ORTHOGONALITY OF THE DICKSON POLYNOMIALS OF THE 
(k + 1)-TH KIND

DIEGO DOMINICI

Abstract. We study the Dickson polynomials of the (k + 1)-th kind over the field of complex numbers. We show that they are a family of co-recursive orthogonal polynomials with respect to a quasi-definite moment functional $L_k$. We find an integral representation for $L_k$ and compute explicit expressions for all of its moments.

Dedicated to V. E. P., in gratitude for her continuous unbounded support (without measure).

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1. Introduction

Let $n \in \mathbb{N}$, $F_q$ be a finite field and $a \in F_q$. The Dickson polynomials $D_n(x; a)$, defined by [44, 9.6.1]

$$D_n(x; a) = \sum_{j=0}^{[n/2]} \binom{n}{j} \binom{n-j}{j} (-a)^j x^{n-2j}, \quad x \in F_q$$

where introduced by Leonard Eugene Dickson (1874 – 1954) in his 1896 Ph.D. thesis “The analytic representation of substitutions on a power of a prime number of letters, with a discussion of the linear group” [19], published in two parts in The Annals of Mathematics [20], [21]. The Dickson polynomials are the unique monic polynomials satisfying the functional equation [44 9.6.3]

$$D_n \left( y + \frac{a}{y}; a \right) = y^n + \left( \frac{a}{y} \right)^n, \quad y \in F_{q^2}.$$ 

See [37] for further algebraic and number theoretic properties of the Dickson polynomials.

Let $N_0$ denote the set $\mathbb{N} \cup \{0\} = 0, 1, 2, \ldots$. In [57], Wang and Yucas extended the Dickson polynomials to a family depending on a new parameter $k \in N_0$, which they called Dickson polynomials of the (k + 1)-th kind. They defined them by
\[ D_{n,k}(x; a) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-kj}{n-j} (-a)^j x^{n-2j}, \]  
(1)

with initial values
\[ D_{0,k}(x; a) = 2 - k, \quad D_{1,k}(x; a) = x. \]  
(2)

They also showed that the polynomials \( D_{n,k}(x; a) \) satisfy the fundamental functional equation
\[ D_{n,k}(y + \frac{a}{y}; a) = y^n + \left( \frac{a}{y} \right)^n + k \frac{ay^n - y^2 \left( \frac{a}{y} \right)^n}{y^2 - a}, \quad y \neq 0. \]  
(3)

Note that
\[ \lim_{y \to \pm \sqrt{a}} \frac{ay^n - y^2 \left( \frac{a}{y} \right)^n}{y^2 - a} = (n - 1) \left( \pm \sqrt{a} \right)^n, \]
and therefore
\[ D_{n,k}(y + \frac{a}{y}; a) = [2 + (n - 1) k] \left( \pm \sqrt{a} \right)^n, \quad y = \pm \sqrt{a}. \]

We clearly have
\[ D_{n,0}(x; a) = D_n(x; a) \quad \text{(Dickson polynomials)} \]
and \([44, 9.6.1]\]
\[ D_{n,1}(x; a) = E_n(x; a) \quad \text{(Dickson polynomials of the second kind)}. \]

In fact, since
\[ \frac{n - kj}{n - j} = k - (k - 1) \frac{n}{n - j}, \]
we have \([57, 2.1]\]
\[ D_{n,k}(x; a) = kE_n(x; a) - (k - 1) D_n(x; a). \]

The polynomials \( D_{n,k}(x; a) \) also satisfy the fundamental recurrence (see \([57, \text{Remark 2.5}]\))
\[ D_{n+2,k} = xD_{n+1,k} - aD_{n,k}, \quad n \in \mathbb{N}_0. \]  
(4)

The first few Dickson polynomials of the \((k + 1)\)-th kind are
\[ D_{2,k}(x; a) = x^2 + a(k - 2), \]
\[ D_{3,k}(x; a) = x^3 + a(k - 3) x, \]
\[ D_{4,k}(x; a) = x^4 + a(k - 4) x^2 + a^2 (2 - k), \]
\[ D_{5,k}(x; a) = x^5 + a(k - 5) x^3 + a^2 (5 - 2k) x. \]
They have zeros at
\[ x = \pm \sqrt{a} \sqrt{2 - k}, \quad \text{if } n = 2, \]
\[ x = 0, \quad \pm \sqrt{a} \sqrt{3 - k}, \quad \text{if } n = 3, \]
\[ x = \pm \sqrt{a} \sqrt{4 - k \pm \sqrt{(k - 2)^2 + 4}}, \quad \text{if } n = 4, \]
\[ x = 0, \quad \pm \sqrt{a} \sqrt{5 - k \pm \sqrt{(k - 1)^2 + 4}}, \quad \text{if } n = 5, \]
as can be verified using a mathematical symbolic computation program such as Mathematica.

**Remark 1.** Note that the polynomials \( D_{n,k}(x; a) \) are monic for \( n \geq 1 \), but \( D_{0,k}(x; a) = 1 \) only for \( k = 1 \).

In this article, we study the polynomials \( D_{n,k}(x; a) \) over the field of complex numbers, with \( a > 0 \) and \( k \in \mathbb{R} \). Our motivation is the three-term recurrence relation (4), which suggests that the Dickson polynomials of the \((k + 1)\)-th kind form a family of orthogonal polynomials with respect to some linear functional \( L_k \). However, from (5) we see that for \( k > 2 \) the polynomials \( D_{n,k}(x; a) \) may have a pair of purely imaginary roots. Also, the polynomials \( D_{3,3}(x; a) \) and \( D_{5,5}(x; a) \) have a triple zero at \( x = 0 \). This implies that the linear functional \( L_k \) is quasi-definite \[16, \text{Theorem 2.4.3}, \quad 27, \text{Theorem 1}]\).

The article is organized as follows: in Section 2, we derive some of the main properties of the Dickson polynomials of the \((k + 1)\)-th kind, including different expressions, a hypergeometric representation, differential equations, and a generating function.

In Section 3, we present some basic results from the theory of orthogonal polynomials that we will need to find the linear functional \( L_k \). We define the co-recursive polynomials associated with a given family of orthogonal polynomials, and list some of their main properties. We also show that a family polynomials related to \( D_{n,k}(x; a) \) are co-recursive polynomials associated with the Chebyshev polynomials of the second kind.

In Section 4 we apply the results of the previous sections to the Dickson polynomials of the \((k + 1)\)-th kind and obtain a representation for their moment functional \( L_k \). We also find explicit expressions for the moments of \( L_k \).

Finally, in Section 5 we summarize our results. In our hope that the results would be of interest to researchers outside the field of orthogonal polynomials and special functions, we have made the paper as self-contained as possible.

## 2. Properties of Dickson polynomials

We begin by checking the initial polynomial \( D_{0,k}(x; a) \). Since it is not clear from the definition (1) that \( D_{0,k}(x; a) = 2 - k \), we consider even and odd degrees and obtain the following result.
Proposition 2. The even and odd Dickson polynomials of the \((k+1)\)-th kind are given by
\[
D_{2n,k}(x; a) = (2 - k)(-a)^n + \sum_{j=1}^{n} \frac{(2 - k)n + kj}{j + n} \binom{n + j}{2j} (-a)^{n-j} x^{2j}
\]
and
\[
D_{2n+1,k}(x; a) = x \sum_{j=0}^{n} \frac{(2 - k)n + kj + 1}{j + n + 1} \binom{n + j + 1}{2j + 1} (-a)^{n-j} x^{2j}.
\]

Proof. From (1), we have
\[
D_{2n,k}(x; a) = \sum_{j=0}^{n} \frac{2n - kj}{2n - j} \binom{2n - j}{j} (-a)^{j} x^{2n-2j},
\]
and switching the index to \(i = n - j\), we get
\[
D_{2n,k}(x; a) = \sum_{i=0}^{n} \frac{(2 - k)n + ki}{i + n} \binom{n + i}{n - i} (-a)^{n-i} x^{2i},
\]
and (6) follows after using the symmetry of the binomial coefficients
\[
\binom{n}{k} = \binom{n}{n-k}.
\]
A similar calculation gives (7). \(\square\)

Next, we will find a representation for \(D_{n,k}(x; a)\) in terms of the generalized hypergeometric function
\[
pFq \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; x \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j x^j}{(b_1)_j \cdots (b_q)_j j!},
\]
where \((u)_j\) denotes the Pochhammer symbol (also called shifted or rising factorial) defined by [45, 5.2.4]
\[
(a)_0 = 1, \\
(a)_j = a(a+1) \cdots (a+j-1), \quad j \in \mathbb{N}.
\]

