Bethe Ansatz Equations

for

the Broken $Z_N$-Symmetric Model

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Abstract

We obtain the Bethe Ansatz equations for the broken $Z_N$-symmetric model by constructing a functional relation of the transfer matrix of $L$-operators. This model is an elliptic off-critical extension of the Fateev-Zamolodchikov model. We calculate the free energy of this model on the basis of the string hypothesis.

KEY WORDS: $Z_2$-symmetry; broken $Z_N$-symmetric model; transfer matrices; functional relation; Bethe Ansatz; string hypothesis; free energy.

1 Introduction

In the two dimensional solvable lattice models with Ising-like edge interaction, the star-triangle relation

\begin{equation}
\rho W(a, b|v, w) \overline{W}(a, c|u, w) W(b, c|u, v) = \sum_d \overline{W}(a, d|u, v) W(d, b|u, w) \overline{W}(d, c|v, w)
\end{equation}
\[ \rho = \rho(u, v, w) \text{ independent of } a, b \text{ and } c \]

plays a central role. In (1.1), the summation on \( d \) is taken over all local states. These are the relations among the two Boltzmann weights \( W(a, b|u, v) \) and \( \mathcal{W}(a, b|u, v) \). They live on the edges in two different directions of the two dimensional planar lattice. The local state variables \( a \) and \( b \) live on the sites. We denote the spectral parameters by \( u \) and \( v \).

\( (\text{Fig. 1 and Fig. 2}) \)

Since Fateev and Zamolodchikov \[1\] obtained an \( N \)-state generalization of the critical Ising model as a solution of the star-triangle relation (STR), there have been known two different off-critical extensions of this model. One is the chiral Potts model \[2\] and the other is the broken \( Z_N \)-symmetric model. Both are Ising-type edge interaction models. The STR for the chiral Potts model was proved in \[3\][4]. Though this model has been still under investigation in \[5\] – \[10\], the lack of a difference-variable parameterization in this model causes difficulties in analysis. Kashiwara and Miwa \[11\] proposed the broken \( Z_N \)-symmetric model, and Hasegawa and Yamada \[12\] proved the STR for this model. Unfortunately the proof in \[11\] was wrong because of the incorrectness of the “ICU lemma” in their paper.

In this Paper, we study the eigenvalues of the transfer matrix \( \Phi(u, v, w) \) of the broken \( Z_N \)-symmetric model,

\[ \Phi(u, v, w)^{b_0b_1\ldots b_{M-1}}_{a_0a_1\ldots a_{M-1}} = \prod_{j=0}^{M-1} W(b_j, a_j|v-w) W(a_j, b_{j+1}|u-w), \quad (1.2) \]

and calculate the free energy of this model.

\( (\text{Fig. 3}) \)

The local state variables take their values in \( Z/NZ \). Throughout the Paper, we deal with the case of \( N \) odd, \( N = 2n + 1 \). The \( Z_2 \)-symmetry of the Boltzmann weights

\[ W(a, b|u) = W(N-a, N-b|u), \quad \mathcal{W}(a, b|u) = \mathcal{W}(N-a, N-b|u), \quad (1.3) \]
ensures that the eigenvalue \( r = \pm 1 \) of the spin reversal operator \( R \) is a good quantum number, where \( R \in \text{End} \left( \left( \mathbb{C}^N \right)^\otimes M \right) \) is defined by

\[
R = R \otimes R \otimes \cdots \otimes R, \quad R v_j^{(N)} = v_{N-j}^{(N)},
\]

which satisfies \( R^2 = 1 \). The vectors \( v_j^{(N)} \) \(( j \in \mathbb{Z}/N\mathbb{Z})\) constitute an orthonormal basis in \( \mathbb{C}^N \). In the homogeneous case \( u = v \), we show first that any eigenvalue \( \varphi(u) \) of \( \Phi(u) = \Phi(u,u,0) \) can be written as

\[
\varphi(u) = \left( \frac{p(0)p(\lambda)}{p(u)p(\lambda - u)} \right)^M \prod_{j=1}^{2nM} \frac{\theta_1(u - u_j|\tau/2)}{\theta_1(u_j|\tau/2)},
\]

\[
p(u) = \prod_{j=1}^{n} \theta_2(u - (2j - 1)\eta|\tau/2), \quad \eta = \frac{n}{N}, \quad \lambda = \frac{1}{2} - \eta.
\]

See Appendix A for the notation of the theta functions. The zeros \( \{u_1, \cdots, u_{2nM}\} \) of \( \varphi(u) \) are described as follows

\[
\left( \frac{\theta_1(v_k + \lambda/2|\tau/2)}{\theta_1(v_k - \lambda/2|\tau/2)} \right)^{2M} = (-1)^{M+1} \prod_{j=1}^{2nM} \frac{\theta_1(v_k - v_j + \eta|\tau/2)}{\theta_1(v_k - v_j - \eta|\tau/2)},
\]

\[
v_k = u_k - \frac{\lambda}{2} \quad \text{for} \quad k = 1, \cdots, 2nM,
\]

\[
\sum_{j=1}^{2nM} v_j \equiv \frac{1 - r}{4} \mod (\mathbb{Z} \oplus \frac{\tau}{2}\mathbb{Z}).
\]

We call the equation (1.7) the Bethe Ansatz equation. The condition (1.9) follows from the double periodicity of \( \varphi(u) \) discussed in Section 4. We obtain the Bethe Ansatz equations above through a functional relation (1.13) for the transfer matrix of \( L \)-operators. These \( L \)-operators \( L(u) \in \text{End} \left( \mathbb{C}^N \otimes \mathbb{C}^2 \right) \) were originally constructed by Sklyanin \[13\][14] as a solution to the relation,

\[
L^{01}(u - v) L^{02}(u - w) R_{8V}^{12}(v - w)
\]

\[
= R_{8V}^{12}(v - w) L^{02}(u - w) L^{01}(u - v) \quad \text{on} \quad \mathbb{C}^N \otimes \mathbb{C}^2 \otimes \mathbb{C}^2,
\]

\((\text{Fig. 4})\)

where the upper indices 0, 1 and 2 mean that \( L^{ij}(u) \) acts only on the \( i \)-th and \( j \)-th components of \( \mathbb{C}^N \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) and as identity on the other component. We denote
the $R$-matrix of the eight-vertex model by $R_{8V}(u)$ \cite{15} \cite{16}. We consider the transfer matrix $\mathcal{L}(u)$ of these $L$-operators,

$$\mathcal{L}(u) = tr_{C^2}(L^0 L^1 \cdots L^{M-1}(u)), \quad (1.11)$$

$$L^0 L^1 \cdots L^{M-1}(u) \in \text{End}(\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N \otimes \mathbb{C}^2). \quad (1.12)$$

We derive the functional relation

$$\mathcal{L}(\lambda - u - 1/4) \Phi(u) = C(u)^M \left( f(u)^M \Phi(u - \eta) + (-f(\lambda - u))^M \Phi(u + \eta) \right), \quad (1.13)$$

$$f(u) = \theta_1(2u|\tau) \frac{\theta_1(u + \eta|\tau/2)}{\theta_2(u|\tau/2)}, \quad (1.14)$$

$$C(u) = [\theta_2 \theta_3 \theta_4](0|\tau) \prod_{j=1}^n \frac{\theta_1(u - 2(j-1)\eta|\tau/2) \theta_2(u + (2j-1)\eta|\tau/2)}{\theta_1(u + 2j\eta|\tau/2) \theta_2(u - (2j-1)\eta|\tau/2)}, \quad (1.15)$$

by the method which Baxter employed to solve the eight-vertex model \cite{15} \cite{16} \cite{17}. The functional relations corresponding in the chiral Potts model and in the RSOS model associated with the eight-vertex were obtained in \cite{18} and \cite{19}, respectively.

We calculate the free energy of the broken $\mathbb{Z}_N$-symmetric model under the hypothesis that the solution of the Bethe Ansatz equations corresponding to the ground state consists of “strings of length $N - 1$”. We show that, in the infinite lattice limit, the centers of these strings are distributed on the imaginary axis with the density $\rho(w)$,

$$\rho(w) = 2N [\theta_2 \theta_3](0|N\tau) \frac{\theta_3(2\sqrt{-1}Nw|N\tau)}{\theta_2(2\sqrt{-1}Nw|N\tau)}, \quad -\frac{\kappa}{4} \leq w < \frac{\kappa}{4}, \quad (1.16)$$

where $\tau = \sqrt{-1}\kappa$, and the free energy per site is

$$F(u) = -\sum_{l=1}^{\infty} \frac{\sinh \left( \frac{2\pi l}{\kappa} u \right) \sinh \left( \frac{2\pi l}{\kappa} \left( \frac{1}{2N} - u \right) \right) \sinh \left( \frac{2\pi l}{N\kappa} n \right)}{l \cosh \left( \frac{\pi l}{\kappa} \right) \cosh^2 \left( \frac{\pi l}{N\kappa} \right)}. \quad (1.17)$$

The last expression agrees with the result of Jimbo, Miwa and Okado \cite{20} obtained by the use of the inversion-trick, and in the trigonometric limit of $\kappa \to \infty$ it recovers the result of Fateev and Zamolodchikov \cite{1} and Albertini \cite{10}.

The organization of this Paper is as follows. In Section 2, we review necessary facts about the $R$-matrix of the eight-vertex model, Sklyanin’s $L$-operator and the
broken \( Z_N \)-symmetric model. We derive the functional relation (1.13) in Section 3. After showing commutation relations among \( \Phi(u), \mathcal{L}(v) \) and \( \mathcal{R} \), we obtain the Bethe Ansatz equations in Section 4. We calculate the free energy of the model under the string hypothesis in Section 5. Finally, in Section 6, we conclude with a brief discussion. We fix the notation and list the formulae for theta functions in Appendix A. Miscellaneous properties of the Boltzmann weights are summarized in Appendix B. We devote Appendix C to the proof of the commutativity between \( \Phi \) and \( \mathcal{L} \).

