Discrete Strings and Deterministic Cellular Strings

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ABSTRACT

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A discrete string theory –a theory of embeddings from $\mathbb{Z} \times \mathbb{Z}_C \rightarrow \mathbb{R}^D$, where $C$ is the number of components of the string– is explored. The closure of the algebra of constraints (‘$\mathbb{Z}_C$-Virasoro algebra’) is exhibited. The $\mathbb{Z}_C$-Virasoro ‘algebra’ is shown to be anomaly free in arbitrary number of target space dimensions. We prove the existence of a (manifestly unitary) light-cone gauge with anomaly free Lorentz algebra in any dimensions. The analog of vertex operators are introduced and the physical spectrum is analysed. There are an infinite number of higher-level states repeating a certain mass pattern and leading to an infinite degeneracy. The connection with the continuum string theory (in $D = 26$) is investigated. Independently, following a method recently introduced by ’t Hooft based on Hilbert-space extension of deterministic systems, a particular one-dimensional cellular automaton submitted to a deterministic evolution is shown to reduce to a massless scalar field theory at long distance scales. This automaton is utilized to define a (‘cellular’) string theory with world-sheet variables evolving under deterministic rules, which in the framework of ‘first-quantization’ corresponds to the $\mathbb{Z}_C$ string theory mentioned above. We show that in this theory also the target space motion of free strings is governed by deterministic laws. Finally, we discuss a model for (off-shell) interacting strings where space-time determinism is fully restored.
1. Introduction

The reduction to discrete variables or to simpler systems has always represented an objective in physics. Numerous physicists have attempted to write the final chapter of discretization, namely the quantization of space and time. Some believe that it is at this level that the determinism in physics should be restored. In particular, in ref. [1] Feynman expressed his insatisfaction with our present understanding of quantum mechanics and speculated that at more fundamental level Nature might resemble a cellular automaton (i.e., an array of elements or cells with associated dynamical variables which, at each time step, evolve according to a given law).

An important step in this direction was recently performed by 't Hooft. From his studies on the quantum physics underlying black hole geometries, 't Hooft argues that the number of mutually orthogonal states in Hilbert space inside a closed surface is finite and given by the total area of the surface [2]. This suggests that at the Planck scale space-time may be discrete. A subclass of discrete theories are, precisely, the deterministic discrete theories, in which there is a basis (primitive basis, in 't Hooft terms), where the evolution matrices $U(t_i, t_j)$ are pure permutation matrices. At successive times $t_1, t_2, t_3, ...$ the basis elements are just permuted, $|e_1\rangle \rightarrow |e_2\rangle \rightarrow |e_3\rangle \rightarrow ...$. For an arbitrary vector $|\psi\rangle$ (a linear superposition of $|e_i\rangle$) one defines the probability of a given state $|e\rangle$ as [3]:

$$P(e) = |\langle e | \psi \rangle|^2. \quad (1.1)$$

It turns out that there is a natural way to reproduce the evolution of $|e\rangle$ by writing a Schrödinger equation for $|\psi\rangle$:

$$\frac{d}{dt} |\psi\rangle = -iH |\psi\rangle. \quad (1.2)$$

At integer time $t$ any basis element $|e\rangle$ evolves into $|e_t\rangle$ in accordance to the law of the cellular automaton, but eq. (1.2) also prescribes how the phase factors evolve.
't Hooft suggests that quantum mechanics may be viewed not as a theory about reality, but as a prescription for making the best possible predictions about the future if we have certain information about the past. The Einstein-Podolsky-Rosen paradox, Bell’s theorems, Gedanken experiments, etc. show that one cannot attach labels to electrons to determine their evolution, but they do not prove that hidden variables associated with the vacuum cannot restore predictability in a formal way. As a matter of fact, ’t Hooft provides explicit cellular automaton examples where determinism is restored in virtue of hidden variables associated with the vacuum [3]. These automata mostly evolve chaotically; their long-distance behaviour can only be treated by using statistics. If in some case an effective quantum field theory emerges with a large fundamental distance scale, at large scales only renormalizable (and super renormalizable) couplings survive, thus one of the renormalizable theories should be reproduced.

An example is the Fermi one-dimensional shift automaton [3, 4] (in sect. 2 we will introduce the bosonic analog). One divides the circle in $C$ cells, $C = 2N + 1$, where $N$ is a natural number. There is a a variable $\sigma_x$ which in each cell can take the values 0 or 1. The evolution equation is taken to be

$$\sigma_{x,t+1} = \sigma_{x-1,t} , \quad \sigma_{C,t} = \sigma_{0,t} ,$$  \hspace{1cm} (1.3)

i.e. the spins shift to the right at constant speed. The Hilbert space is generated by the basis elements $|\sigma_1, ..., \sigma_C\rangle$. Having zeros and ones, one may introduce anticommuting creation and annihilation operators $\hat{\sigma}_x^\pm$ at each site $x$, with

$$\hat{\sigma}_x^- |\sigma_x = 0\rangle = 0 , \quad \hat{\sigma}_x^- |\sigma_x = 1\rangle = |\sigma_x = 0\rangle ,$$  \hspace{1cm} (1.4)

whereas the operation of the $\sigma_x^-$ operators does not depend on the contents of the other cells. Then one applies a Jordan-Wigner transformation

$$\psi_x = (-1)^{\sum_{y<x} \sigma_y} \hat{\sigma}_x^- ,$$  \hspace{1cm} (1.5)
so that
\[ \{ \psi_x, \psi_y \} = 0, \quad \{ \psi_x, \psi_y^\dagger \} = \delta_{xy}, \]

(1.6)

and
\[ \psi_{x,t} = \psi_{x-t,0}. \]

By Fourier transforming
\[ \psi_{x,t} = \frac{1}{\sqrt{C}} \sum_{k=-N}^{N} \tilde{\psi}_{k,t} e^{-2\pi ikx/C}, \]

(1.7)

one learns that the evolution equation (1.3) is generated by a Hamiltonian operator,
\[ H = \frac{2\pi}{C} \sum_{k=-N}^{N} k \tilde{\psi}_{-k}^\dagger \tilde{\psi}_{k}, \]

(1.8)

which in the continuum limit approaches to
\[ \int dx \, \psi_{x}^\dagger(x)i \frac{\partial}{\partial x} \psi(x) \]

i.e. the Hamiltonian of a chiral, right-handed fermion. The ground state \(|0\rangle\) is obtained by filling the Dirac sea with negative-\(k\) modes. This state is a linear combination of all states \(|\sigma_1, ..., \sigma_C\rangle\) of the primitive basis. Physical particles are the excitations above this lowest energy state.

A common problem that arises in discrete-time scenarios and it is of course present here is that energy can only be defined modulo 2\(\pi\). We lack a clear physical understanding on how this may be resolved. Another delicate point is the continuum limit. If we send \(C \to \infty\) before introducing a regulator for continuum expressions, for example, in building \(\psi^\dagger \psi\), we miss the correct Schwinger anomaly in the corresponding current algebra. The reason is a cancellation due to contributions of states of momentum \(k \sim \pm N\). One would like to keep only
low-energy excitations of the field $\psi$ in the process of taking the continuum limit. This is automatically accomplished if a proper regularization is introduced prior to letting $N \to \infty$.

In the last decade string theories have fed an unusual hope for a consistent unification of all forces of Nature including gravity. Despite some remarkable progresses, unfortunately string theory continues being no more than a set of Feynman rules. Confined to a first-quantized formalism, very little can be learned about its origins, and few and weak low-energy predictions can be unambiguously stated. The theory is incomplete and one is urged to the search for fundamental principles.

It is tempting to compare with the large-$N$ QCD analogy. Below the deconfining transition effective strings and continuum Riemann surfaces constitute a correct description of large-$N$ QCD, but above the deconfining transition the Riemann surface picture is totally inadequate. Riemann surfaces are inundated by a sea of holes and they must be replaced by Feynman diagrams. Today we know that quantum chromodynamics is the correct fundamental description.

In the case of string and superstring theories, there have been some dim indications that a more fundamental theory should have some discrete structure.

In particular, in ref. [5] it was shown that one of the phases of a light-cone lattice gauge theory with an infinite number of colors describes free fundamental strings, even if the lattice spacing is not taken to zero.

The fact that a continuum Riemann surface description must break down is evidenced by the presence of the Hagedorn transition. Above the Hagedorn temperature the free energy appears to have a genus zero contribution [6], which implies that the world sheet is no longer simply connected. Indeed, at $T = T_H$ a solitonic mode becomes tachyonic and develops a vacuum expectation value, whereby creating a conglomeration of holes in the Riemann surface. The conclusions of ref. [6] are that continuum world sheets should be replaced by some less continuum structure, and that the density of gauge invariant degrees of freedom of string theories
is much less than any ordinary relativistic field theory. A problem which is quite analogous to the $2\pi$ ambiguity in the energy of the cellular automata is the duality relation $T \rightarrow 1/T$ in the heterotic string. The consequences of this symmetry, if preserved, would be catastrophic, since this symmetry implies that the thermodynamical partition function $Z$ at infinite temperature is equal to $Z(T = 0)$, which would mean that there is no fundamental gauge-invariant degrees of freedom in the theory. The expectation is that this duality symmetry is spontaneously broken.

This work is organized as follows. In sect. 2 we introduce a one-dimensional cellular automaton subject to a simple deterministic law and demonstrate that, in the large-distance regime, this automaton can be described in terms of a quantum field theory of a massless scalar field moving in $1+1$ dimensions. The ground state is expressed in terms of the primitive basis in a non-trivial way. In sect. 3 we introduce the open and closed $Z_C$ discrete string theories. The algebra of the constraints is studied and interpreted. Sect. 4 contains the calculation of the $Z_C$-Virasoro anomaly. The light-cone gauge is investigated in sect. 5. In particular, we show that Lorentz covariance is maintained in any number of space-time dimensions. In sect. 6 the spectrum is analysed both in the light-cone gauge formalism and in the covariant approach. Section 7 contains a discussion on vertex operators, and sect. 8 deals with the connection with the continuum theory and the scattering amplitudes. In sect. 9 we consider the string theory that results upon placing the cellular automaton of sect. 2 to dictate the world-sheet dynamics. The deterministic evolution of free cell strings in target space is disclosed. We argue that deterministic laws governing the evolution of an arbitrary number of interacting cell strings, describing in particular splitting and joining, are possible. To elucidate this, we introduce a model for a fully deterministic cellular string theory. Given an arbitrary initial configuration of closed an open strings, the whole evolution, including possible splitting or joining of the strings and distribution of momenta of emerging strings, is determined by very simple rules. However, in cases where the number of cells in play is large, the evolution of the system becomes so complex that it can be more suitably studied by numerical or statistical methods. In sect.
discuss general aspects of the theory presented in the main text. In addition, some speculations are made on how this theory could find application in realistic models.

