A Note on the Abelian Complexity of the Rudin-Shapiro Sequence

Xiaotao Lü and Pengju Han *

College of Science, Huazhong Agricultural University, Wuhan 430070, China; xiaotaoL.v@mail.hzau.edu.cn
* Correspondence: hanpengju@mail.hzau.edu.cn

Abstract: Let \( \{r(n)\}_{n \geq 0} \) be the Rudin-Shapiro sequence, and let \( \rho(n) := \max\{\sum_{i=1}^{n+1} r(j) \mid i \geq 0\} + 1 \) be the abelian complexity function of the Rudin-Shapiro sequence. In this note, we show that the function \( \rho(n) \) has many similarities with the classical summatory function \( S_r(n) := \sum_{i=0}^{n} r(i) \). In particular, we prove that for every positive integer \( n \), \( \sqrt{3} \leq \frac{\rho(n)}{\sqrt{n}} \leq 3 \). Moreover, the point set \( \{ \frac{\rho(n)}{\sqrt{n}} : n \geq 1 \} \) is dense in \([\sqrt{3}, 3]\).

Keywords: Rudin-Shapiro sequence; abelian complexity; growth order; dense property

1. Introduction

In this note, we are concerned with the abelian complexity \( \rho(n) = \max\{\sum_{i=1}^{n+1} r(j) \mid i \geq 0\} + 1 \) of the Rudin-Shapiro sequence \( r \). The Rudin-Shapiro sequence \( r = r(0)r(1) \cdots \in \{-1,1\}^\mathbb{N} \) is given by the following recurrence relations:

\[
\begin{align*}
r(0) &= 1, \\
r(2n) &= r(n), \\
r(2n + 1) &= (-1)^n r(n) \quad (n \geq 0).
\end{align*}
\]

The Rudin-Shapiro sequence \( r \) is a typical 2-automatic sequence [1]. It has been proved in [2] that the sequence \( \rho(n) \) satisfies \( \rho(1) = 2, \rho(2) = 3, \rho(3) = 4 \) and for every \( n \geq 1 \), \( \rho(4n) = 2\rho(n) + 1, \rho(4n + 1) = 2\rho(n), \rho(4n + 2) = \rho(n) + \rho(n + 1), \rho(4n + 3) = 2\rho(n + 1) \).

Let \( w = w(0)w(1)w(2) \cdots \) be an infinite sequence with \( w(i) \in \mathbb{Z} \) for every \( i \geq 0 \). There are many papers focusing on the summatory function \( S_w(n) := \sum_{i=0}^{n} w(i) \). In [3–5], Brillhart and Morton studied the summatory function \( S_r(n) := \sum_{i=0}^{n} r(i) \) of the Rudin-Shapiro sequence. The sequence \( S_r(n) \) satisfies \( S_r(0) = 1, S_r(1) = 2, S_r(2) = 3, S_r(3) = 2 \) and for every \( n \geq 1 \), \( S_r(4n) = 2S_r(n) + r(n), S_r(4n + 1) = S_r(4n + 1) = 2S_r(n) + 2 = 2S_r(n) + (-1)^n r(n) \).

In detail, Brillhart and Morton proved that for every \( n \geq 1 \),

\[
\sqrt{3/5} \leq \frac{S_r(n)}{\sqrt{n}} \leq \sqrt{\frac{3}{\sqrt{n}}},
\]

and \( \{ \frac{S_r(n)}{\sqrt{n}} : n \geq 1 \} \) is dense in \([\sqrt{3/5}, \sqrt{\frac{3}{\sqrt{n}}}]\). In [6], Lafrance, Rampersad and Yee introduced a Rudin-Shapiro-like sequence \( (l(n))_{n \geq 0} \) which satisfies \( l(0) = 1 \) and for every \( n \geq 0 \),

\[
l(4n) = l(n), \quad l(4n + 1) = l(2n), \quad l(4n + 2) = -l(2n) \quad \text{and} \quad l(4n + 3) = l(n).
\]

They studied the properties of the summatory function \( S_l(N) := \sum_{n=0}^{N} l(n) \). The sequence \( S_l(N) \) satisfies \( S_l(0) = 1 \) and for every \( m \geq 0 \),

\[
S_l(N) = \sum_{i=0}^{N} S_l(i) - \sum_{i=0}^{N} \sum_{j=i}^{N} S_l(j) \sum_{k=1}^{j-i+1} S_l(k).
\]
\[ S_1(4m) = 2S_1(m) - l(m) = S_1(4m + 2), \quad S_1(4m + 1) = 2S_1(m) - l(m) + l(2m), \quad S_1(4m + 3) = 2S_1(m). \]

