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To cite this version:
Serge Bouc, Jacques Thévenaz. The primitive idempotents of the p-permutation ring. 2009. hal-00430256

HAL Id: hal-00430256
https://hal.archives-ouvertes.fr/hal-00430256
Submitted on 6 Nov 2009

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The primitive idempotents of the $p$-permutation ring

Serge Bouc and Jacques Thévenaz

Abstract: Let $G$ be a finite group, let $p$ be a prime number, and let $K$ be a field of characteristic 0 and $k$ be a field of characteristic $p$, both large enough. In this note we state explicit formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} pp_k(G)$, where $pp_k(G)$ is the ring of $p$-permutation $kG$-modules.

AMS Subject Classification: 19A22, 20C20.

Key words: $p$-permutation, idempotent, trivial source.

1. Introduction

Let $G$ be a finite group, let $p$ be a prime number, and let $K$ be a field of characteristic 0 and $k$ be a field of characteristic $p$, both large enough. In this note we state explicit formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} pp_k(G)$, where $pp_k(G)$ is the ring of $p$-permutation $kG$-modules (also called the trivial source ring).

To obtain these formulae, we first use induction and restriction to reduce to the case where $G$ is cyclic modulo $p$, i.e. $G$ has a normal Sylow $p$-subgroup with cyclic quotient. Then we solve the easy and well known case where $G$ is a cyclic $p'$-group. Finally we conclude by using the natural ring homomorphism from the Burnside ring $B(G)$ of $G$ to $pp_k(G)$, and the classical formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} B(G)$.

Our formulae are an essential tool in [2], where Cartan matrices of Mackey algebras are considered, and some invariants of these matrices (determinant, rank) are explicitly computed.

2. $p$-permutation modules

2.1. Notation.

• Throughout the paper, $G$ will be a fixed finite group and $p$ a fixed prime number. We consider a field $k$ of characteristic $p$ and we denote by $kG$ the group algebra of $G$ over $k$. We assume that $k$ is large enough in the sense that it is a splitting field for every group algebra $k(N_G(P)/P)$, where $P$ runs through the set of all $p$-subgroups of $G$. 

• We let $K$ be a field of characteristic 0 and we assume that $K$ is large enough in the sense that it contains the values of all the Brauer characters of the groups $N_G(P)/P$, where $P$ runs through the set of all $p$-subgroups of $G$.

We recall quickly how Brauer characters are defined. We let $\overline{k}$ be an extension of $k$ containing all the $n$-th roots of unity, where $n$ is the $p'$-part of the exponent of $G$. We choose an isomorphism $\theta : \mu_n(\overline{k}) \rightarrow \mu_n(\mathbb{C})$ from the group of $n$-th roots of unity in $\overline{k}$ and the corresponding group in $\mathbb{C}$. If $V$ is an $r$-dimensional $kH$-module for the group $H = N_G(P)/P$ and if $s$ is an element of the set $H_{p'}$ of all $p'$-elements of $H$, the matrix of the action of $s$ on $V$ has eigenvalues $\lambda_1, \ldots, \lambda_r$ in the group $\mu_n(\overline{k})$. The Brauer character $\phi_V$ of $V$ is the central function defined on $H_{p'}$, with values in the field $\mathbb{Q}[\mu_n(\mathbb{C})]$, sending $s$ to $\sum_{i=1}^r \theta(\lambda_i)$. The actual values of Brauer characters may lie in a subfield of $\mathbb{Q}[\mu_n(\mathbb{C})]$ and we simply require that $K$ contains all these values.

2.2. Remark: Let $V$ be as above and let $W$ be a $t$-dimensional $kH$-module. If $s$ has eigenvalues $(\mu_1, \ldots, \mu_t)$ on $W$, its eigenvalues for the diagonal action of $H$ on $V \otimes_k W$ are $(\lambda_i \mu_j)_{1 \leq i \leq r, 1 \leq j \leq t}$. It follows that $\phi_{V \otimes_k W}(s) = \sum_{i=1}^t \sum_{j=1}^r \theta(\lambda_i \mu_j) = \phi_V(s) \phi_W(s)$.

• When $H$ is a subgroup of $G$, and $M$ is a $kG$-module, we denote by $\text{Res}^G_H M$ the $kH$-module obtained by restricting the action of $G$ to $H$. When $L$ is a $kH$-module, we denote by $\text{Ind}^G_H L$ the induced $kG$-module.

• When $M$ is a $kG$-module, and $P$ is a subgroup of $G$, the $k$-vector space of fixed points of $P$ on $M$ is denoted by $M^P$. When $Q \leq P$ are subgroups of $G$, the relative trace is the map $\text{tr}_Q^P : M^Q \rightarrow M^P$ defined by $\text{tr}_Q^P(m) = \sum_{x \in [P/Q]} x \cdot m$.

