Asymptotic Behavior Analysis for a Three-Species Food Chain Stochastic Model with Regime Switching

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This paper discusses the asymptotic behavior of a class of three-species stochastic model with regime switching. Using the Lyapunov function, we first obtain sufficient conditions for extinction and average time persistence. Then, we prove sufficient conditions for the existence of stationary distributions of populations, and they are ergodic. Numerical simulations are carried out to support our theoretical results.

1. Introduction

In recent years, the dynamic relationship between the predator and prey has become one of the research hotspots in ecology and mathematical ecology because of its universal importance. In particular, the predator-prey model is a typical inhibition model, which greatly changes the understanding of the existence and development of basic laws in the biological community. The following model is one of the Volterra models for the three species of the predator-prey system:

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)\left[a_1 - b_{11}x_1(t) - b_{12}x_2(t)\right], \\
\dot{x}_2(t) &= x_2(t)\left[-a_2 + b_{21}x_1(t) - b_{22}x_2(t) - b_{23}x_3(t)\right], \\
\dot{x}_3(t) &= x_3(t)\left[-a_3 + b_{32}x_2(t) - b_{33}x_3(t)\right],
\end{align*}
\]

(1)

where \(x_1(t)\), \(x_2(t)\), and \(x_3(t)\) denote the densities of prey, predator, and top-predator population at time \(t\), respectively. The parameters \(a_1\), \(a_2\), and \(a_3\) are positive constants that stand for the intrinsic growth rate of the species \(x_1(t)\), the death rate of the species \(x_2(t)\), and the death rate of the species \(x_3(t)\), respectively. The coefficient \(b_{11}\), \(b_{22}\), and \(b_{33}\) are the intraspecific competition in the resource, \(b_{21}\) and \(b_{32}\) represent the rate of consumption, and \(b_{23}\) and \(b_{33}\) represent the contribution of prey to the growth of predator. As a matter of fact, there are many extensive studies in the literatures concerned with three-species predator-prey systems (see e.g., [1–6]). For example, Krikorian [1] considered the Volterra predator-prey model in a three-species and explained the global properties of its solution. Zhou [2] investigated the existence and global stability of the positive periodic solutions of delayed discrete food chains with omnivory. Hsu [5] considers a three-species Lotka–Volterra food web model with omnivores, which is defined as feeding on more than one nutritional level.

In addition, the population system is always affected by environmental noise, which is very important for discovering the nature of random system from the biological point of view. Generally speaking, there are various types of environmental noise, e.g., white or color noise. First, let us consider a simple color noise, such as telegraph noise (see e.g., [7–13]). This kind of colored noise can be explained by the transformation between two or more environmental models, which can be considered to be different due to rainfall or nutrition in the population model. Therefore, we can model state switching through a finite-state Markov chain. Let \(r(t)\) be a right-continuous Markov chain on the probability space, taking values in a finite-state space \(S = \{1, 2, \ldots, N\}\) with the generator \(\Gamma = (\gamma_{nm})_{N \times N}\) given by
\[ P[r(t + \Delta t) = v | r(t) = u] = \begin{cases} \gamma_{uv}\Delta t + o(\Delta t), & \text{if } u \neq v, \\ 1 + \gamma_{vv}\Delta t + o(\Delta t), & \text{if } u = v, \end{cases} \]  

where \( \Delta t > 0 \) and \( \gamma_{uv} \) is the transition rate from state \( u \) to state \( v \) and \( \gamma_{vv} \geq 0 \) if \( u \neq v \) while \( \gamma_{uu} = -\sum_{u \neq v} \gamma_{uv} \). Then, we can incorporate the regime switch into the three-species food chain model (1) to obtain

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)\left[a_1(r(t)) - b_{11}(r(t))x_1(t) - b_{12}(r(t))x_2(t)\right], \\
\dot{x}_2(t) &= x_2(t)\left[-a_2(r(t)) + b_{21}(r(t))x_1(t) - b_{22}(r(t))x_2(t) - b_{23}(r(t))x_3(t)\right], \\
\dot{x}_3(t) &= x_3(t)\left[-a_3(r(t)) + b_{31}(r(t))x_1(t) - b_{32}(r(t))x_2(t) - b_{33}(r(t))x_3(t)\right],
\end{align*}
\]

with initial value \( x_i(0) = x_{i0} \geq 0, r(0) = \zeta \in S \). Here, we assume that the coefficients \( a_i(k), b_{ij}(k) \) \( (i, j = 1, 2, 3) \) are all positive for \( k \in S \).

Next, we consider other types of environmental noise, namely, the white noise (see e.g., [14–22]). In particular, Mao [14] showed that different structures of white noise may have different effects on the population systems; Mao et al. [15] revealed that the environmental noise can suppress a potential population explosion. So, we assume the intrinsic growth rate \( a_i(k) \) is disturbed with

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)\left\{a_1(r(t)) - b_{11}(r(t))x_1(t) - b_{12}(r(t))x_2(t)\right\}dt + \sigma_1(r(t))d\omega_1(t), \\
\dot{x}_2(t) &= x_2(t)\left\{-a_2(r(t)) + b_{21}(r(t))x_1(t) - b_{22}(r(t))x_2(t) - b_{23}(r(t))x_3(t)\right\}dt \\
&\quad - \sigma_2(r(t))d\omega_2(t), \\
\dot{x}_3(t) &= x_3(t)\left\{-a_3(r(t)) + b_{31}(r(t))x_1(t) - b_{32}(r(t))x_2(t) - b_{33}(r(t))x_3(t)\right\}dt - \sigma_3(r(t))d\omega_3(t),
\end{align*}
\]

In this paper, we show that system (5) has the following properties:

(i) The solution starting from anywhere in \( R^3 \) will remain in \( R^3 \) with probability 1.

(ii) For any given initial value \( x(0) \in R^3 \) and \( r(0) = \zeta \in S \), there exists a positive constant \( K(p) \) such that the solution \( x(t) \) of system (5) has the following property: \( \lim \sup_t E[x_3(t)^p] \leq K(p), \quad t \geq 0, p \geq 1 \).

(iii) We show that if the noise is sufficiently large, the solution to system (5) will become extinct with probability 1. This is \( \lim_{t \to \infty} x_i(t) = 0, \quad \text{a.s.} \quad i = 1, 2, 3. \)

And we prove that the predator of system (5) will tend to extinction almost surely in some assumptions.

(i) The persistent in time average is investigated under certain conditions, namely, the solution \( x(t) \) of system (5) with any initial value \( x(0) \in R^3, r(0) = \zeta \in S \) has the following property:

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t x_3(s)ds \geq \frac{\sum_{k \in S} \pi_k [b_{13}b_{32}r_1(k) - b_{11}b_{32}r_2(k) - (b_{12}b_{32} + b_{11}b_{33})r_3(k)]}{b_{11}b_{32} + b_{12}b_{32} + b_{11}b_{33}}.
\]

(ii) In the case of noise being relatively small, there is a unique stationary distribution \( \mu(\cdot, \cdot) \) with ergodic property:

\[
P\left( \lim_{t \to \infty} \int_0^t |x(s)|^p ds = \int_{R^3} |x|^p \mu_\pi (dx) \right) = 1, \quad \text{for all } p > 0.
\]
where $\mu(\cdot) = \sum_{k \in \mathbb{S}} \mu(\cdot, k)$ is the marginal stationary distribution of a solution $x(t)$ of system (5). The key method used in this paper is to construct Lyapunov functions. This Lyapunov function analysis for stochastic differential equations has been used by many authors (see [9, 15]).

