KNOTS IN VIRTUALLY FIBERED 3-MANIFOLDS

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Abstract. The theory of virtual covers is extended to knots in manifolds which are not fibered over $S^1$. A theory of virtual covers is presented which is based upon the relation of commensurability of 3-manifolds. The theory is applied to separating knots in fibered link complements and knots in non-fibered, virtually fibered 3-manifolds.

1. Introduction

1.1. Motivation. Knots in compact orientable 3-manifolds can be studied using virtual knot theory. This statement is more of a principle than a theorem. The principle holds when the manifold is a fibered knot complement [8]. Knots living in such manifolds can be distinguished from one another using virtual knot invariants. Moreover, their geometric [8] and combinatorial properties [8, 21] can be detected. The objective is to extend the principle to as large a class of manifolds as possible. The present paper will investigate virtually fibered manifolds [23, 24, 32] and the relation of commensurability [3, 33].

The method may be described in its most general form as follows. Let $N$ be a compact orientable smooth 3-manifold and let $K$ be an oriented knot in $N$. Suppose that $N$ admits a covering space $\Pi: \Sigma \times (0, 1) \to N$, where $\Sigma$ is a compact, connected, orientable surface. Furthermore, suppose that $\mathfrak{t}$ is an oriented knot in $\Sigma \times (0, 1)$ such that $\Pi(\mathfrak{t}) = K$. The knot $\mathfrak{t}$ can be considered as a knot in $\Sigma \times I$, where $I$ is the closed unit interval. As is well-known, there is a one-to-one correspondence $\kappa$ between stability classes of knots in thickened surfaces and virtual knots. Let $v = \kappa(\mathfrak{t})$ be the virtual knot associated to the stability class of $\mathfrak{t}$ in $\Sigma \times I$. A schematic depiction is below.

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The situation described in the preceding paragraph is called a virtual cover with associated virtual knot \( \upsilon \). The virtual cover will be denoted \((\mathbb{E}^\Sigma \times (0,1), \Pi, K^N)\).

If the manifold \( N \) is the complement of a fibered knot \( J \) in \( S^3 \) and \( K \) is a knot in \( N \), then \( K \) has a virtual cover with an associated virtual knot whenever \( \text{lk}(J,K) = 0 \) (see [8]). If \( K \) is close enough to a fiber of the fibration (see Definition 3.1 below), then \( \upsilon \) is an invariant of the ambient isotopy class of \( K \) in \( N \) (compare with [11]). Virtual knot invariants can then be applied to \( \upsilon \) to ascertain properties of \( K \). For example, the technique can be used to separate inequivalent knots, detect non-invertibility, and prove minimality theorems for knot diagrams of \( K \) [21].

More generally, suppose \( N \) has either empty boundary or boundary consisting only of tori. Then \( N \) is said to be fibered if it is the total space of a fiber bundle over \( S^1 \) whose fiber is a compact orientable surface [23]. If \( N \) possesses a finite index covering that is fibered, then \( N \) is said to be virtually fibered. The virtual fibering conjecture [31] of Thurston implies that all non-fibered hyperbolic link complements have finite index coverings by 3-manifolds which are fibered [32]. If \( N \) is a closed hyperbolic 3-manifold, then \( N \) is virtually fibered [1].

It is thus natural to generalize the theory of virtual covers to knots in virtually fibered 3-manifolds. The theory developed in the present paper will be built upon the relation of commensurability. Manifolds \( N_1 \) and \( N_2 \) are said to be commensurable if there is a 3-manifold \( M \) and finite index coverings \( \Pi_1 : M \to N_1, \Pi_2 : M \to N_2 \). Commensurability is an equivalence relation on manifolds. It has proven to be a useful tool in studying virtually fibered manifolds (cf. [3, 23, 32, 33]). In particular, if a commensurability class contains a virtually fibered manifold, then every representative of the class is virtually fibered.

Our interest lies in what happens to virtual covers of knots in manifolds under the relation of commensurability. Suppose that \( N_1, N_2 \) are commensurable 3-manifolds and \( M, \Pi_1, \Pi_2 \) are as above. Let \( K_1, K_2 \) be knots in \( N_1, N_2 \), respectively. Suppose there is a knot \( K \) in \( M \) such that \( \Pi_1(K) = K_1, \Pi_2(K) = K_2 \), and \( K \) is contained in a fundamental region of \( \Pi_1 \) and a fundamental region of \( \Pi_2 \). Then we will say that \( K_1 \) and \( K_2 \) are commensurable knots in manifolds. It will be shown (under certain general hypotheses) that if \( K_1 \) has a virtual cover with associated virtual knot \( \upsilon_1 \), \( K_2 \) has a virtual cover with associated virtual knot \( \upsilon_2 \) and \( K_1, K_2 \) are commensurable knots in manifolds, then \( \upsilon_1 \equiv \upsilon_2 \) as virtual knots.

The main application of this general theorem will be to studying knots in a fixed manifold. Several cases are considered in detail. Virtual knot invariants will be used to detect inequivalent knots in fibered link complements and in non-fibered, virtually fibered link complements. In addition, virtual knot invariants will be used to prove the non-separability of a three component link in \( S^3 \). Other applications of the main theorem will be considered in future papers.
1.2. Statement of Main Theorem. A few definitions and notations will be provided prior to the statement of the main theorem.

**Definition 1.1** (Knot in \( N \), Equivalent Knots). Let \( N \) be a smooth connected oriented manifold with boundary (possibly empty). A smooth embedding \( K : S^1 \to N \) is said to be a knot in \( N \), denoted \( K^N \). \( K_1^N \) and \( K_2^N \) are said to be equivalent in \( N \) if there is an ambient isotopy \( H : N \times I \to N \) taking \( K_1 \) to \( K_2 \) and satisfying the property that \( H_t|_{\partial N} = \text{id}|_{\partial N} \) for all \( t \in I \). If \( K_1^N \) and \( K_2^N \) are equivalent in \( N \), we write \( K_1^N \equiv K_2^N \).

**Notation 1.2** (Superscript Convention). It will frequently occur that we have a knot \( K_i \) in a manifold \( N_i \). This leads to the awkward notation \( K_i^N_i \) where the indices are repeated. Define \( K^i := K_i^{N_i} \). Note that the superscript is given in fraktur. Hence \( K^3 \) denotes a knot \( K_3 \) in a manifold \( N_3 \). In the special case where it is assumed we have a knot in a thickened surface, we will write \( \mathfrak{t}^\bullet \) to mean a knot \( \mathfrak{t}_i \) in a manifold \( \Sigma_i \times (0, 1) \). For example, \( \mathfrak{t}^4 := \mathfrak{t}_4^{\Sigma \times (0, 1)} \).

The definition of a virtual cover with associated virtual knot was given in Section 1.1. It will be useful in our study of commensurability to generalize this definition. Note: Henceforward, some virtual covers will have associated virtual knots whereas others will not.

**Definition 1.3** (Virtual Cover). Let \( K^0 \) and \( K^1 \) be knots. A virtual cover is an orientation preserving covering space projection \( \Pi : N_0 \to N_1 \) such that \( \Pi(K_0) = K_1 \). If \( K_0, K_1 \) are oriented, it is furthermore assumed that \( \Pi \) preserves the orientation of the knots. A virtual cover is denoted by the triple \( (K^0, \Pi, K^1) \).

If the covering space of \( N \) is a thickened surface \( \Sigma \times (0, 1) \), then a virtual knot can be assigned to \( K \). Recall from Section 1.1 that \( \kappa \) denotes the one-to-one correspondence between stability classes of oriented knots in thickened surfaces \( \Sigma \times I \) and oriented virtual knots.

**Definition 1.4** (Associated Virtual Knot). Let \( (\mathfrak{t}^{\Sigma \times (0, 1)}, \Pi, K^N) \) be a virtual cover, where \( \Sigma \) is a compact connected oriented surface and \( \mathfrak{t}, K \) are oriented. The oriented virtual knot \( v = \kappa(\mathfrak{t}^{\Sigma \times I}) \) is called the associated virtual knot to \( (\mathfrak{t}^{\Sigma \times (0, 1)}, \Pi, K^N) \).

The present paper will consider only the case that the knot in the covering space is contained in a fundamental region of the covering. This will be called a fundamental virtual cover. The knots in fibered knot complements considered in \( \mathbb{S} \) are examples of fundamental virtual covers.

