Energy and Angular Momentum in $D$ dimensional Kerr-AdS black holes-new formulation

Emel Altas
Department of Physics, Karamanoğlu Mehmetbey University, 70100, Karaman, Turkey
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Recently it was shown that the conserved charges of asymptotically anti de Sitter spacetimes can be written in an explicitly gauge-invariant way in terms of the linearized Riemann tensor and the Killing vectors. Here we employ this construction to compute the mass and angular momenta of the $D$ dimensional Kerr-AdS black holes, which is one of the most remarkable Einstein metrics generalizing the four dimensional rotating black hole.

I. INTRODUCTION

Recently, we have given a new formulation of conserved charges in cosmological Einstein’s theory [1] and quadratic theories [2] which is explicitly gauge invariant. The formula for asymptotically anti de Sitter (AdS) spacetimes reads

$$Q = \frac{(D-1)(D-2)}{8G_D \Omega_{D-2} \Lambda} \int_{\partial \Sigma} dS_r \left( R^{0 \beta \sigma} \right)^{(1)} \nabla^\beta \xi^\sigma. \tag{1}$$

Here the integral is over the boundary of the spatial hypersurface $\Sigma$, $(R^{0 \beta \sigma})^{(1)}$ is the linearized Riemann tensor, $\xi^\mu$ is a background Killing vector of the AdS spacetime and $G_D$ denotes the Newton’s constant, $\Omega_{D-2}$ is the solid angle. The relation between this formula and the Abbott-Deser [3] formula, which is gauge invariant up to a boundary term, was given in [1]. In [1] the new formula was used to compute the conserved mass and angular momentum of the four dimensional Kerr-AdS black hole and the results are consistent with the other methods [4]. Here we extend the discussion to generic $D$ dimensions for which the computation is much more complex. The relevant solution, that is the $D$ dimensional Kerr-AdS metric, was quite hard to find and in fact it was constructed in 2004 in [5, 6]. Conserved charges of this metric was computed with various techniques including the AD technique [4]. Here our task is twofold: we shall give a computation of the conserved charges for these metrics with the new formula and provide all the relevant details of the computations which are missing in the previous literature. But before that, let us recap some work on conserved charges in gravity theories.

Construction of the conserved quantities has picked up interest for various spacetimes. For an asymptotically flat spacetime one has the Arnowitt-Deser-Misner (ADM) mass [7], which yields exactly the expected energy of an isolated gravitational system. For asymptotically AdS spacetimes, Abbott and Deser generalized the ADM mass and they constructed the Abbott-Deser (AD) charges [3] in cosmological Einstein’s gravity. Another generalization of the conserved charges is the Abbott-Deser-Tekin formulation (ADT) [3, 8], which gives the conserved mass and angular momentum for AdS spacetimes where higher curvature terms generically bring nontrivial contributions to the charges. A detailed review of these constructions and many applications has been recently given in [9].

*Electronic address: emelaltas@kmu.edu.tr
The layout of the paper is as follows: in section II, we construct the energy and angular momentum of the \( D \) dimensional Kerr AdS metric solutions in cosmological Einstein’s gravity using the charge expression given in new formulation (1). Some of the computations are relegated to the Appendices.

II. KERR-ADS BLACK HOLES IN D DIMENSIONS

The mass and angular momentum of the \( D \) dimensional Kerr-AdS black holes are constructed in [4] using the AD formulation [3, 8]. The conserved charges was constructed for the four dimensional Kerr-AdS black holes in [1] using the new formula (1). Here, we extend the discussion and compute the conserved charges of the \( D \)-dimensional Kerr-AdS black holes for asymptotically (anti) de Sitter spacetimes in cosmological Einstein’s gravity. We use the metric given in [5], which is in the Kerr-Schild form \(^1\) [11, 12] as

\[
d s^2 = d\bar{s}^2 + \frac{2MG_D}{U} (k_\mu dx^\mu)^2, \tag{2}
\]

where \( M \) is a real parameter and \( U \) is defined as follows

\[
U := r^\epsilon \sum_{i=1}^{N+\epsilon} \mu_i^2 \prod_{j=1}^{N} \left( r^2 + a_j^2 \right). \tag{3}
\]

Although, the Kerr-Schild metrics are the exact solutions of the cosmological Einstein’s gravity, one can use the perturbation theory expressing the first order expansion\(^2\) of the metric tensor as

\[
h_{\mu\nu} = \frac{2MG_D}{U} k_\mu k_\nu. \tag{4}
\]

The one form \( k_\mu \) is null both with respect to the metrics \( g \) and \( \bar{g} \)

\[
k_\mu k_\nu g^{\mu\nu} = 0 = k_\mu k_\nu \bar{g}^{\mu\nu}, \tag{5}
\]

and it is given with

\[
k_\mu dx^\mu = F dr + W dt - \sum_{i=1}^{N} \frac{a_i \mu_i^2}{1 + \Lambda a_i^2} d\phi_i. \tag{6}
\]

One has the constraint equation \( \sum_{i=1}^{N+\epsilon} \mu_i^2 = 1 \), where \( \epsilon = 1, N = \frac{D-2}{2} \) for even dimensions and \( \epsilon = 0, N = \frac{D-1}{2} \) for odd dimensions. The \( N \) shows the number of rotation parameters. The background metric in (2) is the following de Sitter metric

\[
d \bar{s}^2 = -W \left( 1 - \Lambda r^2 \right) dt^2 + F dr^2 + \sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} d\phi_i^2 + \sum_{i=1}^{N} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} \mu_i d\mu_i^2 + \frac{\Lambda}{W \left( 1 - \Lambda r^2 \right)} \left( \sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} \mu_i d\mu_i \right)^2, \tag{7}
\]

\(^1\) For the exact solutions of Einstein’s theory and the importance of the Kerr-Schild ansatz see [10].

\(^2\) The inverse metric takes the simple form, \( g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} \). The higher order variations of the metric do not survive due to nullity of \( k_\mu \).
where $W$ and $F$ are defined as

$$W := \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{1 + \Lambda a_i^2}, \quad F := \frac{1}{1 + \Lambda a_i^2} \sum_{i=1}^{N+\epsilon} \frac{r^2 \mu_i^2}{r^2 + a_i^2}. \quad (8)$$

For even dimensional case $a_{N+1} = 0$, since the $\phi_{N+1}$ component is missing. The background metric \(\bar{g}\), yields the following expressions

$$\bar{R}_{\mu\nu\alpha\beta} = \Lambda \left( \bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} - \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} \right), \quad \bar{R}_{\mu\nu} = \Lambda (D - 1) \bar{g}_{\mu\nu}, \quad \bar{R} = \Lambda D (D - 1). \quad (9)$$

The charge expression \(\tilde{Q}\), is constructed for (anti) de Sitter background metrics. So, we need to reexpress the conserved charges by rescaling the cosmological constant $\Lambda$ as $\frac{(D-1)(D-2)\Lambda}{2}$. The conserved charge expression then becomes

$$Q = \frac{1}{4G_D \Omega_{D-2} \Lambda (D-3)} \int_{\partial \Sigma} dS_r \left( R^{\mu0}_{\ \ \beta\sigma} \right) \hat{\nabla}^\beta \hat{\xi}^\sigma. \quad (10)$$

Now, we can calculate the energy and angular momentum of the solutions given in \(\tilde{Q}\). For the energy, we use the energy Killing vector, $\bar{\xi}^\mu = (-1, 0)$. Let us compute the integrand in the last equation. Using the symmetries of the Riemann tensor and antisymmetry of the Killing equation, one has

$$\left( R^{\mu0}_{\ \ \beta\sigma} \right) \hat{\nabla}^\beta \hat{\xi}^\sigma = 2\bar{g}^{\mu0} \hat{\nabla}^\beta \hat{\xi}^\sigma \left( \Gamma^r_{\rho\sigma} \right) \hat{\nabla}^\rho (\Gamma^r_{\rho\sigma}) \hat{\nabla}^\sigma (\Gamma^r_{\rho\sigma}), \quad (11)$$

where the covariant derivative of the Killing vector yields

$$\hat{\nabla}^\beta \hat{\xi}^\sigma = \frac{1}{2} \bar{g}^{\mu\nu} \bar{g}^{\gamma\sigma} (\partial_\gamma \bar{\varphi}_{\nu0} - \partial_\nu \bar{\varphi}_{\gamma0}). \quad (12)$$

Then, we can express the integrand as

$$\left( R^{\mu0}_{\ \ \beta\sigma} \right) \hat{\nabla}^\beta \hat{\xi}^\sigma = \left( \bar{g}^{\mu0} \right)^2 \bar{g}^{\beta\nu} \partial_\nu \bar{\varphi}_{00} \left( \hat{\nabla}^\rho (\Gamma^r_{\rho\sigma}) \hat{\nabla}^\sigma (\Gamma^r_{\rho\sigma}) \right) + \partial_\nu \bar{\varphi}_{00} \bar{R}^\nu_{\ \ \rho0} \bar{h}^\rho_{00}. \quad (13)$$

