On Nonzero Kronecker Coefficients and their Consequences for Spectra

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Abstract

A triple of spectra \((r^A, r^B, r^{AB})\) is said to be admissible if there is a density operator \(\rho^{AB}\) with

\[
\text{Spec } \rho^A, \text{Spec } \rho^B, \text{Spec } \rho^{AB} = (r^A, r^B, r^{AB}).
\]

How can we characterise such triples? It turns out that the admissible spectral triples correspond to Young diagrams \((\mu, \nu, \lambda)\) with nonzero Kronecker coefficient \(g_{\mu \nu \lambda}\) \cite{4, 13}. This means that the irreducible representation \(V_\lambda\) is contained in the tensor product of \(V_\mu\) and \(V_\nu\). Here, we show that such triples form a finitely generated semigroup, thereby resolving a conjecture of Klyachko \cite{13}. As a consequence we are able to obtain stronger results than in \cite{4} and give a complete information-theoretic proof of the correspondence between triples of spectra and representations. Finally, we show that spectral triples form a convex polytope.

1 Introduction

A curious connection between representation theory and the spectra of operators was discovered recently. Suppose we are given a bipartite density operator \(\rho^{AB}\), and suppose this has spectrum \(r_{AB} = \text{Spec } (\rho^{AB})\). Let \(r^A\) be the spectrum of the marginal operator \(\rho^A = \text{Tr}_B \rho^{AB}\), and \(r^B\) that of the other marginal operator \(\rho^B\). Then clearly there are restrictions on the possible spectral triples \((r^A, r^B, r^{AB})\) as \(\rho^{AB}\) ranges over all density operators. For instance, if \(\rho^{AB}\) is pure, so \(r^{AB} = (1, 0, ...), \) then \(r^A = r^B\). How does one characterise the set of possible spectral triples? One way to do this is via

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representation theory: there is a correspondence between triples of
spectra and irreducible representations of the symmetric group $V_\mu$, $V_\nu$ and
$V_\lambda$, where

$$V_\lambda \subset V_\mu \otimes V_\nu.$$ (1)

Two rather different methods were used to prove this. In [13] a body of
powerful techniques from invariant theory were harnessed. In [4],
the approach came from the direction of quantum information theory, and
a key ingredient was a theorem relating spectra and Young diagrams due to
Alicki, Rudnicki and Sadowski [16] and Keyl and Werner [11]. The latter
theorem can be given a short and elegant proof (see also [4]) that has
interesting parallels with classical information theory. To those with an in-
formation theory background, therefore, the approach taken in [4] has some
advantages of accessibility. It is shown there that for every density operator
$\rho^{AB}$ there is a sequence of triples $(\mu^j, \nu^j, \lambda^j)$ satisfying relation (1) that
converges to the spectra:

$$\lim_{j \to \infty} (\bar{\mu}^j, \bar{\nu}^j, \bar{\lambda}^j) = (r^A, r^B, r^{AB}),$$

where the bar denotes normalisation. Klyachko [13] proves this as well as a
converse that says that to every $(\mu, \nu, \lambda)$ with $V_\lambda \subset V_\mu \otimes V_\nu$ there is a density
operator with spectra $(\bar{\mu}, \bar{\nu}, \bar{\lambda})$.

One aim of this paper is to show that informational methods can be used
to prove Klyachko’s converse. On our way to this result we prove his con-
jecture [13, Conjecture 7.1.4] that triples $(\mu, \nu, \lambda)$ with $V_\lambda \subset V_\mu \otimes V_\nu$ form a
semigroup. We also prove that the semigroup is finitely generated. Together
with our previous results on the correspondence with spectral triples this will
imply that the set of admissible spectral triples is a convex polytope.

2 Background

Let us consider in more detail the relation between irreducible repre-
sentations and spectra. The irreducible representations of both unitary and
symmetric groups are labelled by Young diagrams. If $\lambda$ denotes a Young di-
agram, its row lengths are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ and its size is $|\lambda| := \sum_{i=1}^d \lambda_i$.
We denote the corresponding irreducible representations of $U(d)$ (or $GL(d)$)
with highest weight $\lambda$ by $U^d_\lambda$ and those of the symmetric group $S_k$ by $V_\lambda$.
Schur-Weyl duality states that $(\mathbb{C}^d)^\otimes k$ decomposes as a direct sum of irre-
ducible representations:

$$(\mathbb{C}^d)^\otimes k \cong \bigoplus_{\lambda \in \text{Par}(k,d)} U^d_\lambda \otimes V_\lambda,$$ (2)

where Par$(k,d)$ indicates the set of partitions of $k$ into $\leq d$ parts; i.e. the
Young diagrams with no more than $d$ rows and a total of $k$ boxes.
Consider a density operator $\rho$ on $\mathbb{C}^d$. We can take $k$ copies of it and measure the label $\lambda$ on $(\mathbb{C}^d)^\otimes k$. The estimation theorem [16, 11] states that, as $k$ increases, the spectrum $r$ of $\rho$ is increasingly well approximated by the normalised row lengths of $\lambda$, i.e. by the distribution $\overline{\lambda} = \lambda/|\lambda|$. Formally:

**Theorem 2.1 (Estimation Theorem)**

Let $P_\lambda$ be the projection onto $U^\lambda_\mu \otimes V_\lambda$. Then

$$\text{Tr} P_\lambda \rho^\otimes k \leq (k + 1)^{d(d-1)/2} \exp \left( -k D(\overline{\lambda}||r) \right)$$

where $D(p||q) = \sum_i p_i \log(p_i/q_i)$ is the Kullback-Leibler distance.

