem (1.3) and (1.0) can be solved by contact discontinuities, vacuum or $\delta$-shocks connecting two constant states $(\rho_\pm, u_\pm)$.

By taking the self-similar transformation $\xi = \frac{t}{T}$, the Riemann problem is reduced to the boundary value problem of the ordinary differential equations:

\[
\begin{cases}
\begin{align*}
-\xi \rho_\xi + (pu)_\xi &= 0, \\
-\xi (pu)_\xi + (pu^2)_\xi &= 0,
\end{align*}
\end{cases}
\tag{2.1}
\]

with $(\rho, u)(\pm \infty) = (\rho_\pm, u_\pm)$. 

For the case \( u_- < u_+ \), there is no characteristic passing through the region \( \{ \xi : u_- < \xi < u_+ \} \), so the vacuum should appear in the region. The solution can be expressed as

\[
(\rho, u)(\xi) = \begin{cases} 
(\rho_-, u_-), & -\infty < \xi \leq u_-, \\
(0, \xi), & u_- < \xi < u_+, \\
(\rho_+, u_+), & u_+ \leq \xi < \infty.
\end{cases}
\]  

(2.2)

For the case \( u_- = u_+ \), it is easy to see that the constant states \((\rho_\pm, u_\pm)\) can be connected by a contact discontinuity.

For the case \( u_- > u_+ \), a solution containing a weighted \( \delta \)-measure on a curve will be constructed. Let \( x = x(t) \) be a discontinuity curve, we consider a piecewise smooth solution of \((\rho, u)(\xi)\) in the form

\[
(\rho, u)(x, t) = \begin{cases} 
(\rho_-, u_-), & x < x(t), \\
(w(t)\delta(x - x(t)), u_3(t)), & x = x(t), \\
(\rho_+, u_+), & x > x(t).
\end{cases}
\]  

(2.3)

In order to define the measure solution as above, like as in [10, 11, 34], the two-dimensional weighted \( \delta \)-measure \( w(t)\delta_S \) supported on a smooth curve \( S = \{(x(s), t(s)) : a \leq s \leq b\} \) should be introduced as follows:

\[
<w(\cdot)\delta_S, \psi(\cdot, \cdot)> = \int_a^b w(s)\psi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2} ds,
\]  

(2.4)

for any \( \psi \in C^\infty_c(R \times R_+) \).

As shown in [34], for any \( \psi \in C^\infty_c(R \times R_+) \), the \( \delta \)-measure solution \((\rho, u)(\xi)\) constructed above satisfies

\[
\begin{align*}
\langle \rho, \psi_t \rangle + \langle \rho \psi \rangle &= 0, \\
\langle \rho \psi \rangle &= 0,
\end{align*}
\]  

(2.5)

in which

\[
\langle \rho, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0 \psi dx dt + \langle w_1(\cdot)\delta_S, \psi(\cdot, \cdot) \rangle,
\]

\[
\langle \rho \psi \rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0 u_0 \psi dx dt + \langle w_2(\cdot)\delta_S, \psi(\cdot, \cdot) \rangle,
\]

where

\[
\rho_0 = \rho_- + [\rho]H(x - \sigma t), \quad \rho_0 u_0 = \rho_- u_- + [\rho u]H(x - \sigma t),
\]

and

\[
w_1(t) = \frac{t}{\sqrt{1 + \sigma^2}}(\sigma[\rho] - [\rho u]), \quad w_2(t) = \frac{t}{\sqrt{1 + \sigma^2}}(\sigma[\rho u] - [\rho u^2]).
\]

Here \( H(x) \) is the Heaviside function given by \( H(x) = 1 \) for \( x > 0 \) and \( H(x) = 0 \) for \( x < 0 \).

Substituting \((2.3)\) into \((2.5)\), one can derive the generalized Rankine-Hugoniot conditions

\[
\begin{align*}
\frac{dx(t)}{dt} &= u_3(t), \\
\frac{du(t)}{dt} &= [\rho]u_3(t) - [\rho u], \\
\frac{d(u(t)u_3(t))}{dt} &= [\rho u]u_3(t) - [\rho u^2]
\end{align*}
\]  

(2.6)

where \([\rho] = \rho_+ - \rho_-\), etc.

Through solving \((2.6)\) with \( x(0) = 0, \ w(t) = 0 \), we obtain

\[
\begin{align*}
u_3(t) &= \sigma = \frac{\sqrt{\rho_- - u_-} + \sqrt{\rho_+ u_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\
x(t) &= \sigma t, \\
w(t) &= -\sqrt{\rho_- - \rho_+}(u_+ - u_-)t,
\end{align*}
\]  

(2.7)

Moreover, the \( \delta \)-measure solution \((\rho, u)(\xi)\) satisfies the \( \delta \)-entropy condition:

\[
u_+ < \sigma < u_-,
\]

which means that all the characteristics on both sides of the \( \delta \)-shock are incoming.
2.2. Riemann problem for the generalized Chaplygin gas equations

In this section, we solve the Riemann problem for the generalized Chaplygin gas equations \( \text{(1.5)} \) with \( \text{(1.6)} \), which one can also see in \( \text{[16, 39]} \).

It is easy to see that \( \text{(1.5)} \) has two eigenvalues

\[
\lambda^B_1 = u - \sqrt{\alpha B \rho} \cdot \frac{\alpha + 1}{2}, \quad \lambda^B_2 = u + \sqrt{\alpha B \rho} \cdot \frac{\alpha + 1}{2},
\]

with corresponding right eigenvectors

\[
\vec{r}_1^B = (-\sqrt{\alpha B \rho} \cdot \frac{\alpha + 1}{2}, \rho)^T, \quad \vec{r}_2^B = (\sqrt{\alpha B \rho} \cdot \frac{\alpha + 1}{2}, \rho)^T.
\]

So \( \text{(1.5)} \) is strictly hyperbolic for \( \rho > 0 \). Moreover, when \( 0 < \alpha < 1 \), we have \( \nabla \lambda_i^B \cdot \vec{r}_i^B \neq 0, \ i = 1, 2 \), which implies that \( \lambda_i^B \) and \( \lambda_i^B \) are both genuinely nonlinear and the associated waves are rarefaction waves and shock waves. When \( \alpha = 1 \), \( \nabla \lambda_i^B \cdot \vec{r}_i^B = 0, \ i = 1, 2 \), which implies that \( \lambda_i^B \) and \( \lambda_i^B \) are both linearly degenerate and the associated waves are both contact discontinuities, see \( \text{[33]} \).

Since system \( \text{(1.5)} \) and the Riemann initial data \( \text{(1.6)} \) are invariant under stretching of coordinates \( (x, t) \rightarrow (\beta x, \beta t) \) (\( \beta \) is constant), we seek the self-similar solution

\[
(\rho, u)(x, t) = (\rho, u)(\xi), \quad \xi = \frac{x}{t}.
\]

Then the Riemann problem \( \text{(1.5)} \) and \( \text{(1.6)} \) is reduced to the following boundary value problem of the ordinary differential equations:

\[
\begin{aligned}
-\xi \rho \xi + (\rho u) \xi &= 0, \\
-\xi (\rho u) \xi + (\rho u^2 - \frac{B}{\rho}) \xi &= 0,
\end{aligned}
\]

with \( (\rho, u)(\pm \infty) = (\rho_\pm, u_\pm) \).

Besides the constant solution, it provides the backward rarefaction wave

\[
\begin{aligned}
\overleftarrow{R}(\rho_-, u_-) : & \quad \xi = \lambda^B_1 = u - \sqrt{\alpha B \rho} \cdot \frac{\alpha + 1}{2}, \\
u - \frac{2\sqrt{\alpha B}}{1 + \alpha} \rho \cdot \frac{\alpha + 1}{2} &= u_-, \quad \rho < \rho_-, \quad u = \frac{\sqrt{\alpha B} \rho - \frac{\alpha + 1}{2}}{1 + \alpha} \rho - \frac{\alpha + 1}{2},
\end{aligned}
\]

and the forward rarefaction wave

\[
\begin{aligned}
\overrightarrow{R}(\rho_-, u_-) : & \quad \xi = \lambda^B_2 = u + \sqrt{\alpha B \rho} \cdot \frac{\alpha + 1}{2}, \\
u + \frac{2\sqrt{\alpha B}}{1 + \alpha} \rho \cdot \frac{\alpha + 1}{2} &= u_-, \quad \rho > \rho_-, \quad u = \frac{\sqrt{\alpha B} \rho - \frac{\alpha + 1}{2}}{1 + \alpha} \rho - \frac{\alpha + 1}{2}.
\end{aligned}
\]

When \( \alpha = 1 \), the backward (forward) rarefaction wave becomes the backward (forward) contact discontinuity.

For a bounded discontinuity at \( \xi = \sigma \), the Rankine-Hugoniot conditions hold:

\[
\begin{aligned}
-\sigma \rho \sigma + [\rho u] &= 0, \\
-\sigma \rho^2 + [\rho u^2 - \frac{B}{\rho}] &= 0,
\end{aligned}
\]

where \([\rho] = \rho - \rho_-\), etc. Together with the Lax shock inequalities, \( \text{(2.11)} \) gives the backward shock wave

\[
\begin{aligned}
\overleftarrow{S}(\rho_-, u_-) : & \quad \sigma^B_1 = \frac{\rho u - \rho_- u_-}{\rho - \rho_-}, \\
u - u_- &= -\sqrt{B(\frac{\rho}{\rho_-} - 1)\left(\frac{1}{\rho} - \frac{1}{\rho_-}\right)}, \quad \rho > \rho_-,
\end{aligned}
\]

and the forward shock wave

\[
\begin{aligned}
\overrightarrow{S}(\rho_-, u_-) : & \quad \sigma^B_2 = \frac{\rho u - \rho_- u_-}{\rho - \rho_-}, \\
u - u_- &= -\sqrt{B(\frac{\rho}{\rho_-} - 1)\left(\frac{1}{\rho} - \frac{1}{\rho_-}\right)}, \quad \rho < \rho_-.
\end{aligned}
\]

When \( \alpha = 1 \), the backward (forward) shock wave becomes the backward (forward) contact discontinuity.
Furthermore, for a given left state \((\rho_-, u_-)\), the backward shock wave \(\vec{S}(\rho_-, u_-)\) has a straight line \(u = u_- - \sqrt{B \rho_-^{-\frac{\alpha+1}{2}}}\), as its asymptote, and for a given right state \((\rho_+, u_+)\), the forward shock wave \(\overline{S}(\rho_+, u_+)\) has a straight line \(u = u_+ + \sqrt{B \rho_+^{-\frac{\alpha+1}{2}}}\) as its asymptote.

