FOUR-DIMENSIONAL GRAVITATIONAL BACKGROUNDS
BASED ON $N = 4, \hat{c} = 4$, SUPERCONFORMAL SYSTEMS

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ABSTRACT

We propose two new realizations of the $N = 4, \hat{c} = 4$ superconformal system based on the compact and non-compact versions of parafermionic algebras. The target space interpretation of these systems is given in terms of four-dimensional target spaces with non-trivial metric and topology different from the previously known four-dimensional semi-wormhole realization. The proposed $\hat{c} = 4$ systems can be used as a building block to construct perturbatively stable superstring solutions with covariantized target space supersymmetry around non-trivial gravitational and dilaton backgrounds.

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1. INTRODUCTION

The study of stable string solutions around non-trivial gravitational backgrounds provides a better understanding of the quantum gravitational phenomena; in principle, it may also shed some light on some physical processes in the presence of a highly curved or even singular space-time (e.g. black-holes, wormholes, singular cosmological solutions or other) [1]–[19].

Approximate classical or semi-classical solutions can be obtained via the $\alpha'$ expansion of the two-dimensional $\sigma$-model [22]. These solutions (in lowest order of $\alpha'$) are identical to those of the classical Einstein equations in the presence of a dilaton field, antisymmetric tensor field and some minimally coupled gauge matter. The $\alpha'$ $\sigma$-model approach is very useful since it provides us with a well-defined method to obtain and study, in low energies, the string-induced effective field theories. The $\alpha'$ expansion however breaks down and it fails to describe the interesting physics of the quantum gravitational phenomena in short distances, in particular all stringy phenomena around highly curved or singular backgrounds. It is therefore necessary to go beyond the $\alpha'$ expansion and try to extend the physically interesting (semi)-classical solutions to some exact string vacua.

In our days we do not know yet of any systematic way to extend the $\sigma$-model classical solutions to some exact (super)-conformal theories. This extension is generally difficult and not obvious at all, since it requests the existence of an exactly solvable (super)-conformal theory defined on the two-dimensional world-sheet. Unfortunately, the number of the known (super)-conformal systems is very limited and even more restricted for the systems with some interesting and non-trivial target-space interpretation.

One way to overcome this difficulty is by constructing directly non-trivial string solutions, via the conformal block-construction method, by tensoring known conformal systems. Then one tries to interpret the obtained string vacua in terms of the target-space backgrounds. In the conformal block construction, the $\alpha'$ expansion is not necessary and is used just as a suitable target-space interpretation.

In this work we will construct two new families of four-dimensional target spaces. The first one is denoted by $\Delta_k^{(4)}$, the second one by $C_k^{(4)}(k = 1, 2,...)$. The $\Delta_k^{(4)}$ space is defined as a direct product of two superconformal models, e.g. two supersymmetric gauged WZW models:

$$\Delta_k^{(4)} = \left(\frac{SU(2)}{U(1)}\right)^{ss}_k \otimes \left(\frac{SL(2,R)}{U(1)}\right)^{ss}_{k'}$$

with,

$$k' = k + 4.$$

(1)

The central charge of $\Delta_k^{(4)}$ is always $c = 6$ ($\hat{c} = 4$) for any value of $k$, thanks to the relation among the $SU(2)$ and $SL(2,R)$ levels. The central-charge deficit coming from the negatively curved $SU(2)/U(1)$ two-dimensional sub-space, it is cancelled by the central charge benefit coming from the positively curved $SL(2,R)/U(1)$ two-dimensional sub-space. The value of the central charge ($c = 6$, or $\hat{c} = 4$) is the same with that of the four-dimensional flat space $F^{(4)}$, with four free super-coordinates.
The $C_k^{(4)}$ space is also a product of two superconformal models:

$$C_k^{(4)} = \left( \frac{SU(2)}{U(1)} \right)_k^{ss} \otimes \left( U(1)_R \otimes U(1)_Q \right)_k^{ss}$$

with

$$Q = \sqrt{\frac{2}{k+2}} \quad \text{and} \quad R = \sqrt{2k}. \quad (2)$$

The $C_k^{(4)}$ has also fixed central charge, $c = 6$ ($\hat{c} = 4$) for any value of $k$ as in $\Delta_k^{(4)}$. Here the central-charge deficit coming from the $SU(2)/U(1)$ two-dimensional sub-space is balanced, owing to the relation among the $SU(2)$ level $k$ and the background charge $Q$ on one of the two Abelian coordinates. The other coordinate is flat but compactified on a torus of radius $R$, which is also fixed by $k$ [see eq. (2)].

What we would like to show is that the $\Delta_k^{(4)}$ and $C_k^{(4)}$ spaces are very special deformations of the $F^{(4)}$ flat space. These deformations are parametrized in $\Delta_k^{(4)}$ by the levels $k$ and $k'$. For $k' = k + 4$ the $N = 4$ superconformal properties of the flat space remain valid in the deformed (curved) space $\Delta_k^{(4)}$. In $C_k^{(4)}$ the deformation is given by the level $k$, the background charge $Q$, and the radius $R$. For the special relations among $k$, $Q$ and $R$ as given in eq. (2), $C_k^{(4)}$ has the $N = 4$ superconformal properties of the flat space.

In order to make the similarities and differences of the $F^{(4)}$, $\Delta_k^{(4)}$ and $C_k^{(4)}$ spaces more transparent, we will first present the realization of the $N = 4, \hat{c} = 4$, in terms of the free super-coordinates, and then present the modifications that are necessary to define the new realizations in the deformed spaces.

2. $N = 4$ SUPERCONFORMAL ALGEBRA BASED ON $F^{(4)}$, $\Delta_k^{(4)}$ and $W_k^{(4)}$

2.1. $F^{(4)}$ realization. Free supercoordinates [23, 24]

The basic operators of the $N = 4$ algebra, (i) the stress tensor $T_B$, (ii) the four supercurrents $G_i$, and (iii) the $SU(2)$ level $n = 1$ currents $S_i$, are constructed in terms of the four free super-coordinates, $(J_i, \Psi_j, i, j = 1, 2, 3, 4)$:

$$J_i(\xi)J_j(\xi') \equiv -\frac{\delta_{ij}}{(\xi - \xi')^2}$$

$$\Psi_i(\xi)\Psi_j(\xi') \equiv -\frac{\delta_{ij}}{(\xi - \xi')}.$$

The (right-moving) basic operators are given as follows:

$$T_B = -\frac{1}{2} \left( J_i^2 - \Psi_i \partial \Psi_i \right)$$
\[ G_1 = +J_1 \Psi_1 + J_2 \Psi_2 + J_3 \Psi_3 + J_4 \Psi_4 \]
\[ G_2 = +J_1 \Psi_2 - J_2 \Psi_1 - J_3 \Psi_4 + J_4 \Psi_3 \]
\[ G_3 = -J_1 \Psi_4 + J_2 \Psi_3 - J_3 \Psi_2 + J_4 \Psi_1 \]
\[ G_4 = -J_1 \Psi_3 - J_2 \Psi_4 + J_3 \Psi_1 + J_4 \Psi_2 \]

\[ S_i = \frac{1}{2} (\Psi_4 \Psi_i + \epsilon_{ijl} \Psi_j \Psi_l) . \quad (4) \]