Let us now return to the case of bipartite states, and consider the Clebsch-Gordan series for the symmetric group:

$$V_\mu \otimes V_\nu \cong \bigoplus_{\lambda \in \text{Par}(k,k)} g_{\mu \nu \lambda} V_\lambda,$$

where the multiplicities $g_{\mu \nu \lambda}$ are known as the Kronecker coefficients (of the symmetric group). Since $V_\lambda \cong V^*_\lambda$, the Kronecker coefficients can also be defined in terms of the $S_k$-invariant subspace of $V_\mu \otimes V_\nu \otimes V_\lambda$, i.e.

$$g_{\mu \nu \lambda} = \dim(V_\mu \otimes V_\nu \otimes V_\lambda)^{S_k}. \quad (4)$$

There is also a way of viewing the Kronecker coefficients in terms of the irreducible representations of $\text{GL}(mn)$, for integers $m,n$ satisfying $m \geq |\mu|$ and $n \geq |\nu|$. It is arrived at by equating the Schur-Weyl decompositions of $(\mathbb{C}^m \otimes \mathbb{C}^n)^\otimes k$ and of $(\mathbb{C}^{mn})^\otimes k$ (see [4]) and reads

$$U^mn_\lambda \downarrow_{\text{GL}(m) \times \text{GL}(n)} \cong \bigoplus_{\mu \in \mathbb{Z}^m_{++}} \bigoplus_{\nu \in \mathbb{Z}^n_{++}} g_{\mu \nu \lambda} U^m_\mu \otimes U^n_\nu,$$

where $\mathbb{Z}^d_{++} := \{ \lambda \in \mathbb{Z}^d : \lambda_1 \geq \ldots \lambda_d \geq 0 \}$ is the set of dominant positive weights for $\text{GL}(d)$. This interpretation of the Kronecker coefficients can equivalently be stated in terms of invariants as

$$g_{\mu \nu \lambda} = \dim(U^m_\mu \otimes U^n_\nu \otimes U^*_\lambda)^{\text{GL}(m) \times \text{GL}(n)}, \quad (5)$$

where $\text{GL}(m) \times \text{GL}(n)$ acts on $U^m_\mu \otimes U^n_\nu$ and simultaneously on $U^*_\lambda$ according to the inclusion $\text{GL}(m) \times \text{GL}(n) \to \text{GL}(m) \otimes \text{GL}(n) \subset \text{GL}(mn)$.

In [4] theorem 2.1 was applied to give the following:

**Theorem 2.2**

For every density operator $\rho^{AB}$, there is a sequence $(\mu^{(j)},\nu^{(j)},\lambda^{(j)})$ of partitions, labeled by natural numbers $j$, such that

$$g_{\mu^{(j)},\nu^{(j)},\lambda^{(j)}} \neq 0 \quad \text{for all } j$$
and
\[
\lim_{j \to \infty} \bar{\mu}^{(j)} = \text{Spec} \rho^A \quad (6)
\]
\[
\lim_{j \to \infty} \bar{\nu}^{(j)} = \text{Spec} \rho^B \quad (7)
\]
\[
\lim_{j \to \infty} \bar{\lambda}^{(j)} = \text{Spec} \rho^{AB} \quad (8)
\]

Klyachko derived a very similar theorem:

**Theorem 2.3**

For a density operator \( \rho^{AB} \) with the rational spectral triple \((r^A, r^B, r^{AB})\) there is an integer \(m > 0\) such that \(g_{mr^A, mr^B, mr^{AB}} \neq 0\).

He also supplied the following converse:

**Theorem 2.4**

Let \(\mu, \nu\) and \(\lambda\) be diagrams with \(k\) boxes and at most \(m, n\) and \(mn\) rows, respectively. If \(g_{\mu\nu\lambda} \neq 0\), then there exists a density operator \(\rho^{AB}\) on \(\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^m \otimes \mathbb{C}^n\) with spectra

\[
\text{Spec } \rho^A = \bar{\mu} \quad (9)
\]
\[
\text{Spec } \rho^B = \bar{\nu} \quad (10)
\]
\[
\text{Spec } \rho^{AB} = \bar{\lambda} \quad (11)
\]

We now give a resumé of our new results.

