BIVARIANT K-THEORY OF GENERALIZED WEYL ALGEBRAS

JULIO GUTIÉRREZ AND CHRISTIAN VALQUI

Abstract. We compute the isomorphism class in $\text{KK}^{\text{alg}}$ of all noncommutative generalized Weyl algebras $A = \mathbb{C}[h](\sigma, P)$, where $\sigma(h) = q h + h_0$ is an automorphism of $\mathbb{C}[h]$, except when $q \neq 1$ is a root of unity. In particular, we compute the isomorphism class in $\text{KK}^{\text{alg}}$ of the quantum Weyl algebra, the primitive factors $B_\lambda$ of $U(\mathfrak{sl}_2)$ and the quantum weighted projective lines $O(\mathbb{WP}_q(k, l))$.

Contents

Introduction 1
1. Basic results on locally convex algebras 3
1.1. Locally convex algebras 4
1.2. Diffeotopies 4
1.3. Extensions of locally convex algebras 5
1.4. The algebra of smooth compact operators and the smooth Toeplitz algebra 7
2. Bivariant K-theory of locally convex algebras 10
2.1. Definition and properties of $\text{kk}^{\text{alg}}$ 10
2.2. Bott Periodicity and Triangulated structure of $\text{KK}^{\text{alg}}$ 11
2.3. Stabilization by Schatten ideals 12
2.4. Weak Morita equivalence 12
2.5. Quasihomomorphisms 13
3. Generalized Weyl algebras 14
4. $\text{kk}^{\text{alg}}$ invariants of generalized Weyl algebras 18
4.1. The Toeplitz algebra of a smooth generalized crossed product 19
4.2. The case where $P$ is a non constant polynomial 20
4.3. The case where $P$ is a constant polynomial 27
References 28

Introduction

In [7], Cuntz defined a bivariant $K$-theory $\text{kk}^{\text{alg}}$ in the category $\text{lc}a$ of locally convex algebras. These are algebras $A$ that are complete locally convex vector spaces over $\mathbb{C}$ with a jointly continuous multiplication. To a pair of locally convex algebras $A$ and $B$, there

2010 Mathematics Subject Classification. Primary 46L87; Secondary 19K35, 58B34.
Key words and phrases. K-theory, kk-theory, smooth generalized crossed products, generalized Weyl algebras.

Julio Gutiérrez was supported by Cienciactiva CG 217-2014.
Christian Valqui was supported by PUCP-DGI-2017-1-0035.
correspond abelian groups \( kk_n^{alg}(A, B), n \in \mathbb{Z} \) and there are bilinear maps
\[
k^1_n^{alg}(A, B) \times kk^m_n(B, C) \to kk^m_{n+m}(A, C)
\]
for every \( A, B \) and \( C \) locally convex algebras and \( m, n \in \mathbb{Z} \). Using this product, we can define a category \( R \mathcal{R}^{alg} \) whose objects are locally convex algebras and whose morphisms are given by the graded groups \( kk^a_{alg}(A, B) \). Then the bivariant \( K \)-theory \( kk^{alg} \) can be seen as a functor \( kk^{alg} : \text{ica} \to R \mathcal{R}^{alg} \). This functor is universal among split exact, diffotopy invariant and stable functors (see Theorem 7.26 in [8]). In particular, an isomorphism in \( R \mathcal{R}^{alg} \) induces an isomorphism in \( R \mathcal{R}^{L^p} \) (see Definition 2.7) and in \( HP \).

Joachim Cuntz initiated the construction of different bivariant \( K \)-theories in several categories (see [6], [7] and [9]), and in [7] he proved that the Weyl algebra \( W = \mathbb{C}\langle x, y \rangle x y - y x = 1 \) is isomorphic to \( C \) in \( R \mathcal{R}^{alg} \). By the universal property of \( R \mathcal{R}^{alg} \) this implies \( R \mathcal{R}^{L^p}(\mathbb{C}, W) = \mathbb{Z} \) and \( R \mathcal{R}^{L^p}(\mathbb{C}, W) = 0 \).

On the other hand, in [10], exact sequences analogous to the Pimsner-Voiculescu exact sequence were constructed for smooth generalized crossed products that satisfy the condition of being tame smooth. We shall consider generalized Weyl algebras over \( \mathbb{C}[h] \) which are smooth generalized crossed products (but are not tame smooth in general).

**Definition 0.1.** Let \( D \) be a ring, \( \sigma \in \text{Aut}(D) \) and \( a \) a central element of \( D \). The generalized Weyl algebra \( D(\sigma, a) \) is the algebra generated by \( x \) and \( y \) over \( D \) satisfying
\[
xd = \sigma(d)x, \ yd = \sigma^{-1}(d)y, \ xy = a \text{ and } xy = \sigma(a)
\]
for all \( d \in D \).

In this article, we compute the isomorphism class in \( R \mathcal{R}^{alg} \) of all non commutative generalized Weyl algebras \( A = \mathbb{C}[h](\sigma, P) \), where \( \sigma(h) = q h + h_0 \) is an automorphism of \( \mathbb{C}[h] \) and \( P \in \mathbb{C}[h] \), except when \( q \neq 1 \) is a root of unity. In the table below we list all possible cases for \( A \) and our results.

| Conditions                  | Results          | Observation     |
|-----------------------------|------------------|-----------------|
| \( P = 0 \)                 | \( A \cong R \mathcal{R}^{alg} \mathbb{C} \) | Prop 4.22       | \( A \) \( \mathbb{N} \)-graded |
| \( P \neq 0 \)              | \( A \cong R \mathcal{R}^{alg} S \mathbb{C} \oplus \mathbb{C} \) | Prop 4.21       | \( A \) tame smooth |
| \( q \) not a root of unity | \( A \cong R \mathcal{R}^{alg} S \mathbb{C} \) | Thm 4.18        |                   |
| \( q = 1 \) \& \( h_0 \neq 0 \) | \( A \cong R \mathcal{R}^{alg} \mathbb{C} \) | Prop 4.20       |                   |
| \( q \neq 1 \), a root of unity | No result       | Thm 4.18        |                   |
| \( q = 1 \) \& \( h_0 = 0 \)      | No result       | A commutative   |                   |

A generalized Weyl algebra \( A = \mathbb{C}[h](\sigma, P) \) is tame smooth if and only if \( P(h) \in \mathbb{C}[h] \) is a non zero constant polynomial (see Remark 4.3). Hence, if \( P \) is a non constant polynomial, \( A \) is a generalized crossed product that is not tame smooth, and so we cannot use the results of [10]. However, in most cases we can construct an explicit faithful representation of \( A \), which allows us to follow the general strategy of [7] and [10], in order to determine the \( R \mathcal{R}^{alg} \) class of \( A \).

Our main result is Theorem 4.18 which computes the isomorphism class of \( A \) in \( R \mathcal{R}^{alg} \) in the following two cases:

- \( q = 1 \) \& \( h_0 \neq 0 \).
- \( q \) is not a root of unity and \( P \) has a root different from \( h_0 \frac{1}{1-q} \).
In each of these cases we construct an exact triangle
\[ SA \to A_1 A_{-1} \to A_0 \to A, \] (0.1)
in the triangulated category \((\mathfrak{R} \mathfrak{K}^{\text{alg}}, S)\), where \(A_n\) is the subspace of degree \(n\) of the \(\mathbb{Z}\)-graded algebra \(A\) (see Lemma 3.2). In order to construct the exact triangle in (0.1) we follow the methods of [10]: we construct a linearly split extension
\[ 0 \to \Lambda A \to T A \to A \to 0 \]
and prove
\[ T A \cong_{\mathfrak{R} \mathfrak{K}^{\text{alg}}} A_0 \quad \text{and} \quad A \cong_{\mathfrak{R} \mathfrak{K}^{\text{alg}}} A_1 A_{-1}. \]

The exact triangle in (0.1) yields \(A \cong_{\mathfrak{R} \mathfrak{K}^{\text{alg}}} A_0 \oplus S(A_1 A_{-1})\). The main result now follows after we prove \(A_1 A_{-1} \cong_{\mathfrak{R} \mathfrak{K}^{\text{alg}}} SC^{-1}\) in Proposition 4.17, since \(A_0 = \mathbb{C}[h] \cong_{\mathfrak{R} \mathfrak{K}^{\text{alg}}} \mathbb{C}\).

The main result allows to compute the isomorphism class in \(\mathfrak{K} \mathfrak{K}^{\text{alg}}\) of the quantum Weyl algebra, the primitive factors \(B_{\lambda}\) of \(U(\mathfrak{sl}_2)\) and the quantum weighted projective lines \(O(\mathbb{WP}_q(k,l))\) (see [4]).

For the sake of completeness we also discuss the case of \(\mathbb{N}\)-graded and the case of tame smooth generalized Weyl algebras.

In the case where \(A = \bigoplus_{n \in \mathbb{N}} A_n\) is an \(\mathbb{N}\)-graded locally convex algebra it can be shown that \(A \cong_{\mathfrak{R} \mathfrak{K}^{\text{alg}}} A_0\). This is the case when
\[ \bullet \quad P \text{ is nonconstant, } q \text{ is not a root of unity and } P \text{ has only } \frac{h_0}{1-q} \text{ as a root or} \]
\[ \bullet \quad P = 0. \]
In these cases we obtain \(A \cong_{\mathfrak{R} \mathfrak{K}^{\text{alg}}} \mathbb{C}\).

In the case where \(P\) is a nonzero constant polynomial, \(A\) is a tame smooth generalized crossed product and the results from [10] apply. In this case there is an exact triangle
\[ SA \to A_0 \to A_0 \to A, \] (0.2)
in the triangulated category \((\mathfrak{R} \mathfrak{K}^{\text{alg}}, S)\) and we obtain \(A \cong_{\mathfrak{R} \mathfrak{K}^{\text{alg}}} SC \oplus \mathbb{C}\).

In the case where \(q = 1\) and \(h_0 = 0\), we have \(\sigma = \text{id}\) and so \(A \cong \mathbb{C}[h,x,y]/(xy - P)\) is a commutative algebra. This case and the case where \(q \neq 1\) a root of unity remain open.

The article is organized as follows. In section 1 we recall basic results on locally convex algebras. Lemma 1.18 is a technical result which asserts that the projective tensor product of the Toeplitz algebra \(\mathcal{T}\) with an algebra with a countable basis is the algebraic tensor product. In section 2 we recall the definition and properties of \(kk^{\text{alg}}\) following [7] and [9]. In section 3 we define generalized Weyl algebras and construct explicit faithful representations when \(q = 1\) and \(h_0 \neq 0\), and when \(q\) is not a root of unity and \(P\) has a root different from \(\frac{h_0}{1-q}\). In section 4 we compute the isomorphism class in \(\mathfrak{R} \mathfrak{K}^{\text{alg}}\) of all noncommutative generalized Weyl algebras \(A = \mathbb{C}[h](\sigma, P)\) where \(\sigma(h) = qh + h_0\) except when \(q \neq 1\) is a root of unity.

1. Basic results on locally convex algebras

In this section we recall some constructions in the category of locally convex algebras that are needed for the definition of the bivariant \(K\)-theory \(kk^{\text{alg}}\). We follow the discussions in [10] and [7]. In Lemma 1.18 we prove that the projective tensor product of the Toeplitz algebra \(\mathcal{T}\) with an algebra with a countable basis is the algebraic tensor product.
1.1. Locally convex algebras.

Definition 1.1. A locally convex algebra $A$ is a complete locally convex vector space over $\mathbb{C}$ which is an algebra such that for any continuous seminorm $p$ in $A$ there is a continuous seminorm $q$ in $A$ such that $p(ab) \leq q(a)q(b)$ for all $a, b \in A$. This is equivalent to requiring the multiplication to be jointly continuous.

A seminorm $p$ of $A$ is called submultiplicative if $p(ab) \leq p(a)p(b)$, for all $a, b \in A$. If the topology of $A$ can be defined by a family of submultiplicative seminorms we say that $A$ is an $m$-algebra.

Morphisms in the category of locally convex algebras are continuous homomorphisms. We denote by $\otimes$ the projective tensor product of locally convex vector spaces (see chapter 43 in [15]). This is a completion of the algebraic tensor product. The projective tensor product of two locally convex algebras is again a locally convex algebra. The following are examples of locally convex algebras.

1. All algebras with a countable basis over $\mathbb{C}$. These are locally convex algebras with the topology generated by all seminorms (Proposition 2.1 in [7]). Examples include the Weyl algebra and generalized Weyl algebras over $\mathbb{C}[h]$ (see Corollary 3.3).
2. $C^\infty([0, 1])$, which is a locally convex algebra with the family of seminorms

$$p_n(f) = \|f\| + \|f'|| + \frac{1}{2}\|f''\| + \cdots + \frac{1}{n!}\|f^{(n)}\|$$

where $\|f\| = \sup\{f(t)|t \in [0, 1]\}$.
3. We define $C[0, 1]$ as the (closed) subalgebra of $C^\infty([0, 1])$ of functions with all derivatives vanishing at 0 and 1. This is a nuclear topological vector space (see Definition 50.1 and Theorem 50.1 in [15]) and therefore, for any locally convex algebra $A$ we have $C[0, 1] \otimes A = A[0, 1]$, where $A[0, 1]$ is the algebra of $C^\infty$ functions with values in $A$ and all derivatives vanishing at 0 and 1. We define $A(0, 1)$ and $A(0, 1)$ as the subalgebras of $A[0, 1]$ of functions that vanish at 1, and at 0 and 1 respectively.

Definition 1.2. We denote by $SA$ and $CA$ the algebras $A(0, 1)$ and $A(0, 1)$ and we call them the suspension and the cone of $A$ respectively.

Note that $S(\cdot)$ is a functor. Given a morphism of locally convex algebras $\phi : A \rightarrow B$, there is a morphism $S(\phi) : SA \rightarrow SB$ defined by $f \mapsto \phi \circ f$. We can iterate this functor $n$ times to obtain $S^nA$ and $S^n(f)$.

1.2. Diffotopies.

An important feature of the bivariant $K$-theory $kk^\alg$ is the invariance with respect to differentiable homotopies. For more details on diffotopies consult Section 6.1 in [8].