The paper is organized as follows. In Section 2, we give the unique existence and boundedness of the solution. In Section 3, we show the sufficient conditions for extinction and persistence in time average, respectively, which have closed relations with the stationary probability distribution of the Markov chain. Then, in Section 4, by using Lyapunov function, the sufficient conditions for the stationary distribution and ergodicity of the solution of system (8) are established. Finally, we illustrate our main results through an example in Section 5.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets). Let $R^3_+$ denote the positive cone of $R^3$, namely, $R^3_+ = \{x \in R^3; x_i > 0, \ i = 1, 2, 3\}$, and $\overline{R^3_+}$ denote the nonnegative cone of $R^3$, this is, $\overline{R^3_+} = \{x \in R^3; x_i \geq 0, \ i = 1, 2, 3\}$. For convenience and simplicity in the following discussion, denote $x(t) = (x_1(t), x_2(t), x_3(t))$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. Let $b_{ij} = \min_{k \in \mathbb{S}} \{b_{ij}(k)\}, b_{ii} = \max_{k \in \mathbb{S}} \{b_{ij}(k)\}, \sigma_{ij} = \min_{k \in \mathbb{S}} \{\sigma_{ij}(k)\}, \sigma_{ii} = \max_{k \in \mathbb{S}} \{\sigma_{ij}(k)\}, \ i, j = 1, 2, 3$. $C_{1,2}^i(\overline{R^3_+} \times \overline{R^3_+} \times S)$ denote the family of all nonnegative real-value function $V(t, x, k)$ which are continuously twice differentiable in $x$ and once in $t$.

Furthermore, as a standing hypothesis, we assume that the Markov chain $r(t)$ is irreducible in this paper. This is very reasonable as it means that the system will switch from any regime to any other regime. This is equivalent to the condition that, for any $u, v \in S$, one can find finite numbers $i_1, i_2, \ldots, i_k \in S$ such that $\gamma_{u,i_1}, \gamma_{i_1,i_2}, \ldots, \gamma_{i_k,v} > 0$. Note that $\Gamma$ always has an eigenvalue $0$. The algebraic interpretation of irreducibility is rank $(1) = N - 1$. Under this condition, the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in R_+^{1 \times N}$ which can be determined by solving the following linear equation $\pi^T = 0$ subject to $\sum_{k=1}^N \pi_k = 1, \pi_k > 0, \ \forall k \in S$.

For convenience, system (5) can be rewritten into the following form:

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))d\omega(t), \tag{8}$$

where

$$f(x(t), r(t)) = \begin{pmatrix} x_1(t)(a_1(r(t)) - b_{11}(r(t))x_1(t) - b_{12}(r(t))x_2(t)) \\ x_2(t)(a_2(r(t)) + b_{21}(r(t))x_1(t) - b_{22}(r(t))x_2(t) - b_{23}(r(t))x_3(t)) \\ x_3(t)(a_3(r(t)) + b_{32}(r(t))x_2(t) - b_{33}(r(t))x_3(t)) \end{pmatrix}, \tag{9}$$

$$g(x(t), r(t)) = \text{diag}(x_1(t)\sigma_1(r(t)), -x_2(t)\sigma_2(r(t)), -x_3(t)\sigma_3(r(t))), \omega(t) = (\omega_1(t), \omega_2(t), \omega_3(t))^T.$$ 

\textbf{Theorem 1.} For any given initial value $x(0) \in R^3_+, r(0) = \zeta \in S$, there is a unique positive solution $x(t)$ of system (5), and the solution will remain in $R^3_+$ with probability 1.

\textbf{Proof.} Note that the coefficients of system (5) are local Lipschitz continuous for the given initial value $x(0) \in R^3_+, r(0) = \zeta \in S$. So, there is a unique maximal local solution $x(t)$ on $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time (see [8, 15]). To show this solution is global, we need to show that $\tau_e = \infty \ a.s.$ Since the initial value is positive and bounded, there is a number $m_0 \geq 0$ large enough such that $x_i(0) \in [1/m_0, m_0], \ i = 1, 2, 3$. For each integer $m \geq m_0$, define the stopping time:

$$\tau_m = \inf\{t \in [0, \infty): x_i(t) \notin \left(\frac{1}{m}, \frac{1}{m} \right), \ \text{for some } i = 1, 2, 3\}, \tag{12}$$

where $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, $\tau_m$ is increasing as $m \rightarrow \infty$. Set $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$, hence $\tau_\infty \leq \tau_e \ a.s.$ If we can show that $\tau_\infty = \infty \ a.s.$, then $\tau_e = \infty \ a.s.$ and $x(t) \in R^3_+$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show that $\tau_\infty = \infty \ a.s.$ If this
Let \( \bar{q} = \max\{c_i(l)/c_j(k): 1 \leq i \leq 3, 1 \leq l, k \leq N \} \). By the definition of \( V(x,k) \), for any \( k, l \in S \), we have

\[
V(x,l) = \sum_{i=1}^{3} c_i(l)(x_i - 1 - \log x_i) \leq \sum_{i=1}^{3} \bar{q}c_i(k)(x_i - 1 - \log x_i) \leq \bar{q}V(x,k).
\]  

(16)

Thus,

\[
\sum_{l=1}^{N} \gamma_{kl}V(x,l) \leq \bar{q}\sum_{l=1}^{N} \gamma_{kl}V(x,k).
\]  

(17)
where $M^* = \max\{M(k), \tilde{q}\sum_{i=1}^{N} y_{kl}\}$ is a positive constant. Then, from Itô formula (10), we have
\[
E V(x(t_m, \omega), t_m) \leq V(x(0, \omega), 0) + M^* E \int_0^{t_m} V(x(t_m, \omega), t_m) dt
\]
It then follows from (20) that
\[
E \left[ V(x(0, \omega), T) e^{M^* T} \right] \geq E \left[ 1_{\Omega_m} (\omega) V(x(t_m, \omega), r(t_m, \omega)) \right]
\]
where $1_{\Omega_m}$ is the indicator function of $\Omega_m$. Letting $m \to \infty$ leads to the contradiction
\[
\infty > \left[ V(x(0, \omega), T) e^{M^* T} \right] = \infty.
\]
This completes the proof.

