The Earliest Detection of a Gradual Change in the Properties of Random Sequences According to the Information of the Meter and the Indicator of the Accompanying Feature

G G Sebryakov, V I Pavlov, S M Muzhichek, P A Motlich, A A Skrynnikov and O V Yermolin

State Research Institute of Aviation Systems, Moscow, 125319, Russia

E-mail: msm19@yandex.ru

Abstract. There is considered a Bayesian procedure for the sequential detection of a “breakdown” of random sequences during registration together with the initial data of the accompanying attribute. The article presented that it is possible and reasonable to use additional information from non-inertia indicator for the determination of the moment of stopping and making decision of an imbalance.

1. Introduction

The task of detecting a gradual change in the properties of random sequences, or the task of detecting gradual discrepancy in process, occurs in many areas, e.g. location of moving objects, state estimation of technical devices deteriorating due to aging and wear of materials, when diagnosing the patients in medicine, etc.

It is obvious that it is necessary to detect a random moment of the discrepancy in the shortest possible time. This requirement is most satisfied with the algorithms constructed with the sequential detection methods. In the overwhelming majority of studies on the discrepancy detection [1-5], only the sequence of observed data is used without additional information. At the same time, the disorder of the observed sequence is usually associated with a change in parameters involved in the generation of data. For example, a change in the parameters of an aircraft path may be due to a change in the mode of engine operation, or a reorientation of the aircraft relative to the center of mass, or a change in the direction of the wind etc.

In many practical cases, along with tracking the informative component of the sequence it is possible to observe changes in the accompanying parameter stochastically associated with a estimated disorder in the basic data sequence. Thus, when using a radar station to determine the changing parameters of the maneuvering aircraft path, the power of the reflected radar signal will also change. Thus, in addition to the data sequence observer, it is possible to use devices which output signals are of different physical nature, but relate stochastically to the estimated disorder in the observed sequence. Such devices it is advisable to call accompanying attribute indicators. The present work on the basis of the general solution given in [6-7], solves the problem of sequential detection of gradual disassembly using the basic data sequence observer and an accompanying attribute indicator.
2. Formulation of the problem
Suppose that for a multi-step process, the output $Z_n$ is observed,

$$Z_n = \phi_j(x_n) + W_n, \quad j = 0, 1; \quad n = 1, N,$$

(1)

where $x_n$ – source data sequence; $\phi_j$ – known homogeneous functions that differ from each other by the value of one of the parameters (for linear functions by the angular coefficient; for harmonic functions, either by amplitude, or frequency, or phase, etc.), $j = 0$ corresponds to the original data in the absence of disorder, $j = 1$ – corresponds to the original data in the presence of a disorder; $\xi_n$ – white discrete centered Gaussian noises of unit intensity; $W$ – noise intensity value; $n$ – numbers of discrete moments of time separated by equal intervals $\Delta t$; $N$ – the maximum number of steps of observation in one series, as well as the output signal of the indicator $Y_n$ of the accompanying feature

$$Y_n = \Psi_{s,n}(Q),$$

(2)

associated with the attribute $Q$ by the operator $\Psi_{s,n}$.

Two values of $s$ correspond to the output signal of the indicator: $s = 0$ corresponds to the absence of the accompanying attribute $Q$ and $s = 1$ – to the presence of the attribute $Q$ when observing the initial data $x_n$.

The operator $\Psi_{s,n}$ is given by the conditional probability of transition

$$\Psi_{s,n} = \Psi(s_n, n) | s_{n-1}, Z_{n-1}, Q, n-1), \quad s = 0, 1; \quad n = 1, N$$

(3)

from the state $s_{n-1}$ to the state $s_n$ [8].

When observing a $Z_n$ sequence, there are two hypotheses: $\Theta = 0 \cup \Theta = 1$, and the hypothesis $\Theta = 1$ means a sequence mismatch that occurs at a random time $k$, $1 < k < \infty$ with probability $P < 1$. Here the mismatch is a gradual change in the properties of a sequence, for example, the change in the mean. We assume that we know the a priori probability of the mismatch $P_{\Theta = 1} = P_1$, and the conditional probability densities of observations $p_\Theta(Z_n) = p(Z_n | \Theta = 0)$, $p_\Theta(Z_n | k) = p(Z_n | \Theta = 1)$ in the absence of the mismatch and its occurrence at the moment $k$, respectively. Observations both before and after the occurrence of the mismatch are independent, and the transition process time is shorter than the time interval $\Delta t$ between successive samples [6]. It is required to make a decision on the presence or absence of a mismatch of the sequence $x_n$ in a sequential manner, using the output signals of the observer (1) and the indicator (2).