Definition 1.3. Let $\phi_0, \phi_1 : A \rightarrow B$ be homomorphisms of locally convex algebras. A diffotopy between $\phi_0$ and $\phi_1$ is a homomorphism $\Phi : A \rightarrow C^\infty([0, 1], B)$ such that $ev_i \circ \Phi = \phi_i$. If there is a diffotopy between $\phi_0$ and $\phi_1$ we call them diffotopic and write $\phi_0 \simeq \phi_1$.

Using a reparameterization of the interval we can assume that all derivatives of $\Phi$ at 0 and 1 vanish and therefore we can assume that a diffotopy is given by a map $\Phi : A \rightarrow B[0, 1]$. With this characterization we can show that diffotopy is an equivalence relation.

Definition 1.4. Given two locally convex algebras $A$ and $B$, we denote by $\langle A, B \rangle$ the set of diffotopy classes of continuous homomorphisms from $A$ to $B$. Given $\phi : A \rightarrow B$ a continuous homomorphism, we denote by $\langle \phi \rangle$ its diffotopy class.
Lemma 1.5. There is a group structure in $\langle A, SB \rangle$ given by concatenation. The group structures in $\langle A, S^n B \rangle$ that we get from concatenation in different variables all agree and are abelian for $n \geq 2$.

Proof. See Lemma 6.4 in [8]. □

Definition 1.6. A locally convex algebra $A$ is called contractible if the identity map is diffeomorphic to 0.

Examples. Examples of contractible locally convex algebras are $hC[h]$ and $CA$. The diffeomorphies are given by $\phi_s : hC[h] \to hC[h], \phi_s(h) = sh$ and $\psi_s : CA \to CA, \psi_s(f)(t) = f(st)$, respectively. Note that the algebras $(h - h_0)C[h]$ are isomorphic to $hC[h]$ and therefore are also contractible.

We conclude this section with a note on $\mathbb{N}$-graded algebras.

Lemma 1.7. Let $A = \bigoplus_{n \in \mathbb{N}} A_n$ be an $\mathbb{N}$-graded locally convex algebra, then $A$ is diffeotopy equivalent to $A_0$. In particular $C[h]$ is diffeotopy equivalent to $\mathbb{C}$.

Proof. The diffeotopy is given by the family of morphisms $\phi_t : A \to A, t \in [0, 1]$, sending an element $a_n \in A_n$ to $t^n a_n$. When $t = 1$ we recover the identity and when $t = 0$ the morphism is a retraction of $A$ onto $A_0$. □

1.3. Extensions of locally convex algebras.

In this section we define extensions of locally convex algebras and their classifying maps. Extensions play a key role in the definition of $kk^{alg}$.

Definition 1.8. An extension of locally convex algebras

$$0 \to I \to E \to B \to 0$$

is linearly split if there is a continuous linear section $s : B \to E$. Similarly we say that an extension of length $n$

$$0 \to I \to E_1 \to \cdots \to E_n \to B \to 0$$

is linearly split if there is a continuous linear map of degree $-1$ such that $ds + sd = id$, where $d$ is the differential of the chain complex.

Example. Let $A$ be a locally convex algebra. The extension

$$0 \to SA \to CA \to A \to 0$$

is called the cone extension of $A$. It is linearly split with continuous linear section $s : B \to E$ given by $a \in A \mapsto f \in CA$ with $f(t) = (1 - \psi(t))a$, where $\psi : [0, 1] \to [0, 1]$ is a $C^\infty$ bijection with $f(0) = 0, f(1) = 1$ and all derivatives vanishing at 0 and at 1.

Now, we define the tensor algebra which has a universal property in the category of locally convex algebras. It is a completion of the usual algebraic tensor algebra. Let $V$ be a complete locally convex vector space. The algebraic tensor algebra is defined as

$$T^{alg}_V = \bigoplus_{n=1}^\infty V\otimes^n.$$

Notice that we are considering a non-unital algebraic tensor algebra. There is a linear map $\sigma : V \to T^{alg}_V$ mapping $V$ into the first summand. We topologize $T^{alg}_V$ with all seminorms of the form $\alpha \circ \phi$, where $\phi$ is any homomorphism from $T^{alg}_V$ into a locally convex algebra $B$ such that $\phi \circ \sigma$ is continuous on $V$ and $\alpha$ is a continuous seminorm on $B$. 
**Definition 1.9.** The tensor algebra $T_V$ is the completion of $T_{alg}V$ with respect to the family of seminorms $\{\alpha \circ \phi\}$ defined above.

The tensor algebra $T_V$ is a locally convex algebra that satisfies the following universal property.

**Proposition 1.10.** Given a continuous linear map $s : V \to B$ from a complete locally convex vector space $V$ to a locally convex algebra $B$ there is a unique morphism of locally convex algebras $\tau : T_V \to B$ such that $\tau \circ \sigma = s$. The morphism $\tau$ is defined by $\tau(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = s(x_1)s(x_2)\ldots s(x_n)$ where $x_i \in V$.

**Proof.** See Lemma 6.9 in [8]. \[\blacksquare\]

In particular, if $A$ is a locally convex algebra, the identity map $id : A \to A$ induces a morphism $\pi : TA \to A$.

We use the universal property of $TA$ to construct a universal extension. There is an extension

$$0 \to JA \to TA \xrightarrow{\gamma} A \to 0$$

where $JA$ is defined as the kernel of $\pi : TA \to A$, which has a canonical continuous linear section given by $\sigma : A \to TA$. This extension is universal in the sense that given any extension of locally convex algebras $0 \to I \to E \to B \to 0$ with continuous linear section $s$ and a morphism $\alpha : A \to B$, there is a morphism of extensions

$$0 \to JA \to TA \xrightarrow{\gamma} A \to 0$$

$$0 \to I \to E \to B \to 0$$

where $\tau : TA \to E$ is the morphism induced by the continuous linear map $s \circ \alpha : A \to E$ and $\gamma : JA \to I$ is the restriction of $\tau$.

Notice that $J(\cdot)$ is a functor. Given a morphism $\alpha : A \to B$, consider the extension $0 \to JB \to TB \to B \to 0$ with its canonical continuous linear section. Then we define $J(\alpha) : JA \to JB$ in the natural way. We can iterate this construction $n$ times to obtain $J^nA$ and $J^n(\alpha)$.

We observe that the map $\gamma : JA \to I$ is unique up to diffotopy. Given two continuous linear sections $s_1$ and $s_2$, the smooth family of continuous linear sections $s_t = ts_1 + (1 - t)s_2$ induces a diffotopy $\gamma_t$ which connects $s_1$ and $s_2$. Hence the corresponding $\gamma_t$'s are diffotopic.

**Definition 1.11.** The morphism $\gamma : JA \to I$ is called the classifying map of the extension $0 \to I \to E \to B \to 0$ and the morphism $\alpha : A \to B$. It is well defined up to diffotopy.

Similarly, we can define the classifying map of an extension

$$0 \to I \to E_1 \to \cdots \to E_n \to B \to 0$$

and a morphism $\alpha : A \to B$ to be the map $\gamma : J^nA \to I$ in

$$0 \to J^nA \xrightarrow{\gamma} T(J^{n-1}A) \to \cdots \to TA \xrightarrow{\tau} A \to 0$$

$$0 \to I \to E_1 \to \cdots \to E_n \to B \to 0$$

which is also unique up to diffotopy.
1.4. The algebra of smooth compact operators and the smooth Toeplitz algebra.

We define the algebra $\mathcal{K}$ of smooth compact operators which play the role of the $\mathcal{C}^*$-algebra of compact operators used in Kasparov’s $KK$-theory. Then we define the smooth Toeplitz algebra $\mathcal{T}$ and prove that the projective tensor product of $\mathcal{T}$ with an algebra with a countable basis is the algebraic tensor product.

**Definition 1.12.** The algebra of smooth compact operators $\mathcal{K}$ is defined as the algebra of $\mathbb{N} \times \mathbb{N}$ matrices $a = (a_{ij})$ such that $p_n(a) = \sum_{i,j \in \mathbb{N}} (1 + i + j)^n |a_{ij}|$ is finite for $n \in \mathbb{N}$. The topology is defined by the seminorms $p_n$.

The algebra $\mathcal{K}$ with the seminorms $p_n$ is a locally convex algebra, which is isomorphic to the space $s'$ of rapidly decreasing sequences as a locally convex vector space.

**Lemma 1.13.** The locally convex spaces $\mathcal{K}$, $s \otimes_\pi s$ and $s \oplus s$ are isomorphic to $s$.

**Proof.** The proofs of these facts can be found in [16] Chapter 3 Section 1.1.

We also define the smooth Toeplitz algebra which plays the role of the Toeplitz $\mathcal{C}^*$-algebra. The Fourier series gives an isomorphism of locally convex spaces between $C^\infty(S^1)$ and the space of rapidly decreasing Laurent series (see Theorem 51.3 in [15])

$$C^\infty(S^1) \cong \left\{ \sum_{i \in \mathbb{Z}} a_i z^i \mid \sum_{i \in \mathbb{Z}} |1 + i|^n |a_i| < \infty, \forall n \in \mathbb{N} \right\},$$

where $z$ corresponds to the function $z : S^1 \to \mathbb{C}$, $z(t) = t$. This space is isomorphic to the space $s$ of rapidly decreasing sequences.

**Definition 1.14.** The smooth Toeplitz algebra $\mathcal{T}$ is defined by the direct sum of locally convex vector spaces $\mathcal{T} = \mathcal{K} \oplus C^\infty(S^1)$. In order to define the multiplication, we define $v_k = (0, z^k)$ and write $x$ for an element $(x, 0)$ with $x \in \mathcal{K}$. We denote the elementary matrices in $\mathcal{K}$ by $e_{ij}$ and set $e_{ij} = 0$ for all $i, j < 0$. The multiplication is defined by the following relations

$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad v_ke_{ij} = e_{(i+k),j}, \quad e_{ij}v_k = e_{i,(j-k)},$$

for all $i, j, k, l \in \mathbb{Z}$ and

$$v_kv_{-l} = \begin{cases} v_{k-l}(1 - e_{00} - e_{11} - \ldots - e_{l-1,l-1}) & l > 0 \\ v_{k-l} & l \leq 0, \end{cases}$$

for all $k, l \in \mathbb{Z}$.

Denote $v_1$ and $v_{-1}$ by $S$ and $S^*$ respectively.

There is a linearly split extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C^\infty(S^1) \to 0$$

where the continuous linear section $C^\infty(S^1) \to \mathcal{T}$ is defined by $z \mapsto S$.

The smooth Toeplitz algebra is generated, as a locally convex algebra, by $S$ and $S^*$. In fact, it satisfies a universal property in the category of $m$-algebras.

**Lemma 1.15** (Satz 6.1 in [3]). $\mathcal{T}$ is the universal unital $m$-algebra generated by two elements $S$ and $S^*$ satisfying the relation $S^*S = 1$ whose topology is defined by a family of submultiplicative seminorms $\{p_n\}_{n \in \mathbb{N}}$ with the condition that there are positive constants $C_n$ such that

$$p_n(S^k) \leq C_n(1 + k^n) \quad \text{and} \quad p_n(S^{*n}) \leq C_n(1 + k^n). \quad (1.1)$$
The following diffeotopy is due to [6]. In the context of $C^*$-algebras a homotopy like this one is used to prove Bott periodicity and the Pimsner-Voiculescu sequence.

**Lemma 1.16** (Lemma 6.2 in [6]). There is a unital diffeotopy $\phi_t : \mathcal{T} \to \mathcal{T} \otimes_\pi \mathcal{T}$ such that

$$\phi_t(S) = S^2 S^* \otimes 1 + f(t)(e \otimes S) + g(t)(Se \otimes 1)$$

$$\phi_t(S^*) = SS^* \otimes 1 + \overline{f(t)}(e \otimes S^*) + \overline{g(t)}(eS^* \otimes 1)$$

where $f, g \in \mathbb{C}[0, 1]$ are such that $f(0) = 0$, $f(1) = 1$, $g(0) = 1$ and $g(1) = 0$.

Note that $\phi_0(S) = S \otimes 1$ and $\phi_1(S) = S^2 S^* \otimes 1 + e \otimes S$. Lemma 1.15 implies that, in order to define a morphism from $\mathcal{T}$ to $\mathcal{T} \otimes_\pi \mathcal{T}$, we only need to check the relations on $S$ and $S^*$ and the bounds of $\mathbb{1}$.

We finish this section with a result for tensoring algebras with a countable basis over $\mathbb{C}$ equipped with the fine topology and the Toeplitz algebra. Although the result is known to experts, we give all the details since it allows us to prove Lemma 1.18 which is a key ingredient in Proposition 4.5, one of our main technical results.

**Lemma 1.17.** The locally convex space $A \otimes_\pi s$ is isomorphic to the space $F$ of sequences $\{x_n\}_{n \in \mathbb{N}} \subseteq A$ such that

$$||x||_{\rho,k} = \sum_{n \in \mathbb{N}} |1 + n|^k \rho(x(n))$$

is finite for all $k \in \mathbb{N}$ and any continuous seminorm $\rho$ on $A$, where the topology on $F$ is defined by the seminorms $|| \cdot ||_{\rho,k}$.

**Proof.** There is an inclusion $\phi : A \otimes s \to F$ defined by $a \otimes \alpha \in A \otimes s \mapsto \{x_n = \alpha_n a\} \in F$. Let $z = \sum_{t=1}^{N} a^{(t)} \otimes \alpha^{(t)}$ be an element of $A \otimes s$. We have

$$||\phi(z)||_{\rho,k} = \sum_{n \in \mathbb{N}} \rho \left( \sum_{t=1}^{N} a^{(t)} \alpha_n^{(t)} \right) |1 + n|^k$$

$$\leq \sum_{n \in \mathbb{N}} \sum_{t=1}^{N} \rho(a^{(t)}) |\alpha_n^{(t)}||1 + n|^k$$

$$= \sum_{t=1}^{N} \rho(a^{(t)}) p_k(\alpha^{(t)}).$$

This implies $||\phi(z)||_{\rho,k} \leq (\rho \otimes p_k)(z)$. We can write $z = \sum_{n \in \mathbb{N}} \sum_{t=1}^{N} a^{(t)} \alpha_n^{(t)} \otimes e_n$ and therefore

$$(\rho \otimes p_k)(z) \leq \sum_{n \in \mathbb{N}} (\rho \otimes p_k) \left( \sum_{t=1}^{N} a^{(t)} \alpha_n^{(t)} \otimes e_n \right)$$

$$= \sum_{n \in \mathbb{N}} \rho \left( \sum_{t=1}^{N} a^{(t)} \alpha_n^{(t)} \right) |1 + n|^k$$

$$= ||\phi(z)||_{\rho,k}.$$
Lemma 1.18. Let $s$ be the locally convex space of rapidly decreasing sequences and $A$ an algebra with a countable basis over $\mathbb{C}$ equipped with the fine topology. Then we have

$$A \otimes \pi s = A \otimes s$$

as locally convex spaces. This implies that

$$A \otimes \pi T = A \otimes T \quad \text{and} \quad A \otimes (T \otimes \pi T) = A \otimes (T \otimes \pi T)$$

as locally convex algebras.

