Ground state of the asymmetric Rabi model in the ultrastrong coupling regime

Li-Tuo Shen, Zhen-Biao Yang, Mei Lu, Rong-Xin Chen, and Huai-Zhi Wu
Lab of Quantum Optics, Department of Physics, Fuzhou University, Fuzhou 350002, China

We study the ground states of the asymmetric single- and two-qubit Rabi models, in which the coupling strengths for the counter-rotating wave and rotating wave interactions are different. We take the transformation method to analytically solve the ground states for both Rabi models and numerically verify it to be valid under a wide range of parameters. We find that the ground state energy in the single- or two-qubit Rabi model has an approximately quadratic dependence on the coupling strengths stemming from different contributions of the counter-rotating wave and rotating wave interactions. For the single-qubit Rabi model, we show the accuracy of results can be further improved by the second-order perturbation correction. Interestingly, for the two-qubit Rabi model, we find that after the ground state entanglement reaches its maximum it decreases to zero with the increase of the coupling strength in the counter-rotating wave or rotating wave interaction, and never increases again when the qubit-oscillator coupling strength is further increased. Furthermore, the maximum of the ground state entanglement in the asymmetric two-qubit Rabi model is far larger than that in the symmetric two-qubit Rabi model.

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I. INTRODUCTION

The Rabi model [1], which describes the interaction between a two-level system and a quantized harmonic oscillator, is a fundamental model in various fields, especially in quantum physics. For the cavity quantum electrodynamics (QED) experiments, the qubit-oscillator coupling strength of the considered Rabi model is always far smaller than the oscillator frequency and the rotating wave approximation (RWA) takes effect, bringing in the ubiquitous Jaynes-Cummins model [2–3]. With recent progresses in Rabi models in the ultrastrong coupling regime [4–13], where the qubit-oscillator coupling strength becomes a considerable fraction of the oscillator and qubit frequencies, the RWA breaks down and relatively complex dynamics arises, leading to many unusual quantum phenomena [14–20].

The explicitly analytic solution to the Rabi model beyond the RWA is very hard to be obtained due to its unclosed Hilbert space. Since the numerical solution [21–23] for the Rabi model to some extent difficult fails to catch its fundamental properties [16–20], various approximately analytic methods for the solution of the ground states of the symmetric Rabi models (SRM) have been tried [24–27]. Especially, Braak [28] used the method based on the \( \mathbb{Z}_2 \) symmetry to analytically determine the spectrum of the single-qubit Rabi model, which is dependent on the composite transcendental function defined through its power series but fails to be used to derive the concrete form of the system’s ground state.

Previous studies consider the ground state in the symmetric Rabi models, i.e., the qubit-oscillator coupling strength of the counter-rotating wave interaction and that of the rotating wave interaction are equal. In this paper, we study the asymmetric Rabi models (ASRM), in which the coupling strengths for the counter-rotating wave and rotating wave interaction terms are different, as to gain deep insight into the dynamics of such models.

We study the single- and two-qubit ASRM and show that the transformed ground state fits well with the exactly numerical simulation for a wide range of parameters, and the ground state energy has an approximately quadratic dependence on the coupling strengths stemming from different contributions of the counter-rotating wave and rotating wave interactions. Besides, in the case of the single-qubit ASRM, we show that when the qubit-oscillator coupling strength increases, the accuracy of the results can be further improved by the correction based on the second-order perturbation theory. While in the two-qubit ASRM, we analytically obtain the negativity. Interestingly, we find that when the coupling strength of the counter-rotating wave or rotating wave interaction is large enough, after the ground state entanglement reaches its maximum it then drops to zero and never increases again as the qubit-oscillator coupling strength increases further. Especially, the maximum of the ground state entanglement in the two-qubit ASRM is far larger than that in the two-qubit SRM. Our study in this paper mainly reveals the collective contribution of the qubit-oscillator coupling strengths of the counter-rotating wave and rotating wave interactions to the ASRM’s ground states. Such an investigation can also be generalized to the complex cases of three- and more-qubit ASRM. Note that the ASRM can be realized by using two unbalanced Raman channels between two atomic ground states induced by a cavity mode and two classical fields [49].
II. THE SINGLE-QUBIT ARSM

