COSET DIAGRAM FOR THE ACTION OF PICARD GROUP
ON $\mathbb{Q}(i, \sqrt{3})$

QaiserMushtaq $^{a}$, Saima Anis $^{b}$

$^{a}$ Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
qmushtaq@isb.apollo.net.pk

$^{b}$ Department of Mathematics, COMSATS Institute of Information Technology,
University Road, Tobe Camp, Abbottabad, Pakistan
saimaanis_pk@yahoo.com

Abstract. The Picard group $\Gamma$ is $PSL(2,\mathbb{Z}[i])$. We have defined coset diagram for the Picard group. It has been observed that some elements of $\mathbb{Q}(i, \sqrt{3})$ of the form $\frac{a+ib\sqrt{3}}{c}$ and their conjugates $\frac{a-b\sqrt{3}}{c}$ over $\mathbb{Q}(i)$ have different signs in the coset diagram for the action of $\Gamma$ on the biquadratic field $\mathbb{Q}(i, \sqrt{3})$, these are called ambiguous numbers. We have noticed that ambiguous numbers in the coset diagram for the action of $\Gamma$ on $\mathbb{Q}(i, \sqrt{3})$ form a unique pattern. It has been shown that there are finite number of ambiguous numbers in an orbit $\Gamma \alpha$, where $\alpha$ is ambiguous, and they form a closed path and it is the only closed path in the orbit $\Gamma \alpha$. We have devised a procedure to obtain ambiguous numbers of the form $\frac{a+kb\sqrt{3}}{c}$, where $k$ is a positive integer.

2000 Mathematics Subject Classification: Primary 20G15, Secondary 11E04 and 05C25.

Keywords: Picard group, biquadratic field, coset diagram and ambiguous number.

Introduction

Picard group $\Gamma$ or $PSL(2,\mathbb{Z}[i])$ is the group of linear fractional transformations $T(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$ and $a,b,c,d \in \mathbb{Z}[i]$.

The finite presentation of $\Gamma$ is

$\langle A, B, C, D : A^2 = B^2 = C^3 = D^2 = (AC)^2 = (AD)^2 = (BD)^2 = 1 \rangle$, where $A, B, C$ and $D$ are linear fractional transformations defined by $A(z) = \frac{1}{z}$, $B(z) = \frac{1}{z}$, $C(z) = \frac{i+z}{z}$ and $D(z) = \frac{1}{z}$, for details see [3], [4] and [10].

In 1988 [6] Q. Mushtaq has defined ambiguous numbers in real quadratic fields and studied these

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1 This paper was written in 2007 after I have given seminar in Algebra Forum (www.algebraforum.org.pk/about/).
It is from my Ph.D. thesis completed in July 2009. I have also given talk of some of its results in 10th International
Pure Mathematics Conference in August 2009 in Islamabad, Pakistan.
numbers for an action of the modular group on real quadratic fields \( \mathbb{Q}(\sqrt{n}) \). Later in 1995 [2] M. Aslam, S.M. Husnine and A. Majeed have found a formula which determines the exact number of ambiguous numbers in the coset diagrams for the action of the modular group on \( \mathbb{Q}(\sqrt{n}) \). Since Picard group is an extension of the modular group, therefore we have investigated ambiguous numbers in biquadratic field by the action of \( \Gamma \) on the biquadratic field.

The Picard group \( \Gamma \) acts on \( \mathbb{Q}(i) \) and its extensions. The fixed points of generators \( A, B, C \) and \( D \) of \( \Gamma \) are \( \pm \frac{\sqrt{3}}{2}, \pm \frac{\sqrt{3}}{2} \pm i \) respectively. They all lie in a biquadratic field \( \mathbb{Q}(i, \sqrt{3}) \), where \( i \) and \( \sqrt{3} \) are zeros of an irreducible polynomial \((t^2 - 3)(t^2 + 1)\) over \( \mathbb{Q} \). The action of \( \Gamma \) on \( \mathbb{Q}(i, \sqrt{3}) \) is different from \( \mathbb{Q}(i, \sqrt{n}) \), where \( n > 1 \) and \( n \neq 3 \) is a square-free integer, and deserves special account. Therefore the closed paths in the coset diagram for the action of \( \Gamma \) on \( \mathbb{Q}(i, \sqrt{3}) \) must be significantly different from the coset diagrams for the action of \( \Gamma \) on \( \mathbb{Q}(i, \sqrt{n}) \) when \( n \neq 3 \).

