RECENT RESULTS FROM LATTICE CALCULATIONS

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Recent results from lattice QCD calculations relevant to particle physics phenomenology are reviewed. They include the calculations of strong coupling constant, quark masses, kaon matrix elements, and $D$ and $B$ meson matrix elements. Special emphasis is on the recent progress in the simulations including dynamical quarks.

1 Introduction

Since it was invented by K. Wilson\footnote{1} in 1974, lattice QCD has grown into an important tool for the analysis of low energy regime of QCD, where the non-perturbative dynamics of quarks and gluons becomes essential. In this talk I review the status of the calculation of several quantities relevant to particle physics phenomenology, such as the determination of fundamental parameters of QCD, and the hadron matrix elements of $K$, $D$ and $B$ mesons.

Although lattice gauge theory starts with first principles in QCD calculations, due to the limitation of computational resources one has to resort to several approximations, such as finite lattice spacing $a$, finite spatial extent $L$ of the lattice, and relatively large quark mass $m_q$. They introduce sources of systematic uncertainties, which can in principle be reduced by performing extrapolations to the appropriate limits.

Among other approximations, the most serious one was the quenched approximation, in which gluon vacuum polarization of by quarks is neglected. It reduces the computational cost by orders of magnitude and so has been used for most lattice calculations, but the systematic uncertainty is not under any quantitative control. The most important development since the last ICHEP conference in 2002, where Lellouch summarized the lattice results\footnote{2}, is that the simulations beyond the quenched approximation—unquenched QCD—have been carried out for many important quantities.

In this review, I first discuss the theoretical issues in dynamical quark simulations, which are related to formulations and algorithms for fermions on the lattice. Then I cover several applications, including the determination of the strong coupling and quark masses, and calculations of weak matrix elements. Other reviews of recent results can be found in \cite{3,4}.

2 Issues in dynamical QCD

Since dynamical fermion simulations involve many inversions of fermion matrix, it is substantially harder to simulate light quarks at their physical mass values. Therefore, an extrapolation, called the chiral extrapolation, in the light quark mass from feasible quark masses to the physical up and down quark masses is necessary. The extrapolation is best controlled with the functional form predicted by the chiral perturbation theory (ChPT), which provides a systematic expansion of low energy QCD for small quark masses.

The question is then whether the region of quark masses in lattice simulations has enough overlap with the convergence radius of ChPT. For the pion decay constant, for instance, at NLO ChPT predicts a nonanalytic behavior $-m_{\pi}^2 \ln m_{\pi}^2$, called the chiral log. The result from the JLQCD collaboration \cite{5,6} clearly shows that there is no signal of curvature for the “pion” mass.
above 550 MeV, which is currently the lowest available pion mass with (improved) Wilson fermion formulations. More recently the MILC collaboration\(^7\) has performed a simulation with the pion mass as low as 250 MeV using the (improved) staggered fermions. They found that the chiral log shows up below 500 MeV, showing a clear advantage of the staggered fermion. In fact, using the simulations including 2+1 (up, down and strange) flavors of staggered quarks, it has been demonstrated that several fundamental physical quantities are in good agreement with the experiment\(^8\).

Wilson fermions explicitly violate chiral symmetry at finite lattice spacing, and the massless quark limit has to be reached by tuning a mass parameter. Because the massless point itself fluctuates statistically in the Monte Carlo simulation, the singularity of the massless limit could show up earlier in the chiral extrapolation, and the dynamical fermion simulation becomes much harder near the chiral limit than expected from naive scaling law. With the staggered fermion, on the other hand, there is an exact U(1) chiral symmetry and the massless limit is fixed.

The price one has to pay for the staggered fermion is the complication of species doubling. The formulation necessarily involves four species (or tastes) of fermions. They mix among themselves at finite lattice spacing. As a result, there are 16 pions on the lattice, only one of which becomes massless in the chiral limit. All other hadrons are also duplicated, and such effects could become a source of systematic uncertainty.

More seriously, one has to take a square-root or fourth-root of the fermion determinant in order to represent degenerate up and down quarks or slightly heavier strange quark using the four-taste staggered fermion. Any local Dirac operator corresponding to the fourth-rooted staggered fermion determinant has not been found so far\(^9\). Without the locality one cannot prove the universality, \textit{i.e.} the continuum limit is the QCD. Therefore, it is potentially a fundamental problem of the calculations employing the staggered fermions\(^6\).