• When $M$ is a $kG$-module, the Brauer quotient of $M$ at $P$ is the $k$-vector space

$$M[P] = M^P / \sum_{Q < P} \text{tr}_Q^P M^Q.$$ 

This $k$-vector space has a natural structure of $kN_G(P)$-module, where as usual $N_G(P) = N_G(P)/P$. It is equal to zero if $P$ is not a $p$-group.

• If $P$ is a normal $p$-subgroup of $G$ and $M$ is a $k(G/P)$-module, denote by $\text{Inf}^G_{G/P} M$ the $kG$-module obtained from $M$ by inflation to $G$. Then there is an isomorphism

$$(\text{Inf}^G_{G/P} M)[P] \cong M$$

of $k(G/P)$-modules.

• When $G$ acts on a set $X$ (on the left), and $x, y \in X$, we write $x =_G y$ if $x$ and $y$ are in the same $G$-orbit. We denote by $[G \backslash X]$ a set of representatives
of $G$-orbits on $X$, and by $X^G$ the set of fixed points of $G$ on $X$. For $x \in X$, we denote by $G_x$ its stabilizer in $G$.

2.3. Review of $p$-permutation modules. We begin by recalling some definitions and basic results. We refer to [3], and to [1] Sections 3.11 and 5.5 for details:

2.4. Definition. A permutation $kG$-module is a $kG$-module admitting a $G$-invariant $k$-basis. A $p$-permutation $kG$-module $M$ is a $kG$-module such that $\text{Res}^G_S M$ is a permutation $kS$-module, where $S$ is a Sylow $p$-subgroup of $G$.

The $p$-permutation $kG$-modules are also called trivial source modules, because the indecomposable ones coincide with the indecomposable modules having a trivial source (see [3] 0.4). Moreover, the $p$-permutation modules also coincide with the direct summands of permutation modules (see [1], Lemma 3.11.2).

2.5. Proposition.

1. If $H$ is a subgroup of $G$, and $M$ is a $p$-permutation $kG$-module, then the restriction $\text{Res}^G_H M$ of $M$ to $H$ is a $p$-permutation $kH$-module.
2. If $H$ is a subgroup of $G$, and $L$ is a $p$-permutation $kH$-module, then the induced module $\text{Ind}^G_H L$ is a $p$-permutation $kG$-module.
3. If $N$ is a normal subgroup of $G$, and $L$ is a $p$-permutation $k(G/N)$-module, the inflated module $\text{Inf}^G_{G/N} L$ is a $p$-permutation $kG$-module.
4. If $P$ is a $p$-group, and $M$ is a permutation $kP$-module with $P$-invariant basis $X$, then the image of the set $X^P$ in $M[P]$ is a $k$-basis of $M[P]$.
5. If $P$ is a $p$-subgroup of $G$, and $M$ is a $p$-permutation $kG$-module, then the Brauer quotient $M[P]$ is a $p$-permutation $k\text{N}_G(P)$-module.
6. If $M$ and $N$ are $p$-permutation $kG$-modules, then their tensor product $M \otimes_k N$ is again a $p$-permutation $kG$-module.

Proof: Assertions 1, 2, 3, and 6 are straightforward consequences of the same assertions for permutation modules. For Assertion 4, see [3] 1.1.(3). Assertion 5 follows easily from Assertion 4 (see also [3] 3.1).

This leads to the following definition:

2.6. Definition. The $p$-permutation ring $\text{pp}_k(G)$ is the Grothendieck group of the category of $p$-permutation $kG$-modules, with relations corresponding to direct sum decompositions, i.e. $[M] + [N] = [M \oplus N]$. The ring structure
on $pp_k(G)$ is induced by the tensor product of modules over $k$. The identity element of $pp_k(G)$ is the class of the trivial $kG$-module $k$.

As the Krull-Schmidt theorem holds for $kG$-modules, the additive group $pp_k(G)$ is a free (abelian) group on the set of isomorphism classes of indecomposable $p$-permutation $kG$-modules. These modules have the following properties:

**2.7. Theorem.** [3 Theorem 3.2]

1. The vertices of an indecomposable $p$-permutation $kG$-module $M$ are the maximal $p$-subgroups $P$ of $G$ such that $M[P] \neq \{0\}$.
2. An indecomposable $p$-permutation $kG$-module has vertex $P$ if and only if $M[P]$ is a non-zero projective $k\overline{N}_G(P)$-module.
3. The correspondence $M \mapsto M[P]$ induces a bijection between the isomorphism classes of indecomposable $p$-permutation $kG$-modules with vertex $P$ and the isomorphism classes of indecomposable projective $k\overline{N}_G(P)$-modules.

**2.8. Notation.** Let $\mathcal{P}_{G,p}$ denote the set of pairs $(P,E)$, where $P$ is a $p$-subgroup of $G$, and $E$ is an indecomposable projective $k\overline{N}_G(P)$-module. The group $G$ acts on $\mathcal{P}_{G,p}$ by conjugation, and we denote by $[\mathcal{P}_{G,p}]$ a set of representatives of $G$-orbits on $\mathcal{P}_{G,p}$.

