Mirror Options

Julián Manzano*
Departament d’Estructura i Constituents
de la Matèria and IFAE,
Universitat de Barcelona,
Diagonal, 647, E-08028 Barcelona

Abstract

In this work we present a new family of options (mirror options) specially crafted to satisfy the necessities of aggressive speculators. The main ideas behind mirror options are: 1) A product that can be adjusted by the holder to agree with his/her market view at any time during its life. 2) The holder’s right to make an arbitrary number of those adjustments without penalizing costs. After defining mirror options as ‘super-versions’ of standard options we derive general formulae for their value in the case where the payoff is a monotonic function of the underlying (which is the case in calls, puts, futures, spreads etc.). We briefly discuss also their valuation for general payoffs and the American case. Finally we analyze the situation where the number of allowed adjustments is restricted and we point out directions for further developments.

*manzano@ecm.ub.es
1 Introduction

Since the first options on stocks appeared in an organized exchange (the CBOE) in 1973 there has been a continuous growth in worldwide option markets. This growth is not only apparent in the increasing volumes of options traded in organized and OTC markets but also in the never ending production of new products providing alternatives to satisfy the necessities of investors, banks and other financial institutions [1, 2]. These necessities may include risk transference, speculative leveraging, portfolio diversification, etc.

The purpose of this article is to present a new family of options specially designed to satisfy the necessities of speculators. Nowadays we have at our disposal a wide spectrum of products that can virtually serve at whatever market view that a speculator may have. The main inconvenient with these ‘standard’ products is that a change in the speculator’s market view may require a change in his/her position forcing unwanted transactions costs. From that perspective, it would be very convenient for aggressive intra-day speculators to have a product that could be adjusted to their changing expectations as many times as desired and without costs associated to these adjustments. This is exactly the spirit of mirror options.

1.1 What is a Mirror option?

Basically a mirror option is a European or American option where the holder has a well-defined payoff function (for example $(S - K)_+$ if we are talking about a call) and the right to make an undetermined number of changes in the real path of the underlying during the life of the option. We will call these changes ‘mirroring’ and we will say that the holder has the right (not the obligation) to ‘mirror’ the underlying an undetermined number of times. In the next section we will state exactly what we mean by ‘mirroring’ but now it will be enough to think that the holder has the right to change the real path of the underlying by a kind of reflected path in the sense that, after the mirroring, downwards movements in the real path become upwards movements in the reflected path and vice-versa. The dates for these mirrorings are not fixed and can be chosen freely by the holder. When the option is exercised, the payoff is calculated not with the real underlying but with the virtual underlying resultant from the mirroring process performed by the holder up to this time.

The freedom to perform mirrorings whenever during the contract allows the holder to adapt the option to his/her particular view of the market and, what is more important, to do it without suffering the associated costs that this policy would imply trading traditional products like futures, calls, puts, etc. Taking this into account we think that mirror options offer a new universe of possibilities that is not available using nowadays standard and OTC products.
The organization of the article is as follows: in section 2 we define the concept of mirroring and immediately after that we state what we understand by a mirror option in mathematical terms. In section 3 we proceed to valuate mirror options within the simplest market hypothesis in order to obtain a general valuation formulae. There we obtain pricing expressions for European mirror options in the case where the payoff is a monotonic function. We also present as examples the cases of the mirror call, mirror put and mirror forward. In section 4 we show explicitly a very important feature of the hedging strategy for mirror options and moved by the conclusions of this section we generalize, in section 5, our valuation formulae to the case where there is a fixed number of allowed mirrorings. Finally in section 6 we discuss some interesting characteristics of our results and point out possible alternatives of further work.

2 Defining mirror options

In order to give a definition of mirror options we have to clearly state the meaning of mirroring and mirror path. By mirroring an underlying \(S\) at a certain time \(t_m\) we understand constructing a new path \(S^*_t\) (mirror path) defined simply by:

\[
S^*_t = \begin{cases} 
S_t & t \leq t_m, \\
S_{t_m}^2 / S_t & t \geq t_m,
\end{cases}
\]  

(1)

We can see how a mirror path looks like in the example of Fig. (1). In Fig. (2) we have plotted the same example in logarithmic scale where we can clearly see the reason for the name ‘mirror’ path. In Eq. (1) we have given the definition of a mirror path when a single mirroring time is allowed. If we allow for a second mirroring at a time \(t'_m\) with \(t'_m \geq t_m\) we can define a new path by just applying Eq. (1) over the former single mirrored path. Note that in the case \(t'_m = t_m\) we recover the original path without mirrorings. In general if we allow for several consecutive mirrorings at times \(t_1 \leq t_2 \leq \cdots \leq t_M\) it is easy to see that

\[
S^*_t = \begin{cases} 
S_{t_1}^2 S_{t_2}^{-2} \cdots S_{t_{2n+1}}^{-2} S_t & t_0 \leq t_1 \leq \cdots \leq t_{2n} \leq t, \\
S_{t_1}^2 S_{t_2}^{-2} \cdots S_{t_{2n+1}}^{-2} S_{t_0} & t_0 \leq t_1 \leq \cdots \leq t_{2n+1} \leq t,
\end{cases}
\]  

(2)

where \(t_0\) is the starting time of the option contract. Note that after an even number of mirrorings \(S^*_t\) equals the value of the underlying \(S_t\) affected by a leveraging factor depending on the historical values taken by the underlying at the mirroring times selected by the holder. After an odd number of mirrorings we have also a path dependent factor but in this case multiplying the inverse value of the underlying. In Fig. (3) we can see the effects of two consecutive mirrorings over the sample path of Fig. (1).

---

\[1\] we have an involution in mathematical language.
Figure 1: A sample path $S$ (darker line) and its mirror path $S^*$ (lighter line after mirroring) plotted against time. The volatility $\sigma$ and drift $\mu$ for the sample path are taken equal to one. Applying Itô lemma we can easily see that the mirror path after the mirroring has volatility $\sigma^*$ and drift $\mu^*$ with $\sigma^* = \sigma$ and $\mu^* = \sigma^2 - \mu$.

Figure 2: Logarithmic plot of Fig.(1) where $X_t := \log \left(\frac{S_t}{S_0}\right)$ and $X^*_t := \log \left(\frac{S^*_t}{S^*_0}\right)$. Here it is easy to see that after the mirroring $X^*$ becomes the specular reflection of $X$. 
Figure 3: The sample path $S$ of Fig. (1) along with the path $S^*$ obtained after two mirrorings. We plot also the mirror path we would have obtained if the second mirroring were not made, that is, the mirror path of Fig. (1). Note that $S^*$ after the second mirroring is just $S$ affected by a multiplicative factor (bigger than one in this example).

Once it is clear what we mean by mirrorings we can define a mirror option as following:

**Definition 1** Given a European or American option with payoff $f$ we define its mirror counterpart as the one with payoff $f^*$ given by

$$f^*(S) \equiv f(S^*),$$

The holder of the mirror option has the right (not the obligation) at each time between $t_0$ (start of the contract) and $T$ (maturity of the contract) to perform mirrorings. The value $S_t^*$ of the mirror underlying at time $t$ is given by Eq. (3) where $t_1 \leq t_2 \leq \cdots \leq t_M$ is the whole set of times where the holder has chosen to perform mirrorings. Besides the right to perform mirrorings the holder has all the rights associated to the standard option counterpart (e.g. early exercise in the American case).