The elements of \( \mathbb{Q}(i, \sqrt{3}) \) are of the form \( u + v \sqrt{3} \), where \( u, v \in \mathbb{Q}(i) \). They can be written as \( \alpha = \frac{(a+bi)+(c+di)\sqrt{3}}{e} \), where \( a, b, c, d, e \in \mathbb{Z} \). The conjugates of \( \alpha \) over \( \mathbb{Q} \) are \( \alpha_1 = \frac{(a+bi)-(c+di)\sqrt{3}}{e} \), \( \alpha_2 = \frac{(a+bi)+(c+di)\sqrt{3}}{e} \) and \( \alpha_3 = \frac{(a+bi)-(c+di)\sqrt{3}}{e} \). The conjugate of \( \alpha \) over \( \mathbb{Q}(i) \) is \( \alpha_1 \) and the conjugate of \( \alpha \) over \( \mathbb{Q}(\sqrt{3}) \) is \( \alpha_2 \). The action of \( \Gamma \) on \( \mathbb{Q}(i, \sqrt{3}) \) shows that certain elements of \( \mathbb{Q}(i, \sqrt{3}) \) of the form \( \frac{a+bi\sqrt{3}}{c} \) behave special under this action. Thus they deserve a classification. As there are two conjugates of \( \alpha = \frac{a+bi\sqrt{3}}{c} \) over \( \mathbb{Q} \), namely, \( \alpha \) and \( \frac{a-b\sqrt{3}}{c} \), and the conjugate of \( \alpha \) is again \( \alpha \) over \( \mathbb{Q}(\sqrt{3}) \), so we have considered conjugate of \( \alpha \) over \( \mathbb{Q}(i) \), that is, \( \bar{\alpha} = \frac{a-b\sqrt{3}}{c} \). A real quadratic irrational number \( \alpha = \frac{a+bi\sqrt{3}}{c} \in \mathbb{Q}(i, \sqrt{3}) \), where \( a, b, c \in \mathbb{Z} \), is called totally positive (negative) if \( \alpha \) and \( \bar{\alpha} \) are both positive (negative). When \( \alpha \) and \( \bar{\alpha} \) have opposite signs, then they are called ambiguous numbers. A general element \( \frac{(a+bi)+(c+di)\sqrt{3}}{e} \) of \( \mathbb{Q}(i, \sqrt{3}) \) does not have any sign that is why we have considered elements of the form \( \frac{a+bi\sqrt{3}}{c} \). We are discussing orbits of \( \mathbb{Q}(i, \sqrt{3}) \) by action of \( \Gamma \) therefore we have written in the statements of our results that ambiguous numbers are in \( \mathbb{Q}(i, \sqrt{3}) \). They play an important role in classifying the orbits of \( \mathbb{Q}(i, \sqrt{3}) \) when \( \Gamma \) acts on it.

A diagrammatic argument, called coset diagram for the action of Picard group \( \Gamma \) on \( \mathbb{Q}(i, \sqrt{3}) \), is defined and used to prove results in this paper. G. Higman and Q. Mushtaq have defined coset diagram for modular group in [5]. One of the applications of coset diagram for modular group is in [1]. We have defined coset diagram for the Picard group \( \Gamma \). They need symbols for the generators as well as a method or pattern to join them. The group \( \Gamma \) consists of four generators, two of order 3 and two of order 2, so it is possible to avoid using colours. The generators \( A \) and \( C \) both have order 3, so the 3-cycles of \( A \) and \( C \) are represented by triangles. But to distinguish generator \( A \) from generator \( C \), we have denoted the 3-cycles of the generator \( C \) by three unbroken edges of a triangle permuted anticlockwise. The 3-cycles of the generator \( A \)
are denoted by three broken edges of a triangle permuted anticlockwise. As generators $B$ and $D$ are involutions so we have represented them by edges and orientation of edges can be avoided. To distinguish generator $B$ from generator $D$, the 2-cycles of generator $B$ have been represented by a bold edge and two vertices which are interchanged by $D$ have been joined by an edge. Fixed points of $A,B,C$ and $D$, if they exist, have been denoted by heavy dots.

