THE EXTERIOR SQUARE $L$-FUNCTION ON GU(2, 2)

AARON POLLACK

Abstract. In this paper we give Rankin-Selberg integrals for the quasisplit unitary group on four variables, GU(2, 2), and a closely-related quasisplit form of GSpin. First, we give a two-variable Rankin-Selberg integral on GU(2, 2). This integral applies to generic cusp forms, and represents the product of the exterior square (degree six) $L$-function and the standard (degree eight) $L$-function. The integral is analogous to a two-variable integral of Bump-Friedberg-Ginzburg on GSp. Then we give a set of integral representations for just the degree six $L$-function on the quasisplit GSpin. These integrals are reinterpretations of an integral originally considered by Gritsenko for Hermitian modular forms, and are analogous to the integrals considered in [14]. We show that they unfold to a model that is not unique, and analyze the integrals via the technique of Piatetski-Shapiro and Rallis.

1. Introduction

The purpose of this paper is to give some Rankin-Selberg integrals for the group GU(2, 2), the quasisplit unitary group in four variables, and a closely-related quasisplit form of GSpin. More precisely, the form of GSpin we consider is defined by the short exact sequence

$$1 \rightarrow \text{GSpin}_6 \rightarrow \text{GU}(2, 2) \rightarrow \text{U}(1) \rightarrow 1,$$

where the map GU(2, 2) → U(1) is given by $g \mapsto \det(g)\nu(g)^{-2}$, where $\nu$ is the similitude. First we give a two-variable integral on GU(2, 2) that applies to generic cusp forms. This integral is modeled on the two-variable integral of Bump-Friedberg-Ginzburg on GSp, and it represents the product of the $L$-functions associated to the six-dimensional exterior square representation of $L_{\text{GU}(2, 2)} = (\text{GL}_1(\mathbb{C}) \times \text{GL}_4(\mathbb{C})) \rtimes \text{Gal}(E/F)$ and the eight-dimensional standard representation of this dual group. In fact, if $E/F$ is the quadratic extension used to define the unitary group, then for primes $p$ of $F$ inert in $E$, the unramified computation for this two-variable integral reduces to the one in [2].

We then turn to a family of Rankin-Selberg convolutions for cusp forms on GSpin in a single variable. The dual group $L_{\text{GSpin}_6}$ continues to have a six-dimensional representation, and the one-variable integrals we give represent the associated $L$-function. These integrals are a reinterpretation and generalization of an integral originally considered by Gritsenko. Gritsenko’s integral is a Hermitian modular form version of the the integral of Kohnen and Skoruppa on GSp, and like the paper [11], his constructions and arguments only work in this holomorphic case. We show that our integrals unfold to a non-unique model, and we analyze the integrals via the technique of Piatetski-Shapiro and Rallis. The constructions here are analogous to those in [14].

Let us now describe the global integrals. First, we write $W_4$ for the four-dimensional $E$ vector space that is the defining representation of $G = \text{GU}(2, 2)$. Then, $W_4$ is equipped with a non-degenerate $E$-valued pairing $\langle \cdot, \cdot \rangle$ satisfying $\langle v, w \rangle = -\langle w, v \rangle$ for all $v, w$ in $W_4$, where $z \mapsto \overline{z}$ is the conjugation of $E$ over $F$. The group $G$ is by definition the group preserving $\langle \cdot, \cdot \rangle$ up to scalar multiple. Now, we denote by $Q$ the parabolic of $G$ that stabilizes an isotropic $E$-line of $W_4$, the so-called Klingen parabolic. We denote by $P$ the Siegel parabolic of $G$, which by definition is the

2010 Mathematics Subject Classification. Primary 11F66; Secondary 11F03; 22E55.
if \( \phi \) stabilizer of an isotropic two-dimensional subspace of \( W_4 \). The two-variable integral is then

\[
I(\phi; s, w) := \int_{\text{GU}(2,2)(F) \backslash \text{GU}(2,2)(A)} \phi(g) E_P(g; w) E_Q(g; s) \, dg
\]

where \( E_P(g; w) \) is the Siegel Eisenstein series in the complex parameter \( w \) and \( E_Q(g; s) \) is the Klingen Eisenstein series in the complex parameter \( s \). The integral unfolds to the Whittaker model of \( \phi \), and we prove that, up to simple factors, the integral represents \( L(\pi, \lambda^2, 3s-1) L(\pi, \text{Std}, 2w-\frac{1}{2}) \) if \( \phi \) is a cusp form in the space of \( \pi \), assumed to be generic.

For the non-unique model integrals we assume the ground field \( F = \mathbb{Q} \) for simplicity. Denote \( H := \text{GSpin}_6 \) the form of \( \text{GSpin}_6 \) defined by the exact sequence above. Analogous to \([14]\), in order to define the integral we consider the six-dimensional \( \mathbb{Q} \)-representation of \( H \). In fact, the group \( \text{GU}(2,2) \) has a six-dimensional \( \mathbb{E} \)-representation, namely \( \wedge^2 W_4(E) \otimes \nu^{-1} \), but not a six-dimensional representation over \( \mathbb{Q} \); the reason we change groups from \( \text{GU}(2,2) \) to \( \text{GSpin}_6 \) is actually to account for this fact. Let \((V_6, q)\) denote this six-dimensional orthogonal representation of \( H \), \( q \) being the invariant quadratic form. This representation may be realized inside of \( \wedge^2 W_4(E) \otimes \nu^{-1} \). We take an element \( v_T \) of \( V_6(\mathbb{Z}) \) with \( q(v_T) \neq 0 \), a Schwartz-Bruhat function \( \alpha \) on \( V_6(\mathbb{A}) \), and then define the special function

\[
P_T^\alpha(h) = \sum_{\gamma \in \text{Stab}(v_T) \backslash \mathbb{Q} \backslash H(\mathbb{Q})} \alpha(v_T \gamma h).
\]

Here \( h \) is in \( H(\mathbb{A}) \), and we are assuming \( H \) acts on the right of \( V_6 \). For a cusp form \( \phi \) on \( H \), the global integral we consider is

\[
J(\phi; s) := \int_{H(\mathbb{Q}) \backslash Z(\mathbb{A}) \backslash H(\mathbb{A})} \phi(g) P_T^\alpha(g) E_Q(g; s) \, dg.
\]

This integral unfolds to a non-unique model, and we show that it represents the partial \( L \)-function \( L^S(\pi, \lambda^2, 3s-1) \). (Because \( \text{GSpin}_6 \) is so closely-related to \( \text{GU}(2,2) \), we continue to write \( \wedge^2 \) for the degree six \( L \)-function.)

When \( E/\mathbb{Q} \) is an imaginary quadratic field, the class number of \( E \) is 1, and \( \phi \) comes from a level one holomorphic Hermitian modular form of scalar weight, we also calculate a corresponding arithmetic integral \( J_\infty(\phi; s) \) in terms of \( \Gamma \)-functions. In fact, in this holomorphic case, the special function \( P_T^\alpha \) takes on a very nice form. Denote by \( Z = X + iY \) the variable on the Hermitian upper half space of genus two, so that \( X, Y \) are two-by-two Hermitian matrices in \( M_2(\mathbb{C}) \) and \( Y \) is positive definite. Denote by \( H_2(O_E) \) the two-by-two Hermitian matrices in \( M_2(O_E) \). Then \( T \) will be a positive definite element of \( H_2(O_E) \), and the modular form \( P_T^\alpha(Z) \) of weight \( r \) is given by the expression

\[
P_T^\alpha(Z) = \sum_{\alpha, \delta \in \mathbb{Z}} \frac{1}{(\alpha \det(Z) + \text{tr}(hZ) + \delta)^r}
\]

where the sum is over \( \alpha, \delta \in \mathbb{Z} \), and \( h \in H_2(O_E) \), satisfying \( \alpha \delta - \det(h) = -\det(T) \). This is the analogue to \( \text{GSpin}_6 \) of the modular and Hilbert modular forms considered by Zagier \([20]\) and the Siegel modular form on \( \text{GSp}_4 \) from \([14]\).

When a global Rankin-Selberg convolution unfolds to a model that is not unique, it can be a difficult problem to determine if the integral represents a partial \( L \)-function, and if so, which \( L \)-function. As explained in \([3]\), because of this difficulty, it is an important problem to decide if Rankin-Selberg integrals associated to non-unique models are a rare phenomenon, or are prevalent. This paper adds to the now-growing literature of Rankin-Selberg integrals associated to non-unique models \([13, 4, 3, 8, 14, 17, 15]\) and thus gives evidence that such integrals might not be so uncommon.

In light of these remarks, any heuristic that can help predict the existence of such non-unique model integrals is of interest. As it turns out, the non-unique model integral in this paper, and
the ones in [14] and [15], were all predicted in the same way, as follows. First, note the similarity between the integrals \( I(\phi; s, w) \) and \( J(\phi; s) \): One changes the Siegel Eisenstein series \( E_P(g; w) \) in the convolution \( I(\phi; s, w) \) to the special function \( P^2_I(g) \), and one gets a new Rankin-Selberg integral that still represents the exterior square \( L \)-function. Furthermore, the Siegel parabolic \( P \cap GSpin_6 \) is the stabilizer of an isotropic line in \( V_6 \), while \( P^2_I \) is defined by summing over an orbit of anisotropic vectors. This relation between two-variable integrals for generic cusp forms and one-variable integrals associated to non-unique models is also present for the integrals in [14] and [15], where now the two-variable integrals are the ones from [2] for \( GSp_4 \) and \( GSp_6 \). Thus in all these cases, by altering slightly a two-variable integral for generic cusp forms, one obtains a one-variable integral associated to a non-unique model, and the \( L \)-function represented by the non-unique model integral is exactly predicted by the \( L \)-functions represented by the two-variable integral.

Finally, let us mention some other Rankin-Selberg integrals for \( GU(2, 2) \) and the related \( SO(6) \). In [5] Furusawa and Morimoto have given a one-variable integral representation for generic cusp forms on \( GU(2, 2) \) for the exterior square \( L \) function. Also, Ginzburg-Piatetski-Shapiro-Rallis [6] have considered the standard \( L \)-functions for \( SO(m) \) via the Bessel model, and Bump-Furusawa-Ginzburg [3] have given a non-unique-model integral for the standard \( L \)-function of \( SO(6n) \). The integrals considered in this paper, perhaps surprisingly, are different from these other integrals.

The layout of the paper is as follows. In section two, we discuss the groups \( GSpin_6 \), \( GU(2, 2) \), their dual groups, and the \( L \)-functions to be studied in this paper. In section three, we give the two-variable integral on \( GU(2, 2) \), and in section four we give the one-variable integral on \( GSpin_6 \). Finally, in the appendix, we explain some simple arithmetic invariant theory of two-by-two Hermitian matrices. This AIT is not needed in the proofs of the main results of the paper, but we have included it because we believe it sheds light on some of the computations used to analyze the integral \( J(\phi; s) \).