2. The Bosonic Cellular Automaton

Let us consider a partition of the circle in \( C \) cells. In each cell \( x \) there are two variables \( V^L_x, V^R_x \), \( x = 1, ..., C \) which take values in \( \mathbb{R} \). The evolution equation is defined by

\[
V^R_{x,t+1} = V^R_{x-1,t}, \quad V^L_{x,t+1} = V^L_{x+1,t},
\]

\[
V^R_0 = V^R_C + \frac{q}{2}, \quad V^L_C = V^L_1 + \frac{q}{2}, \quad q = \text{constant}.
\]

Using that

\[
V^L_0 = V^L_1 = V^L_{C+1,t-1} - \frac{q}{2} = V^L_{C,t} - \frac{q}{2}
\]

one sees that the sum \( V^R_x + V^L_x \) is single-valued:

\[
(V^R + V^L)_{C,t} = (V^R + V^L)_0,t.
\]

The ‘zero mode’ part

\[
v^0_t = \frac{1}{C} \sum_{x=1}^{C} (V^L_x + V^R_x)
\]

will be subject to the law

\[
v^0_{t+1} = v^0_t + \frac{1}{C} q.
\]

Separating the zero mode part we can describe the automaton in terms of single-
valued variables $v^{R}_{x,t}, v^{L}_{x,t}$, i.e. by writing

$$V^{L}_{x,t} + V^{R}_{x,t} = v^{0}_{t} + v^{L}_{x,t} + v^{R}_{x,t},$$

(2.6)

$$v^{R,L}_{x,t} = V^{R,L}_{x,t} - (t \mp x)\frac{q}{2C} + \text{const.},$$

(2.7)

where $v^{R}_{x,t}, v^{L}_{x,t}$ evolve as follows

$$v^{R}_{x,t+1} = v^{R}_{x-1,t}, \quad v^{L}_{x,t+1} = v^{L}_{x+1,t}, \quad v^{R,L}_{x+C,t} = v^{R,L}_{x,t}.$$  

(2.8)

The Hilbert space is spanned by the basis

$$\{|v^{0}\rangle \otimes |v^{L}_{1},...,v^{L}_{C}\rangle \otimes |v^{R}_{1},...,v^{R}_{C}\rangle\}.$$  

(2.9)

Let us introduce operators $\hat{\varphi}_{x}, \hat{\pi}_{x}$ at each site $x$ with $[\hat{\varphi}_{x}, \hat{\pi}_{y}] = i\delta_{xy}$, and the decomposition

$$\hat{\varphi}_{x} = \hat{\varphi}^{L}_{x} + \hat{\varphi}^{R}_{x} + \hat{\varphi}^{0}, \quad \hat{\varphi}^{0} = \frac{1}{C} \sum_{x=1}^{C} \hat{\varphi}_{x},$$

(2.10)

$$\hat{\pi}_{x} = \hat{\pi}^{L}_{x} + \hat{\pi}^{R}_{x} + \frac{1}{C} \hat{\pi}^{0}, \quad \hat{\pi}^{0} = \sum_{x=1}^{C} \hat{\pi}_{x},$$

(2.11)

satisfying

$$[\hat{\varphi}^{R}_{x}, \hat{\pi}^{R}_{y}] = [\hat{\varphi}^{L}_{x}, \hat{\pi}^{L}_{y}] = \frac{C-1}{2C} i\delta_{xy}, \quad [\hat{\varphi}^{0}, \hat{\pi}^{0}] = i,$$

(2.12)

and other commutators equal to zero.
The eigenstates of $\hat{\phi}^L_x, \hat{\phi}^R_x, \hat{\phi}^0$ are given by

$$|v\rangle \equiv |v^0\rangle \otimes |v^L_1, ..., v^L_C\rangle \otimes |v^R_1, ..., v^R_C\rangle = e^{-iv^0\pi^0} \prod_{x=1}^C e^{-i\frac{2C}{C+1}(v^L_x\pi^L_x + v^R_x\pi^R_x)|v = 0\rangle},$$

(2.13)

where $\hat{\phi}_x|v = 0\rangle = 0$. It is easy to verify that

$$\hat{\phi}^L_x|v\rangle = v^L_x|v\rangle, \quad \hat{\phi}^R_x|v\rangle = v^R_x|v\rangle, \quad \hat{\phi}^0|v\rangle = v^0|v\rangle.$$  

(2.14)

Let us now expand $\hat{\phi}^R_x, \hat{\phi}^L_x$ in Fourier components

$$\hat{\phi}^R_x = \frac{i}{2} \sum_{n=1}^N \frac{1}{\sqrt{n}} (a_n \omega^{nx} - a_n^\dagger \omega^{-nx}), \quad \hat{\phi}^L_x = \frac{i}{2} \sum_{n=1}^N \frac{1}{\sqrt{n}} (\bar{a}_n \omega^{-nx} - \bar{a}_n^\dagger \omega^{nx}),$$

(2.15)

$$\omega \equiv e^{i\frac{2\pi}{C}},$$

where $N = (C - 1)/2$ if $C$ is odd, $N = \frac{C}{2} - 1$ if $C$ is even, and $l$ is a constant with dimension of length introduced to match standard string theory conventions in the continuum limit (see below). For more simplicity in the presentation of the formulas, in part of this work we will only refer explicitly to one parity of $C$, namely, $C$ odd $\equiv 2N + 1$, the treatment of the other case being exactly the same.

The momentum operator can be defined in terms of the modes $a_n, a_n^\dagger, \bar{a}_n, \bar{a}_n^\dagger$ as follows

$$\hat{\pi}^R_x = \frac{1}{Cl} \sum_{n=1}^N \sqrt{n} (a_n \omega^{nx} + a_n^\dagger \omega^{-nx}),$$

(2.16)

$$\hat{\pi}^L_x = \frac{1}{Cl} \sum_{n=1}^N \sqrt{n} (\bar{a}_n \omega^{-nx} + \bar{a}_n^\dagger \omega^{nx}).$$

(2.17)

From the canonical commutation relations for $\hat{\phi}$ and $\hat{\pi}$ it follows

$$[a_n, a_m] = [\bar{a}_n, \bar{a}_m] = \delta_{nm},$$

(2.18)

and the remaining commutators vanish.
In the Heisenberg picture we have

\[ \hat{\varphi}_L^{x,t} = \hat{\varphi}^0_L x + t, \quad \hat{\varphi}_R^{x,t} = \hat{\varphi}^0_R x - t, \]

\[ \hat{\varphi}^0_t = \hat{\phi} + \frac{\pi l^2}{C} pt, \quad \hat{\phi} \equiv \hat{\varphi}^0_{t=0}, \quad p \equiv \frac{1}{\pi l^2} q. \]  

(2.19)

It will turn convenient introducing the operators

\[ \alpha_n = \sqrt{n} a_n, \quad \alpha_n^\dagger = \sqrt{n} a_n^\dagger, \quad \bar{\alpha}_n = \sqrt{n} \bar{a}_n, \quad \bar{\alpha}_n^\dagger = \sqrt{n} \bar{a}_n^\dagger, \quad n = 1, ..., N. \]  

(2.20)

From eqs. (2.15), (2.19) and (2.20) we see that the time evolution for $\alpha_n, \bar{\alpha}_n$ is given by

\[ \alpha_{n,t} = U^\dagger(t) \alpha_{n,0} U(t) = \omega^{-nt} \alpha_{n,0}, \quad \bar{\alpha}_{n,t} = U^\dagger(t) \bar{\alpha}_{n,0} U(t) = \omega^{-nt} \bar{\alpha}_{n,0}. \]  

(2.21)

Therefore we take as Hamiltonian

\[ H = \frac{\pi l^2}{C} p \hat{\pi}^0 + \frac{2\pi}{C} \sum_{n=1}^{N} (\alpha_{-n} \alpha_n + \bar{\alpha}_{-n} \bar{\alpha}_n) + \text{const.} \]  

(2.22)

The continuum limit is taken by rescaling $t, x \to t/d, x/d$, and letting $N \to \infty$, $d \to 0$ keeping the length of the circle $C d$ finite, $C d = \pi$. In this limit $\varphi$ becomes

\[ \hat{\varphi} = \hat{\phi} + l^2 p t + \frac{i}{2} \sum_{n=\pm \infty, n \neq 0}^{\infty} \frac{1}{n} (\alpha_n e^{-2 i n x^-} + \bar{\alpha}_n e^{-2 i n x^+}), \quad x^\pm = t \pm x. \]  

(2.23)

This resembles the scalar field in one time plus one compact-space dimensions of string theory, with $p$ representing the center of mass target momentum of the string and $l^2 \equiv 2 \alpha'$. However, there is an important difference: because we have requested the left and right moving components to transform by a given number $p$ (see eq. (2.2)), in eq. (2.23), instead of the operator $\hat{\pi}^0$, the constant $p$ appears. The difference can also be appraised in the zero-mode structure of the automaton Hamiltonian, where $p \hat{\pi}^0$ replaces the usual $(\hat{\pi}^0)^2$ of the string-theory Hamiltonian.
The vacuum state is defined by

\[ \hat{\pi}^0 |0\rangle = 0 , \quad \alpha_n |0\rangle = \bar{\alpha}_n |0\rangle = 0 , \quad n = 1, \ldots, N , \quad (2.24) \]

and

\[ \langle 0 | \hat{\pi}^0 = 0 , \quad \langle 0 | \alpha_n = \langle 0 | \bar{\alpha}_n = 0 , \quad n = -1, \ldots, -N . \quad (2.25) \]

This state is obviously stationary in time. The physical two-dimensional particles are the excitations above the ground state \(|0\rangle\). The state \(|v_L = 0, v_R = 0\rangle\) is obtained from \(|0\rangle\) by the following formula (we omit the usual relation between \(|\pi^0 = 0\rangle\) and \(|v^0 = 0\rangle\))

\[ |v = 0 \rangle = \prod_{n=1}^{N} \exp\left[ \frac{1}{2} a_n^\dagger a_n \omega^{-2nx} \right] \exp\left[ \frac{1}{2} \bar{a}_n^\dagger \bar{a}_n \omega^{2nx} \right] |0\rangle . \quad (2.26) \]

To see this it is sufficient to consider a single oscillator. We have to show that

\[ (ac - a^\dagger \bar{c}) e^{\frac{1}{2} a^\dagger \bar{a} c^2} |0\rangle = 0 , \quad c\bar{c} = 1 . \quad (2.27) \]

By expanding the exponential function and using the algebra (2.18) and the definition of the vacuum \(|0\rangle\), eq. (2.24), we obtain the result stated above:

\[ (ac - a^\dagger \bar{c}) e^{\frac{1}{2} a^\dagger \bar{a} c^2} |0\rangle = \sum_{n=1}^{\infty} \frac{c^{2n-1}}{(n-1)!2^{n-1}} a^\dagger 2n-1 |0\rangle - \sum_{n=0}^{\infty} \frac{c^{2n+1}}{n!2^n} a^\dagger 2n+1 |0\rangle \]

\[ = 0 . \quad (2.28) \]

Using eq. (2.26) one can compute the ‘t Hooft ‘quantum probabilities’ \(P(v) = |\langle v | 0 \rangle|^2\) mentioned in sect. 1.
From the mode operator algebra and the definition of the vacuum state it is easy to compute the correlation functions. We obtain

\[ \langle \hat{\varphi}_R^{x,t} \hat{\varphi}_R^{x',t'} \rangle = \frac{l^2}{4} \sum_{n=1}^{N} \frac{1}{n} e^{-n(x^- - x'^-)} , \quad (2.29) \]

\[ \langle \hat{\varphi}_L^{x,t} \hat{\varphi}_L^{x',t'} \rangle = \frac{l^2}{4} \sum_{n=1}^{N} \frac{1}{n} e^{-n(x^+ - x'^+)} . \quad (2.30) \]

Note that these correlators are finite at coinciding points. In the continuum limit they exactly reproduce the well-known logarithmic expression.

The ‘open’ bosonic cellular automaton is obtained by imposing reflecting boundary conditions on the end cells. Let the cell string be given by \( x = 0, 1, 2, ..., N + 1 \) with variables \( v^R_x, v^L_x \) associated with each cell and the zero mode variable \( v^0 \). The automaton rules are the following:

\[ v^R_{x,t+1} = v^R_{x-1,t} , \quad x = 1, ..., N + 1 , \quad (2.31) \]

\[ v^L_{x,t+1} = v^L_{x+1,t} , \quad x = 0, 1, ..., N , \quad (2.32) \]

\[ v^0_{t+1} = v^0_t + \frac{1}{N+1} q , \quad q = \text{constant} , \quad (2.33) \]

\[ v^R_0 = v^L_1 , \quad v^L_{N+1,t} = v^R_{N+1,t} , \quad (2.34) \]

From eqs. (2.31), (2.32), (2.34) it follows that \( v^R_{0,t+1} = v^L_{1,t} \) and \( v^L_{N+1,t+1} = v^R_{N,t} \).

To obtain the effective long-distance field theory we proceed in the same way as we did in the closed case, that is, we introduce operators \( \hat{\varphi}_x, \hat{\pi}_x \) at each site \( x \) with \( [\hat{\varphi}_x, \hat{\pi}_y] = i\delta_{xy} \), etc. The Fourier expansion which satisfies the boundary
conditions (2.34) is given by
\[ \hat{\phi}_{x,t} = \hat{\phi} + \beta pt + i \frac{l}{2} \sum_{n=-N}^{N} \frac{1}{n} \alpha_n (\xi^{-nx} + \xi^{-nx^+}) , \] (2.35)
where
\[ \xi = e^{i\pi N+1} , \quad \beta = \frac{\pi l^2}{N+1} , \quad p = \frac{1}{\pi l^2 \eta} . \]

As a result, \( \hat{\phi}_x \) is periodic with period \( 2(N+1) \), \( \hat{\phi}_{x+2N+2} = \hat{\phi}_x \). The commutation relations for the \( \alpha_n \) are the same as in the closed case (see eqs. (2.20), (2.18))
\[ [\alpha_n, \alpha_m] = n \delta_{n+m} . \] (2.36)

In a similar way as in the previous case, the continuum limit is taken by rescaling \( t, x \to t/d, x/d \), and letting \( N \to \infty, d \to 0 \) keeping the length \( (N+1)d \) finite, \( (N+1)d = \pi \). The resulting field theory is that of a conformal scalar field satisfying Neumann boundary conditions with a zero-mode constrained by eq. (2.33).

