Power-Law Behaviors in Nonlinearly Coupled Granular Chain under Gravity

Jongbae Hong and Heekyong Kim

Department of Physics Education, Seoul National University, Seoul 151-742, Korea

Abstract

We find power-law behaviors of grain velocity in both propagation and backscattering in a gravitationally compacted granular chain with nonlinear contact force. We focus on the leading peak of the velocity signal which decreases in a power-law $d^{-\alpha}$, where $d$ is the location of the peak, as the signal goes down. The ratio of backscattered to incident leading velocity also follows a power-law $d_i^{-\beta}$, where $d_i$ is the depth of impurity. The up-going backscattered signal is nearly solitary. Therefore, the overall change of the leading velocity peak is given by the power-law $d_i^{-(\alpha+\beta)}$. We find $\alpha = 0.250$ and $\beta = 0.167$ for the Hertzian contact force.

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Recently, physics of granular materials attracts great interest, since the materials are ubiquitous around us and their properties are unique and also useful in many applications[1]. The propagation of sound or elastic wave in granular medium is also one of interesting subjects related to the properties of granular matter[2]. A rather simple system, the granular chain with nonlinear contact force including Hertzian contact [3], has been revived by finding a soliton in transmitting elastic impulse. This soliton found in a loaded horizontal Hertzian chain was first predicted by Nesterenko [4] and its experimental verification was performed by Lazaridi and Nesterenko [5] and recently by Coste et al. [6]. Even though three-dimensional granular systems may not follow simple Hertzian contact force law due to geometrical effect [7], the simple nonlinear contact law is still interesting because the existence of solitary wave may provide a possibility to get information inside granular matter and the one-dimensional system usually provides the starting point for studying more complex systems.

In addition to studying horizontal chain, the gravitationally compacted chain and column of grains have been studied by Sinkovits and Sen [8–10] in terms of molecular dynamics simulations. The two cases are not the same in the following sense that the former shows robust solitary wave within certain amount of loading, while the latter shows solitonlike wave whose waveform disperses little by little as it propagates down. Sen at el. [10] studied backscattering of the solitonlike wave by impurity in a gravitationally compacted chain to get the information on buried impurity such metal-poor land mines. They obtained interesting relations between the phase of grain-velocity signal and the mass of impurity. The result reflects the well-known phase-density relation in the connected string with different densities.

In this Letter, we focus on the propagating and backscattering characteristics of the leading part of grain-velocity signal in the gravitationally compacted granular chain with Hertzian contact force especially. We also treat other types of nonlinear forces. We report interesting power-law behaviors of the velocity signal. This work is attractive because it supplies new laws of signal propagation in the gravitationally compacted granular chain with nonlinear contact force and a possibility of identifying the location of buried impurity by observing the peak-velocity of the leading part of returned signal at surface.
To study the dynamics of grains in the gravitationally compacted Hertzian chain, we solve numerically the equation of motion of a grain at $z_i$,

$$m \ddot{z}_i = \frac{5}{2}a\left[\{\Delta_0 - (z_i - z_{i-1})\}^{3/2} - \{\Delta_0 - (z_{i+1} - z_i)\}^{3/2}\right] - mg,$$

(1)

where $m$ is the mass of grain, $\Delta_0 = R_i + R_{i+1}$, and $a$ is the constant defined in Eq. (2) below. We neglect plastic deformation in Eq. (1). This equation of motion comes from the Hertzian interaction energy between neighboring granular spheres which is given by

$$V(\delta_{i,i+1}) = \frac{2}{5D}\left(\frac{R_i R_{i+1}}{R_i + R_{i+1}}\right)^{1/2} \delta_{i,i+1}^{5/2} \equiv a\delta_{i,i+1}^{5/2},$$

(2)

where $\delta_{i,i+1}$ is the overlap between two adjacent spherical grains and is given by $\delta_{i,i+1} = R_i + R_{i+1} - r_{i,i+1}$, in which $R_i$ is the radius of the sphere whose center is at position $i$ and $r_{i,i+1}$ is the distance between $i$ and $i + 1$, and

$$D = \frac{3}{4}\left(1 - \frac{\sigma_i^2}{E_i} + \frac{1 - \sigma_{i+1}^2}{E_{i+1}}\right),$$

