THE UNIFORM PERFECTNESS OF DIFFEOMORPHISM GROUPS OF OPEN MANIFOLDS

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Abstract. In this paper we study the uniform perfectness, boundedness and uniform simplicity of diffeomorphism groups of compact manifolds with boundary and open manifolds and obtain some upper bounds of their diameters with respect to commutator length, those with support in balls and conjugation-generated norm.

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1. Introduction and statement of results

The algebraic properties of diffeomorphism groups have been studied by Herman [8], Thurston [15], Mather [11], Epstein [6] and others. These fundamental results are about the perfectness and simplicity of groups of diffeomorphisms on closed manifolds (compact, without boundary) or groups of diffeomorphisms with compact support on open manifolds (non-compact, without boundary). For a σ-compact (separable metrizable) $C^\infty$ $n$-manifold $M$ without boundary, let $\text{Diff}^r(M)_0$ denote the group of all $C^r$ diffeomorphisms of $M$ which are isotopic to the identity and $\text{Diff}^r(M)_0$ denote the subgroup of all elements of $\text{Diff}^r(M)$ which are isotopic to the identity by compactly supported isotopies. It is well-known that for $1 \leq r \leq \infty$, $r \neq n+1$ the group $\text{Diff}^r(M)_0$ is perfect, i.e., it coincides with its commutator subgroup, and is simple provided $M$ is connected.

McDuff [12] studied the lattice of normal subgroups of the diffeomorphism group of an open manifold $M$ which is the interior of a compact manifold with nonempty boundary and showed that $\text{Diff}^r(M)_0$ is a perfect group, using the fact due to Ling [10] and Schweitzer, cf. [14], that certain quotient groups of the group by normal subgroups are simple groups.

For an element $g$ of a group $G$ its commutator length $\text{cl}_G(g)$ is defined as the minimum number of commutators whose product is equal to $g$ for $g \in [G,G]$ and as $\text{cl}_G(g) = \infty$ for $g \in G - [G,G]$. The commutator length diameter $\text{cld}G$ of $G$ is defined by $\text{cld}G = \sup_{g \in G} \text{cl}_G(g)$. A group $G$ is said to be uniformly perfect if $\text{cld}G < \infty$, that is, any element of $G$ can be written as a product of a bounded number of commutators.

Recall that a group $G$ is (i) bounded if any conjugation-invariant norm on $G$ is bounded and (ii) uniformly simple ([18]) if there is a positive integer $k$ such that for all $f, g \in G$ with $g \neq e$, $f$ can be written as a product of at most $k$ conjugates of $g$ or $g^{-1}$. Note that every uniformly simple group is simple and bounded and that any bounded perfect group is uniformly perfect.

The problem of the uniform perfectness of diffeomorphism groups has been studied by Burago, Ivanov and Polterovich [4], Tsuboi [17, 19] and others. They showed that the diffeomorphism groups are uniformly perfect for all closed manifolds except for two and four dimensional cases.

**Theorem A.** ([4, 17, 19]) Suppose $M$ is a closed $n$-manifold and $1 \leq r \leq \infty$, $r \neq n+1$.

1. In the case $n = 2m + 1$ ($m \geq 0$) : $\text{clb} \text{Diff}^r(M)_0 \leq 4$.
2. In the case $n = 2m$ ($m \geq 1$) :
   1. $\text{clb} \text{Diff}^r(M)_0 \leq 3$ if $M$ admits a handle decomposition without $m$-handles.
   2. $\text{clb} \text{Diff}^r(M)_0 \leq 4k + 11$ if $m \geq 3$ and $M$ has a $C^\infty$ triangulation with at most $k$ $m$-simplices.

In [17, 19] the estimates of the commutator length diameters in the above theorem were given as $\text{clb} \text{Diff}^r(M) \leq 5$ in (1) and $\text{clb} \text{Diff}^r(M) \leq 4$ in (2)(i). However, in his lecture around 2011, Tsuboi announced that these estimates are improved to those in the above theorem.

The uniform simplicity of diffeomorphism groups has been studied by Tsuboi [18, 19], based on the estimates on the commutator length supported in finite disjoint union of balls ($\text{clb}^f$) and the conjugation-generated norm $\nu$ (cf. Sections 2.3, 6.1).

**Theorem B.** ([18, 19]) Suppose $M$ is a closed connected $n$-manifold ($n \geq 1$) and $1 \leq r \leq \infty$, $r \neq n+1$. $\text{Diff}^r(M)_0$ is uniformly simple in the following cases :

1. $n \neq 2, 4$. 

2. In the case $n = 2m + 1$ ($m \geq 0$) : $\text{clb} \text{Diff}^r(M)_0 \leq 4$.

2. In the case $n = 2m$ ($m \geq 1$) :
   1. $\text{clb} \text{Diff}^r(M)_0 \leq 3$ if $M$ admits a handle decomposition without $m$-handles.
   2. $\text{clb} \text{Diff}^r(M)_0 \leq 4k + 11$ if $m \geq 3$ and $M$ has a $C^\infty$ triangulation with at most $k$ $m$-simplices.
(2) \( n = 2m \) and \( M \) admits a handle decomposition without \( m \)-handles.

The problem of the uniform perfectness and boundedness of groups of equivariant diffeomorphisms, leaf-preserving diffeomorphisms or diffeomorphisms which preserve some submanifolds, etc, has been also studied in \([1, 2, 7, 13, 9]\), etc.

In this paper we extend the strategy in \([4, 17, 18, 19]\) and study the uniform perfectness, boundedness and uniform simplicity of the diffeomorphism groups of compact manifolds with boundary and open manifolds. Our main results are listed below.

Theorem I (Theorems \([4.1, 4.2]\)). Suppose \( n = 2m + 1 \) \( (m \geq 0) \) and \( 1 \leq r \leq \infty \), \( r \neq n + 1 \).

1. If \( M \) is a compact \( n \)-manifold possibly with boundary, then \( \text{cld} \text{Diff}^r(M, \partial) \leq 4 \).

2. If \( M \) is an open \( n \)-manifold, then \( \text{cld} \text{Diff}^r_c(M) \leq 8 \) and \( \text{cld} \text{Diff}^r_c(M) \leq 4 \).

Here, \( \text{Diff}^r(M, \partial) \) is the group of \( C^r \) diffeomorphisms which are isotopic to \( \text{id}_M \) rel a neighborhood of \( \partial M \). See Section 2 for the precise definition.

Theorem II (Corollaries \([5.1, 5.2]\), Propositions \([5.3, 5.4]\)). Suppose \( n = 2m \) \( (m \geq 1) \) and \( 1 \leq r \leq \infty \), \( r \neq n + 1 \).

1. Suppose \( M \) is a compact \( n \)-manifold possibly with boundary and \( m \geq 3 \).

\[
\text{cld} \text{Diff}^r(M, \partial) \leq 2k + 7
\]

if \( M \) has a \( C^\infty \) triangulation \( \mathcal{T} \) which includes at most \( k \) \( m \)-simplices not in \( \partial M \).

2. Suppose \( M \) is an \( n \)-manifold without boundary.

(i) \( \text{cld} \text{Diff}^r(M) \leq 6 \) and \( \text{cld} \text{Diff}^r_c(M) \leq 3 \)

if \( M \) admits a handle decomposition without \( m \)-handles.

(ii) \( \text{cld} \text{Diff}^r(M) \leq 2k + 10 \) and \( \text{cld} \text{Diff}^r_c(M) \leq 2k + 7 \)

if \( m \geq 3 \) and \( M \) admits a handle decomposition \( \mathcal{H} \) such that \( \mathcal{H} \) includes at most \( k \) handles of index \( m \) and each closed \( m \)-cell of the core complex \( P_\mathcal{H} \) for \( \mathcal{H} \) has the strong displacement property for the \( m \)-skeleton of \( P_\mathcal{H} \).

See Section 2.2 for definitions of the displacement property. The estimates in Theorem II (2)(ii) reduce to rough estimates \( 3k+8 \) and \( 3k+5 \) respectively, when each closed \( m \)-cell of \( P_\mathcal{H} \) has the strong displacement property for only itself in \( M \) (Proposition \([5.4]\)). These results induce fine estimates even in the case of closed manifolds (cf. Theorem A (2)(ii)). For example, if \( M \) is the product of two \( m \)-spheres \( (m \geq 3) \), then it has the natural product handle decomposition \( \mathcal{H} \) and both of two closed \( m \)-cells of \( P_\mathcal{H} \) have the strong displacement property for itself in \( M \). This implies \( \text{cld} \text{Diff}^r(M) \leq 11 \) (Example \([5.1]\)).

When an open \( 2m \)-manifold \( M \) has infinitely many \( m \)-handles, if these \( m \)-handles admit a grouping into finitely many classes with appropriate displacement property, then we can induce some estimate on \( \text{cld} \text{Diff}^r(M) \) (Proposition \([5.2]\), Theorem \([5.2]\)). In particular, we obtain some estimates on the commutator length diameter for covering spaces of closed \( 2m \)-manifolds and infinite sums of finitely many compact \( 2m \)-manifolds.

Theorem III (Corollary \([5.3]\)). Suppose \( \pi : \tilde{M} \to M \) is a \( C^\infty \) covering space over a closed \( 2m \)-manifold \( M \) \( (m \geq 3) \) and \( 1 \leq r \leq \infty \), \( r \neq 2m + 1 \). If \( M \) has a \( C^\infty \) triangulation with at most \( k \) \( m \)-simplices, then

\[
\text{cld} \text{Diff}^r(\tilde{M}) \leq 4k + 14 \quad \text{and} \quad \text{cld} \text{Diff}^r_c(\tilde{M}) \leq 2k + 7.
\]
We also obtain a finer estimate of $cld\Diff^r(M)_0$ based on a handle decomposition of $M$ with some displacement property (Theorem 5.3).

**Theorem IV** (Corollary 5.4, Example 5.3). Suppose $M$ is an infinite connected sum $\#_{i=1}^{\infty} N_i$ of a closed 2m-manifold $N$ ($m \geq 3$) and $1 \leq r \leq \infty$, $r \neq 2m+1$. Then, $cld\Diff^r(M)_0 < \infty$ and $cld\Diff^c_r(M)_0 < \infty$.

In Proposition 5.5 we treat a more general class of infinite sums of finitely many compact 2m-manifolds and provide with some explicit estimates on $cld\Diff^r(M)_0$ and $cld\Diff^c_r(M)_0$.

The following is an extension of the results of McDuff [12] and the second author [13]. In Proposition 5.5 we treat a more general class of infinite sums of finitely many compact 2m-manifolds and provide with some explicit estimates on $cld\Diff^r(M)_0$ and $cld\Diff^c_r(M)_0$.

**Theorem V** (Proposition 2.1). Suppose $M$ is the interior of a compact n-manifold $W$ with boundary and $1 \leq r \leq \infty$, $r \neq n+1$. Then $cld\Diff^r(M)_0 \leq \max\{cld\Diff^r(W,\partial)_0,2\} + 2$. In particular, $\Diff^r(M)_0$ is uniformly perfect for $n \neq 2,4$.

For the boundedness and uniform simplicity of diffeomorphism groups we obtain the following results.

**Theorem VI** (Corollary 6.1). Suppose $1 \leq r \leq \infty$, $r \neq n+1$.

1. Suppose $M$ is a compact connected $n$-manifold possibly with boundary and $n \neq 2,4$. Then, $\Diff^r(M,\partial)_0$ is uniformly simple.
2. Suppose $M$ is a connected open $n$-manifold. Then, $\Diff^r(M)_0$ is bounded and $\Diff^c_r(M)_0$ is uniformly simple in the following cases:
   1. $n = 2m + 1$ ($m \geq 0$)
   2. $n = 2m$ ($m \geq 1$) and $M$ satisfies one of the following conditions:
      i. $M$ has a handle decomposition without $m$-handles.
      for $m \geq 3$
      ii. $M$ has a handle decomposition $\mathcal{H}$ with only finitely many $m$-handles and for which the closure of each open $m$-cell of $P_\mathcal{H}$ is strongly displaceable from itself in $M$.
      iii. $M$ is a covering space over a closed 2m-manifold.
      iv. $M$ is an infinite sum of finitely many compact 2m-manifolds (as in Setting 6.4).
3. Suppose $M$ is the interior of a compact $n$-manifold $W$ with boundary. Then, $\Diff^r(M)_0$ is bounded, if $\Diff^r(W,\partial)_0$ is bounded.

Extending the arguments in [18], we introduce the commutator length supported in discrete union of balls ($cl^d$) as a non-compact version of $cl^f$, and discuss the relation between the conjugation-generated norm $\nu$ and $cl^f$, $cl^d$ (Lemma 6.2). Since the norms $cl$, $cl^f$ and $cl^d$ are closely related, we provide with the estimates on these three commutator lengths simultaneously in all steps of Sections 2~5. The estimates on $\nu$ are induced from those on $cl^f$ and $cl^d$ (Section 6.2). Then, Theorem VI follows from the boundedness of $\nu$.

In this paper we have tried to find finer estimates on the commutator lengths $cl$, $cl^f$ and $cl^d$, though this causes complicated expressions in some statements. This is because we are interested in the lower bounds of these norms and also in the relations between the precise values of these norms and the topology of the manifolds.

This paper is organized as follows. In Sections 2.1, 2.2 we fix basic notations used in this paper and recall generalities on extended conjugation-invariant norms on groups. In Section 2.3 we introduce the
commutator length supported in discrete union of balls \((clb^d)\) as a noncompact version of \(clb^f\). In Section 2.4 we separate the notions of absorption and displacement properties for compact subsets in a manifold from the related arguments. We clarify their basic behavior and roles in the basic factorization lemma on diffeomorphisms in [4, 17, 18] and extend the factorization lemma to the case of manifolds with boundary. Section 2.5 includes some observation on factorization of isotopies on noncompact manifolds.

Section 3 is devoted to some basic arguments on handle decompositions and triangulations of manifolds, which are necessary to extend the strategy in [17, 18] to the cases of compact manifolds with boundary and open manifolds. See an introduction in Section 3 for an explanation on this issue. In Section 3.1 we treat \(C^\infty\) handle decompositions. Our device here is that, given a handle decomposition \(\mathcal{H}\) and its dual \(\mathcal{H}^*\) of an open \(n\)-manifold \(M\), instead of a single compact \(n\)-submanifold \(N\) of \(M\) we use a pair \(N_1 \subset N_2\) of compact \(n\)-submanifolds of \(M\) such that \(N_1\) is \(\mathcal{H}\)-saturated and \(N_2\) is \(\mathcal{H}^*\)-saturated. Section 3.2 includes some arguments on \(C^\infty\) triangulations of compact manifolds with boundary. In the bounded case \(C^\infty\) triangulations are important since they naturally induce flows between some subcomplexes and their duals which preserve the boundary of manifolds. Here we use the cylinder structures between complimentary full subcomplexes rather than the induced handle decompositions, since the description of the former is more direct and simpler.

Section 4 includes main results in the odd-dimensional case. In Section 4.1 we recall the basic procedure in [4, 17] for removing crossing points on the track of an isotopy and a factorization of an isotopy. We treat the compact case in Section 4.2 and the open case in Section 4.3, based on the basic procedure and the preliminary lemmas for the bounded/open cases in Section 3.

Section 5 includes main results in the even-dimensional case. In Section 5.1 we recall the basic procedure in [19] for the Whitney trick on the track of an isotopy and a factorization of an isotopy, together with some improvements for finer estimates of \(cl\) and \(clb^f\) in our setting. In Section 5.2 we discuss the compact case based on the basic procedure and the preliminary lemmas for the bounded/open cases in Section 3. Section 5.3 is devoted to the open case. Section 5.3.1 includes some general results on grouping of \(m\)-cells in \(2m\)-manifolds. In Section 5.3.2 we treat three classes: (i) open \(2m\)-manifolds with only finitely many \(m\)-handles, (ii) covering spaces of closed \(2m\)-manifolds and (iii) infinite sum of finitely many compact \(2m\)-manifolds. The results on grouping and the case (i) follow from the compact results in Section 5.2 and some basic lemmas in Section 3, while the cases (ii), (iii) follow from the results on grouping directly.

In the final section 6 we obtain the estimates on the conjugation-generated norm \(\nu\) in the bounded/open cases. First we discuss the relation between \(\nu\) and \(clb^f, clb^d\) as an extension of the arguments in [18]. Then, the estimates on \(\nu\) follow from those on \(clb^f\) and \(clb^d\) in Sections 2 ~ 5. The conclusions on boundedness and uniform simplicity of diffeomorphism groups in the bounded/open cases follow immediately from the boundedness of \(\nu\) according to the basic facts on conjugation-generated norms.

2. Preliminaries

2.1. Notations.

For a subset \(A\) of a topological space \(X\) the symbols Int\(_X\)(\(A\)) and Cl\(_X\)(\(A\)) denote the topological interior and closure of \(A\) in \(X\). For subsets \(A, B\) of \(X\) the symbol \(A \ominus B\) means \(A \subset \text{Int}_X B\) and the symbol \(A_B\) denotes the subset \(A - \text{Int}_X B\). The symbols \(\mathcal{O}(X), \mathcal{F}(X)\) and \(\mathcal{K}(X)\) denote the collections of open subsets, closed subsets and compact subsets of \(X\) respectively. The symbol \(\mathcal{C}(X)\) denotes the collection of connected components of \(X\).
For a family $\mathcal{F} = \{F_\lambda\}_{\lambda \in \Lambda}$ of subsets of a topological space $X$, let $|\mathcal{F}| := \bigcup_{\lambda \in \Lambda} F_\lambda$ and $St(A, \mathcal{F}) := A \cup \left( \bigcup \{F_\lambda \mid \lambda \in \Lambda, A \cap F_\lambda \neq \emptyset \} \right)$ for $A \subset X$. We say that $\mathcal{F}$ is discrete in $X$ if any point of $X$ has a neighborhood $U$ in $X$ which intersects at most one $F_\lambda$ (i.e., $\# \{ \lambda \in \Lambda \mid U \cap F_\lambda \neq \emptyset \} \leq 1$). When $\mathcal{F}$ is discrete in $X$, the following holds: (i) The family $\{C \cap F_\lambda\}_{\lambda \in \Lambda}$ is also discrete in $X$. (ii) $|\mathcal{F}| \in \mathcal{F}(X)$ if $F_\lambda \in \mathcal{F}(X)$ ($\lambda \in \Lambda$). (iii) If $X$ is a paracompact Hausdorff space, then there exists a discrete family $G = \{G_\lambda\}_{\lambda \in \Lambda}$ in $X$ such that $F_\lambda \subset \subset G_\lambda$ ($\lambda \in \Lambda$).

In this paper an $n$-manifold means a $\sigma$-compact (separable metrizable) $C^\infty$ manifold of dimension $n$. A closed/open manifold means a compact/non-compact manifold without boundary.

Suppose $M$ is a $C^\infty$ $n$-manifold possibly with boundary. By $SM(M)$ we denote the collection of $C^\infty$ $n$-submanifolds of $Int \ M$ which are closed in $M$. Let $SM_c(M) := SM(M) \cap K(M)$. We regard $\emptyset$ as an element of $SM_c(M)$. If $L, N \in SM(M)$ and $L \subset N$, then $N_L = N - Int_M L \in SM(M)$. A subcollection $\mathcal{R}$ of $SM_c(M)$ is said to be cofinal if for any $L \in SM_c(M)$ there exists $N \in \mathcal{R}$ with $L \subset N$.

An exhausting sequence in $M$ is a sequence $M_i \in SM_c(M)$ ($i \geq 1$) such that $M_i \subset M_{i+1}$ ($i \geq 1$) and $M = \bigcup_{i \geq 1} M_i$. Such a sequence always exists, since $M$ is $\sigma$-compact. For an exhausting sequence $\{M_i\}_{i \geq 1}$ we use the following notations : $M_i := \emptyset$ for $i \leq 0$ (otherwise specified) and $M_{i,j} := (M_j)_{M_i} \in SM_c(M)$ ($-\infty < i < j < \infty$).

For the manifold $M$, the symbol $\hat{M}$ denotes the $C^\infty$ $n$-manifold $\hat{M} := M \cup (\partial M \times [0,1))$ and for any $O \subset O(M)$ the symbol $\hat{O} \equiv O^\sim$ denotes $\hat{O} := O \cup (\partial O \times [0,1)) \subset O(\hat{M})$.

For the simplicity of notation we always use the symbol $I$ to denote the interval $[0, 1]$. A $C^r$ isotopy on $M$ is a $C^r$ diffeomorphism $F : M \times I \to M \times I$ which preserves the $I$-factor, that is, it takes the form $F(x, t) = (F_t(x), t)$ ($(x, t) \in M \times I$). For a subset $A$ of $M$ the track of $A$ under $F$ is the subset $\bigcup_{t \in I} F_t(A) = \pi_M F(A \times I)$, where $\pi_M : M \times I \to M$ is the projection onto $M$. The support of an isotopy $F$ is defined by $supp F := Cl_M \left( \bigcup_{t \in I} \text{supp} F_t \right)$. Note that an isotopy $F$ on $M$ has compact support if and only if there exists $K \subset K(M)$ such that $F = \text{id}$ on $(M - K) \times I$.

The symbols $\text{Diff}^r(M)$ and $\text{Diff}^c_r(M)$ ($1 \leq r \leq \infty$) denote the group of all $C^r$ diffeomorphisms of $M$ and its subgroup consisting of those with compact support. Similarly, the symbols $\text{Isot}^r(M)$ and $\text{Isot}^c_r(M)$ denote the group of all $C^r$ isotopies of $M$ and its subgroup consisting of those with compact support. There exists a natural group homomorphism $R : \text{Isot}^r(M) \to \text{Diff}^r(M) : R(F) = F_1$. Any subgroup $G$ of $\text{Isot}^r(M)$ induces the subgroup $R(G)$ of $\text{Diff}^r(M)$. For a subset $C$ of $M$ we have the following subgroups of the groups $\text{Isot}^r(M)$ and $\text{Diff}^r(M)$:

- $\text{Isot}^r(M; C)_0 = \{ F \in \text{Isot}^r(M) \mid F_0 = \text{id}_M, F = \text{id}$ on $U \times I$ for some neighborhood $U$ of $C$ in $M \}$,
- $\text{Isot}^c_r(M; C)_0 = \text{Isot}^r(M; C)_0 \cap \text{Isot}^c_r(M),$
- $\text{Diff}^r(M; C)_0 = R(\text{Isot}^r(M; C)_0), \quad \text{Diff}^c_r(M; C)_0 = R(\text{Isot}^c_r(M; C)_0).

Note that $\text{Diff}^c_r(M; C)_0 \subset \subset \text{Diff}^r(M; C)_0 \cap \text{Diff}^c_r(M; C)$ in general. When $C = \partial M$, we simply write as $\text{Diff}^r(M; \partial)_0$, etc. Under this notation, for a compact manifold $W$ the restriction map $\text{Diff}^r(W; \partial)_0 \to \text{Diff}^c_r(\text{Int} W)_0$ is a group isomorphism. For notational simplicity, when $r = \infty$, we usually omit the superscript $\infty$ from the above notations.

2.2. Extended conjugation-invariant norms.

Suppose $G$ is a group. For simplicity we use the notation $b^a := aba^{-1}$ for $a, b \in G$. The next formula is useful in the change of order of factors in multiplications.
Fact 2.1. \( ab = b^a a \) and more generally \( a_1 b_1 \cdots a_s b_s = b_1^{a_1} b_2^{a_1 a_2} b_3^{a_1 a_2 a_3} \cdots b_s^{a_1 \cdots a_s} (a_1 \cdots a_s) \) in \( G \).

We recall basic facts on conjugation-invariant norms [4, 18]. An extended conjugation-invariant norm on \( G \) is a function \( q : G \to [0, \infty] \) which satisfies the following conditions: for any \( g, h \in G \)

(i) \( q(g) = 0 \) iff \( g = e \)  
(ii) \( q(g^{-1}) = q(g) \)  
(iii) \( q(gh) \leq q(g) + q(h) \)  
(iv) \( q(hgh^{-1}) = q(g) \).

A conjugation-invariant norm on \( G \) is an extended conjugation-invariant norm on \( G \) with values in \([0, \infty)\). (Below we abbreviate “an (extended) conjugation-invariant norm” to an (ext.) conj.-invariant norm.)

Note that for any ext. conj.-invariant norm \( q \) on \( G \) the inverse image \( N := q^{-1}([0, \infty)) \) is a normal subgroup of \( G \) and \( q|_N \) is a conj.-invariant norm on \( N \). Conversely, if \( N \) is a normal subgroup of \( G \) and \( q : N \to [0, \infty) \) is conj.-invariant norm on \( N \), then its trivial extension \( q : G \to [0, \infty] \) by \( q = \infty \) on \( G - N \) is an ext. conj.-invariant norm on \( G \). For any ext. conj.-invariant norm \( q \) on \( G \), the \( q \)-diameter of a subset \( A \) of \( G \) is defined by \( q d A := \sup \{ q(g) \mid g \in A \} \).

Example 2.1. (Basic construction)

Suppose \( S \) is a subset of \( G \). Its normalizer is denoted by \( N(S) \). If \( S \) is symmetric \((S = S^{-1})\) and conjugation-invariant \((gSg^{-1} = S \text{ for any } g \in G)\), then \( N(S) = S^{-1} \) and the ext. conj.-invariant norm \( q_{(G,S)} : G \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \) is defined by

\[
q_{(G,S)}(g) = \begin{cases} 
\min\{k \in \mathbb{Z}_{\geq 0} \mid g = g_1 \cdots g_k \text{ for some } g_1, \cdots, g_k \in S\} & (g \in N(S)), \\
\infty & (g \in G - N(S)).
\end{cases}
\]

Here, the empty product \((k = 0)\) denotes the unit element \( e \) in \( G \) and \( S^0 = \{e\} \).

Example 2.2. (Commutator length)

The symbol \( G^c \) denotes the set of commutators in \( G \). Since \( G^c \) is symmetric and conjugation-invariant in \( G \), we have \([G,G] = N(G^c) = (G^c)^\infty \) and obtain the ext. conj.-invariant norm \( q_{(G,G^c)} : G \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \).

We denote \( q_{(G,G^c)} \) by \( cld_{G} \) and call it the commutator length of \( G \). The diameters \( q_{(G,G^c)} dG \) and \( q_{(G,G^c)} dA \) \((A \subseteq G)\) are denoted by \( cld_{G} G \) and \( cld_{G} A \equiv cld_{(A,G)} \) and called the commutator length diameter of \( G \) and \( A \) respectively. Sometimes we write \( cld(g) \leq k \) in \( G \) instead of \( cld_{G}(g) \leq k \).

A group \( G \) is perfect if \( G = [G,G] \), that is, any element of \( G \) is written as a product of commutators in \( G \). We say that a group \( G \) is uniformly perfect if \( cld G < \infty \), that is, any element of \( G \) is written as a product of a bounded number of commutators in \( G \).

More generally, suppose \( S \) is a subset of \( G_c \) and it is symmetric and conjugation-invariant in \( G \). Then, \( N(S) \subseteq [G,G] \) and we obtain \( q_{(G,S)} \), which is denoted by \( cld_{(G,S)} \) and is called the commutator length of \( G \) with respect to \( S \). The diameters \( q_{(G,S)} dG \) and \( q_{(G,S)} dA \) \((A \subseteq G)\) are denoted by \( cld_{S} G \) and \( cld_{(S,G)} A \equiv cld_{S}(A,G) \) respectively.

2.3. Commutator length of diffeomorphism groups.

In this paper we are concerned with the commutator length on diffeomorphism groups. In Section 6 we also study the boundedness and uniform simplicity of diffeomorphism groups, where “the commutator length supported in balls” plays an important role. Since these commutator lengths are closely related, in Sections 2 ~ 5 we treat them simultaneously.

We start with the definition of the commutator length supported in balls. Suppose \( M \) is an \( n \)-manifold possibly with boundary, \( 1 \leq r \leq \infty \) and \( C \in \mathcal{F}(M) \).
Notation. Let $B^r(M, C), B^r_f(M, C)$ and $B^r_d(M, C)$ denote the collections of (i) all $C^r$ $n$-balls in $\text{Int } M - C$, (ii) all finite disjoint unions of $C^r$ $n$-balls in $\text{Int } M - C$ and (iii) all discrete unions of $C^r$ $n$-balls in $\text{Int } M - C$ which are closed in $M$, respectively. The condition (iii) is restated as (iii)$'$ all discrete unions of $C^r$ $n$-balls in $M$ which are included in $\text{Int } M - C$. (cf. the 2nd paragraph in Section 2.1)

Consider the group $G \equiv \text{Diff}^r(M, \partial M \cup C)_0$. In each case where $B = B^r(M, C), B^r_f(M, C)$ or $B^r_d(M, C)$, consider the subset $S$ of $G^c$ defined by

$$S := \bigcup \{ \text{Diff}^r(M, M_D) \mid D \in B \}.$$ 

Then, $S$ is symmetric and conjugation-invariant in $G$. Therefore, we obtain the commutator length $cl_{(G, S)}$ of $G$ with respect to $S$. Depending on the selection of $B$, we denote this commutator length by $clb$, $clb^f$ or $clb^d$ and the commutator length diameter by $clbd$, $clb^fd$ or $clb^dd$, respectively.

Remark 2.1.

1. Since $clb \geq clb^f \geq clb^d$ in $G$ since $B^r(M, C) \subset B^r_f(M, C) \subset B^r_d(M, C)$.

2. (i) If $M - C$ is connected, then $clb = clb^f$ in $G$, since $\text{Int } M - C$ is also connected and any $D \in B^r_f(M, C)$ is included in some $E \in B^r(M, C)$.

(ii) If $M - C$ is relatively compact in $M$, then $B^r_f(M, C) = B^r_d(M, C)$ and $clb^f = clb^d$ in $G$.

3. For $S^r_f := \bigcup \{ \text{Diff}^r(M, M_D) \mid D \in B^r_f(M, C) \}$, the following holds.

   (i) $N(S^r_f) \subset \text{Diff}^r_c(M, \partial M \cup C)_0$ and (ii) $N(S^r_f) = \text{Diff}^r_c(M, \partial M \cup C)_0 \text{ when } r \neq n + 1$.

In the above notations the symbol $C$ is omitted when $C = \emptyset$ and the script $r$ is omitted when $r = \infty$.

Remark 2.2. Suppose $O \in O(M)$ and consider the groups $G := \text{Diff}^r_c(M, M_O)_0$ and $H := \text{Diff}^r_c(O)_0$.

1. The restriction induces the canonical isomorphism $G \cong H$ and the definition of $clb^f_G$ directly implies that $clb^f_G(g) = clb^f_H(g|_O)$ for any $g \in G$.

2. Suppose $M'$ is another $n$-manifold possibly with boundary and $O' \in O(M')$. Let $G' := \text{Diff}^r_c(M', M'_O)_0$ and $H' := \text{Diff}^r_c(O')_0$. If there exists a $C^r$ diffeomorphism $\varphi : O \cong O'$, then it induces the isomorphism by conjugation, $\varphi^* : H \cong H'$, $\varphi^*(h) = \varphi h \varphi^{-1} (h \in H)$ and we have $clb^f_H(h) = clb^f_{H'}(\varphi h \varphi^{-1}) (h \in H)$.

It follows that $clb^f_G(g) = clb^f_{G'}(\chi(g)) (g \in G)$ under the isomorphism $\chi : G \cong H \cong H' \cong G'$.

2.4. Basic estimates on $cl$ and $clb^f$ and the absorption / displacement property.

The estimates of $cl$ and $clb^f$ is based on the next lemma ([17, Theorem 2.1], [19, Proof of Theorem 1.1(1)], [4, 18]). We include its proof to confirm the supports of related diffeomorphisms for later use.

Lemma 2.1. ([4, 17, 18, 19]) Suppose $M$ is an $n$-manifold possibly with boundary, $1 \leq r \leq \infty$, $r \neq n + 1$, $F \in \text{Isot}^c_c(M, \partial)_{0}$ and $f := F \in \text{Diff}^c(M, \partial)_{0}$. Assume that there exist $U \in K(M)$, $V, W \in K(\text{Int } M)$ and $\varphi \in \text{Diff}^r(M, M_U \cup \partial M)_0$ such that $U \supset V \supset W$, $\varphi(V) \subset W$ and $\text{supp } F \subset \text{Int } M \text{ V } W$. Then, the following holds.

1. $cf \leq 2$ in $\text{Diff}^r(M, M_U \cup \partial M)_0$. Moreover, there exists a factorization $f = gh = (ghg^{-1})g$ for some $g \in \text{Diff}^r(M, M_U \cup \partial M)_0 \cap \text{Diff}^r(M, M_V)_0$ and $h \in \text{Diff}^r(M, M_W)_0$.

2. In addition, if $clb^f \varphi \leq k$ in $\text{Diff}^r(M, M_U \cup \partial M)_0$, then $clb^f(f) \leq 2k + 1$ in $\text{Diff}^r(M, M_U \cup \partial M)_0$ and we can take $g$ and $h$ in (1) so that $clb^f(g) \leq 2k$ in $\text{Diff}^r(M, M_U \cup \partial M)_0$ and $clb^f(h) \leq 1$ in $\text{Diff}^r(M, M_W)_0$. 

Proof. Let $O := \text{Int}_M V - W \in \mathcal{O}(\text{Int} M)$. Since $f \in \text{Diff}^r(M, M_O)_{0}$, we have a factorization

$$f = [a_1, b_1] \cdots [a_s, b_s]$$

for some $D_i \in \mathcal{O}(O)$ and $a_i, b_i \in \text{Diff}^r(M, D_i)_{0}$ ($i = 1, \ldots, s$) (\cite{15, 11}). The conjugation by $\varphi$ induces the factorization $f = h g_0 = g h$, where

$$h := [A, B], \quad g_0 := [\varphi, H^{-1}], \quad g := g_0^b$$

and

$$H := \prod_{i=1}^{s} (\varphi^{s-i}a_i b_1) \cdots [a_i, b_i] \varphi^{-s}, \quad A := \prod_{i=0}^{s-1} (\varphi^{s-i}a_i b_1 \varphi^{-i}), \quad B := \prod_{i=0}^{s-1} (\varphi^{s-i}b_i \varphi^{-i}).$$

Note that $H \in \text{Diff}^r(M, M_V)_{0}$ and $A, B \in \text{Diff}^r(M, M_W)_{0}$. Hence, (i) $h \in \text{Diff}^r(M, M_W)_{0}$, (ii) $g_0, g \in \text{Diff}^r(M, M_U \cup \partial M)_{0}$ since $\varphi, H, h \in \text{Diff}^r(M, M_U \cup \partial M)_{0}$ and (iii) $g = f h^{-1} \in \text{Diff}^r(M, M_V)_{0}$.

(2) Let $D := \bigcup_{i=1}^{s} \varphi^{s-i+1}(D_i) \in \mathcal{B}_{f}(\text{Int}_M W)$. Then, $A, B \in \text{Diff}^r(M, M_D)_{0}$ and $h \in \text{Diff}^r(M, M_D)_{0}$, so that $\text{clb}^f(h) \leq 1$ in $\text{Diff}^r(M, M_W)_{0}$. On the other hand, in $\text{Diff}^r(M, M_U \cup \partial M)_{0}$ we have

$$\text{clb}^f(g) = \text{clb}^f(g_0) = \text{clb}^f(\varphi H^{-1} \varphi^{-1} H) \leq \text{clb}^f(\varphi) + \text{clb}^f(H^{-1} \varphi^{-1} H)$$

$$= \text{clb}^f(\varphi) + \text{clb}^f(\varphi^{-1}) = 2 \text{clb}^f(\varphi) \leq 2k,$$

$$\text{clb}^f(f) \leq \text{clb}^f(g) + \text{clb}^f(h) \leq 2k + 1. \quad \square$$

Example 2.3. Suppose $M$ is an $n$-manifold without boundary and $1 \leq r \leq \infty$, $r \neq n + 2$.

(1) $\text{cld} \text{Diff}^r_c(M \times I, \partial)_{0} \leq 2$.