Theoretically, the best solution is to employ the fermions satisfying the Ginsparg-Wilson relation, which have an exact chiral symmetry at finite lattice spacing without sacrificing the flavor symmetry\(^{13}\). Neuberger (overlap) fermions\(^{14,15}\) and domain-wall fermions\(^{16,17,18}\) fall into this class. Exploratory dynamical simulations using the domain-wall fermion have already been performed\(^{19}\).

In the following discussion I assume that the fourth-rooted staggered fermion is a correct, or at least effective, description of QCD and do not quote errors associated with it.

### 3 Fundamental QCD parameters

#### 3.1 Strong coupling constant

The strong coupling constant $\alpha_s(M_Z)$ can be determined using lattice QCD by converting $\alpha^\text{lat}(1/a)$ to the continuum definition $\alpha_s^\text{MS}(M_Z)$ by perturbation theory. The quarkonium spectrum is often used to set $1/a$, because it is insensitive to other systematic errors, such as those from finite volume and chiral extrapolation of light quarks. To improve the perturbative expansion the renormalized coupling\(^{20}\) $\alpha_V(q^*)$, which is defined through the heavy quark potential, is used with an appropriate scale $q^*$.

Figure 1 summarizes lattice results using unquenched simulations. After including the dynamical quark effects, the most important source of systematic error is the unknown higher order perturbation theory. Previous results used two-loop matching, but this year the HPQCD collaboration\(^{26}\) has

\(^{9}\)It was shown that the square-root of the staggered operator is non-local\(^{9,10}\).

\(^{6}\)There is also a positive indication, \textit{i.e.} the eigenvalue spectrum of the staggered operator shows an approximate four-fold degeneracy\(^{11,12}\).
carried out three-loop calculations and reported a (preliminary) result with a substantially reduced error, $\alpha_s(M_Z) = 0.1175(15)$, which is in good agreement with the PDG 2004 average 0.1182(20). They use the simulation data with 2+1 flavors of improved staggered fermions at three lattice spacings and confirm that various input quantities to determine $\alpha_V(q^*)$ give a consistent result within estimated four-loop errors. It should be noted that the results with the Wilson-type fermions is significantly lower. It is likely an unknown higher order effect, but is not fully understood.

3.2 Light quark masses

The light (up, down and strange) quark masses are determined from the pion and kaon masses, e.g., using the PCAC relation such as $m_K^2 = B(m + m_s)$ at the leading order of $m_s$. $\bar{m}$ denotes an average up and down quark mass and $m_s$ is the strange quark mass. Since the effect of chiral log is not significant for $m_s$, most of the calculations use a linear fit in an average light quark mass for interpolation. For the calculation of $\bar{m}$, on the other hand, the NLO effect of ChPT is important.

Because $m_q$ is regularization dependent, the lattice results are usually quoted in the MS scheme by using perturbative matching and sometimes also using non-perturbative techniques at intermediate steps. For the case of the Wilson-type fermions, the determination through axial Ward identity (AWI) and vector Ward identity (VWI) could be different at finite lattice spacing.

In Figure 2, I compile the lattice results for strange quark mass for both quenched and unquenched calculations. In the quenched approximation, systematic studies of the non-perturbative matching and the continuum extrapolation have been extensively studied and the results are in agreement within $\approx 10\%$ quenching error, which appears as a dependence on the input quantity to set the lattice scale, e.g. $m_{\rho}, f_K$, etc.

The CP-PACS$^{36}$ and JLQCD$^{6}$ collaboration...
rations found that $m_s(2\, \text{GeV})$ becomes significantly lower by the effects of two dynamical flavors.$^{4}$ Recently, 2+1-flavor calculations are reported by the CP-PACS/JLQCD$^{49}$ and HPQCD-MILC-UKQCD$^{50,7}$ collaborations. Their results are consistent with each other and slightly lower than the two-flavor data. My average is $m_s(2\, \text{GeV}) = 78 \pm 10\, \text{MeV}$.