It is easy to see that, when \(u_+ + \sqrt{B \rho_+^{-\frac{\alpha+1}{2}}} \leq u_- - \sqrt{B \rho_-^{-\frac{\alpha+1}{2}}}\), the backward shock wave \(\vec{S}(\rho_-, u_-)\) cannot intersect the forward shock wave \(\overline{S}(\rho_+, u_+)\), a delta shock wave must develop in solutions. Under the definition (2.7), a delta shock wave can be introduced to construct the solution of (1.5)–(1.6), which can be expressed as

\[
(\rho, u)(x, t) = \begin{cases} 
(\rho_-, u_-), & x < \sigma^B t, \\
(w^B(t)\delta(x - \sigma^B t), \sigma^B), & x = \sigma^B t, \\
(\rho_+, u_+), & x > \sigma^B t,
\end{cases}
\]  

(2.14)

with

\[
B = \begin{cases} 
\frac{B}{\rho_+}, & x < \sigma^B t, \\
0, & x = \sigma^B t, \\
\frac{B}{\rho_-}, & x > \sigma^B t,
\end{cases}
\]

see [3].

By the weak solution definition in Subsection 2.1, for the system (1.5) we can get the following generalized Rankine-Hugoniot conditions

\[
\begin{align*}
\frac{dx^B(t)}{dt} &= u^B_0(t) = \sigma^B, \\
\frac{d\rho^B(t)}{dt} &= u^B_0(t)\rho - [\rho u], \\
\frac{d(w^B(t)u^B_0(t))}{dt} &= u^B_0(t)[\rho u] - [\rho u^2 - \frac{1}{\rho}].
\end{align*}
\]

(2.15)

where \(x^B(t), w^B(t)\) and \(u^B_0(t)\) are respectively denote the location, weight and propagation speed of the \(\delta\)-shock, \([\rho] = \rho(x^B(t) + 0, t) - \rho(x^B(t) - 0, t)\) denotes the jump of the function \(\rho\) across the \(\delta\)-shock.

Then by solving (2.15) with initial data \(x(0) = 0, w^B(0) = 0\), under the entropy condition

\[
u_+ + \sqrt{\alpha B \rho_+^{-\frac{\alpha+1}{2}}} < \sigma^B < u_- - \sqrt{\alpha B \rho_-^{-\frac{\alpha+1}{2}}},
\]

we can obtain

\[
w^B(t) = \left(\rho_+ \rho_- \left((u_+ - u_-)^2 - \frac{1}{\rho_+} - \frac{1}{\rho_-} \left(\frac{B}{\rho_+^\alpha} - \frac{B}{\rho_-^\alpha}\right)\right)\right)^{\frac{1}{2}} + t,
\]

(2.17)

\[
\sigma^B = \frac{\rho_+ u_+ - \rho_- u_- + \frac{dw^B(t)}{dt}}{\rho_+ - \rho_-},
\]

(2.18)

when \(\rho_+ \neq \rho_-\), and

\[
w^B(t) = (\rho_+ u_+ - \rho_- u_-) t,
\]

(2.19)

\[
\sigma^B = \frac{1}{2}(u_+ + u_-),
\]

(2.20)

when \(\rho_+ = \rho_-\).

In the phase plane \((\rho > 0, u \in \mathbb{R})\), given a constant state \((\rho_-, u_-)\), we draw the elementary wave curves (2.9)–(2.10) and (2.12)–(2.13) passing through this point, which are denoted by \(\overline{R}, \overline{R}, \overline{S}\) and \(\overline{S}\) respectively. The backward shock wave \(\vec{S}\) has an asymptotic line \(u = u_- - \sqrt{B \rho_-^{-\frac{\alpha+1}{2}}}\). In addition, we draw a \(S_{\delta}\) curve, which is determined by

\[
u + \sqrt{\alpha B \rho_+^{-\frac{\alpha+1}{2}}} = u_- - \sqrt{B \rho_-^{-\frac{\alpha+1}{2}}}, \quad \rho > 0.
\]

(2.21)

Then the phase plane can be divided into five parts \(I(\rho_-, u_-), II(\rho_-, u_-), III(\rho_-, u_-), IV(\rho_-, u_-)\) and \(V(\rho_-, u_-)\), see Fig.1.

By the analysis method in the phase plane, one can construct the Riemann solutions for any given \((\rho_+, u_+)\) as follows:
(1) \((\rho_+, u_+) \in I(\rho_-, u_-)\): \(\overrightarrow{R} + \overrightarrow{R}\); (2) \((\rho_+, u_+) \in II(\rho_-, u_-)\): \(\overrightarrow{R} + \overrightarrow{S}\); (3) \((\rho_+, u_+) \in III(\rho_-, u_-)\): \(\overrightarrow{S} + \overrightarrow{R}\); (4) \((\rho_+, u_+) \in IV(\rho_-, u_-)\): \(\overrightarrow{S} + \overrightarrow{S}\); (5) \((\rho_+, u_+) \in V(\rho_-, u_-)\): \(\delta\)-shock.

3. Riemann problem for the extended Chaplygin gas equations

In this section, we first solve the elementary waves and construct solutions to the Riemann problem of (1.1)-(1.2) with (1.6), and then examine the dependence of the Riemann solutions on the two parameters \(A, B > 0\).

The eigenvalues of the system (1.1)-(1.2) are

\[
\lambda_{1,2}^{AB} = u \pm \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}},
\]

with corresponding right eigenvectors

\[
\hat{r}_{1,2}^{AB} = (-\rho, \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}})^T, \quad \hat{r}_{2,2}^{AB} = (\rho, \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}})^T.
\]

Moreover, we have

\[
\nabla\lambda_i^{AB} \cdot \hat{r}_i^{AB} = \frac{An(n+1)\rho^{n+\alpha} + (1-\alpha)\alpha B}{2\sqrt{An\rho^{n+\alpha} + \alpha B}\rho^{\alpha+1}} > 0 \quad (i = 1, 2).
\]

Thus \(\lambda_1^{AB}\) and \(\lambda_2^{AB}\) are genuinely nonlinear and the associated elementary waves are shock waves and rarefaction waves.

For (1.1)-(1.2) with (1.6) are invariant under uniform stretching of coordinates \((x, t) \rightarrow (\beta x, \beta t)\) where constant \(\beta > 0\), we seek the self-similar solution

\[
(\rho, u)(x, t) = (\rho(\xi), u(\xi)), \quad \xi = \frac{x}{t}.
\]

Then the Riemann problem (1.1)-(1.2) with (1.6) is reduced to the boundary value problem of the following ordinary differential equations:

\[
\begin{cases}
-\xi \rho_\xi + (\rho u)_\xi = 0, \\
-\xi (\rho u)_\xi + (\rho u^2 + P)_\xi = 0,
\end{cases}
\quad P = Ap^n - \frac{B}{\rho^n},
\]

with \((\rho, u)(\pm \infty) = (\rho_\pm, u_\pm)\).
Any smooth solutions of (3.1) satisfies
\[
\left( u - \xi \right) \frac{d}{du} \left( \frac{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}{u - \xi} \right) \frac{d\rho}{du} = 0. \tag{3.2}
\]

It provides either the constant state solutions
\[(\rho, u)(\xi) = \text{constant},\]
or the rarefaction wave which is a continuous solutions of (3.1) in the form \((\rho, u)(\xi)\). Then, according to (3.2), for a given left state \((\rho_-, u_-)\), the rarefaction wave curves in the phase plane, which are the sets of states that can be connected on the right by a 1-rarefaction wave or 2-rarefaction wave, are as follows:
\[
R_1(\rho_-, u_-) : \begin{cases}
\xi = \lambda_1 = u - \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}, \\
u - u_- = -\int_{\rho_-}^{\rho} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}}{\rho} d\rho,
\end{cases} \tag{3.3}
\]
and
\[
R_2(\rho_-, u_-) : \begin{cases}
\xi = \lambda_2 = u + \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}, \\
u - u_- = \int_{\rho_-}^{\rho} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}}{\rho} d\rho.
\end{cases} \tag{3.4}
\]

From (3.3) and (3.4), we obtain that
\[
\frac{d\lambda_1^{AB}}{d\rho} = \frac{\partial \lambda_1^{AB}}{\partial u} \frac{du}{d\rho} + \frac{\partial \lambda_1^{AB}}{\partial \rho} = -\frac{An(n+1)\rho^{n-1} + \frac{\alpha(1 - \alpha)B}{\rho^{n+1}}}{2\rho \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}} < 0 \tag{3.5}
\]
\[
\frac{d\lambda_2^{AB}}{d\rho} = \frac{\partial \lambda_2^{AB}}{\partial u} \frac{du}{d\rho} + \frac{\partial \lambda_2^{AB}}{\partial \rho} = \frac{An(n+1)\rho^{n-1} + \frac{\alpha(1 - \alpha)B}{\rho^{n+1}}}{2\rho \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}} > 0 \tag{3.6}
\]
which imply that the velocity of 1-rarefaction (2-rarefaction) wave \(\lambda_1^{AB}\) (\(\lambda_2^{AB}\)) is monotonically decreasing (increasing) with respect to \(\rho\).

