Equivalent theorem of uncertainty relations

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Abstract
We present an equivalence theorem to unify the two classes of uncertainty relations, i.e. the variance-based ones and the entropic forms, showing that the entropy of an operator in a quantum system can be built from the variances of a set of commutative operators. This means that an uncertainty relation in the language of entropy may be mapped onto a variance-based one, and vice versa. Employing the equivalence theorem, alternative formulations of entropic uncertainty relations are obtained for the qubit system that are stronger than the existing ones in the literature, and variance-based uncertainty relations for spin systems are reached from the corresponding entropic uncertainty relations.

Keywords: uncertainty relation, entropy, covariance

(Some figures may appear in colour only in the online journal)

1. Introduction

The renowned uncertainty principle, which was introduced by Heisenberg in the description of microscopic quantum behavior [1], is one of the distinctive features of quantum mechanics, and somewhat similar to the concept of complementarity raised by Bohr [2]. The uncertainty relation (UR) is a mathematical expression for the uncertainty principle, referring to the repulsive nature of incompatible operators, and hence imposing a strong restriction on the outcomes of any joint measurement on these operators. Since the UR has a profound influence on various aspects of quantum information, e.g. quantum nonlocality [3–5], entanglement [6], and quantum cryptography [7], the study of it has never stopped.

It is well known that the most famous UR, the Heisenberg–Robertson UR [8], bounds the product of the variances of observables \( A \) and \( B \) through the expectation value of their commutator, i.e.

\[
\Delta A \Delta B \geq |\langle C \rangle|.
\]

(1)

Here, the state \(|\psi\rangle\) is arbitrary, \( \Delta X = \sqrt{\langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2} \) is the square root of the variance of a given operator \( X \), and \([A, B] = 2iC\) is the commutator of operators \( A \) and \( B \). Note that the relation (1) is applicable to any pair of operators, rather than only to conjugate observables. Though later improvement [9, 10] strengthened the UR, the state-dependent feature of the lower bound remains, which implies the null lower bound triviality [11]. Early attempts at searching for the state-independent lower bounds led only to near-optimal results [12]. In a recent work [13], this triviality problem was solved at length, in which the state-independent optimal trade-off relations for the variances of multiple observables are found to be obtainable—at least in principle; see also [14] for further development of URs involving variances of multiple observables.

It was noticed that the variance is inadequate in quantifying uncertainty, i.e. relabeling of the non-degenerate eigenvalues of an operator may alter the value of its variance [11]. To overcome this problem, the concept of entropy was employed, and a typical UR in entropy is [15]

\[
H(A) + H(B) \geq -2 \ln c_{ab}.
\]

(2)

Here, \( H(A) = -\sum_j p_j \ln p_j \) is the Shannon entropy, with \( p_j \) as the probability distribution of the eigenbasis \(|a_j\rangle\) of operator \( A \) in a measuring system, and similarly for \( H(B) \). The bound \( c_{ab} = \max_{j,k} \langle a_j|b_k \rangle \) is the maximum overlap of the eigenbases of the operators \( A \) and \( B \) and independent of the quantum state. To construct an optimal entropic UR, the key point is to find the best lower bound. This is usually a tough question for general observables [16, 17], and the optimal bounds for the qubit system with Shannon [18–20] and collision entropies [21] have already been obtained. We refer to [22–26] for more involved situations of this problem. The entropic uncertainty relation may in principle apply to multiple observables, of which in fact the state-independent lower bounds have been investigated for mutually unbiased bases [27, 28]; see [29, 30] for recent reviews.

There are actually many types of entropy capable of characterizing the quantum uncertainty [22], e.g. the Rényi entropies \( H_\alpha(A) = \ln(\sum_j p_j^\alpha)/(1 - \alpha) \) with different indices of positive real numbers \( \alpha \in \mathbb{R}^+ \) (when \( \alpha \to 1 \) it is a Shannon entropy). As there is no obvious reason why one type of entropy is superior to others in the context of UR, a new characterization of uncertainty has been introduced: the majorization of the probability distribution [24, 31, 32], which is closely related to the entropy. Since both variance and entropy originate from the probability distribution of measurement, one may naturally ask: are these two types of uncertainty relation relevant or equivalent? We find in this work that the answer is definite.