For $(P,E) \in \mathcal{P}_{G,p}$, let $M_{P,E}$ denote the (unique up to isomorphism) indecomposable $p$-permutation $kG$-module such that $M_{P,E}[P] \cong E$.

**2.9. Corollary.** The classes of the modules $M_{P,E}$, for $(P,E) \in [\mathcal{P}_{G,p}]$ form a $\mathbb{Z}$-basis of $pp_k(G)$.

**2.10. Notation.** The operations $Res^G_H$, $Ind^G_H$, $Inf^G_{G/N}$ extend linearly to maps between the corresponding $p$-permutations rings, denoted with the same symbol.

The maps $Res^G_H$ and $Inf^G_{G/N}$ are ring homomorphisms, whereas $Ind^G_H$ is not in general. Similarly:

**2.11. Proposition.** Let $P$ be a $p$-subgroup of $G$. Then the correspondence $M \mapsto M[P]$ induces a ring homomorphism $Br^G_P : pp_k(G) \to pp_k(\overline{N}_G(P))$.

**Proof:** Let $M$ and $N$ be $p$-permutation $kG$-modules. The canonical bilinear map $M \times N \to M \otimes_k N$ is $G$-equivariant, hence it induces a bilinear map $\beta_P : M[P] \times N[P] \to (M \otimes_k N)[P]$ (see 3 1.2), which is $\overline{N}_G(P)$-equivariant. Now if $X$ is a $P$-invariant $k$-basis of $M$, and $Y$ a $P$-invariant $k$-basis of $N$,
then $X \times Y$ is a $P$-invariant basis of $M \otimes k N$. The images of the sets $X^P$, $Y^P$, and $(X \times Y)^P$ are bases of $M[P]$, $N[P]$, and $(M \otimes_k N)[P]$, respectively, and the restriction of $\beta_P$ to these bases is the canonical bijection $X^P \times Y^P \to (X \times Y)^P$. It follows that $\beta_P$ induces an isomorphism $M[P] \otimes_k N[P] \to (M \otimes_k N)[P]$ of $k\overline{N}_G(P)$-modules. Proposition 2.11 follows.

2.12. Notation. Let $Q_{G,p}$ denote the set of pairs $(P,s)$, where $P$ is a $p$-subgroup of $G$, and $s$ is a $p'$-element of $\overline{N}_G(P)$. The group $G$ acts on $Q_{G,p}$, and we denote by $[Q_{G,p}]$ a set of representatives of $G$-orbits on $Q_{G,p}$.

If $(P,s) \in Q_{G,p}$, we denote by $N_G(P,s)$ the stabilizer of $(P,s)$ in $G$, and by $<Ps>$ the subgroup of $N_G(P)$ generated by $Ps$ (i.e. the inverse image in $N_G(P)$ of the cyclic group $<s>$ of $\overline{N}_G(P)$).

2.13. Remarks :

• When $H$ is a subgroup of $G$, there is a natural inclusion of $Q_{H,p}$ into $Q_{G,p}$, as $\overline{N}_H(P) \leq \overline{N}_G(P)$ for any $p$-subgroup $P$ of $H$. We will consider $Q_{H,p}$ as a subset of $Q_{G,p}$.

• When $(P,s) \in Q_{G,p}$, the group $N_G(P,s)$ is the set of elements $g$ in $N_G(P)$ whose image in $\overline{N}_G(P)$ centralizes $s$. In other words, there is a short exact sequence of groups

\[
1 \to P \to N_G(P,s) \to C_{\overline{N}_G(P)}(s) \to 1.
\]

In particular $N_G(P,s)$ is a subgroup of $N_G(<Ps>)$.

2.15. Notation. Let $(P,s) \in Q_{G,p}$. Let $\tau^G_{P,s}$ denote the additive map from $pp_k(G)$ to $K$ sending the class of a $p$-permutation $kG$-module $M$ to the value at $s$ of the Brauer character of the $\overline{N}_G(P)$-module $M[P]$.

2.16. Remarks :

• It is clear that $\tau^G_{P,s}(M)$ only depends on the restriction of $M$ to the group $<Ps>$. In other words

\[
\tau^G_{P,s} = \tau^{<Ps>}_{P,s} \circ \text{Res}^G_{<Ps>}.
\]

Furthermore, it is clear from the definition that

\[
\tau^G_{P,s} = \tau^{<Ps>/P}_{1,s} \circ \text{Br}^G_{P} \circ \text{Res}^G_{<Ps>}.
\]

• It is easy to see that $\tau^G_{P,s}$ only depends on the $G$-orbit of $(P,s)$, that is, $\tau^G_{P,s,g} = \tau^G_{P,s}$ for every $g \in G$.

The following proposition is Corollary 5.5.5 in [1], but our construction of the species is slightly different (but equivalent, of course). For this reason, we sketch an independent proof :
2.18. Proposition.

