A nodal type polynomial finite element exact sequence over quadrilaterals *

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Abstract

This work proposes two nodal type nonconforming finite elements over convex quadrilaterals, which are parts of a finite element exact sequence. Both elements are of 12 degrees of freedom (DoFs) with polynomial shape function spaces selected. The first one is designed for fourth order elliptic singular perturbation problems, and the other works for Brinkman problems. Numerical examples are also provided.

Keywords: Nodal type; polynomial finite element; exact sequence; quadrilateral meshes.

1 Introduction

Let \( \Omega \subset \mathbb{R}^2 \) be a simply connected Lipschitz domain. The de Rham complex determined by the following exact sequence

\[
0 \rightarrow H^2(\Omega) \xrightarrow{\text{curl}} [H^1(\Omega)]^2 \xrightarrow{\text{div}} L^2(\Omega) \rightarrow 0, \tag{1.1}
\]

also known as the Stokes complex, is well understood and widely applied in the analysis for many problems in solid and fluid mechanics. Typical model problems are biharmonic and Stokes problems, whose solutions can be efficiently approximated by suitable finite element methods. In particular, a divergence-free Stokes element often plays a role in a certain discretization of (1.1) with some biharmonic element. A comprehensive review on this topic can be found in [15]. Generally speaking, there are three types of finite element exact sequences approximating (1.1). The first type are completely conforming, namely, all their components are subspaces of the corresponding forms in (1.1). The second type are semi-conforming, that is, their 0-forms are \( H^2 \)-nonconforming but \( H^1 \)-conforming, and their 1-forms are \( H^1 \)-nonconforming but \( H(\text{div}) \)-conforming. The sequence constructed via the modified Morley element [20, 16] and its higher order extension [13] are of this type. The rectangular Adini element was also

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recently adopted to formulate an exact sequence \[12\] as well as the modified nonconforming Zienkiewicz element \[24\] on triangles. The third type are completely nonconforming. Perhaps the simplest example is the Morley-Crouzeix-Raviart sequence \[17, 8\], whose higher order extension has been recently discovered in \[28\]. Again this construction has also been extended to the rectangular case \[25, 26\]. Although all the three types are successful for the discretization of \((1.1)\), for fourth order elliptic singular perturbation problems and Brinkman problems for porous media flow, only the first two types are sufficiently regular, while the third type might fail if the mesh is not regular and symmetric enough. In such a case, the required finite element sequence must approximate not only \((1.1)\) but also the following de Rham complex

\[
0 \xrightarrow{\text{curl}} H^1(\Omega) \xrightarrow{\text{div}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{} 0. \tag{1.2}
\]

Indeed, the modification \[20, 16\] for the Morley-Crouzeix-Raviart sequence is a compromise for this dilemma.

Note that all the aforementioned examples are designed for triangular or rectangular meshes. However, there are fewer researches on the approximation for \((1.1)\) and \((1.2)\) over general convex quadrilaterals, on which we will give a brief review. For the first type approximation for \((1.1)\), the \(H^2\)-conforming Fraijes de Veubeke-Sander element \[6, 23\] is a successful candidate for biharmonic problems. Owing to a normal aggregation trick, a subspace method, namely, the reduced Fraijes de Veubeke-Sander element was designed \[5\]. For \(H^1\)-conforming approximation of the incompressible flow, Neilan and Sap \[19\] introduced a divergence-free Stokes element from the Fraijes de Veubeke-Sander element. As far as the second type approximation is concerned, Bao et al. \[2\] proposed a \(H^1\)-conforming element for fourth order singular perturbation problems by enriching a spline element space by bubble functions. Note that all these elements are spline-based, and so a cell-refinement procedure cannot be avoided. Comparatively, polynomial shape functions are simple to represent and easy to compute, and therefore they are often more preferred. This has been taken into consideration for the third type approximation. Utilizing the Park-Sheen biharmonic element \[21\], Zhang \[27\] generalized the Morley-Crouzeix-Raviart sequence to general quadrilaterals, but again there is no evidence of its ability to approximate \((1.2)\). Recently, a polynomial modification was proposed by Zhou et al. \[30\], which works for both \((1.1)\) and \((1.2)\). We must point out that, the Adini complex \[12\] and the rectangular Morley complex \[25, 26\] are also successful for the discretization of both \((1.1)\) and \((1.2)\), but their convergence severely relies on the regularity and symmetry of the rectangular cell, and therefore cannot be generalized to arbitrary convex quadrilaterals in an obvious manner. Moreover, we discover that the number of global DoFs of the reduced Fraijes de Veubeke-Sander element \[5\] is significantly less than those of the semi-conforming \[2\] and completely nonconforming counterparts \[30\] benefitting from the nodal type structure.

This work devotes to the construction of a nonconforming finite element exact sequence on general convex quadrilateral meshes for approximating both \((1.1)\) and \((1.2)\), enjoying the advantages that the elements therein are of nodal type structure, and their shape functions are polynomials. In fact, the 0-form dealing with fourth order elliptic singular perturbation problems is, in a pseudo \(H^1\)-conforming manner with respect to \((1.2)\), a direct generalization of the modified nonconforming Zienkiewicz element \[24\] due to Wang, Shi and Xu. The DoFs are values and gradients of at vertices. For the 1-form designed for Brinkman problems, we select vertex values and edge normal means as the DoFs. Optimal and uniform error estimates are also given for both elements with respect to their associated model problems. From some numerical tests, one can observe that the performances of both elements are consistent with our theoretical findings.

The rest of this work is arranged as follows. In Section 2 the nonconforming finite element working for fourth order elliptic singular perturbation problems is defined on quadrilateral meshes. Section 3 introduces the element for Brinkman problems, and shows that both the two elements are parts of a finite element exact sequence. Numerical examples are given in Section 4 to verify the theoretical analysis.

Throughout the work, standard notations in Sobolev spaces are adopted. For a domain \(D \subset \mathbb{R}^2\), \(\mathbf{n}\) and \(\mathbf{t}\) will be the unit outward normal and tangent vectors on \(\partial D\), respectively. The notation \(P_k(D)\)
denotes the usual polynomial space over \( D \) of degree no more than \( k \). The norms and semi-norms of order \( m \) in the Sobolev spaces \( H^m(D) \) are indicated by \( \| \cdot \|_{m,D} \) and \( | \cdot |_{m,D} \), respectively. The space \( H^m_0(D) \) is the closure in \( H^m(D) \) of \( C_0^\infty(D) \). We also adopt the convention that \( L^2(D) := H^0(D) \), where the inner-product is denoted by \( (\cdot, \cdot)_D \). These notations of norms, semi-norms and inner-products also work for vector- and matrix-valued Sobolev spaces, where the subscript \( \Omega \) will be omitted if the domain \( D = \Omega \). Moreover, the positive constant \( C \) independent of the mesh size \( h \) and parameters \( \varepsilon, \nu \) and \( \alpha \) in the model problems might be different in different places.

## 2 Finite element for fourth order elliptic singular perturbation problems

### 2.1 Notations of a quadrilateral and an auxiliary affine transformation

Let \( K \) be an arbitrary convex quadrilateral. The four vertices of \( K \) are given by \( V_1, V_2, V_3, V_4 \) in a counterclockwise order, and the \( i \)th edge of \( K \) is denoted by \( E_i = V_i V_{i+1} \), whose equation is written as \( l_i(x, y) = 0, i = 1, 2, 3, 4 \). Here and throughout the paper, the index \( i \) is taken modulo four. For each \( E_i \), \( M_i \) denotes its midpoint, and \( n_i \) and \( t_i \) will be its unit normal and tangential directions. The equations of lines through \( M_1 M_3, M_2 M_4, V_1 V_3 \) and \( V_2 V_4 \) read as \( m_{13}(x, y) = 0, m_{24}(x, y) = 0, l_{13}(x, y) = 0 \) and \( l_{24}(x, y) = 0 \), respectively. Moreover, we assume that all the aforementioned line equations are uniquely determined by

\[
\begin{align*}
l_1(M_3) = l_2(M_4) &= l_3(M_1) = l_4(M_2) = m_{13}(M_2) = m_{24}(M_3) = l_{13}(V_4) = l_{24}(V_3) = 1. \quad (2.1)
\end{align*}
\]