Using the free-field Operator Product Expansion (OPE), one finds the relations of the \( N=4 \) algebra with \( \hat{c} = 4 \):

\[ G_4(\xi)G_4(\xi') \equiv \frac{\hat{c}}{(\xi - \xi')^3} + \frac{2 T_B(\xi')}{(\xi - \xi')} \]
\[ G_i(\xi)G_j(\xi') \equiv \delta_{ij} \frac{\hat{c}}{(\xi - \xi')^3} - 4 \epsilon_{ijl} \frac{S_l(\xi')}{(\xi - \xi')^2} + 2 \delta_{ij} \frac{T_B(\xi')}{(\xi - \xi')} \]
\[ G_4(\xi)G_i(\xi') \equiv 4 S_i(\xi') \]
\[ S_i(\xi)G_i(\xi') \equiv - \frac{G_4(\xi')}{2(\xi - \xi')} \]
\[ S_i(\xi)G_j(\xi') \equiv \frac{1}{2(\xi - \xi')} (\delta_{ij} G_4(\xi') + \epsilon_{ijl} G_l(\xi')) \]
\[ S_i(\xi)S_j(\xi') \equiv - \delta_{ij} \frac{n}{2(\xi - \xi')^2} + \epsilon_{ijl} \frac{S_l(\xi')}{(\xi - \xi')} . \quad (5) \]

In the above equations the level \( n \) of the \( SU(2) \) currents and the central charge \( \hat{c} \) are related by \( \hat{c} = 4n \), \( 23, 24 \); therefore the level \( n \) is fixed to one \( (n = 1) \) for \( \hat{c} = 4 \).

For later convenience it is useful to adopt a complex notation for the coordinate currents,

\[ P = J_1 + iJ_2 , \quad P^t = -J_1 + iJ_2 , \]
\[ \Pi = J_4 + iJ_3 , \quad \Pi^t = -J_4 + iJ_3 , \]

with

\[ P(\xi)P^t(\xi') \equiv \frac{2}{(\xi - \xi')^2} + 2 T_P \]
\[ \Pi(\xi)\Pi^t(\xi') \equiv \frac{2}{(\xi - \xi')^2} + 2 T_{\Pi} , \quad (6) \]

where \( T_P \) and \( T_{\Pi} \) are the stress tensor of the \( (P, P^t) \) and \( (\Pi, \Pi^t) \) conformal sub-systems.

It is also useful to bosonize the free fermions in terms of two bosons, \( H^+ \) and \( H^- \). The desired bosonization must express the global properties of the \( SO(4) \) level one fermionic currents, \( \Psi_i \Psi_j \).
First, we decompose the $SO(4)_1$ currents in terms of two $SU(2)$ level-one currents using the self-dual and anti-self-dual projections,

$$S_i = \frac{1}{2} \left( +\Psi_4 \Psi_i + \frac{1}{2} \epsilon_{ijl} \Psi_j \Psi_l \right)$$

$$\tilde{S}_i = \frac{1}{2} \left( -\Psi_4 \Psi_i + \frac{1}{2} \epsilon_{ijl} \Psi_j \Psi_l \right).$$

(7)

Then, we parametrize $S_i$ and $\tilde{S}_i$ in terms of the two free bosons, $H^+$ and $H^-$, both of them compactified on the torus with self-dual radius [the $SU(2)$-extended symmetry points]:

$$R_{H^+} = R_{H^-} = \sqrt{2},$$

giving

$$(S_i) = \left( \frac{1}{2} \partial H^+, \ e^{\pm i \sqrt{2} H^+} \right), \quad SU(2)_{H^+},$$

$$(\tilde{S}_i) = \left( \frac{1}{2} \partial H^-, \ e^{\pm i \sqrt{2} H^-} \right), \quad SU(2)_{H^-}.$$  \hspace{1cm} (8)

In terms of $P, P^t, \Pi, \Pi^t, H^+$ and $H^-$ the basic operators of the $N = 4$ algebra [see eq. (4)] become:

$$T_B = -\frac{1}{2} \left( (\partial H^+)^2 + (\partial H^-)^2 - PP^t - \Pi \Pi^t \right)$$

$$G = -\left( \Pi^t e^{-\sqrt{2} H^-} + P^t e^{\sqrt{2} H^+} \right) e^{\pm i \sqrt{2} H^+}$$

$$\tilde{G} = \left( \Pi e^{\sqrt{2} H^-} - P e^{-\sqrt{2} H^+} \right) e^{\pm i \sqrt{2} H^+}$$

$$(S_i) = \left( \frac{1}{2} \partial H^+, \ e^{\pm i \sqrt{2} H^+} \right)$$

where,

$$G = \frac{G_1 + iG_2}{\sqrt{2}}, \quad \tilde{G} = \frac{G_4 + iG_3}{\sqrt{2}}.$$  \hspace{1cm} (9)

The above expressions show clearly that $G$ and $\tilde{G}$ are both doublets under $SU(2)_{H^+}$. The $H^+$ factorization in the supercurrents it is not a particular property of the free-field realization it is a generic property for any $\hat{c} = 4$ system with $N = 4$ symmetry since it follows from the the $N = 4$ superconformal symmetry [25]. As a consequence the supercurrents always have a factorized product form in terms of two kinds of conformal operators; the ones do not have any dependence on the $H^+$ field and they have conformal dimension $\frac{5}{4}$; the others are given uniquely in terms of $H^+$ and have conformal dimension $\frac{1}{4}$ [see the expressions for $G$ and $\tilde{G}$ in eq. (9)].

We will now show that it is possible to replace the four free super-coordinates by those of the $\Delta_k^{(4)}$ curved space, keeping, however, the $N = 4$ symmetry in the deformed system.
2.2. $\Delta^{(4)}_k$ realization. Parafermionic super-coordinates

In the $\Delta^{(4)}_k$ space the coordinate currents are replaced by the $(SU(2))_k$ compact and $(SL(2,\mathbb{R})/U(1))_{k'}$ non-compact parafermions $\chi^k_l(\xi)$ and $\chi^{k'}_{l'}(\xi)$.

In both cases the parafermion algebra is defined by a collection of (non-local) currents $\chi_{\pm l}(\xi)$, $l = 0, 1, 2, \ldots$, [with $\chi^+_l(\xi) = \chi_{-l}(\xi)$ and $\chi^0_l(\xi) = \chi_0(\xi) = 1$]. The parafermion currents satisfy the following OPE relations [26]–[28],

$$\chi_{l_1}(\xi)\chi_{l_2}(\xi') = C^N_{l_1,l_2}(\xi - \xi')^{\Delta_{l_1} + \Delta_{l_2} - 2\Delta_l} \chi_{l_1 + l_2},$$

$$\chi_l(\xi)\chi^l_{l'}(\xi') = (\xi - \xi')^{-2(\Delta_l - 1)} \left( \frac{1}{(\xi - \xi')^2} + \frac{2\Delta_l}{c(N)} T_\chi(\xi') \right),$$

where $T_\chi$ is the stress tensor of the parafermion theory with central charge $c(N) = [3N/(N + 2) - 1]$

$$T_\chi(\xi)T_\chi(\xi') = \frac{c(N)}{2(\xi - \xi')} + \frac{2T_\chi(\xi')}{(\xi - \xi')^2} + \frac{\partial^2 T_\chi(\xi')}{(\xi - \xi')}.$$