### 3 Summary of the Results

Let Kron denote the set of triples \((\mu, \nu, \lambda)\) with nonzero Kronecker coefficients. Our first result is

**Theorem 3.1**

Kron is a semigroup with respect to row-wise addition, i.e. \(g_{\mu \nu \lambda} \neq 0\) and \(g_{\mu' \nu' \lambda'} \neq 0\) implies \(g_{\mu + \mu', \nu + \nu', \lambda + \lambda'} \neq 0\).

This was conjectured in Klyachko’s paper [13, conjecture 7.1.4]. It implies stability of the Kronecker coefficients: i.e. if \(g_{\mu \nu \lambda} \neq 0\) then \(g_{N\mu N\nu N\lambda} \neq 0\), for integers \(N > 0\). This was announced by Kirillov [12, theorem 2.11] but without proof. A simple corollary of stability is that non-vanishing Kronecker coefficients obey entropic relations (as explained in [4]). More importantly, it plays a key role in giving an information-theoretic proof of theorem 2.4. We also present a compact version of the proof of theorem 2.2 which was presented in [4]. In this way we obtain a simple proof for the full correspondence between Kronecker coefficients and admissible spectral triples.

As a third result we will show that
Theorem 3.2
The semigroup Kron is finitely generated.

From this, a straightforward argument shows that

Corollary 3.3
Theorem 2.2 and theorem 2.3 are equivalent.

Using the correspondences to spectral triples, the fact that Kron is a finitely generated semigroup can be given the following geometrical interpretation.

Theorem 3.4
SPEC, the set of admissible spectral triples, is a convex polytope.

4 The Nonzero Kronecker Coefficients form a
Finitely Generated Semigroup

In order to prove theorem 3.1, we introduce a representation of GL(n) known sometimes as the Schwinger representation, or the Cartan product ring:

\[ Q^n := \bigoplus_{\lambda \in \mathbb{Z}^+} U^n_{\lambda}. \] (12)

We assume here that \( \lambda_n \geq 0 \) because we are ultimately interested in combining irreducible representations of \( S_k \), which are only defined for nonnegative \( \lambda \). However, all of our results can be easily generalized for dominant weights \( \lambda \) without the restriction \( \lambda_n \geq 0 \).

To establish \( Q^n \) as a graded ring, we introduce the Cartan product that maps \( U_{\mu} \otimes U_{\nu} \) to \( U_{\mu + \nu} \) by projecting onto the unique \( U_{\mu + \nu} \)-isotypic subspace of \( U_{\mu} \otimes U_{\nu} \). We denote the Cartan product by \( \circ \), so that for \( |u_{\mu}\rangle \in U_{\mu}, |u_{\nu}\rangle \in U_{\nu}, |u_{\mu}\rangle \circ |u_{\nu}\rangle \) is defined to be the projection of \( |u_{\mu}\rangle \otimes |u_{\nu}\rangle \) onto the \( U_{\mu + \nu} \)-isotypic subspace of \( U_{\mu} \otimes U_{\nu} \). Clearly \( Q^n \) is graded under the action of \( \circ \), GL(n) preserves this grading and GL(n) acts properly on products, i.e. \( g(|v\rangle \circ |w\rangle) = (g|v\rangle) \circ (g|w\rangle) \).

The proof of theorem 3.1 now rests on the following lemma:

Lemma 4.1
(a) \( Q^n \) has no zero divisors. That is, if \( |v\rangle, |w\rangle \in Q^n \) are nonzero, then \( |v\rangle \circ |w\rangle \neq 0 \).

(b) \( Q^m \otimes Q^n \otimes (Q^{mn})^* \) has no zero divisors.

Here we have defined \( (Q^n)^* = \bigoplus_{\lambda} (U^n_{\lambda})^* \) with corresponding Cartan product \( (U_{\mu}^n)^* \circ (U_{\nu}^n)^* \rightarrow (U_{\mu + \nu}^n)^* \) and we have extended the Cartan product to tensor products in the natural way.
Proof  Although only statement (b) of the lemma is used in the proof of the
theorem, for ease of exposition we will prove part (a) and then sketch how
similar arguments can establish (b). Our proof is based on the Borel-Weil
theorem [2, p. 115], though the presentation here is mostly self-contained.

Let $|v_\lambda\rangle$ be a highest weight vector for $U_\lambda$. For any $|\alpha\rangle \in U_\lambda$, note that
\[ \langle v_\lambda | g | \alpha \rangle \]
(a) a polynomial in the matrix elements of $g$.
(b) identically zero only if $|\alpha\rangle = 0$ (due to the irreducibility of $U_\lambda$).

Now define the set $X_\alpha := \{ g \in \text{GL}(n) | \langle v_\lambda | g | \alpha \rangle = 0 \}$. The above two claims mean that $X_\alpha$ is a proper closed subset of $\text{GL}(n)$ in the Zariski topology whenever $|\alpha\rangle \neq 0$.