The fragment of a coset diagram in Fig.1 explains clearly the amalgam structure of $\Gamma$, that is, $\Gamma = (A_4 \ast S_3) \ast (S_3 \ast D_2)$, where $M$ is modular group whose finite presentation is $\langle D, C : D^2 = C^3 = 1 \rangle$. A general fragment of the coset diagram for the action of $\Gamma$ on $\mathbb{Q}(i)$ will look as follows.

![Fig.1](image)

The generators $A$ and $C$ together form a Cayley's diagram of $A_4$. Two diagrams of $A_4$ are joined by edges of generator $D$. The edges have joined the 3 vertices of a triangle with broken edges of one diagram of $A_4$ to that of another diagram of $A_4$. The generators $A$ and $D$ together generate $S_3$. A triangle having broken edges is common between the diagram of $A_4$ and $S_3$. Algebraically, $\mathbb{Z}_3$ is an amalgam of $A_4$ and $S_3$. One diagram of $A_4$ can be joined to four other diagrams of $A_4$ by edges representing the generator $D$. We have called the fragment of the coset diagram which consists of diagrams of $A_4$ and edges of $D$ as a fragment $A - C - D$ because this fragment of coset diagram is composed by the use of generators $A, C, D$.

The two fragments $A - C - D$ have joined by bold edges representing the generator $B$ as shown in Fig. 1. The bold edges have joined the 3 vertices of triangle with unbroken edges of one diagram of $A_4$ of one fragment $A - C - D$ to that of another fragment $A - C - D$. The generators $B$ and $C$ together generate $S_3$. The generators $B$ and $D$ generate the group $D_2$. A bold edge is common between the diagram of $S_3$ and $D_2$. Algebraically, $\mathbb{Z}_2$ is an
amalgam of $S_3$ and $D_2$.

Coset diagrams for the orbit of $\Gamma$ on biquadratic field $\mathbb{Q}(i, \sqrt{3})$ give some interesting information. We have shown that there is a finite number of ambiguous numbers in the orbit $\Gamma \alpha$, where $\alpha$ is ambiguous, and that they form a closed path and it is the only closed path in the orbit $\Gamma \alpha$. We have classified all the ambiguous numbers in the orbit. We need Propositions 10 to 9 to obtain one ambiguous number from the other.

**Proposition 1.** If $\alpha = \frac{a + b \sqrt{3}}{c} \in \mathbb{Q}(i, \sqrt{3})$ is a totally positive real quadratic irrational number, then $C(\alpha)$ and $C^2(\alpha)$ are totally negative.

**Proof.** Let $\alpha$ be a totally positive quadratic number. Then there are two possibilities either $a, d, c > 0$ or $a, d, c < 0$, where $d = \frac{a^2 - 3b^2}{c}$. When $a, d, c > 0$, then $C(\alpha) = C(\frac{a + b \sqrt{3}}{c}) = \frac{-a - b \sqrt{3}}{d}$. Here $a_1 = -a - d < 0$, $c_1 = d > 0$, and $d_1 = 2a + c + d > 0$. This shows that $C(\alpha)$ is totally negative. Also

$$C^2(\alpha) = \frac{-1}{1 + a} = \frac{-a + b \sqrt{3}}{2a + c + d}.$$  

Here $a_2 = -a - c < 0$, $c_2 = 2a + c + d > 0$, and $d_2 = \frac{a^2 - 3b^2}{c^2} = \frac{2a^2 + 2ac - 3b^2}{2a + c + d} = \frac{c(2a + c + d)}{2a + c + d} = c > 0$. This shows that $C^2(\alpha)$ is totally negative. Similarly, it can be proved that when $a, d, c < 0$, then $C(\alpha)$ and $C^2(\alpha)$ are totally negative.



If a number $\alpha$ is ambiguous, then $\alpha \alpha = \frac{a^2 - 3b^2}{c^2} < 0$, that is, $a^2 - 3b^2 < 0$. Thus, in other words $\alpha$ is ambiguous when $dc < 0$, where $d = \frac{a^2 - 3b^2}{c^2}$.