1. The map \( \tau_{P,s}^G \) is a ring homomorphism \( pp_k(G) \to K \) and extends to a \( K \)-algebra homomorphism (a species) \( \tau_{P,s}^G : K \otimes \mathbb{Z} pp_k(G) \to K \). These species induce a \( K \)-algebra isomorphism \( T = \prod \left( Q,s \right) \in \mathbb{Q}_G,p \) \( \tau_{P,s}^G : K \otimes \mathbb{Z} pp_k(G) \to \prod\left( P,s \right) \in \mathbb{Q}_G,p \) \( K \).

Proof : By 2.17, to prove Assertion 1, it suffices to prove that \( \tau_{1,s}^G \) is a ring homomorphism, since both \( \text{Res}_{<P,s>}^G \) and \( \text{Br}_{<P,s>}^G \) are ring homomorphisms. In other words, we can assume that \( P = 1 \). Now the value of \( \tau_{1,s}^G \) on the class of a \( kG \)-module \( M \) is the value \( \phi_M(s) \) of the Brauer character of \( M \) at \( s \), so Assertion 1 follows from Remark 2.2.

For Assertion 2, it suffices to prove that \( T \) is an isomorphism. Since \( |P_G,p| \) and \( |Q_G,p| \) have the same cardinality, the matrix \( M \) of \( T \) is a square matrix. Let \( (P,E) \in \mathbb{P}_G,p \), and \( (Q,s) \in \mathbb{Q}_G,p \). Then \( \tau_{Q,s}(M_{P,E}) \) is equal to zero if \( Q \) is not contained in \( P \) up to \( G \)-conjugation, because in this case \( M_{P,E}(Q) = \{0\} \) by Theorem 2.7. It follows that \( M \) is block triangular. As moreover \( M_{P,E}[P] \cong E \), we have that \( \tau_{P,s}(M_{P,E}) = \phi_E(s) \). This means that the diagonal block of \( M \) corresponding to \( P \) is the matrix of Brauer characters of projective \( k\text{N}_G(P) \)-modules, and these are linearly independent by Lemma 5.3.1 of [1]. It follows that all the diagonal blocks of \( M \) are non-singular, so \( M \) is invertible, and \( T \) is an isomorphism.

2.19. Corollary. The algebra \( K \otimes \mathbb{Z} pp_k(G) \) is a split semisimple commutative \( K \)-algebra. Its primitive idempotents \( F_{P,s}^G \) are indexed by \( \mathbb{Q}_G,p \), and the idempotent \( F_{P,s}^G \) is characterized by

\[
\forall (R,u) \in \mathbb{Q}_G,p, \quad \tau_{R,u}^G(F_{P,s}^G) = \left\{ \begin{array}{ll} 1 & \text{if } (R,u) =_G (P,s) \\ 0 & \text{otherwise.} \end{array} \right.
\]

3. Restriction and induction

3.1. Proposition. Let \( H \leq G \), and \( (P,s) \in \mathbb{Q}_G,p \). Then

\[
\text{Res}_{H}^G F_{P,s}^G = \sum_{(Q,t)} F_{Q,t}^H ,
\]

where \( (Q,t) \) runs through a set of representatives of \( H \)-conjugacy classes of \( G \)-conjugates of \( (P,s) \) contained in \( H \).
Proof: Indeed, as Res\textsubscript{G} is an algebra homomorphism, the element Res\textsubscript{G}F\textsubscript{P,s} is an idempotent of \( K \otimes \mathbb{Z} \text{pp}_k(H) \), hence it is equal to a sum of some distinct primitive idempotents \( F\textsubscript{Q,t} \). The idempotent \( F\textsubscript{Q,t} \) appears in this decomposition if and only if \( \tau\textsubscript{Q,t}(\text{Res}\textsubscript{G}F\textsubscript{P,s}) = 1 \). By Remark 2.16

\[
\tau\textsubscript{Q,t}(\text{Res}\textsubscript{G}F\textsubscript{P,s}) = \tau\textsubscript{Q,t}(<Q,t>\text{Res}_{<Q,t>}\text{Res}\textsubscript{G}F\textsubscript{P,s})
\]

\[
= \tau\textsubscript{Q,t}(<Q,t>(\text{Res}\textsubscript{G}F\textsubscript{P,s}))
\]

\[
= \tau\textsubscript{Q,t}(F\textsubscript{Q,t}).
\]

Now \( \tau\textsubscript{Q,t}(F\textsubscript{P,s}) \) is equal to 1 if and only if \((Q,t)\) and \((P,s)\) are \( G \)-conjugate. This completes the proof.