The $\Delta_l$ are the conformal dimensions of $\chi_l$ fields, which are given by

$$\Delta_l = \frac{l(N - l)}{N}.$$  

The $C^N_{l_1,l_2}$ are known structure constants, which are determined by associativity; the conformal dimensions of the fields $\Delta_l$ are constrained to satisfy the recursion relation:

$$\Delta_{l+1} + \Delta_{l-1} + 2\Delta_l - 2\Delta_1 = n_l, \quad n_l = 0, 1, 2\ldots$$

In the case of $(SU(2))_k$ compact parafermions, the parameter $N$ is positive and equal to the level $k$ ($N = k$). In this case the number of $\chi^k_l$ fields is restricted to be finite [$l : 0 \leq l \leq (k - 1)$], and $C^k_{l_1,l_2}$ are given by

$$C^k_{l_1,l_2} = \left[ \frac{\Gamma(k - l_1 + 1)\Gamma(k - l_2 + 1)\Gamma(l_1 + l_2)}{\Gamma(l_1 + 1)\Gamma(l_2 + 1)\Gamma(k - l_1 - l_2)} \right]^{\frac{1}{2}}.$$  

In the non-compact version $(SL(2,\mathbb{R})/U(1))_{k'}$, the parameter $N$ is negative and opposite to the level $k'$ ($N = -k'$). Also, the number of parafermion fields is now infinite ($l : 0 \leq l \leq \infty$). The structure constants $C^{-k'}_{l_1,l_2}$ are also given by eq. (14) via the analytic continuation $k \rightarrow -k'$; one then obtains

$$C^{-k'}_{l_1,l_2} = \left[ \frac{\Gamma(k' + l_1 + l_2)\Gamma(k' + 1)}{\Gamma(l_1 + 1)\Gamma(l_2 + 1)\Gamma(k' + l_1)\Gamma(k' + l_2)} \right]^{\frac{1}{2}}.$$  

In both compact and non-compact cases, their unitary representations and their characters are well known [34], [35]. For our purpose, we need to identify the super-coordinate currents of the $\Delta^{(4)}_k$ space in terms of the parafermion fields $\chi^k_l$, $\chi^{k'}_{l'}$, and in
terms of $H^+$ and $H^-$. As in the free-field realization, the world-sheet fermionic super-partners will still be expressed in terms of $H^+$ and $H^-$, but now with deformed radius (see below).

The coordinate current identification with the parafermions can easily be obtained in the large $k$ and $k'$ limit. Indeed in this limit the modified (non-)local coordinate currents $P_k$ and $\Pi_{k'}$ must approach their flat expressions given by eq. (6), and so they must have conformal dimensions almost equal to one. It is then clear that up to a rescaling $P_k$ and $\Pi_{k'}$ have to be identified with the compact $\chi^k_{l=1}$ and non-compact $\chi^{k'}_{l=1}$ parafermions

$$P(\xi) \longrightarrow P_k(\xi) \equiv \sqrt{\frac{2k}{k+2}} \chi^k_{l=1}(\xi),$$

$$\Pi(\xi) \longrightarrow \Pi_{k'}(\xi) \equiv \sqrt{\frac{2k'}{k' - 2}} \chi^{k'}_{l=1}(\xi).$$

(16)

From eq. (12) the conformal dimensions of the $\Delta^{(4)}_k$ coordinate currents $P_k$ and $\Pi_{k'}$ are equal to $h_P = 1 - \frac{1}{k}$ and $h_{\Pi_{k'}} = 1 + \frac{1}{k'} = 1 + \frac{1}{k+2}$. Owing to the deviation from the free-field dimensionality ($h_P = h_{\Pi} = 1$) the OPE relations among the deformed coordinate currents $P_k$ and $\Pi_{k'}$ are those given by the parafermion algebra (see eqs. (10) for $l = 1$ and the above definitions of $P_k$ and $\Pi_{k'}$):

$$P_k(\xi)P_k(\xi') \equiv \left[ \frac{k}{k+2(\xi - \xi')^2} + 2TP_k(\xi') \right] (\xi - \xi')^\frac{2}{k},$$

$$\Pi_{k'}(\xi)\Pi_{k'}(\xi') \equiv \left[ \frac{k'}{k' - 2(\xi - \xi')^2} + 2T_{\Pi_{k'}}(\xi') \right] (\xi - \xi')^-\frac{2}{k'}. \quad \quad (17)$$

Equations (17) generalize the free-current OPE relations of eq. (6) to those of the $\Delta^{(4)}_k$ curved space.

As we will see below, the anomalous dimensionality of the parafermionic coordinate currents $P_k$ and $\Pi_{k'}$ can be consistently compensated by a suitable modification of their world-sheet fermionic super-partners. Since this modification has to be consistent with the $N = 4, \hat{c} = 4$ algebra, the $H^+$ part is fixed and is identical to that of the free-field realization. The only possible modification that remains consist of deforming the radius of the $H^-$ field by some $\frac{1}{k}$ corrections. The exact value of the deformed $H^-$ radius is found to be

$$R_{H^-} = \alpha \sqrt{2}, \quad \text{with} \quad \alpha = \sqrt{\frac{k+4}{k}}. \quad \quad (18)$$

The above choice of $R_{H^-}$ gives rise to four operators with conformal dimension almost equal to $\frac{1}{2}$ in the large $k$ and $k'$ limit, which are nothing but the deformed world-sheet super-partners of $P_k$ and $\Pi_{k'}$ in $\Delta^{(4)}_k$ space,

$$\Psi_P \longrightarrow e^{i\frac{1}{\sqrt{2}}(-\alpha H^- + H^+)}, \quad \quad \text{with} \quad h_{\Psi_P} = \frac{1}{2} + \frac{1}{k'},$$

$$\Psi_{\Pi} \longrightarrow e^{i\frac{1}{\sqrt{2}}(+\alpha H^- + H^+)}, \quad \quad \text{with} \quad h_{\Psi_{\Pi}} = \frac{1}{2} - \frac{1}{k+4}. \quad \quad (19)$$
Observe that in the limit, \( k \to \infty \), \( \Psi_P \) and \( \Psi_\Pi \) become the two complex worldsheet fermions with canonical dimensionality \( h_{\Psi_P} = h_{\Psi_\Pi} = \frac{1}{2} \). Although in the \( \Delta_k^{(4)} \) space we have anomalous conformal weights for the coordinate currents \( P_k, \Pi_k \), as well as for their super-partners \( \Psi_P, \Psi_\Pi \), we may define without any obstruction the \( N = 4 \) basic operators as follows:

\[
T_B = -\frac{1}{2} \left( (\partial H^+)^2 + (\partial H^-)^2 \right) + T_{P_k} + T_{\Pi_k},
\]

\[
G = -\left( \Pi_{k'} e^{-\frac{i\sqrt{2}}{2} H^+} + P_k e^{\frac{i\sqrt{2}}{2} H^-} \right) e^{\frac{i\sqrt{2}}{2} H^+},
\]

\[
\bar{G} = \left( \Pi_{k'} e^{\frac{i\sqrt{2}}{2} H^-} - P_k e^{-\frac{i\sqrt{2}}{2} H^+} \right) e^{\frac{i\sqrt{2}}{2} H^+},
\]

\[
(S_i) = \left( \frac{1}{2} \partial H^+, \ e^{\pm i\sqrt{2} H^+} \right).
\]

The validity of the \( N = 4 \) superconformal algebra follows from the OPE relations among \( P_k, P_k', \Pi_k', \Pi_k \), given in eq. (17), and those of the free fields \( H^+, H^- \).