3. Discrete String Theory

It is straightforward to extend the previous model to \( D \) scalar fields \( \hat{\phi}^{\mu} \) by allowing for \( 2D \) variables \( V_x^{L\mu}, V_x^{R\mu} \), \( \mu = 0, 1, ..., D-1 \), at each site \( x, x = 1, ..., C \), which evolve according to the deterministic rules eqs. (2.1), (2.2). As will be discussed in sect. 9, in this cellular string theory free cellular strings evolve under deterministic rules in target space. The basic reason is that the target-space momentum \( p^{\mu} \) is a c-number and hence it commutes with the center of mass coordinate. In the ’t Hooft scenario, the reconciliation with target-space quantum mechanics should be achieved by applying the same method that we applied in the
world sheet to target space-time. That is, one defines the Hilbert space of second quantization and looks for a primitive basis where the evolution matrices are just permutation matrices. The objects that we would call ‘particles’ or physical states would be an intrincated combination of the primitive basis elements. In principle, there is no reason why Bell inequalities for the primitive basis elements should imply Bell inequalities for the physical states. An obstacle to accomplish this strategy is the absence of a second-quantized string theory. Furthermore, there is a previous step which has not been performed yet, namely finding a cellular automaton which at large scales describes a massless scalar field theory in $D > 2$ dimensions (the $D = 2$ case is the discussion of sect. 2). We will return to the deterministic cellular string theory in sect. 9. Meanwhile, we come back to the formalism of first-quantization and promote the constant $p^\mu$ to an operator $\hat{p}^\mu$. We have (cf. eqs. (2.15), (2.19), (2.20))

$$\hat{\varphi}^\mu_{x,t} = \hat{\phi}^\mu + \frac{\pi l^2}{C} \hat{p}^\mu t + i \frac{l}{2} \sum_{n=-N}^{N} \frac{1}{n} (\alpha^\mu_n \omega^{-n} + \bar{\alpha}^\mu_n \omega^{-n}) , \quad \omega \equiv e^{i \frac{2\pi}{C}} . \quad (3.1)$$

Aside the zero-mode part, the left and right moving components can still be interpreted in terms of the cellular automaton of sect. 2.

An alternative interpretation is saying that we do not want deterministic evolution and investigating discrete string theories for their own sake, i.e. replace the world-sheet by the lattice $\mathbb{Z} \times \mathbb{Z}_C$, introduce the operator (3.1) and its canonical conjugate $\hat{\pi}^\mu_{x,t}$ acting on the Fock space, etc. Sects. 3 to 8 apply to either interpretations.

The operator $\hat{\varphi}^\mu$ with $\mu = 0$ will be identified with time coordinate in target space so we shall reverse the signs in the appropriate commutators. The nonvanishing commutation relations for the oscillator mode operators are (see eqs. (2.18), (2.20))

$$[\alpha^\mu_n, \alpha^\nu_m] = n\delta_{n+m} \eta^{\mu\nu} , \quad [\bar{\alpha}^\mu_n, \bar{\alpha}^\nu_m] = n\delta_{n+m} \eta^{\mu\nu} , \quad [\hat{\varphi}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu} , \quad (3.2)$$

where $\eta_{\mu\nu}$ is the Minkowski metric with signature $\{-+\ldots\}$. Similarly, the vacuum
state is defined by
\[
\hat{p}^\mu|0\rangle = 0, \quad \alpha_n^\mu|0\rangle = \bar{\alpha}_n^\mu|0\rangle = 0, \quad n = 1, \ldots, N, \quad (3.3)
\]
and
\[
\langle 0|\hat{p}^\mu = 0, \quad \langle 0|\alpha_n^\mu = \langle 0|\bar{\alpha}_n^\mu = 0, \quad n = -1, \ldots, -N, \quad (3.4)
\]
with \(\mu = 0, 1, \ldots, D-1\). To turn this scalar-operator theory into a \(\mathbb{Z}_C\) closed string theory we have to impose Virasoro constraints. Analogously to the continuum theory, these constraints may arise from to a discrete version of reparametrization invariance in some underlying original theory, but here they will be adopted as an \textit{ad hoc} prescription.

The \(\mathbb{Z}_C\)-Virasoro operators are defined as follows
\[
L_n = \frac{1}{2} \sum_{m=-N}^{N} \alpha_{n-m} \cdot \alpha_m, \quad \bar{L}_n = \frac{1}{2} \sum_{m=-N}^{N} \bar{\alpha}_{n-m} \cdot \bar{\alpha}_m \quad n = \pm 1, \ldots, \pm N \quad (3.5)
\]
\[
L_0 = \frac{1}{2} \alpha_0^2 + \sum_{m=1}^{N} \alpha_{-m} \cdot \alpha_m : , \quad \bar{L}_0 = \frac{1}{2} \bar{\alpha}_0^2 + \sum_{m=1}^{N} \bar{\alpha}_{-m} \cdot \bar{\alpha}_m : , \quad (3.6)
\]
Normal ordering is defined as usual by placing annihilation operators to the right.

Throughout this work the constant \(l^2 = 2\alpha'\), if not explicitly displayed, is assumed to be 1.

As we shall see below, the operators \(L_n\) and \(\bar{L}_n\) generate a discrete remnant of conformal transformations. In the continuum limit they become the Fourier modes of the components \(T_{--}\) and \(T_{++}\) of the energy-momentum tensor, whose
discrete analogs are
\[ T_{--} = l^2 \sum_{n=-N}^{N} L_n \omega^{-nx}, \quad T_{++} = l^2 \sum_{n=-N}^{N} \bar{L}_n \omega^{-nx}. \tag{3.7} \]

The physical space is defined to be the space generated by those states in the Fock space satisfying
\[ L_n |\text{phys}\rangle = \bar{L}_n |\text{phys}\rangle = 0, \quad n = 1, 2, \ldots, N \tag{3.8} \]
\[ L_0 |\text{phys}\rangle = \bar{L}_0 |\text{phys}\rangle = a |\text{phys}\rangle \tag{3.9} \]
where \( a \) is a constant which will be studied later.

It is remarkable that these operators \( L_n, \bar{L}_n \), which \textit{prima facie} look like a naive truncation of the usual Virasoro operators, form a closed algebra. Here the word ‘algebra’ is used in a generalized sense. Because indices are ‘angular’ variables, i.e. \( n = \mathbb{Z} \text{ Mod } (2N + 1) \) the structure constant in eq. (3.2) is not well-defined. Indices must be replaced by equivalent classes, \([n] \equiv n \text{ Mod } 2N + 1\). This characteristic arises in many quantum mechanical systems, for example, a point particle on a circle. The exponentiation of the algebra usually permits the obtainment of unambiguous results. Thus the basic commutator is \([\alpha_{[n]}^{\mu}, \alpha_{[m]}^{\nu}] = [n] \delta_{[n+m]}^{\mu\nu}\). For clarity in the formulas we will often omit the brackets of the indices. We proceed as in the usual operator formalism of the bosonic string theory, but a little bit of more care is necessary. Let us first calculate \([L_m, L_n]\) with \( m \neq -n \) so we do not need to keep track of the anomalous terms. The anomaly will be considered in the next section.

\[ [L_m, L_n] = \frac{1}{4} \sum_{k,l=-N}^{N} \left[ \alpha_{m-k} \cdot \alpha_k, \alpha_{n-l} \cdot \alpha_l \right] \]
\[ = \frac{1}{4} \sum_{k,l=-N}^{N} \left( k \alpha_{m-k} \cdot \alpha_l \delta_{k+n-l} + (m-k) \alpha_k \cdot \alpha_l \delta_{m+n-k-l} \right. \]
\[ \left. + k \alpha_{m-k} \cdot \alpha_{n-l} \delta_{k+l} + (m-k) \alpha_k \cdot \alpha_{n-l} \delta_{m-k+l} \right) \tag{3.10} \]
Now we perform the sum over \( l \), bearing in mind that \( \delta_{\left[\pm (2N+1)\right]} = 1 \). We get

\[
[L_m, L_n] = \frac{1}{2} \sum_{k=-N}^{N} k\alpha_{m-k} \cdot \alpha_{n+k} + \frac{1}{2} \sum_{k=-N}^{N} (m-k)\alpha_k \cdot \alpha_{m+n-k} .
\] (3.11)

Changing variables \( k' = k + n \) in the first sum, eq. (3.11) becomes

\[
[L_m, L_n] = \frac{1}{2} \sum_{k'=-N+n}^{N+n} (k' - n)\alpha_{m+n-k'} \cdot \alpha_{k'} + \frac{1}{2} \sum_{k=-N}^{N} (m-k)\alpha_k \cdot \alpha_{m+n-k} .
\] (3.12)

Now we can write (e.g. \( n > 0 \))

\[
\sum_{k'=N+n}^{N+n} = \sum_{k'=N}^{N} - \sum_{k'=N-1}^{N+n} + \sum_{k'=N+1}^{N+n} .
\]

The second and third sums cancel since \( [k] = [k + 2N + 1] \). Therefore the result is

\[
[L_m, L_n] = [m - n]L_{m+n} , \quad m \neq -n .
\] (3.13)

Eq. (3.13) resembles the usual Virasoro algebra with an identification between generators that produce the same action on \( \mathbb{Z}_G \subset S^1 \) (we do not ‘look’ at intermediate points). Strictly speaking, eq. (3.13) is not a Lie algebra because the structure constants are not single-valued; they must be understood as equivalent classes*. This will lead to some ambiguities on physical quantities. In the context of the cellular automaton, this is not a problem: every physical question can be unambiguously answered in terms of the primitive variables. The ambiguities arise in trying to describe the automaton system with the usual tools of quantum mechanics. In other words, in trying to match life in the cellular automaton with the continuum physics. Thus the main problem is the removal of any indefiniteness in the continuum limit.

* For studies on integrable systems with discrete time in the mathematical literature see, e.g., ref. 7.
In the continuum theory eq. (3.13) is recognized as the algebra of infinitesimal
diffeomorphisms of $S^1$. Now a complete basis for deformations $x \rightarrow x + f_x$ of the
discrete circle $\mathbb{Z}_C$ is provided by the operators

$$
\hat{D}_n : \quad \hat{D}_n F_x = \omega^{nx} \sum_{y=1}^{C} F_y D_{xy} , \quad D_{xy} \equiv \frac{1}{C} \sum_{n=-N}^{N} n \omega^{-n(x-y)} , \quad (3.14)
$$

where $F_x$ is an arbitrary map from the discrete circle $\mathbb{Z}_C$ to $\mathbb{R}$. From the definition
eq (3.14) it follows that $\hat{D}_n$ satisfies the Leibnitz rule, i.e. given two arbitrary
maps $F_x$ and $G_x$ from $\mathbb{Z}_C$ to $\mathbb{R}$ one has the relation

$$
\hat{D}_n(F_x G_x) = F_x \hat{D}_n G_x + (\hat{D}_n F_x) G_x .
$$

These operators are readily seen to obey the ‘algebra’

$$
[\hat{D}_m, \hat{D}_n] F_x = [m-n] \hat{D}_{m+n} F_x , \quad (3.15)
$$

which is the same as the algebra (3.13) of the $L_n , \tilde{L}_n$. Introducing $\theta = \frac{2\pi}{C} x$, we find in the continuum $C \rightarrow \infty$ limit

$$
\hat{D}_n \rightarrow i e^{i n \theta} \frac{\partial}{\partial \theta} , \quad (3.16)
$$
i.e. they become the standard basis for diffeomorphisms of $S^1$.

