Electromagnetic beam modulation through transformation optical structures

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Abstract. The transformation media concept based on the form-invariant Maxwell’s equations under coordinate transformations has opened up new possibilities of manipulating electromagnetic (EM) fields. In this paper, we report on application of the finite-embedded coordinate transformation method to the design of EM beam-modulating devices both in Cartesian coordinates and in cylindrical coordinates. By designing the material constitutive tensors of the transformation optical structures through different kinds of coordinate transformations, either the beamwidth of an incident Gaussian plane wave could be modulated by a slab or the wave-propagating direction of an omnidirectional source could be modulated through a cylindrical shell. We present the design procedures and the full-wave EM simulations that clearly confirm the good performance of the proposed beam-modulating devices.

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1. Introduction

Recent progress in the research area of metamaterials has shown a great expansion of the applications of electromagnetic (EM) wave propagation phenomena such as negative refraction [1], perfect imaging [2], image magnification through hyper-lenses [3]–[5] and EM invisibility cloaking [6]–[16], since the EM parameters can now be arbitrarily designed as desired. These extraordinary properties are directly determined by the media parameters, permittivity and permeability. From the point of view of EM device design, a suitable method for regulating the material parameters to obtain the desired device properties is required. Recently, based on the form-invariant form of Maxwell’s equations under certain coordinate transformations, the transformation optics proposed in [6, 8] for controlling the EM fields has been proved to be an effective approach for manipulating the material properties to satisfy the desired ray traces of EM waves [9]. The successful application of transformation optics to invisibility cloaks has triggered more intensive explorations of this idea both theoretically and experimentally, and many interesting theoretical results and practical approaches have been obtained [9]–[16]. Besides the invisibility cloak, EM wave concentrators, rotators and other interesting devices have also been proposed with exotic EM behaviors by utilizing transformation optics [17,18]. Particularly, Rahm et al have expanded transformation optics by the use of finite-embedded coordinate transformation, which adds more flexibility to the transformation design of complex materials and enables the transfer of field manipulations from the transformation optical structures to the surrounding normal medium. Such a technique has been successfully applied in Cartesian coordinates to design a parallel beam shifter and a beam splitter [19].

In this paper, we report on application of the finite-embedded coordinate transformation method to the design of EM beam-modulating devices both in Cartesian coordinates and in cylindrical coordinates. By designing the material constitutive tensors through different kinds of coordinate transformations, the beamwidth of an incident Gaussian EM wave could be modulated by a metamaterial slab, or the wave-propagating direction of an omnidirectional line source could be modulated through a metamaterial cylindrical shell. The finite-embedded coordinate transformation method enables these transformation optical structures to let the modulated beam remain unchanged when leaving the metamaterial region. The good performance of the beam modulation has been verified through two-dimensional (2D) full-wave numerical simulations based on the finite element method. Such EM wave-manipulating devices provide alternative ways either to expand or to compress the EM beamwidth, or to steer EM radiation to specified directions, which would find applications in optical elements, microwave antenna or other potential EM devices.

2. Beam modulation in Cartesian coordinates

We first consider the application of the finite-embedded coordinate transformation in Cartesian coordinates. Unlike the conventional complex optical system, the beam-modulating device we have proposed here is composed of a slab with finite width made of transformation-designed anisotropic metamaterial in which the EM waves are transformed as required. This functional material is embedded in free space with the size of only several wavelengths. To design the functional metamaterial, we utilize the so-called finite-embedded coordinate transformation method as proposed in [19]. The embedded transformations add more flexibility to the design
of transformation optical structures. This method enables the transfer of EM field manipulations from the transformation optical structure to the surrounding medium. The finite-embedded transform structures can also be reflectionless when certain criteria are met.

For simplicity, we restrict the problems to 2D cases. The mathematical formalism used for the calculation of the permittivity and permeability tensors of the transformation optical structure is similar to the one reported previously in [9, 18, 19]. The transformed space filled with this structure and the original virtual space (usually free space) are related by a possible coordinate transformation that meets the requirements of EM beam modulation. To explain clearly, we denote the transformed space as \( \{x'_i\} \), and the original space as \( \{x_i\} \), where \( i = 1, 2, 3 \) indicates the three Cartesian coordinate axes. Once the mapping from the original space to the transformed space is set, the elements of the Jacobian transformation matrix that relates the two coordinate systems could be calculated as

\[
J_{ij} = \frac{\partial x'_i}{\partial x_j}.
\]

(1)

The associated relative permittivity and permeability tensors in the transformed space could be calculated as

\[
\bar{\varepsilon}' = J^{-1} \bar{\varepsilon} J^T \frac{\det(J)}{\det(J)},
\]

(2)

\[
\bar{\mu}' = J^{-1} \bar{\mu} J^T \frac{\det(J)}{\det(J)},
\]

(3)

where \( \bar{\varepsilon} \) and \( \bar{\mu} \) tensors are those for the medium in the original virtual space and \( \det(J) \) denotes the determinant of the Jacobian matrix, respectively.