Moreover, Lafrance, Rampersad and Yee showed that
\[
\limsup_{n \to +\infty} \frac{S_1(n)}{\sqrt{n}} = \sqrt{2}, \quad \liminf_{n \to +\infty} \frac{S_1(n)}{\sqrt{n}} = \frac{\sqrt{3}}{3}.
\]

The sequences \( S_r(n) \) and \( S_1(n) \) are both 2-regular sequences (in the sense of Allouche and Shallit [1]). For the definition and properties of \( k \)-regular sequences, one can refer to [1]. Let \( (s(n))_{n \geq 0} \) be a \( k \)-regular sequence over \( \mathbb{Z} \). It was proved in [1] that there exists a constant \( c \) such that \( s(n) = O(n^c) \). In general, it is a difficult task to compute the exact growth order of sequences satisfying certain recursive relations such as \( k \)-regular sequences.

In [7], Gawron and Ulas obtained the sequence \( \{a(n) \mid n \in \mathbb{N}\} := \{m \in \mathbb{N} \mid c(m) = 1\} \) where \( (c(n))_{n \geq 0} \) is the sequence of coefficients of the compositional inverse of the generating function of the Thue-Morse sequence. The sequence \( (a(n))_{n \geq 0} \) satisfies that \( a(0) = 0, a(1) = 1, a(2) = 2, a(3) = 7 \) and for all \( n \geq 1, a(4n + i) = a(4n - 1) + i + 1 \) with \( i \in [0, 2], a(8n + 3) = a(8n) + 7 \) and \( a(8n + 7) = 4a(4n + 3) + 3 \). They proved that
\[
\liminf_{n \to +\infty} \frac{a(n)}{n^2} = \frac{1}{6}, \quad \limsup_{n \to +\infty} \frac{a(n)}{n^2} = \frac{1}{2}
\]
and \( \{ \frac{a(n)}{n^2} : n \geq 1 \} \) is dense in \([\frac{1}{3}, \frac{1}{2}]\). In [2], Chen, Wen, Wu and the first author studied the maximal digit sum sequence \( M_r(n) := \max \{ \sum_{j=1}^{n-1} r(j) \mid i \geq 0 \} \) and proved that the abelian complexity \( \rho(n) \) of the Rudin-Shapiro sequence satisfies \( \rho(n) = M_r(n) + 1 \) for every \( n \geq 1 \). It is remarkable that the authors in [2] just gave the recursive formulas for the sequence \( M_r(n) \) and proved the 2-regularity of the sequence \( \rho(n) \). It is natural to ask whether the function \( \rho(n) \) has similar properties as the sumatory function \( S_r(n) \). In fact, it is of great interest to study the properties of sequences which satisfy certain recursive formulas.

This note focuses on the growth order of the abelian complexity \( (\rho(n))_{n \geq 1} \) of the Rudin-Shapiro sequence \( r \). Firstly, by studying the maximal and minimal values of the function \( \rho(n) \) in the interval \( I_k := [4^k, 4^{k+1} - 1] \) with \( k \geq 0 \), we got \( \rho(n) = \Theta(\sqrt{n}) \). Then, we investigated two functions \( s(k) := \min \{ n \mid \rho(n) = k \} \) and \( \ell(k) := \max \{ n \mid \rho(n) = k \} \), and obtained the optimal lower and upper bound of the sequence \( \{ \frac{\rho(n)}{\sqrt{n}} \}_{n \geq 1} \). Finally, we showed that \( \rho(n) \) is a quasi-linear function for 4. As a consequence, the set \( \{ \frac{\rho(n)}{\sqrt{n}} : n \geq 1 \} \) is dense between its optimal lower bound and upper bound. In detail, we proved the following theorems.

**Theorem 1.** For every integer \( n \geq 1 \), we have
\[
\sqrt{3} \leq \frac{\rho(n)}{\sqrt{n}} \leq 3.
\]

**Theorem 2.** The set \( \{ \frac{\rho(n)}{\sqrt{n}} : n \geq 1 \} \) is dense in \([\sqrt{3}, 3]\).

The outline of this note is as follows. In Section 2, we compute the maximal and minimal values of the function \( \rho(n) \) in the interval \( I_k := [4^k, 4^{k+1} - 1] \) for every \( k \geq 0 \). In Section 3, we give the proofs of Theorem 1 and Theorem 2.

