EXACTNESS OF LINEAR RESPONSE IN THE QUANTUM HALL EFFECT

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Abstract. In general, linear response theory expresses the relation between a driving and a physical system’s response only to first order in perturbation theory. In the context of charge transport, this is the linear relation between current and electromotive force expressed in Ohm’s law. We show here that, in the case of the quantum Hall effect, all higher order corrections vanish. We prove this in a fully interacting setting and without flux averaging.

1. Introduction

Quantization of the Hall conductance in the bulk is a well understood phenomenon under a spectral gap assumption, both in the absence and presence of interactions between the electrons, see [1, 2, 3, 4, 5] and [6, 7, 8, 9, 10].

The Hall conductance is a linear response coefficient and it can be expressed by Kubo’s formula. In the presence of a spectral gap, the validity of linear response is well established, both in non-interacting [11] and in interacting [12] situations. As was first pointed by Laughlin, the quantum Hall effect has a natural interpretation as a charge pump. In a cylindrical geometry where the current along the cylinder is induced by a time-dependent magnetic flux [16] (see also [17] for a similar idea), the quantum Hall conductance is directly proportional to the charge crossing a fiducial line winding around the cylinder as the flux is slowly increased by one quantum unit. This aspect of the Laughlin argument (the other aspect being the quantization itself) was generalized to the interacting context in [18, 19].

Let $H_{\phi}$ be a smooth family of Hamiltonians parametrized by the magnetic flux $\phi$, and let $P_{\phi}$ be the corresponding ground state projections. Kato constructed in [19] an ‘adiabatic’ propagator $U_{\Lambda}(\phi)$ such that

$$P_{\phi} = U_{\Lambda}(\phi)P_{\phi}U_{\Lambda}(\phi)^*.$$  

We shall work in units where the flux quantum is equal to $2\pi$ and the speed of light is 1. If $\Delta Q_{\Lambda}$ is the expectation value of the charge transported by $U_{\Lambda} = U_{\Lambda}(2\pi)$ across the fiducial line, then the Laughlin argument concludes, formally, that

$$\Delta Q_{\Lambda} = 2\pi \sigma_H \in \mathbb{Z},$$

where $\sigma_H$ is the Hall conductance (for simplicity of the exposition, we assume $\text{Rk}(P_{\phi}) = 1$, corresponding to the integer quantum Hall effect in this introductory section).
In experiments, the quantization is not exact, but the conductance is quantized to nearly one part in a billion \[20\]. That suggests that the universality expressed by the quantization of conductance extends to the charge transport of the full driven Schrödinger equation. If \( \phi \) is changing slowly and smoothly in time, namely \( \phi = \phi(\epsilon t) \) with \( \epsilon \ll 1 \), one may consider the charge transported by the physical propagator \( U_\epsilon(s) \) associated with the time-dependent Hamiltonian \( H_{\phi(s)} \) in rescaled time \( s = \epsilon t \). The adiabatic theorem in the presence of a gap ensures that \( U_A(\phi(s)) \) approximates \( U_\epsilon(s) \) as \( \epsilon \to 0 \). And indeed, \[21\] shows that power-law corrections to Kubo’s formula for the averaged Hall conductance vanish to all orders in \( \epsilon \).

In this work, we prove the result in a fully interacting setting without averaging. Just as in \[21\], the fundamental reason of this exactness is to be found in the adiabatic theorem: If the driving is smooth, then the Schrödinger and adiabatic flows are equal to all orders in adiabatic perturbation theory \[22\] \[23\] \[13\] as soon as the driving has stopped. While \[21\] is geometric in nature, the present interacting result uses the many-body index and relies on locality arguments to leverage on the adiabatic theorem. Accordingly, Kato’s propagator \( U_A \) is replaced with Hastings’ local propagator \( U_\parallel \) introduced in \[24\] \[25\] \[26\] whose action coincides with Kato’s on the ground state space. The corresponding charge transport is denoted \( \Delta Q_\parallel \).

Let us finally comment on the volume dependence of our results. We shall work here in an arbitrary large but finite volume, with errors vanishing faster than any inverse power of the diameter \( L \) of the system. Concretely, that means on the one hand that we have \( \Delta Q_\parallel = 2\pi \sigma_H + O(L^{-\infty}) \) (and it is an integer up to \( O(L^{-\infty}) \)). On the other hand, if \( \Delta Q_\epsilon \) is the charge transported by the physical \( U_\epsilon = U_\epsilon(2\pi) \) across the fiducial line, our main result reads

\[
\Delta Q_\epsilon = 2\pi \sigma_H + O(\epsilon^\infty) + O(L^{-\infty}),
\]

see Theorem \[4.4\]. Note that \( O(\epsilon^\infty) \) is uniform in \( L \). From there, it is a separate question whether \( \sigma_H \) has a well-defined limit as \( L \to \infty \). As argued in \[7\], this can also be answered positively.

2. The Laughlin pump as a many-body index

The Laughlin argument is traditionally exposed in a cylindrical setting with boundaries connected to infinite reservoirs. We shall work in a periodic setting, by glueing the ends of the cylinder into a 2-dimensional torus. Furthermore, we consider a quantum lattice system defined on the torus and work in a large but finite volume. For clarity of the presentation, we denote \( \Gamma \) the set of vertices of the system and \( L \) the diameter of \( \Gamma \), expressed by the graph distance on \( \Gamma \). In this expository section, we ignore errors that vanish fast as \( L \to \infty \).

In this setting, the quantum Hall effect has a natural interpretation as a charge pump and we are interested in the charge transported across a fiducial line \( \nu \) winding across the torus. The physical source of the pumping is a slowly increasing magnetic flux threading the system, see Figure \[1\].

By charge, we mean here that there is a family of operators \( q(Z) \) labelled by subsets of \( \Gamma \) that have diameter smaller than a fixed value \( R_q \); Typically, \( R_q = 1 \). Crucially, these operators have integer spectrum and mutually
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\[ \phi(s) \]

\[ E \]

\[ J_H \]

\[ T - \eta - \eta + B \nu - \]

Figure 1. The Laughlin pump: A slowly varying magnetic flux \( \phi(s) \) threading the torus induces an electromotive force \( E \) along the surface and, in the presence of a magnetic field \( B \) piercing the surface, a Hall current \( J_H \) perpendicular to it. The expectation value of the charge transport \( T^- \) across the fiducial line \( \nu_- \) as \( \phi \) increases by one flux quantum equals the Hall conductance.

The charge in any set \( X \subset \Gamma \) is \( Q_X = \sum_{Z \subset X} q(Z) \) and it has integer spectrum.