Please note that we understand the holder is not allowed to mirror the past history. Only future and therefore non-predictable history is affected by the holder’s mirrorings. Note also that the definition is sufficiently general to accommodate mirror versions of plain-vanilla and exotic options including the path-dependent case. However, we do not intend here to give a static definition. We want instead to make clear which is the spirit of the definition and
therefore allow the reader to adapt it to other possibilities. Just to give an example we can consider mirror exchanges or rainbows [3, 4] where we have a payoff that depends on several underlyings and where the holder is allowed to perform independent mirrorings on each of them.

From the definition it is now clear that depending on the holder ability to foresee the tendency of the underlying the leveraging factor resulting from judiciously chosen mirrorings can generate a much more convenient payoff than the one obtained with the standard option counterpart. For example a holder of a mirror call will try to perform mirrorings in order to maximize the rise of $S^*$ and vice-versa for a mirror put. In the case of options with more general payoffs the holder ability must be directed towards the maximization of the payoff whatever it could be. In some sense mirror options are reminiscent of chooser options [10] with the difference that here the holder ‘chooses’ all the time during the contract.

3 Valuating Mirror Options

For the valuation of mirror options we will assume the simplest market model we can start with. Namely a one factor model consisting of a single underlying $S$, paying the continuous dividend rate $\delta$, together with a riskless bonus $P$. The pair of assets satisfy

$$\begin{align*}
\frac{dS_t}{S_t} &= \mu dt + \sigma dW_t, \\
\frac{dP_t}{P_t} &= r dt,
\end{align*}$$

(3)

where $dW_t$ is a Wiener process and where the drift $\mu$, volatility $\sigma$ and riskless interest rate $r$ are taken constant. We also assume that there are no transactions costs. Therefore we will work with a complete and efficient model without transaction costs. This model can be complicated considerably and can be adapted to the particular underlying or sets of underlyings of the corresponding option. Evidently correlations enter into the game when several underlyings are involved. Moreover, the constancy of the drift and volatility is no longer acceptable when the option is a compounded one and therefore the underlying is another option [11]. Even for a single equity underlying one can consider multifactor models with stochastic volatility and term structure [24, 23, 22, 21], inefficient market models [14, 13, 12, 11, 10, 9], jump-diffusion models [25, 24, 23, 22], models with transaction costs [36, 35, 34] etc. However, in this article we want to present a general valuation formulae obtained with the simplest hypothesis leaving refinements and the treatment of particular cases to further developments.

Using Eq.(2) after $M$ mirrorings we can apply Itô lemma [3, 6] to Eq.(3) to obtain

$$\frac{dS^*_t}{S^*_t} = \mu^* dt + \sigma^* dW_t,$$

where $\mu^*$ and $\sigma^*$ are the drift and volatility of the mirror process $S^*$.
with
\[
\begin{align*}
\sigma^* &= \sigma, \\
\mu^* &= \begin{cases} 
\mu & M \text{ even}, \\
\sigma^2 - \mu & M \text{ odd}.
\end{cases}
\end{align*}
\]

From this fact one can be tempted to make the following incorrect argument:

The volatility of \(S^*\) is unaffected by the mirroring operations made by the holder and by definition the payoff of the mirror option is given by the same function that corresponds to the standard counterpart but computed using \(S^*\) instead of \(S\). Moreover, since in the valuation using a replicating self-financing portfolio the drift of the underlying drops out from the calculations we can naively conclude that the value of the mirror option is the same as the value of its standard counterpart.

As the reader has surely noted the reason why the above argument is wrong is that \(S^*\) is not the value of any tradable. Despite of that, we will see that the value of the mirror option is closely related to the value of its standard counterpart, at least for a family of payoffs where the valuation can be done in a closed form (monotonic payoffs as we will see). Before starting the valuation we need some preliminary results. For \(z \in \mathbb{R}\) and \(\Sigma \in \mathbb{R}^+\) we define
\[
\nu(z, \Sigma) \equiv \frac{1}{\sqrt{2\pi\Sigma}} \int_{-\infty}^{+\infty} f(e^x) e^{-\frac{1}{2\Sigma}(x-z+\frac{1}{2}\Sigma)^2} dx,
\]
\[
\nu(z, 0) \equiv \lim_{\Sigma \to 0} \nu(z, \Sigma) = f(e^z),
\]
where \(f\) is the payoff function. Using the Chapman-Kolmogorov relation satisfied by gaussians we immediately obtain
\[
\nu(z, \Sigma_1 + \Sigma_2) = \frac{1}{\sqrt{2\pi\Sigma_2}} \int_{-\infty}^{+\infty} \nu(x, \Sigma_1) e^{-\frac{1}{2\Sigma_2}(x-z+\frac{1}{2}\Sigma_2)^2} dx.
\]

For \(\phi, \beta \in \mathbb{R}\) and \(\Sigma_1, \Sigma_2 \in \mathbb{R}^+\) we also define
\[
\nu_{\phi, \beta}(z, \Sigma_1, \Sigma_2) \equiv \frac{1}{\sqrt{2\pi\Sigma_2}} \int_{-\infty}^{+\infty} \nu(\phi x + \beta, \Sigma_1) e^{-\frac{1}{2\Sigma_2}(x-z+\frac{1}{2}\Sigma_2)^2} dx,
\]

then performing a change of variables and using Eq.(4) we obtain the relation
\[
\nu_{\phi, \beta}(z, \Sigma_1, \Sigma_2) = \nu \left( \phi z + \beta + \frac{\phi^2 - \phi}{2} \Sigma_2, \Sigma_1 + \phi^2 \Sigma_2 \right).
\]

Now we state the following simple result

**Lemma 1** The payoff \(f\) is a monotonic increasing (decreasing) function if and only if \(\forall \varepsilon \geq 0\) \(\nu(\cdot, \varepsilon)\) is a monotonic increasing (decreasing) function.
Proof. Suppose that $\forall \varepsilon \geq 0 \nu (\cdot, \varepsilon)$ is a monotonic increasing (decreasing) function, then in particular we have $\nu (\cdot, 0) = (f \circ \exp) (\cdot)$ and since the exponential is a monotonic increasing function we immediately obtain that the payoff is a monotonic increasing (decreasing) function. Conversely, if the payoff $f$ is a monotonic increasing (decreasing) function then $f \circ \exp$ is also a monotonic increasing (decreasing) function and finally since $\nu (\cdot, \varepsilon)$ is the convolution of a positively defined function (a gaussian in this case) with $f \circ \exp$ we immediately obtain that $\nu (\cdot, \varepsilon)$ is a monotonic increasing (decreasing) function $\forall \varepsilon \geq 0$. ■

To perform the valuation we take time as discrete points and at the end we will take the continuous limit. We define the time to maturity $\tau$ as $\tau = n \varepsilon$ with $n = 0, 1, 2, \cdots, \infty$ and $\varepsilon$ arbitrary and constant. Since $\varepsilon$ is arbitrary taking the limits $\varepsilon \to 0$, $n \to \infty$ with $\tau = n \varepsilon$ fixed we obtain the value of the option for arbitrary time $t = T - \tau$.

