ON PILOT WAVE QUANTUM COSMOLOGY

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December 20, 1995

Abstract

The de Broglie–Bohm (pilot wave) formulation of quantum theory appears to be free from the conceptual problems specific to quantum mechanics (problem of measurement) and to quantum cosmology (problem of time). We discuss the issue of quantum equilibrium which arises within its context. We then study the extension of this formulation to the case of gravity and demonstrate that the foliation of spacetime by space-like hypersurfaces obtained during the solution of the quantum problem turns out to be distinguished in general. This means that quantum pilot wave dynamics is not invariant with respect to arbitrary change of foliation, or, in other terms, that quantum non-locality takes place. We also discuss general structure of a realistic pilot wave theory which could describe the universe.

PACS numbers: 03.65.Bz, 04.60.-m, 98.80.Hw
1 Introduction

Among the conceptual problems of quantum gravity and quantum cosmology that have not yet been completely resolved there are those specific to quantum mechanics itself and those specific to combination of quantum theory and the theory of gravity. Let us briefly review these problems.

Wavefunctions which we ascribe to our microscopic invisible quantum systems (such as elementary particles) are eventually used to represent probability amplitudes for the macroscopic observable events, for instance, these may be appearences of spots on a photographic plate, or tracks in a cloud chamber. This is clear, but this is not the whole issue. We also wish to understand the operation of a measuring device itself, that is, we wish to build its physical model. We take our device to consist of invisible particles of the very same nature as those which we call elementary and which we describe by wavefunctions. We then ascribe a wavefunction to the collection of such particles and call it the wavefunction of the measuring device. After this, however, our description of the measuring apparatus becomes twofold, since there are two kinds of wavefunctions related to it, with two different but interrelated meanings. The first kind was assigned to a certain microscopic system (the one which the device was constructed to take “measurements” over) and, we repeat, it represents probability amplitudes for certain observable events which can occur with the device. The second kind was assigned to the device itself, and it receives a vague meaning of representing its state, which in its own turn must correspond to observer’s sensible impressions of the device. Such a twofold description of the measuring apparatus is redundant, and this is the source of difficulties and paradoxes of various kinds, the famous Schrödinger cat paradox being one of them. To achieve a coherent physical description of reality this situation must be overcome.

Now to specific problems which arise in attempts to construct a viable quantum theory of gravity and cosmology. Thus, a closed gravitating system, such as “the universe as a whole,” is taken to be described by a wavefunction which obeys Wheeler–De Witt equation and which is independent of any time parameter. In what manner can such a wavefunction correspond to the observed universe which evolves in time? To resolve this problem cosmologists usually tried to reduce the phenomenon of time to simple correlations between physical quantities. They had to choose one of the variables to assign to it the meaning of time. The choice is always made arbitrarily, and even with a specific choice made the time fails to be a universal concept, at best it can be defined in a limited region of the superspace. Moreover, the very meaning of “the wavefunction of the universe” remains highly unclear. What kind of measurements does it correspond to? How do these measurements affect the state of the universe? These and related conceptual difficulties hamper our understanding of an important issue of modern cosmology, namely, the relevance of quantum laws to the properties of the observable large scale structure of the universe.

This procedure, leading to a description of macroscopic objects by a wavefunction which obeys Schrödinger equation and respects superposition principle, may well be not so innocent as it appears to be. In connection with this see [1].
Numerous efforts were aimed at achieving a coherent description of reality preserving both its well-established quantum character on a microscopic level and its macroscopic classical behaviour. Among the most frequently discussed nowadays is the so-called “relative state” formulation of Everett [2] developed further by De Witt [3] and known as “many-worlds,” or “many-minds” theory. Like many other people, we do not find this theory natural, as it demands belief in the “reality” of multitudes of unperceived parallel worlds, and relies on the (hypothetical) physical theory of mind. Recent “consistent history” approach [4, 5, 6] some of its proponents link with this “many worlds” picture [6], and some with the Copenhagen line of thought [7] with its difficulty to account for uniqueness of the observed reality. Another interesting in this respect proposal is the pilot wave formulation of Bohm [7] (for a modern reviews see [8, 9, 10]) which is closely related logically to the well-known ideas of de Broglie [11]. In this theory (highly advertised by Bell [12]) conceptual difficulties of quantum mechanics mentioned above appear to acquire simple resolution. It is this theory which will be the matter of our discussion.

The basic idea of the pilot wave theory is very simple. Any physical system is described by a deterministic evolution of configuration variables (which Bell has called “beables” [12]). These are the same as in classical physics and are just the spatial coordinates of the elementary particles and the field configurations. The only difference between classical and quantum theory is in the dynamics of these configuration variables. In classical physics their dynamics is determined by the principle of extremal action, or by any of the equivalent principles. In quantum physics the evolution of the configuration variables is guided (piloted, in de Broglie’s terminology) by a quantum wave which obeys Schrödinger equation. This formulation of quantum mechanics eventually has been called “ontological interpretation” by its author [8]. We shall call it pilot wave interpretation.