The system’s dynamics is described by a Hamiltonian \( H = \sum_{Z \subset \Gamma} \Phi(Z) \), where the interactions \( \Phi(Z) = \Phi(Z)^\ast \) are uniformly bounded and finite range, namely \( \Phi(Z) = 0 \) whenever the diameter of \( Z \) is larger than \( R_\Phi \). The dynamics generated by \( H \) conserves charge in the sense that \([H, Q_Z] \) is an operator supported along the boundary of \( Z \). It follows that if \( Q_\eta \) denotes the charge in the half system with one boundary \( \eta^- \) (the other one being at the ‘other end of the universe’ \( \eta^+ \)), then the gauge-transformed \( \tilde{H}_\phi = e^{i\phi Q_\eta} H e^{-i\phi Q_\eta} \) differs from \( H \) only along \( \eta_{\pm} \) for all \( \phi \in [0, 2\pi] \). Finally, we consider the family \( H_\phi \) obtained by gauge transforming only those interaction terms in the vicinity of \( \eta^- \), see Section 6.1 in the appendix for details. This is no longer a unitarily equivalent family and \( H_\phi \) corresponds to having a flux \( \phi \) threaded in the torus, see again Figure 1. Integrality of the spectrum of \( Q_\eta \) implies that \( H_{2\pi} = H_0 = H \).

Let \( P_\phi \) be the spectral projection of \( H_\phi \) for the lowest eigenvalue. The result of [24, 25], refined in [26], is the existence of a \( K_\phi \) supported only in a neighbourhood of the line \( \eta_- \), which implements parallel transport on \( P_\phi \). Indeed, the propagator \( U_\parallel(\phi) \) defined by

\[ \partial_\phi U_\parallel(\phi) = iK_\phi U_\parallel(\phi), \quad U_\parallel(0) = \mathbb{1}, \]

satisfies

\[ P_\phi = U_\parallel(\phi) P U_\parallel(\phi)^\ast \]

for all \( \phi \in [0, 2\pi] \). Here we denoted \( P = P_0 \). In particular,

(a) Since the family of Hamiltonians is periodic, \( P = P_{2\pi} \) and so

\[ [U_\parallel, P] = 0, \]

where \( U_\parallel = U_\parallel(2\pi) \).
(b) Since \( K_\phi \) is supported in a neighbourhood of \( \eta_- \), the unitary \( U_\parallel(\phi) \) acts non-trivially only in that same neighbourhood. See again appendix 6.1.

A crucial observation is that in the quantum Hall effect, the conductance, which is a linear response coefficient, is of geometric nature. Concretely, the Kubo formula of linear response is equal to the adiabatic curvature, as was first observed in [4]. In fact, it can be further related directly to the parallel transport corresponding to the addition of a flux quantum. Let \( Q \) be the charge on the ‘orthogonal’ half torus having \( \nu_- \) as one of its boundaries, see Figure 1. The operator of charge transported by \( U_\parallel \) (over a full cycle increasing the flux by \( 2\pi \)) denoted \( T_\parallel = U_\parallel^*QU_\parallel - Q \) is a sum of two contributions \( T_\parallel,\pm \) supported along \( \nu_\pm \). Global charge conservation implies that its expectation value in the invariant state \( P \) vanishes, but in general, this is only due to a cancellation of the influx of charge at one boundary and the outflux at the other one. Focussing on just one of them, it is proved in [9] that the expected charge transport is equal to the Hall conductance

\[
\Delta Q_\parallel = \text{Tr}(P T_\parallel,-) = 2\pi\sigma_H, \tag{2.4}
\]
a fact at the heart of the original Laughlin argument.

3. Charge transport and the adiabatic theorem

The propagator \( U_\parallel \), featuring in Laughlin’s argument, is an approximation of the ‘true’ adiabatic evolution of the system. By this, we mean the solution of the driven Schrödinger equation for the time dependent family of Hamiltonians \( H_\phi(t) \) for a slowly varying flux \( \phi(t) \). In rescaled time \( s = \epsilon t \in [0,1] \), the Schrödinger propagator is the solution of

\[
\begin{align*}
\text{i}\epsilon \partial_s U_\epsilon(s) &= H_\phi(s) U_\epsilon(s), \quad U_\epsilon(0) = \mathbb{1},
\end{align*}
\tag{3.1}
\]

and the adiabatic regime is characterized by \( 0 < \epsilon \ll 1 \).

The adiabatic theorem for gapped many-body systems [13, 14] now goes as follows: For any local observable \( A \),

\[
|\text{Tr}(PU_\epsilon^*(s)AU_\epsilon(s)) - \text{Tr}(PU_\parallel^*(s)AU_\parallel(s))| \leq C\|A\|\text{supp}(A)|^2\epsilon,
\]

where \( U_\parallel(s) = U_\parallel(\phi(s)) \) and \( C \) is independent of the volume of the system. Furthermore, if the driving is smooth and has stopped at \( s = 1 \) (namely \( \phi = 2\pi \)), that is \( \partial_s H_\epsilon \) is compactly supported in \( (0,1) \), then the error is in fact beyond perturbation theory in the sense that

\[
|\text{Tr}(PU_\epsilon^*AU_\epsilon) - \text{Tr}(PU_\parallel^*AU_\parallel)| \leq C_\epsilon_m\|A\|\text{supp}(A)|^2\epsilon^m \tag{3.2}
\]

for all \( m \in \mathbb{N} \).

As in the previous section, we consider the operator of charge transported by \( U_\epsilon \), denoted \( T_\epsilon = U_\epsilon^*QU_\epsilon - Q \) and its contribution \( T_\epsilon,- \) across \( \nu_- \). While (3.2) make the following quite plausible, it is a non-trivial result (as we shall see) that \( \Delta Q_\epsilon = \Delta Q_\parallel + \mathcal{O}(\epsilon^\infty) \). Combined with (2.4), we therefore obtain that

\[
\Delta Q_\epsilon = \sigma_H + \mathcal{O}(\epsilon^\infty). \tag{3.3}
\]
In physical terms, the driving is in this setting the electromotive force $E$ generated by the time dependent flux through Faraday’s law:

$$E = -\partial_t \phi = -\epsilon \phi'(s)$$

Hence, (3.3) can be written as $\Delta Q_\epsilon = \sigma_H + O(|E|^{\infty})$, expressing the exactness of linear response (namely, Ohm’s law) to all orders for the quantum Hall effect.