## 2 Review of the Broken \( Z_N \) Symmetric Model

We fix the notation for matrices. We denote the vector \( t(0, \cdots, 1, \cdots, 0) \) in \( \mathbb{C}^m \) by \( v(0, m) \), \( j = 0, 1, \cdots, m - 1 \), and the matrix elements of \( A \in \text{End}(\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_l}) \) by
\[
A v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_l} = \sum_{j_1=0}^{m_1-1} \sum_{j_2=0}^{m_2-1} \cdots \sum_{j_l=0}^{m_l-1} v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_l} A_{i_1 j_1 \cdots i_l}^{j_1 j_2 \cdots j_l}.
\]
In [13][14], Sklyanin constructed the \( \mathcal{L} \)-operators \( \mathcal{L}(u) \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^2) \) for the eight vertex model satisfying (1.10). The \( \mathcal{R} \)-matrix of the eight-vertex model \( \mathcal{R}^{8V}(u) \) \cite{15} \cite{16} is a solution of the Yang-Baxter equation,
\[
R^{01}_{8V}(u - v) R^{02}_{8V}(u - w) R^{12}_{8V}(v - w) = R^{12}_{8V}(v - w) R^{02}_{8V}(u - w) R^{01}_{8V}(u - v) \quad \text{on} \quad \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2. \tag{2.1}
\]
Its non-zero matrix elements are
\[
R^{00}_{00}(u) = R^{11}_{11}(u) = [\theta_2 \theta_3](\eta)[\theta_2 \theta_3](u)[\theta_1 \theta_4](u + \eta),
\]
\[
R^{01}_{01}(u) = R^{10}_{10}(u) = [\theta_2 \theta_3](\eta)[\theta_1 \theta_4](u)[\theta_2 \theta_3](u + \eta),
\]
\[
R^{01}_{10}(u) = R^{10}_{01}(u) = [\theta_1 \theta_4](\eta)[\theta_2 \theta_3](u)[\theta_2 \theta_3](u + \eta),
\]
\[
R^{10}_{11}(u) = R^{11}_{00}(u) = [\theta_1 \theta_4](\eta)[\theta_1 \theta_4](u)[\theta_1 \theta_4](u + \eta).
\]
(Fig. 5)

The other elements not specified above are all zero. Here we denote \( \theta_1(u) \theta_4(u) \) by \( [\theta_1 \theta_4](u) \) for short. We usually suppress the elliptic modulus \( \tau \). When \( \eta = \frac{n}{N} \),
$N = 2n + 1$, the $L$-operators $L(u)$ in (1.10) have a “cyclic” representation. In this representation, $L(u)$ factorizes elementwise as

$$L_{ai}^{bj}(u) = K_{ia}^{b}(u) K_{ja}^{c}(u), \tag{2.2}$$

where

$$K_{ja}^{b}(u) = G^{-1}_{a} G^{-1}_{b} K_{ja}^{c}(u), \quad G_{a} = \left( \frac{\theta_{4}(2a\eta)}{\theta_{4}(0)} \right)^{1/2} \tag{2.3}$$

for $j = 0, 1, a, b = 0, 1, \cdots, N - 1$ and $\sigma = \pm 1$.

(Fig. 6)

The factors $K_{ia}^{b}(u)$ and $K_{ja}^{c}(u)$ are zero unless $|a - b| = 1$. We can identify these $K(u)$’s as the intertwining vectors appearing in the vertex-face correspondence [21] –[26]. Even in the Fateev-Zamolodchikov model these $K(u)$’s are different from the 3-spin object $V$’s in [3] by definition. Their $V$’s are defined by the Fourier transformed images of the product of two Boltzmann weights. These two objects, however, should have an intimate relationship, because the transfer matrix $L$ in the Fateev-Zamolodchikov model is also constructed from $V$’s [4].

Under the $\mathbb{Z}_{2}$-transformation which sends $a$ to $N - a$, they change as

$$K_{ja}^{a}(u) = (-1)^{(j+\sigma)/2} \left[ \theta_{1+j}\theta_{4-j} \right](u - \sigma a\eta + \frac{1}{4}), \tag{2.4}$$

They satisfy the unitarity relations [22]

$$\sum_{j=0}^{1} G_{a} K_{ja}^{b}(u + \lambda) K_{ja}^{c}(u) = \delta_{ac} \left[ \theta_{2}\theta_{3}\theta_{4} \right](0) G_{b} \theta_{2}(2u), \tag{2.6}$$

$$\sum_{a=0}^{N-1} G_{a} K_{ja}^{b}(u + \lambda) K_{ja}^{c}(u) = \delta_{ij} \left[ \theta_{2}\theta_{3}\theta_{4} \right](0) G_{b} \theta_{2}(2u), \tag{2.7}$$

$$\sum_{b=0}^{N-1} G_{b} K_{ja}^{b}(u) K_{ja}^{b}(u + \lambda) = \delta_{ij} \left[ \theta_{2}\theta_{3}\theta_{4} \right](0) G_{a} \theta_{2}(2u). \tag{2.8}$$

(Fig. 7)
In [12], we determined the Boltzmann weights $W$ and $\overline{W}$ of the broken $\mathbb{Z}_N$-symmetric model by the relations

$$W(a, b|u, v) \sum_{j=0}^{1} K^j_{\bar{a}}(u - w) K^d_j(v - w)$$

$$= \sum_{j=0}^{1} K^j_{\bar{a}}(v - w) K^d_j(v - w) W(c, d|u, v),$$

(2.9)

$$\sum_{b=0}^{N-1} \overline{W}(a, b|u, v) K^e_i(u - w) K^{2c}_b(v - w)$$

$$= \sum_{b=0}^{N-1} K^e_{\bar{a}}(v - w) K^{2d}_i(u - w) \overline{W}(b, c|u, v).$$

(2.10)

(Fig. 8 and Fig. 9)

From the above relations and (2.5), we have the $\mathbb{Z}_2$-symmetry

$$W(a, b|u, v) = W(N - a, N - b|u, v), \quad \overline{W}(a, b|u, v) = \overline{W}(N - a, N - b|u, v).$$

(2.11)

The equation (2.9) implies that $W(a, b|u, v) = W(a, b|u - v)$ and that

$$\frac{W(a + 1, b + 1|u)}{W(a, b|u)} = \frac{\theta_3(u + (a + b + 1)\eta)}{\theta_3(u - (a + b + 1)\eta)},$$

$$\frac{W(a + 1, b - 1|u)}{W(a, b|u)} = \frac{\theta_2(u + (a - b + 1)\eta)}{\theta_2(u - (a - b + 1)\eta)}.$$ 

(2.12)

The crossing symmetry

$$\overline{W}(a, b|u) = G_a G_b W(a, b|\lambda - u)$$

(2.13)

holds in this model. In the paper [27], Hasegawa proved the crossing symmetry only from (2.9) and the unitarity relations (2.7) and (2.8). We thus have

$$\frac{\overline{W}(a + 1, b + 1|u)}{\overline{W}(a, b|u)} = \frac{G_{a+1} G_{b+1}}{G_a G_b} \frac{\theta_4(u - (a + b)\eta)}{\theta_4(u + (a + b + 2)\eta)},$$

$$\frac{\overline{W}(a + 1, b - 1|u)}{\overline{W}(a, b|u)} = \frac{G_{a+1} G_{b-1}}{G_a G_b} \frac{\theta_4(u - (a - b)\eta)}{\theta_4(u + (a - b + 2)\eta)}.$$ 

(2.14)

without directly solving (2.10). We can see from (2.12) and (2.14) that the Boltzmann weights satisfy the reflection symmetry

$$W(a, b|u) = W(b, a|u), \quad \overline{W}(a, b|u) = \overline{W}(b, a|u).$$
Defining \( T_k^{(+)}(\alpha|u) \) and \( T_k^{(-)}(\alpha|u) \) by
\[
T_k^{(+)}(\alpha|u) = \prod_{j=1}^{\alpha} \theta_k(u+(2j-1)\eta) \quad \text{and} \quad T_k^{(-)}(\alpha|u) = T_k^{(+)}(\alpha|\lambda-u),
\]
respectively, the solutions to the recursion relations (2.12) and (2.14) under the normalization \( W(0,0|u) = \overline{W}(0,0|u) = 1 \) are
\[
W(2a,2b|u) = T_2^{(+)}(a-b|u) T_3^{(+)}(a+b|u),
\]
\[
\overline{W}(2a,2b|u) = G_{2a} G_{2b} T_2^{(-)}(a-b|u) T_3^{(-)}(a+b|u).
\]
Here all local state variables are to be read modulo \( N \). See Appendix B for details.

Hasegawa and Yamada in [12] established the star-triangle relation (STR) in the broken \( \mathbb{Z}_N \)-symmetric model,
\[
\rho W(a,b|v-w) \overline{W}(a,c|u-w) W(b,c|u-v)
\]
\[
= \sum_{d=0}^{N-1} \overline{W}(a,d|u-v) W(d,b|u-w) \overline{W}(d,c|v-w),
\]
where \( \rho \) is a scalar function independent of \( a, b \) and \( c \).

### 3 Functional Relation

In this section, we consider the transfer matrix of the \( L \)-operators and construct a functional relation for it. In the course of the calculation, we utilize the factorization property of \( L \) into \( K \)'s (2.2). We define a 2-by-2 matrix \( L(a,b|u,v,w) \) by
\[
L(a,b|u,v,w) = \begin{pmatrix}
K_{0a}^b(u-w) & K_{0a}^b(v-w) & K_{0a}^b(u-w) K_{1a}^b(v-w) \\
K_{1a}^b(u-w) & K_{0a}^b(v-w) & K_{1a}^b(u-w) K_{1a}^b(v-w)
\end{pmatrix}.
\]
Then the transfer matrix \( \mathcal{L}(u,v,w) \) of \( L \)-operators on the lattice of width \( M \) with the periodic boundary condition is
\[
\mathcal{L}(u,v,w)_{a_0b_0|a_Mb_M} = \text{tr} \left( L(a_0,b_0|u,v,w)L(a_1,b_1|u,v,w) \cdots L(a_{M-1},b_{M-1}|u,v,w) \right)
\]
\[
= \prod_{i_0,\ldots,i_{M-1}} K_{ij|a_j}^{b_j}(u-w) K_{ij|a_j}^{b_j}(v-w).
\]
The final goal of this section is to establish the functional relation

\[ L(\lambda - u, \lambda - v, w + 1/4) \Phi(u, v, w) = C(u, v, w)^M \left( f(u, v, w)^M \Phi(u, v, w + \eta) + (-f(\lambda - v, \lambda - u, -w))^M \Phi(u, v, w - \eta) \right), \tag{3.2} \]

where we define \( C(u, v, w) \) and \( f(u, v, w) \) by

\[ C(u, v, w) = [(\theta_2 \theta_3 \theta_4](0) [T_2^{(+)}, T_3^{(+)}, T_3^{(-)}](n|u - w), \tag{3.3} \]

\[ f(u, v, w) = \theta_1(2u - 2w) [\theta_1(0)]^{(v - w + \eta)} [\theta_2 \theta_3](u - w). \tag{3.4} \]