In the following we will present a general scheme on how to build quantitative relations between two prominent uncertainty measures, the variance and the entropy, which indicates that these two classes of UR may actually be unified. The scheme can be sketched as follows: first construct a set of commutative operators for a given physical observable, then reconstruct the probability distribution of the measurement outcomes of the physical observable from the variances of the operators in the set. Based on the quantitative relation, we get various entropic URs for the qubit system from variance-based URs, which give optimal lower and upper bounds for arbitrary entropic measures and for multiple observables beyond mutually unbiased bases. Moreover, new variance-based URs for high spin systems are also obtained from the entropic URs.
2. The equivalence theorem and its applications

In quantum mechanics, a physical system can generally be described by the density operator \( \rho \), which is a positive definite Hermitian matrix; a physical observable is represented by a Hermitian operator and may be expressed through the spectral decomposition

\[
A = \sum_{j=1}^{N} \lambda_j |j\rangle\langle j|, \quad \text{where} \quad |j\rangle \text{ are the eigenvectors of } A \text{ with the corresponding eigenvalues } \lambda_j.
\]

When measuring the observable \( A \) in a quantum system \( \rho \), only its eigenvalues \( \lambda_j \) are attainable with certain probability in every individual measurement. In the \( \rho \)-ensemble, the probability of measuring \( \lambda_j \) reads

\[
|\langle j|\rho|j\rangle|.
\]

This statistical interpretation leads to two uncertainty measures, the variance and entropy, which are mathematically expressed as

\[
V(A) = \frac{1}{2} \sum_{j,k=1}^{N} p_j p_k (\lambda_j - \lambda_k)^2,
\]

\[
H_{\alpha}(A) = \frac{1}{1 - \alpha} \ln \left( \sum_{j=1}^{N} p_j^\alpha \right).
\]

Here, \( V(A) \) signifies variance defined as \( V(A) \equiv \Delta A^2 = \text{Tr}[\rho A^2] - \text{Tr}[\rho A]^2 \), and \( H_{\alpha}(A) \) represents the Rényi entropy. We deal with Rényi entropy throughout this article, and for general forms of the entropic functions we refer to [25]. Notice that subtracting a constant from the operator does not change its variance or entropy, and we are hence legitimate in treating the operators in the following discussion as traceless.

2.1. The equivalence theorem

In the following we exhibit an equivalence theorem, the main result of this work, which may quantitatively relate the different uncertainty measures of discrete systems.

**Theorem 1.** For a given physical observable \( A \) in an \( N \)-dimensional representation with eigenvectors \( |j\rangle \), there exists a set of commutative operators, \( A = \{A_i | A_i = \sum_{j=1}^{N} \lambda^{(i)}_j |j\rangle\langle j|, A_1 = A \} \), whose variances in the quantum state \( \rho \) are

\[
\Delta A_i^2 = \sum_{j,k=1}^{N} p_j p_k s_{jk}^{(i)}, \quad \text{with} \quad s_{jk}^{(i)} = (\lambda^{(i)}_j - \lambda^{(i)}_k)^2,
\]

from which the probability distribution \( p_j = \langle j|\rho|j\rangle \) can be uniquely determined. Here \( A = A_1 \in A \), and the infimum of the cardinality of the set \( A \) lies in \([N - 1, N(N - 1)/2]\).

**Proof.** Let \( l = (j - 1)N + k - (j + 1)j/2 \), then there is a one-to-one correspondence between the integer \( l \) and the integer array \((j, k)\), and equation (5) may be rewritten as

\[
\Delta A_i^2 = \sum_{l=1}^{N(N - 1)/2} G_{il} x_l,
\]

where \( G_{il} = s_{jk}^{(i)} \) and \( x_l = p_j p_k \) with \( k > j \). The number of the linear equation (6) equals the cardinality of the set \( A \), which we denote as \(|A|\). When \(|A| = N(N - 1)/2\), the coefficient matrix \( G_{il} \)
can be constructed as invertible by assigning specific values to \( \lambda_j \) for \( i = 1, 2, \ldots, N(N - 1)/2 \).