(2) $\text{clb}^f \text{Diff}^r_c(M \times I, \partial)_{0} \leq 2 \text{clb}^f(\varphi_{\xi}) + 1$ in $\text{Diff}^r(M \times I, \partial)_{0}$.

if $M$ is a closed $n$-manifold, $\xi \in \text{Diff}^r(I, \partial)_{0} - \{\text{id}_I\}$ and $\varphi_{\xi} := \text{id}_M \times \xi \in \text{Diff}^r(M \times I, \partial)_{0}$.

Proof. (1) The assertion easily follows from Lemma 2.1(1).

(2) Given any $f \in \text{Diff}^r(M \times I, \partial)_{0}$. Take $F \in \text{Isot}^r(M \times I, \partial)_{0}$ with $F_1 = f$. Then, supp $F \subset M \times (\alpha, \beta)$ for some closed interval $J \equiv [\alpha, \beta] \subset (0, 1)$.

Since $\xi \neq \text{id}_I$, there exists a closed interval $K \subset (0, 1)$ such that $\xi(K) \cap K = \emptyset$. Take $\eta \in \text{Diff}^r(I, \partial)_{0}$ with $\eta(K) = J$ and let $\lambda := \eta \xi \eta^{-1} \in \text{Diff}^r(I, \partial)_{0}$. Then, $\lambda(J) \cap J = \emptyset$ and so $[\gamma, \delta] := \lambda(J) \subset (0, \alpha) \cup (\beta, 1)$. Take $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that $\lambda = \text{id} \text{ on } [0, \varepsilon_1] \cup [\varepsilon_2, 1]$ and $J \cup \lambda(J) \subset (\varepsilon_1, \varepsilon_2)$.

Define closed intervals $I_2 \subset I_1 \subset (\varepsilon_1, \varepsilon_2)$ by

$$(I_1, I_2) := \begin{cases} ([\varepsilon_1, \beta], [\varepsilon_1, \delta]) & \text{if } [\gamma, \delta] \subset (0, \alpha), \\ ([\alpha, \varepsilon_2], [\gamma, \varepsilon_2]) & \text{if } [\gamma, \delta] \subset (\beta, 1). \end{cases}$$

Then, $J \subset I_1 - I_2$ and $\lambda(I_1) = I_2$.

For any $\zeta \in \text{Diff}^r(I, \partial)_{0}$ we set $\varphi_{\zeta} := \text{id}_M \times \zeta \in \text{Diff}^r(M \times I, \partial)_{0}$. It follows that $
 \varphi_{\lambda} = \varphi_{\eta} \varphi_{\xi} \varphi_{\eta}^{-1} \in \text{Diff}^r(M \times I, \partial)_{0}$ and $\text{clb}^f \varphi_{\lambda} = \text{clb}^f \varphi_{\xi}$ in $\text{Diff}^r(M \times I, \partial)_{0}$.

Let $(U, V, W) := M \times (I, I_1, I_2)$. Then, it follows that $V, W \in \mathcal{K} \text{Int}_M (M \times I)$, $U \supset V \supset W$, $\varphi_{\lambda}(V) = W$ and supp $F \subset \text{Int}_M V - W$. Therefore, Lemma 2.1(2) implies that $\text{clb}^f f \leq 2 \text{clb}^f \varphi_{\lambda} + 1 = 2 \text{clb}^f \varphi_{\xi} + 1$ in $\text{Diff}^r(M \times I, \partial)_{0}$. \quad \square$

Lemma 2.1 is applied in various situations. The existence of $V, W$ and $\varphi$ is concerned with absorption and displacement of compact subsets by isotopies (cf. \cite{4, 17, 19}). We inspect this point more closely.

Setting 2.1. Suppose $M$ is an $n$-manifold possibly with boundary, $O \in \mathcal{O}(M)$, $K \in \mathcal{K}(M)$ and $L \in \mathcal{F}(M)$. Recall that $\widetilde{M} := M \cup_{\partial M} (\partial M \times [0, 1])$ and $\widetilde{O} := O \cup (\partial O \times [0, 1]) \in \mathcal{O}(\widetilde{M})$.

Definition 2.1. In Setting 2.1 We use the following terminologies:
Consider any condition $\mathcal{P}$ on $\varphi \in \text{Diff}_c(M, M_O)_0$.

(1) For $C \in \mathcal{K}(O)$:
   (i) $C$ is absorbed to $K$ in $O$ with $\mathcal{P}$
       \[ \iff \text{There exists } \varphi \in \text{Diff}_c(M, M_O)_0 \text{ such that } \varphi(C) \subset K \text{ and } \varphi \text{ satisfies the condition } \mathcal{P}. \]
       The diffeomorphism $\varphi$ is called an absorbing diffeomorphism of $C$ to $K$ in $O$ with $\mathcal{P}$.
   (ii) $C$ is weakly absorbed to $K$ in $O$ with $\mathcal{P}$
       \[ \iff C \text{ is absorbed to } \text{any neighborhood of } K \text{ in } M \text{ in } O \text{ with } \mathcal{P}. \]

(2) $K$ has the (weak) absorption property in $O$ with $\mathcal{P}$
    \[ \iff \text{Any } C \in \mathcal{K}(O) \text{ is (weakly) absorbed to } K \text{ in } O \text{ with } \mathcal{P}. \]

(3) When $K \in \mathcal{K}(O)$:
       $K$ has the (weak) neighborhood absorption property in $O$ with $\mathcal{P}$
       \[ \iff \text{Some compact neighborhood of } K \text{ in } O \text{ is (weakly) absorbed to } K \text{ in } O \text{ with } \mathcal{P}. \]

[II] (1) $K$ is displaceable from $L$ in $O$ (or $K$ has the displacement property for $L$ in $O$)
    \[ \iff \text{There exists } \psi \in \text{Diff}_c(M, M_O)_0 \text{ such that } \psi(K) \cap L = \emptyset. \]
    The diffeomorphism $\psi$ is called a displacing diffeomorphism of $K$ from $L$ in $O$.

(2) $K$ is strongly displaceable from $L$ (or $K$ has the strong displacement property for $L$)
    \[ \iff K \text{ is displaceable from } L \text{ in any open neighborhood of } K \text{ in } M \]

Compliment 2.1.

(1) In this paper we are concerned with the following conditions $\mathcal{P}$ on $\varphi$:
   (i) (a) rel $L$ $\iff \varphi(L) \subset L$ \hspace{1cm} (b) keeping $L$ invariant $\iff \varphi(L) = L$ \hspace{1cm} (ii) $clb^f \leq k \iff clb^f \varphi \leq k$ in $\text{Diff}_c(M, M_O)_0$ \hspace{1cm} ($k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$)
   These conditions will play important roles in the estimates of $cl$ and $clb^f$ in Sections 3, 4.

(2) When $K$ has the weak neighborhood absorption property in $O$, we set
   \[ clb^f(K; O) := \min\{k \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \mid (*) \} : \]
   \[ (*)_k K \text{ has the weak neighborhood absorption property in } O \text{ with } clb^f \leq k. \]

(3) In [I](1) consider the case where the condition $\mathcal{P}$ is “rel $L$” or “keeping $L$ invariant”. If $C \in \mathcal{K}(O \cap \text{Int } M)$ and the absorbing diffeomorphism $\varphi$ is constructed as a truncation of a flow which keeps $L$ invariant, then we can cut off the vector field associated to the flow around the boundary and modify $\varphi$ so that $\varphi \in \text{Diff}_c(M, \partial M \cup M_O)_0$ with still keeping the condition $\mathcal{P}$.

(4) In [II](2), if $K \in \mathcal{K}(O \cap \text{Int } M)$, then the isotopy extension theorem is used to modify $\psi$ so that $\psi \in \text{Diff}_c(M, \partial M \cup M_O)_0$ (cf. Lemma 2.7).

(5) The properties defined in Definition 2.1 are preserved by appropriate diffeomorphisms. For example, consider the following condition on $(M, O, K, C)$:
   \[ (*) \quad C \in \mathcal{K}(O) \text{ is weakly absorbed to } K \text{ in } O \subset M \text{ with } clb^f \leq k. \]
   Suppose $M'$ is another $n$-manifold, $O' \in \mathcal{O}(M')$ and $h : O \cong O'$ is a $C^\infty$ diffeomorphism. When $K \in \mathcal{K}(O)$, from Remark 2.2 it follows that $(M, O, K, C)$ satisfies $(*)$ iff $(M', O', h(K), h(C))$ satisfies $(*)$

Example 2.4. Suppose $M$ is an $n$-manifold possibly with boundary. We list some situations in which $K$ is strongly displaceable from $L$ in $M$.

(1) ([19, Lemma 2.1]) Suppose $K$ is a compact $k$-dimensional stratified subset of $\text{Int } M$ and $L$ is an $\ell$-dimensional stratified subset of $M$. 

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Lemma 2.3. \( K \) is weakly absorbed to \( O \) in and \( \chi \in \text{Diff}_c(M,M_O) \) such that \( \chi(K) \subset \text{Int}_M U \) and \( \chi(K) = K \). We can find a compact neighborhood \( V \) of \( K \) in such that \( \chi(V) \subset U \). Then, \( V \) is weakly absorbed to \( K \) in \( O \) with \( clb^f \leq k \). In fact, given any neighborhood \( W \) of \( K \) in \( O \). Since \( \chi(W) \) is a neighborhood of \( K \) in \( O \), the choice of \( U \) implies the existence of \( \varphi \in \text{Diff}_c(M,M_O) \) such that

(i) If \( k + \ell < n \), then \( K \) is strongly displaceable from \( L \).
(ii) In the case \( k + \ell = n \):

(a) If \( K \subset O \subset \mathcal{O}(M) \) and \( Cl_M(K - K^{(k-1)}) \) is displaceable from \( Cl_M(L - L^{(\ell-1)}) \) in \( O \), then \( K \) is displaceable from \( L \) in \( O \).
(b) If \( Cl_M(K - K^{(k-1)}) \) is strongly displaceable from \( Cl_M(L - L^{(\ell-1)}) \) in \( M \), then \( K \) is strongly displaceable from \( L \) in \( M \).

2. Suppose \( K \in \mathcal{K}(\text{Int} M) \) and \( L \in \mathcal{F}(M) \).

(i) If \( U \) is an open \( n \)-disk in \( M \) with \( K \subset U \not\subseteq L \), then \( K \) is displaceable from \( L \) in \( U \).
(ii) If \( K \) has arbitrarily small open \( n \)-disk neighborhoods in \( M \) which is not included in \( L \), then \( K \) is strongly displaceable from \( L \) in \( M \).

3. If \( L \) is a submanifold of \( M \), \( K \) is a compact subset of \( L \) and the normal bundle of \( L \) in \( M \) admits a non-vanishing section over \( K \), then \( K \) is strongly displaceable from \( L \).

We list some basic facts on the absorption/displacement property. The next lemma follows directly from the definitions (and a simple argument using the collar for (1)(ii)).

Lemma 2.2. In Setting 2.1:

1. \( K_1, K_2, K_3 \in \mathcal{K}(O) \). If \( K_1 \) is (weakly) absorbed to \( K_2 \) in \( O \) rel \( L \) and \( K_2 \) is (weakly) absorbed to \( K_3 \) in \( O \) rel \( L \), then \( K_1 \) is (weakly) absorbed to \( K_3 \) in \( O \) rel \( L \).

2. \( K \) has the weak absorption property in \( O \subset M \) (keeping \( K \) invariant), then \( K \) has the weak absorption property in \( \tilde{O} \subset \tilde{M} \) (keeping \( K \) invariant).

3. If \( K \) is displaceable from \( K \cup L \) in \( O \), then a sufficiently small compact neighborhood \( K_0 \) of \( K \) in \( O \) is displaceable from \( K_0 \cup L \).

4. \( K \) is weakly absorbed to \( K' \in \mathcal{K}(O) \) in \( O \) and \( K' \) is displaceable from \( L \) in \( O \), then \( K \) is displaceable from \( L \) in \( O \).

5. If \( K \) is weakly absorbed to \( K' \in \mathcal{K}(O) \) in \( O \) rel \( L \) and \( K' \) is displaceable from \( K' \cup L \) in \( O \), then \( K \) is displaceable from \( K \cup L \) in \( O \).

6. If \( K \) is displaceable from \( L \) in \( O \) and \( K \) has the weak absorption property in \( O \) keeping \( K \) and \( L \) invariant, then \( K \) is strongly displaceable from \( L \) in \( O \).

Lemma 2.3. In Setting 2.3. Suppose \( K \) has the weak neighborhood absorption property in \( O \).

1. Suppose \( K' \in \mathcal{K}(O) \) is weakly absorbed to \( K \) in \( O \) keeping \( K \) invariant.

(i) If \( K \) has the weak neighborhood absorption property in \( O \) with \( clb^f \leq k \), then some compact neighborhood \( V \) of \( K' \) in \( O \) is weakly absorbed to \( K \) in \( O \) with \( clb^f \leq k \).

(ii) If \( K' \subset K \), then \( clb^f(K';O) \leq clb^f(K;O) \).

2. If \( K \) has the weak neighborhood absorption property in \( O \) with \( clb^f \leq k \) and the weak absorption property in \( O \) keeping \( K \) invariant, then \( K \) has the weak absorption property in \( O \) with \( clb^f \leq k \).

Proof. (1) (i) By the assumption \( K \) admits a compact neighborhood \( U \) in \( O \) which is weakly absorbed to \( K \) in \( O \) with \( clb^f \leq k \). By the assumption on \( K' \), there exists \( \chi \in \text{Diff}_c(M,M_O) \) such that \( \chi(K') \subset \text{Int}_M U \) and \( \chi(K) = K \). We can find a compact neighborhood \( V \) of \( K' \) in such that \( \chi(V) \subset U \). Then, \( V \) is weakly absorbed to \( K \) in \( O \) with \( clb^f \leq k \). In fact, given any neighborhood \( W \) of \( K \) in \( O \). Since \( \chi(W) \) is a neighborhood of \( K \) in \( O \), the choice of \( U \) implies the existence of \( \varphi \in \text{Diff}_c(M,M_O) \) such
that \( \varphi(U) \subset \varphi(W) \) and \( clb^f \varphi \leq k \) in \( Diff_c(M, M_O)_0 \). Then, \( \psi := \chi^{-1} \varphi \chi \in Diff_c(M, M_O)_0 \) satisfies the required conditions \( \psi(K') \subset W \) and \( clb^f \psi = clb^f(\chi^{-1} \varphi \chi) = clb^f \varphi \leq k \) in \( Diff_c(M, M_O)_0 \).

The remaining statements readily follow from (1)(i).

\[ \square \]

**Setting 2.2.** Suppose \( M \) is an \( n \)-manifold possibly with boundary, \( 1 \leq r \leq \infty \), \( r \neq n + 1 \), \( O \in \mathcal{O}(Int \ M) \), \( L \in \mathcal{F}(M) \) and \( k \in \mathbb{Z}_{\geq 0} \).

Consider the following statements:

- (\#) \( cld Diff^r_c(M, M_O)_0 \leq 2 \) and any \( f \in Diff^r_c(M, M_O)_0 \) has a factorization \( f = gh \) for some \( g \in Diff^r_c(M, M_O)_0 \) and \( h \in Diff^r_c(M, M_O \cup L)_0 \).
- (b) \( (1) \) (i) \( cld Diff^r_c(M, M_O)_0 \leq 2 \), (ii) \( clb^f \| Diff^r_c(M, M_O)_0 \leq 2k + 1 \).
- \( (2) \) any \( f \in Diff^r_c(M, M_O)_0 \) has a factorization \( f = gh \) such that \( g \in Diff^r_c(M, M_O)_0 \) and \( clb^f(g) \leq 2k \) in \( Diff_c^r(M, M_O)_0 \), \( h \in Diff^r_c(M, M_O \cup L)_0 \) and \( clb^f(h) \leq 1 \) in \( Diff^r_c(M, M_O \cup L)_0 \).

**Lemma 2.4.** In Setting 2.2: Suppose \( f \in Diff^r_c(M, M_O)_0 \).

[I] There exists a factorization \( f = gh \) for some \( g \in Diff^r_c(M, M_O)_0 \) and \( h \in Diff^r_c(M, M_O \cup L)_0 \) if there exists \( F \in Isot^r_c(M, M_O)_0 \) with \( F_1 = f \), \( K \in K(O) \) and a compact neighborhood \( U \) of \( K \) in \( O \) which satisfy one of the following conditions (1) and (2):

- (1) (i) \( \text{supp} F \) is absorbed to \( IntMU - K \) in \( O \) rel \( L \), (ii) \( U \) is weakly absorbed to \( K \) in \( O \).
- (2) \( K \) is strongly displaceable from \( L \) in \( M \).

Then, \( clb^f(f) \leq 2k + 1 \) in \( Diff^r_c(M, M_O)_0 \) and in the factorization \( f = gh \) we can take \( g \) and \( h \) with \( clb^f(g) \leq 2k \) in \( Diff^r_c(M, M_O)_0 \) and \( clb^f(h) \leq 1 \) in \( Diff^r_c(M, M_O \cup L)_0 \).

**Proof.** [I] (1) & [II] (Alterations for [II] are shown in parentheses.) By the condition (i)(a) there exist \( \chi \in Diff_c(M, M_O)_0 \) such that \( \chi(\text{supp} F) \subset IntMU - K \) and \( \chi(L) \subset L \). Consider the open neighborhood \( U_0 := IntMU - \chi_1(\text{supp} F) \) of \( K \) in \( IntMU \). By the condition (ii) there exist \( \eta \in Diff_c(M, M_{U_0})_0 \) such that \( \eta(K) \cap L = \emptyset \). Take a compact neighborhood \( W_0 \) of \( \eta(K) \) in \( U_0 = L \). By the condition (i)(b) \( (ii)(b)' \) there exists \( \psi' \in Diff_c(M, M_O)_0 \) such that \( \psi'(U) \subset \eta^{-1}(IntMU_{W_0}) \) (and \( clb^f(\psi') \leq k \) in \( Diff_c(M, M_O)_0 \)).

Let \( \psi := \eta(\psi')^{-1} \in Diff_c(M, M_O)_0 \). Then we have \( \psi(U) \subset IntMU_{W_0} \) (and \( clb^f(\psi) = clb^f(\eta(\psi')^{-1}) = clb^f(\psi') \leq k \) in \( Diff_c(M, M_O)_0 \)).

Consider \( V := \chi^{-1}(U) \), \( W := \chi^{-1}(W_0) \in K(O) \) and \( \varphi := \chi^{-1} \psi \chi \in Diff_c(M, M_O)_0 \). It follows that \( W \subset IntMV - L \subset O - L \), \( \text{supp} F \subset IntMV - W \), \( \varphi(V) \subset W \) (and \( clb^f(\varphi) = clb^f(\chi^{-1} \psi \chi) = clb^f(\psi') \leq k \) in \( Diff_c(M, M_O)_0 \)).

Since \( Diff^r_c(M, M_{W_0})_0 \subset Diff^r_c(M, M_O \cup L)_0 \), the conclusion follows from Lemmas [2.1].

For (2) By the condition (iii) there exists \( \varphi \in Diff(M, M_{U_0})_0 \) such that \( \varphi(K) \cap (K \cup L) = \emptyset \). We show that the condition (1) is satisfied for \( K_1 := \varphi_1(K) \in K(IntMU) \) instead of \( K \).

Since \( IntMU - K_1 \) is a neighborhood of \( K \), by the condition (i)(a) \( \text{supp} F \) is absorbed to \( IntMU - K_1 \) in \( O \) rel \( L \). From the condition (i)(b) and Lemma [2.2](1) it follows that \( U \) is weakly absorbed to \( K_1 \) in
O. Since \( K_1 \cap L = \emptyset \), obviously \( K_1 \) is strongly displaceable from \( L \) in \( M \). Therefore, the assertion follows from [I] (1).

\[ \square \]

**Lemma 2.5.** In Setting 2.2:

[I] The statement (♯) holds, if the following condition is satisfied:

\( \star \) There exists \( O_0 \in \mathcal{O}(M) \) with \( O_0 \cap \operatorname{Int} M = O, K_0 \in \mathcal{K}(O_0) \) and a compact neighborhood \( U_0 \) of \( K_0 \) in \( \widetilde{O}_0 \) such that

\( (i) \) (a) \( U_0 \) is weakly absorbed to \( K_0 \) in \( \widetilde{O}_0 \),

(b) \( K_0 \) has the weak absorption property in \( O \) rel \( L \) and

(ii) \( K_0 \) is displaceable from \( K_0 \cup L \cup (\partial M \times [0,1]) \) in \( \operatorname{Int} \tilde{M} U_0 \).

[II] The statement (♭) holds, if in [I] \( K_0 \) and \( U_0 \) satisfy the following stronger condition:

\( (i) \) (a') \( U_0 \) is weakly absorbed to \( K_0 \) in \( \widetilde{O}_0 \) with \( \operatorname{clb} f \leq k \) in \( \operatorname{Diff}_c(\tilde{M}, \tilde{M}_{\tilde{O}_0})_0 \).

**Proof.** (Alterations for [II] are shown in parentheses.) Given any \( f \in \operatorname{Diff}_c^r(M, M_0) \). There exists \( F \in \operatorname{Isot}_c^r(M, M_0) \) with \( F_1 = f \). By the condition (ii) there exists \( \psi \in \operatorname{Diff}_c(\tilde{M}, \tilde{M}_{\tilde{O}_0})_0 \) such that \( K := \psi(K_0) \subset \operatorname{Int} M - (K_0 \cup L) \). Note that \( K \in \mathcal{K}(O) \). Since \( \operatorname{supp} F \in \mathcal{K}(O) \) and \( K_0 \subset (\operatorname{Int} \tilde{M} U_0 - K) \cap M \in \mathcal{O}(O_0) \), by the condition (i)(b) there exists \( h \in \operatorname{Diff}_c(M, M_0)_0 \) such that \( h(\operatorname{supp} F) \subset \operatorname{Int} \tilde{M} U_0 - K \) and \( h(L) \subset L \). We can push \( U_0 \) inward using a bicollar of \( \partial O_0 \) in \( \widetilde{O}_0 \) to obtain \( \varphi \in \operatorname{Diff}_c(\tilde{M}, \tilde{M}_{\tilde{O}_0} \cup H_1(\operatorname{supp} F) \cup K)_0 \) such that \( U := \varphi(U_0) \subset O \). Then \( U \) is a compact neighborhood of \( K \) in \( O \).

The conclusions follow from Lemma 2.4 once we show that \( F, K \) and \( U \) satisfies the conditions in Lemma 2.4[I](1) (and [II]). Since \( h(\operatorname{supp} F) \subset \operatorname{Int} M U - K \), it follows that \( \operatorname{supp} F \) is absorbed by \( h \) to \( \operatorname{Int} M U - K \) in \( O \) rel \( L \). Since \( K \cap L = \emptyset \), obviously \( K \) is strongly displaceable from \( L \) in \( M \).

It remains to show that \( U \) is weakly absorbed to \( K \) in \( O \) (with \( \operatorname{clb} f \leq k \)). This follows from the assumption (i)(a) ((i)(a')), Complement 2.1 and the following observation. There exists a collar \( E \cong \partial O_0 \times I \) of \( \partial O_0 \) in \( O_0 - U \) and a \( C^\infty \) diffeomorphism \( \theta : \widetilde{O}_0 \cong O \) such that \( \theta = \text{id} \) on \( O - E \). Then \( (U, K) \) corresponds to \( (U_0, K_0) \) under the \( C^\infty \) diffeomorphism \( \theta \varphi \psi : \widetilde{O}_0 \cong O \). This completes the proof. \( \square \)

Finally we remove the conditions on the neighborhood \( U_0 \) from Lemma 2.5 to obtain a more practical criterion.

**Lemma 2.6.** In Setting 2.2: The statement (♭) holds, if one of the conditions [A] and [B] is satisfied.

[A] There exist \( O_0 \in \mathcal{O}(M) \) with \( O = O_0 \cap \operatorname{Int} M \) and \( K \in \mathcal{K}(O_0) \) such that

\( (i) \) \( \operatorname{clb} f \leq k \).

\( \star \) \( K \) satisfies one of the conditions \((\star)_1 \) and \((\star)_2 \):

\( (\star)_1 \) (i) \( K \) has the weak absorption property (a) in \( O \) rel \( L \) and (b) in \( O_0 \) keeping \( K \) invariant.

(ii) \( K \) is displaceable from \( K \cup L \cup (\partial M \times [0,1]) \) in \( \widetilde{O}_0 \).

\( (\star)_2 \) (i) \( K \) has the weak absorption property (a) in \( O \) rel \( L \) and (b) in \( O_0 \).

(ii) \( K \) is strongly displaceable from \( K \cup L \cup (\partial M \times [0,1]) \) in \( \tilde{M} \).

[B] There exist \( K \in \mathcal{K}(O) \) such that

\( (i) \) \( \operatorname{clb} f \leq k \).

\( \star \) \( K \) satisfies one of the conditions \((\star)_1 \) and \((\star)_2 \):

\( (\star)_1 \) (i) \( K \) has the weak absorption property in \( O \) rel \( L \) and keeping \( K \) invariant.

(ii) \( K \) is displaceable from \( K \cup L \) in \( O \).
exists a vector field \( Y \) such that

\[
\nabla \partial M (\partial M \times [0,1]) \text{ in } \text{Int}_\tilde{M} U.
\]

In the study of commutator

\[
\begin{align*}
G & = GH, \\
H & = G^{-1} F 
\end{align*}
\]

This subsection includes some remarks on factorization of isotopies. In the study of commutator

\[
\begin{align*}
\tilde{M} & \in \text{Diff}_c(\tilde{M}, \tilde{M}_0) \, 0, \\
\tilde{O} & \in \partial M \cap \tilde{O}_0
\end{align*}
\]

The condition \( (\dagger)(i) \) is included as \((*)_1(i)(a)\) and \((*)_2(i)(a)\). For the condition \( (\dagger)(ii) \), first note that the condition \( (\ddagger) \) means that

\[ K \text{ has the weak neighborhood absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

\[ K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

\[ K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

This implies \( (\dagger)(ii)(b) \).

\[ K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

\[ K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

The condition \( (\ddagger) \) is a special case of \( [A] \) (taking \( O_0 = O \)).

\[ (\ddagger) \text{ case : By } (*)_1(ii) \text{ there exists a compact neighborhood } U \text{ of } K \text{ in } \tilde{O}_0 \text{ for which holds } (\ddagger)(ii)(a). \]

By Lemma 2.2(1)(ii) \( K \) has the weak absorption property in \( \tilde{O}_0 \) keeping \( K \) invariant. Hence, by Lemma 2.3(2) \( (\ddagger) \) it follows that

\[ K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

This implies \( (\ddagger)(ii)(b) \).

\[ K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

\[ K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

The condition \( (\ddagger) \) is a special case of \( [A] \) (taking \( O_0 = O \)).

2.5. Factorization of isotopies on manifolds.

This subsection includes some remarks on factorization of isotopies. In the study of commutator length of diffeomorphisms, modification and factorization of isotopies are important subjects. The next factorization lemma is repeatedly applied in the subsequent sections (cf. [19] Lemma 2.3). Recall that \( \pi_M : M \times I \to M \) is the projection onto \( M \).

**Lemma 2.7.** Suppose \( M \) is a compact \( n \)-manifold possibly with boundary, \( K, L \in \mathcal{K}(M) \) and \( F \in \text{Isot}'(M, \partial M \cup (K \cap L))_0 \) satisfies \( \pi_M F((K - L) \times I) \cap L = \emptyset \). Then there exists \( G \in \text{Isot}'(M, \partial M \cup L)_0 \) such that \( G = F \) on \([a \text{ neighborhood of } K] \times I \). It induces a factorization \( F = GH \), where \( H = G^{-1} F \in \text{Isot}'(M, \partial M \cup K)_0 \).

The isotopy \( G \) in Lemma 2.7 is constructed as follows: Let \( X \) be the velocity vector field of \( F \). There exists a vector field \( Y \) on \( M \times I \) such that \( Y = X \) on a neighborhood of \( F(K \times I) \) and \( Y = \frac{\partial}{\partial t} \) on \([a \text{ neighborhood of } \partial M \cup L] \times I \). The isotopy \( G \) is obtained by integrating \( Y \) on \( M \times I \).

**Lemma 2.8.** Suppose \( M \) is an open \( n \)-manifold, \( F \in \text{Isot}'(M)_0 \) and \( R \) is a cofinal subcollection of \( \mathcal{SM}_c(M) \).

1. For any \( K \in \mathcal{SM}_c(M) \) there exists \( L \in R \) such that \( F(K \times I) \subset L \times I \) and \( F(M_L \times I) \subset M_K \times I \).
2. For any \( K_k \in \mathcal{SM}_c(M) \) \( (k \geq 1) \) there exists an exhausting sequence \( \{M_k\}_{k \geq 1} \) of \( M \) such that \( K_k \subset M_k \subset \mathcal{R} \) \( (k \geq 1) \) and \( F(M_{4k,4k+1} \times I) \subset M_{4k-1,4k+2} \times I \) \( (k \geq 0) \). (In the inductive construction of \( \{M_k\}_{k \geq 1} \), given \( M_k \), we can take \( M_{k+1} \) arbitrarily large.)
3. Let \( \{M_k\}_{k \geq 1} \) be an exhausting sequence of \( M \) such that \( F(M_{4k,4k+1} \times I) \subset M_{4k-1,4k+2} \times I \) \( (k \geq 0) \) and set \( M' := \bigcup_{k \geq 0} M_{4k,4k+1} \) and \( M' := \bigcup_{k \geq 0} M_{4k,4k+1} \). Then there exists a factorization \( F = GH \) for some \( G \in \text{Isot}'(M, M')_0 \) and \( H \in \text{Isot}'(M, M'')_0 \).

\[ (\ast)_2 (i) K \text{ has the weak absorption property in } O \text{ rel } L. \]

\[ (\ast)_2 (ii) K \text{ is strongly displaceable from } K \cup L \text{ in } M. \]

**Proof.** [A] By Lemma 2.5 it suffices to show that \( K \) satisfies the following conditions.

\[ (\dagger) (i) K \text{ has the weak absorption property in } O \text{ rel } L, \]

\[ (\dagger)(ii) \text{ there exists a compact neighborhood } U \text{ of } K \text{ in } \tilde{O}_0 \text{ such that } \]

\[ (a) K \text{ is displaceable from } K \cup L \cup (\partial M \times [0,1]) \text{ in } \text{Int}_\tilde{M} U, \]

\[ (b) U \text{ is weakly absorbed to } K \text{ in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

The condition \( (\dagger)(i) \) is included as \((*)_1(i)(a)\) and \((*)_2(i)(a)\). For the condition \( (\dagger)(ii) \), first note that the condition \( (\ddagger) \) means that

\[ (\ddagger) K \text{ has the weak neighborhood absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

\[ (\ddagger) K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]

\[ (\ddagger) K \text{ has the weak absorption property in } \tilde{O}_0 \text{ with } \text{clb} F \leq k \text{ in } \text{Diff}_c(\tilde{M}, \tilde{M}_0) 0. \]
Proof.

(1) Since $F(K \times I), F^{-1}(K \times I) \in \mathcal{K}(M \times I)$, there exists $L \in \mathcal{R}$ such that $F(K \times I) \cup F^{-1}(K \times I) \subseteq L \times I$. Then $L$ satisfies the required conditions.

(2) There exist $M_1, M_2, M_3 \in \mathcal{R}$ such that

$$K_i \in M_i \ (i = 1, 2, 3), \ F(M_1 \times I) \subseteq M_2 \times I \text{ and } M_2 \subseteq M_3.$$ 

Inductively, for $k \geq 1$, given $M_{4k-1} \in \mathcal{R}$, by (1) we can find $M_{4k+i} \in \mathcal{R} \ (i = 0, 1, 2, 3)$ such that

$$K_{4k+i} \in M_{4k+i} \ (i = 0, 1, 2, 3), \ F(M_{4k+i} \times I) \subseteq M_{4k-1} \times I, \ M_{4k} \subseteq M_{4k+1},$$

$$F(M_{4k+1} \times I) \subseteq M_{4k+2} \times I, \ M_{4k+2} \subseteq M_{4k+3}.$$ 

The sequence $\{M_k\}_{k \geq 1}$ is obtained by iteration of this procedure.

(3) For each $k \geq 0$, since $F(M_{4k,4k+1} \times I) \subseteq M_{4k-1,4k+2} \times I$, by the isotopy extension theorem we obtain $G^k \in \text{Isot}^r(M_{4k-1,4k+2}; \partial) \subseteq F$ on a neighborhood of $M_{4k,4k+1} \times I$. The isotopy $G \in \text{Isot}^r(M; M')_0$ is defined by $G = G^k$ on $M_{4k-1,4k+2} \ (k \geq 0)$. Then $G = F$ on a neighborhood of $M^n \times I$ and $H := G^{-1}F \in \text{Isot}^r(M; M''_0)$.

Here we give a simple application of Lemma 2.8.

Lemma 2.9. Suppose $M$ is an open n-manifold.

(1) $\text{clb} \text{Diff}^r(M)_0 \leq \max\{\ell + m, 2m\}$ and $\text{clb} \text{Diff}^r(M)_0 \leq \ell$

if $M$ has an exhausting sequence $\mathcal{L} = \{L_{i}\}_{i \geq 1}$ such that

$$\text{clb} \text{Diff}^r(L_i; \partial)_0 \leq \ell \ (1 \leq i < \infty) \text{ and } \text{clb} \text{Diff}^r(L_{ij}; \partial)_0 \leq m \ (1 \leq i < j < \infty).$$

(2) $\text{clb}^d \text{Diff}^r(M)_0 \leq \max\{\ell + m, 2m\}$ and $\text{clb}^d \text{Diff}^r(M)_0 \leq \ell$

if $M$ has an exhausting sequence $\mathcal{L} = \{L_{i}\}_{i \geq 1}$ such that

$$\text{clb}^d \text{Diff}^r(L_i; \partial)_0 \leq \ell \ (1 \leq i < \infty) \text{ and } \text{clb}^d \text{Diff}^r(L_{ij}; \partial)_0 \leq m \ (1 \leq i < j < \infty).$$

Proof. The assertions for $\text{Diff}^r(M)_0$ are obvious. To obtain the estimates for $\text{Diff}^r(M)_0$, let $q = \text{cl}$ in (1) and $q = \text{clb}^d$ in (2). Given any $f \in \text{Diff}^r(M)_0$. Take $F \in \text{Isot}^r(M)_0$ with $F_1 = f$. By Lemma 2.8 we have

(i) a subsequence $M_k = L_{i(k)} \ (k \geq 1)$ of $\mathcal{L}$ such that $F(M_{4k,4k+1} \times I) \subseteq M_{4k-1,4k+2} \times I \ (k \geq 0)$ and

(ii) a factorization $F = GH$ for some $G \in \text{Isot}^r(M, M')_0$ and $H \in \text{Isot}^r(M, M'')_0$, where

$$M' := \bigcup_{k \geq 0} M_{4k+2,4k+3} \text{ and } M'' := \bigcup_{k \geq 0} M_{4k,4k+1}.$$ 

Hence, the assumption implies that $q(G_1) \leq \max\{\ell, m\}, q(H_1) \leq m$ and $q(f) \leq \max\{\ell, m\} + m$.

Proposition 2.1. Suppose $W$ is a compact n-manifold with boundary and $1 \leq r \leq \infty, r \neq n + 1$.

(1) $\text{clb} \text{Diff}^r(\text{Int} W)_0 \leq \max\{\text{clb} \text{Diff}^r(W; \partial)_0, 2\} + 2$.

(2) $\text{clb}^d \text{Diff}^r(\text{Int} W)_0 \leq \max\{\text{clb}^d \text{Diff}^r(W; \partial)_0, m\} + m,$

where $m := \text{clb}^d \text{Diff}^r(\partial W \times I; \partial)_0 < \infty$ (cf. Example 2.3(2)).