Determination of light quark mass $\bar{m}$ or the ratio $m_s/\bar{m}$ is sensitive to the chiral extrapolation. At the leading order of $m_q$ the ratio is given by the physical meson masses as $m_K^2/m_{\pi}^2 - 1 = 25.9$, and a NLO ChPT analysis yields $24.4 \pm 1.5$. The lattice calculation can be used to improve this estimate. The small quark mass reached by the MILC simulation enabled them to include NLO ChPT terms in the fit$^{50,7}$ as well as the correction terms to describe the taste symmetry breaking$^{52}$ and higher order effects. Their result $27.4 \pm 4.2$ is consistent with the NLO ChPT analysis but slightly higher, suggesting non-negligible higher order effect.

3.3 Heavy quark masses

The charm quark is not too heavy to describe with the $O(a)$-improved Wilson fermion action adopting the naive estimate of discretization effect $O((am_c)^2)$. It can in principle be eliminated by taking the continuum limit, which is feasible in the quenched approximation and precise results $\bar{m}_c(\bar{m}_c) = 1.30(3)$ and $1.32(3)$ GeV are obtained. If one takes the non-relativistic dynamics of charm quark inside the $D_{(s)}$ meson into account$^{55,56}$, the discretization error is not as large as $O((am_c)^2)$. Recent work indicates that the discretization effect is much smaller$^{57}$, and an unquenched calculation is being performed.

For the bottom quark the conventional approach fails for the lattice scale $1/a \sim 2-3$ GeV. Instead, the heavy quark effective theory (HQET) is a good approximation up to corrections of order $O(A_q^2_{\text{QCD}}/m_b) \approx 30\, \text{MeV}$. Higher order perturbation theory is essential for the matching of $m_b$ in order to avoid large corrections due to power divergences. The two-loop calculation was done sometime ago$^{58}$ and the three-loop calculation has been performed recently$^{59}$, reducing the error to the 40 MeV level. Available two-flavor QCD calculations combined with the two-loop matching yield $\bar{m}_b(\bar{m}_b) = 4.21(7)\, \text{MeV}^{60,59}$ and $4.25(11)\, \text{MeV}^{61}$. For the latter, carefully estimated uncertainties in the lattice scale and strange quark mass dominate the error bar, which is expected to be reduced by 2+1-flavor calculations.

Recently, a non-perturbative method to match HQET onto QCD has been formulated and tested on quenched lattices$^{62}$. Another method to calculate $b$ quark mass without recourse to HQET has also been proposed$^{54}$. These methods may enable us to further reduce the systematic error.

4 Kaon physics

4.1 Determination of $|V_{us}|$

The best known method to determine $|V_{us}|$, or the Cabibbo angle, is to use the semi-leptonic $K_{13}$ decays. The relevant form factor $f_+(0)$ is normalized to one in the SU(3) limit ($\bar{m} = m_s$), and the correction starts at the second order in $m_x - \bar{m}$. Calculation of the correction in a quark model yielded $f_+(0) = 0.961(8)^{64}$. Further improvement requires non-perturbative method to calculate $f_+(0)$, and first quenched lattice calculation has been done recently$^{65}$ using double ratios as in the $|V_{cb}|$ calculation$^{66,67}$. They reported $0.960(5)(7)$.

$|V_{us}|$ can also be determined through the leptonic decay $K^\pm \rightarrow \mu^\pm \nu_\mu$, once the decay
constant $f_K$ is known theoretically. This has been attempted\textsuperscript{68} using the recent MILC result $f_K/k = 1.210(4)(13)^7$. It is notable that the error is now comparable to the semi-leptonic determination, i.e. $\sim 1\%$, and the result for $|V_{us}|$ is consistent.

\subsection{4.2 Kaon $B$ parameter}

Calculation of $B_K$ is one of the major goals of lattice QCD calculations. $B_K$ is a matrix element of a $\Delta S = 2$ four-quark operator of $VV + AA$ chiral structure sandwiched by $K$ and anti-$K$ states. In the lattice calculation its chiral structure must be maintained in order to avoid large contamination from wrong chirality operators. Therefore, most lattice calculations have been done using fermion formulations which respect chiral symmetry.

In the quenched approximation, the “benchmark” result was given by the JLQCD collaboration\textsuperscript{69} in 1997 using (unimproved) staggered fermion: $B_K^{N_f=0}(2 \text{ GeV}) = 0.63(4)$. As shown in Figure 3 they calculated $B_K$ at several values of lattice spacing down to $\sim 0.04 \text{ fm}$ and extrapolated to the continuum limit assuming theoretically expected discretization effects. More recent improved staggered fermion results\textsuperscript{70,71} are consistent with the JLQCD’s continuum limit already at large lattice spacings $a \sim 0.1\sim 0.2 \text{ fm}$.