Let us suppose that at time $t = T - n \varepsilon$ the holder has performed a certain number of mirrorings generating a virtual underlying value $S^*_{T-n\varepsilon}$ given by Eq.(2). That value can be very different from the real underlying value $S_{T-n\varepsilon}$ (note also that the value $S^*_{T-n\varepsilon}$ is not affected by a mirroring at time $T - n \varepsilon$). To begin with let us introduce some notation. We define

$$x_t \equiv \log (S_t),$$

$$x^*_t \equiv \log (S^*_t),$$

hence using Eq.(2) we obtain the relation

$$x^*_{t+\varepsilon} = \phi_t x_t + \beta_t,$$

(7)

where $\phi_t$ takes values $\pm 1$ and is equal to $(-1)^{M(t)}$ with $M(t)$ the number of mirroring decisions made by the holder up to time $t$ inclusive. From Eq.(2) we can see that $\beta_t$ also depends on the history of mirrorings up to time $t$ inclusive. Now we need to prove the following result

Lemma 2

$$x^*_t = \phi_t x_t + \beta_t,$$

Proof. From Eq.(2), when passing from time $t$ to time $t + \varepsilon$, we have the following four possibilities:

$$S^*_t = \gamma S_t$$  and no mirroring at time $t$  then  $S^*_{t+\varepsilon} = \gamma S_{t+\varepsilon}$  \hspace{1cm} $\phi_t = 1$,  

(8)

$$S^*_t = \eta S_t^{-1}$$  and mirroring at time $t$  then  $S^*_{t+\varepsilon} = \eta S_t^{-2} S_{t+\varepsilon}$  \hspace{1cm} $\phi_t = -1$,  

(9)

and

$$S^*_t = \eta S_t^{-1}$$  and no mirroring at time $t$  then  $S^*_{t+\varepsilon} = \eta S_t^{-1} S_{t+\varepsilon}$  \hspace{1cm} $\phi_t = 1$,  

(8)

$$S^*_t = \gamma S_t$$  and mirroring at time $t$  then  $S^*_{t+\varepsilon} = \gamma S_t S_{t+\varepsilon}^{-1}$  \hspace{1cm} $\phi_t = -1$.  

(9)
where $\gamma$ and $\eta$ are factors that depend on the values taken by the underlying at mirroring times previous to $t$ (see Eq. (4)). Hence taking logarithms in Eqs. (8) and (9) we obtain

$$x^*_t - x^*_t = \phi_t (x_{t+\varepsilon} - x_t),$$

and using Eq. (7) we immediately obtain

$$x^*_t = x^*_t + \varepsilon + \phi_t (x_t - x^*_t) = \phi_t x_t + \beta_t.$$

With this result and Eq. (6) we are ready to start. We will denote the value of the mirror option at time to maturity $\tau$ by $V^* (\tau)$. It is obvious that $V^* (\tau)$ also has a certain dependence on the history of the underlying up to time $T - \tau$. For example at maturity time we have

$$V^* (0) = f (S_T^* - \tau, 0) = v (x^*_T, 0) = v (\phi_T x_T + \beta_T, 0).$$

(10)

where $\phi_T$ and $\beta_T$ are clearly history dependent. In general to avoid raveling more the notation we will assume such dependence implicitly. However, it is important to realize that because of Eq. (10) mirror options enter into the category of Asian options with the peculiarity that the path dependence is dictated by the holder with his/her mirroring decisions.

Hereafter we will analyze the particular case where the payoff function $f$ is monotonic. Denoting by $V^*_+ (V^*)$ the value of the mirror option whenever the payoff is monotonic increasing (decreasing) we define

$$V^*_\pm (\tau) \equiv e^{-r\tau} q^*_\pm (\tau),$$

(11)

Now will demonstrate by induction the main result of this work, that is:

**Theorem 1** Defining $\alpha \equiv r - \delta$, for the European case we have

$$q^*_\pm (\tau) = \nu \left( x^*_{T-\tau} + \left( \frac{\sigma^2}{2} \pm \left| \alpha - \frac{\sigma^2}{2} \right| \right) \tau, \sigma^2 \tau \right).$$

(12)

**Proof.** Using Eqs. (10) and (11) the result is trivially true for $\tau = 0$. Supposing it valid for $\tau = n\varepsilon$ we will demonstrate it for $\tau = (n+1)\varepsilon$ and therefore for arbitrary $n$ by induction.

According to the standard procedure [2, 7, 8, 9] for our market model (3), the value of a European claim at $\tau = (n+1)\varepsilon$ is given by the $r-$ discounted mean value of the claim at $\tau = n\varepsilon$ using the risk neutral measure ($\mu \rightarrow \alpha$). Let us for a moment remember here the case of an American option. There the value of the option was given by the maximum between
that discounted value and the value obtained by early exercise. Here, like in the American case, we have to choose between two values, namely

\[
q^*_{\pm}((n + 1) \varepsilon) = \max_{\text{mirror}((n + 1)\varepsilon)} \left\{ \frac{1}{\sigma \sqrt{2\pi \varepsilon}} \int_{-\infty}^{+\infty} dxT_{-n\varepsilon} q^*_{\pm} (n\varepsilon) \right\}.
\]

\[
\times \exp \left( -\frac{1}{2\sigma^2 \varepsilon} \left( xT_{-n\varepsilon} - xT_{-(n+1)\varepsilon} - \alpha \varepsilon + \frac{1}{2} \sigma^2 \varepsilon \right)^2 \right),
\]

where \( \max_{\text{mirror}(\tau)} \{ \cdot \} \) means the maximum taken between the values of the expression between braces evaluated with and without mirroring at time to maturity \( \tau \). Note that the risk neutral measure used in Eq.\((12)\) is just the one arising from no-arbitrage arguments based on the standard \( \Delta \)-hedging strategy for our model \((3)\).

Using the inductive hypothesis given by Eq.\((12)\) along with Eq.\((5)\) we have

\[
q^*_{\pm}((n + 1) \varepsilon) = \max_{\text{mirror}((n + 1)\varepsilon)} \left\{ \int_{-\infty}^{+\infty} dxT_{-n\varepsilon} \exp \left( -\frac{1}{2\sigma^2 \varepsilon} \left( xT_{-n\varepsilon} - xT_{-(n+1)\varepsilon} - \alpha \varepsilon + \frac{1}{2} \sigma^2 \varepsilon \right)^2 \right) \right\},
\]

\[
\times \exp \left( -\frac{1}{2\sigma^2 \varepsilon} \left( xT_{-(n+1)\varepsilon} - xT_{-(n+1)\varepsilon} - \alpha \varepsilon + \frac{1}{2} \sigma^2 \varepsilon \right)^2 \right),
\]

now using Eqs.\((3)\) and \((4)\) we can write

\[
q^*_{\pm}((n + 1) \varepsilon) = \max_{\text{mirror}((n + 1)\varepsilon)} \left\{ \nu \left( \phiT_{-(n+1)\varepsilon} xT_{-(n+1)\varepsilon} + \beta T_{-(n+1)\varepsilon} \right) \right\}
\]

\[
\times \left( \alpha \phiT_{-(n+1)\varepsilon} + \left( \frac{\sigma^2}{2} \pm \left| \alpha - \frac{\sigma^2}{2} \right| \right) n + \frac{1 - \phiT_{-(n+1)\varepsilon}}{\sigma^2} \right) \varepsilon, \sigma^2 (n + 1) \varepsilon \right\},
\]

and from Lemma \((3)\) we have

\[
q^*_{\pm}((n + 1) \varepsilon) = \max_{\text{mirror}((n + 1)\varepsilon)} \left\{ \nu \left( xT_{-(n+1)\varepsilon} + \left( \alpha \phiT_{-(n+1)\varepsilon} + \left( \frac{\sigma^2}{2} \pm \left| \alpha - \frac{\sigma^2}{2} \right| \right) n \right) \right\}
\]

\[
\times \left( \frac{1}{\sigma \sqrt{2\pi \varepsilon}} \nu \left( xT_{-(n+1)\varepsilon} + \left( \alpha \phiT_{-(n+1)\varepsilon} + \left( \frac{\sigma^2}{2} \pm \left| \alpha - \frac{\sigma^2}{2} \right| \right) n \right) \right) \right\},
\]

noting that \( x^*_t \) is invariant under mirrorings at time \( t \) we have

\[
q^*_{\pm}((n + 1) \varepsilon) = \max \left\{ \nu \left( xT_{-(n+1)\varepsilon} + \left( \frac{\sigma^2}{2} \right) n + 1 \varepsilon \right) \varepsilon, \sigma^2 (n + 1) \varepsilon \right\},
\]

\[
\times \left( \frac{1}{\sigma \sqrt{2\pi \varepsilon}} \nu \left( xT_{-(n+1)\varepsilon} + \left( \frac{\sigma^2}{2} \right) n + 1 \varepsilon \right) \varepsilon, \sigma^2 (n + 1) \varepsilon \right) \right\},
\]