**Theorem 2.** For any given initial value $x(0) \in R^3$, $r(0) = \zeta \in S$, there is a constant $K(p) > 0$ such that the solution of system (5) satisfies
\[
\lim_{t \to \infty} \sup E \left[ (x_1(t) + x_2(t) + x_3(t))^p \right] \leq K(p), \quad t \geq 0, \quad p > 1.
\]

**Proof.** By Theorem 1, the solution $x(t)$ will remain in $R^3$ for all $t \geq 0$ with probability 1.

Define a function:
\[
V(x, k) = c_1(k)x_1(t) + c_2(k)x_2(t) + c_3(k)x_3(t),
\]
where $c_1(k) = b_{21}(k)b_{32}(k)$, $c_2(k) = b_{12}(k)b_{33}(k)$, and $c_3(k) = b_{12}(k)b_{23}(k)$ are positive constants. By virtue of the generalized Itô’s formula (10), we have
\[
dV(x, k) = \left( c_1(k)a_1(k)x_1(t) - c_2(k)x_2(t) - c_3(k)a_3(k)x_3(t) + \sum_{l=1}^{N} \gamma_{kl} V(x, l) \right) dt
\]
This is
\[
dV(x, k) = \left( c_1(k)a_1(k)x_1(t) - c_2(k)x_2(t) - c_3(k)a_3(k)x_3(t) + \sum_{l=1}^{N} \gamma_{kl} V(x, l) \right) dt
\]
Furthermore, for any given positive constant $p > 1$, we have

\[
dV^p(x, k) = pV^{p-1}(x, k) \left( c_1(k)a_1(k)x_1(t) - c_2(k)a_2(k)x_2(t) - c_3(k)a_3(k)x_3(t) + \sum_{l=1}^{N} y_{kl}V(x, l) \right) dt
\]

\[
+ pV^{p-1}(x, k) \left( -c_1(k)b_{11}(k)x_1^2(t) - c_2(k)b_{22}(k)x_2^2(t) - c_3(k)b_{33}(k)x_3^2(t) \right) dt
\]

\[
+ pV^{p-1}(x, k) \left[ c_1(k)\sigma_1(k)x_1(t) + c_2(k)\sigma_2(k)x_2(t) - c_3(k)\sigma_3(k)x_3(t) \right] dt
\]

\[
+ \frac{1}{2} p(p - 1)V^{p-2}(x, k) \left( c_1^2(k)\sigma_1^2(k)x_1^2(t) + c_2^2(k)\sigma_2^2(k)x_2^2(t) + c_3^2(k)\sigma_3^2(k)x_3^2(t) \right) dt
\]

\[
+ \sum_{l=1}^{N} y_{kl}V^p(x, l) dt.
\]

Note that

\[
c_1(k)b_{11}(k)x_1^2(t) + c_2(k)b_{22}(k)x_2^2(t) + c_3(k)b_{33}(k)x_3^2(t)
\]

\[
\geq \frac{\min\{\tilde{b}_{11}, \tilde{b}_{22}, \tilde{b}_{33}\}}{c_1(k) + c_2(k) + c_3(k)} (c_1(k)x_1(t) + c_2(k)x_2(t) + c_3(k)x_3(t))^2
\]

\[
= \frac{\min\{\tilde{b}_{11}, \tilde{b}_{22}, \tilde{b}_{33}\}}{c_1(k) + c_2(k) + c_3(k)} V^2(x, k).
\]

Similar to (17), we obtain

\[
\sum_{l=1}^{N} y_{kl}V^p(x, l) \leq \tilde{q} \sum_{l=1}^{N} |y_{kl}|V^p(x, k).
\]

Combining (28)–(30), we can obtain

\[
dV^p(x, k) \leq pV^{p-1}(x, k) \left[ a_1(k)V(x, k) - \frac{\min\{\tilde{b}_{11}, \tilde{b}_{22}, \tilde{b}_{33}\}}{c_1(k) + c_2(k) + c_3(k)} V^2(x, k) \right] dt
\]

\[
+ \frac{1}{2} p(p - 1)V^{p-2}(x, k) \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\} V^2(x, k) dt
\]

\[
+ pV^{p-1}(x, k) \left[ c_1(k)\sigma_1(k)x_1(t) + c_2(k)\sigma_2(k)x_2(t) - c_3(k)\sigma_3(k)x_3(t) \right] dt
\]

\[
+ p\tilde{q} \sum_{l=1}^{N} |y_{kl}| V^p(x, k) + \tilde{q} \sum_{l=1}^{N} |y_{kl}| V^p(x, k) dt
\]

\[
\leq \left[ p \left( a_1(k) + \frac{1}{2} (p - 1) \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\} + \left( \tilde{q} + \frac{\tilde{q}^p}{p} \right) \sum_{l=1}^{N} |y_{kl}| \right) V^p(x, k) \right.
\]

\[
- p \frac{\min\{\tilde{b}_{11}, \tilde{b}_{22}, \tilde{b}_{33}\}}{c_1(k) + c_2(k) + c_3(k)} V^{p+1}(x, k) \right] dt
\]

\[
+ pV^{p-1}(x, k) \left[ c_1(k)\sigma_1(k)x_1(t) + c_2(k)\sigma_2(k)x_2(t) - c_3(k)\sigma_3(k)x_3(t) \right] dt.
\]
Hence,
\[
\frac{E[dV^p (x, k)]}{dt} 
\leq p \left( a_1 (k) + \frac{1}{2} (p - 1) \max \{a_1^2, a_2^2, a_3^2\} + \left( \begin{array}{c} \hat{q} + \frac{\hat{q}^2}{p} \end{array} \right) \right) \sum_{i=1}^{N} |\gamma_{kl}| E[V^p (x, k)] 
- p \frac{\min \{\hat{b}_{11}, \hat{b}_{22}, \hat{b}_{33}\}}{c_1 (k) + c_2 (k) + c_3 (k)} E[V^{p+1} (x, k)]
\]
\[
\leq p \left( a_1 (k) + \frac{1}{2} (p - 1) \max \{a_1^2, a_2^2, a_3^2\} + \left( \begin{array}{c} \hat{q} + \frac{\hat{q}^2}{p} \end{array} \right) \right) \sum_{i=1}^{N} |\gamma_{kl}| E[V^p (x, k)] 
- p \frac{\min \{\hat{b}_{11}, \hat{b}_{22}, \hat{b}_{33}\}}{c_1 (k) + c_2 (k) + c_3 (k)} E[V^p (x, k)]^{p+1/p}.
\]

By comparison theorem [9], we obtain
\[
\limsup_{t \to \infty} E[V^p (x, k)] 
\leq \left[ \left( 2a_1 (k) + (p - 1) \max \{a_1^2, a_2^2, a_3^2\} + \left( \begin{array}{c} \hat{q} + \frac{\hat{q}^2}{p} \end{array} \right) \right) \sum_{i=1}^{N} |\gamma_{kl}| \right]^{p} \frac{(c_1 (k) + c_2 (k) + c_3 (k))}{\min \{\hat{b}_{11}, \hat{b}_{22}, \hat{b}_{33}\}}
\]
\[
\hat{\Delta} = L(p),
\]
which implies that there is a $T > 0$ such that
\[
E\left[ \left( x_1 (t) + x_2 (t) + x_3 (t) \right)^p \right] 
\leq \frac{L(p)}{\min_{k \in S} \{c_1 (k), c_2 (k), c_3 (k)\}} \quad t > T.
\]
In addition, $E[V^p (x_1 (t) + x_2 (t) + x_3 (t))^p]$ is continuous and there exists a $T (p)$ such that
\[
E\left[ \left( x_1 (t) + x_2 (t) + x_3 (t) \right)^p \right] \leq \iota (p), \quad t \in [0, T]. \tag{35}
\]
Let $K (p) = \max \{L(p)/\min_{k \in S} \{c_1 (k), c_2 (k), c_3 (k)\}, \iota (p)\}$, then
\[
\limsup_{t \to \infty} E\left[ \left( x_1 (t) + x_2 (t) + x_3 (t) \right)^p \right] \leq K (p), \quad t \geq 0, p > 1. \tag{36}
\]
The proof is complete. \hfill \Box