A. Transformed ground state

The Hamiltonian of the single-qubit ARSM is\[ H_q = \frac{1}{2} w_a \sigma_z + w_b b \dagger b + \frac{\lambda_1}{2} (b \dagger \sigma_+ + b \sigma_+) + \frac{\lambda_2}{2} (b \dagger \sigma_- + b \sigma_-), \] (1)

where \( w_a \) is the transition frequency of the qubit. \( \sigma_+ \) and \( \sigma_- \) are the Pauli matrices, describing the qubit’s energy operator and the spin-flip operators, respectively. \( b \dagger (b) \) is the creation (annihilation) operator of the harmonic oscillator with the frequency \( w_b \). The qubit-oscillator coupling strengths of the rotating wave interaction \( (b \dagger \sigma_+ + b \sigma_-) \) and the counter-rotating wave interaction \( (b \dagger \sigma_- + b \sigma_+) \) are respectively denoted by \( \lambda_1 \) and \( \lambda_2 \). When we perform a rotation around the y axis on the Hamiltonian \( H_q \), the single-qubit Rabi model becomes:

\[ H_{11} = \frac{1}{2} w_a \sigma_z + w_b b \dagger b + \frac{1}{4} (\lambda_1 + \lambda_2) (b \dagger + b) \sigma_z + \frac{i}{4} (\lambda_1 - \lambda_2) (b \dagger - b) \sigma_y, \] (2)

where \( \frac{1}{2} (\sigma_z \pm i \sigma_y) = \sigma_{\pm} \). However, when \( \lambda_1 \neq \lambda_2 \) (here \( \lambda_1, \lambda_2, w_a, w_b \neq 0 \)), there is still no analytical solution to the ground state of the single-qubit ARSM based on the Hamiltonian \( H_{11} \).

Our task in this paper is to determine the ground state energy \( E_g \) and the ground state vector \( |\phi_g \rangle \) for the single- (Section II) or two-qubit (Section III) ARSM, where \( H_{11} |\phi_g \rangle = E_g |\phi_g \rangle \).

To deal with the counter-rotating wave terms in Eq. (2), we apply a unitary transformation to the Hamiltonian \( H_{11} \) \[ 24, 11, 42 \]:

\[ H'_{11} = e^{S_1} H_{11} e^{-S_1}, \] (3)

with

\[ S_1 = \xi_1 (b \dagger - b) \sigma_z, \] (4)

where \( \xi_1 \) is a variable to be determined. Then the transformed Hamiltonian \( H'_{11} \) can be decomposed into three parts:

\[ H'_{11} = H'_{110} + H'_{111} + H'_{112}, \] (5)

with \n
\[ H'_{110} = \frac{1}{4} w_a \eta_1 - \frac{1}{4}(\lambda_1 - \lambda_2) \xi_1 |x\rangle 
+ w_b - \frac{1}{2}(\lambda_1 - \lambda_2) \xi_1 |\sigma_x \rangle b \dagger b 
+ w_a \xi_1^2 - \frac{1}{2}(\lambda_1 - \lambda_2) \xi_1, \] (6)

\[ H'_{111} = \frac{1}{4}(\lambda_1 - \lambda_2) - w_a \xi_1 (b \dagger + b) \sigma_z \]

\[ + \eta_1 \left[ \frac{1}{8}(\lambda_1 - \lambda_2) + w_a \xi_1 \right] (b \dagger - b) i \sigma_y, \] (7)

\[ H'_{112} = \frac{1}{2} w_a \sigma_z \left\{ \cosh [2 \xi_1 (b \dagger - b)] - \eta_1 \right\} 
+ \frac{1}{2} w_a i \sigma_y \left\{ \sinh [2 \xi_1 (b \dagger - b)] - 2 \xi_1 \eta_1 (b \dagger - b) \right\} 
+ \frac{1}{8} (\lambda_1 - \lambda_2) (b \dagger - b) i \sigma_y \left\{ \cosh [2 \xi_1 (b \dagger - b)] - \eta_1 \right\} 
+ \frac{1}{8} (\lambda_1 - \lambda_2) (b \dagger - b) \sigma_z \left\{ \sinh [2 \xi_1 (b \dagger - b)] - b \right\} - 2 \xi_1 \eta_1 (b \dagger - b) \right\} + O(b^2, b^2), \] (8)

where \( \eta_1 = F(0) \cosh[2 \xi_1 (b \dagger - b)] |0 \rangle_F = \exp(-2 \xi_1^2) \) and \( O(b^2, b^2) = \frac{1}{4} (\lambda_1 - \lambda_2) (b \dagger + b)^2 \sigma_x \). The terms \( \cosh[2 \xi_1 (b \dagger - b)] \) and \( \sinh[2 \xi_1 (b \dagger - b)] \) in \( H'_{112} \) have the dominating expansions \[ 12 \]:

\[ \cosh[2 \xi_1 (b \dagger - b)] \approx \eta_1 + O(\xi_1^2), \] (9)

\[ \sinh[2 \xi_1 (b \dagger - b)] \approx 2 \xi_1 \eta_1 (b \dagger - b) + O(\xi_1^3). \] (10)

where \( O(b^2, b^2) \) and \( O(\xi_1^2) \) are higher-order terms of \( b \dagger \) and \( b \), which represent the double- and three-photon transition processes and can be neglected as an approximation for small \( \xi_1 \) and \( |\lambda_1 - \lambda_2| \). Thus, \( H'_{11} \approx H'_{110} + H'_{111} \).