**Lemma 1.** Transformations $B$ and $D$ map an ambiguous number to an ambiguous number.

**Proof.** Let $\alpha = \frac{a + b \sqrt{3}}{c} \in \mathbb{Q}(i, \sqrt{3})$ be an ambiguous number, where $a, b, c \in \mathbb{Z}$. This implies that $\alpha \alpha < 0$, that is, $\frac{a^2 - 3b^2}{c^2} < 0$ which further implies that $a^2 - 3b^2 < 0$. Now

$$B(\alpha) = \frac{1}{\alpha} = \frac{a - b \sqrt{3}}{d},$$

$$D(\alpha) = -\frac{1}{\alpha} = \frac{a - b \sqrt{3}}{-d},$$

imply that $(B(\alpha))(B(\alpha)) = \frac{a^2 - 3b^2}{d^2} = D(\alpha)(D(\alpha))$.

Here $\frac{a^2 - 3b^2}{d^2} < 0$, since $a^2 - 3b^2 < 0$ and $d^2 > 0$. This shows that $B(\alpha)$ and $D(\alpha)$ are ambiguous numbers.

**Remark 1.** The value of $b$ is invariant for the elements of the form $\alpha = \frac{a + b \sqrt{3}}{c}$ in $\Gamma \alpha$, where $\alpha \in \mathbb{Q}(i, \sqrt{3})$.

**Lemma 2.** If $\alpha$ is not ambiguous, then so are $D(\alpha)$ and $B(\alpha)$.
Proof. Suppose \( \alpha = \frac{a+b\sqrt{3}}{c} \in \mathbb{Q}(i, \sqrt{3}) \) is not ambiguous number, where \( a, b, c \in \mathbb{Z} \). This means that \( a\bar{a} \geq 0 \), that is, \( a^2 - 3b^2 \geq 0 \) which further implies that \( a^2 - 3b^2 \geq 0 \) because \( c^2 \neq 0 \). Since \( B(a) = \frac{1}{a} = \frac{-b\sqrt{3}}{d} \), and \( D(a) = \frac{-1}{a} = \frac{-a+b\sqrt{3}}{d} \), so \( (B(a))(B(a)) = \frac{a^2 - 3b^2}{d^2} = D(a)(D(a)) \). Here \( \frac{a^2 - 3b^2}{d^2} \geq 0 \), because \( a^2 - 3b^2 \geq 0 \) and \( d^2 > 0 \). This shows that \( B(a) \) and \( D(a) \) are not ambiguous numbers.

**Proposition 2.** If \( \alpha \) is an ambiguous number, then \( A(\alpha) \) and \( A^2(\alpha) \) are not ambiguous.

Proof. Let \( \alpha = \frac{a+b\sqrt{3}}{c} \) be an ambiguous number. This means that \( a\bar{a} < 0 \), that is, \( a^2 - 3b^2 < 0 \). This implies that \( a^2 - 3b^2 < 0 \). After rationalization of \( A(\alpha) = \frac{1}{a-i} = \frac{c}{a+b\sqrt{3}-ci} \), the imaginary part is \( a^2c^2 + c^4 + 3b^2c^2 - 2abc^2\sqrt{3} \). This means that \( A(\alpha) \) will be ambiguous if and only if \( a^2c^2 + c^4 + 3b^2c^2 - 2abc^2\sqrt{3} \) is zero. Since \( i \neq 0 \), it implies that \( a^2c^2 + c^4 + 3b^2c^2 - 2abc^2\sqrt{3} = 0 \). The term on left hand side will be zero only if \( c = 0 \). But \( c \) cannot be zero because otherwise \( \alpha = \infty \) will be not ambiguous. This shows that \( A(\alpha) \) is not ambiguous.

Also, \( A^2(\alpha) = \frac{1+ia}{a} = \frac{-a+b\sqrt{3}+di}{d} \) will be ambiguous when imaginary part becomes zero, that is, \( d = 0 \). But \( d \) cannot be zero because otherwise \( A^2(\alpha) \) will become \( \infty \). This shows that \( A^2(\alpha) \) is not ambiguous.

**Proposition 3.** If \( \alpha = \frac{a+b\sqrt{3}}{c} \in \mathbb{Q}(i, \sqrt{3}) \) is an ambiguous number, then one of \( C(\alpha) \) and \( C^2(\alpha) \) is ambiguous and the other is totally negative.