This functional relation reduces to (1.13), (1.14) and (1.15) in the homogeneous case, \( u = v \). We achieve this goal by the method à la Baxter \cite{15, 16, 17}, which states the following: Suppose that we can find \( C \)-valued functions \( \phi_j(\nu)(b|u, v, w) \) \((\nu = 0, 1, 2, 3\) and \( b \in \mathbb{Z}/N\mathbb{Z} \)) and matrices \( P_j \in \text{End}(\mathbb{C}^2) \) for \( j \in \mathbb{Z}/M\mathbb{Z} \), which satisfy

\[ P_j^{-1} \sum_{b=0}^{N-1} \phi_j^{(0)}(b|u, v, w) L(a, b|u, v, w) P_{j+1} = \begin{pmatrix} \phi_j^{(1)}(a|u, v, w) & \phi_j^{(3)}(a|u, v, w) \\ 0 & \phi_j^{(2)}(a|u, v, w) \end{pmatrix} \text{ for } a \in \mathbb{Z}/N\mathbb{Z}\text{ and } j \in \mathbb{Z}/M\mathbb{Z}. \tag{3.5} \]

Then we have

\[ \sum_{b_0, \ldots, b_{M-1}} \phi_0^{(0)}(b_0|u, v, w) \phi_1^{(0)}(b_1|u, v, w) \cdots \phi_{M-1}^{(0)}(b_{M-1}|u, v, w) L(u, v, w)^{b_0 b_1 \cdots b_{M-1}} = \prod_{j=0}^{M-1} \phi_j^{(1)}(a_j|u, v, w) + \prod_{j=0}^{M-1} \phi_j^{(2)}(a_j|u, v, w). \tag{3.6} \]

Defining vectors \( \psi^{(\nu)}(u, v, w) \in (\mathbb{C}^N)^\otimes M \) by

\[ \psi^{(\nu)}(u, v, w)_{a_0 a_1 \cdots a_{M-1}} = \prod_{j=0}^{M-1} \phi_j^{(\nu)}(a_j|u, v, w), \tag{3.7} \]

we can write (3.6) as

\[ L(u, v, w) \psi^{(0)}(u, v, w) = \psi^{(1)}(u, v, w) + \psi^{(2)}(u, v, w). \tag{3.8} \]
In the following, we will find a family of solutions to \((3.5)\)

\[ \phi^{(\nu)}(b|u, v, w) = \phi^{(\nu)}(c_j, b, c_{j+1}|u, v, w) \quad (\nu = 0, 1, 2, 3 \text{ and } b \in \mathbb{Z}/N\mathbb{Z}), \]

\[ P_j = P(c_j), \]

labeled by \( \{ (c_0, c_1, \cdots, c_{M-1}) \mid c_j \in \mathbb{Z}/N\mathbb{Z} \text{ for } j \in \mathbb{Z}/M\mathbb{Z} \} \). This gives rise to \( N^M \) vectors \( \psi^{(\nu)} \) labeled as \( \psi^{(\nu)}(u, v, w)^{c_0, c_1, \cdots, c_{M-1}} \). We will also prove that \( \psi \)'s are proportional to the row vectors of the diagonal-to-diagonal transfer matrix \( \Phi \) of the broken \( \mathbb{Z}_N \)-symmetric model,

\[ \psi^{(0)}(\lambda - u, \lambda - v, w + 1/4)^{c_0 c_1 \cdots c_{M-1}} = \Phi(u, v, w)^{c_0 c_1 \cdots c_{M-1}}, \quad (3.9) \]

\[ \psi^{(1)}(\lambda - u, \lambda - v, w + 1/4)^{c_0 c_1 \cdots c_{M-1}} = \left(C(u, v, w) f(u, v, w)\right)^M \Phi(u, v, w - \eta)^{c_0 c_1 \cdots c_{M-1}}, \quad (3.10) \]

\[ \psi^{(2)}(\lambda - u, \lambda - v, w + 1/4)^{c_0 c_1 \cdots c_{M-1}} = \left( -C(u, v, w) f(\lambda - v, \lambda - u, -w)\right)^M \Phi(u, v, w - \eta)^{c_0 c_1 \cdots c_{M-1}}. \quad (3.11) \]

The results \((3.8)\) to \((3.11)\) altogether implies the functional relation \((3.2)\).

Now we start to solve \((3.5)\). We write the matrix elements of \( P_j \) as

\[ P_j = \begin{pmatrix} p_j^{(0)} \\ p_j^{(1)} \\ p_j^{(2)} \end{pmatrix}, \]

and its first column vector \( t(p_j^{(0)}, p_j^{(1)}) \) as \( p_j \). Multiplying \( P_j \) to \((3.5)\) from the left and taking its first column, we have

\[ \sum_{b=0}^{N-1} \phi_j^{(0)}(b|u, v, w) L(a, b|u, v, w) P_{j+1} = \phi_j^{(1)}(a|u, v, w) p_j \quad \text{for } a \in \mathbb{Z}/N\mathbb{Z}. \quad (3.12) \]

For later use, we define the functions \( \Delta_\ast(\pm), \Delta_\ast(\pm), \delta_\ast \) and \( \delta^\ast \) by

\[ \Delta_\ast(\pm)(a, p|u) = p^{(0)} K_1^{a \pm 1}(u) - p^{(1)} K_0^{a \pm 1}(u), \]

\[ \Delta_\ast(\pm)(a, p|u) = K_0^{a \pm 1}(u) p^{(0)} + K_1^{a \pm 1}(u) p^{(1)}, \]

\[ \delta_\ast(a|u) = K_0^{a-1}(u) K_1^{a+1}(u) - K_0^{a+1}(u) K_1^{a-1}(u), \]

\[ \delta^\ast(a|u) = K_0^{a-1}(u) K_1^{a+1}(u) - K_0^{a+1}(u) K_1^{a-1}(u). \]
From (3.15) and (3.16), we can write
\[ \phi \]

The equations (3.12) constitute a system of 2N homogeneous linear equations in \( \phi_j^{(0)}(a|u, v, w) \) and \( \phi_j^{(1)}(a|u, v, w) \) for \( a \in \mathbb{Z}/N\mathbb{Z} \). It has a non-trivial solution if and only if the determinant of its coefficient matrix vanishes. Demanding this condition, we obtain
\[
\prod_{a=0}^{N-1} \Delta_{s(-)}(p_j, a|u - w) \Delta_{s(-)}^*(a, p_{j+1}|v - w) \\
+ \prod_{a=0}^{N-1} \Delta_{s(+)}(p_j, a|u - w) \Delta_{s(+)}^*(a, p_{j+1}|v - w) = 0. \tag{3.13}
\]

Later we find that the equation (3.13) restricts \( p_j \) to a discrete set of values. When we parameterize \( p_j \) as
\[
p_j = p(c), \quad p(c) = \begin{pmatrix}
p_j^{(0)}(c) \\
p_j^{(1)}(c)
\end{pmatrix} = \begin{pmatrix}
[\theta_2 \theta_3](c) \\
-\theta_1 \theta_4(c)
\end{pmatrix}, \tag{3.14}
\]
and denote the dependence on \( p(c) \) simply by \( c \) and \( U_\alpha = u + \alpha \eta \), \( \Delta \)'s and \( \delta \)'s become
\[
\Delta_{s(\pm)}(c, a|u - 1/4) = \frac{[\theta_2 \theta_3](0)}{G_\alpha G_{a \pm 1}} \theta_2(U_{\pm c + a}) \theta_3(U_{\mp c - a}),
\]
\[
\Delta_{s(\pm)}^*(a, c|u - 1/4) = [\theta_2 \theta_3](0) \theta_1(U_{\pm a \pm c + 1}) \theta_4(U_{\pm a \pm c}),
\]
\[
\delta_{s}(a|u - 1/4) = -\frac{[\theta_2 \theta_3 \theta_4](0)}{G_{a - 1} G_{a + 1}} \theta_1(2U_0),
\]
\[
\delta_{s}^*(a|u - 1/4) = [\theta_2 \theta_3 \theta_4](0) G_\alpha^2 \theta_1(2U_1).
\]

Under the parameterization (3.14), the condition (3.13) holds if and only if either \( c_j \)'s are all integers, or all half-integers. We restrict ourselves to the case that \( c_j \)'s are all integers, because only in this case the relation (3.8) gives the functional relation (3.2). The system of equations (3.12) involves not all \( \phi \)'s but only \( \phi_j^{(0)}(a + 1|u, v, w) \), \( \phi_j^{(0)}(a - 1|u, v, w) \) and \( \phi_j^{(1)}(a|u, v, w) \). Expressing \( \phi_j^{(0)}(a + 1|u, v, w) \) and \( \phi_j^{(1)}(a|u, v, w) \) in terms of \( \phi_j^{(0)}(a - 1|u, v, w) \), we obtain
\[
\frac{\phi_j^{(0)}(a + 1|u, v, w)}{\phi_j^{(0)}(a - 1|u, v, w)} = -\frac{\Delta_{s(-)}(c_j, a|u - w) \Delta_{s(-)}^*(a, c_{j+1}|v - w)}{\Delta_{s(+)}(c_j, a|u - w) \Delta_{s(+)}^*(a, c_{j+1}|v - w)}, \tag{3.15}
\]
\[
\frac{\phi_j^{(1)}(a|u, v, w)}{\phi_j^{(0)}(a - 1|u, v, w)} = \delta_{s}(a|u - w) \frac{\Delta_{s(-)}(a, c_{j+1}|v - w)}{\Delta_{s(+)}(a, c_j|u - w)}, \tag{3.16}
\]

From (3.15) and (3.16), we can write \( \phi_j^{(\nu)}(a|u, v, w) \) (\( \nu = 0, 1 \)) as
\[
\phi_j^{(\nu)}(a|u, v, w) = \phi_j^{(\nu)}(c_j, a, c_{j+1}|u, v, w), \tag{3.17}
\]
where the function $\phi^{(\nu)}(c, a, c'|u, v, w)$ is independent of $j$. Taking the determinant of both sides of (3.15), we find

$$
\delta_s(c_j, a|u - w) \delta^s(a, c_j+1|v - w) \frac{\det(P_{j+1})}{\det(P_j)}
= \frac{\phi^{(1)}(c_j, a, c_j+1|u, v, w)}{\phi^{(0)}(c_j, a - 1, c_j+1|u, v, w)} \phi^{(2)}(a|u, v, w) \phi^{(0)}(c_j, a + 1, c_j+1|u, v, w),
$$

(3.18)