In order to describe the construction, we recall an auxiliary affine transformation for each \( K \) generated by decomposing the standard bilinear mapping (see also \([21, 9, 29]\)). The reference square \( \tilde{K} = [-1, 1]^2 \) is determined by its vertices \( \tilde{V}_1 = (-1, -1)^T, \tilde{V}_2 = (1, -1)^T, \tilde{V}_3 = (1, 1)^T \) and \( \tilde{V}_4 = (-1, 1)^T \). The bilinear mapping \( F_K : \tilde{K} \rightarrow K \) such that \( \tilde{V}_i \) is mapped into \( V_i \) for each \( i \) can be decomposed as \( F_K = A_K \circ S_K \) with \( A_K : \tilde{K} \rightarrow K \) and \( S_K : K \rightarrow \tilde{K} \) defined by

\[
A_K(\tilde{x}) = A \tilde{x} + b, \quad S_K(\tilde{x}) = \tilde{x} + \tilde{y} s, \quad \tilde{x} = (\tilde{x}, \tilde{y})^T \in \tilde{K}, \quad \tilde{x} = (\bar{x}, \bar{y})^T \in \tilde{K},
\]

where \( A \) is a \( 2 \times 2 \) matrix, and \( b, d \) and \( s \) are two-dimensional vectors given by

\[
\begin{align*}
A &= \frac{1}{4}(V_3 - V_4 + V_1 + V_2, V_3 + V_4 - V_1 - V_2), \quad d = \frac{1}{4}(V_3 - V_4 - V_1 - V_2), \\
b &= \frac{1}{4}(V_3 + V_4 + V_1 + V_2), \quad s = (s_1, s_2)^T = A^{-1} d.
\end{align*}
\]

Figure 1 gives an example of the intermediate reference element \( \tilde{K} \) and the auxiliary affine transformation \( A_K \). We shall denote a point on \( \tilde{K} \) by \( \tilde{V} = (\tilde{x}, \tilde{y})^T \) if it equals \( A_K^{-1}(V) \) for a point \( V = (x, y)^T \) on \( K \), and an edge in \( \tilde{K} \) by \( \tilde{E} \) if it equals \( A_K^{-1}(E) \) for an edge \( E \) in \( K \). Note that \( \tilde{M}_i \) is also the midpoint of \( \tilde{V}_i \tilde{V}_{i+1} \) and \( \tilde{V}_i = \tilde{V}_i + (1)^{(i+1)} s \). Furthermore, since \( K \) is convex, one must have

\[
|s_1| + |s_2| < 1. \quad (2.3)
\]

Similarly, we write the function \( \tilde{f} = f \circ A_K \) defined over \( \tilde{K} \) for a function \( f \) over \( K \). A simple calculation will derive

\[
\begin{align*}
\tilde{l}_1 &= \frac{1}{2} \left( -\frac{s_2}{s_1 - 1} \tilde{x} + \tilde{y} + 1 \right), \quad \tilde{l}_2 = \frac{1}{2} \left( -\tilde{x} + \frac{s_1}{s_2 + 1} \tilde{y} + 1 \right), \quad \tilde{l}_3 = \frac{1}{2} \left( \frac{s_2}{s_1 + 1} \tilde{x} - \tilde{y} + 1 \right), \\
\tilde{l}_4 &= \frac{1}{2} \left( \tilde{x} - \frac{s_1}{s_2 - 1} \tilde{y} + 1 \right), \quad \tilde{l}_{13} = \frac{\tilde{x} + \tilde{y} + s_1 - s_2}{2(s_1 - s_2 + 1)}, \quad \tilde{l}_{24} = \frac{\tilde{x} + \tilde{y} + s_1 + s_2}{2(s_1 + s_2 + 1)}, \quad \tilde{m}_{13} = \tilde{x}, \quad \tilde{m}_{24} = \tilde{y}. \quad (2.4)
\end{align*}
\]
2.2 An auxiliary 12-DoF finite element

To design the element for fourth order singular perturbation problem, we first introduce an auxiliary element, which extends the rectangular Adini element to general convex quadrilaterals in a pseudo-\(C^0\) manner.

**Definition 2.1.** The quadrilateral finite element \((K,W_K,T_K)\) is defined as follows:

- \(K\) is a convex quadrilateral;
- \(W_K = P_3(K) \oplus \text{span}\{\phi_1, \phi_2\}\) is the shape function space where
  \[
  \phi_1 = (s_2 - 1)(s_2 + 1)l_1l_3m_{24} - s_1s_2l_3m_{24} + s_1l_3m_{24}m_{13},
  \]
  \[
  \phi_2 = (s_1 - 1)(s_1 + 1)l_2l_4m_{24} - s_1s_2l_4m_{24}^2 + s_2l_4m_{13}m_{24}.
  \]

  The parameters \(s_1\) and \(s_2\) are defined via (2.2). 

- \(T_K = \{\tau_j, j = 1, 2, \ldots, 12\}\) is the DoF set where
  \[
  \tau_j(w) = w(V_j), \quad (\tau_{j+4}(w), \tau_{j+8}(w))^T = \nabla w(V_j), \quad j = 1, \ldots, 4.
  \]

Write \(p_1 = l_1l_3l_4, p_2 = l_1l_3l_2, p_3 = l_1l_3l_1, p_4 = \phi_1\) and \(q_1 = l_2l_4l_1, q_2 = l_2l_4l_3, q_3 = l_2l_4l_4, q_4 = \phi_2\). We also define the nodal functionals

\[
\lambda_1(w) = |E_2| \frac{\partial w}{\partial t_2}(V_2), \quad \lambda_2(w) = |E_2| \frac{\partial w}{\partial t_2}(V_3), \quad \lambda_3(w) = |E_4| \frac{\partial w}{\partial t_4}(V_4), \quad \lambda_4(w) = |E_4| \frac{\partial w}{\partial t_4}(V_1),
\]

\[
\mu_1(w) = |E_1| \frac{\partial w}{\partial t_1}(V_1), \quad \mu_2(w) = |E_1| \frac{\partial w}{\partial t_1}(V_2), \quad \mu_3(w) = |E_3| \frac{\partial w}{\partial t_3}(V_3), \quad \mu_4(w) = |E_3| \frac{\partial w}{\partial t_3}(V_4)
\]

and the \(4 \times 4\) matrices \(M\) and \(N\) by setting \(M_{i,j} = \lambda_i(p_j), N_{i,j} = \mu_i(q_j), i, j = 1, 2, 3, 4\). The following lemma is helpful to verify the unisolvency of \((K,W_K,T_K)\).

**Lemma 2.2.** The matrices \(M\) and \(N\) are nonsingular.

**Proof.** For \(i, j = 1, 2, 3, 4\), note that functionals \(|E_i| \frac{\partial w}{\partial t_i}(V_j) = |\tilde{E}_i| \frac{\partial w}{\partial t_i}(\tilde{V}_j)\), therefore we can calculate the entries of \(M\) and \(N\) on \(\tilde{K}\) rather than the physical \(K\) in variables \(s_1\) and \(s_2\). Using (2.4) we set

\[
\begin{align*}
    f_1(s_1, s_2) &= \frac{(s_1 + s_2 - 1)(s_1 - s_2 - 1)}{(s_2 - 1)(s_2 + 1)}, & f_2(s_1, s_2) &= \frac{(s_1 - s_2 + 1)(s_1 + s_2 + 1)}{(s_2 - 1)(s_2 + 1)}, \\
    f_3(s_1, s_2) &= \frac{(s_1 + s_2 + 1)(s_1 - s_2 - 1)}{(s_1 - 1)(s_1 + 1)}, & f_4(s_1, s_2) &= \frac{(s_1 - s_2 + 1)(s_1 + s_2 - 1)}{(s_1 - 1)(s_1 + 1)}.
\end{align*}
\]
and a direct computation gives \( M = (M_1^T, M_2^T)^T \) and \( N = (N_1^T, N_2^T)^T \), where

\[
M_1 = f_3 \begin{pmatrix}
(s_1 + s_2 - 1)/(s_2 - 1) & 0 & (s_1 - s_2 - 1)/(s_1 - s_2 + 1) & -(s_2 + 1)^2(s_1^2 + s_2 - 1) \\
(s_1 - s_2 + 1)/(s_2 - 1) & 0 & -(s_2 + 1)^2(s_1^2 + s_2 - 1) & -(s_2 + 1)^2(s_1^2 + s_2 - 1)
\end{pmatrix},
\]

\[
M_2 = f_4 \begin{pmatrix}
0 & (s_1 + s_2 + 1)/(s_2 + 1) & -(s_2 - 1)^2(s_1^2 - s_2 - 1) & 0 \\
0 & (s_1 - s_2 + 1)/(s_2 + 1) & -(s_2 - 1)^2(s_1^2 - s_2 - 1) & 0
\end{pmatrix},
\]

\[
N_1 = f_5 \begin{pmatrix}
0 & -(s_1 - s_2 + 1)/(s_1 + 1) & -(s_1 + s_2 - 1)/(s_1 + s_2 + 1) & -(s_1 - 1)(-s_2^2 + s_1 + 1) \\
0 & -(s_1 + s_2 + 1)/(s_1 + 1) & 0 & -(s_1 - 1)(-s_2^2 + s_1 + 1)
\end{pmatrix},
\]

\[
N_2 = f_6 \begin{pmatrix}
-(s_1 - s_2 - 1)/(s_1 + 1) & 0 & -(s_1 + 1)^2(s_2^2 + s_1 - 1) & 0 \\
-(s_1 + s_2 - 1)/(s_1 + 1) & 0 & 0 & -(s_1 + 1)^2(s_2^2 + s_1 - 1)
\end{pmatrix}.
\]

