SPECTRAL SEQUENCE OF UNIVERSAL DISTRIBUTION AND
SINNOTT’S INDEX FORMULA

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Abstract. We prove an abstract index formula about Sinnott’s symbol between two
different lattices. We also develop the theory of the universal distribution and predist-
ribution in a double complex point of view. The theory of spectral sequence is used
to interpret the index formula and to analyze the cohomology of the universal distri-
bution. Combing these results, we successfully prove Sinnott’s index formula about
the Stickelberger ideal. In addition, the \{±1\}-cohomology groups of the universal
distribution and the universal predistribution are obtained.

1. Introduction

The theory of universal distribution, with its tremendous application in number theory,
has been well studied in the past thirty years(See Lang \[9\] and Washington \[12\] for
more backgrounds). In \[13\] , Yamamoto studied the \{±1\}-cohomology of the universal
distribution of rank 1(the gap group in \[13\]):

Theorem 1.1. Let \(U_m\) be the universal distribution of rank 1 and level \(m\). Then
\[
H^i(\{±1\}, U_m) = (\mathbb{Z}/2\mathbb{Z})^{2^r - 1}
\]
where \(r\) is the number of distinct prime factors of \(m\).

In his famous paper \[10\], Sinnott successfully obtained the index formula of Stickel-
berger ideal and circular units, which generalized the results of Kummer and Iwasawa.
His result can be stated as

Theorem 1.2. Let \(m\) be a positive integer which is not 2 (mod 4). Let \(G\) be the Galois

group of the cyclotomic extension \(\mathbb{Q}(\zeta_m)/\mathbb{Q}\). Let \(R = \mathbb{Z}[G]\) and let \(S\) be the Stickelberger

deal of \(\mathbb{Q}(\zeta_m)\). Let \(E\) be the group of units in \(\mathbb{Q}(\zeta_m)\) and let \(C\) be the subgroup of circular

units in \(E\). Then
\[
(1). [R^- : S^-] = 2^a h^- ;
(2). [E^+ : C^+] = 2^b h^+ ;
\]
where \(a = b = 0\) if \(r = 1\) and \(a = 2^{r - 2} - 1, b = 2^{r - 2} + 1 - r\) if \(r > 1\), \(h^+\) and \(h^-\) are the
class number of \(\mathbb{Q}(\zeta_m)^+\) and the relative class number of \(\mathbb{Q}(\zeta_m)\) respectively.

Sinnott’s result was a huge success and inspired many followers. Most notably, Ku-
bert \[1\] and \[3\] found the connection of Sinnott’s method and the universal (ordinary)
distribution. By using this connection, he thus showed that Theorem 1.2 is true for the
universal distribution of arbitrary rank.

Sinnott’s computation is very elegant but rather difficult. The motivation to find
an easier proof drives us to this paper. The theory of spectral sequences, though very
popular in topology, algebra and even number theory, had not been able to leave its
mark in the theory of distribution until recently. In [1], Anderson came up with the
idea of using a special double complex to compute the \( \{ \pm 1 \} \)-cohomology of the universal
distribution. With which he proved a conjecture by Yin [14]. Das [4] then used it to
study algebraic monomials and obtained many interesting results. Their method is the
prototype of spectral sequences method used by us here.

In this paper, we prove an abstract index formula about Sinnott’s symbol between two
different lattices. We also develop the theory of the universal distribution and predistrib-
ution in a double complex point of view. The theory of spectral sequence is used to
interpret the index formula and to analyze the cohomology of the universal distribution.
Combing these results, we successfully prove Sinnott’s index formula about the Stick-
elberger ideal (i.e., the first part of Theorem 1.2). In addition, the \( \{ \pm 1 \} \)-cohomology
groups of the universal distribution (i.e., Theorem 1.3) and the universal predistribution
are obtained. Although we only study the rank 1 case in this paper, our method is
capable of generalizing to the higher rank case.

As noted above, this paper is based on my advisor, Professor Greg W. Anderson’s
brilliant idea. I am in debt to his working note [1] which contains the raw form of
the abstract index formula and many other facts stated in this paper. I also benefit
greatly from numerous discussions with him. This paper would be impossible without
his instruction. I thank whole heartedly for his insight, patience and encouragement.

2. The abstract index formula

2.1. Definition of regulator \( \text{reg}(A, B, \lambda) \). Let \( A \) and \( B \) be lattices in a finite dimen-
sional vector space \( V \) over \( \mathbb{R} \). Necessarily there exists some \( \mathbb{R} \)-linear automorphism \( \phi \) of
\( V \) such that \( \phi(A) = B \). Put

\[
(A : B)_V := |\det \phi|,
\]

which is a positive real number independent of the choice of \( \phi \). We call it the Sinnott
symbol of \( A \) to \( B \). We often write it \( \text{reg} \lambda \) in abbreviation.

Here we calculate a few examples of the regulator:

**Example 2.1.** If both \( A \) and \( B \) are finite, then \( \text{reg}(A, B, 0) = \#B/\#A \).
Example 2.2. Let $f : A \to B$ be any homomorphism of finitely generated abelian groups with finite kernel and cokernel, then $\text{reg}(A, B, Rf)$ is exact $\# \text{coker } f / \# \ker f$.

Example 2.3. Let $A$, $B$ and $C$ be finitely generated abelian groups. Let $\lambda : \mathbb{R} A \to \mathbb{R} B$ and $\mu : \mathbb{R} B \to \mathbb{R} C$ be $\mathbb{R}$-linear isomorphisms. Then $\text{reg } \mu \circ \lambda = \text{reg } \mu \cdot \text{reg } \lambda$.

Example 2.4. Let $V$ be a finite dimensional $\mathbb{R}$-vector space. Let $A,B \subseteq V$ be lattices. Let $\alpha : \mathbb{R} A \to V$ and $\beta : \mathbb{R} B \to V$ be the natural isomorphisms induced by the inclusions $A \subseteq V$ and $B \subseteq V$ respectively. Then $\text{reg}(A, B, \beta - 1 \circ \alpha) = (B : A)_V$.

2.2. The abstract index formula. Consider bounded complexes of finitely generated abelian groups

$$(A,d_A) : \cdots \to A^i \to A^{i+1} \to \cdots$$

and

$$(B,d_B) : \cdots \to B^i \to B^{i+1} \to \cdots.$$ 

Given an isomorphism

$$\lambda : \mathbb{R} A \to \mathbb{R} B$$

of bounded complexes of finitely dimensional vector spaces. It naturally induces a map

$$H^i(\lambda) : H^i(\mathbb{R} A) \to H^i(\mathbb{R} B)$$

for every degree $i$. Note that we also have $\mathbb{R} H^i(A) = H^i(\mathbb{R} A)$ and $\mathbb{R} H^i(B) = H^i(\mathbb{R} B)$.

Then we have the following proposition:

Proposition 2.1. With the hypotheses above, then

$$\prod_i (\text{reg } \lambda^i)^{(-1)^i} = \prod_i (\text{reg } H^i(\lambda))^{(-1)^i}.$$ (2.2)

Proof. First we claim that there exist subcomplexes $A' \subseteq A$ and $B' \subseteq B$ satisfying the following conditions:

1. $A'^n$ and $B'^n$ are free abelian groups of the same rank as $A^i$ for all $i$;
2. $H^i(A')$ and $H^i(B')$ are torsion free for all $i$;
3. $A'$ and $B'$ are isomorphic complexes of abelian groups.
4. The sequences

$$0 \to H^i(A') \to H^i(A) \to H^i(A/A') \to 0$$

and

$$0 \to H^i(B') \to H^i(B) \to H^i(B/B') \to 0$$

are exact for all $i$.

This claim can be proved by induction. First since $A$ and $B$ are bounded complexes of finite generated abelian groups, without loss of generality we suppose

$$(A,d_A) : \cdots 0 \to A^{-n} \to \cdots \to A^{-1} \to A^0 \to 0 \cdots$$

and

$$(B,d_B) : \cdots 0 \to B^{-n} \to \cdots \to B^{-1} \to B^0 \to 0 \cdots$$

Consider the subgroup $\text{im}(d_A : A^{-1} \to A^0)$ of $A^0$. Let $r$ be the rank of $\text{im } A^{-1}$ and let $\{e_1, \cdots, e_r\}$ be a maximal independent set in $\text{im } A^{-1}$. We can enlarge it into a maximal
independent set \( E_0 = \{ e_1, \ldots, e_s \} \) of \( A^0 \). Set \( A'^0 \) be the subgroup generated by \( E_0 \).

Then \( A^0 / A'^0 \) is finite. Now consider the inverse image of \( A^0 \), it is a subgroup of \( A^{-1} \).

Moreover, it must have the same rank as \( A^{-1} \). Since \( \ker(d_A : A^{-1} \to A^0) \) is contained in the inverse image of \( A^0 \), so is \( \text{im}(d_A : A^{-2} \to A^{-1}) \). Find \( \{ f_1, \ldots, f_s \} \subseteq A^{-1} \) such that \( d_A(f_i) = e_i \).