In the past few years, many calculations have been performed using the Ginsparg-Wilson fermions (domain-wall and overlap)\textsuperscript{45,75,76,43,77}. The results are shown by triangles in Figure 3. They are slightly lower than the JLQCD’s continuum limit, and discretization error is substantially reduced. For some of these, the renormalization factor is computed non-perturbatively.

Another method to protect the lattice operator from operator mixing is provided by chirally twisted mass lattice QCD\textsuperscript{78}. The numerical result for $B_K^{74}$ is almost flat in $a$ and consistent with the Ginsparg-Wilson fermion results\textsuperscript{6}.

In view of these improved calculations, which have smaller discretization effect and non-perturbative renormalization, I recommend a slightly lower value for a quenched world average, $B_K^{N_f=0}(2 \text{ GeV}) = 0.58(4)$, which is shown in Figure 3 by horizontal lines.

Dynamical quark effects were previously estimated from unimproved staggered simulations\textsuperscript{80,81,82,83} as only slightly negative\textsuperscript{84} or positive\textsuperscript{85}. Due to the large discretization errors inherent in the unimproved staggered fermion, it was difficult to disentangle the sea quark effect from the discretization effect. This year, new unquenched calculations have appeared using the $O(a)$-improved Wilson fermion\textsuperscript{86}, improved staggered fermion\textsuperscript{71}, and domain-wall fermion\textsuperscript{87}.

Figure 4 shows sea quark mass (or mass of pion composed by sea quarks) dependence of $B_K$. The RBC calculation with dynamical domain-wall fermion\textsuperscript{87} is very precise and

\footnote{A related proposal has been made for Wilson fermions\textsuperscript{73} to avoid the wrong chirality operators. A numerical result\textsuperscript{73} suggests large $O(a)$ discretization effect for unimproved Wilson fermion.}
show a trend to decrease toward the chiral limit. The SU(3) breaking $m_s \neq m_d$ effect is estimated to be about $-3\%$. Since the result is still preliminary and the slope in the chiral extrapolation seems to rely on a single point at $m^2_{PS} \approx 0.25 \text{ GeV}^2$, I take this as an indication of quenching error rather than a central value of unquenched QCD, and recommend an average $B_K(2 \text{ GeV}) = 0.58(4)(^{+2}_{-3})$, where the second asymmetric error reflects the quenching error.

5 Heavy quarks

5.1 $D$ meson decays

The CLEO-c and BES III experiments promise to measure the $D_{(s)}$ decays at a few \% accuracy, which provides a stringent test of lattice simulation of heavy quarks. The most relevant hadronic quantities to be calculated on the lattice are the leptonic decay constants and semi-leptonic decay form factors.

As in the charm quark mass calculation the large discretization effect of $O((am_c)^2)$ must be eliminated for precise calculation of $D_{(s)}$ meson decay constants, unless one uses effective theory approaches. Continuum extrapolation has recently been performed in the quenched approximation, yielding $f_{D_s} = 252(9) \text{ MeV}^{88}$ and $240(5)(5) \text{ MeV}^{89}$. They are consistent with the previous world average 230(14) MeV$^{90}$ from the calculations using the Fermilab heavy quark formulation$^{55}$, which applies the idea of HQET for the Wilson-type lattice fermion.

The Fermilab-MILC-HPQCD collaboration has carried out calculations of $f_D$ and $f_{D_s}$ in the presence of 2+1-flavors of staggered sea quarks$^{91}$. Their result is $f_{D_s} = 263(^{+5}_{-3} )(24) \text{ MeV}$. The first error is statistical, and the second error reflects their estimate of systematic error due to matching of the heavy quark action and current to QCD (7\%), discretization effects of the light quark (4\%), charm quark mass determination (4\%), etc. To improve the heavy quark matching, one requires perturbative calculation for complicated lattice actions including many $O(1/m_Q)$ terms, and such work is in progress using automated perturbative calculation technique$^{92}$. Other errors can be reduced by increasing the computing power, and it would be feasible to reduce the total error to the 5\% level.