**Acknowledgments:** We hope it is clear to the reader that this paper builds on [14], where several of the ideas were worked out in the simpler case of the Spin \( L \)-function on \( GSp_4 \). It is thus a pleasure to thank Shrenik Shah for his fruitful collaboration in these matters. While working on the results in this paper, we have also benefited from conversations with Peter Sarnak, Christopher Skinner, Ila Varma, and Xiaoheng Wang. This work has been partially supported through the NSF by grant DMS-1401858.

2. **The groups \( GSpin_6 \), \( GU(2, 2) \), and their \( L \)-functions**

In this section we discuss the groups \( GSpin_6 \), \( GU(2, 2) \), their dual groups, and the \( L \)-functions to be studied in this paper.

2.1. **Notation.** We collect here a few different notations that we will need. For a matrix \( m \), \( ^*m = \overline{m}^t \) denotes the transpose conjugate of \( m \). A matrix is said to be Hermitian if \(^*m = m \). If \( m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is a two-by-two matrix, we denote by \( m' \) the matrix \( \begin{pmatrix} d & -b \\ c & a \end{pmatrix} \). Then \( mm' = \det(m) \) and \( m + m' = \text{tr}(m) \). Set \( J_2 \) the two-by-two matrix \( \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \). Then \( m' = J_2^{-1}mJ_2^{-1} \). We write \( m^\# = ^*m' \). So, \( m^\# = J_2\overline{m}J_2^{-1} \). \( H_2(E) \) denotes Hermitian matrices in \( M_2(E) \), and we write \( H_2(O_E) \) for \( H_2(E) \cap M_2(O_E) \), where \( O_E \) denotes the ring of integers in \( E \).

2.2. **The groups.** Recall that \( E/F \) is a quadratic extension of fields, and that \( (W_4, \langle \; , \; \rangle) \) is the defining representation of \( G = GU(2, 2) \). We pick a symplectic basis \( e_1, e_2, f_1, f_2 \) so that \( \langle e_i, f_j \rangle = \delta_{ij} \). We suppose that \( G \) acts on the right of \( W_4 \). \( G \) is the group of \( (g, \nu) \in \text{GL}(W_4(E)) \times \text{GL}_1 \) satisfying \( \langle vg, wg \rangle = \nu(g)\langle v, w \rangle \) for all \( v, w \in W_4 \). There is a map \( GU(2, 2) \rightarrow U(1) \) given by \( g \mapsto \det(g)\nu(g)^{-2} \). We define \( H \) to be the kernel of this map. Then \( H \) is quasisplit and split over
E. Using Proposition 2.4 of [1], a computation with roots shows that $H$ is the quasisplit form of $\text{GSpin}_6$ defined by $E$.

We now describe the six-dimensional orthogonal representation of $H$. Define $V_6$ to be the $F$ vector subspace of $\wedge^2 W_4(E) \otimes \nu^{-1}$ consisting of elements of the form
\[
\alpha e_1 \wedge e_2 + x f_1 \wedge e_2 + w f_2 \wedge e_2 + \bar{w} e_1 \wedge f_1 + y e_1 \wedge f_2 + \delta f_1 \wedge f_2
\]
where $\alpha, \delta, x, y \in F$ and $w \in E$. Below we check that $H$ preserves this space. We put a symmetric bilinear form $(\cdot, \cdot)$ on $\wedge^2 W_4(E) \otimes \nu^{-1}$ via the wedge product:
\[
v \wedge w = (v, w)e_1 \wedge e_2 \wedge f_1 \wedge f_2.
\]
This is $H$ invariant since $\wedge^2 W_4(E) \otimes \nu^{-2}$ is the trivial representation of $H$, and induces a symmetric bilinear form on $V_6$ via restriction. In fact, we will check below that we have an exact sequence
\[
1 \to \text{GL}_1 \to H \to \text{SO}(V_6, q) \to 1
\]
where the $\text{GL}_1$ is diagonally embedded in $H$, and is the connected component of its center.

For computations, and for the proofs of the above statements, it will be very convenient to have a realization of the representation $V_6$ in terms of matrices in $M_4(E)$, so we give that now. First, we put the basis of $W_4(E)$ in the order $e_1, e_2, f_1, f_2$, so that $\text{GU}(2, 1)$ is the group of $g \in \text{GL}_4/E$ satisfying $gJ_4^J = \nu(g)J_4$, where
\[
J_4 := \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{pmatrix}.
\]
The action of $\text{GU}(2, 1)$ on $\wedge^2 W_4(E) \otimes \nu^{-1}$ can be realized on the set of antisymmetric matrices $m$ in $M_4(E)$ via $m \mapsto \nu(g)^{-1}tgm$. Equivalently, define
\[
V' = \{ v \in M_4(E) : t(\epsilon v) = -\epsilon v \} \text{ where } \epsilon = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}.
\]
The elements of $V'$, in two-by-two block form, are the set of $v = (a \ b' \ c' \ \delta)$, where $a, \delta \in E$, $h \in M_2(E)$, and for a two-by-two matrix $m = (a \ b)$, we write $m' = (\begin{smallmatrix} d & c' \\ -c & a \end{smallmatrix})$. For a four-by-four matrix $g = (A \ B \ C \ D)$ in two-by-two block form, we write $g'' = (A' \ B' \ C' \ \delta')$. Then since $t \epsilon g = g''$, $\text{GU}(2, 1)$ acts on $V'$ by the formula $vg = \nu(g)^{-1}g''vg$.

In this notation, $V_6$ is the set of $v \in V'$ with $a, \delta \in F$, and $h \in H_2(E)$, the set of two-by-two hermitian matrices over $E$. In the identification between the two realizations of the representation $V_6$, the element on line (11) gets sent to $(a \ h' \ \delta)$, where $h = (\begin{smallmatrix} x & y' \\ y & x' \end{smallmatrix})$. Furthermore, for such a $v$, $Q(v) = 2(\alpha \delta - \det(h))$. We define an integral structure on $V_6$ by setting $V_6(\mathcal{O}_F)$ to be the $v \in V_6$ with $a, \delta \in \mathcal{O}_F$ and $h \in H_2(\mathcal{O}_E)$. Hence $(\cdot, \cdot)$ is integrally valued on $V_6(\mathcal{O}_F)$. Let us note that if $m \in \text{GL}_2(E)$, then $M = (\nu^*m^{-1} \ m)$ is in $H$ precisely when $\det(m) \in F^\times$.

**Proposition 2.1.** $H$ preserves the space $V_6$. The induced map $H \to \text{O}(V_6, q)$ yields the exact sequence $1 \to \text{GL}_1 \to H \to \text{SO}(V_6, q) \to 1$.

**Proof.** For $g = (A \ B \ C \ D)$ and $v = (a \ b' \ c' \ \delta)$, one computes
\[
vg = \nu(g)^{-1} \left( \begin{array}{c} \alpha \det(A) + \text{tr}(A'hC) + \delta \det(C) \\ X' \\ \alpha \det(B) + \text{tr}(B'hD) + \delta \det(D) \end{array} \right)
\]
where \[ X = \alpha A'B + C'h'B + A'hD + \delta C'D. \]

Now \( H \) is generated by matrices of the form \((1 \ 0), (0 \ 1)\), with \( u \in H_2(E) \), and \( M \) as above. Thus to check that \( H \) preserves \( V_6 \), it suffices to check that these generating matrices preserve \( V_6 \), and this is easily done with the formula \((2)\). For instance, if \( M = \begin{pmatrix} \nu \cdot m^{-1} & m \\ & \nu \cdot m^{-1} \cdot mh \end{pmatrix} \) with \( \det(m) \in F^\times \), then

\[
\begin{pmatrix} \alpha & h \\ h' & \delta \end{pmatrix} = \begin{pmatrix} \nu \cdot \det(m)^{-1} & \det(m)^{-1} \cdot mh \\ \det(m)^{-1} \cdot mhm & \nu^{-1} \cdot \det(m) \end{pmatrix}.
\]

Since \( H \) preserves \( V_6 \) and the quadratic form \( q \), we obtain a map \( H \to O(V_6, q) \). One can see that the image lands in \( SO(V_6, q) \) by checking this on the generators above. It is also easy to check that the map \( H \to SO(V_6, q) \) has kernel exactly the diagonally embedded \( GL_1 \). What remains is to check the surjectivity \( H \to SO(V_6, q) \). For this, denote by \( \iota : V_6 \to V_6^\delta \) the map defined by \( \iota((\alpha \ h \ \delta)) = \begin{pmatrix} \delta' & -h \\ -h' & \alpha \end{pmatrix} \). Then \( \iota \) generates \( O(V_6) / SO(V_6) \), and furthermore explicit computation shows that \( \iota \) normalizes the image of \( H \) in \( SO(V_6) \). (Again, it suffices to check this on generators, which can be done easily using \((2)\).) The key bit of the proof is now the following formula: If \( v_0 \in V_6 \) with \( q(v_0) = 0 \), then thinking of \( v_0 \) in \( M_4(E) \), we see \( v_0 \) is in fact in \( H \), and one has \( \iota(v) \cdot v_0 = -R_{v_0}(v) \), where \( R_{v_0}(v) = v - 2(q(v_0) v_0)_0 \) is the reflection in \( v_0 \). Thus \( H \times \{1, \iota\} \) surjects onto \( O(V_6) \), from which we conclude that \( H \) surjects onto \( SO(V_6, q) \).

### 2.3. Dual groups and \( L \)-functions.

We now define the exterior square (degree six) and standard (degree eight) \( L \)-functions for \( G = GU(2, 2) \). For everything regarding the two-variable integral, we use the form

\[
J_4 := \begin{pmatrix} 1 & \ 0 \\ -1 & \ 1 \\ \ -1 & \ 1 \\ \ -1 & \ 1 \end{pmatrix}.
\]

That is, we consider \( G \) to be the group \( \{g \in GL_4(E) : gJ_4 \tilde{g} = \nu(g)J_4\} \). Recall that the dual group of \( G \) is \( (GL_1(C) \times GL_4(C)) \times Gal(E/F) \). The nontrivial element \( \theta \) of \( Gal(E/F) \) acts by \((\lambda, g) \mapsto (\lambda \det(g), J_4^{-1}g^{-1}J_4^{-1}) \). For this and the contents of the next paragraph see [10] and [15].