The existence of the \( N = 4 \) symmetry in the \( \Delta_k^{(4)} \) non-trivial space is of main interest, since it gives us the possibility to construct a new class of exact and stable string solutions around non-trivial gravitational and dilaton backgrounds. More explicitly, we can arrange the degrees of freedom of the ten supercoordinates in three superconformal systems [7]–[10]:

\[
\hat{c}(\text{total}) = 10 = [\hat{c} = 2]_0 + [\hat{c} = 4]_1 + [\hat{c} = 4]_2.
\]

The \( \hat{c} = 2 \) subsystem is saturated by two free superfields; in one variation of our solution, one of the two superfields is chosen to be the time-like supercoordinate and the other to be one of the nine space-like supercoordinates. In other variations, the two supercoordinates are Euclidean or even compactified on a one- or two-dimensional torus.

The remaining eight supercoordinates appear in groups of four in \([\hat{c} = 4]_1\) and \([\hat{c} = 4]_2\). Both \([\hat{c} = 4]_A, A = 1, 2\), subsystems show an \( N = 4 \) superconformal symmetry. The non-triviality of our solutions follows from the fact that there exist some realizations of the \( \hat{c} = 4, N = 4 \) superconformal systems, which are based on geometrical and topological non-trivial spaces other than the \( T^{(4)}/\mathbb{Z}_2 \) orbifold and the \( K_3 \) Calabi-Yau space. The \( \Delta_k^{(4)} \) realization I presented above is one new example of such a system.

There is another known non-trivial realization that shares the same superconformal symmetries, namely that of the semi-wormhole four-dimensional space, \( W_k^{(4)} \). For completeness, I will present below the basic operators and fields of the \( W_k^{(4)} \) realization [24],[3] as well as the additional realization based on \( C_k^{(4)} \). It turns out that the \( C_k^{(4)} \) space is the dual of \( W_k^{(4)} \) and that it is also related to some analytic continuations of previous constructions [13],[17].
2.3. $W_k^{(4)}$ semi-wormhole realization. $SU(2)_k \otimes U(1)_Q$ supercoordinates \cite{24,3}

The realization $W_k^{(4)}$ is based on a supersymmetric $SU(2)_k \times U(1)_Q$ WZW model, with a background term $Q = \sqrt{\frac{2}{k+2}}$ in the $U(1)_Q$ current. The four fermions of the model are free and are parametrized by the $H^+$ and $H^-$ fields (via bosonization) as in the free-field realization [eqs. (7), (8)]. The four-coordinate currents are the three $SU(2)_k$ ($J^i, i = 1, 2, 3$) currents and one of the $U(1)_Q$ ($J_4$) currents:

$$J^i(\xi) J^j(\xi') \equiv -\frac{k}{2} \frac{\delta^{ij}}{(\xi - \xi')^2} + e^{ij\ell} \frac{J^\ell}{(\xi - \xi')^2}, \quad i, j = 1, 2, 3;$$

$$J_4(\xi) J_4(\xi') \equiv -\frac{1}{Q^2(\xi - \xi')^2}$$

(22)

(the $Q$ rescaling of $J_4$ is for convenience).

The $T_B, G, \tilde{G}$ and $S_i$ associated to $W_k^{(4)}$ are:

$$T_B = -\frac{1}{2} \left[(\partial H^+)^2 + (\partial H^-)^2 + Q^2 (J_1^2 + J_2^2 + J_3^2 + J_4^2 + \partial J_4)\right]$$

$$G = Q \left[(J_4 - i(J_3 + \sqrt{2}\partial H^-)) e^{-i\sqrt{2}\partial H^+} + (J_4 - iJ_2) e^{-i\sqrt{2}\partial H^-} e^{i\sqrt{2}\partial H^+}\right]$$

$$\tilde{G} = Q \left[(J_4 + i(J_3 + \sqrt{2}\partial H^-)) e^{-i\sqrt{2}\partial H^+} - (J_4 + iJ_2) e^{-i\sqrt{2}\partial H^-} e^{i\sqrt{2}\partial H^+}\right]$$

$$S_0 = \frac{1}{\sqrt{2}} \partial H^+, \quad S_\pm = e^{\pm i\sqrt{2}H^+}.$$  

(23)

In the above expressions the level $k$ of the $SU(2)_k$ and the background charge $Q$ are related because of the $N = 4$ symmetry. Because of this relation, the central charge $\hat{c}(W_k^{(4)}) = 4$ for any value of the level $k$ \cite{24}.

$$\hat{c}[SU(2)_k] = \frac{2}{3} \left[3 - \frac{6}{k + 2} + \frac{3}{2}\right] = 3 - \frac{4}{k + 2},$$

$$\hat{c}[U(1)_Q] = \frac{2}{3} \left[1 + 3Q^2 + \frac{1}{2}\right] = 1 + 2Q^2$$

(the contributions $\frac{3}{2}$ and $\frac{1}{2}$ inside $[\ldots]$ in the first and second line are those of the 3+1 free fermions)

$$\hat{c}[SU(2)_k] + \hat{c}[U(1)_Q] = 4 + 2(Q^2 - \frac{2}{k + 2}),$$

and so $\hat{c}[W_k^{(4)}] = 4$ only if

$$Q = \sqrt{\frac{2}{k + 2}}.$$  

(24)

The existence of $N = 4$, $\hat{c} = 4$ superconformal symmetry with this value of $Q$ is found in ref. \cite{24}. What is extremely interesting is the background interpretation of the $W_k^{(4)}$ space in terms of a four-dimensional (semi-) wormhole space given by Callan, Harvey and Strominger in ref. \cite{3}. Indeed, for large $k$, the three $SU(2)_k$ coordinates define a three-dimensional subspace with a non-trivial topology $S^3$, while the fourth coordinate with a background charge corresponds to the scale factor of the $S^3$ sphere. In the $W_k^{(4)}$ realization [see eq. (23)], both $H^+$ and $H^-$ are compactified on a torus with radius $R_{H^+} = R_{H^-} = \sqrt{2}$ as in the free-field case. There are in total three underlying $SU(2)$ Kac-Moody currents:
(i) the $SU(2)_k$ defined by the coordinate currents
(ii) the $SU(2)^+_k$ defined by the $H^+$ field with $R_{H^+} = \sqrt{2}????
(iii) the $SU(2)^-_k$ defined by the $H^-$ field with $R_{H^-} = \sqrt{2}?????

The background term in $T_B, Q\partial J^4$, comes from a non-trivial dilaton background: $\Phi = QX^4$.

The term $Q\sqrt{2}\partial H^-$ in $G$ and $\tilde{G}$ describes, at the same time, the standard fermionic torsion term $\pm Q\psi^i \psi^j \psi^k (i = 1, 2, 3, 4)$, as well as the fermionic background term $\pm Q\partial \psi^i$. The torsion and background terms in the supercurrents are arranged together in $\sqrt{2}\partial H^-$, which appears as a shift of the $J_3$ current.