**Definition 1.5** (Fundamental Region). Let \( p : M \to N \) be a covering projection and \( G \) the group of covering transformations. Suppose furthermore that \( G \) acts transitively on each fiber of \( p \). A set \( U \subset M \) is said to be a fundamental region of \( p \) if \( U \) has nonempty interior, \( M = \bigcup_{g \in G} g(U) \), and for all \( g_1, g_2 \in G \), where \( g_1 \neq g_2 \), if \( x \in g_1(U) \cap g_2(U) \) then \( x \) is in the topological boundary of \( g_1(U) \) and the topological boundary of \( g_2(U) \).

**Definition 1.6** (Fundamental Virtual Cover). A virtual cover \( (K^0, \Pi, K^1) \) is fundamental if \( \Pi \) is regular and there is a fundamental region \( U \) of \( \Pi \) such that \( \text{im}(K_0) \subseteq \text{int}(U) \).
The precise definition of commensurability for knots in manifolds can now be given. First we define the notion of elementary commensurability. By itself, this will not be an equivalence relation. We will define commensurability to be a finite sequence of elementary commensurabilities. The reader will note that this definition is similar to the definition of virtual knots given in [3].

**Definition 1.7 (Elementary Commensurable Knots in Manifolds).** Let $K^1, K^2$ be oriented knots with $N_1, N_2$ compact, connected, oriented 3-manifolds. $K^1, K^2$ are said to be **elementary commensurable** if there is a knot $K^0$ and for $i = 1, 2$, there is a fundamental virtual cover $(K^0, \Pi_i, K^i)$ such that $N_0$ is compact, connected, and oriented, and $\Pi_i$ is of finite index. If $K^1$ and $K^2$ are elementary commensurable knots, we write $K^1 \equiv K^2$.

**Definition 1.8 (Commensurability of Knots in Manifolds).** If there is a finite sequence $K^0, K^1, \ldots, K^{m-1}, K^m$ of knots in manifolds such that

$$K^0 \equiv K^1 \equiv \ldots \equiv K^{m-1} \equiv K^m,$$

then we say that $K^0$ and $K^m$ are **commensurable**. If $J^1$ and $J^2$ are knots and there is such a finite sequence starting at $J^1$ and ending at $J^2$, then we say that $J^1$ and $J^2$ are **commensurable**. If $J^1$ and $J^2$ are commensurable knots in manifolds, we write $J^1 \approx J^2$.

**Remark 1.1.** This definition is inspired by the idea of commensurable manifolds and commensurable knots in $S^3$ (cf. [3, 23]).

The main theorem describes how the associated virtual knots of fundamental virtual covers behave under the relation of commensurability.

**Theorem 1.** Let $J^1, J^2$ be oriented knots and suppose that $J^1 \equiv J^2$.

1. If $(\mathfrak{k}^1, \Pi_1, J^1)$ is a fundamental virtual cover with associated virtual knot $v_1$, then there is a virtual cover $(\mathfrak{k}^2, \Pi_2, J^2)$ with associated virtual knot $v_2$ such that $v_1 \approx v_2$ as virtual knots.

2. If $(\mathfrak{k}^1, \Pi_1, J^1)$ is a fundamental virtual cover with associated virtual knot $v_1$ and $(\mathfrak{k}^2, \Pi_2, J^2)$ is a fundamental virtual cover with associated virtual knot $v_2$, then $v_1 \approx v_2$ as virtual knots.

Thus, if a fundamental virtual cover with associated virtual knot $v$ can be found for some convenient element of the commensurability class of $K^N$, then the associated virtual knot is an invariant of $K^N$.

An outline of the present paper is as follows. A brief review of virtual knot theory will be given in Section 1.3. The proof of Theorem 1 will be given in Section 2. The first application of Theorem 1 will be to knots in fibered link complements (see Section 3). The second application will be to knots in non-fibered, virtually fibered link complements. In particular, Section 4 will show that our technique can be successfully applied to knots in Gabai’s example [15] of a non-fibered, virtually fibered, hyperbolic 3-manifold.

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1.3. **Review of Virtual Knot Theory.** This review will highlight only a few aspects of virtual knot theory. References are provided so that readers unfamiliar with the field may familiarize themselves with the standard terminology.

1.3.1. **Virtual Knots and Their Invariants.** The four models of virtual knots are:

1. virtual knot diagrams modulo the extended Reidemeister moves \[20\],
2. Gauss diagrams modulo diagrammatic Reidemeister moves \[16\],
3. abstract knot diagrams modulo Kamada–Kamada equivalence \[6, 19\], and
4. knots in thickened surfaces modulo stabilization and destabilization \[22\].

There is a one-to-one correspondence between the equivalence classes of each interpretation (see \[10\]). If we have a representative \(R\) of one of the models (2), (3), or (4), we will also denote by \(\kappa(R)\) the corresponding equivalence class of virtual knot diagrams from model (1). The four models for a single virtual knot are shown below.

![Virtual Knot Models](image)

If a virtual knot has a diagram whose crossings are all classical, then it is said to be a classical knot. If two classical knots are equivalent as virtual knots, then they are equivalent as knots in \(S^3\) \[16\]. A virtual knot is classical if and only if it can be represented as a diagram on a compact planar surface.

Virtual knot invariants can be classified into two types: those that are extensions of non-trivial classical knot invariants and those that are constant on the set of classical knots. For example, the Kauffman bracket polynomial has a natural extension to virtual knots \[20\]. The arrow (or Miyazawa) polynomial \[9\] and the parity bracket \[17\] are two other extensions of the Kauffman bracket. They agree with the Kauffman bracket on the set of classical knots but not on the set of non-classical knots. On the other hand, the Sawollek polynomial \[27\] vanishes on the set of classical knots but is non-trivial on the set on non-classical knots. The same is true for the odd writhe \[20\].

1.3.2. **Parity and the Odd Writhe.** An important tool in virtual knot theory is parity \[17, 25\]. Let \(K\) be a virtual knot diagram and let \(x\) be a classical crossing. Starting at \(x\), traverse the diagram and count the number of times \(N\) that a classical crossing is passed through before returning to \(x\). Then \(x\) is said to be an even crossing or an odd crossing according to whether \(N\) is an even integer or an odd integer, respectively. When all the classical crossings are so labeled, we have the Gaussian parity of the diagram.

The odd writhe \[20\] mentioned above can be constructed from the Gaussian parity. It
is the sum of the local crossing numbers ($\pm 1$, according to the right/left hand rule) of the odd crossings. As a classical knot must have a diagram such that all the crossings are even, the odd writhe can be used to demonstrate that a virtual knot is non-classical.

There are many invariants of virtual knots which arise from parity [18]. Moreover, parity has a well-developed theory of its own [17]. We will use only the odd writhe in the sequel. For more information on how parity is related to virtual covers, the reader is referred to [8, 21].

1.3.3. Related Work. Knots and links in twisted I-bundles can be studied using an analogue of virtual knot theory [4]. In [11], Fiedler introduced an knot invariant that can be thought of as a predecessor of virtual covers. Another approach which uses liftings of knots to covering spaces has been discovered by Carter–Silver–Williams [7]. Knots in thickened orientable closed surfaces can be lifted to the thickened universal cover. The fundamental group of the complement of the lift is called the covering group. The covering group has many useful applications for virtual knot theory. For example, it can be used to determine the virtual genus of a virtual link.

2. Proof of Theorem 1

Theorem 1 states that if two commensurable knots in manifolds have fundamental virtual covers with associated virtual knots, then the associated virtual knots are equivalent as virtual knots. The purpose of this section is to provide a proof of this theorem. Section 2.1 establishes several lemmas that will be needed. Section 2.2 introduces a construction that will simplify the main induction argument. Section 2.3 contains the body of the proof.

2.1. Lemmas for the Proof of Theorem 1

One advantage of commensurability is that equivalent knots in a manifold are commensurable knots in manifolds. Thus to show that two knots in a given manifold are inequivalent, it is sufficient to show that they are not commensurable. This is the content of the following lemma.

Lemma 2. Let $N$ be a compact, connected, and oriented 3-manifold. Let $K^N_1, K^N_2$ be knots. If $K^N_1 \equiv K^N_2$, then $K^N_1 \cong K^N_2$.