2. Basic Properties of the Function \( \rho(n) \)

In this section, we exhibit some basic properties of the abelian complexity function \( \rho(n) \) of the Rudin-Shapiro sequence \( r \).
Following from ([2] Theorem 1 and Lemma 3), the abelian complexity function $\rho(n)$ of the Rudin-Shapiro sequence is given by the following formulas: $\rho(1) = 2$, $\rho(2) = 3$, $\rho(3) = 4$ and for every integer $n \geq 1$,

$$
\begin{align*}
\rho(4n) &= 2\rho(n) + 1, & \rho(4n + 1) &= 2\rho(n), \\
\rho(4n + 2) &= \rho(n) + \rho(n+1), & \rho(4n + 3) &= 2\rho(n+1).
\end{align*}
$$

(1)

Set $\rho(0) := 1$. For every integer $n \geq 0$, let $\Delta \rho(n) := \rho(n+1) - \rho(n)$. Then $\Delta \rho(0) = \Delta \rho(1) = \Delta \rho(2) = \Delta \rho(3) = 1$, and for every integer $n \geq 1$,

$$
\begin{align*}
\Delta \rho(4n) &= -1, & \Delta \rho(4n + 3) &= 1, \\
\Delta \rho(4n + 1) &= \Delta \rho(4n + 2) = \Delta \rho(n).
\end{align*}
$$

(2)

This implies that $\Delta \rho(n) \in \{-1,1\}$ for every integer $n \geq 0$. The first 16 terms of $\rho(n)$, starting with $n = 1$, are listed in Table 1.

**Table 1.** The first 16 terms of the sequence $\rho(n)$.

| $n$ | $\rho(n)$ | $n$ | $\rho(n)$ |
|-----|-----------|-----|-----------|
| 1   | 2         | 6   | 6         |
| 2   | 3         | 7   | 7         |
| 3   | 4         | 8   | 8         |
| 4   | 5         | 9   | 9         |
| 5   | 4         | 10  | 10        |
| 6   | 5         | 11  | 11        |
| 7   | 6         | 12  | 12        |
| 8   | 7         | 13  | 13        |

For simplicity of notation, for every integer $k \geq 0$, put $m_k := \frac{4^{k+1} - 1}{3}$, $M_k := 4^{k+1} - 1$ and $I_k := [4^k, 4^{k+1} - 1]$. Then we have the following two lemmas which give the minimal and maximal values of the function $\rho(n)$ in the interval $I_k$ for every $k \geq 0$.

**Lemma 1.** For every integer $k \geq 0$, the minimum value of $\rho(n)$ in $I_k = [4^k, 4^{k+1} - 1]$ is $2^{k+1}$. Moreover,

$$
\max\left\{n \in I_k : \rho(n) = 2^{k+1}\right\} = m_k.
$$

**Proof.** We will prove this by induction on the variable $k$. For $k = 0$, it follows from Table 1 that this assertion is true. Assume the assertion holds for the interval $I_k$.

We first show that $2^{k+2}$ is the lower bound for $\rho(n)$ in $I_{k+1}$. If $n$ lies in $I_{k+1} = [4^{k+1}, 4^{k+2} - 1]$, then we can write $n = 4m + d$ for some $m \in I_k$ and some $d \in \{0,1,2,3\}$. There are two cases to be considered.

1. When $4^k \leq m \leq 4^{k+1} - 2$, (1) yields that for every $d \in \{0,1,2,3\}$

$$
\rho(n) = \rho(4m+d) \geq 2\min\{\rho(m) : m \in I_k\} = 2^{k+2}.
$$

The last equality is true under the inductive assumption.