With the requirement \(\lambda_1^{AB}(\rho_-, u_-) < \lambda_1^{AB}(\rho, u)\) and \(\lambda_2^{AB}(\rho_-, u_-) < \lambda_2^{AB}(\rho, u)\), noticing (3.5) and (3.6), we get that
\[
R_1(\rho_-, u_-) : \begin{cases}
\xi = \lambda_1 = u - \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}, \\
u - u_- = -\int_{\rho_-}^{\rho} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}}{\rho} d\rho, \quad \rho < \rho_-.
\end{cases} \tag{3.7}
\]
and
\[
R_2(\rho_-, u_-) : \begin{cases}
\xi = \lambda_2 = u + \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}, \\
u - u_- = \int_{\rho_-}^{\rho} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}}{\rho} d\rho, \quad \rho > \rho_-.
\end{cases} \tag{3.8}
\]

For the 1-rarefaction wave, through differentiating \(u\) with respect to \(\rho\) in the second equation in (3.7), we get
\[
u_{\rho} = -\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}} < 0. \tag{3.9}
\]
\[
u_{\rho\rho} = \frac{-An(n - 3)\rho^{n+\alpha} + \alpha(\alpha + 3)B}{2\rho^2 \sqrt{An\rho^{n+\alpha} + \alpha B\rho^{n+1}}}. \tag{3.10}
\]
Thus, it is easy to get $u_{ρ^0} > 0$ for $1 \leq n \leq 3$, i.e., the 1-rarefaction wave is convex for $1 \leq n \leq 3$ in the upper half phase plane ($ρ > 0$).

In addition, from the second equation of (3.17), we have

$$u - u_\sigma = \int^\rho \sqrt{An\rho^{t-1} + \frac{\alpha B}{\rho^t}} d\rho \geq \int^\rho \sqrt{\alpha B \rho - \frac{\alpha + 1}{\alpha + 1} \sqrt{\rho^t + 1}} d\rho = 2\sqrt{\frac{\alpha B}{\alpha + 1}} (\rho - \frac{\alpha + 1}{\alpha + 1} \rho - \frac{\alpha + 1}{\alpha + 1})$$

which means that \( \lim_{ρ \to 0^+} u = +∞ \).

By a similar computation, we have that, for the 2-rarefaction wave, $u_\rho > 0$, $u_{ρ^0} < 0$ for $1 \leq n \leq 3$ and \( \lim_{ρ \to +∞} u = +∞ \). Thus, we can draw the conclusion that the 2-rarefaction wave is concave for $1 \leq n \leq 3$ in the upper half phase plane ($ρ > 0$).

Now we consider the discontinuous solution. For a bounded discontinuity at $ξ = σ$, the Rankine-Hugoniot condition holds:

$$\begin{align*}
\begin{cases}
\sigma^{AB}[ρ] = [ρu], \\
\sigma^{AB}[ρu] = [ρu^2 + P],
\end{cases}
\end{align*}$$

\( P = Aρ^n - \frac{B}{ρ^n} \),

(3.11)

where $[ρ] = ρ_+ - ρ_-$, etc.

Eliminating $σ$ from (3.11), we obtain

$$u - u_\sigma = ± \sqrt{\frac{ρ - ρ_-}{ρ - ρ_0} (A(ρ^n - ρ_-^n) - B(\frac{1}{ρ^n} - \frac{1}{ρ_-^n}))}. \quad (3.12)$$

Using the Lax entropy condition, the 1-shock satisfies

$$\sigma^{AB} < λ_1^{AB}(ρ_-, u_-), \quad λ_1^{AB}(ρ, u) < σ^{AB} < λ_2^{AB}(ρ, u), \quad (3.13)$$

while the 1-shock satisfies

$$λ_1^{AB}(ρ_-, u_-) < σ^{AB} < λ_2^{AB}(ρ_-, u_-), \quad λ_2^{AB}(ρ, u) < σ^{AB}. \quad (3.14)$$

From the first equation in (3.11), we have

$$σ^{AB} = \frac{ρu - ρ_-u_-}{ρ - ρ_-} = u + \frac{ρ_(u - u_-)}{ρ - ρ_-} = u_- + \frac{ρ(u - u_-)}{ρ - ρ_-}. \quad (3.15)$$

Thus, by a simple calculation, (3.13) is equivalent to

$$-ρ \sqrt{Anρ^{-1}} + \frac{αB}{ρ^n} < \frac{ρρ_-(u - u_-)}{ρ - ρ_-} < -ρ \sqrt{Anρ^{-1}} + \frac{αB}{ρ^n}, \quad (3.16)$$

and (3.14) is equivalent to

$$ρ \sqrt{Anρ^{-1}} + \frac{αB}{ρ^n} < \frac{ρρ_-(u - u_-)}{ρ - ρ_-} < -ρ \sqrt{Anρ^{-1}} + \frac{αB}{ρ^n}. \quad (3.17)$$

(3.16) and (3.17) imply that $ρ > ρ_-, u < u_-$ and $ρ < ρ_-, u < u_-$, respectively.

Through the above analysis, for a given left state $(ρ_-, u_-)$, the shock curves in the phase plane, which are the sets of states that can be connected on the right by a 1-shock or 2-shock, are as follows:

$$S_1(ρ_-, u_-) : \begin{cases}
σ_1 = \frac{ρu - ρ_-u_-}{ρ - ρ_-}, \\
u - u_- = -\sqrt{\frac{ρ - ρ_-}{ρ - ρ_0} (A(ρ^n - ρ_-^n) - B(\frac{1}{ρ^n} - \frac{1}{ρ_-^n}))}, \quad ρ > ρ_-,
\end{cases} \quad (3.18)$$

and

$$S_2(ρ_-, u_-) : \begin{cases}
σ_2 = \frac{ρu - ρ_-u_-}{ρ - ρ_-}, \\
u - u_- = -\sqrt{\frac{ρ - ρ_-}{ρ - ρ_0} (A(ρ^n - ρ_-^n) - B(\frac{1}{ρ^n} - \frac{1}{ρ_-^n}))}, \quad ρ < ρ_-,
\end{cases} \quad (3.19)$$
For the 1-shock wave, through differentiating $u$ respect to $\rho$ in the second equation in (3.18), we get
\[
2(u - u_-)u_\rho = \frac{1}{\rho^2} \left( A(\rho^n - \rho_-^n) - B(\rho - \rho_-) \right) + \frac{\rho - \rho_-}{\rho \rho_-} (An\rho^{n-1} + \alpha B) > 0, \tag{3.20}
\]
which means that $u_\rho < 0$ for the 1-shock wave and that the 1-shock wave curve is starlike with respect to $(\rho_-, u_-)$ in the region $\rho > \rho_-$. Similarly, we can get $u_\rho > 0$ for the 2-shock wave and that the 2-shock wave curve is starlike with respect to $(\rho_-, u_-)$ in the region $\rho < \rho_-$. In addition, it is easy to check that $\lim_{\rho \to +\infty} u = -\infty$ for the 1-shock wave and $\lim_{\rho \to 0} u = -\infty$ for the 2-shock wave.

Through the analysis above, for a given left state $(\rho_-, u_-)$, the sets of states connected with $(\rho_-, u_-)$ on the right in the phase plane consist of the 1-rarefaction wave curve $R_1(\rho_-, u_-)$, the 2-rarefaction wave curve $R_2(\rho_-, u_-)$, the 1-shock curve $S_1(\rho_-, u_-)$ and the 2-shock curve $S_2(\rho_-, u_-)$. These curves divide the upper half plane into four parts $R_1R_2(\rho_-, u_-)$, $R_1S_2(\rho_-, u_-)$, $S_1R_2(\rho_-, u_-)$ and $S_1S_2(\rho_-, u_-)$. Now, we put all of these curves together in the upper half plane $(\rho > 0, u \in R)$ to obtain a picture as in Fig.2.

By the phase plane analysis method, it is easy to construct Riemann solutions for any given right state $(\rho_+, u_+)$ as follows:
\[
\begin{align*}
(1) \ (\rho_+, u_+) & \in R_1R_2(\rho_-, u_-) : R_1 + R_2; \quad (2) \ (\rho_+, u_+) \in R_1S_2(\rho_-, u_-) : R_1 + S_2; \\
(3) \ (\rho_+, u_+) & \in S_1R_2(\rho_-, u_-) : S_1 + R_2; \quad (4) \ (\rho_+, u_+) \in S_1S_2(\rho_-, u_-) : S_1 + S_2. 
\end{align*}
\]

4. Formation of $\delta$-shocks and vacuum states as $A, B \to 0$

In this section, we will study the vanishing pressure limit process, i.e., $A, B \to 0$. Since the two regions $S_1R_2(\rho_-, u_-)$ and $R_1S_2(\rho_-, u_-)$ in the $(\rho, u)$ plane have empty interior when $A, B \to 0$, it suffices to analyze the limit process for the two cases $(\rho_+, u_+) \in S_1S_2(\rho_-, u_-)$ and $(\rho_+, u_+) \in R_1R_2(\rho_-, u_-)$.

Firstly, we analyze the formation of $\delta$-shocks in Riemann solutions to the extended Chaplygin gas equations (1.1)-(1.2) with (1.6) in the case $(\rho_+, u_+) \in S_1S_2(\rho_-, u_-)$ as the pressure vanishes.

