A SUPERPOWERED EUCLIDEAN PRIME GENERATOR

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Abstract. When \( k \geq 1 \) and \( n \) is the product of the smallest \( k \) primes, the \((k+1)\)-st smallest prime is the least divisor exceeding 1 of \( n^{n^k} - 1 \). This variant of Euclid’s prime generator is discussed with some of its brethren.

When \( \{p_1, \ldots, p_k\} \) is a finite set of primes, the least divisor exceeding 1 of \( p_1 \cdots p_k + 1 \) is a prime distinct from \( p_1, \ldots, p_k \). In this way, as every schoolchild knows, one sees that there are infinitely many primes: the assumption that there are just finitely many leads to a contradiction. This is essentially the proof attributed to Euclid, who observed that all primes dividing \( p_1 \cdots p_k + 1 \) are distinct from \( p_1, \ldots, p_k \). But is every prime delivered by iterating this algorithm? To be precise, if we put \( p_1 = 2 \) and then define

\[ p_{k+1} = \min \{ d > 1 : d \text{ divides } p_1 \cdots p_k + 1 \} \quad (k \geq 1), \]

is it the case that the sequence \( (p_k)_{k=1}^{\infty} \) contains all the primes? The widely held conjecture that the answer is in the affirmative remains open more than half a century after Mullin \cite{3} posed this question. There are, however, variants of Euclid’s construction that do yield every prime. Given a set of primes \( \{p_1, \ldots, p_k\} \), Pomerance \cite[§1.1.3]{2} defines \( p_{k+1} \) to be the least prime distinct from \( p_1, \ldots, p_k \) that divides a number of the form \( d + 1 \) for some divisor \( d \) of \( n = p_1 \cdots p_k \). He shows that starting with \( p_1 = 2 \), every prime is delivered by this iterative process, and moreover (by extensive computations) that \( p_k \) is the \( k \)-th smallest prime for \( k \geq 5 \). Booker \cite{1} instead considers the prime divisors \( p \) of the integers \( d + n/d \), and shows that at each stage in the iteration, choices for \( d \) and \( p \) may be made so that, taking \( p_{k+1} = p \), every prime is delivered.

The iterative processes of Booker \cite{1} and Pomerance \cite{2} involve some kind of ambiguity, in the latter case involving a choice of the divisor \( d \) of \( n = p_1 \cdots p_k \), and in the former case a choice of both \( d \) and the prime divisor of \( d + n/d \). In this note we present a variant of Euclid’s prime generator in which the sequence of primes is determined in order by a single choice of divisor.

**Theorem 1.** Let \( p_1 = 2 \), and when \( k \geq 1 \), define \( p_{k+1} \) to be the least divisor exceeding 1 of \( n^{n^k} - 1 \), where \( n = p_1 \cdots p_k \). Then for each \( k \), the integer \( p_k \) is the \( k \)-th smallest prime.

This “superpowered” variant of Euclid’s prime generator has computational value that can only be described as rather less than nanoscopic. However, it has the merit of succinctly delivering the \((k+1)\)-st smallest prime in terms of the \( k \) smallest primes. The proof is immediate from the following lemma, the proof of which is reminiscent of the argument underlying the Pollard \( p - 1 \) factorization method (see \cite{4} and \cite[§5.4]{2}).
Lemma 2. When \( n \) is an integer exceeding 1, the least prime divisor of \( n^{n^n} - 1 \) is the smallest prime not dividing \( n \).

Proof. Let \( \pi \) be the smallest prime not dividing \( n \), and let the primes dividing \( n \) be \( p_1, \ldots, p_k \). Then one has \( \pi \leq p_1 \cdots p_k + 1 \leq n + 1 \), as Euclid could have told us. Moreover, all prime divisors of \( \pi - 1 \) lie in \( \{p_1, \ldots, p_k\} \), and since \( p_i^n \geq 2^n \geq n + 1 \geq \pi \) for each \( i \), we find that \( \pi - 1 \) divides \( (p_1 \cdots p_k)^n \), and hence also \( n^n \). But then, defining the integer \( \lambda \) by writing \( n^n = \lambda(\pi - 1) \), and noting that \( \pi \) does not divide \( n \), we find from Fermat’s Little Theorem that \( n^{n^n} = (n^n)^{\pi - 1} \equiv 1 \pmod{\pi} \), which is to say that \( \pi \) divides \( n^{n^n} - 1 \). \( \square \)

The argument just described makes it apparent that less profligate exponents are viable. The conclusion of Theorem 1 remains valid, for example, when \( n^{n^n} - 1 \) is replaced by \( n^{n^m} - 1 \), in which \( m = \lceil (\log n)/(\log 2) \rceil \). In this context, we note also that if \( p_k \) denotes the \( k \)-th smallest prime for each \( k \), and \( n = p_1 \cdots p_k \), then the argument of the proof of Lemma 2 shows that all of the primes \( p \) with \( p_{k+1} \leq p < 2p_{k+1} \) divide \( n^{n^n} - 1 \).

A Euclidean disciple even more orthodox than enthusiasts of Theorem 1 might demand a means of obtaining the next prime without knowing a single one of the previous (smaller) primes. Even zero-knowledge demands such as this can be met by a direct consequence of Lemma 2.

Theorem 3. When \( N \) is an integer exceeding 1, the smallest prime exceeding \( N \) is the least divisor exceeding 1 of \( N!^{N!} - 1 \).

For a proof, simply apply Lemma 2 with \( n = N! \). We encourage readers to entertain themselves by establishing that for each natural number \( N \), the smallest prime exceeding \( N \) is the least divisor exceeding 1 of \( N!^{N!} - 1 \) (the author is grateful to Andrew Booker and Andrew Granville for pointing out this refinement).

References

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