In the open $\mathbb{Z}_C$ string theory $\hat{\varphi}^\mu$ will be given by (cf. eq. (2.35))

$$
\hat{\varphi}_{x,t}^\mu = \hat{\varphi}^\mu + \beta \hat{p}^\mu t + i \frac{l}{2} \sum_{n=-N}^{N} \frac{1}{n} \alpha_n^\mu (\xi^{-nx^-} + \xi^{-nx^+}) , \quad (3.17)
$$

where

$$
\xi = e^{\frac{2\pi}{N+1}}, \quad \beta = \frac{\pi l^2}{N+1} .
$$

It is easy to show that $\hat{\varphi}_{x,t}^\mu$ satisfies the boundary condition $\hat{D}_0 \hat{\varphi}_{x}^\mu = 0$ at the
extremal $x = 0, N + 1$, where $\hat{D}_0$ is defined in eq. (3.14).
The expressions for the $L_n$, $n = -N, \ldots, N$ are the same as in the $\mathbf{Z}_C$ closed string case, eqs. (3.5), (3.6), but with $\alpha^\mu_0 = l p^\mu$. Note that mode operators $\alpha^\mu_n$ and Virasoro operators $L_n$ are identified through $n = \mathbf{Z} \text{Mod} \ (2N + 1)$ except that now the number of cells is $N + 2$, unlike the closed cellular automaton where this relation shows up in a circle with $2N + 1$ or $2N + 2$ cells.

4. Central extension

Let us consider the commutator $[L_n, L_{-n}]$. It has a central term coming from normal ordering contributions

$$[L_n, L_{-n}] = [2n]L_0 + A_n \ .$$

(4.1)

The explicit computation of the cocycle can be performed by accounting for normal ordering in eq.(3.11). Before doing this, it is worth noting some important differences with the $N = \infty$ theory. For illustrative purposes we shall frequently make use of the example provided by the other extremal, $N = 1$. The $\mathbf{Z}_C$-Virasoro operators are given by

$$L_1 = \frac{1}{2} \alpha_1 \cdot \alpha_0 + \frac{1}{2} \alpha_0 \cdot \alpha_1 + \frac{1}{2} \alpha_2 \cdot \alpha_{-1} = \alpha_1 \cdot \alpha_0 + \frac{1}{2} \alpha_{-1} \cdot \alpha_{-1} \ ,$$

(4.2)

$$L_0 = \frac{1}{2} \alpha_0^2 + \alpha_1 \cdot \alpha_{-1} \ ,$$

(4.3)

$$L_{-1} = \alpha_{-1} \cdot \alpha_0 + \frac{1}{2} \alpha_1 \cdot \alpha_1 \ .$$

(4.4)

Note the unusual terms $\frac{1}{2} \alpha_{\pm 1} \cdot \alpha_{\pm 1}$ in $L_{\pm 1}$ (in the case of arbitrary $N$ the corresponding terms in $L_n$, $n = 1, \ldots, N$, are $\frac{1}{2} \sum_{m=-N}^{N-1} \alpha_{n-m} \cdot \alpha_m$, and similarly for $n = -1, \ldots, -N$). As a result $L_1|0\rangle \neq 0$, $\langle 0|L_{-1} \neq 0$. This is related to
the absence of a M"obius residual symmetry in this discrete theory (for a further discussion see sect. 8). From eqs. (4.2), (4.4) one obtains

\[
[L_1, L_{-1}] = \alpha_0^2 - \alpha_{-1} \cdot \alpha_1 - \frac{D}{2}
= \alpha_0^2 + 2\alpha_{-1} \cdot \alpha_1 - \frac{D}{2}
= 2L_0 - \frac{D}{2}.
\] (4.5)

A central term, absent in the corresponding commutator of the continuum string theory, has appeared. To understand its nature, we now consider the general \(N\) case. From eq. (3.11) we obtain (\(0 < n \leq N\))

\[
[L_n, L_{-n}] = \frac{1}{2} \sum_{m=-N}^{N} (m + n) : \alpha_{-m} \cdot \alpha_m : - \frac{D}{2} \sum_{m=-N}^{-1} m(m + n)
- \frac{1}{2} \sum_{m=-N}^{N} m : \alpha_{-n-m-m} \cdot \alpha_{n+m} : + \frac{D}{2} \sum_{m=-N}^{-n-1} m(m + n) + \frac{D}{2} \sum_{m=N-n+1}^{N} m(m + n)
\] (4.6)

Hence

\[
A_n = \frac{D}{2} \sum_{m=-n}^{-1} m(m + n) + \frac{D}{2} \sum_{m'=1}^{n} (m' + N)(m' + N - n)
= \frac{D}{2} nN(N + 1).
\] (4.7)

This is a trivial cocycle which can be removed by redefining

\[
L_0 \to L_0 + \frac{D}{2} \frac{N(N + 1)}{2}.
\]

In fact, this is precisely the number which arises upon normal ordering of \(L_0\):

\[
\frac{1}{2} \sum_{m=-N}^{N} \alpha_{-m} \cdot \alpha_m = \frac{1}{2} \sum_{m=-N}^{N} : \alpha_{-m} \cdot \alpha_m : + \frac{D}{2} \frac{N(N + 1)}{2}.
\] (4.8)

Thus the theory is free of anomalies for any value of \(D\). This may be no surprise, since anomalies are related to short-distance behaviour of correlators and
here correlations functions are well behaved and finite at coincident points. We will find a confirmation of the anomaly freedom in the next section when we study the Lorentz algebra in the light-cone gauge.

Another way to obtain \( A_n \) is by calculating \( \langle 0 | [L_n, L_{-n}] | 0 \rangle \) explicitly using the algebra of mode operators and the definition of the vacuum given in eqs. (3.3), (3.4). One easily finds \( (0 < n \leq N) \)

\[
\langle 0 | L_n L_{-n} | 0 \rangle = \frac{D}{2} \sum_{m=1}^{n-1} m(n - m) = \frac{D}{2} \frac{n^3 - n}{6} \tag{4.9}
\]

where the well-known formulas \( \sum_{m=1}^{n} m = \frac{1}{2} n(n+1) \), \( \sum_{m=1}^{n} m^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \), have been employed. On the other hand, observing that

\[
L_n |0\rangle = \frac{1}{2} \sum_{m=-N}^{-N+n-1} \alpha_{n-m} \cdot \alpha_m |0\rangle \neq 0,
\]

we get

\[
\langle 0 | L_{-n} L_n | 0 \rangle = \frac{D}{2} \frac{n^3 - n}{6} - \frac{D}{2} nN(N + 1) \tag{4.10}
\]

i.e. \( \langle 0 | [L_n, L_{-n}] | 0 \rangle = \frac{D}{2} nN(N + 1) \), in accordance with eq.(4.7).

5. Light-Cone Gauge

The residual symmetry generated by the \( L_n, \bar{L}_n \) can be used to gauge away all the oscillator modes of one ‘target’ coordinate. Indeed, these operators generate the following transformations on the operators \( \phi^{R\mu}, \bar{\phi}^{L\mu} \)

\[
[L_n, \phi^{R\mu}] = -\frac{i}{2} \sum_{m=-N}^{N} \alpha^{\mu}_{n+m} \omega^{mx^-}, \hspace{1cm} [\bar{L}_n, \bar{\phi}^{L\mu}] = -\frac{i}{2} \sum_{m=-N}^{N} \bar{\alpha}^{\mu}_{n+m} \omega^{mx^+}. \tag{5.1}
\]
Thus we use this residual symmetry to set
\[
\hat{\phi}^+ = \phi^+ + \pi l^2 C \frac{p^+ t}{C} , \quad \hat{\phi}^\pm = \frac{\phi^0 \pm \hat{\phi}^{D-1}}{\sqrt{2}} .
\] (5.2)

Eq. (5.2) means, in particular, that in this physical gauge the target time \( \hat{\phi}^+ \) has in some sense a discrete structure. As a result, the target space light-cone energy \( p_- \) will be periodic, i.e.
\[
p_- \cong p_- + \frac{4C}{l^2 p^+} z .
\] (5.3)

It is natural to require that all the target-space coordinates have this type of discrete structure, though we will not explore this possibility here. The consequences and interpretation of eq. (5.3) will be considered later.

Since right and left moving sectors play a symmetrical role, it is sufficient to consider the open string case. The \( Z_C \) Virasoro constraint equations \( L_0 = a, L_n = 0, n = \pm 1, ..., \pm N \) can be solved giving
\[
l_n \equiv p^+ \alpha_n^- = \frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-N}^N \alpha_i^{n-m} \alpha_m^i : -a \delta_n .
\] (5.4)

It is straightforward to show that the \( l_n \) satisfy the algebra
\[
[l_m, l_n] = [m - n] l_{m+n} + \delta_{m+n} \left[ \frac{D-2}{2} m N (N+1) + 2am \right] .
\] (5.5)

The calculation is analogous to the discussions in sects. 2, 3.

While the light-cone formalism is manifestly free of negative-norm states, it is not manifestly covariant. The question is whether the theory is really Lorentz invariant in this gauge. In the standard bosonic string theory, it turns out that only if \( D = 26 \) and \( a = 1 \) the Lorentz algebra is anomaly free (for a review see ref. [8]). For other values of \( D \) and \( a \) it is not possible to fix the light-cone gauge maintaining at the same time the Lorentz covariance of the theory.
The Lorentz generators $J^\mu\nu$ are

$$J^\mu\nu = l^\mu\nu + E^\mu\nu,$$

with

$$l^\mu\nu = \phi^\mu p^\nu - \phi^\nu p^\mu, \quad E^\mu\nu = -i \sum_{n=1}^{N} \frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu).$$

(5.6)

It is therefore important to verify that these operators really generate the Lorentz algebra. Most of the commutators can be carried out in a straightforward way and they give the correct answer. The potential anomaly comes from $[J^i^-, J^j^-]$—which must vanish if Lorentz invariance is to hold—since the transformations generated by $J^{i^-}$ affect the light-cone gauge choice.

By using eq. (5.6) and the algebra of the oscillator mode operators it is easy to prove that

$$[J^i^-, J^j^-] = -(p^+)^{-2}C^{ij}$$

(5.7)

with

$$C^{ij} = 2il_0E^{ij} - [E^i, E^j] - iE^ip^j + iE^jp^i, \quad E^i \equiv p^+ E^{i^-}.$$  

(5.8)

Let us first verify that the classical part vanishes. For the moment we will drop the potential anomalous terms, i.e. all terms quadratic in oscillators. They will be considered in detail later. Using eq. (5.5) and

$$[\alpha^i_m, l_n] = ma^i_{m+n},$$

(5.9)

we find

$$[E^i, E^j] = A + B + C,$$

(5.10)

with

$$A = - \sum_{n,m=1}^{N} \frac{1}{n} \alpha_{-n}^i (\alpha_{-m}^j l_m - \alpha_{-l}^j l_{n+m}) - (i \leftrightarrow j),$$

(5.11)
\[ B = \sum_{n,m=1}^{N} \frac{1}{n} (l_m \alpha_{m-n}^j - l_{n-m} \alpha_{m}^j) \alpha_n^i - (i \leftrightarrow j) , \quad (5.12) \]

\[ C = \sum_{n,m=1}^{N} \left( \frac{1}{n} + \frac{1}{m} \right) \alpha_{-m}^i l_{m-n} \alpha_m^j - \frac{1}{n} \alpha_{-n}^i l_{m} \alpha_{n+m}^j - \frac{1}{m} \alpha_{-n-m}^i \alpha_m l_n - (i \leftrightarrow j) . \quad (5.13) \]

In the above eqs. (5.11), (5.12) and (5.13) some terms cancel out and what remains is

\[ A = -\sum_{n=1}^{N} \left( \sum_{m=1}^{n} - \sum_{m=N+1}^{N+n} \right) \frac{1}{n} \alpha_{-n}^i \alpha_{m-n}^j l_m - (i \leftrightarrow j) , \quad (5.14) \]

\[ B = \sum_{n=1}^{N} \left( \sum_{m=1}^{n} - \sum_{m=N+1}^{N+n} \right) \frac{1}{n} l_{m} \alpha_{m-n}^j \alpha_n^i - (i \leftrightarrow j) , \quad (5.15) \]

\[ C = \sum_{m=1}^{N} \left( \sum_{n=1-m}^{0} - \sum_{n=N-m+1}^{N} \right) \frac{1}{m} \alpha_{-m}^i l_{n} \alpha_m^j \]

\[ + \sum_{n=1}^{N} \left( \sum_{m=1-n}^{0} - \sum_{m=N-n+1}^{N} \right) \frac{1}{n} \alpha_{-n}^i l_{m} \alpha_{m+n}^j - (i \leftrightarrow j) . \quad (5.16) \]