Proof. We prove that the space $F$ from Lemma 1.17 is equal to the algebraic tensor product $A \otimes s$. Let $\{v_n\}_{n \in \mathbb{N}}$ be a countable basis of $A$. Given $\{x_n\}_{n \in \mathbb{N}}$ a sequence of elements in $A$ with $\rho_k(x)$ finite for all $k \in \mathbb{N}$ we have, for $n$ fixed

$$x_n = \sum_{i \in \mathbb{N}} \lambda_n^{(i)} v_i$$

where $\lambda_n^{(i)} \neq 0$ for finitely many $i \in \mathbb{N}$.

First, we prove that span$\{x_n\}_{n \in \mathbb{N}}$ is finite dimensional. Suppose this is not the case. We construct subsequences $\{x_{n_i}\}$ and $\{v_{m_i}\}$ such that $\lambda_{n_i}^{(m_i)} \neq 0$. Choose $n_1$ such that $x_{n_1} \neq 0$ and $m_1$ such that $\lambda_{n_1}^{(m_1)} \neq 0$. Suppose $\{x_{n_1}, \ldots, x_{n_k}\}$ and $\{v_{m_1}, \ldots, v_{m_k}\}$ have been chosen. span$\{x_i\}_{i > n_k}$ is infinite dimensional and therefore it is not contained in span$\{v_i\}_{1 \leq i \leq m_k}$. Choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \notin$ span$\{v_i\}_{1 \leq i \leq m_k}$. We can choose $m_{k+1} > m_k$ such that $\lambda_{n_{k+1}}^{(m_{k+1})} \neq 0$.

Now we define a seminorm in $A$

$$\rho \left( \sum_{i \in \mathbb{N}} c_i v_i \right) = \sum_{i \in \mathbb{N}} |c_i| \alpha_i$$

with $\alpha_i = 0$ for $i \notin \{n_k\}_{k \in \mathbb{N}}$ and $\alpha_{n_k} \geq |\lambda_{n_k}^{(m_k)}|^{-1}$. Thus we have $\rho(x_{n_k}) \geq 1$ and

$$\rho_0(x) = \sum_{i \in \mathbb{N}} \rho(x_i) \geq \sum_{i \in \mathbb{N}} \rho(x_{n_i})$$

diverges. We conclude that span$\{x_n\}_{n \in \mathbb{N}}$ is finite dimensional.

Let $N \in \mathbb{N}$ be such that span$\{x_n\}_{n \in \mathbb{N}} \subseteq$ span$\{v_1, \ldots, v_N\}$. That is

$$x_n = \sum_{i=0}^{N} \lambda_n^{(i)} v_i$$

Then

$$x = \lim_{M \to \infty} \sum_{n=0}^{M} x_n \otimes e_n$$

$$= \lim_{M \to \infty} \sum_{n=0}^{M} \sum_{i=0}^{N} \lambda_n^{(i)} v_i \otimes e_n$$

$$= \lim_{M \to \infty} \sum_{i=0}^{N} v_i \otimes \sum_{n=0}^{M} \lambda_n^{(i)} e_n$$
Consider the seminorm $p_j(\sum c_i v_i) = |c_j|$. Then, since $x \in A \otimes_s s$,

$$\sum_{n \in \mathbb{N}} |1 + nk| p_i(x_n) = \sum_{n \in \mathbb{N}} |1 + nk| |\lambda_i^{(i)}| < \infty$$

for all $k \in \mathbb{N}$. Thus, for a fixed $i$, the sequences $\{\lambda_i^{(i)}\}$ are rapidly decreasing on $n$. Therefore

$$\sum_{n \in \mathbb{N}} |1 + n| p_i(x_n) = \sum_{n \in \mathbb{N}} |1 + n| |\lambda_i^{(i)}|$$

and Consequently $x = \sum_{n \in \mathbb{N}} |1 + n| p_i(x_n)$.

2. Bivariant K-theory of locally convex algebras

2.1. Definition and properties of $kk_{alg}$

The bivariant $K$-theory $kk_{alg}$ is constructed by Cuntz in \[7\]. In this section we give the definition of $kk_{alg}$ and state its main properties. A complete treatise of these constructions in the context of bornological algebras can be found in \[8\]. The proofs translate to the context of locally convex algebras in a straightforward manner.

There is a canonical map

$$\langle J^k A, K \otimes \pi_s B \rangle \to \langle J^{k+1} A, K \otimes \pi_s B \rangle$$

that assigns to each morphism $\alpha$ the classifying map associated with the extension

$$0 \to K \otimes \pi_s B \to K \otimes \pi_s C S^k B \to K \otimes \pi_s S^k B \to 0.$$

Definition 2.1. Let $A$ and $B$ be locally convex algebras. We define

$$kk_{alg}^\ast(A, B) = \lim_{k \in \mathbb{N}} \langle J^k A, K \otimes \pi_s S^k B \rangle$$

and for $n \in \mathbb{Z}$

$$kk_n^\ast(A, B) = \lim_{k \in \mathbb{N}, k+n \geq 0} \langle J^{k+n} A, K \otimes \pi_s S^k B \rangle.$$

The group structure of $kk_n^\ast(A, B)$ is defined using Lemma 1.5.

Lemma 2.2. There is an associative product

$$kk_n^\ast(A, B) \times kk_m^\ast(B, C) \to kk_{n+m}^\ast(A, C)$$

Proof. Follows from Lemma 6.32 in \[8\].

In view of this associative product we can regard locally convex algebras as objects of a category $\mathcal{R}_{alg}$ with morphisms between $A$ and $B$ given by elements of $kk_{alg}^\ast(A, B)$. Any morphism $\phi : A \to B$ of locally convex algebras induces an element $kk(\phi) \in kk_{alg}^\ast(A, B)$ which is associated with the diffeotopy class of $i \circ \phi : A \to B \to K \otimes \pi_s B$, where $i$ is the inclusion of $B$ into the first corner, i.e. $i(b) = e_{b0} \otimes b$. We have $kk(\phi) kk(\psi) = kk(\psi \circ \phi)$ (see Theorem 2.3.1 in \[9\]) and therefore we have a functor

$$kk_{alg}^\ast : lca \to \mathcal{R}_{alg}.$$

In particular the identity of $A$ induces an element $kk(id_A)$ in $kk_{alg}^\ast(A, A)$ which is denoted by $1_A$.

Definition 2.3. A functor $F$ from the category of locally convex algebras to an abelian category $\mathcal{C}$ is called

"..."
(1) diffotopy invariant if $F(f) = F(g)$ whenever $f$ and $g$ are diffotopic,
(2) half exact for linearly split extensions if

$$F(A) \rightarrow F(B) \rightarrow F(C)$$

is exact whenever

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a linearly split extension,
(3) $\mathcal{K}$-stable if the natural inclusion $i : A \rightarrow \mathcal{K} \otimes_\pi A$, sending $a$ to $e_{00} \otimes a$ induces an isomorphism $F(i) : F(A) \rightarrow F(\mathcal{K} \otimes_\pi A)$.

The functor $kk_{alg}^* : \text{lca} \rightarrow \text{KK}_{alg}$ is diffotopy invariant, half exact for linearly split extensions and is $\mathcal{K}$-stable.

2.2. Bott Periodicity and Triangulated structure of $\mathcal{KK}_{alg}$.

The suspension of locally convex algebras determines a functor $S : \mathcal{KK}_{alg} \rightarrow \mathcal{KK}_{alg}$ with $S(A) = SA$.

**Theorem 2.4.** [Bott periodicity] There is a natural equivalence between $S^2$ and the identity functor, hence $\mathcal{KK}_{alg}^{2n}(A,B) \cong \mathcal{KK}_{alg}^{0}(A,B)$ and $\mathcal{KK}_{alg}^{2n+1}(A,B) \cong \mathcal{KK}_{alg}^{1}(A,B)$.

**Proof.** See Corollary 7.25 in [8] and the discussion that follows. ■

By Theorem 2.4, $S$ is an automorphism and $S^{-1} \cong S$. We recall the triangulated structure of $(\mathcal{KK},S)$.

Let $f : A \rightarrow B$ be a morphism in $\text{lca}$. The mapping cone of $f$ is defined as the locally convex algebra

$$C(f) = \{(x,g) \in A \oplus CB \mid f(x) = g(0)\}.$$  

The triangle

$$SB \xrightarrow{kk(\iota)} C(f) \xrightarrow{kk(\pi)} A \xrightarrow{kk(f)} B$$

in $(\mathcal{KK}_{alg},S)$, where $\pi : C(f) \rightarrow A$ is the projection into the first component and $\iota : SB \rightarrow C(f)$ is the inclusion into the first component, is called a mapping cone triangle.

Let $E : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a linearly split extension in $\text{lca}$. This induces an element $kk(E) \in kk_{alg}^1(C,A)$ that corresponds to the classifying map $JC \rightarrow A$ of the extension and hence an element $kk(E) \in kk_{alg}(SC,A)$. The triangle

$$SC \xrightarrow{kk(E)} A \xrightarrow{kk(f)} B \xrightarrow{kk(g)} C$$

in $(\mathcal{KK}_{alg},S)$ is called an extension triangle.

**Proposition 2.5.** The category $\mathcal{KK}_{alg}$ with suspension automorphism $S : \mathcal{KK}_{alg} \rightarrow \mathcal{KK}_{alg}$ and with triangles isomorphic to mapping cone triangles as exact triangles is a triangulated category. Furthermore, extension triangles are exact.

**Proof.** See Propositions 7.22 and 7.23 in [8]. ■
2.3. Stabilization by Schatten ideals.

In [9], Cuntz and Thom define a related bivariant \( K \)-theory in the category \( \mathfrak{ca} \). We recall the definition for the case of the Schatten ideals. Let \( \mathbb{H} \) denote an infinite dimensional separable Hilbert Space.

**Definition 2.6.** The Schatten ideals \( \mathcal{L}_p \subseteq B(\mathbb{H}) \), for \( p \geq 1 \), are defined by

\[
\mathcal{L}_p = \{ x \in B(\mathbb{H}) | Tr|x|^p < \infty \}.
\]

Equivalently, \( \mathcal{L}_p \) consists of the space of bounded operators such that the sequence of its singular values \( \{ \mu_n \} \) is in \( l^p(\mathbb{N}) \).

**Definition 2.7.** Let \( A \) and \( B \) be locally convex algebras and \( p \geq 1 \). We define

\[
kk^\mathcal{L}_p(A, B) = kk^\text{alg}(A, B \otimes \pi \mathcal{L}_p).
\]

The groups \( kk^\mathcal{L}_p(A, B) \), for all \( p \geq 1 \), are isomorphic (Corollary 2.3.5 of [9]). Moreover, when \( p > 1 \), this bivariant \( K \)-theory is related to algebraic \( K \)-theory in the following manner:

**Theorem 2.8** (Theorem 6.2.1 in [9]). For every locally convex algebra \( A \) and \( p > 1 \) we have

\[
kk^\mathcal{L}_p(\mathbb{C}, A) = K_0(A \otimes \pi \mathcal{L}_p).
\]

**Corollary 2.9** (Corollary 6.2.3 in [9]). The coefficient ring \( kk^\mathcal{L}_p(\mathbb{C}, \mathbb{C}) \) is isomorphic to \( \mathbb{Z}[u, u^{-1}] \) with \( \deg(u) = 2 \).

This implies that \( kk^\mathcal{L}_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z} \) and \( kk^\mathcal{L}_1(\mathbb{C}, \mathbb{C}) = 0 \).

Consider the category \( \mathcal{R} \mathcal{L} \) whose objects are locally convex algebras and whose morphisms are given by the graded groups \( kk^\mathcal{L}_p(A, B) \). Since \( kk^\mathcal{L}_p : \mathfrak{ca} \to \mathcal{R} \mathcal{L} \) is diffeotopy invariant, half exact for linearly split extensions and is \( \mathcal{K} \)-stable (see Lemma 7.20 in [8]), by the universal property of \( kk^\text{alg} \) we have a functor \( \mathcal{R} \mathcal{L} \to \mathcal{R} \mathcal{L} \).

2.4. Weak Morita equivalence.

In the context of separable \( C^* \)-algebras, two algebras \( A \) and \( B \) are strong Morita equivalent if and only if \( \mathbb{K} \otimes A \cong \mathbb{K} \otimes B \) (they are stably isomorphic). Therefore a strong Morita equivalence of separable \( C^* \)-algebras induces an equivalence in \( KK \). In the case of locally convex algebras we recall the definition of weak Morita equivalence from [7], which still give us an isomorphism between two objects in \( \mathcal{R} \mathcal{L} \).

A Morita context yields the data required in order to define maps \( A \to \mathbb{K} \otimes \pi B \).

**Definition 2.10.** Let \( A \) and \( B \) be locally convex algebras. A Morita context from \( A \) to \( B \) consists of a locally convex algebra \( E \) that contains \( A \) and \( B \) as subalgebras and two sequences \( (\xi_i)_{i \in \mathbb{N}} \) and \( (\eta_j)_{j \in \mathbb{N}} \) of elements of \( E \) that satisfy

1. \( \eta_j A \xi_i \subset B \) for all \( i, j \).
2. The sequence \( (\eta_j a \xi_i) \) is rapidly decreasing for each \( a \in A \). That is, for each continuous seminorm \( \alpha \) in \( B \), \( \alpha(\eta_j a \xi_i) \) is rapidly decreasing in \( i, j \).
3. For all \( a \in A \), \( (\sum \xi_i \eta_j) a = a \).