Finally, using the payoff monotonicity hypothesis and Lemma \((3)\) we obtain

\[
q^*_{\pm}((n + 1) \varepsilon) = \nu \left( xT_{-(n+1)\varepsilon} + \left( \frac{\sigma^2}{2} \pm \left| \alpha - \frac{\sigma^2}{2} \right| \right) (n + 1) \varepsilon, \sigma^2 (n + 1) \varepsilon \right) .
\]
Now we can trivially take the continuous limit $\varepsilon \to 0$, $n \to \infty$ with $\tau = n\varepsilon$ fixed. In this limit and using Eq. (11) we obtain the value of the mirror option for general monotonic payoff functions as

$$V^*_\pm(\tau) = e^{-r\tau}\nu \left( x^*_{T-\tau} + \left( \frac{\sigma^2}{2} \pm \left| r - \delta - \frac{\sigma^2}{2} \right| \right) \tau, \sigma^2 \tau \right),$$

(14)

This is a remarkable result, remembering that the value $V(\tau)$ for the standard European option is given by

$$V(\tau) = e^{-r\tau}\nu \left( x^*_{T-\tau} + (r - \delta) \tau, \sigma^2 \tau \right),$$

(15)

we immediately note that at the beginning of the contract ($x^*_{t_0} = x_{t_0}$) if we have a monotonic increasing payoff (like in a call for example) we have that the mirror option has the same (for $r - \delta \geq \sigma^2/2$) or bigger (for $r - \delta \leq \sigma^2/2$) value than its standard counterpart. Analogously, if we have a monotonic decreasing payoff (like in a put for example) we have that the mirror option has the same (for $r - \delta \leq \sigma^2/2$) or bigger (for $r - \delta \geq \sigma^2/2$) value than its standard counterpart. Thus we have arrived to the conclusion that, at the beginning of the contract, the deviation of the value of the mirror option from that of its standard counterpart depends critically on the interplay between dividends, interest rates and volatility. Note also that here dividends appear in a non trivial way, that is the rule to incorporate continuous dividends given by $S_t \to S_t \exp (-\delta (T - t))$ is not valid for mirror options. Another point that should be clear is that even though we have derived our expressions for the monotonic payoff case, we can easily carry on the demonstration of Theorem 1 in the limit case where $\sigma = \sqrt{2(r - \delta)}$ for arbitrary payoff functions. In this ‘zero-measure’ case we obtain that Eq. (14) remains valid and therefore at the beginning of the contract we have

$$V^*(T - t_0) = V(T - t_0), \quad (\sigma = \sigma_c),$$

where we define the critical volatility $\sigma_c$ as

$$\sigma_c = \sqrt{2(r - \delta)}.$$

With this definition we can write Eq. (14) in the more suggestive way

$$V^*_\pm(\tau) = e^{-r\tau}\nu \left( x^*_{T-\tau} + (r - \delta) \tau + \left( \sigma^2 - \sigma_c^2 \right) \pm \tau, \sigma^2 \tau \right),$$

where

$$(x)_\pm \equiv \begin{cases} x & \text{if } \text{sign}(x) = \pm 1, \\ 0 & \text{otherwise}. \end{cases}$$

Another point to remark is that now we can clearly see the Asian characteristics of mirror options. From the payoff definition of mirror options we already knew that they were Asian options, but now it is clear also that we have a dependence on $S^*_t$ for any time $t$ during the
contract. Moreover, from Eq.(4), we can interpret \( S_\ast^t \) as a kind of weighted geometric average in a set of ‘mirrored’ values up to time \( t \). This set is, by definition, dictated by the holder with his/her mirrorings decisions at each time up to \( t \).

To illustrate these results we will consider some examples, but before that let us clarify some potentially confusing language. It is standard in financial language to talk about a long option position with payoff \( f \) and about a short option position with payoff \(-f\). Here we will continue with this nomenclature but bearing in mind that unlike what happens with standard contracts a short mirror option position is not equivalent to a sold mirror option position. In the last case we are taking about the position of the issuer of the long mirror option that is not equivalent to any position of a holder because it is only the later who takes the mirroring decisions. Because of this fact holding both a short and a long mirror option position has in general some non-vanishing value. For the monotonic payoff case this value is

\[
(V_{\text{long}}^\ast + V_{\text{short}}^\ast)(T - t) = e^{-r(T-t)} \left\{ \nu \left( x_t^{(\text{long})\ast} + \left( \frac{\sigma^2}{2} \pm \frac{r - \delta}{2} \right) \frac{\tau, \sigma^2 \tau} \right) - \nu \left( x_t^{(\text{short})\ast} + \left( \frac{\sigma^2}{2} \mp \frac{r - \delta}{2} \right) \frac{\tau, \sigma^2 \tau} \right) \right\}, \tag{16}
\]

where the upper (lower) sign corresponds to the case where the payoff \( f \) for the long position is monotonic increasing (decreasing). Note that in this case the holder has the right to perform a certain set of mirrorings for his/her long position and a different set of mirrorings for his/her short position and therefore as time goes by he/she will have in general a mirror path \( x_t^{(\text{long})\ast} \) for the long position and a another mirror path \( x_t^{(\text{short})\ast} \) for the short position. At the beginning of the contract \( (t = t_0) \) we have \( x_{t_0}^{(\text{long})\ast} = x_{t_0}^{(\text{short})\ast} = x_{t_0} \) and therefore using Lemma \[3\] we can clearly see that the value given by Eq.(16) at this time is always positive definite. Moreover the unique possibility for this value to be zero is the limit case where \( \sigma = \sigma_c \) which is the case where all mirror options take the same value as their standard counterparts.

### 3.1 The European Mirror Call

A European long (short) call has the payoff \( f(S) = \pm (S - K)_+ \) where the upper (lower) sign corresponds to the long (short) position. Modeled with Eq.(3) the long (short) call has the value \( V^{(\pm)}(\tau) \) given by the Black-Scholes formula \[2\]

\[
V^{(\pm)}(\tau) = \pm e^{-r\tau} \left\{ F_{\tau} \Phi \left( \frac{\ln \left( \frac{F_{\tau}}{K} \right) + \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) - K \Phi \left( \frac{\ln \left( \frac{F_{\tau}}{K} \right) - \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) \right\},
\]

with

\[
F_{\tau} \equiv S_{T-\tau} e^{(r - \delta)\tau}.
\]
Hence from Eq.(14) and since in this case the payoff is a monotonic increasing (decreasing) function of $S$ we immediately obtain the value of the corresponding European mirror call as

$$V^{(\pm)}(\tau) = \pm e^{-r\tau} \left\{ F^{(\pm)}_\tau \Phi \left( \frac{\ln \left( \frac{F^{(\pm)}_{\tau}}{K} \right) + \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) - K \Phi \left( \frac{\ln \left( \frac{F^{(\pm)}_{\tau}}{K} \right) - \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) \right\},$$

with

$$F^{(\pm)}_\tau \equiv S^{(\pm)}_T e \left( \frac{\sigma^2}{2} \mid r - \delta - \frac{\sigma^2}{2} \mid \right)^{\tau}.$$

Note that necessarily one of the two mirror positions (short or long) has the same value as its standard counterpart at the beginning of the contract.