Proof. Let $H : N \times I \to N$ be the ambient isotopy demonstrating the equivalence of $K_1$ and $K_2$ in $N$. Then $H_1$ is an orientation preserving homeomorphism such that $H_1(K_1) = K_2$. Also, $\text{id}_N : N \to N$ is covering projection taking $K_1$ to $K_1$. Note that $\text{id}_N : N \to N$ and $H_1 : N \to N$ are index one covering projections. In this case, $N$ is itself a fundamental region of $\text{id}_N$ and $H_1$. Thus, $K^N_1 \cong K^N_2$. \hfill $\square$

An ingredient in the proof of Theorem 1 is the reduction to the case of knots in thickened surfaces. Specifically, it is necessary to understand the case of a fundamental virtual cover of a knot in a thickened compact surface by a knot in a thickened surface (not necessarily compact). The following lemma addresses this issue.
Lemma 3. Let $(\Psi, q \times \text{id}, t_i)$ be a fundamental virtual cover, where $\Sigma_i$ is a compact connected oriented surface and $t_i$ are oriented. Suppose furthermore that for $i = 1, 2$, $t_i$ is regular with respect to the canonical projection $\Sigma_i \times (0, 1) \rightarrow \Sigma_i$. Then there is compact connected oriented surface $\Psi_0$ such that $\operatorname{im}(t_0) \subseteq \Psi_0 \times (0, 1)$. Moreover,

$$\kappa(\Psi_0 \times I) \subseteq \kappa(t_1 \times I).$$

Proof. Since $q \times \text{id}$ is a fundamental virtual cover there is a fundamental region $U$ of $q$ such that $U \times (0, 1)$ is a fundamental region of $q \times \text{id}$. The knots $t_0, t_1$ can be represented as diagrams $D_0, D_1$ on $\Sigma_0, \Sigma_1$, respectively.

By hypothesis, $q \times \text{id}$ is one-to-one on $\operatorname{int}(U \times (0, 1))$. Since $\operatorname{im}(t_0) \subseteq \operatorname{int}(U \times (0, 1))$ and $(q \times \text{id})(t_0) = t_1$, there is a one-to-one correspondence between the arcs of $D_0$ and the arcs of $D_1$. In addition, there is a one-to-one correspondence between the crossings of $D_0$ and the crossings of $D_1$. Moreover, two crossings of $D_1$ are connected by an arc in $D_1$ if and only if the corresponding crossings in $D_0$ are connected by the corresponding arc in $D_0$.

Let $\Psi_1$ be a closed regular neighborhood of the immersed curve underlying $D_1$ such that $\Psi_1 \subseteq \operatorname{int}(q(U))$. Since $\Sigma_1$ is compact, $\Psi_1$ is a compact surface. Let $\Psi_0$ be the component of $q^{-1}(\Psi_1)$ containing the diagram $D_0$. It follows that $q|_{\Psi_0} : \Psi_0 \rightarrow \Sigma_1$ is an orientation preserving embedding taking $D_0$ to $D_1$. The Kamada–Kamada interpretation [19] of virtual knots implies that $t_{0 \times I}$ and $t_{1 \times I}$ stabilize to equivalent virtual knots. □

In the proof of Theorem 1 we will combine several different fundamental virtual covers into a single fundamental virtual cover. The following lemma provides a step towards this amalgamation. It states that if a knot in a manifold has two fundamental virtual covers, then those two fundamental virtual covers have a common fundamental virtual cover.

Lemma 4. Let $N_0$ be a smooth, compact, connected, oriented, 3-manifold. For $i = 1, 2$, let $(K^i, \Pi_i, K^i\text{'})$ be fundamental virtual covers, where $K_i$ is oriented and $N_i$ is a smooth oriented, connected, paracompact 3-manifold. Then for $i = 1, 2$, there is a fundamental virtual cover $(K^M, \Delta_i, K^i\text{'})$, where $M$ is a smooth, connected, orientable 3-manifold. The maps satisfy $\Pi_1 \Delta_2 = \Pi_2 \Delta_1$. Also, if $\Pi_1, \Pi_2$ are of finite index and $N_1, N_2$, are compact, then $M$ is compact.

Proof. Let $M_0$ be the pullback of $\Pi_1$ to $N_2$. Then we obtain the commutative diagram below.

\[
\begin{array}{ccc}
M_0 & \to & N_1 \\
\Pi_1' \downarrow & & \downarrow \Pi_1 \\
N_2 & \xrightarrow{\Pi_2} & N_0
\end{array}
\]

Since $N_1$ and $N_2$ are paracompact and Hausdorff, it follows that (see [28], 2.7.14, 2.8.6, and 2.4.10) that $\Pi_1'$ and $\Pi_2'$ are covering projections. Moreover we have that $M_0$ is a smooth orientable 3-manifold.
Recall that $M_0$ may be described as the set of ordered pairs $(x, y)$ in $N_1 \times N_2$ such that $\Pi_1(x) = \Pi_2(y)$. We may suppose that $K_1$ and $K_2$ are parameterized so that their orientations are induced by the counterclockwise orientation of $S^1$. Define $K : S^1 \to M_0$ by $K(z) = (K_1(z), K_2(z))$. Then $K$ is an oriented knot in $M_0$ such that $\Pi'_1(K) = K_2$ and $\Pi'_2(K) = K_1$.

The knot $K$ is contained in a fundamental region of $\Pi'_1$ and a fundamental region of $\Pi'_2$. To see this, let $U_1$ be a fundamental region such that $\text{im}(K_1) \subseteq \text{int}(U_1)$ and let $U_2$ be a fundamental region of $\Pi_2$ such that $\text{im}(K_2) \subseteq \text{int}(U_2)$. Define $U'_1 = (U_1 \times N_2) \cap M_0$ and $U'_2 = (N_1 \times U_2) \cap M_0$.

For $i = 1, 2$, let $G_i$ be the group of covering transformations of $\Pi_i$. The group $G'_1$ of covering transformations of $\Pi'_1$ can be identified with the homeomorphisms of $M_0$ of the form $g \times \text{id}$ for $g \in G_1$. Similarly, the group $G'_2$ of covering transformations of $\Pi'_2$ can be identified with the homeomorphisms of $M_0$ of the form $\text{id} \times g$ for $g \in G_2$. Then $G'_i$ and $G''_i$ act transitively on each fiber. The reader can now verify that $U'_1$ and $U'_2$ are fundamental regions of $\Pi'_1$ and $\Pi'_2$, respectively.

It may occur that $M_0$ is not path connected. Let $M$ be the path component of $M_0$ containing $\text{im}(K)$ and for $i = 1, 2$, let $\Delta_i$ be the restriction of $\Pi'_i$ to $M$. If $M \neq M_0$, then the group of covering transformations and the fundamental regions for $\Pi'_1$ and $\Pi'_2$ are altered as follows. The group $G''_i$ of covering transformations of $\Delta_i$ can be identified with a subgroup of $G'_i$ which acts transitively on each fiber. If $\{g^j_{i,j}\}$ is a set of coset representatives of $G''_i$ in $G'_i$, then $U''_i = M \cap \bigcup_j g^j_{i,j}(U'_i)$ is a fundamental region for $\Delta_i$. Thus we have for $i = 1, 2$ a fundamental virtual cover $(K^M, \Delta_i, K')$.

The last claim follows from the fact that each $N_i$ can be triangulated by a finite number of tetrahedra so small that each tetrahedron is evenly covered by $\Pi'_i$. Under the hypothesis, $\Pi'_i$ is also of finite index. Hence, $M$ is compact. \hfill $\Box$

**Remark 2.1.** Compare with the discussion in [23, 33].

Now consider the case of Lemma 4 where one of the fundamental virtual covers is a fundamental virtual cover with associated virtual knot $v_1$ and the second fundamental virtual cover has finite index. The following lemma shows that in this case, the fundamental virtual cover given by Lemma 4 is a fundamental virtual cover with associated virtual knot $v_2$ and that $v_1 \equiv v_2$ as a virtual knots.

**Lemma 5.** Let $N_0$, $N_1$ be compact connected oriented 3-manifolds and $K^\circ$, $K^\dagger$ oriented knots. Let $(K^\circ, \Delta_1, K^\dagger)$ be a fundamental virtual cover with $\Delta_1$ of finite index. Suppose $(\Pi^\circ, \Pi_1, K^\dagger)$ is a fundamental virtual cover with associated virtual knot $v_1$. Then there is a fundamental virtual cover $(\Pi^\circ, \Pi_0, K^\circ)$ with associated virtual knot $v_0$ such that $v_0 \equiv v_1$ as virtual knots.
Proof. Apply Lemma \ref{lem:virtual_knots} to the fundamental virtual covers \((K^o, \Delta_1, K^1)\) and \((\ell^1, \Pi_1, K^1)\). This gives an oriented knot \(K^M\) and fundamental virtual covers \((K^M, \Delta_1', \ell^1)\), \((K^M, \Pi_1', K^o)\) such that \(\Pi_1\Delta_1' = \Delta_1\Pi_1'\). It will be shown that \(M \approx \Xi \times (0,1)\) where \(\Xi\) is a compact connected orientable surface.