We set $\det(P_j)$ to unity without loss of generality. Then the equations (3.13), (3.16) and (3.18) give

$$
\frac{\phi^{(2)}(c_j, a, c_j+1|u, v, w)}{\phi^{(0)}(c_j, a - 1, c_j+1|u, v, w)} = -\delta^s(a|v - w) \frac{\Delta_{s(-)}(c_j, a|u - w)}{\Delta_{s(+)}(a, c_j+1|v - w)},
$$

(3.19)

where we write $\phi^{(2)}(a|u, v, w)$ as $\phi^{(2)}(c_j, a, c_j+1|u, v, w)$. The relations (3.15), (3.16) and (3.19) recursively determine $\phi^{(\nu)}$'s. We abbreviate $u - w + \alpha \eta$ and $v - w + \gamma \eta$ to $A_\alpha$ and $B_\gamma$ respectively. Comparing (3.15) with (2.12) and (2.14), we have

$$
\phi^{(0)}(a, b + 1, c|u, v, w + 1/4) = W(a, b + 1|A_0) W(b + 1, c|B_0) \phi^{(0)}(a, b - 1, c|u, v, w + 1/4).
$$

(3.20)

Hence we find that $\phi^{(0)}$ is a product of the two Boltzmann weights,

$$
\phi^{(0)}(a, b, c|u, v, w + 1/4) = W(a, b|A_0) W(b, c|B_0).
$$

(3.21)

From (3.15) and (3.16), we obtain

$$
\frac{\phi^{(1)}(a, b + 1, c|u, v, w + 1/4)}{\phi^{(1)}(a, b - 1, c|u, v, w + 1/4)} = \frac{\phi^{(0)}(a, b + 1, c|u - \eta, v - \eta, w + 1/4)}{\phi^{(0)}(a, b - 1, c|u - \eta, v - \eta, w + 1/4)}
$$

(3.22)

The same procedure for $\phi^{(2)}$ yields

$$
\frac{\phi^{(2)}(a, b + 1, c|u, v, w + 1/4)}{\phi^{(2)}(a, b - 1, c|u, v, w + 1/4)} = \frac{\phi^{(0)}(a, b + 1, c|u + \eta, v + \eta, w + 1/4)}{\phi^{(0)}(a, b - 1, c|u + \eta, v + \eta, w + 1/4)}
$$

(3.23)

By (3.21), (3.22) and (3.23), we can write $\phi^{(1)}$ and $\phi^{(2)}$ as

$$
\phi^{(1)}(a, b, c|u, v, w + 1/4) = f_{ac}(u, v, w) W(a, b|A_{-1}) W(b, c|B_{-1}),
$$

(3.24)

$$
\phi^{(2)}(a, b, c|u, v, w + 1/4) = g_{ac}(u, v, w) W(a, b|A_1) W(b, c|B_1),
$$

(3.25)

where $f_{ac}$ and $g_{ac}$ are functions independent of $b$. The equations (3.16), (3.21) and (3.24) determine $f_{ac}(u, v, w)$ as

$$
f_{ac}(u, v, w)
$$
= \left[ \theta_2\theta_3\theta_4 \right](0) \theta_1(2A_0) \frac{G_b}{G_{b-1}} \frac{\theta_1(B_{-b+c+1})\theta_4(B_{-b-c+1})}{\theta_2(A_{a-b})\theta_3(A_{-a-b})} \frac{W(a, b - 1|A_0)W(b - 1, c|B_0)}{W(a, b|A_{-1})W(b, c|B_{-1})}

= C(u, v, w) \theta_1(2A_0) \frac{\theta_1\theta_4(B_1)}{\theta_2\theta_3(A_0)} = C(u, v, w) f(u, v, w),

where \( C(u, v, w) \) and \( f(u, v, w) \) were given in (3.3) and (3.4), respectively. The last equality is due to (3.4). In the same way, we obtain

\[ g_{ac}(u, v, w) = -C(u, v, w) f(\lambda - v, \lambda - u, -w). \]

The equations (3.24) and (3.25) become

\[ \phi^{(1)}(a, b, c|u, v, w + 1/4) = C(u, v, w) f(u, v, w) W(a, b|A_{-1}) W(b, c|B_{-1}), \]

(3.26)

\[ \phi^{(2)}(a, b, c|u, v, w + 1/4) = -C(u, v, w) f(\lambda - v, \lambda - u, -w) W(a, b|A_1) W(b, c|B_1). \]

(3.27)

Substituting (3.21), (3.26) and (3.27) into the definition (3.7) of \( \psi^{(u)}(u, v, w) \) and using the crossing symmetry (2.13), we obtain (3.9), (3.10) and (3.11). We have the functional relation (3.2) as a result.

4 Bethe Ansatz Equations

In this section, we give commutation relations among \( \mathcal{R}, \mathcal{L}(u) \) and \( \Phi(v) \), and reduce the functional relation (3.2) to the functional equation among their eigenvalues. After discussing some properties about the zeros and poles of the eigenvalues of \( \Phi(u) \), we derive the Bethe Ansatz equations (1.7), (1.8) and (1.9) for the broken \( \mathbb{Z}_N \)-symmetric model.

First we have

\[ \mathcal{L}(u, v, w') \Phi(u, v, w) = \Phi(u, v, w) \mathcal{L}(v, u, w'). \]

(4.1)

We give a proof in Appendix C. In the case of the homogeneous system, i.e., \( u = v \) and \( w = w' \), the equation (4.1) means the commutativity of two transfer matrices \( \mathcal{L}(u) = \mathcal{L}(u, u, w) \) and \( \Phi(u) = \Phi(u, u, w) \),

\[ [ \mathcal{L}(u), \Phi(v) ] = 0. \]

(4.2)
The star-triangle relation (2.16) gives

\[
[\Phi(u), \Phi(v)] = 0,
\]
(4.3)

and the \( LLR = RLL \) relation (1.10) guarantees

\[
[\mathcal{L}(u), \mathcal{L}(v)] = 0.
\]
(4.4)

The relations (4.2), (4.3) and (4.4) make it possible to diagonalize \( \mathcal{L}(u) \) and \( \Phi(v) \) simultaneously by eigenvectors independent of the spectral parameters \( u \) and \( v \). Fixing one of the eigenvectors and denoting the corresponding eigenvalues of \( \mathcal{L}(u) \) and \( \Phi(v) \) by \( l(u) \) and \( \varphi(v) \) respectively, we can rewrite the functional relation (3.2) as

\[
l(\lambda - u - 1/4) \varphi(u) = C(u)^M \left( f(u)^M \varphi(u - \eta) + (-1)^M f(\lambda - u)^M \varphi(u + \eta) \right),
\]
(4.5)

where from (3.4) and (3.3), \( f(u) \) and \( C(u) \) are

\[
f(u) = \theta_1(2u) \frac{[\theta_1 \theta_4](u + \eta)}{[\theta_2 \theta_3](u)} \quad \text{and} \quad C(u) = [\theta_2 \theta_3 \theta_4](0) [T_2^{(+)} T_3^{(+)} T_2^{(-)} T_3^{(-)}](n|u).
\]
(4.6)

The next step to derive the Bethe Ansatz equations is to examine the quasi-periodicity property of \( \varphi(u) \). We have the following relations

\[
\Phi(u + 1) = \Phi(u),
\]
(4.7)

\[
\mathcal{R} \Phi(u) = \Phi(u) \mathcal{R} = \Phi(u + \frac{\tau}{2}).
\]
(4.8)

We have defined \( \mathcal{R} \) in (3.4). The periodicity (4.7) is obvious from the definition (1.2) of \( \Phi \) and the periodicity (3.3) of \( \mathcal{W} \) and \( \overline{\mathcal{W}} \). The other periodicity (4.8) follows from (3.6). We also have the commutativity

\[
[\mathcal{R}, \mathcal{L}(u)] = 0,
\]
(4.9)

from the \( \mathbb{Z}_2 \)-symmetry of \( K \)'s (2.5) and the definitions (1.4) and (3.2). Diagonalizing \( \mathcal{R} \) and \( \Phi(u) \) simultaneously with \( \mathcal{L}(v) \), the equations (4.7), (4.8) and (1.4) give

\[
\varphi(u + 1) = \varphi(u), \quad \varphi(u + \frac{\tau}{2}) = r \varphi(u), \quad r = \pm 1,
\]
(4.10)
where \( r \) is an eigenvalue of \( \mathbf{R} \). The poles of \( \varphi(u) \) are coming from only those of the matrix elements of \( \Phi(u) \). We define

\[
p(u) = \left( \frac{\eta(\tau)}{\eta(2\tau)^2} \right)^n \prod_{j=1}^{n} [\theta_2\theta_3](u - (2j - 1)\eta|\tau) \]

\[
= \prod_{j=1}^{n} \theta_2(u - (2j - 1)\eta|\tau/2), \tag{4.11}
\]

which contains all possible poles of \( W(a,b|u) \). The same does \( p(\lambda - u) \) for \( W \) by the crossing symmetry. Hence the set of zeros of \( \left(p(u)p(\lambda - u)\right)^M \) contains all poles of \( \varphi(u) \). By the Lemma in Appendix A and the double periodicity (4.10) of \( \varphi(u) \), we can write \( \varphi(u) \) as

\[
\varphi(u) = (\text{const}) \left( \frac{\prod_{j=1}^{2nM} \theta_1(u - u_j|\tau/2)}{\left(p(u)p(\lambda - u)\right)^M} \right),
\]

\[
\sum_{j=1}^{2nM} u_j \equiv nM\lambda + \frac{1 - r}{4} \mod (\mathbb{Z} \oplus \frac{\tau}{2}\mathbb{Z}).
\]

The initial condition \( \Phi(0) = \text{Id} \) (3.3) determines the normalization of \( \text{const} \),

\[
\varphi(u) = \left( \frac{p(0)p(\lambda)}{p(u)p(\lambda - u)} \right)^M \prod_{j=1}^{2nM} \theta_1(u - u_j|\tau/2) \frac{\prod_{j=1}^{2nM} \theta_1(u - u_j|\tau/2)}{\theta_1(u_j|\tau/2)}. \tag{4.12}
\]

Assuming \( C(u_j) \neq 0 \) in (4.6) and substituting \( u_k \ (k = 1, \cdots, 2nM) \) into (4.5), we have

\[
f(u_k)^M \varphi(u_k - \eta) + (-1)^M f(\lambda - u_k)^M \varphi(u_k + \eta) = 0
\]

for \( k = 1, \cdots, 2nM. \) \( \tag{4.13} \)