(3)

where $\sigma_i$, $\sigma_{i+1}$ and $E_i$, $E_{i+1}$ are Poisson’s ratios and Young’s moduli of the bodies at neighboring positions, respectively [3]. The equilibrium condition

$$g \sum_{j=i+1}^{N} m_j = \frac{5a}{2} \delta_{i,i+1}^{3/2}$$

(4)

has been used for the $i$-th grain of a chain of $N$ grains. A criterion for the initial impulse exists to make Eqs. (1) and (2) valid [6,11]. Therefore, in this work, we choose comparatively weak impulses to satisfy this criterion.

Our system is a vertical chain of $N = 10^3$ grains which has five consecutive impurity grains which are different only in mass from other spherical grains. The mass of impurity is taken as half of the mass of medium grains. As a calculational tool, we use the third-order Gear predictor-corrector algorithm [12] and the same program units as those used in Ref. [10]. That is, the units of distance, mass, and time are $10^{-5}$m, $2.36 \times 10^{-5}$kg, and $1.0102 \times 10^{-3}$s, respectively. These units gives the gravitational acceleration $g = 1$. We set
the grain diameter 100, mass 1, and the constant $a$ of Eq. (1) 5657 for molecular dynamics simulation.

We focus on the peak value of the very leading velocity signals from the moment of initial impulse imposed on the first grain at top of chain to the moment of return to the first grain. Figure 1(a) shows the snap shots of propagating velocity signal passing different depths. Initial impulse velocity 0.1 in program units has been applied to the first grain. One can easily see from Fig. 1(a) that more grains are involved in the signal, i.e., signal disperses as time passes. We find that even though loading on the grain increases in depth due to gravity, total kinetic energy and potential energy of the grains involved in the signal remains constant independently. Therefore, the decrease of peak velocity results from only dispersion of the signal. Fig. 1(a) shows that the propagating signal is not solitary in the gravitationally compacted chain. This is quite different from the case of loaded horizontal chain [4,5].

In this work, we are interested in the effect of gravity which changes loading of chain continuously and the time- or depth-dependent behavior of the leading peak shown in Fig. 1(a). Fig. 1(b) is the log$_{10}$-log$_{10}$ plot of the leading velocity peak versus its location. A clear power-law behavior is seen in Fig. 1(b). The explicit expression for the leading peak velocity is given by

$$\frac{v(d)}{v_i} = Ad^{-\alpha},$$

(5)

where $v_i$ is the initial velocity, $d$ is the distance from top to the leading peak, and $A$ is the intersection which is not equal to one because of boundary effect at the top of chain. But the signal quickly goes to the stable form shown in Fig. 1(a). We will show this in Fig. 2. Figure 1(b) obtained for Hertzian contact gives rise to the power-law index $\alpha = 0.250$. Figure 2 drawn in log$_{10}$-log$_{10}$ plot shows the propagating behaviors of the leading peak of the velocity signal for various nonlinear contact forces, such as $n = 5/2, 3, 7/2$, and 4 of the potential $V(\delta_{i,i+1}) = a\delta_{i,i+1}^n$. The abscissa of Fig. 2 denotes elapsed time after impulse. One can see that the initial impulse quickly follows the power-law of Eq. (5) in 0.1 second.
which corresponds to passing 40th grain in length. Interesting things in Fig. 2 are (1) the slopes are very close to each other even though there is crossing, therefore, one expects similar propagating behaviors for these nonlinear contact forces, and (2) Hertzian case is well-separated from other nonlinear cases. It is unclear right now why Hertzian is special.