3. Extinction and Persistence in Time Average

In the previous section, we have proved that the solution of system (5) with a positive initial value remains in the positive cone $R^3_+$. In order to further study asymptotic properties of system (5), in this section, we investigate the persistence in time average and extinction of system (5) under a certain condition. We first give some assumptions and related definitions.

Assumption 1. $\hat{b}_{12} \geq \hat{b}_{21}, \hat{b}_{23} \geq \hat{b}_{32}$.

Assumption 2. $\sum_{k \in S} \pi_k \{a_0 (k) - 1/2a_1^2 (k)\} > 0$, $\sum_{k \in S} \pi_k \{a_1 (k) - 1/2a_1^2 (k) - \hat{b}_{11}/\hat{b}_{21} (a_2 (k) + 1/2a_2^2 (k))\} < 0$.

Assumption 3. $r_i (k) = a_i (k) - 1/2a_1^2 (k), r_i (k) = a_i (k) + a_2^2 (k)/2$, $i = 2, 3$.

Definition 1. The three species are extinct if $\limsup_{t \to \infty} x_i (t) = 0$, a.s. $i = 1, 2, 3$.

The three species are persistent in time average if $\liminf_{t \to \infty} x_i (t)sds > 0$, a.s.

Theorem 3. If Assumption 1 holds, then for any given initial value $x (0) \in R^3_+$, $r (0) = \zeta \in S$, the solution $x (t)$ of system (5) has the property that
\[
\limsup_{t \to \infty} \frac{\log \left( x_1 (t) + x_2 (t) + x_3 (t) \right)}{t} 
\leq \sum_{k \in S} \pi_k \left[ a_1 (k) - \frac{1}{6} \left( a_1^2 (k) \land a_2^2 (k) \land a_3^2 (k) \right) \right]. \tag{38}
\]
Particularly, if \( \sum_{k=3}^{\infty} |a_1(k) - 1/6 (\sigma_1^2(k) \land \sigma_2^2(k) \land \sigma_3^2(k))| < 0 \) holds, then
\[
\lim_{t \to \infty} x_i(t) = 0 \quad \text{a.s.} \quad i = 1, 2, 3.
\]  

**Proof.** By the generalized Itô formula (10), we yield
\[
d[\log(x_1(t) + x_2(t) + x_3(t))]
= \frac{x_1(t)}{x_1(t) + x_2(t) + x_3(t)} (a_1(k) - b_{11}(k)x_1(t) - b_{12}x_2(t))dt
+ \frac{x_2(t)}{x_1(t) + x_2(t) + x_3(t)} (-a_2(k) + b_{21}(k)x_1(t) - b_{22}(k)x_2(t) - b_{23}(k)x_3(t))dt
+ \frac{x_3(t)}{x_1(t) + x_2(t) + x_3(t)} (-a_3(k) + b_{31}(k)x_2(t) - b_{32}(k)x_3(t))dt
\]
\[
- \frac{1}{2} \frac{\sigma_1^2(k)x_1^2(t) + \sigma_2^2(k)x_2^2(t) + \sigma_3^2(k)x_3^2(t)}{(x_1(t) + x_2(t) + x_3(t))^2} dt
+ \frac{\sigma_1(k)x_1(t)}{x_1(t) + x_2(t) + x_3(t)} d\omega_1(t)
- \frac{\sigma_3(k)x_3(t)}{x_1(t) + x_2(t) + x_3(t)} d\omega_3(t).
\]

By Assumption 1 and basic inequality \((x_1(t) + x_2(t) + x_3(t))^2 \leq 3(x_1^2(t) + x_2^2(t) + x_3^2(t))\), integrating (40) from 0 to \( t \), we obtain
\[
\log(x_1(t) + x_2(t) + x_3(t))
\leq \log(x_1(0) + x_2(0) + x_3(0))
+ \int_0^t \left[ a_1(k) - \frac{1}{6} (\sigma_1^2(k) \land \sigma_2^2(k) \land \sigma_3^2(k)) \right] dt
+ M_1(t) - M_2(t) - M_3(t).
\]

where
\[
M_1(t) = \int_0^t \frac{\sigma_1(k)x_1(t)}{x_1(t) + x_2(t) + x_3(t)} d\omega_1(t),
M_2(t) = \int_0^t \frac{\sigma_2(k)x_2(t)}{x_1(t) + x_2(t) + x_3(t)} d\omega_2(t),
M_3(t) = \int_0^t \frac{\sigma_3(k)x_3(t)}{x_1(t) + x_2(t) + x_3(t)} d\omega_3(t).
\]

Noticing that
\[
\langle M_1, M_1 \rangle = \int_0^t \left[ \frac{\sigma_1(k)x_1(t)}{x_1(t) + x_2(t) + x_3(t)} \right]^2 ds \leq \max_{k \in S} |\sigma_1(k)| t,
\]
by the strong law of large numbers for martingales [9, 15], we therefore have
\[
\lim_{t \to \infty} \frac{M_i(t)}{t} = 0, \quad \text{a.s.}
\]

Similarly, \( \lim_{t \to \infty} M_i(t)/t = 0, \quad \text{a.s.} \quad i = 2, 3 \). It finally follows from (41), by dividing by \( t \) on both sides and then letting \( t \to \infty \), and we obtain
\[
\lim_{t \to \infty} \sup \left( \frac{\log(x_1(t) + x_2(t) + x_3(t))}{t} \right)
\leq \lim_{t \to \infty} \sup \frac{1}{t} \int_0^t \left[ a_1(k) - \frac{1}{6} (\sigma_1^2(k) \land \sigma_2^2(k) \land \sigma_3^2(k)) \right] dt
= \sum_{k \in S} \pi_k \left[ a_1(k) - \frac{1}{6} (\sigma_1^2(k) \land \sigma_2^2(k) \land \sigma_3^2(k)) \right].
\]

So, we obtain the desired assertion. Let us consider the following system:
\[
d\phi(t) = \phi(t) \left[ a_1(k) - b_{11}(k)\phi(t) \right] dt
+ \sigma_1(k)\phi(t)d\omega_1(t), \phi(0) = x_1(0) > 0.
\]

According to Luo [11], we have shown that
By the comparison principle, we know that $x_1(t) \leq \phi(t)$, $t \geq 0$. 