Similarly, if $|\beta\rangle \in U_\lambda'$ and $|v_\lambda\rangle$ is a highest weight vector for $U_{\lambda'}$ then $X_\beta := \{ g \in \text{GL}(n) | \langle v_{\lambda'} | g | \beta \rangle = 0 \}$ is a proper Zariski-closed subset of $\text{GL}(n)$ if and only if $|\beta\rangle \neq 0$.

The fact that $|v_\lambda\rangle$ and $|v_{\lambda'}\rangle$ are highest weight vectors means that $|v_\lambda\rangle \otimes |v_{\lambda'}\rangle = |v_{\lambda+\lambda'}\rangle \in U_{\lambda+\lambda'}$ and thus
\[
\langle v_\lambda | g | \alpha \rangle \langle v_{\lambda'} | g | \beta \rangle
= \langle (|v_\lambda\rangle \otimes |v_{\lambda'}\rangle) g(|\alpha\rangle \otimes |\beta\rangle) \rangle
= \langle v_{\lambda+\lambda'} | g(|\alpha\rangle \circ |\beta\rangle) \rangle.
\] (13)

We are free to replace $|\alpha\rangle \otimes |\beta\rangle$ with $|\alpha\rangle \circ |\beta\rangle$ in the last step because we are taking the inner product with a vector that lies entirely in $U_{\lambda+\lambda'}$. Now suppose $|\alpha\rangle \circ |\beta\rangle = 0$. Then for all $g$ at least one of the terms on the LHS of eq. (13) vanishes, and thus $\text{GL}(n) = X_\alpha \cup X_\beta$. Since $\text{GL}(n)$ is irreducible, it cannot be the union of two proper closed subsets, and we conclude $|\alpha\rangle$ and $|\beta\rangle$ cannot both be nonzero.

The proof of (b) is almost identical, but consider instead $|\alpha\rangle \in U_\mu \otimes U_\nu \otimes U_\lambda', |\beta\rangle \in U_\mu' \otimes U_{\nu'} \otimes U_{\lambda'}$, and the group $\text{GL}(m) \times \text{GL}(n) \times \text{GL}(mn)$ (which is still irreducible). \[ \square \]

Note that we could relax the restriction that $\lambda_n \geq 0$ by multiplying the inner products of the form $\langle v_\lambda | g | \alpha \rangle$ by a high enough power of $\det g$ (guaranteed to be nonzero for $g \in \text{GL}(n)$) to obtain a polynomial in the matrix elements of $g$.

Proof of theorem 3.1: Given any ring $R$ with an action of $G$ on it, let $R^G$ denote the ring of $G$-invariants in $R$. Now recall that if $|\mu| = m$ and $|\nu| = n$, then
\[
g_{\mu\nu\lambda} = \dim(U_\mu \otimes U_\nu \otimes U_\lambda)^{\text{GL}(m) \times \text{GL}(n)},
\] (14)
where $\text{GL}(m) \times \text{GL}(n)$ acts on $U_\lambda$ according to the inclusion $\text{GL}(m) \times \text{GL}(n) \to \text{GL}(m) \otimes \text{GL}(n) \subset \text{GL}(mn)$.
If \( g_{\mu \nu \lambda} \neq 0 \) and \( g_{\mu' \nu' \lambda'} \neq 0 \) then according to eq. (14) there exist non-zero vectors \( |u_{\mu \nu \lambda}\rangle \in (U_\mu \otimes U_\nu \otimes U_\lambda^*)^{\text{GL}(m) \times \text{GL}(n)} \) and \( |u_{\mu' \nu' \lambda'}\rangle \in (U_{\mu'} \otimes U_{\nu'} \otimes U_{\lambda'}^*)^{\text{GL}(m) \times \text{GL}(n)} \). Define \( |u_{\mu+\mu', \nu+\nu', \lambda+\lambda'}\rangle = |u_{\mu, \nu, \lambda}\rangle \otimes |u_{\mu', \nu', \lambda'}\rangle \). Then \( |u_{\mu+\mu', \nu+\nu', \lambda+\lambda'}\rangle \neq 0 \) by part (b) of lemma 1.4 and \( |u_{\mu+\mu', \nu+\nu', \lambda+\lambda'}\rangle \in U_{\mu+\mu'} \otimes U_{\nu+\nu'} \otimes U_{\lambda+\lambda'}^* \) is \( \text{GL}(m) \times \text{GL}(n) \)-invariant, since this property is preserved by the Cartan product. Thus \( (U_{\mu+\mu'} \otimes U_{\nu+\nu'} \otimes U_{\lambda+\lambda'}^*)^{\text{GL}(m) \times \text{GL}(n)} \neq 0 \) and we conclude that \( g_{\mu+\mu', \nu+\nu', \lambda+\lambda'} \neq 0 \).

Next, we prove that the semigroup Kron is finitely generated.