Pilot wave interpretation has been already developed for relativistic theory of particles and bosonic (scalar and vector) fields [8, 9, 10]. It was argued to be consistent with observable special relativity [8, 13]. The case of quantum field theory is only sketched in [8] and still remains to be developed. A straightforward extension to the case of general relativity was made in [10] (minisuperspace pilot wave cosmologies were considered in [14]). In this paper we perform further study of such an extension (Section 4) and discuss possible “beable” formulation of a realistic quantum theory (Section 4) which could describe our universe. Before doing this we recall the basics of the theory in more detail (the following section) and discuss the important question of quantum probabilities (Section 3).

2 Pilot wave quantum dynamics

A set of $N$ nonrelativistic spinless particles are described by their spatial coordinates $\mathbf{x} \equiv \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$. The wavefunction $\psi$ of this system obeys the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \sum_n \frac{1}{m_n} \Delta_n \psi + V \psi,$$  

(1)
where \( V = V(x) \) is the particle interaction potential. If one represents the wavefunction in the polar form as \( \psi = R \exp(iS/\hbar) \) then from (1) it follows that the phase \( S(x, t) \) and the amplitude \( R(x, t) \) satisfy the following system

\[
\frac{\partial S}{\partial t} + \sum_n \frac{1}{2m_n} (\nabla_n S)^2 + V + Q = 0 , \tag{2}
\]

\[
\frac{\partial R^2}{\partial t} + \sum_n m_n \nabla_n \left( R^2 \nabla_n S \right) = 0 , \tag{3}
\]

where

\[
Q = -\sum_n \frac{\hbar^2}{2m_n} \Delta_n R \tag{4}
\]

is the so-called quantum potential. In the pilot wave interpretation of quantum mechanics the evolution of the coordinates \( x \) is governed by the phase \( S(x, t) \) via the guidance equation

\[
m_n \dot{x}_n = \nabla_n S . \tag{5}
\]

Eq. (2) is the quantum generalization of the classical Hamilton–Jacobi equation and differs from the latter only by the presence of the quantum potential \( Q(x, t) \). The guidance condition (5) is just the same as in the classical theory. In the limit in which the quantum potential \( Q \) in (2) can be neglected we recover classical evolution. We thus see that new formulation of quantum theory can be regarded as just a “deformation” of the classical dynamics (general discussion of this analogy with the classical case can be found in [10]).

In the relativistic theory of half-integer spin one continues to describe particles by the same configuration variables, namely, their spatial coordinates. The guidance conditions and the equations for the wave are now different from those of nonrelativistic case [9]. For a system of spin-1/2 particles these are the well known Dirac equations for the multispinor \( \psi_{\alpha_1...\alpha_N}(x_1, \ldots, x_N, t) \):

\[
\dot{\psi}_{\alpha_1...\alpha_N} = \sum_n \left( H_D^{(a)} \right)_{\alpha_1...\alpha_N} , \tag{6}
\]

where \( H_D^{(a)} \) is the usual Dirac matrix Hamiltonian

\[
H_D = -i\gamma^0 \gamma^i \nabla_i + m\gamma^0 , \tag{7}
\]

which acts on the spinor index \( \alpha_n \) and on the corresponding argument \( x_n \) of the multispinor \( \psi_{\alpha_1...\alpha_N} \). The guidance condition is

\[
\frac{dx_n^\mu}{dt} = \frac{\psi^\dagger (\gamma^0 \gamma^\mu)_n \psi}{\psi^\dagger \psi} , \tag{8}
\]

where the label \( n \) numerates the arguments of the multispinor \( \psi \) and \( (\gamma^\mu)_n \) act on the corresponding spinor index \( \alpha_n \). The multispinor \( \psi \) must be chosen antisymmetric with respect to interchange of any pair of its arguments in accordance with Pauli principle.
For integer spin the formulation in which the role of configuration variables would be played by particle coordinates appears to be impossible [8, 9, 10]. Instead one has to consider the field spatial configurations as fundamental configuration variables guided by the corresponding wave functionals. For example, the wave functional \( \chi[\phi(x), t] \) for a scalar field \( \phi \) will obey the standard Schrödinger equation (see Eq. (28) below for the case of curved spacetime background) and the guidance equation will be

\[
\dot{\phi}(x, t) = \frac{\delta}{\delta \phi(x)} S[\phi(x), t] \bigg|_{\phi(x)=\phi(x, t)},
\]

where \( S[\phi(x), t] \) is \( \hbar \) times the phase of the wave functional \( \chi[\phi(x), t] \).

The classical limit in the dynamics of a physical system is achieved for those configuration variables for which quantum potential becomes negligible. It is straightforward to see that in this case such variables evolve in time according to classical laws of motion. We send the reader to the reviews [8, 9, 10] for details. Note that in the new interpretation the temporal dynamics of the particle coordinates and bosonic field configurations completely determine the state of a physical system, be it microscopic or macroscopic. The role of wavefunction in all physical situations is also one and the same, namely, to provide the guidance laws for configuration variables. Therefore the description of all the physical systems has now become unified, and the source of the difficulties, which lied in the twofold character (mentioned in the Introduction) of this description, has been thereby eliminated.