3. Building a detection procedure
The use of additional information from the indicator is possible in two ways: the first is to control the threshold value with which a posteriori mismatch probability is compared, calculated out of the observer signal; the second is to calculate the a posteriori probabilities of hypotheses about the presence or absence of a mismatch based on the output signals of the observer and the indicator. This article discusses the first way.

Let the losses that occur when making a decision $u_n = j$ about the absence ($j = 0$) or the presence ($j = 1$) of a mismatch (discrepancy) at the $n$-th step have the form

$$g(\Theta, u_n, k, n) = \begin{cases} 
  g_{0j}(n) & \text{at } \Theta = 0, n < N, \\
  g_{1j}(n) + C(n-k) & \text{at } \Theta = 0, n < N, \\
  \bar{g}_{0j}(n) & \text{at } \Theta = 0, n = N, \\
  \bar{g}_{1j}(n) & \text{at } \Theta = 1, n = N, 
\end{cases}$$

(4)

where $g_{0j}(n)$ – is the loss associated with the correct non-detection of a discrepancy – the cost of observations; $g_{01}(n)$ – losses caused by false alarm; $g_{11}(n)$ – losses associated with the correct
detection of a discrepancy; $C$ – losses caused by delay in detecting a disorder by one step: $g_{10}(n)$ – losses arising due to the skip of actual mismatch; $g_{0j}(N)$ and $g_{ij}(N)$ – are losses similar in content to $g_{0j}(n)$ and $g_{ij}(n)$, but related to decision making at the last step of the current series of observations and passing to the next one. In the case of gradual degradation due to the smooth change in the characteristics of technical devices for various reasons (wear, aging, changes in the temperature mode of operation, etc.), as a rule, the following ratios are true

\[
\begin{align*}
g_{00}(n) & \ll g_{01}(n) < g_{10}(n); \\
g_{11}(n) + C(n-k) & \ll g_{01}(n) < g_{10}(n); \\
\tilde{g}_{11}(N) & \ll \tilde{g}_{10}(N) < \tilde{g}_{01}(N); \\
\tilde{g}_{0j}(N) & < g_{0j}(n); \tilde{g}_{1j}(n) < \tilde{g}_{1j}(N).
\end{align*}
\]

An optimal Bayesian sequential rule for detecting a discrepancy of a $Z_n$ sequence, $n=1, \ldots, N$, minimizing the average risk $r_n(n) = M[g(\Theta, u_n, k, n)]$, can be found by minimizing the a posteriori risk $r_n(Z_n, u_{n-1}) = r_n(Z_n)$ by choosing $j \in \{0,1\}$. The dependence of the a posteriori risk on $u_t, \ldots, u_{n-1}$ is not taken into account, since all these decisions are connected with the continuation of observations due to the absence of a discrepancy in this series.

\[
r_n(Z_n) = \min \left\{ \inf_{u \in U_{\text{comp}}} M\left[ g(\Theta, u_n, k, n) \mid Z_n, u_n \right], \inf_{u \in U_{\text{cont}}} M\left[ r_n(Z_{n+1}, u_n) \mid Z_n, u_n \right] \right\}
\]

(6)

where $U_{\text{comp}}$ and $U_{\text{cont}}$ – areas of decisions associated with the completion and continuation of observations, respectively.

Using the formula for the total expectation, as well as the loss function (4) and its properties (5), we can determine that

\[
M\left[ g(\Theta, u_n, k, n) \mid Z_n, u_n \right] =
\begin{cases}
g_{10}(n)P_{1,n} + g_{00}(n) & \text{at } u_n = 0, \\
g_{01}(n)(1 - P_{1,n}) + g_{00}(n) & \text{at } u_n = 1,
\end{cases}
\]

(7)

where $P_{1,n} = P(\Theta = 1 \mid Z_n)$ – a posteriori probability of mismatch. The solution $\delta_n(Z_n) = u_n \in U_{\text{comp}}$, at which statistics (7) takes the smallest value

\[
\inf_{u \in U_{\text{comp}}} M\left[ g(\Theta, u_n, k, n) \mid Z_n, u_n \right] = q(P_{1,n}) + g_{00}(n),
\]

(8)

\[
q(P_{1,n}) = \min \left\{ g_{10}(n)P_{1,n}, g_{01}(n)(1 - P_{1,n}) \right\}
\]

(9)

is

\[
\delta_n(Z_n) = \begin{cases} j = 0 & \text{at } g_{10}(n)P_{1,n} < g_{01}(n)(1 - P_{1,n}), \\
& \text{at } g_{10}(n)P_{1,n} > g_{01}(n)(1 - P_{1,n}).
\end{cases}
\]

(10)

