We answer the last question left open in [Z. Kočan, Chaos on one-dimensional compact metric spaces, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 22, article id: 1250259 (2012)] which asks whether there is a relation between an infinite LY-scrambled set and \( \omega \)-chaos for dendrite maps. We construct a continuous self-map of a dendrite without an infinite LY-scrambled set but containing an uncountable \( \omega \)-scrambled set.

1. Introduction

In 2012 in [3] characterized various chaotic properties of maps on dendrites. In this paper he raised three open questions needed to complete characterization of considered notions:

(1) Does the existence of an uncountable \( \omega \)-scrambled set imply distributional chaos?

(2) Does the existence of an uncountable \( \omega \)-scrambled set imply existence of an infinite LY-scrambled set?

(3) Does distributional chaos imply the existence of an infinite LY-scrambled set?

Recently, in [2] Drwięga answered questions (1) and (3). Therefore question (2) is the only which still remains open. In the present paper we use some special properties of Sturmian subshift to construct appropriate example and prove the following Theorem.

**Theorem 1.1.** There exists a continuous self-map \( f \) on dendrite such that:

(1) \( f \) has an uncountable \( \omega \)-scrambled set,

(2) \( f \) does not have an infinite LY-scrambled set.

This provides a negative answer to this question (2).

2. Definitions and notations

Throughout this paper \( \mathbb{N} \) denotes the set \( \{1, 2, 3, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). By a dynamical system we mean a pair \((X, f)\), where \((X, \rho)\) is a compact metric space and \( f \) is a continuous map from \( X \) to itself. The orbit of \( x \in X \) is the set \( O_f(x) := \{f^k(x) : k \geq 0\} \), where \( f^k \) stands for the \( k \)-fold composition of \( f \) with itself. For \( x \in X \), its \( \omega \)-limit set is defined by \( \omega_f(x) := \bigcap_n \bar{O}_f(f^n(x)) \). A set \( A \subset X \) is invariant under \( f \) if \( f(A) \subset A \) and minimal for \( f \) if \( A \) is nonempty, closed, invariant under \( f \), and does not contain any proper subset which satisfies these three conditions. A dynamical system \((X, f)\) is minimal if \( X \) is a minimal set for \( f \). It is known that \((X, f)\) is minimal if and only if every \( x \in X \) has dense orbit or,
equivalently, $\omega_f(x) = X$ for each $x \in X$. By $S^1$ we denote the unit circle identified with the interval $[0, 1]$. For $\alpha \in S^1$ the rotation by $\alpha$ is denoted by $R_\alpha$, that is $R_\alpha(x) = \text{mod}_1(x + \alpha)$ for $x \in S^1$. It is known that if $\alpha$ is irrational, then $(S^1, R_\alpha)$ is minimal (e.g., see [5]).

Let $X$ and $Y$ be compact metric spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. If there is a continuous surjective map $\phi : X \to Y$ with $\phi \circ f = g \circ \phi$, then $f$ and $g$ are semiconjugate (by $\phi$). The map $\phi$ is called a semiconjugacy or a factor map, the map $g$ is called a factor of $f$ and the map $f$ is called an extension of $g$. If $\phi$ in the definition above is a homeomorphism then we call it a conjugacy.

A pair of two different points $(x, y) \in X^2$ is

- **proximal** if 
  $$\liminf_{n \to \infty} \rho(f^n(x), f^n(y)) = 0,$$

- **asymptotic** if 
  $$\lim_{n \to \infty} \rho(f^n(x), f^n(y)) = 0,$$

- **scrambled** or Li-Yorke if it is proximal but not asymptotic.

A set $S \subseteq X$ is LY-scrambled for $f$, if it contains at least two distinct points and every pair of distinct points in $S$ is scrambled. We say that $f : X \to X$ is LY chaotic if there exists a LY-scrambled set (this definition derives from [7]). We say after [6], that a set $S \subseteq X$ is called $\omega$-scrambled for $f$ if it contains at least two points and for any $x, y \in S$ with $x \neq y$, we have

1. $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
2. $\omega_f(x) \cap \omega_f(y)$ is nonempty,
3. $\omega_f(x)$ is not contained in the set of periodic points.