3 Quantum probabilities

The formalism of quantum dynamics outlined above can be readily applied to the case of a single closed quantum system. In practice however we usually deal with what we call quantum ensembles which are collections of identical systems each piloted in a way described in the preceding section. If all these systems are piloted by one and the same wavefunction then an ensemble is called pure. Otherwise it is called mixed. In pilot wave formulation of quantum mechanics the measurement process is regarded as just a partial case of the generic evolution guided by a wave function which obeys Schrödinger equation. Probabilistic character of the measurement outcome arises because of our ignorance of and inability to control the actual (initial) values of particle and field configuration variables in each system of an ensemble as well as in the measuring apparatus.

In order that the probabilities of different measurement outcomes coincide with those calculated in the standard approach it is necessary to assume that the configuration variables of the systems in a pure quantum ensemble are distributed in accord with their wavefunction, so that \( p(x) = |\psi(x)|^2 \), where \( x \) denotes the set of all configuration variables, and \( p(x) \) is their distribution function. Such a condition is sometimes called quantum equilibrium condition [13, 14]: it is a consequence of the Schrödinger equation that provided the equality holds initially for a given ensemble, it will hold at all times (so long as the ensemble remains
closed). However, in the framework of the theory discussed, one has to demonstrate how such a distribution arises. Since this question seems to be very important for the pilot wave interpretational scheme, we shall discuss it in this section.

Among the recent approaches to the problem of quantum equilibrium that we are aware of, one is due to Valentini [13] and another is due to Dürr, Goldstein and Zanghì [14]. To our mind the proofs and demonstrations contained therein are essentially incomplete, for the reasons that follow.

Valentini [13] tries to prove that for a pure ensemble of closed complicated systems the coarse-grained distribution \( \overline{p}(x) \) of the configuration variables will approach the coarse-grained value \( |\overline{\psi}(x)|^2 \) (here overline denotes coarse graining). The corresponding analysis involves the quantity \( \overline{S} = -\int \overline{p} \log \left( \overline{p}/|\overline{\psi}|^2 \right) \) called “subquantum entropy.” By analogy with classical statistical mechanics (Boltzmann’s \( H \)-theorem) it is suggested that this quantity should increase in time approaching its maximum value of zero, thereby leading to coarse-grained quantum equilibrium. Such a “proof” inherits the well-known difficulties of the “proof” of Boltzmann’s \( H \)-theorem. In fact, from time reversibility of the pilot wave formalism it only follows that if the conditions \( p = \overline{p} \) and \( |\psi|^2 = |\overline{\psi}|^2 \) (the conditions of “no fine-grained microstructure,” assumed by Valentini to hold at the initial moment of time) are valid then the above-presented “entropy” \( \overline{S} \) acquires its local minimum at that moment of time. There is no proof that this value will approach zero as time goes to infinity.

Demonstration of Dürr et al. [16] is based on the notion of typicality which is applied to the domain of all possible initial conditions of a model universe. Specifically, the modulus squared \( |\Psi|^2 \) of the universal wave function is taken to represent the measure density of typicality in the domain of configuration variables. The preference of such a measure is based on its property of being time-equivariant. The authors then show that the set of those initial conditions which conform (to certain precision) with all usual quantum mechanical statistical predictions has measure of typicality close to one. To our mind, equivariance alone of the specific subjective measure introduced, although very important property, is not sufficient for regarding this measure as relevant to objective distributions encountered in the experiments.

We think that ergodicity or mixing argument of some kind is also required to justify the quantum equilibrium hypothesis. (In the paper of Dürr et al. [16, section 13] this kind of argument was announced to be of no necessity for reason which is not very clear to us.) To explain this, compare the problem of quantum equilibrium with the problem of classical statistical equilibrium. In classical statistical mechanics it is the (usually hypothetical) ergodicity property of the Hamiltonian flow that can distinguish the invariant Liouville measure. Indeed, as a consequence of Birkhoff–Khinchin ergodic theorem the average time spent by

\[ \text{In the modified pilot wave proposal of Bohm and Vigier [13] (see also [8]) an additional external stochastic force was added to the right-hand side of the guidance equations in order to account for the occurrence of quantum equilibrium. In this paper we consider only the original “minimal” version of the pilot wave theory as it is expressed in [7].} \]

\[ \text{For an introduction to ergodic theory see [17].} \]
an ergodic system in a region Ω of its dynamical variables tends to a value proportional to the invariant measure of this region as time goes to infinity. In the case of Hamiltonian dynamics such an invariant measure is the Liouville measure. Note, that the ergodicity property can be formulated in terms of any measure equivalent\(^4\) to the invariant measure, in this sense ergodicity does not rely on this latter. Justification of microcanonical equilibrium distribution then reduces to the proof (which is usually a non-trivial task) or assumption of ergodicity property of a particular system. All this appears to be of similar relevance to the case of pilot wave quantum mechanics, so that to establish equilibrium property of measure based on \(|\Psi|^2\) one not only has to distinguish this measure on the grounds of its equivariance, but also must relate it to ergodicity of some kind.