As we will see later on, the geometrical as well as the topological structure of the $\Delta^{(4)}_k$ space differs from that of the $W^{(4)}_k$. In $W^{(4)}_k$ there is a non-vanishing torsion due to the WZW term, which is proportional to the $SU(2)$ structure constant. So, in $W^{(4)}_k$ there is a non-vanishing antisymmetric field background $B_{ij}$ with non-trivial field strength $H_{ijk}$.

3. THE TARGET SPACE INTERPRETATION OF THE $\Delta^{(4)}_k$ SPACE AND THE $C^{(4)}_k$ REALIZATION

3.1. The semiclassical limit for $\Delta^{(4)}_k$ space

For large $k$, the $\Delta^{(4)}_k$ space has a dimensional target-space interpretation based on a non-trivial background metric $\tilde{G}_{ij}$ and on the dilaton field $\Phi$ given by $[4]–[10],[13],[15]–[21]$: $ds^2 = k\left\{ (d\alpha)^2 + \tan^2 \alpha d\theta^2 \right\} + k'\left\{ (d\beta)^2 + \tanh^2 \beta d\varphi^2 \right\}$

$2\Phi = \log \cos^2 \alpha + \log \cosh^2 \beta + \text{const.}$, (27)

with $\alpha \in \left[ 0, \frac{\pi}{2} \right] \cup \left[ \pi, \frac{3\pi}{2} \right], \beta \in [0, \infty], \theta, \varphi \in [0, 2\pi]$. The term proportional to $k$ in eq. (26) parametrizes the two-dimensional subspace defined by the $SU(2)_{U(1)}$ parafermionic theory and the term proportional to $k'$ is the two-dimensional subspace that is defined by the non-compact parafermionic theory based on the $SU(2)_{U(1)}$ axial gauged WZW model. It is well known that a different metric $\tilde{G}_{ij}$ and
dilaton function $\tilde{\Phi}$ are obtained if one chooses a vector gauging instead of the axial one \[21\] \( U(1)^V \otimes U(1)^V \):

\[
d\tilde{s}^2 = k\left\{(d\alpha)^2 + \frac{1}{\tan^2 \alpha}d\theta^2\right\} + k'\left\{(d\beta)^2 + \frac{1}{\tanh^2 \beta}d\varphi^2\right\}
\]

\[
2\tilde{\Phi} = \log \sin^2 \alpha + \log \sinh^2 \beta + \text{const.}.
\]

(28)

In both versions of gauging, \( \Delta_k^{(4)} \) has always one non-compact coordinate (\( \beta \)) and three compact ones (\( \alpha, \theta, \varphi \)); \((G_{ij}, \Phi)\) and \((\tilde{G}_{ij}, \tilde{\Phi})\) are related by a generalized duality transformation \[20\]:

\[
R(t) \to \frac{1}{R(t)}, \quad \Phi \to \Phi + \log R(t).
\]

(29)

For later purposes, it is more convenient to use the complex notation:

\[
z = (\sin \alpha)e^{i\theta}, \quad \omega = (\sinh \beta)e^{i\varphi}, \quad \text{(axial case)}
\]

\[
\tilde{z} = (\cos \alpha)e^{i\theta}, \quad y = (\cosh \beta)e^{i\varphi}, \quad \text{(vector case)}.
\]

(30)

In terms of \( z \) and \( w \), \( G_{ij} \) and \( \Phi \) are given by

\[
(ds)^2 = k\frac{dzd\bar{z}}{1 - z\bar{z}} + k'\frac{dwd\bar{w}}{w\bar{w} + 1},
\]

and

\[
2\Phi = \log(1 - z\bar{z}) + \log(w\bar{w} + 1) + \text{const.}.
\]

(31)

The dual metrics \( \tilde{G}_{ij} \) and \( \tilde{\Phi} \) are given in terms of \( x \) and \( y \) as:

\[
(d\tilde{s})^2 = k\frac{dxd\bar{x}}{1 - x\bar{x}} + k'\frac{dyd\bar{y}}{y\bar{y} - 1},
\]

\[
2\tilde{\Phi} = \log(1 - x\bar{x}) + \log(y\bar{y} - 1) + \text{const.}.
\]

(32)

From eqs. (31) and (32), one observes that the \( \left(\frac{SU(2)}{U(1)}\right)_k \) metric and dilaton function are self-dual \( (z \text{ and } x) \) while the \( \left(\frac{SU(2,R)}{U(1)}\right)_{k'} \) are not. The \( w \) subspace is regular while the \( y \) subspace is singular. The \( z \) subspace \( (or \ x) \) defines a \textit{two-dimensional bell}. Its metric \( G_{zz} \), the Ricci tensor \( R_{zz} \), and its scalar curvature \( R^{(z)} \) are singular at the boundary of the bell \( (z = 1)\):

\[
G_{zz} = \frac{k}{1 - z\bar{z}}, \quad R_{zz} = \frac{-1}{(1 - z\bar{z})^2}, \quad R^{(z)} = \frac{-1}{k(1 - z\bar{z})}.
\]

(33)

The \( z \)-bell is negatively curved for any value of \( |z| < 1 \).

The \( w \) subspace is regular everywhere, with finite positive curvature, and is asymptotically flat for \( |w| \to \infty \):

\[
G_{ww} = \frac{k'}{w\bar{w} + 1}, \quad R_{ww} = \frac{+1}{(w\bar{w} + 1)^2}, \quad R^{(w)} = \frac{+1}{k'(w\bar{w} + 1)}.
\]

(34)
It has a *cigar* shape, with maximal curvature at $w = 0$.

The $y$ subspace has a different shape from its $w$-dual. It looks like a *two-dimensional trumpet* with infinite curvature at the boundary ($y = 1$). It is positively curved everywhere ($|y| > 1$) and is asymptotically flat for $|y| \to \infty$:

$$G_{yy} = \frac{k'}{y\bar{y} - 1}, \quad R_{yy} = \frac{+1}{(y\bar{y} - 1)^2}, \quad R(y) = \frac{+1}{k'(y\bar{y} - 1)}. \quad (35)$$

We have therefore two different versions of the $\Delta_k^{(4)}$ space. The first version is the $(z, w)$-bell-cigar four-dimensional space and the second the $(x, y)$-bell-trumpet one.

