Unexpected connections between Burnside Groups and Knot Theory

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Abstract

In Classical Knot Theory and in the new Theory of Quantum Invariants substantial effort was directed toward the search for unknotting moves on links. We solve, in this note, several classical problems concerning unknotting moves. Our approach uses a new concept, Burnside groups of links, which establishes unexpected relationship between Knot Theory and Group Theory. Our method has the potential to be used in computational biology in the analysis of DNA via tangle embedding theory, as developed by D.W.Sumners.

Connections between knot theory and group theory can be traced back to Listing’s pioneering paper of 1847 [Lis], in which he considered knots, and groups of signed permutations. The first, well-established instance of such a connection was provided by W.Wirtinger and M.Dehn [De]. They applied the Poincaré’s fundamental group of a knot exterior to study knots and their symmetries. The connection we describe in this note is, on the one hand, deeply rooted in Poincaré’s tradition and on the other hand it is novel and unexpected. It was discovered in our study of the cubic skein modules of the 3-sphere and it led us to the solution of the twenty year old Montesinos-Nakanishi conjecture.

We outline our main ideas and proofs. The complete exposition of this theory and its applications will be the subject of the sequel paper, [D-P-2].

1 Open problems

Every link can be simplified to a trivial link by crossing changes (\(\begin{array}{c}
\times \\
\times
\end{array}\)). This observation led to many significant developments, in particular, the construction of the Jones polynomial of links and the Reshetikhin-Turaev invariants of 3-manifolds. These invariants had a great impact on modern Knot Theory. The natural generalization of a crossing change, which is addressed in this paper, is a tangle replacement move, that is,
a local modification of a link, \( L \), in which a tangle \( T_1 \) is replaced by a tangle \( T_2 \). Several questions have been asked about which families of tangle moves are unknotting operations.

One such family of moves, which is significant not only in Knot Theory but also in Computational Biology, is the family of rational moves [Sum]. In this note we devote our attention to special classical cases.

(1) Nakanishi 4-move conjecture, 1979.
   Every knot is 4-move equivalent to the trivial knot.

(2) Montesinos-Nakanishi 3-move conjecture, 1981.
   Every link is 3-move equivalent to a trivial link.

(3) Kawauchi’s question, 1985.
   Are link-homotopic links 4-move equivalent?

(4) Harikae-Nakanishi conjecture, 1992.
   Every link is \((2,2)\)-move equivalent to a trivial link.

(5) \((2,3)\)-move question, 1995.
   Is every link \((2,3)\)-move equivalent to a trivial link?

The method of Burnside groups, which we introduce, allows us to answer questions (2), (3), (4) and (5) although Conjecture (1) remains open.

We generalize questions (2), (4) and (5) to the following question:

(6) Rational moves question.
   Is it possible to reduce every link to a trivial link by rational \( \frac{p}{q} \)-moves where \( p \) is a fixed prime and \( q \) is an arbitrary nonzero integer? See Def. 1.1.

To approach our problems we define new invariants of links and call them the Burnside groups of links. These invariants are shown to be unchanged by \( \frac{p}{q} \)-rational moves. The strength of our method lies in the fact that we are able to use the well-developed theory of classical Burnside groups and their associated Lie rings [Mag]. We first describe, in more detail, how our method is applied to rational moves. In particular, we settle the Montesinos-Nakanishi and Harikae-Nakanishi conjectures. Later we answer Kawauchi’s question in detail (Section 3).
Definition 1.1 A rational $\frac{p}{q}$-move\(^1\) refers to changing a link by replacing an identity tangle in it by a rational $\frac{p}{q}$-tangle of Conway (Fig.1.1).

The tangles shown in Figure 1.2 are called rational tangles and denoted by $T(a_1, a_2, ..., a_n)$ in Conway’s notation. A rational tangle is the $\frac{p}{q}$-tangle if $\frac{p}{q} = a_n + \frac{1}{a_{n-1} + ... + a_1}$. Conway proved that two rational tangles are ambient isotopic (with boundary fixed) if and only if their slopes are equal (compare [Kaw]).