When the parameter \( \xi_1 \) satisfies the condition:

\[ e^{2 \xi_1^2} \frac{2 (\lambda_1 + \lambda_2) - 8 w_b \xi_1}{(\lambda_1 - \lambda_2) + 8 w_a \xi_1} = 1, \] (11)

the qubit and the harmonic oscillator are coupled in the following form:

\[ H'_{11} = \frac{1}{2} \left[ (\lambda_1 - \lambda_2) - 4 w_b \xi_1 \right] \times \left( b \dagger - \frac{1}{2} A (\frac{1}{2} + b) \right) \left( b \dagger - \frac{1}{2} A (\frac{1}{2} + b) \right), \] (12)

where \( |- \frac{1}{2} A \rangle \) and \( |\frac{1}{2} A \rangle \) are the eigenstates of \( \sigma_x \), i.e., \( \sigma_x | \frac{1}{2} A \rangle = - | \frac{1}{2} A \rangle \) and \( \sigma_x | \frac{1}{2} A \rangle = | \frac{1}{2} A \rangle \).

Note that \( H_{11} \) in Eq. \[ 12 \] contains no counter-rotating wave interactions in which the atomic excitation (dexcitation) is accompanied by the emission (absorption) of a photon. Therefore, the transformed Hamiltonian \( H'_{11} \) is exactly solvable when we eliminate the counter-rotating wave terms by setting the parameter \( \xi_1 \) to satisfy Eq. \[ 11 \] and by neglecting higher-order transition processes.

It is easy to show that the eigenvector \( |- \frac{1}{2} A \rangle_F \) is the ground state vector of the transformed Hamiltonian \( H_{11} \), with \( |0 \rangle_F \) being the vacuum state of the harmonic oscillator, and the corresponding energy \( E_g \) is:

\[ E_g = \xi_1^2 w_b - \frac{1}{4} \xi_1 (\lambda_1 + 3 \lambda_2) - \frac{1}{2} w_a e^{-2 \xi_1^2}. \] (13)

We see that when \( \lambda_1 = \lambda_2 \), \( E_g \) reduces to the transformed ground state energy shown in Ref. \[ 32 \]. Therefore, the ground state of the original Hamiltonian \[ 11 \] can
is only numerically exact solution when we find that the ground state energy obtained by the method and that by the numerical solution. Especially, the ground state energy obtained by the transformation method increases with the coupling strengths $\lambda_1$ and $\lambda_2$: (a) $w_b = 0.8w_a$; (b) $w_b = w_a$; (c) $w_b = 1.2w_a$. The energy deviation $\Delta E_{g1} = |E_{g1} - E_g|$ versus $\lambda_1$ and $\lambda_2$: (a) $w_b = 0.8w_a$; (c) $w_b = w_a$; (f) $w_b = 1.2w_a$.

Note that the ground state is an entangled state between the harmonic oscillator with the amplitudes $\xi_1$ and $\xi_2$. The energy deviation $\Delta E_{g1} = |E_{g1} - E_g|$ due to the fact that the effect of the neglected double-photon term $O(b^{12}, b^2)$ in Eq. (8) increases with $|\lambda_1 - \lambda_2|$. This is due to the fact that the effect of the neglected double-photon term $O(b^{12}, b^2)$ in Eq. (8) increases with $|\lambda_1 - \lambda_2|$. The energy deviation $\Delta E_{g1} = |E_{g1} - E_g|$ versus $\lambda_1$ and $\lambda_2$: (a) $w_b = 0.8w_a$; (b) $w_b = w_a$; (c) $w_b = 1.2w_a$.

FIG. 1. (Color online) The ground state energy for the single-qubit ASRM obtained by the transformation method $E_0 = E_{g1}$ (red grid) and by the numerical solution $E_0 = E_g$ (blue grid) versus the coupling strengths $\lambda_1$ and $\lambda_2$: (a) $w_b = 0.8w_a$; (b) $w_b = w_a$; (c) $w_b = 1.2w_a$. The energy deviation $\Delta E_{g1} = |E_{g1} - E_g|$ versus $\lambda_1$ and $\lambda_2$: (d) $w_b = 0.8w_a$; (e) $w_b = w_a$; (f) $w_b = 1.2w_a$.

be approximately constructed:

$$|\phi_{g1}\rangle = e^{-S_1} - \frac{1}{2} \mathcal{A}|0\rangle_F$$

$$= \frac{1}{2} \left[ -|\xi_1\rangle_F \langle -\xi_1| + |\xi_1\rangle_F \langle -\xi_1| \right]$$

with $|\xi_1\rangle_F$ and $|\xi_1\rangle_F$ denoting the coherent states of the harmonic oscillator with the amplitudes $\xi_1$ and $-\xi_1$. Note that the ground state is an entangled state between the qubit and the oscillator, with the entanglement depending upon $\xi_1$.