3.2. Proposition. Let \( H \leq G \), and \((Q,t) \in Q_{H,p}\). Then

\[
\text{Ind}\textsubscript{H}^{G}F\textsubscript{Q,t} = [N_{G}(Q,t) : N_{H}(Q,t)]F\textsubscript{Q,t}.
\]

Proof: Since \( K \otimes \mathbb{Z} \text{pp}_k(G) \) is a split semisimple commutative \( K \)-algebra, any element \( X \) in \( K \otimes \mathbb{Z} \text{pp}_k(G) \) can be written

\[
X = \sum_{(P,s) \in [Q_{G,p}]} \tau\textsubscript{P,s}(X)F\textsubscript{P,s}.
\]

and moreover for any \((P,s) \in Q_{G,p}\)

\[
\tau\textsubscript{P,s}(X)F\textsubscript{P,s} = X \cdot F\textsubscript{P,s}.
\]

Setting \( X = \text{Ind}\textsubscript{H}^{G}F\textsubscript{Q,t} \) in this equation gives

\[
\tau\textsubscript{P,s}(\text{Ind}\textsubscript{H}^{G}F\textsubscript{Q,t})F\textsubscript{P,s} = (\text{Ind}\textsubscript{H}^{G}F\textsubscript{Q,t}) \cdot F\textsubscript{P,s}
\]

\[
= \text{Ind}\textsubscript{H}^{G}(F\textsubscript{Q,t} \cdot \text{Res}\textsubscript{G}F\textsubscript{P,s}).
\]

By Proposition 3.1, the element \( \text{Res}\textsubscript{G}F\textsubscript{P,s} \) is equal to the sum of the distinct idempotents \( F\textsubscript{P,s}^{y} \) associated to elements \( y \) of \( G \) such that \(<P,s>^{y} \leq H\). The product \( F\textsubscript{Q,t} \cdot F\textsubscript{P,s}^{y} \) is equal to zero, unless \((Q,t)\) is \( H \)-conjugate to \((P^{y}, s^{y})\), which implies that \((Q,t)\) and \((P,s)\) are \( G \)-conjugate. It follows that the only non zero term in the right hand side of Equation \ref{3.3} is the term corresponding to \((Q,t)\). Hence

\[
\text{Ind}\textsubscript{H}^{G}F\textsubscript{Q,t} = \tau\textsubscript{Q,t}(\text{Ind}\textsubscript{H}^{G}F\textsubscript{Q,t})F\textsubscript{Q,t}.
\]
Now by Remark 2.16 and the Mackey formula
\[
\tau^G_{Q,t}(\text{Ind}_H^G F^H_{Q,t}) = \tau^{<Q>_t} (\text{Res}_{<Q>_t}^H \text{Ind}_H^G F^H_{Q,t}) = \tau^{<Q>_t} \left( \sum_{x \in <Q>_t \cap H} \text{Ind}^{<Q>_t}_{<Q>_t \cap H} x \text{Res}^H_{<Q>_t \cap H} F^H_{Q,t} \right).
\]

By Proposition 3.1, the element \(\text{Res}^H_{<Q>_t \cap H} F^H_{Q,t}\) is equal to the sum of the distinct idempotents \(F^{<Q>_t}_{Q_t}x\) corresponding to elements \(y \in H\) such that \(<Q>_t^y \leq <Q>_t^x \cap H\). This implies \(<Q>_t^y = <Q>_t^x\), i.e. \(y \in N_G(<Q>_t)x\), thus \(x \in N_G(<Q>_t) \cdot H\). This gives
\[
\tau^G_{Q,t}(\text{Ind}_H^G F^H_{Q,t}) = \tau^{<Q>_t} \left( \sum_{x \in N_G(<Q>_t)H/H} \sum_{y \in N_H(Q,t) \cap G(<Q>_t)x} x F^{<Q>_t}_{Q_t^y} \right) = \tau^{<Q>_t} \left( \sum_{x \in N_G(<Q>_t)H/H} \sum_{y \in N_H(Q,t) \cap G(<Q>_t)x} F^{<Q>_t}_{Q_t^y} \right),
\]
where \(z = yx^{-1}\). Finally \(\tau^{<Q>_t} (F^{<Q>_t}_{Q_t^z})\) is equal to 1 if \((Q^z, t^z)\) is conjugate to \((Q, t)\) in \(<Q>_t\), and to zero otherwise.

If \(u \in <Q>_t\) is such that \((Q^z, t^z)^u = (Q, t)\), then \(zu \in N_G(Q, t)\). But since \([<Q>_t, t] \leq Q\), we have \(<Q>_t \leq N_G(Q, t)\), so \(u \in N_G(Q, t)\), hence \(z \in N_G(Q, t)\), and \((Q^z, t^z) = (Q, t)\). It follows that
\[
\tau^G_{Q,t}(\text{Ind}_H^G F^H_{Q,t}) = \left| N_G(Q, t) : N_H(Q, t) \right|,
\]
which completes the proof of the proposition.