Recall that \(\ell_1\) is a knot in \(\Sigma_1 \times (0,1)\), where \(\Sigma_1\) is a compact, connected, oriented, surface. Consider the following commutative diagram (see below for additional definitions).

\[
\begin{array}{ccc}
M & \leftarrow & M'' \\
\downarrow \Delta_1' & & \downarrow \Delta_1'' \\
\Sigma_1 \times (0,1) & \leftarrow & \Sigma_1 \times I \\
\end{array}
\]

The bottom left map is defined by \(f'(s,t) = (s, \frac{1}{s}(x + 1))\). The bottom right map is defined to be the inclusion, i.e \(f''(s,t) = (s,t)\). The bundles \((M'', \Delta_1', \Sigma_1 \times I)\) and \((M'', \Delta_1'', \Sigma_1 \times (0,1))\) are pullbacks over \(f'\) and \(f''\), respectively.

Since \(\Sigma_1 \times I\) and \(\Sigma_1 \times (0,1)\) are paracompact, it follows (see \cite{28}, 2.7.14, 2.8.6, and 2.4.10) that \(\Delta_1'\) and \(\Delta_1''\) are covering projections. Also, we have that \((M'', \Delta_1', \Sigma_1 \times I)\) is equivalent to a bundle of the form \((\Xi \times I, q \times \text{id}, \Sigma_1 \times I)\) (see \cite{30}, Theorem 11.4). The map \(q\) is a covering projection, so that \(\Xi\) is an orientable smooth 2-manifold.

The bundles \((M'', \Delta_1', \Sigma_1 \times (0,1))\) and \((\Xi \times (0,1), q \times \text{id}, \Sigma_1 \times (0,1))\) are equivalent (see \cite{30}, Theorem 10.3). The map \(f' f''\) is homotopic to the identity, so \((M, \Delta_1', \Sigma_1 \times (0,1))\) and \((\Xi \times (0,1), q \times \text{id}, \Sigma_1 \times (0,1))\) are equivalent (\cite{30}, Theorem 11.5). Thus, \(M \approx \Xi \times (0,1)\).

Since \(\Delta_1\) is of finite index, it follows that \(\Delta_1''\) is of finite index. Hence, \(\Xi\) is compact. Thus the equivalence of bundles described above allows for an identification of \(K^M\) with a knot \(\ell^o\), where \(\Sigma_0 = \Xi\) is a compact, connected, orientable, smooth 2-manifold. These observations imply that there are fundamental virtual covers \((\ell^o, \Pi_0, K^o)\) and \((\ell^o, q \times \text{id}, \ell^1)\).

Let \(\nu_0\) be the virtual knot associated to \((\ell^o, \Pi_0, K^o)\). Consider the fundamental virtual cover \((\ell^o, q \times \text{id}, \ell^1)\). By Lemma \ref{lem:virtual_knots} and a general position argument, it follow that \(\nu_0 \equiv \nu_1\) as virtual knots. \(\square\)

### 2.2. Commutative Triangles

This section gives a construction which simplifies the induction argument for Theorem \ref{thm:main}. The idea is that a finite sequence of elementary commensurabilities allows for successive applications of Lemma \ref{lem:virtual_knots} if the resulting spaces and maps are placed in a single commutative diagram, then the result resembles a triangle. Hence we have the following definition of *commutative triangle*.

**Definition 2.1.** Let \(n\) be a non-negative integer. Suppose that for non-negative integers \(m, j\), where \(0 \leq m \leq n + 1\) and \(0 \leq j \leq m\), we have a collection of topological spaces \(X_{m,j}\). Suppose furthermore that for all \(m\) and \(j\), where \(0 \leq m \leq n\) and \(0 \leq j \leq m\), there
are maps $\Delta_{m,j} : X_{m,j} \to X_{m+1,j+1}$ and $\nabla_{m,j} : X_{m,j} \to X_{m+1,j}$, such that for all $m, j$, the following condition is satisfied:

$$\nabla_{m+1,j+1} \Delta_{m,j} = \Delta_{m+1,j} \nabla_{m,j}.$$ 

Then the collection of maps and spaces is called a *commutative triangle*. The subscripts on the maps in a commutative triangle will usually be omitted. The commutativity condition will be abbreviated by the expression $\nabla \Delta = \Delta \nabla$.

**Lemma 6.** Let $K^0, \ldots, K^n$ be a sequence of knots in manifolds such that:

$$K^0 \equiv K^1 \equiv \cdots \equiv K^{n-1} \equiv K^n.$$

For all non-negative integers $m, j$, where $0 \leq m \leq n-1$ and $0 \leq j \leq m$, there is a collection of fundamental virtual covers $(K^{m,j}, \nabla, K^{m+1,j})$, $(K^{m,j}, \Delta, K^{m+1,j+1})$, such that:

1. The spaces $N_{m,j}$ and maps $\nabla, \Delta$ form a commutative triangle, and
2. For $0 \leq i \leq n$, $K^{n,i} = K^i$.

**Proof.** The proof is by induction on $n$. For $n = 1$, we assume that $K^0 \equiv K^1$. In this case, the claim is equivalent to the definition of elementary commensurability.

Assume the result is true up to $n$ and suppose there is a sequence $K^0, \ldots, K^{n+1}$ of knots in manifolds such that $K^0 \equiv \cdots \equiv K^{n+1}$. Define $K^{n+1,i} = K^i$. Since $K^i \equiv K^{i+1}$, there are fundamental virtual covers $(K^{n,i}, \nabla, K^{n+1,i})$ and $(K^{n,i}, \Delta, K^{n+1,i+1})$, where $\nabla$ and $\Delta$ are of finite index. Apply Lemma 4 to $(K^{n,i}, \Delta, K^{n+1,i+1})$ and $(K^{n,i+1}, \nabla, K^{n+1,i+1})$ to obtain fundamental virtual covers $(K^{n-1,i}, \nabla, K^{n,i})$ and $(K^{n-1,i}, \Delta, K^{n,i+1})$.

The conclusion of Lemma 4 guarantees that $\nabla$ and $\Delta$ are of finite index and that $\nabla \Delta = \Delta \nabla$. Moreover, we have that $K^{n,i} \equiv K^{n,i+1}$. The proof is completed by applying the induction hypothesis to the sequence of elementary commensurabilities $K^{n,0} \equiv K^{n,1} \equiv \cdots \equiv K^{n,n}$. \qed

2.3. **Proof of Theorem** Since $J^1 \equiv J^2$, there is a sequence $K^0 \equiv K^1 \equiv \cdots \equiv K^{n-1} \equiv K^n$ such that $K^0 = J^1$ and $K^n = J^2$. By Lemma 6 there is a collection of fundamental virtual covers $(K^{m,j}, \nabla, K^{m+1,j})$, $(K^{m,j}, \Delta, K^{m+1,j+1})$, such that the spaces and maps form a commutative triangle, and for $0 \leq i \leq n$, we have that $K^{n,i} = K^i$. 

---

**Lemma 4.** Let $K^0, \ldots, K^n$ be a sequence of knots in manifolds such that:

$$K^0 \equiv K^1 \equiv \cdots \equiv K^{n-1} \equiv K^n.$$ 

For all non-negative integers $m, j$, where $0 \leq m \leq n-1$ and $0 \leq j \leq m$, there is a collection of fundamental virtual covers $(K^{m,j}, \nabla, K^{m+1,j})$, $(K^{m,j}, \Delta, K^{m+1,j+1})$, such that:

1. The spaces $N_{m,j}$ and maps $\nabla, \Delta$ form a commutative triangle, and
2. For $0 \leq i \leq n$, $K^{n,i} = K^i$.

**Proof.** The proof is by induction on $n$. For $n = 1$, we assume that $K^0 \equiv K^1$. In this case, the claim is equivalent to the definition of elementary commensurability.

Assume the result is true up to $n$ and suppose there is a sequence $K^0, \ldots, K^{n+1}$ of knots in manifolds such that $K^0 \equiv \cdots \equiv K^{n+1}$. Define $K^{n+1,i} = K^i$. Since $K^i \equiv K^{i+1}$, there are fundamental virtual covers $(K^{n,i}, \nabla, K^{n+1,i})$ and $(K^{n,i}, \Delta, K^{n+1,i+1})$, where $\nabla$ and $\Delta$ are of finite index. Apply Lemma 4 to $(K^{n,i}, \Delta, K^{n+1,i+1})$ and $(K^{n,i+1}, \nabla, K^{n+1,i+1})$ to obtain fundamental virtual covers $(K^{n-1,i}, \nabla, K^{n,i})$ and $(K^{n-1,i}, \Delta, K^{n,i+1})$.