A Morita context \( ((\xi_i), (\eta_j)) \) from \( A \) to \( B \) determines a homomorphism \( A \to \mathbb{K} \otimes \pi B \) defined by \( a \mapsto \sum_{i,j \in \mathbb{N}} e_{ij} \otimes \eta_j a \xi_i \). Thus it determines an element \( kk((\xi_i), (\eta_j)) \) of \( kk^\text{alg}(A, B) \).

In the next proposition, we give conditions for a Morita context to determine an equivalence in \( \mathcal{R} \mathcal{L} \).
Proposition 2.11. Let $((\xi_i), (\eta_j))$ be a Morita context from $A$ to $B$ in $E$. If $((\xi_i'), (\eta'_k))$ is a Morita context from $B$ to $A$ in the same locally convex algebra and if $A\xi_i\xi_i' \subset A$ and $\eta_k\eta_j A \subset A$ for all $i, j, k, l$; then

$$kk((\xi_i), (\eta_j)) \cdot kk((\xi_i'), (\eta'_k)) = 1_A.$$ 
Therefore, if we also have $B\xi_i\xi_i' \subset B$ and $\eta_k\eta_j B \subset B$ for all $i, j, k, l$, then $kk((\xi_i), (\eta_j))$ is invertible in $kk^{alg}$.

Proof. See Lemma 7.2 in [7]. □

2.5. Quasihomomorphisms.

The definition of a quasihomomorphism goes back to [5]. We give the definition of a quasi-

homomorphisms.

\begin{definition}
\end{definition}

Definition 2.12. Let $A$, $B$, and $D$ be locally convex algebras with $B$ a closed subalgebra of $D$. A quasihomomorphism from $A$ to $B$ determines an element in $kk^{alg}(A, B)$. As a matter of fact it determines a morphism from $E(A)$ to $E(B)$ for any split-exact functor $E : lca \rightarrow C$ where $C$ is an additive category. The reader can also see Section 4 in [10] and Section 3.3.1 in [8].

\begin{definition}
\end{definition}

Definition 2.13. Let $G \subset A$ be a subset that generates $A$ as a locally convex algebra. If $\alpha(x) - \bar{\alpha}(x) \in B$, $\alpha(x)B \subset B$ and $B\alpha(x) \subset B$ for all $x \in A$. We denote such quasihomomorphism by $(\alpha, \bar{\alpha}) : A \rightarrow D \triangleright B$.

The original definition of quasihomomorphisms required $B$ to be an ideal in $D$ (Definition 2.1 in [5]). Note that if $B$ is an ideal then the conditions $\alpha(x)B \subset B$ and $B\alpha(x) \subset B$ are satisfied automatically. On the other hand we only need to check these conditions in a set of algebraic generators of $A$.

Remark 2.14. Let $C \subset A$ be a subset that generates $A$ as a locally convex algebra. If $\alpha(x) - \bar{\alpha}(x) \in B$, $\alpha(x)B \subset B$ and $B\alpha(x) \subset B$ for all $x \in G$, then the conditions are also satisfied for all $x \in A$.

Next we see how a quasihomomorphism $(\alpha, \alpha') : A \rightarrow D \triangleright B$ determines an element $kk(\alpha, \bar{\alpha}) \in kk^{alg}(A, B)$. As a matter of fact, we work with split exact functors from $lca$ to an additive category $C$. An extension $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $lca$ is split if there is a morphism of locally convex algebras $s : C \rightarrow B$ such that $s = id_C$.

\begin{definition}
\end{definition}

Definition 2.15. Let $C$ be an additive category. A sequence $A \rightarrow B \rightarrow C$ in $C$ is split exact if it is isomorphic to the sequence $A \rightarrow A \oplus C \rightarrow C$ with the natural inclusion and projection. A functor $E : lca \rightarrow C$ is called split exact if it sends split extensions in $lca$ to split exact sequences in $C$.

\begin{lemma}
\end{lemma}

Lemma 2.16. [Section 3.2 in [9]] Let $E$ be a split exact functor from $lca$ to an additive category $C$. Then a quasihomomorphism $(\alpha, \alpha') : A \rightarrow D \triangleright B$ determines a morphism $E(\alpha, \bar{\alpha}) : E(\alpha(a)) \rightarrow E(B)$ in $C$.

Proof. Let $D'$ be the closed subalgebra of $A \oplus D$ generated by all elements $(a, \alpha(a))$ and $(0, b)$ with $a \in A$ and $b \in B$. Then we have an exact sequence

$$0 \rightarrow B \rightarrow D' \rightarrow A \rightarrow 0$$

with the inclusion $B \subseteq D'$ given by $b \mapsto (0, b)$ and the projection $\pi : D' \rightarrow A$ defined by $\pi(a, x) = a$. This extension has two splits $\alpha' : A \rightarrow D'$ defined by $\alpha'(a) = (a, \alpha(a))$ and $\alpha'(a) = (a, \bar{\alpha}(a))$. Because of the split-exactness of $E$, $E(B) \rightarrow E(D')$ is a kernel of
Let Proposition 2.16. Let $E$ be a split exact functor from $\mathfrak{ca}$ to an additive category $\mathfrak{c}$ and $(\alpha, \alpha') : A \rightarrow D \triangleright B$ be a quasihomomorphism from $A$ to $B$ in $D$. We have

(1) $E(\alpha, \bar{\alpha}) = -E(\bar{\alpha}, \alpha)$.

(2) Let $\phi = \alpha - \bar{\alpha}$. If $\phi(x)\bar{\alpha}(y) = \bar{\alpha}(y)\phi(x) = 0$ for all $x, y \in A$, then $\phi$ is a homomorphism and $E(\alpha, \bar{\alpha}) = E(\phi)$.

(3) For any morphism $\phi : A' \rightarrow A$, $(\alpha \circ \phi, \bar{\alpha} \circ \phi) : A' \rightarrow B$ is a quasihomomorphism from $A'$ to $B$ in $D$ and

$$E(\alpha \circ \phi, \bar{\alpha} \circ \phi) = E(\alpha, \bar{\alpha}) \circ E(\phi).$$

(4) If $\psi : D \rightarrow F$ is a morphism such that $\psi|_B : B \rightarrow C \subset F$ and the morphisms $\psi \circ \alpha, \psi \circ \bar{\alpha} : A \rightarrow F$ define a quasihomomorphism from $A$ to $C$ in $F$, then

$$E(\psi \circ \alpha, \psi \circ \bar{\alpha}) = E(\psi|_B) \circ E(\alpha, \bar{\alpha}).$$

(5) Let $\alpha$ and $\bar{\alpha}$ be homomorphisms from $A$ to $D[0, 1]$ such that $\alpha(x) - \bar{\alpha}(x) \in B[0, 1]$, $\alpha(x)B[0, 1] \subset B[0, 1]$ and $B[0, 1] \alpha(x) \subset B[0, 1]$ for all $x \in A$. If $E$ is difftopy invariant, then $E(\alpha_1, \bar{\alpha}_1) = E(\bar{\alpha}_0, \alpha_0)$ (where $\alpha_1 = \alpha \circ \alpha$).

Proof. For (1) – (4) see Proposition 21 in [11].

To prove (5), we consider the evaluation maps $ev_1 : D[0, 1] \rightarrow D$. They restrict to the evaluation maps $ev_1 : B[0, 1] \rightarrow B$. To apply (3) we need to check that the morphisms $ev_1 \circ \alpha, ev_1 \circ \bar{\alpha} : A \rightarrow D$ define a quasihomomorphism from $A$ to $B$ in $D$. First notice that $(ev_1 \circ \alpha)(a) - (ev_1 \circ \bar{\alpha})(a) = (ev_1 \circ (\alpha - \bar{\alpha}))(a)$ is in $B$ because $(\alpha - \bar{\alpha})(a) \in B[0, 1]$. Now consider an element $b \in B$. We want to prove that the product $(ev_1 \circ \alpha)(a)b$ is in $B$. Consider a function $f \in B[0, 1]$ such that $ev_1 \circ f = b$. Then $(ev_1 \circ \alpha)(a)b = ev_1 \circ (\alpha(a)f)$ and $\alpha(a)f \in B[0, 1]$. Similarly, we can prove that $B(ev_1 \circ \alpha)(a) \subset B$. We can now apply (3) and we obtain $E(ev_1 \circ \alpha, ev_1 \circ \bar{\alpha}) = E(ev_1) \circ E(\alpha, \bar{\alpha})$. Since $E$ is difftopy invariant, we have $E(ev_0) = E(ev_1)$, which concludes the result.

3. Generalized Weyl algebras

Generalized Weyl algebras were introduced by Bavula (see [2]) and have been amply studied. Examples of generalized Weyl algebras include the Weyl algebra, the quantum Weyl algebra, the quantum plane, the enveloping algebra of $\mathfrak{sl}_2$ $U(\mathfrak{sl}_2)$, its primitive factors $B_\lambda = U(\mathfrak{sl}_2)/\langle C - \lambda \rangle$ where $C$ is the Casimir element (see Example 4.7 in [12]) and the quantum weighted projective lines $\mathcal{O}(\mathbb{WP}_q(k, l))$ (see [4]).

In our context, generalized Weyl algebras provide a family of examples of $\mathbb{Z}$-graded algebras that are smooth generalized crossed products and do not satisfy the condition of being tame smooth (and therefore they are outside the framework of [10]).

Definition 3.1. Let $D$ be a ring, $\sigma \in \text{Aut}(D)$ and $a$ a central element of $D$. The generalized Weyl algebra $D(\sigma, a)$ is the algebra generated by $x$ and $y$ over $D$ satisfying

$$xd = \sigma(d)x, \quad yd = \sigma^{-1}(d)y, \quad yx = a \quad \text{and} \quad xy = \sigma(a) \quad (3.1)$$

for all $d \in D$.

Examples. The following are examples of generalized Weyl algebras:
(1) The Weyl algebra
\[ A_1(\mathbb{C}) = \mathbb{C}\langle x, y | xy - yx = 1 \rangle \]
is isomorphic to \( \mathbb{C}[h](\sigma, h) \), with \( \sigma(h) = h - 1 \).

(2) The quantum Weyl algebra
\[ A_q(\mathbb{C}) = \mathbb{C}\langle x, y | xy - qyx = 1 \rangle \]
is is isomorphic to \( \mathbb{C}[h](\sigma, h - 1) \), with \( \sigma(h) = qh \).

(3) The quantum plane
\[ \mathbb{C}\langle x, y | xy = qyx \rangle \]
is isomorphic to \( \mathbb{C}[h](\sigma, h) \), with \( \sigma(h) = qh \).

(4) The primitive quotients of \( U(\mathfrak{sl}_2) \) (see Example 3.2 in \([2]\) ),
\[ B_\lambda = U(\mathfrak{sl}_2)/\langle c - \lambda \rangle, \quad \lambda \in \mathbb{C}, \]
are isomorphic to \( \mathbb{C}[h](\sigma, P) \), with \( \sigma(h) = h - 1 \) and \( P(h) = -h(h + 1) - \lambda/4 \).

(5) The quantum weighted projective line or the quantum spindle algebra \( \mathcal{O}(\mathbb{WP}_q(k, l)) \)
is isomorphic to \( \mathbb{C}[h](\sigma, P) \) with \( P(h) = h^k \prod_{l=0}^{l-1}(1 - q^{-2i}h) \) and \( \sigma(h) = q^{2l}h \) (see Theorem 2.1 in \([3]\) and Example 3.8 in \([4]\) ).

**Lemma 3.2.** A generalized Weyl algebra has a \( \mathbb{Z} \)-grading \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) where \( A_0 = D \) and
\[
A_n = \begin{cases} 
Dy^n & n > 0 \\
Dx^n & n < 0 
\end{cases}
\]

**Proof.** Consider the grading in \( A = D(\sigma, a) \) defined by setting the degree of \( y \) equal to 1, the degree of \( x \) equal to \(-1\), and the degree of all elements of \( D \) equal to 0. That is, the degree of the monomial \( \prod_{i=1}^n d_i x^{\alpha_i} y^{\beta_i} \), with \( d_i \in D \), is equal to \( \sum_{i=1}^n \beta_i - \sum_{i=1}^n \alpha_i \). Since the relations defining \( A \) are compatible with the grading, the algebra \( A \) is \( \mathbb{Z} \)-graded.

Now consider the following relations in \( A \). We have
\[
x^n y^n = \sigma^n(a) \sigma^{n-1}(a) \ldots \sigma(a) \\
y^n x^n = \sigma^{-(n-1)}(a) \sigma^{-(n-2)}(a) \ldots a
\]

Using induction on the length of the monomial \( \prod_{i=1}^n d_i x^{\alpha_i} y^{\beta_i} \) we prove (3.2). Note that \( Dy^n = y^n D \) and \( Dx^n = x^n D \).

In the case of generalized Weyl algebras over \( \mathbb{C}[h] \), we have the following result.

**Corollary 3.3.** The generalized Weyl algebra \( A = \mathbb{C}[h](\sigma, P) \), with \( P \in \mathbb{C}[h] \), has a countable basis over \( \mathbb{C} \).

**Proof.** A basis is given by the elements \( h^n, h^n y^n \) and \( h^n x^n \) for \( n \in \mathbb{N}, m \geq 1 \).

There are several ways of writing the same generalized Weyl algebra. The conjugation of \( \sigma \) by an automorphism \( \tau \) of \( D \) gives rise to an isomorphism of generalized Weyl algebras.