If a prime \( p \) of \( F \) is inert in \( E \), then the local group \( G(F_p) \) is still a unitary group. Let us describe the Satake parameters of an unramified representation \( \pi \) of \( G(F_p) \) at such a place \( p \). Denote by \( T \) the diagonal maximal torus of \( G \), so that \( T \) contains \( T' \) the diagonal maximal split torus of \( G \). Suppose \( \alpha \) is an unramified character of \( T(F_p) \) and \( \pi \) is an unramified irreducible quotient of \( \text{Ind}(\delta_B^{1/2} \alpha) \), with \( \delta_B \) the modulus character of uppertriangular Borel of \( G(F_p) \). The elements of \( T \) are of the form \( t = \text{diag}(t_1, t_2, a^{t_1^{-1}}, a^{t_2^{-1}}) \). Thus an unramified character \( \alpha \) of \( T \) is given by characters \( \alpha_0, \alpha_1, \alpha_2 \), where \( \alpha_0 \) is an unramified character of \( F_p^{\times} \), the other \( \alpha_j's \) are unramified characters of \( E_p^{\times} \), and \( \alpha(t) = \alpha_0(a) \alpha_1(t_1) \alpha_2(t_2) \). With these notations, the Frobenius conjugacy class of \( \pi \) in the dual group \( L^G \) is then \((\alpha_0(\varpi), \text{diag}(\alpha_1(\varpi), \alpha_2(\varpi), 1, 1)) \) \( \times \theta \), where \( \varpi \) is a uniformizer of \( F \).

We will now define the exterior square representation of \( L^G \). Set \( V_4 \) the standard 4-dimensional representation of \( GL_4(C) \). Let GL_1(C) \times GL_4(C) act on \( \wedge^2 V_4 \) via \((\lambda, g) \mapsto \lambda \wedge^2 (g) \). Denote this representation by \( p \). Then \( x \mapsto p(x) \) and \( x \mapsto p(\theta^{-1}x \theta) \) are irreducible representations of \( GL_1(C) \times GL_4(C) \) with the same highest weight, and thus are isomorphic. Hence, there is an element \( A \in GL(\wedge^2 V_4) \) satisfying \( p(\theta^{-1}x \theta) = A^{-1}\rho(x)A \). Note that if \( A \) satisfies this identity, then so does \(-A\). It is important that we pick \( A \) with positive trace. (The other choice \(-A \) will not give the \( L\)-function that comes out of the integral representation.) The map that sends \( x = (\lambda, g) \) to
\( \rho(x) \) and \((1,1) \times \theta\) to \(A\) defines a representation of \(L^G = (GL_1(C) \times GL_4(C)) \times Gal(E/F)\). This representation is what we refer to as the exterior square; we denote it by \(\wedge^2\).

The standard representation of \(L^G\) is defined more simply. If \(x = (\lambda, g) \in GL_1(C) \times GL_4(C)\), set \(\rho_4(x) = g \in GL(V_4)\). Then the standard representation \(\rho_{std}\) is the 8-dimensional \(Ind_{E_0}^{GL}(\rho_4)\), where \(L^{G_0} = GL_1(C) \times GL_4(C)\).

The following proposition computes the exterior square and standard \(L\)-function for an unramified representation of \(G(F_p)\) at an inert place \(p\) in terms of \(L\) functions of \(G' = \text{GSp}_4\).

**Proposition 2.2.** Suppose that \(p\) is inert in \(E\), \(\alpha\) is an unramified character of \(T\), and \(\pi\) is an unramified irreducible subquotient of \(Ind_{E}^{GL}(\delta_{B}^{1/2}\alpha)\). Denote by \(\alpha'\) the restriction of \(\alpha\) to \(T'\), and suppose that \(\pi'\) is an unramified irreducible subquotient of \(Ind_{E}^{GL}(\delta_{B}^{1/2}\alpha')\). Then if \(\pi\) has central character \(\omega_\pi\),

\[
L(\pi, \wedge^2, s) = L(\pi', \text{Spin}, s)L(\omega_\pi, 2s).
\]

If \(\pi\) has trivial central character,

\[
L(\pi_p, \text{Std}, w) = \frac{L(\pi_p', \text{Std}, 2w)}{\zeta(2w)}.
\]

Note that the reciprocals of both sides of the first equality are of degree 6 in \(p^{-s}\), while the reciprocals of both sides of the second equality are degree 8 in \(p^{-s}\).

Denote by \(Z_F \simeq GL_1\) the connected component of the center of \(H\), so that \(H/Z_F \simeq SO(V_6,q)\). Since we will consider automorphic representations on \(H\) that transform trivially with respect to \(Z_F\), it suffices to consider the dual group of \(SO(V_6)\). Since \(SO(V_6,q)\) is quasisplit, the dual group \(L SO(V_6)\) may be identified with \(O(V_6(C))\), with the nontrivial element of \(Gal(E/F)\) acting as a reflection. Thus there is again a degree six \(L\)-function, given by the six-dimensional irreducible representation of \(O(V_6(C))\). Because of the close connection to the \(\wedge^2\) \(L\)-function on \(GU(2,2)\), we continue to write \(L(\pi, \wedge^2, s)\) for this \(L\)-function, where now \(\pi\) is an automorphic representation on \(H\) with trivial character on \(Z_F\).

**Proposition 2.3.** Suppose \(p\) is place of \(F\) unramified in \(E\), that \(\alpha\) is an unramified character of the diagonal torus \(T\) of \(H\), trivial on \(Z_F\), and \(\pi_p\) is an irreducible subquotient of \(Ind(\alpha\delta_{B}^{1/2})\). Set \(t_A = \text{diag}(p,p,1,1)\), \(t_B = \text{diag}(p,1,1,p)\), \(t_C = \text{diag}(1,p,p,1)\), and \(t_D = \text{diag}(1,1,p,p)\). If \(p\) is inert in \(E\), then \(L(\pi_p, \wedge^2, s)\)

\[
= \zeta_p(2s)[(1 - \alpha(t_A)|p|^s)(1 - \alpha(t_B)|p|^s)(1 - \alpha(t_C)|p|^s)(1 - \alpha(t_D)|p|^s)]^{-1}.
\]

If \(p\) is split in \(E\), take \(\pi_1, \pi_2\) in \(E_p \simeq F \times F\) so that \(\pi_1 = (p,1)\) and \(\pi_2 = (1,p)\) under this identification. Set \(t_E = \text{diag}(\pi_1, \pi_2, \pi_1, \pi_2)\), \(t_F = \text{diag}(\pi_2, \pi_1, \pi_2, \pi_1)\). Then \(L(\pi_p, \wedge^2, s)\)

\[
= [(1 - \alpha(t_A)|p|^s)(1 - \alpha(t_B)|p|^s)(1 - \alpha(t_C)|p|^s)(1 - \alpha(t_D)|p|^s)(1 - \alpha(t_E)|p|^s)(1 - \alpha(t_F)|p|^s)]^{-1}.
\]

### 3. The two-variable integral

In this section, we state the various results concerning the two-variable Rankin-Selberg integral. As all or almost all the arguments are standard, we omit or sketch very briefly the proofs. Recall also that for this section, we define \(G = GU(2,2)\) in terms of the antidiagonal form,

\[
J_4 := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]

Equivalently, we put the basis of \(W_4\) in the order \(e_1, e_2, f_2, f_1\).
3.1. The global integral. We define the Siegel parabolic $P$ of $G$ to be the stabilizer of the (isotropic) subspace spanned by $f_1, f_2$, and the Klingen parabolic the stabilizer of the line spanned by $f_1$. If $f_P \in \text{Ind}_E^G(\delta_P)$ and $f_Q \in \text{Ind}_Q^G(\delta_Q)$ are standard sections, then the Siegel and Klingen Eisenstein series are, respectively,

$$E_P(g; w) = \sum_{\gamma \in P(F) \setminus G(F)} f_P(\gamma g; w); \quad E_Q(g; s) = \sum_{\gamma \in Q(F) \setminus G(F)} f_Q(\gamma g; s).$$

Following [2], set

$$\nu_P := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}; \quad \nu_Q := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Define $R = (\nu_P^{-1} Q \nu_P) \cap (\nu_Q^{-1} P \nu_Q)$. Then $R$ is the subgroup of $G$ consisting of matrices of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}.$$

**Proposition 3.1.** Suppose $\pi$ has trivial central character. Denote by $Z_E$ the center of $G$, which is $E^\times$ embedded diagonally, $R^0$ the unipotent radical of $R$ and $W_\phi$ the Whittaker model of $\phi$. Then

$$I(\phi; s, w) = \int_{R^0(A)Z_E(A) \setminus G(A)} f_P(\nu_Q g; w)f_Q(\nu_P g; s)W_\phi(g) \, dg.$$

**Proof.** One proves that the double coset $Q(F) \setminus G(F)/P(F)$ is represented by $\{ 1, \nu_P \nu_Q^{-1} \}$, and then the proof is the same as that in [2], Theorem 1.1. \hfill \Box

3.2. Unramified calculation. Define

$$I_p(W; s, w) = \int_{R(F_p)Z_{E_p}(F_p) \setminus G(F_p)} f_P^p(\nu_Q g; w)f_Q^p(\nu_P g; s)W_p(g) \, dg.$$

**Theorem 3.2.** Assume $E/F$ is unramified. Denote by $\epsilon_{E/F}$ the quadratic character associated to $E/F$: i.e., $\epsilon_{E/F}(p)$ is 1 if $p$ is split in $E$ and $-1$ when $p$ is inert in $E$. When all the data is unramified,

$$I_p(W; s, w) = \frac{L(p, \text{Std}, 2w - \frac{1}{2})L(p, \chi^2, 3s - 1)}{L(\epsilon_{E/F}, 4w - 1)\zeta_F(4w)\zeta_E(3s)\zeta_F(6s - 2)}.$$

**Proof.** If $p$ is split in $E$, then this is a calculation with $\text{GL}_4$ and thus is entirely standard. So, let us briefly explain the proof in the case that $p$ is inert in $E$. Write $q$ for the order of the residue field of the integer ring of $F$. Set $U = q^{-(2w - \frac{1}{2})}$, $V = q^{-(3s - 1)}$, and $K = \delta_B^{-1/2}W$, where $W$ is the Whittaker model of $\pi_p$, normalized so that $W(1) = 1$. Define $I_1(W; s, w) = \zeta_F(4w)\zeta_E(3s)I(W; s, w)$. By manipulations similar to the proof of Theorem 1.2 in [2], we get that

$$I_1(W; s, w) = \sum_{n,m \geq 0} V^n U^{2m} \left( \frac{1 - U^{2n+2}}{1 - U^2} \right) \left( \frac{1 - V^{2m+2}}{1 - V^2} \right) K([m, n]),$$

where $K([m, n]) = K(\text{diag}(p^{m+n}, p^n, 1, p^{-m}))$. Since the rational root system of $G$ is that of $\text{GSp}_4$, a fact due to Tamir [18] is that the Casselman-Shalika formula for $G$ reduces to that of $\text{GSp}_4$ in
the sense that $K^G_{p} = K^{GSp}_p$ on the torus of $GSp_4 \subseteq G$. Here $\pi'_p$ is as in Proposition 2.2. Applying this fact, $I_1$ is computed in Bump-Friedberg-Ginzburg [2] Theorem 1.2 to be

$$
L(\pi'_p, \text{Std}, U^2) L(\pi'_p, \text{Spin}, V)
$$

(4)