The second sum in the parentheses is absent in the \( N = \infty \) theory. By using the fact \( n = n \text{ Mod } 2N + 1 \) and conveniently renaming variables \( C \) can be written in the following form

\[ C = \sum_{n=1}^{N} \left( \sum_{m=1}^{n} - \sum_{m=N+1}^{N+n} \right) \frac{1}{n} \alpha_{-n-m}^i l_m \alpha_{m-n}^j + \sum_{n=1}^{N} \left( \sum_{m=1}^{n} - \sum_{m=N+1}^{N+n} \right) \frac{1}{n} l_{m} \alpha_{m-n}^i \alpha_n^j \]

\[ + 2 \sum_{m=1}^{N} \frac{1}{m} \alpha_{-m}^i \alpha_m^j l_0 - \sum_{n=1}^{N} \frac{1}{n} \alpha_{-n}^i (l_n \alpha_{m-n}^j - l_{n} \alpha_{-n}^j) - (i \leftrightarrow j) . \quad (5.17) \]

Therefore

\[ C = -A - B + 2il_0 E^{ij} - iE^i p^j + iE^j p^i . \quad (5.18) \]

Thus, comparing with eqs. (5.7), (5.8) and (5.10) we see that \([J^i^-, J^j^-]|_{\text{classical}} = 0.\]
Let us now consider the computation of the anomaly. We follow the analogous derivation given in ref. [8]. Since the classical part is zero, $C^{ij}$ can only be quadratic in the oscillator modes. In fact it can only be of the form

$$C^{ij} = \sum_{m=1}^{N} \Delta_m (\alpha_m^i \alpha_m^j - \alpha_m^j \alpha_m^i) \ .$$  \hspace{1cm} (5.19)

The coefficients $\Delta_m$ can be determined by evaluating the matrix elements

$$\langle 0| \alpha_m^k E^{ij} \alpha_m^l |0 \rangle = m^2 (\delta^{ik} \delta^{jl} - \delta^{ij} \delta^{kl}) \Delta_m \ .$$ \hspace{1cm} (5.20)

By explicit computation we find

$$\langle 0| \alpha_m^k (-iE^i p^j + iE^j p^i) \alpha_m^l |0 \rangle = -\delta^{ik} p^j p^jm + \delta^{ij} p^j p^km - (i \leftrightarrow j) \ ,$$ \hspace{1cm} (5.21)

and

$$2i \langle 0| \alpha_m^k l_0 E^{ij} \alpha_m^l |0 \rangle = m(p_i^2 - 2a + 2m) (\delta^{jl} \delta^{ki} - \delta^{ik} \delta^{jl}) \ .$$ \hspace{1cm} (5.22)

The calculation of $\langle 0| \alpha_m^k [E^i, E^j] \alpha_m^l |0 \rangle$ is more involved than the evaluation of the corresponding matrix element in the continuum theory because now $l_n$ and $l_{-n}$ do not destroy $|0 \rangle$ and $\langle 0 |$. Let us write

$$\langle 0| \alpha_m^k [E^i, E^j] \alpha_m^l |0 \rangle = A_I + A_{II} + A_{III}$$ \hspace{1cm} (5.23)

where

$$A_I = \langle 0| l_m l_{-m} |0 \rangle (\delta^{jl} \delta^{ki} - \delta^{ik} \delta^{jl}) \ ,$$

$$A_{II} = -\sum_{n=1}^{m} (\delta^{jl} \langle 0| \alpha_m^k l_{n} \alpha_{n-m}^i |0 \rangle + \delta^{ki} \langle 0| \alpha_m^j l_{n} \alpha_{n-m}^l |0 \rangle - (i \leftrightarrow j)) \ ,$$ \hspace{1cm} (5.24)

$$A_{III} = \sum_{n,n' = 1}^{N} \frac{1}{nn'} \langle 0| \alpha_{m-n}^k \alpha_n^i \alpha_{n-m}^j l_{n'} \alpha_{n-m}^l |0 \rangle \ .$$
We find
\[
A_{II} = -\delta^{jl}\delta^{ki}m^2(m-1) - m\delta^{ki}p^jp^l - m\delta^{jl}p^kp^l - (i \leftrightarrow j) .
\] (5.25)

By straightforward algebra we can take \(A_{III}\) to the form
\[
A_{III} = \delta^{il}\delta^{kj} \langle 0|l_m|0 \rangle + \delta^{il} \sum_{n'=1}^{N} \langle 0|\alpha_m^k\alpha_{-m-n'}^l|0 \rangle + \delta^{kj} \sum_{n=1}^{N} \langle 0|l_n\alpha_{m+n}^i\alpha_{-m}^l|0 \rangle
\]
\[
+ \sum_{n,n'=1}^{N} \langle 0|\alpha_m^k\alpha_{-n-n'}^l\alpha_{n+n'}^i\alpha_{-m}^l|0 \rangle - (i \leftrightarrow j) ,
\] (5.26)

obtaining
\[
A_{III} = (m^2 + m^2 - \langle 0|l_m|0 \rangle ) (\delta^{il}\delta^{kj} - \delta^{jl}\delta^{ki}) .
\] (5.27)

Using eqs. (5.24), (5.25), (5.27) and the commutation relation (5.5) we arrive at the following expression:
\[
\langle 0|\alpha_m^k[E^i,E^j]\alpha_{-m}^l|0 \rangle = m \left( \frac{D-2}{2} N(N+1) + 2m + p_i^2 \right) (\delta^{il}\delta^{kj} - \delta^{jl}\delta^{ki})
\]
\[
- m (\delta^{kl}p^jp^l + \delta^{il}p^k p^l - (i \leftrightarrow j)) .
\] (5.28)

Inserting eqs. (5.21), (5.22), (5.28) into eq. (5.20) we find (see also eq. (5.8))
\[
\Delta_m = - \frac{1}{m} \left( \frac{D-2}{2} N(N+1) + 2a \right) .
\] (5.29)

Requiring \(\Delta_m = 0\) gives
\[
a = - \frac{D-2}{2} \frac{N(N+1)}{2} ,
\] (5.30)

with no additional restriction on \(D\).
The value of $a$ is just the same as the value of the constant which emerges in normal ordering the operator $l_0$ and it is in agreement with the value obtained in the usual $N = \infty$ case if one makes use of zeta-function regularization to compute the infinite sum [8], i.e. $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, with $\zeta(-1) = -\frac{1}{12}$, thus giving

$$a = \frac{D - 2}{24},$$

which yields $a = 1$ for $D = 26$. The correspondence with the standard bosonic string theory in the continuum limit is more carefully examined in sect. 8.

6. Spectrum

Along this section we will continue our investigation on properties inherent to the discrete theory, leaving the discussion on the approximation to the continuum theory deferred to sect. 8.

The physical states are much more easily constructed in the light-cone gauge, especially in this discrete theory, since in the covariant formalism physical states will involve states from different levels (see below). The well-known disadvantage of the light-cone gauge is that the states appear as multiplets of $SO(D - 2)$, the transverse rotation group, even though the proof of Lorentz invariance guarantees that the massive levels fill out complete multiplets of $SO(D - 1)$. A general physical state in the closed $\mathbf{Z}_C$ string theory can be written in the form

$$|\text{phys}\rangle = \xi_{\{i,\bar{i}\}}(p) \prod_{n=1}^{N} \{ \alpha_{-n}^{i_1...i_n} \bar{\alpha}_{-n}^{\bar{i}_1...\bar{i}_n} \} |0; p\rangle, \quad (6.1)$$

where $\xi_{\{i,\bar{i}\}}$ is an arbitrary polarization vector and the $\epsilon_n, \bar{\epsilon}_n$ are subject to the constraint

$$\sum_{n=1}^{N} n\epsilon_n = \sum_{n=1}^{N} n\bar{\epsilon}_n,$$

coming from the $l_0 = \bar{l}_0$ condition.
The mass spectrum follows from eq. (5.4):

\[
\frac{1}{8} l^2 m^2 = -a + \sum_{n=1}^N \bar{\alpha}_n \alpha_n = -a + \sum_{n=1}^N \bar{\alpha}_n \alpha_n .
\] (6.2)

or

\[
\frac{1}{8} l^2 m^2 = -a + \sum_{n=1}^N [n] \alpha_n \alpha_n^i ,
\] (6.3)

The above equations imply that \(m^2\) is also an ‘angular’ or periodic variable, i.e.

\[
\frac{1}{8} l^2 m^2 \equiv \frac{1}{8} l^2 m^2 + kC , \quad k = \text{integer},
\] (6.4)

This indefiniteness is also inferred from eq. (5.3).

The fact that time discretization leads to identifications of the form energy \(\approx\) energy \(+ 2\pi n\) is not new. The novelty here is that this property, which is of course present in the world-sheet theory, is also reflected by eq. (6.4) to the target space. In particular, eq. (6.4) implies that in this theory the masses of \(Z_C\)-string excitations of arbitrary spin can be reduced to the interval \(\frac{1}{8} l^2 m^2 = -a, -a + 1, ..., -a + C - 1\).

According to eq. (6.2), the mass of the state (6.1) will be given by

\[
\frac{1}{8} l^2 m^2 = -a + \sum_{n=1}^N n \epsilon_n + ZC .
\] (6.5)

Thus the states with \(\sum_{n=1}^N n \epsilon_n \geq C\) will repeat the same mass pattern. There is no analog of this degeneracy in the continuum string theory. For example, let \(\epsilon_n = C \delta_{n-1}\); this higher-level state has mass \(\frac{1}{8} l^2 m^2 = -a + C = -a\), i.e. the same square mass as the ‘tachyon’ state. For each \(\frac{1}{8} l^2 m^2 = -a, -a + 1, ..., -a + C - 1\), there will be an infinite tower of higher-spin states with the same square mass.
In the covariant formalism the expressions for physical states are complicated, even at the lowest level. Establishing a one-to-one correspondence with light-cone gauge spectrum is important, since this is the basis of the proof of the no-ghost theorem, this gauge being manifestly ghost free.

The mass of a given physical state follows from the mass-shell condition

$$\frac{1}{8} l^2 m^2 = -a + \sum_{n=1}^N \alpha_{-n} \cdot \alpha_n = -a + \sum_{n=1}^N \bar{\alpha}_{-n} \cdot \bar{\alpha}_n ,$$

which is the same as the mass-shell condition one finds in the light-cone gauge treatment, except that now all oscillators contribute to $m^2$.

Let us consider the $N = 1$ example in the case of open strings. The entire spectrum of $N = 1$ open string consists of one scalar field $T$ of mass $m_T^2 = -2a$, one vector field $A_\mu$ of mass $m_A^2 = -2a + 2$, one second-rank tensor $B_{\mu\nu}$ of mass $m_B^2 = -2a + 4$, plus towers of states of masses $-2a + ZC = m_T^2$, $-2a + 2 + ZC = m_A^2$, and $-2a + 4 + ZC = m_B^2$. An interesting problem is deriving the effective field theory action for these states. The linearized equations of motion for all fields are easily obtained from the Virasoro condition $L_1 |\text{phys}\rangle = 0$.