Up to now we have discussed the geometrical structure of the bosonic coordinates of the $\Delta_k^{(4)}$ space. In order to complete the $\sigma$-model description of our solution, we must include the fermionic superpartners of $(z, w)$ coordinates, e.g. the four Weyl-Majorana left-handed ($\psi^a_+, a = 1, 2, 3, 4$), as well as the four Weyl-Majorana right-handed ($\psi^a_-, a = 1, 2, 3, 4$) ones. Because of the $\mathcal{N} = 1$ local supersymmetry, the interactions among fermions are fixed in terms of the two-dimensional $\sigma$-model backgrounds $G_{ij}, B_{ij}$ and $\Phi$. In the superconformal gauge, one has the following generic form for the $\mathcal{N} = 1$ $\sigma$-model action ($B_{ij} = 0$ in $\Delta_k^{(4)}$ space):

$$S = \frac{-1}{2\pi} \int d\xi d\bar{\xi} \left\{ V^a_+ V^a_- - \frac{1}{2} (\psi^a_+ \nabla_- \psi^a_+ - \psi^a_\bar{+} \nabla_+ \psi^a_\bar{+}) - \frac{1}{2} R_{ab,cd} \psi^a_+ \psi^b_- \psi^c_- \psi^d_+ + \Phi R^{(2)} \right\}, \quad (36)$$

where $a = 1, 2, 3, 4$ are local flat indices and

$$V^a_+ = E^a_i \partial X^i, \quad V^a_- = E^a_i \bar{\partial} X^i, \quad \text{with} \quad G_{ij} = E^a_i E^a_j. \quad (37)$$

The $\nabla_-$ and $\nabla_+$ denote the left- and right-handed covariant derivatives acting on left- and right-handed fermions $\psi^a_+, \psi^a_-:

$$\nabla_- \psi^a_+ = \bar{\partial} \psi^a_+ + \Gamma^a_{bc} \psi^b_- V^c_-, \quad \nabla_+ \psi^a_- = \partial \psi^a_- + \Gamma^a_{bc} \psi^b_+ V^c_+. \quad (38)$$

Since $\Delta_k^{(4)}$ is defined as a direct product of two two-dimensional subspaces, the $\sigma$-model action can be written as:

$$S(\Delta_k^{(4)}) = S \left[ \frac{SU(2)}{U(1)} \right]_k + S \left[ \frac{SL(2, R)}{U(1)} \right]_{k'}. \quad (39)$$

Because of the identity

$$R_{ab,cd} \psi^a_+ \psi^b_- \psi^c_- \psi^d_+ = 2 R \psi^1_+ \psi^2_+ \psi^1_- \psi^2_-; \quad (40)$$

valid in both two-dimensional-subspaces, and thanks to the relations

$$-(R^{(z)} + \Gamma_z G^z \Gamma_z) = \frac{1}{k}; \quad \left( \frac{SU(2)}{U(1)} \right)_k,$$

$$-(R^{(w)} + \Gamma_w G^{aw} \Gamma_w) = \frac{-1}{k'}; \quad \left( \frac{SL(2, R)}{U(1)} \right)_{k'}, \quad (41)$$

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it is then possible to rewrite the $\sigma$-model action in a more convenient form, which shows in particular (at least at the classical level) that the fermions can be described in terms of two free bosonic fields compactified in a special radius. Indeed, using eqs. (40) and (41), one finds:

$$
S_{SU(2)} = \frac{1}{4\pi} \int d\xi d\bar{\xi} \left\{ \partial A \partial \bar{A} + t(A)^2 \left( \partial \theta_A - \sqrt{\frac{2}{k}} \psi^1 \psi^2 \right) \left( \partial \bar{\theta}_A - \sqrt{\frac{2}{k}} \bar{\psi}^1 \bar{\psi}^2 \right) 
- (\psi^1 \partial \bar{\psi}^2 + \psi^2 \partial \bar{\psi}^1 + \psi^1 \partial \psi^1 + \psi^2 \partial \psi^2) 
+ \frac{2}{k} (\psi^1 \psi^2) (\bar{\psi}^1 \bar{\psi}^2) + \log C^2(A) R(2) \right\},
$$

(42)

where $A, \theta_A$ are rescaled fields so that, in the large $k$ limit, they are conventionally normalized as two real bosonic fields. In terms of $z$ and $\bar{z}$, the $A$ and $\theta_A$ are defined by the following relations:

$$
\sin^2 \frac{A}{\sqrt{2k}} = \sin^2 \alpha = z\bar{z} 
- i \frac{1}{2} \log \frac{z}{\bar{z}} = \theta = \frac{\theta_A}{\sqrt{2k}} 
$$

$$
t(A)^2 = \frac{z\bar{z}}{1 - z\bar{z}} = \tan^2 \frac{A}{\sqrt{2k}} 
C(A)^2 = 1 - z\bar{z} = \cos^2 \frac{A}{\sqrt{2k}}.
$$

(43)

In a similar way, the

$$
S_{SL(2,R)} = \frac{1}{4\pi} \int d\xi d\bar{\xi} \left\{ \partial B \partial \bar{B} + T(B)^2 \left( \partial \theta_B - \sqrt{\frac{2}{k'}} \psi^3 \psi^4 \right) \left( \partial \bar{\theta}_B - \sqrt{\frac{2}{k'}} \bar{\psi}^3 \bar{\psi}^4 \right) 
- (\psi^3 \partial \bar{\psi}^4 + \psi^4 \partial \bar{\psi}^3 + \psi^3 \partial \psi^3 + \psi^4 \partial \psi^4) 
- \frac{2}{k'} (\psi^3 \psi^4) (\bar{\psi}^3 \bar{\psi}^4) + \log C^2(B) R(2) \right\},
$$

(44)

where now $B$ and $\theta_B$ are defined in terms of $w$ fields:

$$
\sinh^2 \frac{B}{\sqrt{2k'}} = \sin^2 \beta = w\bar{w} 
- i \frac{1}{2} \log \frac{w}{\bar{w}} = \varphi = \frac{\theta_B}{\sqrt{2k'}} 
$$

$$
T(B)^2 = \frac{w\bar{w}}{1 + w\bar{w}} = \tanh^2 \frac{B}{\sqrt{2k'}} 
C(B)^2 = 1 + w\bar{w} = \cosh^2 \frac{B}{\sqrt{2k'}}.
$$

(45)

In both eqs. (42) and (44), the pure fermionic part of the action is given by the free-fermion kinetic terms together with some current-current interaction. This fact permits us
to describe the four left and the four right fermions in terms of free bosonic fields $\phi_A, \phi_B$ compactified in the shifted radii:

$$R^2_A = 1 + \frac{2}{k} = \frac{k + 2}{k}; \quad \left( \frac{SU(2)}{U(1)} \right)_k,$$

$$R^2_B = 1 - \frac{2}{k'} = \frac{k' - 2}{k'}; \quad \left( \frac{SL(2, R)}{U(1)} \right)_{k'}. \quad (46)$$

The deviation from the value $R_A = R_B = 1$ is due to current-current interactions. The decoupling of $\phi_A$ and $\phi_B$ fields can be seen via the bosonization of fermions and the redefinition of the $\partial \theta_A$ and $\partial \theta_B$ bosonic currents order by order in a $\frac{1}{k}$ or $\frac{1}{k'}$ expansion. This statement is indeed exact and it follows from the fact that both $\left( \frac{SU(2)}{U(1)} \right)_k$ and $\left( \frac{SL(2, R)}{U(1)} \right)_{k'}$ are exact $N = (2,2)$ superconformal models.

This property is well known in both $\frac{SU(2)}{U(1)}$ and $\frac{SL(2, R)}{U(1)}$ supersymmetric-coset models in the compact and non-compact parafermionic representations. The $N = 2$ generators, $J(\xi), G(\xi), G'(\xi)$ and $T(\xi)$, are given in terms of free-scalar fields ($\phi_A, \phi_B$) and in terms of (non-local) parafermionic currents ($P_k, \Pi_{k'}$), [26]–[29], [see also eqs. (10), (16) and (17)].