Let us briefly comment on some technicalities to come. The first class of difficulties arise from the fact that the charge operators $Q, Q_\eta$ have both norms and supports that grow with $L$. In view of the error term in (3.2), one may fear that the bound in (3.3) cannot be uniform in the volume. A careful observation will however show that this is not the case, since the effective transport is limited by charge conservation to the vicinity of $\nu_-$, while the current is driven only along $\eta_-$ where the Hamiltonian itself changes (through the step in the gauge potential). It follows that $\sigma_H$ and the operators of charge transport are localized in an $L$-independent region around their intersection. The second difficulty is that while the Schrödinger and parallel transport flows agree on the ground state space to all orders in $\epsilon$ at $\phi = 2\pi$, the charge transport itself happens throughout the full driving, along which the error is only of order $\epsilon$. We bypass this issue by comparing $U_\epsilon(s)PU_\epsilon(s)^*$ only with a dressed ground state projection, both remaining $O(\epsilon^\infty)$-close to each other for all $s$, while the dressed projection merges with the instantaneous ground state projection when the driving stops.

Finally, we compare the present work with [21]. The result itself differs in two respects. Firstly, the error bound we obtain is uniform in the volume, while that in [21] diverges in the number of particles, although that fact is not explicit for example in Theorem A4 of [21]. Secondly, the conductance there is defined through an average over the flux torus, just as in the original work [4]. The methods differ significantly, too. As already pointed out, in order to obtain volume independent bounds, we use a local parallel transport instead of the traditional one of Kato. We further bypass the geometric argument, in particular the Chern-Simons formula and the need for averaging, by using the many-body index of [10].

4. Equality of charge transports

4.1. Spatial setup. We consider a quantum lattice system defined on a large but finite two-dimensional torus. Let $L \in \mathbb{N}$ be even and let $\Gamma = \mathbb{Z}_L^2$, where $\mathbb{Z}_L = \mathbb{Z}/(L\mathbb{Z})$ is identified with $\{-L/2 + 1, \ldots, L/2 - 1, L/2\}$. Note that $|\Gamma| = L^2$ and that $\text{diam}(\Gamma) = L$ in the graph distance. We denote by $\eta$ the ‘horizontal’ strip $\{(x_1, x_2) \in \Gamma : 0 \leq x_2 < L/2\}$ with boundary $\eta_- := \{(x_1, x_2) \in \Gamma : x_2 = 0\}$ and $\eta_+ := \{(x_1, x_2) \in \Gamma : x_2 = L/2 - 1\}$. We similarly denote the ‘vertical’ strip $\nu := \{(x_1, x_2) \in \Gamma : 0 \leq x_1 < L/2\}$ and its boundaries $\nu_- := \{(x_1, x_2) \in \Gamma : x_1 = 0\}$ and $\nu_+ := \{(x_1, x_2) \in \Gamma : x_1 = L/2 - 1\}$.

Let $A$ be the even subalgebra of the CAR algebra generated by $\{1, a_x, a^*_x : x \in \Gamma\}$, which we can think of as acting on the antisymmetric Fock space

$$\mathcal{H} = \mathcal{F}_a(l^2(\Gamma)).$$
An observable $O \in \mathcal{A}$ is said to be supported in $\Lambda \subset \Gamma$ if $O$ can be expressed as an even polynomial in $\{1, a_x, a_x^* : x \in \Lambda\}$, and we denote by $\text{supp}(O)$ the smallest set on which $O$ is supported. Crucially, if $\text{supp}(O_X) \subset X$ and $\text{supp}(O_Y) \subset Y$ and $X, Y \subset \Gamma$ are disjoint, then $[O_X, O_Y] = 0$. Note that for all of what follows, $\mathcal{A}$ could equivalently be taken to be the matrix algebra of a finite quantum spins system defined on $\Gamma$.

For our current purposes, it will be sufficient to consider a notion of support that is weaker than the above. An operator $O$ is almost supported in a set $X$ if there is a sequence $O_r, r \in \mathbb{N}$ with $\text{supp}(O_r) \subset X(r)$ such that

$$\|O - O_r\| = \|O\||X|O(r^{-\infty}), \quad (4.1)$$

where $X(r)$ denotes the $r$-fattening of $X$, namely

$$X(r) := \{x \in \Gamma : \text{dist}(x, X) \leq r\}.$$  

We point out that the constants implicit in the notation $O(r^{-\infty})$ of (4.1) do not depend on the observable $O$. Similarly, here and below, the notation $\mathcal{O}(\cdot)$ is meant to express a bound that is uniform in the size $L$ of the system, see the appendix of [10] for details. The set of operators that are almost localized in $Z$ is denoted by $\mathcal{A}_Z$. With this, (4.1) implies that

$$\|\mathcal{O}(X, O_Y)\| \leq \|O_X\||O_Y\||X|Y|\mathcal{O}(d(X, Y)^{-\infty})$$

whenever $O_X \in \mathcal{A}_X$ and $O_Y \in \mathcal{A}_Y$.

### 4.2. Extensive observables and the Lieb-Robinson bound.

An extensive observable

$$S = \sum_{X \subset \Gamma} \Psi(X) \quad (4.2)$$

is a sum of local terms $\Psi(X) = \Psi(X)^* \in \mathcal{A}$ with $\text{supp}(\Psi(X)) = X$ that satisfy

(i) finite range condition: There is $R_\Psi < \infty$ such that $\Psi(X) = 0$ if $\text{diam}(X) > R_\Psi$,

(ii) finite interaction strength: There is $m_\Psi < \infty$ such that $\|\Psi(X)\| \leq m_\Psi$ for all $X$.

A few remarks. First of all, the above constants are understood to be independent of the system size $L$. Second of all, while the decomposition (4.2) is not unique, we consider it to be fixed for any extensive observable. Thus, there is a natural restriction to a subset $Z \subset \Gamma$ given by $S_Z = \sum_{X \subset Z} \Psi(X)$. Finally (i,ii) imply immediately that $\|S_Z\| \leq Cm_\Psi(R_\Psi^2|Z|$ in the present two-dimensional setting.

The Lieb-Robinson bound for a dynamics $\tau_t(O_X) = e^{itS}O_Xe^{-itS}$ on $\Gamma$ generated by an extensive observable $S$ implies that if $\text{supp}(O_X) = X$, then

$$\|\tau_t(O_X) - \mathbb{E}_{X(t+t\xi)}(\tau_t(O_X))\| \leq C\|\tau_t(O_X)\||X(t)|e^{-\xi t},$$

where $\mathbb{E}_{Z}$ is the partial trace over the complement of the set $Z$ and the positive constants $C, v, \xi$ depend on $S$ but not on the observable, see Section 6.2 in the appendix. This is exactly of the form (4.1), namely

$$\tau_t(\mathcal{A}_X) \subset \mathcal{A}_{X(t)} \quad (4.3)$$

for any $t \in [0, \infty)$. 