The signal performs backscattering when it meets impurity. Figure 3(a) shows typical backscattering by five consecutive impurity grains of mass 0.5 and depth 500 in program units. The total energy of the incident signal is divided into transmitted and reflected parts, i.e., \( v_{\text{inc}}^2 \approx v_{\text{trans}}^2 + v_{\text{back}}^2 \). The backscattered leading peak diminishes abruptly compared with second leading peak in the process of scattering. We call the ratio of backscattered to incident leading peak the reflection coefficient of the leading peak velocity for present analysis. We find a remarkable phenomenon that the reflection coefficient of the leading peak velocity also follows a power-law in depth of impurity. The reflection coefficient, of course, depends on the mass of impurity. We restrict this study to the case of impurity mass 0.5 while the mass of medium grain is 1. Fig. 3(b) is the \( \log_{10}-\log_{10} \) plot of the reflection coefficient, i.e., the ratio of backscattered leading peak to incident one versus the location of impurity \( d_i \). A clear power-law is seen in Fig. 3(b) and the expression is given by

\[
\frac{v_b(d_i)}{v(d_i)} = Bd_i^{-\beta},
\]

where \( v_b(d_i) \) is the leading peak velocity after backscattering by impurities at depth \( d_i \) and \( B \) is the intersection of \( \log_{10}-\log_{10} \) plot. Figure 3(b) obtained for Hertzian contact gives rise to the power-law index \( \beta = 0.167 \) for the impurity mass 0.5. We take the value of \( v_b(d_i) \) at the moment of separation from the tail of incident signal. We confirm that conservation laws of energy and momentum are always retained.

This power-law behavior is an unexpected result. We thought this may stem from the change of loading as the location of impurity changes in the gravitationally compacted chain. We checked this idea for the horizontal chain by changing the amount of loading. We found that there is no change in the reflection coefficient upon loading for the solitary wave in the horizontal chain. We also confirm that the amplitude of incident solitary wave in horizontal
chain cannot change the reflection coefficient either. Therefore, we conclude that the power-law behavior in Eq. (3) is a characteristic of nonsolitariness of the wave in gravitationally compacted granular chain.

We also check the propagation of backscattered wave. It is interesting to see that the backscattered waveform is very solid and it shows negligible change in the height of the leading peak within the depth we considered in this work. Also interesting thing is that the propagating behavior of the backscattered velocity signal is not power-law but linear in depth as shown in Fig. 4 drawn in absolute scale. The saw tooth form stems from finite radius of grain. The trend of changing leading peak in Fig. 4 is flat in the velocity scale used in this analysis. We find that the number of grains participating in the signal increases in the case of signal going down, while it remains nearly constant in the up-going situation. Therefore, there is little dispersion when the signal goes up in the gravitationally compacted chain. In addition, no kinetic and potential energy change either in this case. The linear-law of Fig. 4 does not play a role for the shallow region but play an appreciable role when backscattering produces at a deep enough place.

As a summary of the above analysis, we obtain the overall power-law relation between returned velocity $v_r$ at top and the depth of impurity $d_i$. Since the kinetic and contact potential energy is transferred to the kinetic and gravitational potential energy of the first grain at top, the leading backscattered peak velocity at grain 2, i.e., $v_b(2) \approx -0.004$ in Fig. 3(a), is quite different from the leading peak velocity at grain 1 which we define the returned leading peak velocity $v_r$. Figure 5 shows the log$_{10}$-log$_{10}$ plots of $v_r/v_i$ versus the location of impurity $d_i$ for different contact forces and initial impulse velocities $v_i = 0.1, 0.05$, and 0.01 in program units. One can see a big difference between $v_r$ and $v_b(2)$. We separate artificially the lines of Fig. 5 into two groups by dividing the data of $V(\delta_{i,i+1}) = a\delta_{i,i+1}^3$ by 1.1 to avoid overlap with those of Hertzian interaction $V(\delta_{i,i+1}) = a\delta_{i,i+1}^{5/2}$. One can see that the power-law is independent of initial impulse. The equation of the straight lines of Fig. 5 is written as
\[ \frac{v_r}{v_i} = R(v_i) d_i^{-(\alpha + \beta)}, \]  