**Lemma 3.4.** Let \( \sigma, \tau \) be automorphisms of \( D \) and let \( a \) be a central element of \( D \). Then \( \tau(a) \) is central in \( D \) and
\[
D(\sigma, a) \cong D(\tau \sigma \tau^{-1}, \tau(a)).
\]
Proof. Let \( x' \) and \( y' \) be the generators of \( D(\tau \sigma \tau^{-1}, \tau(a)) \) over \( D \). There is a morphism \( \phi : D(\sigma, a) \rightarrow D(\tau \sigma \tau^{-1}, \tau(a)) \) defined by \( x \mapsto x' \), \( y \mapsto y' \), \( d \mapsto \tau(d) \), for all \( d \in D \). We need to check that \( \phi \) is compatible with the relations (3.1). Using the relations defining \( D(\tau \sigma \tau^{-1}, \tau(a)) \) we have

\[
x' \tau(d) = (\tau \sigma \tau^{-1})(\tau(d))x' = \tau(\sigma(d))x' \\
y' \tau(d) = (\tau \sigma \tau^{-1})\tau(d)y' = \tau(\sigma^{-1}(d))y' \\
x'y' = \tau(a) \\
y'x' = (\tau \sigma \tau^{-1})(\tau(a)) = \tau(\sigma(a)).
\]

\( \phi^{-1} \) is defined by \( x' \mapsto x \), \( y' \mapsto y \), \( d \mapsto \tau^{-1}(d) \) for all \( d \in D \). \( \blacksquare \)

In the case \( D = \mathbb{C}[h] \), we use Lemma 3.4 to write a given generalized Weyl algebra in a canonical form. Any automorphism of \( \mathbb{C}[h] \), is of the form \( \sigma(h) = qh + h_0 \) with \( q, h_0 \in \mathbb{C} \) and \( q \neq 0 \). We have three cases

1. \( \sigma \) is conjugate to \( \text{id} \) if and only if \( \sigma = \text{id} \),
2. if \( q = 1 \) and \( h_0 \neq 0 \), then \( \sigma \) is conjugate to \( h \mapsto h - 1 \),
3. if \( q \neq 1 \), then \( \sigma \) is conjugate to \( h \mapsto qh \).

Combining this with Lemma 3.4, we obtain the following result.

**Proposition 3.5** (Compare with Proposition 2.1.1 in [14].) Let \( A = \mathbb{C}[h](\sigma, P) \), with \( P \in \mathbb{C}[h] \) and \( \sigma(h) = qh + h_0 \) with \( q, h_0 \in \mathbb{C} \) and \( q \neq 0 \). The following facts hold:

1. If \( \sigma = \text{id} \), then \( A \cong \mathbb{C}[h, x, y]/(yx - P) \).
2. If \( q = 1 \) and \( h_0 \neq 0 \) then \( A \cong \mathbb{C}[h](\sigma_1, P_1) \) with \( \sigma_1(h) = h - 1 \) and \( P_1(h) = P(-h_0h) \).
3. If \( q \neq 1 \) then \( A \cong \mathbb{C}[h](\sigma_1, P_1) \) with \( \sigma_1(h) = qh \) and \( P_1(h) = P(h + \frac{h_0}{1-q}) \). \( \blacksquare \)

**Proof.** (1) is straightforward, for (2) and (3) use Lemma 3.4 with \( \tau(h) = -h_0h \) and \( \tau(h) = h + \frac{h_0}{1-q} \), respectively. \( \blacksquare \)

By Proposition 3.5 we can assume that \( \sigma = \text{id} \), \( \sigma(h) = h - 1 \) or \( \sigma(h) = qh \) for some \( q \neq 0 \).

**Proposition 3.6.** Let \( A = \mathbb{C}[h](\sigma, P) \), with \( P \in \mathbb{C}[h] \). The following facts hold:

1. If \( \sigma(h) = h - 1 \) and \( P \) is a non constant polynomial, then \( A \cong \mathbb{C}[h](\sigma, P_1) \) with \( P_1(0) = 0 \).
2. If \( \sigma(h) = qh \) and \( P \) has a nonzero root, then \( A \cong \mathbb{C}[h](\sigma, P_1) \) with \( P_1(1) = 0 \).

**Proof.** Follows from Lemma 3.4. In (1) we set \( \tau(h) = h - \lambda \) for any root \( \lambda \) of \( P \), and in (2) we set \( \tau(h) = \lambda h \), where \( \lambda \) is a non zero root of \( P \). \( \blacksquare \)

Note that \( P_1 \) in Proposition 3.5 has a non zero root if and only if \( P \) has a root different from \( \frac{h_0}{1-q} \).

It is worth mentioning that generalized Weyl algebras over \( \mathbb{C}[h] \) have been classified up to isomorphism (see [2] and [14]).

To finish this section, we construct faithful representations for the generalized Weyl algebras covered in cases (1) and (2) of Proposition 3.6. We define \( V_\mathbb{N} \) as the vector space of sequences of complex numbers indexed by \( \mathbb{N} \). Let \( U_1, U_{-1} \in \text{End}(V_\mathbb{N}) \) be the shift to the
right and the shift to the left respectively. Note that \( U_iU_1 = 1, U_1U_{-1} = 1 - e_{00}. \)

\[
U_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \ddots & \end{bmatrix} \quad U_{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ \vdots & \ddots & \end{bmatrix}
\]

Additionally, we use the following elements \( N = \sum_{i \in \mathbb{N}} (-i) e_{i,i} \) and \( G = \sum_{i \in \mathbb{N}} q^i e_{i,i} \) for \( q \neq 0 \) not a root of unity.

\[
N = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \\ \vdots & \ddots & \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & q & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & q^3 \\ \vdots & \ddots & \end{bmatrix}
\]

**Lemma 3.7.** The following relations are satisfied in \( \text{End}(V_N) \).

1. \( U_1N = (N + 1)U_1 \),
2. \( U_{-1}N = (N - 1)U_{-1} \),
3. \( U_1G = (q^{-1}G)U_1 \),
4. \( U_{-1}G = (qG)U_{-1} \).

As a consequence of Lemma 3.7, we obtain that the subalgebras \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) of \( \text{End}(V_N) \) generated by \( \{U_1, U_{-1}, N\} \) and \( \{U_1, U_{-1}, G\} \), respectively, have countable basis over \( \mathbb{C} \) and therefore are locally convex algebras with the fine topology.

**Lemma 3.8.** We have the following representations for generalized Weyl algebras \( A = \mathbb{C}[h](\sigma, P(h)) \) in the following cases.

1. If \( \sigma(h) = h - 1 \) and \( P \) is a nonzero polynomial with \( P(0) = 0 \), then there is a faithful representation \( \rho : A \to \mathcal{E}_1 \) such that
   \[
   \rho(h) = N, \quad \rho(x) = U_{-1} \quad \text{and} \quad \rho(y) = P(N)U_1 = U_1P(N - 1)
   \]
2. If \( \sigma(h) = qh \) with \( q \neq 0 \) not a root of unity and \( P(1) = 0 \), then there is a faithful representation \( \rho : A \to \mathcal{E}_2 \) such that
   \[
   \rho(h) = G, \quad \rho(x) = U_{-1} \quad \text{and} \quad \rho(y) = P(G)U_1 = U_1P(qG)
   \]

**Proof.** For (1), first we notice that we have an injective homomorphism \( \mathbb{C}[h] \to \text{End}(V_N) \) defined by \( h \mapsto N \).

This homomorphism is injective because all the entries in the diagonal of matrix \( N \) are different. With \( P(0) = 0 \) we will see that the relations of \( \mathbb{C}[h](\sigma, P(h)) \) hold. To prove this, we use the relations of Lemma 3.7. For a polynomial \( \alpha(h) \in \mathbb{C}[h] \) we have

\[
\rho(x\alpha(h)) = U_{-1}\alpha(N) = \alpha(N - 1)U_{-1} = \rho(\alpha(h) - 1)x
\]

\[
\rho(y\alpha(h)) = P(N)U_1\alpha(N) = \alpha(N + 1)P(N)U_1 = \rho(\alpha(h) + 1)y
\]

\[
\rho(yx) = U_1P(N - 1)U_{-1} = U_{-1}U_1P(N) = (1 - e_{00})P(N) = P(N) = \rho(P(h))
\]

\[
\rho(xy) = U_{-1}U_1P(N - 1) = P(N - 1) = \rho(P(h - 1))
\]

We use that \( P(0) = 0 \) in the third row to guarantee \((1 - e_{00})P(N) = P(N)\).
Now we prove that $\rho$ is injective. Let
\[ \alpha = \sum_{n \geq 0} p_n(h)y^n + \sum_{m < 0} q_m(h)x^m \]
be an element of $A$. Then we have
\[ \rho(\alpha) = \sum_{n \geq 0} p_n(P(N))(P(N)\mathcal{U}_1)^n + \sum_{m < 0} q_m(P(N)\mathcal{U}_{m-1}). \]
Note that $(P(N)\mathcal{U}_1)^n = Q_n(N)\mathcal{U}_1^n$ where $Q_n(N) = P(N)P(N+1)\ldots P(N+(n-1))$. Therefore if $\rho(\alpha) = 0$ then $q_m = 0$ and because $Q_n \neq 0$, we have $p_n = 0$. Therefore $\alpha = 0$ and so $\rho$ is injective.

(2) is proved in a similar way: we have an injective homomorphism $\mathbb{C}[h] \hookrightarrow \text{End}(V_N)$ defined by $h \mapsto G$. This homomorphism is injective because $q \neq 0$ not a root of unity imply that all the entries in the diagonal of matrix $G$ are different. Using $P(1) = 0$, it is easy to see that the relations of $D(\sigma,a)$ hold. We also need to use the relations of Lemma 3.7. We prove that $\rho$ is injective in a similar way. In this case we note that $(P(G)\mathcal{U}_1)^n = Q_n(G)\mathcal{U}_1^n$ where $Q_n(G) = P(G)P(q^{-1}G)\ldots P(q^{-(n-1)}G)$.

\section{4. $kk^{alg}$ invariants of generalized Weyl algebras}

In this section, we compute the isomorphism class in $\mathbb{K}K^{alg}$ of generalized Weyl algebras $A = \mathbb{C}[h](\sigma,P)$ where $\sigma(h) = qh + h_0$ is an automorphism of $\mathbb{C}[h]$ and $P \in \mathbb{C}[h]$. We summarize our results in the table below.

| Conditions | Results | Observation |
|------------|---------|-------------|
| $P$ is constant | $P = 0$ | $A \cong \mathbb{K}K^{alg} \mathbb{C}$ | Prop 4.22 A N-graded |
| $P \neq 0$ | $A \cong \mathbb{K}K^{alg} \mathbb{C} \oplus \mathbb{C}$ | Prop 4.21 A tame smooth |
| $P$ is nonconstant with $r$ distinct roots | $q$ not a root of unity | $A \cong \mathbb{K}K^{alg} \mathbb{C}^r$ | Thm 4.18 |
| | $q = 1$ and $h_0 \neq 0$ | $A \cong \mathbb{K}K^{alg} \mathbb{C}^r$ | Thm 4.18 |

In Section 4.1, we recall the definition of $\mathcal{T}_B$ from [10]. Generalized Weyl algebras $A = \mathbb{C}[h](\sigma,P)$ are smooth generalized crossed products and in Proposition 4.5 we construct a linearly split extension
\[ 0 \rightarrow \Lambda_A \rightarrow \mathcal{T}_A \rightarrow A \rightarrow 0. \]
In the case where $P$ is a non constant polynomial, $A$ is a generalized crossed product that is not tame smooth so we cannot apply the results of [10] directly. In this case we follow the methods of [7] and [10] to obtain
\[ \Lambda_A \cong \mathbb{K}K^{alg} A_1 A_{-1} \] (Theorem 4.8) and $\mathcal{T}_A \cong \mathbb{K}K^{alg} A_0$ (Theorem 4.14) in the cases where $P$ is non constant and
\begin{itemize}
  \item $q = 1$ and $h_0 \neq 0$ or
  \item $q$ is not a root of unity and $P$ has a root different from $\frac{h_0}{1-q}$.
\end{itemize}
With these isomorphisms we construct in Theorem 4.15 an exact triangle
\[ SA \rightarrow A_1 A_{-1} \rightarrow A_0 \rightarrow A \]
in the triangulated category \((\mathfrak{R}^{alg}, \mathfrak{S})\) (see Proposition 2.5). This implies
\[
A = A_0 \oplus S(A_1 A_{-1}).
\]

In Proposition 4.17 we prove that \(A_1 A_{-1} \cong \mathfrak{R}^{alg} S \mathbb{C}^{-1}\), and since by Lemma 1.7 we know that \(A_0 \cong \mathfrak{R}^{alg} \mathbb{C}\), we obtain our main result Theorem 4.18 in these cases \(A \cong \mathfrak{R}^{alg} \mathbb{C}^r\).

We also determine the \(\mathfrak{R}^{alg}\)-class of \(A\) when \(A\) is \(\mathbb{N}\)-graded. In this case Lemma 1.7 gives us \(A \cong \mathfrak{R}^{alg} A_0\). This is the case when
- \(P\) is nonconstant, \(q\) is not a root of unity and \(P\) has only \(\frac{h_0}{1-q}\) as a root or
- \(P = 0\).

In both cases we obtain \(A \cong \mathfrak{R}^{alg} \mathbb{C}\) in Propositions 4.22 and 4.20.

If \(P\) is a nonzero constant polynomial, \(A\) is a tame smooth generalized crossed product and the results from [10] apply. In this case we obtain \(A \cong \mathfrak{R}^{alg} S \mathbb{C} \oplus \mathbb{C}\) in Proposition 4.21.

4.1. The Toeplitz algebra of a smooth generalized crossed product.

In [10], Gabriel and Grensing define smooth generalized crossed products. These are involutive locally convex algebras analog to \(C^*\)-algebra generalized crossed products in [1]. In the same article [10], sequences analog to the Pimsner-Voiculescu exact sequence are constructed for smooth generalized crossed products that are tame smooth (see definition 18 in [10]).

**Definition 4.1.** A gauge action \(\gamma\) on a locally convex algebra \(B\) is a pointwise continuous action of \(S^1\) on \(B\). An element \(b \in B\) is called gauge smooth if the map \(t \mapsto \gamma_t(b)\) is smooth.

If we have a gauge action on \(B\), then \(B_n = \{b \in B | \gamma_t(b) = t^n b, \forall t \in S^1\}\) defines a natural \(\mathbb{Z}\)-grading of \(B\).

**Definition 4.2.** A smooth generalized crossed product is a locally convex algebra \(B\) with an involution and a gauge action such that
- \(B_0\) and \(B_1\) generate \(B\) as a locally convex involutive algebra,
- all \(b\) are gauge smooth and the induced map \(B \to C^\infty(S^1, B)\) is continuous.