Consider a slab of thickness \( a \) along the \( x \)-axis for beamwidth compression and expansion. Figure 1 illustrates the mapping from the original space (here the free space) to the transformed space with the mathematical formalism defined by the following equations:

\[
\begin{align*}
{x' = x}, \\
y' &= y + (\eta - 1) \frac{xy}{a}, \quad (0 < x' < a, \text{ within the slab}) \\
z' &= z,
\end{align*}
\]

(4)

\[
\begin{align*}
{x' = x}, \\
y' &= y, \quad \text{(outside the slab region)} \\
z' &= z,
\end{align*}
\]

(5)

where \( \eta \) is the modulating coefficient denoting how much the EM beam would be modulated in the slab. From the coordinate transformations (4) and (5), the permittivity and permeability tensors of the slab could be obtained as

\[
\bar{\varepsilon}' = \bar{\mu}' = \frac{1}{(1 + (\eta - 1) x/a)} \begin{pmatrix} 1 & (\eta - 1) y/a & 0 \\ (\eta - 1) y/a & ((\eta - 1) y/a)^2 + (1 + (\eta - 1) x/a)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

(6)

As indicated in figure 1, the transformation is only applied in the gray region occupied by the slab, and then the slab filled with the transformed medium is embedded in free space. Although the transformation along the right border of the slab is discontinuous as shown in figure 1, as
discussed in [19] the ray traces will maintain their transformed behavior when exiting the slab instead of being forced back to their original positions. This is the main difference between the embedded coordinate transformation and the conventional coordinate transformation, which excludes the abrupt change along the boundary and objectively renders the transformed medium not inherently invisible to the observer in the free space on the right side of the slab.

To confirm the performance of the proposed EM beamwidth modulators, we have carried out EM full-wave analysis of the designed structures using a numerical simulation based on the finite element method. Figure 2 illustrates the computation region for a 2D full-wave simulation. Perfect matched layers (PML) are chosen as the surrounding for the simulation region. A transverse magnetic (TM) Gaussian plane wave (with magnetic field normal to the $x$–$y$-plane) is incident either perpendicularly or obliquely on the slab from the left with a beamwidth $w_0$ of several wavelengths.

Since most of the energy of a Gaussian beam is restricted to within $w_0$ of the beam center, we can use a slab of finite length instead of an infinite slab to modulate the Gaussian beam, which is more reasonable for practical implementation. In the simulation, the slab is chosen with width $a$ along the $x$-axis and length $h$ along the $y$-axis. To simulate the Gaussian beam modulation by a slab of finite length, $h$ is chosen to be much larger than the beamwidth $w_0$ of the Gaussian beam.
Figure 2. Computation domain and details of the full-wave simulations. A slab with finite dimension (of width $a$ and length $h$, light-blue-colored region) is embedded in the free space, which is surrounded by the PML (gray regions). A Gaussian plane wave is considered to impinge on the slab.

Figure 3. Beam modulation of a perpendicular incidence wave by a slab. The normalized transverse magnetic field distribution, power flow lines (white lines) ((a) and (c)) and normalized power density ((b) and (d)) for beamwidth compression ((a) and (b)) or beamwidth expansion ((c) and (d)).

Figure 3 shows two examples of beamwidth modulation by using the idea of embedded coordinate transformation. Both the TM field and the EM power density distribution are illustrated in figure 3. The EM wave is perpendicularly incident on the slab with a width of $a = 3.33\lambda_0$ and a length of $h = 26.66\lambda_0$ ($\lambda_0$ is the wavelength in free space). In figures 3(a) and (b), the Gaussian beam with a beamwidth of $w_0 = 5\lambda_0$ has been compressed to 50% within the transformed medium designed with a compression coefficient $\eta = 0.5$, while in figures 3(c) and (d), the Gaussian beam with a beamwidth of $w_0 = 3.33\lambda_0$ has been expanded within the transformed medium designed with an expansion coefficient of $\eta = 1.5$. We also notice that the simulation confirms that the EM beam retains the modulated width when exiting the
transferred medium, and at both sides of the slab the beam propagation is reflectionless, which is a consequence of the finite-embedded coordinate transformation method used in the design of the slab.

The transformation optical structures should also be effective for any oblique incident waves, since the design procedure does not depend on the incident angles. This has been confirmed in figure 4, which shows the beam modulation for an oblique incident Gaussian plane wave. The physical sizes of the slab are the same as those discussed in figure 3. Either beam compression ($\eta = 0.75$) or beam expansion ($\eta = 1.25$) has been clearly demonstrated in figures 4(a) and (b) or (c) and (d), respectively.