Hence, by a symbolic computation,

\[
det M = 4f_3^2f_4^2(s_1 - s_2 - 1)(s_1^2 + s_2^2 - 1),
\]

\[
det N = 4f_5^2f_6^2(s_1 + s_2 - 1)(s_1^2 + s_2^2 - 1).
\]

It then follows from \((2.3)\) that \( det M \neq 0 \) and \( det N \neq 0 \), which is the desired result. □

**Lemma 2.3.** The element \((K, W_K^-, T_K^-)\) is well-defined.

*Proof.* Set \( r_1 = l_2l_3, r_2 = l_3l_4, r_3 = l_4l_1, r_4 = l_1l_2 \), then our goal is to show \( W_K^- = \text{span}\{p_i, q_i, r_i, i = 1, 2, 3, 4\} \) and that \( \tau_j(w) = 0 \) for \( w \in W_K^- \) will derive \( w = 0 \). First we see all \( p_i, q_i \) and \( r_i \) are linearly independent. Indeed, if

\[
w = \sum_{i=1}^{4} (\alpha_i p_i + \beta_i q_i + \gamma_i r_i) = 0,
\]

for \( \alpha_i, \beta_i, \gamma_i \in \mathbb{R} \), then \( \tau_i(w) = 0 \) for \( i = 1, 2, 3, 4 \). Noting that

\[
\tau_i(p_j) = \tau_i(q_j) = 0, \quad \tau_i(r_j) = r_i(V_j)\delta_{ij}, \quad r_i(V_i) \neq 0, \quad i, j = 1, 2, 3, 4,
\]

we obtain \( \gamma_i = 0, i = 1, 2, 3, 4 \). Moreover, the constructions of all \( q_i \) imply

\[
\lambda_i(q_j) = 0, \quad \mu_i(p_j) = 0, \quad i, j = 1, 2, 3, 4,
\]

and therefore it follows from \( \lambda_i(w) = 0 \) and \( \mu_i(w) = 0 \) that

\[
\sum_{j=1}^{4} \alpha_j \lambda_j(p_j) = 0, \quad \sum_{j=1}^{4} \beta_j \mu_j(q_j) = 0.
\]

Taking \( i = 1, 2, 3, 4 \), we find

\[
M(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T = N(\beta_1, \beta_2, \beta_3, \beta_4)^T = 0,
\]

which implies \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0 \) according to Lemma 2.2. Hence, all \( p_i, q_i \) and \( r_i \) are linearly independent, namely, \( \dim \text{span}\{p_i, q_i, r_i, i = 1, 2, 3, 4\} = \dim W_K^- = 12 \). Moreover, all \( p_i, q_i \) and \( r_i \) are members of \( W_K^- \), and thus these two sets are equal. Repeat the same process above from \((2.6)\) apart from the assumption \( w = 0 \), the unisolvency is derived since \( \tau_j(w) = 0 \) for \( j = 5, \ldots, 12 \) if and only if \( \lambda_i(w) = \mu_i(w) = 0, i = 1, 2, 3, 4 \). □

The following lemma hints a critical property of this element, which partly explains why the coefficients in \( \phi_1 \) and \( \phi_2 \) are necessary.
Lemma 2.4. For all \( w \in W^b_{K} \), it holds that

\[
\frac{1}{|E_i|} \int_{E_i} w \, ds = \frac{1}{2} (w(V_i) + w(V_{i+1})) - \frac{|E_i|}{12} \left( \frac{\partial w}{\partial t_i} (V_{i+1}) - \frac{\partial w}{\partial t_i} (V_i) \right), \quad i = 1, 2, 3, 4. \tag{2.7}
\]

Proof. For each \( i \), let \( \xi_i \in P_1(E_i) \) be taken such that \( \xi_i(V_i) = -1 \) and \( \xi_i(V_{i+1}) = 1 \). Then by integrating by parts, one must have

\[
\frac{1}{|E_i|} \int_{E_i} \frac{\partial w}{\partial t_i} \xi_i \, ds = \frac{1}{6} \left( \frac{\partial w}{\partial t_i} (V_{i+1}) - \frac{\partial w}{\partial t_i} (V_i) \right),
\]

which along with (2.8) leads to (2.7). For \( \xi \in P_3(K) \), the Simpson quadrature rule implies

\[
\frac{1}{|E_i|} \int_{E_i} \frac{\partial w}{\partial t_i} \xi_i \, ds = \frac{1}{6} \left( \frac{\partial w}{\partial t_i} (V_{i+1}) - \frac{\partial w}{\partial t_i} (V_i) \right),
\]

and so the assertion is verified.

\[ \square \]

Remark 2.5. If \( K \) is a rectangle, \( s_1 = s_2 = 0 \). This element degenerates to the traditional Adini element.

2.3 Enriched by bubble functions and normal aggregation

Let us now introduce the following \( H^1_0(K) \)-bubble function space \( W^b_{K} \) over \( K \):

\[
W^b_{K} = \text{span}\{b_0, b_0x, b_0y, b_0l_1l_2l_3l_4\}, \quad b_0 = l_1l_2l_3l_4. \tag{2.9}
\]

The DoFs in \( T^b_{K} \) with respect to these bubble functions are

\[
\tau^b_i(w) = \frac{1}{|E_i|} \int_{E_i} \frac{\partial w}{\partial n_i} \, ds, \quad i = 1, 2, 3, 4.
\]

Lemma 2.6. If \( w \in W^b_{K} \) such that all \( \tau^b_i(w) = 0 \), \( i = 1, 2, 3, 4 \), then \( w = 0 \).

Proof. Over \( K \) we define the following polynomials \( b_1 = 1, b_2 = m_{13}, b_3 = m_{24}, b_4 = l_{13}l_{24} \) and \( b_{i,j} = b_0b_j/l_i \) for \( i, j = 1, 2, 3, 4 \), then we turn to the intermediate reference quadrilateral \( \tilde{K} \). Consider the \( 4 \times 4 \) matrix \( B^- \) determined by

\[
B_{i,j}^- = \frac{1}{|E_i|} \int_{E_i} b_{i,j} \, ds = \frac{1}{|E_i|} \int_{\tilde{E}_i} \tilde{b}_{i,j} \, d\tilde{s}, \quad i, j = 1, 2, 3, 4.
\]

By (2.4) and a direct computation, we find \( B^- = ((B^-_1)^T, (B^-_2)^T, (B^-_3)^T, (B^-_4)^T)^T \), where

\[
B^-_1 = \frac{1}{6} f_1 \left( -s_2(s_1-1), \frac{s_2^2 + 5s_1 + 5}{5(s_1 + 1)}, \frac{s_1 + s_2 - 1}{5(s_1 + s_2 + 1)(s_1 - s_2 + 1)} \right),
\]

\[
B^-_2 = \frac{1}{6} f_2 \left( 1, -\frac{s_2^2 - 5s_1 + 5}{5(s_2 - 1)}, -\frac{s_1 + s_2 - 1}{5(s_1 + s_2 + 1)}, \frac{s_1 - s_2 - 1}{5(s_1 - s_2 + 1)} \right),
\]

\[
B^-_3 = \frac{1}{6} f_3 \left( 1, \frac{s_2(s_1 + 1)}{5(s_1 - 1)}, -\frac{s_1^2 + 5s_1 - 5}{5(s_1 - 1)}, \frac{s_1}{5} \right),
\]

\[
B^-_4 = \frac{1}{6} f_4 \left( 1, -\frac{s_1^2 + 5s_2 + 5}{5(s_2 + 1)}, \frac{s_1(s_2 - 1)}{5(s_2 + 1)}, \frac{s_1 + s_2 - 1}{5(s_1 + s_2 + 1)} \right).
\]
and $f_i, i = 1, 2, 3, 4$ have been given in (2.5). A symbolic computation gives
\[
\det B^- = \frac{f_1 f_2 f_3 f_4 (s_1^6 + s_2^6) - s_1^2 s_2^2 (s_1^2 + s_2^2) + 9(s_1^4 + s_2^4) - 26s_1^2 s_2^2 + 15(s_1^2 + s_2^2) - 25}{20250(s_1 - 1)(s_1 + 1)(s_2 - 1)(s_2 + 1)(s_1 + s_2 + 1)(s_1 - s_2 + 1)}.
\]

Note from (2.3) that the factor of the numerator
\[
(s_1^6 + s_2^6) - s_1^2 s_2^2 (s_1^2 + s_2^2) + 9(s_1^4 + s_2^4) - 26s_1^2 s_2^2 + 15(s_1^2 + s_2^2) - 25
\]
and therefore $B^-$ is nonsingular.