This set is an independent set in the inverse image of \( A^0 \) and has only trivial intersection with \( \ker(d_A : A^{-1} \to A^0) \). We select a maximal independent set in \( \text{im}(A^{-2} \to A^{-1}) \), enlarge it to a maximal independent set in \( \ker(A^{-1} \to A^0) \), together with \( \{ f_1, \ldots, f_s \} \subseteq A^{-1} \), we get a maximal independent set \( E_{-1} \) in the inverse image of \( A^0 \). Set the free subgroup generated by \( E_{-1} \) as \( A'^{-1} \). Continuing this setup, we obtain a subcomplex \( A' \) of \( A \) such that \( A'^0 \) is free, \( (A/A')^i \) is finite and \( H^i(A') \) is torsion free.

Similarly for the complex \( B \), we can construct a subcomplex \( B' \) of \( B \) such that \( B'^0 \) is free, \( (B/B')^i \) is finite and \( H^i(B') \) is torsion free. Hence \( A' \) and \( B' \) satisfy conditions (1) and (2). But (3) and (4) easily follow from (1) and (2). Hence we proved the above claim. Now choose an isomorphism \( \phi : B' \to A' \) of complexes. We have

\[
\prod_i (\text{reg} \lambda^i)^{(-1)^i} \times \prod_i \left( \frac{\det \ker(\phi^i \circ \lambda^i) \cdot \#(B/B')^i}{\#(A/A')^i} \right)^{-1)^i} = \prod_i \left( \frac{\det \ker(H^i(\phi) \circ H^i(A))}{\#H^i(A/A')} \right)^{(-1)^i} = \prod_i (\text{reg} H^i(\lambda))^{(-1)^i}.
\]

Here we use the facts: (1). If \( A \) is a complex of finite abelian group, then

\[
\prod_i (\#H^i(A))^{(-1)^i} = \prod_i (\#A^i)^{(-1)^i}.
\]

(2). If \( V \) is a complex of \( \mathbb{R} \)-vector spaces, \( \phi \) is an automorphism of \( V \), then

\[
\prod_i \| \det \phi^i \|^{(-1)^i} = \prod_i \| \det H^i(\phi) \|^{(-1)^i}.
\]

Now Consider the following data:

- A finite group \( G \).
- A bounded graded finitely generated left \( \mathbb{R}[G] \)-modules
  
  \( V = \bigoplus_i V^i \) such that \( V^i = 0 \) for \( i > 0 \) and \( i = 0 \),

  equipped with two differential structures \( d_1 \) and \( d_2 \).
- An \( \mathbb{R}[G] \)-linear isomorphism \( \phi \) between two cochain complexes \((V, d_1)\) and \((V, d_2)\).
- A lattice \( L = \bigoplus_i L_i \) of \( V \) which is \( G \), \( d_1 \) and \( d_2 \)-stable.
- \( H^i_d(k \cdot L) = H^i_d(k \cdot L) = 0 \) for all \( i \neq 0 \).
- \( H^i_1(L) \) and \( H^i_2(L) \) are free abelian groups.

Now for an arbitrary left ideal \( \theta \subseteq \mathbb{Z}[G] \), by our assumption, we have the following trivial consequences:

- \( H^i_2(V^\theta) = H^i_2(V^\theta) = 0 \) for all \( i \neq 0 \).
- \( L^\theta \) is a lattice in \( L^\theta \) for all \( i \).
- \( H^i_2(L)^\theta \) and \( H^i_2(\phi(L)^\theta) \) are lattices in \( H^i_2(V^\theta) \).

By Proposition 2.1, we have(suggested by Anderson [1]).
Theorem 2.2 (Abstract Index Formula). Under the above assumption, we have
\[(2.3)\quad (H^0_{d_2}(L^\theta) : H^0_{d_2}(\phi L)^\theta) = \prod_i |\det(\phi^i)|^{-1} \cdot I(L, d_1; \theta)^{-1} \cdot I(L, d_2; \theta),\]
where for any complex of \(\mathbb{Z}[G]\)-modules \(A\), we define
\[(2.4)\quad I(A; \theta) := \frac{\# \text{coker}(H^0(A^\theta) \to H^0(A)^\theta)}{\# \text{tor}H^0(A^\theta) \cdot \prod_{i \not= 0} \# H^i(A^\theta)(-1)^i},\]
if the above value is finite.

Proof. Consider the complexes \((L^\theta, d_1)\) and \((L^\theta, d_2)\) with the restriction map \(\phi : V^\theta \to V^\theta\). Note that:
1. \(\text{reg}(L^i\theta, L^i\theta, \phi^i) = |\det(\phi^i|V^\theta)|\) for all \(i\).
2. Since \(H^*_\theta(V^\theta) = H^*_\theta(V^\theta) = 0\) for all \(i \neq 0\), \(H^*_\theta(L^\theta)\) and \(H^*_\theta(L^\theta)\) are both finite and \(H^i(\phi) = 0\). We have \(\text{reg}(H^*_\theta(L^\theta), H^*_\theta(L^\theta), H^i(\phi)) = \# H^*_\theta(L^\theta)/\# H^*_\theta(L^\theta)\) for all \(i \neq 0\).
3. Now for \(i = 0\), consider the map \(\alpha_j : H^0_{d_2}(L^\theta) \to H^0_{d_2}(L^\theta)\). We have \(H^0(\phi) \circ \alpha_1 = \mathbb{R}\alpha_2 \circ H^0(\phi)\). Then
\[
\text{reg}(H^0_{d_2}(L^\theta), H^0_{d_2}(L^\theta), H^0(\phi)) = \text{reg}(\alpha_1) \cdot \text{reg}(\alpha_2)^{-1} \cdot \text{reg}(H^0_{d_2}(L^\theta), H^0_{d_2}(L^\theta), H^0(\phi)),
\]
where
\[
\text{reg}(\alpha_j) = \frac{\# \text{coker}(H^0_{d_2}(L^\theta) \to H^0_{d_2}(L^\theta))}{\# \text{tor}H^0_{d_2}(L^\theta)}
\]
and
\[
\text{reg}(H^0_{d_2}(L^\theta), H^0_{d_2}(L^\theta), H^0(\phi)) = (H^0_{d_2}(L^\theta) : H^0_{d_2}(\phi L)^\theta).
\]
Now applying Formula (2.2) in Proposition 2.1 to the case \(A = (L^\theta, d_1), B = (L^\theta, d_2)\) and \(\lambda = \phi\), we immediately get (2.3). \(\square\)

3. Theory of spectral sequences

3.1. Basic theory of spectral sequences. Let \(G\) be a group and let \(\mathbb{Z}[G]\) be the integral group ring of \(G\). Let \(\theta\) be a left ideal of \(\mathbb{Z}[G]\). For any left module \(M\), let \(M^\theta\) be the subgroup of \(M\) annihilated by \(\theta\). Let
\[(A, d) : \cdots \to A^i \to A^{i+1} \to \cdots\]
be a complex of left \(G\)-modules. Assume
- \(A^i = 0\) for \(i > 0\) and \(i \ll 0\).
- \(H^i(A) = 0\) for \(i \neq 0\).

Let \(M = \mathbb{Z}[G]/\theta\), we have a projective resolution of \(M\):
\[(P, \partial) : \cdots \to P_i \to \cdots \to P_1 \to P_0 \to 0\]
Let \(K^{p,q} = \text{Hom}_G(P_q, A^p)\), then we have a commutative diagram:
\[
\begin{array}{ccc}
K^{p,q+1} & \xrightarrow{d^q} & K^{p+1,q+1} \\
\uparrow \circ \partial & & \uparrow \circ \partial \\
K^{p,q} & \xrightarrow{d^q} & K^{p+1,q}
\end{array}
\]
With abuse of notations, we denote $d\circ \partial$ by $d$ and $(-1)^p \circ \partial$ by $\delta$. Then we get a double complex $K^{*,*} = (K^{p,q}; d, \delta)$. The associate single complex is then defined by

$$K^n = \bigoplus_{p+q=n} K^{p,q}, \quad D = d + \delta.$$  

(3.1)

Recall that we have two filtrations of the double complex $K^{*,*}$

$$'\text{Fil}^p K^{*,*} = \bigoplus_{p' \geq p} K^{p',q},$$

and

$$''\text{Fil}^q K^{*,*} = \bigoplus_{q'' \geq q} K^{p,q''}.$$  

(3.2)

(3.3)