### 3.2 The European Mirror Put

A European long (short) put has the payoff $f(S) = \pm (K - S)_+$ where the upper (lower) sign corresponds to the long (short) position. Modeled with Eq.(3) the long (short) put has the value $V^{(\pm)}(\tau)$ given by the Black-Scholes formula \[2\]

$$V^{(\pm)}(\tau) = \pm e^{-r\tau} \left\{ K \Phi \left( - \frac{\ln \left( \frac{F^{(\pm)}_{\tau}}{K} \right) - \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) - F^{(\pm)}_\tau \Phi \left( - \frac{\ln \left( \frac{F^{(\pm)}_{\tau}}{K} \right) + \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) \right\},$$

with

$$F^{(\pm)}_\tau \equiv S^{(\pm)}_T e^{(r - \delta)\tau}.$$

Hence from Eq.(14) and since in this case the payoff is a monotonic decreasing (increasing) function of $S$ we immediately obtain the value of the corresponding European mirror put as

$$V^{(\pm)*}(\tau) = \pm e^{-r\tau} \left\{ F^{(\pm)*}_\tau \Phi \left( \frac{\ln \left( \frac{F^{(\pm)*}_{\tau}}{K} \right) + \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) - K \Phi \left( \frac{\ln \left( \frac{F^{(\pm)*}_{\tau}}{K} \right) - \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) \right\},$$

with

$$F^{(\pm)*}_\tau \equiv S^{(\pm)*}_T e \left( \frac{\sigma^2}{2} \mid r - \delta - \frac{\sigma^2}{2} \mid \right)^{\tau}.$$

Note again that necessarily one of the two mirror positions (short or long) has the same value as its standard counterpart at the beginning of the contract.

### 3.3 The Mirror Forward

A long (short) forward contract has the payoff $f(S) = \pm (S - K)$ where the upper (lower) sign corresponds to the long (short) position. Using simple no-arbitrage arguments \[2\] we obtain its value $V^{(\pm)}(\tau)$ given by

$$V^{(\pm)}(\tau) = \pm e^{-r\tau} \left( F_{\tau} - K \right),$$
with
\[ F_\tau \equiv S_{T-\tau}e^{(r-\delta)\tau}. \]

Hence from Eq.(14) and since in this case the payoff is a monotonic increasing (decreasing) function of \( S \) we immediately obtain the value of the corresponding mirror forward contract as
\[ V^{(\pm)*}(\tau) = \pm e^{-r\tau} \left( F^{(\pm)*}(\tau) - K \right), \]
with
\[ F^{(\pm)*}(\tau) \equiv S_{T-\tau}^\pm e^{\left( \frac{\sigma^2}{2} + \left| r - \delta - \frac{\sigma^2}{2} \right| \right) \tau}. \] (17)

Like what happens with the standard forward contract here we can also choose the strike \( K \) (the forward price) in order to make the price of the forward contract zero. Note also that from Eq.(17) the forward prices for the long and short positions are different. However, as we have seen, one of them is necessarily equal to the standard value.

We can use this simple result to exemplify the positive definite value arising from holding both a long and a short position at the beginning of the contract. According to Eq.(16) we have
\[ \left( V^{(+)*} + V^{(-)*} \right)(\tau) = e^{-r\tau} \left( S_{T-\tau}^{(long)*} e^{\left( \frac{\sigma^2}{2} + \left| r - \delta - \frac{\sigma^2}{2} \right| \right) \tau} - S_{T-\tau}^{(short)*} e^{\left( \frac{\sigma^2}{2} - \left| r - \delta - \frac{\sigma^2}{2} \right| \right) \tau} \right), \]
where \( S_{T-\tau}^{(long)*} \) (\( S_{T-\tau}^{(short)*} \)) is the value of the mirror underlying for the long (short) position. At time \( t_0 \) (the beginning of the contract) we have \( S_{t_0}^{(long)*} = S_{t_0}^{(short)*} = S_{t_0} \) so for this time we have
\[ \left( V^{(+)*} + V^{(-)*} \right)(T - t_0) = 2e^{\left( \frac{\sigma^2}{2} - r \right)(T - t_0)} \sinh \left( \left| r - \delta - \frac{\sigma^2}{2} \right| (T - t_0) \right) S_{t_0}, \] (18)
which is clearly positive definite as we have already shown in general. From the holder’s perspective the payoff at maturity will be
\[ S_{T}^{(long)*} - S_{T}^{(short)*}, \]
with the break-even given by
\[ \frac{S_{T}^{(long)*} - S_{T}^{(short)*}}{S_{t_0}} = 2e^{\left( \frac{\sigma^2}{2} - r \right)(T - t_0)} \sinh \left( \left| r - \delta - \frac{\sigma^2}{2} \right| (T - t_0) \right). \] (19)

4 Hedging Mirror Options

In this section our aim is to show a very important feature appearing in the hedging of mirror options that is highly relevant when transaction costs and illiquidity of the market are taken
into account. This in turn will provide us with another way to envisage mirror options as a kind of ‘swaps’ of transaction costs between the holder and the issuer.

To calculate the value of mirror options in section 3 we have used the neutral risk probability arising in a ∆-hedging strategy. Let us now calculate explicitly such ∆ from result (14). We define

\[ \Delta \equiv \frac{\partial V}{\partial S}, \]
\[ \Delta^* \equiv \frac{\partial V^*_\pm}{\partial S}, \]

hence using Eqs.(14) and (15) we obtain

\[ \Delta = e^{-r\tau} \frac{1}{S} \frac{\partial \nu}{\partial z}(w, \sigma^2 \tau), \]
\[ \Delta^* \equiv e^{-r\tau} \frac{1}{S} \frac{\partial \nu}{\partial z}(w^* \pm, \sigma^2 \tau) \times \frac{S^*}{S} \frac{\partial S^*}{\partial S}, \]

with

\[ w = x_{T-\tau} + (r - \delta) \tau, \]
\[ w^* \pm = x^*_{T-\tau} + \left( \frac{\sigma^2}{2} \pm \left| r - \delta - \frac{\sigma^2}{2} \right| \right) \tau, \]

and

\[ \frac{S}{S^*} \frac{\partial S^*}{\partial S} = (-1)^{M(t)}. \]

The most important point to note in Eq.(21) is not the difference between \( w \) and \( w^*_\pm \), that was already discussed in the previous sections, but the sign flip given by Eq.(21). In other words we have obtained that, from the hedger perspective, a mirroring made by the holder implies a change in the hedging position in the underlying from a given \( \Delta^* \) to the opposite one. This result is in fact very intuitive; since mirror options were created as instruments allowing the holder costless changes in his/her position it is natural to have the cost of these changes transferred to the issuer (zero in our model). This is, again, very reasonable because issuers are in general large investment banks or financial institutions that are exposed to much lower transaction costs than individual speculators and therefore can absorb such changes in position much more easily. Moreover, such financial institutions do not hedge single instruments individually because the impact of transactions costs is highly non-linear. As it is well known, it is the complete portfolio of options and other products what is hedged as a whole. Hedging each product individually is not realistic because such policy do not profit from cancellations between different positions and reductions of transaction costs due to economies of scale. Therefore the real cost of hedging mirror options is minimized with respect to the total premium received when large quantities of mirror options are issued in the
marker hedging them as a whole along with the rest of products in the portfolio. However, in situations of large illiquidity (like those appearing in crashes or rallies) collective behavior of mirror option holders can lead to large changes in delta that can not be easily canceled out. Because of that it is not unrealistic to conceive that in a real market of mirror options a risk premium depending in some way on the number of allowed mirrorings can appear. In order to admit such correction we need to know the value of mirror options when only a finite number of mirrorings is allowed. This is the subject of the next section.

