EXTREMAL ASPECTS OF THE ERDŐS–GALLAI–TUZA CONJECTURE

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Abstract. Erdős, Gallai, and Tuza posed the following problem: given an $n$-vertex graph $G$, let $\tau_1(G)$ denote the smallest size of a set of edges whose deletion makes $G$ triangle-free, and let $\alpha_1(G)$ denote the largest size of a set of edges containing at most one edge from each triangle of $G$. Is it always the case that $\alpha_1(G) + \tau_1(G) \leq n^2/4$? We also consider a variant on this conjecture: if $\tau_B(G)$ is the smallest size of an edge set whose deletion makes $G$ bipartite, does the stronger inequality $\alpha_1(G) + \tau_B(G) \leq n^2/4$ always hold?

By considering the structure of a minimal counterexample to each version of the conjecture, we obtain two main results. Our first result states that any minimum counterexample to the original Erdős–Gallai–Tuza Conjecture has “dense edge cuts”, and in particular has minimum degree greater than $n/2$. This implies that the conjecture holds for all graphs if and only if it holds for all triangular graphs (graphs where every edge lies in a triangle). Our second result states that $\alpha_1(G) + \tau_B(G) \leq n^2/4$ whenever $G$ has no induced subgraph isomorphic to $K_{3,4}$, the graph obtained from the complete graph $K_4$ by deleting an edge. Thus, the original conjecture also holds for such graphs.

1. Introduction

Given an $n$-vertex graph $G$, say that a set $A \subseteq E(G)$ is triangle-independent if it contains at most one edge from each triangle of $G$, and say that $X \subseteq E(G)$ is a triangle edge cover if $G \setminus X$ is triangle-free. Throughout this paper, $\alpha_1(G)$ denotes the maximum size of a triangle-independent set of edges in $G$, while $\tau_1(G)$ denotes the minimum size of a triangle edge cover in $G$.

Erdős [1] showed that every $n$-vertex graph $G$ has a bipartite subgraph with at least $|E(G)|/2$ edges, which implies that $\tau_1(G) \leq |E(G)|/2 \leq n^2/4$. Similarly, if $A$ is triangle-independent, then the subgraph of $G$ with edge set $A$ is clearly triangle-free; by Mantel’s Theorem, this implies that $\alpha_1(G) \leq n^2/4$. The Erdős–Gallai–Tuza conjecture is a common generalization of these upper bounds:

**Conjecture 1.1** (Erdős–Gallai–Tuza [4]). For every $n$-vertex graph $G$, $\alpha_1(G) + \tau_1(G) \leq n^2/4$.

The conjecture is sharp, if true: consider the graphs $K_n$ and $K_{n/2,n/2}$, where $n$ is even. We have $\alpha_1(K_n) = n/2$ and $\tau_1(K_n) = \binom{n}{2} - n^2/4$, while $\alpha_1(K_{n/2,n/2}) = n^2/4$ and $\tau_1(K_{n/2,n/2}) = 0$. In both cases, $\alpha_1(G) + \tau_1(G) = n^2/4$, but a different term dominates in each case.

The original paper of Erdős, Gallai, and Tuza [4] considered the conjecture only for triangular graphs, which are graphs such that every edge lies in a triangle. Later formulations of the conjecture, such as [2] and [7], dropped the triangularity requirement, and instead stated the conjecture for general graphs; this discrepancy was pointed out by Grinberg on MathOverflow [5], who asked if the two formulations...
were really equivalent. Our results in this paper imply that the two forms of the conjecture are equivalent, settling Grinberg’s question.

Throughout the paper, we use the term minimal counterexample to refer to a vertex-minimal counterexample, that is, a graph $G$ such that the property in question holds for every proper induced subgraph of $G$. When $S \subseteq V(G)$, we write $\overline{S}$ for the set $V(G) - S$, and we write $[S, \overline{S}]$ for the edge cut between $S$ and $\overline{S}$, that is, the set of all edges with one endpoint in $S$ and the other endpoint in $\overline{S}$.