Let us explicitly derive the ‘tachyon’ state. Using eqs. (4.2), (4.3) we see that

$$L_0 |0; \mu\rangle = \frac{1}{2} p^2 |0; \mu\rangle , \quad L_1 |0; \mu\rangle = \frac{1}{2} \alpha_{-1} \cdot \alpha_{-1} |0; \mu\rangle \neq 0 .$$

Therefore we find, somewhat unexpectedly, that the state $|0; \mu\rangle$ is not physical. This is unlike its continuum counterpart. Moreover, how can we explain the discrepancy with the light-cone gauge? In order to clarify this controversy, let us find the correct ‘tachyon’ physical state. From the structure of $L_1$ we see that the ‘tachyon’ state must be of the form
\[ |T(p^\mu)\rangle = (1 + A^{(1)}_{\mu_1\mu_2\mu_3} \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \alpha_{-1}^{\mu_3} + A^{(2)}_{\mu_1...\mu_6} \alpha_{-1}^{\mu_1}...\alpha_{-1}^{\mu_6} + ...)|0; p^\mu\rangle , \quad (6.8) \]

i.e. a linear combination involving all states with \( p^2 = p^2 + ZC, \ C = 3 \). The physical condition \( L_1|T(p^\mu)\rangle = 0 \) is solved with

\[ A^{(1)}_{\mu_1\mu_2\mu_3} = - \frac{1}{6p^2} (p_{\mu_1}\eta_{\mu_2\mu_3} + p_{\mu_2}\eta_{\mu_1\mu_3} + p_{\mu_3}\eta_{\mu_1\mu_2}) + \frac{1}{3p^4} p_{\mu_1}p_{\mu_2}p_{\mu_3} , \quad (6.9) \]

etc. We can also check that the state (6.8) satisfies the \( L_0 \) condition

\[ L_0|T(p^\mu)\rangle = \frac{1}{2} (p^2 + ZC)|T(p^\mu)\rangle + ([3] A^{(1)}_{\mu_1\mu_2\mu_3} \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \alpha_{-1}^{\mu_3} + [6] A^{(2)}_{\mu_1...\mu_6} \alpha_{-1}^{\mu_1}...\alpha_{-1}^{\mu_6} + ...)|0; p^\mu\rangle = \frac{1}{2} p^2 |T(p^\mu)\rangle \quad (6.10) \]

Now we understand what is happening: in the covariant formalism the ‘tachyon’ physical state \( |T(p^\mu)\rangle \) contains, beside \( |0; p^\mu\rangle \), a linear combination of all longitudinal components of the higher-spin states with the same mass \( p^2 = -2a \). In this \( N = 1 \) example they are in fact those having \( kC \) oscillator modes \( \alpha_{-1} \), with \( k = 1, 2, ..., \infty \). In the general case the lowest-level physical state will have the form

\[ |T(p^\mu)\rangle = \sum_{r=0}^{\infty} \sum_{n_1,...,n_r=-N}^{N} \delta_{[n_1+...+n_r]} A_{\mu_1...\mu_r} \alpha_{-n_1}^{\mu_1}...\alpha_{-n_r}^{\mu_r} |0; p^\mu\rangle \quad (6.11) \]

where \( A_{\mu_1...\mu_r} \) are longitudinal components of the states. In the light-cone gauge, clearly, these longitudinal components are automatically gauged away and \( |T(p^\mu)\rangle \) simply reduces to \( |0; p^\mu\rangle \). Correspondence between the light-cone gauge and the covariant formalism requires that these longitudinal states (which, just as in standard string theory, can be expressed as \( \sum_{n=1}^{N} L_{-n}|\chi\rangle \)) decouple from scattering amplitudes.
7. Vertex Operators

In standard string theory conformal invariance permits the mapping of an arbitrary genus Riemann surface with \( p \) tubes extending into the far past and the far future—corresponding to incoming and outgoing strings—to a Riemann surface of the same topology and moduli with \( p \) punctures replacing the tubes, the quantum numbers of the external strings being represented by vertex operators at each puncture. Since the discrete theory is not conformal invariant, in principle it is not clear than an arbitrary external state can be replaced by a vertex operator. Nevertheless, in order to make contact with the continuum theory, it is useful to introduce the discrete analog of these operators.

In the bosonic closed string theory a vertex operator \( W \) for an on-shell string state is a local operator of conformal dimension \((1,1)\), i.e. satisfying

\[
[L_n, W(z, \bar{z})] = z^{n+1} \frac{\partial}{\partial z} W + nz^n W ,
\]

\[
[\bar{L}_n, W(z, \bar{z})] = \bar{z}^{n+1} \frac{\partial}{\partial \bar{z}} W + n\bar{z}^n W ,
\]

which carries the correct Lorentz quantum numbers of the represented particle.

In analogous way we will say that a local operator \( W_{x,t} \) has \( \mathbb{Z}_C \)-conformal dimension \((\lambda, \lambda)\) if it obeys the following commutation relations

\[
[L_n, W_{x^+,x^-}] = -D_n^- W + \lambda n \omega^{nx^-} W ,
\]

\[
[\bar{L}_n, W_{x^+,x^-}] = -D_n^+ W + \lambda n \omega^{nx^+} W ,
\]

where \( D_n^- \), \( D_n^+ \) are operators defined as in eq. (3.14), with respect to \( x^- \), \( x^+ \) respectively, and \( n = -N,...,N \). Similarly, in the case of the open \( \mathbb{Z}_C \) string theory an operator \( W_t \) to be inserted on the boundary is said to have \( \mathbb{Z}_C \)-conformal

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dimension $\lambda$ if it satisfies
\[
[L_n, W_t] = -D^n_t W_t + \lambda n \xi^n W_t, \quad n = -N, \ldots, N. \tag{7.4}
\]
where
\[
D^n_t : D^n_t F_t = \xi^n \sum_{t'=0}^{C} F_{t'} D_{t't'}, \quad D_{t't'} = \frac{1}{C} \sum_{n=-N}^{N} n \xi^{-n(t-t')}. \tag{7.5}
\]

A consequence of the commutation relations (7.1) is that vertex operators can be used to map physical states to physical states, up to spurious states which decouple from scattering amplitudes. This property holds true also in the present case provided the local operator has $\mathbb{Z}_C$-conformal dimension $\lambda = 1$. Indeed, for $\lambda = 1$ one has $[L_n, W_t] = -D^n_t (\xi^n W_t)$, etc.

The candidate for a vertex operator representing the tachyon is
\[
W_T(p^\mu) = : e^{ip \cdot \phi} :. \tag{7.6}
\]
This operator injects momentum $p^\mu$ on a single cell $x, t$. Let us compute the commutator $[L_n, W_T]$. For simplicity we restrain our attention to the open string case. The calculation is similar to the corresponding one of the standard continuum theory, so we will not reproduce all steps. By using the basic commutator
\[
[L_m, e^{p \cdot \alpha-n}] = \frac{1}{2} n(p \cdot \alpha_{m-n} e^{p \cdot \alpha-n} + e^{p \cdot \alpha-n} p \cdot \alpha_{m-n}), \tag{7.7}
\]
on one attains a result of the form (7.4) with $\lambda = 0$ where $W_T$ is not yet normal ordered. The usual anomalous dimension precisely arises in the normal-ordering process. In doing this, we get an additional contribution relative to the continuum case:
\[
[L_m, W_t^T] = -D^T_m W_t^T + \frac{p^2}{2} \xi^n W_t^T \left( \sum_{n=1}^{m} 1 - \sum_{n=-N}^{n=-N+1} 1 \right) \tag{7.8}
\]
that is, the operator (7.5) has no anomalous dimension, fact to be expected in view...
of the absence of short-distance singularities. The $\mathbb{Z}_C$-conformal dimension of the operator $:e^{ip\hat{\varphi}}:$ is simply zero, independent of $p^2$.

Higher-level vertex operators for the closed $\mathbb{Z}_C$ string are obtained in the standard way by successive multiplication of $e^{ip\hat{\varphi}}$ by

$$\frac{i}{4}l^2\hat{p}^\mu + D_0^- \hat{\varphi}^R\mu \equiv D_0^- \varphi^\mu, \quad \frac{i}{4}l^2\hat{p}^\mu + D_0^+ \varphi^L\mu \equiv D_0^+ \varphi^\mu \quad (7.8)$$

The analogous construction for higher-level vertices applies to the open $\mathbb{Z}_C$ string case. The action of the operator $D_0$ is defined on maps which are single-valued on $\mathbb{Z}_C$. The $x^\pm$ are not periodic so in the identification (7.8) the symbolic notation $D_0^\pm x^\pm = iC/2\pi$ is understood (cf. eq. (3.16)). In addition, ‘higher derivative’ operators $(D_0^-)^n \hat{\varphi}^\mu$, $(D_0^+)^n \hat{\varphi}^\mu$ may be present. For example, the graviton vertex operator is given by

$$V_g = \xi_{\mu\nu}(p)D_0^- \varphi^\mu D_0^+ \varphi^\nu e^{ip\hat{\varphi}}, \quad (7.9)$$

where $\frac{1}{8}p^2 = -a + 1$ and $\xi_{\mu\nu}$ is a symmetric tensor obeying the conditions $\xi_{\mu\nu}p^\mu = 0, \xi^\mu_\mu = 0$ which avert the appearance of unwanted additional terms in eqs. (7.2) and (7.3). This vertex has $\mathbb{Z}_C$-conformal dimension (1,1) and thus it can be used to map physical states to physical states. Vertex operators corresponding to levels higher than two will inevitably have $\mathbb{Z}_C$-conformal dimension greater than (1,1).

8. Continuum Limit and Tree-Level Scattering Amplitudes

The theory we were studying hitherto is different from the bosonic string theory in a number of important respects. In this $\mathbb{Z}_C$-string theory, free of short-distance singularities, no anomalous terms have appeared in the algebras. The computational reason behind this is the contribution of excitations $\alpha^\mu_n$ with $|n| \sim N$ which has the opposite sign and leads to exact cancellation of all the anomalies. Indeed,
the well-known conformal anomaly corresponding to $D$ scalar fields

$$\frac{D}{12}(n^3 - n) = -\frac{D}{2} \sum_{m=-n}^{-1} m(m + n),$$

is present in eq. (4.6) but is cancelled by the term

$$\frac{D}{2} \sum_{m=N-n+1}^{N} m(m + n), \quad (8.1)$$

which originated from commutators involving $\alpha^\mu_n$ with $|n| \sim N$. Similarly, the usual anomalous terms in the Lorentz algebra are cancelled from contributions of the type (8.1) and the same fate undergoes the usual anomalous dimension for the operator $e^{ip \cdot \hat{\phi}}$ (cf. eq. (7.7)).

A similar situation emerged in ref. [4] in the context of the Fermi one-dimensional shift automaton. The Schwinger term in the fermionic current algebra is lacking in the finite $N$ theory. In the continuum limit, $(x, t) \to (x/d, t/d)$, $C \to \infty$, $d \to 0$, $Cd = \pi$, one would like to retain only low-energy excitations of the fermions. By the standard $\epsilon$-prescription for distributions the contributions from excitations near $\pm N$ are dampened by a factor $e^{-\epsilon C}$. The processes of letting $\epsilon \to 0$ and taking the continuum limit do not commute. To safely obtain the continuum theory one has to first take $N \to \infty$ and eventually take $\epsilon \to 0$. In doing this all contributions to the anomaly coming from oscillators with frequencies $p \sim \pm N$ will vanish as $N \to \infty$.

Another way to eliminate the non-perturbative contribution of excitations with $n \sim \pm N$ in the continuum limit is by going to Euclidean world-sheet time, $t \to it = \tau$. The operator $\hat{\phi}^\mu$ corresponding to the closed string case takes the form

$$\hat{\phi}^\mu = \hat{\phi}^\mu + i \frac{\pi l^2}{C} \hat{p}^\mu \tau + i \frac{l}{2} \sum_{n=-N}^{N} \frac{1}{n} (\alpha^\mu_n \omega^{-n(i\tau - x)} + \bar{\alpha}^\mu_n \omega^{-n(i\tau + x)}), \quad \omega \equiv e^{i2\pi \sigma}. \quad (8.2)$$

If we are interested in physics below some (world-sheet) energy scale of order $N$, then excitations with $|n| \sim N$ will be frozen, and the ‘effective’ anomalies
corresponding to this long-distance theory, in the absence of terms like e.g. eq. (8.1), will coincide with the continuum values.

The physical mechanism which would effectively implement the $\epsilon$-prescription or any other prescription is unclear to us. We will not attempt any deeper enquiry on this at this primitive stage, though it is interesting to speculate on a vinculum with the Hagedorn phase transition.