3.4. Corollary. Let \((P, s) \in Q_{G,p}\). Then
\[
F^G_{P,s} = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \text{Ind}^G_{<P>_s} F^{<P>_s}_{P,s}.
\]

Proof: Apply Proposition 3.2 with \((Q, t) = (P, s)\) and \(H = <P>_s\). Then \(N_H(Q, t) = <P>_s\), thus by Exact sequence 2.14
\[
\left| N_G(Q, t) : N_H(Q, t) \right| = \frac{|P||C_{\overline{N}_G(P)}(s)|}{|P||s|} = \frac{|C_{\overline{N}_G(P)}(s)|}{|s|},
\]
and the corollary follows.
4. Idempotents

It follows from Corollary 3.4 that, in order to find formulae for the primitive idempotents $F^G_{P,s}$ of $K \otimes_{\mathbb{Z}} pp_k(G)$, it suffices to find the formula expressing the idempotent $F^G_{<Ps>}$ in other words, we can assume that $G = <Ps>$, i.e. that $G$ has a normal Sylow $p$-subgroup $P$ with cyclic quotient generated by $s$.

4.1. A morphism from the Burnside ring. When $G$ is an arbitrary finite group, there is an obvious ring homomorphism $L_G$ from the Burnside ring $B(G)$ to $pp_k(G)$, induced by the linearization operation, sending a finite $G$-set $X$ to the permutation module $kX$, which is obviously a $p$-permutation module. This morphism also commutes with restriction and induction: if $H \leq G$, then

(4.2) $L_H \circ \text{Res}_H^G = \text{Res}_H^G \circ L_G$, \hspace{1cm} $L_G \circ \text{Ind}_H^G = \text{Ind}_H^G \circ L_H$.

Indeed, for any $G$-set $X$, the $kH$-modules $k\text{Res}_H^G X$ and $\text{Res}_H^G(kX)$ are isomorphic, and for any $H$-set $Y$, the $kG$-modules $k\text{Ind}_H^G Y$ and $\text{Ind}_H^G(kY)$ are isomorphic.

Similarly, when $P$ is a $p$-subgroup of $G$, the ring homomorphism $\Phi_P : B(G) \to B(N_G(P))$ induced by the operation $X \mapsto X^P$ on $G$-sets, is compatible with the Brauer morphism $\text{Br}_P^G : pp_k(G) \to pp_k(N_G(P))$:

(4.3) $L_{N_G(P)} \circ \Phi_P = \text{Br}_P^G \circ L_G$.

This is because for any $G$-set $X$, the $kN_G(P)$-modules $k(X^P)$ and $(kX)[P]$ are isomorphic.

The ring homomorphism $L_G$ extends linearly to a $K$-algebra homomorphism $K \otimes_{\mathbb{Z}} B(G) \to K \otimes_{\mathbb{Z}} pp_k(G)$, still denoted by $L_G$. The algebra $K \otimes_{\mathbb{Z}} B(G)$ is also a split semisimple commutative $K$-algebra. Its species are the $K$-algebra maps

$$K \otimes_{\mathbb{Z}} B(G) \to K, \hspace{1cm} X \mapsto |X^H|,$$

where $H$ runs through the set of all subgroups of $G$ up to conjugation. Here we denote by $|X^H|$ the number of $H$-fixed points of a $G$-set $X$ and this notation is then extended $K$-linearly to any $X \in K \otimes_{\mathbb{Z}} B(G)$. The primitive idempotents $e_H^G$ of $K \otimes_{\mathbb{Z}} B(G)$ are indexed by subgroups $H$ of $G$, up to conjugation. They are given by the following formulae, found by Gluck ([4]) and later independently by Yoshida ([5]):

(4.4) $e_H^G = \frac{1}{|N_G(H)|} \sum_{L \leq H} |L| \mu(L, H) G/L$, 

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where $\mu$ denotes the Möbius function of the poset of subgroups of $G$. The idempotent $e^G_H$ is characterized by the fact that for any $X \in K \otimes \mathbb{Z} B(G)$

$$X \cdot e^G_H = |X^H| e^G_H.$$  

4.5. Remark: Since $|X^H|$ only depends on $\text{Res}^G_H X$, it follows in particular that $X$ is a scalar multiple of the “top” idempotent $e^G_G$ if and only if $\text{Res}^G_H X = 0$ for any proper subgroup $H$ of $G$. In particular, if $N$ is a normal subgroup of $G$, then

$$(e^G_G)^N = e^{G/N}_{G/N}.$$  

This is because for any proper subgroup $H/N$ of $G/N$

$$\text{Res}^{G/N}_{H/N} (e^G_G)^N = (\text{Res}^G_H e^G_G)^N = 0.$$  

So $(e^G_G)^N$ is a scalar multiple of $e^{G/N}_{G/N}$. As it is also an idempotent, it is equal to 0 or $e^{G/N}_{G/N}$. Finally

$$|((e^G_G)^N)^{G/N}| = |(e^G_G)^G| = 1,$$

so $(e^G_G)^N$ is non zero.