In Section 2, we prove that if $G$ is a minimal counterexample to Conjecture 1.1, then for every vertex set $S$, the edge cut $[S, \overline{S}]$ has more than $|S| (n - |S|)/2$ edges. A small refinement of the argument shows that $\delta(G) > n/2$ whenever $G$ is a minimal counterexample. Thus, any minimal counterexample is a triangular graph, so if Conjecture 1.1 holds for triangular graphs, then no counterexample exists.

We then consider a variant on Conjecture 1.1. Let $\tau_B(G)$ denote the smallest size of an edge set $X$ such that $G - X$ is bipartite; note that $\tau_B(G) \geq \tau_1(G)$. In [6], the author proposed the following stronger version of Conjecture 1.1:

Conjecture 1.2. For every $n$-vertex graph $G$, $\alpha_1(G) + \tau_B(G) \leq n^2/4$.

A partial result [6] towards Conjecture 1.2 and thus towards Conjecture 1.1 states that $\alpha_1(G) + \tau_B(G) \leq 5n^2/16$ for every graph $G$. In Section 3, we study the properties of a minimal counterexample to Conjecture 1.2, obtaining a “dense cuts” theorem similar to that of Section 2 (but somewhat more complicated to state). This theorem implies that if $G$ has no induced subgraph isomorphic to $K_4^-$, then $\alpha_1(G) + \tau_B(G) \leq n^2/4$. Although this class of graphs is highly constrained, it includes the motivating sharpness examples $K_n$ and $K_{n/2,n/2}$.

2. Dense Cuts in a Minimal Counterexample

Erdős, Gallai, and Tuza [4] showed that $\alpha_1(G) + \tau_1(G) \leq |E(G)|$ for all $G$, via the following argument: if $A \subseteq E(G)$ is triangle-independent, then $E(G) - A$ contains at least 2 edges from each triangle of $G$, so $E(G) - A$ is a triangle edge cover. This argument is “global”, dealing with all edges in $G$; we “localize” it, dealing only with edges in some edge cut $[S, \overline{S}]$ for $S \subseteq V(G)$.

To avoid clutter, we write $f_1(G)$ for the sum $\alpha_1(G) + \tau_1(G)$.

Lemma 2.1. If $S$ is nonempty proper subset of $V(G)$, then

$$f_1(G) \leq f_1(G[S]) + f_1(G[\overline{S}]) + |[S, \overline{S}]|.$$  

Proof. Let $A \subseteq E(G)$ be a largest triangle-independent set in $G$, let $G_1 = G[S]$, and let $G_2 = G[\overline{S}]$. For $i \in \{1, 2\}$, let $A_i = A \cap E(G_i)$, so that $A_i$ is a triangle-independent set in $G_i$, and let $B = A \cap [S, \overline{S}]$. Since $|A_i|$ is a lower bound on $\alpha_1(G_i)$, we have

$$\alpha_1(G) = |A| = |A_1| + |A_2| + |B| \leq \alpha_1(G_1) + \alpha_1(G_2) + |B|.$$  

Next, let $X_i$ be a minimum triangle edge cover in $G_i$ for $i \in \{1, 2\}$, so that $|X_i| = \tau_1(G_i)$, and let $Y = [S, \overline{S}] - A$. We claim that $X_1 \cup X_2 \cup Y$ is a triangle edge cover in $G$. Clearly $X_i$ covers all triangles contained in $V(G_i)$, so it suffices to show that $Y$ covers all triangles intersecting both $S$ and $\overline{S}$. If $T$ is such a triangle, then two edges of $T$ lie in $[S, \overline{S}]$. Since $A$ is triangle-independent, at most one of these edges
is contained in $A$; the other lies in $Y$. Hence $X_1 \cup X_2 \cup Y$ is a triangle edge cover in $G$, and we conclude that

$$\tau_1(G) \leq |X_1| + |X_2| + |Y| = \tau_1(G_1) + \tau_1(G_2) + \left( |S, \overline{S}| - |B| \right).$$

Combining the bounds on $\alpha_1(G)$ and $\tau_1(G)$ yields the desired inequality. \qed

**Corollary 2.2.** Let $G$ be a minimal counterexample to Conjecture 1.1. If $S$ is a proper nonempty subset of $V(G)$, then $|S, \overline{S}| > \frac{1}{2} |S| (n - |S|)$, where $n = |V(G)|$.