Proof. Let $q = \text{cl}$ in (1) and $q = \text{clb}^d$ in (2). Take a collar $N \cong \partial W \times I$ of $\partial W = \partial W \times \{0\}$ in $W$ and consider the exhausting sequence $\mathcal{L} = \{L_i\}_{i \geq 1}$ in $\text{Int} W$ defined by $L_i := W_N \cup (\partial W \times [1/i, 1]) \ (i \geq 1)$. Since $L_i \ (1 \leq i < \infty)$ are diffeomorphic to $W$ and $L_{ij} \ (1 \leq i < j < \infty)$ are diffeomorphic to $\partial W \times I$, it follows that $qd \text{Diff}^r(L_i; \partial)_0 = qd \text{Diff}^r(W; \partial)_0 \ (1 \leq i < \infty)$ and $qd \text{Diff}^r(L_{ij}; \partial)_0 = qd \text{Diff}^r(\partial W \times I; \partial)_0 \ (1 \leq i < j < \infty)$. Hence the conclusion follows from Lemma 2.8 and Example 2.3.

Example 2.5. If $W$ is a compact n-manifold with boundary and has a handle decomposition with no handles of indices greater than $(n - 1)/2$, then $\text{clb} \text{Diff}^r(W; \partial)_0 \leq 2$ ([16] Theorem 4.1]) and hence $\text{clb} \text{Diff}^r(\text{Int} W)_0 \leq 4$ (cf [13] for portable manifolds and related topics).
To explain the aim of this section, first we recall the strategy in [4] [17] [18] [19] for estimate of $cl$ and $clbf$ (i.e., effective factorization to commutators) and understand the trouble which occurs in the bounded case.

Suppose $M$ is a closed $n$-manifold and $f \in \text{Diff}(M)_0$. For simplicity, we assume that $n = 2m + 1$ and consider any handle decomposition $\mathcal{H}$ of $M$, the gradient flow $\varphi_t$ induced from $\mathcal{H}$, the core $m$-skeleton $P$ and the dual $m$-skeleton $Q$. Then, by an argument of removing intersections between $Q$ and a track of $P$ under an isotopy it is shown that $f$ has a factorization $f = agh$ (up to conjugate), where $g \in \text{Diff}(M,Q)_0$, $h \in \text{Diff}(M,P)_0$ and $a \in \text{Diff}(M,M_D)^f$ for some $D \in \mathcal{B}(M)$. Since any compact subset of $M - Q$ is attracted to $P$ under the flow $\varphi_t$ and $P$ is displaceable from itself in $M - Q$, Basic Lemma 2.1 induces a factorization of $g$ into commutators. Similarly, since any compact subset of $M - P$ is attracted to $Q$ under the flow $\varphi_t^{-1}$ and $Q$ is displaceable from itself in $M - P$, Basic Lemma 2.1 induces a factorization of $h$ into commutators. This procedure yields a factorization of $f$ into commutators.

When $M$ is a compact $n$-manifold with nonempty boundary and $\mathcal{H}$ is a usual handle decomposition of $M$ (cf. Convention 3.1), the flow $\varphi_t$ is transverse to $\partial M$ and $Q$ intersects $\partial M$ transversaly, while $P$ is in $\text{Int} M$. Moreover, any compact subset of $\text{Int} M - P$ is attracted to the union of $Q$ and $\partial M$ by $\varphi_t^{-1}$. However, the union $Q \cup \partial M$ is not displaceable from itself even if we attach an outer collar to $M$, though each summand $Q$ and $\partial M$ is displaceable from $Q \cup \partial M$. Thus, the factorization argument breaks down at this point.

This observation means that, in order to apply Basic Lemma 2.1 to both $P$ and $Q$, we need a flow on $M$ with connecting between $P$ and $Q$ and keeping $\partial M$. In this section we see that any triangulation of $M$ induces such flows (Section 3.2). We also observe that, in the case of an open manifold $M$ with a handle decomposition $\mathcal{H}$, the above argument still works if we use a pair $(N, L)$ of compact submanifolds of $M$ such that $N \subset L$, $N$ is $\mathcal{H}$-saturated and $L$ is $\mathcal{H}^*$-saturated (cf. Propositions 3.1 [4] [11] [5.1]). It might be possible to unify these arguments by considering handle decompositions which are transverse to boundary (cf. Section 3.1). We postpone such a general approach to a succeeding work.

3.1. Handle decompositions of manifolds.

3.1.1. Submanifolds saturated with respect to a handle decomposition.

Convention 3.1. Suppose $M$ is a $C^\infty$ $n$-manifold possibly with boundary. In this article a handle decomposition $\mathcal{H}$ of $M$ means a locally finite family of $C^\infty$ handles in $M$ of index $k = 0, 1, \cdots, n$ which satisfies the following conditions: (i) $M$ is constructed from $\mathcal{H}$ by the handle attachment with starting from the 0-handles, that is, there exists a sequence of $n$-submanifolds $M_0 \subset M_1 \subset \cdots \subset M_n = M$, where $M_0$ is the union of the 0-handles in $\mathcal{H}$ and inductively $M_k$ is obtained by attaching the $k$-handles in $\mathcal{H}$ to $M_{k-1}$ (with corners being smoothed), (ii) any two handles in $\mathcal{H}$ of same index are disjoint and (iii) the handles are attached generically so that if $h, k \in \mathcal{H}$, $h \cap k \neq \emptyset$ and $\text{ind} h > \text{ind} k$, then the interior of the attaching region of $h$ intersects the interior of the coattaching region of $k$. Hence, in our convention $\partial M$ occurs only as a portion where no handles are attached.

Setting 3.1. Below we assume that $M$ is an $n$-manifold without boundary and $\mathcal{H}$ is a handle decomposition of $M$. By $\mathcal{H}^*$ we denote the dual handle decomposition of $M$ for $\mathcal{H}$. For $N \in \text{SM}(M)$, let $\mathcal{H}|_N := \{h \in \mathcal{H} \mid h \subset N\}$. We say that $N \in \text{SM}(M)$ is $\mathcal{H}$-saturated if $N = \bigcup \mathcal{H}|_N$ and $\mathcal{H}|_N$...
forms a handle decomposition of \( N \). Let \( \mathcal{SM}(M; \mathcal{H}) := \{ N \in \mathcal{SM}(M) \mid N \text{ is } \mathcal{H} \text{-saturated.} \} \) and \( \mathcal{SM}_c(M; \mathcal{H}) := \mathcal{SM}_c(M) \cap \mathcal{SM}(M; \mathcal{H}) \). We regard as \( \emptyset \in \mathcal{SM}_c(M; \mathcal{H}) \). These notations are also used for \( \mathcal{H}^* \).

The next lemma easily follows from the definitions.

**Lemma 3.1.** In Setting \([3.4]\):

1. For \( N \in \mathcal{SM}(M) \)
   i. \( N \) is \( \mathcal{H} \)-saturated. \( \iff \) (a) \( N = |\mathcal{H}|_N \) and
      (b) if \( h \in \mathcal{H}|_N \), \( k \in \mathcal{H} \), \( h \cap k \neq \emptyset \) and \( \text{ind } k < \text{ind } h \), then \( k \subset N \).
   ii. \( N \) is \( \mathcal{H}^* \)-saturated. \( \iff \) (a) \( N = |\mathcal{H}|_N \) and
      (b) if \( h \in \mathcal{H}|_N \), \( k \in \mathcal{H} \), \( h \cap k \neq \emptyset \) and \( \text{ind } h < \text{ind } k \), then \( k \subset N \).

2. (i) \( N_1, N_2 \in \mathcal{SM}(M; \mathcal{H}) \) \( \implies \) \( N_1 \cap N_2, N_1 \cup N_2 \in \mathcal{SM}(M; \mathcal{H}) \)
   (ii) \( N_1, N_2 \in \mathcal{SM}(M) \) are disjoint, \( N_1 \cup N_2 \in \mathcal{SM}(M; \mathcal{H}) \) \( \implies \) \( N_1, N_2 \in \mathcal{SM}(M; \mathcal{H}) \)
   (iii) (a) \( N \in \mathcal{SM}(M; \mathcal{H}) \) \( \implies \) \( M_N \in \mathcal{SM}(M; \mathcal{H}^*) \)
      (b) \( N \in \mathcal{SM}(M; \mathcal{H}^*) \) \( \implies \) \( M_N \in \mathcal{SM}(M; \mathcal{H}) \)
   (iv) Suppose \( L, N \in \mathcal{SM}(M) \) and \( L \subset N \).
      (a) \( N_L \in \mathcal{SM}(M; \mathcal{H}) \) \( \iff \) \( L \in \mathcal{SM}(M; \mathcal{H}^*) \) and \( N \in \mathcal{SM}(M; \mathcal{H}) \)
      (b) \( N_L \in \mathcal{SM}(M; \mathcal{H}^*) \) \( \iff \) \( L \in \mathcal{SM}(M; \mathcal{H}) \) and \( N \in \mathcal{SM}(M; \mathcal{H}^*) \)

3. For any \( L \in \mathcal{SM}_c(M; \mathcal{H}) \) there exists
   \( N_1 \in \mathcal{SM}_c(M; \mathcal{H}) \) with \( L \subset N_1 \) and \( N_2 \in \mathcal{SM}_c(M; \mathcal{H}^*) \) with \( L \subset N_2 \).

We need a more sophisticated version of Lemma \([2.8](2)\).  

**Lemma 3.2.** Suppose \( M \) is an open \( n \)-manifold, \( \mathcal{H} \) is a handle decomposition of \( M \), \( \mathcal{R} \) is a cofinal subcollection of \( \mathcal{SM}_c(M) \), \( K_k \in \mathcal{SM}_c(M) \) \( (k \geq 1) \) and \( \mathcal{F} \) is a locally finite family of compact subsets of \( M \). Then, for any \( F \in \text{Isot}^*(M)_0 \) the following hold:

[I] There exists an exhausting sequence \( \{ M_k \}_{k \geq 1} \) of \( M \) which satisfies the following conditions:

1. (i) \( K_k \subset M_k \in \mathcal{R} \) \( (k \geq 1) \) (ii) \( F(M_{4k-1,4k+1} \times I) \subset M_{4k-1,4k+2} \times I \) \( (k \geq 0) \)
2. There exist \( N'_k \in \mathcal{SM}_c(M; \mathcal{H}) \) and \( N''_k \in \mathcal{SM}_c(M; \mathcal{H}^*) \) \( (k \geq 0) \) such that for each \( k \geq 0 \)
   (a) \( M_{4k-1,4k+2} \subset N'_k \subset N''_k \subset M_{4k-2,4k+3} \) and
   (b) \( N''_k \cap N''_{k+1} = \emptyset \).
3. There exist \( L'_k \in \mathcal{SM}_c(M; \mathcal{H}) \) and \( L''_k \in \mathcal{SM}_c(M; \mathcal{H}^*) \) \( (k \geq 1) \) such that for each \( k \geq 1 \)
   (a) \( M_{4k-3,4k} \subset L'_k \subset L''_k \subset M_{4k-4,4k+1} \) and
   (b) \( K_1 \cap L'_1 = \emptyset, \; L''_1 \cap L''_{k+1} = \emptyset \).

Here, the notation \( L \subset N \) means that \( St(L, \mathcal{F}) \subset N \).

[II] There exists a factorization \( F = GH \) for some \( G \in \text{Isot}^*(M)_{0} \) and \( H \in \text{Isot}^*(M,M''_0) \), where \( M' := \bigcup_{k \geq 0} M_{4k+2,4k+3} \) and \( M'' := \bigcup_{k \geq 0} M_{4k,4k+1} \).

In addition, we have the following estimates for the commutator lengths \( q = cl \) and \( cl_1^d \):

1. \( q(G_1) \leq \ell_1 \) in \( \text{Diff}^r(M)_0 \) if \( qd(\text{Diff}^r(M,M_{4k-1,4k+2})_0, \text{Diff}^r(M,M''))_0 \leq \ell_1 \) for each \( k \geq 0 \).
2. \( q(H_1) \leq \ell_2 \) in \( \text{Diff}^r(M)_0 \) if \( qd(\text{Diff}^r(M,M_{4k-3,4k})_0, \text{Diff}^r(M,M''))_0 \leq \ell_2 \) for each \( k \geq 1 \).
3. \( q(F_1) \leq \ell_1 + \ell_2 \) in \( \text{Diff}^r(M)_0 \) if \( q(G_1) \leq \ell_1 \) and \( q(H_1) \leq \ell_2 \).

**Proof.** [I] We repeat the proof of Lemma \([2.8](2)\) to achieve the condition (1). In this inductive construction of \( \{ M_k \}_{k \geq 1} \) we can take \( M_{k+1} \) arbitrarily large for a given \( M_k \). To achieve the conditions (2), (3), in the
sequence \( \{M_k\}_{k \geq 1} \) we insert the following compact \( n \)-submanifolds of \( M \), based on Lemma \( 3.1(3) \).

\[
N'_{k,-}, N''_{k,+}, N''_{k,-}, N''_{k,+} \ (k \geq 0), \quad L'_{k,-}, L'_{k,+}, L''_{k,-}, L''_{k,+} \ (k \geq 1)
\]

We choose these submanifolds so to satisfy the following conditions:

\[(a) \quad N'_{0,-} = N''_{0,-} = \emptyset, \quad M_{4k+2} \subseteq_{SLF} N'_{k,+} \subseteq_{SLF} N''_{k,+} \subseteq_{SLF} N''_{k+1,-} \subseteq_{SLF} N'_{k+1,-} \subseteq_{SLF} M_{4k+3} \ (k \geq 0) \]

\[(b) \quad K_1 \subseteq_{SLF} L'_{1,-} \subseteq_{SLF} M_1, \quad M_{4k} \subseteq_{SLF} L'_{k,+} \subseteq_{SLF} L''_{k,+} \subseteq_{SLF} L''_{k+1,-} \subseteq_{SLF} L'_{k+1,-} \subseteq_{SLF} M_{4k+1} \ (k \geq 1) \]

\[(ii) \quad N''_{k,+}, N''_{k,-} \in SM_c(M; \mathcal{H}), \quad N'_{k,-}, N''_{k,+} \in SM_c(M; \mathcal{H}^*) \ (k \geq 0) \]

There exists a Morse function with the properties that the gradient flow satisfies the following properties:

1. For each \( k \geq 0 \) consider \( g_k \in \text{Diff}^r(M, M_{4k-4k+2})_0 \) defined by \( g_k = G_1 \) on \( M_{4k-4k+2} \). Then \( q(g_k) \leq \ell_1 \) in \( \text{Diff}^r(M, M_{N''_k})_0 \) by the assumption. Since \( G_1 \) is “the discrete sum” of \( g_k \ (k \geq 0) \) by \( [I](2) \), it follows that \( q(G_1) \leq \ell_1 \) in \( \text{Diff}^r(M)_0 \).

The statement (2) is shown by a similar argument for \( H \).

3.1.2. The gradient flow and the complex associated to a handle decomposition.

Setting 3.2. Suppose \( M \) is an \( n \)-manifold without boundary and \( \mathcal{H} \) is a handle decomposition of \( M \).

Each handle of index \( m \) in \( \mathcal{H} \) has the core \( m \)-disk and a cocore \((n-m)\)-disk. These disks intersect at the center point transversely. There exists a Morse function \( f \) on \( M \) and a Riemannian metric on \( M \) with the properties that \( f \) has a critical point of index \( m \) at the center point of each handle of index \( m \) and that with respect to the induced (downward) gradient flow \( \varphi_t \) on \( M \) the unstable manifold of each critical point of \( f \) of index \( m \) is an open \( m \)-disk and the total of those disks provides \( M \) with a structure of a \( C^\infty \) CW-complex, which we denote by \( P_H \) and call the core complex of \( \mathcal{H} \) for short. As usual \( P_H^{(k)} \) denotes the \( k \)-skeleton of the complex \( P_H \). The core complex \( P_H^* \) of \( \mathcal{H}^* \) is obtained by considering the stable manifolds of each critical point of \( f \) under the flow \( \varphi_t \). The flow \( \varphi_t \) keeps each cell and skeleton of \( P_H \) and \( P_H^* \) invariant. (When \( \partial M \neq \emptyset \), the partial gradient flow \( \varphi_t \ (t \geq 0) \) is defined and it is transverse to \( \partial M \).) Each \( k \)-handle of \( \mathcal{H} \) determines a unique \( k \)-cell of \( P_H \) and a unique \((n-k)\)-cell of \( P_H^* \). To each \( k \)-cell \( \sigma \) of \( P_H \) the associated \((n-k)\)-cell of \( P_H^* \) is denoted by \( \sigma^* \) and called the dual cell to \( \sigma \).

Setting 3.2. Let \( P = P_H^{(k)} \) and \( Q = P_H^{(n-k-1)} \ (k = 0, 1, \cdots, n) \).

The gradient flow \( \varphi_t \) has the following properties: \( \varphi_t(M - Q) = M - Q \ (t \in \mathbb{R}) \) and \( M - Q \) is attracted to \( P \).

Setting 3.2. Suppose \( N \in SM_c(M; \mathcal{H}) \) and \( P_N := (P_{\mathcal{H}(N)})^{(k)} \) (the \( k \)-skeleton of the core complex of \( \mathcal{H}(N) \)).

Then \( P_N \) is a finite subcomplex of \( P \) and \( P_N \subset \text{Int} N \). The gradient flow \( \varphi_t \) satisfies the following conditions: The flow \( \varphi_t \) is transverse to \( \partial N \), \( \varphi_t(N) \subset N \), \( \varphi_t(N - Q) \subset N - Q \) and \( N - Q \) is attracted to \( P_N \). Therefore, the following holds:
(⁎) $P_N$ has the weak absorption property in Int $N - Q$ keeping $P$ invariant.

More precisely, for any $C \in \mathcal{K}(\text{Int } N - Q)$ and any neighborhood $U$ of $P_N$ in Int $N - Q$ there exists $H \in \text{Isot}(M; M_N \cup Q \cup P_N)_0$ such that $H_1(C) \subset U$, $H_1(P) = P$ and $H_t$ makes each cell of $P_H$ and $P_{H^*}$ invariant. In fact, $H$ is obtained by cutting off the gradient vector field associated to $\varphi_t$ around $M_N \cup Q \cup P_N$ and truncating the induced flow.

To apply Lemma 2.6 in this situation, we need an estimate on $\text{clb}^f(P_N; \text{Int } N - Q)$ (cf. Complement 2.1(2)). The next lemma follows from [18] Proof of Theorem 2.2, Remark 2.1 and provides with this estimate. The following notations are introduced in [18].

**Notation.** Suppose $M$ is an $n$-manifold possibly with boundary.

1. For a stratified subset $P$ of $M$, let $c(P) := \#\{i \in \mathbb{Z}_{\geq 0} \mid P^{(i)} - P^{(i-1)} \neq \emptyset\} \leq \dim P + 1 \leq n + 1$.
2. Suppose $\mathcal{H}$ is a handle decomposition of $M$. The index of a handle $h \in \mathcal{H}$ is denoted by ind $h$.

For any subset $\mathcal{C} \subset \mathcal{H}$, let $c(\mathcal{C}) := \#\{\text{ind } h \mid h \in \mathcal{C}\} \leq n + 1$. Note that $c(\mathcal{H}) = c(P_H)$ and $c(\mathcal{H}^{(k)}) = c(P_{H}^{(k)}) \leq k + 1$, where $\mathcal{H}^{(k)} := \{h \in \mathcal{H} \mid \text{the index of } h \leq k\}$.

**Lemma 3.3.** Suppose $M$ is an $n$-manifold possibly with boundary, $\mathcal{H}$ is a handle decomposition of $M$, $O \subset \mathcal{O}(\text{Int } M)$ and $K \subset \mathcal{K}(O)$. If $K$ is a finite subcomplex of $P_H$, then $\text{clb}^f(K, O) \leq c(K)$.

More precisely, the following holds: There exists a compact neighborhood $U$ of $K$ in $O$ which has the following property.

(⁎) For any neighborhoods $W$ of $K$ and $V$ of $U$ in $O$ there exists $H \in \text{Isot}_{\text{c}}(M, M_V \cup K)_0$ such that
(a) $H_1(U) \subset W$, (b) $H_t(\sigma) = \sigma$ (for $\sigma \in P_H, t \in I$) and
(c) $H$ is a factorization $H = H^{(1)} \cdots H^{(c(K))}$ such that for each $i = 1, 2, \ldots, c(K)$ there exists $D_i \in B_f(\text{Int } M V)$ for which $H^{(i)} \in \text{Isot}(M, M_{D_i} \cup K)_0$ and $H_1^{(i)} \in \text{Diff}(M, M_{D_i})_0$.

The isotopy $H^{(i)}$ is obtained by cutting off the gradient flow $\varphi_t (t \geq 0)$ along the open $k(i)$-cells of $K$ for the dimensions $0 = k(1) < \cdots < k(c(K)) \leq n$ of cells which appear in $K$.

From Lemmas 2.6B and 3.3 we have the following conclusion.

**Lemma 3.4.** In Setting 3.2, 3.2*, 3.2*: If $1 \leq r \leq \infty$, $r \neq n + 1$ and $2k < n$, then the following holds.

1. (i) $\text{clb}^f(M, M_N \cup Q)_0 \leq 2c(P_N)$, (ii) $\text{clb}^d \text{Diff}^r(M, M_N \cup Q)_0 \leq 2c(P_N) + 1$.
2. Any $f \in \text{Diff}^r(M, M_N \cup Q)_0$ has a factorization $f = gh$ such that
   $g \in \text{Diff}^r(M, M_N \cup P)_0$ and $\text{clb}^f(g) \leq 2c(P_N)$ in $\text{Diff}^r(M, M_N \cup Q)_0$,
   $h \in \text{Diff}^r(M, M_N \cup Q)_0$, and $\text{clb}^f(h) \leq 1$ in $\text{Diff}^r(M, M_N \cup P \cup Q)_0$.

**Proof.** We apply Lemma 2.6B(⁎)2 to $O = \text{Int } N - Q$, $L = P$ and $K := P_N \in \mathcal{K}(O)$. Note that $M_O = M_N \cup Q$ and $K \subset L$. It follows that (i) $\text{clb}^f(K, O) \leq c(K)$ by Lemma 3.3, (ii) $K$ has the weak absorption property in $O \cap L$ by Setting 3.2* (⁎) and (iii) $K$ has the strong displacement property for $L$ in $M$, since $\dim K \leq \dim L \leq k$ and $2k < n$ (cf. Example 2.1(1)(i)).

3.1.3. Even-dimensional case

Here we include some basic cases in the even-dimension, which are used in Section 5 to treat the general cases. First we consider the case of handle decompositions without $m$-handles.
Proposition 3.1. Suppose $M$ is a $2m$-manifold without boundary, $1 \leq r \leq \infty$, $r \neq 2m+1$, $\mathcal{H}$ is a handle decomposition of $M$, $N \in \mathcal{SM}_c(M; \mathcal{H})$, $L \in \mathcal{SM}_c(M; \mathcal{H}^*)$ and $N \subset L$. If $\mathcal{H}$ has no $m$-handles in $L$, then

1. $\mathcal{cd}(\text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_L)_0) \leq 3$ and
2. $\mathcal{clb'}d(\text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_L)_0) \leq 2c(\mathcal{H}|_L) + 1$.

Proof. Let $P = P_{\mathcal{H}}^{(m)}$ and $Q = P_x^{(m)}$. By the assumption any open $m$-cell of $P$ does not intersect $L$ and any $m$-cell of $Q$ does not intersect $N$. This implies that

$$P \cap L = P^{(m-1)} \cap L, \quad Q \cap N = Q^{(m-1)} \cap N, \quad P \cap Q \cap L = \emptyset \quad \text{and} \quad M_N \cup P = M_N \cup P^{(m-1)}.$$ 

Given any $f \in \text{Diff}^r(M, M_N)_0$. Take $F \in \text{Isot}^r(M, M_N)_0$ with $F_1 = f$. Then there exists $\Phi \in \text{Isot}(M, M_N \cup P)_0$ such that $Q_1 \cap \pi_M F((P \cap L) \times I) = \emptyset$ for $Q_1 := \phi(Q)$. By Lemma 2.7 we obtain a factorization $F = GH$, where $G \in \text{Isot}^r(M, M_N \cup Q_1)_0$ and $H \in \text{Isot}^r(M, M_N \cup P)_0$.

Applying Lemma 3.4 to $(M, \phi_1(\mathcal{H}), N, P^{(m-1)}, Q_1)$, it follows that $G_1 \in \text{Diff}^r(M, M_N \cup Q_1)_0$ has a factorization $G_1 = g_1g_2$

for some $g_1 \in \text{Diff}^r(M, M_N \cup Q_1)_0$ with $\mathcal{clb'}g_1 \leq 2c(P_{\mathcal{H}|N}^{(m-1)})$ in $\text{Diff}^r(M, M_N \cup Q_1)_0$ and $g_2 \in \text{Diff}^r(M, M_N \cup P \cup Q_1)_0$ with $\mathcal{clb'}g_2 \leq 1$ in $\text{Diff}^r(M, M_N \cup P \cup Q_1)_0$.

Since $g_2H_1 \in \text{Diff}^r(M, M_N \cup P)_0 \subset \text{Diff}^r(M, M_L \cup P)_0$, we can apply Lemma 3.4 to $(M, \mathcal{H}^*, L, Q^{(m-1)}, P)$ to obtain a factorization $g_2H_1 = h_1h_2$ for some $h_1, h_2 \in \text{Diff}^r(M, M_L \cup P)_0$ with $\mathcal{clb'}h_1 \leq 2c(P_{\mathcal{H}|L}^{(m-1)})$ and $\mathcal{clb'}h_2 \leq 1$ in $\text{Diff}^r(M, M_L \cup P)_0$.

1. $f = G_1H_1 = g_1g_2H_1 = g_1h_1h_2$ and $g_1, h_1, h_2 \in \text{Diff}^r(M, M_L \cup P)_0$. Hence, $\mathcal{cl}f \leq 3$ in $\text{Diff}^r(M, M_L)_0$.
2. Since $c(P_{\mathcal{H}|N}^{(m-1)}) + c(P_{\mathcal{H}|L}^{(m-1)}) \leq c(\mathcal{H}|_L)$, it follows that, in $\text{Diff}^r(M, M_L)_0$

$$f = g_1h_1h_2 \quad \text{and} \quad \mathcal{clb'}f \leq 2c(P_{\mathcal{H}|N}^{(m-1)}) + 2c(P_{\mathcal{H}|L}^{(m-1)}) + 1 \leq 2c(\mathcal{H}|_L) + 1. \quad \square$$

Next we consider a basic situation related to $m$-handles. Here we need to force the displacement property for $m$-cells to obtain some estimates on $\mathcal{cl}$ and $\mathcal{clb'}$.

Setting 3.3. Suppose $M$ is a $2m$-manifold without boundary, $\mathcal{H}$ is a handle decomposition of $M$, $P = P_{\mathcal{H}}^{(m)}$, $Q = P_{\mathcal{H}^*}^{(m)}$ and $C, C'$ are some sets of open $m$-cells of $P$ with $C \subset C'$.

The sets of $m$-cells of $P$ and $Q$ are in the 1-1 correspondence $\sigma \longleftrightarrow \sigma^*$, where $\sigma^*$ is the dual $m$-cell to $\sigma$. Hence, the set $C$ determines the set $C^* = \{\sigma^* \mid \sigma \in C\}$ of open $m$-cells of $Q$. For simplicity, we set $Q_C := Q - |C^*|$. In this context, the gradient flow $\varphi_t$ has the following properties: $\varphi_t(M - Q_C) = M - Q_C$ and $M - Q_C$ is attracted to $P^{(m-1)} \cup |C|$.

Setting 3.3*. Suppose $N \in \mathcal{SM}_c(M; \mathcal{H})$, $P_N = (P_{\mathcal{H}|N})^{(m)}$, $C|_N := \{\sigma \in C \mid \sigma \subset N\}$, $O := \text{Int} N - Q_C$, $K := P_{\mathcal{H}|N}^{(m-1)} \cup |C|_N$ and $L = P^{(m-1)} \cup |C'|$.

Then $M_O = M_N \cup Q_C$, $K \in \mathcal{K}(O)$, $L \in \mathcal{F}(M)$, $K \subset L$, $K$ is a subcomplex of $P_N$ and $N \cap |C'| = N \cap |(C|_N)^*|$. The flow $\varphi_t$ has the following properties: The flow $\varphi_t$ is transverse to $\partial N$, $\varphi_t(N) \subset N$, $\varphi_t(N - Q_C) \subset N - Q_C$, $N - Q_C$ is attracted to $K$ and $\varphi_t$ keeps each cell of $P_{\mathcal{H}}$ and $P_{\mathcal{H}^*}$ invariant. Therefore, the following holds:

1. $K$ has the weak absorption property in $O$ keeping $P$, $K$ and $L$ invariant.

More precisely, for any $C \in \mathcal{K}(O)$ and any neighborhood $U$ of $K$ in $O$ there exists $H \in \text{Isot}(M; M_O \cup K)_0$ such that $H_1(C) \subset U$ and $H_t$ keeps each cell of $P_{\mathcal{H}}$ and $P_{\mathcal{H}^*}$ invariant.
Lemma 3.5. In Settings 3.3 3.3\(^{-}\); Suppose \(1 \leq r \leq \infty, r \neq 2m + 1\).
If \(Cl_{M}|C|_{N}\) has the displacement property for \(Cl_{M}|C'|_{N}\) in \(Int\ N - Q_{C}\), then the following holds.

1. (i) \(clb^{\prime} \text{Diff}^{r}(M, M_{N} \cup Q_{C})_{0} \leq 2\),
   (ii) \(clb^{\prime \prime} \text{Diff}^{r}(M, M_{N} \cup Q_{C})_{0} \leq 2c(K) + 1\).

2. Any \(f \in \text{Diff}^{r}(M, M_{N} \cup Q_{C})_{0}\) has a factorization \(f = gh\) such that
   \(g \in \text{Diff}^{r}(M, M_{N} \cup Q_{C})_{0}\) and \(clb^{\prime}(g) \leq 2c(K)\) in \(\text{Diff}^{r}(M, M_{N} \cup Q_{C})_{0}\)
   \(h \in \text{Diff}^{r}(M, M_{N} \cup Q_{C} \cup L)_{0}\) and \(clb^{\prime}(h) \leq 1\) in \(\text{Diff}^{r}(M, M_{N} \cup Q_{C} \cup L)_{0}\).

Proof. We apply Lemma 2.6 [B] \((*)_{1}\) to \(O = Int\ N - Q, L\) and \(K\). It follows that (i) \(clb^{\prime}(K, O) \leq c(K)\) by Lemma 3.3, (ii) \(K\) has the weak absorption property in \(O\) rel \(L\) and keeping \(K\) invariant by Setting 3.3\(^{-}\) \((\dagger)\) and (iii) \(K\) has the displacement property for \(L\) in \(O\) by Example 2.4(1)(ii)(a).

\[\square\]

3.2. Triangulations of manifolds.

3.2.1. Generalities on triangulations of manifolds.

Suppose \(M\) is an \(n\)-manifold possibly with boundary and \(T\) is a \(C^{\infty}\) triangulation of \(M\). For each (closed) simplex \(\sigma \in T\), by the symbol \(\bar{\sigma}\) we denote the interior of \(\sigma\) (the open simplex for \(\sigma\)). The symbol \(T^{(k)}\) \((k = 0, 1, \cdots, n)\) denotes the \(k\)-skeleton of \(T\) and \(|T^{(k)}|\) denotes its realization in \(M\), i.e.,
\(|T^{(k)}| = \{\sigma \mid \sigma \in T^{(k)}\}\) \(\subset M\). Let \(sd^{i}T\) \((i \geq 0)\) denote the \(i\)-th barycentric subdivision of \(T\). Sometimes we do not distinguish a subcomplex \(S\) of \(T\) and its realization \(|S|\) in \(M\).

For each \(k\)-simplex \(\sigma\) of \(T\) its barycenter is denoted by \(b(\sigma)\). The simplex \(\sigma\) determines the dual (closed) \((n - k)\)-cell \(\sigma^{*}\) and the handle \(H_{\sigma}\) of index \(k\). These are (piecewise \(C^{\infty}\)) PL-disks defined by
\[
\sigma^{*} = \bigcup \{b(\sigma^{k}), b(\sigma^{k+1}), \cdots, b(\sigma_{n})\} \mid \sigma = \sigma_{k} < \sigma_{k+1} < \cdots < \sigma_{n} \in T\}
\]
\[
H_{\sigma} = St(b(\sigma), sd^{2}T) = \bigcup \{\tau \in sd^{2}T \mid b(\sigma) \in \tau\}.
\]

Note that, if \(\sigma \subset \partial M\), then \(\sigma^{*}\) is a “half” disk and \(H_{\sigma}\) is a “half” handle cut by \(\partial M\) transversely. The handle \(H_{\sigma}\) has the core disk \(H_{\sigma} \cap \sigma\) and the cocore \(H_{\sigma} \cap \sigma^{*}\). The triangulation \(T\) induces the dual PL cell decomposition \(T^{*} = \{\sigma^{*} \mid \sigma \in T\}\) and the PL handle decomposition \(H_{T} = \{H_{\sigma} \mid \sigma \in T\}\) of \(M\).

The \((n - k - 1)\)-skeleton \((T^{*})^{(n-k-1)}\) of \(T^{*}\) is called the dual \((n - k - 1)\)-skeleton for the \(k\)-skeleton \(T^{(k)}\) of \(T\). Under our convention in Section 3.1 this handle decomposition does not satisfy our requirement. However, in this bounded PL-case we permit this kind of handle decompositions.

Associated to the triangulation \(T\), there exists a (downward) \(C^{\infty}\) gradient flow \(\varphi_{t}\) \((t \in \mathbb{R})\) on \(M\) induced from a Riemannian metric of \(M\) and a Morse function on \(M\) which has a critical point of index \(k\) at \(b(\sigma)\) for each \(k\)-simplex \(\sigma\) of \(T\). The cell decomposition \(T\) is recovered as the unstable cell complex associated to the flow \(\varphi_{t}\) and the dual cell decomposition \(T^{*}\) is approximated by the stable cell complex associated to \(\varphi_{t}\). When \(\partial M \neq \emptyset\), the flow \(\varphi_{t}\) keeps \(\partial M\), while the partial gradient flow associated to a handle decomposition in Section 3.1 is transverse to \(\partial M\).

A \(C^{\infty}\) subpolyhedron of \(M\) means a subcomplex of some \(C^{\infty}\) triangulation of \(M\).

Lemma 3.6. Suppose \(M\) is an \(n\)-manifold possibly with boundary, \(T\) is a \(C^{\infty}\) triangulation of \(M\), \(O \in O(Int\ M)\) and \(K\) is a finite subcomplex of \(T\) with \(K \subset O\). Then, the following holds.

1. (i) \(clb^{\prime}(K, O) \leq \dim K + 1\).

2. If \(K \subset K' \in K(O)\) and \(K'\) is weakly absorbed to \(K\) in \(O\) keeping \(K\) invariant, then
\[
clb^{\prime}(K', O) \leq clb^{\prime}(K; O) \leq \dim K + 1,
\]
For example, the assumption in (2) is satisfied if $K'$ is a finite subcomplex of $T$ with $K \subset K' \subset O$ and $K'$ collapses to $K$ in the PL-sense.

Proof. (1) One may verify this lemma, using the flow $\varphi_t$ as in Lemma 3.3 based on [18, Proof of Theorem 2.2, Remark 2.1]. Here, we use a regular neighborhood of $K$ to obtain a simpler description. Take a fine subdivision $T'$ of $T$ with $St(K, T') \subset O$ and consider the regular neighborhood $U := St(K, sd^2T')$ of $K$ with respect to $T'$. Then $U$ has the following property:

(1) For any neighborhoods $W$ of $K$ and $V$ of $U$ in $O$ there exists $H \in Isot_c(M, M_V \cup K)_0$ such that

(a) $H_1(U) \subset W$,
(b) $H_i(\sigma) = \sigma (\sigma \in T', t \in I)$ and

(c) $H$ has a factorization $H = H'(0)H'(1)\cdots H'(k) (k = \dim K)$ such that for each $i = 0, 1, \cdots, k$ there exists $D_i \in Bf(\operatorname{Int}_M V)$ for which $H'(i) \in Isot(M, M_{D_i} \cup K)_0$ and $H'_i(\sigma) \in \operatorname{Diff}(M, M_{D_i})_0$ (in particular, $clb^f H_1 \leq k + 1$ in $\operatorname{Diff}(M, M_V)_0$).