The first order perturbation of the Christoffel symbol reads

$$\left( \Gamma^r_{\rho\sigma} \right) = \frac{1}{2} \left( \hat{\nabla} h^r_{0} + \hat{\nabla}_\sigma h^r_{\rho} - \hat{\nabla}_\rho h^r_{\sigma} \right). \quad (14)$$

So, one obtains

$$\left( R^{\mu0}_{\ \ \beta\sigma} \right) \hat{\nabla}^\beta \hat{\xi}^\sigma = \frac{1}{2} \partial_\nu \bar{\varphi}_{00} \left( \bar{R}^{\mu0\rho0} \bar{h}^\rho_{\nu0} - \bar{R}^{\mu0\nu0} \bar{h}^\rho_{\rho0} \right) \quad (15)$$

$$+ \frac{1}{2} \left( \bar{g}^{\mu0} \right)^2 \bar{g}^{\beta\nu} \partial_\nu \bar{\varphi}_{00} \left( \hat{\nabla}^\rho \hat{\nabla}_\rho h^0_{\beta\rho} + \hat{\nabla}_\beta \hat{\nabla}_\rho h_{00} - \hat{\nabla}_\omega \hat{\nabla}_0 h_{\beta0} - \hat{\nabla}_\beta \hat{\nabla}_0 h_{\omega0} \right),$$

where the components of the background metric are the functions of both the $r$ and $\mu_i$ components. Hence, in the last equation only $\nu = r$ and $\nu = \mu_i$ survive. One finds\(^3\),

$$\left( R^{\mu0}_{\ \ \beta\sigma} \right) \hat{\nabla}^\beta \hat{\xi}^\sigma = 2M \Delta r^{2-D} (D-3) \left( W(D-1) - 1 \right). \quad (16)$$

Substituting the integrand in \((10)\), one arrives at

$$Q = \frac{M}{2G_D \Omega_{D-2}} \int_{\partial \Sigma} dS_r r^{2-D} \left( W(D-1) - 1 \right). \quad (17)$$

\(^3\) See Appendix C for the details of the construction.
Defining the following functions

\[ \Xi \equiv \prod_{i=1}^{N} \left( 1 + \Lambda a_i^2 \right), \quad \Xi_i \equiv 1 + \Lambda a_i^2, \]  

equation (18)

one ends up with the energy of the even dimensional Kerr-AdS black holes as

\[ E = \frac{M}{\Xi} \sum_{i=1}^{N} \frac{1}{\Xi_i}. \]  

equation (19)

In the general case, the energy of the \(D\)-dimensional rotating black holes can be written as

\[ E = \frac{M}{\Xi} \sum_{i=1}^{N} \left( \frac{1}{\Xi_i} - \frac{1}{2} \left( 1 - \epsilon \right) \right), \]  

equation (20)

which matches with the result given in [4]. Similarly, considering the Killing vector \(\xi_{(i)}\) = \((0, ..., 0, 1_i, 0, ..., 0)\), where \(i\) refers to the \(i^{th}\) azimuthal angle \(\phi_i\), one has

\[ \nabla^{\beta} \xi^\sigma = \frac{1}{2} (g^{\beta\sigma} g_{\phi_j \phi_j} - g^{\beta \phi_j} g^{\sigma \phi_j}) \partial_\rho g_{\phi_i \phi_j}. \]  

equation (21)

Then the integrand becomes

\[ \left( R^{\rho_0 \beta_0} \right)^{(1)}_{\beta_\sigma} \nabla^{\beta} \xi^\sigma = \frac{1}{2} \partial_\sigma g_{\phi_i \phi_j} \left( R^{\rho_0 \beta_0} h_\rho^\beta - R^{\rho_j \tau_0} h_\rho^\beta \right) \]  

equation (22)

\[ + \frac{1}{2} g^{\rho_0 \rho_j} g^{\rho_0 \rho_j} g^{\beta \phi_j} \partial_\rho g_{\phi_i \phi_j} \left( \nabla_\beta \nabla_0 h_{\phi_k \phi_k} + \nabla_{\phi_k} \nabla_\tau h_{\phi_0 \phi_0} - \nabla_\beta \nabla_\tau h_{\phi_k 0} - \nabla_{\phi_k} \nabla_0 h_{\beta \tau} \right). \]

Expressing the covariant derivatives in terms of the partial ones, and computing the corresponding quantities one ends up with

\[ \left( R^{\rho_0 \beta_0} \right)^{(1)}_{\beta_\sigma} \nabla^{\beta} \xi^\sigma = 2 M a_i^{2-D} (D-3)(D-1) \frac{a_i^2}{1 + \Lambda a_i^2}. \]  

equation (23)

Then, the angular momentum can be expressed as

\[ J_i = \frac{M(D-1)}{2 G_D M_D^{-2}} \int d\Omega \sqrt{g} r^{2-D} a_i \frac{a_i^2}{1 + \Lambda a_i^2}. \]  

equation (24)

Using the definitions in [18], one arrives at the angular momentum of the even dimensional Kerr-AdS black holes as

\[ J_i = \frac{M a_i}{\Xi \Xi_i}, \]  

equation (25)

which is same with the expression given in [4]. One can express the total energy in terms of angular momenta as follows

\[ E = \sum_{i=1}^{N} \frac{J_i}{a_i}, \]  

equation (26)

in even dimensions. Also, one has the relation given below

\[ E = \sum_{i=1}^{N} \frac{J_i}{a_i} - \frac{NM}{2 \Xi}, \]  

equation (27)

between the energy and angular momenta for the metric solutions (2) of cosmological Einstein’s gravity in odd dimensional case.
III. CONCLUSIONS

In cosmological Einstein’s theory the conserved charges are explicitly gauge-invariant, under small diffeomorphisms generated by a background vector field, for asymptotically AdS spacetimes. But the current expression, which yields the conserved energy and angular momentum, may be gauge-invariant up to a boundary term. The Abbott-Deser (AD) formulation of the conserved charges obtained from such a current two form, which is not explicitly gauge-invariant. Recently, we have given a new method to construct the conserved charges, in which the starting point is the second Bianchi identity instead of the explicit form of the cosmological Einstein tensor in terms of the metric tensor. Also, we have shown that it is possible to use this formulation to construct the conserved charges in generic gravity theories. In this new formulation, the resulting charge expression involves the linearized Riemann tensor and not only the conserved charges but also the current expression is explicitly gauge-invariant.

The $D$ dimensional Kerr-AdS metric solutions of cosmological Einstein’s gravity are in a complex form which makes the calculation of the conserved charges so difficult. But these solutions are too important since they represent the rotating black holes of the theory for the generic $D$ dimensions. In [4] the conserved charges of the $D$ dimensional Kerr-AdS black holes was constructed with the Abbott-Deser (AD) formulation. In [1] we computed the mass and angular momentum of the four dimensional Kerr-AdS black holes with the new energy formula, which yields the matching results with [4]. In this paper, we extend the discussion and construct the conserved charges of the $D$ dimensional Kerr-AdS black holes using the new formulation. The expressions we obtained for the $D$ dimensional case are consistent with the results given in [4].

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APPENDIX A: DETERMINANT OF THE BACKGROUND METRIC TENSOR

In this part, we compute the determinant of the background metric, de Sitter metric in $D$ spacetime dimensions, assuming $D$ is even. One can express

$$\det g_{\mu \nu} = W(\Lambda r^2 - 1) F \prod_{i=1}^{N} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} \mu_i^2 \det g_{\mu_i \mu_j}. \quad (28)$$

Since we need the $r \to \infty$ limit of the determinant, we can express the last equation as

$$\det g_{\mu \nu} = -W r^{2N} \prod_{i=1}^{N} \frac{\mu_i^2}{1 + \Lambda a_i^2} \det g_{\mu_i \mu_j}. \quad (29)$$

Here we used the large $r$ limit of the $F$ function, defined in [5], which explicitly reads $F = -1/\Lambda r^2$. Obviously, one needs to calculate the determinant of the non-diagonal piece of the background metric. First, let us compute the $g_{\mu_i \mu_j}$ component for the even dimensional case. The non-diagonal piece in (7) involves two terms. Let us compute these terms explicitly and then collect the pieces. We have

$$\sum_{i=1}^{N+1} \frac{r^2 + a_i^2 d\mu_i^2}{1 + \Lambda a_i^2} = \sum_{i=1}^{N} \frac{r^2 + a_i^2 d\mu_i^2}{1 + \Lambda a_i^2} + \frac{r^2 + a_{N+1}^2 d\mu_{N+1}^2}{1 + \Lambda a_{N+1}^2}, \quad (30)$$
where \( a_{N+1} = 0 \). Then, the last equation reduces to

\[
\sum_{i=1}^{N+1} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} d\mu_i^2 = \sum_{i=1}^{N} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} d\mu_i^2 + r^2 d\mu_{N+1}^2. \tag{31}
\]