5 Mirror options with a finite number of allowed mirrorings

In this section we will obtain the value of a mirror option for the case where the holder is allowed to perform \( M \) mirrorings at most. Again we will be able to obtain closed expressions for the monotonic payoff case or, at most, for arbitrary payoffs in the critical point \( \sigma = \sigma_c \).

With the same notational conventions we used in section 3 we define the value of the mirror option with at most \( M \) allowed mirrorings as

\[
V^*_\pm (\tau, M) \equiv e^{-r\tau} q^*_\pm (\tau, M).
\]

In this case we have

**Theorem 2** Defining \( \alpha \equiv r-\delta \), for the European case, where the holder is allowed to perform \( M \) mirrorings at most, we have

\[
q^*_\pm (\tau, M) = \begin{cases} 
\nu \left( x^*_T - \tau + \left( \frac{\sigma^2}{2} \pm \frac{\alpha - \frac{\sigma^2}{2\tau}}{2^M} \right) \tau, \sigma^2 \tau \right) & \text{if } M (T - \tau - \varepsilon) < M, \\
\nu \left( x^*_T - \tau + \left( \frac{\sigma^2}{2} \pm (-1)^M \left( \alpha - \frac{\sigma^2}{2\tau} \right) \right) \tau, \sigma^2 \tau \right) & \text{if } M (T - \tau - \varepsilon) = M,
\end{cases}
\]

where with \( M (T - \tau - \varepsilon) < M \) we mean that at time \( T - \tau \) the holder has at least one mirroring to perform and with \( M (T - \tau - \varepsilon) = M \) we mean that at time \( T - \tau \) the holder has exhausted all his/her allowed mirrorings.

**Proof.** Using Eqs.(10) and (11) the result is trivially true for \( \tau = 0 \). Again, supposing it valid for \( \tau = n\varepsilon \) we will demonstrate it for \( \tau = (n+1)\varepsilon \) and therefore for arbitrary \( n \) by induction. Using the same arguments of theorem 1 we obtain

\[
q^*_\pm ((n+1)\varepsilon, M) = \begin{cases} 
a^*_\pm ((n+1)\varepsilon, M) & \text{if } M (T - (n+2)\varepsilon) < M, \\
b^*_\pm ((n+1)\varepsilon, M) & \text{if } M (T - (n+2)\varepsilon) = M,
\end{cases}
\]

where

\[
a^*_\pm ((n+1)\varepsilon, M) = \max_{\text{mirror}((n+1)\varepsilon)} \left\{ \frac{1}{\sigma \sqrt{2\pi\varepsilon}} \int_{-\infty}^{+\infty} dx_{T-n\varepsilon} q^*_\pm (n\varepsilon, M) \frac{1}{2\sigma^2\varepsilon} \left( x_{T-n\varepsilon} - x_{T-(n+1)\varepsilon} - \frac{\alpha}{2\sigma^2\varepsilon} \right)^2 \right\},
\]
and

\[ b_+^* ((n + 1) \varepsilon, M) = \frac{1}{\sigma \sqrt{2 \pi \varepsilon}} \int_{-\infty}^{+\infty} dx_{T-n\varepsilon} b_+^* (n\varepsilon, M) \times \exp \left( -\frac{1}{2\sigma^2 \varepsilon} \left( x_{T-n\varepsilon} - x_{T-(n+1)\varepsilon} - \alpha \varepsilon + \frac{1}{2} \sigma^2 \varepsilon \right)^2 \right). \]

Using the inductive hypothesis given by Eq. (22) along with Eqs. (5), (6), (7) and Lemma 2 we obtain

\[ b_+^* ((n + 1) \varepsilon, M) = \frac{1}{\sigma \sqrt{2 \pi \varepsilon}} \int_{-\infty}^{+\infty} dx_{T-n\varepsilon} \exp \left( -\frac{1}{2\sigma^2 \varepsilon} \left( x_{T-n\varepsilon} - x_{T-(n+1)\varepsilon} - \alpha \varepsilon + \frac{1}{2} \sigma^2 \varepsilon \right)^2 \right) \times \nu \left( \phi_{T-(n+1)\varepsilon} x_{T-n\varepsilon} + \beta_{T-(n+1)\varepsilon} \left( \frac{\sigma^2}{2} + (-1)^M \left( \alpha - \frac{\sigma^2}{2} \right) n\varepsilon, \sigma^2 n\varepsilon \right) \right) = \nu \left( x_{T-(n+1)\varepsilon}^* + \left( \frac{\sigma^2}{2} + (-1)^M \left( \alpha - \frac{\sigma^2}{2} \right) \right) (n + 1) \varepsilon, \sigma^2 (n + 1) \varepsilon \right). \]

where it is clear that when considering \( b_+^* (\tau, M) \) we have \( \phi_{T-\tau} = (-1)^M \). Note that so far we have not used the payoff monotonicity hypothesis. For \( a_+^* \) using the inductive hypothesis given by Eq. (22) along with Eq. (7) we have

\[ a_+^* ((n + 1) \varepsilon, M) = \max_{\text{mirror}(n+1)\varepsilon} \left\{ \nu \left( x_{T-(n+1)\varepsilon}^* + \beta_{T-(n+1)\varepsilon} \left( \frac{\sigma^2}{2} + \gamma_+ \right) n\varepsilon, \sigma^2 n\varepsilon \right) \times \frac{1}{\sigma \sqrt{2 \pi \varepsilon}} \exp \left( -\frac{1}{2\sigma^2 \varepsilon} \left( x_{T-n\varepsilon} - x_{T-(n+1)\varepsilon} - \alpha \varepsilon + \frac{1}{2} \sigma^2 \varepsilon \right)^2 \right) dx_{T-n\varepsilon} \right\}, \]

where

\[ \gamma_+ = \begin{cases} \pm \left| \alpha - \frac{\sigma^2}{2} \right| & \text{if } M (T - (n + 1) \varepsilon) < M, \\ (-1)^M \left( \alpha - \frac{\sigma^2}{2} \right) & \text{if } M (T - (n + 1) \varepsilon) = M, \end{cases} \]

hence using Eqs. (5), (6) and Lemma 3 we obtain

\[ a_+^* ((n + 1) \varepsilon, M) = \max_{\text{mirror}(n+1)\varepsilon} \left\{ \nu \left( x_{T-(n+1)\varepsilon}^* + \frac{\sigma^2}{2} (n + 1) \varepsilon \right. \\
+ \left. \left( \phi_{T-(n+1)\varepsilon} \left( \frac{\sigma^2}{2} \right) + \gamma_\pm n \right) \varepsilon, \sigma^2 (n + 1) \varepsilon \right) \right\}. \]