Proof. Suppose that \( \alpha \) is a positive ambiguous number. Then by Proposition 1, the information can be tabulated as follows.

| \( \alpha \) | \( C(\alpha) \) | \( C^2(\alpha) \) | \( \bar{\alpha} \) | \( \bar{C}(\alpha) \) | \( \bar{C}^2(\alpha) \) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| +           | -               | -               | +               | -               |
| \(-\)       | +               | -               | \(-\)           | +               |

Similarly, if \( \alpha \) is a negative ambiguous number, then the information about \( C(\alpha) \), \( C^2(\alpha) \), \( \bar{\alpha} \), \( \bar{C}(\alpha) \) and \( \bar{C}^2(\alpha) \) can be tabulated as follows.

| \( \alpha \) | \( C(\alpha) \) | \( C^2(\alpha) \) | \( \bar{\alpha} \) | \( \bar{C}(\alpha) \) | \( \bar{C}^2(\alpha) \) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(-\)       | +               | -               | \(-\)           | +               |
| +           | -               | +               | \(-\)           | +               |

Therefore, from the above tables it can easily be deduced that one of \( C(\alpha) \) and \( C^2(\alpha) \) is ambiguous and the other is totally negative.
Proposition 4. If \( \alpha = \frac{a + b\sqrt{3}}{c} \in \mathbb{Q}(i, \sqrt{3}) \) such that \( d = \frac{a^2 - 3b^2}{c} \) is an integer, then the following hold:

(i) \( d \) of \( C(\alpha) \) and \( C^2(\alpha) \) are integers,
(ii) \( d \) of \( B(\alpha) \) is an integer,
(iii) \( d \) of \( D(\alpha) \) is an integer.

Proof. (i) Let \( \alpha = \frac{a + b\sqrt{3}}{c} \in \mathbb{Q}(i, \sqrt{3}) \) such that \( d = \frac{a^2 - 3b^2}{c} \in \mathbb{Z} \). Then \( C(\alpha) = \frac{-a + b\sqrt{3}}{d} \), where \( d \geq 2a + c + d \in \mathbb{Z} \), and \( C^2(\alpha) = \frac{-a + b\sqrt{3}}{2a + c + d} \), where \( d = c \in \mathbb{Z} \).

(ii) Since \( B(\alpha) = \frac{a - b\sqrt{3}}{d} \), the value of \( d \) of \( B(\alpha) \) is \( \frac{a^2 - 3b^2}{d} = c \in \mathbb{Z} \).

(iii) Also \( D(\alpha) = \frac{-a - b\sqrt{3}}{-d} \) implies that \( d \) of \( D(\alpha) \) is \( \frac{a^2 - 3b^2}{-d} = -c \in \mathbb{Z} \).

A closed path whose all vertices are ambiguous numbers is called the closed path of ambiguous numbers. Lemma 3, Theorem 1 and Theorem 2 show that there is finite number of ambiguous numbers in one orbit and they form a single closed path in the orbit. In this single closed path the value of \( b \) in \( \frac{a + b\sqrt{3}}{c} \) remain invariant.

Lemma 3. Ambiguous numbers in \( \Gamma \alpha \) are finite.

Proof. Let \( \alpha = \frac{a + b\sqrt{3}}{c} \). It will be ambiguous when \( \frac{a^2 - 3b^2}{c^2} < 0 \) or \( a^2 - 3b^2 < 0 \) or \( a^2 < 3b^2 \). This shows that the values of \( a \) is finite which satisfy the condition \( a^2 < 3b^2 \) for constant value of \( b \). By Remark 1, the value of \( b \) remain invariant for the numbers of the form \( \frac{a + b\sqrt{3}}{c} \) in \( \Gamma \alpha \), where \( a, b, c \in \mathbb{Z} \). By Proposition 4, \( d \) is integer, this implies that \( c \) divides \( (a^2 - 3b^2) \), so values of \( c \) are also finite. As values of \( a \) and \( c \) are finite and value of \( b \) is fixed in a orbit so ambiguous numbers of the form \( \frac{a + b\sqrt{3}}{c} \) are also finite in an orbit.

Theorem 1. In a coset diagram for \( \Gamma \alpha \), where \( \alpha \) is an ambiguous number, the ambiguous numbers form a closed path.