The semi-leptonic decay $D \to \pi \ell \nu$, $K \ell \nu$ form factors have also been calculated by the same group, and the results are $f^{D \to \pi}_+ = 0.64(3)(6)$ and $f^{D \to K}_{+} = 0.73(3)(7)$, which can be used to determine $|V_{cs}|$ and $|V_{cd}|$ with experimental inputs from BES$^{93}$ and CLEO$^{94}$. Such determination is currently consistent with the CKM unitarity and the error is dominated by lattice calculation. If we can reduce the error to the 5\% level or better, it will provide interesting test of the CKM unitarity.

5.2 $B$ meson mixings

The mass difference in the $B^0 - \bar{B}^0$ system $\Delta M_d$ is proportional to $f_B^{2} B |V_{td}|^2$, and thus gives a constraint on the CKM element $|V_{td}|$, provided that the corresponding hadronic matrix elements, $B$ meson decay constant $f_B$ and $B$ parameter $B_B$, are theoretically calculated. The analogous mass difference in the
The results from the JLQCD collaboration are consistent with each other. On the other hand, the disagreement between the most recent two calculations, JLQCD \((N_f = 2)\) \(^{109}\) and Wingate \textit{et al.} \((N_f = 2 + 1)\) \(^{110}\), remains even if one uses the common scale such as \(r_0\). At present, it is not clear if it comes from the dynamical strange quark effect or from other (underestimated) systematic errors. For an average of unquenched calculations I recommend \(f_{B_s} = 230 \pm 30\) MeV.

For the heavy-light meson decay constant, ChPT predicts a non-analytic quark mass dependence \(^{111}\) \(\propto (1 + 3g_s^2)m_s^2 \ln m_s^2\), whose effect could become important in the chiral extrapolation of \(f_B\). The coupling \(g\) describe the \(HH^*\pi\) interaction, such as \(BB^*\pi\) or \(DD^*\pi\). For the \(DD^*\pi\) interaction, there is an experimental measurement \(g = 0.59(7)\) \(^{112}\), and quenched lattice calculations \(^{113,114}\) are also consistent with this.

In Figure 6 the chiral extrapolation is shown for a ratio \(\left(f_B M_{B_s}^{1/2}\right)/\left(f_B M_B^{1/2}\right)\) as a function of light and strange quark mass ratio. Lattice data are from JLQCD \(^{109}\) and HPQCD \(^{116}\).

\(^{116}\)The plot is an update of that shown by Kronfeld at Lattice 2003\(^{115}\).
My average

|          | Lellouch², ICHEP 2002 | My average ICHEP 2004 |
|----------|-----------------------|-----------------------|
| $f_{B}$  | 203(27)(±10)          | 189(27)              |
| $f_{B^*}$| 238(31)               | 230(30)              |
| $f_{B^*}/f_B$ | 235(33)(±0.04) | 214(38)              |
| $f_{B^*}/f_B$ | 276(38)               | 262(35)              |
| $\xi$    | 1.22(4)(±0.4)         | 1.22(4)(±0.4)        |
| $\xi/\xi_B$ | 1.18(4)(±0.4)        | 1.23(6)              |

Table 1. My averages of $B$ mixing parameters compared to those by Lellouch at ICHEP 2002. $f_{B^*}/f_B$ and $f_{B^*}/f_B$ are given in units of MeV.

For the HPQCD collaboration¹¹⁶ with the staggered sea quarks become available at smaller quark masses, which are consistent with the expected chiral log behavior, although more statistics is needed to be conclusive. I fit the both data with a function including the chiral log and obtain $f_{B^*}/f_B = 1.22(±0.4)$.

The large uncertainty due to the chiral log can be avoided by constructing a double ratio¹¹⁷ ($f_{B^*}/f_B)/(f_{D^*}/f_D)$, in which the chiral log term cancels at the leading order of $1/M$. The difference is the treatment of the uncertainty due to the chiral extrapolation. Because an indication of the chiral log is already found I take the central value from the fit including the chiral log, while it was given in the asymmetric second error in ².

5.3 $B$ meson decays

The $B$ meson semi-leptonic decay form factors are needed in the determination of $|V_{ub}|$ and $|V_{cb}|$ from exclusive decay modes $B \rightarrow \pi\ell\nu$ and $B \rightarrow D^{(*)}\ell\nu$.