Generalized Weyl algebras \(A = \mathbb{C}[h](\sigma, P)\) are locally convex algebras when given the fine topology. They have an involution defined by \(y^* = x, x^* = y\) and \(d^* = d\) for all \(d \in D\). There is an action of \(S^1\) defined by \(\gamma_t(\omega_n) = t^n \omega_n\) for \(\omega_n \in A_n\). With this action, generalized Weyl algebras over \(\mathbb{C}[h]\) are smooth generalized crossed products.

**Remark 4.3.** Generalized Weyl algebras \(A = \mathbb{C}[h](\sigma, P)\) are only tame smooth when \(P\) is constant (see definition 18 in [10]). If \(P\) is non constant, we have \(A_1 A_{-1} = (P) \subset A_0 = \mathbb{C}[h]\). This implies that \(A\) is not tame smooth because tame smooth generalized crossed products \(B\) have a frame in degree 1 which implies that \(B_1 B_{-1} = B_0\).

**Definition 4.4.** Let \(B\) be a smooth generalized crossed product. We define \(\mathcal{T}_B\) to be the closed subalgebra of \(\mathcal{T} \otimes_\pi B\) generated by \(1 \otimes B_0, S \otimes B_1\) and \(S^* \otimes B_{-1}\).

We tensor the linearly split extension \(0 \to \mathcal{K} \to \mathcal{T} \to C^\infty(S^1) \to 0\) with \(B\) to obtain
\[
0 \to \mathcal{K} \otimes_\pi B \to \mathcal{T} \otimes_\pi B \xrightarrow{\pi_B} C^\infty(S^1) \otimes_\pi B \to 0. \tag{4.1}
\]
which is still a linearly split extension.
Proposition 4.5. Let \( A \) be a generalized Weyl algebra \( \mathbb{C}[h](\sigma, P) \). Then there is a linearly split extension
\[
0 \to \Lambda_A \xrightarrow{\iota} \mathcal{T}_A \xrightarrow{\bar{p}} A \to 0
\]
where \( \Lambda_A \) is the ideal \( \bigoplus_{i,j \geq 0} e_{i,j} \otimes A_{i+1}A_{-(j+1)} \) of \( \mathcal{T}_A \), \( \iota \) is the inclusion of \( \Lambda_A \) in \( \mathcal{T}_A \) and \( \bar{p} \) is the restriction of \( p \) to \( \mathcal{T}_A \).

Proof. By Corollary 3.3, \( A \) has a countable basis over \( \mathbb{C} \). Using Lemma 1.18, we conclude that the projective tensor products of (4.1) are all algebraic. The image of \( \mathcal{T}_A \) is generated by \( 1 \otimes a_0 \), \( z \otimes a_1 \) and \( z^{-1} \otimes A_{-1} \) and it is isomorphic to \( A \) via \( z^n \otimes a_n \mapsto a_n \). The kernel of \( \pi \) is the intersection of \( K \otimes A \) and \( \mathcal{T}_A \). The elements of \( \mathcal{T}_A \) are of the form \( 1 \otimes a_0 + \sum_{k,l \geq 0} S^{k+1}S^{l+1} \otimes a_{k+1}a_{-(l+1)} \). Now we note that
\[
S^{k+1}S^{l+1} = \begin{cases} S^{k-l}(1 - e_{0,0} - \cdots - e_{l,l}) & , k \geq l \\ (1 - e_{0,0} - \cdots - e_{k,k})S^{*(l-k)} & , k < l \end{cases}
\]
Using the vector space decomposition \( \mathcal{T} \otimes A = K \otimes A \oplus C^\infty(S^1) \otimes A \) we note that an element of the kernel is of the form
\[
\sum_{k \geq l} S^{k-l}(-e_{0,0} - \cdots - e_{l,l}) \otimes a_{k+1}a_{-(l+1)} + \sum_{k<l}(-e_{0,0} - \cdots - e_{k,k})S^{*(l-k)} \otimes a_{k+1}a_{-(l+1)}
\]
which is in \( \bigoplus_{i,j \geq 0} e_{i,j} \otimes A_{i+1}A_{-(j+1)} \).

\[\blacksquare\]

4.2. The case where \( P \) is a non constant polynomial.
We consider the short exact sequence
\[
0 \to \Lambda_A \to \mathcal{T}_A \to A \to 0
\]
where \( \mathcal{T}_A \) is the Toeplitz algebra of \( A \). This sequence yields an exact triangle
\[
S\mathcal{A} \to \Lambda_A \to \mathcal{T}_A \to A
\]
in the triangulated category \( (\mathfrak{RAlg}, S) \).

In order to apply Lemma 3.8, we consider generalized Weyl algebras \( A = \mathbb{C}[h](\sigma, P) \) where \( P \) is non constant and
\[\bullet \ q = 1 \text{ and } h_0 \neq 0 \text{ or} \]
\[\bullet \ q \text{ is not a root of unity and } P \text{ has a root different from } \frac{h_0}{1-q}. \]

By Propositions 3.5 and 3.6, in order to cover these cases, it suffices to consider the following two cases:
\[\bullet \ \sigma(h) = h - 1 \text{ and } P \text{ is a non constant polynomial with } P(0) = 0. \]
\[\bullet \ \sigma(h) = qh \text{ where } q \text{ is not a root of unity and } P \text{ is a non constant polynomial with } P(1) = 0. \]

In both cases, we construct an exact triangle
\[
S\mathcal{A} \to A_1A_{-1} \xrightarrow{0} A_0 \to A
\]
in the triangulated category \( (\mathfrak{RAlg}, S) \).

Remark 4.6. We treat the case where \( q \) is not a root of unity and \( P \) has only \( \frac{h_0}{1-q} \) as a root separately in Proposition 4.20.

We start by characterizing the elements of \( A_1A_{-1} \).
Lemma 4.7. The elements of $\Lambda_A$ can be written uniquely as sums $\sum e_{i,j} \otimes y^{i+1}P_{i,j}(h)x^{j+1}$

Proof. Follows from Lemma 3.2.

Define $j_1 : A_1A_{-1} \to \Lambda_A$ by $j_1(a) = e_{00} \otimes a$. We embed $\Lambda_A$ in a suitable algebra so that we can construct a Morita equivalence to its subalgebra $j_1(A_1A_{-1}) = e_{00} \otimes A_1A_{-1}$. Consider the faithful representation $\rho : A \to E$ from Lemma 3.8 (where $E = E_1$ if $q = 1$ and $E = E_2$ if $q \neq 1$). Tensoring with $T_1$ we obtain an injective morphism $1_T \otimes \rho : T \otimes A \to T \otimes E$ which restricts to an injective morphism $\bar{\rho} : T_A \hookrightarrow T \otimes E$.

Now, we show $\Lambda_A \cong_{\text{alg}} A_1A_{-1}$.

Theorem 4.8. There is a Morita equivalence between $\Lambda_A$ and $j_1(A_1A_{-1})$, therefore there is an invertible element $\theta \in kk^{\text{alg}}(\Lambda_A, A_1A_{-1})$ which is an inverse of $kk(j_1)$.

Proof. We write the proof in the case $\sigma(h) = h - 1$ and $P(0) = 0$. The case $\sigma(h) = qh$ and $P(1) = 0$ can be proven in a similar way since the matrices involved in the representation satisfy corresponding algebraic relations (Lemma 3.7).

Using the representation from Lemma 3.8 we obtain a faithful representation $\bar{\rho} : \Lambda_A \to T \otimes E_1$.

The Morita equivalence is given by $\xi_i = \xi'_i = e_{i,0} \otimes U_i$ and $\eta_j = \eta'_j = e_{0,j} \otimes U_{-1}$. We check that these sequences satisfy the conditions in Definition 2.10 and Proposition 2.11.

First, we establish that $\xi_i$, $\eta_j$ defines a Morita context between $\Lambda_A$ and $e_{00} \otimes A_1A_{-1}$ according with Definition 2.10. Let $w = \sum e_{i,j} \otimes y^{i+1}P_{i,j}(h)x^{j+1}$ be an element of $\Lambda_A$.

1. $\eta_j \bar{\rho}(w) \xi_i \in e_{00} \otimes A_1A_{-1}$. We have

   $\eta_j \bar{\rho}(w) \xi_i = e_{00} \otimes U_{-1}^j [P(N)U_i]^{j+1}P_{j,i}(N)U_{-1}^{j+1}U_i$

   Note that $(P(N)U_i)^{j+1} = U_i^{j+1}R_{j+1}(N)$ where

   $R_{j+1}(N) = P(\sigma(N)) \cdots P(\sigma^{j+1}(N)) = P(\sigma(N))R_{j+1}'(N)$

   and therefore we have

   $\eta_j \bar{\rho}(w) \xi_i = e_{00} \otimes U_i R_{j+1}(N)P_{j,i}(N)U_{-1}$

   $= e_{00} \otimes U_i P(\sigma(N))R_{j+1}'(N)P_{j,i}(N)U_{-1}$

   $= e_{00} \otimes P(N)U_i R_{j+1}'(N)P_{j,i}(N)U_{-1}$

   $= \bar{\rho}(e_{00} \otimes yR_{j+1}'(h)P_{j,i}(h)x) \in \bar{\rho}(e_{00} \otimes A_1A_{-1})$

2. The terms $\eta_j \bar{\rho}(w) \xi_i$ are rapidly decreasing. This is because the elements of $\Lambda_A$ are finite sums.

3. $(\sum \xi_i \eta_i) \bar{\rho}(w) = \bar{\rho}(w)$. We have

   $(\sum \xi_i \eta_i) \bar{\rho}(w) = \left(\sum e_{i,j} \otimes U_i U_{-1}^j\right) \left(\sum e_{k,l} \otimes (U_i P(N))^{k+1}P_{k,i}(N)U_{-1}^{l+1}\right)$

   $= \sum e_{k,l} \otimes U_{-1}^k U_i U_{-1}^k U_{-1}^{k+1}R_{k+1}(N)P_{k,i}(N)U_{-1}^{l+1}$

   $= \sum e_{k,l} \otimes U_{-1}^{k+1}R_{k+1}(N)P_{k,i}(N)U_{-1}^{l+1}$

   $= \bar{\rho}(w)$
Now we check the conditions of Proposition 2.11. We show that $\bar{\rho}(w)\xi_k\xi'_l$ and $\eta'_k\eta\bar{\rho}(w)$ are still elements of $\bar{\rho}(\Lambda_A)$.

$$
\bar{\rho}(w)\xi_k\xi'_l = \left( \sum_{i,j} e_{i,j} \otimes (U_1 P(N))^{i+1} P_{i,j}(N) U_{j-1}^{i+1} \right) (e_{k,0} \otimes U_1^k) (e_{l,0} \otimes U_1^l)
$$

which is 0 unless $l = 0$ and in that case we obtain

$$
\bar{\rho}(w)\xi_k\xi'_l = \sum_{e_{i,0}} (U_1 P(N))^{i+1} P_{i,k}(N) U_{-1}^i
$$

$$
= \bar{\rho} \left( \sum_{e_{i,0}} y^{i+1} Q_{i,k}(h)x \right) \in \bar{\rho}(\Lambda_A)
$$

and similarly we compute

$$
\eta'_k\eta\bar{\rho}(w) = (e_{0,k} \otimes U_1^k)(e_{0,l} \otimes U_1^l) \left( \sum_{e_{i,j}} (U_1 P(N))^{i+1} P_{i,j}(N) U_{j-1}^{i+1} \right)
$$

which is 0 unless $k = 0$ and in that case we obtain

$$
\bar{\rho}(w)\xi_k\xi'_l = \sum_{e_{0,j}} U_{1-1}^{i+1}(U_1 P(N))^{i+1} P_{i,j}(N) U_{j-1}^{i+1}
$$

$$
= \bar{\rho} \left( \sum_{e_{0,j}} y^{i+1} Q_{i,j}(h)x^{j+1} \right) \in \bar{\rho}(\Lambda_A).
$$

The Morita context from $e_{00} \otimes A_1 A_{-1}$ to $\Lambda_A$ is defined by $(\xi'_l, \eta'_j)$. So far we have proved $kk((\xi'_l, \eta'_j)) \cdot kk((\xi'_l, \eta'_j)) = 1_{\Lambda_A}$. Let $z = e_{00} \otimes y P_{0,0}x \in e_{00} \otimes A_1 A_{-1}$. Then $\bar{\rho}(z)\xi'_k \xi_l = \eta'_k \eta \bar{\rho}(z) = 0$ unless $l = k = 0$ and in this case $\bar{\rho}(z)\xi'_0 \xi_0 = \bar{\rho}(z)\eta_0 \eta_0' = \bar{\rho}(z)$. Thus we have $kk((\xi'_l, \eta'_j)) \cdot kk((\xi'_l, \eta'_j)) = 1_{\Lambda_0 \otimes A_1 A_{-1}}$.

Next, we show $T_A \cong_{kk^{alg}} A_0$. Define $j_0 : A_0 \to T_A$ by $j_0(a) = 1 \otimes a$. We show that this inclusion induces an invertible element $kk(j_0) \in kk^{alg}(T_A, A_0)$.

**Lemma 4.9.** There is a quasihomomorphism $(id, Ad(S \otimes 1)) : T_A \Rightarrow T \otimes A \triangleright C$, where

$$
C = \bigoplus_{i,j \in \mathbb{N}} e_{i,j} \otimes A_i A_{-j}.
$$

Here $Ad(S \otimes 1)$ is the restriction of $Ad(S \otimes 1) : T \otimes A \to T \otimes A$ defined by $x \mapsto (S \otimes 1)x(S^* \otimes 1)$.

**Proof.** We have $A_i A_{-j} A_j A_{-k} \subseteq A_i A_{-k}$ because $A_{-j} A_j \subseteq A_0$, therefore $C$ is a subalgebra. To prove that the pair $(id, Ad(S \otimes 1))$ defines a quasihomomorphism we check the conditions on the generators. It is clear that $(1 \otimes A_0)C$, $(S \otimes A_1)C$ and $(S^* \otimes A_{-1})C$ are subsets of $C$. Now we let $a_i \in A_i$ and we check

$$
(id - Ad(S \otimes 1))(1 \otimes a_0) = e \otimes a_0 \in C
$$

$$
(id - Ad(S \otimes 1))(S \otimes a_1) = Se \otimes a_1 \in C
$$

$$
(id - Ad(S \otimes 1))(S^* \otimes a_{-1}) = eS^* \otimes a_{-1} \in C.
$$

Define $i_0 : A_0 \to C$ by $i_0(a) = e_{00} \otimes a$.