In the covariant approach the critical values $a = 1$ and $D = 26$ are evidenced by the multiple appearance of zero-norm states. This holds true also in the present theory since e.g. the norms of the states $L_{-1}|0; p^\mu\rangle = \alpha_{-1} \cdot p|0; p^\mu\rangle$, with $p^2 = 0$, and $(L_{-2} + \frac{3}{2}L_{-1})|0; p^\mu\rangle$, with $p^2 = -2$, become zero for $a = 1 \mod \mathbb{Z}$ and $D = 2(13 \mod \mathbb{Z})$ respectively. Thus, in the spirit of making contact with the continuum theory, let us set $a = 1$ and $D = 26$ and define a tree-level scattering amplitude as a correlator of vertex operators associated with the external particles which are present in the scattering process. For scattering of closed $\mathbb{Z}_C$ string states

\[ A_{\text{closed}}^M = \kappa^{M-2}\langle 0|T\{V^1...V^M\}|0\rangle , \tag{8.3} \]

where $\kappa$ is a coupling constant and

\[ V^i = \sum_{t=-\infty}^{\infty} \sum_{x=1}^{C} \omega^{-bx^-} \omega^{-bx^+} W^i_{x,t} , \quad b = \frac{1}{8}p^2 , \tag{8.4} \]

where $W^i_{x,t}$ are the vertex operators discussed in the previous section. The scattering amplitudes for open $\mathbb{Z}_C$ strings are defined in a similar way ($g^2 \sim \kappa$)

\[ A_{\text{open}}^M = g^{M-2} \sum_{t_1 > ... > t_M} \prod_{i=1}^{M} \xi^{-4b_i t_i} \langle 0|W_{t_i}^1...W_{t_M}^M|0\rangle + \text{cyclic perm.} \tag{8.5} \]

The main uncertainty in these ‘scattering amplitudes’ is that only level-one vertices –corresponding to massless particles– have the correct $\mathbb{Z}_C$-conformal dimension 1. As discussed above, only if the effect of excitations with momentum
near ±N is suppressed an anomalous dimension for the operator $e^{ip \cdot \hat{\phi}}$ will arise. It should be emphasized that here the scattering amplitudes (8.3) and (8.5) are introduced with no other scope than developing an approximative method to the continuum theory.

The only remnant of M{"o}bius symmetry left in the process of discretization are discrete translations $\mathbb{Z}_C$. This is not a surprise, since $SL(2, \mathbb{C})$ involve rescaling $\delta x = \lambda x$ and $\delta x = cx^2$ which clearly cannot be symmetries in this $\mathbb{Z} \times \mathbb{Z}_C$ theory. As a matter of fact, the vacuum is not annihilated by $L_{\pm 1}, \bar{L}_{\pm 1}$, as announced in sect. 4, but it is annihilated by $L_0, \bar{L}_0$ which are the generators of $x^-$ and $x^+$ discrete translations.

By construction scattering amplitudes defined by (8.3) will approach the corresponding scattering amplitudes of the continuum theory as $N \to \infty$, since in this limit the basic correlator $\langle \hat{\phi} \hat{\phi} \rangle$ equals the corresponding correlator of the standard string theory, and the Wick theorem is applicable. The only difference is that here the M{"o}bius symmetry $SL(2, \mathbb{C})$ (or $SL(2, \mathbb{R})$ for the open string) is only an approximate symmetry in low-energy processes which do not probe the discrete world-sheet structure. The discrete translation invariance permits to fix the position of one vertex operator. For example, setting $x_M, t_M = 0$, the $M$-tachyon amplitude will be given by

$$
A^M_T = \sum_{t_1 > \ldots > t_{M-1} > 0} \sum_{x_1, \ldots, x_{M-1} = 1}^{C} \left( \prod_{i=1}^{M-1} \omega^{-2b_i t_i} \right) e^{ip_1 \cdot \hat{\phi}_1} \cdots e^{ip_{M-1} \cdot \hat{\phi}_{M-1}} e^{ip_M \cdot \hat{\phi}_M} \langle 0 \vert + \text{permutations} \right) 
$$

$$
= \sum_{t_1 > \ldots > t_{M-1} > 0} \sum_{x_1, \ldots, x_{M-1} = 1}^{C} \left( \prod_{i=1}^{M-1} \omega^{-2b_i t_i} \right) \prod_{i<j} e^{-p_i \cdot p_j \langle \hat{\phi}_i \hat{\phi}_j \rangle} \prod_{i<j} \langle \hat{\phi}_i \hat{\phi}_j \rangle + \text{permutations} ,
$$

(8.6)

where the chiral correlators are given in eq. (2.30) and the zero mode as usual yields the energy-momentum conservation delta function. The fact that only one vertex operator position can be fixed means, in particular, that the three-point function will be non-trivial, i.e. depending on the Mandelstam invariant $p_1 \cdot p_2$, unlike continuum string theory where the three-point function on the sphere is a
constant (proportional to the string coupling constant).

Another question concerns whether we should attribute any physical meaning to the number $C$. In an interacting picture where strings split and join, a possible requisite is to demand the interaction to conserve the number of cells, in other words, that cells are not created or destroyed after each interaction. This means that when two strings with $C_1$ and $C_2$ cells join the resulting string would have $C = C_1 + C_2$. In this scenario $C$ would not be a constant of the theory, but a large number that may vary, just as the number of atoms of a macroscopic body. Clearly, this matter is ignored in the scattering amplitudes (8.3) and (8.5) where the number of cells has been left unaltered after consecutive application of vertex operators. However, even in this scenario, the leading correction to the continuum string amplitudes can still be obtained from (8.3) and (8.5). Indeed, we have just argued that the limit $C \to \infty$ exists and leads to the usual continuum scattering amplitudes. Therefore, in scattering processes involving strings with large values of $C$, any correction to eqs. (8.3) and (8.5) due to cell number variation will have to be of subleading order in a perturbative expansion.

Explicit calculations including the leading corrections to the scattering amplitudes of the continuum string theory are under current investigation. For the moment let us elucidate the nature of the corrections to the propagator. The two-point function $\langle \hat{\phi} \hat{\phi} \rangle$ is of the form

$$\sum_{n=1}^{N} \frac{1}{n} x^n$$

A convenient expression for this sum can be obtained by writing

$$\sum_{n=1}^{N} \frac{1}{n} x^n = -\log(1-x) - \frac{xN}{N} \sum_{n=0}^{\infty} \frac{1}{1 + \frac{n}{N}} x^n , \quad \bar{N} \equiv N + 1 . \quad (8.7)$$
By re-arranging the terms in eq. (8.7) one obtains

\[ \sum_{n=1}^{N} \frac{1}{n} x^n = -\log(1 - x) - \frac{x^N}{N} \sum_{n=0}^{\infty} \left( \frac{1}{N} \right)^n \frac{d^n}{du^n} \frac{1}{1 - e^{-u}} ; \quad u = -\log x \quad (8.8) \]

Thus we see that the standard logarithmic propagator is corrected by a (non-perturbative) term of order \(O(e^{N\log x})\) which multiplies an expansion in powers of \(\frac{1}{N} = \frac{1}{N+1}\).

It is easy to prove that the scattering amplitude (8.5) can also be written as

\[ A^M_{\text{open}} = \text{const.} g^{M-2} (\langle 0 | W_0^1 \Delta_0 W_0^2 \Delta_0 \ldots W_0^{M-1} \Delta_0 W_0^M | 0 \rangle + \text{cyclic perm.} , \quad (8.9) \]

where

\[ \Delta_0 = \sum_{t=0}^{\infty} e^{i(L_0-a)t} , \quad (8.10) \]

while for closed strings

\[ A^M_{\text{closed}} = \text{const.} \kappa^{M-2} \langle 0 | U_0^1 \Delta_c U_0^2 \Delta_c \ldots U_0^{M-1} \Delta_c U_0^M | 0 \rangle + \text{perm.} , \quad (8.11) \]

where

\[ \Delta_c = \sum_{t=0}^{\infty} e^{i(L_0+\bar{L}_0-2a)t} , \quad U_0^i = \sum_{x=1}^{C} W_x^i , \quad (8.12) \]

In this form the factorization property is disclosed, the residue in different channels being manifestly the product of the corresponding tree amplitudes.

The decoupling of longitudinal states could be alleged on the basis of the ‘cancelled-propagator argument’: any longitudinal state can be expressed as a commutator \([H, W']\), where \(W'\) is a lower-level operator, etc. However, a closer inspection shows that here the mechanisms must be different. We leave this issue for future work.
9. Deterministic Cellular Strings

In sect. 3 we promoted \( p^\mu \) to an operator in order to apply the first-quantization procedure. Here we will pursue the pure cellular automaton approach a little bit further. The world-sheet variables of cellular strings are now considered to be fully governed by the automaton laws of sect. 2; \( p^\mu \) is a constant of the two-dimensional theory that determines how left and right moving components of the scalar operator transform under \( x \to x + C \). As discussed at the beginning of sect. 3, to make this compatible with target-space quantum mechanics, one should generalize the techniques applied on the world-sheet to target space-time; the physical states shall be a non-trivial linear combination of all states of the primitive basis for the Hilbert space of second quantization, in such a way that their target-space evolution will look non-deterministic, even though it is deterministic for the fundamental variables.

9.1. Deterministic Motion of Free Strings in Target Space

The existence of a light-cone gauge revealed that the target-space time \( \hat{\phi}^+ \) has a sort of discrete structure. One can arrive at a similar conclusion in the covariant formalism from the ambiguity (6.4) in the mass (6.6). What is more remarkable is the evolution of free cellular strings (i.e. \( g = 0 \)) in \( D \)-dimensional space-time is deterministic and can be described by automaton rules.

To be more explicit, let us consider a closed string and return to the basis in which \( \hat{\phi}^\mu \) are diagonal, as described in sect. 2 (see eq. (2.9))

\[
\{ |v^{0\mu}_L \rangle \otimes |v^L_1, \ldots, v^L_C \rangle \otimes |v^{R\mu}_1, \ldots, v^{R\mu}_C \rangle \} .
\] (9.1)

In this basis we have

\[
\hat{\phi}^R_{x,t} = v^R_{x,t}, \quad \hat{\phi}^L_{x,t} = v^L_{x,t}.
\] (9.2)

It is convenient to work in the light-cone gauge. Under a tick of the automaton
clock, the target time \( \varphi^+ = \phi^+ + \frac{\pi l^2}{C} p^+ t \) changes by

\[
\varphi^+_{t+1} = \varphi^+_t + \frac{\pi l^2}{C} p^+ , \tag{9.3}
\]

and we have the deterministic laws (see eq. (2.1))

\[
\varphi_{x}^{Ri}(\varphi^+ + \frac{\pi l^2}{C} p^+) = \varphi_{x-1}^{Ri}(\varphi^+) , \tag{9.4}
\]

\[
\varphi_{x}^{Li}(\varphi^+ + \frac{\pi l^2}{C} p^+) = \varphi_{x+1}^{Li}(\varphi^+) . \tag{9.5}
\]

The “vacuum” \( |v = 0\rangle \) can be thought of as containing “hidden variables”, being an intricaced superposition of all possible states (cf. eq. (2.26)).

If \( l \) is of the Planck-length order, \( l \sim l_p \), and \( p^+ << 1/l_p \), then

\[
\Delta \varphi^+ = \frac{\pi l^2}{C} p^+ \sim \frac{p^+ l_p}{C} l_p << l_p .
\]

The evolution of \( M \) free closed strings,

\[
\varphi_{x_a}^{ia} = \phi^{ia} + \frac{\pi l^2}{C_a} p^i_a t_a + \varphi_{x_a}^{Ria} + \varphi_{x_a}^{Lia} , \quad i = 1, \ldots, D-2 \, , \quad a = 1, \ldots, M \, , \tag{9.6}
\]

is determined in a similar way. The new configuration after a light-cone time interval \( \Delta \varphi^+ \) will be

\[
\varphi_{x_a}^{ia}(\varphi^+ + \Delta \varphi^+) = \varphi_{x_a-\Delta t_a}^{Ria}(\varphi^+) + \varphi_{x_a+\Delta t_a}^{Lia}(\varphi^+) + \phi^{ia}(\varphi^+) + \frac{\pi l^2}{C_a} p^i_a \delta t_a , \tag{9.7}
\]

\[
\Delta t_a = \left[ \frac{C_a}{\pi l^2 p^+_a} \Delta \varphi^+ \right] ,
\]

where \([ [\ldots] ]\) denotes integer part.
9.2. *Interacting Strings – A Model*

An important question is whether there exist deterministic automaton rules dictating the evolution and interaction of many strings which reproduce the behaviour of usual continuum strings at long-distance scales. In our view, there is no reason to think that this is impossible; for example, in refs. [3,4] it was shown that extremely simple automaton rules can lead to chaotic behaviour. It is conceivable that the cognition of a hypothetical second-quantized string theory should uncover such deterministic laws.

To illustrate the evolution of interacting strings, let us consider a simple example. A possible (cell conserving) interacting rule is the following.

A string $1,\ldots,C$ with variables evolving according to eqs. (2.5), (2.8) or eqs. (2.31)-(2.34) –depending on whether the string is closed or open– breaks if at some time there is a cell $y$ where the following inequality is satisfied

$$\mathcal{E}_{y,y+1} \equiv (v_{y,t}^L - v_{y+1,t}^L)^2 + (v_{y,t}^R - v_{y+1,t}^R)^2 \geq K l^2 , \quad (9.8)$$

where $K$ is a given positive real number, related to the string coupling constant. If the string was closed, the result of the breaking is an open string with ends at the cells $y$ and $y+1$, i.e. the cell string $\{y+1, y+2, \ldots, C, 1, \ldots, y\}$ (see fig. 1).