The solutions of \( x_l \) are the linear functions of \( \Delta A^2_i \), which in turn yields \( N(N - 1)/2 \) equations for \( p_j \)

\[
p_j p_k = x_l(\Delta A^2_i, \ldots, \Delta A^2_{N(N - 1)/2}). \tag{7}
\]

Here \( \Delta A^2_i \) are function arguments of \( x_l(\cdot) \), from which \( p_j \) can also be uniquely determined as functions of \( \Delta A^2_i \).

As equation (7) is an over-determined equation system, we need not know all the \( N(N - 1)/2 \) variables of \( x_l(\cdot) \) to uniquely determine the \( N \) variables \( p_j \). This means that the set \( \mathcal{A} \) may even be constructed with \( |\mathcal{A}| \leq N(N - 1)/2 \). On the other hand, the number of equations constraining \( p_j \) cannot be less than \( N \), otherwise the solution of \( p_j \) will not be unique. Considering the additional constraint \( \sum_{j=1}^N p_j = 1 \), \( |\mathcal{A}| \) must be greater than or equal to \( N - 1 \), the dimension of the Cartan subalgebra of the \( \text{SU}(N) \) group. In all, the cardinality of the set \( |\mathcal{A}| \) lies in \( [N - 1, N(N - 1)/2] \).

Theorem 1 applies to arbitrary physical observables. When the observable is non-degenerate, i.e., \( \forall i \neq j, \lambda_i \neq \lambda_j \), it is possible to construct the commutative set \( \mathcal{A} \) explicitly and the following proposition holds.

**Proposition 1.** For the non-degenerate observable \( A \) in an \( N \)-dimensional representation with the eigenbases \( |i \rangle \), the probability distribution \( p_i = \langle i | \rho | i \rangle \) in a quantum state \( \rho \) may be expressed in terms of the covariance functions

\[
p_i^2 = \frac{\Omega_{ij} \Omega_{jk}}{\Omega_{ik}}, \tag{8}
\]

where \( \Omega_{ij} = -\text{cov}(\ell_i, \ell_j), 1 \leq i < j \leq N \), with the covariance function \( \text{cov}(\ell_i, \ell_j) = \langle \ell_i(A) \ell_j(A) \rangle - \langle \ell_i(A) \rangle \langle \ell_j(A) \rangle \), and the Lagrange basis polynomials \( \ell_j(x) = \prod_{m=1}^N \frac{x - \lambda_m}{\lambda_j - \lambda_m} \).

**Proof.** The least degree polynomial function, assumed to be valued as \( f(\lambda_i) \) for \( N \) distinct \( \lambda_i \), is a linear combination of the Lagrange basis polynomials \( f(x) = \sum_{j=1}^N f(\lambda_j) \ell_j(x) \), where

\[
\ell_j(x) = \prod_{m=1}^N \frac{x - \lambda_m}{\lambda_j - \lambda_m}. \tag{9}
\]

The variance of the operator function \( f(A) \) may be expressed as

\[
\Delta f(A)^2 = \sum_{k>j=1}^N p_j p_k [f(\lambda_j) - f(\lambda_k)]^2, \tag{10}
\]

according to equation (3). By setting \( f(\lambda_i) - f(\lambda_k) = \alpha_k \), we have

\[
\Delta f(A)^2 = \sum_{j=2}^N p_j p_j \alpha_j^2 + \sum_{k>j=1}^N (\alpha_{j-1} - \alpha_k) p_j p_k. \tag{11}
\]