We further assume that \( u_k \ (k = 1, \cdots, 2nM) \) are neither zeros nor poles of \( f(u) \), \( f(\lambda - u) \) and \( \varphi(u \pm \eta) \). Then (4.13) becomes

\[
\left( \frac{f(u_k)p(u_k + \eta)p(\lambda - u_k - \eta)}{f(\lambda - u_k)p(u_k - \eta)p(\lambda - u_k + \eta)} \right)^M = (-1)^M \prod_{j=1}^{2nM} \theta_1(u_k - u_j + \eta|\tau/2) \frac{\prod_{j=1}^{2nM} \theta_1(u_k - u_j - \eta|\tau/2)}{\theta_1(u_k - u_j|\tau/2)}
\]

for \( k = 1, \cdots, 2nM. \) \( \tag{4.14} \)

We can write \( f(u) \) in (4.6) by (A.2) as

\[
f(u) = \frac{\eta(2\tau)}{\eta(\tau)^2} \theta_1(u|\tau/2) \theta_1(u + \eta|\tau/2).
\]
Then the left-hand side of (4.14) reduces to \( \left( \frac{\theta_1(u_k|\tau/2)}{\theta_1(u_k - \lambda|\tau/2)} \right)^{2M} \). After shifting \( u_k \) by \( \lambda/2 \), i.e., putting

\[ v_k = u_k - \frac{\lambda}{2} \quad \text{for} \quad k = 1, \ldots, 2nM, \]

we obtain the Bethe Ansatz equations for the broken \( Z_N \)-symmetric model,

\[
\left( \frac{\theta_1(v_k + \lambda/2|\tau/2)}{\theta_1(v_k - \lambda/2|\tau/2)} \right)^{2M} = (-1)^{M+1} \prod_{j=1}^{2nM} \frac{\theta_1(v_k - v_j + \eta|\tau/2)}{\theta_1(v_k - v_j - \eta|\tau/2)}
\quad \text{for} \quad k = 1, \ldots, 2nM, \quad (4.15)
\]

\[
\sum_{j=1}^{2nM} v_j \equiv \frac{1 - r}{4} \mod (Z \oplus \frac{\tau}{2}Z). \quad (4.16)
\]

## 5 Density Function and Free Energy

In this section, we will calculate the free energy of the broken \( Z_N \)-symmetric model from the Bethe Ansatz equations under the three assumptions concerning the ground state. One is the String Hypothesis below, and the others are about the distribution of string centers \( v_\alpha = \sqrt{-1}w_\alpha \) and the corresponding quantum numbers \( I_\alpha \). We restrict the spectral parameter \( u \) to the region \([0, 1/2N]\), in which all the Boltzmann weights are real and positive, and \( \tau \) to a pure imaginary number, \( \tau = \sqrt{-1}\kappa \) with \( \kappa \) real and positive.

By a string of length \( l \) and parity \( \nu (= 0 \text{ or } 1) \) with its center \( v_\alpha \), we mean the following set,

\[
\left\{ v_{\alpha,j} \left| \begin{array}{c}
 v_{\alpha,j} \equiv v_\alpha + (2j - l - 1)\frac{\eta}{2} + \frac{\nu}{2} \mod (Z \oplus \frac{\tau}{2}Z) \\
 \text{for } j = 1, 2, \ldots, l, \quad \text{and } \quad v_\alpha : \text{pure imaginary}
\end{array} \right. \right\}. \quad (5.1)
\]

We suppose that the following hypothesis holds in the infinite lattice limit \([10][9][28]\).

**String Hypothesis for the ground state**

The solution of the BAE’s \( \{v_j, j = 1, \ldots, 2nM\} \), corresponding to the ground state consists of strings of length \( N - 1 \) and parity \( 1 - (-1)^{n+1}/2 \).
More precisely, for finite systems the solutions of the BAE’s may have deviations from strings. The hypothesis asserts that these deviations vanish in the infinite lattice limit. In the course of the following calculation we deal with the solutions of the BAE’s as if they were genuine strings, since we are interested in thermodynamic quantities. Because all the matrix elements of \( \Phi \) are real and positive, the Perron-Frobenius theorem \cite{29} shows that the ground state belongs to the sector of zero quasi-momentum \cite{10}. The hypothesis implies that the ground state also belongs to the sector \( r = 1 \), and that the corresponding solutions are made up of \( M \) strings of length \( 2n \). We denote them by

\[
v_{\alpha,j} \equiv \sqrt{-1}w_{\alpha} + (N - 2j)\frac{\eta}{2} + \frac{1 - (-1)^{n+1}}{2} \mod (\mathbb{Z} \oplus \frac{\tau}{2}\mathbb{Z})
\]

for \( \alpha = 1, \ldots, M \) and \( j = 1, \ldots, 2n \),

where \( w_{\alpha} \)'s are all real and taken as

\[-\frac{\kappa}{4} \leq w_1 \leq w_2 \leq \cdots \leq w_M < \frac{\kappa}{4}.
\]

Then the BAE's for the ground state becomes

\[
\left( \frac{\theta_1(v_{\beta,k} + \lambda/2|\tau/2)}{\theta_1(v_{\beta,k} - \lambda/2|\tau/2)} \right)^{2M} = (-1)^{M+1} \prod_{\alpha=1}^{M} \prod_{j=1}^{2n} \frac{\theta_1(v_{\beta,k} - v_{\alpha,j} + \eta|\tau/2)}{\theta_1(v_{\beta,k} - v_{\alpha,j} - \eta|\tau/2)}
\]

for \( \beta = 1, \ldots, M \), \( k = 1, \ldots, 2n \). \hspace{1cm} (5.2)

Multiplying (5.2) over \( k = 1, \cdots, 2n \), we have

\[
\frac{\prod_{k=1}^{2n} \chi(w_{\beta}, \frac{n}{2N}(n - 2k + \frac{1}{2n}) + \frac{1 - (-1)^{n+1}}{4})^{2M}}{\prod_{\alpha=1}^{M} \chi(w_{\beta} - w_{\alpha}, 0) \left( \prod_{j=1}^{2n-1} \chi(w_{\beta} - w_{\alpha}, \frac{n}{N}j) \right)^2 \chi(w_{\beta} - w_{\alpha}, \frac{2n^2}{N})} = 1,
\]

where \( \chi(w, a) \) is

\[
\chi(w, a) = \frac{\theta_1(a - \sqrt{-1}w|\tau/2)}{\theta_1(a + \sqrt{-1}w|\tau/2)}.
\]

Taking the logarithm of (5.3) and dividing it by \( 2\sqrt{-1}\pi M \), we have

\[
T(w_{\beta}) = \frac{I_{\beta}}{M} \quad \text{for} \quad \beta = 1, \cdots, M,
\]

(5.4)
where the quantum number $I_\beta$’s are integers and

$$ T(w) = T_1(w) - \frac{1}{2M} \sum_{\alpha=1}^{M} T_2(w - w_\alpha), $$

$$ T_1(w) = \sum_{k=1}^{2n} t\left(w, \frac{n}{2N} \left(n - 2k + \frac{1}{2n} \right) + \frac{1 - (-1)^{n+1}}{4}\right), $$

$$ T_2(w) = t(w, 0) + 2 \sum_{j=1}^{2n} t\left(w, \frac{n}{Nj}\right) - t\left(w, \frac{2n^2}{N}\right), $$

$$ t(w, a) = \frac{1}{\sqrt{-1}} \log \chi(w, a). $$

Albertini et al. numerically investigated the 3-state Fateev-Zamolodchikov model [1]. Their results indicate that the String Hypothesis holds. We assume that the centers $w$ of the strings are distributed densely on the interval $[-\kappa/4, \kappa/4]$ in the limit of $M$ large, and that the quantum numbers $I_\beta$ satisfy

$$ I_{\beta+1} = I_\beta + 1 \quad \text{for} \quad \beta = 1, 2, \cdots, 2nM. \quad (5.5) $$

These are the second and the third assumptions we make. The results by Albertini et al. also supports them. We furthermore conjecture that

$$ w_\alpha = -w_{M-\alpha+1} \quad (5.6) $$

holds exactly for the ground state even in the finite lattice. This conjecture is consistent with their results. If (5.6) is true, we can show that

$$ T(\kappa/4) - T(-\kappa/4) = 1, \quad (5.7) $$

and this implies that $2M$ integers $I_\beta$’s must fill the interval $[-M, M]$ without jumps if all $I_\beta$’s are different. This also supports our assumption. But we do not use the conjecture (5.6) in this Paper.

We now proceed to the calculation. We define the density function for $w$’s by

$$ \rho(w_\beta) = \lim_{M \to \infty} \frac{1}{M(w_{\beta+1} - w_\beta)}, $$

which is positive and for any integrable function $f(x)$,

$$ \lim_{M \to \infty} \frac{1}{M} \sum_{\alpha=1}^{M} f(w_\alpha) = \int_{-\kappa/4}^{\kappa/4} f(w) \rho(w) \, dw \quad (5.9) $$
holds. Considering the difference of (5.4) for $I_{\beta + 1}$ and $I_{\beta}$

$$\frac{1}{M(w_{\beta + 1} - w)} = \frac{I_{\beta + 1} - I_{\beta}}{M(w_{\beta + 1} - w)} = \frac{\mathcal{T}(w_{\beta + 1}) - \mathcal{T}(w_{\beta})}{w_{\beta + 1} - w},$$

and letting $M \to \infty$, we obtain

$$\rho(w) = \frac{dT(w)}{dw} = \frac{dT_1(w)}{dw} - \frac{1}{2} \int_{-\frac{\pi}{\kappa}}^{\frac{\pi}{\kappa}} \frac{dT_2(w - \bar{w})}{dw} \rho(\bar{w}) d\bar{w}. \quad (5.10)$$

By (A.1) and (A.4), we can expand $t(w, a)$ as

$$t(w, a) = \frac{8\pi}{\kappa} \{a\} w + \frac{1}{\sqrt{-1}} \log \theta_1\left(\frac{2\sqrt{-1}}{\kappa} \{a\} + \frac{2}{\kappa} w \big| -\frac{2}{\tau}\right)$$

$$\quad - \theta_1\left(\frac{2\sqrt{-1}}{\kappa} \{a\} - \frac{2}{\kappa} w \big| -\frac{2}{\tau}\right)$$

$$= \frac{4\pi}{\kappa} (2\{a\} - 1) + \sum_{k=1}^{\infty} \frac{\sin \left(\frac{4\pi k}{\kappa} w\right)}{k} \sinh \left(\frac{4\pi}{\kappa} \left(\{a\} - \frac{1}{2}\right)\right).$$

We denote the fractional part of $x$ by $\{x\} = x - [x]$, $[x]$ being the Gauss symbol.