**Theorem 4.** If Assumption 2 holds, then for any initial value $x(0) \in R_3^+$, $r(0) = \zeta \in S$, the predator of system (5) will tend to extinction almost surely.

**Proof.** Note that

$d(48)$

$$dx_1(t) \leq x_1(t) \left[ a_1(k) - b_{11}(k)x_1(t) \right] dt + \sigma_1(k)x_1(t) d\omega_1(t).$$

Integrating both sides of (50) results in

$$\frac{\log x_2(t) - \log x_2(0)}{t} \leq \frac{1}{t} \int_0^t \left[ -a_2(k) - \frac{\sigma_2^2(k)}{2} \right] dt - \frac{\sigma_2(k) \omega_2(t)}{t} + b_{21}(k)x_1(s) ds - \frac{\sigma_2(k) \omega_2(t)}{t}.$$

By the comparison principle and (47), we have

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_1(s) ds \leq \sum_{k \in S} \pi_k \left[ a_1(k) - \frac{1}{2}\sigma_1^2(k) \right] \frac{b_{11}}{\bar{b}_{11}}.$$  

Taking Itô formula to the second equation of system (5), we obtain

$$Merging (49) to (51), by Assumption 2 and$$

$$\lim_{t \to \infty} \omega_2(t)/t = 0,$$

we obtain

$$\limsup_{t \to \infty} \frac{\log x_2(t)}{t} \leq \sum_{k \in S} \pi_k \left[ -a_2(k) - \frac{\sigma_2^2(k)}{2} + \frac{\bar{b}_{21}(k)(a_1(k) - \frac{1}{2}\sigma_1^2(k))}{\bar{b}_{11}} \right] < 0.$$

That is,

$$\lim_{t \to \infty} x_2(t) = 0, \quad a.s. \quad (53)$$

Similarly,

$$\limsup_{t \to \infty} \frac{\log x_3(t)}{t} < 0.$$  

This implies

$$\lim_{t \to \infty} x_3(t) = 0, \quad a.s. \quad (55)$$

The result is confirmed.

**Remark 1.** It can be seen from the proof process of Theorem 4 that when the following conditions are met

By the comparison principle and (47), we have

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_1(s) ds \leq \sum_{k \in S} \pi_k \left[ a_1(k) - \frac{1}{2}\sigma_1^2(k) \right] a.s.$$  

Species $x_1$ and $x_2$ coexist and species $x_3$ is extinct.

**Lemma 1.** If Assumption 3 holds, for any initial value $x(0) \in R_3^+$, $r(0) = \zeta \in S$, the solution $x(t)$ of system (5) has following property:

$$\lim_{t \to \infty} \frac{\log x_i(t)}{t} = 0, \quad i = 1, 2, 3. \quad (57)$$

**Proof.** The proof is a modification of that of Mao [16], Zhu and Yin [12], or Luo [11]. So, we omit it.

**Theorem 5.** If Assumption 3 holds, the solution $x(t)$ of system (5) with any initial value $x(0) \in R_3^+$, $r(0) = \zeta \in S$ has the following property:
where \( x^* = (x_1^*, x_2^*, x_3^*) \) is the only nonnegative solution of the following equation:

\[
\begin{align*}
& a_1(k) - \frac{\sigma_1^2(k)}{2} - b_{11}(k)x_1(t) - b_{12}x_2(t) = 0, \\
& - a_2(k) - \frac{\sigma_2^2(k)}{2} + b_{21}(k)x_1(t) \\
& - b_{22}(k)x_2(t) - b_{23}(k)x_3(t) = 0, \\
& - a_3(k) - \frac{\sigma_3^2(k)}{2} + b_{32}(k)x_2(t) - b_{33}(k)x_3(t) = 0.
\end{align*}
\]

Proof. From system (5), we have

\[
d\left[c_1\log x_1(t) + c_2\log x_2(t) + c_3\log x_3(t)\right]
\]

\[
= c_1\left[a_1(k) - \frac{\sigma_1^2(k)}{2} - b_{11}(k)x_1(t) - b_{12}x_2(t)\right]dt + \sigma_1(k)d\omega_1(t)
\]

\[
+ c_2\left[-a_2(k) - \frac{\sigma_2^2(k)}{2} + b_{21}(k)x_1(t) - b_{22}(k)x_2(t) - b_{23}(k)x_3(t)\right]dt - \sigma_2(k)d\omega_2(t)
\]

\[
+ c_3\left[-a_3(k) - \frac{\sigma_3^2(k)}{2} + b_{32}(k)x_2(t) - b_{33}(k)x_3(t)\right]dt - \sigma_3(k)d\omega_3(t)
\]

\[
\geq \left[c_1r_1(k) - c_2r_2(k) - c_3r_3(k) - (c_2\tilde{b}_{23} + c_3\tilde{b}_{33})x_3(t)\right]dt
\]

\[
+ c_1\sigma_1(k)d\omega_1(t) - c_2\sigma_2(k)d\omega_2(t) - c_3\sigma_3(k)d\omega_3(t),
\]

where \( c_1 = \tilde{b}_{21}, c_2 = \tilde{b}_{11} \) and \( c_3 = \tilde{b}_{13}\tilde{b}_{21} + \tilde{b}_{11}\tilde{b}_{23}\tilde{b}_{32} \).

By Assumption 3, we know

\[
c_1r_1(k) - c_2r_2(k) - c_3r_3(k) > 0.
\]

Integrating the both sides of (60) and dividing by \( t \) on both sides, we yield

\[
\frac{c_1(\log x_1(t) - \log x_1(0)) + c_2(\log x_2(t) - \log x_2(0)) + c_3(\log x_3(t) - \log x_3(0))}{t}
\]

\[
\geq \frac{1}{t}\int_0^t (c_1r_1(k) - c_2r_2(k) - c_3r_3(k))dt - (c_2\tilde{b}_{23} + c_3\tilde{b}_{33})\int_0^t x_3(t)dt
\]

\[
+ \frac{c_1\sigma_1(k)d\omega_1(t) - c_2\sigma_2(k)d\omega_2(t) - c_3\sigma_3(k)d\omega_3(t)}{t}.
\]
According to Lemma 1 and \( \lim_{t \to \infty} \omega_i(t)/t = 0, \)
i = 1, 2, 3,
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t x_3(s) ds \\
\geq \sum_{k \in S} \mu_k [c_1 r_1(k) - c_2 r_2(k) - c_3 r_3(k)] \\
\left(c_1 b_{23} + c_3 b_{33}\right) \\
= \sum_{k \in S} \mu_k \left[b_{33} b_{33} r_1(k) - b_{11} b_{23} r_2(k) - (b_{11} b_{11} + b_{13} b_{23}) r_3(k)\right] = x_3^*.
\]

(63)

The result is confirmed. \( \square \)

**4. Stationary Distribution and Ergodicity**

From Theorem 3, we know that three-species tend to extinct almost surely if the noise intensity is sufficiently large. However, in the section, our main aim here is to find out what happen if the noise is relatively small. Namely, we shall discuss the sufficient conditions under which system (5) admits a unique ergodic stationary distribution. To this end, consider system (8):

\[
dx(t) = f(x(t), r(t)) dt + g(x(t), r(t)) d\omega(t),
\]

where \( x(t) \) is a homogeneous Markov process in 3-dimension Euclidean space \( \mathbb{R}^3 \). The diffusion matrix \( A(x, k) = g(x, k) g^T(x, k) \).