(i) $\left( \frac{SU(2)}{U(1)} \right)_k$

$$J(\xi) = \sqrt{\frac{k}{k+2}} \partial \phi_A$$

$$G(\xi) = P_k e^{i \sqrt{\frac{k^2}{k^2}} \phi_A}$$

$$G'(\xi) = P_{k'}^{(t)} e^{-i \sqrt{\frac{k'^2}{k'^2}} \phi_A}$$

$$T(\xi) = -\frac{1}{2} (\partial \phi_A)^2 + T_{P_k}(\xi). \quad (47)$$

(ii) $\left( \frac{SL(2, R)}{U(1)} \right)_{k'}$

$$J(\xi) = \sqrt{\frac{k'}{k'-2}} \partial \phi_B$$

$$G(\xi) = \Pi_{k'} e^{i \sqrt{\frac{k'^2}{k'^2}} \phi_B}$$

$$G'(\xi) = \Pi_{k'}^{(t)} e^{-i \sqrt{\frac{k'^2}{k'^2}} \phi_B}$$

$$T(\xi) = -\frac{1}{2} (\partial \phi_B)^2 + T_{\Pi_{k'}}(\xi), \quad (48)$$

$\phi_A$ and $\phi_B$ are free bosons compactified on radii $R_A = \sqrt{\frac{k^2}{k}}$ and $R_B = \sqrt{\frac{k'^2}{k'}}$ and parametrize the fermions $\psi^a_\pm$, $a = 1, 2, 3, 4$, which appear in the $\sigma$-model actions in eqs. (42) and (43) according to our previous discussion.

For large $k, k'$, the operators $e^{i \sqrt{\frac{k^2}{k^2}} \phi_A}$ and $e^{i \sqrt{\frac{k'^2}{k'^2}} \phi_B}$ have conformal dimensions of almost $\frac{1}{2}$ ($h_A = \frac{1}{2} + \frac{1}{k}$, $h_B = \frac{1}{2} - \frac{1}{k'}$); $P_k$ and $\Pi_{k'}$ are the parafermionic currents with

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dimensions $h_F = 1 - \frac{1}{k}$ and $h_{\Pi} = 1 + \frac{1}{k'}$, so that $G(\xi)$ in eqs. (47) and (48) are the $N=2$ supercurrents of dimension $\frac{3}{2}$.

In order to show that for $k' = k + 4$ the $N=2$ superconformal symmetry is extended to $N=4$ with $\hat{c} = 4$, it is more convenient to work in terms of the $H^+$ and $H^-$ fields defined in eqs. (7) and (8), instead of the $\phi_A$ and $\phi_B$ given above. Both $(H^+, H^-)$ and $(\phi_A, \phi_B)$ are bosonizations of the same two-dimensional fermions and so they are related:

$$\sqrt{\frac{k+2}{k}} \phi_A = \frac{1}{\sqrt{2}} (-\alpha H^- + H^+)$$

$$\sqrt{\frac{k' - 2}{k'}} \phi_B = \sqrt{\frac{k+2}{k+4}} \phi_B = \frac{1}{\sqrt{2}} \left( + \frac{1}{\alpha} H^- + H^+ \right)$$

with

$$\alpha = \sqrt{\frac{k+4}{k'}}.$$  \hfill (49)

In terms of the $H^+, H^-$ representation the $N=2$ current of the $\Delta^{(4)}_k$ system is given uniquely in terms of the $\partial H^+ U(1)$ current. The extension of the $N=2$ to an $N=4$ symmetry is due to:

(i) the extension of the $U(1)_{H^+}$ current algebra to an $SU(2)_1$ self-dual compactification point with radius $R_{H^+} = \sqrt{2}$;  
(ii) the existence of two (complex) supercurrents $G$ and $\tilde{G}$ as they are given in eqs. (20).
3.2. $C_k^{(4)}$ torus-bell realization. \( \left( \frac{SU(2)}{U(1)} \right)_k \otimes U(1)_R \otimes U(1)_Q \) supercoordinates

The $C_k^{(4)}$ space is based on a supersymmetric gauged WZW model \( \left( \frac{SU(2)}{U(1)} \right)_k \otimes U(1)_R \otimes U(1)_Q \) with a background term $Q = \sqrt{\frac{2}{k+2}}$ in one of the two coordinate currents ($U(1)_Q$). The other free coordinate ($U(1)_R$) is compactified on a torus with radius $R = \sqrt{2k}$.

In this realization the two coordinates ($x_1, x_2$) are those of the two-dimensional bell space ($z, \bar{z}$) of the $SU(2)_k$ parafermions, $P_k, \ P_k^t$ [see eq. (31)]. The other two remaining coordinates ($x_3, x_4$) are free, with a background charge $Q$ on $x_4$:

$$J_i(\xi)J_j(\xi') \equiv \frac{-\delta_{ij}}{(\xi - \xi')^2} \quad \text{with} \quad i, j = 3, 4.$$  

(50)

The fermions are parametrized in terms of the two bosons $\phi_A$ and $\phi_B$ with radii $R_A = \sqrt{\frac{k+2}{k}}$ and $R_B = 1$.

The $N = 2$ superconformal symmetry is manifest in the $C_k^{(4)}$ space since both sub-systems have $N = 2$ superconformal symmetry. The $N = 2$ generators of the $(P^t, P, \phi_A)$ sub-system are given in eq. (47). The $(J_3, J_4, \phi_B)$ Liouville-like sub-system has the following $N = 2$ generators [30]:

$$T_Q(\xi) = -\frac{1}{2} \left[ (\partial \phi_B)^2 + J_3^2 + J_4^2 + Q \partial J_4 \right]$$

$$G(\xi) = \left[ + J_4 + i(J_3 - Q \partial \phi_B) \right] e^{+i\phi_B}$$

$$G^t(\xi) = \left[ - J_4 + i(J_3 - Q \partial \phi_B) \right] e^{-i\phi_B}$$

$$J(\xi) = \partial \phi_B + Q J_3.$$  

(51)

In order to make the $N = 4$ symmetry manifest in the combined system, it is necessary to introduce the $H^+$ field, which defines the $SU(2)_1$ currents of the $N = 4$ algebra. This can be done using the following field redefinitions:

$$\phi_B = \sqrt{\frac{1}{2}}(H^+ + H^-)$$

$$\phi_A = \sqrt{\frac{k}{k+2}} \left[ \sqrt{\frac{1}{2}}(H^+ - H^-) - \sqrt{\frac{2}{k}} \partial \sigma \right]$$

$$J_3 = \sqrt{\frac{2}{k+2}} \left[ \sqrt{\frac{1}{2}}(H^+ - H^-) - \sqrt{\frac{k}{2}} \partial \sigma \right].$$  

(52)

Using the above redefinitions the $N = 4$ generators take the desired factorized form in terms of $P^t, P, J^4, \partial \sigma, H^+$ and $H^-$.