2. When $m = 4^{k+1} - 1$, it follows from (2) that $\Delta(m) = 1$, which implies

$$
\rho(m+1) = \rho(4^{k+1} - 1) + 1 = \rho(m) + 1.
$$

Hence, for every $d \in \{0,1,2,3\}$, using (1) again, we have

$$
\rho(n) = \rho(4m+d) \geq 2\rho(m) > 2\min\{\rho(m) : m \in I_k\} = 2^{k+2}.
$$

(3)
At the same time, using the fact \( m_{k+1} = 4m_k + 1 \), it is easy to check that
\[
\rho(m_{k+1}) = \rho(4m_k + 1) = 2\rho(m_k) = 2^{k+2}.
\]

Now it suffices to show that
\[
m_{k+1} = \max\{n \in I_{k+1} : \rho(n) = 2^{k+2}\}.
\]

Following from the inductive assumption, for every \( m \in I_k \) satisfying \( m > m_k \), we have \( \rho(m) \geq \rho(m_k) + 1 \). By (1), we can get
\[
\rho(m_k + 1) = \rho(4m_k + 2) = \rho(m_k) + \rho(m_k + 1) \geq 2^{k+2} + 1,
\]
\[
\rho(m_k + 2) = \rho(4m_k + 3) = 2\rho(m_k + 1) \geq 2^{k+2} + 2.
\]

Now we only need to consider the case \( n = 4m + d \geq 4m_k + 4 \) with \( d \in \{0, 1, 2, 3\} \). In fact, for every \( m_k + 1 \leq m \leq 4^{k+1} - 2 \) and \( d \in \{0, 1, 2, 3\} \), it follows from (1) that
\[
\rho(4m + d) \geq 2\min\{\rho(m) : m_k + 1 \leq m \leq 4^{k+1} - 1\} \geq 2^{k+2} + 2.
\]

By (3), the case \( n = 4m + d \) for \( m = 4^{k+1} - 1 \) holds, which completes the proof. □

**Lemma 2.** Let \( k \) be a non-negative integer. The maximum value of \( \rho(n) \) in \( I_k = [4^k, 4^{k+1} - 1] \) is \( 3 \cdot 2^{k+1} - 2 \) and this value occurs only at the point \( n = M_k = 4^{k+1} - 1 \).

**Proof.** We will prove this by induction on the variable \( k \). For \( k = 0 \), this assertion holds following from Table 1. Assume the assertion is true for the interval \( I_k \). When \( n \) lies in \( I_{k+1} \), let \( n = 4m + d \) for some \( m \in I_k \) and \( d \in \{0, 1, 2, 3\} \). Similarly with the proof of Lemma 1, we divide it into two cases.

1. When \( 4^k \leq m < 4^{k+1} - 1 \). By (1) and the inductive assumption, we have
   \[
   \rho(n) = \rho(4m + d) \leq 2\max\{\rho(m) : m \in I_k\} + 1 < 3 \cdot 2^{k+2} - 2. \tag{4}
   \]

2. When \( m = M_k = 4^{k+1} - 1 \). Following from (1) and (2), we have
   \[
   \rho(4M_k) = 2\rho(M_k) + 1 = 3 \cdot 2^{k+2} - 3,
   \]
   \[
   \rho(4M_k + 1) = 2\rho(M_k) = 3 \cdot 2^{k+2} - 4,
   \]
   \[
   \rho(4M_k + 2) = \rho(M_k) + \rho(M_k + 1) = 3 \cdot 2^{k+2} - 3,
   \]
   \[
   \rho(4M_k + 3) = 2\rho(M_k + 1) = 3 \cdot 2^{k+2} - 2. \tag{5}
   \]

This implies that \( \rho(n) \leq \rho(4M_k + 3) = \rho(4^{k+2} - 1) = 3 \cdot 2^{k+2} - 2 \).

Following from (4) and (5), we can obtain that \( M_{k+1} = 4M_k + 3 \) is the unique point in \( I_{k+1} \) which attains the maximal value of \( \rho \) in the interval \( I_{k+1} \). This completes the proof. □

**Remark 1.** From Lemma 1, we have that for every \( n \geq 1 \),
\[
\rho(n) \geq 2.
\]

**Remark 2.** If \( n \in I_k = [4^k, 4^{k+1} - 1] \), Lemma 1 gives us
\[
\frac{\rho(n)}{\sqrt{n}} > \frac{2^{k+1}}{\sqrt{4^k}} = 1,
\]
while Lemma 2 implies that
\[
\frac{\rho(n)}{\sqrt{n}} < \frac{3 \cdot 2^{k+1} - 2}{\sqrt{4^k}} < 6.
\]
Thus, for every integer \( n \geq 1 \), \( 1 \leq \frac{\rho(n)}{\sqrt[n]{n}} \leq 6 \), and so \( \rho(n) \) is roughly a constant times \( \sqrt{n} \). However, these bounds are not optimal. Note that \( \rho(n) \geq 2 \) for every \( n \geq 1 \). It is easy to verify that