Next, we give the conditions for the unique existence and of stationary distribution (see [13, 23–26]).

**Lemma 2** (see [23]). System (8) is positive recurrent if the following conditions are satisfied:

1. For \( u \neq v, \gamma_{uv} > 0 \).
2. For each \( k \in S \),
\[
\lambda |\xi|^2 \leq \xi^T A(x, k) \xi \leq \lambda^{-1} |\xi|^2,
\]

with some constant \( \lambda \in (0, 1) \) for all \( x \in \mathbb{R}^3 \),
3. There exists a bounded open subset \( D \) of \( \mathbb{R}^3 \) with a smooth boundary satisfying that, for each \( k \in S \) there exists a nonnegative function \( V(\cdot, k) : D \to \mathbb{R} \) such that \( V(\cdot, k) \) is twice continuously differentiable and that for some \( \kappa > 0 \)
\[
LV(x, k) \leq -\kappa, \text{ for any } (x, k) \in D^c \times S.
\]

Moreover, the positive recurrent process \( (x(t), r(t)) \) has a unique ergodic stationary distribution \( \mu \). That is, if \( Z(x(t), r(t)) \) be a function integrable with respect to the measure \( \mu \), then
\[
P \left( \lim_{t \to \infty} \frac{1}{t} \int_0^t Z(x(s), r(s)) ds = \int \int \int \int Z(x, k) dx \right) = 1.
\]

We assume that the following condition holds:

**Assumption 4**

1. \( u \neq v, \gamma_{uv} > 0, u, v \in S \).

Then, we can prove.

**Theorem 6.** If Assumption 1 and 4 are satisfied, then system (8) with initial value in \( \mathbb{R}^3 \times S \) is positive recurrent and has a unique stationary distribution \( \mu(\cdot, \cdot) \) with ergodic property:
\[
P \left( \lim_{t \to \infty} \int_0^t |x(s)|^p \, ds = \int \int |x|^p \mu(x) \, dx \right) = 1, \text{ for all } p > 0,
\]

where \( \mu(\cdot) = \sum_{k \in S} \mu(x, k) \) is the marginal stationary distribution of a solution \( x(t) \) of system (8).

**Proof.** It is easy to see if all conditions in Lemma 2 are satisfied, then system (8) is positive recurrent. Obviously, we only need to verify (3) and (5). Recall that diffusion matrix is \( A(x, k) = \text{diag} \{a^2_1 x^2_1, a^2_2 x^2_2, a^2_3 x^2_3\} \). Let \( \bar{\sigma} = \min_{k \in S} \sigma^2_i, \quad i = 1, 2, 3 \) and \( \bar{\sigma} = \max_{k \in S} \sigma^2_i, \quad i = 1, 2, 3 \). Hence, we have
\[
\lambda |\xi|^2 \leq \xi^T A(x, k) \xi \leq \lambda^{-1} |\xi|^2,
\]

where \( \lambda = \min \{\bar{\sigma}, (\bar{\sigma})^{-1}, 1\} \), which verifies condition (3) in Lemma 2. To complete the proof, considering the following bounded open subset:
\[
E_p = \left(\frac{1}{\rho} \rho \right) \times \left(\frac{1}{\rho} \rho \right) \times \left(\frac{1}{\rho} \rho \right) \subset \mathbb{R}_+^3,
\]

where \( \rho \) is a sufficiently large number. Then, \( E_p \subset \mathbb{R}_+^3 \).

Define the function \( V : \mathbb{R}_+^3 \times S \to \mathbb{R}_+ \) by
\[
V (x, k) = \sum_{i=1}^{3} (x_i - 1 - \log x_i) + (\omega_k + |\omega|) = V_1 (x) + V_2 (k), \quad (x, k) \in (R^3 - \mathbb{E}_N) \times S. \tag{72}
\]

Here, \(\omega = (\omega_1, \omega_2, \ldots, \omega_N)^T\), \(|\omega| = \sqrt{\omega_1^2 + \cdots + \omega_N^2}\), and \(\omega_k (k \in S)\) are to be determined later and the reason for \(|\omega|\) being here is to make \(\omega_k + |\omega|\) nonnegative.

For \(V_1\), we calculate the following operator to obtain

\[
LV_1 (x) = (x_1 (t) - 1) (a_1 (k) - b_{11} (k)) x_1 (t) - b_{12} (t) - b_{22} (k) x_2 (t) - b_{32} (k) x_3 (t) + \frac{1}{2} \sigma_1^2 (k)
\]
\[
+ (x_2 (t) - 1) (-a_2 (k) + b_{12} (k)) x_2 (t) - b_{22} (k) x_2 (t) - b_{32} (k) x_3 (t) + \frac{1}{2} \sigma_2^2 (k)
\]
\[
+ (x_3 (t) - 1) (-a_3 (k) + b_{32} (k)) x_3 (t) - b_{33} (k) x_3 (t) + \frac{1}{2} \sigma_3^2 (k). \tag{73}
\]

Using Assumption 1, we have

\[
LV_1 (x) \leq -b_{11} (k) \left( x_1 (t) - \frac{a_1 (k) + b_{11} (k) - b_{11} (k)}{2b_{11} (k)} \right)^2
\]
\[
- b_{22} (k) \left( x_2 (t) - \frac{-a_2 (k) + b_{12} (k) + b_{22} (k) - b_{32} (k)}{2b_{22} (k)} \right)^2
\]
\[
- b_{33} (k) \left( x_3 (t) - \frac{-a_3 (k) + b_{32} (k) + b_{23} (k)}{2b_{33} (k)} \right)^2
\]
\[
+ \frac{(a_1 (k) + b_{12} (k) - b_{21} (k))^2}{4b_{11} (k)}
\]
\[
+ \frac{(-a_2 (k) + b_{12} (k) + b_{22} (k) - b_{32} (k))^2}{4b_{22} (k)}
\]
\[
+ \frac{(-a_3 (k) + b_{31} (k) + b_{23} (k))^2}{4b_{33} (k)} - r_1 (k) + r_2 (k) + r_3 (k), \tag{74}
\]

where \(r_1 (k) = a_1 (k) - \sigma_1^2 (k)/2 > 0\), \(r_2 (k) = a_2 (k) + \sigma_2^2 (k)/2\), \(r_3 (k) = a_3 (k) + \sigma_3^2 (k)/2\), \(i = 2, 3\).