$$T_B = -\frac{1}{2} \left[ (\partial H^+)^2 + (\partial H^-)^2 + (\partial \sigma)^2 + J_4^2 + Q \partial J_4 \right] + T_{P_k}$$
\[
G = - \left( -J_4 + i \left( \sqrt{\frac{k}{k + 2}} \partial \sigma - \sqrt{\frac{2}{k + 2}} \partial H^- \right) \right) e^{-i \sqrt{\frac{1}{2}} H^+} + P^t e^{+i \sqrt{\frac{1}{2}} H^- + i \sqrt{\frac{2}{k}} \sigma} \right) e^{i \sqrt{\frac{1}{2}} H^+} \\
\tilde{G} = + \left[ (+J_4 + i \left( \sqrt{\frac{k}{k + 2}} \partial \sigma - \sqrt{\frac{2}{k + 2}} \partial H^- \right) \right) e^{+i \sqrt{\frac{1}{2}} H^+} - P e^{-i \sqrt{\frac{1}{2}} H^- - i \sqrt{\frac{2}{k}} \sigma} \right) e^{i \sqrt{\frac{1}{2}} H^+} \\
S_0 = \frac{1}{\sqrt{2}} \partial H^+ , \quad S_\pm = e^{\pm i \sqrt{2} H^+} .
\]

The closure of the $N = 4$ algebra can be easily verified using the OPE relations of the $P^t, P$ parafermions in eq. (17) and those of the free bosonic fields $H^+, H^-, x_4$ and $\sigma$. The $H^+$ and $H^-$ are compactified on radii $R_{H^+} = R_{H^-} = \sqrt{2} \sqrt{k}$, while the $\sigma$ is compactified with radius $R_\sigma = \sqrt{2} \sqrt{k}$ (or its dual $\tilde{R}_\sigma = \sqrt{2} \sqrt{k}$).

For large $k$ the $C^{(4)}_k$ has a four-dimensional target-space interpretation, which is a product of two sub-spaces; the first sub-space has the shape of a two-dimensional bell while the second sub-space is locally flat with the shape of a two-dimensional cylinder. The metric and the dilaton function on the $C^{(4)}_k$, torus-bell four-dimensional space are:

\[
ds^{2}(C^{(4)}_k) = k \frac{dz \bar{z} + k dw \bar{w}}{1 - z \bar{z}} + 2 \Phi = w + \bar{w} + \log (1 - z \bar{z}) .
\]

What is interesting is that the $C^{(4)}_k$ space can be obtained by performing (supersymmetric) duality transformation on $W^{(4)}_k$ [30], [31]. On $C^{(4)}_k$ the torsion is zero, while it is non-trivial on $W^{(4)}_k$ [see eq. (25)]. The metric of $C^{(4)}_k$ is singular while that of $W^{(4)}_k$ is regular. The question about the relevance of the singularity in the “stringy” constructed models is still an open question.

4. STRING CONSTRUCTIONS USING AS BUILDING BLOCKS THE $F^{(4)}, \Delta^{(4)}_k, W^{(4)}_k$ AND $C^{(4)}_k$ N=4 SPACES.

Having at our disposal non-trivial $N = 4$, $\hat{c} = 4$ superconformal systems, we can use them as building blocks to obtain new classes of exact and stable string solutions around non-trivial backgrounds in both Type II and heterotic superstrings constructions.

In the Type II case, we arrange the degrees of freedom of the ten supercoordinates in three superconformal systems:

\[
\hat{c} = 10 = \{ \hat{c} = 2 \}_0 + \{ \hat{c} = 4 \}_1 + \{ \hat{c} = 4 \}_2 .
\]

The $\hat{c} = 2$ system is saturated by two free superfields (compact or non-compact), and so the background metric is flat:

\[
ds^2 \{ F^2 \} = dx d\bar{x} , \quad (x = x_1 + ix_2) .
\]

The remaining eight supercoordinates appear in a group of four. Both $\{ \hat{c} = 4 \}_{1,2}$ building blocks are $N = 4$ superconformal systems based on $F^{(4)}$, $W^{(4)}_k$, $C^{(4)}_k$ or on the
two versions of $\Delta_k^{(4)}$ spaces. (The $F^{(4)}$ realization includes the compact or non-compact four-dimensional flat space as well as the $T^4/Z_2$ orbifold models.)

The advantage of this approach of constructing new solutions compared with the subclass of models ($F^{(6)} \times W^{(4)}$) studied in the literature (using the $\sigma$-model approach) lies in the complete knowledge of the $\{\hat{c} = 2\}_0$ and $\{\hat{c} = 4\}_A$ superconformal theories. With this knowledge, we are able to study the full string spectrum and derive the partition functions of the models and not only the background solutions for large $k_A$.

The full spectrum around any constructed background is given in terms of several characters of known conformal theories:

(i) in the $W_k^{(4)}$ realization, one uses modular invariant character combination of $SU(2)_k, SU(2)_{H^+}$ and $SU(2)_{H^-}$, together with the $U(1)_Q$ Liouville-type characters;

(ii) in the $\Delta_k^{(4)}$ realization, one uses the compact [34] and non-compact [35] parafermionic characters (string functions), together with the $SU(2)_{H^+}$ and $U(1)_{R_H^-}$ with $R_H = \sqrt{2} \sqrt{\frac{k+4}{k}}$;

(iii) in the $C_k^{(4)}$ realization, one uses the compact parafermionic characters, the $U(1)_Q$ Liouville-type characters, the two $SU(2)_{H^+}$ and $SU(2)_{H^-}$ as well as the $U(1)_{R_e}$ characters (with $R_\sigma = \sqrt{2k}$);

(iv) finally in $F^{(2)}$ and $F^{(4)}$, one uses non-compact flat-space characters, toroidal ones or orbifold ones.

It is important to stress here that the above character combinations are not arbitrary but are dictated by the global existence of $N = 4$ superconformal symmetry as well as the modular invariance of the partition function. These requirements define some generalized GSO projections [7]-[10], similar to that of the fermionic construction [37] and that of the conformal block construction [38] of Gepner. One of these projections is fundamental and guarantees the existence of some space-time supersymmetries via the $N = 4$ spectral flows. It can be expressed in terms of the two isospins of the $N = 4$ ($\hat{c} = 4)_A$, $A = 1, 2$ sub-systems ($[SU(2)_{H^+}]_A [j(s_1), j(s_2)]$)[7]-[10]:

$$2(j(s_1) + j(s_2)) = \text{odd integer}.$$ (55)

Equation (55) guarantees the existence of some covariantized target-space symmetries the proposed non-trivial backgrounds. This stabilizes our solutions (at least perturbatively) and projects out all kinds of tachyonic or complex conformal wave states. This projection phenomenon is similar to the one observed by Kutasov and in the framework of non-critical superstrings with an $N = 2$ globally defined superconformal symmetry [39].

The stability of the vacuum under string-loop perturbation follows from the existence of a globally defined $N = 4$ symmetry, which implies a new realization target space supersymmetry around the non-trivial backgrounds, which are defined by the $W_k^{(4)}, C_k^{(4)}$ and $\Delta_k^{(4)}$ spaces. This new target-space supersymmetry follows from the $N = 4$ spectral flow relations, which imply a spectrum degeneracy among space-time bosonic and space-time fermionic string excitations. This string spectrum degeneracy guarantees the vanishing of the partition function (at least at the one-string loop level), for all values of $k$. 

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Heterotic solutions are simply obtained via a generalized \[ \mathbb{M} \], \[ \mathbb{N} \] heterotic map. Here also the solutions are stable, but the number of covariantized space-time supersymmetries is reduced by a factor of two.

We hope that our explicit construction of a family of consistent and stable solutions will give a better understanding of some fundamental string properties, especially in the case of strongly curved backgrounds (small \( k_A \)), where the notion of space-time dimensionality and topology breaks down.

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