**Proof of theorem 3.2.** Suppose \( V \) is a regular representation of a group \( G \) and \( R = P(V) \) is the ring of polynomials on \( V \). \( G \) acts on \( R \) by \( gp(v) = p(g^{-1}v) \), where \( p \) is a polynomial function applied to \( v \in V \). One can therefore define \( R^G \), the set of \( G \)-invariant elements of \( R \). In 1890, Hilbert proved that \( R^G \) is finitely generated in the case where \( G = \text{SL}(n, \mathbb{C}) \). More generally, the theorem holds whenever \( G \) is reductive; this means that, given a subspace \( W \) of a representation \( V \) that is closed under the action of \( G \), another subspace \( W' \) can be found that is also closed under \( G \) and \( V = W \oplus W' \). For example, \( \text{GL}(d) \) is reductive. A short proof of this finite generation theorem can be found in [6, Theorem 4.1.1].

To prove that Kron is finitely generated, first note that any \( |\alpha\rangle \in U_\lambda \) can be interpreted as the polynomial \( g \to \langle v_\lambda | g | \alpha \rangle \), by the Borel-Weil theorem. The Cartan product takes polynomials to products of polynomials, as we have just seen (Eq. 1.3). This ring is generated by polynomials from the fundamental representations \( U^d_{\omega_i} \), where \( \omega_i = (1^i) \) is the diagram consisting of a single column of height \( i \). This follows because we can write the maximum weight vector \( |v_\lambda\rangle \) for any \( U_\lambda \) as the tensor product of maximum weight vectors \( |v_{\omega_i}\rangle \) of fundamental representations, just by breaking down \( \lambda \) into its columns of various heights. There are, however, relationships amongst these generators; these define an ideal \( I \) closed under \( G \).

Thus we can write \( Q^d = R/I \), where \( R = P(\bigoplus_i U^d_{\omega_i}) \) is the polynomial ring on the direct sum of the fundamental representations. If we take \( G = \text{GL}(d) \), since \( G \) is reductive and \( I \) is closed under \( G \), there is a subspace \( J \) of \( R \) also closed under \( G \), such that \( R = I \oplus J \). Thus any \( G \)-invariant of \( R/I \) defines a \( G \)-invariant of \( J \) and hence of \( R \). So \( (R/I)^G = R^G/I^G \), and \( (R/I)^G \) is finitely generated if \( R \) is. We now apply this argument to \( R = Q^m \times Q^n \times (Q^{mn})^* \) with the reductive group \( G = \text{GL}(m) \times \text{GL}(n) \) acting on it, and conclude that \( R^G \) is finitely generated.

An invariant of \( R^G \) is an element of \( (U_\mu \otimes U_\nu \otimes (U_\lambda)^*)^{\text{GL}(m) \times \text{GL}(n)} \) for some triple \( \langle \mu, \nu, \lambda \rangle \), and this defines an element of Kron. Since the Cartan product in \( R \) corresponds to the sum in Kron, the finite set of generators for \( R^G \) define a finite set of generators for Kron. \( \square \)
5 The Correspondence of Nonzero Kronecker Coefficients to Spectra

Proof of theorem 2.2: Rather than working with the mixed state \( \rho^{AB} \) we will consider a purification \( |\psi\rangle^{ABC} \) of \( \rho^{AB} \), which has Spec \( \rho^C = \text{Spec} \rho^{AB} \). Let \( r^A := \text{Spec} \rho^A, r^B := \text{Spec} \rho^B \) and \( r^C := \text{Spec} \rho^C \). \( P^A_\mu \) denotes the projector onto the Young subspace \( U_\mu \otimes V_\nu \) in system \( A \), and \( P^B_\nu, P^C_\lambda \) are the corresponding projectors onto Young subspaces in \( B \) and \( C \), respectively. As a consequence of theorem 2.1 (see [4, Corollary 2]), for given \( \epsilon > 0 \) one can find a \( k_0 \) such that the following inequalities hold simultaneously for all \( k \geq k_0 \):

\[
\text{Tr} P_X(\rho^A)^{\otimes k} \geq 1 - \epsilon, \quad P_X := \sum_{\mu:\|\bar{\mu} - r^A\|_1 \leq \epsilon} P^A_\mu \tag{15}
\]

\[
\text{Tr} P_Y(\rho^B)^{\otimes k} \geq 1 - \epsilon, \quad P_Y := \sum_{\nu:\|\bar{\nu} - r^B\|_1 \leq \epsilon} P^B_\nu \tag{16}
\]

\[
\text{Tr} P_Z(\rho^C)^{\otimes k} \geq 1 - \epsilon, \quad P_Z := \sum_{\lambda:\|\bar{\lambda} - r^C\|_1 \leq \epsilon} P^C_\lambda. \tag{17}
\]

For \( 0 < \epsilon < \frac{1}{3} \), the estimates (15)-(17) can be combined to yield

\[
\text{Tr} [(P_X \otimes P_Y \otimes P_Z) (|\psi\rangle \langle \psi|^{ABC})^{\otimes k}] \geq 1 - 3\epsilon > 0. \tag{18}
\]