If the string was open, two open strings with $C_1 = y$ and $C_2 = C - y$ cells will emerge (fig. 2). A natural rule prescribing the values of the center of mass momenta carried by each of the resulting strings is attainable by summing up the ‘momenta’ associated with the individual cells. Let us consider an open string with cells $0, 1, \ldots, N+1$. In analogy with the continuum theory, the momentum carried by a single cell is

$$\delta p^\mu_x = \frac{1}{\pi l^2} (\phi^\mu_x - \phi^\mu_{x+1})$$

$$= \frac{1}{N+1} p^\mu + \frac{1}{\pi l^2} (v_{x,t}^R - v_{x,t-1}^R + v_{x,t}^L - v_{x,t-1}^L) , \quad x = 1, \ldots, N . \quad (9.9)$$
The momentum that should be assigned to the end cells can be found by demanding

\[ \sum_{x=0}^{N+1} \delta p_{x,t}^\mu = p^\mu = \text{constant} \quad (9.10) \]

According to the evolution rules (2.33) and the boundary condition (2.34) the quantity

\[ \mathcal{V}^\mu = v_{0,t}^L + v_{N+1,t}^R + \sum_{x=1}^{N} \left( v_{x,t}^R + v_{x,t}^L \right) \quad (9.11) \]

is time-independent, as it can be easily verified. This suggests us to define the end cell momenta as

\[ \delta p_{x,t}^\mu = \frac{1}{2N+2} p^\mu + \frac{1}{2\pi l^2} \left( \frac{v_{x+1,t}^R - v_{x,t}^R + v_{x+1,t}^L - v_{x,t}^L}{v_{x,t}^R - v_{x-1,t}^R} \right), \quad x = 0, N + 1, \quad (9.12) \]

i.e., half the expression corresponding to a non-extremal cell, eq.(9.9). From eq.(9.12) it follows

\[ \sum_{x=0}^{N+1} \delta p_{x,t}^\mu = p^\mu + \frac{1}{\pi l^2} \left( \mathcal{V}_t^\mu - \mathcal{V}_{t-1}^\mu \right) = p^\mu, \]

as desired.

An alternative reflecting boundary condition, which differs from eq. (2.34) by a delay in the reflection process of one time step, is the following:

\[ v_{x+1,t}^R = v_{x,t}^R, \quad v_{x+1,t}^L = v_{x+1,t}^L, \quad x = 0, 1, ..., N + 1, \]

\[ v_{-1,t}^R = v_{0,t}^L, \quad v_{N+2,t}^L = v_{N+1,t}^R. \quad (9.13) \]

Then the eq. (9.9) for \( \delta p_{x,t}^\mu \) as well applies to the end cells. The time-independent
quantity is now

\[ \mathcal{V}^\mu \equiv \sum_{x=0}^{N+1} (v^R_{x,t} + v^L_{x,t}) \quad (9.14) \]

Although both boundary rules give rise to the same theory in the continuum limit, the boundary condition (9.13) may not be very convenient in the first-quantization treatment, since it yields a relative phase factor in the formula connecting left and right mode operators, \( \alpha_n^\mu = \xi n^\mu \).

Now we are ready to pronounce the deterministic rule giving the distribution of momenta of the emerging strings. At the instant of the splitting of the open string 1, ..., \( C \) into the open strings I=1, ..., \( y \) and II=\( y + 1, ..., C \), the total momentum carried by string I is

\[ p^\mu_I = \sum_{x=1}^{y} \delta p^\mu_x = \frac{y}{C-1} p^\mu + \frac{1}{\pi l^2} (\mathcal{V}^\mu_I - \mathcal{V}^\mu_{t-1}) \]

\[ = \frac{y}{C-1} p^\mu + \frac{1}{\pi l^2} (v^L_{y,t} - v^R_{y+1,t}) . \quad (9.15) \]

Similarly

\[ p^\mu_{II} = \sum_{x=y+1}^{C} \delta p^\mu_x = \frac{C-y}{C-1} p^\mu + \frac{1}{\pi l^2} (v^R_{y+1,t} - v^L_{y,t}) . \quad (9.16) \]

In particular, note that energy-momentum is conserved,

\[ p^\mu = p^\mu_I + p^\mu_{II} . \]

Now let us consider joining of strings. The two end cells \( x_1 \) and \( x_2 \) of an open string join if the corresponding \( \mathcal{E}_{x_1,x_2} \) becomes lower than \( K^2 l^2 \), i.e.

\[ \mathcal{E}_{x_1,x_2} = (v^L_{x_1,t} - v^L_{x_2,t})^2 + (v^R_{x_1,t} - v^R_{x_2,t})^2 < K^2 l^2 . \quad (9.17) \]

The result is a closed string with the same value of momentum \( p^\mu \).
In stating the rule for joining of end cells belonging to distinct strings we have to take into account the role of the center of mass coordinate $v^a_{t_0}$, where $a$ labels the different strings. Let us normalize the variables $v^{L\mu}_x, v^{R\mu}_x$ by adding them a global constant such that $\gamma^{\mu a} = 0$. Then we declare that the end cells $x_1, x_2$ of string I and string II join if

$$E_{x_1,x_2} \equiv (v^0_{t_1} - v^0_{t_2})^2 + (v^{L I}_{x_1,t_1} - v^{L II}_{x_2,t_2})^2 + (v^{R I}_{x_1,t_1} - v^{R II}_{x_2,t_2})^2 < K l^2. \quad (9.18)$$

where

$$t_{1,2} = \left[ \frac{C_{I,II} - 1}{\pi l^2 p_{I,II}} \left( \varphi^+ - \varphi^{+I,II} \right) \right], \quad v^{0I,II}_{t_{1,2}} = \varphi^{I,II} + \frac{\pi l^2}{C_{I,II} - 1} p^{I,II}_{t_{1,2}},$$

and $C_{I,II}$ are, respectively, the number of cells of strings I and II (cf. eqs. (2.35)). The result will be an open string with $C_I + C_{II}$ cells.

After each splitting or joining of the strings, the evolution of each of the resulting strings is afresh dictated by eqs. (2.5), (2.8) or eqs. (2.31)-(2.34) until another splitting or joining configuration is encountered.

Since closed strings can turn into open strings and vice versa these rules prescribe a theory which necessarily includes both open and closed (oriented) strings.

Finally, the splitting of a closed string into two closed strings (and similarly for the joining process) is performed when a ‘double’ configuration is met, as depicted in fig. 3. There are two (non-consecutive) cells $x$ and $y$ where eq. (9.8) is satisfied, $E_{x,x+1} \geq K l^2, E_{y,y+1} \geq K l^2$, and at the same time $E_{y,x+1} < K l^2, E_{x,y+1} < K l^2$. Another situation is that the splitting of a closed string into two (or more) closed strings occurs in a finite number of automaton time steps, i.e. by first breaking into two (or more) open strings whose ends join in successive steps (see fig. 4). In the continuum limit the process will appear to happen instantaneously; it should be indistinguishable from an elementary process.

The parameter determining the interaction $E_{x_1,x_2}$ has a simple geometric interpretation in the space of the automata variables as the square distance between the
points \( \{v_0^I, v_{x_1}^{L I}, v_{x_1}^{R I}\} \) and \( \{v_0^{II}, v_{x_2}^{L II}, v_{x_2}^{R II}\} \). Intuitively, the string breaks because when \( E_{x,x+1} \geq K l^2 \) 'it costs too much energy' for the string to keep the cells \( x \) and \( x+1 \) together (in fact, in the continuum limit \( E_{x,x+dx} \) is the energy associated with an infinitesimal segment \( dx \)).

Let us consider the evolution of a single closed string. If the initial configuration is such that \( E_{y,y+1} \) is \( << K l^2 \) for all \( y \), then the values of \( E_{y,y+1} \) will slightly oscillate in time but will never reach the critical value \( K l^2 \), so the string will propagate without ever breaking. If, instead, the initial configuration has some \( E_{y,y+1} \) near the critical value, then splitting may occur during the automaton evolution. Similarly, given \( M \) (open or closed) strings, there are initial configurations such that they will propagate forever without any interaction, but in a generic case splitting and joining will occur. The interaction rate is adjustable by varying \( K \).

We have tested these rules by computer simulations and obtained reasonable behaviour. A thorough study is necessary to decide whether this interacting cellular string theory approximates the dynamics of continuum strings in the long-distance regime.

### 10. Conclusions

The fundamental physical principles of string theory are presently unknown. Considerable progress has been achieved in various areas, but crucial conceptual problems, such as e.g. the off-shell extension, still remain in an opaque and puzzling status. The presence of a Hagedorn phase transition in string theories was interpreted as a limitation in the range of their applicability. Several authors advocated the necessity of a more fundamental theory. In view of the fact that the Hagedorn transition is triggered by a condensation of punctures in the Riemann surface, it was speculated that such a fundamental theory should be formulated on discrete world sheets [6]. However, there are numerous theoretical constraints that a theory of this sort should satisfy. Here we have investigated a ‘minimal’ theory.
which seems to meet all the required properties. In addition, it might hopefully lead to a widely yearned-for result, namely the restoration of determinism in the evolution at Planckian scales.

We have seen, nonetheless, that a discrete world-sheet is intimately linked to a certain discrete target space-time structure. More precisely, we have seen that a discrete world-sheet time implies (in the light-cone gauge) that also the target light-cone time has a discrete structure, leading inevitably to relations of the form (6.4). The problem involved in this relation is the infinite degeneracy of the mass spectrum in virtue of an infinite tower of states for each mass. It is very unclear whether this property represents a problem for the construction of realistic models. Presumably, below the Hagedorn phase transition discrete strings can be effectively replaced by continuum strings and hence these physical states acquire large masses. In any case, the model presented here is relatively simple. \textit{A priori}, other more complex possibilities like, for example, random fractional time steps, or \textit{non-rigid} time steps, in which the lapse of the time step depends on the values of the local variables, cannot be excluded.

Though unavowed in sect. 2, determinism in the evolution of world-sheet variables is still present in the $N \to \infty$ continuum limit, but it no longer admits an interpretation in terms of a cellular automaton. Deterministic systems are found wherever there is a basis in which the wave-function does not spread [3].

We have deliberately separated the issue of discreteness of the world sheet from the issue of determinism in the evolution, since, in principle, the former does not necessitate the latter, though it may be thought of as a prior instance. World-sheet discreteness leads to an indefiniton of the structure constants of the algebras we studied, which translates into ambiguities in physical quantities, such as the square mass of the physical states or the norm of Fock space states. As mentioned in sect. 3, this problem is absent in the fundamental system in terms of automaton variables. The problem arises upon introducing the Hilbert space extension of the system, which is needed to make contact with the usual continuum physics. In
sect. 8 we discussed a prescription to remove such ambiguities in the continuum limit, but the underlying physical mechanisms are unclear.

We have seen that free cellular strings evolve in space-time governed by deterministic rules and speculated that there should as well exist conceivable deterministic rules dictating the evolution of interacting strings. A concrete model was introduced for a string theory where the full evolution, including splitting and joining of an arbitrary number of closed and open cellular strings, is deterministic. While the interaction rules we propose are quite simple, the resulting string dynamics is highly complicated insofar as its description requires numerical calculation or statistical methods.

The reconciliation with target-space quantum mechanics à la ’t Hooft requires a second-quantization framework. One has to define a proper Hilbert space of all possible string states admitting arbitrary occupation numbers, then relate the physical states to the primitive basis, etc. In the context of the interacting model of sect. 9.2, the string amplitudes may be regarded just a first-quantization method to obtain quantum mechanical probabilities, but the inexorable future of the system will have been pre-established at the very moment the initial configuration was given.

There are a number of technical and conceptual points which need detailed investigation. In particular, the properties of the scattering amplitudes are yet to be understood. Of course, it would be inadequate to devote much attention to scattering amplitudes in a system which aspires to be deterministic, but it is necessary to establish a more precise connection with the continuum theory. For the same reason, it may be worth to develop a higher-genus formulation.

Finally, it would also be interesting to consider possible applications of the $\mathbb{Z}_C$-string theory to QCD.

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