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The first extensive observable of interest is the charge, which is denoted 

\[ Q_\Gamma = \sum_{X \subset \Gamma} q(X), \]

where the local charge operator \( q(X) \) is zero if \( \text{diam}(X) > R_q \) and satisfy

\[ \text{Spec}(q(X)) \subset \mathbb{Z}, \quad \text{and} \quad [q(X), q(X')] = 0 \quad \text{for all } X, X'. \]

A natural choice in the present fermionic setting is \( q(X) \neq 0 \) if and only if \( X \) is a singleton in which case \( q(\{x\}) = a_x^* a_x \). In a quantum spin system, \( q(X) \) could be e.g. plaquette of vertex operators.

An extensive operator \( S = \sum_{X \subset \Gamma} \Psi(X) \) is called charge conserving if

\[ [\Psi(X), Q_\Gamma] = 0. \]

Since \( [\Psi(X), Q_Y] = 0 \) for disjoint sets \( X, Y \), we see that \( [\Psi(X), Q_Z] = [\Psi(X), Q_\Gamma] = 0 \) whenever \( \text{dist}(X, Z^c) \geq R_q \). Hence, for any \( Z \subset \Gamma \),

\[ [S, Q_Z] = \sum_{X \subset \Gamma: X \cap Z \neq \emptyset, \text{dist}(X, Z^c) < R_q} [\Psi(X), Q_Z] \in \mathcal{A}_\partial Z \quad (4.4) \]

for any charge conserving extensive observable \( S \), a property that will play an important role in the following.

The second extensive observable we introduce is a periodic family of charge conserving Hamiltonians, parametrized by \( s \in [0, 1] \), namely

\[ H_s = \sum_{X \subset \Gamma} \Phi_s(X), \quad \text{such that} \quad H_0 = H_1. \]

The constants appearing in (i,ii) above are assumed to be independent of the parameter \( s \).

By (4.4),

\[ [H_s, Q_\eta] \in \mathcal{A}_{\eta-} + \mathcal{A}_{\eta+}, \quad [H_s, Q_\nu] \in \mathcal{A}_{\nu-} + \mathcal{A}_{\nu+}. \quad (4.5) \]

4.3. Assumptions. We can now state our assumptions for the following results. As in the rest of this section, we keep the setting more general than the specific Hall cylinder described in the introductory sections, where charge is being pumped by an increase of the magnetic flux. Still, the assumptions below, in particular Assumption 4.3, are made with this example in mind.

The charge transported by parallel transport is not, in general, related to a linear response coefficient, which is why the main theorem does not explicitly refer to a conductance.

Let \( P_s \) denote the orthogonal projection onto the ground state space of \( H_s \). Let \( E_0^s, E_1^s \) be the ground state energy and the energy of the first excited state. \( P_s \) is necessarily finite dimensional, but its rank may grow in the system size \( L \). Our first assumption is that this is not the case. The second part below is the crucial gap assumption.

**Assumption 4.1** (Spectral gap). There is a constant \( \gamma > 0 \) and an \( L_0 \in \mathbb{N} \) such that if \( L > L_0 \), then

(i) \( \text{rk}(P_s) = p \) for all \( s \in [0, 1] \) and

(ii) \( E_1^s - E_0^s \geq \gamma \) for all \( s \in [0, 1] \).
It is important to point out that $\gamma$ is independent of both $s, L$. The second assumption is about smoothness of the function $s \mapsto H_s$ and the fact that the driving is compactly supported in $(0, 1)$.

**Assumption 4.2 (Smooth, compactly supported driving).** The matrix-valued function $s \mapsto \Phi_s(X)$ is infinitely often continuously differentiable for any $X \subset \Gamma$ and $\Phi_s^{(k)}(X) = 0$. Together, these derivatives define extensive observables $H_s^{(k)} = \sum_{X \subset \Gamma} \Phi_s^{(k)}(X)$ in the sense of Section 4.2.

As already pointed out, our last assumption reflects the specific geometry of the quantum Hall cylinder depicted in Figure 1.

**Assumption 4.3 (Localized driving).** The driving extends only along the line $\eta_-$, namely $H_s^{(k)} \in A_{\eta_-}$.

In the case of the Hall effect, this means that the gauge potential describing the threaded flux is constant in space and time away from the line $\eta_-$ and its change, which drives the Hall current, is spatially localized along that same line, see the discussion in Section 2.

Concretely, the last two assumptions are satisfied for the family $H_s = H_{\phi(s)}$ introduced in Sections 2 and 6.1 in the appendix, where $\phi \in C^\infty([0, 1]; \mathbb{R})$ with $\phi(0) = 0, \phi(1) = 2\pi$ and $\phi' \geq 0$ is compactly supported in $(0, 1)$. While Assumption 4.1 is believed to hold for (possibly fractional) quantum Hall systems, there is at the moment of writing no explicit microscopic model where this can be proved, except for perturbations of free systems (and therefore integer conductance), see [8, 27, 28]. See also [29] for the solution of Haldane’s conjecture, proving that an effective model of a $1/3$ fractional quantum Hall effect has a gap.

### 4.4. Conductance, parallel transport and adiabatic evolution.

Let us first recall the two main players transporting charge. On the one hand, $U_{\epsilon}(s)$ is the Schrödinger propagator for the slowly driven system with Hamiltonian $\epsilon^{-1}H_s$, see (3.1). On the other hand $U_{\parallel}(s)$ implements parallel transport of $P_{\phi(s)}$, namely $P_{\phi(s)} = U_{\parallel}(s)PU_{\parallel}(s)^*$. It is generated by

$$K_s = \int W(u)e^{iuH_s} \partial_s H_s e^{-iuH_s} du$$

where $|W(t)| = O(|t|^{-\infty})$. To compare with (2.2), $K_s = \phi'(s)K_{\phi(s)}$. See Section 6.1 and 13 for more details.