Proposition 3. The Dickson polynomials of the \((k+1)\)-th kind admit the hypergeometric representation
\[
D_{n,k}(x; a) = x^n pFq \left( \begin{array}{c} -\frac{n-1}{2}, \ldots, -\frac{n}{2}, 1 - \frac{n}{k} \\ \frac{1}{1-n}, \ldots, \frac{1}{1-n} \end{array} ; \frac{4a}{x^2} \right), \quad k \neq 0.
\]
We also have
\[
D_{n,0}(x; a) = x^n pFq \left( \begin{array}{c} -\frac{n-1}{2}, -\frac{n}{2} \\ \frac{1}{1-n} \end{array} ; \frac{4a}{x^2} \right).
\]
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Proof. Let

\[
D_{n,k}(x; a) = x^n \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c_j,
\]

with

\[
c_j = \frac{n - kj}{n} \binom{n - j}{j} (-a)^j x^{-2j}.
\]  \hfill (9)

We have \(c_0 = 1\) and \(c_j = 0\) for \(j > \frac{n}{2}\). Using (9), we get

\[
\frac{c_{j+1}}{c_j} = \frac{(2j - n + 1) (2j - n) (jk + k - n)}{(j - n + 1) (jk - n) (j + 1)} \frac{a}{x^2}.
\]  \hfill (10)

Let \(k \neq 0\). Then,

\[
\frac{c_{j+1}}{c_j} = \frac{(j - \frac{n}{2} + \frac{1}{2}) (j - \frac{n}{2}) (j + 1 - \frac{n}{k})}{(j - n + 1) (j - \frac{n}{k}) (j + 1)} 4a x^{-2},
\]

and it follows that

\[
c_j = \frac{(-\frac{n-1}{2})_j (-\frac{n}{2})_j (1 - \frac{n}{k})_j}{(1 - n)_j (-\frac{n}{k})_j} \frac{1}{j!} \left( \frac{4a}{x^2} \right)^j.
\]

Thus,

\[
D_{n,k}(x; a) = x^n \sum_{j=0}^{\infty} \frac{(-\frac{n-1}{2})_j (-\frac{n}{2})_j (1 - \frac{n}{k})_j}{(1 - n)_j (-\frac{n}{k})_j} \frac{1}{j!} \left( \frac{4a}{x^2} \right)^j.
\]

If \(k = 0\), we see from (10) that

\[
\frac{c_{j+1}}{c_j} = \frac{4 (j + 1 - \frac{n}{2}) (j - \frac{n}{2})}{(j - n + 1) (j + 1)} \frac{a}{x^2},
\]

and therefore

\[
c_j = \frac{(-\frac{n-1}{2})_j (-\frac{n}{2})_j}{(1 - n)_j} \frac{1}{j!} \left( \frac{4a}{x^2} \right)^j.
\]

Hence,

\[
D_{n,0}(x; a) = x^n \sum_{j=0}^{\infty} \frac{(-\frac{n-1}{2})_j (-\frac{n}{2})_j}{(1 - n)_j} \frac{1}{j!} \left( \frac{4a}{x^2} \right)^j.
\]

\[\square\]

Remark 4. A representation of \(D_{n,2}(x; a)\) in terms of associated Legendre functions of the first and second kinds \[45, 14.3\] was given in ”A representation of the Dickson polynomials of the third kind by Legendre functions” by Neranga Fernando and Solomon Manukure \[arXiv:1604.04682\].

Proposition 5. For \(n \in \mathbb{N}_0\), the Dickson polynomials of the \((k + 1)\)-th kind satisfy the following relations:

\[
D_{n,k+2} - 2D_{n,k+1} + D_{n,k} = 0,
\]
\[-(x^2 - 4a) \left[ (k - 1)n x^2 + a (k - 2)(kn - 2n - k) \right] D''_{n,k} \\
+ x\left[ (k - 1)n x^2 + a (6k + 4n + 3k^2 n - 4kn - 3k^2) \right] D'_{n,k} \\
+ \left[ (k - 1)n^3 x^2 + an (-k - 2n + kn) (-2k - 2n + kn) \right] D_{n,k} = 0,
\]

\[(x^2 - 4a)^2 D''_{n,k}^{(iv)} + 10x(x^2 - 4a)D'''_{n,k} + \left[ (23 - 2n^2)x^2 + 8a(n^2 - 4) \right] D''_{n,k} \\
- 3(2n^2 - 3)x D'_{n,k} + n^2(n^2 - 4)D_{n,k} = 0,
\]

and

\[(x^2 - 4a)D''_{n,k} - 4nD_{n+1,k}D'_{n,k} + (2n + 3)x D'_{n,k} + n(n + 2)D_{n,k} = 0.
\]

**Proof.** All the identities can be automatically found and proved using the hypergeometric representation (8) and the Mathematica package HolonomicFunctions [31]. □

**Remark 6.** The differential equation (11) already appeared in [57, Lemma 2.7].

We can use the recurrence relation (4) to obtain a different representation for the polynomials $D_{n,k}(x;a)$.

**Proposition 7.** For $x \neq \pm 2\sqrt{a}$, the Dickson polynomials of the $(k+1)$-th kind are given by

\[D_{n,k}(x;a) = \left( 1 + k \frac{x - \Delta}{2\Delta} \right) \left( \frac{x + \Delta}{2} \right)^n + \left( 1 - k \frac{x + \Delta}{2\Delta} \right) \left( \frac{x - \Delta}{2} \right)^n, \tag{12}\]

where

\[\Delta = \sqrt{x^2 - 4a}.
\]

We also have,

\[D_{n,k}(\pm 2\sqrt{a};a) = (kn + 2 - k)(\pm \sqrt{a})^n. \tag{13}\]

**Proof.** Let us assume that we can write

\[D_{n,k}(x;a) = R^n, \tag{14}\]

for some function $R(x,k,a)$. Using (14) in the recurrence (4), we obtain

\[R^2 - xR + a = 0,
\]

and therefore

\[R_\pm = \frac{x \pm \Delta}{2}
\]

with

\[\Delta = \sqrt{x^2 - 4a}.
\]

It follows that the general solution of (14) is given by

\[D_{n,k}(x;a) = C_1(x,a,k) \left( \frac{x + \Delta}{2} \right)^n + C_2(x,a,k) \left( \frac{x - \Delta}{2} \right)^n. \tag{15}\]
Using the initial conditions \((2)\) in \((15)\), we get
\[
C_1(x; a, k) + C_2(x; a, k) = 2 - k,
\]
\[
C_1(x; a, k) \left( \frac{x + \Delta}{2} \right) + C_2(x; a, k) \left( \frac{x - \Delta}{2} \right) = x.
\]
Thus, assuming that \(x \neq \pm 2\sqrt{a}\),
\[
C_1(x; a, k) = 1 + k \frac{x - \Delta}{2\Delta}, \quad C_2(x; a, k) = 1 - k \frac{x + \Delta}{2\Delta}.
\]
To verify \((13)\), we replace it in the recurrence \((4)\), and obtain
\[
(kn + 2 + k) \left( \pm \sqrt{a} \right)^{n+2} - \left( \pm \sqrt{a} \right) (kn + 2) \left( \pm \sqrt{a} \right)^{n+1} + a (kn + 2 - k) \left( \pm \sqrt{a} \right)^n = a \left( \pm \sqrt{a} \right)^n \left[ (kn + 2 + k) - 2 (kn + 2) + (kn + 2 - k) \right] = 0.
\]
□

Using \((12)\), we can obtain a generating function for the polynomials \(D_{n,k}(x; a)\).

**Proposition 8.** The ordinary generating function of the polynomials \(D_{n,k}(x; a)\) is given by
\[
G(z; x, k, a) = \sum_{n=0}^{\infty} D_{n,k}(x; a) z^n = \frac{2 - k + (k - 1) x z}{a z^2 - x z + 1}. \tag{16}
\]

**Proof.** From \((12)\), we have (as formal power series)
\[
G(z; x, k, a) = \sum_{n=0}^{\infty} D_{n,k}(x; a) z^n
\]
\[
= \left( 1 + k \frac{x - \Delta}{2\Delta} \right) \sum_{n=0}^{\infty} \left( \frac{x + \Delta}{2} \right)^n + \left( 1 - k \frac{x + \Delta}{2\Delta} \right) \sum_{n=0}^{\infty} \left( \frac{x - \Delta}{2} \right)^n
\]
\[
= \left( 1 + k \frac{x - \Delta}{2\Delta} \right) \frac{1}{1 - \left( \frac{x + \Delta}{2} \right)} + \left( 1 - k \frac{x + \Delta}{2\Delta} \right) \frac{1}{1 - \left( \frac{x - \Delta}{2} \right)}.
\]
Thus,
\[
G(z; x, k, a) = 4 \frac{2 - k + (k - 1) x z}{(x^2 - \Delta^2) z^2 - 4 x z + 4}
\]
and the result follows since
\[
\Delta^2 = x^2 - 4a.
\]
□

**Remark 9.** The same generating function was obtained in [57, Lemma 2.6] using the recurrence \((4)\).
3. Orthogonal polynomials

Let \( \{ \mu_n \} \) be a sequence of complex numbers and \( \mathcal{L} : \mathbb{C}[x] \to \mathbb{C} \) be a linear functional defined by
\[
\mathcal{L} [x^n] = \mu_n, \quad n \in \mathbb{N}_0.
\]
Then, \( \mathcal{L} \) is called the moment functional determined by the moment sequence \( \{ \mu_n \} \).