Now we have to analyze two cases. If \( M (T - (n + 2) \varepsilon) = M - 1 \), remembering that \( \phi_{t} = (-1)^{M(t)} \) and using Lemma 5 we obtain

\[ a_+^* ((n + 1) \varepsilon, M) = \max \left\{ \nu \left( x_{T-(n+1)\varepsilon}^* + \frac{\sigma^2}{2} (n + 1) \varepsilon + (-1)^M \left( \alpha - \frac{\sigma^2}{2} \right) (n + 1) \varepsilon, \sigma^2 (n + 1) \varepsilon \right) \right\}, \]

\[ \nu \left( x_{T-(n+1)\varepsilon}^* + \frac{\sigma^2}{2} (n + 1) \varepsilon - (-1)^M \left( \alpha - \frac{\sigma^2}{2} \right) \varepsilon \pm \left| \alpha - \frac{\sigma^2}{2} \right| n\varepsilon, \sigma^2 (n + 1) \varepsilon \right) \right\} = \nu \left( x_{T-(n+1)\varepsilon}^* + \left( \frac{\sigma^2}{2} \pm \left| \alpha - \frac{\sigma^2}{2} \right| \right) (n + 1) \varepsilon, \sigma^2 (n + 1) \varepsilon \right), \]
and finally if $M \left( T - (n + 2) \varepsilon \right) < M - 1$ we obtain

\[
a^*_\pm \left( (n + 1) \varepsilon, M \right) = \max \left\{ \nu \left( x^*_{T-(n+1)} + \frac{\sigma^2}{2} (n+1) \varepsilon + \left( \alpha - \frac{\sigma^2}{2} \right) \varepsilon \pm \left( \alpha - \frac{\sigma^2}{2} \right) n \varepsilon, \sigma^2 (n+1) \varepsilon \right), \nu \left( x^*_{T-(n+1)} + \frac{\sigma^2}{2} (n+1) \varepsilon - \left( \alpha - \frac{\sigma^2}{2} \right) \varepsilon \pm \left( \alpha - \frac{\sigma^2}{2} \right) n \varepsilon, \sigma^2 (n+1) \varepsilon \right) \right\}
\]

\[
= \nu \left( x^*_{T-(n+1)} + \frac{\sigma^2}{2} (n+1) \varepsilon \pm \left( \alpha - \frac{\sigma^2}{2} \right) \varepsilon \right) (n+1) \varepsilon, \sigma^2 (n+1) \varepsilon \right).
\]

Note that as expected we have

\[
q^*_\pm (\tau, \infty) = q^*_\pm (\tau).
\]

Moreover, from Eq.(22) we can see (at least for the monotonic payoff case) that the value of the mirror option before the holder has exhausted his/her last mirroring do not depend on the allowed number of mirrorings $M$. Immediately after the last allowed mirroring is executed the value of the mirror option always falls to a lower or equal value. The discontinuity arises whenever

\[
\pm \left| \alpha - \frac{\sigma^2}{2} \right| \neq (-1)^M \left( \alpha - \frac{\sigma^2}{2} \right),
\]

where the upper (lower) sign corresponds to the monotonic increasing (decreasing) payoff case. For example we will have a discontinuous fall immediately after the last mirroring in a long mirror call if $M$ is even (odd) and $\sigma > \sigma_c$ ($\sigma < \sigma_c$). Note also that the expression given by Eq.(22) in the trivial case $M = 0$ reduces to the standard result (15).

So far the reader may find that our results are counterintuitive since at the beginning of the contract two mirror options differing only in the number of allowed mirrorings have exactly the same value (if at least 1 mirroring is allowed in both). However in our simplified model our results are correct and must be taken as a base for further developments incorporating the corresponding corrections (see the conclusions for a brief discussion).

6 Conclusions and Outlook

In this work we have created a new family of options specially crafted to satisfy the necessities of those speculators who are convinced of their abilities to call the market and do not want to assume the transaction costs of daily trading. Moreover, we were able to obtain general formulae for the value of these options in the case where the payoff is a monotonic function (which is the case in calls, puts, futures, spreads, asset or nothing options [12], etc.). Moreover, for the general payoff case we have identified a critical volatility $\sigma_c = \sqrt{2 (r - \delta)}$ for which the value of all mirror options, at the beginning of the contract, equals that of their standard counterparts.
We conclude our work with two sets of general remarks. The first set regards general aspects of the pricing of mirror options within the hypothesis of our model. The second set considers the impact other models may have on our results.

From simple financial arguments we know that holding a long call and a long put with the same strike is equivalent to holding a long straddle. Note however that holding a long mirror call and a long mirror put with the same strike is not equivalent to holding a long mirror straddle. This is because in the last case the holder has the right to perform a set of mirrorings for the call and another different set of mirrorings for the put. This is evidently not equivalent to the situation of holding a single mirror straddle. This fact is rather general in the sense that care must be taken when considering the valuation of synthetic options made of mirror options.

Since one long mirror call plus one long mirror put is not one long mirror straddle, which is the price of a long mirror straddle then? Or more generally: which is the price of a mirror option with non-monotonic payoff? The answer can be obtained following the steps of valuation of sections 3 and 5 with the difference that since the straddle has not monotonic payoff we cannot use this property and because of that only a numerical method (the binomial tree for example) can be applied in principle. In general, the difficulty of valuating mirror options with not monotonic payoffs is equivalent to the difficulty of valuating American options where free boundary problems arise. Here the free boundary is given, for each time $t$, by the value $S^*_t$ for which mirroring is equivalent to no mirroring; in a complete analogous way to the free boundary arising in the American option case given by the equivalence of early exercise or not.

In this work we have valued European mirror options, but what about American mirror options? Again, the valuation can be obtained using the steps of sections 3 and 5 but now at each time stage we have to take the maximum over the possibility of mirroring or not and the possibility of early exercise or not. This, also, will in general force us to use numerical methods like for example a binomial tree approach.

An important fact according to our results is that no matter what the market conditions are, all mirror options with monotonic increasing payoffs or all mirror options with monotonic decreasing payoffs will have the same initial value as their standard counterparts. This is a really surprising result since in this case probably nobody would buy the equal value standard options! Moreover when we restrict the holder to make a maximum number of mirrorings $M$, we have obtained that at the beginning of the contract the value of any mirror option with monotonic payoff and $M \geq 1$ does not depend on $M$. Again this seems to be counterintuitive since we expect holders to prefer mirror options with more allowed mirrorings than with
lesser ones. Even though these results are correct in our simplified market model we expect corrections to appear in more involved models.

In particular inefficient market models\[14, 15, 16, 17, 18, 19, 20\], where we have correlations between assets returns at different times, may lead to significant deviations from our results depending on the degree of inefficiency in the model. This can be expected on the intuitive basis that partial knowledge of future returns can be used by the holder to improve his performance with adequate mirrorings.

If inefficiencies can bring about corrections to our formulae the incompleteness of the market \[21, 22, 23, 24\] is not less important (at least equally important as it is in the valuation of standard options). Incompleteness in the market can be generated by several reasons, for example it can appear in mixed jump-diffusion models \[25, 26, 27, 28\], stochastic volatility and term structure models \[29, 30, 31, 32, 33, 34, 35\], etc. In any case both from the point of view of future implementations of mirror options as real OTC products or from the theoretical point of view it is also interesting to consider the modifications these models can produce in our valuation formulae.