The fact that the measure with density \(|\Psi|^2\) is only equivariant rather than also invariant might call for essential modification of the above argument as compared to the classical case. However, if we proceed to the universal level (as suggested in [16]) and take into account general covariance of the complete theory which includes gravity we will find out that the universal wavefunction does not depend on time\(^5\) (which is a well-known fact, see also Section [4] below) so that the corresponding “measure density” is invariant. Instead, here arises the difficulty that the configuration space of the full theory is infinite-dimensional. We could start surmounting this difficulty by approximating our “universe” by a system with finite number of degrees of freedom (minisuperspace approach), each time with zero total energy, and applying ergodicity argument. Of course, there are states which do not lead to ergodic evolution (like, e.g., a state with a real wavefunction). One would have to assume that our universe (or maybe its part which is of relevance) is rather in a state which is close to ergodic. Another serious difficulty is that even in minisuperspace approach the universal wavefunction is usually not square-integrable, hence, cannot define a finite measure. To overcome this we can assume that a large subsystem of the universe is essentially closed and disentangled from the rest of the world, and that its own wavefunction (which necessarily also will not depend on time) is square-integrable (at least on a minisuperspace level). We shall keep in mind these ideas in principle but shall not develop them here in further detail.

Suppose that the above difficulties have been somehow overcome (or did not arise at all, for example, consider a large closed system in Minkowski background non-dynamical spacetime) and that we are left with a system in a stationary state with square-integrable wavefunction. The ergodicity argument could then run as follows. Let \(z = (x, y)\) denote the universal configuration variables, where \(x\) represents the coordinates of the system of interest, and \(y\) the coordinates of the environment (compare with the discussion in [16]). Let the universal wavefunction \(\Psi(z)\) have a structure

\[
\Psi(z) = \psi(x)\phi(y) + \Psi_0(z),
\]

for which \(\phi(y)\) is non-vanishing in a region \(\Omega\), which is complementary to the \(y\)-support

\(^4\)Two measures, \(\mu\) and \(\nu\), with common domain are said to be equivalent if \(\mu(A) = 0 \iff \nu(A) = 0\) for any set \(A\).

\(^5\)In some non-standard proposals, like, e.g., in [13],\(^b\) the universal wavefunction does depend on time and does not respect the Wheeler–De Witt constraint. To such proposals our argument will not be applicable.
of $\Psi_0(z)$. Then each time the corresponding piloted configuration variable $Y$ gets into the region $\Omega$, the configuration variable $X$ is piloted by the wavefunction $\psi(x)$. For $Y$ in $\Omega$, the probability that $X$ will be in a region $\omega$ will be given by the limit of the corresponding mean time ratio as follows

$$\mathbb{P}(X \in \omega \mid Y \in \Omega) = \lim_{T \to \infty} \frac{\int_0^T f_{\omega \times \Omega}(Z(t)) \, dt}{\int_0^T f_{D \times \Omega}(Z(t)) \, dt},$$

where $D$ is the whole domain of $x$, and $f_M$ denotes the characteristic function of the set $M$. If the evolution is ergodic, then, according to Birkhoff–Khinchin ergodic theorem (see [17]), the limit in (11) exists for almost every initial value of $Z$, and the probability (11) is equal to

$$\mathbb{P}(X \in \omega \mid Y \in \Omega) = \frac{\mu_\Psi(\omega \times \Omega)}{\mu_\Psi(D \times \Omega)} = \mu_\psi(\omega) \equiv \int_\omega |\psi(x)|^2 \, dx,$$

where $\mu_\Psi$ and $\mu_\psi$ are the measures in the domains, respectively, of $z$, and of $x$ with densities determined by the corresponding wavefunctions, and we took the wavefunction $\psi(x)$ to be normalized. The characteristics of the region $\Omega$ dissappear from the result (12), and we can apply a formal limit of infinite-dimensional domain of $y$. To a certain extent the equality (12) constitutes the proof of quantum equilibrium.

Certain cases, e.g., measurements of coordinates, appear to be tractable in a rather simple way. Consider, for example, the classical two-slit experiment (see Fig. 1). A system of collimating instruments and velocity selectors (not shown in the figure) to the left of the slit $A$ filters out particle wavefunctions, letting those with wavelengths in a sufficiently narrow band to pass to reach the slit $A$. The role of the slit $A$ is to produce spherical monochromatic waves in the space to the right of $A$, for this its dimensions have to be much
smaller than the wavelength of the wavefunction in the space to the left of \(A\). The spherical waves produced are then diffracted on a pair of slits \(B\) and \(C\). Now, the appearance of the familiar interference pattern on the screen \(D\) will take place provided particles fall onto the slit \(A\) uniformly within the slit’s range (since the wavefunction is also uniform on the scale of the slit dimensions, the condition \(p = |\psi|^2\) will hold in the vicinity of the slit \(A\), hence, by equivariance argument it will hold also in the space to the right of \(A\), in particular, on the screen \(D\)). But this last condition can be easily granted since the dimensions of the slit \(A\) are small. Thus, whatever of continuous particle distributions is realized to the left of \(A\) (its spatial scale of continuity is comparable to the spatial scale of particle wavefunctions), the familiar interference pattern will appear on the screen.