Rational tangles can also be viewed as tangles, which are obtained by applying a finite number of consecutive twists of the neighboring endpoints to the elementary tangle [0].

Rational $\frac{p}{q}$-tangles were used by Sumners and Ernst in their mathematical model of DNA recombination [E-S]. This is a very promising development in Computational Biology.

The following definition establishes a connection between two classical theories, Knot

\(^1\)This move was first considered by J.M.Montesinos [Mon].
Theory and the Theory of Burnside Groups\textsuperscript{2}. Burnside groups of a link play a crucial role in our research and they can contribute significantly to the applications of Knot Theory in Computational Biology.

**Definition 1.2** Let $D$ be a diagram of a link $L$. We define the associated core group $\Pi_D^{(2)}$ of $D$ by the following presentation: generators of $\Pi_D^{(2)}$ correspond to arcs of the diagram. Any crossing $v_s$ yields the relation $r_s = y_iy_j^{-1}y_iy_k^{-1}$ where $y_i$ corresponds to the overcrossing and $y_j, y_k$ correspond to the undercrossings at $v_s$ (see Fig.1.3).

**Remark 1.3** In the above presentation of $\Pi_L^{(2)}$ one relation may be dropped since it is a consequence of others.

![Fig. 1.3](image)

**Definition 1.4** (1) The $n$th Burnside group of a link is the quotient of the fundamental group of the double branched cover of $S^3$ with the link as the branch set by its subgroup that is generated by all relations of the form $w^n = 1$. Succinctly: $B_L(n) = \pi_1(M_L^{(2)})/(w^n)$.

(2) The unreduced $n$th Burnside group of the unoriented link $L$ is the quotient group $\hat{B}_L(n) = \Pi_L^{(2)}/(w^n)$, where $\Pi_L^{(2)}$ is the associated core group of $L$.

The relation to the fundamental group of a double branched cover, mentioned before, is formulated below. An elementary proof using only Wirtinger presentation and also valid for tangles is presented in [Pr].

\textsuperscript{2}These groups were first considered by W.Burnside in 1902, when he asked when the group, in modern notation $B(r,n)$, is finite. Here $B(r,n)$ is the quotient group of the free group on $r$ generators modulo the subgroup generated by all words of the form $w^n$. It was shown that $B(r,n)$ is finite for $n = 2, 3, 4, 6$. On the other hand, Novikov and Adjan proved that $B(r,n)$ is infinite for $r \geq 2$ and $n$ odd and sufficiently large [N-A]. It is an open problem whether the group $B(2,5)$ is infinite, as most of experts predict, or finite, in which case it would have $5^{34}$ elements.
Theorem 1.5 (Wada) If \( D \) is a diagram of a link (or a tangle) \( L \), then \( \Pi_D^{(2)} = \pi_1(M^{(2)}_L) \). Furthermore, if we put \( y_i = 1 \) for any fixed generator, then \( \Pi_D^{(2)} \) reduces to \( \pi_1(M_L) \).

The next theorem allows us to use Burnside groups to analyze elementary moves on links.

**Theorem 1.6** The groups \( B_L(n) \) and \( \hat{B}_L(n) \) are preserved by rational \( \frac{a}{q} \)-moves. In particular the \( n \)th Burnside group is preserved by \( n \)-moves.

**Proof:** Let \( L' \) be obtained from \( L \) by a \( \frac{a}{q} \)-move. Then \( M^{(2)}_L \) is obtained from \( M^{(2)}_{L'} \) by performing the \( \frac{a}{n} \)-surgery. Such a surgery can be easily proved to preserve the \( n \)th Burnside group of the fundamental group of the manifold. \( \Box \)

## 2 Reductions by rational moves

We show that the answer to the rational move question, Problem (6) is negative.

**Theorem 2.1**  
(1) The closure, \( \Delta^4_3 \), of the 3-braid \( \Delta^4_3 = (\sigma_1\sigma_2)^6 \) (Figure 2.1) is not \( \frac{a}{q} \)-move reducible to a trivial link for any prime number \( p \geq 5 \).