On the other hand, the function \( \tilde{f}(x) \equiv f(x) - f(\lambda_i) \) has the values \( f(\lambda_i) - f(\lambda_i) = -\alpha_i \) for \( N \) distinct \( \lambda_i \), therefore \( \tilde{f}(x) = -\sum_{i=2}^N \alpha_i \ell_i(x) \). Because \( \Delta f(A)^2 = \Delta \tilde{f}(A)^2 \), we have
\[ \Delta f(A)^2 = \sum_{i,j=2}^{N} \alpha_{i-1}\alpha_{j-1}(\langle \ell_i(A)\ell_j(A) \rangle - \langle \ell_i(A) \rangle \langle \ell_j(A) \rangle) \]

\[ = \sum_{i=2}^{N} \alpha_i^2 \text{cov}(\ell_i, \ell_i) + \sum_{n>m=2}^{N} (\alpha_{m-1} + \alpha_{n-1}^2 - (\alpha_{m-1} - \alpha_{n-1})^2) \text{cov}(\ell_m, \ell_n) \]

\[ = \sum_{i=2}^{N} \alpha_i^2 \sum_{j=2}^{N} \text{cov}(\ell_i, \ell_j) - \sum_{n>m=2}^{N} (\alpha_{m-1} - \alpha_{n-1})^2 \text{cov}(\ell_m, \ell_n). \quad (11) \]

Here, \( \text{cov}(\ell_i, \ell_j) = (\ell_i(A)\ell_j(A) - \langle \ell_i(A) \rangle \langle \ell_j(A) \rangle) \). The expectation value \( \langle \ell_i(A) \rangle = \text{Tr}[\rho\ell_i(A)] \) when mixed states are involved. The equivalence of equations (10) and (11) does not depend on the values of \( \alpha_i \), hence

\[ p_j p_j = \sum_{i=2}^{N} \text{cov}(\ell_j, \ell_k); p_j p_k = -\text{cov}(\ell_j, \ell_k), \quad k > j. \quad (12) \]

Using the fact that \( \sum \ell_i(x) = 1 \) we have \( p_j p_j = -\text{cov}(\ell_i, \ell_j), \) for \( i < j \), and equation (8) is obtained. \( \square \)

Proposition 1 gives the most direct relations between the probabilities and covariances which are inherited from the characteristic functions that relate the probability distribution and high-order momentums in probability theory. Next, we shall illustrate the extraordinary function of the equivalence theorem in bridging the prevailing variance-based and entropic URs through concrete examples of spin systems.

2.2. Uncertainty relations for qubits

The qubit system might be the most investigated system in quantum information, and it possesses enormous potential in applications. In such systems, any physical observable may be represented by a \( 2 \times 2 \) traceless Hermitian matrix, and therefore the eigenvalues of an operator may be assigned as \( \lambda_1 = -\lambda_2 = \lambda \). According to proposition 1 the following corollary holds.

**Corollary 1.** In a qubit system, there exists the following monotonic functional relations between the entropy and the variance

\[ H_\alpha(A) = f_\alpha(\Delta A^2) = \frac{1}{1 - \alpha} \ln(a_+^\alpha + a_-^\alpha), \quad (13) \]

\[ \Delta A^2 = f_{\alpha}^{-1}[H_\alpha(A)] \equiv g_\alpha(A), \quad (14) \]

where \( a_{\pm} \equiv (1 \pm \sqrt{1 - \Delta A^2})/2 \) with the eigenvalues of \( A \) being absorbed into its variance \( \Delta A^2/\lambda^2 \rightarrow \Delta A^2 \), and \( f_{\alpha}^{-1} \) being the inverse function of \( f_\alpha \).

**Proof.** For a qubit system where \( N = 2 \), we have \( \ell_1(A) = (A - \lambda_2)/(\lambda_1 - \lambda_2), \) \( \ell_2(A) = (A - \lambda_1)/(\lambda_2 - \lambda_1), \) and

\[ p_1 p_2 = -\text{cov}(\ell_1, \ell_2) = \frac{\Delta A^2}{4\lambda_2^2}. \quad (15) \]

Absorbing \( \lambda^2 \) into \( \Delta A^2 \), and considering \( p_1 + p_2 = 1 \), then
Substituting equation (16) into the definition of the Rényi entropy equation (4), we have
\[
H_\alpha(A) = f_\alpha(\Delta A^2) = \frac{1}{1 - \alpha} \ln(a_+^\alpha + a_-^\alpha),
\]
where \(a_\pm \equiv (1 \pm \sqrt{1 - \Delta A^2})/2\). Equation (17) is a monotonic function for \(\Delta A^2 \in [0, 1]\), and therefore
\[
\Delta A^2 = f_\alpha^{-1}[H_\alpha(A)] \equiv g_\alpha(A).
\]
Here, \(f_\alpha^{-1}\) is the inverse function of the Rényi entropy with index \(\alpha\).