To what extent such kind of argument can be applied to other quantum experiments such as, for example, Stern–Gerlach experiment, remains to be an open question which we do not attempt to discuss in this paper.

We would only like to point out that sometimes the strong condition of quantum equilibrium achieved on a universal level is not necessary. And, to support the agreement between the real quantum experiments and the predictions of the pilot wave theory it is only necessary that quantum equilibrium distribution arises in preparation processes (natural or artificial). For example, in the case of the two-slit experiment (Fig. 1) quantum equilibrium distribution arises in the space to the right of the slit \(A\), even though it may not take place to the left of \(A\).

If the probabilities as predicted by pilot wave theory turn out to be the same as in the ordinary quantum theory, then the problem of relativistic invariance of the pilot wave dynamics will also be removed. The problem arises because quantum dynamics of an individual system does not respect relativistic invariance of the corresponding classical theory: there is a distinguished reference frame in which the guidance condition (8) or (9) holds. This feature must be called quantum non-locality. Below we show that the same feature takes place also for the case of general relativity. Nevertheless, as far as statistical predictions of the theory are concerned (which are the only testable predictions), these (if) coinciding in the ordinary and in the pilot wave formulations of the quantum theory turn out to be in accord with the corresponding symmetries (see \([8, 10]\) for discussion).

### 4 Pilot wave quantum gravity

We now proceed to the second topic of this work, namely, to the question of pilot wave quantum gravity and quantum cosmology. The theory of gravity being a particular case of the theory of bosonic fields, it is readily described in terms of the general formalism outlined above with modifications caused by the presence of constraints in the theory. The classical action for a general relativistic system of bosonic fields in the ADM form looks like follows

\[
I = \int_M d^3x \sqrt{-g} \left( \pi^{ab} \pi_{ab} + \pi \Phi - \mathcal{N} \mathcal{H} - \mathcal{N}^{\alpha} \mathcal{H}_\alpha \right),
\]  

(13)
where $\mathcal{H}$ and $\mathcal{H}_a$ are correspondingly the so-called Hamiltonian and momentum constraints, and $\mathcal{N}$ and $\mathcal{N}^a$ are the Lagrange multipliers. The symbol $\Phi$ denotes the set of bosonic fields, $g_{ab}$ is a positive definite three-metric with the determinant $g$, $\pi_\Phi$ and $\pi^{ab}$ are the corresponding generalized momenta. General relativistic constraints have the following form\(^6\) (we assume for simplicity that there are no other constraints)

\[
\mathcal{H} \equiv \frac{1}{2\mu} g_{abcd} \pi^{ab} \pi^{cd} + \mu \sqrt{g} \left( 2\Lambda - (3)^{\mathcal{R}} \right) + \mathcal{H}^\Phi \approx 0, \tag{14}
\]

\[
\mathcal{H}_a \equiv -2\nabla_b \pi^b_a + \mathcal{H}^\Phi_a \approx 0, \tag{15}
\]

where we have written explicitly only the gravitational parts of the constraints, $\mathcal{H}^\Phi$ and $\mathcal{H}^\Phi_a$ are their $\Phi$-parts which will not be specified here, $\mu = (16\pi G)^{-1}$, $G$ is the Newton’s constant,

\[
g_{abcd} = \frac{1}{\sqrt{g}} \left( g_{ac} g_{bd} + g_{ad} g_{bc} - g_{ab} g_{cd} \right) \tag{16}
\]

is Wheeler’s supermetric, $\nabla_a$ denotes covariant derivative with respect to the three-metric $g_{ab}$. \(^{(3)}\mathcal{R}$ is the scalar three-curvature of this metric, and $\Lambda$ is the cosmological constant.

The classical equations of motion, together with the constraint equations, are obtained by varying the action \(^{13}\) over all its variables. For example, equations of motion for the metric are

\[
\dot{g}_{ab} = \frac{\mathcal{N}}{\mu} g_{abcd} \pi^{cd} + \nabla_a \mathcal{N}_b + \nabla_b \mathcal{N}_a, \tag{17}
\]

where the overdot denotes the derivative with respect to time $t$.