When we write the Fourier expansions of $\frac{dT_j(w)}{dw}$, $\frac{dT_2(w)}{dw}$ and $\rho(w)$ as

$$\frac{dT_j(w)}{dw} = \sum_{k=-\infty}^{\infty} A_{jk} \exp\left(\frac{4\sqrt{-1} \pi k}{\kappa} w\right) \quad \text{for} \quad j = 1, 2,$$

$$\rho(w) = \sum_{k=-\infty}^{\infty} \rho_k \exp\left(\frac{4\sqrt{-1} \pi k}{\kappa} w\right),$$

the integral equation (5.10) gives

$$\rho_k = \frac{A_{1,k}}{1 + \frac{\kappa}{4} A_{2,k}}.$$

The coefficients $A_{jk}$ are

$$A_{1,k} = \begin{cases} \frac{4n}{N\kappa} \exp\left(\frac{\pi k}{N\kappa} (N - 1)\right) \cosh\left(\frac{\pi k}{N\kappa} (N + 1)\right) & k = 0, \\
\frac{4n}{N\kappa} \frac{4 \sinh\left(\frac{\pi k}{N\kappa} (N - 1)\right) \cosh\left(\frac{\pi k}{N\kappa} (N + 1)\right)}{\kappa \sinh\left(\frac{2\pi k}{\kappa}\right) \cosh\left(\frac{2\pi k}{\kappa}\right)} & k \neq 0, \end{cases}$$

$$1 + \frac{\kappa}{4} A_{2,k} = \begin{cases} \frac{2n}{N} \exp\left(\frac{\pi k}{N\kappa} (N - 1)\right) \cosh\left(\frac{\pi k}{N\kappa} (N + 1)\right) & k = 0, \\
\frac{2n}{N} \frac{4 \sinh\left(\frac{\pi k}{N\kappa} (N - 1)\right) \cosh\left(\frac{\pi k}{N\kappa} (N + 1)\right)}{\sinh\left(\frac{2\pi k}{\kappa}\right)} & k \neq 0. \end{cases}$$
We obtain the density function for strings,

\[ \rho(w) = \frac{2}{\kappa} \sum_{k=-\infty}^{\infty} \exp\left(\frac{4\sqrt{-1} \pi k w}{\kappa}\right) \cosh\left(\frac{\pi k}{N\kappa}\right). \]

Using (A.1) and (A.5), we can rewrite it as

\[ \rho(w) = 2 \kappa [\theta_3 \theta_4](0|N\tau) \theta_3 \left(2\sqrt{-1}Nw|N\tau\right)/\theta_2 \left(2\sqrt{-1}Nw|N\tau\right). \] (5.11)

The free energy per site \( F(u) \) of the model is defined by

\[ F(u) = \lim_{M \to \infty} \left( -\frac{1}{M} \log \varphi(u) \right), \] (5.12)

where \( \varphi(u) \) is the eigenvalue of the transfer matrix \( \Phi(u) \) corresponding to the ground state. We can write \( \varphi(u) \) as

\[ \varphi(u) = \left(\frac{p(0)p(\lambda)}{p(u)p(\lambda - u)}\right)^{2n} \prod_{j=1}^{M} D_j(u, w), \]

\[ D_j(u, w) = \frac{\theta_1(\sqrt{-1}w + \beta_j - u|\tau/2)}{\theta_1(\sqrt{-1}w + \beta_j|\tau/2)}, \]

\[ \beta_j = \{\gamma_j\}, \quad \gamma_j = \frac{n}{2N}(N - 2j + 1) + \frac{1 - (-1)^{n+1}}{4}, \]

by (4.12), (5.6) and the String Hypothesis. Then the free energy per site is

\[ F(u) = -\log \left(\frac{p(0)p(\lambda)}{p(u)p(\lambda - u)}\right) - \sum_{j=1}^{2n} \lim_{M \to \infty} \frac{1}{M} \sum_{\alpha=1}^{M} \log D_j(u, w) \]

\[ = -\log \left(\frac{p(0)p(\lambda)}{p(u)p(\lambda - u)}\right) - \frac{1}{2} \sum_{j=1}^{2n} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\log D_j(u, w)\right)\rho(w) dw. \]

Since \( \rho(w) \) is an even function, it is enough to integrate the even part of \( \log D_j(u, w) \).

We hence have

\[ F(u) = -\log \left(\frac{p(0)p(\lambda)}{p(u)p(\lambda - u)}\right) \]

\[ -\frac{1}{2} \sum_{j=1}^{2n} \log D_j^{(e)}(u, \frac{\kappa}{4}) + \frac{1}{2} \sum_{j=1}^{2n} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d \log D_j^{(e)}(u, w)}{dw} \rho^{(-1)}(w) dw, \]
where \( D_j^{(e)}(u, w) \) and \( \rho^{(-1)}(w) \) are
\[
\rho^{(-1)}(w) = \int_{-\frac{\kappa}{2}}^{w} \rho(\omega) \, d\omega,
\]
\[
D_j^{(e)}(u, w) = \left( D_j(u, w)D_j(u, -w) \right)^{1/2}.
\]
A little cumbersome calculation yields
\[
- \log \left( \frac{p(0)p(\lambda)}{p(u)p(\lambda - u)} \right) = E_1(u) + E_2(u),
\]
\[
- \frac{1}{2} \sum_{j=1}^{2n} \log D_j^{(e)}(u, \frac{\kappa}{4}) = - E_1(u) + E_3(u),
\]
\[
\frac{1}{2} \sum_{j=1}^{2n} \int_{-\frac{\kappa}{2}}^{\frac{\kappa}{2}} \frac{d \log D_j^{(e)}(u, w)}{dw} \rho^{(-1)}(w) \, dw = - E_3(u) + E_4(u),
\]
where
\[
E_1(u) = \frac{2\pi n}{\kappa} u \left( \frac{1}{N} - 2u \right),
\]
\[
E_2(u) = 4 \sum_{l=1}^{\infty} \frac{\sinh \left( \frac{2\pi l}{\kappa} u \right) \sinh \left( \frac{2\pi l}{\kappa} \left( \frac{1}{2N} - u \right) \right) \cosh \left( \frac{\pi l}{\kappa} \right) \sinh \left( \frac{2\pi l}{N\kappa} n \right)}{l \sinh \left( \frac{2\pi l}{\kappa} \right) \sinh \left( \frac{2\pi l}{N\kappa} \right)},
\]
\[
E_3(u) = -4 \sum_{l=1}^{\infty} (-1)^l \frac{\sinh \left( \frac{2\pi l}{\kappa} u \right) \sinh \left( \frac{2\pi l}{\kappa} \left( \frac{1}{2N} - u \right) \right) \cosh \left( \frac{2\pi l}{N\kappa} (n + 1) \right) \sinh \left( \frac{2\pi l}{N\kappa} n \right)}{l \sinh \left( \frac{2\pi l}{\kappa} \right) \sinh \left( \frac{2\pi l}{N\kappa} \right)},
\]
\[
E_4(u) = 4 \sum_{l=1}^{\infty} \frac{\sinh \left( \frac{2\pi l}{\kappa} u \right) \sinh \left( \frac{2\pi l}{\kappa} \left( u - \frac{1}{2N} \right) \right) \cosh \left( \frac{2\pi l}{N\kappa} (n + 1) \right) \sinh \left( \frac{2\pi l}{N\kappa} n \right)}{l \sinh \left( \frac{2\pi l}{\kappa} \right) \sinh \left( \frac{2\pi l}{N\kappa} \right) \cosh \left( \frac{\pi l}{\kappa} \right) \cosh \left( \frac{\pi l}{N\kappa} \right)}.
\]
Now \( F(u) \) has the final expression
\[
F(u) = E_2(u) + E_4(u)
\]
\[
= - \sum_{l=1}^{\infty} \frac{\sinh \left( \frac{2\pi l}{\kappa} u \right) \sinh \left( \frac{2\pi l}{\kappa} \left( \frac{1}{2N} - u \right) \right) \sinh \left( \frac{2\pi l}{N\kappa} n \right)}{l \cosh \left( \frac{\pi l}{\kappa} \right) \cosh \left( \frac{\pi l}{N\kappa} \right)}.
\] (5.13)
It agrees with the result of Jimbo, Miwa and Okado \[20\] obtained by the use of the inversion-trick. In the trigonometric limit \( \kappa \to +\infty \), the free energy formula (5.13) reduces to the integral
\[
\lim_{\kappa \to \infty} F(u) = - \int_{0}^{\infty} dx \frac{\sinh (N\pi xu) \sinh (N\pi x(\frac{1}{2N} - u)) \sinh (n\pi x)}{x \cosh \left( \frac{N\pi x}{2} \right) \cosh \left( \frac{\pi x}{2} \right)},
\]
which also agrees with the results of Fateev and Zamolodchikov themselves \cite{1} and by Albertini \cite{10}. The former was obtained by the inversion-trick, and the latter by the Bethe Ansatz method.

6 Discussion

The first main goal of this Paper is the functional relation (3.2). We obtain it as a functional relation for $\mathcal{L}(u)$. The diagonal-to-diagonal transfer matrix $\Phi(u)$ of the broken $\mathbb{Z}_N$-symmetric model appears naturally in this relation. We obtain the Boltzmann weights $W$ and $\overline{W}$ of the broken $\mathbb{Z}_N$- symmetric model in the algebraic way different from that of \cite{12}. We obtain $W$ and $\overline{W}$ as the solutions to the relation (3.12). In \cite{12}, they are the solutions to the relations (2.9) and (2.10).

The Bethe Ansatz equations (4.15) are the second goal of this Paper. The commutativity (4.1) between $\mathcal{L}(u)$ and $\Phi(v)$ is essential to get the Bethe Ansatz equations (4.15) from the functional relation (3.2). It is notable that the unitarity relations (2.6) and (2.7) guarantee this commutativity. This contrasts with the usual situation where the commutativity of the transfer matrices is derived from the STR or the $LLR = RLL$ type relations.