While $K_s$ arises from the extensive observable $\partial_s H_s$, it does not satisfy the finite range condition (i) in Section 4.2. However, the Lieb-Robinson bound for $e^{-iuH_s}$ and the fast decay of $|W|$ imply that $K_s = \sum_{X \subset \Gamma} k_s(X)$ with

$$\sup_{s \in [0, 1]} \sup_{x, y \in \Gamma} \sum_{X \subset \Gamma, X \ni x, y} \frac{\|k_s(X)\|}{f(d(x, y))} < \infty$$

for a positive, decreasing function $f$ such that $f(r) = O(r^{-\infty})$. We refer to [26] for details.
The subexponential decay of $f$ carries over to the Lieb-Robinson bound and must be weakened to the following: for any $\alpha > 1$ and any $c > 0$,

$$
\tau_t(A_X) \subset A_{X(e^{ct\alpha})},
$$

(4.8)

for all $t \in [0, \infty)$, see Section 6.2 in the appendix. This difference is irrelevant for times of order 1, it is essential at the adiabatic time scale $\epsilon^{-1}$. Since the exact value of $\alpha$ will bear no effect on the final result, we chose $\alpha = 2$ for the rest of this paper.

We shall now denote $Q = Q_\nu$. Charge conservation, namely

$$
[K_s, Q] \in A_{\nu-} + A_{\nu+},
$$

(4.9)

and the Lieb-Robinson bound (4.8) imply that for $U_\parallel = U_\parallel(1)$

$$
U_\parallel^* QU_\parallel - Q = i \int_0^1 U_\parallel(s)^*[K_s, Q]U_\parallel(s)ds = T_{\parallel,-} + T_{\parallel,+}
$$

(4.10)

where $T_{\parallel,-} \in A_{\nu-}$ and $T_{\parallel,+} \in A_{\nu+}$. We immediately note, and shall use it later, that $K_s \in A_{\eta-}$ implies that $T_{\parallel,\pm} \in A_{\nu\pm \cap \eta-}$. The same holds for $U_\epsilon = U_\epsilon(1)$, in the sense that

$$
U_\epsilon^* QU_\epsilon - Q = i \int_0^1 U_\epsilon(s)^*[H_s, Q]U_\epsilon(s)ds = T_{\epsilon,-} + T_{\epsilon,+},
$$

(4.11)

see (4.5). However, since the time evolution runs over a long time $\epsilon^{-1}$, the transport observables $T_{\epsilon,\pm}$ are almost localized in an $(\nu\epsilon^{-1})$-fattening of $\nu\pm$, namely $T_{\epsilon,\pm} \in A_{(\nu\pm)/(\nu\epsilon^{-1})}$, see (4.3).

The main result of this work is now:

**Theorem 4.4.** Let Assumptions (4.1, 4.2, 4.3) hold. If $\epsilon^{-2} < L/2$, then

$$
\text{Tr}(PT_{\parallel,-}) = \text{Tr}(PT_{\epsilon,-}) + \mathcal{O}(\epsilon^\infty) + \mathcal{O}(L^{-\infty}).
$$

As pointed out earlier, the two error terms are uniform in the other parameter, provided $\epsilon^{-2} < L/2$ is satisfied. As is physically most relevant, we prefer to think of $\epsilon$ as an arbitrarily small but fixed parameter and let $L \to \infty$ first.

The theorem expresses in general the equality of two charge transports, independently of the fact that the left hand side is a linear response coefficient. In the more specific case of the Laughlin setting described in Section 2, then the left hand side is, up to a factor $2\pi p$, the Hall conductance

$$
\text{Tr}(PT_{\parallel,-}) = p2\pi \sigma_H + \mathcal{O}(L^{-\infty}),
$$

(4.12)

and it is an integer, see [9]. In that case, the theorem states, as announced, that this linear response coefficient expresses the full charge transport, to all orders in the adiabatic parameter $\epsilon$, and equivalently to all orders in the driving.

5. DRESSED GROUND STATES AND PROOFS

While the adiabatic theorem briefly discussed in the previous sections is the fundamental reason for the validity of the theorem, it will not appear in the proofs below per se. In fact, we shall in the following rather revisit the derivation of the adiabatic theorem presented in [13], with an additional
5.1. Dressing the ground state projection. A key player in the proofs are the dressed ground state projections $\Pi_{n,\epsilon}(s)$, $(n \in \mathbb{N})$. On the one hand, they follow the driven projection $U_\epsilon(s)PU_\epsilon(s)^*$ to order $O(\epsilon^{n-2})$ for all $s \in [0,1]$. On the other hand, they follow the instantaneous (namely, parallel transported) ground state $P_s$ only to order $O(\epsilon)$ for all $s \in [0,1]$, but $\Pi_{n,\epsilon}(1) = P_1 = P$ exactly when the driving has stopped, see Assumption 4.2.

We briefly recall the construction of $\Pi_{n,\epsilon}(s)$, see [13]. There exist extensive observables $\{A_j(s) : j \in \mathbb{N}\}$ such that if $S_{n,\epsilon}(s) = \sum_{j=1}^{n} e^{iA_j(s)}$, then

$$\Pi_{n,\epsilon}(s) = e^{iS_{n,\epsilon}(s)}P_s e^{-iS_{n,\epsilon}(s)}.$$ 

Moreover, $A_j(s)$ are functions of $H_\epsilon$ and its derivatives, all at the same epoch $s$. On the one hand, this implies that $A_j(s)$ are charge conserving. On the other hand, whenever the derivatives vanish, so do the $A_j$, so that $A_j(0) = 0$ and $A_j(1) = 0$. Hence,

$$\Pi_{n,\epsilon}(0) = P = \Pi_{n,\epsilon}(1).$$

There are now two possibilities for comparing the driven $U_\epsilon(s)PU_\epsilon(s)^*$ with the dressed $\Pi_{n,\epsilon}(s)$. Firstly, we shall use

$$W_{n,\epsilon}(s) = U_\epsilon(s)^* e^{iS_{n,\epsilon}(s)} U_{\parallel}(s). \quad (5.1)$$