4.1. Limit behavior of the Riemann solutions as $A, B \to 0$

When $(\rho_+, u_+) \in S_1S_2(\rho_-, u_-)$, for fixed $A, B > 0$, let $(\rho^{AB}_*, u^{AB}_*)$ be the intermediate state in the sense that $(\rho_-, u_-)$ and $(\rho^{AB}_*, u^{AB}_*)$ are connected by 1-shock $S_1$ with speed $\sigma^{AB}_1$ and $(\rho_+, u_+)$ are connected by 2-shock $S_2$ with speed $\sigma^{AB}_2$. Then it follows
\[
S_1: \begin{cases} 
\sigma^{AB}_1 = \frac{\rho^{AB}_* u^{AB}_*}{\rho^{AB}_* - \rho_- u_-}, \\
u^{AB}_* - u_- = -\sqrt{\frac{\rho^{AB}_* - \rho_-}{\rho^{AB}_* - \rho_+}} \left( A(\rho^{AB}_*) - B(\frac{1}{\rho^{AB}_*} - \frac{1}{\rho_-}) \right), \quad \rho^{AB}_* > \rho_-.
\end{cases} \tag{4.1}
\]
\[
S_2: \begin{cases} 
\sigma^{AB}_2 = \frac{\rho_+ u_+ - \rho^{AB}_* u^{AB}_*}{\rho^{AB}_* - \rho_+}, \\
u_+ - u^{AB}_* = -\sqrt{\frac{\rho^{AB}_* - \rho_+}{\rho^{AB}_* - \rho^{AB}_*}} \left( A(\rho^{AB}_*) - B(\frac{1}{\rho^{AB}_*} - \frac{1}{\rho_+}) \right), \quad \rho_+ < \rho^{AB}_*.
\end{cases} \tag{4.2}
\]

In the following, we give some lemmas to show the limit behavior of the Riemann solutions of system (1.1)-(1.2) with (2.1) as $A, B \to 0$. 
Lemma 4.1. \( \lim_{A,B \to 0} \rho_{AB}^* = +\infty. \)

Proof. Eliminating \( u_{AB}^* \) in the second equation of (4.1) and (4.2) gives

\[
\begin{aligned}
   u_+ - u_- &= - \sqrt{\frac{\rho_{AB}^* - \rho_-}{\rho_{AB}^* - \rho_+}} \left( A((\rho_{AB}^*)^n - \rho_n^*) - B \left( \frac{1}{(\rho_{AB}^*)^\alpha} - \frac{1}{\rho_n^*} \right) \right) \\
   &\quad - \sqrt{\frac{\rho_+ - \rho_{AB}^*}{\rho_+ - \rho_-}} \left( A(\rho_1^* - (\rho_{AB}^*)^n) - B \left( \frac{1}{\rho_1^*} - \frac{1}{(\rho_{AB}^*)^\alpha} \right) \right). 
\end{aligned}
\]

(4.3)

If \( \lim_{A,B \to 0} \rho_{AB}^* = M \in (\max\{\rho_-, \rho_+\}, +\infty) \), then by taking the limit of (4.3) as \( A, B \to 0 \), we obtain that \( u_+ - u_- = 0 \), which contradicts with \( u_+ < u_- \). Therefore we must have \( \lim_{A,B \to 0} \rho_{AB}^* = +\infty. \)

By Lemma 4.1 from (4.3) we immediately have the following lemma.

Lemma 4.2. \( \lim_{A,B \to 0} A(\rho_{AB}^*)^n = \frac{\rho_- \rho_+}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}(u_- - u_+)^2. \)

Lemma 4.3.

\[
\begin{aligned}
   \lim_{A,B \to 0} u_{AB}^* &= \lim_{A,B \to 0} \sigma_{AB}^1 = \lim_{A,B \to 0} \sigma_{AB}^2 = \sigma. 
\end{aligned}
\]

(4.4)

Proof. From the first equation of (4.1) and (4.2) for \( S_1 \) and \( S_2 \), by Lemma 4.1, we have

\[
\begin{aligned}
   \lim_{A,B \to 0} \sigma_{AB}^1 &= \lim_{A,B \to 0} \frac{\rho_{AB}^* u_{AB}^* - \rho_-}{\rho_{AB}^* - \rho_-} = \lim_{A,B \to 0} \frac{u_{AB}^* - \rho_- u_-}{1 - \rho_-} = \lim_{A,B \to 0} u_{AB}^*, \\
   \lim_{A,B \to 0} \sigma_{AB}^2 &= \lim_{A,B \to 0} \frac{\rho_+ u_+ - \rho_{AB}^* u_{AB}^*}{\rho_+ - \rho_{AB}^*} = \lim_{A,B \to 0} \frac{\rho_+ u_+}{\rho_{AB}^*} - u_{AB}^* = \lim_{A,B \to 0} u_{AB}^*, 
\end{aligned}
\]

which immediately lead to \( \lim_{A,B \to 0} u_{AB}^* = \lim_{A,B \to 0} \sigma_{AB}^1 = \lim_{A,B \to 0} \sigma_{AB}^2. \)

From the second equation of (4.1), by Lemma 4.1 4.2 we get

\[
\begin{aligned}
   \lim_{A,B \to 0} u_{AB}^* &= u_- - \lim_{A,B \to 0} \sqrt{\frac{\rho_{AB}^* - \rho_-}{\rho_{AB}^* - \rho_+}} \left( A((\rho_{AB}^*)^n - \rho_n^*) - B \left( \frac{1}{(\rho_{AB}^*)^\alpha} - \frac{1}{\rho_n^*} \right) \right) \\
   &= u_- - \sqrt{\rho_-} \frac{\rho_- - \rho_+}{\rho_- + \sqrt{\rho_+}} \left( u_- - u_+ \right)^2 \\
   &= u_- - \sqrt{\rho_-} \frac{\rho_+}{\rho_- + \sqrt{\rho_+}} \left( u_- - u_+ \right) \\
   &= \sqrt{\rho_-} u_- + \sqrt{\rho_+} u_+ = \sigma.
\end{aligned}
\]

The proof is completed.

Lemma 4.4.

\[
\begin{aligned}
   \lim_{A,B \to 0} \int_{\sigma_{AB}^1}^{\sigma_{AB}^2} \rho_{AB}^* d\xi &= \sigma[\rho] - [\rho u], \\
   \lim_{A,B \to 0} \int_{\sigma_{AB}^1}^{\sigma_{AB}^2} \rho_{AB}^* u_{AB}^* d\xi &= \sigma[\rho u] - [\rho u]^2. 
\end{aligned}
\]

(4.5)

(4.6)
**Proof.** The first equations of the Rankine-Hugoniot condition (3.11) for $S_1$ and $S_2$ read

\[
\begin{align*}
\sigma_1^{AB} (\rho^{AB} - \rho_-) &= \rho_s^{AB} u_s^{AB} - \rho_- u_-, \\
\sigma_2^{AB} (\rho_+ - \rho_s^{AB}) &= \rho_+ u_+ - \rho_s^{AB} u_s^{AB},
\end{align*}
\]

from which we have

\[
\lim_{A,B \to 0} \rho_s^{AB} (\sigma_2^{AB} - \sigma_1^{AB}) = \lim_{A,B \to 0} (-\sigma_1^{AB} - \sigma_2^{AB}) = \sigma[\rho] - [\rho u].
\]

Similarly, from the second equations of the Rankine-Hugoniot condition (3.11) for $S_1$ and $S_2$

\[
\begin{align*}
\sigma_1^{AB} (\rho_s^{AB} u_s^{AB} - \rho_- u_-) &= \rho_s^{AB} (u_s^{AB})^2 - \rho_- u_-^2 + A((\rho_s^{AB})^\gamma - \rho_-^\gamma) - B\left(\frac{1}{\rho_s^{AB}} - \frac{1}{\rho_-}\right), \\
\sigma_2^{AB} (\rho_+ u_+ - \rho_s^{AB} u_s^{AB}) &= \rho_+ u_+^2 - \rho_s^{AB} (u_s^{AB})^2 + A(\rho_+^\gamma - (\rho_s^{AB})^\gamma) - B\left(\frac{1}{\rho_+^\gamma} - \frac{1}{\rho_s^{AB}}\right),
\end{align*}
\]

we obtain

\[
\lim_{A,B \to 0} \rho_\alpha^{AB} u_s^{AB} (\sigma_2^{AB} - \sigma_1^{AB})
= \lim_{A,B \to 0} (-\sigma_1^{AB} - \sigma_2^{AB})
= \sigma[\rho u] - [\rho u^2].
\]

Thus, from (4.8) and (4.10) we immediately get (4.5) and (4.6). The proof is finished.

**Remark 4.1.** The above lemmas show that, as $A, B \to 0$, $S_1$ and $S_2$ coincide, the intermediate density $\rho_*^{AB}$ becomes singular, the velocities $\sigma_1^{AB}, \sigma_2^{AB}$ and $u_*^{AB}$ for Riemann solutions of (1.1) approach to $\sigma$, which are consistent with the velocity and the density of the $\delta$-shock solution to the transport equations (1.3) with the same Riemann data $(\rho_\pm, u_\pm)$ in Section 2.

### 4.2. $\delta$-shocks and concentration

Now we show the following theorem which is similar to Theorem 3.1 in [10] and characterizes the vanishing pressure limit in the case $(\rho_\pm, u_\pm) \in \mathbb{N}(\rho_-, u_-)$.

**Theorem 4.1.** Let $u_- > u_+$ and $(\rho_-, u_-) \in \mathbb{N}(\rho_-, u_-)$. For any fixed $A, B > 0$, assume that $(\rho^{AB}, u^{AB})$ is the two-shock Riemann solution of (1.1) with Riemann data $(\rho_\pm, u_\pm)$ constructed in section 3. Then as $A, B \to 0$, $\rho^{AB}$ and $\rho^{AB} u^{AB}$ converge in the sense of distributions, and the limit functions of $\rho^{AB}$ and $\rho^{AB} u^{AB}$ are the sums of a step function and a $\delta$-measure with weights

\[
\frac{t}{\sqrt{1 + \sigma^2}} (\sigma[\rho] - [\rho u]) \quad \text{and} \quad \frac{t}{\sqrt{1 + \sigma^2}} (\sigma[\rho u] - [\rho u^2]),
\]

respectively, which form a $\delta$-shock solution of (1.3) with the same Riemann data $(\rho_\pm, u_\pm)$.