In fact, we can write as $U = \bigcup \{H_\sigma \mid \sigma \in K'\}$, where $K' = T'|_K$ (the subdivision of $K$ induced by $T'$) and $H_\sigma = St(b(\sigma), sd^2T')$ (the handle associated to $\sigma$ in $T'$). Then for each $i$-simplex $\sigma$ of $K'$ (with $H_\sigma \not\in \operatorname{Int}_M K$) we can compress the handle $H_\sigma$ toward the union of $H_\sigma \cap K$ and the attaching region of $H_\sigma$ in a small disk neighborhood of $H_\sigma$. The isotopy $H'(i)$ is obtained by this procedure and the modification due to [18, Remark 2.1].

The statement (2) follows from Lemma 2.3(1)(ii).

3.2.2. Double mapping cylinder structures between complimentary full subcomplexes.

We further observe that the triangulation $T$ of a manifold $M$ provides $M$ with a more rigid structure than the induced handle decomposition $H_T$ and the gradient flow $\varphi_t$. Especially, around any subcomplex of $T$ we can take a regular neighborhood with a mapping cylinder structure. This structure also induces some flow, which has a simple description and is used to construct absorbing diffeomorphisms. This section is devoted to this approach, which yields some basic estimates on $cl$ and $clb^f$ in compact manifolds possibly with boundary.

Setting 3.4. Suppose $M$ is an $n$-manifold possibly with boundary, $T$ is a $C^\infty$ triangulation of $M$, $S$ is a full subcomplex of $T$ and $U := \{\sigma \in T \mid \sigma \cap |S| = \emptyset\}$.

Then $U$ is also a full subcomplex of $T$ and $S = \{\sigma \in T \mid \sigma \cap |U| = \emptyset\}$. In this case $S$ and $U$ are said to be complimentary to each other in $T$. Also note that any $\sigma \in T$ is represented as $\sigma = (\sigma \cap S) * (\sigma \cap U)$ (the join of $(\sigma \cap S)$ and $(\sigma \cap U)$). Consider the standard PL regular neighborhoods $St(|S|, sdT)$ of $|S|$ and $St(|U|, sdT)$ of $|U|$. These regular neighborhoods have the structure of PL mapping cylinder with the base $|S|$ and $|U|$ respectively, and $M$ has the structure of PL double mapping cylinder as the union of these mapping cylinders. This structure respects $\partial M$. From these observations we have a $C^\infty$ flow $\psi_t$ on $M$ which fixes $|S|$ and $|U|$ pointwise and keeps each simplex of $T$ invariant and under which $M - |U|$ is attracted to $|S|$. The associated vector field $X_\psi$ on $M$ is constructed inductively on a neighborhood of $T^{(k)} - (S \cup U)$ for $k = 1, \cdots, n$, so that $X_\psi$ is tangent to each simplex in $T - (S \cup U)$. Therefore, the following holds:

(2) For any $C \in \mathcal{K}(M - |U|)$ and any neighborhood $U$ of $|S|$ in $M - |U|$ there exists $H \in Isot_c(M; |S| \cup |U|)_0$ such that $H_1(C) \subset U$ and $H_t(\sigma) = \sigma (\sigma \in T, t \in I)$. If $C \in \mathcal{K}(\operatorname{Int} M - |U|)$, then we can take $H$ in $Isot_c(M; \operatorname{Int} M - |U|)_0$.  

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Note that, if $P$ is any subcomplex of $\mathcal{T}$, then $sdP$ is a full subcomplex of $sd\mathcal{T}$ and we can apply the above argument in $sd\mathcal{T}$.

**Lemma 3.7.** Suppose $M$ is an $n$-manifold possibly with boundary, $1 \leq r \leq \infty$, $r \neq n + 1$, $\mathcal{T}$ is a $C^\infty$ triangulation of $M$, $P$ is a finite full subcomplex of $\mathcal{T}$, $Q := \bigcup \{\sigma \in \mathcal{T} \mid \sigma \cap P = \emptyset\}$, $L$ is a subcomplex of $\mathcal{T}$ with $P \subseteq L$, $O_0 := M - Q$ and $\tilde{O}_0 := O_0 \cup (\partial O_0 \times [0, 1))$.

If $clb^f(P, \tilde{O}_0) \leq \ell < \infty$ and $P$ is displaceable from $L \cup (\partial M \times [0, 1))$ in $\tilde{O}_0$, then the following holds.

1. (i) $cld Diff^c_\ell(M, \partial M \cup Q)_0 \leq 2$, (ii) $clb^f Diff^c_\ell(M, \partial M \cup Q)_0 \leq 2\ell + 1$.

2. Any $f \in Diff^c_\ell(M, \partial M \cup Q)_0$ has a factorization $f = gh$ such that

$$
g \in Diff^c_\ell(M, \partial M \cup Q)_0^c \quad \text{and} \quad clb^f(g) \leq 2\ell \text{ in } Diff^c_\ell(M, \partial M \cup Q)_0,
$$

$$
h \in Diff^c_\ell(M, \partial M \cup Q \cup L)_0^c \quad \text{and} \quad clb^f(h) \leq 1 \text{ in } Diff^c_\ell(M, \partial M \cup Q \cup L)_0.
$$

Proof. Let $O := Int M - Q$. Then, $M_O = \partial M \cup Q$ and by Setting 3.4(i) the following holds;

$(*)_1$ (i) $P$ has the weak absorption property (a) in $O$ rel $L$ and (b) in $O_0$ keeping $P$ invariant.

Therefore, the conclusion follows from Lemma 2.6[A] $(*)_1$ (taking $K := P$) and the assumptions. □

We list two important examples of complimentary full subcomplexes used in the subsequent sections.

**Example 3.1.** Suppose $M$ is an $n$-manifold possibly with boundary and $\mathcal{T}$ is a $C^\infty$ triangulation of $M$.

1. The pair $(|\mathcal{T}^{(k)}|, |(\mathcal{T}^*)^{(n-k-1)}|)$ of the $k$-skeleton of $\mathcal{T}$ and its dual $(n-k-1)$-skeleton is a pair of complimentary full subcomplexes in $sd\mathcal{T}$, that is, $|(\mathcal{T}^*)^{(n-k-1)}| = \bigcup \{\tau \in sd\mathcal{T} \mid \tau \cap |\mathcal{T}^{(k)}| = \emptyset\}$.

2. Let $(P, Q) := (|\mathcal{T}^{(k)}|, |(\mathcal{T}^*)^{(n-k-1)}|)$ or $(|(\mathcal{T}^*)^{(k)}|, |\mathcal{T}^{(n-k-1)}|)$, $(k = 0, 1, \ldots, n)$ and $O_0 := M - Q$. Let $L$ be a subcomplex of $sd\mathcal{T}$ with $P \subseteq L$. When $M$ is compact, $P$ is a finite full subcomplex of $sd\mathcal{T}$ and the next assertions follow from Lemma 3.6 and the general position argument, respectively.

(i) $clb^f(P, \tilde{O}_0) \leq k + 1$. Moreover, if $K$ is a compact $C^\infty$ subpolyhedron in $M$ with $K \subseteq P$ and $P$ is weakly absorbed to $K$ in $\tilde{O}_0$ keeping $K$ invariant, then $clb^f(P, \tilde{O}_0) \leq clb^f(K, \tilde{O}_0) \leq \dim K + 1$.

(ii) If $2k < n$ and $\dim L \leq k$, then $P$ is displaceable from $L \cup (\partial M \times [0, 1))$ in $\tilde{O}_0$.

**Example 3.2.** (Even-dimensional case)

Suppose $M$ is a compact $2m$-manifold possibly with boundary, $\mathcal{T}$ is a $C^\infty$ triangulation of $M$, and $(\mathcal{C}, \mathcal{D})$ is a partition of the set of $m$-simplices of $\mathcal{T}$. Let $\mathcal{T}[m]$ and $\mathcal{T}^*[m]$ denote the sets of $m$-simplices of $\mathcal{T}$ and $m$-cells of $\mathcal{T}$ respectively.

1. The sets $\mathcal{T}[m]$ and $\mathcal{T}^*[m]$ are in the 1-1 correspondence $\sigma \leftrightarrow \sigma^*$, and the a partition $(\mathcal{C}, \mathcal{D})$ of $\mathcal{T}[m]$ induces the partition $\mathcal{C}^* = \{\sigma^* \mid \sigma \in \mathcal{C}\}$, $\mathcal{D}^* = \{\sigma^* \mid \sigma \in \mathcal{D}\}$ of $\mathcal{T}^*[m]$.

Consider the subcomplexes $\mathcal{S} := \mathcal{T}^{(m-1)} \cup \mathcal{C}$ of $\mathcal{T}$ and $\mathcal{U} := (\mathcal{T}^*)^{(m-1)} \cup \mathcal{D}^* = (\mathcal{T}^*)^{(m)} - \mathcal{C}^*$ of $\mathcal{T}^*$.

Then, $|\mathcal{S}|$, $|\mathcal{U}|$ underlie complimentary full subcomplexes of $sd\mathcal{T}$, that is, $|\mathcal{U}| = \bigcup \{\sigma \in sd\mathcal{T} \mid \sigma \cap |\mathcal{S}| = \emptyset\}$.

2. Suppose $\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{T}[m]$ and $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{T}[m]$. Consider the following cases:

- **I.** $(P, Q) = (|\mathcal{S}|, |\mathcal{U}|)$, $C_0 = |\mathcal{C}|$, $L = |\mathcal{T}^{(m-1)}| \cup |\mathcal{C}'|$, $L_0 = |\mathcal{C}'|$

- **II.** $(P, Q) = (|\mathcal{U}|, |\mathcal{S}|)$, $C_0 = |\mathcal{D}|$, $L = |(\mathcal{T}^*)^{(m-1)}| \cup |\mathcal{D}'|$, $L_0 = |\mathcal{D}'|$

In each case, $P, Q$ are complimentary finite full subcomplexes of $sd\mathcal{T}$, $L$ is a subcomplex of $sd\mathcal{T}$ and $P \subseteq L$, $P = P^{(m-1)} \cup C_0$, $C_0 = Cl_M(P - P^{(m-1)})$ and $L = L^{(m-1)} \cup L_0$, $L_0 = Cl_M(L - L^{(m-1)})$.

Let $O_0 := M - Q$. The next assertion (i) follows from Lemma 3.6

(i) $clb^f(P, \tilde{O}_0) \leq m + 1$. Moreover, if $K$ is a compact $C^\infty$ subpolyhedron in $M$ with $K \subseteq P$ and $P$ is weakly absorbed to $K$ in $\tilde{O}_0$ keeping $K$ invariant, then $clb^f(P, \tilde{O}_0) \leq clb^f(K, \tilde{O}_0) \leq \dim K + 1$. 23
(ii) (a) If $C_0$ is displaceable from $L_0 \cup (\partial M \times [0, 1))$ in $\tilde{O}_0$, then $P$ is displaceable from $L \cup (\partial M \times [0, 1))$ in $\tilde{O}_0$.
(b) If $C_0$ is weakly absorbed to a $(m-1)$-dimensional compact subpolyhedron in $\tilde{O}_0$, then $C_0$ is displaceable from $L_0 \cup (\partial M \times [0, 1))$.

The assertion (ii)(a) is easily verified by the same argument as in Example 2.3(1)(ii)(a). The assertion (ii)(b) follows from Lemma 2.2(2)(ii)(a).

4. DIFFEOMORPHISM GROUPS OF $(2m+1)$-MANIFOLDS

4.1. Factorization of isotopies on $(2m+1)$-manifolds.

4.1.1. Removing crossing points.

We recall the trick in [4, Section 3.3], [17, Lemma 6.5] to remove crossing points between some $m$-strata and the tracks of some compact $m$-submanifold under an isotopy on a $(2m+1)$-manifold. Consider the following situation.

Setting 4.1.

(i) $M$ is an $n$-manifold possibly with boundary ($n = 2m+1, m \geq 1$) and $N \in SM_c(\text{Int } M)$.
(ii) $E$ is a compact $C^\infty$ $m$-submanifold of $\text{Int } N$,

$D \subset D' \subset D'' \subset E$ are compact $C^\infty$ $m$-submanifolds of $E$.
(iii) $\Lambda$ is an $m$-dimensional stratified subset in $M - E$.
(iv) $H \in \text{Isot}'(M, M_N)_0(1 \leq r \leq \infty)$ and $W$ is an open neighborhood of $\pi_M H(D'' \times I)$ in $\text{Int } N$.

Suppose $\Lambda^{(m-1)} \cap \pi_M H(E \times I) = \emptyset$ and $H = \text{id}$ on $[\text{a neighborhood of } (E - \text{Int } D)] \times I$.

Removing crossing points. ([4, Section 3.3], [17, Lemma 6.5])

(Claim.) There exists $\overline{H} \in \text{Isot}'(M, M_N)_0, U \in B_f(W - E)$ and $A \in \text{Isot}(M, M_U)_0$ such that

(i) $\overline{H}$ is a $C^r$-approximation of $H$ and $\overline{H} = H$ on $(M - W) \times I$,
(ii) $U \cap \pi_M \overline{H}((E - \text{Int } D) \times I) = \emptyset$,
(iii) $\pi_M A \overline{H}(E \times I) \cap \Lambda = \emptyset$ and $A_1 \in \text{Diff}(M, M_U)_0$.

(Construction.)

(1) Take an open neighborhood $W_0$ of $D''$ in $W$ such that $\pi_M H(W_0 \times I) \subset W$. Then, we can find a $C^r$-approximation $\overline{H} \in \text{Isot}'(M, M_N)_0$ of $H$ such that $\overline{H} = H$ on $(M - W_0) \times I$ and

(i) $\pi_M \overline{H}|_{D' \times I} : D' \times I \rightarrow W$ is a $C^\infty$-immersion outside of a finite subset, which has no double points on $(D' - \text{Int } D) \times I$ in $D' \times I$.

If $\overline{H}$ is sufficiently close to $H$, then

(ii) (a) $\pi_M \overline{H}((E - \text{Int } D) \times I) \cup (E \times \{0, 1\})) \cap \Lambda = \emptyset$, $\pi_M \overline{H}(D \times I) \cap \pi_M \overline{H}((E - \text{Int } D') \times I) = \emptyset$,
(b) $\pi_M \overline{H}(E \times I) \cap \Lambda^{(m-1)} = \emptyset$.

Let $W' := W - \pi_M \overline{H}((E - \text{Int } D) \times I)$. Note that $\pi_M \overline{H}(D'' \times I) \subset W$ and

(iii) $\pi_M \overline{H}(\text{Int } D \times I) \cap \pi_M \overline{H}((E - \text{Int } D) \times I) = \emptyset$ and $\pi_M \overline{H}(\text{Int } D \times I) \subset W'$.

We also choose the immersion $\pi_M \overline{H}|_{D' \times I}$ in generic with respect to $\Lambda$ and the self-intersections, so that the arguments below work well. It follows that

(iv) $\pi_M \overline{H}(E \times I)$ and $\Lambda$ intersect transversely at finitely many points $\{\pi_M \overline{H}(v_i, s_i)\}_{i=1, \ldots, p}$, where $\{v_i\}_{i=1, \ldots, p}$ are distinct points in $\text{Int } D$, $s_i \in (0, 1) (i = 1, \cdots, p)$ and $\pi_M \overline{H}$ is an embedding on $\bigcup_{i=1}^p \{v_i\} \times [s_i, 1]$. 

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(v) $\pi_M \overline{P}_{|D \times I}$ has finitely many double point curves, which are in general position with respect to the curves $\pi_M \overline{P}(\{v\} \times I)$ (v $\in D$).

(2) We construct a disjoint union of closed $n$-disks $U = \bigcup_{i=1}^p U_i$ in $W' - E$ and $A \in \text{Isot}(M, M_U)_0$ such that $\pi_M A \overline{P}(D \times I) \cap \Lambda = \emptyset$.

[1] the case that $m \geq 2$:

In this case, generically each arc $\pi_M \overline{P}(\{v_i\} \times [s_i, 1])$ does not contain any double points and does not intersect $E$. For each $i = 1, \ldots, p$ take a thin closed $n$-disk neighborhood $U_i$ of $\pi_M \overline{P}(v_i \times [s_i, 1])$ in $W' - E$ so that $\{U_i\}_{i=1,\ldots,p}$ are disjoint. The isotopy $A$ is obtained by pushing the track $\pi_M \overline{P}(D \times I)$ along the arcs $\pi_M \overline{P}(v_i \times [s_i, 1])$ (i = 1, \ldots, p) in $U$. To achieve the required condition $\pi_M A \overline{P}(D \times I) \cap \Lambda = \emptyset$, we choose a small parameter $t_0 \in (0, 1)$ such that $\pi_M \overline{P}(D \times [0, t_0]) \cap U = \emptyset$, and take $A$ so that $A_t$ ($t \in [t_0, 1]$) pushes a thin neighborhood of each arc $\pi_M \overline{P}(v_i \times [s_i, 1])$ ahead of the point $\pi_M \overline{P}(v_i, s_i)$ in $U_i$.

[2] the case that $m = 1$:

In this case the double point curves may intersect each other and possibly there are at most finitely many triple points and cusps. Also the arcs $\pi_M \overline{P}(\{v_i\} \times [s_i, 1])$ may intersect the double point curves. To remove the crossing points $\pi_M \overline{P}(E \times I) \cap \Lambda$, for each crossing point $\pi_M \overline{P}(v_i, s_i)$ we construct a tree $T_i \subset \pi_M \overline{P}(\text{Int } D \times [0, 1])$ rooted at $\pi_M \overline{P}(v_i, s_i)$ which serves as a guide to construct the isotopy $A$.

In general, for a curve $\pi_M \overline{P}(\{v\} \times [s, 1])$ ($v \in \text{Int } D$), consider the double points on this curve of the form $\pi_M \overline{P}(v, t) = \pi_M \overline{P}(v', t')$ with $v' \in \text{Int } D, s < t < t' < 1$. To each such double point $\pi_M \overline{P}(v, t)$ consider the curve $\pi_M \overline{P}(\{v'\} \times [t', 1])$, which is called a branch on the curve $\pi_M \overline{P}(\{v\} \times [s, 1])$. Generically, $\pi_M \overline{P}(\{v\} \times [s, 1])$ is an arc including no triple points, no non-transverse double points and no cusps, and it has only finitely many branches $\pi_M \overline{P}(\{v'_j\} \times [s'_j, 1])$ rooted at a double point $\pi_M \overline{P}(v, s_j) = \pi_M \overline{P}(v', s'_j)$ with $s < s_j < s'_j < 1$ ($j = 1, \ldots, \ell$). Generically, these branches are arcs with similar properties.

If the arc $\pi_M \overline{P}(\{v\} \times [s, 1])$ is a branch on an arc $\pi_M \overline{P}(\{v\} \times [s, 1])$ rooted at $\pi_M \overline{P}(v', s') = \pi_M \overline{P}(v, s)$ ($v' < s$) and has no branches on itself, then we can find a thin closed $n$-disk neighborhood $O$ of this arc and $A' \in \text{Isot}(M, M_O)_0$ such that $\pi_M A' \overline{P}(\{v\} \times [s, 1]) = \pi_M \overline{P}(\{v'\} \times [s, 1])$ and this arc do not have any branch corresponding to $\pi_M \overline{P}(\{v\} \times [s, 1])$ with respect to $A' \overline{P}$ (so the number of branches decreases by one). To define $A'$ we have to respect the times $t'_k, t_k$ which appear in the remaining double points $\pi_M \overline{P}(v, t_k) = \pi_M \overline{P}(v'_k, t'_k)$ on $\pi_M \overline{P}(\{v\} \times [s, 1])$ with $s < t'_k < t_k < 1$ ($k = 1, \ldots, q$). Take $\varepsilon > 0$ such that $t'_k < t_k - 2\varepsilon$ ($k = 1, \ldots, q$). Then $A'_\tau$ ($\tau \in [s, 1]$) pushes $O'[s, \tau]$ ahead of the point $\pi_M \overline{P}(v, s)$ and remains $O'[\tau + \varepsilon, 1]$ fixed, where $O'$ is a shrink of $O$ and $O'[a, b]$ denotes the part of $O'$ corresponding to $\pi_M \overline{P}(\{v\} \times [a, b])$. Since $\overline{H}_\tau(D)$ may appear in $O'[\tau + \varepsilon, 1]$, if we push this part across $O'[s]$, then it may happen that $A'_\tau \overline{P}(D)$ touches $\pi_M \overline{P}(\{v\} \times [s, 1])$ and generates a new branch instead of $\pi_M \overline{P}(\{v\} \times [s, 1])$.

The tree $T_i = \bigcup_{j=0}^{\ell_i} T_i^{(j)}$ is defined as follows : Let $T_i^{(0)} = \pi_M \overline{P}(\{v_i\} \times [s_i, 1])$ and inductively $T_i^{(j)}$ is defined as the union of branches on the arcs in $T_i^{(j-1)}$. Generically, this procedure ends in a finite step $\ell_i$ and $T_i$ forms a tree and $\{T_i\}_{i=1}^p$ are disjoint. We call $T = \bigcup_{i=1}^p T_i$ the intersection graph for $\overline{P}$ for short. Let $T^{(j)} = \bigcup_{i=1}^p T_i^{(j)}$ ($j = 0, 1, \ldots, \ell$), where $\ell = \max_i \ell_i$ and $T_i^{(j)} = \emptyset$ for $j > \ell_i$. Generically, $T \cap E = \emptyset$ and we may assume that the points $w$’s in $\text{Int } D$ which appear in the following way are all distinct : $w = v_i$ for some $i = 1, \ldots, p$ or $\pi_M \overline{P}(v, s) = \pi_M \overline{P}(w, t)$, where $\pi_M \overline{P}(v, s)$ is a point of a
branch arc in $T$ (including the arcs $\pi_M\overline{H}(\{v_i\} \times [s_i, 1])$ and $\pi_M\overline{H}(w, t)$ is a point on a double points curve (including the case $s > t$).

For each $i = 1, \cdots, p$ take a thin closed $n$-disk neighborhood $U_i$ of $T_i$ in $W' - E$ so that $\{U_i\}_{i=1,\cdots,p}$ are disjoint. Let $U = \bigcup_{i=1}^p U_i$ and for each $j = 0, 1, \cdots, \ell$ let $U^{(j)}$ denote a disjoint union of thin closed $n$-disk neighborhoods of the arcs in $T^{(j)}$ in $U$. Then, in the backward order we can inductively construct $A^{(j)} \in \text{Isot}(M, M_{U_{(j)}})_0$ ($j = 0, 1, \cdots, \ell$) such that $A^{(k)}A^{(k+1)}\cdots A^{(\ell)}\overline{H}$ has the intersection graph $\bigcup_{j=1}^{k-1} T^{(j)}$ for each $k = 0, 1, \cdots, \ell$. Finally, let $A := A^{(0)}A^{(1)}\cdots A^{(\ell)} \in \text{Isot}(M, M_U)_0$. Then $\overline{H}$ has the empty intersection graph, which means $\pi_M\overline{H}(D \times I) \cap \Lambda = \emptyset$.

(3) Since $U \subset W'$, it follows that $\pi_M\overline{H}(E - \text{Int } D) \times I) = \pi_M\overline{H}(E - \text{Int } D) \times I)$, which does not intersect $\Lambda$ by (1)(ii)(a). Therefore, we have $\pi_M\overline{H}(E \times I) \cap \Lambda = \emptyset$. Finally, by [18] Remark 2.1, we can modify $A \in \text{Isot}(M, M_U)_0$ so that $A_1 \in \text{Diff}(M, M_U)^c_0$. This completes the construction.

4.1.2. Factorization of isotopies on $(2m + 1)$-manifolds.

Next we recall the strategy in [1], [17] to factor isotopies on compact $(2m + 1)$-manifolds.

**Setting 4.2.** Suppose $M$ is a $(2m + 1)$-manifold possibly with boundary ($m \geq 1$), $N \in SM_c(\text{Int } M)$, $F \in \text{Isot}^r(M, M_N)_0$ and $P, Q$ are disjoint $m$-dimensional stratified subsets of $M$.

**Review of the arguments in [1], [17].**

(Step 1) First we remove the intersections between low dimensional parts of $Q$ and $\pi_M F(P \times I)$.

1. By the general position argument we obtain an arbitrarily small isotopy $K \in \text{Isot}(M, M_N \cup P)_0$ with support in an arbitrarily small neighborhood of $Q$ such that $Q_1 := K_1(Q)$ satisfies the following conditions:

$$\pi_M F(P \times I) \cap Q_1^{(m-1)} = \emptyset \text{ and } \pi_M F((P^{(m-1)} \times I) \cup (P \times \{0, 1\})) \cap Q_1 = \emptyset.$$  

2. By (1) and Lemma [2.7] there exists a factorization $F = GH$ for some $G \in \text{Isot}^r(M, M_N \cup Q_1)_0$ and $H \in \text{Isot}^r(M, M_N \cup P^{(m-1)})_0$. From (1) it follows that

(i) $\pi_M H(P \times I) \cap Q_1^{(m-1)} = \emptyset$ and

(ii) $\pi_M H((P^{(m-1)} \times I) \cup (P \times \{0, 1\})) \cap Q_1 = \emptyset$.

(Step 2) Next we remove the intersection $\pi_M H(P \times I) \cap Q_1$.

1. Take an open neighborhood $V$ of $M_N \cup P^{(m-1)}$ in $M$ such that $H = \text{id}$ on $V \times I$. Since $O := P - (M_N \cup P^{(m-1)})$ is an $m$-manifold without boundary and $P - V$ is a compact subset of $O$, there exists a compact $m$-submanifold $E$ of $O$ with $P - V \subset \text{Int } E$. Let $D \subset D' \subset D'' \subset E$ be shrink of $E$ with $P - V \subset \text{Int } D$. Note that $E - \text{Int } D \subset P - \text{Int } D \subset P \cap V$. Since $\pi_M H(D'' \times I) \subset \text{Int } N - (P - \text{Int } E)$, there exists an open neighborhood $W$ of $\pi_M H(D'' \times I)$ in $W$ with $C|M|W \subset \text{Int } N - (P - \text{Int } E)$.

2. By Removing crossing points in Subsection 4.1.1 (applied to $M, N, E, Q_1, H, W$), there exists $\overline{H} \in \text{Isot}^r(M, M_N)_0$, $U \in B_f(W - E)$ and $A \in \text{Isot}(M, M_U)_0$ such that

(i) $\overline{H}$ is a $C^\infty$-approximation of $H$ and $\overline{H} = H$ on $(M - W) \times I$,

(ii) $U \cap \pi_M \overline{H}(E - \text{Int } D) \times I = \emptyset$,

(iii) $\pi_M A\overline{H}(E \times I) \cap Q_1 = \emptyset$ and $A_1 \in \text{Diff}(M, M_U)^c_0$.

Then, it follows that

(iv) (a) $H' := A\overline{H} \in \text{Isot}^r(M, M_N \cup P^{(m-1)})_0$ and (b) $\pi_M H'(P \times I) \cap Q_1 = \emptyset$. 

In fact, since $U \subset W - E$, we have $U \subset \text{Int } N - P$ and $A \in \text{Isot}_r(M, M_N \cup P)_0$. From (i)(a) it follows that $\overline{T} \in \text{Isot}_r(M, M_N \cup P^{(m-1)})_0$. This implies (iv)(a). Since $P - E \subset V \cap (M - W)$, it follows that $\overline{T}_t = H_t = \text{id} = A_t$ and $H'_t = \text{id}$ on $P - E$, so that $\pi_M H^*((P - E) \times I) = P - E$. This and (iii) means (iv)(b).

By (iv)(b) and Lemma 2.7 there exists a factorization

$$H' = G'H''$$

for some $G' \in \text{Isot}_r(M, M_N \cup Q_1)_0$ and $H'' \in \text{Isot}_r(M, M_N \cup P)_0$.

(Step 3) Factorization of $f$

We denote the 1-levels of the isotopies $G, H, \overline{T}, A, H', G'$ and $H''$ by the corresponding letters $g, h, \overline{h}, a, h', g'$ and $h''$ respectively. (Step 1) and (Step 2) lead to the following factorization of $f$.

1. Since $\overline{h}^{-1} h \in \text{Diff}_r(M, M_N)_0$ is sufficiently close to $\text{id}_M$, for the open cover $\{\text{Int } N - Q_1, \text{Int } N - P\}$ of $\text{Int } N$ we have a factorization

$$\overline{h}^{-1} h = \hat{h} \hat{g} \quad \text{for some } \hat{g} \in \text{Diff}_r(M, M_N \cup Q_1)_0 \quad \text{and} \quad \hat{h} \in \text{Diff}_r(M, M_N \cup P)_0.$$  

2. Since $\overline{h} = a^{-1} g'h''$, it follows that

$$f = gh = g \overline{h} (\overline{h}^{-1} h) = g (a^{-1} g'h'') (h \hat{h}) = \hat{g}^{-1} [(\hat{g} g a^{-1} g h) (h' h)] \hat{g} = \hat{g}^{-1} [\hat{a} g h] \hat{g},$$

where (a) $\hat{g} := \hat{gg}' \in \text{Diff}_r(M, M_N \cup Q_1)_0$, (b) $\overline{h} := h' \hat{h} \in \text{Diff}_r(M, M_N \cup P)_0$, (c) $\overline{a} := \hat{gg}^{-1} g h \in \text{Diff}_r(M, M_N \cup P)_0$.  

4.2. Compact manifold case.

**Theorem 4.1.** Suppose $M$ is a compact $(2m+1)$-manifold possibly with boundary ($m \geq 0$) and $1 \leq r \leq \infty$, $r \neq 2m + 2$. Then $\text{cld Diff}_r(M, \partial)_0 \leq 4$ and $\text{clb}_d \text{Diff}_r(M, \partial)_0 \leq 4m + 6$.

**Compliment 4.1.** In Theorem 4.1 if $\mathcal{T}$ is a $C^\infty$ triangulation of $M$, $(P, Q) \equiv (|\mathcal{T}^{(m)}|, |(\mathcal{T}^*)^{(m)}|)$ and $\alpha := \text{clb}_d (P, (M - Q)^c)$ and $\beta := \text{clb}_d (Q, (M - P)^c)$, then $\text{clb}_d \text{Diff}_r(M, \partial)_0 \leq 2(\alpha + \beta + 1)$.

Recall that $O^c \equiv \widehat{O} := O \cup (\partial O \times [0, 1)) \in \mathcal{O}(\widetilde{M})$ for $O \in \mathcal{O}(M)$.

In the proof of Theorem 4.1 we use a triangulation (Example 3.1) instead of a handle decomposition and apply Lemma 3.1 to obtain the estimates on $cl$ and $clb$. This is our refinement for the existence of $\partial M$.

**Proof of Theorem 4.1 and Compliment 4.1.**

1. If $m = 0$, then $M$ is a finite disjoint union of closed intervals and circles. Since

$$\text{clb}_d \text{Diff}_r([0, 1], \partial)_0 \leq 2 \quad \text{and} \quad \text{clb}_d \text{Diff}_r(S^1, \partial)_0 \leq 3 \quad \text{(cf. [IS] Remark 3.1)},$$

it follows that

$$\text{cld Diff}_r(M, \partial)_0 \leq \text{clb}_d \text{Diff}_r(M, \partial)_0 \leq 3.$$  

Below we assume that $m \geq 1$.

2. Take any $C^\infty$ triangulation $\mathcal{T}$ of $M$ and let $(P, Q) = (|\mathcal{T}^{(m)}|, |(\mathcal{T}^*)^{(m)}|)$ (the $m$-skeleton of $\mathcal{T}$ and its dual $m$-skeleton). Given any $f \in \text{Diff}_r(M, \partial)_0$, there exists $F \in \text{Isot}_r(M, \partial)_0$ with $F_1 = f$.

For notational simplicity, we set $D_A := \text{Diff}_r(M, \partial M \cup A)_0$ for any subset $A$ of $M$.

We apply the argument in Subsection 4.1.2 to the data $\widetilde{M}, M, F, P, Q$. Here, we can identify as $\text{Isot}_r(M, \partial)_0 = \text{Isot}_r(\widetilde{M}, \widetilde{M}_M)_0$ and $\text{Diff}_r(M, \partial)_0 = \text{Diff}_r(\widetilde{M}, \widetilde{M}_M)_0$ canonically.

(i) In (Step 1) we obtain an isotopy $K \in \text{Isot}(M, \partial M \cup P)_0$ and $Q_1 := K_1(Q)$.

(ii) (Step 3) yields a factorization of $f$ in the following form: $f = \hat{g}^{-1} (agh) \hat{g}$, where

(a) $g, \hat{g} \in D_{Q_1}$,  
(b) $h \in D_P$,  
(c) $a \in \text{Diff}_r(M, M_U)_0$ for some $U \in B_f(\text{Int } M)$.

(iii) Let $k := K_1 \in D_P$. Since $Q_1 = k(Q)$, we obtain $g' := g k^{-1} \equiv k^{-1} g k \in D_Q$.  

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We apply Lemma 3.4 to the factors of the factorization of $f$ in (2)(ii).

From Example 3.1, it follows that

(i) $(P, Q) = ([T]^{(m)}), ([T^*]^{(m)}))$ is a pair of complimentary finite full subcomplexes of $sd T$,

(ii) $P$ is displaceable from $P \cup (\partial M \times [0,1])$ in $(M - Q^\sim)$ and $\alpha := clb^f (P, (M - Q^\sim)) \leq m + 1$,

(iii) $Q$ is displaceable from $Q \cup (\partial M \times [0,1])$ in $(M - P^\sim)$ and $\beta := clb^f (Q, (M - P^\sim)) \leq m + 1$.

Hence, we can apply Lemma 3.7 to the factors of the factorization of $f$.

We apply the argument in Subsection 4.1.2 to the data $M$.

Next, we apply Lemma 3.7 to $sd T$, $(Q, P)$, $L = P$ and $g' \in D_Q$ to obtain a factorization

$$g' = g_1 g_2$$

with $g_1 \in (D_Q)^c$, $clb^f (g_1) \leq 2\alpha$ in $D_Q$ and $g_2 \in (D_{Q \cup P})^c$, $clb^f (g_2) \leq 1$ in $D_{Q \cup P}$.

This induces the factorization $g = g^h = g_1^k g_2^k$. Since $k(P, Q) = (P, Q_1)$, it follows that

(iv) $g_1^k \in (D_{Q_1})^c$, $clb^f (g_1^k) \leq 2\alpha$ in $D_{Q_1}$ and $g_2^k \in (D_{Q_1 \cup P})^c$, $clb^f (g_2^k) \leq 1$ in $D_{Q_1 \cup P}$.

Next, we apply Lemma 3.7 to $sd T$, $(Q, P)$, $L = Q$ and $g^h \in D_P$ to obtain a factorization

(v) $g^h = h_1 h_2$ with $h_1 \in (D_P)^c$, $clb^f (h_1) \leq 2\beta$ in $D_P$, $h_2 \in (D_{P \cup Q})^c$, $clb^f (h_2) \leq 1$ in $D_{P \cup Q}$.

Therefore, in $Diff^r (M, \partial M)_0$ it follows that

(vi) $f = \tilde{g}^{-1} (agh) \tilde{g} = \tilde{g}^{-1} (ag^h_1 h_1^2) \tilde{g}$,

$$cl f \leq 4$$

and

$$clb f = clb (ag_1^h h_1^2) = clb f + clb h_1 + clb h_2$$

$$\leq 1 + 2\alpha + 2\beta + 1 \leq 4m + 6.$$

\[\square\]

Remark 4.1. In the last paragraph of Proof of Theorem 4.1, we decomposed the composition $\tilde{g} \tilde{h}$ instead of $\tilde{h}$. (The same argument has already appeared in the proof of Proposition 3.1.) We call this device an incorporation of a factor to the next factor.

For a $(2m + 1)$-manifold with a handle decomposition we have the following relative estimate on $clb^f$.