With Assumption 4.2 and the discussion above, $W_{n,\epsilon}(0) = 1$, while at $s = 1$,

$$W_{n,\epsilon} = U_{\parallel}^* U_{\parallel} \quad (5.2)$$

reduces to a direct comparison of the driven Schrödinger propagator with the implementation of parallel transport. Secondly, there is a unitary propagator $V_{n,\epsilon}(s)$ such that

$$\Pi_{n,\epsilon}(s) = V_{n,\epsilon}(s) P V_{n,\epsilon}(s)^*.$$ 

To emphasize the difference with $e^{iS_{n,\epsilon}(s)}$, we note that the right hand side contains $P$ and not $P_\epsilon$. Importantly, $V_{n,\epsilon}(s)$ is obtained as the solution of

$$i\epsilon \partial_s V_{n,\epsilon}(s) = (H_\epsilon + R_{n,\epsilon}(s)) V_{n,\epsilon}(s), \quad V_{n,\epsilon}(0) = 1,$$

for an extensive observable $R_{n,\epsilon}(s)$ which is small in the sense that

$$||R_{n,\epsilon}(s),O|| \leq C ||O|| \text{supp}(O) \epsilon^{n+1}, \quad (5.3)$$

see again [13] for an explicit construction. Moreover, $R_{n,\epsilon}(s)$ is obtained from multicommutators of $H_\epsilon$ and its derivatives $\{H^{(j)}_\epsilon : j = 1,\ldots,n\}$ so that $R_{n,\epsilon}(s) \in \mathcal{A}_{\eta_\epsilon}$ by Assumption 4.3 and it is charge conserving. We define

$$\tilde{W}_{n,\epsilon}(s) = U_\epsilon(s)^* V_{n,\epsilon}(s) \quad (5.4)$$

and note that

$$\tilde{W}_{n,\epsilon}(s) P \tilde{W}_{n,\epsilon}(s)^* = W_{n,\epsilon}(s) PW_{n,\epsilon}(s)^*.$$ 

While $\tilde{W}_{n,\epsilon}(s)$ and $W_{n,\epsilon}(s)$ act equally on the ground state space, they are distinct unitary operators.
5.2. Charge transports. All unitaries introduced so far are functions of the Hamiltonian and its derivatives. Therefore, they are all charge conserving in the sense of (4.10, 4.11). It follows that

\[ W_{n,e}(s)^* Q W_{n,e}(s) - Q = T_-(s) + T_+(s) \]

where

\[
T_-(s) = T_{\|,-}(s) + U_{\|}(s)^* T^S(s) U_{\|}(s) - U_{\|}(s)^* e^{-iS_{n,e}(s)} T_{\|,-}(s) e^{iS_{n,e}(s)} U_{\|}(s)
\]

and \( T^S(s) \) denotes the charge transport operator associated with \( e^{iS_{n,e}(s)} \).

As noted earlier, the first term \( T_{\|,-}(s) \in A_{\nu,-} \). The same applies to \( T^S(s) \) and hence to the second term by the Lieb-Robinson bound \( (4.18) \) for \( U_{\|}(s) \). Finally, \( T_{\|,-}(s) \in A_{(\nu,-)_{|\nu,-|^{-1}}} \). Overall, \( T_-(s) \in A_{(\nu,-)_{|\nu,-|^{-1}}} \), namely the charge transport operator associated with \( W_{n,e}(s) \) is an observable that is extended in a neighbourhood of width of order \( O(\epsilon^{-1}) \) along \( \nu_- \). The minus sign in front of the last term arises from the exchange of \( U_\epsilon(s) \) with \( U_\epsilon(s)^* \) from \( (4.11) \).

The decomposition (5.6) of the charge transport operator and (5.2) imply the following comparison lemma. Recall that if \( s \) is omitted, the functions are evaluated at \( s = 1 \); For the projector, \( P = P_0 = P_1 \).

**Lemma 5.1.** Under the assumptions of Theorem 4.4.

\[
\text{Tr}(PT_-) = \text{Tr}(P T_{\|,-}) - \text{Tr}(P T_{\epsilon,-}).
\]

**Proof.** By Assumption 4.2, \( S_{n,e} = 0 \) and so \( T^S = 0 \). Therefore, (5.6) reduces to

\[
T_- = T_{\|,-} - U_{\|}^* T_{\epsilon,-} U_{\|}.
\]

The assertion now follows by cyclicity of the trace since \( U_{\|} P U_{\|}^* = P \).

With this, Theorem 4.4 is an immediate consequence of the following proposition.

**Proposition 5.2.** Under the assumptions of Theorem 4.4.

\[
\text{Tr}(P T_-) = O(\epsilon^\infty) + O(L^{-\infty}).
\]

5.3. Proof of Proposition 5.2. In this section, we shall not repeat the running assumptions of Theorem 4.4. Unless otherwise specified, our notation \((\cdot \cdot \cdot)_-\) takes into account a fattening of order \( O(\epsilon^{-2}) \), namely the observable belongs to \( A_{(\nu,-)_{(\epsilon^{-2}/3}}} \) or \( A_{(\eta,-)_{(\epsilon^{-2}/3)}} \). The assumption \( \epsilon^{-2} < L/2 \) ensure that the two strips are spatially separated by a distance that is proportional to \( L \).

The first lemma provides a bound on the charge transport operator associated with the auxiliary unitary \( \tilde{W}_{n,A}(s)^* \). It relies on the fact that the difference between the dressed unitary \( V_{n,e}(s) \) and the Schrödinger propagator \( U_\epsilon(s) \) is small in norm, which is also at the heart of the proof of the adiabatic theorem. This fact is also at the heart of the proof of the adiabatic theorem. Here, we shall moreover use the fact that the driving is localized along \( \eta_- \), Assumption 4.3.
Lemma 5.3. For all $s \in [0, 1]$ and $n \in \mathbb{N}$, $n > 8$, 
\[
\|(\tilde{W}_{n, \epsilon}(s)Q\tilde{W}_{n, \epsilon}(s)^{*} - Q)_{-}\| = \mathcal{O}(\epsilon^{n-8}) .
\]