Proof. Suppose \( \alpha_1 \) is an ambiguous number in \( \Gamma \alpha \). By \( \alpha_1^k \) we mean ambiguous number \( \alpha_1 \) in \( k \)th triangle. Then by Proposition 3, either \( C(\alpha_1^k) \) or \( C^2(\alpha_1^k) \) is ambiguous. Let \( C(\alpha_1^k) = \alpha_2 \) be ambiguous. This means each triangle with unbroken edges in the coset diagram for \( \Gamma \) contains two ambiguous numbers. By Proposition 10, \( B(\alpha_1^k) = \alpha_1^m \), \( B(\alpha_2^k) = \alpha_2^m \) and \( D(\alpha_2^k) = \alpha_3^k, D(\alpha_1^k) = \alpha_2^k \) are also ambiguous. So within the \( k \)th triangle, the generators \( D \) or \( B \) are used to reach to the next ambiguous numbers as shown in Fig. 2. Since \( (BC)^2 = 1 \), there fore there are just two paths, one from \( k \)th triangle and another from \( m \)th triangle, and generator \( B \) connects these two paths.
We can continue in this way as shown in Fig.2 and since by Lemma 3, there are only a finite number of ambiguous numbers, therefore after a finite number of steps we reach the vertex, $a_1^{k+n} = a_1^{k-1}$. Hence ambiguous numbers form a closed path in the coset diagram. ■

**Theorem 2.** If $\alpha$ is an ambiguous number, then in $\Gamma \alpha$ there is only one closed path of ambiguous numbers.

**Proof.** In a diagram of $A_4$ there are four triangles with unbroken edges. So the diagram of $A_4$ in which $\alpha = \alpha_0$ is a vertex of one of the triangles with unbroken edges contains three more triangles with unbroken edges. According to Proposition 3, if $\alpha_0$ is an ambiguous number, then one of $C(\alpha_0)$ and $C^2(\alpha_0)$ is also an ambiguous number. Let us denote this ambiguous number by $\alpha_1$. Then by Lemma 1, these two ambiguous numbers are joined with other ambiguous numbers by generators $B$ and $D$. Therefore suppose $B(\alpha_0) = \alpha_2$, $B(\alpha_1) = \alpha_3$ and $D(\alpha_0) = \alpha_4$, $D(\alpha_1) = \alpha_5$.

As the vertices of other three triangles with unbroken edges are also vertices of triangles with broken edges, according to Proposition 2 unambiguous, $A(\alpha_j)$ and $A^2(\alpha_j)$ are not ambiguous numbers but they contain $i$, where $i = 0, 1$. By applying transformations $C$ and $C^2$ on them, one gets not ambiguous numbers having $i$. So this diagram for $A_4$ contains only one triangle having two ambiguous numbers. If we expand this diagram and apply generators $B$ and $D$ on numbers which are not ambiguous, then by Lemma 2, no further ambiguous numbers are found. Since these numbers which are not ambiguous contain $i$, so by applying transformations $C$ and $C^2$ on them, one gets numbers which are not ambiguous but contain $i$. According to
Lemma 1 and Proposition 3, the generators $B, C$ and $D$ map $a_0$ and $a_1$ to other ambiguous numbers, namely, $a_2, a_3, a_4, \ldots, a_n$, since they are finite according to Lemma 3. By Theorem 1 they form a closed path. As one cannot find any further ambiguous numbers from the numbers which are not ambiguous, this means that there is only one closed path of ambiguous numbers in $\Gamma a_0$.

**Theorem 3.** If $a$ is an ambiguous number of the form $\frac{a+k\sqrt{3}}{c} \in \mathbb{Q}(i, \sqrt{3})$, where $a, c \in \mathbb{Z}$ and $k$ is a constant integer, then

(i) $a^2 < 3k^2$ ,

(ii) $c$ is a divisor of $a^2 - 3k^2$ .

**Proof.** (i) Let $a = \frac{a+k\sqrt{3}}{c}$ be an ambiguous number. This means that $a\bar{a} < 0$ that is $\frac{a^2 - 3k^2}{c^2} < 0$. Since $c^2 > 0$, this implies that $a^2 - 3k^2 < 0$, which further implies that $a^2 < 3k^2$.