**Proposition 4.1.** Suppose $M$ is a $(2m + 1)$-manifold without boundary $(m \geq 0)$, $1 \leq r \leq \infty$, $r \neq 2m + 2$ and $\mathcal{H}$ is a handle decomposition of $M$.

1. $clb^f d (Diff^r (M, M_N)_0, Diff^r (M, M_{N_1})_0) \leq 2c(\mathcal{H}) + 2 \leq 2c(\mathcal{H}) + 2$ for any $N \in SM_c (M, \mathcal{H})$ and $N_1 \in SM_c (M, \mathcal{H}^*)$ with $N \subset N_1$.

2. If $M$ is closed, then $clb^f d Diff^r (M, M)_0 \leq 2c(\mathcal{H}) + 2$.

The estimate in [II] is compared with that in [I], that is,

$$clb^f Diff^r (M, M)_0 \leq 4c(\mathcal{H}) + 3 \quad ([I], \text{Proof of Theorem 1.5} \; \text{p.54}).$$

**Proof.** [I] The proof is similar to that of Theorem 4.1. We apply Lemma 3.4 instead of Lemma 3.7.

(1) If $m = 0$, then as in Proof of Theorem 4.1, $N$ is a finite disjoint union of closed intervals and circles and $clb^f d Diff^r (M, M_{N_0})_0 \leq 3$, while $c(\mathcal{H}) = 2$. Below we assume that $m \geq 1$.

(2) Let $(P, Q) = (P_{\mathcal{H}^*}^{(m)}, P_{\mathcal{H}^*}^{(m)})$. For notational simplicity, we set

$$\mathcal{D}_A := Diff^r (M, M_N \cup A)_0$$

and $\mathcal{D}_{1, A} := Diff^r (M, M_{N_1} \cup A)_0$ for $A \subset M$.

Given any $f \in Diff^r (M, M_N)_0$. There exists $F \in Isot^r (M, M_{N_0})_0$ with $F_1 = f$.

We apply the argument in Subsection 4.1.2 to the data $M$, $N$, $F$, $P$, $Q$.

(i) In (Step 1) we obtain an isotopy $K \in Isot (M, M_N \cup P)_0$ and $Q_1 := K_1 (Q)$.

(ii) (Step 3) yields a factorization of $f$ in the following form:

- (a) $g, \tilde{g} \in D_{Q_1}$,
- (b) $h \in D_P$,
- (c) $a \in Diff^r (M, U)_0$ for some $U \in B (\text{Int } N)$.

(iii) Let $k := K_1 \in D_P$. Since $k(M_N \cup Q) = M_N \cup Q_1$, we obtain $g' := g^{k-1} = k^{-1} gk \in D_Q$.

(3) We apply Lemma 3.4 to the factors of the factorization of $f$ in (2)(ii). We put
(i) \( \alpha := c(H_N^{(m)}) = c((P_{H_N})^{(m)}) \leq m + 1 \) and \( \beta := c(H_N^{(m)}) = c((P_{H_N})^{(m)})^{(m)} \leq m + 1 \).

Since \( H_N^{(m)} \subset H_N^{(m)} \) and \( H_N^{(m)} = \{h \in H_N \mid \text{the index of } h \in H \geq m + 1\} \), it follows that

(ii) \( \alpha + \beta \leq c(H_N^{(m)}) \).

First we apply Lemma \( \star \) to \( M, N, H, (P,Q) \) and \( g' \in D_Q \) to obtain a factorization

\[ g' = g_1g_2 \quad \text{with} \quad g_1 \in (D_Q)^c, \quad clb^f(g_1) \leq 2\alpha \quad \text{in} \quad D_Q \quad \text{and} \quad g_2 \in (D_{P,Q})^c, \quad clb^f(g_2) \leq 1 \quad \text{in} \quad D_{P,Q}. \]

This induces the factorization \( g = g^{k_f} = g_k^fg_k^g. \) Since \( k(P,Q) = (P,Q_1) \), it follows that

(iii) \( g_k^f \in (D_Q)^c, \quad clb^f(g_k^f) \leq 2\alpha \quad \text{in} \quad D_Q, \quad \text{and} \quad g_2^k \in (D_{Q_1,P})^c, \quad clb^f(g_2^k) \leq 1 \quad \text{in} \quad D_{Q_1,P}. \)

Next we apply Lemma \( \star \) to \( M, N, H, (Q,P) \) and \( g_k^f h \in D_{1,P} \) to obtain a factorization \( g_k^f h = h_1h_2, \)

(iv) \( h_1 \in (D_{1,P})^c, \quad clb^f(h_1) \leq 2\beta \quad \text{in} \quad D_{1,P} \quad \text{and} \quad h_2 \in (D_{1,P})^c, \quad clb^f(h_2) \leq 1 \quad \text{in} \quad D_{1,P}. \)

Therefore, in \( \text{Diff}^r(M,N_1) \) it follows that

(v) \( f = g^{-1}(agh)g = g^{-1}(ag_k^f h_1h_2)g, \)

\[ \quad cl f \leq 4 \quad \text{and} \quad clb^f f = clb^f(a g_k^f h_1h_2) = clb^f a + clb^f g_k^f h_1 + clb^f h_2 \]

\[ \leq 1 + 2\alpha + 2\beta + 1 \leq 2c(H_N^{(m)}) + 2. \]

Finally, the assertion \( \star \) follows from \( \star \) by taking \( N = N_1 = M. \)

\[ \square \]

4.3. Open manifold case.

**Theorem 4.2.** Suppose \( M \) is an open \((2m+1)\)-manifold \((m \geq 0), 1 \leq r \leq \infty, r \neq 2m+2 \) and \( H \) is any handle decomposition of \( M \). Then,

1. (i) \( cl\text{Diff}^r(M)_0 \leq 8 \) and (ii) \( clb^d\text{Diff}^r(M)_0 \leq 4c(H) + 4 \leq 8m + 12, \)
2. (i) \( cl\text{Diff}^e(M)_0 \leq 4 \) and (ii) \( clb^d\text{Diff}^e(M)_0 \leq 2c(H) + 2 \leq 4m + 6. \)

**Proof.** (1) Take any \( f \in \text{Diff}^r(M)_0 \). There exists \( F \in \text{Isot}^r(M)_0 \) with \( F_1 = f. \)

(i) By Lemma \( \star \) there exists an exhausting sequence \( \{M_k\}_{k \geq 1} \) in \( M \) such that

(a) \( M_k \in SM_c(M) \quad (k \geq 1) \quad \text{and} \quad F(M_{4k,4k+1} \times I) \subset M_{4k-1,4k+2} \times I \quad (k \geq 0). \)

Set \( M' := \bigcup_{k \geq 0} M_{4k+2,4k+3} \) and \( M'' := \bigcup_{k \geq 0} M_{4k,4k+1}. \) Then there exists a factorization \( F = GH \) for some \( G \in \text{Isot}^r(M,M')_0 \) and \( H \in \text{Isot}^r(M,M'')_0. \) This induces the factorization \( f = gh, \) where \( g = G_1 \in \text{Diff}^r(M,M')_0 \) and \( h = H_1 \in \text{Diff}^r(M,M'')_0. \)

For each \( k \geq 0 \) we have \( G|_{M_{4k-1,4k+2} \times I} \in \text{Isot}^r(M_{4k-1,4k+2},\partial)_0 \) and \( g|_{M_{4k-1,4k+2}} \in \text{Diff}^r(M_{4k-1,4k+2},\partial)_0 \).

By Theorem \( \star \) it follows that \( clg|_{M_{4k-1,4k+2}} \leq 4 \) in \( \text{Diff}^r(M_{4k-1,4k+2},\partial)_0 \). This implies that \( cl f \leq 4 \) in \( \text{Diff}^r(M,M'_0). \) Similarly \( cl f \leq 4 \) in \( \text{Diff}^r(M,M'')_0. \) Therefore \( cl f \leq 8 \) in \( \text{Diff}^r(M)_0. \)

(ii) We trace the argument in (i) with necessary modifications. For notational simplicity we set \( \alpha := 2c(H) + 2 \) and \( D_A := \text{Diff}^r(M,M_A)_0 \) for \( A \subset M. \) For \( N \in SM_c(M) \) and \( \eta \in \text{Diff}^r(N,\partial)_0 \) let \( \eta \in \text{Diff}^r(M,M)_0 \) denote the canonical extension of \( \eta \) by \( id. \)

By Lemma \( \star \) we may assume that the exhausting sequence \( \{M_k\}_{k \geq 1} \) satisfies the following additional conditions with respect to the handle decomposition \( H: \)

(a) There exist \( N_k' \in SM_c(M;H) \) and \( N_k'' \in SM_c(M;H') \) \((k \geq 0)\) such that for each \( k \geq 0 \)

\[ M_{4k-4k+2} \subset N_k' \subset N_k'' \subset M_{4k-2,4k+3} \quad \text{and} \quad N_k'' \cap N_k' = \emptyset. \]

(c) There exist \( L_k' \in SM_c(M;H) \) and \( L_k'' \in SM_c(M;H') \) \((k \geq 1)\) such that for each \( k \geq 1 \)

\[ M_{4k-3,4k} \subset L_k' \subset L_k'' \subset M_{4k-4,4k+1} \quad \text{and} \quad L_k'' \cap L_k' = \emptyset. \]
By Proposition 4.1 we have \( clb^f d(D_{N_k}^t, D_{N_k}) \leq \alpha (k \geq 0) \) and \( clb^f d(D_{L_k}^t, D_{L_k}) \leq \alpha (k \geq 1) \). Then, it follows that \( clb^f (g|_{M_{4k-1,4k+2}})^\sim \leq \alpha \) in \( D_{N_k}^t \) and \( clb^f (h|_{M_{4k-3,4k+1}})^\sim \leq \alpha \) in \( D_{L_k}^t \). This implies that \( clb^f g \leq \alpha \) and \( clb^f h \leq \alpha \) in \( \text{Diff}^r(M)_0 \). Therefore \( clb^f f \leq 2\alpha \) in \( \text{Diff}^r(M)_0 \).

(2) The assertions follow directly from Theorem 4.1 and Proposition 4.1.

5. Diffeomorphism groups of 2m-manifolds.

5.1. Factorization of isotopies on 2m-manifolds.

5.1.1. Whitney trick.

First we recall the Whitney trick used in [19] to remove inessential intersections between some m-strata and the tracks of some m-disks under an isotopy on a 2m-manifold. Below we assume that \( m \geq 3 \).

Setting 5.1. Consider the following situation:

(i) \( M \) is a 2m-manifold possibly with boundary \( (m \geq 3) \) and \( N \in SM_c(\text{Int} M) \).

(ii) \( E = \bigcup_{i=1}^s E_i \) is a disjoint union of closed m-disks in \( \text{Int} N \),

\[
D = \bigcup_{i=1}^s D_i, \quad D' = \bigcup_{i=1}^s D'_i, \quad D'' = \bigcup_{i=1}^s D''_i, \quad \text{where} \quad D_i \subseteq D_i' \subseteq D_i'' \subseteq E_i \quad \text{are closed m-subdisks of} \quad E_i \quad (i = 1, \cdots , s).
\]

(The case for \( s = 1 \) is described in [19].)

(iii) \( \Lambda \) is a m-dimensional stratified subset in \( M - E \).

(iv) \( \tilde{H} \in \text{Isot}(M, M_N)_0 \) and \( W \) is an open neighborhood of \( \pi_M \tilde{H}(D'' \times I) \) in \( \text{Int} N \).

Suppose \( \Lambda^{(m-1)} \cap \pi_M \tilde{H}(E \times I) = \emptyset \) and \( \tilde{H} = \text{id} \) on [a neighborhood of \( (E - \text{Int} D) \times I \)].

Basic Whitney trick. ([19] p161 - 165, Lemma 3.6])

1. Take a \( C^\infty \) approximation \( \tilde{H}' \in \text{Isot}(M, M_N)_0 \) of \( \tilde{H} \) such that (a) \( \tilde{H}' = \tilde{H} \) on \( (M - W) \times I \),

(b) \( \pi_M \tilde{H}'|_{D' \times I} \) is a \( C^\infty \) immersion outside a 1-dimensional subset and it is generic with respect to \( D, \Lambda \) and self-intersections, and \( \pi_M \tilde{H}'((D' - \text{Int} D) \times I) \) has no double point,

(c) \( \pi_M \tilde{H}'((E - \text{Int} D) \times I) \cap \pi_M \tilde{H}'((\text{Int} D) \times I') = \emptyset \), \( \pi_M \tilde{H}'((E - \text{Int} D) \times I) \cap \Lambda = \emptyset \) and \( \tilde{H}' = \text{id} \) on [a neighborhood of \( (E - \text{Int} D'') \times I \)].

2. In this situation the intersection \( \Gamma := \Lambda \cap \pi_M \tilde{H}'(D \times I) \) is a compact 1-manifold as a trace of finite points \( \Lambda \cap \tilde{H}'(D) \) (\( t \in I \)). Since \( \Lambda \cap D = \emptyset \), the algebraic intersection number between \( \Lambda \) and each level \( \tilde{H}'_t(D) \) is zero and there are finitely many generations and cancellations of intersection points.

3. A pre-Whitney disk is constructed by tracing a pair of intersection points starting from each generation point of intersection and connecting them in each level by the image of a geodesic segment in \( D \), where each \( D_i \) is regarded as a round m-disk with a flat metric. If two such pairs of intersection points meet at a cancellation point of intersection in a level \( \tilde{H}'_t(D) \), the remaining two intersection points from those pairs are coupled and their traces are connected by a geodesic segment in each level. The gap among three geodesic segments in the level \( \tilde{H}'_t(D) \) is fulfilled by a geodesic triangle in this level.

(i) If the trace of a pair of intersection points reaches the 1-level \( \tilde{H}'_t(D) \), we obtain an arc component of \( \Gamma \) and a disk bounded by this arc and an arc in \( \tilde{H}'_t(D) \). The Whitney disk for this arc component is obtained by smoothing this disk near each geodesic triangle.

(ii) If the trace of a pair of intersection points self-intersects at a cancellation point \( \tilde{H}'_t(v) \) in a level \( \tilde{H}'_t(D) \), then we obtain a circle component of \( \Gamma \) and a disk bounded by this circle. The Whitney disk associated to this circle component is obtained by smoothing this disk near each geodesic triangle and add a thin band along the arc \( \pi_M \tilde{H}'(\{v\} \times [t,1]) \) in \( \pi_M \tilde{H}'(D \times I) \).
Since $\tilde{H}'$ is generic, it follows that

(iii) these Whitney disks are pairwise disjoint and embedded in $\pi_M \tilde{H}(D \times I) - D$, and in $\pi_M \tilde{H}(D \times I)$ they have (a) no double points if $m \geq 4$ and (b) finitely many double points if $m = 3$. In the case (b), at each double point we take the branch arc of the form $\pi_M \tilde{H}'(\{v_i\} \times [t_i, 1])$, which has no double points except $\pi_M \tilde{H}'(v_i, t_i)$.

(4) Let $\tilde{U}$ be the disjoint union of small $2m$-disk neighborhoods of these Whitney disks (together with the branch arcs in the case $m = 3$) in $W = (E \cup \pi_M \tilde{H}'((E - \text{Int} D) \times I))$. Then there exists $\tilde{A} \in \text{Isot}(M, M_U)_0$ such that $\pi_M \tilde{A} \tilde{H}'(D \times I) \cap \Lambda = \emptyset$ and $\tilde{A}_1 \in \text{Diff}(M, M_U)_0$. Note that $\pi_M \tilde{A} \tilde{H}'((E \times I) \cap \Lambda = \emptyset$.

5.1.2. Factorization of isotopies on $2m$-manifolds.

**Setting 5.2.** Suppose $M$ is an $2m$-manifold possibly with boundary $(m \geq 1)$, $N$ is a compact $2m$-submanifold of $\text{Int} M$ and $F \in \text{Isot}^r(M, M_N)_0$. Suppose $P$ and $Q$ are $m$-dimensional stratified subsets of $M$ such that $P^{(m-1)} \cap Q = P \cap Q^{(m-1)} = \emptyset$.

**Review of the arguments in [19] and some refinements in our setting.**

**Step 1** First we remove the intersections between low dimensional skeletons of $Q$ and $\pi_M F(P \times I)$.

1. There exists an arbitrarily small isotopy $K \in \text{Isot}(M, M_N \cup P^{(m-1)})_0$ with support in an arbitrarily small neighborhood of $Q$ such that $Q_1 := K_1(Q)$ satisfies the following conditions:
   
   0) $Q_1^{(m-1)} \cap P = \emptyset$
   
   1) $Q_1 \cap \pi_M F((P^{(m-2)} \times I) \cup (P^{(m-1)} \times \{0, 1\})) = \emptyset$
   
   2) $Q_1^{(m-1)} \cap \pi_M F(P^{(m-1)} \times I) = \emptyset$
   
   3) $Q_1^{(m-2)} \cap \pi_M F(P \times I) = \emptyset$.

2. By (1)(i) and Lemma [2.7] there exists a factorization $F = GH$ for some $G \in \text{Isot}^r(M, M_N \cup Q_1)_0$ and $H \in \text{Isot}^r(M, M_N \cup P^{(m-2)})_0$. From (1)(ii), (iii) it follows that
   
   1) $Q_1^{(m-1)} \cap \pi_M H(P^{(m-1)} \times I) = \emptyset$
   
   2) $Q_1^{(m-2)} \cap \pi_M H(P \times I) = \emptyset$.

* Below assume that $m \geq 2$.

**Step 2** Next we remove the intersection $Q_1 \cap \pi_M H(P^{(m-1)} \times I)$. (cf. Proof of Theorem [4.1] Case [I])

1. Choose a $C^\infty$ approximation $\overline{P} \in \text{Isot}(M, M_N \cup P^{(m-2)})_0$ of $H$ which is generic with respect to $P$ and $Q_1$ (so that (2) holds). If $\overline{P}$ is sufficiently close to $H$, then
   
   1) $Q_1^{(m-1)} \cap \pi_M \overline{P}(P^{(m-1)} \times I) = \emptyset$
   
   2) $Q_1^{(m-2)} \cap \pi_M \overline{P}(P \times I) = \emptyset$.

2. There exists $U \in B_f(\text{Int} N - (P^{(m-1)} \cup Q_1^{(m-1)}))$ and $A \in \text{Isot}(M, M_U)_0$ such that
   
   $Q_1 \cap \pi_M A \overline{P}(P^{(m-1)} \times I) = \emptyset$ and $A_1 \in \text{Diff}(M, M_U)_0$. Let $H' := A \overline{P} \in \text{Isot}(M, M_N \cup P^{(m-2)})_0$.

3. There exists a factorization $H' = G'H''$ for some
   
   $G' \in \text{Isot}(M, M_N \cup Q_1)_0$ and $H'' \in \text{Isot}(M, M_N \cup P^{(m-1)})_0$.
   
   We have $Q_1^{(m-2)} \cap \pi_M H''(P \times I) = \emptyset$ since $Q_1^{(m-2)} \cap \pi_M H'(P \times I) = \emptyset$.

* From here we assume that $m \geq 3$.

**Step 3** We remove the intersection $Q_1^{(m-1)} \cap \pi_M H''(P \times I)$. (cf. Proof of Theorem [4.1] Case [I])

1. Take a $C^\infty$ approximation $\overline{H''} \in \text{Isot}(M, M_N \cup P^{(m-1)})_0$ of $H''$ which is generic with respect to $P$ and $Q_1$. We have $Q_1^{(m-2)} \cap \pi_M \overline{H''}(P \times I) = \emptyset$.

2. There exists $U'' \in B_f(\text{Int} N - (P \cup Q_1^{(m-2)}))$ and $A'' \in \text{Isot}(M, M_U^{(m-1)})_0$ such that
   
   $Q_1^{(m-1)} \cap \pi_M A'' \overline{H''}(P \times I) = \emptyset$ and $A_1'' \in \text{Diff}(M, M_U^{(m-1)})_0$. Let $H''' := A'' \overline{H''} \in \text{Isot}(M, M_N \cup P^{(m-1)})_0$. 

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In [19] Step 3 is incorporated to the inductive argument in Step 4 (2) below and repeated in each inductive step.

Until now we have obtained \( H^{(0)} := H^m \in \text{Isot}(M, M_N \cup P^{(m-1)})_0 \) with \( Q_1^{(m-1)} \cap \pi_M H^{(0)}(P \times I) = \emptyset \).

**Setting 5.2**. Let \( S \) denote the set of all \( m \)-strata of \( P \) and we put
\[
S_0 := \{ \sigma \in S \mid H^{(0)} \equiv \text{id on } [\text{a neighborhood of } \sigma] \times I \} \quad \text{and} \quad S_c := \{ \sigma \in S - S_0 \mid \sigma \text{ is an open } m\text{-disk}, \sigma \subset \text{Int } N \}.
\]

Suppose \( \mathcal{E} \) is a finite subset of \( S_c \) and \( \mathcal{E} = \bigcup_{j=1}^k \mathcal{E}_j \) is a partition of \( \mathcal{E} \) \((k \geq 1)\). Let
\[
\Sigma_0 := P^{(m-1)} \cup |S_0| \quad \text{and} \quad \Sigma_j := P^{(m-1)} \cup |S_0| \cup \left( \bigcup_{i=1}^j |\mathcal{E}_i| \right) \quad (j = 1, \ldots, k).
\]

Let \( U \) denote the set of all \( m \)-strata of \( Q_1 \) and let
\[
\Lambda_j := Q_1^{(m-1)} \cup \{ \tau \in U \mid \tau \cap |\mathcal{E}_j| = \emptyset \} \quad (j = 1, \ldots, k).
\]

Note that

(i) \( \Sigma_j = \Sigma_{j-1} \cup |\mathcal{E}_j| \) (a disjoint union),
(ii) \( \Lambda_j^{(m-1)} = Q_1^{(m-1)} \), \( \Lambda_j \cap |\mathcal{E}_j| = \emptyset \) and \( \Sigma_j \cap (\Sigma_{j-1} \cup \Lambda_j) = \Sigma_{j-1} \).
(iii) \( H^{(0)} \in \text{Isot}(M, M_N \cup S_0)_0 \) and \( \pi_M H^{(0)}(P \times I) \cap Q_1^{(m-1)} = \emptyset \).

(Step 4) This is the main step in which the Whitney trick is used to remove inessential intersections between \( Q_1 \) and \( \pi_M H^{(0)}(P \times I) \) over \( m \)-strata.

Starting from \( H^{(0)} \), inductively we construct isotopies \( H^{(j)} \), \( A^{(j)} \), \( H'^{(j)} \), \( G^{(j)} \), \( H^{(j)} \) \((j = 1, \ldots, k)\) as follows. Suppose \( H^{(j-1)} \in \text{Isot}(M, M_N \cup \Sigma_{j-1})_0 \) is obtained so that \( \pi_M H^{(j-1)}(P \times I) \cap Q_1^{(m-1)} = \emptyset \).

[1] We apply Basic Whitney trick to the isotoype \( H^{(j-1)} \), the \( m \)-dimensional stratified subset \( \Lambda_j \), disjoint unions of closed \( m \)-disks \( D \subset D' \subset D'' \subset E \subset |\mathcal{E}_j| \) and an open neighborhood \( W \) of \( \pi_M H^{(j-1)}(D'' \times I) \) in \( \text{Int } N \). Here, \( D, D', D'', E \) and \( W \) are chosen as follows:

(i) There exists an open neighborhood \( V \) of \( M_N \cup \Sigma_{j-1} \) in \( M \) such that \( H^{(j-1)}|_{V \times I} = \text{id} \). Let \( \mathcal{E}_j = \{ \sigma_i \}_{i=1}^8. \) (If \( \mathcal{E}_j = \emptyset \), then the argument below is formally skipped.) Then each \( \sigma_i \) is an open \( m \)-disk in \( \text{Int } N \) and \( \sigma_i - V \) is a compact subset of \( \sigma_i \). Hence we can take closed \( m \)-disks \( D_i \subset D'_i \subset E_i \) in \( \sigma_i \) with \( \sigma_i - V \subset D_i \). Let \( D := \bigcup_{i=1}^8 D_i \), \( D' := \bigcup_{i=1}^8 D'_i \), \( D'' := \bigcup_{i=1}^8 D''_i \) and \( E := \bigcup_{i=1}^8 E_i \). These satisfy the condition (vi) in Setting 5.1 since \( \pi_M H^{(j-1)}(P \times I) \cap Q_1^{(m-1)} = \emptyset \) and \( E - \text{Int } D \subset V \).
(ii) From the assumption and (i) it follows that
\[
\pi_M H^{(j-1)}(D'' \times I) \subset O := \text{Int } N - (\Sigma_{j-1} \cup (|\mathcal{E}_j| - \text{Int } E) \cup Q_1^{(m-1)}) \in \mathcal{O}(M).
\]
We take the neighborhood \( W \) so that \( Cl_M W \subset O \).

[2] Basic Whitney trick yields the following data:

(a) \( H^{(j)} \in \text{Isot}(M, M_N)_0 \) : a \( C^\infty \) approximation of \( H^{(j-1)} \) with \( H^{(j)} = H^{(j-1)} \) on \( (M-W) \times I \),
(b) \( U_j \in \mathcal{B}_f(W - E) \),
(c) \( A^{(j)} \in \text{Isot}(M, M_{U_j})_0 \) such that \( \pi_M A^{(j)} H^{(j)}(E \times I) - \Lambda_j = \emptyset \) and \( A^{(j)} \in \text{Diff}(M, M_{U_j})_0 \).

The isotopies \( H^{(j)} \), \( A^{(j)} \) and \( H'^{(j)} := A^{(j)} H^{(j)} \) satisfy the following conditions:

(i) \( M_{U_j} \supset M - W \supset M - Cl_M W \supset M_N \cup \Sigma_{j-1} \cup (|\mathcal{E}_j| - \text{Int } E) \cup Q_1^{(m-1)} \)

since \( U_j \subset W \subset Cl_M W \subset O \).

V \supset |\mathcal{E}_j| - \text{Int } E \quad \text{since } \text{Int } E \supset D_i \supset \sigma_i - V \quad (i = 1, \ldots, s).

Hence, \( V - Cl_M W \supset M_N \cup \Sigma_{j-1} \cup (|\mathcal{E}_j| - \text{Int } E) \).

(ii) On \( (V - W) \times I \), we see that \( H^{(j)} = H^{(j-1)} = \text{id}, A^{(j)} = \text{id} \) by (i), and so \( H'^{(j)} = \text{id} \).
Hence \( H^{(j)}, A^{(j)}, H'^{(j)} \in \text{Isot}(M, M_N \cup \Sigma_{j-1})_0 \) by (i).
(iii) \( \pi_M H^{n(j)}(\mathcal{E}_j \times I) \cap \Lambda_j = \emptyset \). This follows from (c) and the next observation:

\[
\pi_M H^{n(j)}((|\mathcal{E}_j| - E) \times I) = |\mathcal{E}_j| - E \subset M - \Lambda_j
\]

since \( H^{n(j)} = \text{id} \) on \((V - W) \times I\) by (ii) and \( \Lambda_j \cap |\mathcal{E}_j| = \emptyset \) by Setting 5.2(ii).

(iv) \( \pi_M H^{n(j)}(P \times I) \cap Q_1^{(m-1)} = \emptyset \). In fact, by the assumption \( H^{(j-1)} \) satisfies the same condition and so does \( H^{(j)} \) as a fine \( C^\infty \) approximation of \( H^{(j-1)} \). Since \( A^{(j)} = \text{id} \) on \( Q_1^{(m-1)} \times I \) by (i),

the conclusion holds.

[3] To obtain a factorization of \( H^{n(j)} \) based on [2](iii) we apply Lemma 2.7 to \( N, K = \Sigma_j \cap N, L = (\Sigma_{j-1} \cup \Lambda_j) \cap N \) and \( H^{n(j)}|_{N \times I} \). Note that

(i) \( K \cap L = \Sigma_{j-1} \cap N \) and \( K - L = |\mathcal{E}_j| \cap N \) by Setting 5.2(i), (ii),

(ii) \( H^{n(j)} \in \text{Isot}(M, M_N \cup \Sigma_{j-1})_0 \) and \( \pi_M H^{n(j)}(|\mathcal{E}_j| \times I) \cap (\Sigma_{j-1} \cup \Lambda_j) = \emptyset \) by [2](ii), (iii).

Lemma 2.7 induces a factorization \( H^{n(j)} = G^{(j)} \circ H^{(j)} \) for some

\[ G^{(j)} \in \text{Isot}(M, M_N \cup \Sigma_{j-1} \cup \Lambda_j)_0 \quad \text{and} \quad H^{(j)} \in \text{Isot}(M, M_N \cup \Sigma_j)_0. \]

From [2](iv) and the facts that \( G^{(j)} = \text{id} \) on \( \Lambda_j \times I \) and \( Q_1^{(m-1)} \subset \Lambda_j \), it follows that \( H^{(j)} = (G^{(j)})^{-1} H^{n(j)} \) also satisfies the next condition:

(iii) \( \pi_M H^{(j)}(P \times I) \cap Q_1^{(m-1)} = \emptyset \).

This completes the inductive step.

In [19] the inductive argument in Setp 4 is applied to each open \( m \)-cell in \( S_c \). The grouping of \( m \)-cells in Setting 5.2 may decrease the number \( k \) of inductive steps in Setp 4 effectively. This advantage is essentially used to treat open \( 2m \)-manifolds in Section 5.3.

(Step 5) Factorization of \( f \)

We denote the 1-levels of the isotopies appeared in Steps 1 - 4 by the corresponding small letters.

As a summary of the previous steps, we have the following factorizations:

1. Steps 1 - 3:

   \[
   f = gh, \quad \overline{h} = h, \quad a\overline{h} = h' = g'h'', \quad \overline{h'} = h''', \quad \overline{h''} = a'\overline{h'''}.
   \]

   \[
   \therefore f = gh = g\overline{h}(\overline{h}^{-1}h) = g(a^{-1}h')(\overline{h}^{-1}h) = g^{-1}g'h''(\overline{h}^{-1}h) = g^{-1}g'(a''h''')(\overline{h}^{-1}h).
   \]

2. Step 4:

   \[
   h^{(0)} = h'', \quad h^{(j)} = h^{(j-1)}, \quad a^{(j)} = g^{(j)}h^{(j)} \quad (j = 1, \ldots, k).
   \]

   (i) \( h^{(j-1)} = h^{(j)}(h^{(j-1)}h^{(j-1)}) = (a^{(j)}g^{(j)})h^{(j)}(h^{(j-1)}h^{(j-1)}) \)

   (ii) \( h^{(m)} = (a^{(1)}g^{(1)})h^{(1)}(h^{(1)}h^{(1)}h^{(0)}) \)

   \[
   = (a^{(1)}g^{(1)})(a^{(2)}g^{(2)})h^{(1)}(h^{(2)}h^{(1)}h^{(0)}) = (a^{(1)}g^{(1)})(a^{(2)}g^{(2)})(a^{(3)}g^{(3)})h^{(1)}h^{(2)}h^{(1)}h^{(0)}.
   \]

   \[
   = \cdots = [\prod_{j=1}^{k}(a^{(j-1)}g^{(j)})h^{(k)}[\prod_{j=k}^{l}(h^{(j)}h^{(j-1)})].
   \]

   (iii) \( f = g^{-1}g'g''(\prod_{j=1}^{k}(a^{(j-1)}g^{(j)})h^{(k)}[\prod_{j=k}^{l}(h^{(j)}h^{(j-1)})]. \)

3. \( f_1 := g^{-1}g'a''(\prod_{j=1}^{k}(a^{(j-1)}g^{(j)})) \in \text{Diff}^r(M, M_N)_0 \):

   (i) \( f_1 = [g^{-1}g''(gg')\prod_{j=1}^{k}(a^{(j-1)}g^{(j)}) \)

   (ii) Let \( b_{-1} = g^{-1}g', \quad b_0 = gg'a''(gg')^{-1}, \quad g_0 = gg' \) and

   \[
   g_j = g^{(j)}, \quad a_j = a^{(j)}^{-1}, \quad b_j = a_{j+1}g_{j+1} \quad (j = 1, \ldots, k).
   \]

From Fact 2.1 it follows that
\begin{align*}
(gg') \prod_{j=1}^{k} (a^{(j)^{-1}}g^{(j)}) &= g_0 \prod_{j=1}^{k} (a_j g_j) = \left[ \prod_{j=1}^{k} (g_{j-1} a_j) \right] g_k \\
&= \left[ \prod_{j=1}^{k} b_j \right] g_0 g_1 \cdots g_k = \left[ \prod_{j=1}^{k} b_j \right] (gg') \left[ \prod_{j=1}^{k} g^{(j)} \right] \quad \text{and} \\
f_1 &= \left[ \prod_{j=1}^{k} b_j \right] (gg') \left[ \prod_{j=1}^{k} g^{(j)} \right]
\end{align*}

(iii) Note that \( gg' \in \Diff^r(M, M_N \cup Q_1)_0 \), \( g^{(j)} \in \Diff^r(M, M_N \cup \Sigma_{j-1} \cup \Lambda_j)_0 \) \((j = 1, \ldots, k)\) and \( b_j \in \Diff^r(M, M_{V_j})_0^1\) for some \( V_j \in B_f(\text{Int } N) \) \((j = -1, 0, 1, \ldots, k)\).

(4) \( f_2 := \left[ \prod_{j=1}^{k} (h^{(j)^{-1}} - h^{(j-1)}) \right] (h^{r-1} h') \in \Diff^r(M, M_N_0) \):

(i) \( f_2 \) is chosen arbitrarily close to \( \text{id}_M \).

(ii) If \( P \cap Q_1 \cap \text{Int } N \) is a finite set, then there is \( \hat{U} \in B_f(\text{Int } N) \) with \( P \cap Q_1 \cap \text{Int } N \subset \text{Int } \hat{U} \).

By (i) for the open cover \{ \text{Int } N - P, \text{Int } \hat{U}, \text{Int } N - Q_1 \} \) of \( \text{Int } N \), there is a factorization \( f_2 = \hat{h} \hat{a} \hat{g} \) such that \( \hat{h} \in \Diff^r(M, M_N \cup P)_0 \), \( \hat{a} \in \Diff^r(M, M_\hat{U})_0 \), \( \hat{g} \in \Diff^r(M, M_N \cup Q_1)_0 \).

By \([18] \) Remark 2.1 we can modify \( \hat{a} \) and \( \hat{g} \), so that \( \hat{a} \in \Diff^r(M, M_\hat{U})_0 \).

(5) The case where \( P \cap Q_1 \cap \text{Int } N \) is a finite set :

(i) There is a factorization \( f_2 = \hat{h} \hat{a} \hat{g} \) as in (4)(ii). Let

\[
\begin{align*}
b_{-2}' &= \hat{a}, \quad b_j' := b_j \quad (j = -1, 0, \ldots, k) \quad \text{and} \\
g^{(1)} &= \hat{g}(gg')^{(1)}, \quad g^{(j)} := g^{(j)} \quad (j = 2, \ldots, k), \quad h^{(k)} := h^{(k)} \hat{h}.
\end{align*}
\]

Then it follows that \( \hat{g} \left[ \prod_{j=1}^{k} b_j \right] = \hat{g} \left[ \prod_{j=1}^{k} b_j \right] \hat{g}^{-1} \hat{g} = \left[ \prod_{j=1}^{k} b_j \right] \hat{g} \) and

\[
f = f_1 h^{(k)} f_2 = \left[ \prod_{j=1}^{k} b_j \right] (gg') \left[ \prod_{j=1}^{k} g^{(j)} \right] h^{(k)} \hat{a} \hat{g}
\]

(ii) Note that \( g^{(j)} \in \Diff^r(M, M_N \cup \Lambda_j)_0 \) \((j = 1, \ldots, k)\), \( h^{(k)} \in \Diff^r(M, M_N \cup \Sigma_k)_0 \) and \( b_j' \in \Diff^r(M, M_{V_j})_0 \) for some \( V_j' \in B_f(\text{Int } N) \) \((j = -2, -1, 0, 1, \ldots, k)\)

In the subsequent subsections we obtain some estimates of \( clf \) and \( clb^f f \) from the factorization of \( f \) in Step 5(5).