Next, we turn to the matrix $B$ with $B_{i,j} = \tau^b_i(b_0 b_j)$. Our aim is to show $B$ is nonsingular as $W_K = \text{span}\{b_0 b_j, j = 1, 2, 3, 4\}$. For each $i$ and $j$, since
\[
B_{i,j} = \frac{1}{|E_i|} \left( \int_{E_i} \frac{\partial l_i}{\partial n_i} b_{i,j} \, ds + \int_{E_i} \frac{\partial (b_{i,j})}{\partial n_i} l_i \, ds \right) = \frac{\partial l_i}{\partial n_i} B^-_{i,j},
\]
then
\[
\det B = \left( \prod_{i=1}^{4} \frac{\partial l_i}{\partial n_i} \right) \det B^- \neq 0,
\]
which completes the proof.

We are in a position to propose the element for fourth order elliptic singular perturbation problems by the normal aggregation strategy. The DoFs are at vertices over a quadrilateral, which is a nodal type construction.

Definition 2.7. The quadrilateral finite element $(K, W_K, T_K)$ is defined by:

- $K$ is a convex quadrilateral;
- $W_K$ is the shape function space:
\[
W_K = \left\{ w \in W_K^- \oplus W_K^+ : \frac{1}{|E_i|} \int_{E_i} \frac{\partial w}{\partial n_i} \, ds = \frac{1}{2} \left( \frac{\partial w}{\partial n_i} (V_i) + \frac{\partial w}{\partial n_i} (V_{i+1}) \right), \ i = 1, 2, 3, 4 \right\}; \quad (2.10)
\]
- $T_K = T_K^-$ is the DoF set.

Here $W_K^b$ and $(K, W_K, T_K^-)$ have been given in (2.9) and Definition 2.1 respectively.

Theorem 2.8. The element $(K, W_K, T_K)$ is well-defined. Moreover, (2.7) holds for all $w \in W_K$ and $P_2(K) \subset W_K$.

Proof. The four relations in (2.10) hint that $\dim W_K \geq 12$. It suffices to show if $w \in W_K$ fulfilling $\tau_j(w) = 0$ for $j = 1, 2, \ldots, 12$ then $w = 0$. Write $w = w^- + w^b$ with $w^- \in W_K^-$ and $w^b \in W_K^b$.

The assumption above gives $\tau_j(w^-) = 0$ as $\tau_j(w^b) = 0$ for all $j$, and by Lemma 2.3, we find $w^- = 0$. Moreover, the relations in (2.10) will lead to $\tau^b_i(w) + \tau^b_i(w^b) = 0, i = 1, 2, 3, 4$, and by Lemma 2.6, we get $w^b = 0$, which means $w = 0$ and the unisolvency has been derived. The next assertion is trivial as $w^b$ is bubble functions. By noting that the relations in (2.10) hold for $P_2(K)$, the last assertion immediately follows.

The nodal basis representation can be obtained as follows. Let $\psi_j^- \in W_K^-$, $j = 1, 2, \ldots, 12$ be the nodal basis of $(K, W_K, T_K^-)$ and $\psi_j^b \in W_K^b$, $j = 1, 2, 3, 4$ satisfy $\tau^b_i(\psi_j^b) = \delta_{ij}, i = 1, 2, 3, 4$. Then
\[
\psi_j = \psi_j^- + \sum_{i=1}^{4} c_{i,j} \psi_i^b, \ j = 1, 2, \ldots, 12
\]
Moreover, for \(E\) and \(V\) are correspondingly denoted by the sets of all vertices, interior vertices, boundary vertices, edges, interior edges and boundary edges.

Let \(\Omega \subset \mathbb{R}^2\) be a polygonal domain and \(\partial \Omega\) be its boundary. For a given \(f \in L^2(\Omega)\), the fourth order elliptic singular perturbation problem appears as: Find \(u\) such that

\[
\varepsilon^2 \Delta^2 u - \Delta u = f \quad \text{in} \quad \Omega,
\]

\[
u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega,
\]

where \(\varepsilon\) is the singular perturbation parameter tending to zero. A weak formulation is to find \(u \in H^2_0(\Omega)\) such that

\[
\varepsilon^2 (\nabla^2 u, \nabla^2 v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H^2_0(\Omega).
\]

2.4 Applied to fourth order elliptic singular perturbation problems

Let \(\Omega \subset \mathbb{R}^2\) be a polygonal domain and \(\partial \Omega\) be its boundary. For a given \(f \in L^2(\Omega)\), the fourth order elliptic singular perturbation problem appears as: Find \(u\) such that

\[
\varepsilon^2 \Delta^2 u - \Delta u = f \quad \text{in} \quad \Omega,
\]

\[
u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega,
\]

where \(\varepsilon\) is the singular perturbation parameter tending to zero. A weak formulation is to find \(u \in H^2_0(\Omega)\) such that

\[
\varepsilon^2 (\nabla^2 u, \nabla^2 v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H^2_0(\Omega).
\]

Let \(\{T_h\}\) be a family of quasi-uniform and shape-regular partitions of \(\Omega\) consisting of convex quadrilaterals. For a cell \(K \in T_h\), \(h_K\) denotes the diameter of \(K\), and so the parameter \(h := \max_{K \in T_h} h_K\).

The sets of all vertices, interior vertices, boundary vertices, edges, interior edges and boundary edges are correspondingly denoted by \(V_h\), \(V^i_h\), \(V^b_h\), \(E_h\), \(E^i_h\) and \(E^b_h\). For each \(E \in E_h\), \(n_E\) is a fixed unit vector perpendicular to \(E\) and \(t_E\) is a vector obtained by rotating \(n_E\) by ninety degree counterclockwisely. Moreover, for \(E \in E^i_h\), the jump of a function \(v\) across \(E\) is defined as \([v]^E = v|_{K_1} - v|_{K_2}\), where \(K_1\) and \(K_2\) are the cells sharing \(E\) as a common edge, and \(n_E\) points from \(K_1\) to \(K_2\).

For \(E \in E^b_h\), we set \([v]^E = v|_{K}\) if \(E\) is an edge of \(K\).

We now define the finite element space \(W_h\) by setting

\[
W_h = \left\{ w \in L^2(\Omega) : w|_K \in W_K, \forall K \in T_h, \text{ and } \nabla w \right. \text{ are continuous at all } V \in \mathcal{V}_h^i \text{ and vanishes at all } V \in \mathcal{V}_h^b \}.\]

Then the finite element approximation of (2.12) is: Find \(u_h \in W_h\) fulfilling

\[
\varepsilon^2 a_h(u_h, v_h) + b_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in W_h,
\]

where

\[
a_h(u_h, v_h) = \sum_{K \in T_h} (\nabla^2 u_h, \nabla^2 v_h)_K, \quad b_h(u_h, v_h) = \sum_{K \in T_h} (\nabla u_h, \nabla v_h)_K.
\]

Moreover, we define a discrete semi-norm by setting

\[
\|v\|^2_{\varepsilon,h} = \varepsilon^2 |v|^2_{L,h} + |v|^2_{L^1,h} \text{ with } |v|^2_{m,h} = \sum_{K \in T_h} |v|^2_{m,K}, \quad m = 1, 2.
\]

Clearly, owing to the definition of \(W_h\), we observe that \(\|\cdot\|_{\varepsilon,h}\) is a norm on \(V_h\). Thus, by the Lax-Milgram lemma, the problem (2.13) has unique solution.