The constraint equation \( \sum_{i=1}^{N+1} \mu_i^2 = 1 \), can be rewritten as follows

\[
\mu_{N+1}^2 = 1 - \sum_{i=1}^{N} \mu_i^2. \tag{32}
\]

Taking the derivative of the last expression, one obtains

\[
d\mu_{N+1} = -\frac{\sum_{i=1}^{N} \mu_i d\mu_i}{1 - \sum_{i=1}^{N} \mu_i^2}. \tag{33}
\]

and equation (31) becomes

\[
\sum_{i=1}^{N+1} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} d\mu_i^2 = \sum_{i=1}^{N} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} d\mu_i^2 - \frac{r^2}{\mu_{N+1}^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \mu_j d\mu_i d\mu_j. \tag{34}
\]

Similarly, we can express the remaining non-diagonal piece in (7) as

\[
(\sum_{i=1}^{N+1} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} \mu_i d\mu_i)^2 = (\sum_{i=1}^{N} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} \mu_i d\mu_i - r^2 \mu_i d\mu_i)^2, \tag{35}
\]

where we have used the derivative of the constraint equation (33), and \( a_{N+1} = 0 \). We can express the last equation as

\[
(\sum_{i=1}^{N+1} \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} \mu_i d\mu_i)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \mu_j d\mu_i d\mu_j \left( \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} - r^2 \right) \left( \frac{r^2 + a_j^2}{1 + \Lambda a_j^2} - r^2 \right). \tag{36}
\]

Collecting the pieces, the \( \overline{g}_{i\mu,j\mu} \) component reads

\[
\overline{g}_{i\mu,j\mu} = \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} \delta_{ij} + \mu_i \mu_j \left( \frac{r^2}{\mu_{N+1}^2} + \frac{\Lambda}{W(1 - \Lambda r^2)} \left( \frac{r^2 + a_i^2}{1 + \Lambda a_i^2} - r^2 \right) \left( \frac{r^2 + a_j^2}{1 + \Lambda a_j^2} - r^2 \right) \right). \tag{37}
\]

Now, let us consider the \( r \to \infty \) limit of the \( \overline{g}_{i\mu,j\mu} \). One ends up with

\[
\overline{g}_{i\mu,j\mu} = \frac{r^2 \delta_{ij}}{1 + \Lambda a_i^2} + \frac{r^2 \mu_i \mu_j}{\mu_{N+1}^2} - \frac{\Lambda^2 r^2}{W(1 + \Lambda a_i^2)(1 + \Lambda a_j^2)} \frac{a_i^2 a_j^2 \mu_i \mu_j}{W(1 + \Lambda a_i^2)(1 + \Lambda a_j^2)}. \tag{38}
\]

Note that \( i, j, \ldots = 1, 2, \ldots N \), where \( N = \frac{D}{2} \), and there is no summation over the indices. For simplicity, let us express the result in a compact form as

\[
\overline{g}_{i\mu,j\mu} = A_i \delta_{ij} + B_i B_j + C_i C_j, \tag{39}
\]

where \( A_i, B_i \) and \( C_i \) functions are defined as follows

\[
A_i := \frac{r^2}{1 + \Lambda a_i^2}, \quad B_i := \frac{r \mu_i}{\mu_{N+1}}, \quad C_i := \frac{\Lambda r}{\sqrt{W(1 + \Lambda a_i^2)}}. \tag{40}
\]
Computing the determinant of a metric, which is in the form \([39]\) is not so easy. To simplify the calculation, let us consider the lower dimensional cases and then generalize the results. Start with the \(N = 2\) case, we can express the determinant as
\[
\det \mathcal{g}_{\mu_i \mu_j} = A_1 A_2 \left(1 + \frac{B_1^2}{A_1} + \frac{B_2^2}{A_2} + \frac{C_i^2}{A_i} + \frac{C_j^2}{A_j} + \frac{1}{A_1 A_2} (B_1^2 C_2^2 + B_2^2 C_1^2 - 2B_1 B_2 C_1 C_2) \right).
\]
\[\text{(41)}\]
Note that \(N = 2\) corresponds to the six dimensional spacetime. When \(N = 3\), one has \(D = 8\) and the determinant yields
\[
\det \mathcal{g}_{\mu_i \mu_j} = A_1 A_2 A_3 \left(1 + \frac{B_1^2}{A_1} + \frac{B_2^2}{A_2} + \frac{B_3^2}{A_3} + \frac{C_i^2}{A_i} + \frac{C_j^2}{A_j} + \frac{1}{A_1 A_2} (B_1^2 C_2^2 + B_2^2 C_1^2 - 2B_1 B_2 C_1 C_2) + \frac{1}{A_2 A_3} (B_2^2 C_3^2 + B_3^2 C_2^2 - 2B_2 B_3 C_2 C_3) \right).
\]
\[\text{(42)}\]
Then, we can generalize the results as follows
\[
\det \mathcal{g}_{\mu_i \mu_j} = \prod_{k=1}^{N} A_k \left(1 + \sum_{j=1}^{N} \left(\frac{B_j^2}{A_j} + \frac{C_j^2}{A_j}\right) + \sum_{i=1}^{N} \sum_{j \neq i} \left(\frac{B_i C_j}{A_i A_j} - \frac{B_i B_j C_i C_j}{A_i A_j} \right) \right).
\]
\[\text{(43)}\]
Now, we need to express this result in terms of the \(r \) and \(\mu_i \) components. Inserting the functions given in \([40]\), one obtains
\[
\frac{B_j^2}{A_j} + \frac{C_j^2}{A_j} = \frac{1}{W \mu_{N+1}^2} \left(W \mu_j^2 (1 + \Lambda a_j^2) - \Lambda^2 \mu_{N+1}^2 \frac{\mu_j^2 \lambda_j^2}{1 + \Lambda a_j^2} \right).
\]
\[\text{(44)}\]
Also, the last two terms in \([43]\) yield the following
\[
\frac{B_i^2 C_j^2}{A_i A_j} - \frac{B_i B_j C_i C_j}{A_i A_j} = \frac{\Lambda^2 \mu_i^2 \mu_j^2 \lambda_i \lambda_j (a_i^2 - a_j^2)}{W \mu_{N+1}^2 (1 + \Lambda a_j^2)}.
\]
\[\text{(45)}\]
Using the equations \([43\text{ }45]\), we arrive at
\[
1 + \sum_{j=1}^{N} \left(\frac{B_j^2}{A_j} + \frac{C_j^2}{A_j}\right) + \sum_{i=1}^{N} \sum_{j \neq i} \left(\frac{B_i C_j}{A_i A_j} - \frac{B_i B_j C_i C_j}{A_i A_j} \right) = \frac{1}{W \mu_{N+1}^2} \left(W \mu_j^2 (1 + \Lambda a_j^2) - \Lambda^2 \mu_{N+1}^2 \sum_{j=1}^{N} \frac{\mu_j^2 \lambda_j^2}{1 + \Lambda a_j^2} + \Lambda^2 \sum_{i=1}^{N} \sum_{j \neq i} \frac{\mu_i^2 \mu_j^2 \lambda_i \lambda_j (a_i^2 - a_j^2)}{W \mu_{N+1}^2 (1 + \Lambda a_j^2)} \right).
\]
\[\text{(46)}\]
One can get rid of the \(\mu_{N+1}^2\) terms, substituting the constraint \(\mu_{N+1}^2 = 1 - \sum_{i=1}^{N} \mu_i^2\) in the last equation. One obtains
\[
1 + \sum_{j=1}^{N} \left(\frac{B_j^2}{A_j} + \frac{C_j^2}{A_j}\right) + \sum_{i=1}^{N} \sum_{j \neq i} \left(\frac{B_i C_j}{A_i A_j} - \frac{B_i B_j C_i C_j}{A_i A_j} \right) = \frac{1}{W \mu_{N+1}^2} \left(W + \sum_{j=1}^{N} \mu_j^2 \lambda_j^2 \frac{\mu_j^2 \lambda_j^2}{1 + \Lambda a_j^2} - \Lambda^2 \sum_{j=1}^{N} \frac{\mu_j^2 \lambda_j^2 \lambda_j (a_i^2 - a_j^2)}{W \mu_{N+1}^2 (1 + \Lambda a_j^2)} \right).
\]
\[\text{(47)}\]
which can be reduced further expressing the function \(W\), defined in equation \([8]\), as
\[
W = 1 - \sum_{i=1}^{N} \frac{\mu_i^2 \lambda_i^2}{1 + \Lambda a_i^2}.
\]
\[\text{(48)}\]
Inserting \( W \) in (47), one ends up with

\[
1 + \sum_{j=1}^{N} \left( \frac{B_j^2}{A_j} + \frac{C_j^2}{A_j} \right) + \sum_{i<j}^{N} \left( \frac{B_i^2 C_j^2}{A_i A_j} - \frac{B_i B_j C_i C_j}{A_i A_j} \right) = \frac{1}{W^{\mu_{N+1}}}.
\]

From (43), one arrives the determinant of the \( \overline{g}_{\mu_\nu} \) as

\[
\det \overline{g}_{\mu_\nu} = \frac{1}{W^{\mu_{N+1}}} \prod_{i=1}^{N} A_i = \frac{r^{2N}}{W^{\mu_{N+1}}} \prod_{i=1}^{N} \frac{1}{1 + \Lambda a_i^2}.
\]

As a final step, we need to insert this expression in (29). We end up with

\[
\det \overline{g}_{\mu_\nu} = - \left( \frac{r^{2N}}{\mu_{N+1}} \prod_{i=1}^{N} \frac{\mu_i}{1 + \Lambda a_i^2} \right)^2
\]

which yields

\[
\sqrt{-\overline{g}} = \frac{r^{2N}}{\mu_{N+1}} \prod_{i=1}^{N} \frac{\mu_i}{1 + \Lambda a_i^2}
\]

for the even dimensional case. For the odd dimensional case, one can carry out the similar calculation.