Since \( (|\psi\rangle^{ABC})^{\otimes k} \) is evidently invariant under permutation of its \( k \) subsystems, it takes the form

\[
(|\psi\rangle^{ABC})^{\otimes k} = \sum_{\mu\nu\lambda} |\alpha_{\mu\nu\lambda}\rangle,
\]

where \( |\alpha_{\mu\nu\lambda}\rangle \in U_\mu \otimes U_\nu \otimes U_\lambda \otimes (V_\mu \otimes V_\nu \otimes V_\lambda)^{S_k} \). Equation (18) then implies that there must be at least one triple \((\mu, \nu, \lambda)\) with \( \|\bar{\mu} - r^A\|_1 \leq \epsilon, \|\bar{\nu} - r^B\|_1 \leq \epsilon, \|\bar{\lambda} - r^C\|_1 \leq \epsilon \) and \( |\alpha_{\mu\nu\lambda}\rangle \neq 0 \). Thus \( (V_\mu \otimes V_\nu \otimes V_\lambda)^{S_k} \neq 0 \) implies that \( g_{\mu\nu\lambda} \neq 0 \). It remains to pick a sequence of decreasing \( \epsilon_j \) with corresponding triples \((\mu^{(j)}, \nu^{(j)}, \lambda^{(j)})\). \( \square \)

It has been observed in different contexts that the speed of convergence in theorem 2.1 and consequently in theorem 2.2 is proportional to \( 1/\sqrt{k} \) [16, 17].

We will now prove corollary 3.3: the equivalence of theorem 2.2 and 2.3.

Proof of corollary 3.3: We start by showing how theorem 2.3 follows from theorem 2.2.

Let \( (r^A, r^B, r^{AB}) \in \text{SPEC} \), the set of admissible spectral triples. According to theorem 2.2 there is a sequence of elements in Kron, whose normalised values converge to \( (r^A, r^B, r^{AB}) \). By theorems 3.1 and 3.2 the set Kron is a
finitely generated semigroup. With a finite set of generators \((\mu^{(i)}, \nu^{(i)}, \lambda^{(i)})\) of Kron we can therefore express \((r^{A}, r^{B}, r^{AB})\) in the form

\[
(r^{A}, r^{B}, r^{AB}) = \sum_{i} x_{i}(\bar{\mu}^{(i)}, \bar{\nu}^{(i)}, \bar{\lambda}^{(i)}),
\]  

(19)

for a set of nonnegative numbers \(x_{i}\) which sum to one. Since the union of the \(t + 1\)-vertex simplices equals the whole polytope, every point in it can be taken to be the sum of just \(t + 1\) normalised generators \((\bar{\mu}^{(i)}, \bar{\nu}^{(i)}, \bar{\lambda}^{(i)})\) (cf. Carathéodory’s theorem). From the set of \(m + n + mn\) equations in the variables \(x_{i}\) in equation (19), choose a set of \(t\) linearly independent ones, add the \((t + 1)\)th constraint \(\sum x_{i} = 1\) and write the set of equations as \(M\vec{x} = \vec{r}\), i.e. \(\vec{r} = (r_{1}, \ldots, r_{t}, 1)\) for \(r_{j} \in \{r_{1}^{A}, \ldots, r_{m}^{A}, r_{1}^{B}, \ldots, r_{n}^{B}, r_{1}^{AB}, \ldots, r_{mn}^{AB}\}\) and \(x = (x_{1}, \ldots, x_{t+1})\).

If \((r^{A}, r^{B}, r^{AB})\) is rational, the \(x_{i}\) will be rational as well, since \(M\) is rational. This shows that \((r^{A}, r^{B}, r^{AB}) = \sum_{i} n_{i}(\bar{\mu}^{(i)}, \bar{\nu}^{(i)}, \bar{\lambda}^{(i)})\), where we set \(x_{i} = \frac{n_{i}}{n}\) for \(n_{i}, n \in \mathbb{N}\). Multiplication by \(|\mu|n\) results in

\[
|\mu|n(r^{A}, r^{B}, r^{AB}) = \sum_{i} n_{i}(\mu^{(i)}, \nu^{(i)}, \lambda^{(i)}).
\]

Since the right hand side of this equation is certainly an element of Kron this shows that for rational \((r^{A}, r^{B}, r^{AB})\) there is a number \(m := |\mu|n\) such that \(g_{m^{A},m^{B},m^{AB}} \neq 0\).

It remains to show that theorem 2.2 follows from theorem 2.3. Suppose \((r^{A}, r^{B}, r^{AB})\) is a spectral triple corresponding to some \(\rho^{AB}\). Then we can construct a series of rational triples \((r^{A(j)}, r^{B(j)}, r^{AB(j)})\) that approaches \((r^{A}, r^{B}, r^{AB})\) and by theorem 2.3 there exists a series \((\mu(j), \nu(j), \lambda(j))\) such that \((\bar{\mu}^{(j)}, \bar{\nu}^{(j)}, \bar{\lambda}^{(j)})\) approaches \((r^{A}, r^{B}, r^{AB})\) and \(g_{\mu^{(j)},\nu^{(j)},\lambda^{(j)}} \neq 0\) for all \(j\).