The map $f$ is $\omega$-chaotic if there is at least a two-point $\omega$-scrambled set for $f$.

An arc is any topological space homeomorphic to the interval $[0, 1]$. A continuum is a nonempty connected compact metric space. A dendrite is a locally connected continuum containing no subset homeomorphic to the circle. Let $X$ be a compact arcwise connected metric space and $v \in X$. We say that $S \subseteq X$ is an $n$-star with center $v \in S$ if there is a continuous injection $\varphi : S \to \mathbb{C}$ such that $\varphi(v) = 0$ and $\varphi(S) = \{r \exp(2k\pi i/n) : r \in [0, 1], k = 1, 2, \ldots, n\}$. The valence of $v$ in $X$, denoted by $\text{val}(v)$, is the number (which may be $\infty$)

$$\sup \{n \in \mathbb{N} : \text{there exists an } n\text{-star with center } v \text{ contained in } X\}.$$

The point $v$ is called an endpoint of $X$ if $\text{val}(x) = 1$, and a branching point of $X$ if $\text{val}(x) \geq 3$. The Gehman dendrite is a dendrite $G$ having the set of end points homeomorphic to the Cantor ternary set in $[0, 1]$ such that all branching points $G$ have valence 3.

Finally, let us present some standard notation related to symbolic dynamics. Let $A$ be any finite set (an alphabet) and let $A^*$ denote the set of all finite words over $A$ including the empty word. For any word $w \in A^*$ we denote by $|w|$ the length of $w$, that is the number of letters which form this word. If $w$ is the empty word then we put $|w| = 0$. An infinite word is a mapping $w : \mathbb{N} \to A$, in other words it is an infinite sequence $w_1w_2w_3 \ldots$ where $w_i \in A$ for any $i \in \mathbb{N}$. The set of all infinite words over an alphabet $A$ is denoted by $A^\mathbb{N}$. We endow $A^\mathbb{N}$ with the product topology of discrete topology on $A$. By $0^\infty$ we will denote the infinite word $0^\infty = 000 \cdots \in A^\mathbb{N}$. If $x \in A^\mathbb{N}$ and $i, j \in \mathbb{N}$ with $i \leq j$ then we denote $x_{[i, j]} = x_ix_{i+1}\ldots x_{j-1}$ (we agree with that $x_{[i, i]}$ is empty word) and given $X \subset A^\mathbb{N}$ by $\mathcal{L}(X)$ we denote the language
of $X$, that is, the set $L(X) := \{x_{1,k} : x \in X, k > 0\}$. We write $L_n(X)$ for the set of all words of length $n$ in $L(X)$. If $u_k$ is a sequence of words such that $|u_k| \to \infty$ then we write $z = \lim_{k \to \infty} u_k$ if the limit $z = \lim_{k \to \infty} u_k0^\infty$ exists in $A^\mathbb{N}$. Let $n \in \mathbb{N}$ and $\sigma$ a shift map defined on $A^\mathbb{N}$ by

$$(\sigma(x))_i = x_{i+1} \text{ for } i \in \mathbb{N}.$$

By $\Sigma^+_n$ we denote a dynamical system formed by $\left\{(0, \ldots, n-1)^\mathbb{N}, \sigma\right\}$. If $S \subset A^\mathbb{N}$ is nonempty, closed and $\sigma$-invariant then $S$ together with the restriction $\sigma|_S : S \to S$ (or even the set $S$) is called a subshift of $\Sigma^+_n$. Recall that the space $A^\mathbb{N}$ is metrizable by the metric $\rho : A^\mathbb{N} \times A^\mathbb{N} \to \mathbb{R}$ given for $x, y \in A^\mathbb{N}$ by

$$\rho(x, y) = \begin{cases} 2^{-k}, & \text{if } x \neq y, \\ 0, & \text{otherwise} \end{cases}$$

where $k$ is the length of maximal common prefix of $x$ and $y$, that is $k = \max\{i \geq 1 : x[1,i) = y[1,i)\}$.