Finally we cannot overlook deviations appearing in our results in the presence of transaction costs \[36, 37, 38\]. In particular we have to remember that even though we have derived our formulae in the absence of such costs it is just in the real world \textit{with transaction costs} where mirror options are attractive! Therefore we have to consider that absence just as a simplification to the valuation procedure, in the same way as with standard options.

Here it is worth to note the following: when analyzing the hedging of mirror options in section \[4\] we have realized that each mirroring made by the holder implies a change of sign in the $\Delta$ of the hedger. This is in fact intuitive since mirror options can be thought as a way to transfer transactions costs from the holder to the issuer. However, since the issuer deals in general with large portfolios where cancellations between different positions occur, we are in fact effectively transferring only a fraction of those costs. In general such fraction will be lower the bigger the portfolio of mirror and standard options the issuer has. However as we warned in section \[4\] in situations of large illiquidity (like those appearing in crashes or rallies) collective behavior of holders of mirror option can lead to large changes in delta that can not be easily canceled out. Because of that we have considered in section \[5\] the valuation of mirror options when only a finite number $M$ of mirrorings is allowed. Then we can take the value of a real mirror option to be the one of section \[5\] (or the numerical result if the payoff is not monotonic) plus a correction depending on $M$ and other factors incorporating the inefficiency, incompleteness and transactions costs of the market. However, as it well known, when considering transaction costs it is \textit{the hedger's whole portfolio of options} (mirror and standard options) what should be valued \[38\]. What it is not well known, to our knowledge,
is how to ascribe a fraction of such value to each component of the portfolio. This is necessary if we want to obtain a realistic correction to our results due to transaction costs.

Considering the above remarks we can fairly say that a lot of work can still be done in order to have practical implementations of mirror options. However, because of their attractive characteristics we believe that innovative financial institutions can make real OTC mirror options available in the near future.

7 Acknowledgments

J.M. would like to thank J. Masdemont, A. Gisbert, J. Masoliver and J. Perelló for their useful comments. The author also acknowledges a fellowship from Generalitat de Catalunya, grant 1998FI-00614.

References

[1] Lawrence and Larry McMillan, Options As a Strategic Investment, Prentice Hall (1992)
[2] Hull J.C., Options, Futures, & Other Derivatives, Prentice Hall (2000)
[3] Rubinstein M., One for Another, RISK 4, 30-32 (July-August 1991)
[4] Rubinstein M., Somewhere Over the Rainbow, RISK 4, 63-66 (November 1991)
[5] B. Øksendal, Stochastic Differential Equations, Springer-Verlag (1998)
[6] I. Karatzas, S. E. Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag (1988)
[7] Baxter M., Rennie A., Financial Calculus, Cambridge University Press (1998)
[8] Harrison J. M., Pliska S. R., Martingales and Stochastic Integrals in the Theory of Continuous Trading, Stochastic Processes and Their Applications, 11 (1981), 215-260
[9] Cox J., Ross S., and Rubinstein, 1979, Option Pricing: A Simplified Approach, Journal of Financial Economics 7, 229-263
[10] Rubinstein M., Options for the Undecided, RISK 4, 70-73 (April 1991)
[11] Rubinstein M., Double Trouble, in RISK 5, 73 (December 1991-January 1992)
[12] Rubinstein M., Unscrambling the Binary Code, RISK 4, 75-83 (October 1991)
[13] Wilmott P., *Option Pricing*, Oxford Financial Press (1997)

[14] Fama E. F., *Efficient capital markets: a review of theory and empirical work*, J. Finance 25, 383-417 (1970)

[15] Lo A. W., Wang J., *Implementing option pricing models when asset returns are predictable*, Journal of Finance 50, 87-129 (1995)

[16] Baviera R., Pasquini M., Serva M., Vergni D., Vulpiani A., *Efficiency in foreign exchange markets*, http://xxx.lanl.gov/abs/cond-mat/9901225 (1999)

[17] Zhang Y. -C., *Towards a Theory of Marginally Efficient Markets*, http://xxx.lanl.gov/abs/cond-mat/9901243 (1999)

[18] Cornalba L., Bouchaud J. P., Potters M., *Option Pricing and Hedging with temporal correlations*, http://xxx.lanl.gov/abs/cond-mat/0011213 (2001)

[19] Bouchaud J. P., Matacz A., Potters M., *The leverage effect in financial markets: retarded volatility and market panic*, http://xxx.lanl.gov/abs/cond-mat/0101120 (2001)

[20] Masoliver J., Perelló J., *Option pricing and perfect hedging on correlated stocks*, Elsevier Preprint and http://xxx.lanl.gov/abs/cond-mat/0012014 (2001)

[21] Hofmann N., Platen E., Schweizer M., *Option pricing under incompleteness and stochastic volatility*, Math. Finance 2(3), 153-187 (1992)

[22] El Karoui N., Quenez M., *Dynamical programming and pricing of contingent claims in an incomplete market*, SIAM J. Control and Optim. 33, 29-66 (1995)

[23] Britten-Jones M., Neuberger A., *Arbitrage pricing with incomplete markets*, Appl. Math. Finance 3, 347-363 (1996)

[24] Aurell E., Baviera R., Hammarlid O., Serva M., Vulpiani A., *A general methodology to price and hedge derivatives in incomplete markets*, International Journal of Theoretical and Applied Finance 3, 1-24 (2000)

[25] Merton R. C., *Option pricing when underlying stock returns are discontinuous*, J. of Financial Economics 3, 125-144 (1976)

[26] Scott L. O., *Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: applications of Fourier inversion methods*, Mathematical Finance 7, 413-426 (1997)
[27] Aase K. K., Contingent claims valuation when the security price is combination of an Ito process can a random point process, Stoch. Proc. Appl. 28, 185-220 (1998)

[28] Kou S. G., A Jump Diffusion Model for Option Pricing with Three Properties: Leptokurtic Feature, Volatility Smile, and Analytical Tractability, Preprint, Columbia University (1999)

[29] Hull J. C., White A., The pricing of options with stochastic volatilities, Journal of Finance 42, 281-300 (1987)

[30] Scott L. O., Option pricing when the variance changes randomly: Theory, estimation, and application, Journal of Financial and Quantitative Analysis 22, 419-438 (1987)

[31] Stein E. M., Stein J. C., Stock price distributions with stochastic volatility: An analytic approach, Review of Financial Studies 4, 727-752 (1991)

[32] Heston S., A closed-form solution for options with stochastic volatility to bond and currency options, Review of Financial Studies 6, 327-343 (1993)

[33] Ghysels E., Harvey A. C., Renault E., Stochastic volatility, Statistical Methods in Finance, North Holland (1996)

[34] Derman E., Kani I., Stochastic Implied Trees: Arbitrage pricing with stochastic term and strike structure of volatility, Goldman Sachs, (1997)

[35] Heston S., Nandi S., A closed form GARCH option valuation model, Review of Financial Studies 13, 585-625 (2000)

[36] Leland H. E., Option pricing and replication with transaction costs, J. Finance 40, 1283-1301 (1985)

[37] Hodges S. D., Neuberger A., Optimal replication of contingent claims under transaction costs, Rev. Future Markets 8, 222-239 (1993)

[38] Dewynne J. N., Whalley A. E., and Wilmott, Path-dependent options and transaction costs, Mathematical Models in Finance, Edited by S. D. Howison, F. P, Kelly and P. Wilmott, Chapman & Hall, London (1995)