The value of $\xi_1$ is obtained by numerically solving the nonlinear equation (11). In Fig. 1, we compare the ground state energy obtained by the transformation method and that by the numerical solution. Especially, we find that the ground state energy obtained by the transformation method coincides very well with the numerically exact solution when $|\lambda_1 - \lambda_2| \leq 0.1w_a$. For example, when $\lambda_1 = 0.02w_a$ and $\lambda_2 = 0.12w_a$, the error is only $0.0004991 \approx 0.1\%$ at the resonant case. Therefore, when $\lambda_1 \leq w_a$ and $\lambda_2 \leq w_a$, the transformed ground state energy $E_{g1}$ is approximately:

$$E_{g1} \simeq -\frac{1}{2}w_a - \frac{(\lambda_1 + 3\lambda_2)^2}{64w_a(w_a + w_b)}$$

which shows that the ground state energy has an approximately quadratic dependence on the coupling strengths and the contribution of the counter-rotating wave interaction is larger than that of the rotating wave interaction. This result differs from that of the single-qubit SRM (12). Fig. 1 (a) - (c) show the error of result obtained by the transformation method increases with $|\lambda_1 - \lambda_2|$. The result shows that the transformation method functions very well for small $\xi_1$ and $|\lambda_1 - \lambda_2|$, but obviously deviates from the exact value when $|\lambda_1 - \lambda_2|$. Therefore, it is necessary to correct the result when the value $|\lambda_1 - \lambda_2|$ becomes large.

Consider the fidelity $F_1$ for the ground state $|\phi_{g1}\rangle$, defined here as $F_1 = \langle \phi_{g1}|\phi_g\rangle$ where $|\phi_g\rangle$ is the ground state obtained through numerical solution (12). $F_1$ is plotted as a function of the coupling strengths $\lambda_1$ and $\lambda_2$ under different detunings in Fig. 2. The result shows that the fidelity is higher than 99.9999% when $\lambda_1 \leq 0.5w_a$ and $\lambda_2 \leq 0.5w_a$. Furthermore, the fidelity under the positive detuning $w_b - w_a > 0$ decreases slowest among all the detuning cases in Fig. 2 (a) - (c) when $\lambda_1$ and $\lambda_2$ increase.

B. Correction for the transformed ground state

The merit of using the transformation method is that the ground state can be corrected explicitly by taking into account the higher-order transition processes. Fig. 1 shows that the transformation method functions very well for small $\xi_1$ and $|\lambda_1 - \lambda_2|$, but obviously deviates from the exact value when $|\lambda_1 - \lambda_2|$. Therefore, it is necessary to correct the result when the value $|\lambda_1 - \lambda_2|$ becomes large.

For $|\lambda_1 - \lambda_2| > 0.1w_a$, the $O(b^{12}, b^2)$ term can not be ignored. To make the correction, we calculate the eigenenergy $E_{g1,2}$ of the second excited eigenstate from $H_{t10} + H_{t11}$ as following:

$$E_{g1,2} = \omega_b(\xi_1^2 + \frac{3}{2}) + (\lambda_1 - \lambda_2)\xi_1 - \frac{1}{2} \sqrt{\frac{1}{2} \Omega_1^2 + \frac{1}{16} \Omega_2^2}$$

where $\Omega_1 = (\lambda_1 - \lambda_2) - 4\omega_b\xi_1$ and $\Omega_2 = 7(\lambda_1 - \lambda_2)\xi_1 - 2\omega_b e^{-2\xi_1^2 + 2w_b}$. Based on the second-order perturbation theory, we obtain the corrected ground state energy and treat the $O(b^{12}, b^2)$ term as a perturbation which couples the transformed ground state $|-\frac{1}{2}\rangle_A|0_F\rangle$ to the second excited state $|-\frac{1}{2}\rangle_A|2_F\rangle$ ($|2_F\rangle$ is the two-photon state of the oscillator):

$$E'_{g1} = E_{g1} + \frac{\mathcal{F}(2)A(-\frac{1}{2})O(b^{12}, b^2)|-\frac{1}{2}\rangle_A|0_F\rangle}{E_{g1} - E_{g1,2}}$$

$$= \xi_1^2 w_b - \frac{1}{4} \xi_1(\lambda_1 + 3\lambda_2) - \frac{1}{2} \omega_a e^{-2\xi_1^2}$$
and $E_{q1}'$ is plotted as a function of the coupling strengths $\lambda_1$ and $\lambda_2$ under different detunings in Fig. 3 (a) - (c). After the correction, the energy deviation $\Delta E_{q1}$ from the numerical result is plotted in Fig. 3 (d) - (f), which shows that the error is greatly reduced after the correction. The discrepancy between $E_{q1}'$ and $E_q$ is almost invisible when there is a negative detuning, i.e., $w_b - w_a < 0$.