Corollary 1 predicts that an entropic UR may be straightforwardly converted into a variance-based UR. For example, putting equation (16) into the entropic UR equation (2) we get
\[
\frac{a_+^{a_+} - a_-^{a_-}}{b_+^{b_+} - b_-^{b_-}} \leq c_{ab}^2,
\]
where the quantities \(a_\pm = (1 \pm \sqrt{1 - \Delta A^2})/2\), \(b_\pm = (1 \pm \sqrt{1 - \Delta B^2})/2\). There is also the majorized UR [31], \(\tilde{p}(\rho) \otimes \tilde{q}(\rho) \prec \tilde{\omega}\), where \(\tilde{p}(\rho)\) and \(\tilde{q}(\rho)\) are the probability distributions for two observables in a quantum state \(\rho\), and \(\tilde{\omega}\) is a state independent vector. Taking equation (16) into this majorized UR we have
\[
(1 + \sqrt{1 - \Delta A^2})(1 + \sqrt{1 - \Delta B^2}) \leq (1 + c_{ab})^2.
\]
Here, \(c_{ab}\) is defined in equation (2).

On the other hand, the variance-based UR may also be transformed into an entropic one. However, the state-dependence of the lower bounds of the variance-based URs leads to trivial entropy relations, and the non-trivial results only exist for the state-independent ones. For example, we have the variance-based UR from theorem 1 of [13],
\[
[a^2(p^2 - 1) + \Delta A^2][b^2(p^2 - 1) + \Delta B^2] \geq (\sqrt{a^2 - \Delta A^2} \sqrt{b^2 - \Delta B^2} - \kappa p^2)^2.
\]
Taking equation (14) into this variance-based UR we have
\[
[a^2(p^2 - 1) + g_\beta(A)][b^2(p^2 - 1) + g_\beta(B)] \geq \left(\sqrt{a^2 - g_\beta(A)} \sqrt{b^2 - g_\beta(B)} - \kappa p^2\right)^2.
\]
Here, \(a^2 = \text{Tr}[A^2]/2\), \(b^2 = \text{Tr}[B^2]/2\), \(p^2 = 2\text{Tr}[\rho^2] - 1\), and \(\kappa = \text{Tr}[AB]/2\); \(\alpha\) and \(\beta\) are independent Rényi indices. Equation (21) gives both the optimal lower and upper bounds for arbitrary entropic measures, and is tight: equation (21) is satisfied for all the quantum states; for all the values of entropies of operators \(A\) and \(B\) satisfying equation (21) there is the quantum state corresponding to them. This provides a better analytic result compared to the existing ones [18–22]. To show this more explicitly, we take a pure quantum system with operators \(A = \sigma \cdot \tilde{n}_a\), \(B = \sigma \cdot \tilde{n}_b\) and the Shannon entropies of \(\alpha = \beta = 1\) as an example. In this case, equation (21) becomes
\[
g_\beta(A)g_\beta(B) \geq \left(\sqrt{1 - g_\beta(A)} \sqrt{1 - g_\beta(B)} - \cos \theta_{ab}\right)^2,
\]
where $\theta_{ab}$ is the angle between unit vectors $\mathbf{n}_a$ and $\mathbf{n}_b$. Figure 1 illustrates the allowed regions for the Shannon entropies of operators $A$ and $B$ predicted by equation (22). These figures are consistent with the recent results obtained by analyzing the parameters of the state space of the qubit [33].