Now consider the following diagram:

\[
\begin{array}{ccccccc}
\vdots & \rightarrow & K^{p-1,q+1} & \delta & \rightarrow & K^{p,q+1} & \delta & \rightarrow & K^{p+1,q+1} & \delta & \rightarrow & K^{p,q+2} & \delta & \rightarrow & \vdots \\
\vdots & \rightarrow & K^{p-1,q} & \delta & \rightarrow & K^{p,q} & \delta & \rightarrow & K^{p+1,q} & \delta & \rightarrow & K^{p,q+1} & \delta & \rightarrow & \vdots \\
\vdots & \rightarrow & K^{p-1,q-1} & \delta & \rightarrow & K^{p,q-1} & \delta & \rightarrow & K^{p+1,q-1} & \delta & \rightarrow & K^{p,q} & \delta & \rightarrow & \vdots \\
\end{array}
\]

We have

$$H^q_\delta(K^{p,*}) = \text{Ext}^q_G(M, A^p).$$

(3.4)

and

$$H^p_\delta(K^{*,q}) = \begin{cases} 0, & \text{if } p \neq 0; \\ \text{Hom}_G(P_q, H^0(A)), & \text{if } p = 0. \end{cases}$$

(3.5)

Therefore we can compute the $E_2$ terms of the related spectral sequences. For the first one,

$$'E_2^{p,q} = H^p(\text{Ext}^q_G(M, A));$$

(3.6)

for the second one,

$$''E_2^{p,q} = \begin{cases} 0, & \text{if } p \neq 0; \\ \text{Ext}^q_G(M, H^0(A)), & \text{if } p = 0. \end{cases}$$

(3.7)

Since the second case collapses at $p = 0$, we have

$$H^i(K^*) = \text{Ext}^i_G(M, H^0(A)).$$

(3.8)
From now on we will focus only on the first case. We omit the symbol $'$ from our notations. Then
\[(3.9) \quad E_2^{p,q} = H^p(\text{Ext}^q_G(M,A)) \Rightarrow \text{Ext}^{p+q}_G(M,H^0(A)).\]

Set $q = 0$, then
\[(3.10) \quad E_2^{p,0} = H^p(\text{Ext}^0_G(M,A)) = H^p(A^\theta).\]

Because $\text{Fil}^1 K^*$ is trivial, we have
\[E_\infty^{0,0} = \text{Fil}^0 H^0(K^*) = \text{im} \left( H^0(\text{Fil}^0 K^*) \to H^0(K^*) \right).\]

Since $\text{Fil}^0 K^*$ is nothing but the complex
\[0 \to \text{Hom}_G(P_0, A^0) \to \text{Hom}_G(P_1, A^0) \to \cdots \to \text{Hom}_G(P_q, A^0) \to \cdots,\]
we have $H^0(\text{Fil}^0 K^*) = A^0$ and
\[E_\infty^{0,0} = \text{im} \left( A^0 \to H^0(A^\theta) \right).\]

We show further it factors through $H^0(A^\theta)$. First note that $H^0(A^\theta) = \text{coker}(A^{-1} \theta \to A^0)$, therefore we only need to show that $A^{-1} \theta$ is contained in the boundary of $K^0$. This follows immediately from the diagram
\[
\begin{array}{cccccc}
0 & \to & A^0 & \to & K^0 & \to & K^1 \\
& & \uparrow{d} & & \uparrow{d} & & \uparrow{d} \\
0 & \to & A^{-1} & \to & K^{-1} & \to & K^{-1} \\
\end{array}
\]
which is exact at the two rows. Combining the above arguments, we have
\[(3.11) \quad E_\infty^{0,0} = \text{im} \left( H^0(A^\theta) \to H^0(A^\theta) \right).\]

3.2. Application to the abstract index formula. By the results obtained in the above subsection, we can express $I(A, \theta)$ in terms of the order of $E_r$. We give here an important special case:

**Proposition 3.1.** If one has
\[(3.12) \quad \# \text{Ext}^1_G(M, H^0(A)) = \prod_q \# H^{1-q}(\text{Ext}^q_G(M, A)),\]
then
\[(3.13) \quad I(A; \theta) = \prod_{\substack{p+q \leq 0 \\text{q} \leq 0}} \# H^p(\text{Ext}^q_G(M, A))^{(-1)^{p+q}} = \prod_{\substack{p+q \leq 0 \\text{q} \leq 0}} (\# E_2^{p,q})^{(-1)^{p+q}}.\]

**Proof.** First note that the given identity (3.12) is nothing but
\[\prod_q \# E_\infty^{1-q,q} = \prod_q \# E_2^{1-q,q}.\]

Since for the spectral sequence, $H^*(E_r) = E_{r+1}$, we always have
\[\# E_2^{p,q} \geq \# E_3^{p,q} \geq \cdots \geq \# E_\infty^{p,q}.\]

Hence
\[\# E_2^{1-q,q} = \# E_3^{1-q,q} = \cdots = \# E_\infty^{1-q,q},\]
which means that for \( r \geq 2, \)
\[
\text{im}(d_r : E_r^{1-q-r,q+r-1} \to E_r^{1-q,q}) = \text{im}(d_r : E_r^{1-q,q} \to E_r^{1-q+r,q-r+1}) = 0.
\]
Therefore we have a shorter complex:
\[
\cdots \to E_r^{1-q-2r,q+2r-2} \to E_r^{1-q-r,q+r-1} \to 0.
\]
Now we set to prove the following fact:
\[
\prod_{p+q \leq 0} \frac{\#E_{p,q}^r}{(p,q) \neq (0,0)} (-1)^{p+q} \cdot \# \text{tor} E_r^{0,0} = \text{Constant}.
\]
Observe that the set \( \{ E_{p,q}^r : p + q \leq 0, q \geq 0 \} \), the only term not finite is \( E_{0,0}^r \). If we substitute it by its torsion, we still get a group of complexes composed of finite abelian groups and with differential \( d_r \). The cohomology groups are \( E_{r+1}^{0,0} \) (or \( \text{tor} E_r^{0,0} \)). By the invariance of Euler characteristic under cohomology, (3.14) is proved. Note that \( E_{\infty,0}^r \) is free and
\[
\prod_{p+q \leq 0} \frac{\#E_{p,q}^{\infty}}{(p,q) \neq (0,0)} (-1)^{p+q} = \# \text{coker}(H_0(A^0) \to H_0(A^0)).
\]
The formula (3.13) now follows immediately. \( \square \)

4. The universal distribution and predistribution

4.1. Definitions and basic properties. Let \( A \) be the free abelian group generated by the symbols \([a]\) with \( a \in \mathbb{Q}/\mathbb{Z} \). We call the elements which are linear combinations of \([a] - \sum_{n|a}[b]\) distribution relations in \( A \) and the elements which are linear combinations of \( \sum_{n|a}[b]\) predistribution relations in \( A \). Let \( U \) be the quotient group of \( A \) modulo the distribution relations and let \( O \) the quotient group of \( A \) modulo the predistribution relations. We call \( U \) and \( O \) the (rank 1) universal distribution and the (rank 1) universal predistribution respectively. Now for the subgroup \( A_m = [a] : a \in \frac{1}{m} \mathbb{Z}/\mathbb{Z} > \) of \( A \), put
\[
U_m = A_m/ < [a] - \sum_{n|a}[b], n|m, a \in \frac{n}{m} \mathbb{Z}/\mathbb{Z} >,
\]
and
\[
O_m = A_m/ < \sum_{n|a}[b], n|m, a \in \frac{n}{m} \mathbb{Z}/\mathbb{Z} >.
\]
We call \( U_m \) and \( O_m \) the universal distribution and the universal predistribution of level \( m \) respectively.

In [10], Sinnott introduced an \( \mathbb{R}[G] \)-module \( U_m \) and used it to compute the index of the Stickelberger ideal and the circular units. Kubert [3] then proved that Sinnott’s module are actually isomorphic to the one we defined above. We have

Proposition 4.1.

1. \( U_m \cong U_{\text{Sinnott}} \);
2. \( O_m \cong O_{\text{Kubert}} \).
Proof. (1). See Kubert [7].
(2). Define
\[ e_m : A_m \to O_K, \quad \sum n_i [a_i] \mapsto \sum n_i \exp(2\pi i a_i) \]
It is routine to check that \( e_m \) is actually an isomorphism. Since we don’t need this fact in the latter context, we omit it here. \( \square \)

4.2. **The connecting map \( \phi_m \).** Let
\[ \phi_m : \mathbb{R} \otimes A_m \to \mathbb{R} \otimes A_m, \quad [x] \mapsto \sum_{n \mid m} \frac{[nx]}{n}. \]
Then \( \phi_m \) is an automorphism of \( \mathbb{R} \)-vector space \( \mathbb{R}A_m \), the inverse map is given by
\[ \phi_{m}^{-1} : [x] \mapsto \sum_{n \mid m} \frac{\mu(n) [nx]}{n}, \]
where
\[ \mu(n) = \begin{cases} (-1)^i, & \text{if } n \text{ is a product of } i \text{ distinct prime numbers;} \\ 0, & \text{otherwise.} \end{cases} \]
is the Möbius function. Now if we enlarge the definition of distribution and predistribution relations to \( \mathbb{R}A_m \), then we have

**Proposition 4.2.** \( \phi_m \) maps distribution relations to predistribution relations. In other words, \( \phi_m \) induces an isomorphism from \( \mathbb{R}U_m \) to \( \mathbb{R}O_m \).