The reflectionless characteristics of the proposed beam modulators also enable us to cascade them to obtain a large compression or expansion of the EM power density. Figure 5 shows an example of beamwidth compression through two cascaded slabs, resulting in more than four times enhancement of the output EM power density, which could be applicable to some optical elements.

### 3. Beam modulation in cylindrical coordinates

Next we extend the above idea of beam modulation to cylindrical coordinates. The finite-embedded coordinate transformation method is now applied to the cylindrical system. Instead of EM traces modulating in the $y$-direction as discussed in the previous section, we consider beam modulation in the circumferential direction in the cylindrical system. By filling a shell region with transformation-designed functional material surrounding an EM source, such a beam modulation in the circumferential direction could be used to squeeze an omnidirectional cylindrical wave in some special directions, either to increase the radiation directivity of the source or to render the original source confused to the outsider observer.

To modulate the cylindrical wave, a coordinate transformation is required between two cylindrical coordinates, under which the uniformly distributed radial traces in the original cylindrical system could be squeezed in the circumferential direction to redistribute within
Figure 5. The normalized transverse magnetic field distribution (a), power flow lines (white lines) and normalized power density (b) for beamwidth compression by two cascaded slabs.

Figure 6. Coordinate mapping from a cylindrical shell region (gray region on the left) in the original virtual space to a shell region (gray region on the right) in the coordinate transformed space with the outer radius $b$ and the inner radius $a$. The uniformly distributed radial traces within $\pi \leq \theta \leq -\pi$ in the original shell have been squeezed into $\pi/2 \leq \theta' \leq -\pi/2$ within the shell. Certain specified directions in the transformed cylindrical system. Different mappings between the two coordinates would yield different beam modulating effects and would result in different field distributions. Figure 6 shows one of the 2D mappings from a cylindrical shell region (gray region on the left) in the original virtual space to a shell region (gray region on the right) in the coordinate transformed space. Generally, the coordinate transformation formalisms could be further explored to obtain more optimal beam modulating effects.
be described as

\[
\begin{align*}
\rho' &= \rho, \\
\theta' &= F(\rho, \theta, a, b), & (a < \rho < b, \text{within the shell region}) \\
z' &= z,
\end{align*}
\]

\( (7) \)

\[
\begin{align*}
\rho' &= \rho, \\
\theta' &= \theta, & \text{(outside the shell region)} \\
z' &= z,
\end{align*}
\]

\( (8) \)

where \( F(\rho, \theta, a, b) \) is a specified function that determines the transformation of the space in the circumferential direction and it should satisfy \( F(\rho, \theta, a, b)|_{\rho=a} = \theta \) to ensure the continuity of the radial traces at the inner boundary of the shell. However, no limitation is required for the angle coordinate at the outer boundary. Therefore, the medium property for the transformed region could be designed by applying the finite-embedded coordinate transformation method to this region and then embedding in the free space, similar to the previous section.

To demonstrate the design procedure, one simple example of the transformation is given in the following:

\[
\begin{align*}
\rho' &= \rho, \\
\theta' &= a\theta/\rho, & \text{(within the shell region)} \\
z' &= z,
\end{align*}
\]

\( (9) \)

\[
\begin{align*}
\rho' &= \rho, \\
\theta' &= \theta, & \text{(outside the shell region)} \\
z' &= z,
\end{align*}
\]

\( (10) \)

If \( b = 2a \), the above transformation indicates that the uniformly distributed radial traces within \( \pi \leq \theta \leq -\pi \) in the original shell have been squeezed into \( \pi/2 \leq \theta' \leq -\pi/2 \) upon the outer boundary of the transformed shell as shown in figure 6. The material constitutive parameters of the shell in the transformed space are derived through the Jacobian transformation matrix in the cylindrical coordinates

\[
J_{\rho'\rho}(\rho, \theta, z) = \begin{pmatrix}
1 & 0 & 0 \\
-a\theta/\rho^2 & a/\rho & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

\( (11) \)

To represent the material tensors in Cartesian coordinates, we need to include the transformation and the inverse transformation between Cartesian coordinates and their corresponding cylindrical coordinates, which are described as

\[
\begin{align*}
x' &= \rho' \cos \theta', \\
y' &= \rho' \sin \theta', \\
z' &= z', \\
\rho &= \sqrt{x^2 + y^2}, \\
\theta &= \tan^{-1}(y/x),
\end{align*}
\]

\( (12) \)

The permittivity tensor is then obtained through equation (2), and the Jacobian transformation matrix is calculated as

\[
J_{x'x} = J_{x'\rho} J_{\rho\rho} J_{\rho x},
\]