3. The Gehman Dendrite

Let us recall the construction of a continuous dendrite map from [4]. Let $G$ be the Gehman dendrite and fix any point $p \in G$ with $val(p) = 2$. For any distinct $a, b \in G$ we will denote by $[a, b]$ the unique arc in $G$ joining these points. It is easy to see that branching points in $G$ can be arranged in such a way that each of the following intervals contains branching points (or point $p$) exactly at endpoints: $B_0 = [p, p_0], B_1 = [p, p_1]$, and for every $n \in \mathbb{N}$, $B_{i_1i_2\ldots i_n} = [p_{i_1i_2\ldots i_n}, p_{i_1i_2\ldots i_n+1}]$ where every $i_k$ is either 0 or 1. It is well known that by the above construction we will cover all points of $G$ but endpoints (see Figure 1). Furthermore, every point $x \in E$ can be uniquely associated to a sequence of zeros and ones $i_1i_2i_3\ldots$ in such way that it is the limit of the codes of the arcs converging to the point.

![Figure 1. The Gehman dendrite](image)

We define a continuous map $g$ on a dendrite $G$ in the following way. Let $g(B_0) = g(B_1) = \{p\}$. For every $i_1, i_2, \ldots, i_n$, let $g|_{B_{i_1i_2\ldots i_n}} : B_{i_1i_2\ldots i_n} \to B_{i_2i_3\ldots i_n}$ be a homeomorphism such that $g(p_{i_1i_2\ldots i_n}) = p_{i_2i_3\ldots i_n}$, and let $g$ act on $E$ as the shift map on the space $\Sigma^+_2$. Let $X$ be a closed $g$-invariant subset of $E$. Denote

$$D_X = \bigcup_{x \in X} [x, p]$$
and

\[ f = g|_{D_X}. \]

Now, let us recall useful lemmas from [1] about the Gehman dendrite.

**Lemma 3.1.** If \( X \subset \Sigma^+_n \) is a subshift then the set \( D_X \) is an \( f \)-invariant subdendrite of the Gehman dendrite \( G \).

**Lemma 3.2.** If \( X \) is a closed and nonempty subset of \( \Sigma^+_2 \) without isolated points, then the set \( D_X = \bigcup_{x \in X} [x_L, p] \subset G \) is homeomorphic with the Gehman dendrite.

4. **STURMIAN SUBSHIFT AND THE MAIN EXAMPLE**

A Sturmian subshift is an extension of an irrational rotation \((S^1, R_\alpha)\) generated by a particular interval cover (see [5]). Every Sturmian subshift is minimal and has zero topological entropy.

Let \( \alpha \in (0, 1) \) be irrational number and fix cover \([0, 1/4], [1/4, 1)\) of \( S^1 \). Denote by \( S_\alpha = \overline{O(A(\alpha))} \) the Sturmian subshift defined by the itinerary \( A(\alpha) \) of a point \( \alpha \), that is the infinite sequence of symbols such that

\[ A(\alpha)_i = 0 \iff R^i_\alpha(\alpha) \in [0, 1/4) \]

and

\[ A(\alpha)_i = 1 \iff R^i_\alpha(\alpha) \in [1/4, 1) \]

for \( i \in \mathbb{N} \).

**Lemma 4.1.** Let \( C \subset \mathbb{R} \setminus \mathbb{Q} \) be a Cantor set. Then the set

\[ S = \bigcup_{\alpha \in C} S_\alpha \]

is closed. Furthermore, if \( x_n \in S_{\alpha_n} \), \( \lim_{n \to \infty} \alpha_n = \alpha \in C \) and \( \lim_{n \to \infty} x_n = x \) exists, then \( x \in S_\alpha \).

**Proof.** Fix any sequence \((x_n) \subset S\) such that \( \lim_{n \to \infty} x_n = x \) exists. There exists a sequence \((\alpha_n) \subset C\) such that \( x_n \in S_{\alpha_n} \) and going to a subsequence we may also assume that the following limit exists \( \lim_{n \to \infty} \alpha_n = \alpha \in C \).