On the contrary,

\[
LV_2 (k) = \sum_{u \in S} (\omega_u - \omega_u), \tag{75}
\]

Next, we define a vector \(\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_N)^T\) with

\[
\Xi_k = r_1 (k) - r_2 (k) - r_3 (k) - \frac{(a_1 (k) + b_{11} (k) - b_{21} (k))^2}{4b_{11} (k)}
\]
\[
- \frac{(-a_2 (k) + b_{12} (k) + b_{22} (k) - b_{32} (k))^2}{4b_{22} (k)}
\]
\[
- \frac{(-a_3 (k) + b_{31} (k) + b_{23} (k))^2}{4b_{33} (k)} \tag{76}
\]

Since the generator matrix \(\Gamma\) is irreducible and Lemma 2.3 in [13], for \(\Xi_k\) there exists a solution,

\[
\omega = (\omega_1, \omega_2, \ldots, \omega_N)^T, \tag{77}
\]

for the following system:

\[
\Gamma \omega - \Xi_k = \sum_{k=1}^{N} \pi_k \Xi_k, \tag{78}
\]

Thus, we obtain

\[
\sum_{u \in S} (\omega_u - \omega_u) - \left[ \begin{array}{c}
- \frac{r_1 (k) - r_2 (k) - r_3 (k) - \frac{(a_1 (k) + b_{11} (k) - b_{21} (k))^2}{4b_{11} (k)}}{4b_{22} (k)}
\end{array} \right]
\]
\[
- \left[ \begin{array}{c}
\frac{(-a_2 (k) + b_{12} (k) + b_{22} (k) - b_{32} (k))^2}{4b_{22} (k)}
\end{array} \right]
\]
\[
- \left[ \begin{array}{c}
\frac{(-a_3 (k) + b_{31} (k) + b_{23} (k))^2}{4b_{33} (k)}
\end{array} \right] \tag{79}
\]

\[
= - \sum_{k \in S} \pi_k \left[ \begin{array}{c}
- \frac{r_1 (k) - r_2 (k) - r_3 (k) - \frac{(a_1 (k) + b_{11} (k) - b_{21} (k))^2}{4b_{11} (k)}}{4b_{22} (k)}
\end{array} \right]
\]
\[
- \left[ \begin{array}{c}
\frac{(-a_2 (k) + b_{12} (k) + b_{22} (k) - b_{32} (k))^2}{4b_{22} (k)}
\end{array} \right]
\]
\[
- \left[ \begin{array}{c}
\frac{(-a_3 (k) + b_{31} (k) + b_{23} (k))^2}{4b_{33} (k)}
\end{array} \right].
\]
Combining (74)–(79), we obtain
\[
LV(x) \leq -\Pi - b_{11}(k)\left( x_1(t) - \frac{a_1(k) + b_{11}(k) - b_{21}(k)}{2b_{11}(k)} \right)^2
- b_{22}(k)\left( x_2(t) - \frac{-a_2(k) + b_{12}(k) + b_{22}(k) - b_{32}(k)}{2b_{22}(k)} \right)^2
- b_{33}(k)\left( x_3(t) - \frac{-a_3(k) + b_{33}(k) + b_{23}(k)}{2b_{33}(k)} \right)^2
\]
\[
\Leftrightarrow \Phi(x,k) = \Phi(x,k).
\]

It then follows from Assumption 4 that
\[
\lim_{|x| \to 0} \Phi(x,k) = \lim_{|x| \to \infty} \Phi(x,k) = -\infty.
\]

Therefore, from (81), there exists a sufficiently small \( \rho \) such that
\[
LV(x,k) \leq -1, \quad \text{for all } (x,k) \in \left( \mathbb{R}^3 - \mathbb{T}_\rho \right) \times S.
\]

It then follows Lemma 2 that the solution of system (8) is positive recurrent with respect to the domain \( E_\rho \) and has a unique stationary distribution. Moreover, the ergodic property follows from the moments estimation of Theorem 2. \( \square \)

5. Numerical Simulations

In order to verify the results above, we numerically simulate the solution of the system (5). By Euler–Maruyama scheme [27, 28], we use discretized Brownian paths over \([0,T]\) and write efficient Matlab code, then obtain the simulation figures.

Example 1. Let us assume that the Markov chain \( r(t) \) is on the state space \( S = \{1, 2\} \) with the generator \( \Gamma = \begin{pmatrix} -4 & 4 \\ 6 & -6 \end{pmatrix} \).

Then, we can get the stationary distribution of \( r(t) \) is \( \pi = (0.6, 0.4) \). Take the step size \( \Delta t = 0.005 \) and the following setting for parameters.

Case 1. When \( r(t) = 1 \),
\[
a_1(1) = 0.01, b_{11}(1) = 0.2, b_{12}(1) = 0.6, \sigma_1(1) = 0.3,
\]
\[
a_2(1) = 0.8, b_{21}(1) = 0.3, b_{22}(1) = 0.2, b_{32}(1) = 0.8, \sigma_2(1) = 0.28,
\]
\[
a_3(1) = 0.5, b_{23}(1) = 0.8, b_{33}(1) = 0.1, \sigma_3(1) = 0.3,
\]

when \( r(t) = 2 \),
\[
a_1(1) = 0.02, b_{11}(1) = 0.2, b_{12}(1) = 0.5, \sigma_1(1) = 0.29,
\]
\[
a_2(1) = 0.6, b_{21}(1) = 0.4, b_{22}(1) = 0.2, b_{32}(1) = 0.8, \sigma_2(1) = 0.3,
\]
\[
a_3(1) = 0.1, b_{23}(1) = 0.6, b_{33}(1) = 0.1, \sigma_3(1) = 0.3,
\]

The above parameters satisfy the condition of Theorem 3, that is, \( \bar{b}_{12} = 0.3 \times \bar{b}_{13} = 0.2 \times \sum_{k \in \mathbb{N}} \frac{a_1(k) - 1/6 \sigma_1^2(k) \wedge \sigma_2^2(k) \wedge \sigma_3^2(k)}{a_2(k) + 1/2 \sigma_2^2(k) - \bar{b}_{21}(k)} \) is negative. As can be seen from Figure 1, when Assumption 1 holds, all three species are extinct.

Case 2. When \( r(t) = 1 \),
\[
a_1(1) = 0.5, b_{11}(1) = 0.2, b_{12}(1) = 0.6, \sigma_1(1) = 0.2,
\]
\[
a_2(1) = 0.8, b_{21}(1) = 0.3, b_{22}(1) = 0.8, b_{32}(1) = 0.8, \sigma_2(1) = 0.28,
\]
\[
a_3(1) = 0.5, b_{23}(1) = 0.8, b_{33}(1) = 0.1, \sigma_3(1) = 0.3,
\]

when \( r(t) = 2 \),
\[
a_1(1) = 0.5, b_{11}(1) = 0.2, b_{12}(1) = 0.5, \sigma_1(1) = 0.25,
\]
\[
a_2(1) = 0.9, b_{21}(1) = 0.3, b_{22}(1) = 0.2, b_{32}(1) = 0.8, \sigma_2(1) = 0.3,
\]
\[
a_3(1) = 0.1, b_{23}(1) = 0.6, b_{33}(1) = 0.1, \sigma_3(1) = 0.3,
\]

It is easy to calculate
\[
\sum_{k \in \mathbb{N}} \pi_k \left[ a_1(k) - \frac{1}{2} \sigma_1^2(k) \right] = 0.4755 > 0,
\]
\[
\sum_{k \in \mathbb{N}} \pi_k \left[ a_1(k) - \frac{1}{2} \sigma_1^2(k) - \frac{\bar{b}_{11}(k)}{b_{21}(k)} \left( a_2(k) + \frac{1}{2} \sigma_2^2(k) \right) \right] = -0.11218 < 0.
\]

Figure 2 shows that, under the condition of Assumption 2, predators are almost inevitably extinct, and only the lowest predators survive.