Note that there are two ways in which Klyachko’s theorem 2.3 does not quite give the full strength of theorem 2.2. First, it does not guarantee the speed of convergence. Second, it does not imply that, in the case of rational triples \((r^{A}, r^{B}, r^{AB})\), there is an increasing sequence of values of \(k\) for which \(g_{kr^{A},kr^{B},kr^{AB}} \neq 0\); this follows from theorem 2.2 and can be thought of as a sort of stability obtained without appeal to theorem 3.1.

We will now turn out attention to theorem 2.4. This theorem is a consequence of three things: compactness of the set of density matrices on \(\mathbb{C}^{m} \otimes \mathbb{C}^{n}\), the semigroup property (theorem 3.1) and the following lemma, which may be thought of as a simpler and more quantitative version of theorem 2.3 [13, Theorem 5.3.1].

**Lemma 5.1**

Let \(\mu, \nu\) and \(\lambda\) be diagrams with \(k\) boxes and at most \(m, n\) and \(mn\) rows, respectively. If \(g_{\mu\nu\lambda} \neq 0\), then there exists a density operator \(\rho^{AB}\) on \(\mathbb{C}^{m} \otimes \mathbb{C}^{n}\)
with spectra
\[ \|\text{Spec } \rho^A - \bar{\mu}\|_1 \leq \delta \]  \hspace{1cm} (20)
\[ \|\text{Spec } \rho^B - \bar{\nu}\|_1 \leq \delta \]  \hspace{1cm} (21)
\[ \|\text{Spec } \rho^{AB} - \bar{\lambda}\|_1 \leq \delta \]  \hspace{1cm} (22)

for \( \delta = O(mn \sqrt{(\log k)}/k) \)

Here if \( p, q \) are probability distributions then \( \|p - q\|_1 := \sum_x |p(x) - q(x)| \).

**Proof** It will suffice to construct a pure state \( |\varphi\rangle_{ABC} \in \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^{mn} \) with \( \|\text{Spec } \varphi^A - \bar{\mu}\|_1 \leq \delta \), \( \|\text{Spec } \varphi^B - \bar{\nu}\|_1 \leq \delta \) and \( \|\text{Spec } \varphi^C - \bar{\lambda}\|_1 \leq \delta \), since \( \text{Spec } \varphi^{AB} = \text{Spec } \varphi^C \).

By our assumption, there exists a unit vector \( |\psi\rangle \in (V_\mu \otimes V_\nu \otimes V_\lambda)^{\otimes k} \). Now extend \( |\psi\rangle \) to a unit vector \( |\psi\rangle \in (U_\mu \otimes U_\nu \otimes U_\lambda) \otimes (V_\mu \otimes V_\nu \otimes V_\lambda)^{\otimes k} \). By Schur-Weyl duality (Eq. 2) we can embed this space in \((\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^{mn})^{\otimes k}\). Thus
\[ |\psi\rangle \in ((\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^{mn})^{\otimes k})^{\otimes k} \cong ((\mathbb{C}^{m^2n^2})^{\otimes k})^{\otimes k} \]

Again by Schur-Weyl duality, \((\mathbb{C}^{m^2n^2})^{\otimes k}\) is an irreducible representation of \( \text{GL}(m^2n^2) \), which we denote by \( U_{(k)}^{m^2n^2} \) to emphasize which \( \text{GL}(\cdot) \) we are using. Denote the projector onto \( U_{(k)}^{m^2n^2} \otimes V_{(k)} \subset (\mathbb{C}^{m^2n^2})^{\otimes k} \) by \( P_{(k)}^{m^2n^2} \). Note that \( \text{Tr } P_{(k)}^{m^2n^2} = \dim U_{(k)}^{m^2n^2} = \binom{k+m^2n^2-1}{m^2n^2-1} \leq k^{m^2n^2} \). Fix a vector \( |\phi_0\rangle \in \mathbb{C}^{m^2n^2} \) and let \( dU \) denote a Haar measure for \( U(m^2n^2) \) with normalisation \( \int dU = 1 \). Then by Schur’s lemma
\[ P_{(k)}^{m^2n^2} = \dim U_{(k)}^{m^2n^2} \int_{U \in U(m^2n^2)} dU \langle \phi_0 | U \rangle \langle U | \phi_0 \rangle^{\otimes k} \]

Thus
\[
1 = \text{Tr } P_{(k)}^{m^2n^2} |\psi\rangle \langle \psi| = \dim U_{(k)}^{m^2n^2} \int_{U \in U(m^2n^2)} dU \text{Tr } |\psi\rangle \langle \psi| (U |\phi_0\rangle \langle \phi_0| U^{\dagger})^{\otimes k} \leq \dim U_{(k)}^{m^2n^2} \max_{U \in U(m^2n^2)} \text{Tr } |\psi\rangle \langle \psi| (U |\phi_0\rangle \langle \phi_0| U^{\dagger})^{\otimes k}
\]