\[
\lim_{k \to \infty} \frac{\rho(M_k)}{\sqrt{M_k}} = \lim_{k \to \infty} \frac{3 \cdot 2^{k+1} - 2}{\sqrt{4^{k+1} - 1}} = 3
\]

and

\[
\lim_{k \to \infty} \frac{\rho(m_k)}{\sqrt{m_k}} = \lim_{k \to \infty} \frac{2^{k+1}}{\sqrt{4^{k+1} - 3}} = \sqrt{3}.
\]

In other words, 3 and \( \sqrt{3} \) are two accumulation points of the set \( \{\frac{\rho(n)}{\sqrt[n]{n}} : n \geq 1\} \). In the following section, we will prove that 3 and \( \sqrt{3} \) are the optimal upper and lower bound for the sequence \( \{\frac{\rho(n)}{\sqrt[n]{n}}\}_{n \geq 1} \) respectively.

### 3. Proofs of Theorems 1 and 2

Following that \( M_k \) and \( m_k \) both go to infinity with \( k \) tending to infinity (by Lemmas 1 and 2), we can see that there are only finite number of places \( n \) such that \( \rho(n) \) has a fixed value \( k \). When \( \rho(n) = k \), for a fixed \( k \), the ratio \( \rho(n) / \sqrt[n]{n} \) will be the smallest when \( n \) is largest while it will be largest if \( n \) is smallest. This leads us to the following idea: for a fixed \( k \in \mathbb{N} \) with \( k \geq 1 \), let us focus on the smallest and largest values of \( n \) such that \( \rho(n) = k \). For this purpose, we introduced two auxiliary functions \( s(k) \) and \( \ell(k) \).

**Definition 1.** Given an integer \( k \geq 1 \), let \( s(k) \) and \( \ell(k) \) be the smallest and largest values of \( n \) such that \( \rho(n) = k \) respectively, i.e.,

\[
s(k) := \min\{n : \rho(n) = k\},
\]
\[
\ell(k) := \max\{n : \rho(n) = k\}.
\]

Following from (1), Table 1 and \( \rho(0) = 1 \), the initial 8 terms of the sequences \( s(k) \) and \( \ell(k) \) are given in Table 2.

**Table 2.** The initial values for the sequences \( s(k) \) and \( \ell(k) \).

| \( k \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|--------|----|----|----|----|----|----|----|----|
| \( s(k) \) | 0  | 1  | 2  | 3  | 4  | 5  | 7  | 8  |
| \( \ell(k) \) | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 9  |

For the sequences \( s(k) \) and \( \ell(k) \), we have the following results.

**Lemma 3.** The sequence \( \ell(k) \) satisfies \( \ell(1) = 0 \) and for every integer \( k \geq 1 \),

\[
\ell(2k) = 4\ell(k) + 1, \quad \ell(2k + 1) = 4\ell(k) + 2.
\]

**Proof.** The assertion holds for \( k = 1 \) by Table 2. Fix some integer \( k \geq 2 \), assume \( \ell(k) = n \). Then \( \rho(n) = k \). By the definition of \( \ell(n) \) and (2), for every integer \( m > n \), \( \rho(m) \geq \rho(n) + 1 = k + 1 \) and

\[
\rho(n + 1) = \rho(n) + 1 = k + 1.
\]

Therefore, for every \( m > n \) and \( d \in \{0, 1, 2, 3\} \),

\[
\rho(4m + d) \geq 2 \min\{\rho(m) : m > n\} = 2k + 2.
\]
At the same time, when \( m = n \), it is obvious that

\[
\begin{align*}
\rho(4n) &= 2\rho(n) + 1 = 2k + 1, \\
\rho(4n + 1) &= 2\rho(n) = 2k, \\
\rho(4n + 2) &= \rho(n) + \rho(n + 1) = 2k + 1, \\
\rho(4n + 3) &= 2\rho(n + 1) = 2k + 2. 
\end{align*}
\]

This implies that \( \ell(2k) = 4n + 1 = 4\ell(k) + 1 \) and \( \ell(2k + 1) = 4n + 2 = 4\ell(k) + 2 \). □