5.2. Compact manifold case.

5.2.1. Compact manifold case I — Triangulations.

Setting 5.3. Suppose \( N \) is a compact \( 2m \)-manifold possibly with boundary \((m \geq 3), 1 \leq r \leq \infty, r \neq 2m + 1\), \( T \) is a \( C^\infty \) triangulation of \( N \), \((P, Q) := (|T^{(m)}|, |T^{*m}|)\), \( S \) is the set of \( m \)-simplices of \( T \), \( S' := \{ \sigma \in S \mid \sigma \not\subseteq \partial N \} \) and \( \bar{N} := N \cup \partial N (\partial N \times [0, 1]) \).

Setting 5.3. Suppose \( S \supset F \supset S' \), \((F_j)_{j=1}^{k} \) is a finite cover of \( F \) (as a set), \( G_j := S - F_j, (P_{F_j}, Q_{G_j}) := (P^{(m-1)} \cup |F_j|, Q^{(m-1)} \cup |G_j|) \), \( O_j := N - Q_{G_j} \) and \( \ell_j := clb^f (P_{F_j}, \bar{O}_j) \quad (j = 1, \ldots, k) \).

**Theorem 5.1.** In Setting 5.3.5.3.:

[1] \( cld \Diff^r(N, \partial N)_0 \leq 3k + 5 \) \quad and \quad \( clb^d \Diff^r(N, \partial N)_0 \leq 2 \left( \sum_{j=1}^{k} \ell_j + k + m + 2 \right) \leq 2(m + 2)(k + 1) \)

if each \( |F_j| \) \((j = 1, \ldots, k)\) is strongly displaceable from \( |F_j| \cup (\partial N \times [0, 1]) \) in \( \bar{N} \).
[II] \( \text{cld} \text{Diff}^r(N, \partial N)_0 = 2k + 7 \) and \( \text{clb}^d \text{Diff}^r(N, \partial N)_0 \leq 2 \sum_{j=1}^k \ell_j + k + 2m + 6 \leq (2m + 3)(k + 1) + 3 \)

if each \( |\mathcal{F}_j| \quad (j = 1, \ldots, k) \) is strongly displaceable from \( |S| \cup (\partial N \times [0, 1)) \) in \( \tilde{N} \).

Compliment 5.1.

(1) (i) \( \ell_j \leq m + 1 \).

(ii) If \( K_j \) is a compact \( C^\infty \) subpolyhedron in \( N \) with \( K_j \subset P_{\mathcal{F}_j} \) and \( P_{\mathcal{F}_j} \) is weakly absorbed to \( K_j \) in \( \tilde{O}_j \) keeping \( K_j \) invariant, then \( \ell_j \leq \text{clb}^d(K_j, \tilde{O}_j) \leq \dim K_j + 1 \).

(2) The condition in the case [II] holds, for example, if each \( |\mathcal{F}_j| \) has an arbitrarily small 2m-disk neighborhood in \( \tilde{N} \).

Corollary 5.1. In Setting 5.3, Let \( \ell := \#S' \) (i.e., the number of \( m \)-simplices of \( \mathcal{T} \) not in \( \partial N \)). Then

\[
\text{cld} \text{Diff}^r(N, \partial)_0 \leq 2\ell + 7 \quad \text{and} \quad \text{clb}^d \text{Diff}^r(N, \partial N)_0 \leq (2m + 1)(\ell + 1) + 5.
\]

When \( N \) is a closed 2m-manifold, the estimates in Corollary 5.1 are compared with those in [19], that is, \( \text{cl Diff}^r(N)_0 \leq 4\ell + 11 \) (the last sentence in [19, Proof of Theorem 1.2] (p.166)),

\[
\text{clb}^d \text{Diff}^r(N)_0 \leq 4(\ell + 4)m + 3\ell + 7 \quad ([19, Proof of Corollary 1.3] (p.173)).
\]

Proof of Theorem 5.1.

We apply the argument in Subsection 5.1.2 in \( M \equiv \tilde{N} \). Take any \( f \in \text{Diff}^r(N, \partial N)_0 \equiv \text{Diff}^r(M, M_N)_0 \) and \( F \in \text{Isot}^r(M, M_N)_0 \) with \( F_1 = f \). Our task is to estimate \( \text{cl} f \) and \( \text{clb}^d f \) in \( \text{Diff}^r(M, M_N)_0 \).

The cell complex \( Q \) underlies a subcomplex of \( sd \mathcal{T} \). We denote this subcomplex by \( Q \). To apply (Step 4) we prefer \( \tilde{Q} \) consisting of smooth closed simplices rather than \( Q \) itself consisting of piecewise smooth cells. For each \( m \)-simplex \( \sigma \) of \( P \) its dual \( \sigma^* \) underlies a subcomplex of \( \tilde{Q} \) which we denote by \( \tilde{Q}^* \). Note that \( Q^{(m-1)} \subsetneq Q^{(m-1)} \tilde{Q} \) and \( P \cap Q^{(m-1)} \neq \emptyset \).

(1) We define \( m \)-dimensional stratified subsets \( P_0 \) and \( Q_0 \) of \( N \) as follows:

(i) Let \( P_0 := |\mathcal{T}^{(m-1)}| \cup |S'| \). This is a subcomplex of \( P \) and \( M_N \cup P_0 = M_N \cup P \).

(ii) Take a small neighborhood \( U \) of the finite set \( P_0 \cap Q = \bigcup \{ \sigma \cap \sigma^* \mid \sigma \in S' \} \) in \( \text{Int} N \) \( (P^{(m-1)} \cup Q^{(m-1)}) \) which is a finite disjoint union of closed 2m-disks. Then, there exists \( K' \in \text{Isot}(M, M_N)_0 \) such that each \( \sigma \in S' \) intersects a unique open \( m \)-simplex in \( K'_1(\tilde{\sigma}^*) \) transversely at one point. Let \( Q_0 := K'_1(Q) \).

Note that \( P_0^{(m-1)} \cap Q_0 = P_0 \cap Q_0^{(m-1)} = \emptyset \).

(2) We apply (Step 1) \( \sim \) (Step 3) in Subsection 5.1.2 to \( (M, N, F, P_0, Q_0) \) to obtain the corresponding factorization of the isotopy \( F \).

(i) In (Step 1) (1) we obtain the isotopy \( K \) and \( Q_1 = K_1(Q_0) \).

In Setting 5.2 (replacing open simplices by closed simplices) we have

(ii) [the set of \( m \)-simplices of \( P_0 \)] = \( S' \),

\[ S_0 = \{ \sigma \in S' \mid H^{(0)} \equiv \text{id} \text{ on } [\text{a neighborhood of } \hat{\sigma} ] \times I \} \quad \text{and} \quad S_c = S' - S_0 \].

Let \( \mathcal{E} = S_c \) and take a partition \( \mathcal{E} = \bigcup_{j=1}^k \mathcal{E}_j \) such that \( \mathcal{E}_j \subset \mathcal{F}_j \) \( (j = 1, \cdots, k) \). (For example, let \( \mathcal{E}_j := \mathcal{E} \cap \mathcal{F}_j - \bigcup_{i=1}^{j-1} \mathcal{F}_i \).) Then, it follows that

(iii) (a) \( S_k = P_0 \), (b) \( \mathcal{U} = \) the set of \( m \)-simplices of \( Q_1 \), and

(c) \( A_j := Q_1^{(m-1)} \cup \bigcup \{ \tau \in \mathcal{U} \mid \hat{\sigma} \cap \mathcal{E}_j = \emptyset \} \).

We apply (Step 4) and (Step 5) to \( \{ \mathcal{E}_j \}_{j=1}^k \) to obtain a factorization of \( f \) of the following form:
(iv) \( f = (\hat{a} \hat{g})^{-1} \left[ \prod_{j=1}^{k} g^{(j)} \right] \hat{a} \hat{g}, \) where

(a) \( g^{(j)} \in \text{Diff}^r(M, M_N \cup \Lambda_j)_0 \) \((j = 1, \cdots, k),\)
(b) \( h^{(k)} \in \text{Diff}^r(M, M_N \cup P)_0,\)
(c) \( b^{(j)}_j \in \text{Diff}^r(M, M_N)_0 \) for some \( V_j \in \mathcal{B}_f(\text{Int } N) \) \((j = -2, -1, 0, 1, \cdots, k).\)

(3) Next we apply Examples 5.1, 5.2 and Lemma 5.7 to this situation.

Let \( L_j := P_{F_j} \) in \([I]\) and \( L_j = P \) in \([II]\) \((j = 1, \cdots, k).\) It follows that

(i) \( cld \text{Diff}^r(N, \partial N \cup P)_0 \leq 2 \) and \( clb^f d \text{Diff}^r(N, \partial N \cup P)_0 \leq 2m + 1,\)
(ii) any \( g \in \text{Diff}^r(N, \partial N \cup Q_{g, j})_0 \) has a factorization \( g = g_1 g_2 \) such that

\[
\begin{align*}
&g_1 \in \text{Diff}^r(N, \partial N \cup Q_{g, j})_0 \quad \text{and} \quad \text{clb}^f (g_1) \leq 2 \ell_j \text{ in } \text{Diff}^r(N, \partial N \cup Q_{g, j})_0, \\
&g_2 \in \text{Diff}^r(N, \partial N \cup Q_{g, j} \cup L_j)_0 \quad \text{and} \quad \text{clb}^f (g_2) \leq 1 \text{ in } \text{Diff}^r(N, \partial N \cup Q_{g, j} \cup L_j)_0.
\end{align*}
\]

The assertion (i) follows from Example 3.1 and Lemma 3.7 for \( (Q^{(m-1)}, P). \) The assertion (ii) follows from Example 3.2, Lemma 3.7 for \((P_{F_j}, Q_{g, j})\) and the assumption on \( F_j. \) We also note that Complement 5.1(1) follows from Example 3.2 for \((P_{F_j}, Q_{g, j}).\)

(4) In order to apply the estimates on \( cl \) and \( cb^f \) in (3) to (2)(iv), we need to transform them to the corresponding estimates for \( Q_{g, j}, \) using the diffeomorphism \( \varphi := K_1 K'_1 \in \text{Diff}(M, M_N)_0.\)

We use the following notations :

(i) \((P_1, Q_1, Q_{g_1}) := \varphi(P_0, Q, Q_{g_0})\) (as a cell complex), \( L_{1,j} := \varphi(L_j) \) \((j = 1, \cdots, k),\)

\( \sigma^b := \varphi(\sigma^*) \) for \( \sigma \in \mathcal{S} \) and \( \mathcal{C}^b := \{ \sigma^b \mid \sigma \in \mathcal{C} \} \) for \( \mathcal{C} \subset \mathcal{S}.\)

(ii) Note that (a) \( Q_{g_1} \) is a subdivision of the cell complex \( Q_1 \) into a simplicial complex,

(b) \( Q_{g_1} = Q_1^{(m-1)} \cup |\mathcal{G}_j| \subset \Lambda_j \) and (c) \( M_N \cup L_{1,j} = M_N \cup P_1 \) in \([II].\)

(iii) The choice of \( K' \) and \( K \) implies that

(a) \( \sigma^b \) is sufficiently close to \( \sigma^* \) so that \((P - \hat{\sigma}) \cap \sigma^b = \emptyset,\) and

(b) \( P_1 \) differs from \( P_0 \) only in \( U \) and there exists \( \eta \in \text{Diff}(M, M_V)_0 \) with \( P_1 = \eta(P_0).\)

By [13] Remark 2.1 we may assume that \( \eta \in \text{Diff}(M, M_V)_0.\)

Since the tuple \((N, P_0, Q_{g_0}, L_j)\) corresponds to \((N, P_1, Q_{1,g_1}, L_{1,j})\) under the diffeomorphism \( \varphi \) and \( \partial N \cup P = \partial N \cup P_0, \) from (3) we have the following conclusions :

(i) \( cld \text{Diff}^r(N, \partial N \cup P)_0 \leq 2 \) and \( clb^f d \text{Diff}^r(N, \partial N \cup P)_0 \leq 2m + 1,\)
(ii) any \( g \in \text{Diff}^r(N, \partial N \cup Q_{1,g_1})_0 \) has a factorization \( g = g_1 g_2 \) such that

\[
\begin{align*}
&g_1 \in \text{Diff}^r(N, \partial N \cup Q_{1,g_1})_0 \quad \text{and} \quad \text{clb}^f (g_1) \leq 2 \ell_j \text{ in } \text{Diff}^r(N, \partial N \cup Q_{1,g_1})_0, \\
&g_2 \in \text{Diff}^r(N, \partial N \cup Q_{1,g_1} \cup L_{1,j})_0 \quad \text{and} \quad \text{clb}^f (g_2) \leq 1 \text{ in } \text{Diff}^r(N, \partial N \cup Q_{1,g_1} \cup L_{1,j})_0.
\end{align*}
\]

(5) Finally we deduce the estimates of \( \text{cl } f \) and \( \text{clb}^f f \) in \( \text{Diff}^r(M, M_N)_0. \) First we note that

(i) \( g^{(j)} \in \text{Diff}^r(M, M_N \cup \Lambda_j)_0 \subset \text{Diff}^r(M, M_N \cup Q_{1,g_1})_0 \) by (4)(ii)(b), and

\( \text{cl } g^{(j)} \leq 2, \text{ clb}^f g^{(j)} \leq 2 \ell_j + 1 \) by (4)(ii).

(b) \( \text{cl } h^{(k)} \leq 2, \text{ clb}^f h^{(k)} \leq 2m + 1 \) by (3)(i).

(c) \( \text{cl } b^{(j)}_j \leq 1, \text{ clb}^f b^{(j)}_j \leq 1, \) so that \( \text{cl } (\prod_{j=2}^{k} b^{(j)}_j) \leq k + 3, \text{ clb}^f (\prod_{j=2}^{k} b^{(j)}_j) \leq k + 3. \)

Case \([I]\) : From (i) it follows that \( \text{cl } f \leq (k + 3) + 2k + 2 = 3k + 5 \) and

\[
\text{clb}^f f \leq (k + 3) + \sum_{j=1}^{k} (2 \ell_j + 1) + 2m + 1 = 2 \left( \sum_{j=1}^{k} \ell_j + k + m + 2 \right) \leq 2(m + 2)(k + 1) = (n + 4)(k + 1).
\]
Case [II]: In this case we can further refine the factorization of \( f \) in (2)(iv).

(ii) By (4)(ii)’ and (4)(ii)(c) each \( g^{(j)} \) in \( \text{Diff}^r(M, M_N \cup Q_1, g_j) \) has a factorization \( g^{(j)} = g_j h_j \)

such that \( g_j \in \text{Diff}^r(M, M_N \cup Q_1, g_j) \) and \( clb^j(h_j) \leq 2 \ell_j \) in \( \text{Diff}^r(M, M_N \cup Q_1, g_j) \).

Let \( g_1 = g_1' := g_j h_1 \cdots h_{j-1}(j = 2, \ldots, k) \) and \( h' := h_1 \cdots h_k \). Then, it follows that

\( g_j \in \text{Diff}^r(M, M_N) \) \( (j = 1, \ldots, k) \) and \( h' \in \text{Diff}^r(M, M_N \cup P_1) \),

\( \prod_{j=1}^k g^{(j)} = \prod_{j=1}^k (g_j h_j) = g_1 h_1 \cdots h_{k-1} \).

Let \( \eta' := \eta^{-1} h' \in \text{Diff}^r(M, M_U) \) for \( U' := h'(U) \in B_j(\text{Int} N) \).

Since \( \eta(P_0) = P_1 \), we have \( h^{(k)} \eta' \in \text{Diff}^r(M, M_N \cup P_1) \), which implies that

\( h := h^{(k)} \eta' \in \text{Diff}^r(M, M_N \cup P_1) \) and \( cl h \leq 2 \), \( clb^j h \leq 2m + 1 \) by (4)(i)’.

Let \( (h')^n = \eta' h' \), it follows that \( (h'h^{(k)})^n = (h')^n (h^{(k)})^n = \eta' h \) and

\( clb^j (h'h^{(k)}) = clb^j (\eta' h) \leq 1 + 1 + 2m + 1 = 2m + 3 \).

(iv) From (ii) and (iii) it follows that \( cl f \leq (k + 3) + k + 4 = 2k + 7 \) and

\[ clb^j f \leq (k + 3) + \sum_{j=1}^k (2 \ell_j + 2m + 3) = 2 \sum_{j=1}^k \ell_j + k + 2m + 6 \leq (2m + 3)(k + 1) + 3 = (n + 3)(k + 1) + 3. \]

This completes the proof. \( \square \)

**Proof of Corollary 5.1.**

(1) Let \( S' = \{ \sigma_1, \ldots, \sigma_\ell \} \) and take \( F := S' \) and its cover \( F_j = \{ \sigma_j \} \) \( (j = 1, \ldots, \ell) \). Since each \( \{ F_j \} = \sigma_j \) has an arbitrarily small 2m-disk neighborhood in \( \tilde{N} \), by Theorem 5.1[II] it follows that

\[ cld \text{Diff}^r(N, \partial N) \leq 2 \ell + 7 \] and

\[ clb^d \text{Diff}^r(N, \partial N) \leq 2 \sum_{j=1}^\ell \ell_j + \ell + 2m + 6. \]

(2) Next we show that \( \ell_j \leq m \) \( (j = 1, \ldots, \ell) \). Recall that \( P_{F_j} = P^{(m-1)}(\partial \sigma_j) \). Take any face \( \tau \) of \( \sigma_j \) and let \( K_j := P^{(m-1)} - \tau \). Then \( P_{F_j} \) elementarily collapses to the subcomplex \( K_j \) in the PL-sense.

Hence, \( P_{F_j} \) is weakly absorbed to \( K_j \) in \( \tilde{O}_j \) keeping \( K_j \) invariant. Thus, \( \ell_j \leq clb^j (K_j, \tilde{O}_j) \leq m \) by Compliment 5.1(1)(ii).

(3) From (1), (2) we have \( clb^d \text{Diff}^r(N, \partial N) \leq (2m + 1)(\ell + 1) + 5. \) \( \square \)

5.2.2. Compact manifold case II — Handle decompositions.

**Setting 5.4.** Suppose \( M \) is a 2m-manifold without boundary \( (m \geq 3), 1 \leq r \leq \infty, r \neq 2m + 1, \mathcal{H} \) is a handle decomposition of \( M \), \( (P, Q) = (P^m, P^m_r) \) (the \( m \)-skeletons of the core complexes of \( \mathcal{H} \) and \( \mathcal{H}^* \)) and \( S \) is the set of all open \( m \)-cells of \( P \). Suppose \( N \in SM_c(M), N_1 \in SM_c(M, \mathcal{H}), N_2 \in SM_c(M, \mathcal{H}^*), N \subseteq N_1 \subseteq N_2 \) and \( \text{Int} N \) includes any \( \sigma \in S \) with \( \sigma \cap \text{Int} N \neq \emptyset \).

**Setting 5.4**. Suppose \( \mathcal{C} \) is a subset of \( S \) such that \( \{ \sigma \in S \mid \sigma \cap \text{Int} N \neq \emptyset \} \subset \mathcal{C} \subset \{ \sigma \in S \mid \sigma \cap \text{Int} N_1 \} \)

and \( \{ \mathcal{C}_j \}_{j=1}^k \) is a finite cover of \( \mathcal{C} \) (as a set).

**Proposition 5.1.** In Setting 5.4 [5.4]:
\[ \text{I] } \text{cld}(\text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_{N_2})_0) \leq 3k + 5 \quad \text{and} \quad \text{cld}^d(\text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_{N_2})_0) \leq 2(\ell \leq c(\mathcal{H} + (k - 1)c(\mathcal{H}^{(m)}) + k + 2) \leq 2(m + 2)(k + 1) \quad \text{if each } Cl_M|C_j| \ (j = 1, \ldots, k) \text{ is strongly displaceable from itself in } M. \]

\[ \text{II] } \text{cld}(\text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_{N_2})_0) \leq 2k + 7 \quad \text{and} \quad \text{cld}^d(\text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_{N_2})_0) \leq 2(\ell \leq c(\mathcal{H} + (k - 1)c(\mathcal{H}^{(m)}) + k + 6 \leq (2m + 3)(k + 1) + 3 \quad \text{if each } Cl_M|C_j| \ (j = 1, \ldots, k) \text{ is strongly displaceable from } Cl_M|S| \text{ in } M. \]

**Proof.** The proof is similar to that of Theorem 5.1 and is based on Lemmas 3.4 and 3.5.

Take any \( f \in \text{Diff}^r(M, M_N)_0 \) and \( F \in \text{Isot}^r(M, M_N)_0 \) with \( f = F_1 \). Our goal is to estimate \( cf \) and \( cbl f \) in \( \text{Diff}^r(M, M_{N_2})_0 \).

(1) First we apply (Step 1) \( \sim \) (Step 3) in Subsection 5.1.2 to \((M, N), P, Q \) and \( F \). Then we obtain

(i) \( K \in \text{Isot}(M, M_N \cup P^{(m-1)})_0, Q_1 = K_1(Q) \) and \( H^{(0)} \in \text{Isot}^r(M, M_N \cup P^{(m-1)})_0. \)

To apply (Step 4) we replace \( N \) by \( N_1 \). In Setting 5.2 we have

(ii) \([\text{the set of open m-cells of } P] = S, \quad S_0 := \{\sigma \in S \mid H^{(0)} \equiv \text{id on } [a \text{ neighborhood of } \sigma \times I] \}, \quad S_2 = S - S_0 \subset C. \)

Let \( E \subset S \) and take a partition \( E = \bigcup_{j=1}^k E_j \) such that \( E_j \subset C_j \ (j = 1, \ldots, k) \). Then we have

(iii) \( \Sigma_j := \text{Diff}^{(m-1)} \cup |S_0| \cup \bigcup_{j=1}^k |E_j| \ (j = 1, \ldots, k), \quad \Sigma_j = P, \quad U := \text{the set of open m-cells of } Q_1 \),

\( \Lambda_j := Q^{(m-1)} \cup \bigcup \{t \in U \mid t \cap |E_j| = \emptyset \} \ (j = 1, \ldots, k). \)

We apply (Step 4) to \((M, N_1), P, Q_1, H^{(0)}\) and \( \{E_j \}_j \). Then (Step 5) yields the factorization

(iv) \( f = (\tilde{a} \tilde{g})^{-1} \left[ \left[ \prod_{j=1}^k g^{(j)} \right] \left[ \prod_{j=1}^k h^{(k)} \right] \right] \tilde{a} \tilde{g}, \)

where

(a) \( g^{(j)} \in \text{Diff}^{(m)}(M, M_{N_1} \cup A_j)_0 \) \( (j = 1, \ldots, k) \), \quad (b) \( h^{(k)} \in \text{Diff}^{(m)}(M, M_{N_1} \cup P)_0, \)

(c) \( b_j \in \text{Diff}^{(m)}(M, M_{V_j})_0 \) for some \( V_j \in B_j(\text{Int } N_1) \) \( (j = -2, -1, 0, 1, \ldots, k). \)

(2) Next we apply Lemmas 3.4, 3.5. Let \((K_j, Q_{C_j}) := (P^{(m-1)}_N \cup |C_j|, Q - |C_j^*|), L_j := P_{C_j} \) in \([I] \) and \( L_j := P \) in \([II] \) and \( \ell_j := c(K_j), \ell := c(P^{(m-1)}_{\mathcal{H}^*}(N_2)) \) \( (j = 1, \ldots, k). \) It follows that

(i) \( \text{any } g \in \text{Diff}^{(m)}(M, M_{N_1} \cup Q_{C_j})_0 \) \( (j = 1, \ldots, k) \) has a factorization \( g = g_1 g_2 \) such that

\( g_1 \in \text{Diff}^{(m)}(M, M_{N_1} \cup Q_{C_j})_0 \) \( \quad \text{and } \quad \text{cld}^d(g_1) \leq 2 \ell_j \text{ in Diff}^{(m)}(M, M_{N_1} \cup Q_{C_j})_0, \)

\( g_2 \in \text{Diff}^{(m)}(M, M_{N_1} \cup Q_{C_j} \cup L_j)_0 \) \( \quad \text{and } \quad \text{cld}^d(g_2) \leq 1 \text{ in Diff}^{(m)}(M, M_{N_1} \cup Q_{C_j} \cup L_j)_0. \)

(ii) \( \text{cld}^{(m)}(M, M_{N_2} \cup P)_0 \leq 2 \quad \text{and} \quad \text{cld}^{d}(M, M_{N_2} \cup P)_0 \leq 2 \ell + 1. \)

The assertion (i) follows from Lemma 3.5 for the data:

\( M, \mathcal{H}, (P, Q), N_1, C_j, C_j^* := C_j \) in \([I] \) and \( S \) in \([II] \), \((K_j, Q_{C_j}), L_j, O_j := \text{Int } N_1 - Q_{C_j} \) and \( \ell_j \).

By the assumption, \( Cl_M|C_j| \) is displaceable from \( Cl_M|S| \) in \( O_j \). The assertion (ii) follows from Lemma 3.4 for the data \( M, \mathcal{H}^*, (Q^{(m-1)}, P), N_2 \) and \( \ell \). It is easily seen that

(iii) \( \ell_j \leq c(P^{(m)}_{\mathcal{H}}) = c(\mathcal{H}^{(m)}) \leq m + 1, \quad \ell \leq c(P^{(m-1)}_{\mathcal{H}^*}) = c(\mathcal{H}^{(m-1)}) \leq m \quad \text{and} \)

\( \ell_j + \ell \leq c(\mathcal{H}^{(m)}) + c(\mathcal{H}^{(m-1)}) = c(\mathcal{H}) \leq 2m + 1. \)

(3) We transform the estimates in (2) to those associated to \( Q_1 \), using \( K_1 \in \text{Diff}(M, M_N)_0 \). We set

(i) \( (P_1, Q_1, \ell_1, L_1, j) := K_1(P, Q_{C_j}, L_j), \sigma^b := K_1(\mathcal{S}) \) for \( \sigma \in S \) and \( D^b := \{\sigma^b \mid \sigma \in D\} \) for \( D \subset S. \)

The choice of \( K \) in (Step 1) means that

(ii) \( \sigma^b \cap \tau \neq \emptyset \text{ iff } \sigma = \tau \ (\sigma, \tau \in S), \text{ hence } A_j = Q_1 - |C_j| \supset Q_1 - |C_j^*| = K_1(Q - |C_j^*|) \equiv Q_1, C_j. \)
(b) $P_1$ differs from $P$ only in some $U \in B_f(Int N)$, which is a small neighborhood of $P \cap \Omega \cap \text{Int } N$ in $\text{Int } N$, and there exists $\eta \in \text{Diff}(M, M_U)_{0}$ with $P_1 = \eta(P)$.

By [9], Remark 2.1 we may assume that $\eta \in \text{Diff}(M, M_U)_{0}$.

Since the tuple $(N_2, N_1, P, Q_{c_1}, L_j)$ corresponds to $(N_2, N_1, P_1, Q_{1,c_1}, L_{1,j})$ under the diffeomorphism $K_1$, from (2) we have the following conclusions :

(i') any $\varphi \in \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1})_0$ ($j = 1, \cdots, k$) has a factorization $\varphi = \varphi_1 \varphi_2$ such that

\[
\varphi_1 \in \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1})_0 \quad \text{and} \quad \text{cl} b^f_1(\varphi_1) \leq 2 \ell_j \text{ in } \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1})_0,
\]

\[
\varphi_2 \in \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1} \cup L_{1,j}, \bar{\nu}_0) \quad \text{and} \quad \text{cl} b^f_2(\varphi_2) \leq 1 \text{ in } \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1} \cup L_{1,j}, \bar{\nu}_0),
\]

(ii') $\text{cl} \text{Diff}^r(M, M_{N_2} \cup P_1)_{0} \leq 2 \quad \text{and} \quad \text{cl} b^f \text{Diff}^r(M, M_{N_2} \cup P_1)_{0} \leq 2 \ell + 1$.

(4) It remains to estimate the contribution of $g^{(j)}$, $h^{(k)}$ and $b'_j$ to $cl \ f$ and $cl b^f \ f$ in $\text{Diff}^r(M, M_N)_{0}$.

(i) $g^{(j)} \in \text{Diff}^r(M, M_{N_1} \cup \Omega_{1,c_1})_0 \subset \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1})_0$ by (3)(ii)(a), and

\[
\text{cl} \ g^{(j)} \leq 2, \quad \text{cl} b^f g^{(j)} \leq 2 \ell_j + 1 \text{ by (3)(i)'},
\]

(b) $h^{(k)} \leq 2$, $\text{cl} b^f h^{(k)} \leq 2 \ell + 1$ by (2)(ii).

(c) $\text{cl} b'_j \leq 1, \quad \text{cl} b^f b'_j \leq 1$, so that $\text{cl} \left( \prod_{j=-2}^{k} b'_j \right) \leq k + 3$, $\text{cl} b^f \left( \prod_{j=-2}^{k} b'_j \right) \leq k + 3$.

Case [I] : From (i) it follows that $\text{cl} \ f \leq (k + 3) + 2k + 2 = 3k + 5$ and

\[
\text{cl} b^f \ f \leq (k + 3) + \sum_{j=1}^{k} (2 \ell_j + 1) + 2 \ell + 1 = 2 \left( \sum_{j=1}^{k} \ell_j + \ell + k + 2 \right)
\]

\[
\leq 2 \left( c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)}) + k + 2 \right) \leq 2(m + 2)(k + 1) = (n + 4)(k + 1).
\]

Case [II] : We refine the factorization of $f$ in (1)(iv).

(ii) By (3)(i)' each $g^{(j)} \in \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1})_0$ has a factorization $g^{(j)} = g_j h_j$

such that $g_j \in \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1})_0$ and $\text{cl} b^f (g_j) \leq 2 \ell_j \text{ in } \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1})_0$,

$h_j \in \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1} \cup P_1)_{0}$ and $\text{cl} b^f (h_j) \leq 1 \text{ in } \text{Diff}^r(M, M_{N_1} \cup Q_{1,c_1} \cup P_1)_{0}$.

Let $g'_j = g_j, g'_j := g_j^{h_1 \cdots h_j - 1}$ ($j = 2, \cdots , k$) and $h' := h_1 \cdots h_k$. Then, it follows that

(a) $g'_j \in \text{Diff}^r(M, M_{N_1})_0$ ($j = 1, \cdots, k$) and $h' \in \text{Diff}^r(M, M_{N_1} \cup P_1)_{0} \subset \text{Diff}^r(M, M_{N_2} \cup P_1)_{0}$,

(b) $\prod_{j=1}^{k} g^{(j)} = \prod_{j=1}^{k} (g_j h_j) = g_1 g_2 h_1 g_3 h_2 \cdots g_k h_{k-1}(h_1 \cdots h_k) = (\prod_{j=1}^{k} g'_j) h'$.

(c) $\text{cl} b'_j \leq 1, \quad \text{cl} b^f b'_j \leq 2 \ell_j$ and $\text{cl} h' \leq 2, \quad \text{cl} b^f h' \leq 2 \ell + 1$ by (3)(ii)'.

(d) $\text{cl} (h' h^{(k)}) \leq \text{cl} h' + \text{cl} h^{(k)} \leq 2 + 2 = 4$ by (i)(b) and (c).

(iii) To obtain a finer estimate on $\text{cl} b^f (h' h^{(k)})$ we decompose the composition $h' h^{(k)}$ using $\eta \in \text{Diff}^r(M, M_{U})_{0}$ given in (3)(ii)(b). We note that

(a) $\eta' := (\eta^{-1})' \in \text{Diff}^r(M, M_{U'})_{0}$ for $U' := h'(U) \in B_f(Int N_1)$.

Since $\eta(P) = P_1$, we have $(h^{(k)})' \eta \in \text{Diff}^r(M, M_{N_1} \cup P_1)_0$, which implies that

(b) $h := h'(h^{(k)})' \eta \in \text{Diff}^r(M, M_{N_1} \cup P_1)_0$ and $\text{cl} h \leq 2, \quad \text{cl} b^f h \leq 2 \ell + 1$ by (3)(ii)'.

Since $(h')' \eta = (\eta')' h'$, it follows that $(h' h^{(k)})' \eta = (h')' (h^{(k)})' \eta = (\eta')' h'$ and

(c) $\text{cl} b^f (h' h^{(k)}) = \text{cl} b^f (\eta')' h' \leq 1 + 1 + 2 \ell + 1 = 2 \ell + 3$.

(iv) From (ii) and (iii) it follows that $\text{cl} f \leq (k + 3) + k + 4 = 2k + 7$ and

\[
\text{cl} b^f f \leq (k + 3) + \sum_{j=1}^{k} (2 \ell_j) + 2 \ell + 3 = 2 \sum_{j=1}^{k} \ell_j + 2 \ell + k + 6
\]

\[
\leq 2 \left( c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)}) \right) + k + 6 \leq (2m + 3)(k + 1) + 3 = (n + 3)(k + 1) + 3.
\]

This completes the proof. \qed
When \( M \) is a closed \( 2m \)-manifold, we can select \( N = N_1 = N_2 = M \). Then, Proposition 5.1 reduces to the following form.

**Corollary 5.2.** Suppose \( M \) is a closed \( 2m \)-manifold (\( m \geq 3 \)), \( 1 \leq r \leq \infty \), \( r \neq 2m + 1 \), \( \mathcal{H} \) is a handle decomposition of \( M \), \( \mathcal{C} \) is the set of all open \( m \)-cells of \( P_\mathcal{H} \) and \( \{ C_j \}_{j=1}^k \) is a finite cover of \( \mathcal{C} \).

[I] \( \text{cld} \text{Diff}^r(M)_0 \leq 3k + 5 \) and
\[
\text{clb}'' \text{d} \text{Diff}^r(M)_0 \leq 2\left( c(\mathcal{H}) + (k-1)c(\mathcal{H}^{(m)}) + k + 2 \right) \leq 2(m+2)(k+1)
\]
if each \( \text{Cl}_M|C_j| \) \( (j = 1, \ldots, k) \) is strongly displaceable from itself in \( M \).

[II] \( \text{cld} \text{Diff}^r(M)_0 \leq 3k + 7 \) and
\[
\text{clb}'' \text{d} \text{Diff}^r(M)_0 \leq 2\left( c(\mathcal{H}) + (k-1)c(\mathcal{H}^{(m)}) \right) + k + 6 \leq (2m+3)(k+1) + 3
\]
if each \( \text{Cl}_M|C_j| \) \( (j = 1, \ldots, k) \) is strongly displaceable from \( \text{Cl}_M|\mathcal{C}| \) in \( M \).

**Example 5.1.** Consider the product of two \( m \)-spheres \( M = S^m \times S^m \). Each factor \( S^m_k \) \( (k = 1, 2) \) has a handle decomposition with one 0-handle and one \( m \)-handle. By taking the products of these handles and smoothing them, we obtain a handle decomposition \( \mathcal{H} \) of \( M \) with one 0-handle, two \( m \)-handles and one \( 2m \)-handle. The core complex \( P_\mathcal{H} \) of \( \mathcal{H} \) has two open \( m \)-cells \( \sigma_j \) \( (j = 1, 2) \), for which we may assume that \( \text{Cl}_M|\sigma_j| \) \( (j = 1, 2) \) are smooth \( m \)-spheres with a trivial normal bundle and these intersect at the unique 0-cell (i.e., the center point of the 0-handle) transversely. Hence, each \( \text{Cl}_M|\sigma_j| \) \( (j = 1, 2) \) is strongly displaceable from itself in \( M \) (cf. Example 2.4 (3)), but it is not displaceable from \( \text{Cl}_M(\sigma_1 \cup \sigma_2) \). Therefore, if \( m \geq 3 \) and \( 1 \leq r \leq \infty \), \( r \neq 2m + 1 \), from Corollary 5.2 [I] it follows that
\[
\text{cld} \text{Diff}^r(M)_0 \leq 11 \quad \text{and} \quad \text{clb}'' \text{d} \text{Diff}^r(M)_0 \leq 18.
\]

5.3. **Open manifold case.**

This section is devoted to the estimates of commutator length diameter of diffeomorphism groups for some classes of open manifolds.