Proof. By its definition (5.4), the unitary $\tilde{W}_{n, \epsilon}(s)$ is the unique solution of the following initial value problem 
\[
\text{ic} \partial_{s} \tilde{W}_{n, \epsilon}(s) = U_{\epsilon}(s)^{*}R_{n, \epsilon}(s)U_{\epsilon}(s)\tilde{W}_{n, \epsilon}(s) , \quad \tilde{W}_{n, \epsilon}(0) = I . \tag{5.8}
\]
Hence, 
\[
\tilde{W}_{n, \epsilon}(s)Q\tilde{W}_{n, \epsilon}(s)^{*} - Q = -\frac{i}{\epsilon} \int_{0}^{s} [U_{\epsilon}(r)^{*}R_{n, \epsilon}(r)U_{\epsilon}(r), \tilde{W}_{n, \epsilon}(r)Q\tilde{W}_{n, \epsilon}(r)^{*}]dr
\]
\[
= -\frac{i}{\epsilon} \int_{0}^{s} \tilde{W}_{n, \epsilon}(r)[\tilde{R}_{n, \epsilon}(r), Q]\tilde{W}_{n, \epsilon}(r)^{*}dr \tag{5.9}
\]
where $\tilde{R}_{n, \epsilon}(r) = V_{n, \epsilon}(r)^{*}R_{n, \epsilon}(r)V_{n, \epsilon}(r)$. As already pointed out, $R_{n, \epsilon}(r)$ is an extensive observable conserving charge and $R_{n, \epsilon}(r) \in \mathcal{A}_{\nu}$ by Assumption 4.3. It further follows from the Lieb-Robinson bound (4.8) for $V_{n, \epsilon}(r)$ that $R_{n, \epsilon}(r)$ is again an extensive observable, but $\tilde{R}_{n, \epsilon}(r) \in \mathcal{A}_{(\eta_{-})_{(n-2)}}$. Since it is moreover charge conserving, $[\tilde{R}_{n, \epsilon}(r), Q]$ is a sum of two contributions localized along $\nu_{\pm}$ as in (4.9) and hence, 
\[
([\tilde{R}_{n, \epsilon}(r), Q]_{-})_{\mathcal{A}_{(\eta_{-})_{(n-2)}}} .
\]
Since $|\langle \eta_{-} \cap \nu_{-} \rangle_{(n-2)}| \leq C\epsilon^{-4}$, it follows from (5.9) and (5.3) that 
\[
\|(\tilde{W}_{n, \epsilon}(s)Q\tilde{W}_{n, \epsilon}(s)^{*} - Q)_{-}\| \leq \epsilon^{-1} \sup_{r \in [0, 1]} \|[[\tilde{R}_{n, \epsilon}(r), Q]_{-}\| \leq C\epsilon^{n-8}
\]
uniformly in $L$.

We note that the fundamental reason underlying the validity of the lemma is that both unitaries $U_{\epsilon}(s), V_{n, \epsilon}(s)$ being compared within $\tilde{W}_{n, \epsilon}(s)$ correspond to propagators over an adiabatic time $\epsilon^{-1}$, see (5.8). This would not be the case for the a priori more natural $W_{n, \epsilon}(s)$ which mixes the two time scales 1 and $\epsilon^{-1}$.

With this in hand, we turn to a key step of the argument, despite its simplicity. It allows to bypass the geometric picture of [21], replacing it with the many-body index 
\[
\text{Ind}_{P}(U) = \text{Tr}(P(U^{*}QU - Q)_{-})
\]
of [10].

Lemma 5.4. $\text{Ind}_{P}(\tilde{W}_{n, \epsilon}(s)^{*}W_{n, \epsilon}(s)) = \mathcal{O}(L^{-\infty})$ for all $s \in [0, 1]$.

Proof. We first show that $\text{Ind}_{P}(\tilde{W}_{n, \epsilon}(s)^{*}W_{n, \epsilon}(s))$ is well-defined for any $s \in [0, 1]$. The unitary $\tilde{W}_{n, \epsilon}(s)^{*}W_{n, \epsilon}(s)$ conserves charge. For $\epsilon^{-2} < L/2$, the strips $(\nu_{\pm})_{(\epsilon^{-2}/3)}$ remain $\mathcal{O}(L)$ apart, so that locality arguments involving $\tilde{W}_{n, \epsilon}(s)^{*}W_{n, \epsilon}(s)$ continue to hold. The observation (5.5) can be written as 
\[
[\tilde{W}_{n, \epsilon}(s)^{*}W_{n, \epsilon}(s), P] = 0
\]
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exactly, namely without any error in either \( \epsilon \) or \( L^{-1} \). Hence all assumptions of the index theorem in [10] hold, yielding an index associated with \( P \) and \( \tilde{W}_{n,\epsilon}(s)^*W_{n,\epsilon}(s) \).

The map \( s \mapsto \tilde{W}_{n,\epsilon}(s)^*W_{n,\epsilon}(s) \) being differentiable, it is a fortiori continuous. It follows that \( s \mapsto \text{Ind}_P(\tilde{W}_{n,\epsilon}(s)^*W_{n,\epsilon}(s)) \) is constant up to \( O(L^{-\infty}) \), see [9] Proposition 2.2. The statement then follows from \( W_{n,\epsilon}(0)^*W_{n,\epsilon}(0) = 1 \) and \( \text{Ind}_P(1) = 0 \). □

We are now equipped to finish the proof of Proposition 5.2.

Proof of Proposition 5.2. Recalling the definition (5.6) of the charge transport operator \( T_-(s) = (W_{n,\epsilon}(s)^*QW_{n,\epsilon}(s) - Q)_- \), we have by the Lieb-Robinson bound (4.8) for \( W_{n,\epsilon}(s) \) that

\[
\text{Ind}_P(\tilde{W}_{n,\epsilon}(s)^*W_{n,\epsilon}(s)) = \text{Tr}(PT_-(s)) + \text{Tr}(PW_{n,\epsilon}(s)^*(\tilde{W}_{n,\epsilon}(s)Q\tilde{W}_{n,\epsilon}(s)^* - Q)_-W_{n,\epsilon}(s)).
\]

Moreover,

\[
|\text{Tr}(PW_{n,\epsilon}(s)^*(\tilde{W}_{n,\epsilon}(s)Q\tilde{W}_{n,\epsilon}(s)^* - Q)_-W_{n,\epsilon}(s))| \leq p\left\| (\tilde{W}_{n,\epsilon}(s)Q\tilde{W}_{n,\epsilon}(s)^* - Q)_- \right\|
\]

which is of order \( O(\epsilon^n) \) by Lemma 5.3. Plugging this in (5.10) at \( s = 1 \) and using Lemma 5.4, we conclude that

\[
\text{Tr}(PT_-) = O(L^{-\infty}) + O(\epsilon^{n-8})
\]

for all \( n \in \mathbb{N} \), which is the claim we had set out to prove. □

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6. Appendix

6.1. On parallel transport. The reader may have noticed that the formula (4.12) expressing the equality of conductance with an index appears in fact in a different fashion in the cited [9]. The proposition we wish to prove in this appendix shows that they are indeed the same.

In order to avoid confusion, we insist that all indices \((\cdot)_-\) appearing in the text to refer to the boundary of \( \eta \), not of \( \nu \). Consequently, we define the ‘twist Hamiltonian’ \( \tilde{H}_\phi \) which arises from \( H \) by a gauge transformation. By contrast, the ‘twist Hamiltonian’ \( H_{\phi} = \sum_{X \subset \Gamma} \Phi_\phi(X) \) is obtained from \( H = \sum_{X \subset \Gamma} \Phi(X) \) by defining

\[
\Phi_\phi(X) = \begin{cases} e^{i\phi Q_\eta}\Phi(X)e^{-i\phi Q_\eta} & \text{if } X \cap \eta_- \neq \emptyset \text{ and } X \cap \eta_-^c \neq \emptyset \\ \Phi(X) & \text{otherwise} \end{cases}
\]
It follows that $H_\phi$ and $H$ differ from each other only along $\eta_-$, while $\tilde{H}_\phi$ and $H_\phi$ differ from each other only along $\eta_+$.