(2) The closure of the 5-braid \( \Delta^4_5 = (\sigma_1\sigma_2\sigma_3\sigma_4)^{10} \) (Figure 2.2) is not 3-move reducible to a trivial link.

**Sketch of the proof:** We use Sanov’s theorem about the structure of the Lie algebra associated to the Burnside group of prime exponent \( p \geq 5 \) [San, V-Lee]. For \( p = 3 \) we observe that the third Burnside group \( B_{\Delta^3_5}(3) \) is the quotient of the free Burnside group, \( B(4,3) \), by the normal subgroup generated by relations \( Q_i x_i^{-1} \), where the words \( Q_i \) can be computed from the Fig.2.2 using the core relations (Fig.1.3)\(^3\). To see that relations are non-trivial in the free Burnside group we use the theorem of Levi and van der Waerden about the structure of the associated Lie rings of Burnside groups of exponent 3 [L-W].

\[ \Delta^4_3 = \]

\begin{center}
Figure 2.1
\end{center}

\(^3\)One gets \( Q_i = x_1x_2^{-1}x_3x_4^{-1}x_5x_1^{-1}x_2x_3^{-1}x_4x_5^{-1}x_4x_5^{-1}x_3x_2^{-1}x_5x_3^{-1}x_1 \) (see [D-P-1]).
The closed braid $\hat{\Delta}_5^4$ is 3-move equivalent to a link of 20 crossings (closure of the 5-string braid $(\sigma_3\sigma_4^{-1}\sigma_3^{-1}\sigma_2)^4$). It is still an open problem whether the Montesinos-Nakanishi conjecture holds for links up to 19 crossings. Q. Chen proved that it holds for links up to 12 crossings [Ch].

The negative answers to Problems (2), (3) and (5) follow from Theorem 2.1 for $p = 3, 5$ and 7, respectively (see [D-P-2] for details).

3 Kawauchi’s Question on 4-moves

In this section, we use the 4th Burnside group of links to show that there is an obstruction to 4-move reducibility of links which are link homotopically trivial. Therefore the answer to Kawauchi’s question is negative.

Let $W$ denote the “half” 2-cabling of the Whitehead link, which is link homotopy equivalent to the trivial link of 3 components (Figure 3.1).

**Theorem 3.1** The link $W$ is not 4-move equivalent to a trivial link.

Our proof uses the obstruction in the Burnside group $B(2, 4)$. The obstruction lies in the last nontrivial term of the lower central series, $\gamma_5$, of $B(2, 4)$ or equivalently in the fifth term of the associated graded Lie ring of $B(2, 4)$. In fact we have:
Lemma 3.2 $B_W(4) = B(2, 4)/\gamma_5$.

Figure 3.2 Computation of relations of $B_W(4)$.

Sketch of the proof: The lower central series of $B(2, 4)$ is known to be of class 5, with the last term $\gamma_5 = L_5$ isomorphic to $Z_2 \oplus Z_2$, [V-Lee]. First we compute $\pi_1(M^{(2)}_W)$ using a presentation of the associated core group of $W$ by putting the generator $z = 1$. From the diagram we obtain the relations: $Q_1 x^{-1}$ and $Q_2 y^{-1}$ (Fig.3.2), and further the equivalent relations

\begin{align*}
R_1 &= x^{-1}(x^2y^{-2}x^2y^{-2}x^2y^{-2})x(x^2y^{-2}x^2y^{-2}) = x^{-1}(x^2y^{-2})^4xx^{-1}(y^2x^{-2})^2x(x^2y^{-2})^2 \\
&= x^{-1}(x^2y^{-2})^4x[x, (x^2y^{-2})^2] = [x, (x^2y^{-2})^2] = [x, [x^2, y^2]] = [x, y, x, y, x] \in \gamma_5 \text{ (see [V-Lee]).}
\end{align*}

Those elements form a basis of $\gamma_5 = Z_2 \oplus Z_2$ considered as a $Z_2$ linear space. We verified this fact using programs GAP, Magnus, and Magma. These calculations can be done manually, however they strongly depend on the unpublished Ph.D. thesis of J.J. Tobin [Tob], as it was pointed out to us by M.Vaughan-Lee [V-Lee-2].