FIG. 3. (Color online) The corrected ground state energy obtained by the transformation method(red grid) and the numerical solution (blue grid) versus the coupling strengths $\lambda_1$ and $\lambda_2$: (a) $w_b = 0.8w_a$; (b) $w_b = w_a$; (c) $w_b = 1.2w_a$. The energy deviation $\Delta E_{q1} = |E_{q1}' - E_q|$ versus $\lambda_1$ and $\lambda_2$: (d) $w_b = 0.8w_a$; (e) $w_b = w_a$; (f) $w_b = 1.2w_a$.

III. THE TWO-QUBIT ARSM

A. Transformed ground state

When a $\frac{\pi}{2}$ rotation around the $y$ axis is performed, the Hamiltonian of the two-qubit ARSM is [50]:

$$H_{t_2} = \frac{1}{2}w_a(J_+ + J_-) + w_bb^\dagger b + (g_1 + g_2)(b^\dagger + b)J_z$$
$$+ \frac{(g_1 - g_2)}{2}(b^\dagger - b)(J_+ - J_-),$$

(18)

where $w_a$ is the transition frequency of each qubit. $J_t\{t = \pm, z\}$ describes the collective qubit operator of a spin-1 system, satisfying the angular momentum commutation relations: $[J_z, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_z$. $b^\dagger (b)$ is the creation (annihilation) operator of the harmonic oscillator with the frequency $w_b$. The qubit-oscillator coupling strengths of the rotating and the counter-rotating wave interactions are $g_1$ and $g_2$, respectively. We denote the eigenstates of $J_z$ by $| \pm 1\rangle_A$, $|0\rangle_A$, and $|1\rangle_A$, i.e., $J_z|m\rangle_A = m|m\rangle_A$ ($m = 0, \pm 1$). $|0\rangle_F$ is the vacuum state for the harmonic oscillator, and $|X\rangle_F$ denotes the coherent state field with the amplitude $X$.

To transform the Hamiltonian $H_{t_2}$ into a mathematical form without the counter-rotating wave terms, we apply a unitary transformation to the Hamiltonian $H_{t_2}$:

$$H_{t_2}' = e^{S_2}H_{t_2}e^{-S_2},$$

(19)

with

$$S_2 = \xi_2(b^\dagger - b)J_z,$$

(20)

where $\xi_2$ is a variable to be determined. Therefore, the transformed Hamiltonian $H_{t_2}'$ is decomposed into three parts:

$$H_{t_2}' = H_{t_220}' + H_{t_221}' + H_{t_222}',$$

(21)

with

$$H_{t_220}' = w_bb^\dagger b + \left[w_a\eta_2 - (g_1 - g_2)\eta_2\xi_2\right]J_x$$
$$+ \left[w_b\xi_2^2 - 2\xi_2(g_1 + g_2)\right]J_z^2,$$

(22)

$$H_{t_221}' = \left[(g_1 + g_2) - w_b\xi_2\right](b^\dagger + b)J_z$$
$$+ i\left[w_b\eta_2\xi_2 + (g_1 - g_2)\eta_2\right](b^\dagger - b)J_y,$$

(23)

$$H_{t_222}' = w_aJ_x\left\{\cosh[\xi_2(b^\dagger - b)] - \eta_2\right\}$$
$$+ iw_aJ_y\left\{\sinh[\xi_2(b^\dagger - b)] - \eta_2\xi_2(b^\dagger - b)\right\}$$
$$+ (g_1 - g_2)(b^\dagger - b)J_x\left\{\sinh[\xi_2(b^\dagger - b)]\right\}$$
$$- \eta_2\xi_2(b^\dagger - b) + i(g_1 - g_2)(b^\dagger - b)J_y$$
$$\times\left\{\cosh[\xi_2(b^\dagger - b)] - \eta_2\right\} + O(b^{1,2}, b^2),$$

(24)

where $\eta_2 = F_\omega(0)\cosh[\xi_2(b^\dagger - b)]|0\rangle_F = \exp(-\frac{\xi_2^2}{2})$ and $O(b^{1,2}, b^2) = (g_1 - g_2)\eta_2\xi_2J_x(b^\dagger - 2b^\dagger b - b^2)$. As shown in the single-qubit ARSM, for small $\xi_2$ and $|g_1 - g_2|$, $H_{t_222}'$ can be neglected, thus $H_{t_2} \approx H_{t_220}' + H_{t_221}'$.