The Fateev-Zamolodchikov model is the trigonometric limit of the broken $\mathbb{Z}_N$-symmetric model. It has the $\mathbb{Z}_N$-symmetry besides the $\mathbb{Z}_2$-symmetry. Hence the $\mathbb{Z}_N$-charge $q \in \{-n, -n + 1, \ldots, n - 1, n\}$ is a good quantum number, where $\exp(2\sqrt{-1}\pi \frac{q}{N})$ is an eigenvalue of the $\mathbb{Z}_N$-charge operator $Q \in \text{End}(\mathbb{C}^N \otimes M)$,

$$Q = Q \otimes Q \otimes \cdots \otimes Q,$$

$$Q v_j^{(N)} = \exp\left(2\sqrt{-1}\pi \frac{j}{N}\right) v_j^{(N)} \quad \text{for} \quad j = 0, 1, \ldots, N - 1. \quad (6.1)$$

Albertini \cite{10} obtained the Bethe Ansatz equations and the formula for the eigenvalue $\varphi_{FZ}(u)$ of the diagonal-to-diagonal transfer matrix for the Fateev-Zamolodchikov
model. They are

$$
\left(\frac{s(v_k + \lambda/2)}{s(v_k - \lambda/2)}\right)^{2M} = (-1)^{M+1} \prod_{j=1}^{2nM-2|q|} \frac{s(v_k - v_j + \eta)}{s(v_k - v_j - \eta)} \quad \text{for } k = 1, \cdots, 2nM, \quad (6.2)
$$

$$
\varphi_{FZ}(u) = \left(\frac{p_\infty(0)p_\infty(\lambda)}{p_\infty(u)p_\infty(\lambda - u)}\right)^M \prod_{j=1}^{2nM-2|q|} \frac{s(u - u_j)}{s(u_j)}, \quad p_\infty(u) = \lim_{\kappa \to \infty} p(u), \quad (6.3)
$$

where \(s(u) = \sin(\pi u)\) and \(p(u)\) is given in (4.14). The notations are slightly changed from [10] to compare to our case. The main difference is the number of factors in the right-hand sides of the BAE's and in the expressions of the eigenvalues \(\varphi(u)\) and \(\varphi_{FZ}(u)\). It is always \(2nM\) in the broken \(\mathbb{Z}_N\)-symmetric model, and in the Fateev-Zamolodchikov model it is \(2nM - 2|q|\), which depends on the sector of the \(\mathbb{Z}_N\)-charge operator \(Q\). This difference originates in that the \(\mathbb{Z}_N\)-symmetry holds only in the Fateev-Zamolodchikov model and that it breaks away from the criticality. The BAE’s (4.13) for the broken \(\mathbb{Z}_N\)-symmetric model should coincide with those for the Fateev-Zamolodchikov model in the trigonometric limit \(\kappa \to \infty\). We conjecture that the situation is the following. In the solution \(\{v_1, \cdots, v_{2nM}\}\) to the BAE’s (4.13) for the broken \(\mathbb{Z}_N\)-symmetric model, some of them diverge to \(\pm \sqrt{-1}\infty\) all in the same order in \(\kappa\) when the trigonometric limit is taken. The half of them diverge to \(\sqrt{-1}\infty\), and the other half to \(-\sqrt{-1}\infty\). The number of them are always even and between 0 and \(2n\). Let \(2q\) be this number. Then \(q\) determines the sector of the \(\mathbb{Z}_N\)-charge operator in which this eigenvalue falls. In this situation, the BAE’s (4.13) and the eigenvalue \(\varphi(u)\) in (4.12) surely become the BAE’s (6.2) and \(\varphi_{FZ}(u)\) in (6.3) respectively in the trigonometric limit.

The free energy (5.13) agrees with the result of [20] by the inversion-trick. The string hypothesis for the ground state in Section 5 is consistent with their result. In our formulation, it is manifest that the free energy \(F(u)\) is doubly periodic in \(u\),

$$
F(u + 1) = F(u + \tau/2) = F(u),
$$

from (4.10), (5.12) and the fact that the ground state belongs to the sector of \(r = 1\). This result of \(F(u)\) also gives the ground state energy of the one dimensional spin
chain Hamiltonian $\mathcal{H}$ corresponding to the broken $\mathbb{Z}_N$-symmetric model,

$$\log \Phi(u) = Id + u\mathcal{H} + O(u^2).$$

The Hamiltonian $\mathcal{H}$ itself is modular invariant, $\mathcal{H}(\tau) = \mathcal{H}(-\frac{1}{\tau})$. We will report on these accounts elsewhere.

## A Theta Function

We summarize the necessary facts about theta functions in this Appendix. See [30] [31] for proofs. We define $\theta_1(u|\tau)$ by

$$\theta_1(u|\tau) = 2q^{1/4} \sin(\pi u) \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos(2\pi u) + q^{4n}) (1 - q^{2n}),$$

where $q = \exp(\sqrt{-1}\pi \tau)$. It is an odd function in $u$ and has the quasi-periodicity

$$\theta_1(u + 1|\tau) = \theta_1(u|\tau),
\theta_1(u + \tau|\tau) = -q^{-1} \exp(2\sqrt{-1}\pi u) \theta_1(u|\tau),$$

and satisfies

$$\theta_1(u|\tau) = \sqrt{-1} \left( \frac{\sqrt{-1}}{\tau} \right)^{1/2} \exp(-\sqrt{-1}\pi u^2) \theta_1\left(\frac{u}{\tau} - \frac{1}{\pi}\right). \quad (A.1)$$

The other theta functions $\theta_2, \theta_3$ and $\theta_4$ are defined by

$$\theta_2(u|\tau) = \theta_1(u + 1/2|\tau),$$
$$\theta_3(u|\tau) = -q^{1/4} \exp(\sqrt{-1}\pi u) \theta_1(u + 1/2 + \tau/2|\tau),$$
$$\theta_4(u|\tau) = -\sqrt{-1}q^{1/4} \exp(\sqrt{-1}\pi u) \theta_1(u + \tau/2|\tau).$$

We abbreviate a product of theta functions of the same argument to, for example,

$$[\theta_1 \theta_2](u|\tau) = \theta_1(u|\tau) \theta_2(u|\tau),$$
$$[\theta_2 \theta_3 \theta_4](0|\tau) = \theta_2(0|\tau) \theta_3(0|\tau) \theta_4(0|\tau).$$

In this notation,

$$[\theta_1 \theta_4](u|\tau) = \frac{\eta(2\tau)}{\eta(\tau)} \theta_1(u|\tau/2), \quad [\theta_2 \theta_3](u|\tau) = \frac{\eta(2\tau)}{\eta(\tau)} \theta_2(u|\tau/2),$$
$$[\theta_1 \theta_2](u|\tau) = \frac{\eta(2\tau)}{\eta(4\tau)} \theta_1(2u|2\tau), \quad [\theta_3 \theta_4](u|\tau) = \frac{\eta(2\tau)}{\eta(4\tau)} \theta_4(2u|2\tau). \quad (A.2)$$
hold, where

\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \]

is the Dedekind eta function. The necessary addition formulae are

\[ [\theta_2 \theta_3](u|\tau) [\theta_2 \theta_3](v|\tau) \pm [\theta_1 \theta_4](u|\tau) [\theta_1 \theta_4](v|\tau) = [\theta_2 \theta_3](0|\tau) \theta_2(u \mp v|\tau) \theta_3(u \pm v|\tau), \]
\[ [\theta_1 \theta_4](u|\tau) [\theta_2 \theta_3](v|\tau) \pm [\theta_2 \theta_3](u|\tau) [\theta_1 \theta_4](v|\tau) = [\theta_2 \theta_3](0|\tau) \theta_1(u \pm v|\tau) \theta_4(u \mp v|\tau). \]

(A.3)

In Section 5, we use the Fourier expansions,

\[ \frac{1}{\sqrt{-1}} \log \frac{\theta_1(w + v|\tau)}{\theta_1(w - v|\tau)} = -2\pi \{v\} + 2 \sum_{k=1}^{\infty} \frac{\sin(2\pi k v) \sin(2\pi k(w - \tau/2))}{k \sin(\pi k \tau)}, \]
\[ \frac{\theta_3(u|\tau)}{\theta_4(u|\tau)} = \frac{1}{[\theta_3 \theta_4](0|\tau)} \sum_{k=-\infty}^{\infty} \frac{\exp(\sqrt{-1} \pi k u)}{\cos(\pi k \tau)}, \]

(A.4)

(A.5)

which are valid for \(0 < \text{Im}(w) < \tau\) and \(v\) real. We are denoting the fractional part of \(x\) by \(\{x\}\). The expansion (A.5) is essentially the same as that of the Jacobi elliptic function \(dn(u, k)\)

\[ dn(u, k) = \frac{\pi}{2K} \sum_{l=-\infty}^{\infty} \frac{\exp(\sqrt{-1} \pi l u/K)}{\cos(\pi l \tau)}, \quad K = \frac{\pi}{2} \theta_3(0|\tau)^2. \]

The next lemma is fundamental.

**Lemma 1** Let \(f(u)\) be a meromorphic function which is not identically zero and has the quasi-periodicity property

\[ f(u + 1) = \exp(-2\sqrt{-1} \pi B) f(u), \]
\[ f(u + \tau) = \exp(-2\sqrt{-1} \pi (A_1 + A_2 u)) f(u). \]

Denoting the zeros and poles of \(f(u)\) by \(u_1, u_2, \ldots, u_n\) and \(v_1, v_2, \ldots, v_m\) respectively, then we have

\[ n - m = A_2, \quad \sum_{j=1}^{n} u_j - \sum_{j=1}^{m} v_j \equiv \frac{1}{2} A_2 - A_1 + B \tau \mod (\mathbb{Z} \oplus \tau \mathbb{Z}), \]

and

\[ f(u) = C \exp(\sqrt{-1} \pi (A_2 - 2B u)) \frac{\prod_{j=1}^{n} \theta_1(u - u_j|\tau)}{\prod_{j=1}^{m} \theta_1(u - v_j|\tau)}, \]

with \(C\) independent of \(u\).
B Boltzmann Weights

In this Appendix, we list some formulae about the Boltzmann weights. Solving the recursion relations (2.12) and (2.14) under the normalization of $W(0, 0|u) = \overline{W}(0, 0|u) = 1$, we have, for $a, b = 0, 1, \cdots, n$,

$$W(2a, 2b|u) = W(N - 2a, N - 2b|u) = T_2^{(+))(a - b|u)} T_3^{(+))(a + b|u)},$$

$$W(2a, N - 2b|u) = W(N - 2a, 2b|u) = T_2^{(+))(a + b|u)} T_3^{(+))(a - b|u)},$$

$$\overline{W}(2a, 2b|u) = \overline{W}(N - 2a, N - 2b|u) = G_{2a} G_{2b} T_2^{(-)(a - b|u)} T_3^{(-)(a + b|u)},$$

$$\overline{W}(2a, N - 2b|u) = \overline{W}(N - 2a, 2b|u) = G_{2a} G_{2b} T_2^{(-)(a + b|u)} T_3^{(-)(a - b|u)}.$$  