The dependences similar to (7)-(10) will be valid for the second component to be minimized in (6) at the final time instant $N$ of the current observation series

\[
\rho_N(Z_n) = M\left[ r_n(Z_{n+1}, u_n) \mid Z_n, u_n \right] =
\]

\[
= \min \left\{ q(P_{1,n}) + \tilde{g}_{00}(N), M\left[ r_n(Z_{n+1}, u_n) \mid Z_n, u_n \right] \right\}, \quad n = N-1, \ldots, 1,
\]
According to the physical meaning, $\rho_n(Z_n)$ is the smallest future a posteriori risk. The moment of termination of observations is defined as

$$
\tau_{\text{comp}} = \inf \{ n \Delta : q(P_{1,n}) = \rho_n(Z_n) \}.
$$

The optimal rule for detecting gradual mismatch can be represented as [6]

$$
\delta^* = \begin{cases} 
  u_n & \text{at } n \Delta < \tau_s; \\
  j = 0 & \text{at } n \Delta = \tau_s, \; P_{1,n} < c_j; \\
  j = 1 & \text{at } n \Delta = \tau_s, \; P_{1,n} > c_j,
\end{cases}
$$

where the threshold value $c_j$ at the time of the termination of observations is defined as

$$
c_j = g_{10}(n) / \left[ g_{10}(n) + g_{00}(n) \right].
$$

Whatever the solutions in (12) might be, a new series of observations $n = 1, N$ begins after time (11).

The posterior probability $P_{1,n}$ of the mismatch of the $Z_n$ sequence in the $n$-th step is determined on the basis of the Bayes formula

$$
P_{1,n} = \frac{P_{1,n-1} \Lambda_n}{\left[ P_{1,n-1} \Lambda_n + (1 - P_{1,n-1}) \right]}, \quad n = 1, N, \; P_1(n = 1) = P_{1,1},
$$

where $\Lambda_n = p_{1,n}(Z_n | Z_{n-1}) / p_{0,n}(Z_n | Z_{n-1})$.

If the indicator of the accompanying attribute is inertia-free, then we can assume that if at moment $k$ the indicator output changes the mismatch occurs with probability $P(\Theta = 1) = 1$. In this case we may truncate the detection rule. The posterior probability of mismatch (14) must be compared with the new threshold value

$$
c_j = \tilde{g}_{10}(N) / \left[ \tilde{g}_{10}(N) + \tilde{g}_{00}(N) \right],
$$

and the rule for detecting mismatch will be

$$
\delta_{\text{trunc}}^* = \begin{cases} 
  j = 0 & \text{at } P_{1,k} < c_{\text{trunc}}; \\
  j = 1 & \text{at } P_{1,k} > c_{\text{trunc}},
\end{cases}
$$

Comparing the values of the thresholds (13) and (15) and taking into account relations (5) we can conclude that the use of information from the indicator of the accompanying attribute leads to a reduction in the delay in determining frustration due to a lower threshold.

4. Conclusion

The procedure of detecting the gradual discrepancy in random sequences is developed in Bayesian formulation. It makes possible to reduce the delay in detection by using additional information from the indicator of the accompanying attribute. The value of the delay depends on the losses assigned by the physical sense to the wrong decisions and on the dynamic characteristics of the indicator. The indicator is considered to be non-inertia. The procedure is developed for the case of an non-controlled observation process and a low measurement cost. It is worth noting that the presence of a accompanying attribute in the source data does not necessarily mean that a discrepancy is present, and, conversely, a discrepancy may occur if the attribute is not registered. In this regard, it is advisable
when analyzing long-term processes, to pre-assign the duration of a series of observations. There must be expected a definite gain in using several indicators of related traits. When using each specific indicator, due to its “correlation” with the process under study, the loss function (5), which affects the threshold (15), will be individual.

References
[1] Bakut P, Zhulina, Yu and Ivanchuk N 1980 Detection of moving objects (Moscow: Soviet radio) (In Russian)
[2] Maltsev A and Silaev A 1985 Detection of abrupt changes in parameters and optimal estimation of the state of discrete dynamic systems Automation and Remote Control 1 48–58
[3] Kligene N and Telksnis L 1983 Methods for detecting moments of change in the properties of random processes (review) Automation and Remote Control 10 5–56
[4] Sosulin Y and Fishman M 1985 Theory of Sequential Decisions and Its Applications (Moscow: Radio and communication) (In Russian)
[5] Shiryaev A 1996 Minimax optimality of the cumulative sum method (CUSUM) in the case of continuous time Russian Mathematical Surveys 51 173–174
[6] Tartakovsky A 1987 Optimal detection of changes in the properties of random sequences. Sequential Detection Automation and Remote Control 7 106–113
[7] Pavlov V 1998 Optimal detection of changes in the properties of random sequences according to the information of the meter and indicator Automation and Remote Control 1 54–59
[8] Bukhalev V 1992 Recurrent algorithms for recognition and estimation of the state of a dynamic object using information from gauges and indicators The Journal of Computer and System Sciences International 1 148–156