**APPENDIX B: INTEGRAL EXPRESSIONS**

Here, we compute the integral expressions, that we have used in the construction of the conserved charges of the \( D \) dimensional Kerr-AdS metrics. Let us calculate the integral given below

\[
I_1 := \int_{-1}^{1} \frac{\prod_{i=1}^{n} \mu_i d\mu_i}{\sqrt{1 - \sum_{i=1}^{n} \mu_i^2}}.
\]

Note that, the integer \( n \) has not specified as being odd or even. Considering the following parameterizations of the \( \mu_i \)'s

\[
\begin{align*}
\mu_1 &= \cos \theta_1 \\
\mu_2 &= \sin \theta_1 \cos \theta_2 \\
&. \\
&. \\
\mu_{n-1} &= \sin \theta_1 \ldots \sin \theta_{n-2} \cos \theta_{n-1} \\
\mu_n &= \sin \theta_1 \ldots \sin \theta_{n-1} \cos \theta_n
\end{align*}
\]

one can calculate the integral easily. First, let us compute the denominator. We have

\[
1 - \sum_{i=1}^{n} \mu_i^2 = 1 - \cos^2 \theta_1 - \sin^2 \theta_1 \cos^2 \theta_2 - \ldots - \sin^2 \theta_1 \ldots \sin^2 \theta_{n-2} \cos^2 \theta_{n-1} - \sin^2 \theta_1 \ldots \sin^2 \theta_{n-1} \cos^2 \theta_n
\]
which reduces to the following simple expression

\[ 1 - \sum_{i=1}^{n} \mu_i^2 = \sin^2 \theta_1 \sin^2 \theta_2 \ldots \sin^2 \theta_{n-1} \sin^2 \theta_n. \]  (56)

Then, we can express the denominator of the integrand as

\[ \sqrt{1 - \sum_{i=1}^{n} \mu_i^2} = \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-1} \sin \theta_n. \]  (57)

Now let us compute the numerator. The multiplication of the \( \mu_i \) parameters, \( \mu_1 \mu_2 \ldots \mu_n \), in terms of the \( \theta_i \)'s yields

\[ \prod_{i=1}^{n} \mu_i = \cos \theta_1 \sin^{n-1} \theta_1 \cos \theta_2 \sin^{n-2} \theta_2 \ldots \cos \theta_{n-1} \sin \theta_{n-1} \cos \theta_n. \]  (58)

Also, we need to calculate the \( d\mu_1 d\mu_2 \ldots d\mu_n \) term. Since

\[ \prod_{i=1}^{n} d\mu_i = \det \begin{vmatrix} \frac{\partial \mu_1}{\partial \theta_1} & \frac{\partial \mu_1}{\partial \theta_2} & \ldots & \frac{\partial \mu_1}{\partial \theta_n} \\ \frac{\partial \mu_2}{\partial \theta_1} & \frac{\partial \mu_2}{\partial \theta_2} & \ldots & \frac{\partial \mu_2}{\partial \theta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mu_n}{\partial \theta_1} & \frac{\partial \mu_n}{\partial \theta_2} & \ldots & \frac{\partial \mu_n}{\partial \theta_n} \end{vmatrix} d\theta_1 d\theta_2 \ldots d\theta_n, \]  (59)

one ends up with

\[ \prod_{i=1}^{n} d\mu_i = \sin^n \theta_1 \sin^{n-1} \theta_2 \ldots \sin^2 \theta_{n-1} \sin \theta_n d\theta_1 d\theta_2 \ldots d\theta_n. \]  (60)

Collecting the pieces \((58) [60]\), one arrives at the numerator as

\[ \prod_{i=1}^{n} \mu_i d\mu_i = \cos \theta_1 \sin^{2n-1} \theta_1 \cos \theta_2 \sin^{2n-3} \theta_2 \ldots \cos \theta_{n-1} \sin^3 \theta_{n-1} \cos \theta_n \sin \theta_n d\theta_1 d\theta_2 \ldots d\theta_n. \]  (61)

Then, the integrand can be expressed in terms of the new parameters. We have

\[ \frac{\prod_{i=1}^{n} \mu_i d\mu_i}{\sqrt{1 - \sum_{i=1}^{n} \mu_i^2}} = \cos \theta_1 \sin^{2n-2} \theta_1 \cos \theta_2 \sin^{2n-4} \theta_2 \ldots \cos \theta_{n-1} \sin^2 \theta_{n-1} \cos \theta_n \sin \theta_n d\theta_1 d\theta_2 \ldots d\theta_n. \]  (62)

Now, let us consider the boundaries of the integral. We can write

\[ I_1 = \int_{-1}^{1} \frac{\prod_{i=1}^{n} \mu_i d\mu_i}{\sqrt{1 - \sum_{i=1}^{n} \mu_i^2}} = 2 \int_{0}^{1} \frac{\prod_{i=1}^{n} \mu_i d\mu_i}{\sqrt{1 - \sum_{i=1}^{n} \mu_i^2}}. \]  (63)

since the integrand is an even function of the \( \mu_i \)'s. The integral then becomes

\[ I_1 = 2 \int_{-\pi/2}^{0} \int_{-\pi/2}^{0} \cos \theta_1 \sin^{2n-2} \theta_1 \cos \theta_2 \sin^{2n-4} \theta_2 \ldots \cos \theta_{n-1} \sin^2 \theta_{n-1} \cos \theta_n d\theta_1 d\theta_2 \ldots d\theta_n. \]  (64)
So, one obtains the following equation

\[
I_1 = 2 \int_{-\pi/2}^{0} \cos \theta_1 \sin^{2n-2} \theta_1 d\theta_1 \int_{-\pi/2}^{0} \cos \theta_2 \sin^{2n-4} \theta_2 d\theta_2 \ldots \int_{-\pi/2}^{0} \cos \theta_{n-1} \sin^2 \theta_{n-1} d\theta_{n-1} \int_{-\pi/2}^{0} \cos \theta_n d\theta_n,
\]

which is now in a familiar form. The \( \theta_n \) integral yields one and then one has

\[
I_1 = 2 \int_{-\pi/2}^{0} \cos \theta_1 \sin^{2n-2} \theta_1 d\theta_1 \int_{-\pi/2}^{0} \cos \theta_2 \sin^{2n-4} \theta_2 d\theta_2 \ldots \int_{-\pi/2}^{0} \cos \theta_{n-1} \sin^2 \theta_{n-1} d\theta_{n-1}.
\]

Defining the \( y_i \) functions \( y_i := \sin \theta_i \), where \( i = 1, 2, \ldots, n-1 \), we can rewrite the last equation as

\[
I_1 = 2 \int_{-1}^{0} y_1^{2n-2} dy_1 \int_{-1}^{0} y_2^{2n-4} dy_2 \ldots \int_{-1}^{0} y_{n-1}^{2} dy_{n-1}.
\]

Finally, we can express the integral given in (53) as

\[
I_1 = \frac{1}{(2n-1)(2n-3) \ldots 3} \frac{2}{(2n-1)!!}.
\]

We also need to compute the following integral

\[
I_2 := \int_{-1}^{1} \frac{1}{\sqrt{1 - \sum_{i=1}^{n} \mu_i^2}} \prod_{i=1}^{n} \mu_i d\mu_i \sum_{j=1}^{n} \alpha_j \mu_j^2,
\]

where the \( \prod_{i=1}^{n} \mu_i d\mu_i \) piece was given in equation (62). The remaining piece reads

\[
\sum_{j=1}^{n} \alpha_j \mu_j^2 = \alpha_1 \cos^2 \theta_1 + \alpha_2 \sin^2 \theta_1 \cos^2 \theta_2 + \ldots + \alpha_n \sin^2 \theta_1 \ldots \sin^2 \theta_{n-1} \cos^2 \theta_n.
\]