(ii) The ambiguous number in the closed path containing $k\sqrt{3}$ are of the form $\frac{a+k\sqrt{3}}{c}$. We know that $d$ of $k\sqrt{3}$ is $-3k^2$. By Proposition 4, $d$ of other ambiguous numbers in the closed path containing $k\sqrt{3}$ are also integer. We have $d = \frac{a^2 - 3k^2}{c}$, it will be integer if and only if $c$ divides $a^2 - 3k^2$. This shows that $c$ is the divisor of $a^2 - 3k^2$.

We denote the greatest common divisor of $a, b$ and $c$ by $(a, b, c)$, where $a, b, c \in \mathbb{Z}$. If $a$ divides $b$, then we denote it by $a | b$.

**Proposition 5.** Let $m \neq k$ be a factor of $k \in \mathbb{Z}^+$ and $(a, k, c) \neq 1$. If $a = \frac{\dot{a} + m\sqrt{3}}{\dot{c}}$ such that $(\dot{a}, m, \dot{c}) = 1$ and $\dot{c} | \dot{a}^2 - 3m^2$, then ambiguous numbers of the form $\frac{a + m\sqrt{3}}{c}$ do not exist in the closed path of ambiguous numbers of the form $\frac{a+k\sqrt{3}}{c}$.

**Proof.** Let $m \neq k$ be a factor of $k$. Let $a = \frac{a+k\sqrt{3}}{c}$ be an ambiguous number such that $(a, k, c) \neq 1$. This means $a$ can be written as $\frac{\dot{a} + m\sqrt{3}}{\dot{c}}$ such that $(\dot{a}, m, \dot{c}) = 1$. For $q \in \mathbb{Z}$ we have $\dot{a}q = a, \dot{c}q = c$ and $mq = k$. Since $a$ is ambiguous number, this means $\dot{a}^2 < 3m^2$ and $\dot{c} | \dot{a}^2 - 3m^2$. This means $a$ is an ambiguous number of the form $\frac{\dot{a} + m\sqrt{3}}{\dot{c}}$ and by Theorem 1, it forms a closed path of ambiguous numbers. By Remark 1, they are of the form $\frac{a+k\sqrt{3}}{c}$. By Theorem one, it is the only closed path of ambiguous numbers of the form $\frac{\dot{a} + m\sqrt{3}}{\dot{c}}$ in an orbit. So if $m \neq k$, then in the closed path all ambiguous numbers will be of the form $\frac{a + m\sqrt{3}}{c}$. Hence the ambiguous numbers of forms $\frac{a+k\sqrt{3}}{c}$ and $\frac{a+m\sqrt{3}}{c}$ lie in different closed paths.

By Proposition 5, one can check whether an element of the form $\frac{a+k\sqrt{3}}{c}$, where $(a, k, c) \neq 1$, belongs to the closed path of ambiguous numbers of form $\frac{a+k\sqrt{3}}{c}$ or of the form $\frac{a+m\sqrt{3}}{c}$, where
such that \( m \neq k \) is a factor of \( k \). Let \( \alpha = \frac{a + k \sqrt{3}}{c} \) such that \((a,k,c) \neq 1\), then \( \alpha \) can be written as \( \frac{a + m \sqrt{3}}{c} \) such that \((a,m,c) = 1\), where \( m \) is a factor of \( k \). If \( c \mid \alpha^2 - 3m^2 \), then \( \alpha \) is of the form \( \frac{a + k \sqrt{3}}{c} \) otherwise \( \frac{a + k \sqrt{3}}{c} \). In the later case \( c \) should divide \( \alpha^2 - 3k^2 \). In other words if \( \alpha = \frac{a + k \sqrt{3}}{c} \) is in its simplest form and \( c \mid a^2 - 3k^2 \), then \( \alpha \) occurs in the closed path of the form \( \frac{a + k \sqrt{3}}{c} \).