The notation $L(\pi_p, \text{Std}, Z)$ means $\sum_{k \geq 0} \text{tr}(\text{Sym}^k \text{Std}(A_p)) Z^k$, where $A_p$ is the conjugacy class in $L^G$ corresponding to $\pi_p$, and similarly for the other $L$ functions. Via Proposition 2.2, (4) is equal to

$$
\frac{\zeta(U^2) L(\pi_p, \text{Std}, U) L(\pi_p, \wedge^2, V)}{\zeta(U^4) \zeta(V^2) \zeta(U^2) \zeta(V^2)}
$$

(5)

The theorem follows. \hfill \Box

Remark 3.3. The normalized Siegel Eisenstein series is $E^*(g; w) = L(\epsilon_{E/F}, 4w - 1) \zeta_F(4w) E_P(g; w)$; see, for instance, Tan [19]. The normalized Klingen Eisenstein series is $\zeta(3s) \zeta_F(6s - 2) E_Q(g; s)$. These facts can be verified using the work of Lai [12], together, of course, with Langlands' theory of the constant term. Thus the $L$-functions in the denominators of the local unramified calculation disappear upon the insertion of the normalized Eisenstein series.

4. The one-variable integral

In this section, we consider the one-variable integral for the degree six $L$-function of $H(A)$. As remarked in the introduction, it unfolds to a non-unique model. Throughout the rest of the paper, $F = \mathbb{Q}$ and $E$ is an imaginary quadratic extension of $\mathbb{Q}$.

4.1. Global constructions. We will now define the special function $P^2_T(h)$ precisely. For a matrix $T \in H_2(E)$, we set $v_T$ the element of $V_6$ represented by the matrix $(T, T^T) \in V_6 \subseteq M_4(E)$. For simplicity, we will assume $-\det(T) \in F$ is not a norm from $E$. This is equivalent to assuming that the Hermitian form on $E^2$ given by $T$, namely $\langle u, v \rangle_T := u^T \overline{v}$, has no nonzero isotropic vectors.

Define $G_T$ to be the stabilizer of $v_T$ inside $H$. Then $G_T$ is a quaternion unitary group on two variables. The quaternion algebra $B_T$ in terms of which it is defined is the set of $m$ in $M_2(E)$ that satisfy $m = Tm^\# T^{-1}$, where recall that $m^\# = \overline{m}^t$. The assumption that $-\det(T)$ is not a norm from $E$ implies that the quaternion algebra $B_T$ is not split. An element $(1, 1)$ in $H$ is in $G_T$ if and only if $\text{tr}(Tu) = 0$. Similarly, an element $M = (\lambda m^\# \quad m)$ in $H$ is in $G_T$ if and only if $m$ is in $B_T$, i.e.,

$$
\begin{pmatrix}
\lambda m^\# \\
\quad m
\end{pmatrix} \in G_T \iff m = Tm^\# T^{-1}.
$$

(5)

For a Schwartz-Bruhat function $\alpha$ on $V_6(A)$, we set

$$
P^2_T(h) = \sum_{\gamma \in G_T(H(A) \setminus H(Q))} \alpha(v_T \gamma h).
$$

Let us also define the normalized Eisenstein series. Pick a Schwartz-Bruhat function $\Phi$ on $W_4(A_E)$. Pick a nonzero vector $f$ in $Ef_1 \oplus Ef_2$ and set

$$
f^{\Phi, *}(g; s) = \zeta^S(6s - 2) |\nu(g)|^{3s}_F \int_{\text{GL}_4(A_E)} \Phi(tfg)|t|^{3s}_E \, dt.
$$

Then if $Q$ denotes the stabilizer of the line $Ef$ in $H$, $f^{\Phi, *} \in \text{Ind}(\delta_Q^s)$, and for almost all $p$, $f^{\Phi, *}_p(1; s) = \zeta_p(6s - 2) \zeta_{E_p}(3s)$. We set $E^{\Phi, *}_Q(g; s) = \sum_{\gamma \in Q(F) \setminus G(F)} f^{\Phi, *}(\gamma g; s)$. The following three lemmas will trivialize the unfolding.

Lemma 4.1. The double coset space $Q(F) \setminus H(F)/G_T(F)$ is a singleton.
Proof. The coset space $Q(F)\backslash H(F)$ is in bijection with the isotropic $E$ lines in $W_4(E)$. Using that $-\det(T)$ is not a norm from $E$, it is easy to check via explicit matrix computations that all the isotropic lines are in one orbit under $G_T$. □

Definition 4.2. Consider $V_E := Ef_1 \oplus Ef_2$. Define a conjugation on $V_E$ as $af_1 + bf_2 = \overline{a}f_1 + \overline{b}f_2$, $a, b \in E$. Thinking of $V_E$ as row vectors, we have a right action on $V_E$ by $T$ via $v \mapsto vT$, $v \in V_E$. For a nonzero $f$ in $Ef_1 \oplus Ef_2$, set $f_T = \overline{T}F(1_{-1})$. Denote $(\cdot, \cdot)_T$ the Hermitian form defined by $T$ on $V_E$. That is, if $u, v$ are in $Ef_1 \oplus Ef_2$, thought of as row vectors, then $(u, v)_T = u^T\overline{v}$.

We note for later use that $f \left(\left(-1\right)^{-1}f_T\right) = -\left(f, f\right)_T$, and that
\[
\begin{pmatrix}
\langle f, fT \rangle_T & \langle f, fT \rangle_T \\
\langle fT, f \rangle_T & \langle fT, fT \rangle_T
\end{pmatrix} = (f, f)_T \begin{pmatrix} 1 & \det(T) \end{pmatrix}.
\]
Since $f \left(\left(-1\right)^{-1}f_T\right) = -\left(f, f\right)_T \neq 0$, $f, f_T$ make a basis of $V_E$.

Lemma 4.3. If $g \in Q \cap G_T$, and $fg = \lambda f$, then $fTg = \overline{\lambda}f_T$. Conversely, if $x \in \GL(V_E)$ satisfies $fx = \lambda f$, $fTx = \overline{\lambda}f_T$, then $(x^#_x)$ is in $G_T$.

Proof. The defining relation for stabilizing $v_T$ is
\[
\epsilon^{-1}TgevTg = \nu(g)v_T.
\]
Note that $J^*_4gJ^{-1}_4 = \nu(g)g^{-1}$, $J^*_4v_TJ^{-1}_4 = -v_T$, and $J^*_4\epsilon J^{-1}_4 = -\epsilon$. Take \([6] \), apply conjugate transpose to it, and conjugate by $J_4$. We thus obtain that the defining relation is equivalent to
\[
g^{-1}v_T\epsilon J_4 = v_T\epsilon J\overline{\gamma}^{-1}.
\]
Now suppose $g \in Q \cap G_T$. Then so is $g^{-1}$, so
\[
\lambda fv_T\epsilon J_4 = fgv_T\epsilon J_4 = f\nu_T\epsilon J\overline{\gamma}.
\]
Multiplying this out gives
\[
\lambda f T \left(\left(-1\right)\right) = f T \left(\left(1_{-1}\right)\right) \overline{\gamma},
\]
or $\overline{\lambda}f_T = f_Tg$, as desired. The same equations prove the converse. □

Define $D_E \subset G_T$ to be the set of $(x^#_x)$, where $x \in \GL_2(E)$ satisfies $fx = \lambda f$, $fTx = \overline{\lambda}f_T$, for some $\lambda$ in $E^\times$. Then $D_E \cong \GL_1/E$. It follows immediately from Lemma [4,3] that

Lemma 4.4. $G_T \cap Q$ is the set of matrices in $H$ of the form \( \begin{pmatrix} \lambda x^# & x \\ x & \end{pmatrix} \) where $fx = \lambda f$, $f Tx = \overline{\lambda}f_T$, $\lambda \in \GL_1/E$.

Now we define the Fourier coefficient that will appear in the unfolded integral. Denote by $U_P$ the unipotent radical of the Siegel parabolic of $H$, so that $U_P$ consists of the matrices $U = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ in $H$. We define $\chi : U_P(A) \to \C^\times$ via $\chi(U) = \psi(\tr(Tu))$ where $u$ is the upper right two-by-two block of $U$. Note that $D_E$ centralizes $\chi$ by [5]. We set
\[
\phi_{\chi,N}(g) = \int_{U_P(F)\backslash U_P(A)} \chi^{-1}(u)\phi(ug)\, du; \quad \text{and} \quad \phi_{\chi}(g) = \int_{D_E(F)\backslash Z_E(A)\backslash D_E(A)} \phi_{\chi,N}(xg)\, dx.
\]
For $N_T := U_P \cap G_T$, define
\[
\phi_{\chi,T}(v_Tg) = \int_{N_T(A)\backslash U_P(A)} \chi(u)\phi(v_Tug)\, du.
\]
Proof. Observe that

\[ J_{\Phi,*}^{\phi}(\phi; s) := \int_{H(\mathbb{Q})Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(g)P_T^\phi(g)E_Q^{\Phi,*}(g; s) \, dg. \]

Then

\[ J_{\Phi,*}^{\phi}(\phi; s) = \zeta^\phi(6s-2) \int_{U_F(\mathbb{A})\backslash H(\mathbb{A})} \alpha_\phi(vTg)|\nu(g)|^{3s}\Phi(fg)\phi_\chi(g) \, dg. \]

Then

\[ J_{\Phi,*}^{\phi}(\phi; s) = \zeta^\phi(6s-2) \int_{U_F(\mathbb{A})\backslash H(\mathbb{A})} \alpha_\phi(vTg)|\nu(g)|^{3s}\Phi(fg)\phi_\chi(g) \, dg. \]

Performing an inner integral over \( D_E \) in the expression

\[ \int_{N_T(\mathbb{A})\backslash H(\mathbb{A})} \alpha_\phi(vTg)|\nu(g)|^{3s}\Phi(fg)\phi_\chi(g) \, dg \]

we get \( \Box \), proving the proposition.