(7)

where the coefficient \( R(v_i) \), the intersection of the \( \log_{10}\log_{10} \) plot, contains the energy transfer at top and the coefficients \( A \) and \( B \) of Eqs. (5) and (6), respectively. \( R(v_i) \)'s weakly depends on the amount of initial impulse and the type of contact force law as shown in Fig. 5. The power-law index \( \alpha + \beta = 0.417 \) for Hertzian contact and 0.378 for another. From the analysis above, we know that the index is composed of two different sources, i.e., the index for downward propagation \( \alpha \) and the index for backscattering \( \beta \). One can confirm this by adding the slopes of Fig. 1(b) and Fig. 2(b) for Hertzian chain.  

In conclusion, we solve numerically the many-body equations of motion of the gravitationally compacted granular chain coupled by nonlinear contact forces and analyze the propagating behaviors of the leading peak of the velocity signal and the backscattering characteristic by buried impurity. We find two power-law relations in Eqs. (5) and (6) for propagation and backscattering, respectively. The former contains only the effect of dispersion of nonsolitary wave as it propagates. The latter, however, has nothing to do with dispersion but the phenomenon occurring at the point of scattering. Comparing with the case of solitary wave in loaded horizontal chain, we conclude that Eq. (5) is a special feature of nonsolitary property of the gravitationally compacted granular chain. The backscattered up-going wave, on the other hand, is a nearly solitary wave and does not follow power-law but follow very weak linear-law unlike the down-going wave. We conclude that this interesting feature is the characteristic of the gravitationally compacted granular chain with nonlinear contact force. The solitary behavior of the up-going wave in the gravitationally compacted granular chain may be useful to find buried impurity like nonmetallic land mine.  

In Eq. (7), we obtain the combined power-law on the location of impurity for the returned leading peak velocity at top of chain. This power-law also exists for other nonlinear contact forces as well as Hertzian as we show in Fig. 5. We therefore conclude that above features and power-laws are generic for any nonlinear contact forces of type \( F(\delta_{i,i+1}) \propto \delta_{i,i+1}^{n-1} \) in the gravitationally compacted granular chain.
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Figure Captions

Fig. 1: (a) Snap shots of propagating wave forms in a gravitationally compacted granular chain with Hertzian contact law. (b) Log$_{10}$-log$_{10}$ plot of $v(d)$ vs. depth $d$ for the leading peak in (a). Slope of the straight line is $-0.250$.

Fig. 2: Log$_{10}$-log$_{10}$ plot of $v(d)$ vs. $d$ for the leading peak for various interaction potentials, i.e., $n = 5/2, 3, 7/2,$ and $4$ of $V(\delta_{i,i+1}) = a\delta_{i,i+1}^n$.

Fig. 3: (a) Snap shot of scattering by five light impurities at 500. (b) Log$_{10}$-log$_{10}$ plot of $v_b(d_i)/v(d_i)$ vs. $d_i$. Slope of the straight line is $-0.167$.

Fig. 4: Plot of leading peak vs. its location for the backscattered part of the velocity signal. Absolute scales are used for both abscissa and ordinate. Five impurities are at 700.

Fig. 5: Log$_{10}$-log$_{10}$ plot of $v_r/v_i$ vs. $d_i$ for the leading peak for various initial velocities, $v_i = 0.1, 0.05, 0.01$. Upper three lines are Hertzian ($n = 5/2$) cases. Lower three lines ($n = 3$) are lowered artificially. Slopes are $0.416$ ($n = 5/2$) and $0.378$ ($n = 3$).
(a)

velocity vs. depth
\( \alpha = 0.250 \)
\[ v_0(d) = \beta \frac{v(d)}{v(0)} \]

\[ \beta = 0.167 \]

The graph shows the relationship between the depth of impurity and the normalized values of \( v_0(d) \) and \( v(d) \). The equation \( \beta = 0.167 \) is indicated on the graph.
$v_i = 0.1$
$v_i = 0.05$
$v_i = 0.01$
$v_i = 0.1$
$v_i = 0.05$
$v_i = 0.01$