Then, we can express the second integral (69) as

\[
I_2 = 2 \left( \alpha_1 \int_{-\pi/2}^{0} \cos^3 \theta_1 \sin^{2n-2} \theta_1 d\theta_1 \int_{-\pi/2}^{0} \cos \theta_2 \sin^{2n-4} \theta_2 d\theta_2 \ldots \int_{-\pi/2}^{0} \cos \theta_{n-1} \sin^2 \theta_{n-1} d\theta_{n-1} 
+ \alpha_2 \int_{-\pi/2}^{0} \cos \theta_1 \sin^{2n-2} \theta_1 d\theta_1 \int_{-\pi/2}^{0} \cos^3 \theta_2 \sin^{2n-4} \theta_2 d\theta_2 \ldots \int_{-\pi/2}^{0} \cos \theta_{n-1} \sin^2 \theta_{n-1} d\theta_{n-1} 
+ \ldots + \alpha_n \int_{-\pi/2}^{0} \cos \theta_1 \sin^{2n-2} \theta_1 d\theta_1 \ldots \int_{-\pi/2}^{0} \cos \theta_{n-1} \sin^2 \theta_{n-1} d\theta_{n-1} \int_{-\pi/2}^{0} \cos^3 \theta_n d\theta_n \right).
\]

For simplicity, let us focus on the first integral on the right hand side of the last equation. Using the \( y_i \) functions, \( y_1 = \sin \theta_1, y_2 = \sin \theta_2 \ldots y_{n-1} = \sin \theta_{n-1} \), again this piece yields \( \frac{2}{(2n+1)!!} \). Similarly, the additional pieces give the same value. So then, one finds

\[
I_2 = 2 \left( \frac{2}{(2n+1)!!} + \frac{2}{(2n+1)!!} + \ldots + \frac{2}{(2n+1)!!} \right) = \frac{4}{(2n+1)!!} \sum_{j=1}^{n} \alpha_j.
\]
Note that, due to the equality of the pieces, we can express the following result

\[ I_3 := \int \frac{1}{\sqrt{1 - \sum_{i=1}^{n} \mu_i^2}} \alpha_j \beta_j^2 = \frac{4}{(2n + 1)!!} \alpha_j, \]  

(73)

which also denotes any arbitrary piece of the second integral.

**APPENDIX C: ENERGY AND ANGULAR MOMENTA IN D DIMENSIONS**

In this section, we give the construction of the energy and angular momentum of the for the even dimensional Kerr-AdS metric solutions \([2]\) in cosmological Einstein theory. To compute the energy, we need to use the energy Killing vector, \(\bar{\xi}^\mu = (-1, \vec{0})\) and compute the integrand in charge expression \([1]\) for the given Killing vector. We have

\[ \left(R^{\prime 0}_{\beta \sigma}\right)^{(1)} \nabla^\beta \bar{\xi}^\sigma = \left((R^r_{\rho \beta \sigma})^{(1)} \mathfrak{g}^{\rho 0} - \mathcal{R}^r_{\rho \beta \sigma} h^{\rho 0}\right) \nabla^\beta \bar{\xi}^\sigma, \]  

(74)

where the linearized Riemann tensor reads

\[ (R^r_{\rho \beta \sigma})^{(1)} = \nabla^\rho \left(\Gamma^r_{\rho \beta}^{(1)}\right) - \nabla_\beta \left(\Gamma^r_{\rho \beta}^{(1)}\right), \]  

(75)

and the first order expansion of the Christoffel symbol is

\[ (\Gamma^r_{\rho \beta})^{(1)} = \frac{1}{2} \left(\nabla h^\rho_{\sigma} + \nabla h^\sigma_{\rho} - \nabla h^\rho_{\sigma}\right). \]  

(76)

Using the antisymmetry of the indices, \(\beta\) and \(\sigma\), one obtains

\[ \left(R^{\prime 0}_{\beta \sigma}\right)^{(1)} \nabla^\beta \bar{\xi}^\sigma = 2 \mathfrak{g}^{00} \nabla^\beta \bar{\xi}^\sigma \nabla^\rho \left(\Gamma^r_{\rho \beta}^{(1)}\right) - \mathcal{R}^r_{\rho \beta \sigma} h^{\rho 0} \nabla^\beta \bar{\xi}^\sigma, \]  

(77)

where

\[ \nabla^\beta \bar{\xi}^\sigma = \frac{1}{2} \mathfrak{g}^{\beta \mu} \mathfrak{g}^{\gamma \rho} \left(\partial_\gamma \mathfrak{g}_{\mu 0} - \partial_\mu \mathfrak{g}_{\gamma 0}\right) \]  

(78)

for the Killing vector \(\bar{\xi}^\mu = (-1, \vec{0})\). Then one has

\[ \left(R^{\prime 0}_{\beta \sigma}\right)^{(1)} \nabla^\beta \bar{\xi}^\sigma = \left(\mathfrak{g}^{00}\right)^2 \mathfrak{g}^{\beta \mu} \partial_\mu \mathfrak{g}_{00} \left(\nabla_\rho \left(\Gamma^r_{0 \beta}^{(1)}\right) - \nabla_\beta \left(\Gamma^r_{0 \rho}^{(1)}\right)\right) + \partial_\rho \mathfrak{g}_{00} \mathcal{R}^r_{\rho \beta \sigma} h^{\rho 0}. \]  

(79)

Expressing the linearized Christoffel connection in terms of the linear order metric perturbations, one arrives at

\[ \left(R^{\prime 0}_{\beta \sigma}\right)^{(1)} \nabla^\beta \bar{\xi}^\sigma = \frac{1}{2} \partial_\rho \mathfrak{g}_{00} \left(R^{00}_{\rho 0} h^\rho_0 - \mathcal{R}^{00}_{\rho 0} h^\rho_0\right) \]  

(80)

\[ + \frac{1}{2} \left(\mathfrak{g}^{00}\right)^2 \mathcal{R}^r_{\rho \beta \sigma} h^{\rho 0} \left(\nabla_\rho \nabla_0 h_{\beta \rho} + \nabla_\rho \nabla_0 h_{\beta 0} - \nabla_\beta \nabla_0 h_{\rho 0}\right). \]

Since \(\mathfrak{g}_{00}\) is a function of \(r\) and \(\mu_i\)’s, we obtain the following

\[ \left(R^{\prime 0}_{\beta \sigma}\right)^{(1)} \nabla^\beta \bar{\xi}^\sigma = \frac{1}{2} \partial_\rho \mathfrak{g}_{00} \left(R^{00}_{\rho 0} h^\rho_0 - \mathcal{R}^{00}_{\rho 0} h^\rho_0\right) + \frac{1}{2} \partial_\mu \mathfrak{g}_{00} \left(R^{00}_{\mu 0} h^\rho_0 - \mathcal{R}^{00}_{\mu 0} h^\rho_0\right) \]  

(81)

\[ + \frac{1}{2} \left(\mathfrak{g}^{00}\right)^2 \partial_\rho \mathfrak{g}_{00} \left(\nabla_\rho \nabla_0 h_\rho + \nabla_\rho \nabla_0 h_0 - \nabla_\rho \nabla_0 h_{\rho 0}\right) \]  

\[ + \frac{1}{2} \left(\mathfrak{g}^{00}\right)^2 \partial_\rho \mathfrak{g}_{00} \left(\nabla_\rho \nabla_0 h_{\mu 0} + \nabla_\rho \nabla_0 h_{\mu 0} - \nabla_\rho \nabla_0 h_{\mu 0}\right). \]
We have \( \bar{g}_{00} = W r^2 \) and \( \bar{g}_{rr} = -\frac{1}{r^2 \Lambda^2} \) when we take the \( r \to \infty \) limit. Using the constraint \( \sum_{i=1}^{N+1} \mu_i^2 = 1 \), one obtains \( U = r^{D-3} \) and from the equations (6, 11) we can express

\[
\begin{align*}
    h_{00} &= 2MW^2 r^{3-D}, \\
    h_{rr} &= \frac{2M}{\Lambda^2} r^{-D-1}.
\end{align*}
\]  

Note that, from equation (6) we have \( k_{\mu_i} = 0 \), which yields \( h_{\nu\mu_i} = 0 \). Let us calculate the right hand side of the equation (81) term by term. The third and the fourth terms vanish and the first two terms yield

\[
\frac{1}{2} \partial_\nu \bar{g}_{00} \left( \bar{R}^{0\nu\rho}_{\rho} h^\nu_{\rho} - \bar{R}^{0\nu\rho}_{\rho} h^0_{\rho} \right) = 2M \Lambda r^{2-D}(1 - W).
\]  