Proposition 4.5. Set \( J_{\Phi,*}^{\phi}(\phi; s) \) the Rankin-Selberg integral with the normalized Klingen Eisenstein series, i.e.,

\[ J_{\Phi,*}^{\phi}(\phi; s) := \int_{H(\mathbb{Q})Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(g)P_T^\phi(g)E_Q^{\Phi,*}(g; s) \, dg. \]

Then

\[ J_{\Phi,*}^{\phi}(\phi; s) = \zeta^\phi(6s-2) \int_{U_F(\mathbb{A})\backslash H(\mathbb{A})} \alpha_\phi(vTg)|\nu(g)|^{3s}\Phi(fg)\phi_\chi(g) \, dg. \]

Proof. Observe that

(7) \[ \int_{N_T(\mathbb{A})\backslash H(\mathbb{A})} \phi(ng) \, dn = \sum_{\lambda \in F^\times} \phi_\chi(N(d_\lambda g) \]

where \( d_\lambda = \text{diag}(\lambda, \lambda, 1, 1) \).

First unfold \( P_T^\phi \), and then unfold the Eisenstein series using Lemma 4.1. Now, integrating over \( N_T \), and then using \( \Box \) we obtain

\[ \frac{J_{\Phi,*}^{\phi}(\phi; s)}{\zeta^\phi(6s-2)} = \int_{D_E(\mathbb{Z})N_T(\mathbb{A})Z_F(\mathbb{A})\backslash H(\mathbb{A})} \alpha_\phi(vTg)f_Q^\phi(g; s)\phi_\chi,N(g) \, dg \]

\[ = \int_{D_E(\mathbb{A})N_T(\mathbb{A})\backslash H(\mathbb{A})} \alpha_\phi(vTg)f_Q^\phi(g; s)\phi_\chi(g) \, dg. \]

Below we make the unramified computation, Theorem 4.10, which implies

Theorem 4.6. For a sufficiently large finite set \( S \) of places, including the archimedean place, we have \( J_{\Phi,*}^{\phi}(\phi; s) = J_{S,*}^{\phi}(\phi; s)L^S(\pi, \lambda^2, 3s-1) \), where

\[ J_{S,*}^{\phi}(\phi; s) = \int_{U_P(\mathbb{A}_S)\backslash H(\mathbb{A}_S)} \alpha_\phi(vTg)|\nu(g)|^{3s}\Phi(fg)\phi_\chi(g) \, dg \]

and \( \mathbb{A}_S := \prod_{v \in S} Q_v \).

When \( \phi \) comes from a level one Hermitian modular form of weight \( r \geq 7 \), \( E \) has class number one, and \( T \) is positive definite, we can also pick the data \( \alpha_\infty \) and \( \Phi_\infty \) so as to compute \( J_{s,*}^{\phi}(\phi; s) \) in terms of \( \Gamma \)-functions, at least under some simplifying assumptions. This is Theorem 4.16. Combining this with Theorem 4.6 and Proposition 4.15 which controls the local integrals at the bad finite places, we obtain

Theorem 4.7. Suppose \( E \) has class number one, \( T \) is positive definite, and \( \phi \) comes from a level one Hermitian modular form \( f_\phi \) of weight \( r \geq 7 \), as made precise in section 4.4. Assume moreover that \( T \in H_2(O_E) \), \( f \in O_Ef_1 \oplus O_Ef_2 \), and \( \langle f, f \rangle_T = 1 \). Then there exists \( \phi_1 \) in the space of \( \pi \), and data \( \alpha \) for \( P_T^\alpha \), \( \Phi \) for \( E_Q^{\Phi,*} \) so that

\[ J_{\Phi,*}^{\phi}(\phi_1; s) = a(T)(2\pi)^{-gs}\Gamma(3s-1)\Gamma(3s)\Gamma(3s+r-3)L^S(\pi, \lambda^2, 3s-1), \]

where \( a(T) \) is the (constant) Fourier coefficient of \( f_\phi \) corresponding to \( T \) and \( S \) is the set of primes ramified in \( E \).
4.2. **Unramified computation.** In this section we give the unramified computation. Everything here is local, so we write $E$ in place of $E_p$ and $F$ in place of $F_p$. We assume throughout that $p$ is unramified in $E$, so that either $E$ is $F \times F$ or $E$ is the unique unramified quadratic field extension of $F$.

Let us first relate the degree six $L$-function of $H = \text{GSpin}_6$ to Hecke operators.

**Definition 4.8.** Define $\Delta^s(g) = |\nu(g)|^s \text{char}(g)$ where the characteristic function is that of $M_4(O_E)$.

Set $K = H(E) \cap \text{GL}_4(O_E)$. The function $\Delta^s(g)$ is bi-$K$-invariant. The following theorem may be extracted from Hina-Sugano [9] and Gritsenko [7].

**Theorem 4.9.** Denote $\omega_\pi$ the Macdonald spherical function of the irreducible representation $\pi$ of $H(F)$, and $\omega_\pi$ the central character of $\pi$. Then

$$\int_{H(F)} \omega_\pi(g) \Delta^s(g) \, dg = \frac{L(\pi, \lambda^2, s - 2)}{L(\omega_\pi, 2s - 4)L(\omega_\pi, 2s - 2)}.$$

Using this theorem, we now give the unramified computation. Let $\ell : \pi \to \mathbb{C}$ be a $(U_P, \chi)$ functional. (We do not need to hypothesize the extra invariance under $D_E$). Let $v_0$ be the spherical vector of $\pi_p$, and assume $\Phi$ is the characteristic function of $W_4(O_E)$. Assume moreover $\alpha$ is the characteristic function of $V_6(O_F)$. Set

$$J^s(\ell; s) = \zeta(6s - 2) \int_{U_P(F) \backslash H(F)} \alpha_{\chi}(v_Tg)|\nu(g)|^{3s} \Phi(fg) \ell(\pi(g)v_0) \, dg.$$

**Theorem 4.10.** Assume $T$ is integral, $f \in W_4(O_E)$, and $(f, f)_T \in O_F^\times$. Then

$$J^s(\ell; s) = L(\pi, \lambda^2, 3s - 1)\ell(v_0).$$

**Remark 4.11.** Note that since we assume $E/F$ is unramified, the inverse different is trivial, and thus the pairing $(h_1, h_2) \mapsto \text{tr}(h_1h_2)$ on $H_2(E)$ identifies $H_2(O_E)$ with its own $O_F$-linear dual. Also note that the condition $(f, f)_T \in O_F^\times$ implies $p^{-1}T$ is not integral.

Define

$$I^\Delta(\ell; s) = \int_{H(F)} \Delta^s(g) \ell(\pi(g)v_0) \, dg.$$

The proof will be to show $J^{\Phi,s}(\ell; s) = \zeta(6s - 2)\zeta(6s)I^\Delta(\ell; 3s + 1)$. Before doing this, we need a couple preliminaries.

**Definition 4.12.** For $m \in \text{GL}_2(E)$ with $\det(m) \in F^\times$, define $\Xi_T(m) = \text{char}(m^{-1}Tm^\#)$ integral. That is, $\Xi_T(m)$ is 1 if $m^{-1}Tm^\#$ is integral, and zero otherwise.

**Proposition 4.13.** Suppose $M = \binom{\lambda m^\#}{m}$ is in $H$. Define $Y(M) = \text{char}(|\lambda| \leq 1)$. Then $\alpha_\chi(v_TM) = Y(M)|\lambda|\Xi_T(m)$.

**Proof.** Suppose $M$ as above, and $u = \binom{1 h}{1}$ $U_P$. We have

$$v_TuM = \binom{m^{-1}Tm^\#}{\lambda^{-1}\text{tr}(Th)}\left(\binom{m^{-1}Tm^\#}{\lambda^{-1}\text{tr}(Th)}\right).$$

Hence, for $\alpha(v_TM) \neq 0$, we need $m^{-1}Tm^\#$ integral. Furthermore, it is clear from this expression that the set of $u \in U_P$ for which $\alpha(vTOTYPE) \neq 0$ is an abelian group. Hence, the integral defining $\alpha_\chi$ will vanish unless $\chi$ is trivial on this set. It follows that for $\alpha_\chi(v_TM)$ to not vanish, we need $|\lambda| \leq 1$, and furthermore that the integral

$$\int_{N_T \backslash U_P} \chi(u)\alpha(v_TM) \, du$$

in this case is $|\lambda|$. This gives the proposition. \qed
Proof of Theorem 4.10. By the Iwasawa decomposition,
\[ I^\Delta(\ell; s) = \int_{M_P} \delta_P^{-1}(M)|\nu(M)|^s \ell(\pi(M)v_0) \left( \int_{U_P} \chi(u) \text{char}(uM) du \right) dM, \]
where \( M_P = \left( \ast_s \right) \subseteq H \) is the Levi of the Siegel parabolic in \( H \). We compute this inner integral as a function of \( M \in M_P \).

Suppose \( M = \left( \lambda_m \right) \). Define \( U_m = \{ u \in H_2(E) : um \in M_2(O_E) \} \). Then \( U_m \) is an abelian group, so the inner integral vanishes unless

\[ \{ u \mapsto \psi(\text{tr}(Tu)) \} \text{ is trivial when restricted to } U_m. \]

If condition \( \mathbb{2} \) is satisfied then the inner integral is equal to the measure of \( U_m \). We claim
\[ \int_{H_2(E)} \psi(\text{tr}(Tu)) \text{char}(um \in M_2(O_E)) du = |\det(m)|^{-1} \text{char}(\text{val}_p(m) = 0) \Xi_T(m). \]

Here, for a matrix \( m \), \( \text{val}_p(m) \) is defined by the equality \( m = p^{\text{val}_p(m)}m_0 \) with \( m_0 \) in \( M_2(Z_p) \setminus pM_2(Z_p) \).

Suppose \( m \) is such that the left-hand side of \( \mathbb{10} \) is nonzero. First, because \( p^{-1}T \) is not integral, we get \( \text{val}_p(m) = 0 \). Changing variables \( u \mapsto m^\# um^{-1} \), we see that the unipotent integral is equal, up to a nonzero constant, to
\[ \int_{H_2(E)} \psi(\text{tr}(m^{-1}Tm^\# u)) \text{char}(m^\# u \in M_2(O_E)) du. \]

Since \( m^\# \) is still integral, we see \( \text{tr}(m^{-1}Tm^\# u) \) must be integral for all integral \( u \). Hence \( m^{-1}Tm^\# \) is integral, by Remark \( \mathbb{4,11} \).

Now suppose \( \text{val}_p(m) = 0 \) and \( m^{-1}Tm^\# \) is integral. We will show \( \text{tr}(Tu) \) is integral for all \( u \) in \( U_m \). To see this, take such a \( u \) in \( U_m \). Then obviously \( Tuu \) is integral, since \( T \) is integral. Since \( m^\# u \) is integral, we also have
\[ m'Tu = (m^{-1}Tm^\#)(m^\# u) \]

is integral. It follows that \( (Tu)'m \) is integral. Adding \( Tum \) and \( (Tu)'m \) together, we get \( \text{tr}(Tu)m \) is integral, since \( Tu + (Tu)' \) is integral. Since \( \text{val}_p(m) = 0 \), we conclude \( \text{tr}(Tu) \) is integral.