The eigenvalues $\nu_k\{k = 1, 2, 3\}$ and the eigenvectors $|\psi_k\rangle_A$ of the Hamiltonian $H_{t_2}'' = H_{t_220}' - w_bb^\dagger b$ are:

$$\nu_1 = \frac{A}{2} - \frac{1}{2}\sqrt{A^2 + 8B^2},$$

$$|\psi_1\rangle_A = \frac{1}{N_1}\left\{|-1\rangle_A - \frac{(A + \sqrt{A^2 + 8B^2})}{2B}|0\rangle_A + |1\rangle_A\right\},$$

$$\nu_2 = A,$$

$$|\psi_2\rangle_A = \frac{1}{N_2}\left\{|-1\rangle_A + |1\rangle_A\right\},$$

$$\nu_3 = \frac{A}{2} + \frac{1}{2}\sqrt{A^2 + 8B^2},$$

$$|\psi_3\rangle_A = \frac{1}{N_3}\left\{|-1\rangle_A - \frac{(A - \sqrt{A^2 + 8B^2})}{2B}|0\rangle_A + |1\rangle_A\right\},$$

(25)

with

$$A = w_b\xi_2^2 - 2\xi_2(g_1 + g_2),$$

(26)
The value of nonlinear equation (28). We find that when \( \nu_1 < \nu_2 < \nu_3 \). Then \( H'_{2}\) can be expanded in terms of the renormalized eigenvectors:

\[
H'_{2} \approx \sum_{k=1}^{3} \nu_k |\varphi_k\rangle A|\varphi_k\rangle + [(D_1 b + D_2 b^\dagger)|\varphi_1\rangle A|\varphi_2\rangle + (D_3 b + D_4 b^\dagger)|\varphi_2\rangle A|\varphi_3\rangle + H.c.] + w_b b^\dagger b, \tag{27}
\]

where \( D_x \) \((x = 1, 2, 3, 4)\) is the coefficient depending on the variable \( \xi \).

After transforming the Hamiltonian \( H_{2}\) into \( H'_{2}\), we can eliminate the counter-rotating wave terms describing the coupling between the lowest eigenstates by setting:

\[
D_1 = \eta_2 \left[ w_a \xi_2 + (g_1 - g_2) \right] \left( A + \sqrt{A^2 + 8B^2} \right)
- 2\sqrt{2}B \left[ (g_1 + g_2) - w_b \xi_2 \right] = 0. \tag{28}
\]

The value of \( \xi_2 \) is obtained by numerically solving the nonlinear equation (28). We find that when \( g_1 \leq 0.5w_a \) and \( g_2 \leq 0.5w_a \), \( \xi_2 \) has an approximate relation with the coupling strengths as:

\[
\xi_2 \approx \frac{(w_b - w_a)g_1 + (w_b + w_a)g_2}{w_b^2 + w_a^2}. \tag{29}
\]

In Fig. 4, we compare the ground state energy obtained by the transformation method and that obtained by numerical solution. We find that when \( g_1 \leq 0.25w_a \) and \( g_2 \leq 0.25w_a \), the ground state energy through the transformation method coincides very well with the exact value even for \( |g_1 - g_2| = 0.24w_a \). For example,

\[
B = \frac{1}{\sqrt{2}}[w_a \eta_2 - \eta_2 \xi_2 (g_1 - g_2)], \tag{26}
\]

where \( N_k \) is the normalization factor for the eigenvector \(|\varphi_k\rangle\). Here the eigenvalues are arranged in the decreasing order: \( \nu_1 < \nu_2 < \nu_3 \). Then \( H'_{2}\) can be expanded in terms of the renormalized eigenvectors:

\[
H'_{2} \approx \sum_{k=1}^{3} \nu_k |\varphi_k\rangle A|\varphi_k\rangle + [(D_1 b + D_2 b^\dagger)|\varphi_1\rangle A|\varphi_2\rangle + (D_3 b + D_4 b^\dagger)|\varphi_2\rangle A|\varphi_3\rangle + H.c.] + w_b b^\dagger b, \tag{27}
\]

where \( D_x \) \((x = 1, 2, 3, 4)\) is the coefficient depending on the variable \( \xi_2 \).

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\[
D_1 = \eta_2 \left[ w_a \xi_2 + (g_1 - g_2) \right] \left( A + \sqrt{A^2 + 8B^2} \right)
- 2\sqrt{2}B \left[ (g_1 + g_2) - w_b \xi_2 \right] = 0. \tag{28}
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B = \frac{1}{\sqrt{2}}[w_a \eta_2 - \eta_2 \xi_2 (g_1 - g_2)], \tag{26}
\]

where \( N_k \) is the normalization factor for the eigenvector \(|\varphi_k\rangle\). Here the eigenvalues are arranged in the decreasing order: \( \nu_1 < \nu_2 < \nu_3 \). Then \( H'_{2}\) can be expanded in terms of the renormalized eigenvectors:

\[
H'_{2} \approx \sum_{k=1}^{3} \nu_k |\varphi_k\rangle A|\varphi_k\rangle + [(D_1 b + D_2 b^\dagger)|\varphi_1\rangle A|\varphi_2\rangle + (D_3 b + D_4 b^\dagger)|\varphi_2\rangle A|\varphi_3\rangle + H.c.] + w_b b^\dagger b, \tag{27}
\]

where \( D_x \) \((x = 1, 2, 3, 4)\) is the coefficient depending on the variable \( \xi_2 \).