**Proof.** Let $G_1 = G[S]$ and let $G_2 = G[\overline{S}]$. Since $G$ is a minimal counterexample, we have

$$\alpha_1(G_1) + \tau_1(G_1) \leq |S|^2 / 4,$$

$$\alpha_1(G_2) + \tau_1(G_2) \leq (n - |S|)^2 / 4.$$

By Lemma 2.1 it follows that

$$\alpha_1(G) + \tau_1(G) \leq \frac{n^2}{4} - \frac{|S| (n - |S|)}{2} + |S, \overline{S}|.$$

Since $\alpha_1(G) + \tau_1(G) > n^2 / 4$, the claim follows. \qed

Applying Corollary 2.2 to a set consisting of a vertex of minimum degree yields the lower bound $\delta(G) > (n - 1)/2$. Parity considerations allow us to obtain the stronger bound $\delta(G) > n/2$.

**Theorem 2.3.** If $G$ is a minimal counterexample to Conjecture 1.1, then $\delta(G) > n/2$, where $n = |V(G)|$.

**Proof.** Let $v$ be a vertex of minimum degree in $G$, and let $G_0 = G - v$. By minimality, $\alpha_1(G_0) + \tau_1(G_0) \leq (n - 1)^2 / 4$. By Lemma 2.1, since $G$ is a counterexample we have

$$\frac{n^2}{4} < \alpha_1(G_0) + \tau_1(G_0) + d(v).$$

We split into cases according to the parity of $n$.

**Case 1:** $n$ is odd. Using Equation (1), we have

$$\frac{2n - 1}{4} = \frac{n^2 - (n - 1)^2}{4} < d(v),$$

so $d(v) > \frac{n}{2} - \frac{1}{4}$, which implies $d(v) > n/2$ since $n$ is odd.

**Case 2:** $n$ is even. Since $\alpha_1(G_0) + \tau_1(G_0)$ is an integer, the condition $\alpha_1(G_0) + \tau_1(G_0) \leq (n - 1)^2 / 4$ implies

$$\alpha_1(G_0) + \tau_1(G_0) \leq \frac{n^2 - 2n}{4} = \frac{n^2}{4} - \frac{n}{2}.$$

Therefore, Equation (1) implies

$$\frac{n^2}{4} < \frac{n^2}{4} - \frac{n}{2} + d(v),$$

which again easily yields $d(v) > n/2$. \qed

**Corollary 2.4.** If Conjecture 1.1 holds for all triangular graphs, then Conjecture 1.1 holds for all graphs.
Lemma 3.1. Let \( G \) be a minimal counterexample to Conjecture 1.1. By Lemma 2.3, \( \delta(G) > n/2 \). Thus, any two vertices have a common neighbor, so \( G \) is triangular. It follows that if Conjecture 1.1 holds for all triangular graphs, then no minimal counterexample exists, and so the conjecture holds for all graphs. \( \square \)

3. Dense Cuts in the \( \tau_B \) Variant

In this section, we consider Conjecture 1.2, which deals with the sum \( \alpha_1(G) + \tau_B(G) \). We again focus on edge cuts in a minimal counterexample to the conjecture \( \alpha_1(G) + \tau_B(G) \leq n^2/4 \). The development is analogous to Section 2, with some differences.

For shorthand, let \( f_B(G) = \alpha_1(G) + \tau_B(G) \).

Lemma 3.1. Let \( G \) be a graph, and let \( A \) be a triangle-independent set of edges in \( G \). If \( S \) is a proper nonempty subset of \( V(G) \), then

\[
f_B(G) \leq f_B(G[S]) + f_B(G[S]) + \frac{1}{2} |S, \overline{S}| + |S, \overline{S} \cap A|.
\]

Proof. Clearly, \( \alpha_1(G) \leq \alpha_1(G[S]) + \alpha_1(G[\overline{S}]) + |S, \overline{S} \cap A| \), since \( A \cap G[S] \) and \( A \cap G[\overline{S}] \) are triangle-independent sets in \( G[S] \) and \( G[\overline{S}] \) respectively. The bound \( \tau_B(G) \leq \tau_B(G[S]) + \tau_B(G[\overline{S}]) + \frac{1}{2} |S, \overline{S}| \) follows by considering the two different ways to join the partition sets of a largest bipartite subgraph in \( G[S] \) with those of one in \( G[\overline{S}] \). \( \square \)

Since the conclusion of Lemma 3.1 deals with both the graph \( G \) and a triangle-independent set \( A \), it is difficult to draw blanket conclusions about the structure of a minimal counterexample \( G \). However, we can draw some conclusions if we impose restrictions on the structure of \( G[S] \).