Now we compute the measure of \( U_m \). To do this, write \( m = kdk' \) with \( k, k' \) in \( GL_2(O_E) \) and \( d = \text{diag}(d_1, d_2) \) diagonal. (Note that one can do this both when \( E \) is a field, and when \( E = F \times F \).) By changing \( k, k' \) if necessary, we may assume \( \det(d) \in F^\times \). A change of variable gives \( \text{meas}(U_m) = \text{meas}(U_d) \), so we compute the measure of \( U_d \). First assume \( E \) is a field. Since \( \text{val}_p(m) = 0 \), we may assume \( d_3 = 1 \). A matrix multiplication \( ud \) now immediately gives \( \text{meas}(U_d) = |d_1|^{-1} = |\det(m)|^{-1} \). Now assume \( E = F \times F \). We have \( d = \text{diag}((d_1^A, d_2^A), (d_1^B, d_2^B)). \) Since \( \text{val}_p(m) = 0 \), we may assume \( d_3^A = 1 \), and then since \( \det(d) \in F^\times \), we get \( d_1^A = d_1^B d_2^B \). Now the matrix multiplication \( ud \) gives \( \text{meas}(U_d) = |d_1^B d_2^B|^{-1} = |\det(m)|^{-1} \).

Hence,
\[ I^\Delta(\ell; s) = \int_{M_P} \delta_P^{-1}(M)|\nu(M)|^s |\det(m)|^{-1} \text{char}(\text{val}_p(m) = 0) \Xi_T(m)\ell(\pi(M)v_0) dM. \]

Since \( \ell(\pi(M)v_0) \neq 0 \) implies \( |\lambda| \leq 1 \), this is
\[ \int_{M_P} \delta_P^{-1}(M)|\nu(M)|^s |\det(m)|^{-1} \text{det}(Y(M) \text{char}(\text{val}_p(m) = 0) \Xi_T(m)\ell(\pi(M)v_0) dM. \]

Thus \( \zeta(6s) I^\Delta(\ell; 3s + 1) \)
\[ \int_{M_P} \delta_P^{-1}(M)|\nu(M)|^{3s+1} |\det(m)|^{-1} \text{det}(Y(M) \text{char}(m) \Xi_T(m)\ell(\pi(M)v_0) dM. \]
On the other hand, from Proposition 4.13 we get

\[ \frac{J^{\Phi,*}(\ell; s)}{\xi(6s - 2)} = \int_M \delta_F^{-1}(M)\nu(M)|^{3s}\Phi(fM)|\lambda|\Xi_T(m)\ell(\pi(M)v_0)\, dM. \]

Since \(|\lambda| = |\nu| \det(m)|^{-1}\), and since \(\text{char}(fM)\Xi_T(m) = \text{char}(m)\Xi_T(m)\) by Lemma 4.14 below, we are done.

Recall \(f_T\) from Definition 4.2.

**Lemma 4.14.** If \(m^{-1}Tm^\#$ \in M_2(O_E) \cap H_2(E), f_m \) is in \(O_Ef_1 \oplus O_Ef_2 =: V(O_E), \) and \(\langle f, f \rangle_T \in O_F^\times, \) then \(m \in M_2(O_E).\)

**Proof.** Since \(f(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}) f_T = -\langle f, f \rangle_T \in O_F^\times \) is a unit, \(f \) and \(f_T\) form a basis of the free \(O_E\) module \(V(O_E).\) Hence to check that \(m \in M_2(O_E), \) it suffices to check that \(f_m \) and \(f_Tm\) are in \(V(O_E), \) and thus we must only check the latter condition. Set \(J_2 = (1 - 1).\) Then

\[ f_Tm = fTJ_2m = fT(J_2mJ_2^{-1})J_2 = fTm^#J_2 = fM \left( m^{-1}Tm^# \right) J_2. \]

Hence \(f_Tm \in V(O_E),\) as desired. \(\square\)

4.3. **Ramified computation.** In this section we control the local integral \(J(\ell; s)\) at bad finite places. Everything below is local at \(p.\)

**Proposition 4.15.** Let \(v_0\) in the space of \(\pi_p\) be given. Then, there exist \(v\) in the space of \(\pi_p,\) a Schwartz function \(\Phi\) on \(W_4(E),\) and a Schwartz-Bruhat function \(\alpha\) on \(V_0(F),\) all depending only on \(v_0,\) so that for all \((\ell, \chi)\) models \(\ell,\)

\[ J(\ell; v, s) := \int_{N_T \backslash H} \alpha(v_Tg)|\nu(g)|^{3s}\Phi(fg)\ell(\pi(g)v_0)\, dg = \ell(v_0). \]

**Proof.** As this is similar to [14] Proposition 3.4, we omit the proof. \(\square\)

4.4. **Holomorphic Hermitian modular forms.** In this section, we make the archimedean computation for holomorphic Hermitian modular forms, under some simplifying assumptions. The calculations here are very similar to those in the analogous section of [14]. We assume \(E\) is an imaginary quadratic field, that \(T\) is positive definite, and that \(\phi\) comes from a scalar weight Hermitian modular form. Additionally, our simplifying assumptions are that \(E\) has class number one, \(\phi\) is level one, \(T\) is in \(H_2(O_E),\) and \(\langle f, f \rangle_T = fT^*f = 1.\)

Denote by \(H\) the Hermitian upper half space, consisting of \(Z = X + iY \) in \(M_2(\mathbb{C})\) with \(X, Y\) Hermitian and \(Y\) positive definite. Then \(H^+(\mathbb{R})\) (the elements with positive similitude) act transitively on \(H\) via the formula \(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\) takes \(Z\) to \((AZ + B)(CZ + D)^{-1}.\) Set \(K_\infty\) to be the elements of \(H^+(\mathbb{R})\) that stabilize \(i\) and have similitude 1. (So, \(K_\infty\) sits inside of \(SU(2, 2).\) For \(\gamma\) as above, \(Z, E, \) define \(j(\gamma, Z) = \det(CZ + D).\)

For a finite prime \(p\) of \(F,\) we denote by \(K_p\) the open compact subgroup of \(H(Q_p)\) equal to \(H(Q_p) \cap \text{GL}_4(O_{E_p}).\) We assume our cusp form \(\phi\) satisfies

1. \(\phi(gk_fk_\infty) = j(k_\infty, i)^{-r}\phi(g)\) for all \(k_f \in \prod_p K_p, k_\infty \in K_\infty.\)
2. The function \(f_\phi : H \to \mathbb{C}\) well-defined by \(f_\phi(g_\infty(i)) = \nu(g_\infty)^{-r}j(g_\infty, i)^r\phi(g_\infty)\) for \(g_\infty\) in \(H(\mathbb{R})^+\) is a holomorphic Hermitian modular form.

It follows from these assumptions that

\[ f_\phi(g_\infty \cdot i) = \sum_{H \in H_2(O_E)\backslash H_2(\mathbb{C})} a(H)e^{2\pi i \operatorname{tr}(Hg_\infty \cdot i)}, \]
where $a(H)$ are the (constant) Fourier coefficients of $f_\phi$ and $H_2(O_E)^\vee$ denotes the elements $H$ of $H_2(E)$ satisfying $\text{tr}(HR) \in \mathbb{Z}$ for all $R \in H_2(O_E)$. (Compare Remark 4.11). Since $U_P(A) = U_P(Q)U_P(R)U_P(\mathbb{Z})$, we have for $m \in M^+(\mathbb{R})$,

$$\nu(m)^{-r}j(m,i)^r \int_{U_P(Q)\backslash U_P(A)} \psi^{-1}(\text{tr}(Tu))\phi(um) du = a(T)e^{2\pi i \text{tr}(Tm \cdot i)}.$$ 

To compute the model $\phi_\chi$ that appears in the unfolded integral, we still must integrate over $D_E/Z_F$. Since $fJ_2^1 f_T = (f,f)_T$, which is 1 by assumption, the elements $f,f_T$ extend to a symplectic, $O_E$-basis of $W_4(O_E)$. Hence, under the identification $D_E \cong \text{GL}_{1/E}$ defined by the action on the vector $f$, $D_E(\mathbb{Z}) := D_E(A_f) \cap \text{Stab}(W_4(O_E) \otimes \mathbb{Z} \mathbb{Z})$ becomes exactly $\text{GL}_1(\mathbb{O}_E)$. Since we assume $E$ has class number one, the natural map

$$\text{GL}_1(E) \text{GL}_1(A)/\text{GL}_1(\mathbb{O}_E) \to \text{GL}_1(\mathbb{R})/\text{GL}_1(E)(\mathbb{R})$$

is a bijection, and hence so is

$$D_E(Q)Z_F(A)\backslash D_E(A)/D_E(\mathbb{Z}) \to Z_F(R)\backslash D_E(R).$$

Thus, for $m \in M^+(\mathbb{R})$,

$$\phi_\chi(m) = a(T)\nu(m)^{-r}j(m,i)^{-r} \int_{\text{GL}_1(\mathbb{R})\backslash D_E(\mathbb{R})=U(1)(\mathbb{R})} e^{2\pi i \text{tr}(Tz \cdot m \cdot i)} dz,$$

since $\nu(z) = j(z,Z) = 1$ for $z \in U(1)(\mathbb{R})$. Since for $z \in D_E$, $\text{tr}(Tz \cdot Z) = \text{tr}(TZ)$, we get that

$$\phi_\chi(m) = a(T)\nu(m)^{-r}j(m,i)^{-r}e^{2\pi i \text{tr}(Tm \cdot i)}$$

up to a nonzero constant. If $m$ has negative similitude, $\phi_\chi(m) = 0$.

We switch notation from the previous section, and instead consider the global integral to be

$$J^\Phi,*_\infty(\phi; s) = \int_{H(Q)Z_F(A)/H(A)} \phi(g)P_T^\infty(g)E_Q^\Phi,*_\infty(g; s) dg,$$

i.e., we replace $P_T^\infty$ by its complex conjugate. It now follows from Theorem 4.10 and Proposition 4.13 that we may choose another cusp form $\phi_1$ in the space of $\pi$ so that $J^\Phi,*_\infty(\phi_1; s) = J^\Phi,*_\infty(\phi; s)L^S(\pi,\lambda^2,3s-1)$ where the archimedean integral

$$J^\Phi,*_\infty(\phi; s) = \Gamma_R(6s-2) \int_{U_P(\mathbb{R})\backslash H(\mathbb{R})} \alpha_\chi(g)||\nu(g)||^{3s}\Phi(fg)\phi_\chi(g) dg.$$

Here $\Gamma_R(s) = \pi^{-s/2}\Gamma(s/2)$. Note that the archimedean integral in this equality involves the Fourier coefficient of the cusp form $\phi$, not $\phi_1$. We will show

**Theorem 4.16.** For a choice of data $\Phi_\infty$ and $\alpha_\infty$ made below, the archimedean integral $J^\Phi,*_\infty(\phi; s)$ is equal to

$$a(T)(2\pi)^{-9s}\Gamma(3s - 1)\Gamma(3s)\Gamma(3s + r - 3)$$

up to a nonzero constant.