For each \(K \in T_h\) and \(s > 0\), we define the interpolation operator \(I_K : H^{2+s}(K) \to W_K\) according to Theorem 2.3 such that \(\tau_j(I_K w) = \tau_j(w), \quad j = 1, 2, \ldots, 12\). Since \(I_K v = v\) for all \(v \in P_2(K)\) and \(\{T_h\}\) is quasi-uniform and shape regular, we find

\[
|w - I_K w|_{j,K} \leq C h^{k-j} |w|_{k,K}, \quad \forall w \in H^k(K) \cap H^{2+s}(K), \quad j = 0, 1, 2, k = 2, 3.
\]
Then the global interpolation operator \( I_h : H^2(\Omega) \cap H^{2+\epsilon}(\Omega) \to W_h \) is set as \( I_h|_K = I_K \). Moreover, owing to Theorem 2.8 one has
\[
\int_E [w_h]_E \, ds = \int_E \left[ \frac{\partial w_h}{\partial n}_E \right] \, ds = 0, \forall E \in \mathcal{E}_h
\]
via (2.7) and (2.10). The Strang lemma says
\[
lip(u - u_h)_{\epsilon,h} \leq C\left( \inf_{v_h \in W_h} \lip u - v_h \lip_{\epsilon,h} + \sup_{w_h \in W_h} \frac{E_{\epsilon,h}(u, w_h)}{\lip w_h \lip_{\epsilon,h}} \right)
\]
with the consistency error
\[
E_{\epsilon,h}(u, w_h) = \epsilon^2 a_h(u, w_h) + b_h(u, w_h) - (f, w_h).
\]
Hence, applying the proof of Theorem 1 in Chen et al.’s work [4] and invoking (2.14), (2.15), we get
\[
\inf_{v_h \in W_h} \lip u - v_h \lip_{\epsilon,h} \leq Ch\lip [u - I_h u] \lip_{\epsilon,h} \leq Ch(\epsilon|u|_3 + |u|_2),
\]
\[
E_{\epsilon,h}(u, w_h) \leq Ch(\epsilon|u|_3 + |u|_2 + \lip f \lip_0)\lip w_h \lip_{\epsilon,h}, \forall w_h \in W_h,
\]
which leads to the following convergence result.

**Theorem 2.9.** Let \( u \in H^3(\Omega) \) and \( u_h \in W_h \) be the solutions of (2.12) and (2.13), respectively. Then
\[
\lip u - u_h \lip_{\epsilon,h} \leq Ch(\epsilon|u|_3 + |u|_2 + \lip f \lip_0).
\]

From Theorem 2.9, this element ensures a linear convergence order with respect to \( h \), uniformly in \( \epsilon \), provided that \( \epsilon|u|_3, |u|_2 \) are uniformly bounded. However these terms might blow up when \( \epsilon \) tends to zero. The next result, following a similar line of Theorem 4.3 in [25], guarantees a uniform convergence rate under the impact of such boundary layers.

**Theorem 2.10.** Assume \( \Omega \) is a convex domain. Let \( u \in H^3(\Omega) \) and \( u_h \in W_h \) be the solutions of (2.12) and (2.13), respectively. Then it holds the uniform error estimate
\[
\lip u - u_h \lip_{\epsilon,h} \leq Ch^2 \lip f \lip_0.
\]

### 3 Finite element for Brinkman problems and the associated exact sequence

#### 3.1 Construction of the finite element

Let us turn to the construction of a vector-valued nodal type finite element. As in the scalar case, we shall prove the unisolvency of two auxiliary elements. For a general convex quadrilateral \( K \), the element \((K, V^b_K, \Sigma^b_K)\) and the \( H_0(\text{div}; K)\)-bubble element \((K, V^b_K, \Sigma^b_K)\) are defined through
\[
V^b_K = [P_1(K)]^2 \oplus \text{span}\{\text{curl} x^3, \text{curl} x^2 y, \text{curl} x y^2, \text{curl} y^3, \text{curl} \phi_1, \text{curl} \phi_2\},
\]
\[
\Sigma^b_K = \{\sigma_j, j = 1, 2, \ldots, 12\}, \quad V^b_K = \text{curl} W^b_K, \Sigma^b_K = \{\sigma_j^b, j = 1, 2, 3, 4\},
\]
where \( \phi_1, \phi_2 \) have been given in Definition 2.1 and the DoFs are
\[
\sigma_j(v) = \int_{E_j} v \cdot n_j \, ds, \quad (\sigma_{j+4}(v), \sigma_{j+8}(v))^T = v(V_j), \quad \sigma_j^b(v) = \int_{E_j} v \cdot t_j \, ds, \quad j = 1, 2, 3, 4.
\]
Lemma 3.1. Both \((K, V_K^-, \Sigma_K^+)\) and \((K, V_K^+, \Sigma_K^-)\) are well-defined.

Proof. We only deal with \((K, V_K^-, \Sigma_K^+)\) as the latter is much simpler. Define \(v_0 = (x, y)^T\), then
\[
V_K^- = \text{span}\{v_0\} \oplus \text{span}\{\text{curl} \ w : w \in W_K^\perp\}. \tag{3.1}
\]
Suppose \(v = cv_0 + \text{curl} \ w \in V_K^\perp\) for some \(w \in W_K^\perp\) such that \(\sigma_j(v) = 0\) for all \(j\), then
\[
c\sigma_j(v_0) + \sigma_j(\text{curl} \ w) = 0, \ j = 1, 2, \ldots, 12. \tag{3.2}
\]
However, we notice that \(\text{div} \ v_0 = 2\), and by Green’s formula
\[
\sum_{i=1}^{4} \sigma_i(\text{curl} \ w) = \int_K \text{div} \ w \, dx = 0, \quad \sum_{i=1}^{4} \sigma_i(v_0) = \int_K \text{div} \ v_0 \, dx = 2|K| \neq 0.
\]
Summing over \(3.2\) for \(j = 1, 2, 3, 4\) gives \(c = 0\), and therefore \(\sigma_j(\text{curl} \ w) = 0, j = 1, 2, \ldots, 12\). Hence, it suffices to show \(\text{curl} \ w = 0\). Indeed, we can select \(w\) such that \(w(V_1) = 0\) without changing the value of each \(\sigma_j(\text{curl} \ w)\). Then
\[
w(V_{i+1}) = w(V_i) + \int_{E_i} \frac{\partial w}{\partial t} \, ds = w(V_i) + \sigma_i(\text{curl} \ w) = 0, \ i = 1, 2, 3. \tag{3.3}
\]
Moreover,
\[
\nabla w(V_j) = (-\sigma_{j+8}(\text{curl} \ w), \sigma_{j+4}(\text{curl} \ w))^T = 0, \ j = 1, 2, 3, 4. \tag{3.4}
\]
As a consequence of \(3.3, 3.4\) and Lemma 2.3, we find \(w = 0\), and so \(\text{curl} \ w = 0\), which implies \(v = 0\).

The proof is done.

Parallel to Lemma 2.7, the following fact is crucial for the convergence in the Darcy limit.

Lemma 3.2. For all \(v \in V_K^\perp\), it holds that
\[
\frac{1}{|E_i|} \int_{E_i} v \cdot n \xi_i \, ds = \frac{1}{6} (v(V_{i+1}) - v(V_i)) \cdot n, \ i = 1, 2, 3, 4, \tag{3.5}
\]
where \(\xi_i \in P_3(E_i)\) has been defined in the proof of Lemma 2.4.

Proof. According to \(3.1\), we shall verify this relation for \(v_0\) and all \(\text{curl} \ w, w \in W_K^\perp\). Since \(v_0 \in [P_3(K)]^2\), the Simpson quadrature rule ensures \(3.4\). On the other hand, if \(w \in W_K^\perp\), substituting \(2.7\) into \(2.8\) will derive \(3.5\) for \(v = \text{curl} \ w\), which completes the proof.

Now we introduce the nodal type vector-valued element for Brinkman problems.

Definition 3.3. The finite element \((K, V_K, \Sigma_K)\) is determined through:

- \(K\) is a convex quadrilateral;
- \(V_K\) is the shape function space:
\[
V_K = \left\{ v \in V_K^- \oplus V_K^+: \frac{1}{|E_i|} \int_{E_i} v \cdot t_i \, ds = \frac{1}{2} (v(V_i) + v(V_{i+1})) \cdot t_i, \ i = 1, 2, 3, 4 \right\}, \tag{3.6}
\]
- \(\Sigma_K = \Sigma_K^+\) is the DoF set.

Theorem 3.4. The element \((K, V_K, \Sigma_K)\) is well-defined. Moreover, \(3.5\) holds for all \(v \in V_K\) and \([P_3(K)]^2 \subset V_K\).