Let \( U \in U(m^2n^2) \) be the unitary operator achieving the above maximisation, and define \( |\varphi\rangle := U |\phi_0\rangle \). Then
\[
|\langle \psi| (|\varphi\rangle^{\otimes k})|^2 \geq \frac{1}{\dim U_{(k)}^{m^2n^2}} \geq k^{-m^2n^2}.
\]

Let \( P_{\mu}^{m}, P_{\nu}^{n} \) and \( P_{\lambda}^{mn} \) denote the projectors onto \( U_\mu^m \otimes V_\mu \subset (\mathbb{C}^m)^{\otimes k}, U_\nu^n \otimes V_\nu \subset (\mathbb{C}^n)^{\otimes k} \) and \( U_\lambda^m \otimes V_\lambda \subset (\mathbb{C}^{mn})^{\otimes k} \), respectively. Then by construction
\((P^m \otimes P^n \otimes P^{mn}) |\psi\rangle = |\psi\rangle\), so \(|\psi\rangle\langle\psi| \leq P^m \otimes P^n \otimes P^{mn}\) and

\[
\text{Tr}\left[(P^m \otimes P^n \otimes P^{mn}) |\varphi\rangle\langle\varphi|^{\otimes k}\right] \geq \text{Tr}\left(|\psi\rangle\langle\psi|\right)^k = \frac{1}{\dim U_{(k)}} \geq k^{-m^2n^2}.
\]

Focussing for now on the \(A\) subsystem, we have

\[
\text{Tr} P^m (\varphi^A)^{\otimes k} \geq k^{-m^2n^2}. \tag{23}
\]

On the other hand, theorem 2.1 (Spectrum Estimation) states that

\[
\text{Tr} P^m (\varphi^A)^{\otimes k} \leq (k + 1)^{m(m-1)/2} \exp(-kD(\bar{\mu}||\text{Spec } \varphi^A)). \tag{24}
\]

Combining eqns. (23) and (24), we find that

\[
D(\bar{\mu}||\text{Spec } \varphi^A) \leq \frac{\frac{1}{2}m(m-1)\log(k+1) + m^2n^2 \log k}{k}
\]

and for \(k > 1\), we can apply Pinsker’s inequality [15] (which states that \(\|p - q\|_1^2 \leq D(p||q)\) for any probability distributions \(p, q\)) to bound

\[
\|\bar{\mu} - \text{Spec } \varphi^A\|_1 \leq 3mn\sqrt{(\log k)/k}.
\]

This proves eq. (20). Eqs. (21) and (22) follow by repeating this argument (starting with eq. (23)) for \(P^n\) and \(P^{mn}\).

\[\square\]

6 Convexity

Let us now gather together some implications of the theorems. Let KRON denote the normalised triples \((\bar{\mu}, \bar{\nu}, \bar{\lambda})\), where \((\mu, \nu, \lambda) \in \text{Kron}\). From theorem 2.2 we know that any admissible spectral triple, i.e. any point in SPEC, can be approximated by a sequence in KRON, and therefore lies in KRON; thus SPEC \(\subseteq\) KRON. From theorem 2.1 we know that KRON \(\subseteq\) SPEC, and hence, since SPEC is closed, KRON \(\subseteq\) SPEC. Thus we have

\[\text{SPEC} = \text{KRON}.
\]

Note that KRON consists of rational points (normalised row lengths of diagrams) and there are certainly operators with irrational spectra. So KRON, unlike its closure, is a proper subset of SPEC.

Theorems 3.1 and 3.2 allow us to say more about KRON, and hence SPEC. The semigroup property of Kron (theorem 3.1) implies that if \((\bar{\mu}, \bar{\nu}, \bar{\lambda}), (\bar{\mu}', \bar{\nu}', \bar{\lambda}') \in \text{KRON}\), then

\[
(p\bar{\mu} + (1 - p)\bar{\mu}', p\bar{\nu} + (1 - p)\bar{\nu}', p\bar{\lambda} + (1 - p)\bar{\lambda}') \in \text{KRON},
\]

11
for every $p$ with $0 \leq p \leq 1$. Thus $\mathbf{KRON}$ is convex. Furthermore, theorem 3.2 implies that there is a finite set of generators $(\mu^{(i)}, \nu^{(i)}, \lambda^{(i)})$ of Kron, so any $(\bar{\mu}, \bar{\nu}, \bar{\lambda}) \in \mathbf{KRON}$ can be written

$$(\bar{\mu}, \bar{\nu}, \bar{\lambda}) = \sum_i x_i (\bar{\mu}^{(i)}, \bar{\nu}^{(i)}, \bar{\lambda}^{(i)}).$$

Thus $\mathbf{KRON}$ is a convex polytope. We enshrine this in theorem 3.4.

An alternative proof for theorem 3.4 that makes use of Kirwan’s convexity theorem for moment maps can be found in [3, Chapter 2.3.6].

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