**Lemma 4.** The sequence \( s(k) \) satisfies \( s(1) = 0, s(2) = 1, s(3) = 2 \) and for every \( k \geq 2 \),

\[
s(2k) = 4s(k) - 1, \quad s(2k + 1) = 4s(k) - 1.
\]

**Proof.** The initial values \( s(1), s(2) \) and \( s(3) \) can be easily verified by Table 2. For an integer \( k \geq 2 \), suppose \( s(k) = n \) for some integer \( n \). Then \( \rho(n) = k \). By the definition of \( s(k) \) and (2), we have that \( \rho(m) < \rho(n) \) whenever \( m < n \) and

\[
\rho(n - 1) = \rho(n) - 1 = k - 1.
\]

Therefore, following from (1), for every \( 0 \leq m < n - 1 \) and \( d \in \{0, 1, 2, 3\} \), we have

\[
\rho(4m + d) \leq 2 \max\{\rho(m) : 0 \leq m \leq n - 1\} + 1 = 2k - 1.
\]

Note that the value of \( 4m + d \) ranges from 0 to \( 4n - 5 \). At the same time, it follows from (1) that

\[
\begin{align*}
\rho(4n - 4) &= 2\rho(n - 1) + 1 = 2k - 1, \\
\rho(4n - 3) &= 2\rho(n - 1) = 2k - 2, \\
\rho(4n - 2) &= \rho(n - 1) + \rho(n) = 2k - 1, \\
\rho(4n - 1) &= 2\rho(n) = 2k.
\end{align*}
\]

This implies that \( s(2k) = 4n - 1 = 4s(k) - 1 \). The other formula \( s(2k + 1) = 4s(k) \) follows from the fact that \( \rho(4n) = 2\rho(n) + 1 = 2k + 1 \) and \( \rho(i) \leq 2k \) for every \( 0 \leq i \leq 4n - 1 \). This ends the proof. □

The following two propositions show the upper bound for the sequence \( \ell(k) \) and the lower bound for the sequence \( s(k) \). For the sake of simplicity, for every integer \( k \geq 2 \), let

\[
k = \sum_{j=0}^{m} k_j2^j := [k_m k_{m-1} \cdots k_0]_2
\]

be the binary expansion of \( k \) with \( m \geq 1 \) and \( k_m = 1 \). For every \( x \geq 0 \), let \( \lfloor x \rfloor \) be the greatest integer which is no more than \( x \).

**Proposition 1.** For every integer \( k \geq 2 \), we have

\[
\ell(k) \leq \frac{k^2}{3}.
\]

**Proof.** For every \( k \geq 2 \), let the binary expansion of \( k \) be \([k_m k_{m-1} \cdots k_0]_2\). Following from Lemma 3, we have

\[
\ell(k) = \ell([k_m k_{m-1} \cdots k_0]_2) = 4\ell([k_m k_{m-1} \cdots k_1]_2) + k_0 + 1.
\]
Now we apply (6) by replacing $k$ with $[k_m k_{m-1} \cdots k_1]_2$, which yields
\[
\ell([k_m k_{m-1} \cdots k_1]_2) = 4\ell([k_m k_{m-1} \cdots k_2]_2) + k_1 + 1.
\]

Repeating this progress $m$ times, by the fact that $k_j = \lfloor \frac{k}{2^j} \rfloor - 2\lfloor \frac{k}{2^j+1} \rfloor$ for every $0 \leq j \leq m-1$, we can obtain that
\[
\ell(k) = \ell([k_m k_{m-1} \cdots k_0]_2) = 4\ell([k_m k_{m-1} \cdots k_1]_2) + k_0 + 1 = 4^2\ell([k_m k_{m-1} \cdots k_2]_2) + 4(k_1 + 1) + k_0 + 1 = \cdots = 4^m \ell(k_m) + \sum_{j=0}^{m-1} 4^j(k_j + 1)
\]
\[
= 4^m \ell(1) + \sum_{j=0}^{m-1} 4^j(\lfloor \frac{k}{2^j} \rfloor - 2\lfloor \frac{k}{2^j+1} \rfloor + 1)
\]
\[
= \frac{4^m - 1}{3} + \sum_{j=0}^{m-1} 4^j(\lfloor \frac{k}{2^j} \rfloor - 2\lfloor \frac{k}{2^j+1} \rfloor)
\]
\[
\leq \frac{4^m}{3} + k - 2\lfloor \frac{k}{2} \rfloor + 4\lfloor \frac{k}{2} \rfloor - 8\lfloor \frac{k}{2^2} \rfloor + \cdots + 4^{m-1}\lfloor \frac{k}{2^m-1} \rfloor - 2 \cdot 4^{m-1}\lfloor \frac{k}{2^m} \rfloor - \cdots - 2 \cdot 4^{m-1} + k + 2 \sum_{j=1}^{m-1} 4^{j-1}\lfloor \frac{k}{2^j} \rfloor
\]
\[
= \frac{4^m}{3} - 2 \cdot 4^{m-1} + k + 2 \sum_{j=1}^{m-1} 4^{j-1}\lfloor \frac{k}{2^j} \rfloor
\]
\[
= 2^{m-1}k - \frac{2}{3} \cdot 4^{m-1}.
\]