5.3.1. **Grouping of \( m \)-cells in open \( 2m \)-manifolds.**

In this subsection we obtain basic results related to grouping of infinitely many \( m \)-cells in open \( 2m \)-manifolds based on Theorem 5.1 and Proposition 5.1 in the compact \( 2m \)-manifold case. These results are used in the next subsection to treat some important classes of open \( 2m \)-manifolds (for example, covering spaces, infinite connected sums, etc.).

First we consider the case related to exhausting sequences and triangulations.

**Setting 5.5.** Suppose \( M \) is an open \( 2m \)-manifold, \( \{ M_k \}_{k \geq 0} \) is an exhausting sequence of \( M \) \( (M_k = \emptyset \) \( (k < 0) \) \) and \( \mathcal{T} \) is a \( C^\infty \) triangulation of \( M \) such that each \( M_k \) underlies a subcomplex of \( \mathcal{T} \). Let \( \mathcal{S} \) and \( \mathcal{S}_k \) \( (k \geq 0) \) denote the sets of \( m \)-simplices of \( \mathcal{T} \) and \( \mathcal{T}|_{M_k} \) respectively. Suppose \( p \geq 1, q \geq 0 \) and \( \{ \mathcal{S}_{0,j} \}_{j=1}^{p+q} \) is a finite cover of \( \mathcal{S}_0 \) and \( \{ \mathcal{S}_{k,j} \}_{j=1}^p \) is a finite cover of \( \mathcal{S}_k \) for each \( k \geq 1 \) such that
\[
(\exists) \quad \{ |\mathcal{S}_{k,j}| \}_{k \geq 0} \text{ is a disjoint family for each } j = 1, \ldots, p.
\]

Recall that \( \widetilde{\mathcal{N}} := N \cup \partial N \ (\partial N \times [0,1)) \) for any manifold \( N \).

**Proposition 5.2.** In Setting 5.5: Suppose \( m \geq 3 \) and \( 1 \leq r \leq \infty \), \( r \neq 2m + 1 \).

[I] \( \text{cld} \text{Diff}^r(M)_0 \leq 3(2p + q) + 10, \quad \text{clb}'' \text{d} \text{Diff}^r(M)_0 \leq 2(m+2)(2p + q + 2), \)
\[
\text{cld} \text{Diff}^r_c(M)_0 \leq 3(p + q) + 5, \quad \text{clb}'' \text{d} \text{Diff}^r_c(M)_0 \leq 2(m+2)(p + q + 1),
\]
if (a) each $|S_{0,j}|$ $(j = 1, \cdots, p + q)$ is strongly displaceable from $|S_{0,j}| \cup (\partial M_0 \times [0,1))$ in $\tilde{M}_0$ and
(b) each $|S_{k,j}|$ $(k \geq 1, j = 1, \cdots, p)$ is strongly displaceable from $|S_{k,j}| \cup (\partial M_{k-1,k} \times [0,1))$ in $\tilde{M}_{k-1,k}$.

\[ \| \text{cld Diff}^r(M)_0 \leq 2(2p + q) + 14, \quad \text{cld Diff}^r_c(M)_0 \leq (2m + 3)(2p + q + 2) + 6, \]
\[ \text{cld Diff}^r(M)_0 \leq 2(p + q) + 7, \quad \text{cld Diff}^r_c(M)_0 \leq (2m + 3)(p + q + 1) + 3, \]

if (a) each $|S_{0,j}|$ $(j = 1, \cdots, p + q)$ is strongly displaceable from $|S_0| \cup (\partial M_0 \times [0,1))$ in $\tilde{M}_0$ and
(b) each $|S_{k,j}|$ $(k \geq 1, j = 1, \cdots, p)$ is strongly displaceable from $|S_k| \cup (\partial M_{k-1,k} \times [0,1))$ in $\tilde{M}_{k-1,k}$.

**Proof.** We consider the two cases (†) $q = q' = cl$ and (‡) $q = clb^d$, $q' = clb^d$ simultaneously. We set $a, b \in \mathbb{Z}_{\geq 1}$ as follows:

(i) in the case (†): $$(a, b) := \begin{cases} (3(p + q) + 5, 3p + 5) & \text{in [I]} \\ (2(p + q) + 7, 2p + 7) & \text{in [II]} \end{cases}$$

(ii) in the case (‡): $$(a, b) := \begin{cases} (2(m + 2)(p + q + 1), 2(m + 2)(p + 1)) & \text{in [I]} \\ ((2m + 3)(p + q + 1) + 3, (2m + 3)(p + 1) + 3) & \text{in [II]} \end{cases}$$

Our goal is to show that (b) $qd \text{Diff}^r_c(M)_0 \leq a$ and $qd \text{Diff}^r_c(M)_0 \leq a + b$ in each case.

Below, we show the following claim:

(∗) $qd \text{Diff}^r_c(M, M_{M_k})_0 \leq a$ and $qd \text{Diff}^r_c(M, M_{M_k})_0 \leq b$ for any $0 \leq k < \ell < \infty$.

This claim and Lemma 2.3 imply the assertion (b).

The estimates (∗) and (**) are deduced by Theorem 5.1.

1. Let $S_{M_k}$ and $S_{M_k,\ell}$ denote the sets of $m$-simplices of $T|_{M_k}$ and $T|_{M_k,\ell}$ respectively. For each $k \geq 0$ the set $S_{M_k} = \bigcup_{i=0}^{k} S_i$ has the cover $\{ F_j^k \}_{j=1}^{p+q}$ defined by

(i) $F_j^k := \bigcup_{i=0}^{k} S_{i,j} \ (j = 1, \ldots, p)$, $F_j^\ell := S_{0,j} \ (j = p + 1, \ldots, p + q)$.

For each $0 \leq k < \ell \leq \infty$ the set $S_{M_k,\ell} = \bigcup_{i=k+1}^{\ell} S_i$ has the covering $\{ G_j^{k,\ell} \}_{j=1}^{p}$ defined by

(ii) $G_j^{k,\ell} := \bigcup_{i=k+1}^{\ell} S_{i,j} \ (j = 1, \ldots, p)$.

2. The estimate (∗) is obtained by applying Theorem 5.1 to $M_k$, $T|_{M_k}$, $S_{M_k}$ and $\{ F_j^k \}_{j=1}^{p+q}$.

In each case of [I] and [II], by the assumption (ii) it is easily seen that

each $|F_j^k| \ (j = 1, \ldots, p + q)$ is strongly displaceable from $|F_j^\ell| \cup (\partial M_k \times [0,1))$ in [I] and $|S_{M_k}| \cup (\partial M_k \times [0,1))$ in [II] in $\tilde{M}_k$.

3. Similarly, (**) is obtained by applying Theorem 5.1 to $M_{k,\ell}$, $T|_{M_{k,\ell}}$, $S_{M_{k,\ell}}$ and $\{ G_j^{k,\ell} \}_{j=1}^{p}$.

Next we consider the case concerned with handle decompositions.

**Setting 5.6.** Suppose $M$ is a $2m$-manifold without boundary $(m \geq 3)$, $1 \leq r \leq \infty$, $r \neq 2m + 1$, $\mathcal{H}$ is a handle decomposition of $M$, $P = P^m_H$ (the $m$-skeleton of the core complex of $\mathcal{H}$) and $Q = P^{2m}_H$ (the $m$-skeleton of the core complex of $\mathcal{H}^r$).

**Setting 5.6'.** Suppose $\mathcal{S}$ is the set of all open $m$-cells of $P$, $\{ S_j \}_{j=1}^{k}$ $(k \geq 1)$ is a finite cover of $\mathcal{S}$ and for each $j = 1, \ldots, k$, $\{ S_{j,\lambda_j} \}_{\lambda_j \in \Lambda_j}$ is a covering of $S_j$ such that each $S_{j,\lambda_j}$ is a finite set and $\{ |S_{j,\lambda_j}| \}_{\lambda_j \in \Lambda_j}$ is a locally finite family in $M$. For any subset $\Delta \subset \Lambda_j$ let $S_{j,\Delta} := \bigcup_{\lambda_j \in \Delta} S_{j,\lambda_j} \subset S_j$.

**Theorem 5.2.** In Setting 5.6, 5.6' :

[I] If $Cl_M|_{S_j,\Delta}$ is strongly displaceable from itself in $M$ for each $j = 1, \ldots, k$ and any finite subset $\Delta \subset \Lambda_j$, then
The estimates for \( \text{Diff}_c^r(M) \) are given as:

\[
\begin{align*}
(i) & \quad cld \text{Diff}_c^r(M)_0 \leq 3k + 5, \\
(ii) & \quad cld \text{Diff}_c^r(M)_0 \leq 2(c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)}) + k + 2) \leq 2(m + 3)(k + 1), \\
(iii) & \quad 4(c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)}) + k + 2) \leq 4(m + 2)(k + 1).
\end{align*}
\]

Proof. To treat various cases simultaneously, we use the following notations:

Let \( q = q' = cl \) or \( q = clb^l, \ q' = clb^d \). We set \( a \in \mathbb{Z}_{\geq 1} \) as follows:

\[
\begin{align*}
(i) & \quad \text{in the case } (q) : \quad a := \begin{cases} 3k + 5 & \text{in } [I], \\ 2k + 7 & \text{in } [II]. \end{cases} \\
(ii) & \quad \text{in the case } (q') : \quad a := \begin{cases} 2(c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)}) + k + 2) & \text{in } [I], \\ 2(c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)})) + k + 6 & \text{in } [II]. \end{cases}
\end{align*}
\]

(1) First we show the following claim based on Proposition 5.1.

Claim. If \( N \in \mathcal{SM}_c(M), \ U \in \mathcal{SM}_c(M, \mathcal{H}), \ N \subseteq N_1 \subseteq N_2 \) and \( \text{Int} N_1 \) includes any \( \{S_{j,\lambda_j}: j = 1, \ldots, k, \lambda_j \in \Lambda_j \} \) with \( |S_{j,\lambda_j}| \cap \text{Int} N \neq \emptyset, \) then

\[
qd(\text{Diff}_c^r(M, M_N)_0, \text{Diff}_c^r(M, M_{N_2})_0) \leq a.
\]

Proof of Claim. We apply Setting 5.4, 5.4 and Proposition 5.1.

(i) For Setting 5.4 : If \( \sigma \in S \) and \( \sigma \cap \text{Int} N \neq \emptyset, \) then \( \sigma \subset \text{Int} N_1. \) In fact, \( \sigma \in S_{j,\lambda_j} \) for some \( j = 1, \ldots, k \) and \( \lambda_j \in \Lambda_j \) and \( |S_{j,\lambda_j}| \cap \text{Int} N \neq \emptyset. \) Hence, the assumption on \( N_1 \) implies \( \text{Int} N_1 \supset \{S_{j,\lambda_j}: j = 1, \ldots, k, \lambda_j \in \Lambda_j \}. \)

(ii) For Setting 5.4 : For each \( j = 1, \ldots, k \) we set

\[
\Delta_j := \{\lambda_j \in \Lambda_j \mid |S_{j,\lambda_j}| \cap \text{Int} N \neq \emptyset\} \quad \text{and} \quad C_j := S_{j,\Delta_j} \equiv \bigcup_{\lambda_j \in \Delta_j} S_{j,\lambda_j} \subset S_j.
\]

Then, (a) \( \Delta_j \) is a finite set since \( \{S_{j,\lambda_j}: \lambda_j \in \Lambda_j \} \) is a locally finite family, (b) \( |C_j| \subset \text{Int} N_1 \) by the assumption on \( N_1 \) and \( C_j \) satisfies the condition:

\[
\{\sigma \in S \mid \sigma \cap \text{Int} N \neq \emptyset\} \subset C \subset \{\sigma \in S \mid \sigma \subset \text{Int} N_1\}.
\]

(iii) In the case \([I]\) each \( Cl_M[S_j,\Delta_j] \) is strongly displaceable from itself in \( M.\) Hence, the conclusion follows from Proposition 5.1.

In the case \([II]\) each \( Cl_M[S_j,\Delta_j] \) is strongly displaceable from \( Cl_M[S]\) in \( M.\) Since \( Cl_M[C] \subset Cl_M[S],\) the conclusion follows from Proposition 5.1.

(2) The estimates for \( \text{Diff}_c^r(M)_0 \) in \([I]\)(i) and \([II]\)(i) are obtained as follows.

We have to show that \( qd(\text{Diff}_c^r(M)_0 \leq a. \) Take any \( f \in \text{Diff}_c^r(M)_0 \) and \( F \in \text{Isot}_c^r(M)_0 \) with \( f = F_1. \) Since \( \text{supp} F \) is compact and \( F = \{Cl_M[S_{j,\lambda_j}] \mid j = 1, \ldots, k, \lambda_j \in \Lambda_j\} \) is a locally finite family of compact subsets of \( M, \) we can find a triple \( N, N_1, N_2 \) as in Claim with \( \text{supp} F \subset N. \) From Claim it
follows that \( qd(\text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_{N_2})_0) \leq a \).

Since \( f \in \text{Diff}^r(M, M_N)_0 \), it follows that \( q(f) \leq a \) in \( \text{Diff}^r(M, M_{N_2})_0 \) and hence \( q(f) \leq a \) in \( \text{Diff}^r(M)_0 \).

(3) For the estimates for \( \text{Diff}^r(M)_0 \) in [I](ii) and [II](ii), we have to show that
\[
q'd(\text{Diff}^r(M)_0) \leq 2a, \quad \text{when } M \text{ is noncompact}.
\]

Take any \( f \in \text{Diff}^r(M)_0 \) and \( F \in \text{Isot}^r(M)_0 \) with \( f = F_1 \). By Lemma 3.2[I] there exists an exhausting sequence \( \{M_k\}_{k \geq 1} \) of \( M \) which satisfies the following conditions:

(i) \( M_k \in S_c(M) \) (\( k \in \mathbb{Z} \)) and \( F(M_{4k+1} \times I) \subseteq M_{4k-1, 4k+2} \times I \) (\( k \geq 0 \)).

(ii) There exist \( N_k^1 \in SM_c(M; \mathcal{H}) \) and \( N_k^2 \in SM_c(M; \mathcal{H}^*) \) (\( k \geq 0 \)) such that for each \( k \geq 0 \)
\[(a) \quad M_{4k-1, 4k+2} \subseteq N_k^1 \subseteq M_{4k-2, 4k+3}, \quad (b) \quad N_k^2 \cap N_{k+1}^2 = \emptyset \quad \text{and}
(c) \quad \text{Int } N_k^1 \text{ includes any } |S_{j, \lambda_j}^0| \quad (j = 1, \cdots , k, \lambda_j \in \Lambda_j) \text{ with } |S_{j, \lambda_j}^0| \cap \text{Int } M_{4k-1, 4k+2} \neq \emptyset.
\]

(iii) There exist \( L_k^1 \in SM_c(M; \mathcal{H}) \) and \( L_k^2 \in SM_c(M; \mathcal{H}^*) \) (\( k \geq 1 \)) such that for each \( k \geq 1 \)
\[(a) \quad M_{4k-3, 4k} \subseteq L_k^1 \subseteq M_{4k-2, 4k+1}, \quad (b) \quad L_k^2 \cap L_{k+1}^2 = \emptyset \quad \text{and}
(c) \quad \text{Int } L_k^1 \text{ includes any } |S_{j, \lambda_j}^0| \quad (j = 1, \cdots , k, \lambda_j \in \Lambda_j) \text{ with } |S_{j, \lambda_j}^0| \cap \text{Int } M_{4k-3, 4k} \neq \emptyset.
\]

The conditions (ii)(c) and (iii)(c) are achieved by taking the locally finite family \( \mathcal{F} = \{C_M|S_{j, \lambda_j}^0| \quad j = 1, \cdots , k, \lambda_j \in \Lambda_j \}. \) From Claim and (ii), (iii) it follows that
\[(iv) \quad q'd(\text{Diff}^r(M, M_{M_{4k-1, 4k+2}})_0, \text{Diff}^r(M, M_{N_k^2})_0) \leq a \quad (k \geq 0) \quad \text{and}
q'd(\text{Diff}^r(M, M_{M_{4k-3, 4k}})_0, \text{Diff}^r(M, M_{L_k^2})_0) \leq a \quad (k \geq 1).
\]

Lemma 3.2[II] we have a factorization
\[(v) \quad F = GH \quad \text{for some } G \in \text{Isot}^r(M, M')_0 \text{ and } H \in \text{Isot}^r(M, M')_0, \]
where \( M' := \bigcup_{k \geq 0} M_{4k+2, 4k+3} \) and \( M'' := \bigcup_{k \geq 0} M_{4k, 4k+1} \).

By (iv) it follows that in \( \text{Diff}^r(M)_0 \)
\[(vi) \quad q'(G_1) \leq a, \quad q'(H_1) \leq a \quad \text{and hence } q'(f) \leq 2a \quad \text{as required}.
\]

This completes the proof.

\[\square\]

5.3.2. Some classes of open 2m-manifolds \( M \) with cld \( \text{Diff}^r(M)_0 < \infty \).

In this subsection we show the uniform perfectness of diffeomorphism groups for some important classes of open 2m-manifolds, including covering spaces and infinite connected sums.

[1] Open 2m-manifolds with finitely many m-handles

Proposition 5.3. Suppose \( M \) is a 2m-manifold without boundary and \( 1 \leq r \leq \infty, r \neq 2m + 1 \). If \( M \) admits a handle decomposition \( \mathcal{H} \) without m-handles, then
\[(i) \quad \text{cld } \text{Diff}^r(M)_0 \leq 6, \quad \text{cld } b^d \text{Diff}^r(M)_0 \leq 4c(\mathcal{H}) + 2 \leq 8m + 2,
(ii) \quad \text{cld } \text{Diff}_c^r(M)_0 \leq 3, \quad \text{cld } b^d \text{Diff}_c^r(M)_0 \leq 2c(\mathcal{H}) + 1 \leq 4m + 1.
\]

Proposition 5.4. Suppose \( M \) is a 2m-manifold without boundary (\( m \geq 3 \)), \( 1 \leq r \leq \infty, r \neq 2m + 1 \), \( \mathcal{H} \) is a handle decomposition of \( M \) with only finitely many m-handles, \( S \) is the set of all open m-cells of \( P_\mathcal{H} \) and \( \{S_j\}_{j=1}^k \) (\( k \geq 1 \)) is a finite cover of \( S \).

[1] If \( C(M)|S_j| \) is strongly displaceable from itself in \( M \) for each \( j = 1, \cdots , k \), then
\[(i) \quad \text{cld } \text{Diff}_c^r(M)_0 \leq 3k + 8, \quad \text{cld } b^d \text{Diff}_c^r(M)_0 \leq 2 \left(2c(\mathcal{H}) + (k-1)c(\mathcal{H}^{(m)})\right) + 2k + 3 \leq 2(m + 2)(k + 3) - 7,
(ii) \quad \text{cld } \text{Diff}_c^r(M)_0 \leq 3k + 5, \quad \text{cld } b^d \text{Diff}_c^r(M)_0 \leq 2 \left(c(\mathcal{H}) + (k-1)c(\mathcal{H}^{(m)})\right) + 2k + 4 \leq 2(m + 2)(k + 1).
\]
Proof of Propositions 5.3, 5.4

The closed manifold cases are included in Proposition 3.1 and Corollary 5.2. Below we assume that $M$ is an open manifold. We only verify the case of $\text{Diff}^r(M)_0$. The case of $\text{Diff}^c(M)_0$ follows from simpler versions of the arguments (2) and (3) below (cf. Proof (2) of Theorem 5.2).

Take any $f \in \text{Diff}^r(M)_0$ and $F \in \text{Isot}^r(M)_0$ with $F_1 = f$.

(1) Let $K := \text{the union of all } m\text{-handles in } \mathcal{H}$. By Lemma 3.2 [II] there exists an exhausting sequence 

$$\{M_i\}_{i \geq 1} \text{ of } M \text{ which satisfies the following conditions :}$$

(i) $K \subset M_1$, 

(ii) There exist $N_i' \subset \mathcal{SM}_c(M; \mathcal{H})$ and $N_i'' \subset \mathcal{SM}_c(M; \mathcal{H}^r)$ (i $\geq 0$) such that for each $i \geq 0$

$$a) \ M_{4i-1,4i+2} \subset N_i' \subset N_i'' \subset M_{4i-2,4i+3} \text{ and } b) \ N_i' \cap N_i'' = \emptyset.$$

(iii) There exist $L_i' \subset \mathcal{SM}_c(M; \mathcal{H}^r)$ and $L_i'' \subset \mathcal{SM}_c(M; \mathcal{H}^r)$ (i $\geq 1$) such that for each $i \geq 1$

$$a) \ M_{4i-3,4i} \subset L_i' \subset L_i'' \subset M_{4i-4,4i+1} \text{ and } b) \ K \cap L_i'' = \emptyset, \ L_i'' \cap L_{i+1}'' = \emptyset.$$

By Lemma 3.2 [II] there exists a factorization $F = GH$ for some $G \in \text{Isot}^r(M, M')_0$ and $H \in \text{Isot}^r(M, M'')_0$, where $M' := \bigcup_{i \geq 0} M_{4i+2,4i+3}$ and $M'' := \bigcup_{i \geq 0} M_{4i,4i+1}$.

(2) in Proposition 5.3:

Let $q = cl$ or $clb^d$ and set $\ell := 3$ for $q = cl$ and $\ell := 2c(\mathcal{H}) + 1$ for $q = clb^d$. Since $\mathcal{H}$ has no $m$-handles, by Proposition 5.1 if $N \subset \mathcal{SM}_c(M; \mathcal{H})$, $L \subset \mathcal{SM}_c(M; \mathcal{H}^r)$ and $N \subset L$, then

$$qd(\text{Diff}^r(M, M_N)_0, \text{Diff}^r(M, M_L)_0) \leq \ell.$$

Thus, by (1)(ii),(iii) we have

(i) $qd(\text{Diff}^r(M, M_{N_i'})_0, \text{Diff}^r(M, M_{N_i''})_0) \leq \ell$ (i $\geq 0$) and

(ii) $qd(\text{Diff}^r(M, M_{L_i'})_0, \text{Diff}^r(M, M_{L_i''})_0) \leq \ell$ (i $\geq 1$).

Therefore, from Lemma 3.2 [II] it follows that $q(G_1), q(H_1) \leq \ell$ and $q(f) \leq 2\ell$ in $\text{Diff}^r(M)_0$.

(3) in Proposition 5.4:

Let $q = cl$ or $clb^d$ and set $\ell$ and $\ell'$ as follows;

- for $q = cl$ : $\ell := 3k + 5$ in [I], $\ell := 2k + 7$ in [II], $\ell' := 3$

- for $q = clb^d$ : $\ell := \begin{cases} 2c(\mathcal{H}) + (k - 1)c(\mathcal{H}(m)) + 2k + 4 \leq (2m + 2)(k + 1) \text{ in [I]} \\ 2(2c(\mathcal{H}) + (k - 1)c(\mathcal{H}(m))) + k + 6 \leq (2m + 3)(k + 1) + 3 \text{ in [II]} \end{cases}$

$$\ell' := 2c(\mathcal{H}) - 1 \leq 4m + 1.$$

Since $K \subset N_i' \subset \mathcal{SM}_c(M; \mathcal{H})$, we have $|S| \subset N_0'$. Hence, we can apply Proposition 5.1 to $M_2 \subset N_0' \subset N''_0, S$ and $\{S_{j\ell}^k\}_{j=1}^k$ so to obtain the following estimate.

(i) $qd(\text{Diff}^r(M, M_{M_2})_0, \text{Diff}^r(M, M_{N_0''})_0) \leq \ell$

Since $N_i''$ and $L_i''$ (i $\geq 1$) do not intersect $K$, by Proposition 3.1 and (1)(ii), (iii) it follows that

(ii) $qd(\text{Diff}^r(M, M_{N_i'})_0, \text{Diff}^r(M, M_{N_i''})_0) \leq \ell' \leq \ell$ (i $\geq 1$),

$$qd(\text{Diff}^r(M, M_{L_i'})_0, \text{Diff}^r(M, M_{L_i''})_0) \leq \ell' \text{ (i $\geq 1$).}$$

Therefore, from Lemma 3.2 [II] it follows that

$$qd(\text{Diff}^r(M, M_{N_i'})_0, \text{Diff}^r(M, M_{N_i''})_0) \leq \ell' \text{ (i $\geq 1$).}$$
(iii) \( q(G_1) \leq \ell, q(H_1) \leq \ell' \) and \( q(f) \leq \ell + \ell' \) in \( \text{Diff}^r(M)_0 \). \( \square \)

[2] Covering spaces of closed 2m-manifolds

**Setting 5.7.** Suppose \( \pi : \widetilde{M} \to M \) is a \( C^\infty \) covering space over a closed 2m-manifold \( M \) (\( m \geq 3 \)) and \( 1 \leq r \leq \infty, r \neq 2m + 1 \).

**Setting [5.7]⁺.** Suppose \( \mathcal{H} \) is a handle decomposition of \( M \), \( \mathcal{S} \) is the set of open \( m \)-cells of \( P_\mathcal{H} \) and \( \{ S_j \}_{j=1}^k \) \( (k \geq 1) \) is a finite cover of \( \mathcal{S} \) which satisfies the following condition:

(i) Each \( Cl_M|S_j| \ (j = 1, \cdots, k) \) has an open neighborhood \( U_j \) in \( M \) which is evenly covered by \( \pi \).

**Theorem 5.3.** In Setting [5.7] [5.7]⁺:

[I] Suppose \( Cl_M|S_j| \) is strongly displaceable from itself in \( M \) for each \( j = 1, \cdots, k \).

(i) \( cld\text{Diff}^r_\pi(\widetilde{M})_0 \leq 3k + 5 \),

\[ cld^bd\text{Diff}^r_\pi(\widetilde{M})_0 \leq 2(c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)})) + 2 \leq 2(m + 2)(k + 1). \]

(ii) \( cld\text{Diff}^r(\widetilde{M})_0 \leq 6k + 10 \),

\[ cld^bd\text{Diff}^r(\widetilde{M})_0 \leq 4(c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)})) + 2 \leq 4(m + 2)(k + 1). \]

[II] Suppose \( Cl_M|S_j| \) is strongly displaceable from \( Cl_M|S| \) in \( M \) for each \( j = 1, \cdots, k \).

(i) \( cld\text{Diff}^r_\pi(\widetilde{M})_0 \leq 2k + 7 \),

\[ cld^bd\text{Diff}^r_\pi(\widetilde{M})_0 \leq 2(c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)})) + 6 \leq (2m + 3)(k + 1) + 3. \]

(ii) \( cld\text{Diff}^r(\widetilde{M})_0 \leq 4k + 14 \),

\[ cld^bd\text{Diff}^r(\widetilde{M})_0 \leq 4(c(\mathcal{H}) + (k - 1)c(\mathcal{H}^{(m)})) + 2k + 12 \leq 2(2m + 3)(k + 1) + 6. \]

**Corollary 5.3.** In Setting [5.7]: Suppose \( M \) has a \( C^\infty \) triangulation with at most \( \ell \) \( m \)-simplices.

(i) \( cld\text{Diff}^r(\widetilde{M})_0 \leq 2\ell + 7 \) and \( cld^bd\text{Diff}^r(\widetilde{M})_0 \leq (2m + 3)(\ell + 1) + 3 \).

(ii) \( cld\text{Diff}^r(\widetilde{M})_0 \leq 4\ell + 14 \) and \( cld^bd\text{Diff}^r(\widetilde{M})_0 \leq 2(2m + 3)(\ell + 1) + 6 \).

**Proof of Theorem 5.3.** We apply Setting [5.6] [5.6]⁺ and Theorem [5.2] in \( \widetilde{M} \).

(1) For Setting [5.6]:

The lifts and inverse images in \( \widetilde{M} \) of various objects on \( M \) are denoted by the symbol \( \widetilde{\cdot} \). Then we have the following data on \( \widetilde{M} \):

\( \mathcal{H}, \mathcal{H}^*, \mathcal{S}, \mathcal{S}_j \) and \( \widetilde{P} := \pi^{-1}(P), \widetilde{Q} := \pi^{-1}(Q), \widetilde{U}_j := \pi^{-1}(U_j) \).

Here, \( \mathcal{H} \) and \( \mathcal{H}^* \) are the handle decompositions of \( M \) consisting of the lifts of handles of \( \mathcal{H} \) and \( \mathcal{H}^* \) respectively, \( \mathcal{S} \) and \( \mathcal{S}_j \) are the sets of lifts of open \( m \)-cells in \( \mathcal{S} \) and \( \mathcal{S}_j \) respectively. It follows that

(i) \( \widetilde{P} \) is the \( m \)-skeleton of the core complex of \( \mathcal{H} \),

\( \mathcal{S} \) is the set of all open \( m \)-cells of \( \widetilde{P} \),

\( \mathcal{Q} \) is the \( m \)-skeleton of the core complex of \( \mathcal{H}^* \) and \( \mathcal{H}^* = (\mathcal{H})^* \),

(ii) \( \mathcal{S} = \bigcup_{j=1}^k \mathcal{S}_j \), \( |\mathcal{S}| = |\mathcal{S}_j| = |\mathcal{S}_j| \), \( \mathcal{S}_j = (\mathcal{S}_j)^* \), \( |\mathcal{S}_j^*| = |(\mathcal{S}_j)^*| \).

(2) For Setting [5.6]⁺:

We can take \( U_j \) so that there exists an open neighborhood \( U_j' \) of \( Cl_M U_j \) in \( M \) which is evenly covered by \( \pi \). Then \( \widetilde{U}_j' = \pi^{-1}(U_j') \) has a (at most countable) disjoint open cover \( \{ \widetilde{U}_j'|_{\lambda_j} \}_{\lambda_j \in \Lambda_j} \) such that the
restrictions \( \pi : \tilde{U}_{j,\lambda_j} \to U_j \) (\( \lambda_j \in \Lambda_j \)) are diffeomorphisms. Let \( \tilde{U}_{j,\lambda_j} := \tilde{U}_j \cap \tilde{U}_{j,\lambda_j} \) (\( \lambda_j \in \Lambda_j \)) and \( \tilde{S}_{j,\lambda_j} := \{ \sigma \in \tilde{S}_j \mid \sigma \subset \tilde{U}_{j,\lambda_j} \} \) (the lift of \( S_j \) on \( \tilde{U}_{j,\lambda_j} \)). Then, the following holds for each \( j = 1, \ldots, k \).

(i) \( \{ \tilde{U}_{j,\lambda_j} \}_{\lambda_j \in \Lambda_j} \) is a discrete family of relatively compact open subsets of \( \tilde{M} \).

(ii) \( \pi : \tilde{U}_{j,\lambda_j} \to U_j \) is a diffeomorphism. 

(iii) From (i)(b) and (ii)(c) it follows that

(iv) Hence, for any finite subset \( \Delta \subset \Lambda_j \) the disjoint finite union \( \text{Cl}_M | \tilde{S}_{j,\Delta} | = \bigcup_{\lambda_j \in \Delta} \text{Cl}_M | \tilde{S}_{j,\lambda_j} | \) has the same property as in (iii), that is,

\[
\text{Cl}_M | \tilde{S}_{j,\lambda_j} | \text{ is strongly displaceable from itself in } \tilde{M} \text{ in the case [I].}
\]

\[
\text{Cl}_M | \tilde{S}_{j,\Delta} | \text{ is strongly displaceable from } \text{Cl}_M | \tilde{S} | \text{ in } \tilde{M} \text{ in the case [II].}
\]

Since \( c(\tilde{H}) = c(H) \) and \( c(\tilde{H}^{(m)}) = c(H^{(m)}) \), the conclusions now follow from Theorem 5.2 \( \square \)

3 Infinite sums of finitely many compact 2m-manifolds

**Setting 5.8.** Consider a finite family \( \mathcal{N} = \{(N_i, T_i)\}_{i=0}^\ell \), where each \( N_i \) is a compact 2m-manifold such that \( \partial N_i \) (possibly empty) is a disjoint union of two \((2m-1)\)-manifolds \( \partial_\pm N_i \) and \( T_i \) is a \( C^\infty \) triangulation of \( N_i \). Assume that \( \partial_- N_0 = \emptyset \). Let \( C_i \) denote the set of \( m \)-simplices of \( T_i \) for \( i = 0, 1, \ldots, \ell \).

We consider the class of open 2m-manifolds which are infinite sums of compact manifolds in \( \mathcal{N} \).

**Setting 5.8**. Suppose \( M \) is an open 2m-manifold \((m \geq 3), 1 \leq r \leq \infty, r \neq 2m + 1, \{ M_k \}_{k \geq 0} \) is an exhausting sequence of \( M \) and \( T \) is a \( C^\infty \) triangulation of \( M \) such that each \( M_k \) underlies a subcomplex of \( T \). Let \( S \) denote the set of \( m \)-simplices of \( T \) and if \( N \subset M \) underlies a subcomplex of \( T \), then \( S|_N \) denotes the set of \( m \)-simplices of \( T|_N \).

Let \( L_k := M_{k-1,k} (k \geq 0) \) and set \( \partial_- L_k := \partial M_{k-1} \) and \( \partial_+ L_k := \partial M_k \). Suppose

(a) \( L_0 \equiv M_0 \) admits a \( C^\infty \) diffeomorphism \( \varphi_0 : (L_0, \partial L_0, T|_{L_0}) \approx (N_0, \partial N_0, T_0) \) which is a simplicial isomorphism and

(b) each \( L_k (k \geq 1) \) is a disjoint union of compact 2m-manifolds \( L_{k,s} (s = 1, \ldots, n_k) \) and each \( L_{k,s} \) admits some \( i \equiv i(k, s) \in \{ 1, \ldots, \ell \} \) and a \( C^\infty \) diffeomorphism

\[
\varphi_{k,s} : (L_{k,s}, L_{k,s} \cap \partial_- L_k, L_{k,s} \cap \partial_+ L_k, T|_{L_{k,s}}) \approx (N_i, \partial_- N_i, \partial_+ N_i, T_i),
\]

which is a simplicial isomorphism.

In this case we say that \( M \) (or \( (M, \{ M_k \}_{k \geq 0}, T) \)) is an infinite sum of the model manifolds \( \mathcal{N} = \{(N_i, T_i)\}_{i=0}^\ell \).

**Setting 5.8**. Suppose \( C_0 \) has a finite cover \( \{ C_{0,j} \}_{j=1}^{\ell} \) and each \( C_i \) \((i = 1, \ldots, \ell)\) has a finite cover \( \{ C_{i,j} \}_{j=1}^{p} \), where \( p \geq 1 \) and \( q \geq 0 \). We assume that for each \( j = 1, \ldots, p \) the family \( \{ C_{i,j} \}_{i=1}^{\ell} \) satisfies the following condition :

(*) if \( |C_{i,0,j}| \cap \partial_+ N_{i_0} \neq \emptyset \) for some \( i_0 = 0, 1, \ldots, \ell \), then \( |C_{i,j}| \cap \partial_- N_i = \emptyset \) for any \( i = 1, \ldots, \ell \).
Proof of Corollary 5.4. In Settings 5.5, 5.8++:

\[ cld Diff^r(M)_0 \leq 2(2p + q) + 14, \quad cld Diff^r_c(M)_0 \leq 2(a + b) + 7, \quad cld Diff^r_q(M)_0 \leq 2(m + 3)(a + b + 1) + 3, \]

if (i) each \(|C_{0,j}| (j = 1, \ldots, p + q)\) is strongly displaceable from \(|C_{0,j}| \cup (\partial N_0 \times [0,1])\) in \(\tilde{N}_0\) and

(ii) each \(|C_{i,j}| (i = 1, \ldots, \ell, j = 1, \ldots, p)\) is strongly displaceable from \(|C_{i,j}| \cup (\partial N_i \times [0,1])\) in \(\tilde{N}_i\).