**Proof.** 1. Set $\xi = \frac{t}{\sigma}$, for any fixed $A, B > 0$, the two-shock Riemann solution can be written as

\[
(\rho^{AB}, u^{AB})(\xi) = \begin{cases} 
(\rho_- u_-), & -\infty < \xi < \sigma_1^{AB}, \\
(\rho^{AB}_s u^{AB}_s), & \sigma_1^{AB} < \xi < \sigma_2^{AB}, \\
(\rho_+ u_+), & \sigma_2^{AB} < \xi < \infty,
\end{cases}
\]

which satisfies the following weak formulations:

\[
- \int_{-\infty}^{\infty} (u^{AB}(\xi) - \xi) \rho^{AB}(\xi) \psi'(\xi) d\xi + \int_{-\infty}^{\infty} \rho^{AB}(\xi) \psi(\xi) d\xi = 0,
\]

\[
- \int_{-\infty}^{\infty} (u^{AB}(\xi) - \xi) \rho^{AB}(\xi) u^{AB}(\xi) \psi'(\xi) d\xi + \int_{-\infty}^{\infty} \rho^{AB}(\xi) u^{AB}(\xi) \psi(\xi) d\xi
= \int_{-\infty}^{\infty} \left( A(\rho^{AB}(\xi))^n - \frac{B}{(\rho^{AB}(\xi))^n} \right) \psi'(\xi) d\xi,
\]
for any test function $\psi \in C_0^\infty (\mathbb{R})$.

2. By using the weak formulation (4.11), we can obtain the limit of $\rho^{AB}$, which is denoted by the following identities:

$$
\lim_{A,B \to 0} \int_{-\infty}^{\infty} \left( \rho^{AB}(\xi) - \rho_0(\xi - \sigma) \right) \psi(\xi) d\xi = (\sigma[\rho] - [\rho u]) \psi(\sigma),
$$

(4.13)

for any test function $\psi \in C_0^\infty (\mathbb{R})$, where

$$
\rho_0(\xi) = \rho_- + [\rho] \chi(\xi),
$$

and $\chi(\xi)$ is the characteristic function. Since the proof of (4.13) is the same as step 2 in the proof of Theorem 3.1 in [10], we omit it.

3. Now we turn to justify the limit of $\rho^{AB}u^{AB}$ by using the weak formulation (4.12). The first integral on the left hand side of (4.14) can be decomposed into

$$
- \left\{ \int_{-\infty}^{\sigma^{AB}_1} + \int_{\sigma^{AB}_1}^{\sigma^{AB}_2} + \int_{\sigma^{AB}_2}^{\infty} \right\} (u^{AB}(\xi) - \xi) \rho^{AB}(\xi) u^{AB} (\xi) \psi'(\xi) d\xi.
$$

(4.14)

The sum of the first and last term of (4.14) is

$$
- \int_{-\infty}^{\sigma^{AB}_1} (u_- - \xi) \rho_- u_- \psi'(\xi) d\xi - \int_{\sigma^{AB}_2}^{\infty} (u_+ - \xi) \rho_+ u_+ \psi'(\xi) d\xi
$$

$$
= -\rho_- u_-^2 \psi(\sigma^{AB}_1) + \rho_+ u_+^2 \psi(\sigma^{AB}_2) + \rho_- u_- \sigma^{AB}_1 \psi(\sigma^{AB}_1) - \rho_+ u_+ \sigma^{AB}_2 \psi(\sigma^{AB}_2) - \int_{-\infty}^{\sigma^{AB}_2} \rho_- u_- \psi(\xi) d\xi - \int_{\sigma^{AB}_2}^{\infty} \rho_+ u_+ \psi(\xi) d\xi,
$$

which converges as $A,B \to 0$ to

$$
([\rho u^2] - \sigma[\rho u]) \psi(\sigma) - \int_{-\infty}^{\infty} (\rho_0 u_0)(\xi - \sigma) \psi(\xi) d\xi.
$$

The second term of (4.14) is

$$
-\rho^{AB}_* u^{AB}_* \int_{\sigma^{AB}_1}^{\sigma^{AB}_2} (u^{AB}_* - \xi) \psi'(\xi) d\xi
$$

$$
= -\rho^{AB}_* u^{AB}_* \left( (u^{AB}_* - \sigma^{AB}_2) \psi(\sigma^{AB}_2) - (u^{AB}_* - \sigma^{AB}_1) \psi(\sigma^{AB}_1) + \int_{\sigma^{AB}_1}^{\sigma^{AB}_2} \psi(\xi) d\xi \right)
$$

$$
= -\rho^{AB}_* u^{AB}_* \left( \frac{u^{AB}_* \psi(\sigma^{AB}_2) - \psi(\sigma^{AB}_1)}{\sigma^{AB}_2 - \sigma^{AB}_1} - \frac{\psi(\sigma^{AB}_2) - \psi(\sigma^{AB}_1)}{\sigma^{AB}_2 - \sigma^{AB}_1} \right)
$$

$$
+ \frac{1}{\sigma^{AB}_2 - \sigma^{AB}_1} \int_{\sigma^{AB}_1}^{\sigma^{AB}_2} \psi(\xi) d\xi,
$$

which converges as $A,B \to 0$ to

$$
-(\sigma[\rho u - \rho u^2]) \left( \sigma \psi'(\sigma) - \sigma \psi'(\sigma) - \psi(\sigma) + \psi(\sigma) \right) = 0,
$$

by Lemma 4.3.
Now we compute the integral on the right hand side of (4.12), by Lemma 4.14.3, we obtain
\[
\int_{-\infty}^{\infty} \left( A(\rho^{AB}(\xi))^{n} - \frac{B}{(\rho^{AB}(\xi))^{\alpha}} \right) \psi'(\xi) d\xi
= \left\{ \int_{-\infty}^{\sigma_{AB}^{-}} + \int_{\sigma_{AB}^{-}}^{\sigma_{AB}^{+}} + \int_{\sigma_{AB}^{+}}^{\infty} \right\} \left( A(\rho^{AB}(\xi))^{n} - \frac{B}{(\rho^{AB}(\xi))^{\alpha}} \right) \psi'(\xi) d\xi
= \int_{-\infty}^{\sigma_{AB}^{-}} \left( A\rho_{n}^{n} - \frac{B}{\rho_{n}^{\alpha}} \right) \psi'(\xi) d\xi + \int_{\sigma_{AB}^{-}}^{\sigma_{AB}^{+}} \left( A(\rho_{n}^{AB})^{n} - \frac{B}{(\rho_{n}^{AB})^{\alpha}} \right) \psi'(\xi) d\xi + \int_{\sigma_{AB}^{+}}^{\infty} \left( A\rho_{n}^{n} - \frac{B}{\rho_{n}^{\alpha}} \right) \psi'(\xi) d\xi
= \left( A\rho_{n}^{n} - \frac{B}{\rho_{n}^{\alpha}} \right) \psi(\sigma_{1}^{AB}) - \left( A\rho_{n}^{n} - \frac{B}{\rho_{n}^{\alpha}} \right) \psi(\sigma_{2}^{AB}) + \left( A(\rho_{n}^{AB})^{n} - \frac{B}{(\rho_{n}^{AB})^{\alpha}} \right) (\psi(\sigma_{2}^{AB}) - \psi(\sigma_{1}^{AB}))
\]
which converge to 0 as $A, B \to 0$.

Then, the integral identity (4.14) yields
\[
\lim_{A, B \to 0} \int_{-\infty}^{\infty} \left( (\rho^{AB}u^{AB})(\xi) - (\rho u)(\xi - \alpha) \right) \psi(\xi) d\xi = (\sigma[\rho u] - [\rho u^{2}]) \psi(\sigma), \tag{4.15}
\]
for any test function $\psi \in C_{0}^{\infty}(-\infty, \infty)$.

4. Finally, we are in the position to study the limits of $\rho^{AB}$ and $\rho^{AB}u^{AB}$ by tracking the time-dependence of the weights of the $\delta$-measure as $A, B \to 0$.

Let $\phi(x, t) \in C_{0}^{\infty}(-\infty, \infty) \times [0, \infty)$ be a smooth test function and $\tilde{\phi}(\xi, t) = \phi(\xi, t)$. Then we have
\[
\lim_{A, B \to 0} \int_{0}^{\infty} \int_{-\infty}^{\infty} \rho^{AB}(\frac{\phi}{t}) \phi(x, t) dx dt = \lim_{A, B \to 0} \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \rho^{AB}(\xi) \tilde{\phi}(\xi, t) d\xi \right) dt.
\]

On the other hand, from (4.13), we have
\[
\lim_{A, B \to 0} \int_{-\infty}^{\infty} \rho^{AB}(\xi) \tilde{\phi}(\xi, t) d\xi = \int_{-\infty}^{\infty} \rho_{0}(\xi - \sigma) \tilde{\phi}(\xi, t) d\xi + (\sigma[\rho] - [\rho u]) \tilde{\phi}(\sigma, t)
= \frac{1}{t} \int_{-\infty}^{\infty} \rho_{0}(x - \sigma t) \phi(x, t) dx + (\sigma[\rho] - [\rho u]) \phi(\sigma, t).\]

Combining the two relations above yields
\[
\lim_{A, B \to 0} \int_{0}^{\infty} \int_{-\infty}^{\infty} \rho^{AB}(\frac{\phi}{t}) \phi(x, t) dx dt = \int_{0}^{\infty} \int_{-\infty}^{\infty} \rho_{0}(x - \sigma t) \phi(x, t) dx dt + \int_{0}^{\infty} t(\sigma[\rho] - [\rho u]) \phi(\sigma, t) dt.
\]
The last term, by definition, equals to
\[
\langle w_{1}(\cdot) \delta_{S}, \phi(\cdot, \cdot) \rangle,
\]
with
\[
w_{1}(t) = \frac{t}{\sqrt{1 + \sigma^{2}}}(\sigma[\rho] - [\rho u]).\]

Similarly, from (4.13) we can show that
\[
\lim_{A, B \to 0} \int_{0}^{\infty} \int_{-\infty}^{\infty} (\rho^{AB}u^{AB})(\frac{\phi}{t}) \phi(x, t) dx dt = \int_{0}^{\infty} \int_{-\infty}^{\infty} (\rho_{0}u_{0})(x - \sigma t) \phi(x, t) dx dt + \langle w_{2}(\cdot) \delta_{S}, \phi(\cdot, \cdot) \rangle,
\]
with
\[
w_{2}(t) = \frac{t}{\sqrt{1 + \sigma^{2}}}(\sigma[\rho u] - [\rho u^{2}]).\]

Then we complete the proof of Theorem 4.1.
4.3. Formation of vacuum states

In this subsection, we show the formation of vacuum states in the Riemann solutions to (1.1) in the case \((r_+, u_+) \in R_1 R_2 (r_-, u_-)\) with \(u_- < u_+\) and \(\rho_\pm > 0\) as the pressure vanishes.