It suffices to show that
\[
2^{m-1}k - \frac{2}{3} \cdot 4^{m-1} \leq \frac{k^2}{3}.
\]

Note that $2^m \leq k < 2^{m+1}$. This implies that
\[
\frac{k}{4} < 2^{m-1} \leq \frac{k}{2}.
\]

Consider the function $f_k(x) = kx - \frac{2}{3}x^2$. It is not hard to check that $f_k(x)$ is strictly increasing on the interval $(\frac{k}{3}, \frac{k}{2})$ with fixed $k \geq 1$. Hence
\[
kx - \frac{2}{3}x^2 \leq f_k(\frac{k}{2}) = \frac{1}{3}k^2,
\]
which is the desired result. \qed

**Proposition 2.** For every integer $k \geq 2$, we have
\[
s(k) \geq \frac{k^2}{9}.
\]

**Proof.** The assertion holds when $k = 2$ and $k = 3$ since $s(2) = 1 > 4/9$ and $s(3) = 2 > 1$. For every integer $k \geq 4$, let the binary expansion of $k$ be $[k_m k_{m-1} \cdots k_0]_2$ with $m \geq 2$ and $k_m = 1$. Following from Lemma 4, we have
\[
s(k) = s([k_m k_{m-1} \cdots k_0]_2) = 4s([k_m k_{m-1} \cdots k_1]_2) + k_0 - 1.
\]
Arguing analogously as in the proof of Proposition 1, we have

\[ s(k) = s([k_m k_{m-1} \cdots k_0]_2) \]

\[ = 4^{m-1} s([k_m k_{m-1}]_2) + \sum_{j=0}^{m-2} 4^j (k_j - 1) \]

\[ = 4^{m-1} s(2 + k_{m-1}) + \sum_{j=0}^{m-2} 4^j (\lfloor \frac{k}{2^j} \rfloor - 2 \lfloor \frac{k}{2^{j+1}} \rfloor - 1) \]

\[ = 4^{m-1} s(2 + k_{m-1}) - \frac{4^{m-1} - 1}{3} + \sum_{j=0}^{m-2} 4^j (\lfloor \frac{k}{2^j} \rfloor - 2 \lfloor \frac{k}{2^{j+1}} \rfloor) \]

\[ = 4^{m-1} s(2 + k_{m-1}) - \frac{4^{m-1} - 1}{3} - 2 \cdot 4^{m-2} \lfloor \frac{k}{2^{m-1}} \rfloor + k + 2 \sum_{j=1}^{m-2} 4^{j-1} \lfloor \frac{k}{2^j} \rfloor \]

\[ \geq 4^{m-1} s(2 + k_{m-1}) - \frac{4^{m-1} - 1}{3} - 2 \cdot 4^{m-2} \lfloor \frac{k}{2^{m-1}} \rfloor + k + 2 \sum_{j=1}^{m-2} 4^{j-1} (\frac{k}{2^j} - 1) \]

\[ \geq 4^{m-1} s(2 + k_{m-1}) - 2 \cdot 4^{m-2} (3 + k_{m-1}) + 2^{m-2} k \]

\[ = \begin{cases} 2^{m-2} k - 2 \cdot 4^{m-2} & \text{if } k_{m-1} = 0, \\ 2^{m-2} k & \text{if } k_{m-1} = 1. \end{cases} \]

The last equality uses the fact that \( s(2) = 1 \) and \( s(3) = 2. \)