The first unquenched calculations of the $B \rightarrow \pi\ell\nu$ form factors have recently been presented by the Fermilab lattice¹¹² and the HPQCD¹¹² collaborations, which both worked on 2+1-flavor gauge field ensembles produced by the MILC collaboration. Figure 7 shows their results together with the previous quenched calculations¹¹²,¹¹³,¹¹⁴,¹¹⁵,¹¹⁶. We do not observe any significant difference between quenched and unquenched calcula-
tions within relatively large statistical errors.

Since lattice calculations are possible only in the high $q^2$ region where the recoil momentum of the daughter pion is small, the extraction of $|V_{ub}|$ using these lattice results requires experimental data in the same kinematical region. Such analysis was done by the CLEO collaboration\textsuperscript{127}. They provided the data of the decay rate in three $q^2$ bins, and the highest bin (above 16 GeV$^2$) corresponds to the region of the lattice calculation. The result using an average of the four quenched lattice calculations $|V_{ub}| = (2.88 \pm 0.55 \pm 0.30 \pm 0.45 \pm 0.18) \times 10^{-3}$ still has large errors (due to statistical, systematic, lattice, and $\rho \ell \nu$ form factor dependence, in the order given), but we expect better measurements from BaBar and Belle in near future. In fact, a preliminary result with higher purity (and thus with smaller experimental systematic error) was reported at this conference by the Belle collaboration\textsuperscript{128}.

The Fermilab lattice collaboration has updated\textsuperscript{121} the calculation of $B \to D \ell \nu$ form factor at zero recoil, which can be used in the precise determination of $|V_{cb}|$. Their new result with 2+1 flavors of dynamical quarks, $\mathcal{F}_{B \to D}(1) = 1.075(18)(15)$, is consistent with the previous quenched calculation\textsuperscript{66}. Similar calculation of the $B \to D^{(*)} \ell \nu$ form factor\textsuperscript{67} is in progress.

6 Conclusions

The most important progress in lattice QCD calculations in the past few years is the inclusion of sea quark effects. By performing the real simulations with dynamical quarks, the \textit{unknown} errors due to quenching in the previous calculations are being eliminated for many of the quantities discussed here.

In unquenched calculations, the non-analytic quark mass dependence appearing from the pion loops must be taken into account in the chiral extrapolation. Without enough data points below $m_{\rho S} \simeq 500$ MeV, the extrapolation induces large systematic uncertainty, since we do not know where the chiral log effect becomes important. In this respect the advantage of staggered fermion is clear: the simulation is the fastest and therefore one can reach the chiral regime. Employing the improved staggered fermions, the HPQCD-MILC-UKQCD-Fermilab group have recently presented several results with 2+1 flavors of dynamical quarks.

The drawback of the staggered fermion is its complicated taste structure, in particular the fourth-root trick introduced to represent single dynamical flavor. It is not proved that the fourth-root of the staggered fermion determinant is the same as a local quantum field theory. Such a proof is essential, as it would provide a theoretical basis that these calculations correspond to the real QCD. Until such a proof is available, the simulation with other fermion formulations without such taste structure should be pursued.

The constraints on the CKM unitarity triangle using the recent lattice results are shown in Figure 8. Although the error bars are not significantly reduced over the last several years, the uncertainty due to quenching is now basically eliminated, and the calculations are from the first principles. Much more work is needed to reduce other systematic errors, and the ideas are already being tested within the quenched approximation.

Figure 8. Constraints on the CKM unitarity triangle using the most recent lattice results for $B_K$, $f_B$, $B_{1/2}$, and $\xi$. Plot is from the UTfit collaboration\textsuperscript{129}. 
Acknowledgments

I would like to thank I. Allison, Y. Aoki, C. Bernard, N. Christ, C. Dawson, J. Flynn, E. Gamiz, A. Gray, R. Horsley, T. Iijima, T. Izubuchi, J. Laiho, C.J.D. Lin, Q. Mason, C. Maynard, F. Mescia, J. Noaki, M. Okamoto, C. Pena, M. Pierini, G. Schierholz, J. Shigemitsu, J. Simone, A. Soni, A. Stocchi, S. Tamhanker, M. Wingate, H. Wittig, and the members of the CP-PACS and JLQCD collaborations for correspondence and discussions. I apologize that some of their works could not be included in this review. I also thank Andreas Kronfeld for carefully reading the manuscript. This work is supported in part by the Grant-in-Aid of the Ministry of Education (No. 14540289).

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