The number \( \mu_n \) is called the moment of order \( n \).

A moment functional \( \mathcal{L} \) is called positive-definite if \( \mathcal{L} [r(x)] > 0 \) for every polynomial \( r(x) \) that is not identically zero and is non-negative for all real \( x \). Otherwise, \( \mathcal{L} \) is called quasi-definite.

A sequence \( \{ P_n \} \subset \mathbb{C}[x] \), with \( \deg(P_n) = n \) is called an orthogonal polynomial sequence with respect to \( \mathcal{L} \) provided that
\[
\mathcal{L} [P_n P_m] = h_n \delta_{n,m}, \quad n, m \in \mathbb{N}_0,
\]
where \( h_n \neq 0 \) and \( \delta_{n,m} \) is Kronecker’s delta.

One of the fundamental properties of orthogonal polynomials is that they satisfy a three-term recurrence relation.

**Theorem 10.** Let \( \mathcal{L} \) be a moment functional and let \( \{ P_n \} \) be the sequence of monic orthogonal polynomials associated with it. Then, there exist \( \beta_n \in \mathbb{C} \) and \( \gamma_n \in \mathbb{C} \setminus \{0\} \) such that the polynomials \( P_n(x) \) are a solution of the 3-term recurrence relation
\[
P_{n+1} = (x - \beta_n) P_n - \gamma_n P_{n-1}, \quad n \in \mathbb{N},
\]
with initial conditions
\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0.
\]

**Proof.** See [13, Theorem 4.1]. \(\square\)

A linearly independent solution of (17) with initial conditions
\[
P_0^*(x) = 0, \quad P_1^*(x) = 1,
\]
is given by the so-called associated orthogonal polynomials \( P_n^*(x) \) [13, 4.3]. Note that \( \deg(P_n^*(x)) = n - 1 \).

The converse of Theorem 10 is known as Favard’s Theorem.

**Theorem 11.** Let \( \{ P_n \} \) be a sequence of polynomials satisfying the 3-term recurrence relation (17) with \( \beta_n \in \mathbb{C} \) and \( \gamma_n \in \mathbb{C} \setminus \{0\} \). Then, there exists a unique linear functional \( \mathcal{L} \) such that
\[
\mathcal{L} [P_0] = 1, \quad \mathcal{L} [P_n P_m] = h_n \delta_{n,m},
\]
with
\[
h_0 = P_0, \quad h_1 = \gamma_1, \quad h_n = \gamma_n h_{n-1}, \quad n = 2, 3, \ldots
\]

**Proof.** See [13, Theorem 4.4]. \(\square\)
Remark 12. It follows from (2) and (4) that (at least for \( k \neq 2 \)) \( \{D_{n,k}\} \) is a sequence of monic (for \( n \geq 1 \)) orthogonal polynomials with respect to a moment functional \( L_k \) satisfying

\[
L_k[D_{n,k}D_{m,k}] = h_n(k) \delta_{n,m},
\]

with

\[
h_0(k) = 2 - k, \quad h_n(k) = a^n, \quad n \in \mathbb{N}.
\]

In Section 4 we will find a representation for the moment functional \( L_k \).

Proposition 13. Let \( \mathcal{L} \) be a moment functional and \( \{P_n\} \) be the sequence of monic orthogonal polynomials associated with it. Then, the following are equivalent:

(a) All the moments of odd order are zero,

\[
\mathcal{L}[x^{2n+1}] = 0, \quad n \in \mathbb{N}_0.
\]

(b) The polynomials \( P_n(x) \) satisfy

\[
P_n(-x) = (-1)^n P_n(x), \quad n \in \mathbb{N}_0.
\]

Proof. See [13, Theorem 4.3]. \( \square \)

Proposition 14. Let \( k \neq 2 \) and \( \mu_n(k) \) denote the moments of the linear functional defined by (21). Then, we have

\[
\mu_0(k) = \frac{1}{2-k},
\]

\[
\mu_{2n+1}(k) = 0, \quad n \in \mathbb{N}_0,
\]

and

\[
\mu_{2n} = -\sum_{j=0}^{n-1} \frac{(2-k)n+kj}{j+n} \left(\frac{n+j}{2j}\right)(-a)^{n-j} \mu_{2j}, \quad n \in \mathbb{N}.
\]

The first few nonzero moments are

\[
\mu_2(k) = a, \quad \mu_4(k) = -a^2(k-3), \quad \mu_6(k) = a^3(k^2 - 6k + 10), \quad \mu_8(k) = -a^4(k^3 - 9k^2 + 29k - 35).
\]

Proof. From (22), we see that

\[
2 - k = h_0 = L_k[D_{0,k}^2] = D_{0,k}^2 L_k[1] = (2-k)^2 \mu_0,
\]

from which (23) follows.

Using (1), it is clear that

\[
D_{n,k}(-x; a) = (-1)^n D_{n,k}(x; a),
\]

and Proposition [13] gives

\[
\mu_{2n+1}(k) = 0, \quad n = 0, 1, \ldots.
\]

In the remainder of the paper \( L_k \) will denote the moment functional associated with the polynomials \( D_{n,k}(x; a) \).
From (6) we have
\[ D_{2n,k}(x; a) = \sum_{j=0}^{n} \frac{(2-k)n+kj}{j+n} \left( \frac{n+j}{2j} \right) (-a)^{n-j} x^{2j}, \]
and therefore
\[ 0 = L_k[D_{2n,k}] = \sum_{j=0}^{n} \frac{(2-k)n+kj}{j+n} \left( \frac{n+j}{2j} \right) (-a)^{n-j} \mu_{2j}(k) \]
\[ = \sum_{j=0}^{n-1} \frac{(2-k)n+kj}{j+n} \left( \frac{n+j}{2j} \right) (-a)^{n-j} \mu_{2j}(k) + \mu_{2n}(k). \]

\[ \square \]

**Remark 15.** In Section 4 we will find a closed-form expression for \( \mu_{2n}(k) \).

The task of finding an explicit integral representation for the functional \( \mathcal{L} \) is called a **moment problem** [11,32,49]. A moment functional \( \mathcal{L} \) is called **determinate** if there exists a unique (up to an additive constant) distribution \( \psi(x) \) such that
\[ \mathcal{L}[x^n] = \int_{\Lambda} x^n d\psi(x), \quad (26) \]
where the set \( \Lambda \) is called the support of the distribution \( \psi \). Otherwise, \( \mathcal{L} \) is called **indeterminate** [7, 54].

A criteria to decide if the moment functional \( \mathcal{L} \) is determinate is due to Torsten Carleman [49, P 59]: If \( \gamma_n > 0 \) and
\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{\gamma_n}} = \infty, \]
then \( \mathcal{L} \) is determinate. Since for the Dickson polynomials we have \( \gamma_n = a \), it follows that the moment problem is determinate.

One method to find a distribution function satisfying (26) is given by Markov’s theorem.

**Theorem 16.** Let the moment functional \( \mathcal{L} \), supported on the set \( \Lambda \subset \mathbb{C} \), be determinate, \( \{P_n\} \) be the monic orthogonal polynomials with respect to \( \mathcal{L} \) and \( \{P_n^*\} \) be the associated polynomials. Then,
\[ \lim_{n \to \infty} \mathcal{L} \left[ \frac{P_n^*(z)}{P_n(z)} \right] = \mathcal{L} \left[ \frac{1}{z-x} \right], \quad z \notin \Lambda, \quad (27) \]
where the convergence is uniform on compact subsets of \( \mathbb{C} \setminus \Lambda \).