1. If \( k_{m-1} = 0 \), then we have \( 2^m \leq k < 3 \cdot 2^{m-1} \), and it follows that

\[ \frac{k}{6} < 2^{m-2} < \frac{k}{4}. \]

Put \( g_k(x) := xk - 2x^2 \). It is obvious that \( g_k(x) \) is strictly increasing on the interval \( (\frac{k}{6}, \frac{k}{4}) \) with fixed \( k \). Hence

\[ 2^{m-2} k - 2 \cdot 4^{m-2} \geq g_k \left( \frac{k}{6} \right) = \frac{k^2}{9}. \]

2. If \( k_{m-1} = 1 \), then we have \( 3 \cdot 2^{m-1} \leq k < 2^{m+1} \), and it follows that

\[ \frac{k}{8} < 2^{m-2} < \frac{k}{6}, \]

which implies

\[ 2^{m-2} k \geq \frac{k^2}{8} = \frac{k^2}{9}. \]

This ends the proof. \( \square \)

Now, combining Proposition 1 and Proposition 2, we can prove Theorem 1.

**Proof of Theorem 1.** For every integer \( n \geq 1 \), assume \( \rho(n) = k \) for some integer \( k \). It follows from Remark 1 that \( k \geq 2 \). By Proposition 1 and the definition of \( \ell(k) \), we have

\[ \frac{\rho(n)}{\sqrt{n}} \geq \frac{k}{\sqrt{\ell(k)}} \geq \sqrt{3}. \]

Following from Proposition 2 and the definition of \( s(k) \), we have

\[ \frac{\rho(n)}{\sqrt{n}} \leq \frac{k}{\sqrt{s(k)}} \leq 3. \]
This completes the proof. □

By Remark 2 and Theorem 1, we get the following corollary.

**Corollary 1.** The numbers $\sqrt{3}$ and 3 are the optimal lower and upper bounds for the set $\{\rho(n)/\sqrt{n}: n \geq 1\}$ respectively.

To prove Theorem 2, we need an auxiliary notation which was firstly introduced in [8]. Let $b \geq 2$ be an integer and $f : \mathbb{N} \to \mathbb{Z}$ be a discrete function (or an integer sequence). For every $x \geq 0$, let

$$\delta_f(x) := \limsup_{n \to +\infty} \frac{|f(n)|}{n^x}.$$ 

Write $\Delta f(n) := f(n + 1) - f(n)$. Set

$$\alpha = \alpha(f) := \inf \{x \geq 0 \mid \delta_f(x) = 0\} \text{ and } \beta = \beta(f) := \inf \{x \geq 0 \mid \delta_{\Delta f}(x) = 0\}.$$  

**Definition 2.** (Quasi-linear function) Let $f : \mathbb{N} \to \mathbb{Z}$. If $\alpha(f) > \beta(f)$ and there exists a constant $C_1 > 0$ and an integer $b \geq 2$ such that for all positive integers $n$ and $0 \leq i \leq b - 1$,

$$|f(bn + i) - b^\alpha f(n)| \leq C_1 n^\beta,$$

then we call $f$ a quasi-linear function for $b$.

For the quasi-linear functions, we have the following lemma. For more details about this, see [8].

**Lemma 5 ([8]).** Given an integer function $f(n)$, set $\alpha := \alpha(f)$. Suppose $a_1$ and $a_2$ are two accumulation points of $\{f(n)/n^\alpha : n \geq 1\}$. If $f$ is a quasi-linear function, then $\{f(n)/n^\alpha : n \geq 1\}$ is dense in $[a_1, a_2]$.

**Proof of Theorem 2.** Following from the fact that $\Delta \rho(n) \in \{-1, 1\}$ for every $n \geq 0$, we have $\beta(\rho) = 0$. By Theorem 1, $\alpha(\rho) = \frac{1}{2}$. Moreover, for every $0 \leq i \leq 3$ and $n \geq 1$, (1) yields

$$|\rho(4n + i) - 2\rho(n)| \leq 2|\rho(n + 1) - \rho(n)| = 2.$$ 

Therefore $\rho(n)$ is a quasi-linear function for $b = 4$. It follows from Remark 2 that $\sqrt{3}$ and 3 are two accumulation points of the set $\{\rho(n)/\sqrt{n}: n \geq 1\}$. Hence we obtained the desired result by Lemma 5. □

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