For observables of more than two, the following corollary exists:

**Corollary 2.** In a qubit system, for three independent observables $A = \sigma \cdot \mathbf{n}_a$ and $B = \sigma \cdot \mathbf{n}_b$ and $C = \sigma \cdot \mathbf{n}_c$, where $\mathbf{n}_a$, $\mathbf{n}_b$, and $\mathbf{n}_c$ are not co-plane, the entropic UR involving $H_\alpha(A)$, $H_\beta(B)$, and $H_\gamma(C)$ where $\alpha, \beta, \gamma \in \mathbb{R}^+$, take the form of an equality.

Taking equation (14) into proposition 1 of [13], one may easily notice that the corollary 2 holds, and the equality form of the entropic URs could also be obtained explicitly from the variance-based URs for multiple observables [14]. As an illustration, we take the Pauli operators of the qubit system as an example. The variance-based uncertainty equality $\Delta \sigma_x^2 + \Delta \sigma_y^2 + \Delta \sigma_z^2 = 4 - 2\text{Tr}[\rho^2]$ (see [13, 14]) leads to the following entropic uncertainty equality

$$g_\alpha(\sigma_x) + g_\beta(\sigma_y) + g_\gamma(\sigma_z) = 4 - 2\text{Tr}[\rho^2].$$

Here, the function of entropy $g_\alpha$ is defined in equation (14). This gives out an optimal equality form of trade-off relations for $H_\alpha(\sigma_x)$, $H_\beta(\sigma_y)$ and $H_\gamma(\sigma_z)$ in the arbitrary qubit state, while

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Figure 1. The allowed regions for the Shannon entropies of the operators $A = \sigma \cdot \mathbf{n}_a$ and $B = \sigma \cdot \mathbf{n}_b$ in pure states with (a) $\theta_{ab} = 90^\circ$, (b) $\theta_{ab} = 45^\circ$, (c) $\theta_{ab} = 30^\circ$ and (d) $\theta_{ab} = 0^\circ$ respectively. Here, $\theta_{ab}$ is the angle between the unit vectors $\mathbf{n}_a$ and $\mathbf{n}_b$. The obtained entropic uncertainty relation is optimal: 1) for every point in the shaded area there is a quantum state that gives the corresponding values of $H(A)$ and $H(B)$; 2) for every quantum state, the values of $H(A)$ and $H(B)$ lie in the shaded region.
the results given in [28, 34, 35] provide upper and/or lower bounds in the special case of \( \alpha = \beta = \gamma = 1 \). For the collision entropy with \( \alpha = \beta = \gamma = 2 \), equation (23) gives the following uncertainty equality:

\[
e^{-H_a(\gamma)} + e^{-H_b(\gamma)} + e^{-H_c(\gamma)} = 1 + \text{Tr}[\rho^2],
\]

(24)

where the monotonic relation \( \Delta \mathcal{A}^2 = g_2(\mathcal{A}) = 2 - 2e^{-H_2(\mathcal{A})} \) is employed.

2.3. Uncertainty relations for spin-1 and even higher

Proposition 1 is generally applicable to arbitrary non-degenerate observables; here, we take the spin systems as examples for high dimensional systems. For spin-1 operators \( \vec{J}_a = \vec{J} \cdot \vec{n}_a \) with eigenvalues \( \lambda_1 = 1, \lambda_2 = 0 \) and \( \lambda_3 = -1 \) (assume \( h = 1 \)) we have

\[
\ell_1(J_a) = (J_a^2 + J_b^2)2, \quad \ell_2(J_a) = 1 - J_a^2, \quad \ell_3(J_a) = (J_a^2 - J_b^2)/2.
\]

(25)