4.7. The case of a cyclic $p'$-group. Suppose that $G$ is a cyclic $p'$-group, of order $n$, generated by an element $s$. In this case, there are exactly $n$ group homomorphisms from $G$ to the multiplicative group $k^\times$ of $k$. For each of these group homomorphisms $\varphi$, let $k_{\varphi}$ denote the $kG$-module $k$ on which the generator $s$ acts by multiplication by $\varphi(s)$. As $G$ is a $p'$-group, this module is simple and projective. The (classes of the) modules $k_{\varphi}$, for $\varphi \in \hat{G} = \text{Hom}(G, k^\times)$, form a basis of $\text{pp} k(G)$.

Since moreover for $\varphi, \psi \in \hat{G}$, the modules $k_{\varphi} \otimes_k k_{\psi}$ and $k_{\varphi \psi}$ are isomorphic, the algebra $K \otimes_{\mathbb{Z}} \text{pp} k(G)$ is isomorphic to the group algebra of the group $\hat{G}$. This leads to the following classical formula:

4.8. Lemma. Let $G$ be a cyclic $p'$-group. Then for any $t \in G$,

$$F^G_{1,t} = \frac{1}{n} \sum_{\varphi \in \hat{G}} \hat{\varphi}(t^{-1}) k_{\varphi},$$

where $\hat{\varphi}$ is the Brauer character of $k_{\varphi}$.
Proof: Indeed for \( s, t \in G \)
\[
\tau_{s,t}^G \left( \frac{1}{n} \sum_{\varphi \in \hat{G}} \varphi(s^{-1})k_{\varphi} \right) = \frac{1}{n} \sum_{\varphi \in \hat{G}} \varphi(s^{-1})\varphi(t) = \delta_{s,t},
\]
where \( \delta_{s,t} \) is the Kronecker symbol.

4.9. The case \( G = <Ps> \). Suppose now more generally that \( G = <Ps> \), where \( P \) is a normal Sylow \( p \)-subgroup of \( G \) and \( s \) is a \( p' \)-element. In this case, by Proposition 3.3, the restriction of \( F_{P,s}^G \) to any proper subgroup of \( G \) is equal to zero. Moreover, since \( N_G(P,t) = G \) for any \( t \in G/P \), the conjugacy class of the pair \((P,t)\) reduces to \( \{(P,t)\} \).

4.10. Lemma. Suppose \( G = <Ps> \), and set \( E_G^G = L_G(e_G^G) \). Then
\[
E_G^G = \sum_{<t> = <s>} F_{P,t}^G.
\]

Proof: By 4.2 and by Remark 4.5, the restriction of \( E_G^G \) to any proper subgroup of \( G \) is equal to zero. Let \( (Q,t) \in Q_{G,p} \), such that the group \( L = <Qt> \) is a proper subgroup of \( G \). By Proposition 3.2, there is a rational number \( r \) such that
\[
F_{Q,t}^G = r \text{Ind}_L^G F_{Q,t}^L.
\]
It follows that
\[
E_G^G \cdot F_{Q,t}^G = r \text{Ind}_L^G ((\text{Res}_L E_G^G) \cdot F_{Q,t}^L) = 0.
\]
Now \( E_G^G \) is an idempotent of \( K \otimes_{\mathbb{Z}} ppk(G) \), hence it is a sum of some of the primitive idempotents \( F_{Q,t}^G \) associated to pairs \((Q,t)\) for which \( <Qt> = G \). This condition is equivalent to \( Q = P \) and \( <t> = <s> \).

It remains to show that all these idempotents \( F_{P,t}^G \) appear in the decomposition of \( E_G^G \), i.e. equivalently that \( \tau_{P,t}^G(E_G^G) = 1 \) for any generator \( t \) of \( <s> \). Now by 4.6 and Remark 2.16
\[
\tau_{P,t}^G(E_G^G) = \tau_{1,t}^{G/P}(\text{Br}_P(E_G^G)) = \tau_{1,t}^{<s>}(E_{<s>}^{<s>}).
\]
Now the value at \( t \) of the Brauer character of a permutation module \( kX \) is equal to the number of fixed points of \( t \) on \( X \). By \( K \)-linearity, this gives
\[
\tau_{1,t}^{<s>}(E_{<s>}^{<s>}) = \left| (e_{<s>}^{<s>})^t \right|,
\]
and this is equal to 1 if \( t \) generates \( <s> \), and to 0 otherwise, as was to be shown.
4.11. Proposition. Let \((P, s) \in \mathcal{Q}_{G,p}\), and suppose that \(G = <Ps>\). Then
\[
F^G_{P,s} = E^G_P \cdot \text{Inf}^G_{G/P} F^G_{1,s}.
\]

Proof: Set \(E_s = E^G_P \cdot \text{Inf}^G_{G/P} F^G_{1,s}\). Then \(E_s\) is an idempotent of \(K \otimes \mathbb{Z}_{pp_k}(G)\), as it is the product of two (commuting) idempotents. Let \((Q, t) \in \mathcal{Q}_{G,p}\). If \(<Qt> \neq G\), then \(\tau_{Q,t}^G(E^G_s) = 0\) by Lemma 4.10, thus \(\tau_{Q,t}(E_s) = 0\). And if \(<Qt> = G\), then \(Q = P\) and \(<t> = <s>\). In this case
\[
\tau_{Q,t}^G(E_s) = \tau_{P,t}^G(E^G_s) = \tau_{P,t}^G(\text{Inf}^G_{G/P} F^G_{1,s}).
\]