Let \( \pi_n: S_{\alpha_n} \to S^1 \) be the standard projection (almost 1-1 conjugacy) and let \( \pi: S_\alpha \to S^1 \). Denote \( z_n = \pi_n(x_n) \) and going to a subsequence assume that \( z = \lim_{n \to \infty} z_n \). We may also assume that all \( z_n \) belong to a small one sided neighborhood of \( z \) and converge monotonically with respect to that neighborhood. Let us consider two cases.

If \( R^i_\alpha(z) \notin \{0, 1/4\} \) for every \( i \in \mathbb{N}_0 \) then there exists a unique point \( y \) such that \( \pi(y) = z \). Furthermore, for every \( N \) there exists \( k \) such that if \( i \leq k \) and \( n \geq N \) then \( R^i_\alpha(z_n) \notin \{0, 1/4\} \). Clearly, for \( n \) sufficiently large (or in other words, \( z_n \) sufficiently close to \( z \)) we have \( x_n(i) = y(i) \). This immediately implies that \( y = x \) and so the claim holds.

For the second case, assume that \( R^i_\alpha(z) \in \{0, 1/4\} \) for some \( i \). Since \( \alpha \) is irrational number, there is at most one such integer \( i \). For simplicity assume that \( i = 0 \). Then, by arguments as above we see that for every \( i \neq 0 \) there is \( N \) such that \( x_n(i) = y(i) \) for every \( n \geq N \). This shows that \( x(i) = y(i) \) for every \( i \neq 0 \). But since \( z \in \{0, 1/4\} \), there exists \( \tilde{y} \in S_\alpha \) such that \( \tilde{y}(i) = y(i) \) for \( i \neq 0 \) and \( y(0) = 1 - \tilde{y}(0) \). In particular, either \( x = y \) or \( x = \tilde{y} \). In any case \( x \in S_\alpha \), completing the proof. \( \square \)
Remark 4.2. Sturmian system $S_\alpha$ does not have Li-Yorke pairs. Simply, if $\pi: S_\alpha \to S^1$ is factor map and $x, y \in S_\alpha$ is proximal then $\pi(x) = \pi(y)$. But in that case the pair is asymptotic.

Let us recall Lemma 3.1 from [8] which is a slight generalization of Lemma 2.2 in [8]:

Lemma 4.3. Let $a = a_1a_2a_3\ldots, b = b_1b_2b_3\ldots \in \{0,1\}^\mathbb{N}$. Define the following operation:

$$a \circ b := a_1b_1a_2b_1a_3b_1a_4b_1a_5b_1a_6b_1\ldots$$

Then

(1) $$\omega(a \circ b) \supseteq \overline{O(a)} \cup \overline{O(b)},$$

(2) $$\omega(a \circ b) \subseteq \overline{O(a)} \cup \overline{O(b)} \cup \{a_1\ldots a_jb_1: j \geq i \geq 1\}$$

$$\cup \{b_1\ldots b_i a_2: j \geq i \geq 1\}.$$ 

Now we are ready to start our construction of a subshift with a special structure. Fix any Cantor set $A \subset [0,1/2) \setminus \mathbb{Q}$ and $\beta \in [0,1/2) \setminus (\mathbb{Q} \cup A)$. Denote $S := \bigcup_{\alpha \in A} S_\alpha$ and $Z = S_\beta$. By Lemma 4.1 the set $S$ is a subshift and it is also clear that $S \cap Z = \emptyset$.

Since map $\psi: (Z, \sigma) \to (S^1, R)$ is $1 - 1$ on a residual set and every uncountable Borel set contains a Cantor set (e.g. see [9]), there exists a Cantor set $\hat{A} \subset Z$.

But any two Cantor sets are homeomorphic, we can index elements of $\hat{A}$ in the following way $\hat{A} = \{b_\alpha: \alpha \in A\} \approx A$.