The conclusion of Lemma 4 guarantees that $\nabla$ and $\Delta$ are of finite index and that $\nabla \Delta = \Delta \nabla$. Moreover, we have that $K^{n,i} \equiv K^{n,i+1}$. The proof is completed by applying the induction hypothesis to the sequence of elementary commensurabilities $K^{n,0} \equiv K^{n,1} \equiv \cdots \equiv K^{n,n}$. \qed
Let \((\mathcal{E}_n, \Pi_n, K')\) be a fundamental virtual cover with associated virtual knot \(v_0\). By successive application of Lemma 5 to the maps \(\nabla\), we obtain a sequence of fundamental virtual covers \((\mathcal{E}_n, \Pi_n, K')\), \((\mathcal{E}_{n-1}, \Pi_{n-1}, K'_{n-1})\), \ldots, \((\mathcal{E}_1, \Pi_1, K'_{1})\), \((\mathcal{E}_0, \Pi_0, K'^{\circ,0})\) such that the diagram below commutes.

\[
\begin{array}{c}
\Sigma_{0,0} \times (0,1) \longrightarrow \Sigma_{1,0} \times (0,1) \longrightarrow \cdots \longrightarrow \Sigma_{n-1,0} \times (0,1) \longrightarrow \Sigma_{n,0} \times (0,1) \\
\Pi_{0,0} \downarrow \quad \Pi_{1,0} \downarrow \quad \cdots \downarrow \quad \Pi_{n-1,0} \downarrow \quad \Pi_{n,0} \downarrow \\
N_{0,0} \quad N_{1,0} \quad \cdots \quad N_{n-1,0} \quad N_{n,0}
\end{array}
\]

If \(v'_0\) is the virtual knot associated with \((\mathcal{E}_0, \Pi_{0,0}, K'^{\circ,0})\), it follows that \(v'_0 \approx v_0\) as virtual knots.

To prove the first claim, define \(\Pi_{n,n} : \Sigma_{0,0} \times (0,1) \rightarrow N_{n,n}\) by \(\Pi_{n,n} = \Delta \Delta \cdots \Delta \Pi_{0,0}\). It follows that \((\mathcal{E}_n, \Pi_{n,n}, K'^{n,n})\) satisfies the conclusion of the first part of the theorem.

To prove the second claim, let \((\mathcal{E}_n, \Pi_{n,n}, K'^{n,n})\) be a fundamental virtual cover with associated virtual knot \(v_n\). Perform successive applications of Lemma 5 to the maps \(\Delta\), to obtain a sequence of fundamental virtual covers \((\mathcal{E}_n, \Pi_{n,n}, K'^{n,n})\), \((\mathcal{E}_{n-1}, \Pi_{n-1,n-1}, K'^{n-1,n-1})\), \ldots, \((\mathcal{E}_1, \Pi_{1,1}, K'^{1,1})\), \((\mathcal{E}_0, \Pi_{0,0}, K'^{\circ,0})\) such that the diagram below commutes.

\[
\begin{array}{c}
\Gamma \times (0,1) \longrightarrow \Sigma_{1,1} \times (0,1) \longrightarrow \cdots \longrightarrow \Sigma_{n-1,n-1} \times (0,1) \longrightarrow \Sigma_{n,n} \times (0,1) \\
\Pi \downarrow \quad \Pi_{1,1} \downarrow \quad \cdots \downarrow \quad \Pi_{n-1,n-1} \downarrow \quad \Pi_{n,n} \downarrow \\
N_{0,0} \quad \Delta \quad N_{1,1} \quad \Delta \quad \cdots \quad \Delta \quad N_{n-1,n-1} \quad \Delta \quad N_{n,n}
\end{array}
\]

Recall that \(\Gamma\) is a compact, connected, oriented surface. If \(v\) is the virtual knot associated to \((\mathcal{E}_0, \Pi_{0,0}, K'^{\circ,0})\), it follows that \(v \approx v_n\) as virtual knots.

The proof of the second claim will be completed by showing that \(v_0 \approx v'_0 \approx v \approx v_n\). Only the middle equivalence has yet to be established. Apply Lemma 4 to the fundamental virtual covers \((\mathcal{E}_0, \Pi_0, K'^{\circ,0})\) and \((\mathcal{E}_0 \times (0,1), \Pi, K'^{\circ,0})\) to obtain fundamental virtual covers \((M', \Pi', \mathcal{E}_0^{\circ,0})\) and \((M, \Pi', \mathcal{E}_0^{\circ,0})\). It can be shown that \(M \approx \Xi \times (0,1)\) where \(\Xi\) is a connected oriented surface (not necessarily compact). Indeed, just reread the first four paragraphs of the proof of Lemma 4 and apply the argument to either \((M, \Pi', \mathcal{E}_0^{\circ,0})\) or \((M, \Pi_0, \mathcal{E}_0 \times (0,1))\).

It may therefore be assumed, without loss of generality, that we have fundamental virtual covers \((h, \mathcal{E}_0 \times (0,1), q_1 \times \text{id}, \mathcal{E}_0^{\circ,0})\) and \((h, \Xi \times (0,1), q_2 \times \text{id}, \mathcal{E}_0 \times (0,1))\). In addition, we may assume that \(h, \mathcal{E}_0\) and \(\mathcal{E}_0\) are regular with respect to the canonical projections onto the surfaces \(\Xi, \Sigma_0, \) and \(\Gamma\), respectively.

Let \(\Psi_1\) be the compact connected oriented surface obtained from the application of Lemma
Thus, \( v_0 \equiv v_n \) and the second claim is proved. \( \Box \)

3. Application: Knots in Fibered Link Complements

3.1. Fibered Link Complements and Virtual Covers. A fibered link complement has a covering space that is homeomorphic to \( \Sigma \times \mathbb{R} \), where \( \Sigma \) is a Seifert surface of the link \([5]\). The details are as follows.

Let \( J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_n \) be a fibered link of \( n \) components in \( S^3 \). For \( i = 1 \) to \( n \), we will use \( J_i \) to denote both the smooth embedding \( J_i : S^1 \to S^3 \) and the image of this map.

Let \( V(J_i) \) be an open regular neighborhood of \( J_i \). The neighborhoods may be chosen so that \( V(J_i) \cap V(J_j) = \emptyset \) for \( i \neq j \). Let \( V(J) = \bigcup_{i=1}^n V(J_i) \).

Since \( J \) is a fibered link, there is a fiber bundle \( p : S^3 \setminus J \to S^1 \) with fiber \( F \). Fibered links have the additional requirement that for each \( i \) there is an open regular neighborhood \( V'(J_i) \approx S^1 \times D^2 \) such that \( J_i \) is identified with \( S^1 \times \{0\} \) and \( p|_{V'(J_i)}(z, b) = b/|b| \) for \( b \neq 0 \) (see \([26]\)). Throughout, we identify \( V'(J_i) = V(J_i) \). Let \( N_J = S^3 \setminus V(J) \). We denote also by \( p \) the restriction of \( p : S^3 \setminus J \to S^1 \) to \( N_J \).

Let \( \exp : \mathbb{R} \to S^1 \subset \mathbb{C} \) be the covering map defined by \( \exp(t) = e^{2\pi i t} \). Let \( \Pi_J : M_J \to N_J \) denote the pullback of \( \exp \) over \( p \). Then \( M_J \approx \Sigma_J \times \mathbb{R} \), where \( \Sigma_J \) is a compact orientable surface \([26]\). Moreover, \( \Sigma_J \) is ambient isotopic in \( S^3 \) to a Seifert surface of \( J \) (more exactly, it is a Seifert surface of a parallel of \( J \) on \( \partial N_J \)).

The following definitions allow knots in fibered link complements to be placed into the context of Theorem \([5]\).

**Definition 3.1** (Special Surface Form). Let \( N \) be a compact connected orientable 3-manifold and \( K : S^1 \to \text{int}(N) \) be a knot. Let \( \Sigma \) be an embedded compact orientable surface in \( N \) such that \( \partial \Sigma \subset \partial N \). Suppose that there are real numbers \( 0 = t_0 < \ldots < t_{2p-1} < t_{2p} = 1 \) and pairwise disjoint 3-cells \( B_1, B_2, \ldots, B_p \) in \( N \) that satisfy the following properties:

1. for all \( i \), \( B_i \cap \Sigma = D_i \), where \( D_i \) is a closed embedded disk \( D^2 \) in \( N \),
2. for all \( i \), \( \partial(D_i) \) consists of two embedded open disks \( C_{i,1}, C_{i,2} \),
3. \( K(\exp([t_{2i}, t_{2i+1}])) \subset \Sigma \setminus \bigcup_{i=1}^p \text{int}(B_i) \), for \( 0 \leq i \leq p - 1 \),
4. \( K(\exp((t_{2j-1}, t_{2j}))) \subset \bigcup_{i=1}^p \partial(D_i) \), for \( 1 \leq j \leq p \),
(5) for each $i$, $1 \leq i \leq p$, and $j = 1, 2$, there is exactly one $k$, $1 \leq k \leq p$, such that

$$K(\exp((t_{2k-1}, t_{2k})) \subset C_{i,j}.$$ 

Then $K$ is said to be in special surface form relative to $\Sigma$ in $N$. See Figure 1.