To obtain the second line, first express the covariant derivatives in terms of the partial derivatives and the background Christoffel symbol. The first piece yields

\[
\nabla_0 \nabla_0 h_{rr} = -\Gamma^{0}_{00} \partial_r h_{rr} + 2\Gamma^{0}_{00} \Gamma^{r}_{rr} h_{rr} + 2\Gamma^{0}_{r0} \Gamma^{0}_{00} h_{rr} + 2\Gamma^{0}_{r0} \Gamma^{0}_{0r} h_{00}
\]  

and the second one reads

\[
\nabla_r \nabla_r h_{00} = \partial_r \partial_r h_{00} - 2\partial_r (\Gamma^{0}_{r0} h_{00}) - \Gamma^{0}_{rr} \partial_r h_{00} - 2\Gamma^{0}_{0r} \partial_r h_{00} + 2\Gamma^{0}_{r0} \Gamma^{0}_{rr} h_{00} + 4\Gamma^{0}_{r0} \Gamma^{0}_{0r} h_{00},
\]  

also the third and the last pieces respectively yield

\[
\nabla_0 \nabla_r h_{0r} = \Gamma^{0}_{r0} \Gamma^{0}_{00} h_{rr} + 3\Gamma^{0}_{0r} \Gamma^{0}_{rr} h_{00} - \Gamma^{0}_{00} \partial_r h_{rr} + 2\Gamma^{0}_{00} \Gamma^{0}_{rr} h_{rr} - \Gamma^{0}_{rr} \partial_r h_{00},
\]  

and

\[
\nabla_r \nabla_r h_{0r} = 2\Gamma^{0}_{r0} \Gamma^{0}_{00} h_{rr} + 2\Gamma^{0}_{r0} \Gamma^{0}_{0r} h_{00} + \Gamma^{0}_{0r} \Gamma^{0}_{rr} h_{rr} + \Gamma^{0}_{r0} \Gamma^{0}_{rr} h_{00} - \partial_r (\Gamma^{0}_{00} h_{rr}) - \partial_r (\Gamma^{0}_{r0} h_{00}).
\]  

Inserting the pieces, one arrives at

\[
\nabla_0 \nabla_0 h_{rr} + \nabla_r \nabla_r h_{0r} - \nabla_0 \nabla_r h_{0r} - \nabla_r \nabla_0 h_{0r} = \partial_r \partial_r h_{00} - \Gamma^{0}_{rr} \partial_r h_{00} - 2\Gamma^{0}_{0r} \partial_r h_{00} + \Gamma^{0}_{00} \partial_r h_{rr} + \Gamma^{0}_{rr} \partial_r h_{00} + \Gamma^{0}_{r0} \Gamma^{0}_{00} h_{rr} + \Gamma^{0}_{r0} \Gamma^{0}_{0r} h_{00}.
\]  

where

\[
\Gamma^\nu_{\nu \rho} = \frac{1}{2} g^{\nu \sigma} (\partial_\rho \bar{g}_{0\sigma} + \partial_0 \bar{g}_{\nu \sigma} - \partial_\sigma \bar{g}_{\nu 0}).
\]  

After a straightforward calculation, one ends up with

\[
\nabla_0 \nabla_0 h_{rr} + \nabla_r \nabla_r h_{00} - \nabla_0 \nabla_r h_{0r} - \nabla_r \nabla_0 h_{0r} = 2MW r^{1-D} (WD^2 - 4WD + 4W + 2 - D).
\]  

So then, the second line in equation (81) becomes

\[
\frac{1}{2} (g^{00} g^{rr})^2 \partial_0 \bar{g}_{00} \left( \nabla_0 \nabla_0 h_{rr} + \nabla_r \nabla_r h_{00} - \nabla_0 \nabla_r h_{0r} - \nabla_r \nabla_0 h_{0r} \right) = 2M \Lambda r^{2-D} (WD^2 - 4WD + 4W + 2 - D).
\]  

Now, let us compute the last line. The first two pieces read

\[
\nabla_0 \nabla_0 h_{\mu \nu} = \Gamma^{0}_{00} \Gamma^{\rho}_{\nu \rho} h_{rr} + \Gamma^{0}_{0r} \Gamma^{0}_{00} h_{rr} + 2\Gamma^{0}_{r0} \Gamma^{0}_{0r} h_{00}.
\]
In terms of the components of the background metric tensor, the last equation can be written as

\[
\nabla_{\mu_j} \nabla_r h_{00} = \partial_{\mu_j} \partial_r h_{00} - 2 \partial_{\mu_j} (\Gamma_{r0}^0 h_{00}) - \Gamma_{r \mu_j}^{\mu_k} \partial_{\mu_k} h_{00} + 2 \Gamma_{r \mu_j}^0 \Gamma_0^{\mu_k} h_{00} - 2 \Gamma_0^{\mu_k} \partial_r h_{00} + 4 \Gamma_0^{\mu_k} \Gamma_0^{\mu_0} h_{00}. \tag{93}
\]

The third and the last terms yield

\[
\nabla_0 \nabla_r h_{0 \mu_j} = 3 \Gamma_{r0}^0 \Gamma_0^{\mu_j} h_{00} - \Gamma_0^{\mu_j} \partial_r h_{00}, \tag{94}\]

and

\[
\nabla_{\mu_j} \nabla_0 h_{0 \sigma} = - \partial_{\mu_j} (\Gamma_{0 \sigma}^0 h_{0r}) - \partial_{\mu_j} (\Gamma_{0 \sigma}^0 h_{00}) + 2 \Gamma_{0 \mu_j}^0 \Gamma_0^{\sigma r} h_{0r} + 2 \Gamma_{0 \mu_j}^0 \Gamma_0^{\sigma 0} h_{00} + \Gamma_{r \mu_j}^0 \Gamma_0^{\sigma_k} h_{00}. \tag{95}\]

Collecting the pieces, we end up with

\[
\nabla_0 \nabla_0 h_{\mu_j r} + \nabla_{\mu_j} \nabla_r h_{00} - \nabla_0 \nabla_r h_{\mu_j 0} - \nabla_{\mu_j} \nabla_0 h_{0r} = 3 \Gamma_{r0}^0 \Gamma_0^{\mu_j} h_{00} - \Gamma_0^{\mu_j} \partial_r h_{00} \tag{96}\]

\[
\Gamma_0^{\mu_k} \Gamma_0^{\sigma j} h_{00} - \nabla_0 \nabla_r h_{\mu_j 0} - \nabla_{\mu_j} \nabla_0 h_{0r} = \frac{2}{\tau} \partial_{\mu_j} W \tag{97}\]

Collecting the pieces, we end up with

\[
\nabla_0 \nabla_0 h_{\mu_j r} + \nabla_{\mu_j} \nabla_r h_{00} - \nabla_0 \nabla_r h_{\mu_j 0} - \nabla_{\mu_j} \nabla_0 h_{0r} = 3 \Gamma_{r0}^0 \Gamma_0^{\mu_j} h_{00} - \Gamma_0^{\mu_j} \partial_r h_{00} \tag{96}\]

Then, we can express the last line in (91) as

\[
\frac{1}{2} (f^{00})^2 g^{rr} F^{\mu j} \partial_{\mu_j} \bar{g}_{00} \left( \nabla_0 \nabla_0 h_{\mu_j r} + \nabla_{\mu_j} \nabla_r h_{00} - \nabla_0 \nabla_r h_{\mu_j 0} - \nabla_{\mu_j} \nabla_0 h_{0r} \right) = -\frac{3 M W r^{2-D}}{2 W} (1 - D) F^{\mu j} \partial_{\mu_j} W. \tag{98}\]

We need to compute the $F^{\mu j} \partial_{\mu_j} W \partial_{\mu_j} W$ term, which is complicated due to the inverse metric. In order to simplify the calculation, we can use the explicit form of the Riemann tensor. Let us consider the $R^0_{\phi_0 \phi_j}$ term. We can express

\[
R^0_{\phi_0 \phi_j} = \Lambda g^{\phi_0 \phi_j} = \Gamma_{0 \phi_j}^0 + \Gamma_{0 \phi_k}^0 \Gamma_{\phi_k \phi_j}^0, \tag{99}\]

where the first equality comes from the equation (90). Using $\Gamma_{0 \phi_j}^0 \Gamma_{\phi_k \phi_j}^0 = \Lambda g^{\phi_0 \phi_j}$, we arrive at

\[
\Gamma_{0 \phi_k}^0 \Gamma_{\phi_k \phi_j}^0 = 0. \tag{100}\]

In terms of the components of the background metric tensor, the last equation can be written as

\[
\Gamma_{0 \mu_k}^0 \Gamma_{\phi_k \phi_j}^0 = -\frac{1}{4 \Lambda} \partial_j W F^{\mu j} \partial_{\mu_j} \bar{g}_{\phi_0 \phi_j} = 0, \tag{101}\]

which yields the following identity

\[
F^{\mu j} \partial_{\mu_j} \bar{g}_{\phi_0 \phi_j} = 0. \tag{102}\]