For instance let \( k = 2 \). The prime decomposition of \( k \) is obviously \( 2 = 2 \times 1 \). Therefore ambiguous numbers of the form \( \frac{a + \sqrt{3}}{c} \) already exist in a closed path containing \( \sqrt{3} \). So they will not exist in the closed path containing \( 2\sqrt{3} \). By using Theorem 3 and Proposition 5, it can be shown easily that the closed path containing \( 2\sqrt{3} \) has 32 ambiguous numbers. For \( a = \frac{a + 2\sqrt{3}}{c} \) to be ambiguous, \( a^2 < 12 \) which implies that \( a = 0, \pm 1, \pm 2, \pm 3 \). Let \( d = \frac{a^2 - 12}{c} \). When \( a = 0 \), then \( d = \frac{-12}{c} \) will be integer if and only if \( c = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \). When \( a = 0 \) and \( c = 2 \), then \( \alpha = \frac{2\sqrt{3}}{2} = \sqrt{3} \). Since \( \sqrt{3} \) already exist in the closed path of ambiguous numbers of the form \( \frac{a + \sqrt{3}}{c} \), we discard \( c = \pm 2 \) and its cofactor \( \pm 6 \). Also when \( c = 6 \), then \( \alpha = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \) and \( 3 \) divides \( 3 \). This shows that \( \alpha \) belongs to the closed path of the form \( \frac{a + \sqrt{3}}{c} \), therefore we also discard the cofactor \( 6 \). Eventually, we have eight values of \( c \), this means we have 8 values of \( \alpha \).

When \( a = \pm 1 \), then \( d = \frac{1 - 12}{c} = \frac{-11}{c} \), \( d \in \mathbb{Z} \) if and only if \( c = \pm 1, \pm 11 \). So we have four values of \( c \). When \( a = 2 \), \( d = \frac{4 - 12}{c} = \frac{-8}{c} \), \( d \in \mathbb{Z} \) if and only if \( c = \pm 1, \pm 2, \pm 4, \pm 8 \). When \( a = \pm 2 \) and \( c = \pm 2 \), \( \alpha = \frac{\pm 2 + 2\sqrt{3}}{2} = \pm 1 + \sqrt{3} \) is of the form \( \frac{a + \sqrt{3}}{c} \). So we discard \( 2 \) and \( 4 \) (its cofactor). When \( a = \pm 3 \), then \( d = \frac{9 - 12}{c} = \frac{-3}{c} \) implies that \( c = \pm 1, \pm 3 \). Here we have twelve values of \( c \) for \( a = \pm 1, \pm 2, \pm 3 \). This means that there are \( 24 \) values of \( \alpha \) because of the positive and negative values of \( a \). Thus the total ambiguous numbers of the form \( \frac{a + 2\sqrt{3}}{c} \) in an orbit of the coset diagram obtained by action of \( \Gamma \) on \( \mathbb{Q}(i, \sqrt{3}) \) are \( 24 + 8 = 32 \).

A fragment of the coset diagram obtained by action of \( \Gamma \) on \( \mathbb{Q}(i, \sqrt{3}) \) containing closed path of ambiguous numbers of form \( \frac{a + 2\sqrt{3}}{c} \), is shown below. In Fig.4, here are two layers of fragment \( A - C - D \). These two layers are connected by bold edges. The distinction between two layers are the conjugates, that is, if \( \alpha = \frac{a + 2\sqrt{3}}{c} \) occurs in one layer, then \( \bar{\alpha} = \frac{a - 2\sqrt{3}}{c} \) occurs in the second layer. Thus there is a mapping, say \( S \), from one layer to the other, defined by \( S : \alpha \mapsto \bar{\alpha} \). It can be noted here that in each diagram of \( A_4 \), there are exactly two ambiguous numbers.
In this diagram we have shown only the closed path of ambiguous numbers of the form $\frac{a+2\sqrt{3}}{c}$ out of the fragment of Fig.4. In Fig.5, we apply repeatedly the generators $C, C^2$ and $D$ to obtain the ambiguous numbers in the path from $\frac{-3-2\sqrt{3}}{3}$ to $-3+2\sqrt{3}$ in one layer. The
ambiguous numbers in the path from \(\frac{3 + 2\sqrt{3}}{3} \) to \(3 - 2\sqrt{3}\) are in second layer. These two layers are connected by bold edges. For instance, \(-3 + 2\sqrt{3}\) and \(\frac{3 + 2\sqrt{3}}{3}\) are connected by a bold edge as shown below. This is the only closed path of ambiguous numbers in the orbit \(\Gamma\left(2\sqrt{3}\right)\).

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