Proof. The proof is very similar to that of Theorem 2.3 and thus omitted.
3.2 Applied to Brinkman problems

Consider the following Brinkman problem of porous media flow over $\Omega$: For given $f \in [L^2(\Omega)]^2$ and $g \in L^2_0(\Omega)$, find the velocity $u$ and the pressure $p$ satisfying

$$
-\text{div}(\nu \nabla u) + \alpha u + \nabla p = f \quad \text{in } \Omega,
$$

$$
\text{div } u = g \quad \text{in } \Omega,
$$

$$
u u = 0 \quad \text{on } \partial \Omega.
$$

(3.7)

Here we assume that parameters $\nu, \alpha \geq 0$ are constants but $\nu \alpha \neq 0$. A weak formulation of (3.7) is to find $(u, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega)$ satisfying

$$
a(u, v) - b(v, p) = (f, v), \quad \forall v \in [H^1_0(\Omega)]^2,
$$

$$
b(u, q) = (g, q), \quad \forall q \in L^2_0(\Omega),
$$

with the bilinear forms

$$
a(u, v) = \nu (\nabla u, \nabla v) + \alpha (u, v), \quad b(v, q) = (\text{div } v, q).
$$

(3.8)

This problem has a unique solution due to the following inf-sup condition

$$
\sup_{v \in [H^1_0(\Omega)]^2} \frac{b(v, q)}{\|v\|_1} \geq C\|q\|_0, \quad \forall q \in L^2_0(\Omega)
$$

(3.9)

according to [3] for all possible $\nu$ and $\alpha$.

Let $\{\mathcal{T}_h\}$ be given as in Subsection 2.2. We select the following finite element spaces $V_h$ and $P_h$:

$$
V_h = \left\{ v \in [L^2(\Omega)]^2 : v|_K \in V_K, \forall K \in \mathcal{T}_h, \int_E [v \cdot n_E] E \, ds = 0 \text{ for all } E \in \mathcal{E}_h, \right. \left. \text{and } v \text{ is continuous at all } V \in V_h \text{ and vanishes at all } V \in V_h \right\}.
$$

$$
P_h = \left\{ q \in L^2_0(\Omega) : q|_K \in P_0(K), \forall K \in \mathcal{T}_h \right\}.
$$

If we write $\text{div}_h|_K = \text{div}$ on $K$, then we have the divergence-free condition $\text{div}_h V_h \subset P_h$. A discrete formulation of (3.8) will be given as: Find $(u_h, p_h) \in V_h \times P_h$, such that

$$
a_h(u_h, v_h) - b_h(v_h, p_h) = (f, v_h), \quad \forall v_h \in V_h,
$$

$$
b_h(u_h, q_h) = (g, q_h), \quad \forall q_h \in P_h,
$$

(3.10)

where $a_h(\cdot, \cdot)$ and $b_h(\cdot, \cdot)$ are discrete versions of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, respectively:

$$
a_h(u_h, v_h) = \nu \sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v_h)_K + \alpha (u_h, v_h),
$$

$$
b_h(u_h, q_h) = (\text{div}_h v_h, q_h).
$$

Moreover, the norm $\| \cdot \|_{1,h}$ and the semi-norms $| \cdot |_{1,h}, \| \cdot \|_{a_h}$ are equipped by

$$
\| v_h \|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} \| v_h \|_{1,K}^2, \quad | v_h |_{1,h}^2 = \sum_{K \in \mathcal{T}_h} | v_h |_{1,K}^2, \quad \| v_h \|_{a_h}^2 = a_h(v_h, v_h).
$$

Clearly, $\| \cdot \|_{a_h}$ is a norm on $V_h$.

For each $K \in \mathcal{T}_h$ and $s > 0$, the nodal interpolation operator $\Pi_K : [H^{1+s}(K)]^2 \to V_K$ is defined via

$$
\sigma_j(\Pi_K v) = \sigma_j(v), \quad j = 1, 2, \ldots, 12. \quad \text{Like the scalar case, we have from Theorem 3.4 that}
$$

$$
| v - \Pi_K v |_{j,K} \leq C h^{k-j} | v |_{k,K}, \quad \forall v \in [H^k(K) \cap H^{1+s}(K)]^2, \quad j = 0, 1, k = 1, 2.
$$

(3.11)
The global interpolation operator $\Pi_h : [H^1_0(\Omega) \cap H^{1+s}(\Omega)]^2 \to V_h$ is naturally set as $\Pi_h|_K = \Pi_K$. Since $\Pi_K$ preserves normal integral on $E \subset \partial K$ for all $K$, we find through integrating by parts that

$$b_h(\Pi_h v, q_h) = b(v, q_h), \quad \forall v \in [H^1_0(\Omega) \cap H^{1+s}(\Omega)]^2, \quad \forall q_h \in P_h. \tag{3.12}$$

Owing to the Scott-Zhang smoothing strategy [22], $\Pi_h$ can be modified into $\overline{\Pi}_h$ interpolating continuously from $[H^1_0(\Omega)]^2$ to $V_h$. Meanwhile, (3.12) holds for all $v \in [H^1_0(\Omega)]^2$ if $\Pi_h$ is replaced by $\overline{\Pi}_h$. Hence, by Fortin’s trick and (3.9), the following discrete inf-sup condition is derived:

$$\sup_{v_h \in V_h} b_h(v_h, q_h) \geq \sup_{v \in [H^1_0(\Omega)]^2} \frac{b_h(\overline{\Pi}_h v, q_h)}{\|\overline{\Pi}_h v\|_{1,h}} \geq \frac{b(v, q_h)}{C\|v\|_1} \geq C\|q_h\|_0, \quad \forall q_h \in P_h. \tag{3.13}$$

Then by Theorem 3.1 in [30], (3.10) has a unique solution $(u_h, p_h) \in V_h \times P_h$, and

$$\|u - u_h\|_{a,h} \leq C \left( \inf_{v_h \in Z_h(g)} \|u - v_h\|_{a,h} + \sup_{w_h \in V_h} \frac{E_h(u, p, w_h)}{\|w_h\|_{a,h}} \right),$$

$$\|p - p_h\| \leq C \left( \|p - p_h\|_0 + M^{1/2} \left( \inf_{v_h \in Z_h(g)} \|u - v_h\|_{a,h} + \sup_{w_h \in V_h} \frac{E_h(u, p, w_h)}{\|w_h\|_{a,h}} \right) \right), \tag{3.14}$$

where $P_h$ is the $L^2$-projection operator from $L^2_0(\Omega)$ to $P_h$, $M = \max\{\nu, \alpha\}$ and

$$Z_h(g) = \{v_h \in V_h : b_h(v_h, q_h) = (g, q_h), \quad \forall q_h \in P_h\},$$

$$E_h(u, p, w_h) = \sum_{K \in T_h} \left( -\nu \int_{\partial K} \frac{\partial u}{\partial n} \cdot w_h \, ds \right) + \nu \int_{\partial K} \frac{\partial p_h}{\partial n} \cdot w_h \, ds. \tag{3.11}$$

Now we are in a position to estimate each term in (3.14). To this end, let $u \in [H^2(\Omega) \cap H^1_0(\Omega)]^2$ be the weak velocity solution of (3.8). It follows from (3.12) that $\Pi_h u \in Z_h(g)$, and therefore by (3.11)

$$\inf_{v_h \in Z_h(g)} \|u - v_h\|_{a,h} \leq \|u - \Pi_h u\|_{a,h} \leq Ch(\nu^{1/2} + \alpha^{1/2})\|u\|_2. \tag{3.15}$$

On the other hand, by (3.9) and (3.10), Theorem 3.3 ensures

$$\int_E |q| |n_E|_E \, ds = 0, \quad \forall q \in P_1(E), \quad \int_E |v| |t_E|_E \, ds = 0, \quad \forall E \in \mathcal{E}_h.$$ 

If $p \in H^2(\Omega)$, then following the spirit of the consistency error analysis in [30], we have

$$E_h(u, p, w_h) \leq \begin{cases} Ch(\nu^{1/2} |u|_2 + \nu^{-1/2} |p|_2), & \text{if } \nu \neq 0; \\ Ch(\nu^{1/2} |u|_2 + \alpha^{-1/2} |p|_2), & \text{if } \alpha \neq 0. \end{cases} \tag{3.16}$$

Substituting (3.15) and (3.10) into (3.14), we will obtain the following convergence result.

**Theorem 3.5.** Let $(u, p) \in ([H^1_0(\Omega) \cap H^2(\Omega)]^2 \times (L^2_0(\Omega) \cap H^2(\Omega))$ be the weak solution of (3.8). The discrete solution of (3.17) is given by $(u_h, p_h) \in V_h \times P_h$. Then the following error estimates hold:

$$\|u - u_h\|_{a,h} \leq Ch \left( (\nu^{1/2} + \alpha^{1/2})\|u\|_2 + \min\{C_1 \nu^{-1/2}h, C_2 \alpha^{-1/2}\} |p|_2 \right),$$

$$\|p - p_h\| \leq Ch \left( |p|_1 + M^{1/2} \left( (\nu^{1/2} + \alpha^{1/2})\|u\|_2 + \min\{C_1 \nu^{-1/2}h, C_2 \alpha^{-1/2}\} |p|_2 \right) \right), \tag{3.17}$$

where we set $\alpha^{-1/2} = +\infty$ if $\alpha = 0$, and $\nu^{-1/2} = +\infty$ if $\nu = 0$. 