**Proof.** See [56].

The function
\[ S(z) = \mathcal{L} \left[ \frac{1}{z-x} \right], \quad (28) \]
is called the *Stieltjes transform* of $L$ \cite{55} and it has the asymptotic behavior \cite{28} Section 12.9
\[
\mathcal{S}(z) \sim \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \cdots, \quad z \to \infty.
\] (29)

3.1. **Co-recursive polynomials.** Let $\{q_n\}$ be a sequence of polynomials satisfying (17) with initial conditions
\[
q_0(x) = u, \quad q_1(x) = x - v.
\] (30)
The polynomials $q_n(x)$ are called *co-recursive* with parameters $(u,v)$. They were introduced by T. Chihara in \cite{12}, where he considered the case $u = 1$. See also \cite{5}, \cite{36}, \cite{47}, and \cite{50}.

Note that when $u = 1$ we could consider the polynomials $q_n(x)$ to be solutions of the perturbed 3-term recurrence relation
\[
xq_n = q_{n+1} + \tilde{\beta}_n q_n + \gamma_n q_{n-1}, \quad n \in \mathbb{N}_0
\]
with initial conditions
\[
q_{-1}(x) = 0, \quad q_0(x) = 1,
\]
where
\[
\tilde{\beta}_n = \beta_n + (v - \beta_0) \delta_{n,0}.
\]
Orthogonal polynomials that are solutions of 3-term recurrence relations with finite perturbations of the recurrence coefficients are called *co-polynomials*. See \cite{11}, \cite{35}, and \cite{38}.

Let $\{P_n\}$ and $\{P^*_n\}$ be the linearly independent solutions of (17) with initial conditions \cite{18} and \cite{19}. Then, we can represent $q_n(x)$ as a linear combination
\[
q_n(x) = AP_n(x) + BP^*_n(x).
\]
Using (30), we see that
\[
A = u, \quad B = (1 - u) x + u\beta_0 - v,
\]
and therefore
\[
q_n(x) = uP_n(x) + [(1 - u) x + u\beta_0 - v] P^*_n(x), \quad n \in \mathbb{N}_0. \] (31)
Linear combinations of orthogonal polynomials have been studied by many authors. See \cite{3}, \cite{33}, \cite{40}, \cite{41}, and \cite{42}.

Note that the associated polynomials for both sequences are the same, i.e., $q^*_n(x) = P^*_n(x)$. Thus, we have
\[
\frac{q_n(z)}{q^*_n(z)} = \frac{P_n(z)}{P^*_n(z)} + (1 - u) z + u\beta_0 - v,
\]
and assuming that the moment problem is determined, we can use Markov’s theorem and obtain
\[
\frac{L_q[1]}{S_q(z)} = \frac{uL_P[1]}{S_P(z)} + (1 - u) z + u\beta_0 - v,
\]
where \( S_q(z), S_p(z) \) and \( L_q, L_p \) denote the Stieltjes transforms and linear functionals associated with the sequences \( \{q_n\} \) and \( \{p_n\} \), respectively. Solving for \( S_q(z) \), we obtain

\[
S_q(z) = \frac{L_q[1] S_P(z)}{(1-u) z + u\beta_0 - v | S_P(z) + uL_P[1]}. \quad (32)
\]

The rational transformation (32) is a particular case of the general spectral transformations of the Stieltjes transform \[58\]

\[
\tilde{S}(z) = \frac{A(z) S(z) + B(z)}{C(z) S(z) + D(z)},
\]

where \( A(z), B(z), C(z), D(z) \) are polynomials. See also [8], [10], [23], [34], and [46].

Suppose that \( L_P[1] = 1 \), and let’s denote the polynomial \( (1-u) z + u\beta_0 - v \) appearing in (32) by \( Q(z) \). If the function \( S_P(z) \) is a solution of the quadratic equation

\[
A Q_S^2 P + (BQ + uA - C) S_P + uB = 0, \quad (33)
\]

for some polynomials \( A(z), B(z), C(z) \), then we have

\[
\frac{S_q(z)}{L_q[1]} = \frac{S_P(z)}{Q(z) S_P(z) + u} = \frac{A(z) S_P(z) + B(z)}{C(z)}.
\]

Stieltjes transforms satisfying quadratic equations with polynomial coefficients are called second degree forms. See [2], [6], [43], and [48].

The linear transformation

\[
\frac{S_q(z)}{L_q[1]} = \frac{A(z) S_P(z) + B(z)}{C(z)}
\]

can be written as a composition of three basic transformations:

1. The Uvarov transformation, [39], [53], defined by

\[
\frac{L_U[r]}{\lambda_U} = L_P[r] + Mr(\omega), \quad r \in \mathbb{C}[x],
\]

or by

\[
\frac{S_U(z)}{\lambda_U} = S(z) + \frac{M}{z - \omega},
\]

where \( M + L_P[1] \neq 0 \) and

\[
\lambda_U = \frac{L_U[1]}{L_P[1] + M}.
\]

2. The Christoffel transformation, [9], [14] defined by

\[
\frac{L_C[r]}{\lambda_C} = L_P[(x - \omega)r(x)], \quad r \in \mathbb{C}[x],
\]

or by

\[
\frac{S_C(z)}{\lambda_C} = (z - \omega) S(z) - L_P[1],
\]
where \( L_P [x - \omega] \neq 0 \) and

\[
\lambda_C = \frac{L_C [1]}{L_P [x - \omega]}.
\]

(3) The Geronimus transformation \[17, 18, 25, 26\], defined by

\[
\frac{L_G [r]}{\lambda_G} = L_P \left[ \frac{r(x)}{x - \omega} \right] + Mr(\omega), \quad r \in \mathbb{C} [x],
\]

or by

\[
(z - \omega) \frac{S_G (z)}{\lambda_G} = S (z) - S (\omega) + M,
\]

where \( M - S(\omega) \neq 0 \) and

\[
\lambda_G = \frac{L_G [1]}{M - S(\omega)}.
\]

If the coefficients in the 3-term recurrence relation \([17]\) are constant, then we see that \( P^*_n (x) \) and \( P_{n-1} (x) \) satisfy the same recurrence, and have the same initial conditions. Therefore, they are identical \( P^*_n (x) = P_{n-1} (x) \) and we obtain

\[
x = \frac{P_{n+1}}{P_n} + \beta + \gamma \frac{P_{n-1}}{P_n} = \frac{P_{n+1}}{P_{n+1}^*} + \beta + \gamma \frac{P_n^*}{P_n}.
\]

Using Markov’s theorem, we conclude that

\[
z = \frac{L_P [1]}{S_P (z)} + \beta + \gamma \frac{S_P (z)}{L_P [1]}.
\]

Assuming that \( L_P [1] = 1 \), we find that \( S_P (z) \) is the solution of the quadratic equation

\[
\gamma S_P^2 - (z - \beta) S_P + 1 = 0. \tag{34}
\]

Multiplying \([33]\) by \( \gamma \), \([34]\) by \( AQ \), and subtracting, we get

\[
[\gamma (BQ + uA - C) + (z - \beta) AQ] S_P + u\gamma B - AQ = 0.
\]

Therefore,

\[
A = u\gamma, \quad B = Q, \quad C = Q^2 + u(z - \beta)Q + u^2\gamma,
\]

or replacing \( Q (z) \) by \( (1 - u)z + u\beta - v, \)

\[
B = (1 - u)z + u\beta - v, \quad C = (z - v) [(1 - u)z + u\beta - v] + u^2\gamma,
\]

We conclude that

\[
\frac{S_q (z)}{L_q [1]} = \frac{u\gamma S_P (z) + (1 - u)z + u\beta - v}{(z - v) [(1 - u)z + u\beta - v] + u^2\gamma}. \tag{35}
\]

Polynomial solutions of 3-term recurrence relations with constant coefficients

\[
x q_n = q_{n+1} + \beta q_n + \gamma q_{n-1}, \quad q_0 (x) = u, \quad q_1 (x) = x - v,
\]
were analyzed in [15], where the authors concluded that the linear functional $L_q$ was of the form

$$
\frac{L_q [r]}{L_q [f]} = \frac{1}{2\pi\gamma} \int_{\beta-2\sqrt{\gamma}}^{\beta+2\sqrt{\gamma}} r(x) \sqrt{4\gamma - (x - \beta)^2} \frac{f(x)}{dx} + M_1 r(y_1) + M_2 r(y_2), \quad r \in \mathbb{C}[x],
$$

with

$$
f(x) = (x - v) [(1 - u)x + u\beta - v] + u^2\gamma = (1 - u)(x - y_1)(x - y_2).
$$