**Proposition 4.10.** There is a Morita equivalence between $C$ and $i_0(A_0)$. Therefore there is an invertible element $\kappa \in kk^{alg}(C, A_0)$.
Proof. Using Lemma 3.8 we think of $C$ represented in $T \otimes E$ (where $E = E_1$ if $q = 1$ and $E = E_2$ if $q \neq 1$). The Morita equivalence is given by $\xi_i = \xi_i' = e_{i,0} \otimes U_i^1$ and $\eta_j = \eta_j' = e_{0,j} \otimes U_{i-1}^1$. The proof that these elements determine a Morita equivalence is similar to the proof of Theorem 4.8. The Morita context $((\xi_i), (\eta_j))$ determines a morphism $C \rightarrow K \otimes \eta_i(A_0)$ that in turn determines the element $kk((\xi_i), (\eta_j)) \in kk_0^{alg}(C, A_0)$. We define $\kappa = kk((\xi_i), (\eta_j))kk(i_0)^{-1}$ where $i_0 : A_0 \rightarrow e_{00} \otimes A_0$.

\[ \text{Proposition 4.11. Let } \kappa \in kk(C, A_0) \text{ as in Proposition 4.10, then} \]
\[ kk(j_0)kk(id, Ad(S \otimes 1))\kappa = 1_{A_0}. \]

This implies that $kk(j_0)$ has a right inverse and that $kk(id, Ad(S \otimes 1))$ has a left inverse.

Proof. We have
\[ (id - Ad(S \otimes 1))(j_0(a_0)) = e_{00} \otimes a_0, \]
thus $kk(j_0)kk(id, Ad(S \otimes 1)) = kk(i_0)$. By Proposition 4.10 $kk(i_0)\kappa = 1_{A_0}$. \[ \]

To show that $kk(j_0)$ is invertible, we construct a right inverse for $kk(id, Ad(S \otimes 1))$. In order to do this, we construct a difftopic family of quasihomomorphism between $T_A$ and a subalgebra $\tilde{C}$ of $(T \otimes_A T) \otimes A$ and prove that $\tilde{C}$ is Morita equivalent to $T_A$. To construct this difftopic family we use the difftopy $\phi_t : T \rightarrow T \otimes_A T$ of Lemma 1.16.

Consider the map $\Phi_t = \phi_t \otimes id_A : T \otimes A \rightarrow (T \otimes_A T) \otimes A$ where $\phi_t$ is the difftopy of lemma 1.16. Since $\phi_0(S) = S \otimes 1$, then $\Phi_0(x \otimes a) = x \otimes 1 \otimes a$.

\[ \text{Lemma 4.12. There is a difftopic family of quasihomomorphisms} \]
\[ (\Phi_t, \Phi_0 \circ Ad(S \otimes 1)) : T_A \rightarrow (T \otimes_A T) \otimes A \rightarrow \tilde{C}. \]

Where $\tilde{C}$ is the subalgebra
\[ \bigoplus_{i,j,p,q \in \mathbb{N}} e_{i,j} \otimes S^p S^q \otimes A_{i+p}^j A_{(-j+q)}. \]

Proof. We check that $(\Phi_t, \Phi_0 \circ Ad(S \otimes 1))$ define quasihomomorphisms using the generators of $T_A$. First we note that $\Phi_t(1 \otimes A_0)\tilde{C}, \Phi_t(S \otimes A_1)\tilde{C}$ and $\Phi_t(S^* \otimes A_{-1})\tilde{C}$ are subsets of $\tilde{C}$. Finally, we compute
\[ (\Phi_t - \Phi_0 \circ Ad(S \otimes 1))(1 \otimes a_0) = e \otimes 1 \otimes a_0 \in \tilde{C} \]
\[ (\Phi_t - \Phi_0 \circ Ad(S \otimes 1))(S \otimes a_1) = f(t)(e \otimes S \otimes a_1) + g(t)(Se \otimes 1 \otimes a_1) \in \tilde{C} \]
\[ (\Phi_t - \Phi_0 \circ Ad(S \otimes 1))(S^* \otimes a_{-1}) = \tilde{f}(t)(e \otimes S^* \otimes a_{-1}) + \tilde{g}(t)(eS^* \otimes 1 \otimes a_{-1}) \in \tilde{C}. \]

Define $\eta : T_A \rightarrow \tilde{C}$ as the restriction of the injective morphism $T \otimes A \rightarrow (T \otimes_A T) \otimes A$ given by $\eta(x \otimes a) = e \otimes x \otimes a$.

\[ \text{Proposition 4.13. There is a Morita equivalence between } \tilde{C} \text{ and } \eta(T_A). \text{ Therefore } kk(\eta) \in kk_0^{alg}(T_A, \tilde{C}) \text{ is invertible.} \]

Proof. Using Lemma 3.8 we have an injective morphism $\tilde{C} \rightarrow (T \otimes_A T) \otimes E$ (where $E = E_1$ if $q = 1$ and $E = E_2$ if $q \neq 1$) The Morita equivalence is given by $\xi_i = \xi_i' = e_{i,0} \otimes 1 \otimes U_i^1$ and $\eta_j = \eta_j' = e_{0,j} \otimes 1 \otimes U_{i-1}^1$. The proof is similar to the proof of Theorem 4.8.

\[ \text{Theorem 4.14. } kk(j_0) \in kk_0^{alg}(A_0, T_A) \text{ is invertible.} \]
Proof. By 4.11, we know that $kk(j_0)$ has a right inverse and $kk(id, Ad(S \otimes 1))$ has a left inverse. Now, we prove that $k(id, Ad(S \otimes 1))$ has a right inverse which completes the proof.

Since $\phi_0(S) = S \otimes 1$, then if $a_i \in A_i$ and $a_{-j} \in A_{-j}$, we have

$$\Phi_0(e_{i,j} \otimes a_i a_{-j}) = e_{i,j} \otimes 1 \otimes a_i a_{-j} \in \mathcal{C}$$

and therefore $\Phi_0(C) \subseteq \mathcal{C}$, thus by item (4) of Proposition 2.16 we have

$$kk(id, Ad(S \otimes 1))kk(\Phi_0|_C) = kk(\Phi_0, \Phi_0 \circ Ad(S \otimes 1)).$$

By item (5) of Proposition 2.16 we obtain

$$kk(\Phi_0, \Phi_0 \circ Ad(S \otimes 1)) = kk(\Phi_1, \Phi_0 \circ Ad(S \otimes 1)).$$

We have $\phi_1(S) = S^2 S^* \otimes 1 + e \otimes S$ and therefore $\Phi_1 - \Phi_0 \circ Ad(S \otimes 1) = \eta$. By item (2) of Proposition 2.16 $kk(\Phi_1, \Phi_0 \circ Ad(S \otimes 1)) = kk(\eta)$ and by Lemma 4.13 $kk(\eta)$ is invertible.

With the isomorphisms in $\mathbf{R}\mathbf{A}^{alg}$ from Theorems 4.8 and 4.11 we construct the desired exact triangle.

**Theorem 4.15.** For a generalized Weyl algebra $A = \mathbb{C}[h](\sigma, P(h))$ with $P$ a non constant polynomial and

- $q = 1$ and $h_0 \neq 0$ or
- $q$ is not a root of unity and $P$ has a root different from $\frac{h_0}{1-q}$

there is an exact triangle

$$SA \to A_1 A_{-1} \to A_0 \to A.$$

Proof. The linearly split extension

$$0 \to \Lambda_A \stackrel{i}{\to} \mathcal{T}_A \stackrel{p}{\to} A \to 0 \tag{4.2}$$

yields an exact triangle

$$SA \xrightarrow{kk(E)} \Lambda_A \xrightarrow{kk(i)} \mathcal{T}_A \xrightarrow{kk(p)} A,$$

where $kk(E) \in kk_{alg}(A, \Lambda_A) = kk_{alg}(SA, A)$ is the element defined by the extension (4.2).

By Theorem 4.8 the inclusion $j_1 : A_1 A_{-1} \to \Lambda_A$ defined by $j_1(x) = e_{00} \otimes x$ induces an invertible element $kk(j_1) \in kk_{alg}^{alg}(A_1 A_{-1}, \Lambda_A)$. By Theorem 4.14 the inclusion $j_0 : A_0 \to \mathcal{T}_A$ defined by $j_0(a) = 1 \otimes a$ induces an invertible element $kk(j_0) \in kk_{alg}(A_0, \mathcal{T}_A)$. We define $\phi$ by the commutative diagram in $\mathbf{R}\mathbf{A}^{alg}$

$$\begin{array}{ccc}
\Lambda_A & \xrightarrow{kk(i)} & \mathcal{T}_A \\
kk(j_1) \downarrow & & \downarrow kk(j_0)^{-1} \\
A_1 A_{-1} & \xrightarrow{\phi} & A_0
\end{array}$$

and claim that

$$\phi = kk(i) - kk(\sigma). \tag{4.3}$$

For this we use Proposition 4.11 to obtain

$$kk(j_0)^{-1} = kk(id, Ad(S \otimes 1))$$

and therefore

$$kk(j_1)kk(\iota)kk(j_0)^{-1} = kk(j_1)kk(\iota)kk(id, Ad(S \otimes 1))^\kappa.$$
By the Chinese Reminder Theorem, there is an isomorphism $\phi$. Proof. Let $h$ since $(\phi, \psi) : A_{1}A_{-1} \Rightarrow T \otimes A \otimes C$, where $\phi(x) = e_{00} \otimes x$ and $\psi(x) = e_{11} \otimes x$. Since $\phi$ and $\psi$ are orthogonal, we obtain $kk(\phi, \psi) = kk(\phi) - kk(\psi)$. Now we multiply this difference by $\kappa$ which is given by the Morita equivalence of Proposition 4.10. Thus we have that $kk(\phi)\kappa$ and $kk(\psi)\kappa$ are determined by maps $A_{1}A_{-1} \rightarrow C \rightarrow K \otimes A_{0}$ that send $x \mapsto e_{00} \otimes x$ and $x \mapsto e_{00} \otimes \rho^{-1}(U_{1}R(G)U_{1}) = e_{00} \otimes R(\sigma(h))$ (here we use the representation $\rho$ of Lemma 3.8). Thus we conclude $\phi = kk(i) - kk(\sigma)$, proving (4.3).

Now we prove that $\phi = 0$. Both $i_{1}$ and $\sigma$ factor through a contractible subalgebra of $C[h]$. This is because we have $i_{1}(A_{1}A_{-1}) = P(h)C[h]$ and $\sigma(A_{1}A_{-1}) = P(\sigma(h))C[h]$ and the polynomials $P(h)$ and $P(\sigma(h))$ have some linear factors $L(h)$ and $L(\sigma(h))$. Thus the morphisms $i_{1}$ and $\sigma$ factor through the subalgebras $L(h)C[h]$ and $L(\sigma(h))C[h]$ which are contractible. Therefore we have $kk(i_{1}) = kk(\sigma) = 0$.]

**Lemma 4.16.** Let $(\Sigma, \Sigma)$ be a triangulated category. If there is an exact triangle

$$
\Sigma X \rightarrow Y \rightarrow Z \rightarrow X
$$

then $X \cong Z \oplus \Sigma^{-1}Y$.

**Proof.** See Corollary 1.2.7 in [13].

Now we compute the isomorphism class of $A_{1}A_{-1}$ in $\mathfrak{R}^{alg}$.

**Proposition 4.17.** Let $A = C[h](\sigma, P)$ where $P$ is a nonconstant polynomial with $r$ different roots, then

$$
A_{1}A_{-1} \cong_{\mathfrak{R}^{alg}} C^{r-1}.
$$

**Proof.** Let $P(h) = c(h - h_{1})^{n_{1}} \ldots (h - h_{r})^{n_{r}}$. Without loss of generality we can assume $c = 1$. Since $A_{1}A_{-1} = (P(h))$ we have a linearly split extension

$$
0 \rightarrow A_{1}A_{-1} \rightarrow C[h] \overset{\sigma}{\rightarrow} C[h]/(P(h)) \rightarrow 0.
$$

By the Chinese Reminder Theorem, there is an isomorphism

$$
\phi : C[h]/(P(h)) \rightarrow \prod_{i=1}^{r} C[h]/(h - h_{i})^{n_{i}}.
$$

We have the following commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & (h - h_{i})^{n_{i}}C[h] \\
\downarrow & & \downarrow \mu_{i} \\
0 & \longrightarrow & (h - h_{i})C[h] \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
C[h] & \longrightarrow & C[h]/(h - h_{i})^{n_{i}} \\
\mu_{i} & \longrightarrow & \mu_{i} \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & 0 \\
C & \longrightarrow & C \\
& & \mu_{i} \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & 0
\end{array}
$$

Since $(h - h_{i})^{n_{i}}C[h]$ and $(h - h_{i})C[h]$ are contractible, $kk(q_{i})$ and $kk(\sigma_{h_{i}})$ are invertible, therefore $kk(\mu_{i}) \in kk_{alg}(C[h]/(h - h_{i})^{n_{i}}, C)$ is invertible. By the additivity of $\mathfrak{R}^{alg}$, the homomorphism $\mu : \prod_{i=1}^{r} C[h]/(h - h_{i})^{n_{i}} \rightarrow C^{r}$ given by $\mu_{i}$ in the $i$-th component induces an invertible element $kk(\mu)$. Note that $\mu \circ \pi : C[h] \rightarrow C^{r}$ is given by $\sigma_{h_{i}}$ in the $i$-th component.
Since all evaluation maps \( ev_{h_0} \) induce the same \( kk^{alg} \)-isomorphism \( kk(ev_0) \) in \( kk^{alg}(\mathbb{C}[h], \mathbb{C}) \), we have the commutative diagram in \( \mathfrak{R}^{alg} \)

\[
\begin{array}{ccc}
\mathbb{C}[h] & \xrightarrow{kk(\pi)} & \mathbb{C}[h]/P(h) \\
kk(ev_0) \downarrow & & \downarrow kk(\mu) \\
\mathbb{C} & \xrightarrow{kk(\Delta)} & \mathbb{C}^r
\end{array}
\]

where \( \Delta : \mathbb{C} \to \mathbb{C}^r \) is the diagonal morphism \( \Delta(1) = (1, \ldots, 1) \). Replacing \( \mathbb{C}[h] \) by \( \mathbb{C} \) and \( \mathbb{C}[h]/(P(h)) \) by \( \mathbb{C}^r \) in the exact triangle corresponding to (4.4), we obtain an exact triangle in \( \mathfrak{R}^{alg} \)

\[
SC^r \to A_1 A_{-1} \to \mathbb{C} \xrightarrow{kk(\Delta)} \mathbb{C}^r.
\]  

(4.5)

The linearly split extension \( 0 \to \mathbb{C} \xrightarrow{\Delta} \mathbb{C}^r \to \mathbb{C}^{r-1} \to 0 \) yields an exact triangle

\[
SC^{r-1} \to C \xrightarrow{kk(\Delta)} \mathbb{C}^r \to \mathbb{C}^{r-1}.
\]

Permuting this triangle we obtain the exact triangle

\[
SC^r \to SC^{r-1} \to \mathbb{C} \xrightarrow{kk(\Delta)} \mathbb{C}^r.
\]  

(4.6)

Since both triangles (4.5) and (4.6) complete the morphism \( kk(\Delta) : \mathbb{C} \to \mathbb{C}^r \), by the axiom TR3 of triangulated categories we have \( A_1 A_{-1} \cong_{\mathfrak{R}^{alg}} SC^{r-1} \).