Proof. By straightforward calculation. \( \square \)

Note. From now on we denote by \( \varphi_m \) the above induced map.

5. **The cochain complexes \( (L_m, d_{1m}) \) and \( (L_m, d_{2m}) \)**

5.1. **Set up.** In this section and sequel, we fix the following notations:
- \( K_m = \mathbb{Q} (\zeta_m), G_m = \text{Gal}(\mathbb{Q} (\zeta_m) / \mathbb{Q}) = (\mathbb{Z} / m \mathbb{Z})^\times; \)
- \( \sigma = -1 \) is the complex conjugation in \( G_m, \theta = 1 + c, J = \{1, c\}; \)
- \( L_m = \langle [x, g] : g \mid m, g \text{ square free, } x \in \mathbb{Z} / \mathbb{Z} \rangle; \)
- \( L_{m, g} = \langle [x, g] : x \in \mathbb{Z} / \mathbb{Z} \rangle \text{ for a fixed square free factor } g; \)
- \( L^i_m = \bigoplus L_{m, g} \text{ for all square free } g \mid m \text{ such that } i = -\# \text{Supp } g; \)
- \( V_m = \mathbb{R} \otimes L_m, V_{m, g} = \mathbb{R} \otimes L_{m, g}, V^i_m = \mathbb{R} \otimes L^i_m. \)

For any square free positive integer \( g, \) suppose that \( g = p_1 \cdots p_r, p_1 < \cdots < p_r, \) is the prime factorization of \( g. \) Put
\[ \epsilon(g, p) = \begin{cases} (-1)^i, & \text{if } p = p_i; \\ 0, & \text{otherwise.} \end{cases} \]

Now we define
\[ d_{1m} : L^i_m \to L^{i+1}_m, [x, g] \mapsto \sum_{i=1}^{r} \epsilon(g, p_i) ([x, g/p_i] - \sum_{p_i \mid y = x} [y, g/p_i]). \]
and

\[
(5.2) \quad d_{2m} : L_m^i \to L_{m+1}^i, [x,g] \mapsto \sum_{i=1}^{r} \epsilon(g,p_i)(-\sum_{p, y=x} [y,g/p_i]),
\]

By straightforward calculation, we have \(d_{1m}^2 = d_{2m}^2 = 0\). Therefore \(V_m\) is equipped with two cochain complexes structure, we write them \((V_m, d_{1m})\) and \((V_m, d_{2m})\) respectively.

In the next section, we are going to study the cohomology groups.

5.2. Connecting map again. In this subsection, we define a connecting map \(\phi_m\) between \((V_m, d_{1m})\) and \((V_m, d_{2m})\), generalizing the one defined in §4.2. We put

\[
\phi_m : V_m \to V_m, [x,g] \mapsto \sum_{n|m} \frac{[nx,g]}{n}.
\]

Then \(\phi_m\) is an automorphism of the vector space \(V_m\). Furthermore, the inverse map of \(\phi_m\) is given by

\[
\phi_m^{-1} : V_m \to V_m, [x,g] \mapsto \sum_{n|m} \frac{\mu(n) [nx,g]}{n}.
\]

The following proposition establishes the connection between \((V_m, d_{1m})\) and \((V_m, d_{2m})\).

**Proposition 5.1.** \(\phi_m\) is an isomorphism from cochain complex \((V_m, d_{1m})\) to cochain complex \((V_m, d_{2m})\), i.e.,

\[
d_{2m} \phi_m = \phi_m d_{1m}.
\]

**Proof.** By direct calculation. 

**Remark.** Under the apparent isomorphism from \(V_m^0\) to \(\mathbb{R} \otimes A_m\), we can see that the connecting map \(\phi\) defined in §3.2 is the same map \(\phi\) defined on \(V_m^0\). Later we will see that \(\varphi = H^0(\tilde{\phi})\).

Now we try to calculate the determinant of \(\phi_m\). We have

**Proposition 5.2.**

\[
(5.3) \quad \prod_i \det(\phi_m : V_m^i)^{(-1)^i} = \prod_{p|m} \prod_{\chi \in \hat{G}_m} (1 - \chi(p)^{-1})^{-1}.
\]

**Proof.** First notice that \(V_{m,g}\) is invariant under \(\phi_m\). Moreover, let \(h = m/g\), for any \(f \mid h\), define

\[
V_{m,g}^f = \mathbb{R} \otimes <[x,g]: fx = 0>,
\]

then clearly \(V_{m,g}^f\) is invariant under \(\phi_m\). By definition, we have \(V_{m,g}^h = V_{m,g}\). Put

\[
V_{m,g}^{(f)} = V_{m,g}^f / \sum_{p \mid f} V_{m,g}^{f/p},
\]
We can see that $V_{m,g}^{(f)}$ is a real vector space with a basis \{[$a, g] : (a, f) = 1$\}. Furthermore $V_{m,g}^{(f)}$ has a natural $\mathbb{R}[G_f]$-module structure. Actually it is a free $\mathbb{R}[G_f]$-module of rank 1. $\phi_m$ induces an automorphism in $V_{m,g}^{(f)}$.

$$\phi_m : V_{m,g}^{(f)} \rightarrow V_{m,g}^{(f)}$$

$$[x, g] \mapsto \sum_{n|\infty \ \sigma_n, (n, fg) = 1} \frac{[nx, g]}{n}.$$ 

We calculate its determinant first. Let $S_{f,g} = \text{Supp } m - \text{Supp } f \cup \text{Supp } g$.

For $p \in S_{f,g}$, define

$$\tau_p : V_{m,g}^{(f)} \rightarrow V_{m,g}^{(f)}$$

$$[x, g] \mapsto \sum_{n|p} \frac{[nx, g]}{n}.$$ 

Note that $\tau_{p_i} \circ \tau_{p_j} = \tau_{p_j} \circ \tau_{p_i}$ and

$$\phi_m|_{V_{m,g}^{(f)}} = \tau_{p_1} \circ \cdots \circ \tau_{p_s}$$

where $p_i \in S_{f,g}$. Then we have

$$\det(\phi_m : V_{m,g}^{(f)}) = \prod_{p \in S_{f,g}} \det \tau_p.$$ 

For any $p \in S_{f,g}$, let $c_{p,f}$ be the smallest number satisfying $p^{c_{p,f}} \equiv 1 \pmod{f}$. Since the map $\tau_p$ can be regarded as the left multiplication by the group ring element $\sum_i \frac{\sigma_i^p}{p^i}$ in $\mathbb{R}[G_f]$, then by [11] Lemma 1.2(b), we have

$$\det(\tau_p) = \prod_{\chi \in \hat{G}_f} \chi\left(\sum_i \frac{\sigma_i^p}{p^i}\right) = (1 - p^{-c_{p,f}})^{-\varphi(f)\epsilon_{p,f}} := a_{p,f},$$

and

$$\det(\phi_m : V_{m,g}^{(f)}) = \prod_{p \in S_{f,g}} a_{p,f}.$$ 

Now by the Inclusion-Exclusion Principle, we have

$$\det(\phi_m : V_{m,g}^{(f)}) = \prod_{p | f} \det(\phi_m : V_{m,g}^{(f)}/p) = \prod_{f' | f, f' \neq 1} \det(\phi_m : V_{m,g}^{(f)}/f')^{-\mu(f')}.$$ 

Hence

$$\prod_{p \in S_{f,g}} a_{p,f} = \prod_{f' | f} \det(\phi_m : V_{m,g}^{(f)}/f')^{\mu(f')}.$$

By the Möbius inverse formula,

$$\det(\phi_m : V_{m,g}) = \prod_{f' | f} \prod_{p \in S_{f,g}} a_{p,f}$$
Therefore we have
\[ \prod_i \det(\phi_m: V_m^i)^{(-1)^i} = \prod \left( \prod_{g|m} \prod_{f|m} a_{p,f} \right)^{\mu(g)}. \]

Now let’s look at the right hand side of the above identity. The exponent of \( a_{p,f} \) is
\[ \sum_{g|m, (p,g)=1} \mu(g) = \sum_{g|m} \mu(g) = \begin{cases} 1, & \text{if } \frac{m}{fp} = 1; \\ 0, & \text{otherwise}. \end{cases} \]
here \( p^a \parallel m \). Write \( m = m_p \cdot p^a \), then
\[ \prod_i \det(\phi_m: V_m^i)^{(-1)^i} = \prod_{p|m} a_{p,m_p} = \prod_{\chi \text{ odd } p|m} \left( 1 - \chi(p)p^{-1} \right)^{-1}, \]
which is exact the right hand side of the identity (5.3). \( \square \)