Proceeding to quantization, first, we recall that in Schrödinger representation the general relativistic quantum system is described by the wave functional $\Psi[g_{ab}(x), \Phi(x)]$ where $g_{ab}(x)$ and $\Phi(x)$ are field configurations on a three-manifold $\Sigma$ with the coordinates $x$. The wave functional obeys the quantum constraint equations

\[
\hat{\mathcal{H}}_\mu \Psi = 0, \tag{18}
\]

in which $\hat{\mathcal{H}}_\mu$ are obtained from their classical counterparts $\mathcal{H}_0 \equiv \mathcal{H}$ and $\mathcal{H}_a$ after replacement of the generalized momenta $\pi^{ab}$ and $\pi^\Phi$ by the corresponding variational derivative operators\(^7\) The wave functional $\Psi$, which is called the wavefunction of the universe, does not depend on time variable $t$. This well-known fact, which we mentioned in the beginning of this paper, remains to be one of the main obstacles in attempts to give meaning to the wavefunction $\Psi$. In the pilot wave interpretation which we discuss here this problem does not arise.

To give pilot wave interpretation of the wavefunction of the universe (see also \(^{10}\)) let us write it in the standard polar form as $\Psi = R \exp (iS/\hbar)$ and substitute into the constraint

\(^6\)The symbol “≈” in \(^{(14)}\) and \(^{(13)}\) below does not mean approximation, but indicates that the equality to zero is a constraint.

\(^7\)In this paper we ignore the question of operator ordering and regularization.
The momentum constraints $H$ will mean reparametrization invariance of both the amplitude and the phase of the wave functional. This means that these functionals can be considered as functionals of three-geometry $(^{(3)}G)$ and of field configuration $\Phi$ regarded in a coordinate-invariant manner. The Hamiltonian constraint $H$ will give birth to two equations which we shall write in a symbolic form as

$$\frac{1}{2\mu} \delta S \circ \delta S + \mu \sqrt{g} \left(2\Lambda - (^{(3)}R)\right) = \frac{\hbar^2}{2\mu} \frac{\delta \circ \delta R}{R} + \frac{\Re \left(\bar{\Psi} \hat{H}\Phi \Psi\right)}{R^2} = 0,$$

(19)

$$\delta \circ \left(R^2 \delta S\right) - \frac{2\mu}{\hbar} \Im \left(\bar{\Psi} \hat{H}\Phi \Psi\right) = 0,$$

(20)

where $\Re$ and $\Im$ denote real and imaginary parts respectively, $\delta$ symbolizes the variational derivative $\delta/\delta g_{ab}(x)$, and the circle “$\circ$” the contraction with respect to Wheeler’s supermetric (16). Note that in the formal classical limit $\hbar \to 0$ the equation (19) reduces to the classical Einstein–Hamilton–Jacobi equation.

According to the general guidance rules quantum evolution of the configuration variables $g_{ab}$ is now given by the equations (17) with the substitution

$$\pi^{ab}(x) \to \frac{\delta S}{\delta g_{ab}(x)} \bigg|_{g_{ab}(x)=g_{ab}(x,t)}.$$

(21)

As for the Lagrange multipliers $N$ and $N^a$ which enter the equations (17), they remain undetermined and are to be specified arbitrarily. This situation is analogous to the situation in classical theory in which the arbitrariness in the choice of the Lagrange multipliers $N$ and $N^a$ reflects reparametrization freedom. Thus to get a solution $g_{ab}(x, t), \Phi(x, t)$ to the quantum dynamics one must first solve the constraint equations (18), then specify the initial configurations (say, at $t = 0$) for the fields $g_{ab}$ and $\Phi$, specify in an arbitrary way the functions $N(x, t)$ and $N^a(x, t)$ and then solve the guidance equations (17) and the analogous equations for $\Phi$. Solution thus obtained will represent a four-geometry foliated by spatial hypersurfaces $\Sigma(t)$ on which the three-metric induced is $g_{ab}(x, t)$, the lapse function is $N(x, t)$, the shift vector is $N^a(x, t)$ and the field configuration is $\Phi(x, t)$.

In the classical theory the choice of the Lagrange multipliers $N$ and $N^a$ affects not the physical solution but only the way in which it is foliated by a family of hypersurfaces $\Sigma(t)$ and the way in which coordinates are chosen on each of these hypersurfaces. If we examine the guidance equations (17), (21) we will notice that the role of the shift vector $N^a$ in quantum dynamics is actually analogous to that of the classical theory. Namely, its form is related only to the way in which the spatial coordinates $x$ are chosen on each of the hypersurfaces $\Sigma(t)$. This follows from the fact that the wave functional $\Psi$ is reparametrization invariant. However the role of the lapse function $N$ in the classical and in the quantum cases is different. Because of nonlocal character of the equation (19), which involves variational derivative of a nonlocal functional $R[g_{ab}(x), \Phi(x)]$, physical character of a solution (four-geometry, for example) will in general depend on the specification of the Lagrange multiplier $N$, regarded
as a three-scalar on every hypersurface $\Sigma(t)$. This dependence will be negligible, however, in the classical limit when the quantum potential in (19) can be neglected.