\( (13) \)
where

\[
\mathbf{J}_{x'} = \begin{bmatrix}
\cos \theta' & -\rho' \sin \theta' & 0 \\
\sin \theta' & \rho' \cos \theta' & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

\[
\mathbf{J}_{\rho x} = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta / \rho & \cos \theta / \rho & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(14)
correspond to the Jacobian matrix for the transformations indicated by (12). The resulting permittivity tensor is

\[
\mathbf{\tilde{\varepsilon}'} = \mathbf{J}_{x'x} \mathbf{\tilde{\varepsilon}} \mathbf{J}_{x'}^T / \det(\mathbf{J}_{x'x})
\]

\[
= \frac{1}{a / \rho} \begin{bmatrix}
\cos \theta' & \sin \theta' & 0 \\
\sin \theta' & \cos \theta' & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & -a \theta / \rho & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\cos \theta' & \sin \theta' & 0 \\
0 & -a \theta / \rho & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

(15)

where

\[
\begin{align*}
\rho &= \rho' = \sqrt{x'^2 + y'^2}, \\
\theta &= \theta' / a = \sqrt{x'^2 + y'^2} \tan^{-1}(y' / x') / a, \\
\cos \theta' &= x' / \sqrt{x'^2 + y'^2}, \\
\sin \theta' &= y' / \sqrt{x'^2 + y'^2},
\end{align*}
\]

and \(x'\) and \(y'\) are the physical coordinates; \(a\) is the inner radius.

The permeability tensor is obtained through (3) and is of the same form as the permittivity. A similar discontinuity occurs at the outer boundary of the shell owing to the use of the finite-embedded coordinate transformation method.

To verify the performance of the beam modulation in the circumferential direction, full-wave simulations have been carried out and are displayed in figure 7. A linear line source is located in the origin, which radiates cylindrical waves omnidirectionally in the free space (figures 7(a) and (b)). To modulate the beam, the line source is surrounded by a shell with outer radius \(b = 3.33\lambda_0\) and inner radius \(a = b / 2 = 1.67\lambda_0\), which is filled with the transformed medium designed through (11)–(16). As shown in figures 7(c) and (d), the omnidirectional cylindrical waves radiated from the line source are condensed into the right half-space, leaving the other half-space with almost no radiation because of the transformed medium in the shell. The squeezed beam keeps the bended behavior when exiting the shell into the outer free space. Owing to the non-uniform coordinate transformation formalism of (9), the EM power density is not distributed uniformly in the shell with more power squeezed near the directions of \(\theta' = \pm \pi / 2\). The simulation result clearly demonstrates the beam modulation from omnidirectional to radiation in the half-space as desired from the transformation (9) at \(a = b / 2\).

To demonstrate the arbitrary beam modulation ability by using the transformation-designed shell, three examples have been given in figure 8 showing different modulating effects. The
omnidirectional radiation could be split into double beams (figures 8(a) and (b)), treble beams (figures 8(c) and (d)) or even quadruple beams (figures 8(e) and (f)) radiating in four perpendicular directions. Both the near-field distribution and the power density have clearly confirmed the different designs of modulating beams from omnidirectionality to some special directionality. The transformation optical structures are designed based on geometrical optics and ray tracing; therefore a small discrepancy exists between the design and the simulation because of some undesired dispersion of the EM fields resulting from the diffractions.

4. Conclusions

In this paper, we have successfully applied the finite-embedded coordinate transformation method to design EM beam-modulating devices both in Cartesian coordinates and in cylindrical coordinates. We show two types of examples to describe how to design the transformation optical structures by determining the permittivity and permeability tensors of
Figure 8. Transverse electric field distribution, power flow lines (white lines) ((a), (c) and (e)) and the power density distribution ((b), (d) and (e)) for a double beam splitter ((a) and (b)), a treble beam splitter ((c) and (d)) and a quadruple beam splitter ((e) and (f)) (the black lines denote the boundary of the shell filled with the transformation optical structures).
the structures through different kinds of coordinate transformations. In Cartesian coordinates, the beamwidth of the Gaussian plane waves could be either compressed or expanded by a slab of the transformation optical structure, whereas in cylindrical coordinates, the radiation of an omnidirectional line source could be modulated to specified directions through a cylindrical shell of the transformation optical structures. Both the EM field distributions and the power density flows through 2D full-wave numerical simulations have confirmed the performance of the beam modulation. Such EM wave manipulating devices and the design method not only provide alternative ways either to expand or compress EM beamwidth or to squeeze EM radiation to particular directions, but also find applications in optical elements, the microwave antenna system or other potential EM devices. We believe that by utilizing the transformation optics method, more sophisticated beam modulation devices could be proposed considering that there are unlimited coordinate transformations we could use (linear or nonlinear, transformation for one coordinate or for all three coordinates, or even using different coordinate transformations for different regions).

Acknowledgments

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