Now let \( V_m^\theta = \{ x \in V_m : \theta \cdot x = 0 \} \). \( V_m^\theta \) has a basis consisting of \( \{ [x,g] - [-x,g] : 0 < x < 1/2 \} \subseteq V_m \). Denote by \( \phi_m^\theta \) the restriction of \( \phi_m \) on \( V_m^\theta \). Then \( \phi_m^\theta \) is an automorphism of \( V_m^\theta \). We have

**Proposition 5.3.**

(5.4) \[ \prod_i \det(\phi_m^\theta: V_m^i)^{(-1)^i} = \prod_{\chi \text{ odd } p|m} \prod_{\chi \text{ odd } p|m} \left( 1 - \chi(p)p^{-1} \right)^{-1}. \]

**Proof.** The proof is similar to the proof of Proposition 5.2. Note that \( V_{m,g}^{(f)_\theta} \) is a real vector space with a basis \( \{ [x,g] - [-x,g] : (a,f) = 1, 0 < a < f/2 \} \). On the quotient space \( V_{m,g}^{(f)_\theta} \),
\[ \phi_m^\theta : [x,g] - [-x,g], \quad \mapsto \sum_{n|m, \infty \wedge Z = m} \frac{[nx,g] - [-nx,g]}{n}. \]
Now the restriction of \( \tau_p \) on \( V_{m,g}^{(f)_\theta} \) is
\[ \tau_p^\theta : [x,g] - [-x,g], \quad \mapsto \sum_{n|p, \infty} \frac{[nx,g] - [-nx,g]}{n}. \]
We still have
\[ \phi_m^\theta|_{V_{m,g}^{(f)_\theta}} = \tau_{p_1}^\theta \circ \cdots \circ \tau_{p_s}^\theta, \]
where \( p_i \in S_{f,g} \). Similar to the calculation of \( \det \tau_p \) in Proposition 5.2, we have
\[ \det \tau_p^\theta := b_{p,f} = \begin{cases} (1 - p^{-c_{p,f}})^{-\varphi(f)/2c_{p,f}}, & \text{if } c_{p,f} \text{ odd}; \\ (1 + p^{-c_{p,f}/2})^{\varphi(f)/c_{p,f}}, & \text{if } c_{p,f} \text{ even}. \end{cases} \]
We have
\[ \prod_i \det(\phi_m^\theta: V_m^i)^{(-1)^i} = \prod_{p|m} b_{p,m_p} = \prod_{\chi \text{ odd } p|m} \prod_{\chi \text{ odd } p|m} \left( 1 - \chi(p)p^{-1} \right)^{-1}. \] \( \square \)
6. Computation of $H^*(L_m, d_{1m})$ and $H^*(L_m, d_{2m})$

This section is dedicated to the computation of the cohomology groups $H^*(L_m, d_{1m})$ and $H^*(L_m, d_{2m})$. We first introduce module structures on $A$ and $L = \cup L_m$. Using this structure, we find new bases for $A$ and $L$, which is applied to study the cohomology groups.

Let $\Lambda = \mathbb{Z}[X_2, X_3, \cdots, X_p, \cdots]$ be the polynomial ring generated by indeterminants $X_p$ for all prime number $p$. For every positive integer $n = \prod p^v$, put
\[
X_n = \prod X_p^{n_p}, \quad Y_n = \prod (1 - X_p)^{n_p},
\]
then the set $\{X_n, n \in \mathbb{N}\}$ is a $\mathbb{Z}$-basis of $\Lambda$, so is $\{Y_n, n \in \mathbb{N}\}$. Now $A$ and $L$ are equipped with $\Lambda$-module structures by the following rules:
\[
X_n[a] = \sum_{nb=a} [b], \quad X_n[a, g] = \sum_{nb=a} [b, g].
\]
Put
\[
d_1[a, g] := \sum_{p | g} \epsilon(g, p) Y_p[a, g/p],
\]
and
\[
d_2[a, g] := \sum_{p | g} \epsilon(g, p) \cdot (-X_p)[a, g/p].
\]
Then $d_1^2 = d_2^2 = 0$ and $L$ is equipped with a cochain complex structure by $d_1$ or $d_2$.

Furthermore
\[
d_1|_{L_m} = d_{1m}, \quad d_2|_{L_m} = d_{2m}.
\]

For any $a \in \mathbb{Q}/\mathbb{Z}$, we can uniquely write
\[
a \equiv \sum_p \sum_v a_{pv} \frac{1}{p^v} \pmod{\mathbb{Z}},
\]
where $0 \leq a_{pv} < p$ for each pair of any prime number $p$ and any positive integer $v$. Note that $a_{pv} = 0$ for all but finite any $\{p, v\}$. For each nonnegative integer $k$ we define $\mathcal{R}_k$ to be the set of $a \in \mathbb{Q}/\mathbb{Z}$ such that there exist at most $k$ prime numbers $p$ such that $a_{p1} = p - 1$. In particular, $\mathcal{R}_0$ is the set of $a \in \mathbb{Q}/\mathbb{Z}$ such that such that $a_{p1} \neq p - 1$ for all prime numbers $p$.

**Proposition 6.1.** (1) For each positive integer $m$, the collection
\[
\{X_n[a] : n \mid m, \ a \in \mathcal{R}_0 \cap \frac{n}{m}\mathbb{Z}/\mathbb{Z}\}
\]
constitutes a basis for the free abelian group $A_m$.

(2) The collection $\{X_n[a] : n \in \mathbb{N}, a \in \mathcal{R}_0\}$ constitutes a basis for the free abelian group $A$.

(3) As a $\Lambda$-module $A$ is free with a $\Lambda$-basis $\{[a] : a \in \mathcal{R}_0\}$.

(4) In (1) and (2), if we change $X_n$ by $Y_n$, the related results are still true.

**Proof.** First note that $(1) \Rightarrow (2) \Rightarrow (3)$. For $(1)$, since
\[
|\mathcal{R}_0 \cap \frac{1}{m}\mathbb{Z}/\mathbb{Z}| = \varphi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|,
\]
it suffices to show that the given collection generates \( A_m \). This can be easily deduced by induction to \( R_k \), with the fact that for any \( a \in A_m \),

\[
[a] = -\sum_{i=1}^{p-1} \frac{a - i}{p} + X_p[p^a].
\]

For (4), note that the identity

\[
X_n - (-1)^n Y_n = \sum_{l|n} c_{nl} t_l,
\]

holds for any \( n = \prod p_i^{n_i} \) and integer constants \( c_{nl} \), therefore (4) follows immediately from (1) and (2).

By Proposition 6.2, we have

**Proposition 6.2.** (1). For each positive integer \( m \), the collection

\[
\{ X_n[a,g] : ng \mid m, \; n \in \mathbb{N}, \; g \text{ squarefree}, \; a \in R_0 \cap \frac{ng}{m} \mathbb{Z}/\mathbb{Z} \}
\]

constitutes a basis for the free abelian group \( L_m \).

(2). The collection \( \{ X_n[a,g] : n \in \mathbb{N}, \; g \text{ squarefree}, \; a \in R_0 \} \) constitutes a basis for the free abelian group \( L \).

(3). In (1) and (2), if we change \( X_n \) by \( Y_n \), the related results are still true.

With the help of Proposition 6.2, we can compute the cohomology groups of \((L_m, d_{1m})\) and \((L_m, d_{2m})\).

**Theorem 6.3.** (1). The complex \((L, d_1)\) and \((L, d_2)\) are acyclic in negative degree, moreover, \( H^0(L, d_1) \) is the universal distribution \( U \) and \( H^0(L, d_2) \) is the universal predistribution \( \mathcal{O} \).

(2). The complex \((L_m, d_{1m})\) and \((L_m, d_{2m})\) are acyclic in negative degree, moreover, \( H^0(L_m, d_{1m}) \) is \( U_m \) and \( H^0(L_m, d_{2m}) \) is \( \mathcal{O}_m \). In both cases, the natural map \( H^0(L_m) \to H^0(L) \) is injective.