According to proposition 1, the covariances of the operators \( \ell_1(J_a) \) and \( \ell_3(J_a) \) can be evaluated and the probability distribution is recovered

\[
\begin{align*}
\rho_1 &= \frac{\Omega_{12} \Omega_{13}}{\Omega_{23}} = \frac{|V(J_a^2) + (J_a^2 - (J_a^2)(J_a))/2||V(J_a) - V(J_a^2)|}{4|V(J_a^2) - (J_a^2)(J_a))/2|}, \\
\rho_2 &= \frac{\Omega_{12} \Omega_{23}}{\Omega_{13}} = \frac{|V(J_a^2) - (J_a^2)(J_a))/2||V(J_a) - V(J_a^2)|}{V(J_a) - V(J_a^2)}, \\
\rho_3 &= \frac{\Omega_{13} \Omega_{23}}{\Omega_{12}} = \frac{|V(J_a^2) - (J_a^2)(J_a))/2||V(J_a) - V(J_a^2)|}{4|V(J_a^2) + (J_a^2)(J_a))/2|}.
\end{align*}
\]

(26, 27, 28)

The collision entropy may now be expressed as

\[
H_2(J_a) = -\ln[1 - 2(\rho_1 \rho_2 + \rho_1 \rho_3 + \rho_3^2)] = -\ln[1 - \frac{1}{2}V(J_a) - \frac{3}{2}V(J_a^2)],
\]

(29)

For the two operators \( J_a \) and \( J_b = \vec{J} \cdot \vec{n}_b \), the entropic UR

\[
H_2(J_a) + H_2(J_b) \geq c
\]

(30)

immediately leads to the following variance-based UR

\[
[2 - V(J_a) - 3V(J_a^2)][2 - V(J_b) - 3V(J_b^2)] \leq 4e^{-c}.
\]

(31)

Here, the lower bound of equation (31) is optimal if the \( c \) in equation (30) is optimal, and the tightness inherits that of equation (30). Puchała, et al [32] found a simple bound for equation (30), i.e.

\[
c = -\ln[\frac{1 + e_{ab}}{2} + (1 - \frac{1 + e_{ab}}{2})^2],
\]

(32)

where \( e_{ab} \) is the maximum overlap of eigenbases of operators \( J_a \) and \( J_b \). Considering equation (32) for the case of the angular momentum operators along the \( x \) and \( z \) axes, equation (31) becomes
A numerical evaluation of the above inequality shows that \( V(J_\perp) + V(J_z) \geq 7/16 \), which is consistent with that of [36] for a spin-1 system. A similar expression to equation (29) may also be obtained for the spin-3/2 system, of which the collision entropy reads

\[
H_{C}(J_{\perp}) = -\ln \left\{ 1 - \left[ \frac{5}{9} V(J_{\perp}^{3}) + \frac{1}{4} V(J_{z}^{2}) + \frac{365}{144} V(J_{\perp}) - \frac{41}{18} \langle J_{z}^{3} \rangle - \langle J_{\perp}\rangle \langle J_{z}^{2} \rangle \right] \right\}.
\]  

(34)

In principle, there is also no difficulty getting similar relations to equation (31) for even higher spin systems by applying proposition 1.

To summarize, we have built a one-to-one correspondence between the variance and entropy in a qubit system, as has been shown in section 2.2. For high-dimensional systems, different covariance functions are needed to build the probability distributions, see section 2.3. The measurement of covariance functions involves the measurements of high-order moments of an operator which are compatible with the measurement of its variance (the operator is commuted with its powers). This indicates that high-order momentum may be a necessity for further understanding of the entropic and covariance-based uncertainty relations.

3. Conclusions

In this work we find an equivalence theorem to unify the superficially different classes of variance- and entropy-based uncertainty relations. For non-degenerate observables, the probability distributions are recovered from the covariance functions of the operators. Among the various applications of this theorem, optimal entropic uncertainty relations containing multiple observables are obtained from the variance-based uncertainty relations for the qubit system, where when the observables are more than two, the obtained entropic uncertainty relations are in equality form. Explicit functional relations between the variance and entropy are constructed for a higher spin system. While interesting in their own right, these results may also have direct applications in the study of quantum nonlocality, as the uncertainty relations are employed to determine the strength of quantum correlations [4, 5]. Another important impact of the equivalence theorem is on the structure of the uncertainty relation in the presence of quantum memory [37], which is crucial for the security of quantum key distribution. Finally, since the theorem generally applies to an arbitrary dimensional discrete system, it constitutes the basis for further studies of different uncertainty measures and relations.

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