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As the scalar case, boundary layers might appear if $\nu \to 0$. In such a Darcy limit, $|u|_2$, $|p|_1$ and $|p|_2$ might explode. We need a uniform convergence result instead of Theorem 3.5. To this end, $\Omega$ is assumed to be a convex polygonal domain with vertices $x_j, j = 1, \ldots, N$ on $\partial \Omega$. We also introduce the space

$$H^1_+(\Omega) = \left\{ q \in H^1(\Omega) \cap L^2_0(\Omega) : \int_\Omega \frac{|q(x)|^2}{|x-x_j|^2} \, dx < \infty, \ j = 1, \ldots, N \right\}$$

with the norm

$$\|q\|_{1,+} = \|q\|_1 + \sum_{j=1}^N \int_\Omega \frac{|q(x)|^2}{|x-x_j|^2} \, dx.$$ 

The following result is an analogue counterpart of Theorem 3.3 in [30], whose proof will be omitted.

**Theorem 3.6.** Assume that $\Omega$ is convex, and $\alpha = 1, \nu \leq 1$ in (3.8). Moreover, the known terms $f \in [H^1(\Omega)]^2$ and $g \in H^1_+(\Omega)$. Let $(u, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega)$ be the weak solution of (3.8). The discrete solution of (3.10) is given by $(u_h, p_h) \in V_h \times P_h$. Then we have the following uniform error estimate

$$\|u - u_h\|_{1,h} + \|p - p_h\|_0 \leq Ch^{1/2} (\|f\|_1 + \|g\|_{1,+}).$$

### 3.3 Finite element exact sequence

In this section, we will see that the finite element spaces $W_h, V_h$ and $P_h$ constitute a discrete de Rham complex.

**Theorem 3.7.** The following finite element sequence is exact.

$$0 \longrightarrow W_h \xrightarrow{\text{curl}_h} V_h \xrightarrow{\text{div}_h} P_h \longrightarrow 0,$$

where $\text{curl}_h|_K = \text{curl}$ on $K$.

**Proof.** We have shown $\text{div}_h V_h \subset P_h$. Moreover, we know that $\text{div}_h$ is surjective due to the discrete inf-sup condition (3.13). Next, we show $\text{curl}_h W_h \subset V_h$. On one hand, for all $K \in \mathcal{T}_h$, owing to the definitions of $V^K_h$ and $W^K_h$, one has $\text{curl}(W^K_h \oplus V^K_h) \subset V^K_h \oplus W^K_h$. Furthermore, by the proof of Lemma 3.2, the relation (3.6) holds for $v = \text{curl}_h w, \forall w \in W^K_h$. This way we find $\text{curl}_h W^K_h \subset V^K_h, \forall K \in \mathcal{T}_h$. On the other hand, $\forall w_h \in W_h$, the definition of $W_h$ ensures the continuous conditions in the definition of $W_h$ for $v_h = \text{curl}_h w_h$, which gives $\text{curl}_h W_h \subset V_h$. To verify $\text{curl}_h W_h = Z_h := \{ v_h \in V_h : \text{div}_h v_h = 0 \}$, it suffices to show the dimensions of this two spaces are the same since $\text{curl}_h W_h \subset Z_h$. Let $N_V, N_1$ and $N_K$ be the numbers of interior vertices, interior edges and cells in $\mathcal{T}_h$, respectively. Then by using Euler’s formula $N_V - N_1 + N_K = 1$, we have

$$\dim Z_h = \dim V_h - \dim (\text{div} V_h) = \dim V_h - \dim P_h = (2N_V + N_1) - (N_K - 1) = 3N_1 = \dim W_h = \dim (\text{curl}_h W_h),$$

which implies the exactness of the sequence. 

**Remark 3.8.** With a Scott-Zhang smoothing trick [22] acting on $\nabla H^2_0(\Omega)$, the interpolation operator $\Pi_h$ in (2.14) can be modified into $\tilde{\Pi}_h$ to work on the whole $H^2_0(\Omega)$ rather than $H^2_0(\Omega) \cap H^{2+s}(\Omega)$ (see e.g. [14]). As a consequence, we have the following commutative diagram:

$$\begin{array}{c}
0 \longrightarrow & H^2_0(\Omega) \xrightarrow{\text{curl}} & [H^1_0(\Omega)]^2 \xrightarrow{\text{div}} & L^2_0(\Omega) \longrightarrow & 0 \\
\downarrow \tilde{\Pi}_h & & \downarrow \Pi_h & & \\
0 \longrightarrow & W_h \xrightarrow{\text{curl}_h} & V_h \xrightarrow{\text{div}_h} & P_h \longrightarrow & 0.
\end{array}$$

(3.19)
Remark 3.9. We end this section by remarking that, the exact sequence (3.18) and commuting diagram (3.19) can be adapted to a more general mesh type, namely, mixed meshes consisting of both triangles and quadrilaterals, in light of the pseudo-\(C^0\) property of \(W_h\) and the pseudo-\(H(\text{div})\) property of \(V_h\). In fact, for a quadrilateral cell \(K\), we can still select \(W_K\) as in (2.10) and \(V_K\) as in (3.6). But if \(K\) is a triangle, the modified nonconforming Zienkiewicz element space due to Wang et al. [24] will be a successful candidate for \(W_K\), and \(V_K\) can also be obtained in a similar manner as in Definition 3.3 from \(W_K\). Then \(W_h, V_h\) are formulated as before, and analogous counterparts of the error estimates Theorems 2.9, 2.10, 3.5 and 3.6 are also appropriate.

4 Numerical examples

Some numerical examples are provided in this section. Let the solution domain \(\Omega\) be the unit square \([0,1]^2\), where three types of convex quadrilateral meshes are considered. As for the first type, each mesh \(T_h\) is generated by an \(n \times n\) uniform rectangular partition. Figure 2(a) provides an example. Meshes of the second type consists of uniform trapezoids, see Figure 2(b). As shown in Figure 2(c) the random partitions are demonstrated as well, which are generated by stochastically deforming the first-type partitions with at most 20\%. In the following examples, the 16-node Gauss quadrature rule is adopted when the entries of stiffness matrices are accumulated for all meshes.

![Figure 2: Three types of quadrilateral partitions of \(\Omega\).](image)

Before the numerical experiment, we give a brief analysis in terms the computational cost in comparison with some other elements working for the same problems over the same meshes reviewed in the introduction part. For fourth order elliptic singular perturbation problems, we have reviewed two \(H^2\)-nonconforming elements in literature, see [2] for the \(H^1\)-conforming construction and [30] for the \(H^1\)-nonconforming one. For our tested \(n \times n\) meshes, both elements are edge-based and the numbers of global DoFs are about \(5n^2\). As far as \(W_h\) in this work is concerned, this number will be about \(3n^2\), reducing the computational costs in some degree benefiting from the nodal type structure. The reduced \(H^2\)-conforming Fraijes de Veubeke-Sander element [5] has the same DoFs as ours, but the shape function space is spline-based, which is less preferred in practical applications than our polynomial selection. For Brinkman problems, we investigate the \(H^1\)-conforming construction in [19] and a nonconforming one in [30]. Again, the number of global DoFs of \(V_h\) in this work is about \(4n^2\), much less than those of the two aforementioned examples: \(8n^2\) for the element in [19] and \(6n^2\) for the other.