**Proof.** For each prime number \( p \), we define operators \( d_2^p \), \( T_2^p \) and \( \pi_2^p \) on \( L \) by the rules:

\[
d_2^p X_n[a,g] = \begin{cases} 
-\epsilon(g,p) X_{np}[a,g/p], & \text{if } p \mid g, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
T_2^p X_n[a,g] = \begin{cases} 
-\epsilon(gp,p) X_{np}[a,gp], & \text{if } (p,g) = 1 \& p \mid n, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
\pi_2^p X_n[a,g] = \begin{cases} 
X_n[a,g], & \text{if } (p,ng) = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

where \( X_n[a,g] \) runs through the basis given in Theorem 6.2(2). It is easy to check:

\[
d_2^p T_2^q = T_2^q d_2^p = \delta_{pq}(1 - \pi_2^p)
\]

where \( \delta_{pq} \) is the Kronecker symbol, and

\[
\pi_2^p \pi_2^q = \pi_2^q \pi_2^p = \pi_2^q \pi_2^p.
\]

By Proposition 6.2(1), for any fixed positive integer \( m \), we can check that

\[
d_2^p L_m, \; T_2^p L_m, \; (1 - \pi_2^p)L_m \subseteq \begin{cases} 
L_m, & \text{if } p \mid m, \\
0, & \text{if } (p,m) = 1.
\end{cases}
\]
Now for a fixed positive integer $m$, we define $d_{2m}$, $T_{2m}$ and $\pi_{2m}$ on $L_m$ by the rules:

$$d_{2m} := \sum_{p|m} d_2^p,$$
$$T_{2m} := \sum_{p|m} (\sum_{q<p} \pi_2^q) T_2^p,$$
$$\pi_{2m} := \prod_{p|m} \pi_2^p,$$

here we abuse the notations $d_2^p$, $T_2^p$ and $\pi_2^p$ with their restrictions on $L_m$. It is easy to check the definition $d_{1m}$ here coincides the one we defined in §4. We can also check that

$$\pi_{2m} X_n[a, g] = \begin{cases} [a, 1], & \text{if } n = 1 & g = 1; \\ 0, & \text{otherwise.} \end{cases}$$

for any elements $X_n[a, g]$ of the basis given in Theorem 3.2.(1). Now we have

$$d_{2m}T_{2m} + T_{2m}d_{2m} = (\sum_{\ell|m} d_2^\ell) (\sum_{p|m} (\sum_{q<p} \pi_2^q) T_2^p) + (\sum_{p|m} (\sum_{q<p} \pi_2^q) T_2^p) (\sum_{\ell|m} d_2^\ell)$$
$$= \sum_{p} \sum_{\ell} \left( \sum_{q<p} \pi_2^q \right) (d_2^\ell T_2^p + T_2^p d_2^\ell)$$
$$= \sum_{p} (\sum_{q<p} \pi_2^q) (1 - \pi_2^p)$$
$$= \sum_{p} \left( \sum_{q<p} \pi_2^q - \pi_2^p \right)$$

$$= 1 - \pi_{2m}.$$

By the above argument, the cochain map $\text{id} : (L_m, d_{2m}) \to (L_m, d_{2m})$ is homotopic to the cochain map $\pi_{2m}$. But in the negative degree, $\pi_{2m}$ is nothing but the zero map. Therefore we showed that $(L_m, d_{2m})$ is acyclic in negative degrees. Now since for the map $d_{2m}$ (resp. $T_{2m}$, $\pi_{2m}$), $d_{2m}|_{L_m \cap L_m'} = d_{2m'}|_{L_m \cap L_m'}$ (resp. $T_{2m'}$, $\pi_{2m'}$), there exists a unique operator $d_2$ (resp. $T_2$, $\pi_2$) with restriction at $L_m$ the operator $d_{2m}$ (resp. $T_{2m}$, $\pi_{2m}$). $\pi_2$ is a cochain map homotopic to the identity map of the cochain complex $(L, d_2)$ and vanishes at negative degrees. Therefore $(L, d_2)$ is acyclic at negative degrees. Now for $n = 0$, the cohomology groups $H^0(L, d_2)$ and $H^0(L_m, d_{2m})$ easily follow from the definitions of $d_2$ and $d_{2m}$.

Now by a parallel argument to $d_1$ and $Y_n$, we construct $T_1$, $\pi_1$ and $T_{1m}$, $\pi_{1m}$ respectively. the remaining assertions follow immediately.

Remark 1. The proof here is given by Anderson in the preprint version of [2].

2. In the higher rank case, we also have similar result by applying essentially the same trick.

7. More spectral sequences

In the following sections we are going to use the spectral sequence method to attack the cochain complexes introduced in §4. First recall:

- $J = \mathbb{Z}/2\mathbb{Z} = \{1, c\} \subseteq G_m$, $\theta = 1 + c \in \mathbb{Z}[J]$;
- $L_m = \langle [a, g] : g \mid m, a \in \frac{2m}{m}\mathbb{Z}/\mathbb{Z} \rangle >, d = d_1$ or $d_2$;
- $H^0_{d_1}(L_m) = U_m, H^0_{d_2}(L_m) = O_m$;
- $r = \#\text{Supp } m, g \mid m, g$ square free, $p = -\#\text{Supp } g$. 

Now let $M = \text{coker}(Z[J] \overset{1+c}{\longrightarrow} Z[J])$. Consider two chain complexes

$$(P, \partial) : \cdots \overset{\partial_{q+1}}{\longrightarrow} Z[J]_{q+1} \overset{\partial_q}{\longrightarrow} Z[J]_q \overset{\partial_{q-1}}{\longrightarrow} \cdots \overset{\partial_0}{\longrightarrow} Z[J]_0 \longrightarrow 0$$

and

$$(F, \partial) : \cdots \overset{\partial_{q+1}}{\longrightarrow} Z[J]_{q+1} \overset{\partial_q}{\longrightarrow} Z[J]_q \overset{\partial_{q-1}}{\longrightarrow} \cdots \overset{\partial_0}{\longrightarrow} Z[J]_0 \longrightarrow 0$$

where $Z[J]_q = Z[J]$ and $\partial_q = 1 + (-1)^q \cdot c$. It is clear that $P$ is a projective resolution of $M$ and $F$ is an exact sequence. Regard the cochain complex $(L_m, d)$ as $A$ in §2, we can construct two double complexes by

$$K^{p,q} = \begin{cases} \text{Hom}_G(Z[J]_q, L^p_m) := (L^p_m, q), & \text{if } q \geq 0 \\ 0, & \text{if } q < 0 \end{cases}$$

and

$$F^{p,q} = (L^p_m, q)$$

where the induced differentials $\delta_q : (x, q) \mapsto ((-1)^p(1 + (-1)^q)c)x + 1)$ and $d : (x, q) \mapsto (d(x), q)$. The two filtrations for the double complex $K^{**}$ are

$$'\text{Fil}^p K^{**} = \bigoplus_{p' \geq p} K^{p',q}$$

and

$$''\text{Fil}^q K^{**} = \bigoplus_{q' \geq q} K^{p,q''}.$$
We consider and calculate the cohomology groups one by one: $X_{J,L}^q = \hat{H}^q(J,L_m^p)$, if $q$ odd; $H^2(J,L_m^p)$, if $q$ even.

Now for the Galois cohomology $\hat{H}^q(J,L_m^p)$, first since (Note that $m \neq 2 \mod 4$ by our assumption)

$$L_{m,g} = \begin{cases} \bigoplus_{a \neq 0} \mathbb{Z}[a,g] \bigoplus \mathbb{Z}[-a,g] \bigoplus \mathbb{Z}[a,0] \bigoplus \mathbb{Z}[a,g], & \text{if } m \text{ even;} \\ \bigoplus_{2a \neq 0} \mathbb{Z}[a,g] \bigoplus \mathbb{Z}[-a,g] \bigoplus \mathbb{Z}[0,g], & \text{if } m \text{ odd.} \end{cases}$$

Then

$$\hat{H}^q(J, L_{m,g}) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } q \text{ odd, } m \text{ even;} \\ (\mathbb{Z}/2\mathbb{Z}), & \text{if } q \text{ odd, } m \text{ odd;} \\ 0, & \text{if } q \text{ even.} \end{cases}$$

Hence

$$\hat{H}^q(J, L_m^p) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 \langle - \rangle, & \text{if } q \text{ odd, } m \text{ even;} \\ (\mathbb{Z}/2\mathbb{Z}) \langle - \rangle, & \text{if } q \text{ odd, } m \text{ odd;} \\ 0, & \text{if } q \text{ even.} \end{cases}$$

Denote by $X_g$ the cocycle represented by $[0,g]$ and by $Y_g$ the cocycle represented by $[1/2,g]$, then for $q > 0,$

$$'E_1^{p,q} = \hat{H}^q(J, L_m^p) = \begin{cases} \bigoplus_g (\langle X_g \rangle \bigoplus \langle Y_g \rangle), & \text{if } q \text{ odd, } m \text{ even;} \\ \bigoplus_g \langle X_g \rangle, & \text{if } q \text{ odd, } m \text{ odd;} \\ 0, & \text{if } q \text{ even.} \end{cases}$$