Case 3. When \( r(t) = 1 \),
\[
a_1(1) = 0.9, b_{11}(1) = 0.2, b_{12}(1) = 0.6, \sigma_1(1) = 0.15,
\]
\[
a_2(1) = 0.3, b_{21}(1) = 0.3, b_{22}(1) = 0.2, b_{32}(1) = 0.8, \sigma_2(1) = 0.1,
\]
\[
a_3(1) = 0.2, b_{23}(1) = 0.8, b_{33}(1) = 0.1, \sigma_3(1) = 0.1,
\]

when \( r(t) = 2 \),
\[
a_1(1) = 0.9, b_{11}(1) = 0.2, b_{12}(1) = 0.6, \sigma_1(1) = 0.15,
\]
\[
a_2(1) = 0.3, b_{21}(1) = 0.3, b_{22}(1) = 0.2, b_{32}(1) = 0.8, \sigma_2(1) = 0.1,
\]
\[
a_3(1) = 0.2, b_{23}(1) = 0.8, b_{33}(1) = 0.1, \sigma_3(1) = 0.1,
\]
\[ a_1(1) = 0.95, b_{11}(1) = 0.2, b_{12}(1) = 0.5, \sigma_1(1) = 0.25, \]
\[ a_2(1) = 0.2, b_{21}(1) = 0.3, b_{22}(1) = 0.2, b_{23}(1) = 0.8, \sigma_2(1) = 0.12, \]
\[ a_3(1) = 0.1, b_{32}(1) = 0.7, b_{33}(1) = 0.1, \sigma_3(1) = 0.1, \]

It is easy to see that

\[ \sum_{k \in N} \pi_k \left[ a_1(k) - \frac{1}{2} \sigma_1^2(k) \right] = 0.90075 > 0, \]

\[ \sum_{k \in S} \pi_k \left[ r_1(k) - \frac{\tilde{b}_{11} r_2(k)}{b_{21}} - \frac{\tilde{b}_{12} b_{21} + \tilde{b}_{11} b_{22} r_3(k)}{b_{21} b_{32}} \right] = 0.5689608 > 0. \]
and setting is as follows.

Let us assume that the Markov chain \( r(t) \) is on the state space \( S = \{1, 2\} \) with the generator \( \Gamma = \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \) and setting is as follows.

When \( r(t) = 1 \),
\[
\begin{align*}
a_1(1) &= 0.9, b_{11}(1) = 0.6, b_{12}(1) = 0.6, \sigma_1(1) = 0.15, \\
a_2(1) &= 0.1, b_{21}(1) = 0.5, b_{22}(1) = 0.2, b_{23}(1) = 0.8, \sigma_2(1) = 0.1, \\
a_3(1) &= 0.1, b_{32}(1) = 0.2, b_{33}(1) = 0.8, \sigma_3(1) = 0.1, \\
\end{align*}
\]
when \( r(t) = 2 \),
\[
\begin{align*}
a_1(1) &= 0.95, b_{11}(1) = 0.6, b_{12}(1) = 0.5, \sigma_1(1) = 0.25, \\
a_2(1) &= 0.1, b_{21}(1) = 0.5, b_{22}(1) = 0.2, b_{23}(1) = 0.8, \sigma_2(1) = 0.12, \\
a_3(1) &= 0.1, b_{32}(1) = 0.1, b_{33}(1) = 0.8, \sigma_3(1) = 0.1.
\end{align*}
\]

Figure 3 shows that when Assumption 3 holds, the three species live for an average of time.

In order to further expound the influence of the regime switching on the stationary distribution of system (5), we give the following example.

**Example 2.** Let us assume that the Markov chain \( r(t) \) is on the state space \( S = \{1, 2\} \) with the generator \( \Gamma \) and setting is as follows.

When \( r(t) = 1 \),
\[
\begin{align*}
a_1(1) &= 0.9, b_{11}(1) = 0.6, b_{12}(1) = 0.6, \sigma_1(1) = 0.15, \\
a_2(1) &= 0.1, b_{21}(1) = 0.5, b_{22}(1) = 0.2, b_{23}(1) = 0.8, \sigma_2(1) = 0.1, \\
a_3(1) &= 0.1, b_{32}(1) = 0.2, b_{33}(1) = 0.8, \sigma_3(1) = 0.1, \\
\end{align*}
\]
when \( r(t) = 2 \),
\[
\begin{align*}
a_1(1) &= 0.95, b_{11}(1) = 0.6, b_{12}(1) = 0.5, \sigma_1(1) = 0.25, \\
a_2(1) &= 0.1, b_{21}(1) = 0.5, b_{22}(1) = 0.2, b_{23}(1) = 0.8, \sigma_2(1) = 0.12, \\
a_3(1) &= 0.1, b_{32}(1) = 0.1, b_{33}(1) = 0.8, \sigma_3(1) = 0.1.
\end{align*}
\]

Then, we can get the stationary distribution of \( r(t) \) is \( \pi = (0.5, 0.5) \). It is easy to check that Assumption 4 is satisfied and \( \pi = 0.1229 > 0 \). The stationary distribution of the Markov chain implies that the populations have an equal chance living in the component-wise environment 1 or 2. Figure 4 shows that the stable distribution of Markov chain shows that species \( x_1 \) can survive in different environments. In Figure 4(a), the blue line, red line, and black line, respectively, represent the density track of species \( x_1 \) under two switching states. In Figure 4(b), the black line, red line, and blue line, respectively, represent the probability density function curve of species \( x_1 \) in two switching states. Figure 5 shows the density trajectory and probability density function.
curve of predator $x_2$ under Markov chain switching. Figure 6 shows the density trajectory and probability density function curve of predator $x_3$ under Markov chain switching.

6. Conclusion

Three-species food chain model has been studied by many scholars recently. In particular, many asymptotic estimates on the sample average in time have been obtained (see e.g., [8, 29]). However, to the best of our knowledge, there is rare result about the stationary distribution of a three-species food chain stochastic model under regime switching. In this paper, we develop and analyse a three-species food chain stochastic model, which takes both white and colored noises into account. We first prove the existence of the global positive solution of the model. Then, using the stochastic Lyapunov functions, we investigate extinction in probability and persistence in time average. Furthermore, we obtain a stationary distribution of the solution. Moreover, some interesting questions deserve further investigation, such as incorporating intervention strategies into the system. We leave this for future consideration.

Data Availability

In addition, the numerical simulation in our paper is a verification of our conclusion. At present, no data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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