Noting that $T_k^{(\sigma)(a|u)}$ in (2.13) satisfies

$$T_k^{(\sigma)(0|u)} = T_k^{(\sigma)(N|u)},$$

$$T_k^{(\sigma)(N - a|u)} = T_k^{(\sigma)(a|u)},$$

we can extend the domain of the first argument of $T_k^{(\sigma)}$ to all integers by periodicity. With this convention we can rewrite the above expression of the Boltzmann weights simply as

$$W(2a, 2b|u) = T_2^{(+))(a - b|u)} T_3^{(+))(a + b|u)},$$

$$\overline{W}(2a, 2b|u) = G_{2a} G_{2b} T_2^{(-)(a - b|u)} T_3^{(-)(a + b|u)}.$$  

We have in particular at $u=0$ and $\lambda$,

$$W(a, b|0) = G_a^{-1} G_b^{-1} \overline{W}(a, b|\lambda) = 1, \quad G_a G_b W(a, b|\lambda) = \overline{W}(a, b|0) = \delta_{ab}. \quad (B.3)$$

When we write $u + a\eta$ as $U_a$, the next identity for $T_k^{(\sigma)}$

$$\frac{T_k^{(\sigma_1)(n - a|u)}}{T_k^{(\sigma_1)(a|u + \sigma_2 \eta)}} = \left( \frac{\theta_k(U_{(1+\sigma_1)/2})}{\theta_k(U_{-2\sigma_2 + (1+\sigma_1)/2})} \right)^{\sigma_1 \sigma_2} T_k^{(\sigma_1)(n|u)}$$

gives

$$\frac{W(a, b - 1|u)}{W(a, b|u + \sigma \eta)} = \left( \frac{\theta_2(U_{\sigma(-a+b)}) \theta_3(U_{\sigma(a+b)})}{\theta_1(U_{\sigma(a+b)+1})} \right)^{-\sigma} \frac{[T_2^{(+)T_3^{(+)}}]}{[T_2^{(-)T_3^{(-)}]}(n|u)}, \quad \frac{W(a, b - 1|u)}{W(a, b|u + \sigma \eta)} = \frac{G_{b-1}}{G_b} \left( \frac{\theta_1(U_{\sigma(-a+b)+1})}{\theta_1(U_{\sigma(a+b)+1})} \right)^{\sigma} \frac{[T_2^{(-)T_3^{(-)}]}(n|u).} \quad (B.4)$$
$T_k^{(\sigma)}$ has the quasi-periodicity

$$T_k^{(\sigma)}(a|u + 1) = \exp(4\sqrt{-1}\pi a^2 \eta) T_k^{(\sigma)}(a|u + \tau) = T_k^{(\sigma)}(a|u),$$

$$T_k^{(\sigma)}(a|u + \tau/2) = \exp(-2\sqrt{-1}\pi a^2 \eta) T_k^{(\sigma)}(a|u), \; k + k' \equiv 0 \mod 5.$$

Hence we have

$$W(a, b, |u + 1) = W(a, b|u), \; \overline{W}(a, b, |u + 1) = \overline{W}(a, b|u), \; \text{(B.5)}$$

and

$$W(a, b|u + \tau/2) = \exp(-4\sqrt{-1}\pi (a^2 + b^2) \eta) W(a, N - b|u), \; \overline{W}(a, b|u + \tau/2) = \exp(4\sqrt{-1}\pi (a^2 + b^2) \eta) \overline{W}(a, N - b|u).$$

We note

$$W(a, b|u) \overline{W}(b, c|u) = \exp(-4\sqrt{-1}\pi (a^2 - c^2) \eta) W(a, N - b|u + \tau/2) \overline{W}(N - b, c|u + \tau/2). \; \text{(B.6)}$$

## C Commutation Relation between $\mathcal{L}$ and $\Phi$

In this appendix, we give a proof of (4.1). A graphical representation of this proof for the case of $M = 2$ is illustrated in Fig. 11.

$$\left(\Phi(u, v, w) \mathcal{L}(u, v, w')\right)_{\alpha_0 \cdots \alpha_{M-1}}^{c_0 \cdots c_{M-1}} = \sum_{b_0 \cdots b_{M-1}} \Phi(u, v, w)_{b_0 \cdots b_{M-1}} \mathcal{L}(u, v, w')_{a_0 \cdots a_{M-1}}^{b_0 \cdots b_{M-1}} = \sum_{b_0 \cdots b_{M-1}} \left( \prod_{j=0}^{M-1} W(b_{j-1}, c_j|u - w) \overline{W}(c_j, b_j|v - w) K_{ij}^{b_j} (u - w') K_{ij}^{b_j} (v - w') \right).$$

By the unitarity relation (2.7), inserting

$$1 = \sum_{a'\alpha'} \delta_{a'a} \frac{\sum G_{a'} K_{a'a}^{i \alpha} (w - w' + \lambda) K_{\alpha a}^{i \alpha} (w - w')}{[\theta_2 \theta_3 \theta_4](0) G_{c_0} \theta_2 (2w - 2w')},$$

we have

$$= \left( [\theta_2 \theta_3 \theta_4](0) G_{c_0} \theta_2 (2w - 2w') \right)^{-1} \sum_{b_0 \cdots b_{M-1}} \sum_{i' a'} G_{a'} K_{i' a'}^{c_0} (w - w' + \lambda) \cdots$$

27
\[ \times \left( \prod_{j=0}^{M-1} W(c_j, b_j|v - w) W(b_{j-1}, c_j|u - w) \right) K^{i_0 c_0}_{a_0^i} (w - w') K^{b_{M-1}}_{a_{M-1}} (u - w') \]

\[ \times \left( \prod_{j=1}^{M-1} K^{i_j b_j}_{a_j} (v - w') K^{b_{j-1}}_{i_j a_{j-1}} (u - w') \right) K^{i_0 b_0}_{a_0^i} (v - w'). \]

Successive use of (2.9) and (2.10) yields

\[
= \left( [\theta_2 \theta_3 \theta_4](0) G_{c_0} \theta_2(2w - 2w') \right)^{-1} \\
\times \left( \sum_{b_0 \cdots b_{M-1} \atop i_0^1 \cdots i_{M-1}} G_{a_i} K^{c_0}_{i a'} (w - w' + \lambda) K^{i a'}_{b_0} (w - w') \right) \times \left( \prod_{j=0}^{M-2} K^{c_j}_{i_j b_j} (v - w') K^{i_j c_{j+1}}_{b_{j+1}} (u - w') K^{c_{M-1}}_{i_{M-1} b_{M-1}} (v - w') K^{c_{M-1} c_0}_{a'} (u - w') \right) \times \left( \prod_{j=0}^{M-2} W(b_j, a_j|v - w) W(a_j, b_{j+1}|u - w) \right) \times W(b_{M-1}, a_{M-1}|v - w) W(a_{M-1}, a'|u - w). \]

By the unitarity relation (2.6), we have

\[ \sum_{i'} G_{a_0} K^{c_0}_{i a'} (w - w' + \lambda) K^{i a'}_{b_{M-1}} (w - w') \]

\[
= \left[ [\theta_2 \theta_3 \theta_4](0) G_{c_0} \theta_2(2w - 2w') \right] = \delta_{a' b_{M-1}}, \]

then the above formula reduces

\[
= \sum_{b_0 \cdots b_{M-1} \atop i_0^1 \cdots i_{M-1}} \left( \prod_{j=0}^{M-1} K^{c_j}_{i_j b_j} (v - w') K^{i_j c_{j+1}}_{b_{j+1}} (u - w') \right) W(b_j, a_j|v - w) W(a_j, b_{j+1}|u - w) \]

\[
= \sum_{b_0 \cdots b_{M-1}} \mathcal{L}(u, v, w') c_0 \cdots c_{M-1} \Phi(u, v, w) b_0 \cdots b_{M-1} \]

\[
= \left( \mathcal{L}(u, v, w') \Phi(u, v, w) \right) c_0 \cdots c_{M-1}. \]

Now we obtain

\[ \Phi(u, v, w) \mathcal{L}(u, v, w') = \mathcal{L}(u, v, w') \Phi(u, v, w). \]

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Figure caption

Fig1 Graphical representation of
\[ W(a, b|u, v) \text{ and } \overline{W}(a, b|u, v) \]

Fig2 Graphical representation of the STR

Fig3 Graphical representation of \( \Phi(u, v, w) \)

Fig4 Graphical representation of (1.10)

Fig5 Graphical representation of the YBE

Fig6 Graphical representations of
\[ K_{ia}^b(u - w) \text{ and } K_{ja}^b(v - w) \]

Fig7 Graphical representations of
unitary relations between \( K \)'s

Fig8 Graphical representation of (2.9)

Fig9 Graphical representation of (2.10)

Fig10 Graphical representations of \( \mathcal{L}(u, v, w) \)

Fig11 Graphical representation of a proof of
\[ \mathcal{L}(u, v, w') \Phi(u, v, w) = \Phi(u, v, w) \mathcal{L}(u, v, w') \]
\[ W(a, b|u, v) = \]

\[ \mathcal{W}(a, b|u, v) = \]

Fig.1 Graphical representations of \( W(a, b|u, v) \) and \( \mathcal{W}(a, b|u, v) \)
Fig. 2 Graphical representation of the STR
\[ \Phi(u, v, w) b_0 b_1 \cdots b_{M-1} = \Phi(u, v, w) a_0 a_1 \cdots a_{M-1} = \]

**Fig. 3** Graphical representation of \( \Phi(u, v, w) \)
Fig. 4 Graphical representation of (1.10)
\[ \mathcal{R}^{kl}_{ij}(u - v) = u \]

Fig. 5 Graphical Representation of the YBE
Fig. 6 Graphical representations of $K_{ia}^b(u - w)$ and $K_{ja}^b(v - w)$
Fig. 7 Graphical representations of unitary relations between $K$'s
Fig. 8 Graphical representation of (2.9)
Fig. 9  Graphical representation of (2.10)
\[ \mathcal{L}(u, v, w)^{b_0b_1\cdots b_{M-1}}a_0a_1\cdots a_{M-1} = \]

Fig. 10  Graphical representations of \( \mathcal{L}(u, v, w) \)
Fig. 11 Graphical representation of a proof of $\mathcal{L}(u, v, w') \Phi(u, v, w) = \Phi(u, v, w) \mathcal{L}(v, u, w')$