The $r \to \infty$ limit of the $\bar{g}_{\phi_0 \phi_j}$ components reads $\bar{g}_{\phi_0 \phi_j} = \frac{x^2}{x + \Lambda x^2} \delta_{ij}$, where $i = 1, \ldots, N$. Then one finds, $F^{\mu j} \partial_{\mu_j} W \partial_{\mu_j} W = 0$. So, the third line in equation (101) has no contribution to the energy. Adding the first two terms and the second line, the non-vanishing terms, we arrive at

\[
\left( R^0_{\beta \sigma} \right)^{(1)} \nabla^\beta \bar{z}^\sigma = 2 M \Lambda r^{2-D} (D - 3) (W (D - 1) - 1). \tag{103}\]
Substituting this result in (10), one can express the energy as

\[ Q = \frac{M}{2G_D \Omega_{D-2}} \int_{\partial \Sigma} dS_r r^{2-D} \left( W(D-1) - 1 \right), \]  

(104)

where one can write

\[ W(D-1) - 1 = D - 2 - (D-1) \sum_{i=1}^{N} \frac{\Lambda \mu_i^2 a_i^2}{1 + \Lambda a_i^2}, \]  

(105)

and in the even dimensional case the solid angle reads

\[ \Omega_{D-2} = \frac{2 \pi^{\frac{D}{2} - 1}}{(D-3)!!}. \]  

(106)

Taking \( G_D = 1 \) and inserting the \( \sqrt{-g} \), it was given in equation (52) in Appendix A, one has

\[ Q = \frac{M(D - 3)!!}{2^{D+1} \pi^{\frac{D}{2} - 1}} \int_{-1}^{1} \prod_{i=1}^{N} \frac{\mu_i d\mu_i}{\sqrt{1 - \sum_{i=1}^{N} \mu_i^2}} \left( D - 2 - (D-1) \sum_{j=1}^{N} \frac{\Lambda \mu_j^2 a_j^2}{1 + \Lambda a_j^2} \right) \int_{0}^{2\pi} \prod_{k=1}^{N} d\phi_k, \]  

(107)

where the \( \phi_k \) integrals yield \((2\pi)^N = (2\pi)^{\frac{D}{2} - 1}\). Defining the function \( \Xi \)

\[ \Xi \equiv \prod_{i=1}^{N} \left( 1 + \Lambda a_i^2 \right), \]  

(108)

we can express the energy corresponds to the Killing vector \( \bar{\xi}^\mu = (-1, \bar{0}) \) as

\[ Q = \frac{M(D - 3)!!}{4 \Xi} \int_{-1}^{1} \prod_{i=1}^{N} \frac{\mu_i d\mu_i}{\sqrt{1 - \sum_{i=1}^{N} \mu_i^2}} \left( D - 2 - (D-1) \sum_{j=1}^{N} \frac{\Lambda \mu_j^2 a_j^2}{1 + \Lambda a_j^2} \right). \]  

(109)

To take the integral, use the equations (53, 69) given in Appendix B. We can write

\[ \int_{-1}^{1} \prod_{i=1}^{N} \frac{\mu_i d\mu_i}{\sqrt{1 - \sum_{i=1}^{N} \mu_i^2}} = \frac{2}{(2N - 1)!!} = \frac{2}{(D-3)!!}, \]  

(110)

and also

\[ \int_{-1}^{1} \prod_{i=1}^{N} \frac{\mu_i d\mu_i}{\sqrt{1 - \sum_{i=1}^{N} \mu_i^2}} \sum_{j=1}^{N} \frac{\Lambda \mu_j^2 a_j^2}{1 + \Lambda a_j^2} = \frac{4}{(2N + 1)!!} \sum_{j=1}^{N} \frac{\Lambda a_j^2}{1 + \Lambda a_j^2} = \frac{4}{(D-1)!!} \sum_{j=1}^{N} \frac{\Lambda a_j^2}{1 + \Lambda a_j^2}. \]  

(111)

Then, the energy can be written as

\[ E = \frac{M}{\Xi} \left( \frac{D - 2}{2} - \sum_{j=1}^{N} \frac{\Lambda a_j^2}{1 + \Lambda a_j^2} \right), \]  

(112)

where we can express

\[ \frac{D - 2}{2} = \sum_{j=1}^{N} \frac{1 + \Lambda a_j^2}{1 + \Lambda a_j^2}. \]  

(113)
Finally, for the even dimensional case one ends up with

\[ E = \frac{M}{\Xi} \sum_{i=1}^{N} \frac{1}{\Xi_i} \]  

(114)

where we have defined the \( \Xi_i \) as follows

\[ \Xi_i = 1 + \Lambda \alpha_i^2. \]  

(115)

Similarly, one can compute the energy of the odd dimensional Kerr-Ads black holes. Combining the results, we arrive at the energy of the \( D \)-dimensional rotating black hole as

\[ E = \frac{M}{\Xi} \sum_{i=1}^{N} \left( \frac{1}{\Xi_i} - \frac{1}{2} (1 - \epsilon) \right). \]  

(116)

To compute the angular momentum, one needs to perform a similar computation. This time, we consider the Killing vector \( \xi^{\mu}_{(i)} = (0, ..., 0, 1_i, 0, ..., 0) \), where the only non-zero term is the \( i \)th \( \phi \) component. One has

\[ \nabla^\beta \xi^\sigma = \frac{1}{2} (g^{\beta \gamma} g^{\tau \phi_j} - g^{\beta \phi_j} g^{\tau \nu}) \partial_\nu g_{\phi_i \phi_j}, \]  

(117)

where only \( r \) and \( \mu_i \) derivatives of the \( g_{\phi_i \phi_j} \) component survive. Then we obtain

\[ \left( R^{r0} \right)_{\beta \sigma}^{(1)} \nabla^\beta \xi^\sigma = \frac{1}{2} \partial_r g_{\phi_i \phi_j} \left( R^{\phi_i 0 \rho} h^r - R^{\phi_j \rho r} h^0 \right) + \frac{1}{2} \partial_\rho g_{\phi_i \phi_j} \left( R^{\mu_i \phi_j 0 \rho} h^r - R^{\mu_j \phi_i \rho} h^0 \right) \]

\[ + \frac{1}{2} \left( g^{\rho \phi_j} g^{0 \phi_i \phi_j} \partial_r g_{\phi_i \phi_j} \right) \left( \nabla_r \nabla_0 h_{\phi_k r} + \nabla_{\phi_k} \nabla_r h_{0 r} - \nabla_r \nabla_{\phi_k} h_{0 0} \right) \]

\[ + \frac{1}{2} g^{\rho \phi_j} g^{0 \phi_i \phi_j} g_{\mu_i \mu m} \partial_{\mu m} g_{\phi_i \phi_j} \left( \nabla_{\mu_i} \nabla_0 h_{\phi_k r} + \nabla_{\phi_k} \nabla_{\mu_i} h_{0 0} - \nabla_{\phi_k} \nabla_{0 \mu r} \right). \]  

(118)

The first four terms on the right hand side of the last equation reads

\[ \frac{1}{2} \partial_r g_{\phi_i \phi_j} \left( R^{\phi_j 0 \rho} h^r - R^{\phi_i \rho r} h^0 \right) + \frac{1}{2} \partial_\rho g_{\phi_i \phi_j} \left( R^{\mu_j \phi_i 0 \rho} h^r - R^{\mu_i \phi_j \rho} h^0 \right) = 2 M \Lambda r^{2-D} k_{\phi_i}. \]  

(119)

Let us compute the remaining terms. The first piece in the second line yields

\[ \nabla_r \nabla_0 h_{\phi_k r} = -\partial_r \left( \Gamma_{00}^0 h_{\phi_k 0} \right) + \Gamma_{0r}^0 \Gamma_{00}^0 h_{\phi_k 0} + \Gamma_{r0}^0 \Gamma_{00}^0 h_{\phi_k 0} + \Gamma_{rr}^0 \Gamma_{00}^0 h_{\phi_k 0}, \]  

(120)

and the second one gives

\[ \nabla_{\phi_k} \nabla_r h_{0 r} = -\Gamma_{00}^0 \partial_r h_{\phi_k 0} + 2 \Gamma_{0r}^0 \Gamma_{r0}^0 h_{\phi_k 0} + \Gamma_{r0}^0 \Gamma_{00}^0 h_{\phi_k 0}. \]  

(121)

The third and the fourth terms respectively read

\[ \nabla_r \nabla_r h_{\phi_k 0} = \partial_r \partial_r h_{\phi_k 0} - \partial_r \left( \Gamma_{00}^0 h_{\phi_k 0} \right) - \partial_r \left( \Gamma_{0r}^0 h_{\phi_k 0} \right) - \Gamma_{r0}^0 \partial_r h_{\phi_k 0} - \Gamma_{0r}^0 \partial_r h_{\phi_k 0} - \Gamma_{r0}^0 \partial_r h_{\phi_k 0} + \Gamma_{r0}^0 \Gamma_{0r}^0 h_{\phi_k 0} + \Gamma_{rr}^0 \Gamma_{00}^0 h_{\phi_k 0} + 2 \Gamma_{r0}^0 \Gamma_{00}^0 h_{\phi_k 0} + \Gamma_{r0}^0 \Gamma_{00}^0 h_{\phi_k 0}, \]  