We consider $'E_1^{p,q}$ as a finite dimensional $\mathbb{Z}/2\mathbb{Z}$ vector space. Immediately we have $'E_2^{p,q} = 0$ for $q$ even. Now for $q$ odd, if $m$ is odd, the induced differential $d^1$ is

$$X_g \xrightarrow{d^1} 0, \quad X_g \xrightarrow{d^1} \sum_{i=1}^{-p} X_{g/p_i};$$

if $m$ is even, the induced differential $d^1$ is

$$X_g \xrightarrow{d^1} \delta_{2p_1} Y_{g/2}, \quad Y_g \xrightarrow{d^1} \delta_{2p_1} Y_{g/2}.$$

and

$$X_g \xrightarrow{d^1} \sum_{i=1}^{-p} X_{g/p_i} + \delta_{2p_1} Y_{g/2},$$

$$Y_g \xrightarrow{d^1} \sum_{i=1}^{-p} Y_{g/p_i} + \delta_{2p_1} Y_{g/2}.$$

Set $X_m^p = H^2(J, L_m^p)$. Let $X_m^\bullet$ be the cochain complex formed by $X_m^p$ and $d^1$. By definition, for any even positive $q$, $'E_2^{p,q}$ is just the $p$-th cohomology group $(X_m^\bullet,d^1)$. We calculate the cohomology groups one by one:

1. $m$ is odd and $d^1 = d^1_{-1}$. This is trivial:

$$'E_2^{p,q}, d_1 = 'E_1^{p,q} = (\mathbb{Z}/2\mathbb{Z}) \langle - \rangle.$$
(2). \( m \) is odd and \( d^1 = d_2^1 \). In this case, if \( m = p^n \), it is easy to see that \( H^0(X_{m^*}, d_1^1) = H^{-1}(X_{m^*}, d_1^1) = 0 \). Now if \( m = m_1m_2 \) and \( (m_1, m_2) = 1 \), we can check

\[
(X_{m_1*}, d_1^{2m_1}) = (X_{m_1*}, d_1^{1}) \otimes (X_{m_2*}, d_1^{1}).
\]

By Küneth’s formula, \( H^p(X_{m*}, d_2^1) = 0 \). Therefore we have

\[
(\prime E_2^{p,q}, d_2) = \cdots = (\prime E_\infty^{p,q}, d_2) = 0.
\]

(3). \( m \) is even and \( d^1 = d_1^1 \). Since \( X_{m^*} \) is a \( \mathbb{Z}/2\mathbb{Z} \)-vector space, by the formula above about \( d_1^1 \), we always have

\[
\dim_{\mathbb{Z}/2\mathbb{Z}} \text{im}(X_m^p \rightarrow X_m^{p+1}) = \binom{r-1}{p-1},
\]

therefore

\[
\dim_{\mathbb{Z}/2\mathbb{Z}} \ker(X_m^p \rightarrow X_m^{p+1}) = 2 \binom{r}{p} - \binom{r-1}{p-1}.
\]

Hence

\[
\dim_{\mathbb{Z}/2\mathbb{Z}} H^p(X_{m^*}, d_1^1) = 2 \binom{r}{p} - \binom{r-1}{p-1} - \binom{r-1}{p-1} = \binom{r}{p}.
\]

Or we have

\[
(\prime E_2^{p,q}, d_1) = (\mathbb{Z}/2\mathbb{Z})^{(\cdot)}.
\]

(4). \( m \) is even and \( d^1 = d_2^1 \). In this case, if \( m = 2^k \),

\[
H^0(X_{2^k^*}, d_2^1) = H^{-1}(X_{2^k^*}, d_2^1) = \mathbb{Z}/2\mathbb{Z}.
\]

Now if \( m = 2^km' \), \( m' > 1 \) odd, set

\[
X_{m'}^{p^m} = \bigoplus_{g|m'} (< X_g > \bigoplus < Y_g >)
\]

and

\[
d'_2 : X_g \mapsto \sum_{i=1}^{-p} X_{g/p_i}, \ Y_g \mapsto \sum_{i=1}^{-p} Y_{g/p_i}.
\]

Then we have

\[
(X_{m*}^*, d_2^1) = (X_{2^k*}^*, d_2^1) \otimes (X_{m'j*}^*, d_2^1).
\]

Similar to the case (2), we can see \( H^p(X_{m*}^*, d'_2) = 0 \). By Küneth’s formula again,\n
\[
(\prime E_2^{p,q}, d_2) = H^p(X_{m*}^*, d_2^1) = 0.
\]

Combining all the cases above, for \( d = d_1, q > 0 \), we have

\[
(\prime E_2^{p,q}, d_1) = \begin{cases} 
(\mathbb{Z}/2\mathbb{Z})^{(\cdot)}, & \text{if } q \text{ odd}; \\
0, & \text{otherwise}.
\end{cases}
\]

For \( d = d_2, q > 0 \), we have

\[
(\prime E_2^{p,q}, d_2) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & \text{if } q \text{ odd}, \ m = 2^k, \ p = 0 \text{ or } -1; \\
0, & \text{otherwise}.
\end{cases}
\]
Similarly for $F^{*,*}$, for $d = d_1$, $q \in \mathbb{Z}$, we have

\[
\langle \epsilon^{p,q}_{2,F}, d_1 \rangle = \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & \text{if } q \text{ odd; } \\
0, & \text{otherwise.}
\end{cases}
\]

For $d = d_2$, $q \in \mathbb{Z}$, we have

\[
\langle \epsilon^{p,q}_{2,F}, d_2 \rangle = \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & \text{if } q \text{ odd, } m = 2^k, p = 0 \text{ or } -1; \\
0, & \text{otherwise.}
\end{cases}
\]

Our next task is to show that $F^{*,*}$ collapses at $\epsilon^{p,q}_{2,F}$. For this purpose, we define a subcomplex $SF^{*,*}$ of $F^{*,*}$:

\[
SF^{p,q} = \begin{cases} 
(L^p_m, q), & \text{if } q \text{ even; } \\
(\beta(L^p_m), q), & \text{if } q \text{ odd.}
\end{cases}
\]

where

\[
\beta([a, g]) = \begin{cases} 
[a, g], & \text{if } 2a \neq 0; \\
2[a, g], & \text{if } 2a = 0.
\end{cases}
\]

It is easy to verify $d$ and $\delta$ are well defined in $SF^{*,*}$. Moreover, $H^0(SF^{*,*}) = 0$ for every pair $(p, q)$. Therefore the total cohomology group of $SF^*$ is trivial. Hence to study $F^{*,*}$, it suffices to study the double complex $QF^{*,*} = F^{*,*}/SF^{*,*}$. Note that $QF^{*,*}$ vanishes at the odd rows and at the even rows, $QF^{p,q}$ is nothing but $\epsilon^{p,q}_{1,F}$ we just got above. Moreover, the induced differential $d$ in $QF^{*,*}$ is nothing but $d^1$ in $\epsilon^{*,*}_{1,F}$. On one hand

\[
H^n(QF^*) = H^n(F^*) = \hat{H}^n(J, H^0_d(L_m)) = \bigoplus_q \epsilon^{n-q,q}_{E_{\infty,F}};
\]

on the other hand, we have

\[
H^n(QF^*) = \frac{\ker (d + \delta : \bigoplus_q QF^{n-q,q} \to \bigoplus_q QF^{n+1-q,q})}{\im (d + \delta : \bigoplus_n QF^{n-1-q,q} \to \bigoplus_q QF^{n-q,q})}
\]

\[
= \bigoplus_q \frac{\ker (d : QF^{n-q,q} \to QF^{n+1-q,q})}{\im (d : QF^{n-1-q,q} \to QF^{n-q,q})}
\]

\[
= \bigoplus_q \epsilon^{n-q,q}_{E_{2,F}}.
\]

Therefore for any pair $(p, q)$,

\[
\epsilon^{p,q}_{2,F} = \epsilon^{p,q}_{E_{\infty,F}}.
\]

In particular, for $n = 1$,

\[
\hat{H}^1(J, H^0_d(L_m)) = \bigoplus_q \epsilon^{1-q,q}_{2,F} = \bigoplus_q \epsilon^{1-q,q}_{E_2}.
\]

By results of (7.1) and (7.2), we easily have

**Theorem 7.1.** The group $J = \{1, c\}$ acts trivially on the cohomology groups $H^i(J, \mathcal{O}_m)$ and $H^i(J, U_m)$ for $i = 1$ or $2$, moreover,

\[
H^1(J, \mathcal{O}_m) = H^2(J, \mathcal{O}_m) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & \text{if } m = 2^k; \\
0, & \text{otherwise.}
\end{cases}
\]
and
\[ H^1(J, U_m) = H^2(J, U_m) = (\mathbb{Z}/2\mathbb{Z})^{2r-1}. \]

**Remark.** 1. The second statement is first proved in Yamamoto [13]. The spectral sequence method employed here makes the calculation significantly simpler than those in [13] and in [10]. Moreover, this same spectral sequence method can also be applied to the universal distribution of higher rank, thus recover the results in Kubert [8].

2. For any cyclic group \( C \in G \) which has trivial intersection with \( G_{p_i} \) for \( p_i \lVert m \), we can also obtain similar result without any extra difficult.