They only considered the case where $u > 0$ and all parameters (including the roots of $f$) are real numbers. For the masses $M_1, M_2$ they obtained

$$
M_i = 2 \frac{u\gamma}{|y_i - v| - |y_i - v|} + u \neq 1,
$$

and

$$
M_1 = M_2 = \left[ 1 - \frac{\gamma}{(\beta - v)^2} \right]_+, \quad u = 1,
$$

where

$$
[x]^+ = \frac{x + |x|}{2}.
$$

The solution of (34) having the right asymptotic behavior, is

$$
S_P (z) = \frac{z - \beta - \sqrt{(z - \beta)^2 - 4\gamma}}{2\gamma} \sim \frac{1}{z} + \frac{\beta}{z^2}, \quad z \to \infty,
$$

and it follows that an integral representation of the lineal functional $L_P$ is given by

$$
L_P [r] = \frac{1}{2\pi\gamma} \int_{\beta-2\sqrt{\gamma}}^{\beta+2\sqrt{\gamma}} r(x) \sqrt{4\gamma - (x - \beta)^2} dx, \quad r \in \mathbb{C}[x].
$$

Using the change of variables

$$
x = \beta + 2y\sqrt{\gamma},
$$

we obtain

$$
L_P [r] = \frac{2}{\pi} \int_{-1}^{1} r(\beta + 2y\sqrt{\gamma}) \sqrt{1 - y^2} dy.
$$

Therefore, it is enough to study the linear functional

$$
L_U [r] = \frac{2}{\pi} \int_{-1}^{1} r(y) \sqrt{1 - y^2} dy,
$$

associated to the Chebyshev polynomials of the second kind, which we will define in the next subsection.
3.2. Chebyshev polynomials. The monic *Chebyshev polynomials of the second kind* $U_n(x)$ are defined by \[ U_n(x) = 2^{-n} (n+1) \binom{3}{2} (-n, n+2; 1; 2). \]

They are a solution of the recurrence relation
\[ U_{n+1} - x U_n + \frac{1}{4} U_{n-1} = 0, \quad (37) \]
with initial conditions \[ U_0(x) = 1, \quad U_1(x) = x. \quad (38) \]

Note that \[ U_{-1}(x) = 0. \quad (39) \]

The polynomials $U_n(x)$ satisfy the orthogonality relation \[ L_U[U_n U_m] = \frac{2}{\pi} \int_{-1}^{1} U_n(x) U_m(x) \sqrt{1-x^2} dx = \delta_{n,m}. \quad (40) \]

The Stieltjes transforms of the linear functional $L_U$ is given by \[ S_U(z) = \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{z-x} dx = 2z \left(1 - \sqrt{1-z^{-2}}\right), \quad z \in \mathbb{C} \setminus [-1, 1], \quad (41) \]

where here and in the rest of the paper \[ \sqrt{\cdot}: \mathbb{C} \to \left\{ z \in \mathbb{C} | -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2} \right\} \]
denotes the principal branch of the square root. Note that \[ \sqrt{1-z^{-2}} \sim 1, \quad z \to \infty. \]

It is clear from the three-term recurrence relation \[ D_n(x; a) \] that the polynomials $D_{n,k}(x; a)$ are related to the polynomials $U_n(x)$. Let’s introduce the scaled polynomials $d_n(x; k)$ defined by
\[ d_n(x; k) = (2-\sqrt{a})^{-n} D_{n,k} \left(2\sqrt{ax}\right). \quad (42) \]

The polynomials $d_n(x; k)$ are a solution of the recurrence \[ (37) \] satisfied by the polynomials $U_n(x)$, with initial conditions \[ d_0(x; k) = 2-k, \quad d_1(x; k) = x. \]

Therefore, we see that the scaled polynomials $d_n(x; k)$ are co-recursive polynomials (with respect to the Chebyshev polynomials of the second kind) with parameters $u = 2-k, \quad v = 0$.

Using \[ (31) \], we have \[ d_n(x; k) = (2-k) U_n(x) + (k-1) x U_{n-1}(x), \quad (43) \]

since \[ d_n^*(x; k) = U_n^*(x) = U_{n-1}(x). \]
Remark 17. If we use the values of the monic Chebyshev polynomials at \(x = 0\),

\[ U_n(0) = 2^{-n} \cos \left( \frac{n\pi}{2} \right), \]

and the representation (43), we get

\[ D_{n,k}(0; a) = (2\sqrt{a})^n d_n(0; k) = \left( \sqrt{a} \right)^n (2 - k) \cos \left( \frac{n\pi}{2} \right), \]

in agreement with (6)-(7).

If \(k = 0, 1, 2\), (42) and (43) give

\[ D_{n,k}(x) = (2\sqrt{a})^n \left[ (2 - k) U_n \left( \frac{x}{2\sqrt{a}} \right) + (k - 1) \frac{x}{2\sqrt{a}} U_{n-1} \left( \frac{x}{2\sqrt{a}} \right) \right], \]

\[ D_{n,0}(x; a) = (2\sqrt{a})^n \left[ 2U_n \left( \frac{x}{2\sqrt{a}} \right) - \frac{x}{2\sqrt{a}} U_{n-1} \left( \frac{x}{2\sqrt{a}} \right) \right], \]

\[ D_{n,1}(x; a) = (2\sqrt{a})^n U_n \left( \frac{x}{2\sqrt{a}} \right), \]

\[ D_{n,2}(x; a) = (2\sqrt{a})^{n-1} xU_{n-1} \left( \frac{x}{2\sqrt{a}} \right), \]

and in particular, for \(a = 1\), we have

\[ D_n(2x) = D_{n,0}(2x; 1) = 2^{n+1} U_n(x) - x2^n U_{n-1}(x), \]

\[ E_n(2x) = D_{n,1}(2x; 1) = 2^n U_n(x), \]

as it was observed in [57].

Remark 18. Stoll [51] studied second order recurrences with constant coefficients and general initial conditions and found polynomial decompositions in terms of Chebyshev polynomials.

4. Main results

In this section we find a representation for the linear functional \(L_k\) defined by (21). Although this seems to be something already considered in [15], we have found that the range of parameters of the polynomials \(D_{n,k}(x)\) requires a different analysis.

Let \(k \neq 2\), \(l_k[r]\) denote the linear functional satisfying

\[ l_k[1] = \frac{1}{2 - k}, \quad l_k[d_n d_m] = 4^{-n} \delta_{n,m}, \quad n, m \in \mathbb{N}, \]

(45)

and \(s(z; k)\) be it’s Stieltjes transform. From (35) and (41), we have

\[ s(z; k) = \frac{2z (k - 2) \sqrt{1 - z^{-2}} + k}{2 - k 4 (k - 1) z^2 + (k - 2)^2}, \]

since \(v = \beta = 0\).

Our next objective is to represent the function \(s(z; k)\) as the Stieltjes transform of a distribution. We begin with a couple of lemmas.
Lemma 19. Let $\mathcal{L}$ be a linear functional with Stieltjes transform $S(z)$ and $f(x) = (x - \omega_1)(x - \omega_2)$.

Then,

$$\mathcal{L} \left[ \frac{1}{f(x)(z - x)} \right] = \frac{S(z)}{f(z)} + \frac{[S(\omega_2) - S(\omega_1)]z + \omega_2S(\omega_1) - \omega_1S(\omega_2)}{(\omega_1 - \omega_2)f(z)},$$

where we always assume that the functional $\mathcal{L}$ acts on the variable $x$.

Proof. Since

$$\frac{1}{(x - \omega_1)(x - \omega_2)(z - x)} = \frac{1}{(z - \omega_1)(z - \omega_2)(z - x)} + \frac{1}{(\omega_1 - \omega_2)(z - \omega_2)(\omega_2 - x)} - \frac{1}{(\omega_1 - \omega_2)(z - \omega_1)(\omega_1 - x)},$$

we get

$$\mathcal{L} \left[ \frac{1}{f(x)(z - x)} \right] = \frac{S(z)}{(z - \omega_1)(z - \omega_2)} + \frac{S(\omega_2)}{(\omega_1 - \omega_2)(z - \omega_2)} - \frac{S(\omega_1)}{(\omega_1 - \omega_2)(z - \omega_1)},$$

and the conclusion follows. \[\square\]

For the particular case of the linear functional $L_U$ defined by \[40\] with Stieltjes transform $S_U(z)$ given in \[41\] and $f(x) = x^2 - b^2$, Lemma \[19\] gives

$$L_U \left[ \frac{1}{(x^2 - b^2)(z - x)} \right] = 2z\frac{\sqrt{1 - b^{-2}} - \sqrt{1 - z^{-2}}}{z^2 - b^2}, \quad z, b \in \mathbb{C} \setminus [-1, 1], \quad (47)$$

because

$$\frac{S_U(z)}{z^2 - b^2} - \frac{S_U(b)}{2b(z - b)} + \frac{S_U(-b)}{2b(z + b)} = \frac{2z(1 - \sqrt{1 - z^{-2}})}{z^2 - b^2} + \frac{2b(1 - \sqrt{1 - b^{-2}})}{2b(b - z)} + \frac{2(-b)(1 - \sqrt{1 - b^{-2}})}{2b(b + z)}.$$