We also define $K_\infty$-invariant Hermitian inner products $|| \cdot ||$ on $W_4(\mathbb{C})$ and $\Lambda^2_W(\mathbb{C}) \otimes \nu^{-1}$. The inner product on $W_4(\mathbb{C})$ is defined so as to make $e_1,e_2,f_1,f_2$ orthonormal, and the one on $\Lambda^2_W(\mathbb{C}) \otimes \nu^{-1}$ is induced from this inner product on $W_4(\mathbb{C})$. The inner products are $\mathbb{C}$-linear in the first variable. If $v = W(e_1 \wedge f_1) - W^*(e_2 \wedge f_2) + Be_1 \wedge f_2 + Ce_2 \wedge f_1 + Ge_1 \wedge e_2 + Ef_1 \wedge f_2$ then

$$||v||^2 = 2||W||^2 + B^2 + C^2 + G^2 + E^2.$$ 

Define $r_* = -(e_1 - if_1) \wedge (e_2 - if_2) \in \Lambda^2_W(\mathbb{C}) \otimes \nu^{-1}$. In terms of the matrix representation of $V_0$, $r_* = \begin{pmatrix} -1 & 0 \\ i & 1 \end{pmatrix}$. The following proposition is analogous to [14], Lemma 6.1.

**Proposition 4.17.** The element $r_*$ satisfies the following properties:
(1) \(|(r_s, v)|^2 = |v|^2 - (v, v)\) for \(v \in V_6(\mathbb{R})\).
(2) \(r_s k_\infty = j(k_\infty, i)^{-1} r_s\).
(3) If \(g_\infty \cdot i = Z = X + iY\), then

\[
j(g_\infty, i)^{-1} \nu(g_\infty) r_s g_\infty^{-1} = \begin{pmatrix} -1 & Z \\ Z' & \det(Z) \end{pmatrix}.
\]

(4) If \(v = (\alpha, \beta, \delta)\), then \(j(g_\infty, i)^{-1} \nu(g_\infty)(r_s, v g_\infty) = - (\alpha \det(Z) + \text{tr}(h' Z) + \delta) =: Q_v(Z)\).

Proof. We explain the proof of (3). For this, note that \(j(g_\infty, i)^{-1} \nu(g_\infty) r_s g_\infty^{-1}\) is right \(K_\infty\) invariant by (2). Hence to compute it, we may assume \(g = u M\) is in \(P\), with \(M = (\nu^* m^{-1} \cdot m)\), and \(u = (1 X)\), where \(Y = v^* m^{-1} m^{-1}\). Then by (3),

\[
r_s M^{-1} = \begin{pmatrix} -\nu^{-1} \det(m) & \det(m) m^{-1} i \nu m^{-1} \\ (\det(m) m^{-1} i \nu m^{-1})^* \\
\nu \det(m)^{-1} & 1 \end{pmatrix}.
\]

Hence

\[
j(M, i)^{-1} \nu(M) r_s M^{-1} = \begin{pmatrix} -1 & iY \\ iY^* & - \det(Y) \end{pmatrix},
\]

since \(\det(Y) = (\nu \det(m)^{-1})^2\). Acting again by \(u^{-1} = (1 - X)\), one gets the desired formula.

We define \(\alpha_\infty(v_T g) = (r_s, v_T g)^{-r}\). Note that since \(\det(T) > 0\), the first part of the above proposition implies \(\alpha_\infty(v_T g)\) is well-defined, and that the sum defining \(P^\alpha_{\Gamma}(t)\) converges absolutely for \(r \geq 7\). In the notation of holomorphic Hermitian modular forms, we have for \(Z \in \mathcal{H}\),

\[
P^\alpha_{\Gamma}(Z) = \sum_{v \in V_6(\mathbb{Z}), q(v) = -2 \det(T)} \frac{1}{Q_v(Z)}.
\]

(Recall that for an element \(v = (\alpha, \beta, \delta)\) of \(V_6\), \(q(v) = 2(\alpha \delta - \det(h))\).) The following proposition computes the complex conjugate of \(\alpha_\lambda(g)\), and is proved exactly as [14], Lemma 6.2.

**Proposition 4.18.** Suppose \(m \in M^+(\mathbb{R})\), so that \(m(i) = i Y\) for some \(Y\). Then

\[
\int_{\mathcal{N}\setminus U_P(\mathbb{R})} e^{-2 \pi i \text{tr}(T X)} \alpha_\infty(v_T u(X) m) \, dX = \frac{(2 \pi i)^r}{(r - 1)!} \nu(m)^r j(m, i)^{-r} e^{-2 \pi \text{tr}(T Y)}.
\]

We have defined

\[
f^{\Phi,*}(g; s) = \Gamma(6s - 2) |\nu(g)|^{3s} \int_{\mathcal{GL}_1(C)} |t|^{3s} \Phi(t f g) \, dt.
\]

Define \(\Phi\) by \(\Phi(v) = e^{-2 \pi ||v||^2}\). Then we find

\[
f^{\Phi,*}(g; s) = \Gamma(6s - 2) \Gamma_C(3s) |\nu(g)|^{3s} (||fg||^2)^{-3s}
\]

up to a nonzero constant, where \(\Gamma_C(s) = 2(2 \pi)^{-s} \Gamma(s)\). To compute \(|\nu(g)||fg||^{-2}\), it suffices to take \(g\) in \(M^+(\mathbb{R})\), since the quantity is right \(K_\infty\)-invariant and left \(U_P(\mathbb{R})\)-invariant. Suppose \(g = (\lambda m, m)\), so that \(\nu(g) = \lambda \det(m)\), and \(g \cdot i = i Y\), with \(Y = \lambda m^# m^{-1}\). Then if \(m = (a b \ c d)\), one finds

\[
Y = \frac{\lambda}{\det(m)} \begin{pmatrix} |d|^2 + |c|^2 & -(a\overline{c} + b\overline{d}) \\ -(\overline{a}c + \overline{b}d) & |a|^2 + |b|^2 \end{pmatrix}.
\]

Now set \(L_f = J_2^{-1} \int f \overline{f} J_2^{-1}\), so that \(L_f\) is a rank one Hermitian matrix. (Recall \(J_2 = (1 - 1)\).) If \(f = u f_1 + v f_2\), then

\[
\text{tr}(L_f Y) = |u|^2 y_{22} - \text{tr}(u \overline{v} y_{12}) + |v|^2 y_{11} = ( - \overline{v} \ \overline{u} ) Y \begin{pmatrix} -v \\ u \end{pmatrix} > 0.
\]
One gets

\[ |\nu(g)||fg||^{-2} = \det(Y) \left( \text{tr}(L_f Y) \right)^{-1}. \]

Since this expression is right \( K_\infty \)-invariant and left \( U_P \)-invariant, it is true for all \( g \) in \( H(\mathbb{R}) \).

We have \( dm = \delta_P(m)^{-1}dY, \delta_P(m) = \det(Y)^2 \), and \( |\nu(m)j(m,i)|^{-1} = \det(Y) \). Combining these identities with Proposition 4.18 and (11), we obtain, up to a nonzero constant

\[
J^{\Phi,*}(\phi; s) = \int_{Z(\mathbb{R}) \backslash M(\mathbb{R})} \delta_P^{-1}(m) \alpha(x(m)) f^{\Phi,*}(m; s) \phi(x(m)) \, dm
\]

\[ = a(T) \Gamma_\mathbb{R}(6s - 2) \Gamma_C(3s) \int \left| \text{tr}(L_f Y) \right|^{r-2} \frac{\det Y}{\text{tr}(L_f Y)}^{3s+r-2} e^{-4\pi \text{tr}(TY)} d^* Y \]

where \( d^* Y = \det(Y)^{-2} dY \). Now we make a variable change using the basis \( f, f_T \). So, define \( F \in \text{GL}_2(E) \) via \( f_1 F = f, f_2 F = f_T \). We make the variable change \( Y = \langle f, f \rangle_T^{-1} F^* Y F \). Then

\[
\langle f, f \rangle_T^{-1} F T F^* = \begin{pmatrix} 1 & \det(T) \\ 0 & 1 \end{pmatrix}
\]

and \( \langle f, f \rangle_T^{-1} \text{tr}(Y) F_T F^* = \langle f, f \rangle_T \left( \begin{array}{c} \text{tr}(Y) \\ 0 \end{array} \right) \).

Hence \( \text{tr}(TY) = \tilde{y}_{11} + \det(T) \tilde{y}_{22} \) and \( \text{tr}(L_f Y) = \langle f, f \rangle_T \tilde{y}_{22} \). Thus \( J^{\Phi,*}(\phi; s) \) is

\[
= a(T) \Gamma_\mathbb{R}(6s - 2) \Gamma_C(3s) \int \left( \langle f, f \rangle_T \tilde{y}_{22} \right)^{r-2} \left( \frac{\det(Y)}{\langle f, f \rangle_T \tilde{y}_{22}} \right)^{3s+r-2} e^{-4\pi \tilde{y}_{11} + \det(T) \tilde{y}_{22}} d^* Y
\]

\[ = a(T) \langle f, f \rangle_T^{-3s} \Gamma_\mathbb{R}(6s - 2) \Gamma_C(3s) \int \frac{dy_{22}}{t>0; 0>y_{12}>0} dy_{22} \frac{dt}{t} \frac{dy_{12}}{y_{12}}
\]

making the variable change \( t = \det(Y)/y_{22} = y_{11} - |y_{12}|^2/y_{22} > 0 \). This last integral is, up to a nonzero constant, equal to

\[ a(T)(4\pi)^{-9s} \Gamma_\mathbb{R}(6s - 2) \Gamma_C(3s) \Gamma(3s + r - 3) \]

up to constant factors. This proves Theorem 4.16.

**APPENDIX A. ARITHMETIC INVARIANT THEORY OF TWO-BY-TWO HERMITIAN MATRICES**

The purpose of this appendix is to explain some simple arithmetic invariant theory of two-by-two Hermitian matrices. Although the arithmetic invariant theory is not needed in the proofs above, we think its presence adds a useful perspective on the matrix computations that appear in the unramified computation for the non-unique model integral \( J^{\Phi,*}(\phi; s) \). In particular, this section explains the arithmetic significance of the quantities \( \tilde{m}^{-1} T m^# \) and \( f_T \) appearing in the proofs above.