**Proposition 7.2.**
\[
I(L_m, d_1; \theta) = \begin{cases} 
2, & \text{if } r = 1; \\
2^{2r-2}, & \text{if } r > 1.
\end{cases}
\]
\[
I(L_m, d_2; \theta) = \begin{cases} 
2, & \text{if } m = 2^k; \\
1, & \text{otherwise}.
\end{cases}
\]

**Proof.** By the identity (7.3), the condition in Proposition 3.1 is satisfied. For \( d = d_1 \), by (7.1), then the exponent of 2 in \( I(L_m, d_1; \theta) \) is equal to
\[
\sum_{p+q \leq r, q > 0, \text{odd}} (-1)^{p+1} \binom{r}{-p} = \sum_{p=r-q, q \leq 0, \text{odd}} (-1)^{p+1} \binom{r}{-p} = \sum_{p=1}^{r} \binom{p+1}{2} (-1)^{p} \binom{r}{p} = \sum_{k} k \binom{r}{2k-1} - \sum_{k} k \binom{r}{2k} = \begin{cases} 
1, & \text{if } r = 1; \\
2^{2r-2}, & \text{if } r > 1.
\end{cases}
\]
The case \( d = d_2 \) immediately follows from Proposition 3.1 and (7.2).

8. Sinnott’s index formula

In this section we set to prove the following theorem:

**Theorem 8.1** (See [10], Theorem). Let \( R = \mathbb{Z}[\zeta_m] \) and let \( S \) be the Stickelberger ideal of \( \mathbb{Q}(\zeta_m) \). Then
\[
[R^{-} : S^{-}] = 2^a h^{-},
\]
where \( a = 0 \) if \( r = 1 \) and \( a = 2^{r-2} - 1 \) if \( r > 1 \).

**Note.** In this section, the subscript \( m \) is omitted from our notations(i.e., \( G \) is the Galois group \( G_m \) and so on). \( p \) is always regarded as a prime factor of \( m \). The superscript “\(^{-}\)”, is in accordance with the superscript “\(^{-}\)” in the previous sections.

**Proof.** We consider the following diagram:
where \( s > 1 \) and \( \varphi^- = \varphi|_{\mathcal{U}^-} \),

\[
\psi^{(s)}([x] - [-x]) = \sum_{(n, m) = 1} \frac{[nx] - [-nx]}{n^s},
\]

\[
\beta([x] - [-x]) = \frac{1}{2\pi i} \sum_{t \in (\mathbb{Z}/m\mathbb{Z})^*} (\exp(2\pi i t x) - \exp(-2\pi i t x)) \sigma_t^{-1}
\]

and

\[
\alpha^{(s)} = \beta \circ \psi^{(s)} \circ \varphi^-.
\]

\( \psi^{(s)} \) is well defined and all the above maps are isomorphisms of vector spaces. Then we have

\[
(R^- : \alpha^{(s)}(U^-)) = \begin{cases} 
(2\pi)^{-\varphi(m)/2} \sqrt{d(K_m)/d(K_m^+)}, & \text{if } m \neq 2^k; \\
\frac{1}{2}(2\pi)^{-\varphi(m)/2} \sqrt{d(K_m)/d(K_m^+)}, & \text{if } m = 2^k.
\end{cases}
\]

**Proof of Lemma 8.2.** We first consider the following diagram which is exact at the rows:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \mathcal{O}^+ & \longrightarrow & \mathcal{O} & \longrightarrow & \operatorname{im}(1 - c) & \longrightarrow & 0 \\
& & \uparrow \text{id} & & \uparrow i & & \uparrow i & & \\
0 & \longrightarrow & \mathcal{O}^+ & \longrightarrow & \mathcal{O}^+ \oplus \mathcal{O}^- & \longrightarrow & 2\mathcal{O}^- & \longrightarrow & 0
\end{array}
\]

where \( i \) is the natural inclusion map. By Theorem 7.4, if \( m \) is not a power of 2, \( \mathcal{O}^- = \operatorname{im}(1 - c); \) if \( m \) is a power of 2, then \( \mathcal{O}^- \slash \operatorname{im}(1 - c) = \mathbb{Z}/2\mathbb{Z}. \) Therefore,

\[
(\mathcal{O} : \mathcal{O}^+ \oplus \mathcal{O}^-) = (\operatorname{im}(1 - c) : 2\mathcal{O}^-) = \begin{cases} 
2^{\varphi(m)/2}, & \text{if } m \neq 2^k; \\
2^{\varphi(m)/2 - 1}, & \text{if } m = 2^k.
\end{cases}
\]

Now let \( T \) be the map from \( \mathbb{C}\mathcal{O} \) to \( \mathbb{C}[G] \) such that \( T([x]) = \sum_i \exp(2\pi i tx) \sigma_t^{-1} \), and then we have \( T|_{\mathcal{O}^-} = 2\pi i \beta|_{\mathcal{O}^-}. \) Then on one hand,

\[
(R^+ \oplus R^- : T(\mathcal{O}^+ \oplus \mathcal{O}^-)) = (R^+ : T(\mathcal{O}^+)) \cdot (R^- : T(\mathcal{O}^-))
\]

on the other hand,

\[
(R^+ \oplus R^- : T(\mathcal{O}^+ \oplus \mathcal{O}^-)) = (R^+ \oplus R^- : R) \cdot (R : T(\mathcal{O})) \cdot (\mathcal{O} : \mathcal{O}^+ \oplus \mathcal{O}^-).
\]

But we know \( (R^+ \oplus R^- : R) = 2^{-\varphi(m)/2}, \) and by the definition of \( T, (R : T(\mathcal{O})) = \sqrt{d(K)} \) and \( (R^+ : T(\mathcal{O}^+)) = \sqrt{d(K^+)}. \) Now the lemma follows from the above results and

\[
(R^- : \beta(\mathcal{O}^-)) = (2\pi)^{-\varphi(m)/2}(R^- : 2\pi i \beta(\mathcal{O}^-)).
\]
Lemma 8.3. Let $S = \{ p : p | m \}$, then

\[ (\mathcal{O}^- : \psi(s)(\mathcal{O}^-)) = \prod_{\chi \text{ odd}} L_S(s, \chi). \]  

(8.3)

Proof of Lemma 8.3. Note that if we let

\[ \Theta_S(s) = \sum_{(n,m)=1} \frac{\sigma_n}{n^s}, \]

then $\psi(s)$ is just the left multiplication of $\Theta_S(s)$ on $\mathbb{R}\mathcal{O}^-$. By Lemma 1.2(b), we have

\[ (\mathcal{O}^- : \psi(s)(\mathcal{O}^-)) = \prod_{\chi \text{ odd}} \chi(\Theta_S(s)) = \prod_{\chi \text{ odd}} L_S(s, \chi). \]

Lemma 8.4.

\[ (\mathcal{O}^- : \varphi^-(U^-)) = \begin{cases} 2^{-2r-2} \prod_{p|m} \prod_{\chi \text{ odd}} (1 - \chi(p)^{-1})p^{-1}, & \text{if } r > 1; \\ \frac{1}{2}, & \text{if } r = 1, p \neq 2; \\ 1, & \text{if } m = 2^k. \end{cases} \]

(8.4)

Proof of Lemma 8.4. This follows from the abstract index formula (2.3), Proposition 5.3 and Proposition 7.2.

Now let $s$ approach 1, then

\[ \lim_{s \to 1} \alpha(s)([x] - [-x]) = \lim_{s \to 1} \beta \psi(s) H^0(\varphi)([x] - [-x]) \]

\[ = \frac{1}{2\pi i} \sum L_{\chi} \sum_{n \in \mathbb{N}} \exp(2\pi i nt) - \exp(-2\pi i nt) \]

\[ = \sum (\frac{1}{2} - \{nt\})\sigma_t^{-1}. \]

(8.5)

If we let $\alpha = \lim_{s \to 1} \alpha(s)$, by (8.1), (8.2), (8.3) and (8.4), with the class number formula,

\[ h^- = (2\pi)^{-\varphi(m)/2} \prod_{\chi \text{ odd}} L(1, \chi) \frac{d(K_m)}{d(K_m^+)} \omega Q, \]

and since $(U^- : (1 - c)U) = 2^{2r-1}$, then we have

\[ (R^- : \alpha((1 - c)U)) = \lim_{s \to 1} (R^- : \alpha(s)(U^-)) \cdot (U^- : (1 - c)U) \]

\[ = \begin{cases} \frac{h^-}{\omega Q} \cdot 2^{2r-2}, & \text{if } r > 1; \\ \frac{h^-}{\sqrt{2}}, & \text{if } r = 1. \end{cases} \]

(8.6)

But by (8.5), $\alpha((1 - c)U)$ is nothing but $e^{-S'}$ in \[ ] and by \[ ], Lemma 3.1, we have $\omega$. This is enough to finish the proof of the theorem...
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