This fact, that the solution to the quantum dynamics depends on the specification of the lapse function, signifies that quantum dynamics breaks foliation-invariance of the classical theory, or, in other terms, that quantum non-locality takes place. We shall clarify this point by posing the following question. Suppose that the four-geometry $(4)\mathcal{G}$ arises as a solution

$$(4)\mathcal{G} \equiv \{g_{ab}(x, t), N^\mu(x, t)\}, \quad (22)$$

to the quantum pilot wave geometrodynamics of pure gravity described above. If we change in an arbitrary way the space-like foliation of the four-geometry obtained $(4)\mathcal{G}$ we will obtain different representation

$$(4)\mathcal{G} \equiv \{\tilde{g}_{ab}(x, t), \tilde{N}^\mu(x, t)\}, \quad (23)$$

doing the same four-geometry. The question is whether this new representation is a solution of the quantum pilot wave geometrodynamics (perhaps with a new solution for the wave functional). The answer is: generally speaking, no, and this answer distinguishes the space-like foliation (22) from a generic one (23). A simple argument can be given to justify our statement. Although the classical Hamiltonian constraint equation (14) is not satisfied for the four-geometry (22), the classical momentum constraint equation (15) is satisfied. Then, however, because the four-geometry (22) is not a solution to classical field equations, if we change foliation and proceed to (23) we must expect the classical momentum constraint equation (15) to be no longer valid. Hence there is no possibility to obtain (23) as a solution to the quantum dynamics.

To illustrate this conclusion consider a real solution $\Psi$ to the Wheeler–De Witt equation, and pick a solution for configuration variables with arbitrary initial three-geometry $(3)\mathcal{G}$, arbitrary lapse $N$, and shift $N^a \equiv 0$. Since the wave functional is real, the three geometry will remain constant in time. Now consider an arbitrary family of space-time foliations $\{\Sigma_\lambda(t)\}$ parametrized by $\lambda$ from a real interval containing zero, such that $\Sigma_0(t)$ coincides with the original foliation. For each foliation from the family $\{\Sigma_\lambda\}$ there will be induced $(3)\mathcal{G}_\lambda$, $N_\lambda$ and $N^a_\lambda$ in spacetime. To simplify situation we can choose spatial coordinates on the hypersurfaces of each family in such a way that $N^a_\lambda \equiv 0$ (this is always possible). Now, the value of the momentum constraint $H_a(\lambda)$ with respect to the foliation $\Sigma_\lambda$ will not necessarily be zero, in fact one can easily calculate its derivative at $\lambda = 0$ to be

$$\frac{\partial H_a(\lambda)}{\partial \lambda} \bigg|_{\lambda=0} = 2\mu \sqrt{g} \left( N^{(3)} R^b_a \nabla_b \xi - \nabla_a \nabla^b N \nabla_b \xi + (3) \Delta N \nabla_a \xi \right), \quad (24)$$

where $\xi$ is the temporal component of the vector field $\partial/\partial \lambda$ at $\lambda = 0$ generated by the family of foliations $\{\Sigma_\lambda\}$. Obviously, the expression (24) will in general be non-zero. Therefore change of foliation will lead to a representation (23) which cannot be a solution to quantum dynamics.

For an example in a finite form consider again a real solution $\Psi$ for the wave functional. Choose in an arbitrary way the initial three-geometry $(3)\mathcal{G}(0)$, and choose $N^a \equiv 0$ for all
t, and $N \equiv 1$ for $0 \leq t \leq t_1$, $N \neq \text{const}$ for $t > t_1$. Again, since the wave functional is real the solution will be $(^3G(t)) \equiv (^3G(0))$. Now change the foliation in such a way that $\tilde{N}^a \equiv 0$ for all $t$, $\tilde{N}(t) \equiv N(t_1 + t)$ for some interval $0 \leq t \leq t_2 < t_1$, and $\tilde{N} \equiv N$ for $t > t_1$ (this is always possible in general). The new three-geometry $(^3G(t))$ will be different from the old one in the interval $0 \leq t \leq t_2$ but will remain unchanged in the region $t \geq t_1$. We will have then $(^3G(0)) = (^3G(t_1))$ (identical initial conditions for time intervals $t \geq 0$ and $t \geq t_1$), $\tilde{N}^\mu(t) \equiv \tilde{N}^\mu(t_1 + t)$ in a certain interval of $t \geq 0$ (identical lapse and shift in the corresponding time intervals), but $(^3G(t)) \neq (^3G(t_1 + t))$ (different solutions in the corresponding time intervals) which shows that new foliation cannot in principle be a solution of quantum dynamics.