Our setup is as follows. \( F \) is a field, either a number field or a finite extension of \( \mathbb{Q}_p \). \( E \) denotes a quadratic extension of \( F \), so that \( E \) is either \( F \times F \) or a quadratic field extension of \( F \). We denote by \( \mathcal{O}_F, \mathcal{O}_E \) the rings of integers in \( F \) and \( E \), and \( x \mapsto \overline{x} \) denotes the conjugation of \( E \) over \( F \).

Let \( V \) be a free \( \mathcal{O}_E \)-module of rank two. Denote by \( V^\vee \) the \( \mathcal{O}_E \)-linear dual of \( V \). We let \( \overline{V}, c : V \to \overline{V} \) denote the conjugate \( \mathcal{O}_E \)-module. This means that \( \overline{V} \) is the set of symbols \( \overline{v} \), with \( v \in V \), and the addition in \( \overline{V} \) is given by \( \overline{v} + \overline{w} = \overline{v + w} \). The \( \mathcal{O}_E \)-module structure on \( \overline{V} \) is given by \( m \overline{v} = \overline{mv} \), where \( c : V \to \overline{V} \) denotes the \( \mathcal{O}_E \)-conjugate-linear bijection \( v \mapsto \overline{v} \). Similarly, \( \overline{V^\vee} \) denotes the conjugate of \( V^\vee \), which is naturally identified with the dual of \( V \). If \( L : M \to N \) is an \( \mathcal{O}_E \)-module map of two \( \mathcal{O}_E \)-modules, \( \overline{L} : \overline{M} \to \overline{N} \) denotes the conjugate map, which is uniquely determined by the equality \( cL = \overline{L}c : M \to N \). The canonical pairing between \( V \) and \( V^\vee \) is denoted \( \langle \cdot, \cdot \rangle \); we use the same notation for the canonical pairing between \( \overline{V} \) and \( \overline{V^\vee} \).
We are given the data of a Hermitian form $T$ on $V$, and a symplectic form $J$ on $V^\vee$. We think of $J$ as a map $J : V^\vee \to V$ that satisfies $t^*J = -J$. We think of $T$ as an $O_E$-linear map $T : V \to V^\vee$ that satisfies $t^*T = T$. To be consistent with the body of the paper, we write the maps $T, J$ on the right their respective domains; e.g., if $v \in V$, then $vT$ denotes the image of $v$ under $T$ in $V^\vee$.

Picking an $O_E$-basis $f_1, f_2$ for $V$, we may compute the four numbers $T_{ij} = \langle f_i, T f_j \rangle$. We define

$$\text{det}(T) = T_{11}T_{22} - T_{12}T_{21}$$

the determinant of the associated two-by-two matrix. If $\delta_1, \delta_2$ denote the basis of $V^\vee$ dual to $f_1, f_2$, we may compute the norm of the Pfaffian of $J$, $N(Pf(J)) = N((\delta_1 J, \delta_2 J))$. While both $\text{det}(T)$ and $N(Pf(J))$ depend on the basis, the product $N(Pf(J)) \text{det}(T)$ does not, and does not depend on the choice of Pfaffian.

Associated to this data of $(T : V \to V^\vee, J : V^\vee \to V)$, we will define a quaternion ring $H_{T,J}$, and a left $H_{T,J}$-module structure on $V^\vee$, as follows. First, the quaternion ring $H_{T,J} := O_E \oplus O_ES$. The ring structure on $H_{T,J}$ is determined by the equalities $S^2 = -N(Pf(J)) \text{det}(T)$, and for $x \in O_E$, $Sx = xS$. The conjugation on $H_{T,J}$ is given by

$$x + yS \mapsto \overline{x} + \overline{yS} = \overline{x} - S\overline{y} = \overline{x} - yS.$$

We let $S$ act on $V^\vee$ via the map $\delta \mapsto \overline{\delta J T}$. We let $O_E$ act on $V^\vee$ just by the $O_E$-module structure. Then since $S$ acts conjugate-linearly, the relation $Sx = xS$ is satisfied. Since $S^2 = JTcJTc = JT\overline{JT} = JTJ^\vee T$, and the modules are free, this composite may be computed in matrices, and is found to be $-N(Pf(J)) \text{det}(T)$. Diagrammatically,

$$S^2 : V^\vee \xrightarrow{J} V \xrightarrow{T} V^\vee \xrightarrow{T} V^\vee \xrightarrow{J} V^\vee.$$

Hence $V^\vee$ is naturally an $H_{T,J}$-module.

Now we explain the quantities $m^{-1}Tm^#$ and $f_T = \overline{JT}$. First, note that an element $f \in V$ gives a linear map $V^\vee \to O_E$, $\delta \mapsto \langle f, \delta \rangle$. Now, the map $\delta \mapsto -\langle f, S(\delta) \rangle$ is also a linear map on $V^\vee$, hence there is an $f_T \in V$ so that $\langle f_T, \delta \rangle = -\langle f, S(\delta) \rangle$ for all $\delta \in V^\vee$. Since

$$-\langle f, S(\delta) \rangle = -\langle f, \overline{\delta J T} \rangle = -\langle \overline{f}, \delta J T \rangle = -\langle \overline{f}^* T^\dagger J, \delta JT \rangle = \langle \overline{f}^* T^\dagger J, \delta JT \rangle,$$

we get $f_T = \overline{JT}$, as defined in the body of the paper.

Suppose now we are given an element $m \in \text{GL}(V_E)$, $V_E := V \otimes_{O_E} E$. Associated to $m$ we have an $O_E$ module $V(m) := V^{\vee^t m}$ of $V_E^\vee$.

**Lemma A.1.** Suppose $J$ is invertible. Then, the $O_E$-module $V(m)$ of $V_E^\vee$ is an $H_{T,J}$ module if and only if $\text{det}(m)m^{-1}T^*m^{-1} = m^{-1}Tm^#$ is in $H_2(O_E)$.

**Proof.** We must check that $S(V(m))$ is contained in $V(m)$. But

$$t^*mS^t m^{-1} = t^*mJTc^t m^{-1} = J(J^{-1} t^*mJ)T^*m^{-1}c = \text{det}(m)Jm^{-1}T^*m^{-1}c.$$

Hence $V(m)$ is closed under the action of $H_{T,J}$ precisely when $\text{det}(m)Jm^{-1}T^*m^{-1}$ is integral. Since we assume $J$ is invertible, this is equivalent to $m^{-1}Tm^#$ being integral, as desired. \qed

Finally, let us reconsider the situation in Lemma A.1 from the perspective of $V(m)$. We are given that this module is taken to itself under the action of $S$, and we have $f \in V$ with $\langle f, V(m) \rangle$ in $O_E$. Hence, so is $\langle f_T, V(m) \rangle$, since

$$\langle f_T, V(m) \rangle = \langle f, SV(m) \rangle \subseteq \langle f, V(m) \rangle \subseteq O_E.$$

This was the motivation for the definition of the element $f_T = \overline{JT}$ and the proof of Lemma A.1.
References

[1] Mahdi Asgari. Local $L$-functions for split spinor groups. *Canad. J. Math.*, 54(4):673–693, 2002.
[2] Daniel Bump, Solomon Friedberg, and David Ginzburg. Rankin-Selberg integrals in two complex variables. *Math. Ann.*, 313(4):731–761, 1999.
[3] Daniel Bump, Masaaki Furusawa, and David Ginzburg. Non-unique models in the Rankin-Selberg method. *J. Reine Angew. Math.*, 468:77–111, 1995.
[4] Daniel Bump and Jeffrey Hoffstein. Some Euler products associated with cubic metaplectic forms on GL(3). *Duke Math. J.*, 53(4):1047–1072, 1986.
[5] Masaaki Furusawa and Kazuki Morimoto. Shalika periods on GU(2, 2). *Proc. Amer. Math. Soc.*, 141(12):4125–4137, 2013.
[6] D. Ginzburg, I. Piatetski-Shapiro, and S. Rallis. $L$ functions for the orthogonal group. *Mem. Amer. Math. Soc.*, 128(611):viii+218, 1997.
[7] V. A. Gritsenko. Jacobi functions and Euler products for Hermitian modular forms. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 183(Modul. Funktsii i Kvadrat. Formy. 1):77–123, 165–166, 167–168, 1990.
[8] Nadya Gurevich and Avner Segal. The Rankin-Selberg integral with a non-unique model for the standard $L$-function of $G_2$. *J. Inst. Math. Jussieu*, 14(1):133–152, 2015.
[9] Tatsuo Hina and Takashi Sugano. On the local Hecke series of some classical groups over $p$-adic fields. *J. Math. Soc. Japan*, 35(1):133–152, 1983.
[10] Henry H. Kim and Muthukrishnan Krishnamurthy. Twisted exterior square lift from GU(2, 2) to GL$_6$/F. *J. Ramanujan Math. Soc.*, 23(4):381–412, 2008.
[11] Wilfried Kohnen and Nils-Peter Skoruppa. A certain Dirichlet series attached to Siegel modular forms of degree two. *Invent. Math.*, 95(3):541–558, 1989.
[12] K. F. Lai. Tamagawa number of reductive algebraic groups. *Compositio Math.*, 41(2):153–188, 1980.
[13] Ilya Piatetski-Shapiro and Stephen Rallis. A new way to get Euler products. *J. Reine Angew. Math.*, 392:110–124, 1988.
[14] A. Pollack and S. Shah. On the Rankin-Selberg integral of Kohnen and Skoruppa. *ArXiv e-prints*, October 2014.
[15] A. Pollack and S. Shah. The Spin $L$-function on GSp$_4$ via a non-unique model. *ArXiv e-prints*, March 2015.
[16] Jonathan D. Rogawski. Analytic expression for the number of points mod $p$. In *The zeta functions of Picard modular surfaces*, pages 65–109. Univ. Montréal, Montreal, QC, 1992.
[17] A. Segal. A family of new-way integrals for the standard $L$-function of cuspidal representations of the exceptional group of type $G_2$. *ArXiv e-prints*, January 2015.
[18] Boaz Tamir. On $L$-functions and intertwining operators for unitary groups. *Israel J. Math.*, 73(2):161–188, 1991.
[19] Victor Tan. Poles of Siegel Eisenstein series on $U(n, n)$. *Canad. J. Math.*, 51(1):164–175, 1999.
[20] Don Zagier. Modular forms associated to real quadratic fields. *Invent. Math.*, 30(1):1–46, 1975.

Department of Mathematics, Stanford University, Stanford, CA USA
E-mail address: aaronjp@stanford.edu