Remark 3.1. In [8], special surface form was called special Seifert form. The name has been altered here to reflect the general case of a fibered manifold in which the fiber is not a Seifert surface of a knot.

**Figure 1.** A knot in special surface form.

**Definition 3.2** (Upper/Lower Hemisphere). Let $K$ be in special surface form relative to $\Sigma$ in $N$. Let $t_i$, $B_i$, $D_i$, $C_{i,j}$ be as in Definition 3.1. Choose an orientation of $\Sigma$, consisting of a section $\Gamma$ of the normal bundle. We say that $C_{i,j}$ is the upper hemisphere of $B_i$ if every smooth path $\tau : [0, 1] \to C_{i,j} \cup \text{bd}(D_i)$ such that $\tau(0) \in \text{bd}(D_i)$ and $\tau'(0) = \Gamma(\tau(0))$, there is an $\varepsilon > 0$ such that $\tau((0, \varepsilon)) \subset C_{i,j}$. If $C_{i,j}$ is not the upper hemisphere of $B_i$, then it is called the lower hemisphere. In Figure 1, the upper hemispheres are shaded green.

**Definition 3.3** (Diagram of a special surface form). Let $N$ be a compact connected manifold, $\Sigma$ a compact connected oriented surface in $N$, and $K$ a knot in special surface form relative to $\Sigma$ in $N$. The diagram of $K$ on $\Sigma$, denoted $[K, \Sigma]$, is the diagram consisting of the arcs of $\text{im}(K) \cap \Sigma$ together with crossings in each 2-disk $D_i$ which connect the ends of two arcs in $D_i$ if and only if those ends are connected by an arc in $\text{im}(K) \cap \partial B_i$. For each $i$, if two ends of arcs in $D_i$ are connected by an arc of $\text{im}(K) \cap B_i$ in the upper (lower) hemisphere, then these ends correspond to the over-crossing (respectively, under-crossing) arc of the crossing in $D_i$.

**Theorem 7.** Let $J$ be a fibered link and $K$ a knot in $N = N_J$. Suppose that $K$ is in special surface form relative to $\Sigma = \Sigma_J$.

1. There is a fundamental virtual cover $(t^{\Sigma \times (0, 1)}, \Pi_J, K^N)$ with associated virtual knot $\kappa([K, \Sigma])$.

2. Suppose that there is a fundamental virtual cover $(t^{\Sigma \times (0, 1)}, \Pi, L^N)$ with associated virtual knot $\nu$ and $K^N \Rightarrow L^N$. Then $\kappa([K, \Sigma]) = \nu$ as virtual knots.
Proof. Using the homotopy long exact sequence of a fibration, one can show that the covering \( \Pi_J : \Sigma \times (0, 1) \to N \) is regular and that the group of covering transformations is infinite cyclic. Since \( K \) is in special surface form, it follows that \( K \) lifts to a knot \( \mathfrak{t} \) in \( \Sigma \times (0, 1) \) that is in special surface form relative to \( \Sigma \) in \( \Sigma \times (0, 1) \) and having diagram \([K, \Sigma]\). Clearly, \( \mathfrak{t} \) may be taken to be in a fundamental region for \( \Pi_J \). The second claim follows immediately from Theorem 1. \( \square \)

Remark 3.2. The case of fibered knots is given in Theorem 4 of [8]. The proof presented here is ultimately a consequence of the theory of fiber bundles. In contrast, the proof of Theorem 4 in [8] is a consequence of the isotopy extension theorem.

3.2. Example: Non-invertibility in Fibered Knot Complements. Figure 2 shows a Seifert surface of \( 6_2 \). The knot \( 6_2 \) is fibered and a fiber of a fibration can be found using Murasugi sums [29]. Indeed, the depicted surface is a Murasugi sum of a fiber of a fibration of the Hopf link complement and a fiber of a fibration of a torus link complement. The green band on the left picture depicts this summation.

The right hand side depicts an ambient isotopic Seifert surface \( \Sigma \) of \( 6_2 \). We will use \( \Sigma \) and the corresponding fibration in our example. Let \( J = \partial \Sigma \), so that \( J \leftrightarrow 6_2 \). Let \( N = N_J \).

A knot \( K \) (in red) in special surface form relative to this fiber is given on the left hand side in Figure 3. By Theorem 7, there is a fundamental virtual cover \((\mathfrak{t}^{\Sigma \times (0,1)}, \Pi_J, K^N)\) with associated virtual knot \( \nu \). The knot in \( S^3 \) is given in Figure 4. This knot is equivalent to the unknot. The unknot is invertible in \( S^3 \).

The associated virtual knot \( \nu \) is given on the right of Figure 3. The reduced Sawollek polynomial [27] for each of the two possible orientations of \( \nu \) is given below:

\[
\tilde{Z}_v(x, y) = -2 + x + 2x^2 - x^3 - \frac{2x}{y} + \frac{3x^2}{y} - \frac{x^3}{y} - 2y + 3xy - x^2y,
\]

\[
\tilde{Z}_{v^{-1}}(x, y) = 1 - 2x - x^2 + 2x^3 + \frac{x}{y} - \frac{3x^2}{y} + \frac{2x^3}{y} + y - 3xy + 2x^2y.
\]

By Theorem 7 we have that for any orientation \( \bar{K} \) of \( K \), \( \bar{K} \) is not equivalent to its inverse \( \bar{K}^{-1} \) in the complement of \( J \leftrightarrow 6_2 \). For other examples in the case of fibered knot complements, see [8].

3.3. Example: Detecting Non-separable Links in \( S^3 \). Virtual covers can also be used to study links in \( S^3 \). In [8], it was shown that virtual covers can be used to prove that a two component link is not separable. Theorem 4 allows for a strengthening of this result.

Definition 3.4 (Separable Links in \( S^3 \)). Let \( L_1, L_2, \) and \( L_1 \sqcup L_2 \) be links in \( S^3 \). Then \( L_1 \) and \( L_2 \) are said to be separable if there is a 3-cell \( A \) in \( S^3 \) such that \( L_1 \subset \text{int}(A) \) and \( L_2 \subset \text{int}(S^3 \setminus A) \).
Figure 2. (Left) A fiber of the complement of $6_2$. The fiber is a Murasugi sum of a Hopf link and a torus link. (Right) A fiber $\Sigma$ obtained from an ambient isotopy of the fiber on the left.

Figure 3. (Left) A knot $K$ in special surface form relative to the fiber of Figure 2. The associated virtual knot (Right) is not invertible. Hence, $K$ is not invertible in $N_{6_2}$.

Figure 4. In $S^3$, with $J \cong 6_2$ removed, the knot $K$ is unknotted.
Theorem 8. Let $J$ be a fibered link in $S^3$ and $K$ a knot in $N = N_J$. If $J$ and $K$ are separable and $\left(\Sigma \times (0,1), \Pi, K^N\right)$ is a fundamental virtual cover with associated virtual knot $\nu$, then $\nu \leftrightarrow K$ as virtual knots.

Proof. Suppose that $J$ and $K$ are separable. Let $\Sigma = \Sigma_J$ be a fiber of $N_J$. It follows from the definition that $K$ has a fundamental virtual cover $\left(\Sigma \times (0,1), \Pi_J, K^N\right)$ with associated virtual knot $K$. By Theorem 7 the associated virtual knot is unique among fundamental virtual covers. Thus $\nu \leftrightarrow K$. □

Consider for example the link in Figure 5. The link drawn in black is a Hopf link $J$. The Hopf link is fibered. The knot $K$ drawn in red is a figure eight knot. The left hand side of Figure 6 shows $K$ drawn in special surface form relative to a fiber $\Sigma$ of $N = N_J$.

By Theorem 7 there is a virtual cover $\left(\Sigma \times (0,1), \Pi_J, K^N\right)$ with associated virtual knot $\kappa([K, \Sigma])$. A diagram of $\kappa([K, \Sigma])$ is drawn on the right hand side of Figure 6. It is easily seen that this diagram is equivalent to the trivial knot. Since $K \cong 4_1$ is nontrivial as a knot in $S^3$, it follows from Theorem 8 that $J$ and $K$ are not separable in $S^3$.

![Figure 5](image.png)  
**Figure 5.** The link $J$ in black is a Hopf link. The knot $K$ in red $4_1$.