For every $\alpha \in A$ let $\pi_\alpha: S_\alpha \to S^1$ be the standard factor map, and let $a_\alpha = \pi_\alpha^{-1}(1/8)$. Note that $a_\alpha$ is uniquely defined, since $R^i_\alpha(1/8) \notin \{0,1/4\}$ for every $i \in \mathbb{N}_0$. Finally put

$$x_\alpha = a_\alpha \circ b_\alpha.$$ 

By Lemma 4.3 we see that

$$\omega(x_\alpha) = \{S_\alpha, Z\} \cup \{a_1\ldots a_j b_1: j \geq i \geq 1\} \cup \{b_1\ldots b_j a_2: j \geq i \geq 1\}.$$ 

where $a_\alpha = a_1a_2\ldots$ and $b_\alpha = b_1b_2\ldots$.

Lemma 4.4. For every $\alpha \in A$ the set $\overline{O(x_\alpha)}$ does not contain scrambled set with more than two points.

Proof. It is clear, that any two points $x, y \in O(x_\alpha)$ are not proximal, because $x_\alpha$ is not eventually periodic. Similarly, if $x \in S_\alpha$ and $y \in Z$, they cannot be proximal. This implies that the only possibility when $x, y$ are Li-Yorke pair, is that $x \in O(x_\alpha)$ and $y \in \omega(x_\alpha)$. Indeed, there is no three points scrambled set. \qed

Denote

(4.1) $$\mathbb{X} = \bigcup_{\alpha \in A} O(x_\alpha)$$

Lemma 4.5. The shift $\mathbb{X}$ has an uncountable $\omega$-scrambled set and does not have an infinite scrambled set.
Proof. First we claim that

\[ X = \bigcup_{x_n} O(x_n) = \bigcup_{x_n} O(x_n) \cup \omega(x_n) = \bigcup_{x_n} O(x_n). \]

By the argument in the proof of Lemma 4.1 we see that if \( x_{\alpha_n} \to x \) and \( \alpha = \lim_{n \to \infty} \alpha_n \) then \( \lim_{n \to \infty} x_{\alpha_n} = x \), mainly because all points \( x_{\alpha_n} \) project onto point \( 1/8 \in S^1 \) which has singleton fiber (we can repeat argument from the proof of Lemma 4.1).

Now, fix any sequence of points \( (x_n) \subset \bigcup_{x_n \in A} O(x_n) \) and let \( x = \lim_{n \to \infty} x_n \). We assume, going to a subsequence if necessary that \( x_n \in O(x_{\alpha_n}) \) and \( \lim_{n \to \infty} \alpha_n = \alpha \).

There are four possible cases (after going to a subsequence):

1. For every \( n \) there exists \( \alpha_n \in S_\alpha \) such that \( d(\alpha_n, x_n) < 1/n \). In this case, we may also assume that \( \lim_{n \to \infty} \alpha_n = \alpha \) exists. But then \( \alpha \in S_\alpha \) by the arguments in Lemma 4.1 and we also must have \( a = x \), so the claim holds.

2. For every \( n \) there exists \( b_n \in Z \) such that \( d(b_n, x_n) < 1/n \), and then \( x \in Z \).

3. For every \( n \) there exist \( i_n < j_n \) such that \( x_n = a_{i_n}^{\alpha_n} \ldots a_{j_n}^{\alpha_n} b_n \) where \( b^n = b_n \in Z \) and \( a^{\alpha_n} = a_{\alpha_n} \in S_{\alpha_n} \) are points from the definition. First suppose that \( j_n - i_n \to \infty \). Then \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} \sigma^{i_n}(a_{\alpha_n}) \in S_{\alpha_n} \) by Lemma 4.1. In the second case, when \( i_n - j_n \leq k \) for every \( n \), there exists a word \( w \) (which is a subword of every \( x_n \), so also a subword of \( x_\alpha \)), such that \( x_n = w b_n \sigma^{i_n} \) and then \( \lim_{n \to \infty} x_n = w b_1 b_2 \ldots \) where \( b_n = b_1 b_2 \ldots \) so also in this case the claim holds.

4. For every \( n \) there exists \( i_n > j_n \) such that \( x_n = b_{i_n}^n \ldots b_n a_1^\alpha a_2^\alpha \). This case is analogous to the previous one.