Let us now consider the standard treatment (see, e.g., [18] and references therein) of the classical limit for gravitation and assume the wavefunction to acquire an approximate shape

$$\Psi[^3G, \Phi] \approx R[^3G] \exp \left( \frac{iS[^3G]}{\hbar} \right) \chi[^3G, \Phi], \quad (25)$$

in which real $S[^3G]$ is the solution of the classical purely gravitational Einstein–Hamilton–Jacobi equation (which is just the equation (19) without the quantum potential for gravity and without the $\Phi$-part), and real $R[^3G]$ is chosen so that it obeys the equation

$$\delta \circ (R^2 \delta S) = 0, \quad (26)$$

the notation of which was explained above. If we assume the dependence of the functional $\chi[^3G, \Phi]$ on the three-geometry to be sufficiently weak, then from the equations (19) and (20) with the quantum potential for gravity neglected we will obtain the following equation for this functional

$$\frac{i\hbar}{\mu} \delta S \circ \delta \chi = \hat{H}^\Phi \chi. \quad (27)$$

In the classical limit the role of the wave functional $\chi$ for gravity is negligible. The guidance equations (17), (21) for the metric will then determine the four-geometry which will obey the classical Einstein equations (because the functional $S[^3G]$ in (25) obeys the classical Einstein–Hamilton–Jacobi equation). And the functional $\chi[t, \Phi] \equiv \chi[^3G(t), \Phi]$ will evolve on this background four-geometry according to the standard Schrödinger equation

$$i\hbar \dot{\chi} = \int _\Sigma d^3x \mathcal{N}^\mu \hat{H}^\Phi_\mu \chi, \quad (28)$$

which follows from (27). We thus see how naturally the limit of quantum field theory on a classical geometric background is attained in the pilot wave formulation.

To end this section we note that in the pilot wave framework considered here one can consistently formulate a hybrid theory in which only part of the degrees of freedom are quantized, and other remain classical. To this end it is only sufficient to drop the quantum potential term in Eq. (19) for those degrees of freedom which are to be classical and leave Eq. (20) unchanged. For example, if one does not find it appropriate to quantize gravity one simply should drop the explicit quantum potential term for gravity in Eq. (19).
5 Extension to a complete theory and discussion

A few points must be taken about the extension of the proposal considered to the case of full quantum theory (see also some reasoning in [8, Section 12.3]). In constructing such a theory one has first to resolve the problem of fermionic negative energy levels. A straightforward way of doing this will be to assume, following Dirac, that all such energy levels are occupied. This requires infinite number of particles, hence, infinite number of arguments of the wavefunction $\Psi$. This number will be countable if we also assume the spatial section $\Sigma$ to be compact, for example, we could take it to be topologically a three-sphere. After this we must learn to describe interactions between particles and bosonic fields. Presumably this will require considering wave functionals of type $\Psi_{\alpha_1...\alpha_n...}[G, \Phi, x_1, \ldots, x_n, \ldots]$ which are simultaneously antisymmetric multispinors with a countable set of spinor arguments. Gravity will now have to be described in terms of variables adjusted for coupling to spinors, for example, in terms of Ashtekar variables (see [19]). The wavefunction would obey constraint equation of the following type

$$\left( H + \sum_n H_D^{(n)} + \sum_n H_{\text{int}}^{(n)} \right) \Psi = 0,$$

where $H$ is the Hamiltonian constraint of the bosonic fields, which acts on the field arguments of the wavefunction, $H_D^{(n)}$ is the Dirac Hamiltonian constraint, and $H_{\text{int}}^{(n)}$ is the interaction Hamiltonian constraint which act on the $n$-th fermionic particle argument. Interactions will make possible particle transitions from the negative energy levels to the positive ones and backwards, which processes will correspond to creation and annihilation of pairs. Such a theory still remains to be elaborated.

We end this paper by a few more remarks. First, it is clear (see also the general discussion in [8]) that the wavefunction of the universe need not be a member of a Hilbert space, that is, it need not be square-integrable with respect to all its arguments. The role of the wavefunction in the pilot wave interpretation is to provide guidance for the particle and field configuration variables and this clearly does not require normalization. Probabilistic nature arises here as a secondary concept and requires at most only square-integrability over part of its arguments. This fact in a simple way removes the second difficulty (besides the problem of time) of the standard quantum gravity, namely, lack of its probabilistic interpretation. However, this same fact creates obstacles on the way of justifying the quantum equilibrium hypothesis (see Section 3).

Our second remark concerns the choice of boundary conditions for the wavefunction of the universe. In the pioneer formulation [20] Hartle and Hawking considered their no-boundary conditions as leading to the real wavefunction (their treatment was afterwards more correctly specified [21] and shown to produce besides real also complex solutions). From the viewpoint of the pilot wave interpretation a real wavefunction of the universe is implausible as it leads to the evolution which is hardly compatible with the observed one. It seems that in the approach considered the wavefunction with tunneling boundary conditions [24, 25] is a good
candidate for describing the real universe, as it is essentially complex \([23]\) and can attain classical behaviour in certain regions of superspace (see very similar remarks in \([24]\)).

Finally, it is not unlikely that pilot wave interpretation of quantum theory will help deeper understanding and resolving the famous information loss paradox \([25]\) as well as other problems of the same nature which might involve the effects of quantum gravity (see in this respect \([24]\)).

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