![Figure 6](image.png)  
**Figure 6.** (Left) The knot $K$ in special surface form relative to a fiber $N_J$ (see Figure 5). (Right) The associated virtual knot is trivial.
3.4. Example: Separating Knots in Fibered Link Complements. The Borromean rings $J$ are a fibered three component link. A fiber of $N_J$ is depicted on the left hand side of Figure 7. An ambient isotopic surface is depicted on the right hand side of Figure 7. This surface will be used as a fiber $\Sigma = \Sigma_J$ of $N_J$.

Two knots, $K_1$ and $K_2$ are shown in Figure 8 in special surface form relative to $\Sigma$. In $S_3$, both are equivalent to $5_2$ (see Figure 9).

Associated virtual knots $\nu_i$ for the virtual covers given by Theorem 7 for $K_i$ are shown in Figure 10. For the odd writhe $\theta_{\nu_i}$, we have $\theta(K_1) = 2$ and $\theta(K_2) = 4$. It follows from Theorem 7 that $K_1$ and $K_2$ are inequivalent as knots in $N_J$.

Figure 7. (Left) A fiber of the Borromean rings complement. (Right) A fiber obtained via ambient isotopy from the fiber on the left.

Figure 8. Knots $K_1$ (Left) and $K_2$ (Right) in the complement of the Borromean rings.

4. Application: Knots in Virtually Fibered Manifolds

4.1. Gabai’s Example. There are many known examples of non-fibered, virtually fibered 3-manifolds [2, 15, 23, 32]. For an excellent overview of the literature on the topic with
Figure 9. \(K_1\) (Left) and \(K_2\) (Right) are equivalent to \(5_2\) as knots in \(S^3\).

Figure 10. The associated virtual knot \(v_1\) for \(K_1\) (Left) and the associated virtual knot \(v_2\) for \(K_2\) (Right).

many additional references, see [12]. We will use just one of these as an on-going example. The relevant details in the construction of Gabai’s example [15] will be provided in full for the convenience of the reader.

Let \(J\) be the two component link on the right hand side of Figure 11 and \(L\) the three component link on the left hand side of Figure 11. It will be shown that \(J\) is virtually fibered by showing that \(N_J\) has a twofold covering by \(N_L\) and that \(N_L\) is fibered. For a proof that \(J\) is non-fibered and hyperbolic, see [15].

4.1.1. The twofold covering: Figure 12 shows the fundamental region of the twofold cover without any identifications. The knot complement \(N_J\) can be obtained by identifying annuli and punctured disks in boundary of the fundamental region as follows. On the left hand side of Figure 13 two annuli are distinguished: one is shaded red and the other is shaded gold. The identification of these annuli produces the picture on the right hand side of Figure 13. The red annulus on the right shows the image of the annuli under the identification.

Finally, identify the blue and green twice punctured discs on the right hand side of Figure
The result is the complement of $J$ in $S^3$, i.e., the link complement $N_J$. If the corresponding identifications are made in the twofold cover given in Figure 12, then the result is the complement of $L$ in $S^3$, i.e., the link complement $N_L$.

Figure 11. The links $L$ (Left) and $J$ (Right). The complement $N_J$ admits a twofold cover by $N_L$. Both pictures were drawn in SnapPy.

Figure 12. Gabai’s double cover $N_L$ of $N_J$ is constructed by identifying rectangles on the boundaries of the fundamental regions.
4.1.2. A fiber of $N_L$:

A fiber of $N_L$ can be found using a combination of Murasugi’s sums [14] and Stalling’s twisting [29].

Figure 14 illustrates an ambient isotopy of $S^3$ taking $L$ to a link which bounds the fiber $\Sigma_L$ (as yet undetermined). Of course, performing the isotopy has changed the covering of $N_J$. Relabeling the newly obtained link as $L$, we retain a double covering $N_L \to N_J$.

The first step in the construction of Gabai’s fiber consists of Murasugi’s sum and Stalling’s twisting. The top row of Figure 15 illustrates Murasugi’s sum applied to two fibers of a fibration of the Hopf link. The bottom right of this figure shows a dotted circle drawn of the resultant fiber. Stalling’s twist is applied to this circle to obtain the fiber on the bottom left of Figure 15.

The second step in the construction of Gabai’s fiber consists of two applications of Murasugi’s sum. On the top left of Figure 16, we have a torus link and a fiber represented as a Seifert surface of this link (alternatively, one can use Futer’s method [13]). First, Murasugi’s sum is applied to the fiber of the torus link and the fiber from Figure 15. Murasugi’s sum is again applied to the resultant fiber and a second copy of the fiber for the torus link. The final fiber is depicted on the right of Figure 16.

Observe that the fiber in Figure 16 is a Seifert surface of the link $L$ given in Figure 14.
From this it can be concluded that $L$ is a fibered link with fiber $\Sigma_L$ as depicted on the right in Figure 16.

**Figure 14.** Gabai’s fiber is obtained by an ambient isotopy of $S^3$ taking $L$ to the link in the bottom right.

**Figure 15.** The first step in constructing Gabai’s fiber uses Murasugi’s sums (Top Row) and Stalling’s twistings (Bottom Row)
4.2. Knots in Virtually Fibered 3-manifolds. The following immediate corollary of Theorem 4 demonstrates how virtual covers can be used to separate inequivalent knots in virtually fibered 3-manifolds.

**Corollary 9.** Suppose that $K_N^1$, $K_N^2$ are oriented knots such that $N$ is connected, compact, and oriented. Suppose that $N$ admits a finite index regular covering $\Delta : M \to N$ where $M$ is compact, connected, and oriented, and $\Delta$ preserves orientation. Suppose that $(J^M_1, \Delta, K^N_1)$ and $(J^M_2, \Delta, K^N_2)$ are fundamental virtual covers. Furthermore, suppose that $(\varphi^1, \Pi_1, J^M_1)$ and $(\varphi^2, \Pi_2, J^M_2)$ are fundamental virtual covers with associated virtual knots $\nu_1$ and $\nu_2$, respectively. If $K_N^1 \cong K_N^2$, then $\nu_1 \cong \nu_2$ as virtual knots.

**Proof.** The hypotheses imply that $J^M_1 \cong K^N_1 \cong K^N_2 \cong J^M_2$. The claim now follows from Theorem 4. □

**Remark 4.1.** The above corollary implies that if $K_1 = K_2$ and $J_1 \neq J_2$ are different lifts of $K_1$ to $M$, then the associated virtual knots are equivalent. This means that the associated virtual knot is independent of the choice of the lift.

Suppose that $N$ is virtually fibered, $K_N^1, K_N^2$ are knots and $\Delta : M \to N$ is a finite index cover. Suppose that $M$ is a fibered link complement and $(L^M_1, \Delta, K^N_1)$, $(L^M_2, \Delta, K^N_2)$ are fundamental virtual covers. If $L_1$ and $L_2$ are in special surface form relative to some (possibly different) fibers, then Theorem 7 says that $L_1$ and $L_2$ have fundamental virtual covers with associated virtual knots $\nu_1$ and $\nu_2$, respectively. Corollary 9 implies that if $K_N^1 \cong K_N^2$, then $\nu_1 \cong \nu_2$.

The next two sections will apply these observations to knots in the manifold described in the preceding section (Gabai’s example). Let $N = N_J$, $M = N_L$ and $\Delta : M \to N$ the two-fold covering constructed in the previous section. Since $\Delta$ has index two, it follows that $\Delta$ is a regular covering.

4.3. Example: Separating Knots in Virtually Fibered 3-Manifolds. Let $K_1$ be the oriented knot in $N = N_J$ drawn in red in the top right of Figure 17. Let $K_2$ be the oriented...
The top left of Figure 17 illustrates one of two possible choices for the knot $L_1$ in $N_L$. Similarly, the top left illustrates one of two possibilities for $L_2$ in $N_L$. By Remark 4.1 any choice of the lift will ultimately produce an equivalent associated virtual knot. The previous construction of Gabai’s example shows that $(L_1^M, \Delta, K_1^N)$ and $(L_2^M, \Delta, K_2^N)$ are fundamental virtual covers.

The middle of Figure 17 indicates $L_1$ as a diagram on a fiber $\Sigma_L$. Note that the ambient isotopy depicted in Figure 14 has been used to position $L_1$ into special surface form relative to this fiber. The middle of Figure 18 similarly depicts $L_2$ as a diagram on $\Sigma_L$. By Theorem 7 $L_1$ and $L_2$ have fundamental virtual covers.