Corollary 5.4. In Settings 5.5, 5.8++:

\[ cld Diff^r(M)_0 \leq 2(2p + q) + 14, \quad cld Diff^r_c(M)_0 \leq 2(m + 3)(2p + q + 2) + 6, \]

\[ cld Diff^r_q(M)_0 \leq 2(m + 3)(p + q + 1) + 3, \]

if (i) each \(|C_{0,j}| (j = 1, \ldots, p + q)\) is strongly displaceable from \(|C_{0}| \cup (\partial N_0 \times [0,1])\) in \(\tilde{N}_0\) and

(ii) each \(|C_{i,j}| (i = 1, \ldots, \ell, j = 1, \ldots, p)\) is strongly displaceable from \(|C_{i}| \cup (\partial N_i \times [0,1])\) in \(\tilde{N}_i\).

Note that the condition (\(\zet\)) is obviously satisfied if \(a_1, a_2 \geq \max\{\#C_i \mid i = 0, 1, \ldots, \ell\}\).

Proof of Proposition 5.5. We apply Setting 5.5 and Proposition 5.2.

1. For Setting 5.5: By Setting 5.8++ (\(\alpha\), \(\beta\)),

   (i) \(S_0 := S|_{L_0}\) inherits a finite cover \(\{S_{0,j}\}_{j=1}^{p+q}\) corresponding with the finite cover \(\{C_{0,j}\}_{j=1}^{p+q}\) of \(C_0\),

   (ii) for each \(k \geq 1\)

      (a) each \(S|_{L_{k,s}} (s = 1, \ldots, n_k)\) inherits a finite cover \(\{S_{k,s,j}\}_{j=1}^{p}\) corresponding with the finite cover \(\{C_{i,j}\}_{j=1}^{p}\) of \(C_i\),

      (b) \(S := S|_{L_k} = \bigcup_{s=1}^{n_k} S|_{L_{k,s}}\) has a finite cover \(\{S_{k,s,j}\}_{j=1}^{p}\) defined by \(S_{k,j} := \bigcup_{s=1}^{n_k} S_{k,s,j}\).

   From Setting 5.8++ (\(\ast\)) it follows that (\(\zet\)) \(\{S_{k,j}\}_{k \geq 0}\) is a disjoint family for each \(j = 1, \ldots, p\).

2. The assumptions in [I] and [II] imply the corresponding conditions in [I] and [II] in Proposition 5.2.

   Therefore, the conclusions follow from Proposition 5.2.

Proof of Corollary 5.4. We apply Setting 5.8++ and Proposition 5.5 to the following situation:

1. We use the following notations:

   \(C_0 = \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_h\}\),

   where \(\alpha_u \cap \partial N_0 = \emptyset\) \((u = 1, \ldots, g)\) and \(\beta_v \cap \partial N_0 \neq \emptyset\) \((v = 1, \ldots, h)\),

   \(C_i = \{\sigma_{i,1}, \ldots, \sigma_{i,x_i}, \tau_{i,1}, \ldots, \tau_{i,y_i}\} (i = 1, \ldots, \ell)\),

   where \(\sigma_{i,u} \cap \partial N_i \neq \emptyset\) \((u = 1, \ldots, x_i)\) and \(\tau_{i,v} \cap \partial N_i = \emptyset\) \((v = 1, \ldots, y_i)\).

   It follows that \(g+h \leq a+b, h \leq a_2+b\) and \(x_i \leq a_1, y_i \leq a_2 (i = 1, \ldots, \ell)\).

2. For Setting 5.8++: Define the finite covers \(\{C_{0,j}\}_{j=1}^{a+b}\) of \(C_0\) and \(\{C_{i,j}\}_{j=1}^{a}\) of \(C_i (i = 1, \ldots, \ell)\) by
\[ C_{0,j} = \begin{cases} \{\alpha_j\} & (j = 1, \ldots, g) \\ \emptyset & (j = g + 1, \ldots, a + b - h) \end{cases} \quad C_{i,j} = \begin{cases} \{\sigma_{i,j}\} & (j = 1, \ldots, x_i) \\ \emptyset & (j = x_i + 1, \ldots, a - y_i) \end{cases} \quad C_{i,j} = \begin{cases} \{\beta_i\} & (j = a + b - h + t) \end{cases} \quad (i = 1, \ldots, \ell). \]

The conditions \( g \leq a + b - h \) and \( x_i \leq a - y_i \) assure the well-definedness. The condition (i) follows from the assumptions (i) \sim (iii). In fact, if \( |C_{i,o,j}| \cap \partial_+ N_{i,o} \neq \emptyset \), then \( j > a_1 \geq x_i \). These covers obviously satisfy the condition [II] in Proposition 5.5 from which follows the conclusion.

Example 5.3. Another example is given by the class of “the complement \( M \) of the intersection of a nested sequence \( \{C_k\}_{k=0}^\infty \) of compact \( n \)-submanifolds in a closed \( n \)-manifold \( N \).” This means that

\[ M = N - C_\infty \quad \text{and} \quad N \equiv C_0 \supset C_1 \supset \cdots \supset C_k \supset C_{k+1} \supset \cdots \supset C_\infty := \bigcap_{k=0}^\infty C_k. \]

The open \( n \)-manifold \( M \) has the exhausting sequence \( M_k := N_{C_k} (k \geq 1) \), whose \( k \)-th part is given by \( L_k := M_{k-1,k} = (C_{k-1})_C k \geq 1 \). If there exist finitely many model embeddings between compact \( n \)-manifolds \( F_i \subset \text{Int} E_i \subset E_i (i = 1, \ldots, \ell) \) and if each \( C_k (k \geq 0) \) is a disjoint union of compact \( 2m \)-manifolds \( C_{k,s} (s = 1, \ldots, t_k) \) each of which admits a \( C^\infty \) diffeomorphism \( (C_{k,s}, C_{k,s} \cap C_{k+1}) \approx (E_i, F_i) \) onto some pair \( (E_i, F_i) \), then \( M \) is expressed as an infinite sum of finitely many compact \( n \)-manifolds \( L_0 \) and \( (E_i)_{F_i} (i = 1, \ldots, \ell) \).

6. Conjugation-generated norm on diffeomorphism groups

6.1. Conjugation-generated norm.

First we recall basic facts on conjugation-generated norms on groups [1]. Suppose \( G \) is a group. Let \( G^\times := G - \{e\} \). For \( g \in G \) let \( C(g) \) denote the conjugacy class of \( g \) in \( G \) and let \( C_g := C(g) \cup C(g^{-1}) \). Since \( C_g \) is a symmetric and conjugation invariant subset of \( G \), it follows that \( N(g) = N(C_g) = \bigcup_{k \geq 0} (C_g)^k \) and we obtain the ext. conj.-invariant norm \( q_{G,C_g} \) on \( G \). This norm is denoted by \( \nu_g \) and called the conjugation-generated norm with respect to \( g \). Note that \( C_h = C_g \) and \( \nu_h = \nu_g \) for any \( h \in C_g \). We also consider the quantity

\[ \nu(G) := \min\{k \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \mid \nu_g(f) \leq k \text{ for any } g \in G^\times \text{ and } f \in G\}. \]

A group \( G \) is called uniformly simple [18] if \( \nu(G) < \infty \), that is, there is \( k \in \mathbb{Z}_{\geq 0} \) such that for any \( f \in G \) and \( g \in G^\times \), \( f \) can be expressed as a product of at most \( k \) conjugates of \( g \) or \( g^{-1} \).

A group \( G \) is called bounded if any conj.-invariant norm on \( G \) is bounded (or equivalently, any bi-invariant metric on \( G \) is bounded). Every bounded perfect group is uniformly perfect.

Fact 6.1. (1) If \( G \) is uniformly simple, then \( G \) is simple and \( \nu_g \) is bounded for any \( g \in G^\times \).

(2) If \( \nu_g \) is bounded for some \( g \in G^\times \), then \( G \) is bounded. More precisely, if \( g \in G^\times \) and \( \nu_g \leq k \) for some \( k \in \mathbb{Z}_{\geq 0} \), then \( q \leq kq(g) \) for any ext. conj.-invariant norm \( q \) on \( G \).
Next we clarify relations between the conjugation-generated norm and the commutator length supported in balls in diffeomorphism groups \[4,18\]. Recall that \(\mathcal{C}(X)\) denotes the set of connected components of a topological space \(X\).

**Definition 6.1.** Suppose \(M\) is an \(n\)-manifold possibly with boundary and \(g \in \text{Diff}^r(M)\). We say that

1. \(g\) is component-wise non-trivial if \(g|_U \neq \text{id}_U\) for any \(U \in \mathcal{C}(M)\),
2. \(g\) is component-wise end-non-trivial if
   a. \(g|_U \neq \text{id}_U\) for any \(U \in \mathcal{C}(M) \cap \mathcal{K}(M)\) and
   b. \(g|_V \neq \text{id}_V\) for any \((U, K, V)\) with \(U \in \mathcal{C}(M) - \mathcal{K}(M)\), \(K \in \mathcal{K}(U)\) and \(V \in \mathcal{C}(U - K)\) such that \(Cl_U V\) is not compact.

**Compliment 6.1.** The condition (b) is equivalent to the following practical condition (♯):

(♯) There exists an exhausting sequence \(M_i \in \mathcal{SM}_c(M)\) \((i \geq 1)\) in \(M\) for which
\[g|_L \neq \text{id}_L\] for any \(L \in \bigcup_{i \geq 1} \mathcal{C}(M_{i-1, i})\).

Compliment 6.1 is verified by a routine argument on compact \(n\)-submanifolds of \(M\). Note that any exhausting sequence \(\{M_i\}_{i \geq 1}\) in \(M\) admits \(g \in \text{Diff}(M, \partial)_0\) such that \(g|_L \neq \text{id}_L\) for any \(L \in \bigcup_{i \geq 1} \mathcal{C}(M_{i-1, i})\).

**Setting 6.1.** Suppose \(M\) is an \(n\)-manifold possibly with boundary and \(\mathcal{G}\) is a subgroup of \(\text{Diff}^r(M)_0\). For \(A \subset M\), let \(\mathcal{G}_A := \mathcal{G} \cap \text{Diff}^r(M, M_A)_0\). Note that \(g(\mathcal{G}_A)g^{-1} = \mathcal{G}_{g(A)}\) for any \(g \in \mathcal{G}\).

**Lemma 6.1.** (cf. \[4, 18\] Lemma 3.1) et al.) In Setting 6.1: Suppose \(g \in \mathcal{G}\), \(A \subset M\) and \(g(A) \cap A = \emptyset\).

1. \((\mathcal{G}A)^c \subset C^4_g\) in \(\mathcal{G}\). More precisely, for any \(a, b \in \mathcal{G}_A\) the following identity holds:
   \[
   [a, b] = g(g^{-1})^c g^{bc}(g^{-1})^b \text{ in } \mathcal{G}, \quad \text{where } c := a^{-1} \in \mathcal{G}_{g^{-1}(A)}.\]
   Note that \(cb = bc\).
2. \((\mathcal{G}B)^c \subset C^4_g\) in \(\mathcal{G}\), if \(B \subset M\) and there exists \(h \in \mathcal{G}\) with \(h(B) \subset A\).

**Proof.** (2) Let \(k := h^{-1}gh \in \mathcal{G}\). Then, \(k(B) \cap B = \emptyset\) and by (1) we have \((\mathcal{G}B)^c \subset C^4_k = C^4_g\). \(\square\)

**Lemma 6.2.** Suppose \(M\) is an \(n\)-manifold possibly with boundary and \(g \in \text{Diff}^r(M, \partial)_0\) is component-wise non-trivial.

1. \((i)\) \(\nu_g \leq 4clb^f\) in \(\text{Diff}^r(M, \partial)_0\).
2. \((ii)\) \(\nu_g \leq 4clb^f\) in \(\text{Diff}^r_c(M, \partial)_0\) if \(g \in \text{Diff}^r_c(M, \partial)_0\) \((c, \mathcal{M}\) is a finite set).

**Proof of Lemma 6.2**

1. Let \(\mathcal{G} = \text{Diff}^r(M, \partial)_0\) in (i) and \(\mathcal{G} = \text{Diff}^r_c(M, \partial)_0\) in (ii).

(1) First we show that \((\mathcal{G}D)^c = \text{Diff}^r(M, M_D)_0 \subset C^4_g\) for any \(D \in \mathcal{B}_f(M)\).

For any \(U \in \mathcal{C}(M)\), since \(g|_U \neq \text{id}_U\), there exists \(E_U \in \mathcal{B}(U)\) such that \(g(E_U) \cap E_U = \emptyset\).

Let \(E := \bigcup \{E_U \mid U \in \mathcal{C}(M)\} \in \mathcal{B}_d(M)\). Then, \(g \in \mathcal{G}\) and \(g(E) \cap E = \emptyset\). Hence, by Lemma 6.1(2) it suffices to find \(h \in \text{Diff}^r_c(M, \partial)_0 \subset \mathcal{G}\) with \(h(D) \subset E\). There exists \(U_1, \cdots, U_m \in \mathcal{C}(M)\) such that \(D \subset \bigcup_{i=1}^m U_i\). For each \(i = 1, \cdots, m\), since \(D \cap U_i \in \mathcal{B}_f(U_i)\) and \(U_i\) is connected, there exists \(h_i \in \text{Diff}^r_c(U_i, \partial)_0\) with \(h_i(D \cap U_i) \subset E_{U_i}\). Then, \(h\) is defined by \(h = h_i\) on \(U_i\) and \(h = \text{id}\) on \(M - \bigcup_{i=1}^m U_i\).
(2) Given any \( f \in \mathcal{G} \). If \( \text{clb}^d(f) = \infty \), then the assertion is trivial. Suppose \( k := \text{clb}^d(f) \in \mathbb{Z}_{\geq 0} \). Then 
\[ f = f_1 \cdots f_k \] for some \( D_i \in \mathcal{B}^\nu_f(M) \) and \( f_i \in \text{Diff}^r(M, M_{D_i})_0 \) \((i = 1, \ldots, k)\). Since \( \nu_g(f_i) \leq 4 \) \((i = 1, \ldots, k)\), it follows that \( \nu_g(f) \leq 4k \).

[II] Take an exhausting sequence \( \{M_i\}_{i \geq 1} \) in \( M \) as in Compliment 6.1(2). Let \( L_i := M_{i-1}, i \geq 1 \) \( \mathcal{C} := \bigcup_{i \geq 1} \mathcal{C}(L_i) \) and \( \mathcal{G} := \text{Diff}^r(M, \partial)_0 \). We show the following claims in order.

1. There exists a family \( E_L \in \mathcal{B}(L) \) \((L \in \mathcal{C})\) such that \( g(E) \cap E = \emptyset \) for \( E := \bigcup_{L \in \mathcal{C}} E_L \in \mathcal{B}_d(M) \).

   In fact, inductively, we can find a family of points \( p_L \in \text{Int} L \) \((L \in \mathcal{C})\) such that \( g(A) \cap A = \emptyset \) for \( A := \{p_L \mid L \in \mathcal{C}\} \subset M \). Since \( A \) is closed in \( M \), there exists an open neighborhood \( U \) of \( A \) in \( M \) with \( g(U) \cap U = \emptyset \). For each \( L \in \mathcal{C} \) choose an n-ball neighborhood \( E_L \) of \( p_L \) in \( \text{Int} L \cap U \).

   Then, \( \{E_L\}_{L \in \mathcal{C}} \) is discrete since \( E_L \in \mathcal{B}(L) \) \((L \in \mathcal{C})\), and \( g(E) \cap E = \emptyset \) since \( E \subset U \).

2. Given any \( D = \bigcup_{j \in J} D_j \in \mathcal{B}^\nu_d(M) \) \((i.e., D \subset Int M \) and \( \{D_j\}_{j \in J} \) is a discrete family of n-balls in \( M \). Then, there exists \( h \in \mathcal{G} \) such that \( h(D) \subset E \).

   The required \( h \) is obtained as the composition \( h = h_2h_1 \) of \( h_1, h_2 \in \mathcal{G} \) such that

   - (i) for each \( j \in J \) there exists \( i(j) \geq 1 \) with \( h_1(D_j) \subset L_{i(j)} \) and \( (ii) h_2(h_1(D)) \subset E \).

   - (i) There exists a discrete family \( \{C_j\}_{j \in J} \) of n-balls in \( M \) such that \( D_j \subset C_j \subset Int M \) \((j \in J)\).

   For each \( j \in J \) there exists \( i(j) \geq 1 \) with \( D_j \cap \text{Int} L_{i(j)} \neq \emptyset \). Then, we can push each \( D_j \) into \( \text{Int} L_{i(j)} \) in \( C_j \). This procedure yields the desired \( h_1 \).

   - (ii) For each \( L \in \mathcal{C} \), since \( h_1(D) \cap L \in \mathcal{B}^\nu_f \) \((L) \) and \( L \) is connected, there exists \( h_L \in \text{Diff}^r(M, M_{\text{Int} L})_0 \) such that \( h_L(h_1(D) \cap L) \subset E_L \). Then, \( h_2 \) is defined by \( h_2|_L = h_L \) \((L \in \mathcal{C})\).

3. \( (\mathcal{G}_D)^c = \text{Diff}^r(M, M_{D})_0 \subset C_4^d \) in \( \mathcal{G} \) for any \( D \in \mathcal{B}^\nu_d(M) \).

   This follows from (2) and Lemma 6.1(2).

4. \( \nu_g \leq 4\text{clb}^d \) in \( \mathcal{G} \).

   Given any \( f \in \mathcal{G} \). If \( \text{clb}^d(f) = \infty \), then the assertion is trivial. Suppose \( k := \text{clb}^d(f) \in \mathbb{Z}_{\geq 0} \). Then 
\[ f = f_1 \cdots f_k \] for some \( D_i \in \mathcal{B}^\nu_d(M) \) and \( f_i \in \text{Diff}^r(M, M_{D_i})_0 \) \((i = 1, \ldots, k)\). Since \( f_i \in C_4^d \) \((i = 1, \ldots, k)\), it follows that \( f \in C_4^d \) and \( \nu_g(f) \leq 4k \).

6.2. Estimates of conjugation-generated norm on diffeomorphism groups.

The estimates on \( \text{clb}^d \) and \( \text{clb}^d \) obtained in Sections 4, 5 lead to the following conclusions for \( \nu_g \).

6.2.1. Odd-dimensional case.

**Theorem 6.1.** Suppose \( M \) is a compact \((2m + 1)\)-manifold possibly with boundary \((m \geq 0)\), \( 1 \leq r \leq \infty \), \( r \neq 2m + 2 \). If \( g \in \text{Diff}^r(M, \partial)_0 \) and \( \text{clb}^r \neq \text{id}_U \) for any \( U \in C(M) \), then the following holds.

1. \( \nu_g \text{Diff}^r(M, \partial)_0 \leq 4\text{clb}^d \text{Diff}^r(M, \partial)_0 \leq 16m + 24 \).

2. If \( M \) is closed, then 
\[ \nu_g \text{Diff}^r(M)_0 \leq 4\text{clb}^d \text{Diff}^r(M)_0 \leq 8c(\mathcal{H}) + 8 \]
for any handle decomposition \( \mathcal{H} \) of \( M \).

**Theorem 6.2.** Suppose \( M \) is an open \((2m + 1)\)-manifold \((m \geq 0)\), \( 1 \leq r \leq \infty \), \( r \neq 2m + 2 \) and \( \mathcal{H} \) is any handle decomposition of \( M \).

1. \( \nu_g \text{Diff}^r(M)_0 \leq 4\text{clb}^d \text{Diff}^r(M)_0 \leq 16c(\mathcal{H}) + 16 \leq 32m + 48 \)
if \( g \in \text{Diff}^r(M)_0 \) is component-wise end-non-trivial.

2. \( \nu_g \text{Diff}^r_c(M)_0 \leq 4\text{clb}^d \text{Diff}^r_c(M)_0 \leq 8c(\mathcal{H}) + 8 \leq 16m + 24 \)
if \( g \in \text{Diff}^r_c(M)_0 \) is component-wise non-trivial.
Theorem 6.4. Suppose $1 \leq \nu \leq (2)$ follow from Theorem 4.2 and Lemma 6.2 [II], [I](ii).

Theorem 6.3. Even-dimensional case.

6.2.2. Suppose $M_{\nu}$ is a non-trivial. For a closed $2m$-manifold with a handle decomposition, we can use Corollary 5.2 to obtain a fine estimate on $\nu_g$ Diff$^r(M_\nu)$ when $\nu = 2m \neq 2m + 1$, $\mathcal{T}$ is a $C^\infty$ triangulation of $M$, $\mathcal{S}$ is the set of $m$-simplices of $\mathcal{T}$, $\mathcal{S} \supset \mathcal{F} \supset \{\sigma \in \mathcal{S} | \sigma \not\subset \partial M\}$ and $\{\mathcal{F}_j\}_{j=1}^k$ is a finite cover of $\mathcal{F}$ (as a set). If $g \in$ Diff$^r(M, \partial_0)$ is component-wise non-trivial, then the following holds.

[I] $\nu_g d$ Diff$^r(M, \partial_0) \leq 4 clb^d$ Diff$^r(M, \partial_0) \leq 8(m + 2)(k + 1)$

if each $|\mathcal{F}_j|$ $(j = 1, \ldots, k)$ is strongly displaceable from $|\mathcal{F}_j| \cup (\partial M \times [0, 1))$ in $\widetilde{M}$.

[II] $\nu_g d$ Diff$^r(M, \partial_0) \leq 4 clb^d$ Diff$^r(M, \partial_0) \leq 4(2m + 3)(k + 1) + 12$

if each $|\mathcal{F}_j|$ $(j = 1, \ldots, k)$ is strongly displaceable from $|\mathcal{S}| \cup (\partial M \times [0, 1))$ in $\widetilde{M}$.

[III] $\nu_g d$ Diff$^r(M, \partial_0) \leq 4 clb^d$ Diff$^r(M, \partial_0) \leq 4(2m + 1)(\ell + 1) + 20$ for $\ell := \#\{\sigma \in \mathcal{S} | \sigma \not\subset \partial M\}$.

Recall that $\widetilde{M} := M \cup_{\partial M} (\partial M \times [0, 1))$. Theorem 6.3 follows from Theorem 5.1 Corollary 5.1 and Lemma 6.2 [I](i). When $M$ is a closed $2m$-manifold, the estimate in Theorem 6.3 [III] is compared with that in [19], that is,

$\nu_g$ Diff$^r(M) \leq 16(\ell + 4)m + 12\ell + 28$ (19 Proof of Corollary 1.3) (p.173).

For a closed $2m$-manifold with a handle decomposition, we can use Corollary 5.2 to obtain a fine estimate of $\nu_g$.

In Section 5.3 we obtained some estimates on $clb^d$ for some important classes of open $2m$-manifolds (covering spaces, infinite sums etc.) The consequence on $\nu_g$ of these results are listed below.

Proposition 6.1. Suppose $M$ is an open $2m$-manifold, $1 \leq r \leq \infty$, $r \neq 2m + 1$, $\mathcal{H}$ is a handle decomposition of $M$, $g \in$ Diff$^r(M_0)$ is component-wise end-non-trivial and $h \in$ Diff$^r_c(M_0)$ is component-wise non-trivial.

1. If $\mathcal{H}$ includes no $m$-handles, then

$I$ $\nu_g d$ Diff$^r(M_0) \leq 4 clb^d$ Diff$^r(M_0) \leq 16d(\mathcal{H}) + 8 \leq 32m + 8$,

$\nu_h d$ Diff$^r_c(M_0) \leq 4 clb^d$ Diff$^r(M_0) \leq 8c(\mathcal{H}) + 4 \leq 16m + 4$.

2. Suppose $m \geq 3$, $\mathcal{H}$ includes only finitely many $m$-handles, $\mathcal{S}$ is the set of all open $m$-cells of $P_\mathcal{H}$ and $\{\mathcal{S}_j\}_{j=1}^k (k \geq 1)$ is a finite cover of $\mathcal{S}$.

$I$ $\nu_g d$ Diff$^r(M_0) \leq 4 clb^d$ Diff$^r(M_0) \leq 8(m + 2)(k + 3) - 28$,

$\nu_h d$ Diff$^r_c(M_0) \leq 4 clb^d$ Diff$^r_c(M_0) \leq 8(m + 2)(k + 1)$.

if $Cl_M|\mathcal{S}_j|$ is strongly displaceable from itself in $M$ for each $j = 1, \ldots, k$.

[II] $\nu_g d$ Diff$^r(M_0) \leq 4 clb^d$ Diff$^r(M_0) \leq 4(2m + 3)(k + 3) - 8$,

$\nu_h d$ Diff$^r_c(M_0) \leq 4 clb^d$ Diff$^r_c(M_0) \leq 4(2m + 3)(k + 1) + 12$.

if $Cl_M|\mathcal{S}_j|$ is strongly displaceable from $Cl_M|\mathcal{S}|$ in $M$ for each $j = 1, \ldots, k$.

Theorem 6.4. Suppose $\pi : \widetilde{M} \to M$ is a $C^\infty$ covering space over a closed $2m$-manifold $M$ ($m \geq 3$),

$1 \leq r \leq \infty$, $r \neq 2m + 1$, $g \in$ Diff$^r(\widetilde{M})$ is component-wise end-non-trivial and $h \in$ Diff$^r_c(\widetilde{M})$ is component-wise non-trivial.
Suppose $\mathcal{H}$ is a handle decomposition of $M$, $\mathcal{S}$ is the set of open $m$-cells of $\mathcal{H}$ and 
\( \{ S_j \}_{j=1}^k \) (\( k \geq 1 \)) is a finite cover of $\mathcal{S}$ such that
each $\text{Cl}_M | S_j |$ (\( j = 1, \ldots, k \)) has an open neighborhood in $M$ which is evenly covered by $\pi$.

[I] If $\text{Cl}_M | S_j |$ is strongly displaceable from itself in $M$ for each $j = 1, \ldots, k$, then
(i) $\nu_b d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 8(m + 2)(k + 1),$
(ii) $\nu_g d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 16(m + 2)(k + 1)$.

[II] If $\text{Cl}_M | S_j |$ is strongly displaceable from $\text{Cl}_M | S |$ in $M$ for each $j = 1, \ldots, k$, then
(i) $\nu_b d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 4(2m + 3)(k + 1) + 12,$
(ii) $\nu_g d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 8(2m + 3)(k + 1) + 24.$

[B] Suppose $M$ has a $C^\infty$ triangulation with at most $\ell m$-simplices.
(i) $\nu_b d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 4(2m + 3)(\ell + 1) + 12,$
(ii) $\nu_g d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 8(2m + 3)(\ell + 1) + 24.$

Setting 6.2. Suppose $M$ is an open $2m$-manifold ($m \geq 3$), $1 \leq r \leq \infty$, $r \neq 2m + 1$, $g \in \text{Diff}^r(M)$ is component-wise end-non-trivial and $h \in \text{Diff}_c(M)$ is component-wise non-trivial. Assume that
(i) $\mathcal{N} = \{ (N_i, T_i) \}_{i=0}^\infty$ is a family of compact $2m$-manifolds as in Setting 5.8 and it satisfies the condition (\( \ast \)) in Setting 5.8+ with respect to some finite covers \{ $C_{0,j}$ \}_{j=1}^p of $C_0$ and \{ $C_{i,j}$ \}_{j=1}^{p^1}$ of $C_i$ ($i = 1, \ldots, \ell$).
(ii) $(M, \{ M_k \}_{k \geq 0}, T)$ is an infinite sum of the model manifolds in $\mathcal{N}$ as in Setting 5.8+.

Proposition 6.2. In Setting 6.2:

[I] $\nu_b d \text{Diff}(M) \leq 4c b^d d \text{Diff}(M) \leq 8(m + 2)(2p + q + 2),$
$\nu_g d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 8(m + 2)(p + q + 1)$
if \{ $C_{0,j}$ \}_{j=1}^p and \{ $C_{i,j}$ \}_{j=1}^{p^1}$ (\( i = 1, \ldots, \ell \)) satisfy the condition in Proposition 5.5[I].

[II] $\nu_b d \text{Diff}(M) \leq 4c b^d d \text{Diff}(M) \leq 4(2m + 3)(2p + q + 2) + 24$
$\nu_g d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 4(2m + 3)(p + q + 1) + 12$
if \{ $C_{0,j}$ \}_{j=1}^p and \{ $C_{i,j}$ \}_{j=1}^{p^1}$ (\( i = 1, \ldots, \ell \)) satisfy the condition in Proposition 5.5[II].

[III] $\nu_b d \text{Diff}(M) \leq 4c b^d d \text{Diff}(M) \leq 4(2m + 3)(2a + b + 2) + 24$
$\nu_g d \text{Diff}_c(M) \leq 4c b^d d \text{Diff}_c(M) \leq 4(2m + 3)(a + b + 1) + 12$
if $C_0$ and $C_i$ (\( i = 1, \ldots, \ell \)) satisfy the condition (\( \ast \)) in Corollary 5.4.

The estimates in Proposition 6.1, Theorem 6.4 and Proposition 6.2 are immediately obtained by Lemma 6.2[II] and the results in Section 5.3, i.e., Propositions 5.3, 5.4, Theorem 5.3, Corollary 5.3 and Proposition 5.5 + Corollary 5.4, respectively.

6.2.3. Some special cases.

The next lemma is the $\nu$-version of Lemma 2.1.

Lemma 6.3. Suppose $M$ is an n-manifold possibly with boundary, $1 \leq r \leq \infty$, $r \neq n+1$, $F \in \text{Isot}_c^r(M, \partial) \text{ and } f := F_1 \in \text{Diff}_c^r(M, \partial)$.
Assume that there exist $V, W \in \mathcal{K}(\text{Int} M)$ and $\varphi \in \text{Diff}_c^r(M, \partial) \text{ such that} V \supset W$, $\varphi(V) \subset W$, supp $F \subset \text{Int} M V - W$ and $\varphi|_N \neq \text{id}_N$ for any $N \in \mathcal{C}(M)$.
Then, $\nu_{\varphi}(f) \leq 6$ in $\text{Diff}_c^r(M, \partial)$. 

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\textbf{Proof.} We take }U \in \mathcal{K}(\text{Int }M)\text{ such that }V \subset U \text{ and } \varphi \in \text{Diff}^r(M, M_U)_0, \text{ and repeat the proof of Lemma 2.1. Then, we work in } \text{Diff}^r_c(M, \partial)_0. \text{ It follows that } f = hg_0, \quad g_0 = [\varphi, H^{-1}] = \varphi(H^{-1}\varphi^{-1}H) \text{ and } \text{clb}^d(h) \leq 1. \text{ Hence, } \nu_{\varphi}(g_0) \leq 2 \text{ and } \nu_{\varphi}(h) \leq 4 \text{clb}^d(h) \leq 4 \text{ by Lemma 6.2[I(ii)]. This implies that } \nu_{\varphi}(f) \leq 6 \text{ in } \text{Diff}^r_c(M, \partial)_0. \qed

\textbf{Example 6.1.} Suppose }M\text{ is a closed }n\text{-manifold, }1 \leq r \leq \infty, r \neq n + 2. \text{ Then}
\nu_{\varphi_\xi} \text{clb}^d(\text{Diff}^r(M \times I, \partial)_0) \leq 6 \text{ for any } \xi \in \text{Diff}^r(I, \partial)_0 - \{\text{id}_I\} \text{ and } \varphi_\xi := \text{id}_M \times \xi \in \text{Diff}^r(M \times I, \partial)_0.

This example follows from the same argument as in Example 2.3, using Lemma 6.3. The next proposition follows from Lemma 6.2(I) and Proposition 2.1(2).

\textbf{Proposition 6.3.} Suppose }W\text{ is a compact connected }n\text{-manifold with boundary, }1 \leq r \leq \infty, r \neq n + 1 \text{ and } g \in \text{Diff}^r(\text{Int }W)_0 \text{ is component-wise end-non-trivial. Let } m := \text{clb}^d(\text{Diff}^r(\partial W \times I; \partial)_0 < \infty. \text{ Then,}
\nu_g \text{clb}^d(\text{Diff}^r(\text{Int }W)_0) \leq 4 \text{clb}^d(\text{Diff}^r(\text{Int }W)_0) \leq 4 \max\{\text{clb}^d(\text{Diff}^r(W; \partial)_0, m\} + 4m.

In this case, }g\text{ is component-wise end-non-trivial if and only if }g\text{ satisfies the following condition;}
(i) \ g|_{L \times [0, 1]} \neq \text{id} \text{ for any } L \in \mathcal{C}(\partial W) \text{ and any collar neighborhood } L \times [0, 1] \text{ of } L = L \times \{0\} \text{ in } W.

6.3. Boundedness and uniform simplicity of diffeomorphism groups.

In this final section we summarize the conclusions on boundedness and uniform simplicity of diffeomorphism groups, which are immediately induced from the estimates on }\nu_g\text{ in Section 6.2 (together with the estimates on }\text{clb}^d\text{ and }\text{clb}^d\text{ in the previous sections). The estimates on }\text{clb}^d\text{ in the previous sections lead us to some boundedness result on the group }\text{Diff}^r(M)_0\text{ even for a non-compact manifold }M. \text{ However, the investigation of (uniform) simplicity is restricted to the subgroup }\text{Diff}^r_c(M; \partial)_0, \text{ since the diffeomorphism group }\text{Diff}^r(M)_0\text{ of any }n\text{-manifold }M\text{ includes the proper normal subgroups (a) }\text{Diff}^r(M, \partial)_0\text{ if }\partial M \neq \emptyset\text{ and (b) }\text{Diff}^r_c(M)_0\text{ if }M\text{ is non-compact. The next lemma follows from Lemma 6.2 together with Fact 2.1(2) and Remark 2.1(3).}

\textbf{Lemma 6.4.} Suppose }M\text{ is an }n\text{-manifold possibly with boundary.}

(1) \text{Diff}^r(M, \partial)_0 \text{ is bounded if } \text{clb}^d \text{ is bounded in } \text{Diff}^r(M, \partial)_0.

(2) \quad (i) \text{ If } \text{clb}^d \text{ is bounded in } \text{Diff}^r_c(M, \partial)_0 \text{ and } \# \mathcal{C}(M) < \infty, \text{ then } \text{Diff}^r_c(M, \partial)_0 \text{ is bounded.}

(ii) \text{ If } \text{Diff}^r_c(M, \partial)_0 \text{ is bounded and } r \neq n + 1, \text{ then } \text{clb}^d \text{ is bounded in } \text{Diff}^r_c(M, \partial)_0.

The following is the main result in this section, which follows immediately from the results in Section 6.2.

\textbf{Corollary 6.1.} \text{Suppose }1 \leq r \leq \infty, r \neq n + 1.

[1] Suppose }M\text{ is a compact }n\text{-manifold possibly with boundary. When }n \neq 2, 4,

(i) \text{Diff}^r(M, \partial)_0 \text{ is bounded and (ii) } \text{Diff}^r(M, \partial)_0 \text{ is uniformly simple if }M\text{ is connected.}

[2] Suppose }M\text{ is an open }n\text{-manifold. Then,}

(i) \text{Diff}^r(M)_0 \text{ is bounded,}

(ii) \text{Diff}^r_c(M)_0 \text{ is (a) bounded if } \# \mathcal{C}(M) < \infty \text{ and (b) uniformly simple if }M\text{ is connected in the following cases :}

(1) \ n = 2m + 1 \text{ (}m \geq 0\text{)}

(2) \ n = 2m \text{ (}m \geq 1\text{)} \text{ and }M\text{ satisfies one of the following conditions:}

\text{for }m \geq 1 :
(a) $M$ has a handle decomposition without $m$-handles.

for $m \geq 3$:

(b) $M$ has a handle decomposition $H$ with only finitely many $m$-handles and for which

the closure of each open $m$-cell of $P_H$ is strongly displaceable from itself in $M$.

(c) $M$ is a $C^\infty$ covering space over a closed $2m$-manifold.

(d) $M$ is an infinite sum of finitely many compact $2m$-manifolds (as in Setting 6.2).

[3] Suppose $W$ is a compact $n$-manifold with boundary. Then, Diff$^r$(Int $W$)$_0$ is bounded, if clb$^f$ is bounded in Diff$^r$(W,$\partial$)$_0$ (equivalently, Diff$^r$(W,$\partial$)$_0$ is bounded).

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