At this moment, for fixed \(A, B > 0\), let \((\rho_{AB}^*, u_{AB}^*)\) be the intermediate state in the sense that \((\rho_-, u_-)\) and \((\rho_{AB}^*, u_{AB}^*)\) are connected by 1-rarefaction wave \(R_1\) with speed \(\lambda_{1AB}^*\), \((\rho_{AB}^*, u_{AB}^*)\) and \((\rho_+, u_+)\) are connected by 2-rarefaction wave \(R_2\) with speed \(\lambda_{2AB}^*\). Then it follows

\[
R_1 : \begin{cases}
\xi = \lambda_{1AB}^* = u - \sqrt{A\rho_-^{n-1} + \frac{\alpha B}{\rho_-^{n+1}}}, \\
u - u_- = -\int_{\rho_-}^{\rho} \frac{\sqrt{A\rho'^{n-1} + \frac{\alpha B}{\rho'^{n+1}}}}{\rho'} \, d\rho, \quad \rho_{AB}^* \leq \rho \leq \rho_-. 
\end{cases}
\]

\[
R_2 : \begin{cases}
\xi = \lambda_{2AB}^* = u + \sqrt{A\rho_-^{n-1} + \frac{\alpha B}{\rho_-^{n+1}}}, \\
u_+ - u = \int_{\rho}^{\rho_+} \frac{\sqrt{A\rho'^{n-1} + \frac{\alpha B}{\rho'^{n+1}}}}{\rho'} \, d\rho, \quad \rho_{AB}^* \leq \rho \leq \rho_+. 
\end{cases}
\]

Now, from the second equations of (4.16) and (4.17), using the following integral identity

\[
\int_{\rho}^{\rho_-} \frac{\sqrt{A\rho'^{n-1} + \frac{\alpha B}{\rho'^{n+1}}}}{\rho'} \, d\rho = \frac{2}{\alpha + 1} \left( -\sqrt{A\rho_-^{n-1} + \frac{\alpha B}{\rho_-^{n+1}}} + \sqrt{A\rho_-^{n-1} \ln(\sqrt{A\rho_-^{n-1} \rho^{n+1}} + \alpha B)} \right) \bigg|_{\rho}^{\rho_-},
\]

it follows that the intermediate state \((\rho_{AB}^*, u_{AB}^*)\) satisfies

\[
u_+ - u_- = \int_{\rho_{AB}^*}^{\rho_-} \frac{\sqrt{A\rho'^{n-1} + \frac{\alpha B}{\rho'^{n+1}}}}{\rho'} \, d\rho + \int_{\rho_-}^{\rho_{AB}^*} \frac{\sqrt{A\rho'^{n-1} + \frac{\alpha B}{\rho'^{n+1}}}}{\rho'} \, d\rho \leq \int_{\rho_{AB}^*}^{\rho_-} \frac{\sqrt{A\rho'^{n-1} + \frac{\alpha B}{\rho'^{n+1}}}}{\rho'} \, d\rho + \int_{\rho_-}^{\rho_{AB}^*} \frac{\sqrt{A\rho'^{n-1} + \frac{\alpha B}{\rho'^{n+1}}}}{\rho'} \, d\rho = \frac{2}{\alpha + 1} \left( -\sqrt{A\rho_-^{n-1} + \frac{\alpha B}{\rho_-^{n+1}}} + \sqrt{A\rho_-^{n-1} \ln(\sqrt{A\rho_-^{n-1} \rho^{n+1}} + \alpha B)} \right) \bigg|_{\rho}^{\rho_-},
\]

\[
+ \sqrt{A\rho_-^{n-1} + \frac{\alpha B}{(\rho_{AB}^*)^{n+1}}} - \sqrt{A\rho_-^{n-1} \ln(\sqrt{A\rho_-^{n-1} (\rho_{AB}^*)^{n+1}} + \alpha B) + \sqrt{A\rho_-^{n-1} (\rho_{AB}^*)^{n+1}}} + \sqrt{A\rho_-^{n-1} (\rho_{AB}^*)^{n+1}} + \sqrt{A\rho_-^{n-1} \rho^{n+1}} + \sqrt{A\rho_-^{n-1} (\rho_{AB}^*)^{n+1}} + \sqrt{A\rho_-^{n-1} (\rho_{AB}^*)^{n+1}} + \sqrt{A\rho_-^{n-1} (\rho_{AB}^*)^{n+1}} + \sqrt{A\rho_-^{n-1} (\rho_{AB}^*)^{n+1}},
\]

which implies the following result.

**Theorem 4.2.** Let \(u_- < u_+\) and \((\rho_+, u_+) \in I(\rho_-, u_-)\). For any fixed \(A, B > 0\), assume that \((\rho_{AB}^*, u_{AB}^*)\) is the two-rarefaction wave Riemann solution of (1.1) with Riemann data \((\rho_+, u_+)\) constructed in section 3. Then as \(A, B \to 0\), the limit of the Riemann solution \((\rho_{AB}^*, u_{AB}^*)\) is two contact discontinuities connecting the constant states \((\rho_\pm, u_\pm)\) and the intermediate vacuum state as follows:

\[
(\rho, u)(\xi) = \begin{cases}
(\rho_-, u_-), & -\infty < \xi \leq u_-; \\
(0, \xi), & u_- \leq \xi \leq u_+; \\
(\rho_+, u_+), & u_+ \leq \xi < \infty,
\end{cases}
\]
which is exactly the Riemann solution to the transport equations (1.2) with the same Riemann data $(\rho_\pm, u_\pm)$.

Indeed, if \( \lim_{A,B \to 0} \rho_{AB}^* = K \in (0, \min\{\rho_-, \rho_+\}) \), then (4.18) leads to \( u_+ - u_- = 0 \), which contradicts with \( u_- < u_+ \). Thus \( \lim_{A,B \to 0} \rho_{AB}^* = 0 \), which just means vacuum occurs. Moreover, as \( A,B \to 0 \), one can directly derive from (4.10) and (4.17) that \( \lambda_1^{AB}, \lambda_2^{AB} \to u \) and two rarefaction waves \( R_1 \) and \( R_2 \) tend to two contact discontinuities \( \xi = \frac{x}{t} = u_\pm \), respectively. These reach the desired conclusion.

5. Formation of \( \delta \)-shocks and two-rarefaction wave as \( A \to 0 \)

In this section, we discuss the limit behaviors of Riemann solutions of (1.1)-(1.2) with \( 1.6 \) as the pressure approaches the generalized Chaplygin gas pressure, i.e., \( A \to 0 \).

From Section 2 and 3, we can easily check that, as \( A \to 0 \), the backward (forward) rarefaction wave curve \( R_1(R_2) \) of (1.1)-(1.2) tends to the backward (forward) rarefaction wave curve \( \overline{R}(\overline{R}) \) of (1.5), and the backward (forward) shock wave curve \( S_1(S_2) \) of (1.1)-(1.2) tends to the backward (forward) contact discontinuity curve of (1.5), and the backward (forward) shock wave curve \( S_1(S_2) \) of (1.1)-(1.2) tends to the backward (forward) contact discontinuity curve of (1.5) when \( \alpha = 1 \) (see Fig.3).

5.1. Formation of \( \delta \)-shocks

In this subsection, we study the formation of the delta shock waves in the limit as \( A \to 0 \) of solutions of (1.1)-(1.2) with (1.0) in the case \( (\rho_+, u_+) \in V(\rho_-, u_-) \), i.e., \( u_+ + \sqrt{B\rho_+^{\alpha+1}} \leq u_- - \sqrt{B\rho_-^{\alpha+1}} \).

**Lemma 5.1.** When \( (\rho_+, u_+) \in V(\rho_-, u_-) \), there exists a positive parameter \( A_0 \) such that \( (\rho_+, u_+) \in S_1S_2(\rho_-, u_-) \) when \( 0 < A < A_0 \).

**Proof.** From \( (\rho_+, u_+) \in V(\rho_-, u_-) \), we have

\[
u_+ + \sqrt{B\rho_+^{\alpha+1}} \leq u_- - \sqrt{B\rho_-^{\alpha+1}}, \tag{5.1}\]

then

\[
(u_- - u_+)^2 \geq \left( \sqrt{B\rho_+^{\alpha+1}} + \sqrt{B\rho_-^{\alpha+1}} \right)^2
\]

\[
= B(\rho_+^{\alpha-1} + \rho_-^{\alpha-1} + 2\rho_+^{\alpha-1} \rho_-^{\alpha-1})
\]

\[
>B(\rho_+^{\alpha-1} + \rho_-^{\alpha-1} - \rho_+^{\alpha-1} \rho_-^{\alpha-1} - \rho_+^{\alpha-1} \rho_-^{\alpha-1})
\]

\[
=B\left( \frac{1}{\rho_+} - 1 \right) \left( \frac{1}{\rho_-} - 1 \right).
\tag{5.2}
\]
All the states \((\rho, u)\) connected with \((\rho_-, u_-)\) by a backward shock wave \(S_1\) or a forward shock wave \(S_2\) satisfy
\[
-u_- = \sqrt{\frac{\rho - \rho_-}{\rho_+ - \rho_-}} A((\rho^A_1)^n - \rho^n) - B\left(\frac{1}{(\rho^A_1)^n} - \frac{1}{\rho^n}\right), \quad \rho > \rho_-,
\] (5.3)
or
\[
-u_- = \sqrt{\frac{\rho - \rho_-}{\rho_+ - \rho_-}} A((\rho^A_1)^n - \rho^n) - B\left(\frac{1}{(\rho^A_1)^n} - \frac{1}{\rho^n}\right), \quad \rho < \rho_-.
\] (5.4)

When \(\rho_+ = \rho_-\), the conclusion is obviously true. When \(\rho_+ \neq \rho_-\), by taking
\[
(u_+ - u_-)^2 = \frac{\rho_+ - \rho_-}{\rho_+ - \rho_-} \left(A((\rho^A_1)^n - \rho^n) - B\left(\frac{1}{(\rho^A_1)^n} - \frac{1}{\rho^n}\right)\right),
\] (5.5)
we have
\[
A_0 = \frac{\rho_+ \rho_-}{(\rho_+ - \rho_-)(\rho^A_1 - \rho^n)} \left((u_+ - u_-)^2 - B\left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right)(\frac{1}{\rho^A_1} - \frac{1}{\rho^n})\right),
\] (5.6)
which together with Lemma 5.2 gives the conclusion. The proof is completed.