The claim is proved. Now the rest of proof is simple. Namely, if \( D \) is a scrambled set with at least three points, then by Lemma 4.4 we see that there are distinct \( \alpha, \gamma \in A \) such that \( x \in O(x_\alpha) \) and \( y \notin O(x_\gamma) \) form a Li-Yorke pair (that is \( x \not\in O(x_\gamma) \) and \( y \notin O(x_\alpha) \)). Since \( \text{dist}(S_\alpha, S_\beta) > 0 \), the only possibility is that points \( (x, y) \) are proximal through set \( Z \), that is, there is \( z \in Z \) and \( j_n \) such that \( d(\sigma^{j_n}(x), z) < 1/n \) and \( d(\sigma^{j_n}(y), z) < 1/n \). But then the only possibility by Lemma 4.3 is that \( \sigma^{j_n}(x) = \sigma^{j_n+p}(x_\alpha) \) and \( \sigma^{j_n}(y) = \sigma^{j_n+q}(x_\gamma) \), for some numbers \( p, q \in \mathbb{N}_0 \). But it is impossible, since when \( x_\alpha \) is close to \( Z \) it follows the trajectory of \( b_n \) while \( x_\gamma \) follows the trajectory of \( b_\gamma \). But \( b_n \) and \( b_\gamma \) belong to distinct trajectories from singleton fibers through projection of \( Z \) onto \( S^1 \), therefore their itineraries \( \sigma^p(b_n), \sigma^q(b_\gamma) \) are distal, in particular cannot approach common point \( z \) simultaneously. Indeed, there is not scrambled set with more than two points.

On the other hand, the set \( C = \{ x_\alpha : \alpha \in A \} \) is \( \omega \)-scrambled, because \( Z \subset \bigcap_{\alpha \in A} \omega(x_\alpha) \) and for \( \alpha \neq \gamma \) we have \( S_\alpha \subset \omega(x_\alpha) \setminus \omega(x_\gamma) \). The proof is completed. \( \square \)

Proof of Theorem 1.1. Let \( X \) be provided by (4.1). If we put \( D = D_X \) then by Lemma 3.1 \( D \) is a Gehman dendrite and the induced map \( f \colon D \to D \) is well defined continuous surjection. We may view \( X \subset D \) as a set of endpoints of \( D \) invariant for \( f \). Furthermore, there is a unique fixed point \( p \in D \setminus X \) such that, if \( x \not\in X \) then \( \lim_{n \to \infty} f^n(x) = p \). Then it is clear that the only Li-Yorke pairs for \( f \) are these contained in \( X \subset D \). The statement of Lemma 4.5 completes the proof. \( \square \)
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REFERENCES

[1] D. Arévalo, W. Charatonik, P. Pellicer Covarrubias, L. Simón, Dendrites with a closed set of end points. Topology Appl. 115 (2001), no. 1, 1–17.

[2] T. Drwięga, Dendrites and chaos, to appear in Internat. J. Bifur. Chaos Appl. Sci. Engrg., to appear

[3] Z. Kočan, Chaos on one-dimensional compact metric spaces, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 22, article id: 1250259 (2012).

[4] Z. Kočan, V. Kornecká-Kurková & M. Málek Entropy, horseshoes and homoclinic trajectories on trees, graphs and dendrites, Ergod. Th. Dyn. Sys. 31, (2011) 165–175, Erratum: 177–177.

[5] P. Kůrka Topological and Symbolic Dynamics, Societe Mathematique de France, 2003.

[6] S. H. Li, ω-chaos and topological entropy, Trans. Amer. Math. Soc. 339 (1993), 243–249.

[7] T. Y. Li, J. Yorke, Period three implies chaos, Amer. Math. Monthly 82 (1975), 985–992.

[8] R. Pikula, On some notions of chaos in dimension zero, Colloq. Math. 107 (2007), 167–177.

[9] S. M. Srivastava, A Course on Borel Sets, Graduate Texts in Mathematics, vol. 180, Springer-Verlag, New York, 1998.