**Theorem 4.18.** Let \( A = \mathbb{C}[h](\sigma, P(h)) \) be generalized Weyl with \( \sigma(h) = qh + h_0 \) and \( P \) a non constant polynomial such that

- \( q = 1 \) and \( h_0 \neq 0 \) or
- \( q \) is not a root of unity and \( P \) has a root different from \( \frac{h_0}{1-q} \),

then \( A \cong_{\mathfrak{R}^{alg}} \mathbb{C}^r \).

**Proof.** The result follows from Theorem 4.15, Lemma 4.16 and Proposition 4.17.

**Corollary 4.19.** Let \( A \) be as in Theorem 4.18. Then \( A \cong \mathbb{C}^r \) in \( \mathfrak{R}^{alg} \) and so

\[
kk_0^{\mathcal{L}_p}(\mathcal{C}, A) = \mathbb{Z}^r \quad \text{and} \quad kk_1^{\mathcal{L}_p}(\mathcal{C}, A) = 0.
\]

Corollary 4.19 implies \( K_0(A \otimes \pi \mathcal{L}_p) = \mathbb{Z}^r \). This is compatible with Theorem 4.5 of [12], which computes \( K_0(A) = \mathbb{Z}^r \) for \( A = \mathbb{C}[h](\sigma, P) \) when \( \sigma(h) = h - 1 \) and \( P \) has \( r \) simple roots.

**Examples.** We apply Theorem 4.18 in the following cases.

1. The quantum Weyl algebra \( A_q \) with \( q \neq 1 \) not a root of unity, is isomorphic to \( \mathbb{C} \) in \( \mathfrak{R}^{alg} \).
2. In the case of the primitive factors \( B_\lambda \) of \( U(\mathfrak{sl}_2) \), we have \( P(h) = -h(h+1) - \lambda/4 \). If \( \lambda = 1 \), then \( B_\lambda \cong \mathbb{C} \) in \( \mathfrak{R}^{alg} \). If \( \lambda \neq 1 \), then \( B_\lambda \cong \mathbb{C}^2 \) in \( \mathfrak{R}^{alg} \). This implies \( kk_0^{\mathcal{L}_p}(\mathcal{C}, B_\lambda) = \mathbb{Z} \oplus \mathbb{Z} \) and \( kk_1^{\mathcal{L}_p}(\mathcal{C}, B_\lambda) = 0 \).
3. The quantum weighted projective line \( \mathcal{O}(\mathbb{WP}_q(k, l)) \) is isomorphic to \( \mathbb{C}[h](\sigma, P) \) with \( \sigma(h) = q^{2l}h \) and

\[
P(h) = h^k \prod_{i=0}^{l-1} (1 - q^{-2i}h).
\]
In the case $q \neq 1$ is not a root of unity, we have $\mathcal{O}(\mathbb{WP}_q(k, l)) \cong \mathbb{C}^{l+1}$ in $\mathfrak{AR}^{alg}$. This implies $kk^{L_0}_0(\mathbb{C}, \mathcal{O}(\mathbb{WP}_q(k, l))) = \mathbb{Z}^{l+1}$ and $kk^{L_1}_1(\mathbb{C}, \mathcal{O}(\mathbb{WP}_q(k, l))) = 0$. (Compare with Corollary 5.3 of [1].)

In the case where $q$ is not a root of unity and $P$ has only $\frac{h_0}{1-q}$ as a root we have the following result.

**Proposition 4.20.** The generalized Weyl algebra $A = \mathbb{C}[h](\sigma, P(h))$, with $\sigma(h) = qh + h_0$ such that $q \neq 1$ and $P$ has only $\frac{h_0}{1-q}$ as a root, is isomorphic to $\mathbb{C}$ in $\mathfrak{AR}^{alg}$.

**Proof.** By Proposition 3.35, $A$ is isomorphic to $\mathbb{C}[h](\sigma_1, P_1)$ with $\sigma_1(h) = qh$ and $P_1(h) = ch^n$ with $c \in \mathbb{C}^*$ and $n \geq 1$. The algebra $\mathbb{C}[h](\sigma_1, P_1)$ is $\mathbb{N}$ graded with $\deg h = 2$, $\deg x = n$ and $\deg y = n$. To prove this we check that the defining relations

$$xh = qhx, \ yh = q^{-1}hy, \ yx = ch^n \text{ and } xy = cq^n h^n$$

are compatible with the grading.

The result follows from Lemma 1.7, since the degree 0 subalgebra of $A$ is equal to $\mathbb{C}$. \hfill \blacksquare

**Example.** The quantum plane $\mathbb{C}[h](\sigma, h)$ with $\sigma(h) = qh$ is isomorphic to $\mathbb{C}$ in $\mathfrak{AR}^{alg}$.

### 4.3. The case where $P$ is a constant polynomial.

If $P$ is a nonzero constant polynomial, then $A = \mathbb{C}[h](\sigma, P)$ is a tame smooth generalized crossed product and we can apply the results from [10].

**Proposition 4.21.** Let $A = \mathbb{C}[h](\sigma, P)$ where $P \neq 0$ is a constant polynomial, then $A \cong SC \times \mathbb{C}$ in $\mathfrak{AR}^{alg}$. This implies $A \cong SC \times \mathbb{C}$ in $\mathfrak{AR}^{alg}$ and therefore we have $kk^{L_0}_0(\mathbb{C}, A) = \mathbb{Z}$ and $kk^{L_1}_1(\mathbb{C}, A) = \mathbb{Z}$.

**Proof.** In this case $A$ is a tame smooth generalized crossed product with frame $\xi_i = y^i$ and $\bar{\xi}_i = x^i$ for $i \in \mathbb{N}$. This frame satisfies the conditions of Definition 18 in [10], therefore we have a linearly split extension

$$0 \to \Lambda_A \xrightarrow{\iota} T_A \xrightarrow{\bar{\iota}} A \to 0,$$

that yields an exact triangle

$$SA \xrightarrow{kk(E)} \Lambda_A \xrightarrow{kk(\iota)} T_A \xrightarrow{kk(\bar{\iota})} A.$$

By Theorem 27 of [10], $j_1 : \mathbb{C}[h] \to \Lambda_A$, defined by $j_1(x) = e_{00} \otimes x$ induces an invertible element $kk(j_1)$ and by Theorem 33 of [10], $j_0 : \mathbb{C}[h] \to T_A$ defined by $j_0(x) = 1 \otimes x$ induces an invertible element $kk(j_0)$. We have a commutative diagram in $\mathfrak{AR}^{alg}$

$$\begin{array}{ccc}
\Lambda_A & \xrightarrow{kk(\iota)} & T_A \\
\downarrow{kk(j_1)} & & \downarrow{kk(j_0)} \\
\mathbb{C}[h]kk(j_1) & \xrightarrow{\alpha} & \mathbb{C}[h].
\end{array}$$

We prove that $\alpha = 1_{\mathbb{C}[h]} - kk(\sigma)$ and that $1_{\mathbb{C}[h]} = kk(\sigma)$, thus concluding that $\alpha = 0$. By Theorem 33 of [10], we have $kk(j_0)^{-1} = kk(id, Ad(S \otimes 1))kk(j_1)^{-1}$. The product $kk(j_1)kk(\iota)kk(1, Ad(S \otimes 1))$ corresponds to a quasihomomorphism

$$(\phi, \psi) : \mathbb{C}[h] \Rightarrow T \otimes A \triangleright \mathcal{C},$$
where \( \phi(Q) = e_{00} \otimes Q \) and \( \psi(Q) = e_{11} \otimes Q \) for all \( Q \in \mathbb{C}[h] \). Since \( \phi \) and \( \psi \) are orthogonal \( \langle \phi, \psi \rangle = \langle \phi \rangle - \langle \psi \rangle \). We now compose \( \langle \phi \rangle \) and \( \langle \psi \rangle \) with \( \langle \mu \rangle \). Theorem 27 of [10] characterizes \( \langle \mu \rangle \) as given by a Morita equivalence defined by

\[
\Xi_i = S^i \otimes y^i \quad \text{and} \quad \Xi_i = S^i \otimes x^i.
\]

therefore \( \langle \mu \rangle \) is defined by the morphism \( Q \mapsto Q \) and \( \langle \mu \rangle \) is defined by \( Q \mapsto xQy = \sigma(Q) \). This implies that \( \alpha = 1_\mathbb{C}[h] - \langle \mu \rangle \).

The commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}[h] & \xrightarrow{\sigma} & \mathbb{C}[h] \\
\downarrow{\text{ev}_0} & & \downarrow{\text{ev}_0} \\
\mathbb{C} & \xrightarrow{\text{id}} & \mathbb{C},
\end{array}
\]

implies that \( \langle \mu \rangle = 1_\mathbb{C}[h] \) and thus \( \alpha = 0 \).

This implies the existence of an exact triangle in \( \mathfrak{R}^{alg} \)

\[
\mathbb{A} \to \mathbb{C} \to \mathbb{C} \to \mathbb{A}.
\]

Using Lemma [4.16] we obtain \( A \cong S\mathbb{C} \oplus \mathbb{C} \) in \( \mathfrak{R}^{alg} \).

In the case where \( P = 0 \) we have the following result.

**Proposition 4.22.** The generalized Weyl algebra \( A = \mathbb{C}[h] \langle \sigma, P(h) \rangle \) with \( P = 0 \) is isomorphic to \( \mathbb{C} \) in \( \mathfrak{R}^{alg} \).

**Proof.** The relations

\[
xh = \sigma(h)x, \; yh = \sigma^{-1}(h)y, \; yx = 0 \quad \text{and} \quad xy = 0
\]

are compatible with the grading determined by \( \deg h = 0, \deg x = 1 \) and \( \deg y = 1 \), therefore the algebra \( A \) is \( \mathbb{N} \)-graded. The result follows from Lemma [1.7] and the fact that the degree 0 subalgebra of \( A \) is equal to \( \mathbb{C}[h] \).

**References**

[1] Beatriz Abadie, Søren Eilers, and Ruy Exel, *Morita equivalence for crossed products by Hilbert C*-bimodules*, Trans. Amer. Math. Soc. 350 (1998), no. 8, 3043–3054.

[2] V. V. Bavula and D. A. Jordan, *Isomorphism problems and groups of automorphisms for generalized Weyl algebras*, Trans. Amer. Math. Soc. 353 (2001), no. 2, 769–794.

[3] Tomasz Brzeziński, *Circle and line bundles over generalized Weyl algebras*, Algebr. Represent. Theory 19 (2016), no. 1, 57–69.

[4] Tomasz Brzeziński and Simon A. Fairfax, *Quantum teardrops*, Comm. Math. Phys. 316 (2012), no. 1, 151–170.

[5] Joachim Cuntz, *Generalized homomorphisms between C*-algebras and KK-theory*, Dynamics and processes (Bielefeld, 1981), Lecture Notes in Math., vol. 1031, Springer, Berlin, 1983, pp. 31–45.

[6] ———, *Bivariante K-Theorie für lokalkonveexe Algebren und der Chern-Connes-Charakter*, Doc. Math. 2 (1997), 139–182 (German, with English summary).

[7] ———, *Bivariant K-theory and the Weyl algebra*, K-Theory 35 (2005), no. 1-2, 93–137.

[8] Joachim Cuntz, Ralf Meyer, and Jonathan M. Rosenberg, *Topological and bivariant K-theory*, Oberwolfach Seminars, vol. 36, Birkhäuser Verlag, Basel, 2007.

[9] Joachim Cuntz and Andreas Thom, *Algebraic K-theory and locally convex algebras*, Math. Ann. 334 (2006), no. 2, 339–371.

[10] Olivier Gabriel and Martin Grensing, *Six-term exact sequences for smooth generalized crossed products*, J. Noncommut. Geom. 7 (2013), no. 2, 499–524.
[11] Martin Grensing, *Universal cycles and homological invariants of locally convex algebras*, J. Funct. Anal. **263** (2012), no. 8, 2170–2204.

[12] Timothy J. Hodges, *Noncommutative deformations of type-A Kleinian singularities*, J. Algebra **161** (1993), no. 2, 271–290.

[13] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[14] Lionel Richard and Andrea Solotar, *Isomorphisms between quantum generalized Weyl algebras*, J. Algebra Appl. **5** (2006), no. 3, 271–285.

[15] François Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967.

[16] Manuel Valdivia, *Topics in locally convex spaces*, North-Holland Mathematics Studies, vol. 67, North-Holland Publishing Co., Amsterdam-New York, 1982. Notas de Matemática [Mathematical Notes], 85.

Instituto de Matemática y Ciencias Afines (IMCA) Calle Los Biólogos 245. Urb San César. La Molina, Lima 12, Perú.

E-mail address: julio.gutierrez@imca.edu.pe

Pontificia Universidad Católica del Perú, Sección Matemáticas, PUCP, Av. Universitaria 1801, San Miguel, Lima 32, Perú.

E-mail address: cvalqui@pucp.edu.pe