When \(0 < A < A_0\), the Riemann solution of (1.1)-(1.2) with \(1.6\) includes a backward shock wave \(S_1\) and a forward shock wave \(S_2\) with the intermediate state \((\rho^*_A, u^*_A)\) besides two constant states \((\rho_\pm, u_\pm)\). We then have

\[
S_1 : \quad \begin{align*}
\sigma^A_{1B} &= \frac{\rho^A_1 u^A_1 - \rho_- u_-}{\rho^A_1 - \rho_-}, \\
u^A_1 - u_- &= -\sqrt{\frac{\rho^A_1 - \rho_-}{\rho^A_1 \rho_-}} \left(\rho^A_1 ((\rho^A_1)^n - \rho^n) - B\left(\frac{1}{(\rho^A_1)^n} - \frac{1}{\rho^n}\right)\right), \quad \rho^A_1 > \rho_-,
\end{align*}
\] (5.7)
and
\[
S_2 : \quad \begin{align*}
\sigma^A_{2B} &= \frac{\rho_+ u_+ - \rho^A_1 u^A_1}{\rho_+ - \rho^A_1}, \\
u_+ - u^A_1 &= -\sqrt{\frac{\rho_+ - \rho^A_1}{\rho_+ \rho^A_1}} \left(\rho^A_1 ((\rho^A_1)^n - \rho^n) - B\left(\frac{1}{(\rho^A_1)^n} - \frac{1}{\rho^n}\right)\right), \quad \rho_+ < \rho^A_1.
\end{align*}
\] (5.8)

Here \(\sigma^A_1\) and \(\sigma^A_2\) are the propagation speed of \(S_1\) and \(S_2\), respectively. Similar to that in Section 4, in the following, we give some lemmas to show the limit behavior of the Riemann solutions of system (1.1)-(1.2) with (1.6) as \(A \to 0\).

**Lemma 5.2.** \(\lim_{A \to 0} \rho^A_1 = +\infty\).

**Proof.** Eliminating \(u^{AB}_1\) in the second equation of (5.7) and (5.8) gives
\[
u_- - u_+ = \sqrt{\frac{\rho^A_1 - \rho_-}{\rho^A_1 \rho_-} \left(\rho^A_1 ((\rho^A_1)^n - \rho^n) - B\left(\frac{1}{(\rho^A_1)^n} - \frac{1}{\rho^n}\right)\right)}
\] (5.9)
\[
+ \sqrt{\frac{\rho_+ - \rho^A_1}{\rho_+ \rho^A_1} \left(\rho^A_1 ((\rho^A_1)^n - \rho^n) - B\left(\frac{1}{(\rho^A_1)^n} - \frac{1}{\rho^n}\right)\right)},
\] (5.10)
If \(\lim_{A \to 0} \rho^A_1 = K \in (\max\{\rho_-, \rho_+\}, +\infty)\), then by taking the limit of (5.9) as \(A \to 0\), we obtain that
\[
u_- - u_+ = \sqrt{B} \left\{\sqrt{\frac{1}{\rho_+} - \frac{1}{K}} + \sqrt{\frac{1}{\rho_-} - \frac{1}{K}}\right\}
\] (5.10)
which contradicts with Lemma 5.1. Therefore we must have \(\lim_{A \to 0} \rho^A_1 = +\infty\).

By Lemma 5.2 from (5.9) we immediately have the following lemma.
Lemma 5.3. \( \lim_{A \to 0} A(\rho^A)^n < \rho_-(u_+ - u_-)^2. \)

Lemma 5.4. Let \( \lim_{A \to 0} u^A_* = \sigma^B, \) then

\[
\lim_{A \to 0} u^A_* = \lim_{A \to 0} \sigma^A_1 = \lim_{A \to 0} \sigma^A_2 = \sigma^B \in \left( u_+ + \sqrt{\alpha B \rho_+ - \frac{\alpha + 1}{2}}, u_- - \sqrt{\alpha B \rho_- - \frac{\alpha + 1}{2}} \right). \quad (5.11)
\]

Proof. From the second equation of \((5.7)\) for \( S_1, \) by Lemma 4.2 and 4.3 we have

\[
\lim_{A \to 0} u^A_* = u_- - \lim_{A \to 0} \sqrt{\frac{\rho^A_+ - \rho^A_-}{\rho^A_+ \rho^A_-}} \left( A(\rho^A)^n - B \left( \frac{1}{(\rho^A_+)^n} - \frac{1}{(\rho^A_-)^n} \right) \right)
\]

\[
= u_- - \sqrt{\frac{1}{\rho_-} \left( \lim_{A \to 0} A(\rho^A)^n + B \rho^A_+ \right)}
\]

\[
< u_- - \sqrt{\alpha B \rho_- - \frac{\alpha + 1}{2}}.
\]

Similarly, from the second equation of \((5.8)\) for \( S_2, \) we have

\[
\lim_{A \to 0} u^A_* = u_+ + \lim_{A \to 0} \sqrt{\frac{\rho^A_+ - \rho^A_-}{\rho^A_+ \rho^A_-}} \left( A(\rho^A)^n - B \left( \frac{1}{(\rho^A_+)^n} - \frac{1}{(\rho^A_-)^n} \right) \right)
\]

\[
= u_+ + \sqrt{\frac{1}{\rho_+} \left( \lim_{A \to 0} A(\rho^A)^n + B \rho^A_- \right)}
\]

\[
> u_+ + \sqrt{\alpha B \rho_+ - \frac{\alpha + 1}{2}}.
\]

Furthermore, similar to the analysis as Lemma 4.3 we can obtain \( \lim_{A \to 0} u^A_* = \lim_{A \to 0} \sigma^A_1 = \lim_{A \to 0} \sigma^A_2 = \sigma^B. \) The proof is complete.

Similar to Lemma 4.4 we have the following lemma.

Lemma 5.5.

\[
\lim_{A \to 0} \int_{\sigma^A_1} \rho^A_* d\xi = \sigma^B [\rho] - [\rho u], \quad (5.14)
\]

\[
\lim_{A \to 0} \int_{\sigma^A_1} \rho^A_* u^A_* d\xi = \sigma^B [\rho u] - [\rho u^2 - B \rho^A]. \quad (5.15)
\]

Lemma 5.6. For \( \hat{\sigma}^B \) mentioned in Lemma 5.4

\[
\hat{\sigma}^B = \sigma^B = \frac{\rho_+ u_+ - \rho_- u_- - \left( \frac{1}{\rho_+} - \frac{1}{\rho_-} \right) \left( B \rho_+ + B \rho_- \right) \right)^{\frac{1}{2}}}{\rho_+ - \rho_-}, \quad (5.16)
\]

as \( \rho_+ \neq \rho_- \), and

\[
\hat{\sigma}^B = \sigma^B = \frac{u_+ + u_-}{2}, \quad (5.17)
\]

as \( \rho_+ = \rho_- \).

Proof. Let \( \lim_{A \to 0} A(\rho^A)^n = L, \) by Lemma 5.3 from \((5.12)\) and \((5.13)\) we have

\[
\lim_{A \to 0} u^A_* = u_- - \sqrt{\frac{1}{\rho_-} \left( L + B \rho_+ \right)} = u_+ + \sqrt{\frac{1}{\rho_+} \left( L + B \rho_- \right)} = \hat{\sigma}^B,
\]
which leads to
\[ L + \frac{B}{\rho_+^\alpha} = \rho_-(u_- - \sigma^B)^2, \]  
\[ L + \frac{B}{\rho_-^\alpha} = \rho_+(u_+ - \sigma^B)^2. \]  
Eliminating \( L \) from (5.18) and (5.19), we have
\[ (\rho_+ - \rho_-)(\sigma^B)^2 - 2(\rho_+u_+ - \rho_-u_-)\sigma^B + \rho_+u_+^2 - \rho_-u_-^2 - B\left(\frac{1}{\rho_+^\alpha} - \frac{1}{\rho_-^\alpha}\right) = 0. \]  
From (5.20), noticing \( \alpha B \in \left(u_+ + \sqrt{\alpha B \rho_+^{\frac{\alpha+1}{\alpha}}}, u_- - \sqrt{\alpha B \rho_-^{\frac{\alpha+1}{\alpha}}}\right) \), we immediately get (5.16) and (5.17). The proof is finished.

**Remark 5.1.** The above Lemmas 5.2-5.5 shows that, as \( A \to 0 \), the intermediate density \( \rho_+^A \) becomes unbounded, the velocities \( \sigma_1^A \) and \( \sigma_2^A \) of shocks \( S_1 \) and \( S_2 \) and the intermediate velocity \( u_+^A \) for the Riemann solution of (1.1)-(1.2) approach to \( \sigma^B \), and the intermediate density becomes a singular measure simultaneously, which are consistent with the velocity and the density of the \( \delta \)-shock solution to the generalized Chaplygin gas equations (1.5) with the same Riemann data \((\rho_\pm, u_\pm)\) in Section 2. Thus similar to Theorem 4.1, we draw the conclusion as follows.