After transforming the Hamiltonian \( H_{2}\) into \( H'_{2}\), we can eliminate the counter-rotating wave terms describing the coupling between the lowest eigenstates by setting:

\[
D_1 = \eta_2 \left[ w_a \xi_2 + (g_1 - g_2) \right] \left( A + \sqrt{A^2 + 8B^2} \right)
- 2\sqrt{2}B \left[ (g_1 + g_2) - w_b \xi_2 \right] = 0. \tag{28}
\]

The value of \( \xi_2 \) is obtained by numerically solving the nonlinear equation (28). We find that when \( g_1 \leq 0.5w_a \) and \( g_2 \leq 0.5w_a \), \( \xi_2 \) has an approximate relation with the coupling strengths as:

\[
\xi_2 \approx \frac{(w_b - w_a)g_1 + (w_b + w_a)g_2}{w_b^2 + w_a^2}. \tag{29}
\]

In Fig. 4, we compare the ground state energy obtained by the transformation method and that obtained by numerical solution. We find that when \( g_1 \leq 0.25w_a \) and \( g_2 \leq 0.25w_a \), the ground state energy through the transformation method coincides very well with the exact value even for \( |g_1 - g_2| = 0.24w_a \). For example,
From Eq. (35), we see that the two-qubit entanglement of the two-qubit ASRM decreases to zero and never increases again, and the maximum negativity is about 0.104 which is only 0.035 in the two-qubit SRM. In conclusion, we have used the transformation method to obtain the approximately analytical ground states of the single- and two-qubit ASRM, and show that the results coincide well with those obtained by numerical simulation for a wide range of parameters. We find that the ground state energy in the single- or two-qubit ASRM has an approximately quadratic dependence on the qubit-oscillator coupling strengths, and the effect of the counter-rotating wave interaction on the ground state energy is larger than that of the rotating wave interaction. In the single-qubit ASRM, we find that the accuracy of the obtained ground state energy can be further improved by the second-order perturbation correction. Interestingly, we observe that the ground state entanglement of the two-qubit ASRM decreases to zero and never increases again as long as the qubit-oscillator coupling strengths are large enough. Furthermore, the maximum of the ground state entanglement in the two-qubit ASRM is far larger than that in the two-qubit SRM, and the ground state entanglement mainly appears when the coupling strength of the rotating wave interaction is bigger than that of the counter-rotating wave interaction, which is because the contribution to the ground state entanglement from the counter-rotating wave interaction is larger than that from the rotating wave interaction. As seen from Fig. 7, when $g_1 > 1.11w_a$ or $g_2 > 0.88w_a$ at $w_b = w_a$, $M_{\rho_A}$ decreases to zero and never increases again, and the maximum negativity is about 0.104 which is only 0.035 in the two-qubit SRM.
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[1] I. I. Rabi, Phys. Rev. 49, 324 (1936); 51, 652 (1937).
[2] E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).
[3] B. W. Shore and P. L. Knight, J. Mod. Opt. 40, 1195 (1993).
[4] S. Osnaghi, P. Bertet, A. Auffeves, et al., Phys. Rev. Lett. 87, 037902 (2001).
[5] S. M. Spillane, T. J. Kippenberg, K. J. Vahala, et al., Phys. Rev. A 71, 013817 (2005).
[6] A. A. Abdumalikov Jr, O. Astafiev, Y. Nakamura, Y. A. Pashkin, and J. S. Tsai, Phys. Rev. B 78, 180502(R) (2008).
[7] A. A. Anappara, S. D. Liberato, A. Tredicucci, C. Ciuti, G. Biasiol, L. Sorba, and F. Beltram, Phys. Rev. B 79, 201303(R) (2009).