We now check the performance of the finite element space \(W_h\) applied to fourth order elliptic singular perturbation problems. The exact solution of (2.11) is arranged as

\[
    u = \sin^2(2\pi x) \sin^2(2\pi y). \tag{4.1}
\]
In Table 1, we list the errors in the energy norm $\| u - u_h \|_{\varepsilon,h}$ with different values of $\varepsilon$ and $h$. The results for the biharmonic equation $\Delta^2 u = f$ as well as the Poisson problem $-\Delta u = f$ with pure Dirichlet boundary conditions are also presented. As predicted in Theorem 2.3, the first order convergence rate is observed for all possible $\varepsilon$.

| $\varepsilon$ | $n = 4$ | $n = 8$ | $n = 16$ | $n = 32$ | $n = 64$ | order |
|---------------|--------|--------|--------|--------|--------|-------|
| rectangular meshes: | | | | | | |
| biharmonic | 2.909E0 | 1.315E0 | 5.913E-1 | 2.804E-1 | 1.368E-1 | 1.04 |
| 1 | 2.913E0 | 1.315E0 | 5.914E-1 | 2.804E-1 | 1.368E-1 | 1.04 |
| $2^{-6}$ | 1.323E-1 | 3.136E-2 | 1.052E-2 | 4.537E-3 | 2.156E-3 | 1.07 |
| $2^{-12}$ | 1.236E-1 | 2.354E-2 | 5.019E-3 | 1.173E-3 | 2.866E-4 | 2.03 |
| Poisson | 1.236E-1 | 2.354E-2 | 5.017E-3 | 1.171E-3 | 2.847E-4 | 2.04 |
| trapezoidal meshes: | | | | | | |
| biharmonic | 3.153E0 | 1.928E0 | 9.336E-1 | 4.562E-1 | 2.251E-1 | 1.02 |
| 1 | 3.158E0 | 1.929E0 | 9.337E-1 | 4.563E-1 | 2.251E-1 | 1.02 |
| $2^{-6}$ | 1.161E-1 | 4.455E-2 | 1.658E-2 | 7.393E-3 | 3.551E-3 | 1.06 |
| $2^{-12}$ | 1.026E-1 | 3.177E-2 | 7.448E-3 | 1.833E-3 | 4.799E-4 | 1.93 |
| Poisson | 1.026E-1 | 3.177E-2 | 7.444E-3 | 1.829E-3 | 4.772E-4 | 1.94 |
| randomly perturbed meshes: | | | | | | |
| biharmonic | 2.677E0 | 1.534E0 | 7.210E-1 | 3.553E-1 | 1.741E-1 | 1.03 |
| 1 | 3.229E0 | 1.485E0 | 7.169E-1 | 3.563E-1 | 1.743E-1 | 1.03 |
| $2^{-6}$ | 1.286E-1 | 3.424E-2 | 1.292E-2 | 5.698E-3 | 2.749E-3 | 1.05 |
| $2^{-12}$ | 1.175E-1 | 2.662E-2 | 5.998E-3 | 1.453E-3 | 3.715E-4 | 1.97 |
| Poisson | 1.164E-1 | 2.713E-2 | 5.971E-3 | 1.445E-3 | 3.687E-4 | 1.97 |

Table 1: The errors $\| u - u_h \|_{\varepsilon,h}$ produced by $W_h$ applied to the given fourth order elliptic singular perturbation problem through (4.1) over three kinds of meshes.

We then turn to the performance of the mixed finite element pair $V_h \times P_h$ applied to the Brinkman problem. We fix the parameter $\alpha = 1$ in (3.7) and test different $\nu \in (0,1]$. The cases for the pure Darcy problem ($\nu = 0$, $\alpha = 1$) and Stokes problem ($\nu = 1$, $\alpha = 0$) are also investigated. The exact solution of (3.7) is determined by

$$u = \text{curl} \left( \sin^2(\pi x) \sin^2(\pi y) \right), \quad p = \sin(\pi x) - 2/\pi. \quad (4.2)$$

Tables 2 and 3 show the velocity and pressure errors, respectively. The optimal convergence rate is also achieved for all possible parameters.

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Table 2: The velocity errors $\|u - u_h\|_{a_h}$ produced by $V_h \times P_h$ applied to the Brinkman problem through (4.2) over three kinds of meshes.

| $\nu^{1/2}$ | $n = 4$ | $n = 8$ | $n = 16$ | $n = 32$ | $n = 64$ | order |
|-------------|---------|---------|---------|---------|---------|-------|
| rectangular meshes: | | | | | | |
| Stokes | 3.186E0 | 1.503E0 | 6.926E-1 | 3.324E-1 | 1.631E-1 | 1.03 |
| 1 | 3.190E0 | 1.503E0 | 6.927E-1 | 3.324E-1 | 1.631E-1 | 1.03 |
| $2^{-6}$ | 1.340E-1 | 3.340E-2 | 1.194E-2 | 5.327E-3 | 2.564E-3 | 1.05 |
| $2^{-12}$ | 1.236E-1 | 2.355E-2 | 5.019E-3 | 1.174E-3 | 2.874E-4 | 2.03 |
| Darcy | 1.236E-1 | 2.354E-2 | 5.017E-3 | 1.171E-3 | 2.847E-4 | 2.04 |
| trapezoidal meshes: | | | | | | |
| Stokes | 3.345E0 | 2.103E0 | 1.025E0 | 5.029E-1 | 2.486E-1 | 1.02 |
| 1 | 3.349E0 | 2.103E0 | 1.025E0 | 5.029E-1 | 2.486E-1 | 1.02 |
| $2^{-6}$ | 1.175E-1 | 4.662E-2 | 1.789E-2 | 8.105E-3 | 3.916E-3 | 1.05 |
| $2^{-12}$ | 1.027E-1 | 3.181E-2 | 7.497E-3 | 1.884E-3 | 5.282E-4 | 1.84 |
| Darcy | 1.026E-1 | 3.181E-2 | 7.492E-3 | 1.880E-3 | 5.257E-4 | 1.84 |
| randomly perturbed meshes: | | | | | | |
| Stokes | 3.276E0 | 1.617E0 | 8.079E-1 | 4.022E-1 | 1.991E-1 | 1.01 |
| 1 | 3.294E0 | 1.711E0 | 8.244E-1 | 3.953E-1 | 1.990E-1 | 0.99 |
| $2^{-6}$ | 1.378E-1 | 3.720E-2 | 1.392E-2 | 6.439E-3 | 3.117E-3 | 1.05 |
| $2^{-12}$ | 1.183E-1 | 2.676E-2 | 5.933E-3 | 1.492E-3 | 3.689E-4 | 2.02 |
| Darcy | 1.317E-1 | 2.709E-2 | 6.026E-3 | 1.468E-3 | 3.778E-4 | 1.96 |

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| \( \nu^{1/2} \) | \( n = 4 \) | \( n = 8 \) | \( n = 16 \) | \( n = 32 \) | \( n = 64 \) | order |
|---|---|---|---|---|---|---|
| **rectangular meshes:** Stokes | 4.593E-1 | 0.201E-1 | 5.810E-2 | 2.223E-2 | 1.027E-2 | 1.11 |
| 1 | 4.616E-1 | 0.202E-1 | 5.827E-2 | 2.225E-2 | 1.027E-2 | 1.11 |
| \( 2^{-6} \) | 1.586E-1 | 7.995E-2 | 4.005E-2 | 2.003E-2 | 1.001E-2 | 1.00 |
| \( 2^{-12} \) | 1.586E-1 | 7.995E-2 | 4.005E-2 | 2.003E-2 | 1.001E-2 | 1.00 |
| Darcy | 1.586E-1 | 7.995E-2 | 4.005E-2 | 2.003E-2 | 1.001E-2 | 1.00 |
| **trapezoidal meshes:** Stokes | 1.478E-1 | 5.967E-1 | 2.443E-1 | 1.158E-1 | 5.680E-2 | 1.03 |
| 1 | 1.480E-1 | 5.978E-1 | 2.445E-1 | 1.158E-1 | 5.681E-2 | 1.03 |
| \( 2^{-6} \) | 1.569E-1 | 7.906E-2 | 3.960E-2 | 1.981E-2 | 9.907E-3 | 1.00 |
| \( 2^{-12} \) | 1.569E-1 | 7.906E-2 | 3.960E-2 | 1.981E-2 | 9.907E-3 | 1.00 |
| Darcy | 1.569E-1 | 7.906E-2 | 3.960E-2 | 1.981E-2 | 9.907E-3 | 1.00 |
| **randomly perturbed meshes:** Stokes | 6.333E-1 | 2.721E-1 | 1.163E-1 | 5.723E-2 | 2.674E-2 | 1.10 |
| 1 | 5.373E-1 | 3.083E-1 | 1.231E-1 | 5.165E-2 | 2.633E-2 | 0.97 |
| \( 2^{-6} \) | 1.620E-1 | 8.170E-2 | 4.115E-2 | 2.054E-2 | 1.030E-2 | 1.00 |
| \( 2^{-12} \) | 1.646E-1 | 8.191E-2 | 4.122E-2 | 2.056E-2 | 1.031E-2 | 1.00 |
| Darcy | 1.585E-1 | 8.231E-2 | 4.119E-2 | 2.058E-2 | 1.030E-2 | 1.00 |

Table 3: The pressure errors \( \| p - p_h \|_0 \) produced by \( V_h \times P_h \) applied to the Brinkman problem through (4.2) over three kinds of meshes.

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