Note that since \[35\]

$$\frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \frac{1}{z - x} \, dx = \frac{2}{z\sqrt{1 - z^{-2}}}, \quad z \in \mathbb{C} \setminus [-1, 1], \quad (48)$$

we have

$$\lim_{b^2 \to 1^-} L_U \left[ \frac{1}{(x^2 - b^2)(z - x)} \right] = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \frac{1}{z - x} \, dx$$

$$= -\frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \frac{1}{z - x} \, dx = \frac{-2}{z\sqrt{1 - z^{-2}}}$$

$$= -2z\frac{\sqrt{1 - z^{-2}}}{z^2 - 1} = \lim_{b^2 \to 1^-} 2z\frac{\sqrt{1 - b^{-2}} - \sqrt{1 - z^{-2}}}{z^2 - b^2},$$
and in particular

\[ \frac{1}{2z\sqrt{1 - z^{-2}}} = \frac{1}{4} L_U \left[ \frac{1}{(1 - x^2)(z - x)} \right]. \] (49)

**Lemma 20.** Let \( \omega(k) \) be defined by

\[ \omega(k) = \frac{1}{2} \frac{k - 2}{\sqrt{k - 1}} i, \quad k \neq 1, \] (50)

where \( i^2 = -1 \). Then,

\[ \omega(k) \in \mathbb{C} \setminus [-1, 1], \quad k \in \mathbb{R} \setminus \{0, 1, 2\}. \]

**Proof.** The result follows immediately from the definition (50), since

\[ \omega(k) \in (-\infty, -1), \quad k \in (-\infty, 0) \cup (0, 1), \]
\[ i\omega(k) \in \mathbb{R}, \quad k \in (1, \infty), \]
\[ \omega(0) = -1, \quad \omega(2) = 0. \]

The function \( s(z; k) \) defined in (46) has a branch cut on the segment \([-1, 1]\) and (perhaps removable) poles at \( z = \pm \omega \) if \( k \neq 1 \). In the next theorem we split \( s(z; k) \) in two parts, one analytic in \( \mathbb{C} \setminus [-1, 1] \) and the other analytic in \( \mathbb{C} \setminus \{\pm \omega\} \).

**Theorem 21.** Let \( k \neq 2 \) and \( z \in \mathbb{C} \setminus [-1, 1] \). Then, we have

\[ s(z; k) = s_c(z; k) + \chi(k) s_d(z; k), \] (51)

where \( \chi(k) \) is the characteristic function defined by

\[ \chi(k) = \begin{cases} 0, & k \in [0, 2] \\ 1, & k \in \mathbb{R} \setminus [0, 2] \end{cases}, \] (52)

\( s_c(z; k) \) is the continuous part of \( s(z; k) \)

\[ s_c(z; k) = \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - x^2}}{4(k - 1) x^2 + (k - 2)^2} \frac{1}{z - x} dx, \] (53)

and \( s_d(z; k) \) is the discrete part of \( s(z; k) \)

\[ s_d(z; k) = \frac{4k}{2 - k} \frac{z}{4(k - 1) z^2 + (k - 2)^2}. \] (54)

**Proof.** Let \( k \in \mathbb{R} \setminus \{0, 1, 2\} \). From (46), we have

\[ s(z; k) = 2z \frac{k}{x^2 - \sqrt{1 - z^{-2}}} \frac{1}{4(k - 1) z^2 + (k - 2)^2}. \]

Using (50), we get

\[ \frac{(k - 2)^2}{4(k - 1)} = -\omega^2, \quad 1 - \omega^{-2}(k) = \left( \frac{k}{2 - k} \right)^2. \] (55)
Hence,
\[ s(z; k) = \frac{2z}{4(k-1)} \left( \frac{\sqrt{1 - \omega^2} - \sqrt{1 - z^2} + \frac{k}{2-k} - \frac{k}{2-k}}{z^2 - \omega^2} \right). \]

Since we know from Lemma 20 that \( \omega(k) \in \mathbb{C} \setminus [-1, 1] \), we can use (47) with \( b = \omega \) and obtain
\[ s(z; k) = \frac{1}{4(k-1)} LU \left[ \frac{1}{(x^2 - \omega^2)(z-x)} \right] + \frac{2z}{4(k-1)} \frac{k}{2-k} - \frac{k}{2-k} \frac{1}{z^2 - \omega^2}. \]

But
\[ \frac{k}{2-k} - \frac{k}{2-k} = \begin{cases} 0, & k \in [0, 2) \\ \frac{2k}{2-k}, & k \in \mathbb{R} \setminus [0, 2] \end{cases}, \]

and therefore
\[ s(z; k) = \begin{cases} s_c(z; k), & k \in (0, 2) \setminus \{1\} \\ s_c(z; k) + s_d(z; k), & k \in \mathbb{R} \setminus [0, 2] \end{cases}, \]

where
\[ s_c(z; k) = LU \left[ \frac{1}{4(k-1)x^2 + (k-2)^2 z^2 - x} \right], \]
and
\[ s_d(z; k) = \frac{2z}{4(k-1)} \frac{2k}{2-k} \frac{1}{z^2 - \omega^2}. \]

If \( k = 0 \), then we have from (46)
\[ s(z; 0) = z \frac{\sqrt{1 - z^2}}{2(z^2 - 1)} = \frac{1}{2z\sqrt{1 - z^2}}, \]
and using (49), we get
\[ \frac{1}{2z\sqrt{1 - z^2}} = \frac{1}{4} LU \left[ \frac{1}{(1-x^2)(z-x)} \right] = s_c(z; 0). \]

If \( k = 1 \), then we have from (46)
\[ s(z; 1) = 2z \left( 1 - \sqrt{1 - z^2} \right), \]
and using (41) we get
\[ s(z; 1) = \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{z-x} dx = s_c(z; 1). \]

\[ \square \]

**Remark 22.** Orthogonal polynomials with linear functionals of the form
\[ \mathcal{L}[r] = \int_{-1}^{1} r(x) \frac{(1-x)^\alpha (1+x)^\beta}{f(x)} dx, \quad \alpha, \beta = \pm \frac{1}{2}, \]
where \( f(x) \) is a polynomial, are called Bernstein–Szegö polynomials \([45, 18.31], [52, 2.6]\). These polynomials are examples of the Geronimus transformation applied to the Jacobi polynomials, see \([4, 24, 29, 2.7.3], and 58\).

**Corollary 23.** Let \( k \neq 2 \), the linear functional \( l_k \) be defined by \((43)\), the characteristic function \( \chi(k) \) be defined by \((53)\), and the function \( \omega(k) \) de defined by \((50)\).

Then, for all \( r(x) \in \mathbb{C}[x] \) we have

\[
l_k[r] = l_k^{(c)}[r] + \chi(k) l_k^{(d)}[r],
\]

where

\[
l_k^{(c)}[r] = \frac{2}{\pi} \int_{-1}^{1} \frac{r(x) \sqrt{1-x^2}}{4(k-1)x^2 + (k-2)^2} dx,
\]

\[
l_k^{(d)}[r] = k \frac{r(\omega) + r(-\omega)}{2(k-1)(2-k)},
\]

and we assume that

\[
\lim_{k \to 1} \frac{\chi(k)}{k-1} = 0.
\]

**Proof.** The result is a direct consequence of Theorem \(21\) since

\[
\frac{4k}{2-k} \frac{z}{4(k-1)z^2 + (k-2)^2} = \frac{k}{(2-k)(k-1)z^2 - \omega^2} = \frac{1}{2(k-1)(2-k)} \left( \frac{1}{\omega} + \frac{1}{\omega + \omega} \right).
\]

For \( k = 1 \), we see from \((43)\) that

\[
d_n(x; 1) = U_n(x),
\]

and therefore

\[
l_1[r] = L_U[r] = \frac{2}{\pi} \int_{-1}^{1} r(x) \sqrt{1-x^2} dx,
\]

which agrees with \((56)\) for \( k = 1 \) if we use \((59)\). \(\square\)

**Remark 24.** For \( k = 2 \), we see from \((43)\) that

\[
d_n(x; 2) = xU_{n-1}(x),
\]

and therefore we can interpret \( l_2 \) as the linear functional

\[
l_2[r] = L_U \left[ \frac{r(x)}{4x^2} \right] = \frac{1}{2\pi} \int_{-1}^{1} r(x) \frac{\sqrt{1-x^2}}{x^2} dx,
\]

defined for all polynomials \( r(x) \) such that \( r(x) = x^2 p(x) \), \( p(x) \in \mathbb{C}[x] \). It follows that \( d_n(x; 2) \) will be a family of orthogonal polynomials for \( n \geq 1 \).
4.1. The Dickson polynomials. We can now use the previous results to the Dickson polynomials of the \((k + 1)\)-th kind \(D_{n,k}(x; a)\), related to the scaled polynomials \(d_n(x; k)\) by (42).