[8] G. Günter, A. A. Anappara, J. Hees, et al., Physica A 458, 178 (2009).
[9] T. Niemczyk, F. Deppe, H. Huebl, et al., Phys. Rev. A 77, 033808 (2012).
[10] P. Forn-Díaz, J. Lisenfeld, D. Marcos, J. J. Garcíá-Ripoll, E. Solano, C. J. P. M. Harmans, and J. E. Mooij, Phys. Rev. Lett. 105, 237001 (2010).
[11] Y. Todorov, A. M. Andrews, R. Colombelli, et al., Phys. Rev. A 87, 033815 (2013).
[12] T. Schwartz, J. A. Hutchison, C. Genet, and T. W. Ebbesen, Phys. Rev. Lett. 106, 196405 (2011).
[13] G. Scalari, C. Maissen, D. Turčinková, et al., Science 335, 1323 (2012).
[14] A. Crespi, S. Longhi, and R. Osellame, Phys. Rev. Lett. 108, 163601 (2012).
[15] S. Hayashi, Y. Ishigaki, and M. Fujii, Phys. Rev. B 86, 045408 (2012).
[16] X. Cao, J. Q. You, H. Zheng, and F. Nori, New J. Phys. 13, 073002 (2011).
[17] A. Ridolfo, M. Leib, S. Savasta, and M. J. Hartmann, Phys. Rev. Lett. 109, 196402 (2012).
[18] S. Ashhab, Phys. Rev. A 87, 033826 (2013).
[19] H. P. Zheng, F. C. Lin, Y. Z. Wang, and Y. Segawa, Phys. Rev. A 59, 4589 (1999).
[20] S. B. Zheng, X. W. Zhu, and M. Feng, Phys. Rev. A 62, 033807 (2000).
[21] E. K. Irish, J. Gea-Banacloche, I. Martin, and K. C. Schwab, Phys. Rev. B 72, 195410 (2005).
[22] C. Ciuti and I. Carusotto, Phys. Rev. A 74, 033811 (2006).
[23] D. Wang, T. Hansson, Å. Larson, H. O. Karlsson, and J. Larson, Phys. Rev. A 77, 053808 (2008).
[24] X. F. Cao, J. Q. You, H. Zheng, A. G. Kofman, and F. Nori, Phys. Rev. A 82, 022119 (2010).
[25] P. Natapfel and C. Ciuti, Phys. Rev. Lett. 107, 190402 (2011).
[26] V. V. Albert, Phys. Rev. Lett. 108, 180402 (2012).
[27] J. D. Ferrer, L. I. Komarov, and A. P. Ulyanenkov, J. Phys. A: Math. Gen. 29, 4035 (1996).
[28] Q. H. Chen, T. Liu, Y. Y. Zhang, and K. L. Wang, Eur. Phys. Lett. 96, 14003 (2011).
[29] H. Chen, Y. M. Zhang, and X. Wu, Phys. Rev. B 40, 11326 (1989).
[30] J. Stolze and L. Müller, Phys. Rev. B 42, 6704 (1990).
[31] E. K. Irish, Phys. Rev. Lett. 99, 173601 (2007).
[32] T. Liu, K. L. Wang, and M. Feng, Eur. Phys. Lett. 86, 54003 (2009).
[33] D. Zuoco, G. M. Reuther, S. Kohler, and P. Hänggi, Phys. Rev. A 80, 033846 (2009).
[34] J. Casanova, G. Romero, I. Lizuain, J. J. García-Ripoll, and E. Solano, Phys. Rev. Lett. 105, 263603 (2010).
[35] M. J. Hwang and M. S. Choi, Phys. Rev. A 82, 025802 (2010).
[36] J. Song, Y. Xia, X. D. Sun, Y. Zhang, B. Liu, and H. S. Song, Eur. Phys. J. D 66, 1 (2012).
[37] L. X. Yu, S. Q. Zhu, Q. F. Liang, G. Chen, and S. T. Jia, Phys. Rev. A 83, 015803 (2012).
[38] S. Agarwal, S. M. H. Rafaanjanii, and J. H. Eberly, Phys. Rev. A 85, 043815 (2012).
[39] Q. H. Chen, C. Wang, S. He, T. Liu, and K. L. Wang, Phys. Rev. A 86, 023822 (2012).
[40] H. Zheng, Eur. Phys. J. B 38, 559 (2004).
[41] Z. G. Lü and H. Zheng, Phys. Rev. B 75, 054302 (2007).
[42] E. K. Irish and H. Zheng, Eur. Phys. J. D 59, 473 (2010).
[43] M. C. Lee and C. K. Law, Phys. Rev. A 86, 015803 (2012).
[44] S. Agarwal, S. M. H. Rafaanjanii, and J. H. Eberly, Phys. Rev. A 85, 043815 (2012).
[45] F. Altintas and R. Eryigit, Phys. Rev. A 87, 022124 (2013).
[46] L. H. Du, X. F. Zhou, Z. W. Zhou, X. Zhou, and G. C. Guo, Phys. Rev. A 86, 014303 (2012).
[47] H. H. Zhong, Q. T. Xie, and C. H. Lee, arXiv:1305.6782 (2013).
[48] D. Braak, Phys. Rev. Lett. 107, 100401 (2011).
[49] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, Phys. Rev. A 75, 013804 (2007).
[50] F. T. Hioe, Phys. Rev. A 8, 1440 (1973).
[51] G. Vidal, R. F. Werner, Phys. Rev. A 65, 032314 (2002).