**Proposition 6.1.** Let $U_\parallel$ be the solution of

$$
\partial_s U_\parallel(s) = iK_s U_\parallel(s), \quad U_\parallel(0) = \mathbb{1},
$$

at $s = 1$. Here, $K_s$ is given by (4.6) with the concrete Hamiltonian $H_{\phi(s)}$.

Let

$$
U = e^{2\pi i(\tilde{K}_- - Q_\eta)}
$$

where

$$
\tilde{K}_- = \int W(u)e^{iu\tilde{H}_\phi(\partial_\phi \tilde{H}_\phi)}e^{-iu\tilde{H}_\phi}du \bigg|_{\phi=0}
$$

Then $U_\parallel = U + \mathcal{O}(L^{-\infty})$. In particular, (4.13) holds.

**Proof.** By construction,

$$
\partial_\phi H_\phi = (i[Q_\eta, \tilde{H}_\phi])_- = (\partial_\phi \tilde{H}_\phi)_-.
$$

Hence $\partial_s H_s = (\partial_\phi \tilde{H}_s)_-$ so that

$$
K_s = \int W(u)e^{iu\tilde{H}_s(\partial_\phi \tilde{H}_s)}e^{-iuH_s}du
$$

is a gauge covariant family, see (2.1), hence

$$
(\tilde{K}_\phi)_- = e^{i\phi Q_\eta} \tilde{K}_- e^{-i\phi Q_\eta}
$$

is supported along $\eta_-$ as well.

This shows that $\partial_\phi(s)(\tilde{K}_\phi(s))_-$ generates the propagator $e^{i\phi(s)Q_\eta}e^{i\phi(s)(\tilde{K}_- - Q_\eta)}$.

We conclude by (6.1) and the uniqueness of the solution of ODEs that

$$
U_\parallel(s) = e^{i\phi(s)Q_\eta}e^{i\phi(s)(\tilde{K}_- - Q_\eta)} + \mathcal{O}(L^{-\infty})
$$

for all $s \in [0, 1]$. In particular,

$$
U_\parallel(1) = e^{2\pi i(\tilde{K}_- - Q_\eta)} + \mathcal{O}(L^{-\infty}) \quad (6.2)
$$

by integrability of the spectrum of charge.

Theorem 3.2 of [9] proves the identity (4.12) with $T_-$ associated with $U = e^{2\pi i(\tilde{K}_- - Q_\eta)}$ instead of $U_\parallel$, hence (6.2) concludes the proof of (4.12). \qed

**6.2. Lieb-Robinson bounds.** The Lieb-Robinson bound for a local dynamics $\tau_\xi$ on $\Gamma$ generated by an extensive observable satisfying the decay condition (4.7) implies that for $\text{supp}(O_X) = X$,

$$
\|\tau_\xi(O_X) - E_{X(\tau)}(\tau_\xi(O_X))\| \leq \int_{U(X(\tau))} \||\tau_\xi(O_X), U\||\mu(U)
$$

$$\leq \||\tau_\xi(O_X)||X|e^{\xi f(r)}
$$

see [30], where $E_Z : \mathcal{A} \to \mathcal{A}_Z$ denotes the normalized partial trace over $Z^c = \Gamma \setminus Z$, and $U(Z^c)$ is the unitary group in $\mathcal{A}_Z$ equipped with its Haar
measure \( \mu \). The function \( f \) and the constant \( \xi > 0 \) depend on the generator of \( \tau \) but neither on \( \Gamma \) nor on the observable \( O_X \). As already noted, \( f \) is a positive, decreasing function decaying faster than any inverse power.

Let us first consider the case \( f(r) = C e^{-\xi r} \), which happens for a dynamics generated by an extensive observable satisfying the finite range condition (i) of Section 4.2. By picking \( r = \frac{\xi}{\zeta} t + s \), we obtain

\[
\| \tau_t(O_X) - E_{X(t)}(\tau_t(O_X)) \| \leq C \| \tau_t(O_X) \| X_{(ct)} e^{-\zeta s}
\]

where \( v = \frac{\xi}{\zeta} \) and the constant \( C \) is uniform in \( t \). This is (4.1) for \( \tau_t(O_X) \in \mathcal{A}_{X_{(ct)}} \), namely (4.3) for all \( t \in [0, \infty) \). Note that we have replaced \( |X| \) by the larger \( |X_{(ct)}| \) to match the claimed support.

In the present context, the generator \( K \) as well as the \( A_j \)’s and hence also \( R_{n,t} \) all have slower decay, expressed concretely as \( D_n(r) = r^k e^{-C r / \ln(\ln(r))} \) for some \( k = k(n) \), see [12]. No affine choice \( r(t) \) as above will be such that \( e^{ct} D(r(t)) \) is uniformly bounded on \( [0, \infty) \). In order to deal with that in a rather explicit fashion, we note that \( D_n(r) \leq C_n \zeta^n e^{-C n \zeta} \) for any \( 0 < \beta < 1 \) and any \( \zeta > 0 \). In this case, we pick

\[
r = \frac{1}{2} \left( \frac{2 \xi t}{\zeta} \right)^{1/\beta} + s.
\]

By midpoint concavity,

\[
\zeta r^{\beta} \geq \xi t + 2^{\beta-1} \zeta s^{\beta}
\]

and hence

\[
e^{ct} D_n(r) \leq C e^{ct} e^{-C n \zeta} \leq C e^{-2^{\beta-1} \zeta s^{\beta}} = O(s^{-\infty}).
\]

We conclude that, for a dynamics \( \sigma_t \) generated by an extensive observable that satisfies (4.7) with only a subexponential \( f \), we have

\[
\| \sigma_t(O_X) - E_{X_{(ct)}}(\sigma_t(O_X)) \| \leq C \| \sigma_t(O_X) \| X_{(ct^{1/\beta})} e^{-2^{\beta-1} \zeta s^{\beta}}
\]

where \( c = \frac{\xi}{2^{\beta-1} \zeta} \) and the constant \( C \) is uniform in \( t \). This is (4.4) for \( \sigma_t(O_X) \in \mathcal{A}_{X_{(ct^{1/\beta})}} \), namely (4.8) with \( \alpha = 1/\beta \), for all \( t \in [0, \infty) \).

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