By Lemma 4.10, the first factor in the right hand side is equal to 1. The second factor is equal to
\[
\tau_{P,t}^G(\text{Inf}^G_{G/P} F^G_{1,s}) = \tau_{P,t}^G(\text{Br}^G(\text{Inf}^G_{G/P} F^G_{1,s})) = \tau_{P,t}^G(F^G_{1,s}) = \delta_{t,s},
\]
where \(\delta_{t,s}\) is the Kronecker symbol. Hence \(\tau_{P,t}(E_s) = \delta_{t,s}\), and this completes the proof. \(\square\)

4.12. Theorem. Let \(G\) be a finite group, and let \((P, s) \in \mathcal{Q}_{G,p}\). Then the primitive idempotent \(F^G_{P,s}\) of the \(p\)-permutation algebra \(K \otimes \mathbb{Z}_{pp_k}(G)\) is given by the following formula:
\[
F^G_{P,s} = \frac{1}{|P||s||C_{N_G(P)}(s)|} \sum_{\varphi \in \mathcal{S}} \tilde{\varphi}(s^{-1})|L|\mu(L, <Ps>) \text{Ind}_L^G k^G_{L,\varphi}.\]

where \(k^G_{L,\varphi} = \text{Res}_L^G \text{Inf}_{<s>} <Ps>\).

Proof: By Corollary 3.4, and Proposition 4.11
\[
F^G_{P,s} = \frac{|s|}{|C_{N_G(P)}(s)|} \text{Ind}^G_{<Ps>}(E^G_{<Ps>} \cdot \text{Inf}_{<s>} F^G_{1,s}).
\]

By Equation 4.4, this gives
\[
F^G_{P,s} = \frac{|s|}{|C_{N_G(P)}(s)|} \text{Ind}^G_{<Ps>} \frac{1}{|P||s|} \sum_{L \leq <Ps>} |L|\mu(L, <Ps>) \text{Ind}_L^G k^G_{L,\varphi} \cdot \text{Inf}_{<s>} F^G_{1,s}.\]
Moreover for each $L \leq <Ps>$

\[ \text{Ind}_{L}^{<Ps>} k \cdot \text{Inf}_{<s>}^{<Ps>} F_{1,s}^{<s>} \cong \text{Ind}_{L}^{<Ps>} (\text{Res}_{L}^{<Ps>} \text{Inf}_{<s>}^{<Ps>} F_{1,s}^{<s>}) \]

\[ \cong \text{Ind}_{L}^{<Ps>} \text{Inf}_{L/L \cap P}^{L/L \cap P} \text{Iso}_{L/P}^{L/P} \text{Res}_{L/P}^{<s>} F_{1,s}^{<s>}. \]

Here we have used the fact that if $L$ and $P$ are subgroups of a group $H$, with $P \unlhd H$, then there is an isomorphism of functors

\[ \text{Res}_{L}^{H} \circ \text{Inf}_{H/P}^{L} \cong \text{Inf}_{L/L \cap P}^{L/L \cap P} \circ \text{Iso}_{L/P}^{L/P} \circ \text{Res}_{L/P}^{H/P}, \]

which follows from the isomorphism of $(L,H/P)$-bisets

\[ H \times_{H} (H/P) \cong L(H/P) \cong (L/L \cap P) \times_{L/L \cap P} (L/P) \times_{L/P} (H/P). \]

Now Proposition 3.1 implies that $\text{Res}_{L/P}^{<s>} F_{1,s}^{<s>} = 0$ if $LP/P \nleq <s>$, i.e. equivalently if $PL \nleq <Ps>$. It follows that

\[ F_{P,s}^{G} = \frac{1}{|P||C_{N_{G}(P)}(s)|} \sum_{L \leq <Ps>, PL = <Ps>} |L| \mu(L, <Ps>) \text{Ind}_{L}^{G} (\text{Res}_{L}^{<Ps>} \text{Inf}_{<s>}^{<Ps>} F_{1,s}^{<s>}). \]

By Lemma 4.8, this gives

\[ F_{P,s}^{G} = \frac{1}{|P||s||C_{N_{G}(P)}(s)|} \sum_{\varphi \in <s>, L \leq <Ps>, PL = <Ps>} \tilde{\varphi}(s^{-1}) |L| \mu(L, <Ps>) \text{Ind}_{L}^{G} k_{L,\varphi}^{<Ps>}, \]

where $k_{L,\varphi}^{<Ps>} = \text{Res}_{L}^{<Ps>} \text{Inf}_{<s>}^{<Ps>} k_{\varphi}$, as was to be shown.

\[ \square \]

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