**Lemma 25.** Let \(\mathcal{L}_k : \mathbb{C}[x] \to \mathbb{C}\) be the linear functional defined by
\[
\mathcal{L}_k [r(x)] = l_k \left[ r(2\sqrt{a}x) \right],
\]
where \(l_k\) is the linear functional defined by (45). Then, \(\mathcal{L}_k\) satisfies
\[
\mathcal{L}_k [D_{0,k}^2] = 2 - k, \quad \mathcal{L}_k [D_{n,k}D_{m,k}] = a^n \delta_{n,m}, \quad n, m \in \mathbb{N},
\]
and for \(k \neq 2\), its Stieltjes transform is given by
\[
S(z; k, a) = \frac{z}{2(2-k)} \frac{(k-2)\sqrt{1-4az^{-2}+k}}{(k-1)z^2+(k-2)^2}, \quad z \in \mathbb{C} \setminus [-2\sqrt{a}, 2\sqrt{a}].
\]

**Proof.** Using (42) in (60), we have
\[
\mathcal{L}_k [D_{n,k}(x)D_{m,k}(x)] = l_k \left[ D_{n,k}(2\sqrt{a}x)D_{m,k}(2\sqrt{a}x) \right]
= (2\sqrt{a})^{n+m} l_k \left[ d_n(x; k)d_m(x; k) \right].
\]
Therefore, (45) gives
\[
\mathcal{L}_k [D_{0,k}^2] = l_k \left[ (2-k)^2 \right] = (2-k)^2 l_k [1] = 2 - k,
\]
and
\[
\mathcal{L}_k [D_{n,k}(x)D_{m,k}(x)] = (2\sqrt{a})^{n+m} 4^{-n} \delta_{n,m}, \quad n, m \in \mathbb{N}.
\]
But since
\[(2\sqrt{a})^{n+m} 4^{-n} \delta_{n,m} = a^n \delta_{n,m},\]
(61) follows.

Using (46) in (60), we get
\[
\mathcal{L}_k \left[ \frac{1}{z-x} \right] = l_k \left[ \frac{1}{z-2\sqrt{a}x} \right] = \frac{1}{2\sqrt{a}} l_k \left[ \frac{1}{\frac{z}{2\sqrt{a}} - x} \right]
= \frac{1}{2\sqrt{a}} \left( \frac{z}{2\sqrt{a}} \right)^{k} \frac{2\sqrt{a}}{2\sqrt{a}} \frac{(k-2)\sqrt{1-4az^{-2}+k}}{(k-1)\frac{z^2}{4a}+(k-2)^2},
\]
and (62) follows. \(\square\)

**Corollary 26.** The linear functional \(\mathcal{L}_k\) defined by (60) is identical to the linear functional \(L_k\) satisfying (21).

Next, we find a representation for the linear functional \(L_k\).

**Theorem 27.** Let \(k \neq 2\) and \(L_k\) be the linear functional defined by (21). Then, \(L_k\) admits the representation
\[
L_k [r] = L_k^{(c)} [r] + \chi (k) L_k^{(d)} [r],
\]
where $\chi(k)$ was defined in (52),

$$L_k^{(c)}[r] = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{r(t) \sqrt{4a-t^2}}{(k-1)t^2 + (k-2)^2a} dt$$ (63)

and

$$L_k^{(d)}[r] = k \frac{r(\Omega)+r(-\Omega)}{2(k-1)(2-k)}$$ (64)

with $\Omega(k)$ defined by

$$\Omega(k) = 2\sqrt{a}\omega(k) = \sqrt{\frac{k-2}{k-1}}i, \quad k \neq 1.$$ (65)

**Proof.** Changing variables to $t = 2\sqrt{ax}$ in the integral

$$I = \frac{1}{4} \int_{-1}^{1} \frac{r(x) \sqrt{1-x^2}}{4(k-1)x^2 + (k-2)^2a} dx,$$

we obtain

$$I = \frac{1}{4} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{r\left(\frac{t}{2\sqrt{a}}\right) \sqrt{4a-t^2}}{(k-1)t^2 + (k-2)^2a} dt.$$

Therefore, from (57) we get

$$l_k^{(c)}[r(2\sqrt{at})] = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{r(t) \sqrt{4a-t^2}}{(k-1)t^2 + (k-2)^2a} dt.$$

Also, from (58) and (65) we have

$$l_k^{(d)}[r(2\sqrt{at})] = k \frac{r(2\sqrt{a}\omega) + r(-2\sqrt{a}\omega)}{2(k-1)(2-k)} = k \frac{r(\Omega) + r(-\Omega)}{2(k-1)(2-k)}.$$

Therefore, using (56) in (60) we see that

$$L_k[r] = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{r(t) \sqrt{4a-t^2}}{(k-1)t^2 + (k-2)^2a} dt + \chi(k)k \frac{r(\Omega) + r(-\Omega)}{2(k-1)(2-k)}.$$

$\square$

Although Theorem 27 seems to be valid only when $k \neq 2$, we can see that $L_2$ is well defined.

**Lemma 28.** Let $k \neq 1$ and $\Omega(k)$ be defined by (65). Then,

$$D_{n,k}(\Omega; a) = (2-k) \left(-i\sqrt{\frac{a}{k-1}}\right)^n.$$ (66)
Proof. Let’s assume that
\[ D_{n,k}(\Omega; a) = b_0 B^n, \]
for some functions \( b_0(k,a) \) and \( B(k,a) \). Using (4), we have
\[ 0 = b_0 B^{n+2} - \Omega b_0 B^{n+1} + a b_0 B^n = b_0 B^n \left( B^2 - \Omega B + a \right). \]
Using (2), we get
\[ b_0 = D_{0,k}(\omega; a) = 2 - k \]
and
\[ \Omega = D_{1,k}(\Omega; a) = (2 - k) B(k,a). \]
Thus,
\[ B(k,a) = \frac{\Omega}{2 - k} = -\frac{\sqrt{a}}{k - 1}, \]
and clearly
\[ B^2 - \Omega B + a = 0. \]
\[ \square \]
It follows from the previous Lemma that \( L_2^{(d)} \) is well defined.

**Proposition 29.** Let \( \Omega(k) \) be defined by (65) and \( L_k^{(d)} \) be defined by (64). Then, for \( k \neq 1 \)
\[ L_k^{(d)} [D_{n,k} D_{m,k}] = \frac{(2 - k) k}{k - 1} \left[ 1 + \left( \frac{-1}{2} \right)^{n+m} \right] \left( i \sqrt{\frac{a}{k - 1}} \right)^{n+m}. \]

Proof. From (66) we have
\[ D_{n,k}(\Omega; a) D_{m,k}(\Omega; a) = (2 - k)^2 \left( -i \sqrt{\frac{a}{k - 1}} \right)^{n+m}. \]
Using (25), we get
\[ D_{n,k}(\Omega; a) D_{m,k}(\Omega; a) + D_{n,k}(\Omega; a) D_{m,k}(\Omega; a) \]
\[ = (2 - k)^2 \left( i \sqrt{\frac{a}{k - 1}} \right)^{n+m} \left[ 1 + (-1)^{n+m} \right]. \]
Thus,
\[ L_k^{(d)} [D_{n,k} D_{m,k}] = \frac{(2 - k) k}{k - 1} \left[ 1 + \left( \frac{-1}{2} \right)^{n+m} \right] \left( i \sqrt{\frac{a}{k - 1}} \right)^{n+m}. \]
\[ \square \]
We can now extend Theorem 27 to all values of \( k \).
Corollary 30. Let $h_n(k)$ be defined by (22) and $\chi(k)$ be defined by (52). Then,
\[
\frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{\sqrt{4a - t^2} D_{n,k}(t) D_{m,k}(t)}{(k - 1) t^2 + a (k - 2)^2} dt = h_n(k) \delta_{n,m}, \quad n, m \in \mathbb{N}_0.
\]

Remark 31. If we set $k = 2$ in (67), we obtain
\[
\frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{\sqrt{4a - t^2} D_{n,2}(t) D_{m,2}(t) dt = h_n(k) \delta_{n,m},}
\]
which seems to make no sense, since the integrand is singular at $t = 0$. However, if we use (44) we have
\[
D_{n,2}(t; a) = a^{-\frac{n-1}{2}} t U_{n-1}\left(\frac{t}{2\sqrt{a}}\right),
\]
and we can write (68) as
\[
a^{-\frac{n+m-1}{2}} \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{\sqrt{4a - t^2} U_{n-1}\left(\frac{t}{2\sqrt{a}}\right) U_{m-1}\left(\frac{t}{2\sqrt{a}}\right) dt = h_n(k) \delta_{n,m},}
\]
or changing variables to $\tau = \frac{t}{2\sqrt{a}}$
\[
a^{\frac{n+m}{2}} \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - \tau^2} U_{n-1}(\tau) U_{m-1}(\tau) d\tau = h_n(k) \delta_{n,m}.
\]
This agrees with (40), since we have $U_{-1} = 0$ from (39) and $h_0(2) = 0$ from (22), while for $n, m \geq 1$ we know from (22) that
\[
a^{-\frac{n+m}{2}} h_n(k) \delta_{n,m} = \delta_{n,m}.
\]
Finally, we will use the function $S(z; k, a)$ to find explicit expressions for the moments of $L_k$.