Analogously, $R_2 = [y, x, y, x, y] \in \gamma_5$.

It follows from Lemma 3.2 and the fact that $\gamma_5$ is in the center of $B(2, 4)$, that $|B_W(4)| = 2^{10}$. On the other hand the abelianization of $B_W(4)$ is isomorphic to $Z_4 \oplus Z_4$. So if $W$ were 4-reducible into a trivial link then the trivial link would have 3 components. But $B_{T_3}(4) = B(2, 4)$ has $2^{12}$ elements, as predicted by Burnside and verified by Tobin.
Remark 3.3 Nakanishi showed, using Alexander modules, that the Borromean rings, $BR$, cannot be reduced to a trivial link by 4-moves. Our Burnside obstruction method also works in this case. Knowing that $|B(2, 4)| = 2^{12}$ we can verify that $|B_{BR}(4)| = 2^5$. In addition, we conclude that $W$ and $BR$ are not 4-move equivalent.

By Coxeter’s theorem [Cox] the quotient group $B_3/(\sigma_i^4)$ is finite. Therefore, for closed 3-braids, we can list all possible 4th Burnside groups. This allows us to find all 4-move equivalence classes of closed 3-braids [D-P-2].

4 Limitations of the Burnside group invariant

The method based on the Burnside group invariant has been quite successful in the study of the unknotting property of several classes of tangle replacement moves. As we saw in Sections 2 and 3, for any fixed prime number $p \geq 3$ and an arbitrary nonzero integer $q$, rational $\frac{p}{q}$-moves are not unknotting operations. However our method has its limitations. We have been unable to find, by our method, obstructions for $nq$-reduction of a link $L$, to a trivial link if the abelianization of the $n$th Burnside group of the link, $(B_L(n))^{(ab)} = H_1(M_L^{(2)}, \mathbb{Z}_n)$, is a cyclic group (i.e. \{1\} or $\mathbb{Z}_n$). This is explained in Theorem 4.1.

Define the restricted Burnside group of a link, $R_L(n)$, as the quotient group $B_L(n)/N$, where $N$ is the intersection of all normal subgroups of $B_L(n)$ of finite indexes.

**Theorem 4.1** Let $n$ be a power of a prime number. Assume that the abelianization of $B_L(n)$ is a cyclic group. Then the restricted Burnside group, $R_L(n)$, is isomorphic to $H_1(M_L^{(2)}, \mathbb{Z}_n)$. In particular, if $B_L(n)$ is finite (e.g. for $n = 2, 3, 4$), then $B_L(n)$ is a cyclic group.

**Proof:** It was proved by E. Zelmanov that $R(r, n)$ is finite for any $n$. It follows that $R_L(n)$ is finite for any $n$.

Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_i \geq \cdots$ be the lower central series of $B_L(n)$. If $(B_L(n))^{(ab)}$ (and therefore also $(R_L(n))^{(ab)}$) is a cyclic group then $\gamma_2 = \gamma_3 = \gamma_4 \ldots$. Since $R_L(n)$ is a finite nilpotent group, therefore $R_L(n) = (B_L(n))^{(ab)}$.

If $B_L(n)$ is finite (as it is in the case of $n = 2, 3, 4$), then $R_L(n) = B_L(n)$, so $B_L(n)$ is a cyclic group. $\square$

Therefore, the method based on the Burnside group invariant will not produce any
obstructions for the Nakanishi 4-move conjecture and the Kawauchi 4-move question for a link of two components.

5 Conclusion

Our interest in the analysis of rational moves on links was inspired by our long pursuit of a program for understanding a 3-dimensional manifold by the knot theory which it supports. The method we introduced, the Burnside group of links, not only settles classical conjectures (e.g., the Montesinos-Nakanishi conjecture) but also has clear potential to be used in computational biology in an analysis of DNA, its recombination, action of topoisomers, and analysis of protein folding and protein evolution.

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