Corollary 3.2. Let \( G \) be a minimal counterexample to Conjecture 1.2 and let \( S \subseteq V(G) \). If \( G[S] \) has independence number \( t \), then

\[
|S, \overline{S}| > (|S| - 2t)(n - |S|),
\]

where \( n = |V(G)| \).

Proof. Let \( A \) be any triangle-independent set of edges in \( G \). Since \( A \) is triangle-independent, the set \( N_A(v) \) is independent in \( G \) for every vertex \( v \). Thus, \( |N_A(v) \cap S| \leq t \) for each \( v \in \overline{S} \), which yields \( |S, \overline{S} \cap A| \leq t(n - |S|) \).

Since \( G \) is a minimal graph satisfying \( \alpha_1(G) + \tau_B(G) > |V(G)|^2/4 \), Lemma 3.1 implies that

\[
\frac{1}{2} |S, \overline{S}| + |S, \overline{S} \cap A| > \frac{|S| (n - |S|)}{2}.
\]

Hence

\[
|S, \overline{S}| > |S| (n - |S|) - 2 |S, \overline{S} \cap A| \geq (|S| - 2t)(n - |S|).
\]

\( \square \)

Corollary 3.2 allows us to prove a general result about maximal cliques in minimal counterexamples.

Lemma 3.3. If \( G \) is a minimal counterexample to Conjecture 1.2 and \( S \) is a maximal clique in \( G \), then \( S \) is contained in an induced copy of \( K_{|S| + 1} \).
Proof. By Corollary B.3, we have $|S, \overline{S}| > (|S| - 2)(n - |S|)$. Thus, $\overline{S}$ contains a vertex $v$ such that $|N(v) \cap S| \geq |S| - 1$. Since $S$ is maximal, $G[S \cup \{v\}] \cong K_{|S|+1}$. □

Before proving that Conjecture 1.2 holds for graphs with no induced copy of $K_4^-$, we prove that it holds for all triangle-free graphs. (This is not trivial, since it is possible that $\tau_B(G) > 0$ even though $G$ is triangle-free.)

Lemma 3.4. If $G$ is a triangle-free $n$-vertex graph, then $\alpha_1(G) + \tau_B(G) \leq n^2/4$.

Proof. This follows from a result of Erdős, Faudree, Pach, and Spencer [3], who showed that if $G$ is an $n$-vertex triangle-free graph with $m$ edges, then $\tau_B(G) \leq m - 4m^2/n^2$. This inequality immediately implies $\alpha_1(G) + \tau_B(G) \leq 2m - 4m^2/n^2$; maximizing the upper bound over $m$ yields $\alpha_1(G) + \tau_B(G) \leq n^2/4$. □

Theorem 3.5. If $G$ is an $n$-vertex graph with no induced copy of $K_4^-$, then $\alpha_1(G) + \tau_B(G) \leq n^2/4$.

Proof. If not, let $G$ be a minimal graph with no induced copy of $K_4^-$ for which $f_B(G) > n^2/4$. If $G'$ is an induced subgraph of $G$ with $n'$ vertices, then $G'$ also has no induced copy of $K_4^-$, so $f_B(G') \leq (n')^2/4$. Thus, $G$ is a minimal counterexample to Conjecture 1.2.

Let $S$ be a clique of maximum size in $G$. By Lemma 3.3 we have $|S| \geq 3$. By Lemma 3.3 it follows that $S$ is contained in some induced copy of $K_{|S|+1}^-$, which contradicts the assumption that $G$ has no induced copy of $K_4^-$. □

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