Let $\nu_1 = \kappa([L_1, \Sigma])$ be the associated virtual knot for this fundamental virtual cover of $L_1$. A diagram of $\nu_1$ is depicted on the bottom left of Figure 18. Similarly, the bottom left of Figure 18 depicts a diagram of the associated virtual knot $\nu_2 = \kappa([L_2, \Sigma])$ for the indicated fundamental virtual cover of $K_2$. Computing the odd writhe [20] gives us that $\theta(\nu_1) = 2$ and $\theta(\nu_2) = -2$. Thus, $K_1$ and $K_2$ are not equivalent as knots in $N_J$.

Note that $K_1$ and $K_2$ can not be separated by knot invariants of $S^3$. As a knot in $S^3$, $K_1$ is the figure eight knot depicted in the bottom right of Figure 17. Similarly, as a knot in $S^3$, $K_2$ is the figure eight knot depicted on the bottom right of Figure 18. Since the figure eight knot is both invertible and amphicheiral, $K_1$ and $K_2$ are equivalent as knots in $S^3$.

4.4. Example: Non-invertibility in Virtually Fibered 3-Manifolds. Virtual covers can also be used to detect geometric properties of knots in virtually fibered manifolds. In this section we will consider an example where a virtual cover is used to detect the non-invertibility of a knot in $N_J$.

Let $K$ be the knot in $N_J$ depicted in red in the middle of Figure 19. It will be shown that $K$ is non-invertible in $N_J$. In other words, given an orientation of $K$, there is no equivalence of knots in $N_J$ taking $K$ to the knot $K^{-1}$ obtained from $K$ by reversing its orientation.

Let $T$ be the red knot in the top panel of Figure 19. From our construction of Gabai’s example, it follows that $(T^M, \Delta, K^N)$ is a fundamental virtual cover. $T$ is in special surface form relative to $\Sigma = \Sigma_L$. By Theorem 7 it follows that there is a fundamental virtual cover with associated virtual knot $\nu = \kappa([T, \Sigma])$.

The diagram of $\nu$ in the bottom left of Figure 19 is easily seen to be equivalent to a classical knot. It can be shown with a little effort that this classical knot is $8_{17}$ from Rolfsen’s table [26]. Note that $8_{17}$ is non-invertible in $S^3$. 
Figure 17. The knot $K_1$ in $N = N_J$ (top right, red), the knot $L_1$ in $M = N_L$ (top left, red), $L_1$ as a diagram on $\Sigma_L$ (middle, red), associated virtual knot $v_1$ (bottom left), and $K_1$ depicted as a figure eight knot in $S^3$. 
Figure 18. The knot $K_2$ in $N = N_J$ (top right, red), the knot $L_2$ in $M = N_L$ (top left, red), $L_2$ as a diagram on $\Sigma_L$ (middle, red), associated virtual knot $v_2$ (bottom left), and $K_2$ depicted as a figure eight knot in $S^3$. 
Let $\vec{K}$ be an orientation of $K$. This gives an orientation $\vec{v}$ of $v$. Applying the above work to $\vec{K}^{-1}$, we see that $\vec{K}^{-1}$ has a fundamental virtual cover with associated virtual knot $\vec{v}^{-1}$. Suppose by way of contradiction that $K$ is invertible. Then $\vec{K}^N \cong (\vec{K}^{-1})^N$. By Corollary 9 it follows that $\vec{v} \cong (\vec{v})^{-1}$. But this contradicts the fact that $8_{17}$ is non-invertible. Thus, $K$ is non-invertible in $N_J$.

The result of this section would not be very interesting if $K$ turned out to be non-invertible when considered as a knot in $S^3$. If this were the case, then any number of previous methods could have been used to establish the result. However, this is not the case. The bottom right of Figure 19 shows $K$ as a knot in $S^3$. This knot can be seen to be a left handed trefoil. It is well-known that the left handed trefoil is invertible in $S^3$.

**References**

[1] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.

[2] I. Agol, S. Boyer, and X. Zhang. Virtually fibered Montesinos links. *J. Topol.*, 1(4):993–1018, 2008.

[3] M. Boileau, S. Boyer, R. Cebanu, and G. S. Walsh. Knot commensurability and the Berge conjecture. *Geom. Topol.*, 16(2):625–664, 2012.

[4] M. O. Bourgoin. Twisted link theory. *Algebr. Geom. Topol.*, 8(3):1249–1279, 2008.

[5] G. Burde and H. Zieschang. *Knots*, volume 5 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2003.

[6] J. S. Carter, S. Kamada, and M. Saito. Stable equivalence of knots on surfaces and virtual knot cobordisms. *J. Knot Theory Ramifications*, 11(3):311–322, 2002. Knots 2000 Korea, Vol. 1 (Yongpyong).

[7] J. S. Carter, D. S. Silver, and S. G. Williams. Invariants of Links in Thickened Surfaces. *arXiv:1304.4655v1[math.GT]*, April 2013.

[8] M. W. Chrisman and V. O. Manturov. Fibered knots and virtual knots. *J. Knot Theory Ramifications*, 22(12):1341003, 23, 2013.

[9] H. A. Dye and L. H. Kauffman. Virtual crossing number and the arrow polynomial. *J. Knot Theory Ramifications*, 18(10):1335–1357, 2009.

[10] R. Fenn, C. Rourke, and B. Sanderson. The rack space. *Trans. Amer. Math. Soc.*, 359(2):701–740 (electronic), 2007.

[11] T. Fiedler. A small state sum for knots. *Topology*, 32(2):281–294, 1993.

[12] S. Friedl and T. Kitayama. The virtual fibering theorem for 3-manifolds. *ArXiv e-prints*, October 2012.

[13] D. Futer. Fiber detection for state surfaces. *Algebr. Geom. Topol.*, 13(5):2799–2807, 2013.

[14] D. Gabai. The Murasugi sum is a natural geometric operation. In *Low-dimensional topology (San Francisco, Calif., 1981)*, volume 20 of *Contemp. Math.*, pages 131–143. Amer. Math. Soc., Providence, RI, 1983.

[15] D. Gabai. On 3-manifolds finitely covered by surface bundles. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 145–155. Cambridge Univ. Press, Cambridge, 1986.

[16] M. Goussarov, M. Polyak, and O. Viro. Finite-type invariants of classical and virtual knots. *Topology*, 39(5):1045–1068, 2000.

[17] D.M. Ilyutko and V. O. Manturov. *Virtual Knot Theory:State of the Art*, volume 51 of *Series on Knots and Everything*. World Scientific, 2013.

[18] D.P. Ilyutko, V. O. Manturov, and I. M. Nikonov. Virtual knot invariants arising from parities. *arXiv:1102.5081v1[math.GT]*, 2011.

[19] N. Kamada and S. Kamada. Abstract link diagrams and virtual knots. *Journal of Knot Theory and its Ramifications*, 9:93–106, 2000.
L. H. Kauffman. Virtual knot theory. *European J. Combin.*, 20(7):663–690, 1999.

V. A. Krasnov and V. O. Manturov. Graph-valued invariants of virtual and classical links and minicity problem. *J. Knot Theory Ramifications*, 22(12):1341006, 14, 2013.

G. Kuperberg. What is a virtual link? *Algebraic and Geometric Topology*, 3:587–591, 2003.

C. J. Leininger. Surgeries on one component of the Whitehead link are virtually fibered. *Topology*, 41(2):307–320, 2002.

C. J. Leininger. Erratum to: “Surgeries on one component of the Whitehead link are virtually fibered” [Topology 41 (2002), no. 2, 307–320; MR1876892 (2002j:57037)]. *Topology*, 42(2):507–508, 2003.

V. O. Manturov. Free knots and parity. In *Introductory lectures on knot theory*, volume 46 of *Ser. Knots Everything*, pages 321–345. World Sci. Publ., Hackensack, NJ, 2012.

D. Rolfsen. *Knots and links*. Publish or Perish Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7.

J. Sawollek. On Alexander-Conway Polynomials for Virtual Knots and Links. *arXiv:9912173[math.GT]*, December 1999.

E. H. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

J. R. Stallings. Constructions of fibre knots and links. In *Algebraic and geometric topology* (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII, pages 55–60. Amer. Math. Soc., Providence, R.I., 1978.

N. Steenrod. *The Topology of Fibre Bundles*. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.

W. P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

G. S. Walsh. Great circle links and virtually fibered knots. *Topology*, 44(5):947–958, 2005.

G. S. Walsh. Orbifolds and commensurability. *arXiv e-prints*, March 2010.
Figure 19. The knot $K$ in $N = N_J$ (middle, red), the knot $T$ in $M = N_L$ (top, red), associated virtual knot $\nu \simeq 8_{17}$ (bottom left), and $K$ depicted as a left handed trefoil in $S^3$ (bottom right).