**Theorem 5.1.** Let \((\rho_+, u_+) \in V(\rho_-, u_-)\). For any fixed \( A > 0 \), assume that \((\rho^A, u^A)\) is the two-shock Riemann solution of (1.1)-(1.2) with Riemann data \((\rho_\pm, u_\pm)\) for \( 0 < A < A_0 \) constructed in section 3. Then as \( A \to 0 \), \( \rho^A \) and \( \rho^A u^A \) converge in the sense of distributions, and the limit functions of \( \rho^A \) and \( \rho^A u^A \) are the sums of a step function and a \( \delta \)-measure with weights
\[ \frac{t}{\sqrt{1 + (\sigma^B)^2}}(\sigma^B[p] - [pu]) \quad \text{and} \quad \frac{t}{\sqrt{1 + (\sigma^B)^2}}(\sigma^B[p]\mu - [pu^2 - \frac{B}{\rho^\alpha}]), \]
respectively, which form a \( \delta \)-shock solution of (1.1)-(1.2) with the same Riemann data \((\rho_\pm, u_\pm)\).

**5.2. Formation of two-rarefaction-wave solutions**

Now consider the formation of the two-rarefaction-wave (two-contact-discontinuity) solution of (1.1)-(1.2) with (1.6) in the case \((\rho_+, u_+) \in I(\rho_-, u_-)\) for \( 0 < \alpha < 1 (\alpha = 1) \) as \( A \to 0 \).

**Lemma 5.7.** When \((\rho_+, u_+) \in I(\rho_-, u_-)\), there exists a positive parameter \( A_1 \) such that \((\rho_+, u_+) \in R_1R_2(\rho_-, u_-)\) when \( 0 < A < A_1 \).

**Proof.** All the states \((\rho, u)\) connected with \((\rho_-, u_-)\) by a backward shock wave \( R_1 \) or a forward shock wave \( R_2 \) satisfy
\[ u - u_- = -\int_{\rho_-}^{\rho} \sqrt{\frac{A\rho^{n-1} + \frac{\alpha B}{\rho^\alpha}}{\rho}} \, d\rho, \quad \rho < \rho_- \] 
or
\[ u - u_- = \int_{\rho_-}^{\rho} \sqrt{\frac{A\rho^{n-1} + \frac{\alpha B}{\rho^\alpha}}{\rho}} \, d\rho, \quad \rho > \rho_- \]
When \( \rho_+ = \rho_- \), the conclusion is obviously true. When \( \rho_+ \neq \rho_- \), if \( \rho_+ > \rho_- \), by taking \( \rho > \rho_+ \) in (5.21), we have
\[ u_+ - u_- = \int_{\rho_-}^{\rho_+} \sqrt{\frac{A\rho^{n-1} + \frac{\alpha B}{\rho^\alpha}}{\rho}} \, d\rho \\
> \int_{\rho_-}^{\rho_+} \sqrt{\frac{A\rho^{n-1}}{\rho}} \, d\rho \\
= \frac{2\sqrt{An}}{n-1} (\rho_+^{\frac{n-1}{n}} - \rho_-^{\frac{n-1}{n}}), \]
from which we can get
\[ A < \frac{(n - 1)^2(u_+ - u_-)^2}{4n(\rho_+^{\frac{n}{n-1}} - \rho_-^{\frac{n}{n-1}})^2}. \]

Similarly, for \( \rho_+ < \rho_- \), we can get the same inequality as (5.23). So we take
\[ A_1 = \frac{(n - 1)^2(u_+ - u_-)^2}{4n(\rho_+^{\frac{n}{n-1}} - \rho_-^{\frac{n}{n-1}})^2}. \]

The proof is finished.

When \( 0 < A < A_1 \), the Riemann solution of (1.1)-(1.2) with \( (1.6) \) includes a backward rarefaction wave \( R_1 \) and a forward rarefaction wave \( R_2 \) with the intermediate state \( (\rho^*_+, u^*_+) \) besides two constant states \( (\rho_\pm, u_\pm) \). We then have

\[
R_1 : \begin{cases}
\xi = \lambda_1^{AB} = u - \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}, \\
u - u_- = - \int_{\rho_-}^{\rho_+} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}}{\rho} d\rho, \quad \rho^*_A \leq \rho \leq \rho_-,
\end{cases}
\]

and

\[
R_2 : \begin{cases}
\xi = \lambda_2^{AB} = u + \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}, \\
u_+ - u = \int_{\rho_-}^{\rho_+} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}}{\rho} d\rho, \quad \rho^*_A \leq \rho \leq \rho_+.
\end{cases}
\]

Here \( \rho^*_A \) is determined by
\[
u_+ - u_- = \int_{\rho_-}^{\rho_+} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}}{\rho} d\rho + \int_{\rho_+}^{\rho_+} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{n+1}}}}{\rho} d\rho.
\]

Furthermore, setting \( (\rho_*, u_*) = \lim_{A \to 0} (\rho^*_A, u^*_A) \), we obtain
\[
\rho_+^{\frac{n}{n+1}} = \frac{(\alpha + 1)(u_+ - u_-)}{4\sqrt{\alpha B}} + \frac{1}{2}(\rho_+^{\frac{n}{n+1}} + \rho_-^{\frac{n}{n+1}}), \quad u_* = \frac{u_+ - u_-}{2} + \frac{\sqrt{\alpha B}}{\alpha + 1}(\rho_+^{\frac{n}{n+1}} - \rho_-^{\frac{n}{n+1}}),
\]

Letting \( A \to 0 \) in (5.25) and (5.26), then for \( 0 < \alpha < \frac{1}{1(\alpha = 1)} \), \( R_1 \) and \( R_2 \) become the backward rarefaction wave (the backward contact discontinuity) \( \vec{R} \) and the forward rarefaction wave (the forward contact discontinuity) \( \vec{R} \), respectively, as follows:

\[
\begin{align*}
\vec{R} : \begin{cases}
\xi = \lambda_1^{B} = u - \sqrt{\alpha B\rho^{n+1}}, \\
u - u_- = u_- - \frac{2\sqrt{\alpha B}}{1+\alpha} \rho_-^{\frac{n+1}{2}}, \quad \rho_- \leq \rho \leq \rho_*,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\vec{R} : \begin{cases}
\xi = \lambda_2^{B} = u + \sqrt{\alpha B\rho^{n+1}}, \\
u_+ - u_- = u_+ + \frac{2\sqrt{\alpha B}}{1+\alpha} \rho_+^{\frac{n+1}{2}}, \quad \rho_+ \leq \rho \leq \rho_*.
\end{cases}
\end{align*}
\]

As a conclusion, for the case \( (\rho_+, u_+) \in I(\rho_-, u_-), \) as \( A \to 0 \), the two rarefaction wave \( R_1 \) and \( R_2 \) in (5.25) and (5.26) approach the two rarefaction waves (contact discontinuities) \( \vec{R} \) and \( \vec{R} \) in (5.29) and (5.30) for \( 0 < \alpha < \frac{1}{1(\alpha = 1)} \), and the intermediate state \( (\rho^*_A, u^*_A) \) tends to the state \( (\rho_*, u_*) \) in (5.28). In summary, in this case, we have the following result.

**Theorem 5.2.** Let \( (\rho_+, u_+) \in I(\rho_-, u_-) \). For any fixed \( A > 0 \), assume that \( (\rho^*_A, u^*_A) \) is the two-shock Riemann solution of (1.1)-(1.2) with Riemann data \( (\rho_\pm, u_\pm) \) for \( 0 < A < A_1 \) constructed in section 3. Then as \( A \to 0 \), the limit of the Riemann solution \( (\rho^*_A, u^*_A) \) is two rarefaction waves (contact discontinuities) connecting the constant states \( (\rho_\pm, u_\pm) \) and the intermediate nonvacuum state as follows:

\[
\lim_{A \to 0} (\rho, u)(\xi) = \begin{cases}
(\rho_-, u_-), & -\infty < \xi \leq u_- - \sqrt{\alpha B\rho_-^{\frac{n+1}{2}}}, \\
(\rho_*, u_*), & u_- - \sqrt{\alpha B\rho_-^{\frac{n+1}{2}}} \leq \xi \leq u_+ + \sqrt{\alpha B\rho_+^{\frac{n+1}{2}}}, \\
(\rho_+^*, u_+^*), & u_+ + \sqrt{\alpha B\rho_+^{\frac{n+1}{2}}} \leq \xi < \infty,
\end{cases}
\]

which is exactly the Riemann solution to the (generalized) Chaplygin gas equations with the same Riemann data \( (\rho_\pm, u_\pm) \) for \( 0 < \alpha < \frac{1}{1(\alpha = 1)} \).
6. Conclusions and discussions

In this paper, we have considered two kinds of the flux approximation limit of Riemann solutions to extended Chaplygin gas equations and studied the concentration and the formation of delta shock during the limit process. Moreover, we have proved that the vanishing pressure limit of the Riemann solutions to extended Chaplygin gas equations is just the corresponding ones to transport equations, and when extended Chaplygin pressure approaches the generalized Chaplygin pressure, the limit of the Riemann solutions to extended Chaplygin gas equations is just the corresponding ones to the generalized Chaplygin gas equations. In fact, one can further prove that when the extended Chaplygin pressure approaches the pressure for the perfect fluid, i.e., $B \to 0$ for fixed $A$, the limit of the Riemann solutions to the extended Chaplygin gas equations is just the corresponding ones to the Euler equations for perfect fluids.

On the other hand, recently, Shen and Sun have studied the Riemann problem for the nonhomogeneous transport equations, and the nonhomogeneous (generalized) Chaplygin gas equations with coulomb-like friction, see [31, 32, 36]. Similarly, we will also consider the Riemann problem for the nonhomogeneous extended Chaplygin gas equations with coulomb-like friction. Furthermore, we will consider its flux approximation limit and analyze the relations Riemann solutions among the nonhomogeneous extended Chaplygin gas equations, the generalized Chaplygin gas equations and the nonhomogeneous transport equations. These will be left for our future work.

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