Proposition 32. Let $L_k$ be the linear functional defined by (22). Then, the moments of $L_k$ of even order
\[
\mu_{2n}(k) = L_k \left[ x^{2n} \right],
\]
are given by
\[
\mu_{2n}(1) = 2^{2n+1} \left( \frac{1}{n+1} \right) (-a)^n, \quad n = 0, 1, \ldots,
\]
\[
\mu_{2n}(2) = -2^{2n-1} \left( \frac{1}{n} \right) (-a)^n, \quad n = 1, 2, \ldots,
\]
\[
\mu_{2n}(3) = \ldots
\]
and if $k \neq 1, 2$,
\[
\mu_{2n}(k) = \frac{1}{2} \left( \frac{k - 2}{k - 1} \right)^{2n+1} (-a)^n \left( \frac{k}{k - 2} + \sum_{j=0}^{n} \left( \frac{1}{2} \right) \left( \frac{4(k - 1)}{(k - 2)^2} \right)^j \right).
\]  
(71)

**Proof.** From (29) and (62), we have
\[
\sum_{j=0}^{\infty} \mu_j(k) z^{j+1} = -\frac{z}{2(k - 1) z^2 + a(k - 2)^2}. \tag{72}
\]
Using (24), we get
\[
\sum_{j=0}^{\infty} \mu_{2j}(k) z^{2j} = -\frac{1}{2} \frac{\sqrt{1 - 4az^{-2} + \frac{k}{k - 2}}}{k - 1 + a(k - 2)^2 z^{-2}}.
\]
Letting $u = z^{-2}$, we see that
\[
\sum_{j=0}^{\infty} \mu_{2j}(k) u^j = -\frac{1}{2} \frac{\sqrt{1 - 4au + \frac{k}{k - 2}}}{k - 1 + a(k - 2)^2 u},
\]
and therefore
\[
\sqrt{1 - 4au + \frac{k}{k - 2}} = -2 \left[ k - 1 + a(k - 2)^2 u \right] \sum_{j=0}^{\infty} \mu_{2j} u^j
= -\sum_{j=0}^{\infty} 2(k - 1) \mu_{2j} u^j - \sum_{j=1}^{\infty} 2a(k - 2)^2 \mu_{2(j-1)} u^j.
\]
Since
\[
\sqrt{1 - 4au} = \sum_{j=0}^{\infty} \left( \frac{1}{2} \right) (-4au)^j,
\]
we obtain
\[
1 + \frac{k}{k - 2} = -2(k - 1) \mu_0,
\]
and
\[
\left( \frac{1}{2} \right) (-4a)^j = -2(k - 1) \mu_{2j} - 2a(k - 2)^2 \mu_{2(j-1)}, \quad j = 1, 2, \ldots.
\]
If $k = 1$, we get
\[
\left( \frac{1}{2} \right) (-4a)^j = -2a \mu_{2(j-1)}, \quad j = 1, 2, \ldots,
\]
or
\[
\mu_{2n}(1) = 2^{2n+1} \left( \frac{1}{2} \right) \left( \frac{n}{n + 1} \right) (-a)^n, \quad n = 0, 1, \ldots.
\]
If $k = 2$, we have
\[
\mu_{2n}(2) = -2^{2n-1} \left( \frac{1}{2} \right) (-a)^n, \quad n = 1, 2, \ldots.
\]
If \( k \neq 1,2 \), we set \( y_j = \mu_{2j} \), and obtain the recurrence

\[
y_{j+1} = -\frac{a(k-2)^2}{k-1} y_j - \frac{(4a)^{j+1}}{2(k-1)} \left( \frac{1}{j+1} \right),
\]
with

\[
y_0 = \frac{1}{2 - k}.
\]

As it is well known, the general solution of the initial value problem

\[
y_{n+1} = c_n y_n + g_n, \quad y_{n_0} = y_0,
\]
is [22, 1.2.4]

\[
y_n = y_0 \prod_{j=n_0}^{n-1} c_j + \sum_{k=n_0}^{n-1} g_k \prod_{j=k+1}^{n-1} c_j.
\]

Thus,

\[
y_n = \frac{1}{2 - k} \left[ -\frac{a(k-2)^2}{k-1} \right]^n - \sum_{j=0}^{n-1} \frac{(-4a)^{j+1}}{2(k-1)} \left( \frac{1}{j+1} \right) \left[ -\frac{a(k-2)^2}{k-1} \right]^{n-j-1},
\]
or

\[
y_n = -\frac{1}{2(k-1)} \left[ -\frac{a(k-2)^2}{k-1} \right]^n \left( \frac{k}{k-2} + \sum_{j=0}^{n} \left( \frac{1}{2} \right) \left[ \frac{4(k-1)}{(k-2)^2} \right]^j \right)
\]
and the result follows. \( \square \)

**Remark 33.** If \( k = 0 \), we get from [71]

\[
\mu_{2n}(0) = \frac{(4a)^n}{2} \sum_{j=0}^{n} (-1)^j \left( \frac{1}{2} \right),
\]
and using the identity [45, 26.3.10]

\[
\sum_{j=0}^{n} (-1)^j \left( \alpha \right) = (-1)^n \left( \alpha - 1 \right),
\]
we obtain

\[
\mu_{2n}(0) = 2^{2n-1} \left( -\frac{1}{n} \right) (-a)^n.
\]

This agrees with [63], since

\[
\mu_{2n}(0) = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n} \frac{t^{2n}}{\sqrt{4a - t^2}} dt.
\]

When \( k = 1 \), we have from [63]

\[
\mu_{2n}(1) = \frac{1}{2\pi a} \int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n} \sqrt{4a - t^2} dt,
\]
and therefore (69) gives
\[ \frac{1}{2\pi a} \int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n} \sqrt{4a - t^2} dt = 2^{2n+1} \left( \frac{1}{n+1} \right) (-a)^n, \]
which can be verified directly.

When \( k = 2 \), we can write (see Remark 37)
\[ \mu_{2n} (2) = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2n} \sqrt{4a - t^2} dt = \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} t^{2(n-1)} \sqrt{4a - t^2} dt, \]
where \( n = 1, 2, \ldots \). Hence,
\[ \mu_{2n} (2) = a \mu_{2(n-1)} (1) = a 2^{2(n-1)+1} \left( \frac{1}{2} \right) (-a)^{n-1}, \quad n = 1, 2, \ldots, \]
in agreement with (70).

5. Conclusions

We have shown that the Dickson polynomials of the \((k+1)\)-th kind defined by
\[ D_{n,k}(x; a) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-kj}{n-j} \binom{n-j}{j} (-a)^j x^{n-2j} \]
satisfy the orthogonality relation
\[ \frac{1}{2\pi} \int_{-2\sqrt{a}}^{2\sqrt{a}} \sqrt{4a - t^2} D_{n,k}(t) D_{m,k}(t) (k-1) t^2 + a (k-2)^2 dt + \chi(k) \left[ \frac{1 + (-1)^{n+m}}{2} \right] \frac{(2-k)k}{k-1} \left( \frac{1}{k-1} \right)^{n+m} = h_n(k) \delta_{n,m}, \]
where \( a > 0, k \in \mathbb{R} \),
\[ \chi(k) = \begin{cases} 0, & k \in [0,2] \\ 1, & k \in \mathbb{R} \setminus [0,2] \end{cases}, \]
and
\[ h_0(k) = 2 - k, \quad h_n(k) = a^n, \quad n = 1, 2, \ldots. \]

We hope that this work will outline some connections between finite fields and orthogonal polynomials, and that it would be of interest to researchers in both areas.

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