(122)

and

\[ \nabla_{\phi_k} \nabla_0 h_{r r} = 2 \Gamma_{r0}^0 \Gamma_{00}^0 h_{\phi_k 0}. \]  

(123)
Substituting the pieces, we obtain

\[ \nabla_r \nabla_0 h_{\phi r} + \nabla_{\phi k} \nabla_r h_{0r} - \nabla_r \nabla_r h_{\phi 0} - \nabla_{\phi k} \nabla_0 h_{rr} = \Gamma^0_{r0k} \Gamma^0_{\phi r} h_{\phi 0} - \partial_r \partial_r h_{\phi 0} \]  
(124)

\[ + h_{\phi 0} \partial_r \Gamma^0_{r0k} + \Gamma^0_{r0k} \partial_r h_{\phi 0} + \Gamma^0_{0r} \partial_r h_{\phi 0} - \Gamma^0_{r0k} \Gamma^0_{0r} h_{\phi 0} - \Gamma^0_{r0k} \Gamma^0_{0r} h_{\phi 0}. \]

From the equations (4) and (6) one has

\[ h_{\phi 0} = 2MW^{3-D}k_\phi. \]  
(125)

Using \( \partial_r \phi_{\phi r} = 2\Gamma^0_{r0} \phi_{\phi r} \) and taking the \( r \) and \( \mu_i \) derivatives, one arrives at

\[ \nabla_r \nabla_0 h_{\phi r} + \nabla_{\phi k} \nabla_r h_{0r} - \nabla_r \nabla_r h_{\phi 0} - \nabla_{\phi k} \nabla_0 h_{rr} = 2MW^{1-D}k_k(-D^2 + 4D - 4). \]  
(126)

Since one has \( (\sqrt{g})^2 \phi_{\phi r} \phi_{\phi r} \nabla_r \phi_{\phi r} = 2W^{-1}r\phi_{\phi r} \), the second line in (118) can be written as

\[ \frac{1}{2} (\sqrt{g})^2 \phi_{\phi r} \phi_{\phi r} \nabla_r \phi_{\phi r} \nabla_0 h_{\phi r} + \nabla_{\phi k} \nabla_r h_{0r} - \nabla_r \nabla_r h_{\phi 0} - \nabla_{\phi k} \nabla_0 h_{rr} = 2MA^{r-3}k_k(-D^2 + 4D - 4). \]  
(127)

Now, let us compute the last line in (118). We have

\[ \nabla_{\mu} \nabla_0 h_{\phi r} = -\partial_\mu (\Gamma^0_{r0} h_{\phi 0}) + \Gamma^0_{0\mu} \Gamma^0_{r0} h_{\phi 0} + \Gamma^0_{0\mu} \Gamma^0_{r0} h_{\phi 0} + \Gamma^0_{0\mu} \Gamma^0_{r0} h_{\phi 0}. \]  
(128)

and

\[ \nabla_{\phi k} \nabla_r h_{0\mu} = \Gamma^0_{r0k} \Gamma^0_{\mu\phi r} h_{\phi 0} - \Gamma^0_{\mu\phi r} \partial_r h_{\phi 0} + \Gamma^0_{\mu\phi r} \Gamma^0_{0\phi r} h_{\phi 0} + \Gamma^0_{\mu\phi r} \Gamma^0_{0\phi r} h_{\phi 0}. \]  
(129)

and also

\[ \nabla_{\mu_{\alpha-1}} \nabla_r h_{\phi 0} = \partial_\mu \partial_r h_{\phi 0} - \partial_\mu (\Gamma^0_{r0} h_{\phi 0}) - \partial_\mu (\Gamma^0_{r0} h_{\phi 0}) - \Gamma^0_{r0k} \partial_r h_{\phi 0} + \Gamma^0_{r0k} \partial_r h_{\phi 0} + \Gamma^0_{r0k} \partial_r h_{\phi 0} + \Gamma^0_{r0k} \partial_r h_{\phi 0}. \]  
(130)

Collecting the results, we obtain

\[ \nabla_{\mu} \nabla_0 h_{\phi r} + \nabla_{\phi k} \nabla_r h_{0\mu} - \nabla_{\mu} \nabla_r h_{\phi 0} - \nabla_{\phi k} \nabla_0 h_{\mu r} = \Gamma^0_{r0k} \Gamma^0_{\mu\phi r} h_{\phi 0} - \partial_\mu \partial_r h_{\phi 0}, \]

which yields

\[ \nabla_{\mu} \nabla_0 h_{\phi r} + \nabla_{\phi k} \nabla_r h_{0\mu} - \nabla_{\mu} \nabla_r h_{\phi 0} - \nabla_{\phi k} \nabla_0 h_{\mu r} = M r^{2-D} (D-1)(2W \partial_\mu h_{\phi 0} + k_\phi \partial_\mu W) \]  
(133)

using (125). To compute the last line, we use

\[ \frac{1}{2} (\sqrt{g})^2 \phi_{\phi r} \phi_{\phi r} \nabla_\mu \phi_{\phi r} = -\frac{1}{2W} \phi_{\phi r} \phi_{\phi r} \nabla_\mu \phi_{\phi r}. \]  
(134)
and then, we can express
\[
\frac{1}{2} \eta^{0 \beta \gamma} \phi_0 \eta^{\mu \nu} \partial_{\mu \nu} \phi_0 \phi_j \left( \nabla_{\mu} \nabla_0 h_{\phi_k r} + \nabla_{\phi_k} \nabla_r h_{0 \mu} - \nabla_{\mu} \nabla_r h_{\phi_k 0} - \nabla_{\phi_k} \nabla_0 h_{\mu r} \right)
\]
\[
= - \frac{M}{2W} r^{2-D} (D-1) \eta^{0 \beta \gamma} \eta^{\mu \nu} \partial_{\mu \nu} \phi_0 \phi_j (2W \partial_{\mu} k_{\phi_k} + k_{\phi_k} \partial_{\mu} W).
\]
(135)

Remember the equation (102), which reads
\[
\eta^{0 \beta \gamma} \partial_{\mu} \phi_0 \phi_j \eta^{\mu \nu} \partial_{\mu \nu} \phi_0 = 0.
\]
So, the last term in the last expression vanishes. After a straightforward calculation, one can show the vanishing of the first one too. Then, the last line in (118) has no contribution to the angular momentum. Inserting the results, we find
\[
\left( R^0_{\beta \sigma} \right)^{(1)} \nabla^\beta \tilde{\xi}^\sigma = -2 M \Lambda r^{2-D} (D-3)(D-1) k_{\phi_i},
\]
where we have \( k_{\phi_i} = -\frac{a_i \mu_i^2}{1 + \Lambda a_i^2} \), from the equation (106). So then, the integrand yields
\[
\left( R^0_{\beta \sigma} \right)^{(1)} \nabla^\beta \tilde{\xi}^\sigma = 2 M \Lambda r^{2-D} (D-3)(D-1) \frac{a_i \mu_i^2}{1 + \Lambda a_i^2},
\]
and the angular momentum becomes
\[
J_i = \frac{M(D-1)!}{4} \int_{-1}^{1} \prod_{j=1}^{N} \frac{\mu_j d\mu_j}{(1 + \Lambda a_j^2)} \frac{a_i \mu_i^2}{1 - \sum_{k=1}^{N} \mu_k^2}.
\]
(136)
(137)

Using the definition of \( \Xi \), given in (108), one can express the angular momentum as
\[
J_i = \frac{M(D-1)!}{4 \Xi} \frac{a_i}{1 + \Lambda a_i^2} \int_{-1}^{1} \prod_{j=1}^{N} \frac{\mu_j d\mu_j}{\sqrt{1 - \sum_{k=1}^{N} \mu_k^2}} \mu_i^2.
\]
(138)
(139)

From equation (73) in Appendix B, the integral reads
\[
\int_{-1}^{1} \prod_{j=1}^{N} \frac{\mu_j d\mu_j}{\sqrt{1 - \sum_{k=1}^{N} \mu_k^2}} \mu_i^2 = \frac{4}{(2N+1)!} = \frac{4}{(D-1)!}.
\]
(140)

One ends up with the angular momentum of the even dimensional Kerr-AdS black holes corresponding to the Killing vector \( \xi_{(i)}^\mu = (0, ..., 0, 1_i, 0, ..., 0) \) as
\[
J_i = \frac{M a_i}{\Xi \Xi_i}.
\]
(141)

in cosmological Einstein’s gravity. Note that, in odd spacetime dimensions one also ends up with the same expression.

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